Advanced Classical Mechanics: Relativistic Fields

PROBLEM SHEET 2

LAGRANGIAN FIELD THEORY

1. [LORENTZ LAW]

a) Electrostatic lagrangian

$$L = \frac{1}{2} m |\underset{\sim}{v}|^2 - q V(\underset{\sim}{r})$$

$$= \frac{1}{2} m \left((v^1)^2 + (v^2)^2 + (v^3)^2 \right)$$

$$= \frac{1}{2} m \, \delta_{ij} v^i v^j$$

For each $j = 1, 2, 3$, we have the EL equation

$$\frac{\partial L}{\partial r^j} - \frac{d}{dt} \frac{\partial L}{\partial v^j} = 0 \qquad v_j = \dot{r}_j$$

$$\frac{\partial L}{\partial r^j} = -q \frac{\partial V}{\partial r^j}, \qquad \frac{\partial L}{\partial v^j} = m v^j \quad \left(\text{from middle equality}\right)$$

$$\frac{d}{dt} \frac{\partial L}{\partial v^j} = m \dot{v}^j$$

$$= m \ddot{r}^j$$

Thus $\quad m \ddot{r}^j = -q \dfrac{\partial V}{\partial r^j} \quad$ for $j = 1, 2, 3$

This agrees with the form discussed in the lectures.

b) As discussed in the lectures,

$$\frac{\partial(\underset{\sim}{v}\cdot\underset{\sim}{A})}{\partial\underset{\sim}{r}} - \frac{d}{dt}\frac{\partial(\underset{\sim}{v}\cdot\underset{\sim}{A})}{\partial\underset{\sim}{v}} = -\dot{\underset{\sim}{A}} + \underset{\sim}{v}\times\underset{\sim}{B}$$

If $q\,\underset{\sim}{v}\cdot\underset{\sim}{A}$ is added to the lagrangian the RHS of this equation times q should be added to the LHS of the previous EL eq,

$$-q\underbrace{\frac{\partial V}{\partial\underset{\sim}{r}}}_{\nabla V} - m\ddot{\underset{\sim}{r}} = 0$$

ie

$$-q\nabla V - m\ddot{\underset{\sim}{r}} + q(-\dot{\underset{\sim}{A}} + \underset{\sim}{v}\times\underset{\sim}{B}) = 0$$

newtonian force

ie $$\underset{\sim}{F} = q(\underbrace{-\nabla V - \dot{\underset{\sim}{A}}}_{\underset{\sim}{E}} + \underset{\sim}{v}\times\underset{\sim}{B})$$

$$= q(\underset{\sim}{E} + \underset{\sim}{v}\times\underset{\sim}{B}),$$ as req^d.

c) $L = -\dfrac{mc^2}{\gamma} - \dfrac{qc}{\gamma} U^{\mu} A_{\mu}$

$$= -\frac{mc^2}{\gamma} - \frac{qc}{\gamma}\left(\frac{\gamma V}{c} - \gamma \underset{\sim}{v} \cdot \underset{\sim}{A}\right)$$

$$= -mc^2\sqrt{1-\frac{v^2}{c^2}} - qV + q\underset{\sim}{v} \cdot \underset{\sim}{A}$$

Since the frame is fixed, we use cartesian, rather than 4-vector notation

Thus $\dfrac{\partial}{\partial \underset{\sim}{v}} \sqrt{1-\dfrac{v^2}{c^2}} = -\dfrac{\underset{\sim}{v}}{c^2}\gamma$

So $\dfrac{\partial L}{\partial \underset{\sim}{v}} = m\gamma\underset{\sim}{v} + q\underset{\sim}{A}$

defines the CANONICAL RELATIVISTIC 3-MOMENTUM (cf $m\gamma\underset{\sim}{v}$, the usual 'kinetic' relativistic 3-momentum)

The hamiltonian H is defined

$$H = \sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L$$

$$\overset{\text{here}}{=} \frac{\partial L}{\partial \underset{\sim}{v}} \cdot \underset{\sim}{v} - L$$

$$= m\gamma v^2 + \cancel{q\underset{\sim}{v} \cdot \underset{\sim}{A}} + \frac{mc^2}{\gamma} + qV - \cancel{q\underset{\sim}{v} \cdot \underset{\sim}{A}}$$

