Advanced Classical Mechanics: Relativistic Fields

PROBLEM SHEET 3

RELATIVISTIC

ELECTRODYNAMICS

1. [NON-COVARIANCE OF EM UNDER GAL. TRANS.]

a) It is not entirely obvious HOW we would expect $\underset{\sim}{E}$ and $\underset{\sim}{B}$ to change under galilean transformations.

However, for two frames related by a galilean boost in the x-direction, $\bar{t}=t$, $\bar{x}=x-vt$

so for a scalar $f(t,x) = f(\bar{t},\bar{x})$,

$$\partial_{\bar{x}} f = \partial_x f, \quad \partial_t f = \partial_{\bar{t}} f - v\partial_{\bar{x}} f$$

ie, in general, $\partial_t = \partial_{\bar{t}} - v\partial_{\bar{x}}$

So, for example in M2, if this galilean boost would be covariant, then

$$\partial_t \underset{\sim}{B} = \partial_{\bar{t}} \underset{\sim}{B} - v\partial_{\bar{x}} \underset{\sim}{B}$$

Suggesting spatial derivatives of $\underset{\sim}{B}$ in Faraday's law; it is unnatural to have $\bar{B}$ as a function of derivatives of the field (we assume covariance implies LINEAR transformations) so are led to the conclusion that $\underset{\sim}{B}$ itself (and, by equivalent arguments, $\underset{\sim}{E}$), cannot transform covariantly under galilean boosts.

Other arguments about non-galilean covariance can be constructed about the other Maxwell equations.

b) The plane wave solution

$$\underset{\sim}{E} = \underset{\sim}{E}_0 \, e^{i(\underset{\sim}{k}\cdot\underset{\sim}{r} - \omega t)}$$

$$\underset{\sim}{B} = \underset{\sim}{B}_0 \, e^{i(\underset{\sim}{k}\cdot\underset{\sim}{r} - \omega t)}$$

with $(\underset{\sim}{E}_0, \underset{\sim}{B}_0, \underset{\sim}{k})$ a right-handed triad of constant vectors, and $\frac{\omega}{|\underset{\sim}{k}|} = c$, is a solution of Maxwell's equations in the absence of sources, with $c = 1/\sqrt{\epsilon_0 \mu_0}$ fixed

Therefore, whatever it does to the field vectors, a galilean boost by v in the $\underset{\sim}{k}$ direction should give a new plane wave speed of $c - v$ in the boosted frame. If Maxwell's equations apply in the new frame, the plane wave speed again appears to be c, at odds with the galilean prediction.

Special relativity deals with these two problems

- how $\underset{\sim}{E}, \underset{\sim}{B}$ transform (using each other's components)
- how plane waves appear to all inertial observers.

It turns out that basing covariance on LORENTZ TRANSFORMATIONS (assuming all observers see plane EM waves travelling at c) is enough to resolve all these problems

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2. [EINSTEIN'S MOVING MAGNET & CONDUCTOR]

Consider the case of a magnet and SINGLE charge approaching each other (with a slight offset, to reduce extra effects from symmetry)

FRAME 1: Charge stationary, moving magnet

At the position of the charge, B is changing in time as the field lines move with the magnet

So $\dot{\underset{\sim}{B}} \neq 0$ at charge x, and the change is in the xy components of the paper

Thus, by Faraday's law M2, for xy components,

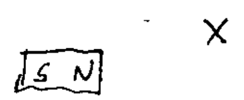
$$0 \neq \dot{\underset{\sim}{B}} = -\nabla \times \underset{\sim}{E}$$

ie we expect there to be a nonzero E_z component, whose xy-variation corresponds to the changing B-field (there will also typically be z-variation in E_x, E_y contributing to this curl).

If we only have a single charge, there is no electrostatic contribution to E_z, ie this field component, and its contribution $\frac{1}{2}\varepsilon_0 E_z^2$ to the electric energy, comes from the time-varying magnetic field.

At the position of the charge, therefore there is typically a nonzero E_z component, so the charge experiences a force in the z-direction, $F_z = q E_z$, directly from the variation in time of the B-field.

In FRAME 2:



FRAME 2

The charge is now moving with velocity $\underset{\sim}{v}$, in the $-x$-direction. Since the $\underset{\sim}{B}$-field it is moving in has x and y components, the Lorentz force due to the motion, $\underset{\sim}{F}_{motion} = q\, \underset{\sim}{v} \times \underset{\sim}{B}$ has a nonzero z-component, from the nonzero y-component of $\underset{\sim}{B}$.

Thus the net effect, an electromotive force on the charge out of the plane, is the same as the other frame, although we have not introduced an extra $\underset{\sim}{E}$ component, or changed the electromagnetic energy density.

This argument extends to all the free charges in the conductor, by superposition of the fields.

Many textbooks (eg Feynman Lectures, vol II, lecture 18) have a discussion of this problem

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3. [EM UNDER C, P, T REVERSAL]

Recall, under
$$C;\ q \to -q$$
$$P:\ \mathbf{r} \to -\mathbf{r} \qquad \text{(spatial inversion in 3D)}$$
$$T:\ t \to -t$$

So, for instance, since $\mathbf{v} = \dot{\mathbf{r}} = d\mathbf{r}/dt$,

$$C: \mathbf{v} \to \mathbf{v}, \quad P: \mathbf{v} \to -\mathbf{v}, \quad T: \mathbf{v} \to -\mathbf{v}.$$

a) ρ is a scalar charge density, so changes sign under C, P, T like charge q:

$$C: \rho \to -\rho, \quad P: \rho \to \rho, \quad T: \rho \to \rho$$

If we consider $\mathbf{J}$ as the product of charge density and velocity, it transforms like $\rho \mathbf{v}$:

$$C: \mathbf{J} \to -\mathbf{J}, \quad P: \mathbf{J} \to -\mathbf{J}, \quad T: \mathbf{J} \to -\mathbf{J}$$

b) Nonrelativistic Lorentz force $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

Since $\mathbf{F} = m\mathbf{a} = m\dot{\mathbf{v}}$,
$$C: \mathbf{F} \to \mathbf{F}$$
$$P: \mathbf{F} \to -\mathbf{F}$$
$$T: \mathbf{F} \to \mathbf{F}$$

If Lorentz force does not change form under C, P, T, then each of $q\mathbf{E}$, $q\mathbf{v}\times\mathbf{B}$ must change like $\mathbf{F}$, ie

$\mathbf{E}$ changes like $q\mathbf{F}$

$\mathbf{B}$ changes like $q\ \mathbf{v}\times\mathbf{F}$

(or simply change in q times change in $\mathbf{v}$ times change in $\mathbf{F}$

ie
$$
\begin{array}{lll}
C: \underset{\sim}{E} \rightarrow -\underset{\sim}{E} & \qquad & C: \underset{\sim}{B} \longrightarrow -\underset{\sim}{B} \\
P: \underset{\sim}{E} \rightarrow -\underset{\sim}{E} & \qquad & P: \underset{\sim}{B} \longrightarrow +\underset{\sim}{B} \quad \text{(pseudo-vector)} \\
T: \underset{\sim}{E} \rightarrow +\underset{\sim}{E} & \qquad & T: \underset{\sim}{B} \longrightarrow -\underset{\sim}{B}
\end{array}
$$

c) (M1) $\nabla \cdot \underset{\sim}{B}$ only has one term, so automatically satisfies transformations

