Functions of One Complex Variable

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References: Wu, Tan, Fubianhanshu
(GTM) Conway, Functions of one complex variable I

Contents

§1 Complex numbers and Complex plane

Before entering complex numbers, let's consider the real plane $\mathbb{R}^2$. We might think $\mathbb{C}$ and $\mathbb{R}^2$ are similar at first sight, but they are totally different in fact.

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$,

$$f(x, y) = (u(x, y), v(x, y)),$$

if we require some smooth conditions in the sense of complex numbers, we are actually requiring the *Cauchy-Riemann equation*:

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

Therefore functions of complex variables have much better properties than the real functions, as we'll see in this course.

§1.1 Complex numbers

Now let's start from the real numbers to construct complex numbers. We know that real numbers form a *field*.

> **Question 1.1.1.** Is $(\mathbb{R} \cup \{-\infty\}, \max, +)$ a field? Which axiom is missing? Rigorously derive first *beautiful* fact that it does not satisfy.

Definition 1.1.2 (Complex field)**.** Define the set of complex numbers $\mathbb{C} := \mathbb{R} \times \mathbb{R}$, and we define two operations on $\mathbb{C}$: For $a, b, c, d \in \mathbb{R}$,

$$+ : \quad (a, b) + (c, d) := (a + c, b + d),$$

$$\times : \quad (a, b) \times (c, d) := (ac - bd, bc + ad).$$

We can check that $(\mathbb{C}, +, \times)$ is a field.

Just like we did when constructing real numbers, we need to embed the real numbers into complex field, i.e.

$$\mathbb{R} \simeq \{(a, 0) : a \in \mathbb{R}\} \subset \mathbb{C}, \quad a \mapsto (a, 0).$$

For simplicity reasons, we'll write (a, b) as $a + ib$.

Definition 1.1.3. Let $i := (0, 1)$, called the **imaginary unit**. Let $z = (a, b) =: a + ib$ for $a, b \in \mathbb{R}$.

Define the **real part** of z as $\operatorname{Re} z := a$, the **imaginary part** of z as $\operatorname{Im} z := b$, and ib is called a **(purely) imaginary number**.

Definition 1.1.4. For $z = a + ib$, $a, b \in \mathbb{R}$, the **(complex) conjugate** of z is defined as $\overline{z} = a - ib$.

The **absolute value** or **norm** of z is

$$|z| := z\overline{z} = \sqrt{a^2 + b^2}.$$

The geometry meaning of norm is the distance between (a, b) and the origin in the Eucild plane $\mathbb{R}^2$.

Proposition 1.1.5
Some easy properties of conjugates:

- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$,
- $\overline{z_1 z_2} = \overline{z_1} + \overline{z_2}$,
- $\operatorname{Re} z = \frac{z+\overline{z}}{2}$, $\operatorname{Im} z = \frac{z-\overline{z}}{2i}$.

Definition 1.1.6. Let $z \in \mathbb{C} \setminus \{0\}$. The **argument**

$$\operatorname{Arg} z := \{\theta \in \mathbb{R} : z = |z| \cos\theta + i|z| \sin\theta\},$$

and the **principal argument** $\arg z \in \operatorname{Arg}(z) \cap [0, 2\pi)$.
The **complex plane** is the identification $\mathbb{C} \cong \mathbb{R}^2$.

Hence we have the triangle inequality:

Proposition 1.1.7 (Triangle inequality)
Let $z_1, z_2, z_3 \in \mathbb{C}$,

$$|z_1 - z_2| \leq |z_1 - z_3| + |z_3 - z_2|.$$

Proof. Left as exercise. □

There are some other trivial facts about complex numbers:

Proposition 1.1.8 (Triangular form of complex numbers)
If $z_i = r_i \cos\theta_i + i r_i \sin\theta_i$, $i = 1, 2$, then

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)).$$

Proposition 1.1.9 (Lines and circles in complex plane)
The line $ax + by + c = 0$, $a, b, c \in \mathbb{R}$ is equivalent to

$$\overline{B}z + B\overline{z} + c = 0, \quad B := \frac{a + ib}{2}.$$

The circle $|z - B| = R > 0$ is equivalent to

$$-z\overline{z} + \overline{B}z + B\overline{z} + c = 0, \quad c := -B\overline{B} + R^2.$$

§1.2 Metric and topological properties of $\mathbb{C}$

Obviouly $\mathbb{C}$ is a metric space with metric $d(z_1, z_2) := |z_1 - z_2|$, i.e. the metric of Euclid plane $\mathbb{R}^2$.

Remark 1.2.1 (Notations) — Let $z \in \mathbb{C}$, $r \geq 0$, denote the open and closed balls as

$$B(z,r) := \{w \in \mathbb{C} : |z-w| < r\}, \quad \overline{B}(z,r) := \{w \in \mathbb{C} : |z-w| \leq r\}.$$

Also let $B_0(z,r) := B(z,r) \setminus \{z\}$.

(Here the teacher defines the open subsets in metric spaces, but basic topology knowlege is assumed, I won't bother to write it down)

In this course, for a subset A of some topological space X, we use $\mathring{A}$ or $intA$ to denote the *interior* of A, $\overline{A}$ to denote the *closure* of A, and ∂A to denote the *boundary* of A.

(Again I omitted the definition of connected and path-connected space)

Theorem 1.2.2

Let $G \subset \mathbb{C}$ be an open subset. TFAE:

(1) G is connected.

(2) G is path-connected.

(3) $\forall x, y \in G$, there is a piecewise linear path that connects x and y.

Proof. It suffices to prove (1) $\implies$ (3).

Fix $z \in G$, let

$$\Omega = \{w \in G : w \text{ is connected to } z \text{ by a piecewise linear path}\}.$$

We need to show Ω is both closed and open (i.e. *clopen*).

Clearly Ω is open and Ω^c is open (by definition), so we're done. □

(definitions of convergence, Cauchy sequence and complete space)

Proposition 1.2.3

$\mathbb{C}$ is complete.

Proof. Left as exercise. □

Theorem 1.2.4 (Cantor's theorem)

A metric space (X,d) is complete iff for each $\{F_n\}_{n\geq 1}$ sequence of non-empty closed sets with $F_1 \supseteq F_2 \supseteq \dots$ and $\lim_{n\to\infty} \operatorname{diam}_d(F_i) = 0$, we have

$$\operatorname{card}\left(\bigcap_{n=1}^{\infty} F_n\right) = 1.$$

Proof. Take $x_n \in F_n$ arbitarily, since $\operatorname{diam}(F_n) \to 0$, we have that $\{x_n\}$ is a Cauchy sequence.

Then the convergence point $x \in \bigcap F_n$ and it is the only point since $\operatorname{diam}(F_n) \to 0$.

Conversely, let $\{x_n\}$ be a Cauchy sequence, take $F_n := \overline{\{x_k, k \geq n\}}$, we use the condition directly to conclude (check it yourself). □

Proposition 1.2.5
Let (X,d) be a complete metric space and $Y \subset X$ is a subset. Then (Y,d) is complete iff Y is closed in X.

Proof. By Cantor's theorem above. □

(Definition of compactness and some basic properties)

Proposition 1.2.6
Every compact metric space is complete.

Theorem 1.2.7 (Lebesgue's covering lemma)
If (X,d) is sequentially compact and $\mathcal{G}$ is an open cover of X, then $\exists \varepsilon > 0$, such that for all $x \in X$, $\exists G \in \mathcal{G}$ such that $B(x,\varepsilon) \subset G$.

(Equivalence of compactness, sequential compactness, complete & totally bounded in a metric space)

Theorem 1.2.8 (Heine-Borel)
A subset K of $\mathbb{R}^n$ is compact iff K is closed and bounded.

Definition 1.2.9. A continuous map $\gamma : [0,1] \to \mathbb{C}$ is called a **continuous curve(path)**, if $\gamma \in C^1$ and $\forall t \in [0,1]$, $\gamma'(t) \neq (0,0)$, we say γ is a **smooth curve(path)**. Also we can define *piecewise smooth curves* and **retifiable curves** as we did in previous courses.

A **Jordan curve (simple closed curve)** $\gamma : [0,1] \to \mathbb{C}$ is a continuous curve with $\gamma(0) = \gamma(1)$, and $\gamma(t) \neq \gamma(s)$ for all $0 < t < s \leq 1$.

Definition 1.2.10. A path-connected open set $\Omega \subset \mathbb{C}$ is called a **region**. (Connected and path-connected is equivalent in $\mathbb{C}$)

Theorem 1.2.11 (Jordan curve theorem)
For each Jordan curve γ in $\mathbb{C}$, $\mathbb{C} \setminus \gamma([0,1])$ has exactly two connected components.

Proof. This theorem is surprisingly hard to prove. See GTM Conwey II. □

§1.3 Limits and continuous functions in $\mathbb{C}$

Consider $S \subset \mathbb{C}$ and $f : S \to \mathbb{C}$.

Definition 1.3.1 (Limits)**.** Suppose z_0 is a limit point of S. If there exists $a \in \mathbb{C}$ such that

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \forall z \in B_0(z_0,\delta) \cap S, \quad f(z) \in B(a,\varepsilon),$$

then a is the limit of $f(z)$ as $z \in S$ tends to z_0, write $\lim_{z \to z_0} f(z) = a$.

A simple fact is that

$$\lim_{z \to z_0} f(z) = a \iff \lim_{z \to z_0} \operatorname{Re} f(z) = \operatorname{Re} a \text{ and } \lim_{z \to z_0} \operatorname{Im} f(z) = \operatorname{Im} a.$$

Definition 1.3.2. We say f is **continuous** at $z_0 \in S$, if either z_0 is not a limit point of S or $\lim_{z \to z_0} f(z) = f(z_0)$.

f is **continuous** in S if f is continuous at z_0 for all $z_0 \in S$.

Proposition 1.3.3

Easy properties of continuous functions:

- Above definition is equivalent to the topological definition.
- Image of connected / compact sets are connected /compact.
- The norm function $f(z) = |z|$ is continuous.
- If $S \subset \mathbb{C}$ is compact, $f \in C^0(S, \mathbb{C})$, then $|f(z)|$ is bounded, and both $\sup |f(z)|$ and $\inf |f(z)|$ are attained.

Proof. Left as an exercise. □

§1.4 Extended complex plane

Definition 1.4.1 (Extended complex plane)**.** Let $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ be the **extended complex plane**. The addition and multiplication with ∞ is exactly what you might guess. Note that $\infty \pm \infty, \frac{0}{0}, 0 \cdot \infty$ are meaningless and $\frac{\infty}{\infty} = 0$.

Definition 1.4.2 (Topology of extended complex plane)**.** The space $\widehat{\mathbb{C}}$ is homeomorphic to the Riemann sphere S^2 via **stereographic projection**, i.e. the projection centered at $(0, 0, 1) \in \mathbb{R}^3$ which projects the unit sphere onto the xy-plane. The point $(0, 0, 1)$ is mapped to ∞.

In fact this is the **1d complex projective space** $\mathbb{C}P^1$.

Definition 1.4.3 (Chordal metric)**.** For $z_1, z_2 \in \widehat{\mathbb{C}}$, define the distance $d(z_1, z_2)$ to be the Euclidean distance on the unit sphere in $\mathbb{R}^3$, namely

$$d(z_1, z_2) = \frac{2|z_1 - z_2|}{\sqrt{(1 + |z_1|^2)(1 + |z_2|^2)}}, \quad d(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}}.$$

There are also *spherical metric* on extended complex plane, which we won't dig into the details.

§2 Analytic functions

§2.1 Definitions

Finally we can come to the main part:

Definition 2.1.1. Consider an open set $G \subset \mathbb{C}$, and a function $f : G \to \mathbb{C}$. We say f is **differentiable** at a point $a \in G$ if

$$f'(a) := \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \text{ exists and is finite.}$$

The value $f'(a)$ is called the derivative of f at a.

If f is differentiable at every point in G, we say that f is differentiable (or **analytic**, **holomorphic**) on G.

Remark 2.1.2 — For historical reasons, the different names of differentiable functions are induced by different definitions, but later people found out that they are in fact equivalent.

Proposition 2.1.3

If $f : G \to \mathbb{C}$ is differentiable at $a \in G$, then f is continuous at a.

Proof.

$$\lim_{z\to a} |f(z) - f(a)| = \lim_{z\to a} \frac{|f(z) - f(a)|}{|z-a|} |z-a| = |f'(a)| \lim_{z\to a} |z-a| = 0.$$

□

The complex derivatives have many same properties as the real case, like linearity, Leibniz's law and the chain rule. The proof is identical, so we leave them out.

Definition 2.1.4. Given a set $A \subset \mathbb{C}$, we say that f is **analytic** on A, if f is analytic on an open neighborhood of A.

Definition 2.1.5. Let $G \subset \mathbb{C}$ be an open set. An analytic function $f : G \to \mathbb{C}$ is a **univalent** function if f is injective. (also called **conformal mapping**)

If an analytic $f : G_1 \to G_2$ is bijective, f^{-1} is also holomorphic, we say f is **biholomorphic** function. (Also called **conformal equivalence/homeomorphism**)

Theorem 2.1.6

Let $f : G \to \mathbb{C}$ be a univalent function. Then

- $f'(z) \neq 0$, $\forall z \in G$.
- $f(G)$ is open in $\mathbb{C}$.
- $f^{-1} : f(G) \to G$ is analytic and

$$(f^{-1})'(f(z)) = \frac{1}{f'(z)}.$$

§2.2 Cauchy-Riemann equation and harmonic functions

Recall that earlier we said complex analytic function satisfies Cauchy-Riemann equation:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Let $f = u + iv$, $z_0 = x_0 + iy_0$, since the limit in complex numbers can be approached by any direction, using it on the x-direction and y-direction we get

$$\frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0) = f'(z_0) = \frac{\partial v}{\partial y}(z_0) - i\frac{\partial u}{\partial y}(z_0).$$

Conversely, we have the following theorem:

Theorem 2.2.1

Assume $u, v \in C^1(G, \mathbb{R})$. Then $f = u + iv$ is analytic on G iff C-R equation hold on G.

Proof. Let $h = \Delta x + i\Delta y$ and compute f' directly. □

Moreover, note that if $u, v \in C^2$ and satisfy C-R equation, we have

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial y \partial x} - \frac{\partial^2 v}{\partial x \partial y} = 0.$$

This means that u is a **harmonic function**. In fact v is harmonic as well. (Harmonic naturally induces C^2)

Definition 2.2.2. Let $u, v : G \to \mathbb{R}$ be harmonic functions, we say v is a **harmonic conjugate** of u if $f = u + iv$ is analytic on $\mathbb{C}$.

Remark 2.2.3 — u is a harmonic conjugate of v iff v is a harmonic conjugate of $-u$.

Theorem 2.2.4

If $\Omega \subset \mathbb{C}$ is a simply connected region, then each harmonic function $u : \Omega \to \mathbb{R}$ has a harmonic conjugate.

Proof. Fix $p_0 = (x_0, y_0) \in \Omega$. For each piecewise linear curve γ_1, γ_2 from $p_0 \to z$, by Green's formula (simply connected condition is used here),

$$\int_{\gamma_1 - \gamma_2} \frac{\partial u}{\partial x}\,\mathrm{d}y - \frac{\partial u}{\partial y}\,\mathrm{d}x = \iint_D \Delta u \,\mathrm{d}x\,\mathrm{d}y = 0.$$

where D is the oriented regions enclosed by γ_1 and γ_2.

Hence let

$$v(z) := \int_{\gamma_1} \frac{\partial u}{\partial x}\,\mathrm{d}y - \frac{\partial u}{\partial y}\,\mathrm{d}x.$$

By above computation, v is well-defined. Now we only need to check C-R equation, which is straightforward. □

Remark 2.2.5 — $u = \ln(x^2 + y^2)$ is harmonic in $\mathbb{C} \setminus \{0\}$ but it doesn't have harmonic conjugates.

