Math 337 - Elementary Differential Equations

Lecture Notes – Systems of Two First Order Equations: Part B

Joseph M. Mahaffy,
⟨jmahaffy@sdsu.edu⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://jmahaffy.sdsu.edu

Fall 2019

Outline

Introduction

Introduction

- This is the second part of notes for **Systems of Two 1^{st} Order Differential Equations**
- Part A has the topics below
  - A motivating example of a Greenhouse/Rockbed system of passive heating
  - Solutions for the example above - illustrating key techniques
  - Graphs for **direction fields** and **phase portraits**
  - **MatLab** and **Maple** introduced for these problems
- Part B has the following topics
  - Definitions and theorems for **Systems of Two 1^{st} Order Differential Equations**
  - Superposition and linear independence
  - Solving with **eigenvalue techniques**
  - Analysis of different cases with their phase portraits

General Linear System - 2D

General System of Two 1^{st} Order Linear DEs

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} p_{11}(t)x_1 + p_{12}(t)x_2 + g_1(t) \\ p_{21}(t)x_1 + p_{22}(t)x_2 + g_2(t) \end{pmatrix}, \tag{1}$$

which can be written

$$\dot{\mathbf{x}} = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t),$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix}, \quad \text{and} \quad \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

System (1) is a 1^{st} **order linear system of DEs of dimension 2**

If $\mathbf{g}(t) = \mathbf{0}$, then System (1) is **homogeneous**; otherwise it is **nonhomogeneous**

Two 1^{st} Order Linear DEs

Existence and Uniqueness for Two 1^{st} Order Linear DEs

Theorem (Existence and Uniqueness)

Let each of the functions $p_{11},\dots,p_{22}$, g_1, *and* g_2 *be continuous on an open interval* $I = \{t | t \in (\alpha, \beta)\}$, *let* t_0 *be any point in* I, *and let* x_{10} *and* x_{20} *be any given numbers. Then there exists a unique solution to the system (1):*

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} p_{11}(t)x_1 + p_{12}(t)x_2 + g_1(t) \\ p_{21}(t)x_1 + p_{22}(t)x_2 + g_2(t) \end{pmatrix},$$

that also satisfies the initial conditions

$$x_1(t_0) = x_{10}, \qquad x_2(t_0) = x_{20}.$$

Further the solution exists throughout the interval I.

Linear Autonomous System

Linear Autonomous System: If the coefficient matrix $\mathbf{P}$ and vector function $\mathbf{g}$ are independent of time, *i.e.*, **constants**, then we have the **linear autonomous system**

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b},$$

with constant matrix $\mathbf{A}$ and constant vector $\mathbf{b}$.

The **equilibrium solutions** or **critical points** are found by solving:

$$\mathbf{A}\mathbf{x}_e = -\mathbf{b} \qquad \text{or} \qquad \mathbf{x}_e = -\mathbf{A}^{-1}\mathbf{b}.$$

The change of variables $\mathbf{y} = \mathbf{x} - \mathbf{x}_e$ allows us to concentrate on the **homogeneous linear system with constant coefficients**

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$$

Superposition Principle

Theorem (Superposition Principle)

Suppose that $\mathbf{x}_1(t)$ *and* $\mathbf{x}_2(t)$ *are solutions of the equation*

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t).$$

Then the expression

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t),$$

where c_1 *and* c_2 *are arbitrary constants, is also a solution.*

We use the linearity of differentiation and matrices to show this

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \frac{d}{dt}\left(c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)\right) = c_1\dot{\mathbf{x}}_1(t) + c_2\dot{\mathbf{x}}_1(t) \\ &= c_1\mathbf{A}\mathbf{x}_1(t) + c_2\mathbf{A}\mathbf{x}_2(t) = \mathbf{A}\left(c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)\right) = \mathbf{A}\mathbf{x}(t) \end{aligned}$$

Wronskian and Linear Independence

Definition (Wronskian)

Suppose that $\mathbf{x}_1(t) = [x_{11}(t), x_{21}(t)]^T$ and $\mathbf{x}_2(t) = [x_{12}(t), x_{22}(t)]^T$. The **Wronskian** of the solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ is given by the determinant

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} x_{11}(t) & x_{12}(t) \\ x_{21}(t) & x_{22}(t) \end{vmatrix}$$

Definition (Linear Independence of Solutions)

Suppose that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are solutions of $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ on some interval I. We say that $\mathbf{x}_1$ and $\mathbf{x}_2$ are **linearly dependent** if there exists a constant k such that

$$\mathbf{x}_1(t) = k\mathbf{x}_2(t), \qquad \text{for all } t \text{ in } I.$$

Otherwise, $\mathbf{x}_1$ and $\mathbf{x}_2$ are **linearly independent**.

