[03-60-231] Assignment 3

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October 28, 2013

1 Question 3.1

1.1 Part b

Prove that if $A \subset B$, and $B \subseteq C$, then $A \subset C$
Proof: (Direct Proof) Suppose $A \subset B$ and $B \subseteq C$
We are to prove $A \subset C$ by definition this is equivalent to $A \subseteq C$ and $A \neq C$

1. First we prove $A \subseteq C$
   (a) $A \subseteq B \land A \neq B$ — (assumption $A \subset B$), (definition of $\subset$)
   (b) $A \subseteq B$ — a, I2
   (c) $A \subseteq C$ — (assumpution $B \subseteq C$), b, Lemma 3.2.3 (iii)
2. $\therefore A \subseteq C$
3. Now we prove $A \neq C$
   (a) $(\exists x)(x \in B \land x \notin A)$ — (assumption $A \subset C$), Lemma 3.2.6
   (b) $a \in B \land a \notin A$ — a, EI, a is a new constant
   (c) $a \in B$ — b, I2
   (d) $a \notin A \land a \in B$ — b, E9
   (e) $a \notin A$ — d, I2
   (f) $(\forall x)(x \in B \Rightarrow x \in C)$ — (assumption $B \subseteq C$), (definition of $\subseteq$)
   (g) $a \in B \Rightarrow a \in C$ — f, UI
   (h) $a \in C$ — c, g, I3
   (i) $a \notin A \land a \in C$ — e, h, I6
   (j) $(\exists x)(x \notin A \land x \in C)$ — i, EQ
   (k) $(\exists x)(x \notin A \land x \in C) \lor (\exists x)(x \in A \land x \notin C)$ — j, I1
   (l) $(\exists x)(x \in A \land x \notin C) \lor (\exists x)(x \notin A \land x \in C)$ — k, E10
   (m) $A \neq C$ — l, (defintion of $\neq$)
4. $\therefore A \neq C$
5. $\therefore A \subseteq C \land A \neq C$ — 2, 4, I6
6. $\therefore A \subset C$ — 5, (definition of $\subset$)

hence $A \subset B \land B \subseteq C \Rightarrow A \subset C$ $\square$

2 Question 4.5

2.1 Part c

Prove that $(\overline{A} \cap B \cap \overline{C} \cap D) \cup (A \cap \overline{C}) \cup (\overline{B} \cup \overline{C}) \cup (\overline{C} \cap \overline{D}) = \overline{C}$
Proof: (Bidirectional proof)

1. $\Leftrightarrow (\overline{A} \cap B \cap \overline{C} \cap D) \cup (A \cap \overline{C}) \cup (\overline{B} \cap \overline{C}) \cup (\overline{C} \cap \overline{D})$
2. $\Leftrightarrow (\overline{A} \cap B \cap \overline{C} \cap D) \cup (\overline{C} \cap A) \cup (\overline{B} \cap \overline{C}) \cup (\overline{C} \cap \overline{D})$ — 1, Thm 4.2.2 (iii)
3. $\Leftrightarrow (\overline{A} \cap B \cap \overline{C} \cap D) \cup (\overline{C} \cap A) \cup (\overline{C} \cap \overline{B}) \cup (\overline{C} \cap \overline{D})$ — 2, Thm 4.2.2 (iii)
4. $\Leftrightarrow (\overline{A} \cap B \cap \overline{C} \cap D) \cup (\overline{C} \cap A) \cup (\overline{C} \cap (\overline{B} \cup \overline{D}))$ — 3, Thm 4.2.3 (ii)
5. $\Leftrightarrow (\overline{A} \cap B \cap \overline{C} \cap D) \cup (\overline{C} \cap A) \cup (\overline{C} \cap (\overline{D} \cup \overline{B}))$ — 4, Thm 4.2.2 (iii)
6. $\Leftrightarrow (\overline{A} \cap B \cap \overline{C} \cap D) \cup (\overline{C} \cap (A \cup (\overline{D} \cup \overline{B})))$ — 5, Thm 4.2.3 (ii)
7. $\Leftrightarrow (\overline{C} \cap D \cap \overline{A} \cap B) \cup (\overline{C} \cap ((A \cup \overline{D}) \cup \overline{B}))$ — 6, Thm 4.2.2 (iv)
8. $\Leftrightarrow (\overline{C} \cap (D \cap \overline{A} \cap B)) \cup (\overline{C} \cap (\overline{D} \cup A \cup \overline{B}))$ — 7, Thm 4.2.2 (iii)
9. $\Leftrightarrow (\overline{C} \cap (D \cap \overline{A} \cap B)) \cup (\overline{C} \cap (\overline{D} \cup \overline{\overline{A}} \cup \overline{B}))$ — 8, Thm 4.3.7 (i)
10. $\Leftrightarrow (\overline{C} \cap (D \cap \overline{A} \cap B)) \cup (\overline{C} \cap \overline{(D \cap \overline{A} \cap B)})$ — 9, Thm 4.3.6 (ii)x2
11. $\Leftrightarrow (\overline{C} \cap ((D \cap \overline{A} \cap B)) \cup \overline{(D \cap \overline{A} \cap B)})$ — 10, Thm 4.2.3 (ii)
12. $\Leftrightarrow (\overline{C} \cap U)$ — 11, Thm 4.3.7 (iii)
13. $\Leftrightarrow \overline{C}$ — 12, definition of $\cap$ and definition of U

