MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #25

Limit Cycles, Closed Orbits and Poincare-Bendixson Theorem in 2D Systems

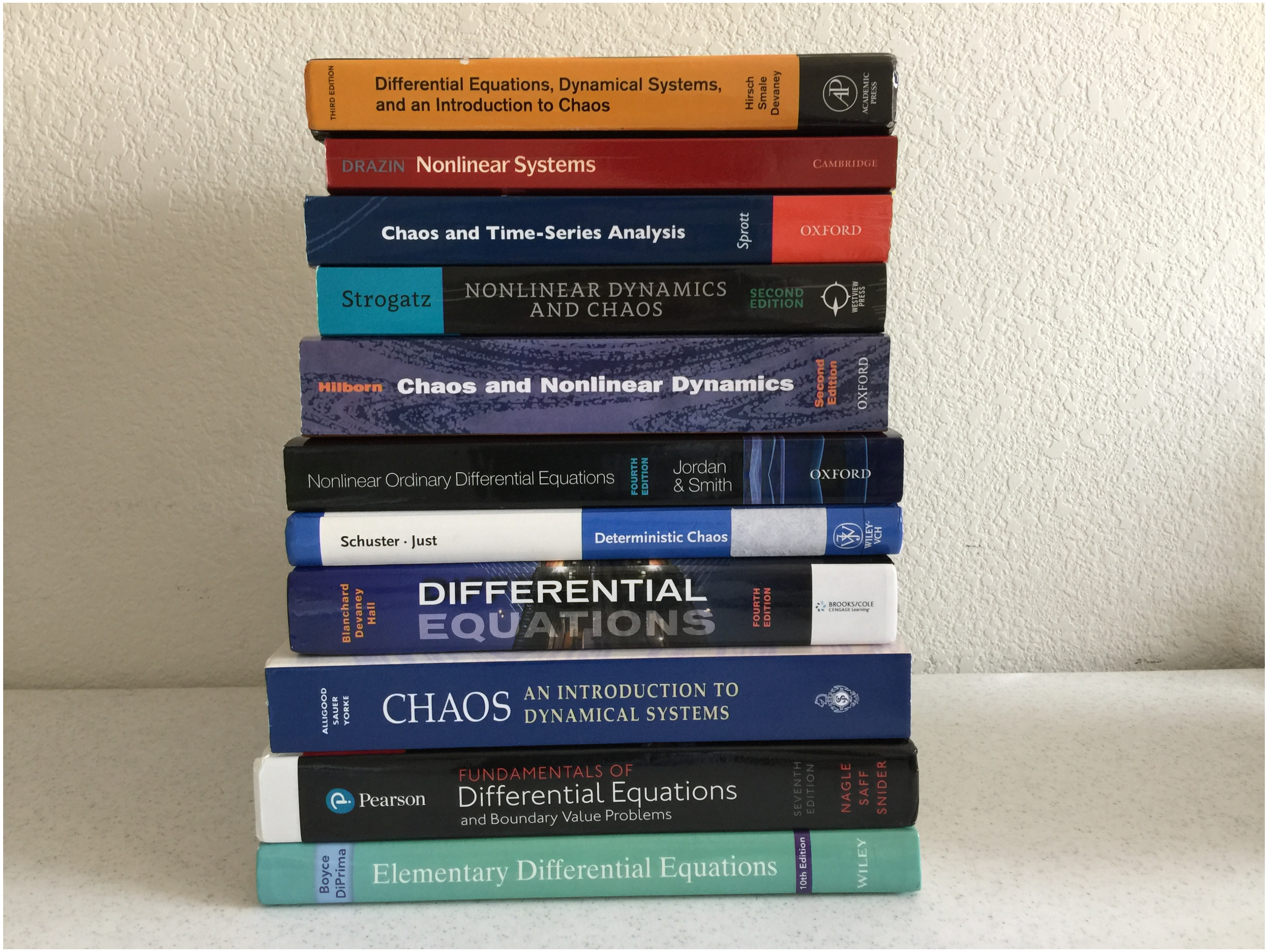
Instructor: Dr. Bo-Wen Shen*

Department of Mathematics and Statistics
San Diego State University

Outline

- Introduction
  - Limit cycle
  - A mini review of vector calculus
  - Gradient system ($\vec{v} = \nabla\phi$) for irrotational flow ($\nabla\times\vec{v} = 0$)
  - Hamiltonian system for incompressible flow ($\nabla \cdot \vec{v} = 0$)
- Energy methods for stability analysis near a critical point
  - The limit cycle of van dan Pol Equation
- Methods for ruling out closed orbits
  - I. Existence of a Laypunov function
  - II. Zero curl ($\nabla\times\vec{v} = 0$)
  - III. Positive ($\nabla \cdot \vec{v} > 0$) or negative ($\nabla \cdot \vec{v} < 0$) divergence
- Poincare-Bendixson Theorem
- Summary
  - Eigenvalue analysis for gradient and Hamiltonian Systems
  - Poincare-Bendixson Theorem

References


THIRD EDITION
Differential Equations, Dynamical Systems, and an Introduction to Chaos
Hirsch Smale Devaney
ACADEMIC PRESS
DRAZIN Nonlinear Systems
CAMBRIDGE
Chaos and Time-Series Analysis
Sprott
OXFORD
Strogatz
NONLINEAR DYNAMICS AND CHAOS
SECOND EDITION
WESTVIEW PRESS
Hilborn Chaos and Nonlinear Dynamics
Second Edition
OXFORD
Nonlinear Ordinary Differential Equations
FOURTH EDITION
Jordan & Smith
OXFORD
Schuster · Just
Deterministic Chaos
WILEY-VCH
Blanchard Devaney Hall
DIFFERENTIAL EQUATIONS
FOURTH EDITION
BROOKS/COLE
ALLIGOOD SAUER YORKE
CHAOS AN INTRODUCTION TO DYNAMICAL SYSTEMS
Pearson
FUNDAMENTALS OF Differential Equations and Boundary Value Problems
SEVENTH EDITION
NAGLE SAFF SNIDER
Boyce DiPrima
Elementary Differential Equations
10th Edition
WILEY

Poincare-Bendixson Theorem

TBD

The Poincare–Bendixson Theorem essentially determines all of the possible limiting behaviors of a planar flow (HSD).

Consider a particular trajectory starting in R. The Poincare-Bendixson Theorem states that there are only two (three) possibilities for that trajectory (Hilborn, p101) (R is a closed bounded subset of the plane):

1. The trajectory approaches a fixed point of the system as $t \to \infty$.

2. The trajectory approaches a limit cycle as $t \to \infty$.

3. The trajectory is a limit cycle.

Hilborn

Poincare-Bendixson Theorem

TBD

- The theorem works only in two dimensions because only in two dimensions does a closed curve separate the space into a region "inside" the curve and a region "outside."
- Thus a trajectory starting inside the limit cycle can never get out and a trajectory starting outside can never get in.
- from the Poincare-Bendixson Theorem we arrive at an important result: Chaotic trajectories (in a bounded system) cannot occur in a state space of two dimensions.
- For systems described by differential equations, we need at least three state-space dimensions for chaos.

Hilborn

Terminology

- Limit cycle: an isolated closed path

Consider $X' = F(X)$, here $X' = (x', y') = \vec{v}$.

- Gradient system ($\vec{v} = \nabla\phi$) for irrotational flow ($\nabla\times\vec{v} = 0$)

  ϕ : velocity potential; $\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right)$

- Hamiltonian system for incompressible flow ($\nabla \cdot \vec{v} = 0$)

  ψ: streamfunction; $\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \left(\frac{\partial\psi}{\partial y}, \frac{-\partial\psi}{\partial x}\right)$

Limit Cycles

- An isolated closed path is called a limit cycle: "Isolated" in the sense that there is no other closed path in its immediate neighborhood (Jordan and Smith).

Nagle et al.

A new edition

Limit Cycle

Definition 5. A nontrivial† closed trajectory with at least one other trajectory spiraling into it (as time approaches plus or minus infinity) is called a **limit cycle.**

An old edition

Limit Cycle

Definition 5. A nontrivial† closed trajectory that is isolated is called a **limit cycle.**

ω-limit set and α-limit set

- A point z in R^n is in the ω-limit set $\omega(v_0)$ of the solution curve $\mathrm{F}(t, v_0)$ if there is a sequence of points increasingly far out along the orbit (that is, $t \to \infty$) which converges to z.
- Specifically, z is in $\omega(v_0)$ if there exists an unbounded increasing sequence $\{t_n\}$ of real numbers with $\lim_{n \to \infty} \mathrm{F}(t_n, v_0) = z$.
- A point z in R^n is in the α-limit set $\alpha(v_0)$ if there exists an unbounded decreasing sequence $\{t_n\}$ of real numbers ($t_n \to \infty$) with $\lim_{n \to \infty} \mathrm{F}(t_n, v_0) = z$.

Alligood et al.

ω-Limit Cycle: Definition

- If γ is an ω-limit cycle, there exists $X \notin \gamma$ such that

$$\lim_{t\to\infty} d(\phi_t(X), \gamma) = 0.$$

- Geometrically this means that some solution $(\phi_t(X))$ spirals toward γ as $t \to \infty$.

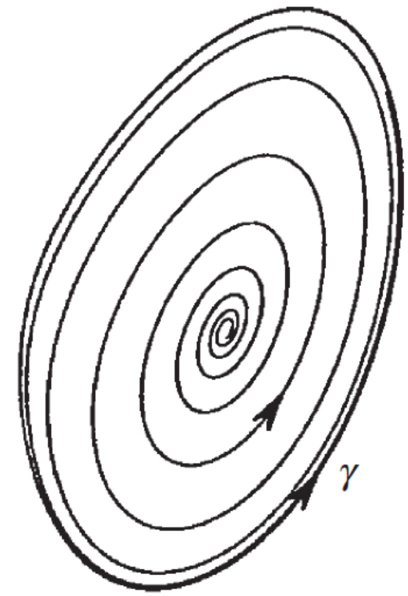


Figure 10.10 A solution spiraling toward a limit cycle.

