[03-60-231] Assignment 2

Stephen Nusko
103693282

October 9, 2013

1 Question 2.1

1.1 Part c

Evaluate $(\forall x)(\exists y)(P(x) \Rightarrow Q(x,y))$

1. Let $x = 1$. We obtain $(\exists y)(P(1) \Rightarrow Q(1,y))$
   (a) Let $y = 2$. Then $P(1) \equiv T$ and $Q(1,2) \equiv T$
   (b) Therefore $(\exists y)(P(1) \Rightarrow Q(1,y)$ is evaluated to true.
2. Let $x = 2$. We obtain $(\exists y)(P(2) \Rightarrow Q(2,y)$
   (a) Since $P(2) \equiv F$
   (b) $\therefore (\exists y)(P(2) \Rightarrow Q(2,y))$ is evaluated to true.
3. $\therefore (\forall x)(\exists y)(P(x) \Rightarrow Q(x,y))$ is evaluated to true.

1.2 Part i

Evaluate $(\exists x)(\exists y)(Q(x,a) \Rightarrow Q(b,y)$

1. Let $x = 1$. We obtain $(\exists y)(Q(1,a) \Rightarrow Q(b,y)$
   (a) Let $y = 2$. We obtain $(Q(1,a) \Rightarrow Q(b,2)$
      i. Since $Q(1,a) \equiv Q(1,2) \equiv T \equiv Q(b,2) \equiv Q(1,2)$
      ii. $\therefore Q(1,2) \Rightarrow Q(1,2)$ is evaluated to true.
   (b) $\therefore (\exists y)(Q(1,a) \Rightarrow Q(b,y))$ is evaluated to true.
2. $\therefore (\exists x)(\exists y)(Q(x,a) \Rightarrow Q(b,y))$ is evaluated to true.

2 Question 2.2

2.1 Part c

Prove $((\forall x)\alpha(x) \vee (\forall x)\beta(x)) \Rightarrow (\forall x)(\alpha(x) \vee \beta(x))$
Solution: (indirect proof)

1. $\neg((\forall x)(\alpha(x) \vee \beta(x))$	Hypothesis
2. $(\exists x)\neg(\alpha(x) \vee \beta(x))$	1, FE7
3. $(\exists x)(\neg\alpha(x) \wedge \neg\beta(x))$	2, E17

4. $\neg\alpha(c) \wedge \neg\beta(c)$ — 3, EI, c is a constant
5. $\neg\alpha(c)$ — 4, I2
6. $\neg\beta(c) \wedge \neg\alpha(c)$ — 4, E9
7. $\neg\beta(c)$ — 6, I2
8. $(\exists x)\neg\alpha(x)$ — 5, EQ
9. $(\exists x)\neg\beta(x)$ — 7, EQ
10. $(\exists x)\neg\alpha(x) \wedge (\exists x)\neg\beta(x)$ — 8, 9, I6
11. $\neg(\forall x)\alpha(x) \wedge (\exists x)\neg\beta(x)$ — 10, FE7
12. $\neg(\forall x)\alpha(x) \wedge \neg(\forall x)\beta(x)$ — 11, FE7
13. $\neg((\forall x)\alpha(x) \vee (\forall x)\beta(x))$ — 12, E17

$\therefore \neg((\forall x)(\alpha(x) \vee \beta(x)) \Rightarrow \neg((\forall x)\alpha(x) \vee (\forall x)\beta(x))$
$\therefore ((\forall x)\alpha(x) \vee (\forall x)\beta(x)) \Rightarrow (\forall x)(\alpha(x) \vee \beta(x))$ $\square$

2.2 Part d

Prove $(\forall x)(\alpha(x) \Rightarrow \beta(x)) \Rightarrow ((\forall x)\neg\beta(x) \Rightarrow (\forall x)\neg\alpha(x))$
Solution: (indirect proof)