Hence, $(\overline{A} \cap B \cap \overline{C} \cap D) \cup (A \cap \overline{C}) \cup (\overline{C} \cap \overline{D}) = \overline{C}$ □

3 Question 4.6

3.1 Part a

Let $X \cup Y = X$ for and set X, then $(\forall X)(X \cup Y = X)$. Prove that $Y = \emptyset$
Proof: (contridiction)

1. suppose $Y \neq \emptyset$, then $(\exists x)(x \in Y)$ — assumption
2. $x \in Y$ — 1, EI
3. $x \in Y \vee x \in \emptyset$ — 2, I1
4. $x \in \emptyset \vee x \in Y$ — 3, E10
5. $x \in (\emptyset \cup Y)$ — 4, definition of $\cup$
6. $\emptyset \cup Y = \emptyset$ — definition of sets X and Y, UI
7. $x \in \emptyset$ — 5, 6, $\text{sub}_=$
8. $false$ — 7, definition of $\emptyset$

Hence if $X \cup Y = X$ for all set X, then $Y = \emptyset$ □

4 Question 4.8

4.1 Part c

Prove that $(A \cup B) - C = (A - C) \cup (B - C)$
This is equivalent to proving $(\forall x)(x \in ((A \cup B) - C)) \Leftrightarrow (\forall x)(x \in ((A - C) \cup (B - C)))$
Proof: (Bidirectional proof)

1. $\Leftrightarrow (\forall x)(x \in ((A \cup B) - C))$
2. $\Leftrightarrow x \in ((A \cup B) - C)$ UI
3. $\Leftrightarrow x \in (A \cup B) \wedge x \notin C$ (definition of $-$)
4. $\Leftrightarrow (x \in A \vee x \in B) \wedge x \notin C$ (definition of $\cup$)
5. $\Leftrightarrow x \notin C \wedge (x \in A \vee x \in B)$ E9
6. $\Leftrightarrow (x \notin C \wedge x \in A) \vee (x \notin C \wedge x \in B)$ E13
7. $\Leftrightarrow (x \in A \wedge x \notin C) \vee (x \in B \wedge x \notin C)$ E10, E10
8. $\Leftrightarrow (x \in (A - C)) \vee (x \in (B - C))$ (definition of $-$), (defintion of $-$)
9. $\Leftrightarrow x \in ((A - C) \cup (B - C))$ (definition of $\cup$)
10. $\Leftrightarrow (\forall x)(x \in ((A - C) \cup (B - C)))$ gen

Hence, $(A \cup B) - C = (A - C) \cup (B - C)$ $\square$

5 Question 4.9

5.1 Part a

Prove that $A \subseteq B \Leftrightarrow A \cap \overline{B} = \emptyset$
We must prove $(A \subseteq B \Rightarrow A \cap \overline{B}) \wedge (A \cap \overline{B} \Rightarrow A \subseteq B)$
First we will prove that $A \subseteq B \Rightarrow A \cap \overline{B} = \emptyset$
Proof: (contridiction) Assume $A \subseteq B$ and $A \cap \overline{B} \neq \emptyset$

1. $(\forall x)(x \in A \Rightarrow x \in B)$ assumption, (definition of $\subseteq$)
2. $(\exists x)(x \in A \cap \overline{B})$ assumption, (definition of $\neq \emptyset$)
3. $x \in A \Rightarrow x \in B$ 1, UI
4. $x \in A \cap \overline{B}$ 2, EI
5. $x \in A \wedge x \in \overline{B}$ 2, (definition of $\cap$)
6. $x \in A$ 5, I2
7. $x \in B$ 6, 3, I3
8. $x \in \overline{B} \wedge x \in A$ 4, E9
9. $x \in \overline{B}$ 7, I2
10. $x \in B \wedge x \in \overline{B}$ 7, 9, I6
11. $false$ 10, E1

$\therefore A \subseteq B \Rightarrow A \cap \overline{B} = \emptyset$
Now we prove $A \cap \overline{B} = \emptyset \Rightarrow A \subseteq B$
Proof: (Direct proof)
Assume $A \cap \overline{B} = \emptyset$

