# Induction

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November 25, 2019

1. Prove by induction that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

*Solution:*

Step 1: Base case where $k = 1$

$$\begin{aligned}\sum_{k=1}^{1} k = 1 &= \frac{1(1+1)}{2} \\ &= 1\end{aligned}$$

Step 2: Assume true for $n$

Step 3: Prove for $n + 1$

$$\begin{aligned}\sum_{k=1}^{n+1} k &= \sum_{k=1}^{n} k + n + 1 \\ &= \frac{n(n+1)}{2} + n + 1 \\ &= \frac{n^2+n)}{2} + \frac{2n+2}{2} \\ &= \frac{n^2+3n+2}{n} \\ &= \frac{(n+1)(n+2)}{2} \\ &= \frac{(n+1)\big((n+1)+1\big)}{2}\end{aligned}$$

Therefore by induction, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

2. Prove by induction that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

*Solution:*

Step 1: Base case where $k = 1$

$$\begin{aligned}\sum_{k=1}^{1} k^2 = 1^2 &= \frac{1(1+1)(2\cdot 1+1)}{6} \\ &= 1\end{aligned}$$

Step 2: Assume true for $n$

Step 3: Prove for $n+1$

$$\begin{aligned}\sum_{k=1}^{n+1} k^2 &= \sum_{k=1}^{n} k^2 + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= (n+1)\left(\frac{n(2n+1)}{6} + \frac{6(n+1)}{6}\right) \\ &= (n+1)\left(\frac{2n^2+n)}{6} + \frac{6n+6}{6}\right) \\ &= (n+1)\left(\frac{2n^2+7n+6}{6}\right) \\ &= (n+1)\left(\frac{(n+2)(2n+3)}{6}\right) \\ &= \frac{(n+1)\big((n+1)+1\big)\big(2(n+1)+1\big)}{6}\end{aligned}$$

Therefore by induction, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

3. Prove by induction that $n^3 - n$ is divisible by 3 for all positive integers.

*Solution:*

Step 1: Let $n = 1$ then $1^3 - 1 = 0$ which is divisible by 3.

Step 2: Assume true for $n$.

Step 3: Prove for $n + 1$

$$\begin{aligned}(n+1)^3 - (n+1) &= (n^3 + 3n^2 + 3n + 1) - (n+1) \\ &= n^3 - n + 3n^2 + 3n\end{aligned}$$

By our assumption $n^3 - n$ is divisible by 3 and $3n^2 + 3n$ is divisible by 3, therefore $n^3 - n + 3n^2 + 3n$ is divisible by 3. Therefore $(n+1)^3 - (n+1)$ is divisible by 3.

So by induction $n^3 - n$ is divisible by 3 for all positive integers.

4. Prove that the sequence $x_{n+1} = \dfrac{x_n + \sqrt{3x_n}}{2}$ is an increasing sequence where $x_1 = 1$.

*Solution:*

Step 1: Base case $x_2 = \dfrac{x_1 + \sqrt{3x_1}}{2} = \dfrac{1+\sqrt{3}}{2} > 1 = x_1$

Step 2: Assume up $x_n > x_{n-1}$

Step 3: Prove $x_{n+1} > x_n$

Since:

$$\begin{aligned} x_n &> x_{n-1} \\ \Rightarrow 3x_n &> 3x_{n-1} \\ \Rightarrow \sqrt{3x_n} &> \sqrt{3x_{n-1}} \end{aligned}$$

Because $\sqrt{x}$ is a monotonically increasing function.

Then

$$\begin{aligned} x_{n+1} &= \frac{x_n + \sqrt{3x_n}}{2} \\ &> \frac{x_{n-1} + \sqrt{3x_{n-1}}}{2} = x_n \end{aligned}$$

Therefore $x_{n+1} > x_n$. Thus by induction the sequence $\{x_n\}$ is an increasing sequence.

5. Prove that $n! > 2^n$ for all positive integers greater than or equal to 4.

*Solution:*

Step 1: Base case $4! = 4 \cdot 3 \cdot 2 = 24$, and $2^4 = 16$. Therefore $4! > 2^4$.

Step 2: Assume up to $n$.

