Today: 4.2 ~ Chain Rule Proof

4.3 ~ Mean Value Theorem & Consequences.

Chain Rule 4.14

Suppose that I is a nbhd of x_0 and $f: I \to \mathbb{R}$ is differentiable at x_0.
Suppose J is an open interval such that $f(I) \subseteq J$ and $g: J \to \mathbb{R}$ is differentiable at $f(x_0)$. Then $(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$

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proof: Let $y_0 = f(x_0) \in J$.

Define $h: J \to \mathbb{R}$ by

$$h(y) = \begin{cases} \dfrac{g(y) - g(y_0)}{y - y_0}, & \text{if } y \neq y_0 \\ g'(y_0), & \text{if } y = y_0. \end{cases}$$

Remark ①: Notice that

$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = g'(y_0) = g'(f(x_0)).$$

so h is continuous at $y_0 = f(x_0)$.

② For every $y \in J$, $g(y) - g(y_0) = h(y)(y - y_0)$.

So for every $x \in I$, $g(f(x)) - g(f(x_0)) = h(f(x))(f(x) - f(x_0))$.

Now compute

$$\lim_{x \to x_0} \frac{(g \circ f)(x) - (g \circ f)(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0}$$

$$= \lim_{x \to x_0} h(f(x)) \frac{(f(x) - f(x_0))}{x - x_0}.$$

$$= h(f(x_0)) \cdot f'(x_0) \quad \text{by limit laws \& continuity.}$$

$$= g'(f(x_0)) \cdot f'(x_0).$$

$\boxed{\mathcal{Z}}$

4.3 MVT & consequences.

Def. Suppose $x_0 \in D$ and $f: D \to \mathbb{R}$.

We say x_0 is a "local maximizer" of f

iff

$\exists \delta > 0$ s.t. $\forall x \in (x_0 - \delta, x_0 + \delta)$, $f(x) \le f(x_0)$.

Lemma 4.16

Suppose $f: D \to \mathbb{R}$ and $x_0 \in D$ is a local maximizer of f. If f is differentiable at x_0, then $f'(x_0) = 0$.

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proof: Let $\{x_n\} \subseteq I$ (a nbhd of x_0 contained in D) such that $x_n < x_0$ and $\lim_{n\to\infty} x_n = x_0$.

Thus for all n, $x_n - x_0 < 0$ and $f(x_n) - f(x_0) \le 0$

Thus for all n, $$\frac{f(x_n) - f(x_0)}{x_n - x_0} \ge 0.$$

Thus $$f'(x_0) = \lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} \ge 0.$$

Similarly, let $\{z_n\} \subseteq I$ such that

$$z_n > x_0 \quad \text{and} \quad \lim_{n\to\infty} z_n = x_0.$$

So for all n, $$\frac{f(z_n) - f(x_0)}{z_n - x_0} \le 0.$$

Thus $$f'(x_0) = \lim_{n\to\infty} \frac{f(z_n) - f(x_0)}{z_n - x_0} \le 0.$$

Thus $f'(x_0) = 0$. $\boxtimes$

Thm 4.17 Rolle's Thm

Suppose $f:[a,b]\to\mathbb{R}$ is continuous and differentiable on (a,b). Suppose also $f(a)=f(b)$.
Then $\exists x_0\in(a,b)$ s.t. $f'(x_0)=0$.

proof: By the Extreme Value Theorem f attains a maximum and minimum on $[a,b]$.

case 1: Suppose the max & min are attained at $x=a$ and $x=b$.
Then $\forall x\in[a,b]$, $f(x)=f(a)=f(b)$
You show $f'(x)=0$ for all $a<x<b$.

case 2: Suppose the max or min is attained on (a,b)
W.L.O.G. suppose $a<x_0<b$ and $f(x_0)$ is a local max.
Then $f'(x_0)=0$ by Lemma 4.16.

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Thm 4.18 Mean Value Theorem:

Suppose $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b).

Then $\exists\, x_0 \in (a,b)$ s.t. $f'(x_0) = \dfrac{f(b) - f(a)}{b - a}$

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proof: Let $h(x) = f(x) - \dfrac{f(b) - f(a)}{b - a} \cdot x$

for $h : [a,b] \to \mathbb{R}$.

Notice that h is continuous on $[a,b]$ and differentiable on (a,b).

$$h(a) = f(a) - \frac{f(b) - f(a)}{b-a} \cdot a.$$

$$= \frac{(b-a)\, f(a)}{b-a} - \frac{a \cdot f(b) - a \cdot f(a)}{b-a},$$

$$= \frac{b\, f(a) - a\, f(b)}{b-a}.$$

$$h(b) = f(b) - \frac{f(b)-f(a)}{b-a} \cdot b.$$

$$= \frac{(b-a)\,f(b)}{b-a} - \frac{b\cdot f(b) - f(a)\cdot b}{b-a}$$

$$= \frac{f(a)\,b - a\,f(b)}{b-a} = h(a).$$

Apply Rolle's Thm to h and $\exists\, x_0 \in (a,b)$ s.t. $h'(x_0) = 0$.

Note $h'(x) = f'(x) - \dfrac{f(b)-f(a)}{b-a}$

Thus $f'(x_0) = \dfrac{f(b)-f(a)}{b-a}$. $\boxtimes$

Lemma 4.19 Let $f : (a,b) \to \mathbb{R}$ be differentiable.

$$f \text{ is constant on } (a,b)$$
$$\text{iff}$$
$$\forall x \in (a,b),\ f'(x) = 0.$$

Proof: $(\Rightarrow)$ Suppose $\exists\, c \in \mathbb{R}$ s.t. $f(x) = c$ for $x \in (a,b)$.

Fix $x_0 \in (a,b)$.

Consider $\displaystyle\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} 0 = 0.$

$(\Leftarrow)$ Suppose $\forall x \in (a,b),\ f'(x) = 0$.

Suppose $\exists\, x_1, x_2 \in (a,b)$ and $f(x_1) \neq f(x_2)$ and $x_1 < x_2$.

Notice that f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2).

Apply MVT to $f : [x_1, x_2] \to \mathbb{R}$ and $\exists\, x_0 \in (x_1, x_2)$

s.t. $\displaystyle f'(x_0) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0$, $\Rightarrow\Leftarrow$.

Cor. 4.21 Suppose $f:(a,b)\to\mathbb{R}$ is differentiable.

If $\forall x\in(a,b)$, $f'(x)>0$, then f is strictly increasing on (a,b).

I.e. $\forall\, u,v\in(a,b)$, if $u<v$, then $f(u)<f(v)$.

proof: Suppose $\forall x\in(a,b)$, $f'(x)>0$.

Let $u,v\in(a,b)$ and suppose $u<v$.

Notice f is continuous on $[u,v]$ and differentiable on (u,v) since f is differentiable on (a,b).

Apply MVT to f on $[u,v]$ and $\exists x_0\in(u,v)\subseteq(a,b)$

s.t.
$$f'(x_0)=\frac{f(v)-f(u)}{v-u}>0.$$

Thus $f(v)>f(u)$. $\square$