Numerical results from large deflection beam and frame problems analyzed by means of elliptic Integrals

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NUMERICAL RESULTS FROM LARGE DEFLECTION BEAM AND FRAME PROBLEMS ANALYSED BY MEANS OF ELLIPTIC INTEGRALS

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SUMMARY

Numerical evaluations of elliptic integral solutions of some large deflection beam and frame problems are presented. The values are given in tabular form with up to six significant figures. The numerical technique used for evaluating the elliptic integrals is described.

INTRODUCTION

When developing finite element methods, there is always a need for checking the results against an exact solution. In the case of finite element methods for geometrically nonlinear beam, frame and shell structures, elliptic integral solutions of large deflection beam and frame problems offer such exact solutions. A number of papers presenting formulae for such problems have been published during the last decades. The interested reader can find much of the work in this field summarized in Frisch-Fay.[1]

One big problem for the user is, however, that the numerical results are usually presented in diagram form. The accuracy obtained by measuring in a diagram is not sufficient for the present purpose. In those few cases for which results are presented in tables, these are often given with too small accuracy, due to the fact that at the time the calculations were performed no digital computers were available. Thus, iterations were performed manually by trial and error procedures and the elliptic integrals were evaluated from tables using interpolation.

The main purpose of this paper is to present highly accurate results in tabular form for some large deflection beam and frame problems analysed by means of elliptic integrals. The structural analysis is not presented here but can be found in Mattiasson[2] or in the original papers.[1,5–8] However, the numerical technique for the special problem of evaluating the elliptic integrals is described here.

Basic assumptions

For the examples presented in this paper the following assumptions are made:

1. The material is linearly elastic.
2. The axial and the shear deformations are ignored.
3. The members are initially straight and have uniform cross-sections.
4. The plane of loading coincides with the plane of bending.

ELLIPTIC INTEGRALS

A detailed treatment on the subject of elliptic integrals is found, for instance, in Bowman.[3] For the present purpose only the first and second kinds are of interest.

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Definitions

The incomplete elliptic integral of the first kind is defined as

$$F(p, \phi) = \int_0^{\phi} \frac{d\Phi}{\sqrt{(1-p^2 \sin^2 \Phi)}} \qquad 0<p<1 \tag{1}$$

where ϕ is the amplitude of $F(p, \phi)$ and p is its modulus.

The complete elliptic integral of the first kind is defined by

$$K(p) = F(p, \pi/2) = \int_0^{\pi/2} \frac{d\Phi}{\sqrt{(1-p^2 \sin^2 \Phi)}} \qquad 0<p<1 \tag{2}$$

The incomplete elliptic integral of the second kind is given by

$$E(p, \phi) = \int_0^{\phi} \sqrt{(1-p^2 \sin^2 \Phi)}\, d\Phi \qquad 0<p<1 \tag{3}$$

and the complete elliptic integral of the second kind is defined by

$$E(p) = E(p, \pi/2) = \int_0^{\pi/2} \sqrt{(1-p^2 \sin^2 \Phi)}\, d\Phi \qquad 0<p<1 \tag{4}$$

Numerical evaluation of the elliptic integrals

The following method for evaluating elliptic integrals is described in detail by King.[4] It is well adapted for machine computation and has shown excellent convergence properties for all possible values of the amplitude ϕ and the modulus p. With given ϕ and p the method is accomplished as follows:

According to Landen's scale of increasing amplitudes values $\phi_1, \phi_2, \ldots, \phi_i, \phi_{i+1}$ are calculated according to the following recurrence formula starting with $\phi_0 = \phi$:

$$\tan(\phi_{i+1} - \phi_i) = (b_i/a_i) \tan \phi_i \qquad \phi_{i+1} > \phi_i \tag{5}$$

where a_i and b_i are obtained from the so-called scale of arithmetic-geometrical means:

$$\begin{array}{lll}
a_0 = 1 & b_0 = \sqrt{(1-p^2)} & c_0 = p \\
a_1 = \frac{1}{2}(a_0 + b_0) & b_1 = \sqrt{(a_0 b_0)} & c_1 = \frac{1}{2}(a_0 - b_0) \\
\ldots & \ldots & \ldots \\
a_{i+1} = \frac{1}{2}(a_i + b_i) & b_{i+1} = \sqrt{(a_i b_i)} & c_{i+1} = \frac{1}{2}(a_i - b_i) \\
\ldots & \ldots & \ldots \\
a_n & b_n & c_n
\end{array}$$

The a's and b's tend to the same limit and the c's tend to zero with extraordinary rapidity, even when a_0 and b_0 are numbers of very different magnitudes.

