**Stephen Kim**

**MAT 220 – Homework 4**

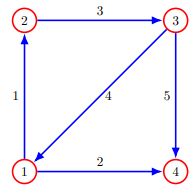
**Part A:**

1. **7.3 – Trimming a vector. Find a matrix A for which Ax = (x2, . . . , xn−1), where x is an n-vector. (Be sure to specify the size of A, and describe all its entries.)**

Ax = (n – 1 – 1) x 1 🡪 n – 2 x 1

An-2 x n \* = //Remove first and last row of identity matrix

A = n – 2 x n

1. **7.6 – Rows of incidence matrix. Show that the rows of the incidence matrix of a graph are always linearly dependent. Hint. Consider the sum of the rows.**

Aij =

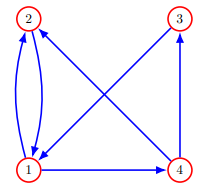
1 2 3 4 5

A =

All of the columns have a sum of 0

A1 + A2 + A3 + A4 = [-1 + 1 + 0 + 0] + [-1 + 0 + 0 + 1] + [0 – 1 + 1 + 0] + [0 + 0 – 1 + 1] = 0 + 0 + 0 + 0 = 0

A1, A2, …, An are linearly dependent

1. **7.7 – Incidence matrix of reversed graph. (See exercise 6.5.) Suppose A is the incidence matrix of a graph. The reversed graph is obtained by reversing the directions of all the edges of the original graph. What is the incidence matrix of the reversed graph? (Express your answer in terms of A.)**

Aij = ARij =

2

5

4

1

3

5

4

1

3

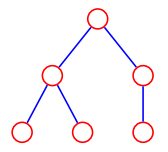
1 2 3 4 5 6

A = AR = Ax =

6

AR = (-1)A

The reverse of an incidence matrix is the original matrix multiplied by -1.

1. **7.9 – Social network graph. Consider a group of n people or users, and some symmetric social relation among them. This means that some pairs of users are connected, or friends (say). We can create a directed graph by associating a node with each user, and an edge between each pair of friends, arbitrarily choosing the direction of the edge. Now consider an n-vector v, where vi is some quantity for user i, for example, age or education level (say, given in years). Let D(v) denote the Dirichlet energy associated with the graph and v, thought of as a potential on the nodes.**
   1. **Explain why the number D(v) does not depend on the choice of directions for the edges of the graph.**

The Dirichlet energy does not take the direction of edges into consideration, it measures the variability of the distance between nodes. It is taking into consideration the ages of each user as opposed to the direction of the edge.

* 1. **Would you guess that D(v) is small or large? This is an open-ended, vague question; there is no right answer. Just make a guess as to what you might expect, and give a short English justification of your guess.**

I would guess that D(v) is small because if the edges represent pairs of friends, I would expect all connected graphs to be groups of friends. I assume that most groups of friends have many people of similar ages, although there are exceptions. If there are many people of similar ages in the graph, then the potential differences would be lower.

1. **7.12 – Some properties of convolution. Suppose that a is an n-vector.** 
   1. **Convolution with 1. What is 1 ∗ a? (Here we interpret 1 as a 1-vector.)**

a = b =

a \* b = = =

* 1. **Convolution with a unit vector. What is ek ∗ a, where ek is the kth unit vector of dimension q? Describe this vector mathematically (i.e., give its entries), and via a brief English description. You might find vector slice notation useful.**

a = ek = 🡪 ek \* a = c = 🡪 cx =

c1 = = = = (0)a1 = 0

c2 = = = = (0)a2 + (0)a1 = 0

ck = = = (0)ak + (0)ak-1 + … + (1)a1 = a1

cq+n-1 = = = aq+n-k

c = The result is a vector of length q+n-k, where ek is multiplied by a, within the dimension q.

