1. (6 pts)

(1) Consider the initial value problem given by the system of differential equations:

$$\dot{x} = \begin{bmatrix} 27 & 14 \\ -42 & -22 \end{bmatrix} x, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -9 \\ 15 \end{bmatrix}.$$

Find the eigenvalues and eigenvectors of the matrix above. If the eigenvalue is repeated, then write this value in both entries for λ_i and use the second vector for any generalized eigenvectors for this particular case. For complex values use i to represent the imaginary part.

$$\lambda_1 = \_\_, \vec{v}_1 = \begin{bmatrix} \_\_ \\ \_\_ \end{bmatrix}, \text{ and } \lambda_2 = \_\_, \vec{v}_2 = \begin{bmatrix} \_\_ \\ \_\_ \end{bmatrix}$$

In the written work show the key steps required to find the eigenvalues and eigenvectors (not how a computer program finds them). Also, give the general real-valued solution to the linear system of differential equations.

(2) Find the unique solution to the initial value problem given above (where t is the independent variable) and in your written work show how you find the arbitrary constants (without a computer):

$x_1(t) =$ ____________________
$x_2(t) =$ ____________________

(3) In the written part of this problem include a reasonable Phase Portrait for this system of differential equations. The Phase Portrait needs to include all equilibria, show the position and direction of flow for all real eigenvectors or show the direction of flow with clockwise or counter-clockwise flow for complex eigenvalue problems. Include several typical trajectories (at least 4), including the one for the given initial condition. Describe the qualitative behavior, such as stable node, unstable spiral, etc.

Answer(s) submitted:

- `6`
- `(-3)(2)e^(6t) + (-3)(1)e^(-t)`
- `(-3)(-3)e^(6t) + (-3)(-2)e^(-t)`

(correct)

Correct Answers:

- `-1; -1; 2; 6; 2; -3`
- `3 * -1 e^(-1 t) + -3 * 2 e^(6 t)`
- `3 * 2 e^(-1 t) + -3 * -3 e^(6 t)`

2. (6 pts)

(1) Consider the initial value problem given by the system of differential equations:

$$\dot{y} = \begin{bmatrix} 6 & 1 \\ -1 & 4 \end{bmatrix} y, \quad \begin{matrix} y_1(0) \\ y_2(0) \end{matrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}.$$

Find the eigenvalues and eigenvectors of the matrix above. If the eigenvalue is repeated, then write this value in both entries for λ_i and use the second vector for any generalized eigenvectors for this particular case. For complex values use i to represent the imaginary part.

$$\lambda_1 = \_\_, \vec{v}_1 = \begin{bmatrix} \_\_ \\ \_\_ \end{bmatrix}, \lambda_2 = \_\_, \vec{v}_2 = \begin{bmatrix} \_\_ \\ \_\_ \end{bmatrix}$$

In the written work show the key steps required to find the eigenvalues and eigenvectors (not how a computer program finds them). Also, give the general real-valued solution to the linear system of differential equations.

(2) Find the unique solution to the initial value problem given above (where t is the independent variable) and in your written work show how you find the arbitrary constants (without a computer):

$y_1(t) =$ __________
$y_2(t) =$ __________

(3) In the written part of this problem include a reasonable Phase Portrait for this system of differential equations. The Phase Portrait needs to include all equilibria, show

the position and direction of flow for all real eigenvectors or show the direction of flow with clockwise or counter-clockwise flow for complex eigenvalue problems. Include several typical trajectories (at least 4), including the one for the given initial condition. Describe the qualitative behavior, such as stable node, unstable spiral, etc.

Answer(s) submitted:

- `5`
- `3(1)e^(5t) + (-1)te^(5t)`

(correct)

Correct Answers:

- `5; 1; -1; 5; 0; 1`
- `3*e^(5*t)+(-1)*t*e^(5*t); 3*[-e^(5*t)]+(-1)*(-t+1)*e^(5*t)`

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3. (6 pts)

(1) Consider the initial value problem given by the system of differential equations:

$$\dot{y} = \begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix} y, \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

Find the eigenvalues and eigenvectors of the matrix above. If the eigenvalue is repeated, then write this value in both entries for λ_i and use the second vector for any generalized eigenvectors for this particular case. For complex values use i to represent the imaginary part.

$$\lambda_1 = \_\_\_, \vec{v}_1 = \begin{bmatrix} \_\_\_ \\ \_\_\_ \end{bmatrix}, \text{ and } \lambda_2 = \_\_\_, \vec{v}_2 = \begin{bmatrix} \_\_\_ \\ \_\_\_ \end{bmatrix}$$

In the written work show the key steps required to find the eigenvalues and eigenvectors (not how a computer program finds them). Also, give the general real-valued solution to the linear system of differential equations.