Wronskian and Linear Independence

Theorem (Wronskian and Linear Independence)

Suppose that

$$\mathbf{x}_1(t) = \begin{pmatrix} x_{11}(t) \\ x_{21}(t) \end{pmatrix} \qquad \text{and} \qquad \mathbf{x}_2(t) = \begin{pmatrix} x_{12}(t) \\ x_{22}(t) \end{pmatrix}$$

*are solutions of $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ on an interval I. Then $\mathbf{x}_1$ and $\mathbf{x}_2$ are **linearly independent** if and only if the **Wronskian***

$$W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0, \qquad \text{for all } t \text{ in } I.$$

The two linearly independent solutions of $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ are often called a **fundamental set of solutions**

Fundamental Solutions

Theorem (Fundamental Solutions)

Suppose that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are two solutions of

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \tag{2}$$

*and that their Wronskian is not zero on an interval I. Then $\mathbf{x}_1$ and $\mathbf{x}_2$ form a **fundamental set of solutions** for (2), and the general solution is given by*

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t),$$

where c_1 and c_2 are arbitrary constants. If there is a given initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, where $\mathbf{x}_0$ is any constant vector, then this condition determines the constants c_1 and c_2 uniquely.

Solving $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$

Consider the general problem

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t),$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

We attempt a solution of the form

$$\mathbf{x} = e^{\lambda t}\mathbf{v}, \qquad \text{so} \qquad \lambda e^{\lambda t}\mathbf{v} = \mathbf{A}e^{\lambda t}\mathbf{v}$$

Since $e^{\lambda t}$ is never zero,

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \qquad \text{or} \qquad (\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0},$$

where $\mathbf{I}$ is the 2×2 identity matrix

This is the classic **eigenvalue problem**

Eigenvalue Problem

Thus, solving the homogeneous DE $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ is equivalent to solving the **eigenvalue problem**

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0} \qquad \text{with} \qquad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

From Linear Algebra (Math 254) the **eigenvalues** are found by solving

$$\det|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0.$$

This gives the **characteristic equation**

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$$

This is a quadratic equation, so easily solved for λ_1 and λ_2

Each λ_i is inserted into $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$, and the corresponding **eigenvectors**, $\mathbf{v}_i$ are found

Real and Different Eigenvalues

Consider $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ and assume that the **eigenvalue problem** $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ has **real and different eigenvalues**, λ_1 and λ_2

The two solutions are

$$\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1 \qquad \text{and} \qquad \mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{v}_2,$$

so the **Wronskian** is

$$W[\mathbf{x}_1(t), \mathbf{x}_2(t)](t) = \begin{vmatrix} v_{11}e^{\lambda_1 t} & v_{12}e^{\lambda_2 t} \\ v_{21}e^{\lambda_1 t} & v_{22}e^{\lambda_2 t} \end{vmatrix} = \begin{vmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{vmatrix} e^{(\lambda_1+\lambda_2)t}$$

Since $e^{(\lambda_1+\lambda_2 t)t}$ is nonzero, the Wronskian is nonzero if and only if $\det|\mathbf{v}_1, \mathbf{v}_2| = 0$.

Recall if the Wronskian is nonzero, then $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ form a **fundamental set of solutions** to the system of DEs

Linear Algebra Result

Theorem

Let $\mathbf{A}$ *have real or complex eigenvalues,* λ_1 *and* λ_2*, such that* $\lambda_1 \neq \lambda_2$*, and let the corresponding eigenvectors be*

$$\mathbf{v}_1 = \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix}.$$

If $\mathbf{V}$ *is the matrix formed from* $\mathbf{v_1}$ *and* $\mathbf{v_2}$ *with*

$$\mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix},$$

then

$$\det|\mathbf{V}| = \begin{vmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{vmatrix} \neq 0.$$

Real and Different Eigenvalues 1

The two previous slides show that if $\mathbf{A}$ has **real and different eigenvalues**, λ_1 and λ_2, then the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

has a **fundamental set of solutions**

$$\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1 \qquad \text{and} \qquad \mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{v}_2,$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the corresponding eigenvectors for λ_1 and λ_2, respectively

It follows that the general solution can be written

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2.$$

Real and Different Eigenvalues 2

Example 1: Consider the example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -0.5 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Find the general solution to this problem and create a phase portrait.

From above we need to find the eigenvalues and eigenvectors, so solve

$$\det \begin{vmatrix} -0.5-\lambda & 2 \\ 0 & -1-\lambda \end{vmatrix} = (\lambda+0.5)(\lambda+1) = 0,$$

which is the **characteristic equation** with solutions $\lambda_1 = -0.5$ and $\lambda_2 = -1$

Real and Different Eigenvalues 3

Example 1 (cont): For $\lambda_1 = -0.5$ we have:

$$\begin{pmatrix} -0.5-\lambda_1 & 2 \\ 0 & -1-\lambda_1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & -0.5 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This results in the eigenvector $\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Similarly, for $\lambda_2 = -1$ we have:

$$\begin{pmatrix} -0.5-\lambda_2 & 2 \\ 0 & -1-\lambda_2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0.5 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This results in the eigenvector $\xi^{(2)} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

