dwz 数理方程 期末考点

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参考 zfx xgg 的讲义（爱心 × ∞）

一、简答类

①化简方程（S-L形式） ②积分变换 ③求Green函数
④写出定解的条件极值.

二、大题类

Chap. 17~20 各一道.

Chap 17. 分离变量法总结

考点：(1)化简方程 & (2)求本征值

(1). 将以下方程化为 S-L 标准型.

* S-L：$\frac{d}{dx}[p(x)\frac{dy}{dx}]+[\lambda\rho(x)-q(x)]y=0$

即：$y''+\frac{p'(x)}{p(x)}y'+\frac{\lambda\rho(x)-q(x)}{p(x)}y=0$

依次化简即可.

[22]：$y''-x^4y'+\lambda y=0$

解. 对比 $y''+\frac{p'(x)}{p(x)}y'+\frac{\lambda\rho(x)-q(x)}{p(x)}y=0$

$\Rightarrow \frac{p'(x)}{p(x)}=-x^4$. $p(x)=e^{\int -x^4dx}=e^{-\frac{1}{5}x^5}$.

且 $\rho(x)=p(x)$, $q(x)=0$

∴ 化为：$\frac{d}{dx}[e^{-\frac{x^5}{5}}\frac{dy}{dx}]+\lambda e^{-\frac{x^5}{5}}y=0$.

[20]. $y''-xy'+\lambda y=0$

解. 对比 $y''+\frac{p'(x)}{p(x)}y'+\frac{\lambda\rho(x)-q(x)}{p(x)}y=0$

$\frac{p'(x)}{p(x)}=-x$. $p(x)=e^{\int -xdx}=e^{-\frac{x^2}{2}}$.

$\rho(x)=p(x)$, $q(x)\equiv 0$

∴ S-L：$\frac{d}{dx}(e^{-\frac{x^2}{2}}\frac{dy}{dx})+\lambda e^{-\frac{x^2}{2}}y=0$

[18]：$(1-x^2)y''(x)-2xy'(x)+\lambda y(x)=0$

解：对比 $y''+\frac{p'(x)}{p(x)}y'+\frac{\lambda\rho(x)-q(x)}{p(x)}y=0$；

$\frac{p'(x)}{p(x)}=\frac{-2x}{1-x^2}$，$\rho(x)=\frac{p(x)}{1-x^2}$, $q(x)=0$

∴ $p(x)=e^{\int\frac{2x}{x^2-1}dx}=x^2-1 \Rightarrow \rho(x)=-1$

∴ 化为 $\frac{d}{dx}[(x^2-1)\frac{dy}{dx}]-\lambda y=0$

(2) 求本征值，化为 $y''+\lambda y=0$ 型方程或其他形式方程.

[22]：权当作练习，各位自己做去（太水了）

[21] $y''+\lambda y=0$，$y(0)=0$，$y(1)+y'(1)=0$

(1) 证明 $\lambda_n>0$ (2) 证/写出 本征函数

解：(1) (左乘) y^*，积分.

$-\int y^*y''dx=\lambda\int\|y\|^2dx$

$y^*y'\Big|_L-\int y'y^{*\prime}dx=-\int\|y'\|^2dx$

$\Leftrightarrow \lambda_n=\frac{\int\|y'\|^2dx}{\int\|y\|^2dx}>0$

(2) 很显然，$y_n=\sin nx$ 与 $\cos nx$ 的线性组合

由 $y(0)=0\Rightarrow$ 可设 $y=\sum_{i=1}^{\infty}\sin nx$

其中 $n=\sqrt{\lambda_n}$.

由 $y(1)+y'(1)=0\Rightarrow \sin n+n\cos n=0$

即 $\underline{n=-\tan n}$

λ_n 为 $\sqrt{\lambda_n}=-\tan\sqrt{\lambda_n}$ 的解（第n个解）

$\int_0^1 y_n^2dx=\int_0^1\sin^2\sqrt{\lambda}x\,dx$

$=\frac{1}{2}(1-\frac{1}{2\sqrt{\lambda}}\sin 2\sqrt{\lambda})$

∴ 归一化正交本征函数

$$y_n=\sqrt{\frac{2}{1-\frac{1}{2\sqrt{\lambda_n}}\sin 2\sqrt{\lambda_n}}}\sin\sqrt{\lambda_n}x.$$

[20]. $xy''+2y'+\lambda xy=0$ 求本征值，本征函数及正交关系

解：对比：$\begin{cases} y''+\frac{2}{x}y'+\lambda y=0 \\ y''+\frac{p'}{p}y'+\frac{\lambda\rho-q}{p}y=0 \end{cases}$

$\frac{p'(x)}{p(x)}=\frac{2}{x}$. $p(x)=e^{\int\frac{2}{x}dx}=x^2$

$\Rightarrow \frac{d}{dx}[x^2\frac{dy}{dx}]+\lambda x^2y=0$

$\Rightarrow \frac{1}{\lambda x^2}\frac{d}{dx}[x^2\frac{dy}{dx}]+y=0$

$\Rightarrow \frac{1}{(\sqrt{\lambda}x)^2}\frac{d}{d(\sqrt{\lambda}x)}\cdot[(\sqrt{\lambda}x)^2\cdot\frac{dy}{d(\sqrt{\lambda}x)}]+y=0$

$\frac{1}{\tau^2}\frac{d}{d\tau}[\tau^2\frac{dy}{d\tau}]+y=0$. 为球bessel方程. 0阶.

$\Rightarrow y=Aj_0(\sqrt{\lambda}x)+Bn_0(\sqrt{\lambda}x)$

$y(0)$ 有界 $\Rightarrow B=0$.

$ay'(a)+y(a)=0 \Leftrightarrow$

$-a\sqrt{\lambda}j_1(\sqrt{\lambda}a)+j_0(\sqrt{\lambda}a)=0$

$\Rightarrow \frac{\sin\sqrt{\lambda}a}{\sqrt{\lambda}a}=\sqrt{\lambda}a\cdot\frac{1}{\lambda a^2}[\sin\sqrt{\lambda}a-a\sqrt{\lambda}\cos(\sqrt{\lambda}a)]$

$\Rightarrow \sqrt{\lambda}a\cos(\sqrt{\lambda}a)=0$

$\begin{cases} \lambda=0 \Rightarrow y_0=1 \\ \lambda\neq0 \Rightarrow \lambda_n=(\frac{2n+1}{2a}\pi)^2.\ y_n=j_0(\frac{2n+1}{2a}\pi x) \end{cases}$

$n=0 \rightarrow \int_0^a y_0^2x^2dx=\frac{a^3}{3}$

$n\neq0 \Rightarrow \int_0^a y_0^2x^2dx=\frac{2a^3}{(2n+1)^2\pi^2}$

不同本征函数有 $\int_0^a y_0(\sqrt{\lambda_i}x)y_0(\sqrt{\lambda_j}x)x^2dx$

$=\delta_{ij}\cdot\frac{2a^3}{(2n+1)^2\pi^2}$ $(i.j\neq0)$.

[18]. $xy''+y'+\lambda xy=0$. $(0<x<l)$. $y(0)$有界. $y'(l)=0$

解：对比 $\begin{cases} y''+\frac{1}{x}y'+\lambda y=0 \\ y''+\frac{p'}{p}y'+\frac{\lambda\rho-q}{p}y=0 \end{cases}$

$\Rightarrow \frac{p'}{p}=\frac{1}{x}$. $p(x)=e^{\int\frac{1}{x}dx}=x$

$\therefore \frac{d}{dx}[x\frac{dy}{dx}]+\lambda xy=0$

$\Rightarrow \frac{1}{x}\frac{d}{dx}(x\frac{dy}{dx})+\lambda y=0$

令 $t=\sqrt{\lambda}x \Rightarrow \frac{1}{t}\frac{d}{dt}(t\frac{dy}{dt})+y=0$

为 0阶 Bessel 方程 $\Rightarrow$

$y=A\cdot J_0(\sqrt{\lambda}x)+B\cdot N_0(\sqrt{\lambda}x)$.