The calculations are repeated until c_n is equal to zero to a desired degree of accuracy. The values of the elliptic integrals are then obtained as

$$F(p, \phi) = \phi_n/(a_n 2^n) \tag{6}$$

$$K(p) = \pi/(2a^n) \tag{7}$$

$$E(p) = K(p)\left[1 - \frac{1}{2} \sum_{i=0}^{n} 2^i c_i^2\right] \tag{8}$$

$$E(p, \phi) = \frac{E(p)}{K(p)} F(p, \phi) + \sum_{i=1}^{n} c_i \sin \phi_i \tag{9}$$

Some notes on the numerical calculations

The numerical calculations have been performed on an IBM 360/65 computer using double precision (approximately 15 decimal digits precision). When evaluating the elliptic integrals according to the recurrence procedure described above, the break condition for the calculations has been $c_n < 10^{-10}$.

All problems analysed involve iterations; in some cases one iteration loop is contained within another. These iterations were all performed according to the secant method. The break condition for the iterations has been chosen to be $|\Delta X/X| < 10^{-8}$, where X is the value of a certain variable and ΔX is the iterative additional contribution to the same variable.

NUMERICAL RESULTS

Introductory remarks

In a previous study by the author[2] the problems summarized schematically in Figure 1 were analysed. Due to the space limitations of the present paper not all the numerical data given in Reference 2 can be reproduced herein. Numerical results for problems a (for $n = 0$), e, f, g and h have been selected to be presented in the following sections.

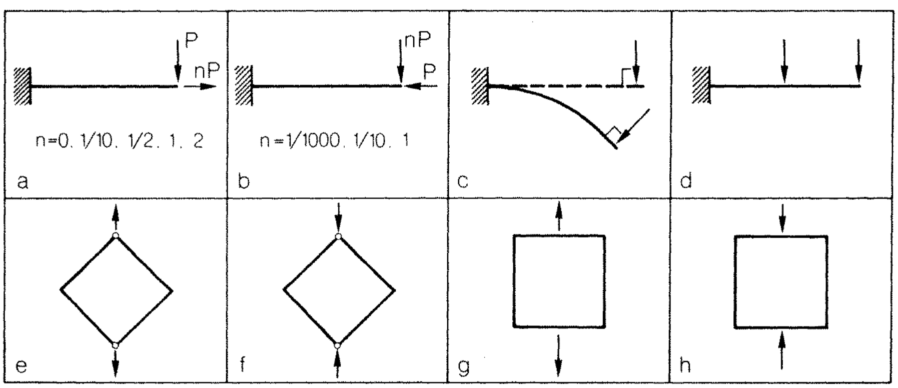

Figure 1. Problems analysed in Reference 2

Cantilever beam with a transversal point load

The problem of the large deflection of a cantilever beam with a transversally acting point load at the free end has been analysed by Barten,[5] Bisshopp and Drucker[6] and Frisch-Fay.[1] It has probably been the 'test problem' most commonly used in examination of finite element procedures for geometrically nonlinear beam and frame analysis. (See Figures 2 and 3 and Table I.)