Real and Different Eigenvalues 4

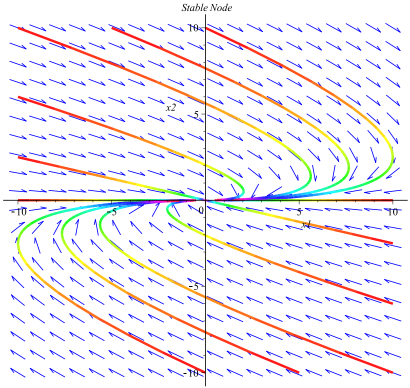
Example 1 (cont): The results above give the general solution

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-0.5t} + c_2 \begin{pmatrix} 4 \\ -1 \end{pmatrix} e^{-t},$$

which is a solution exponentially decaying toward the origin.

This is a **sink** or **stable node**.

Solutions move rapidly in the direction $\xi^{(2)} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$, while decaying more slowly in the direction $\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



Real and Different Eigenvalues with IVP 1

Example 2: Consider the initial value problem (**IVP**):

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Find the general solution to this problem, create a phase portrait, and solve the initial value problem (**IVP**).

From above we need to find the eigenvalues and eigenvectors, so solve

$$\det \begin{vmatrix} -\lambda & 1 \\ -3 & 4-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) = 0,$$

which is the **characteristic equation** with solutions $\lambda_1 = 1$ and $\lambda_2 = 3$

Real and Different Eigenvalues with IVP 2

Example 2 (cont): For $\lambda_1 = 1$ we have:

$$\begin{pmatrix} -\lambda_1 & 1 \\ -3 & 4-\lambda_1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This results in the eigenvector $\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Similarly, for $\lambda_2 = 3$ we have:

$$\begin{pmatrix} -\lambda_2 & 1 \\ -3 & 4-\lambda_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This results in the eigenvector $\xi^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

Real and Different Eigenvalues with IVP 3

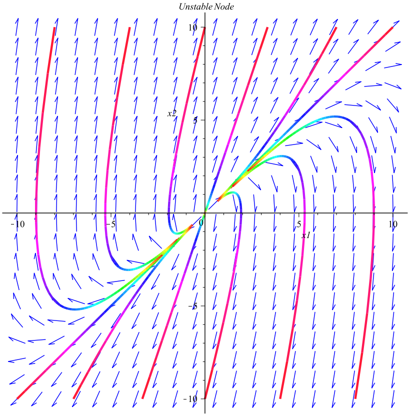
Example 2 (cont): The results above give the general solution

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t},$$

which is a solution exponentially growing away from the origin.

This is a **source** or **unstable node**.

Solutions first move away from the origin in the direction $\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then asymptotically parallel the direction $\xi^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ for larger t



Real and Different Eigenvalues with IVP 4

Example 2 (cont): Finally, we solve the initial value problem (**IVP**) with the general solution and initial condition:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t}, \qquad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

From the ***initial condition*** this gives the linear system to solve:

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

With standard techniques (such as ***Gaussian elimination***) it is easy to find the solution, $c_1 = 3$ and $c_2 = -2$.

It follows that the **unique solution** to this **IVP** is:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t - 2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t}.$$

Real and Different Eigenvalues 1

Example 3: Consider the example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Find the general solution to this problem and create a phase portrait.

From above we need to find the eigenvalues and eigenvectors, so solve

$$\det \begin{vmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 - 4 = (\lambda - 2)(\lambda + 2) = 0,$$

which is the **characteristic equation** with solutions $\lambda_1 = 2$ and $\lambda_2 = -2$

Real and Different Eigenvalues 2

Example 3 (cont): For $\lambda_1 = 2$ we have:

$$\begin{pmatrix} 1-\lambda_1 & 3 \\ 1 & -1-\lambda_1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This results in the eigenvector $\xi^{(1)} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

Similarly, for $\lambda_2 = -2$ we have:

$$\begin{pmatrix} 1-\lambda_2 & 3 \\ 1 & -1-\lambda_2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This results in the eigenvector $\xi^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

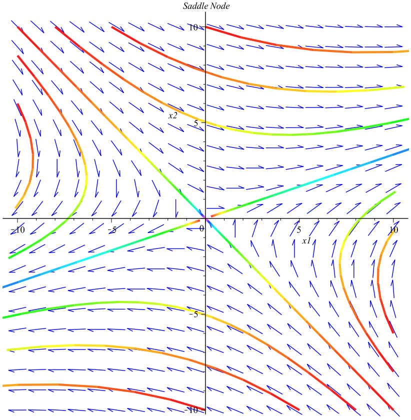
Real and Different Eigenvalues 3

Example 3 (cont): The results above give the general solution

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}.$$

This is a **saddle node**.