(M2) $\nabla \times \underset{\sim}{E} = -\dot{\underset{\sim}{B}}$

LHS, $\nabla \times \underset{\sim}{E}$ transforms like $\underset{\sim}{r} \times \underset{\sim}{E}$
$\begin{array}{ll} C & - \\ P & + \\ T & + \end{array}$
RHS $-\dot{\underset{\sim}{B}}$ transforms like $t\underset{\sim}{B}$
$\begin{array}{ll} C & - \\ P & + \\ T & + \end{array}$

✓ both sides agree

(M3) $\varepsilon_0 \nabla \cdot \underset{\sim}{E} = \rho$

LHS $\varepsilon_0 \nabla \cdot \underset{\sim}{E}$
$\begin{array}{ll} C & - \\ P & + \\ T & + \end{array}$
RHS ρ
$\begin{array}{ll} C & - \\ P & + \\ T & + \end{array}$

✓ both sides agree

(M4) $\mu_0^{-1} \nabla \times \underset{\sim}{B} - \varepsilon_0 \dot{\underset{\sim}{E}} = \underset{\sim}{J}$

$\mu_0 \nabla \times \underset{\sim}{B}$
$\begin{array}{ll} C & - \\ P & - \\ T & - \end{array}$
$\varepsilon_0 \dot{\underset{\sim}{E}}$
$\begin{array}{ll} C & - \\ P & - \\ T & - \end{array}$
$\underset{\sim}{J}$
$\begin{array}{ll} C & - \\ P & - \\ T & - \end{array}$

✓ all terms agree.

So Maxwell's equations all transform covariantly under C, P & T transformations

d) The relativistic Lorentz law

$$\frac{d}{ds} p^\mu = q F^\mu{}_\nu U^\nu$$

proper time

ie 0-component $\frac{d\mathcal{E}}{ds} = \underbrace{\gamma q \underset{\sim}{E} \cdot \underset{\sim}{v}}_{C+\ P+\ T-}$

C+ P+ T−

since $\mathcal{E}, \gamma$ do not change under C, P, T ✓

1,2,3-component $\frac{d(m\gamma \underset{\sim}{v})}{ds} = \frac{q\gamma}{c} (\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B})$

No change in C, P, T from nonrelativistic version since γ does not change.

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4. [COVARIANCE OF MAXWELL EQUATIONS]

a) In tensorial form, mixed form $F^\mu{}_\nu$ of Faraday tensor transforms like

$$F^{\bar{\alpha}}{}_{\bar{\beta}} = \Lambda^{\bar{\alpha}}{}_\mu F^\mu{}_\nu \Lambda^\nu{}_{\bar{\beta}}$$

As a matrix, $F^\mu{}_\nu$ is written

$$\underset{\approx}{F}_{\text{mixed}} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & +B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

and the transformed form is similar, with $E_j \to \bar{E}_j$, $B_j \to \bar{B}_j$.

Furthermore, since $\Lambda^{\bar{\alpha}}{}_{\mu}$ is represented by the given matrix
$$\underline{\underline{\Lambda}} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and $\Lambda^{\bar{\alpha}}{}_{\mu} \Lambda^{\mu}{}_{\bar{\beta}} = \delta^{\bar{\alpha}}_{\bar{\beta}}$ (ie identity), $\Lambda^{\mu}{}_{\bar{\beta}}$ should be represented by the inverse $\underline{\underline{\Lambda}}^{-1} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Thus $\bar{\underset{\approx}{F}}_{mixed} = \underline{\underline{\Lambda}}\ \underline{\underline{F}}_{mixed}\ \underline{\underline{\Lambda}}^{-1}$ and

straightforward but tedious matrix gives the required forms. (omitted here).

b) If $J^{\mu} = (c\rho, \underset{\sim}{J})$, then
$$J^{\bar{\mu}} = \Lambda^{\bar{\mu}}{}_{\nu}\ J^{\nu}$$

ie
$$\begin{pmatrix} c\bar{\rho} \\ \bar{\underset{\sim}{J}} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\rho \\ \underset{\sim}{J} \end{pmatrix}$$

ie
$$\left.\begin{aligned} c\bar{\rho} &= \gamma\,(c\rho - \beta\,\underset{\sim}{J}) \\ \bar{J}_x &= \gamma\,(J_x - \beta c\,\rho) \\ \bar{J}_y &= J_y \\ \bar{J}_z &= J_z \end{aligned}\right\}$$ giving the required transformations.

c) In tensorial form, $\partial_{\bar{\mu}} = \Lambda^{\nu}_{\bar{\mu}}\, \partial_{\nu}$

ie $\begin{pmatrix} c^{-1}\partial_{\bar{t}} \\ \bar{\nabla} \end{pmatrix} = \underline{\underline{\Lambda}}^{-1} \begin{pmatrix} c^{-1}\partial_t \\ \nabla \end{pmatrix}$

ie $\partial_{\bar{t}} = \gamma(\partial_t + v\partial_x)$, $\partial_{\bar{x}} = \gamma\left(\frac{v}{c^2}\partial_t + \partial_x\right)$

$\partial_{\bar{y}} = \partial_y$, $\partial_{\bar{z}} = \partial_z$, as required.

d) Want to show transformed forms of Maxwell's equations hold, if they hold in unbarred frame

M1: $\bar{\nabla} \cdot \underset{\sim}{B} = \partial_{\bar{x}} \bar{B}_x + \partial_{\bar{y}} \bar{B}_y + \partial_{\bar{z}} \bar{B}_z$

$$= \gamma\left(\frac{v}{c^2}\partial_t + \partial_x\right) B_x$$

$$+ \partial_y\left(\gamma\left(B_y + \frac{v}{c^2}E_z\right)\right) + \partial_z \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

$$= \gamma\left[\underbrace{\partial_x B_x + \partial_y B_y + \partial_z B_z}_{=0 \text{ M1}}\right]$$

$$+ \gamma\frac{v}{c^2}\left(\underbrace{\dot{B}_x + \partial_y E_z - \partial_z E_y}_{=0 \;\; x\text{-component of M2}}\right)$$

$$= 0, \text{ as required.}$$

M2) $\quad \partial_{\bar{t}} \bar{\underset{\sim}{B}} + \bar{\nabla} \times \bar{\underset{\sim}{E}}$

x-component

$$\partial_{\bar{t}} \bar{B}_x + \partial_{\bar{y}} \bar{E}_z - \partial_{\bar{z}} \bar{E}_y$$

$$= \gamma(\partial_t + v\partial_x) B_x + \partial_y \gamma (E_z + vB_y) - \partial_z \gamma (E_y - vB_z)$$

$$= \gamma \underbrace{(\dot{B}_x + \partial_y E_z - \partial_z E_y)}_{=0 \quad M2_x} + v \underbrace{\nabla \cdot B}_{=0 \quad M1}$$

$$= 0$$

y-component

$$\partial_{\bar{t}} \bar{B}_y + \partial_{\bar{z}} \bar{E}_x - \partial_{\bar{x}} \bar{E}_z$$

$$= \gamma(\partial_t + v\partial_x)\gamma(B_y + \frac{v}{c^2} E_z) + \partial_z E_x -$$

$$- \gamma(\frac{v}{c^2}\partial_t + \partial_x)\gamma(E_z + vB_y)$$

$$= \gamma^2 (\dot{B}_y + \cancel{\partial_z E_x} - \partial_x E_z) + (1-\gamma^2)\partial_z E_x$$

$$+ \frac{\gamma^2 v}{c^2}(\cancel{\dot{E}_z} - \cancel{\dot{E}_z}) + \gamma^2 v (\cancel{\partial_x B_y} - \partial_x B_y)$$

$$+ \frac{\gamma^2 v^2}{c^2} \partial_x E_z$$

$$= 0, \text{ as required.}$$

The argument for the z-component of M2 is almost identical to the y-component.

