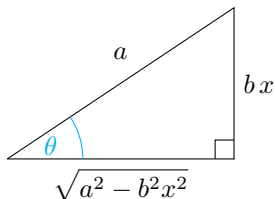


Trig Substitution

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For integrals of the form $\int (a^2 - b^2 x^2)^n dx$ use $x = \frac{a \sin(\theta)}{b}$ as your substitution. From there we get the triangle:



A useful trig identity for these questions is $r^2 \sin^2(x) + r^2 \cos^2(x) = r^2$.

Example 1: Find $\int \frac{x-2}{\sqrt{9+4x-x^2}} dx$

Solution:

$$\begin{aligned} \int \frac{x-2}{\sqrt{9+4x-x^2}} dx &= \int \frac{x-2}{\sqrt{9-4+(4x-x^2)}} dx \\ &= \int \frac{x-2}{\sqrt{9-(x-2)^2}} dx \end{aligned}$$

Let $x-2 = 3 \sin(\theta)$, then $dx = 3 \cos(\theta) d\theta$.

$$\begin{aligned} \int \frac{x-2}{\sqrt{9+4x-x^2}} dx &= \frac{1}{3} \int \frac{3 \sin(\theta) \cos(\theta)}{\sqrt{9-(3 \sin(\theta))^2}} d\theta \\ &= \int \frac{\sin(\theta) \cos(\theta)}{\sqrt{9-9 \sin^2(\theta)}} d\theta \\ &= \int \frac{\sin(\theta) \cos(\theta)}{\sqrt{9 \cos^2(\theta)}} d\theta \\ &= \frac{1}{3} \int \frac{\sin(\theta) \cos(\theta)}{\cos(\theta)} d\theta \\ &= \frac{1}{3} \int \sin(\theta) d\theta \\ &= -\frac{1}{3} \cos(\theta) + C \\ &= \frac{-\sqrt{9-(x-2)^2}}{27} + C \end{aligned}$$

1. $\int \sqrt{4 - 25x^2} \, dx$

Solution:

Let $x = \frac{2}{5} \sin(\theta)$, then $dx = \frac{2}{5} \cos(\theta) \, d\theta$.

$$\begin{aligned}
 \int \sqrt{4 - 25x^2} \, dx &= \frac{2}{5} \int \sqrt{4 - 25 \left(\frac{2}{5} \sin(\theta) \right)^2} \cos(\theta) \, d\theta \\
 &= \frac{2}{5} \int \sqrt{4 - 25 \left(\frac{2}{5} \sin(\theta) \right)^2} \cos(\theta) \, d\theta \\
 &= \frac{2}{5} \int \sqrt{4 - 4 \sin^2(\theta)} \cos(\theta) \, d\theta \\
 &= \frac{2}{5} \int \sqrt{4 \cos^2(\theta)} \cos(\theta) \, d\theta \\
 &= \frac{4}{5} \int \cos^2(\theta) \, d\theta \\
 &= \frac{4}{5} \int \frac{1 + \cos(2\theta)}{2} \, d\theta \\
 &= \frac{2}{5} \int 1 + \cos(2\theta) \, d\theta \\
 &= \frac{2}{5} \left(\theta + \frac{\sin(2\theta)}{2} \right) + C \\
 &= \frac{2}{5} \left(\theta + \frac{2 \sin(\theta) \cos(\theta)}{2} \right) + C \\
 &= \frac{2}{5} \left(\theta + \sin(\theta) \cos(\theta) \right) + C \\
 &= \frac{2}{5} \left(\sin^{-1} \left(\frac{5x}{2} \right) + \frac{\sqrt{4 - 25x^2}}{2} \frac{5x}{2} \right) + C \\
 &= \frac{2}{5} \sin^{-1} \left(\frac{5x}{2} \right) + \frac{x \sqrt{4 - 25x^2}}{2} + C
 \end{aligned}$$

2. $\int \frac{1}{\sqrt{1-9x^2}} dx$

Solution:

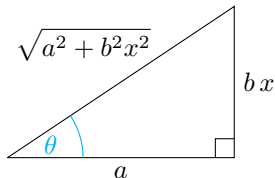
3. $\int \frac{1}{x^4 \sqrt{9-x^2}} dx$

Solution:

4. $\int \frac{\sqrt{16-x^2}}{x^2} dx$

Solution:

For integrals of the form $\int (a^2 + b^2 x^2)^n dx$ use $x = \frac{a \tan(\theta)}{b}$ as your substitution. The triangle we get from this substitution is:



A useful identity for these questions is $r^2 + r^2 \tan^2(x) = r^2 \sec^2(x)$.