由 $y(0)$ 有界 $\Rightarrow B=0$.

$y'(l)=0 \Rightarrow \sqrt{\lambda}J_0'(\sqrt{\lambda}l)=0 \Rightarrow -J_1(\sqrt{\lambda})=0$

$\Rightarrow \lambda_i=(\frac{\mu_i}{l})^2$

$y_i=J_0(\frac{\mu_i^{(1)}}{l}x)$.

归一化：$y_i=\frac{\sqrt{2}}{J_0(\mu_i^{(1)})}\cdot J_0(\mu_i^{(1)}x)$.

Chap 18. 积分变换法.

一. 简答类：求Fourier 变换或写出 $U(k,p)$ 方程.

[22]. 求 $e^{-x}\eta(x)$ 的 Fourier 变换：

$F(e^{-x}\eta(x))=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-x}\eta(x)\cdot e^{-ikx}dx$

$=\frac{1}{\sqrt{2\pi}}\cdot\int_0^\infty e^{-(1+ik)x}dx$

$=\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{1+ik}$

[21]. $(1-x^2)^{\nu-\frac{1}{2}}\eta(1-x^2)$ 的Fourier 变换：

$F((1-x^2)^{\nu-\frac{1}{2}}\eta(1-x^2))=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(1-x^2)^{\nu-\frac{1}{2}}\cdot\eta(1-x^2)\cdot e^{-ikx}dx$

$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)^{\nu-\frac{1}{2}}e^{-ikx}dx$. 令 $x=\cos\theta$

$\therefore$ 原式 $=\frac{1}{\sqrt{2\pi}}\int_0^\pi(\sin\theta)^{2\nu-1}\cdot e^{-ik\cos\theta}\cdot\sin\theta d\theta$

利用 $e^{-ik\cos\theta}=\sum_{l=0}^{\infty}(-i)^l\cdot(2l+1)j_l(kr)P_l(\cos\theta)$

上式 $=\frac{1}{\sqrt{2\pi}}\sum_{l=0}^{\infty}(-i)^l\cdot(2l+1)\cdot j_l(kr)\cdot\int_0^\pi(\sin\theta)^{2\nu}\cdot P_l(\cos\theta)d\theta$

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[21.1(2)]

一、接上：

原式 $=\frac{1}{\sqrt{2\pi}}\sum_{l=0}^{\infty}(i)^l\cdot(2l+1)\cdot j_l(kr)\cdot\int_0^{\pi}(\sin\theta)^{2\nu}\cdot P_l(\cos\theta)d\theta$

利用 $P_l(\cos\theta)=\frac{1}{2^l\cdot l!}\frac{d^l}{d(\cos\theta)^l}\cdot(\cos^2\theta-1)^l$

上式 $=\frac{1}{\sqrt{2\pi}}\sum_{l=0}^{\infty}(i)^l\cdot(2l+1)j_l(kr)\cdot\int_0^{\pi}\cdot\frac{1}{2^l l!}\cdot(\sin\theta)^{2\nu}\cdot\frac{d^l}{d(\cos\theta)}\cdot(\sin\theta)^{2l}d\theta$

着手于求下式：$\int_{-1}^{1}(1-x^2)^{\nu-\frac{1}{2}}\cdot\frac{d^l}{dx^l}\cdot(1-x^2)^l\,dx$

之后暂时不会。^^ ...

[20]. $\begin{cases}\frac{\partial^2u}{\partial t^2}-c^2\frac{\partial^2u}{\partial x^2}=e^{-ax^2} & ,t>0.\ -\infty<x<\infty\\ u|_{t=0}=0, & \frac{\partial u}{\partial t}|_{t=0}=0\end{cases}$

作 Fourier 变换.

记 $U(k,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,t)e^{-ikx}dx$

则 $F(\frac{\partial^2u}{\partial t^2})=\frac{d^2U}{dt^2}$, $F(\frac{\partial^2u}{\partial x^2})=-k^2U$

$F(e^{-ax^2})=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ax^2-ikx}dx$

$=\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{k^2}{4a}}\cdot\int_{-\infty}^{\infty}e^{-a(x-\frac{ik}{2a})^2}dx$

$=\frac{1}{\sqrt{2a}}e^{-\frac{k^2}{4a}}$

两边对 t 作得：$\frac{d^2U}{dt^2}+c^2k^2U=\frac{1}{\sqrt{2a}}e^{-\frac{k^2}{4a}}$.

$U|_{t=0}=0$. $\frac{dU}{dt}|_{t=0}=0$

[18]. ~~$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=$~~

$\frac{\partial^2u}{\partial t^2}-\frac{\partial^2u}{\partial x^2}=0$. $u|_{x=0}=0$. $\frac{\partial u}{\partial x}|_{x=1}=\sin t$.

$u|_{t=0}=0$, $\frac{\partial u}{\partial t}|_{t=0}=0$

设 $U(x,p)=\int_0^{\infty}u(x,t)e^{-pt}dt$

$\Rightarrow\quad p^2U-\frac{d^2U}{dx^2}=0\Rightarrow\frac{d^2U}{dx^2}-p^2U=0$

$U|_{x=0}=0$, $\frac{\partial U}{\partial x}|_{x=1}=\frac{1}{p^2+1}$

$U|_{t=0}=0$, $\frac{dU}{dt}|_{t=0}=0$

[17]. 仅大题，先不读/没有问答.

二、计算题.

[22]. $\frac{\partial u}{\partial t}-\kappa\frac{\partial^2u}{\partial x^2}=0$. $0<x<l$. $t>0$

$u|_{x=0}=A\sin\omega t$. $u|_{x=l}=0$ $(t>0)$

$u|_{t=0}=0$ $(0\le x\le l)$

求 Laplace 变换像函数 $U(x,p)$.

解. 设 $U(x,p)=\int_0^{\infty}u(x,t)e^{-pt}dt$

则 $\mathcal{L}\{\frac{\partial u}{\partial t}\}=pU\Rightarrow$

$pU(x,p)-\kappa\frac{d^2U}{dx^2}=0$

即 $\frac{d^2U}{dx^2}-\frac{p}{\kappa}U=0$.

$U(x,p)=A'e^{\sqrt{\frac{p}{\kappa}}x}+Be^{-\sqrt{\frac{p}{\kappa}}x}$

由 $U|_{x=0}=\frac{A\omega}{p^2+\omega^2}$. $U|_{x=l}=0$

$\Rightarrow\begin{cases}A'+B=\frac{A\omega}{p^2+\omega^2}\\ A'\cdot e^{\sqrt{\frac{p}{\kappa}}l}+B\cdot e^{-\sqrt{\frac{p}{\kappa}}l}=0\end{cases}$

即 $B=-A'\cdot e^{2\sqrt{\frac{p}{\kappa}}l}$

$\Rightarrow\begin{cases}A'=\frac{A\omega}{(p^2+\omega^2)\cdot(1-e^{2\sqrt{\frac{p}{\kappa}}l})}\\ B=\frac{-A\omega e^{2\sqrt{\frac{p}{\kappa}}l}}{(p^2+\omega^2)(1-e^{2\sqrt{\frac{p}{\kappa}}l})}\end{cases}$

$\therefore U(x,p)=\frac{A\omega}{(p^2+\omega^2)(1-e^{2\sqrt{\frac{p}{\kappa}}l})}\cdot\left[e^{\sqrt{\frac{p}{\kappa}}x}-e^{2\sqrt{\frac{p}{\kappa}}l-\sqrt{\frac{p}{\kappa}}x}\right]$

[21]. 一维热传导用 Laplace 解.

