MATH 537, Fall 2020
Ordinary Differential Equations

Lecture #2

Chapter 1 First Order Equations

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Abbreviations

BVP: boundary value problems

DE: differential equations

EP: equilibrium points

IC: initial conditions

IVP: initial value problems

LI: linearly independent

ODE: ordinary differential equations

PDE: partial differential equations

1.1 Important Concepts

TBD

1. Bifurcation;
2. Critical points, $f(x_c) = 0$;
3. (equilibrium points = fixed points = critical points)
4. Derivative tests
5. General solution
6. Initial Value Problem (IVP)
7. Particular solution
8. Phase Line
9. Separable ODEs
10. Sink vs. Source
11. Stable vs. Unstable Solutions, $f'(x_c)$.
12. Structurally Stable vs. Unstable (i.e., with bifurcation)

Existence Theorem: A Quick Look

TBD

Existence Theorem

Let the right side $f(x, y)$ *of the ODE in the initial value problem*

$$(1) \qquad y' = f(x, y), \qquad y(x_0) = y_0$$

$$y' = \frac{dy}{dx}$$

be continuous at all points (x, y) *in some rectangle*

$$R: |x - x_0| < a, \qquad |y - y_0| < b \qquad \text{(Fig. 26)}$$

and **bounded** *in* R*; that is, there is a number* K *such that*

$$(2) \qquad \boxed{|f(x, y)| \leqq K} \qquad \textit{for all } (x, y) \textit{ in } R.$$

Then the initial value problem (1) *has at least one solution* $y(x)$*. This solution exists at least for all* x *in the subinterval* $|x - x_0| < \alpha$ *of the interval* $|x - x_0| < a$*; here,* α *is the smaller of the two numbers* a *and* b/K.

To be discussed within Chapter 7

Uniqueness Theorem: A Quick Look

TBD

Uniqueness Theorem

Let f and its partial derivative $f_y = \partial f/\partial y$ be continuous for all (x, y) in the rectangle R (Fig. 26) and bounded, say,

$$(3) \qquad \text{(a)} \quad |f(x, y)| \leqq K, \qquad \text{(b)} \quad |f_y(x, y)| \leqq M \qquad \textit{for all } (x, y) \textit{ in } R.$$

Then the initial value problem (1) has at most one solution $y(x)$. Thus, by Theorem 1, the problem has precisely one solution. This solution exists at least for all x in that subinterval $|x - x_0| < \alpha$.

To be discussed within Chapter 7

Terminology

- We will study equations of the following form:

$$x' = f(x, t; a) \qquad \text{(ordinary differential equation)}$$

and

$$x \rightarrow g(x; a), \qquad \text{(difference equation)}$$

with $x \in U \subset R^n$, $t \in R^1$, and $a \in R^p$. We refer to $x, t,$ and a ad dependent variables, independent variables and parameter.

- By a solution of the above differential equation, we mean a map, x, from some interval, $I \in R^1$ into R^n, written as follows:

$$x: I \rightarrow R^n$$

$$t \rightarrow x(t).$$

- System with $f = f(x; a)$ that is not a function of time is referred to as autonomous systems.

Maps vs. Flows

TBD

(difference equation)	(differential equation)
Maps	Flows
Discrete time	*Continuous* time
Variables change *abruptly*	Variables change *smoothly*
Described by *algebraic* equations	Described by *differential* equations
Complicated 1-D dynamics	*Simple* 1-D dynamics
$X_{n+1} = f(X_n)$	$dx/dt = f(x)$
Capital letters	Lower case letters
$Example\colon X_{n+1} = rX_n$	$Example\colon dx/dt = \lambda x$
$Solution\colon X_{n+1} = r^n X_0$	$Solution\colon x = x_0 e^{\lambda t}$
$Growth\ for\ r > 1$	$Growth\ for\ \lambda > 0$
Decay $for\ r < 1$	Decay $for\ \lambda < 0$
$n \to t \Longrightarrow r = e^{\lambda}$	$t \to n \Longrightarrow \lambda = \ln(r)$

1.1 The Simplest Example & Initial Value Problems (IVPs)

$$\boxed{x' = ax}$$

t: *indepedent variable*
x: *depedent variable*
a: *parameter*

assume $\quad x = ke^{\lambda t} \qquad a \neq 0$

plug in $\quad (\lambda - a)ke^{\lambda t} = 0$

sol-1 $\quad k = 0 \qquad x = 0$

trivial solution

sol-2 $\quad \lambda = a \qquad x = ke^{at}$

general solution

verify $\quad x' = ake^{at} = ax$

apply an IC $\quad \boxed{x(t = 0) = u_0}$

$$\boxed{x = u_0 e^{at}}$$

Initial value problem

$$x' = ax; \qquad x(0) = u_0$$

$$x = u_0 e^{at}$$

vs.
general solution:
a collection of all solutions of
a differential equation (DE)

1.1 Analysis of Solutions

$$x' = ax; \quad x(0) = u_0$$
$$a \neq 0$$
$$x = u_0 e^{at}$$

$a > 0$

$u_o > 0$

$u_o < 0$

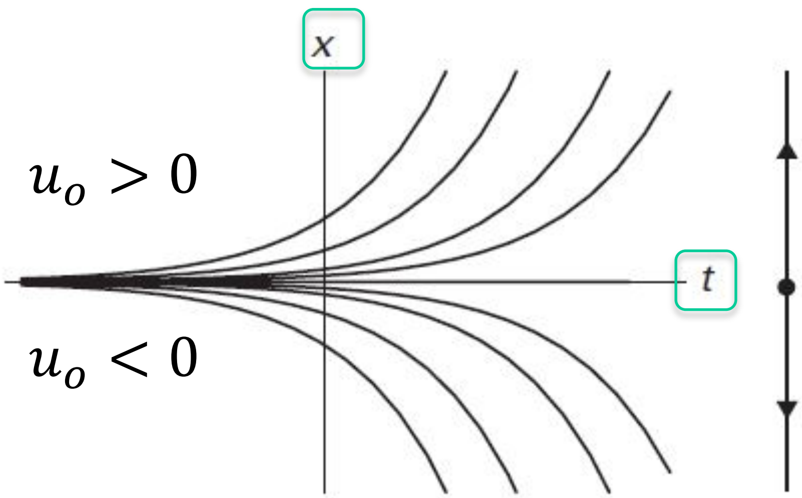


Figure 1.1 The solution graphs and phase line for $x' = ax$ for $a > 0$. Each graph represents a particular solution.

