Surface Area of Rotations

Stephen Styles

September 27, 2020

Let $y = f(x) \geq 0$, if $f(x)$ is a smooth (continuously differentiable) function on the interval $[a, b]$, then the surface area generated by revolving the function about the line $y = r$ can be calculated by

$$S = \int_a^b 2\pi(f(x) - r)\sqrt{1 + \left(f'(x)\right)^2}\, dx.$$

Similarly, if $x = g(y) \geq 0$, where $g(y)$ is a smooth function on $[c, d]$, then the surface area generated by revolving the function around the line $x = r$ can be calculated by

$$S = \int_c^d 2\pi(g(y) - r)\sqrt{1 + \left(g'(y)\right)^2}\, dy.$$

Examples:

1. Find the surface area of the solid created by the rotating the function $f(x) = \sqrt{x}$ around the x-axis for $2 \leq x \leq 6$.

Solution:

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$S = \int_2^6 2\pi\sqrt{x}\sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2}\, dx$$

$$= \int_2^6 2\pi\sqrt{x}\sqrt{1 + \frac{1}{4x}}\, dx$$

$$= \int_2^6 2\pi\sqrt{x}\sqrt{\frac{4x + 1}{4x}}\, dx$$

$$= \int_2^6 \pi\sqrt{4x + 1}\, dx$$

$$= \frac{\pi}{6}(4x + 1)^{3/2}\Big|_2^6$$

$$= \frac{\pi}{6}\left(125 - 27\right)$$

$$= \frac{49\pi}{3}$$

2. Find the surface area of the solid created by the rotating the function $f(x) = \sqrt{16 - x^2}$ around the x-axis for $-1 \leq x \leq 1$.

Solution:

$$f'(x) = \frac{1}{\sqrt{16 - x^2}}(-2x)$$
$$= \frac{-2x}{\sqrt{16 - x^2}}$$

$$S = \int_{-1}^{1} 2\pi\sqrt{16 - x^2}\sqrt{1 + \left(\frac{-2x}{\sqrt{16 - x^2}}\right)^2}\, dx$$
$$= \int_{-1}^{1} 2\pi\sqrt{16 - x^2}\sqrt{1 + \frac{4x^2}{16 - x^2}}\, dx$$
$$= \int_{-1}^{1} 2\pi\sqrt{16 - x^2}\sqrt{\frac{16 - x^2 + 4x^2}{16 - x^2}}\, dx$$
$$= \int_{-1}^{1} 2\pi\sqrt{16 + 3x^2}\, dx$$
$$= 4\pi \int_{0}^{1} \sqrt{16 + 3x^2}\, dx$$

Let $x = \dfrac{4}{\sqrt{3}}\tan(\theta)$, then $dx = \dfrac{4}{\sqrt{3}}\sec^2(\theta)\, d\theta$.

$$S = 4\pi \int_{a}^{b} \sqrt{16 + 3\left(\frac{4}{\sqrt{3}}\tan(\theta)\right)^2} \frac{4}{\sqrt{3}}\sec^2(\theta)\, d\theta$$
$$= \frac{16\pi}{\sqrt{3}} \int_{a}^{b} \sqrt{16 + 16\tan^2(\theta)}\sec^2(\theta)\, d\theta$$
$$= \frac{16\pi}{\sqrt{3}} \int_{a}^{b} \sqrt{16\sec^2(\theta)}\sec^2(\theta)\, d\theta$$
$$= \frac{16\pi}{\sqrt{3}} \int_{a}^{b} 4\sec(\theta)\sec^2(\theta)\, d\theta$$
$$= \frac{64\pi}{\sqrt{3}} \int_{a}^{b} \sec^3(\theta)\, d\theta$$
$$= \frac{64\pi}{\sqrt{3}} \left(\frac{\sec(\theta)\tan(\theta) + \ln\left|\sec(\theta) + \tan(\theta)\right|}{2}\right)\Bigg|_{a}^{b}$$
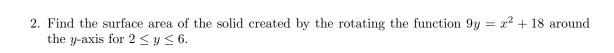$$= \frac{64\pi}{\sqrt{3}} \left(\frac{\frac{\sqrt{16+3x^2}}{4}\frac{\sqrt{3}x}{4} + \ln\left|\frac{\sqrt{16+3x^2}}{4} + \frac{\sqrt{3}x}{4}\right|}{2}\right)\Bigg|_{0}^{1}$$
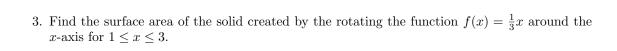$$= \left(2\pi x\sqrt{16 + 3x^2} + \frac{32\pi}{\sqrt{3}}\ln\left|\frac{\sqrt{16 + 3x^2}}{4} + \frac{\sqrt{3}x}{4}\right|\right)\Bigg|_{0}^{1}$$
$$= 2\pi\sqrt{19} + \frac{32\pi}{\sqrt{3}}\ln\left(\frac{\sqrt{19} + \sqrt{3}}{4}\right)$$

Questions:

1. Find the surface area of the solid created by the rotating the function $f(x) = x^3$ around the x-axis for $0 \leq x \leq 1$

2. Find the surface area of the solid created by the rotating the function $9y = x^2 + 18$ around the y-axis for $2 \le y \le 6$.

3. Find the surface area of the solid created by the rotating the function $f(x) = \frac{1}{3}x$ around the x-axis for $1 \leq x \leq 3$.

4. Find the surface area generated by revolving the curve $y = \frac{x^3}{3} + \frac{1}{4x}$, $1 \le x \le 3$ about the line $y = -2$.

5. Find the surface area of the solid created by the rotating the function $x = e^{-2y}$ around the y-axis for $0 \leq x \leq 1$. (Trig Sub Question. After you solve the integral, do not try to simplify after plugging in the values of the integral)