Step 3: Prove for $n+1$.

$$\begin{aligned}(n+1)! &= (n+1) \cdot n! \\ &> (n+1) \cdot 2^n \\ &> 2 \cdot 2^n \\ &> 2^{n+1}\end{aligned}$$

Therefore by induction $n! > 2^n$ for all positive integers greater than or equal to 4.

6. Prove that for any real number $x > -1$ and any positive integer $n$, $(1+x)^n \geq 1 + nx$.

*Solution:*

Step 1: Base case for $n = 1$, $(1+x)^1 = 1 + x = 1 + 1 \cdot x$.

Step 2: Assume up to $n$.

Step 3: Prove for $n+1$

$$\begin{aligned}
(1+x)^{n+1} &= (1+x)^n(1+x) \\
&\geq (1+nx)(1+x) \\
&= 1 + x + nx + nx^2 \\
&= 1 + (n+1)x + nx^2 \\
&> 1 + (n+1)x \text{ since } nx^2 > 0
\end{aligned}$$

Therefore by induction for any real number $x > -1$ and any positive integer $n$, $(1+x)^n \geq 1 + nx$.

7. Using induction, prove that the sequence $a_{n+1} = \dfrac{2a_n}{3 + a_n}$ is monotone with $a_1 = 1$ and bounded below by 0.

*Solution:*

Step 1: Base case

$$\begin{aligned} a_2 &= \frac{2a_1}{3 + a_1} \\ &= \frac{2}{3 + 1} \\ &= \frac{1}{2} \\ &> 0 \end{aligned}$$

Step 2: Assume true for $n$, i.e. $a_n > 0$.

Step 3: Prove $a_{n-1} > 0$

$$a_{n+1} = \frac{2a_n}{3 + a_n}$$

Where $2a_n > 0$ and $3 + a_n > 0$, therefore $a_{n+1} > 0$. So by induction we know that all $a_n > 0$.

To solve that the sequence is monotone we check

$$\begin{aligned} a_{n+1} - a_n &= \frac{2a_n}{3 + a_n} - a_n \\ &= \frac{2a_n}{3 + a_n} - \frac{3a_n + {a_n}^2}{3 + a_n} \\ &= \frac{-a_n - {a_n}^2}{3 + a_n} \\ &< 0 \end{aligned}$$

Therefore, we have a sequence $\{a_n\}$ where each $a_n > 0$ and $a_{n-1} - a_n < 0$. So our sequence is monotonically decreasing.

8. A sequence $\{a_n\}$ is given by $a_1 = \sqrt{2}$, $a_n = \sqrt{2 + a_{n-1}}$

   (a) Show by induction that $\{a_n\}$ is increasing and bounded above by 3.

*Solution:*
Step 1: Base case $a_1 = \sqrt{2}$, $a_2 = \sqrt{2 + a_1} = \sqrt{2 + \sqrt{2}}$
Thus $a_1 < a_2 < 3$.

Step 2: Assume $a_{n-1} < a_n < 3$

Step 3: Prove $a_n < a_{n+1} < 3$

$$\begin{aligned} a_{n-1} &< a_n < 3 \\ 2 + a_{n-1} &< 2 + a_n < 2 + 3 \\ \sqrt{2 + a_{n-1}} &< \sqrt{2 + a_n} < \sqrt{3 + 2} \\ a_n &< a_{n+1} < \sqrt{5} < 3 \end{aligned}$$

Therefore by induction we see that $\{a_n\}$ is an increasing sequence bounded above by 3.

   (b) Find $\lim\limits_{a \to \infty} a_n$.

*Solution:*
Since our sequence is always increasing and has an upper bound, by the Monotone Convergence Theorem, a limit must exist.

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \sqrt{2 + a_{n-1}} \\ &= \sqrt{2 + \lim_{n\to\infty} a_{n-1}} \\ \Rightarrow L &= \sqrt{2 + L} \\ \Rightarrow L^2 &= L + 2 \\ \Rightarrow L^2 - L - 2 &= 0 \\ \Rightarrow (L+1)(L-2) &= 0 \end{aligned}$$

Therefore, the limit either equals $-1$ or 2. But since all sequence starts at $\sqrt{2}$ and is always increasing, we see that our limit must be 2.
That is $\lim\limits_{n\to\infty} a_n = 2$.