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MAT 220 – Furno

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**Homework 1**

1. **1.5 – Interpreting sparsity. Suppose the n-vector x is sparse, i.e., has only a few nonzero entries. Give a short sentence or two explaining what this means in each of the following contexts.**
   1. **x represents the daily cash flow of some business over n days.**

Over a period of n days, daily cash flow is very low. There are many days without any cash flow. The business might not be active very often.

* 1. **x represents the annual dollar value purchases by a customer of n products or services.**

Out of the n products and services, most have an annual dollar value equal to 0. Perhaps the customer does not purchase very many products.

* 1. **x represents a portfolio, say, the dollar value holdings of n stocks.**

Out of n stocks, only a few have a portfolio, meaning only a few have a dollar value holding. This may indicate that most stocks have no quantities.

* 1. **x represents a bill of materials for a project, i.e., the amounts of n materials needed.**

Out of n materials, most materials in a bill of materials require an amount of 0 for the specified project. Not many materials are needed for the project.

* 1. **x represents a monochrome image, i.e., the brightness values of n pixels.**

Out of n pixels, most pixels of a monochrome image have a brightness value of 0, perhaps indicating that the image is very dark.

* 1. **x is the daily rainfall in a location over one year.**

The amount of rainfall in the specified location is very low over one year. There are very few days with any rainfall at all.

1. **1.7 – Suppose that x is a Boolean vector with entries that are 0 or 1, and y is a vector encoding the same information using the values -1 and +1. Express y in terms of x using vector notation. Also express x in terms of y using vector notation.**

Ex. x = (0,1,1,0) y = (-1,1,1,-1)

Find function such that: f(0) = -1 & f(1) = 1

y = 2x – 1 x = (y + 1)/2

y = (2x1 – 1, 2x2 – 1, … 2xn – 1)

x = ((y1 + 1)/2, (y2 + 1)/2, … (yn + 1)/2)

1. **1.9 – Symptoms vector. A 20-vector s records whether each of 20 different symptoms is present in a medical patient, with si = 1 meaning the patient has the symptom and si = 0 meaning she does not. Express the following using vector notation.**
   1. **The total number of symptoms the patient has.**

The sum of all elements in the vector 🡪 Σ20i = 1 (si)

* 1. **The patient exhibits five out of the first ten symptoms.**

The sum of the first 10 elements in the vector is 5 🡪 Σ10i = 1 (si) = 5

1. **1.11 – Word count and word count histogram vectors. Suppose the n-vector w is the word count vector associated with a document and a dictionary of n words. For simplicity we will assume that all words in the document appear in the dictionary.** 
   1. **What is 1Tw?**

This represents the sum of the elements in the word count vector, or the total number of words in the document.

* 1. **What does w282 = 0 mean?**

The 282nd word in the dictionary appears 0 times in the document.

* 1. **Let h be the n-vector that gives the histogram of the word counts, i.e., hi is the fraction of the words in the document that are word i. Use vector notation to express h in terms of w. (You can assume that the document contains at least one word.)**

Each word count is divided by the total number of words in the document to get the relative frequency of each word.

hi = (w1/1Tw, w2/1Tw, … wi/1Tw)

As long as the document contains at least one word, h can be expressed in terms of w as:

h = w/1Tw

1. **1.13 – Average age in a population. Suppose the 100-vector x represents the distribution of ages in some population of people, with xi being the number of i – 1 year olds, for i = 1, … , 100. (You can assume that x ≠ 0, and that there is no one in the population over age 99.) Find expressions, using vector notation, for the following quantities.**
   1. **The total number of people in the population.**

The sum of the elements in the vector, representing the total number of people in the population can be written as: 1Tx.

* 1. **The total number of people in the population age 65 and over.**

This can be represented as the sum of the elements in the 100-vector x in the range of (65 + 1) to 100. The total number of people in the population over 65 is: 1Tx66:100.

* 1. **The average age of the population. (You can use ordinary division of numbers in your expression.)**

The average age of the population can be represented by the sum of all the ages divided by the number of people in the population.

The sum of all the ages can be found by taking the sum of each age multiplied by the number of people in that age group.: Σ100i =1 xi (i– 1)

The total number of people in the population is: 1Tx.

