[03-60-231] Assignment 4

Stephen Nusko
103693282

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1 Question 5.3

1.1 Part a

Prove or disprove If R and S are asymmetric, then (i) $R \cup S$ is a symmetric relation; (ii) $R \cap S$ is a symmetric relation
We must prove or disprove R and S are asymmetric $\Rightarrow R \cup S$ is a symmetric relation,
as well as R and S are asymmetric $\Rightarrow R \cap S$ is a symmetric relation
We will disprove both statements, first R and S are asymmetric $\Rightarrow R \cup S$ is a symmetric relation
Let $R = \{(a, b)\}$
Let $S = \{(a, b), (c, d)\}$
Then both R and S are asymmetric
first we disprove R and S are asymmetric $\Rightarrow R \cup S$ is a symmetric relation.
and $R \cup S = \{(a, b), (c, d)\}$
By inspection $R \cup S$ is not symmetric
$\therefore$ If R and S are asymmetric $\Rightarrow R \cup S$ is a symmetric relation is false
Now we disprove R and S are asymmetric $\Rightarrow R \cap S$ is a symmetric relation
$R \cap S = \{(a, b)\}$
By inspection $R \cap S$ is not symmetric.
$\therefore$ If R and S are asymmetric $\Rightarrow R \cap S$ is a symmetric relation is false $\square$

2 Question 5.9

Let $\mathbb{N}$ be the set of all positive integers and R is a relation in $\mathbb{N}$ such that
$R = \{(a, b) \in \mathbb{N} \text{ x } \mathbb{N} \mid$ the sum of the decimal digits in $a =$ the sum of the decimal digits in $b\}$.
$\forall a, b \in \mathbb{N}$ let $n, m \in$ be the number of digits in a, b respectively, and let $i, j \in \mathbb{B}$ be the individual digits at the respective indexes.
Then $R = \{(a, b) \in \mathbb{N} \text{ x } \mathbb{B} \mid \sum\limits_{i=1}^{n} a = \sum\limits_{j=1}^{m} b\}$
Prove that R is an equivalence relation.
To prove the R is an equivalence relation we must prove that R is reflexive, symmetric, and transitive.
First we will prove that R is reflexive, ie $(\forall x \in \mathbb{N})(x, x) \in R$
Proof: (direct)

1. Let $x \in \mathbb{N}$ — hypothesis
2. $\sum\limits_{i=1}^{n} x = \sum\limits_{i=1}^{n} x$ — high school algebra
3. $\sum\limits_{i=1}^{n} x = \sum\limits_{j=1}^{m} x$ — change of variable
4. $(x, x) \in R$ — definition of R

$\therefore (\forall x \in \mathbb{N})(x, x) \in R$ $\square$
$\therefore R$ is reflexive
Next we prove that R is symmetric, ie if $(x, y) \in R \Rightarrow (y, x) \in R$
Proof: (direct)

1. Let $(x, y) \in R$ Hypothesis
2. $\sum_{i=1}^{n} x = \sum_{j=1}^{m} y$ definition of R
3. $\sum_{i=1}^{n} x = \sum_{i=1}^{n} x$ 2, $\text{sub}_=$
4. $\sum_{j=1}^{m} y = \sum_{i=1}^{n} x$ 2, $\text{sub}_=$
5. $(y, x) \in R$ definition of R

$\therefore (x, y) \in R \Rightarrow (y, x) \in R$ $\square$
$\therefore R$ is symmetric
Finally we prove that R is transitive, ie if $(x, y) \in R \wedge (y, z) \in R \Rightarrow (x, z) \in R$
Proof: (direct)

1. Let $(x, y) \in R \wedge (y, z) \in R$ Hypothesis
2. $(x, y) \in R$ and $(y, z) \in R$ 1, I2, E9, I2
3. $\sum_{i=1}^{n} x = \sum_{j=1}^{m} y$ and $\sum_{j=1}^{m} y = \sum_{k=1}^{o} z$ definition of R twice
4. $\sum_{i=1}^{n} x = \sum_{k=1}^{o} z$ 3, $\text{sub}_=$
5. $(x, z) \in R$ definition of R

$\therefore (x, y) \in R \wedge (y, z) \in R \Rightarrow (x, z) \in R$ $\square$
$\therefore R$ is transitive
$\therefore R$ is reflexive, symmetric, and transitive.
$\therefore R$ is an equivalence relation. $\square$

The equivalence class of 98 are all positive integers such that the sum of their digits is equal to 17
for valus from in $[0, 50]$ that belong to the equivalence class $[98]/R = \emptyset$

3 Question 5.12

Let R be a reflexive and transitive relation in X.
Let S be a relation in X such that $(x, y) \in S \Leftrightarrow (x, y) \in R \wedge (y, x) \in R$