1. $\neg(\exists x)(x \in A \cap \overline{B})$ — assumption, (definition of $= \emptyset$)
2. $(\forall x)\neg(x \in A \cap \overline{B})$ — 1, FE8
3. $\neg(x \in A \cap x \in \overline{B})$ — 2, UI
4. $\neg(x \in A \wedge x \in \overline{B})$ — 3, (definition of $\cap$)
5. $\neg(x \in A) \vee x \notin \overline{B}$ — 4, E16
6. $\neg(x \in A) \vee x \in \overline{\overline{B}}$ — 5, Lemma 4.3.5
7. $\neg(x \in A) \vee x \in B$ — 6,
8. $x \in A \Rightarrow x \in B$ — 7, E18
9. $(\forall x)(x \in A \Rightarrow x \in B)$ — 8, gen
10. $A \subseteq B$ — 9, (definition of $\subseteq$)

$\therefore A \cap \overline{B} = \emptyset \Rightarrow A \subseteq B$
$\therefore (A \subseteq B \Rightarrow A \cap \overline{B} = \emptyset) \wedge (A \cap \overline{B} = \emptyset \Rightarrow A \subseteq B)$ I6
Hence, $A \subseteq B \Leftrightarrow A \cap \overline{B} = \emptyset$ $\square$

6 Question 4.11

6.1 Part b

We are prove or disprove that $P(A \cup B) = P(A) \cup P(B)$
To disprove we must show either $P(A \cup B) \not\subseteq P(A) \cup P(B)$ or $P(A) \cup P(B) \not\subseteq P(A \cup B)$
We will show $P(A \cup B) \not\subseteq P(A) \cup P(B)$
Disproof: (by counter example)
Let $A = a, b$, and $B = 0, 1$
$A \cup B = \{a, b, 0, 1\}$
$P(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
$P(B) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$
$P(A) \cup P(B) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{0\}, \{1\}, \{0, 1\}\}$
$P(A \cup B) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{0\}, \{1\}, \{0, 1\}, \{a, 0\}, \{a, 1\}, \{b, 0\}, \{b, 1\}, \{a, b, 0\}, \{a, b, 1\}, \{0, 1, a\}, \{0, 1, b\}, \{a, b, 0, 1\}\}$
since $\{a, b, 0, 1\} \in P(A \cup B)$ and $\{a, b, 0, 1\} \notin P(A) \cup P(B)$
$\therefore P(A \cup B) \not\subseteq P(A) \cup P(B)$
Hence, $P(A \cup B) \neq (A) \cup P(B)$ $\square$

7 Question 4.14

7.1 Part a

Prove $\bigcup_{x \in \{A\}} X = A$
We are to prove $(\forall x)(x \in \bigcup_{x \in \{A\}} X \Leftrightarrow x \in A)$
First we will prove $(\forall x)(x \in \bigcup_{x \in \{A\}} X \Rightarrow x \in A)$
Proof: (Direct proof)

1. Let $x \in \bigcup_{x\in\{A\}} X$,then $(\exists X)(X \in \{A\} \wedge x \in X)$ — assumption, UI, (defintion of $\bigcup$)
2. Since $A = A$, then $A \in \{A\}$ — Lemma 3.2.1, (definition of $\{A\}$)
3. $A \in \{A\} \wedge x \in A$ — 1, 2, EI
4. $x \in A \wedge A \in \{A\}$ — 3, E9
5. $x \in A$ — 4, I2
6. $(\forall x)(x \in A)$ — 5, gen

$\therefore (\forall x)(x \in \bigcup_{x\in\{A\}} X \Rightarrow x \in A)$

Now we prove $(\forall x)(x \in A \Rightarrow x \in \bigcup_{x\in\{A\}} X)$ Proof: (Direct proof)

1. Let $x \in A$ — Assumption, UI
2. $A \in \{A\}$ — Lemma 3.2.1, (definition of $\{A\}$)
3. $A \in \{A\} \wedge x \in A$ — 2, 1, I6
4. $(\exists X)(X \in \{A\} \wedge x \in X)$ — 3, EQ
5. $x \in \bigcup_{x\in\{A\}} X$ — 4, (definition of $\bigcup$)
6. $(\forall x)(x \in \bigcup_{x\in\{A\}} X)$ — 5, gen

$\therefore (\forall x)(x \in A \Rightarrow x \in \bigcup_{x\in\{A\}} X)$

$\therefore (\forall x)((x \in \bigcup_{x\in\{A\}} X \Rightarrow x \in A) \wedge (x \in A \Rightarrow x \in \bigcup_{x\in\{A\}} X))$ — I6

$\therefore (\forall x)(x \in \bigcup_{x\in\{A\}} X \Leftrightarrow x \in A)$ — E20

Hence, $\bigcup_{x\in\{A\}} X = A$ $\square$