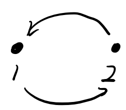
How to Describe the Microstates of a Multi-Body Quantum System

N particles $\Psi(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_N)$

Identical Particles

E.g. 2 particles



$$|\Psi(x_1, x_2)|^2 = |\Psi(x_2, x_1)|^2$$

$\hat{P}_{12}$ exchange particles 1 & 2

$$\hat{P}_{12}\Psi(x_1, x_2) = \Psi(x_2, x_1)$$

$$\hat{P}_{12}^2\Psi(x_1, x_2) = \Psi(x_1, x_2)$$

$$P_{12} = \pm 1 \begin{cases} +1 & \text{bosons} \\ -1 & \text{fermions} \end{cases}$$

N-particles: $\mathcal{P} \in S_N \rightarrow$ permutation group

Boson: $\mathcal{P}\Psi(x_1, x_2, \cdots, x_N) = \Psi(x_1, \cdots, x_N)$

Fermion: $\mathcal{P}\Psi(x_1, x_2, \cdots, x_N) = \underset{\text{parity of permutation}}{(-1)^P}\Psi(x_1, \cdots, x_N)$

* Non-interacting system

$\Psi(x_1, x_2, \cdots, x_N)$ can be built out of a single particle ψ

$$\hat{H} = \hat{H}_1 + \hat{H}_2$$

$$\begin{cases} \hat{H}_1\psi_{k_1}(x) = \varepsilon_{k_1}\psi_{k_1}(x) \\ \hat{H}_2\psi_{k_2}(x) = \varepsilon_{k_2}\psi_{k_2}(x) \end{cases}$$

$$\Psi(x_1, x_2) = \psi_{k_1}(x_1)\psi_{k_2}(x_2), \quad \hat{H}\Psi(x_1, x_2) = (\varepsilon_{k_1} + \varepsilon_{k_2})\Psi(x_1, x_2)$$

$$\mathcal{P}_{12}\Psi(x_1,x_2) = \psi_{k_1}(x_2)\psi_{k_2}(x_1) \neq \pm\Psi(x_1,x_2) \quad \text{unless } k_1 = k_2$$

Need to symmetrize or anti-symmetrize

$$\Rightarrow \Psi(x_1,x_2) = \frac{1}{\sqrt{2}}[\psi_{k_1}(x_1)\psi_{k_2}(x_2) \pm \psi_{k_1}(x_2)\psi_{k_2}(x_1)]$$

$+$ for bosons
$-$ for fermions

$\Psi_F(x_1,x_2) = 0$ if $k_1 = k_2$ (Pauli exclusive principle)

N particles:

$$\Psi_B(x_1,x_2,\ldots,x_N) = A\sum_{P\in S_N} \psi_{k_1}(x_{p_1})\psi_{k_2}(x_{p_2})\cdots\psi_{k_N}(x_{p_N})$$

$$\Psi_F(x_1,x_2,\ldots,x_N) = A\sum_{P\in S_N} (-1)^P \psi_{k_1}(x_{p_1})\psi_{k_2}(x_{p_2})\cdots\psi_{k_N}(x_{p_N})$$

It's redundant to label particles as $\{x_1,x_2,\ldots,x_N\}$ and then (anti) symmetric over the labels

$\Rightarrow$ Essential information: single particle states $\{k_1,k_2,\ldots,k_N\}$

$\Rightarrow$ $\boxed{\text{occupation number of single-particle states}}$ $\{n_k\}$ quantum #

$|n_{k_1}, n_{k_2}, \ldots, n_{k_N}\rangle \Rightarrow$ Fock basis

boson: $n_{k_i} = 0,1,2,\ldots$ "second quantization"

fermion: $n_{k_i} = 0,1$

e.g. free particles in a box

$$\phi_k(\vec{x}) = \frac{1}{\sqrt{V}} e^{i\vec{k}\cdot\vec{x}}$$ $\vec{k}$ is quantized according to BC

Non-interacting:

$$N = \sum_k n_k \qquad E = \sum_k \varepsilon_k n_k$$ single-particle energy (no interaction)

$$Z(T, V, N) = \sum_E e^{-\beta E(\{n_k\})}$$

$$= \sum_{\{n_k\}}{}' e^{-\beta \sum_k \varepsilon_k n_k} \qquad \text{constrained to } \sum_k n_k = N$$

$$\Xi(T, V, \mu) = \sum_{N=1}^{\infty} e^{\beta \mu N} Z(T, V, N)$$

$$= \sum_{N=1}^{\infty} \sum_{\{n_k\}}{}' e^{\beta(\mu \sum_k n_k - \sum_k \varepsilon_k n_k)}$$

$$= \sum_{\{n_k\}} e^{-\beta \sum_k (\varepsilon_k - \mu) n_k}$$

$$= \sum_{\{n_k\}} \prod_k e^{-\beta(\varepsilon_k - \mu) n_k}$$

$$= \prod_k \sum_{n_k} e^{-\beta(\varepsilon_k - \mu) n_k}$$

$\sum_{n_k}$ bosons: $\sum_{n_k=0}^{\infty}$, $\quad \Xi = \prod_k \overbrace{\frac{1}{1 - e^{-\beta(\varepsilon_k - \mu)}}}^{\Xi_k} = \prod_k \Xi_k$

fermions: $\sum_{n_k=0}^{1}$, $\quad \Xi = \prod_k \left[1 + e^{-\beta(\varepsilon_k - \mu)}\right]$

$$\ln \Xi = \begin{cases} -\sum_k \ln\left[1 - e^{-\beta(\varepsilon_k - \mu)}\right] & \text{bosons} \\ \sum_k \ln\left[1 + e^{-\beta(\varepsilon_k - \mu)}\right] & \text{fermions} \end{cases} \qquad \ln \Xi = \sum_k \ln \Xi_k$$

$$\bar{N} = \frac{\partial \ln \Xi}{\partial(\beta\mu)} = \begin{cases} \sum_k \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1} = \sum_k \{n_k\} & \text{bosons} \\ \sum_k \frac{1}{e^{\beta(\varepsilon_k - \mu)} + 1} = \sum_k \{n_k\} & \text{fermions} \end{cases}$$

Average occupation number of a single particle state

$$\langle n_k \rangle = \begin{cases} \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1} & \text{bosons} \quad \text{B-E distribution} \\ \frac{1}{e^{\beta(\varepsilon_k - \mu)} + 1} & \text{fermions} \quad \text{F-D distribution} \end{cases} \qquad (\mu \sim T \ln T)$$

Classical M-B distribution $\langle n_k \rangle_{M.B.} = e^{-\beta(\varepsilon_k - \mu)}$

In the limit $e^{-\beta\mu} \gg 1$, $\langle n_k \rangle_{M.B.} = \langle n_k \rangle_{B.E.} = \langle n_k \rangle_{F.D.}$ $(z \ll 1)$

Physically $\langle n_k \rangle \ll 1$. the difference between distinguishable and identical particles is unimport

Non-degenerate limit $\mapsto$ particles are very far away from each other

$e^{\beta\mu} = n\lambda^3 \to 0$, $\beta\mu \xrightarrow{T\to\infty} -\infty$

high-temperature limit $\begin{cases} \ll 1 & \text{QUANTUM} \\ \text{otherwise} & \text{CLASSICAL} \end{cases}$

T must be large enough to overgap the energy gap $k_B T \gg \Delta\varepsilon_k$

thus we'll neglect the discretation of energy level