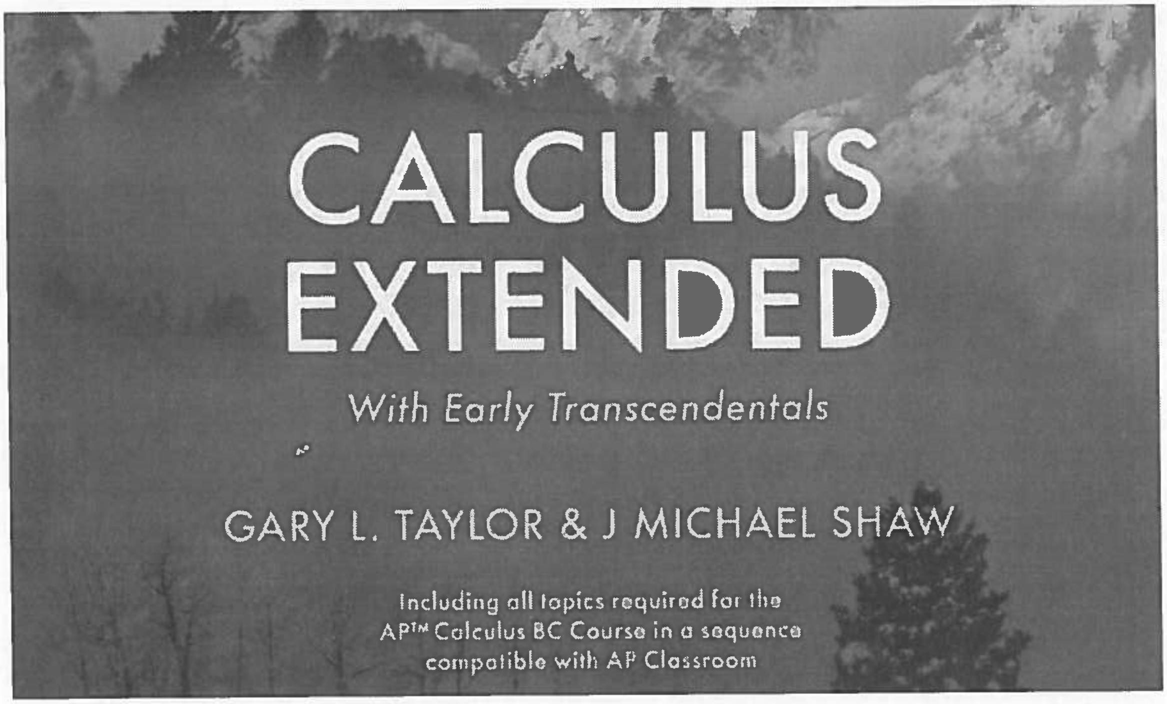

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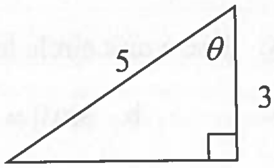
Lesson 1.1 Trig Review, Limits, Continuity

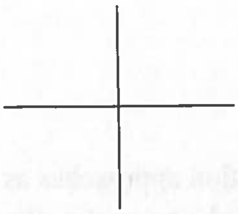
Trig Review

Example 1: Use the triangle at right to find



a. $\sin\theta$ b. $\tan\theta$ c. $\sec\theta$

Example 2: Find the following, if θ is an angle in standard position whose terminal side passes through the point $(-5,2)$.



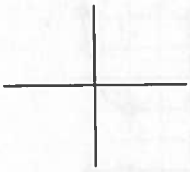
$\sin\theta =$

$\csc\theta =$

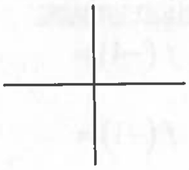
$\cot\theta =$

Example 3: Draw angles in standard position and make "reference triangles" to find:

a. $\cos 210° =$



b. $\tan 315° =$



Example 4: Since 2π radians $= 360°$, it follows that $\pi^R = 180°$, and the following common radian measures should be easy to think about in degrees. Convert each common radian measure to degrees.

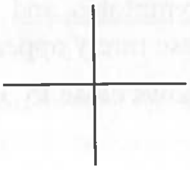
a. $\frac{\pi}{2} =$ b. $\frac{\pi}{4} =$ c. $\frac{\pi}{3} =$ d. $\frac{\pi}{6} =$

Example 5: Convert from radians to degrees or degrees to radians without using a calculator.

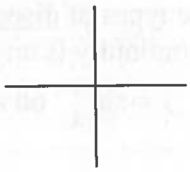
a. $\frac{5\pi}{4} =$ b. $270° =$ c. $-120° =$

Examples: Draw angles in standard position, and make "reference triangles" to find the following without using a calculator:

6. $\cos\left(\frac{-3\pi}{4}\right) =$

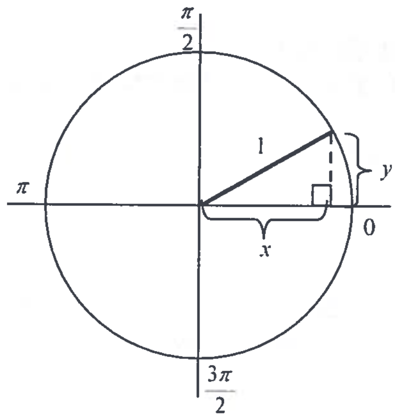


7. $\csc\frac{5\pi}{3} =$



A unit circle is created by letting $r=1$ when dealing with the circular trig functions.

Then, $\sin\theta = y$, $\cos\theta = x$, and $\tan\theta = \frac{y}{x}$.



Example 8: Use a unit circle to find:

a. $\sin\frac{\pi}{6} =$ b. $\sin 0 =$ c. $\cos 0 =$

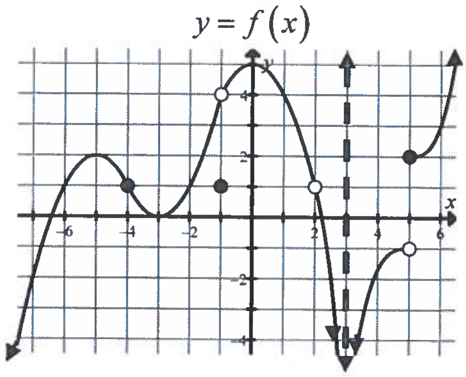
d. $\sin\frac{\pi}{2} =$ e. $\tan\left(\frac{-\pi}{2}\right) =$ f. $\tan\pi =$

g. $\csc\frac{3\pi}{2} =$ h. $\cos\frac{3\pi}{2} =$

Limits

Informally, a **limit is a y-value** which a function approaches as x approaches some value.
$\lim_{x\to c} f(x) = L$ means as x approaches c, $f(x)$ approaches the y-value of L.

Examples



limits:

9. $\lim_{x\to -4} f(x) =$
11. $\lim_{x\to -1} f(x) =$
13. $\lim_{x\to 2} f(x) =$
15. $\lim_{x\to 3} f(x) =$
17. $\lim_{x\to 5} f(x) =$

function values:

10. $f(-4) =$
12. $f(-1) =$
14. $f(2) =$
16. $f(3) =$
18. $f(5) =$

one-sided limits:

19. $\lim_{x\to 5^-} f(x) =$
20. $\lim_{x\to 5^+} f(x) =$

Continuity

Informally, a function is continuous where it can be drawn without lifting a pencil. Roughly, continuous means "connected."
Formally, a function is continuous where its limit and function value are the same.

Definition of Continuity

A function f is continuous at an x-value c if and only if $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$.

In this course, we will work with three types of discontinuities: holes, vertical asymptotes, and jumps (breaks). A fourth type of discontinuity is an oscillating discontinuity (these rarely appear). To investigate this fourth type, graph $y = \sin\frac{1}{x}$ on a calculator and look at windows close to $x=0$.

Example 21. List the x-values of the discontinuities of the function $y = f(x)$ graphed on page 2.

All discontinuities can be classified as removable or nonremovable.
Removable discontinuities occur when the function has a limit (holes in the graph).
Nonremovable discontinuities occur when the limit of the function does not exist (jumps, vertical asymptotes, or oscillations).

Example 22. Which of the discontinuities from Example 21 are removable?

At x-values where a function is continuous, limits can be found by **direct substitution**.

Examples:

23. $\lim_{x\to 3}\left(3x^2+2\right)=$

24 $\lim_{x\to 1}\dfrac{x^2+x}{x+1}=$

25. $\lim_{x\to \frac{\pi}{3}}\cos(2x)=$

For **piecewise functions**, one-sided limit evaluation is often necessary.

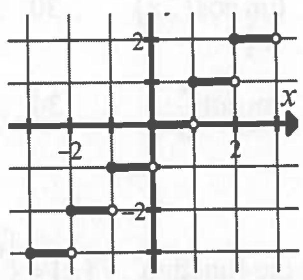
Examples:

26. If $f(x)=\begin{cases} 4-x, & x\le 1 \\ 4x-x^2, & x>1 \end{cases}$, $\lim_{x\to 1} f(x)=$

27. If $g(x)=\begin{cases} 3x-x^3, & x\le 1 \\ 2x^2-1, & x>1 \end{cases}$, $\lim_{x\to 1} g(x)=$

28. For this same g function, $\lim_{x\to -1} g(x)=$

Another function requiring one-sided limit analysis is a step function called the **Greatest Integer Function** also known as the **Floor Function**. $f(x)=\lfloor x\rfloor=$ the greatest integer less than or equal to x. The graph is shown at the right.



Examples: Find the following limits.

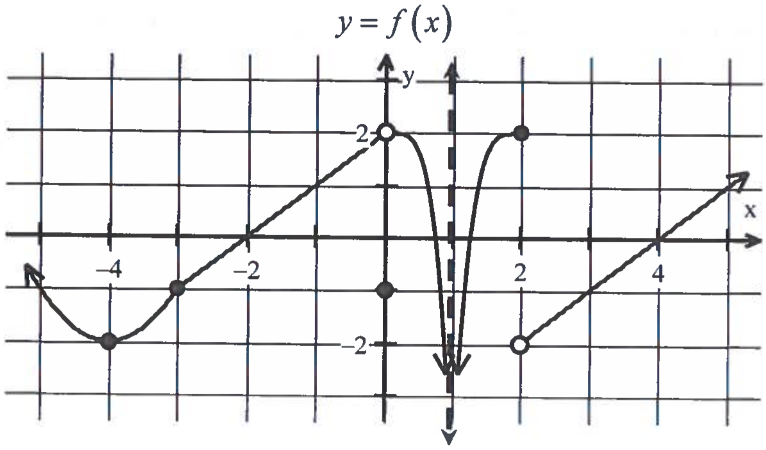
29. $\lim_{x\to \frac{1}{2}}\lfloor x\rfloor=$

30. $\lim_{x\to 1}\lfloor x\rfloor=$

31. $\lim_{x\to 5^-}\lfloor 2x-3\rfloor=$

Assignment 1.1

Use the appearance of the graph shown at the right to find the following limit and function values.



1. $\lim_{x\to -4} f(x)$ 2. $\lim_{x\to -1} f(x)$

3. $\lim_{x\to 0} f(x)$ 4. $f(0)$

5. $\lim_{x\to 0^-} f(x)$ 6. $\lim_{x\to 1} f(x)$

7. $f(1)$ 8. $\lim_{x\to 2} f(x)$

9. $\lim_{x\to 2^-} f(x)$ 10. $\lim_{x\to 2^+} f(x)$ 11. $f(2)$ 12. $\lim_{x\to 4} f(x)$ 13. $\lim_{x\to 4^+} f(x)$

Use the function $g(x) = \begin{cases} 2x-3, & x \le 0 \\ -x-3, & 0 < x \le 2 \\ 3x, & x > 2 \end{cases}$ for problems 14-20.

14. Sketch an accurate graph without using a calculator.

15. $\lim_{x\to 0} g(x) =$ 16. $\lim_{x\to 2} g(x) =$ 17. $\lim_{x\to 2^-} g(x) =$

18. $\lim_{x\to 2^+} g(x) =$ 19. $g(2) =$ 20 $\lim_{x\to -2} g(x) =$

Find each of the following limits without using a calculator. Simplify your answers.

21. $\lim_{x\to 0} (2x-5)$ 22. $\lim_{x\to -3} (x^2 - 5x + 4)$ 23. $\lim_{x\to 2} \dfrac{2x-5}{\sqrt{x+7}}$ 24. $\lim_{x\to -2} |3x+5|$

25. $\lim_{x\to 2} \dfrac{3x-6}{\sqrt{x+6}}$ 26. $\lim_{x\to \pi} \sin x$ 27. $\lim_{x\to \frac{\pi}{2}} \cos x$ 28. $\lim_{x\to \pi} \tan x$

29. $\lim_{x\to \frac{\pi}{2}} \cos(2x)$ 30. $\lim_{x\to 2} \cos \dfrac{\pi x}{3}$ 31. $\lim_{x\to 3} \sec \dfrac{\pi x}{4}$ 32. $\lim_{x\to 7} \csc \dfrac{\pi x}{6}$

33. $\lim_{x\to \pi} \cot \dfrac{x}{6}$ 34. $\lim_{x\to 5\pi} \cos \dfrac{x}{3}$

Use the function $f(x) = \begin{cases} 6x - 3x^3, & x \le 2 \\ 4x - x^4, & x > 2 \end{cases}$ for problems 35, 36.

35. $\lim_{x\to 2^+} f(x)$ 36. $\lim_{x\to 2} f(x)$

Use the function $g(x)=\begin{cases} 2\sin\dfrac{3x}{2}, & x\le\pi \\ \sec\dfrac{11x}{6}, & x>\pi \end{cases}$ for problems 37-39.

37. $\lim\limits_{x\to\pi^+} g(x)$ 38. $\lim\limits_{x\to\pi^-} g(x)$ 39. $\lim\limits_{x\to\pi} g(x)$

Use the functions $g(x)=3x^2-5x$ and $f(x)=\sqrt[3]{3x+5}$ for problems 40-42.

40. $\lim\limits_{x\to 2} g(x)$ 41. $\lim\limits_{x\to 1} f(x)$ 42. $\lim\limits_{x\to 3} f(g(x))$

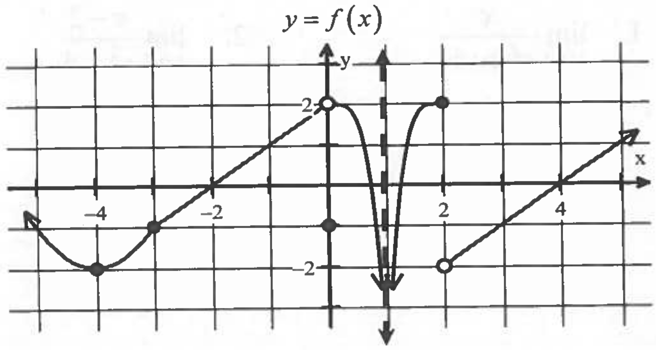
If $\lim\limits_{x\to 3} h(x)=5$ and $\lim\limits_{x\to 3} k(x)=3$ find the following limits.

43. $\lim\limits_{x\to 3}(h(x)+k(x))$ 44. $\lim\limits_{x\to 3}(h(x)k(x))$ 45. $\lim\limits_{x\to 3}\dfrac{h(x)}{k(x)}$

The symbol $\lfloor\ \rfloor$ is used to represent the Greatest Integer Function in the following problems. Find these limits without using a calculator or state that the limit does not exist.

46. $\lim\limits_{x\to 3^-}\lfloor x-1\rfloor$ 47. $\lim\limits_{x\to 3^+}\lfloor x-1\rfloor$ 48. $\lim\limits_{x\to 3}\lfloor x-1\rfloor$ 49. $\lim\limits_{x\to 3}\left\lfloor \frac{x}{2}-1\right\rfloor$ 50. $\lim\limits_{x\to 3^-}\lfloor 4x-1\rfloor$

51. Identify each x-value at which the function shown appears to be discontinuous and classify each as removable or nonremovable.

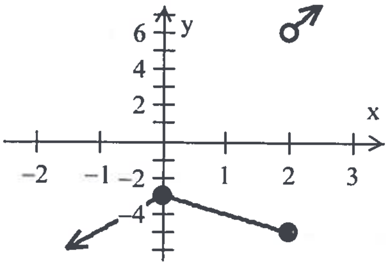


Find all discontinuities for the following functions and classify each as removable or nonremovable. Do not use a calculator.

52. $f(x)=\begin{cases} 3x^3+4x, & x\le -2 \\ x^4+16, & x>-2 \end{cases}$ 53. $f(x)=\begin{cases} 3x^3+4x, & x\le 2 \\ x^4+16, & x>2 \end{cases}$

54. $g(x)=\begin{cases} 2\sin\dfrac{\pi x}{2}, & x\le 1 \\ \cos\dfrac{\pi x}{3}, & x>1 \end{cases}$ 55. $g(x)=\begin{cases} \cos x, & x\le 0 \\ -x+1, & 0<x\le 2 \\ \sin\dfrac{\pi x}{2}, & x>2 \end{cases}$

56. $f(x)=\lfloor x+5\rfloor$ 57. $h(x)=\left\lfloor \frac{x}{3}\right\rfloor$

Selected Answers:
1. -2   2. 1   3. 2   5. 2   6. DNE or $-\infty$ 7. DNE   9. 2   10. -2   11. 2   12. 0
14.


15. -3   16 DNE   17. -5   19. -5   20. -7   21. -5
22. 28   23. $-\frac{1}{3}$ 24. 1   26. 0   27. 0   29. -1
30. $-\frac{1}{2}$ 31. $-\sqrt{2}$ 32. -2   34. $\frac{1}{2}$ 35. -8   36. DNE
38. -2   40. 2   41. 2   42. $\sqrt[3]{41}$ 43. 8   44. 15
45. $\frac{5}{3}$ 46. 1   48. DNE   50. 10
51. $x=0$ remov., $x=1$ nonrem., $x=2$ nonrem.   52. $x=-2$ nonrem.   53. no discontinuities
54. $x=1$ nonrem.   56. nonremovable discontinuity at every integer

Lesson 1.2 More Limits, Continuity, Intermediate Value Theorem, Graphing Adjustments

If direct substitution does not give an answer to a limit problem because an indeterminate form is obtained (usually $\frac{0}{0}$), use algebraic techniques to change the form of the limit.

Examples:

1. $\lim_{x\to 0}\frac{x}{x(x+1)}$

2. $\lim_{x\to 2}\frac{x-2}{x^2-4}$

3. $\lim_{x\to 1}\frac{x^3-1}{x-1}$

4. $\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}$

5. $\lim_{x\to 2^+}\frac{x}{x-2}$

6. Discuss the continuity of

$$f(x)=\begin{cases}\dfrac{x^2-2x-3}{x-3}, & x\neq 3\\ 5, & x=3\end{cases}$$

7. If $g(x)=\begin{cases}3x^2+a, & x>2\\ x-3, & x\leq 2\end{cases}$ is a continuous function, find the value of a.

8. Use a calculator to fill in the tables to help find these limits if $f(x)=\dfrac{x^3-3x^2+x+2}{x^3-2x^2-x+2}$.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$						

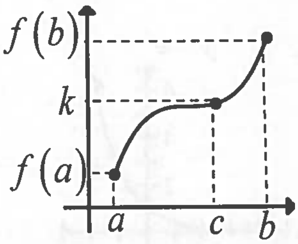
a. $\lim\limits_{x\to 1} f(x)=$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

b. $\lim\limits_{x\to 2} f(x)=$

c. Can a table like these be used to find limits with absolute certainty?

Intermediate Value Theorem



If f is continuous on $[a,b]$ and k is any y-value between $f(a)$ and $f(b)$, then there is at least one x-value c between a and b such that $f(c)=k$.
In other words, f takes on every y-value between $f(a)$ and $f(b)$.

Examples:

9. Does the Intermediate Value Theorem guarantee a c-value on the given interval.

a. $f(x)=x^2-x,$
$f(c)=12,\quad [0,5]$

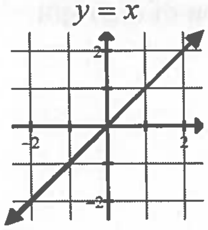
b. $g(x)=\dfrac{x^2-4}{x-2},$
$g(c)=4,\quad [0,3]$

10. Find the value of c in Example 9a.

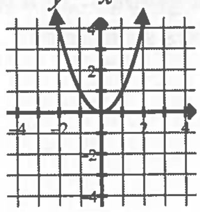
Parent Graphs

These graphs occur so frequently in this course that it would be worth your time to learn (memorize) them.

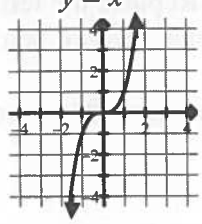
$y=x$



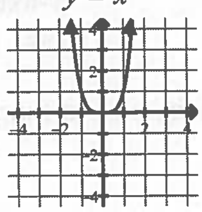
$y=x^2$



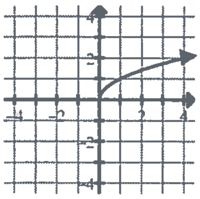
$y=x^3$



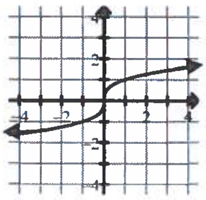
$y=x^4$



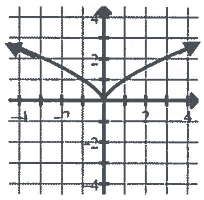
$y=\sqrt{x}$



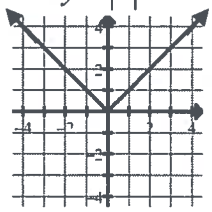
$y=\sqrt[3]{x}$



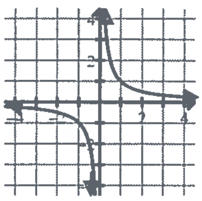
$y=x^{\frac{2}{3}}$



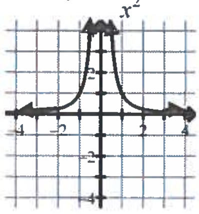
$y=|x|$



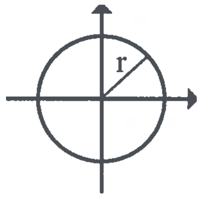
$y=\frac{1}{x}$



$y=\frac{1}{x^2}$



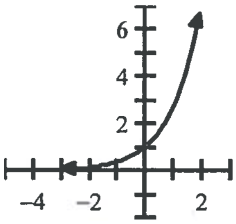
$x^2+y^2=r^2$



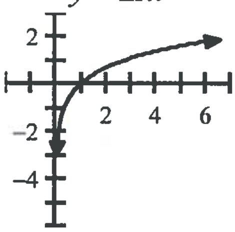
$y=\sin x$

$y=\cos x$

$y=e^x$



$y=\ln x$

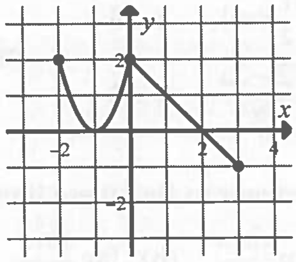


<u>Graphing Adjustments to $y=f(x)$</u>

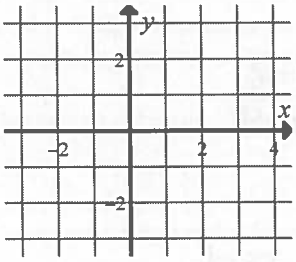
1. $y=-f(x)$ — reflect across the x-axis
2. $y=f(-x)$ — reflect across the y-axis
3. $y=f(x)+d$ — shift up if $d>0$, shift down if $d<0$
4. $y=f(x+c)$ — shift left if $c>0$, shift right if $c<0$
5. $y=a\cdot f(x)$ — vertical stretch if $a>1$, vertical squeeze if $a<1$
   (assumes a is positive, if a is negative a reflection is needed)
6. $y=f(b\cdot x)$ — horizontal squeeze if $b>1$, horizontal stretch if $b<1$
   (assumes b is positive, if b is negative a reflection is needed)
7. $y=|f(x)|$ — reflect all points below the x-axis across the x-axis. Leave points above the x-axis alone.
8. $y=f(|x|)$ — eliminate completely all points left of the y-axis. Leave points right of the y-axis alone. Replace the left half of the graph with a reflection of the right half. Your graph should then show y-axis symmetry.

<u>Note:</u> Adjustments to functions always produce functions.

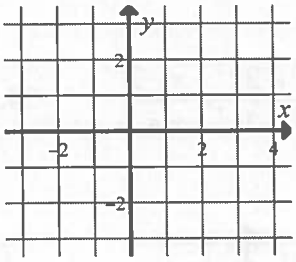
Examples: Use the graph of $y = f(x)$ shown to sketch the following:



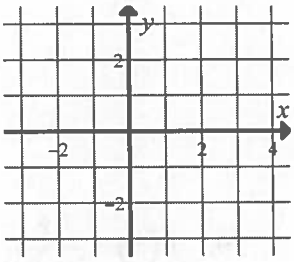
11. $y = f(x+2)$



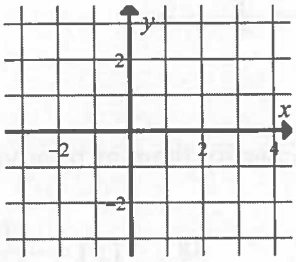
12. $y = -f(x)+2$



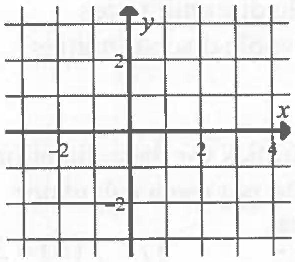
13. $y = \frac{1}{2} f(-x)$



14. $y = \left| f(2x) \right|$



15. $y = f(|x|)$



Assignment 1.2

Find the indicated limits without using a calculator. **Show steps using correct <u>limit symbolism</u>!**

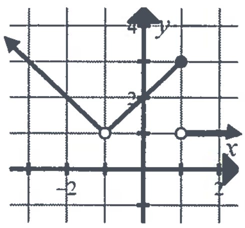
1. $\lim\limits_{x \to 1} \dfrac{x^2-1}{x-1}$
2. $\lim\limits_{x \to -1} \dfrac{2x^2-x-3}{x+1}$
3. $\lim\limits_{x \to -1} \dfrac{x^2-1}{x-1}$
4. $\lim\limits_{x \to -1} \dfrac{x^3+1}{x+1}$
5. $\lim\limits_{x \to -1} \dfrac{x}{x^2+1}$
6. $\lim\limits_{x \to 5^+} \dfrac{x-5}{x^2-25}$
7. $\lim\limits_{x \to -5} \dfrac{x-5}{x^2-25}$
8. $\lim\limits_{x \to 2} \dfrac{2-x}{x^2-4}$
9. $\lim\limits_{x \to 3} \dfrac{x^2-9}{x^2-6x+9}$
10. $\lim\limits_{x \to -2} \sqrt[3]{x^2+4}$
11. $\lim\limits_{x \to 0} \dfrac{x}{x-1}$
12. $\lim\limits_{x \to 2^-} \dfrac{1}{x^2-4}$
13. $\lim\limits_{x \to 1} \dfrac{x}{x^2+1}$
14. $\lim\limits_{x \to 4} \dfrac{\sqrt{x}-2}{x-4}$
15. $\lim\limits_{x \to 3} \dfrac{x-3}{\sqrt{x+1}-2}$
16. $\lim\limits_{x \to 0^-} \dfrac{|x|}{x}$
17. $\lim\limits_{x \to 0} \dfrac{|x|}{x}$
18. $\lim\limits_{x \to \frac{\pi}{2}} \sin x$
19. $\lim\limits_{x \to \pi} \sec x$
20. $\lim\limits_{x \to \frac{\pi}{2}} \cos(3x)$
21. $\lim\limits_{x \to 5} \csc \dfrac{\pi x}{6}$
22. $\lim\limits_{x \to 3^+} \lfloor x-1 \rfloor$
23. $\lim\limits_{x \to 3^-} \lfloor x-1 \rfloor$
24. $\lim\limits_{x \to 3} \lfloor x-1 \rfloor$
25. $\lim\limits_{x \to 2} \lfloor x+6 \rfloor$
26. $\lim\limits_{x \to 3} \left\lfloor \frac{x}{2} \right\rfloor$
27. $\lim\limits_{x \to 5^-} \lfloor 2x-3 \rfloor$
28. $\lim\limits_{x \to 5} \dfrac{\frac{1}{x}-\frac{1}{5}}{x-5}$

29. $\lim_{x\to 3}\begin{cases}\frac{1}{2}x+1, & x\le 3\\ \frac{12-2x}{3}, & x>3\end{cases}$ 30. $\lim_{x\to 1}\begin{cases}x^2+1, & x<1\\ x^3+1, & x\ge 1\end{cases}$ 31. $\lim_{x\to 2}\begin{cases}x-2, & x\le 0\\ x+2, & x>0\end{cases}$

Use a calculator to find these limits.

32. (a) $\lim_{x\to 1}\frac{\sin x}{6x}$ (b) $\lim_{x\to 0}\frac{\sin x}{6x}$ 33. $\lim_{x\to -2}\frac{x^3+2x^2-x-2}{x^3+2x^2+x+2}$ 34. $\lim_{x\to 2}\frac{|2-x|}{25x-50}$

35. a. $h(-1)$ b. $h(1)$
c. $\lim_{x\to -1}h(x)$ d. $\lim_{x\to 1^-}h(x)$
e. $\lim_{x\to 1^+}h(x)$ f. $\lim_{x\to 1}h(x)$
g. removable discontinuities
h. nonremovable discontinuities



$h(x)=\begin{cases}-x, & x<-1\\ x+2, & -1<x\le 1\\ 1, & x>1\end{cases}$

Find all discontinuities for these functions and classify them as removable or nonremovable. Do not use a calculator.

36. $f(x)=\frac{2x-3}{x+1}$ 37. $f(x)=2x-3$ 38. $f(x)=\frac{1}{x^2-9}$ 39. $f(x)=\frac{x}{x^2+x}$

40. $f(x)=\frac{x^2-9}{x+3}$ 41. $f(x)=\begin{cases}x^2, & x\le 0\\ x, & x>0\end{cases}$ 42. $f(x)=\begin{cases}x-3, & x\le 1\\ x, & x>1\end{cases}$

43. $f(x)=\begin{cases}2x-5, & x>3\\ x^2-8, & x<3\end{cases}$ 44. $f(x)=\lfloor x-1\rfloor$ 45. $f(x)=\left\lfloor \frac{x}{2}\right\rfloor$

Use a calculator to find all discontinuities for these functions and classify them as removable or nonremovable.

46. $f(x)=\frac{10x}{6x^3-31x^2+23x+20}$ 47. $f(x)=\frac{x}{x^3+4x}$ 48. $f(x)=\left\lfloor \frac{x}{4}\right\rfloor$

49. If the function $f(x)=\begin{cases}x^3, & x\le 2\\ ax^2, & x>2\end{cases}$ is continuous, find the value of a.

50. Find the values of a and b so that $f(x)=\begin{cases}x-1, & x\le -1\\ ax+b, & -1<x<1\\ 2x+1, & x\ge 1\end{cases}$ is continuous.

Determine whether the Intermediate Value Theorem would guarantee a c-value on the given interval.

51. $f(x)=x^2+x-1$, $f(c)=11$, $[0,5]$

52. $f(x)=\frac{x}{x-1}$, $f(c)=1$, $[0,2]$

53. $f(x)=|x|,$ $\qquad f(c)=3,$ $\qquad [-4,1]$

54. $f(x)=\begin{cases} x, & x\le 1 \\ 3, & x>1 \end{cases},$ $\qquad f(c)=2,$ $\qquad [0,4]$

55. $f(x)=\dfrac{x^2+x}{x-1},$ $\qquad f(c)=6,$ $\qquad \left[\frac{5}{2},4\right]$

56. Find the c-value in Problem 51.
57. Find the c-value in Problem 53.
58 Find the c-value in Problem 55.

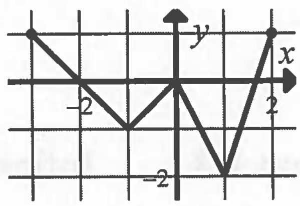
59. Find an equation of the line which intersects the graph of $f(x)=\begin{cases} x^2+1, & x<1 \\ x^3+1, & x\ge 1 \end{cases}$ when $x=-2$ and again when $x=2$.

60. Use the <u>parent graph</u> of $y=\sqrt{x}$ to graph the following.
a. $y=\sqrt{x}+2$ $\quad$ b. $y=-\sqrt{x}$ $\quad$ c. $y=2\sqrt{x}$

61. Use the parent graph of $y=e^x$ to graph the following.
a. $y=e^x+2$ $\quad$ b. $y=e^{-x}$

62. Use the parent graph of $y=\ln x$ to graph the following.
a. $y=|\ln x|$ $\quad$ b. $y=\ln|x|$

Use the graph of $y=f(x)$ to draw **accurate** graphs of the following.



63. $y=-f(x)$ $\quad$ 64. $y=|f(x)|$ $\quad$ 65. $y=f(|x|)$

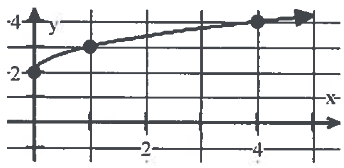
66. $y=f(x)-1$ $\quad$ 67. $y=\frac{1}{2}f(x)$ $\quad$ 68. $y=f\left(\frac{1}{2}x\right)$

Selected Answers:

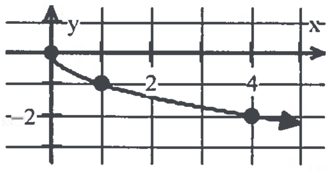
1. 2 $\quad$ 2. -5 $\quad$ 3. 0 $\quad$ 4. 3 $\quad$ 6. $\frac{1}{10}$ $\quad$ 7. DNE $\quad$ 8. $-\frac{1}{4}$ $\quad$ 9. DNE $\quad$ 10. 2 $\quad$ 11. 0
12. DNE or $-\infty$ $\quad$ 14. $\frac{1}{4}$ $\quad$ 15. 4 $\quad$ 16. -1 $\quad$ 18. 1 $\quad$ 19. -1 $\quad$ 21. 2 $\quad$ 22. 2 $\quad$ 24. DNE $\quad$ 26. 1
27. 6 $\quad$ 28. $-\frac{1}{25}$ $\quad$ 29. DNE $\quad$ 30. 2 $\quad$ 32a. .140 $\quad$ b. .166 or .167 $\quad$ 34. DNE $\quad$ 35a. DNE
35b. 3 $\quad$ c. 1 $\quad$ e. 1 $\quad$ g. $x=-1$ $\quad$ 36. $x=-1$ nonremovable $\quad$ 38. $x=\pm 3$ nonremovable
39. $x=0$ removable, $x=-1$ nonremovable $\quad$ 40. $x=-3$ removable
42. $x=1$ nonremovable $\quad$ 44. nonremovable discontinuity at every integer
45. nonremovable discontinuity at every even integer $\quad$ 47. $x=0$ removable $\quad$ 49. $a=2$
50. $a=\frac{5}{2}, b=\frac{1}{2}$ $\quad$ 51. yes $\quad$ 52. no $\quad$ 53. yes $\quad$ 54. no $\quad$ 56. $c=3$ $\quad$ 57. $c=-3$
59. $y-5=x+2$ or $y-9=x-2$ or $y=x+7$

Selected Answers continued:

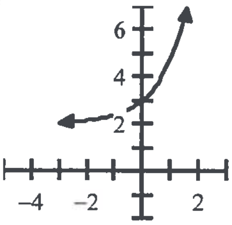
60a.



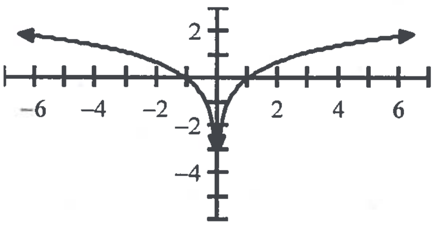
60b.



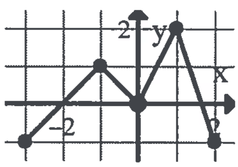
61a.



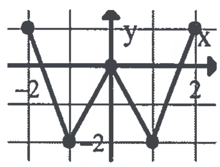
62b.



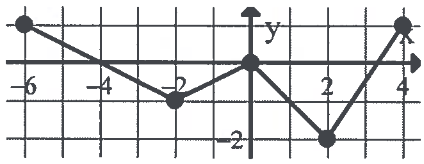
63.



65.



68.



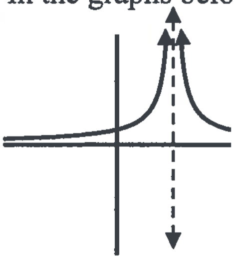
Lesson 1.3 Infinite Limits, Limits at Infinity, Curve Sketching

Review:

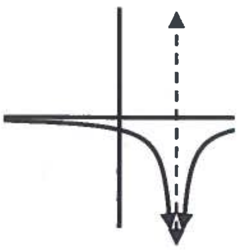
The graph of the function $f(x)=\dfrac{x-1}{(x-1)(x-2)^2(x-4)^3}$ has a **hole** at $x=$ ______,

an **even vertical asymptote** at $x=$ ______,
and an **odd vertical asymptote** at $x=$ ______.

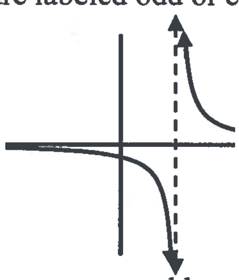
In the graphs below, the vertical asymptotes are labeled odd or even.



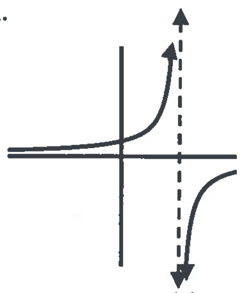
even



even



odd



odd

Infinite Limits

You have seen examples where a limit does not exist at a vertical asymptote. Such non-existent limits can be expressed as infinite limits if the vertical asymptote is even or if you are finding one-sided limits. We will write $\lim_{x\to c} f(x)=\infty$ or $\lim_{x\to c} f(x)=-\infty$.

The examples below make use of your knowledge of even and odd vertical asymptotes as well as holes.

Examples:

1. $\lim_{x\to 2^+} \frac{x+3}{x-2} =$
2. $\lim_{x\to 2^-} \frac{x+3}{x-2} =$
3. $\lim_{x\to 1^+} \frac{x-2}{(x-1)^2} =$
4. $\lim_{x\to 1^-} \frac{x-2}{(x-1)^2} =$
5. $\lim_{x\to 2} \frac{x^2-4}{x-2} =$
6. $\lim_{x\to 0^+} \ln x =$

Limits at Infinity

If the graph of a function $f(x)$ approaches a horizontal asymptote to the left and/or the right, $f(x)$ is said to have a limit at infinity. If the asymptote is $y = L$ then $\lim_{x\to\infty} f(x)=L$. In other words, limits at infinity give us end behaviors for graphs of functions. For "large" values of x, the highest degree terms in the numerator and denominator dominate the other terms and are the only terms you need to consider.

Review Examples: Find the horizontal asymptotes.

7. $f(x)=\frac{5x^4-3x^2+2}{10x^4+3}$
8. $g(x)=\frac{5x^4-3x^2+2}{10x^5+3}$
9. $h(x)=\frac{5x^4-3x^2+2}{10x^3+3}$

Examples: Find the following limits.

10. $\lim_{x\to\infty} \frac{5x^4-3x^2+2}{10x^4+3} =$
11. $\lim_{x\to -\infty} \frac{5x^4-3x^2+2}{10x^5+3} =$
12. $\lim_{x\to\infty} \frac{5x^4-3x^2+2}{10x^3+3} =$
13. $\lim_{x\to\infty} \frac{(2x+3)(x-1)^2}{(x+2)(3x-1)^2} =$

Rational functions like those above have at most one horizontal asymptote, so the limit is the same whether x approaches ∞ or $-\infty$. However, radical functions frequently have two horizontal asymptotes.

Examples: Find these limits.

14. $\lim_{x\to\infty} \frac{\sqrt{4x^2-3}}{x} =$
15. $\lim_{x\to -\infty} \frac{\sqrt{4x^2-3}}{x} =$

Use the end behavior of $y=e^x$ to find these limits.

16. $\lim\limits_{x\to-\infty} e^x$ 17. $\lim\limits_{x\to\infty} e^x$ 18. $\lim\limits_{x\to\infty}\left(e^{-x}+\dfrac{2x^2-x}{x^2+3}\right)$ 19. $\lim\limits_{x\to\infty}\left(\dfrac{2x^2+4x}{e^x+3x^2}\right)$

Curve Sketching

Examples: For Examples 20-22 give the domain, reduce the function, find vertical asymptotes, holes, and end behavior.

20. $f(x)=\dfrac{x+2}{x^2-2x}$

Do:

V.A.:

E.B.:

21. $g(x)=\dfrac{2x^3}{(x+3)^2}$

Do:

V.A.:

E.B.:

22. $h(x)=\dfrac{x^2+2x-8}{x^2-4}$

$h(x)=\dfrac{(x+4)(x-2)}{(x+2)(x-2)}$

Do:

$h_{red}(x)=$

V.A.:

Hole:

E.B.:

Curve Sketching Recipe:

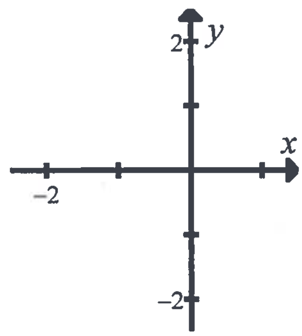
1. Give the domain (watch for denom. restrictions, radical restrictions).
2. Reduce $f(x)$. Oftentimes, you must factor before you can reduce.
3. Find vertical asymptotes (denom. restr. after reducing) and holes (denom. restr. which reduce away).
4. Give x- and y-intercepts.
5. Find the end behavior (horizontal asymptotes or other) using highest degree terms of num. and denom.
6. (if needed) Find a starting point.
7. Graph.

Examples: Follow the **Curve Sketching Recipe** to graph.

23. $f(x)=x(x-1)(x+2)^2$

Do: x-int.:

y-int.: E.B.:



24. $g(x)=\dfrac{x(x-1)^2(x+3)^3}{x^2(x-1)(x-3)^2}$

Do.:

$g_{red}(x)=$

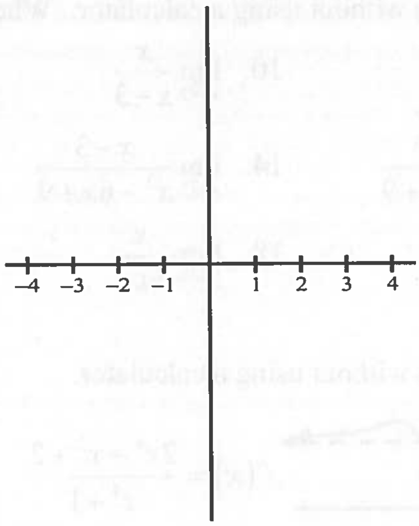
V.A.:

Holes:

x-int.:

y-int.:

E.B.:



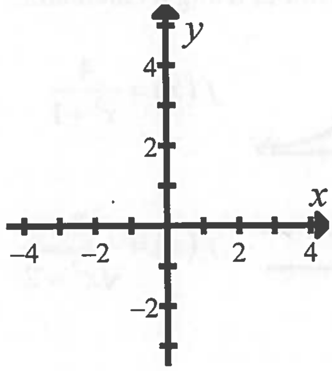
25. $y=\dfrac{x+1}{\sqrt{x^2-4}}$

Do.: V.A.:

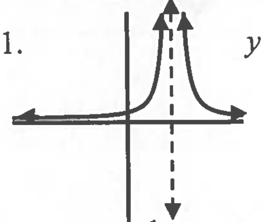
Holes: x-int.:

y-int.: E.B.:

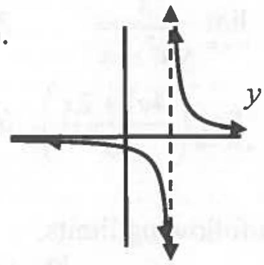
Starting Point:



Assignment 1.3
Use the graphs to find these limits (answer ∞ or -∞).

1.  $y=\dfrac{1}{(x-2)^2}$

a. $\lim\limits_{x\to 2^-}\dfrac{1}{(x-2)^2}$ b. $\lim\limits_{x\to 2^+}\dfrac{1}{(x-2)^2}$

2.  $y=\dfrac{1}{x-2}$

a. $\lim\limits_{x\to 2^-}\dfrac{1}{x-2}$ b. $\lim\limits_{x\to 2^+}\dfrac{1}{x-2}$

Find the vertical asymptotes, if any, without using a calculator, and classify each of them as even or odd.

3. $f(x)=\dfrac{1}{x^2}$ 4. $f(x)=\dfrac{x}{x(x-1)^2}$ 5. $f(x)=\dfrac{x}{x^2-4}$ 6. $f(x)=\dfrac{x}{x^2-x-2}$

7. $g(x)=\dfrac{x^3-1}{x-1}$ 8. $g(x)=\csc(\pi x)$

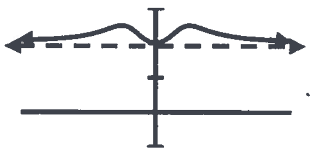
Find these limits without using a calculator. Whenever appropriate answer ∞ or $-\infty$.

9. $\lim\limits_{x\to 3^-} \dfrac{x}{x-3}$ 10. $\lim\limits_{x\to 3} \dfrac{x}{x-3}$ 11. $\lim\limits_{x\to 1^+} \dfrac{x}{x^2-x}$ 12. $\lim\limits_{x\to 0} \dfrac{x}{x^2-x}$

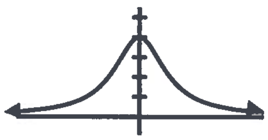
13. $\lim\limits_{x\to 3} \dfrac{x+3}{x^2-6x+9}$ 14. $\lim\limits_{x\to 3} \dfrac{x-3}{x^2-6x+9}$ 15. $\lim\limits_{x\to 0} \dfrac{x^2-2x}{x^3}$ 16. $\lim\limits_{x\to 0^-} \left(\dfrac{1}{x}-10\right)$

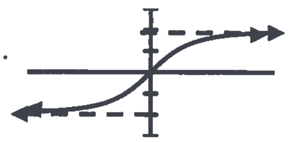
17. $\lim\limits_{x\to \frac{\pi}{2}^+} \dfrac{3}{\cos x}$ 18. $\lim\limits_{x\to \pi} \dfrac{x}{\csc x}$

Find these limits without using a calculator.

19.  $f(x)=\dfrac{2x^4+x^2+2}{x^4+1}$ a. $\lim\limits_{x\to\infty} f(x)$ b. $\lim\limits_{x\to-\infty} f(x)$

Find these limits without using a calculator.

20.  $f(x)=\dfrac{4}{x^2+1}$ a. $\lim\limits_{x\to\infty} f(x)$ b. $\lim\limits_{x\to-\infty} f(x)$

21.  $f(x)=\dfrac{2x}{\sqrt{x^2+2}}$ a. $\lim\limits_{x\to\infty} f(x)$ b. $\lim\limits_{x\to-\infty} f(x)$

22. $\lim\limits_{x\to \infty} \dfrac{2x+5}{3x-4}$ 23. $\lim\limits_{x\to -\infty} \dfrac{1-5x^3}{10x^3-x^2}$ 24. $\lim\limits_{x\to \infty} \dfrac{x(2x-1)^2}{3x(x-3)^2}$ 25. $\lim\limits_{x\to -\infty} \dfrac{4x^2+3}{2x}$

26. $\lim\limits_{x\to -\infty} \dfrac{x}{\sqrt{x^2+x}}$ 27. $\lim\limits_{x\to \infty} \dfrac{x}{\sqrt{x^2+x}}$ 28. $\lim\limits_{x\to -\infty} \dfrac{2-x}{\sqrt{x^2-3}}$ 29. $\lim\limits_{x\to -\infty} \dfrac{2x^2-2}{\sqrt{x^4}}$

30. $\lim\limits_{x\to \infty} \dfrac{\sin x}{x+1}$ 31. $\lim\limits_{x\to -\infty} \left(\dfrac{4e^x+2x}{3x}\right)$ 32. $\lim\limits_{x\to \infty} \left(x^5e^x+2\right)$

Use a calculator to find the following limits.

33. $\lim\limits_{x\to \infty} \left(x^5e^{-x}+2\right)$ 34. $\lim\limits_{x\to -\infty} \dfrac{|2x+5|}{x-7}$

Follow the **Curve Sketching Recipe** to graph each function without using a calculator. List intercepts, asymptotes, holes, end behavior, etc. Show accurate graphs.

35. $f(x)=(x+2)(x-1)^2$ 36. $f(x)=\dfrac{x-2}{x+2}$ 37. $f(x)=\dfrac{x(x-1)^3}{x^2(x-1)}$

38. $f(x)=\dfrac{1}{\sqrt{x}}$ 39. $f(x)=\dfrac{-x}{\sqrt{x^2-1}}$

40. If $f(x)=\begin{cases} 2ax-6, & x\le 2 \\ x^2+a, & x>2 \end{cases}$ is a continuous function, find the value of a.

Use a calculator to find all discontinuities.

41. $f(x)=\begin{cases} \dfrac{\cos x-1}{x}, & x<0 \\ 5x, & x>0 \end{cases}$

42. $f(x)=\dfrac{x^2-4}{x^3-2x^2-2x+4}$

Does the Intermediate Value Theorem guarantee a value of c in the given interval? If so, find the c-value. If not, explain why not.

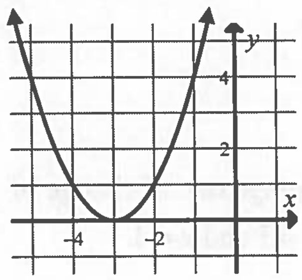
43. $f(x)=\dfrac{x^2-x}{x}$, $f(c)=-1$ on $[-2,2]$

44. $f(x)=x^2-x$, $f(c)=-1$ on $[-2,2]$

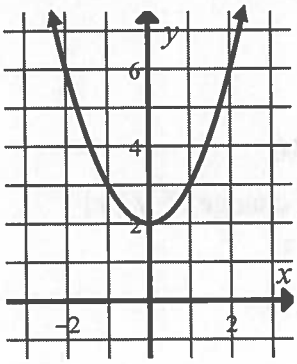
45. $f(x)=x^2-x$, $f(c)=5$ on $[-2,2]$

46. Use the parent graph of $y=x^2$ to determine an equation for each graph.

a.



b.



Selected Answers:

3. $x=0$ (even) 5. $x=\pm 2$ (both odd) 7. none 8. 0, ± 1, ± 2, ± 3,... (all odd) 9. $-\infty$
10. DNE 11. ∞ 12. -1 14. DNE 15. $-\infty$ 16. $-\infty$ 18. 0 19a. 2 b. 2 21a. 2 b. -2
22. $\frac{2}{3}$ 23. $-\frac{1}{2}$ 25. $-\infty$ or DNE 26. -1 27. 1 30. 0 31. $\frac{2}{3}$ 33. 2

35. x-int: $(-2,0)$ (odd)
$(1,0)$ (even)
y-int: $(0,2)$
EB: like $y=x^3$

36. Do: $x\ne -2$
VA: $x=-2$ (odd)
x-int: $(2,0)$
y-int: $(0,-1)$
EB: HA $y=1$

40. $a=\frac{10}{3}$ 41. $x=0$
43. No, f is disc. at $x=0$
46a. $y=(x+3)^2$

Lesson 1.4 Rate of Change, Squeeze Theorem, Limits of Compositions of Discontinuous Functions

Rate of Change:
Another meaning for slope is rate of change. In this course there will be two situations where you will use slopes (rates of change).

1. **Average Rate of Change** This is the slope between two points on a graph or a rate of change between two points in time. It is found algebraically using a method from previous courses. $\text{AROC} = m = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$ or $\text{AROC} = \dfrac{\Delta y}{\Delta t} = \dfrac{y_2 - y_1}{t_2 - t_1}$

2. **Instantaneous Rate of Change** This is the slope at a single point on a curve or a rate of change at a single instant in time. It can be approximated using one or more average rates of change or found exactly using a Calculus technique that will be shown in the next unit.

Examples:
1. Given $f(x) = x^3 - 2x^2 - 10$

   a. find the average rate of change of $f(x)$ between $x = 2$ and $x = 3$.

   b. find the average rate of change of $f(x)$ between $x = 3$ and $x = 4$.

   c. Which of these is likely to be a better estimate of the instantaneous rate of change at $x = 2.4$?

2. The data in the table shows the mileage from the start of a four hour car trip recorded at one hour intervals. Assume the car continued in the same straight line.

time in hours	0	1	2	3	4
miles from start	0	55	120	180	250

   a. Find the average rate of change (average speed) of the car for the final two hours of the trip.

   b. Estimate the instantaneous speed at the 1.5 hour instant.

   c. During which hour does the data suggest the car reached the greatest instantaneous speed?

Squeeze Theorem (Sandwich Theorem)

If $f(x) \le g(x) \le h(x)$ for all $x \ne c$ in some interval containing c and if $\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x)$,

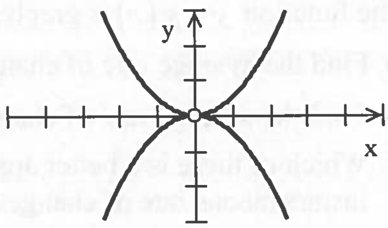
then $\lim_{x\to c} g(x) = L$.

Informally: If a function g is squeezed (sandwiched) between two other functions with the same limit then g also approaches that same limit.

Examples:

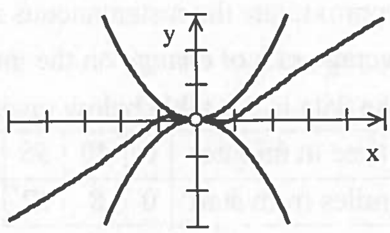
3. The graphs of $f(x) = \dfrac{x^3}{2x}$ and $g(x) = \dfrac{-x^3}{2x}$ are shown.

   Find $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$.



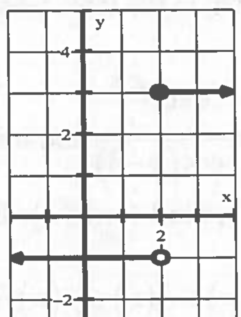
4. The graph of a third function $k(x)$ is shown along with the two functions from example 3.

   If $g(x) \le k(x) \le f(x)$ find $\lim_{x\to 0} k(x)$. Explain.

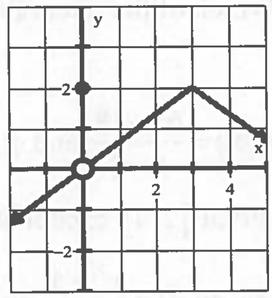


Use the functions graphed to find the following limits.

$y = f(x)$



$y = g(x)$



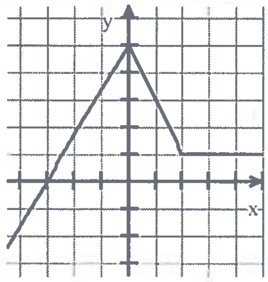
5. $\lim_{x\to 3} \dfrac{(f(x))^2}{g(x)+1} =$

6. $\lim_{x\to 2.5} g(f(x)) =$

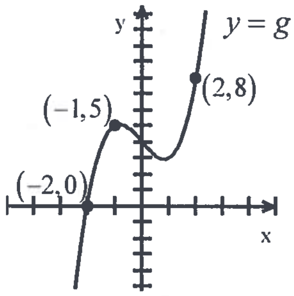
7. $\lim_{x\to 3} f(g(x)) =$

Assignment 1.4

1. The function $y = f(x)$ graphed at the right is a piecewise linear function. Find the instantaneous rate of change at each of the following x-values.
   a. $x = -1$
   b. $x = 1$
   c. $x = 4$



2. The function $y = g(x)$ is graphed at the right.
   a. Find the average rate of change on the interval $[-2,-1]$.
   b. Find the average rate of change on the interval $[-1,2]$.
   c. Which of these is a better approximation for the instantaneous rate of change of $g(x)$ at $x = -1.5$?



3. Approximate the instantaneous rate of change of $y = 3e^x + 5\sin x$ at $x = 3.3$ by finding the average rate of change on the interval $[3,4]$ accurate to three decimal places.
4. The data in the table below gives times and distances for a marathoner at selected points in the race.

time in minutes	0	40	55	95	129
miles from start	0	8	12	20	26

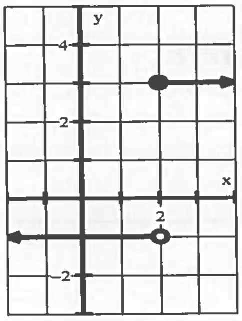
   a. Find the runner's average rate of change (speed in miles per minute) for the 26 miles included in the table.
   b. Approximate the instantaneous speed at the half-marathon spot (13.1 miles).
   c. Which of the intervals shown in the table was the slowest for the runner?

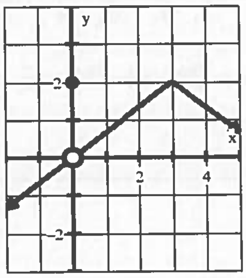
5. If $f(x) = \dfrac{6x-18}{x-3}$ and $g(x) = \dfrac{6\sin\frac{\pi x}{6}}{\cos(x-3)}$ and it is known that $f(x) \le h(x) \le g(x)$ on the interval $[2,4]$ except at $x = 3$. Find $\lim\limits_{x\to 3} h(x)$. Explain your reasoning.

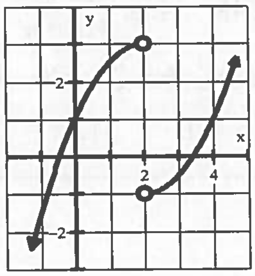
6. Given $f(x) = \dfrac{x^2-4}{x+2}$ and $f(x) \le h(x) \le j(x)$ for all x except $x = -2$. If $\lim\limits_{x\to -2} h(x)$ can be found by using the Squeeze Theorem what is $\lim\limits_{x\to -2} j(x)$?

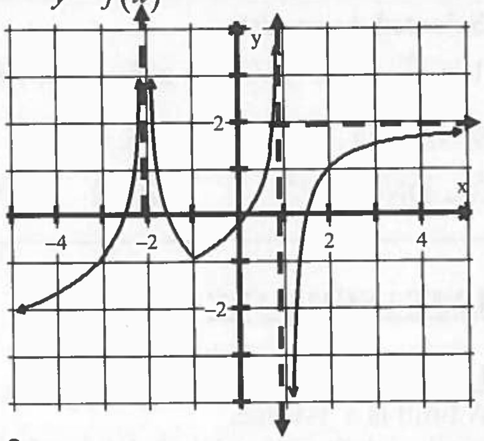
Use the four functions graphed below to find the limits shown or state that the limit does not exist.

$y = f(x)$ $\quad$ $y = g(x)$ $\quad$ $y = h(x)$ $\quad$ $y = j(x)$









7. $\lim\limits_{x\to -2} j(x)$ $\quad$ 8. $\lim\limits_{x\to 1} j(x)$ $\quad$ 9. $\lim\limits_{x\to -1} \dfrac{f(x)-2}{\left(j(x)\right)^2}$ $\quad$ 10. $\lim\limits_{x\to \infty} h\left(j(x)\right)$

11. $\lim\limits_{x\to -1} g\left(f(x)+1\right)$ $\quad$ 12. $\lim\limits_{x\to 0} f\left(|x|+2\right)$ $\quad$ 13. $\lim\limits_{x\to 0} \left(g(x)\bullet f(x+2)\right)$ $\quad$ 14. $\lim\limits_{x\to -2} j\left(j(x)\right)$

15. Find the equation of the horizontal asymptote for the function $g(x) = \dfrac{x^3+x}{e^x+x}$ without using a calculator.

16. Find $\lim\limits_{x\to 0} \dfrac{\sin x + 2e^x}{\cos x}$ without using a calculator.

17. Find the values of a and b so that $f(x) = \begin{cases} x-1, & x \le -1 \\ ax+b, & -1 < x < 1 \\ 2x+1, & x \ge 1 \end{cases}$ is continuous.

18. Use a calculator to find this limit $\lim\limits_{x\to 2} \dfrac{|2-x|}{25x-50}$.

19. Determine whether the Intermediate Value Theorem would guarantee a c-value where $f(c) = 6$, for the function $f(x) = \dfrac{x^2+x}{x-1}$ on the interval $\left[\frac{5}{2}, 4\right]$.

20. If your answer to problem 19 is yes, find the c-value. If your answer is no, try it again.

Find the following limits without using a calculator.

21. $\lim\limits_{x\to -2} (3x-3)$ $\quad$ 22. $\lim\limits_{x\to 2^-} \left\lfloor \frac{x}{2} - 4 \right\rfloor$ $\quad$ 23. $\lim\limits_{x\to 2} \left\lfloor \frac{x}{2} - 4 \right\rfloor$ $\quad$ 24. $\lim\limits_{x\to 3^+} \left\lfloor \frac{x}{2} - 4 \right\rfloor$

Use the graph of $y = f(x)$ for Problems 25-31.

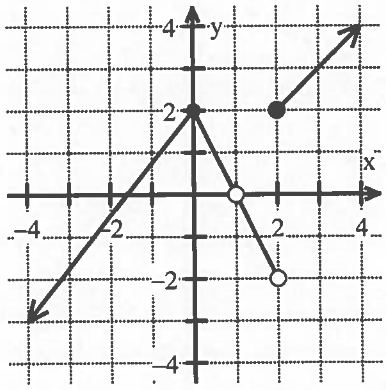
Find the following limits and function values.

25. $\lim\limits_{x\to 2} f(x)$ $\quad$ 26. $f(2)$ $\quad$ 27. $\lim\limits_{x\to 2^-} f(x)$

28. $\lim\limits_{x\to 1} f(x)$ $\quad$ 29. $\lim\limits_{x\to 0} f(x)$

30. List all removable discontinuities of $f(x)$.

31. List all nonremovable discontinuities of $f(x)$.



Selected Answers:							
1. a. $\frac{5}{3}$ b. -2 c. 0	2. a. 5	3. 99.048	4. a. $\frac{26}{129}\frac{\text{mi}}{\text{min}}$	5. 6	6. -4	8. DNE	
9. -3	11. 2	13. 0	16. 2	17. $a=\frac{5}{2}, b=\frac{1}{2}$	20. 3	22. -4	24. -3
25. DNE	27. -2	29. 2	30. $x=1$	31. $x=2$			

UNIT 1 SUMMARY

Limits:
A limit is a y-value.
Analyze left and/or right behavior. Use direct substitution.

Definition of Continuity:
A function f is continuous at an x-value c if and only if $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$.

Discontinuities: holes, vertical asymptotes, and jumps (breaks).
Removable (holes). Nonremovable (jumps and vertical asymptotes).

Limit at infinity: (end behavior)
Consider the highest degree terms in the numerator and denominator.

Curve sketching:
Vertical Asymptotes: denominator restrictions from the **reduced** function (write equations in the form $x = a$)
Holes: denominator restrictions from the **original** function which are no longer restricted in the reduced function (plug into the reduced function to find the y-value and write as ordered pairs)

Average Rate of Change: (the slope between two points) AROC = $\dfrac{y_2 - y_1}{x_2 - x_1}$

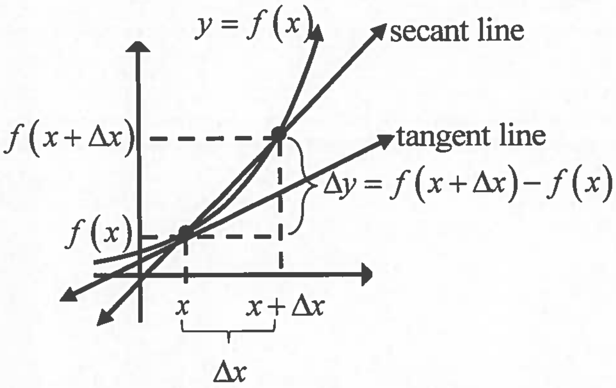
Instantaneous Rate of Change: can be approximated with an average rate of change

Squeeze Theorem (Sandwich Theorem):

If $f(x) \le g(x) \le h(x)$ for all $x \ne c$ in some interval containing c and if $\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x)$, then $\lim_{x\to c} g(x) = L$.

Lesson 2.1 Limit Definition of the Derivative, Alternate Form, Trig Review

Any nonvertical line has the same slope at every point. In Calculus we frequently deal with the slope of a curve. The slope of a curve is defined to be the same as the slope of the curve's tangent line at a given point. To find the slope of a tangent line we use a limit of the slope of a secant line.



$$m_{\text{sec}} = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$m_{\tan} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

The slope of a tangent line is called the **derivative** of the function at a given x-value. The most commonly used symbol for the derivative is $f'(x)$. Here are some other notations you will encounter (assume $y=f(x)$).

$$\boxed{f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx}f(x) = m_{\tan} = \lim_{\Delta x \to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{h \to 0}\frac{f(x+h)-f(x)}{h}}$$

A vertical tangent line has no slope, so a curve has no derivative at any point where it has a vertical tangent line. Differentiation is the process of finding derivatives. If a derivative exists at a point on a curve, the function is said to be differentiable at that point.

Examples:

1. If $f(x) = x^2 + 2$

a. find $f'(x)$.

$f'(x) = \lim_{\Delta x \to 0}$ ________________

b. use your answer from part a. to find $f'(-3)$.

2. If $y=\sqrt{x}$, find y'.

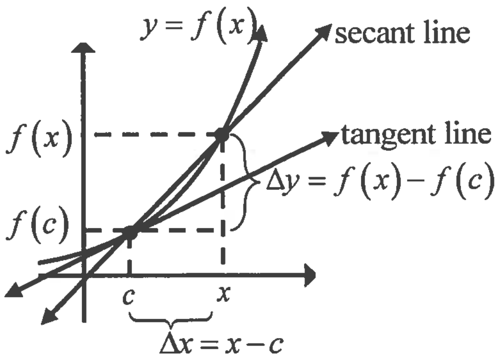
$y' =$

3. Given $y=f(t)=\dfrac{2}{t}$,

find the derivative of y with respect to t.

$$\frac{dy}{dt}=f'(t)=$$

Alternate Form of the Limit Definition of the Derivative
(Gives the value of the derivative at a single point.)



$$m_{\tan}=\lim_{x\to c} m_{\sec}$$

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

Example 4. If $f(x)=x^3$, use the alternate form of the derivative to find $f'(3)$.

$f'(3)=$

Solving Trig Equations

Example 5: Solve the following trig equations without using a calculator. Find all of the solutions in the interval $[0, 2\pi)$.

a. $\csc x = \frac{-2}{\sqrt{3}}$

b. $\cot\theta = \sqrt{3}$

c. $2\cos^2\theta - 1 = 0$

d. $\cos^2\theta - \cos\theta = 0$

Assignment 2.1

Use the limit definition of the derivative to find $f'(x)$ or $f'(t)$. Show correct limit symbolism.

1. $f(x) = -3x$ 2. $f(x) = x^2 - 1$ 3. $f(x) = \dfrac{1}{x-1}$ 4. $f(t) = t^3 - 12t$ 5. $f(x) = 3$

Use the alternate form of the limit definition of the derivative to find the indicated derivative.

6. $f(x) = x^2 - 1$ Find $f'(2)$.
7. $f(x) = x^3 - 2x^2 - 1$ Find $f'(2)$.
8. $f(x) = \dfrac{1}{x}$ Find $f'(3)$.
9. $f(x) = (x-1)^{\frac{2}{3}}$ Find $f'(1)$.

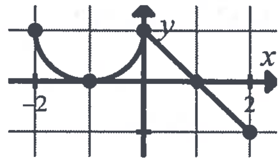
10. If $y = x^2 - x$, use the limit definition of the derivative to find y'.
11. If $y = x^3 + 1$, use the limit definition of the derivative to find $\frac{dy}{dx}$.
12. If $f(x) = 2x^2 + 4$, use the limit definition of the derivative to find $f'(x)$. Then find $f'(4)$.
13. If $f(x) = 2x^2 + 4$, use the alternate form of the limit definition of the derivative to find $f'(4)$.

For Problems 14-17, solve for θ, where $0 \le \theta < 2\pi$, without using a calculator.

14. $\sec^2\theta - 4 = 0$ 15. $\sin^2\theta = \cos^2\theta$ 16. $\tan\theta - \sin\theta = 0$ 17. $2\sin^2\theta = \cos\theta + 1$

18. Use a calculator to solve for x on the interval $[0, 2\pi)$ for $\tan x = \csc^2 x - 2$.

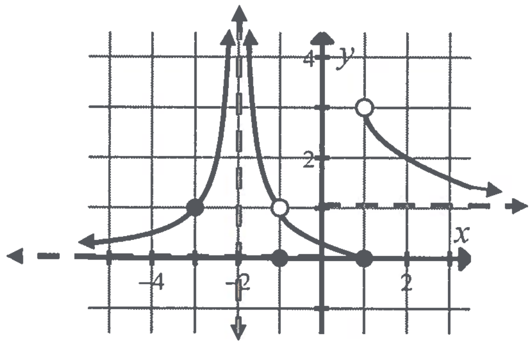
Use the graph of $y = f(x)$ shown to graph the following.



19. $y = |f(x)|$ 20. $y = f(x-2)+1$ 21. $y = -2f(x)$

22. Find the domain, vertical asymptotes, holes, intercepts, end behavior, and graph for the function $y = \dfrac{x(x-1)}{x^2-1}$.

Use the graph of $y = f(x)$ for Problems 23-32.
Find the following limits and function values.



23. $\lim\limits_{x\to -1} f(x)$ 24. $f(-1)$ 25. $\lim\limits_{x\to 1} f(x)$

26. $\lim\limits_{x\to 1^+} f(x)$ 27. $\lim\limits_{x\to -3} f(x)$ 28. $\lim\limits_{x\to \infty} f(x)$

29. $\lim\limits_{x\to -\infty} f(x)$ 30. $\lim\limits_{x\to -2} f(x)$

31. List all removable discontinuities of $f(x)$.

32. List all nonremovable discontinuities of $f(x)$.

Find the following limits without using a calculator.

33. $\lim\limits_{x\to 2}(5x-3)$ 34. $\lim\limits_{x\to 3}\dfrac{x^2-9}{3-x}$ 35. $\lim\limits_{t\to -3}\dfrac{3+t}{t^2-9}$ 36. $\lim\limits_{t\to 2}\dfrac{t^2-4}{t^2-3t+2}$ 37. $\lim\limits_{x\to 0^+}\left(x+\dfrac{1}{x^3}\right)$

38. $\lim\limits_{x\to \frac{1}{2}}\dfrac{4x-2}{2x-1}$ 39. $\lim\limits_{x\to 1}\dfrac{x-1}{x^4-1}$ 40. $\lim\limits_{x\to 1}\dfrac{x^2-2x+1}{x+1}$ 41. $\lim\limits_{x\to 1}\dfrac{x^2-2x+1}{x-1}$ 42. $\lim\limits_{x\to 2}\dfrac{3x+5}{\tan\frac{\pi x}{4}}$

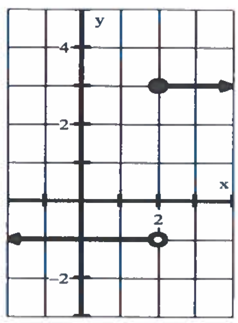
Use the functions graphed below to find the limits shown or state that the limit does not exist.

43. $\lim\limits_{x\to 0} f(h(x))$

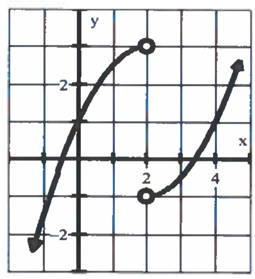
44. $\lim\limits_{x\to 2}\left((f(x)-1)^2-6\right)$

45. $\lim\limits_{x\to 2}(h(x)+f(x))$

$y = f(x)$



$y = h(x)$



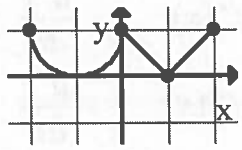
Selected Answers:

1. $f'(x)=-3$ 2. $f'(x)=2x$ 3. $f'(x)=-\frac{1}{(x-1)^2}$ 4. $f'(t)=3t^2-12$ 5. $f'(x)=0$

6. $f'(2)=4$ 7. $f'(2)=4$ 8. $f'(3)=-\frac{1}{9}$ 9. $f'(1)$ is undefined 10. $y'=2x-1$

11. $\frac{dy}{dx}=3x^2$ 12. $f'(x)=4x$, $f'(4)=16$ 13. $f'(4)=16$ 15. $\theta=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$

16. $\theta=0$, π 17. $\theta=\frac{\pi}{3},\pi,\frac{5\pi}{3}$ 19.



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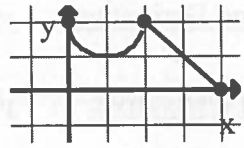

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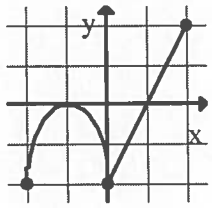
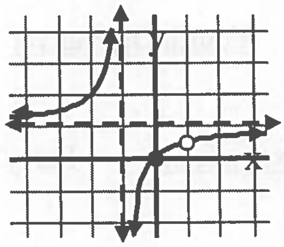


22. Do: $x\neq\pm1$ VA: $x=-1$ (odd)

$y_{red}=\frac{x}{x+1}$ x-int: $(0,0)$ (odd)

Hole: $\left(1,\frac{1}{2}\right)$ y-int: $(0,0)$

EB: HA $y=1$



23. 1 24. 0 25. DNE 26. 3 27. 1 28. 1 29. 0 30. ∞ or DNE 31. $x=-1$

32. $x=-2, 1$ 33. 7 34. -6 35. $-\frac{1}{6}$ 36. 4 37. ∞ or DNE 38. 2 39. $\frac{1}{4}$

40. 0 41. 0 42. 0 43. -1 44. -2

Lesson 2.2 Differentiation Rules (shortcuts), Tangent Lines, Differentiability, Rates of Change

Derivative Rules:

Power Rule: $\frac{d}{dx}x^n=nx^{n-1}$

Constant Rule: If c is any constant, $\frac{d}{dx}c=0$.

Scalar Multiple Rule: If c is any constant, $\frac{d}{dx}(c\,f(x))=c\,f'(x)$.

Sum Rule: $\frac{d}{dx}(f(x)+g(x))=f'(x)+g'(x)$

Examples: Differentiate.

1. $f(x)=x^4$ 2. $y=x^{-\frac{2}{3}}+3$ 3. $h(t)=5-\frac{1}{2t^3}$ 4. $f(x)=\frac{5}{(2x)^3}$

$f'(x)=$ $y'=$ $h(t)=$

$h'(t)=$

Higher-Order Derivatives
Since the derivative of a function is another function, we can repeat the differentiation process to find the derivative of a derivative. The result is still another function which could again be differentiated. These derivatives are called higher-order derivatives.

Notation:

First Derivative:	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}f(x)$
Second Derivative:	y''	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx^2}f(x)$
Third Derivative:	y'''	$f'''(x)$	$\frac{d^3y}{dx^3}$	$\frac{d^3}{dx^3}f(x)$
Fourth Derivative:	$y^{(4)}$	$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$\frac{d^4}{dx^4}f(x)$

Example 5. For $f(x)=\dfrac{1}{2\sqrt[3]{x^2}}$, find $f'(1)$ and $f''(-8)$.

Equation of a Tangent Line:

Since the derivative of a function gives us a slope formula for tangent lines to the graph of the function, the derivative can be used to find equations of tangent lines.

