Calculating Volumes Using Cylindrical Shells

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The formula to calculate the volume using cylindrical shells is:

$$V = \int_a^b 2\pi\, r(x) f(x)\, dx$$

Where $r(x)$ is the radius to the line you are rotating about. Using we just consider rotations around the y-axis, so in this cause $r(x) = x$. However, if we consider rotations around the line $x = a$, then our radius would be $r(x) = x - a$.

This can also be extended to rotations around the x-axis or a line $y = a$ using the formula:

$$V = \int_c^d 2\pi\, r(y) f(y)\, dy$$

Where $r(y) = y$ if we are rotation around the x-axis or $r(y) = y - a$ if we are rotating around the line $y = a$

Examples:

1. Find the volume of the solid generated by rotating the region bounded by the curves $y = x$ and $y = x^2$, about the y-axis.

Solution:

$$x = x^2 \text{ if and only if } x = 0, 1$$

$$\begin{aligned} V &= \int_0^1 2\pi x(x - x^2)\, dx \\ &= 2\pi \int_0^1 x^2 - x^3\, dx \\ &= 2\pi \left(\frac{x^3}{3} - \frac{x^4}{4} \right) \Bigg|_0^1 \\ &= \frac{\pi}{6} \end{aligned}$$

2. Find the volume of the solid generated by rotating the region bounded by the curves $y = \sqrt{x}$ and $y = x$, about the line $x = -4$.

Solution:

$$x = \sqrt{x} \text{ if and only if } x = 0, 1$$

$$
\begin{aligned}
V &= \int_0^1 2\pi(x+4)(\sqrt{x}-x)\,dx \\
&= 2\pi\int_0^1 x^{3/2} - x^2 + 4\sqrt{x} - 4x\,dx \\
&= 2\pi\left(\frac{2x^{5/2}}{5} - \frac{x^3}{3} + \frac{8x^{3/2}}{3} - 2x^2\right)\Bigg|_0^1 \\
&= \frac{22\pi}{15}
\end{aligned}
$$

3. Find the volume of the solid generated by rotating the region bounded by the curves $y = \dfrac{1}{x\sqrt{x^2+1}}$, $x = 1$, and $x = 2$ about the y-axis.

Solution:

$$
\begin{aligned}
V &= \int_1^2 2\pi x \frac{1}{x\sqrt{x^2+1}}\,dx \\
&= 2\pi\int_0^1 \frac{1}{\sqrt{x^2+1}}\,dx \\
&= 2\pi \sinh^{-1}(x)\Big|_1^2 \\
&= 2\pi\big(\sinh^{-1}(2) - \sinh^{-1}(1)\big)
\end{aligned}
$$

4. Determine the volume of the solid obtained by rotating the region bounded by $x = (y-1)^2$ and $x = 5y - 11$ about the line $y = -1$.

Solution:

$$
\begin{aligned}
5y - 11 &= (y-1)^2 \\
\Rightarrow 5y - 11 &= y^2 - 2y + 1 \\
\Rightarrow 0 &= y^2 - 7y + 12 \\
\Rightarrow 0 &= (y-3)(y-4) \\
\Rightarrow y &= 3, 4
\end{aligned}
$$

Therefore we know these curves will intersect at the points $(1, 2)$ and $(0, 1)$

$$
\begin{aligned}
V &= 2\pi\int_3^4 (y+1)\big((5y-11) - (y-1)^2\big)\,dx \\
&= 2\pi\int_3^4 -y^3 + 6y^2 - 5y - 12\,dx \\
&= 2\pi\left(\frac{-y^4}{4} + 2y^3 - \frac{5y^2}{2} - 12y\right)\Bigg|_3^4 \\
&= \frac{3\pi}{2}
\end{aligned}
$$

Questions:

1. Calculate the volume of the solid generated by rotating the region bounded by the curves $y = \dfrac{1}{x}$, $x = 1$, and $x = 3$ about the y-axis.

Solution:

$$\begin{aligned} V &= \int_1^3 2\pi x \frac{1}{x}\, dx \\ &= \int_1^3 2\pi\, dx \\ &= 2\pi x \Big|_1^3 \\ &= 4\pi \end{aligned}$$

2. Calculate the volume of the solid generated by rotating the region bounded by the curves $y = \left(\dfrac{x}{2}\right)^2$, $y = 4$, and $x = 0$ about the x-axis.

Solution:

$$\begin{aligned} y &= \left(\frac{x}{2}\right)^2 \\ \Rightarrow \sqrt{y} &= \frac{x}{2} \\ \Rightarrow 2\sqrt{y} &= x \end{aligned}$$

$$\begin{aligned} V &= \int_0^4 2\pi y 2\sqrt{y}\, dy \\ &= 4\pi \int_0^4 y^{3/2}\, dy \\ &= 4\pi \frac{2}{5} y^{5/2} \Big|_0^4 \\ &= \frac{256\pi}{5} \end{aligned}$$

3. Determine the volume of the solid obtained by rotating the region bounded by $x = (y-3)^2$ and $x = -4y + 9$ about the line $y = -2$.

Solution:

$$\begin{aligned} -4y + 9 &= (y-3)^2 \\ \Rightarrow -4y + 9 &= y^2 - 6y + 9 \\ \Rightarrow y^2 - y &= 0 \\ \Rightarrow y(y-2) &= 0 \\ \Rightarrow y &= 0, 2 \end{aligned}$$

$$\begin{aligned} V &= 2\pi \int_0^2 (y+2)\Big((-4y+9) - (y-3)^2\Big)\,dx \\ &= 2\pi \int_0^2 (y+2)(2y - y^2)\,dy \\ &= 2\pi \int_0^2 2y^2 - y^3 + 4y - 2y^2\,dy \\ &= 2\pi \int_0^2 4y - y^3\,dy \\ &= 2\pi \left(2y^2 - \frac{y^4}{4}\right)\Bigg|_0^2 \\ &= 8\pi \end{aligned}$$

4. Determine the volume of the solid generated by rotating the region bounded by the curve $y = \sin(x^2)$ from $x = 0$ to $x = \sqrt{\pi}$, about the y-axis.

Solution:

$$\begin{aligned} V &= \int_0^{\sqrt{\pi}} 2\pi x \sin(x^2)\,dx \\ &= -\pi \cos(x^2)\Big|_0^{\sqrt{\pi}} \\ &= 2\pi \end{aligned}$$