Solutions move toward the origin in the direction $\xi^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and move away from origin in the direction $\xi^{(1)} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ for larger t



Real and Different Eigenvalues 4

Example 4: Consider the example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Find the general solution to this problem and create a phase portrait.

If we seek equilibria, then

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_{1e} \\ x_{2e} \end{pmatrix}$$

However, any solution of the form $x_{1e} = 2x_{2e}$ is a **critical point**, giving a line of **equilibria**

Our method from before still applies, so seek $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$, which gives the **eigenvalue problem** below

$$\det \begin{vmatrix} -2-\lambda & 4 \\ 1 & -2-\lambda \end{vmatrix} = \lambda^2 + 4\lambda = \lambda(\lambda+4) = 0,$$

has the **characteristic equation** with eigenvalues $\lambda = 0, -4$

Real and Different Eigenvalues

Example 4 (cont): For $\lambda_1 = 0$ we have:

$$\begin{pmatrix} -2-\lambda_1 & 4 \\ 1 & -2-\lambda_1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This results in the eigenvector $\xi^{(1)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Similarly, for $\lambda_2 = -4$ we have:

$$\begin{pmatrix} -2-\lambda_2 & 4 \\ 1 & -2-\lambda_2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This results in the eigenvector $\xi^{(2)} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

Real and Different Eigenvalues 6

Example 4 (cont): The **eigenvalue problem** gives two solutions to the DE

$$\mathbf{x}_1(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{x}_2(t) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-4t}$$

The **Wronskian** satisfies

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \det \begin{vmatrix} 2 & 2e^{-4t} \\ 1 & -e^{-4t} \end{vmatrix} = -4e^{-4t} \neq 0,$$

so these do form a **fundamental set of solutions**

Thus the general solution is given by

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-4t}.$$

Real and Different Eigenvalues

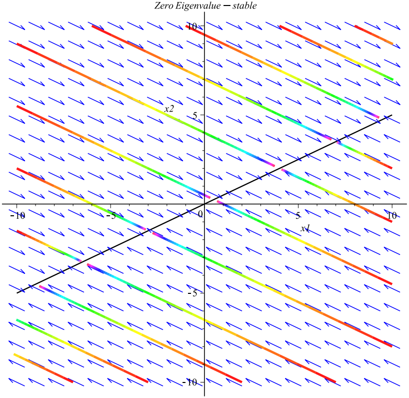
Example 4 (cont): The phase portrait for

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-4t}.$$

This is a **degenerate** case where the line $x_1 = 2x_2$ all form **equilibria**.

All solutions **exponentially approach** one of the equilibria along lines parallel to the line $x1 = -2x_2$

Note: There is an **unstable case**, which we omit, where the eigenvalues satisfy $\lambda_1 = 0$ and $\lambda_2 > 0$



Complex Eigenvalues 1

Consider a **system of two linear homogeneous differential equations**:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x},$$

where $\mathbf{A}$ is a real-valued matrix.

With a solution of the form $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$, there are **eigenvalues**, λ, with corresponding **eigenvectors**, $\mathbf{v}$ satisfying

$$\det|\mathbf{A} - \lambda\mathbf{I}| = 0 \quad \text{and} \quad (\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$$

The **characteristic equation** for the **eigenvalues** is a quadratic equation.

Assume the eigenvalues are complex, then $\lambda = \mu \pm i\nu$, since $\mathbf{A}$ is real-valued

Complex Eigenvalues 2

Assume the DE, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, has eigenvalues $\lambda_1 = \mu + i\nu$ and $\lambda_2 = \bar{\lambda}_1 = \mu - i\nu$

Assume $\mathbf{v}_1$ is an eigenvector corresponding to λ_1, so

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v}_1 = \mathbf{0}$$

Taking **conjugates** (with $\mathbf{A}$, $\mathbf{I}$, and $\mathbf{0}$, real)

$$(\mathbf{A} - \bar{\lambda}_1\mathbf{I})\bar{\mathbf{v}}_1 = (\mathbf{A} - \lambda_2\mathbf{I})\bar{\mathbf{v}}_1 = \mathbf{0}$$

This gives two complex solutions to the system of DEs

$$\mathbf{x}_1(t) = e^{(\mu+i\nu)t}\mathbf{v}_1 \qquad \text{and} \qquad \mathbf{x}_2(t) = e^{(\mu-i\nu)t}\bar{\mathbf{v}}_1$$

We use **Euler's formula** to separate the solutions into real and imaginary parts

$$e^{i\nu t} = \cos(\nu t) + i\sin(\nu t)$$

Complex Eigenvalues 3

Assume the **eigenvector**, $\mathbf{v}_1 = \mathbf{a} + i\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are real-valued, then

$$\begin{aligned} \mathbf{x}_1(t) &= (\mathbf{a} + i\mathbf{b})e^{\mu t}(\cos(\nu t) + i\sin(\nu t)) \\ &= e^{\mu t}(\mathbf{a}\cos(\nu t) - \mathbf{b}\sin(\nu t)) + ie^{\mu t}(\mathbf{a}\sin(\nu t) + \mathbf{b}\cos(\nu t)) \end{aligned}$$