Sometimes we will want to find a line perpendicular to the tangent line at a certain point. Such a line is called a normal line.

$$m_{\text{normal}}=\frac{-1}{m_{\text{tangent}}}$$

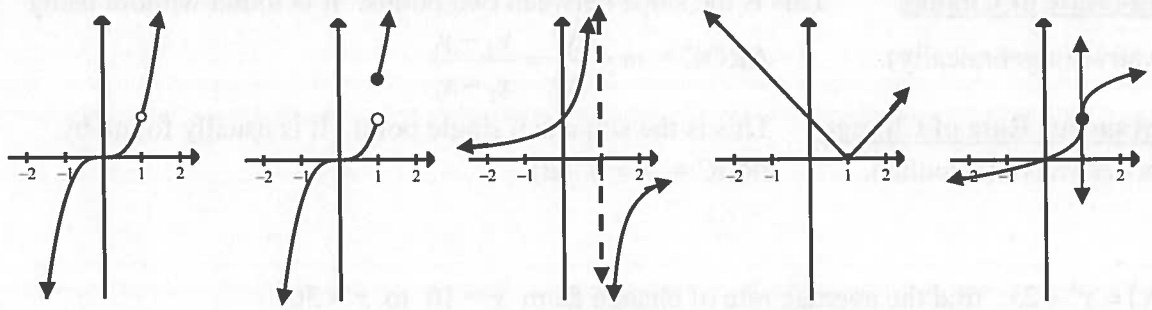
Examples:

6. Find an equation of the line tangent to the graph of $f(x)=4x^5-3x^2+5$ at the point (1,6).

7. Find an equation of the normal line to the same curve at the same point.

NONDIFFERENTIABILITY (when a derivative does not exist)

Each of these functions has no derivative when $x = 1$.



hole | jump | vertical asymptote | sharp turn | vertical tangent

These five characteristics destroy differentiability:

1. Holes
2. Jumps (breaks)
3. Vert. Asymptotes

} discontinuities

4. Sharp Turns
5. Vert. Tangent Lines

Note:
If a function is not continuous, it is not differentiable (see the first three figures above).
A function may be continuous and still not be differentiable (see the last two figures above).

Examples: Find the x-values where $f(x)$ is not differentiable. Give a reason for each.

8. $f(x) = |x|$

9. $f(x) = \begin{cases} x^2, & x \le 0 \\ x, & x > 0 \end{cases}$

10. $f(x) = \begin{cases} x^2, & x \le 0 \\ x^2 + 1, & x > 0 \end{cases}$

11. $f(x) = \dfrac{x}{x(x-1)}$

12. $f(x) = \sqrt[3]{x}$

Rate of Change:

Another meaning for slope is rate of change. We now have two ways to find slopes (rates of change).

1. **Average Rate of Change** This is the slope between two points. It is found without using a derivative (algebraically). AROC = $m=\dfrac{\Delta y}{\Delta x}=\dfrac{y_2-y_1}{x_2-x_1}$
2. **Instantaneous Rate of Change** This is the slope at a single point. It is usually found by using a derivative (calculus). IROC = $m=f'(c)$

Examples:

13. If $f(x)=x^3+2x$, find the average rate of change from $x=10$ to $x=30$.

14. If $f(x)=x^3+2x$, find the instantaneous rate of change when $x=10$.

Assignment 2.2

Find the derivative. Use correct symbolism.

1. $y=2$
2. $f(x)=x^2$
3. $g(x)=x^3+1$
4. $y=t+2$
5. $f(t)=-2t^2-3t+2$
6. $f(x)=-\frac{1}{3}x^2-\frac{2}{5}x+\frac{5}{2}$

Find the value of the derivative of the function at the given point. Show steps with correct symbolism.

7. $f(x)=3x^{-2}$ at $(1,3)$
8. $g(x)=x^2-2x$ at $(2,0)$
9. $h(x)=x^3-1$ at $(1,0)$
10. $f(x)=2-x^3$ at $(2,-6)$

Differentiate each function. Show steps with correct symbolism.

11. $y=\dfrac{1}{x}$
12. $f(x)=x^2-\dfrac{4}{x^2}$
13. $y=(2x-1)^2$
14. $g(x)=x(x^2+1)$
15. $y=\dfrac{\sqrt{x}}{x}$
16. $y=\sqrt[3]{x}+\sqrt{x^3}$
17. $f(t)=\dfrac{t^2-2t}{t}$
18. $f(x)=\dfrac{1}{\sqrt[3]{x^2}}$
19. $y=\dfrac{1}{3x^2}$
20. $y=\dfrac{1}{(3x)^2}$
21. $f(x)=\dfrac{x^2-x-1}{\sqrt{x}}$
22. $y=(3x^2-5)(x+7)$

Find the indicated value or expression. Show steps with correct symbolism.

23. $y=3x^2, \quad y''=?$ 24. $f(x)=\sqrt{x}+2, \quad f'(4)=?$ 25. $f(t)=2-\frac{2}{t}, \quad f''(2)=?$

26. $y=x(x-2), \quad \frac{d^2y}{dx^2}=?$ 27. $f^{(3)}(x)=2x-1, \quad f^{(5)}(3)=?$ 28. $\frac{d}{dx}\left(x^3+5\right)=?$

29. $\frac{d^2}{dx^2}\left(3x-x^{-1}\right)=?$

30. Find the second derivative of $f(x)=\dfrac{x^2-4x-6}{2x}$.

Find an equation of a line with the following characteristics.

31. tangent to the graph of $f(x)=x^2-1$ at the point (2,3)

32. tangent to the graph of $f(x)=\frac{2}{x}$ when $x=1$

33. normal to the graph of $f(x)=\frac{2}{x}$ when $x=1$

34. tangent to the graph of $y=x^2-2x+3$ when $x=1$

35. Find the x-values of all points where the graph of $f(x)=3x^3+2x-2$ has a slope of 11.

36. Find the x-values of all points where the graph of $y=x^4-3x^2+2$ has a horizontal tangent line.

37. Find the point(s) where the graph of $y=\dfrac{1}{x}$ has a slope of $-\frac{1}{4}$.

38. Find the average rate of change of the function $f(x)=3x^3-4$ between $x=2$ and $x=4$.

39. Find the instantaneous rate of change of the function $f(x)=3x^3-4$ at $x=3$.

40. Find the average rate of change of $y=\dfrac{x}{x+2}$ on the interval $[1,4]$.

41. Find the rate of change of $y=\dfrac{x^2-x}{x^2}$ at the point (1,0).

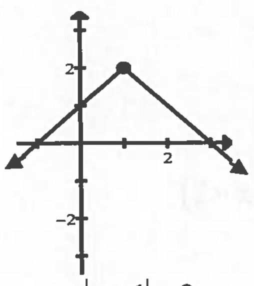
42. If $f(x)=2x^3-3x+2$ find:

a. the average rate of change on the interval $[0,3]$.

b. the instantaneous rate of change at $x=3$.

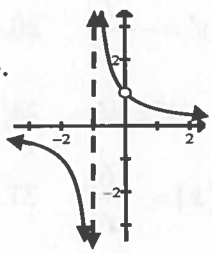
Find the x-values where the function is <u>not</u> differentiable. Give a reason for each value.

43.



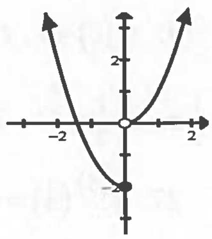
$y=-|x-1|+2$

44.



$y=\dfrac{x}{x^2+x}$

45.



$y=\begin{cases} x^2-2, & x\le 0 \\ x^2, & x>0 \end{cases}$

Find the x-values where the function is <u>not</u> differentiable. Give a reason for each value.

46. $f(x)=x^{\frac{2}{3}}$

47. $f(x)=3x^{\frac{1}{3}}$

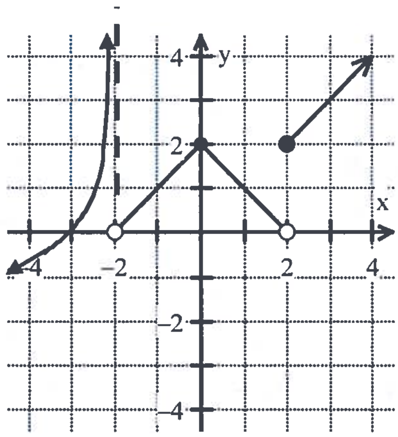
For each of the following piecewise functions:
a. Find any x-values at which the function is discontinuous.
b. Differentiate the function.
c. Find any x-values at which the function is not differentiable.

48. $f(x)=\begin{cases}3x^2-x, & x\le 1\\ 5x-3, & x>1\end{cases}$

49. $f(x)=\begin{cases}3x^2-x, & x\le 1\\ 5x-2, & x>1\end{cases}$

50. $f(x)=\begin{cases}3x^2-x, & x\le 1\\ 4x-2, & x>1\end{cases}$

51. a. Identify any x-values at which the function shown is not continuous.
b. Identify any x-values at which the function shown is not differentiable.



52. Use the limit definition of the derivative to find $f'(x)$ if $f(x)=2x^2-5$.

53. Use the alternate form of the limit definition of the derivative to find $f'(1)$ if $f(x)=x^2+2x$.

54. Find $\lim\limits_{h\to 0}\dfrac{(x+h)^3-x^3}{h}$ mentally without showing any steps.

Selected Answers:

1. $y'=0$ 3. $g'(x)=3x^2$ 5. $f'(t)=-4t-3$ 6. $f'(x)=-\frac{2}{3}x-\frac{2}{5}$ 7. $f'(1)=-6$

9. $h'(1)=3$ 10. $f'(2)=-12$ 11. $y'=-\dfrac{1}{x^2}$ 12. $f'(x)=2x+\dfrac{8}{x^3}$ 14. $g'(x)=3x^2+1$

15. $y'=-\frac{1}{2}x^{-\frac{3}{2}}$ 17. $f'(t)=1,\ t\ne 0$ 19. $y'=-\dfrac{2}{3x^3}$ 20. $y'=-\dfrac{2}{9x^3}$

21. $f'(x)=\frac{3}{2}x^{\frac{1}{2}}-\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}x^{-\frac{3}{2}}$ 22. $y'=9x^2+42x-5$ 24. $f'(4)=\dfrac{1}{4}$

25. $f''(2)=-\dfrac{1}{2}$ 27. $f^{(5)}(3)=0$ 30. $f''(x)=-\dfrac{6}{x^3}$ 31. $y-3=4(x-2)$

Selected Answers continued:

33. $y-2=\frac{1}{2}(x-1)$ 34. $y=2$ 35. $x=\pm 1$ 36. $x=0,\pm\sqrt{\frac{3}{2}}$ 38. AROC = 84

39. $f'(3)=81$ 41. $y'(1)=1$ 42a. AROC = 15 b. $f'(3)=51$ 43. $x = 1$ (sharp turn)

44. $x=-1$ (vert. asymp.), $x=0$ (hole) 47. $x = 0$ (vert. tang.)

48a. always continuous b. $f'(x)=\begin{cases} 6x-1, & x\le 1 \\ 5, & x>1 \end{cases}$ c. always differentiable

49a. discontinuous at $x=1$ b. $f'(x)=\begin{cases} 6x-1, & x<1 \\ 5, & x>1 \end{cases}$ c. not differentiable at $x = 1$

Lesson 2.3 Position → Velocity → Acceleration, Calculator Differentiation

Important Terms

Position Function	gives the location of an object at time t, usually $s(t)$, $x(t)$, or $y(t)$
Velocity Function	the rate of change (derivative) of position, usually $v(t)$ Velocity is positive for upward or rightward motion and negative for downward or leftward motion.
Acceleration Function	the rate of change (derivative) of velocity, usually $a(t)$
Initial Position	starting position (at $t=0$), s_o
Initial Velocity	starting velocity (at $t=0$), v_o
Speed	the absolute value of velocity
Displacement	the net change in position, (final pos. − original pos.)
Total Distance	total distance traveled by the object in the time interval (takes into account all direction changes)

Example 1. If $s(t)=t^3+t$, find $v(t)$ and $a(t)$.

Examples: Use the position function $s(t)=16t^3-36t^2+24$ of an object moving on a horizontal line for Examples 2-11. Distance units are measured in feet and time units are measured in seconds.

2. What is the initial position of the object?

3. What is the velocity of the object at $t=1$ second?

4. What is the speed of the object at $t=1$ second?

5. What is the acceleration of the object at $t=1$ second?

6. When is the object at rest?

7. When is the object moving right?

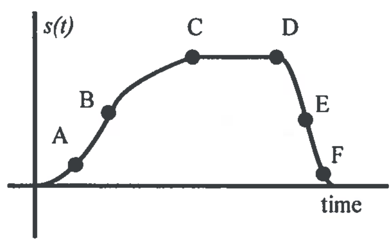
8. When is the object moving left?

9. When is the velocity of the object equal to 54 $^{ft}/_{sec}$?

10. What is the displacement of the object between $t=0$ and $t=2$ seconds?

11. What is the total distance traveled by the object between $t=0$ and $t=2$ seconds?

The graph shows the position function of a radio controlled model car. Answer these questions and explain.



12. Was the car going faster at A or at B?

13. When was the car stopped?

14. At which point was the car's velocity the greatest?

15. At which point was the car's speed the greatest?

Vertical Motion Examples:

Suppose $s(t) = -16t^2 + 48t + 160$ gives the position (in feet) above the ground for a ball thrown into the air from the top of a high cliff (where time is measured in seconds).

16. Find the initial velocity.

17. At what time does the ball hit the ground?

18. At what time does the ball reach its maximum height?

Calculator Differentiation

Some CAS calculators can find derivatives of functions in symbolic form. This is not one of the uses of calculators that are allowed on the Advanced Placement test and will not be helpful. All graphing calculators can be used to find the value of a derivative at a specific point and this is allowed and required on the Advanced Placement test.
For example a TI84 can find a derivative at a point using nDeriv( ) in the Math menu.
Use a graphing calculator to find the following. As always give answers accurate to three decimal places.

Examples:

19. If $f(x) = x^3 + 3^x$, find $f'(2)$.

$f'(2) =$

20. If $g(x) = \ln(x^2 - 3)$, find $g'(2)$, $g'(4)$, and sketch a graph of $g'(x)$.

Hint: To save time and avoid confusing parentheses, let $y_1 = \ln(x^2 - 3)$.

$g'(2) =$ $\qquad$ $g'(4) =$

To graph $g'(x)$, let $y_2 = \left.\frac{d}{dx}(y_1)\right|_{x=x}$.

21. If $f(x) = |x|$, find $f'(0)$.

Some calculators are unable to give a correct answer to this last problem. Make sure you understand the limitations of your calculator.

Assignment 2.3

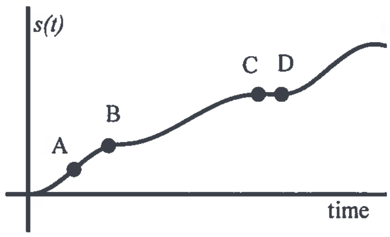
You may use a calculator for these questions.

1. The position, in meters, of a particle moving in a straight line is given by $x(t) = 4t^3 + 6t + 2.5$ (where t is measured in seconds).
   a. Find the velocity function.
   b. Find the velocity at time $t = 2$ seconds.
   c. Find the acceleration function.
   d. Find the acceleration at time 3 seconds.
   e. When is the velocity of the particle 18 meters per second?
   f. Find the velocity when the position of the particle is 25 meters.
   g. Find the initial position.
   h. Find the particle's displacement from 0 to 1.5 seconds.

2. A helium balloon rises so that its height (position) is given by $s(t) = t^2 + 3t + 5$ (where height is measured in feet and time is measured in seconds). Assume $t \geq 0$.
   a. When is the balloon 45 feet high?
   b. How fast is the balloon rising at time 1 second?
   c. How fast is the balloon rising at time 4 seconds?
   d. What is the balloon's velocity when it is 45 feet high?

3. A ball rolls on an inclined plane with position function $s(t) = 2t^3 + 3t^2 + 5$ (where position is measured in centimeters and time is measured in seconds).
   a. Find the ball's velocity at time 2 seconds.
   b. When is the velocity of the ball 30 centimeters per second?

4. The graph at the right shows the position function of a car. Answer these questions and explain each answer.
   a. What was the car's initial position?
   b. Was the car going faster at A or at B?
   c. Was the car speeding up or slowing down at B?
   d. What happened between C and D?

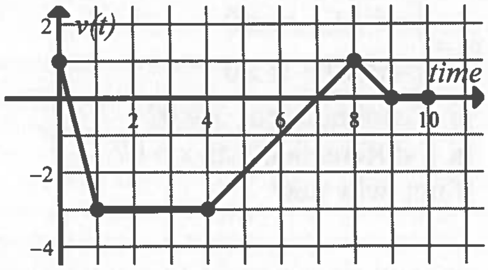


5. A particle moves along a horizontal line with position function $x(t) = t^3 - 3t^2$ (where position is measured in centimeters and time is measured in minutes).
   a. Find the particle's displacement between $t = 0$ minutes and $t = 5$ minutes.
   b. Find the particle's velocity when $t = 4$ minutes.
   c. Find the particle's acceleration when $t = 4$ minutes.
   d. At what time does the particle change direction?
   e. What is the total distance traveled by the particle between 0 and 5 minutes?

$$\textbf{Average Velocity} = \frac{\text{displacement}}{\text{elapsed time}} \qquad \textbf{Average Speed} = \frac{\text{total distance}}{\text{elapsed time}}$$

   f. Find the particle's average velocity (average rate of change of position) between $t = 0$ and $t = 5$ minutes.
   g. Find the particle's average speed between $t = 0$ and $t = 5$ minutes.

6. The graph at the right shows the velocity function of a particle moving horizontally.
   a. When does the particle move left?
   b. When is the particle's acceleration positive?
   c. When is the speed greatest?
   d. When does the particle stop for more than an instant?



7. The position at time t seconds of a pebble dropped from an initial height of 600 feet is given by $s(t)=-16t^2+600$.
   a. At what time will the pebble hit the ground?
   b. What is the pebble's velocity when it hits the ground?
   c. What is the pebble's speed when it hits the ground?

Do not use a calculator on problems 8-17.
Find $f'(x)$.

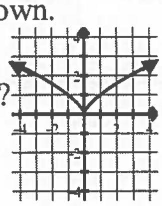
8. $f(x)=2x-\dfrac{3}{x^3}$ 9. $f(x)=(2x+3)^2$

Evaluate the derivative of $f(x)$ at the indicated point for Problems 10 and 11.

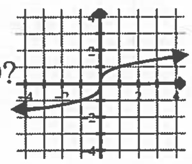
10. $f(x)=2x\sqrt{x}$ at $(4,16)$ 11. $f(x)=\sqrt[3]{x^2}$ at $(-8,4)$

12. If $y=x(x-2)$ find $\dfrac{d^2y}{dx^2}$.
13. Find an equation of a line tangent to the graph of $f(x)=2x^4-3x^3$ when $x=1$.
14. Find a <u>point</u> on the graph of $f(x)=x^4+3$ where a tangent line has a slope of -4.
15. Use the limit definition of the derivative to find $f'(x)$ if $f(x)=3x^2-x$.
16. If $f(x)=x^3+5$, find the instantaneous rate of change at $x=1$.
17. If $f(x)=x^3+5$, find the average rate of change between $x=0$ and $x=2$.
18. If $f(x)=7^x$, use a calculator to find $f'(3)$.
19. If $g(x)=\sin x^3+4x^3$, find $g'(2)$, $g'(-4)$, and $g''(1)$.

20. The graph of $f(x)=x^{\frac{2}{3}}$ is shown.
   a. Is f continuous at $x=0$?
   b. Is f differentiable at $x=0$? If not, why not?



21 The graph of $f(x)=x^{\frac{1}{3}}$ is shown.
   a. Is f continuous at $x=0$?
   b. Is f differentiable at $x=0$? If not, why not?



22. $f(x)=\begin{cases} x^2+1, & x\le 0 \\ x^2+x+1, & x>0 \end{cases}$
   a. Is f continuous at $x=0$?
   b. Is f differentiable at $x=0$? If not, why not?

23. $f(x)=\begin{cases} x^2+1, & x\le 0 \\ -x^2+2, & x>0 \end{cases}$
   a. Is f continuous at $x=0$?
   b. Is f differentiable at $x=0$? If not, why not?

24. $f(x)=\begin{cases} x^2+1, & x\le 0 \\ -x^2+1, & x>0 \end{cases}$

a. Is f continuous at $x=0$?
b. Is f differentiable at $x=0$?
If not, why not?

25. $f(x)=\left\lfloor \frac{x}{2}+1 \right\rfloor$

a. Is f continuous at $x=0$?
b. Is f differentiable at $x=0$?
If not, why not?

Evaluate the following limits without using a calculator.

26. $\lim\limits_{x\to 2}\dfrac{\frac{1}{2}-\frac{1}{x}}{x-2}$ 27. $\lim\limits_{x\to\infty}\dfrac{x^2-9}{2x^2+9}$ 28 $\lim\limits_{x\to\infty}\dfrac{(2x-1)^2}{x^2-9}$ 29. $\lim\limits_{x\to-\infty}\dfrac{3x^2-5}{x+1}$ 30. $\lim\limits_{x\to-\infty}\dfrac{-\sqrt{x^2}}{x}$

31. $\lim\limits_{x\to 2}\left\lfloor \frac{x}{3}+5 \right\rfloor$ 32. $\lim\limits_{x\to 3^-}\left\lfloor \frac{x}{3}+5 \right\rfloor$ 33. $\lim\limits_{x\to 3}\left\lfloor \frac{x}{3}+5 \right\rfloor$ 34. $\lim\limits_{x\to-\infty}\left(e^x-\dfrac{3x}{x^2+x}\right)$

35. If $f(x)=\begin{cases} x+1, & 1<x<3 \\ x^2+ax+b, & x\le 1 \text{ or } x\ge 3 \end{cases}$ is continuous, find the values of a and b.

Selected Answers

1. b. $V(2)=54\frac{m}{sec}$ d. $A(3)=72\frac{m}{sec^2}$ e. $t=\pm 1$ sec f. $V(1.5)=33\frac{m}{sec}$ h. 22.5 m

2. a. $t=5\ sec$ b. $V(1)=5\frac{ft}{sec}$ d. $V(5)=13\frac{ft}{sec}$ 3. b. $t=-2.791,\ 1.791\ sec$

4. c. slowing down (the slope of S is decreasing) 5. a. disp. = 50 cm

5. b. $V(4)=24\frac{cm}{min}$ d. t = 0,2 min e. TD = 58 cm f. $V_{\text{avg}}=10\frac{cm}{min}$

5. g. avg. speed = $\frac{58}{5}\frac{cm}{min}$ 6. a. $\frac{1}{4}<t<7$ c. $1\le t\le 4$

7. a. $t=6.123$ or 6.124 sec b. $V(6.123)=-195.959\frac{ft}{sec}$ 8. $f'(x)=2+\dfrac{9}{x^4}$

9. $f'(x)=8x+12$ 10. $f'(4)=6$ 11. $f'(-8)=-\frac{1}{3}$ 12. $y''=2$

13. $y+1=-1(x-1)$ 14. $(-1,4)$ 16. $f'(1)=3$ 17. AROC = 4

18. $f'(3)=$ 667.447 or 667.448 20. a. yes b. no, sharp turn 21. a. yes b. no, vert. tan.

22. a. yes b. no, sharp turn 23. a. no b. no, jump 24. a. yes b. yes 25. a. no b. no

26. $\frac{1}{4}$ 27. $\frac{1}{2}$ 28. 4 29. $-\infty$ or DNE 30. 1 31. 5 32. 5 33. DNE

35. $a=-3,\ b=4$

Lesson 2.4 Review of Logarithms, Derivatives of $\sin x$, $\cos x$, e^x, and $\ln x$

Change of Form Definition:

Exponential form $\left\{\begin{array}{l} x = e^y \leftrightarrow y = \ln x \\ x = a^y \leftrightarrow y = \log_a x \end{array}\right\}$ Logarithmic form

Example1: Change the following equations from exponential form to log form or vice versa.

a. $3^4 = 81$ b. $\log(.1) = -1$

Example 2:

a. Since $e^0 = 1$, $\ln 1 =$ b. Since $e^1 = e$, $\ln e =$ c. $\ln e^n = e^{\ln n} =$

Example 3: Use the inverse idea from Example 3c. to simplify.

a. $\ln e^{\sqrt{2}} =$ b. $e^{\ln(3x)} =$

Properties of Logarithms:

1. $\ln(ab) = \ln a + \ln b$
2. $\ln \dfrac{a}{b} = \ln a - \ln b$
3. $\ln a^n = n \ln a$

These properties work for any bases, but only if $a > 0$ and $b > 0$

Example 4: Expand using Logarithm Properties 1-3 above.

a. $\ln \frac{5}{8} =$ b. $\ln \sqrt[3]{x^2+1} =$

Example 5: Condense into a single logarithm. $(x > 0$ and $y > 0)$

a. $-3\ln x + 5\ln y$ b. $\frac{1}{2}\ln x + \ln(x+1) - 3\ln y$

Example 6: Solve for x.

$\log_2 x - \log_2(x-8) = 3$

Change of Base Formula: $\log_a x = \dfrac{\log_b x}{\log_b a}$

Since the only two logarithmic bases on your calculator are 10 (log key) and e (ln key), you will change bases on your calculator in one of two ways:

$$\log_a x = \frac{\log x}{\log a} \qquad \text{or} \qquad \log_a x = \frac{\ln x}{\ln a}$$

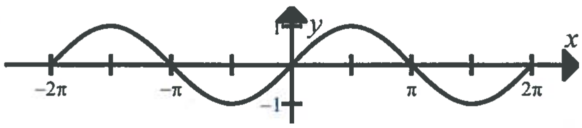
Example 7: Use your calculator to find $\log_7 112$ to 3 or more decimal places.

Example 8:

a. Find an exact value for x, if $3^{x+2} = 6$.

b. Use your calculator to find a decimal value for your answer from Part a.

The graph of $f(x) = \sin x$ is shown at the right.



Example 9:
Estimate slopes for the graph of $f(x) = \sin x$ at $x = -2\pi,\ \frac{-3\pi}{2},\ -\pi,\ \frac{-\pi}{2},\ 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}$, and 2π.

Plot those slopes in the coordinate plane at right, and connect them to make a smooth continuous curve. This is the graph of $f'(x)$.

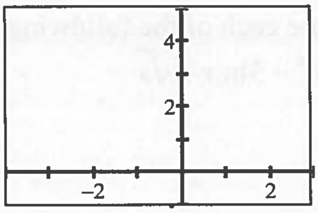
$f'(x) =$

Example 10:
Use your calculator to sketch a graph of the derivative of $y = \cos x$. Can you identify your result?

Derivatives of Two Trigonometric Functions:

$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$

Example 11:
Use your calculator to graph $y = e^x$ and its derivative in the same coordinate plane. What do you notice?

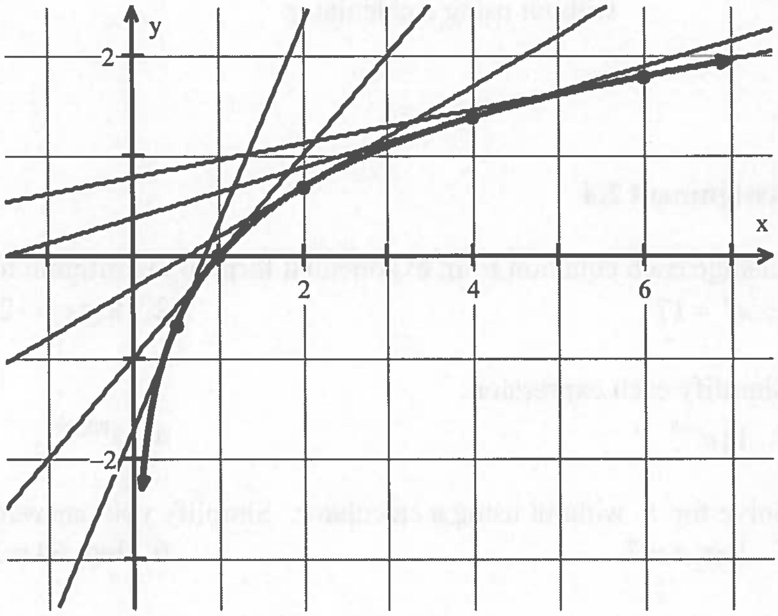


e is the only base for which the basic exponential function and its derivative are the same.

Differentiating the Exponential Function:
$$\frac{d}{dx}e^x = e^x$$

Example 12:
The function $y = \ln x$ is graphed at the right with some tangents shown.

Fill in the table by estimating slopes at the points indicated with as much precision as possible.



x	$\frac{d}{dx}\ln x$
$\frac{1}{2}$	
1	
2	
4	
6	

Can you make a conclusion about the derivative of the natural logarithm function?

Differentiating the Natural Logarithmic Function:
$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Examples:
Differentiate each of the following functions using correct notation without using a calculator.

13. $y=2e^x-3\ln x+\sqrt{x}$

14. $g(\theta)=3\sin\theta-5\cos\theta$

Examples:
Find the indicated value without using a calculator.

15. If $h(x)=15\sin(x)+4e^x$ find $h'(0)$.

16. If $f(t)=4\ln t-3e^t$ find $f''(5)$.

Example 17: Find an equation of the line tangent to the graph of $f(x)=-3\cos x$ when $x=\frac{\pi}{4}$ without using a calculator.

Assignment 2.4

Change each equation from exponential form to logarithmic form or vice versa.

1. $e^x=17$

2. $\log x=-2$

Simplify each expression.

3. $\ln e^{a+b}$

4. $3^{\log_3 m^2}$

Solve for x without using a calculator. Simplify your answers.

5. $\log_2 x=3$

6. $\log_x 64=3$

Use Properties of Logarithms to expand the expressions. (Assume all variables are positive.)

7. $\ln\frac{a}{bc}$

8. $\log(xy^2)$

Use Properties of Logarithms to condense the expressions into single logarithms. (All variables represent positive quantities).

9. $3\ln x-\frac{1}{2}\ln y$

10. $\ln a-(2\ln b-\ln c)$

Solve for t without using a calculator.

11. $\ln e^{t^2-t}=6$

12. $e^{2t-1}-3=0$

13. Find the value of $\log_3 20$. (Express answers to 3 or more decimal place accuracy.)

Differentiate each of the following without using a calculator. Show steps and answers using correct notation.

14. $f(x)=2\sqrt{x}+3e^x$
15. $y=\dfrac{2}{x}-\ln x$
16. $g(\theta)=3\sin(\theta)$
17. $f(x)=\dfrac{3x^2+4x-2x\cos x}{2x}$
18. $y=ex^2-2e^x$
19. $f(t)=3\ln t+\sin(t)$
20. $y=f(x)+g(x)$

Find the indicated value without using a calculator. Show steps and answers using correct notation.

21. $g(x)=x+2\cos x$ find $g'\left(\frac{\pi}{4}\right)$
22. $f(x)=x^2+5x-\ln x$ find $f'(1)$
23. $f(x)=7e^x+\sin(x)$ find $f'(0)$
24. $h(t)=\dfrac{5t^4+9t^3-6t^2e^t}{3t^2}$ find $h''(2)$
25. $f(x)=\pi\sin x-\pi^2\ln x$ find $f''\left(\frac{\pi}{2}\right)$

26. Find an equation of a line tangent to $g(\theta)=\frac{3}{2}+\sin(\theta)$ at $\theta=\frac{\pi}{6}$ without using a calculator.

27. Find the x-coordinate(s) of point(s) at which the graph of $g(x)=x-e^x$ has a tangent line parallel to the graph of $y=7$ without using a calculator.

The following two limits can now be found with very little work.

28. $\displaystyle\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$
29. $\displaystyle\lim_{h\to 0}\frac{2e^{x+h}-2e^x}{h}$

30. Use the Limit Definition of the Derivative to find $f'(x)$ if $f(x)=3x^2+x$.

31. Use the Alternate Form of the Limit Definition of the Derivative to find $f'(3)$ if $f(x)=3x^2-2$.

32. If $f(x)=2x^3-3x+2$ find:
   a. the average rate of change on the interval $[0,3]$.
   b. the instantaneous rate of change at $x=3$.

Differentiate without using a calculator.

33. $f(x)=2x(x^2+3)$
34. $g(x)=\dfrac{3x^2-6x+9}{3x}$
35. $f(x)=\dfrac{1}{4x^4}$

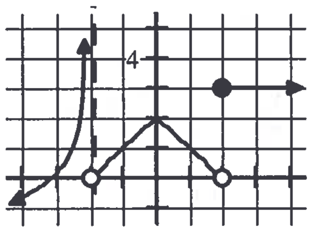
36. Use the appearance of the graph at the right to evaluate the following limits.

a. $\lim\limits_{x\to -3} f(f(x))$

b. $\lim\limits_{x\to -2^-} f(f(x))$

c. $\lim\limits_{x\to 0} f(f(x))$

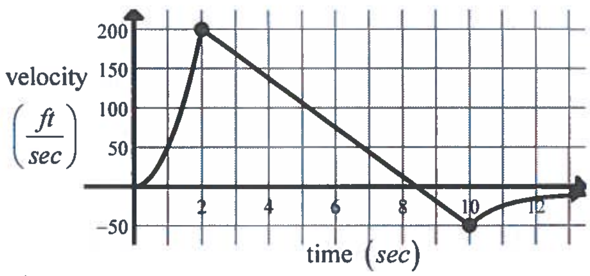
$y = f(x)$



37. If the Squeeze Theorem can be applied and $g(x) \le f(x) \le h(x)$ except at $x = 5$ where $g(x) = \dfrac{x^2 - 25}{x - 5}$, find $\lim\limits_{x\to 5} h(x)$ and $\lim\limits_{x\to 5} f(x)$.

38. A model rocket is fired straight upward. The engine burns for two seconds. The rocket continues to coast upward then starts to fall. A parachute is released ten seconds after the launch. The graph shows the rocket's velocity.

a. What was the rocket's greatest velocity?
b. Estimate the velocity at time 7 *sec*.
c. Was the rocket moving upward or downward at time 7 *sec*.?
d. Estimate the rocket's speed when the parachute was released.
e. Estimate when the rocket started to fall.
f. Estimate when the acceleration was the greatest.
g. When was the acceleration constant? Estimate the value of this constant acceleration.



Selected Answers:

1. $\ln 17 = x$ 3. $a+b$ 5. $x = 8$ 7. $\ln a - \ln b - \ln c$ 9. $\ln \dfrac{x^3}{\sqrt{y}}$ 10. $\ln \dfrac{ac}{b^2}$

11. $t = -2, 3$ 12. $t = \dfrac{\ln 3 + 1}{2}$ 13. 2.726 or 2.727 14. $f'(x) = x^{-\frac{1}{2}} + 3e^x$

16. $g'(\theta) = 3\cos\theta$ 17. $f'(x) = \frac{3}{2} + \sin x$ 19. $f'(t) = \frac{3}{t} + \cos t$ 21. $g'\left(\frac{\pi}{4}\right) = 1 - \sqrt{2}$

22. $f'(1) = 6$ 24. $h''(2) = \frac{10}{3} - 2e^2$ 25. $f''\left(\dfrac{\pi}{2}\right) = 4 - \pi$ 26. $y - 2 = \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right)$

27. $x = 0$ 28. $\cos x$ 30. $f'(x) = 6x + 1$ (using the limit definition)

31. $f'(3) = 18$ (using the alternate form) 33. $f'(x) = 6x^2 + 6$ 35. $f'(x) = -x^{-5}$

36a. 2 38b. $V(7) \approx 40 \frac{ft}{sec}$ d. $|V(10)| = 50 \frac{ft}{sec}$ f. just before $t = 2$ *sec*

Lesson 2.5 Product Rule, Quotient Rule, Trig Rules

Product Rule: $\frac{d}{dx}\left(f(x)\bullet g(x)\right)=f(x)g'(x)+g(x)f'(x)$ or $\boxed{\frac{d}{dx}(f\bullet s)=fs'+sf'}$

Quotient Rule: $\frac{d}{dx}\frac{f(x)}{g(x)}=\frac{g(x)f'(x)-f(x)g'(x)}{\left(g(x)\right)^2}$ or $\boxed{\frac{d}{dx}\frac{t}{b}=\frac{bt'-tb'}{b^2}}$

Examples: Differentiate.

1. $f(x)=\left(3x^2-2\right)\left(2x+3\right)$

2. $y=\frac{2x^2-4x+3}{2-3x}$

3. $y=\frac{-9}{5x^2}$

4. If $f(2)=3$ and $f'(2)=-4$ find $g'(2)$ when $g(x)=x^2f(x)$.

5. $g(x)=3e^x\sin(x)$

Use the trig identities $\sin^2 x+\cos^2 x=$ $\tan x=$ —— $\sec x=$ ——

and the quotient rule to derive formulas for:

$\frac{d}{dx}\tan x=$

$\frac{d}{dx}\sec x=$

Derivations for derivative formulas for $\cot x$ and $\csc x$ are very similar.

Derivatives of the Trigonometric Functions:

1.	$\frac{d}{dx}\sin x = \cos x$	2.	$\frac{d}{dx}\cos x =$
3.	$\frac{d}{dx}\tan x =$	4.	$\frac{d}{dx}\cot x = -\csc^2 x$
5.	$\frac{d}{dx}\sec x = \sec x \tan x$	6.	$\frac{d}{dx}\csc x = -\csc x \cot x$

From now on these formulas can be used making it unnecessary to use the quotient rule process for these simple functions. By far the most common errors using them involve incorrect signs. Can you see a quick way to remember which derivatives need a negative sign?

Example 6: Find the indicated derivative.

$\frac{d}{dx}\left(4\cot(x)\right) =$

Examples: Differentiate the following without using a calculator.

7. $y = e^x \tan(x)$

8. $g(\theta) = \sec\theta \bullet \ln\theta$

Example 9: Without a calculator, find the slope of the graph of $y = -3\tan(x)$, where $x = \frac{\pi}{4}$.

Assignment 2.5

1. Use the Product Rule to differentiate. Simplify your answer. $f(x) = (x^2 - 2)(4x + 3)$
2. Differentiate without using the Product Rule. $f(x) = (x^2 - 2)(4x + 3)$
3. Use the Quotient Rule to differentiate. $f(x) = \dfrac{2x+1}{x^2+2}$
4. Differentiate without using the Quotient or Product Rules. $f(x) = \dfrac{x^2-4}{x+2}$
5. Differentiate without using the Quotient or Product Rules. $g(x) = \dfrac{2}{5x^2}$

Differentiate by any method you wish.

6. $y=\dfrac{12x^2-4}{4}$ 7. $f(t)=\dfrac{1}{t^2}(t^3-t^2)$ 8. $g(x)=2(x^2+5x-3)$ 9. $f(x)=\dfrac{2x-3}{3x-2}$

10. $y=x^2\sin x$ 11. $f(x)=\ln x\bullet\cot x$ 12. $f(x)=\dfrac{x^2-c}{x^2+c}$, c is a constant

13. $f(\theta)=\dfrac{e^\theta}{\sin\theta}$ 14. $g(x)=\dfrac{2x-4}{2\sqrt{x}}$ 15. $y=\dfrac{2(1-\sin x)}{3\cos x}$

Find the indicated derivative value. Simplify.

16. $g(x)=x\cos x$ find $g'\left(\frac{\pi}{4}\right)$ 17. $f(x)=x^2+5x-\tan x$ find $f'(0)$

18. $f(x)=\dfrac{x^2-9}{x-2}$ find $f'(1)$ 19. $h(t)=\dfrac{\sec t}{t^2}$ find $h'(\pi)$

20. Find the second derivative of $y=\dfrac{4x^3}{3}$.

21. Find an equation of the tangent line to the graph of $f(x)=\dfrac{x}{x+1}$ at the point $(-2,2)$.

22. Find the x-coordinate(s) of point(s) at which the graph of $g(x)=(2x-1)(x^2+3)$ has a tangent line parallel to the graph of $y=6x+1$.

23. Find the x-coordinate(s) of point(s) at which the graph of $f(x)=\dfrac{x^2}{x+1}$ has a horizontal tangent line.

24. Find the average rate of change of $f(x)=\dfrac{x}{\sin x}$ on the interval $\left[\frac{\pi}{6},\frac{\pi}{2}\right]$.

25. Find the rate of change of $f(x)=\dfrac{x}{\sin x}$ when $x=\frac{\pi}{6}$.

26. Use the values given in the table to find the following.

t	$r(t)$	$r'(t)$	$s(t)$	$s'(t)$
3	−2	3	2	5

a. $\frac{d}{dt}(2s(t)+4r(t))$ at $t=3$ b. $\frac{d}{dt}(r(t)\bullet s(t))$ at $t=3$ c. $\dfrac{d}{dt}\dfrac{r(t)}{s(t)}$ at $t=3$

27. Use a calculator to write an equation for the line tangent to the graph of $f(x)=\ln(|\cos x|+2)$ at the point where $x=.821$.

28. The volume formula of a cube is $V=s^3$. Find the rate of change of the volume with respect to the side length when $s=4$.

29. A particle moves horizontally according to the equation $s=t^2-5t+4$. When is the particle moving left?

30. The height in feet of a rock thrown vertically on the moon is given by the equation $h=-\frac{27}{10}t^2+27t+6$ (where time is measured in seconds). When does the rock reach its greatest height? How high is it?

31. $f(x)=4^x-3x$ (Use a calculator for these problems.)
   a. Find $f'(3)$. b. Find $f''(1)$. c. Graph $f'(x)$.

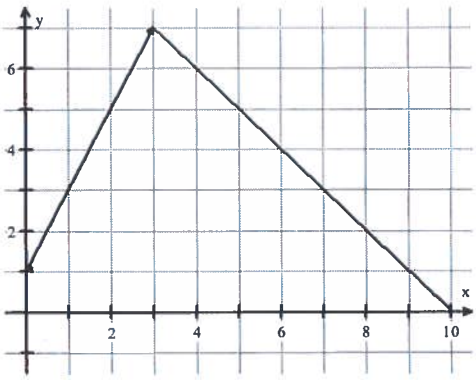
32. If $f(x)$ is an n^{th} degree polynomial, find $f^{(n+1)}(x)$.

33. True or False? If $y=f(x)g(x)$, then $y'=f'(x)g'(x)$.

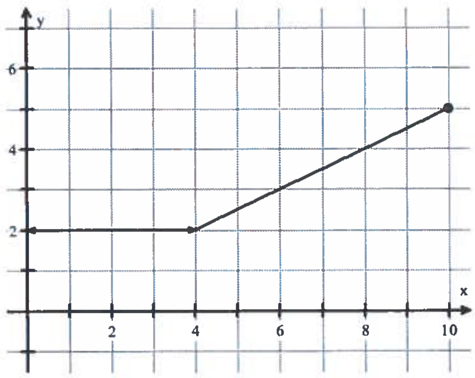
34. Find the total distance (in meters) traveled between $t=0$ and $t=4$ seconds by a particle whose position equation is $s(t)=\frac{1}{4}t^4-3t^3+4t^2-4$.

35. $f(x)$ and $g(x)$ are piecewise linear functions graphed below.

$y=f(x)$



$y=g(x)$



a. If $h(x)=f(x)\bullet g(x)$ find $h'(2)$. b. If $j(x)=\dfrac{f(x)}{2g(x)}$ find $j'(6)$.

36. The position function of a particle moving horizontally along the x-axis is $x(t)=2t^3-3t^2+2$.
   a. Find the velocity function.
   b. Find the acceleration function.
   c. When is the particle at rest?
   d. When is the particle moving left?
   e. When is the particle's velocity equal to 12?
   f. Find the particle's speed when the acceleration is equal to zero.
   g. Find the particle's displacement from $t=-1$ to $t=3$.
   h. Find the total distance traveled by the particle between $t=-1$ and $t=3$.

Selected Answers:

1. $f'(x)=12x^2+6x-8$ 3. $f'(x)=\dfrac{(x^2+2)2-(2x+1)2x}{(x^2+2)^2}$ 4. $f'(x)=1,\ x\neq -2$

5. $g'(x)=\dfrac{-4}{5x^3}$ 7. $f'(t)=1,\ t\neq 0$ 9. $f'(x)=\dfrac{5}{(3x-2)^2}$ 10. $y'=x^2\cos x+2x\sin x$

13. $f'(\theta)=\dfrac{e^{\theta}\sin\theta-e^{\theta}\cos\theta}{\sin^2\theta}$ 14. $g'(x)=\frac{1}{2}x^{-\frac{1}{2}}+x^{-\frac{3}{2}}$

16. $g'\left(\dfrac{\pi}{4}\right)=\dfrac{-\pi}{4\sqrt{2}}+\dfrac{1}{\sqrt{2}}$ 17. $f'(0)=4$ 18. $f'(1)=6$ 20. $y''=8x$

21. $y-2=1(x+2)$ 22. $x=0,\ \frac{1}{3}$ 23. $x=0$, -2 24. AROC $=\frac{1}{2}$ 25. $f'\left(\frac{\pi}{6}\right)=\dfrac{\frac{1}{2}-\frac{\pi}{6}\left(\frac{\sqrt{3}}{2}\right)}{\frac{1}{4}}$

26a. 22 b. −4 c. 4 28. $\left.\dfrac{dv}{ds}\right|_{s=4}=48$ 29. moving left on $\left(-\infty,\frac{5}{2}\right)$

30. $t=5\ sec$, $h(5)=73.5\ ft$ 32. 0 34. TD = 66.5 m 36a. $v(t)=6t^2-6t$

36b. $a(t)=12t-6$ c. $t=0,1$ d. $(0,1)$ e. $t=2,-1$ f. $\left|v\left(\frac{1}{2}\right)\right|=\frac{3}{2}$ g. disp. = 32 h. TD = 34

UNIT 2 SUMMARY

Limit Definition of the Derivative: $f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

Alternate Form of the Limit Definition of the Derivative: $f'(c)=\lim_{x\to c}\dfrac{f(x)-f(c)}{x-c}$
(Gives the value of the derivative at a single point.)

Power Rule: (short-cut derivative)
$\dfrac{d}{dx}x^n=nx^{n-1}$

Equation of a Tangent Line: Use the derivative to find m. $y-y_1=m(x-x_1)$

Nondifferentiability: (where the derivative does not exist)

1. Discontinuities 2. Sharp Turns 3. Vertical Tangent Lines

Average Rate of Change: (the slope between two points) AROC = $\dfrac{y_2-y_1}{x_2-x_1}$

Instantaneous Rate of Change: (slope at a single point) IROC = $f'(c)$

Pos.→Vel.→Acc. (differentiate)
Speed (the absolute value of velocity)
Displacement the net change in position, (final pos. − original pos.)
Total Distance total distance traveled by the object in the time interval (takes into account all direction changes)

Calculator Derivative: $f'(c)=\left.\dfrac{d}{dx}(f(x))\right|_{x=c}$ (Math 8 on TI 84)

Differentiating the Exponential Function: $\dfrac{d}{dx}e^x=e^x$

Differentiating the Natural Logarithmic Function: $\dfrac{d}{dx}\ln x=\dfrac{1}{x}$

Product Rule: $\dfrac{d}{dx}\left(f(x)\bullet g(x)\right)=f(x)g'(x)+g(x)f'(x)$ or $\boxed{\dfrac{d}{dx}(f\bullet s)=fs'+sf'}$

Quotient Rule: $\dfrac{d}{dx}\dfrac{f(x)}{g(x)}=\dfrac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$ or $\boxed{\dfrac{d}{dx}\dfrac{t}{b}=\dfrac{bt'-tb'}{b^2}}$

Derivatives of the Trigonometric Functions:

1. $\dfrac{d}{dx}\sin x=\cos x$
2. $\dfrac{d}{dx}\cos x=-\sin x$
3. $\dfrac{d}{dx}\tan x=\sec^2 x$
4. $\dfrac{d}{dx}\cot x=-\csc^2 x$
5. $\dfrac{d}{dx}\sec x=\sec x\tan x$
6. $\dfrac{d}{dx}\csc x=-\csc x\cot x$

Lesson 3.1 Chain Rule

Discovery Example: Use the product rule to differentiate.

$\frac{d}{dx}\left(x^3-3\right)^2=\frac{d}{dx}\left(\left(x^3-3\right)\left(x^3-3\right)\right)$

This can be generalized to the following rule.

Chain Rule: (used to differentiate any composition of functions)

$\frac{d}{dx}f\left(g(x)\right)=f'\left(g(x)\right)g'(x)$ or in another form $\frac{d}{dx}f(u)=f'(u)u'$ where u is a function of x.

Examples: Differentiate.

1. $f(x)=\left(3x^3-5x\right)^4$

2. $y=\sin(3x)$

3. $g(x)=\sqrt{\left(2x^2-x\right)^3}$

4. $h(\theta)=3\cos^2\theta$

5. $g(x)=\dfrac{1}{2x+1}$

6. $f(x)=3x^2\sqrt[3]{9-4x^2}$

7. $f(t)=\sin^3\left(4t^2\right)$

8. Given this data find the following:

x	$p(x)$	$q(x)$	$p'(x)$	$q'(x)$
2	5	3	7	6
3	4	$\frac{1}{2}$	8	$\frac{3}{2}$

a. If $f(x)=p(x)\bullet q(x)$ find $f'(2)$.

b. If $h(x)=p\left(q(x)\right)$ find $h'(2)$.

Note: You must quickly learn to distinguish between the Chain Rule and the Product Rule!

Assignment 3.1

Find the derivative without using a calculator.

1. $y=(3x+5)^3$
2. $f(x)=3(7x+5)^4$
3. $y=\sqrt{2-3x}$
4. $f(t)=\dfrac{1}{(1-t)^2}$
5. $y=\sqrt[3]{(x^2+1)^2}$
6. $g(x)=x(2x+3)^3$
7. $y=\dfrac{1}{\sqrt{x+1}}$
8. $f(x)=\dfrac{3x-2}{x+1}$
9. $g(x)=\sec(4x)$
10. $y=4\tan(2x)$
11. $f(\theta)=\frac{1}{2}\sin^2(3\theta)$
12. $y=\sqrt{\dfrac{4x^3-2x}{2x}}$

Find an equation of the line tangent to the graph of f at the given point without using a calculator.

13. $f(x)=\sqrt{2x^2+2}$ at $(-1,2)$
14. $f(x)=\dfrac{x+4}{x}$ at $(2,3)$
15. $f(x)=\dfrac{1}{\sqrt{(9x)^3}}$ at $\left(\frac{1}{4},\frac{8}{27}\right)$
16. $f(x)=\dfrac{1}{x^2}+\sqrt{\cos x}$ at $\left(2\pi,\dfrac{1}{4\pi^2}+1\right)$

Find the indicated derivatives.

17. $\frac{d}{dx}(2\sin x-3)^4$
18. $\dfrac{d^2}{dt^2}(t^2-1)^{\frac{3}{2}}$

19. Find the point(s) at which a line tangent to the graph of $f(x)=(2x-3)^3$ is parallel to the graph of $y=24x-7$. You may use a calculator.

20. If $g(x)=(f(x))^3$, $f(1)=2$, and $f'(1)=4$, find $g'(1)$.

21. Given these values

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	3	2	-1	4
3	-2	$\frac{1}{2}$	6	5

find the following derivatives.

a. $\frac{d}{dx}g(f(x))$ at $x=2$

b. $\frac{d}{dx}(g(x)f(x))$ at $x=2$

c. $\frac{d}{dx}\sqrt{g(x)}$ at $x=2$

d. $\frac{d}{dx}\dfrac{g(x)}{f(x)}$ at $x=2$

22. Find an equation of the line tangent to the graph of $y=3^x$ when $x=1.2$. You will need a calculator.

23. If $f(x)=5x^2-3x+2$, use the alternate form of the limit definition of the derivative to find $f'(1)$. Show steps with correct limit symbolism.

24. The position function of an object is $s(t)=\frac{2}{3}t^3-\frac{5}{2}t^2+3$. Find the **displacement** of the object between time $t=0$ and time $t=3$. Find the **total distance** traveled by the object between time $t=0$ and time $t=3$.

25. A sprinter in a 100 meter dash is clocked every 10 meters as shown in the table.

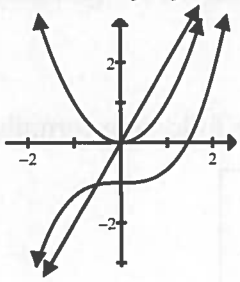
meters covered	0	10	20	30	40	50	60	70	80	90	100
time in seconds	0	1.4	2.5	3.5	4.5	5.4	6.6	7.6	8.5	9.3	10.2

a. How long does it take the sprinter to finish the race?
b. What is the sprinter's average speed over the first 50 meters?
c. What is the sprinter's average speed over the second 50 meters?
d. What is the sprinter's average speed between 20 and 30 meters?
e. What is the sprinter's average speed between 30 and 40 meters?
f. What is the sprinter's approximate speed as he passes the 30 meter mark?
g. During which ten meter segment of the race is the sprinter running the fastest?
h. During which portion of the race is the sprinter accelerating the fastest?

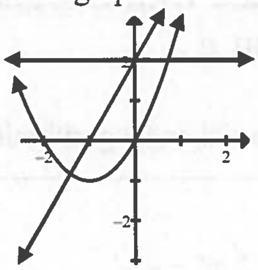
26. A particle is moving vertically on the y-axis with position $y(t)=\frac{1}{3}t^3+t^2-3t+7$. When is the particle moving upward? Do not use a calculator.

The functions f, f', and f'' are shown on the same graph. Determine which is which.

27.



28.



29. The height (position) in feet of a ball thrown straight down from a tall building is given by $h=-16t^2-22t+220$ (where t is measured in seconds).
a. What is the ball's initial height?
b. What is the ball's initial velocity?
c. What is the speed of the ball 3 seconds after it was thrown?
d. How far does the ball travel in the first 3 seconds?

Selected Answers:

1. $y'=9(3x+5)^2$ 3. $y'=-\frac{3}{2}(2-3x)^{-\frac{1}{2}}$ 4. $f'(t)=\dfrac{2}{(1-t)^3}$

6. $g'(x)=6x(2x+3)^2+(2x+3)^3$ 7. $y'=-\frac{1}{2}(x+1)^{-\frac{3}{2}}$ 8. $f'(x)=\dfrac{5}{(x+1)^2}$

9. $g'(x)=4\sec(4x)\tan(4x)$ 10. $y'=8\sec^2(2x)$ 11. $f'(\theta)=3\sin(3\theta)\cos(3\theta)$

12. $y'=\dfrac{2x}{\sqrt{2x^2-1}}$ 13. $y-2=-1(x+1)$ 15. $y-\frac{8}{27}=-\frac{16}{9}\left(x-\frac{1}{4}\right)$

More Selected Answers:

16. $y-\left(\frac{1}{4\pi^2}+1\right)=-\frac{1}{4\pi^3}(x-2\pi)$ 17. $8\cos x(2\sin x-3)^3$ 18. $3t^2(t^2-1)^{-\frac{1}{2}}+3(t^2-1)^{\frac{1}{2}}$

19. $\left(\frac{5}{2},8\right),\left(\frac{1}{2},-8\right)$ 20. $g'(1)=48$ 21a. -5 b. 10 c. $\sqrt{2}$ d. $\frac{14}{9}$

22. $y-3.737=4.105(x-1.2)$ or $y-3.737=4.106(x-1.2)$

24. disp. = -4.5, TD = 5.916 or 5.917 25b. Avg. Speed = 9.259 $\frac{m}{sec}$ e. 10 $\frac{m}{sec}$

25g. between 80 and 90 m 26. $t<-3$ or $t>1$

27. f looks like a cubic, f' looks like a parabola, and f'' looks like a line

29a. $h(0)=220\ ft$ c. $|v(3)|=118\ \frac{ft}{sec}$

Lesson 3.2 Chain Rule with Exponentials and Logs including bases other than e

Applying the chain rule to exponential and logarithmic functions gives us the following formulas.

Given u is a function of x.

Since $\frac{d}{dx}e^x=e^x$, it follows that $\frac{d}{dx}e^u=e^u u'$.

Since $\frac{d}{dx}\ln x=\frac{1}{x}$, it follows that $\frac{d}{dx}\ln u=\frac{1}{u}u'$ or more simply $\frac{d}{dx}\ln u=\frac{u'}{u}$.

Using the inverse relationship of logs and exponentials we can derive a formula for differentiating exponentials with bases other than e.

$\frac{d}{dx}a^x=\frac{d}{dx}e^{(\ln a^x)}$	inverse property of exponentials and logs
$=\frac{d}{dx}e^{(x\ln a)}$	log property
$=$	chain rule derivative
$=$	same log property in reverse
$=$	same inverse property of exponentials and logs in reverse

$\frac{d}{dx}a^x=a^x\ln a$ and the chain rule form $\frac{d}{dx}a^u=a^u u'\ln a$

Using the change of base formula $\log_a x = \dfrac{\ln x}{\ln a}$ we can derive a formula for differentiating logs with bases other than e.

$\frac{d}{dx}\log_a x = \frac{d}{dx}\frac{\ln x}{\ln a}$ change of base formula

$= \frac{1}{\ln a}\frac{d}{dx}\ln x$ since $\ln a$ is a constant

$=$ derivative of $\ln x$

$$\boxed{\frac{d}{dx}\log_a x = \frac{1}{x\ln a} \text{ and the chain rule form } \frac{d}{dx}\log_a u = \frac{u'}{u\ln a}}$$

<u>Examples:</u>

Differentiate each of the following.

1. $f(x) = e^{x^2+3x}$

2. $g(x) = \ln(3x)$ use $\frac{d}{dx}\ln u = \frac{u'}{u}$

3. $g(x) = \ln(3x)$ use $\ln(3x) = \ln 3 + \ln x$

4. $y = \ln(t^2 + t)$

5. $h(x) = x\ln x$

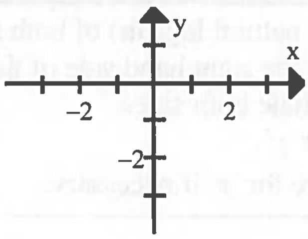
6. $g(t) = e^{\frac{-3}{t}}$

7. $f(v) = 3^{\sqrt{v}}$

<u>Example 8:</u>

a. Graph $y = \ln|x|$ in the coordinate plane at right.

b. For the right half of the graph the absolute value has no effect, so $\frac{d}{dx}\ln|x| = \frac{1}{x}$ for $x > 0$.



What is $\frac{d}{dx}\ln|x|$ when $x < 0$?

Plot some slopes and think about it.

$\frac{d}{dx}\ln|x| =$ for $x < 0$.

When differentiating the natural log function, $\ln|x|$ and $\ln|u|$ the absolute value can be ignored. In these cases, absolute value may change the domain of the function – but not the derivative.

Example 9:

Find $\frac{d}{dy}\ln\left|5-2y^3\right|$

When possible, expand logarithmic functions before differentiating them.

Example 10: Differentiate $y=\ln\dfrac{x\sqrt{2x+1}}{x^2+1}$.

First, rewrite as $y=$

Then, $y'=$

Example 11: If $y=\log_2\left(x^2+1\right)$, find $y'(2)$

So far, we have not encountered any functions having both a variable base and a variable exponent (for example, $y=x^x$, $y=\left(x^2+1\right)^{x-1}$, $y=x^{\ln x}$). Such functions are neither power functions nor exponential functions. They can only be differentiated using a process called **logarithmic differentiation**.

Procedure for Logarithmic Differentiation: y = variable base$^{\text{variable exponent}}$

1. Take the natural log (**ln**) of both sides of the equation.
2. Simplify the right hand side of the equation (Log Property 3).
3. Differentiate both sides.
4. Solve for y'.
5. Substitute for y if necessary.

Example 12: Differentiate $y=(x^2+1)^{x-1}$. Express your answer in terms of x.

Assignment 3.2

Differentiate in Problems 1-18 without using a calculator.

1. $f(x)=e^{5x+1}$
2. $g(y)=2^{-5y}$
3. $h(t)=5^{t^2-2t}$
4. $y=\ln x^5$
5. $y=e^{\sqrt{x}-2}$
6. $y=e^{2\sin x}$
7. $y=\ln\left(x^2-5x\right)$
8. $f(x)=2x^2e^{3x}$
9. $g(x)=\dfrac{e^x}{\sin x^2}$
10. $g(y)=(\ln y)^4$
11. $y=\ln\dfrac{x^2}{x-1}$
12. $h(x)=(e^{-2x}-1)^3$
13. $f(x)=\ln\left|x^3+2x\right|$
14. $f(x)=\log_5\left|x^2-1\right|$
15. $g(t)=\ln\sqrt{t^3-t}$
16. $h(x)=\tan x\ln x$
17. $y=x^{3x}$
18. $y=(2x-1)^{\ln x}$

19. For $y=\dfrac{e^{-3x}}{3}+2e^{2x}$, find $\dfrac{d^2y}{dx^2}$ without using a calculator.

20. Write an equation for the line tangent to the graph of $f(x)=e^{x^2-4x+3}$ at the point where $x=1$ without using a calculator.

21. Find an equations for the tangent line to $y=x^2-\ln(x+1)+1$ at the point (0, 1) without using a calculator.

22. Use the data given in the table to find the following:

a. $\frac{d}{dx}\left(f(x)\bullet g(x)\right)$ at $x=1$ b. $\frac{d}{dx}\dfrac{f(x)}{g(x)}$ at $x=1$

c. $\frac{d}{dx}f\left(g(x)\right)$ at $x=1$ d. $\frac{d}{dx}g\left(f(x)\right)$ at $x=1$

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	2	5	1	3
2	4	−6	−2	7

Differentiate each function in Problems 23-26. Assume C and c are constants.
<u>Note</u> the differences in each problem.

23. $y=C^c$
24. $y=x^c$
25. $y=c^x$
26. $y=x^x$

Use the function $g(x)=\begin{cases}3x-1, & x\le 0\\ \frac{4}{27}x^3-1, & 0<x<3\\ 4, & x\ge 3\end{cases}$ for Problems 27-34.

27. Sketch a graph of $g(x)$.

Find the following limits.

28. $\lim\limits_{x\to 0}g(x)$
29. $\lim\limits_{x\to 3}g(x)$
30. $\lim\limits_{x\to 3^-}g(x)$
31. $\lim\limits_{x\to \frac{3}{\sqrt[3]{4}}}g(x)$

32. List all discontinuities of $g(x)$.

33. Find $g'(x)$

34. List all x-values at which g is **not** differentiable.

Selected Answers:

1. $f'(x)=5e^{5x+1}$ 3. $h'(t)=5^{t^2-2t}(2t-2)\ln 5$ 4. $y'=\frac{5}{x}$ 6. $y'=2e^{2\sin x}\cos x$

8. $f'(x)=6x^2e^{3x}+4xe^{3x}$ 11. $y'=\frac{2}{x}-\frac{1}{x-1}$ 13. $f'(x)=\frac{3x^2+2}{x^3+2x}$

15. $g'(t)=\frac{1}{2}\bullet\frac{3t^2-1}{t^3-t}$ 16. $h'(x)=\frac{\tan x}{x}+\ln x\sec^2 x$ 17. $y'=(3+3\ln x)x^{3x}$

18. $y'=\left(\ln x\frac{2}{2x-1}+\ln(2x-1)\frac{1}{x}\right)y$ 19. $\frac{d^2y}{dx^2}=3e^{-3x}+8e^{2x}$ 20. $y-1=-2(x-1)$

21. $y=-x+1$ 22a. 11 b. -1 c. 15 d. 35 23. $y'=0$ 25. $y'=c^x\ln c$

26. $y'=(1+\ln x)x^x$ 28. -1 30. 3 31. 0

Lesson 3.3 Implicit Differentiation

All the derivatives you have done to this point have been of <u>explicit equations</u>. For example $y=x^2$, $y=\frac{x}{x-1}$, and $y=\sqrt{2x+1}$ all <u>explicitly</u> express y in terms of x.
In this lesson you will be working with <u>implicit equations</u> where the relationship between x and y is only <u>implied</u>. $x^2+y^2=1$, $xy+y^2=3$, and $xy=1$ are all examples of <u>implicit</u> equations.
It is possible to differentiate implicit equations using **<u>implicit differentiation</u>**.

<u>Procedure:</u>
1. Differentiate both sides with respect to x.
(Remember the y' "chain rule factor" for any term involving y.)
2. Collect all y' (or $\frac{dy}{dx}$) terms on one side of the equation.
3. Factor out y'.
4 Divide to solve for y'.

<u>Warm-up Examples:</u> Differentiate.

1. $y=x$ 2. $y=x^2$ 3. $y=(2x-1)^2$ 4. $y=(f(x))^2$ 5. $x=y^2$

Examples:

1. Given $x^2 - 2y^3 + 3x = 6$, find y'.

2. Find the slope of the line tangent to the graph of $x^2 + 4y^2 = 25$ at (3,2).

3. Given $x^3 - 2xy + y^3 = 5x$, find $\frac{dy}{dx}$ and evaluate at the point (1,2).

4. Given $x^2 + y^2 = 3$, find y'' in terms of y.

5. Given $\cot y = x - y$ find $\frac{dy}{dx}$.

Assignment 3.3

Use implicit differentiation to find y'.

1. $x^2 + y^2 = 4$
2. $xy = 7$
3. $x^2y^3 + y = x^3$
4. $\cos x - 2\sin(2y) = 4$
5. $\tan(xy) = y^2$
6. $\sqrt{x} + \sqrt{y} = 9$
7. $y^2 = \dfrac{x}{x+1}$

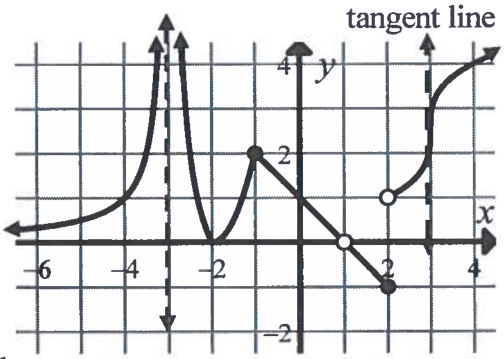
Find $\frac{dy}{dx}$ and evaluate the derivative at the given point.

8. $x^3 - xy = 3$ at $(1,-2)$
9. $e^y + \ln\left(\frac{1}{2}y\right) - x^2 = 0$ at $(e,2)$
10. $x^3 + y = 2xy$ at $(1,1)$
11. $\sec(x+y) = x+1$ at $(0,0)$

12. Use implicit differentiation to find y' and evaluate the derivative at the point $(3,-4)$ if $x^2+y^2=25$.
13. Use explicit differentiation to find the slope of the graph of $x^2+y^2=25$. at the point $(3,-4)$. You must first solve for y. If your answer does not match the correct answer to Problem 12, find your mistake.
14. Given $x^2+xy=4$, find $\frac{d^2y}{dx^2}$ in terms of x and y.
15. Find an equation of the line tangent to the circle $x^2+y^2=169$ at the point $(-12,5)$.
16. Find the point(s) at which the graph of $x^2+2y^2-4y-6=0$ has a horizontal tangent line.
17. Find the point(s) at which the graph of $x^2+2y^2-4y-6=0$ has a vertical tangent line.

18. If f is an unknown differentiable function of y where $f(3)=0$ and $f'(3)=2$ and the ordered pair (1,3) is a point on the curve $\frac{f(y)}{x}=x^3-y^2+8$ find $\left.\frac{dy}{dx}\right|_{(1,3)}$.

19. The volume formula for a sphere is $V=\frac{4}{3}\pi r^3$. Find the rate of change of the volume with respect to the radius $\left(\frac{dV}{dr}\right)$, when the radius is 4.



Use the figure at the right for Problems 20-27.

20. Find $\lim_{x\to 1} f(x)$.
21. Find $\lim_{x\to -3} f(x)$.
22. Find $\lim_{x\to -\infty} f(x)$.
23. Find $\lim_{x\to 2} f(x)$.
24. Find $\lim_{x\to 2^+} f(x)$.
25. List the x-values of all discontinuities of $f(x)$.
26. Which of these discontinuities are removable?
27. List the x-values where $f(x)$ is not differentiable.

28. An object's velocity in meters per second is $v=2t^3-9t^2+12t-5$. Find the object's speed each time the acceleration is zero.

29. Find $\frac{d}{dx}\left(\frac{x^2-2x}{x^2}\right)$ without using a calculator.

30. If $f(x)=3^{x^2}-\log_3(x^2+x)$ find $f'(x)$ without using a calculator.

Use the function $g(x)=\begin{cases} x-1, & x\le 0 \\ x^2-1, & 0<x<2 \\ 4, & x\ge 2\end{cases}$ for Problems 31-38.

31. Sketch a graph of $g(x)$.

Find the following limits.

32. $\lim\limits_{x\to 0} g(x)$ 33. $\lim\limits_{x\to 2} g(x)$ 34. $\lim\limits_{x\to 2^-} g(x)$ 35. $\lim\limits_{x\to 1} g(x)$

36. List all discontinuities of $g(x)$.

37. Find $g'(x)$

38. List all x-values at which g is **not** differentiable.

Selected Answers

1. $y'=-\dfrac{x}{y}$ 2. $y'=-\dfrac{y}{x}$ 3. $\dfrac{dy}{dx}=\dfrac{3x^2-2xy^3}{3x^2y^2+1}$ 4. $y'=\dfrac{\sin x}{-4\cos(2y)}$ 6. $y'=-\sqrt{\dfrac{y}{x}}$

8. $\dfrac{dy}{dx}=\dfrac{y-3x^2}{-x}=\dfrac{3x^2-y}{x}$, $\left.\dfrac{dy}{dx}\right|_{(1,-2)}=5$ 9. $\dfrac{dy}{dx}=\dfrac{2x}{e^y+\frac{1}{y}}=\dfrac{2xy}{ye^y+1}$; $\left.\dfrac{dy}{dx}\right|_{(e,2)}=\dfrac{4e}{2e^2+1}$

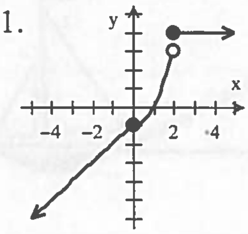
10. $y'=\dfrac{2y-3x^2}{1-2x}$; $y'(1,1)=1$ 12. $y'(3,-4)=\dfrac{3}{4}$ 14. $y''=\dfrac{2x+2y}{x^2}$

15. $y-5=\frac{12}{5}(x+12)$ 16. $(0,3),(0,-1)$ 17. $\left(\pm\sqrt{8},1\right)$ 18. $\left.\frac{dy}{dx}\right|_{(1,3)}=\frac{3}{8}$

19. $\frac{dV}{dr}(4)=64\pi$ 28. $|v(1)|=0\,\frac{m}{sec}$, $|v(2)|=1\,\frac{m}{sec}$ 29. $\dfrac{2}{x^2}$

30. $f'(x)=3^{x^2}2x\ln 3-\dfrac{2x+1}{\ln 3\left(x^2+x\right)}$

31.



32. -1 33. DNE 34. 3 35. 0 36. $x=2$

Lesson 3.4 Related Rates Story Problems

In related rate story problems, the idea is to find a rate of change (with respect to time) of one quantity by using the rate of change (with respect to time) of a related quantity.

Procedure for **Related Rate Problems**

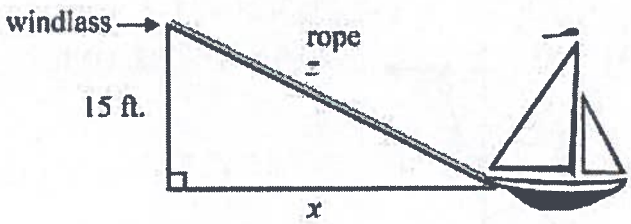
1. Draw a figure (if necessary) and choose variables for all unknowns.
2. Write what is given and what is to be found using your variables and $\frac{d}{dt}$ symbols.
3. Write an equation relating the variables.
   (a) If a quantity is **changing** it must be represented with a variable letter.
   (b) If a quantity is **constant** it must be represented with a number value.
   (c) Look for secondary relationships between quantities to reduce the number of variables.
4. Implicitly differentiate both sides with respect to t.
5. Substitute number values and solve.
6. Write a sentence to explain the meaning of your answer.

Examples:

1. In this problem, the first three steps of the procedure are done. We need to complete only the last two steps.

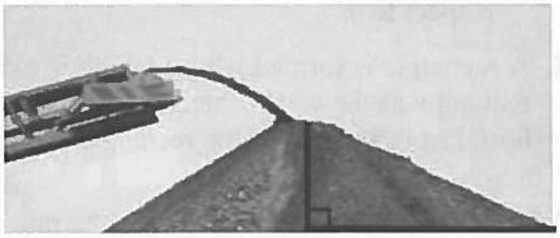
   Given: $\frac{dx}{dt} = -6, \; y = 4, \; xy = 12$ Find: $\frac{dy}{dt}$

2. A windlass is used to tow a boat to a dock. The rope is attached to the boat at a point 15 feet below the level of the windlass. If the windlass pulls in rope at the rate of 30 feet per minute, at what rate is the boat approaching the dock when there is 25 feet of rope out? Write a sentence explaining the meaning of the answer.



3. A policeman traveling south toward an intersection spots a speeding car traveling east away from the intersection. When the policeman is .6 mi from the intersection and the car is .8 mi from the intersection, the policeman's radar shows the distance between them is increasing at the rate of 20 mph. If the speed of the police car is 60 mph, what is the speed of the car? Write a sentence explaining the meaning of the answer.

4. Gravel is falling on a conical pile at the rate of $10\,\frac{ft^3}{\min}$. At all times, the radius of the cone is twice the height of the cone. Find the rate of change of the height of the pile when the radius of the pile is $6\,ft$. $V=\frac{1}{3}\pi r^2 h$ Write a sentence explaining the meaning of the answer.



Assignment 3.4

1. Given: $\frac{dx}{dt}=3,\ x=4,\ y=x^2-2x$ Find: $\frac{dy}{dt}$
2. Given: $\frac{dx}{dt}=10,\ y=5,\ x^2+y^2=169$ Find: $\frac{dy}{dt}$

Write sentences explaining the meaning on problems 3-12.

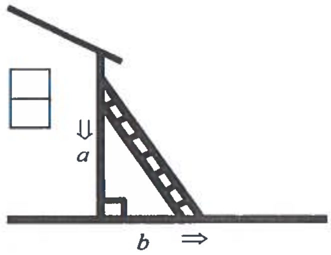
3. The radius of a circle is increasing at the rate of $3\,{}^{cm}\!/_{sec}$. At the instant the radius is 4 cm., find:
   a. the rate of change of the area of the circle.
   b. the rate of change of the circumference of the circle.

4. A large spherical balloon is inflated at the rate of $18\,\frac{ft^3}{min}$. When the radius is 2 feet, how fast is the radius changing? $V = \frac{4}{3}\pi r^3$

5. All edges of a cube are expanding at the rate of $2\,\frac{in}{sec}$. At the instant the edges are all 5 inches long find:
a. the rate of change of the surface area of the cube.
b. the rate of change of the volume of the cube.

6. A point is moving on the curve $y = \sqrt{x}$ so that the x-coordinate is changing at the rate of $3\,{}^{cm}\!/_{min}$. When the y-coordinate is 25 centimeters, find the rate of change of the y-coordinate.

7. A 10 foot ladder is leaning against a house as shown. The base of the ladder is pulled away from the house at the rate of $1.5\,\frac{ft}{sec}$. At the instant the base of the ladder is 6 feet from the house, answer these questions.



a. Find the rate at which the top of the ladder is moving.
b. Find the rate at which the area of the triangle formed is changing.
c. Is the area from part b increasing or decreasing? Would your answer be the same at all times?
d. Let $\theta =$ the angle formed by the ladder and the house, find the rate of change of θ with respect to t.

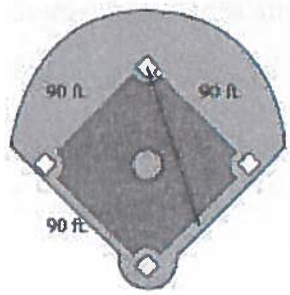
8. A rectangle is formed whose length is twice the width. It is enlarged to a similar (same shape) rectangle as the width changes at the rate of 4 inches per minute. When the width is 10 inches, how fast is the area of the rectangle changing?

9. Sand is added to a conical pile at the rate of $8\,\frac{ft^3}{\text{min}}$. If the diameter of the cone is twice the height of the cone, find the rate of change of the height of the pile when the volume of the pile is 1125π cubic feet. $V = \frac{1}{3}\pi r^2 h$

10. A bicyclist and a jogger are moving on two perpendicular intersecting streets. The bicyclist is moving north toward the intersection at the rate of 60 feet per second and the jogger is moving west away from the intersection at the rate of 15 feet per second. What is the rate at which the straight line distance between them is changing when the bicyclist is 120 feet from the intersection and the jogger is 50 feet from the intersection?

11. At halftime of a football game, two fan volunteers are attached together with a bungee cord and positioned on the sideline at the 50 yard line. One fan runs along the sideline at $4\,\frac{yd}{sec}$ and the other runs on the 50 yard line toward midfield at $3\,\frac{yd}{sec}$. At what rate is the bungee cord being stretched when the sideline runner reaches the 30 yard line?

12. A base runner who is running at the rate of 25 feet per second is halfway to first base. Find the rate of change of his distance from second base.