Since $m\gamma v^2 + \frac{mc^2}{\gamma} = m\gamma c^2 \underbrace{\left(\frac{v^2}{c^2} + \frac{1}{\gamma^2}\right)}_{1}$

$$= m\gamma c^2,$$

$H = \underbrace{m\gamma c^2}$ $+ qV$ ← potential energy of particle

↑ usual relativistic 'kinetic' energy of particle (including rest mass)

If V, $\underset{\sim}{A}$ do not have any explicit dependence on t, then $\frac{\partial L}{\partial t} = 0$, and so the hamiltonian H is a constant of the motion, ie $m\gamma c^2 + qV$ is constant in time.

d) EL equation is, in the fixed frame (cartesian)

$$\frac{\partial L}{\partial \underset{\sim}{r}} - \frac{d}{dt}\frac{\partial L}{\partial \underset{\sim}{v}} = 0$$

Since $\frac{\partial (mc^2/\gamma)}{\partial \underset{\sim}{r}} = 0$,

$$\frac{\partial L}{\partial \underset{\sim}{r}} = \frac{\partial (q\underset{\sim}{v}\cdot\underset{\sim}{A})}{\partial \underset{\sim}{r}} - q\nabla V$$

$$\frac{d}{dt}\frac{\partial L}{\partial \underset{\sim}{v}} = \frac{d}{dt}\left(m\gamma \underset{\sim}{v}\right) + \frac{d}{dt}\left(\frac{\partial q\, \underset{\sim}{v}\cdot \underset{\sim}{A}}{\partial \underset{\sim}{v}}\right)$$

Thus, EL eq becomes

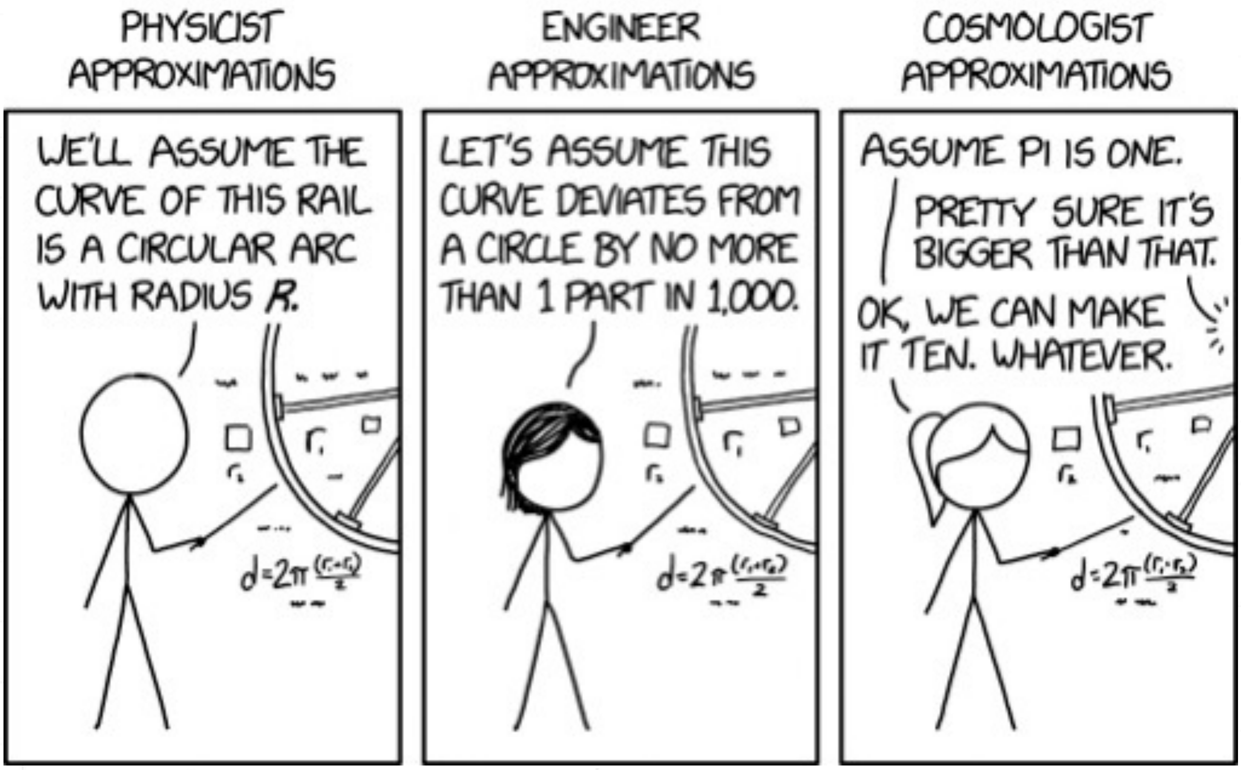
$$-\frac{d}{dt}\left(m\gamma \underset{\sim}{v}\right) - q\nabla V + q\underbrace{\left[\frac{\partial(\underset{\sim}{v}\cdot \underset{\sim}{A})}{\partial \underset{\sim}{r}} - \frac{d}{dt}\frac{\partial(\underset{\sim}{v}\cdot \underset{\sim}{A})}{\partial \underset{\sim}{v}}\right]}_{-\dot{\underset{\sim}{A}} + \underset{\sim}{v}\times \underset{\sim}{B} \text{ from above}} = 0$$