Denote the real and imaginary parts of $\mathbf{x}_1(t) = \mathbf{u}(t) + i\mathbf{w}(t)$

$$\mathbf{u}(t) = e^{\mu t}(\mathbf{a}\cos(\nu t) - \mathbf{b}\sin(\nu t)) \quad \text{and} \quad \mathbf{w}(t) = e^{\mu t}(\mathbf{a}\sin(\nu t) + \mathbf{b}\cos(\nu t))$$

A similar calculation gives

$$\mathbf{x}_2(t) = \mathbf{u}(t) - i\mathbf{w}(t),$$

so $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are complex conjugates.

The desire is to show that $\mathbf{u}(t)$ and $\mathbf{w}(t)$ are real-valued solutions forming a **fundamental set** for $\dot{\mathbf{x}} = A\mathbf{x}$

Complex Eigenvalues 4

Since $\mathbf{x}_1(t) = \mathbf{u}(t) + i\mathbf{w}(t)$ is a solution to the DE $\dot{\mathbf{x}}_1 = \mathbf{A}\mathbf{x}_1$, we have

$$\begin{aligned} \mathbf{0} &= \dot{\mathbf{x}}_1 - \mathbf{A}\mathbf{x}_1 = (\dot{\mathbf{u}} + i\dot{\mathbf{w}}) - \mathbf{A}(\mathbf{u} + i\mathbf{w}) \\ &= (\dot{\mathbf{u}} - \mathbf{A}\mathbf{u}) + i(\dot{\mathbf{w}} - \mathbf{A}\mathbf{w}) \end{aligned}$$

This vector is zero if and only if the real and imaginary parts are zero, so

$$\dot{\mathbf{u}} - \mathbf{A}\mathbf{u} = \mathbf{0} \qquad \text{and} \qquad \dot{\mathbf{w}} - \mathbf{A}\mathbf{w} = \mathbf{0}$$

or $\mathbf{u}(t)$ and $\mathbf{w}(t)$ are real-valued solutions of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$

It remains to show $\mathbf{u}(t)$ and $\mathbf{w}(t)$ form a **fundamental set of solutions**, which is done with the **Wronskian**

Complex Eigenvalues 5

The two solutions are

$$\mathbf{u}(t) = e^{\mu t}(\mathbf{a}\cos(\nu t) - \mathbf{b}\sin(\nu t)) \quad \text{and} \quad \mathbf{w}(t) = e^{\mu t}(\mathbf{a}\sin(\nu t) + \mathbf{b}\cos(\nu t)),$$

so let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, then the **Wronskian** satisfies

$$\begin{aligned} W[\mathbf{u},\mathbf{w}](t) &= \begin{vmatrix} e^{\mu t}(a_1\cos(\nu t) - b_1\sin(\nu t)) & e^{\mu t}(a_1\sin(\nu t) + b_1\cos(\nu t)) \\ e^{\mu t}(a_2\cos(\nu t) - b_2\sin(\nu t)) & e^{\mu t}(a_2\sin(\nu t) + b_2\cos(\nu t)) \end{vmatrix} \\ &= (a_1b_2 - a_2b_1)e^{2\mu t} \end{aligned}$$

Assume $\nu \neq 0$ and the eigenvectors are $\mathbf{v}_1 = \mathbf{a} + i\mathbf{b}$ and $\mathbf{v}_2 = \mathbf{a} - i\mathbf{b}$,

$$\begin{vmatrix} a_1 + ib_1 & a_1 - ib_1 \\ a_2 + ib_2 & a_2 - ib_2 \end{vmatrix} = -2i(a_1b_2 - a_2b_1) \neq 0$$

by our Theorem from Linear Algebra

Thus, the **Wronskian** shows $\mathbf{u}(t)$ and $\mathbf{w}(t)$ form a **fundamental set of solutions** to our problem

Complex Eigenvalues 6

Example 5: Consider the example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Find the general solution to this problem and create a phase portrait.

From above we need to find the eigenvalues and eigenvectors, so solve

$$\det \begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 5 = 0,$$

which is the **characteristic equation** with solutions $\lambda = 1 \pm 2i$ (complex eigenvalues)

Complex Eigenvalues

Example 5 (cont): For $\lambda_1 = 1 + 2i$ we have:

$$\begin{pmatrix} 3-\lambda_1 & -2 \\ 4 & -1-\lambda_1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This results in the eigenvector $\xi^{(1)} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$.

