Discrete Math 1510 Assignment 3

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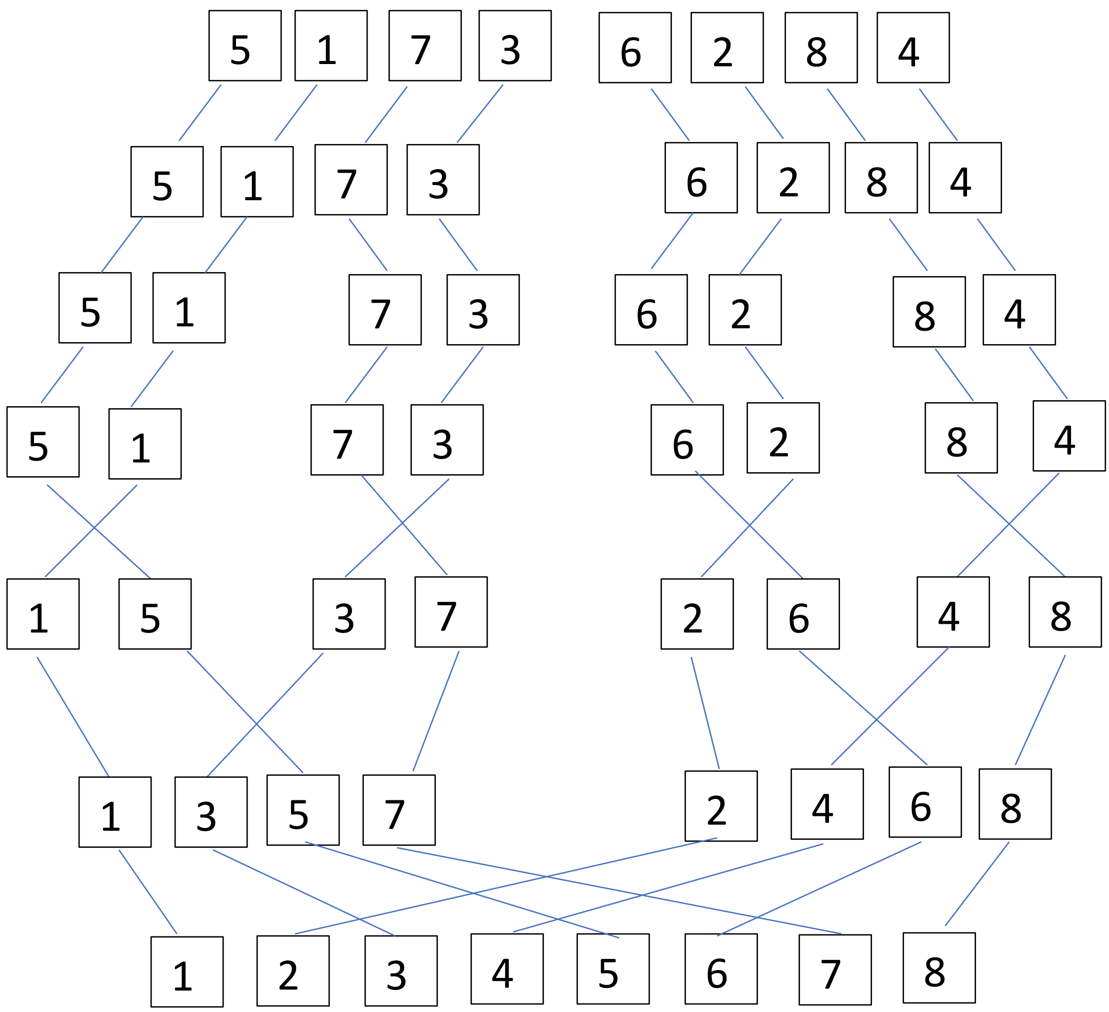
Question 1:

Apply the Merge Sort Algorithm to sort the finite sequence 5, 1, 7, 3, 6, 2, 8, 4 into increasing order



Question 2:

A recurrence relation which expresses the worst case number of comparisons required to implement the Merge Sort Algorithm for an input list of length n = 2^K is $c_k = 2c_{k-1} + 2^K - 1$ where C0 = 0.

(a) Check that this recurrence produces the right answer for k = 1, 2, 3 (which corresponds to n = 2, 4, 8).

As above in question one we can note that when merging both sides always had more than one element. This allows us to make a case for n =2 ,4, 8.