Theorem 2.2.6

Let $u, v \in C^1(G, \mathbb{R})$, then $f = u + iv$ is analytic iff $\frac{\partial f}{\partial \bar{z}} \equiv 0$.

Here $\frac{\partial}{\partial \bar{z}} := \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$.

Proof. Computation. □

Theorem 2.2.7 (Existence of logarithm)

Let Ω be a simply connected region. Let $u, v \in C^2(\Omega, \mathbb{R})$, and $f = u + iv$ is an analytic function with $f(z) \neq 0$, $\forall z \in \Omega$. Then there exists an analytic function $g : \Omega \to \mathbb{C}$ s.t. $e^{g(z)} = f(z)$ for all $z \in \Omega$.

Proof. Set $u_0(z) := \ln|f(z)|$. We can check that $\Delta u_0(z) = 0$. Thus $\exists v_0$ such that $h := u_0 + iv_0$ is analytic on Ω.

We have
$$\left|e^{h(z)}\right| = \left|e^{u_0(z)}\right| = |f(z)|, \quad \forall z \in \Omega,$$
so $e^{h(z)} = f(z)e^{ic}$ for some $c \in \mathbb{R}$ (since $z \mapsto e^{h(z)}f(z)^{-1}$ gives an analytic map F from Ω to S^1, if $\exists z$ s.t. $F'(z) \neq 0$, then $F(z+h) = F(z) + F'(z)h + o(h)$, taking suitable h will imply $|F(z+h)| \neq 1$), which means $g = h - ic$ is the desired function. □

Theorem 2.2.8

Let Ω be a simply connected region, $u, v \in C^2(\Omega, \mathbb{R})$. If analytic function $f = u + iv \neq 0$ for all $z \in \Omega$, then for all $n \in \mathbb{N}$, there exists an analytic function $h : \Omega \to \mathbb{C}$ such that $(h(z))^n = f(z)$, $\forall z \in \Omega$.

Proof. Let $g(z)$ satisfy $e^{g(z)} = f(z)$, so $h(z) := e^{\frac{1}{n}g(z)}$ is a desired function. □

Theorem 2.2.9

Let $f = u + iv$ be an analytic function on a region Ω, then the Jacobian of $f : (x, y) \to (u, v)$ is $|f'(z)|^2$.

Proof. We have
$$J_f = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = |f'(z)|^2.$$
□

Corollary 2.2.10

Let $f : \Omega \to \mathbb{C}$ be analytic. If $f' \in C(\Omega, \mathbb{C})$, and $f'(z_0) \neq 0$, then there exists an open neighborhood D of z_0 such that

- $f(D)$ is open.
- $f : D \to f(D)$ is bijective.
- $f^{-1} : f(D) \to D$ is also analytic, with $(f^{-1})'(f(z)) = \frac{1}{f'(z)} \neq 0$.

i.e. $f : D \to f(D)$ is univalent/biholomorphic.

Proof. Inverse function theorem. □

§2.3 Power Series

First let's recall some trivial properties inherited from real numbers:

Definition 2.3.1. Let $a_n \in \mathbb{C}, n \in \mathbb{N}$ be a complex sequence. The series $\sum_{n=1}^{\infty} a_n$ **converges** to z iff $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, $\forall n \geq N$, $|A - \sum_{i=1}^{n} a_n| < \varepsilon$.

The series $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if $\sum_{n=1}^{\infty} |a_n|$ converges. Obviously if $\sum_{n=1}^{\infty} a_n$ converges absolutely, it must converge.

Definition 2.3.2 (Power series)**.** A **power series about** $a \in \mathbb{C}$ is a series $\sum_{n=0}^{\infty} a_n (z-a)^n$.

We say a series $\sum_{i=1}^{\infty} f_i(z)$ **converges uniformly** on a set $S \subset \mathbb{C}$ to $g(z)$ if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \forall z \in S, \quad \left| g(z) - \sum_{i=1}^{\infty} f_i(z) \right| < \varepsilon.$$

Proposition 2.3.3

Absolutely convergent series have all the nice properties in real analysis, like they can switch summation order, they can perform addition and multiplication etc.

Theorem 2.3.4

For a power series $\sum_{n=0}^{\infty} a_n (z-a)^n$, define $R \in [0, +\infty]$ by

$$\frac{1}{R} := \limsup_{n \to \infty} |a_n|^{\frac{1}{n}}.$$

This R is called the **radius of convergence**. We have

- If $|z - a| < R$, the series converges absolutely.
- If $|z - a| > R$, the series becomes unbounded.
- For all $r \in (0, R)$, the series converges uniformly on $\overline{B(a, r)}$.

Proof. WLOG $a = 0$. Fix $r \in [0, R)$, let $r_0 \in (r, R)$. Thus there exists $N \in \mathbb{N}$ s.t.

$$\forall n \geq N, \quad |a_n|^{\frac{1}{n}} < \frac{1}{r_0}.$$

For all $z \in \overline{B(0,r)}$, $\forall n \geq N$,

$$|a_n z^n| < \left(\frac{|z|}{r_0}\right)^n \leq \left(\frac{r}{r_0}\right)^n.$$

This means the series is controlled by a geometric series, so it must converges uniformly and absolutely.

For the second statement, just note that the terms of the series will eventually become unbounded, hence so do the series. ☐

Proposition 2.3.5

For a power series $\sum_{n=0}^{\infty} a_n(z-a)^n$, if $\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$ exists, then it's equal to R.

Proof. Left as an exercise. ☐

For $z \in \mathbb{C}$, define

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

to be the exponent function. This is an extension of exponent function of real numbers. From this definition, clearly we have

$$e^{ix} = \cos x + i \sin x$$

by Taylor expansions of $\cos x$ and $\sin x$.

Proposition 2.3.6

If $\sum a_n(z-a)^n$ and $\sum b_n(z-a)^n$ have radii of convergence $\geq r > 0$, let $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, then for all $z \in B(a, r)$,

$$\sum_{n=0}^{\infty}(a_n + b_n)(z-a)^n = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=0}^{\infty} b_n(z-a)^n,$$

$$\sum_{n=0}^{\infty} c_n(z-a)^n = \sum_{n=0}^{\infty} a_n(z-a)^n \cdot \sum_{n=0}^{\infty} b_n(z-a)^n.$$

They both have radii of convergence at least r.

Proof. Left as an exercise. ☐

Proposition 2.3.7 (Power series are analytic)
Let $f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n$ with $R>0$, we have

- $\forall k \in \mathbb{N}$,
$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z-a)^{n-k}$$
has radius of convergence R.

- f is infinitely differentiable on $B(a,R)$ and the k-th derivative is precisely the series above, for all $k \in \mathbb{N}$ and $z \in B(a,R)$.

In particular, $a_n = \frac{1}{n!} f^{(n)}(a)$.

Proof. WLOG $a=0$. By induction, it suffices to prove the case $k=1$. For convergence part, just compute
$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \limsup_{n\to\infty} |na_n|^{\frac{1}{n-1}}.$$
For the derivative, fix $r \in B(0,R)$, for $z, z' \in B(a,r)$,
$$\left| \frac{f(z)-f(z')}{z-z'} - \sum_{n=1}^{\infty} na_n(z'-a)^{n-1} \right| \le \left| \sum_{n=0}^{k} a_n \frac{(z-a)^n-(z'-a)^n}{z-z'} - na_n(z'-a)^{n-1} \right| + \sum_{n=k+1}^{\infty} |\dots| + |\dots|.$$

The tail part can be controlled:
$$\left| a_n \frac{(z-a)^n-(z'-a)^n}{z-z'} \right| \le |a_n| \sum_{i=1}^{n} |z-a|^{i-1}|z'-a|^{n-i} \le n|a_n|r^{n-1}.$$
and $|na_n(z-a)^{n-1}| \le n|a_n|r^{n-1}$, which converges as $r<R$. Thus for k sufficiently large, the tail part is less than $\frac{\varepsilon}{2}$.

The main part can be computed directly, and we can prove that it converges to 0 as $z \to z'$. (Check it yourself!) □

§2.4 Multivalued functions

Definition 2.4.1. A map $F : S \to \mathcal{P}(\mathbb{C})$ is called a **multivalued function** on S. Here $\mathcal{P}(\mathbb{C})$ is the power set of $\mathbb{C}$. It is also called a *correspondence*.

Remark 2.4.2 — If $f : D \to \mathbb{C}$ is a function, then f^{-1} is a multivalued function on $f(D)$.

Definition 2.4.3. Let F be a multivalued function on a region Ω, if there exists an analytic function $f : \Omega \to \mathbb{C}$ s.t. $f(z) \in F(z)$ for all $z \in \Omega$, then f is an **analytic branch** of F.

Example 2.4.4

Let $f(z) = e^z$ be an analytic function on $\mathbb{C}$, Then $f^{-1}(z) = Ln(z) : \mathbb{C} \setminus \{0\} \to \mathcal{P}(\mathbb{C})$ with

$$z \mapsto Ln(z) := \ln|z| + i\,\mathrm{Arg}(z) = \{\ln|z| + i(\arg(z) + 2k\pi) : k \in \mathbb{Z}\}.$$

In fact, $Ln(z)$ can be viewed as a function from a Riemann surface (a covering space of $\mathbb{C}\setminus\{0\}$ with covering group $\mathbb{Z}$) to $\mathbb{C}$.

Definition 2.4.5 (Riemann surfaces)**.** A connected Hausdorff space R is a **Riemann surface** if there exists a set of **charts** $\{(U_\alpha, \phi_\alpha) : \alpha \in A\}$ s.t.

- $(U_\alpha)_{\alpha\in A}$ is an open cover of R.
- $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset \mathbb{C}$ is a homeomorphism.
- If $U_\alpha \cap U_\beta \neq \emptyset$, then
$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$
is a *biholomorphism*.

In other words, it's a 2-dimensional manifold with smoothness requirements of complex analyticity.

Definition 2.4.6. Let R be a Riemann surface with atlas $\{(U_\alpha, \phi_\alpha)\}$, a function $f : R \to \mathbb{C}$ is **holomorphic** if $f \circ \varphi_\alpha^{-1}$ is holomorphic on $\varphi_\alpha(U_\alpha)$ for all $\alpha \in A$.

Let R_1, R_2 be Riemann surfaces with altases $\{(U_\alpha, \varphi_\alpha)\}$, $\{(V_\beta, \phi_\beta\}$. A function $f : R_1 \to R_2$ is **holomorphic** if $\phi_\beta \circ f \circ \varphi_\alpha^{-1}$ is holomorphic in $\varphi_\alpha(U_\alpha \cap V_\beta)$ for all $\alpha \in A$, $\beta \in B$.

Remark 2.4.7 — Since I learned differential manifolds before, thus the notes for this part will lack some explanations of details.

An immediate example of Riemann surface is $\widehat{\mathbb{C}}$ and the unit sphere in $\mathbb{R}^3$.

Example 2.4.8

The map $z^n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ extends to a holomorphic map on $\widehat{\mathbb{C}}$.

§2.5 Mobius transformations

Earlier we say that analytic maps are also called "conformal maps", which means they preserves angles.

Proposition 2.5.1

Let f be an analytic function, γ_1, γ_2 are two paths $[-1, 1] \to \mathbb{C}$ with $\gamma_1(0) = \gamma_2(0) = z$. Let $\theta := \arg\gamma_1'(0) - \arg\gamma_2'(0)$, then

$$\theta = \arg((f \circ \gamma_1)'(0)) - \arg((f \circ \gamma_2)'(0)).$$

Proof. Since $(f \circ \gamma_1)'(0) = f'(z)\gamma_1'(0)$, $\arg((f \circ \gamma_1)'(0)) = \arg f'(z) + \arg\gamma_1'(0)$, which gives the desired result. □

Definition 2.5.2 (Mobius transformation)**.** A mapping $S(z) = \frac{az+b}{cz+d}$ is called a **linear fractional transformation**. If $ad - bc \neq 0$, then $S(z)$ is called a **Mobius transform**, denoted by $S \in Mob$.

Remark 2.5.3 — Many of this part are covered in the course Geometry I, but since I didn't take notes of that course, I'm still taking all the notes.

Proposition 2.5.4

Let $S(z) = \frac{az+b}{cz+d}$,

- $S^{-1}(z) = \frac{dz-b}{-cz-a}$.
- For all $S_1, S_2 \in Mob$, $S_1 \circ S_2 \in Mob$.

Observe that when a, b, c, d is multiplied by a same constant, the map $S(z)$ remains the same. Thus there's a group isomorphism

$$\mathrm{PGL}(2, \mathbb{C})/\{\pm 1\} \to Mob, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{az+b}{cz+d}.$$

Remark 2.5.5 — For all $S \in Mob$, it extends to a biholomorphism map $S : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$.

Definition 2.5.6. **Elementary linear fractional transformation** consists of

- Rotation: $z \mapsto ze^{i\theta}$, $\theta \in \mathbb{R}$.
- Dilation: $z \mapsto az$, $a > 0$.
- Inversion: $z \mapsto \frac{1}{z}$.
- Translation: $z \mapsto z + b$, $b \in \mathbb{C}$.

They generate all the Mobius transformations as a group. (Check it yourself)

Proposition 2.5.7

If $S_1, S_2 \in Mob$ coincide on 3 different points, then $S_1 = S_2$. This means Mobius transformations are determined by the image of 3 points.

Proof. This can be computed directly. □

Definition 2.5.8 (Cross ratio)**.** Let $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}}$ be distinct points, define the **cross ratio**

$$(z_1, z_2, z_3, z_4) := \left(\frac{z_1 - z_3}{z_1 - z_4}\right) \cdot \left(\frac{z_2 - z_3}{z_2 - z_4}\right)^{-1}.$$

It is the image of z_1 under the unique Mobius transformation which maps z_2, z_3, z_4 to $1, 0, \infty$ respectively.

Proposition 2.5.9
Let $T \in Mob$, for all $z \in \widehat{\mathbb{C}}$,

$$(z, z_2, z_3, z_4) = (T(z), T(z_2), T(z_3), T(z_4)).$$

Proof. Let $S(z) = (z, z_2, z_3, z_4)$, then ST^{-1} is the unique map which maps $T(z_2)$ to 1, $T(z_3)$ to 0 and $T(z_4)$ to ∞, hence $ST^{-1}(T(z)) = S(z)$ gives the result. □

Proposition 2.5.10
Let $z_2, z_3, z_4 \in \widehat{\mathbb{C}}$ be distinct points, $w_2, w_3, w_4 \in \widehat{\mathbb{C}}$ are also distinct, then there exists a unique $S \in Mob$ which sends z_i to w_i.

Proof. This map is namely $(z, w_2, w_3, w_4)^{-1}(z, z_2, z_3, z_4)$. □

Proposition 2.5.11
Let $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}}$ be distinct, they lie on a circle in $\widehat{\mathbb{C}}$ iff

$$(z_1, z_2, z_3, z_4) \in \mathbb{R}.$$

Proof. It suffices to show Mobius transform S sends circles to circles (hence we can send z_1, z_2, z_3 to real numbers, and z_4 lie on the same circle iff $z_4 \in \mathbb{R}$).

Since circles in $\widehat{\mathbb{C}}$ can be written as

$$Az\overline{z} + \overline{B}z + B\overline{z} + C = 0,$$

where $A, C \in \mathbb{R}$, $B\overline{B} - AC > 0$. You can either compute directly or prove the generators of the Mobius transformation group preserves circles, i.e. rotations, dilations, translations and inversions preserves circles. □

Definition 2.5.12. Let Γ be a circle through distinct points $z_2, z_3, z_4 \in \widehat{\mathbb{C}}$. We say z and z^* are **symmetric** w.r.t. Γ if

$$(z^*, z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)}$$

When the circle is a straight line, this coincides with the symmetry w.r.t. a line; and this definition is well-defined (i.e. independent of the choice of z_2, z_3, z_4).

Proposition 2.5.13
Let Γ be a circle, $S \in Mob$, and z, z^* are symmetric w.r.t. Γ. Then $S(z)$, $S(z^*)$ are also symmetric w.r.t. $S(\Gamma)$.