M3) $\bar{\nabla} \cdot \bar{\underset{\sim}{E}} = \partial_{\bar{x}} \bar{E}_x + \partial_{\bar{y}} \bar{E}_y + \partial_{\bar{z}} \bar{E}_z$

$$= \gamma\left(\frac{v}{c^2}\partial_t + \partial_x\right) E_x + \partial_y \gamma (E_y - v B_z) + \partial_z \gamma (E_z + v B_y)$$

$$= \gamma (\partial_x E_x + \partial_y E_y + \partial_z E_z) + \gamma v \left(\frac{\partial_t E_x}{c^2} - \partial_y B_z + \partial_z B_y \right)$$

$$= \gamma \frac{\rho}{\varepsilon_0} + \gamma v \mu_0 \left(\varepsilon_0 \dot{E}_x - \mu_0^{-1} (\partial_y B_z - \partial_z B_y) \right)$$

$$= \frac{\gamma}{\varepsilon_0} \left(\rho - \frac{v}{c^2} J_x \right), \quad \text{as req}^d$$

M4) x - component

$$\mu_0^{-1}(\partial_{\bar{y}} \bar{B}_z - \partial_{\bar{z}} \bar{B}_y) - \varepsilon_0 \, \partial_{\bar{t}} E_x$$

$$= \mu_0^{-1} \partial_y \gamma (B_z - \tfrac{v}{c^2} E_y) - \mu_0^{-1} \partial_z \gamma (B_y + \tfrac{v}{c^2} E_z) - \varepsilon_0 \gamma (\partial_t + v \partial_x) E_x$$

$$= \gamma (\mu_0 (\partial_y B_z - \partial_z B_y) - \varepsilon_0 \dot{E}_x) - v \varepsilon_0 \gamma \; \nabla \cdot \underset{\sim}{E}$$

$$= \gamma (J_x - v \rho) \qquad \text{as req}^d$$

y - component

$$\mu_0^{-1}(\partial_{\bar{z}} \bar{B}_x - \partial_{\bar{x}} \bar{B}_z) - \varepsilon_0 \partial_{\bar{t}} \bar{E}_y$$

$$= \mu_0^{-1} \partial_z B_x - \mu_0^{-1} \gamma\left(\frac{v}{c^2}\partial_t + \partial_x\right) \gamma \left(B_z - \frac{v}{c^2} E_y\right)$$

$$- \varepsilon_0 \gamma (\partial_t + v\partial_x) \gamma (E_y - v B_z)$$

$$= \gamma^2 \left(\mu_0^{-1} (\partial_z B_x - \partial_x B_z) - \varepsilon_0 \dot{E}_y \right)$$

$$+ (1-\gamma^2) \mu_0^{-1} \partial_z B_x - \mu_0^{-1} \gamma^2 \frac{v}{c^2} (\dot{B}_z - \partial_x E_y)$$

$$+ \mu_0^{-1} \frac{v^2}{c^4} \gamma^2 \dot{E}_y - \gamma^2 \varepsilon_0 v (\partial_x E_y - \dot{B}_z)$$

$$+ \varepsilon_0 \gamma^2 v^2 \partial_x B_z$$

$$= \gamma^2 J_y - \gamma^2 \frac{v^2}{c^2} \left((\partial_z B_x - \partial_x B_z)\mu_0^{-1} + \varepsilon_0 \dot{E}_y \right)$$

$$= \gamma^2 (1-\beta^2) J_y$$

$$= J_y, \quad \text{as required.}$$

The z-component of M4 is very similar to the y-component.

5. [$\varepsilon_{\mu\nu\rho\sigma}$ & DUAL FARADAY TENSOR]

a) If $\varepsilon_{\mu\nu\rho\sigma} = -\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} -1 & \text{if } (\mu\nu\rho\sigma) \text{ odd} \\ +1 & \text{if } (\mu\nu\rho\sigma) \text{ even} \\ 0 & \text{otherwise} \end{cases}$

(μ,ν,ρ,σ) even (+)		(μ,ν,ρ,σ) odd (−)	
0123	1032	0132	1023
0231	1320	0213	1302
0312	1203	0321	1230
2013	3021	2031	3012
2130	3210	2103	3201
2301	3102	2310	3120

b) If $\Lambda^{\mu}_{\ \bar{\nu}}$ corresponds to a proper Lorentz transformation (boost or rotation), with corresponding matrix (on covariant components) $\underline{\underline{\Lambda}}$, in the barred frame, if $\varepsilon_{\mu\nu\rho\sigma}$ is a tensor,

$$\varepsilon_{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}} = \varepsilon_{\mu\nu\rho\sigma}\Lambda^{\mu}_{\ \bar{\mu}}\Lambda^{\nu}_{\ \bar{\nu}}\Lambda^{\rho}_{\ \bar{\rho}}\Lambda^{\sigma}_{\ \bar{\sigma}}$$

By showing that $\varepsilon_{\bar{\mu}\bar{\nu}\bar{\rho}\bar{\sigma}}$ has the same components as $\varepsilon_{\mu\nu\rho\sigma}$, we will show it is a tensor,

Consider first $(\bar{\mu},\bar{\nu},\bar{\rho},\bar{\sigma}) = (\bar{0},\bar{1},\bar{2},\bar{3})$ (ie an even permutation)

Then

$$\varepsilon_{\bar{0}\bar{1}\bar{2}\bar{3}} = \varepsilon_{\mu\nu\rho\sigma} \Lambda^{\mu}_{\ \bar{0}} \Lambda^{\nu}_{\ \bar{1}} \Lambda^{\rho}_{\ \bar{2}} \Lambda^{\sigma}_{\ \bar{3}}$$

$$= \det \underline{\underline{\Lambda}} \quad \text{from given information}$$

$$= +1$$

If we try an odd permutation, eg $(\bar{\mu}, \bar{\nu}, \bar{\rho}, \bar{\sigma}) = (\bar{0}, \bar{1}, \bar{3}, \bar{2})$, we have

$$\varepsilon_{\bar{0}\bar{1}\bar{3}\bar{2}} = \varepsilon_{\mu\nu\rho\sigma} \Lambda^{\mu}_{\ \bar{0}} \Lambda^{\nu}_{\ \bar{1}} \Lambda^{\rho}_{\ \bar{3}} \Lambda^{\sigma}_{\ \bar{2}}$$

$$= \varepsilon_{\mu\nu\rho\sigma} \Lambda^{\mu}_{\ \bar{0}} \Lambda^{\nu}_{\ \bar{1}} \Lambda^{\sigma}_{\ \bar{2}} \Lambda^{\rho}_{\ \bar{3}} \quad \text{reordering terms}$$

$$= -\varepsilon_{\mu\nu\sigma\rho} \Lambda^{\mu}_{\ \bar{0}} \Lambda^{\nu}_{\ \bar{1}} \Lambda^{\sigma}_{\ \bar{2}} \Lambda^{\rho}_{\ \bar{3}} \quad \text{odd permutation of } (\mu,\nu,\rho,\sigma)$$

$$= -\det \underline{\underline{\Lambda}} = -1$$

So for $(\bar{0}, \bar{1}, \bar{3}, \bar{2})$, an odd permutation, we get -1.

