**Homework 4**
**Discrete Dynamical Systems and Chaos**
**Math 538**
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**Problem 2.3:** Let $g(x,y) = (x^2 - 5x + y, x^2)$. Find and classify the fixed points of $g$ as sinks, sources, or saddles.

To find the fixed points, we set $g(x,y) = (x,y)$ such that:

$$\begin{aligned} x^2 - 5x + y &= x \\ x^2 &= y \\ x^2 - 6x + x^2 &= 0 \\ 2x^2 - 6x &= 0 \\ 2x(x-3) &= 0 \end{aligned}$$

Thus we get the fixed points:

$$(x_1, y_1) = (0,0) \qquad (x_2, y_2) = (3,9)$$

To find the classification, we find the eigenvalues of the evaluated Jacobian:

$$Dg(x,y) = \begin{pmatrix} 2x-5 & 1 \\ 2x & 0 \end{pmatrix}$$

(a) $(x_1, y_1) = (0,0)$ - Saddle

$$|Dg(0,0) - \lambda I| = \left| \begin{pmatrix} -5-\lambda & 1 \\ 0 & 0-\lambda \end{pmatrix} \right| = (-5-\lambda)(-\lambda) = 0 \qquad \lambda = 0, -5$$

(b) $(x_2, y_2) = (3,9)$ - Source

$$|Dg(3,9) - \lambda I| = \left| \begin{pmatrix} 1-\lambda & 1 \\ 6 & 0-\lambda \end{pmatrix} \right| = (1-\lambda)(-\lambda) - 6 = \lambda^2 - \lambda - 6 = 0 \qquad \lambda = 3, -2$$

**Problem 2.5:** Let $f(x,y,z) = (x^2y, y^2, xz+y)$ be a map on $\mathbb{R}^3$. Find and classify the fixed points of $f$.

To find the fixed points, we set $f(x,y,z) = (x,y,z)$ such that:

$$\begin{aligned} x^2y &= x & \qquad\qquad x^2y - x &= 0 \\ y^2 &= y & y^2 - y &= 0 \\ xz + y &= z & (x-1)z + y &= 0 \end{aligned}$$

Looking at the following equation, we get the following values:

$$y^2 - y = 0 \quad \rightarrow \quad y = 0 \text{ or } y = 1$$

Notice the cases:

(a) y = 0

$$x^2y - x = -x = 0 \quad \rightarrow \quad x = 0 \qquad (x-1)z + y = -z = 0 \quad \rightarrow \quad z = 0$$

such that a fixed point would be:

$$(x_1, y_1, z_1) = (0,0,0)$$

(b) y = 1

$$x^2y - x = x^2 - x = 0 \quad \rightarrow \quad x = 0 \text{ or } x = 1$$

(i) x = 0

$$(x-1)z + y = -z + 1 = 0 \quad \rightarrow \quad z = 1$$

such that another fixed point would be:

$$(x_2, y_2, z_2) = (0,1,1)$$

(ii) x = 1

$$(x-1)z + y = 0 \quad \rightarrow \quad 1 = 0$$

From this, we do not get any fixed point when $x = 1, y = 1$

To find the classification, we find the eigenvalues of the evaluated Jacobian:

$$Df(x,y,z) = \begin{pmatrix} 2xy & x^2 & 0 \\ 0 & 2y & 0 \\ z & 1 & x \end{pmatrix}$$

(a) $(x_1, y_1) = (0,0,0)$ - Sink

$$|Df(0,0,0) - \lambda I| = \left| \begin{pmatrix} 0-\lambda & 0 & 0 \\ 0 & 0-\lambda & 0 \\ 0 & 1 & 0-\lambda \end{pmatrix} \right| = (-\lambda)^3 = 0 \qquad \lambda = 0$$

(b) $(x_2, y_2) = (0,1,1)$ - Saddle

$$|Df(0,1,1) - \lambda I| = \left| \begin{pmatrix} 0-\lambda & 0 & 0 \\ 0 & 2-\lambda & 0 \\ 1 & 1 & 0-\lambda \end{pmatrix} \right| = (-\lambda)(2-\lambda)(-\lambda) = 0 \qquad \lambda = 0, 2$$

**Problem 2.6:** Let $f(x,y) = (\sin\frac{\pi}{3}x, \frac{y}{2})$. Find all fixed points and their stability. Where does the orbit of each initial value go?

To find the fixed points, we set $f(x,y) = (x,y)$ such that:

$$\sin\frac{\pi x}{3} = x$$
$$\frac{y}{2} = y$$
$$y = 0$$

By observation, we can see the fixed point:

$$(x_1, y_1) = \left(\frac{1}{2}, 0\right) \qquad (x_2, y_2) = \left(-\frac{1}{2}, 0\right) \qquad (x_3, y_3) = (0, 0)$$

To find the classification, we find the eigenvalues of the evaluated Jacobian:

$$Df(x,y) = \begin{pmatrix} \frac{\pi}{3}\cos\frac{\pi}{3}x & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