Proof. Trivial by definition. □

§3 Complex integration

§3.1 Riemann-Stieltjes integral

Definition 3.1.1 (BV functions)**.** A function $\gamma : [a,b] \to \mathbb{C}$ is of **bounded variation** if $\exists M > 0$, $\forall$ partition $p : a = t_0 < t_1 < \cdots < t_m = b$ of $[a,b]$, we have

$$v(\gamma; p) := \sum_{k=1}^{m} |\gamma(t_k) - \gamma(t_{k-1})| \leq M.$$

The **total variation** of γ is

$$V(\gamma) := \sup_p v(\gamma; p).$$

BV curves are also called **rectifiable curves**.

Proposition 3.1.2

Let $\gamma : [a,b] \to \mathbb{C}$ be of bounded variation,

- if $P \subset Q$ are both partitions of $[a,b]$, then $v(\gamma; P) \leq v(\gamma; Q)$.
- The sum of BV functions is still BV with

$$V(\alpha\gamma_1 + \beta\gamma_2) \leq |\alpha| V(\gamma_1) + |\beta| V(\gamma_2), \quad \alpha, \beta \in \mathbb{C}.$$

Proposition 3.1.3

If $\gamma : [a,b] \to \mathbb{C}$ is piecewise smooth, then γ is BV with $V(\gamma) = \int_a^b |\gamma'(t)| \,\mathrm{d}t$.

Theorem 3.1.4

Let $\gamma : [a,b] \to \mathbb{C}$ be of bounded variation, and $f : [a,b] \to \mathbb{C}$ continuous. There exists $I \in \mathbb{C}$, s.t. $\forall \varphi > 0$, $\exists \delta > 0$, for all partition p of $[a,b]$ with $mesh(p) := \max_k |t_k - t_{k-1}| \leq \delta$, we have

$$\left| I - \sum_{k=1}^{m_p} f(\tau_k)(\gamma(t_k) - \gamma(t_{k-1})) \right| \leq \varepsilon$$

for all choices of $\tau_k \in [t_{k-1}, t_k]$, $k = 1, \ldots, m_p$.

Proof. The proof of above propositions should be already covered in the course Analysis I, II, III. □

Definition 3.1.5 (Integrals)**.** In the notations of the above theorem, we say $I =: \int_a^b f(t) \,\mathrm{d}\gamma(t)$ is the **Riemann-Stieltjes integral** of f with respect to γ over $[a,b]$.

Some easy properties: Let $f, g \in C[a,b]$, $\gamma_1, \gamma_2 \in BV[a,b]$, $\alpha, \beta \in \mathbb{C}$, we have

$$\int_a^b (\alpha f + \beta g) \,\mathrm{d}\gamma_1 = \alpha \int_a^b f \,\mathrm{d}\gamma_1 + \beta \int_a^b g \,\mathrm{d}\gamma_1.$$

$$\int_a^b f \, \mathrm{d}(\alpha\gamma_1 + \beta\gamma_2) = \alpha \int_a^b f \, \mathrm{d}\gamma_1 + \beta \int_a^b f \, \mathrm{d}\gamma_2.$$

$$\int_a^b f \, \mathrm{d}\gamma = \sum_{k=1}^n \int_{t_{k-1}}^{t_k} f \, \mathrm{d}\gamma,$$

where $a = t_0 < \cdots < t_n = b$.

Proposition 3.1.6

If γ is piecewise smooth, $f \in C[a, b]$, then

$$\int_a^b f \, \mathrm{d}\gamma = \int_a^b f(t)\gamma'(t) \, \mathrm{d}t.$$

Proof. Consider the real and imaginary parts separately, and apply mean value theorems. □

Definition 3.1.7 (Line integral)**.** Let $\gamma : [a, b] \to \mathbb{C}$ be a rectifiable curve, $f : \gamma([a, b]) \to \mathbb{C}$ continuous, then the **line integral** of f along γ is

$$\int_\gamma f(z) \, \mathrm{d}z := \int_a^b f(\gamma(t)) \, \mathrm{d}\gamma(t).$$

Example 3.1.8

Let $\gamma = e^{it}$ on $[0, 2\pi]$, $f(z) = \frac{1}{z}$,

$$\int_\gamma f(z) \, \mathrm{d}z = \int_0^{2\pi} e^{-it} i e^{it} \, \mathrm{d}t = 2\pi i.$$

Let $g(z) = z^m$, $m \geq 0$,

$$\int_\gamma g(z) \, \mathrm{d}z = \int_0^{2\pi} e^{imt} i e^{it} \, \mathrm{d}t = i \int_0^{2\pi} e^{i(m+1)t} \, \mathrm{d}t = 0.$$

Proposition 3.1.9

The line integral is well-defined, i.e. for all $\varphi : [c, d] \to [a, b]$ continuous and non-decreasing with $\varphi(c) = a, \varphi(d) = b$, then $\forall f \in C(\gamma([a, b]))$,

$$\int_\gamma f \, \mathrm{d}z = \int_{\gamma\circ\varphi} f \, \mathrm{d}z.$$

In other words, it is invariant under reparametrization.

Proof. Chain rule. (for continuous functions, we need $\varepsilon - \delta$ arguments) □

Proposition 3.1.10

Let $\gamma \in BV[a,b]$, $-\gamma(t) := \gamma(a+b-t)$, f continuous on $\gamma([a,b])$, we have

- $\int_\gamma f \,\mathrm{d}z = -\int_{-\gamma} f \,\mathrm{d}z$,
- $|\int_\gamma f \,\mathrm{d}z| \le \int_\gamma |f(z)||\,\mathrm{d}z| \le V(\gamma) \sup f(z)$, where $|\,\mathrm{d}z| := \mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$.

Theorem 3.1.11

Let $G \subset \mathbb{C}$ be an open set, $\gamma \in BV[a,b]$ in G is a path from α to β. If $f, F \in C(G)$ satisfying $F'(z) = f(z)$, then

$$\int_\gamma f(z) \,\mathrm{d}z = F(\beta) - F(\alpha).$$

Proof. When γ is piecewise smooth, we can directly prove it:

$$\int_\gamma f \,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t) \,\mathrm{d}t = \int_a^b (F \circ \gamma)'(t) \,\mathrm{d}t = F(\beta) - F(\alpha).$$

For rectifiable curves, the idea is to approxiamte γ with piecewise smooth curves (the details are too complicated and hence omitted). □

§3.2 Power series and analytic functions

Theorem 3.2.1

Let γ be a rectifiable curve, $f_n : \gamma \to \mathbb{C}$ is a uniformly convergent series of functions to f, then

$$\lim_{n\to\infty} \int_\gamma f_n \,\mathrm{d}z = \int_\gamma f \,\mathrm{d}z.$$

Now recall the Stokes formula in multi-dimensional calculus:

Theorem 3.2.2 (Green's formula)

Let $\Theta \subset \mathbb{C}$ be a bounded region, $\partial\Omega$ is a finite union of piecewise smooth curves, let $u(x,y)$, $v(x,y)$ be C^1 on some open neighborhood of $\overline{\Omega}$, then

$$\int_{\partial\Omega} u \,\mathrm{d}x + v \,\mathrm{d}y = \int_\Omega \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\right) \mathrm{d}x \,\mathrm{d}y.$$

Apply this for a complex-valued function f,

$$\int_{\partial\Omega} f \,\mathrm{d}z = \int_{\partial\Omega} (u+iv)(\mathrm{d}x + i \,\mathrm{d}y) = \int_\Omega \left(-\frac{\partial}{\partial y} + i\frac{\partial}{\partial x}\right) u + \left(-\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) v \,\mathrm{d}x \,\mathrm{d}y = 2i \int_\Omega \frac{\partial}{\partial \overline{z}} f \,\mathrm{d}x \,\mathrm{d}y.$$

Recall that when f is analytic, $\frac{\partial}{\partial \overline{z}} f = 0$, this attempts us to write

Theorem 3.2.3 (Cauchy's theorem)

If $\Omega \subset \mathbb{C}$ satisfies the conditions of Green's formula, $f : \overline{\Omega} \to \mathbb{C}$ is continuous and f is analytic on Ω, then

$$\int_{\partial\Omega} f(z)\,\mathrm{d}z = 0.$$

However, this theorem cannot be proved directly using Green's formula (the condition is weaker), we need to prove it explicitly.

Lemma 3.2.4

Let D be a triangular region of $\mathbb{C}$, and f is an analytic function on an open neighborhood of $\overline{D}$, then

$$\int_{\partial D} f\,\mathrm{d}z = 0.$$

Proof. Let $D_i (i = 1, 2, 3, 4)$ be the small triangles splitted by the mid-line of D.

Considering the orientations of D_i, we have

$$\int_{\partial D} f\,\mathrm{d}z = \sum_{i=1}^{4} \int_{\partial D_i} f\,\mathrm{d}z.$$

Assume by contradiction that $|\int_{\partial D} f\,\mathrm{d}z| = M > 0$, then there exists i s.t. $|\int_{\partial D_i} f\,\mathrm{d}z| \geq \frac{M}{4}$. Continuing this process, let Δ_k denote the triangle be obtain on the k-th step,

$$\left|\int_{\partial\Delta_k} f\,\mathrm{d}z\right| \geq \frac{M}{4^k}.$$

However, since $f(z) = f(z_0) + f'(z_0)(z - z_0) + o(z - z_0)$ as $z \to z_0$, in fact we can compute that $\int_{\partial D} z\,\mathrm{d}z = \int_{\partial D} 1\,\mathrm{d}z = 0$, hence the integral of the first two terms is zero.

$$\begin{aligned} \left|\int_{\partial\Delta_k} f\,\mathrm{d}z\right| &= \left|\int_{\partial\Delta_k} o(z - z_0)\,\mathrm{d}z\right| \\ &\leq \max_{z\in\partial\Delta_k} \left|\frac{o(z - z_0)}{z - z_0}\right| \cdot \mathrm{diam}(\Delta_k) \cdot |\partial\Delta_k| \\ &\leq \max_{z\in\partial\Delta_k} \left|\frac{o(z - z_0)}{z - z_0}\right| \cdot \frac{C}{2^k 2^k}. \end{aligned}$$

This is a contradiction since when z_0 is the convergence point of Δ_k, we have $M \leq \max_{z\in\Delta_k} |\frac{o(z-z_k)}{z-z_k}| \to 0$. □

Proof of Theorem 3.2.3. The idea is to approximate the region Ω by polygons, Since polygons have triangulation, we can use above lemma to prove the integral on each polynomial is zero.

Therefore it suffices to prove that $\forall \varepsilon > 0$, there exists a polygonal curve $\partial D \subseteq \Omega$ and $\overline{D} \subset \Omega$ such that

$$\left|\int_{\partial\Omega} f(z)\,\mathrm{d}z - \int_{\partial D} f(z)\,\mathrm{d}z\right| < \varepsilon.$$

Exercise 3.2.5. Prove this claim.

□

Theorem 3.2.6 (Cauchy's integral formula)
Let Ω be a bounded region with $\partial\Omega$ being a finite union of piecewise smooth curves.
Let $f \in C(\overline{\Omega})$ be analytic on Ω, then for all $z \in \Omega$,

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(w)}{w - z} \, \mathrm{d}w.$$

Proof. Let $B(z, \varepsilon) \subset \Omega$ be an open disk centered at z, denote $\Omega' := \Omega \setminus B(z, \varepsilon)$.
By Cauchy's theorem on Ω' and $g(w) := \frac{f(w)}{w-z}$,

$$\int_{\partial\Omega'} g(w) \, \mathrm{d}w = 0 \implies \int_{\partial\Omega} g(w) \, \mathrm{d}w = \int_{\partial B(z,\varepsilon)} g(w) \, \mathrm{d}w.$$

Let ε be sufficiently small, we have $f(w) = f(z) + f'(z)(w - z) + o(w - z)$ for $w \in \partial B(z, \varepsilon)$,

$$\begin{aligned} \int_{\partial B(z,\varepsilon)} g(w) \, \mathrm{d}w &= \int_{|w-z|=\varepsilon} \left(\frac{f(z)}{w - z} + f'(z) + \frac{o(w - z)}{w - z} \right) \mathrm{d}w \\ &= 2\pi i f(z) + \int_{|w-z|=\varepsilon} (f'(z) + o(1)) \, \mathrm{d}w. \end{aligned}$$

Thus when $\varepsilon \to 0$, the latter term tends to 0, which gives the desired. □

With this formula, we finially come to one of the fundamental facts in complex analysis:

Theorem 3.2.7
Let f be analytic on $B(z_0, R)$, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_n)^n$$

for $|z - z_n| < R$, where $a_n = \frac{1}{n!} f^{(n)}(z_0)$ and the radius of convergence of this series is $\geq R$.

Proof. Fix $r \in (0, R)$, it suffices to prove the theorem for all $r < R$. We have

$$f(z) = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{w - z} \, \mathrm{d}w.$$

Since $|w - z_0| = r$, for all $|z - z_0| < r$,

$$\frac{1}{w - z} = \frac{1}{w - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{w - z_0}} = \frac{1}{w - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^n.$$

Now it's tempting to substitute the series into the integral above, which involves a commutation of integral and summations. (the details need uniform convergence of the series, which is left out)

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{|w-z_0|=r} \frac{f(w)}{(w - z_0)^{n+1}} \, \mathrm{d}w \cdot (z - z_0)^n.$$

Therefore $f(z)$ is a power serie, the rest of the theorem is trivial by the properties of power series. □

Corollary 3.2.8
If $f : G \to \mathbb{C}$ is differentiable, then f is infinitely differentiable.

Corollary 3.2.9
If $f : G \to \mathbb{C}$ is analytic, $\overline{B}(z_0, r) \subset G$. For all n,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{(w-z)^{n+1}} \,\mathrm{d}w.$$

Corollary 3.2.10 (Cauchy's estimate)
Let f be analytic on $B(z_0, R)$, $|f(z)| \leq M$ for all $z \in B(z_0, R)$. We have

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!M}{R^n}.$$

Proof. Just take the norm on the both sides of the previous corollary for all $r < R$. □

§3.3 Zeros of analytic functions

Definition 3.3.1. Let $f : G \to \mathbb{C}$ be analytic and G an open subset of $\mathbb{C}$. If $a \in G$ satisties $f(a) = 0$, we say a is a **zero of f of multiplicity $m \geq 1$** if $\exists g : G \to \mathbb{C}$ analytic s.t.

$$g(a) \neq 0, \quad f(z) = (z-a)^m g(z), \forall z \in G.$$

Proposition 3.3.2
Let f, g be analytic functions on a region G, we have $f \equiv g$ iff the set

$$\{z \in G : f(z) = g(z)\}$$

has a limit point in G.

Sketch of proof. WLOG $g \equiv 0$. Let the limit point be a, since f can be written as a power series in a neighborhood of a, we only need to show all the coefficients are 0, which can be easily proved by considering the first nonzero coefficient.

Now we can use a standard technique to show that the set $\{z : f(z) = 0\}$ is the entire G by the connectedness of G, which is left as an exercise.

Hint: for all $z \in G$, there's a piecewise linear path γ connecting z_0 and z, condier $\sup\{t \in [0,1] : f(\gamma(s)) = 0, \forall s \leq t\}$, show that it must be 1. □

Corollary 3.3.3

Let f be analytic on a region Ω and not identically 0, then for all $a \in \Omega$ with $f(a) = 0$, there exists $g : \Omega \to \mathbb{C}$ analytic and $n \in \mathbb{N}$ s.t.

$$g(a) \neq 0, \quad f(z) = (z-a)^n g(z), \forall z \in \Omega.$$

Proof. Let $g(z) = \frac{f(z)}{(z-a)^n}$ when $z \neq a$, and $g(a) = \frac{1}{n!} f^{(n)}(z)$, where n is the minimum number s.t. $f^{(n)}(z) \neq 0$.