Differentiate.

13. $f(x)=\dfrac{x^3-3x}{x^2-4}$

14. $y=\sqrt{2x-1}\,\sqrt[3]{3x+1}$

15. Find an equation of the tangent line to the graph of $y=\left(3x^2+2\right)^4$ when $x=0$.

16. If $f(x)=\dfrac{x^2+3}{x}$
   a. find the average rate of change from $x=1$ to $x=3$.
   b. find the instantaneous rate of change at $x=3$.

Answer these same three questions for Problems 17-20.
(a) What is the derivative of the function from the left at $x=1$?
(b) What is the derivative of the function from the right at $x=1$?
(c) Is the function differentiable at $x=1$?

17. $f(x)=\begin{cases}(x-1)^3, & x\le 1\\ (x-1)^2, & x>1\end{cases}$

18. $f(x)=\begin{cases}\frac{1}{2}x^2-\frac{1}{2}, & x\le 1\\ x-1, & x>1\end{cases}$

19. $f(x)=\begin{cases}\frac{1}{2}x^2-\frac{1}{2}, & x<1\\ x-1, & x>1\end{cases}$

20. $f(x)=\begin{cases}x^2-2x, & x\le 1\\ x^2-2x+1, & x>1\end{cases}$

21. If $f(x)=x^2-x$, use the limit definition of the derivative to find $f'(x)$.

22. If $y=\dfrac{1}{x^2}+\sin(4x)$ find $\dfrac{d^2y}{dx^2}$.

23. Find an equation of the tangent line to the graph of $y=\ln|\sec x|$ at the point where $x=\pi$.

24. If $r(3)=-2$, $s(3)=2$, $r'(3)=3$, $s'(3)=5$, and $r'(2)=4$, find $\frac{d}{dt}r(s(t))$ at $t=3$

Selected Answers

1. $\dfrac{dy}{dt}=18$ 2. $\dfrac{dy}{dt}=\pm 24$

Write sentence answers for 3-12.

3a. $\dfrac{dA}{dt}=24\pi\dfrac{\text{cm}^2}{\text{sec}}$ b. $\dfrac{dC}{dt}=6\pi\dfrac{\text{cm}}{\text{sec}}$ 4. $\dfrac{dr}{dt}=\dfrac{9}{8\pi}\dfrac{\text{ft}}{\text{min}}$ 5a. $\dfrac{dA}{dt}=120\dfrac{in^2}{sec}$ b. $\dfrac{dV}{dt}=150\dfrac{in^3}{sec}$

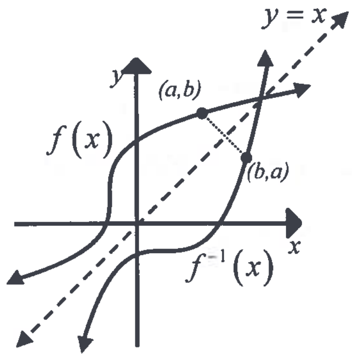
6. $\dfrac{dy}{dt}=\dfrac{3}{50}\dfrac{cm}{min}$ 7a. $\dfrac{da}{dt}=-\dfrac{9}{8}\dfrac{ft}{sec}$ b. $\dfrac{dA}{dt}=\dfrac{21}{8}\dfrac{ft^2}{sec}$ c. increasing, no d. $\dfrac{d\theta}{dt}=\dfrac{3}{16}\dfrac{radians}{sec}$

8. $\dfrac{dA}{dt}=160\dfrac{in^2}{min}$ 9. $\dfrac{dh}{dt}=\dfrac{8}{225\pi}\dfrac{ft}{min}$ 10. $-49.615\frac{ft}{sec}$ 11. $5\frac{yd}{sec}$ 12. $-11.180\frac{ft}{sec}$

15. $y=16$ 16a. AROC $=0$ b. $f'(3)=\dfrac{2}{3}$ 17a. 0 b. 0 c. yes 18a. 1 b. 1 c. yes

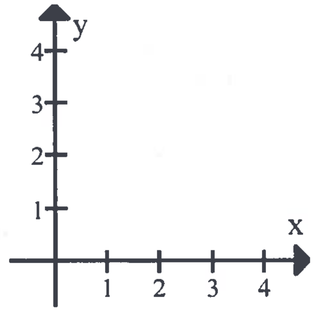
22. $y''=6x^{-4}-16\sin(4x)$ 24. 20

Lesson 3.5 Derivatives of Inverse Functions



At the right are the graphs of a function $f(x)$ and its inverse $f^{-1}(x)$. Remember that if the graph of f contains the point (a, b), then the graph of f^{-1} contains the point (b, a). Also, the graph of f^{-1} is the reflection of the graph of f across the line $y = x$.

From the graphs, do you see a relationship between the slope of the graph of f at (a, b) and the slope of the graph of f^{-1} at (b, a)?



Example 1: Let $f(x) = \sqrt{x}$.

a. Sketch the graph of $f(x)$.

b. Find $f^{-1}(x)$. Hint: You must list a domain restriction.

c. Sketch the graph of $f^{-1}(x)$ in the same coordinate plane as the graph of $f(x)$.

d. Differentiate both $f(x)$ and $f^{-1}(x)$.

e. Find the slope of the graph of $f(x)$ at (4, 2) and the slope of the graph of $f^{-1}(x)$ at (2, 4).

f. What conclusion can you make about these slopes?

Since slope $= m = \dfrac{\Delta y}{\Delta x}$, it should make sense that switching x and y (for inverse functions) should produce reciprocal slopes for inverse functions.

Derivatives of Inverse Functions:

If (a, b) is a point on f, then (b, a) is a point on f^{-1}, and $(f^{-1})'(b) = \dfrac{1}{f'(a)}$

or if f and g are inverse functions, then $g'(x) = \dfrac{1}{f'(g(x))}$.

Derivatives of inverses have reciprocal slopes at "image points" (points reflected across $y = x$). (a, b) and (b, a) are image points.

Note: When finding derivatives of inverse functions, do not use the same x-value for both f and f^{-1}. This hardly ever works. (It only works when the x- and y-values of the ordered pairs are the same.)

Example 2: Let f and g be inverse functions such that f has the function and derivative values shown in the table.

x	$f(x)$	$f'(x)$
-1	1	$\frac{3}{2}$
0	2	2
1	5	$\frac{1}{2}$

From the given information, find each of the following if possible. The table below is a good way to organize the given information.

a. $g'(1)$ b. $g'(2)$ c. $g'(3)$

d. $g'(0)$ e. $g'(5)$

$f(x)$ points		$g(x)$ points	
(,)	$m=$	(,)	$m=$
(,)	$m=$	(,)	$m=$
(,)	$m=$	(,)	$m=$

Example 3:

If $f(3)=28$ where $f(x)=x^3-2x+7$ find $\left(f^{-1}\right)'(28)$.

Assignment 3.5

1. $f(x)=x^3-1$. Let $g(x)=f^{-1}(x)$.
   a. Find $g(x)$.
   b. Graph $f(x)$ and $g(x)$ in the same coordinate plane.
   c. Find $f'(x)$ and $g'(x)$.
   d. Find $f'(1)$ and $g'(0)$.
   e. What is the relationship between the slopes in Part d?

2. $f(x)=\sqrt{x+1}$. Let $g(x)=f^{-1}(x)$
   a. Find $g(x)$.
   b. Graph $f(x)$ and $g(x)$ in the same coordinate plane.
   c. Find $f'(x)$ and $g'(x)$.
   d. Find $f'(3)$ and $g'(2)$.
   e. What is the relationship between the slopes in Part d?

3. Let f and g be inverse functions such that: $\begin{cases} f(-1)=0,\ f(0)=1, \text{ and } f(1)=3 \\ f'(-1)=\frac{4}{3},\ f'(0)=\frac{1}{5}, \text{ and } f'(1)=2 \end{cases}$

Find each of the following (if possible).

a. $g'(-1)$ b. $g'(0)$ c. $g'(1)$ d. $g'(2)$ e. $g'(3)$

4. If $f(2)=3$ and $f'(2)=4$, find $\left(f^{-1}\right)'(3)$.

5. If $(1, 2)$ is a point on $f(x)=x^3+2x-1$, find $\left(f^{-1}\right)'(2)$.

6. If $f(x)=x^3-\frac{4}{x}$ $(x>0)$, find $\left(f^{-1}\right)'(6)$.

f and g are inverse functions in Problems 7-9. Find g' at the given value.

7. $f(2)=5$
$f'(2)=\frac{-2}{3}$
$g'(5)=$

8. $f(x)=x^5+2x^3-1$
$f(1)=2$
$g'(2)=$

9. $f(x)=e^{x-2}$
$g'(1)=$

10. Find $\left(f^{-1}\right)'(5)$ if $f(x)=x^3-4x^2+3x-7$ on the interval $[3,\infty)$.

11. f and g are inverse functions. The graph of g passes through the points $(-1,2)$, and $(2,-1)$. $f'(-1)=-2$ and $f'(2)=-1$. Find:

a. $g'(-1)$ b. $g'(2)$

Differentiate in Problems 12-15 without using a calculator.

12. $g(t)=-4\cot\left(3t^2\right)$ 13. $y=\dfrac{\tan t}{t}$ 14. $f(x)=e^{\sec x}$ 15. $h(\theta)=2\theta\cos\theta-\sin\theta$

Find the indicated derivatives for Problems 16-25 without using a calculator..

16. $y=x^5+\dfrac{1}{x}$, find $\dfrac{d^2y}{dx^2}$

17. $f(x)=\dfrac{x^2}{e^x}$, find $f'(x)$

18. $g(t)=(2t-1)^5$, find $g''(t)$

19. $\frac{d}{dx}\left(6^{2x}-3\right)^4=$

20. $\frac{d}{dx}\ln\left|\sin^3 x\right|=$

21. $\sin\left(t^2\right)=ye^t$, find $\frac{dy}{dt}$

22. $\sin(y-2x)=x^2-10$, find $\frac{dy}{dx}$

23. $y=e^{x^2-1}$, find $\dfrac{d^2y}{dx^2}$

24. $\dfrac{d}{dx}\left(\dfrac{\ln\left|x^2-1\right|}{x}\right)=$

25. $y^3-x=y\ln x$, find $\frac{dy}{dx}$

26. An object moves along a vertical path with its position at time t (in seconds), according to the equation $y(t)=te^{t+1}$ (where y is measured in centimeters (cm)).
   a. Find the object's position at time $t=-2\,sec.$
   b. Find the equation for the object's velocity.
   c. Find the object's velocity at time $t=-2\,sec.$
   d. Find an equation for the object's acceleration.
   e. Find $a(-2)$.
   f. For what interval(s) of time is the object moving downward?

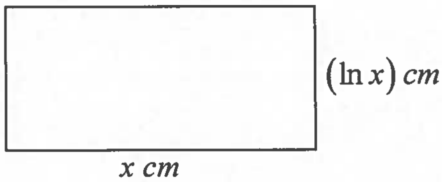
Use a calculator to evaluate problems 27, 28.

27. find $f'(1.237)$ for $f(x)=\sqrt{e^x+5}$ 28. find $f''(1.237)$ for $f(x)=\sqrt{e^x+5}$

For Problems 29-32, find each limit (if it exists) without using a calculator.

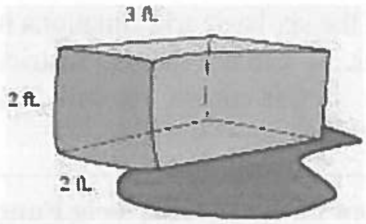
29. $\lim\limits_{x\to -3}\dfrac{2x-1}{3x+4}$ 30. $\lim\limits_{t\to 2}\dfrac{t^2-4}{t^2-3t+2}$ 31. $\lim\limits_{x\to -1}\dfrac{x+1}{\sqrt{x^2+1}}$ 32. $\lim\limits_{x\to -\infty}\dfrac{x+1}{\sqrt{x^2+1}}$

33. If the area of the rectangle shown is increasing at the rate of $4\ {cm^2}/{sec}$, find $\dfrac{dx}{dt}$ when $x=e\ cm.$



34. A point moves along the curve $y=\sqrt{x}$ so that the y-coordinate is increasing at the rate of $4\,\frac{cm}{sec}$. Find the rate of change of the x-coordinate with respect to time when $y=2\ cm.$

35. A block of ice is exposed to heat in such a way that the block maintains a similar shape as it melts. The block of ice is initially 2 feet wide, 2 feet high, and 3 feet long, as shown at right. If the rate of change in the width of the ice is $-\frac{1}{3}$ ft/hr, find:



   a. the rate of change in the volume of the block of ice when the width is 1 ft.
   b. the amount of time it will take for the block of ice to completely melt.

Selected Answers:

1a. $g(x)=\sqrt[3]{x+1}$ c. $f'(x)=3x^2$, $g'(x)=\frac{1}{3}(x+1)^{-\frac{2}{3}}$ d. $f'(1)=3$, $g'(0)=\frac{1}{3}$

1e. They are reciprocals. 2a $g(x)=x^2-1,\ x\ge 0$ c. $f'(x)=\dfrac{1}{2\sqrt{x+1}}$, $g'(x)=2x$

3a. not possible b. $\frac{3}{4}$ c. 5 4. $\frac{1}{4}$ 5. $\frac{1}{5}$ 6. $\frac{1}{13}$ 8. $\frac{1}{11}$ 9. 1 10. $\frac{1}{19}$

More Selected Answers:

11. a. -1 b. $-\frac{1}{2}$ 12. $g'(t)=24t\csc^2(3t^2)$ 14. $f'(x)=e^{\sec x}\sec x\tan x$

16. $\frac{d^2y}{dx^2}=20x^3+2x^{-3}$ 17. $\boxed{f'(x)=\frac{e^x\bullet 2x-x^2e^x}{\left(e^x\right)^2}}=\frac{2x-x^2}{e^x}$ 19. $8\left(6^{2x}-3\right)^3 6^{2x}\ln 6$

20. $3\frac{\cos x}{\sin x}=3\cot x$ 21. $\frac{dy}{dt}=\frac{2t\cos\left(t^2\right)-ye^t}{e^t}$ 22. $y'=\frac{2x}{\cos(y-2x)}+2$

23. $y''=2e^{x^2-1}+4x^2e^{x^2-1}$ 24. $\frac{\frac{2x^2}{x^2-1}-\ln\left|x^2-1\right|}{x^2}$ 25. $y'=\frac{\frac{y}{x}+1}{3y^2-\ln x}=\frac{y+x}{3xy^2-x\ln x}$

26a. $y(-2)=-\frac{2}{e}$ cm b. $v(t)=te^{t+1}+e^{t+1}$ e. $a(-2)=0\frac{cm}{sec^2}$ 28. .471 or .472

30. 4 32. -1 33. $2\,{}^{cm}\!/_{sec}$ 34. $\frac{dx}{dt}=16\frac{cm}{sec}$ 35a. $-\frac{3}{2}\frac{ft^3}{hr}$ b. 6 hours

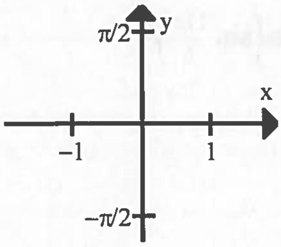
Lesson 3.6 Inverse Trigonometric Functions, Differentiating Inverse Trigonometric Functions

None of the six basic trig functions is one-to-one, so none of them have an inverse function. However, we can use domain restrictions to make the trig functions one-to-one, so that they do have inverses. In this course, we will deal only with the inverse trig functions for the sine, cosine, and tangent functions.

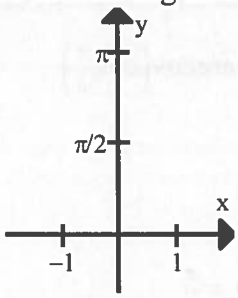
Definition of the Inverse Trig Functions:

Function	Domain (x values)	***Range (y values)**
$\underline{y=\arcsin x}\leftrightarrow \sin y=x$	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$\underline{y=\arccos x}\leftrightarrow \cos y=x$	$[-1,1]$	$[0,\pi]$
$\underline{y=\arctan x}\leftrightarrow \tan y=x$	$(-\infty,\infty)$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

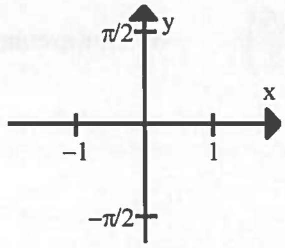
Example 1: Graph the indicated inverse trig functions in the coordinate planes below:



$y = \arcsin x = \sin^{-1} x$

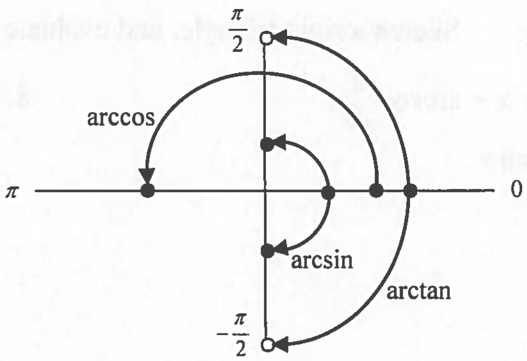


$y = \arccos x = \cos^{-1} x$



$y = \arctan x = \tan^{-1} x$

*A geometric representation of the range values for each inverse trig function is shown in the coordinate plane at right.



Remember that the answer to an inverse trig problem must fall in the correct range and that there is only one correct answer.

Example 2:

a. $\sin\frac{-\pi}{6} =$ b. $\sin\frac{7\pi}{6} =$ c. $\sin\frac{11\pi}{6} =$ d. $\arcsin\left(-\frac{1}{2}\right) =$

Example 3: Evaluate without a calculator.

a. $\arctan 1$ b. $\arccos(-1)$ c. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ d. $\arcsin 2$

Example 4: Use a calculator to evaluate.

a. $\arcsin(0.3)$ b. $\arctan\left(-\frac{5}{2}\right)$

Example 5: Simplify without a calculator.

a. $\sin\left(\arcsin \frac{\sqrt{3}}{2}\right)$ b. $\tan(\arctan 3)$ c. $\arccos\left(\cos \frac{\pi}{3}\right)$ d. $\arcsin\left(\sin \frac{11\pi}{6}\right)$

Example 6: Solve for x. $\arcsin(x^2 - 3) = \frac{\pi}{2}$

Examples: Sketch a right triangle, and evaluate without a calculator.

7. Given $x = \arccos \frac{2}{\sqrt{5}}$, find $\tan x$.

8. Given that $y = \arcsin x$, find $\cos y$.

Example 9: Use your work from Example 8 to find $\frac{d}{dx}\arcsin x$.

Derivatives of the Inverse Trig Functions:

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arcsin u = \frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arccos u = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\arctan u = \frac{u'}{1+u^2}$$

(where u is a function of x)

Examples: Differentiate.

10. $g(y) = \arctan(2y-1)$ 11. $f(x) = \arcsin\sqrt{x}$ 12. $h(t) = \cos^{-1}(\ln t)$

ASSIGNMENT 3.6

Evaluate the expressions in Problems 1-4 without using a calculator.

1. $\arcsin\frac{\sqrt{3}}{2}$ 2. $\arctan(-1)$ 3. $\arccos\left(\frac{-1}{2}\right)$ 4. $\tan^{-1}\sqrt{3}$

Use a calculator to evaluate in Problems 5-8.

5. $\arctan(-3)$ 6. $\arccos(.8)$ 7. $\arcsin\left(\frac{-1}{3}\right)$ 8. $\arccos\left(\sqrt{2}-1\right)$

Simplify the expressions in Problems 9-12 without using a calculator.

9. $\cos\left(\arccos\left(\frac{-2}{3}\right)\right)$ 10. $\tan(\arctan(2x+3))$

11. $\arcsin\left(\cos\frac{\pi}{2}\right)$ 12. $\arctan\left(\tan\frac{4\pi}{3}\right)$ Be careful!

In Problems 13 and 14, solve for x without using a calculator.

13. $\arctan(3-x) = \frac{-\pi}{4}$ 14. $\arccos\left(x^2-2\right) = \pi$

For Problems 15 and 16, evaluate without using a calculator. First, sketch a triangle for each problem.

15. Find $\cos y$, given that $y = \arcsin\left(\frac{-4}{5}\right)$. 16. Find $\sin x$, given that $x = \arctan(3)$.

Differentiate in Problems 17-28 without using a calculator.

17. $y = 2\arctan(3x)$ 18. $f(x) = \arcsin\left(x^2-1\right)$ 19. $g(y) = \arcsin e^{-y}$

20. $h(t) = \arctan t^{\frac{3}{2}}$ 21. $y = x^2\cos^{-1}x$ 22. $f(\theta) = \arctan(\ln\theta)$

23. $y = \cos\left(\ln t^2\right)$ 24. $f(x) = \tan(x)\ln|x-1|$ 25. $g(t) = 2\cos^2\sqrt{t}$

26. $f(x) = \dfrac{x^2-3}{\tan x}$ 27. $y = x\sin(-3x)$ 28. $h(y) = (\ln(\sec y))^3$

29. $y = \ln\left(x^2\sqrt[3]{x+1}\right)$ 30. $y = 4\left(\log_6\left(x^2+1\right)\right)^3$

31. For $y = x\sin x$, evaluate $\dfrac{d^2y}{dx^2}$ at $x = \dfrac{\pi}{4}$ without using a calculator.

32. For $\sec y - xy = x + 2$, find $\frac{dy}{dx}$ without using a calculator.

33. Write an equation for the line tangent to $y = e^{-x} - 3$ when $x = 0$ without using a calculator.

34. Find an equation of the line tangent to the graph of $y = \sqrt{\cos\left(x - \frac{\pi}{4}\right)}$ when $x = \dfrac{\pi}{4}$ without using a calculator.

35. Without using a calculator, find the x-values where $y = e^x \sin x$ has horizontal tangents on the interval $[-\pi, \pi]$.

36. Find a and b so that $f(t)$ is <u>differentiable</u> at $t = -1$.

$$f(t) = \begin{cases} at^3 + bt^2 - 2, & t \le -1 \\ -bt^2 + at - 4, & t > -1 \end{cases}$$

37. $y(t) = t^3 - t^{-3}$ represents the position of a point on the y-axis at time $t > 0$. $v(t)$ represents the velocity and $a(t)$ represents the acceleration of the point.

Without a calculator, find

a. $y(2)$ b. $v(2)$ c. $a(2)$

38. If $g(x) = e^x + 3x$, find $(g^{-1})'(1)$.

39. Use a calculator to find $(g^{-1})'(2)$ for $g(x) = \sqrt{x^3 + 2x + 5}$.

Find the limits in Problems 40-42, without using a calculator.

40. $\displaystyle\lim_{x\to\infty} \frac{x(2x-3)^2}{x^3+10}$ 41. $\displaystyle\lim_{x\to\infty} \frac{x(2x-3)}{x^3+10}$ 42. $\displaystyle\lim_{x\to 1} \frac{\arctan(x) + \frac{\pi}{4}}{x+1}$

Selected Answers:

1. $\frac{\pi}{3}$ 2. $-\frac{\pi}{4}$ 3. $\frac{2\pi}{3}$ 5. -1.249 7. $-.339$ or $-.340$ 9. $-\frac{2}{3}$ 11. 0 12. $\frac{\pi}{3}$

13. $x=4$ 14. $x=\pm 1$ 15. $\cos y=\frac{3}{5}$ 16. $\sin x=\frac{3}{\sqrt{10}}$ 17. $y'=\frac{6}{1+9x^2}$

19. $g'(y)=\frac{-e^{-y}}{\sqrt{1-\left(e^{-y}\right)^2}}$ 20. $h'(t)=\frac{3\sqrt{t}}{2\left(1+t^3\right)}$ 21. $\frac{-x^2}{\sqrt{1-x^2}}+2x\arccos x$

23. $y'=\frac{-2\sin\left(\ln t^2\right)}{t}$ 25. $g'(t)=\frac{-2\cos\sqrt{t}\sin\sqrt{t}}{\sqrt{t}}$ 27. $y'=-3x\cos(-3x)+\sin(-3x)$

28. $h'(y)=3(\ln(\sec y))^2\tan y$ 30. $\frac{dy}{dx}=12\left(\log_6\left(x^2+1\right)\right)^2\frac{2x}{\left(x^2+1\right)\ln 6}$

31. $y''\left(\frac{\pi}{4}\right)=-\frac{\pi}{4\sqrt{2}}+\frac{2}{\sqrt{2}}$ 32. $y'=\frac{1+y}{\sec y\tan y-x}$ 33. $y=-x-2$ 34. y = 1

35. $x=-\frac{\pi}{4},\frac{3\pi}{4}$ 36. $a=-2, b=-1$ 37b. $12\frac{3}{16}$ c. $\boxed{12-12\bullet 2^{-5}}=11\frac{5}{8}$ 38. $\frac{1}{4}$

39. 1.528 or 1.529 40. 4 41. 0 42. $\frac{\pi}{4}$

UNIT 3 SUMMARY

Chain Rule: $\frac{d}{dx}f(g(x))=f'(g(x))g'(x)$

Differentiation Rules:

$\frac{d}{dx}e^x=e^x$ $\frac{d}{dx}a^x=a^x\ln a$ $\frac{d}{dx}\ln x=\frac{1}{x}$ $\frac{d}{dx}\ln|x|=\frac{1}{x}$

$\frac{d}{dx}e^u=e^u u'$ $\frac{d}{dx}a^u=a^u u'\ln a$ $\frac{d}{dx}\ln u=\frac{u'}{u}$ $\frac{d}{dx}\ln|u|=\frac{u'}{u}$

$\frac{d}{dx}\log_a x=\frac{1}{x\ln a}$ $\frac{d}{dx}\arcsin x=\frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}\arccos x=\frac{-1}{\sqrt{1-x^2}}$ $\frac{d}{dx}\arctan x=\frac{1}{1+x^2}$

$\frac{d}{dx}\log_a u=\frac{u'}{u\ln a}$ $\frac{d}{dx}\arcsin u=\frac{u'}{\sqrt{1-u^2}}$ $\frac{d}{dx}\arccos u=\frac{-u'}{\sqrt{1-u^2}}$ $\frac{d}{dx}\arctan u=\frac{u'}{1+u^2}$

Implicit Differentiation: differentiate both sides with respect to x, remember the y' "chain rule factor" and remember to use the Product Rule for xy terms

Related Rates: (story problems)

Procedure for Related Rate Problems

1. Draw a figure (if necessary) and choose variables for all unknowns.
2. Write what is given and what is to be found using your variables and $\frac{d}{dt}$ symbols.
3. Write an equation relating the variables.
   (a) If a quantity is **changing** it must be represented with a variable letter.
   (b) If a quantity is **constant** it must be represented with a number value.
   (c) Look for secondary relationships between quantities to reduce the number of variables.
4. Implicitly differentiate both sides with respect to t.
5. Substitute number values and solve.

Geometry Formulas

Right Triangle	Circle	Sphere	Cube	Cone	Rectangle
$a^2+b^2=c^2$	$A=\pi r^2$	$V=\frac{4}{3}\pi r^3$	$V=e^3$	$V=\frac{1}{3}\pi r^2 h$	$A=lw$
$A=\frac{1}{2}bh$	$C=2\pi r$	$A=4\pi r^2$	$A=6e^2$		$P=2l+2w$

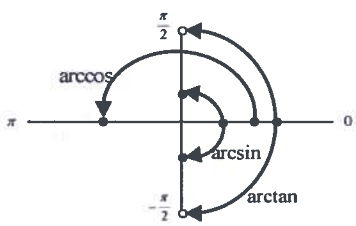
Derivatives of Inverse Functions:

If f and g are inverse functions, then $f'(a)=\dfrac{1}{g'(b)}$ where (a,b) is a point on the graph of f and (b,a) is the "image point" on the graph of g.

Definition of the Inverse Trig Functions:

Function	Domain (x values)	***Range (y values)**
$y=\arcsin x \leftrightarrow \sin y = x$	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$y=\arccos x \leftrightarrow \cos y = x$	$[-1,1]$	$[0,\pi]$
$y=\arctan x \leftrightarrow \tan y = x$	$(-\infty,\infty)$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

*A geometric representation of the range values for each inverse trig function is shown in the coordinate plane at right.

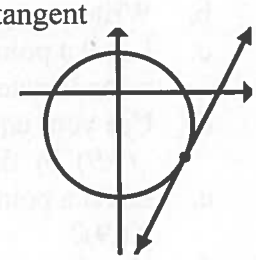


Lesson 4.1 Approximating with the Tangent Line Applications of Rates of Change

In many instances, finding a value of a function is difficult or impossible. With the use of Calculus techniques, we can approximate the function value by finding a y-value on a tangent line to the function. Since this method involves using a linear function (the tangent line function) at a nearby point, it is sometimes called a <u>local linearization</u> approximation.

<u>Examples:</u>

1. If $(2,-2)$ is a point on the graph of $x^2+y^2+2y=4$, use the equation of a tangent line passing through the point $(2,-2)$ to approximate a y-coordinate

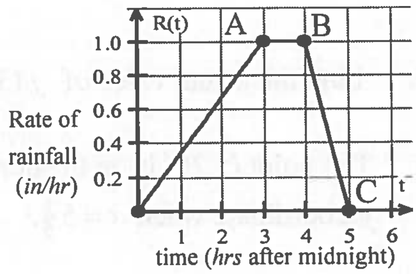


   (a) when the x-coordinate is 2.1.

   (b) when the x-coordinate is 1.9.

2. If $f(2)=3$ and $f'(2)=-2$, use local linearization to approximate $f(2.01)$,

<u>Examples</u>:
The graph at right models the rate of rainfall in inches per hour from midnight until 6:00 A.M. during a tropical rainstorm.

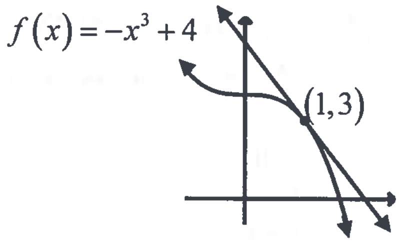


3. Write a complete sentence to explain what Point A on the graph represents. Include numbers and units in your answer.

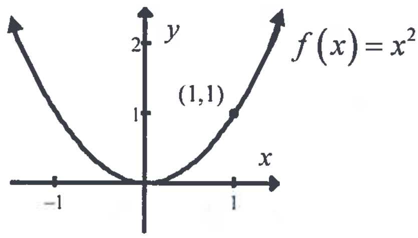
4. What is the slope of the graph between Points A and B?
5. Write a complete sentence to explain the meaning of your answer to Example 4.

6. What is the slope of the graph between Points B and C?
7. Write a complete sentence to explain the meaning of your answer to Example 6.

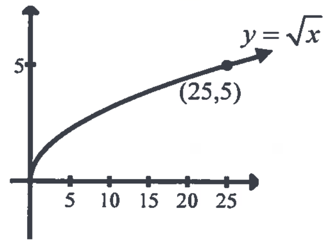
Assignment 4.1



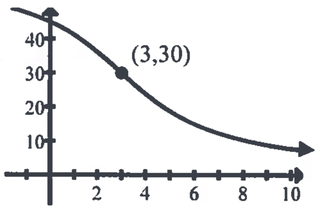
1. a. Write an equation of the tangent line shown.
   b. Use this tangent line equation to approximate $f(1.1)$.
   c. What is the actual value of $f(1.1)$?



2. Make a large copy of the graph on your own paper.
   a. Draw the tangent line at the point (1,1).
   b. Write an equation of this tangent line.
   c. Label a point on your tangent line with an x-coordinate of .9 as point A.
   d. Use your equation of the tangent line to approximate $f(.9)$ by finding the y-coordinate of your point A.
   e. Label a point B on the parabola with an x-coordinate of .9. What is the actual value of $f(.9)$?
   f. Use the same tangent line to approximate $f(.6)$. How accurate is your approximation?



3. Approximate $\sqrt{26}$ using the equation of a tangent line. You must choose your own equation and point. The graph shown should help.



4. The graph of a function $y=f(x)$ is shown. If $f'(3)=-9$, use local linearization to approximate $f(3.1)$.

5. Find the actual value of $f(3.1)$ from problem 4 or explain why it cannot be found.

6. The point (5,20) is on the curve $y=x\sqrt{x^2-9}$. Use a tangent line to approximate the y-coordinate when $x=5\frac{1}{5}$.

7. The length of one side of a square is found to be 8 inches with a possible measurement error of $\frac{1}{16}$ inch.
   a. Instead of using the actual area formula ($A=s^2$), approximate the area of the square using a local linearization of the area formula if the length of the side is really $8\frac{1}{16}$ inches (without using a calculator).
   b. Find the approximate area if the side is actually $7\frac{15}{16}$ inches.
   c. Use your answers from parts a and b to give an approximate range of values for the area of the square.

8. Use a tangent line equation to approximate $f(8.01)$ if $f(x)=\sqrt[3]{x}$ (without using a calculator).

9. The point (1,2) is on the graph of $x^3+xy+y^4=19$. Use the equation of a tangent line to approximate a y-coordinate when $x=1.1$.

10. Use a calculator to find an actual y-coordinate on the graph of the curve from problem 9 when $x=1.1$. Show the equation you are solving.

11. Given the function $y=x^3$

   a. use the equation of a tangent line to approximate $\left(2\frac{1}{6}\right)^3$ without using a calculator.

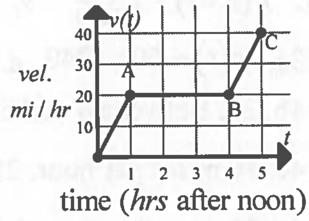
   b. find the actual value of $\left(2\frac{1}{6}\right)^3$?

12. Use the data in the table shown to find a linear approximation of $g(4.2)$.

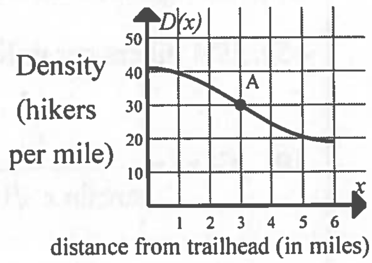
x	$g(x)$	$g'(x)$
4	3	−2

13. An interstate driver is traveling 420 miles across a state from south to north without stopping. At noon she notices her speed is 60 miles per hour and her position is at interstate mile marker 240. Note: Interstate mile markers increase from south to north.
   a. Use this data to write a linear function (local linearization) which could be used to estimate her position as a function of time. Assume $t=0$ at noon.
   b. Approximate her position at 2:00 pm.
   c. Approximate her position at 10:30 am.
   d. What is the domain on which your linear function can be applied?

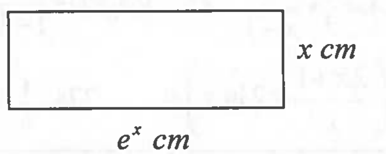
14. The graph at the right models the velocity of a car.
   a. Tell what Point A represents.
   b. Find the slope between B and C and tell what it represents.
   c. What is the velocity at 12:30 pm? at 4:15 pm?
   d. What is the acceleration at 12:30 pm? at 2:00 pm?



15. The graph at the right represents the density of hikers on a trail.
   a. Tell what Point A represents including numbers and units.
   b. If $D'(3)=-6$, tell what this means in the context of the problem using numbers and units.
   c. If $D'(3)=-6$, use local linearization to approximate the density of hikers 3.1 miles from the trailhead.



16. If the area of the rectangle shown is increasing at the rate of $3\ {cm^2}/{sec}$, find $\frac{dx}{dt}$ when $x=2\ cm$.



17. Find the average rate of change of $f(x)=\sin x+\cos x$ on the interval $\left[\frac{\pi}{4},\frac{5\pi}{4}\right]$.

Differentiate in Problems 18-26 without using a calculator .

18. $y=2\arctan\left(e^{2x}\right)$ 19 $f(x)=\ln|\arcsin x|$ 20. $y=\dfrac{t^2}{\ln t}$

21. $y=3^{2t-1}t^2$ 22. $f(y)=\dfrac{e^{\sqrt{y}}}{y^2}$ 23. $f(x)=e^x\ln x$

24. $f(x)=\ln(x-1)^{\frac{2}{3}}$ 25. $g(y)=\ln|(1-\ln y)|$ 26. $\ln y=(2x+1)\ln x$, $\frac{dy}{dx}=?$

27. Find these limits without using a calculator.

a. $\lim\limits_{x\to 3}\dfrac{x-3}{x^2-9}$ b. $\lim\limits_{x\to 3}\dfrac{x+3}{x^2-9}$ c. $\lim\limits_{x\to\infty}\dfrac{3x^3+x-3}{2x^3-9}$ d. $\lim\limits_{x\to\infty}\dfrac{3x^3+x-3}{2x^5-9}$

Selected Answers:

1a. $y=-3x+6$ b. $f(1.1)\approx 2.7$ c. $f(1.1)=2.669$ 2b. $y=2x-1$

3. $\sqrt{26}\approx 5.1$ 4. $f(3.1)\approx 29.1$ 6. $y\approx 22.05$ 7a. A $\approx$ 65 sq. in.

8. $f(8.01)\approx 2\frac{1}{1200}$ 9. $y\approx 1.984$ or 1.985 11a. $\left(2\frac{1}{6}\right)^3\approx 10$ 12. $y\approx 2.6$

13a. $s(t)\approx 60t+240$ c. mile 150 14a. At 1 pm the velocity of the car is 20 miles per hour.

14b. 20, Between 4 and 5 pm the car is accelerating at the rate of 20 miles per hour per hour.

14c. 10 miles per hour, 25 miles per hour d. $20\dfrac{mi}{hr^2}$, $0\dfrac{mi}{hr^2}$

15a. Three miles from the trailhead the density of hikers is 30 hikers per mile. b. Three miles from the trailhead the density of hikers is decreasing at the rate of 6 hikers per mile per mile.

15c. 29.4 hikers per mile 16. $\dfrac{1}{e^2}\dfrac{\text{cm}}{\text{sec}}$ 17. AROC $=\dfrac{-2\sqrt{2}}{\pi}$ 18. $y'=\dfrac{4e^{2x}}{1+e^{4x}}$

19. $f'(x)=\dfrac{1}{\arcsin x\,\sqrt{1-x^2}}$ 22. $f'(y)=\dfrac{y^2e^{\sqrt{y}}\bullet\frac{1}{2}y^{-\frac{1}{2}}-e^{\sqrt{y}}\bullet 2y}{y^4}$ 23. $f'(x)=\dfrac{e^x}{x}+e^x\ln x$

24. $f'(x)=\frac{2}{3}\bullet\dfrac{1}{x-1}$ 25. $g'(y)=\dfrac{-\frac{1}{y}}{1-\ln y}=-\dfrac{1}{y(1-\ln y)}$

26. $y'=\left(\dfrac{2x+1}{x}+2\ln x\right)y$ 27a. $\frac{1}{6}$ b. DNE c. $\frac{3}{2}$ d. 0

Lesson 4.2 L'Hospital's Rule

Some limits cannot be found using algebraic methods. If direct substitution produces one of these two indeterminate forms $\left(\frac{0}{0} \text{ or } \frac{\pm\infty}{\pm\infty}\right)$, then a rule known as **L'Hospital's Rule** may help you find the limit.

L'Hospital's Rule

If $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$ or if both of these limits are $\pm\infty$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$

To use L'Hospital's Rule, you need the limit of an expression written in fractional form.

Examples: Evaluate.

1. $\lim_{x\to\infty} \frac{x}{e^x}$

2. $\lim_{x\to\infty} \frac{e^x}{x}$

3. $\lim_{x\to 0} \frac{x}{e^x}$

4. $\lim_{x\to 0} \frac{3-3e^{3x}}{x}$

5. $\lim_{x\to -1} \frac{2x^2-2}{x+1}$ (w/o LR)

6. $\lim_{x\to -1} \frac{2x^2-2}{x+1}$

Do not use L'Hospital's Rule just because a problem "looks like" a candidate for the rule. Example 3 is not a candidate for L'Hospital's Rule, because direct substitution does not produce $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.

If using L'Hospital's Rule leaves you with the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then you can use the rule again. It is a process which can be repeated as many times as necessary. Just remember to use direct substitution at each step to make sure the rule can be used (check for $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ form).

Examples: Evaluate.

7. $\lim_{x\to-\infty} \frac{x^2}{e^{-x}}$

8. $\lim_{x\to 1} \frac{\ln x - x + 1}{x^2 - 2x + 1}$

The following limits are not $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ forms. Identify the form and tell which are indeterminate.

9. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

10. $\lim_{x\to 0^+} x^x$

11. $\lim_{x\to\infty} x^{\frac{1}{x}}$

12. $\lim_{x\to 0^+} (\sin x)^{\frac{1}{x}}$

Find the following limits.

13. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

14. $\lim_{x\to\infty} x^{\frac{1}{x}}$

15. Sometimes limits can be evaluated quickly using the concept of relative growth rates. Rank the following in order of rate of growth as x approaches infinity from slowest to fastest.

$$y = x^2, \quad y = 1, \quad y = x, \quad y = \ln x, \quad y = x^{10}, \quad y = e^x$$

Use the concept of relative growth rates to evaluate the following when possible.

16. $\lim\limits_{x\to\infty} \dfrac{\ln x}{e^x + x^3}$

17. $\lim\limits_{x\to 0^+} \dfrac{e^x}{\ln x}$

Assignment 4.2

For Problems 1-18, find the indicated limits without using a calculator.
<u>Hint</u>: Not all problems will require the use of L'Hospital's Rule.

1. $\lim\limits_{x\to 2} \dfrac{x^2-4}{x+2}$

2. $\lim\limits_{x\to 1} \dfrac{x^2+2x-3}{x-1}$

3. $\lim\limits_{x\to\infty} \dfrac{e^x}{x^2}$

4. $\lim\limits_{x\to 0} \dfrac{x}{x-(1-e^x)}$

5. $\lim\limits_{x\to 0^+} \dfrac{|x|}{x}$

6. $\lim\limits_{x\to 0^-} \dfrac{|x|}{x}$

7. $\lim\limits_{x\to 0} \dfrac{|x|}{x}$

8. $\lim\limits_{x\to 2} \dfrac{x^2-4}{\ln(x-1)}$

9. $\lim\limits_{x\to 0} \dfrac{2(e^x-1)}{x^2}$

10. $\lim\limits_{x\to\infty} \dfrac{2x^5-x^2}{3x^5+x^4-5x}$

11. $\lim\limits_{x\to\infty} \dfrac{2x^4-x^2}{3x^5+x^4-5x}$

12. $\lim\limits_{x\to\infty} \dfrac{3x^5+x^4-5x}{2x^4-x^2}$

13. $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+5}}{x}$

14. $\lim\limits_{x\to-\infty} \dfrac{\sqrt{x^2+5}}{x}$

15. $\lim\limits_{x\to 5} \dfrac{2x-10}{5x}$

16. $\lim\limits_{x\to 0} \dfrac{\tan x}{x \sec x}$

17. Find $\lim\limits_{\theta\to\frac{\pi}{2}} \dfrac{-2\cos\theta}{e^{\theta-\frac{\pi}{2}}-1}$

18. $\lim\limits_{\theta\to\frac{\pi}{2}} \dfrac{2-\cos\theta}{\theta}$

19. $\lim\limits_{x\to 1} \dfrac{\arctan(x)-\frac{\pi}{4}}{x-1}$

20. $\lim\limits_{x\to 0^+} \left(e^x+x\right)^{\frac{3}{x}}$

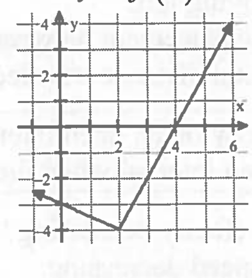
21. $\lim\limits_{x\to 0^+} (1+x)^{\frac{1}{x}}$

22. $\lim\limits_{x\to\infty} (2+x)^{\frac{1}{x}}$

23. $\lim\limits_{x\to 1^+} (\ln x)^{1-x}$

24. Using the graph and table at the right, find the following limit. $\lim\limits_{x\to 4} \dfrac{f(x)}{2g(x)-6}$

$y = f(x)$



$g(x)$ and $g'(x)$ are both continuous

x	$g(x)$	$g'(x)$
4	3	-2

25. Use the concept of relative growth rate to evaluate the following.

a. $\lim\limits_{x\to\infty} \dfrac{e^x - x}{x^2}$ b. $\lim\limits_{x\to\infty} \dfrac{\ln x}{x}$

26. Use a calculator to find $\lim\limits_{x\to 1.5} \dfrac{2x^3 - 3x^2 - 8x + 12}{6x^3 - 25x^2 + 34x - 15}$.

27. If $y = \dfrac{x}{x+5}$, find the equation of the tangent line when $x = -6$ and use it to approximate the y-coordinate when $x = -6\frac{1}{5}$.

Differentiate without using a calculator.

28. $g(t) = \sin(\arctan t)$ 29. $x^2 + 2\ln y = y$ $\frac{dy}{dx} = ?$ 30. $y = x\sqrt{3x^2 + 2}$

31. $y = \dfrac{3x-1}{x^2+2}$ 32. $x^2 - 2xy + y = 8$ $\frac{dy}{dx} = ?$

33. Given: $f(4) = 2$, $g(4) = 3$, $f'(4) = -1$, $f'(3) = -2$, and $g'(4) = 5$,

a. If $h(x) = f(x) \bullet g(x)$, find $h'(4)$. b. If $j(x) = (f(x))^3$, find $j'(4)$.

c. If $k(x) = f(g(x))$, find $k'(4)$.

34. Given: $f(x) = \begin{cases} 2x + a, & x \le 2 \\ x - b, & 2 < x < 3 \\ x^2 - 1, & x \ge 3 \end{cases}$ is a continuous function, find: a and b.

35. Find the point where the tangent line to the graph of $f(x) = 3x^2 - 2x + 5$ is parallel to the graph of $y = 10x - 3$.

36. The position (in *cm*) of an object moving on a horizontal line is given by $s(t) = 2t^3 - 3t^2 - t + 8$ (where time is measured in seconds). Answer the following questions. You may use a calculator.

a. What is the object's velocity equation?
b. What is the object's initial velocity?
c. What is the object's acceleration equation?
d. What is the object's acceleration at $t = 3$ seconds?
e. What is the object's speed at $t = 1$ second?
f. When is the object moving left?
g. What is the object's displacement between zero and two seconds?
h. What is the object's total distance traveled between zero and two seconds?

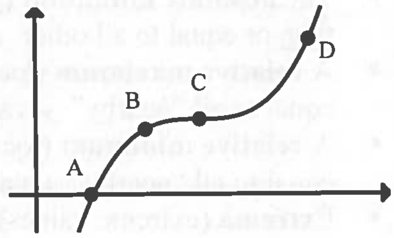
An object has **increasing velocity** on an open interval when its acceleration is greater than zero. It has **increasing speed** on an open interval when the velocity and acceleration have the same sign.

i. When is the object's velocity decreasing?
j. When is the object's speed decreasing.

37. A particle is moving on the curve $4x^2+16y^2=100$. When the particle is in the fourth quadrant with an x-coordinate of 3 cm, the y-coordinate is decreasing at the rate of $\frac{1}{2}\frac{\text{cm}}{\text{sec}}$. Find the rate of change of the x-coordinate and tell what your answer means about the motion of the particle.

38. Which is greater – the average rate of change between points A and B or the instantaneous rate of change at B?

39. Copy the figure on your own paper and sketch a tangent line which intersects the curve between points C and D whose slope is equal to the average rate of change between C and D.



Selected Answers:

1. 0 2. 4 3. ∞ or DNE 4. $\frac{1}{2}$ 6. -1 8. 4 10. $\frac{2}{3}$ 11. 0 13. 1

15. 0 16. 1 17. 2 18. $\frac{4}{\pi}$ 19. $\frac{1}{2}$ 20. e^6 21. e 22. 1 23. 1

24. $-\frac{1}{2}$ 27. $y \approx 5$ 29. $y' = \dfrac{2xy}{y-2}$ 30. $y' = \dfrac{3x^2}{\sqrt{3x^2+2}} + \sqrt{3x^2+2}$

31. $y' = \dfrac{-3x^2+2x+6}{\left(x^2+2\right)^2}$ 32. $\dfrac{dy}{dx} = \dfrac{-2x+2y}{-2x+1}$ 33a. 7 b. -12 c. -10 34. $a=3,\ b=-5$

35. $(2,13)$ 36 a. $v(t)=6t^2-6t-1$ 36b. $v(0)=-1\frac{\text{cm}}{\text{sec}}$ c. $a(t)=12t-6$

36d. $a(3)=30\frac{\text{cm}}{\text{sec}^2}$ e. $|v(1)|=1\frac{\text{cm}}{\text{sec}}$ f. $(-.145\text{sec}, 1.145\text{sec})$ g. 2 cm

36h. 6.151 or 6.152 cm i. $t<\frac{1}{2}$sec j. $(-\infty,-.145\text{sec}), \left(\frac{1}{2}\text{sec}, 1.145\text{sec}\right)$

37. $\dfrac{dx}{dt} = -\frac{4}{3}\frac{\text{cm}}{\text{sec}}$ This means the particle is moving leftward at the rate of $\frac{4}{3}\frac{\text{cm}}{\text{sec}}$.

LESSON 4.3 Absolute Extrema and the Mean Value Theorem

Definitions (informal)

- The **absolute maximum** (global maximum) of a function is the y-value that is greater than or equal to all other y-values in the function.
- The **absolute minimum** (global minimum) of a function is the y-value that is less than or equal to all other y-values in the function.
- A **relative maximum** (local maximum) of a function is a y-value that is greater than or equal to all "nearby" y-values in the function.
- A **relative minimum** (local minimum) of a function is a y-value that is less than or equal to all "nearby" y-values in the function.
- **Extrema** (extreme values) are either maximum values (maxima) or minimum values (minima).
- **Critical Numbers** are x-values at which $f(x)$ exists but $f'(x)$ is either zero or undefined.

Extreme Value Theorem (EVT)

If f is continuous on $[a,b]$ then f has both an absolute (global) minimum and an absolute (global) maximum on the interval.

In practice, the standard method of finding these max/min values is by the **candidate test**.

THE CANDIDATE TEST
PROCEDURE FOR FINDING ABSOLUTE (GLOBAL) EXTREMA:

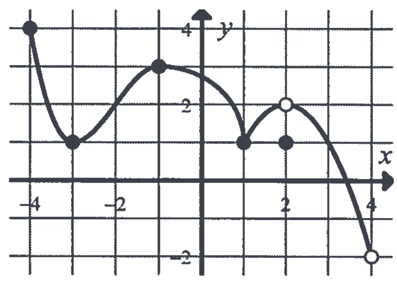
1. Find all critical numbers of the function.
2. Find y-values at each critical number **and** at each endpoint of the interval.
3. Choose the least and greatest y-values as absolute extrema.

Note:

Absolute extrema can occur either at critical numbers or endpoints.
Relative extrema can occur only at critical numbers. We will not consider endpoint extrema to be relative extrema although some textbooks allow this.

Examples: Use the figure of $y = f(x)$ at the right to answer these questions.

1. What is the absolute maximum of f?
2. At what x-value does f have an absolute maximum?
3. What is the absolute maximum point on f?
4. What is the absolute minimum of f?
5. At what x-value(s) does f have a relative minimum?
6. At what x-value(s) does f have a relative maximum?



Examples:

7. Find the global extrema of $f(x)=\frac{1}{3}x^3-2x^2$ on the interval $[-1,3]$.

8. Find the absolute maximum and minimum values of $f(x)=|x-2|$ on the interval $[0,5]$.

9. Find the extrema of $f(x)=3x^{\frac{2}{3}}-2x$ on $[-1,3]$.

10. Find the global maximum and minimum of $g(x)=2\sin x-\cos(2x)$ on the interval $[0,2\pi]$.

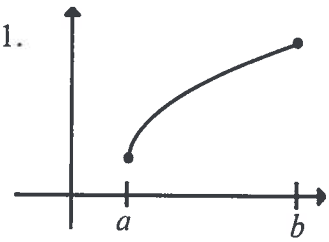
Discovering the Mean Value Theorem

For Examples 11-16 draw these lines (if possible).

(a) Draw the secant line between the two points $(a, f(a))$ and $(b, f(b))$.

(b) Draw all tangent lines parallel to the secant line.

11.



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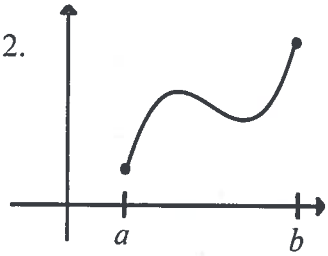

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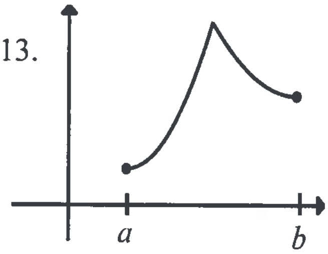

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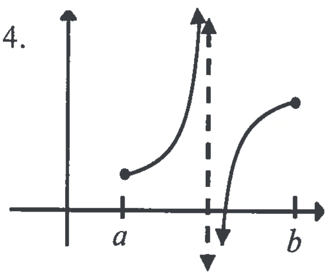

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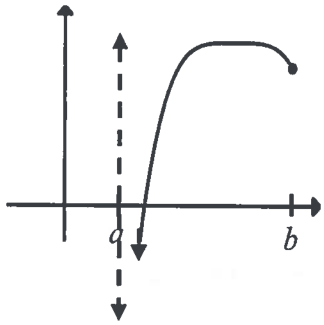

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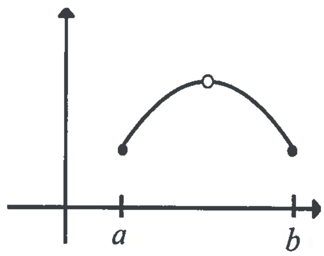


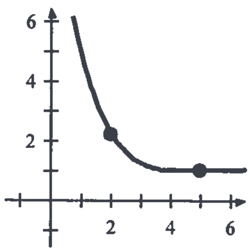
For examples 17-19

(a) Draw the secant line between the two points $(2, f(2))$ and $(5, f(5))$.

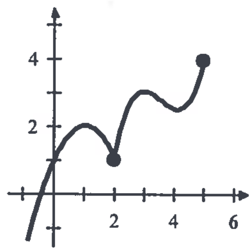
(b) Draw all tangent lines parallel to the secant line at some point on the interval (2,5).

(c) Estimate the value of c where $(c, f(c))$ is a point of tangency.

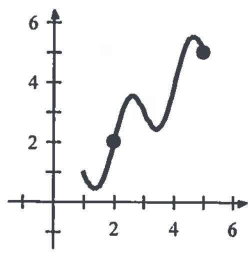
17. $c \approx$ ______



18. $c \approx$ ______



19. $c \approx$ ______

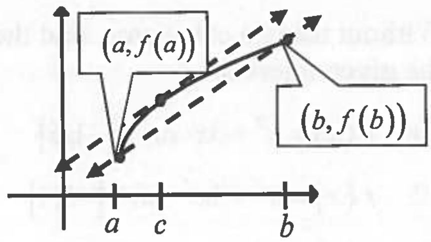


MEAN VALUE THEOREM: If f is continuous on $[a,b]$ and differentiable on (a,b), then there is a number c in (a,b)

such that $$f'(c)=\frac{f(b)-f(a)}{b-a}$$

tangent slope (inst. rt. of ch.)

secant slope (avg. rt. of ch.)



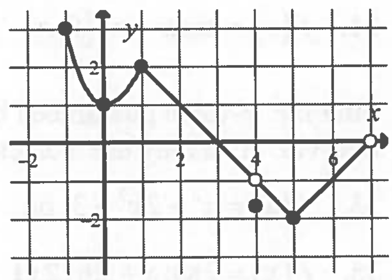
Informally: The Mean Value Theorem states that given the right conditions of continuity and differentiability, there will be at least one tangent line parallel to the secant line.

In still other words: The instantaneous rate of change (slope of tangent) will equal the average rate of change (slope of secant) at least once.

Example 20. Given $f(x)=3-\frac{6}{x}$, find all c which satisfy the Mean Value Theorem on the interval $[3,6]$.

Assignment 4.3

Use the graph of $y=f(x)$ at the right for Problems 1-4.



1. What is the absolute maximum value of $f(x)$?
2. At what point does $f(x)$ reach a global minimum?
3. At what x-value(s) does $f(x)$ have a relative minimum?
4. At what x-value(s) does $f(x)$ have a relative maximum?

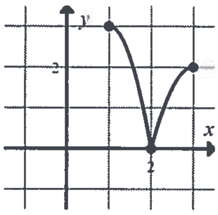
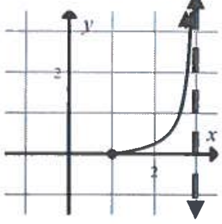
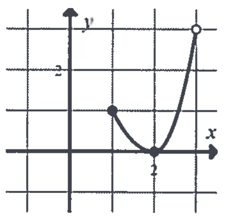
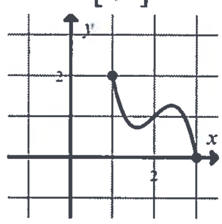
Find the critical numbers of these functions without using a calculator.

5. $y=x^2-3x$
6. $f(x)=6x^{\frac{2}{3}}-2x$
7. $y=-2x+3$
8. $y=|x+2|$
9. $f(x)=\cos^2 x+\sin x$ on $[0,2\pi]$

Without using a calculator, find the absolute maximum and absolute minimum values of $f(x)$ on the given interval.

10. $f(x) = x^2 - 3x$ on $[-1,5]$
11. $f(x) = 6x^{\frac{2}{3}} - 2x$ on $[-1,27]$
12. $f(x) = x^3 + 3x^2$ on $[-3,1]$
13. $f(x) = x^3 + 3x^2$ on $\left[-1,\frac{1}{2}\right]$
14. $f(x) = |x+2|$ on $[-3,0]$
15. $f(x) = 3(x-2)$ on $[-3,1]$
16. $f(x) = \dfrac{x^2}{x^2+2}$ on $[-1,2]$
17. $f(x) = \cos^2 x + \sin x$ on $[0,2\pi]$ see Problem 9.
18. $f(x) = \begin{cases} -x^2+2, & x \le 0 \\ -x+2, & 0 < x < 6 \\ -4 & x \ge 6 \end{cases}$ on $[-1,8]$

Find the absolute minimum and maximum values of the functions graphed on $[1,3]$.

19. 
20. 
21. 
22. 

Does the Mean Value Theorem apply to the given function on the given interval? If it does, find the c-value. If it does not, explain why not. Do not use a calculator.

23. $f(x) = |x|$ on $[-1,3]$
24. $f(x) = x^2 - 2x$ on $[1,3]$
25. $f(x) = x^2 - 3x + 2$ on $[1,2]$
26. $f(x) = x^{\frac{2}{3}}$ on $[-2,2]$
27. $f(x) = x^{\frac{2}{3}}$ on $[0,1]$
28. $f(x) = \dfrac{1}{x-4}$ on $[2,6]$
29. $f(x) = \dfrac{x^2-x}{x}$ on $[-1,1]$
30. $f(x) = x^2 - 2x$ on $[0,2]$
31. $f(x) = \sin x$ on $[0,\pi]$
32. $f(x) = \tan x$ on $[0,\pi]$

Find the c-value guaranteed by the Mean Value Theorem for the given function on the given interval. You may use a calculator.

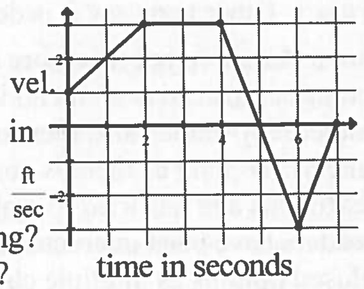
33. $f(x) = x^3 - 2x^2 + 3$ on $[-1,2]$
34. $f(x) = \dfrac{1}{x-1}$ on $[2,3]$
35. $f(x) = 2\sin x + \sin(2x)$ on $[0,\pi]$

36. The height, in feet, of an object at time t seconds is given by $h = -16t^2 + 200$.
    a. Find the average velocity of the object during the first 3 seconds.
    b. Use the Mean Value Theorem to find the time at which the object's instantaneous velocity equals this average velocity.

Find the following limits.

37. $\lim\limits_{t \to \frac{\pi}{4}} \dfrac{\cos t - \sin t}{2 - 2\tan t}$ 38. $\lim\limits_{x \to -\infty} x^2 e^x$ 39. $\lim\limits_{x \to \infty} \dfrac{e^x - 3}{x^2 + x + 2}$ 40. $\lim\limits_{x \to \infty} \dfrac{5x^4 - 3x^2 + 5}{x^4 + 3x - 7}$

Use the graph of a velocity function model for an object moving horizontally shown at the right for problems 41-46.



41. Find the object's acceleration at time 5 seconds.
42. Find the speed of the object at time 6 seconds.
43. On which interval of time is the object moving right?
44. On which open interval(s) of time is the object's velocity increasing?
45. On which open interval(s) of time is the object's speed increasing?

Selected Answers:

1. 3 3. $x = 0, 4, 5$ 5. $x = \frac{3}{2}$ 6. $x = 0, 8$ 8. $x = -2$ 9. $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$

10. max. $f = 10$, min. $f = -\frac{9}{4}$ 11. max. $f = 8$, min. $f = 0$

13. max. $f = 2$, min. $f = 0$ 15. max. $f = -3$, min. $f = -15$

16. max. $f = \frac{2}{3}$, min. $f = 0$ 17. max. $f = \frac{5}{4}$, min. $f = -1$

19. max. value = 3, min. value = 0 21. no max. value, min. value = 0

24. c = 2 25. $c = \frac{3}{2}$ 26. MVT does not apply. f is not diff. at $x = 0$ (sharp turn)

27. $c = \frac{8}{27}$ 30. c = 1 31. $c = \frac{\pi}{2}$ 33. $c = -.215, 1.548$ or 1.549 34. c = 2.414

36b. $t = \frac{3}{2} sec$ 37. $\frac{\sqrt{2}}{4}$ 39. ∞ 40. 5 41. $a(5) = -3\frac{ft}{sec^2}$ 42. $speed(6) = 3\frac{ft}{sec}$

43. $[0,5)$ seconds 44. $(0,2)$sec, $(6,7)$sec 45. $(0,2)$sec, $(5,6)$sec

LESSON 4.4 Increasing/Decreasing Functions, First Derivative Test for Relative Extrema

PROCEDURE: **(Increasing/ Decreasing** and **First Derivative Test)**

1. Find **domain** restrictions.
2. Find all **critical numbers** (where $f'(x) = 0$ or $f'(x)$ is undefined – but domain restrictions cannot be critical numbers).
3. Locate critical numbers and domain restrictions on an f' **number line.** Label critical numbers CN.
4. Test the sign of $f'(x)$ in each interval and **label the signs** on the number line.
5. List **increasing/decreasing intervals** and/or identify **relative min/max** x-values. If requested, find y-values or points.

Increasing/Decreasing Intervals
A frequent area of confusion for students is the question of whether to include endpoints on intervals. Some textbooks would say the function $y=x^2$ is decreasing on $(-\infty,0)$ and increasing on $(0,\infty)$. Other texts say it is decreasing on $(-\infty,0]$ and increasing on $[0,\infty)$. Using closed intervals is not as intuitive but it is more correct. The concept of increasing/decreasing is not based on a single point but instead is about an interval. The precalculus definition of increasing is that as x-values increase, y-values also increase and if this is true on an open interval it is also true on the closed interval so long as there is continuity. The College Board is well aware of this discrepancy among textbooks and has tried to make sure that students were not penalized. On a free response question, readers have been instructed to count either answer correct. They will never list both open and closed options as multiple choice responses. It is more likely that multiple choice answers will be closed intervals.
The question about whether to include zero when listing where $y=x^3$ is increasing is a completely separate question. All texts would say it is increasing on $(-\infty,\infty)$ even though the derivative is zero at $x=0$. Any other answer would not receive credit on an AP test.

Examples: Find the intervals on which these functions are increasing and decreasing and find all local extrema points.

1. $f(x)=(x^2-9)^{\frac{2}{3}}$

2. $y=x-2\sin x$ on $(0,2\pi)$

3. $f(x)=\dfrac{x^4+3}{3x}$

4. $f(x)=-xe^{-2x^2}$

5. $f(x)=x^3-3x^2+3x$

The function in Example 5 is a strictly monotonic function. Strictly increasing or strictly decreasing functions are called monotonic.

Example 5 illustrates two **important points.**

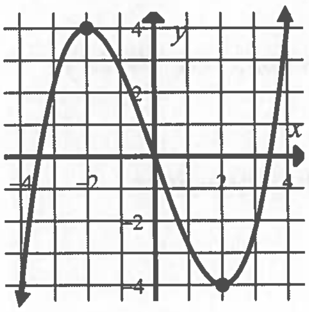
1. Not every critical number produces a relative maximum or minimum.
2. Even though the slope at $x = 1$ is zero, the function is always increasing.

Assignment 4.4

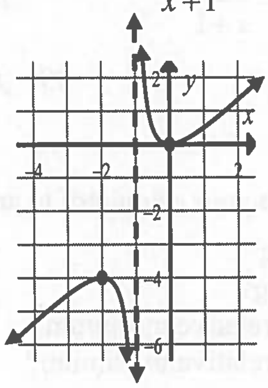
Use the graphs shown with each function to answer these four questions.

a. On which intervals is f increasing?
b. On which intervals is f decreasing?
c. At which point(s) does f have a relative minimum?
d. At which point(s) does f have a relative maximum?

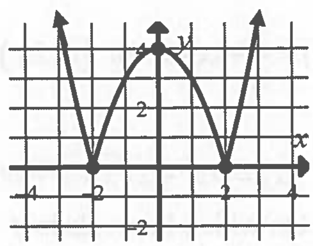
1. $f(x) = \frac{1}{4}x^3 - 3x$



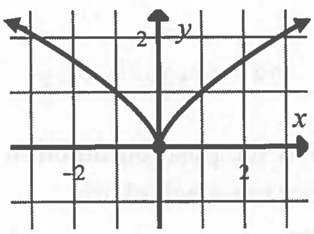
2. $f(x) = \dfrac{x^2}{x+1}$



3. $f(x) = |x^2 - 4|$

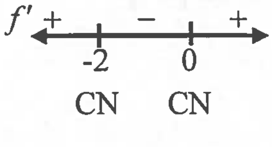


4. $f(x) = x^{\frac{2}{3}}$

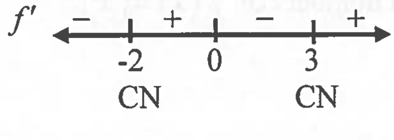


Use the f' number lines (sign charts) shown to answer these four questions.

a. On which intervals is f increasing?
b. On which intervals is f decreasing?
c. At which x-values does f have a relative minimum?
d. At which x-values does f have a relative maximum?

5. 

6. Domain: $x \neq 0$



Which of the functions in Problems 1-4 would have the following f' number lines?

7. f': $+$ | -2 (CN) | $-$ | -1 | $-$ | 0 (CN) | $+$

8. f': $-$ | -2 (CN) | $+$ | 0 (CN) | $-$ | 2 (CN) | $+$

Without using a calculator, answer these four questions for Problems 9-19. Show organized work and an f' number line to support your answers.

a. On which intervals is f increasing?
b. On which intervals is f decreasing?
c. At which point(s) does f have a relative minimum?
d. At which point(s) does f have a relative maximum?

9. $f(x)=x^2-8x$
10. $f(x)=-2x^2+4x+6$
11. $f(x)=2x^3+3x^2-12x$
12. $f(x)=\frac{3}{5}x^{\frac{5}{3}}-\frac{3}{2}x^{\frac{2}{3}}+1$
13. $f(x)=x^3+1$
14. $f(x)=\dfrac{x}{x^2-9}$
15. $f(x)=(x+1)^{\frac{2}{3}}$
16. $f(x)=\dfrac{x-1}{x+1}$
17. $f(x)=x^4-2x^3$
18. $f(x)=\dfrac{x}{2}+\cos x$ on $[0,2\pi)$
19. $f(x)=\sin x+\cos x$ on $(0,2\pi)$

20. If $f'(x)=5x^3-2\sqrt{x+5}+\sin(x^2)$, use your calculator to answer these four questions.
    a. On which intervals is f increasing?
    b. On which intervals is f decreasing?
    c. At which x-values does f have a relative minimum?
    d. At which x-values does f have a relative maximum?

Determine whether the following functions are strictly monotonic on the interval $(0,\infty)$. Do not use a calculator.

21. $f(x)=x^2$
22. $g(x)=x^{\frac{2}{3}}$
23. $h(x)=\frac{1}{3}x^3-x$

For Problems 24-26 the height, in feet, of a ball is given by the position function $s(t)=-16t^2+64t+6$. Assume $0\le t\le 4$ seconds. Do not use a calculator.

24. On which interval of time is the ball moving upward?
25. What is the maximum height of the ball?
26. For what open intervals of time is
    a. the velocity of the ball increasing?
    b. the speed of the ball increasing?
27. If the function $f(x)=x^2+ax+b$ has a relative minimum point at $(2,-4)$, solve for a and b.
28. Without a calculator find the absolute extrema of $f(x)=4x^3-12x+1$ on the interval $[-1,3]$.
29. Without using a calculator find the absolute extrema of $f(x)=|x-2|+2$ on $[1,4]$.
30. Find all critical numbers of $f(x)=\lfloor x\rfloor$.

31. Can the Mean Value Theorem be used for the function $f(x)=\dfrac{x}{x-2}$ on the interval $[0,3]$? If it can be used, find the c-value. If it cannot, explain why not.

32. Find the c-value guaranteed by the Mean Value Theorem for the function $f(x)=x^3-x^2-2x$ on the interval $[-1,1]$. You may use a calculator.

Find the following limits.

33. $\lim\limits_{\theta\to\frac{\pi}{2}}\dfrac{\sin(2\theta)}{\cos\theta}$

34. $\lim\limits_{x\to 2}\dfrac{3e^{x-2}-3}{4x-8}$

35. A triangular lawn is being mowed so that the base of the triangle of unmown grass is decreasing at the rate of 2 feet per minute. If the altitude of the triangle is always three times the base, how is the area changing when the base is 100 feet?

Selected Answers:

1. a. $(-\infty,-2],[2,\infty)$ b. $[-2,2]$ c. $(2,-4)$ d. $(-2,4)$

3. a. $[-2,0],[2,\infty)$ b. $(-\infty,-2],[0,2]$ c. $(\pm 2,0)$ d. $(0,4)$

5. a. $(-\infty,-2],[0,\infty)$ b. $[-2,0]$ c. $x=0$ d. $x=-2$

7. problem 2 9. a. $[4,\infty)$ b. $(-\infty,4]$ c. $(4,-16)$ d. none

10. a. $(-\infty,1]$ b. $[1,\infty)$ c. none d. $(1,8)$

12. a. $(-\infty,0],[1,\infty)$ b. $[0,1]$ c. $\left(1,\frac{1}{10}\right)$ d. $(0,1)$

13. a. $(-\infty,\infty)$ b. none c. none d. none

14. a. none b. $(-\infty,-3),(-3,3),(3,\infty)$ c. none d. none

16. a. $(-\infty,-1),(-1,\infty)$ b. none c. none d. none

18. a. $\left[0,\frac{\pi}{6}\right],\left[\frac{5\pi}{6},2\pi\right)$ b. $\left[\frac{\pi}{6},\frac{5\pi}{6}\right]$ c. $\left(\frac{5\pi}{6},\frac{5\pi}{12}-\frac{\sqrt{3}}{2}\right)$ d. $\left(\frac{\pi}{6},\frac{\pi}{12}+\frac{\sqrt{3}}{2}\right)$

20. a. $[.936,\infty)$ b. $(-\infty,.936]$ c. $x=.936$ d. none 21. yes 23. no

24. $[0\,\text{sec},2\,\text{sec})$ 26. a. never 27. $a=-4$, $b=0$

28. max. $f=73$, min. $f=-7$ 32. $c=-.333$ 33. 2

34. $\frac{3}{4}$ 35. The area is decreasing at the rate of $600\,\frac{\text{ft}^2}{\text{min}}$.

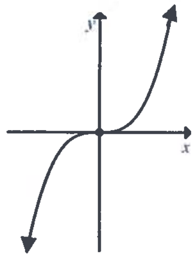
LESSON 4.5 Concavity and Points of Inflection, Second Derivative Test for Relative Extrema

A graph with this shape is called **concave upward.** The tangent lines lie **below** the graph. The slopes of the tangent lines are increasing which means $f''(x) \geq 0$.

A graph with this shape is called **concave downward.** The tangent lines lie **above** the graph. The slopes of the tangent lines are decreasing which means $f''(x) \leq 0$.

Nonmathematical Memory Device:
Concave upward ↔ positive ↔ smiley face ↔ 

Concave downward ↔ negative ↔ frowny face ↔ 

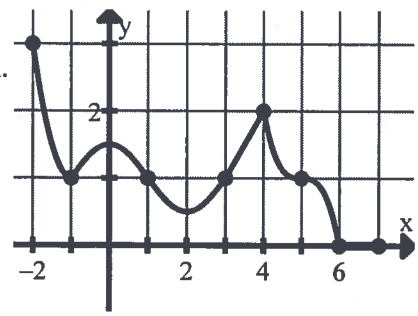


On the graph of $y = x^3$ shown, the point (0,0) is a point of inflection.

A point on a graph is a **point of inflection** if:
1. the graph has a tangent line at that point **and**
2. the graph changes concavity at that point.

Note: Some textbooks use other definitions for a point of inflection that do not require the existence of a tangent line. The AP™ test is carefully written so that this is not an issue.

Examples: Use the graph at the right to answer these questions. Base your answers on appearances of the graph.



1. On which intervals is the graph concave upward?

2. On which intervals is the graph concave downward?

3. On which intervals does the graph have no concavity?

4. What are the points of inflection?

Analytically we find concavity intervals and points of inflection by using a second derivative number line.
The procedure is parallel to the procedure used in the last lesson to find increasing/decreasing intervals and relative extrema by using a first derivative number line.

Most textbooks use open intervals for concavity. However, there is no universal definition of concavity so there is some chance for ambiguity. A few texts define concavity in terms of where the first derivative is increasing/decreasing so the same issue arises about including/excluding endpoints.

Examples:

5. Determine the points of inflection and discuss the concavity for the graph of $f(x) = x^4 + x^3 - 3x^2 + 1$.

6. If $f(x) = \dfrac{x^2+1}{x^2-4}$ and $f''(x) = \dfrac{10(3x^2+4)}{(x^2-4)^3}$, list the intervals where the graph of f is concave upward, concave downward, and list the points of inflection.

THE SECOND DERIVATIVE TEST FOR RELATIVE EXTREMA

This test does not require a second derivative number line. It does not find points of inflection. It is used to find relative extrema (max/min).

Procedure:
1. Use f' to find critical numbers.
2. Plug critical numbers into f'' and analyze concavity to determine if the function has a relative minimum or maximum.

Note: The Second Derivative Test does not always give an answer (when $f''(x) = 0$). Use it only when the directions require it or when the given information requires it.

Examples:

7. Use the Second Derivative Test to find the relative minimum and relative maximum points for the graph of $f(x) = -3x^4 + 6x^2$.

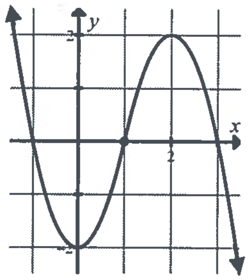
8. $g(x)$ is a function with some derivative values shown in the table. find the x-values of the relative maximum and relative minimum points when possible.

x	$g'(x)$	$g''(x)$
-3	0	4
-1	0	-1
0	0	0
1	3	5
2	0	3

Assignment 4.5

Use the appearance of these graphs to answer these three questions.

a. On which interval(s) is the graph of the function concave upward?
b. On which interval(s) is the graph of the function concave downward?
c. What are the points of inflection?

1.


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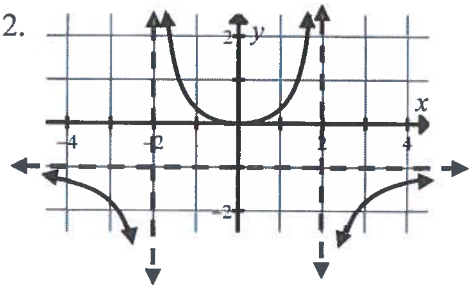

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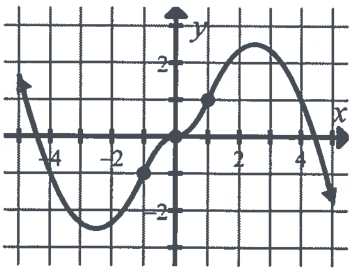

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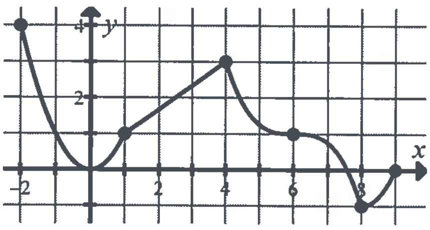


Show organized steps and an f'' number line to answer the same three questions for these functions without using a calculator.

