Classwork 6
Abstract Algebra
Math 320

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Problem 1: Let $f(x), g(x), h(x) \in F[x]$ with $f(x)$ and $g(x)$ relatively prime. If $h(x)|f(x)$, prove that $h(x)$ and $g(x)$ relatively prime.

Let $h(x)|f(x)$, so the following is true:

$$f(x) = h(x)q(x)$$

Let $d(x) = (h(x), g(x))$, so $d(x)|h(x)$ and $d(x)|g(x)$

$$h(x) = d(x)a_1(x)$$
$$g(x) = d(x)a_2(x)$$

Putting it all together gets us:

$$f(x) = h(x)q(x) = d(x)a_1(x)q(x) = d(x)b_1(x)$$
$$g(x) = d(x)a_2(x)$$

Because $f(x)$ and $g(x)$ are relatively prime, they do not share any factors. And because they both share $d(x)$, then $d(x) = 1$. Because $d(x)$ is also $(h(x), g(x))$, $h(x)$ and $g(x)$ are relatively prime.

Problem 2: Express $x^4 - 4$ as a product of irreducibles in $\mathbb{Q}[x], \mathbb{R}[x], \mathbb{C}[x]$.

$$\begin{aligned} x^4 - 4 &= (x^2 - 2)(x^2 + 2) \in \mathbb{Q}[x] \\ &= (x - \sqrt{2})(x + \sqrt{2})(x^2 + 2) \in \mathbb{R}[x] \\ &= (x - \sqrt{2})(x + \sqrt{2})(x - \sqrt{2}i)(x + \sqrt{2}i) \in \mathbb{C}[x] \end{aligned}$$

Problem 3: Show that $x^7 - x$ factors in $\mathbb{Z}_7[x]$ as $x(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)$ without doing any polynomial multiplication/division.

We can show this by using our knowledge of roots. If we plug in $0, 1, 2, 3, 4, 5, 6$ into $f(x) = x^7 - x$, then the result should be 0.

$$\begin{aligned}
f(0) &= 0 \\
f(1) &= 0 \\
f(2) &= 7(18) = 0 \\
f(3) &= 7(312) = 0 \\
f(4) &= 7(2340) = 0 \\
f(5) &= 7(11160) = 0 \\
f(6) &= 7(39990) = 0
\end{aligned}$$

Because we have proved that $0, 1, 2, 3, 4, 5, 6$ are roots of $f(x) = x^7 - x$, then $f(x)$ can be factored into $x(x-1)(x-2)(x-3)(x-4)(x-5)(x-6)$.

Bonus (a): Find all polynomials of degree 2 in $\mathbb{Z}_2[x]$

Notice in $\mathbb{Z}_2[x]$, all numbers are equal to either 0 or 1. All polynomials can be written as $ax^2 + bx + c$. So by looking at the different a, b, c values, we get:

$$\begin{aligned}
&x^2 \\
&x^2 + x + 1 \\
&x^2 + 1 \\
&x^2 + x
\end{aligned}$$

Bonus (b): Find all *irreducible* polynomials of degree 2 in $\mathbb{Z}_2[x]$

To find the following, all we need to do is reduce all the reducible polynomials from part (a) to find the irreducible ones.

$$\begin{aligned}
x^2 &= x(x) \\
x^2 + x + 1 & \\
x^2 + 1 &= (x+1)^2 \\
x^2 + x &= x(x+1)
\end{aligned}$$

Because we were able to reduce all polynomials except $x^2 + x + 1$, $x^2 + x + 1$ is irreducible