$\frac{\partial u}{\partial t}-\kappa\cdot\frac{\partial^2u}{\partial x^2}=0$. $0<x<l$. $t>0$

$u|_{x=0}=0$. $u|_{x=l}=0$. $u|_{t=0}=u_0x(l-x)$.

此时 $u|_{t=0}$ 存在非零状况.

此定解问题，设 $U(x,p)=\mathcal{L}(u)=\int_0^{\infty}u(x,t)e^{-pt}dt$

则：$pU-\kappa\cdot\frac{d^2U}{dx^2}-u_0x(l-x)=0$

$\Rightarrow\quad\frac{d^2U}{dx^2}-\frac{p}{\kappa}U=-\frac{u_0}{\kappa}x(l-x)$

由解的唯一性定出 $U(x,p)=\frac{u_0}{p}\left[x(l-x)-\frac{2\kappa}{p}\right]$

这个特解使得 $U|_{x=0}=U|_{x=l}=-\frac{2\kappa u_0}{p^2}$

反代求 $U(x,p)$ 得

$$U(x,p)=\frac{u_0}{p}\left[x(l-x)-\frac{2k}{p}\right]+\frac{2ku_0}{p\sinh\sqrt{\frac{p}{k}}l}\cdot\left[\sinh\sqrt{\frac{p}{k}}x+\sinh\sqrt{\frac{p}{k}}(l-x)\right]$$

下面进行反演：

$$\mathcal{L}^{-1}\left(\frac{u_0}{p}\left[x(l-x)-\frac{2k}{p}\right]\right)=\underline{u_0x(l-x)-2ku_0t}$$

对 $\frac{2ku_0}{p\sinh\sqrt{\frac{p}{k}}l}\left[\sinh\sqrt{\frac{p}{k}}x+\sinh\sqrt{\frac{p}{k}}(l-x)\right]$

$$=2ku_0\,\mathrm{res}\left\{\frac{e^{pt}}{p^2}\cdot\frac{\sinh\sqrt{\frac{p}{k}}x+\sinh\sqrt{\frac{p}{k}}(l-x)}{\sinh\sqrt{\frac{p}{k}}l}\right\}$$

分母零点/极点 时仅一阶，$p=-(\frac{n\pi}{l})^2k$

<u>经过计算</u>（此处略去 qwq）

$$u(x,t)=u_0x(l-x)-2ku_0t+\frac{8u_0}{l}\sum_{n=0}^{\infty}\frac{e^{-(\frac{2n+1}{l}\pi)^2kt}}{(\frac{(2n+1)\pi}{l})^3}\cdot\sin\left(\frac{2n+1}{l}\pi x\right).$$

[20]. $\frac{\partial^2u}{\partial t^2}-a^2\frac{\partial^2u}{\partial x^2}=0$. $u|_{x=0}=0$. $\frac{\partial u}{\partial x}|_{x=l}=0$

$u|_{t=0}=u_0\sin\left(\frac{\pi x}{2l}\right)$, $\frac{\partial u}{\partial t}|_{t=0}=0$.

解：采用 Laplace 变换.

设 $U(x,p)=\int_0^\infty u(x,t)e^{-pt}dt$

代入初值条件的方程：

$$p^2U-u_0\sin\left(\frac{\pi x}{2l}\right)p-a^2\frac{d^2U}{dx^2}=0$$

$$\Rightarrow\frac{d^2U}{dx^2}-\frac{p^2}{a^2}U=-\frac{u_0p}{a^2}\sin\left(\frac{\pi x}{2l}\right)$$

设 $U(x,p)=A(p)\cdot\sin kx+B(p)\cdot\cos kx$

则：~~$-(A+B)k^2$~~

$$-Ak^2\sin kx-Bk^2\cos kx-\frac{p^2}{a^2}A\sin kx-\frac{p^2}{a^2}B\cos kx=-\frac{u_0p}{a^2}\sin\left(\frac{\pi}{2l}x\right)$$

代入得 $B=0$，$A\left(k^2+\frac{p^2}{a^2}\right)\sin kx=\frac{u_0p}{a^2}\sin\left(\frac{\pi}{2l}x\right)$

∴取 $k=\frac{\pi}{2l}$，$A(p)=\frac{u_0p}{(\frac{\pi a}{2l})^2+p^2}$

即特解 $U_1(p,x)=\frac{u_0p}{(\frac{\pi a}{2l})^2+p^2}\sin\left(\frac{\pi}{2l}x\right)$

这样使得 $U|_{x=0}=0$，$\frac{dU}{dx}|_{x=l}=0$ 符合条件

$$\therefore U(x,p)=\frac{u_0p}{p^2+\frac{\pi^2a^2}{4l^2}}\sin\frac{\pi}{2l}x$$

反演 $u=u_0\cos\left(\frac{\pi a}{2l}t\right)\sin\frac{\pi x}{2l}$.

[18]

※注意 $\mathcal{L}\{f''(t)\}=\underline{p^2F(p)-pf(0)-f'(0)}$

$\frac{\partial^2u}{\partial t^2}-\frac{\partial^2u}{\partial x^2}=0$. $u|_{t=0}=u_0e^{-(\frac{x}{a})^2}$, $\frac{\partial u}{\partial t}|_{t=0}=0$

Fourier 变换解此题：

设 $U(k,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,t)e^{-ikx}dx$

得到：$\frac{d^2U}{dt^2}+k^2U=0$.

①：$U=Ae^{ikt}+Be^{-ikt}$

由 $e^{-(\frac{x}{a})^2}$ 的 Fourier 变换：

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{x^2}{a^2}-ikx}dx=\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{a^2k^2}{4}}\int_{-\infty}^{\infty}e^{-(\frac{x}{a}+\frac{ika}{2})^2}dx$$

$$=\frac{1}{\sqrt{2\pi}}\cdot ae^{-\frac{a^2k^2}{4}}\cdot\sqrt{\pi}=\frac{a}{\sqrt{2}}e^{-\frac{a^2k^2}{4}}$$

$\Rightarrow\frac{dU}{dt}|_{t=0}=0$. $U|_{t=0}=\frac{u_0a}{\sqrt{2}}e^{-\frac{a^2k^2}{4}}$

得 $U=\frac{u_0a}{2\sqrt{2}}e^{-\frac{a^2k^2}{4}}\cdot(e^{ikt}+e^{-ikt})$

反演，

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\cdot\frac{u_0a}{2\sqrt{2}}\int_{-\infty}^{\infty}e^{-\frac{a^2k^2}{4}}\cdot(e^{ikt}+e^{-ikt})\cdot e^{ikx}dk$$

$$=\frac{u_0a}{4\sqrt{\pi}}\cdot\frac{2}{a}\cdot\left[e^{-(\frac{x+t}{a})^2}+e^{-(\frac{x-t}{a})^2}\right]\cdot\sqrt{\pi}$$

$$=\frac{u_0}{2}\left[e^{-(\frac{x+t}{a})^2}+e^{-(\frac{x-t}{a})^2}\right]$$

② (provided by HimzFX).

取 $U=A\sin kt+B\cos kt$. 由 $\frac{dU}{dt}|_{t=0}=0$ 得 $A=0$.

