Today: 10/22

- Note: No office hour Thursday, 10/24
  No class on 10/31
- Today: ε-δ criterion & sequence def for continuity
  3.2 Extreme Value Theorem.

Example 1: Suppose $f(x) = 3x - 4$ where $f: \mathbb{R} \to \mathbb{R}$.
Prove f is continuous at $x_0 = 10$ using ε-δ criterion.

proof: Let $\varepsilon > 0$.
Let $\delta = \frac{\varepsilon}{3} > 0$.
Let $x \in \mathbb{R}$ and suppose $|x - 10| < \delta$.
~~Let x0~~ So $|x - 10| < \frac{\varepsilon}{3}$.
And $|3x - 4 - 26| < \varepsilon$.

SIDE: $|f(x) - f(x_0)| < \varepsilon$. **GOAL:** $|x - x_0| < \underset{\varepsilon}{\delta}$.

$$|3x - 4 - 26| < \varepsilon.$$

$$|3x - 30| < \varepsilon$$

$$|x - 10| < \frac{\varepsilon}{3}.$$

δ depends on

- x_0
- ε

Example: Suppose $g(x) = 2x^2 - 5x$ where $g: \mathbb{R} \to \mathbb{R}$.

Show g is continuous at $x_0 = 5$ using ε-δ criterion.

(GOAL)

SIDE: $|x-5| <$ ___

$|g(x) - g(5)| < \varepsilon.$

$|2x^2 - 5x - 25| < \varepsilon.$

$|(x-5)(2x+5)| < \varepsilon.$

$|(x-5)(2x+5)| <$ nicer $< \varepsilon.$

$\boxed{x \text{ is NEAR } 5}$

$4.9 < x < 5.1$

$2(4.9)+5 < 2x+5 < 15.2$

δ must be ≤ 0.1.

$|(x-5)(2x+5)| \leq |(x-5)(15.2)| < \varepsilon$

$|x-5| < \frac{\varepsilon}{15.2}.$

proof: Let $\varepsilon > 0$.

Let $\delta = \min\{0.1, \frac{\varepsilon}{15.2}\}$.

Let $x \in \mathbb{R}$ and suppose $|x-5| < \delta$.

So $4.9 < x < 5.1$ and $|x-5| < \frac{\varepsilon}{15.2}$.

So $2x+5 < 15.2$ and $(15.2)|x-5| < \varepsilon$.

Thus $|(2x+5)(x-5)| < 15.2|x-5| < \varepsilon$.

Thus $|2x^2 - 5x - 25| < \varepsilon$. $\square$

$|g(x) - g(5)| < \varepsilon$

Example 3: Suppose $h(x) = \dfrac{x^2+4}{x-1}$ for $h: \mathbb{R} \setminus \{1\} \to \mathbb{R}$.

Prove h is continuous at $5 = x_0$ using the sequential definition.

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proof: Let $\{a_n\} \subseteq \mathbb{R} \setminus \{1\}$ and suppose $\lim_{n\to\infty} a_n = 5$.

Notice $f(a_n) = \dfrac{a_n^2+4}{a_n - 1}$.

Using limit Laws of 2.1, we have $\lim_{n\to\infty} a_n^2 + 4 = 29$.

Also $\lim_{n\to\infty} a_n - 1 = 4$. So $\lim_{n\to\infty} \dfrac{a_n^2+4}{a_n-1} = \dfrac{29}{4} = f(5)$. $\blacksquare$

3.2 Extreme Value Theorem.

Old Calculus Problem:

"Given $f(x)$ on $[a,b]$, find absolute max/min values."

EVT - existence of solutions to this problem

~ doesn't assure us the routine you know works.

Definitions: Suppose $f: S \to \mathbb{R}$ where $S \subseteq \mathbb{R}$.

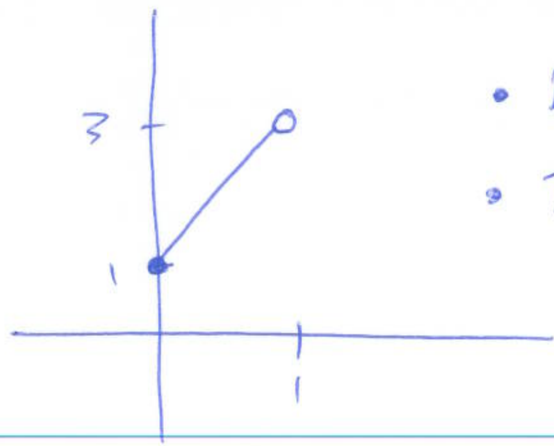
Let $x_0 \in S$. We say $f(x_0)$ is a maximum value and x_0 is a maximizer

iff

$\forall x \in S,\ f(x) \le f(x_0)$.

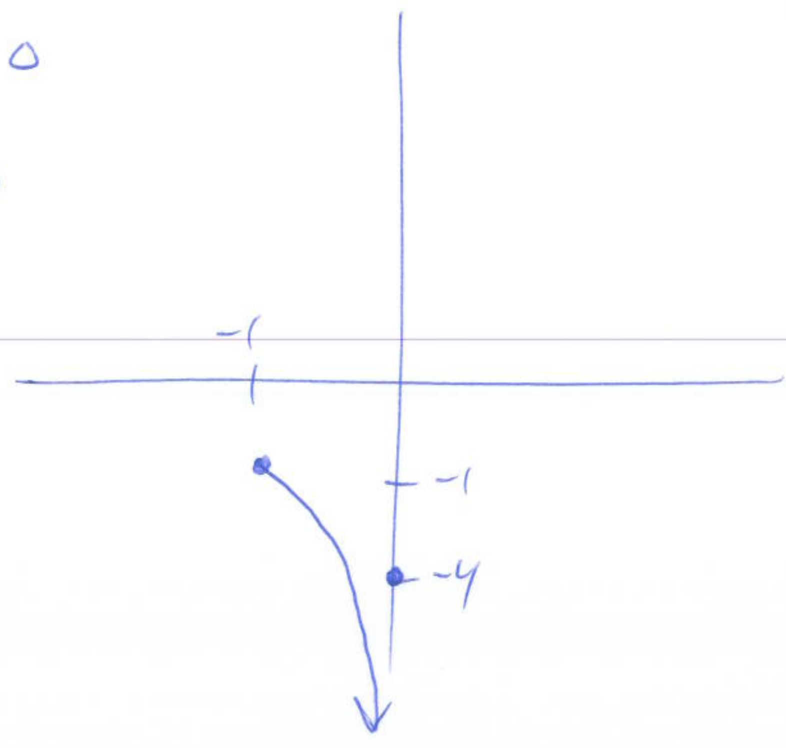
Non-Examples:

① Suppose $f(x) = 2x+1$ where $f : [0,1) \to \mathbb{R}$.



- Does not attain a maximum value.
- The image $f([0,1)) = [1,3)$ is bounded above.

② $f(x) = \begin{cases} \frac{1}{x}, & -1 \le x < 0 \\ -4, & \text{if } x = 0 \end{cases}$



- Does not attain a minimum.
- Does attain a maximum.
- Not continuous at $x_0 = 0$.

Thm 3.9 EVT.

A continuous function $f : [a,b] \to \mathbb{R}$ attains a max & min value.

Lemma 3.10 Suppose $f : [a,b] \to \mathbb{R}$ is continuous.

Then $\exists M \in \mathbb{R}$ s.t. $\forall x \in [a,b]$, $f(x) \leq M$.

(The image of f is bounded).

proof: Suppose not.

So $\forall M \in \mathbb{R}$, $\exists x \in [a,b]$ s.t. $f(x) > M$.