Example 2: Find $\int \frac{e^{2x}}{\sqrt{e^{4x} + 1}} dx$

Solution:

Let $e^{2x} = \tan(\theta)$, then $2e^{2x} dx = \sec^2(\theta) d\theta$.

$$\begin{aligned} \int \frac{e^{2x}}{\sqrt{e^{4x} + 1}} dx &= \frac{1}{2} \int \frac{\sec^2(\theta)}{\sqrt{\tan^2(\theta) + 1}} d\theta \\ &= \frac{1}{2} \int \frac{\sec^2(\theta)}{\sqrt{\sec^2(\theta)}} d\theta \\ &= \frac{1}{2} \int \sec(\theta) d\theta \\ &= \frac{1}{2} \ln |\sec(\theta) + \tan(\theta)| + C \\ &= \frac{1}{2} \ln \left| \sqrt{e^{4x} + 1} + e^{2x} \right| + C \end{aligned}$$

Note: This integral can also be solved using inverse hyperbolic trig functions.

Let $u = e^{2x}$, then $du = 2e^{2x} dx$.

$$\begin{aligned} \int \frac{e^{2x}}{\sqrt{e^{4x} + 1}} dx &= \frac{1}{2} \int \frac{1}{\sqrt{u^2 + 1}} du \\ &= \frac{1}{2} \sinh^{-1}(u) + C \\ &= \frac{1}{2} \sinh^{-1}(e^{2x}) + C \end{aligned}$$

1. $\int \frac{\sqrt{9+x^2}}{x^4} dx$

Solution:

2. $\int \frac{3x}{x^2+10x+29} dx$

Solution:

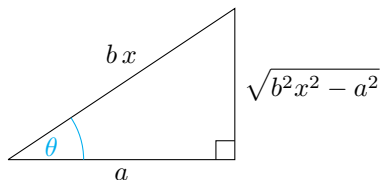
3. $\int_{-1}^1 \frac{1}{(1+x^2)^2} dx$

Solution:

4. $\int \frac{1}{\sqrt{25x^2 + 16}} dx$

Solution:

For integrals of the form $\int (b^2x^2 - a^2)^n dx$ use $x = \frac{a \sec(\theta)}{b}$ as your substitution. The triangle we get from this substitution is:



A useful identity for these questions is $r^2 \sec^2(x) - r^2 = r^2 \tan^2(x)$

Example 3: Find $\int x^2 \sqrt{x^2 - 10x} dx$

Solution:

$$\begin{aligned} \int (x - 5) \sqrt{x^2 - 10x} dx &= \int x^2 \sqrt{x^2 - 10x + 25 - 25} dx \\ &= \int x^2 \sqrt{(x - 5)^2 - 25} dx \end{aligned}$$

Let $x - 5 = 5 \sec(\theta)$, then $dx = 5 \sec(\theta) \tan(\theta) d\theta$.

$$\begin{aligned} \int (x - 5) \sqrt{x^2 - 10x} dx &= 25 \int \sec(\theta) \sqrt{(5 \sec(\theta))^2 - 25} \sec(\theta) \tan(\theta) d\theta \\ &= 25 \int \sec^2(\theta) \sqrt{25 \sec^2(\theta) - 25} \tan(\theta) d\theta \\ &= 25 \int \sec^2(\theta) \sqrt{25 \tan^2(\theta)} \tan(\theta) d\theta \\ &= 125 \int \sec^2(\theta) \tan^2(\theta) d\theta \end{aligned}$$

Let $u = \tan(\theta)$, then $du = \sec^2(\theta) d\theta$.

$$\begin{aligned} \int (x - 5) \sqrt{x^2 - 10x} dx &= 125 \int u^2 du \\ &= \frac{125u^3}{3} + C \\ &= \frac{125 \tan^3(\theta)}{3} + C \\ &= \frac{125}{3} \left(\frac{x^2 - 10x}{5} \right)^{3/2} + C \\ &= \frac{(x^2 - 10x)^{3/2}}{3} + C \end{aligned}$$

1. $\int \frac{1}{\sqrt{25x^2 - 1}} dx$

Solution:

2. $\int \frac{\sqrt{16x^2 - 9}}{x} dx$

Solution:

3. $\int \frac{1}{x^2\sqrt{x^2-36}} dx$

Solution:

4. $\int \frac{x}{\sqrt{x^2-4}} dx$

Solution: