1 Inputs and Outputs

As input, we take an undirected graph $G = (V_G, E_G)$. Each vertex $s \in V_G$ represents a node in a network and is annotated with location information. We therefore can compute quantities like $\mathrm{GCL}(s, s')$, the Great Circle Latency between s and s'.

Additionally, an edge $\{s, s'\} \in E_G$ has an associated measured Round Trip Time $\mathrm{RTT}(s, s')$. In practice, latencies are collected for almost every pair of nodes, so G is nearly complete.

We also take several hyperparameters. First is the residual latency threshold ϵ. Next are weighting parameters for each component of the loss function. Finally, there are a few hyperparameters describing the output.

We return a triangle mesh $M = (V_M, E_M)$, stored in a doubly connected edge list format. As hinted at above, the actual vertex-edge connectivity is to be selected before running the optimization. That said, each vertex has coordinates in $\mathbb{R}^3$ which are to be chosen by the optimization algorithm. For the purposes of the algorithm and mesh regularity, we parameterize each vertex position with a single number. In our current implementation, each vertex $v \in V_M$ can be broken into parts as (p_v, z_v), where p_v is a latitude-longitude pair, and z_v is an altitude. The optimization algorithm then determines the best z values.

Note that the above parameterization implies we can map vertices $s \in V_G$ to positions on the mesh $\pi(s)$. For the purposes of the following computations, assume the stronger statement $\pi(s) \in V_M$. While this assumption is not strictly necessary, it significantly simplifies the geodesic distance computation. Furthermore, provided our mesh is fine enough, the stronger assumption will lead to minimal numerical error.

2 Laplacian

Here and in the subsequent sections, computations can take serious advantage of vectors and matrices. Therefore, while notationally inelegant, we will assign indices to the vertices in V_M.

On that note, if i and j are two indices for which $(v_i, v_j) \in E_M$, let $\mathrm{nxt}(i, j)$ be the index such that $v_i \to v_j \to v_{\mathrm{nxt}(i,j)}$ traces a triangle counterclockwise. Note that this index exists and is unique assuming we have a mesh without boundary. On a mesh with boundary, if no $\mathrm{nxt}(i, j)$ exists, then the half-edge (v_i, v_j) lies on the boundary.

We also write ∂M to represent the boundary of our mesh. Abusing notation, we can write things like $v_i \in \partial M$ or $(v_i, v_j) \in \partial M$.

We define the following variables:

$N_{i,j}$	Outward normal of triangle $v_i \to v_j \to v_{\mathrm{nxt}(i,j)}$
$A_{i,j}$	Area of triangle $v_i \to v_j \to v_{\mathrm{nxt}(i,j)}$
$D_{i,j}$	Vertex triangle areas; diagonal
$\theta_{i,j}$	Measure of $\angle v_i v_{\mathrm{nxt}(i,j)} v_j$
L_C^{N}	Cotangent operator with zero-Neumann boundary condition
L_C^{D}	Cotangent operator with zero-Dirichlet boundary condition

2.1 Forward Computation

We have the following (standard) definition of the Laplace-Beltrami operator on a mesh:

$$
\begin{aligned}
N_{i,j} &= \left(v_i - v_{\mathrm{nxt}(i,j)}\right) \times \left(v_j - v_{\mathrm{nxt}(i,j)}\right), \\
A_{i,j} &= \frac{1}{2}\left\|N_{i,j}\right\|_2, \\
D_{i,j} &= \begin{cases} \frac{1}{3} \displaystyle\sum_{\substack{k \\ (v_i, v_k) \in E_M}} A_{i,k} & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}
$$

$$\cot\left(\theta_{i,j}\right) = \frac{\left(v_i - v_{\mathrm{nxt}(i,j)}\right)\cdot\left(v_j - v_{\mathrm{nxt}(i,j)}\right)}{2A_{i,j}},$$

$$\left(L_C^{\mathrm{N}}\right)_{i,j} = \begin{cases} \frac{1}{2}\cot\left(\theta_{i,j}\right) & \text{if } \left(v_i, v_j\right) \in \partial M, \\ \frac{1}{2}\cot\left(\theta_{j,i}\right) & \text{if } \left(v_j, v_i\right) \in \partial M, \\ \frac{1}{2}\left(\cot\left(\theta_{i,j}\right) + \cot\left(\theta_{j,i}\right)\right) & \text{if } \left(v_i, v_j\right), \left(v_j, v_i\right) \in E_M, \\ -\frac{1}{2}\left(\sum_{\substack{k \\ \left(v_i, v_k\right)\in E_M}} \cot\left(\theta_{i,k}\right) + \sum_{\substack{k \\ \left(v_k, v_i\right)\in E_M}} \cot\left(\theta_{k,i}\right)\right) & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

$$\left(L_C^{\mathrm{D}}\right)_{i,j} = \begin{cases} \frac{1}{2}\left(\cot\left(\theta_{i,j}\right) + \cot\left(\theta_{j,i}\right)\right) & \text{if } \left(v_i, v_j\right) \in E_M,\ v_i \notin \partial M, \text{ and } v_j \notin \partial M, \\ -\frac{1}{2}\sum_{\substack{k \notin \partial M \\ \left(v_i, v_k\right)\in E_M \\ \left(v_k, v_i\right)\in E_M}} \left(\cot\left(\theta_{i,k}\right) + \cot\left(\theta_{k,i}\right)\right) & \text{if } i = j \text{ and } v_i \notin \partial M, \\ 0 & \text{otherwise.} \end{cases}$$

