Surface Area of Rotations

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Let $y = f(x) \geq 0$, if $f(x)$ is a smooth (continuously differentiable) function on the interval $[a, b]$, then the surface area generated by revolving the function about the line $y = r$ can be calculated by

$$S = \int_a^b 2\pi(f(x) - r)\sqrt{1 + \left(f'(x)\right)^2}\, dx.$$

Similarly, if $x = g(y) \geq 0$, where $g(y)$ is a smooth function on $[c, d]$, then the surface area generated by revolving the function around the line $x = r$ can be calculated by

$$S = \int_c^d 2\pi(g(y) - r)\sqrt{1 + \left(g'(y)\right)^2}\, dy.$$

Examples:

1. Find the surface area of the solid created by the rotating the function $f(x) = \sqrt{x}$ around the x-axis for $2 \leq x \leq 6$.

Solution:

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$\begin{aligned}
S &= \int_2^6 2\pi\sqrt{x}\sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2}\, dx \\
&= \int_2^6 2\pi\sqrt{x}\sqrt{1 + \frac{1}{4x}}\, dx \\
&= \int_2^6 2\pi\sqrt{x}\sqrt{\frac{4x+1}{4x}}\, dx \\
&= \int_2^6 \pi\sqrt{4x+1}\, dx \\
&= \frac{\pi}{6}(4x+1)^{3/2}\Big|_2^6 \\
&= \frac{\pi}{6}\Big(125 - 27) \\
&= \frac{49\pi}{3}
\end{aligned}$$

2. Find the surface area of the solid created by the rotating the function $f(x) = \sqrt{16 - x^2}$ around the x-axis for $-1 \leq x \leq 1$.

Solution:

$$
\begin{aligned}
f'(x) &= \frac{1}{\sqrt{16 - x^2}}(-2x) \\
&= \frac{-2x}{\sqrt{16 - x^2}}
\end{aligned}
$$

$$
\begin{aligned}
S &= \int_{-1}^{1} 2\pi\sqrt{16 - x^2}\sqrt{1 + \left(\frac{-2x}{\sqrt{16 - x^2}}\right)^2}\,dx \\
&= \int_{-1}^{1} 2\pi\sqrt{16 - x^2}\sqrt{1 + \frac{4x^2}{16 - x^2}}\,dx \\
&= \int_{-1}^{1} 2\pi\sqrt{16 - x^2}\sqrt{\frac{16 - x^2 + 4x^2}{16 - x^2}}\,dx \\
&= \int_{-1}^{1} 2\pi\sqrt{16 + 3x^2}\,dx \\
&= 4\pi\int_{0}^{1} \sqrt{16 + 3x^2}\,dx
\end{aligned}
$$

Let $x = \frac{4}{\sqrt{3}}\tan(\theta)$, then $dx = \frac{4}{\sqrt{3}}\sec^2(\theta)\,d\theta$.

$$
\begin{aligned}
S &= 4\pi\int_{a}^{b} \sqrt{16 + 3\left(\frac{4}{\sqrt{3}}\tan(\theta)\right)^2}\,\frac{4}{\sqrt{3}}\sec^2(\theta)\,d\theta \\
&= \frac{16\pi}{\sqrt{3}}\int_{a}^{b} \sqrt{16 + 16\tan^2(\theta)}\sec^2(\theta)\,d\theta \\
&= \frac{16\pi}{\sqrt{3}}\int_{a}^{b} \sqrt{16\sec^2(\theta)}\sec^2(\theta)\,d\theta \\
&= \frac{16\pi}{\sqrt{3}}\int_{a}^{b} 4\sec(\theta)\sec^2(\theta)\,d\theta \\
&= \frac{64\pi}{\sqrt{3}}\int_{a}^{b} \sec^3(\theta)\,d\theta \\
&= \frac{64\pi}{\sqrt{3}}\left(\frac{\sec(\theta)\tan(\theta) + \ln\left|\sec(\theta) + \tan(\theta)\right|}{2}\right)\Bigg|_{a}^{b} \\
&= \frac{64\pi}{\sqrt{3}}\left(\frac{\frac{\sqrt{16+3x^2}}{4}\frac{\sqrt{3}x}{4} + \ln\left|\frac{\sqrt{16+3x^2}}{4} + \frac{\sqrt{3}x}{4}\right|}{2}\right)\Bigg|_{0}^{1} \\
&= \left(2\pi x\sqrt{16 + 3x^2} + \frac{32\pi}{\sqrt{3}}\ln\left|\frac{\sqrt{16 + 3x^2}}{4} + \frac{\sqrt{3}x}{4}\right|\right)\Bigg|_{0}^{1} \\
&= 2\pi\sqrt{19} + \frac{32\pi}{\sqrt{3}}\ln\left(\frac{\sqrt{19} + \sqrt{3}}{4}\right)
\end{aligned}
$$