This argument can clearly be generalized to any odd or even permutation. If we are considering an even permutation, we have an EVEN number of pairwise exchanges of $(\bar{0}, \bar{1}, \bar{2}, \bar{3})$, giving a net $+1 \times \det \underline{\underline{\Lambda}}$. If odd, we have an odd number of exchanges, and net $-1 \times \det \underline{\underline{\Lambda}}$.

It remains to show that if we have a repeated index for ε in the bared frame, we get 0.

Try $(\bar{\mu}, \bar{\nu}, \bar{\rho}, \bar{\sigma}) = (\bar{0}, \bar{0}, \bar{2}, \bar{3})$

$$\varepsilon_{\bar{0}\bar{0}\bar{2}\bar{3}} = \varepsilon_{\mu\nu\rho\sigma} \Lambda^{\mu}_{\bar{0}} \Lambda^{\nu}_{\bar{0}} \Lambda^{\rho}_{\bar{2}} \Lambda^{\sigma}_{\bar{3}}$$

$$= -\varepsilon_{\nu\mu\rho\sigma} \Lambda^{\mu}_{\bar{0}} \Lambda^{\nu}_{\bar{0}} \Lambda^{\rho}_{\bar{2}} \Lambda^{\sigma}_{\bar{3}} \quad \text{odd permutation of } (\mu\nu\rho\sigma)$$

$$= -\varepsilon_{\mu\nu\rho\sigma} \Lambda^{\nu}_{\bar{0}} \Lambda^{\mu}_{\bar{0}} \Lambda^{\rho}_{\bar{2}} \Lambda^{\sigma}_{\bar{3}} \quad \text{relabel dummy indices } \mu \to \nu,\ \nu \to \mu$$

The final equality is clearly $-$ the first one, so the expression must be zero, as required.

We conclude $\varepsilon_{\mu\nu\rho\sigma}$ is a tensor, at least when considering boosts & rotations (it switches sign under P and T transformations, however).

c) We begin by noting that any even permutation of the symbols (μ, ν, ρ, σ), has the same sign of ε as $\varepsilon^{\mu\nu\rho\sigma}$ itself.

eg $\varepsilon^{\sigma\mu\nu\rho} = \varepsilon^{\sigma\nu\rho\mu} = \varepsilon^{\sigma\rho\mu\nu}$

So, for instance,

$$\varepsilon^{\sigma\mu\nu\rho} = \frac{1}{3}\left(\varepsilon^{\sigma\mu\nu\rho} + \varepsilon^{\sigma\nu\rho\mu} + \varepsilon^{\sigma\rho\mu\nu}\right)$$

Thus, for $\sigma = 0, 1, 2, 3$

$$\varepsilon^{\sigma\mu\nu\rho} \partial_\mu F_{\nu\rho} = \frac{1}{3} \partial_\mu F_{\nu\rho} \left(\varepsilon^{\sigma\mu\nu\rho} + \varepsilon^{\sigma\nu\rho\mu} + \varepsilon^{\sigma\rho\mu\nu} \right)$$
$$= \frac{1}{3} \varepsilon^{\sigma\mu\nu\rho} \left(\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} \right)$$

by systematically relabelling dummy indices in the sum. For our usual form of the homogeneous Maxwell equation, this is 0, so

$$\varepsilon^{\sigma\mu\nu\rho} \partial_\mu F_{\nu\rho} = 0, \qquad \sigma = 0, 1, 2, 3$$

as required.

d) If $G_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$, we have

$$G_{01} = \frac{1}{2} \varepsilon_{01\rho\sigma} F^{\rho\sigma}$$
$$= \frac{1}{2} \Big(\underbrace{\varepsilon_{0123}}_{+} F^{23} + \underbrace{\varepsilon_{0132}}_{-} \underbrace{F^{32}}_{-F^{23}} \Big)$$
$$= F^{23}, \qquad \text{etc}$$

Clearly, this argument can be extended to all components of $G_{\mu\nu}$, whose $\mu\nu$th covariant component corresponds to the $\rho\sigma$th contravariant component of F, $F^{\rho\sigma}$, when $\mu\nu\rho\sigma$ is an even permutation of 0123.

Thus

$$G_{01} = F^{23}, \quad G_{02} = F^{31}, \quad G_{03} = F^{12}$$

$$G_{12} = F^{03}, \quad G_{13} = F^{20}, \quad G_{23} = F^{01}$$

and $G_{\mu\nu}$ is antisymmetric, since

$$G_{\mu\nu} + G_{\nu\mu} = \frac{1}{2} F^{\rho\sigma} (\varepsilon_{\mu\nu\rho\sigma} + \underbrace{\varepsilon_{\mu\nu\sigma\rho}}_{-\varepsilon_{\mu\nu\rho\sigma}})$$

$$= 0$$

So

$$G_{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z/c & E_y/c \\ B_y & E_z/c & 0 & -E_x/c \\ B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

as expected

e) From earlier arguments,

$$0 = \varepsilon^{\sigma\mu\nu\rho} \partial_\mu F_{\nu\rho}$$

$$= \partial_\mu (\varepsilon^{\sigma\mu\nu\rho} F_{\nu\rho})$$

Since this equation is tensorially covariant, we can raise/lower ALL indices and get a valid equation, ie $$0 = \partial^\mu (\underbrace{\varepsilon_{\sigma\mu\nu\rho} F^{\nu\rho}}_{G_{\sigma\mu}})$$

$$= \partial^\mu G_{\sigma\mu} \qquad \sigma = 0, 1, 2, 3$$

Since G is antisymmetric, and relabelling $\sigma \to \nu$, we have

$$\partial^\mu G_{\mu\nu} = 0 \quad , \quad \text{as required.}$$

The other Maxwell equation's proof requires some preparation.

First, if $G_{\alpha\beta} = \frac{1}{2} \varepsilon_{\alpha\beta\tau\sigma} F^{\tau\sigma}$

ie $G_{01} = \varepsilon_{0123} F^{23}$

so $F^{23} = - \varepsilon^{0123} G_{01}$

ie

$$F^{\tau\sigma} = -\frac{1}{2} \varepsilon^{\alpha\beta\tau\sigma} G_{\alpha\beta}$$

So, if $\partial_\tau F^{\tau\sigma} = \mu_0 J^\sigma$

Then $$\mu_0 J^\sigma = \partial_\tau G_{\alpha\beta} \times \left(-\frac{1}{2} \varepsilon^{\alpha\beta\tau\sigma}\right)$$
$$= \partial_\tau G_{\alpha\beta} \times \left(\frac{1}{2} \varepsilon^{\sigma\alpha\beta\tau}\right)$$

Since 3012 is an odd permutation of 0123.