2.2 Reverse Computation

We compute

$$\frac{\partial v_i}{\partial z_\ell} = \begin{cases} e_3 & \text{if } \ell = i, \\ 0 & \text{otherwise,} \end{cases}$$

$$\frac{\partial N_{i,j}}{\partial z_\ell} = \begin{cases} \left(v_{\mathrm{nxt}(i,j)} - v_j\right) \times \frac{\partial v_\ell}{\partial z_\ell} & \text{if } \ell = i, \\ \left(v_i - v_{\mathrm{nxt}(i,j)}\right) \times \frac{\partial v_\ell}{\partial z_\ell} & \text{if } \ell = j, \\ \left(v_j - v_i\right) \times \frac{\partial v_\ell}{\partial z_\ell} & \text{if } \ell = \mathrm{nxt}(i,j), \\ 0 & \text{otherwise,} \end{cases}$$

$$\frac{\partial A_{i,j}}{\partial z_\ell} = \frac{1}{4A_{i,j}} N_{i,j} \cdot \frac{\partial N_{i,j}}{\partial z_\ell},$$

$$\left(\frac{\partial D}{\partial z_\ell}\right)_{i,j} = \begin{cases} \frac{1}{3}\sum_{\substack{k \\ \left(v_i, v_k\right)\in E_M}} \frac{\partial A_{i,k}}{\partial z_\ell} & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

$$\frac{\partial}{\partial z_\ell}\cot\left(\theta_{i,j}\right) = \begin{cases} \dfrac{\left(v_j - v_{\mathrm{nxt}(i,j)}\right)\cdot \frac{\partial v_\ell}{\partial z_\ell} - 2\cot\left(\theta_{i,j}\right)\frac{\partial A_{i,j}}{\partial z_\ell}}{2A_{i,j}} & \text{if } \ell = i, \\ \dfrac{\left(v_i - v_{\mathrm{nxt}(i,j)}\right)\cdot \frac{\partial v_\ell}{\partial z_\ell} - 2\cot\left(\theta_{i,j}\right)\frac{\partial A_{i,j}}{\partial z_\ell}}{2A_{i,j}} & \text{if } \ell = j, \\ \dfrac{\left(2v_{\mathrm{nxt}(i,j)} - v_i - v_j\right)\cdot \frac{\partial v_\ell}{\partial z_\ell} - 2\cot\left(\theta_{i,j}\right)\frac{\partial A_{i,j}}{\partial z_\ell}}{2A_{i,j}} & \text{if } \ell = \mathrm{nxt}(i,j), \\ 0 & \text{otherwise,} \end{cases}$$

$$
\left(\frac{\partial L_C^{\mathrm{N}}}{\partial z_\ell}\right)_{i,j} = \begin{cases} \frac{1}{2}\frac{\partial}{\partial z_\ell}\cot\left(\theta_{i,j}\right) & \text{if } \left(v_i, v_j\right) \in \partial M, \\ \frac{1}{2}\frac{\partial}{\partial z_\ell}\cot\left(\theta_{j,i}\right) & \text{if } \left(v_j, v_i\right) \in \partial M, \\ \frac{1}{2}\left(\frac{\partial}{\partial z_\ell}\cot\left(\theta_{i,j}\right) + \frac{\partial}{\partial z_\ell}\cot\left(\theta_{j,i}\right)\right) & \text{if } \left(v_i, v_j\right), \left(v_j, v_i\right) \in E_M, \\ -\frac{1}{2}\left(\displaystyle\sum_{\substack{k \\ \left(v_i, v_k\right) \in E_M}} \frac{\partial}{\partial z_\ell}\cot\left(\theta_{i,k}\right) + \sum_{\substack{k \\ \left(v_k, v_i\right) \in E_M}} \frac{\partial}{\partial z_\ell}\cot\left(\theta_{k,i}\right)\right) & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}
$$

$$
\left(\frac{\partial L_C^{\mathrm{D}}}{\partial z_\ell}\right)_{i,j} = \begin{cases} \frac{1}{2}\left(\frac{\partial}{\partial z_\ell}\cot\left(\theta_{i,j}\right) + \frac{\partial}{\partial z_\ell}\cot\left(\theta_{j,i}\right)\right) & \text{if } \left(v_i, v_j\right) \in E_M,\ v_i \notin \partial M, \text{ and } v_j \notin \partial M, \\ -\frac{1}{2}\displaystyle\sum_{\substack{k \notin \partial M \\ \left(v_i, v_k\right) \in E_M \\ \left(v_k, v_i\right) \in E_M}} \left(\frac{\partial}{\partial z_\ell}\cot\left(\theta_{i,k}\right) + \frac{\partial}{\partial z_\ell}\cot\left(\theta_{k,i}\right)\right) & \text{if } i = j \text{ and } v_i \notin \partial M, \\ 0 & \text{otherwise.} \end{cases}
$$

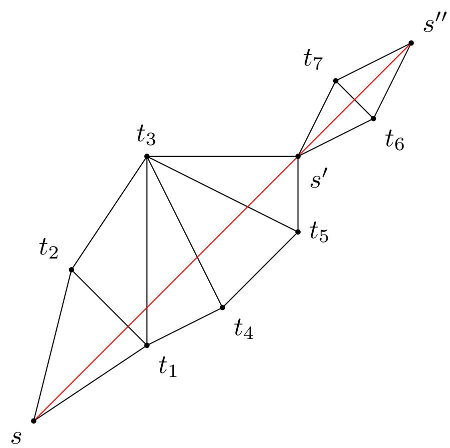

Figure 1: Example unfolded section of a mesh in black with a geodesic between s and s'' shown in red.

