Math 337 - Elementary Differential Equations

Lecture Notes – Linear Differential Equations

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Spring 2020

Outline

Introduction

Introduction

- Classical Differential Equation from Newton
  - Falling cat
- Linear Differential Equation
  - Previously, Malthusian growth, Radioactive decay, Newton's law of cooling
  - Solve general 1^{st} order linear differential equation
- Applications
  - Mercury build-up in fish
  - Lake pollution (time varying)

Falling Cat 1

Falling Cat

- Cats evolved to be stealthy animals with quick reflexes and an extremely good jumping ability
- The cat family has the best mammalian predators on this planet
- Smaller cats are adapted to hunting in trees
- Cats evolved a very flexible spine, which aids in their ability to spring for prey, absorb shock from their lightning fast strikes, and rapidly rotate their bodies in mid air
- With their very sensitive inner ear for balance, which is combined with quick reflexes and a flexible spine, a cat that falls is capable of righting itself very rapidly, insuring that it lands on the ground feet first

Falling Cat

Falling Cat - Scientific Studies

- This property of falling feet first has been admired by humans for many years
- There was a study in the Annals of Improbable Research (1998) on the number of times a particular cat ended up on its feet when dropped from several different heights
- There was a scientific study of cats falling out of New York apartments, where paradoxically the cats falling from the highest apartments actually fared better than ones falling from an intermediate height

[1] Jared M. Diamond (1988), Why cats have nine lives, Nature 332, pp 586-7

Falling Cat

- Left is the dynamics of a cat falling from an inverted position and ending on its feet
- The full dynamics involve complex partial differential equations
- A cat can react sufficiently fast that this inversion process happens in **about 0.3 seconds**
- **With this information, determine the minimum height from which a cat can be dropped to insure that it lands on its feet**

Falling Cat

Model for the Falling Cat: Newton's law of motion

- **Mass times acceleration is equal to the sum of all the forces** acting on the object
    - Equation for **Falling Cat**

$$ma = -mg \qquad \text{or} \qquad a = -g$$

    - m is the mass of the cat
    - a is the acceleration
    - $-mg$ is the force of gravity (assuming up is positive)
    - Ignore other forces (air resistance)
- g is a constant ($g = 979$ cm/sec^2 at a latitude like San Diego when you add centripetal acceleration to the standard value given for g, which is 980.7 cm/sec^2)

Falling Cat

Height, Velocity, and Acceleration

- Let $h(t)$ be the height (or position) of the cat at any time t
- **Velocity** and **Acceleration** satisfy:

$$\frac{dh}{dt} = v(t) \qquad \text{and} \qquad \frac{d^2h}{dt^2} = \frac{dv}{dt} = a$$

- The **initial conditions** for falling off a limb:

$$h(0) = h_0 > 0 \qquad \text{and} \qquad v(0) = 0$$

Falling Cat

Differential Equation: Velocity

- The velocity of the falling cat satisfies the first order linear differential equation

$$v'(t) = -g \quad \text{with} \quad v(0) = 0$$

- Integrate for the solution

$$v(t) = -\int g\,dt = -gt + c$$

- The initial condition gives $v(0) = c = 0$, so

$$v(t) = -gt$$

Falling Cat

Differential Equation: Height Since $\frac{dh}{dt} = v(t)$

- The height of the falling cat satisfies the first order linear differential equation

$$h'(t) = -gt \quad \text{with} \quad h(0) = h_0$$

- Integrate for the solution

$$h(t) = -\int gt\, dt = -g\frac{t^2}{2} + c$$

- The initial condition gives $h(0) = c = h_0$, so

$$h(t) = h_0 - g\frac{t^2}{2}$$

Falling Cat

Solution - Height of Cat:

- The height of the cat any time t satisfies

$$h(t) = -\frac{gt^2}{2} + h_0$$

- With $g = 979$ cm/sec^2, the height in cm is

$$h(t) = h_0 - 489.5\,t^2$$

- At $t = 0.3$ sec

$$h(0.3) = h_0 - 489.5(0.3)^2 = h_0 - 44.055 \text{ cm} > 0$$

- Thus, the cat must be higher than 44.1 cm for it to have sufficient time to right itself before hitting the ground (This is about 1.5 feet)

Differential Equation with Only Time Varying Function

Definition (Differential Equation with Time Varying Function)

The simplest first order (linear) differential equation has only a time varying nonhomogeneous function, $f(t)$,

$$\frac{dy}{dt} = f(t). \tag{1}$$

Theorem (Solution)

Consider the differential equation with only a time varying nonhomogeneous function, (1). Provided $f(t)$ is integrable, the solution satisfies:

$$y(t) = \int f(t)\,dt.$$

Differential Equation Example

DE Example: Initial Value Problem

$$\frac{dy}{dt} = 2\,t - \sin(t), \qquad y(0) = 3$$

Solution:

$$y(t) = \int (2\,t - \sin(t))\,dt = t^2 + \cos(t) + C$$

$$y(0) = 1 + C = 3, \quad \text{so} \quad C = 2$$

$$y(t) = t^2 + \cos(t) + 2$$

General Linear Differential Equation

Definition (General Linear Differential Equation)

A differential equation that can be written in the form

$$\frac{dy}{dt} + p(t)y = g(t) \tag{2}$$

is said to be a **first order linear differential equation** with dependent variable, y, and independent variable, t.