Now, contract each side with $\varepsilon_{\sigma\mu\nu\rho}$:

$$\mu_0 \, \varepsilon_{\sigma\mu\nu\rho} J^\sigma = \frac{1}{2} \partial_\tau G_{\alpha\beta} \left(\varepsilon_{\sigma\mu\nu\rho} \varepsilon^{\sigma\alpha\beta\tau} \right)$$

Now, if

$$\begin{aligned}
\varepsilon_{\sigma\mu\nu\rho} \varepsilon^{\sigma\alpha\beta\tau} = & - \delta^\alpha_\mu \delta^\beta_\nu \delta^\tau_\rho - \delta^\alpha_\nu \delta^\beta_\rho \delta^\tau_\mu \\
& - \delta^\alpha_\rho \delta^\beta_\mu \delta^\tau_\nu + \delta^\alpha_\mu \delta^\beta_\rho \delta^\tau_\nu \\
& + \delta^\alpha_\nu \delta^\beta_\mu \delta^\tau_\rho + \delta^\alpha_\rho \delta^\beta_\nu \delta^\tau_\mu
\end{aligned}$$

then

$$\begin{aligned}
\mu_0 \varepsilon_{\sigma\mu\nu\rho} J^\sigma &= \frac{1}{2} \partial_\tau G_{\alpha\beta} \quad (\text{— above —}) \\
&= \frac{1}{2} \big(- \partial_\rho G_{\mu\nu} + \partial_\rho G_{\nu\mu} \\
&\qquad - \partial_\mu G_{\nu\rho} + \partial_\mu G_{\rho\nu} \\
&\qquad - \partial_\nu G_{\rho\mu} + \partial_\nu G_{\mu\rho} \big) \\
&= - \left(\partial_\rho G_{\mu\nu} + \partial_\mu G_{\nu\rho} + \partial_\nu G_{\rho\mu} \right)
\end{aligned}$$

ie $$\partial_\rho G_{\mu\nu} + \partial_\mu G_{\nu\rho} + \partial_\nu G_{\rho\mu} = - \mu_0 \varepsilon_{\sigma\mu\nu\rho} J^\sigma$$

as required

6. [VECTOR POTENTIALS]

a) Assume $\underset{\sim}{A} = \frac{B}{2}(-y, x, 0)$

Therefore $\nabla \times \underset{\sim}{A} = \frac{B}{2}\begin{vmatrix} \hat{\underset{\sim}{x}} & \hat{\underset{\sim}{y}} & \hat{\underset{\sim}{z}} \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix}$

$$= \frac{B}{2}(-\partial_z x, -\partial_z y, \partial_x x + \partial_y y)$$

$$= (0, 0, B), \quad \text{as required}$$

Under a translation of the spatial origin,

$$\underset{\sim}{A} \longrightarrow \bar{\underset{\sim}{A}} = \frac{B}{2}(-y - y_0, x + x_0, 0)$$

if $x \longrightarrow x + x_0$, $y \longrightarrow y + y_0$

Since x_0, y_0 are constants, the curl of $\underset{\sim}{A}$ is not affected, and still

$$\nabla \times \bar{\underset{\sim}{A}} = (0, 0, B).$$

b) $$\underset{\sim}{A} = \frac{B}{2\pi(x^2+y^2)}(-y,\ x,\ 0)$$

Note that
$$\left.\begin{aligned} \partial_x \frac{y}{x^2+y^2} &= -\frac{2xy}{(x^2+y^2)^2} \\ \partial_y \frac{y}{x^2+y^2} &= \frac{(x^2-y^2)}{(x^2+y^2)^2} \end{aligned}\right\} \text{provided } (x,y)\neq(0,0)$$

and similarly for x/x^2+y^2,

$$\nabla \times \underset{\sim}{A} = \Big(\underbrace{0,\ \ 0}_{\text{as before}},\ \partial_x \frac{x}{(x^2+y^2)} + \partial_y \frac{y}{(x^2+y^2)}\Big)$$

$$= \left(0,\ 0,\ \frac{(y^2-x^2)}{(x^2+y^2)^2} + \frac{(x^2-y^2)}{(x^2+y^2)^2}\right)$$

$$= \underset{\sim}{0}, \qquad \text{provided} \qquad (x,y)\neq(0,0).$$

This proves the first part.

Now, we consider the flux of $\underset{\sim}{B} = \nabla \times \underset{\sim}{A}$ through a disk D, of radius R, in the $z=0$ plane centred at the origin.

The perpendicular to the disk is therefore in the z-direction.

By Stokes' theorem, therefore

$$\int_D d^2\underset{\sim}{r} \quad B_z = \oint_{\partial D} \underset{\sim}{A} \cdot d\underset{\sim}{r} \qquad (*)$$

where, on the RHS, ∂D is the boundary circle of D with radius R, and $d\underset{\sim}{r}$ is the line element on this boundary,

Parametrizing the boundary by angle ϕ, so

$$\left.\begin{aligned} x &= R \cos\phi \\ y &= R \sin\phi \end{aligned}\right\} \text{ on } \partial D,$$

$$d\underset{\sim}{r} = \frac{d\underset{\sim}{r}}{d\phi} d\phi = R(-\sin\phi, \cos\phi, 0)\, d\phi$$

On ∂D, $$\underset{\sim}{A} = \frac{B}{2\pi R^2}(-R\sin\phi, R\cos\phi, 0)$$
$$= \frac{B}{2\pi R}(-\sin\phi, \cos\phi, 0)$$

So
$$\oint_{\partial D} \underset{\sim}{A} \cdot d\underset{\sim}{r} = \int_0^{2\pi} d\phi \, \frac{B}{2\pi R}(-\sin\phi, \cos\phi, 0) \cdot R \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}$$

$$= \frac{B}{2\pi} \int_0^{2\pi} d\phi \, \underbrace{(\sin^2\phi + \cos^2\phi)}_{1}$$

$$= B$$

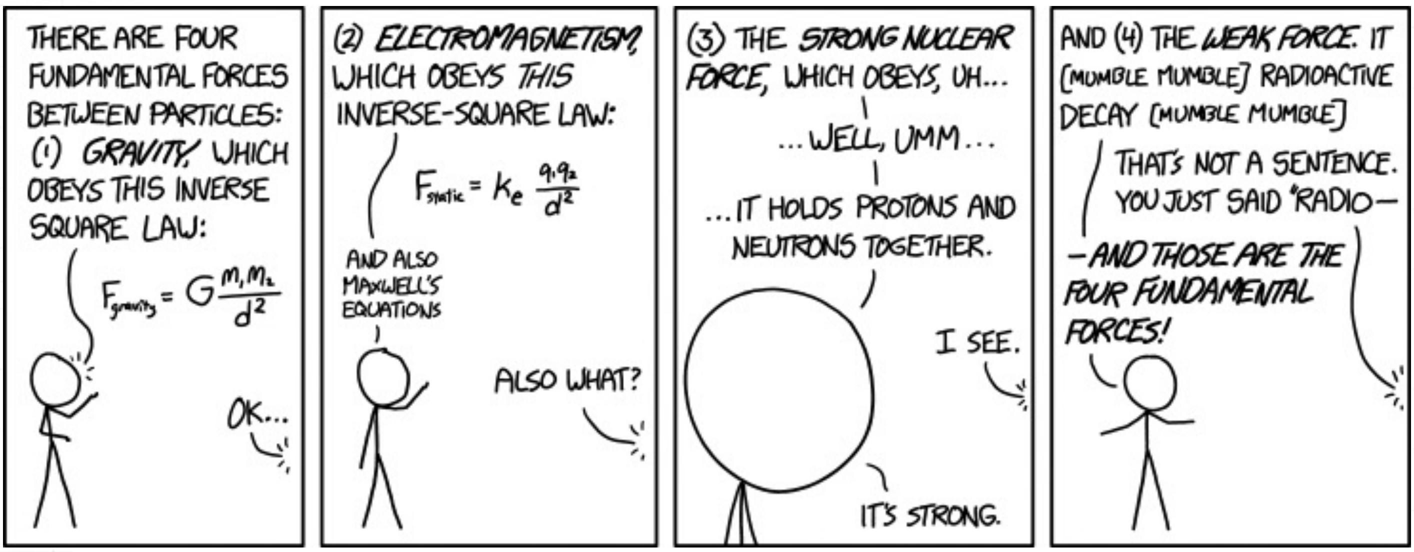
Since the RHS of $(*)$ is nonzero, so must be the left. We have already shown that B_z must be zero unless $(x,y) = (0,0)$.