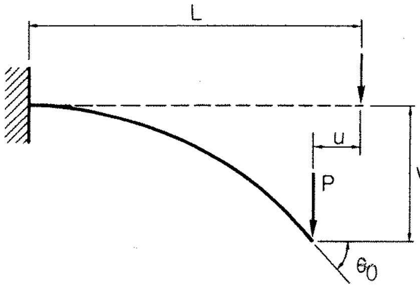

Figure 2. Horizontal cantilever with a vertical point load at the free end

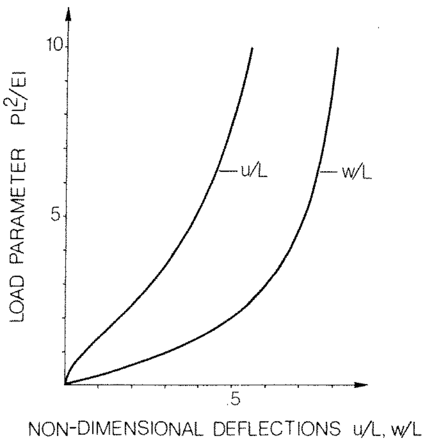

Figure 3. Load–deflection curves for the cantilever beam in Figure 2

Table I. Deflections at the free end for the cantilever beam in Figure 2

PL^2/EI	w/L	u/L	θ_0	PL^2/EI	w/L	u/L	θ_0
0.2	0.06636	0.00265	0.09964	4.0	0.66996	0.32894	1.12124
0.4	0.13098	0.01035	0.19716	4.5	0.69397	0.35999	1.17228
0.6	0.19235	0.02249	0.29074	5.0	0.71379	0.38763	1.21537
0.8	0.24945	0.03817	0.37906	5.5	0.73042	0.41236	1.25211
1.0	0.30172	0.05643	0.46135	6.0	0.74457	0.43459	1.28370
1.2	0.34901	0.07640	0.53730	6.5	0.75676	0.45468	1.31107
1.4	0.39147	0.09732	0.60698	7.0	0.76737	0.47293	1.33496
1.6	0.42941	0.11860	0.67065	7.5	0.77670	0.48957	1.35593
1.8	0.46326	0.13981	0.72876	8.0	0.78498	0.50483	1.37443
2.0	0.49346	0.16064	0.78175	8.5	0.79239	0.51886	1.39084
2.5	0.55566	0.20996	0.89500	9.0	0.79906	0.53182	1.40547
3.0	0.60325	0.25442	0.98602	9.5	0.80510	0.54383	1.41854
3.5	0.64039	0.29394	1.06012	10.0	0.81061	0.55500	1.43029

Pinned-fixed square diamond frame

The problem of the large deflection of diamond-shaped frames has been analysed by Jenkins, Seitz and Przemieniecki[7] by means of elliptic integrals. (See Figures 4 and 5 and Tables II and III.)

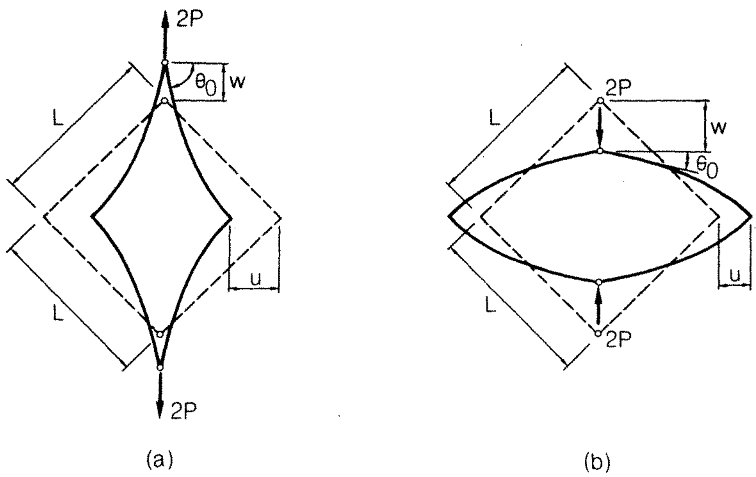

Figure 4. Pinned-fixed square diamond frame loaded in (a) tension and (b) compression

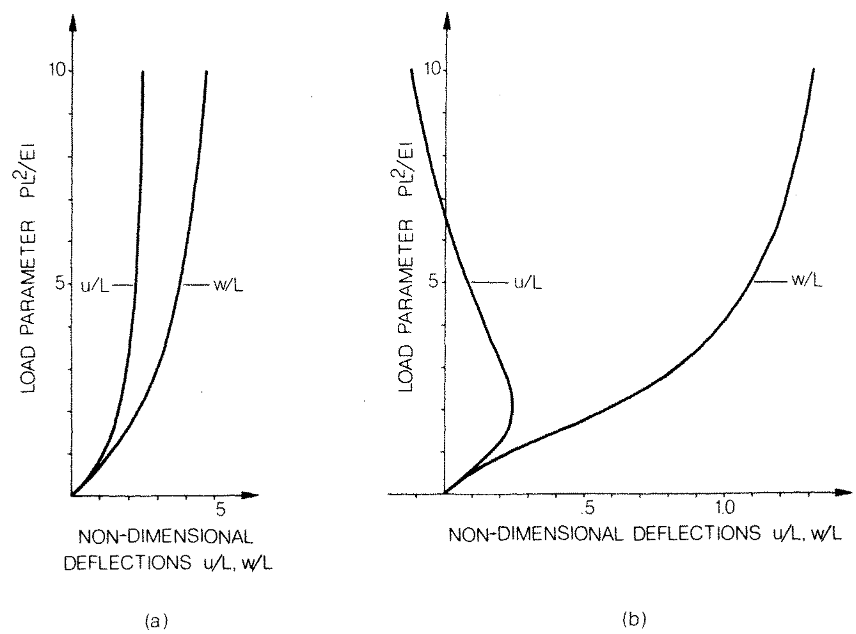

Figure 5. Load–deflection curves for a square diamond frame loaded in (a) tension according to Figure 4(a), and (b) compression according to Figure 4(b).