Integrating Factor

Definition (Integrating Factor)

Consider an undetermined function $\mu(t)$ with

$$\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}y.$$

The function $\mu(t)$ is an **integrating factor** for (2) if it satisfies the differential equation

$$\frac{d\mu(t)}{dt} = p(t)\mu(t).$$

Solving a Linear DE 1

Consider the **Linear Differential Equation**

$$\frac{dy}{dt} - 2y = 4 - t.$$

Multiply the equation by the undetermined function, $\mu(t)$, so

$$\mu(t)\frac{dy}{dt} - 2\mu(t)y = \mu(t)(4 - t).$$

If $\mu(t)$ is an integrating factor, then

$$\frac{d\mu(t)}{dt} = -2\mu(t) \qquad \text{or} \qquad \mu(t) = e^{-2t}$$

Solving a Linear DE 2

With the **integrating factor**, our example can be write

$$e^{-2t}\frac{dy}{dt} - 2e^{-2t}(t)y = \frac{d}{dt}\left[e^{-2t}y\right] = (4-t)e^{-2t}.$$

The quantity $\frac{d}{dt}\left[e^{-2t}y\right]$ is a total derivative, so we integrate both sides giving:

$$e^{-2t}y(t) = \int (4-t)e^{-2t}dt + C = \frac{1}{4}(2t-7)e^{-2t} + C,$$

so

$$y(t) = \frac{1}{4}(2t-7) + Ce^{2t}.$$

General Integrating Factor

The differential equation for the **integrating factor** is

$$\frac{d\mu(t)}{dt} = p(t)\mu(t) \qquad \text{or} \qquad \frac{1}{\mu(t)}\frac{d\mu(t)}{dt} = p(t).$$

Note that $\frac{d(\ln(\mu(t)))}{dt} = \frac{1}{\mu(t)}\frac{d\mu(t)}{dt}$. It follows that

$$\ln(\mu(t)) = \int p(t)dt.$$

The **general integrating factor** satisfies

$$\mu(t) = e^{\int p(t)dt}.$$

1^{st} Order Linear DE Solution

Thus, the 1^{st} **Order Linear DE Solution**

$$\frac{dy}{dt} + p(t)y = g(t) \qquad \text{with} \qquad \mu(t) = e^{\int p(t)dt}$$

is integrated to produce

$$\mu(t)y(t) = \int \mu(t)g(t)\,dt + C.$$

Theorem (Solution of 1^{st} Order Linear DE)

With the 1^{st} Order Linear DE given above and assuming integrability of $p(t)$ and $g(t)$, then the solution is given by

$$y(t) = e^{-\int p(t)dt}\left[\int e^{\int p(t)dt} g(t)\,dt + C\right].$$

Linear DE –Example 1

Consider the **Linear DE Solution**

$$t\frac{dy}{dt} - y = 3t^2\sin(t).$$

1. Put this equation into standard form, so divide by t and obtain

$$\frac{dy}{dt} - \left(\frac{1}{t}\right)y = 3t\sin(t). \tag{3}$$

2. Observe $p(t) = -\frac{1}{t}$, so find integrating factor

$$\mu(t) = e^{\int(-1/t)dt} = e^{-\ln(t)} = \frac{1}{t}.$$

3. Multiply (3) by $\mu(t)$ giving

$$\left(\frac{1}{t}\right)\frac{dy}{dt} - \left(\frac{1}{t^2}\right)y = \frac{d}{dt}\left(\frac{y}{t}\right) = 3\sin(t).$$

Linear DE –Example 1

The previous slide showed the transformation of

$$t\frac{dy}{dt} - y = 3t^2\sin(t)$$

with the integrating factor $\mu(t) = \frac{1}{t}$ to

$$\frac{d}{dt}\left(\frac{y}{t}\right) = 3\sin(t).$$

4. Integrate this equation

$$\left(\frac{1}{t}\right)y(t) = 3\int\sin(t)dt + C = -3\cos(t) + C,$$

which gives the **solution**

$$y(t) = -3t\cos(t) + Ct.$$

Pollution in a Lake

Pollution in a Lake: Introduction

- An urgent problem in modern society is how to reduce pollution in our water sources
- These are complex issues, requiring a multidisciplinary approach, and are often politically intractable because of the key role that water plays in human society and the many competing interests
- Here we examine a very simplistic model for pollution of a lake
- The model illustrates some basic elements from which more complicated models can be built and analyzed