Questions:

1. Find the surface area of the solid created by the rotating the function $f(x) = x^3$ around the x-axis for $0 \leq x \leq 1$

Solution:

$$f'(x) = 3x^2$$

$$\begin{aligned} S &= \int_0^1 2\pi x^3 \sqrt{1 + \left(3x^2\right)^2}\, dx \\ &= \int_0^1 2\pi x^3 \sqrt{1 + 9x^4}\, dx \end{aligned}$$

Let $u = 1 + 9x^4$, then $du = 36x^3\, dx$

$$\begin{aligned} S &= \frac{\pi}{18} \int_a^b \sqrt{u}\, du \\ &= \frac{\pi}{18} \frac{2}{3} u^{3/2} \Big|_a^b \\ &= \frac{\pi}{27} (1 + 9x^4)^{3/2} \Big|_0^1 \\ &= \frac{\pi}{27} \left(10^{3/2} - 1\right) \end{aligned}$$

2. Find the surface area of the solid created by the rotating the function $9y = x^2 + 18$ around the y-axis for $12 \le y \le 20$.

Solution:

$$\begin{aligned}
x^2 + 18 &= 9y \\
\Rightarrow x^2 &= 9y - 18 \\
\Rightarrow x &= \sqrt{9y - 18} \\
&= 3\sqrt{y - 2}
\end{aligned}$$

$$g'(y) = \frac{3}{2\sqrt{y-2}}$$

$$\begin{aligned}
S &= \int_{12}^{20} 2\pi \cdot 3\sqrt{y-2}\sqrt{1 + \left(\frac{3}{2\sqrt{y-2}}\right)^2}\, dy \\
&= 6\pi \int_{12}^{20} \sqrt{y-2}\sqrt{1 + \frac{9}{4(y-2)}}\, dy \\
&= 6\pi \int_{12}^{20} \sqrt{y-2}\sqrt{\frac{4(y-2)+9}{4(y-2)}}\, dy \\
&= 6\pi \int_{12}^{20} \sqrt{4(y-2)+9}\, dy \\
&= 6\pi \int_{12}^{20} \sqrt{4y+1}\, dy \\
&= 6\pi \cdot \frac{2}{3}(4y+1)^{3/2}\Big|_{12}^{20} \\
&= 4\pi(729 - 343) \\
&= 1544\pi
\end{aligned}$$

3. Find the surface area of the solid created by the rotating the function $f(x) = \frac{1}{3}x$ around the x-axis for $1 \le x \le 3$.