- The magnitudes $|x|$ are monotonically increasing functions.

1.1 Analysis of Solutions

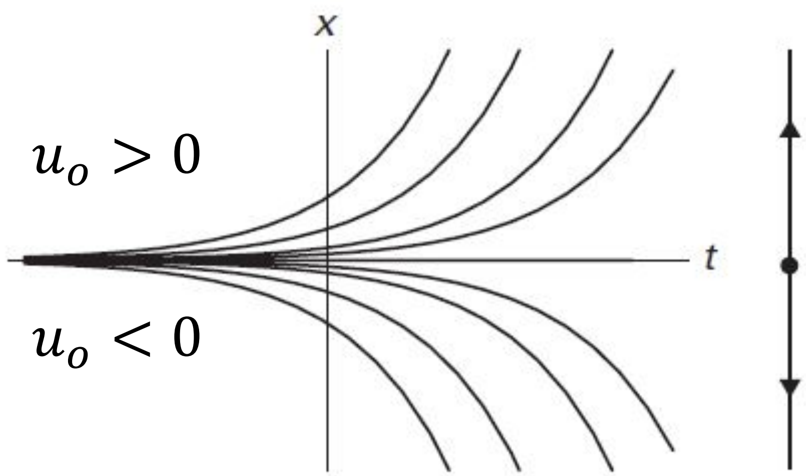
$$x' = ax; \quad x(0) = u_0$$

$$a \neq 0$$

$$x = u_0 e^{at}$$

$a > 0$

$a < 0$



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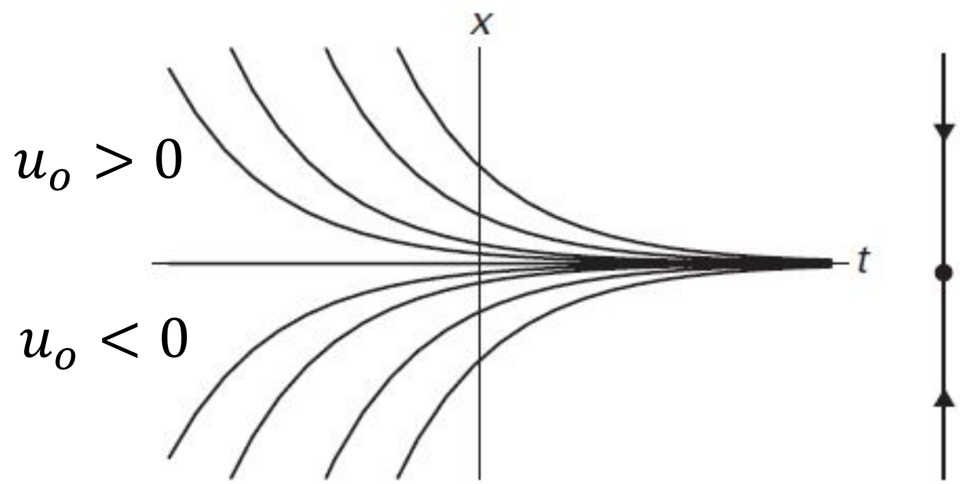


Figure 1.2 The solution graphs and phase line for $x' = ax$ for $a < 0$.

- The magnitudes $|x|$ are monotonically increasing functions.
- The magnitudes $|x|$ are monotonically decreasing functions.

1.1 Analysis of Solutions

$$x' = ax; \quad x(0) = u_0$$

$$\mathrm{a} = 0$$

$$x = u_0$$

$$x = u_0 e^{at}$$

1.1 Equilibrium Points (Fixed Points)

- Given $x' = f(x; a)$, equilibrium points, also known as fixed points or critical points, are defined when $f(x_c) = 0$.

- Example 1: Consider $x' = ax$. $x = 0$ is a critical point.

- Example 2: Consider $x' = x - x^{\mathbf{2}}$ (i.e., the Logistic Equation). $f(x_c) = 0$ leads to $x - x^{\mathbf{2}} = 0$. Thus, $x = 0$ and $x = 1$ are critical points.

- Example 3: Similarly, within $x' = x - x^{\mathbf{3}}$, three critical points are $x = 0$, $x = 1$ and $x = -1$.

1.1 Unstable vs. Stable Solutions

$a > 0$

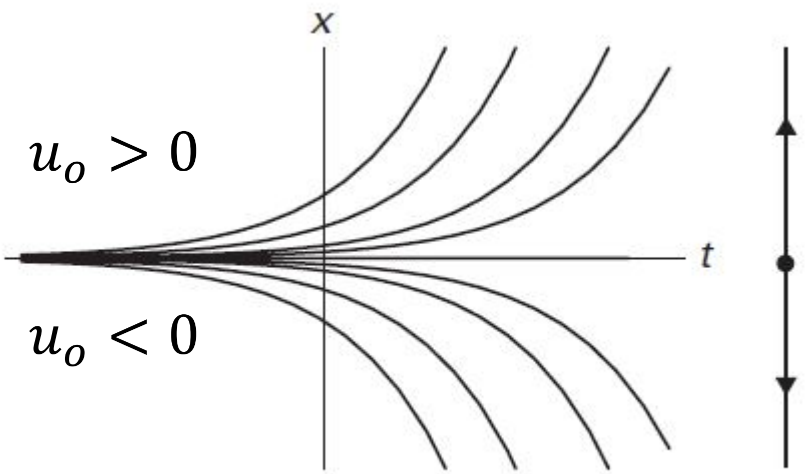


Figure 1.1 The solution graphs and phase line for $x' = ax$ for $a > 0$. Each graph represents a particular solution.