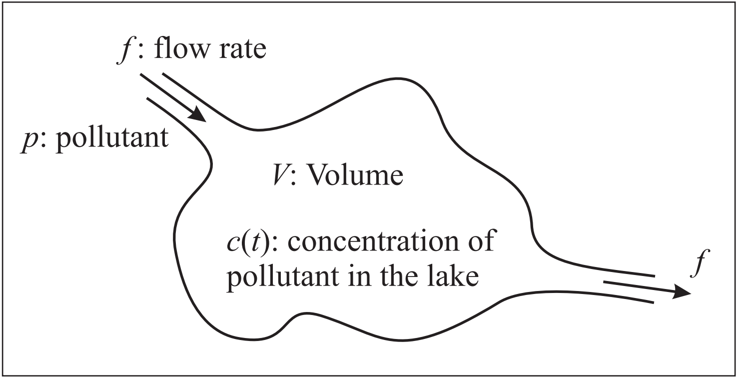
Pollution in a Lake

Pollution in a Lake: Problem set up

- Consider the scenario of a new pollutant appearing upstream from a clean lake with volume V
- Assume that the inflowing river has a concentration of the new pollutant, $p(t)$
- Assume that the river flows into the lake at a rate, $f(t)$
- Assume that the lake is well-mixed and maintains a constant volume by having a river exiting the lake with the same flow rate, $f(t)$, of the inflowing river

Pollution in a Lake

Diagram for Lake Problem: Model with a linear first order DE for the concentration of the pollutant in the lake, $c(t)$



Pollution in a Lake

Differential Equation for Pollution in a Lake

- Set up a differential equation for the mass balance of the pollutant
- **The change in amount of pollutant =**
  Amount entering - Amount leaving
- The amount entering is the concentration of the pollutant, $p(t)$, in the river times the flow rate of the river, $f(t)$
- The amount leaving has the same flow rate, $f(t)$
- Since the lake is assumed to be well-mixed, the concentration in the outflowing river will be equal to the concentration of the pollutant in the lake, $c(t)$
- The product $f(t)c(t)$ gives the amount of pollutant leaving the lake per unit time

Pollution in a Lake

Differential Equations for Amount and Concentration of Pollutant

- The change in **amount of pollutant** satisfies the model

$$\frac{da(t)}{dt} = f(t)p(t) - f(t)c(t)$$

- Since the lake maintains a constant volume V, then $c(t) = a(t)/V$, which also implies that $c'(t) = a'(t)/V$
- Dividing the above differential equation by the volume V,

$$\frac{dc(t)}{dt} = \frac{f(t)}{V}(p(t) - c(t))$$

- This is a **Linear First Order DE**
- If the lake is initially clean, then $c(0) = 0$

Pollution in a Lake

Solution of the DE: Rewrite the DE for the concentration of pollutant as

$$\frac{dc(t)}{dt} + \frac{f(t)}{V}c(t) = \frac{f(t)p(t)}{V} \quad \text{with} \quad c(0) = 0$$

- This DE has the integrating factor

$$\mu(t) = e^{\int (f(t)/V)dt}$$

- With the integrating factor, the DE becomes

$$\frac{d}{dt}\left(\mu(t)c(t)\right) = \frac{\mu(t)f(t)p(t)}{V}$$

Pollution in a Lake

Solution of the DE (cont):

- The DE is integrated to produce

$$\mu(t)c(t) = \int (\mu(t)f(t)p(t)/V)\,dt + C$$

- With the initial condition, $c(0) = 0$, we have

$$c(t) = \mu^{-1}(t)\int_0^t (\mu(s)f(s)p(s)/V)\,ds$$

- or

$$c(t) = e^{-\int (f(t)/V)dt}\int_0^t \left(e^{\int (f(s)/V)ds} f(s)p(s)/V\right) ds$$

Basic Example: Pollution in a Lake 1

Basic Example: Pollution in a Lake Part 1

- Suppose that you begin with a 100,000 m^3 clean lake
- Assume the river entering (and flowing out) has a constant flow, $f = 100$ m^3/day
- Assume the concentration of some pesticide in the river is constant at $p = 5$ ppm (parts per million)
- Form the differential equation describing the concentration of pollutant in the lake at any time t and solve it
- Find out how long it takes for this lake to have a concentration of 2 ppm

Basic Example: Pollution in a Lake 2

Solution: This example follows the model derived above with $V = 10^5$, $f = 100$, and $p = 5$, so the differential equation for the concentration of pollutant is

$$\frac{dc(t)}{dt} = -\frac{f}{V}(c(t) - p) = -0.001(c(t) - 5) \quad \text{with} \quad c(0) = 0$$

