Homework 4
Ordinary Differential Equations
Math 537
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Problem 1: Consider the Lorenz model:

$$\frac{dX}{dt} = -\sigma X + \sigma Y$$
$$\frac{dY}{dt} = -XZ + rX - Y$$
$$\frac{dZ}{dt} = XY - \beta Z$$

(a) Find the Jacobian matrix at the trivial critical point $(X, Y, Z) = (0, 0, 0)$.

Let $f_1 = \frac{dX}{dt}$, $f_2 = \frac{dY}{dt}$, $f_3 = \frac{dZ}{dt}$. Notice the following Jacobian:

$$J(X,Y,Z) = \begin{bmatrix} \frac{\partial f_1}{\partial X} & \frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial Z} \\ \frac{\partial f_2}{\partial X} & \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial Z} \\ \frac{\partial f_3}{\partial X} & \frac{\partial f_3}{\partial Y} & \frac{\partial f_3}{\partial Z} \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ -Z + r & -1 & -X \\ Y & X & -\beta \end{bmatrix}$$

Now we get the following Jacobian at the trivial critical point:

$$J(0,0,0) = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix}$$

(b) Choose $\sigma = 10$. Perform a (linear) stability analysis in r, β-space using the matrix in (a). [Hint: Describe the regions where the Jacobian matrix has real and/or complex eigenvalues.]

Notice the following Jacobian at the trivial critical point with $\sigma = 10$:

$$J(0,0,0) = \begin{bmatrix} -10 & 10 & 0 \\ r & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix}$$

Notice we can find the eigenvalues by solving $det(J - \lambda I) = 0$:

$$(-10 - \lambda)(-1 - \lambda)(-\beta - \lambda) - 10r(-\beta - \lambda) = 0$$
$$-(\beta + \lambda)\Big((10 + \lambda)(1 + \lambda) - 10r\Big) = 0$$
$$-(\beta + \lambda)(\lambda^2 + 11\lambda + 10(1 - r)) = 0$$

Thus we get the following eigenvalues:

$$\lambda_1 = -\beta, \qquad \lambda_{2,3} = \frac{-11 \pm \sqrt{121 - 4(10)(1 - r)}}{2}$$

Notice we can the following:

$$\begin{aligned} 121 - 4(10)(1-r) &= 0 \\ 121 - 40 + 40r &= 0 \\ r &= \frac{-81}{40} \end{aligned} \qquad\qquad \begin{aligned} 121 - 4(10)(1-r) &= 121 \\ (1-r) &= 0 \\ r &= 1 \end{aligned}$$

Notice the linear stability analysis at the specified r, β spaces.

(a) Let $r < \frac{-81}{40}$

We get the following:

$$\lambda_{2,3} = \frac{-11 \pm i\sqrt{|121 - 4(10)(1-r)|}}{2}$$

If we let $\beta > 0$, we get $\lambda_1 < 0$, thus giving us 1 negative real eigenvalue and 2 complex eigenvalues with negative real parts. This results in a **Spiral Sink**.

If we let $\beta < 0$, we get $\lambda_1 > 0$, thus giving us 1 positive real eigenvalue and 2 complex eigenvalues with negative real parts. This results in a **Saddle Focus**.

(b) Let $r = \frac{-81}{40}$

We get the following:

$$\lambda_{2,3} = \frac{-11}{2}$$

If we let $\beta > 0$, we get $\lambda_1 < 0$, thus giving us 3 negative eigenvalues. This results in a **Sink**.

If we let $\beta < 0$, we get $\lambda_1 > 0$, thus giving us 1 positive eigenvalue and 2 negative eigenvalues. This results in a **Saddle**.

(c) Let $\frac{-81}{40} < r < 1$

We get the following:

$$\lambda_2 = \frac{-11 + \sqrt{121 - 4(10)(1-r)}}{2} < 0, \qquad \lambda_3 = \frac{-11 - \sqrt{121 - 4(10)(1-r)}}{2} < 0$$

If we let $\beta > 0$, we get $\lambda_1 < 0$, thus giving us 3 negative eigenvalues. This results in a **Sink**.

If we let $\beta < 0$, we get $\lambda_1 > 0$, thus giving us 1 positive eigenvalue and 2 negative eigenvalues. This results in a **Saddle**.

(d) Let $r = 1$

We get the following:

$$\lambda_2 = \frac{-11 + 11}{2} = 0, \qquad \lambda_3 = \frac{-11 - 11}{2} = -11$$

If we let $\beta > 0$, we get $\lambda_1 < 0$, thus giving us a zero eigenvalue, and 2 negative eigenvalues. This results in a **Attractive Line of Equilibrium**

If we let $\beta < 0$, we get $\lambda_1 > 0$, thus giving us a zero eigenvalue, 1 positive eigenvalue, and 1 negative eigenvalue. This results in a **Saddle Around a Line of Equilibrium**.