We have $\lambda_2 = \bar{\lambda}_1$ and $\xi^{(2)} = \bar{\xi}^{(1)}$

Thus,

$$\begin{aligned} \mathbf{x}_1(t) &= \begin{pmatrix} 1 \\ 1-i \end{pmatrix} e^t(\cos(2t) + i\sin(2t)) = \\ \mathbf{u}(t) + i\mathbf{w}(t) &= \begin{pmatrix} e^t\cos(2t) \\ e^t(\cos(2t)+\sin(2t)) \end{pmatrix} + i\begin{pmatrix} e^t\sin(2t) \\ e^t(\sin(2t)-\cos(2t)) \end{pmatrix} \end{aligned}$$

Complex Eigenvalues

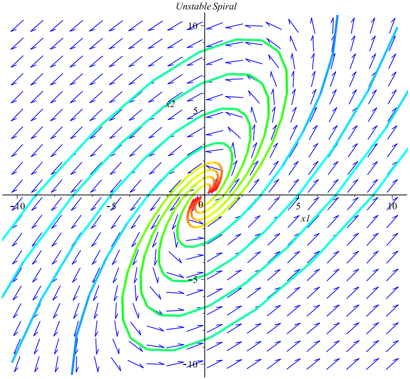
Example 5 (cont): From above the general solution is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} e^t\cos(2t) \\ e^t(\cos(2t)+\sin(2t)) \end{pmatrix} + c_2 \begin{pmatrix} e^t\sin(2t) \\ e^t(\sin(2t)-\cos(2t)) \end{pmatrix}.$$

This is an **unstable spiral**.

All solutions spiral away from the origin.

Solutions with complex eigenvalues with negative real parts spiral toward the origin, creating a **stable spiral**



Imaginary Eigenvalues

Example 6: Consider the example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Find the general solution to this problem and create a phase portrait.

From above we need to find the eigenvalues and eigenvectors, so solve

$$\det \begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = \lambda^2 + 1 = 0,$$

which is the **characteristic equation** with solutions $\lambda = \pm i$ (purely imaginary eigenvalues)

Imaginary Eigenvalues

Example 6 (cont): For $\lambda_1 = i$ we have:

$$\begin{pmatrix} 2-\lambda_1 & -5 \\ 1 & -2-\lambda_1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This results in the eigenvector $\xi^{(1)} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$.

We have $\lambda_2 = \bar{\lambda}_1$ and $\xi^{(2)} = \bar{\xi}^{(1)}$

Thus,

$$\begin{aligned} \mathbf{x}_1(t) &= \begin{pmatrix} 2+i \\ 1 \end{pmatrix}(\cos(t) + i\sin(t)) = \\ \mathbf{u}(t) + i\mathbf{w}(t) &= \begin{pmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{pmatrix} + i\begin{pmatrix} 2\sin(t) + \cos(t) \\ \sin(t) \end{pmatrix} \end{aligned}$$

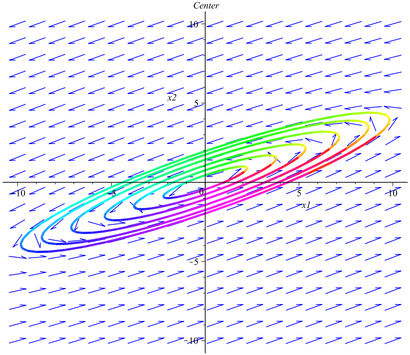
Imaginary Eigenvalues

Example 6 (cont): From above the general solution is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 2\cos(t) - \sin(t) \\ \cos(t) \end{pmatrix} + c_2 \begin{pmatrix} 2\sin(t) + \cos(t) \\ \sin(t) \end{pmatrix}.$$

This is a **center**.

All solutions form ellipses around the origin.



Repeated Eigenvalues 1

Example 7: Consider the example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Find the general solution to this problem and create a phase portrait.

From above we need to find the eigenvalues and eigenvectors, so solve

$$\det \begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (\lambda-2)^2 = 0,$$

which has the **characteristic equation** with solutions $\lambda = 2$ with an **algebraic multiplicity** of **2**

Repeated Eigenvalues 2

Example 7 (cont): For $\lambda_1 = \lambda_2 = 2$ we have:

$$\begin{pmatrix} 2-\lambda_1 & 0 \\ 0 & 2-\lambda_1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

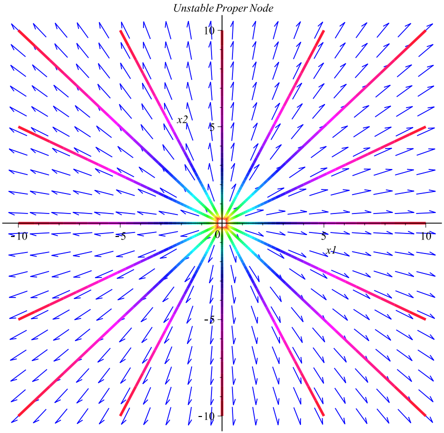
Thus, $\lambda = 2$ has a **geometric multiplicity** of **2**, so the **eigenspace** for $\lambda = 2$ has dimension 2.