HSD

Limit Sets: Ex 1

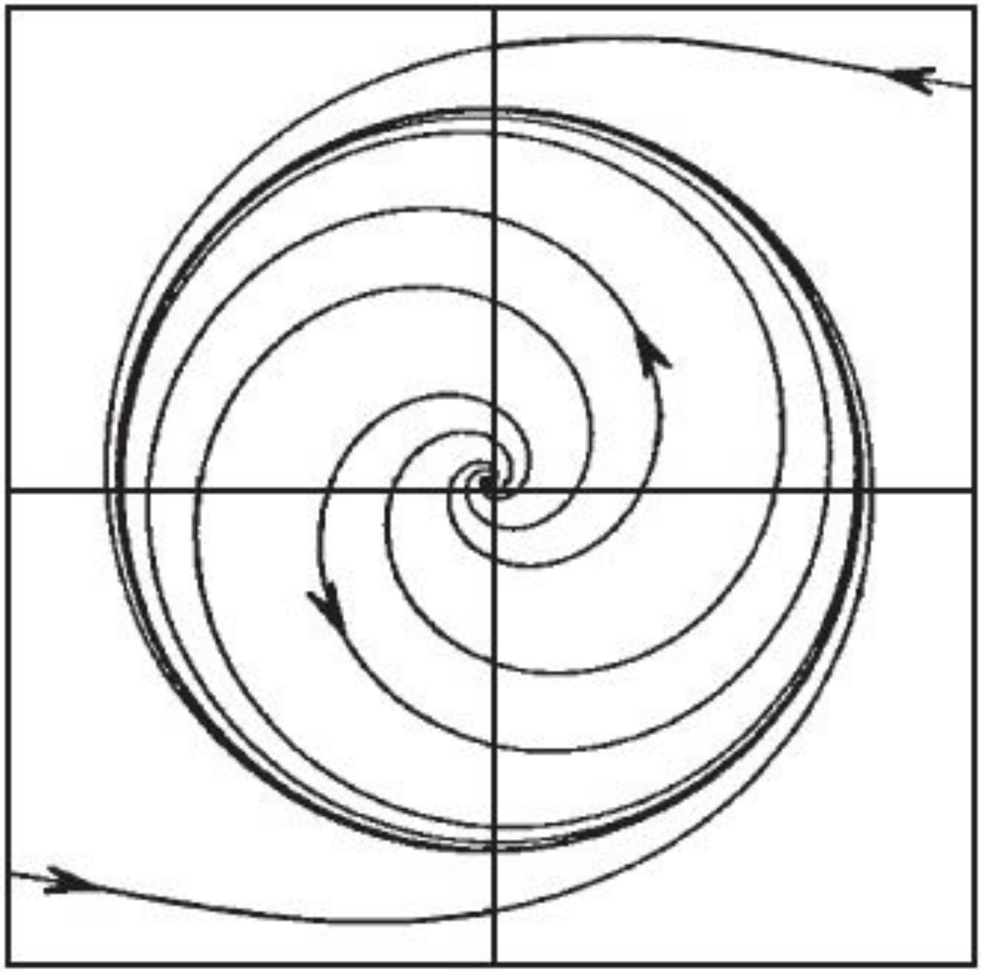


Figure 10.1 The phase plane for $r' = \quad (r - r^3)$, $\theta' = 1$.

$$\frac{dx}{dt} = ax - y - x(x^2 + y^2),$$

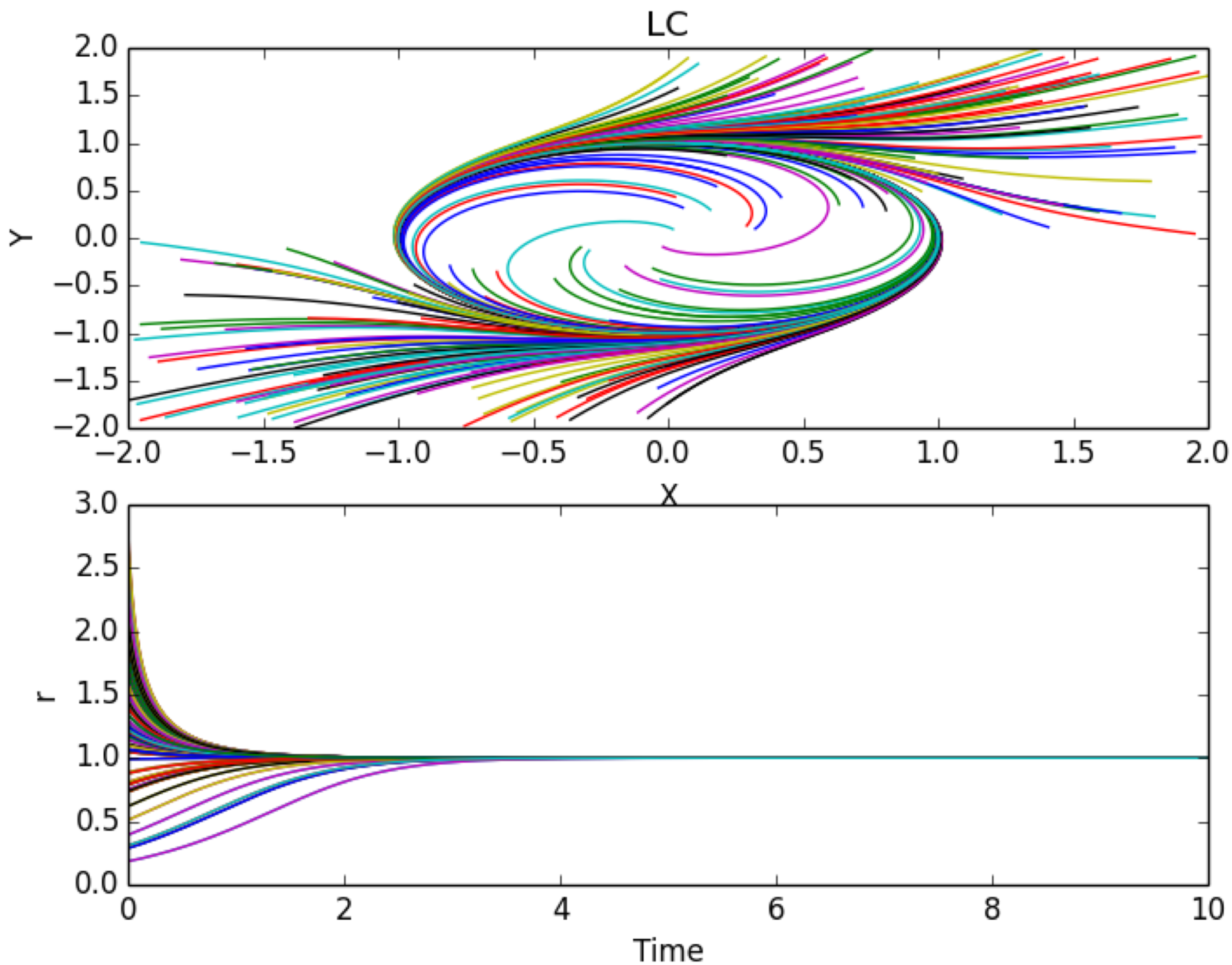
$$\frac{dy}{dt} = x + ay - y(x^2 + y^2),$$

where $a = 1.0$ and $t \in [0, 10]$.

$$r' < 0 \; when \; r > 1$$

$$r' > 0 \; when \; r < 1$$

Limit Cycle: Ex 1 (cont.)



2: [35 points] $dx/dt = -(ax + x^3)$ for $x \geq 0$ and $x(t=0) = x_o$. [Hint: set $r = x^2$, solve for r and discuss the results when $a < 0$, $a = 0$ or $0 < a$.]

$$dx/dt = x - x^3 \quad \text{when a=-1}$$

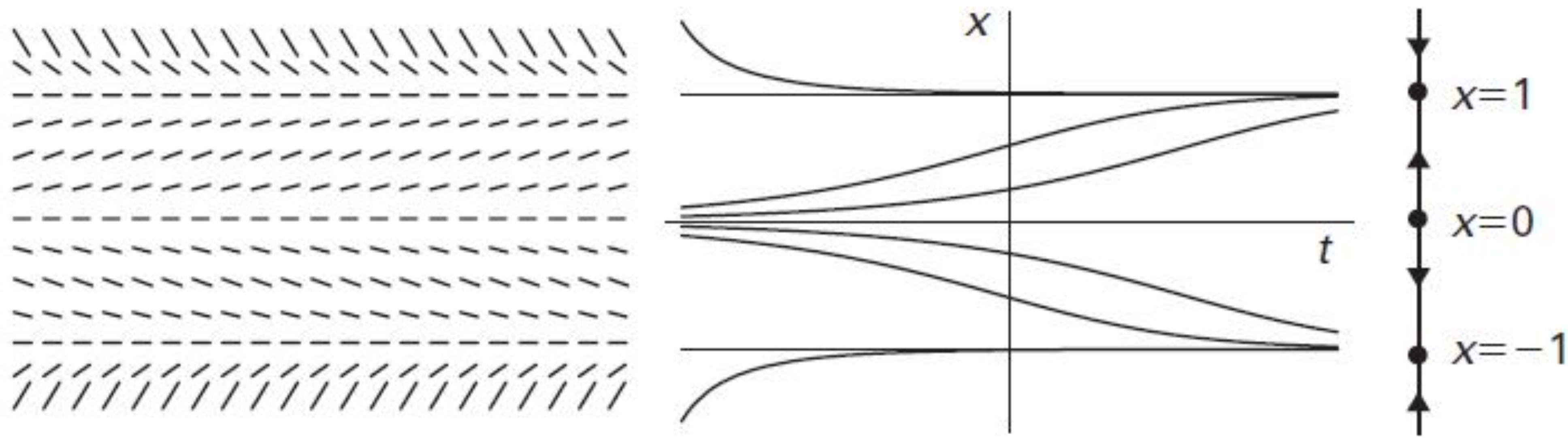


Figure 1.5 The slope field, solution graphs, and phase line for $x' = x - x^3$.

Limit Sets: An Alternative "Ex1"

Exercise 4-32 Consider the system

$$\dot{x} = y + x(1 - x^2 - y^2) \qquad \dot{y} = -x + y(1 - x^2 - y^2) \tag{4-72}$$

Using the polar coordinates

$$r^2 = x^2 + y^2 \qquad \theta = \arctan \frac{y}{x}$$

$$r\dot{r} = x\dot{x} + y\dot{y} \qquad r^2\dot{\theta} = x\dot{y} - y\dot{x}$$

reduce the system to

$$\dot{r} = r(1 - r^2) \qquad \dot{\theta} = -1$$

and verify that the general solution is given by

$$r = \frac{1}{(1 + ce^{-2t})^{1/2}} \qquad \theta = -(t - t_0)$$

With $t_0 = 0$, we have

$$x = \frac{\cos t}{(1 + ce^{-2t})^{1/2}} \qquad y = -\frac{\sin t}{(1 + ce^{-2t})^{1/2}}$$