3 Geodesic Distance

For this section, due to complexities with the reverse computation, we only have a single variable definition:

$$
\phi \;\Big|\; \text{Vector of geodesic distances in } \mathbb{R}^{|E_G|}
$$

Notation-wise, we will talk about geodesic paths between s and s', both in E_G. This actually means we are interested in projecting s and s' to the mesh surface, moving them to the nearest mesh vertices, then finding the geodesic path between those two mesh vertices.

3.1 Forward Computation

We use MeshUtility's implementation of the fast marching method to compute geodesic paths between s and s' for every edge $\left\{s, s'\right\} \in E_G$. This method returns a sequence of vertices and edges the geodesic path passes through, as well as where across the edges the path passes. Automatically, this gives an easy way to compute the geodesic distance.

3.2 Reverse Computation

The reverse computation is unfortunately rather complicated and requires some casework. The overall strategy is to first compute the derivative of the geodesic distance between two vertices with respect to the length of any edge in E_M, and then use the chain rule to find the partials we actually want.

To start, a geodesic path is a straight line on an unfolded representation of the mesh, as in Figure 1. In fact, we can assign coordinates in $\mathbb{R}^2$ to the vertices such that the geodesic distance is the Euclidean distance between the start and end points. In the interest of keeping the notation in this section readable, we will use the name of vertices (s, t_1, etc.) to denote these two-dimensional coordinates.

Each geodesic path can be partitioned by the vertices it passes through. If we have the partials of the lengths of each of these segments with respect to each of the mesh's edges, then we automatically have the partials of the length of the entire geodesic simply by addition. In our example, the geodesic path passes through s, s', and s''. Our analysis below will focus on the segment between s and s'.

An extremely useful tool is the law of cosines, as well as its derivative. That is, for any three vectors u_1, u_2, and u_3, we have

$$\|u_2 - u_3\|_2^2 = \|u_1 - u_2\|_2^2 + \|u_1 - u_3\|_2^2 - 2(u_1 - u_2) \cdot (u_1 - u_3),$$

where the last term can be rewritten in terms of the cosine of the angle between $u_1 - u_2$ and $u_1 - u_3$.

Define M to be the length of the distance from s to s' (that is, $M = \left\|s' - s\right\|_2$), and denote $\theta_{abc} = m\angle abc$ for any a, b, and c. The partials can be broken into four cases:

- Edges on the boundary. For example, edge $\{t_2, t_3\}$. We can then simplify the diagram to Figure 2.

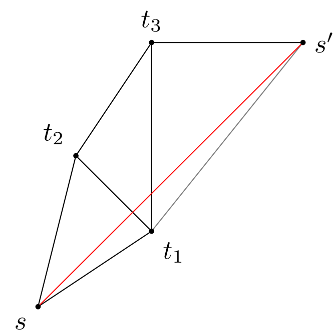

Figure 2: Simplification of Figure 1 when considering $\{t_2, t_3\}$.

Let $m = \|t_2 - t_3\|_2$. Applying the law of cosines to $\angle st_1s'$ and $\angle t_2t_1t_3$ and differentiating, we have

$$\begin{aligned} M\frac{\partial M}{\partial m} &= \|t_1 - s\|_2 \cdot \left\|t_1 - s'\right\|_2 \cdot \sin\left(\theta_{st_1s'}\right)\frac{\partial \theta_{st_1s'}}{\partial m}, \\ m &= \|t_1 - t_2\|_2 \cdot \|t_1 - t_3\|_2 \cdot \sin\left(\theta_{t_2t_1t_3}\right)\frac{\partial \theta_{t_2t_1t_3}}{\partial m}. \end{aligned}$$

Now we notice that $\partial\theta_{st_1s'}/\partial m = \partial\theta_{t_2t_1t_3}/\partial m$, so

$$\frac{\partial M}{\partial m} = \frac{\|t_2 - t_3\|_2 \cdot \left\|(t_1 - s) \times \left(t_1 - s'\right)\right\|_2}{\left\|s' - s\right\|_2 \cdot \left\|(t_1 - t_2) \times (t_1 - t_3)\right\|_2}.$$

- Edges in the interior where the start and end are on "opposite sides." Edge $\{t_1, t_3\}$, as shown in Figure 3, exemplifies this case.

  The computation here is unfortunately complex, but the idea is similar to that of the previous case. Let $m = \|t_1 - t_3\|_2$. We first apply the law of cosines to $\angle t_3t_2t_1$, and $\angle t_3t_2s$. Then we differentiate each result. This yields

$$\frac{\partial\theta_{t_3t_2t_1}}{\partial m} = \frac{\|t_3 - t_1\|_2}{\left\|(t_2 - t_1) \times (t_2 - t_3)\right\|_2},$$

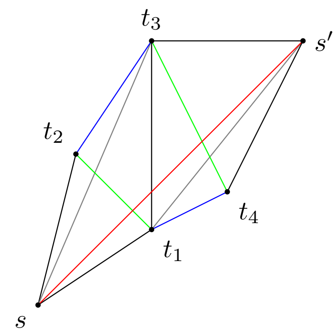

Figure 3: Simplification of Figure 1 when considering $\{t_1, t_3\}$. Note that the triangles $\triangle st_1t_2$ and $\triangle s't_3t_4$ are connected to opposite sides of the quadrilateral $\square t_1t_4t_3t_2$. Alternatively, the two triangles share no vertices.