It follows that we can select the standard basis vectors as our eigenvectors, which gives the general solution

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}.$$

Repeated Eigenvalues 3

Example 7 (cont): This DE produces an **unstable proper node** or **star node** with all solutions following straight paths away from the origin



Repeated Eigenvalues 4

Example 8: Consider the example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Find the general solution to this problem and create a phase portrait.

This is an **upper triangular matrix**, so its eigenvalues are the diagonal elements.

Thus, $\lambda = -1$ with an **algebraic multiplicity** of **2**

$$\begin{pmatrix} -1-\lambda & 1 \\ 0 & -1-\lambda \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This system only has the **1** eigenvector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Repeated Eigenvalues 5

Example 8 (cont): Since there is only one eigenvector, we obtain the one solution

$$\mathbf{x}_1(t) = \mathbf{v_1}e^{-t} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t}$$

Thus, $\lambda = -1$ has a **geometric multiplicity** of **1**, so the **eigenspace** for $\lambda = -1$ has dimension 1.

If we examine the scalar equations, then

$$\dot{x}_1 = -x_1 + x_2 \qquad \text{and} \qquad \dot{x}_2 = -x_2$$

Thus, $x_2(t) = c_2 e^{-t}$, so

$$\dot{x}_1 + x_1 = c_2 e^{-t} \qquad \text{with} \qquad \mu(t) = e^t$$

This has the solution

$$x_1(t) = c_2 t e^{-t} + c_1 e^{-t}$$

Repeated Eigenvalues 6

Example 8 (cont): Combining the results above we see

$$\begin{aligned}\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} &= \begin{pmatrix} c_1 + c_2 t \\ c_2 \end{pmatrix} e^{-t} \\ &= c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] e^{-t}\end{aligned}$$

The second solution has the form

$$\mathbf{x}_2(t) = \mathbf{v} t e^{-t} + \mathbf{w} e^{-t}$$

Upon differentiation

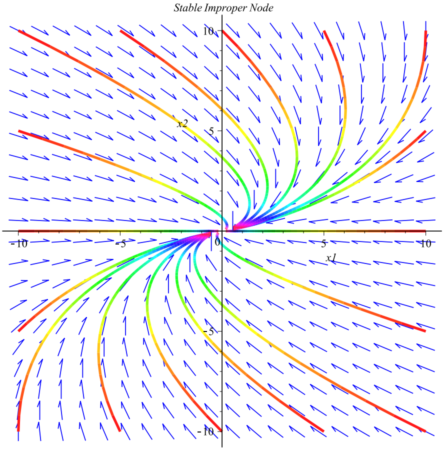
$$\dot{\mathbf{x}}_2(t) = \mathbf{v}(1-t)e^{-t} - \mathbf{w}e^{-t} = \mathbf{A}\mathbf{x}_2 = \mathbf{A}(\mathbf{v}te^{-t} + \mathbf{w}e^{-t})$$

Since $(\mathbf{A}+\mathbf{I})\mathbf{v} = \mathbf{0}$, this reduces to solving for $\mathbf{w}$

$$(\mathbf{A}+\mathbf{I})\mathbf{w} = \mathbf{v} \quad \text{or} \quad \mathbf{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + k \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Repeated Eigenvalues

Example 8 (cont): This DE produces a **stable improper node** with all solutions moving toward the origin



Repeated Eigenvalues - General

Repeated Eigenvalues - Two Dimensional Null Space

Suppose the 2×2 matrix $\mathbf{A}$ has a repeated eigenvalue λ.

If the eigenspace spanned by the eigenvectors has dimension 2, $\mathbf{v}_1$ and $\mathbf{v}_2$, then the solution is simply

$$\mathbf{x}(t) = c_1 \mathbf{v}_1 e^{\lambda t} + c_2 \mathbf{v}_2 e^{\lambda t}$$

Repeated Eigenvalues - General

Repeated Eigenvalues - One Dimensional Null Space If the 2×2 matrix $\mathbf{A}$ has only one eigenvector $\mathbf{v}$ associated with λ, then one solution is

$$\mathbf{x}_1(t) = \mathbf{v}e^{\lambda t}$$

We attempt a second solution of the form

$$\mathbf{x}_2(t) = \mathbf{v}te^{\lambda t} + \mathbf{w}e^{\lambda t},$$

which upon differentiation gives

$$\dot{\mathbf{x}}_2(t) = \mathbf{v}(\lambda t + 1)e^{\lambda t} + \lambda \mathbf{w}e^{\lambda t} = \mathbf{A}\mathbf{x}_2 = \mathbf{A}(\mathbf{v}te^{\lambda t} + \mathbf{w}e^{\lambda t})$$

Since $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$, this reduces to solving for $\mathbf{w}$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{w} = \mathbf{v}$$

This gives the second linearly independent solution, $\mathbf{x}_2(t)$, above, where $\mathbf{w}$ solves this **higher order null space problem**, which will include a particular solution and any multiple, $k\mathbf{v}$