ie

$$\frac{d}{dt}\left(m\gamma \underset{\sim}{v}\right) = q\left(\underset{\sim}{E} + \underset{\sim}{v}\times \underset{\sim}{B}\right)$$

as req^d

In the low-velocity limit, $\gamma \to 1$, so the expression on the LHS tends to the usual expression for the newtonian force.



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2. [LAPLACE'S EQUATION]

Electrostatic energy density

$$\mathcal{E} = \frac{1}{2} \varepsilon_0 |\underset{\sim}{E}|^2 = \frac{1}{2} \varepsilon_0 (\nabla V)^2$$

for V electrostatic potential.

Total energy is, over all space

$$\int_{\substack{\text{all}\\\text{space}}} d^3\underset{\sim}{r}\ \mathcal{E} = \frac{\varepsilon_0}{2}\int_{\substack{\text{all}\\\text{space}}} d^3\underset{\sim}{r}\ (\nabla V)^2$$

Assume that V makes the value of the total energy stationary, with respect to variations $V + \delta V$ of order δ.

Thus, total energy of varied potential $= \dfrac{\varepsilon_0}{2}\displaystyle\int_{\substack{\text{all}\\\text{space}}} d^3\underset{\sim}{r}\ (\nabla V + \nabla\delta V)^2$

$$= \frac{\varepsilon_0}{2}\int d^3\underset{\sim}{r}\ (\nabla V)^2 + \varepsilon_0\int d^3\underset{\sim}{r}\ \nabla V\cdot\nabla\delta V + O(\delta^2)$$

where integrals are over all space.

If the total energy is stationary, this implies

$$\int_{\text{all space}} d^3\underset{\sim}{r}\ \nabla V\cdot\nabla\delta V = 0$$

This integral over all space is realized as the limit of some large volume $\mathcal{V}\to\infty$. When $\mathcal{V}$ has finite volume, it has boundary $\partial\mathcal{V}$.

Since $\nabla(\delta V \nabla V) = \nabla(\delta V)\cdot\nabla V + \delta V \nabla^2 V$

Thus, for finite volume $\mathcal{V}$,

$$\int_{\mathcal{V}} d^3\underset{\sim}{r} \; \nabla V \cdot \nabla \delta V = \int_{\mathcal{V}} d^3\underset{\sim}{r} \; \nabla(\delta V \nabla V)$$

$$+ \int_{\mathcal{V}} d^3\underset{\sim}{r} \; \delta V \nabla^2 V$$

In the limit $\mathcal{V} \to \infty$, whe have the divergence theorem in 3D

$$\int_{\mathcal{V}} d^3\underset{\sim}{r} \; \nabla(\delta V \nabla V) = \int_{\partial\mathcal{V}} d^2 S \; \underset{\sim}{\hat{n}} \cdot \nabla V \, \delta V$$

$$= \int_{\partial\mathcal{V}} d^2 S \; \underset{\sim}{n} \cdot \underset{\sim}{E} \, \delta V$$

$$\longrightarrow 0 \quad \text{since } \underset{\sim}{E} \to 0 \text{ at } \infty.$$

Thus, in the limit,

$$0 = \int_{\substack{\text{all}\\ \text{space}}} d^3\underset{\sim}{r} \; \nabla V \cdot \nabla \delta V = \int_{\substack{\text{all}\\ \text{space}}} d^3\underset{\sim}{r} \; \delta V \nabla^2 V$$

If the variation vanishes for all variations δV, then

$$\nabla^2 V = 0 \quad \text{over all space,}$$

ie V satisfies Laplace's equation

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3. [D'ALEMBERT EQUATION IS COVARIANT]

