Advanced Classical Mechanics: *Relativistic Fields*

Problem sheet 3 - Relativistic Electrodynamics

1. [**Non-covariance of electromagnetism under Galilean transformation**]
   Let's reconsider some of the issues we discussed back at the start of the course:

   (a) Prove that the electric and magnetic fields cannot transform covariantly under Galilean transformation.

   (b) Argue that the electromagnetic field cannot be covariant with respect to Galilean transformations, using the speed of electromagnetic waves.

2. [**Einstein's 'moving magnet and conductor problem'**]
   Einstein opened his famous 1905 paper 'On the electrodynamics of moving bodies' with the following discussion [1]

   > It is known that Maxwell's electrodynamics—as usually understood at the present time—when applied to moving bodies, lead to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there arises in the neighbourhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neighbourhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is not corresponding energy, but which gives rise – assuming equality of relative motion in the two cases discussed – to electric currents of the same path and intensity as those produced by the electric forces in the former case.

   By treating the conductor as a collection of free charges, try to quantify this argument yourself from Maxwell's equations and the nonrelativistic Lorentz force law, illustrating the fact that working in the frame of the magnet and frame of the the conductor both give the same observable effect (an electromotive force), but via apparently different physical arguments.

3. [**Symmetry of Maxwell's equations under C, P and T reversal**]
   In this question, assume we are in the nonrelativistic regime. Recall from problem sheet 1 how the mass m, charge q, position $\boldsymbol{r}$, velocity $\boldsymbol{v}$ and acceleration $\boldsymbol{a}$ of a particle transform under charge inversion C, the parity transformation P, and time inversion T.

   (a) Using the nonrelativistic picture of current density $\boldsymbol{J}$ as charge density ρ times a velocity field, deduce how $\rho, \boldsymbol{J}$ transform under C, P and T.

[1] `The Principle of Relativity`, Lorentz, Einstein et al, Dover 1952.

(b) Assuming invariance of the Lorentz force law under the three transformations, deduce how $\boldsymbol{E}$ and $\boldsymbol{B}$ change under C, P and T.

(c) Hence show that Maxwell's equations are covariant under charge, parity and time reversal.

(d) Does anything change in the relativistic version of this argument (i.e. using the relativistic Lorentz law)?

4. [**Covariance of Maxwell equations under Lorentz transformations**]
Work in two frames with coordinates (ct, x, y, z), $(c\bar{t}, \overline{x}, \overline{y}, \overline{z})$, and assume the barred frame is reached from the unbarred frame by a Lorentz boost by velocity v, in the x-direction, given by the matrix

$$\boldsymbol{\Lambda} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$.

(a) Assuming the Faraday tensor transforms covariantly under Lorentz transformations, show that under the transformation above, the electric and magnetic field components transform as follows

$$\begin{array}{rclcrcl} \overline{E}_x & = & E_x & \qquad & \overline{B}_x & = & B_x \\ \overline{E}_y & = & \gamma(E_y - vB_z) & & \overline{B}_y & = & \gamma(B_y + vc^{-2}E_z) \\ \overline{E}_z & = & \gamma(E_z + vB_y) & & \overline{B}_z & = & \gamma(B_z - vc^{-2}E_y) \end{array}$$

Show that this is consistent with the form given for the transformed fields given in the lectures.
(Hint: Calculate using matrices, and argue that the transformed, mixed Faraday tensor $F^{\overline{\mu}}{}_{\overline{\nu}} = \Lambda^{\overline{\mu}}_{\rho} F^{\rho}{}_{\sigma} \Lambda^{\sigma}_{\overline{\nu}}$ *can be represented by the matrix product* $\boldsymbol{\Lambda}\mathbf{F}_{\text{mixed}}\boldsymbol{\Lambda}^{-1}$.