This can be solved like we did by substitution for Newton's Law of Cooling. Alternately, we use an integrating factor

$$\frac{dc(t)}{dt} + 0.001c = 0.005 \qquad \text{with} \qquad \mu(t) = e^{\int 0.001dt} = e^{0.001t}$$

so

$$\frac{d}{dt}\left(e^{0.001t}c(t)\right) = 0.005e^{0.001t} \qquad \text{or} \qquad e^{0.001t}c(t) = 5e^{0.001t} + C$$

Basic Example: Pollution in a Lake 3

Solution: From the integration before and multiplying by $\mu^{-1}(t)$,

$$c(t) = 5 + Ce^{-0.001t} \qquad \text{with} \qquad c(0) = 0$$

Thus, the solution is

$$c(t) = 5 - 5e^{-0.001t}.$$

Solving

$$c(t) = 2 = 5 - 5e^{-0.001t} \qquad \text{gives} \qquad e^{0.001t} = \frac{5}{3}$$

It follows that the concentration reaches 2 ppm when $t = 1000 \ln\left(\frac{5}{3}\right) \approx 510.8$ days.

Example 2: Pollution in a Lake 1

Example 2: Pollution in a Lake Varying flow, $f(t)$, and pollutant entering $p(t)$

- Again start with a constant volume, $V = 100,000$ m^3 clean lake
- Assume the river entering (and flowing out) has a seasonal flow, $f(t) = 100 + 60\sin(0.0172t)$ m^3/day
- If there is a point source pollutant dumped at $t = 0$ upstream, then a reasonable model for its concentration in the river is $p(t) = 8e^{-0.002t}$ ppm (parts per million)
- Form the differential equation describing the concentration of pollutant in the lake at any time t and solve it
- Graph the solution and approximate how long it takes for this lake to have a concentration of 2 ppm

Example 2: Pollution in a Lake 2

Solution: This model follows the original derivation above with $V = 10^5$, $f(t) = 100 + 60\sin(0.0172t)$, and $p(t) = 8e^{-0.002t}$, so the DE for the concentration of pollutant is

$$\begin{aligned} \frac{dc(t)}{dt} &= -\frac{f(t)}{V}(c(t) - p(t)) \quad \text{with} \quad c(0) = 0 \\ &= -(0.001 + 0.0006\sin(0.0172t))\left(c(t) - 8e^{-0.002t}\right) \end{aligned}$$

This requires use of an integrating factor

$$\mu(t) = e^{\int(0.001+0.0006\sin(0.0172t))dt} = e^{0.001t-0.0349\cos(0.0172t)}$$

so

$$\frac{d}{dt}\left(e^{0.001t-0.0349\cos(0.0172t)}c(t)\right) = (0.008+0.0048\sin(0.0172t))e^{-0.001t-0.0349\cos(0.0172t)}$$

Example 2: Pollution in a Lake 3

Solution: From before,

$$\frac{d}{dt}\left(e^{0.001t-0.0349\cos(0.0172t)}c(t)\right) = (0.008+0.0048\sin(0.0172t))e^{-0.001t-0.0349\cos(0.0172t)}$$

The integrating gives

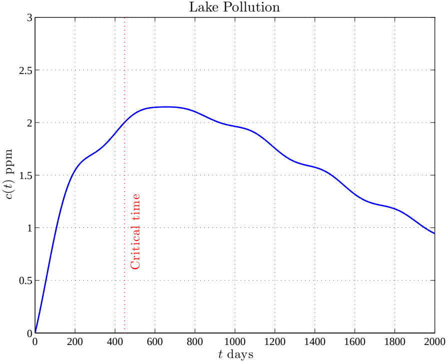
$$e^{0.001t-0.0349\cos(0.0172t)}c(t) = \int (0.008 + 0.0048\ \sin\left(0.0172\,t\right))\ \mathrm{e}^{-0.001\,t-0.0349\ \cos(0.0172\,t)}\,dt.$$

This last integral cannot be solved, even with **Maple**.

Numerical methods are needed to solve and graph this problem, and our preferred method is MatLab

Example 2: Pollution in a Lake

MatLab Solution: The pollution problem is integrated numerically (ode23). MatLab finds that the pollution exceeds 2 ppm after $t = 447.4$ days. Below shows a graph. (Programs are provided on **Lecture page**.)



Pollution in a Lake: Complications

Pollution in a Lake: Complications The above examples for pollution in a lake fail to account for many significant complications

- There are considerations of irregular variations of pollutant entering, stratification in the lake, and uptake and reentering of the pollutant through interaction with the organisms living in the lake
- The river will have varying flow rates, and the leeching of the pollutant into river is highly dependent on rainfall, ground water movement, and rate of pollutant introduction
- Obviously, there are many other complications that would increase the difficulty of analyzing this model