3.1 Part a

Prove that S is an equivalence relation.
To prove that S is an equivalence relation we must prove that S is reflexive, symmetric, and transitive.
First we will prove that S is reflexive, ie $(\forall x \in X)(x, x) \in S$
Proof: (direct)

1. Let $x \in X$ hypothesis
2. $(x, x) \in R$ R is reflexive

3. $(x,x) \in R \land (x,x) \in R$ 2, E3

4. $(x,x) \in S$ definition of S

$\therefore (\forall x \in X)(x,x) \in S$ $\square$
$\therefore S$ is reflexive
Next we prove that S is symmetric, ie if $(x,y) \in S \Rightarrow (y,x) \in S$
Proof: (direct)

1. Let $(x,y) \in S$ hypothesis
2. $(x,y) \in R \land (y,x) \in R$ definition of S
3. $(y,x) \in R \land (x,y) \in R$ 2, E9
4. $(y,x) \in S$ definition of S

$\therefore (x,y) \in S \Rightarrow (y,x) \in S$ $\square$
$\therefore S$ is symmetric
Finally we prove that S is transitive, ie if $(x,y) \in S \land (y,z) \in S \Rightarrow (x,z) \in S$
Proof: (direct)

1. Let $(x,y) \in S \land (y,z) \in S$ Hypothesis
2. $(x,y) \in S$ and $(y,z) \in S$ 1, I2, E9, I2
3. $(x,y) \in R \land (y,x) \in R$ and $(y,z) \in R \land (z,y) \in R$ definition of S twice
4. $(x,y) \in R$ and $(y,x) \in R$ and $(y,z) \in R$ and $(z,y) \in R$ (3, I2, E9, I2) twice
5. $(x,y) \in R \land (y,z) \in R$ and $(z,y) \in R \land (y,x) \in R$ 4, I6 twice
6. $(x,z) \in R$ and $(z,x) \in R$ 5, R is transitive twice
7. $(x,z) \in R \land (z,x) \in R$ 6, I6
8. $(x,z) \in S$ definition of S

$\therefore (x,y) \in S \land (y,z) \in S \Rightarrow (x,z) \in S$ $\square$
$\therefore S$ is transitive.
$\therefore S$ is reflexive, symmetric, and transitive.
$\therefore S$ is an equivalence relation. $\square$

3.2 Lemma equivalence

Before proving part b we will first prove a helpful lemma equivalence, if $(x,y) \in S \Rightarrow [x]/S = [y]/S$
We need to show that $(\forall a)(a \in [x]/S \Rightarrow a \in [y]/S) \land (\forall a)(a \in [y]/S \Rightarrow a \in [x]/S)$
Proof: (direct)
Assume $(x,y) \in S$
First we prove that $(\forall a)(a \in [x]/S \Rightarrow a \in [y]/S)$

1. $(x,y) \in S$ Hypothesis
2. Let $a \in [x]/S$ Hypothesis
3. $(x,a) \in S$ definition of $[x]/S$
4. $(a,x) \in S$ S is symmetric
5. $(a,x) \in S \land (x,y) \in S$ 4, 1, I6
6. $(a,y) \in S$ S is transitive

7. $(y, a) \in S$ — S is symmetric
8. $a \in [y]/S$ — definition of $[y]/S$

$\therefore (\forall a)(a \in [x]/S \Rightarrow a \in [y]/S)$
Now we prove that $(\forall a)(a \in [y]/S \Rightarrow a \in [x]/S)$

1. $(x, y) \in S$ — Hypothesis
2. $(y, x) \in S$ — S is symmetric
3. Let $a \in [y]/S$ — Hypothesis
4. $(y, a) \in S$ — definition of $[y]/S$
5. $(a, y) \in S$ — S is symmetric
6. $(a, y) \in S \wedge (y, x) \in S$ — 5, 2, I6
7. $(a, x) \in S$ — S is transitive
8. $(x, a) \in S$ — S is symmetric
9. $a \in [x]/S$ — definition of $[x]/S$

$\therefore (\forall a)(a \in [y]/S \Rightarrow a \in [x]/S)$
$\therefore (\forall a)(a \in [x]/S \Rightarrow a \in [y]/S) \wedge (\forall a)(a \in [y]/S \Rightarrow a \in [x]/S)$
$\therefore$ If $(x, y) \in S$, then $[x]/S = [y]/S$

3.3 Part b

Let $\widetilde{R}$ be a relation in $[X]/S$ such that $\widetilde{R} = \{([x]/S, [y]/S) \mid (x, y) \in R\}$. Prove that $\widetilde{R}$ is a partial order.
To prove that $\widetilde{R}$ is a partial order, we must prove that $\widetilde{R}$ is reflexive, antisymmetric, and transitive.
First we will prove that $\widetilde{R}$ is reflexive, ie $(Y \in [X]/S)(Y, Y) \in \widetilde{R}$