Lagrange Density

$$\mathcal{L} = c^{-2}\dot{\varphi}^2 - |\nabla\varphi|^2 = \partial_\mu \varphi\, \partial^\mu \varphi$$

If $\mathcal{L}$ is a Lorentz scalar, it should take the same form in all inertial frames

Since ∂_μ is a covariant 4-vector operator, changing frames by a Lorentz transformation,

$$\mathcal{L} = \partial_\mu \varphi\, \partial^\mu \varphi = \Lambda^{\bar{\nu}}_{\mu} \Lambda^{\mu}_{\bar{\rho}}\, \partial_{\bar{\nu}} \varphi\, \partial^{\bar{\rho}} \varphi$$

$$= \underbrace{\delta^{\bar{\nu}}_{\bar{\rho}}}_{\text{identity Lorentz transform in barred frame}}\, \partial_{\bar{\nu}} \varphi\, \partial^{\bar{\rho}} \varphi$$

$$= \partial_{\bar{\nu}} \varphi\, \partial^{\bar{\nu}} \varphi$$

Thus $\mathcal{L}$ takes the same form in the barred frame, so is a Lorentz scalar.

d'Alembert equation

$$c^{-2}\ddot{\varphi} - \nabla^2 \varphi = 0$$

ie $$\partial_\mu \partial^\mu \varphi = 0$$

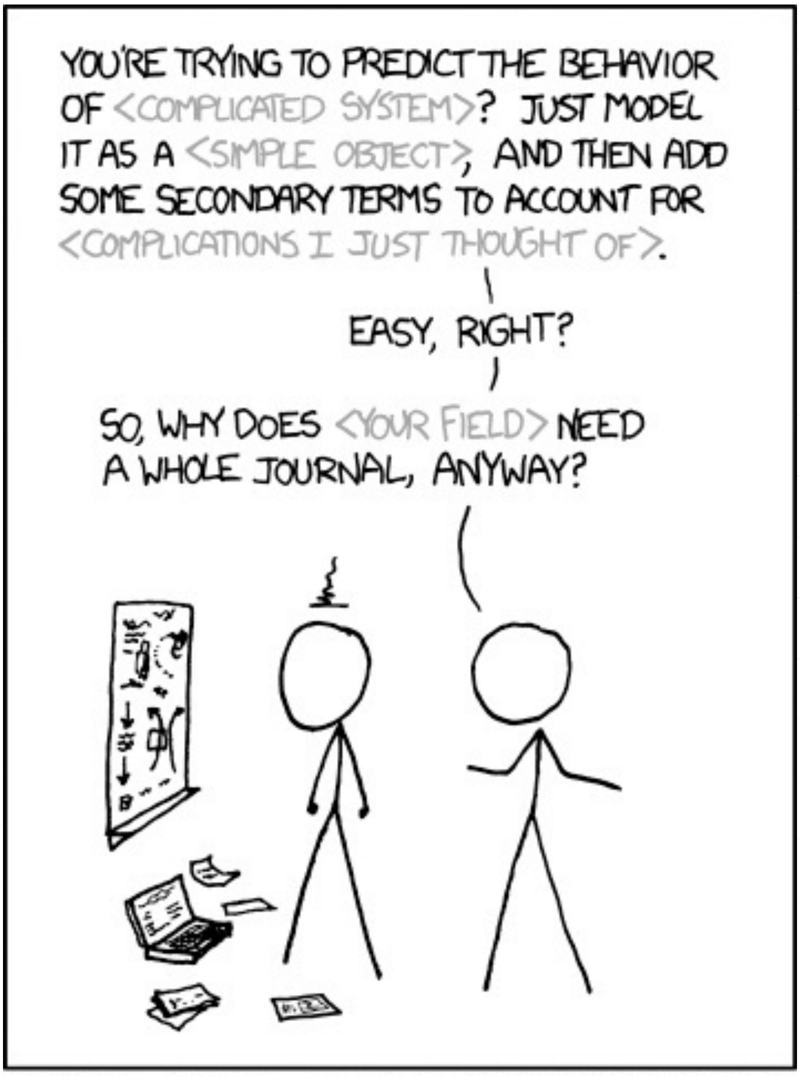
Under a Lorentz transformation, LHS is

$$\partial_\mu \partial^\mu \varphi = \partial_{\bar{\nu}} \partial^{\bar{\nu}} \varphi$$ by an identical argument to the above.

