Final Project - Complex Variable Analysis
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Find the function, which is analytic throughout circle C and it's interior, whose center is at the origin and whose radius is unity, and has the value:

$$\frac{a-\cos(\theta)}{a^2-2a\cos(\theta)+1}+i\frac{\sin(\theta)}{a^2-2a\cos(\theta)+1} \tag{1}$$

(note: $a > 1$ and θ is the vectorial angle)

at points on the circumference of C.

$$\begin{aligned}
f^{(n)}(0) &= \frac{n!}{2\pi i}\int_C \frac{f(z)}{z^{n+1}}dz && (2)\\
&= \frac{n!}{2\pi i}\int_0^{2\pi} e^{-ni\theta}\cdot id\theta\cdot\frac{a-\cos(\theta)+i\sin(\theta)}{a^2-2a\cos(\theta)+1} && (3)\\
&\text{let} z = e^{i\theta} && (4)\\
&= \frac{n!}{2\pi i}\int_0^{2\pi} \frac{e^{-ni\theta}}{a-e^{i\theta}}d\theta && (5)\\
&= \frac{n!}{2\pi i}\int_C \frac{dz}{z^n(a-z)} && (6)\\
&= \frac{d^n}{dz^n}\frac{1}{a-z}\bigg|_{z=0} && (7)\\
&= \frac{n!}{a^{n+1}} && (8)
\end{aligned}$$

Therefore with MacLaurin's Theorem

$$\begin{aligned}
f(z) &= f(0)+zf'(0)+\frac{z^2}{2!}f''(0)+...+\frac{z^n}{n!}f^{(n)}(0). && (9)\\
f(z) &= \sum_{n=0}^{\text{inf}}\frac{z^n}{a^{n+1}} && (10)\\
&= (a-z)^{-1} && (11)\\
&\forall x\in C && (12)
\end{aligned}$$

State why the components of velocity can be obtained from the stream function by means of the equations:

$$p(x,y) = \psi_y(x,y) \qquad\qquad q(x,y) = -\psi_x(x,y) \tag{13}$$

Let the vector $\vec{V}(x,y) = p(x,y) + iq(x,y)$ represent the velocity of a particle of a fluid at a point (x,y).

Let $\phi(x,y)$ represents the velocity potential of a particle of fluid at a point (x,y).

$$\phi(x,y) = \int_{(x_0,y_0)}^{(x,y)} p(s,t)ds + \int_{(x_0,y_0)}^{(x,y)} q(s,t)dt \tag{14}$$

When we differentiate both sides with respect to x and with respect to y, we get

$$\phi_x(x,y) = p(x,y) \tag{15}$$
$$\phi_y(x,y) = q(x,y) \tag{16}$$

So $\vec{V}(x,y) = \phi_x(x,y) + i\phi_y(x,y)$, which makes $\vec{V}(x,y)$ the gradient of $\phi(x,y)$.

Let $\psi(x,y)$ be a harmonic conjugate to ϕ (x,y). That is, Let $\psi(x,y)$ and $\phi(x,y)$ be harmonic such that they satisfy the Cauchy-Riemann Equations:

$$\psi_x(x,y) = \phi_y(x,y) \qquad\qquad \psi_y(x,y) = -\phi_x(x,y) \tag{17}$$

The Complex Potential is: $F(z) = \phi(x,y) + i\psi(x,y)$

$$\begin{aligned} F'(z) &= \phi_x(x,y) + i\psi_x(x,y) && (18)\\ &= \phi_y(x,y) - i\psi_y(x,y) && (19) \end{aligned}$$

$\vec{V}$(x,y) is the Complex Conjugate of F'(z):

$$\begin{aligned} \vec{V}(x,y) &= \bar{F}'(z) && (20)\\ &= \phi_x(x,y) + i\phi_y(x,y) && (21)\\ &= -1(\psi_y(x,y) + i\psi_x(x,y)) && (22)\\ &= i^2 && (23) \end{aligned}$$