HW2
Math 537 Ordinary Differential Equations
Due Sep 25, 2020

Student Name: ______________________________ **ID** ______________

1: [25 points] Consider the following second-order ordinary differential equations (ODEs) for linear pendulum oscillations:

$$\frac{d^2x}{dt^2} + c\frac{dx}{dt} + Kx = 0, \tag{1}$$

which is a linearized version of the nonlinear system:

$$\frac{d^2x}{dt^2} + c\frac{dx}{dt} + Ksin(x) = 0.$$

Assume $c = 5$ and $K = 4$.

(a) Solve Eq. (1) for solutions.

(b) Convert Eq. (1) into a system of first-order ODEs by introducing $y = dx/dt$. Solve the system of the first-order ODEs.

2: [25 points] Consider the following system of linear ODEs:

$$\frac{dx}{dt} = \alpha y, \tag{2a}$$

$$\frac{dy}{dt} = -\beta x. \tag{2b}$$

Discuss the region in the $\alpha\beta$-plane where this system has different types of eigenvalues.

3: [25 points] Consider the following linearized Lorenz model (Lorenz, 1963):

$$\frac{dX}{dt} = -\sigma X + \sigma Y, \tag{3a}$$

$$\frac{dY}{dt} = rX - Y. \tag{3b}$$

Perform a stability analysis for $\sigma > 0$ (i.e., discuss the cases with $r > 1$, $r = 1$, and $r < 1$, respectively.)

4: [25 points] Consider the following epidemic model (Kermack and McKendrick, 1927), which is called the "SIR" model:

$$\frac{dS}{dt} = -\frac{\beta}{N} SI, \tag{4.1}$$

$$\frac{dI}{dt} = \frac{\beta}{N} SI - \nu I, \tag{4.2}$$

$$\frac{dR}{dt} = \nu I. \tag{4.3}$$

Here, S, I, and R denote susceptible, infected, and recovered individuals, respectively. Three parameters, $\beta > 0$, $\nu > 0$, and $N > 0$, represent a transmission rate, a recovery rate, and a fixed population ($N = S + I + R$), respectively. Complete the following derivations to convert Eqs. (4.1)-(4.3) into the following equations:

$$S = S(0) e^{-\frac{\beta}{N\nu}(R(t) - R(0))}, \tag{4.4}$$

$$I = N - S(0) e^{-\frac{\beta}{N\nu}(R(t) - R(0))} - R, \tag{4.5}$$

$$\frac{dR}{dt} = \nu \left(N - R - S(0) e^{-\frac{\beta}{N\nu}(R(t) - R(0))} \right), \tag{4.6}$$

where $S(0)$ and $R(0)$ represent the initial values of S and R, respectively.

(a) Show

$$S + I + R = constant = N \tag{4.7}$$

(i.e., $\frac{d(S+I+R)}{dt} = 0$).

(b) Apply Eqs (4.1) and (4.3) to obtain the following:

$$\frac{S'}{S} = -\frac{\beta}{N\nu} R'.$$

Integrate the above Eq. to obtain Eq. (4.4), yielding $S = S(R)$.

(c) Apply Eqs. (4.4) and (4.7) to find Eq. (4.5) for I, which is a function of R.

(d) Apply the above to obtain Eq. (4.6).

(e) Briefly discuss how to analyze Eq. (4.6) to reveal the characteristics of the solution.

Note that based on Eqs. (4.4)-(4.6), we can obtain the solutions by solving a single first order ODE in Eq. (4.6) for $R(t)$, and then compute $S(t)$ and $R(t)$ using Eqs. (4.4) and (4.5), respectively.