Also, under the same transformation, $O \to O$
so, combining LHS & RHS,

$$\partial_{\bar{\nu}} \partial^{\bar{\nu}} \varphi = 0$$

So the d'Alembert equation takes the same form in all inertial frames, ie is covariant.



LIBERAL-ARTS MAJORS MAY BE ANNOYING SOMETIMES, BUT THERE'S *NOTHING* MORE OBNOXIOUS THAN A PHYSICIST FIRST ENCOUNTERING A NEW SUBJECT.

4. [O(3) - COVARIANT D'ALEMBERT THEORY]

a) $\mathcal{L} = \delta^{ij} \eta^{\mu\nu} \partial_\mu \varphi_i \partial_\nu \varphi_j$

This is manifestly Lorentz covariant, since $\eta^{\mu\nu}$, ∂_μ (the objects with greek indices) are known to be covariant, and each φ_i is a Lorentz scalar.

Under Lorentz transformation,

$$\eta^{\mu\nu} \longrightarrow \eta^{\bar{\alpha}\bar{\beta}}$$
$$\left.\begin{aligned} \partial_\mu \varphi_i &\rightarrow \partial_{\bar{\alpha}} \varphi_i \\ \partial_\nu \varphi_j &\rightarrow \partial_{\bar{\beta}} \varphi_j \end{aligned}\right\} \text{note } \varphi_i \rightarrow \varphi_i \text{ since Lorentz scalar}$$

Since δ^{ij} has no relativistic indices, we treat each of the 9 euclidean components as a Lorentz scalar.

In the move to the transformed frame, therefore

$$\mathcal{L} \longrightarrow \delta^{ij} \eta^{\bar{\alpha}\bar{\beta}} \partial_{\bar{\alpha}} \varphi_i \partial_{\bar{\beta}} \varphi_j$$
$$= \delta^{ij} \Lambda^{\bar{\alpha}}_{\mu} \Lambda^{\bar{\beta}}_{\nu} \eta^{\mu\nu} \Lambda^{\rho}_{\bar{\alpha}} \partial_\rho \varphi_i \Lambda^{\sigma}_{\bar{\beta}} \partial_\sigma \varphi_j$$
$$= \underbrace{\Lambda^{\bar{\alpha}}_{\mu} \Lambda^{\rho}_{\bar{\alpha}}}_{\delta^{\rho}_{\mu}} \quad \underbrace{\Lambda^{\bar{\beta}}_{\nu} \Lambda^{\sigma}_{\bar{\beta}}}_{\delta^{\sigma}_{\nu}} \delta^{ij} \eta^{\mu\nu} \partial_\rho \varphi_i \partial_\sigma \varphi_j$$

identity transform

$$= \delta^{ij} \eta^{\mu\nu} \partial_\mu \varphi_i \partial_\nu \varphi_j$$
$$= \mathcal{L}, \quad \text{as required.}$$

We will consider the EL equation for the φ_k component of the $(\varphi_1, \varphi_2, \varphi_3)$ triplet.

$$\frac{\partial \mathcal{L}}{\partial \varphi_k} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\tau \varphi_k)} = \frac{\partial}{\partial(\partial_\tau \varphi_k)} \Big[(\partial_0 \varphi_1)^2 - (\partial_1 \varphi_1)^2 - (\partial_2 \varphi_1)^2 - (\partial_3 \varphi_1)^2 + (\partial_0 \varphi_2)^2 - (\partial_1 \varphi_2)^2 - (\partial_2 \varphi_2)^2 - (\partial_3 \varphi_3)^2 + (\partial_0 \varphi_3)^2 - (\partial_1 \varphi_3)^2 - (\partial_2 \varphi_3)^2 - (\partial_3 \varphi_3)^2 \Big]$$

$$= \begin{cases} 2\partial_\tau \varphi_k & \tau = 0 \\ -2\partial_\tau \varphi_k & \tau = 1,2,3 \end{cases}$$

$$= 2\,\partial^\tau \varphi_k,$$ as expected.

Thus EL equation for φ_k is

$$\cancel{\frac{\partial \mathcal{L}}{\partial \varphi_k}} - \partial_\tau \frac{\partial \mathcal{L}}{\partial(\partial_\tau \varphi_k)} = 0$$

ie
$$2\,\partial_\tau \partial^\tau \varphi_k = 0$$

ie

$$\Box^2 \varphi_k = 0$$

So each component of the field satisfies the d'Alembert equation, as required.

b) Consider now a rotation $R^{i}_{\bar{j}}$ in field space, where

$$\varphi_i \rightarrow \varphi_{\bar{j}} = R^{i}_{\bar{j}}\, \varphi_i$$

for $i, \bar{j} = 1, 2, 3$.

Lagrange density transforms to

$$\mathcal{L} \Longrightarrow \delta^{\bar{k}\bar{l}}\, \eta^{\mu\nu}\, \partial_\mu \varphi_{\bar{k}}\, \partial_\nu \varphi_{\bar{l}}$$

euclidean metric unchanged by rotn

spacetime indices unchanged by rot^n in field space

$$= R^{\bar{k}}_{i} R^{\bar{l}}_{j}\, \delta^{ij}\, \eta^{\mu\nu}\, R^{m}_{\bar{k}}\, \partial_\mu \varphi_m\, R^{n}_{\bar{l}}\, \partial_\nu \varphi_n$$

$$= \underbrace{R^{\bar{k}}_{i} R^{m}_{\bar{k}}}_{\delta^{m}_{i}}\; \underbrace{R^{\bar{l}}_{j} R^{n}_{\bar{l}}}_{\delta^{n}_{j}}\; \delta^{ij}\, \eta^{\mu\nu}\, \partial_\mu \varphi_m\, \partial_\nu \varphi_n$$

identity rotations

$$= \delta^{ij}\, \eta^{\mu\nu}\, \partial_\mu \varphi_i\, \partial_\nu \varphi_j = \mathcal{L}$$

so $\mathcal{L}$ is invariant, as req^d.

c) Under the stated transformation,

$$\Phi^{(3)}_{1} = \frac{\partial}{\partial \phi} (\cos\phi\, \varphi_1 + \sin\phi\, \varphi_2) \Big|_{\phi=0}$$

$$= \varphi_2$$

$$\Phi^{(3)}_{2} = \frac{\partial}{\partial \phi} (-\sin\phi\, \varphi_1 + \cos\phi\, \varphi_2) \Big|_{\phi=0} = -\varphi_1$$

$$\Phi_3^{(3)} = \frac{\partial(\varphi_3)}{\partial \phi}\Big|_{\phi=0} = 0$$

Thus we have

$$j_3^\mu = \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_1)}}_{2\partial^\mu \varphi_1} \Phi_1^{(3)} + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_2)}}_{2\partial^\mu \varphi_2} \Phi_2^{(3)} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_3)} \Phi_3^{(3)}$$

$$= 2(\varphi_2 \partial^\mu \varphi_1 - \varphi_1 \partial^\mu \varphi_2)$$

as in O(2) example in lecture notes.

d) Can find quantities ~~conserved~~ associated with RH rotations about other axes by cyclic permutation:

$$\Phi_1^{(1)} = 0, \quad \Phi_2^{(1)} = \varphi_3, \quad \Phi_3^{(1)} = -\varphi_2$$

$$j_1^\mu = 2(\varphi_3 \partial^\mu \varphi_2 - \varphi_2 \partial^\mu \varphi_3)$$

and

$$\Phi_1^{(2)} = -\varphi_3, \quad \Phi_2^{(2)} = 0, \quad \Phi_3^{(2)} = \varphi_1$$

$$j_2^\mu = 2(\varphi_1 \partial^\mu \varphi_3 - \varphi_3 \partial^\mu \varphi_1)$$

e) General rotation matrix

$$\underset{\approx}{M} = \exp(\tau\, \underset{\sim}{\hat{n}} \cdot \underset{\approx}{S})$$

$$= \underbrace{\underset{\approx}{1}_3}_{\substack{3\times 3\\ \text{identity}}} + \tau\, \underset{\sim}{\hat{n}} \cdot \underset{\approx}{S} + O(\tau^2)$$

and $\underset{\approx}{S} = (\underset{\approx}{S}^{(1)}, \underset{\approx}{S}^{(2)}, \underset{\approx}{S}^{(3)})$ 3-vector of 3x3 matrices

Since, if $\underset{\sim}{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$, the transformation associated with $\underset{\approx}{M}$ is

$$T_\tau : \underset{\sim}{\varphi} \longrightarrow \underset{\sim}{\bar{\varphi}} = \underset{\approx}{M}\, \underset{\sim}{\varphi}$$

and

$$\left.\frac{\partial \underset{\approx}{M}}{\partial \tau}\right|_{\tau=0} = \underset{\sim}{\hat{n}} \cdot \underset{\approx}{S} \quad \text{from expansion above}$$

$$= \hat{n}_j\, \underset{\approx}{S}^{(j)} \quad \text{summing over cartesian indices in field space.}$$

We therefore have a 3-vector of Φ's,

$$\underset{\sim}{\Phi} = \left.\frac{\partial \underset{\sim}{\bar{\varphi}}}{\partial \tau}\right|_{\tau=0} = \hat{n}_j\, \underset{\approx}{S}^{(j)}\, \underset{\sim}{\varphi}$$

Sum of matrix x vector multiplications

ie

$$\Phi_k = \hat{n}_j\, S^{(j)}{}^{\overset{\text{column}}{i}}{}_{\underset{\text{row}}{k}}\, \varphi_i$$

Conserved noetherian 4-current is

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_1)} \Phi_1 + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_2)} \Phi_2 + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_3)} \Phi_3$$

$$= \delta_{kl} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_l)} \Phi_k$$

$$= \delta_{kl} \; 2 \partial^\mu \varphi_l \; \hat{n}_j \; S^{(j)\,i}{}_k \; \varphi_i$$

$$= 2 \hat{n}_j \; (\partial^\mu \underset{\sim}{\varphi}) \cdot \underset{\approx}{S}^{(j)} \underset{\sim}{\varphi}$$ in 3-vector form

Now, $2 (\partial^\mu \underset{\sim}{\varphi}) \cdot \underset{\approx}{S}^{(1)} \underset{\sim}{\varphi} = 2 (\varphi_2 \partial^\mu \varphi_3 - \varphi_3 \partial^\mu \varphi_2)$

$$= j_1^\mu$$

$$2 (\partial^\mu \underset{\sim}{\varphi}) \cdot \underset{\approx}{S}^{(2)} \underset{\sim}{\varphi} = j_2^\mu$$

$$2 (\partial^\mu \underset{\sim}{\varphi}) \cdot \underset{\approx}{S}^{(3)} \underset{\sim}{\varphi} = j_3^\mu$$

ie Noether current for general transformation

$$j^\mu = \delta_{ik} \hat{n}_i \, j_k^\mu$$

ie a sum of the Noether currents due to rotational invariance about 1, 2, 3 axes.

5. [RELATIVISTIC WAVEPACKET]

$$\varphi = f(x-ct),$$

a) $\partial_x \varphi = f', \quad \partial_x^2 \varphi = f''$

$\partial_t \varphi = -cf', \quad \partial_t^2 \varphi = c^2 f''$

$\partial_y \varphi = \partial_z \varphi = 0$

So $$\Box^2 \varphi = c^{-2}\partial_t^2 \varphi - \partial_x^2 \varphi$$

$$= \underbrace{c^{-2}c^2}_{1} f'' - f''$$

$$= 0, \quad \text{as required.}$$

b) We choose now, in 1+1 dimensions, to work with (t, x) and (ω, k), rather than covariant forms of 4-vectors and factors of c.

Therefore

$$\tilde{\varphi}(\omega, k) = \frac{1}{(2\pi)^2} \int_{\substack{\text{all} \\ \text{1+1 space}}} dt\, dx\, \varphi(t,x)\, e^{i(\omega t - kx)}$$

$$= \frac{1}{(2\pi)^2} \int dt\, dx\, f(x-ct)\, e^{i(\omega t - kx)}$$

Now make the substitution

$x \longrightarrow u = x - ct, \qquad dx = du,$
integrating over all u, $-\infty < u < \infty$.

Also, $\omega t - kx = \omega t - k(x - ct) - ckt$

$$= t(\omega - ck) - ku,$$

and FT of f is

$$\tilde{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} du \; f(u)\, e^{-iuk}.$$

↑ u dummy variable here

Thus
$$\tilde{\varphi}(\omega,k) = \frac{1}{(2\pi)^2}\int_{\substack{1+1\\ \text{space}}} dt\, du\; f(u)\, e^{it(\omega - ck) - iuk}$$

$$= \tilde{f}(k) \times \frac{1}{2\pi}\int_{-\infty}^{\infty} dt \; e^{it(\omega - ck)}$$

$$= \tilde{f}(k)\; \delta(\omega - ck)$$

↑ Dirac δ-fn

Now,
$$\tilde{\varphi}(-\omega,-k) = \tilde{f}(-k)\; \delta(-\omega + ck)$$

$$= \tilde{f}^*(k)\; \delta(\omega - ck)$$

from known properties of 1D Fourier transforms, and the fact that δ is symmetric.

