

**Homework 6**  
**Numerical Matrix Analysis**  
**Math 543**  
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**Problem TB-18.1:**

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \\ 1 & 1.0001 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 0.0001 \\ 4.0001 \end{bmatrix}$$

(a) What are the matrices  $A^+$  and  $P$  for this example? Give exact answers.

$$\begin{aligned} A^+ &= (A^*A)^{-1}A^* = \left( \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.0001 & 1.0001 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.0001 & 1.0001 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 3.0002 \\ 3.0002 & 3.00040002 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.0001 & 1.0001 \end{bmatrix} \\ &= \frac{1}{2 * 10^{-8}} \begin{bmatrix} 3.00040002 & -3.0002 \\ -3.0002 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.0001 & 1.0001 \end{bmatrix} \\ &= 50,000,000 \begin{bmatrix} .00020002 & -.0001 & -.0001 \\ -.0002 & .0001 & .0001 \end{bmatrix} \\ &= \begin{bmatrix} 10,001 & -5,000 & -5,000 \\ -10,000 & 5,000 & 5,000 \end{bmatrix} \\ P &= AA^+ = \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \\ 1 & 1.0001 \end{bmatrix} \begin{bmatrix} 10,001 & -5,000 & -5,000 \\ -10,000 & 5,000 & 5,000 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \end{aligned}$$

(b) Find the exact solutions  $x$  and  $y = Ax$  to the least squares problem  $Ax \approx b$

$$\begin{aligned} A^+Ax &= x = A^+b = \begin{bmatrix} 10,001 & -5,000 & -5,000 \\ -10,000 & 5,000 & 5,000 \end{bmatrix} \begin{bmatrix} 2 \\ 0.0001 \\ 4.0001 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ y &= Ax = AA^+b = Pb = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 2 \\ 0.0001 \\ 4.0001 \end{bmatrix} = \begin{bmatrix} 2 \\ 2.0001 \\ 2.0001 \end{bmatrix} \end{aligned}$$

(c) What are  $\kappa(A)$ ,  $\theta$ , and  $\eta$ ? From here on, numerical answers are acceptable.

$$\begin{aligned}\kappa(A) &= \|A\| \|A^+\| \approx 42429.2354161703 \\ \theta &= \cos^{-1} \frac{\|y\|}{\|b\|} \approx 0.684702873261185^R \\ \eta &= \frac{\|A\| \|x\|}{\|y\|} \approx 1.000000000555537\end{aligned}$$

(d) What are the four condition numbers of Theorem 18.1?

	$y$	$x$		$y$	$x$
$b$	$\frac{1}{\cos \theta}$	$\frac{\kappa(A)}{\eta \cos \theta}$	$=$	$b$	1.290977236078942
$A$	$\frac{\kappa(A)}{\cos \theta}$	$\kappa(A) + \frac{\kappa(A)^2 \tan \theta}{\eta}$		$A$	54775.17706651028
					1469883252.863362

(e) Give examples of perturbations  $\delta b$  and  $\delta A$  that approximately attain these four condition numbers.

Let  $\delta b = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$ , with  $a \in \mathbb{R}$ , Now notice that  $P\delta b = \delta b$ , and  $y = Pb$

$$\begin{aligned}\kappa_{b \rightarrow y} &= \frac{\|P(b + \delta b) - Pb\|}{\|y\|} \bigg/ \frac{\|\delta b\|}{\|b\|} \\ &= \frac{\|P\delta b = \delta b\|}{\|y\|} * \frac{\|b\|}{\|\delta b\|} \\ &= \frac{\|b\|}{\|y\|} \\ &= 1.290977236078942\end{aligned}$$

Let  $\delta b = \begin{bmatrix} 1 \\ -0.5 \\ -0.5 \end{bmatrix}$ , so  $\|A^+ \delta b\| = 21213.91056005508 = \|A^+\| \|\delta b\|$ , and  $x = A^+ b$ .

$$\kappa_{b \rightarrow x} = \frac{\|A^+ \delta b\|}{\|A^+ b\|} \bigg/ \frac{\|\delta b\|}{\|b\|} = \frac{21213.91056005508}{1.414213562373095} * \frac{4.472225399060293}{1.224744871391589} = 54775.17703608065$$

$$\text{Let } \delta A = \begin{bmatrix} 10^{-12} & -10^{-12} \\ -10^{-12} & 10^{-12} \\ 10^{-12} & -10^{-12} \end{bmatrix}, \text{ so } \tilde{A} = A + \delta A, \text{ and } \tilde{y} = \tilde{A}(\tilde{A}^* \tilde{A})^{-1} \tilde{A}^* b = \begin{bmatrix} 2.000000080004251 \\ 2.000099959999875 \\ 2.000099960003150 \end{bmatrix}$$

$$\begin{aligned} \kappa_{A \rightarrow y} &= \frac{\|\tilde{y} - y\|}{\|y\|} \bigg/ \frac{\|\delta A\|}{\|A\|} = \frac{9.798024764246804 * 10^{-8}}{3.464217086160112} * \frac{2.449571394482489}{2.449489742783178 * 10^{-12}} \\ &= 28284.46119148417 \end{aligned}$$

$$\text{Let the above be true, so } \tilde{x} = \tilde{A}^+ b = \begin{bmatrix} 1.001200092248810 \\ 0.998799987752254 \end{bmatrix}.$$

$$\begin{aligned} \kappa_{A \rightarrow x} &= \frac{\|\tilde{x} - x\|}{\|x\|} \bigg/ \frac{\|\delta A\|}{\|A\|} = \frac{0.001697102831166}{1.414213562373095} * \frac{2.449571394482489}{2.449489742783178 * 10^{-12}} \\ &= 1200072922.383391 \end{aligned}$$

**Problem PB-14.1:** We could use these matrices ( $A_k$ ) to least-squares-fit polynomials (of matching degree  $k$ ) to some data-set with 101 measurements. Is it necessarily better to have more model parameters (i.e. fitting a higher degree polynomial)? — Discuss.

Based on my observations, it is not necessarily better to have more model parameters as the Vandermonde Matrix is very ill-conditioned. So as  $k$  increases, (the degree of the polynomial gets higher), we reach larger and larger condition numbers showing us that the matrix is ill conditioned, meaning we lose a lot of accuracy when trying to use it. As you can see from the plot, you get insanely high condition numbers such as  $10^{27}$