We only need to show g is analytic at a, which can be proved by written g as a power series:

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k, \quad g(z) = \sum_{k=n}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^{k-n}.$$

the radius of convergence is positive (same as f), thus g is analytic at a. □

Definition 3.3.4. An **enrite function** is an analytic function $f : \mathbb{C} \to \mathbb{C}$.

Proposition 3.3.5

If f is an entire function, then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with radius of convergence $R = +\infty$.

With this proposition, we can prove some famous theorems in a few lines:

Proposition 3.3.6 (Liouville's theorem)

Let f be a bounded entire function, then f must be a constant.

Proof. By Cauchy's estimate, if $\sup |f| = M < \infty$, we have $|f'(z)| \leq \frac{M}{r}$ for all $r > 0$. (Since the radius of convergence is $+\infty$) □

Theorem 3.3.7 (The Fundamental Theorem of Algebra)

Let $p(z)$ be a non-constant polynomial, then $\exists a \in \mathbb{C}$ such that $p(a) = 0$.

Proof. Suppose $p(z) \neq 0$ for all z, then $\frac{1}{p(z)}$ is an entire function. But we have

$$\lim_{z \to +\infty} |p(z)| = +\infty \implies \lim_{z \to \infty} \frac{1}{|p(z)|} = 0,$$

therefore it is a bounded entire function, which must be a constant by Liouville's theorem. □

§3.4 Index of a closed curve

First let's look at the final result of this section:

Theorem 3.4.1 (Maximum Modulus Theorem)
Let $f : \Omega \to \mathbb{C}$ be analytic, such that there exsits $a \in \Omega$ satisfying

$$|f(a)| \geq |f(z)|, \quad \forall z \in \Omega,$$

then f must be a constant.

Theorem 3.4.2 (Open mapping property)
Let $f : \Omega \to \mathbb{C}$ be an analytic function. If it's not a constant, then it's an *open map*, i.e. $\forall G \subset \Omega$ open, $f(G)$ is also open.

Proof of Theorem 3.4.1. If we assume open mapping property is true, then $\forall a \in \Omega$, $f(B(a, r))$ is open, contradicting with the maximum condition. □

As you can see, these properties are surprisingly good. All we need to do now is to prove Theorem 3.4.2.

To prove the open mapping property, we need to develop the concept of the index.

Proposition 3.4.3
Let $\gamma : [0, 1] \to \mathbb{C}$ be a closed rectifiable curve where $\gamma(0) = \gamma(1)$, and $a \in \mathbb{C} \setminus \gamma$. Then

$$\frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{z - a} \in \mathbb{Z}.$$

This quantity is called the **index** of γ w.r.t. a, denoted by $n(\gamma; a)$.

Remark 3.4.4 — Geometrically, it is the **winding number** of the loop with respect to a.

Proof. Here we only prove the case when γ is smooth. (we can prove the rectifiable case can be approached by the smooth case, like we do in the proof of Cauchy's theorem)

Define $g : [0, 1] \to \mathbb{C}$ as

$$g(t) := \int_0^t \frac{\gamma'(s)}{\gamma(s) - a} \, \mathrm{d}s.$$

Then $g(0) = 0$, $g(1) = \int_\gamma \frac{1}{z-a} \, \mathrm{d}z$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{-g}(\gamma - a)) = e^{-g}(\gamma' - g'(\gamma - a)) = 0.$$

Hence $e^{-g}(\gamma - a)$ is a constant, which implies

$$e^{-g(0)}(\gamma(0) - a) = e^{-g(1)}(\gamma(1) - a) \implies e^0 = e^{-g(1)}.$$

Be careful that when taking logarithm in complex field, we have multiple solutions: $g(1) = 2k\pi i$, $k \in \mathbb{Z}$. □

Example 3.4.5
The index of the unit circle with respect to 0 is 1 or -1 (with different orientations)

Some basic properties of winding numbers are

- $n(\gamma; a) = -n(-\gamma; a)$.
- $n(\gamma_1 + \gamma_2; a) = n(\gamma_1; a) + n(\gamma_2; a)$, here the "addition" is the concatenation of curves.
- $n(\gamma; z)$ is constant on each connected component of $\mathbb{C} \setminus \gamma$. (Since $z \mapsto n(\gamma; z)$ is continuous)
- $n(\gamma; z) = 0$ for z in the *unbounded* component. (As $|a| \to \infty$, we have $n(\gamma; a) \le \frac{C}{d(a,\gamma)} \to 0$)

Lemma 3.4.6
Consider a rectifiable curve γ, $\varphi : \gamma \to \mathbb{C}$ continuous, $m \in \mathbb{N}$. Then

$$F_m(z) := \int_\gamma \frac{\varphi(w)}{(w-z)^m} \, \mathrm{d}w$$

is analytic on $z \in \mathbb{C} \setminus \gamma$, and $F'_m(z) = mF'_{m+1}(z)$.

Proof. First we prove that F_m is continuous: fix $a \in \mathbb{C} \setminus \gamma$,

$$\frac{1}{(w-z)^m} - \frac{1}{(w-a)^m} = \frac{z-a}{(w-a)(w-z)} \left(\frac{1}{(w-a)^{m-1}} + \cdots + \frac{1}{(w-z)^{m-1}} \right).$$

Therefore we can compute

$$\frac{F_m(z) - F_m(a)}{z-a} = \int_\gamma \frac{\varphi(w)}{(w-a)(w-z)} \sum_{k=0}^{m-1} \frac{1}{(w-a)^{m-1-k}(w-z)^k} \, \mathrm{d}w.$$

Therefore taking the limit $z \to a$ we get the desired. □

Theorem 3.4.7 (Cauchy's integral formula v1)
Let $G \subset \mathbb{C}$ open, $f : G \to \mathbb{C}$ analytic, γ is a closed rectifiable curve in G s.t. $n(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus G$. Then for all $a \in G \setminus \gamma$,

$$n(\gamma; a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} \, \mathrm{d}z.$$

Proof. Define $\varphi : G \times G \to \mathbb{C}$ by

$$\varphi(z, w) = \begin{cases} \dfrac{f(z) - f(w)}{z-w} & \text{if } z \neq w, \\ f'(z) & \text{if } z = w. \end{cases}$$

We can check that φ is continuous on $G \times G$ and analytic for all $w \in G$.

Define $g : \mathbb{C} \to \mathbb{C}$ by

$$g(z) := \begin{cases} \displaystyle\int_\gamma \varphi(z, w)\,\mathrm{d}w & z \in G, \\ \displaystyle\int_\gamma \frac{f(w)}{w - z}\,\mathrm{d}w & n(\gamma; z) = 0. \end{cases}$$

We need to check it's well-defined: for $z \in G$ with $n(\gamma; z) = 0$, the two definition coincides since $f(z) \int_\gamma \frac{1}{z-w}\,\mathrm{d}w = 0$.

Now g is entire and $\int_\gamma \frac{f(w)}{w-z}\,\mathrm{d}w \to 0$ as $|z| \to +\infty$. So g is bounded $\implies$ g is a constant by Liouville's theorem. Hence $g \equiv 0$ for all $a \in G \setminus \gamma$, which gives

$$0 = \int_\gamma \frac{f(w)}{w - a}\,\mathrm{d}w - f(a) \int_\gamma \frac{1}{w - a}\,\mathrm{d}w.$$

□

Theorem 3.4.8 (Cauchy's integral formula v2)

Let G, f be as before. Let $\gamma_1, \ldots, \gamma_m$ be closed rectifiable curves in G s.t. $\sum_{i=1}^m n(\gamma_i; w) = 0$ for all $w \notin G$. Then for all $a \in G \setminus \bigcup_{i=1}^m \gamma_i$,

$$f(a) \sum_{k=1}^m n(\gamma_k; a) = \frac{1}{2\pi i} \sum_{k=1}^m \int_{\gamma_k} \frac{f(z)}{z - a}\,\mathrm{d}z.$$

Proof. Consider the formal sum of curves $\gamma_1 + \gamma_2 + \cdots + \gamma_m$ (just like we do in algebraic topology), it satisfies the condition of Cauchy integral formula version 1. □

Corollary 3.4.9

Let $f, G, \gamma_1, \ldots, \gamma_m$ as above,

$$\sum_{k=1}^m \int_{\gamma_k} f\,\mathrm{d}z = 0.$$

Proof. Put $g(z) = f(z) \cdot (z - a)$ and use Cauchy's integral formula. □

Corollary 3.4.10

Let $f, G, \gamma_1, \ldots, \gamma_m$ as above, for all $k \in \mathbb{N}$ and $a \in G \setminus \bigcup_{j=1}^m \gamma_j$,

$$f^{(k)}(a) \sum_{j=1}^m n(\gamma_j; a) = \frac{k!}{2\pi i} \sum_{j=1}^m \int_{\gamma_j} \frac{f(z)}{(z - a)^{k+1}}\,\mathrm{d}z.$$

Proof. Use the lemma above and taking the derivatives with respect to a. □

§3.5 Counting zeros

Now we can proceed on proving the open mapping theorem 3.4.2.

Theorem 3.5.1

Let $f : \Omega \to \mathbb{C}$ be an analytic function with zeros $a_1, a_2, \ldots, a_n$, repeated according to multiplicity. If γ is a closed rectifiable curve in G avoiding the zeros and satisfying $n(\gamma; w) = 0, \forall w \notin G$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_{k=1}^m n(\gamma; a_k)$$

Remark 3.5.2 — Earlier we proved that there can't be a zero with multiplicity infinity, unless $f \equiv 0$.

This theorem reveals the relation between $n(f \circ \gamma; 0)$ and $n(\gamma; a_k)$, since $\int_\gamma \frac{f'}{f}\,\mathrm{d}z = \int_{f\circ\gamma} \frac{1}{z}\,\mathrm{d}z$.

Proof. Write $f = (z - a_1)\cdots(z - a_n)g(z)$ where $g(z)$ is a nonzero analytic function on G. Since

$$\frac{f'(z)}{f(z)} = \frac{1}{z - a_1} + \cdots + \frac{1}{z - a_n} + \frac{g'(z)}{g(z)}$$

by Cauchy's theorem we get the desired equality. □

Theorem 3.5.3

Suppose $f : B(a, R) \to \mathbb{C}$ is an analytic function with $f(a) = \alpha$. If a is a zero of multiplicity $m \geq 1$ of $f(z) - \alpha$, then

$$\exists \varepsilon > 0, \exists \delta > 0, \forall \beta \in B_0(\alpha, \delta), \quad f(z) - \beta \text{ has exactly } m \text{ simple zeros in } B(a, \varepsilon).$$

Remark 3.5.4 — This theorem says that analytic functions behaves like z^m near the zeros of multiplicity m, or zeros of high multiplicity can split into simple zeros under a small disturbance.

Proof. Firstly we can take $\varepsilon \in (0, \frac{R}{2})$ s.t.

$$\forall z \in B_0(a, 2\varepsilon), \quad f(z) \neq \alpha, f'(z) \neq 0.$$

(this is because the zeros of $f - \alpha$ has no limit points, for f' we need to discuss whether $f'(a) = 0$ or not)

Define $\gamma(t) := a + \varepsilon e^{2\pi i t}$, $t \in [0, 1]$, let $\sigma(t) := f \circ \gamma(t)$. We have $\alpha \notin \sigma$, hence $\exists \delta > 0$ s.t. $B(\alpha, \delta) \cap \sigma = \emptyset$. For $\beta \in B(\alpha, \delta)$, we have

$$n(\sigma; \alpha) = n(\sigma; \beta) = \frac{1}{2\pi i}\int_{f\circ\gamma} \frac{1}{z - \beta}\,\mathrm{d}z = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z) - \beta}\,\mathrm{d}z = \sum_{k=1}^p n(\gamma; z_k(\beta)),$$

where $z_k(\beta)$ are zeros of $f(z) - \beta$.

With the same technique we can compute $n(\sigma;\alpha) = m$, since $f'(z_k(\beta)) \neq 0$ as $z_k(\beta) \in B(a, 2\varepsilon)$, the multiplicity of $z_k(\beta)$ must be 1.

Also note that γ is a circle centered at a, $n(\gamma; z_k(\beta))$ can only be 0 or 1 (1 if $z_k(\beta) \in B(a, \varepsilon)$), hence we must have $p \geq m \implies f(z) - \beta$ has m simple zeros inside $B(a, \varepsilon)$. □

Proof of Theorem 3.4.2. For all $a \in U$, $\exists \varepsilon, \delta > 0$ s.t. $B(f(a), \delta) \subset f(B(a, \varepsilon))$ by above theorem since $\forall \beta \in B(f(a), \delta)$, $f(z) - \beta$ has zeros in $B(a, \varepsilon)$. □

Corollary 3.5.5

Let $f : \Omega \to \mathbb{C}$ be univalent with $f(\Omega) = \Omega'$. Then $\forall z \in \Omega$, $f'(z) = 0$.

Moreover $f^{-1}(z)$ is analytic on Ω' and $f'(z)(f^{-1})'(f(z)) = 1$.

Proof. If $f'(z_0) = 0$, $f(z) - f(z_0)$ has a zero of multiplicity at least 2, contradicts with the injective condition of f by above theorem.

Now we have f^{-1} is continuous since f is open and univalent, the rest follows from the inverse function theorem (Conway prop. III 2.20). □

Next we'll show some simple and nice properties of analytic functions.

Theorem 3.5.6 (Weierstrass)

Let f be a nonconstant entire function, then $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Proof. Suppose not, there exists $z_0 \in \mathbb{C}$ s.t. $B(z_0, r) \notin f(\mathbb{C})$.

Then $g(z) := \frac{1}{f(z) - z_0}$ is entire, and $|g(z)| \leq \frac{1}{r}$. Now by Liouville's theorem, g must be a constant, contradiction! □

Theorem 3.5.7 (Picard's little theorem)

Let f be a transcendental (i.e. non-polynomial) entire function, then $\#(\mathbb{C} \setminus f(\mathbb{C})) \leq 1$.

Theorem 3.5.8 (Mean value theorem)

Let $f : \Omega \to \mathbb{C}$ be analytic, then for all $z_0 \in \Omega$, $\forall r > 0$ with $\overline{B(z_0, r)} \subset \Omega$, we have

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) \, \mathrm{d}\theta.$$

Proof. This is immediate by Cauchy's integral formula. □

Corollary 3.5.9

We have
$$|f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})|\,\mathrm{d}\theta.$$
$$|f(z_0)| \le \frac{1}{\pi r^2}\int_{B(z_0,r)} |f(z)|\,\mathrm{d}z.$$

§3.6 Schwarz's lemma and hyperbolic geometry

Theorem 3.6.1 (Schwarz's lemma)

Let $f : B(0,1) \to B(0,1)$ be analytic, $f(0) = 0$. Then

- $\forall z \in B(0,1)$, $f(z) \le |z|$, $|f'(z)| \le 1$.
- TFAE:
  (a) $\exists z_0 \neq 0$ s.t. $|f(z_0)| = |z_0|$.
  (b) $|f'(0)| = 1$.
  (c) $f(z) = ze^{i\theta}$ for some $\theta \in [0, 2\pi]$.

Proof. Let
$$f(z) = a_1 z + a_2 z^2 + \cdots$$
then $\frac{f(z)}{z}$ is analytic on $B(0,1)$. By maximum modulus theorem, $|\frac{f(z)}{z}|$ takes its maximum value at the boundary.

Hence for $z_0 \in B(0,1)$,
$$\left|\frac{f(z_0)}{z_0}\right| \le \frac{1}{r}, \quad \forall r < 1.$$
therefore $|f(z_0)| \le |z_0|$, and $|f'(0)| = \lim_{z\to 0} |\frac{f(z)}{z}| \le 1$.

For the second part, if $|f(z_0)| = |z_0|$ for some z_0, again by maximum modulus theorem, $\frac{f(z)}{z}$ must be a constant C with $|C| = 1$. Therefore $(a) \implies (c)$.