The average age of the population is: Σ100i =1 xi (i– 1) or (1/n)Tx

1Tx

1. **1.16 – Inner product of nonnegative vectors. A vector is called nonnegative if all its entries are nonnegative.**
   1. **Explain why the inner product of two nonnegative vectors is nonnegative.**

When you multiply two nonnegative numbers together, the product is nonnegative. The inner product is the sum of the products of corresponding entries, therefore the inner product of two nonnegative vectors is always nonnegative.

* 1. **Suppose the inner product of two nonnegative vectors is zero. What can you say about them? Your answer should be in terms of their respective sparsity patterns, i.e., which entries are zero and nonzero.**

If the inner product of two nonnegative vectors is zero, then the product of each corresponding entry is zero. This can be achieved if each set of corresponding vector entries includes at least one zero, ensuring that only zero’s will be added together. This also means that at least 50% of the total entries of both vectors must be zero, so the vectors would be relatively sparse.

1. **1.17 – Linear combinations of cash flows. We consider cash flow vectors over T time periods, with a positive entry meaning a payment received, and negative meaning a payment made. A (unit) single period loan, at time period t, is the T-vector lt, that corresponds to a payment received of $1 in period t and a payment made of $(1 + r) in period t + 1, with all other payments zero. Here r > 0 is the interest rate (over one period). Let c be a $1 T – 1 period loan, starting at period 1. This means that $1 is received in period 1, $1(1 + r)T – 1 is paid in period T, and all other payments are zero. Express c as a linear combination of single period loans.**

T = 4, c = (1, 0, 0, -(1+r)3

l1 = (1, -(1+r), 0, 0) 🡪 α1 + 0 + 0 = 1

l2 = (0, 1, -(1+r), 0) 🡪 -(1+r) + α2 + = 0

l3 = (0, 0, 1, -(1+r)) 🡪 0 + -(1+r)(1+r) + α3 = 0

T = n, c = l1 + (1+r)l2 + (1+r)2l3 + … + (1+r)n-1ln

1. **2.1 – Linear or not? Determine whether each of the following scalar valued functions of n-vectors is linear. If it is a linear function, give its inner product representation, i.e., an n-vector a for which f(x) = aTx for all x. If it is not linear, give specific x, y, α, and β for which superposition fails,**

**i.e., f(αx + βy) ≠ αf(x) + βf(y).**

* 1. **The spread of values of the vector, defines as f(x) = maxkxk – minkxk.**

α = 1, β = 1, x = (1, 0, 0, 0), y = (-1, 0, 0, 0)

f(αx + βy) = f(x + y) = f((1, 0, 0, 0) + (-1, 0, 0, 0)) = f(0, 0, 0, 0) = 0 – 0 = 0

αf(x) + βf(y) = f(x) + f(y) = f(1, 0, 0, 0) + f(-1, 0, 0, 0) = (1 – 0) + (0 – (-1)) = 1 + 1 = 2

f(αx + βy) ≠ αf(x) + βf(y) therefore the function is not linear.

* 1. **The difference of the last element and the first, f(x) = xn – x1.**

α = 1, β = 1, x = (1, 0, 0, 0), y = (-1, 0, 0, 0)

f(αx + βy) = f(x + y) = f((1, 0, 0, 0) + (-1, 0, 0, 0)) = f(0, 0, 0, 0) = 0 – 0 = 0

αf(x) + βf(y) = f(x) + f(y) = f(1, 0, 0, 0) + f(-1, 0, 0, 0) = (0 – 1) + (0 – (-1)) = -1 + 1 = 0

f(αx + βy) = αf(x) + βf(y) therefore the function is linear.

* 1. **The median of an n-vector, where we will assume n = 2k + 1, is odd. The median of the vector x is defined as the (k + 1)st largest number among the entries of x. For example, the median of (-7.1, 3.2, -1.5) is -1.5.**

f(x) = median(x), n = 2k + 1

α = 1, β = 1, x = (1, 2, 3), y = (7, 3, 9)

f(αx + βy) = f(x + y) = f(8, 5, 12) = 8

αf(x) + βf(y) = f(x) + f(y) = f(1, 2, 3) + f(7, 3, 9) = 2 + 7 = 9

f(αx + βy) ≠ αf(x) + βf(y) therefore the function is not linear.