(a) $(x_1, y_1) = \left(\frac{1}{2}, 0\right)$ - Sink

$$\left|Df\left(\frac{1}{2}, 0\right) - \lambda I\right| = \left|\begin{pmatrix} \frac{\sqrt{3}\pi}{6} - \lambda & 0 \\ 0 & \frac{1}{2} - \lambda \end{pmatrix}\right| = \left(\frac{\sqrt{3}\pi}{6} - \lambda\right)\left(\frac{1}{2} - \lambda\right) = 0 \qquad \lambda = \frac{\sqrt{3}\pi}{6}, \frac{1}{2}$$

(b) $(x_2, y_2) = \left(-\frac{1}{2}, 0\right)$ - Sink

$$\left|Df\left(-\frac{1}{2}, 0\right) - \lambda I\right| = \left|\begin{pmatrix} \frac{\sqrt{3}\pi}{6} - \lambda & 0 \\ 0 & \frac{1}{2} - \lambda \end{pmatrix}\right| = \left(\frac{\sqrt{3}\pi}{6} - \lambda\right)\left(\frac{1}{2} - \lambda\right) = 0 \qquad \lambda = \frac{\sqrt{3}\pi}{6}, \frac{1}{2}$$

(c) $(x_3, y_3) = (0, 0)$ - Saddle

$$\left|Df(0,0) - \lambda I\right| = \left|\begin{pmatrix} \frac{\pi}{3} - \lambda & 0 \\ 0 & \frac{1}{2} - \lambda \end{pmatrix}\right| = \left(\frac{\pi}{3} - \lambda\right)\left(\frac{1}{2} - \lambda\right) = 0 \qquad \lambda = \frac{\pi}{3}, \frac{1}{2}$$

**Problem T2.2:** Show that the map in (2.14) has exactly two fixed points, $(0, 0)$ and $(-0.6, -0.6)$.

$$f(x, y) = (-x^2 + 0.4y, x) \tag{2.14}$$

Notice to find the fixed points, we do the following:

$$\begin{aligned} -x^2 + 0.4y &= x \\ x &= y \\ -x^2 + 0.4x &= x \\ x^2 + 0.6x &= 0 \\ x(x + 0.6) &= 0 \end{aligned}$$

Such that our fixed points are:

$$x = y = 0, -0.6 \qquad \rightarrow \qquad (x_1, y_1) = (0, 0) \quad (x_2, y_2) = (-0.6, -0.6)$$

**Problem T2.3:**

(a) Verify equation (2.20).

$$A^n = a^{n-1} \begin{pmatrix} a & n \\ 0 & a \end{pmatrix} \tag{2.20}$$

Notice that for $n = 2$, we have:

$$A^2 = AA = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a^2 & 2a \\ 0 & a^2 \end{pmatrix}$$

Notice that for any $k$, we have:

$$A^k = \begin{pmatrix} a^k & ka^{k-1} \\ 0 & a^k \end{pmatrix}$$

Now notice that this can be applied for any $k+1$:

$$AA^k = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \begin{pmatrix} a^k & ka^{k-1} \\ 0 & a^k \end{pmatrix} = \begin{pmatrix} a^{k+1} & (k+1)a^k \\ 0 & a^{k+1} \end{pmatrix} = a^k \begin{pmatrix} a & k+1 \\ 0 & a \end{pmatrix}$$

(b) Use equation (2.21) to show that the fixed point $(0, 0)$ is a sink if $|a| < 1$ and a source if $|a| > 1$.

$$A^n \begin{pmatrix} x \\ y \end{pmatrix} = a^{n-1} \begin{pmatrix} ax + ny \\ ay \end{pmatrix} \tag{2.21}$$

Notice, we can factor out another $a$, such that:

$$A^n \begin{pmatrix} x \\ y \end{pmatrix} = a^n \begin{pmatrix} x + \frac{n}{a}y \\ y \end{pmatrix}$$

Now if we notice that if $|a| < 1$, we get that the entire equation goes to 0 (Sink), as this equation is most heavily influenced by the $a^n$ term as it is exponential whereas the other values are linear.

Now if we notice that if $|a| > 1$, we get that the entire equation goes to $\infty$ (Source), as this equation is most heavily influenced by the $a^n$ term as it is exponential whereas the other values are linear.

**Problem T2.4:** Verify that multiplication by $A$ rotates a vector by $\arctan(b/a)$ and stretches by a factor of $\sqrt{a^2+b^2}$

Let the following be true:

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \qquad r = \sqrt{a^2+b^2}$$

We can then factor out $r$ such that:

$$A = r\begin{pmatrix} a/r & -b/r \\ b/r & a/r \end{pmatrix} = \sqrt{a^2+b^2}\begin{pmatrix} a/\sqrt{a^2+b^2} & -b/\sqrt{a^2+b^2} \\ b/\sqrt{a^2+b^2} & a/\sqrt{a^2+b^2} \end{pmatrix} = \sqrt{a^2+b^2}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Now we can notice that $A$ is being stretched by $r$ and we can also see the rotation of $\theta$

$$r = \sqrt{a^2+b^2} \qquad \theta = \arctan(b/a)$$

**Problem T2.5:** Prove that the Henon map has a period-two orbit if and only if $4a > 3(1-b)^2$.

Notice the solutions for the fixed points of the Henon Map:

$$\begin{aligned} a - x^2 + by &= x \\ x &= y \\ a - x^2 + bx &= x \\ a - x^2 + (b-1)x &= 0 \\ x^2 + (1-b)x - a &= 0 \end{aligned}$$

