Arc Length

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Let $y = f(x)$ be a smooth function (twice differentiable) on the interval $[a, b]$ then the arc length of $f(x)$ on the interval is given by

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

We can also consider functions $x = g(y)$. Let $g(y)$ be a smooth function on the interval $[c, d]$ then the arc length of $g(y)$ is given by

$$s = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$

Note: The general equation for arc length is

$$s = \int \sqrt{dx^2 + dy^2}.$$

Examples:

1. Find the arc length of $f(x) = \dfrac{x^{3/2}}{3}$ from 0 to 5.

Solution:

$$f'(x) = \frac{x^{1/2}}{2}$$

$$\begin{aligned} s &= \int_0^5 \sqrt{1 + \left(\frac{\sqrt{x}}{2}\right)^2}\, dx \\ &= \int_0^5 \sqrt{1 + \frac{x}{4}}\, dx \\ &= \int_0^5 \frac{1}{2}(4 + x)^{1/2}\, dx \\ &= \frac{1}{3}(4 + x)^{3/2}\Big|_0^5 \\ &= \frac{19}{3} \end{aligned}$$

2. Find the arc length of a function whose derivative is given by $f'(x) = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)$ from $\frac{1}{2}$ to 2.

Solution:

$$\begin{aligned}
s &= \int_{1/2}^{2} \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2}\, dx \\
&= \int_{1/2}^{2} \sqrt{1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}}\, dx \\
&= \int_{1/2}^{2} \sqrt{\frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4}}\, dx \\
&= \int_{1/2}^{2} \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2}\, dx \\
&= \int_{1/2}^{2} \frac{x^2}{2} + \frac{1}{2x^2}\, dx \\
&= \frac{x^3}{6} - \frac{1}{2x}\bigg|_{1/2}^{2} \\
&= \frac{33}{16}
\end{aligned}$$

3. Find the arc length of $g(y) = y^2 - \frac{1}{8}\ln(y)$ from 1 to 2.

Solution:

$$g'(y) = 2y - \frac{1}{8y}$$

$$\begin{aligned}
s &= \int_{1}^{2} \sqrt{1 + \left(2y - \frac{1}{8y}\right)^2}\, dy \\
&= \int_{1}^{2} \sqrt{1 + 4y^2 - \frac{1}{2} + \frac{1}{64y^2}}\, dy \\
&= \int_{1}^{2} \sqrt{4y^2 + \frac{1}{2} + \frac{1}{64y^2}}\, dy \\
&= \int_{1}^{2} \sqrt{\left(2y + \frac{1}{8y}\right)^2}\, dy \\
&= \int_{1}^{2} 2y + \frac{1}{8y}\, dy \\
&= y^2 + \frac{\ln(y)}{8}\bigg|_{1}^{2} \\
&= 3 + \frac{\ln(y)}{8}
\end{aligned}$$

Questions:

1. Find the arc length of $f(x) = \ln(\cos(x))$ from 0 to $\dfrac{\pi}{4}$.

2. Find the arc length of $f(x) = \cosh(x)$ from 0 to 1.