Math 531 - Partial Differential Equations

Heat Conduction —
in a One-Dimensional Rod

Joseph M. Mahaffy,
⟨jmahaffy@mail.sdsu.edu⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://jmahaffy.sdsu.edu

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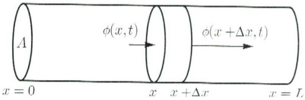
Outline

Heat Conduction in a One-Dimensional Rod

Heat in a Rod: Consider a rod of length L with cross-sectional area A, which is perfectly insulated on its lateral surface.

Below is a diagram of this rod



We examine the heat transfer through a small slice of the rod

- Define $e(x,t)$ = ***thermal energy density***
- **Heat energy** in the small slice $= e(x,t)A\Delta x$
- Define $\phi(x,t)$ = ***heat flux*** (amount of thermal energy per unit time flowing to the right per unit surface area)

Heat Conduction in a One-Dimensional Rod

Conservation of Heat Energy: With insulated lateral edges, the basic conservation equation for **heat** in our small slice satisfies

Rate of change of heat energy in time	=	Heat energy flowing across boundaries per unit time	+	Heat energy generated inside per unit time

The ***rate of change of heat energy*** satisfies

$$\frac{\partial}{\partial t}\left(e(x,t)A\Delta x\right)$$

The ***heat flux across the boundaries*** satisfies

$$\phi(x,t)A - \phi(x+\Delta x,t)A$$

(***heat*** entering on left and leaving on right)

Heat Conduction in a One-Dimensional Rod

Heat sources/sinks: Define $Q(x,t) =$ ***heat energy per unit volume generated per unit time***, accounting for any sources or sinks of ***heat*** inside the thin rod

Conservation of heat energy (thin slice) combining elements above:

$$\frac{\partial}{\partial t}\left(e(\xi_1,t)A\Delta x\right) = \phi(x,t)A - \phi(x+\Delta x,t)A + Q(\xi_2,t)A\Delta x,$$

where by the **Intermediate Value Theorem** assuming continuity of both $e(x,t)$ and $Q(x,t)$, there are $\xi_1, \xi_2 \in (x, x+\Delta x)$ providing equality above.

Rearranging we have

$$\frac{\partial e(\xi_1,t)}{\partial t} = \frac{\phi(x,t) - \phi(x+\Delta x,t)}{\Delta x} + Q(\xi_2,t),$$

which by taking the limit as $\Delta x \to 0$ gives

$$\frac{\partial e(x,t)}{\partial t} = -\frac{\partial \phi(x,t)}{\partial x} + Q(x,t).$$

Alternate Integral Derivation

Alternate Integral Derivation: Use the ***conservation of heat energy*** on any interval $[a,b]$, then

$$\frac{d}{dt}\int_a^b e(x,t)dx = \phi(a,t) - \phi(b,t) + \int_a^b Q(x,t)dt.$$

However, by **Leibnitz's rule of differentiation of an integral** and the **Fundamental Theorem of Calculus**, we have

$$\frac{d}{dt}\int_a^b e(x,t)dx = \int_a^b \frac{\partial e(x,t)}{\partial t} \quad \text{and} \quad \phi(a,t)-\phi(b,t) = -\int_a^b \frac{\partial \phi(x,t)}{\partial x}dx$$

It follows that for any interval $[a,b]$

$$\int_a^b \left(\frac{\partial e(x,t)}{\partial t} + \frac{\partial \phi(x,t)}{\partial x} - Q(x,t) \right) dx = 0,$$

so the integrand is zero, giving the same equation as before.

Heat and Temperature

Temperature and Specific heat: Define $u(x,t)$ as the temperature of a material and $c(x)$ as the specific heat of a material (the heat energy required to raise a unit mass of a material a unit of temperature)

Mass density: Define $\rho(x)$ as the mass density (per unit volume)

Thermal energy: From the definitions above, we have

$$e(x,t) = c(x)\rho(x)u(x,t)$$

Fourier's Law: Heat flows proportional to the negative gradient of the temperature (hot to cold) or

$$\phi(x,t) = -K_0(x)\frac{\partial u(x,t)}{\partial x}$$

Heat Equation

From the **heat conduction** equation

$$\frac{\partial e(x,t)}{\partial t} = -\frac{\partial \phi(x,t)}{\partial x} + Q(x,t),$$

we obtain the **heat equation**

$$c(x)\rho(x)\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u(x,t)}{\partial x}\right) + Q(x,t).$$

If the material in the rod is consistent, $c(x)$, $\rho(x)$, and $K_0(x)$ are constant. Also, if there are no sources or sinks, $Q(x,t) = 0$. Then the **heat equation** has the form:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

where $k = K_0/(c\rho)$ is the ***thermal diffusivity***.