$$\frac{\partial\|t_3 - s\|_2}{\partial m} = \frac{\big\|(t_2 - s) \times (t_2 - t_3)\big\|_2}{\|t_3 - s\|_2} \cdot \frac{\partial\theta_{t_3t_2s}}{\partial m}.$$

Noting that $\partial\theta_{t_3t_2s}/\partial m$ and $\partial\theta_{t_3t_2t_1}/\partial m$ are equivalent, we obtain.

$$\frac{\partial\|t_3 - s\|_2}{\partial m} = \frac{\|t_3 - t_1\|_2 \cdot \big\|(t_2 - s) \times (t_2 - t_3)\big\|_2}{\|t_3 - s\|_2 \cdot \big\|(t_2 - t_1) \times (t_2 - t_3)\big\|_2}.$$

Similarly,

$$\frac{\partial\big\|t_1 - s'\big\|_2}{\partial m} = \frac{\|t_1 - t_3\|_2 \cdot \Big\|\big(t_4 - s'\big) \times (t_4 - t_1)\Big\|_2}{\|t_1 - s'\|_2 \cdot \big\|(t_4 - t_3) \times (t_4 - t_1)\big\|_2}.$$

Now we apply the law of cosines to $\angle st_1s'$ and differentiate. This yields

$$\begin{aligned}
\Big\|s' - s\Big\|_2 \frac{\partial M}{\partial m} &= \Big\|t_1 - s'\Big\|_2 \cdot \frac{\partial\big\|t_1 - s'\big\|_2}{\partial m} - \frac{(t_1 - s) \cdot \big(t_1 - s'\big)}{\|t_1 - s'\|_2} \cdot \frac{\partial\big\|t_1 - s'\big\|_2}{\partial m} \\
&\quad + \Big\|(t_1 - s) \times \Big(t_1 - s'\Big)\Big\|_2 \cdot \frac{\partial\theta_{st_1s'}}{\partial m} \\
&= \left(1 - \frac{(t_1 - s) \cdot \big(t_1 - s'\big)}{\|t_1 - s'\|_2^2}\right) \cdot \frac{\|t_1 - t_3\|_2 \cdot \Big\|\big(t_4 - s'\big) \times (t_4 - t_1)\Big\|_2}{\big\|(t_4 - t_3) \times (t_4 - t_1)\big\|_2} \\
&\quad + \Big\|(t_1 - s) \times \Big(t_1 - s'\Big)\Big\|_2 \cdot \frac{\partial\theta_{st_1s'}}{\partial m}.
\end{aligned}$$

Applying the same process to $\angle t_1s't_3$, $\angle s't_3s$, and $\angle t_3st_1$, we get

$$\begin{aligned}
\|t_1 - t_3\|_2 &= \left(1 - \frac{\big(t_3 - s'\big) \cdot \big(t_1 - s'\big)}{\|t_1 - s'\|_2^2}\right) \cdot \frac{\|t_1 - t_3\|_2 \cdot \Big\|\big(t_4 - s'\big) \times (t_4 - t_1)\Big\|_2}{\big\|(t_4 - t_3) \times (t_4 - t_1)\big\|_2} \\
&\quad + \Big\|\Big(t_1 - s'\Big) \times \Big(t_3 - s'\Big)\Big\|_2 \cdot \frac{\partial\theta_{t_1s't_3}}{\partial m}, \\
\Big\|s' - s\Big\|_2 \frac{\partial M}{\partial m} &= \left(1 - \frac{(t_3 - s) \cdot \big(t_3 - s'\big)}{\|t_3 - s'\|_2^2}\right) \cdot \frac{\|t_1 - t_3\|_2 \cdot \big\|(t_2 - s) \times (t_2 - t_3)\big\|_2}{\big\|(t_2 - t_1) \times (t_2 - t_3)\big\|_2}
\end{aligned}$$

$$+ \left\| (t_3 - s) \times \left(t_3 - s'\right) \right\|_2 \cdot \frac{\partial \theta_{s' t_3 s}}{\partial m},$$

and

$$\|t_1 - t_3\|_2 = \left(1 - \frac{\left(t_1 - s'\right) \cdot \left(t_3 - s'\right)}{\|t_3 - s'\|_2^2}\right) \cdot \frac{\|t_1 - t_3\|_2 \cdot \left\|(t_2 - s) \times (t_2 - t_3)\right\|_2}{\left\|(t_2 - t_1) \times (t_2 - t_3)\right\|_2}$$
$$+ \left\|(t_1 - s) \times (t_3 - s)\right\|_2 \cdot \frac{\partial \theta_{t_3 s t_1}}{\partial m}.$$

We now recognize

$$\begin{aligned}
\frac{\partial \theta_{s t_1 s'}}{\partial m} + \frac{\partial \theta_{t_1 s' t_3}}{\partial m} + \frac{\partial \theta_{s' t_3 s}}{\partial m} + \frac{\partial \theta_{t_3 s t_1}}{\partial m} &= \frac{\partial}{\partial m}\left(\theta_{s t_1 s'} + \theta_{t_1 s' t_3} + \theta_{s' t_3 s} + \theta_{t_3 s t_1}\right) \\
&= \frac{\partial}{\partial m}(2\pi) \\
&= 0.
\end{aligned}$$