Table II. Deflections of a square diamond frame loaded in tension according to Figure 4(a)

PL^2/EI	w/L	u/L	θ_0	PL^2/EI	w/L	u/L	θ_0
0.2	0.03065	0.03233	0.85206	4.0	0.20839	0.33940	1.35835
0.4	0.05648	0.06248	0.91116	4.5	0.21436	0.35742	1.38202
0.6	0.07829	0.09036	0.96354	5.0	0.21931	0.37322	1.40209
0.8	0.09677	0.11603	1.01005	5.5	0.22348	0.38720	1.41926
1.0	0.11252	0.13960	1.05144	6.0	0.22703	0.39966	1.43409
1.2	0.12601	0.16124	1.08840	6.5	0.23011	0.41086	1.44698
1.4	0.13765	0.18112	1.12154	7.0	0.23279	0.42097	1.45825
1.6	0.14774	0.19941	1.15134	7.5	0.23515	0.43016	1.46817
1.8	0.15655	0.21627	1.17825	8.0	0.23726	0.43855	1.47694
2.0	0.16429	0.23184	1.20263	8.5	0.23914	0.44625	1.48473
2.5	0.17997	0.26594	1.25448	9.0	0.24084	0.45335	1.49168
3.0	0.19183	0.29447	1.29613	9.5	0.24239	0.45992	1.49791
3.5	0.20104	0.31865	1.33015	10.0	0.24380	0.46601	1.50351

Table III. Deflections of a square diamond frame loaded in compression according to Figure 4(b)

PL^2/EI	w/L	u/L	θ_0	PL^2/EI	w/L	u/L	θ_0
0.2	0.03630	0.03419	0.71042	4.0	0.98775	0.14019	-0.81646
0.4	0.07901	0.06958	0.62668	4.5	1.04356	0.10788	-0.90650
0.6	0.12862	0.10512	0.53425	5.0	1.08927	0.07735	-0.98149
0.8	0.18507	0.13932	0.43403	5.5	1.12730	0.04893	-1.04470
1.0	0.24754	0.17046	0.32789	6.0	1.15940	0.02265	-1.09854
1.2	0.31440	0.19689	0.21852	6.5	1.18684	-0.00158	-1.14482
1.4	0.38343	0.21743	0.10899	7.0	1.21058	-0.02393	-1.18493
1.6	0.45233	0.23161	0.00216	7.5	1.23134	-0.04455	-1.21994
1.8	0.51910	0.23964	-0.09976	8.0	1.24966	-0.06362	-1.25070
2.0	0.58236	0.24224	-0.19539	8.5	1.26596	-0.08129	-1.27787
2.5	0.72091	0.23108	-0.40404	9.0	1.28058	-0.09770	-1.30199
3.0	0.83140	0.20519	-0.57200	9.5	1.29378	-0.11298	-1.32351
3.5	0.91852	0.17335	-0.70695	10.0	1.30578	-0.12724	-1.34277

Square frame loaded at the midpoints of a pair of opposite sides

The elliptic integral solution for the deflections and bending moments in a square frame loaded at the midpoints of a pair of opposite sides is given by Kerr.[8] (See Figures 6 and 7 and Tables IV and V.)