Heat Equation

The first PDE that we'll solve is the **heat equation**

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}.$$

This ***linear PDE*** has a domain $t > 0$ and $x \in (0, L)$.
In order to solve, we need ***initial conditions***

$$u(x, 0) = f(x),$$

and ***boundary conditions (linear)***

- **Dirichlet** or **prescribed**: *e.g.*, $u(0, t) = u_0(t)$
- **Neumann: Insulated**: *e.g.*, $u_x(0, t) = 0$
- **Neumann: Prescribed flux**: *e.g.*, $-K_0 u_x(0, t) = \phi(t)$
- **Robin** or **mixed**: *e.g.*, Newton's cooling:
  $K_0 u_x(0, t) = H(u(0, t) - u_E(t))$

Heat Equation Equilibrium

Consider the **heat equation**:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

with the initial condition and **Dirichlet boundary conditions**

$$u(x,0) = f(x), \qquad u(0,t) = T_1(t) \quad \text{and} \quad u(L,t) = T_2(t).$$

Suppose that the boundary conditions (BCs) are constant, $T_1(t) = T_1$ and $T_2(t) = T_2$.

Examine the **steady-state** or **equilibrium** solution, which implies that

$$\frac{\partial u}{\partial t} = 0, \qquad \text{so} \quad u(x,t) = u(x).$$

Heat Equation Equilibrium

The **equilibrium heat equation (ODE)** problem reduces to

$$\frac{d^2u}{dx^2} = 0 \qquad \text{with} \quad u(0) = T_1 \quad \text{and} \quad u(L) = T_2.$$

The solution of the ODE is

$$u(x) = c_1 x + c_2.$$

Since $u(0) = T_1$, we have $c_2 = T_1$.

Also, $u(L) = T_2$ implies $T_2 = c_1 L + T_1$ or $c_1 = \frac{T_2 - T_1}{L}$, giving the solution

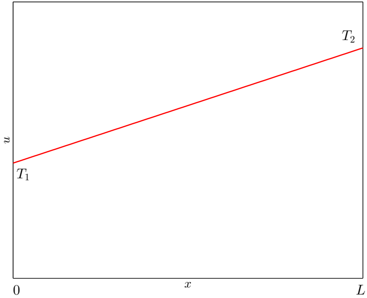
$$u(x) = \frac{T_2 - T_1}{L} x + T_1.$$

Heat Equation Equilibrium

The **equilibrium solution** for the **heat equation** with fixed temperatures at each end is

$$u(x) = \frac{T_2 - T_1}{L}x + T_1.$$

Thus, the temperature equilibrates to a linear function connecting the two end temperatures



Heat Equation Equilibrium – Insulated

Consider the **heat equation** with the initial condition and **Neumann boundary conditions**:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = f(x), \qquad u_x(0,t) = 0 \quad \text{and} \quad u_x(L,t) = 0.$$

As before, the equilibrium problem is

$$\frac{d^2u}{dx^2} = 0 \qquad \text{with} \quad u'(0) = 0 \quad \text{and} \quad u'(L) = 0.$$

The general solution of the ODE is

$$u(x) = c_1 x + c_2.$$

But $u'(x) = c_1$, so either BC implies $c_1 = 0$.

The BC gives **no information** about c_2

Heat Equation Equilibrium – Insulated

From above the ODE has the solution

$$u(x) = c_2.$$

So what is c_2?

Since the lateral sides and the ends are ***insulated***, then the ***thermal energy*** is conserved

$$\frac{d}{dt}\int_0^L c\rho u(x)dx = -K_0\frac{\partial u}{\partial x}(0,t) + K_0\frac{\partial u}{\partial x}(L,t) = 0.$$

The initial ***thermal energy*** is

$$c\rho\int_0^L f(x)dx = c\rho\int_0^L u(x)dx = c\rho\int_0^L c_2 dx = c\rho L c_2.$$

It follows that

$$u(x) = c_2 = \frac{1}{L}\int_0^L f(x)dx.$$