Similarly since $\frac{f(z)}{z}$ is defined as $f'(0)$ when $z = 0$, again by maximum modulus theorem we have $(b) \implies (c)$. ☐

Theorem 3.6.2

Let $g : B(0,1) \to B(0,1)$ be a biholomorphic function. We have
$$g(z) = e^{i\theta}\frac{z - z_0}{1 - \overline{z}_0 z}$$
for some $z_0 \in B(0,1)$, $\theta \in (0, 2\pi]$.

Proof. Set $l_{z_0}(z) := \frac{z - z_0}{1-\overline{z}_0 z}$. It is analytic for all $z_0 \in B(0,1)$, and its inverse map is l_{-z_0}.

Denote $l_{g(0)}$ by l, and set $f := l(g(z))$, then $f(0) = 0$ is a biholomorphism.

By Schwarz's lemma, $|f'(0)| \le 1$ and $|(f^{-1})'(0)| \le 1$. This means $|f'(0)| = 1 \implies f(z) = e^{i\theta}z$ for some $\theta \in (0, 2\pi]$. We have

$$e^{i\theta}z = \frac{g(z) - g(0)}{1 - \overline{g(0)}g(z)} \implies g(z) = e^{i\theta}\frac{z - g(0)}{1 - \overline{g(0)}z}.$$

□

Therefore we have the group of biholomorphic maps on $B(0,1)$:

$$A(B(0,1)) := \left\{ e^{i\theta}\frac{z - z_0}{1 - \overline{z}_0 z} : \theta \in (0, 2\pi], z_0 \in B(0,1) \right\}.$$

Its action on $B(0,1)$ is transitive.

Theorem 3.6.3 (Schwarz's lemma, another version)
Let $f : B(0,1) \to B(0,1)$ be analytic. Then for all $z_1, z_2 \in B(0,1)$,

$$\frac{|f(z_1) - f(z_2)|}{|1 - \overline{f(z_1)}f(z_2)|} \le \frac{|z_1 - z_2|}{|1 - \overline{z}_1 z_2|}.$$

The equality holds iff f is biholomorphic.

Proof. Consider the biholomorphic map $l_{f(z_2)} \circ f \circ l_{z_2}^{-1}$, it maps 0 to 0. Therefore by Schwarz's lemma we have

$$|l_{f(z_2)} \circ f \circ l_{z_2}^{-1}(z_1)| \le |z_1|,$$

which simplifies to the desired inequality. When equality holds, the map must be a rotation, hence f must be a biholomorphism. □

Corollary 3.6.4
Taking $z_1 \to z_2 = z$, we have

$$\frac{|f'(z)|}{1 - |f(z)|^2} \le \frac{1}{1 - |z|^2}.$$

Equality iff f biholomorphic.

Definition 3.6.5 (Poincare metric)**.** Define the **Poincare (hyperbolic) metric** on $B(0,1)$ to be

$$\mathrm{d}s := \frac{|\,\mathrm{d}z|}{1 - |z|^2}.$$

The length of a (rectifiable) curve $\gamma : [a, b] \to B(0,1)$ is defined as

$$l_p(\gamma) := \int_\gamma \mathrm{d}s = \int_a^b \frac{|\,\mathrm{d}\gamma(t)|}{1 - |\gamma(t)|^2}.$$

Note that by above corollary, biholomorphisms preserves the hyperbolic length.

Remark 3.6.6 — In geometry course we're told that circles orthogonal to $\partial B(0,1)$ are all the geodesic lines in Poincare disk.

Now we can prove it by finding all the geodesic lines through the origin, and using biholomorphisms to move it around.

Hence we can define the distance

$$d_p(z_1, z_2) := l_\gamma,$$

where γ is the (shortest) geodesic line connecting z_1 and z_2. (In hyperbolic geometry the geodesic line connecting two given points is unique)

Using the language of hyperbolic geometry, we can restate Schwarz's lemma as below:

Theorem 3.6.7 (Schwarz's lemma)
Let $f : B(0,1) \to B(0,1)$ be an analytic map, then for all (smooth) $\gamma : [0,1] \to B(0,1)$, we have

$$l_p(f \circ \gamma) \le l_p(\gamma),$$

hence $d_p(f(x), f(y)) \le d_p(x,y)$ for all $x, y \in B(0,1)$.

§4 Singularity

§4.1 Classification of singularities

Definition 4.1.1. We say a function f has an **isolated singularity** at $z = a$ if f is defined and analytic on $B_0(a,R)$ but not on $B(a,R)$.

We say a is a **removable singularity** if $\exists g : B(a,R) \to \mathbb{C}$ analytic s.t. $f = g$ on $B_0(a,R)$.

Example 4.1.2
$e^{\frac{1}{z}}$, $\frac{1}{z}$, $\frac{\sin z}{z}$ have an isolated singularity at $z = 0$. The singularity of $\frac{\sin z}{z}$ is removable since you can write down the power series.

Theorem 4.1.3
If f has an isolated singularity at $z = a$, then a is a removable singularity iff

$$\lim_{z \to a} (z-a) f(z) = 0.$$

Proof. If a is removable, this is trivial.

Let's consider the other direction. Let $g(z) = (z-a)f(z)$.

Claim 4.1.4. g is analytic (at $z = a$).

Using this claim, since a is a zero of g, there exists $h : B(a,R) \to \mathbb{C}$ analytic s.t. $g(z) = (z-a)h(z)$ for all $z \in B(a,R)$, hence $h = f$ on $B_0(a,R)$. ☐

In order to prove this claim, we need some preparations.

Theorem 4.1.5 (Morera's theorem)
Let $f: G \to \mathbb{C}$ be continuous, if $\int_\Delta f \,\mathrm{d}z = 0$ for every triangular curve Δ, then f is analytic.

Proof. WLOG $G = B(a, R)$, it suffices to show that there exists F s.t. $F' = f$.
Define
$$F(z) := \int_\gamma d \,\mathrm{d}z, \quad z \in B(a, R),$$
where γ is the line segment from a to z.
By the condition, denote the line segment by $[z_0, z_1]$, as $z_1 \to z_0$, we have
$$\begin{aligned}\left|\frac{F(z_1) - F(z_0)}{z_1 - z_0} - f(z_0)\right| &= \frac{1}{|z_1 - z_0|}\left|\int_{[z_0, z_1]} (f(z) - f(z_0)) \,\mathrm{d}z\right| \\ &\le \sup |f(z) - f(z_0)| \to 0\end{aligned}$$
Hence $F' = f$ on $B(a, R)$, as desired. □

Proof of the claim. On $B(a, R)$, for each triangular curve, we can use Cauchy's theorem to make it arbitarily small, and since $\lim_{z\to a} g(z) = 0$, the integral of g on the curve must be arbitarily small, hence it must be 0.
By Morera's theorem we get g is analytic. □

Definition 4.1.6. Let $z = a$ be an isolated singularity of f,

- If $\lim_{z\to a} f(z) = \infty$, then a is a **pole** of f.
- If a is neither a pole nor a removable singularity, we say a is an **esstential singularity**.

Like the zeros, we can also define multiplicity for poles.

Proposition 4.1.7
Let G be a region, $f : G \setminus \{a\} \to \mathbb{C}$ is analytic with a pole at $z = a$. Then $\exists m \in \mathbb{N}$ and $g : G \to \mathbb{C}$ analytic s.t.
$$f(z) = \frac{g(z)}{(z-a)^m}, \quad \forall z \in G \setminus \{a\}.$$

Proof. Consider $h(z) = \frac{1}{f(z)}$, it has a removable singularity at $z = a$. We can extend $h(a) = 0$, it's analytic with a zero at a, then m is precisely the multiplicity of the zero. □

Definition 4.1.8. Let f be an analytic function with a pole at $z = a$. If $m \in \mathbb{N}$ is the smallest integer s.t. $f(z)(z-a)^m$ has a removable singularity at $z = a$, we say f has **a pole of order m** at $z = a$.

Definition 4.1.9. Let $\{z_n\}_{n\in\mathbb{Z}}$ be a **doubly infinite** sequence of complex numbers. We say $\sum_{n=-\infty}^{+\infty} z_n$ is (absolutely) convergent if both $\sum_{n=0}^{+\infty} z_n$ and $\sum_{n=1}^{+\infty} z_{-n}$ are (absolutely) convergent.
For functions $\{f_n\}_{n\in\mathbb{Z}}$, we can define the convergence similarly.

Theorem 4.1.10

Let $0 \le R_1 < R_2 \le +\infty$, $a \in \mathbb{C}$. Define $ann(a, R_1, R_2) := B(a, R_2) \setminus \overline{B(a, R_1)}$, if f is analytic on $ann(a, R_1, R_2)$, then we can write

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-a)^n,$$

where the bi-infinite series converges absolutely and uniformly on $\overline{ann(a, r_1, r_2)}$ for all $R_1 < r_1 < r_2 < R_2$.

Here $a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} \, \mathrm{d}z$ is unique, γ is a circle $|z-a| = r$ for any $r \in (R_1, R_2)$.

Proof. Firstly, a_n is well-defined by Cauchy's theorem.

By Cauchy's integral formula, let γ_i be the circle $|z-a| = r_i$, $i = 1, 2$,

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(w)}{w-z} \, \mathrm{d}w - \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{w-z} \, \mathrm{d}w.$$

denote the above two terms by $f_2(z)$ and $f_1(z)$, respectively ($f_1(z)$ contains the negative sign).

Since $f_2(z)$ is analytic on $B(a, r_2)$, it can be written as

$$f_2(z) = \sum_{n=0}^{\infty} a_n (z-a)^n.$$

Claim 4.1.11. f_1 is analytic on $\mathbb{C} \setminus \overline{B(a, r_1)}$.

Define $g(z) := f_1(a + \frac{1}{z})$ for $z \in B_0(0, \frac{1}{r_1})$. We can check $z = 0$ is a removable singularity of g with extension $g(0) = 0$.

Let $g(z) = \sum_{n=1}^{\infty} B_n z^n$, we have

$$f_1(z) = \sum_{n=1}^{\infty} B_n (z-a)^{-n}.$$

These series converges absolutely and uniformly on $ann(a, r_1, r_2)$, it remains to check $a_{-n} = B_n$ and the uniqueness, which is left as an exercise. □

The series above is called the **Laurent series** or **Laurent expansion** of the function f.

Let $z = a$ be an isolated singularity of a and let

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-a)^n$$

be its Laurent expansion in $ann(a, 0, R)$. We have three corollaries corresponding to three kinds of singularities:

Corollary 4.1.12

The followings are equivalent:

(1) a is removable.

(2) $\lim_{z\to a} f(z)$ exists and is in $\mathbb{C}$.

(3) $f(z)$ is bounded on some $B_0(a,\varepsilon)$.

(4) $a_n = 0$ for all $n \leq -1$.

Proof. (1) $\implies$ (2) $\implies$ (3) is trivial.

(3) $\implies$ (4) since

$$|a_n| \leq \frac{1}{2\pi}\int_{|w-z|=\varepsilon} \frac{M}{\varepsilon^{n+1}}\,\mathrm{d}s = M\varepsilon^{-n} \to 0$$

for any fixed $n \leq -1$ as $\varepsilon \to 0$.

(4) $\implies$ (1) since the series already defines an analytic function on $B(a,R)$ (it converges on $B_0(a,R)$ and exists at a), hence a is removable. □

Corollary 4.1.13

a is a pole of order $m \in \mathbb{N}$ iff $a_{-m} \neq 0$ and for all $n \leq -m-1$, $a_n = 0$.

Proof. This is esstentially (1) $\iff$ (4) of the previous corollary on $(z-a)^m f(z)$. □

Corollary 4.1.14

a is an essential singularity iff $a_n \neq 0$ for infinitely many negative n.

Proof. Follows by definition. □

Theorem 4.1.15 (Casorati-Weierstrass theorem)

Let f be an analytic function with an essential singularity at $z = a$, then for every $\delta > 0$ (sufficiently small), $f(B_0(a,\delta))$ is dense in $\mathbb{C}$.

Proof. Suppose it is not dense for some $\delta > 0$. Let $b \in \mathbb{C}\setminus\overline{f(B_0(a,\delta))}$, and $B(b,\varepsilon) \subset \mathbb{C}\setminus\overline{f(B_0(a,\delta))}$ as well.

Let $g(z) := \frac{1}{f(z)-b}$, then $|g(z)| \leq \frac{1}{\varepsilon}$ on $B_0(a,\delta)$. By above corollary on removable singularities, $g(z)$ has a removable singularity at $z = a$ (i.e. it can extend to a analytically).

However, $f(z) = \frac{1}{g(z)} + b$, so $f(z)$ has either a removable singularity or a pole at $z = a$, contradiction! □

Theorem 4.1.16 (Great Picard theorem)
Let f be an analytic function on $B_0(a, R)$. If $z = a$ is an essential singularity of $f(z)$, then $\forall \varepsilon \in (0, R)$,

$$\#(\mathbb{C} \setminus f(B_0(a, \varepsilon))) \leq 1.$$

The proof of this theorem is beyond the scope of this course.

Now we'll implement the definition of singularities at ∞.

Definition 4.1.17. If ∞ is an isolated singularity of $f(z)$, let $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ be the Laurant expansion at ∞,

- ∞ is removable iff $a_n = 0$ for all $n \geq 1$.
- ∞ is a pole iff only finitely many of $\{a_1, a_2, \dots\}$ are nonzero.
- ∞ is an essential singularity iff infinitely many of them are not zero.

If f is entire, the power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is the Larent expansion of f at $z = \infty$ by the uniqueness of Larent expansion.

Theorem 4.1.18
A function $f : \mathbb{C} \to \mathbb{C}$ is biholomorphic iff $f(z) = az + b$ for some $a, b \in \mathbb{C}$, $a \neq 0$.

Proof. If f is a biholomorphism, then f is entire, it has an isolated singularity at ∞.

It is not removable (otherwise f is bounded $\implies$ f is constant), and it's not essential by Casorati-Weierstrass theorem.

Therefore ∞ is a pole, which means f is a polynomial by looking at the Larent expansion at ∞. Since f is bijective, the degree of f must be 1. □

§4.2 Meromorphic functions

Definition 4.2.1. Let $\Omega \subset \widehat{\mathbb{C}}$ be a region. If a function $f : \Omega \to \mathbb{C}$ is analytic except for poles, we call f a **meromorphic function**.

Remark 4.2.2 — If f is a meromorphic function, we can define $f(a) = \infty$ for all poles a, then f is an analytic function between Riemann surfaces. In fact analytic functions between Riemann spheres except constant ∞ are all the meromorphic functions.

In this note, when we say "analytic", we're actually saying " $\mathbb{C}$-valued analytic" unless specified otherwise.

Theorem 4.2.3
Let $\Omega \subset \mathbb{C}$ be a region, $f, g : \mathbb{C} \to \widehat{\mathbb{C}}$ are meromorphisms. If $f(z) = g(z)$ on a sequence $\{z_n\}$ which has an accumulation point in Ω, then $f \equiv g$.

Proof. By continuity, $f - g$ must be 0 at the accumulation point, therefore it has a neighborhood in which $f - g$ has no pole.

By the result on analytic functions, $f - g \equiv 0$ on this neighborhood, hence the rest is trivial by connectedness of Ω. □

Next we state an important theorem on the existence of meromorphic functions without proof.

Theorem 4.2.4 (Mittag-Leffler)

Given distinct points $z_1, z_2, \ldots, z_n, \ldots$ in $\mathbb{C}$ with $\lim_{n\to\infty} |z_n| = +\infty$, $m_n \in \mathbb{N}$, and

$$L_n(z) = \frac{a_{n,1}}{z - z_n} + \cdots + \frac{a_{n,m_n}}{(z - z_n)^{m_n}},$$

there exists a meromorphic function $f : \mathbb{C} \to \widehat{\mathbb{C}}$ whose poles are exactly $z_1, z_2, \cdots$, and the Laurent expansion of f at z_n coincides with $L_n(z)$ for negative powers of $z - z_n$ for each $n \in \mathbb{N}$.

