Math 532: Homework 9
Due 11/20/19
Everyone turns in an individual copy.

Book Problems

1. 5.65.5
2. 5.68.4
3. 5.68.5
4. 5.68.8
5. 5.68.9

More Problems

Throughout this section, we will use the fact that if a sequence of complex numbers $\{z_n\}_{n=0}^{\infty}$ is a Cauchy sequence, then there exists some $z \in \mathbb{C}$ such that

$$\lim_{n\to\infty} z_n = z.$$

In more techincal terms, this means that if I can show that for all $\epsilon > 0$, there exits a natural number N such that

$$|z_n - z_m| < \epsilon,\ n, m > N,$$

then I know the sequence has a limit. We also know if a sequence converges it is Cauchy since if $z_n \to z$ we can find N for $\epsilon/2$ so that

$$|z_n - z| < \frac{\epsilon}{2},\ n \geq N.$$

Therefore we see that if we take $m, n \geq N$, then we have that

$$|z_m - z_n| = |z_m - z - (z_n - z)| \leq |z_m - z| + |z_n - z| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We often use Cauchy convergence when we are not certain of where we are already going as we see in the following problems.

1. Suppose the series

$$\sum_{j=0}^{\infty} a_j(z-z_0)^j$$

converges absolutely, i.e.

$$\lim_{N\to\infty} \sum_{j=0}^{N} \left|a_j(z-z_0)^j\right| = S < \infty.$$

Show that the original series converges, i.e.

$$\lim_{N\to\infty} \sum_{j=0}^{N} a_j(z-z_0)^j = \tilde{S} \in \mathbb{C},\ \tilde{S} \neq \infty.$$

Hint: The trick here is to choose $N_2 > N_1$ and then control the difference

$$\begin{aligned}\left|\sum_{j=0}^{N_2} a_j(z-z_0)^j - \sum_{j=0}^{N_1} a_j(z-z_0)^j\right| &\leq \sum_{j=N_1+1}^{N_2} \left|a_j(z-z_0)^j\right| \\ &\leq \sum_{j=0}^{N_2} \left|a_j(z-z_0)^j\right| - \sum_{j=0}^{N_1} \left|a_j(z-z_0)^j\right|.\end{aligned}$$

2. *Weierstrass' M-Test*: Suppose there is a sequence of complex functions $\{g_j(z)\}_{j=0}^{\infty}$, such that $|g_j(z)| \leq M_j$ for $z \in D \subset \mathbb{C}$ and where the positive real bounds M_j are such that

$$\sum_{j=0}^{\infty} M_j < \infty.$$

Show then that the series $\sum_{j=0}^{\infty} g_j(z)$ converges absolutely for all $z \in D$. Hint, again, let $N_2 > N_1$ and control

$$\left|\sum_{j=0}^{N_2} |g_j(z)| - \sum_{j=0}^{N_1} |g_j(z)|\right| \leq \sum_{j=N_1+1}^{N_2} |g_j(z)|$$

3. We can expand the impact of Weierstrass' M-Test by showing that we also get the *uniform convergence* of the series

$$G(z) = \sum_{j=0}^{\infty} g_j(z).$$

By uniform convergence, we mean that *for all* $z \in D$, we can choose *one* value of $\epsilon > 0$ and *one* corresponding $N > 0$ such that

$$\left| G(z) - \sum_{j=0}^{n} g_j(z) \right| < \epsilon, \ n \geq N, \ z \in D.$$

Using your work from the previous problem, show that

(a) If the sequence $\{g_j(z)\}_{j=0}^{\infty}$ satisfies the conditions of Weierstrass' M-Test for $z \in D$, then the corresponding series is uniformly Cauchy, i.e. we can choose one ϵ and one corresponding N such that

$$\left| \sum_{j=0}^{N_2} g_j(z) - \sum_{j=0}^{N_1} g_j(z) \right| < \epsilon, \ N_2 > N_1 \geq N, \ z \in D.$$

(b) Given the existence of the limit of the partial sums, use

$$\left| G(z) - \sum_{j=0}^{N_1} g_j(z) \right| \leq \left| G(z) - \sum_{j=0}^{N_2} g_j(z) \right| + \left| \sum_{j=0}^{N_2} g_j(z) - \sum_{j=0}^{N_1} g_j(z) \right|$$

with $N_2 > N_1$ to show that you get the uniform convergence of the partial sums.

4. Suppose that the series

$$\sum_{j=0}^{\infty} a_j (z - z_0)^j$$

converges when $z = z_1$. Show that the series converges absolutely and uniformly for all z such that $|z - z_0| \leq R < |z_1 - z_0|$.

To do this, first recall that we must necessarily have that

$$\lim_{j \to \infty} a_j (z_1 - z_0)^j = 0.$$

Show that this implies there exists some $M > 0$ such that

$$\left| a_j (z_1 - z_0)^j \right| \leq M, \ j \geq 0.$$

Then use the Weierstrass' M-Test.

5. The Riemann Zeta function is given by the series

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}.$$

(a) Show that if $z = x + iy$ then

$$n^{-z} = n^{-x} e^{-iy \ln n}$$

(b) Show the Riemann Zeta function converges absolutely and uniformaly for $x \geq 1 + \epsilon$, $\epsilon > 0$.

(c) Show $\zeta(1) = \infty$.

(d) Do we have uniform convergence for $x > 1$?