1. $\neg((\forall x)\neg\beta(x) \Rightarrow (\forall x)\neg\alpha(x))$ — Hypothesis
2. $\neg(\neg((\forall x)\neg\beta(x)) \vee (\forall x)\neg\alpha(x))$ — 1, E18
3. $\neg\neg(\forall x)\neg\beta(x) \wedge \neg(\forall x)\neg\alpha(x))$ — 2, E17
4. $(\forall x)\neg\beta(x) \wedge \neg(\forall x)\neg\alpha(x)$ — 3, E15
5. $(\forall x)\neg\beta(x)$ — 4, I2
6. $\neg(\forall x)\neg\alpha(x) \wedge (\forall x)\neg\beta(x)$ — 4, E9
7. $\neg(\forall x)\neg\alpha(x)$ — 7, I2
8. $(\exists x)\neg\neg\alpha(x)$ — 8, FE7
9. $\neg\neg\alpha(c)$ — 8, EI
10. $\neg\beta(c)$ — 5, UI
11. $\neg\neg\alpha(c) \wedge \neg\beta(c)$ — 9, 10, I6
12. $(\exists x)(\neg\neg\alpha(x) \wedge \neg\beta(x))$ — 11, EQ
13. $(\exists x)\neg(\neg\alpha(x) \vee \beta(x))$ — 12, E17
14. $\neg((\forall x)(\neg\alpha(x) \vee \beta(x)))$ — 13, FE7
15. $\neg((\forall x)(\alpha(x) \Rightarrow \beta(x)))$ — 14, E18

$\therefore \neg((\forall x)\neg\beta(x) \Rightarrow (\forall x)\neg\alpha(x)) \Rightarrow \neg((\forall x)(\alpha(x) \Rightarrow \beta(x)))$
$\therefore (\forall x)(\alpha(x) \Rightarrow \beta(x)) \Rightarrow (\forall x)\neg\beta(x) \Rightarrow (\forall x)\neg\alpha(x)$ $\square$

3 Question 2.3

3.1 Part d

P1:	$(\exists x)(P(x) \land (\forall y)((R(y) \land S(x,y)) \Rightarrow Z(x,y))$
P2:	$(\forall x)(P(x) \Rightarrow (\exists y)(R(y) \land \neg U(x,y) \land T(x,y))$
P3:	$(\forall x)(\forall y)((P(x) \land R(y) \land T(x,y)) \Rightarrow S(x,y)$
C:	$(\exists x)(\exists y)(P(x) \land R(x) \land Z(x,y) \land \neg U(x,y))$

Prove: $((P1 \land P2 \land P3) \Rightarrow C)$
Solution: (direct proof)

1. $(\exists x)(P(x) \land (\forall y)((R(y) \land S(x,y)) \Rightarrow Z(x,y)))$ Hypothesis from P1
2. $(\forall x)(P(x) \Rightarrow (\exists y)(R(y) \land \neg U(x,y) \land T(x,y)))$ Hypothesis from P2
3. $(\forall x)(\forall y)((P(x) \land R(y) \land T(x,y)) \Rightarrow S(x,y))$ Hypothesis from P3
4. $P(c) \land (\forall y)((R(y) \land S(c,y)) \Rightarrow Z(c,y)$ 1, EI, c is a constant
5. $P(c)$ 4, I2
6. $(\exists y)(R(y) \land \neg U(c,y) \land T(c,y)$ 5, 2, I3
7. $R(k) \land \neg U(c,k) \land T(c,k)$ 6, EI, k is a constant
8. $R(k) \land \neg U(c,k)$ 7, I2
9. $R(k)$ 8, I2
10. $\neg U(c,k) \land R(k)$ 9, E9
11. $\neg U(c,k)$ 10, I2
12. $T(c,k) \land (R(k) \land \neg U(c,k))$ 7, E9
13. $T(c,k)$ 12, I2
14. $P(c) \land R(k)$ 5, 9, I6
15. $P(c) \land R(k) \land T(c,k)$ 14, 13, I6
16. $S(c,k)$ 15, 3, I3
17. $(\forall y)((R(y) \land S(c,y) \Rightarrow Z(c,y)) \land P(c)$ 4, E9
18. $(\forall y)((R(y) \land S(c,y) \Rightarrow Z(c,y))$ 17, I2
19. $(R(k) \land S(x,k)) \Rightarrow Z(c,k)$ 18, UI
20. $R(k) \land S(c,y)$ 9, 16, I6
21. $Z(c,k)$ 20, 19, I3
22. $P(c) \land R(k) \land Z(c,k)$ 14, 21, I6
23. $P(c) \land R(k) \land Z(c,k) \land \neg U(c,k)$ 22, 11, I6
24. $(\exists y)(P(c) \land R(y) \land Z(c,y) \land \neg U(c,y)$ 23, EQ
25. $(\exists x)(\exists y)(P(x) \land R(y) \land Z(x,y) \land \neg U(x,y)$ 24, EQ