Remark 4.2.5 — If $f, g : \Omega \to \widehat{\mathbb{C}}$ are meromorphic and their Laurent expansions satisfy $a_n = b_n$ for all $n \leq -1$ at each poles, then $f - g : \Omega \to \mathbb{C}$ is an analytic function (you can write the power series expansion of $f - g$ at each point).

This means the meromorphic function above is unique up to an analytic function.

Theorem 4.2.6

A meromorphic function f on $\widehat{\mathbb{C}}$ is a *rational function*, i.e. $f = \frac{P}{Q}$ for some polynomials P and Q.

Proof. Since $\widehat{\mathbb{C}}$ is compact, WLOG let $z_1, z_2, \ldots, z_l, \infty$ be the poles (poles are isolated). Let

$$L_k(z) = \sum_{j=1}^{m_k} \frac{a_{k,j}}{(z - z_k)^j}, \quad k = 1, \ldots, l,$$

$$L_\infty(z) = b_1 z + \cdots + b_{m_\infty} z^{m_\infty}$$

be the corresponding singular parts (i.e. negative powers in Larent series), then $f(z) - L_1(z) - \cdots - L_l(z) - L_\infty(z)$ is an analytic function $\widehat{\mathbb{C}} \to \mathbb{C}$ by above remark.

Thus by Maximum modulus principle it must be constant, which proves the theorem. □

Corollary 4.2.7

A biholomorphism $\widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a Mobius transformation.

Proof. Let f be a biholomorphism, it is meromorphic $\implies$ it is a rational function $\frac{P(z)}{Q(z)}$.

Note that if $\deg P \neq 1$ or $\deg Q \neq 1$, f is not a bijection since

$$f(z) = \alpha \iff P(z) - \alpha Q(z) = 0,$$

by Fundamental theorem of algebra we can take α s.t. $P - \alpha Q$ has more than one zeros.

Hence $f(z) = \frac{az+b}{cz+d}$, we can easily prove $ad \neq bc$, so we're done. □

Since poles are somewhat an opposite to zeros, for zeros we have similar results:

Theorem 4.2.8

Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers with $\lim_{n\to\infty}|z_n| = +\infty$. Let $\{m_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$, then there exsits an analytic function $f : \mathbb{C} \to \mathbb{C}$ with zeros z_n of multiplicity m_n.

The proof can be found on the textbook. If we only require finitely many zeros, the theorem is trivial. Since the zeros can't have an accumulation point in $\mathbb{C}$, the condition in the theorem is the best nontrivial case.

Corollary 4.2.9

Let $g : \mathbb{C} \to \mathbb{C}$ be a meromorphic function, then there exists $f, h : \mathbb{C} \to \mathbb{C}$ analytic such that $g = \frac{f}{h}$.

Proof. Let the poles of g be $\{z_n\}$ with multiplicity $\{m_n\}$, by above theorem we can construct an analytic function h with zeros at $\{z_n\}$ with multiplicity $\{m_n\}$.

Set $f := gh$, we can check that f only has removable singularities. □

§4.3 Residues

Definition 4.3.1 (Residue)**.** Let f be a function with isolated singularity at $z = a$, suppose its Laurent series at $z = a$ is

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n(z-a)^n.$$

Then the **residue** of f at a is defined as a_{-1}, denoted by $Res(f, a)$.

The motivation to consider this coefficient is as follows: if we want to know $\int_\gamma f$, where γ is a closed curve not passing through a, then only the coefficient a_{-1} will change its value.

Theorem 4.3.2 (Residue theorem)

Let f be analytic in a region $\Omega \subset \widehat{\mathbb{C}}$ except for the isolated singularities $a_1, \ldots, a_m$.

If $\gamma_1, \ldots, \gamma_l$ are closed rectifiable curves in Ω without passing through $a_1, \ldots, a_m$, satifying

$$\sum_{i=1}^{l} n(\gamma_i; z) = 0, \quad \forall z \notin \Omega,$$

then

$$\sum_{j=1}^{l} \frac{1}{2\pi i}\int_{\gamma_j} f(z)\,\mathrm{d}z = \sum_{k=1}^{m} Res(f, a_k)\sum_{j=1}^{l} n(\gamma_j; a_k).$$

Proof. Case 1. If ∞ is not a singularity, let

$$\sigma_k(t) = a_k + r_k e^{-2\pi i t \sum_{j=1}^{l} n(\gamma_j, a_k)}$$

we can check the winding number condition, so by Cauchy's theorem,

$$\sum_{j=1}^{l} \frac{1}{2\pi i}\int_{\gamma_j} f\,\mathrm{d}z = -\sum_{k=1}^{m} \frac{1}{2\pi i}\int_{\sigma_k} f\,\mathrm{d}z.$$

Now we use the Laurent expansion of f at $z = a_k$, recall that

$$\int_{\sigma_k} (z - a_k)^n \, \mathrm{d}z = 0, \quad \forall n \neq -1,$$

therefore

$$-\frac{1}{2\pi i} \int_{\sigma_k} f \, \mathrm{d}z = -\frac{1}{2\pi i} \int_{\sigma_k} \frac{Res(f, a_k)}{z - a_k} \, \mathrm{d}z = \sum_{j=1}^{l} n(\gamma_j; a_k) Res(f, a_k).$$

Case 2. If ∞ is a singularity, the proof is similar. Be careful that $Res(f, \infty) = -\frac{1}{2\pi i} \int_{|z|=\rho} f \, \mathrm{d}z$ for ρ sufficiently large, the negative sign comes from the issue of orientation. □

Corollary 4.3.3

If $f : \mathbb{C} \setminus \{z_1, \ldots, z_n\} \to \mathbb{C}$ is analytic, then

$$\sum_{k=1}^{n} Res(f, z_k) + Res(f, \infty) = 0.$$

Proposition 4.3.4

If f has a pole of order m at $z = a$, and $g(z) := (z - a)^m f(z)$, then

$$Res(f, a) = \frac{1}{(m-1)!} g^{(m-1)}(a).$$

Proof. Consider the power series expansion of g. □

Example 4.3.5

Prove that

$$\int_0^{+\infty} \frac{\sin x}{x} = \frac{\pi}{2}.$$

Proof. Let $f(z) = \frac{e^{iz}}{z}$, it has a simple pole at $z = 0$. Take $0 < r < 1 < R < +\infty$, we can draw a curve in the complex plane consisting of two semicircles (in the upper plane) of radii r and R, and two line segments connecting them.

By Cauchy's theorem,

$$\begin{aligned} 0 &= \int_\gamma f(z) \, \mathrm{d}z \\ &= \int_r^R \frac{e^{ix}}{x} \, \mathrm{d}x + \int_{-R}^{-r} \frac{e^{ix}}{x} \, \mathrm{d}x + \int_{\gamma_R} \frac{e^{iz}}{z} \, \mathrm{d}z - \int_{\gamma_r} \frac{e^{iz}}{z} \, \mathrm{d}z. \end{aligned}$$

The first two term can be easily computed as $2i \int_r^R \frac{\sin x}{x} \, \mathrm{d}x$, and we can compute

$$\lim_{r \to 0+} \int_{\gamma_r} \frac{e^{iz}}{z} \, \mathrm{d}z = \lim_{r \to 0+} \int_{\gamma_r} \frac{1}{z} \, \mathrm{d}z = \pi i.$$

Hence it suffices to prove $\int_{\gamma_R} f(z)\,\mathrm{d}z = 0$.

$$\begin{aligned}\left|\int_{\gamma_R} \frac{e^{iz}}{z}\,\mathrm{d}z\right| &= \left|\int_0^\pi \exp(iRe^{i\theta})\,\mathrm{d}\theta\right| \\ &\le \int_0^\pi |\exp(iRe^{i\theta})|\,\mathrm{d}\theta \\ &= \int_0^\pi \exp(-R\sin\theta)\,\mathrm{d}\theta = 2\int_0^{\frac{\pi}{2}} \exp(-R\sin\theta)\,\mathrm{d}\theta.\end{aligned}$$

We estimate it in two parts:

$$\int_\varepsilon^{\frac{\pi}{2}} \exp(-R\sin\theta)\,\mathrm{d}\theta \le \frac{\pi}{2}\exp(-R\sin\varepsilon) \to 0,$$

and $\int_0^\varepsilon \exp(-R\sin\theta)\,\mathrm{d}\theta \le \varepsilon \to 0$. Hence we're done. □

Theorem 4.3.6 (Argument Principle)

Let f be a meromorphic function on G, let the poles be $p_1, \ldots, p_m$ and zeros be $z_1, \ldots, z_m$, counted with multiplicity.

If γ is a closed rectifiable curve on G not containing any poles or zeros, and $n(\gamma; z) = 0$ for all $z \in \mathbb{C} \setminus G$. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_{k=1}^n n(\gamma; z_k) - \sum_{k=1}^m n(\gamma; p_k).$$

Proof. Let

$$g(z) := f(z)\prod_{j=1}^m (z-p_j)\prod_{k=1}^n (z-z_k)^{-1}.$$

Then g is analytic and nonzero in G. Clearly

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^m \frac{1}{z-z_k} - \sum_{j=1}^m \frac{1}{z-p_j} + \frac{g'(z)}{g(z)},$$

so we're done by Cauchy's theorem. □

Theorem 4.3.7 (Rouche's theorem)

Let $\Omega_0 \Subset \Omega$ be bounded regions and $\partial\Omega_0 = \Gamma$ is a Jordan curve. Let f, g be meromorphic functions in Ω with no zeros or poles on Γ.

Let Z_f, Z_g, P_f, P_g be the number of zeros and poles of f, g inside Γ, respectively, counted according to multiplicity. If

$$|f(z) + g(z)| < |f(z)| + |g(z)|, \quad \forall z \in \Gamma,$$

then $Z_f - P_f = Z_g - P_g$.

Proof. Since we have

$$\left|\frac{f(z)}{g(z)}+1\right| < \left|\frac{f(z)}{g(z)}\right| + 1$$

on Γ, $\frac{f}{g} \notin [0,+\infty]$ on Γ.

Let $\log(z)$ be a branch of logarithm function on $\mathbb{C}\backslash\mathbb{R}_{\geq 0}$, then $\log\frac{f}{g}$ is analytic in a neighborhood of Γ.

We have

$$0 = \frac{1}{2\pi i}\int_\Gamma \left(\log\frac{f}{g}\right)' \mathrm{d}z = \frac{1}{2\pi i}\int_\Gamma \left(\frac{f'}{f} - \frac{g'}{g}\right) \mathrm{d}z = (Z_f - P_f) - (Z_g - P_g).$$

□

Corollary 4.3.8

If f, g are analytic on Ω and $|g| < |f|$ on Γ, then $Z_f = Z_{f+g}$.

Rouche's theorem can also be used to prove open mapping theorem, but we'll not show it here.

Theorem 4.3.9

Let f be a meromorphic function on an open set G with poles $p_1, \ldots, p_m$ and zeros $z_1, \ldots, z_n$ (counted with multiplicity). If g is analytic in G and γ is a closed rectifiable curve in G not containing any of the poles or zeros.

If $n(\gamma; z) = 0$ for all $z \notin G$, then

$$\frac{1}{2\pi i}\int_\gamma g\frac{f'}{f}\,\mathrm{d}z = \sum_{i=1}^m g(z_i)n(\gamma; z_i) - \sum_{i=1}^m g(p_i)n(\gamma; p_i).$$

Proof. Note that $Res(g\frac{f'}{f}, z_j) = m_j g(z_j)$, where m_j is the multiplicity of z_j. For poles we have similar equalities, thus the result follows from the residue theorem. □

§5 Analytic continuation

In this section we'll discuss whether we can extend an analytic function to some larger region.

Definition 5.0.1. Let $D \subset \mathbb{C}$ be a region, $f: D \to \mathbb{C}$ is analytic. If there is a region $\Omega \supsetneq D$ and $F: \Omega \to \mathbb{C}$ analyitc s.t. $f(z) = F(z)$ in D, then F is an **analytic continuation** of f, and we say f can be **(analytically) extended** to Ω.

Example 5.0.2

Let $f(z) = \sum_{n=0}^{\infty} z^n$, then the radius of convergence is 1, but we can extend f to $\mathbb{C} \setminus \{1\}$ by $\frac{1}{1-z}$.

§5.1 Natural boundary

Definition 5.1.1. Let $f : D \to \mathbb{C}$ be an analytic function. If f cannot be analytically extended to a larger region, then we call ∂D the **natural boundary** of f.

Theorem 5.1.2

Let $f : D \to \mathbb{C}$ be analytic, ∂D is the natural boundary of f iff $\forall z_0 \in D$, the Taylor series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z)}{n!}(z - z_0)^n$$

has a radius of convergence $R = d(z_0, \partial D)$.

Remark 5.1.3 — Warning! This theorem is false, we need to modify its statement. (Bonus question)

Proof. If ∂D is the natural boundary, fix a $z_0 \in D$.

Let $r_0 = d(z_0, \partial D)$, since f is analytic in $B(z_0, r_0)$, we have $R \geq r_0$. If $R > r_0$ we can extend f to a larger region, thus $R = r_0$.

Conversely, suppose ∂D is not the natural boundary, then $\exists \Omega \supsetneq D$ s.t. f can be extended to Ω.

Fix $z_1 \in \partial D$ and r_1 s.t. $B(z_1, r_1) \subset \Omega$. Hence we can take $z_0 \in B(z_1, r_1) \cap D$ and $r < r_1$ s.t. $z_1 \in B(z_0, r)$, this means f is analytic in $B(z_0, r)$, which implies the radius of convergence at $z = z_0$ is at least $r > |z_0 - z_1| \geq d(z_0, \partial D)$. □

Definition 5.1.4. Let D be a disk and $f : D \to \mathbb{C}$ analytic. For $w \in \partial D$, we say w is a **regular point** of f if $\exists r > 0$ and $g : B(w, r) \to \mathbb{C}$ s.t. $f(z) = g(z)$ on $B(w, r) \cap D$. Otherwise we say w is a **singular point** of f.

In the case of regular points, we say g is a **direct continuation** of f.

Theorem 5.1.5

If a power series $f(z) = \sum_n a_n z^n$ has radius of convergence R, then there exists a singular point $w \in \partial B(0, R)$ of f.

Proof. By the compactness of $\partial B(0, R)$, if there's no singular points, we can find finitely many direct continuation $g_{w_n} : B(w_n, r_n) \to \mathbb{C}$, such that $\bigcup B(w_n, r_n) \supseteq \partial B(0, R)$.

Now we simply define $F(z) = f(z)$ or $g_{w_j}(z)$, it suffices to check the compatibility, i.e. $g_{w_j}(z) = g_{w_k}(z)$ in $B(w_j, r_j) \cap B(w_k, r_k)$.

When $z \in B(0, R)$, they are clearly equal by definition, consequencely, they are equal on the entire $B(w_j, r_j) \cap B(w_k, r_k)$. Therefore $d(0, D \cup \bigcup_j B(w_j, r_j)) > R$. □

§5.2 Continuation along curves

Definition 5.2.1. A **function element** is a pair (f, G) where G is a region and f is an analytic function in G.

Two function pairs (f, D_0) and (g, D_1) are **direct continuation** of each other if $D_0 \cap D_1 \neq \emptyset$ and $f(z) = g(z)$ for all $z \in D_0 \cap D_1$. In this case, we write $(f, D_0) \sim (g, D_1)$.

Definition 5.2.2. A chain $\mathcal{C}$ is a finite sequence of disks, say $\mathcal{C} = (D_0, D_1, \dots, D_n)$ s.t. $D_{i-1} \cap D_i \neq \emptyset$ for all $i = 1, \dots, n$. If $(f_i, D_i), i = 0, 1, \dots, n$ are function elements satisfying $(f_{i-1}, D_{i-1}) \sim (f_i, D_i)$, then we say (f_n, D_n) is the **analytic continuation of (f_0, D_0) along $\mathcal{C}$**.