$a < 0$

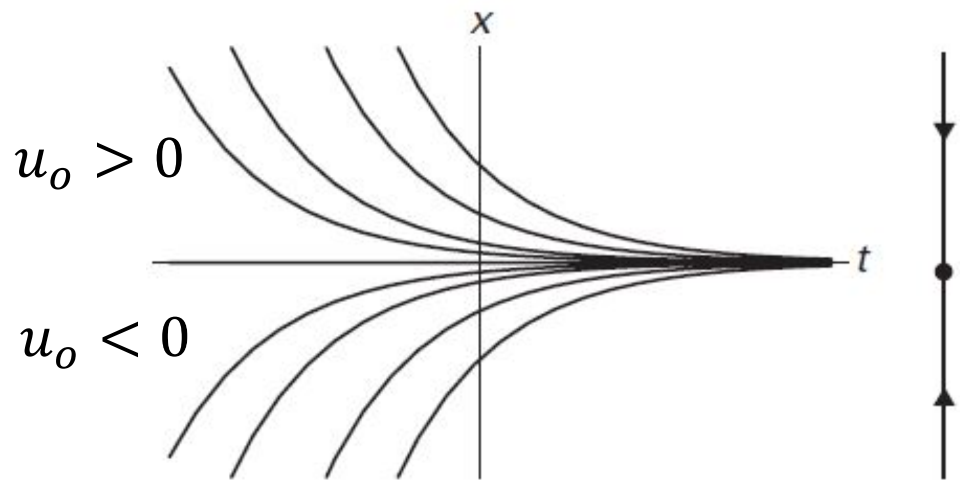


Figure 1.2 The solution graphs and phase line for $x' = ax$ for $a < 0$.

- Unstable solutions with $a > 0$
- moving away from an equilibrim point, $\mathrm{x} = 0$.
- $\mathrm{x} = 0$ is a source.

- Stable solutions with $a > 0$
- moving toward an equilibrim point, $\mathrm{x} = 0$.
- $\mathrm{x} = 0$ is a sink.

1.1 Phase Line(s)

The phase line:

- as the solution is a function of time, we may view it as a particle moving along the real line, which is called a phase line.
- A line represents intervals of the domain of the derivatives. An interval over which the derivative is positive has an arrow pointing in the positive direction along the line.

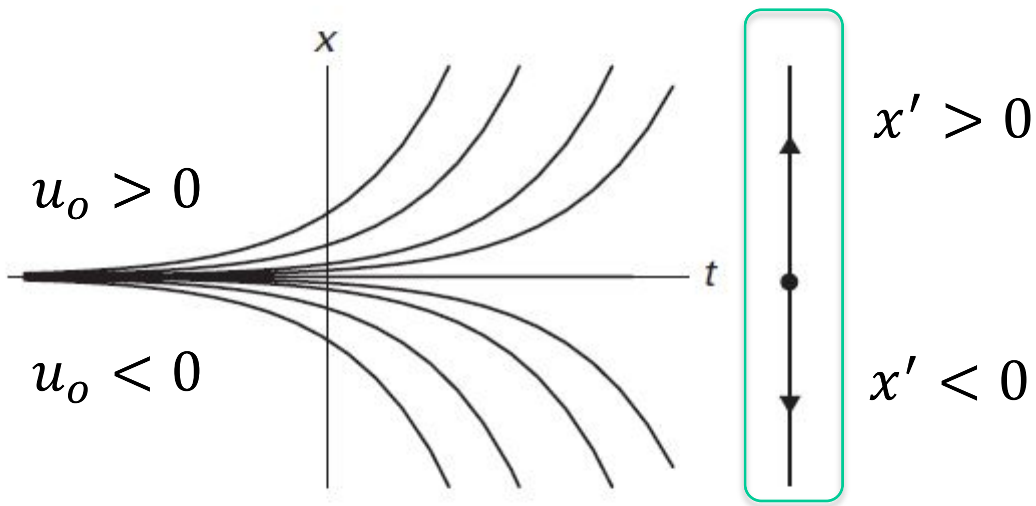
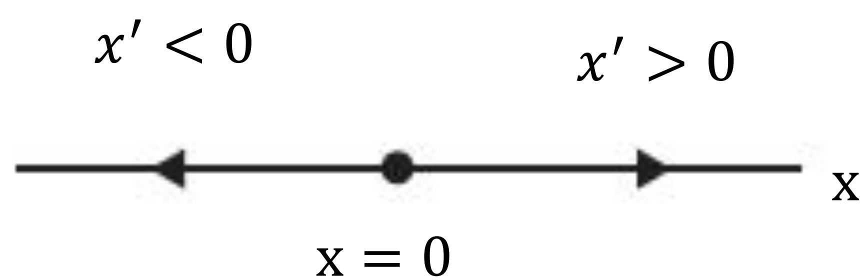


Figure 1.1 The solution graphs and phase line for $x' = ax$ for $a > 0$. Each graph represents a particular solution.

$$\boxed{x' = ax}$$



1.1 Phase Line(s)

- The phase line: as the solution is a function of time, we may view it as a particle moving along the real line.
- $x' = ax$

- unstable solutions, moving away from an equilibrim point, $x = 0$.

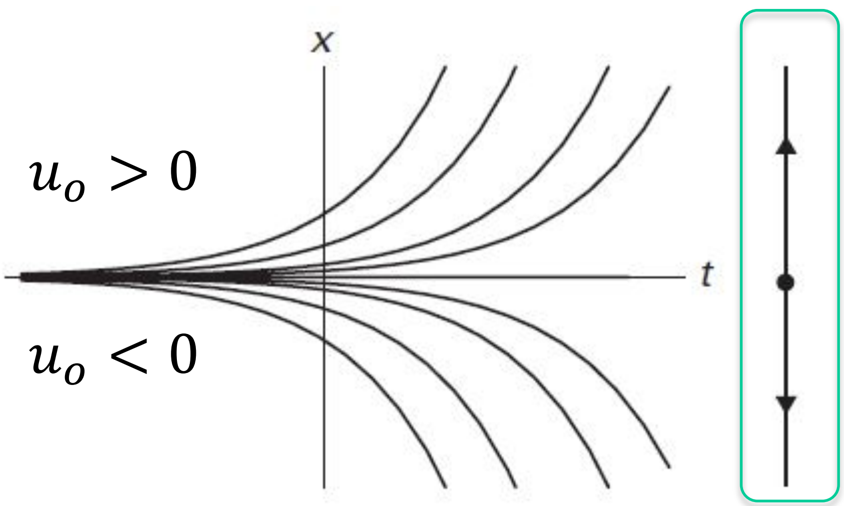
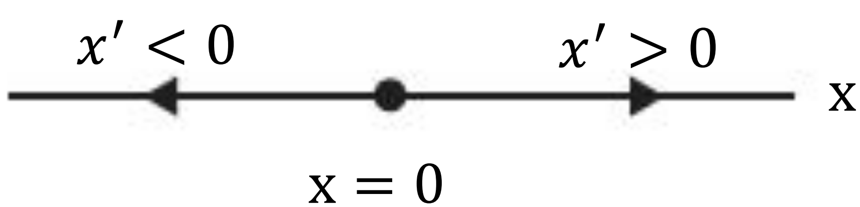


Figure 1.1 The solution graphs and phase line for $x' = ax$ for $a > 0$. Each graph



- stable solutions, moving toward an equilibrim point, $x = 0$.

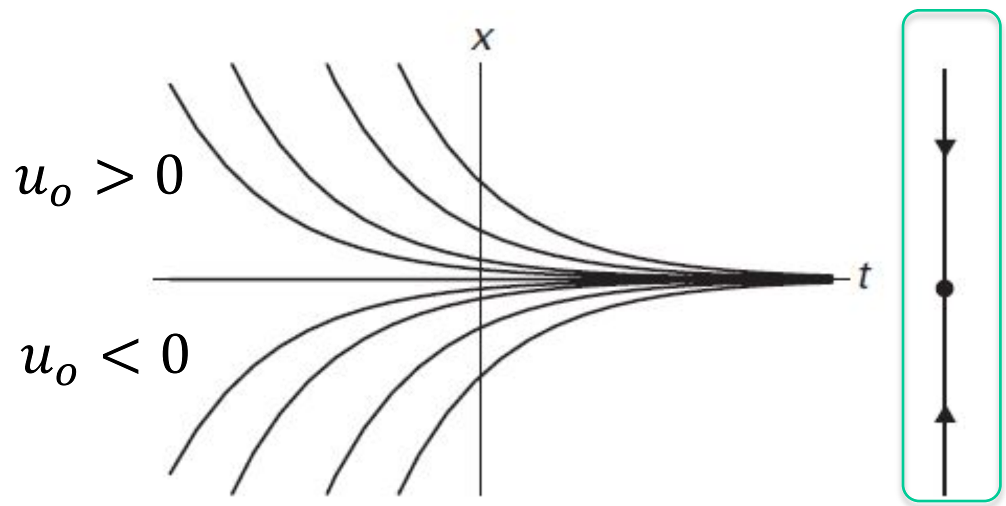
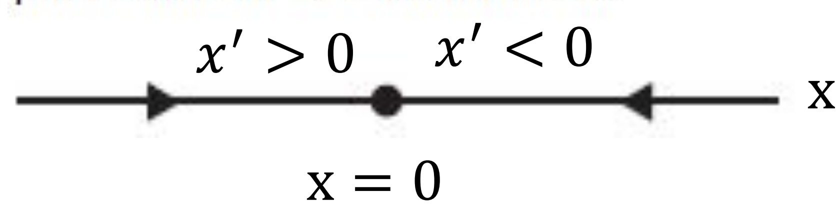


Figure 1.2 The solution graphs and phase line for $x' = ax$ for $a < 0$.



1.1 Linear (Local) Stability Analysis for 1st Order ODEs

consider a general case $$\frac{dx}{dt} = f(x)$$

find critical points $$f(x_c) = 0$$

linearize $f(x)$ wrt a critical pt $$\frac{dx}{dt} = f(x) = f(x_c) + f'(x_c)(\mathrm{x} - \mathrm{x_c}) + \cdots$$

$$\frac{dx}{dt} = f'(x_c)(\mathrm{x} - \mathrm{x_c}) + \cdots$$

$$\boxed{x' = ax}$$

find solution $$\mathrm{x} - \mathrm{x_c} = \mathrm{c_0}\exp(f'(x_c)t)$$

stability

the critical point is stable if $f'(x_c) < 0$	a sink
the critical point is unstable if $f'(x_c) > 0$	a source

Haberman (2013)

1.1 Linear (Local) Stability Analysis for 1st Order ODEs

$$\frac{dx}{dt} = f'(x_c)(\mathrm{x} - \mathrm{x_c}) + \cdots$$

$f'(x_c) < 0$

$\frac{dx}{dt} > 0$ $\frac{dx}{dt} < 0$

$\mathrm{x} = x_c$

$f'(x_c) < 0$

$\frac{dx}{dt} < 0$ $\frac{dx}{dt} > 0$

$\mathrm{x} = x_c$

the critical point is stable if $f'(x_c) < 0$	a sink
the critical point is unstable if $f'(x_c) > 0$	a source

Haberman (2013)

1.1 Local Stability Analysis at Fixed Points

- Given $x' = f(x)$, equilibrium points or also known as fixed points or critical points are defined when $f(x_c) = 0$.

