Discrete Math 1510 Assignment 2

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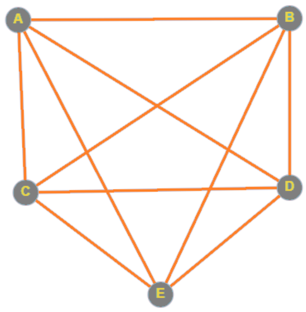
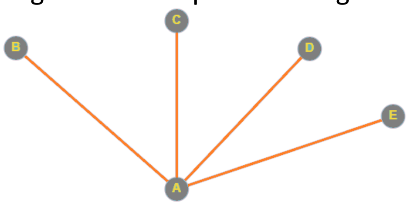
Question 1

For each of the following graph properties, find out if there is any simple graph on 5 vertices with that property. If you claim that there is no such graph, provide an argument supporting this claim, otherwise specify a graph with the corresponding property

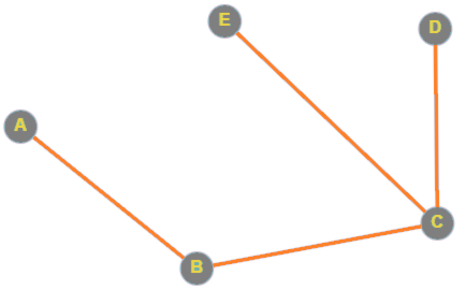
a) The following graph has five vertices. Each vertex has longest shortest path of 1 edge giving the graph a diameter of 1.



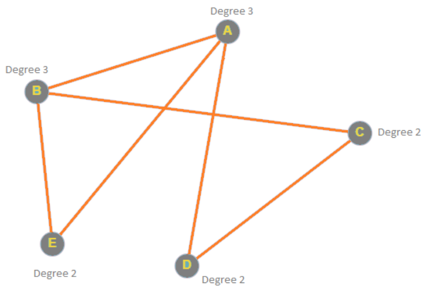
b) The following graph below of 5 vertices has the property of a diameter of 2 as each vertex has the longest shortest path of 2 edges.



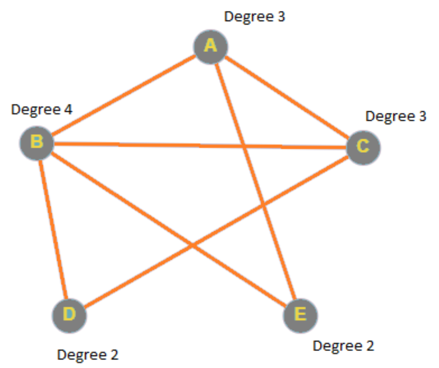
c) The following graph below of 5 vertices has the property of a diameter of 3 as vertex A needs to travel longest shortest path of 3 edges to D and E.



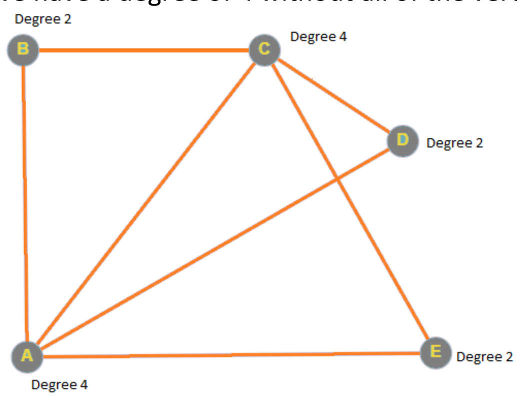
d) The following graph has 5 vertices with degree sequence 3,3,2,2,2



e) The following graph has 5 vertices with degree sequence 4,3,3,2,2



f) The following graph has 5 vertices with degree sequence 4,4,3,2,1
This graph is not possible as demonstrated below because you cannot have two vertices out of the five have a degree of 4 without all of the vertices having a degree equal to two or higher.

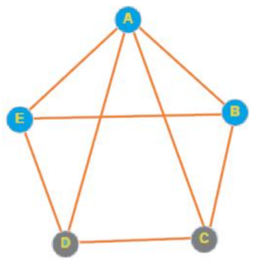


Question 2
(a) Can a bipartite graph contain K3 as a subgraph? Prove your answer.
K_3 is not a bipartite graph therefore cannot be a subgraph of a bipartite graph

(b) Consider the graph represented by the adjacency matrix below. Is it bipartite? Prove your answer.

Adjacency matrix =
(0, 1, 1, 1, 1)
(1, 0, 1, 0, 1)
(1, 1, 0, 1, 0)
(1, 0, 1, 0, 1)
(1, 1, 0, 1, 0)

The matrix above creates this graph



This graph is not Bipartite as the blue vertices show that k_3 is a sub graph. As explained in question 2a, k_3 is not Bipartite therefore, any graph with k_3 as a sub graph is not bipartite as well.

Question 3

Let G = (V, E) be the graph with vertex set V = {a, b, c, d} and edge set E = {e1, e2, e3, e4, e5, e6}, where

- **e1 has endpoints a and b.**
- **e2 has endpoints a and c.**
- **e3 has endpoints b and c.**
- **e4 has endpoints c and d.**
- **e5 has endpoints c and d.**
- **e6 has endpoints d and d.**

(a) Identify any parallel edges or loops in G

Parallel edges: parallel edges Exists between c and d as it connects e4 and e5.
Loops: There is also a loop on d as e6 has the endpoints d and d.

(b) Write down the adjacency matrix for G (indexed alphabetically)

a b c d
a (0, 1, 1, 0)
b (1, 0, 1, 0)
c (1, 1, 0, 2)
d (0, 0, 2, 2)

(c) Write down an adjacency listing for G (listed alphabetically)

Vertex	Adjacent To
A	b,c
B	a,c
C	a,b,d,d
D	c,c,d

(d) Write down the incidence matrix for G (indexed alphabetically)

e1 e2 e3 e4 e5 e6
a (1, 1, 0, 0, 0 0)
b (1, 0, 1, 0, 0, 0)
c (0, 1, 1, 1, 1, 0)
d (0, 0, 0, 1, 1, 2)

(e) Identify any cut vertices or bridges in G

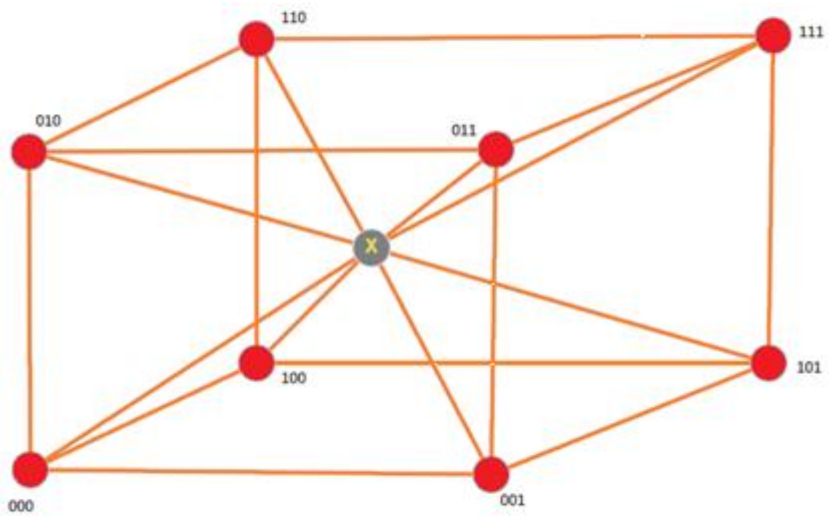
Cut vertex: There is one cut vertex because d becomes disconnected if you remove vertex c
Bridges: No bridges exist for this graph.

(f) How many edges in G?

Six edges.

Question 4:
Construct a graph G as follows. Start with Q3, which is the graph of a cube, with eight vertices labelled 000, 001, 010, 011, 100, 101, 110, 111 and twelve edges with the property that two vertices are connected if their labels differ in just one place. (E.g. there is an edge from 110 to 111 because they differ just in the rightmost place; but there is no edge from 100 to 111, because they differ in two places.) Now, append one extra vertex labelled x, and include an edge between x and each of the original vertices in Q3. Thus the graph G is now a simple graph containing exactly nine vertices and twenty edges.

Below is the graph with one added vertex giving it 9 vertex and 20 edges.