Remark 5.2.3 — This definition actually suppose that the continuation along $\mathcal{C}$ is unique, which is trivial by induction. This is to say that f_n is uniquely determined by f_0 and the chain $\mathcal{C}$.

Definition 5.2.4. We say a chain $\mathcal{C} = (D_0, \dots, D_n)$ **covers** a path $\gamma : [0, 1] \to \mathbb{C}$ if

- $\gamma(0)$ is the center of D_0, $\gamma(1)$ is the center of D_n.
- $\exists 0 = s_0 < s_1 < \cdots < s_n = 1$ s.t.

$$\gamma([s_i, s_{i+1}]) \subset D_i, \quad \forall i \in \{0, 1, \dots, n-1\}.$$

If (f_n, D_n) is the analytic continuation of (f_0, D_0) along $\mathcal{C}$ and $\mathcal{C}$ covers $\gamma : [0, 1] \to \mathbb{C}$, then we say that (f_n, D_n) is an **analytic continuation of (f_0, D_0) along γ**.

In fact the continuation is unique if we fix D_n, i.e. if (g, D_n') is another continuation and $D_n \subset D_n'$, then $f_n \equiv g$ in D_n.

Theorem 5.2.5 (Monodromy)

Assume that D is a simply connected region region, $\Delta = B(a, r) \subset D$ and $(f(z), \Delta)$ is a function element. If (f, D) has an analytic continuation along a curve γ for each curve γ in D staring from a, then there exists a unique analytic function $F : D \to \mathbb{C}$ s.t. $f(z) = F(z)$ for all $z \in \Delta$.

Proof. First we prove the case when $D = B(0, 1)$, $\Delta = B(0, r)$. Then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with radius of convergence $R \geq r$.

We argue by contradiction and asumme that $R < 1$, since there exists a singular point $z_0 \in \partial B(0, R)$ of f, (f, Δ) does not have an analytic continuation along 0 to z_0, contradiction!

Case 2. $D = \mathbb{C}$ and $\Delta = B(0, r)$, the proof is the same.

Case 3. D is a simply connected region. Then by the Riemann Mapping theorem (discuss soon), there exists a univalent function $g : D \to B(0, 1)$ s.t. $g(a) = 0$.

By the corollary to the open mapping theorem, g is biholomorphic, note that $g(B(a, r))$ is open, take $B(0, r_0) \subset g(B(a, r))$, by case 1 on $f \circ g^{-1}$, there exists analytic function F with $F \equiv f \circ g^{-1}$ on $B(0, r_0)$, hence $F \circ g$ is the desired function. $\square$

§5.3 Schwarz Reflection Principle

For a region $\Omega \subset \mathbb{C}$, denote

$$\Omega_+ := \Omega \cap \{z : \operatorname{Im} z > 0\}, \quad \Omega_- := \Omega \cap \{z : \operatorname{Im} z < 0\},$$

$$\Omega_0 := \Omega \cap \mathbb{R}, \quad \Omega^* := \{\overline{z} : z \in \Omega\}.$$

Theorem 5.3.1

Let Ω be a region satidfying $\Omega^* = \Omega$, consider $f : \Omega_+ \cup \Omega_0 \to \mathbb{C}$ which is continuous on $\Omega_+ \cup \Omega_0$ and analytic in Ω_+. If $f(\Omega_0) \subset \mathbb{R}$, then f can be extended to Ω.

More explicitly,

$$F(z) = \begin{cases} f(z), & z \in \Omega_+ \cup \Omega_0 \\ \overline{f(\overline{z})}, & z \in \Omega_-. \end{cases}$$

Proof. First we prove the analyticity in Ω_-, which is straight forward: for all $z_0 \in \Omega_-$, we have

$$\lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} = \lim_{z \to z_0} \overline{\left(\frac{f(\overline{z}) - f(\overline{z}_0)}{\overline{z} - \overline{z}_0} \right)} = \overline{f'(\overline{z}_0)}.$$

Next we prove the continuity on Ω_0, which is also trivial by direct discussion.

Finally, we use Morera's theorem to show F is analytic on Ω. Consider a triangular curve γ, if γ passes through the real axis, we can draw a small trapezoid s.t. its upper and lower edge lies on the line $\operatorname{Im} z = \pm\varepsilon$. Denote the upper and lower edge as $\widetilde{\gamma}_+$ and $\widetilde{\gamma}_-$, and the entire trapezoid as $\widetilde{\gamma}$.

By Cauchy's theorem we have

$$\left| \int_\gamma F(z) \,\mathrm{d}z \right| = \left| \int_{\widetilde{\gamma}} F(z) \,\mathrm{d}z \right| \le \left| \int_{\widetilde{\gamma}_+ + \widetilde{\gamma}_-} \operatorname{Re} F \,\mathrm{d}z \right| + \varepsilon + \left| \int_{\widetilde{\gamma}} \operatorname{Im} F \,\mathrm{d}z \right|$$

Since F maps real number to real numbers, $\operatorname{Im} F \to 0$ uniformly, and the integral on $\widetilde{\gamma}_+$ and $\widetilde{\gamma}_-$ canceal out each other, thus the integral must be 0. □

Note that the real axis is a circle on $\widehat{\mathbb{C}}$, we can similarly consider any other circle.

Let $D = B(a, r)$, Ω a region, denote

$$\Omega_D := \Omega \cap D, \quad \Omega_{D^c} := \Omega \cap D^c,$$

$$\Omega_{\partial D} := \Omega \cap \partial D, \quad \Omega_D^* := \{z^* : z \text{ and } z^* \text{ are symmetric w.r.t. } \partial D, z \in \Omega\}.$$

Theorem 5.3.2

Let $D = B(a, r)$, $D' = B(b, R)$, Ω a region satisfying $\Omega = \Omega_D^*$. Let $f : \Omega_D \cup \Omega_{\partial D} \to \mathbb{C}$ be a continuous function and analytic in Ω_D.

Assume $f(\Omega_{\partial D}) \subset \partial D'$. If $b \notin f(\Omega_D)$, then f can be analytically extended to Ω. Otherwise f can be meromorphically extended to Ω.

Same arguments holds for $g : \Omega_{D^c} \cup \Omega_{\partial D} \to \mathbb{C}$.

Proof. Let $\psi, \varphi \in Mob$ s.t. $\varphi(\mathbb{R}) \subset \partial D$, and $\psi(\partial D') = \mathbb{R} \cup \{\infty\}$.

Define $\widetilde{f} := \psi \circ f \circ \varphi$. Note that $\widetilde{f}$ satisfies the condition of the previous theorem, the conclusion follows immediately. □

§5.4 Riemann mapping theorem

For simply connected region $\Omega \subset \mathbb{C}$, Riemann mapping theorem states that there is only two possiblities:

- $\Omega = \mathbb{C}$,

- Ω is biholomorphic to $B(0,1)$.

If Ω is in $\widehat{\mathbb{C}}$, then there is another possibility that $\Omega = \widehat{\mathbb{C}}$.

Recall that biholomorphic functions have the following property we once proved:

Theorem 5.4.1

Let $f : \Omega \to \mathbb{C}$ be a univalent function, then $\forall z \in \Omega$, $f'(z) \neq 0$. Conversely, if $f'(z_0) \neq 0$ for some $z_0 \in \Omega$, then $\exists \rho > 0$ s.t. f is univalent on $B(z_0, \rho)$.

Theorem 5.4.2

If $\Omega \subset \mathbb{C}$ is a simply connected region, $f : \Omega \subset \mathbb{C}$ is univalent, then $f(\Omega)$ is simply connected.

Proof. This proof needs Jordan curve theorem (more detailed version). It states that for any Jordan curve γ, $\mathbb{C} \setminus \gamma$ has two connected component, called the inside and outside of γ, the inside is bounded.

Let γ be a Jordan curve in $G := f(\Omega)$, let Γ be a piecewise smooth Jordan curve in G satisfying $ins\gamma \subset ins\Gamma$, ins means the inside of a curve. It suffices to show that $ins\Gamma \subset G$.

(We can construct Γ by using finite many disks to cover $ins\gamma$ and take the outer boundary of these disks)

Let $\Gamma' := f^{-1}(\Gamma) \subset \Omega$, we have $ins\Gamma' \subset \Omega$, hence we only need to show $f(ins\Gamma') \supseteq ins\Gamma$. By the Argument principle, for each $w_0 \in ins\Gamma$, the number N of zeros of $f(z) - w_0$ in $ins\Gamma'$ is

$$N = \frac{1}{2\pi i} \int_{\Gamma'} \frac{f'(z)}{f(z) - w_0} \, \mathrm{d}z = \frac{1}{2\pi i} \int_{f(\Gamma')} \frac{\mathrm{d}w}{w - w_0} = n(\Gamma, w_0) = 1.$$

The last equality is by Jordan curve theorem. Hence we're done. $\square$

To prove Riemann mapping theorem, we need some topological tools and Arzela-Ascoli theorem.

Lemma 5.4.3

If $G \subset \mathbb{C}$ is open, we can find compact sets $\{K_i\}_{i\in\mathbb{N}}$ s.t.

- $K_n \subset \mathring{K}_{n+1}$,
- $\bigcup_{i=1}^{\infty} K_n = G$,
- $K \subset G$ compact $\implies K \in K_n$ for some n.
- Every connected component of $\widehat{\mathbb{C}} \setminus K_n$ contains a connected component of $\widehat{\mathbb{C}} \setminus G$.

Proof. Let $K_n := \{z \in \mathbb{C}, |z| \leq n\} \cap \{z \in \mathbb{C} : d(z, \mathbb{C} \setminus G) \geq \frac{1}{n}\}$ for all sufficiently large n, they satisfy the desired conditions. $\square$

Define for $f, g \in C(G, X)$, where G open, (X, d) is a metric space,

$$\rho_n(f, g) := \sup\{d(f(z), g(z)) : z \in K_n\}, \quad n \in \mathbb{N}.$$

and

$$\rho(f,g)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\rho_n(f,g)}{\rho_n(f,g)+1}.$$

Note that ρ is a metric in $C(G,X)$.

Proposition 5.4.4

$(C(G,X),\rho)$ is a complete metric space, and its topology is independent of the choice of $\{K_n\}_{n\in\mathbb{N}}$.

Proposition 5.4.5

A sequence $\{f_n\}$ in $(C(G,X),\rho)$ converges to $f\in C(G,X)$ iff $\{f_n\}$ **converges uniformly on compact subsets of G**.

Definition 5.4.6. A set $\mathcal{F}\subset C(G,X)$ is **normal** if for each sequence $\{f_n\}$ in $\mathcal{F}$, there exists a subsequence $\{f_{n_k}\}$ that converges uniformly on compact subsets of G.

Proposition 5.4.7

A set $\mathcal{F}\subset C(G,X)$ is normal iff $\mathcal{F}$ is **precompact**, i.e. $\overline{\mathcal{F}}$ is compact in $C(G,X)$.

Definition 5.4.8. A set $\mathcal{F}\subset C(G,X)$ is **equicontinuous** at a point $z_0\in G$ if

$$\forall\varepsilon>0,\exists\delta>0,\forall z\in B(z_0,\delta),f\in\mathcal{F},\quad d(f(z),f(z_0))<\varepsilon.$$

$\mathcal{F}$ is **equicontinuous over a set $E\subset G$** if $\forall\varepsilon>0$, $\forall z,z'\in E$, $\forall f\in\mathcal{F}$,

$$|z-z'|<\delta\implies|f(z)-f(z')|<\varepsilon.$$

Proposition 5.4.9

If $\mathcal{F}\subset C(G,X)$ is equicontinuous at each point of G, then $\mathcal{F}$ is equicontinuous over each compact subset of G.

Proof. Lebesgue's covering lemma. □

Theorem 5.4.10 (Arzela-Ascoli theorem)

A set $\mathcal{F}\subset C(G,X)$ is normal iff the followings hold:

- $\forall z\in G$, $\{f(z):f\in\mathcal{F}\}$ has compact closure in X.
- $\mathcal{F}$ is equicontinuous at each point of G.

Proof. The proof is left to the readers. You can refer to books on real analysis. □

For an open set $G \subset \mathbb{C}$, denote

$$H(G) := \{f \in C(G, \mathbb{C}) : f \text{ is analytic.}\}, A(G) := \{f \in C(\overline{G}, \mathbb{C}) : f|_G \in H(G)\}.$$

Definition 5.4.11. A set $\mathcal{F} \in H(G)$ is **locally bounded** if $\forall a \in G$, $\exists M > 0$, $\exists r > 0$,

$$|f(z)| < M, \quad \forall f \in \mathcal{F}, z \in G \cap B(a, r).$$

Lemma 5.4.12

A set $\mathcal{F} \in H(G)$ is locally bounded iff $\mathcal{F}$ is uniformly bounded on each compact subset of G.

Proof. This is easy. □

Theorem 5.4.13 (Montel)

A set $\mathcal{F} \in H(G)$ is normal iff $\mathcal{F}$ is locally bounded.

Proof. By the definition of locally bounded, the first condition of AA theorem is satisfied.

Fix $a \in G$, $\varepsilon > 0$. There exists $r > 0$ and M s.t. $\overline{B}(a, r) \in G$ and the functions in $\mathcal{F}$ are bounded by M in this ball.

Thus for $z \in B(a, \frac{1}{2}r)$,

$$\begin{aligned} |f(a) - f(z)| &= \frac{1}{2\pi} \left| \int_{|w-a|=\frac{1}{2}r} \frac{f(w)}{w-a} \,\mathrm{d}w - \int_{|w-a|=\frac{1}{2}r} \frac{f(w)}{w-z} \,\mathrm{d}w \right|. \\ &= \frac{1}{2\pi} \left| \int_{|w-a|=\frac{1}{2}r} \frac{f(w)(a-z)}{(w-a)(w-z)} \right| \\ &\leq \frac{1}{2\pi} 2\pi \frac{r}{2} \frac{M|z-a|}{r \cdot \frac{r}{2}} \\ &= \frac{2M}{r} |z-a|. \end{aligned}$$

Hence for $\delta = \min\{\frac{1}{2}r, \frac{r}{2M}\varepsilon\}$ we have $|z - a| < \delta \implies |f(a) - f(z)| < \varepsilon$, which means $\mathcal{F}$ is equicontinuous at every point. By AA theorem we know that $\mathcal{F}$ is normal.

Conversely, suppose $\mathcal{F}$ is normal but not locally bounded, then there exists $K \subset G$ compact s.t. $\exists \{f_n\}_{n \in \mathbb{N}} \in \mathcal{F}$ s.t.

$$\sup\{f_n(z) : z \in K\} \geq n.$$

But by the normal property, $\{f_n\}$ has a convergent subsequence that uniformly converges to some continuous function f in K, this leads to a contradiction since f must be bounded in K. □

Theorem 5.4.14

If $\{f_n\} \in H(G)$ uniformly converges on compact subsets of G to $f \in C(G, \mathbb{C})$, then $f \in H(G)$ and $\{f_n^{(k)}\}$ converges to $f^{(k)}$ in $C(G, \mathbb{C})$ for all $k \in \mathbb{N}$.

Proof. By Morera's theorem, we have

$$\int_\Delta f(z)\,\mathrm{d}z = \lim_{n\to\infty}\int_\Delta f_n(z)\,\mathrm{d}z = 0$$

for every triangular path Δ, hence $f \in H(G)$.

It suffices to show that $f_n^{(k)}$ converges to $f^{(k)}$ uniformly on each closed ball $\overline{B}(a,r) \subset G$. Take $R > r$ s.t. $\overline{B}(a,R) \subset G$, for $|z-a| \le r$ we have

$$\left|f_n^{(k)}(z) - f^{(k)}(z)\right| \le \frac{k!}{2\pi}\left|\int_{\partial B(a,R)} \frac{f_n(w)-f(w)}{(w-z)^{n+1}}\,\mathrm{d}w\right| \le C \sup_{w\in\partial B(a,R)} |f_n(w)-f(w)| \to 0.$$

□

Theorem 5.4.15 (Hurwitz)

Given a region $G \subset \mathbb{C}$ and a sequence $\{f_n\} \subset H(G)$ converging to f in $C(G,\mathbb{C})$.