- A local solution near the critical point is
  - stable for $f'(x_c) < 0$ and
  - unstable for $f'(x_c) > 0.$

- Consider $x' = ax$. $x = 0$ is a critical point. $f'(0) = \mathrm{a}$. A local solution is
  - stable $a < 0$ for and
  - unstable $a > 0.$

1.1 Dependence on Parameters

- The "equation", $x' = ax$, is stable if $a \neq 0$.
- More precisely, if a is replaced by another constant b with a sign that is the same as a, the qualitative behavior of the solutions does not change.
- On the other hand, for $a = 0$, the slightest change in a leads to a radical change in the behavior of solutions.
- We therefore say that we have a bifurcation at $a = 0$ in the one-parameter family of equations $x' = ax$.

1.1 Bifurcation

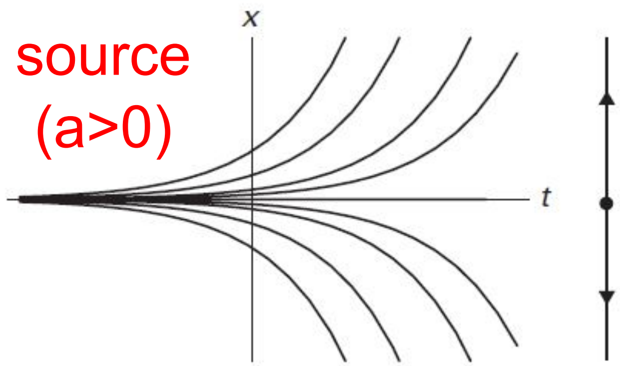


Figure 1.1 The solution graphs and phase line for $x' = ax$ for $a > 0$. Each graph represents a particular solution.

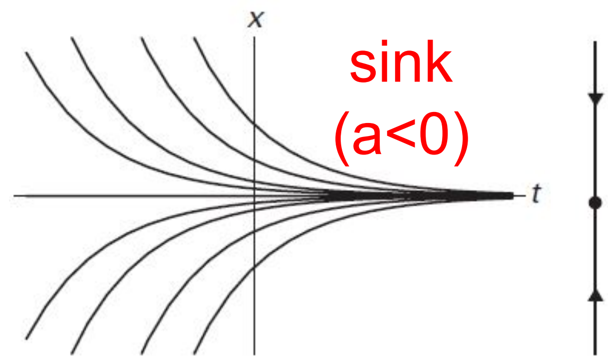


Figure 1.2 The solution graphs and phase line for $x' = ax$ for $a < 0$.

- A bifurcation occurs when there is a "significant" change in the structure of the solutions of the system as "a" varies.
- In the previous example, solutions are unstable for $a > 0$ but stable for $a < 0$. Thus, we have a bifurcation at $a = 0$.
- The simplest types of bifurcations occur when the number of equilibrium solutions changes as "a" varies. For example, within $x' = 1 - ax^2$, there are two critical points for $a > 0$ but no critical points for $a \leq 0$.

Definition: Bifurcation Points

$$\frac{dx}{dt} = f(x, a)$$

critical points	$f(x, a) = 0$
bifurcation points	$\boxed{f(x, a) = 0 \ \& \ f_x(x, a) = 0}$
example	$\dfrac{dx}{dt} = ax$
critical points	$ax = 0 \quad \rightarrow x = 0$
bifurcation points	$f_x(x, a) = a \ \rightarrow a = 0$

1.1 Important Concepts

1. Bifurcation;
2. Critical points, $f(x_c) = 0$;
3. (equilibrium points = fixed points = critical points)
4. General solution
5. Initial Value Problem (IVP)
6. Particular solution
7. Phase Line;
8. Separable ODEs
9. Sink vs. Source
10. Stable vs. Unstable Solutions, $f'(x_c)$.
11. Structurally Stable vs. Unstable (i.e., with bifurcation)

Sect. 1.2: the Logistic Equation

$$\boxed{x' = ax}$$

- linear population model if $a > 0$
- x: population (i.e., assume $\mathrm{x} > 0$).
- $\frac{\mathrm{dx}}{\mathrm{dt}}$: the rate of growth of the population, (called a growth rate, or an exponential growth rate)
- $\frac{\mathrm{dx}}{\mathrm{dt}}$ is proportional to x

$$\boxed{x' = ax\,(1 - \frac{\mathrm{x}}{\mathrm{N}})}$$

- $\frac{\mathrm{dx}}{\mathrm{dt}}$ is proportional to x for small x (and $\mathrm{x} < N$).
- $\frac{\mathrm{dx}}{\mathrm{dt}}$ becomes negative for large x (i.e., $\mathrm{x} > N$).
- N is called carrying capacity.

We choose $\mathrm{N} = 1$ (see Quiz II)

$$\boxed{x' = ax\,(1 - x)}$$

$$\boxed{\equiv f_a(x)}$$

- first order, nonlinear, separable
- autonomous, ($f(x) = ax\,(1 - x)$ is not an explicit function of time).

