Advanced Classical Mechanics: Relativistic Fields

PROBLEM SHEET 1

SPECIAL RELATIVITY

& LORENTZ COVARIANCE

1. [TWIN PARADOX]

Consider 3 frames

S earth frame
$\bar{S}$ 'kidnap' frame, velocity $+v$ with respect to S
S' return frame, velocity $-v$ with respect to S

We have coordinates $\begin{cases} (t, x) & \text{in } S \\ (\bar{t}, \bar{x}) & \text{in } \bar{S} \\ (t', x') & \text{in } S' \end{cases}$

Assume that $S, \bar{S}$ have a common origin $O, \bar{O}$ at the moment of departure from earth, so with $\beta = v/c$, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$,

$$\begin{pmatrix} c\bar{t} \\ \bar{x} \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

Assume spacecraft changes direction at P, which has coordinates $(x = D,\ t = D/v)$ in S

Therefore, in $\bar{S}$, P has coordinates

$$\begin{pmatrix} c\bar{t} \\ \bar{x} \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} D/\beta \\ D \end{pmatrix} = \begin{pmatrix} \gamma/_\beta\, D\,(1-\beta^2) \\ 0 \end{pmatrix}$$

ie time elapsed in $\bar{S}$ is $D/_{v\gamma}$ when time elapsed in S is D/v

Now, since S' has the same relative velocity to S as $\bar{S}$, and we assume that spacetime is isotropic, the journey from P back to earth $(x=0)$ must also take the same time, $D/v\gamma$ in S', but

D/v in S.

Thus Bobbie, the kidnapped twin, has aged $2D/v\gamma$ over the journey, whereas Alice, the earthbound twin, has aged $2D/v$.

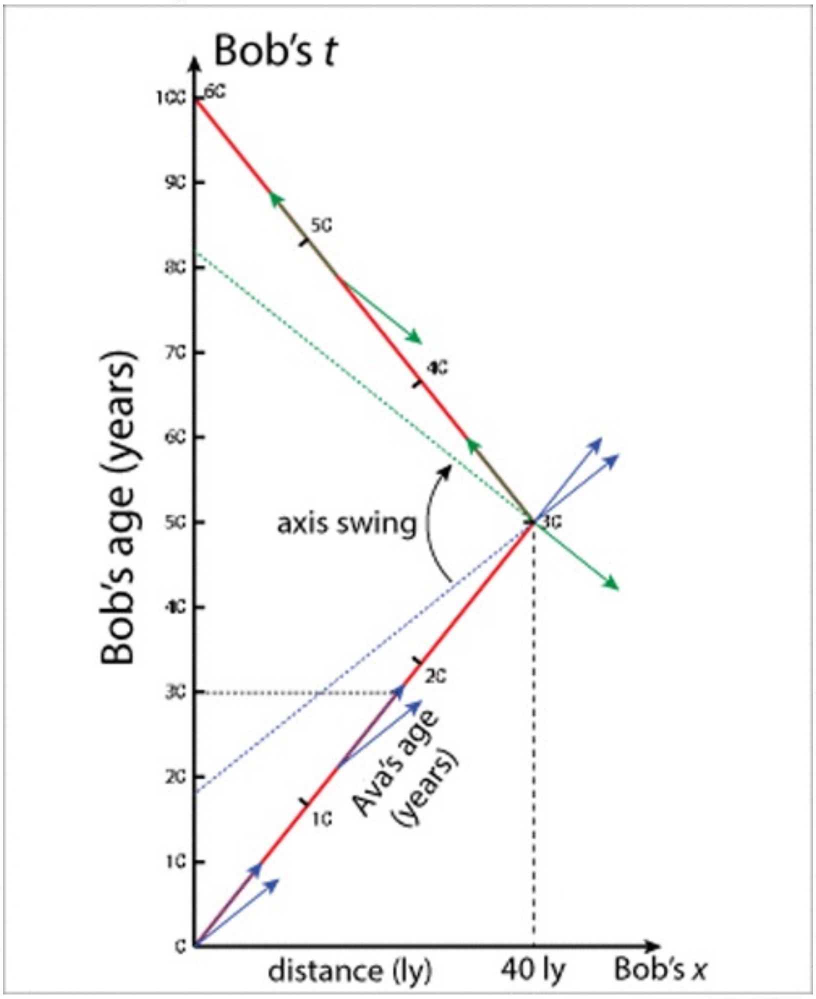
Bobbie is therefore younger by a factor $1/\gamma$, due to relativistic time dilation.