For $\overline{B}(a,R) \in G$, if f is not a constant and $f(z) \ne 0$ for $|z-a| = R$, then $\exists N \in \mathbb{N}$, $\forall n \ge N$, f and f_n have the same number of zeros in $B(a,R)$.

Proof. We use Rouché's theorem.

Let $\delta := \inf\{|f(z)| : z \in \partial\overline{B}(a,R)\} > 0$, there exists $N \in \mathbb{N}$, $\forall n \ge N$ we have

$$|f(z) - f_n(z)| < \frac{\delta}{2} < |f(z)|, \forall z \in \overline{B}(a,R).$$

Hence we can use Rouché's theorem directly on f and $f_n - f$. □

> **Question 5.4.16.** Use above Hurwitz's theorem to prove the version of Hurwitz's theorem in [Wu-Tan, Thm 8.14].

Corollary 5.4.17

Given a region $G \subset \mathbb{C}$ and a sequence $\{f_n\} \subset H(G)$ convergent to $f \in C(G,\mathbb{C})$. If each f_n never vanishes on G, then $f \equiv 0$ or f never vanished.

Now we are ready to prove Riemann mapping theorem.

Theorem 5.4.18 (Riemann Mapping Theorem)

Let $G \subsetneq \mathbb{C}$ be a simply connected region. Let $a \in G$, there exists a unique conformal homeomorphism $f : G \to B(0,1)$ s.t. $f(a) = 0$ and $f'(a) > 0$.

Proof. First we prove the uniqueness.

Assume that f, g are two such conformal homeomorphisms, then $h := f \circ g^{-1} : B(0,1) \to B(0,1)$ satisfies

$$h(0) = 0, \quad h'(0)(h^{-1})'(0) = 1.$$

By Schwarz's lemma, $|h'(0)| \leq 1$ and $|(h^{-1})'(0)| \leq 1$, thus the equality holds, and we have

$$h(z) = cz, \quad c \in \mathbb{C}, |c| = 1.$$

Then $f(z) = cg(z)$ and $0 < f'(a) = cg'(a)$ implies $c = 1$.

For the existence part, define

$$\mathcal{F} := \{f \in H(G) : f : G \hookrightarrow B(0,1), f(a) = 0, f'(a) > 0\}.$$

Clearly $\mathcal{F}$ is (locally) bounded, hence by Montel's theorem, $\mathcal{F}$ is normal.

> **Claim 5.4.19.** $\mathcal{F} \neq \emptyset$, and $\overline{\mathcal{F}} = \mathcal{F} \cup \{0\}$ in $C(G, \mathbb{C})$.

Assuming the claim, note that $D : H(G) \to \mathbb{C}$ by $f \mapsto f'(a)$ is continuous in the metric space $C(G, \mathbb{C})$ (check it). Hence $\exists f \in \overline{\mathcal{F}}$ maximizing $|D(f)|$ as $\overline{\mathcal{F}}$ is compact by normality.

By above claim this f must lie in $\mathcal{F}$. It suffices to show that $f(G) = B(0,1)$.

Suppose not, let $w \in B(0,1) \setminus f(G)$, then

$$h_2(z) := \frac{f(z) - w}{1 - \overline{w} f(z)} \in H(G),$$

and it never vanishes.

Since G is simply connected, there exists $h \in H(G)$ s.t.

$$(h(z))^2 = h_2(z), \quad \forall z \in G.$$

Recall that $\varphi_w(z) := \frac{z-w}{1-z\overline{w}}$ is a conformal homeomorphism in $B(0,1)$, define $g \in H(G)$ by

$$g(z) := \frac{|h'(a)|}{h'(a)} \frac{h(z) - h(a)}{1 - \overline{h(a)} h(z)},$$

then $g(G) \subset B(0,1)$, $g(a) = 0$, and g is univalent (check it). We have $g'(a) = \frac{|h'(a)|}{1-|h(a)|^2} > 0$, thus $g \in \mathcal{F}$. But

$$2h(a)h'(a) = h_2'(a) = f'(a)(1 - |w|^2),$$

and $|h(a)|^2 = |h_2(a)| = |w|$, we have

$$|g'(a)| = \frac{|f'(a)|(1 - |w|^2)}{2\sqrt{|w|}(1 - |w|)} = |f'(a)| \frac{1 + |w|}{2\sqrt{|w|}} > |f'(a)|,$$

which contradicts with the maximality of $|f'(a)|$. □

Now we only need to prove Claim 5.4.19.

Proof of Claim 5.4.19. Take $b \in \mathbb{C} \setminus G$, we can find $g \in H(G)$ s.t. $g^2(z) = z - b$.

The basic idea is to use this g to map G into a half plane, and using a well-known map to turn this half plane to the unit circle. (Actually g doesn't map G to a half plane, but the idea is similar)

Clearly g is univalent since $g(z_1) = \pm g(z_2) \implies z_1 - b = z_2 - b$. Since g is an open map, we can take r s.t. $B(g(a), r) \subset g(G)$.

If $g(a) \neq 0$, take $r < |g(a)|$, now $B(-g(a), r) \cap g(G) = \emptyset$ by $g(z_1) = \pm g(z_2) \implies z_1 = z_2$. Otherwise $g(a) = 0 \implies 0 = (g(a))^2 = a - b$, contradiction!

This means $g(G)$ avoids a disk in $\mathbb{C}$, hence we can use the map $\frac{r}{z+g(a)}$ to map $g(G)$ into $B(0,1)$. Next we'll use Möbius transformation to make g satisfy the conditions on $g(a)$ and $g'(a)$, this proves that $\mathcal{F} \neq \emptyset$.

For the latter part, let $f_n \in \mathcal{F}$ and $f_n \to f$ in $C(G, \mathbb{C})$. We've proven that $f \in H(G)$, so $f(a) = 0$, $f'(a) \geq 0$. Since $f(G) \subset \overline{B}(0,1)$ is open, we must have $f(G) \subset B(0,1)$.

Fix $z_0, z_1 \in G$ and $\overline{B}(z_0, r) \subset G$, $z_1 \notin \overline{B}(z_0, r)$.

$$0 \neq f_n(z) - f_n(z_1) \rightrightarrows f(z) - f(z_1), \quad \forall z \in \overline{B}(z_0, r),$$

by corollary of Hurwitz's theorem, $f \equiv f(z_1)$ or $f - f(z_1)$ never vanishes on $\overline{B}(z_0, r)$.

The former one implies $f \equiv 0$ (since $f(a) = 0$), and the latter one implies that f is univalent, thus $f'(a) \neq 0$, $f \in \mathcal{F}$. □

Until now, we've fully proven Riemann Mapping Theorem. There are many results based on it, and even new branches of mathematics started from variations of Riemann Mapping theorem.

We can also discuss the boundary of the map:

Theorem 5.4.20

In the settings of Riemann mapping theorem, if ∂G is a Jordan curve, then f extends to a map
$$g \in C(\overline{G}, \overline{B}(0,1)).$$

Further reading on "prime ends" and the Carathéodory theorem from [Garnett-Marshall, *Harmonic measures*].

§5.5 Harmonic functions and Dirichlet problem

Recall that given $G \subset \mathbb{C}$ open, a harmonic function $u : G \to \mathbb{R}$ is a C^2 function that satisfies Laplace's equation
$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Definition 5.5.1. Let
$$P_r(\theta) := \sum_{n=-\infty}^{+\infty} r^{|n|} e^{in\theta}$$
for $r \in [0,1)$, $\theta \in \mathbb{R}$, this is called the **Poisson kernel**.

Lemma 5.5.2

For $r \in [0,1)$, $\theta \in \mathbb{R}$, we have
$$P_r(\theta) = \operatorname{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) = \frac{1-r^2}{1-2r\cos\theta + r^2}.$$

Proof. Let $z := re^{i\theta}$, then
$$\operatorname{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) = \operatorname{Re}((1+z)(1+z+z^2+\cdots)) = 1 + 2\operatorname{Re}\left(\sum_{n=1}^{\infty} z^n\right).$$

This is because the series $1 + z + z^2 + \cdots$ is absolutely convergent, we can expand the product directly. This proves the first equality.

The second equality is just direct computation. □

Proposition 5.5.3

Some properties of Poisson kernel:

- $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\,\mathrm{d}\theta = 1$.
- $P_r(\theta) > 0$ for all $\theta \in \mathbb{R}$.
- $P_r(\theta) < P_r(\delta)$ for $0 < \delta < |\theta| < \pi$.
- $\forall \delta \in (0, \pi)$,
$$\lim_{r\to 1-} P_r(\theta) = 0,$$
this convergence is uniform when $|\theta| \in [\delta, \pi]$.

Dirichlet problem asks that which regions G in $\mathbb{C}$ satisfy for each $f \in C(\partial G, \mathbb{C})$, there exists $u \in C(G, \mathbb{C})$ s.t. $u|_{\partial G} = f$ and u is harmonic in G? Such regions are called **Dirichlet regions**.

Remark 5.5.4 — This problem is from Johann Peter Gustas Lejeune Dirichlet (1805-1859) when studying the stability of the solar sysmtem.

Theorem 5.5.5

$D := B(0, 1)$ is a Dirichlet region. Moreover u is unique and is given by

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it})\,\mathrm{d}t$$

for $r \in [0, 1)$, $\theta \in [0, 2\pi]$.

Proof. To prove u is harmonic in D, let $z = re^{i\theta}$,

$$u(z) = \frac{1}{2\pi}\operatorname{Re}\left(\int_{-\pi}^{\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} f(e^{it})\,\mathrm{d}t\right).$$

Denote this integral by I.

Recall that for a function $\varphi(w, z) : \gamma \times G \to \mathbb{C}$ differentiable to z, we have the integral

$$\int_\gamma \varphi(w, z)\,\mathrm{d}w$$

is analytic, and the derivative is $\int_\gamma \frac{\partial\varphi}{\partial z}(w, z)\,\mathrm{d}w$.

Using this proposition, we know that I is analytic, hence u is harmonic.

For the continuity, let z be a point in the neighborhood of $e^{i\alpha}$, WLOG $\alpha = 0$, we have $r \to 1-$, thus

$$|u(z) - f(1)| \le \frac{1}{2\pi}\left|\int_{|t|\le\delta} P_r(\theta - t)(f(e^{it}) - f(1))\,\mathrm{d}t\right| + \varepsilon.$$

Here we used the last property (uniform convergence) in the above proposition. Now since $|f(e^{it})-f(1)|<\varepsilon$ and $|P_r(\theta-t)|<M_r$,

For the uniqueness, if u_1, u_2 are two different solution, we have $u_1-u_2=0$ on the boundary, by mean value property of harmonic functions, we must have $u_1-u_2\equiv 0$. □

Theorem 5.5.6 (Mean value theorem)
Let $u:G\to\mathbb{R}$ be a harmonic function, $\overline{B}(a,r)\subset G$, then

$$u(a)=\frac{1}{2\pi}\int_0^{2\pi}u(a+re^{i\theta})\,\mathrm{d}\theta.$$

Proof. There exists f analytic s.t. $u=\operatorname{Re}(f)$ on $\overline{B}(a,r)$, by Cauchy's integral formula this is trivial. □

Theorem 5.5.7 (Maximum Principle)
Let $u:G\to\mathbb{R}$ (G is connected and open) be a continuous function that satisfies the conclusion of Mean value theorem. If $a\in G$ s.t. $\forall z\in G$, $u(a)\geq u(z)$, then u is a constant function.

Similarly there's a minimum principle, and the proof is quite the same as the complex version (just some topology arguments).

Remark 5.5.8 — In fact for a continuous $u:G\to\mathbb{R}$, it is harmonic iff u satisfies mean value principle.

Definition 5.5.9. Let $\varphi:G\to\mathbb{R}$ continuous, we say φ is a **subharmonic function** if for all $\overline{B}(a,r)\subset G$,

$$\varphi(a)\leq\frac{1}{2\pi}\int_0^{2\pi}\varphi(a,re^{i\theta})\,\mathrm{d}\theta.$$

Similarly we define **superharmonic functions**.

Theorem 5.5.10 (Harnack's inequality)
If $u:\overline{B}(0,R)\to[0,+\infty)$ is continuous and harmonic in $B(a,R)$, then $\forall r\in[0,R)$,

$$\frac{R-r}{R+r}u(a)\leq u(a+re^{i\theta})\leq\frac{R+r}{R-r}u(a).$$

Proof. Since we have (by the solution on Dirichlet problem)

$$u(a+re^{i\theta})=\frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2rR\cos\theta+r^2}u(a+Re^{it})\,\mathrm{d}t,$$

the result follows immediately. □

§6 Bonus section

§6.1 Riemann zeta function

Recall that Riemann zeta function is defined as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 - p^s)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Hence zeta function is related to the prime distributions.

Let $\pi(T) := \#\{p \text{ prime}, p \leq T\}$, we have

$$\pi(T) \sim Li(T) := \int_2^T \frac{1}{\log u} \, \mathrm{d}u \sim \frac{T}{\log T}.$$

Here the notation $f(T) \sim g(T)$ means $\lim_{T\to\infty} \frac{f(T)}{g(T)} = 1$.

This result is the famous Prime Number Theorem by Ch. J. de la Valléc-Pousin & J. Hadamard.

H. van Koch in 1901 proved that

$$\pi(T) = Li(T) + O(\sqrt{T} \log T) \iff \text{Riemann hypothesis.}$$

§6.2 Dynamic systems

Let $g : X \to X$ be a map on a topological space X. We use g^n to denote the n-th iteration of g. Let $\psi : X \to \mathbb{C}$, we write

$$Z^{(n)}_{g,-\psi}(s) := \sum_{x \in P_{1,g^n}} e^{-sS_n\psi(x)}, \quad n \in \mathbb{N}, s \in \mathbb{C},$$

where

$$S_n\psi(x) := \sum_{i=0}^{n-1} \psi(g^i(x)), \quad P_{i,g} := \{x \in X : g^i(x) = x, g^k(x) \neq x, k \in \{1, 2, \ldots, i-1\}\}.$$

Definition 6.2.1. Define the (Rvelle) zeta function

$$\zeta_{g,-\psi}(s) := \exp\left(\sum_{n=1}^{\infty} \frac{Z^{(n)}_{g,-\psi}(s)}{n}\right), \quad s \in \mathbb{C}.$$

This is a formal series since we don't care about its convergence.

Definition 6.2.2. Let $w : X \to \mathbb{C}$ be a function, define

$$\mathcal{D}_{g,-\psi,w}(s) := \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{x \in P_{1,g^n}} e^{-sS_n\psi(x)} \prod_{i=0}^{n-1} w(g^i(x))\right).$$

This is called **Dynamic Dirichlet series** with coefficient W.

Lemma 6.2.3

For above g, ϕ, w, fix $a \in \mathbb{R}$, suppose that

- $|P_{1,g^n}| < +\infty$, $\forall n \in \mathbb{N}$.
- $$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{x\in P_{1,g^n}} e^{-aS_n\phi(x)} \prod_{i=0}^{n-1} |w(g^i(x))| < 0.$$

Then $\mathcal{D}_{g,-\phi,w}(s)$ as an infinite product converges uniformly and absolutely on $\{s \in \mathbb{C} : \operatorname{Re}(s) = a\}$, and with $l_\phi(\tau) := \sum_{x\in\tau} \phi(x)$, we have the Euler product formula

$$\mathcal{D}_{g,-\phi,w}(s) = \prod_{\tau\in\mathcal{P}(g)} \left(1 - e^{-sl_\phi(\tau)} \prod_{x\in\tau} w(x)\right)^{-1}.$$

Here $\mathcal{P}(g)$ is the primitive periodic orbits of g, i.e.

Note that in this case $\mathcal{D}$ is similar to Riemann zeta function

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}.$$

Don't want to take notes anymore :(