Advanced Classical Mechanics: *Relativistic Fields*

Problem sheet 2 - Lagrangian Field Theory

1. [**Relativistic and non-relativistic Lorentz force law**]
   In this question, assume the frame of reference is fixed.

   (a) Using cartesian index notation, show that Hamilton's principle, based on the Lagrangian

   $$L = \tfrac{1}{2}m|\boldsymbol{v}|^2 - qV(\boldsymbol{r}), \qquad \boldsymbol{v} = \dot{\boldsymbol{r}} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r},$$

   yields the usual equations of motion for a particle of mass m and charge q, moving in an electrostatic field, with electric field $\boldsymbol{E}$ described by an electrostatic potential $V(\boldsymbol{r})$. Be careful to be consistent with upper and lower indices.
   (Note: we have written the potential energy here as q times electrostatic potential V, i.e. PE is proportional to the electrostatic potential with charge as proportionality constant.)

   (b) Now include a vector potential $\boldsymbol{A}$, which you will recall from earlier courses (or Part III of this course), giving the new Lagrangian

   $$L = \tfrac{1}{2}m|\boldsymbol{v}|^2 - qV(\boldsymbol{r}) + q\boldsymbol{v}\cdot\boldsymbol{A}.$$

   Verify that this Lagrangian does indeed give the Lorentz force law, where as usual $\boldsymbol{E} = -\nabla V - \partial_t \boldsymbol{A}$, and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$.

   (c) We now seek a relativistic version of the Lorentz force law. Keeping the frame of interest fixed, define the relativistic Lagrangian

   $$L = -\frac{mc^2}{\gamma} - \frac{qc}{\gamma}U^\mu A_\mu,$$

   where $\gamma = \gamma(|\boldsymbol{v}|)$, the 4-velocity $U^\mu = (\gamma, \gamma\boldsymbol{v}/c)$, and the *4-potential* is defined $A^\mu = (V/c, \boldsymbol{A})$. Write down a form of L that depends explicitly on $\boldsymbol{v}, V$ and $\boldsymbol{A}$. What is the canonical relativistic 3-momentum? Show that if $V, \boldsymbol{A}$ have no explicit dependence on time, then the Hamiltonian corresponds to the expected expression for relativistic energy of the particle.

   (d) Show that applying the Euler-Lagrange equations for time t in the fixed reference frame gives

   $$m\frac{\mathrm{d}}{\mathrm{d}t}(\gamma\boldsymbol{v}) = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}),$$

   Show that this equation reduces to the non-relativistic Lorentz force law in the low-velocity limit.

You might recognise this relativistic form of the equation of motion as that used for fast-moving charged particles in electromagnetic fields, such as in particle accelerators and plasma physics. Note that it is only the LHS, describing the motion of the *particle*, which looks different to the usual non-relativistic Lorentz force law – the RHS, involving fields.

2. **[Laplace's equation from a variational argument]**
The energy density of an electrostatic field is $\frac{1}{2}\varepsilon_0|\boldsymbol{E}|^2$, where ε_0 is the constant permittivity of free space, and $\boldsymbol{E}$ is given by an electrostatic potential V by $\boldsymbol{E} = -\nabla V$. Assume that there are no charges and that $V, \boldsymbol{E} \to 0$ as $\boldsymbol{r} \to \infty$. Show that if the integrated energy density of the field is stationary, than V must satisfy Laplace's equation $\nabla^2 V = 0$.

3. **[The d'Alembert equation is covariant, and its Lagrange density is a Lorentz scalar]**
The d'Alembert equation with propagation speed c is
$$c^{-2}\ddot{\varphi} - \nabla^2\varphi = 0,$$
and which follows by making stationary the lagrangian density
$$\mathcal{L} = c^{-2}\dot{\varphi}^2 - |\nabla\varphi|^2.$$
By rewriting these in 4-index notation, show explicitly that when φ is a Lorentz scalar, so is $\mathcal{L}$ and the left-hand side of the d'Alembert equation. Since the scalar function 0 at all x^μ is Lorentz invariant, this shows that the d'Alembert equation is Lorentz covariant.

4. **[Noether currents for an O(3)-covariant d'Alembert field theory]**
We can generalize the two-component field of the lectures to a set of three Lorentz scalar fields $\varphi_1, \varphi_2, \varphi_3$, for which the physics is invariant with respect to rotations in $(\varphi_1, \varphi_2, \varphi_3)$-space. We therefore treat these as covariant components under rotations in this abstract 3D space, and use the Euclidean metric and abstract cartesian index notation in this 3D field space. We will consider the Lagrangian density
$$\mathcal{L} = \delta^{ij}\eta^{\mu\nu}\partial_\mu\varphi_i\partial_\nu\varphi_j,$$
and assume the components φ_j are unchanged under Lorentz transformations.

   (a) Show that $\mathcal{L}$ is a Lorentz scalar, and show that the Euler-Lagrange equations give the d'Alembert equation for each field component.

   (b) Show that $\mathcal{L}$ is invariant to any rotation in 3D field space.

   (c) Consider specifically azimuthal rotation about the 3-axis in field space
$$T_\phi^{(3)} : \begin{cases} \varphi_1 \longrightarrow & \cos\phi\,\varphi_1 + \sin\phi\,\varphi_2 \\ \varphi_2 \longrightarrow & -\sin\phi\,\varphi_1 + \cos\phi\,\varphi_2 \\ \varphi_3 \longrightarrow & \varphi_3 \end{cases}$$
   Find the divergenceless Noetherian current j_3^μ associated with this symmetry.

   (d) Write down the Noetherian currents j_1^μ, j_2^μ, corresponding to right-handed rotation about the 1-axis and 2 axis.

   (e) The matrix representing a rotation about axis $\hat{\boldsymbol{n}}$ by angle τ can be written
$$\mathbf{M} = \exp(\tau\hat{\boldsymbol{n}}\cdot\mathbf{S}),$$
   where $\mathbf{S}$ is the 3-vector of matrices
$$\mathbf{S} = \left(\mathbf{S}^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \mathbf{S}^{(2)} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \mathbf{S}^{(3)} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right).$$
   You should interpret the exponential of a matrix as defined by Taylor expansion of the exponential. Find the Noetherian current following from rotational invariance by a rotation about axis $\boldsymbol{n}$. Show that this current is a sum of j_1^μ, j_2^μ and j_3^μ.

This shows that there can be a complicated interplay between spacetime and internal symmetries of fields.

5. **[Relativistic 1+1 wavepackets]**

   Consider the field
   $$\varphi(t,x) = f(x - ct)$$
   for some twice-differentiable and square-integrable function f, and $ct = x^0, x = x^1$.

   (a) Show that φ satisfies the d'Alembert equation $\Box^2\varphi = 0$.

   (b) Working in $1+1$ dimensions, find the Fourier transform $\tilde{\varphi}(\omega, k)$ in terms of the Fourier transform of f, and show that $\tilde{\varphi}(-\omega,-k) = \tilde{\varphi}^*(\omega,k)$.

   (c) Using $\tilde{\varphi}$, show that φ consists only of forward-propagating waves (i.e. waves propagating in the $+x$-direction).

6. **[Superluminal phase velocity]**

   Assume that in some reference frame, we have a d'Alembert field given by the plane wave superposition
   $$\varphi = \mathrm{e}^{-\mathrm{i}K^{(+)}_\mu x^\mu} + \mathrm{e}^{-\mathrm{i}K^{(-)}_\mu x^\mu},$$
   where $K^{(\pm)\mu} = (\Omega, K\cos\theta_0, \pm K\sin\theta_0, 0)$ and $0 < \theta_0 < \pi/2$.

   (a) Sketch the wavevectors in Fourier space.

   (b) Show that, in the x-direction, the ratio of angular frequency to spatial frequency in the 1-direction is $c/\cos\theta_0$. This suggests that the 'phase velocity' of the superposition, in the 1-direction, is superluminal.

   (c) Why does this not violate special relativity?

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