Arc Length

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Let $y = f(x)$ be a smooth function (twice differentiable) on the interval $[a, b]$ then the arc length of $f(x)$ on the interval is given by

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

We can also consider functions $x = g(y)$. Let $g(y)$ be a smooth function on the interval $[c, d]$ then the arc length of $g(y)$ is given by

$$s = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$

Note: The general equation for arc length is

$$s = \int \sqrt{dx^2 + dy^2}.$$

Examples:

1. Find the arc length of $f(x) = \dfrac{x^{3/2}}{3}$ from 0 to 5.

Solution:

$$f'(x) = \frac{x^{1/2}}{2}$$

$$\begin{aligned} s &= \int_0^5 \sqrt{1 + \left(\frac{\sqrt{x}}{2}\right)^2}\, dx \\ &= \int_0^5 \sqrt{1 + \frac{x}{4}}\, dx \\ &= \int_0^5 \frac{1}{2}(4 + x)^{1/2}\, dx \\ &= \frac{1}{3}(4 + x)^{3/2}\Big|_0^5 \\ &= \frac{19}{3} \end{aligned}$$

2. Find the arc length of a function whose derivative is given by $f'(x) = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)$ from $\frac{1}{2}$ to 2.

Solution:

$$
\begin{aligned}
s &= \int_{1/2}^{2} \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2}\, dx \\
&= \int_{1/2}^{2} \sqrt{1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}}\, dx \\
&= \int_{1/2}^{2} \sqrt{\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}}\, dx \\
&= \int_{1/2}^{2} \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2}\, dx \\
&= \int_{1/2}^{2} \frac{x^2}{2} + \frac{1}{2x^2}\, dx \\
&= \frac{x^3}{6} - \frac{1}{2x}\Big|_{1/2}^{2} \\
&= \frac{33}{16}
\end{aligned}
$$

3. Find the arc length of $g(y) = y^2 - \frac{1}{8}\ln(y)$ from 1 to 2.

Solution:

$$
g'(y) = 2y - \frac{1}{8y}
$$

$$
\begin{aligned}
s &= \int_{1}^{2} \sqrt{1 + \left(2y - \frac{1}{8y}\right)^2}\, dy \\
&= \int_{1}^{2} \sqrt{1 + 4y^2 - \frac{1}{2} + \frac{1}{64y^2}}\, dy \\
&= \int_{1}^{2} \sqrt{4y^2 + \frac{1}{2} + \frac{1}{64y^2}}\, dy \\
&= \int_{1}^{2} \sqrt{\left(2y + \frac{1}{8y}\right)^2}\, dy \\
&= \int_{1}^{2} 2y + \frac{1}{8y}\, dy \\
&= y^2 + \frac{\ln(y)}{8}\Big|_{1}^{2} \\
&= 3 + \frac{\ln(2)}{8}
\end{aligned}
$$

Questions:

1. Find the arc length of $f(x) = \ln(\cos(x))$ from 0 to $\frac{\pi}{4}$.

Solution:

$$\begin{aligned} f'(x) &= \frac{1}{\cos(x)} - \sin(x) \\ &= -\tan(x) \end{aligned}$$

$$\begin{aligned} s &= \int_0^{\frac{\pi}{4}} \sqrt{1 + \big(-\tan(x)\big)^2}\, dx \\ &= \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2(x)}\, dx \\ &= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2(x)}\, dx \\ &= \int_0^{\frac{\pi}{4}} \sec(x)\, dx \\ &= \ln|\sec(x) + \tan(x)|\Big|_0^{\frac{\pi}{4}} \\ &= \ln(\sqrt{2} + 1) \end{aligned}$$

2. Find the arc length of $f(x) = \cosh(x)$ from 0 to 3π.

Solution:

$$f'(x) = \sinh(x)$$

$$\begin{aligned} s &= \int_0^{3\pi} \sqrt{1 + \sinh^2(x)}\, dx \\ &= \int_0^{3\pi} \sqrt{\cosh^2(x)}\, dx \\ &= \int_0^{3\pi} \cosh(x)\, dx \\ &= \sinh(x)\Big|_0^{3\pi} \\ &= \sinh(3\pi) \end{aligned}$$

3. Find the arc length of $f(x) = \dfrac{2}{3x^{1/3}}$ from 1 to 8.

Solution:

$$f'(x) = \frac{2}{3x^{1/3}}$$

$$\begin{aligned} s &= \int_1^8 \sqrt{1 + \left(\frac{2}{3x^{1/3}}\right)^2}\, dx \\ &= \int_1^8 \sqrt{1 + \frac{4}{9x^{2/3}}}\, dx \\ &= \int_1^8 \sqrt{\frac{9x^{2/3}+4}{9x^{2/3}}}\, dx \\ &= \frac{1}{3}\int_1^8 \frac{\sqrt{9x^{2/3}+4}}{x^{1/3}}\, dx \end{aligned}$$

Let $u = 9x^{2/3} + 4$ then $du = \dfrac{6}{x^{1/3}} dx$

$9(1)^{2/3} + 4 = 13,\ 9(8)^{2/3} + 4 = 40$

$$\begin{aligned} s &= \frac{1}{18}\int_{13}^{40} u^{1/2}\, du \\ &= \frac{1}{18}\frac{2}{3}u^{3/2}\Big|_{13}^{40} \\ &= \frac{1}{27}\left(40^{3/2} - 13^{3/2}\right) \end{aligned}$$

4. Find the arc length of $g(y) = 2y^3 + \dfrac{1}{24y}$ from $y = 1$ to $y = 3$.

Solution:

$$g'(y) = 6y^2 - \frac{1}{24y^2}$$

$$\begin{aligned}
s &= \int_1^3 \sqrt{1 + \left(6y^2 - \frac{1}{24y^2}\right)^2}\,dy \\
&= \int_1^3 \sqrt{1 + \left(6y^2\right)^2 - \frac{1}{2} + \left(\frac{1}{24y^2}\right)^2}\,dy \\
&= \int_1^3 \sqrt{\left(6y^2\right)^2 + \frac{1}{2} + \left(\frac{1}{24y^2}\right)^2}\,dy \\
&= \int_1^3 \sqrt{\left(6y^2 + \frac{1}{24y^2}\right)^2}\,dy \\
&= \int_1^3 6y^2 + \frac{1}{24y^2}\,dy \\
&= 2y^3 - \frac{1}{24y}\Big|_1^3 \\
&= \frac{1873}{36}
\end{aligned}$$