We can thus combine the last four law of cosine computations and cancel out all of the unknown derivatives except $\partial M / \partial m$. The final result is rather complicated expression. Due to formatting issues, it will be presented in two parts: the numerator and denominator of a fraction. The numerator is

$$\begin{aligned}
&\left(1 - \frac{(t_1 - s) \cdot \left(t_1 - s'\right)}{\|t_1 - s'\|_2^2}\right) \cdot \frac{\|t_1 - t_3\|_2 \cdot \left\|\left(t_4 - s'\right) \times (t_4 - t_1)\right\|_2}{\left\|(t_4 - t_3) \times (t_4 - t_1)\right\|_2 \cdot \left\|(t_1 - s) \times \left(t_1 - s'\right)\right\|_2} \\
&\quad + \left(1 - \frac{\left(t_3 - s'\right) \cdot \left(t_1 - s'\right)}{\|t_1 - s'\|_2^2}\right) \cdot \frac{\|t_1 - t_3\|_2 \cdot \left\|\left(t_4 - s'\right) \times (t_4 - t_1)\right\|_2}{\left\|(t_4 - t_3) \times (t_4 - t_1)\right\|_2 \cdot \left\|\left(t_1 - s'\right) \times \left(t_3 - s'\right)\right\|_2} \\
&\quad + \left(1 - \frac{(t_3 - s) \cdot \left(t_3 - s'\right)}{\|t_3 - s'\|_2^2}\right) \cdot \frac{\|t_1 - t_3\|_2 \cdot \left\|(t_2 - s) \times (t_2 - t_3)\right\|_2}{\left\|(t_2 - t_1) \times (t_2 - t_3)\right\|_2 \cdot \left\|(t_3 - s) \times \left(t_3 - s'\right)\right\|_2} \\
&\quad + \left(1 - \frac{\left(t_1 - s'\right) \cdot \left(t_3 - s'\right)}{\|t_3 - s'\|_2^2}\right) \cdot \frac{\|t_1 - t_3\|_2 \cdot \left\|(t_2 - s) \times (t_2 - t_3)\right\|_2}{\left\|(t_2 - t_1) \times (t_2 - t_3)\right\|_2 \cdot \left\|(t_1 - s) \times (t_3 - s)\right\|_2} \\
&\quad - \frac{\|t_1 - t_3\|_2}{\left\|\left(t_1 - s'\right) \times \left(t_3 - s'\right)\right\|_2} - \frac{\|t_1 - t_3\|_2}{\left\|(t_1 - s) \times (t_3 - s)\right\|_2}.
\end{aligned}$$

and the denominator is

$$\frac{\left\|s' - s\right\|_2}{\left\|(t_1 - s) \times \left(t_1 - s'\right)\right\|_2} + \frac{\left\|s' - s\right\|_2}{\left\|(t_3 - s) \times \left(t_3 - s'\right)\right\|_2}.$$

- Edges in the interior where the start and end are on the "same side." In our example, edge $\{t_3, t_4\}$ satisfies this, shown in Figure 4.

  Let $m = \|t_3 - t_4\|_2$ and $q = \|t_1 - t_5\|_2$. Once again, the strategy is to use the law of cosines and differentiate. Using the angles $\angle t_1 t_4 t_5$, $\angle t_4 t_5 t_3$, $\angle t_5 t_3 t_1$, and $t_3 t_1 t_4$, we get

$$\begin{aligned}
\|t_1 - t_5\|_2 \frac{\partial q}{\partial m} &= \left\|(t_4 - t_1) \times (t_4 - t_5)\right\|_2 \cdot \frac{\partial \theta_{t_1 t_4 t_5}}{\partial m}, \\
\|t_3 - t_4\|_2 &= \left\|(t_5 - t_3) \times (t_5 - t_4)\right\|_2 \cdot \frac{\partial \theta_{t_4 t_5 t_3}}{\partial m},
\end{aligned}$$

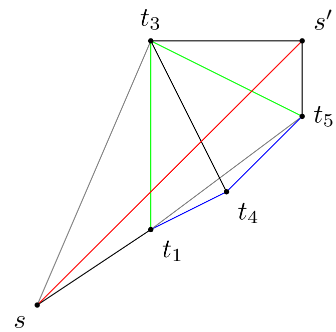

Figure 4: Simplification of Figure 1 when considering $\{t_3, t_4\}$. Note that the triangles $\triangle s t_1 t_3$ and $\triangle s' t_3 t_5$ are connected to adjacent sides of the quadrilateral $\square t_1 t_4 t_5 t_3$. Alternatively, the two triangles share the vertex t_3.

$$\begin{aligned}\|t_1 - t_5\|_2 \frac{\partial q}{\partial m} &= \big\|(t_3 - t_1) \times (t_3 - t_5)\big\|_2 \cdot \frac{\partial \theta_{t_5 t_3 t_1}}{\partial m}, \\ \|t_3 - t_4\|_2 &= \big\|(t_1 - t_3) \times (t_1 - t_4)\big\|_2 \cdot \frac{\partial \theta_{t_3 t_1 t_4}}{\partial m}.\end{aligned}$$

We now again apply the trick where the sum of the angle is constant 2π, meaning the sum of the partials is 0. Scaling the equations, adding them, and solving, we get

$$\frac{\partial q}{\partial m} = -\frac{\|t_3 - t_4\|_2 \cdot \left(\frac{1}{\|(t_5 - t_3)\times(t_5 - t_4)\|_2} + \frac{1}{\|(t_1 - t_3)\times(t_1 - t_4)\|_2}\right)}{\|t_1 - t_5\|_2 \cdot \left(\frac{1}{\|(t_4 - t_1)\times(t_4 - t_5)\|_2} + \frac{1}{\|(t_3 - t_1)\times(t_3 - t_5)\|_2}\right)}.$$

We finish this case by noting

$$\frac{\mathrm{d}M}{\mathrm{d}q} = \frac{\|t_1 - t_5\|_2 \cdot \Big\|(t_3 - s) \times \big(t_3 - s'\big)\Big\|_2}{\|s - s'\|_2 \cdot \big\|(t_3 - t_1) \times (t_3 - t_5)\big\|_2}.$$

From the chain rule, we thus have

$$\frac{\partial M}{\partial m} = -\frac{\|t_3 - t_4\|_2 \cdot \Big\|(t_3 - s) \times \big(t_3 - s'\big)\Big\|_2 \cdot \left(\frac{1}{\|(t_5 - t_3)\times(t_5 - t_4)\|_2} + \frac{1}{\|(t_1 - t_3)\times(t_1 - t_4)\|_2}\right)}{\|s - s'\|_2 \cdot \left(\frac{\|(t_3 - t_1)\times(t_3 - t_5)\|_2}{\|(t_4 - t_1)\times(t_4 - t_5)\|_2} + 1\right)}.$$

- Edges not incident to faces through which the geodesic path passes. The lengths of these edges have no effect on the length of the geodesic path (locally speaking, at least), so the partials here are 0.

One might think that there is a fifth case missing from the above analysis. In particular, cases similar to that seen in Figure 5 aren't immediately similar to any of the situations referenced above. However, if we make the identifications

$$\begin{aligned} &s \leftarrow s' && t_1 \leftarrow s' && t_3 \leftarrow t_7 \\ &t_4 \leftarrow t_6 && t_5 \leftarrow s'' && s' \leftarrow s'', \end{aligned}$$

we see that Figure 5 is just a degenerate instance of Figure 4.

With the partials of the geodesic distances with respect to the length of the edges of the mesh in hand, we need only compute the partials of the edge lengths with respect to the mesh parameters. We find

$$\frac{\partial \big\|v_i - v_j\big\|_2}{\partial z_\ell} = \begin{cases} \frac{v_i - v_j}{\|v_i - v_j\|_2} \cdot \frac{\partial v_\ell}{\partial z_\ell} & \text{if } \ell = i, \\ \frac{v_j - v_i}{\|v_i - v_j\|_2} \cdot \frac{\partial v_\ell}{\partial z_\ell} & \text{if } \ell = j, \\ 0 & \text{otherwise.} \end{cases}$$

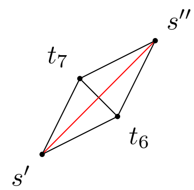

Figure 5: Simplification of Figure 1 when considering $\{t_6, t_7\}$.

All that remains is to use the chain rule to compute the partials of the geodesic distance with respect to the mesh parameters.

4 Curvature

We will define the following:

E_G^ϵ	The set of network edges at threshold ϵ
κ_e^{R}	The Ollivier-Ricci curvature of the edge E_G^ϵ
$\widetilde{\kappa^{\mathrm{G}}}_i$	The discrete Gaussian curvature at v_i, scaled by vertex area
κ_i^{G}	The discrete Gaussian curvature at v_i
$\widetilde{N}_i$	An outward pointing vector at v_i
$\widetilde{\kappa^{\mathrm{H}}}_i$	The mean curvature normal at v_i
κ_i^{H}	The mean curvature at v_i
κ_i^{+}	The first principal curvature at v_i
κ_i^{-}	The second principal curvature at v_i

4.1 Ollivier-Ricci Curvature

We use the GraphRicciCurvature library to compute the Ollivier-Ricci curvatures of the edges of the graph $\left(V_G, E_G^\epsilon\right)$. Here, $E_G^\epsilon \subseteq E_G$ is the set of edges whose RTTs are at most ϵ milliseconds higher than their GCLs.

Note that κ_e^{R} is then only defined for edges in E_G^ϵ, as opposed to being defined for all edges in E_G.

4.2 Forward Computation

For these computations (particularly the mean curvature one), consider v as a matrix of vertex positions, where each row corresponds to a vertex. We will also use e_i to denote the ith standard basis vector. We have

$$
\begin{aligned}
\theta_{i,j} &= \arctan\left(\frac{1}{\cot\left(\theta_{i,j}\right)}\right) \bmod \pi, \\
\widetilde{\kappa^{\mathrm{G}}}_i &= 2\pi - \sum_{\substack{k \\ \left(v_i, v_k\right) \in E_M}} \theta_{k, c\left(i,k\right)}, \\
\kappa^{\mathrm{G}} &= D^{-1} \widetilde{\kappa^{\mathrm{G}}}, \\
\widetilde{N}_i &= \sum_{\substack{k \\ \left(v_i, v_k\right) \in E_M}} N_{i,k}, \\
\widetilde{\kappa^{\mathrm{H}}}_i &= -\frac{1}{2} e_i^{\intercal} D^{-1} L_C^{\mathrm{N}} v, \\
\kappa_i^{\mathrm{H}} &= \operatorname{sgn}\left(\widetilde{N}_i^{\intercal} \widetilde{\kappa^{\mathrm{H}}}_i\right) \left\| \widetilde{\kappa^{\mathrm{H}}}_i \right\|_2,
\end{aligned}
$$

$$
\begin{aligned}
\kappa_i^{+} &= \kappa_i^{\mathrm{H}} + \sqrt{\left(\kappa_i^{\mathrm{H}}\right)^2 - \kappa_i^{\mathrm{G}}},\\
\kappa_i^{-} &= \kappa_i^{\mathrm{H}} - \sqrt{\left(\kappa_i^{\mathrm{H}}\right)^2 - \kappa_i^{\mathrm{G}}}.
\end{aligned}
$$

4.3 Reverse Computation

Differentiating,

$$
\begin{aligned}
\frac{\partial \theta_{i,j}}{\partial z_\ell} &= -\frac{\partial \cot\left(\theta_{i,j}\right)}{\partial z_\ell} \cdot \frac{1}{1+\cot^2\left(\theta_{i,j}\right)},\\
\frac{\partial \widetilde{\kappa^{\mathrm{G}}}_i}{\partial z_\ell} &= -\sum_{\substack{k\\ \left(v_i, v_k\right) \in E_M}} \frac{\partial \theta_{k, c(i,k)}}{\partial z_\ell},\\
\frac{\partial \kappa^{\mathrm{G}}}{\partial z_\ell} &= D^{-1}\left(\frac{\mathrm{d}\widetilde{\kappa^{\mathrm{G}}}}{\partial z_\ell} - \frac{\mathrm{d}D}{\partial z_\ell}\kappa^{\mathrm{G}}\right),\\
\frac{\partial \widetilde{N}_i}{\partial z_\ell} &= \sum_{\substack{k\\ \left(v_i, v_k\right) \in E_M}} \frac{\partial N_{i,k}}{\partial z_\ell},\\
\frac{\partial \widetilde{\kappa^{\mathrm{H}}}_i}{\partial z_\ell} &= -\frac{1}{2} e_i^{\intercal} D^{-1}\left(\left(\frac{\partial L_C^{\mathrm{N}}}{\partial z_\ell} - \frac{\partial D}{\partial z_\ell} D^{-1} L_C^{\mathrm{N}}\right) v + L_C^{\mathrm{N}} \frac{\partial v}{\partial z_\ell}\right),\\
\frac{\partial \kappa_i^{\mathrm{H}}}{\partial z_\ell} &= \frac{\operatorname{sgn}\left(\widetilde{N}_i^{\intercal} \widetilde{\kappa^{\mathrm{H}}}_i\right)}{\left\|\widetilde{\kappa^{\mathrm{H}}}_i\right\|_2} \widetilde{\kappa^{\mathrm{H}}}_i^{\intercal} \frac{\partial \widetilde{\kappa^{\mathrm{H}}}_i}{\partial z_\ell},\\
\frac{\partial \kappa_i^{+}}{\partial z_\ell} &= \frac{2\kappa_i^{+} \frac{\partial \kappa_i^{\mathrm{H}}}{\partial \rho_i} - \frac{\partial \kappa_i^{\mathrm{G}}}{\partial \rho_i}}{\kappa_i^{+} - \kappa_i^{-}},\\
\frac{\partial \kappa_i^{-}}{\partial z_\ell} &= \frac{\frac{\partial \kappa_i^{\mathrm{G}}}{\partial \rho_i} - 2\kappa_i^{-} \frac{\partial \kappa_i^{\mathrm{H}}}{\partial \rho_i}}{\kappa_i^{+} - \kappa_i^{-}}.
\end{aligned}
$$

5 Geodesic Loss

We will define the following in this section:

t	Vector of RTTs in $\mathbb{R}^{\lvert E_G \rvert}$
$\mathbf{1}$	Vector of all ones in $\mathbb{R}^{\lvert E_G \rvert}$
ν_0	Intermediate value
ν_1	Intermediate value
δ	Intermediate value
β_0	The constant term in the least squares linear estimator between ϕ and t
β_1	The linear term in the least squares linear estimator between ϕ and t
$\mathcal{L}_{\mathrm{geodesic}}(M)$	The sum of squared residuals when using β as an estimator, scaled to be unitless

5.1 Forward Computation

We make the following computations:

$$
\nu_0 = \left(\mathbf{1}^{\intercal} t\right)\left(\phi^{\intercal} \phi\right) - \left(t^{\intercal} \phi\right)\left(\mathbf{1}^{\intercal} \phi\right),
$$

$$
\begin{aligned}
\nu_1 &= |E_G| t^\intercal \phi - \big(\mathbf{1}^\intercal \phi\big)\big(\mathbf{1}^\intercal t\big),\\
\delta &= |E_G| \phi^\intercal \phi - \big(\mathbf{1}^\intercal \phi\big)^2,\\
\beta_0 &= \frac{\nu_0}{\delta},\\
\beta_1 &= \frac{\nu_1}{\delta},\\
\mathcal{L}_{\text{geodesic}}(M) &= \frac{1}{t^\intercal t}\big\|t - (\beta_0 \mathbf{1} + \beta_1 \phi)\big\|_2^2.
\end{aligned}
$$

5.2 Reverse Computation

The partials of the above quantities are as follows:

$$
\begin{aligned}
\frac{\partial \nu_0}{\partial z_\ell} &= \Big(2\big(\mathbf{1}^\intercal t\big)\phi - \big(\mathbf{1}^\intercal \phi\big)t - \big(t^\intercal \phi\big)\mathbf{1}\Big)^\intercal \frac{\partial \phi}{\partial z_\ell},\\
\frac{\partial \nu_1}{\partial z_\ell} &= \Big(|E_G| - \big(\mathbf{1}^\intercal t\big)\mathbf{1}\Big)^\intercal \frac{\partial \phi}{\partial z_\ell},\\
\frac{\partial \delta}{\partial z_\ell} &= 2|E_G| \phi^\intercal \frac{\partial \phi}{\partial z_\ell},\\
\frac{\partial \beta_0}{\partial z_\ell} &= \frac{1}{\delta}\left(\frac{\partial \nu_0}{\partial z_\ell} - \beta_0 \frac{\partial \delta}{\partial z_\ell}\right),\\
\frac{\partial \beta_1}{\partial z_\ell} &= \frac{1}{\delta}\left(\frac{\partial \nu_1}{\partial z_\ell} - \beta_1 \frac{\partial \delta}{\partial z_\ell}\right),\\
\frac{\partial\big(\mathcal{L}_{\text{geodesic}}(M)\big)}{\partial z_\ell} &= -\frac{2}{t^\intercal t}\big(t - (\beta_0 \mathbf{1} + \beta_1 \phi)\big)\left(\frac{\partial \beta_1}{\partial z_\ell}\phi + \beta_1 \frac{\partial \phi}{\partial z_\ell}\right).
\end{aligned}
$$