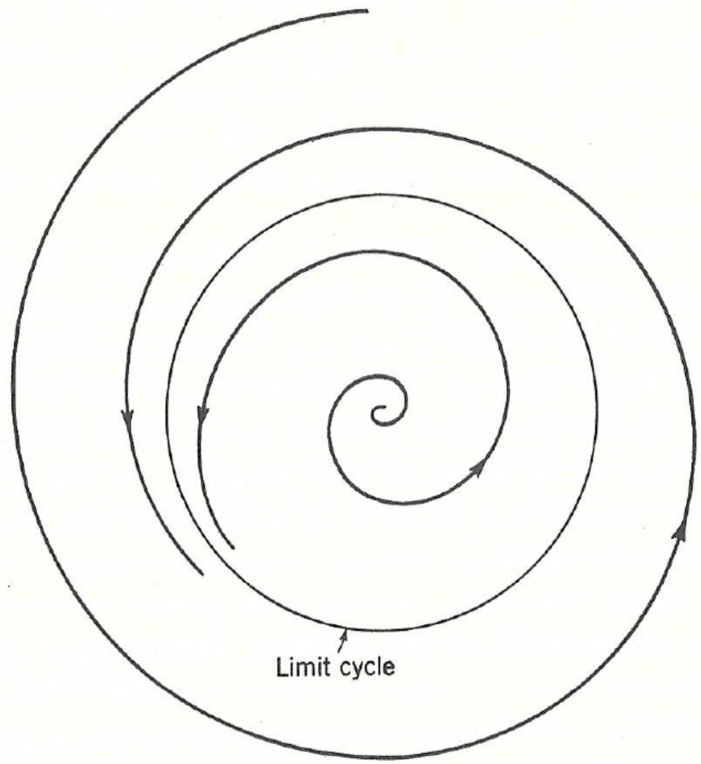


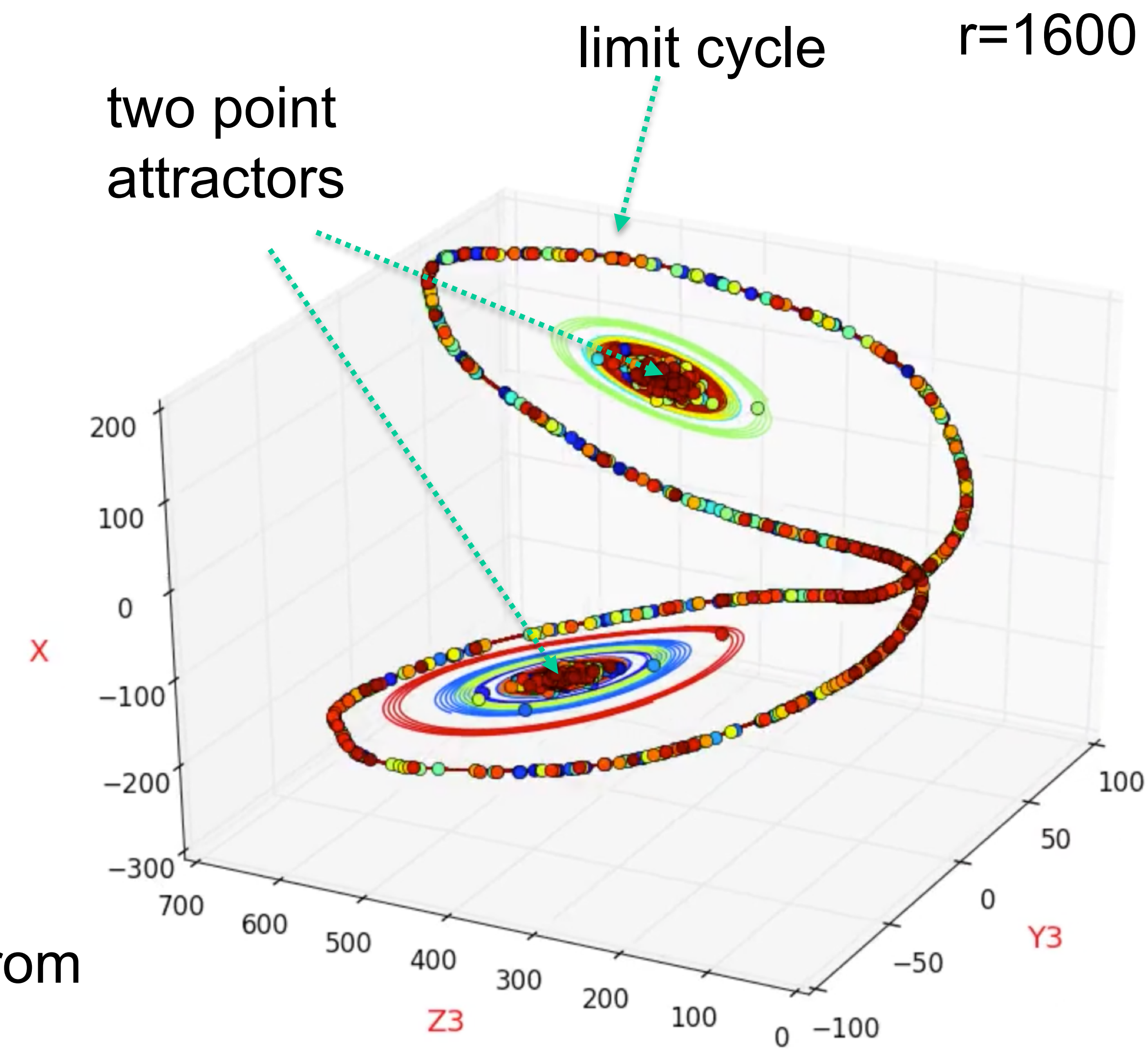
Fig. 4-5

Boyce and DiPrima

The Second Kind of Attractor Coexistence

- For the 2nd kind of attractor coexistence, a limit cycle (LC) that is an isolated closed orbit coexists with point attractors.

- Time evolution of 2,048 orbits in the X-Y_3-Z_3 space using the 9DLM.
- The total simulation time is τ = 3.5.
- Transient orbits are only kept for the last 0.25 time units, i.e. for the time interval of [max (0, T-0.25), T] at a given time T.



- The animation is available from https://goo.gl/sMhoUb.

A Brief Review on Calc III

Learning Outcomes

Formulas for Grad, Div, Curl, and the Laplacian

	Cartesian (x, y, z) $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors in the directions of increasing x, y, and z. **P, Q, and R** are the scalar components of $\mathbf{F}(x, y, z)$ in these directions.
Gradient	$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$
Divergence	$\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$
Curl	$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$
Laplacian	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

The Fundamental Theorem of Line Integrals

1. Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ be a vector field whose components are continuous throughout an open connected region D in space. Then there exists a differentiable function f such that

$$\mathbf{F} = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

if and only if for all points A and B in D the value of $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path joining A to B in D.

2. If the integral is independent of the path from A to B, its value is

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

Green's Theorem (Tangential Form)
$$\iint_R \nabla \times \vec{F} \cdot \vec{k}\,dxdy = \oint \vec{F} \cdot d\vec{r}$$

Stokes' Theorem
$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

Green's Theorem (Normal Form)
$$\iint_R \nabla \cdot \vec{F}\,dxdy = \oint \vec{F} \cdot \vec{n}\,ds$$

Divergence Theorem
$$\iiint_E \operatorname{div} \mathbf{F}\,dV = \iint \vec{F} \cdot \vec{n}\,dS$$

Curl and Divergence

- $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, a "meta" vector: $F = (P, Q, R)$

- Gradient:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (f_x, f_y, f_z)$$

- Curl ("a Cross product of ∇ and F"):

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z, P_z - R_x, Q_x - P_y)$$

- Divergence ("a Dot product of ∇ and F"):

$$\nabla \bullet F = (P_x + Q_y + R_z)$$

Curl and Divergence (2D)

- $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$, a "meta" vector: $F = (P, Q)$

- Gradient: $\nabla f = (f_x, f_y)$

- Curl ("a Cross product of ∇ and F"):

$$\nabla \times F \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = (0, 0, Q_x - P_y)$$

$$\nabla \times \vec{F} \cdot \vec{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

- Divergence ("a Dot product of ∇ and F"):

$$\nabla \cdot F = P_x + Q_y$$

Vector Calculus Identities (2D)

- curl (gradient f)=0

$$f = \phi$$

$$\nabla \times (\nabla \phi) = 0$$

$$\boxed{\vec{v} = \nabla \phi}$$

$$(u, v) = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right)$$

$$\phi: velocity\ potential$$

$$\boxed{\nabla \times \vec{v} = 0}$$

$$irrotational\ flow$$

- div (curl F)=0

$$F = (P, Q, R)$$

$$\nabla \times F = \left(R_y - Q_z, P_z - R_x, Q_x - P_y\right)$$

$$F = (0, 0, \psi); \quad \nabla \times F = \left(\psi_y, -\psi_x, 0\right)$$

$$\boxed{\vec{v} = \nabla \times F}$$

$$(u, v) = \left(\frac{\partial \psi}{\partial y}, \frac{-\partial \psi}{\partial x}\right)$$

$$\psi: streamfunction$$

$$\boxed{\nabla \cdot \vec{v} = 0}$$

$$incompressible\ flow$$

Path Independence

$$\boxed{\nabla \times \vec{v} = 0}$$

$$irrotational\ flow$$

$$\boxed{\nabla \cdot \vec{v} = 0}$$

$$incompressible\ flow$$

Green's Theorem (Tangential Form)

$$\oint \vec{V} \cdot d\vec{r} = \iint \nabla \times \vec{V} \cdot \vec{k}\, dxdy$$

$$\oint \vec{V} \cdot d\vec{r} = 0$$

$$\oint Pdx + Qdy = 0$$

Green's Theorem (Normal Form)

$$\oint \vec{V} \cdot \vec{n}\, ds = \iint \nabla \cdot \vec{V} dxdy$$

$$\oint \vec{V} \cdot \vec{n}\, ds = 0$$

$$\oint Pdy - Qdx = 0$$

Review: Path Independence

$\vec{F} = (P,Q)$ $\quad \vec{T}$ tangent vector

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r} \text{ (circulation)}$$

$$= \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt}\right)dt$$

$$\overset{?}{=} \int_a^b \nabla f\cdot d\vec{r} = \int_a^b \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$\nabla\times\vec{F} = 0$ $\quad$ Irrotational (e.g., A2)

$$\nabla\times\vec{F}\cdot\vec{k} = 0 = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$= \int_a^b df = f(b) - f(a) \quad \text{potential function}$$

$\vec{n}$ normal vector

$$\int \vec{F}\cdot\vec{n}\,ds = \int P\,dy - Q\,dx \text{ (flux)}$$

$$= \int_a^b \left(P\frac{dy}{dt} - Q\frac{dx}{dt}\right)dt$$

$$\overset{?}{=} \int_a^b \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy$$

$\nabla\cdot\vec{F} = 0$ $\quad$ incompressible (e.g., A1)

$$\vec{F} = (P,Q) = \left(\frac{\partial\psi}{\partial y}, \frac{-\partial\psi}{\partial x}\right)$$

$$= \int_a^b d\psi = \psi(b) - \psi(a) \quad \text{stream function}$$

Review: Complex Potential

$$\Phi = \phi + i\psi$$

$$\frac{d\Phi}{dz} = \frac{\partial \Phi}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = \phi_x - i\phi_y$$

$$\frac{d\Phi}{dz}dz = \left(\phi_x - i\phi_y\right)\left(dx + idy\right)$$

$$= \left(\phi_x dx + \phi_y dy\right) + i\left(\phi_x dy - \phi_y dx\right)$$

$$= \left(Pdx + Qdy\right) + i\left(Pdy - Qdx\right)$$

$$= \vec{F}\cdot\vec{T} + i\vec{F}\cdot\vec{n}$$ Circulation + i Flux

Cauchy Riemann Equations

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \frac{\partial \phi}{\partial y} = \frac{-\partial \psi}{\partial x}$$

$$\vec{F} = \left(P,Q\right) = \left(\phi_x,\phi_y\right)$$

$$\phi = \ln\sqrt{x^2 + y^2}$$

(A4)

$$\psi = \tan^{-1}\left(\frac{y}{x}\right)$$

(A3)

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Energy Methods for Stability Analysis

$$\frac{d^2x}{dt^2} + h\left(x, \frac{dx}{dt}\right) + g(x) = 0$$ (nonlinear)

multiply both sides by $\frac{dx}{dt}$

$$x' \frac{d^2x}{dt^2} + x'h + x'g(x) = 0$$

$$\underbrace{\frac{1}{2}\frac{d}{dt}(x')^2}_{d(KE)/dt} + x'h + \underbrace{\frac{d}{dt}\int g(x)\,dx}_{d(PE)/dt} = 0$$

$$\frac{d}{dt}(KE + PE) = -x'h$$

Nagle et al.