1. **7.15 – Channel equalization. We suppose that u1, . . . , um is a signal (time series) that is transmitted (for example by radio). A receiver receives the signal y = c ∗ u, where the n-vector c is called the channel impulse response. (See page 138.) In most applications n is small, e.g., under 10, and m is much larger. An equalizer is a k-vector h that satisfies h ∗ c ≈ e1, the first unit vector of length n + k − 1. The receiver equalizes the received signal y by convolving it with the equalizer to obtain z = h ∗ y.** 
   1. **How are z (the equalized received signal) and u (the original transmitted signal) related? Hint. Recall that**

**h ∗ (c ∗ u) = (h ∗ c) ∗ u.**

z = h \* y y = c \* u

z = h \* (c \* u) 🡪 z = (h \* c) \* u 🡪 z = ≈e1 \* u; z = received signal + equalizer

* 1. **(Use** [**https://www.random.org/integers/**](https://www.random.org/integers/) **to get the random +1’s and −1’s for your answer in part b. You might need to get random 0’s and 1’s and then transform them into −1’s and +1’s.)**

**Numerical example. Generate a signal u of length m = 50, with each entry a random value that is either −1 or +1. Plot u and y = c ∗ u, with c = (1, 0.7, −0.3). Also plot the equalized signal z = h ∗ y, with**

**h = (0.9, −0.5, 0.5, −0.4, 0.3, −0.3, 0.2, −0.1).**

u = [1,-1,-1,1,-1,1,-1,1,-1,-1,-1,-1,-1,-1,1,-1,1,-1,1,1,1,-1,1,-1,-1,-1,1,1,-1,-1,-1,1,-1,1,-1,1,1,-1,-1,-1,-1,-1,-1,1,-1,1,1,1,-1,-1]

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **#** | **u** | **c** | **y=c\*u** | **h** | **z** |
| 1 | 1 | 1 | 1 | 0.9 | 0.9 |
| 2 | -1 | 0.7 | -0.7 | -0.5 | 0.35 |
| 3 | -1 | -0.3 | 0.3 | 0.5 | 0.15 |
| 4 | 1 | 1 | 1 | -0.4 | -0.4 |
| 5 | -1 | 0.7 | -0.7 | 0.3 | -0.21 |
| 6 | 1 | -0.3 | -0.3 | -0.3 | 0.09 |
| 7 | -1 | 1 | -1 | 0.2 | -0.2 |
| 8 | 1 | 0.7 | 0.7 | -0.1 | -0.07 |
| 9 | -1 | -0.3 | 0.3 | 0.9 | 0.27 |
| 10 | -1 | 1 | -1 | -0.5 | 0.5 |
| 11 | -1 | 0.7 | -0.7 | 0.5 | -0.35 |
| 12 | -1 | -0.3 | 0.3 | -0.4 | -0.12 |
| 13 | -1 | 1 | -1 | 0.3 | -0.3 |
| 14 | -1 | 0.7 | -0.7 | -0.3 | 0.21 |
| 15 | 1 | -0.3 | -0.3 | 0.2 | -0.06 |
| 16 | -1 | 1 | -1 | -0.1 | 0.1 |
| 17 | 1 | 0.7 | 0.7 | 0.9 | 0.63 |
| 18 | -1 | -0.3 | 0.3 | -0.5 | -0.15 |
| 19 | 1 | 1 | 1 | 0.5 | 0.5 |
| 20 | 1 | 0.7 | 0.7 | -0.4 | -0.28 |
| 21 | 1 | -0.3 | -0.3 | 0.3 | -0.09 |
| 22 | -1 | 1 | -1 | -0.3 | 0.3 |
| 23 | 1 | 0.7 | 0.7 | 0.2 | 0.14 |
| 24 | -1 | -0.3 | 0.3 | -0.1 | -0.03 |
| 25 | -1 | 1 | -1 | 0.9 | -0.9 |
| 26 | -1 | 0.7 | -0.7 | -0.5 | 0.35 |
| 27 | 1 | -0.3 | -0.3 | 0.5 | -0.15 |
| 28 | 1 | 1 | 1 | -0.4 | -0.4 |
| 29 | -1 | 0.7 | -0.7 | 0.3 | -0.21 |
| 30 | -1 | -0.3 | 0.3 | -0.3 | -0.09 |
| 31 | -1 | 1 | -1 | 0.2 | -0.2 |
| 32 | 1 | 0.7 | 0.7 | -0.1 | -0.07 |
| 33 | -1 | -0.3 | 0.3 | 0.9 | 0.27 |
| 34 | 1 | 1 | 1 | -0.5 | -0.5 |
| 35 | -1 | 0.7 | -0.7 | 0.5 | -0.35 |
| 36 | 1 | -0.3 | -0.3 | -0.4 | 0.12 |
| 37 | 1 | 1 | 1 | 0.3 | 0.3 |
| 38 | -1 | 0.7 | -0.7 | -0.3 | 0.21 |
| 39 | -1 | -0.3 | 0.3 | 0.2 | 0.06 |
| 40 | -1 | 1 | -1 | -0.1 | 0.1 |
| 41 | -1 | 0.7 | -0.7 | 0.9 | -0.63 |
| 42 | -1 | -0.3 | 0.3 | -0.5 | -0.15 |
| 43 | -1 | 1 | -1 | 0.5 | -0.5 |
| 44 | 1 | 0.7 | 0.7 | -0.4 | -0.28 |
| 45 | -1 | -0.3 | 0.3 | 0.3 | 0.09 |
| 46 | 1 | 1 | 1 | -0.3 | -0.3 |
| 47 | 1 | 0.7 | 0.7 | 0.2 | 0.14 |
| 48 | 1 | -0.3 | -0.3 | -0.1 | 0.03 |
| 49 | -1 | 1 | -1 | 0.9 | -0.9 |
| 50 | -1 | 0.7 | -0.7 | -0.5 | 0.35 |