(b) Argue that since the 4-current $J^\mu = (c\rho, \boldsymbol{J})$ is a contravariant 4-vector, in the transformed coordinate frame we have

$$\overline{\rho} = \gamma(\rho - \beta J_x/c), \quad \overline{J}_x = \gamma(J_x - v\rho), \quad \overline{J}_y = J_y, \quad \overline{J}_z = J_z.$$

(c) Show that under transformation the spacetime derivatives take on the following forms:

$$\partial_{\bar{t}} = \gamma(\partial_t + v\partial_x), \quad \partial_{\overline{x}} = \gamma(vc^{-2}\partial_t + \partial_x), \quad \partial_{\overline{y}} = \partial_y, \quad \partial_{\overline{z}} = \partial_z.$$

(d) Show that $\overline{\boldsymbol{E}}, \overline{\boldsymbol{B}}$ satisfy Maxwell's equations in the transformed frame.

Lorentz initially justified the Lorentz transformations by a very similar argument to the one above, as the set of transformations which maintain the form of Maxwell's equations. You can read it for yourself, for instance reprinted in `The Principle of Relativity`, Lorentz, Einstein et al, Dover 1952.

We have not proved covariance under a *general* boost, since the Lorentz transformation for a boost in a general direction (i.e. not along a coordinate axis) is complicated, which complicates the transformed frame derivatives, etc.

5. **[The four-dimensional antisymmetric symbol and the dual Faraday tensor]**
In four dimensions, the Levi-Civita symbol is generalized to be rank four:

$$\varepsilon_{\mu\nu\rho\sigma} = -\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} -1 & \text{if } (\mu,\nu,\rho,\sigma) \text{ is an odd permutation of } (0,1,2,3) \\ +1 & \text{if } (\mu,\nu,\rho,\sigma) \text{ is an even permutation of } (0,1,2,3) \\ 0 & \text{otherwise} \end{cases}$$

Recall (e.g. from determinant theory) that a permutation is *even* if it can be reached by an even number of pairwise exchanges, and *odd* if it requires an odd number. For instance, $(1,3,2,0)$ is an even permutation, and $(0,1,3,2)$ is odd. In fact, if $\mathbf{M}$ is a 4-dimensional matrix, then $\det \mathbf{M} = \varepsilon_{\mu\nu\rho\sigma} M^{\mu 0} M^{\nu 1} M^{\rho 3} M^{\sigma 4}$.

(a) Write down the 12 values of (μ,ν,ρ,σ) where $\varepsilon_{\mu\nu\rho\sigma} = +1$, and the 12 values of (μ,ν,ρ,σ) where $\varepsilon_{\mu\nu\rho\sigma} = -1$.

(b) Show that the components of $\varepsilon_{\mu\nu\rho\sigma}$ are the same under rotations and boosts, i.e. it transforms as a rank-4 covariant tensor.
(Hint: Use the fact that the determinant of every proper Lorentz transformation matrix is $+1$.)

(c) Show that the homogeneous tensorial Maxwell equations $\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$ can be expressed more compactly as

$$\varepsilon^{\sigma\mu\nu\rho} \partial_\mu F_{\nu\rho} = 0 \qquad \text{for } \sigma = 0,1,2,3.$$

(d) Show that the components of the *dual Faraday tensor* $G_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$ are given by

$$G_{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z/c & E_y/c \\ B_y & E_z/c & 0 & -E_x/c \\ B_z & -E_y/c & E_x/c & 0 \end{pmatrix}.$$

(e) Show that the equations for the dual Faraday tensor

$$\partial^\mu G_{\mu\nu} = 0, \qquad \partial_\mu G_{\nu\rho} + \partial_\nu G_{\rho\mu} + \partial_\rho G_{\mu\nu} = -\mu_0 \varepsilon_{\sigma\mu\nu\rho} J^\sigma$$

are equivalent to Maxwell's equations. You will probably find the following identity useful:

$$\varepsilon_{\sigma\mu\nu\rho}\varepsilon^{\sigma\alpha\beta\tau} = -\delta^\alpha_\mu \delta^\beta_\nu \delta^\tau_\rho - \delta^\alpha_\nu \delta^\beta_\rho \delta^\tau_\mu - \delta^\alpha_\rho \delta^\beta_\mu \delta^\tau_\nu + \delta^\alpha_\mu \delta^\beta_\rho \delta^\tau_\nu + \delta^\alpha_\nu \delta^\beta_\mu \delta^\tau_\rho + \delta^\alpha_\rho \delta^\beta_\nu \delta^\tau_\mu.$$

6. **[Vector potential examples]**
In this question, we find forms for vector potentials $\boldsymbol{A}$ giving rise to some simple magnetic fields, through $\boldsymbol{B} = \nabla \times \boldsymbol{A}$.

(a) (Constant B-field) Show that the constant field $(0,0,B)$ is the curl of $\boldsymbol{A} = \frac{1}{2}(-By, Bx, 0)$. What happens to both fields under a translation of the spatial origin?

(b) (B-field concentrated on z-axis) Show that the magnetic field $\boldsymbol{B} = B\delta(x)\delta(y)\hat{\boldsymbol{z}}$ arises as the curl of the vector potential

$$\boldsymbol{A} = \frac{B}{2\pi(x^2+y^2)}(-y, x, 0).$$

(Hint: you should check two parts. The first is to verify that $\nabla \times \boldsymbol{A} = 0$ for all nonzero x, y. The second, harder part, follows from integrating the flux through a small area of the x,y-plane including the origin, and using Stokes' theorem.)

In both parts, the $\boldsymbol{A}$-field is circulating around the z-axis, but the different radial dependence gives rise to very different magnetic fields. The $\boldsymbol{B}$-field in the second part is that of an infinitely narrow, infinitely long solenoid up the z-axis.

7. [**The Riemann-Silberstein vector**]
In this question, we explore an alternative way of combining the electric and magnetic fields other than by the Faraday tensor. It turns out that it is surprisingly elegant to approach electromagnetism using a complex 3-vector, called the *Riemann-Silberstein vector*,[2] which is defined
$$\boldsymbol{F} = \sqrt{\varepsilon_0}(\boldsymbol{E} + \mathrm{i}c\boldsymbol{B}).$$
   (a) Show that, with this definition, the dimensions of $\boldsymbol{F}$ are $\texttt{[M]}^{1/2}\ \texttt{[L]}^{-1/2}\ \texttt{[T]}^{-1}$. Why might this be a natural physical dimension for the electromagnetic field?
   (b) Show that Maxwell's equations are equivalent to the following two equations for $\boldsymbol{F}$:
$$\begin{aligned} \nabla \cdot \boldsymbol{F} &= \varepsilon_0^{-1/2}\rho \\ \mathrm{i}c^{-1}\partial_t \boldsymbol{F} &= \nabla \times \boldsymbol{F} - \mathrm{i}\mu_0^{1/2}\boldsymbol{J}. \end{aligned}$$
   (c) Find a 3×3 matrix operator '$\triangleright$', with which the second of these equations can be put in a simple matrix form.
   (d) What are the energy density W and Poynting vector $\boldsymbol{S}$ in terms of $\boldsymbol{F}$?
   (e) What are the electromagnetic invariants a and b in terms of $\boldsymbol{F}$?

8. [**Noether current for electromagnetic gauge symmetry**]
Recall that the homogeneous electromagnetic lagrangian is
$$\mathcal{L}_{\mathrm{EM}} = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4\mu_0}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu).$$
Consider a fixed gauge transformation $A_\mu \to \overline{A}_\mu = A_\mu - \partial_\mu \chi$ for a fixed but arbitrary Lorentz scalar field χ.
   (a) Show that $\mathcal{L}_{\mathrm{EM}}$ is unchanged by the gauge transformation.
   (b) Generalizing the gauge transformation to a continuous set of transformations parametrized by τ, find the field quantities Φ_μ associated with this transformation.
   (c) Write down the Noether current corresponding to your transformation. Can you prove that it is divergenceless without Noether's theorem?

9. [**Gauge symmetry and electromagnetic action**]
This question follows from the previous one, where we have to extend our notion of noetherian symmetry to incorporate the gauge symmetry of the inhomogeneous Maxwell equation, which has the interacting electromagnetic lagrangian
$$\mathcal{L}_{\mathrm{EM,int}} = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - A_\sigma J^\sigma.$$

[2] Although the vector was considered by Silberstein in 1907, the term was coined recently by Iwo Bialynicki-Birula.