Energy Methods for Stability Analysis

$$\frac{d^2x}{dt^2} + h\left(x, \frac{dx}{dt}\right) + g(x) = 0$$ (nonlinear)

$$F = -\nabla G$$

$$G(x) = \int g(x)\, dx, \textit{potential energy}$$

$$E(x, v) = \frac{1}{2}v^2 + G(x), \textit{total energy}$$

$$v = \frac{dx}{dt}$$

$$\frac{dE}{dt} = -\frac{dx}{dt}h = -vh;$$

- $\frac{dE}{dt} = 0$ when $h = 0$; *conservative*
- $\frac{dE}{dt} > 0$ when $vh < 0$
- $\frac{dE}{dt} < 0$ when $vh > 0$

Nagle et al.

Energy Methods for Stability Analysis

Consider $h = a(x)\frac{dx}{dt}$

$$\frac{d^2x}{dt^2} + a(x)\frac{dx}{dt} + g(x) = 0$$

$$G(x) = \int g(x)\,dx, potential\ energy$$

$$E(x, v) = \frac{1}{2}v^2 + G(x), total\ energy$$

$$\frac{dE}{dt} = -\frac{dx}{dt}h = -a\left(\frac{dx}{dt}\right)^2 = -av^2; \qquad v = \frac{dx}{dt}$$

- $\frac{dE}{dt} = 0$ when $a = 0$; $e.g., conservative\ when\ a \equiv 0$
- $\frac{dE}{dt} < 0$ when $a > 0$; positive damping
- $\frac{dE}{dt} > 0$ when $a < 0$; $negative\ damping$

Nagle et al.

The Limit Cycle of van dan Pol Equation: Ex 2

3: [30 points] Consider the following differential equation:

$$\frac{d^2X}{dt^2} + b(X^2-1)\frac{dX}{dt} + X = 0.$$

$$\frac{dE}{dt} = -hX' = -b(X^2-1)v^2; \qquad v = \frac{dX}{dt}$$

Assume b ($b = 0.1$) is positive.

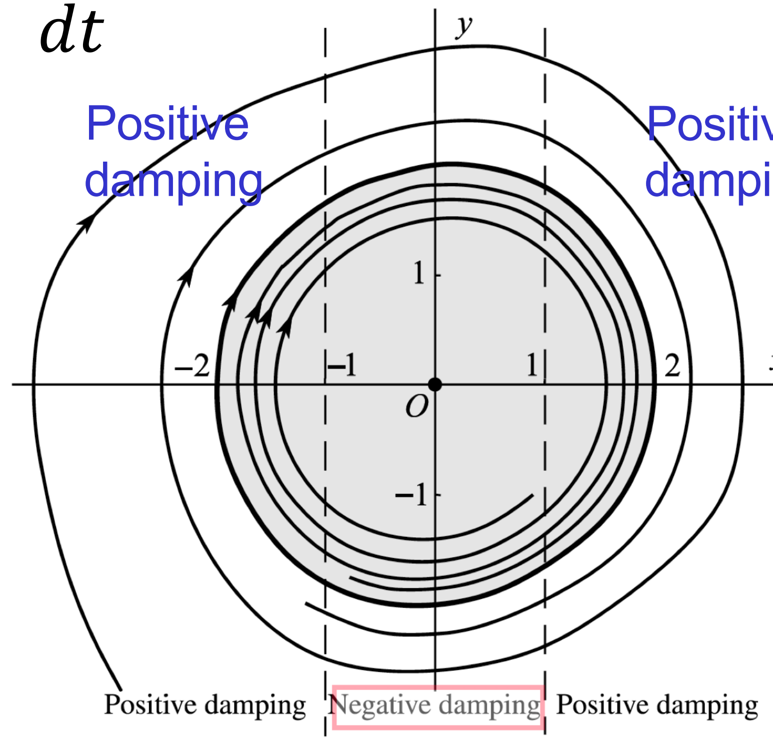
- $\frac{dE}{dt} = 0$ when $x = 1$;
- $\frac{dE}{dt} < 0$ when $|X| > 1$; "*sink*"
- $\frac{dE}{dt} > 0$ when $|X| < 1$; "*source*"

Jordan and Smith (p126)

Negative damping

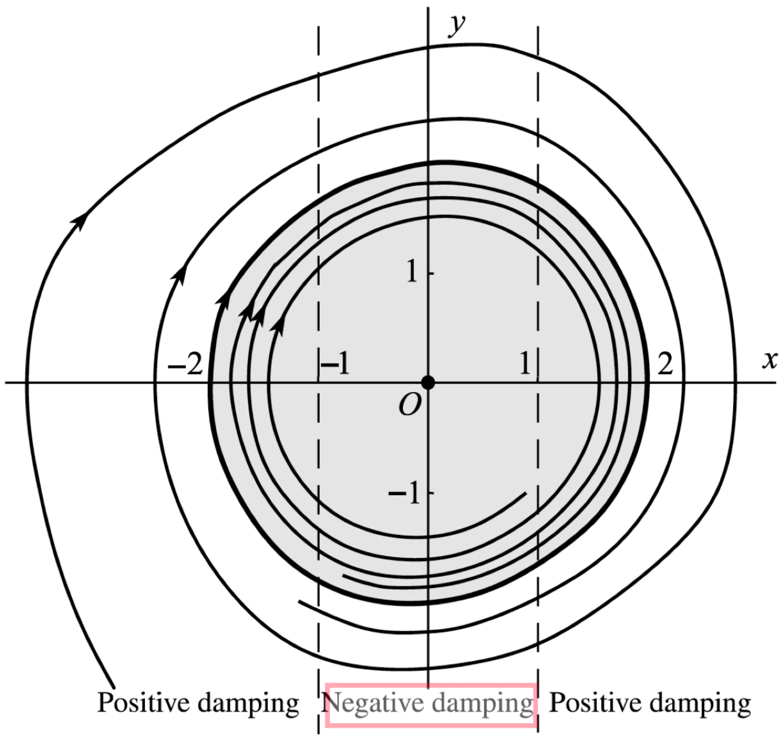
Positive damping

Positive damping



The Limit Cycle of van dan Pol Equation (cont.)

$b = 0.1$ $\qquad$ $b = 0.5$

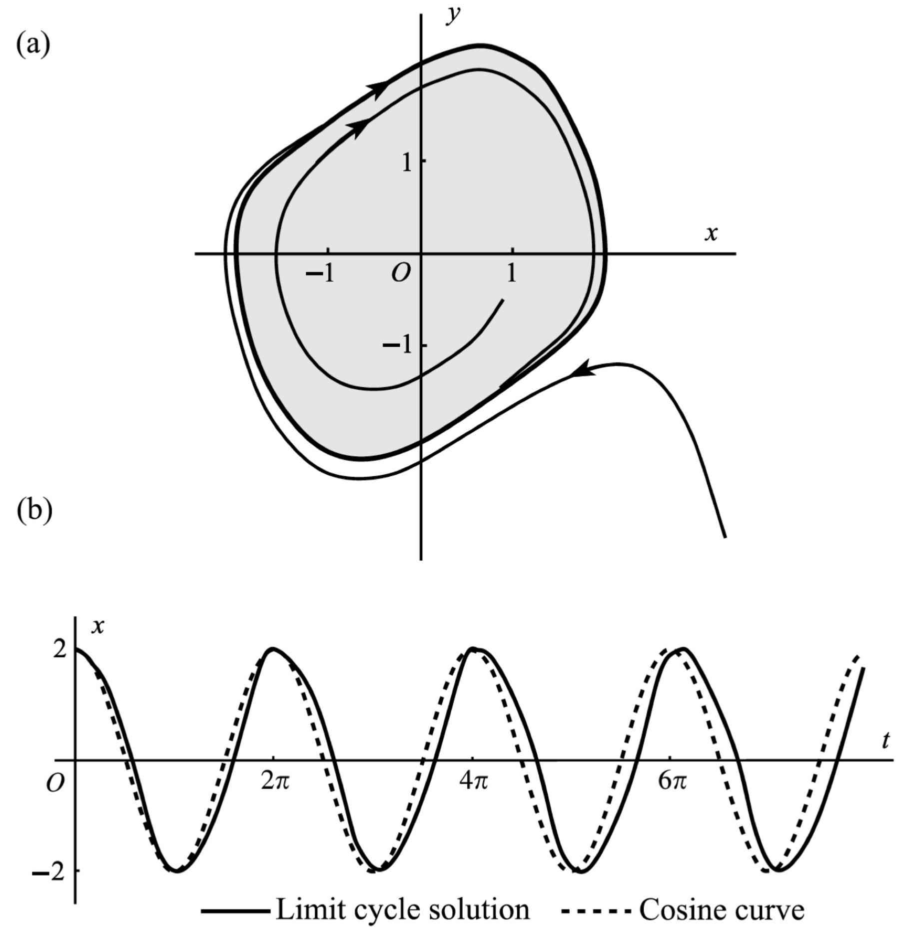


Negative damping

Positive damping

Positive damping

Jordan and Smith



(Spectral Analysis?)

An Estimate of the Amplitude for a LC

When b is small, $b = \epsilon$, we have $x'' + \epsilon(x^2 - 1)x' + x = 0$

By ignoring the dissipative term, we have

$$\boxed{x(t) \approx a\cos t, \quad y(t) \approx -a\sin t} \qquad y = \frac{dx}{dt}, T \approx 2\pi$$

$$h = \epsilon(x^2 - 1)x'; \qquad \frac{dE}{dt} = -hx' = -\epsilon(x^2 - 1)(x')^2;$$

$$\int_0^{2\pi} \frac{dE}{dt} dt = E(2\pi) - E(0) = 0 \quad \Longrightarrow \int_0^{2\pi} (x^2 - 1)(x')^2 dt = 0$$

$$\cos^2(t) = \frac{1 + \cos(2t)}{2}$$

$$\Longrightarrow \int_0^{2\pi} (a^2 cos^2(t) - 1)a^2 \sin^2(t)\, dt = 0$$

$$\sin^2(t) = \frac{1 - \cos(2t)}{2}$$

$$\int_0^{2\pi} (a^2 \frac{1 + \cos(2t)}{2} - 1) \frac{1 - \cos(2t)}{2} dt = 0 \qquad \int_0^{2\pi} (a^2 \frac{1 - \cos^2(2t)}{4} - \frac{1}{2})\, dt = 0$$

An Estimate of the Amplitude for a LC

When b is small, $b = \epsilon$, we have $x'' + \epsilon(x^2 - 1)x' + x = 0$

By ignoring the dissipative term, we have

$$\boxed{x(t) \approx a\cos t, \quad y(t) \approx -a\sin t} \qquad y = \frac{dx}{dt}, T \approx 2\pi$$

$$h = \epsilon(x^2 - 1)x'; \qquad \frac{dE}{dt} = -hx' = -\epsilon(x^2 - 1)(x')^2;$$

$$\int_0^{2\pi} \frac{dE}{dt}dt = E(2\pi) - E(0) = 0 \quad \Longrightarrow \int_0^{2\pi} (x^2 - 1)(x')^2 dt = 0$$

$$\Longrightarrow \int_0^{2\pi} (a^2 cos^2(t) - 1)a^2 \sin^2(t)\,dt = 0 \qquad \int_0^{2\pi} (a^2 \frac{1 - \cos^2(2t)}{4} - \frac{1}{2})\,dt = 0$$

$$\Longrightarrow \int_0^{2\pi} (a^2 \frac{2 - (1 + \cos(4t))}{8} - \frac{1}{2})\,dt = 0 \qquad \boxed{\frac{1}{4}a^2 - 1 = 0}$$