y = u \* [1, 0.7, -0.3]

1. **8.1 - Sum of linear functions. Suppose f : Rn → Rm and g : Rn → Rm are linear functions. Their sum is the function h : Rn → Rm, defined as h(x) = f(x)+g(x) for any n-vector x. The sum function is often denoted as h = f + g. (This is another case of overloading the + symbol, in this case to the sum of functions.) If f has matrix representation f(x) = Fx, and g has matrix representation g(x) = Gx, where F and G are m × n matrices, what is the matrix representation of the sum function h = f + g? Be sure to identify any + symbols appearing in your justification.**

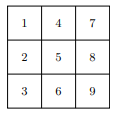
f: Rn 🡪 Rm, where f(x) = Fx |F = m x n g: Rn 🡪 Rm, where g(x) = Gx |G = m x n

h: Rn 🡪 Rm , where h(x) = f(x) + g(x) |matrix of h(x) H = m x n

f = g =

h = + =

1. **8.4 – Linear functions of images. In this problem we consider several linear functions of a monochrome image with N × N pixels. To keep the matrices small enough to work out by hand, we will consider the case with N = 3 (which would hardly qualify as an image). We represent a 3 × 3 image as a 9-vector using the ordering of pixels shown below.**

**x =**

**(This ordering is called column-major.) Each of the operations or transformations below defines a function y = f(x), where the 9-vector x represents the original image, and the 9-vector y represents the resulting or transformed image. For each of these operations, give the 9 × 9 matrix A for which y = Ax.**

* 1. **Turn the original image x upside-down.**

🡪 y = = A A =

* 1. **Rotate the original image x clockwise 90◦ .**

🡪 y = = A A =

* 1. **Translate the image up by 1 pixel and to the right by 1 pixel. In the translated image, assign the value yi = 0 to the pixels in the first column and the last row.**

🡪 y = = A A =

* 1. **Set each pixel value yi to be the average of the neighbors of pixel i in the original image. By neighbors, we mean the pixels immediately above and below, and immediately to the left and right. The center pixel has 4 neighbors; corner pixels have 2 neighbors, and the remaining pixels have 3 neighbors.**

🡪 🡪 y = = A A =

1. **10.2 – Ones matrix. There is no special notation for an m × n matrix all of whose entries are one. Give a simple expression for this matrix in terms of matrix multiplication, transpose, and the ones vectors 1m, 1n (where the subscripts denote the dimension).**

🡪 Am x n = 1m x 1 \* 11 x n = 1m x 1 \* 1Tn x 1 = 1m1Tn

1. **10.4 – Block matrix notation. Consider the block matrix**

**A =** ,

**where B is 10 × 5. What are the dimensions of the four zero matrices and the identity matrix in the definition of A? What are the dimensions of A?**

**A =**  (10 + 5 + 10) x (10 + 5 + 10) = 25 x 25

The dimension of A is 25 x 25.

