Trig Substitution

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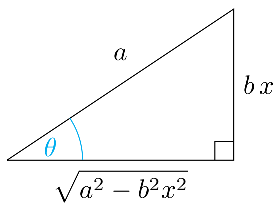
For integrals of the form $\int (a^2 - b^2x^2)^n \, dx$ use $x = \dfrac{a\sin(\theta)}{b}$ as your substitution. From there we get the triangle:



A useful trig identity for these questions is $r^2\sin^2(x) + r^2\cos^2(x) = r^2$.

Example 1: Find $\displaystyle\int \frac{x-2}{\sqrt{9+4x-x^2}}\,dx$

Solution:

$$\begin{aligned}
\int \frac{x-2}{\sqrt{5+4x-x^2}}\,dx &= \int \frac{x-2}{\sqrt{9-4+4x-x^2}}\,dx \\
&= \int \frac{x-2}{\sqrt{9-(x-2)^2}}\,dx
\end{aligned}$$

Let $x - 2 = 3\sin(\theta)$, then $dx = 3\cos(\theta)\,d\theta$.

$$\begin{aligned}
\int \frac{x-2}{\sqrt{5+4x-x^2}}\,dx &= \frac{1}{3}\int \frac{3\sin(\theta)\cos(\theta)}{\sqrt{9-(3\sin(\theta))^2}}\,d\theta \\
&= \int \frac{\sin(\theta)\cos(\theta)}{\sqrt{9-9\sin^2(\theta)}}\,d\theta \\
&= \int \frac{\sin(\theta)\cos(\theta)}{\sqrt{9\cos^2(\theta)}}\,d\theta \\
&= \frac{1}{3}\int \frac{\sin(\theta)\cos(\theta)}{\cos(\theta)}\,d\theta \\
&= \frac{1}{3}\int \sin(\theta)\,d\theta \\
&= -\frac{1}{3}\cos(\theta) + C \\
&= \frac{-\sqrt{9-(x-2)^2}}{27} + C
\end{aligned}$$

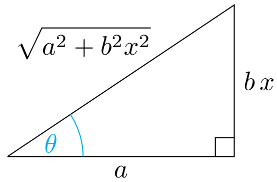
1. $\int \sqrt{4 - 25x^2}\, dx$

2. $\int \frac{1}{\sqrt{1 - 9x^2}}\, dx$

3. $\displaystyle\int \frac{1}{x^4\sqrt{9-x^2}}\,dx$

4. $\displaystyle\int \frac{\sqrt{16-x^2}}{x^2}\,dx$

For integrals of the form $\int (a^2 + b^2x^2)^n \, dx$ use $x = \dfrac{a\tan(\theta)}{b}$ as your substitution. The triangle we get from this substitution is:



A useful identity for these questions is $r^2 + r^2\tan^2(x) = r^2\sec^2(x)$.

Example 2: Find $\displaystyle\int \frac{e^{2x}}{\sqrt{e^{4x}+1}}\, dx$

Solution:

$$\text{Let } e^{2x} = \tan(\theta), \text{ then } 2\,e^{2x}\,dx = \sec^2(\theta)\,d\theta.$$

$$\begin{aligned}
\int \frac{e^{2x}}{\sqrt{e^{4x}+1}}\, dx &= \frac{1}{2}\int \frac{\sec^2(\theta)}{\sqrt{\tan^2(\theta)+1}}\, d\theta \\
&= \frac{1}{2}\int \frac{\sec^2(\theta)}{\sqrt{\sec^2(\theta)}}\, d\theta \\
&= \frac{1}{2}\int \sec(\theta)\, d\theta \\
&= \frac{1}{2}\ln\left|\sec(\theta) + \tan(\theta)\right| + C \\
&= \frac{1}{2}\ln\left|\sqrt{e^{4x}+1} + e^{2x}\right| + C
\end{aligned}$$

Note: This integral can also be solved using inverse hyperbolic trig functions.

$$\text{Let } u = e^{2x}, \text{ then } du = 2\,e^{2x}\,dx.$$

$$\begin{aligned}
\int \frac{e^{2x}}{\sqrt{e^{4x}+1}}\, dx &= \frac{1}{2}\int \frac{1}{\sqrt{u^2+1}}\, du \\
&= \frac{1}{2}\sinh^{-1}(u) + C \\
&= \frac{1}{2}\sinh^{-1}\left(e^{2x}\right) + C
\end{aligned}$$

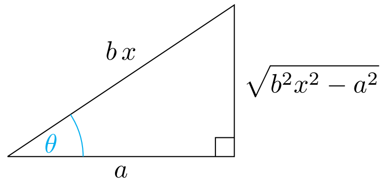
1. $\displaystyle\int \frac{\sqrt{9+x^2}}{x^4}\,dx$

2. $\displaystyle\int \frac{3x}{x^2+10x+29}\,dx$

3. $\displaystyle\int_{-1}^{1} \frac{1}{(1+x^2)^2}\,dx$

4. $\displaystyle\int \frac{1}{\sqrt{25x^2+16}}\,dx$

For integrals of the form $\int (b^2x^2 - a^2)^n \, dx$ use $x = \dfrac{a\sec(\theta)}{b}$ as your substitution. The triangle we get from this substitution is:



A useful identity for these questions is $r^2\sec^2(x) - r^2 = r^2\tan^2(x)$

Example 3: Find $\displaystyle\int x^2\sqrt{x^2 - 10x}\, dx$

Solution:

$$\begin{aligned}\int (x-5)\sqrt{x^2 - 10x}\, dx &= \int x^2\sqrt{x^2 - 10x + 25 - 25}\, dx \\ &= \int x^2\sqrt{(x-5)^2 - 25}\, dx\end{aligned}$$

Let $x - 5 = 5\sec(\theta)$, then $dx = 5\sec(\theta)\tan(\theta)\, d\theta$.

$$\begin{aligned}\int (x-5)\sqrt{x^2 - 10x}\, dx &= 25\int \sec(\theta)\sqrt{(5\sec(\theta))^2 - 25}\,\sec(\theta)\tan(\theta)\, d\theta \\ &= 25\int \sec^2(\theta)\sqrt{25\sec(\theta)^2 - 25}\,\tan(\theta)\, d\theta \\ &= 25\int \sec^2(\theta)\sqrt{25\tan(\theta)^2}\,\tan(\theta)\, d\theta \\ &= 125\int \sec^2(\theta)\tan^2(\theta)\, d\theta\end{aligned}$$

Let $u = \tan(\theta)$, then $du = \sec^2(\theta)\, d\theta$.

$$\begin{aligned}\int (x-5)\sqrt{x^2 - 10x}\, dx &= 125\int u^2\, du \\ &= \frac{125u^3}{3} + C \\ &= \frac{125\tan^3(\theta)}{3} + C \\ &= \frac{125}{3}\left(\frac{x^2 - 10x}{5}\right)^3 + C \\ &= \frac{(x^2 + 10x)^{3/2}}{3} + C\end{aligned}$$

1. $\int \frac{1}{\sqrt{25x^2 - 1}}\,dx$

2. $\int \frac{\sqrt{16x^2 - 9}}{x}\,dx$

3. $\displaystyle\int \frac{1}{x^2\sqrt{x^2-36}}\,dx$

4. $\displaystyle\int \frac{x}{\sqrt{x^2-4}}\,dx$