A Perturbative Analysis of the Solution

Consider the following Van der Pol equation

$$\frac{d^2X}{dt^2} + b(X^2 - 1)\frac{dX}{dt} + X = 0, \tag{1}$$

which can be written as follows:

$$\frac{d^2X}{dt^2} + X = \epsilon\frac{dX}{dt} - \epsilon X^2\frac{dX}{dt}, \tag{2}$$

where ϵ is introduced to replace b in order to perform a perturbative analysis.
We seek a first-order expansion for the solution in the form

$$X = X_o + \epsilon X_1 + \dots \tag{3}$$

Plugging the above into Eq. (2), we have

$$\left(X_o'' + \epsilon X_1''\right) + \left(X_o + \epsilon X_1\right) = \epsilon\left(\frac{dX_o}{dt} + \epsilon\frac{dX_1}{dt}\right) - \epsilon\left(X_o + \epsilon X_1\right)^2\left(\frac{dX_o}{dt} + \epsilon\frac{dX_1}{dt}\right). \tag{4}$$

A Perturbative Analysis of the Solution

$$\left(\boxed{X_o''} + \epsilon X_1''\right) + \boxed{(X_o} + \epsilon X_1) = \epsilon \left(\frac{dX_o}{dt} + \epsilon \frac{dX_1}{dt} \right) - \epsilon \left(X_o + \epsilon X_1 \right)^2 \left(\frac{dX_o}{dt} + \epsilon \frac{dX_1}{dt} \right). \tag{4}$$

Considering terms with ϵ^0, we have

$$\boxed{X_o'' + X_o = 0.} \tag{5}$$

A Perturbative Analysis of the Solution

$$\left(X_o'' + \boxed{\epsilon X_1''}\right) + \left(X_o + \boxed{\epsilon X_1}\right) = \epsilon\left(\boxed{\frac{dX_o}{dt}} + \epsilon\frac{dX_1}{dt}\right) - \epsilon\left(\boxed{X_o} + \epsilon X_1\right)^2\left(\boxed{\frac{dX_o}{dt}} + \epsilon\frac{dX_1}{dt}\right). \tag{4}$$

Considering terms with ϵ^0, we have

$$X_o'' + X_o = 0. \tag{5}$$

Considering terms with ϵ^1, we have

$$\boxed{X_1'' + X_1} = \boxed{X_o'} - \boxed{X_o^2 X_o'}. \tag{6}$$

From Eq. (5), we have the solution of X_o as follows:

$$X_o = a\cos(t+\beta), \tag{7}$$

A Perturbative Analysis of the Solution

$$X_1'' + X_1 = \left(-a + \frac{1}{4} a^3\right) sin(t+\beta) + \frac{1}{4} a^3 sin(3(t+\beta)). \tag{12}$$

Consider $X_1 = u + v$, which satisfy the following:

$$u'' + u = \left(-a + \frac{1}{4} a^3\right) sin(t+\beta), \tag{13}$$

$$v'' + v = \frac{a^3}{4} sin(3(t+\beta)). \tag{14}$$

A particular solution of Eq. (13) is:

$$u_p = \frac{at}{2}\left(1 - \frac{1}{4} a^2\right) cos(t+\beta), \tag{15}$$

A particular solution of Eq. (14) is:

$$v_p = \frac{1}{32} a^3 sin(3(t+\beta)). \tag{16}$$

A red circle indicates a nonlinear term.

A Perturbative Analysis of the Solution

From Eqs. (3), (7), (15-16), we have

$$X = acos(t+\beta) + \epsilon\left[\frac{at}{2}\left(1-\frac{1}{4}\,a^2\right)cos(t+\beta) + \frac{1}{32}a^3 sin(3(t+\beta))\right] + \cdot \qquad (17)$$

As a result of the presence of the mixed-secular term, the above expansion is non-uniform for $t \geq O(\epsilon^{-1})$ because the correction term is the order or larger than the first term. The mixed-secular term in Eq. (17) disappears if

$$a(1-\frac{1}{4}a^2) = 0, \qquad \boxed{\frac{1}{4}a^2 - 1 = 0} \qquad (18)$$

leading to $a = 0$ $a = \pm 2$, the latter of which provides an estimate on the amplitude (a). Therefore, the solution becomes

$$X = 2cos(t+\beta) + \frac{1}{4}\epsilon sin(3(t+\beta)) + \cdots \qquad (19)$$

The Limit Cycle of van dan Pol Equation: A Proof

We conclude this section with a statement of a theorem on **Lienard equations** due to N. Levinson and O. K. Smith,[†] which can be used to show that van der Pol's equation has a unique nonconstant periodic solution (see Problem 24).

Levinson and Smith's Theorem

Theorem 8. Let $f(x)$ and $g(x)$ be continuous functions and let

$$F(x) := \int_0^x f(s)\,ds \ , \qquad G(x) := \int_0^x g(s)\,ds \ .$$

The Lienard equation

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0$$

has a unique nonconstant periodic solution whenever all of the following conditions hold:

(a) $f(x)$ is even.

(b) $F(x) < 0$, for $0 < x < a$;
$F(x) > 0$, for $x > a$, for some a.

(c) $F(x) \to +\infty$ as $x \to +\infty$, monotonically for $x > a$.

(d) $g(x)$ is an odd function with $g(x) > 0$ for $x > 0$.

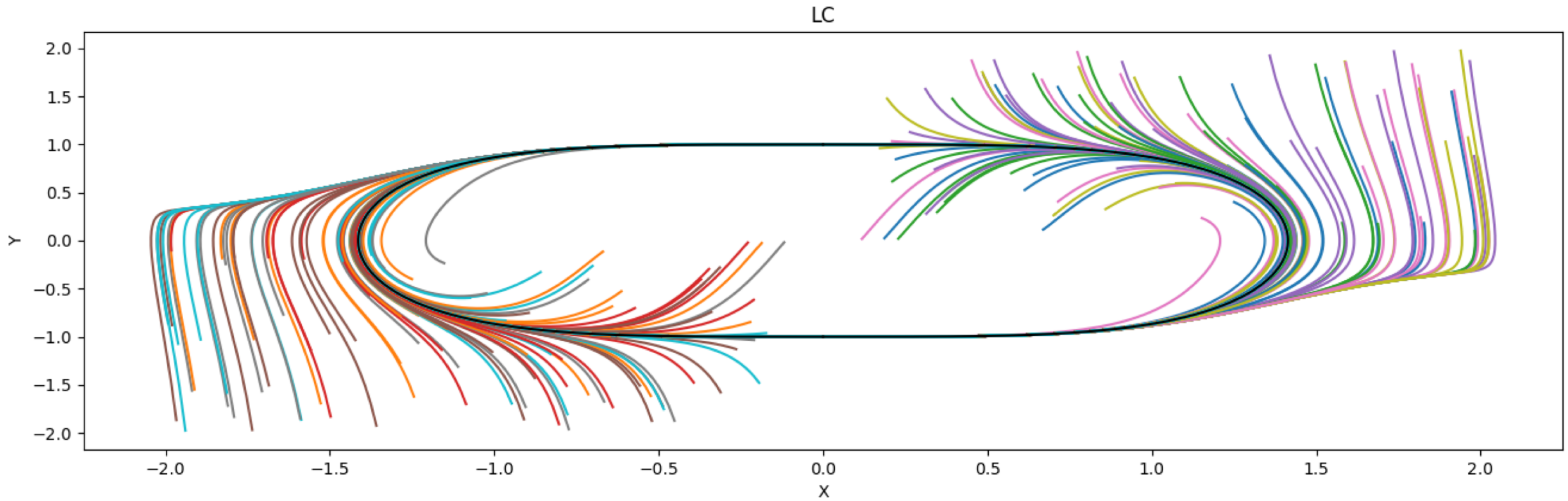
(e) $G(x) \to +\infty$ as $x \to +\infty$.

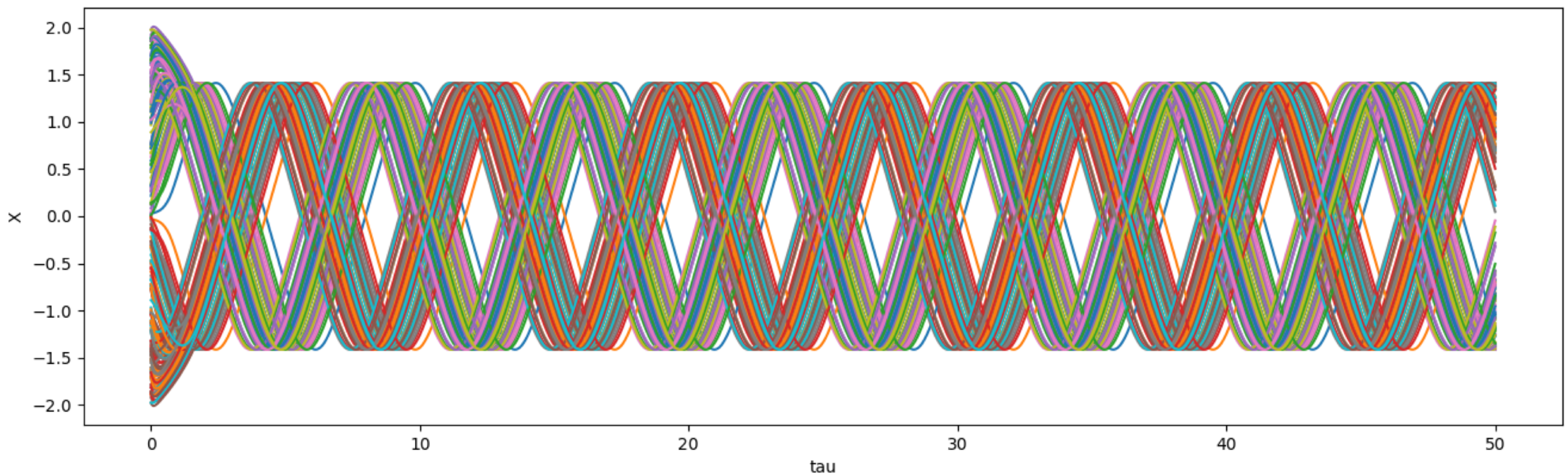
Nagle et al.