Note that we used a word-based argument involving the basic postulates of relativity, rather than explicitly compute a Lorentz transformation to S'.

This kind of argument is allowed, and it is often preferable to setting up a complicated argument when the answer is obvious by symmetry.

The Lorentz transformation to S' would have been complicated, as it does not naturally have the same spacetime origin as S and $\bar{S}$

An example space-time diagram of the "paradox":



2. [Parity reversal]

a) If $P^{\bar{i}}_{\ j} = -\delta^{\bar{i}}_{\ j}$,

then $r^{\bar{i}} = P^{\bar{i}}_{\ j} \, r^j = -\delta^{\bar{i}}_{\ j} \, r^j$

so sign of components is reversed.

Similarly, If $\bar{x} = -x$, $\frac{\partial}{\partial \bar{x}} = -\frac{\partial}{\partial x}$

so $\nabla \to -\nabla$ under parity reversal.

b) If $\underset{\sim}{u}$, $\underset{\sim}{v}$ are true vectors,

$$\underset{\sim}{u} \to -\underset{\sim}{u}$$
$$\underset{\sim}{v} \to -\underset{\sim}{v}$$

Scalar product $\underset{\sim}{u} \cdot \underset{\sim}{v} \xrightarrow{P} (-\underset{\sim}{u}) \cdot (-\underset{\sim}{v})$

$= \underset{\sim}{u} \cdot \underset{\sim}{v}$, ie scalar product unchanged by parity reversal.

c) $\underset{\sim}{w} = \underset{\sim}{u} \times \underset{\sim}{v} \xrightarrow{P} (-\underset{\sim}{u}) \times (-\underset{\sim}{v}) = \underset{\sim}{u} \times \underset{\sim}{v}$

ie $\underset{\sim}{w}$ unchanged by parity reversal.

d) $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$ true vectors,

$$\underset{\sim}{a} \cdot \underset{\sim}{b} \times \underset{\sim}{c} \xrightarrow{P} (-\underset{\sim}{a}) \cdot (-\underset{\sim}{b}) \times (-\underset{\sim}{c})$$

$$= -\underset{\sim}{a} \cdot \underset{\sim}{b} \times \underset{\sim}{c}$$

ie scalar triple product changes sign.

e) We know $\underset{\sim}{r}$ is a true vector

Time t is a true scalar, and does not change sign under P

So $\dot{\underset{\sim}{r}}$, $\ddot{\underset{\sim}{r}}$, ... are all true vectors

Since m does not change sign under P, $m\underset{\sim}{a} = m\ddot{\underset{\sim}{r}}$, and $m\underset{\sim}{v} = m\dot{\underset{\sim}{r}} = \underset{\sim}{p}$, are all true vectors, as required.

f) Lorentz law $\underset{\sim}{f} = q(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B})$

$\underset{\sim}{f}$ is a true vector, and q a true scalar.

Thus $\underset{\sim}{E}$ must be a true vector, since $q\underset{\sim}{E}$ should change sign under P.

Thus $\underset{\sim}{v} \times \underset{\sim}{B}$ should change sign under P
Since $\underset{\sim}{v}$ is a true vector, $\underset{\sim}{B}$ must be a pseudovector.

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3. [Parity & reflection]

a) A rotation by π about the z-axis is represented by $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Thus, when multiplied (on either side) by $-\underset{\sim}{1}_3$, we get $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, as required.

In general, a rotation changes ALL vectors apart from those along the rotation axis, which are fixed.

A rotation by π, in this case about $\underset{\sim}{\hat{z}}$, acts as a 2D parity reversal in the xy-plane (in 2D, parity is not distinct from rotation as $\det(-\underset{\sim}{1}_2) = 1$.

Therefore, the 3D parity reversal restores the correct sign to the xy directions, but reverses $\hat{\underset{\sim}{z}}$ (which was unchanged by the rotation).

b) We can therefore generalize the previous argument. To get a reflection in a direction $\hat{\underset{\sim}{n}}$, we rotate by π about the $\hat{\underset{\sim}{n}}$ axis (which is effectively a parity reversal in the plane perpendicular to $\hat{\underset{\sim}{n}}$, but leaves $\hat{\underset{\sim}{n}}$ unchanged), then we apply parity reversal.

c) An orthogonal matrix is defined to be a matrix $\underset{\approx}{O}$ such that $\underset{\approx}{O}^T = \underset{\approx}{O}^{-1}$, equivalent to the condition that its columns, as vectors, are orthogonal,

$$\underset{\approx}{O} = \begin{pmatrix} \underset{\sim}{e}_1 & \underset{\sim}{e}_2 & \underset{\sim}{e}_3 \end{pmatrix} \qquad \underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = \delta_{ij}$$

(Nb not component, metric notation)

Thus, if $\underset{\sim}{e}_1 \cdot \underset{\sim}{e}_2 \times \underset{\sim}{e}_3 = +1$, $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$ are a positively-oriented basis for 3D space, and hence the transformation of $(\underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z})$ under a rotation.

If, on the other hand, $\underset{\sim}{e}_1 \cdot \underset{\sim}{e}_2 \times \underset{\sim}{e}_3 = -1$, then we can apply parity reversal $\underline{\underline{P}}$ to $\underline{\underline{O}}$, giving

$$\underline{\underline{P}}\,\underline{\underline{O}} = \begin{pmatrix} -\underset{\sim}{e}_1 & -\underset{\sim}{e}_2 & -\underset{\sim}{e}_3 \end{pmatrix}$$

and $(-\underset{\sim}{e}_1) \cdot (-\underset{\sim}{e}_2) \times (-\underset{\sim}{e}_3) = -\underset{\sim}{e}_1 \cdot \underset{\sim}{e}_2 \times \underset{\sim}{e}_3 = +1$,

so, by the previous argument, $\underline{\underline{P}}\,\underline{\underline{O}} = \underline{\underline{R}}$, a rotation matrix,

Since $\underline{\underline{P}}^2 = \underline{\underline{1}}_3$, premultiplying by $\underline{\underline{P}}$ gives

$$\underline{\underline{O}} = \underline{\underline{P}}^2\,\underline{\underline{O}} = \underline{\underline{P}}\,\underline{\underline{R}},$$ as required.

4. [P & T in Minkowski space]

a) Under parity, Minkowski metric transforms

$$\eta_{\bar\mu\bar\nu} P^{\bar\mu}_{\ \rho} P^{\bar\nu}_{\ \sigma} = \begin{cases} (+1)\times(+1)\times(+1) = +1 & \rho=\sigma=0 \\ (-1)\times(-1)\times(-1) = -1 & \rho=\sigma=1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

ie $\eta_{\bar\mu\bar\nu} P^{\bar\mu}_{\ \rho} P^{\bar\nu}_{\ \sigma} = \eta_{\rho\sigma}$, as required

Since $T^{\bar\mu}_{\ \nu} = -P^{\bar\mu}_{\ \nu}$, an identical argument holds for time reversal.

b)

	P	T
proper time s	invariant	changes sign
s^2	invariant	invariant
$U^\mu = (\gamma, \gamma \underset{\sim}{v})$	$(\gamma, -\gamma \underset{\sim}{v})$	$(+\gamma, -\gamma \underset{\sim}{v})$ since s changes sign
a^μ	a^0 unchanged a^i change sign	a^0 changes sign a^i unchanged
P^μ	$(E/c, -\underset{\sim}{P})$	$(E/c, -\underset{\sim}{P})$
J^μ	$(\rho/c, -\underset{\sim}{J})$	$(\rho/c, -\underset{\sim}{J})$

5. [Inhomogeneous Lorentz transformations]

a) Spacetime interval

$$s^2 = (x^\mu - y^\mu)(x^\nu - y^\nu)\,\eta_{\mu\nu}$$

Under inhomogeneous transformation,

$$x^{\bar\rho} = \Lambda^{\bar\rho}_{\ \mu}\,(x^\mu + X^\mu)$$
$$y^{\bar\rho} = \Lambda^{\bar\rho}_{\ \mu}\,(y^\mu + X^\mu)$$

So

$$(x^{\bar\rho} - y^{\bar\rho})(x^{\bar\sigma} - y^{\bar\sigma})\,\eta_{\bar\rho\bar\sigma}$$

$$= \Lambda^{\bar\rho}_{\ \mu}\,(x^\mu - X^\mu - y^\mu + X^\mu)\,\Lambda^{\bar\sigma}_{\ \nu}\,(x^\nu - X^\nu - y^\nu + X^\nu)\,\eta_{\bar\rho\bar\sigma}$$

$$= \underbrace{\Lambda^{\bar\rho}_{\ \mu}\,\Lambda^{\bar\sigma}_{\ \nu}\,\eta_{\bar\rho\bar\sigma}}_{\eta_{\mu\nu}}\,(x^\mu - y^\mu)(x^\nu - y^\nu)$$

$$= \eta_{\mu\nu}\,(x^\mu - y^\mu)(x^\nu - y^\nu) = s^2$$

as required.