$U|_{t=0}=B=\phi(x)$

逆变换取卷积式应用：$u=u_0e^{-(\frac{x}{a})^2}*F^{-1}(\cos kt)$

$$\therefore u=u_0\cdot\frac{1}{\sqrt{2\pi}}\cdot\frac{\sqrt{2\pi}}{2}\int_{-\infty}^{+\infty}[\delta(\xi+t)+\delta(\xi-t)]e^{-(\frac{x-\xi}{a})^2}d\xi$$

(利用 $F^{-1}(\cos kt)=\frac{\sqrt{2\pi}}{2}[\delta(x+t)+\delta(x-t)]$)

得 $u=\frac{u_0}{2}\left[e^{-(\frac{x+t}{a})^2}+e^{-(\frac{x-t}{a})^2}\right]$ 与上同.

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[1]. $\frac{\partial u}{\partial t}-k\frac{\partial^2 u}{\partial x^2}=0$. $0<x<a$, $t>0$

$\frac{\partial u}{\partial x}\Big|_{x=0}=0$, $\frac{\partial u}{\partial x}\Big|_{x=a}=1$. $u\Big|_{t=0}=\frac{x^2}{2a}$

解:(本题为积分变换大题之模板)

① 观察范围: $0<x<a$ // $t>0$. 对有界的 x 与半无界 t:

使用 Laplace 变换. 令 $U(x,p)=\int_0^\infty u(x,t)e^{-pt}dt$

$\mathcal{L}\left(\frac{\partial u}{\partial t}\right)=pU-u\big|_{t=0}=pU-\frac{x^2}{2a}$.

$\Rightarrow pU-\frac{x^2}{2a}-k\frac{d^2U}{dx^2}=0$. 同时 $\frac{dU}{dx}\Big|_{x=0}=0$

$\frac{d^2U}{dx^2}-\frac{p}{k}U=-\frac{x^2}{2ak}$, $\frac{dU}{dx}\Big|_{x=a}=\frac{1}{p}$

(由微分方程解之唯一性)先找特解

设 $U_1(x,p)=Ax^2+B$

$\Rightarrow 2A-\frac{p}{k}(Ax^2+B)=-\frac{x^2}{2ak}$

$\therefore \frac{p}{k}A=\frac{1}{2ak}$, $A=\frac{1}{2ap}$.

$2A=\frac{p}{k}B$. $B=\frac{k}{ap^2}$

$\therefore U_1(x,p)=\frac{x^2}{2ap}+\frac{k}{ap^2}$

这使得: $\frac{\partial U_1}{\partial x}\Big|_0=0$. $\frac{\partial U_1}{\partial x}\Big|_a=\frac{1}{p}$

再解: $\frac{d^2U}{dx^2}-\frac{p}{k}U=0$. $\frac{\partial U}{\partial x}\Big|_0=0$. $\frac{\partial U}{\partial x}\Big|_a=0$

$\Rightarrow U=Ae^{\sqrt{\frac{p}{k}}x}+Be^{-\sqrt{\frac{p}{k}}x}$

$\Rightarrow \begin{cases} A=B \\ A\sqrt{\frac{p}{k}}e^{\sqrt{\frac{p}{k}}a}-B\sqrt{\frac{p}{k}}e^{\sqrt{\frac{p}{k}}a}=0 \end{cases}$

$\Rightarrow A=\cancel{-\frac{1}{p\sqrt{\frac{p}{k}}(e^{\sqrt{\frac{p}{k}}a}-e^{\sqrt{\frac{p}{k}}a})}}\ 0=B$

由此得 $U(x,p)=\cancel{-\frac{\cosh\sqrt{\frac{p}{k}}x}{(p\sqrt{\frac{p}{k}})\cdot\sinh\sqrt{\frac{p}{k}}a}}\ \frac{x^2}{2ap}+\frac{k}{ap^2}$.

下进行反演:(由普遍反演公式. 见[3])

$u(x,t)=\frac{x^2}{2a}+\frac{k}{a}t$

[总结] 积分变换法先看用哪种变换.

无界就 Fourier, 有界就 Laplace.

边界, 初始条件也要变换别错了

反演公式请牢记.

Chap 19. Green 函数

一、问答类.

[2]. 求解 $\left[\frac{d^2}{dx^2}+1\right]g(x;\xi)=\delta(x-\xi)$, $0<x,\xi<1$

$g(0;\xi)=0$, $\frac{dg(x;\xi)}{dx}\Big|_{x=1}=0$

(这就是数理上的内容)

解: 当 $x<\xi$ 时. $\delta(x-\xi)=0$

$\left[\frac{d^2}{dx^2}+1\right]g(x;\xi)=0$. $g(x;\xi)=A\sin x+B\cos x$

由 $g(0;\xi)=0$ 得 $B=0$.

同理. 当 $x>\xi$ 时. $\delta(x-\xi)=0$.

$\left[\frac{d^2}{dx^2}+1\right]g(x;\xi)=0$. $g(x;\xi)=C\sin x+D\cos x$

由 $\frac{dg}{dx}\Big|_{x=1}=0$. 得. $C\cos 1-D\sin 1=0$, $\underline{C=D\tan 1}$

整合有: $g(x;\xi)=\begin{cases} A\sin x. & x<\xi \\ D\tan 1\cdot\sin x+D\cdot\cos x. & x>\xi \end{cases}$

对微分方程在 $\xi-0^+$ 至 $\xi+0^+$ 积分.

$\frac{dg}{dx}\Big|_{\xi-0}^{\xi+0}=D\tan 1\cdot\cos\xi+D\sin\xi-A\cos\xi=1$

由于 $g(x;\xi)$ 在 $[\xi-0,\xi+0]$ 连续:

$A\sin\xi=D(\tan 1\cdot\sin\xi+\cos\xi)$

化简得 $D=-\frac{\sin\xi}{\cos^2\xi}$, $C=-\frac{\sin\xi}{\cos^2\xi}\tan 1$.

$A=-\frac{\sin\xi}{\cos^2\xi}\tan 1-\frac{1}{\cos\xi}$

$\therefore g(x;\xi)=\begin{cases} -\frac{\sin\xi\tan 1}{\cos^2\xi}\sin x-\frac{1}{\cos\xi}\sin x, & x<\xi \\ -\frac{\sin\xi}{\cos^2\xi}\tan 1\sin x-\frac{\sin\xi}{\cos^2\xi}\cos x, & x>\xi. \end{cases}$

☆ 求解 Green 函数的两大要点:

① $g(x;\xi)$ 在 $(\xi-0,\xi+0)$ 上连续

② 对 $\xi-0\sim\xi+0$ 积分 $\boxed{\text{整个微分方程}}$, 得导数关系

对①② 联立即可.

[21] 写出球内 Helmholtz 方程定解问题及 Green 函数满足的方程.

答: 球内 Helmholtz 方程:

$$\nabla^2 u + k^2 u = 0 \quad . \quad u\big|_{\Sigma} = f(\theta,\varphi)$$

Green 函数满足的方程:

$$\nabla^2 G(\vec r;\vec r') + k^2 G = \delta(\vec r-\vec r').$$

$$G\big|_{r=a} = 0$$

[20]. 球内 Helmholtz 第二类边值问题及 Green 函数的定解问题:

$$\nabla^2 u + k^2 u = 0 . \qquad u\big|_{r=a} = f(\theta,\varphi);$$

Green: $\nabla^2 G(\vec r;\vec r') + k^2 G = \delta(\vec r-\vec r')$ (可以加 $\frac{-1}{\varepsilon_0}$)

$$\frac{\partial G}{\partial r}\Big|_{r=a} = 0.$$

[18]. $\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \\ u\big|_{x=0},\ \frac{\partial u}{\partial x}\big|_{x=1} = \mu(y), \quad u\big|_{y=0} = 0 .\ \frac{\partial u}{\partial y}\big|_{y=1} = 0 \end{cases}$

写出 Green 定解问题

答: $\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = \delta(x-x')\,\delta(y-y').$

$$G\big|_{x=0} = G\big|_{y=0} = \frac{\partial G}{\partial x}\Big|_{x=1} = \frac{\partial G}{\partial y}\Big|_{y=1} = 0.$$

[17]. 只有大题.