Mercury in Fish 1

Introduction - Fishing and Mercury in the Great Lakes Region

- Mercury, a heavy metal, is a dangerous neurotoxin that is very difficult to remove from the body
- It concentrates in the tissues of fish, particularly large predatory fish such as Northern Pike, Lake Trout, Bass, and Walleye
- The primary sources of mercury in the Great Lakes region
  - Runoff of different minerals that are mined
  - Incinerators that burn waste, especially batteries
  - Most batteries no longer contain mercury
- Bacteria converts mercury into the highly soluable methyl mercury
  - Enters fish by simply passing over their gills
  - Larger fish consume small fish and concentrate mercury

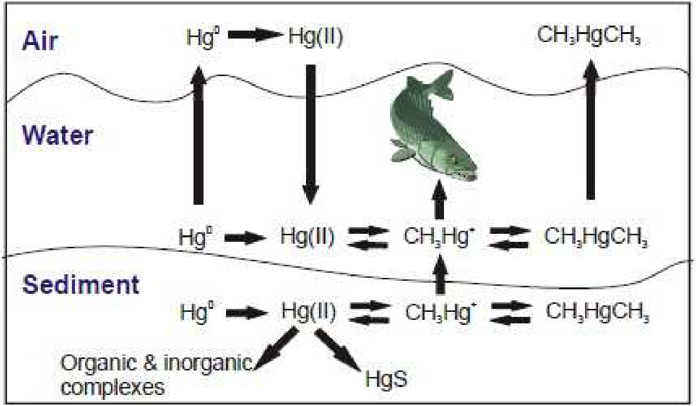
Mercury in Fish 2

Introduction - Mercury and Health

- Higher levels of Hg in fish may cause children problems in their developing neural system, resulting learning disorders
- Michigan Department of Health warns that young children and pregnant women should limit their consumption of fish, especially the larger predatory fishes

Mercury in Fish

How Hg gets into fish



Lyndsay Marie Doetzel, An investigation of the factors affecting mercury accumulation in lake trout, *Salvelinus namaycush*, in Northern Canada, University of Saskatchewan, Saskatoon, 2007

Mercury in Fish

Introduction - Mercury Buildup in Fish

- So why do fish build up the dangerous levels of Hg in their tissues?
- Hg is not easily removed from the system, so when ingested it tends to remain in the body
- Heavy metals are eliminated with chelating agents
- Mathematically, this build up is seen as the integral of the ingested Hg over the lifetime of the fish
- Thus, older and larger fish should have more Hg than the younger fish

Modeling Mercury in Fish

Modeling Mercury in Fish - Outline for the Model

- Classic model for growth of a fish (length) is the **von Bertalanffy equation**

$$\frac{dL}{dt} = b(L_* - L) \quad \text{with} \quad L(0) = 0$$

- Develop an allometric model relating weight to length, *i.e.*,

$$W = \alpha L^k$$

  - Standard dimensional analysis gives an integer appropriate integer k

- Assumptions are made for accumulation of mercury (Hg) in the fish and a DE for the amount of Hg in the fish is formulated
- All models are fit to data with MatLab

von Bertalanffy Equation

von Bertalanffy Equation: Growth of the length of fish satisfies

$$\frac{dL}{dt} = b(L_* - L) \qquad \text{with} \qquad L(0) = 0$$

- Rewrite DE and obtain integrating factor

$$\frac{dL}{dt} + bL = bL_* \qquad \text{or} \qquad \mu(t) = e^{\int b\,dt} = e^{bt}$$

- Thus,

$$\frac{d}{dt}\left(e^{bt}L\right) = bL_* e^{bt}$$

- Integration yields

$$e^{bt}L(t) = L_* e^{bt} + C \qquad \text{or} \qquad L(t) = L_* + Ce^{-bt}$$

- With $L(0) = 0$, the solution is

$$L(t) = L_*\left(1 - e^{-bt}\right)$$

Lake Trout Data for Length vs Age

Lake Trout Data for Length vs Age (Lake Superior, 1997)

age (yr)	length (cm)	age (yr)	length (cm)	age (yr)	length (cm)	age (yr)	length (cm)
6	56.6	8	58.9	9	58.9	13	75.4
7	57.7	8	60.2	9	78.5	14	83.8
7	56.6	8	71.4	10	75.2	15	87.4
7	51.8	9	54.9	11	80.3	18	76.5
8	55.4	9	85.6	13	78.5		

Find the nonlinear least squares fit to the **von Bertalanffy equation**

$$L(t) = L_* \left(1 - e^{-bt}\right)$$

for some parameters L_* and b

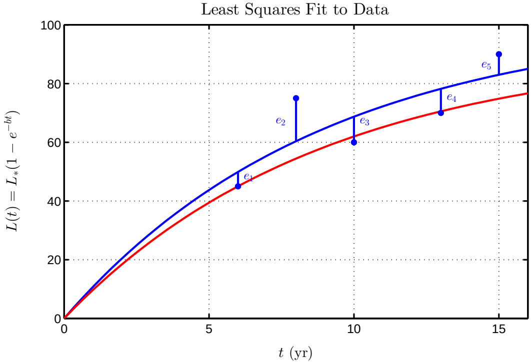
Kory Groetsch, Total Mercury and Copper Concentrations in Lake Trout and Whitefish Fillets, Activity: 19-23, From Lake Superior, Environmental Section, Biological Services Division, 1998