This now shows, therefore that $B_z = B_0\,\delta(x)\,\delta(y)$,

and $$\underbrace{\int_D d^2r \; B_0\,\delta(x)\,\delta(y)}_{B_0} = B$$

ie

$$\underset{\sim}{B} = B\,\delta(x)\,\delta(y)\,\hat{\underset{\sim}{z}},$$ as required



7. [RIEMANN-SILBERSTEIN VECTOR]

$$\underset{\sim}{F} = \sqrt{\varepsilon_0}\,(\underset{\sim}{E} + ic\,\underset{\sim}{B})$$

a) Lorentz force: $MLT^{-2} = Q\,[E]$
$= Q\,[\text{velocity}]\,[B]$

Thus $[\underset{\sim}{E}]$ & $[c\underset{\sim}{B}]$ in ( ) have the same dimensions, required for consistency.

Now, $[\text{energy density}] = [\varepsilon_0 E^2]$

$$= ML^{-1}T^{-2}$$

and $[\underset{\sim}{F}]$ is the square root of this,

ie $[\underset{\sim}{F}] = M^{1/2}L^{-1/2}T^{-1}$, as reqd.

Since the lagrangian density is a quadratic function of field amplitudes, it is natural to give the field dimensions of square root energy density.

b) $$\nabla\cdot\underset{\sim}{F} = \sqrt{\varepsilon_0}\,(\underbrace{\nabla\cdot\underset{\sim}{E}}_{\rho/\varepsilon_0\ (M3)} + ic\,\underbrace{\nabla\cdot\underset{\sim}{B}}_{0\ (M1)})$$

$$= \rho/\sqrt{\varepsilon_0}$$ as reqd.

$$i c^{-1} \partial_t \underset{\sim}{F} = i c^{-1} \sqrt{\varepsilon_0} \left(\partial_t \underset{\sim}{E} + i c \partial_t \underset{\sim}{B} \right)$$

$$= i c^{-1} \sqrt{\varepsilon_0} \left\{ \varepsilon_0^{-1} \left(\mu_0^{-1} \nabla \times \underset{\sim}{B} - \underset{\sim}{J} \right)^{(M4)} + i c \left(- \nabla \times \underset{\sim}{E} \right)^{(M2)} \right\}$$

$$= i \sqrt{\varepsilon_0} \underbrace{c^{-1} (\varepsilon_0 \mu_0)^{-1}}_{c} \nabla \times \underset{\sim}{B}$$

$$- i \underbrace{c^{-1} \varepsilon_0^{-1/2}}_{\sqrt{\mu_0}} \underset{\sim}{J} + \sqrt{\varepsilon_0} \nabla \times \underset{\sim}{E}$$

$$= \nabla \times \underset{\sim}{F} - i \sqrt{\mu_0} \underset{\sim}{J} \quad \text{as reqd.}$$

c) The second equation can be rewritten

$$i c^{-1} \underset{\text{identity matrix}}{1} \partial_t \underset{\sim}{F} - \underbrace{\nabla \times \underset{\sim}{F}}_{\begin{pmatrix} 0 & -\partial_z & \partial_y \\ \partial_z & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix} \underset{\sim}{F}} = -i \sqrt{\mu_0} \underset{\sim}{J}$$

ie $\quad D \underset{\sim}{F} = -i \sqrt{\mu_0} \underset{\sim}{J}$

with $\quad D = \begin{pmatrix} i c^{-1} \partial_t & \partial_z & -\partial_y \\ -\partial_z & i c^{-1} \partial_t & \partial_x \\ \partial_y & -\partial_x & i c^{-1} \partial_t \end{pmatrix}$

d) $$W = \frac{1}{2}\left(\varepsilon_0 |\underset{\sim}{E}|^2 + \mu_0^{-1} |\underset{\sim}{B}|^2\right)$$

Now, $$\underset{\sim}{F}^* \cdot \underset{\sim}{F} = \varepsilon_0 |\underset{\sim}{E}|^2 + \underbrace{\varepsilon_0 c^2}_{\mu_0^{-1}} |\underset{\sim}{B}|^2$$

so $$W = \frac{\underset{\sim}{F}^* \cdot \underset{\sim}{F}}{2}$$

$$\underset{\sim}{S} = \mu_0^{-1}\, \underset{\sim}{E} \times \underset{\sim}{B}$$

$$= \mu_0^{-1} \left(\frac{\mathrm{Re}\, \underset{\sim}{F}}{\sqrt{\varepsilon_0}}\right) \times \left(\frac{\mathrm{Im}\, \underset{\sim}{F}}{c\sqrt{\varepsilon_0}}\right)$$

$$= \underbrace{\mu_0^{-1} \varepsilon_0^{-1} c^{-1}}_{c} \left(\frac{\underset{\sim}{F} + \underset{\sim}{F}^*}{2}\right) \times \left(\frac{\underset{\sim}{F} - \underset{\sim}{F}^*}{2i}\right)$$

$$= \frac{c}{4i}\left\{ \cancel{\underset{\sim}{F} \times \underset{\sim}{F}} + \underset{\sim}{F}^* \times \underset{\sim}{F} - \underset{\sim}{F} \times \underset{\sim}{F}^* - \cancel{\underset{\sim}{F}^* \times \underset{\sim}{F}^*} \right\}$$

ie

$$\underset{\sim}{S} = c\, (\mathrm{Re}\, \underset{\sim}{F}) \times (\mathrm{Im}\, \underset{\sim}{F})$$

$$= \frac{c}{2i}\, \underset{\sim}{F}^* \times \underset{\sim}{F}$$

automatically imaginary

e) $a = \frac{E^2}{c^2} - B^2$, $b = -\frac{2}{c} \underset{\sim}{E} \cdot \underset{\sim}{B}$

Now, $\underset{\sim}{F} \cdot \underset{\sim}{F} = \varepsilon_0 (\underset{\sim}{E} + ic \underset{\sim}{B}) \cdot (\underset{\sim}{E} + ic \underset{\sim}{B})$

no *

$$= \varepsilon_0 \{ E^2 - c^2 B^2 + 2ic \underset{\sim}{E} \cdot \underset{\sim}{B} \}$$

$$= \mu_0^{-1} \left\{ \frac{E^2}{c^2} - B^2 + \frac{2i}{c} \underset{\sim}{E} \cdot \underset{\sim}{B} \right\}$$

ie $a = \mu_0^{-1} \, \mathrm{Re}(\underset{\sim}{F} \cdot \underset{\sim}{F})$

$b = -\mu_0^{-1} \, \mathrm{Im}(\underset{\sim}{F} \cdot \underset{\sim}{F})$

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8. [NOETHER CURRENT FOR EM GAUGE SYM]

a) As shown above,

$$F_{\mu\nu} \longrightarrow (\partial_\mu A_\nu - \partial_\mu \partial_\nu \chi) - (\partial_\nu A_\mu - \partial_\nu \partial_\mu \chi)$$

$$= F_{\mu\nu} \underbrace{- \partial_\mu \partial_\nu \chi + \partial_\mu \partial_\nu \chi}_{0}$$

Since the Faraday tensor is unchanged, so is the lagrange density.

b) Generalize the gauge transformation to

$$A_\mu \longrightarrow \bar{A}_\mu = A_\mu - \tau \partial_\mu \chi$$

for χ a fixed Lorentz scalar field, τ independent of space time position.