1.2 Logistic Equation: Solutions

separable ODE

$$x' = a(x - x^2)$$

$$\frac{dx}{x - x^2} = adt$$

$$\frac{dx}{x(1 - x)} = adt$$

the method of partial fractions

$$\left(\frac{1}{x} + \frac{1}{1 - x}\right) dx = adt$$

$x \in (0,1)$ (see Quiz II)

$$\ln(x) - \ln(1 - x) = at + C$$

$$\ln\left(\frac{x}{1 - x}\right) = at + C$$

$$\ln\left(\frac{x}{1 - x}\right) = at + C$$

$$\left(\frac{x}{1 - x}\right) = C_0 e^{at}$$

$$x = C_0 e^{at}(1 - \mathrm{x})$$

$$(1 + C_0 e^{at})x = C_0 e^{at}$$

$$x = \frac{C_0 e^{at}}{1 + C_0 e^{at}}$$

$$x \rightarrow 1 \; as \; t \; \rightarrow \infty$$

1.2 Analysis of Solutions (sigmoid function)

$$\frac{dx}{dt} = 3(x - x^2),$$

$$1 > x_0 > 0$$

general solution $\quad x = \frac{C_0 e^{at}}{1+C_0 e^{at}} \qquad$ $x \in (0,1)$

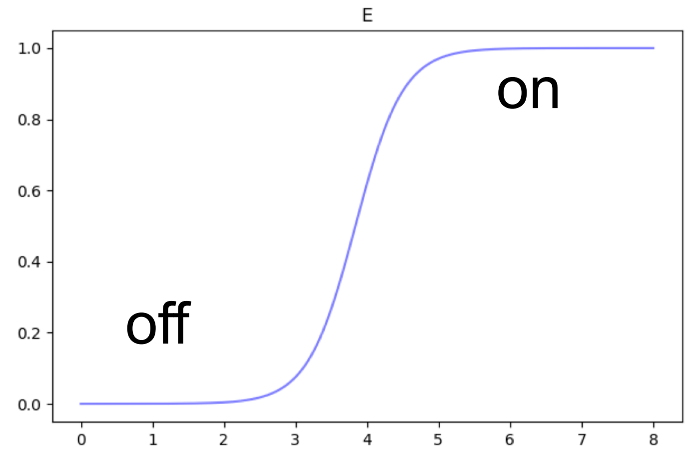
apply an IC $\quad x(0) = \frac{C_0}{1+C_0} = x_0$

$$C_0 = \frac{x_0}{1 - x_0}$$

$$\boxed{x = \frac{x_0 e^{at}}{1-x_0+x_0 e^{at}}}$$

$x_0 > 0$

sigmoid function



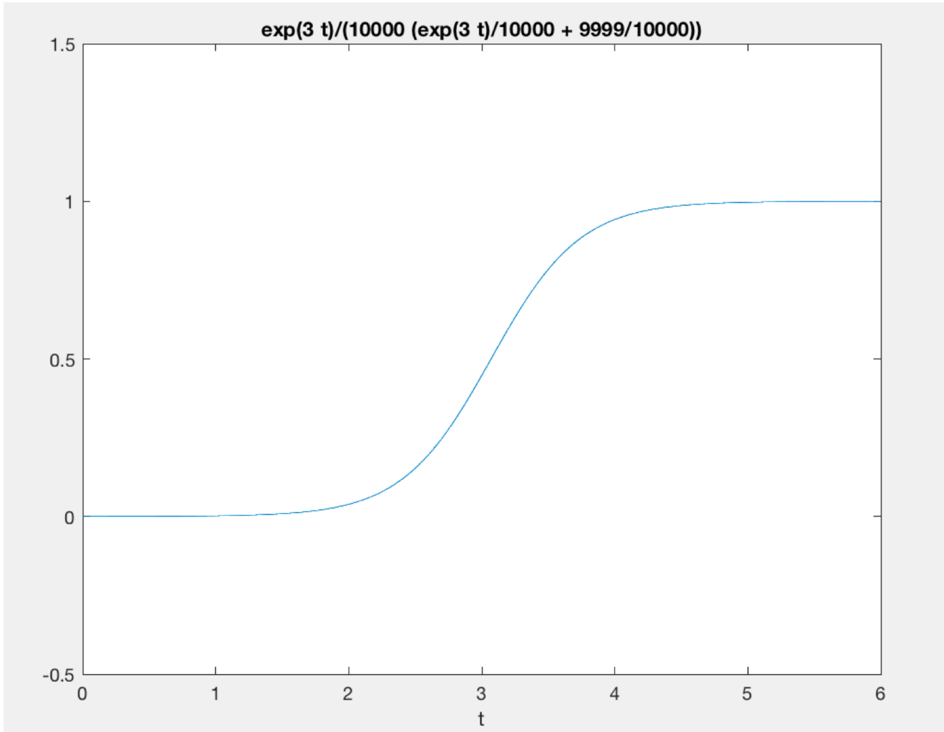
$x \to 1 \; as \; t \; \to \infty \; (forward \; in \; time)$ on