二. 计算类

[22]. $\left.\begin{aligned} \nabla^2 u &= r^l Y_{lm}(\theta,\varphi) \\ u\big|_{r=a} &= 0 \end{aligned}\right\}$ 最终 $u(\vec r) = \frac{-(a^2-r^2)\, r^l Y_{lm}(\theta,\varphi)}{2(2l+3)}.$

利用 Green 函数来求解

(个人认为本题难度较高)

首先直接写出球内 Poisson 方程的 Green 函数:

$$G(\vec r,\vec r') = \frac{1}{|\vec r-\vec r'|} - \frac{a}{r'}\,\frac{1}{|\vec r-(\frac{a}{r'})^2\vec r'|}$$

其满足 $\nabla^2 G = -4\pi\,\delta(\vec r-\vec r')$

$$\nabla'^2 G = -4\pi\,\delta(\vec r-\vec r') . \qquad G\big|_{r'=a} = 0 \quad (\Delta)$$

且(改写) $\nabla'^2 u = r'^l Y_{lm}(\theta',\varphi')$, $u\big|_{r'=a} = 0$ (*)

作操作

$$\iiint_{V|r'<a} (u\nabla'^2 G - G\nabla'^2 u)\, dV' = -4\pi u(\vec r) - \iiint_{r'<a} G\cdot r'^l Y_{lm}(\theta',\varphi')\, dV'$$

左式.化为面积分:

$$LHS = \oiint_{\Sigma} [u\nabla' G - G\nabla u]\cdot d\vec\Sigma = 0$$

$$\therefore u(\vec r) = \frac{-1}{4\pi}\iiint_{r'<a} G\cdot r'^l Y_{lm}(\theta',\varphi')\, dV'$$

代入 $G(\vec r,\vec r')$ 的表达式; 其中

$$\left.\begin{aligned} \frac{1}{|\vec r-\vec r'|} &= \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} , \\ \frac{1}{|\vec r-\frac{a^2}{r'^2}\vec r'|} &= \frac{1}{\sqrt{r^2+\frac{a^4}{r'^2}-2r\frac{a^2}{r'}\cos\gamma}} \end{aligned}\right\}$$ 利用 γ 夹角相同

我们逐项计算:

第一项 $\int_0^r r'^2 dr' \int_0^\pi \sin\theta'\cdot d\theta' \cdot \frac{r'^l Y_{lm}(\theta',\varphi')}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} \int_0^{2\pi} d\varphi'$

对分母展开. 上式 $= \int_0^r r'^2 dr' \int_0^{2\pi} d\varphi' \cdot \int_0^\pi \sin\theta' d\theta'\, r'^l Y_{lm}(\theta',\varphi') .\ \frac{1}{r}\sum_{k=0}^{\infty}\left(\frac{r'}{r}\right)^k \cdot \frac{4\pi}{2k+1}\cdot\sum_{m=-k}^{k} Y_{km}\cdot Y_{km}^*$ ①

$+ \int_r^a r'^2 dr' \int_0^{2\pi} d\varphi' . \int_0^\pi \sin\theta' d\theta'\, r'^l Y_{lm}(\theta',\varphi') .\ \frac{1}{r'}\sum_{k=0}^{\infty}\left(\frac{r}{r'}\right)^k \cdot \frac{4\pi}{2k+1}\sum_{m=-k}^{k} Y_{km}\cdot Y_{km}^*$ ②

上式 $= \int_0^r r'^{2+l} \cdot \frac{1}{r}\left(\frac{r'}{r}\right)^l \cdot \frac{4\pi}{\cancel{2k}\,2l+1}\, Y_{lm}(\theta,\varphi)\, dr'$

$+ \int_r^a r'^{2+l}\, \frac{1}{r'}\left(\frac{r}{r'}\right)^l \frac{4\pi}{2l+1}\, Y_{lm}(\theta,\varphi)\, dr'$

$= \frac{4\pi}{(2l+1)(2l+3)} Y_{lm}(\theta,\varphi)\, r^{l+2} + \frac{2\pi}{2l+1} r^l Y_{lm}(\theta,\varphi) \cdot (a^2-r^2)$

再看下项: $\iiint_{r'<a} \frac{\frac{a}{r'} r'^l Y_{lm}(\theta',\varphi')}{\sqrt{r^2+\frac{a^4}{r'^2}-2r\frac{a^2}{r'}\cos\gamma}}\, dV'$

$$= \int_0^a r'^2 dr' \int_0^{2\pi} d\varphi' \int_0^\pi \sin\theta' d\theta' \cdot \frac{1}{a} r'^l Y_{lm}(\theta',\varphi') \cdot \sum_{k=0}^{\infty}\left(\frac{rr'}{a^2}\right)^k \sum_{m=-k}^{k}\frac{4\pi}{2k+1}\cdot Y_{km} Y_{km}^*$$

$$= \int_0^a r'^2\, r'^l \cdot r'^l dr' \cdot \left[\frac{4\pi}{2l+1}\cdot\frac{r^l}{a^{2l+1}} Y_{lm}(\theta,\varphi)\right]$$

$$= \frac{4\pi a^2}{(2l+1)(2l+3)} Y_{lm}(\theta,\varphi)\, r^l .$$

$\therefore$ 全代入: $u(\vec r) = (a^2-r^2)\left[\frac{Y_{lm}(\theta,\varphi)\, r^l}{(2l+1)(2l+3)} - \frac{r^l\, Y_{lm}(\theta,\varphi)}{2(2l+1)}\right]$

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Green 函数.(接上).

[21].(太恶心了,先跳过)/和22年一样 ✓

[20]. 球内 Laplace 方程第一类边值问题的 Green 函数:

$$G(\vec r,\vec r') = \frac{1}{|\vec r-\vec r'|} - \frac{a}{r'}\frac{1}{|\vec r - \frac{a^2}{r'^2}\vec r'|}$$

(1) 计算 $\nabla^2 G(\vec r,\vec r')$ 在球内的值

(2) 利用之求解: $\begin{cases}\nabla^2 u = A+Br^2\sin2\theta\cos\phi\\ u|_{r=a}=0\end{cases}$,

A.B 为已知常数.