5. $f(x)=x^4-4x^3+2$ 6. $g(x)=\cos x+\sin x$ on $[0,2\pi]$ 7. $f(x)=3x^5-5x^4$

8. $f(x)=x^{\frac{2}{3}}-3$ 9. $f(x)=-2\sin x-\frac{1}{2}x^2+2x+1$ on $[0,2\pi]$

Use the Second Derivative Test to find the relative extrema points (see Example 7 on the previous page).

10. $f(x)=x^3-3x^2+6$ 11. $f(x)=-\frac{1}{4}x^4+\frac{9}{2}x^2+5$ 12. $g(x)=2\sin x+3$ on $[0,2\pi]$

For problems 13 and 14, find the x-values of relative minimum points and the x-values of relative maximum points.

13. -3, 1, and 3 are critical numbers of f and $f''(-3)=-2$, $f''(1)=0$, and $f''(3)=2$.

14. $f'(-2)=f'(0)=f'(4)=0$

f'': $+$ | -1 | $-$ | 2 | $+$

15. Find the relative extrema points and points of inflection for the graph of $y=xe^x-e^x$ without using a calculator.

Without using a calculator, find local maximum and minimum points and points of inflection. Then sketch a graph. It is not necessary to find the x-intercepts. Show organized steps with f' and f'' number lines to support your answers.

16. $f(x)=2x^3-3x^2-12x+5$

17. $f(x)=4x^3-x^4$

18. Use the following information to sketch a possible graph of f. f is a continuous function, $f(-1)=3$, $f(2)=0$, $f(-2)=-1$, $f'(2)$ does not exist.

f': $+$ | -1 | $-$ | 2 | $+$

f'': $+$ | -2 | $-$ | 2 | $-$

19. Use a calculator to determine if the function $f(x)=4x^3+\sin(5x)$ is concave upward or downward on an interval including $x=.523$. Be sure to use radians mode.

20. Find the <u>absolute</u> maximum and <u>absolute</u> minimum for the function $f(x)=x^4-2x^2$ on the interval $[-2,1]$. Do not use a calculator.

21. Find the c-value guaranteed by the Mean Value Theorem for the function $f(x)=\sqrt{x-2}$ on the interval $[2,6]$.

22. If the graph of $y=ax^3+bx^2+cx+d$ has a point of inflection at the point (0,2) and a relative maximum at the point $(-1,4)$, find the values of a, b, c, and d.

23. a. Sketch a smooth curve whose slope is always positive and whose slope is increasing.
    b. Sketch a smooth curve whose slope is always positive and whose slope is decreasing.
    c. Sketch a smooth curve whose slope is always negative and whose slope is increasing.
    d. Sketch a smooth curve whose slope is always negative and whose slope is decreasing.

Use the f' number line shown for Problems 24-26.
Fill in the blank with $>$ or $<$.

f': $+$ | -3 | $-$ | 5 | $+$

24. If $g(x)=f(x)+2$, then $g'(4)$____0.

25. If $g(x)=-2f(x)$, then $g'(4)$____0.

26. If $g(x)=f(x-10)$, then $g'(4)$____0.

27. Given: $x^2+xy+y=-3$ Do not use a calculator on this problem.

a. Find the value of $\frac{dy}{dx}$ at the point $(1,-2)$. b. Find the value of $\frac{d^2y}{dx^2}$ at $(1,-2)$.

c. Use the equation of a tangent line to approximate a y-value when $x=0.9$.

d. Is your approximation in part c an underestimate or an overestimate of the exact value of y? Explain.

e. Does the curve have a relative maximum, a relative minimum, or neither at $(1,-2)$? Explain.

28. Given $f'(x)=\ln\frac{x+1}{x+2}-\cos\left(0.3x^2+5\right)$ and $f''(x)=\frac{1}{x+1}-\frac{1}{x+2}+0.6x\sin\left(0.3x^2+5\right)$, use a calculator to find the following using the domain $(-1,4)$:

a. On what interval(s) within the domain is f increasing?

b. At what x-value(s) within the domain does f have local extrema?

c. At what x-value(s) within the domain does f have points of inflection?

29. $x(t)=t^3-27t+50$ is the position function of a particle moving along a horizontal line.

a. On what open interval is the velocity of the particle increasing?

b. On what open interval is the speed of the particle increasing?

c. What is the displacement of the particle on the interval $[0,4]$?

d. What is the total distance traveled by the particle on the interval $[0,4]$?

Selected Answers:

1a. $(-\infty,1)$ b. $(1,\infty)$ c. $(1,0)$ 2a. $(-2,2)$ b. $(-\infty,-2)$, $(2,\infty)$ c. none

4a. $(-2,1)$, $(4,6)$, $(8,9)$ b. $(6,8)$ c. $(6,1)$ 5a. $(-\infty,0)$, $(2,\infty)$ b. $(0,2)$

5c. $(0,2)$, $(2,-14)$ 6a. $\left(\frac{3\pi}{4},\frac{7\pi}{4}\right)$ b. $\left[0,\frac{3\pi}{4}\right)$, $\left(\frac{7\pi}{4},2\pi\right]$ c. $\left(\frac{3\pi}{4},0\right)$, $\left(\frac{7\pi}{4},0\right)$

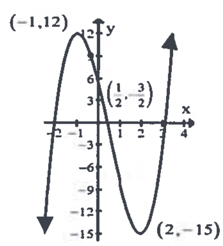
8a. none b. $(-\infty,0)$, $(0,\infty)$ c. none 9a. $\left(\frac{\pi}{6},\frac{5\pi}{6}\right)$ b. $\left[0,\frac{\pi}{6}\right)$, $\left(\frac{5\pi}{6},2\pi\right]$

9c. $\left(\frac{\pi}{6},\frac{-\pi^2}{72}+\frac{\pi}{3}\right)$, $\left(\frac{5\pi}{6},\frac{-25\pi^2}{72}+\frac{5\pi}{3}\right)$ 10. rel. max. pt. (0,6) rel. min. pt. (2,2)

11. rel. max. pt. $\left(\pm3,\frac{101}{4}\right)$ rel. min. pt. (0,5) 12. rel. max. pt. $\left(\frac{\pi}{2},5\right)$ rel. min. pt. $\left(\frac{3\pi}{2},1\right)$

13. rel. max. at $x=-3$, rel. min. at $x=3$

15. rel. min. at $(0,-1)$, PI at $\left(-1,\frac{-2}{e}\right)$

16. 

19. concave down ($f''(.523)=-.013$)

20. max. $f=8$, min. $f=-1$

21. c = 3 22. $a=1, b=0, c=-3, d=2$ 24. <

27a. 0 b. -1 c. $y\approx-2$

28a. $[3.198,4)$ or $[3.199,4)$ b. local min at $x=3.198$ or 3.199

UNIT 4 SUMMARY

Approximations using a tangent line:
Find the equation of a tangent line at a convenient point. Plug in a new x-value to find a new y-value on the tangent line which is close to a y-value on the curve.

L'Hospital's Rule

If $\lim\limits_{x\to c} f(x)=0$ and $\lim\limits_{x\to c} g(x)=0$ or if both of these limits are $\pm\infty$, then $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to c}\dfrac{f'(x)}{g'(x)}$.

Definition of **Critical Numbers**: x-values where f' is zero or undefined (cannot be domain restrictions)

Absolute Extrema: Compare y-values at endpoints and critical numbers (**Candidate Test**).

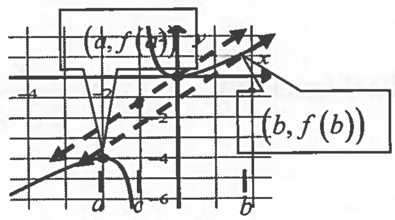
Relative (Local) Extrema:
First Derivative Test (best way): Find critical numbers, and make an f' number line. Domain restrictions must be on all number lines – but cannot be max/min points. An f' number line also gives increasing/decreasing intervals.
Second Derivative Test (Use only when you have to.): Use f' to find the critical numbers – then plug them into f'', and use concavity to see if they are at a maximum or a minimum.

Points of Inflection and Concavity: Find possible points of inflection (where f'' is zero or undefined), and make an f'' number line. Remember, domain restrictions cannot be points of inflection but must be on all number lines.

Mean Value Theorem: f must be continuous on $[a,b]$ and differentiable on (a,b).

inst. rt. ch. = avg. rt. ch.
IROC = AROC
tangent slope = secant slope

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$



MVT is used to find the c-value.

Lesson 5.1 Curve Sketching with Extrema and Inflection Points

Curve Sketching Recipe:

1. Give the domain.
2. Reduce $f(x)$.
3. Find vertical asymptotes and holes.
4. Give x- and y-intercepts.
5. Find the end behavior (horizontal asymptotes or other).
6. Find increasing/decreasing intervals and relative extrema points (show an f' number line).
7. Find concavity and points of inflection (show an f'' number line).
8. Graph.

1. (a rational function) $f(x)=\dfrac{3x-2}{x^2-2x+1}$, $f'(x)=\dfrac{-3x+1}{(x-1)^3}$, $f''(x)=\dfrac{6x}{(x-1)^4}$

Do.:

V.A.:

Holes:

x-int.:

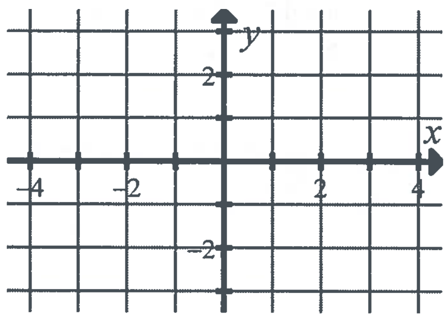
y-int.:

E.B.:

Rel. Max. Pts.:

Rel. Min. Pts.:

P.I.:



3. (a radical function) $f(x)=\dfrac{x}{\sqrt{x^2+2}}$, $f'(x)=\dfrac{2}{\sqrt{(x^2+2)^3}}$, $f''(x)=\dfrac{-6x}{\sqrt{(x^2+2)^5}}$

Do.:

V.A.:

Holes:

x-int.:

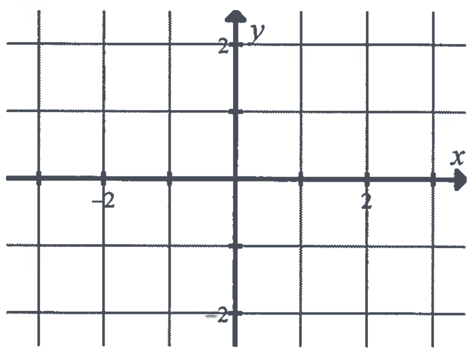
y-int.:

E.B.:

Rel. Max. Pts.:

Rel. Min. Pts.:

P.I.:



Assignment 5.1

Without using a calculator, find local extrema points, points of inflection, and sketch a graph. Show organized steps and justification. It is not necessary to find x-intercepts and there are no domain restrictions or asymptotes. However, an end behavior analysis will be helpful.

1. $y = x^3 - 3x^2 + 5$ 2. $y = 1 - x - x^3$ 3. $y = x^4 - 4x^3 + 16$

Find intercepts and relative extrema points and graph these functions without using a calculator.

4. $f(x) = x^2 - 2x - 8$ 5. $g(x) = \left|x^2 - 2x - 8\right|$

Find relative extrema points, points of inflection, and end behavior and graph without a calculator.

6. $y = \dfrac{2x^2}{x^2+3}$, $y'' = \dfrac{-36(x^2-1)}{(x^2+3)^3}$

Find the domain, relative extrema points, asymptotes, and end behavior and graph without a calculator. There are no points of inflection.

7. $f(x) = \dfrac{x^2+1}{2x}$

Find the domain, reduced function, hole, intercepts, relative extrema points, and points of inflection. Then graph without using a calculator.

8. $f(x) = \dfrac{x^2\sqrt{4-x}}{x}$, $f' = \dfrac{8-3x}{2\sqrt{4-x}}$, $f'' = \dfrac{3x-16}{4(4-x)^{\frac{3}{2}}}$, $f\left(\frac{8}{3}\right) = 3.079$

9. **Without a calculator, find the <u>domain</u>, x-intercepts, and relative extrema points. Then graph $f(x)$. There are no points of inflection.** $f(x) = \sqrt{9-x^2}$

10. **Find the x- and y-intercepts, relative extrema points, and points of inflection for $y = \sin x + \cos x$ on $[0, 2\pi)$. Then sketch the graph of y without using a calculator.**

11. **Find the domain, x- and y-intercepts, local extrema points, and points of inflection for the graph of $y = xe^x - e^x$. Then sketch its graph without using a calculator.**

12. **Find the domain, local extrema points, and points of inflection for the graph of $y = x - \ln x$. Then sketch the graph without using a calculator.**

13. **True or False? If $f'(x) > 0$ for all real x-values, then $\lim\limits_{x\to\infty} f(x) = \infty$. Show a graph to illustrate your answer.**

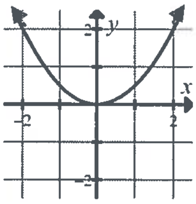
14. **Find an equation of the line tangent to the graph of $g(x) = \dfrac{4^{x-2} + \ln x^2}{\sin(x-3)}$ at the point where $x = 5$. You may use a calculator.**

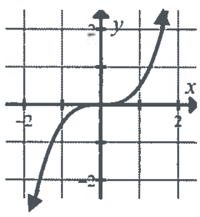
15. a. Find the points at which the graph of $x^2+4y+2y^2=6$ has horizontal tangent lines.
    b. Determine whether each of these points is at a local minimum or a local maximum. (Show organized work using the Second Derivative Test.)

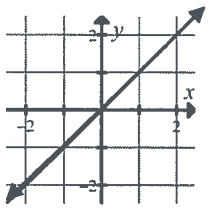
16. Find the points at which the graph of $x^2+4y+2y^2=6$ has vertical tangent lines.

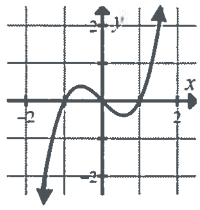
17. Given: $f(4)=2,\ g(4)=3,\ f'(4)=-1,$ and $g'(4)=5,$
    a. If $h(x)=f(x)\bullet g(x)$, find $h'(4)$.
    b. If $j(x)=\left(f(x)\right)^3$, find $j'(4)$.

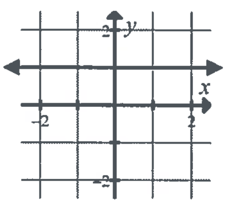
Match the graph of f in the top row with the appropriate graph of f' in the bottom row.

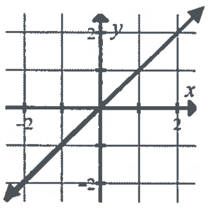
18. 

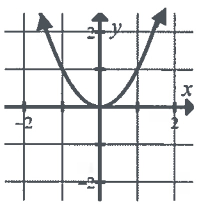
19. 

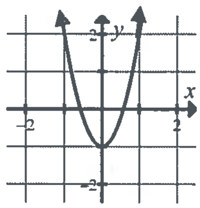
20. 

21. 

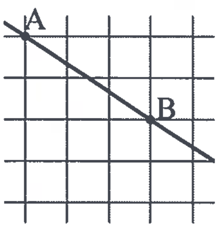
A. 

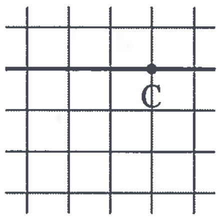
B. 

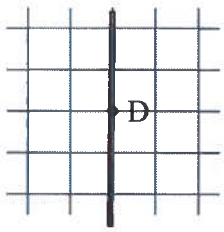
C. 

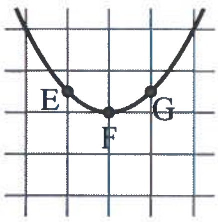
D. 

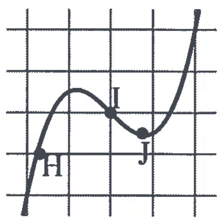
22. Find (or estimate) the slope of each graph at each lettered point.











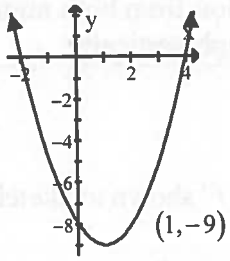
23. A point moves along the curve $y=\sqrt[3]{x}$ so that the y-coordinate is increasing at the rate of two units per second. At what rate is the x-coordinate changing when $x=8$ units?

24. Find $\left(f^{-1}\right)'(5)$ if $f(x)=x^3-4x^2+3x-7$ on the interval $[3,\infty)$.

Selected Answers:

1. rel. max. pt. $(0,5)$ rel. min. pt. $(2,1)$ PI $(1,3)$ 2. no rel. extrema, PI $(0,1)$

3. rel. min. pt. $(3,-11)$ PI $(0,16),(2,0)$ 4.



6. rel. min. pt. $(0,0)$, P.I. $\left(\pm 1,\frac{1}{2}\right)$,

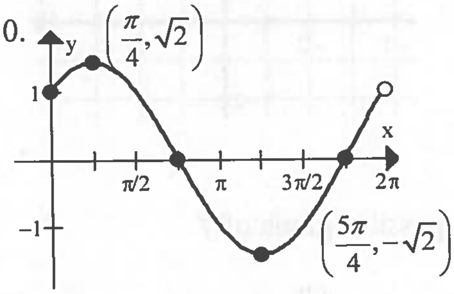
E.B. horiz. asymp. $y=2$

7. Do: $x\neq 0$, rel. max. pt. $(-1,-1)$, rel. min. pt. $(1,1)$,

V. A. $x=0$ (odd), E. B. like $y=\frac{1}{2}x$

9. Do: $-3\leq x\leq 3$, x-int. $(\pm 3,0)$, rel. max. pt. $(0,3)$

10.



11. Rel. Min. at $(0,-1)$, P.I. at $\left(-1,-\frac{2}{e}\right)$

12. Do.: $x>0$, Rel. Min. pt.: $(1,1)$, no P.I.

15.a. $(0,-3),(0,1)$ 16. $\left(\pm\sqrt{8},-1\right)$

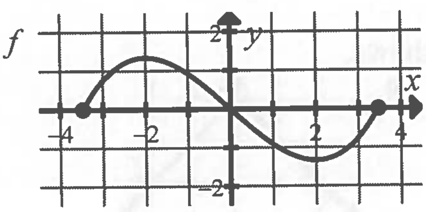
17a. 7 18. B 22. (A) $-\frac{2}{3}$, (G) 1

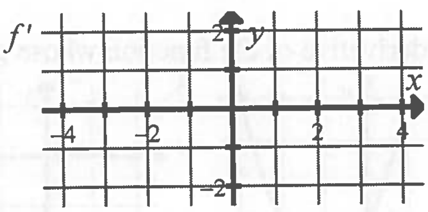
23. $\frac{dx}{dt}=24$ *units per sec.* 24. $\frac{1}{19}$

Lesson 5.2 Graphing Derivatives and Antiderivatives from Graphs

Derivatives: f graph $\rightarrow f'$ graph (or f' graph $\rightarrow f''$ graph)
Find (or estimate) slopes and plot them as points.

Example: 1. Use the graph of f shown to sketch a graph of f'.



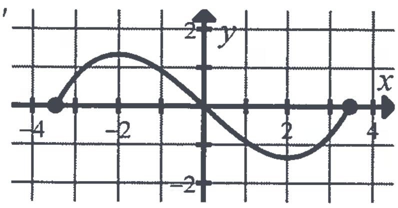


Antiderivatives: f' graph $\rightarrow$ f graph
1. Make an f' number line by using the location or position of the points on the f' graph. This does not involve the slopes of f'.
2. Make an f'' number line by using the slopes of the f' graph.
3. Combine information from both number lines to graph f. If no starting point is given, you are free to shift the graph vertically.

Examples:

2. Use the graph of f' shown to sketch a graph of f with a starting point of $(0,1)$.

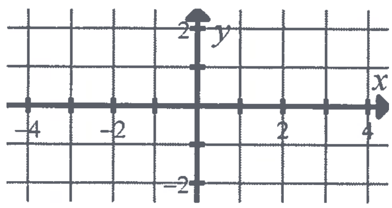
f'



f' ____________

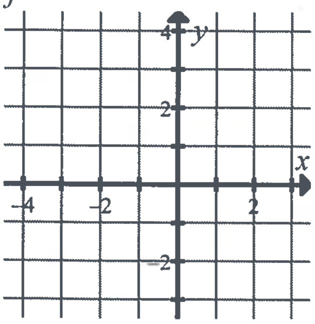
f'' ____________

f



3. Use the graph of f' shown to sketch a graph of f'' and a possible graph of f.

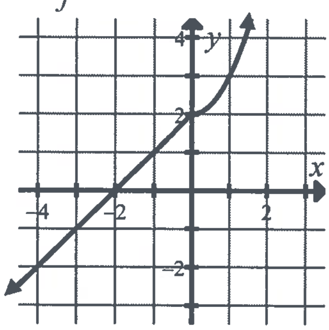
f



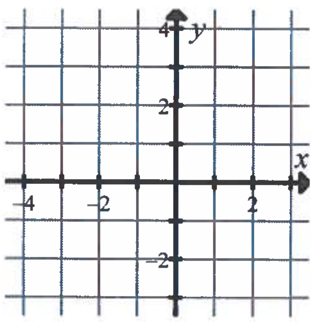
f' ⟷

f'' ⟷

f'



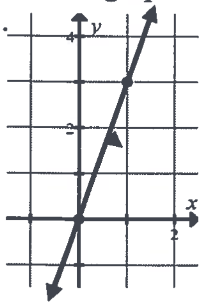
f''



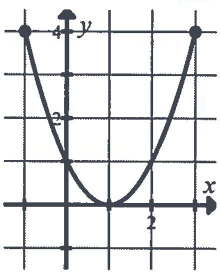
Assignment 5.2

Sketch a graph of the derivative of the function whose graph is shown.

1.



2.



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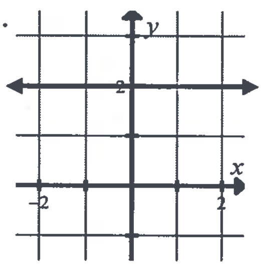

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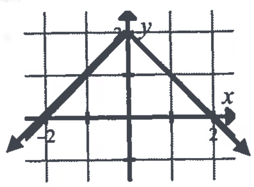

Use the graph of f' shown and the given starting point to graph f (the antiderivative).

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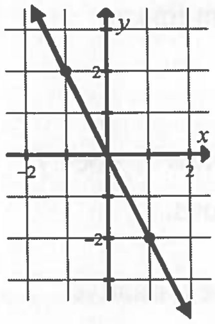
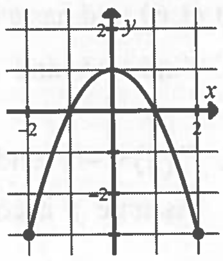
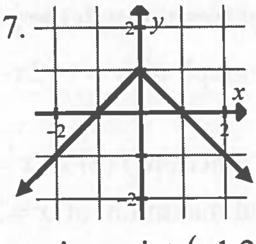


starting point (0,0)

6.


starting point (1,2)

7.


starting point $(-1,2)$

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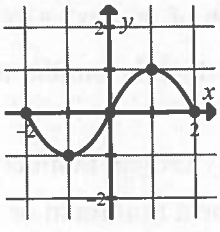

starting point (0,0)

Use the graph of f' shown to sketch a graph of f'' and a possible graph of f.

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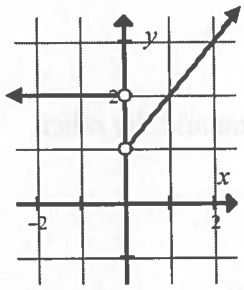

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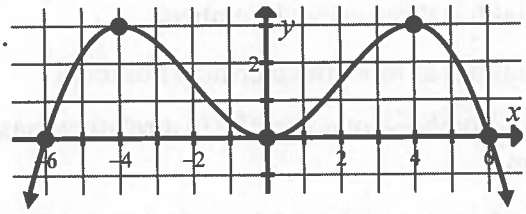

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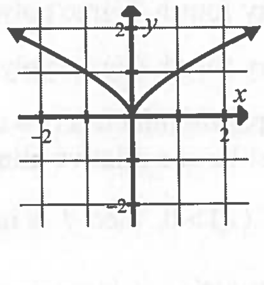


12. Find the vertical asymptotes, end behavior, and relative extrema points. Then graph $f(x)=\dfrac{1}{x^2-2x-8}$ without using a calculator.

13. If $f(x)=x(x-4)^3$, find relative extrema points and points of inflection. Then graph f without using a calculator. Hint: $f''(x)=12(x-4)(x-2)$.

14. Use the <u>Second Derivative Test</u> to find the relative extrema points of $f(x)=x^3-3x^2-5$.

15. Use the following information to sketch a possible graph of f.
$f(0)=f(4)=0,\ f(2)=-2,$
$f'(x)<0$ when $x<2,\ f'(x)>0$ when $x>2,\ f'(2)$ does not exist,
$f''(x)<0$ when $x\neq 2$

16. Find the c-value guaranteed by the Mean Value Theorem for $f(x)=x^3-2x+3$ on the interval $[0,2]$.

17. Find the absolute minimum and absolute maximum of the function $f(x)=x^3-12x-2$ on the interval $[0,4]$ without using a calculator.

18. Without using a calculator, sketch a graph of $f(x)=\left|-x^2-6x\right|$.

19. The graph of $y=ax^2+bx+c$ passes through the point $(1,6)$ and has a tangent line at $(0,16)$ which is parallel to the graph of $y=-12x-2$. Find a, b, and c.

20. If the only critical number of a function $f(x)$ is $x=3$, $f'(2)=-6$, and $f'(4)=7$, does f have a local minimum or a local maximum at $x=3$? Assume f is continuous.

21. If $x=3$ is a critical number of a function $g(x)$ and $g''(3)=-6$, does g have a relative minimum or a relative maximum at $x=3$?

Answer True or False on problems 22-26.

22. Every fourth degree polynomial has three critical numbers.
23. Every fourth degree polynomial has at most three critical numbers.
24. If a polynomial has two critical numbers, one must be at a relative maximum and the other must be at a relative minimum.
25. If $f'(2)>0$, then f is increasing on some interval containing $x=2$.
26. If a function f is increasing on an interval containing $x=2$, then $f'(2)>0$.

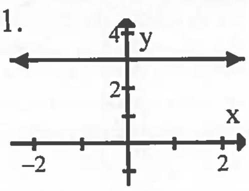
Determine if the Mean Value Theorem can be applied to $f(x)$ on the given interval. If it can be applied, find the c-value. If it cannot be applied, explain why not. You may use a calculator. Answer with three or more decimal place accuracy.

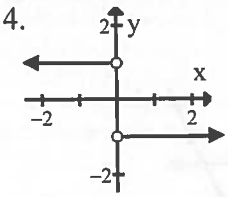
27. $f(x)=x\sqrt{2x-1}$ on $[1,5]$

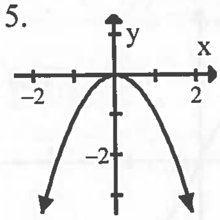
28 $f(x)=\begin{cases} x^2, & x\le 0 \\ x, & x>0 \end{cases}$ on $[-1,3]$

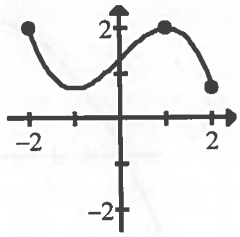
29. Use the implicit relation $2xy+y^4=x^2+1$ containing the point (2,1) on the graph for the following. Do not use a calculator.
    a. Find $\frac{dy}{dx}$.
    b. Is the curve increasing or decreasing on an interval containing the point (2,1)?
    c. Use the equation of a tangent line to approximate the y-coordinate when $x=1.9$.
    d. Is the curve concave upward or downward on an interval containing the point (2,1)?
    e. Is your approximation in part c an underestimate or an overestimate. Explain.

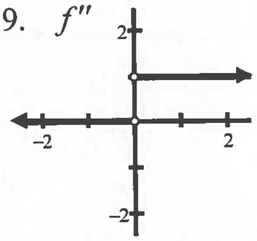
Selected Answers:

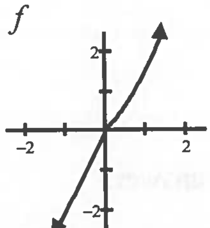
1. 

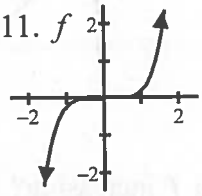
4. 

5. 

6. 

9. f'' 

f 

11. f 

These f graphs could be shifted vertically.

12. V.A.: $x=4,-2$, H.A.: $y=0$, Rel. Max. pt.: $\left(1,-\frac{1}{9}\right)$, no Rel. Min.

13. Rel. Min. pt.: $(1,-27)$, no Rel. Max., P.I.: $(2,-16)$, $(4,0)$

14. Rel. Max. pt.: $(0,-5)$, Rel. Min. pt.: $(2,-9)$ 16. $c=\sqrt{\frac{4}{3}}$

17. abs. min. $f=-18$, abs. max. $f=14$ 19. $a=2,\ b=-12,\ c=16$

27. $c=2.877$ 28. MVT does <u>not</u> apply. (sharp turn at $x=0$)

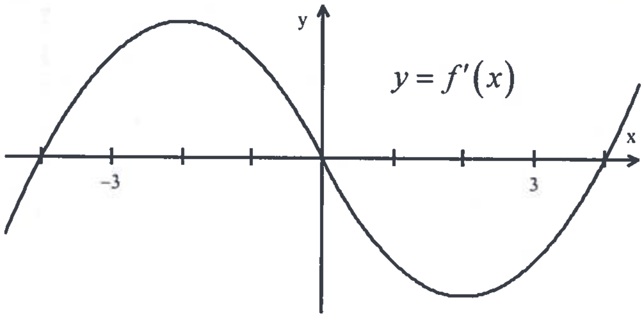
29a. $\dfrac{dy}{dx}=\dfrac{x-y}{x+2y^3}$ b. increasing c. $y\approx\frac{39}{40}$ d. upward e. underestimate since $y''>0$ on $[1.9,2]$

Lesson 5.3 Using Graphs of the First Derivative with Justification

An extremely common type of Free Response Question on the AP™ Calculus test is one where a graph of a first derivative is given and students are asked make conclusions about the original function with justification. It is very important that students reference the given graph by name in their justifications. Typically, these justifications require a short sentence. It is unnecessary and unwise to write more than one sentence of justification. If a student justifies correctly but continues by writing a false statement the justification point is lost.

Many students find it helpful to draw first or second derivative number lines (sign charts). These number lines do **not** count as justification but may help the student answer the question correctly which is, of course, the first priority.

The graph below is a graph of a differentiable function $f'(x)$ the derivative of a function $y=f(x)$. This graph of $f'(x)$ has horizontal tangents only at $x=\pm 2$ and its domain is $(-\infty,\infty)$.



Examples:

1. On what **open** interval(s) is f increasing? Justify your answer.

2. On what interval(s) is f decreasing? Justify your answer.

3. At what x-value(s) does f have a relative maximum. Justify your answer.

4. At what x-value(s) does f have a local minimum. Justify your answer.

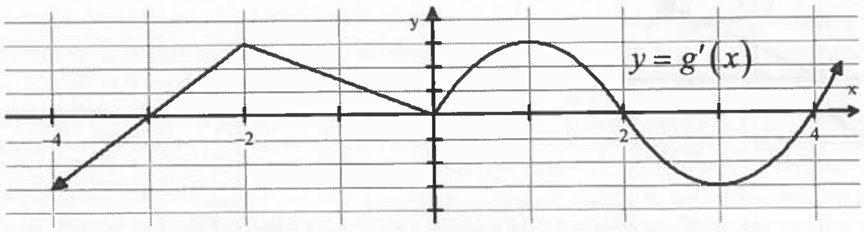
5. On what interval(s) is f concave down? Justify your answer.

6. At what x-value(s) does f have a point of inflection. Justify your answer.

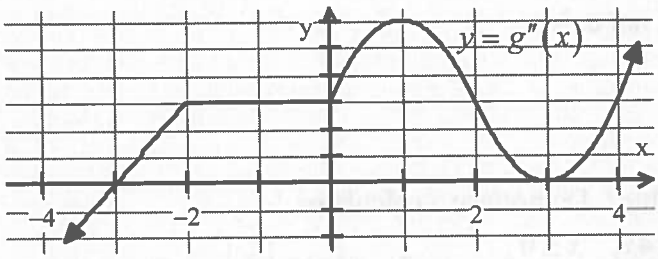
7. On what interval(s) is f both decreasing and concave upward? Justify your answer.

Assignment 5.3

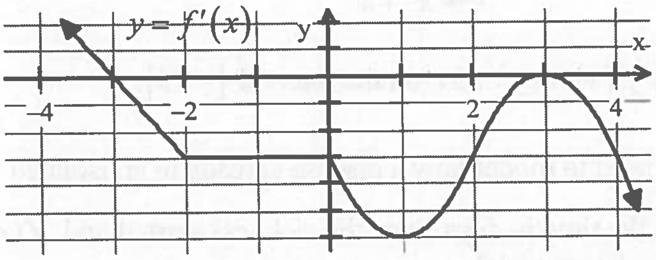
Use the graph shown in each problem to answer the questions and justify your answers. Assume there is no hidden behavior.

1. 

   a. On what interval(s) is g decreasing? Justify.
   b. On what interval(s) is g increasing? Justify.
   c. At what x-value(s) does g have a relative minimum? Justify.
   d. On what interval(s) is g concave upward? Justify.
   e. At what x-value(s) does g have a point of inflection. Justify.

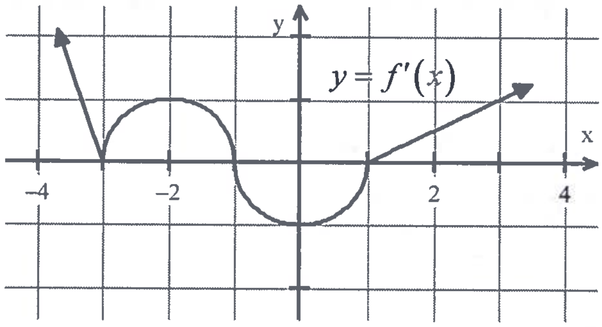
2. 

   a. On what interval(s) is g concave upward? Justify.
   b. At what x-value(s) does g have a point of inflection. Justify.

3. 

   a. On what interval(s) is f decreasing? Justify.
   b. At what x-value does f have a local minimum. Justify.
   c. On what interval(s) is f concave downward? Justify.
   d. At what x-value(s) does f have a point of inflection. Justify.
   e. On what interval(s) is f both increasing and concave upward? Justify.

4. $y = f'(x)$ graphed below is a piecewise function consisting of two linear pieces and two semicircles as shown.



a. On what interval(s) is f decreasing? Justify.
b. On what interval(s) is f concave upward? Justify.
c. At what x-value(s) does f have a point of inflection. Justify.
d. At what x-value(s) does f have a relative minimum? Justify.
e. Find $f''(-3)$ or explain why it does not exist.
f. Find $f''(-2)$ or explain why it does not exist.
g. Find $f''(-1)$ or explain why it does not exist.
h. Find $\lim\limits_{x\to 3} \dfrac{3x-9}{f'(x)-1}$.

Find the discontinuities. Which are removable? Do not use a calculator.

5. $f(x) = \dfrac{x-2}{x^2-3x+2}$ 6. $f(x) = \begin{cases} x^2-4x, & x \le 0 \\ x-1, & x > 0 \end{cases}$ 7. $f(x) = \left\lfloor \frac{x}{3} \right\rfloor$

Find these limits without a calculator.

8. $\lim\limits_{x\to 2} \dfrac{x-2}{x^2-4}$ 9. $\lim\limits_{x\to 2} \dfrac{x+2}{x^2-4}$ 10. $\lim\limits_{x\to \infty} \dfrac{x+2}{x^2+4}$

11. Find the absolute extrema of the function $f(x) = -x^2+3x$ on the interval $[-1,3]$.

12. The function $f(x) = \dfrac{4000}{1+100e^{-0.5x}}$ can be used to model how a disease spreads in an isolated population of 4000 people. x represents the time in days since the sickness started and $f(x)$ represents the number of people who have become sick.
Use a calculator to help answer the questions below.
a. How many people have become sick by the tenth day?
b. How fast was the disease spreading on the tenth day?
c. Use the maximum function on your calculator to find the maximum point on f'. What does the x-coordinate represent? What does the y-coordinate represent?
Note: Finding a maximum or minimum with a calculator will not be allowed on the AP™ Calculus test.
d. How many people have caught the disease when the curve is the steepest?
e. Why would the slope of the curve decrease after a period of time?
f. When was the rate of the spread of the disease increasing the fastest?

13. Without using a calculator, find vertical asymptotes, relative extrema points, and end behavior, and then sketch a graph of $f(x)=\dfrac{x^2+2}{x^2-9}$.

14. Use the intercepts, vertical asymptotes, relative extrema points, and end behavior to graph f, if $f(x)=\dfrac{x+1}{x^2-4x+4}$ and $f'(x)=\dfrac{-x-4}{(x-2)^3}$. Do not use a calculator.

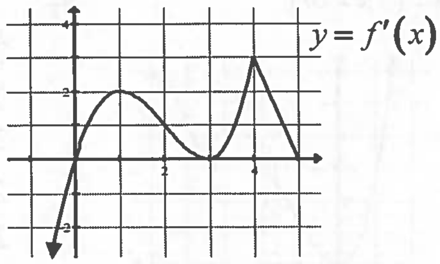
15. Find all points of inflection of $f(x)=\frac{1}{4}x^4-2x^3+2x-6$ without using a calculator.

16. Find all relative extrema points on the graph of $y=\dfrac{x}{x^3+4x}$ without using a calculator.

17. Find the c-value guaranteed by the Mean Value Theorem for the function $y=-3x^3-2x^2+3x-5$ on the interval $[-2,0]$. You may use a calculator.

18. Without using a calculator, find the domain, the intercepts, the vertical asymptote, the end behavior, the relative extrema points, and the points of inflection. Then sketch a graph of $f(x)=\dfrac{x^3+2}{x}$. Hint: $f'(x)=\dfrac{2x^3-2}{x^2}$ and $f''(x)=\dfrac{2x^3+4}{x^3}$.

Use the graph of $f'(x)$ shown for problems 19 and 20.



19. Sketch a graph of $f''(x)$.

20. Sketch a graph of $f(x)$ which contains the point $(0,0)$.

21. Given $g'(x)=2x^3-0.1e^x+x$. You may use a calculator.
   a. On what interval(s) is g increasing?
   b. At what x-value(s) does g have a relative minimum?
   c. Find $g''(x)$.
   d. On what interval(s) is g concave upward?
   e. At what x-value(s) does g have a point of inflection?

Selected Answers:

1. a. $(-\infty,-3]$ and $[2,4]$ because $g'\le 0$ b. $[-3,2]$ and $[4,\infty)$ because $g'\ge 0$
   c. $x=-3,\ 4$ because g' changes from negative to positive
   d. $(-\infty,-2)$, $(0,1)$, and $(3,\infty)$ because g' is increasing
   e. $x=-2,0,1,3$ because g' changes between increasing and decreasing
2. a. $(-3,\infty)$ because $g''\ge 0$ b. $x=-3$ because g'' changes sign
3. a. $[-3,\infty)$ because $f'\le 0$ d. $x=1,\ 3$ because f' has relative extrema
   e. none because f' is never positive and increasing

More Selected Answers:

4f. 0 g. DNE because $f' \leq 0$ has a vertical tangent h. 6

5. $x=1$ nonremovable, $x=2$ removable 8. $\frac{1}{4}$ 9. DNE 10. 0

11. max $f = \frac{9}{4}$, min $f = -4$

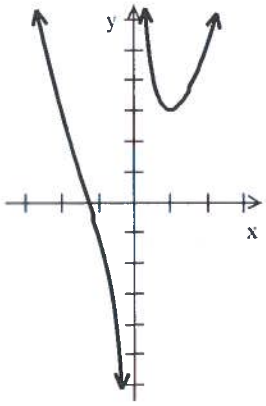
12. a. 2389 or 2390 people b. $481.009 \frac{people}{day}$ c. (9.210 , 500.000)

13. V. A.: $x = \pm 3$ (odd), Rel. Max. pt.: $\left(0, -\frac{2}{9}\right)$, no Rel. Min., H. A.: $y = 1$

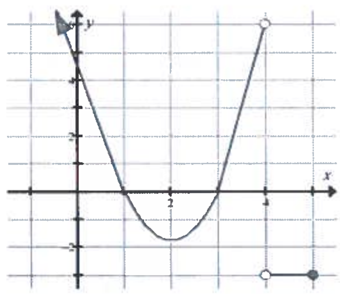
14. V.A.: $x = 2$ (even), x-int.: $(-1, 0)$ (odd), y-int.: $\left(0, \frac{1}{4}\right)$, Rel. Min. pt.: $\left(-4, -\frac{1}{12}\right)$, H.A: $y = 0$

17. $c = -1.190$ or -1.191

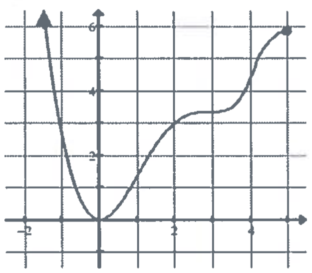
18. Do.: $x \neq 0$
V. A.: $x = 0$ (odd)
x-int.: $\left(-\sqrt[3]{2}, 0\right)$
no y-int.
E. B.: like $y = x^2$
Rel. Min. pt.: $(1,3)$
no Rel. Max.
P. I.: $\left(-\sqrt[3]{2}, 0\right)$



19.



20.



21a. $[.108, 9.869]$ or $[.109, 9.869]$
21b. $x = .108$ or $.109$
21c. $g''(x) = 6x^2 - 0.1e^x + 1$
21d. $(-\infty, 8.338)$ or $(-\infty, 8.339)$
21e. $x = 8.338$ or 8.339

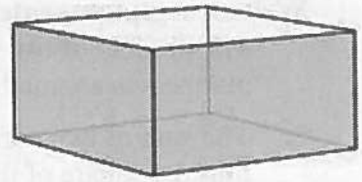
Lesson 5.4 Max/Min Applications (Optimization)

One of the most common applications of Calculus involves determining maximum or minimum values.

<u>Procedure:</u>
1. Choose variables and/or draw a labeled figure.
2. Write a primary equation. Isolate whatever is to be maximized or minimized.
3. Rewrite with only one variable on each side. This may require a secondary equation.
4. Find the domain.
5. Take the derivative, find critical numbers, make a number line, etc.

Examples: **Answer with a complete sentence.**

1. A box with no lid is to be made from 48 cm^2 of material. If the box must have a square base, find the dimensions that produce a maximum volume.



2. The sum of two nonnegative numbers is 30. Find both numbers if the sum of twice the first plus the square of the second is

a. a maximum.

b. a minimum.

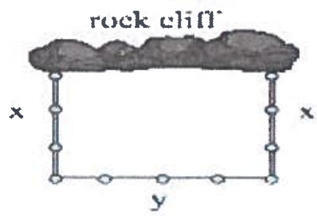
Assignment 5.4

Write a complete sentence explaining the meaning of each answer for problems 1-12.

1. The product of two positive numbers is 100. Find the two numbers so that the sum of the numbers is as small as possible.

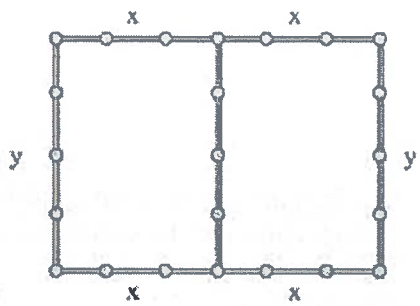
2. The sum of two nonnegative numbers is 25. Find the two numbers so that the sum of the first plus the square of the second is a maximum.

3. A rancher plans to fence in three sides of a rectangular pasture, with the fourth side being against a rock cliff. He needs to enclose 320,000 square meters of pasture. What dimensions would require the least amount of fence material?

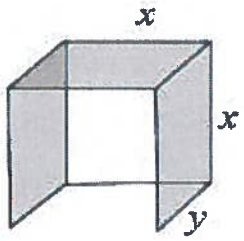


4. The perimeter of a rectangle is 80 feet. Find the length and width so that the rectangle has a maximum area.

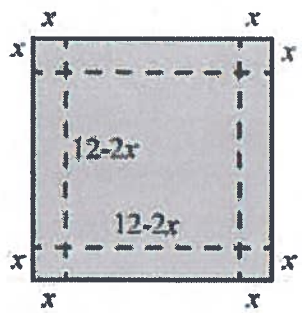
5. Two adjacent rectangular corrals are to be made using 240 feet of fencing. The fence must extend around the outer perimeter and across the middle as shown in the diagram. Find the dimensions so that the total enclosed area is as large as possible.



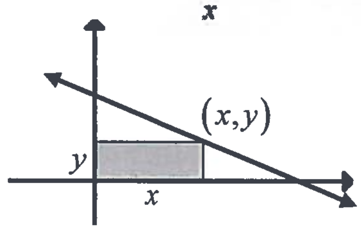
6. A shelter at a bus stop is to be made with three Plexiglas sides and a Plexiglas top. If the volume of the shelter is 486 cubic feet, find the dimensions that require the least amount of Plexiglas.



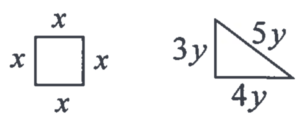
7. A box is made by cutting small squares from each corner of a piece of square material 12 inches on each side and then folding up the flaps. Find the length of each side of the square cutouts that will produce the greatest volume box.



8. A rectangle is positioned with one vertex on the line $y=-\frac{1}{2}x+3$ in the first quadrant as shown. Find the point so that the rectangle has a maximum area.



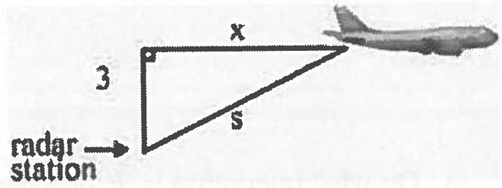
9. 36 total inches of wire is to be used to form the perimeter(s) of a square and/or a 3-4-5 ratio right triangle. Find the dimensions of the figure(s) that enclose both a maximum and minimum total area.



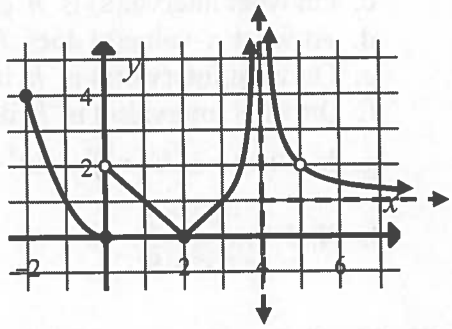
10. A box with an open top has a square base. If the volume of the box is 4000 cubic centimeters, what dimensions minimize the amount of material used?

11. The volume formula for a cone is $V = \frac{1}{3}\pi r^2 h$. If $\frac{dr}{dt} = 3\frac{in}{min}$ and $h = 3r$, find $\frac{dV}{dt}$ when $r = 6$ inches.

12. An airplane flying at an altitude of 3 miles flies directly over a radar station. When the plane is 5 miles away from the station, the radar shows the distance s is changing at the rate of 300 miles per hour. What is the plane's speed?



13. Use the graph of $y = f(x)$ at the right for these problems.



a. Find $\lim\limits_{x\to 0} f(x)$. b. Find $\lim\limits_{x\to 0^+} f(x)$.

c. Find $\lim\limits_{x\to 4} f(x)$. d Find $\lim\limits_{x\to\infty} f(x)$.

e. Find $\lim\limits_{x\to 3} f(x)$. f. List the discontinuities of f.

g. Which of these discontinuities are removable?

h. Find the absolute maximum of $f(x)$ on $[-2,3]$.

i. Find the absolute minimum of $f(x)$ on $[-2,3]$.

j. Find $f'(1)$. k. Find $f''(1)$.

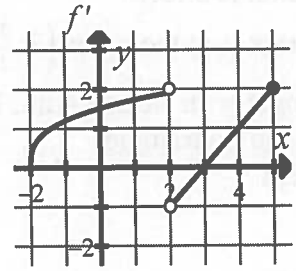
l. List all x-values where $f'(x)$ does not exist.

m. List all x-values at which $f(x)$ has a local minimum.

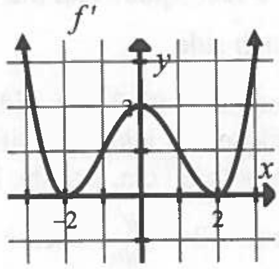
n. List all x-values at which $f(x)$ has a local maximum.

Use these graphs of f' to graph f''.

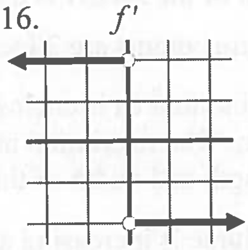
14. f'



15. f'

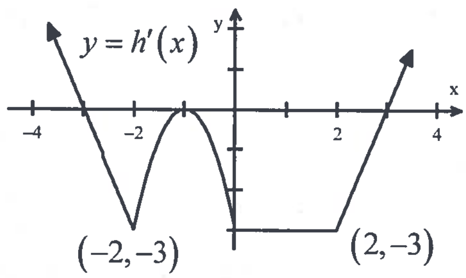


16. f'



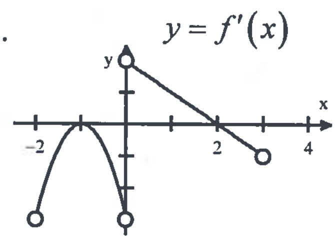
17. Use the graph of f' in Problem 14 to sketch a possible graph of f.
18. Use the graph of f' in Problem 15 to sketch a graph of f with the starting point $(-2,-2)$.
19. Use the graph of f' in Problem 16 to sketch a continuous graph of f with the starting point $(0,2)$.

20. The graph of $y = h'(x)$ is shown with no hidden behavior.



a. On what interval(s) is h decreasing? Justify your answer.
b. At what x-value(s) does h have a relative minimum? Justify your answer.
c. On what interval(s) is h concave upward? Justify your answer.
d. At what x-value(s) does h have a point of inflection. Justify your answer.
e. On what interval(s) is h increasing and concave downward? Justify your answer.
f. On what interval(s) is h decreasing and linear? Justify your answer.
g. If $f(x) = 3(h'(x))^2 + 4x^3$ find $f'(-2.5)$. You may use a calculator.
h Find $\lim\limits_{x \to -3} \dfrac{h'(x)}{x^2 - 9}$.

21. Use the graph of $y = f'(x)$ shown, given f is continuous on $(-2,3)$.

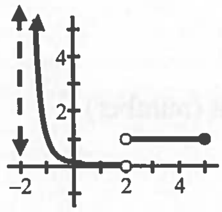


a. On what interval(s) is f decreasing? Justify.
b. At what x-value(s) does f have a relative minimum? Justify.
c. On what interval(s) is f concave upward? Justify.
d. At what x-value(s) does f have a point of inflection. Justify.

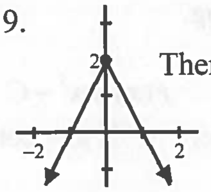
Selected Answers:
1. The numbers are both 10.
2. The first number is 0 and the second is 25.
3. The sides perpendicular to the rock are both 400 m and the other side is 800 m.
4. The rectangle's length and width are both 20 *ft*.
5. The shared side is 40 feet and the other dimension of each corral is 30 feet.
6. The back of the shelter is a 9 feet by 9 feet square and the other dimension is 6 feet.
7. The square cutouts are 2 inches on each side.
8. The vertex is at the point $\left(3, \frac{3}{2}\right)$.
9. The minimum area is enclosed by a square with 3.6 in. sides and a triangle with sides 5.4 in., 7.2 in., and 9 in. The maximum area is enclosed by a square with 9 in. sides and no triangle.
10. The length and width of the box are both 20 cm. and the height is 10 cm.
11. The volume is increasing at the rate of $324\pi \frac{in^3}{min}$ when $r = 6$ in.
12. The plane's speed is 375 *mph* when it is 5 miles away.
13a. DNE b. 2 c. DNE or ∞
13d. 1 e. 1 f. $x = 0,4,5$ g. $x = 5$ h. 4 i. 0 j. -1 k. 0
13l. $x = 0,2,4,5$ (also possibly $x = 3$) m. $x = 0,2$ n. none

More Selected Answers:

14. f''



19.



There could also be a jump at $x = 0$.

20. a. $[-3,3]$ because $h' \le 0$ b. $x = 3$ because h' changes from negative to positive.
   c. $(-2,-1)$ and $(2,\infty)$ because h' is increasing
   d. $x = -2, -1$ because h' changes between increasing and decreasing
   e. $(-\infty,-3)$ because h' is positive and decreasing
   f. $[0,2]$ because h' is negative and horizontal g. 102 h. $\frac{1}{2}$

21. a. $(-2,0]$, $[2,3)$ because $f'(x) \le 0$
   b. $x = 0$ because $f'(x)$ changes from negative to positive
   c. $(-2,-1)$ because $f'(x)$ is increasing
   d. $x = -1$ because $f'(x)$ has a relative maximum (because $f'(x)$ changes from inc. to dec)

UNIT 5 SUMMARY

Curve Sketching:

Precalculus: domain, intercepts, vert. asymptotes, holes, end behavior, symmetry
Calculus: f' number line → inc./decr. and max./min.
f'' number line → concavity, pts. of infl.

Graph to Graph:

$f' \to f''$ Find slopes on f' and plot them as points on f''.
$f' \to f$ Make an f' number line using the **location** of points on the f' graph.
Make an f'' number line using the **slope** at points on the f' graph.
Use both number lines to sketch a graph of f.

Max/Min Applications:

Procedure:
1. Choose variables and/or draw a labeled figure.
2. Write a primary equation. Isolate whatever is to be maximized or minimized.
3. Rewrite with only one variable on each side. This may require a secondary equation.
4. Find the domain.
5. Take the derivative, find critical numbers, make a number line, etc.

Lesson 6.1 Antidifferentiation, Indefinite Integrals

Warm-up Examples: Differentiate each of the following.

1. $f(x) = x^3$

2. $f(x) = x^3 - 10$

3. $f(x) = x^3 + C$
   where C is any constant (number)

So what should you get when you antidifferentiate $f'(x) = 3x^2$? $f(x) =$ ______

This problem can be written as $\int 3x^2 dx =$

The symbol $\int$ is called an integral symbol and tells you to integrate (antidifferentiate) the expression which follows it. That expression is called an integrand. dx indicates that you are integrating with respect to the variable x but does not affect the integration process. C is called the constant of integration and must be written as part of your answer when you are antidifferentiating.

Integration Rules:

Power Rule: $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + C, \quad n \neq -1$

Constant Rule: If k is any constant, $\int k \, dx = kx + C$

Scalar Multiple Rule: If k is any constant, $\int k f(x)\, dx = k \int f(x)\, dx$
(Constants may be "factored out" of the integral expression. **NEVER** "factor out" a variable.)

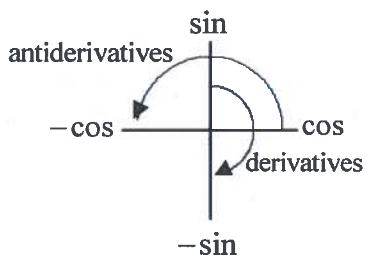
Sum Rule: $\int \big(f(x) + g(x)\big)\, dx = \int f(x)\, dx + \int g(x)\, dx$

Exponential Rules: $\int e^x dx = e^x + C$ $\qquad \int a^x dx = \dfrac{a^x}{\ln a} + C$

Trig Rules:
$\int \cos x \, dx =$ $\qquad \int \sin x \, dx =$
$\int \sec^2 x \, dx =$ $\qquad \int \csc^2 x \, dx =$
$\int \sec x \tan x \, dx =$ $\qquad \int \csc x \cot x \, dx =$

The most common errors made with the trig functions are sign errors.

The simple diagram at the right, reminiscent of the unit circle, may aid in remembering rules for these two functions which are used far more than any other trig functions. Rotate clockwise for derivatives. Rotate counterclockwise (anticlockwise) for antiderivatives.



Examples: Evaluate (Integrate as indicated).

4. $\int x^3\,dx$ 5. $\int 2\,dx$ 6. $\int (t^4+2)\,dt$

7. $\int (2y^2+4y+1)\,dy$ 8. $\int \frac{\sqrt{x}+1}{x^2}\,dx$ 9. $\int -3\sin x\,dx$

10. $\int \left(e^x-2^x\right)dx$ 11. $\int \left(5\sqrt[3]{t^2}+3\cos t-1\right)dt$ 12. $\int \frac{\sec y\tan y}{4}\,dy$

Note: Put $+C$ when you integrate, but never when you differentiate.

Sometimes an initial condition is given which makes it possible to solve for C.

Example 13: If $f'(x)=(\csc x)^2$ and $f\left(\frac{5\pi}{4}\right)=3$, find $f(x)$.

If we know the acceleration equation for an object, and if we are given initial conditions for the object's velocity and position, integration allows us to find the velocity and position equations for the object.

Remember: Pos. → Vel. → Acc. (Differentiate), so Acc. → Vel. → Pos. (Integrate).

Example 14: The acceleration of a particle at time t is given by $a(t)=4t-3$.
$v(1)=6$ and $s(2)=5$.

a. Find the velocity equation. $v(t)=$

b. Find the position equation. $s(t)=$

Example 15: Given that on earth, the acceleration of an object due to gravity is approximately $-32\ ft/sec^2$ (negative indicates downward), develop

a. the equation for the velocity of the object.
$v_o =$ initial velocity

$v(t) =$

b. the equation for the position of the object.
$s_o =$ initial position

$s(t) =$

Note: The two equations $v(t) = -32t + v_0$ and $s(t) = -16t^2 + v_0 t + s_0$ may be used for any motion affected only by the earth's gravity.

Assignment 6.1
Do not use a calculator on this entire assignment.

For Problems 1-5, rewrite the integrand and then integrate.

1. $\int \frac{1}{x^3}\,dx$ 2. $\int \sqrt[4]{t}\,dt$ 3. $\int (x+1)(x-2)\,dx$ 4. $\int \frac{2y}{\sqrt{y}}\,dy$ 5. $\int \left(\frac{5^{3x}}{5^{2x}}\right)dx$

Evaluate (integrate) each integral in Problems 6-20.

6. $\int (2x^3 - x^2 + 1)\,dx$ 7. $\int \left(5x^{\frac{1}{4}} - x^{\frac{-2}{3}}\right)dx$ 8. $\int \frac{1}{3x^2}\,dx$ 9. $\int \frac{1}{(3x)^2}\,dx$

10. $\int \left(\frac{2}{e^{-x}} + \sqrt{x}\right)dx$ 11. $\int \frac{2\sqrt{t}-1}{\sqrt{t}}\,dt$ 12. $\int (\sin y - 2\cos y)\,dy$ 13. $\int (3t-10)^2\,dt$

14. $\int (\sec^2\theta - 2)\,d\theta$ 15. $\int \frac{4x^4 - xe^x}{x}\,dx$ 16. $\int \left(\theta^2 + 6\sec\theta\tan\theta\right)d\theta$ 17. $\int \frac{1}{\sin^2 y}\,dy$

18. $\int \left(6^x - \frac{2}{x^4}\right)dx$ 19. $\int \sqrt{9e^{2x}}\,dx$ 20. $\int \sqrt{y}\left(y^2 + 2\sqrt{y}\right)dy$

21. If $f'(x) = 3x^2 - 4x + 2$ and $f(1) = -3$, find $f(x)$.

22. The derivative of a function is $\frac{dy}{dt} = \frac{-3}{t^2} + 1$. If the graph of the function contains the point (3, 10), find the equation of the function.

23. a. Find an equation for the family of functions whose derivative is $y' = 3\sqrt{x}$.
    b. Find the particular function from the family in Part a. whose curve passes through the point (4, 0).

24. Find $g(x)$, given that: $g''(x) = 2x - 3$, $g'(0) = -5$, and $g(-1) = 2$.

25. The acceleration of an object moving along a horizontal path is given by the equation $a(t) = 6t - 4$. The object's initial velocity is 5, and its initial position is -2.

a. Find a velocity equation for the object.
b. Find the velocity of the object when $t = 2$.
c. Find a position equation for the object.
d. Find the object's position when $t = 2$.

26. The velocity of an object moving along a vertical path is given by the equation $v(t) = \sqrt{t} + 1, \quad t \geq 0$.

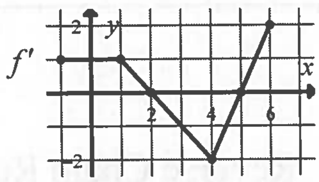
a. Find an equation for the object's acceleration.
b. Find the acceleration of the object when $t = 9$.
c. The object's position at $t = 9$ is 20. Find an equation for the object's position.

27. A ball is dropped from a bridge which is 160 feet above a river. How long will it take the ball to hit the water? Use the equation $s(t) = -16t^2 + v_o t + s_o$.

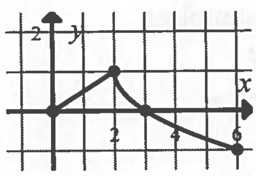
28. Find $f(x)$, given that: $f'(x) = \begin{cases} 2x+8, & x \leq 2 \\ 3x^2, & x > 2 \end{cases}$, $f(1) = 1$.

29. For the first 4 seconds of a race, a sprinter accelerates at a rate of 3 meters per second per second (3 m/sec^2). He then continues to run at the constant speed that he has attained for the rest of the race.
a. Write a piecewise function to express the sprinter's velocity $v(t)$ as a function of time.
b. Find $v(2)$, $v(4)$, and $v(6)$.
c. Write a piecewise function to express the sprinter's position $s(t)$ as a function of time.
d. How far does the sprinter run during the first 4 seconds of the race?
e. How long will it take the sprinter to run 100 m?

30. Use the graph of f' shown to graph f'' and a possible graph of f.



31. Use the graph of f' shown to graph f with the starting point (0,2).



32. Find an equation of a line tangent to the curve $y = x^{\frac{2}{3}}$ which is parallel to the line $2x - 6y = 5$.

33. Find the cubic function of the form $y = ax^3 + bx^2 + cx + d$ which has a relative maximum point at (0, 2) and a point of inflection at $(-1, -2)$.

34. The point $(1,-2)$ is on the graph of $y^2 - x^2 + 2x = 5$.

a. Find the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(1,-2)$.

b. Does the graph have a local maximum, a local minimum, or neither at this point? Explain.

Selected Answers:

1. $-\frac{1}{2x^2}+C$ 2. $\frac{4}{5}t^{\frac{5}{4}}+C$ 3. $\frac{x^3}{3}-\frac{x^2}{2}-2x+C$ 4. $\frac{4}{3}y^{\frac{3}{2}}+C$ 5. $\frac{5^x}{\ln 5}+C$ 8. $-\frac{1}{3x}+C$

9. $-\frac{1}{9x}+C$ 10. $2e^x+\frac{2}{3}x^{\frac{3}{2}}+C$ 11. $2t-2t^{\frac{1}{2}}+C$ 13. $3t^3-30t^2+100t+C$

14. $\tan\theta-2\theta+C$ 15. x^4-e^x+C 18. $\frac{6^x}{\ln 6}+\frac{2}{3}x^{-3}+C$ 20. $\frac{2}{7}y^{\frac{7}{2}}+y^2+C$

21. $f(x)=x^3-2x^2+2x-4$ 22. $y=\frac{3}{t}+t+6$ 23a. $y=2x^{\frac{3}{2}}+C$ b. $y=2x^{\frac{3}{2}}-16$

24. $g(x)=\frac{1}{3}x^3-\frac{3}{2}x^2-5x-\frac{7}{6}$ 25a. $v(t)=3t^2-4t+5$ c. $s(t)=t^3-2t^2+5t-2$

26c. $s(t)=\frac{2}{3}t^{\frac{3}{2}}+t-7$ 27. $t=\sqrt{10}$ sec 28. $f(x)=\begin{cases} x^2+8x-8, & x\le 2 \\ x^3+4, & x>2 \end{cases}$

29a. $v(t)=\begin{cases} 3t, & 0\le t\le 4 \\ 12, & t>4 \end{cases}$ c. $s(t)=\begin{cases} \frac{3}{2}t^2, & 0\le t\le 4 \\ 12t-24, & t>4 \end{cases}$ e. 10.333 *sec*

32. $y-4=\frac{1}{3}(x-8)$ 33. $y=-2x^3-6x^2+2$

34a. $\frac{dy}{dx}(1,-2)=0$ and $\frac{d^2y}{dx^2}(1,-2)=-\frac{1}{2}$ b. local maximum by the second derivative test

Lesson 6.2 Reverse Chain Rule, u-Substitution

<u>Warm-up Examples</u>:

Differentiate

1. $\frac{d}{dx}(1+5x)^4 =$ 2. $\frac{d}{dx}\sin(1+5x)=$

Now, integrate

3. $\int 5(1+5x)^3\,dx =$ 4. $\int 5\cos(1+5x)\,dx =$

<u>**Note**</u>: You must insert the <u>chain rule factor</u>, the <u>derivative of the inside function</u> in Examples 1 and 2, so you had to delete the <u>derivative of the inside function</u> in Examples 3 and 4.

Each of the Integration Rules from the last lesson can now be generalized as Reverse Chain Rule integrals.

	***x* form**	***u* form** (Reverse Chain Rule)
Power Rule:	$\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C,\quad n \neq -1$	$\int u^n u'\,dx = \dfrac{u^{n+1}}{n+1} + C,\quad n \neq -1$
Exponential Rules:	$\int e^x\,dx = e^x + C$	$\int e^u u'\,dx = e^u + C$
	$\int a^x\,dx = \dfrac{a^x}{\ln a} + C$	$\int a^u u'\,dx = \dfrac{a^u}{\ln a} + C$
Trig Rules:	$\int \cos x\,dx = \sin x + C$	$\int \cos u\,u'\,dx = \sin u + C$
	$\int \sin x\,dx = -\cos x + C$	$\int \sin u\,u'\,dx = -\cos u + C$
	$\int \sec^2 x\,dx = \tan x + C$	$\int \sec^2 u\,u'\,dx = \tan u + C$
	$\int \csc^2 x\,dx = -\cot x + C$	$\int \csc^2 u\,u'\,dx = -\cot u + C$
	$\int \sec x \tan x\,dx = \sec x + C$	$\int \sec u \tan u\,u'\,dx = \sec u + C$
	$\int \csc x \cot x\,dx = -\csc x + C$	$\int \csc u \cot u\,u'\,dx = -\csc u + C$

Examples: Integrate.

5. $\int (3x-1)^{10}\,dx$

6. $\int (3t^2+2t)(t^3+t^2)\,dt$

7. $\int \dfrac{6x^2}{\sqrt{4x^3-5}}\,dx$

8. $\int (y^3+1)^2\,dy$

9. $\int \sin(4x)\,dx$

10. $\int 3\theta\cos\theta^2\,d\theta$

11. $\int \sin^2 x\cos x\,dx$

u-Substitution

For more complicated integration problems, simple rules for integration might fail, and you may have to make some type of substitution to be able to integrate. In this course, a common substitution will be to let $u =$ the radicand$\left(\sqrt{\text{radicand}}\right)$ part of the expression and to change the variable throughout the integral before integrating. You should use this method of substitution (called u-substitution) only when simpler methods don't work. It should be your last resort.

Procedure for u-substitution: (for $\int \_\_\_\_ \, dx$ problems requiring the method)

- Let $u =$ radicand, inside the $\sqrt{\ }$ symbol, or some other "inside" function.
- Solve for x (in terms of u).
- Differentiate the equation from Step 2.
- Find dx.
- Substitute u-expressions for x-expressions in the integral.
  Note: Most often, $dx \neq du$. Don't forget to substitute for dx.
- Integrate.
- Substitute back, so that your final answer is again in terms of x.

Sometimes it is easier to do Step 3 before Step 2. These two steps are reversible.

Examples: Integrate.

12. $\int x\sqrt{x-1}\, dx$

13. $\int \frac{2x-1}{(2x+3)^9}\, dx$

Assignment 6.2

Do not use a calculator on this entire assignment.

Evaluate (integrate) in Problems 1-8.

1. $\int 3(3x-2)^5\, dx$
2. $\int (5t-3)^8\, dt$
3. $\int \frac{x^2}{\sqrt{4-x^3}}\, dx$
4. $\int \sqrt{y^2-3}\; y\, dy$
5. $\int \frac{5x^2}{(x^3+2)^6}\, dx$
6. $\int \frac{-3}{\sqrt{1-v}}\, dv$
7. $\int (2x^2-3x)^4(4x-3)\, dx$
8. $\int \frac{x-1}{(2x^2-4x)^5}\, dx$

Evaluate (integrate) in Problems 9-20.

9. $\int \frac{\left(\sqrt{t}-4\right)^{10}}{\sqrt{t}}\,dt$ 10. $\int \frac{1}{\sqrt[3]{5x}}\,dx$ 11. $\int (2u+1)^2\,du$ 12. $\int y^3 4^{3y^4}\,dy$

13. $\int \frac{x^2-3}{x^2}\,dx$ 14. $\int \frac{3x^2+x-2}{\sqrt{x}}\,dx$ 15. $\int \left(2+\frac{1}{x}\right)^4 \frac{1}{x^2}\,dx$ 16. $\int \pi \sin(\pi\theta)\,d\theta$

17. $\int \frac{e^{\frac{1}{x}}}{x^2}\,dx$ 18. $\int \sec(2x-1)\tan(2x-1)\,dx$ 19. $\int \tan^5\theta \sec^2\theta\,d\theta$ 20. $\int \frac{\csc^2 t}{\cot^4 t}\,dt$

Use u-substitution to evaluate in Problems 21-23.

21. $\int 30x\sqrt{x+1}\,dx$ 22. $\int \frac{3x-5}{\sqrt{\frac{1}{2}x-1}}\,dx$ 23. $\int (5x-8)(1-x)^{11}\,dx$

24. If $f''(x)=x^{\frac{-4}{3}}$, $f'(8)=\frac{3}{2}$, and $f(27)=5$, find $f(x)$.

25. The derivative of a function is $\frac{dy}{dx}=6x\sqrt{x^2-3}$. Find the function if $(2,5)$ is a point on the graph of the function.

26. If $f'(x)=\cos(3x)$ and $f\left(\frac{\pi}{6}\right)=2$, find $f(x)$.

27. Evaluate $\frac{d}{dx}\int (x^2-3)^4\,dx$.

28. The velocity of a particle moving along a vertical line is given by the equation $v(t)=\left(\frac{1}{3}t-1\right)^2$.

The particle's position at time zero is 4.

a. Find an equation for the particle's acceleration $a(t)$.
b. Find an equation for the particle's position $y(t)$.
c. At what time(s) is the particle at rest?
d. At what time(s) is the particle moving upward?
e. For what value(s) of t does the particle's speed equal the particle's velocity?
f. Find the total distance traveled by the particle from $t=3$ to $t=9$.
g. Find the interval(s) of time for which the speed of the particle is increasing

29. Find an equation for the line tangent to the graph of $y=\sqrt{3x-5}$ when $x=2$.

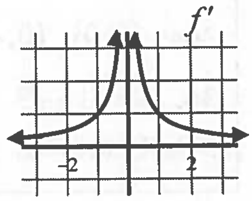
30. Differentiate $2x^2+y^2=4y$ implicitly to find the point(s) where the curve has

a. horizontal tangents. b. vertical tangents.

31. (2, 7) is a point on the curve of $f(x)=x^3-3x+5$. Use a tangent line to approximate $f(2.1)$.

32. The graph of $f'(x)$ is shown at right.

a. Use the given graph to make f' and f'' number lines.
b. Sketch a graph of f which passes through the points $(1,-1)$ and $(-1,1)$.



33. Find the instantaneous rate of change for $f(t)=\dfrac{t}{t+1}$ when $t=1$.

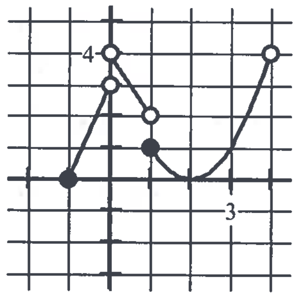
34. Find the average rate of change for $f(t)=\dfrac{t}{t+1}$ on $[0, 2]$.

35. Which of the rates of change from Problems 33 and 34 represents:

a. the slope of a secant line for the graph of $f(t)$?
b. the slope of a tangent line for the graph of $f(t)$?

36. Find the value of c in $[0, 2]$ such that $f'(c)=$ the average rate of change of $f(t)=\dfrac{t}{t+1}$ on $[0, 2]$. It is at this t-location that the slopes of what two lines are the same? (MVT).

37. Use the graph of $y=f(x)$ at right to find:



a. $\lim\limits_{x\to 1} f(x)$ b. $f(1)$

c. $\lim\limits_{x\to 4^-} f(x)$ d. $f(4)$

e. $\lim\limits_{x\to 0} f(1-x^2)$ f. $\lim\limits_{x\to 2} f(f(x))$

g. $\lim\limits_{x\to 0} f(1+|x|)$ h. $\lim\limits_{x\to 0}(f(x)-3.5)^2$

38. Use the alternate form of the limit definition of the derivative to find $f'(2)$ for $f(x)=x^2+1$.

Selected Answers:

1. $\frac{1}{6}(3x-2)^6+C$ 2. $\frac{1}{45}(5t-3)^9+C$ 3. $-\frac{2}{3}\sqrt{4-x^3}+C$ 5. $-\frac{1}{3}\left(x^3+2\right)^{-5}+C$

6. $6\sqrt{1-v}+C$ 8. $-\frac{1}{16}\left(2x^2-4x\right)^{-4}+C$ 9. $\frac{2}{11}\left(\sqrt{t}-4\right)^{11}+C$

10. $\dfrac{3}{2\sqrt[3]{5}}x^{\frac{2}{3}}+C$ or $\frac{3}{10}(5x)^{\frac{2}{3}}+C$ 11. $\frac{4}{3}u^3+2u^2+u+C$ or $\frac{1}{6}(2u+1)^3+C$

13. $x+\dfrac{3}{x}+C$ 14. $\frac{6}{5}x^{\frac{5}{2}}+\frac{2}{3}x^{\frac{3}{2}}-4x^{\frac{1}{2}}+C$ 16. $-\cos(\pi\theta)+C$ 18. $\frac{1}{2}\sec(2x-1)+C$

20. $\frac{1}{3}(\cot t)^{-3}+C$ 21. $12(x+1)^{\frac{5}{2}}-20(x+1)^{\frac{3}{2}}+C$ 22. $8\left(\frac{1}{2}x-1\right)^{\frac{3}{2}}+4\left(\frac{1}{2}x-1\right)^{\frac{1}{2}}+C$

23. $\frac{1}{4}(1-x)^{12}+\frac{5}{13}(1-x)^{13}+C$ 24. $-\frac{9}{2}x^{\frac{2}{3}}+3x-\frac{71}{2}$ 25. $y=2\left(x^2-3\right)^{\frac{3}{2}}+3$

26. $f(x)=\frac{1}{3}\sin(3x)+\frac{5}{3}$ 28a. $a(t)=\frac{2}{9}t-\frac{2}{3}$ b. $y(t)=\left(\frac{1}{3}t-1\right)^3+5$ e. at all times f. 8

30a. $(0,0)$, $(0,4)$ b. $\left(\pm\sqrt{2},2\right)$ 31. $f(2.1)\approx 7.9$ 33. $\frac{1}{4}$ 34. $\frac{1}{3}$

36. $c=-1+\sqrt{3}$ Tangent and secant lines have the same slope.