Thus, we get the following solutions for the fixed points:

$$x_{1-1/2} = y_{1-1/2} = \frac{-(1-b) \pm \sqrt{(1-b)^2 + 4a}}{2}$$

Now notice the solutions for the period 2 orbits of the Henon Map:

$$\begin{aligned} a - (a - x^2 + by)^2 + bx &= x \\ a - x^2 + by &= y \end{aligned}$$

Simplifying this, we get:

$$\left(x^2 - (1-b)x - a + (1-b)^2\right)\left(x^2 + (1-b)x - a\right) = 0$$

We can see that the right factor is the previous fixed point equations, so we can find the period-two orbits from the left factor such that:

$$x_{2-1/2} = \frac{(1-b) \pm \sqrt{(1-b)^2 - 4(-a + (1-b)^2)}}{2} = \frac{(1-b) \pm \sqrt{4a - 3(1-b)^2}}{2}$$

We can see that we only get real valued period 2 orbits when:

$$4a - 3(1-b)^2 > 0 \qquad \rightarrow \qquad 4a > 3(1-b)^2$$

**Problem T2.7:** Set $b = 0.4$.

(a) Prove that for $0.09 < a < 0.27$, the Henon map $f$ has one sink fixed point and one saddle fixed point.

To find the fixed points, we set $f(x, y) = (x, y)$ such that:

$$\begin{aligned} a - x^2 + 0.4y &= x \\ x &= y \\ a - x^2 + 0.4x &= x \\ a - x^2 - 0.6x &= 0 \\ x^2 + 0.6x - a &= 0 \end{aligned}$$

To get real valued fixed points, we get:

$$x = \frac{-0.6 \pm \sqrt{0.36 + 4a}}{2} = -0.3 \pm \sqrt{0.09 + a} \qquad 0.36 + 4a > 0 \quad \rightarrow \quad a > -0.9$$

(b) Find the largest magnitude eigenvalue of the Jacobian matrix at the first fixed point when $a = 0.27$. Explain the loss of stability of the sink.

Notice the Jacobian of the Henon:

$$Df(x, y) = \begin{pmatrix} -2x & 0.4 \\ 1 & 0 \end{pmatrix}$$

Now when $a = 0.27$, we get the fixed points:

$$x = y = -0.3 \pm \sqrt{0.09 + 0.27} = -0.9, 0.3$$

Now we can evaluate the Jacobian at $(x_1, y_1) = (-0.9, -0.9)$:

$$|Df(-0.9, -0.9) - \lambda I| = \left| \begin{pmatrix} 1.8 - \lambda & 0.4 \\ 1 & 0 - \lambda \end{pmatrix} \right| = (1.8 - \lambda)(-\lambda) - 0.4 = \lambda^2 - 1.8\lambda - 0.4 = 0$$

Solving for $\lambda$ gives us:

$$\lambda = \frac{1.8 \pm \sqrt{3.24 - 4(-0.4)}}{2} = 0.9 \pm 1.1$$

The largest magnitude eigenvalue gets us:

$$\lambda = 1$$

At this value, the sink is at the edge case between a sink ($\lambda < 1$) and a source ($\lambda > 1$)

(c) Prove that for $0.27 < a < 0.85$, $f$ has a period-two sink.

Notice the period-two orbits:

$$x_{2-1/2} = \frac{0.6 \pm \sqrt{4a - 3(0.6)^2}}{2} = \frac{0.6 \pm \sqrt{4a - 1.08}}{2}$$

To get real valued period-two orbits, the following must be true:

$$4a - 1.08 > 0 \qquad a > 0.27$$

(d) Find the largest magnitude eigenvalue of $Df^2$, the Jacobian of $f^2$ at the period-two orbit, when $a = 0.85$.

Notice the Jacobian of the Henon Squared with $a = 0.85, b = 0.4$:

$$Df^2(x, y) = \begin{pmatrix} 4x(0.85 - x^2 + 0.4y) + 0.4 & 0.8(0.85 - x^2 + 0.4y) \\ -2x & 0.4 \end{pmatrix}$$

Using the result from the previous problem, we can see the period-2 orbit with $a = 0.85, b = 0.4$:

$$x_{2-1/2} = \frac{0.6 \pm \sqrt{4(0.85) - 3(0.6)^2}}{2} = -0.461577310586, 1.06157731059$$

$$y_{2-1/2} = \frac{0.85 - x_{2-1/2}^2}{0.4} = -0.138473193176, 0.318473193176$$

Now we can evaluate the Jacobian at $(x_1, y_1) = (-0.461577310586, -0.138473193176)$:

$$|Df^2(x_1, y_1) - \lambda I| = \begin{pmatrix} -1.0112 - \lambda & -0.265868530898 \\ -2.12315462117 & 0.4 - \lambda \end{pmatrix} \qquad \lambda = -1.33630430, 0.72510430$$