C0 = 0
C1 = 2C0 + 21 − 1 = 2 × 0 + 2 − 1 = 1
C2 = 2C1 + 22 − 1 = 2 × 1 + 4 − 1 = 5
C3 = 2C2 + 23 − 1 = 2 × 5 + 8 − 1 = 17

This gives us the comparisons 1, 5, 17 for n = 2 ,4 ,8

(b) Come up with a justification why this recurrence relation is correct in general.

This recurrence relation is correct because if we look at the current problem it is made up of two sub problems being the $2c_{k-1}$ term(right hand side). When merging 2k comparisons are made using c_{k-1} term. The −1 is due to not needing an more comparisons when one list has only one element and the other is empty.

Question 3

Consider the recurrence relation $c_k = 2c_{k-1}$ + 2k − 1. Solve the recurrence as follows:

(a) Find the general solution to the homogeneous part of the recurrence.

This Question was shown in lecture 15 The homogeneous recurrence is $c_k = 2c_{k-1}$ and the solution is $c_k = \alpha 2^k$

(b) Guess a specific solution of the form $c_k = k2^k + d$ for an arbitrary constant d. Substitute back into the recurrence to solve for d, to give a particular solution to the recurrence.

My guess would be $c_k = k2^k$ + d because the Ck term is how we define our guess, and the Ck−1 term, which we just replace every k by k − 1 as it's just a single step back in our sequence. This gives us $c_{k-1} = (k-1)2^{k-1}$ + d. When we sub back into the recurrence, we get
$k2^k + d$ = 2((k − 1)2^{k-1} +d) + 2^k− 1.
When solving for d we get

$k2^k + d$ = 2((k − 1)2^{k-1} +d) + 2^k− 1
$k2^k + d$ = 2(k − 1)2^{k-1} +2d + 2^k− 1
$k2^k + d$ = (k − 1)2^k +2d + 2^k− 1
$k2^k + d$ = $k2^k - 2^k$ +2d + 2^k− 1
$k2^k + d$ = $k2^k$+2d + 2^k− 1
d = 2d − 1
d =1

we get d = 1 and the sub back in the final solution $c_k = k2^k + 1$
(c) Add together the general solution to the homogeneous part, and the specific solution to the whole recurrence, to obtain a general solution to the original recurrence.

Adding both together we get $c_k = \alpha 2^k + k2^k + 1$

(d) Solve the recurrence relation subject to initial condition c0 = 0.
Using the initial condition c_o= 0 we simply substitute k_0 = 0 and then we get
$c_o = \alpha 2^0 + 0 \text{ x } 2^0 + 1$
Substituting in what we know about c_o and simplifying the right hand side, we get 0 = α + 1 which gives α = −1. Substituting this value for α back into our general solution we get
$c_k = -2^k + k2^k + 1$ which is the solution to the recurrence relation
$c_k = 2C_{k-1} + 2^k - 1$ with initial value c_o = 0.

Question 4

Solve the recurrence relation $\boldsymbol{a_n = 2a_{n-1} + 8a_n - 2}$ with initial conditions $\boldsymbol{a_0}$ = 4 and $\boldsymbol{a_1}$ = 10.

Given $a_n = 2a_{n-1} + 8a_n - 2$ and using the characteristic equation $r^2 - 2r - 8 = 0$ we factorise this and get $r_1 = -2 \; and \; r_2 = 4$ These will now produce the general solution
$a_n = \alpha(-2)^n + \beta(4)^n$. Now using the two initial conditions a_0 = 4 and a_1 = 10 we get two equations being
4 = α+β
10 = −2α + 4β
Solving these simultaneously gives α = 1, and β = 3.
Therefore, the solution to the recurrence is $a_n = (-2)^n + 3(4)^n$

Question 5

Solve the recurrence relation $\boldsymbol{b_n = 8b_{n-1} - 16b_{n-2}}$ with initial conditions $\boldsymbol{b_0}$ = 1 and $\boldsymbol{b_n}$ = 8.
Given $b_n = 8b_{n-1} - 16b_{n-2}$ and using the characteristic equation $r^2 - 8r + 16 = 0$ we factorise this and get repeated root 4. These will now produce the general solution $b_n = \alpha(4)^n + \beta n(4)^n$
Now using the two initial conditions $b_0 = 1$ and $b_1 =$ 8 we get two equations being
1 = α and 8 = 4α + 4β
Solving these simultaneously gives α = 1, and β = 1.
Therefore, the solution to the recurrence is $b_n = 4^n + n4^n - 4^n(n+1)$