(e) Let $r > 1$

We get the following:

$$\lambda_2 = \frac{-11 + \sqrt{121 - 4(10)(1-r)}}{2} > 0, \qquad \lambda_3 = \frac{-11 - \sqrt{121 - 4(10)(1-r)}}{2} < 0$$

If we let $\beta > 0$, we get $\lambda_1 < 0$, thus giving us 1 positive eigenvalues and 2 negative eigenvalue. This results in a **Saddle**

If we let $\beta < 0$, we get $\lambda_1 > 0$, thus giving us 2 positive eigenvalues and 1 negative eigenvalue. This results in a **Saddle**

Problem 2: Consider the non-dissipative Lorenz model:

$$\frac{dX}{dt} = \sigma Y$$
$$\frac{dY}{dt} = -XZ + rX$$
$$\frac{dZ}{dt} = XY$$

(a) Find critical points.

Notice we can find the critical points by solving:

$$\begin{bmatrix} \frac{dX}{dt} \\ \frac{dY}{dt} \\ \frac{dZ}{dt} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & \sigma & 0 \\ r & 0 & -X \\ 0 & X & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

This results in the critical points $\boldsymbol{(0,0,B),(C,0,r)}$ $\forall B, C \in \mathbb{R}$.

(b) Find the Jacobian matrix at critical points(s).

Let $f_1 = \frac{dX}{dt}$, $f_2 = \frac{dY}{dt}$, $f_3 = \frac{dZ}{dt}$. Notice the following Jacobian:

$$J(X,Y,Z) = \begin{bmatrix} \frac{\partial f_1}{\partial X} & \frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial Z} \\ \frac{\partial f_2}{\partial X} & \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial Z} \\ \frac{\partial f_3}{\partial X} & \frac{\partial f_3}{\partial Y} & \frac{\partial f_3}{\partial Z} \end{bmatrix} = \begin{bmatrix} 0 & \sigma & 0 \\ -Z+r & 0 & -X \\ Y & X & 0 \end{bmatrix}$$

Now we get the following Jacobian at the critical points:

$$J(0,0,B) = \begin{bmatrix} 0 & \sigma & 0 \\ -B+r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad J(C,0,r) = \begin{bmatrix} 0 & \sigma & 0 \\ 0 & 0 & -C \\ 0 & C & 0 \end{bmatrix}$$

(c) Perform a linear stability analysis at each of the critical points.

For $J(0,0,B)$, we get the following eigenvalues:

$$-\lambda^3 - \sigma(-B+r)(-\lambda) = 0$$
$$-\lambda\left(\lambda^2 - \sigma(-B+r)\right) = 0$$
$$\lambda_1 = 0, \quad \lambda_{2,3} = \pm\sqrt{\sigma(-B+r)}$$

If $\sigma(-B+r) > 0$, then we get 1 zero eigenvalue, 1 negative eigenvalue, and 1 positive eigenvalue. This would lead to a **Saddle Around a Line of Equilibrium**.

If $\sigma(-B+r) < 0$, then we get 1 zero eigenvalue, and two complex eigenvalues with zero real parts. This would lead to a **Center**

For $J(C,0,r)$, we get the following eigenvalues

$$-\lambda\left(\lambda^2 + C^2\right) = 0$$
$$\lambda_1 = 0, \quad \lambda_{2,3} = \pm i|C|$$

If $C \neq 0$, we get 1 zero eigenvalue, and 2 complex eigenvalues with zero real parts. This would lead to a **Center**

If $C = 0$, we get 3 zero eigenvalues with **No Phase Portrait**.

Problem 3: Consider the following harmonic oscillators:

$$\frac{d^2x_1}{dt^2} = -k_1x_1$$
$$\frac{d^2x_2}{dt^2} = -k_2x_2$$

Let $k_1 = 4\omega_1^2$ and $k_2 = \omega_2^2$.

(a) Convert the above equations into a linear system with four first-order differential equations. Find the matrix A that represents the 4D system.

Let $\frac{dx_1}{dt} = y_1$ and $\frac{dx_2}{dt} = y_2$ such that:

$$y_1 = \frac{dx_1}{dt}, \qquad y_1' = \frac{d^2x_1}{dt^2}, \qquad\qquad y_2 = \frac{dx_2}{dt}, \qquad y_2' = \frac{d^2x_2}{dt^2}$$

Using this and letting $k_1 = 4\omega_1^2$ $k_2 = \omega_2^2$, we get the following:

$$X' = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{d^2x_1}{dt^2} \\ \frac{dx_2}{dt} \\ \frac{d^2x_2}{dt^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4\omega_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_2^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix} = Ax$$

(b) Find the eigenvalues and eigenvectors of A in the 4-D phase space.

Notice we can find the eigenvalues by solving $det(J - \lambda I) = 0$:

$$-\lambda\Big(- \lambda(\lambda^2 + \omega_2^2)\Big) + 4\omega_1^2(\lambda^2 + \omega_2^2) = 0$$
$$(\lambda^2 + 4\omega_1^2)(\lambda^2 + \omega_2^2) = 0$$

Thus we get $\boldsymbol{\lambda_{1,2} = \pm 2i\omega_1}$ and $\boldsymbol{\lambda_{3,4} = \pm i\omega_2}$.