Limit Cycle: Ex 5

Shen

$$\ddot{X} + \left[X^4 + 4\dot{X}^2 - 4\right]\dot{X} + \frac{X^3}{2} = 0 \qquad \dot{X} = Y$$





Outline

- Introduction
    - Limit cycle
    - A mini review of vector calculus
    - Gradient system ($\vec{v} = \nabla\phi$) for irrotational flow ($\nabla\times\vec{v} = 0$)
    - Hamiltonian system for incompressible flow ($\nabla\cdot\vec{v} = 0$)
- Energy methods for stability analysis near a critical point
    - The limit cycle of van dan Pol Equation
- Methods for ruling out closed orbits
    I. Existence of a Laypunov function
    II. Zero curl ($\nabla\times\vec{v} = 0$)
    III. Positive ($\nabla\cdot\vec{v} > 0$) or negative ($\nabla\cdot\vec{v} < 0$) divergence
- Poincare-Bendixson Theorem
- Summary
    - Eigenvalue analysis for gradient and Hamiltonian Systems
    - Poincare-Bendixson Theorem

Methods for Ruling Out Closed Orbits

I. Existence of a Laypunov function

II. Zero curl ($\nabla \times \vec{v} = 0$)

III. Positive ($\nabla \cdot \vec{v} > 0$) or negative ($\nabla \cdot \vec{v} < 0$) divergence

Strogatz

Lyapunov Functions

- The Lyapunov function is constructed as an energy-like function in order to analyze local stability near a critical point.
- Let $E(\vec{X})$ and $\vec{X}_c$ be a Lyapunov function and a critical point.
- The Lyapunov function E: $R^n \to R$ for $\vec{X}_c$ in some neighborhood D of $\vec{X}_c$ has the following properties:

(1) $E(\vec{X}_c) = 0$ and $E(\vec{X}) > 0$ for all $X \neq X_c$ in D, and

(2) $\dot{E}(\vec{X}) \leq 0$ for all X in D,

here, $\dot{E}(\vec{X})$ represents the rate of change of E along a solution trajectory. Therefore, the above properties suggest that the Lyapunov function is positive and decreases along the trajectory. Mathematically, $\dot{E}(\vec{X})$ can be written as follows:

$$\dot{E}(\vec{X}) = \frac{\partial E}{\partial X}\frac{dX}{dt} + \frac{\partial E}{\partial Y}\frac{dY}{dt} + \frac{\partial E}{\partial Z}\frac{dZ}{dt} + \dots \quad \text{when } \vec{X} = (X, Y, Z, ...), or \qquad (E1)$$

$$\boxed{\frac{dE}{dt} = \nabla E \cdot \left(\frac{dX}{dt}, \frac{dY}{dt}, \frac{dZ}{dt}\right) < 0}$$

An obtuse angle between ∇E and the direction of a flow.

No Closed Orbits When a Lyapunov Function Exist

Liapunov Functions

Even for systems that have nothing to do with mechanics, it is occasionally possible to construct an energy-like function that decreases along trajectories. Such a function is called a Liapunov function. If a Liapunov function exists, then closed orbits are forbidden, by the same reasoning as in Example 7.2.2.

To be more precise, consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with a fixed point at $\mathbf{x}^*$. Suppose that we can find a ***Liapunov function***, i.e., a continuously differentiable, real-valued function $V(\mathbf{x})$ with the following properties:

1. $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{x}^*$, and $V(\mathbf{x}^*) = 0$. (We say that V is *positive definite*.)
2. $\dot{V} < 0$ for all $\mathbf{x} \neq \mathbf{x}^*$. (All trajectories flow "downhill" toward $\mathbf{x}^*$.)

Then $\mathbf{x}^*$ is globally asymptotically stable: for all initial conditions, $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$. In particular the system has no closed orbits. (For a proof, see Jordan and Smith 1987.)

The intuition is that all trajectories move monotonically down the graph of $V(\mathbf{x})$ toward $\mathbf{x}^*$ (Figure 7.2.1).

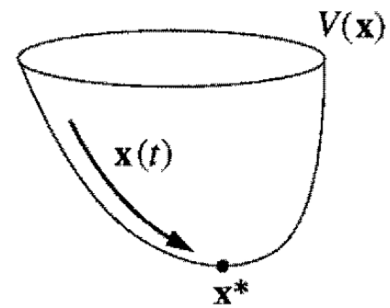


Figure 7.2.1

Large value

small value

- ∇V points in the direction of greatest increase of a function
- $\frac{dV}{\mathrm{dt}} = \nabla V \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$
- $\frac{dV}{\mathrm{dt}} < 0,$ the trajectory moves in the decrease of the function

Strogatz

Strict Lyapunov Functions

Corollary 6. *If L is a strict Liapunov function for a planar system, then there are no limit cycles.* ■

- Liapunov functions not only detect stable equilibria; they can also be used to estimate the size of the basin of attraction of an asymptotically stable equilibrium, as the preceding example shows. (HSD, p199).

HSD

Review: Green's Theorem

Formulas for Grad, Div, Curl, and the Laplacian

	Cartesian (x, y, z) **i**, **j**, and **k** are unit vectors in the directions of increasing x, y, and z. **P, Q, and R** are the scalar components of $\mathbf{F}(x, y, z)$ in these directions.
Gradient	$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$
Divergence	$\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$
Curl	$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$
Laplacian	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

The Fundamental Theorem of Line Integrals

1. Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ be a vector field whose components are continuous throughout an open connected region D in space. Then there exists a differentiable function f such that
$$\mathbf{F} = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$
if and only if for all points A and B in D the value of $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path joining A to B in D.
2. If the integral is independent of the path from A to B, its value is
$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

Green's Theorem (Tangential Form) $$\iint_R \nabla \times \vec{F} \cdot \vec{k}\,dxdy = \oint \vec{F} \cdot d\vec{r}$$

Stokes' Theorem $$\iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

Green's Theorem (Normal Form) $$\iint_R \nabla \cdot \vec{F}\,dxdy = \oint \vec{F} \cdot \vec{n}\,ds$$

Divergence Theorem $$\iiint_E \text{div}\,\mathbf{F}\,dV = \iint \vec{F} \cdot \vec{n}\,dS$$

Terminology

- Limit cycle: an isolated closed path

Consider $X' = F(X)$, here $X' = (x', y') = \vec{v}$.

- Gradient system ($\vec{v} = \nabla\phi$) for irrotational flow ($\nabla\times\vec{v} = 0$)

  ϕ : velocity potential; $\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right)$

- Hamiltonian system for incompressible flow ($\nabla \cdot \vec{v} = 0$)

  ψ: streamfunction; $\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \left(\frac{\partial\psi}{\partial y}, \frac{-\partial\psi}{\partial x}\right)$

Gradient System $\left(\nabla\times\left(\frac{dx}{dt},\frac{dy}{dt}\right)=\mathbf{0}\right)$

- Gradient system ←→ $\left(\frac{dx}{dt},\frac{dy}{dt}\right)=\nabla\phi$
- Gradient system → potential function (ϕ)
- The gradient of the potential function is parallel to the direction of flow.
- (therefore, the flow is across contour lines).
- (Note that the contour line does not represent a streamfuntion)

HSD

No Closed Orbits when Curl is Zero

Gradient Systems

Suppose the system can be written in the form $\dot{\mathbf{x}} = -\nabla V$, for some continuously differentiable, single-valued scalar function $V(\mathbf{x})$. Such a system is called a ***gradient system*** with ***potential function*** V.

Theorem 7.2.1: Closed orbits are impossible in gradient systems.

$$\begin{aligned}\Delta V &= \int_0^T \frac{dV}{dt}\,dt \\ &= \int_0^T (\nabla V \cdot \dot{\mathbf{x}})\,dt \\ &= -\int_0^T \|\dot{\mathbf{x}}\|^2\,dt \\ &< 0\end{aligned}$$

$$\dot{\mathbf{x}} = -\nabla V$$

potential function V.

(unless $\dot{\mathbf{x}} \equiv \mathbf{0}$, in which case the trajectory is a fixed point, not a closed orbit). This contradiction shows that closed orbits can't exist in gradient systems. ■

Strogatz

Gradient System $\left(\nabla\times\left(\frac{dx}{dt},\frac{dy}{dt}\right)=0\right)$

$\nabla\times\vec{V}=0$ ➔ curl vs. circulation

Green's Theorem (Tangential Form) $\oint\vec{V}\cdot d\vec{r}=\iint\nabla\times\vec{V}\cdot\vec{k}\,dxdy$

$X'=\vec{V};\ Since\ \vec{V}=\nabla\phi, we\ have\ X'=\nabla\phi\ and\ \nabla\times\vec{V}=0.$

$$RHS=\iint\nabla\times\vec{V}\ \mathrm{dA}=0.$$

$$LHS=\int\vec{V}\cdot d\vec{r}=\int\nabla\phi\cdot d\vec{r}$$

$$=\int\left(\frac{dx}{dt},\frac{dy}{dt}\right)\cdot\left(\frac{dx}{dt},\frac{dy}{dt}\right)\mathrm{dt}$$

$$=\int\left(\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right)\mathrm{dt}\geq 0$$

Contradiction ➔no closed orbit

- Gradient flow,
- Velocity is parallel to the gradient
- Gradient, pointing in the direction of greatest increase of a function
- (across the contour lines of the potential function)

Gradient System: Eigenvalue Analysis

$$\frac{dx}{dt} = G_x$$
$$\frac{dy}{dt} = G_y$$

$$J(G_x, G_y) = \frac{\partial(G_x, G_y)}{\partial(x, y)} = \begin{bmatrix} G_{xx} & G_{xy} \\ G_{yx} & G_{yy} \end{bmatrix}$$

$$A = \begin{bmatrix} G_{xx} & G_{xy} \\ G_{yx} & G_{yy} \end{bmatrix}$$

Matrix A is symmetric
→ eigenvalues are real
→ no closed orbit (or spiral source/sink)

Blandchart et al.

Gradient System $\left(\nabla\times\left(\frac{dx}{dt},\frac{dy}{dt}\right)=\mathbf{0}\right)$

Theorem. (Properties of Gradient Systems) *For the system* $X' = -\text{grad}\, V(X)$*:*

1. *If c is a regular value of V, then the vector field is perpendicular to the level set* $V^{-1}(c)$.
2. *The critical points of V are the equilibrium points of the system.*
3. *If a critical point is an isolated minimum of V, then this point is an asymptotically stable equilibrium point.* $\square$

Vector field in perpendicular to the tangent line of the contour line

Proposition. *For a gradient system* $X' = -\text{grad}\, V(X)$*, the linearized system at any equilibrium point has only real eigenvalues.* $\square$

The Jacobian matrix is symmetric → real eigenvalues

→ no closed orbits

HSD

No Closed Orbits when Divergence Is of One Sign

Bendixson Negative Criterion

Theorem 6. Let $f(x, y)$ and $g(x, y)$ have continuous first partial derivatives in the simply connected domain†† D and assume that

$$f_x(x, y) + g_y(x, y)$$

is of one sign in D. Then there are no (nonconstant) periodic solutions to

$$\frac{dx}{dt} = f(x, y) \ , \tag{4}$$

$$\frac{dy}{dt} = g(x, y) \tag{5}$$

that lie entirely in D.

$$\nabla \cdot (f, g) > 0 \ or < 0$$

Green's Theorem (Normal Form) $\oint \vec{V} \cdot \vec{n}\, ds = \iint \nabla \cdot \vec{V} dxdy$ Nagle et al.

No Closed Orbits when Divergence Is of One Sign

Theorem 9.7.2 Let the functions F and G have continuous first partial derivatives in a simply connected domain D of the xy-plane. If $F_x + G_y$ has the same sign throughout D, then there is no closed trajectory of the system (15) lying entirely in D.