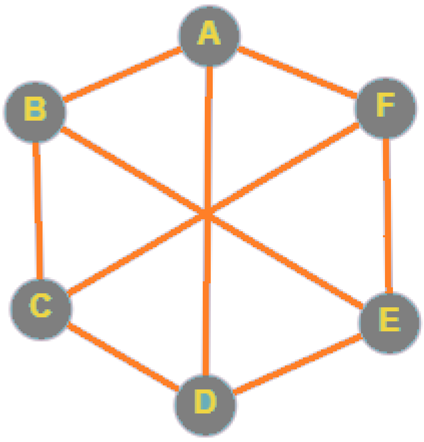


G

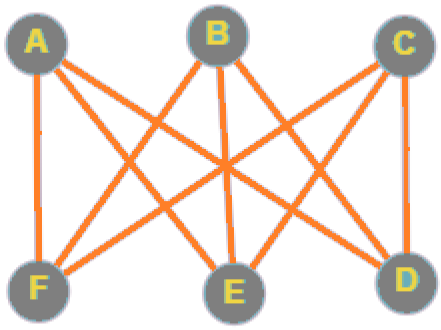
(a) Is G Hamiltonian? If yes, specify an explicit Hamiltonian cycle in order to prove it. If no, explain why not.
Yes. The Hamiltonian cycle for this graph is
000, 100, 110, 111, 101, x, 010, 011, 001 , 000

(b) Is G Eulerian? If yes, specify an explicit Eulerian cycle in order to prove it. If no, explain why not.
Yes. The Eulerian cycle for this graph is
000, x, 111, 101, x, 100, 101, 001, x, 110, 111, 011, x, 010, 011, 001, 000, 010, 110, 100, 000

Question 5:
Which two of the following three graphs are isomorphic? (The graphs are given diagrammatically first, and then below that by their adjacency matrices.) Specify an isomorphism between the isomorphic pair, and prove it is an isomorphism. Provide a clear argument why the third one is not isomorphic to the other two.



G_1



G_3

Graphs G_1 and G_3 are isomorphic. We prove this by the following:
G_1 and G_3 with be isomorphic if and only if the is bijection α from vertex set G_1 to vertex set G_3 such that α(u)α(v) ε E(H) <=> uv ε E(g)
G_1 will be isomorphic to G_3 <=> A_g = P. $A_h.P^T$ or in this case G_1 = P. $G_3.P^T$
Given both graphs and adjacency matrixes and using the follow alpha mapping
α:V(G_1)->(G_3)
α = (a,b,c,d,e,f,) G_1
(c,f,b,e,a,d) G_3

we can now create matrix **P** using the alpha mapping giving the result of

a b c d e f
a(0, 0, 1, 0, 0, 0)
b(0, 0, 0, 0, 0, 1)
c (0, 1, 0, 0, 0, 0)
d (0, 0, 0, 0, 1, 0)
e (1, 0, 0, 0, 0, 0)
f (0, 0, 0, 1, 0, 0)

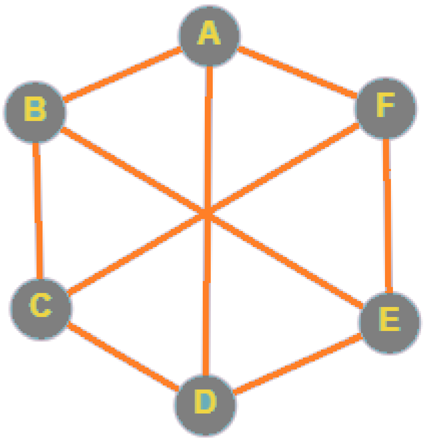
Now we create P^T Swapping row for columns using p matrix giving the result

a b c d e f
a(0, 0, 0, 0, 1, 0)
b(0, 0, 1, 0, 0, 0)
c (1, 0, 0, 0, 0, 0)
d (0, 0, 0, 0, 0, 1)
e (0, 0, 0, 1, 0, 0)
f (0, 1, 0, 0, 0, 0)

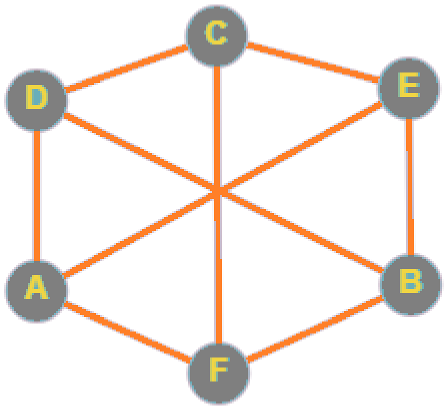
Now using matrix multiplication. We multiply P. G_3. P^T, giving the result.

a b c d e f
a(0, 1, 0, 1, 0, 1)
b(1, 0, 1, 0, 1, 0)
c (0, 1, 0, 1, 0, 1)
d (1, 0, 1, 0, 1, 0)
e (0, 1, 0, 1, 0, 1)
f (1, 0, 1, 0, 1, 0)

Which gives us the original matrix of the graph G_1 proving their isomorphism. You can also visually redraw the graph G_3 to be the same shape as G_1 as show below.