Notice we can find V_1 by solving $(A - \lambda I)V_1 = 0$ with $\lambda = 2i\omega_1$

$$\begin{pmatrix} -2i\omega_1 & 1 & 0 & 0 \\ -4\omega_1^2 & -2i\omega_1 & 0 & 0 \\ 0 & 0 & -2i\omega_1 & 1 \\ 0 & 0 & -\omega_2^2 & -2i\omega_1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Notice we get that $\boldsymbol{V_1 = (1,\ 2i\omega_1,\ 0,\ 0)^T}$. Because $\lambda_2 = -2i\omega_1$ is the conjugate of λ_1, then V_2 is the conjugate of V_1, such that $\boldsymbol{V_2 = (1,\ -2i\omega_1,\ 0,\ 0)^T}$.

Notice we can find V_3 by solving $(A - \lambda I)V_3 = 0$ with $\lambda = i\omega_2$

$$\begin{pmatrix} -i\omega_2 & 1 & 0 & 0 \\ -4\omega_1^2 & -i\omega_2 & 0 & 0 \\ 0 & 0 & -i\omega_2 & 1 \\ 0 & 0 & -\omega_2^2 & -i\omega_2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Notice we get that $\boldsymbol{V_3 = (0,\ 0,\ 1,\ i\omega_2)^T}$. Because $\lambda_4 = -i\omega_2$ is the conjugate of λ_3, then V_4 is the conjugate of V_3, such that $\boldsymbol{V_4 = (0,\ 0,\ 1,\ -i\omega_2)^T}$.

(c) Find the linear map T using (b) and compute $T^{-1}AT$.

Notice the following:

$$T = (V_1, V_2, V_3, V_4) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2i\omega_1 & -2i\omega_1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & i\omega_2 & -i\omega_2 \end{pmatrix}$$

$$\begin{aligned} T^{-1}AT &= \begin{pmatrix} 1/2 & 1/4i\omega_1 & 0 & 0 \\ 1/2 & -1/4i\omega_1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2i\omega_2 \\ 0 & 0 & 1/2 & -1/2i\omega_2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -4\omega_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_2^2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2i\omega_1 & -2i\omega_1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & i\omega_2 & -i\omega_2 \end{pmatrix} \\ &= \begin{pmatrix} i\omega_1 & 1/2 & 0 & 0 \\ -i\omega_1 & 1/2 & 0 & 0 \\ 0 & 0 & i\omega_2/2 & 1/2 \\ 0 & 0 & -i\omega_2/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2i\omega_1 & -2i\omega_1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & i\omega_2 & -i\omega_2 \end{pmatrix} \\ &= \begin{pmatrix} 2i\omega_1 & 0 & 0 & 0 \\ 0 & -2i\omega_1 & 0 & 0 \\ 0 & 0 & i\omega_2 & 0 \\ 0 & 0 & 0 & -i\omega_2 \end{pmatrix} \end{aligned}$$

Notice this is in the form of $T^{-1}AT$ being a matrix with its diagonal consisting of its eigenvalues and the other entries being zero.

Problem 4: Consider the following matrix:

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

(a) Find the eigenvector(s) and generalized eigenvector(s) associated with the matrix A.

Notice that A is an upper triangular matrix. This means that we get a repeated eigenvalue of $\boldsymbol{\lambda = 2}$.

Notice, we can find V_1 by solving $(A - \lambda I)V_1 = 0$:

$$\begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Because of the second and third row, we get that $y = z = 0$, so we let $x = 1$, such that $\boldsymbol{V_1 = (1, 0, 0)^T}$.

Notice, we can find V_2 by solving $(A - \lambda I)V_2 = V_1$:

$$\begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Because we need $3y = 1$, we get $y = 1/3$. Because of the second row, we get that $z = 0$. Lastly, we can let $x = 1$, such that $\boldsymbol{V_2 = (1, 1/3, 0)^T}$.

Notice, we can find V_3 by solving $(A - \lambda I)V_3 = V_2$:

$$\begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1/3 \\ 0 \end{pmatrix}$$

Because we need $3y = 1$, we get $y = 1/3$. Because we need $-z = 1/3$, we get $z = -1/3$. Lastly, we can let $x = 1$, such that $\boldsymbol{V_2 = (1, 1/3, -1/3)^T}$.

(b) Construct a linear map T using the eigenvector(s) and generalized eigenvector(s) in (a) and compute $T^{-1}AT$.

Notice the linear map T and $T^{-1}AT$:

$$T = \begin{pmatrix} V_1 & V_2 & V_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1/3 & 1/3 \\ 0 & 0 & -1/3 \end{pmatrix}$$

$$T^{-1}AT = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} 2 & 3 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1/3 & 1/3 \\ 0 & 0 & -1/3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Notice that $T^{-1}AT$ is in the form of case (iii) with its diagonal consisting of the repeated eigenvalue, $\lambda = 2$ and its superdiagonal consisting of ones.