1. Let $Y \in [X]/S$ — Hypothesis
2. $(\exists v)(v \in X \wedge Y = [v]/S)$ — 1, definition of $[X]/S$
3. $v \in X \wedge Y = [v]/S$ — 2, EI
4. $v \in X$ and $Y = [v]/S$ — 3, I2, E9, I2
5. $(v, v) \in R$ — R is reflexive
6. $([v]/S, [v]/S) \in \widetilde{R}$ — definition of $\widetilde{R}$
7. $(Y, Y) \in \widetilde{R}$ — 6, $\text{sub}_=$

$\therefore \widetilde{R}$ is reflexive.
Next we will prove that $\widetilde{R}$ is antisymmetric, ie if $(A, B) \in \widetilde{R} \wedge (B, A) \in \widetilde{R} \Rightarrow A = B$
Saying $(A, B) \in \widetilde{R} \equiv (\exists a, b \in X)(((a, b) \in R \wedge A = [a]/S \wedge B = [b]/S)$, by definition of $\widetilde{R}$
Proof: (direct)

1. Let $(A, B) \in \widetilde{R} \wedge (B, A) \in \widetilde{R}$ — Hypothesis
2. $(A, B) \in \widetilde{R}$ and $(B, A) \in \widetilde{R}$ — 1, I2
3. $(a, b) \in R \wedge A = [a]/S \wedge B = [b]/S$ and $(b, a) \in R \wedge B = [b]/S \wedge A = [a]/)S$ — definition of $\widetilde{R}$ twice
4. $(a, b) \in R$ and $A = [a]/S$ and $B = [b]/S$ and $(b, a) \in R$ — 3, I2, E9

5. $(a, b) \in R \wedge (b, a) \in R$ 4, 4, I6
6. $(a, b) \in S$ 5, definition of S
7. $[a]/S = [b]/S$ 6, Lemma equivalence
8. $A = B$ 7, sub$_=$ twice

$\therefore \widetilde{R}$ is antisymmetric
Finally we will prove $\widetilde{R}$ is transitive, ie if $(A, B) \in \widetilde{R} \wedge (B, C) \in \widetilde{R} \Rightarrow (A, C) \in \widetilde{R}$
Proof: (direct)

1. $(A, B) \in \widetilde{R} \wedge (B, C) \in \widetilde{R}$ Hypothesis
2. $(A, B) \in \widetilde{R}$ and $(B, C) \in \widetilde{R}$ 1, I2, E9, I2
3. $(a, b) \in R \wedge A = [a]/S \wedge B = [b]/S$ and $(b, c) \in R \wedge B = [b]/S \wedge C = [c]/S$ 2, I2, E9, I2
4. $(a, b) \in R$ and $A = [a]/S$ and $B = [b]/S$ and $(b, c) \in R$ and $C = [c]/S$ 3, I2, E9
5. $(a, b) \in R \wedge (b, c) \in R$ 4, 4, I6
6. $(a, c) \in R$ R is transitive
7. $([a]/S, [c]/S) \in \widetilde{R}$ definition of $\widetilde{R}$
8. $(A, C) \in \widetilde{R}$ sub$_=$

$\therefore \widetilde{R}$ is transitive.
$\therefore \widetilde{R}$ is reflexive, antisymmetric, and transitive.
$\therefore \widetilde{R}$ is a partial order $\square$

4 Question 5.18

4.1 Part c

Let R be a relation in X. Prove that R is symmetric iff $R = R^{-1}$
We are to prove that R is symmetric $\Leftrightarrow R = R^{-1}$
This is the same as proving that R is symmetric $\Rightarrow R = R^{-1} \wedge R = R^{-1} \Rightarrow R$ is symmetric
First we will prove R is symmetric $\Rightarrow R = R^{-1}$
Proof: (direct)
Assume R is symmetric.
We are to prove that $R = R^{-1}$
This is the same as proving $(x, y) \in R \Rightarrow (x, y) \in R^{-1} \wedge (x, y) \in R^{-1} \Rightarrow (x, y) \in R$
First we will prove $(x, y) \in R \Rightarrow (x, y) \in R^{-1}$
Proof: (direct)

1. Let $(x, y) \in R$ Hyposthesis
2. $(y, x) \in R$ 1, R is symmetric
3. $(x, y) \in R^{-1}$ 2, definition of R^{-1}

$\therefore (x, y) \in R \Rightarrow (x, y) \in R^{-1}$
Now we prove $(x, y) \in R^{-1} \Rightarrow (x, y) \in R$
Proof: (direct)