解、(1) $\nabla^2 G(\vec r,\vec r') = -4\pi\delta(\vec r-\vec r')$ (利用电像法)

(2): 我们将 $A+Br^2\sin2\theta\cos\phi$ 转化为 $r^l Y_{lm}(\theta,\phi)$ 型:

$$Y_{lm}(\theta,\varphi) = \sqrt{\frac{(l-|m|)!}{(l+|m|)!}\cdot\frac{2l+1}{4\pi}}\cdot P_{l|m|}(\cos\theta)e^{im\phi}$$

$$Y_{00}=\sqrt{4\pi}$$

$A+Br^2\sin2\theta\cos\varphi$

$$= A+Br^2\cdot 2\cos\theta(1-\cos^2\theta)^{\frac12}\cdot\frac{e^{i\varphi}+e^{-i\varphi}}{2}$$

$$= A+Br^2\cos\theta(1-\cos^2\theta)^{\frac12}(e^{i\varphi}+e^{-i\varphi})$$

$$P_{lm}(\cos\theta) = \frac{(-1)^m}{2^l\cdot l!}(\sin\theta)^m\frac{d^{l+m}}{d(\cos\theta)^{l+m}}(\cos^2\theta-1)^l$$

取 $l=2$, $m=1$

$$Y_{21}(\theta,\varphi) = \sqrt{\frac{5}{24\pi}}\cdot P_{21}(\cos\theta)e^{i\varphi}$$

$$=\sqrt{\frac{5}{24\pi}}\cdot\frac{-1}{8}\sin\theta\cdot\frac{d^3}{d(\cos\theta)^3}(\cos^4\theta-2\cos^2\theta+1)\cdot e^{i\varphi}$$

$$=\sqrt{\frac{5}{24\pi}}\cdot(-3\sin\theta\cos\theta)\cdot e^{i\varphi}$$

$$=\sqrt{\frac{5}{24\pi}}\cdot\left(-\frac32\sin2\theta\, e^{i\varphi}\right)$$

∴ $Y_{21}+Y_{2-1} = \sqrt{\frac{5}{24\pi}}\cdot(-3\sin2\theta\cos\varphi)$

∴ $A+Br^2\sin2\theta\cos\varphi$

$$=A\sqrt{4\pi}\,Y_{00}(\theta,\varphi) - \sqrt{\frac{8\pi}{15}}(Y_{21}+Y_{2-1})\cdot Br^2$$

$$=A\sqrt{4\pi}\,Y_{00} - B\sqrt{\frac{8\pi}{15}}(Y_{21}+Y_{2-1})r^2$$

下应用[22]之结论:

$\nabla^2 u = r^l Y_{lm}(\theta,\varphi)$ 的解为

$$u=\frac{(r^2-a^2)r^l Y_{lm}(\theta,\varphi)}{2(2l+3)}$$

代入此处:

$$u(\vec r) = \frac{A\sqrt{4\pi}\cdot(r^2-a^2)Y_{00}}{6} + \frac{B\sqrt{\frac{8\pi}{15}}(a^2-r^2)r^2(Y_{21}+Y_{2-1})}{14}$$

$$= \frac{A(r^2-a^2)}{6} + \frac{B\sqrt{\frac{8\pi}{15}}(a^2-r^2)r^2}{14}\cdot\sqrt{\frac{5}{24\pi}}(-3\sin2\theta\cos\varphi)$$

$$=\frac{A(r^2-a^2)}{6}+\frac{B\sin2\theta\cos\varphi\cdot r^2(r^2-a^2)}{14}$$ ✓

[18].(书本作业原题)

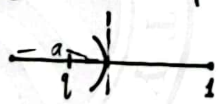
用电像法求出球外 Laplace 方程第一类边值问题的 Green 函数,并由此求球面 $r=a$ 上感应电荷分布 $\sigma(\theta,\varphi)$

解:先写出该定解问题满足的方程:

$$\nabla^2 G(\vec r,\vec r') = -\frac{1}{\varepsilon_0}\delta(\vec r-\vec r'),\quad r,r'>a$$

$$G|_{r=a}=0$$

电像法思想:假设可等效为球内一电荷,连线同轴.



(如左图)

在这种情况下. $G(\vec r,\vec r') = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} + \frac{q}{\sqrt{r^2+r''^2-2rr''\cos\gamma}}\right]$

代入 $G|_{r=a}=0$ 得:

$$\frac{1}{r'}\sum_{l=0}^{\infty}\left(\frac{a}{r'}\right)^l P_l(\cos\theta) = -\frac{q}{a}\sum_{l=0}^{\infty}\left(\frac{r''}{a}\right)^l P_l(\cos\theta)$$

由 $P_l(x)$ 的正交性:

$l=0$ 时, $q=-\frac{a}{r'}$;

$l\neq0$ 时 只能有 $\frac{a}{r'}=\frac{r''}{a} \Rightarrow r''=\frac{a^2}{r'}$

∴ $G=\frac{1}{4\pi\varepsilon_0}\left[\frac{1}{|\vec r-\vec r'|} - \frac{a}{r'}\cdot\frac{1}{|\vec r-\frac{a^2}{r'^2}\vec r'|}\right]$

再下求: $\sigma(\theta,\varphi) = -\varepsilon_0\frac{\partial G}{\partial r}\Big|_{r=a}$

$$=\frac{1}{4\pi a}\cdot\frac{a^2-r'^2}{(r'^2+a^2-2ar'\cos\theta)^{\frac32}}$$

[17]. 同[18].

Green 还有补充. 见后附纸.

Chap 20. 变分法初步.

一、问答类

复习: E~L方程. $\frac{\partial F}{\partial y}-\frac{d}{dx}\frac{\partial F}{\partial y'}=0$

$\frac{d}{dx}[p(x)\frac{dy}{dx}]+q(x)y(x)=f(x)$

为 $J[y]=\int_{x_0}^{x_1}\{\frac{1}{2}[p(x)(\frac{dy}{dx})^2-q(x)y^2(x)]+f(x)y(x)\}dx$

取极值的条件.

[22]. $\int_{x_0}^{x_1}\sqrt{1+y'^2}\,dx$. 取极值的E-L方程

若曲线过 (x_0,y_0) 和 (x_1,y_1), 求曲线方程.

解. $\frac{\partial F}{\partial y}=0 \Rightarrow \frac{d}{dx}\frac{\partial F}{\partial y'}=0$

即: $\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}}$

$=\frac{y''\sqrt{1+y'^2}-\frac{y'}{\sqrt{1+y'^2}}\cdot y'\cdot y''}{1+y'^2}$

$=(1+y'^2)^{-\frac{3}{2}}\cdot[y''(1+y'^2)-y'^2y'']=0$

$\Leftrightarrow \quad y''=0$

这即代表: $y=Ax+B$

由于过 (x_0,y_0) 和 (x_1,y_1)

$\therefore \quad y=\frac{y_1-y_0}{x_1-x_0}(x-x_0)+y_0$

[21]. $\int_{x_0}^{x_1}\sqrt{1+x}\sqrt{1+y'^2}\,dx$.

$\frac{\partial F}{\partial y}-\frac{d}{dx}\frac{\partial F}{\partial y'}=0 \Leftrightarrow$ 由于 $\frac{\partial F}{\partial y}=0 \Rightarrow \frac{d}{dx}\frac{\partial F}{\partial y'}=0$

$\therefore \frac{d}{dx}[\sqrt{1+x}\cdot\frac{y'}{\sqrt{1+y'^2}}]=0$

由[22]. $\frac{d}{dx}(\frac{y'}{\sqrt{1+y'^2}})=\frac{y''}{(1+y'^2)^{\frac{3}{2}}}$

$\therefore$ 上式 $=\frac{1}{2\sqrt{1+x}}\cdot\frac{y'}{\sqrt{(1+y'^2)}}+\sqrt{1+x}\cdot\frac{y''}{(\sqrt{1+y'^2})^3}=0$

$\Leftrightarrow \quad y'(1+y'^2)=-y''\cdot 2(1+x)$

即: $2(1+x)\cdot y''+(y')^3+y'=0$

本2阶常微分方程不含y. 故设 $u=y'$

$\Rightarrow 2(1+x)\cdot u'+u^3+u=0$

在 $u\neq0$ 时. 由常微分方程解的唯一性

(如果复数域?) $u=\pm i$; 但不行

若 $u'\neq0 \Rightarrow$ 考虑 $u=Ax^{-\frac{1}{2}}$

$2(1+x)\cdot(-\frac{1}{2}Ax^{-\frac{3}{2}})+A^3x^{-\frac{3}{2}}+Ax^{-\frac{1}{2}}=0$