G_1



$G_3\ Re-drawn.$

The third option from the graphs G_2 is not isomorphic as because of the following three reasons:

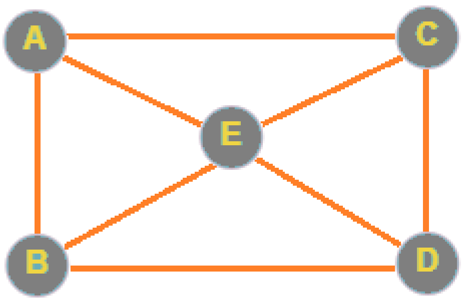
- G_2 compared to G_1 have different cycle lengths. G_1 has a minimal cycle length of 4 where G_2 has a minimal cycle length of 3. (A cycle length is defined as starting at any chosen vertex and taking the shortest path back to the same vertex.)

- No alpha mapping combination can be found that results in satisfying our condition
  G_2 and G_3 will be isomorphic if and only if the is bijection α from vertex set G_2 to vertex set G_3 such that α(u)α(v) ε E(H) <=> uv ε E(g)
  G_2 will be isomorphic to G_1 <=> A_g = P. $A_h.P^T$ or in this case G_2 = P. $G_1.P^T$
  And
  G_2 and G_1 will be isomorphic if and only if the is bijection α from vertex set G_2 to vertex set G_1 such that α(u)α(v) ε E(H) <=> uv ε E(g)
  G_2 will be isomorphic to G_1 <=> A_g = P. $A_h.P^T$ or in this case G_2 = P. $G_3.P^T$

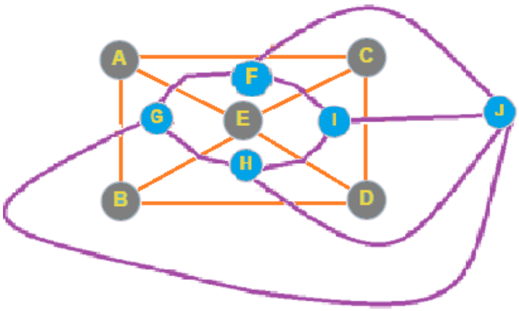
- Looking at G_2 diagrammatically it cannot be rearranged in any way to match the other graphs therefore failing to meet all conditions that make a graph isomorphic.

Question 6

Using this planer graph, I have created.

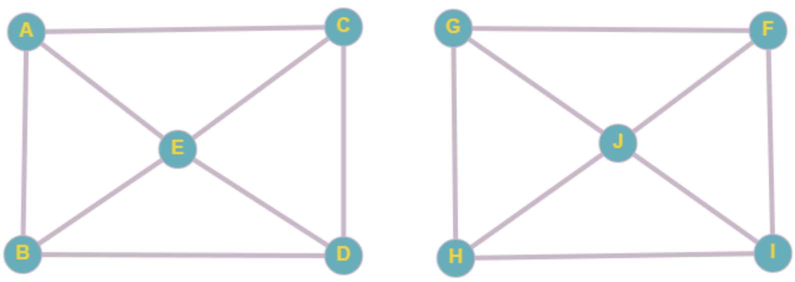


By placing a vertex on each face and connecting the corresponding vertices we get the following.



And by using the bijection of
F -> C
G -> A
H -> B
I -> D
J -> E

The graphs are isomorphic and re-drawn would look like this



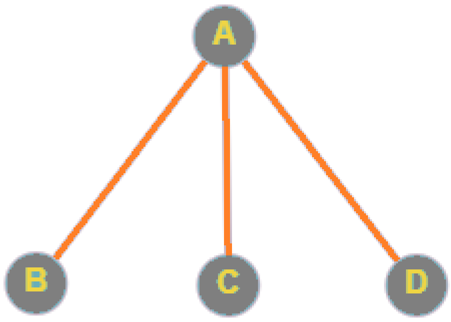
Question 7
a) Prove by induction on n: If d1, . . . , dn are any positive integers satisfying d1 + · · · + dn = 2(n − 1) then there exists a tree with degree sequence d1, . . . , dn