Then corresponding $\Phi_\mu = \frac{\partial \bar{A}_\mu}{\partial \tau}\Big|_{\tau=0}$

$$= - \partial_\mu \chi .$$

c) Noether current

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \Phi_\nu$$

$$= \underbrace{-\frac{1}{\mu_0} F^{\mu\nu}}_{\text{from lectures}} \times (-\partial_\nu \chi)$$

$$= \frac{1}{\mu_0} F^{\mu\nu} \partial_\nu \chi .$$

Divergence: $\partial_\mu j^\mu = \frac{1}{\mu_0} \partial_\nu \chi \, \partial_\mu F^{\mu\nu} + F^{\mu\nu} \partial_\mu \partial_\nu \chi$

The first term is zero as we have the homogeneous, source-free EM lagrangian (so $\partial_\mu F^{\mu\nu} = 0$ from Maxwell's equations). The second term is zero as it is a contraction of a symmetric tensor, $\partial_\mu \partial_\nu \chi$, and the antisymmetric $F^{\mu\nu}$.

9. [EM ACTION, GAUGE SYMMETRY]

a) If $\mathcal{L}_{EM,int} = -\frac{1}{4\mu_0} F_{\mu\nu}F^{\mu\nu} - A_\sigma J^\sigma$,

under the transformation $A_\mu \longrightarrow \bar{A}_\mu = A_\mu - \partial_\mu \chi$,

$$\mathcal{L}_{EM,int} \longrightarrow \bar{\mathcal{L}}_{EM,int} = -\frac{1}{4\mu_0} F_{\mu\nu}F^{\mu\nu} - A_\sigma J^\sigma + J^\sigma \partial_\sigma \chi$$

$F^{\mu\nu}$ invariant

$$= \mathcal{L}_{EM,int} + J^\sigma \partial_\sigma \chi$$

that is, the lagrangian density is gauge-dep.

b) If we consider the gauge transformation $A_\mu \rightarrow A_\mu - \partial_\mu \chi$ as a variation of A_μ, then we can assume that $\partial_\mu \chi = 0$ on the spacetime boundary $\partial\Omega$

Under the gauge transformation, from the previous part,

$$\mathcal{S}[A_\mu] \longrightarrow \mathcal{S}[A_\mu - \partial_\mu \chi]$$
$$\parallel$$
$$\mathcal{S}[A_\mu] + \int_\Omega d^4x \; J^\sigma \partial_\sigma \chi$$

Thus $S[\bar{A}_\mu] = S[A_\mu]$ if

$$\int_\Omega d^4x \; J^\sigma \partial_\sigma \chi = 0$$

(product rule / integration by parts)

$$= -\int_\Omega d^4 x \; \chi \, \partial_\sigma J^\sigma + \underbrace{\int_\Omega d^4x \; \partial_\sigma (\chi J^\sigma)}_{\text{div theorem}}$$

$$\int_{\partial\Omega} d^3S \; \chi \, n_\sigma J^\sigma$$

The second term becomes an integral over $\partial\Omega$ of the flux of χJ^σ. Since we have assumed $\partial_\mu \chi |_{\partial\Omega} = 0$, we have χ constant on $\partial\Omega$, say $\chi|_{\partial\Omega} = \chi_0$

Consider now gauge transformations with $\chi_0 = 0$. Since χ is otherwise arbitrary inside Ω, the only way $\int d^4x \, \chi \partial_\sigma J^\sigma$ vanishes for all χ with $\chi_0 = 0$ is if $\partial_\sigma J^\sigma = 0$ at all points inside Ω, that is, if the EM source current satisfies charge continuity.

If this is the case, then the second term above must vanish, $\int_{\partial\Omega} d^3S \; n_\sigma J^\sigma = 0$, as any charges entering the spacetime volume Ω must leave it.

Thus, the action is invariant to the

gauge transformation if and only if $\partial_\mu J^\mu = 0$ everywhere in spacetime.

Thus we conclude
gauge symmetry $\longleftrightarrow$ charge conservation.

c) The interaction term in $\mathcal{L}_{EM,int}$ is independent of $\partial_\mu A_\nu$, so, as in Q 13 c),

$$j^\mu = \frac{\partial \mathcal{L}_{EM,int}}{\partial(\partial_\mu A_\nu)} (-\partial_\nu \chi)$$

$$= \frac{1}{\mu_0} F^{\mu\nu} \partial_\nu \chi \text{, as above}$$

Divergence:

$$\partial_\mu j^\mu = \frac{1}{\mu_0} \partial_\mu F^{\mu\nu} \partial_\nu \chi + \cancel{\frac{1}{\mu_0} F^{\mu\nu} \partial_\mu \partial_\nu \chi}^{\text{as previous Q}}$$

$$= J^\nu \partial_\nu \chi \text{ by Maxwell eq}$$

Since $J^\nu \neq 0$ typically, j^μ is not zero for an EM field with sources.

This does not contradict Noether's theorem, since $\bar{\mathcal{L}} \neq \mathcal{L}$ here (this is an assumption to prove Noether's theorem), and the action must be invoked directly, as in (b) above, to find the conservation law corresponding to gauge symmetry

* Additional notes on Q9

Solution above is a bit convoluted because in earlier interations of the course Noether's theorem was only discussed for invariant Lagrangians. Here we have under a gauge transform:

$$\delta \mathcal{L} = \bar{\mathcal{L}}_{EM,int} - \mathcal{L}_{EM,int} = J^\sigma \partial_\sigma \chi$$

As it stands $\delta \mathcal{L}$ is not in the form of a 4-div field required for a symmetry. But:

$$J^\sigma \partial_\sigma \chi = \partial_\sigma (J^\sigma \chi) - \chi \partial_\sigma J^\sigma$$

If we assume $\partial_\sigma J^\sigma = 0$, so J^σ is a conserved charge, then $\delta \mathcal{L}$ is a genuine symmetry

$$\delta \mathcal{L} = \partial_\mu \mathcal{J}^\mu = \partial_\mu (J^\mu \chi)$$

we identify our "curly J" from this as:

$$\mathcal{J}^\mu = J^\mu \chi$$

From video 21 we know that Noether's theorem is then:

$$j^\mu = \frac{\partial \mathcal{L}_{EM,int}}{\partial \partial_\mu A_\nu} (-\partial_\nu \chi) - \mathcal{J}^\mu$$

$$= \frac{1}{\mu_0} F^{\mu\nu} \partial_\nu \chi - J^\mu \chi$$

$$= \partial_\nu \left(\frac{1}{\mu_0} F^{\mu\nu} \chi \right) - \frac{\chi}{\mu_0} \partial_\nu F^{\mu\nu} - J^\mu \chi$$

$$= \partial_\nu \left(\frac{1}{\mu_0} F^{\mu\nu} \chi \right) + \frac{\chi}{\mu_0} \partial_\nu F^{\nu\mu} - J^\mu \chi$$

$$= \partial_\nu \left(\frac{1}{\mu_0} F^{\mu\nu} \chi \right) + \cancel{J^\mu \chi} - \cancel{J^\mu \chi}$$

As required by Noether's theorem j^μ is conserved since:

$$\partial_\mu j^\mu = \partial_\mu \partial_\nu \left(\frac{1}{\mu_0} F^{\mu\nu} \chi \right) = 0$$

due to the antisymmetry of $F^{\mu\nu}$.