$\Rightarrow -Ax^{-\frac{3}{2}}-Ax^{-\frac{1}{2}}+A^3x^{-\frac{3}{2}}+Ax^{-\frac{1}{2}}=0$

$\Rightarrow A^3-A=0.\ A(\neq0)=\pm1$

$\therefore u=\pm\frac{1}{\sqrt{x}}$. (此处解答有误. 参见P9)

代入. $y=\int\pm\frac{1}{\sqrt{x}}dx=\pm2\sqrt{x}$ ✓

[20]. $\int_{x_0}^{x_1}y^2y'^2\,dx$. 写E-L方程及初积分

E-L方程给出: $\frac{\partial F}{\partial y}-\frac{d}{dx}\frac{\partial F}{\partial y'}=0$

依题. $2y\cdot y'^2-\frac{d}{dx}(2y'\cdot y^2)=0$

$2y\cdot y'^2-2[y''y^2+2y\cdot y'^2]=0$

$\Rightarrow y'^2-y''y-2y'^2=0$

$\Leftrightarrow y'^2=-y''y$

$\frac{y''}{y'}=-\frac{y'}{y}$

与出初积分. $\ln y'=-\ln y+C$, 即

$y'y=e^C$, $y'y=C$ ✓

或: 初积分: 在F不显含x时.

$\frac{d}{dx}(y'\frac{\partial F}{\partial y'}-F)=0$

$\Rightarrow \frac{d}{dx}(y^2y'^2)=0$. 即

$y\cdot y'=\text{Const}$ ✓

[18]. $I=\int_{x_0}^{x_1}y\cdot\sqrt{1+y'^2}\,dx$

由 $\frac{\partial F}{\partial y}-\frac{d}{dx}\cdot\frac{\partial F}{\partial y'}=0$ 得:

$\sqrt{1+y'^2}-\frac{d}{dx}(y\cdot\frac{y'}{\sqrt{1+y'^2}})=0$

$\sqrt{1+y'^2}-y'^2/\sqrt{1+y'^2}-y\cdot\frac{y''\sqrt{1+y'^2}-\frac{y'^2y''}{\sqrt{1+y'^2}}}{1+y'^2}=0$

$\Rightarrow \frac{1}{\sqrt{1+y'^2}}-\frac{y\cdot y''}{(1+y'^2)^{\frac{3}{2}}}=0$

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变分法.(接上)

即为 $1+y'^2 - y\cdot y''=0$

⇒ $y''y = 1+y'^2$ 即为其微分方程.

[17]. 只有大题.

二、计算类

复习: Rayleigh-Ritz 变分法:

本征值 → 泛函条件极值 $\xrightarrow{\text{试探函数}}$ 求解.

[22]. $y''+\frac{1}{x}y'+\lambda y=0$. 试探函数 $\alpha(1-x^2)$

先化为 S-L 型方程.

上式 = $\frac{d}{dx}\left[x\frac{dy}{dx}\right]+\lambda x y=0$

∴泛函可以写作:

(参考: $\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)+q(x)y(x)=f(x) \rightsquigarrow \int\left\{\frac12\left[p(x)\left(\frac{dy}{dx}\right)^2-q(x)y^2(x)\right]+f(x)y(x)\right\}dx$)

⇒ $\frac12\delta\int_0^1 \left(x\cdot\left(\frac{dy}{dx}\right)^2-\lambda x y^2\right)dx=0$

将 λ 视作 Lagrange 乘子:

$J[y]=\int_0^1 x\cdot y'^2dx$, $J_1[y]=\int_0^1 xy^2dx=1$

由 $y=\alpha_1(1-x^2)$

⇒ $J[y]=\int_0^1 x\cdot 4x^2\alpha_1^2dx=\alpha_1^2$

$J_1[y]=\int_0^1\frac12\alpha_1^2(1-t)^2dt=\frac{\alpha_1^2}{2}\cdot\frac13=\frac{\alpha_1^2}{6}=1$

∴ $\alpha_1=\sqrt6$, $\overline{\lambda_1}=\alpha_1^2=6$.

$(y_i,\bar{y}_1)=\frac{\sqrt2}{J_1(\mu_i)}\cdot\int_0^1 J_0(\mu_i x)\cdot\sqrt6(1-x^2)x\,dx$

$=\frac{2\sqrt3}{J_1(\mu_i)}\cdot\int_0^1 J_0(\mu_i x)\cdot(x-x^3)\,dx$

下计算此积分: (由 chpp17-P):

$\int_0^1(1-x^2)\cdot\frac{1}{\mu_i}\frac{d}{dx}\left(xJ_1(\mu_i x)\right)dx$

$=\frac{2}{\mu_i}\int_0^1 x^2 J_1(\mu_i x)\,dx=\frac{2}{\mu_i^2}J_2(\mu_i)$

∴ $(y_i,\bar{y}_1)=\frac{2\sqrt3}{J_1(\mu_i)}\cdot\frac{2}{\mu_i^2}J_2(\mu_i)=\frac{4\sqrt3}{\mu_i^2}\cdot\frac{J_2(\mu_i)}{J_1(\mu_i)}$

P8. [21] 问答补拉.

直接猜解不好, 应用首积分.

$F(x,y,y')=\sqrt{1+x}\cdot\sqrt{1+y'^2}$

$\frac{\partial F}{\partial y}=0$ ⇒ $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$ 故有:

$\frac{d}{dx}\left(\sqrt{1+x}\cdot\frac{y'}{\sqrt{1+y'^2}}\right)=0$. 即

$\sqrt{1+x}\cdot y'=c\cdot\sqrt{1+y'^2}$

即 $(1+x)y'^2=c^2(1+y'^2)$, 记 $c^2=C_1$

⇒ $y'^2=\frac{C_1}{1+x-C_1}$, $y'=\frac{\pm\sqrt{C_1}}{\sqrt{1+x-C_1}}$

∴反推, $y=\pm2\sqrt{C_1}\cdot\sqrt{1-C_1+x}+C_2$

C_1, C_2 为常数.

[21]. $(1+x)y''+\lambda y=0$ 最小本征值估计. $y=cx(1-x)$

先化为 S-L 方程:

$y''+\frac{\lambda}{1+x}y=0$

⇒ $\frac{d}{dx}\left(\frac{dy}{dx}\right)+\frac{\lambda}{1+x}y=0$

其泛函极值条件, $\frac12\delta\int_0^1\left(\left(\frac{dy}{dx}\right)^2-\frac{\lambda}{1+x}\cdot y^2\right)dx=0$

将 λ 视作 Lagrange 乘子:

$J[y]=\int_0^1 y'^2dx$. $J_1[y]=\int_0^1\frac{y^2}{1+x}dx=1$

由 $y=cx(1-x)$.

$y'=c(1-2x)$, $J[y]=\int_0^1c^2(4x^2-4x+1)\,dx$

$=\frac13c^2$.

$J_1[y]=\int_0^1\frac{c^2x^2(1-x)^2}{1+x}dx$

$=c^2\int_0^1\left(x^3-3x^2+4x-4+\frac{4}{1+x}\right)dx=c^2\left(-\frac{11}{4}+4\ln2\right)$

而 $J_1[y]=1$ ⇒ $c=\frac{1}{\sqrt{4\ln2-\frac{11}{4}}}$

由 $\frac{\partial(J-\lambda J_1)}{\partial c}=0$ ⇒ $\frac23c-2\lambda c\left(-\frac{11}{4}+4\ln2\right)=0$

即 $\lambda=\frac{1}{3\left(4\ln2-\frac{11}{4}\right)}$ 为最小本征值!

