

Surface Area of Rotations

Stephen Styles

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Let $y = f(x) \geq 0$, if $f(x)$ is a smooth (continuously differentiable) function on the interval $[a, b]$, then the surface area generated by revolving the function about the line $y = r$ can be calculated by

$$S = \int_a^b 2\pi(f(x) - r)\sqrt{1 + (f'(x))^2} dx.$$

Similarly, if $x = g(y) \geq 0$, where $g(y)$ is a smooth function on $[c, d]$, then the surface area generated by revolving the function around the line $x = r$ can be calculated by

$$S = \int_c^d 2\pi(g(y) - r)\sqrt{1 + (g'(y))^2} dy.$$

Examples:

1. Find the surface area of the solid created by the rotating the function $f(x) = \sqrt{x}$ around the x -axis for $2 \leq x \leq 6$.

Solution:

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x}} \\ S &= \int_2^6 2\pi\sqrt{x}\sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2} dx \\ &= \int_2^6 2\pi\sqrt{x}\sqrt{1 + \frac{1}{4x}} dx \\ &= \int_2^6 2\pi\sqrt{x}\sqrt{\frac{4x+1}{4x}} dx \\ &= \int_2^6 \pi\sqrt{4x+1} dx \\ &= \frac{\pi}{6}(4x+1)^{3/2}\bigg|_2^6 \\ &= \frac{\pi}{6}(125 - 27) \\ &= \frac{49\pi}{3} \end{aligned}$$

2. Find the surface area of the solid created by the rotating the function $f(x) = \sqrt{16 - x^2}$ around the x -axis for $-1 \leq x \leq 1$.

Solution:

$$\begin{aligned} f'(x) &= \frac{1}{\sqrt{16 - x^2}}(-2x) \\ &= \frac{-2x}{\sqrt{16 - x^2}} \end{aligned}$$

$$\begin{aligned} S &= \int_{-1}^1 2\pi \sqrt{16 - x^2} \sqrt{1 + \left(\frac{-2x}{\sqrt{16 - x^2}} \right)^2} dx \\ &= \int_{-1}^1 2\pi \sqrt{16 - x^2} \sqrt{1 + \frac{4x^2}{16 - x^2}} dx \\ &= \int_{-1}^1 2\pi \sqrt{16 - x^2} \sqrt{\frac{16 - x^2 + 4x^2}{16 - x^2}} dx \\ &= \int_{-1}^1 2\pi \sqrt{16 + 3x^2} dx \\ &= 4\pi \int_0^1 \sqrt{16 + 3x^2} dx \end{aligned}$$

Let $x = \frac{4}{\sqrt{3}} \tan(\theta)$, then $dx = \frac{4}{\sqrt{3}} \sec^2(\theta) d\theta$.

$$\begin{aligned} S &= 4\pi \int_a^b \sqrt{16 + 3 \left(\frac{4}{\sqrt{3}} \tan(\theta) \right)^2} \frac{4}{\sqrt{3}} \sec^2(\theta) d\theta \\ &= \frac{16\pi}{\sqrt{3}} \int_a^b \sqrt{16 + 16 \tan^2(\theta)} \sec^2(\theta) d\theta \\ &= \frac{16\pi}{\sqrt{3}} \int_a^b \sqrt{16 \sec^2(\theta)} \sec^2(\theta) d\theta \\ &= \frac{16\pi}{\sqrt{3}} \int_a^b 4 \sec(\theta) \sec^2(\theta) d\theta \\ &= \frac{64\pi}{\sqrt{3}} \int_a^b \sec^3(\theta) d\theta \\ &= \frac{64\pi}{\sqrt{3}} \left(\frac{\sec(\theta) \tan(\theta) + \ln |\sec(\theta) + \tan(\theta)|}{2} \right) \Big|_a^b \\ &= \frac{64\pi}{\sqrt{3}} \left(\frac{\frac{\sqrt{16+3x^2}}{4} \frac{\sqrt{3}x}{4} + \ln \left| \frac{\sqrt{16+3x^2}}{4} + \frac{\sqrt{3}x}{4} \right|}{2} \right) \Big|_0^1 \\ &= \left(2\pi x \sqrt{16 + 3x^2} + \frac{32\pi}{\sqrt{3}} \ln \left| \frac{\sqrt{16 + 3x^2}}{4} + \frac{\sqrt{3}x}{4} \right| \right) \Big|_0^1 \\ &= 2\pi \sqrt{19} + \frac{32\pi}{\sqrt{3}} \ln \left(\frac{\sqrt{19} + \sqrt{3}}{4} \right) \end{aligned}$$

Questions:

1. Find the surface area of the solid created by the rotating the function $f(x) = x^3$ around the x -axis for $0 \leq x \leq 1$

2. Find the surface area of the solid created by the rotating the function $9y = x^2 + 18$ around the y -axis for $2 \leq y \leq 6$.

3. Find the surface area of the solid created by the rotating the function $f(x) = \frac{1}{3}x$ around the x -axis for $1 \leq x \leq 3$.

4. Find the surface area generated by revolving the curve $y = \frac{x^3}{3} + \frac{1}{4x}$, $1 \leq x \leq 3$ about the line $y = -2$.

5. Find the surface area of the solid created by the rotating the function $x = e^{-2y}$ around the y -axis for $0 \leq x \leq 1$. (Trig Sub Question)