Since we have a lagrangian density, we can say that the potential field A_μ makes the *electromagnetic action*

$$\mathcal{S}[A_\mu] = \int_\Omega \mathrm{d}^4 x \mathcal{L}_{\text{EM,int}}$$

stationary, where Ω is a spacetime volume, and A_μ is not varied on the boundary $\partial\Omega$.

(a) Write down how $\mathcal{L}_{\text{EM,int}}$ changes under the gauge transformation in the previous question.

(b) By considering the gauge transformation now as a variation of the electromagnetic action, show that the value of the action is invariant to gauge transformation. Can you interpret the electromagnetic gauge symmetry in the light of this?

(c) Follow the procedure to construct a 'Noether current' from $\mathcal{L}_{\text{EM,int}}$ by the family of gauge transformations you considered in the previous question. Is this divergenceless?

10. [**Liénard-Wiechert potentials for a moving point charge**]
A moving point charge contributes to an EM field at the event $t, \boldsymbol{r}$ where its spacetime trajectory touches the past lightcone of $t, \boldsymbol{r}$, which occurs at the so-called *retarded time* t_{ret}. The retarded time is defined mathematically to be $t_{\mathbf{ret}}(t, \boldsymbol{r}) = t - |\boldsymbol{r} - \boldsymbol{R}(t)|/c$. Assume a point charge q follows the path $\boldsymbol{R}(t)$; we will assume the frame is fixed. The current and charge are given by $\rho(t, \boldsymbol{r}) = q\delta(\boldsymbol{r} - \boldsymbol{R}(t)), \boldsymbol{J}(t, \boldsymbol{r}) = q\dot{\boldsymbol{R}}(t)\delta(\boldsymbol{r} - \boldsymbol{R}(t))$. In the Lorenz gauge, V and $\boldsymbol{A}$ each satisfy the inhomogeneous d'Alembert equation with point sources given by ρ and $\boldsymbol{J}$; propagator methods can be used to show

$$V(t, \boldsymbol{r}) = \frac{q}{4\pi\varepsilon_0} \int \mathrm{d}t' \, \frac{\delta(t' - t + |\boldsymbol{r} - \boldsymbol{R}(t')|/c)}{|\boldsymbol{r} - \boldsymbol{R}(t')|}, \; \boldsymbol{A}(t, \boldsymbol{r}) = \frac{\mu_0 q}{4\pi} \int \mathrm{d}t' \, \dot{\boldsymbol{R}}(t) \frac{\delta(t' - t + |\boldsymbol{r} - \boldsymbol{R}(t')|/c)}{|\boldsymbol{r} - \boldsymbol{R}(t')|}.$$

(a) Check that the charge continuity equation holds for these functions, and verify that, when $\boldsymbol{R}(t)$ is constant (the particle is not moving), the potentials agree with what you expect. Interpret the integral expressions for $\boldsymbol{A}$ and V.

(b) Using properties of δ-functions, and with $\boldsymbol{n}(t') = (\boldsymbol{r} - \boldsymbol{R}(t'))/|\boldsymbol{r} - \boldsymbol{R}(t')|$, the spatial unit vector between $\boldsymbol{r}$ and $\boldsymbol{R}(t')$, show that

$$V(t, \boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{|\boldsymbol{r} - \boldsymbol{R}|(1 - \boldsymbol{n} \cdot \dot{\boldsymbol{R}}/c)}, \quad \boldsymbol{A}(t, \boldsymbol{r}) = \frac{\mu_0}{4\pi} \frac{q\dot{\boldsymbol{R}}}{|\boldsymbol{r} - \boldsymbol{R}|(1 - \boldsymbol{n} \cdot \dot{\boldsymbol{R}}/c)},$$

where $\boldsymbol{R}$, $\dot{\boldsymbol{R}}$ and $\boldsymbol{n}$ are evaluated at t_{ret}. These *Liénard-Wiechert potentials* demonstrate the difficulty in solving electrodynamics problems beyond finding the appropriate equations.

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