Homework 1
Linear Algebra
Math 524
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Section 1.A Problem 5: Show that $(\alpha+\beta)+\lambda = \alpha+(\beta+\lambda), \quad \forall\ \alpha,\ \beta,\ \lambda\ \in \mathbb{C}$.

Solution 1.A Problem 5: Let $\alpha = a+bi$, $\beta = c+di$, $\lambda = e+fi$

$$\begin{aligned}
(\alpha+\beta)+\lambda &= (a+bi+c+di)+e+fi && (1)\\
&= a+bi+c+di+e+fi && (2)\\
&= a+bi+(c+di+e+fi) && (3)\\
&= \boldsymbol{\alpha+(\beta+\lambda)} && (4)
\end{aligned}$$

Section 1.A Problem 6: Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda), \quad \forall\ \alpha,\ \beta,\ \lambda\ \in \mathbb{C}$.

Solution 1.A Problem 6: Let $\alpha = a+bi$, $\beta = c+di$, $\lambda = e+fi$

$$\begin{aligned}
(\alpha\beta)\lambda &= ((a+bi)(c+di))(e+fi) && (5)\\
&= ((ac-bd)+i(ad+bc))(e+fi) && (6)\\
&= (ac-bd)e+(ad+bc)ei+(ac-bd)fi-(ad+bc)f && (7)\\
&= ace-bde+adei+bcei+acfi-bdfi-adf-bcf && (8)
\end{aligned}$$

$$\begin{aligned}
(\alpha\beta)\lambda &= (a+bi)((c+di)(e+fi)) && (9)\\
&= (a+bi)((ce-df)+i(cf+ed)) && (10)\\
&= (ce-df)a+(cf+ed)ai+(ce-df)bi-(cf+ed)b && (11)\\
&= ace-adf+acfi+aedi+bcei-bdfi-bcf-bde && (12)
\end{aligned}$$

So $\boldsymbol{(\alpha\beta)\lambda = \alpha(\beta\lambda)}$

Section 1.B Problem 1: Prove that $-(-v) = v, \qquad \forall v \in V$

Solution 1.B Problem 1:

$$\begin{aligned}
-(-v)+(-v) &= 0 && (13)\\
-(-v)+(-v)+v &= v && (14)\\
-(-v)+0 &= v && (15)\\
\boldsymbol{-(-v)} &= \boldsymbol{v} && (16)
\end{aligned}$$

Section 1.B Problem 3: Suppose v, $w \in V$. Explain why $\exists! x \in V$ such that $v + 3x = w$

Solution 1.B Problem 3: Let v, $w \in V$. Suppose $\exists\ x_1$, $x_2 \in V$ such that $v + 3x_1 = w$ and $v + 3x_2 = w$

$$
\begin{aligned}
w = v + 3x_1 &= v + 3x_2 && (17)\\
v + 3x_1 &= v + 3x_2 && (18)\\
\boldsymbol{x_1} &= \boldsymbol{x_2} && (19)
\end{aligned}
$$

Because $v + 3x = w$ resembles a linear equation in terms of x, there is only a single input per each output, w.

Section 1.C Problem 10: Suppose U_1 and U_2 are subspaces of V. Prove that the intersection $U_1 \cap U_2$ is a subspace of V.

Solution 1.C Problem 10: Let $u_1, u_2 \in U_1 \cap U_2$

$$
\begin{aligned}
&\text{Additive Identity: } 0 \in U_1 \text{ and } 0 \in U_2 \text{, so } 0 \in U_1 \cap U_2 && (20)\\
&\text{Closed under Addition: } u_1 + u_2 \in U_1 \text{ and } u_1 + u_2 \in U_2 \text{, so } u_1 + u_2 \in U_1 \cap U_2 && (21)\\
&\text{Closed under Scalar Multi: } cu_1 \in U_1 \text{ and } cu_1 \in U_2 \text{, so } cu_1 \in U_1 \cap U_2 \quad \forall c \in \mathbb{C} && (22)
\end{aligned}
$$

Thus $U_1 \cap U_2$ is a subspace of V as it follows the given conditions

Section 1.C Problem 20: Suppose $U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$. Find a subspace W of $\mathbb{F}^4$ such that $\mathbb{F}^4 = U \oplus W$.

Solution 1.C Problem 20:

$$
\begin{aligned}
&\text{Let } (w, x, y, z) \in \mathbb{F}^4 && (23)\\
&\text{Let } (x - w, 0, 0, y - z) \in W && (24)\\
&\text{So } U \oplus W = (x, x, y, y) \in \mathbb{F}^4 && (25)
\end{aligned}
$$

$$
\begin{aligned}
&\text{Let } UW = (uw_1, uw_2, uw_3, uw_4) \in U \cap W. && (26)\\
&\text{Because } UW \in W,\ uw_2, uw_3 = 0 && (27)\\
&\text{Because } UW \in U,\ uw_1 = uw_2 = 0 \text{ and } uw_3 = uw_4 = 0 && (28)\\
&\text{Thus } U \cap W = \{\emptyset\} && (29)
\end{aligned}
$$

Because W and U meet the given conditions of direct sum, W is a subspace of $\mathbb{F}^4$ such that $\mathbb{F}^4 = U \oplus W$.