37b. 1 d. DNE e. 2 f. 4 38. $f'(2)=4$

Lesson 6.3 Definite Integrals, Calculator Integration, The Fundamental Theorem of Calculus

A Definite Integral is written with upper and lower limits attached to an integration expression. The value of a definite integral $\left(\int_a^b f(x)\,dx\right)$ may be thought of as a "signed area" from the lower limit a (usually a left side boundary) to the upper limit b (usually a right-side boundary), and between the curve of $f(x)$ and the x-axis. The value may be positive, negative, or zero.
Unlike the previous integration process which produced an indefinite integral (an antiderivative) representing a family of curves, a definite integral represents **a number value**.

Calculator Integration: A TI-84 calculator can be used to find the value of a definite integral from a to b by using $\int f(x)\,dx$ in the calculate menu or fnInt in the math menu. The calculate menu shows a graphical representation of the "signed area" together with the value of the definite integral.

Examples:
Use the calculate menu to evaluate the following definite integrals.

1. $\int_{-3}^{1}(x^3-6x)\,dx$
2. $\int_{-\sqrt{6}}^{\sqrt{6}}(x^3-6x)\,dx$
3. $\int_{-5}^{5}\left|x^3-6x\right|dx$

The math menu only provides the value of the definite integral, but that is usually all that we need. The math menu gives a more accurate answer. fnInt is recommended for all problems from now on.
Note: Newer operating systems have a MATHPRINT setting that simplifies this process.

Use the math menu to evaluate:

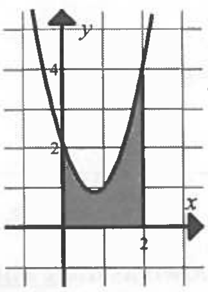
4. $\int_{-5}^{5}\left|x^3-6x\right|dx =$

fnInt$(\text{abs}(x^3-6x), x, -5, 5)$

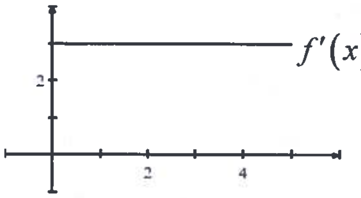
This is the syntax for a TI83.

5. Use the idea of "signed area" to evaluate $\int_0^3 |2x-1|\,dx$ without using a calculator.

6. Set up a definite integral which could be used to find the area of the region bounded by the graph of $y=2x^2-3x+2$ (shown at right), the x-axis, and the vertical lines $x=0$ and $x=2$.

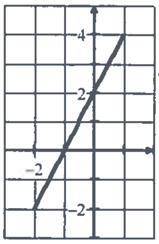


Discovering the Fundamental Theorem of Calculus

7. 
$f'(x)$

a. Find $\int_0^5 f'(x)\,dx$ b. Write an equation for $f'(x)$ on $[0,5]$.

c. Find $f(x)$ if $f(1)=4$. d. Find $f(5)-f(0)$

8. 
$f'(x)$

a. Find $\int_{-2}^{1} f'(x)\,dx$ b. Write an equation for $f'(x)$ on $[-2,1]$.

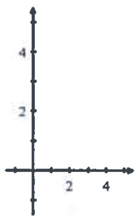
c. Find $f(x)$ if $f(1)=0$. d. Find $f(1)-f(-2)$

9. Given $x(t)=-\frac{1}{2}t^2+4t-3$ is the position equation for an object moving on the x-axis.

a. Find the displacement of the object on the interval $[1,4]$. b. Find the velocity equation.

$v(t)=$

c. Sketch a graph of $v(t)$.



d. Find $\int_1^4 v(t)\,dt$

Notice for each of these, the answers to parts a and d are the same.

If f' is a continuous function on $[a, b]$, then $\int_a^b f'(x)\,dx = f(x)\big\vert_a^b = f(b)-f(a)$ This relationship is known as the **<u>Fundamental Theorem of Calculus</u>**.

Evaluate using the Fundamental Theorem of Calculus without using a calculator.

10. $\int_0^4 \left(2\sqrt{y}+1\right) dy$ 11. $\int_0^1 (4t+1)^5\, dt$

12. $\int_1^5 \frac{x}{\sqrt{2x-1}}\, dx$ 13. $\int_0^{\frac{\pi}{2}} \cos(2x)\, dx$

START PLUS ACCUMULATION METHOD

Since $\int_a^b f'(x)\,dx = f(b)-f(a)$, it follows that $\boxed{f(b)=f(a)+\int_a^b f'(x)\,dx}$.

This means a function value at an endpoint can be found as a starting value plus a definite integral. Although this is only a slight variation of the Fundamental Theorem of Calculus, it gives us a different way to approach problems. This is an extremely common AP™ Calculus type problem.

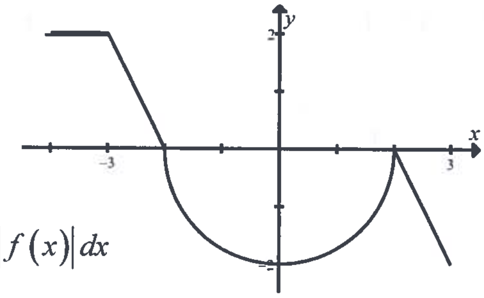
Examples:

14. If $f'(x)=3x^2+3$ and $f(0)=4$, find $f(2)$ without a calculator.

15. If an object's velocity is $v(t)=2^{3t^2-5}$ and $s(1)=8$ find $s(2)$.

Assignment 6.3

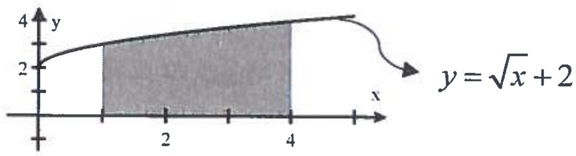
The graph of the function $y = f(x)$ consists of line segments and a semicircle as shown. Evaluate the following using geometry formulas.



1. $\int_{-4}^{-2} f(x)\,dx$ 2. $\int_{0}^{2} f(x)\,dx$

3. $\int_{-4}^{3} f(x)\,dx$ 4. $\int_{-2}^{2} \left(f(x)+2\right)dx$ 5. $\int_{-4}^{3} \left|f(x)\right|dx$

6. Show an integral setup and evaluate to find the area shaded in the figure shown without using a calculator.



For Problems 7 and 8 sketch a graph for each function, and use the idea of "signed areas" to evaluate these definite integrals using geometry formulas without using a calculator.

7. $f(x) = x-1$

a. $\int_{-2}^{2} f(x)\,dx$ b. $\int_{-2}^{2} |f(x)|\,dx$

8. $g(x) = 2x+3$

a. $\int_{-2}^{0} g(x)\,dx$ b. $\int_{-2}^{0} |g(x)|\,dx$

Evaluate the definite integrals in Problems 9-22 without using a calculator.

9. $\int_{0}^{2} (1-2y)\,dy$ 10. $\int_{0}^{1} (t^2-1)^4\,t\,dt$ 11. $\int_{0}^{1} x(4x-3)^2\,dx$ 12. $\int_{-2}^{-1} \left(\dfrac{1-3x^4}{x^2}\right)dx$

13. $\int_{1}^{4} \dfrac{2\sqrt{x}-1}{\sqrt{x}}\,dx$ 14. $\int_{1}^{8} \left(u^{\frac{2}{3}} + u^{\frac{-1}{3}}\right)du$ 15. $\int_{1}^{2} \dfrac{dx}{2\sqrt{3x-2}}$ 16. $\int_{0}^{\pi} \sin x\,dx$

17. $\int_{0}^{\pi} \cos x\,dx$ 18. $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sec^2\theta\,d\theta$ 19. $\int_{0}^{4} xe^{3x^2}\,dx$ 20. $\int_{1}^{3} 3^{6x-1}\,dx$

21. $\int_{0}^{\frac{\pi}{2}} \sin\left(\frac{2x}{3}\right)dx$ 22. $\int_{0}^{\pi} \left(3\sin x + \sin(2x)\right)dx$ ** 23. $\int_{0}^{4} \left|x^2-4\right|dx$

*24. $\int_{0}^{1} \dfrac{2x}{(x+1)^6}\,dx$ * 25. $\int_{1}^{2} x\sqrt[3]{x-1}\,dx$

*(Hint: Problems 24 and 25 require u-substitution.)

**(Hint: For Problem 23 sketch a graph and split the integral into two integrals without absolute value.)

Use your calculator to evaluate the definite integrals in Problems 26-28. Express answers to 3 or more decimal place accuracy.

26. $\int_{1}^{12} \dfrac{1}{x}\,dx$ 27. $\int_{0}^{6} \sqrt{y^3+1}\,dy$ 28. $\int_{0}^{\frac{\pi}{3}} \sin^2\theta\,d\theta$

29. Given $f'(x) = \left|8\sin(10x)\right|$,

a. use a calculator to find $\int_{0}^{4} f'(x)\,dx$. b. if $f(0)=3$, find $f(4)$.

30. If an object's acceleration is $a(t)=\cos t^2$ and $v(1)=5\frac{\text{ft}}{\text{sec}}$, find $v(3)$.

31. $x(t)=t^3-3t^2+1$ represents the position equation for a particle moving along the x-axis.

a. Find the velocity equation.
b. Find the acceleration equation.
c. Find the velocity at $t=1$.
d. Find the speed at $t=1$.
e. When is the particle's velocity decreasing?
f. Find the displacement on $[1, 4]$.
g. Find the total distance traveled from $t=1$ to $t=4$. (Show a velocity number line).
h. Find $\int_1^4 v(t)\,dt$ without using a calculator. Compare your answer to Part f.
i. Use your calculator to find $\int_1^4 |v(t)|\,dt$. Compare your answer to Part g.

> *You now have two ways to find <u>displacement</u> and <u>total distance</u>. Using definite integrals, **displacement** $=\int_a^b v(t)\,dt$ and **total distance** $=\int_a^b |v(t)|\,dt$ on the interval $[a, b]$. Given a choice of methods, always do total distance by evaluating a definite integral on your calculator.

32. Find the area between $f(x)=\ln(2x+5)$ and the x-axis on the interval $[1, 3]$. Show an integral set up, and evaluate using a calculator.

33. If $\int_0^4 f(x)\,dx=3$ and $\int_0^4 g(x)\,dx=-2$ find $\int_0^4 (4f(x)-3g(x))\,dx$.

34. If $\int_0^4 f(x)\,dx=3$ and $\int_0^4 g(x)\,dx=-2$ find $\int_4^0 f(x)\,dx-\int_0^4 g(x)\,dx$.

Find the limits in Problems 35-37 without using a calculator.

35. $\lim\limits_{x\to\frac{\pi}{2}}\left(\dfrac{\cos x}{x-\frac{\pi}{2}}\right)$ 36. $\lim\limits_{x\to 0}\dfrac{\sin(2x)}{\sin(-5x)}$ 37. $\lim\limits_{\theta\to\frac{\pi}{2}}\dfrac{-2\cos\theta}{e^{\theta-\frac{\pi}{2}}-1}$

Selected Answers:

1. 3 2. $-\pi$ 4. $8-2\pi$ 5. $4+2\pi$ 6. $\boxed{\left(\frac{16}{3}+8\right)-\left(\frac{2}{3}+2\right)}=\frac{32}{3}$

7. a. -4 b. 5 8. a. 2 9. -2 10. $\frac{1}{10}$ 11. $\frac{1}{2}$ 12. $\boxed{(1+1)-\left(\frac{1}{2}+8\right)}=-\frac{13}{2}$

14. $\boxed{\left(\frac{3}{5}\bullet 32+\frac{3}{2}\bullet 4\right)-\left(\frac{3}{5}+\frac{3}{2}\right)}=\frac{231}{10}$ 15. $\frac{1}{3}$ 16. 2 17. 0 18. $2\sqrt{3}$

19. $\frac{1}{6}e^{48}-\frac{1}{6}$ 20. $\dfrac{3^{17}-3^5}{6\ln 3}$ 21. $\frac{3}{4}$ 22. 6

23. $\boxed{\left(\frac{-8}{3}+8\right)+\left(\left(\frac{64}{3}-16\right)-\left(\frac{8}{3}-8\right)\right)}=16$ 24. $\boxed{\left(-\frac{1}{2}\bullet 2^{-4}+\frac{2}{5}\bullet 2^{-5}\right)-\left(-\frac{1}{2}+\frac{2}{5}\right)}=\frac{13}{160}$

25. $\boxed{\frac{3}{7}+\frac{3}{4}}=\frac{33}{28}$ 26. 2.484 or 2.485 29. a. 20.533 or 20.534 b. 23.533 or 23.534

31. c. -3 f. 18 g. 22 33. 18 34. -1 35. -1 37. 2

Lesson 6.4 The Second Fundamental Theorem of Calculus Integration Involving the Natural Log Function

The following examples serve as an informal guide toward discovering the Second Fundamental Theorem of Calculus.

1. $\int_{10}^{x} f'(t)\,dt =$

2. $\frac{d}{dx}\int_{10}^{x} f'(t)\,dt =$

Second Fundamental Theorem of Calculus:

For any constant a, $\frac{d}{dx}\int_{a}^{x} f(t)\,dt = f(x)$ (if f is continuous from a to x).

Now find:

3. $\int_{0}^{x^2} f'(t)\,dt =$

4. $\frac{d}{dx}\int_{0}^{x^2} f'(t)\,dt =$

5. $\int_{x^3}^{2x} f'(t)\,dt =$

6. $\frac{d}{dx}\int_{x^3}^{2x} f'(t)\,dt =$

Second Fundamental Theorem (Chain Rule Version):

If u and v are functions of x, then $\frac{d}{dx}\int_{u}^{v} f(t)\,dt = f(v)\,v' - f(u)\,u'$

(if f is continuous from u to v). Note the "chain rule factors" v' and u'.

Examples: Find each of the following without integrating.

7. $\frac{d}{dx}\int_{x}^{0} (2t-3)\,dt =$

8. $\frac{d}{dx}\int_{2}^{5} (2t-3)\,dt =$

9. $\frac{d}{dx}\int_{-1}^{x^3} (t^2+2t)\,dt =$

10. If $f(x) = \int_{0}^{3x^2} (1-t^2)^{10}\,dt$, then $f'(x) =$

Differentiation and integration are inverse operations.

So if $\frac{d}{dx}\ln|x| = \frac{1}{x}$, then $\int \frac{1}{x}\,dx = \ln|x| + C$, and if $\frac{d}{dx}\ln|u| = \frac{u'}{u}$, then $\int \frac{u'}{u}\,dx = \ln|u| + C$.

Log Rules: $\int \frac{1}{x}\,dx = \ln|x| + C$ and $\int \frac{u'}{u}\,dx = \ln|u| + C$

Note: Although it is true that both $\frac{d}{dx}\ln x = \frac{1}{x}$ and $\frac{d}{dx}\ln|x| = \frac{1}{x}$, $\int \frac{1}{x}\,dx = \ln|x|$ only. **Why?**

Examples: Integrate

11. $\int \frac{-3}{x}\,dx$

12. $\int \frac{P}{P^2+1}\,dP$

13. $\int \frac{\sec^2 x}{\tan x}\,dx$

Example 14: Integrate $\int \frac{\ln x}{x}\,dx$

Example 15: Integrate $\int \frac{1}{x(2-\ln x)^3}\,dx$

Examples: Rewrite as a fraction using a trig identity.

16. Integrate $\int \cot x\,dx$

17. Integrate $\int \tan x\,dx$

So,

$$\int \tan x\,dx = -\ln|\cos x| + C \qquad \int \cot x\,dx = \ln|\sin x| + C$$

$$\int \tan u\,u'dx = -\ln|\cos u| + C \qquad \int \cot u\,u'\,dx = \ln|\sin u| + C$$

Example 18: Integrate $\int \tan(2x)\,dx$

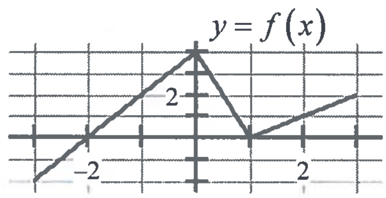
Assignment 6.4

Use the Second Fundamental Theorem of Calculus to evaluate in Problems 1-6.

1. $\frac{d}{dx}\int_{-3}^{x}(t^2-t+1)\,dt$ 2. $\frac{d}{dx}\int_{2}^{x}\sqrt[3]{t^2-5t}\,dt$ 3. $\frac{d}{dt}\int_{6}^{t}\frac{2}{x-5}\,dx,\ t>6$ 4. $\frac{d}{dx}\int_{x}^{3}(1-t)^4\,dt$

5. $\frac{d}{dx}\int_{x^2}^{3}(1-t)^4\,dt$ 6. $\frac{d}{dt}\int_{3t}^{4t}2^x\,dx$ 7. $\int_{1}^{a}t(3t^2-1)^5\,dt$

8. If $g(x)=\int_0^{2x}f(t)\,dt$ find each of the following:



a. $g'(x)$ b. $g'(0)$

c. x-value(s) where g has a relative minimum. Justify.

d. x-value(s) where g has a point of inflection. Justify.

e. $g''\left(\frac{1}{4}\right)$

Evaluate each integral in Problems 9-22 without using a calculator.

9. $\int_1^{e}\frac{2}{x}\,dx$ 10. $\int_1^{3}\frac{4}{2t-1}\,dt$ 11. $\int e^{x^2-1}x\,dx$

12. $\int\frac{x}{x^2-1}\,dx$ 13. $\int\frac{x^2-1}{x}\,dx$ 14. $\int\frac{4y-6}{y^2-3y+2}\,dy$

15. $\int\frac{-1}{(x+1)^3}\,dx$ 16. $\int\frac{3u}{\sqrt[3]{u^2+1}}\,du$ 17. $\int\frac{\sqrt{\ln x}}{x}\,dx$

18. $\int\frac{2}{x(1+\ln x)^5}\,dx$ 19. $\int\frac{\cos y}{\sin y-2}\,dy$ 20. $\int\frac{e^{2y}}{e^{2y}-2}\,dy$

21. $\int\cot(5x)\,dx$ 22. $\int y\tan(y^2)\,dy$

23. Use the substitution $u=f(x)$ to rewrite $\int_1^3 f'(x)\sin(f(x))\,dx$ as an integral with respect to u if $f(1)=4$ and $f(3)=9$. Do not integrate.

24. If $g'(x)=\ln\left|\cos x^2\right|$ and $g(4.23)=5.192$ find $g(2.159)$.

Differentiate in Problems 25-29 without using a calculator.

25. $y=\frac{t}{\ln t}$ 26. $f(y)=\arctan(3y)$ 27. $y=3^{2x+1}$

28. $\frac{d}{dx}\ln\left|\sin^3 x\right|=$ 29. $y=4\arcsin(x^2)$

30. Use the alternate form of the limit definition of the derivative to find $f'(3)$, if $f(t)=2t^2-3$.

31. Find $a, b,$ and c for $f(x)=ax^2+bx+c$, such that $f(1)=10$, and $f(x)$ has a relative minimum at $(-1,2)$.

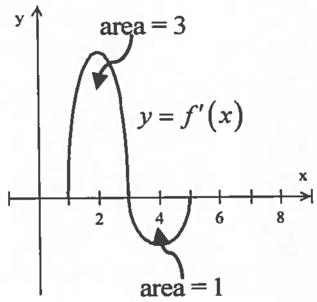
32. If $f(x) = (x-1)^{\frac{2}{3}}$, the Mean Value Theorem does not apply to which interval?
a. $[0, 2]$ b. $[1, 9]$ Why?

33. For Problem 32, use the interval on which the MVT does apply to find the c-value(s) where $f'(c) = \dfrac{f(b) - f(a)}{b-a}$

Evaluate the following inverse trig functions. The range values for inverse trig functions are shown on pages 70 and 71.

34. $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$ 35. $\arcsin(1)$ 36. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ 37. $\arctan 1$

The graph shown is a graph of $y = f'(x)$. The two enclosed regions have areas of 3 and 1 as shown. Use this figure for problems 38-40.



38. If $f(1) = 4$, find $f(5)$.
39. If $f(3) = 4$, find $f(1)$.
40. If $f(5) = 4$, find $f(1)$.

Selected Answers:

1. $x^2 - x + 1$ 4. $-(1-x)^4$ 5. $-2x(1-x^2)^4$ 7. $\frac{1}{36}(3a^2-1)^6 - \frac{1}{36}\cdot 2^6$

8a. $g'(x) = 2f(2x)$ c. g has a rel. min. at $x = -1$ because $2f(2x)$ changes from neg. to pos.

8e. -16 9. 2 10. $2\ln 5$ 12. $\frac{1}{2}\ln|x^2-1| + C$ 13. $\frac{1}{2}x^2 - \ln|x| + C$

14. $2\ln|y^2 - 3y + 2| + C$ 16. $\frac{9}{4}(u^2+1)^{\frac{2}{3}} + C$ 18. $-\dfrac{1}{2(1+\ln x)^4} + C$

19. $\ln|\sin y - 2| + C$ 21. $\frac{1}{5}\ln|\sin(5x)| + C$ 24. 6.732 25. $y' = \dfrac{\ln t - 1}{(\ln t)^2}$

26. $f'(y) = \dfrac{3}{1+9y^2}$ 28. $3\cot x$ 30. 12 31. $a=2,\ b=4,\ c=4$

33. $c = \frac{91}{27} = 3.370$ 34. $-\frac{\pi}{6}$ 35. $\frac{\pi}{2}$ 36. $-\frac{\pi}{4}$ 37. $\frac{\pi}{4}$ 38. 6 39. 1 40. 2

Lesson 6.5 Integration Involving Inverse Trig Functions, Advanced Integration Techniques

When integrating a fraction where the degree of the numerator ≥ the degree of the denominator, you will have to use long division (or creative thinking) to "split the fraction."

Example 1: $\int \frac{x^2-4x+2}{x^2+2}\,dx$ Example 2: $\int \frac{1}{\sqrt{x}-1}\,dx$

Since $\frac{d}{dx}\arcsin u = \frac{u'}{\sqrt{1-u^2}}$ and $\frac{d}{dx}\arctan u = \frac{u'}{1+u^2}$, it follows that

$\int \frac{u'}{\sqrt{1-u^2}}\,dx = \arcsin u + C$ and $\int \frac{u'}{1+u^2}\,dx = \arctan u + C$ (where u is a function of x).

Extending these integration rules gives us these more general **integration rules**.

1. $\int \frac{u'}{\sqrt{a^2-u^2}}\,dx = \arcsin\frac{u}{a} + C$
2. $\int \frac{u'}{a^2+u^2}\,dx = \frac{1}{a}\arctan\frac{u}{a} + C$

Note: Since $\frac{d}{dx}\arcsin x$ and $\frac{d}{dx}\arccos x$ differ only in sign, it is not necessary to have a third integration rule which integrates into $\arccos x$.

Warm-up Example: Differentiate $y = \arcsin\frac{x}{2}$.

Examples: Integrate.

3. $\int \frac{1}{\sqrt{4-x^2}}\,dx$ 4. $\int \frac{dx}{4x^2+25}$ 5. $\int_0^{\frac{1}{2}} \frac{8}{3+4x^2}dx$

6. $\int \frac{8x}{3+4x^2}\,dx$ 7. $\int \frac{8x^2}{3+4x^2}\,dx$ 8. $\int \frac{x+4}{\sqrt{4-x^2}}\,dx$

Example 9. Complete the square to find $\int \frac{1}{x^2+4x+8}\,dx$.

Example 10: $\int \frac{3-x}{\sqrt{1-x^2}}\,dx$ Example 11: $\int \frac{1}{e^x+1}\,dx$

Assignment 6.5 Do not use a calculator on this assignment.

Evaluate the integrals in Problems 1-6.

1. $\int_0^{\frac{1}{4}} \frac{1}{\sqrt{1-4x^2}}\,dx$ 2. $\int_0^{\frac{5}{3}} \frac{2}{9x^2+25}\,dx$ 3. $\int_0^1 \frac{x^2}{x+1}\,dx$

4. $\int \frac{8}{2+(2t+1)^2}\,dt$ 5. $\int \frac{w^2}{\sqrt{4-w^6}}\,dw$ 6. $\int \frac{dx}{x\sqrt{16-(\ln x)^2}}$

Complete the square to evaluate Problems 7 and 8.

7. $\int \frac{1}{t^2 - 10t + 32}\, dt$

8. $\int_{-3}^{-1} \frac{1}{\sqrt{7 - x^2 - 6x}}\, dx$

Evaluate the integrals in Problems 9-16.

9. $\int \frac{\sqrt{\arctan \theta}}{1 + \theta^2}\, d\theta$

10. $\int \frac{5x}{x^2 + 1}\, dx$

11. $\int \frac{5x^2}{x^2 + 1}\, dx$

12. $\int \frac{2x^2 - 4}{x + 1}\, dx$

13. $\int \frac{2}{1 - e^{2y}}\, dy$
Hint: See Example 11

14. $\int \frac{3 - 4t}{t^2 + 9}\, dt$
Hint: Split

15. $\int \frac{e^{-2v}}{3 + e^{-4v}}\, dv$

16. $\int \frac{2x}{\sqrt[3]{3x + 1}}\, dx$ **Hint**: Think about the method of last resort.

Simplify the expressions in Problems 17-19.

17. $\frac{d}{dt} \int_1^t (-2x + \ln x)\, dx$

18. $\frac{d}{dx} \int_{x^2}^5 \sin(2t)\, dt$

19. $\int_0^1 x e^{x^2}\, dx$

Differentiate.

20. $g(y) = \ln\left|(1 - \ln y)\right|$

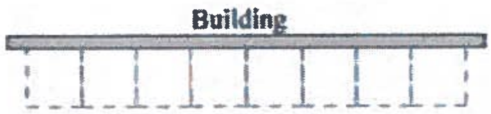
21. $x^2 + 2\ln y = y$ (solve for $\frac{dy}{dx}$)

22. Sketch a possible graph for $f(x)$, given the following characteristics:

$f(0) = 1$, $f(1)$ does not exist, $f(2) = 2$
$f'(x) < 0$ for $x < 1$ and $1 < x < 2$, $f'(x) > 0$ for $x > 2$
$f''(x) < 0$ for $x < 1$, $f''(x) > 0$ for $x > 1$

23. A small dog kennel with 8 individual rectangular holding pens of equal size is to be constructed using 144 *ft* of chain link fencing material. One side of the kennel is to be placed against a building and requires no fencing, as shown in the figure below.

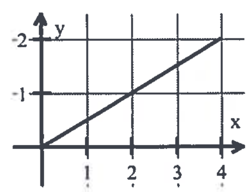
a. Find the dimensions (for each holding pen) that produce a maximum area for each pen.
b. What is that maximum area for <u>each</u> holding pen?



24. The velocity of an object moving along a horizontal path is given by the equation $v(t) = 4\sin t + 3t^2$.

a. Find an equation for the object's acceleration.
b. Find an equation for the object's position if the initial position is 3.
c. Find the object's position at $t = \pi$.

25. Find $f(4)$ if $f(2) = 3$ and the graph of $y = f'(x)$ is shown.



Without a calculator, sketch graphs and use geometry to evaluate.

26. $\int_0^4 |3x - 2|\, dx$

27. $\int_{-2}^{1} (2 - |x|)\, dx$

Selected Answers:

1. $\frac{\pi}{12}$ 2. $\frac{\pi}{30}$ 3. $-\frac{1}{2}+\ln 2$ 4. $\frac{4}{\sqrt{2}}\arctan\frac{2t+1}{\sqrt{2}}+C$ 6. $\sin^{-1}\left(\frac{\ln x}{4}\right)+\text{C}$

7. $\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{t-5}{\sqrt{7}}\right)+\text{C}$ 8. $\frac{\pi}{6}$ 9. $\frac{2}{3}(\arctan\theta)^{\frac{3}{2}}+C$ 11. $5x-5\arctan x+C$

12. $x^2-2x-2\ln|x+1|+C$ 13. $-\ln\left|e^{-2y}-1\right|+C$ or $2y-\ln\left|1-e^{2y}\right|+C$

14. $\arctan\frac{t}{3}-2\ln\left(t^2+9\right)+C$ 15. $-\frac{1}{2\sqrt{3}}\arctan\frac{e^{-2v}}{\sqrt{3}}+C$

16. $\frac{2}{15}\left(\sqrt[3]{3x+1}\right)^5-\frac{1}{3}\left(\sqrt[3]{3x+1}\right)^2+C$ 18. $-2x\sin\left(2x^2\right)$ 19. $\frac{1}{2}e-\frac{1}{2}=.859$

20. $g'(y)=\dfrac{-\frac{1}{y}}{1-\ln y}=-\dfrac{1}{y(1-\ln y)}$ 21. $y'=\dfrac{2xy}{y-2}$ 23a. 8 *ft.* by 9 *ft.* b. 72 ft^2

24a. $a(t)=4\cos t+6t$ b. $x(t)=-4\cos t+t^3+7$ 26. $\frac{52}{3}$ 27. 3.5

Lesson 6.6 Approximation Using Riemann Sums and Trapezoids

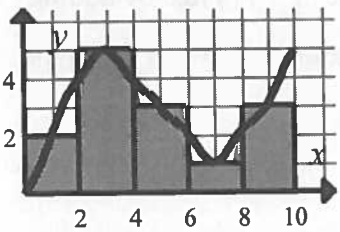
Some functions cannot be integrated, and sometimes you are given data or a graph – but not an actual function. It is still possible to <u>approximate</u> "areas." One method of approximating a definite integral is to add areas of rectangles. This is called a **<u>Riemann Sum</u>**.

<u>Example 1</u>:

Approximate $\int_0^{10} f(x)\,dx$ by adding the areas of the five rectangles shown.

This is a <u>Midpoint Riemann Sum.</u>

$y=f(x)$



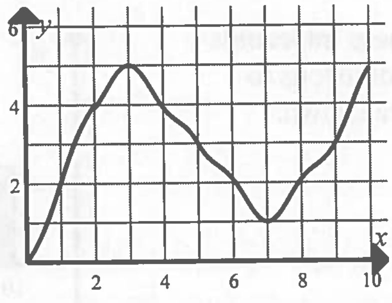
x	$f(x)$
0	0
1	2
2	4
3	5
4	4
5	3
6	2
7	1
8	2
9	3
10	5

<u>Example 2:</u>

Approximate $\int_0^{10} f(x)\,dx$ by using 5 rectangles of equal width $(n=5)$ and a <u>Left Riemann Sum</u>.

Draw rectangles on the figure.

$y=f(x)$



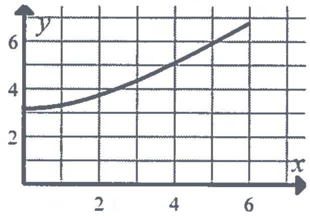
Example 3:

Approximate $\int_0^{10} f(x)\,dx$ by using a Right Riemann Sum with four subdivisions using the data in the table.

x	0	2	5	9	10
$f(x)$	3	8	2	−1	0

Example 4:

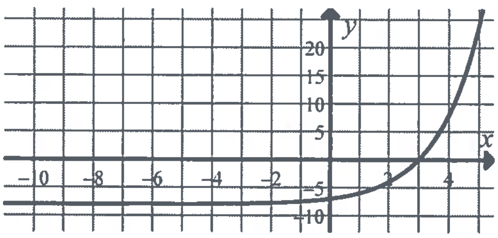
Approximate $\int_0^{6} \sqrt{x^2+10}\,dx$ using a Midpoint Riemann Sum with 3 equal subdivisions.



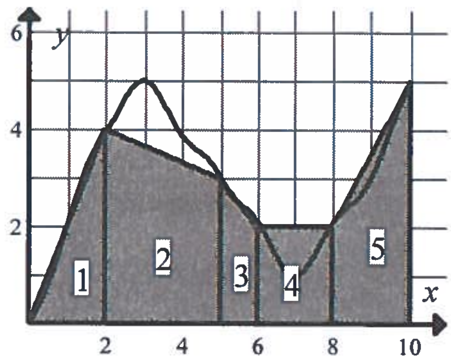
Example 5: Approximate $\int_{-10}^{5} (2^x - 8)\,dx$

by using five Right hand rectangles whose widths are determined by the intervals separating the following x values:

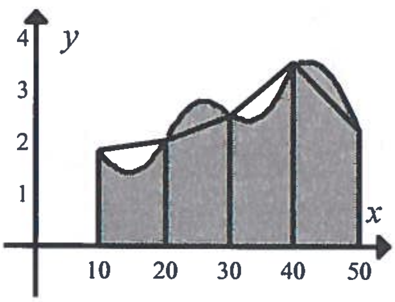
$x=-10$, $x=-4$, $x=0$, $x=2$, $x=3$, and $x=5$.



Example 6: Approximate $\int_0^{10} f(x)\,dx$ by adding the areas of the 5 "trapezoids" shown in the graph at the right.



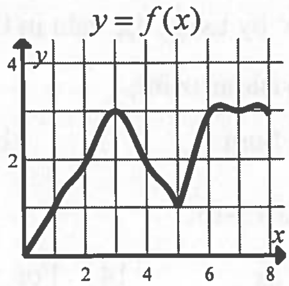
Example 7: Use these trapezoids with four equal subdivisions to approximate the area shown.



x	y
10	1.8
20	2
30	2.5
40	3.5
50	2.2

Assignment 6.6 Show set ups on all Riemann Sums and Trapezoidal approximations.

1. Use the graph of $y = f(x)$ at right to approximate $\int_0^8 f(x)\,dx$ using
   a. a Midpoint Riemann Sum with 4 equal subdivisions.
   b. a Left hand Riemann Sum with 8 equal subdivisions.
   c. a Trapezoidal approximation with 4 equal subdivisions.



x	$f(x)$
0	0
1	1
2	2
3	3
4	2
5	1
6	3
7	3
8	3

2. Use the data in the table below to approximate the area between the graph of $f(t)$ and the t-axis, from $t = 1$ to $t = 13$, using a Midpoint Riemann Sum with 6 rectangles of equal width. Plot the data, sketch a graph, and draw rectangles first.

t	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$f(t)$	0	5	8	11	12	15	17	18	15	13	12	9	6	4

3. Use the data in the table shown to approximate $\int_2^4 f(x)\,dx$ with a Midpoint Riemann Sum with 2 equal subintervals.

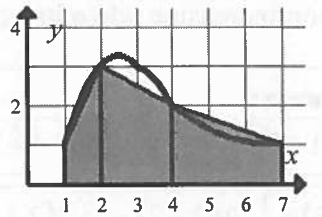
x	2.0	2.5	3.0	3.5	4.0
$f(x)$	3	2	4	3	5

4. Approximate $\int_0^{16}\left(\sqrt{x}+1\right)dx$ using a Right Riemann Sum with 4 rectangles of equal width. Draw an accurate sketch (without using a calculator if possible), and show your rectangles.

5. Is you answer from Problem 4 an underestimate or an overestimate of the actual value of the integral? What characteristic of the square root function makes your answer true?

6. Would a Left Riemann Sum approximation of an integral of an increasing function be an underestimate or an overestimate?

7. Use the trapezoids shown to approximate $\int_1^7 f(x)\,dx$.



8. Approximate $\int_{-3}^{5}\left(\sqrt[3]{x}-\frac{1}{2}\right)dx$ by using 4 regions whose widths are determined by the intervals separating $x = -3$, $x = 0$, $x = 1$, $x = 2$, and $x = 5$ using the following methods.
   a. a Riemann Sum with Left hand rectangles
   b. trapezoids

9. Use trapezoids with four equal subdivisions to approximate $\int_{-2}^{2}\sqrt{x^4+1}\,dx$.

10. Look at a graph of the square root function from Problem 9 with a calculator. Is the trapezoidal approximation an underestimate or an overestimate? What characteristic of the graph determines this?

11. Approximate $\int_2^8 \frac{1}{x}\,dx$ using a Right Riemann Sum with 3 equal subdivisions.

12. Approximate $\int_0^{10} f(x)\,dx$ by using the data in the table with two unequal subdivisions using

x	0	2	4	7	10
$f(x)$	7	-2	0	4	10

a. a Midpoint Riemann Sum. b. trapezoids of width 4 and 6.

Use a calculator for Problems 13-15.

13. Evaluate $\int_{-4}^{5} (3x^3-4)^{10}\,dx$

14. For $f(x)=\sin^2(3x^2)$, find $f'(1.63)$

15. Find the area between $f(x)=\ln(\sin(x)+1)$ and the x-axis on the interval $[0,\pi]$. First show an integral set up.

Evaluate the expressions in Problems 16-19 without using a calculator.

16. $\frac{d}{dx}\int_0^x \sqrt{t}\,dt$

17. $\frac{d}{dx}\int_{2x}^0 \sqrt{t}\,dt$

18. $\frac{d}{dx}\int_0^{x^2} \sqrt{t}\,dt$

19. $\int_0^x \sqrt{t}\,dt$

Evaluate the following without using a calculator.

20. $\int_{-1}^{0} \frac{1}{\sqrt{3-x^2-2x}}\,dx$ Hint: Comp. Sq.

21. $\int \frac{4x}{\sqrt{1-x^4}}\,dx$

22. $\int \frac{x+5}{x^2+16}\,dx$

23. $\int \frac{\sqrt{x-1}}{x}\,dx$ Hint: Let $u = \sqrt{x-1}$

24. $\int \frac{e^t \cos e^t}{\sin e^t}\,dt$

25. $\int \frac{y}{y+2}\,dy$

26. $\int_{-2}^{2} |x^3|\,dx$ Hint: Draw a graph..

27. $\int \frac{e^{5x}-e^x+2}{e^{2x}}\,dx$

28. $\int (x^2-2x)^5(x-1)\,dx$

29. A spherical balloon is expanding at the rate of 5 cm^3/sec. How fast is the <u>diameter</u> of the balloon increasing when its volume is 36π cm^3? $\left(V=\frac{4}{3}\pi r^3\right)$

Selected Answers:

1a. $\boxed{2(1+3+1+3)}=16$ b. $\boxed{1(0+1+2+3+2+1+3+3)}=15$

1c. $\boxed{\frac{1}{2}\cdot 2(0+2)+\frac{1}{2}\cdot 2(2+2)+\frac{1}{2}\cdot 2(2+3)+\frac{1}{2}\cdot 2(3+3)}=17$

2. $\boxed{2(8+12+17+15+12+6)}=140$ 4. $4\bullet\sqrt{5}+4\bullet 3+4\bullet\sqrt{13}+4\bullet\sqrt{17}$

7. $\boxed{\frac{1}{2}\cdot 1(1+3)+\frac{1}{2}\cdot 2(3+2)+\frac{1}{2}\cdot 3(2+1)}=\frac{23}{2}$

8a. $\boxed{3\left(\sqrt[3]{-3}-\frac{1}{2}\right)+1\left(\sqrt[3]{0}-\frac{1}{2}\right)+1\left(\sqrt[3]{1}-\frac{1}{2}\right)+3\left(\sqrt[3]{2}-\frac{1}{2}\right)}$

9. $\frac{1}{2}\bullet 1\left(\sqrt{17}+\sqrt{2}\right)+\frac{1}{2}\bullet 1\left(\sqrt{2}+1\right)+\frac{1}{2}\bullet 1\left(1+\sqrt{2}\right)+\frac{1}{2}\bullet 1\left(\sqrt{2}+\sqrt{17}\right)$

More Selected Answers:

12a. $4(-2)+6\bullet 4=16$ 15. 1.486 16. $\sqrt{x}$ 17. $-2\sqrt{2x}$ 19. $\frac{2}{3}x^{\frac{3}{2}}$

20. $\frac{\pi}{6}$ 21. $2\sin^{-1}x^2+C$ 22. $\frac{1}{2}\ln\left(x^2+16\right)+\frac{5}{4}\arctan\frac{x}{4}+C$

23. $2\sqrt{x-1}-2\arctan\sqrt{x-1}+C$ 25. $y-2\ln|y+2|+C$ 26. 8 27. $\frac{1}{3}e^{3x}+e^{-x}-e^{-2x}+C$

28. $\frac{1}{12}\left(x^2-2x\right)^6+C$ 29. $\frac{5}{18\pi}\frac{cm}{sec}$

Lesson 6.7 Summation Notation, Integration by Parts

As the number of subdivisions increases, the accuracy of a Riemann Sum approximation improves. To achieve perfect accuracy we need to approach infinitely many subdivisions. This is the **limit definition of a definite integral.**

$$\int_a^b f(x)\,dx=\lim_{n\to\infty}\sum_{k=1}^{n}\left(f(a+k\Delta x)\,\Delta x\right)=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\underbrace{f\left(a+k\frac{b-a}{n}\right)}_{\text{right-hand heights}}\overbrace{\frac{b-a}{n}}^{\text{width}}\right)$$

where n is the number of subdivisions.

Example 1: Write $\int_1^4 x^5\,dx$ as an infinite Right Riemann Sum.

Example 2: Write $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sin\left(\frac{\pi}{2}+k\bullet\frac{\pi}{n}\right)\frac{\pi}{n}\right)$ as a definite integral.

Note: We will focus on Right Riemann Sums for these. A Left Sum would be the same except the summation would be from $k=0$ to n-1.

Integration by parts is a method of integration used mainly for products of algebraic and transcendental functions (such as $\int xe^x\,dx$) or products of two transcendental functions (such as $\int e^x\sin x\,dx$).

Development of the formula for integration by parts: If u and v are both functions of x, then

$\frac{d}{dx}(uv)=$

Formula for integration by parts:
$\int uv'\,dx = uv - \int vu'\,dx$ or $\int u\,dv = uv - \int v\,du$

Strategy: Let u be the part whose derivative is "simpler" (or at least no more complicated) than u itself. Let dv be the more complicated part (or the part which can easily be integrated). Also, remember that you typically have only two choices. If one choice doesn't work, try the other.

Example 3: $\int xe^x\,dx$

Let $u =$

$du =$

Let $dv =$

$v =$

$\int xe^x\,dx =$

Example 4: $\int x\sin(3x)\,dx$

$\int x\sin(3x)\,dx =$

Example 5: $\int \arcsin x\,dx$

$\int \arcsin x\,dx =$

Example 6: $\int x^2\sin(2x)\,dx$

Example7: $\int_1^e x^2 \ln x\,dx$

Assignment 6.7

Write each Riemann Sum as a definite integral and each definite integral as a right Riemann Sum. Do <u>not</u> evaluate.

1. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left(f\left(2+k\bullet\frac{3}{n}\right)\frac{3}{n}\right)$ 2. $\int_1^5 f(x)\,dx$ 3. $\int_2^4 \sin x\,dx$ 4. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left(\cos\left(0+\frac{k\pi}{n}\right)\frac{\pi}{n}\right)$

5. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left(\sqrt{3+\frac{2k}{n}}\bullet\frac{2}{n}\right)$ 6. $\int_4^5 (x+1)^2\,dx$ 7. $\int_0^5 \left(x^2+1\right)dx$

Integrate without using a calculator. Integration by parts will be used on most, but **<u>not all</u>** problems.

8. $\int x\sin x\,dx$ 9. $\int x\cos(2x)\,dx$ 10. $\int 4xe^{2x}\,dx$ 11. $\int x^2e^{x^3}\,dx$

12. $\int \frac{x}{e^x}\,dx$ 13. $\int \ln x\,dx$ 14. $\int \frac{\ln x}{x^2}\,dx$ 15. $\int \frac{(\ln x)^2}{x}\,dx$

16. $\int \arctan x\,dx$ 17. $\int x^2\cos x\,dx$ 18. $\int x^2e^{2x}\,dx$ 19. $\int_0^1 \frac{x}{1+x^2}\,dx$

20. $\int_{-3}^{-1}\frac{1}{\sqrt{7-x^2-6x}}\,dx$ 21. $\int_0^1 xe^{3x}\,dx$ 22. $\int_1^{e^2} x\ln x\,dx$ 23. $\int x\arctan x\,dx$ *

$*\int \frac{x^2}{1+x^2}\,dx$ can be integ. using long division

Evaluate in Problems 24-26.

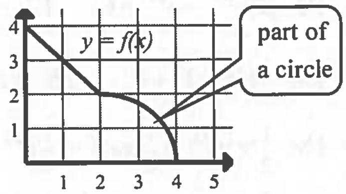
24. $\frac{d}{dx}\int_0^{2x}\sin^4 t\,dt$ 25. $\frac{d}{dx}\int_{x^2}^{5}\sin\left(t^2\right)dt$ 26. $\frac{d}{dt}\int_t^{2t} 2^{x^2}\,dx$

27. Use a calculator to find $\int_{\frac{\pi}{2}}^{\pi}\sin^3(3x-1)\,dx$. (You must be in radian mode.)

28. If $f'(x)=\sin x^3$ and $f(1.2)=6.25$, find $f(3.6)$.

29. Use geometry to find $\int_0^4 f(x)\,dx$ for the function shown at right.

$$f(x)=\begin{cases}4-x, & 0\le x\le 2\\ \sqrt{4x-x^2}, & 2<x\le 4\end{cases}$$



30. Use the data in the table shown to approximate $\int_2^{14} f(x)\,dx$ with four subdivisions using:
a. a right Riemann Sum. b. a left Riemann Sum. c. trapezoids.

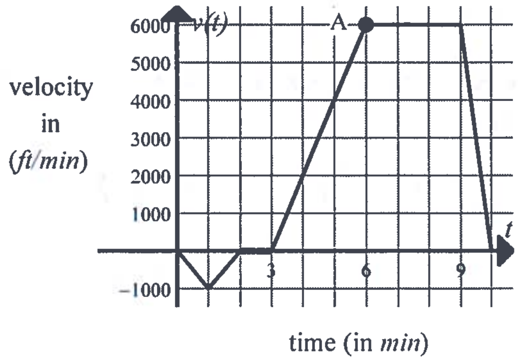
x	2	7	9	10	14
$f(x)$	0	3	8	2	−2

For Problems 31-32, write an equation for a line tangent to each curve at the given point. Do not use a calculator.

31. $y=2\sin(-x)$ at the point where $x=\frac{\pi}{4}$ 32. $f(x)=4e^x$ at the point where $x=0$

33. Use a calculator to write an equation for the line tangent to the graph of $f(x)=\ln\left(|\cos x|+2\right)$ at the point where $x=.821$.

34. The "rate graph" at right represents the velocity of a car during a 10 minute factory test drive along a straight path.
   a. On what interval(s) of time was the car moving backward (reverse)? forward? at rest?
   b. Write a sentence telling what Point A represents.
   c. Find the <u>speed</u> of the car at $t=1$, $t=2$, and $t=4$ min.



   d. Find the <u>acceleration</u> of the car on the time interval (3, 6).
   e. On what time interval(s) is the car's acceleration the greatest?
   f. On what time interval(s) is the absolute value of the car's acceleration the greatest?
   g. On what time interval(s) was the car speeding up (increasing in speed)?
   h. Find $\int_0^{10} v(t)\,dt$ i. Find $\int_0^{10} |v(t)|\,dt$
   j. Write a sentence telling what $\int_0^{10} v(t)\,dt$ represents. Include numbers and units.
   k. Write a sentence telling what $\int_0^{10} |v(t)|\,dt$ represents. Include numbers and units.

Selected Answers:

1. $\int_2^5 f(x)\,dx$ (Other answers are possible on these if the function is shifted horizontally)

2. $\lim\limits_{n\to\infty}\sum_{k=1}^{n}\left(f\left(1+k\bullet\frac{4}{n}\right)\frac{4}{n}\right)$ 3. $\lim\limits_{n\to\infty}\sum_{k=1}^{n}\left(\sin\left(2+\frac{2k}{n}\right)\frac{2}{n}\right)$ 5. $\int_3^5 \sqrt{x}\,dx$

7. $\lim\limits_{n\to\infty}\sum_{k=1}^{n}\left(\left(\left(\frac{5k}{n}\right)^2+1\right)\frac{5}{n}\right)$ 8. $-x\cos x+\sin x+C$ 9. $\frac{1}{2}x\sin(2x)+\frac{1}{4}\cos(2x)+C$

10. $2xe^{2x}-e^{2x}+C$ 12. $-xe^{-x}-e^{-x}+C$ 13. $x\ln x-x+C$ 14. $-\frac{1}{x}\ln x-\frac{1}{x}+C$

15. $\frac{1}{3}(\ln x)^3+C$ 16. $x\arctan x-\frac{1}{2}\ln\left(1+x^2\right)+C$ 17. $x^2\sin x+2x\cos x-2\sin x+C$

18. $\frac{1}{2}x^2e^{2x}-\frac{1}{2}xe^{2x}+\frac{1}{4}e^{2x}+C$ 19. $\frac{1}{2}\ln 2$ 20. $\frac{\pi}{6}$ 21. $\frac{2}{9}e^3+\frac{1}{9}$ 22. $\frac{3}{4}e^4+\frac{1}{4}$

23. $\frac{1}{2}x^2\arctan x-\frac{1}{2}x+\frac{1}{2}\arctan x+C$ 24. $2\sin^4(2x)$ 28. 6.294 or 6.295   29. $6+\pi$

30a. $\boxed{5\bullet3+2\bullet8+1\bullet2+4(-2)}=25$ b. $\boxed{5\bullet0+2\bullet3+1\bullet8+4\bullet2}=22$

31. $y+\sqrt{2}=-\sqrt{2}\left(x-\frac{\pi}{4}\right)$ 32. $y=4x+4$

33. $y-.986=-.272(x-.821)$ or $y-.986=-.273(x-.821)$

34a. reverse $(0,2)$, forward $(3,10)$, at rest $[2,3]$

34b. Six minutes after the start of the test drive the car's velocity was 6000 feet per minute.

34d. 2000 *ft/min/min*   f. $(9,10)$ h. 29,000 *ft*   i. 31,000 *ft*

Lesson 6.8 Partial Fractions, Improper Integrals

We have often simplified an expression like $\frac{1}{x-4}-\frac{1}{x-3}$ by getting a common denominator and combining the two fractions into one. By a reverse process we can sometimes split a single fraction in two to make integration easier.

Example1:

$$\int \frac{1}{x^2-7x+12}\,dx$$

Example 2:

$$\int \frac{5x-3}{x^2-2x-3}\,dx$$

Example 3:

$$\int \frac{2x-2}{x^2-2x-3}\,dx$$

Example 4:

$$\int \frac{x^3-x+2}{x^2+x-2}\,dx$$

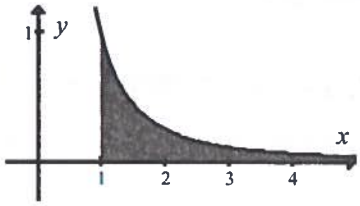
An integral is called **improper** if
1. one or both limits of integration are infinite
2. the function has an infinite discontinuity (a vertical asymptote) at or between the limits

Examples: Explain why each of the following is improper.

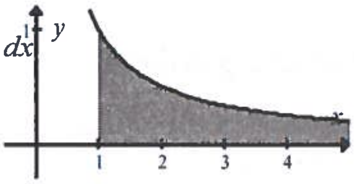
5. $\int_1^\infty \frac{1}{x}\,dx$ 6. $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$ 7. $\int_1^5 \frac{1}{\sqrt{x-1}}\,dx$ 8. $\int_{-2}^{2} \frac{1}{(x+1)^2}\,dx$

<u>Examples:</u> Evaluate the following improper integrals. Identify those which diverge.

9. $\int_1^{\infty} \frac{1}{x^2}\, dx$



10. $\int_1^{\infty} \frac{1}{x}\, dx$



11. $\int_0^{\infty} \cos x\, dx$

12. $\int_1^{\infty} xe^{-x}\, dx$

13. $\int_{-1}^{2} \frac{dx}{x^3}$

Assignment 6.8

Integrate without using a calculator.

1. $\int \frac{1}{x^2-1}\, dx$

2. $\int \frac{3}{x^2-x-2}\, dx$

3. $\int \frac{5x-2}{2x^2-x-1}\, dx$

4. $\int \frac{2x^2+2x-2}{x^3-x}\, dx$

5. $\int 3x \ln x\, dx$

6. $\int \frac{1}{t^2-10t+32}\, dt$

7. $\int \frac{2x+12}{x^2+4x}\, dx$

Which of the integrals in Problems 8-11 is/are improper. For any which are improper identify why the integral is improper. **Do not evaluate the integrals.**

8. $\int_0^{9} \frac{x+3}{\sqrt[3]{x}}\, dx$

9. $\int_0^{1} \frac{x^3}{3x-2}\, dx$

10. $\int_0^{4} \frac{1}{x^2-4x-5}\, dx$

11. $\int_0^{\infty} \frac{x}{e^x}\, dx$

Evaluate these improper integrals or show that the integral diverges without using a calculator. Show correct limit symbolism.

12. $\int_0^9 \frac{1}{\sqrt{x}}\,dx$ 13. $\int_2^{11} \frac{1}{\sqrt{(x-2)^3}}\,dx$ 14. $\int_0^\infty \frac{4}{e^x}\,dx$ 15. $\int_{-\infty}^0 e^{3x}\,dx$

16. $\int_1^\infty \frac{1}{\sqrt[3]{x}}\,dx$ 17. $\int_0^\infty \frac{x}{e^x}\,dx$ 18. $\int_e^\infty \frac{1}{x(\ln x)^4}\,dx$ 19. $\int_0^\infty \frac{e^x}{e^x+3}\,dx$

20. $\int_0^\infty \sin x\,dx$ 21. $\int_0^3 \frac{6}{x^2-9}\,dx$ 22. $\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx$ 23. $\int_0^4 \frac{1}{(x-1)^2}\,dx$

Write each Riemann Sum as a definite integral and each definite integral as a right Riemann Sum. Do not evaluate.

24. $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(3\left(\frac{k}{n}+4\right)+2\right)\frac{1}{n}\right)$ 25. $\int_2^7 \left(2x^2+5x\right)dx$ 26. $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(\cos\left(\frac{k\pi}{3n}\right)\right)\frac{\pi}{3n}\right)$

27. Find $\frac{d}{dx}\int_{x^3}^{x} \cos^2(2t+1)\,dt$.

Without a calculator, sketch graphs and use geometry to evaluate Problems 28 and 29.

28. $\int_0^3 |x-2|\,dx$ 29. $\int_{-1}^2 (|x|-2)\,dx$

30. **Sketch graphs and show shaded areas representing the values of the following .**

I. $\int_{-2}^2 |x^3+x|\,dx$ II. $\int_{-2}^2 |x^2+5x+6|\,dx$ III. $\int_{-2}^2 |x^2+5x-6|\,dx$ IV. $\int_{-2}^2 |x+1|\,dx$

Match each of the integrals to one of the descriptions below.

a. The integral can be evaluated geometrically using areas of triangles, so that no actual integration is necessary.
b. Absolute value is not even necessary for the given limits of integration.
c. Use of symmetry for the graph allows the problem to be done using only one integral that does not involve absolute value.
d. The integral can only be done by using more than one integral. That is, the problem must be split into two or more integrals to eliminate the absolute value.

31. a. Set up integrals that do not involve absolute value which could be used to integrate the integrals shown in Problem 30 I, II, and III.
b. Evaluate the integral in Problem 30 IV using areas of triangles.

Use a calculator for Problems 32-35. For Problems 32-35, $f(x)=3x^2+\ln|x|$.

32. Find $f'(4)$ 33. Find $\int_1^7 f(x)\,dx$ 34. Solve $3x^2+\ln|x|=0$

35. f is discontinuous when $x=0$. Is the discontinuity a hole, an asymptote, or a jump?

Selected Answers:

1. $\frac{1}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|+C$ 2. $\ln\left|\frac{x-2}{x+1}\right|+C$ 3. $\frac{3}{2}\ln|2x+1|+\ln|x-1|+C$

4. $2\ln|x|+\ln|x-1|-\ln|x+1|+C$ 5. $\frac{3}{2}x^2\ln x-\frac{3}{4}x^2+C$ 7. $3\ln|x|-\ln|x+4|+C$

8. improper (V.A. at $x=0$) 9. improper (V.A. at $x=\frac{2}{3}$) 10. not improper

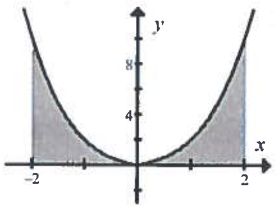
11. improper (infinite limit) 12. 6 13. diverges 14. 4 15. $\frac{1}{3}$ 17. 1

18. $\frac{1}{3}$ 19. diverges 21. diverges 22. $\frac{9}{2}$ 23. diverges 24. $\int_4^5(3x+2)\,dx$

25. $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(2\left(2+\frac{5k}{n}\right)^2+5\left(2+\frac{5k}{n}\right)\right)\frac{5}{n}\right)$ 26. $\int_0^{\frac{\pi}{3}}\cos x\,dx$ 27. $\cos^2(2x+1)-3x^2\cos^2(2x^3+1)$

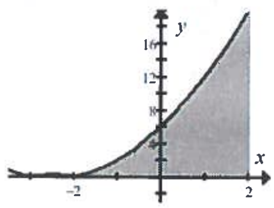
28. 2.5 29. -3.5

30.I.



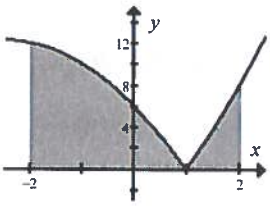
matches with c

30. II.



matches with b

30. III.



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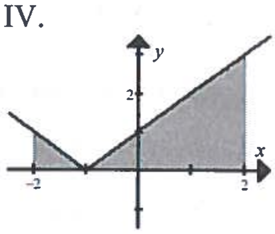
30. IV.



matches with a

31a. I $\int_{-2}^{2}|x^3+x|\,dx=2\int_0^2(x^3+x)\,dx$ II $\int_{-2}^{2}|x^2+5x+6|\,dx=\int_{-2}^{2}(x^2+5x+6)\,dx$

III $\int_{-2}^{2}|x^2+5x-6|\,dx=\int_{-2}^{1}(-x^2-5x+6)\,dx+\int_1^2(x^2+5x-6)\,dx$

31b. IV $\int_{-2}^{2}|x+1|\,dx=5$

32. $f'(4)=24.250$ 33. 349.621 34. $x=\pm.488$ or $\pm.489$ 35. an asymptote

UNIT 6 SUMMARY

Indefinite Integrals $\int f'(x)\,dx = f(x) + C$ You might have an initial condition and be able to solve for C.

Definite Integrals (Fundamental Theorem of Calculus) $\int_a^b f'(x)\,dx = f(x)\Big|_a^b = f(b) - f(a)$

Start Plus Accumulation $f(b) = f(a) + \int_a^b f'(x)\,dx$

Second Fundamental Theorem (Chain Rule Version):
If u and v are functions of x, then $\frac{d}{dx}\int_u^v f(t)\,dt = f(v)\,v' - f(u)\,u'$
(if f is continuous from u to v). Note the "chain rule factors" v' and u'.

Reverse Chain Rule for Integrals
$\int f'(u)u'\,dx = f(u) + C$ (Where u is a function of x)

Five Ways to Integrate
1. Term by term. 2. Reverse Chain Rule 3. u-Substitution
4. Long Division (numerator degree $\geq$ denominator degree)
5. Complete the square in denominator

Procedure for u-substitution:
1. Let $u =$ radicand (part inside the $\sqrt{\ }$ symbol).
2. Solve for x.
3. Differentiate the equation from Step 2.
4. Find dx.
5. Substitute u-expressions for x-expressions in the integral.
6. Integrate.
7. Substitute back, so that your final answer is again in terms of x.

Calculator Integration Math 9 on a TI84

Integrals involving absolute value: draw a graph, use geometry.

Definite Integral $\leftrightarrow$ Infinite Riemann Sum

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n}\big(f(a+k\Delta x)\,\Delta x\big) = \lim_{n\to\infty}\sum_{k=1}^{n}\left(\underbrace{f\left(a+k\frac{b-a}{n}\right)}_{\text{right-hand heights}}\overbrace{\frac{b-a}{n}}^{\text{width}}\right)$$

where n is the number of subdivisions.

	x **form**	u **form** (Reverse Chain Rule)
Power Rule:	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$	$\int u^n u'\,dx = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
Exponential Rules:	$\int e^x\,dx = e^x + C$	$\int e^u u'\,dx = e^u + C$
	$\int a^x\,dx = \frac{a^x}{\ln a} + C$	$\int a^u u'\,dx = \frac{a^u}{\ln a} + C$
Trig Rules:	$\int \cos x\,dx = \sin x + C$	$\int \cos u\, u'\,dx = \sin u + C$
	$\int \sin x\,dx = -\cos x + C$	$\int \sin u\, u'\,dx = -\cos u + C$
	$\int \sec^2 x\,dx = \tan x + C$	$\int \sec^2 u\, u'\,dx = \tan u + C$
	$\int \csc^2 x\,dx = -\cot x + C$	$\int \csc^2 u\, u'\,dx = -\cot u + C$
	$\int \sec x \tan x\,dx = \sec x + C$	$\int \sec u \tan u\, u'\,dx = \sec u + C$
	$\int \csc x \cot x\,dx = -\csc x + C$	$\int \csc u \cot u\, u'\,dx = -\csc u + C$
Log Rules:	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$	$\int \frac{u'}{u}\,dx = \ln\lvert u\rvert + C$
Inverse Trig	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$	$\int \frac{u'}{\sqrt{a^2-u^2}}\,dx = \arcsin\frac{u}{a} + C$
	$\int \frac{1}{1+x^2}\,dx = \arctan x + C$	$\int \frac{u'}{a^2+u^2}\,dx = \frac{1}{a}\arctan\frac{u}{a} + C$

Formula for Integration by Parts:
$\int u\,dv = uv - \int v\,du$

Partial Fractions: Example: Write fractions like $\frac{2}{(x-3)(x-4)}$ as $\frac{A}{(x-3)} + \frac{B}{(x-4)}$ then find A and B in order to integrate.

Complete the Square: When the denominator of an integrand fraction is not factorable completing the square may allow you to use an inverse trig formula.

Improper Integrals
1. one or both limits of integration are infinite
2. the function has an infinite discontinuity (a vertical asymptote) at or between the limits

Lesson 7.1 Solving Differential Equations, Verifying Solutions, Exponential Growth and Decay

Differential Equations are equations with derivatives in them. In this course, you will only learn how to solve the simplest type of differential equations, in which you can separate variables. You may be asked to find a general solution of the differential equation (which gives you a family of curves) or a particular solution (which gives you a single curve).

Procedure for Solving Differential Equations

1. Rewrite y' as $\frac{dy}{dx}$ (if necessary).
2. Multiply both sides of the equation by dx (if necessary).
3. Separate variables. (This is the most crucial step.)
4. Integrate both sides of the equation. (Remember to add C to one side.)
5. Solve for y (if necessary).
6. Use an initial condition to solve for C (if an initial condition is given).

Steps 5 and 6 are interchangeable.

Example 1: Find a general solution of $x+2yy'=0$

First, rewrite as $x+2y\frac{dy}{dx}=0$. Then,

Write your solution to Example 1 as a pair of possible functions (in the form $y=f(x)$) for the particular solutions to the differential equation. $y=$ or $y=$

Example 2: Find an equation of a function in the form $y=f(x)$ which contains the point $(0,-3)$, and whose slope is $\frac{xe^{x^2}}{y}$ for each point (x, y) on the curve.

Example 3: Solve the differential equation $y-2=x\frac{dy}{dx}$ if $y(1)=\frac{1}{2}$. Express your answer in the form $y=f(x)$.

An algebra equation like $x^3+x^2+4=0$ cannot be solved using techniques you have learned without a calculator. However, you should be able to answer the following question.

Example 4: (algebra warm-up) Is $x = -2$ a solution of $x^3+x^2+4=0$?

Example 5: Is $y=\frac{1}{2}e^x+e^{-x}$ a solution of the differential equation $y'=e^x-y$?

Exponential Growth and Decay

Mathematical models in which the rate of change of a variable is proportional to the variable itself are common in both the business and scientific worlds.

Suppose that the rate of change of y (with respect to time) is proportional to y itself.

$\frac{dy}{dt}$	=	k	•	y

Rate of change of y with respect to time	=	constant of proportionality	•	amount of substance y present at time t (y is a function of t)

Example 6: Separate variables and solve the differential equation $\frac{dy}{dt} = ky$ from the previous page.

The equation from Example 6 is called the **Basic Law of Exponential Growth or Decay**:

$$y = Ce^{kt}$$

Constants:

C is the initial value (the amount of substance present at time $t = 0$)
k is the constant of proportionality ($k > 0$ for growth and $k < 0$ for decay)

Variables:

t is the variable for time
y is the amount of substance present at time t. (y is a function of t.)

Example 7: What is the rate of growth of the population in a city whose population triples every 100 years? Assume that the population growth can be modeled by the Basic Law of Exponential Growth, and express your answer as a percent (rounded to the nearest hundredth of a percent).

Example 8: Let y represent the mass, in pounds, of a radioactive element whose half-life is 4000 years. If there are 200 pounds of the element in an inactive mine, how much will still remain in 1000 years? Express your answer to 3 or more decimal place accuracy.

Assignment 7.1

For Problems 1-4, find a general solution of each differential equation.

1. $y' = \dfrac{x^2 - 1}{2y^2 + 3}$
2. $e^x y \frac{dy}{dx} = 1$ Solve for y.
3. $2xy' = y + 1$
4. $(x-2)\frac{dy}{dx} = 2y$ Solve for y.

For Problems 5,6, find a particular solution of the differential equation with the given initial condition. (Remember to write your solutions in the form $y = f(x)$.)

5. $\frac{dy}{dx} = \dfrac{-2x}{y}$ and $y(2) = -4$
6. $y = -3x\frac{dy}{dx}$ and $y(1) = e$

7. Find an equation of a function which contains the point $(-2,1)$ and whose slope is $\dfrac{x}{2y}$ for each point on the graph of the function.

Determine whether each of the following is a solution of the differential equation $y'' - 9y = 0$. Show organized work.

8. $y = \sin(3x)$
9. $y = e^{3x}$
10. $y = \cos(3x)$
11. $y = e^{-3x}$

12. \$1000 is placed into a certificate of deposit (CD) in which interest is compounded continuously at a rate of $5\frac{1}{2}\%$ per year (actual rate of return will be higher due to compounding of interest). Use your calculator and the formula $A = Pe^{rt}$ to find:

    a. the amount that the CD would be worth in 1 year. 5 years. 10 years.
    b. the time it would take the CD to be worth \$1,200.
    c. the time it would take the CD to double in value.

13. Suppose 200 bacteria are introduced into a culture to study their rate of growth. Two days later, the culture is found to contain 300 bacteria. Assuming the rate of growth is proportional to the number of bacteria present ($y=Ce^{kt}$), how many bacteria will be present in 3 more days (5 days after the start)?

14. Find the half-life of a radioactive isotope if 4.92 grams out of an initial 5 grams of the isotope remain after 10 years.

15. An isotope of carbon (C^{14}) is used for estimating how long ago certain living organisms were on earth. (The method is called carbon dating.) The half-life of C^{14} is approximately 5730 years. If the skull of an ancient primate contains 10% (.1) of the C^{14} present in the skull of a modern primate of a similar species, estimate how long ago the ancient primate lived (to the nearest thousand years).

16. A worker at a hazardous waste plant was accidentally exposed to toxic chemicals which were absorbed into his bloodstream. Upon feeling ill, the worker went to a hospital and had some blood drawn for testing. The concentration of chemical in the drawn blood was found to be .0158 *mg/ml*. Expensive medication was administered to counter the effects of the chemical in the blood, but the doctor on duty knew that the concentration of the chemical in the bloodstream would have to decrease gradually over time according to the Basic Law of Exponential Decay ($y=Ce^{kt}$). Medication would have to be administered every hour until the concentration was below .0050 *mg/ml*. Two hours later, blood was again drawn, and it was found to contain a chemical concentration of .0126 *mg/ml*. The doctor asked a lab technician to do the following. (You do the same):

   a. Write the particular solution for exponential decay for the chemical in the patient's blood. (Let $t = 0$ represent the time that blood was first drawn.)
   b. Sketch a graph of the function from Part a.
   c. Find out how long it will be before the patient can be taken off medication.
   d. When the patient has only a negligible amount of chemical in his bloodstream (less than .0001 *mg/ml*), he can be released from the hospital. Find out how long the patient has to be hospitalized (from the time he first came to the hospital and had his blood drawn).
   e. Occasionally, patients exposed to this chemical suffer damage to their central nervous systems. A maximum concentration of .020 *mg/ml* requires a follow-up examination. The doctor estimated that the maximum concentration of the chemical in the worker's bloodstream occurred 1 hour after exposure. The patient estimated 1 hour after exposure would have been about 3 hours prior to his blood being drawn for the first time. Should the patient be asked to return for a follow-up exam? Why or why not?
   f. Find the half-life for the chemical in the bloodstream for the patient.

Differentiate in Problems 17-19.

17. $y = 3^{2t-1}t^2$ 18. $f(y) = \dfrac{e^{\sqrt{y}}}{y^2}$ 19. $f(x) = e^x \ln x$

Antidifferentiate in Problems 20-22.

20. $y' = \dfrac{e^{-x}}{1+e^{-x}}$ 21. $g'(x) = \dfrac{2x-4}{x}$ 22. $y' = \dfrac{(\ln t)^3}{t}$

23. Find the area of the region bounded by $y=\left(\frac{1}{2}\right)^x$, $y=0$, $x=-2$, and $x=0$ without using a calculator.

24. Use a calculator to find the area between the curve $y=|2\cos x+\cos(2x)|$ and the x-axis, from $x=0$ to $x=\pi$. Show an integral set up and an answer.

25. For a particle moving along a straight path with velocity $v(t)=e^{-1.3t}-t\ln(.37t)$, $t>0$, use your calculator to find:
   a. the time when the particle is at rest.
   b. the speed of the particle at time $t=4$.
   c. the acceleration of the particle at time $t=5$.
   d. the total distance traveled by the particle on the interval $[1,5]$.

26. If a particle moves along the curve $y=x^{\frac{2}{3}}$, such that $\frac{dx}{dt}=3$ for all x, find:

   a. $\frac{dy}{dt}$ when $x=-1$ b. $\frac{dy}{dt}$ when $x=8$

   c. $\lim\limits_{x\to\infty}\frac{dy}{dt}$ d. $\lim\limits_{x\to 0^-}\frac{dy}{dt}$

27. a. Use a tangent line to the graph of $y=x^{\frac{2}{3}}$ to approximate $(8.1)^{\frac{2}{3}}$.

   b. Why could a tangent line to $y=x^{\frac{2}{3}}$ at $x=0$ not be used to approximate $(.1)^{\frac{2}{3}}$?

28. f and g are inverse functions. The graph of g passes through the points $(-1,2)$, and $(2,-1)$ $f'(-1)=-2$ and $f'(2)=-1$. Find:

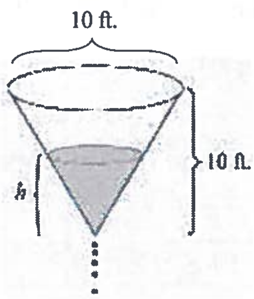
   a. $g'(-1)$ b. $g'(2)$

Find the indicated limits in Problems 29-31.

29. $\lim\limits_{x\to a}\frac{x-a}{a^2-x^2}$ 30. $\lim\limits_{x\to\infty}\frac{(3x-2)(x+1)^2}{(2x+1)^2(x-5)}$ 31. $\lim\limits_{x\to 0}\frac{1-e^x}{\sin(4x)}$

32. If $f'(t)=\lim\limits_{h\to 0}\frac{t+h+\sqrt{t+h}-(t+\sqrt{t})}{h}$, find $f(t)$.

33. A conical tank, as shown at right, has a hole in its bottom and is leaking water at the rate of 1 cubic foot per minute. Find the rate of change in the height, h, of the water in the tank when $h=4\ ft$? $V=\frac{1}{3}\pi r^2 h$
Write appropriate units for your answer.
Hint: Find a relationship between r and h. $r=$ ___ h



Selected Answers:

1. $\frac{2}{3}y^3+3y=\frac{1}{3}x^3-x+C$ 2. $y=\sqrt{-2e^{-x}+C}$ or $y=-\sqrt{-2e^{-x}+C}$

3. $\ln|y+1|=\frac{1}{2}\ln|x|+C$ or $\ln|y+1|=\frac{1}{2}\ln|2x|+C$ 4. $y=C(x-2)^2$

5. $y=-\sqrt{-2x^2+24}$ 6. $y=ex^{-\frac{1}{3}}$ 7. $y=\sqrt{\frac{1}{2}x^2-1}$ 8. not a solution

12a. $A(1)=\$1056.54$, $A(5)=\$1316.53$, $A(10)=\$1733.25$ c. 12.602 or 12.603 *yrs*

13. 551 bacteria 14. 429.741 or 429.742 *yrs* 15. 19,000 *yrs*

18. $f'(y)=\dfrac{y^2e^{\sqrt{y}}\bullet\frac{1}{2}y^{-\frac{1}{2}}-e^{\sqrt{y}}\bullet 2y}{y^4}$ 19. $f'(x)=\frac{e^x}{x}+e^x\ln x$ 20. $y=-\ln(1+e^{-x})+C$

23. $\frac{3}{\ln 2}$ 24. 4.403 or 4.404 25a. $t=2.731$ c. $a(5)=-1.617$

25d. $TD=4.510$ or 4.511 26a. -2 c. 0 27 a. $4\frac{1}{30}$ 28b. $-\frac{1}{2}$ 29. $-\frac{1}{2a}$

30. $\frac{3}{4}$ 31. $-\frac{1}{4}$ 33. $\frac{dh}{dt}=-\frac{1}{4\pi}\frac{ft}{min}$

Lesson 7.2 Slope Fields, Euler's Method

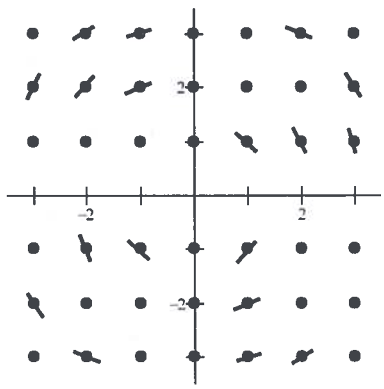
A **slope field** is a graphical representation of a set of slopes obtained from a differential equation. Remember that a differential equation involves a derivative. That derivative represents the slopes for a function. In Lesson 7.1 you learned to solve differential equations by separating variables. Even if you cannot separate variables and integrate, you can still use a differential equation to plot the slopes for a function.

Example 1: Find the slopes given by the differential equation $\frac{dy}{dx}=\frac{-x}{y}$ at the following points:

a. (3, 2) b. (−1, 3) c. (−2, −1) d. (2, −2)

Why can't you find slopes when $y=0$?

Example 2: Find and plot the slopes given by $\frac{dy}{dx} = \frac{-x}{y}$ for each remaining marked point (dot) in the coordinate plane at the right.



Example 3: In Example 2, you made what is known as a slope field. Starting at the point $(0, 1)$, follow the flow of the slopes to sketch the solution curve containing $(0, 1)$. Your graph should be "parallel" to the slope lines and be like an "average of slopes" whenever it goes between lines. Your solution curve must represent a function whose domain is the largest possible open interval containing the given point. Sketch a solution curve passing through $(-1, 1)$ and one passing through $(0, -3)$. What type of graph does this differential equation seem to be producing?

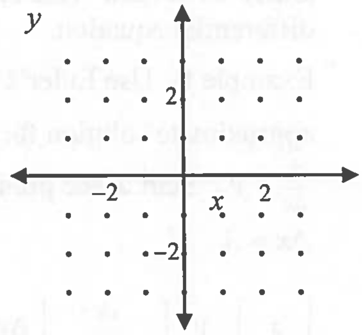
Note: The most common student error in sketching a particular solution to a differential equation is to extend the sketch too far and create a graph which is not a function. It is important to set appropriate "boundaries" for your sketch. Why is the x-axis a "boundary" for the differential equation from Examples 1 and 2?

Example 4: Solve the differential equation $\frac{dy}{dx} = \frac{-x}{y}$.

a. Find the particular solution in the form $y = f(x)$ for the differential equation $\frac{dy}{dx} = \frac{-x}{y}$ whose graph passes through the point $(0,1)$.

b. Find the particular solution in the form $y = f(x)$ whose graph passes through the point $(0,-3)$.

Example 5: For the differential equation $y' = \dfrac{1}{y}$



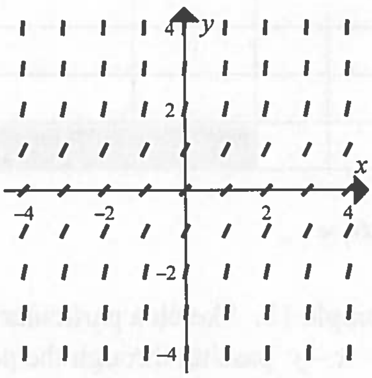
a. Draw the slope field in the dot coordinate plane at right.

b. Graph the particular solutions passing through the points $(-2,-1)$ and $(2,2)$ as functions of x.

c. Solve the differential equation.
Write as a function the particular solution for the differential equation whose graphs pass through $(-2,-1)$.

d. Write as a function the particular solution for the differential equation whose graphs pass through $(2,2)$.

