

Arc Length

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Let $y = f(x)$ be a smooth function (twice differentiable) on the interval $[a, b]$ then the arc length of $f(x)$ on the interval is given by

$$s = \int_a^b \sqrt{1 + [f'(x)]^2} dx.$$

We can also consider functions $x = g(y)$. Let $g(y)$ be a smooth function on the interval $[c, d]$ then the arc length of $g(y)$ is given by

$$s = \int_c^d \sqrt{1 + [g'(y)]^2} dy.$$

Note: The arc length for a parametric equation can be calculated by

$$s = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Examples:

1. Find the arc length of $f(x) = \frac{x^{3/2}}{3}$ from 0 to 5.

Solution:

$$\begin{aligned} f'(x) &= \frac{x^{1/2}}{2} \\ s &= \int_0^5 \sqrt{1 + \left(\frac{\sqrt{x}}{2}\right)^2} dx \\ &= \int_0^5 \sqrt{1 + \frac{x}{4}} dx \\ &= \int_0^5 \frac{1}{2} (4 + x)^{1/2} dx \\ &= \frac{1}{3} (4 + x)^{3/2} \Big|_0^5 \\ &= \frac{19}{3} \end{aligned}$$

2. Find the arc length of a function whose derivative is given by $f'(x) = \frac{1}{2} \left(x^2 - \frac{1}{x^2} \right)$ from $\frac{1}{2}$ to 2.

Solution:

$$\begin{aligned}
 s &= \int_{1/2}^2 \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2} \right)^2} dx \\
 &= \int_{1/2}^2 \sqrt{1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}} dx \\
 &= \int_{1/2}^2 \sqrt{\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}} dx \\
 &= \int_{1/2}^2 \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2} \right)^2} dx \\
 &= \int_{1/2}^2 \frac{x^2}{2} + \frac{1}{2x^2} dx \\
 &= \frac{x^3}{6} - \frac{1}{2x} \Big|_{1/2}^2 \\
 &= \frac{33}{16}
 \end{aligned}$$

3. Find the arc length of $g(y) = y^2 - \frac{1}{8} \ln(y)$ from 1 to 2.

Solution:

$$\begin{aligned}
 g'(y) &= 2y - \frac{1}{8y} \\
 s &= \int_1^2 \sqrt{1 + \left(2y - \frac{1}{8y} \right)^2} dy \\
 &= \int_1^2 \sqrt{1 + 4y^2 - \frac{1}{2} + \frac{1}{64y^2}} dy \\
 &= \int_1^2 \sqrt{4y^2 + \frac{1}{2} + \frac{1}{64y^2}} dy \\
 &= \int_1^2 \sqrt{\left(2y + \frac{1}{8y} \right)^2} dy \\
 &= \int_1^2 2y + \frac{1}{8y} dy \\
 &= y^2 + \frac{\ln(y)}{8} \Big|_1^2 \\
 &= 3 + \frac{\ln(2)}{8}
 \end{aligned}$$

Questions:

1. Find the arc length of $f(x) = \ln(\cos(x))$ from 0 to $\frac{\pi}{4}$.

2. Find the arc length of $f(x) = \cosh(x)$ from 0 to 1.