1. Let $(x, y) \in R^{-1}$ Hypothesis
2. $(y, x) \in R$ 1, definition of R^{-1}

3. $(x, y) \in R$ 2, R is symmetric

$\therefore (x, y) \in R^{-1} \Rightarrow (x, y) \in R$
$\therefore (x, y) \in R \Rightarrow (x, y) \in R^{-1} \wedge (x, y) \in R^{-1} \Rightarrow (x, y) \in R$ I6
$\therefore (x, y) \in R \Leftrightarrow (x, y) \in R^{-1}$ E20
$\therefore R = R^{-1}$ principle of extension
$\therefore R$ is symmetric $\Rightarrow R = R^{-1}$
Next we prove that $R = R^{-1} \Rightarrow R$ is symmetric.
Proof: (direct)
Assume $R = R^{-1}$

1. Let $(x, y) \in R$ Hypothesis
2. $(y, x) \in R^{-1}$ 1, definition of R^{-1}
3. $(y, x) \in R$ 2, $\text{sub}_=$

$\therefore (x, y) \in R \Rightarrow (y, x) \in R$
$\therefore R$ is symmetric.
$\therefore R = R^{-1} \Rightarrow R$ is symmetric.
$\therefore R$ is symmetric $\Rightarrow R = R^{-1} \wedge R = R^{-1} \Rightarrow R$ is symmetric.
$\therefore R$ is symmetric $\Leftrightarrow R = R^{-1}$ E20
Hence, R is symmetric iff $R = R^{-1}$ $\square$

5 Question 6.2

Let $\mathbb{N}$ be the set of all positive integers. Let $g : \mathbb{N}$ x $\mathbb{N} \to \mathbb{N}$ such that
$g((i, j)) = 2^i 3^j$
Prove that g is a one-to-one function, Is it onto? First we will prove that g is a one-to-one function, ie that if $g((a, b)) = g((c, d))$, then $(a, b) = (c, d)$ Proof: (direct)

1. Let $g((a, b)) = g((c, d))$ Hypothesis
2. $2^a 3^b = 2^c 3^d$ definition of g twice
3. $\frac{2^a 3^b}{2^c 3^d} = 1$ Highschool algebra
4. $2^{a-c} 3^{b-d} = 1$ Highschool algebra
5. $\forall n \in \mathbb{Z} 2^n > 0$ and $\forall m \in \mathbb{Z} 3^m > 0$ Highschool algebra
6. $\therefore 2^{a-c} > 0$ and $3^{b-d} > 0$ 5, UI
7. $\because 2, 3$ are both primes $\therefore 2^{a-c} = 1$, and $3^{b-d} = 1$ 6, Highschool algebra
8. $2^{a-c} = 1 \Rightarrow a - c = 0$ Highschool algebra
9. $a - c = 0$ 7, 8, I3
10. $a = c$ highschool algebra
11. $3^{b-d} = 1 \Rightarrow b - d = 0$ Highschool algebra
12. $b - d = 0$ 7, 11, I3
13. $b = d$ 12, Highschool algebra
14. $a = c \wedge b = d$ 10, 13 I6
15. $(a, b) = (c, d)$ definition

$\therefore g$ is a one-to-one function. Next we will show that g is not onto.
Proof: (counterexample)
$2 \in \mathbb{N}$ 2 is a positive integer.
$0 \notin \mathbb{N}$, 0 is not positive integer.
$2^1 3^0 = 2$ 2 is a prime, and is its only factor.
however $\neg(\exists i, j \in \mathbb{N}) g((i, j)) = 2$, because j would have to be equal to 0, and $0 \notin \mathbb{N} \therefore g((i, j))$ is not onto.

6 Question 6.5

6.1 Part b

Prove that if $I_X \subseteq f$, then $f = I_X$
Proof: (direct)
Assume $I_X \subseteq f$
We are to prove that $f = I_X$, ie that $f \subseteq I_X \wedge I_X \subseteq f$
We already have the second by our assumption so we must prove that $f \subseteq I_X$
Proof: (direct)

1. Let $(x, y) \in f$ Hypothesis
2. $I_X \subseteq f$ Hypothesis
3. $(\exists z \in X)(x, z) \in I_X$ $Domain(f) = Domain(I) = X$
4. $(x, z) \in I_X$ EI
5. $(x, z) \in f$ 2, 4, definition of $\subseteq$
6. $(x, z) \in f \wedge (x, y) \in f$ 1, 5, I6
7. $z = y$ definition of function
8. $(x, y) \in I_X$ 4, sub$_=$

$\therefore f \subseteq I_X$
$\therefore f \subseteq I_X \wedge I_X \subseteq f$
$\therefore f = I_X$ □