Since δ is also real, we have

$$\tilde{\varphi}(-\omega, -k) = \tilde{\varphi}^*(\omega, k),$$ as required.

c). The Fourier transform of the field δ can be considered as the Fourier superposition

$$\varphi(t, x) = \int_{\substack{\text{null}\\ \text{vectors}}} d\omega\, dk\; \tilde{\varphi}(\omega, k)\, e^{i(kx - \omega t)}$$

$$= \int d\omega\, dk\; \tilde{f}(k)\, \delta(\omega - ck)\, e^{i(kx - \omega t)}$$

The δ-function $\delta(\omega - ck) = \delta(c(k - \frac{\omega}{c}))$,

implying all k values are positive, when ω is positive. Wave velocity is $\omega/k > 0$, and in fact, $\omega/k = c$, so all Fourier components are in the $+x$ direction, as required.

Note that there may still be Fourier components with $k < 0$: it is $\omega/k > 0$ that determines the speed of the wave.

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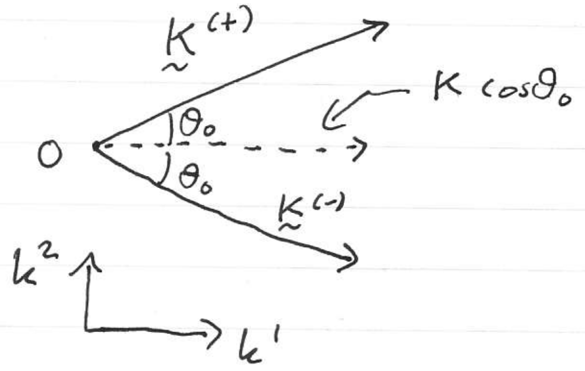
6. [SUPERLUMINAL PHASE VELOCITY]

$$\varphi = e^{-i K^{(+)}_\mu x^\mu} + e^{-i K^{(-)}_\mu x^\mu}$$

$$K^{(\pm)\,\mu} = (\Omega, K\cos\theta_0, \pm K\sin\theta_0, 0)$$

a)

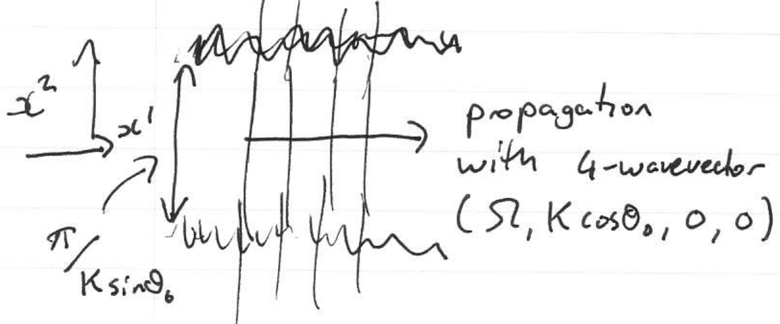


b) Rewrite the field explicitly in terms of spacetime components

$$\varphi = e^{i(Kx^1\cos\theta_0 + Kx^2\sin\theta_0 - \Omega t)} + e^{i(Kx^1\cos\theta_0 - Kx^2\sin\theta_0 - \Omega t)}$$

$$= e^{i(x^1 K\cos\theta_0 - \Omega t)} \times 2\cos(Kx^2\sin\theta_0)$$

This field looks like a plane wave, with 4-wavevector $(\Omega, K\cos\theta_0, 0, 0)$, modulated by a cosine variation in the x^2 direction, with spatial period $2\pi / K\sin\theta_0$



The 'phase velocity' of the plane wave-like factor is $\frac{\Omega}{K\cos\theta_0} = \frac{c}{\cos\theta_0}$, since the original waves were null, so $c = \Omega / K$

Thus the phase fronts move in the 1-direction with velocity $c/\cos\theta_0 > c$.

c) The superposition clearly does not violate special relativity, as it is a perfectly allowed superposition of two plane waves (provided, of course, that we do not force square-integrability / normalizability).

The example illustrates that wave-related objects, such as the phase fronts here, CAN travel faster than c, provided the plane wave components do not.

Although superluminal variation is allowed, INFORMATION CANNOT BE TRANSFERRED BY THEM: they are merely an interference phenomenon and there are many other examples of superluminal interference features of waves.