(2) Find the unique solution to the initial value problem given above (where t is the independent variable) and in your written work show how you find the arbitrary constants (without a computer):

$y_1(t) =$ ________________
$y_2(t) =$ ________________

(3) In the written part of this problem include a reasonable Phase Portrait for this system of differential equations. The Phase Portrait needs to include all equilibria, show the position and direction of flow for all real eigenvectors or show the direction of flow with clockwise or counter-clockwise flow for complex eigenvalue problems. Include several typical trajectories (at least 4), including the one for the given initial condition. Describe the qualitative behavior, such as stable node, unstable spiral, etc.

Answer(s) submitted:

- `0`
- `2(2) + (-2)(1)e^t`
- `2(-3) + (-2)(-2)e^t`

(correct)

Correct Answers:

- `0; 2; -3; 1; -1; 2`
- `2 * 2 + 2 * -1 e^(1 t)`
- `2 * -3 + 2 * 2 e^(1 t)`

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4. (8 pts)

(1) Consider the first order linear system of differential equations with the real parameter α, which is given by:

$$\dot{y} = \begin{bmatrix} 1 & \alpha - 9 \\ 1 & 1 \end{bmatrix} y.$$

In the written work give the characteristic equation and eigenvalues in terms of α for this system.

(2) There are two critical values of α ($\alpha_1 < \alpha_2$), where the qualitative nature of the phase portrait changes. (For example, unstable node to unstable spiral or saddle node to stable node.) Determine values of α where the type of node changes for the origin.

$\alpha_1 =$ _____ and $\alpha_2 =$ _____.

In the written work, characterize the values of the eigenvalues for $\alpha < \alpha_1$, $\alpha = \alpha_1$, $\alpha \in (\alpha_1, \alpha_2)$, $\alpha = \alpha_2$, and $\alpha > \alpha_2$. State clearly the type of behavior (such as a STABLE NODE) for each of these five values or ranges of α.

(3) For $\alpha = 0$, the linear system above has the form

$$\dot{y} = \begin{bmatrix} 1 & -9 \\ 1 & 1 \end{bmatrix} y, \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix},$$

where some initial conditions are also specified. Find the eigenvalues and eigenvectors of the matrix above. If the eigenvalue is repeated, then write this value in both entries for λ_i and use the second vector for any generalized eigenvectors for this particular case. For complex values use i to represent the imaginary part.

$$\lambda_1 = \_\_,\ \vec{v}_1 = \begin{bmatrix} \_\_ \\ \_\_ \end{bmatrix}, \text{ and } \lambda_2 = \_\_,\ \vec{v}_2 = \begin{bmatrix} \_\_ \\ \_\_ \end{bmatrix}$$

In the written work show the key steps required to find the eigenvalues and eigenvectors (not how a computer program finds them). Also, give the general real-valued solution to the linear system of differential equations.

(4) Find the unique solution to the initial value problem given above (where t is the independent variable) and in your written work show how you find the arbitrary constants (without a computer):

$y_1(t) =$ __________________
$y_2(t) =$ __________________

(5) In the written part of this problem include a reasonable Phase Portrait for this system of differential equations. The Phase Portrait needs to include all equilibria, show the position and direction of flow for all real eigenvectors or show the direction of flow with clockwise or counter-clockwise flow for complex eigenvalue problems. Include several typical trajectories (at least 4), including the one for the given initial condition. Describe the qualitative behavior, such as stable node, unstable spiral, etc.

Answer(s) submitted:

- `9`
- `10`
- `1 + 3i`
- `1( -3sin(3t) )e^t + (-1)( 3cos(3t) )e^t`
- `1( cos(3t) )e^t + (-1)( sin(3t) )e^t`

(correct)

Correct Answers:

- `9`
- `10`
- `1+3i; 3i; 1; 1-3i; -3i; 1`
- `3 * e^(1 t)*(-1*cos(3 t) - 1*sin(3 t))`
- `e^(1 t)*(1*cos(3 t) + -1*sin(3 t))`

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5. (7 pts)

(1) Consider the nonhomogeneous system of differential equations given by:

$$\dot{x} = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} x + \begin{bmatrix} -7 \\ 11 \end{bmatrix}.$$

Find the equilibrium, $(x_{1e}, x_{2e})^T$, for this nonhomogeneous system of differential equations.

$x_{1e} =$ ____ and $x_{2e} =$ ____.

In the written work, show how you found your equilibria.

(2) In the written work, show that the change of variables, $y_i(t) = x_i(t) - x_{ie}, i = 1, 2$, transforms the nonhomogeneous system of differential equations into the homogeneous system of differential equations:

$$\dot{y} = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} y, \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -3 \\ 10 \end{bmatrix},$$

where some initial conditions are given. Find the eigenvalues and eigenvectors of the matrix above. If the eigenvalue is repeated, then write this value in both entries for λ_i and use the second vector for any generalized eigenvectors for this particular case. For complex values use i to represent the imaginary part.

$$\lambda_1 = \_\_,\ \vec{v}_1 = \begin{bmatrix} \_\_ \\ \_\_ \end{bmatrix}, \text{ and } \lambda_2 = \_\_,\ \vec{v}_2 = \begin{bmatrix} \_\_ \\ \_\_ \end{bmatrix}$$

In the written work show the key steps required to find the eigenvalues and eigenvectors (not how a computer program finds them). Also, give the general real-valued solution to the linear system of differential equations.