6 Curvature Loss

We will define the following:

$B_r(e)$	The ball of radius r around e
$\mathcal{L}_{\text{curvature}}(M)$	Sum of squares of the differences between vertices actual and desired curvatures

6.1 Determining the Ball Around an Edge

Before tackling the loss functional, we must determine what it means for a point to be close to an edge. Suppose s and s' are vertices in V_G, and let $e = \{s, s'\}$. Recall that p_s (similarly $p_{s'}$) is the point in $\mathbb{R}^2$ corresponding to s. Define $\pi(e)$ to be the edge connecting p_s and $p_{s'}$. We say that $v \in B_r(e)$ when p_v is within distance r of $\pi(e)$. Figure 6 shows this setup.

To determine whether a vertex is in the ball, we break the problem into three parts:

- Determine whether $p_v \in B_r(p_s)$. This is the case when $\|p_v - p_s\|_2 < r$.
- Determine whether the distance from p_v to $\pi(e)$ is less than r. To do this, we first project p_v onto the line containing $\pi(e)$ to get $p_{v,e}$:

  $$p_{v,e} = p_s + \frac{(p_{s'} - p_s)^\intercal (p_v - p_s)}{\|p_{s'} - p_s\|_2^2}(p_{s'} - p_s).$$

  To determine whether this projection lies on the actual segment and that the projection is not too far away from the original point, we simultaneously check the three conditions

  $$(p_v - p_s)^\intercal (p_{s'} - p_s) \geq 0,$$

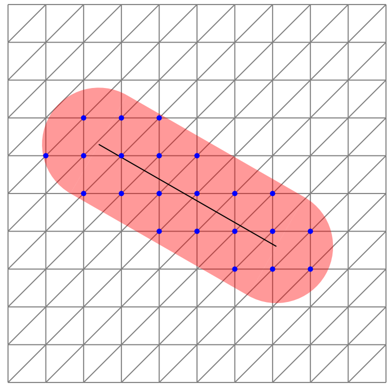

Figure 6: An overhead visualization of $B_r(e)$. The mesh is in gray, $\pi(e)$ is in black, and $B_r(e)$ is in blue.

$$
\begin{aligned}
(p_v - p_{s'})^\intercal (p_s - p_{s'}) &\geq 0, \\
\left\| p_v - p_{v,e} \right\|_2 &< r.
\end{aligned}
$$

- Determine whether $p_v \in B_r(p_{s'})$. This is the case when $\|p_v - p_{s'}\|_2 < r$.

If any of the three above parts yields a positive response, then $v \in B_r(e)$.

6.2 Forward Computation

We have

$$\mathcal{L}_{\text{curvature}}(M) = \frac{1}{\left|E_G^\epsilon\right|} \sum_{e \in E_G^\epsilon} \sum_{\substack{k \\ v_k \in B_r(e)}} \left(\kappa_e^{\mathrm{R}} - \kappa_k^{\mathrm{G}}\right)^2.$$

6.3 Reverse Computation

Differentiating,

$$\frac{\partial\big(\mathcal{L}_{\text{curvature}}(M)\big)}{\partial \rho_\ell} = \frac{1}{|E_G|} \sum_{e \in E_G} \sum_{\substack{k \\ v_k \in B_\epsilon(e)}} -2\left(\kappa_e^{\mathrm{R}} - \kappa_k^G\right) \frac{\partial \kappa_k^G}{\partial z_\ell}.$$

7 Smoothness Loss

We will define the following:

$\mathcal{L}_{\text{smooth}}(M)$	The surface area independent $\mathrm{MVS}_{\text{cross}}$ energy

7.1 Forward Computation

We use the

$$\mathcal{L}_{\text{smooth}}(M) \propto -\left(\kappa^+\right)^\intercal L_C^{\mathrm{N}} \kappa^+ - \left(\kappa^-\right)^\intercal L_C^{\mathrm{N}} \kappa^-.$$

In terms of scaling, we divide by the surface area of the mesh when $z = 0$ (that is, the area of a flat plane, a sphere, or similar).

7.2 Reverse Computation

Differentiating, and using matrix symmetry,

$$\frac{\partial\big(\mathcal{L}_{\mathrm{smooth}}(M)\big)}{\partial z_\ell} \propto -\Big(\kappa^+\Big)^{\mathsf{T}}\left(\frac{\partial L_C^{\mathrm{N}}}{\partial z_\ell}\kappa^+ + 2L_C^{\mathrm{N}}\frac{\partial \kappa^+}{\partial z_\ell}\right) - \Big(\kappa^-\Big)^{\mathsf{T}}\left(\frac{\partial L_C^{\mathrm{N}}}{\partial z_\ell}\kappa^- + 2L_C^{\mathrm{N}}\frac{\partial \kappa^-}{\partial z_\ell}\right).$$