Solution:

$$f'(x) = \frac{1}{3}$$

$$\begin{aligned} S &= \int_1^3 2\pi \frac{1}{3} x \sqrt{1 + \left(\frac{1}{3}\right)^2}\, dx \\ &= \frac{2\pi}{3} \int_1^3 x \sqrt{1 + \frac{1}{9}}\, dx \\ &= \frac{2\pi}{3} \sqrt{\frac{10}{9}} \int_1^3 x\, dx \\ &= \frac{\pi}{3} \sqrt{\frac{10}{9}} x^2 \Big|_1^3 \\ &= \frac{8\sqrt{10}\pi}{9} \end{aligned}$$

4. Find the surface area generated by revolving the curve $y = \dfrac{x^3}{3} + \dfrac{1}{4x}$, $1 \le x \le 3$ about the line $y = -2$.

Solution:

$$f'(x) = x^2 - \frac{1}{4x^2}$$

$$\begin{aligned}
S &= \int_1^3 2\pi\left(\frac{x^3}{3} + \frac{1}{4x} + 2\right)\sqrt{1 + \left(x^2 - \frac{1}{4x^2}\right)^2}\,dx \\
&= 2\pi \int_1^3 \left(\frac{x^3}{3} + \frac{1}{4x} + 2\right)\sqrt{1 + \left(x^2\right)^2 - \frac{1}{2} + \left(\frac{1}{4x^2}\right)^2}\,dx \\
&= 2\pi \int_1^3 \left(\frac{x^3}{3} + \frac{1}{4x} + 2\right)\sqrt{\left(x^2\right)^2 + \frac{1}{2} + \left(\frac{1}{4x^2}\right)^2}\,dx \\
&= 2\pi \int_1^3 \left(\frac{x^3}{3} + \frac{1}{4x} + 2\right)\sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2}\,dx \\
&= 2\pi \int_1^3 \left(\frac{x^3}{3} + \frac{1}{4x} + 2\right)\left(x^2 + \frac{1}{4x^2}\right)dx \\
&= 2\pi \int_1^3 \frac{x^5}{3} + \frac{x}{12} + \frac{x}{4} + \frac{1}{16x^3} + 2x^2 + \frac{1}{2x^2}\,dx \\
&= 2\pi \int_1^3 \frac{x^5}{3} + 2x^2 + \frac{x}{3} + \frac{1}{2x^2} + \frac{1}{16x^3}\,dx \\
&= 2\pi\left(\frac{x^6}{18} + \frac{2x^3}{3} + \frac{x^2}{6} - \frac{1}{2x} - \frac{1}{32x^2}\right)\Bigg|_1^3 \\
&= \frac{2141\pi}{18}
\end{aligned}$$

5. Find the surface area of the solid created by the rotating the function $x = e^{-2y}$ around the y-axis for $0 \leq x \leq 1$. (Trig Sub Question. After you solve the integral, do not try to simplify after plugging in the values of the integral)

Solution:

$$g'(y) = -2e^{-2y}$$

$$S = \int_0^1 2\pi e^{-2y} \sqrt{1 + \big(- 2e^{-2y}\big)^2}\, dy$$

Let $e^{-2y} = \dfrac{1}{2}\tan(\theta)$, then $-2e^{-2y}\, dy = \dfrac{1}{2}\sec^2(\theta)\, d\theta$

$$\begin{aligned}
S &= \frac{-\pi}{2} \int_a^b \sqrt{1 + \tan^2(\theta)}\, \sec^2(\theta)\, d\theta \\
&= \frac{-\pi}{2} \int_a^b \sqrt{\sec^2(\theta)}\, \sec^2(\theta)\, d\theta \\
&= \frac{-\pi}{2} \int_a^b \sec^3(\theta)\, d\theta \\
&= \frac{-\pi}{2} \left(\frac{\sec(\theta)\tan(\theta) + \ln\big|\sec(\theta) + \tan(\theta)\big|}{2} \right)\Bigg|_a^b \\
&= \frac{-\pi}{2} \left(\frac{\sqrt{1 + 4e^{-4y}}e^{-2y} + \ln\big|\sqrt{1 + 4e^{-4y}} + e^{-2y}\big|}{2} \right)\Bigg|_0^1 \\
&\approx 4.21593
\end{aligned}$$