For n = 1 and n = 2 the claim is obviously true, as a single node has degree sequence sum 0 = 2 × 1 − 2, and two nodes can have at most one edge, so has degree sequence sum of 1 + 1 = 2 × 2 − 2 = 2 Now we will assume that it is true for some n ≥ 2, and let the degrees d1, . . . , dk, dk+1 of k + 1 vertices be given with d1 + · · · + dk + dk+1 = 2(k + 1) − 2. Since the degrees cannot all be 1 we can assume without loss of generality that dk+1 > 1. Then the degrees d1+· · ·+dk+dk+1−2 satisfy the condition of the induction hypothesis, so there is a tree with k vertices with these degrees. In that tree, add a (k+1)-th vertex, take the k-th vertex of degree dk+dk+1−2 remove dk+1−1 of its neighbours, attach them to the (k+1)-th vertex instead and join the k-th and the (k + 1)-th vertices by an edge. This new tree has the correct degree sequence.

b)

Explicitly describe all the trees (up to isomorphism) with degree sequence d1, d2, d3, d4 such that d1 + d2 + d3 + d4 = 6.

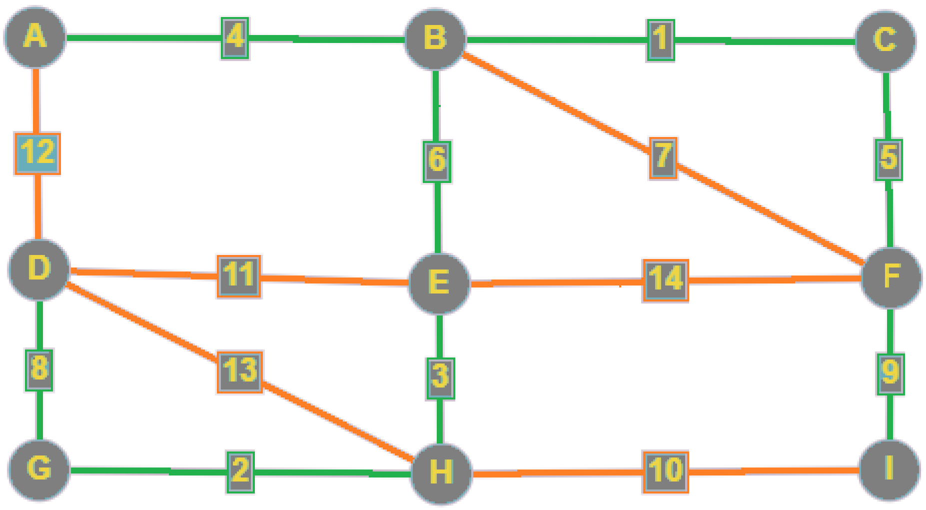
There are only two cases. The first case below has degree sequence 3, 1, 1, 1.



The second case below has degree sequence 1, 2, 2, 1

Question 8

a) Use Prim's and Kruskal's algorithms to find minimum spanning trees in the following weighted graph, which is specified first by its diagram and then by its formal definition.



b)) **Do the algorithms find the same tree or not? Is the weight the same (it should be).**

Yes both algorithms find the same tree.

Yes both weights are the same, the weight is 38.

C) Which algorithm do you prefer? Which is easier to implement by the means you are likely to use?

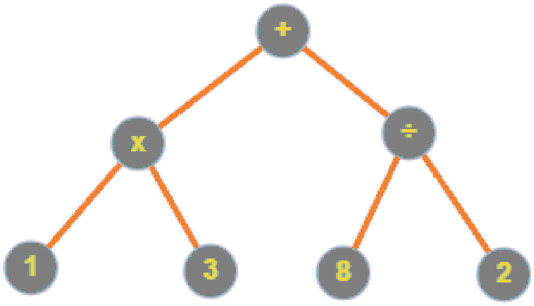
I find Prim's easier to implement but I find Kruskal's to be my preferred method of choice.

Question 9

The following is a prefix expression for an arithmetic expression involving single digit numbers.

+ × 1 3 ÷ 8 2

(a) Reconstruct the expression tree that the expression comes from.



(b) Read off the in-order expression, bracketing as appropriate.

(1 × 3) + (8 ÷ 2)

(c) Evaluate the expression: what number do you get?

(1 × 3) + (8 ÷ 2) = (3)+(4) = 7