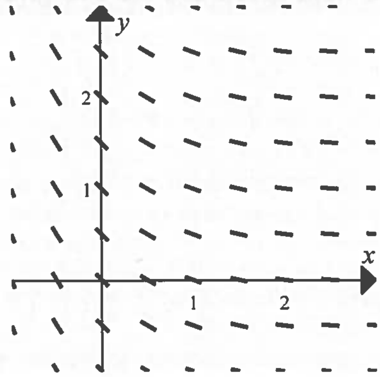
Example 6: Which of the differential equations below matches the slope field shown at right?



a. $y' = x$ b. $y' = y$ c. $y' = x - y$

d. $y' = 1 + y^2$ e. $y' = 1 + x^2$

Example 7: The slope field for a certain differential equation is shown at the right. Which of the following could be a specific solution to the differential equation?



a. $y = e^x$ b. $y = e^{-x}$ c. $y = -e^x$

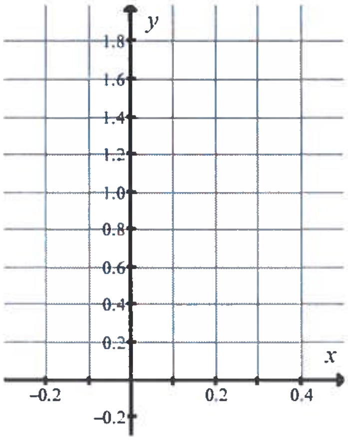
d. $y = -\ln x$ e. $y = \ln x$

Euler's Method This is a more precise method of graphing an approximate solution to a differential equation.

Example 8. Use Euler's method to construct an approximate solution for the differential equation $\frac{dy}{dx} = y$. Start at the point $(0,1)$ and use step size $\Delta x = .1$

x	y	$\frac{dy}{dx}$	$\Delta y = (\text{slope})\,\Delta x$

$y(.3) \approx$



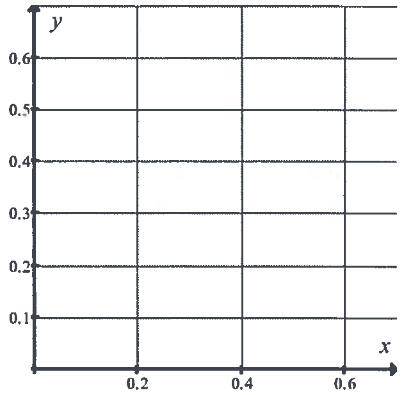
Example 9. Solve $\frac{dy}{dx} = y$ algebraically. Fill in the table with the actual values of y.

x	y
0	1
.1	
.2	
.3	

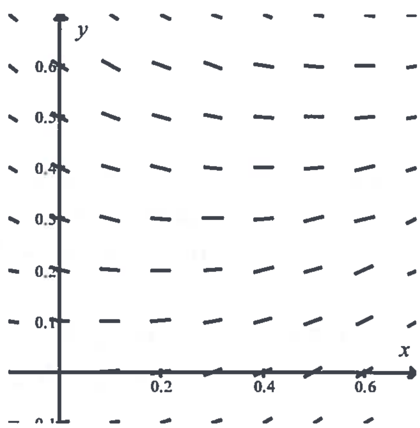
Example 10. Use Euler's Method to approximate the particular solution of the diff. eq. $y' = x - y$ passing through the point $(0,0.5)$. Let $\Delta x = .2$ and do three steps (n = 3). Graph the points.

x	y	$\frac{dy}{dx}$	$\Delta y = (\text{slope})\,\Delta x$

$y(.6) \approx$



Example 11. Sketch a particular solution of the diff. eq. $y' = x - y$ passing through the point $(0,0.5)$ using the slope field given. Do the two graphs coincide?



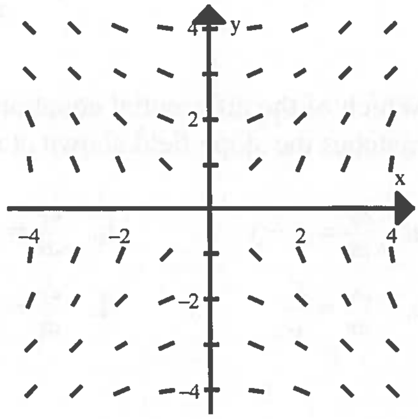
Assignment 7.2 A tear-out sheet is provided on page 167 for your work.

1. Find the slopes given by the differential equation $y' = \dfrac{x^2}{y-2}$ at each of the following points:

   a. $(0,0)$ b. $(1,1)$ c. $(-2,4)$ d. $(4,-2)$ e. $(-3,-3)$ f. $(5,12)$

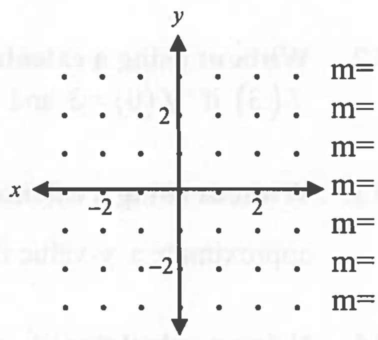
2. For the differential equation in Problem 1, why are there no slopes when $y = 2$?

3. The slope field for $y' = \dfrac{x}{y}$ is shown at right.



   a. Plot the following points on the slope field:
      i. $(1, 2)$ ii. $(3, 1)$ iii. $(0, 3)$
      iv. $(0,-2)$ v. $(-2,-1)$
   b. Plot a separate solution curve through each of the points from Part a. Remember that the curves have to be <u>functions</u>.
   c. What would a solution curve containing $(2, 2)$ look like?
   d. Solve the differential equation $y' = \dfrac{x}{y}$.

4. For the differential equation $\frac{dy}{dx} = y$



   a. Draw the slope field for the differential equation.
   b. Graph the particular solutions passing through the points $(0,1)$ and $(0,-1)$.
   c. Solve the differential equation, <u>and</u> find the particular solutions that contain the points $(0, 1)$ and $(0,-1)$.

5. For the differential equation $\frac{dy}{dx} = \dfrac{x}{2}$

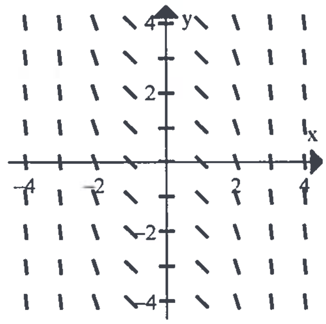
   a. Draw the slope field for the differential equation.
   b. Graph the particular solution passing through the point $(-1, 1)$.
   c. Solve the differential equation, and find the particular solution that contains the point $(-1, 1)$.

6. Repeat the three parts of Problem 5 for the differential equation $y' = \dfrac{1}{2y}$. For this problem, draw your graph as and write your solution as a <u>function of x</u>.

7. Repeat the three parts of Problem 5 for the differential equation $y'y^2 = 1$.

8. Repeat the three parts of Problem 5 for the differential equation $\frac{dy}{dx} = 2x(y-1)$, but use the <u>origin</u> (instead of $(-1, 1)$) for Parts b. and c.

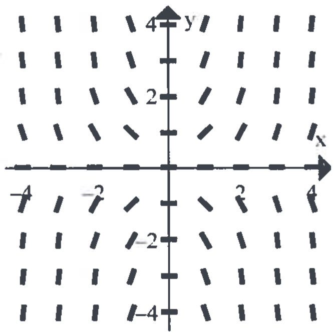
9. The slope field for a certain differential equation is shown at right. Which of the following could be a specific solution to that differential equation.



a. $\frac{1}{3}x^3 + y = 2$ b. $x^2 + y^2 = 4$

c. $x^2 - y^2 = 4$ d. $y = \frac{4}{x}$ e. $\frac{y}{x} = 4$

10. Which of the differential equations below matches the slope field shown at right?



a. $\frac{dy}{dx} = x - y$ b. $\frac{dy}{dx} = y - x$

c. $\frac{dy}{dx} = \frac{x}{y}$ d. $\frac{dy}{dx} = \frac{y}{x}$ e. $\frac{dy}{dx} = xy$

11. Show that $y = \frac{1}{2}\sin x - \frac{1}{2}\cos x + e^x$ is a solution of the differential equation $y' - y = \cos x$.

12. **Without using a calculator,** use Euler's Method with a step size of 0.1 to approximate $f(.3)$ if $f(0) = 3$ and $f'(x) = x + y$.

13. **Without using a calculator,** use Euler's Method with 3 steps each with a size of $\frac{1}{2}$ to approximate a y-value if $y(0) = 2$ and $y' = 2x - 3y$.

14. **Using a calculator,** if $y(1) = 2$ and $y' = e^{xy}$ use 4 steps of Euler's Method to approximate $y(0.8)$.

15. Use the acceleration graph at right to find the following for an object moving along a straight path.

a. $a(1)$ b. $a(6)$ c. $a(9)$ d. $a(12)$

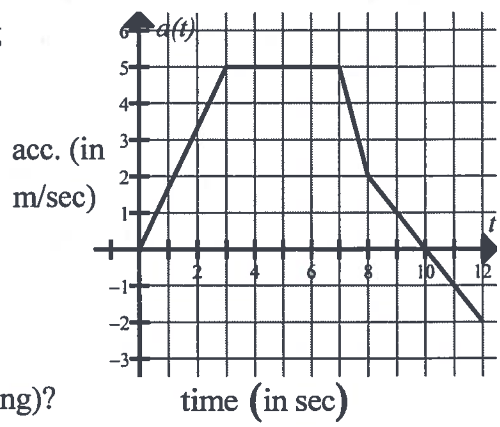
Suppose $v(0) = 2$ *m/sec*. Find:

e. $v(3)$ f. $v(7)$ g. $v(10)$ h. $v(12)$

i. At what time on [0,12] was the object moving the fastest? Justify your answer.

j. At what time on [0,12] was the object moving the slowest? Justify your answer.

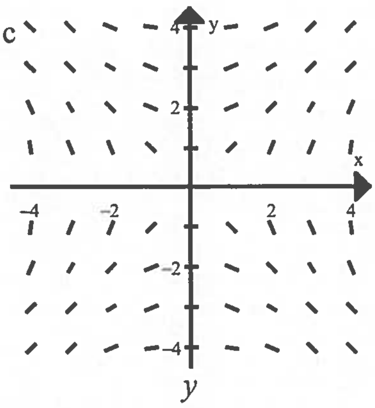
k. When was the object slowing down (speed decreasing)?



Assignment 7.2 Tear-out Sheet Name ____________________________

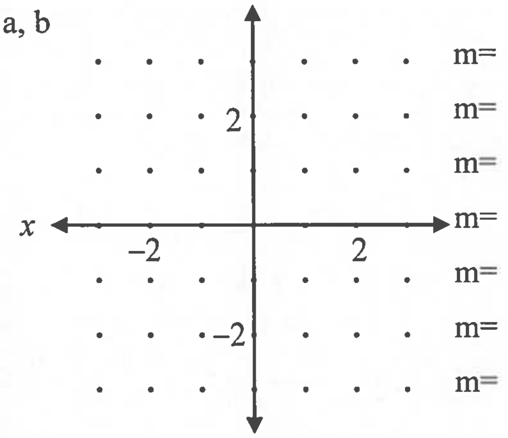
1. a. ____ b. ____ c. ____ d. ____ e. ____ f. ____ 2.

3. a, b, c



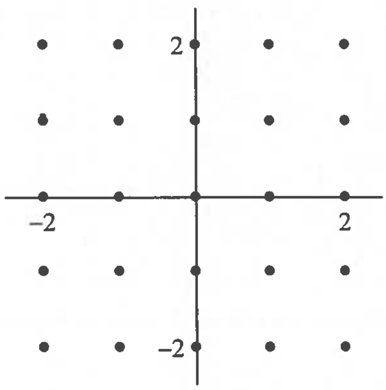
3. d. solve $y' = \dfrac{x}{y}$

4. a, b



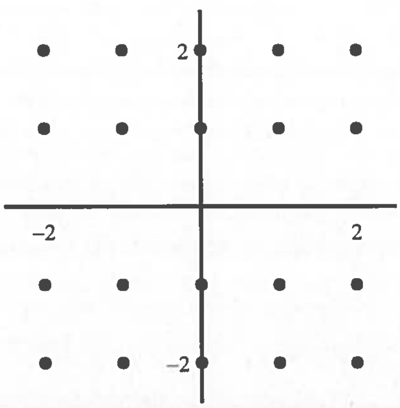
4. c.

5. a, b



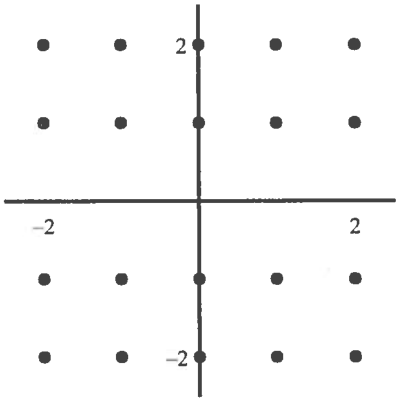
5. c.

6. a, b



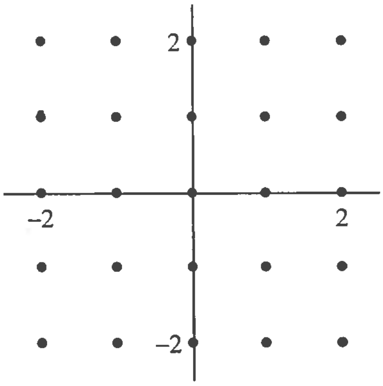
6. c.

7. a, b



7. c.

8. a, b



8. c.

9. ____ 10. ____

Find the x-value(s) where each of the functions in Problems 16-19 is not differentiable. Give a reason why each function is not differentiable for those values of x. Do not use a calculator.

16. $f(x)=\left|x^2-9\right|$

17. $p(x)=\dfrac{x^2(x-2)}{x(x+1)}$

18. $q(x)=x-x^{\frac{1}{3}}$

19. $h(x)=\begin{cases} 2x+1, & x\le 1 \\ \frac{1}{2}x^2+x+\frac{3}{2}, & x>1 \end{cases}$

20. The product of two positive numbers is 80. Find the numbers so that the sum of the first number and five times the second number is a minimum. Do not use a calculator.

21. Use the f' and f'' number lines below to sketch a possible graph of a continuous function f.

f': $-$ | 0 | $+$ | 2 | $-$ f'': $+$ | -2 | $-$ | 0 | $-$

Evaluate the integrals in Problems 22-29 without using a calculator.

22. $\displaystyle\int \frac{x+1}{x-1}\,dx$

23. $\displaystyle\int \frac{\cos y}{\sin^3 y}\,dy$

24. $\displaystyle\int \sec^5 x\,\tan x\,dx$

25. $\displaystyle\int \frac{3}{x^2-6x+18}\,dx$

26. $\displaystyle\int \frac{4}{2+\sqrt{x}}\,dx$

27. $\displaystyle\int_0^{\infty} xe^{-x}\,dx$

28. $\displaystyle\int_0^{3} \frac{1}{x-1}\,dx$

29. $\displaystyle\int_{\sqrt{3}}^{2} \frac{1}{\sqrt{1-\frac{1}{4}x^2}}\,dx$

The velocity of a moving object is given by $v(t)=\dfrac{1}{4}t^3-2t+1$.

30. If the position at $t=2$ is given by $x(2)=3$, find $x(t)$.

31. Find the total distance traveled by the object on the interval $[0,4]$.

32. $f(x)=\dfrac{x}{x^2+1}$, $f'(x)=\dfrac{1-x^2}{\left(x^2+1\right)^2}$, and $f''(x)=\dfrac{2x\left(x^2-3\right)}{\left(x^2+1\right)^3}$.

a. Without using a calculator, list the domain, asymptotes, and intercepts for $f(x)$.

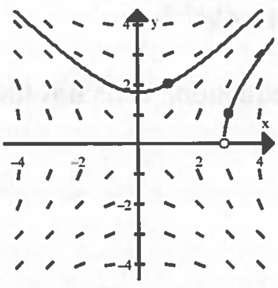
b. Find the relative extrema points of $f(x)$.

c. Find the points of inflection of $f(x)$.

d. Sketch $f(x)$ without using a calculator.

Selected Answers:

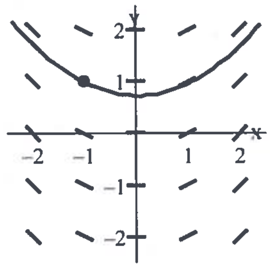
1a. 0   c. 2   e. $-\frac{9}{5}$ 3b. i,ii.



d. $y=\sqrt{x^2+C}$ or $y=-\sqrt{x^2+C}$

More Selected Answers:

4c. $y=e^x$ for $(0,1)$ 5a,b.



c. $y=\frac{1}{4}x^2+\frac{3}{4}$

6c. $y=\sqrt{x+2}$ 7c. $y=\sqrt[3]{3x+4}$ 8c. $y=-e^{x^2}+1$ 12. $f(0.3)\approx 4.024$

13. $y\left(\frac{3}{2}\right)\approx\frac{1}{2}$ 14. $y(0.8)\approx 1.078$ or 1.079 15a. $\frac{5}{3}\frac{m}{sec^2}$ c. $1\frac{m}{sec^2}$ e. $9.5\frac{m}{sec}$ f. $29.5\frac{m}{sec}$

15i. $10\,sec.$ j. $0\,sec.$ 16. $x=\pm 3$ (sharp turns) 17. $x=0$ (hole), $x=-1$ (VA)

18. $x=0$ (vertical tangent) 20. The first number is 20 and the second is 4.

22. $x+2\ln|x-1|+C$ 23. $-\frac{1}{2}(\sin y)^{-2}+C$ 24. $\frac{1}{5}(\sec x)^5+C$ 25. $\tan^{-1}\left(\frac{x-3}{3}\right)+C$

26. $8\sqrt{x}-16\ln\left(\sqrt{x}+2\right)+C$ 27. 1 28. diverges

29. $\frac{\pi}{3}$ 30. $x(t)=\frac{1}{16}t^4-t^2+t+4$ 31. 7.128 or 7.129

32a. Do.: all reals, V.A.: none, H.A.: y = 0, x-int.: (0,0) odd, y-int.: (0,0)

b. rel. min. pt. $\left(-1,-\frac{1}{2}\right)$, rel. max. pt. $\left(1,\frac{1}{2}\right)$ c. P.I. $(0,0)$, $\left(\sqrt{3},\frac{\sqrt{3}}{4}\right)$, $\left(-\sqrt{3},\frac{-\sqrt{3}}{4}\right)$

Lesson 7.3 Logistic Equations

Exponential growth modeled by $y=Ce^{kt}$ assumes unlimited growth and is unrealistic for most population growth. More typically the growth rate decreases as the population grows and there is a maximum population M called the carrying capacity. This is modeled by the **logistic** differential equation $\frac{dP}{dt}=kP(M-P)$.

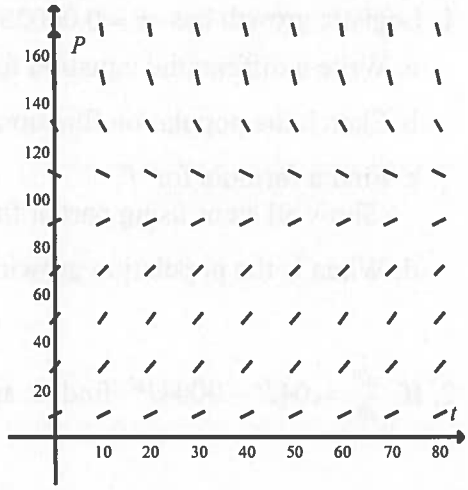
The solution equation is of the form $P=\dfrac{M}{1+Ce^{-kMt}}$.

Note: Unlike in the exponential growth equation, C is **not** the initial amount.

Example: A national park is capable of supporting no more than 100 grizzly bears. We model the equation with a logistic differential equation with $k = 0.001$.

a. Write the differential equation.

b. The slope field for this differential equation is shown. Where does there appear to be a horizontal asymptote?



What happens if the starting point is above this asymptote?

What happens if the starting point is below this asymptote?

c. If the park begins with ten bears, sketch a graph of $P(t)$ on the slope field.

d. Solve the differential equation to find $P(t)$ with this initial condition.

e. Instead of solving the differential equation, use the general form of a logistic equation $P = \dfrac{M}{1 + Ce^{-kMt}}$ to find the same solution.

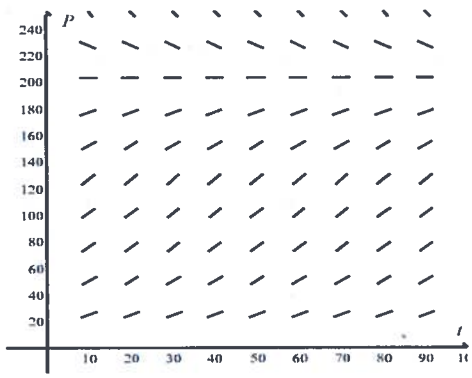
f. Find $\lim\limits_{t \to \infty} P(t)$

g. When will the bear population reach 50?

h. When is the bear population growing the fastest.

Assignment 7.3

1. Logistic growth has $k = 0.00025$, $M = 200$, and $P(0) = 10$.
   a. Write a differential equation for the population.
   b. Sketch the population function on the slope field.
   c. Find a formula for P.
      Show all steps using partial fractions.
   d. When is the population growing the fastest?

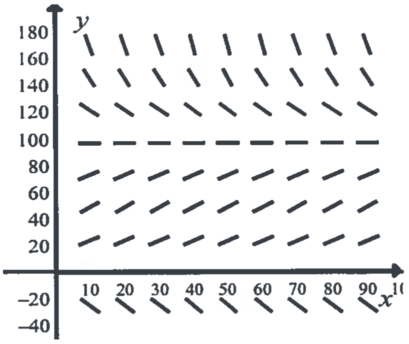


2. If $\frac{dP}{dt} = .04P - .0004P^2$ find k and the carrying capacity.

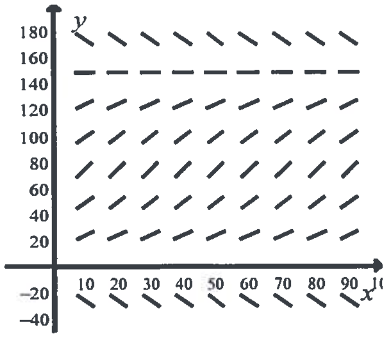
Match each of these differential equations with one of slope fields shown.

3. $\frac{dy}{dx} = .065y$
4. $\frac{dy}{dx} = .0006y(100 - y)$
5. $\frac{dy}{dx} = .06y\left(1 - \frac{y}{150}\right)$

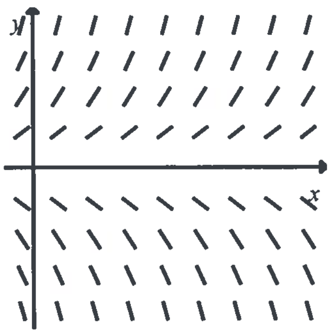
A.



B.



C.



6. Given the logistic equation $P(t) = \dfrac{2000}{1 + 19e^{-0.6t}}$:
   a. find the carrying capacity.
   b. find the value of k.
   c. find the initial population.
   d. find the time at which the population reaches 500.
   e. give the logistic differential equation.

7. Given the logistic differential equation $\frac{dP}{dt} = .03P(100 - P)$:
   a. find the value of k.
   b find the carrying capacity.
   c. find the value of P when $\frac{dP}{dt}$ is the greatest.

8. Given the logistic differential equation $\frac{dy}{dt} = 2y\left(1 - \frac{y}{50}\right)$:
   a. find the value of k.
   b. find the value of M.
   c. give the logistic equation if $y(0) = 10$.

9. A 200 gallon tank can support no more than 150 guppies. Six guppies are introduced into the tank. Assume that the rate of growth of the population is $\frac{dP}{dt} = .0015P(150-P)$, where time t is measured in weeks.

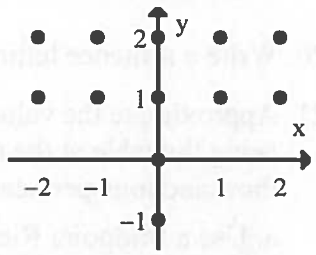
   a. Find a formula for the guppy population in terms of t.

   b. How long will it take for the guppy population to be 100? 125?

10. The amount of food placed daily into a biology lab enclosure can support no more than 200 fruit flies. A biologist releases 25 flies into the enclosure. Four days later she counts 94 flies.
    a. Give the logistic equation.
    b. Find the number of flies on the 7th day.
    c. When will there be 175 flies?
    d. Find the logistic differential equation.
    e. Find the population and the time at which the growth is the fastest.
    f. Starting with 94 flies on day 4 and a step size of 1 day, use Euler's Method to approximate the number of flies on the 7th day.

11. Find the particular solution of $4y' = 3x^2 + 2x$, if $y(2) = 5$.

12. Consider the differential equation given by $\frac{dy}{dx} = \frac{xy}{2}$.



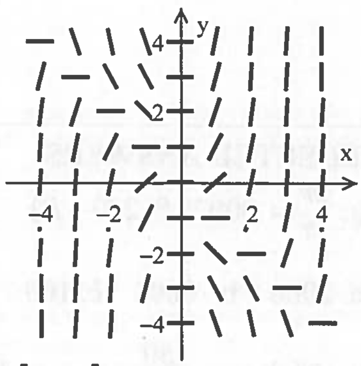
    a. Sketch a slope field on your own paper for the given differential equation at the twelve indicated points, and sketch a particular solution containing the point $(0, 1)$.

    b. Use Euler's Method starting at the point $(0,1)$ with step size $\Delta x = 1$ to find an approximate y-value when $x = 2$.

    c. Solve the differential equation $\frac{dy}{dx} = \frac{xy}{2}$ and find the particular solution containing the point $(0, 1)$. Give your answer in the form $y = f(x)$.

    d. Use your solution to find the exact y-value when $x = 2$.

13. Which of the following differential equations is represented by the slope field shown?



    A. $\frac{dy}{dx} = x(x-y)$ B. $\frac{dy}{dx} = x(x+y)$ C. $\frac{dy}{dx} = y(x+y)$

    D. $\frac{dy}{dx} = \frac{x}{(x+y)}$ E. $\frac{dy}{dx} = xy$

14. Sketch a slope field for the differential equation $\frac{dy}{dx} = x + y$. Use $[-2,2]$ for both your x-interval and your y-interval.

15. For the differential equation from Problem 14, sketch a particular solution which passes through the point $(1,-1)$.

16. If $(2,6)$ is a point on the solution curve of the differential equation $y + xy' = 5$, determine the concavity of the solution curve at that point.

17. Solve the differential equation $\frac{dH}{dt} = \frac{t^2}{H^2}$ to find an expression for $H(t)$ if $H(-2) = 3$.

18. Which of the following is a solution of the differential equation $y' + y = e^x$?

A. $y = e^x + 2e^{-x}$ B. $y = \frac{1}{2}e^x - 2e^{-x}$ C. $y = e^x - 3e^{-x}$ D. $y = \frac{1}{2}e^{-x} + 3e^x$

19. The rate of coal production, $R(t)$, from a coal mine was 2,000,000 tons per year at the beginning of the year 2000. By the beginning of the year 2006, the coal production had decreased to 1,200,000 tons per year. The mine will be profitable until less than 200,000 tons of coal are produced in a given year. Assuming that the decline in the amount of coal mined per year closely models the equation for exponential decay $(y = Ce^{kt})$, find the following:

a. the particular equation for the amount of coal mined per year. (Let $t = 0$ for the beginning of the year 2000.)
b. the half-life for the production of coal (from your model in Part a.).
c. the year when it will no longer be profitable to mine coal.

20. Write a sentence telling what $\int_0^6 R(t)\,dt$ (from Problem 19) represents.

21. Approximate the value of the integral from Problem 20 using the table at the right. (R values are to the nearest thousand tons per year.)

a. Use a Midpoint Riemann Sum with 3 equal subdivisions.
b. Use a Trapezoidal approximation with widths of 1,2, and 3 in that order.

t	$R(t)$
0	2000
1	1837
2	1687
3	1549
4	1423
5	1307
6	1200

22. Use your calculator to find the actual value of $\int_0^6 R(t)\,dt$ from Problem 20 to the nearest thousand tons.

SELECTED ANSWERS

1a. $\frac{dP}{dt} = .00025P(200 - P)$ c. $P(t) = \frac{200}{1 + 19e^{-.05t}}$ 2. $k = .0004$, $M = 100$

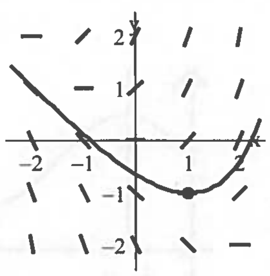
6a. 2000 b. .0003 c. 100 d. 3.076 e. $\frac{dP}{dt} = .0003P(2000 - P)$ 8c. $y = \frac{50}{1 + 4e^{-2t}}$

9a. $P(t) = \frac{150}{1 + 24e^{-.225t}}$ 10a. $P(t) = \frac{200}{1 + 7e^{-.456t}}$ b. 155 flies c. 8.526 days

10d. $\frac{dP}{dt} = .002P(200 - P)$ 11. $y = \frac{1}{4}x^3 + \frac{1}{4}x^2 + 2$ 12c. $y = e^{\frac{1}{4}x^2}$

More Selected Answers:

14,15



16. Concave upward since $y''\big|_{(2,6)} = \frac{1}{2} > 0$

17. $H(t) = \sqrt[3]{t^3 + 35}$ 18. B

19a. $R(t) = 2{,}000{,}000e^{-.085t}$ b. 8.141 yrs

19c. 27.045 *yrs* (early in 2027)

20. $\int_0^6 R(t)\,dt$ gives the total amount of coal (in tons) produced by the mine from the beginning of 2000 through 2005.

21a. 9,386,000 tons b. 9,428,000 tons 22. 9,397,000 tons

UNIT 7 SUMMARY

Differential Equations: (Equations involving derivatives.)

Procedure for Solving Differential Equations

1. Rewrite y' as $\frac{dy}{dx}$ (if necessary).
2. Multiply both sides of the equation by dx (if necessary).
3. Separate variables. (This is the most crucial step.)
4. Integrate both sides of the equation. (Remember to add C to one side.)
5. Solve for y (if necessary).
6. Use an initial condition to solve for C (if an initial condition is given).

Steps 5 and 6 are interchangeable.

Exponential Growth and Decay: $y = Ce^{kt}$

Slope Fields: Be able to:

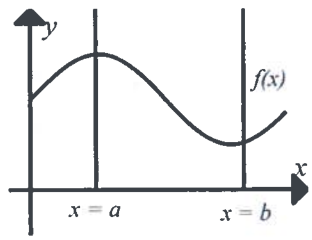
Sketch a slope field	given:	a differential equation.
Sketch a solution curve	given:	a differential equation and a starting point.
Match a slope field	with:	a differential equation.
Match a slope field	with:	the solution of a differential equation.

Euler's Method: This is a method of finding an approximate y-value on a solution to a differential equation.

x	y	$\frac{dy}{dx}$	$\Delta y = (\text{slope})\Delta x$

Logistic Equations: (M is the carrying capacity) $\frac{dP}{dt} = kP(M-P)$ $P = \dfrac{M}{1+Ce^{-kMt}}$

Lesson 8.1 Average Value of a Function, Arc Length Accumulation of Rate Functions



Discovering the Formula for the Average Value of a Function.
The average value of a function represents its average "height."

1. Draw a horizontal segment from $x = a$ to $x = b$ in the figure at right which could represent the average height of $f(x)$ on $[a, b]$.

2. Find the area of the rectangle formed.

   $A = \underbrace{\qquad}_{\text{width}} \bullet \textit{ height}$

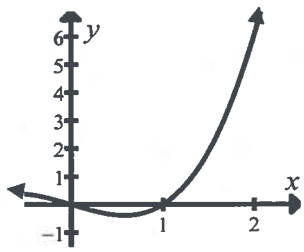
3. $\int_a^b f(x)\,dx = (b-a) \bullet \textit{height}$ Now solve for the average height.

Average Value of a Function:

Average value of $f(x)$ on $[a, b]$ = $f(c)$ = $\boxed{f_{avg} = \dfrac{\overset{\text{"area" of region}}{\int_a^b f(x)\,dx}}{\underset{\text{"width" of region}}{b-a}}}$ or $\dfrac{1}{b-a}\int_a^b f(x)\,dx$

$f(x)$ must be continuous on $[a, b]$.

Example 4: Find the average value of $f(x) = x^3 - x$ on the interval [0, 2] without using a calculator.

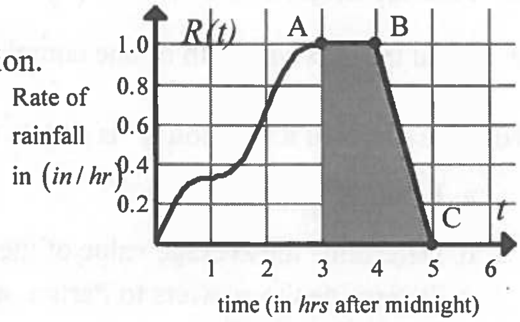


Example 5: Use your calculator to find the value of c in the interval [0, 2] where $f(c)$ = the average value found in Example 4.

Arc Length Formula: Arc Length $= \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\,dx$

Example 6: Find the length of the arc of the curve $f(x) = x^3 - x$ from Example 4 on the interval [0,2].

Accumulation of "Rate Functions":
In Lesson 4.1 you interpreted rate function models. We can now revisit that topic and include the concept of accumulation.



Examples:
The graph at right models the rate of rainfall in inches per hour from midnight until 6:00 A.M. during a tropical rainstorm.

6. Write a complete sentence to explain what Point A on the graph represents. Include numbers and units in your answer to this Example and Examples 8, 10, and 12.

7. What is the slope of the graph between Points B and C?

8. Write a complete sentence to explain the meaning of your answer to Example 7.

9. Find $\int_3^5 R(t)\,dt$.

10. Write a complete sentence to explain the meaning of your answer to Example 9.

11. Approximate the value of $\int_0^6 R(t)\,dt$ using geometrical regions. Show computations.

12. Write a complete sentence to explain the meaning of your answer to Example 11.

ASSIGNMENT 8.1

For Problems 1-5, find the average value of each function on the given interval.

1. $f(x) = x^3$ on $[0,2]$ No calculator.

2. $g(t) = \dfrac{1}{(t-1)^2}$ on $[2,5]$ No calculator.

3. $f(y) = 2y - \sqrt{y}$ on $[1,4]$ No calculator.

4. $f(t) = \dfrac{t^2-1}{\sqrt{t}+1}$ on $[.4, 3.2]$ Use a calculator.

5. $h(x) = .5^x$ on $[-2,1]$ Use a calculator.

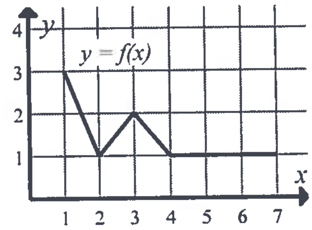
6. Find the exact x-value where the function in Problem 1 equals its average value.

7. Use a calculator to solve for c for the equation $h(c) = h_{avg}$ in Problem 5. Your answer should be expressed to 3 or more decimal place accuracy.

8. Find the length of the curve $f(x)=3x^2-2x+3$ on the interval $[1,4]$.

9. Find the curved length of one complete cycle of the graph of $y=\cos x$.

10. The graph of a function f is shown in the figure at right.

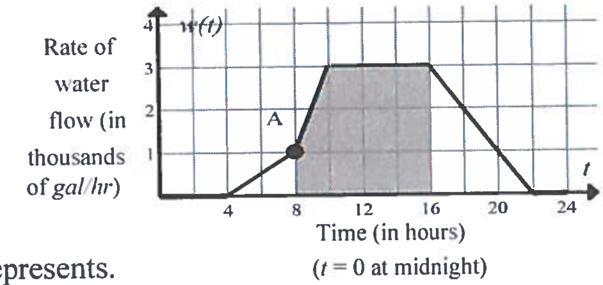


a. Evaluate $\int_1^7 f(x)\,dx$.

b. Determine the average value of the function on the interval $[1,7]$.

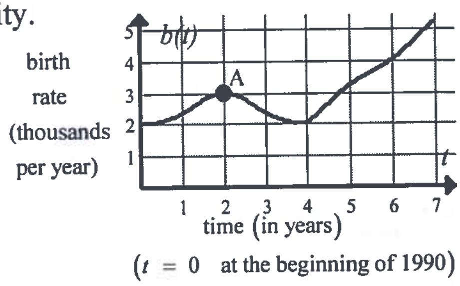
c. Determine the answers to Parts a. and b. if the graph is shifted two units upward.

11. The graph at the right represents the rate of flow of irrigation water (in thousands of gallons/hour) from a reservoir during a 24 hour period.



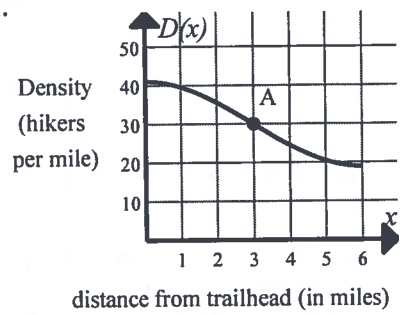
a. Write a sentence telling what Point A represents.
b. Find the shaded area. Show computations.
c. Write a sentence telling what the shaded area represents. Include numbers and units.
d. Set up an integral which represents the total amount of water released from the reservoir during the day shown.
e. Find the value of the integral in Part d.
f. Find the average rate of water flow during the 24 hours. Label units.

12. The graph at the right models the birth rate in a Utah city.



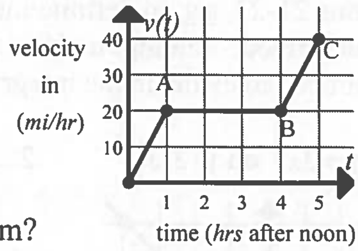
a. Tell what Point A represents.
b. Approximate $\int_2^6 b(t)\,dt$ using geometrical shapes. Show computations.
c. Tell what your answer to Part b. represents.
d. When was the birth rate the lowest, and what was the birth rate at that time?

13. The graph at the right represents the density of hikers on a trail.

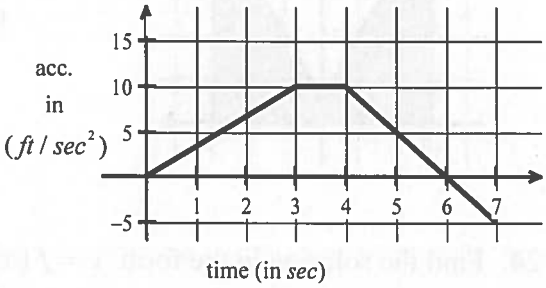


a. Tell what Point A represents.
b. $\int_0^3 D(x)\,dx = 110$. Write a sentence with numbers and units stating what this represents.
c. If $D'(3) = -6$, use <u>local linearization</u> to approximate the density of hikers 3.1 miles from the trailhead.

14. The graph at right models velocity.
    a. Tell what Point B represents.
    b. Tell what the slope between B and C represents.
    c. What is the velocity at 4:30 pm?
    d. What is the acceleration at 4:30 pm?
    e. What distance is traveled between noon and 5:00 pm?
    f. What is the average velocity between noon and 5:00 pm?



15. The graph at right models acceleration
    a. What is the acceleration at $t = 2\ sec$?
    b. When is the acceleration $10\ ft/sec^2$?
    c. What is the minimum acceleration?
    d. If the initial velocity is zero, what is the velocity at $t = 6\ sec$?
    e. If the initial velocity is $20\ ft/sec$, what is the velocity at $t = 6\ sec$?
    f. If the initial velocity is $20\ ft/sec$, what is the velocity at $t = 7\ sec$?



16. The rate at which oil is leaking from an old storage tank is modeled by the function $L(t) = 5.2e^{0.3t} - 4t$ where t is measured in hours after midnight. At the same time oil is being added to the tank at the rate of $A(t) = \left| 56\sin\left(\dfrac{3t^2+5}{6}\right) \right|$ where both functions are measured in liters per hour. At midnight there were 380 liters of oil in the tank.
    a. How much oil has leaked from the tank between midnight and 6 am?
    b. How much oil is left in the tank at 6 am?

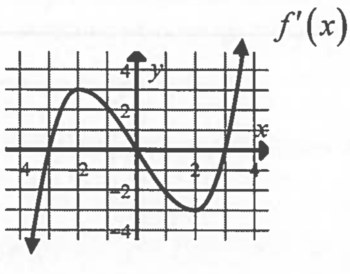
For Problems 17 and 18, sketch graphs and approximate the values of the intervals using
a. a Midpoint Riemann Sum. b. a Trapezoidal approximation
where $n =$ the number of equal subdivisions.

17. $\int_0^8 x^3\,dx$, $n = 4$

18. $\int_0^{12} -\sqrt{x}\,dx$, $n = 3$

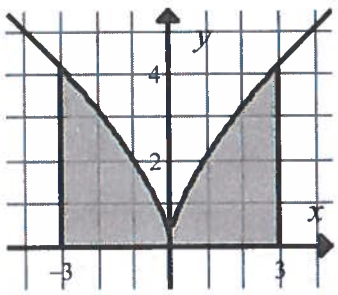
19. Two men in a search party begin walking from Search Headquarters at the same time. One man walks North at a rate of $4\ ft/sec$, while the other man walks West at a rate of $3\ ft/sec$. After both men have walked for one minute, find
    a. the distance separating the two men.
    b. the rate at which the distance between the two men is changing.

20. Use the graph of f' at right to sketch
    a. a graph of f''.
    b. a graph of f which passes through the origin of the coordinate plane.
    Use a separate coordinate plane for each graph.



For Problems 21-23, set up definite integrals which could be used to find the areas of the regions shown or described. Attempt to draw the graphs for Problems 22 and 23 without using a calculator. You do not need to evaluate the integrals that you set up.

21. $f(x)=2x^{\frac{2}{3}}$ on $[-3,3]$



22. Region bounded by

$y=\frac{x}{x^2+1}$,
$x=0$, $x=4$, and
$y=0$

23. Region bounded by

$g(y)=y^2+1$,
$x=0$, $y=-2$, and
$y=3$

24. Find the solution in the form $y=f(x)$ of the differential equation $\frac{x}{2}\frac{dy}{dx}-1=y^2$ if $f(1)=1$.

Selected Answers:

1. 2   2. $\frac{1}{4}$ 3. $\boxed{\frac{1}{3}\left(\left(16-\frac{16}{3}\right)-\left(1-\frac{2}{3}\right)\right)}=\frac{31}{9}$ 4. 1.105   5. 1.683

6. $x=\sqrt[3]{2}$ 7. c = –.751   8. 39.141   10 a. 8 b. $\frac{4}{3}$

11a. At 8:00 AM water is flowing from the reservoir at the rate of 1000 $\frac{gal}{hr}$. b. 22000 *gal*

11e. 33000 *gal*   12b. approximately 11000 c. Approximately 11,000 babies were born in the Utah city during 1992, 1993, 1994, and 1995.   13c. $D(3.1)\approx 29.4\frac{hikers}{mile}$

14c. $v(4.5)=30\frac{mi}{hr}$ 14e. 100 miles f. $v_{avg}=20\frac{mi}{hr}$ 15a. $\frac{20}{3}\frac{ft}{sec^2}$ d. $35\frac{ft}{sec}$ e. $55\frac{ft}{sec}$

15f. $52.5\frac{ft}{sec}$ 16a. 15.527 liters b. 595.531 or 595.532 liters

17b. $\boxed{\frac{1}{2}\bullet 2\left(0+2^3\right)+\frac{1}{2}\bullet 2\left(2^3+4^3\right)+\frac{1}{2}\bullet 2\left(4^3+6^3\right)+\frac{1}{2}\bullet 2\left(6^3+8^3\right)}=1088$

18a. $\boxed{4\left(-\sqrt{2}-\sqrt{6}-\sqrt{10}\right)}=-28.103$ or -28.104 19a. 300 ft. b. $5\frac{ft}{sec}$

21. $\int_{-3}^{3}2x^{\frac{2}{3}}\,dx$ or $2\int_{0}^{3}2x^{\frac{2}{3}}\,dx$ 23. $\int_{-2}^{3}\left(y^2+1\right)dy$ 24. $y=\tan\left(2\ln x+\frac{\pi}{4}\right)$

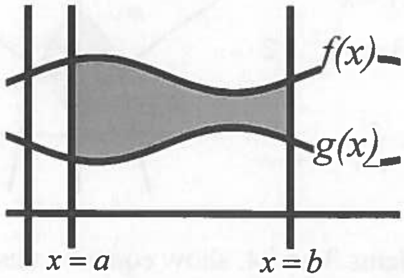
Lesson 8.2 Area Between Curves

$\int_a^b f(x)\,dx$ produces a value ("signed area") which may be positive, negative, or zero.

However, if you are asked to find an <u>actual area</u>, that area <u>cannot be negative</u>.

<u>Area of a Region Between Two Curves</u>

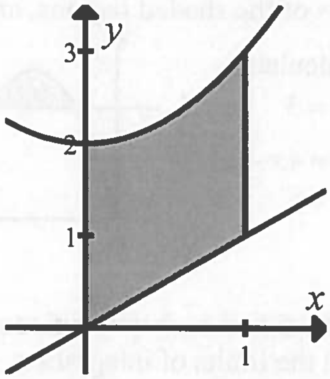
$A = \int_a^b \big(f(x) - g(x)\big)\,dx$ if $f(x)$ and $g(x)$ are continuous and $f(x) \ge g(x)$ on $[a, b]$.



For functions of x, $A = \int_a^b (\text{top curve} - \text{bottom curve})\,dx$

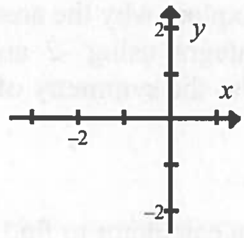
For functions of y, $A = \int_a^b (\text{right curve} - \text{left curve})\,dy$.

<u>Example 1</u>: Find the area of the region bounded by $y = x^2 + 2$, $y = x$, $x = 0$, and $x = 1$.

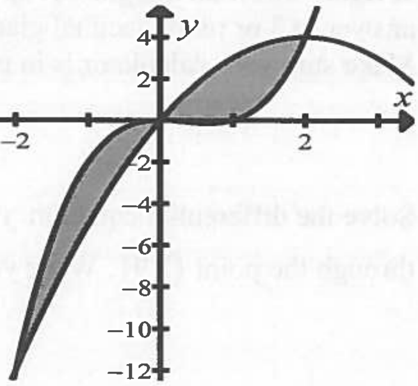


Sometimes you have to find where two curves intersect to determine "boundaries" for your region(s). These intersections will provide you with limits of integration for your integral(s). You <u>must</u> show an equation set up, even when using a calculator to find the intersections.

<u>Example 2</u>: (Functions of y)
Find the area of the region bounded by $x = y^2 - 3$ and $y = x + 1$.



<u>Example 3</u>: Set up integrals for the total area of the regions located between the two curves as shown. You may use a calculator.
$f(x) = x^3 - x^2$ and $g(x) = -x^2 + 4.1x$

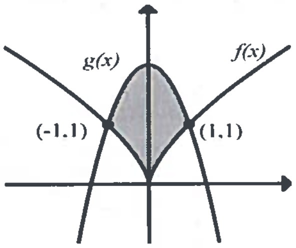


ASSIGNMENT 8.2

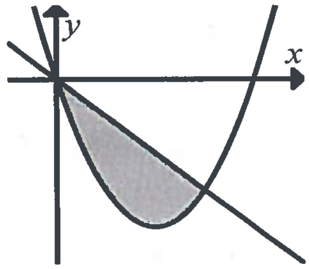
For Problems 1 and 2, set up integrals that could be used to find the areas of the shaded regions. Do not integrate. Show the equation(s) used to find the limits of integration for Problem 2 without using a calculator.

1. $f(x) = x^{\frac{2}{3}}$
   $g(x) = -x^2 + 2$

   $A = \int_{-1}^{1}$ ______

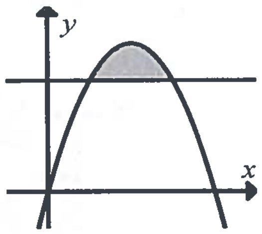


2. $f(x) = x^2 - 4x$
   $g(x) = -x$

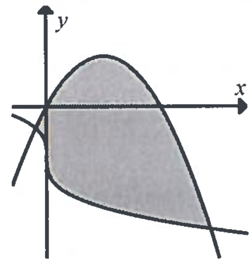


For Problems 3 and 4, show equations used to find the limits of integration, show integral set ups for the areas of the shaded regions, and then find the areas.

3. No calculator.
   $f(x) = 3$
   $g(x) = 4x - x^2$



4. Use a calculator.
   $f(x) = 2x - x^2$
   $g(x) = -\sqrt[3]{x} - 1$



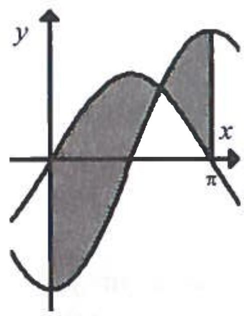
For Problems 5 and 6, sketch regions bounded by the graphs of the given equations, show equations used to find the limits of integration, show integral set ups, and find the areas.

5. Use a calculator.
   $y = 3^t$
   $y = \sqrt{t+2}$

6. No calculator. Hint: Write as functions of y. (Isolate x)
   $x = y^2 - 3$ $\quad$ $y = -\frac{1}{2}x$

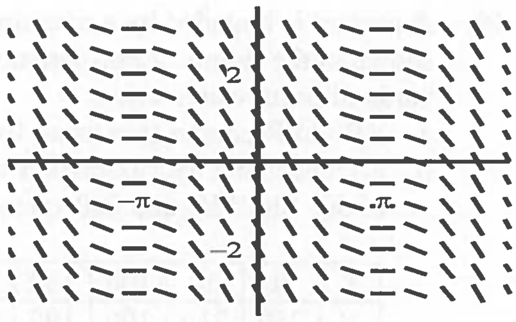
7. a. Sketch graphs of $y = -4x$ and $y = -x^3$ in one coordinate plane.
   b. Explain why the area bounded between the curves cannot be written as a single integral using -2 and 2 as limits of integration.
   c. Use the symmetry of the graphs to write a single integral for the area.

8. Use a calculator to find the shaded area between the curves $f(x) = 2\sin x$ and $g(x) = -3\cos x$ on $[0, \pi]$ as shown at right. Show an integral set up and express your final answer to 3 or more decimal place accuracy. Make sure your calculator is in radian mode.



9. Solve the differential equation $y' = -3x^2y^2$ to find the particular solution passing through the point $(1,9)$. Write your answer in the form $y = f(x)$.

10. Which of these differential equations corresponds to the slope field shown?



A. $\dfrac{dy}{dx}=-\sin x$ B. $\dfrac{dy}{dx}=\cos x+1$

C. $\dfrac{dy}{dx}=\sin x$ D. $\dfrac{dy}{dx}=-\cos x-1$

11. Find:
a. the instantaneous rate of change of $f(x)=-2x+\ln x$ at $x=1$ and at $x=2$.
b. the average rate of change of $f(x)=-2x+\ln x$ on the interval $[1,2]$.

12. Find the value of c where $f'(c)=\dfrac{f(b)-f(a)}{b-a}$, for $f(x)=-2x+\ln x$ on $[1,2]$.

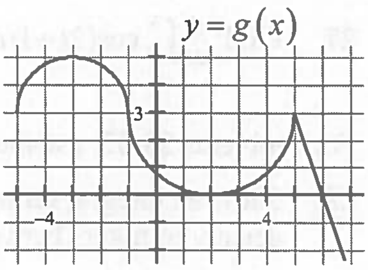
13. Find the average value of $f(x)=-2x+\ln x$ on $[1,2]$.

14. Without using a calculator, find the maximum value of $f(x)=-2x+\ln x$.

15. Use the equation of a tangent line at $x=1$ to approximate $f(.9)$ on the graph of $f(x)=-2x+\ln x$.

16. Approximate $\int_1^4(-2x+\ln x)\,dx$ by using a Trapezoidal approximation with 3 equal subdivisions.

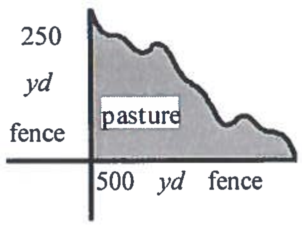
17. The graph at the right is of $y=g(x)$ consisting of two semicircles and a segment as shown. If $f(x)=\int_{-1}^{x}g(t)\,dt$ find each of the following:



a. an expression for $f'(x)$.
b. $f'(-1)$.
c. the interval(s) on which f is increasing. Justify.
d. the x-value where f has a local maximum. Justify.
e. the x-value(s) where $f''(x)$ is undefined. Explain.
f. the x-values of the point(s) of inflection of f. Justify.
g. the exact value of $f(6)$.
h. the exact value of $f(-5)$.

18. If $h'(x)=e^{3x^2-4}$ and $h(7)=9$, find $h(3)$.

19. A pasture is bounded by a river and two fences as shown in the figure. Approximate the total square yards of pasture area using
   a. a Right Riemann sum using five equal subdivisions.
   b. a Trapezoidal approximation with four unequal widths of 50, 100, 150, and 200 yards in that order.



x	0	50	100	150	200	250	300	350	400	450	500
y	250	210	190	190	170	120	80	60	60	40	0

Differentiate the functions in Problems 20 and 21.

20. $y = \dfrac{3x^3 - x + 1}{x^2}$

21. $f(x) = \left(x - \sqrt{x}\right)^{10}$

Antidifferentiate in Problems 22-23.

22. $y' = \dfrac{(x-2)^2}{\sqrt{x}}$

23. $f'(x) = x(x^2+3)^5$

24. Evaluate the following improper integral $\displaystyle\int_0^1 \frac{1}{\sqrt{3-x^2-2x}}\,dx$.

Find the indicated limits for Problems 25 and 26.

25. $\displaystyle\lim_{x\to 2^-} \frac{|x-2|}{x-2}$

26. $\displaystyle\lim_{x\to 2} \frac{|x-2|}{x-2}$

27. Find $\frac{d}{dx}\int_{x^3}^{x} \cos(2t+1)\,dt$ without integrating.

For Problems 28-32, use the graph and table at the right.

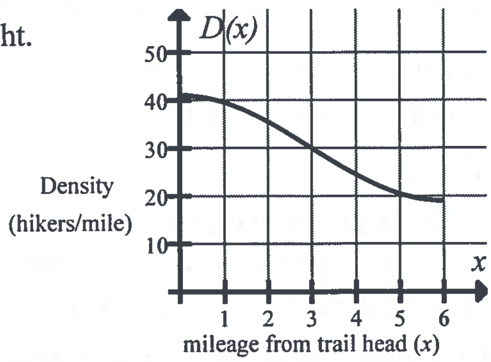
28. Show an integral set up for the average density of hikers between mile 1 and mile 5.

29. Use a Midpoint Riemann Sum with $n = 2$ to approximate this same average density.

30. Use four equal width trapezoids to approximate this average density on the same interval.

31. Estimate where on the trail the density of hikers is equal to this same average density.

32. Write a complete sentence using numbers and units to tell what $\int_1^5 D(x)\,dx$ represents.



x	1	2	3	4	5
D	40	35	30	25	20

33. Find the length of the curve on the graph of $y = 3x^2 + 5x - 1$ between the x-intercepts.

Selected Answers:

2. area $= \int_0^3 \left(g(x) - f(x)\right) dx$ 3. $\frac{4}{3}$ 4. 6.356   5. 1.167 or 1.168

6. $\boxed{\left(-1-\frac{1}{3}+3\right)-\left(-9+9-9\right)} = \frac{32}{3}$ 8. 7.211   9. $f(x) = \dfrac{1}{x^3 - \frac{8}{9}} = \dfrac{x^2}{9x^3 - 8}$

10. D   11a. $f'(1) = -1$, $f'(2) = -\frac{3}{2}$ 12. $c = \frac{1}{\ln 2} = 1.442$ or 1.443

13. -2.613 or -2.614 14. $-1 + \ln\frac{1}{2}$ 15. $f(.9) \approx -1.9$

16. $\boxed{\frac{1}{2} \bullet 1\left[\left(-2+\ln 1\right)+2\left(-4+\ln 2\right)+2\left(-6+\ln 3\right)+\left(-8+\ln 4\right)\right]} = -12.515$

17a. $f'(x) = g(x)$ c. f is increasing on the interval $[-5,6]$ because $g \geq 0$.

17f. f has points of inflection at $x = -3, 2, 5$ because g changes between incr. and decr.

17g. $\frac{39}{2} - \frac{9}{2}\pi$

19b. $\boxed{\frac{1}{2} \bullet 50(250+210) + \frac{1}{2} \bullet 100(210+190) + \frac{1}{2} \bullet 150(190+80) + \frac{1}{2} \bullet 200(80+0)} = 59750$

21. $f'(x) = 10\left(x - \sqrt{x}\right)^9 \left(1 - \frac{1}{2}x^{-\frac{1}{2}}\right)$ 22. $y = \frac{2}{5}x^{\frac{5}{2}} - \frac{8}{3}x^{\frac{3}{2}} + 8x^{\frac{1}{2}} + C$

23. $f(x) = \frac{1}{12}\left(x^2+3\right)^6 + C$ 24. $\frac{\pi}{3}$ 27. $\cos(2x+1) - 3x^2\cos\left(2x^3+1\right)$

28. $D_{avg} = \dfrac{\int_1^5 D(x)\,dx}{5-1}$ 29. $D_{avg} \approx 30 \dfrac{hikers}{mile}$ 30. $D_{avg} \approx 30 \dfrac{hikers}{mile}$

31. 3 miles from the trailhead   32. $\int_1^5 D(x)\,dx$ represents the total number of hikers on the trail at a distance between 1 and 5 miles from the trailhead.   33. 6.666 or 6.667

Lesson 8.3 Volumes of Solids with Known Cross Sections (including discs and washers)

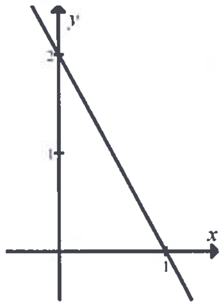
Area formulas for common cross sections:

Square	Rectangle	Semicircle	Triangle	Equilateral Triangle
$A = s^2$	$A = bh$	$A = \frac{1}{2}\pi r^2$	$A = \frac{1}{2}bh$	$A = \frac{\sqrt{3}}{4}s^2$

Volumes of Solids with Known Cross Sections
For cross sections <u>perpendicular to the x-axis:</u>
$V = \int_a^b A\,dx$, where A is a function of x and gives the area of a representative cross section.

For cross sections <u>perpendicular to the y-axis:</u>
$V = \int_a^b A\,dy$, where A is a function of y.

<u>Examples</u>:



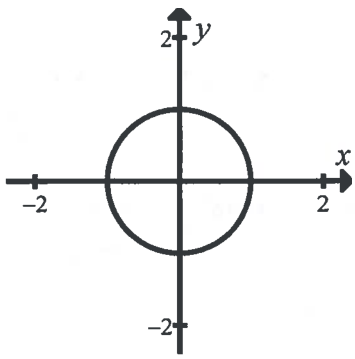
1. Find the volume of the solid whose base is a triangle bounded by $y = -2x + 2$, $x = 0$, and $y = 0$, and whose cross sections are <u>squares</u> which are perpendicular to the x-axis.

$$V = \int_a^b A\,dx$$

$$V = \int_0^1 s^2\,dx =$$

2. <u>Set up</u> (but do not integrate) integrals for the volumes of the solids with the same base as in Example 1, but whose cross sections are semicircles perpendicular to the x-axis.

3. <u>Set up</u> (but do not integrate) an integral for the volume of a solid whose base is bounded by $y = -x^2 + 2$ and $y = x$ and whose cross sections are rectangles of height $\frac{1}{4}$ perpendicular to the x-axis.



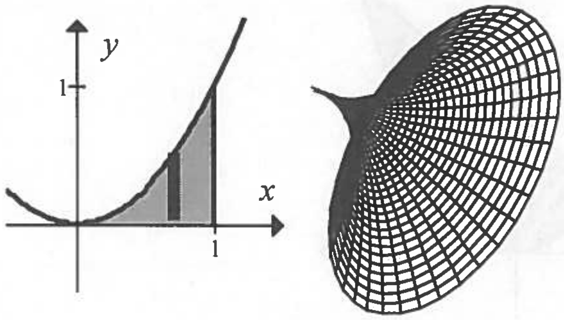
4. <u>Set up</u> integrals for the volumes of the solids whose base is the circle $x^2 + y^2 = 1$ and whose cross sections are:
   a. Equilateral triangles perpendicular to the y-axis.

   b. Rectangles whose heights are three times their bases and whose bases are perpendicular to the y-axis.

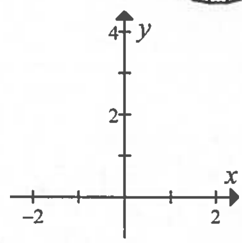
Circular cross sections can be formed by revolving very thin (essentially no width) rectangles about an axis of revolution. These circular cross sections are more commonly called discs.

Disc Formula:	$V = \pi \int_a^b r^2 \ (dx \text{ or } dy)$

Example 5: Set up an integral for the volume of the solid formed by revolving the region bounded by $y = x^2$, $y = 0$, and $x = 1$ about the x-axis.



Example 6: Find the volume of the solid formed by revolving the region in Quadrant I bounded by $y = x^2$, $x = 0$, and $y = 4$ about the y-axis.

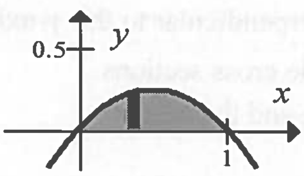


When a region is revolved about a line which is not one of its boundaries, its volume is formed from a sum of volumes of washers (at least some of the discs have holes in them).

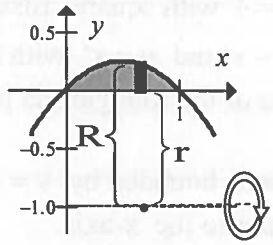
Washer Formula:	$V = \pi \int_a^b (R^2 - r^2) \ (dx \text{ or } dy)$	R = Outer radius (from the axis of revolution)
		r = Inner radius (from the axis of revolution)

Example 7: Set up integrals for the volumes of the solids formed by revolving the region bounded by $y = -x^2 + x$ and $y = 0$

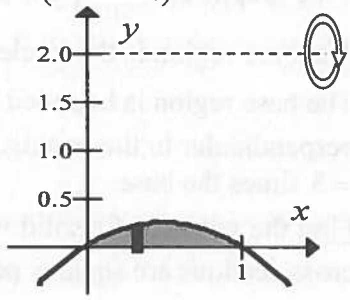
a. about the x-axis
(Discs)



b. about $y = -1$
(Washers)

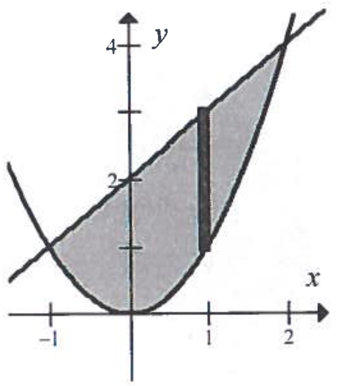


c. about $y = 2$
(Washers)

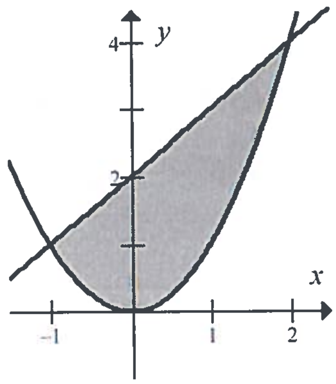


Example 8: Set up integrals for the volumes of the solids formed by revolving the region bounded by $y=x^2$ and $y=x+2$

a. about the x-axis



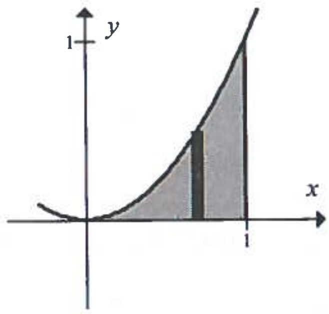
b. about the line $y=4$



Assignment 8.3

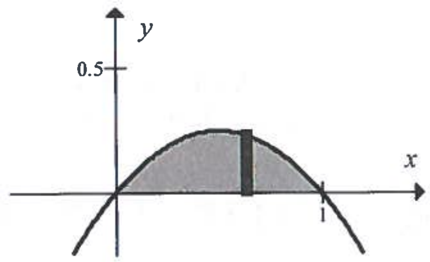
Set up (but do not integrate) integrals for evaluating the volumes of the solids formed by the given cross sections.

1. Base region bounded by $y=x^2$, $y=0$, and $x=1$ as shown.



   a. Squares
   b. Semicircles
   c. Rectangles with height 2

2. Base region bounded by $y=-x^2+x$ and $y=0$ as shown.



   a. Squares
   b. Equilateral triangles
   c. Rectangles whose heights are half of their bases

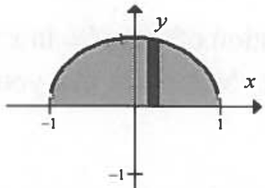
Sketch the graphs of the regions and set up the integrals used for finding the volumes.

3. The base region is the circle $x^2+y^2=4$ with square cross sections perpendicular to the y-axis.
4. The base region is bounded by $y=2-x$ and $y=x^2$ with right triangle cross sections perpendicular to the x-axis. The base of the triangle sits in the region and the height $=3$ times the base.
5. Find the volume of a solid whose base is bounded by $y=\sqrt{x}$, $y=0$, and $x=4$, and whose cross sections are squares perpendicular to the x-axis.

6. <u>Find</u> the volume of a solid whose base is bounded by $y = 2x$, $y = 0$, and $x = 3$, and whose cross sections are semicircles perpendicular to the x-axis.

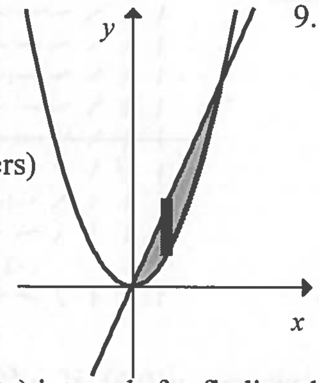
Use discs or washers to <u>set up</u> (but not integrate) integrals for finding volumes of the solids described.

7. Region bounded by $y = \sqrt{1 - x^2}$ and $y = 0$
   a. revolved about the x-axis
   b. revolved about $y = -1$



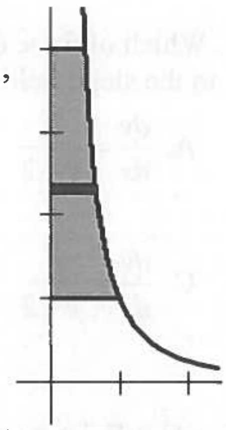
8. Region bounded by

   $y = x^2$ and $y = 2x$
   a. revolved about the x-axis (Washers)
   b. revolved about $y = 5$ (Washers)



9. Region bounded by $y = \dfrac{1}{x^2}$,

   $x = 0$, $y = 1$, and $y = 4$
   a. revolved about the y-axis (Discs)
   b. revolved about $x = -1$ (Washers)



<u>Set up</u> (but do not integrate) integrals for finding the volumes of the solids described. <u>Sketch</u> each region and show at least one representative rectangle.

10. Region bounded by
    $y = \dfrac{1}{x}$, $y = 0$, $x = 1$, and $x = 4$
    a. revolved about the x-axis
    b. revolved about $y = -3$
    c. revolved about $y = 5$

11. Region bounded by
    $y = \sqrt{x}$, $x = 0$, and $y = 2$
    a. revolved about the y-axis
    b. revolved about $x = -2$
    c. revolved about $x = 4$

12. Without using a calculator, <u>find the volume</u> of the solid formed when the region bounded by $y = \sqrt[3]{x}$, $x = 0$, and $y = 2$ is revolved about:
    a. the x-axis
    b. the y-axis

13. Sketch the region, and use a calculator to <u>find</u> the volumes of the solids formed when the region bounded by $f(x) = 2^{x-1}$ and $g(x) = 2 - x^2$ is revolved about:
    a. the x-axis
    b. $y = 3$

14. Sketch the region, and <u>set up</u> integrals for the volumes of the solids formed by the given cross sections. The base region is bounded by $f(x) = |x|$, $y = 0$, $x = -2$, and $x = 3$.
    a. squares perp. to the x-axis
    b. semicircles perp. to the x-axis

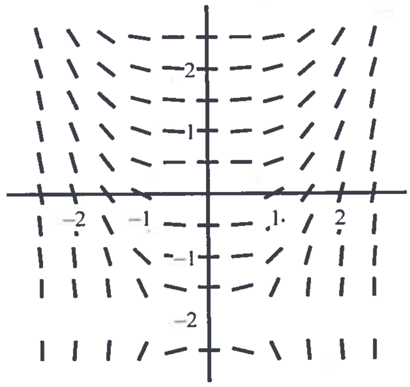
Sketch the regions bounded by the given equations, and find the areas of the regions. Show your integral set ups, and do not use a calculator.

15. $f(x) = 2x^2 + 3x$ and $g(x) = 2$

16. $f(y) = y^2 - 2y$ and $g(y) = 2 - y$

17. Find the particular solution of $2xyy' = \ln x$, if the graph of the particular solution contains the point $(e,1)$. Make sure that your answer expresses y as a function of x. $(y = f(x))$

18. Which of these differential equations corresponds to the slope field shown?



A. $\frac{dy}{dx} = \frac{x^2}{y+2}$ B. $\frac{dy}{dx} = \frac{x^2}{y-2}$

C. $\frac{dy}{dx} = \frac{x^3}{y-2}$ D. $\frac{dy}{dx} = \frac{x^3}{y+2}$

19. Use Euler's method with three steps to approximate $y(0.6)$ if $y(0) = 2$ on the solution of the differential equation $y' = x + y$. Do **not** use a calculator.

20. If the growth rate of lowland gorillas in a wild animal preserve is modeled by the differential equation $\frac{dP}{dt} = .0004P(250 - P)$ where t is measured in years.

a. Find $\lim_{t\to\infty} P(t)$.

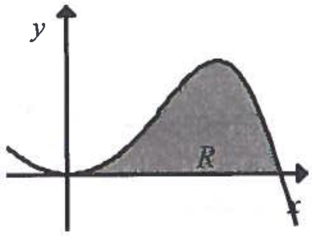
b. Find $\lim_{t\to\infty} \frac{dP}{dt}$.

c. If 50 gorillas are introduced what is the population when it is growing the fastest?

d. If 300 gorillas are introduced what conclusion can be made about $\frac{dP}{dt}$?

Use a calculator for Problems 21-23

21. Region R is shown bounded by the graph of $y = \sin x^2$ and the x-axis. Find the perimeter of region R.



22. Find the average (mean) value of $f(x) = \log(x^2 + 5)$ on [1, 4].

23. Find the x-value on the interval [1, 4] which produces the average value in Problem 22.

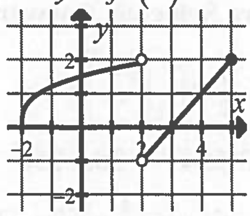
Differentiate the functions in Problems 24 and 25.

24. $g(y) = y^3(2y-1)^{\frac{2}{3}}$

25. $h(t) = \frac{2t-1}{3t^2+5}$

26. Use the graph of $y = f'(x)$ shown at the right to sketch:
    a. a graph of $f''(x)$.
    b. a graph of $f(x)$ that is continuous on the interval $[-2,5]$ and which contains the point $(2,1)$.

$y = f'(x)$



27. $v(t) = \dfrac{t-1}{t^2+1}$ is the velocity equation for an object moving along a horizontal path when $t \geq 0$.
    Use a calculator to find:
    a. the velocity of the object at $t = 2.3$
    b. the acceleration of the object at $t = 2.3$
    c. the displacement of the object from $t = 0$ to $t = 3$.
    d. the total distance traveled by the object from $t = 0$ to $t = 3$.

Show integral set ups for Parts c. and d.

28. If the graph of $f(x)$ is concave upward on (a,b), does a Trapezoidal approximation give a value larger or smaller than the actual value of $\int_a^b f(x)\,dx$?

29. If $f''(x) < 0$ on (a,b), does a Trapezoidal approximation give a value larger or smaller than the actual value of $\int_a^b f(x)\,dx$?

30. If $f''(x) = 0$ on (a,b), what must be true about the value found using a Trapezoidal approximation and the actual value of $\int_a^b f(x)\,dx$?

Selected Answers:

1a. $V = \int_0^1 \left(x^2\right)^2 dx$ b. $V = \int_0^1 \frac{1}{2}\pi\left(\frac{1}{2}x^2\right)^2 dx$ c. $V = \int_0^1 2x^2\,dx$

2a. $V = \int_0^1 \left(-x^2+x\right)^2 dx$ 3. $V = \int_{-2}^2 \left(2\sqrt{4-y^2}\right)^2 dy$

5. 8 6. $\frac{9}{2}\pi$ 7a. $\pi\int_{-1}^1 \left(\sqrt{1-x^2}\right)^2 dx$ b. $\pi\int_{-1}^1 \left(\left(1+\sqrt{1-x^2}\right)^2 - 1^2\right) dx$

8a. $\pi\int_0^2 \left((2x)^2 - \left(x^2\right)^2\right) dx$ b. $\pi\int_0^2 \left(\left(5-x^2\right)^2 - (5-2x)^2\right) dx$

9b. $\pi\int_1^4 \left(\left(1+\frac{1}{\sqrt{y}}\right)^2 - 1^2\right) dy$ 10b. $\pi\int_1^4 \left(\left(3+\frac{1}{x}\right)^2 - 3^2\right) dx$ c. $\pi\int_1^4 \left(5^2 - \left(5-\frac{1}{x}\right)^2\right) dx$

11a $\pi\int_0^2 \left(y^2\right)^2 dy$ b. $\pi\int_0^2 \left(\left(2+y^2\right)^2 - 2^2\right) dy$ c. $\pi\int_0^2 \left(4^2 - \left(4-y^2\right)^2\right) dy$

12a. $\pi\left(32 - \frac{3}{5}\bullet 8^{\frac{5}{3}}\right)$ b. $\pi\frac{2^7}{7}$ 13a. 16.302 or 16.303 b. 28.688 or 28.689

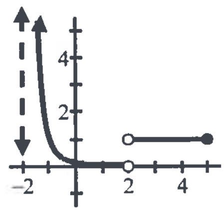
More Selected Answers:

14b. $\frac{1}{2}\pi\int_{-2}^{3}\left(\frac{|x|}{2}\right)^2 dx$ 15. $\boxed{\left(1-\frac{1}{12}-\frac{3}{8}\right)-\left(-4+\frac{16}{3}-6\right)}=\frac{125}{24}$ 17. $y=\sqrt{\frac{1}{2}(\ln x)^2+\frac{1}{2}}$

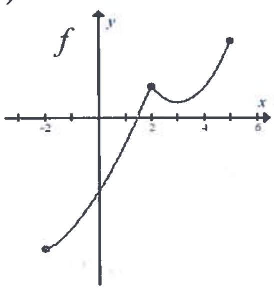
19. 3.584 20a. 250 c. 125 21. 4.602 22. 1.049 or 1.050 23. 2.492

24. $g'(y)=\frac{4}{3}y^3(2y-1)^{-\frac{1}{3}}+3y^2(2y-1)^{\frac{2}{3}}$

26a. f''



b. f



27a. $v(2.3)=.206$ or $.207$ b. $a(2.3)=.007$ or $.008$ c. disp. = $-.097$ or $-.098$

27d. T. D. = .779 or .780 28. larger 29. smaller 30. They are equal.

