HW 5, Math 330

Due Tuesday, November 19

1. Suppose $\{x_n\}, \{y_n\} \subseteq \mathbb{R}$ and suppose $\lim_{n\to\infty} (x_n - y_n) = 0$. Prove that if $\lim_{n\to\infty} x_n = x_0$, then $\lim_{n\to\infty} y_n = x_0$.
   Hint: for all n we have $|y_n - x_0| = |y_n - x_n + x_n - x_0|$.

2. Suppose $S \subseteq \mathbb{R}$ is bounded above.
   (a) Prove $\exists \{x_n\} \in S$ such that $\lim_{n\to\infty} x_n = \sup S$.
   (b) Prove that $\sup S$ is a maximum of S or is a limit point of S.
   (c) Give examples of sets S for each of the following cases:
      i. $\sup S$ is a maximum of S and a limit point of S
      ii. $\sup S$ is a limit point of S and not a maximum of S
      iii. $\sup S$ is a maximum of S and not a limit point of S

3. The following statement is false. Negate it and prove the negation.
   $\forall \{a_n\}, \{b_n\} \subseteq \mathbb{R}$, if $\{a_n b_n\}$ converges, then $\{a_n\}$ converges and $\{b_n\}$ converges.

4. Use the $N - \epsilon$ definition of sequence convergence to prove: $\lim_{n\to\infty} \dfrac{4n^2 + n}{n^2 + 3n} = 4$

5. Use $\epsilon - \delta$ definition arguments to prove:
   (a) $\lim_{x\to 2} x^2 - 4x = -4$
   (b) $\lim_{x\to 3} \dfrac{2}{x} = \dfrac{2}{3}$.

6. Here is a definition for "converges to $+\infty$." Suppose that $f : D \to \mathbb{R}$ and x_0 is a limit point of D. We write $\lim_{x\to x_0} f(x) = +\infty$ iff $\forall M \in \mathbb{N}^+, \exists \delta > 0$ such that $\forall x \in D$, if $0 < |x - x_0| < \delta$, then $f(x) > M$.
   Suppose $f : (-\infty, 2) \cup (2, \infty) \to \mathbb{R}$ by $f(x) = \dfrac{3}{(x-2)^2}$. Prove that $\lim_{x\to 2} f(x) = +\infty$.

7. Prove that $f : [1, \infty) \to \mathbb{R}$ by $f(x) = \sqrt{x}$ is uniformly continuous.