Green's Theorem (Normal Form) $\oint \vec{V} \cdot \vec{n}\, ds = \iint \nabla \cdot \vec{V}\, dxdy$

$$RHS = \iint \nabla \cdot \vec{V} \ \mathrm{dA} \neq 0.$$

$$LHS = \int \vec{V} \cdot d\vec{n} = \int \left(\frac{dx}{dt}, \frac{dy}{dt}\right) \cdot (dy, -dx) = 0.$$

assume that a flow is on a circle

Contradiction
➔no closed orbit

Boyce_n_DiPrima

2D Hamiltonian System:

$$\left(\nabla \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = 0\right)$$

(zero divergence)

2D Hamiltonian System

By analogy with the form of Hamilton's canonical equations in mechanics, a system

$$\dot{x} = X(x, y), \qquad \dot{y} = Y(x, y) \tag{2.69}$$

is called a **Hamiltonian system** if there exists a function $H(x, y)$ such that

$$X = \frac{\partial H}{\partial y} \quad \text{and} \quad Y = \frac{\partial H}{\partial x}. \tag{2.70}$$

Then H is called the **Hamiltonian function** for the system. A necessary and sufficient condition for (2.69) to be Hamiltonian is that

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0. \tag{2.71}$$

$$\left(\nabla \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = 0\right)$$

Hamiltonian System: Conservative $\left(\nabla \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = 0\right)$

Proposition. *For a Hamiltonian system in $\mathbb{R}^2$, H is constant along every solution curve.* □

Proposition. *Suppose (x_0, y_0) is an equilibrium point for a planar Hamiltonian system. Then the eigenvalues of the linearized system are either $\pm\lambda$ or $\pm i\lambda$ where $\lambda \in \mathbb{R}$.* □

$$x' = \frac{\partial H}{\partial y}(x, y)$$

$$y' = -\frac{\partial H}{\partial x}(x, y)$$

$$\dot{H} = \frac{\partial H}{\partial x} x' + \frac{\partial H}{\partial y} y' = \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} + \frac{\partial H}{\partial y}\left(-\frac{\partial H}{\partial x}\right) = 0.$$

$$J = \begin{bmatrix} H_{yx} & H_{yy} \\ -H_{xx} & -H_{xy} \end{bmatrix}$$

- The direction of the flow (dx/dt, dy/dt) and the gradient of H are orthogonal.
- The tangent vector of the contour line of H represents the direction of the flow.
- H is called the Hamiltonian function, and is a streamfunction.
- $a_{11} + a_{22} = 0 \Rightarrow \lambda_1 + \lambda_2 = 0 \Rightarrow$ *saddle point (witn no closed orbit) or center*

Hamiltonian System: An Example

A Hamiltonian function is

$$H(x,y) = \frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2} + \frac{1}{4}.$$

The constant value $1/4$ is irrelevant here; we choose it so that H has minimum value 0, which occurs at $(\pm 1, 0)$, as is easily checked. The only other equilibrium point lies at the origin. The linearized system is

One Saddle
Two Centers

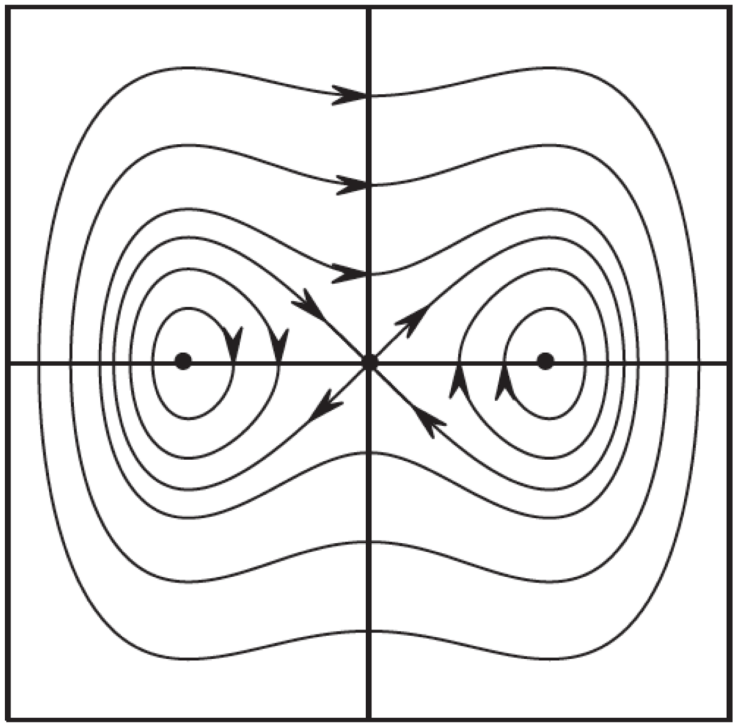
$$X' = \begin{pmatrix} 0 & 1 \\ 1 - 3x^2 & 0 \end{pmatrix} X.$$

At $(0,0)$, this system has eigenvalues ± 1, so we have a saddle. At $(\pm 1, 0)$, the eigenvalues are $\pm\sqrt{2}i$, so we have a center, at least for the linearized system.

Plotting the level curves of H and adding the directions at nonequilibrium points yields the phase portrait shown in Figure 9.11. Note that the equilibrium points at $(\pm 1, 0)$ remain centers for the nonlinear system. Also note that the stable and unstable curves at the origin match up exactly. That is, we have solutions that tend to $(0,0)$ in both forward and backward time. Such solutions are known as *homoclinic solutions* or *homoclinic orbits*. ■

HSD

Hamiltonian System: An Example (cont.)



Center --- Saddle --- Center

Figure 9.11 Phase portrait for
$x' = y,\ y' = -x^3 + x$.

- H is called the Hamiltonian function, and is a streamfunction.
- The tangent vector of the contour line of H represents the direction of the flow.

HSD

MT Part A

2: [25 points] A nonlinear, non-dissipative Lorenz model is written as follows:

$$\frac{d^2X}{dt^2} - (\sigma r + C)\,X + \frac{X^3}{2} = 0. \tag{2}$$

Here, we assume that both σ and r are positive, and choose $C = 0$ for convenience. Complete the following problems.

(a) [3 points] Transform the 2nd order ODE in Eq. (2) into a system of the first order ODEs, (i.e., $Y = X'$).

(b) [3 points] Find critical points in the above 2D system in problem (2a).

(c) [6 points] Compute the Jacobian matrix of the above 2D system.

(d) [13 points] Perform a linear stability analysis for all of the critical points.

MEDIAN	MAXIMUM	MEAN
76.5	100.0	71.88

Outline

- Introduction
  - Limit cycle
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  - Gradient system ($\vec{v} = \nabla\phi$) for irrotational flow ($\nabla\times\vec{v} = 0$)
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  - I. Existence of a Laypunov function
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  - Eigenvalue analysis for gradient and Hamiltonian Systems
  - Poincare-Bendixson Theorem

Poincare-Bendixson Theorem

Poincaré–Bendixson Theorem

Theorem 7. Let f and g have continuous first partial derivatives on the closed bounded region R and assume the system

$$\frac{dx}{dt} = f(x, y) \ , \quad \frac{dy}{dt} = g(x, y)$$

has no critical points in R. If a solution $x = \phi(t)$, $y = \psi(t)$ to the system exists for all $t \geq t_0$ and its trajectory $\Gamma(t) := \big(\phi(t), \psi(t)\big)$ remains inside R for $t \geq t_0$, then either Γ is a limit cycle or it spirals toward a limit cycle in R. In either case, the system has a nonconstant periodic solution.

- R has a hole in it (Nagle et al., p765)
- R cannot be simply connected; it must have a hole (Boycy-DiPrima, p569)
- [In the example by Strogatz (p204), a ring-shaped region is used.]

Nagle et al.

Poincare-Bendixson Theorem

Theorem. (Poincaré–Bendixson) *Suppose that Ω is a nonempty, closed, and bounded limit set of a planar system of differential equations that contains no equilibrium point. Then Ω is a closed orbit.*

- If γ is an ω-limit cycle, there exists $X \notin \gamma$ such that

$$\lim_{t \to \infty} d(\phi_t(X), \gamma) = 0.$$

- Geometrically this means that some solution spirals toward γ as $t \to \infty$.

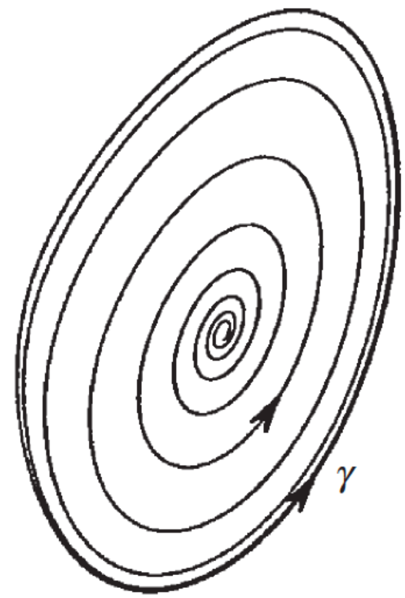


Figure 10.10 A solution spiraling toward a limit cycle.

The same as that in Nagle et al.

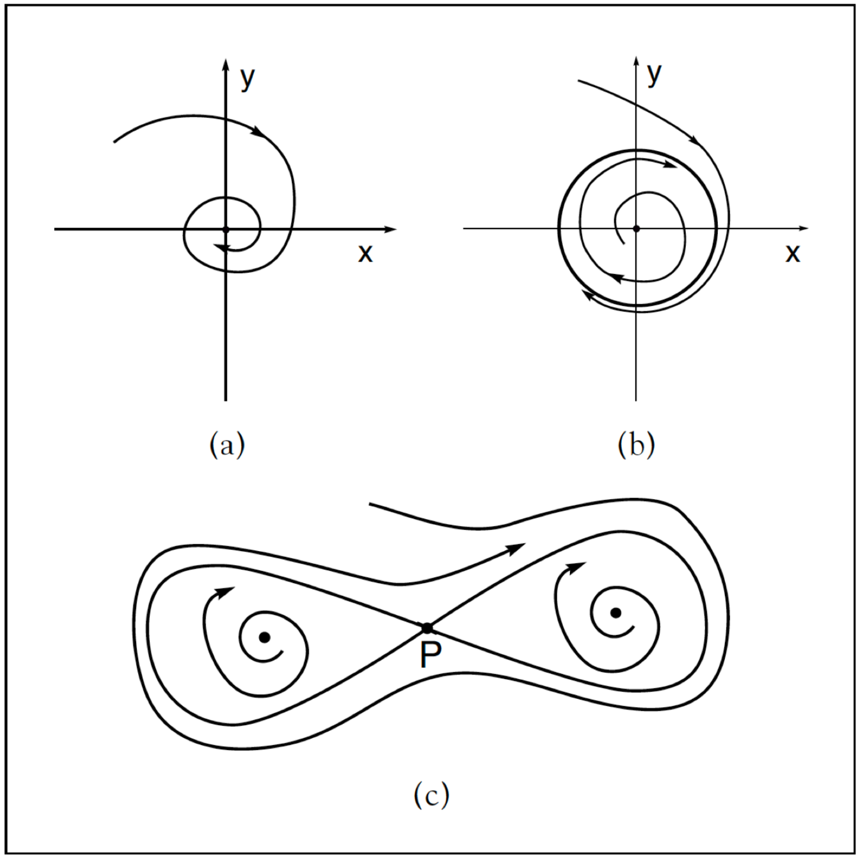
Three Cases of Poincare-Bendixson Theorem

Alligood et al.

Steady state
(a point)

Limit Cycle

2D!



Limit Cycle, enclosing three critical points (BWS)

Figure 8.4 Planar limit sets.
The three pictures illustrate the three cases of the Poincaré-Bendixson Theorem. (a) The limit set is one point, the origin. (b) The limit set of each spiraling trajectory is a circle, which is a periodic orbit. (c) The limit set of the outermost trajectory is a figure eight. This limit set must have an equilibrium point P at the vertex of the "eight". It consists of two connecting arcs plus the equilibrium. Trajectories on the connecting arcs tend to P as $t \to \infty$ and as $t \to -\infty$.