UNIT 8 SUMMARY

Area Between Two Curves: $A=\int_a^b\left(f(x)-g(x)\right)dx$ or $A=\int_a^b\left(f(y)-g(y)\right)dy$

top curve – bottom curve right curve – left curve

Arc Length Formula: Arc Length $=\int_a^b\sqrt{1+\left(f'(x)\right)^2}\,dx$

Volumes with Known Cross Sections: $V=\int_a^b A\,dx$

$A=$ area of cross section

Volumes of Revolution (Discs): $V=\pi\int_a^b r^2\,dx$

Volumes of Revolution (Washers): $V=\pi\int_a^b\left(R^2-r^2\right)dx$

Average Value of a Function: $f_{avg}=\dfrac{\int_a^b f(x)\,dx}{b-a}$

Lesson 9.1 Series Definitions, Geometric Series, n^{th} Term Test

Factorial Definition: $n! = n(n-1)(n-2)(n-3)\cdots 1$

Example 1: Simplify $\dfrac{(n+2)!}{n!} =$

Definition: A **series** is a **sum** of numbers.

An infinite series can be represented as $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$

Example 2: Write out the first five terms of the series $\sum_{n=1}^{\infty} \dfrac{n}{n+1} =$

Examples: Write an expression for the nth term of the following series.

3. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$

4. $\dfrac{3}{1} + \dfrac{3}{2} + \dfrac{3}{6} + \dfrac{3}{24} + \dfrac{3}{120} + \cdots$

Examples: Write an expression for the following series using sigma notation.

5. $\dfrac{2}{1} + \dfrac{4}{3} + \dfrac{6}{5} + \dfrac{8}{7} + \cdots =$

6. $2 - 6 + 18 - 54 + 162 + \cdots =$

Example 7: What happens if we add more and more terms of a series like

$\sum_{n=0}^{\infty} (2n+1) = 1 + 3 + 5 + 7 + \cdots$ Show a sequence of partial sums.

This sequence of partial sums is approaching infinity. When this happens, the series is called **divergent**.

Example 8: What happens if we add more and more terms of $\sum_{n=1}^{\infty} \dfrac{3}{10^n} =$

This is an example of a **convergent** geometric series.

Geometric Series:
If consecutive terms in a series have a common ratio r, the series is called a **geometric series.**

$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots$ is the general form of a geometric series.

If the geometric series converges it is possible to find the sum even though it has infinitely many terms.

Let $S = a + ar + ar^2 + ar^3 + \cdots$

then $rS = \quad ar +$

subtracting $S - rS =$

factoring $S(1-r) =$

$S =$

first term!

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$ if the geometric series converges.

The Geometric Series Test: for a geometric series $\sum_{n=0}^{\infty} ar^n$

If $|r| \geq 1$ the geometric series **diverges**. If $|r| < 1$ the series **converges**.

Examples: Determine if these series converge or diverge and, if possible, find the sum of the series.

9. $\sum_{n=0}^{\infty} \frac{3}{2^n}$

10. $\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$

11. $\sum_{n=1}^{\infty} 4\left(-\frac{1}{2}\right)^n$

Example 12: Find the fraction form of the repeating decimal $.\overline{08}$ using a geometric series.

Example 13: The series $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right) = (1+1)+\left(1+\frac{1}{2}\right)+\left(1+\frac{1}{3}\right)+\cdots$ does **not** converge.

Show a sequence of partial sums.

A series **cannot** converge unless the terms approach a limit of **zero**.

n^{th} Term Test for Divergence:

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges. This test is inconclusive if $\lim_{n\to\infty} a_n = 0$.

Example 14: Show that $\sum_{n=1}^{\infty} \frac{n!}{2n!+1}$ diverges.

Assignment 9.1

Simplify without using a calculator.

1. $\dfrac{7!}{10!}$ 2. $\dfrac{(2n+1)!}{(2n-1)!}$

Write an expression for the *nth* term of each series. Use $n = 1,2,3,\cdots$

3. $\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\frac{6}{5}+\cdots$ 4. $-1+\frac{1}{4}-\frac{1}{9}+\frac{1}{16}-\cdots$ 5. $\frac{2}{1}+\frac{4}{3}+\frac{8}{7}+\frac{16}{15}+\cdots$ 6. $\frac{3}{1}+\frac{3}{2}+\frac{3}{6}+\frac{3}{24}+\cdots$

Use sigma notation to write an equivalent expression for each series. Use $n = 0,1,2,3,\cdots$

7. $\frac{1}{2}+\frac{x}{6}+\frac{x^2}{24}+\frac{x^3}{120}+\cdots$ 8. $-1+1+3+5+\cdots$ 9. $\frac{1}{1}+\frac{4}{3}+\frac{9}{9}+\frac{16}{27}+\cdots$

Determine whether each of the following infinite series converges or diverges. Show justification and name the test being used. In addition, find the sum of the series, if possible.

10. $\displaystyle\sum_{n=0}^{\infty}5\left(\frac{2}{3}\right)^n$ 11. $\displaystyle\sum_{n=1}^{\infty}5\left(\frac{2}{3}\right)^n$ 12. $\displaystyle\sum_{n=0}^{\infty}5\left(\frac{3}{2}\right)^n$ 13. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+2}{n^2}$

14. $\displaystyle\sum_{n=2}^{\infty}\frac{n^2}{\ln n}$ 15. $5+\frac{5}{2}+\frac{5}{4}+\frac{5}{8}+\cdots$ 16. $3+\frac{9}{2}+\frac{27}{4}+\frac{81}{8}+\cdots$ 17. $1+0.2+0.04+0.008+\cdots$

18. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{\sin n}$ 19. $\displaystyle\sum_{n=1}^{\infty}\frac{3^n+2}{3^{n+2}}$ 20. $\displaystyle\sum_{n=0}^{\infty}\frac{e^n}{\pi^{n+1}}$ 21. $\frac{1}{2}+\frac{2}{4}+\frac{6}{8}+\frac{24}{16}+\frac{120}{32}+\cdots$

22. $\displaystyle\sum_{n=1}^{\infty}\left(\sin e^{10}\right)^n$ 23. $\displaystyle\sum_{n=1}^{\infty}\frac{(-5)^n}{6}$ 24. $\displaystyle\sum_{n=1}^{\infty}\left(-\frac{5}{6}\right)^n$ 25. $18-12+8-\frac{16}{3}+\frac{32}{9}-\cdots$

26. $\displaystyle\sum_{n=1}^{\infty}\frac{2n+3}{3n+2}$ 27. $\displaystyle\sum_{n=0}^{\infty}\frac{n!}{e^n}$ 28. $\displaystyle\sum_{n=0}^{\infty}\frac{4}{3^n}$ 29. $\displaystyle\sum_{n=1}^{\infty}4^{-n}$

30. Given the series $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+2}{n^3}$:

a. Find $\lim\limits_{n\to\infty} a_n$.

b. Explain why the *nth* Term Test **cannot** be used to conclude the series converges.

31. Find the x- and y-intercepts, relative extrema, and points of inflection for $y=\sin x+\cos x$ on $[0,2\pi)$. Then sketch the graph of y without using a calculator.

Differentiate implicitly to find $\frac{dy}{dx}$.

32. $\cos(y-x)=x^3+2$ 33. $2xy=\tan y^2$

Find the limits without using a calculator.

34. $\displaystyle\lim_{x\to\frac{\pi}{2}}\left(\frac{\cos x}{x-\frac{\pi}{2}}\right)$ 35. $\displaystyle\lim_{x\to 0}\frac{\sin(2x)}{\sin(-5x)}$ 36. $\displaystyle\lim_{x\to 0}\frac{\sin(3x)}{x}$ 37. $\displaystyle\lim_{t\to 0}(2t\sec t)$

38. Without using a calculator find $\int_0^{\infty} \frac{e^x}{1+e^x}\,dx$.

For Problems 39-42, a region is in the 1st quadrant bounded by $y=3\cos(2x)$, $y=3x$, and $x=0$.

39. Use a calculator to find the area of the region. Show an integral set up and an answer

40. Set up (but do not integrate) an integral for finding the volume of the solid formed by revolving the region about the x-axis.

41. Set up (but do not integrate) an integral for finding the volume of the solid formed by revolving the region about $y=4$.

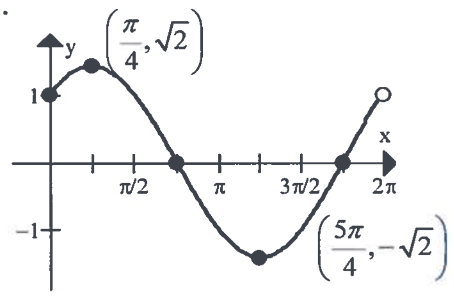
42. Set up (but do not integrate) an integral for finding the volume of the solid formed by using rectangular cross sections whose bases are in the region and are perpendicular to the x-axis. The heights of the rectangles are always half of their bases.

Selected Answers:

1. $\frac{1}{720}$ 4. $\frac{(-1)^n}{n^2}$ 6. $\frac{3}{n!}$ 7. $\sum_{n=0}^{\infty} \frac{x^n}{(n+2)!}$ 10. converges by GST, Sum = 15

11. conv. by GST, Sum = 10 12. div. by GST or nTT 14. div. by nTT
15. conv. by GST, Sum = 10 16. div. by GST or nTT 17. conv. by GST, Sum = 1.25
18. div. by nTT 20. conv. by GST, Sum = $\frac{1}{\pi - e}$ 22. conv. by GST, Sum = -.407 or -.408
23. div. by GST or nTT 24. conv. by GST, Sum = $-\frac{5}{11}$ 26. div. by nTT
27. div. by nTT 29. conv. by GST, Sum = $\frac{1}{3}$

31.



32. $y' = \frac{3x^2}{-\sin(y-x)} + 1 = \frac{3x^2 - \sin(y-x)}{-\sin(y-x)}$

33. $y' = \frac{-2y}{2x - 2y\sec^2 y^2}$ 34. -1 36. 3

38. diverges 39. .888 41. $\pi\int_0^{.5149}\left((4-3x)^2 - (4-3\cos(2x))^2\right)dx$

42. $\int_0^{.5149} \frac{1}{2}\left(3\cos(2x) - 3x\right)^2 dx$

Lesson 9.2 Power Series, Geometric Power Series, Integration and Differentiation of Power Series

A series with variable terms like $1+x+x^2+x^3+\cdots+x^n+\cdots$ is called a **power series.** Note that this series is a **geometric power series.** If it converges, what must be true about the variable x?

For these x-values $1+x+x^2+x^3+\cdots+x^n+\cdots=$

This means for these x-values, the function $f(x)=\dfrac{1}{1-x}$ can be written as $f(x)=\sum_{n=0}^{\infty}x^n$.

Examples: Find a power series for each of the following functions. Show four terms and the general term. Also give the series using sigma notation and give the interval of convergence.

1. $f(x)=\dfrac{1}{1+x}$

2. $g(x)=\dfrac{3}{1-2x}$

3. $h(x)=\dfrac{1}{3x}$

Power Series by Substitution:

Examples:

4. Using the power series from Example 1, make a new power series for $f(x^2)$.

5. Using the power series from Example 2, make a new series for $g(\sqrt{x})$.

Power Series by Differentiation:

Examples:

6. Use the power series for $f(x)=\dfrac{1}{1-x}=1+x+x^2+x^3+\cdots$ above to write a power series for $f'(x)$.

7. If $j(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots+\frac{x^n}{n!}+\dots$ find $j'(x)$.

Can you identify the function $j(x)$?

Power Series by integration:

8. Since $\frac{1}{1+t}=1-t+t^2-t^3+\dots+(-1)^n t^n+\dots$ (see Example 1)

$\int_0^x \frac{1}{1+t}dt=\int_0^x \left(1-t+t^2-t^3+\dots+(-1)^n t^n+\dots\right)dt$ Now integrate both sides.

9. Use the result of Example 8 to write a power series for $f(x)=\ln(x)$.

10. Now write a series for $g(x)=x^2\ln x$.

Assignment 9.2

Find a geometric power series for each of the following functions. Show four terms and the general term. Also give the series using sigma notation and give the interval of convergence.

1. $\frac{1}{1-3x}$ 2. $\frac{2}{1-x^3}$ 3. $\frac{x}{1+x}$
4. $\frac{1}{1+(-x-3)}$ 5. $\frac{3}{4x}$ 6. $\frac{1}{2-2x}$

Find a function for each of the following geometric power series. Also give the interval of convergence.

7. $\sum_{n=0}^{\infty}(2x)^n$ 8. $\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n x^n$ 9. $\sum_{n=0}^{\infty}4(x-1)^n$
10. $\sum_{n=1}^{\infty}(x^2)^n$ 11. $\sum_{n=0}^{\infty}(\sin x)^n$

12. Use the result of Example 7 at the top of the previous page to write a power series for $f(x)=e^{x^2}$. Show four terms and the general term.

13. Use the result of Example 7 on the previous page to write a power series for $g(x)=xe^x$. Show four terms and the general term.

14. Find a geometric power series for $g(x)=\dfrac{1}{1+x}$. Show four terms and the general term.

15. Use the answer to Problem 14 to find a power series for $\dfrac{1}{1+x^2}$.

16. Use integration to find a power series for $\arctan x$.

Use the function $f(x)=\sum_{n=1}^{\infty}\dfrac{x^n}{n}$ to find the following. Answer using sigma notation.

17. $f(-x)$ 18. $f'(x)$ 19. $\int_0^t f(x)\,dx$

Use the function $g(x)=1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\cdots+\dfrac{(-1)^n x^{2n}}{(2n)!}+\cdots$ to find the following. Show four terms and the general term.

20. $g(\sqrt{x})$ 21. $g'(x)$ 22. $\int_0^x g(t)\,dt$

23. Use the power series for $f(x)=\ln(x)$ in Example 9 on the previous page to find a simplified answer for $f(1)$.

Determine whether the following series converge or diverge. Find the sum when possible without using a calculator.

24. $\sum_{n=0}^{\infty}\left(\dfrac{e}{\pi}\right)^n$ 25. $\sum_{n=0}^{\infty}\left(\dfrac{\pi}{e}\right)^n$ 26. $\sum_{n=0}^{\infty}\dfrac{2n+1}{n+1}$ 27. $\sum_{n=2}^{\infty}\dfrac{n!}{(n-2)!}$

Use sigma notation to write an equivalent expression for each series. Use $n=1, 2, 3, \cdots$

28. $3+7+11+15+\cdots$ 29. $1-\frac{1}{2}+\frac{1}{6}-\frac{1}{24}+\frac{1}{120}-\cdots$ 30. $2+6+18+54+\cdots$

Integrate the following without using a calculator:

31. $\int t^2\tan(t^3)\,dt$ 32. $\int\dfrac{2x}{9+x^2}\,dx$ 33. $\int\dfrac{\ln x+3}{x}\,dx$ 34. $\int\dfrac{(e^{2y}-1)^2}{e^y}\,dy$

35. $\int\dfrac{dx}{x\sqrt{\ln x}}$ 36. $\int\dfrac{e^x+e^{-x}}{e^x-e^{-x}}\,dx$ 37. $\int\dfrac{x^2-2x-1}{x-1}\,dx$

Selected Answers:

1. $1+3x+9x^2+27x^3+\cdots+(3x)^n+\cdots=\sum_{n=0}^{\infty}(3x)^n$; $\left(-\frac{1}{3},\frac{1}{3}\right)$

2. $2+2x^3+2x^6+2x^9+\cdots+2\left(x^3\right)^n+\cdots=\sum_{n=0}^{\infty}2\left(x^3\right)^n$; $(-1,1)$

3. $x-x^2+x^3-x^4+\cdots+(-1)^n x^{n+1}+\cdots=\sum_{n=0}^{\infty}(-1)^n x^{n+1}=\sum_{n=1}^{\infty}(-1)^{n+1}x^n$; $(-1,1)$

5. $3+3(1-4x)+3(1-4x)^2+3(1-4x)^3+\cdots+3(1-4x)^n+\cdots=\sum_{n=0}^{\infty}3(1-4x)^n$; $\left(0,\frac{1}{2}\right)$

6. $\frac{1}{2}+\frac{1}{2}x+\frac{1}{2}x^2+\frac{1}{2}x^3+\cdots+\frac{1}{2}x^n+\cdots=\sum_{n=0}^{\infty}\frac{1}{2}x^n$; $(-1,1)$

7. $\dfrac{1}{1-2x}$; $\left(-\frac{1}{2},\frac{1}{2}\right)$ 8. $\dfrac{1}{1+\frac{1}{2}x}=\dfrac{2}{2+x}$; $(-2,2)$ 9. $\dfrac{4}{1-(x-1)}=\dfrac{4}{2-x}$; $(0,2)$

10. $\dfrac{x^2}{1-x^2}$; $(-1,1)$ 12. $e^{x^2}=1+x^2+\dfrac{x^4}{2!}+\dfrac{x^6}{3!}+\cdots+\dfrac{x^{2n}}{n!}+\cdots$

14. $1-x+x^2-x^3+\cdots+(-1)^n x^n+\cdots$ 15. $1-x^2+x^4-x^6+\cdots$

16. $x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+\cdots$ 17. $f(-x)=\sum_{n=1}^{\infty}\dfrac{(-1)^n x^n}{n}$ 18. $f'(x)=\sum_{n=1}^{\infty}x^{n-1}$

19. $\int_0^t f(x)\,dx=\sum_{n=1}^{\infty}\dfrac{t^{n+1}}{n(n+1)}$ 20. $g\left(\sqrt{x}\right)=1-\dfrac{x}{2!}+\dfrac{x^2}{4!}-\dfrac{x^3}{6!}+\cdots+\dfrac{(-1)^n x^n}{(2n)!}+\cdots$

21. $g'(x)=-x+\dfrac{x^3}{3!}-\dfrac{x^5}{5!}+\cdots+\dfrac{(-1)^n x^{2n-1}}{(2n-1)!}+\cdots$ 24. conv. by GST, Sum = $\dfrac{1}{1-\frac{e}{\pi}}$

25. div. by GST or nTT 27. div. by nTT 28. $\sum_{n=1}^{\infty}(4n-1)$

31. $-\frac{1}{3}\ln\left|\cos t^3\right|+C$ 32. $\ln\left(9+x^2\right)+C$ 33. $\dfrac{(\ln x+3)^2}{2}+C$

34. $\frac{1}{3}e^{3y}-2e^y-e^{-y}+C$ 35. $2(\ln x)^{\frac{1}{2}}+C$ 36. $\ln\left|e^x-e^{-x}\right|+C$

37. $\dfrac{x^2}{2}-x-2\ln|x-1|+C$

Lesson 9.3 Taylor Series

In this section you will be finding polynomial functions that can be used to approximate transcendental functions. If $P(x)$ is a polynomial function used to approximate some other function $f(x)$, they must contain the same point with some x-value c. That means $P(c)=f(c)$. To be a better approximation they should have the same slope at that point. This means $P'(c)=f'(c)$. For even greater accuracy, $P''(c)=f''(c)$ and so on. Putting this together gives us the **Taylor Polynomial Expansion:** If $f(x)$ has derivatives of all orders it can be approximated by the polynomial function shown.

$$\boxed{P_n(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f'''(c)}{3!}(x-c)^3+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n}$$

This is called an n^{th} degree or n^{th} order Taylor Polynomial centered at c or expanded about c.

When the center is at $c=0$ the Taylor polynomial is called a **Maclaurin Polynomial** which can be written as :

$$\boxed{P_n(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots+\frac{f^{(n)}(0)}{n!}x^n}$$

Example 1. Use the formula for a Maclaurin Polynomial to find the fifth degree Maclaurin Polynomial for $f(x)=e^x$.

$f(x)=$	$f(0)=$
$f'(x)=$	$f'(0)=$
$f''(x)=$	$f''(0)=$
$f'''(x)=$	$f'''(0)=$
$f^{(4)}(x)=$	$f^{(4)}(0)=$
$f^{(5)}(x)=$	$f^{(5)}(0)=$

$P_5(x)=$

This polynomial is a good approximation for $f(x)=e^x$. By extending the pattern into an infinite series it becomes exactly correct instead of an approximation.

$f(x)=e^x=$

The general form of Taylor and Maclaurin Polynomials can be extended to Taylor and Maclaurin Series.

Taylor Series (provided $f(x)$ has derivatives of all orders)

$$f(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f'''(c)}{3!}(x-c)^3+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n+\cdots$$

Maclaurin Series

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots$$

These formulas allow us to form a power series for functions that cannot be written as geometric power series.

Example 2. Use the formula for a Taylor Polynomial to find the fourth order Taylor Polynomial and the Taylor Series for $f(x)=\cos x$ centered at $c=\pi$.

$f(x)=$	$f(\pi)=$
$f'(x)=$	$f'(\pi)=$
$f''(x)=$	$f''(\pi)=$
$f'''(x)=$	$f'''(\pi)=$
$f^{(4)}(x)=$	$f^{(4)}(\pi)=$

$P_4(x)=$

$f(x)=$

Example 3. Use your Taylor Polynomial from example 2 to approximate $\cos 3$.

$\cos 3 \approx$

Example 4. Use the formula for a Maclaurin Series to find the Maclaurin series for $f(x) = \sin x$.

$f(x) =$	$f(0) =$
$f'(x) =$	$f'(0) =$
$f''(x) =$	$f''(0) =$
$f'''(x) =$	$f'''(0) =$
$f^{(4)}(x) =$	$f^{(4)}(0) =$
$f^{(5)}(x) =$	$f^{(5)}(0) =$

$f(x) = \sin x =$

Example 5. Find a power series for $f(x)$ centered at $c = 1$ if $f(1) = 2$ and $f^{(n)}(1) = n!$.

Assignment 9.3

1. Use the formula to find a fourth degree Maclaurin Polynomial for $f(x) = \frac{1}{e^x}$. Show all derivatives.
2. Use the polynomial from Problem 1 to approximate $\frac{1}{\sqrt{e}}$
3. Use the formula to find a fourth degree Maclaurin Polynomial for $f(x) = e^{3x}$. Show all derivatives.
4. Use the formula to find a fifth degree Maclaurin Polynomial for $g(x) = xe^x$. Show all derivatives.
5. Use the formula to find a third degree Taylor Polynomial centered at $c = 1$ for $f(x) = \sqrt[3]{x}$. Show all derivatives.
6. Use the formula to find a fourth degree Taylor Polynomial centered at $c = 1$ for $h(x) = \ln x$. Show all derivatives.

7. Use the polynomial from Problem 6 to approximate $\ln 1.3$.

8. Use the formula to find a Taylor Series centered at $c = \frac{\pi}{4}$ for $f(x) = \sin x$. Show all derivatives.

9. Use the formula to find a Taylor Series centered at $c = 0$ for $f(x) = \cos(2x)$. Show four terms (Zero terms don't count.) and a general term. Show all derivatives.

10. Use the formula to write a Maclaurin Series for $f(x) = \frac{1}{1+x}$. Show four terms and the general term. Show all derivatives.

11. Write a geometric series expansion for $f(x) = \frac{1}{1+x}$. Also give the interval of convergence.

12. Write four terms and the general term of the Taylor series expansion of $f(x) = \frac{1}{x-1}$ about $x = 2$.

13. Use the series from Problem 12 to find four terms and the general term of the series expansion about $x = 2$ for $\ln|x-1|$.

14. The Taylor Series of a function about $x = 3$ is given by
$f(x) = 1 + \frac{3(x-3)}{1!} + \frac{5(x-3)^2}{2!} + \frac{7(x-3)^3}{3!} + \cdots + \frac{(2n+1)(x-3)^n}{n!} + \cdots$ What is the value of $f'''(3)$?

15. Let $f(x)$ be a function such that $f(0) = 2$, $f'(x) = 3f(x)$, and the n^{th} derivative of f is given by $f^{(n)}(x) = 3f^{(n-1)}(x)$.
    (a) Give the first four terms and the general term of the Taylor Series for f centered at $x = 0$.
    (b) Find $f(x)$ by solving the differential equation $f'(x) = 3f(x)$ (that is $y' = 3y$) with the initial condition $f(0) = 2$.

16. Let f be the function defined by the power series $f(x) = 2 + 2x + 2x^2 + 2x^3 + \cdots + 2x^n + \cdots$
If $g'(x) = f(x)$ and $g(0) = 2$, then $g(x) = ?$ Show four terms and the general term.

17. The Taylor series for a function f about $x = 0$ is $2 + \frac{4}{3}x + \frac{8}{9}x^2 + \frac{16}{27}x^3 + \cdots + \frac{2^{n+1}x^n}{3^n} + \cdots$ for $-1 < x < 1$. Calculator Allowed.
    (a) Write the first four nonzero terms and the general term for f', the derivative of f.
    (b) Using the appropriate second-degree Taylor polynomial approximate $f(0.2)$ and $f'(0.2)$.
    (c) Use the values found in part (b) to approximate the equation of the tangent line to f at $x = 0.2$.

18. The Maclaurin series for $f(x)$ is given by $1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots + \frac{x^n}{(n+1)!} + \cdots$
    (a) Find $f'(0)$. (b) Find $f^{(15)}(0)$.

19. Find a geometric power series for $f(x) = \frac{x}{1+4x^2}$. Show four terms and the general term. Also give the series using sigma notation and give the interval of convergence.

20. Find a function for the geometric power series $f(x) = \sum_{n=0}^{\infty} 4(3x)^n$. Also give the interval of convergence.

Determine whether the following series converge or diverge. Find the sum when possible without using a calculator.

21. $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$ 22. $\sum_{n=0}^{\infty} (1.3)^n$ 23. $\sum_{n=0}^{\infty} \frac{n^2-1}{n+1}$

24. The derivative of a function is $\frac{dy}{dt} = \frac{-3}{t^2} + 1$. If the graph of the function contains the point (3, 10), find the equation of the function.

25. a. Find an equation for the family of functions whose derivative is $y' = 3\sqrt{x}$.
b. Find the particular function from the family in Part a. whose curve passes through the point (4, 0).

Evaluate the following definite integrals without a calculator.

26. $\int_1^4 \frac{2\sqrt{x}-1}{\sqrt{x}}\,dx$ 27. $\int_1^2 \frac{dx}{2\sqrt{3x-2}}$ 28. $\int_0^3 \frac{2x}{\sqrt{x+1}}\,dx$

Differentiate without using a calculator.

29. $h(x) = \sin(\pi x) + \pi x^2$ 30. $f(\theta) = \frac{-2\theta}{\sin\theta}$

31. $f(x) = \tan\left(\ln\left(x^2 - 2x\right)\right)$ 32. $h(x) = e^{\sin x \cos x}$

33. Find the values of a and b which make $g(x) = \begin{cases} ax^3 - bx + 7, & x \le 1 \\ -bx^2 + 3bx, & x > 1 \end{cases}$ continuous and differentiable.

34. If $3x^2 - 2y^2 = 4xy + 20$, find $\frac{dy}{dx}$.

35. Find the <u>point(s)</u> at which the graph from Problem 34 has <u>vertical</u> tangents.

36. Find an equation of the line tangent to the graph from Problem 34 at the point $\left(-\sqrt{\frac{20}{3}}, 0\right)$.

Selected Answers:

1. $f(x) \approx 1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4$ 2. .60677 3. $f(x) \approx 1 + 3x + \frac{9}{2}x^2 + \frac{27}{3!}x^3 + \frac{81}{4!}x^4$

4. $g(x) \approx x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5$ 5. $f(x) \approx 1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{10}{27 \bullet 3!}(x-1)^3$

7. .261975 8. $f(x) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{\sqrt{2}} \bullet \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 - \frac{1}{\sqrt{2}} \bullet \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 + \frac{1}{\sqrt{2}} \bullet \frac{1}{4!}\left(x - \frac{\pi}{4}\right)^4 + \cdots$

10. $f(x) = 1 - x + x^2 - x^3 + \bullet\bullet\bullet + (-1)^n x^n + \bullet\bullet\bullet$

12. $f(x) = 1 - (x-2) + (x-2)^2 - (x-2)^3 + \bullet\bullet\bullet + (-1)^n (x-2)^n + \bullet\bullet\bullet$

13. $\ln|x-1| = (x-2) - \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 - \frac{1}{4}(x-2)^4 + \bullet\bullet\bullet + (-1)^n \frac{1}{n+1}(x-2)^{n+1} + \bullet\bullet\bullet$

14. $f'''(3) = 7$ 15a. $f(x) = 2 + 6x + 9x^2 + 9x^3 + \bullet\bullet\bullet + \dfrac{2(3^n)}{n!}x^n + \bullet\bullet\bullet$ b. $f(x) = 2e^{3x}$

16. $g(x) = 2 + \left(2x + x^2 + \frac{2}{3}x^3 + \bullet\bullet\bullet + \frac{2}{n+1}x^{n+1} + \bullet\bullet\bullet\right)$ 18a. $f'(0) = \frac{1}{2}$ b. $f^{(15)}(0) = \frac{1}{16}$

19. $f(x) = x - 4x^3 + 16x^5 - 64x^7 + \bullet\bullet\bullet + (-1)^n 4^n x^{2n+1} + \bullet\bullet\bullet = \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n+1}$, $-\frac{1}{2} < x < \frac{1}{2}$

21. converges to 3 23. diverges 24. $y = \dfrac{3}{t} + t + 6$

25 a. $y = 2x^{\frac{3}{2}} + C$ b. $y = 2x^{\frac{3}{2}} - 16$ 27. $\frac{1}{3}$ 28. $\boxed{\left(\frac{4}{3} \bullet 8 - 8\right) - \left(\frac{4}{3} - 4\right)} = \frac{16}{3}$

29. $h'(x) = \pi \cos(\pi x) + 2\pi x$ 30. $f'(\theta) = \dfrac{-2\sin\theta + 2\theta\cos\theta}{\sin^2\theta}$

31. $f'(x) = \sec^2\left(\ln\left(x^2 - 2x\right)\right)\dfrac{2x-2}{x^2-2x}$ 32. $h'(x) = e^{\sin x \cos x}\left(-\sin^2 x + \cos^2 x\right)$

33. $a = 2,\ b = 3$ 34. $y' = \dfrac{3x - 2y}{2y + 2x}$ 35. $(2,-2)$, $(-2,2)$ 36. $y = \frac{3}{2}\left(x + \sqrt{\frac{20}{3}}\right)$

Lesson 9.4 Elementary Series, Alternating Series

In this section you will be using the four elementary power series. You are expected to know them from memory.

$\sin x =$

$\cos x =$

$e^x =$

$\ln x =$

Creating new power series.
Examples: Using these elementary series, find a power series for each of the following functions. Show four terms and the general term.

1. $\sin x^2 =$

2. $\cos\sqrt{x} =$

3. $xe^x =$

4. $\dfrac{1}{x} =$

5. $\int_1^t \ln x \, dx =$

An **Alternating Series** is a series whose terms alternate between positive and negative. Three of the four elementary series are alternating series.

Alternating Series Test for Convergence:

The alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ and $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converge if

1. The terms of the series alternate.
2. $a_{n+1} \le a_n$ for all n after a certain n (terms never increase in absolute value) and
3. $\lim_{n\to\infty} a_n = 0$

Examples: Determine the convergence or divergence.

6. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$

7. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+1}{n}$

We now have three tests for Convergence/Divergence.

n^{th} Term Test for Divergence: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

This test is inconclusive if $\lim_{n\to\infty} a_n = 0$. (cannot be used to show conv.)

Geom. Series Test: $|r| \geq 1 \rightarrow$ **diverges**, $|r| < 1 \rightarrow$ **converges** and $\sum_{n=0}^{\infty} ar^n = \dfrac{a}{1-r}$.

Alternating Series Test for Convergence: converges if

1. alternating terms
2. $a_{n+1} \leq a_n$ for all n after a certain n (terms never increase in absolute value)
3. $\lim_{n\to\infty} a_n = 0$

Assignment 9.4

Use an elementary series to give a series for each of the following functions. Show four terms and the general term.

1. $f(x) = e^{-4x}$
2. $g(x) = \cos(3x)$
3. $f(x) = 2\sin x^2$
4. $h(x) = (x-1)\ln x$

Write each of the following series as a function using elementary functions.

5. $x - \dfrac{x^3}{2!} + \dfrac{x^5}{4!} - \dfrac{x^7}{6!} + \cdots$
6. $1 - x^2 + \dfrac{x^4}{2!} - \dfrac{x^6}{3!} + \dfrac{x^8}{4!} - \cdots$
7. $\dfrac{x-1}{x} - \dfrac{(x-1)^2}{2x} + \dfrac{(x-1)^3}{3x} - \dfrac{(x-1)^4}{4x} + \cdots$
8. $1 + x - \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!} - \cdots$

Find the value of each of the following using an elementary function.

9. $1 - \dfrac{\pi^2}{2!} + \dfrac{\pi^4}{4!} - \dfrac{\pi^6}{6!} + \cdots$
10. $\left(\frac{1}{e} - 1\right) - \dfrac{\left(\frac{1}{e}-1\right)^2}{2} + \dfrac{\left(\frac{1}{e}-1\right)^3}{3} - \dfrac{\left(\frac{1}{e}-1\right)^4}{4} + \cdots$
11. $5 + \dfrac{4^2}{2!} + \dfrac{4^3}{3!} + \dfrac{4^4}{4!} + \cdots$
12. $-\dfrac{2^3}{3!} + \dfrac{2^5}{5!} - \dfrac{2^7}{7!} + \dfrac{2^9}{9!} - \cdots$

Determine the convergence or divergence of each series. Show justification and name the test used. If possible, find the sum of the series without using a calculator.

13. $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{n}$
14. $\sum_{n=0}^{\infty} \dfrac{(-1)^n \sqrt{n}}{n^2+1}$
15. $\sum_{n=1}^{\infty} \dfrac{(-1)^n n^3}{n^3+2}$
16. $\sum_{n=2}^{\infty} \dfrac{(-1)^n n}{\ln n}$
17. $\sum_{n=1}^{\infty} \dfrac{\cos(n\pi)}{n}$
18. $\sum_{n=1}^{\infty} \left(\dfrac{3}{4}\right)^n$
19. $\sum_{n=1}^{\infty} \dfrac{4n+2}{5n-1}$
20. $\sum_{n=1}^{\infty} 2\pi^{-n}$
21. $\sum_{n=1}^{\infty} \dfrac{(-1)^n n}{5n^2-1}$
22. $\sum_{n=0}^{\infty} \dfrac{(-2)^n}{5^{n+1}}$
23. $1 - \dfrac{\left(\frac{\pi}{4}\right)^2}{2!} + \dfrac{\left(\frac{\pi}{4}\right)^4}{4!} - \dfrac{\left(\frac{\pi}{4}\right)^6}{6!} + \cdots$

Given $f(x)=1-\frac{4}{3}(x-2)^2+\frac{16}{5}(x-2)^4-\frac{2^6}{7}(x-2)^6+\cdots$ is a Taylor Series expansion for $f(x)$ find:

24. a general term for the series.
25. the center of the series.
26. $f(2)$
27. $f'(2)$
28. $f''(2)$
29. $f^{(11)}(2)$
30. $f^{(12)}(2)$
31. $f'(x)$
32. Is the point $(2,1)$ on this same function a local minimum, a local maximum, or neither. Justify.

Use sigma notation to write an equivalent expression for each series. Use $n = 1,2,3,\cdots$

33. $5+\frac{5}{2}+\frac{5}{6}+\frac{5}{24}+\frac{5}{120}+\cdots$
34. $\frac{1}{4}+\frac{1}{7}+\frac{1}{10}+\frac{1}{13}+\cdots$

35. Find an equation of a line tangent to the curve $y=x^{\frac{2}{3}}$ which is parallel to the line $2x-6y=5$.

36. Find the cubic function of the form $y=ax^3+bx^2+cx+d$ which has a relative maximum point at (0, 2) and a point of inflection at $(-1,-2)$.

37. Find the value of $2-\frac{2}{3}+\frac{2}{9}-\frac{2}{27}+\cdots$

38. Find a third degree Taylor Polynomial for $f(x)=\tan x$ centered at $c=\frac{\pi}{4}$.

39. Find a series for the function $f(x)=\cos x$ centered at $c=\frac{3\pi}{4}$. Show four terms.

40. Write the following function as a geometric series showing four terms and the general term and find the interval of convergence. $f(x)=\dfrac{3}{1+2x}$

41. Find a function for the series $g(x)=1-\frac{3}{4}x+\frac{9}{16}x^2-\frac{27}{64}x^3+\cdots$ and determine its domain.

Give a series representation for these functions. Show four terms and a general term.

42. $f(x)=\int_0^x \frac{\sin t}{t}\,dt$
43. $g(x)=\int_0^x \frac{\cos t-1}{t}\,dt$

Evaluate these integrals without using a calculator.

44. $\int_0^\infty \frac{e^{-\frac{1}{x}}}{x^2}\,dx$
45. $\int_0^3 \frac{x-28}{x^2-x-6}\,dx$

Selected Answers:

1. $1-4x+8x^2-\frac{32}{3}x^3+\bullet\bullet\bullet+\dfrac{(-1)^n 4^n x^n}{n!}+\bullet\bullet\bullet$

2. $1-\dfrac{(3x)^2}{2!}+\dfrac{(3x)^4}{4!}-\dfrac{(3x)^6}{6!}+\bullet\bullet\bullet+\dfrac{(-1)^n 3^{2n} x^{2n}}{(2n)!}+\bullet\bullet\bullet$

4. $(x-1)^2-\dfrac{(x-1)^3}{2}+\dfrac{(x-1)^4}{3}-\dfrac{(x-1)^5}{4}+\bullet\bullet\bullet+\dfrac{(-1)^{n+1}(x-1)^{n+1}}{n}+\bullet\bullet\bullet$

5. $x\cos x$ 6. e^{-x^2} 7. $\dfrac{\ln x}{x}$ 9. -1 10. -1 11. e^4 12. -1.091

13. converge by AST 14. converge by AST 15. diverge by nTT
17. converge by AST 18. converge by GST, Sum = 3 19. diverge by nTT

20. converge by GST, Sum = $\dfrac{2}{\pi-1}$ 21. converge by AST

22. converge by GST, Sum = $\frac{1}{7}$ 23. converge by AST, Sum = $\dfrac{1}{\sqrt{2}}$

25. $c=2$ 27. 0 28. $-\frac{8}{3}$ 29. 0

31. $f'(x)=-\frac{8}{3}(x-2)+\frac{64}{5}(x-2)^3-\dfrac{2^6\bullet 6}{7}(x-2)^5+\cdots$ 33. $\sum_{n=1}^{\infty}\frac{5}{n!}$ 35. $y-4=\frac{1}{3}(x-8)$

36. $y=-2x^3-6x^2+2$ 37. $\frac{3}{2}$ 38. $\tan x\approx 1+2\left(x-\frac{\pi}{4}\right)+\frac{4}{2!}\left(x-\frac{\pi}{4}\right)^2+\frac{16}{3!}\left(x-\frac{\pi}{4}\right)^3$

40. $f(x)=3-6x+12x^2-24x^3+\bullet\bullet\bullet+(-1)^n 3\bullet 2^n x^n+\bullet\bullet\bullet$, IOC: $-\frac{1}{2}<x<\frac{1}{2}$

41. $g(x)=\dfrac{1}{1+\frac{3}{4}x}$, Do: $-\frac{4}{3}<x<\frac{4}{3}$

42. $g(x)=x-\dfrac{x^3}{3\bullet 3!}+\dfrac{x^5}{5\bullet 5!}-\dfrac{x^7}{7\bullet 7!}+\bullet\bullet\bullet+\dfrac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}+\bullet\bullet\bullet$ 44. 1 45. diverges

Lesson 9.5 Error Approximations

Alternating Series Remainder:
For a convergent alternating series when approximating the sum of a series by using only the first n terms, the error will be less than or equal to the absolute value of the $n+1^{st}$ term (this is the next term or the first unused term).

Example 1. Approximate the sum of $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n!}$ by using the first 6 terms .

Example 2. Find the upper bound for the remainder for the approximation from Example 1.

Example 3. Find upper and lower bounds for the actual sum of the series in Example 1.

Example 4. Approximate $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}$ with an error of less than .001.

Example 5. Use an elementary series to find the actual value of the series in Example 4.

Alternating Series Remainder $\le \left|\dfrac{f^{(n+1)}(c)(x-c)^{n+1}}{(n+1)!}\right|$ First unused term!	This expression looks far more complicated than it really is. It is simply the next term.

If a non-alternating series is approximated, the method is slightly different and slightly harder. It is called the **Lagrange Remainder or Taylor's Theorem Remainder**.

Lagrange Remainder $\le \left|\dfrac{f^{(n+1)}(z)(x-c)^{n+1}}{(n+1)!}\right|$	**Where z is the x-value between x and c inclusive which makes $\left|f^{(n+1)}(z)\right|$ a maximum.**

As in an alternating series remainder the $(n+1)^{st}$ term of the Taylor series is used however, the $(n+1)^{st}$ derivative factor is carefully chosen.

Choose a value of z which makes the $\left|f^{(n+1)}(z)\right|$ factor a maximum. This may be at the center, at the x-value where f is to be evaluated, or you may know the maximum value in advance (sine and cosine functions have a maximum value of 1).

Example 6. Estimate e^2 using a Maclaurin polynomial of degree 10 for e^x.

Example 7. Use the Lagrange form of the remainder (error) to estimate the accuracy of using this partial sum.

Example 8. If $f^{(5)}(x)=700\sin x$ and if $x=.7$ is in the convergence interval for the power series of f centered at $x=0$, find an upper limit for the error when the fourth-degree Taylor polynomial is used to approximate $f(.7)$.

Example 9. If $f^{(6)}(x)$ is a positive decreasing function, find the error bound when a 5th degree Taylor polynomial centered at $x=4$ is used to approximate $f(4.1)$. Assume the series converges for $x=4.1$.

Assignment 9.5

1. Approximate the sum of the alternating series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}$ with an error less than or equal to 0.001.
2. If the first four terms are used to approximate the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n^3-1}$ find an upper bound for the remainder.
3. Approximate e^{-1} with a sixth degree Maclaurin Polynomial and find an upper limit of the Alternating Series Remainder.

4. How many terms of a Maclaurin Polynomial are needed to approximate $\sin 1$ with an error of less than 0.001?

5. How many terms of a Maclaurin Polynomial are needed to approximate $\sin 2$ with an error of less than 0.001?

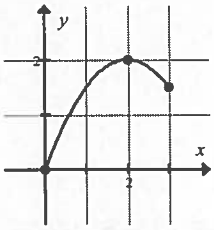
6. If a Taylor Polynomial centered at 1 is used to approximate $\ln 2$ with an error of less than 0.001, how many terms are needed?

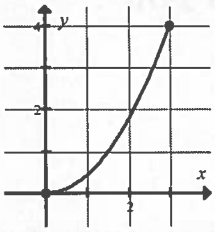
7. If $\left|f^{(4)}(x)\right| \le 4$ find the Lagrange error bound if a third degree Taylor Polynomial centered at $x=1$ is used to approximate $f(2)$. Assume the series converges for $x = 2$.

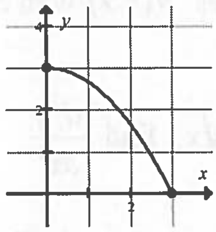
8. If $P_3(2)=5$ for the function from problem 7, find the range of possible values for $f(2)$.

9. If $f^{(6)}(x)=200\sin x$ and $x=.5$ is in the interval of convergence of the power series for f, then find the error when a fifth-degree Taylor polynomial, centered at $x=0$ is used to approximate $f(.5)$.

10. If a sixth degree Taylor Polynomial centered at $x=0$ is used to approximate $f(3)$, find the Lagrange error bound for each of the following if the graph shown is a portion of the graph of $f^{(7)}(x)$. Assume the series converges for $x = 3$.

a. 

b. 

c. 

11. Assuming the function from problem 10 is represented by an alternating series, which of the three answers would be the same using an alternating series error bound?

12. The function $f(x)=e^{-2x}$ is approximated by the polynomial $f(x)\approx 1-2x+2x^2-\frac{4}{3}x^3$. Use the alternating series error bound to determine positive x-values for which this approximation has an error of less than $\frac{27}{8}$ without using a calculator.

13. For $f(x)=\ln x,\ c=1$:

a. Write a Taylor Polynomial $P_4(x)$.

b. Write a power series for $f(x)$ using Σ notation.

c. Approximate $f(1.3)$ using $P_4(1.3)$.

d. Find the actual value of $f(1.3)$.

e. Find the Lagrange error (remainder) bound, $R_4(1.3)$.

f. Find the number of terms from the Taylor Polynomial needed to approximate $f(1.3)$ with an error (remainder) less than .001.

14. Find an upper limit for the error when the Taylor polynomial $T(x)=x-\frac{x^3}{3!}$ is used to approximate $f(x)=\sin x$ at $x=0.5$.

15. Let $f(x)$ be a function whose Taylor series converges for all x. If $\left|f^{(n)}(x)\right|<1$ what is the minimum number of terms of the Taylor series, centered at $x=1$, necessary to approximate $f(1.2)$ with an error less than 0.00001? Assume the series has no zero terms.

16. (calculator allowed)

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
2	3	4	5	6	7

Let h be a function having derivatives of all orders for $x>0$, selected values of h and its first four derivatives are indicated in the table above.

a. Write the first-degree Taylor polynomial for h about $x=2$ and use it to approximate $h(1.9)$.

b. Write the third-degree Taylor polynomial for h about $x=2$ and use it to approximate $h(1.9)$.

c. Assuming the fourth derivative of h is a positive increasing function, use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x=2$ approximates $h(1.9)$ with error less than 3×10^{-5}.

17. If $y=\frac{3}{x}-2\sqrt{x}$, find $\frac{d^2y}{dx^2}$.

18. Find the <u>point(s)</u> where the line(s) tangent to the graph of $f(x)=\frac{1}{3}x^3-x^2+x+3$ is/are parallel to the graph of $y-x=5$.

19. $\frac{d}{dx}\int_{2x}^{3x}\sin t^2 dt=?$

20. If $f(2)=-15$ and $f'(x)=\ln(x+2)+e^x$ find $f(6)$. You may use a calculator.

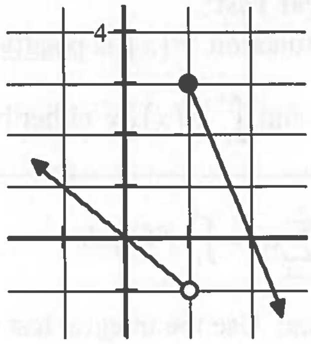
21. Find $\int\frac{\arcsin\frac{x}{2}}{\sqrt{4-x^2}}dx$ without using a calculator.

22. Find a general solution of the differential equation $t\frac{dy}{dt}-2y=\frac{dy}{dt}$. Solve for y.

23. Without using a calculator, find the volume of the solid formed by square cross sections perpendicular to the y-axis, whose base is the region bounded by $xy=4$, $x=0$, $y=1$, and $y=4$.

24. Find the equation of a tangent line to $x^2+3yx=y^2+3$ at (1,2).

25. The graph of $y=f(x)$ shown consists of two linear pieces.
Find the following limits.



a. $\lim\limits_{x\to 0} f\left(1-x^2\right)$

b. $\lim\limits_{x\to 0}\dfrac{f(x)}{4x}$

c. $\lim\limits_{x\to 2}\dfrac{f(x)}{x^2-4}$

d. $\lim\limits_{x\to 1}\dfrac{f(x)-3}{x^2}$

26. Find $\lim\limits_{\theta\to\frac{\pi}{2}}\dfrac{-5\cos\theta}{1-e^{\theta-\frac{\pi}{2}}}$

Selected Answers:

1. .902 2. $R\le\frac{1}{249}$ 3. .368, $R\le\frac{1}{5040}$ 4. three terms 5. five terms

6. 1000 terms 7. $R\le\frac{1}{6}$ 8. $4\frac{5}{6}\le f(2)\le 5\frac{1}{6}$

9. $R\le .0043$ or $R\le .00208$ (better answer)

10a. $R\le .867$ or .868 b. $R\le 1.735$ or 1.736 c. $R\le 1.301$ or 1.302

11. c 12. $0<x<\frac{3}{2}$

13a. $P_4=(x-1)-\dfrac{(x-1)^2}{2}+\dfrac{(x-1)^3}{3}-\dfrac{(x-1)^4}{4}$, b. $f(x)=\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n+1}(x-1)^n}{n}$,

c. $f(1.3)\approx P_4(1.3)=.261975$, d. $f(1.3)=\ln(1.3)=.262364$,

e. $R_4(1.3)\le .000486$, f. four terms

14. Alt Series Error $\le .00026$ or Lagrange Error $\le .0026$ 15. five terms

16 (a) $h(x)\approx 3+4(x-2)$, $h(1.9)\approx 2.6$,

(b) $h(x)\approx 3+4(x-2)+\frac{5}{2}(x-2)^2+(x-2)^3$, $h(1.9)\approx 2.624$

(c) $R\le .000029<.00003$

17. $y''=6x^{-3}+\frac{1}{2}x^{-\frac{3}{2}}$ 18. $(0,3)$, $\left(2,\frac{11}{3}\right)$ 19. $3\sin\left(9x^2\right)-2\sin\left(4x^2\right)$

20. 388.130 21. $\frac{1}{2}\left(\arcsin\frac{x}{2}\right)^2+C$ 22. $y=C(t-1)^2$

23. 12 24. $y-2=8(x-1)$ 25b. $-\frac{1}{4}$ c. $-\frac{3}{4}$ 26. -5

Lesson 9.6 Integral Test and p-Series

Integral Test:
If the function $f(x)$ is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$ then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ either both converge or both diverge.

Note: $\sum_{n=1}^{\infty} a_n \neq \int_1^{\infty} f(x)\,dx$

Examples: Use the integral test to determine convergence or divergence.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

2. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Examples: Use the integral test to determine convergence or divergence of these series.

3. $\sum_{n=1}^{\infty} \frac{1}{n}$

4. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

5. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

***p*-Series and Harmonic Series:**

If p is a positive constant then $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$ is called a ***p*-series.**

The last three examples are all p-series. Each of them could have been done using the following test.

p-series Test
If $p>1$ then the p-series **converges.** If $0<p\le 1$ then the p-series **diverges.**

The **harmonic series** is the p-series in which $p=1$. $\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\cdots$ (Example 3 above)

Examples: Use the p-series test to determine convergence or divergence of these series.

6. $1+\dfrac{1}{2\sqrt{2}}+\dfrac{1}{3\sqrt{3}}+\dfrac{1}{4\sqrt{4}}+\cdots$

7. $\sum_{n=1}^{\infty} n^3\sqrt[3]{n^{-11}}$

Assignment 9.6

Use the Integral Test to show convergence or divergence.

1. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n+2}$

2. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{e^n}$

3. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}\left(\sqrt{n}+2\right)}$

4. $1+\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}+\dfrac{1}{\sqrt{4}}+\cdots$

5. $\dfrac{\ln 2}{4}+\dfrac{\ln 3}{9}+\dfrac{\ln 4}{16}+\dfrac{\ln 5}{25}+\cdots$

Explain why the Integral Test cannot be used for these series.

6. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n+2}$

7. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n-2}$

8. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{n+2}$

Use the p-series Test to show convergence or divergence.

9. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}}$

10. $1+\dfrac{1}{2\sqrt{2}}+\dfrac{1}{3\sqrt{3}}+\dfrac{1}{4\sqrt{4}}+\cdots$

11. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^e}$

Determine the convergence or divergence by any method.

12. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{e}\right)^{\ln n}$

13. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[4]{n^5}}$

14. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5}{4}\right)^n$

15. $\displaystyle\sum_{n=1}^{\infty}\frac{2n}{\sqrt{n^2+2}}$

16. $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$

17. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n^2}+\frac{1}{n^3}\right)$

18. $\displaystyle\sum_{n=1}^{\infty}\frac{e^n}{3^{n+1}}$

19. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt[3]{n}}$

20. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(n+1)}{\sqrt[3]{n}}$

21. If $f(x)=\dfrac{1}{1+x}$

a. Write a fourth degree Maclaurin Polynomial. Hint: It is **not** necessary to find any derivatives.
b. Write a power series for the function using Σ notation.
c. Approximate $f(.2)$ using the polynomial from part a.
d. Find the Alternating Series error (remainder) bound for this polynomial.
e. Find the actual value of $f(.2)$.
f. Find the actual error in the approximation from part c.
g. Find the number of terms from the Maclaurin Polynomial needed to approximate $f(.2)$ with an error (remainder) less than .001.

22. Use one of the elementary functions to find the **simplified** value of the following series.

$$\left(e^{-2}-1\right)-\frac{\left(e^{-2}-1\right)^2}{2}+\frac{\left(e^{-2}-1\right)^3}{3}-\frac{\left(e^{-2}-1\right)^4}{4}+\frac{\left(e^{-2}-1\right)^5}{5}+\cdots$$

23. Write the first four nonzero terms and the general term of the Taylor series for $g(x)=\dfrac{3x^2}{1+x^3}$ about $x=0$.

24. Find the interval of convergence of the series in Problem 23.

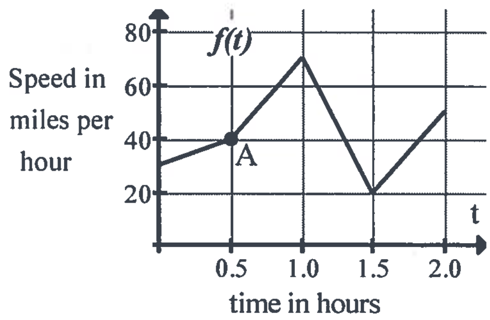
25. Use integration and your series from Problem 23 to find four terms of a series for $f(x)=\ln\left(1+x^3\right)$.

26. Use your series from Problem 25 to find an approximation for $\ln\frac{9}{8}$ with an error of less than 0.001.

27. Let $f(x)$ be a function whose Taylor series converges for all x. If $\left|f^{(n)}(x)\right|<4$ what is the minimum number of terms of the Taylor series, centered at $x=2$, necessary to approximate $f(2.3)$ with an error less than 0.001? Assume there are no zero terms.

28. The graph at the right shows a speed function for a car trip.

a. Write a sentence telling what the coordinates of point A represent in words. (Include numbers and units.)
b. Using the graph shown, if $\int_0^2 f(t)\,dt=80$, write a sentence telling what this means about the car trip in words. (Include numbers and units.)



Find the indicated antiderivatives. Do not use a calculator.

29. $y'=\dfrac{\ln(x)-x^2}{x}, \quad y=?$

30. $f'(t)=e^{3t^2+3t}(2t+1), \quad f(t)=?$

Differentiate. Do not use a calculator.

31. $y=\ln(\arctan(2x))$

32. $f(x)=\ln(x)\arctan(2x)$

33. $g(x)=\left(\ln\left(\arcsin x^2\right)\right)^3$

34. For $\sin\left(t^2\right)=ye^t$, find $\frac{dy}{dt}$

35. Without a calculator, find the volume of the solid formed by revolving the regions bounded by $f(x)=\sqrt{\sin x}$ and $y=0$ on $[0,\pi]$, about the x-axis.

36. Evaluate $\int_0^6 \frac{4}{\sqrt{6-x}}\,dx$. Do not use a calculator.

37. Use the Trapezoidal Rule to approximate $\int_0^{\frac{\pi}{4}} e^{\tan x}\,dx$ using three equal subdivisions.

Selected Answers:

1. diverges 2. converges 3. diverges 5. converges
6. The series is not always positive. 8. The series is not decreasing. 9. diverges by p-sT
10. converges by p-sT 12. diverges (harmonic series) 13. converges by p-sT
14. diverges by GST 16. converges by IT 18. converges by GST
19. converges by AST 21a. $f(x)\approx 1-x+x^2-x^3+x^4$ b. $f(x)=\sum_{n=0}^{\infty}(-1)^n x^n$
21c. $f(.2)\approx .8336$ d. $ASR\le .00032$ e. $f(.2)=.8333\cdots$ f. error $=.0002666\cdots$ g. 5 terms
23. $f(x)=3x^2-3x^5+3x^8-3x^{11}+\dots+(-1)^{n+1}3x^{3n-1}+\dots$
25. $\ln\left(1+x^3\right)=x^3-\frac{1}{2}x^6+\frac{1}{3}x^9-\frac{1}{4}x^{12}+\cdots$ 26. .1171 27. 5 terms
29. $y=\frac{1}{2}(\ln x)^2-\frac{1}{2}x^2+C$ 31. $y'=\dfrac{2}{\left(1+4x^2\right)\arctan(2x)}$
32. $f'(x)=\ln x\,\dfrac{2}{1+4x^2}+\arctan(2x)\,\dfrac{1}{x}$ 33. $g'(x)=3\left(\ln\left(\arcsin x^2\right)\right)^2\,\dfrac{2x}{\arcsin x^2\sqrt{1-x^4}}$
34. $y'=\dfrac{2t\cos t^2-ye^t}{e^t}$ 35. 2π 37. $\boxed{\frac{1}{2}\bullet\frac{\pi}{12}\left(e^{\tan 0}+2e^{\tan\frac{\pi}{12}}+2e^{\tan\frac{\pi}{6}}+e^{\tan\frac{\pi}{4}}\right)}=1.295$

Lesson 9.7 Direct Comparison Test, Limit Comparison Test

Direct Comparison Test: Let $0 < a_n \le b_n$ for all n after a certain n.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. 2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Informally:

1. If the "larger" series converges, then the "smaller" series must also converge.
2. If the "smaller" series diverges, then the "larger" series must also diverge.

Note: The series must have positive terms.

Examples: Determine the convergence or divergence of the following.

1. $\sum_{n=1}^{\infty} \frac{1}{1+2^n}$
2. $\sum_{n=3}^{\infty} \frac{1}{n-2}$
3. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$

Limit Comparison Test: If $a_n > 0$, $b_n > 0$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$

where L is **finite and positive**, then both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or they both diverge.

This limit comparison works well when comparing a messy algebraic series to an easier p-series. To choose the *p*-series, disregard all but the highest powers of n in the numerator and denominator.

Examples: Determine the convergence or divergence of the following.

4. $\sum_{n=2}^{\infty} \frac{1}{3n^2-4}$
5. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+7}$
6. $\sum_{n=1}^{\infty} \frac{n}{4n^3+n^2+5}$

Assignment 9.7

Use the Direct Comparison Test to determine convergence or divergence.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ 2. $\sum_{n=5}^{\infty} \frac{1}{n-4}$ 3. $\sum_{n=2}^{\infty} \frac{2}{\sqrt{n}-1}$ 4. $\sum_{n=1}^{\infty} \frac{1}{5^n+1}$

5. $\sum_{n=1}^{\infty} \frac{3^{n+1}}{4^n+2}$ 6. $\sum_{n=1}^{\infty} \frac{\ln n}{n+2}$ 7. $\sum_{n=1}^{\infty} \frac{1}{n!}$

Use the Limit Comparison Test to determine convergence or divergence.

8. $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$ 9. $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n^2+2}}$ 10. $\sum_{n=2}^{\infty} \frac{2n^2-3n+2}{3n^5+n-4}$ 11. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+3}}$

12. $\sum_{n=1}^{\infty} \frac{4}{2n+\sqrt{n^2+3}}$ 13. $\sum_{n=1}^{\infty} \sin\frac{1}{n}$

14. Use the integral test to determine the convergence of $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$.

15. Use the p-series test to determine the convergence of $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n}$.

Determine the convergence of each of the following by any convergence test.
Do not use the same test twice.

16. $\sum_{n=0}^{\infty} 4\left(\frac{1}{6}\right)^n$ 17. $\sum_{n=1}^{\infty} \frac{1}{3^n-4}$ 18. $\sum_{n=1}^{\infty} \frac{1}{3^n+4}$ 19. $\sum_{n=1}^{\infty} \frac{n}{3n+4}$ 20. $\sum_{n=2}^{\infty} (-1)^n \frac{5}{n-1}$

21. Given $f(x)=e^{-x}$:

a. Write a fourth degree Taylor Polynomial centered at $c = 1$.
b. Write a power series for $f(x)$ using Σ notation.
c. Approximate $f(1.1)$ using the Taylor polynomial from part a.
d. Find the actual value of $f(1.1)$.
e. Find the upper limit for the error (remainder) for your approximation in part c.
f. Find the number of terms from the Taylor Polynomial needed to approximate $f(1.1)$ with an error (remainder) less than .001.

22. Let f be the function given by $f(x)=\sin\left(3x+\frac{\pi}{4}\right)$.

a. Find $T(x)$ the third-degree Taylor polynomial for f about $x = 0$.
b. Find the coefficient of x^{18} in the Taylor series for f about $x = 0$.
c. Use the Lagrange error bound to show that $|f(0.1)-T(0.1)| < 0.0003$.
d. Write a third-degree Taylor polynomial for $f'(x)$ about $x = 0$.

23. Let f be the function defined by $f(x)=\frac{1}{x-2}$. Without using a calculator:

a. Write the first four terms and the general term of the Taylor series expansion of $f(x)$ about $x=3$.

b. Use the result of Part (a) to find the first four terms and the general term of the series expansion about $x=3$ for $\ln|x-2|$.

c. Use the series in Part (b) to find an approximation for $\ln 1.5$ with an error of less than 0.02.

24. The acceleration of an object moving along a horizontal path is given by the equation $a(t)=6t-4$. The object's initial velocity is 5, and its initial position is -2.

a. Find a velocity equation for the object.
b. Find the velocity of the object when $t=2$.
c. Find a position equation for the object.
d. Find the object's position when $t=2$.

Integrate without using a calculator.

25. $\int_0^1 \frac{3y^3+y^2+3y+2}{y^2+1}\,dy$ 26. $\int \frac{\csc^2(\pi x)}{\cot(\pi x)}\,dx$ 27. $\int \frac{\cos y}{\sin y-2}\,dy$

28. $\int \frac{\cot^2 x-1}{\cot x}\,dx$ 29. $\int \frac{e^{\tan x-1}}{\cos^2 x}\,dx$

Selected Answers:

1. conv. 2. div. 3. div. 4. conv. 5. conv. 6. div. 7. conv. 8. div.
9. div. 10. conv. 11. conv. 12. div. 13. div. 14. conv. 15. div.
16. conv. GST 17. conv. LCT 18. Conv. DCT 19. div. nTT

20. conv. AST 21a. $f(x)\approx P_4(x)=\frac{1}{e}-\frac{1}{e}(x-1)+\frac{1}{e}\frac{(x-1)^2}{2!}-\frac{1}{e}\frac{(x-1)^3}{3!}+\frac{1}{e}\frac{(x-1)^4}{4!}$

21b. $f(x)=\sum_{n=0}^{\infty}\frac{1}{e}\frac{(-1)^n(x-1)^n}{n!}$ c. .3328711139 d. $f(1.1)=e^{-1.1}=.3328710837$

e. error $\le 3.06566201\times10^{-8}$ f. 3 terms

22a. $f(x)\approx\frac{1}{\sqrt{2}}+\frac{3}{\sqrt{2}}x-\frac{9}{2\sqrt{2}}x^2-\frac{27}{6\sqrt{2}}x^3$ b. $\frac{-3^{18}}{18!\sqrt{2}}$

22d. $f'(x)\approx\frac{3}{\sqrt{2}}-\frac{9}{\sqrt{2}}x-\frac{27}{2\sqrt{2}}x^2+\frac{81}{6\sqrt{2}}x^3$

23a. $f(x)=1-(x-3)+(x-3)^2-(x-3)^3+\cdots+(-1)^n(x-3)^n+\cdots$

23b. $\ln|x-2|=(x-3)-\frac{(x-3)^2}{2}+\frac{(x-3)^3}{3}-\frac{(x-3)^4}{4}+\cdots+\frac{(-1)^n(x-3)^{n+1}}{n+1}+\cdots$

23c. $\ln 1.5\approx .41666$ 24a. $v(t)=3t^2-4t+5$ c. $s(t)=t^3-2t^2+5t-2$

25. $\frac{5}{2}+\frac{\pi}{4}$ 27. $\ln|\sin y-2|+C$ 28. $\ln|\sin x|+\ln|\cos x|+C$ 29. $e^{\tan x-1}+C$

Lesson 9.8 Ratio Test

Ratio Test: (useful for series involving factorials or exponentials)

1. $\sum a_n$ converges if $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|<1$ 2. $\sum a_n$ diverges if $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|>1$ or $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\infty$

3. The Ratio Test is inconclusive if $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=1$

Examples: Determine the convergence or divergence.

1. $\displaystyle\sum_{n=0}^{\infty}\frac{n!}{3^n}$ 2. $\displaystyle\sum_{n=1}^{\infty}\frac{3^{n+1}}{4^n n^2}$

3. $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n\sqrt{n}}{n-1}$ 4. $\displaystyle\sum_{n=1}^{\infty}\frac{(2n+1)!!}{3^n(2n-1)n!}=\sum_{n=1}^{\infty}\frac{(2n+1)(2n-1)(2n-3)\cdots5\cdot3\cdot1}{3^n(2n-1)n!}$

Handling Mixed Problems:

1. Does the n^{th} term approach zero? If not, the series diverges (n^{th} term test).
2. Is the series a special type: geometric $\left(r^n\right)$, p-series $\left(\frac{1}{n^p}\right)$, or alternating?

$f(x)$ can be integrated

factorials or exponentials

3. Can you use the Integral Test or the Ratio Test
4. Can you compare to a special type using DCT or LCT?

compare to larger or smaller series

messy algebraic

Convergence/Divergence Tests

n^{th} term test	div. if $\lim_{n\to\infty} a_n \neq 0$ (cannot be used to show convergence)
Geom. series test	$\sum_{n=0}^{\infty} ar^n \qquad \lvert r\rvert<1 \to \text{conv.}, \quad \lvert r\rvert \geq 1 \to \text{div.}, \quad S=\frac{a}{1-r}$
p-series	$\sum_{n=1}^{\infty}\frac{1}{n^p} \qquad p>1\to\text{conv.}, \ p\leq 1\to\text{div.}$
Alternating series	alternating with decr. terms and $\lim_{n\to\infty} a_n = 0 \to \text{conv.}$
Integral test	$f(x)$ must be positive, continuous, and decreasing $a_n = f(n) \quad \sum_{n=1}^{\infty} a_n$ conv. if $\int_1^{\infty} f(x)\,dx$ conv., $\sum_{n=1}^{\infty} a_n$ div. if $\int_1^{\infty} f(x)\,dx$ div.
Ratio test	$\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert<1\to\text{conv.}, \quad \lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert>1\to\text{div.}$, (inconclusive if $\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert=1$) (works well for factorials and exponentials)
Direct Comparison	a series with terms **smaller** than a known convergent series also converges a series with terms **larger** than a known divergent series also diverges (both series must be positive)
Limit Comparison	if $\lim_{n\to\infty}\frac{a_n}{b_n}$ is finite and positive both series converge or both diverge (use with "messy" algebraic series, usually compared to a p-series) (both series must be positive)

Mixed Examples: Determine the convergence or divergence.

5. $\sum_{n=1}^{\infty}\frac{n-1}{2n+1}$

6. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

7. $\sum_{n=1}^{\infty}\left(\frac{e}{3}\right)^n$

8. $\sum_{n=1}^{\infty}\frac{1}{4n+5}$

9. $\sum_{n=1}^{\infty}(-1)^n\frac{3}{n^2}$

10. $\frac{1}{10}+\frac{1\cdot 2}{10^2}+\frac{1\cdot2\cdot3}{10^3}+\frac{1\cdot2\cdot3\cdot4}{10^4}+\frac{1\cdot2\cdot3\cdot4\cdot5}{10^5}+\cdots$

Assignment 9.8

Use the ratio test to determine convergence or divergence if possible.

1. $\sum_{n=1}^{\infty}\frac{n!}{n^3}$ 2. $\sum_{n=0}^{\infty}n\left(\frac{2}{3}\right)^n$ 3. $\sum_{n=1}^{\infty}\frac{3^n}{n^3}$ 4. $\sum_{n=1}^{\infty}\frac{(-3)^n}{n!}$

5. $\sum_{n=1}^{\infty}\frac{(2n)!}{n3^n}$ 6. $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ 7. $\sum_{n=0}^{\infty}\frac{(2n+1)!!}{n!}=\sum_{n=1}^{\infty}\frac{(2n+1)(2n-1)(2n-3)\cdots3\bullet1}{n(n-1)(n-2)\cdots2\cdot1}$

Determine convergence or divergence using any test.

8. $\sum_{n=1}^{\infty}(-1)^n\frac{3}{n}$ 9. $\sum_{n=1}^{\infty}\frac{3}{n}$ 10. $\sum_{n=0}^{\infty}\frac{1}{3^n}$ 11. $\sum_{n=1}^{\infty}\frac{4}{n\sqrt{n}}$

12. $\sum_{n=1}^{\infty}(-1)^n\frac{3n}{n+1}$ 13. $\sum_{n=1}^{\infty}\frac{3n+2}{n^2+2n-4}$ 14. $\sum_{n=2}^{\infty}\frac{2^n}{\ln n}$ 15. $\sum_{n=1}^{\infty}\frac{|\cos n|}{4^n}$

16. $\sum_{n=1}^{\infty}4\left(\frac{5^n}{3^{n+1}}\right)$ 17. $\sum_{n=3}^{\infty}\frac{(n-2)3^n}{n!}$

18. Which of the following series is/are equivalent to $\sum_{n=1}^{\infty}\frac{2n}{n+1}$?

a. $\sum_{n=0}^{\infty}\frac{2(n+1)}{n+2}$ b. $\sum_{n=0}^{\infty}\frac{2n}{n+1}$ c. $1+\sum_{n=2}^{\infty}\frac{2n}{n+1}$ d. $\sum_{n=1}^{\infty}\left(2-\frac{2}{n+1}\right)$ e. $\frac{7}{3}+\sum_{n=3}^{\infty}\frac{2n}{n+1}$

19. The function f has derivatives of all orders for all real numbers x. Given $f(3)=-2$, $f'(3)=4$, $f''(3)=2$, and $f'''(3)=-9$.

a. Write the third degree Taylor polynomial for f about $x=3$.
b. Use your part a answer to approximate $f(2.5)$.
c. Given $\left|f^{(4)}(x)\right|\le 3$ use the Lagrange error bound on the approximation to $f(2.5)$ found in part b to find a range of possible values for $f(2.5)$.
d. Write the fourth degree Taylor polynomial for $g(x)=f(x^2+3)$ about $x=0$.
e. Use your answer to part d to determine if g has a local minimum or a local maximum at $x=0$.

20. Given the series $f(x)=3-15x+75x^2-375x^3+\cdots$ (calculator allowed)

a. Write an expression for $f(x)$ using sigma notation.
b. Find the interval of convergence of the geometric series.
c. Find the exact value of $f(.1)$.
d. Can the exact value of $f(.3)$ be found? If it can, find it. If not, explain why not.
e. Approximate $f(.1)$ using the first four terms of the series.
f. Find the alternating series error bound for this approximation.
g. Find the actual error for this approximation.

21. Calculator Allowed

The table shown is a record of the velocity of a car traveling in a straight line. At time 35 seconds the car attains its absolute maximum velocity and at time 45 seconds the car has a relative minimum velocity. There are no other local or absolute extrema for the car's velocity between 0 and 50 seconds.

time (seconds)	$v(t)$ (feet per second)
0	0
5	15
10	21
15	29
20	51
25	65
30	75
35	80
40	75
45	55
50	70

a. During what intervals of time is the acceleration of the car positive?

b. Find the average acceleration of the car over the interval $0 \le t \le 50$ seconds. Label units.

c. Find an approximation for the acceleration of the car at $t = 32.5$ seconds. Show the computations you used to arrive at your answer and label units.

d. Approximate $\int_0^{50} v(t)\,dt$ with a Riemann sum, using midpoints of five equal subintervals.

e. Using correct units, explain the meaning of your answer to part d.

Differentiate.

22. $y = \csc\frac{1}{x} - 3\sqrt{x}$ 23. $P(t) = \sqrt{\cos t - \sin t}$ 24. $g(x) = \ln|\sec(3x)|$ 25. $y = e^{\sin x}e^{\cos x}$

26. $f''(x) = 10x^{\frac{-1}{3}}$, $f'(8) = 50$, and $f(1) = 30$. Find an equation for the curve $f(x) =$

Selected Answers:

1. ∞, div. 2. $\frac{2}{3}$, conv. 3. 3, div. 4. 0, conv. 5. ∞, div. 6. 1, inconclusive

7. 2, div. 8. conv AST 10. conv. GST 11. conv. pST 13. div. LCT 14. div. nTT

16. div. GST 17. conv. RT 19a. $f(x) \approx -2 + 4(x-3) + (x-3)^2 - \frac{3}{2}(x-3)^3$

19b. $f(2.5) \approx -3.5625$ d. $g(x) \approx -2 + 4x^2 + x^4$ 20a. $\sum_{n=0}^{\infty} 3(-5x)^n$ b. $-.2 < x < .2$ c. 2

20e. 1.875 21b. 1.4 ft/sec^2

21e. The car traveled a total distance of approximately 2440 feet between 0 and 50 seconds.

23. $P'(t) = \frac{-\sin t - \cos t}{2\sqrt{\cos t - \sin t}}$ 24. $g'(x) = 3\tan(3x)$ 25. $y' = e^{\sin x}e^{\cos x}(\cos x - \sin x)$

26. $f(x) = 9x^{\frac{5}{3}} - 10x + 31$

Lesson 9.9 Interval of Convergence (by Ratio Test)

The **radius of convergence** of a power series is the distance from the center c at which the series will converge. The **interval of convergence** is the range of x-values within which the series will converge.

Examples: Find the interval of convergence and the radius of convergence of the following power series.

1. $\sum_{n=1}^{\infty} \frac{2^n x^n}{n}$

2. $\sum_{n=0}^{\infty} \frac{(x+1)^n}{2^n}$

3. $\sum_{n=0}^{\infty} n!x^n$

4. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

Note: In Lesson 9.1 we found intervals of convergence for Geometric Series. It was not necessary to check endpoints because Geometric Series diverge when $r = 1$. However the ratio test is inconclusive when the ratio is 1. Therefore when using the Ratio Test to find intervals of convergence it is necessary to check endpoints using one of the other tests (although it is not necessary to check when asked to find only the radius of convergence).

Assignment 9.9

Find the radius of convergence.

1. $\displaystyle\sum_{n=0}^{\infty}\frac{(2x)^n}{n+1}$ 2. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n^2}$.

Find the interval of convergence. Remember to check endpoints when using the Ratio Test.

3. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n}$ 4. $\displaystyle\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n$ 5. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n (x+3)^n}{n(n+1)}$ 6. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}$

7. $\displaystyle\sum_{n=0}^{\infty}(2n)!x^{2n}$ 8. $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{4^{n+1}}$ 9. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n (x-3)^n}{n3^n}$ 10. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^{n+1} x^n}{n+1}$

11. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n (x-5)^n}{2^{n+1}}$ 12. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2n}$ 13. $\displaystyle\sum_{n=0}^{\infty}\frac{(n+1)!x^n}{n!}$ 14. $\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{n}(6x)^n$

15. Given $f(x)=\dfrac{1}{2!}+\dfrac{x}{3!}+\dfrac{x^2}{4!}+\dfrac{x^3}{5!}+\cdots+\dfrac{x^n}{(n+2)!}+\cdots$

a. For what values of x does the given series converge? Show work.
b. Let $g(x)=x^2 f(x)$. Write the Maclaurin series for $g(x)$, showing the first four terms and the general term.
c. Write $g(x)$ in terms of an elementary function.
d. Write $f(x)$ in terms of the same elementary function.

16. Let f be the function given by $f(x)=e^{-x^2}$.

a. Find the first four terms and the general term of the power series for $f(x)$ about $x=0$.
b. Find the interval of convergence of this power series for $f(x)$.
c. Use the first four nonzero terms of $f(x)$ to approximate $f(0.8)$.
d. Show this approximation has an error less than 0.007.

17. Let f be the function given by $f(t)=\dfrac{3}{1+t^2}$.

a. Find the first four nonzero terms and the general term for the power series expansion of $f(t)$ about $t=0$.
b. Given $g(x)=\int_0^x f(t)\,dt$, find the first four nonzero terms and the general term for the power series expansion of $g(x)$ about $x=0$.
c. Find the interval of convergence of the power series for $g(x)$.