Parameters and Model

There are two parameters L_* and b to fit in the **von Bertalanffy equation**

$$L(t; L_*, b) = L_* \left(1 - e^{-bt}\right)$$

The graph below shows the model with two different parameter sets red and blue along with the errors, e_i, between the data and the blue model



Sum of Square Errors

- The model $L(t; L_*, b) = L_*\left(1 - e^{-bt}\right)$ is a function of t, depending nonlinearly on the parameters L_* and b
- Assume there are N data points $(t_i, L_i), i = 1..N$
- Define the error between the measured length, L_i, at time t_i and the model evaluated at t_i as

$$e_i = L_i - L_*\left(1 - e^{-bt_i}\right)$$

- The **Sum of Squares Error** function satisfies

$$J(L_*, b) = \sum_{i=1}^{N} (L_i - L(t; L_*, b))^2 = \sum_{i=1}^{N} e_i^2$$

Nonlinear Least Squares

- The **Sum of Square Errors** function, $J(L_*, b)$ is at a minimum when
$$\frac{\partial J(L_*, b)}{\partial L_*} = 0 \qquad \text{and} \qquad \frac{\partial J(L_*, b)}{\partial b} = 0$$
- This generally requires solving two nonlinear equations
  - The equations could have multiple local minima
  - Often difficult or impossible to solve analytically - Not the case for **Linear Models**
  - Handled by MatLab with special function *fminsearch*

Least Squares Error with MatLab

Define a MatLab function for the sum of square errors between the data and the model $L(t; L_*, b) = L_* \left(1 - e^{-bt}\right)$

```
function J = sumsq_vonBert(p,tdata,ldata)
% Function computing sum of square errors for von
    Bertalanffy model
model = p(1)*(1 - exp(-p(2)*tdata));
error = model - ldata;
J = error*error';
end
```

If the data are stored in *tdfish* and *ldfish*, then apply the MatLab function *fminsearch* with function *sumsq_vonBert*

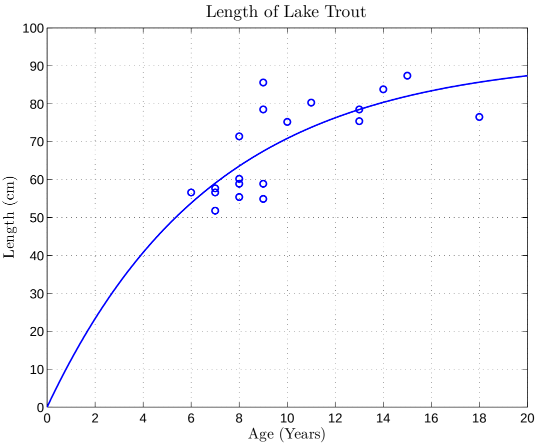
```
[p1,J,flag] = fminsearch(@sumsq_vonBert,[100,0.1],[],tdfish
    ,ldfish)
```

The result are the best fitting parameters

$$L_* = 92.401 \qquad \text{and} \qquad b = 0.14553$$

Graph of Length of Lake Trout

Graph for Length of Lake Trout: Shows data and best fitting von Bertalanffy model



Lake Trout Data for Weight vs Length

Lake Trout Data for Weight vs Length (Lake Superior, 1997)

length (cm)	Weight (g)	length (cm)	Weight (g)	length (cm)	Weight (g)	length (cm)	Weight (g)
29.0	200	57.7	1520	69.6	2800	78.5	3500
51.8	1600	58.9	1600	71.4	3050	80.3	4500
54.9	1450	58.9	1800	75.2	3920	83.8	5000
55.4	1300	60.2	2200	75.4	3980	85.6	4350
56.6	1350	62.5	1800	76.5	3980	86.6	4500
56.6	1660	68.1	3400	78.5	3629	87.4	4650
57.4	1550						

Find the best fitting **allometric model**

$$W(L) = kL^a$$

for some parameters k and a

Allometric Models 1

Allometric Models: Relationship between Length and Weight of Lake Trout using a **Power Law Relationship** – $W(L) = kL^a$

- Examine **3** versions of the **Allometric** or **Power Law** model
  - Best fit through the logarithms of the data
  - Nonlinear least squares best fit
  - Dimensional analysis modeling
- This is algebraic and not a differential equation
- Dimensional considerations important in differential equations
- Show a variety of **MatLab** programming methods