But j^μ is Gauge dependent!

Gauge transformations do not represent physical symmetry operations, but rather probe the redun-dancies with a model. So j^μ does not add any new physical quantities that are conserved. For example if $\chi = \text{const}$ over space-time (a global or rigid gauge transform) then

$$j^\mu = -\chi \frac{1}{\mu_0} \partial_\nu F^{\nu\mu} = -\chi J^\mu$$

So $j^\mu \propto J^\mu$, the conserved charge we assumed!

10. [LIÉNARD - WIECHERT POTENTIALS]

a) If $\rho = q\, \delta(\underset{\sim}{r} - \underset{\sim}{R}(t))$

$\underset{\sim}{J} = q\, \dot{\underset{\sim}{R}}(t)\, \delta(\underset{\sim}{r} - \underset{\sim}{R}(t))$,

with

$\delta(\underset{\sim}{r}) = \delta(x)\,\delta(y)\,\delta(z)$

Treating δ like a usual function for derivatives, from the chain rule,

$$\dot{\rho} = -q\, \dot{\underset{\sim}{R}}(t) \cdot \nabla \delta(\underset{\sim}{r} - \underset{\sim}{R}(t))$$

$$\nabla \cdot \underset{\sim}{J} = q\, \underbrace{\dot{\underset{\sim}{R}}(t)}_{\text{ind of } x,\, y,\, z} \cdot \nabla \delta(\underset{\sim}{r} - \underset{\sim}{R}(t))$$

Thus $\dot{\rho} + \nabla \cdot \underset{\sim}{J} = 0$ for these sources.

Now, if $R(t) =$ constant (say at origin, $\underset{\sim}{R} = 0$), then $\dot{\underset{\sim}{R}} = 0$

From $\dot{\underset{\sim}{R}} = 0$, $A(t, \underset{\sim}{r}) = 0$ ~~in~~ from formula. This is expected for an electrostatic field.

For V, there is some $t' < t$ for which $t' = t - |\underset{\sim}{r}|/c$

($t - t'$ is the time taken for EM radiation to reach $\underset{\sim}{r}$ from $\underset{\sim}{0}$)

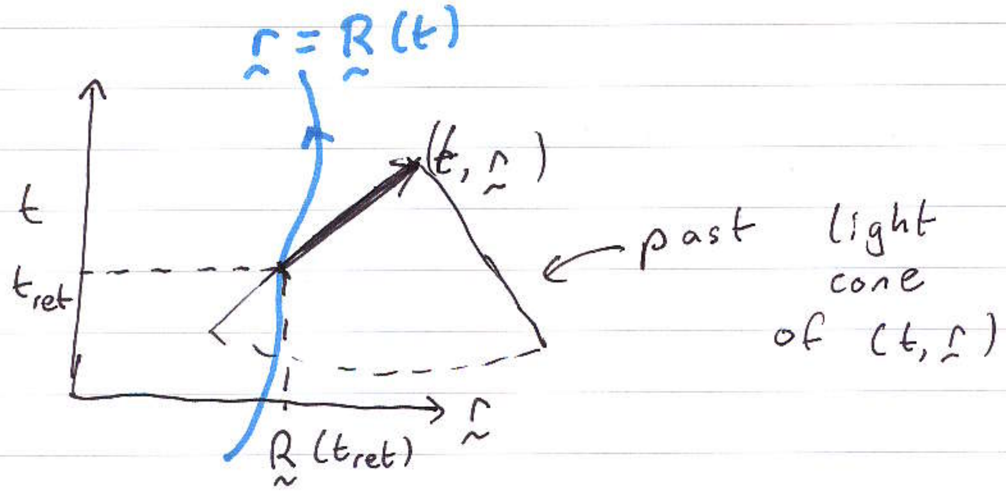
In this case,

$$V = \frac{q}{4\pi\varepsilon_0} \int dt' \, \frac{\delta(t' - t + |\underset{\sim}{r}|/c)}{|\underset{\sim}{r}|}$$

contributes when $t' = t - |\underset{\sim}{r}|/c$ ie at $t' = t_{ret}$

$$= \frac{q}{4\pi\varepsilon_0 |\underset{\sim}{r}|}$$

which has been seen previously as the potential of a point charge q at the origin.



The formula for V implies that the spacetime event on the worldline of the charge which contributes to $(t, \underset{\sim}{r})$ is the (unique) spacetime point $(t_{ret}, \underset{\sim}{R}(t_{ret}))$ <u>on</u> the past lightcone of $(t, \underset{\sim}{r})$

(This point must be unique as $\dot{\underset{\sim}{R}} < c$)

Otherwise the formula for V looks like the usual formula for the Coulomb potential (but shows the field propagates at c)

Since, for the source, $\underset{\sim}{J} = \dot{\underset{\sim}{R}}\, \rho$, and $\hat{A}$ is related to $\underset{\sim}{J}$ as V is to ρ, the integral expression for $\hat{A}$ is very similar to that for V, but with the integrand weighted by $\dot{\underset{\sim}{R}}(t')$

Thus the vector potential comes from the same spacetime point at t_{ret}, and is the scalar potential weighted by the velocity of the charge.

b) We recall from the properties of δ-functions that

$$\int_{-\infty}^{\infty} g(t')\, \delta(f(t'))\, dt' = \frac{g(t_0)}{|f'(t_0)|} \qquad f' = \frac{df}{dt}$$

if $f(t_0) = 0$ (and t_0 is the only such zero)

[If in difficulty, consider an expansion of $f(t')$ around t_0]

Here, $f(t') = t' - t + |\underset{\sim}{r} - \underset{\sim}{R}(t')|/c$

and t_0 is t_{ret}

Thus $f' = 1 - \dot{\underset{\sim}{R}} \cdot \frac{(\underset{\sim}{r} - \underset{\sim}{R})}{|\underset{\sim}{r} - \underset{\sim}{R}|}/c$

$= 1 - \frac{\dot{\underset{\sim}{R}}}{c} \cdot \underset{\sim}{n}$

and $f' > 0$ always since $|\underset{\sim}{n}| = 1$, $\frac{\dot{\underset{\sim}{R}}}{c} < 1$

For V, $\quad g(t') = \dfrac{1}{|\underset{\sim}{r} - \underset{\sim}{R}(t')|}$

so, integrating the expression for V, giving a δ- contribution at t_{ret},

$$V = \frac{q}{4\pi\varepsilon_0} \frac{1}{|\underset{\sim}{r} - \underset{\sim}{R}(t_{ret})|} \frac{1}{1 - \underset{\sim}{n} \cdot \dot{\underset{\sim}{R}}(t_{ret})/c}$$

as required

The argument for $\underset{\sim}{A}$ is very similar, with g now given by $\dfrac{\dot{\underset{\sim}{R}}(t')}{|\underset{\sim}{r} - \underset{\sim}{R}(t')|}$

Thus

$$\underset{\sim}{A} = \frac{\mu_0 q}{4\pi} \frac{\dot{\underset{\sim}{R}}(t_{ret})}{|\underset{\sim}{r} - \underset{\sim}{R}(t_{ret})|} \frac{1}{1 - \underset{\sim}{n} \cdot \dot{\underset{\sim}{R}}(t_{ret})/c}$$

again as req'd