2D Chaotic System: Dixon System

The systems in this chapter were inspired by the work of Dixon *et al.* (1993) in which they transformed a set of three ODEs introduced by Cummings *et al.* (1992) to model the dynamical behavior of the magnetic field of a neutron star. The transformation reduced the system to a two-dimensional flow that nevertheless preserves the chaotic behavior in apparent violation of the Poincaré–Bendixson theorem (Hirsch *et al.*, 2004), which states that the attractor for any smooth two-dimensional bounded continuous-time autonomous system is either a stable equilibrium or a limit cycle.

The system derived by Dixon *et al.* (1993) is given by

$$\begin{aligned} \dot{x} &= \frac{xy}{x^2+y^2} - \alpha x \\ \dot{y} &= \frac{y^2}{x^2+y^2} - \beta y + \beta - 1 \end{aligned} \tag{5.1}$$

and is singular at the origin ($x = y = 0$) and thus does not satisfy the smoothness condition required for the Poincaré–Bendixson theorem to apply. All orbits are attracted to the singularity in finite time, and as a result

Sprott, 2010, Elegant Chaos

2D Chaotic System: Dixon System

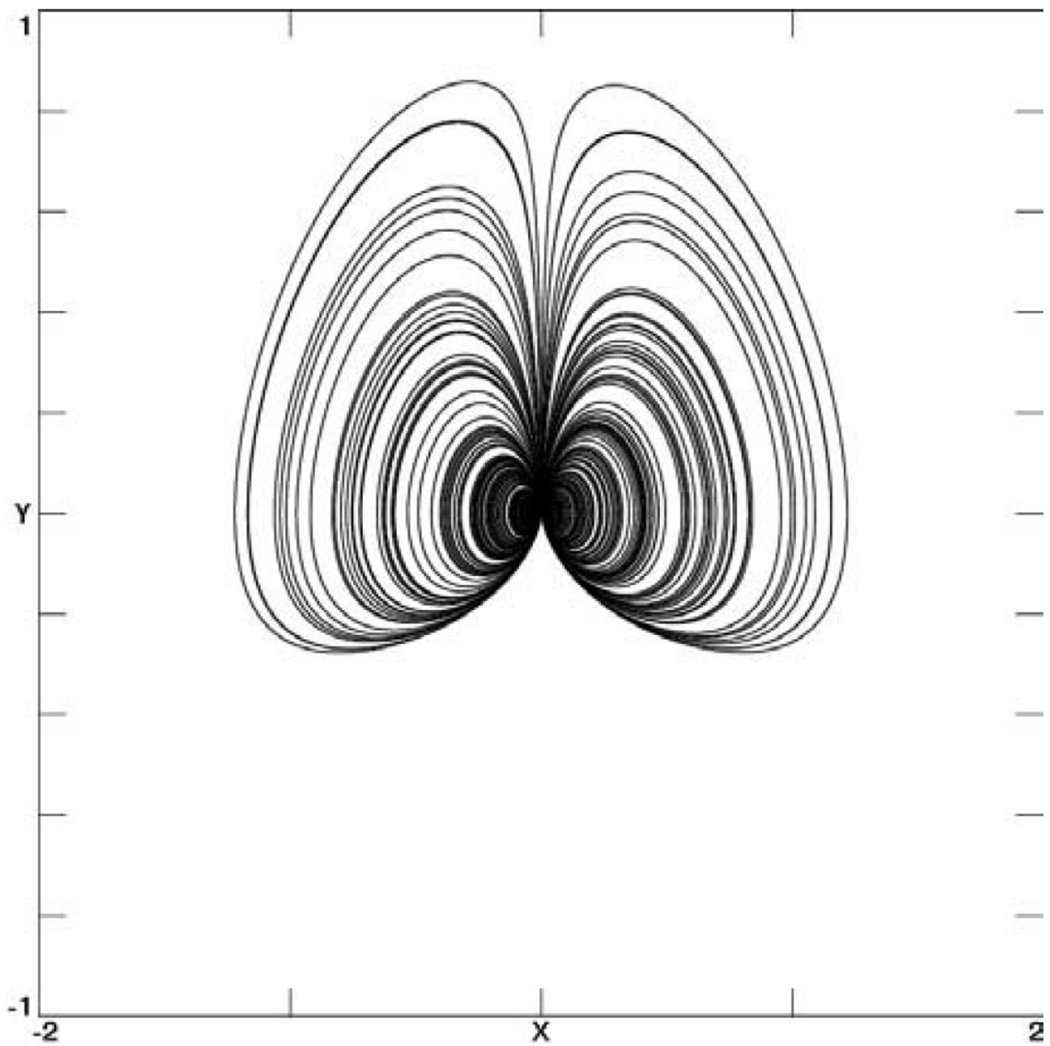


Fig. 5.1 State space plot for the Dixon system in Eq. (5.1) with $(\alpha, \beta) = (0, 0.7)$ for $(x_0, y_0) = (1, 0)$.

2D Chaotic System: Dixon System

they are sensitive to even the smallest nonzero perturbation, including one arising from the numerical method. The system displays what Dixon *et al.* (1993) call 'S-chaos' (singularity-chaos) for a range of parameters including $\alpha = 0$ and $\beta = 0.7$. The state space plot for this case is shown in Fig. 5.1. A longer calculation of the trajectory would show that it densely fills a region of the xy-plane, and hence it is two-dimensional. Since the resulting plot does not have fractal structure, it is not a proper strange attractor, and the dynamics perhaps should not be considered truly chaotic as pointed out by Alvarez-Ramirez, *et al.* (2005). For the same reason, Lyapunov exponents are not quoted in this chapter because they are not well-defined, and their calculation in the vicinity of the singularity is problematic.

Summary: Eigenvalue Analysis (2D Systems)

- curl (gradient ϕ)=0

$$Gradient\ System$$

$$irrotational\ flow$$

$$\phi: velocity\ potential$$

$$\vec{v} = \nabla\phi = (\phi_x, \phi_y)$$

$$J(\phi_x, \phi_y) = \begin{bmatrix} \phi_{xx} & \phi_{xy} \\ \phi_{yx} & \phi_{yy} \end{bmatrix}$$

- J is a symmetric matrix.
- Its eigenvalues are real.
- There are no center or spiral source/sink

- div (curl F)=0

$$Hamiltonian\ System$$

$$incompressible\ flow$$

$$\psi: streamfunction$$

$$F_o = (0,0,\psi)$$

$$\vec{v} = \nabla \times F_o = (\psi_y, -\psi_x)$$

$$J(\psi_y, -\psi_x) = \begin{bmatrix} \psi_{yx} & \psi_{yy} \\ -\psi_{xx} & -\psi_{xy} \end{bmatrix}$$

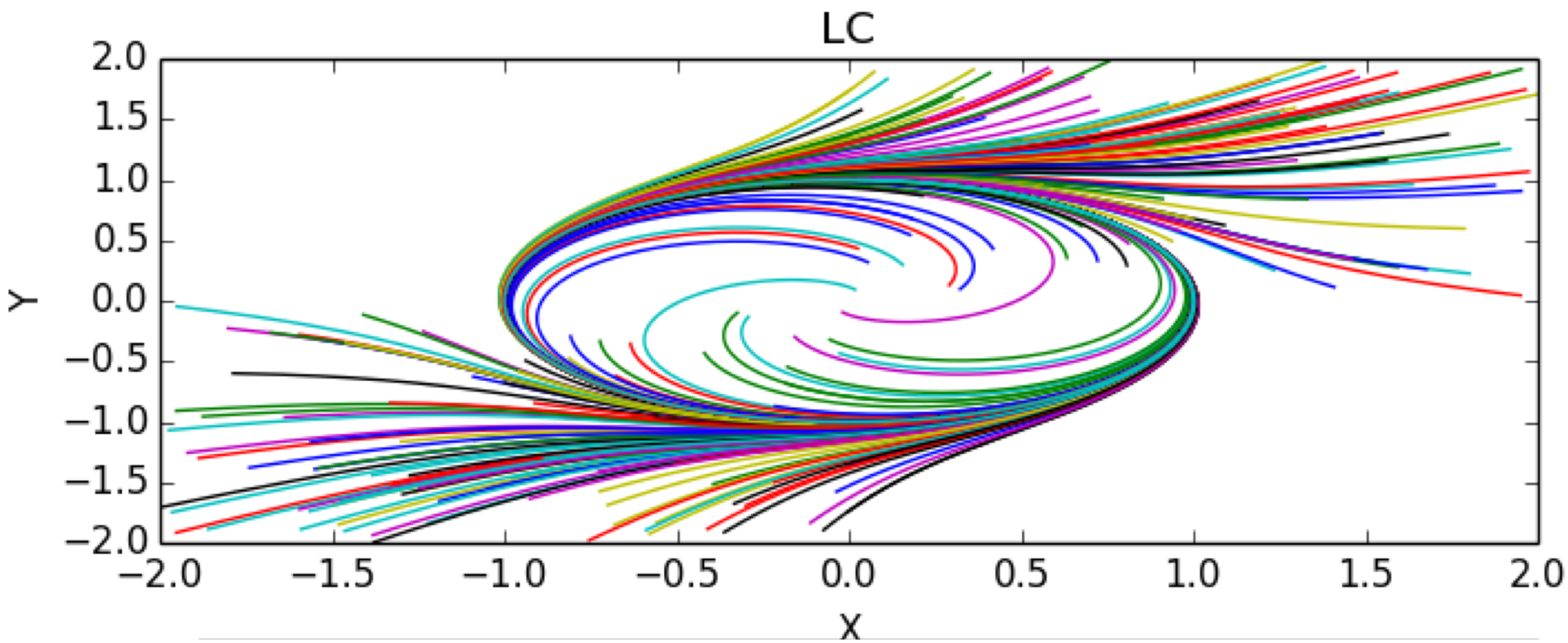
- $Trace = \lambda_1 + \lambda_2 = 0.$
- $Det = \lambda_1 \lambda_2$
- $\lambda_{1,2} = \pm\, \alpha$ (saddle) when Det<0.
- $\lambda_{1,2} = \pm\, i\beta$ (center) when Det>0.

Limit Cycle

- An isolated closed path is called a limit cycle

Limit Cycle

Definition 5. A nontrivial† closed trajectory with at least one other trajectory spiraling into it (as time approaches plus or minus infinity) is called a **limit cycle.**



Summary

- Limit cycle: An isolated closed path is called a limit cycle
- Gradient system ($\vec{v} = \nabla\phi$) for irrotational flow ($\nabla\times\vec{v} = 0$)
- Hamiltonian system for incompressible flow ($\nabla \cdot \vec{v} = 0$)
- No closed orbits when the system meets one of the following conditions:
    I. existence of a Laypunov function;
    II. zero curl ($\nabla\times\vec{v} = 0$)
    III. positive ($\nabla \cdot \vec{v} > 0$) or negative ($\nabla \cdot \vec{v} < 0$) divergence
- Poincare-Bendixson Theorem (2D, bounded, autonomous)
    - There are only two possibilities for that trajectory
    - The trajectory approaches a fixed point of the system as $t \to \infty$.
    - The trajectory approaches a limit cycle as $t \to \infty$.
    - Chaotic trajectories (in a bounded system) cannot occur in a state space of two dimensions (within differential equations).