Allometric Models 2

Logarithm of Allometric Model: – $W(L) = kL^a$

$$\ln(W) = a\ln(L) + \ln(k)$$

- Let $y = \ln(W)$ be the dependent variable and $x = \ln(L)$ be the independent variable
  - With $b = \ln(k)$ this **logarithmic** form is a linear relation,

$$y = ax + b$$

- Easy formulas for finding a and b for data y vs x
- Take logarithms of the length and weight data, $\ln(L)$ and $\ln(W)$
  - Use **MatLab** to find linear least squares fit to these logarithmic data
  - Obtain **Allometric model** exponent, a, and coefficient, $k = e^b$

Allometric Models

Define a MatLab function for the sum of square errors between the logarithm of the length (*ltdfish*) and weight (*wtdfish*) data and the logarithmic model $\ln(W) = a\ln(L) + \ln(k)$

```
function [k,a] = powerfit(ldata,wdata)
% Power law fit for model W = k*L^a
% Uses linear least squares fit to logarithms of data
Y = log(wdata);         % Logarithm of W-data
X = log(ldata);         % Logarithm of L-data
p = polyfit(X,Y,1);     % Linear fit to X and Y with p = [slope, intercept]
a = p(1);               % Value of exponent
k = exp(p(2));          % Value of leading coefficient
end
```

Apply this MatLab function to obtain $k = 0.015049$ and $a = 2.8591$, giving a best allometric model

$$W(L) = 0.015049\,L^{2.8591}$$

Allometric Models

Nonlinear Least Squares Fit: – $W(L) = kL^{a}$

- This uses the nonlinear best fit to the length and weight data using a **MatLab program** almost identical to the one used for the time and length data for the von Bertalanffy model
- Create a sum of square errors function and use MatLab's *fminsearch* function
- Produces best fitting model with smallest sum of square error $J_2 = 2.8683 \times 10^6$ given by

$$W(L) = 0.068695\, L^{2.5052}$$

Allometric Models

Dimensional Analysis for $W(L) = kL^a$

- Two previous models indicate $a \approx 3$
- **Similarity argument**
  - Lake Trout look similar at most ages
  - Increasing length scales the width and height similarly or $V \propto L^3$
  - Since weight is proportional to volume, $W \propto L^3$
- Create **MatLab program** to find best k to the model

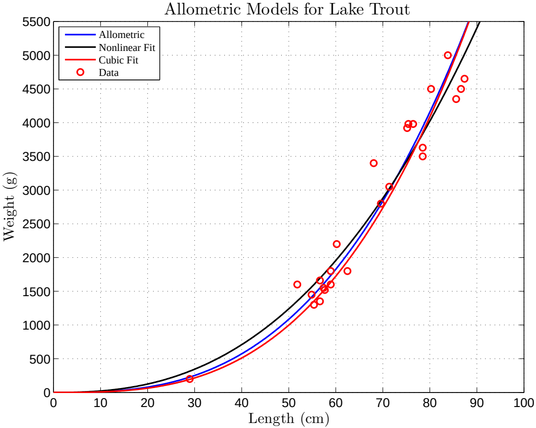
$$W(L) = kL^3$$

- Program finds best fitting model as

$$W(L) = 0.00799791\, L^3$$

Graph of Allometric Models

Graph of Allometric Models: Shows data and **3** Allometric Models, which are all very close to each other (similar least sum of square errors)



Mercury Accumulation in Lake Trout 1

Mercury Accumulation in Lake Trout

- Mercury (Hg) accumulates in fish from feeding and water passing over the gills
- Since fish are cold-blooded, their energy expenditure (balanced by food and O_2 intake) should be roughly proportional to the weight of the fish
- Mostly, Hg stays in the body once it enters
- The rate of Hg entering the body of a fish should be proportional to the weight of the fish
- Resulting differential equation:

$$\frac{dH}{dt} = \kappa W(t)$$

Mercury Accumulation in Lake Trout 2

Mercury Accumulation Model: We select the **cubic weight model** as the simplest of similar models

$$\frac{dH}{dt} = \kappa W_* \left(1 - e^{-bt}\right)^3$$

- This is a **time-varying** only DE
- Solve by integration or

$$H(t) = \kappa W_* \int \left(1 - e^{-bt}\right)^3 dt \quad \text{with} \quad H(0) = 0$$

- Integrate by expanding cubic expression

$$H(t) = \kappa W_* \int \left(1 - 3e^{-bt} + 3e^{-2bt} - e^{-3bt}\right) dt$$

Mercury Accumulation in Lake Trout 4

- Integrating the DE:

$$H(t) = \kappa W_* \int \left(1 - 3e^{-bt} + 3e^{-2bt} - e^{-3bt}\right) dt$$

- Gives

$$H(t) = \kappa W_* \left(t + \frac{3e^{-bt}}{b} - \frac{3e^{-2bt}}{2b} + \frac{e^{-3bt}}{3b}\right) + C$$

- The initial condition $H(0) = 0$ gives

$$C = \kappa W_* \left(-\frac{3}{b} + \frac{3}{2b} - \frac{1}{3b}\right) = -\frac{11\kappa W_*}{6b}$$