Bifurcation Example 1

Bifurcation Example: Consider the example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \alpha & 2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

which contains a parameter α that affects the behavior of this system

We want to determine the different **qualitative behaviors** for different values of α

The eigenvalues satisfy

$$\det \begin{vmatrix} \alpha - \lambda & 2 \\ -2 & -\lambda \end{vmatrix} = \lambda^2 - \alpha\lambda + 4 = 0$$

Thus, the eigenvalues satisfy

$$\lambda = \frac{\alpha \pm \sqrt{\alpha^2 - 16}}{2}$$

Bifurcation Example 2

Bifurcation Example: For

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \alpha & 2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{3}$$

The eigenvalues are $\lambda = \frac{\alpha \pm \sqrt{\alpha^2 - 16}}{2}$

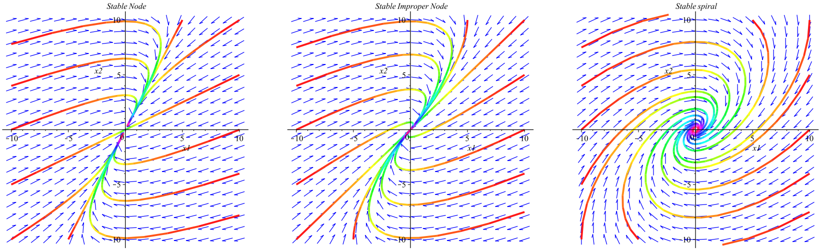
Classifications as α varies are:

- For $\alpha < -4$, System (3) is a **Stable Node**
- For $\alpha = -4$, System (3) is a **Stable Improper Node**
- For $-4 < \alpha < 0$, System (3) is a **Stable Spiral**
- For $\alpha = 0$, System (3) is a **Center**
- For $0 < \alpha < 4$, System (3) is a **Unstable Spiral**
- For $\alpha = 4$, System (3) is a **Unstable Improper Node**
- For $\alpha > 4$, System (3) is a **Unstable Node**

Bifurcation Example

Bifurcation Example: Phase Portraits ($\alpha < 0$)

Observe a smooth transition as eigenvalues change from negative to complex with negative real part

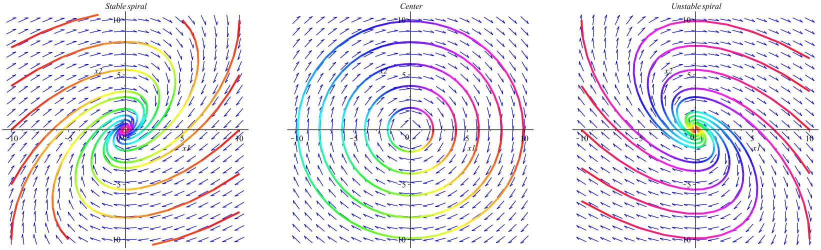


$\alpha = -5$ $\alpha = -4$ $\alpha = -2$

Bifurcation Example 4

Bifurcation Example: Phase Portraits ($-4 < \alpha < 4$)

Observe the transitions as complex eigenvalues change from negative real part to positive real part - This is a significant part of a **Hopf bifurcation**

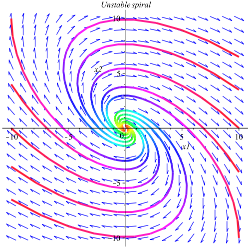


$\alpha = -2$ $\qquad$ $\alpha = 0$ $\qquad$ $\alpha = 2$

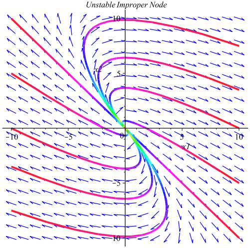
Bifurcation Example

Bifurcation Example: Phase Portraits ($\alpha > 0$)

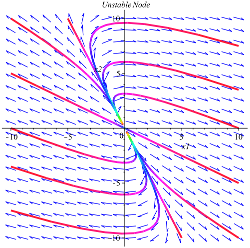
Observe a smooth transition as eigenvalues change from complex with positive real part to positive real values



$\alpha = 2$



$\alpha = 4$



$\alpha = 5$

Stability Diagram

Consider the system

$$\dot{\mathbf{x}} = \mathbf{J}\mathbf{x}$$

Let λ_1 and λ_2 be eigenvalues of $\mathbf{J}\mathbf{x}$

Results from Linear Algebra give $tr(\mathbf{J}) = \lambda_1 + \lambda_2$, $\det|\mathbf{J}| = \lambda_1 \cdot \lambda_2$, and $D = (j_{11} - j_{22})^2 + 4 j_{12} j_{21}$

The figure shows the **Stability Diagram** for $\dot{\mathbf{x}} = \mathbf{J}\mathbf{x}$ with axes of $tr(\mathbf{J})$ vs $\det|\mathbf{J}|$