1. **10.6 – Product of rotation matrices. Let A be the 2 × 2 matrix that corresponds to rotation by θ radians, defined in (7.1), and let B be the 2 × 2 matrix that corresponds to rotation by ω radians. Show that AB is also a rotation matrix, and give the angle by which it rotates vectors. Verify that AB = BA in this case, and give a simple English explanation.**

A = B =

AB = =

= , which is a rotations matrix that rotates by ( radians

BA = =

=

AB = BA. Multiplying A and B together increases the radians of the rotation to the radians of A plus the radians of B. It does not matter which occurs first in the expression.

**Part B:**

1. ***Rainfall and river height.* The *T*-vector *r* gives the daily rainfall in some region over a period of *T* days. The vector *h* gives the daily height of a river in the region (above its normal height). By careful modeling of water flow, or by fitting a model to past data, it is found that these vectors are (approximately) related by convolution: *h* = *g* ∗ *r*, where**

***g* = (0*.*2*,*0*.*7*,*0*.*4*,*0*.*1)*.***

**Give a short story in English (with no mathematical terms) to approximately describe this relation. For example, you might mention how many days after a day of heavy rainfall the river height is most affected or how many days it takes for the river height to return to the normal height once the rain stops.**

h1 = 0.2r1

h2 = 0.7r1 + 0.2r2

h3 = 0.4r1 + 0.7r2 + 0.2r3

h4 = 0.1r1 + 0.4r2 + 0.7r3 + 0.2r4

h5 = 0.1r2 + 0.4r3 + 0.7r4 + 0.2r5

h6 = 0.1r3 + 0.4r4 + 0.7r5 + 0.2r6

According to the relationship by convolution, one day of rainfall makes the river height go up a small amount. The river height increases rapidly and is the highest on the second day of rainfall, but begins to drop immediately afterwards. The river height is almost normal again after 3 days.

1. **Verify that *f*(*x*1*,x*2*,x*3) = (*x*1 + 2*x*2*,*3*x*3 − 2*x*2) satisfies *f*(*x* + *y*) = *f*(*x*) + *f*(*y*) for all vectors *x,y* ∈ R3 and satisfies *f*(*αx*) = *αf*(*x*) for all *α* ∈ R and all *x* ∈ R3. Then find a matrix *A* such that *f*(*x*) = *Ax*, where *x* is a column vector.**

*f*(*x*1*,x*2*,x*3) = (*x*1 + 2*x*2*,*3*x*3 − 2*x*2)

*f*(*x* + *y*) = *f*(*x*) + *f*(*y*) for all vectors *x,y* ∈ R3

x = (a1, a2, a3) y = (b1, b2, b3) x + y = (a1 + b1, a2 + b2, a3 + b3)

f(x + y) = f(a1 + b1, a2 + b2, a3 + b3) = [(a1 + b1) + 2(a2 + b2), 3(a3 + b3) – 2(a2 + b2)]

= (a1 + b1 + 2a2 +2b2, 3a3 + 3b3 – 2a2 – 2b2)

f(x) + f(y) = (a1 + 2a2, 3a3 – 2a2) + (b1 + 2b2, 3b3 – 2b2) = (a1 + 2a2 + b1 + 2b2, 3a3 – 2a2 + 3b3 – 2b2)

= (a1 + b1 + 2a2 + 2b2, 3a3 + 3b3 – 2a2 – 2b2

*f*(*αx*) = *αf*(*x*) for all *α* ∈ R and all *x* ∈ R3

*α* = (a1) x = (b1, b2, b3)

*f*(*αx*) = f(a1b1, a1b2, a1b3) = (a1b1 + 2a1b2, 3a1b3 – 2a1b2)

*αf*(*x*) = a1[b1 + 2b2, 3b3 – 2b2] = (a1b1 + 2a1b2, 3a1b3 – 2a1b2)

**3.**

**A = B = C =**  **D =**

**Calculate the following or state that it is not defined:**

1. ***AB***

= =

1. ***BA***

= =

1. ***BC***

= =

1. ***CB***

= =

1. ***ABC***

=

= =

1. ***AD***

= =

1. ***DA***

= =

1. ***BD***

= =

1. ***DB***

= =

1. ***CD***

= =

1. ***DC***

=

=

1. **tr(*AB*)**

= 🡪 tr = 3 + 8 = 11

1. ***CT***

T =

1. **(*ABA*)*T***

= =

= =

T =