- The solution for Hg accumulation is

$$H(t) = \frac{\kappa W_*}{6b}\left(6bt + 18e^{-bt} - 9e^{-2bt} + 2e^{-3bt} - 11\right)$$

Mercury Accumulation in Lake Trout 5

Lake Trout Data for Hg concentration vs Age (Lake Superior, 1997)

age (yr)	Hg (ppm)	age (yr)	Hg (ppm)	age (yr)	Hg (ppm)	age (yr)	Hg (ppm)
6	0.17	8	0.2	9	0.15	13	0.53
7	0.17	8	0.14	9	0.4	14	0.39
7	0.18	8	0.2	10	0.34	15	0.33
7	0.1	9	0.13	11	0.39	18	0.52
8	0.19	9	0.46	13	0.39		

The solution of the DE, $H(t)$, gives the total amount of Hg in Lake Trout

Find the concentration of Hg in Lake Trout, which satisfies

$$c(t) = \frac{H(t)}{W(t)}$$

Mercury Accumulation in Lake Trout 6

Concentration of Hg in Lake Trout requires Weight vs Age

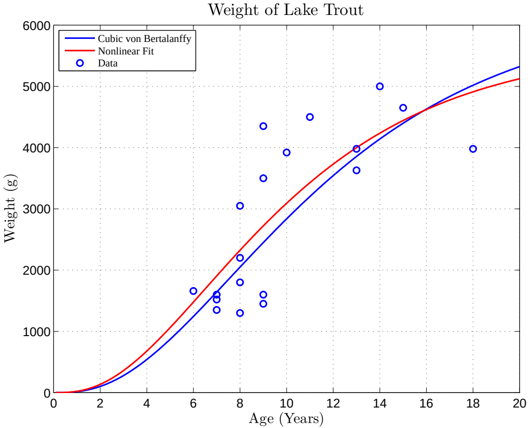
- For integration we assume a weight model of the form

$$W(t) = W_* \left(1 - e^{-bt}\right)^3$$

- The **2** parameters W_* and b are fit to time and weight data on Lake Trout
  - We can use the parameters from the von Bertalanffy fit with $b = 0.14553$ giving $W_* = 6295.4$
  - We can fit both parameters to the time weight data, giving $b = 0.16960$ giving $W_* = 5677.67$
- The von Bertalanffy model fits existing length data best
- Fitting both matches weight/time data best
- Similar graphs and least square errors

Modeling Mercury in Fish

Weight vs Age of Lake Trout: Two models presented above are graphed



Mercury Accumulation in Lake Trout

Concentration of Hg in Lake Trout: $c(t) = \frac{H(t)}{W(t)}$

- The solution for Hg accumulation is

$$H(t) = \frac{\kappa W_*}{6b}\left(6bt + 18e^{-bt} - 9e^{-2bt} + 2e^{-3bt} - 11\right)$$

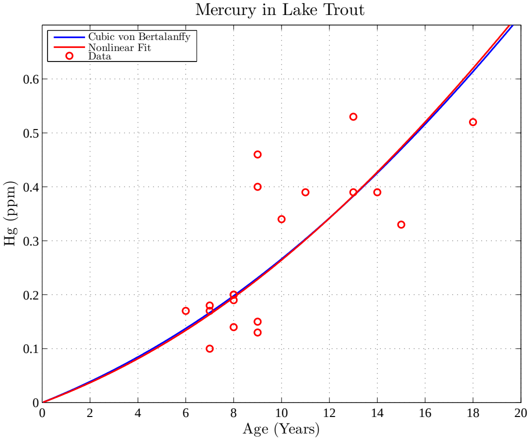
- The weight model satisfies

$$W(t) = W_*\left(1 - e^{-bt}\right)^3$$

- **MatLab** is used to fit the $c(t)$ of Hg vs time data (See **MatLab Programming** file)
- The **2** parameters sets W_*, b, and κ are fit to Hg vs time data
  - From the von Bertalanffy fit, $b = 0.14553$, $W_* = 6295.4$, and $\kappa = 0.071406$
  - From the time/weight data fit, $b = 0.16960$, $W_* = 5677.67$, and $\kappa = 0.066953$

Modeling Mercury in Fish

Concentration of Mercury in Lake Trout The concentration of Hg is measured in ppm (of mercury) $c(t) = \frac{H(t)}{w(t)}$



Mercury Accumulation in Lake Trout

Discussion of Model for Mercury in Lake Trout:

- Observe data and model show accelerating accumulation of Hg
- Data are more scattered for Hg concentration, showing variability in environment
- It is clear why the Michigan Department of Health advises against eating larger fish
- **Model weaknesses**
  - Kleiber's Law suggests food intake $\propto W^{3/4}$, which decreases accumulation
  - Larger (older) Lake Trout eat larger prey containing higher concentrations of Hg, which increases accumulation
  - Spatial variation of Hg concentration occurs in Lake Superior (PDE?)