Class Work 9
Abstract Algebra
Math 320

Stephen Giang, William Diebolt
Sobhan Ahmadi Pishkouhi

Problem 1: For parts (a) and (b), determine if the given ring is a field. If it is, explain why. If not, explain why and provide one zero divisor.

(a) $\mathbb{Q}[x]/(x^6 - 144)$

Notice that $x^3 - 12$ and $x^3 + 12$ are both in $\mathbb{Q}/(x^6 - 144)$ as they both have degree less than 6. When multiplied, they equal $(x^6 - 144)$. So the following is a zero divisor:

$$[x^3 - 12][x^3 + 12] = [x^6 - 144] = [0]$$

(b) $\mathbb{Z}_3[x]/(2x^3 + x + 1)$

Notice that all constants of $\mathbb{Z}_3$ are $0, 1, 2$. Notice that all zero divisors will be factors of the given polynomial. Also notice because of the congruence class, the only factors will be of degree 2 and degree 1 at the same time. Because degree one, then its factor is a root. Notice:

$$\begin{aligned} f(x) &= 2x^3 + x + 1 \\ f(0) &= 1 \neq 0 \\ f(1) &= 1 \neq 0 \\ f(2) &= 1 \neq 0 \end{aligned}$$

Problem 2: Find the multiplicative inverse of $[x-1]$ in $\mathbb{Q}[x]/(x^2-3)$ using the following method:

Let $(x-1, x^2-3) = 1$, such that the following is true:

$$(x-1)u(x) + (x^2-3)v(x) = 1$$

We assume that $u(x)$ and $v(x)$ are both degree 1. So we can write the following:

$$u(x) = ax + b \qquad\qquad v(x) = cx + d$$

We now have

$$\begin{aligned} (x-1)(ax+b) + (x^2-3)(cx+d) &= 1 \\ ax^2 + (b-a)x - b + cx^3 + dx^2 - 3cx - 3d &= 1 \\ cx^3 + (a+d)x^2 + (b-a-3c)x + (-b-3d) &= 1 \end{aligned}$$

So we get the systems of equation:

$$\begin{aligned} c &= 0 \\ a + d &= 0 \\ b - a - 3c &= 0 \\ -b - 3d &= 1 \end{aligned}$$

So after reducing the system, we get $a = \frac{1}{2}, b = \frac{1}{2}, c = 0, d = \frac{-1}{2}$. So now we get the following:

$$\begin{aligned} \frac{1}{2}(x-1)(x+1) - \frac{-1}{2}(x^2-3) &= 1 \\ \frac{1}{2}(x-1)(x+1) &= 1 \end{aligned}$$

Thus $\frac{1}{2}(x+1) = [x-1]^{-1}$

Problem 3:

(a) Prove that the set $\{(2a, 0) : a \in \mathbb{Z}\}$ is an ideal of $\mathbb{Z} \times \mathbb{Z}$

Notice the following:

Let $(2a, 0) \in \mathbb{Z} \times \mathbb{Z}$. If we let $a = 0$, then $(0, 0) = \mathbb{Z} \times \mathbb{Z}$. Thus $\mathbb{Z} \times \mathbb{Z}$ contains the zero element.

Let $x = (2a, 0)$ and $y = (2b, 0)$.

$$x - y = (2a, 0) - (2b, 0) = (2(a - b), 0) \in \mathbb{Z} \times \mathbb{Z}$$

Thus closed under subtraction.

Let $x = (2a, 0)$ and $y = (b, c)$

$$xy = (2a, 0)(b, c) = (2ab, 0) = (2ba, 0) = (b, c)(2a, 0) = yx$$

Thus absorption property proven.

(b) Let F be a field, $c \in F$, and consider the set K_c consisting of polynomials that have c as a root:

Notice the following:

Let $f(x) = 0$. Notice that $f(c) = 0_F \in K_c$.

Let $f(x), g(x) \in K_c$.

$$f(x) - g(x) = (f - g)(x), \qquad f(c) - g(c) = (f - g)(c) = 0_F \in K_c$$

Let $f(x) \in K_c$ and $g(x) \in F[x]$.

$$f(c)g(c) = 0_F = g(c)f(c)$$

Thus the absorption property is proven