b) Since $x^{\bar{\mu}} = \Lambda^{\bar{\mu}}_{\nu} x^{\nu} + \Lambda^{\bar{\mu}}_{\nu} X^{\nu}$

so $\Lambda^{\bar{\mu}}_{\nu} x^{\nu} = x^{\bar{\mu}} - \Lambda^{\bar{\mu}}_{\nu} X^{\nu}$

and multiplying by $\Lambda^{\rho}_{\bar{\mu}}$ on each side,

$$\underbrace{\Lambda^{\rho}_{\bar{\mu}} \Lambda^{\bar{\mu}}_{\nu}}_{\delta^{\rho}_{\nu}} x^{\nu} = \Lambda^{\rho}_{\bar{\mu}} (x^{\bar{\mu}} - \Lambda^{\bar{\mu}}_{\nu} X^{\nu})$$

$$= \Lambda^{\rho}_{\bar{\mu}} x^{\bar{\mu}} - \underbrace{\Lambda^{\rho}_{\bar{\mu}} \Lambda^{\bar{\mu}}_{\nu}}_{\delta^{\rho}_{\nu}} X^{\nu}$$

ie $x^{\nu} = \Lambda^{\nu}_{\bar{\mu}} x^{\bar{\mu}} - X^{\nu}$

Thus the translation vector is $-X^{\nu}$ in the unbarred frame, whereas the translation vector for the original transformation was $\Lambda^{\bar{\mu}}_{\nu} X^{\bar{\mu}} = +X^{\bar{\mu}}$ (in the barred frame).

c) Using transforms given:

$$x^{\sigma'} = \hat{\Lambda}^{\sigma'}_{\bar{\mu}} (\Lambda^{\bar{\mu}}_{\nu} (x^{\nu} + X^{\nu}) + Y^{\bar{\mu}})$$

$$= \hat{\Lambda}^{\sigma'}_{\bar{\mu}} (\Lambda^{\bar{\mu}}_{\nu} (x^{\nu} + X^{\nu}) + \underbrace{\Lambda^{\bar{\mu}}_{\nu} \Lambda^{\nu}_{\bar{\rho}}}_{= \delta^{\bar{\mu}}_{\bar{\rho}}} Y^{\bar{\rho}})$$

$$= \underbrace{\hat{\Lambda}^{\sigma'}_{\bar{\mu}} \Lambda^{\bar{\mu}}_{\nu}}_{\hat{\hat{\Lambda}}^{\sigma'}_{\nu}} (x^{\nu} + \underbrace{X^{\nu} + \Lambda^{\nu}_{\bar{\rho}} Y^{\bar{\rho}}}_{Z^{\nu}})$$

$$= \hat{\hat{\Lambda}}^{\sigma'}_{\nu} (x^{\nu} + Z^{\nu})$$ which is another inhomogeneous LT

6. [Eigenvalues & eigenvectors]

a) We will show that, if $T_{\mu\nu} v^{\nu} = a\, \eta_{\mu\sigma} v^{\sigma}$,

then an equivalent equation holds in the barred frame, $T_{\bar{\alpha}\bar{\beta}} v^{\bar{\beta}} = a\, \eta_{\bar{\alpha}\bar{\rho}} v^{\bar{\rho}}$

We will do this by starting with the LHS of the barred equation, transform to unbarred coordinates, then transform back.