$\therefore ((P1 \land P2 \land P3) \Rightarrow C)$ $\square$

4 Question 2.4

4.1 Part f

The universe of discourse are poems.
Let I(x) denote an interesting poem.
Let M(x) denote a modern poem.
Let F(x) denote a poem written by me.
Let S(x) denote a poem written about a soap bubble.
Let R(x) denote a poem that is popular with people of real taste.
Let A(x) denote a poem that is affected.

P1:	$\neg(\exists x)(I(x) \wedge \neg R(x))$
P2:	$\neg(\exists x)(M(x) \wedge \neg B(x))$
P3:	$(\forall x)(F(x) \Rightarrow S(x))$
P4:	$\neg(\exists x)(B(x) \wedge R(x))$
P5:	$\neg(\exists x)(\neg M(x) \wedge S(x))$
C:	$(\forall x)(F(x) \Rightarrow \neg I(x))$

Prove $((P1 \wedge P2 \wedge P3 \wedge P4 \wedge P5) \Rightarrow C)$
Solution: (direct proof)

1. $\neg(\exists x)(I(x) \wedge \neg R(x))$ Hypothesis from P1
2. $\neg(\exists x)(M(x) \wedge \neg B(x))$ Hypothesis from P2
3. $(\forall x)(F(x) \Rightarrow S(x))$ Hypothesis from P3
4. $\neg(\exists x)(B(x) \wedge R(x))$ Hypothesis from P4
5. $\neg(\exists x)(\neg M(x) \wedge S(x))$ Hypothesis from P5
6. $(\forall x)\neg(I(x) \wedge \neg R(x))$ 1, FE8
7. $(\forall x)(\neg I(x) \vee \neg\neg R(x))$ 6, E16
8. $(\forall x)(\neg\neg R(x) \vee \neg I(x))$ 7, E10
9. $(\forall x)(\neg R(x) \Rightarrow \neg I(x))$ 8, E18
10. $\neg R(x) \Rightarrow \neg I(x)$ 9, UI
11. $(\forall x)\neg(\neg M(x) \wedge S(x))$ 5, FE8
12. $(\forall x)(\neg\neg M(x) \vee \neg S(x))$ 11, E16
13. $(\forall x)(\neg S(x) \vee \neg\neg M(x))$ 12, E10
14. $(\forall x)(\neg S(x) \vee M(x))$ 13, E15
15. $(\forall x)(S(x) \Rightarrow M(x))$ 14, E18
16. $S(x) \Rightarrow M(x)$ 15, UI
17. $(\forall x)\neg(M(x) \wedge \neg B(x))$ 2, FE8
18. $(\forall x)(\neg M(x) \vee \neg\neg B(x))$ 17, E16
19. $(\forall x)(\neg M(x) \vee B(x))$ 18, E15
20. $(\forall x)(M(x) \Rightarrow B(x))$ 19, E18
21. $M(x) \Rightarrow B(x)$ 20, UI

22. $(\forall x)\neg(B(x) \wedge R(x))$ 4, FE8
23. $(\forall x)(\neg B(x) \vee \neg R(x))$ 22, E16
24. $(\forall x)(B(x) \Rightarrow \neg R(x))$ 23, E18
25. $B(x) \Rightarrow \neg R(x)$ 24, UI
26. $S(x) \Rightarrow B(x)$ 16, 21, I5
27. $S(x) \Rightarrow \neg R(x)$ 26, 25, I5
28. $S(x) \Rightarrow \neg I(x)$ 27, 10, I5
29. $F(x) \Rightarrow \neg I(x)$ 3, 28, I5

$\therefore ((P1 \wedge P2 \wedge P3 \wedge P4 \wedge P5) \Rightarrow C)$ $\square$