Area of Ellipse
Stephen Giang

Proof. Notice the following equation of an ellipse centered at the origin in rectangular coordinates

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a \neq 0$ is the radius along the x-axis and $b \neq 0$ is the radius along the y-axis. Now we can rewrite the equation and solve for x.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad\qquad b^2x^2 = a^2b^2 - a^2y^2$$

$$b^2x^2 + a^2y^2 = a^2b^2 \qquad\qquad x^2 = a^2 - \frac{a^2y^2}{b^2}$$

Thus we get the following equation:

$$x = \pm\sqrt{a^2 - \frac{a^2}{b^2}y^2}$$

Now we can integrate the following to find the Area:

$$A = \int_{-b}^{b} \sqrt{a^2 - \frac{a^2}{b^2}y^2}\, dy \;=\; 2\int_0^b \sqrt{a^2 - \frac{a^2}{b^2}y^2}\, dy$$

We can now make the following substitutions:

$$y = b\sin\theta \qquad dy = b\cos\theta\, d\theta$$

With these substitutions, we can see the bounds change to $y = 0$ and $y = \pi/2$.

$$\begin{aligned} A &= 2\int_0^{\pi/2} \sqrt{a^2 - \frac{a^2}{b^2}\left(b^2\sin^2\theta\right)}\, b\cos\theta\, d\theta \\ &= 2\int_0^{\pi/2} \sqrt{a^2\left(1 - \sin^2\theta\right)}\left(b\cos\theta\right)\, d\theta \\ &= 2\int_0^{\pi/2} \left(a\cos\theta\right)\left(b\cos\theta\right)\, d\theta \\ &= 2ab\int_0^{\pi/2} \cos^2\theta\, d\theta \end{aligned}$$

$$\begin{aligned} &= 2ab\int_0^{\pi/2} \frac{(1+\cos 2\theta)}{2}\, d\theta \\ &= ab\int_0^{\pi/2} 1 + \cos 2\theta\, d\theta \\ &= ab\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} \end{aligned}$$

Evaluating this, we get our final result:

$$\boldsymbol{A = \pi\, a\, b}$$

Notice, a circle is simply an ellipse with $a = b = r$, thus we also get the area of a circle:

$$\boldsymbol{A = \pi\, r^2}$$

$\square$