(3) Find the unique solution to the initial value problem in $y(t)$ given above (where t is the independent variable) and in your written work show how you find the arbitrary constants (without a computer):

$y_1(t) =$ __________________
$y_2(t) =$ __________________

(4) In the written part of this problem include a reasonable Phase Portrait for the original system of differential equations in $x(t)$. The Phase Portrait needs to include the equilibrium, show the position and direction of flow for all real eigenvectors or show the direction of flow with clockwise or counter-clockwise flow for complex eigenvalue problems. Include several typical trajectories (at least 4. Describe the qualitative behavior, such as stable node, unstable spiral, etc.

Answer(s) submitted:

- `1`
- `2`
- `i`
- `(3/2)(-2cos(t)) + (-11/2)(-2sin(t))`

(correct)

Correct Answers:

- `1`
- `2`
- `i; 3+i; -5; -i; 3-i; -5`
- `-2*[3*cos(1*t)-sin(1*t)]+3*[cos(1*t)+3*sin(1*t)]; -2*(-5)*cos(1*t)+3*(-5)*sin(1*t)`

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6. (11 pts) Consider the model given by the equations:

$$\frac{dX}{dt} = X(0.49 - 0.033X - 0.0358Y),$$

$$\frac{dY}{dt} = Y(0.4 - 0.017Y - 0.0379X)$$

(1) In the written work, give a brief explanation of each species' ecological behavior. Describe each term on the right hand side of the differential equations.

(2) Determine all possible equilibria.

Extinction: $(X_e, Y_e) = ($ ____, ____ $)$.

Only Species X: $(X_e, Y_e) = ($ ____, ____ $)$.

Only Species Y: $(X_e, Y_e) = ($ ____, ____ $)$.

Coexistence: $(X_e, Y_e) = ($ ____, ____ $)$.

In the written work, state or show how you obtained these values.

(3) In the written work, give your Jacobian matrix for this system. Write the Jacobian matrix with the numerical values at each of the four equilibria. Perform a linear stability analysis, giving eigenvalues and eigenvectors at each equilibrium. (If you have complex eigenvalues, then always list the one with negative imaginary part first). For your answers on the eigenvectors, you will either make one of the components equal to 1 or will be given which component of the eigenvector is 1.

Linearization at Extinction equilibrium: (List the smallest eigenvalue first).
$\lambda_1 =$ ____, Eigenvector, $\xi_1 =$ [____, ____]
$\lambda_2 =$ ____, Eigenvector, $\xi_2 =$ [____, ____]

Linearization at Only Species X: (List the smallest eigenvalue first).
$\lambda_1 =$ ____, Eigenvector, $\xi_1 =$ [____, ____]
$\lambda_2 =$ ____, Eigenvector, $\xi_2 =$ [1, ____]

Linearization at Only Species Y: (List the smallest eigenvalue first).
$\lambda_1 =$ ____, Eigenvector, $\xi_1 =$ [____, ____]
$\lambda_2 =$ ____, Eigenvector, $\xi_2 =$ [____, 1]

Linearization at Coexistence Equilibrium: (List the smallest eigenvalue first).
$\lambda_1 =$ ____, Eigenvector, $\xi_1 =$ [1, ____]

$\lambda_2 =$ ____, Eigenvector, $\xi_2 =$ [1, ____]

In the written work show the key steps required to find the eigenvalues and eigenvectors (not how a computer program finds them) for each of these cases. Describe the qualitative behavior, such as stable node, unstable spiral, etc., at each of these equilibria.

(4) In the written work, create a graph of the phase plane. Show all equilibria and draw the nullclines. Introduce arrows to show representative directions of the trajectories. (You must show at least 4 representative trajectories.) Also, include the direction of flow along each of the species axes. From this diagram explain whether this model exhibits 'Competitive Exclusion' or 'Coexistence' of the two species.

Answer(s) submitted:

- `0`
- `0`
- `.49/.033`
- `0`
- `0`
- `.4/.017`

- 7.5268
- 6.7490
- .4
- 0
- 1
- .49
- 1
- 0
- -0.4900
- 1
- 0
- -0.1628
- -0.3272 / 0.5316
- -0.4000
- 0
- 1
- -0.3524
- -0.0475999998000000/ 0.891764705700000
- -0.4525
- 0.2041 / 0.2695
- 0.0893
- -0.3377 / 0.2695

(correct)

Correct Answers:

- 0
- 0
- 14.8484848484848
- 0
- 0
- 23.5294117647059
- 7.52682767459978
- 6.7490135960393
- 0.4
- 0
- 1
- 0.49
- 1
- 0
- -0.49
- 1
- 0
- -0.162757575757576
- -0.615608254474974
- -0.4
- 0
- 1
- -0.35235294117647
- -0.053430079155673
- -0.45246582575595
- 0.757367276247658
- 0.0893472813614893
- -1.25336619437004