Math 532: Homework 5
Due 10/02/19
Everyone turns in an individual copy.

Notes

Remember that an open disc $D_R(z)$ is defined by

$$\begin{aligned} D_R(z) &= \{w \in \mathbb{C} : |w - z| < R\} \\ &= \left\{(\tilde{x}, \tilde{y}) \in \mathbb{R}^2 : \sqrt{(\tilde{x} - x)^2 + (\tilde{y} - y)^2} < R\right\} \end{aligned}$$

where we have let $z = x + iy$ and $w = \tilde{x} + i\tilde{y}$. Similarly, we define a closed disc $\bar{D}_R(z)$ so that

$$\bar{D}_R(z) = \{w \in \mathbb{C} : |w - z| \leq R\}$$

Thus we think of discs in $\mathbb{C}$ or $\mathbb{R}^2$ interchangeably, and we will write $D_R(z)$ and $D_R((x, y))$ interchangeably. Similarly, when we define a domain $D \subset \mathbb{C}$, we are implicitly defining a domain $D \subset \mathbb{R}^2$.

As to differentiability requirements involving the real and imaginary parts of analytic functions, for a domain $D \subset \mathbb{C}$, a bit of notation honestly makes this so much easier. First define the set of continuous functions over D which we denote as $C(D)$ or

$$C(D) = \{g : D \to \mathbb{R} \text{ such that } g(x, y) \text{ is continuous for } (x, y) \in D\}.$$

In the same vein, we can define $C^1(D)$ so that

$$C^1(D) = \{g : D \to \mathbb{R} \text{ such that } g_x,\ g_y \in C(D)\}.$$

For now, we state without proof that if $f(z) = u + iv$ is analytic for $z \in D$, then $u, v \in C^2(D)$, where

$$C^2(D) = \{g : D \to \mathbb{R} \text{ such that } g_{xx},\ g_{yy},\ g_{xy},\ g_{yx} \in C(D)\}.$$

Note, if $u \in C^2(D)$, then $u_{xy} = u_{yx}$. Also keep in mind that continuity of second derivatives ensures the continuity of first derivatives and so forth, which is to say that

$$C^2(D) \subset C^1(D) \subset C(D).$$

Likewise, over closed discs $\bar{D}((x,y))$, keep in mind that $u \in C\left(\bar{D}((x,y))\right)$ implies there is some positive real value $M > 0$ such that

$$|u(\tilde{x},\tilde{y})| \leq M, \ (\tilde{x},\tilde{y}) \in \bar{D}((x,y)),$$

i.e. continuous functions are bounded over closed and bounded sets, though we only need the result over discs right now. Note, this implies that if $u \in C^2\left(\bar{D}((x,y))\right)$, then not only is u bounded on $\bar{D}((x,y))$, but every derivative up to the second ones are as well.

Problems

1. (2pts each) 2.26.4

2. (5pts) 2.26.6

3. (5pts) 2.27.1. Note, for $f(z) = u + iv$ analytic in $z \in D$, we again take for granted that $u, v \in C^2(D)$, which is what makes 2.27.1 work. Moreover, we see that by defining the Laplacian $\Delta u = u_{xx} + u_{yy}$, we have that
$$\Delta u = 0.$$
Using an identical argument, we can show that $\Delta v = 0$. A function with zero Laplacian is called *harmonic*. Thus 2.27.1 shows us that in polar coordinates
$$\Delta u = r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$

4. (5pts) Suppose $u(\tilde{x},\tilde{y})$ is harmonic, or $\Delta u = 0$, on the closed disc $\bar{D}_R((x,y))$ for some $R > 0$. For $(\tilde{x},\tilde{y}) \in \bar{D}_R((x,y))$, if we let
$$\tilde{x} = x + r\cos(\theta), \ \tilde{y} = y + r\sin(\theta), \ 0 \leq r \leq R,$$
then 2.27.1 tells us that
$$\Delta u = r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$
Defining
$$u_a(x,y,r) = \int_0^{2\pi} u(x + r\cos(\theta), y + r\sin(\theta))d\theta,$$
by integrating $\Delta u = 0$ in θ from 0 to 2π, show that
$$u_a(x,y,r) = c(x,y),$$
i.e. $u_a(x,y,r)$ does not depend on the radius r. Note, for
$$g \in C^1\left(\bar{D}_R((x,y))\right),$$

you will need to use the general identity

$$\int_0^{2\pi} \partial_r g(x + r\cos(\theta), y + r\sin(\theta))d\theta = \partial_r \int_0^{2\pi} g(x + r\cos(\theta), y + r\sin(\theta))d\theta,$$

which says that one can move a derivative through an integral when the integral is along a different coordinate. Also remember that if

$$u \in C^2\left(\bar{D}((x, y))\right),$$

then

$$u \in C\left(\bar{D}((x, y))\right),$$

and thus $u(\tilde{x}, \tilde{y})$ must be bounded on $\bar{D}((x, y))$.

5. (5pts) So, harmonic functions are interesting for a variety of reasons. Suppose that $u(x, y)$ is a harmonic function over some domain D. Show then that

$$\begin{aligned} u(x, y) =& \frac{1}{\pi R^2} \iint_{\bar{D}_R((x,y))} u(\tilde{x}, \tilde{y})\ dA \\ =& \frac{1}{\pi R^2} \int_0^R \int_0^{2\pi} u(x + r\cos(\theta), y + r\sin(\theta))rd\theta dr, \end{aligned}$$

or that the average of $u(x, y)$ over any closed disc $\bar{D}_R(z) \subset D$ is exactly itself. To do this

- Using the previous problem, show that the function $u_A(x, y, R)$ where

$$u_A(x, y, R) = \frac{1}{\pi R^2} \int_0^R \int_0^{2\pi} u(x + r\cos(\theta), y + r\sin(\theta))rd\theta dr$$

  does not depend on R and thus show the identity

$$u_A(x, y, R) = u_A(x, y, 0).$$

- Given that u is harmonic, we have that $u \in C^2\left(\bar{D}((x, y))\right)$. Thus we have the Mean-Value Theorem result that

$$u(x+r\cos(\theta), y+r\sin(\theta)) = u(x, y)+u_x(\xi, \eta)r\cos(\theta)+u_y(\xi, \eta)r\sin(\theta),$$

  where $(\xi, \eta) \in D_r((x, y))$. Moreover, since u must have bounded derivatives on the closed disc, we have that there exist two positive constants M_x and M_y such that

$$|u_x(\xi, \eta)r\cos(\theta)| \leq M_x r, \ \ |u_y(\xi, \eta)r\sin(\theta)| \leq M_y r.$$

Using this show that

$$\lim_{R\to 0} u_A(x,y,R) = u(x,y),$$

and thus that we have $u(x,y) = u_A(x,y,R)$ for any R such that $\bar{D}_R((x,y)) \subset D$.

6. (5pts) Let D be a bounded domain with boundary ∂D, and let u be harmonic on D, and let u be continuous on $\bar{D} = D \cup \partial D$, i.e. u is continuous on the closure of the bounded domain D. Define

$$M = \max_{(x,y)\in\bar{D}} u(x,y).$$

Using the previous problem, show that if $u(x,y) = M$ for (x,y) an interior point of $\bar{D}$, then u is a constant on $\bar{D}$. Likewise, show that for

$$m = \min_{(x,y)\in\bar{D}} u(x,y).$$

if $u(x,y) = m$ for (x,y) an interior point of $\bar{D}$, then u is a constant on $\bar{D}$. Using this, show that if u is not constant, then it must attain its maximum and minimum on the boundary ∂D.

Note, your argument will need to go something like this. Suppose $u(x,y) = M$ for an interior point (x,y). Then for any closed disc

$$\bar{D}_R((x,y)) \subset D,\ R > 0,$$

from the previous problem we must have that

$$u(x,y) = \frac{1}{\pi R^2}\iint_{\bar{D}_R((x,y))} u(\tilde{x},\tilde{y})\ dA.$$

But then

$$\frac{1}{\pi R^2}\iint_{\bar{D}_R((x,y))} \left(u(x,y) - u(\tilde{x},\tilde{y})\right)\ dA = 0,$$

while $u(x,y) - u(\tilde{x},\tilde{y}) \geq 0$, and thus we must have

$$u(\tilde{x},\tilde{y}) = u(x,y) = M.$$

Now, cover $\bar{D}$ by a finite number of discs and use path-connectivity in the domain and you have the maximum part of the proof. You can do the rest.