Advanced Classical Mechanics: *Relativistic Fields*

Problem sheet 1 - Special Relativity & Lorentz Covariance

1. [**Special relativity revision: the twin paradox**]
   A pair of identical twins, Alice and Bob, are separated at birth. Alice stays on earth, at rest in frame S. Bob is kidnapped, and sent off in a spacecraft leaving earth at sublight speed, with velocity v. The rest frame of the spacecraft is $\overline{S}$. At a distance D from earth in the frame of the earth, the kidnappers have a change of heart and make an instantaneous reversal of direction to travel at velocity $-v$ with respect to the earth frame S. (You can assume the new rest frame of the spacecraft is S'.) Assuming nothing unusual happens in the instantaneous change of direction, what is the age difference of Alice and Bob when the spacecraft eventually returns to earth?

2. * [**The parity reversal transformation for cartesian vectors**]
   In the lectures, we showed that rotations are orthogonal matrices, but we did not show that orthogonal matrices are rotations. One important orthogonal transformation which is not a rotation is the *parity reversal transformation* $\mathbf{P}$, which in index notation is $P^{\bar{i}}_{j} = -\delta^{\bar{i}}_{j}$. We distinguish the reflections from the rotations, since *not all vectors transform in the same way under parity transformations.*

   (a) Show that the position vector $\boldsymbol{r} = (x, y, z)$ and gradient operator $\nabla = (\partial_x, \partial_y, \partial_z)$ switch sign under a parity transformation. Vectors (and vector indices on tensors) which switch sign under parity transformation are called *true vectors* or *polar vectors*: this is seen as 'transforming correctly'.

   (b) Assume that $\boldsymbol{u}$ and $\boldsymbol{v}$ are true vectors. Show that $\boldsymbol{u} \cdot \boldsymbol{v}$ does not change sign under a parity transformation. Thus *true scalars are invariant to parity transformation.*

   (c) Show that, if $\boldsymbol{w}$ is defined by the cross product of true vectors $\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v}$, then $\boldsymbol{w}$ does not switch sign under parity transformation. Vectors which do not switch sign under parity transformation are called *pseudovectors* or *axial vectors.*
   (Note: the definition of the antisymmetric symbol ε_{ijk} holds in all coordinate frames.)

   (d) Show that the scalar triple product $\boldsymbol{a} \cdot \boldsymbol{b} \times \boldsymbol{c}$ of true vectors changes sign under parity reversal. If a scalar changes sign under parity transformation, it is called a *pseudoscalar.*

   (e) Newton's force law $\boldsymbol{f} = m\boldsymbol{a} = \partial_t \boldsymbol{p}$ gives the force $\boldsymbol{f}$ on a particle in terms of its mass m times its acceleration $\boldsymbol{a}$, and as the rate of change of its momentum $\boldsymbol{p}$. Show that these quantities are all true vectors and scalars.

   (f) The Lorentz force law, giving the force $\boldsymbol{f}$ on a charged particle of charge q in terms of the particle velocity $\boldsymbol{v}$, in an electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$, is given by

   $$\boldsymbol{f} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}).$$

   Assuming q is a true scalar, determine whether $\boldsymbol{E}$, and $\boldsymbol{B}$ are vectors or pseudovectors.

3. [**Parity reversal and reflection**]
Using the parity transformation from the question above, we can realise the missing orthogonal transformations, the reflections, as combinations of rotations with parity reversal. You should note that the determinant of any rotation $R^{\bar{i}}_{j}$ is $+1$, whereas the determinant of $P^{\bar{i}}_{j}$ is -1.

   (a) Show that the transformation for reflection in the z-plane, represented by the matrix $\mathbf{Z} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, can be represented by the product of parity reversal with a rotation. Can you interpret this geometrically?

   (b) Give an argument (not necessarily with mathematics) that we can find a reflection matrix in *any* plane as the product of parity reversal with a suitable rotation matrix.

   (c) Show that any orthogonal matrix can be written either as a rotation matrix, or as the product of the parity reversal matrix and a rotation matrix.

The set of all orthogonal matrices in three dimensions (combining with matrix multiplication) is a group, called the *orthogonal group* $O(3)$. The set of parity-preserving orthogonal matrices, i.e. rotation matrices of determinant $+1$, is also a group, called the *special orthogonal group* $SO(3)$. Since the $SO(3)$ group is contained inside the $O(3)$ group, it is called a *subgroup*.