$x \to 0 \; as \; t \; \to -\infty \; (backward \; in \; time)$ off

$x \in \; [0, 1]$

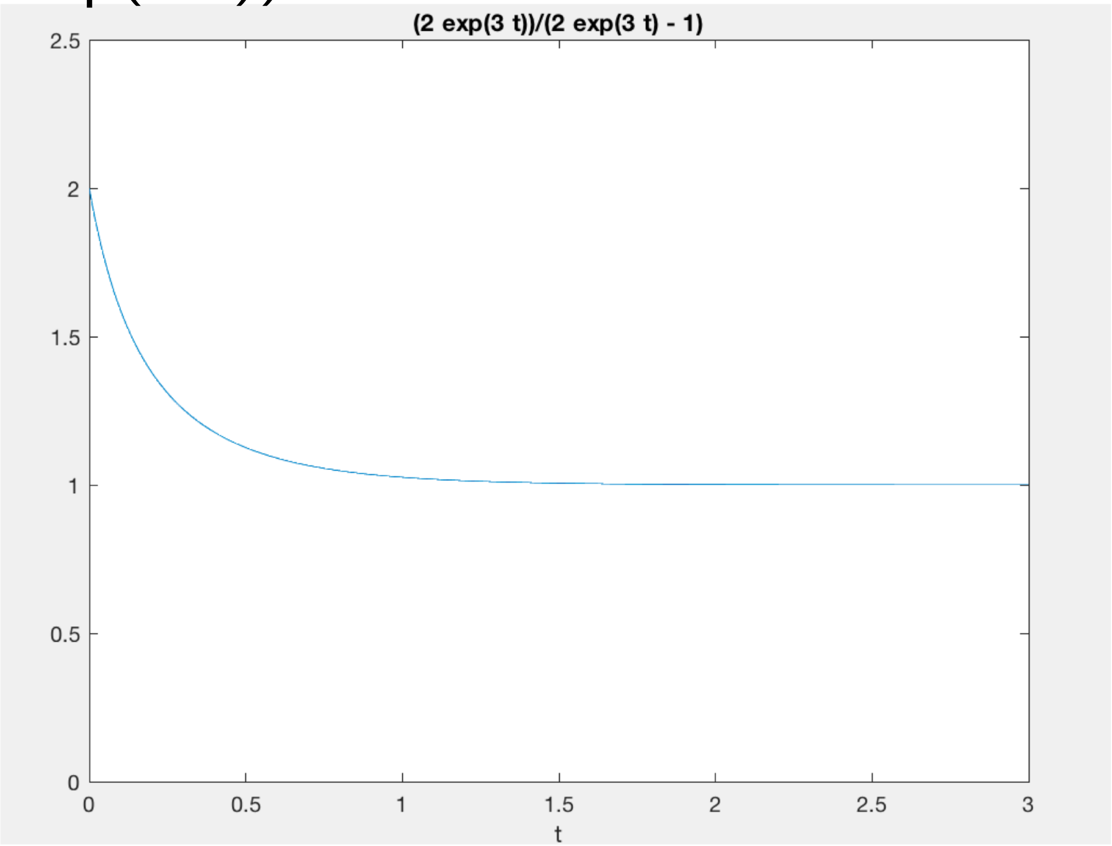
1.2 Symbolic Plotting

```
syms t x0 a
a=3
x0=0.0001
fun=x0*exp(a*t)/(1-x0+x0*exp(a*t))
ezplot (fun, [0, 6, -0.5, 1.5])
```



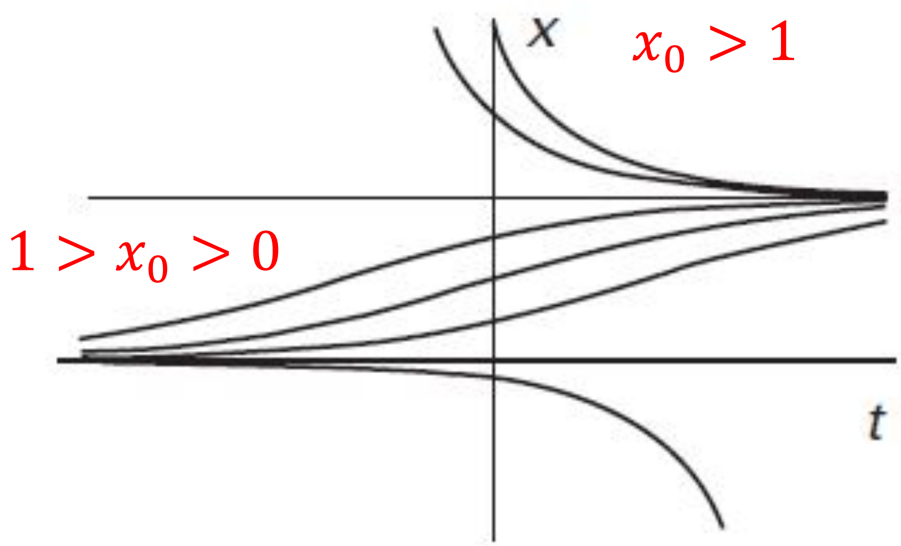
1.2 Symbolic Plotting

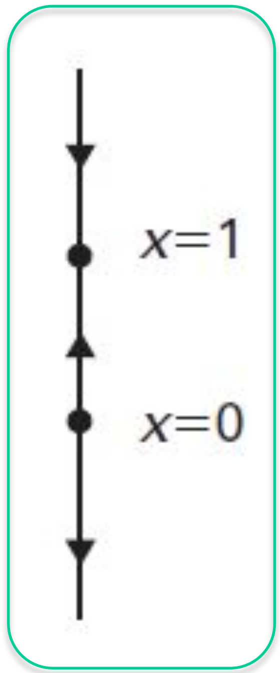
```
syms t x0 a
a=3
x0=2
fun=x0*exp(a*t)/(1-x0+x0*exp(a*t))
ezplot (fun, [0, 3, 0, 2.5])
```



$$x' = a(x - x^2)$$

$$x = \frac{x_0 e^{at}}{1 - x_0 + x_0 e^{at}}$$





To be discussed

1.2 Stability Analysis: Derivative Tests

$$x' = a(x - x^2)$$

$$\boxed{f_a(x) = a(x - x^2)}$$

critical points $\boxed{f_a(x) = 0}$

$$\boxed{x = 0 \; or \; x = 1}$$

1st derivative $f'(x) = a - 2ax$

$x = 0$ $\quad f'(0) = a > 0$

unstable

x = 0

$x = 1$ $\quad f'(1) = -a < 0$

stable

x = 1

1.2 Stability Analysis: Perturbation Method

$\boxed{x = x_c + \varepsilon(t)}$ $\boxed{total\ field = critical\ point\ value + small\ value}$

