Systems 2D
Differential Equations
Math 337
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Problem 11 (c):

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Solution 11 (c): Let $\begin{vmatrix} 0-\lambda & 2 \\ -3 & 5-\lambda \end{vmatrix} = 0$

$$\begin{aligned} (\lambda)(\lambda-5)+6 = \lambda^2 - 5\lambda + 6 &= 0 \\ = (\lambda-3)(\lambda-2) &= 0 \\ \lambda &= 3, 2 \end{aligned}$$

Let $\lambda_1 = 3$

$$\begin{pmatrix} 0-3 & 2 \\ -3 & 5-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Let $\lambda_2 = 2$

$$\begin{pmatrix} 0-2 & 2 \\ -3 & 5-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\boldsymbol{\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}}$$

Because the eigenvalues are both positive, we have an unstable node. As $t \to \infty$, the phase portrait is going away from the origin, which is why it creates an unstable node.

Problem 12 (c):

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -12 & -10 \\ 15 & 13 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Solution 12 (c): Let $\begin{vmatrix} -12-\lambda & -10 \\ 15 & 13-\lambda \end{vmatrix} = 0$

$$\begin{aligned} (\lambda+12)(\lambda-13)+150 &= \lambda^2-\lambda+6=0 \\ &= (\lambda-3)(\lambda+2)=0 \\ &\lambda = 3, -2 \end{aligned}$$

Let $\lambda_1 = 3$

$$\begin{pmatrix} -12-3 & -10 \\ 15 & 13-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -15 & -10 \\ 15 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

Let $\lambda_2 = -2$

$$\begin{pmatrix} -12--2 & -10 \\ 15 & 13--2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -10 & -10 \\ 15 & 15 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\boldsymbol{\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}}$$

Because the eigenvalues have opposite signs, we have a saddle point. As $t \to \infty$, the phase portrait is going away from the origin along one eigenvector and going towards the origin along the other eigenvector, which is why it creates a saddle point.

Problem 13 (c):

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -25 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Solution 13 (c): Let $\begin{vmatrix} 0-\lambda & -25 \\ 1 & 0-\lambda \end{vmatrix} = 0$

$$\begin{aligned} (\lambda)(\lambda) + 25 = \lambda^2 + 25 &= 0 \\ = \lambda^2 &= -25 \\ \lambda &= \pm 5i \end{aligned}$$

Let $\lambda_1 = 5i$

$$\begin{pmatrix} 0-5i & -25 \\ 1 & 0-5i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -5i & -25 \\ 1 & -5i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5i \\ 1 \end{pmatrix}$$

$$x_1(t) = \begin{pmatrix} 5i \\ 1 \end{pmatrix} (\cos(5t) + i\sin(5t))$$

$$u(t) + iw(t) = \begin{pmatrix} -5\sin(5t) \\ \cos(5t) \end{pmatrix} + i \begin{pmatrix} 5\cos(5t) \\ \sin(5t) \end{pmatrix}$$

$$\boldsymbol{\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} -5\sin(5t) \\ \cos(5t) \end{pmatrix} + c_2 \begin{pmatrix} 5\cos(5t) \\ \sin(5t) \end{pmatrix}}$$

Because the eigenvalues' real part is 0, we have a center or ellipse. As $t \to \infty$, the phase portrait moves in a counter clockwise rotation.

Problem 14 (c):

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 6 & -9 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Solution 14 (c): Let $\begin{vmatrix} 6-\lambda & -9 \\ 1 & 6-\lambda \end{vmatrix} = 0$

$$\begin{aligned} (\lambda-6)(\lambda-6)+9 = \lambda^2 - 12\lambda + 45 &= 0 \\ \lambda &= \frac{12 \pm \sqrt{144-180}}{2} \\ &= 6 \pm 3i \end{aligned}$$

Let $\lambda_1 = 6+3i$

$$\begin{pmatrix} 6-(6+3i) & -9 \\ 1 & 6-(6+3i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3i & -9 \\ 1 & -3i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3i \\ 1 \end{pmatrix}$$

$$\begin{aligned} x_1(t) &= \begin{pmatrix} 3i \\ 1 \end{pmatrix} (\cos(3t) + i\sin(3t))\, e^{6t} \\ u(t) + iw(t) &= \begin{pmatrix} -3\sin(3t) \\ \cos(3t) \end{pmatrix} e^{6t} + i \begin{pmatrix} 3\cos(3t) \\ \sin(3t) \end{pmatrix} e^{6t} \end{aligned}$$

$$\boldsymbol{\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} -3\sin(3t) \\ \cos(3t) \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} 3\cos(3t) \\ \sin(3t) \end{pmatrix} e^{6t}}$$

Because the eigenvalues' real part is positive, we have a spiral source. As $t \to \infty$, the phase portrait is going away from the origin, which is why it creates a spiral source.