* 1. **The average of the entries with odd indices, minus the average of the entries with even indices. You can assume that n = 2k is even.**

f(x) = avgOddIndices(x) – avgEvenIndices(x), n = 2k

α = 1, β = 1, x = (1, 2, 3, 4), y = (1, 9, 8, 4)

f(αx + βy) = f(x + y) = f(2, 11, 11, 8) = (2 + 11/2) – (11 + 8/2) = 13/2 – 19/2 = -3

αf(x) + βf(y) = f(x) + f(y) = f(1, 2, 3, 4) + f(1, 9, 8, 4) = [4/2 – 6/2] + [9/2 – 13/2] = -1 + (-2) = -3

f(αx + βy) = αf(x) + βf(y) therefore the function is linear.

* 1. **Vector extrapolation, defined as xn + (xn – xn-1), for n ≥ 2. (This is a simple prediction of what xn+1 would be, based on a straight line drawn through xn and xn-1.)**

f(xn) = xn + (xn – xn-1), n ≥ 2

α = 1, β = 1, x = (1, 2, 3), y = (8, 4, 7), n = 3

f(αx + βy) = f(x + y) = f(9, 6, 10) = 10 + (10 – 6) = 14

αf(x) + βf(y) = f(x) + f(y) = f(1, 2, 3) +f(8, 4, 7) = [3 + (3 – 1)] + [7 + (7 – 4)] = 5 + 10 = 15

f(αx + βy) ≠ αf(x) + βf(y) therefore the function is not linear.

1. **2.2 – Processor powers and temperature. The temperature T of an electronic device containing three processors is an affine function of the power dissipated by the three processors, P = (P1, P2, P3). When all three processors are idling, we have P = (10, 10, 10), which results in a temperature T = 30. When the first processor operates at full power and the other two are idling, we have P = (100, 10, 10), and the temperature rises to T = 60. When the second processor operates at full power and the other two are idling, we have P = (10, 100, 10) and T = 70. When the third processor operates at full power and the other two are idling, we have P = (10, 10, 100) and T = 65. Now suppose that all three processors are operated at the same power Psame. How large can Psame be, if we require that T ≤ 85? Hint. From the given data, find the 3-vector a and number b for which T = aTP + b.**

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **P1** | **P2** | **P3** | **Measured Temp** | **Predicted Temp** |
| 10 (0) | 10 (0) | 10 (0) | 30 |  |
| 100 (90) | 10 (0) | 10 (0) | 60 |  |
| 10 (0) | 100 (90) | 10 (0) | 70 |  |
| 10 (0) | 10 (0) | 100 (90) | 65 |  |
| ? | ? | ? | - | ≤ 85 |

P1 coefficient = (60 – 30)/(100 – 10) = 30/90 = 1/3

P2 coefficient = (70 – 30)/(100 – 10) = 40/90 = 4/9

P3 coefficient = (65 – 30)/(100 – 10) = 35/90 = 7/18

1/3(P1 – 10) + 4/9(P2 – 10) + 7/18(P3 – 10) = 85 – 30

(P – 10) = 55/(1/3 + 4/9 + 7/18) 🡪 P = [55/(21/18)] +10 = 1200/21 = 400/7

Psame = 400/7

1. **2.4 – Linear function? The function φ : R3 → R satisfies**

**φ(1, 1, 0) = −1, φ(−1, 1, 1) = 1, φ(1, −1, −1) = 1.**

**Choose one of the following, and justify your choice: φ must be linear; φ could be linear; φ cannot be linear.**

|  |  |  |  |
| --- | --- | --- | --- |
| **X1** | **X2** | **X3** | **Result** |
| 1 | 1 | 0 | -1 |
| -1 | 1 | 1 | 1 |
| 1 | -1 | -1 | 1 |

See if function is linear:

φ(-1, 1, 1) = 1 φ(1, -1, -1) = 1

φ(-1, 1, 1) = φ( -1 (-1, 1, 1))

= -1 φ (-1 ,1 ,1)

= (-1) 1

1 ≠ -1

Φ cannot be linear