$x' = a(x - x^2)$ $\boxed{total\ field = basic\ state + perturbation}$

$\boxed{x_c = 0\ or\ x = 1}$

$\boxed{x = 1 + \varepsilon}$

$$\varepsilon' = a\left(x - x^2\right) = a(1+\varepsilon)(-\ \varepsilon) \approx -a\varepsilon$$

$\varepsilon = 0$

$\boxed{x = 0 + \varepsilon}$

$$\varepsilon' = a\varepsilon(1 - \varepsilon) \approx +a\varepsilon$$

$\varepsilon = 0$

1.2 Stability Analysis: Another Test

$$x' = x - x^3 = f(x)$$

critical points $\boxed{f(x) = 0}$

$$\boxed{x = 0 \ or \ x = \pm 1}$$

1st derivative $f'(x) = 1 - 3x^2$

$x = 0$ $\quad f'(0) = 1 > 0$

unstable

$\mathrm{x} = 0$

$x = \pm 1$ $\quad f'(\pm 1) = -2 < 0$

stable

$\mathrm{x} = \pm 1$

1.2 Stability Analysis: Perturbation Method

$$x = x_c + \varepsilon(t)$$

$$\boxed{total\ field = critical\ point\ value + small\ value}$$

$$\boxed{x' = x - x^3}$$

$$\boxed{total\ field = basic\ state + perturbation}$$

$$\boxed{x_c = 0\ or\ x = \pm 1}$$

$$\boxed{x = 1 + \varepsilon}$$

$$\varepsilon' = x(1 - x^2) = x(1 - x)(1 + x) = -\varepsilon(1 + \varepsilon)(2 + \varepsilon) \approx -2\varepsilon$$

$$\varepsilon = 0$$

$$\boxed{x = 0 + \varepsilon}$$

$$\varepsilon' = \varepsilon(1 - \varepsilon)(1 + \varepsilon) \approx +\varepsilon$$

$$\varepsilon = 0$$

1.1 Bifurcation

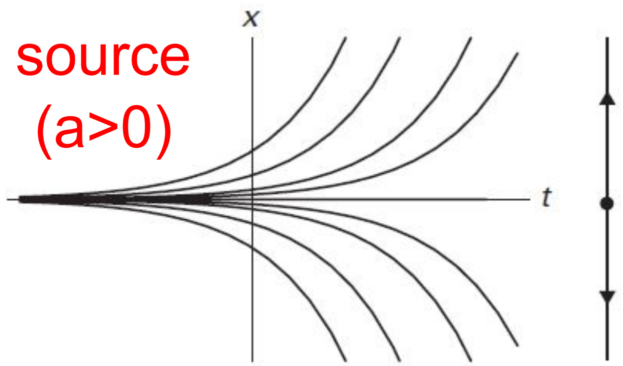


Figure 1.1 The solution graphs and phase line for $x' = ax$ for $a > 0$. Each graph represents a particular solution.

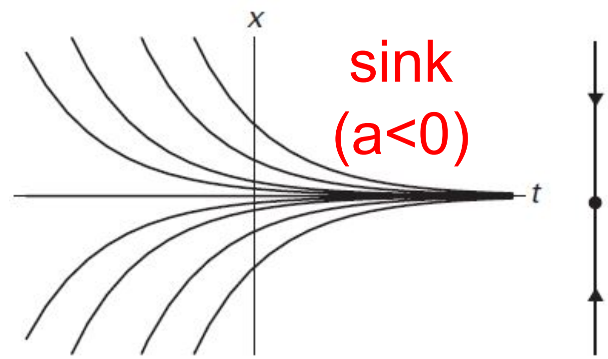


Figure 1.2 The solution graphs and phase line for $x' = ax$ for $a < 0$.

- A bifurcation occurs when there is a "significant" change in the structure of the solutions of the system as "a" varies.
- In the previous example, solutions are unstable for $a > 0$ but stable for $a < 0$. Thus, we have a bifurcation at $a = 0$.
- The simplest types of bifurcations occur when the number of equilibrium solutions changes as "a" varies. For example, within $x' = 1 - ax^2$, there are two critical points for $a > 0$ but no critical points for $a \leq 0$.

Bifurcation Points: Another Example

$$\frac{dx}{dt} = f(x, a)$$

critical points

$$f(x, a) = 0$$

bifurcation points

$$\boxed{f(x, a) = 0 \;\&\; f_x(x, a) = 0}$$

example

$$\frac{dx}{dt} = a - x^2$$

critical points

$0 = a - x^2$ $\qquad$ $x = \pm\sqrt{a}$ as $a \geq 0$

bifurcation points

$f_x(x, a) = 0 \rightarrow x = 0 \rightarrow a = 0$ in the above Eq.

Bifurcation Points: Logistic Equation

$$\frac{dx}{dt} = f(x, a)$$

critical points

$$f(x, a) = 0$$

bifurcation points

$$\boxed{f(x, a) = 0 \ \& \ f_x(x, a) = 0}$$

example

$$\frac{dx}{dt} = a(x - x^2)$$

critical points

$$a(x - x^2) = 0 \qquad a = 0 \ or \ x = 0 \ or \ 1$$

bifurcation points

$f_x(x, a) = a(1 - 2x) \rightarrow x = \frac{1}{2} \rightarrow a = 0$ in the above Eq.