[20]. $y''+\lambda y=0$. $y(0)=0$. $y(1)=0$.

取 $y=c_1x(1-x)+c_2x^2(1-x)$.

求最低两个本征值的近似值.

解: $y''+\lambda y=0 \Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right)+\lambda\cdot y=0$

即 $\frac{1}{2}\delta\int_0^1\left((y')^2-\lambda y^2\right)dx=0$.

即 $I=\int_0^1 y'^2dx$, $I_1=\int_0^1 y^2dx=1$

代入 $y=c_1x(1-x)+c_2x^2(1-x)$:

$y'=c_1(1-2x)+c_2x(2-3x)$.

$$I=\int_0^1 c_1^2(1-2x)^2+c_2^2x^2(2-3x)^2\,dx+\int_0^1 2c_1c_2x(1-2x)(2-3x)$$

$\left(\frac{9}{5}-\frac{12}{4}+\frac{4}{3}=\frac{9}{5}-\frac{5}{3}=\frac{2}{15}\right)$

$$=\frac{1}{3}c_1^2+c_2^2\int_0^1 9x^4-12x^3+4x^2\,dx+2c_1c_2\int_0^1 x(1-2x)(2-3x)\,dx$$

$$=\frac{1}{3}c_1^2+\frac{2}{15}c_2^2+\frac{c_1c_2}{3}$$

$$\lambda I_1=\int_0^1\lambda\cdot y^2dx=\lambda\int_0^1\left[c_1^2x^2(1-x)^2+c_2^2x^4(1-x)^2\right]dx+\lambda\cdot\int_0^1 2c_1c_2x^3(1-x)^2dx$$

$$=\lambda\cdot\left(c_1^2\int_0^1 x^4-2x^3+x^2dx+c_2^2\int_0^1 x^6-2x^5+x^4dx\right)+\lambda\cdot 2c_1c_2\int_0^1(x^5-2x^4+x^3)dx$$

$$=\lambda\left(\frac{c_1^2}{30}+\frac{c_2^2}{105}\right)+\frac{\lambda c_1c_2}{30}=\lambda$$

由极值条件:

$\frac{\partial(I-\lambda I_1)}{\partial c_1}=0$, $\frac{2}{3}c_1+\frac{c_2}{3}-\lambda\left[\frac{c_1}{15}+\frac{c_2}{30}\right]=0$

$\frac{\partial(I-\lambda I_1)}{\partial c_2}=0$, $\frac{4}{15}c_2+\frac{c_1}{3}-\lambda\left(\frac{2c_2}{105}+\frac{c_1}{30}\right)=0$

$$\Rightarrow\begin{cases}\left(\frac{2}{3}-\frac{\lambda}{15}\right)c_1+\left(\frac{1}{3}-\frac{\lambda}{30}\right)c_2=0\\ \left(\frac{1}{3}-\frac{\lambda}{30}\right)c_1+\left(\frac{4}{15}-\frac{2\lambda}{105}\right)c_2=0.\end{cases}$$

$$\det\begin{vmatrix}\frac{2}{3}-\frac{\lambda}{15} & \frac{1}{3}-\frac{\lambda}{30}\\ \frac{1}{3}-\frac{\lambda}{30} & \frac{4}{15}-\frac{2\lambda}{105}\end{vmatrix}=0$$

$$\Rightarrow(10-\lambda)\left(4-\frac{2}{7}\lambda\right)=\left(5-\frac{\lambda}{2}\right)^2$$

即 $\frac{2}{7}\lambda^2-4\lambda-\frac{20}{7}\lambda+40=\frac{1}{4}\lambda^2+25-5\lambda$

$$\Rightarrow\frac{1}{28}\lambda^2-\frac{13}{7}\lambda+15=0$$

得 $\lambda=\dfrac{\frac{13}{7}\pm\sqrt{\frac{169}{49}-\frac{15}{7}}}{\frac{1}{14}}$

$=26\pm16=\underline{10\text{ or }42}$

即最小两本征值近似为 10, 42.

[18]. $y''+\lambda y=0$, $y(0)=y(1)=0$.

试探函数 $y(x)=a(x-x^2)$, 求误差及估计解.

前过程类似[20]: $I=\int_0^1 y'^2dx$, $I_1=\int_0^1 y^2dx=1$

代入 $y(x)$: $I=\frac{a^2}{3}$, $I_1=\frac{1}{30}a^2=1$

$\Rightarrow a=\pm\sqrt{30}$ (取正值).

$\bar{\lambda}_1=\frac{1}{3}a^2=10$,

$\bar{y}_1=\sqrt{30}(x-x^2)$

而精确解: $\underline{y_1=2\sin\pi x}$

$$\|y_1-\bar{y}_1\|^2=\int_0^1\left(2\sin\pi x-\sqrt{30}(x-x^2)\right)^2dx$$

$$=3-\frac{16\sqrt{30}}{\pi^3}$$

$$\approx0.1736$$

[17] $\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\lambda R=0$.

$R(0)$ 有界, $R(1)=0$.

取试探函数 $R(r)=a(1-r^2)+b(1-r^2)^2$

解: 先化简: $\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\lambda r^2R=0$

它的泛函极值条件为:

$$\frac{1}{2}\delta\int_0^1\left(r^2(R'(r))^2-\lambda r^2R^2\right)dr=0$$

即: $I=\int_0^1 r^2R'^2dr$, $I_1=\int_0^1 r^2R^2dr=1$

代入: $R'=-2ar+(-4r+4r^3)b$

$$I=\int_0^1 4r^4a^2+r^4(2-3r+4r^2)^2b^2-4abr^3(2r-3r^2+4r^3)\,dr$$

$$=\frac{4}{5}a^2+\frac{352}{315}b^2-\frac{66}{35}ab$$

$$I_1=\int_0^1 a^2(r^6-2r^4+r^2)+b^2(r^{10}-4r^8+6r^6-4r^4+r^2)+2ab(r^2-3r^4+3r^6-r^8)\,dr$$

$$=\frac{8}{105}a^2+b^2\left(\frac{1}{11}-\frac{4}{9}+\frac{6}{7}-\frac{4}{5}+\frac{1}{3}\right)+2ab\left(\frac{1}{3}-\frac{3}{5}+\frac{3}{7}-\frac{1}{9}\right)$$

（接上）

$\therefore I = \int_0^1 (-2ar^2 + 4br^4 - 4br^2)^2 dr$

$= \frac{4}{5}a^2 + \frac{128}{315}b^2 + \frac{32}{35}ab$

$I_1 = \int_0^1 r^2 \left[a(1-r^2) + b(1-r^2)^2 \right]^2 dr$

$= \frac{8}{105}a^2 + \frac{32}{315}ab + \frac{128}{3465}b^2$

由：$\frac{\partial(I-\lambda I_1)}{\partial a} = 0 \Rightarrow \left(\frac{8}{5} - \frac{16}{105}\lambda\right)a + \left(\frac{32}{35} - \frac{32}{315}\lambda\right)b = 0$

$\frac{\partial(I-\lambda I_1)}{\partial b} = 0 \Rightarrow \left(\frac{32}{35} - \frac{32}{315}\lambda\right)a + \left(\frac{256}{315} - \frac{256}{3465}\lambda\right)b = 0$

解 $\det\begin{vmatrix} \sim & \sim \\ \sim & \sim \end{vmatrix} = 0$ 得

$\lambda = 3(10 \pm 3\sqrt{5})$

$\therefore \lambda_1 = 3(10 - 3\sqrt{5}) \approx 9.87539.$

（ws上省略一张纸的计算）.