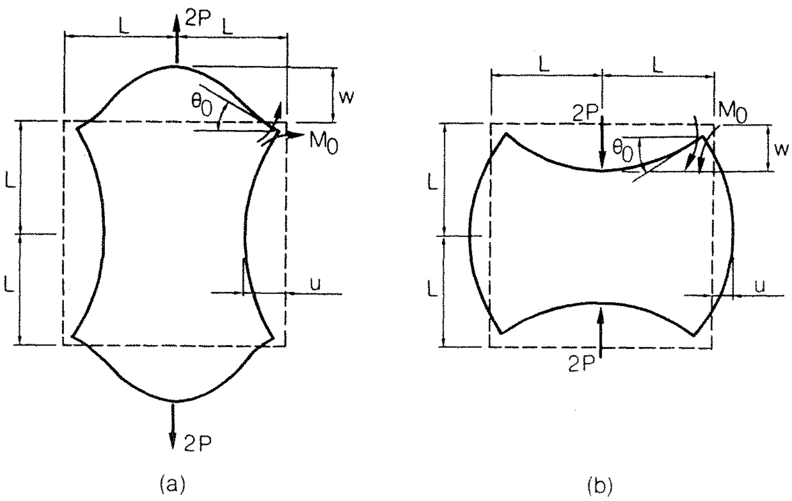

Figure 6. Square frame loaded in (a) tension and (b) compression

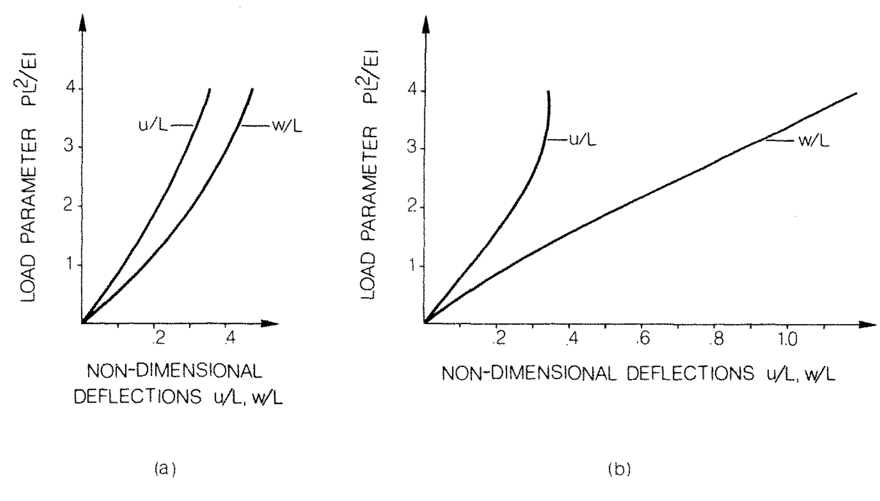

Figure 7. Deflections for a square frame loaded in (a) tension according to Figure 6(a), and (b) compression according to Figure 6(b)

Table IV. Deflections and corner moments for a square frame loaded in tension according to Figure 6(a). Note: $\mu_0 = M_0/PL$

PL^2/EI	w/L	u/L	θ_0	μ_0
0.2	0.04043	0.02473	0.04835	0.25764
0.4	0.07843	0.04887	0.09346	0.26399
0.6	0.11410	0.07233	0.13548	0.26917
0.8	0.14755	0.09505	0.17454	0.27332
1.0	0.17889	0.11699	0.21082	0.27655
1.2	0.20825	0.13813	0.24449	0.27899
1.4	0.23575	0.15845	0.27573	0.28073
1.6	0.26152	0.17795	0.30471	0.28188
1.8	0.28567	0.19664	0.33161	0.28252
2.0	0.30833	0.21453	0.35658	0.28272
2.5	0.35917	0.25595	0.41156	0.28171
3.0	0.40287	0.29298	0.45752	0.27916
3.5	0.44073	0.32611	0.49618	0.27561
4.0	0.47375	0.35581	0.52892	0.27146

Table V. Deflections and corner moments for a square frame loaded in compression according to Figure 6(b). Note: $\mu_0 = M_0/PL$

PL^2/EI	w/L	u/L	θ_0	μ_0
0.2	0.04293	0.02523	0.05168	0.24094
0.4	0.08840	0.05084	0.10674	0.23033
0.6	0.13645	0.07670	0.16520	0.21810
0.8	0.18708	0.10264	0.22699	0.20416
1.0	0.24025	0.12850	0.29199	0.18851
1.2	0.29591	0.15408	0.36000	0.17116
1.4	0.35397	0.17914	0.43072	0.15220
1.6	0.41429	0.20344	0.50378	0.13181
1.8	0.47668	0.22668	0.57869	0.11021
2.0	0.54087	0.24854	0.65486	0.08770
2.5	0.70665	0.29483	0.84627	0.02999
3.0	0.87339	0.32561	1.02992	-0.02479
3.5	1.03230	0.33921	1.19656	-0.07183
4.0	1.17703	0.33754	1.34181	-0.10926

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