4. * [**Parity and time reversal in Minkowski space**]
Parity reversal $P^{\bar{\mu}}_{\nu}$ and time reversal $T^{\bar{\mu}}_{\nu}$ also occur as transformations in Minkowski space, with the definitions

$$P^{\bar{\mu}}_{\nu} = \begin{cases} +1 & \bar{\mu} = \nu = 0 \\ -1 & \bar{\mu} = \nu = 1,2,3 \\ 0 & \bar{\mu} \neq \nu \end{cases} , \quad T^{\bar{\mu}}_{\nu} = \begin{cases} -1 & \bar{\mu} = \nu = 0 \\ +1 & \bar{\mu} = \nu = 1,2,3 \\ 0 & \bar{\mu} \neq \nu \end{cases} .$$

   (a) Show that the components of the Minkowski metric are unchanged by parity and time reversal, i.e. they can be considered as Lorentz transformations.

   (b) Establish how the following change under parity or time reversal: proper time s, spacetime interval s^2, spacetime position x^μ, 4-velocity U^μ, 4-acceleration a^μ, 4-momentum P^μ, electromagnetic 4-current j^μ You may assume electric charge and rest mass are true scalars: they do not change sign under either parity or time reversal.

The full Lorentz group has a complicated subgroup structure based on these transformations. The full Lorentz group itself includes all Minkowski metric-preserving transformations, including parity and time reversal. The *orthochronous Lorentz group* is the subgroup of Lorentz transformations in which time reversal is not allowed, although parity reversal is. The *proper Lorentz group* is the subgroup of these transformations in which parity reversal is not allowed, but time reversal is. The *restricted Lorentz group* is the subgroup of transformations in which neither parity reversal nor time reversal is allowed. This group is the one we usually consider, and it consists only of rotations and boosts, and its matrices have determinant $+1$.[1]

[1] Distinguishing between these different kinds of transformations is important: we normally assume physics are invariant under the full Lorentz group, but certain particle physics experiments have shown that this is not the case for some physical phenomena, when parity reversal is combined with charge reversal C, given by $q \to -q$. The *'CPT' theorem* is a result of quantum field theory, which says that physics must be invariant to the combined transformation *CPT*.

5. **[Inhomogeneous Lorentz transformations]**
You may have noticed that we have not included *all* possible Lorentz transformations, as transforming by $\boldsymbol{\Lambda}$ keeps the origin invariant, and since space is homogeneous, we should be able to translate the spacetime origin by a translation X^μ. An *inhomogenous Lorentz transformation* can be defined
$$x^{\overline{\mu}} = \Lambda^{\overline{\mu}}_{\nu}(x^\nu + X^\nu).$$
This can be thought of as 'first translate by X^μ, then perform a regular Lorentz transformation $\Lambda^{\overline{\mu}}_{\nu}$'.

   (a) Check that this transformation keeps the spacetime interval between two events x^μ, y^ν invariant.

   (b) Find the inverse inhomogeneous transformation to this Lorentz transformation.

   (c) Show that the result of performing a second inhomogeneous Lorentz transformation,
$$x^{\sigma'} = \hat{\Lambda}^{\sigma'}_{\overline{\mu}}(x^{\overline{\mu}} + Y^{\overline{\mu}})$$
   for $\hat{\boldsymbol{\Lambda}}$ a different Lorentz matrix and Y^μ a different translation 4-vector, gives an overall net inhomogeneous Lorentz transformation, and identify its translation vector.

We will keep things simple in this course by considering origin-preserving homogeneous Lorentz transformations. The set of inhomogeneous Lorentz transforms also forms a group, called the *inhomogeneous Lorentz group* or *Poincaré group.* The homogeneous Lorentz group is a subgroup of this group. The Poincaré group is the full symmetry group of special relativistic spacetime.

6. * **[Eigenvalues and eigenvectors of rank 2 tensors]**
A rank 2 tensor $T_{\mu\nu}$ has eigenvalue a, with eigen-4-vector v^ν, if $T_{\mu\nu}v^\nu = a\eta_{\mu\sigma}v^\sigma$.

   (a) Show that this equation is covariant, i.e. it is valid in any coordinate frame with appropriately transformed tensor indices.

   (b) Show that if the tensor $T_{\mu\nu}$ is symmetric with real components, and the eigen-4-vector is not null, then the eigenvalue a must be real.

   (c) Suppose now that $T_{\mu\nu}$ is symmetric and has another eigenvalue b, $b \neq a$, with eigen-4-vector u^μ, i.e. $T_{\mu\nu}u^\nu = b\eta_{\mu\sigma}u^\sigma$. Show that $u_\mu v^\mu = 0$.

   (d) Assume now that $T^{\mu\nu}$ is the energy-momentum tensor for relativistic dust, $T^{\mu\nu}_{\text{dust}}(x) = n(x)U^\mu(x)U^\nu(x)$. By considering what happens in the local rest frame, find the eigenvalues and eigen-4-vectors for this tensor. What happens to them in a general frame?

SRC 01/26