Thus

$$T_{\bar{\alpha}\bar{\beta}} v^{\bar{\beta}} = \Lambda^{\mu}_{\bar{\alpha}} \underbrace{\Lambda^{\nu}_{\bar{\beta}} \Lambda^{\bar{\beta}}_{\tau}}_{\delta^{\nu}_{\tau}} T_{\mu\nu} v^{\tau}$$

$$= \Lambda^{\mu}_{\bar{\alpha}} T_{\mu\nu} v^{\nu}$$

$$= \Lambda^{\mu}_{\bar{\alpha}}\, a\, \eta_{\mu\sigma} v^{\sigma} \quad \text{eigen-eqn}$$

$$= a\, \Lambda^{\mu}_{\bar{\alpha}}\, \delta^{\kappa}_{\sigma}\, \eta_{\mu\kappa} v^{\sigma}$$

introduce $\delta^{\kappa}_{\sigma} = \Lambda^{\kappa}_{\bar{\rho}} \Lambda^{\bar{\rho}}_{\sigma}$

$$= a\, \Lambda^{\mu}_{\bar{\alpha}} \Lambda^{\kappa}_{\bar{\rho}} \Lambda^{\bar{\rho}}_{\sigma}\, \eta_{\mu\kappa} v^{\sigma}$$

$$= a\, \eta_{\bar{\alpha}\bar{\rho}} v^{\bar{\rho}}$$

as required.

b) Eigenvalue eq.

$$T_{\mu\nu} V^{\nu} = a \eta_{\mu\sigma} V^{\sigma}$$

Assume $T_{\mu\nu}$ is real valued, but eigenvalues and eigenvectors in general can be complex.

Consider:

$$T_{\mu\nu} (V^{\mu})^{*} V^{\nu} = \left(T_{\mu\nu} V^{\mu} (V^{\nu})^{*} \right)^{*}$$

using $T_{\mu\nu}^{*} = T_{\mu\nu}$. Now use that it is also symmetric $T_{\mu\nu} = T_{\nu\mu}$ then

$$T_{\mu\nu} (V^{\mu})^{*} V^{\nu} = \left(T_{\nu\mu} (V^{\nu})^{*} V^{\mu} \right)^{*}$$

$$\Downarrow$$

$$= \left(T_{\mu\nu} (V^{\mu})^{*} V^{\nu} \right)^{*}$$

relabelling $\mu \leftrightarrow \nu$

Thus $T_{\mu\nu} (V^{\mu})^{*} V^{\nu}$ is real.

Then contract $(V^{\mu})^{*}$ with eigenvalue eq. to get

$$\underbrace{T_{\mu\nu} (V^{\mu})^{*} V^{\nu}}_{\text{real}} = a \eta_{\mu\sigma} V^{\sigma} (V^{\mu})^{*}$$

$$= a \underbrace{V_{\mu} (V^{\mu})^{*}}$$

$V^{0}(V^{0})^{*} - V^{1}(V^{1})^{*} - V^{2}(V^{2})^{*} - V^{3}(V^{3})^{*}$ is manifestly real also

$\therefore$ a is real

older answer which is less clear:

b) If $T_{\mu\nu}$ is symmetric, then

$$T_{\mu\nu} v^\nu = a\, \eta_{\mu\sigma} v^\sigma$$

and

$$T_{\mu\nu} v^\mu = a\, \eta_{\nu\rho} v^\rho$$

Thus, contracting with v on both sides,

$$\underbrace{T_{\mu\nu} v^\mu v^\nu}_{\text{real}} = a\, \eta_{\rho\sigma} v^\rho v^\sigma = a \underbrace{v_\rho v^\rho}_{\neq 0,\ \text{real}}$$

Therefore a must be real, as required.

c) If $T_{\mu\nu} v^\nu = a\, \eta_{\mu\sigma} v^\sigma$, $T_{\mu\nu} u^\mu = b\, \eta_{\nu\rho} u^\rho$,

Contract T with u and v,

ie $$T_{\mu\nu} u^\mu v^\nu = a\, \eta_{\mu\sigma} u^\mu v^\sigma = a\, u_\sigma v^\sigma$$
$$= b\, \eta_{\nu\rho} u^\rho v^\nu = b\, u_\rho v^\rho$$

ie if $u_\mu v^\mu \neq 0$, then $a = b$, which is contrary to assumption. Thus $u_\mu v^\mu = 0$, as required.

d) In local rest frame,

$$U^\mu = (1, \underset{\sim}{0}), \quad \text{so}$$

$$T^{\mu\nu}_{dust} = \begin{pmatrix} n_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

eigenvalues: n_0 (rest particle density) with eigen-4-vector $(1,0,0,0)$

0 with threefold degeneracy, and any basis in space in the rest frame, say $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$.

In a general frame, this gives a way of finding the rest frame and rest particle density, as the eigenvalue and unit proper time vector in an arbitrary frame, and a set of spacelike orthogonal 4-vectors to the proper time direction