Selected Answers:

1. $\frac{1}{2}$ 2. 1 3. $(-1,1]$ 4. $(-3,3)$ 5. $[-4,-2]$ 6. $(-\infty,\infty)$ 8. $(-4,4)$

9. $(0,6]$ 10. $(-1,1]$ 11. $(3,7)$ 12. $[-1,1]$ 13. $(-1,1)$

15a. $(-\infty,\infty)$ b. $g(x)=\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\dfrac{x^4}{4!}+\dfrac{x^5}{5!}+\cdots+\dfrac{x^{n+2}}{(n+2)!}+\cdots$ c. $g(x)=e^x-1-x$

16a. $e^{-x^2}=1-x^2+\dfrac{x^4}{2!}-\dfrac{x^6}{3!}+\cdots+\dfrac{(-1)^n x^{2n}}{n!}+\cdots$ b. $(-\infty,\infty)$ c. .5211 d. $.00699<.007$

17a. $f(t)=3-3t^2+3t^4-3t^6+\cdots+3(-1)^n t^{2n}+\cdots$

b. $g(x)=3x-x^3+\dfrac{3x^5}{5}-\dfrac{3x^7}{7}+\cdots+\dfrac{3(-1)^n x^{2n+1}}{2n+1}+\cdots$ c. $-1\le x\le 1$

Lesson 9.10 Absolute vs Conditional Convergence, Review

Many convergent series have negative terms, alternating or some other pattern. Taking the absolute value of each term of a convergent series with some negative terms makes the new positive series **less** likely to converge since the sum will be greater without negative terms. If the new positive series is still convergent the original series is called **absolutely convergent.** If, on the other hand, the new positive series diverges the original convergent series with negative terms is called **conditionally convergent.**

If a series converges after taking the absolute value of its terms it is guaranteed to also converge with no absolute value.

Examples: Determine whether each of the following series is absolutely convergent, conditionally convergent, or divergent.

1. $\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}$ 2. $\sum_{n=1}^{\infty}\dfrac{\cos n}{n^2}$ 3. $\sum_{n=1}^{\infty}\dfrac{(-e)^n}{n^e}$

Assignment 9.10

Determine whether each series is absolutely convergent, conditionally convergent, or divergent.

1. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ 2. $\sum_{n=0}^{\infty}\frac{(-1)^n n^2}{n^3-4}$ 3. $\sum_{n=1}^{\infty}\frac{(-1)^n (n-1)}{n}$ 4. $\sum_{n=1}^{\infty}\frac{(\sin n)^3}{n^3}$

Determine the convergence or divergence of the series. Whenever possible, find the sum of the series.

5. $\sum_{n=1}^{\infty}5\frac{2^n}{3^{n+2}}$ 6. $\sum_{n=2}^{\infty}\frac{(n+1)^2}{n\ln n}$ 7. $\sum_{n=0}^{\infty}3\left(\frac{\sqrt{5}}{2}\right)^n$ 8. $\sum_{n=0}^{\infty}\frac{3^n}{n!}$

9. $\sum_{n=1}^{\infty}\frac{\sqrt{\ln n}}{n}$ 10. $\sum_{n=1}^{\infty}\frac{1}{\sqrt[5]{n^4}}$ 11. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^3+3n}}$ 12. $\sum_{n=3}^{\infty}\frac{n+1}{n(n-2)}$

13. $\sum_{n=0}^{\infty}\frac{1}{4^n-3}$ 14. $\sum_{n=1}^{\infty}\frac{e^n}{n!}$ 15. $\sum_{n=1}^{\infty}\frac{3^n}{n^3}$ 16. $\sum_{n=0}^{\infty}\frac{(-1)^n 3^{2n+1}}{(2n+1)!}$

17. $1+\frac{1}{2}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{n!}+\cdots$

18. Find a third degree Maclaurin Polynomial for $f(x)=e^{-2x}$.

19. Find a third degree Taylor Polynomial centered at $c=\frac{-3\pi}{4}$ for $f(x)=\tan x$.

20. Use a Taylor Polynomial to approximate $\sin(0.5)$ accurate to the third decimal place (error less than 0.001).

21. Use a Taylor Polynomial to approximate $e^{-0.2}$ accurate to the third decimal place.

22. Find the largest possible value of $|f(1.9)-p(1.9)|$ if $p(x)$ is a second degree Taylor polynomial of $f(x)$ centered at $x=2$ and $-9\le f'''(x)\le 5$.

23. Show four terms and a general term of a Taylor series centered at $c=-1$ for the function $g(x)=\frac{1}{x}$.

24. Show four terms and a general term of a Taylor series centered at $c=0$ for the function $f(x)=2^x$.

25. Show four terms of a Taylor series centered at $c=0$ for the function $f(x)=\frac{1}{(x+1)^4}$.

Find the interval of convergence for each series.

26. $\sum_{n=0}^{\infty}\frac{(-1)^n (x-3)^n}{(n+1)^2}$ 27. $\sum_{n=0}^{\infty}\left(\frac{2x}{5}\right)^n$ 28. $\sum_{n=1}^{\infty}\frac{2^n (x-3)^n}{n}$ 29. $\sum_{n=0}^{\infty}n!(x-5)^n$

30. Find the series corresponding to the integral $\int_0^x \frac{\sin t}{t}\,dt$. Show four terms and a general term.

31. Use an appropriate series to approximate $\int_0^1 \sin\sqrt{x}\,dx$ within 0.001.

32. Do not use a calculator. $\int_0^{\pi}\left(x^2-\frac{x^4}{3!}+\frac{x^6}{5!}-\frac{x^8}{7!}+\cdots\right)dx=?$

33. Given $f(x)=\cos x^2$. Do not use a calculator.

a. Write the first four nonzero terms and the general term of the Taylor series for f about $x=0$.

b. Using this series find $\lim_{x\to 0}\frac{1-\frac{1}{2}x^4-f(x)}{x^8}$.

c. Write the first four nonzero terms of the Taylor series for $\int_0^x \cos t^2\,dt$ about $x=0$.

d. Approximate $\int_0^1 \cos t^2\,dt$ using the first two terms of your answer to part c.

e. Show that the approximation found in part d differs from the actual value of $\int_0^1 \cos t^2\,dt$ by less than $\frac{1}{200}$.

Answers:

1. converges absolutely 2. converges conditionally 3. diverges

4. converges absolutely 5. converges, GST; Sum $=\frac{10}{9}$ 6. diverges, nTT 7. div. GST

8. conv. RT 9. div. IT or DCT 10. div. p-sT 11. conv. DCT (or LCT)

12. div. DCT (or LCT) 13. conv. LCT 14. conv. RT 15. div. nTT (or RT)

16. conv. AST; Sum = 0.141 17. conv. RT (or DCT); Sum = $e-1$

18. $f(x)\approx 1-2x+2x^2-\frac{4}{3}x^3$ 19. $f(x)\approx 1+2\left(x+\frac{3\pi}{4}\right)+2\left(x+\frac{3\pi}{4}\right)^2+\frac{8}{3}\left(x+\frac{3\pi}{4}\right)^3$

20. 0.479 21. 0.818 or 0.819 22. 0.0015

23. $f(x)=-1-(x+1)-(x+1)^2-(x+1)^3-\cdots-(x+1)^n-\cdots$

24. $f(x)=1+(\ln 2)x+\frac{(\ln 2)^2}{2!}x^2+\frac{(\ln 2)^3}{3!}x^3+\cdots+\frac{(\ln 2)^n}{n!}x^n+\cdots$

25. $f(x)=1-4x+10x^2-20x^3+\cdots$ 26. $[2,4]$ 27. $\left(-\frac{5}{2},\frac{5}{2}\right)$ 28. $\left[\frac{5}{2},\frac{7}{2}\right)$

29. Conv. only when $x=5$. 30. $x-\frac{x^3}{3!3}+\frac{x^5}{5!5}-\frac{x^7}{7!7}+\cdots+\frac{(-1)^n x^{2n+1}}{(2n+1)!(2n+1)}+\cdots$

31. $\approx\frac{2}{3}-\frac{1}{15}+\frac{1}{420}=.602$ 32. π 33a. $f(x)=1-\frac{x^4}{2}+\frac{x^8}{4!}-\frac{x^{12}}{6!}+\cdots+\frac{(-1)^n x^{4n}}{(2n)!}+\cdots$

33b. $-\frac{1}{24}$ c. $x-\frac{1}{10}x^5+\frac{1}{4!9}x^9-\frac{1}{6!13}x^{13}+\cdots$ d. $\frac{9}{10}$

33e. alternating series error $<\frac{1}{216}<\frac{1}{200}$

UNIT 9 SUMMARY

Taylor Series Expansion:

$$\boxed{f(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\frac{f'''(c)}{3!}(x-c)^3+\cdots+\frac{f^{(n)}(c)}{n!}(x-c)^n+\cdots}$$

Maclaurin Series :

$$\boxed{f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots}$$

Elementary Series

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+\frac{(-1)^n x^{2n+1}}{(2n+1)!}+\cdots$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots+\frac{(-1)^n x^{2n}}{(2n)!}+\cdots$$

$$\ln x=(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\cdots+\frac{(-1)^{n+1}(x-1)^n}{n}+\cdots$$

Alternating Series Remainder $\leq$ the first unused term

Lagrange Remainder $\leq \left|\dfrac{f^{(n+1)}(z)(x-c)^{n+1}}{(n+1)!}\right|$

Where z is the x-value between x and c inclusive which makes $\left|f^{(n+1)}(z)\right|$ a maximum.

If $\sum_{n=1}^{\infty}|a_n|$ converges, $\sum_{n=1}^{\infty}a_n$ converges absolutely.

If $\sum_{n=1}^{\infty}a_n$ converges but $\sum_{n=1}^{\infty}|a_n|$ diverges, $\sum_{n=1}^{\infty}a_n$ converges conditionally.

Handling Mixed Problems:

1. Does the n^{th} term approach zero? If not, the series diverges (n^{th} term test).
2. Is the series a special type: geometric (r^n), p-series$\left(\frac{1}{n^p}\right)$, or alternating?

$f(x)$ can be integrated

factorials or exponentials

3. Can you use the Integral Test or the Ratio Test
4. Can you compare to a special type using DCT or LCT?

compare to larger or smaller series

messy algebraic

Convergence/Divergence Tests

n^{th} term test	div. if $\lim\limits_{n\to\infty} a_n \neq 0$ (cannot be used to show convergence)
Geom. series test	$\sum\limits_{n=0}^{\infty} ar^n$ $\quad |r|<1 \to$ conv. , $|r|\geq 1 \to$ div. , $\mathrm{S}=\dfrac{\mathrm{a}}{1-r}$
p-series	$\sum\limits_{n=1}^{\infty} \dfrac{1}{n^p}$ $\quad p>1 \to$ conv. , $p\leq 1 \to$ div.
Alternating series	alternating with decr. terms and $\lim\limits_{n\to\infty} a_n = 0 \to$ conv.
Integral test	$f(x)$ must be positive, continuous, and decreasing $a_n = f(n)$ $\quad \sum\limits_{n=1}^{\infty} a_n$ conv. if $\int_1^{\infty} f(x)\,dx$ conv., $\sum\limits_{n=1}^{\infty} a_n$ div. if $\int_1^{\infty} f(x)\,dx$ div.
Ratio test	$\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| < 1 \to$ conv. , $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| > 1 \to$ div. , (inconclusive if $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 1$) (works well for factorials and exponentials)
Direct Comparison	a series with terms **smaller** than a known convergent series also converges a series with terms **larger** than a known divergent series also diverges (both series must be positive)
Limit Comparison	if $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n}$ is finite and positive both series converge or both diverge (use with "messy" algebraic series, usually compared to a p-series) (both series must be positive)

Lesson 10.1 Parametric Equations

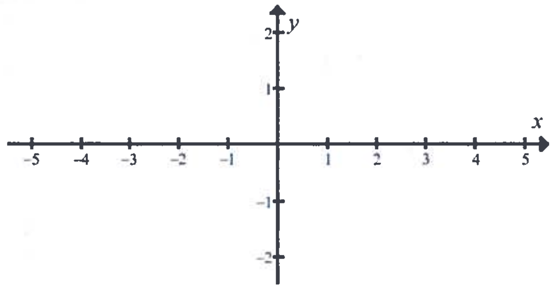
Example 1. Plot points to sketch the curve described by the parametric equations. Mark the orientation on the curve.

$x = t^2 - 5$

$y = \frac{t}{2}$

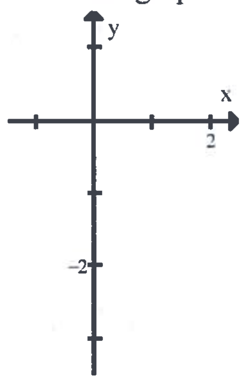
$-3 \le t \le 2$

t						
x						
y						



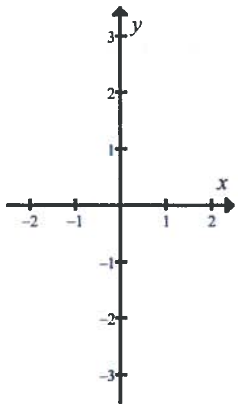
Example 2. Change the following to rectangular form by eliminating the parameter. Then graph.

$x = \dfrac{1}{\sqrt{t+1}}$ and $y = \dfrac{t}{t+1}$, $t > -1$



Example 3. Eliminate the parameter to sketch the curve.

$x = \cos\theta$ and $y = 3\sin\theta$, $0 \le \theta \le 2\pi$



Even though equations are given in terms of a parameter, it is possible to find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ by differentiating and then dividing.

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}}$$

Example 4. If $x = \cos t$ and $y = 3\sin t$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Example 5. Find the slope and concavity of $x = \sqrt{t}$ and $y = \frac{1}{4}t^2 - 1,\ t \geq 0$ at the point $(2,3)$.

Example 6. Write an equation of a tangent line to the curve defined by $x = t - 1$ and $y = \frac{1}{t} + 1$ at the point when $t = 1$.

Arc Length: If a curve is smooth and does not intersect itself the length of an arc is given by

$$\boxed{\text{arc length} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

Example 7. Using the parametric equations from example 6, find the arc length on the interval $1 \leq t \leq 3$.

Assignment 10.1

Using these parametric equations, eliminate the parameter to write the corresponding rectangular equation. Sketch the curve indicating the orientation without using a calculator.

1. $x = 2t - 3,\ y = \frac{2}{3}t + 4$ 2. $x = t^3,\ y = t^2$ 3. $x = \sqrt{t},\ y = 4 - t$ 4. $x = t^4,\ y = 4\ln t$

5. Use a calculator set in parametric mode to graph the curve represented by the parametric equations $x = -3 + 4\cos\theta$ and $y = 1 + 2\sin\theta$. Then eliminate the parameter.

6. Given the parametric equations $x = -2t + 1$ and $y = t^3 + 3$:

   a. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

   b. Find an equation of the tangent line when $t = 1$.

   c. Use concavity to determine if the tangent line is above the curve or below the curve.

7. Given the parametric equations $x = 3\cos\theta$ and $y = 3\sin\theta$:

   a. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

   b. Find an equation of the tangent line when $\theta = \frac{\pi}{4}$.

   c. Use concavity to determine if the tangent line is above the curve or below the curve.

8. Use a calculator to graph the curve represented by the parametric equations $x = 3\sin(2t)$ and $y = 2\sin t$. The curve crosses itself at the point (0,0). Find equations of all tangent lines at that point.

Without using a calculator, find all points at which each curve has horizontal and vertical tangents.

9. $x = 2t + 1,\ y = t^2$ 10. $x = t^2 + 1,\ y = t^2 + 4t$

11. $x = t^2 - t + 3,\ y = 4t^3 - 12t$ 12. $x = \tan\theta,\ y = \sec\theta$

Given the parametric equations $x = -3t - 5$ and $y = t^3 - 12t + 3$ (without using a calculator):

13. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t.

14. Use the second derivative test to determine if the curve has a local maximum, a local minimum, or neither when $t = 2$.

15. Use the second derivative test to determine if the curve has a local maximum, a local minimum, or neither at the point $(1,19)$.

16. Use the second derivative test to determine if the curve has a local maximum, a local minimum, or neither at the point $(-8,-8)$.

Show an integral setup and find the length of each arc on the given interval.

17. $x = 3t - t^2,\ y = 4t^{\frac{3}{2}}$ $1 \le t \le 2$ 18. $x = t + \cos t,\ y = t - \sin t$ $0 \le t \le \pi$

19. $x = \arccos t,\ y = \ln\sqrt{1 + t^2}$ $0 \le t \le \frac{1}{2}$

20. Find the length of the arc between the two y-intercepts of $x=t^2-1$ and $y=2t$.

21. Differentiate $\sin(y-2x)=x^2-10$ to find $\frac{dy}{dx}$.

Integrate each of the following without using a calculator.

22. $\displaystyle\int\frac{\left(\sqrt{t}-4\right)^{10}}{\sqrt{t}}\,dt$ 23. $\displaystyle\int y^3\left(y-\frac{1}{y}\right)dy$ 24. $\displaystyle\int\frac{3x^2+x-2}{\sqrt{x}}\,dx$

25. $\displaystyle\int\frac{6-4x}{\sqrt{2x-1}}\,dx$ 26. $\displaystyle\int_0^{\frac{2\pi}{3}}\tan\left(\frac{x}{2}\right)dx$ 27. $\displaystyle\int\sin(e^{-t})e^{-t}\,dt$

28. $\displaystyle\int_{-2}^{-1}\left(\frac{1-3x^4}{x^2}\right)dx$

29. If $f''(x)=x^{\frac{-4}{3}}$, $f'(8)=\frac{3}{2}$, and $f(27)=5$, find $f(x)$.

30. $v(t)=-\sqrt{t+4}$ represents the velocity equation of an object moving along a vertical path for $t\geq-4$. Let $a(t)$ represent the acceleration and $y(t)$ represent the position of the object at time t. Find:

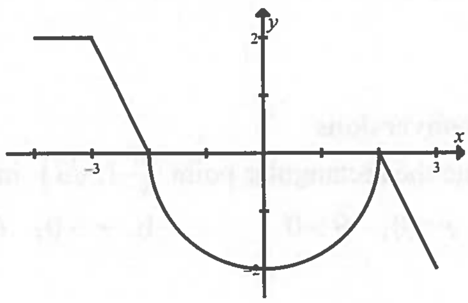
a. an equation for the acceleration of the object at time t. $a(t)=?$

b. an equation for the position of the object at time t if $y(0)=-\frac{10}{3}$. $y(t)=?$

c. $y(5)$ d. $v(5)$ e. $a(5)$

f. the speed of the object at time $t=5$.

The graph of the function $y=f(x)$ consists of line segments and a semicircle as shown. Evaluate the following using geometry formulas.



31. $\displaystyle\int_{-4}^{3}f(x)\,dx$ 32. $\displaystyle\int_{-2}^{2}\left(f(x)+2\right)dx$

Selected Answers:

1. $y=\frac{1}{3}x+5$ 2. $y=x^{\frac{2}{3}}$ 4. $y=\ln x$ 5. $\left(\frac{x+3}{4}\right)^2+\left(\frac{y-1}{2}\right)^2=1$

6a. $\frac{dy}{dx}=-\frac{3}{2}t^2$, $\frac{d^2y}{dx^2}=\frac{3}{2}t$ b. $y-4=-\frac{3}{2}(x+1)$ c. below

7a. $\frac{dy}{dx}=-\cot\theta$, $\frac{d^2y}{dx^2}=-\frac{1}{3}\csc^3\theta$ b. $y-\frac{3}{\sqrt{2}}=-\left(x-\frac{3}{\sqrt{2}}\right)$ c. above 8. $y=\pm\frac{1}{3}x$

9. H.T: (1,0) V.T: none 10. H.T: $(5,-4)$ V.T: (1,0)

11. H.T: (3,-8), (5,8) V.T: $\left(\frac{11}{4},-\frac{11}{2}\right)$ 12. H.T: $(0,\pm1)$ V.T: none

13. $\frac{dy}{dx}=-t^2+4$, $\frac{d^2y}{dx^2}=\frac{2}{3}t$ 14. min. 15. max. 17. 7.336 or 7.337

More Selected Answers:

18. 3.678 19. .538 20. 4.591 22. $\frac{2}{11}\left(\sqrt{t}-4\right)^{11}+C$

24. $\frac{6}{5}x^{\frac{5}{2}}+\frac{2}{3}x^{\frac{3}{2}}-4x^{\frac{1}{2}}+C$

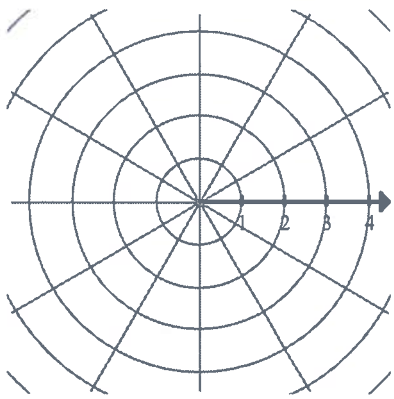
25. $4\sqrt{2x-1}-\frac{2}{3}\left(\sqrt{2x-1}\right)^{3}+C$ or $(6-4x)\sqrt{2x-1}+\frac{4}{3}(2x-1)^{\frac{3}{2}}+C$

26. $-2\ln\frac{1}{2}=\ln 4$ 28. $\boxed{(1+1)-\left(\frac{1}{2}+8\right)}=-\frac{13}{2}$ 29. $-\frac{9}{2}x^{\frac{2}{3}}+3x-\frac{71}{2}$

30a. $a(t)=-\frac{1}{2}(t+4)^{-\frac{1}{2}}$ b. $y(t)=-\frac{2}{3}(t+4)^{\frac{3}{2}}+2$ c. -16 d. -3 e. $-\frac{1}{6}$ f. 3

32. $8-2\pi$

Lesson 10.2 Polar Graphs



Plotting points in polar form

1. Use the polar grid to plot these polar points.

A $\left(3,\frac{\pi}{6}\right)$ B $(4,\pi)$

C $\left(3,-\frac{5\pi}{6}\right)$ D $\left(-2,\frac{3\pi}{2}\right)$

Point conversions

2. Write the rectangular point $\left(-1,\sqrt{3}\right)$ in polar form such that:

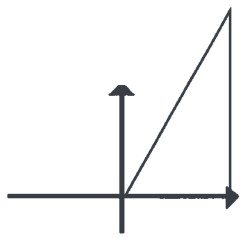
a. $r>0,\quad \theta>0$ b. $r>0,\quad \theta<0$ c. $r<0,\quad \theta>0$ d. $r<0,\quad \theta<0$

3. Change the point $(2,\pi)$ to rectangular form.

4. Change the point $(-3,3)$ to polar form.

Conversion Equations

$x^2+y^2=$	$x=$
$\tan\theta=$	$y=$



Equation conversions (rectangular to polar)

5. $y = 4$
6. $3x - y + 2 = 0$
7. $x^2 + y^2 - 2x = 0$

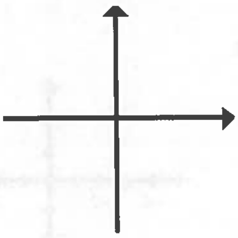
Equation conversions (polar to rectangular)

8. $r = -2$
9. $r = 3\cos\theta$
10. $r = 2\csc\theta$

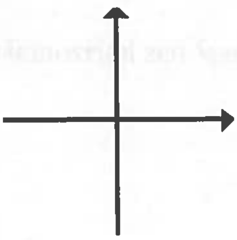
Sketching polar graphs (use a calculator on those in **bold**)

11. Circles

a. $\boldsymbol{r = 2\cos\theta}$

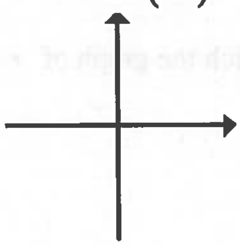


b. $r = 5\sin\theta$

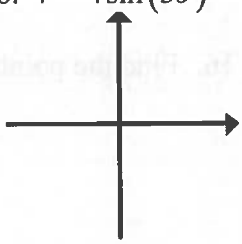


12. Rose petal curves

a. $\boldsymbol{r = 3\cos(2\theta)}$

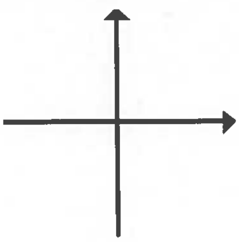


b. $r = 4\sin(3\theta)$

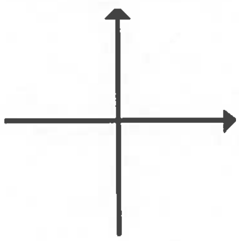


13. Limaçons

a. $\boldsymbol{r = 2 - 3\cos\theta}$

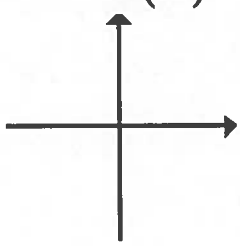


b. $r = 3 + 3\sin\theta$



14. Lemniscate

$\boldsymbol{r^2 = 9\sin(2\theta)}$



Sketching polar graphs:

Circles: $r = d\cos\theta$ (x-axis symmetry), $r = d\sin\theta$ (y-axis symmetry) $\Big\}$ d is the diameter

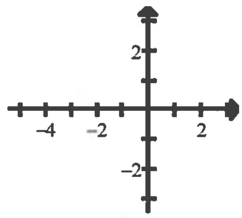
Rose petal curves: $r = a\cos(n\theta)$ (x-axis symmetry), $r = a\sin(n\theta)$ (y-axis symmetry) $\Big\}$ a is the maximum r, n petals if n is odd, $2n$ petals if n is even

Limaçons: $r = a \pm b\cos\theta$ (x-axis symmetry), $r = a \pm b\sin\theta$ (y-axis symmetry) $\Big\}$ may have inner loop, may or may not include the pole

Tangent lines

15. Find an equation of the tangent line to the graph of $r = 2(1-\sin\theta)$ at the point $(2,0)$.

16. Find the points at which the graph of $r = 2-2\cos\theta$ has horizontal tangents.



Assignment 10.2

Without using a calculator, accurately plot each of the following polar coordinate points on a separate graph. Give the rectangular coordinates of the point.

1. $\left(5,\frac{\pi}{2}\right)$ 2. $\left(-3\sqrt{2},\frac{3\pi}{4}\right)$ 3. $\left(4,-\frac{\pi}{3}\right)$ 4. $\left(-1,\frac{7\pi}{6}\right)$

Without using a calculator, plot each of the following rectangular points and give two sets of polar coordinates for $0\le\theta<2\pi$.

5. $(5,-5)$ 6. $\left(-1,\sqrt{3}\right)$ 7. $(-5,0)$ 8. $\left(-5,-5\sqrt{3}\right)$

9. Use a calculator to give rectangular coordinates for the polar point (-7.2,4.5).

10. Use a calculator to give two sets of polar coordinates ($0\le\theta<2\pi$) for the rectangular point (-2,5).

Match each of the following equations with one of the descriptions given without using a calculator.

11. $r=3\sin(2\theta)$ 12. $r=4\cos\theta$ 13. $r\cos\theta=4$ 14. $r=4+2\cos\theta$

a. a circle with y-axis symmetry
b. a four petal rose
c. a vertical line
d. a limaçon with y-axis symmetry
e. a circle with x-axis symmetry
f. a horizontal line.
g. a limaçon with x-axis symmetry

Convert the following rectangular equations to polar (solve for r) and sketch the graph.

15. $y=5$ 16. $2x+y+5=0$ 17. $y^2=2x$

Convert the following polar equations to rectangular and sketch the graph.

18. $r=5$ 19. $\theta=\frac{3\pi}{4}$ 20. $r=3\sec\theta$

Use a calculator to graph. Determine if the interval $0\le\theta<2\pi$ produces a complete graph.

21. $r=5\sin\left(\dfrac{3\theta}{2}\right)$ 22. $r=5-6\cos\theta$ 23. $r=\theta$

24. Without using a calculator graph $r=2+4\sin\theta$. Find an equation of the line tangent to the curve at the point $(2,0)$.

25. Without using a calculator graph $r=4\cos\theta$. Find an equation of the line tangent to the curve at the polar point $\left(2\sqrt{3},\frac{\pi}{6}\right)$.

26. Without using a calculator graph $r=1-\sin\theta$. Find the points at which the curve has horizontal tangent lines.

27. Find the length of the arc of the curve defined by $x=\ln t$ and $y=t+1$ on the interval $1\le t\le 6$.

28. Find an equation of the line tangent to the curve defined by $x=\ln t$ and $y=t+1$ when $t=1$.

29. Is the curve defined by $x = \ln t$ and $y = t+1$ concave upward or downward when $t = 2$?

30. Give a rectangular equation for the curve defined by $x = \ln t$ and $y = t+1$.

31. Sketch a graph of a function $f(x)$ having the following characteristics:
f is continuous. $f(-2) = f(0) = 0$.
$f'(0)$ is undefined. $f'(x) > 0$ for $x < -1$ and for $x > 0$. $f'(x) < 0$ on $(-1, 0)$.
$f''(x) < 0$ for $x < 0$, and $f''(x) > 0$ for $x > 0$.

32. Evaluate without using a calculator. $\int_0^\infty e^{-x}\,dx$

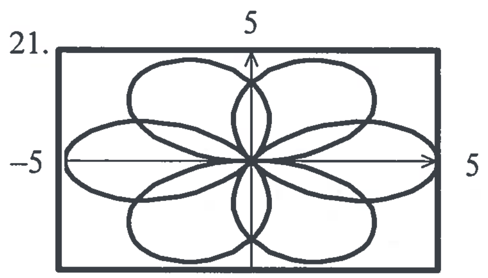
Selected Answers:

show graphs for 1-8 1. (0,5) 2. (3,-3) 3. $\left(2, -2\sqrt{3}\right)$ 5. $\left(5\sqrt{2}, \frac{7\pi}{4}\right)$, $\left(-5\sqrt{2}, \frac{3\pi}{4}\right)$

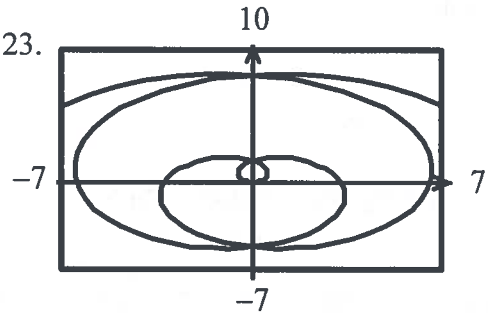
7. $(5, \pi)$, $(-5, 0)$ 8. $\left(10, \frac{4\pi}{3}\right)$, $\left(-10, \frac{\pi}{3}\right)$ 9. (1.517 or 1.518 , 7.038)

11. b 13. c 15. $r = 5\csc\theta$ 16. $r = \dfrac{-5}{2\cos\theta + \sin\theta}$ 17. $r = \dfrac{2\cos\theta}{\sin^2\theta}$

19. $y = -x$ 20. $x = 3$

21. 

$0 \le \theta < 4\pi$ shows a complete graph.

23. 

No interval shows a complete graph.

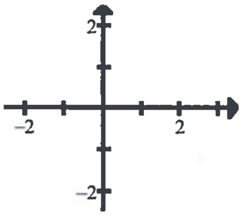
24. $y = \frac{1}{2}(x-2)$ 25. $y - \sqrt{3} = -\frac{1}{\sqrt{3}}(x-3)$ 26. $\left(2, \frac{3\pi}{2}\right)$, $\left(\frac{1}{2}, \frac{\pi}{6}\right)$, $\left(\frac{1}{2}, \frac{5\pi}{6}\right)$

27. 5.384 28. $y = x+2$ 30. $y = e^x + 1$ 32. 1

Lesson 10.3 Polar Area

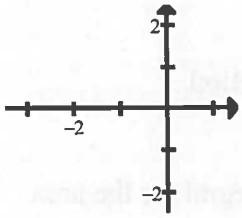
$$\text{Polar area} = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta$$

Example 1. Find the area of one petal of the curve $r = 3\cos(3\theta)$.



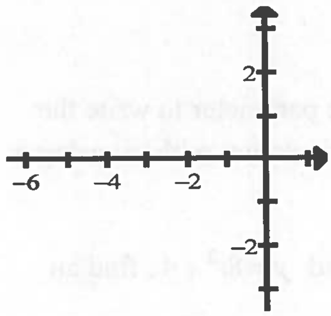
Intersections of Polar Graphs

Example 2. Find the points of intersection of the graphs of $r = 1 - 2\cos\theta$ and $r = 1$.

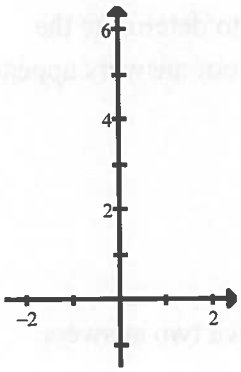


Area between two curves

Example 3. Find the area of the region common to the two regions bounded by $r = -6\cos\theta$ and $r = 2 - 2\cos\theta$.



Example 4. Find the area between the loops of $r = 2(1 + 2\sin\theta)$.



Assignment 10.3

Without using a calculator, graph the following polar curves and find the points of intersection.

1. $r = 1 + \sin\theta$ and $r = 1 - \sin\theta$
2. $r = 1 + \sin\theta$ and $r = 1 - \cos\theta$

3. Use a calculator to graph the following curves. Then find the points of intersection.
   $r = 6 - 8\sin\theta$ and $r = 2$

Graph the following polar curves without using a calculator. Set up a definite integral for the area of the indicated region. Use a calculator to evaluate the integral.

4. the interior of $r = 1 - \cos\theta$
5. one petal of $r = 4\sin(3\theta)$
6. one petal of $r = 3\cos(2\theta)$
7. the common interior of $r = 3 - 2\cos\theta$ and $r = -3 + 2\cos\theta$

Use a calculator to graph the following curves. Set up a definite integral for the area of the indicated region. Use a calculator to evaluate the integral.

8. between the loops of $r = 1 + 2\sin\theta$
9. inside $r = 3\cos\theta$ and outside $r = 2 - \cos\theta$
10. common interior of $r = 3$ and $r = 6\sin(2\theta)$
11. region bounded by $r = \theta + \sin(3\theta)$ and the x-axis for $0 \le \theta \le \pi$

12. Given the parametric equations $x = 4t - 1$ and $y = 8t - 4$, eliminate the parameter to write the corresponding rectangular equation. Sketch the curve indicating the orientation without using a calculator.

13. Without using a calculator given the parametric equations $x = 3t + 5$ and $y = 8t^2 + 4$, find an equation of the line tangent to the curve when $x = 2$.

14. Without using a calculator given the parametric equations $x = 4\cos\theta$ and $y = 8\sin\theta$, determine the concavity on an interval containing $\theta = \frac{7\pi}{6}$.

15. Given the parametric equations $x = 2 + 2\cos\theta$ and $y = 1 + \sin\theta$, show work to determine the points of horizontal and vertical tangency. Graph with a calculator to see if your answers appear correct.

16. Without a calculator convert the polar point $\left(3, \frac{3\pi}{2}\right)$ to rectangular form.

17. Without a calculator convert the polar point $\left(4, \frac{2\pi}{3}\right)$ to rectangular form.

18. Without a calculator convert the rectangular point $(-5, -5)$ to polar form. Give two answers such that $0 \le \theta < 2\pi$.

19. Using a calculator convert the rectangular point $(-1.372\ ,\ 5.164)$ to polar form. Give two answers such that $0 \le \theta < 2\pi$.

20. Give the rectangular form of the polar equation $r = 2\sin\theta$.
21. Given the polar equation $r = 1 - 2\sin\theta$:
    a. Use a calculator to sketch a graph.
    b. Without using a calculator, find an equation of the line tangent to the curve at the point $(1,0)$.
    c. Find the points at which the curve has vertical tangents. You may use a calculator.

Selected Answers:

1. $(1,0),\ (1,\pi),\ (0,0)$ 2. $\left(1+\frac{1}{\sqrt{2}}, \frac{3\pi}{4}\right),\ \left(1-\frac{1}{\sqrt{2}}, \frac{7\pi}{4}\right),\ (0,0)$ 4. 4.712
5. 4.188 or 4.189 6. 3.534 7. 10.557 or 10.558 8. 8.337 or 8.338 9. 5.196
10. 22.110 or 22.111 11. 7.000 12. $y = 2x - 2$ 13. $y - 12 = -\frac{16}{3}(x-2)$
15. V.T: $(4,1),\ (0,1)$ H.T: $(2,2),\ (2,0)$ 17. $\left(-2, 2\sqrt{3}\right)$ 18. $\left(5\sqrt{2}, \frac{5\pi}{4}\right),\ \left(-5\sqrt{2}, \frac{\pi}{4}\right)$
19. $(5.343, 1.830),\ (-5.343, 4.972)$ 21b. $y - 0 = -\frac{1}{2}(x-1)$

Lesson 10.4 Polar Arc Length, Vector Definitions

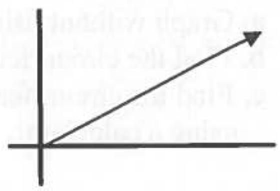
Arc Length

Example 1. Find the length of the arc from $\theta = 0$ to $\theta = 2\pi$ for the curve $r = 2 - 2\cos\theta$.

$$\text{Arc Length} = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Vectors (definitions):

Vector-



Magnitude-

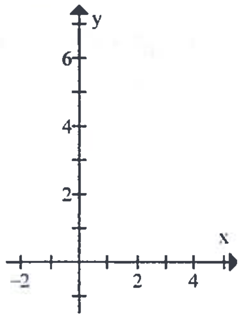
Direction-

Equivalent Vectors-

Component form-

Example 2. The initial point of a vector is $(-1,3)$ and its terminal point is $(2,7)$

a. Graph the vector.



b. Graph the vector in standard position on the same axes.

c. Give the component form of the vector.

d. Find the magnitude of the vector.

e. Find the direction of the vector.

Example 3. Find the direction of a vector given by $\langle -3\,,\,5\rangle$.

Example 4. If the magnitude of a vector v is $\|v\| = 6$ and its direction is $\theta = \frac{2\pi}{3}$, write the vector in component form.

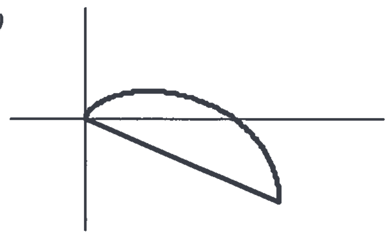
Assignment 10.4

1. Given the polar curve $r = 4\sin\theta$.
   a. Graph without using a calculator.
   b. Find the circumference using a geometry formula.
   c. Find the circumference showing a polar arc length integral setup and integrate without using a calculator.

2. Graph $r = 4\cos(2\theta)$ without a calculator. Then use a calculator to find the length of the arc forming one petal.

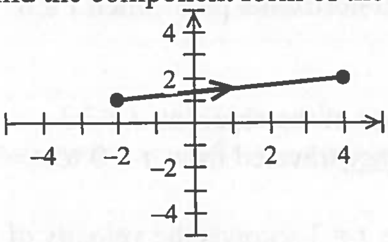
3. Use a calculator to graph $r = e^{\frac{\theta}{2}}$ on the interval $0 \le \theta \le \frac{3\pi}{2}$ and find the length of the curve.

4. The region shown is bounded by the polar curve $r = 1 - \sin\theta$ and the line $\theta = -\frac{\pi}{6}$.
   a. Find the area of the region.
   b. Find the perimeter of the region.

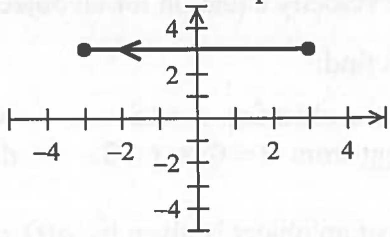


Find the component form of the vector and sketch it with the initial point at the origin.

5.



6.



7. Find the component form of the u and v vectors whose initial and terminal points are given. Show that u and v are equivalent. u: $(3,-2)$, $(5,2)$ $\quad v$: $(-1,-4)$, $(1,0)$

8. The initial and terminal points of a vector are $(-1,3)$ and $(3,6)$.

a. Sketch the vector.
b. Write the component form.
c. Sketch the vector with the initial point at the origin.

9. If the initial point of vector v is $(5,-2)$ and $v=\langle -2,4\rangle$, find the terminal point.

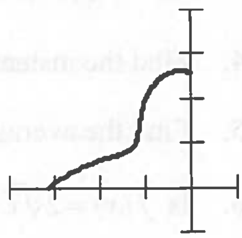
10. Find the magnitude of the vector $v=\langle -4,3\rangle$.

Find the component form of each vector given the magnitude and the direction without using a calculator.

11. $\|v\|=5, \theta=0.$ $\qquad$ 12. $\|v\|=6, \theta=\dfrac{4\pi}{3}.$ $\qquad$ 13. $\|v\|=4, \theta=\dfrac{7\pi}{4}.$

14. Use a calculator to find the magnitude and the direction of the vector $v=\langle -8,-13\rangle$.

15. The graph at the right shows the polar curve $r=\theta-\sin(3\theta)$ on the interval $\frac{\pi}{2}\le\theta\le\pi$.



a. Find the area of the region bounded by the curve, the x-axis, and the y-axis.

b. Find $\frac{dr}{d\theta}$ at $\theta=\frac{3\pi}{4}$ without using a calculator.

c. Use your answer to part b to determine if r is increasing or decreasing on an interval containing $\theta=\frac{3\pi}{4}$.

d. Find the value of θ on $\frac{\pi}{2}\le\theta\le\pi$ at which the curve is closest to the pole.

e. Find the x-coordinate of the point on the curve when $\theta=\frac{3\pi}{4}$.

f. Find $\frac{dx}{d\theta}$ at $\theta=\frac{3\pi}{4}$ using a calculator.

g. Use your answer to part f to determine if x is increasing or decreasing on an interval containing $\theta=\frac{3\pi}{4}$.

16. $v(t)=\dfrac{t-1}{t^2+1}$ is the velocity equation for an object moving along a horizontal path when $t\ge 0$.
Use a calculator to find:

a. the velocity of the object at $t=2.3$
b. the acceleration of the object at $t=2.3$
c. the displacement from $t=0$ to $t=3$.
d. the total distance traveled from $t=0$ to $t=3$.

17. If the acceleration of an object is given by $a(t)=e^{\sin t}$ and at time $t=3$ seconds the velocity of the object is 15 feet per second, find the object's velocity at $t=10$ seconds.

18. $f(x)=2\sqrt{x}-x$. Without using a calculator, find:
   a. the domain of $f(x)$.
   b. the x-intercept(s) for the graph of $f(x)$.
   c. the maximum and minimum function values of $f(x)$.
   d. the range of $f(x)$.

19. Show that the graph of $f(x)=2\sqrt{x}-x$ has no points of inflection by building an f'' number line which indicates the concavity for the graph of f.

20. Use the results from Problems 18 and 19 to sketch a graph of $f(x)=2\sqrt{x}-x$ without using a calculator.

21. Find $\int_0^4 (2\sqrt{x}-x)\,dx$ without using a calculator.

22. Find the average value of the function $f(x)=2\sqrt{x}-x$ on $[0,4]$.

23. Use a calculator to find the value of c where $f(c)=f_{avg}$ for Problem 22.

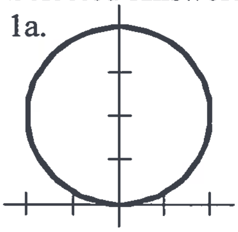
24. Find the instantaneous rate of change of $f(x)=2\sqrt{x}-x$ at $x=2$.

25. Find the average rate of change of $f(x)=2\sqrt{x}-x$ on $[0,4]$.

26. Is $f(x)=2\sqrt{x}-x$ continuous on [0, 4] and differentiable on (0, 4)?

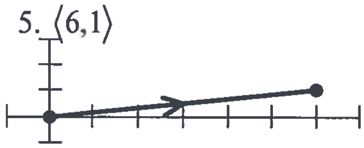
27. If your answer to Problem 26 was yes, then the MVT guarantees that
$f'(c)=\dfrac{f(b)-f(a)}{b-a}$ (the answer to Problem 25) for some c in (0, 4). Find c.
If your answer to Problem 26 was no, then do it again.

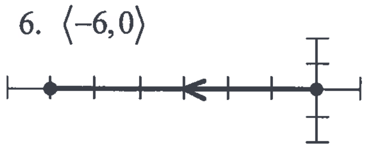
Selected Answers:

1a.


b. 4π 2. 9.688 3. 21.356

4a. .596 or .597 b. 3.499 or 3.500

5. $\langle 6,1\rangle$


6. $\langle -6,0\rangle$


More Selected Answers

7. $u = \langle 2,4 \rangle = v$ 8b. $\langle 4,3 \rangle$ 9. $(3,2)$ 10. $\|v\| = 5$ 11. $v = \langle 5,0 \rangle$

12. $v = \left\langle -3, -3\sqrt{3} \right\rangle$ 13. $v = \left\langle 2\sqrt{2}, -2\sqrt{2} \right\rangle$ 14. $\|v\| = 15.264$, $\theta = 4.160$ or 4.161

15a. 3.756 b. $1 - \frac{3}{\sqrt{2}}$ d. 2.504 or 2.505 e. -1.166 f. -.373

16a. $v(2.3) = .206$ or $.207$ b. $a(2.3) = .007$ or $.008$ c. disp. = $-.097$ or $-.098$

16d. T. D. = .779 or .780 17. 23.547 18b. x-int.: $(0,0)$, $(4,0)$

18d. range: $y \le 1$ 21. $\boxed{\left(\frac{4}{3} \bullet 8 - \frac{1}{2} \bullet 16\right) - 0} = \frac{8}{3}$ 22. $\frac{2}{3}$ 23. $c = .178$ or $.179$, 2.488

25. 0 27. c = 1

Lesson 10.5 Calculus of Vector-Valued Functions

Position vector-

Speed-

Velocity vector-

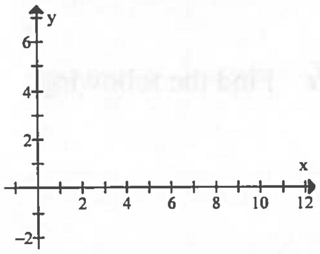
Distance traveled-

Acceleration vector-

"at rest"

Example 1. Given a position vector $\left\langle 3t^2, t^3 - 3t^2 + 4 \right\rangle$ for a particle moving in the xy-plane find the following.

a. graph the path of the particle on the interval $0 \le t \le 2$



b. the velocity vector at time $t = 1$

c. the speed of the particle at time $t = 1$

d. the distance traveled between $t = 0$ and $t = 3$

e. the time(s) when the particle is at rest

f. the acceleration vector at time $t = 2$

g. the direction of the particle at time $t = 1$ and when $t = 2$

Example 2. A particle is moving in the xy-plane with acceleration vector $a(t)=\langle -2\cos t,-3\sin t\rangle$. At time $t=0$ its velocity vector is $v(0)=\langle 0,3\rangle$ and its position vector is $s(0)=\left\langle 2,\frac{\sqrt{2}}{2}\right\rangle$. Find :

a. the velocity vector when $t=\frac{\pi}{4}$

b. the position vector when $t=\frac{\pi}{4}$

c. the speed when $t=\frac{\pi}{4}$

d. the time(s) when the particle is at rest

e. the direction of the particle when $t=\frac{\pi}{4}$

f. the distance traveled between $t=0$ and $t=2$

Assignment 10.5

1. The position of a particle in the xy-plane is given by $x=4t^2$ and $y=\sqrt{t}$. Find the following:

   a. the velocity vector at $t=4$
   b. the acceleration vector at $t=4$
   c. the speed of the particle at $t=4$
   d. the distance the particle moves between $t=0$ and $t=4$
   e. the direction of the particle at $t=4$

2. The position of a particle is given by $x=t^2$ and $y=t^3$. Find the following:

   a. the speed of the particle at $t=2$
   b. the direction of the particle at $t=2$
   c. the distance the particle moves between $t=1$ and $t=4$
   d. the velocity vector at $t=3$
   e. the acceleration vector at $t=4$

3. A particle moves in the xy-plane so that at time t its velocity vector is $v(t) = \langle t^3, \cos(\pi t) \rangle$ and the particle's position vector at time $t = 0$ is $\langle 2,1 \rangle$.
   a. What is the position vector of the particle when $t = 2$? Do not use a calculator.
   b. What is the acceleration vector of the particle when $t = 2$? Do not use a calculator.
   c. What is the direction of the particle when $t = 1$?
   d. What is the distance the particle travels between $t = 0$ and $t = 2$?
   e. When is the particle at rest?
   f. What is the speed of the particle when $t = 2$?

4. A particle moves on the xy-plane so that at time t its coordinates are $x = t^3 + t$ and $y = t^5 - 2t^2$. Find its velocity vector at time $t = 2$.

5. A calculator is allowed on this problem.
   The position of an object moving on a curve is $(x(t), y(t))$ at time t.
   Given $\frac{dx}{dt} = \sqrt{\frac{t}{2+t}}$ and $\frac{dy}{dt} = \cos(t^2 - 1)$. At time $t = 1$, the position of the object is $(2,4)$.
   a. Find the position of the object at time $t = 3$.
   b. Find the speed of the object at time $t = 3$.
   c. Find the total distance traveled by the object over the interval $1 \le t \le 3$.
   d. Find an equation of the line tangent to the curve at time $t = 3$.
   e. Find the acceleration vector at time $t = 3$.

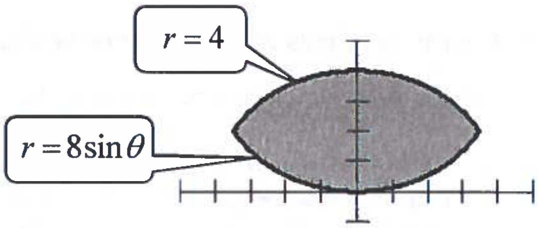
6. Given a parametric curve defined by $x = e^t$ and $y = t + 1$.
   a. Find the length of the arc of the curve on the interval $1 \le t \le 6$.
   b. Find an equation of the line tangent to the curve when $t = 1$.
   c. Is the curve concave upward or downward when $t = 2$?
   d. Give a rectangular equation for the curve.
   e. Show an integral setup with respect to the variable x that gives the length of the same arc as that in part a.

7. Find all points of horizontal and vertical tangency to the curve $x = 2 - 2\cos\theta$, $y = 2\sin(2\theta)$.

8. Find the area common to the interiors of $r = 4\cos\theta$ and $r = 2$.

9. Given the function $f(x) = \ln(x+1)$:
   a. Write a power series for f showing four terms and the general term.
   b. Find the interval of convergence of this power series.
   c. Approximate $\ln(1.2)$ by using a fourth degree Taylor polynomial of f.
   d. Using your answer for part c and the alternating series remainder, give an upper and lower limit for the actual value of $\ln(1.2)$

10. The figure at the right shows a shaded region bounded by the polar curves $r=4$ and $r=8\sin\theta$.



a. Find the area of the shaded region.
b. Find the perimeter of the shaded region.
c. Convert the two polar equations to rectangular form.
d. Set up an integral with respect to the variable x and find the area of the shaded region.

Selected Answers:

1a. $v(4)=\left\langle 32,\frac{1}{4}\right\rangle$ b. $a(4)=\left\langle 8,-\frac{1}{32}\right\rangle$ c. $\|v(4)\|=32.001$ d. dist. = 64.413 e. $\theta=.031$

2a. $\|v(2)\|=12.649$ b. $\theta=1.107$ c. dist. = 64.949 d. $v(3)=\langle 6,27\rangle$ e. $a(4)=\langle 2,24\rangle$

3a. $s(2)=\langle 6,1\rangle$ b. $a(2)=\langle 12,0\rangle$ c. $\theta=.418$ d. dist. = 4.567 or 4.568 e. never

3f. $\|v(2)\|=8.062$ 4. $\langle 13,72\rangle$ 5a. $s(3)=\langle 3.394, 4.280 \text{ or } 4.281\rangle$ b. $\|v(3)\|=.788$

5c. dist. = 1.960 or 1.961 d. $y-4.280=-.187(x-3.394)$ or $y-4.281=-.188(x-3.394)$

5e. $a(3)=\langle .051 \text{ or } .052,-5.936\rangle$ 6a. 400.891 b. $y-2=\frac{1}{e}(x-e)$ c. concave down

6d. $y=\ln x+1$ 7. Horiz. $\left(2\pm\sqrt{2},\pm 2\right)$, Vert. $(0,0)$, $(4,0)$

8. 4.913 9a. $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+\frac{(-1)^{n+1}x^n}{n}+\cdots$ b. $(-1,1]$ c. .18227

10a. 19.653 or 19.654 b. 16.755 c. $x^2+y^2=16$, $x^2+y^2=8y$ d. 19.653 or 19.654

Assignment 10.6 Review

1. Given the parametric equations $x=t+3$, $y=t^2$, eliminate the parameter to write the corresponding rectangular equation. Sketch the curve indicating the orientation without using a calculator.

2. Given the parametric equations $x=\frac{1}{2}t^2$, $y=\frac{1}{3}t^3-\frac{1}{2}t^2$:
   a. Find the point(s) at which the curve has a horizontal tangent.
   b. Use the second derivative test to determine if the curve has a local maximum, a local minimum, or neither at the point found in part a.
   c. Find an equation of the tangent line when $t=6$.

3. Given the parametric equations $x=3\cos\theta$, $y=2\sin\theta$, find the length of the arc on the interval $0\le\theta\le\frac{\pi}{2}$,

4. Convert the polar equation $r=4\sin\theta$ to rectangular form. Sketch a graph without using a calculator.

5. Convert the polar equation $r=-3-3\cos\theta$ to rectangular form. Sketch a graph without using a calculator.

6. Graph $r=2-3\cos\theta$ without using a calculator.

7. Without using a calculator, find an equation of the line tangent to the graph of $r=2-3\cos\theta$ at $\theta=\frac{\pi}{2}$.

8. Find all points of intersection of $r=2-3\cos\theta$ and $r=-5\cos\theta$.

9. Use a calculator to graph $r^2=10\sin(2\theta)$ and $r=\sqrt{5}$ and find the area common to the two interiors.

10. Set up an integral for the length of the arc in the first quadrant on the curve $r^2=10\sin(2\theta)$. Why is this integral improper?

11. The position of a particle moving in the *xy*-plane at any time t, $0\le t<2\pi$, is given by the parametric equations $x=4\cos(2t)$ and $y=4\sin t$. Do not use a calculator.
    a. Find the velocity vector for the particle at any time t.
    b. Find the velocity vector for the particle at time $t=\pi$.
    c. Find the acceleration vector for the particle at time $t=\pi$.
    d. Find the speed when $y=-4$.
    e. For what values of t is the particle at rest?
    f. Find the direction of the particle when $t=\frac{\pi}{2}$.
    g. Set up an integral for the distance traveled on the interval $1\le t\le 5$.

12. A particle moves along the graph of $y=\sin x$. If the x-component of acceleration is always 3 and at time $t=0$, the position of the particle is the point $(\pi,0)$ and the velocity vector of the particle is $\langle 2,-2\rangle$. Without using a calculator. Find the x- and y-coordinates of the position of the particle in terms of t.

13. Calculator Allowed
    A particle moving along a curve in the *xy*-plane has position $(x(t),y(t))$ with $\frac{dx}{dt}=\sin t^2$ and $\frac{dy}{dt}=e^{t^2}$. The position of the particle is $(-1,4)$ at time $t=3$.
    a. Find the acceleration vector at time $t=2$.
    b. Find the position point of the particle at time $t=0$.
    c. At what time does the speed of the particle reach 15 when $t>0$.

14. Calculator Allowed
    Let g be a function that has derivatives of all orders for all real numbers. Assume $g(0)=2$, $g'(0)=-1$, $g''(0)=4$, and $g'''(0)=3$.
    a. Write the third-degree Taylor polynomial for g about $x=0$.
    b. Write the fourth-degree Taylor polynomial for $g(x^2)$.
    c. Write the third-degree Taylor polynomial for h, where $h(x)=\int_0^x g(t)\,dt$, about $x=0$.

15. Euler's Formula

a. Use the elementary series for e^x to write at least six terms of the series for $e^{i\pi}$.

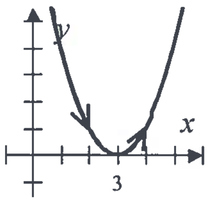
b. Simplify your answer using the relationship for powers of i. ($i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, etc.)

c. Separate the terms of your simplified series into two infinite series (one with the odd power terms and one with the even power terms). Use elementary series for sine and cosine to evaluate.

Note: This result is sometimes written as the equation $e^{i\pi} + 1 = 0$ called Euler's Formula. This mathematically elegant equation contains the five most "important" numbers in mathematics.

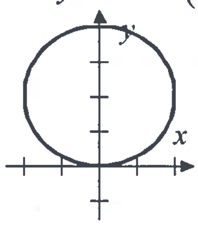
Answers:

1. $y = (x-3)^2$

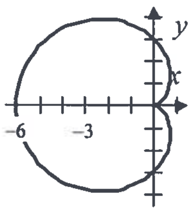


2a. $\left(\frac{1}{2}, -\frac{1}{6}\right)$ b. local minimum c. $y - 54 = 5(x - 18)$

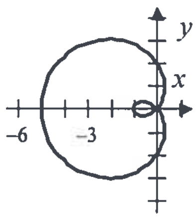
3. 3.966 4. $x^2 + y^2 = 4y$



5.



6.



7. $y - 2 = -\frac{3}{2}(x - 0)$

8. $(5, \pi)$, $(0,0)$ 9. 6.575 or 6.576

10. $\int_0^{\frac{\pi}{2}} \sqrt{10\sin(2\theta) + \left(\frac{10\cos(2\theta)}{\sqrt{10\sin(2\theta)}}\right)^2}\, d\theta$

This integral is improper since $\frac{dr}{d\theta}$ is undefined at both endpoints.

11a. $v(t) = \langle -8\sin(2t), 4\cos t\rangle$ b. $v(\pi) = \langle 0, -4\rangle$ c. $a(\pi) = \langle -16, 0\rangle$ d. $\left\| v\left(\frac{3\pi}{2}\right)\right\| = 0$

11e. $t = \frac{\pi}{2}, \frac{3\pi}{2}$ f. $\theta = \frac{3\pi}{4}$ g. $\int_1^5 \sqrt{(-8\sin(2t))^2 + (4\cos t)^2}\, dt$

12. $x(t) = \frac{3}{2}t^2 + 2t + \pi$, $y(t) = \sin\left(\frac{3}{2}t^2 + 2t + \pi\right)$ 13a. $a(2) = \langle -2.614 \text{ or } -2.615, 218.393\rangle$

13b. $(-1.773 \text{ or } -1.774, -1440.545)$ c. $t = 1.645$

14a. $g(x) \approx 2 - x + 2x^2 + \frac{1}{2}x^3$ b. $g(x^2) \approx 2 - x^2 + 2x^4$ c. $h(x) \approx 2x - \frac{1}{2}x^2 + \frac{2}{3}x^3$

15a. $e^{i\pi} = 1 + (i\pi) + \frac{(i\pi)^2}{2!} + \frac{(i\pi)^3}{3!} + \frac{(i\pi)^4}{4!} + \frac{(i\pi)^5}{5!} + \frac{(i\pi)^6}{6!} + \frac{(i\pi)^7}{7!} + \cdots$

b. $e^{i\pi} = 1 + (i\pi) - \frac{\pi^2}{2!} - \frac{i\pi^3}{3!} + \frac{\pi^4}{4!} + \frac{i\pi^5}{5!} - \frac{\pi^6}{6!} - \frac{i\pi^7}{7!} + \cdots$

c. -1

UNIT 10 SUMMARY

Parametric derivatives: $\dfrac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$ $\qquad \dfrac{d^2y}{dx^2}=\dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$

Arc Length $=\int_a^b\sqrt{1+\left(f'(x)\right)^2}\,dx=\underbrace{\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt}_{\text{parametric}}=\underbrace{\int_{\theta_1}^{\theta_2}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta}_{\text{polar}}$

Polar Conversions: $r^2=x^2+y^2$, $\quad x=r\cos\theta$, $\quad y=r\sin\theta$, $\quad \theta=\arctan\frac{y}{x}$

Sketching polar graphs:

Circles: $\left.\begin{array}{ll} r=d\cos\theta & (x\text{-axis symmetry}) \\ r=d\sin\theta & (y\text{-axis symmetry}) \end{array}\right\}$ d is the diameter

Rose petal curves: $\left.\begin{array}{ll} r=a\cos(n\theta) & (x\text{-axis symmetry}) \\ r=a\sin(n\theta) & (y\text{-axis symmetry}) \end{array}\right\}$ a is the maximum r, n petals if n is odd, $2n$ petals if n is even

Limaçons: $\left.\begin{array}{ll} r=a\pm b\cos\theta & (x\text{-axis symmetry}) \\ r=a\pm b\sin\theta & (y\text{-axis symmetry}) \end{array}\right\}$ may have inner loop, may or may not include the pole

Polar Area $=\frac{1}{2}\int_{\theta_1}^{\theta_2}r^2\,d\theta$

Vectors:

Magnitude = $\sqrt{x^2+y^2}$ $\qquad$ Direction = $\theta=\arctan\frac{y}{x}$

Speed = $|v(t)|=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$ $\qquad$ Total Dist. = $\int_{t_1}^{t_2}|v(t)|\,dt=\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

LIST OF DIFFERENTIATION FORMULAS

Power Rule

$\frac{d}{dx} x^n = n x^{n-1}$

General Power Rule

$\frac{d}{dx} u^n = n u^{n-1} u'$ (where u is a function of x)

Chain Rule

$\frac{d}{dx} f(g(x)) = f'(g(x)) g'(x)$

Product Rule: $\frac{d}{dx}(f \bullet s) = fs' + sf'$

Quotient Rule: $\frac{d}{dx} \frac{t}{b} = \frac{bt' - tb'}{b^2}$

Exponential and Logarithmic Rules

$\frac{d}{dx} e^x = e^x$

$\frac{d}{dx} a^x = a^x \ln a$

$\frac{d}{dx} \ln x = \frac{1}{x}$

$\frac{d}{dx} \ln|x| = \frac{1}{x}$

$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$

$\frac{d}{dx} e^u = e^u u'$

$\frac{d}{dx} a^u = a^u u' \ln a$

$\frac{d}{dx} \ln u = \frac{u'}{u}$

$\frac{d}{dx} \ln|u| = \frac{u'}{u}$

$\frac{d}{dx} \log_a u = \frac{u'}{u \ln a}$

Trigonometric and Inverse Trigonometric Rules

$\frac{d}{dx} \sin x = \cos x$

$\frac{d}{dx} \cos x = -\sin x$

$\frac{d}{dx} \tan x = \sec^2 x$

$\frac{d}{dx} \cot x = -\csc^2 x$

$\frac{d}{dx} \sec x = \sec x \tan x$

$\frac{d}{dx} \csc x = -\csc x \cot x$

$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$

$\frac{d}{dx} \sin u = \cos u \, u'$

$\frac{d}{dx} \cos u = -\sin u \, u'$

$\frac{d}{dx} \tan u = \sec^2 u \, u'$

$\frac{d}{dx} \cot u = -\csc^2 u \, u'$

$\frac{d}{dx} \sec u = \sec u \tan u \, u'$

$\frac{d}{dx} \csc u = -\csc u \cot u \, u'$

$\frac{d}{dx} \arcsin u = \frac{u'}{\sqrt{1-u^2}}$

$\frac{d}{dx} \arccos u = \frac{-u'}{\sqrt{1-u^2}}$

$\frac{d}{dx} \arctan u = \frac{u'}{1+u^2}$

LIST OF INTEGRATION FORMULAS

Power Rule for Integrals

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

General (u) Forms (Where u is a function of x)

$$\int u^n\,u'\,dx = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$$

Reverse Chain Rule

$$\int f'\left(g(x)\right)g'(x)\,dx = f\left(g(x)\right) + C$$

Exponential and Logarithmic Integrals

$$\int e^x\,dx = e^x + C \qquad \int e^u\,u'\,dx = e^u + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C \qquad \int a^u\,u'\,dx = \frac{a^u}{\ln a} + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C \qquad \int \frac{u'}{u}\,dx = \ln|u| + C$$

Trigonometric Integrals

$$\int \cos x\,dx = \sin x + C \qquad \int \cos u\,u'\,dx = \sin u + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \sin u\,u'\,dx = -\cos u + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \sec^2 u\,u'\,dx = \tan u + C$$

$$\int \csc^2 x\,dx = -\cot x + C \qquad \int \csc^2 u\,u'\,dx = -\cot u + C$$

$$\int \sec x\tan x\,dx = \sec x + C \qquad \int \sec u\tan u\,u'\,dx = \sec u + C$$

$$\int \csc x\cot x\,dx = -\csc x + C \qquad \int \csc u\cot u\,u'\,dx = -\csc u + C$$

$$\int \tan x\,dx = -\ln|\cos x| + C \qquad \int \tan u\,u'\,dx = -\ln|\cos u| + C$$

$$\int \cot x\,dx = \ln|\sin x| + C \qquad \int \cot u\,u'\,dx = \ln|\sin u| + C$$

$$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C \qquad \int \tan^2 u\,u'\,dx = \tan u - x + C$$

Integrals Involving Inverse Trig Functions

$$\int \frac{u'}{\sqrt{1-u^2}}\,dx = \arcsin u + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$\int \frac{u'}{\sqrt{a^2-u^2}}\,dx = \arcsin\frac{u}{a} + C$$

$$\int \frac{u'}{1+u^2}\,dx = \arctan u + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

$$\int \frac{u'}{a^2+u^2}\,dx = \frac{1}{a}\arctan\frac{u}{a} + C$$

Lesson A-1 Shell Method Volume

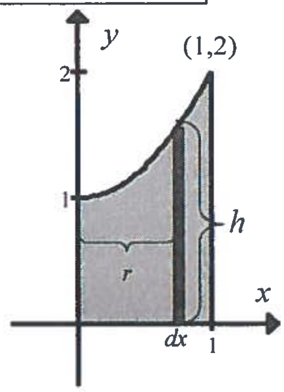
Key: The representative element (rectangle) must be <u>parallel</u> to the axis of revolution. Recall that for discs or washers, the element had to be perpendicular to the axis of revolution..
Remember: **<u>PARASHELL</u> vs <u>PERPENDISCULAR</u>**

Revolving rectangular elements about a parallel axis produces cylindrical shells (like the wrappings around a toilet paper roll).

The volume formula for the shell method is: $\boxed{V = 2\pi\int_a^b rh\,dx \text{ (or } dy)}$ $r > 0,\ h > 0$

<u>Examples</u>:

1. Find the volume of the solid formed by revolving the region bounded by $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about the y-axis.

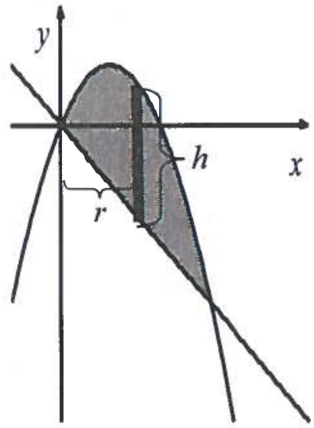


$r = x$

$h = x^2 + 1$

V =

Why would using the disc method for this problem be much harder?



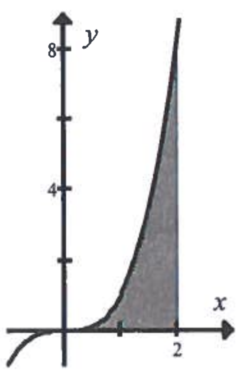
2. <u>Set up</u> (but do not integrate) an integral giving the volume of the solid formed by revolving the region bounded by $y = 2x - x^2$ and $y = -x$ about the y-axis.

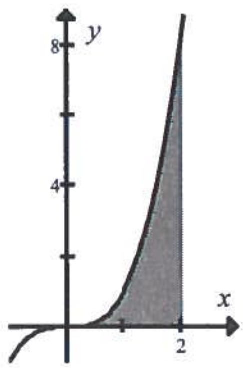
$r =$

$h =$

V =

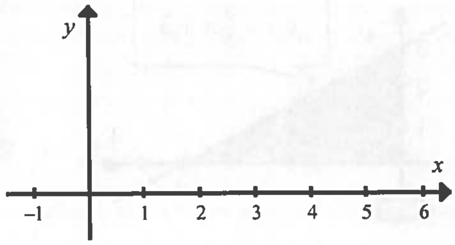
3. Use both the <u>shell method</u> and the <u>disc method</u> to find the volume formed by revolving the region bounded by $y = x^3$, $x = 2$, and $y = 0$ about the x-axis.





For revolutions about lines other than the x- and y-axes, the formula is still $V = 2\pi\int_a^b rh\,dx$ (or dy),

but r is slightly harder to find.
(Remember that both r and h must be nonnegative.)

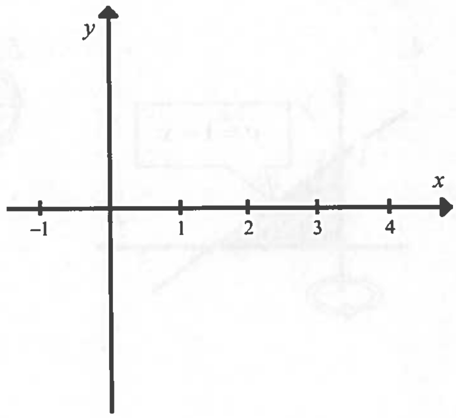


Example 4: Use the Shell Method to set-up integrals which could be used to find the volumes of the solids formed when the region bounded by $y=\sqrt{x-1}$, $y=0$, and $x=5$ is revolved about:

a. the y-axis

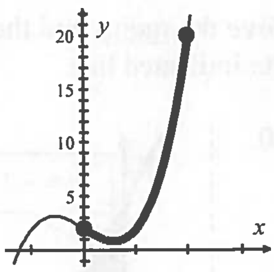
b. $x=-2$

c. $x=5$



Example 5: Find the volume of the solid formed by revolving the region bounded by $x = y^2$ and $x=4$ about the line $y=-3$.

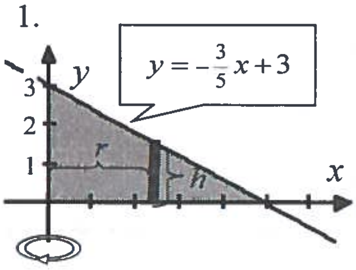
Example 6: Set up an integral (but do not integrate) which could be used to find the volume of the solid which would be formed if the region from Example 5 were revolved about the line $y=3$,



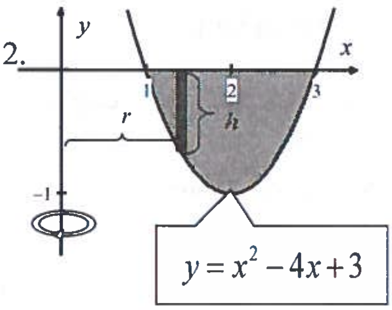
Assignment A-1

Give the radius and the height of the shell formed when the representative element is revolved about the indicated axis.

1.



2.



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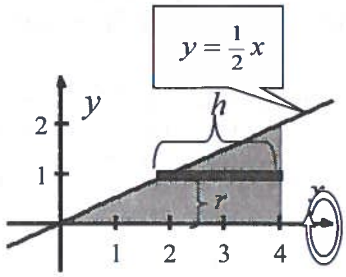
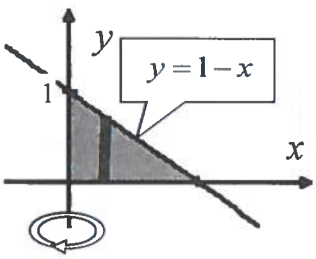
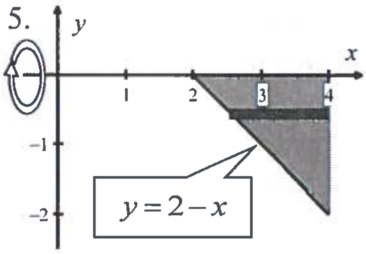


For problems 4-7, **set up** integrals (but do not integrate) which could be used to find the volumes formed by revolving the plane regions described or shown about the indicated axes. Use the **shell method**.

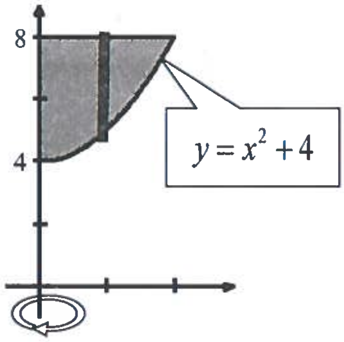
4.



5.



6.



7. Region bounded by $y=\sqrt{x}$, $y=0$, and $x=5$ revolved about the x-axis.

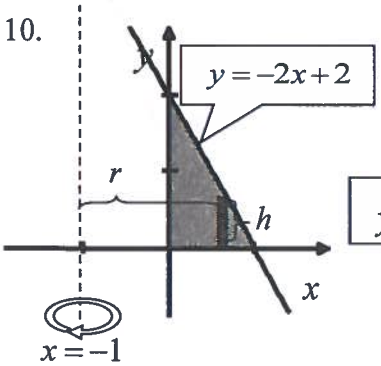
Use **discs, washers, or shells** to **find the volumes** indicated in problems 8 and 9 without a calculator.

8. Region bounded by $y=x$ and $y=x^3$ in Quadrant I revolved about the x-axis.

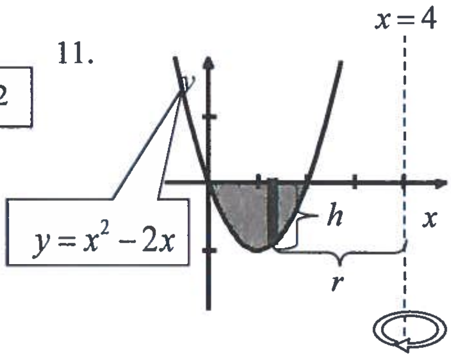
9. Region bounded by $y=\sqrt{x}+2$, $x=0$, $y=0$, and $x=4$ revolved about the y-axis.

Give the radius and the height of the shell formed when the representative element is revolved about the indicated line.

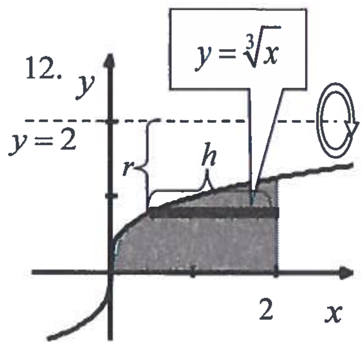
10.



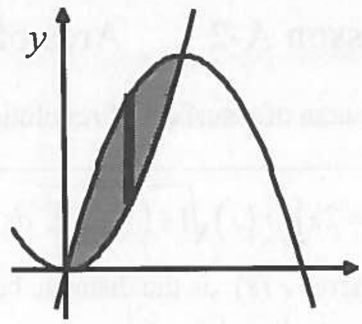
11.



12.



For problems 13-15, **set up** integrals (but do not integrate) which could be used to find the volumes formed by revolving the plane region bounded by $y = x^2$ and $y = 4x - x^2$ about the indicated axis of revolution. Use **any method**.



13. about the x-axis

14. about $x = 4$

15. about $x = -1$

For problems 16 and 17 draw accurate sketches and **set up** integrals (but do not integrate) which could be used to find the volumes formed by revolving the plane region bounded by $x = y - y^2$ and $x = 0$ about the indicated axes of revolution. Use **any method**.

16. about $y = 2$

17. about $y = -3$

18. **Set up** integrals (but do not integrate) which could be used to find the volumes formed by revolving the plane region bounded by $y = \frac{1}{2}x$, $y = 0$, and $x = 6$ about the line $y = -2$. Use the indicated method.
    a. washers
    b. shells

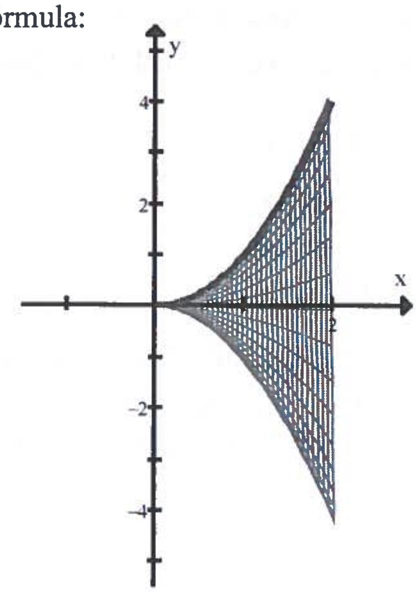
19. **Find** the volumes of the solids formed by revolving the region bounded by $y = \dfrac{1}{x^2}$, $y = 0$, $x = 1$, and $x = 3$ about the indicated axes. You may use a calculator.
    a. about the y-axis
    b. about the line $x = 4$

Lesson A-2 Area of a Surface of Revolution

The area of a surface of revolution can be found by using the formula:

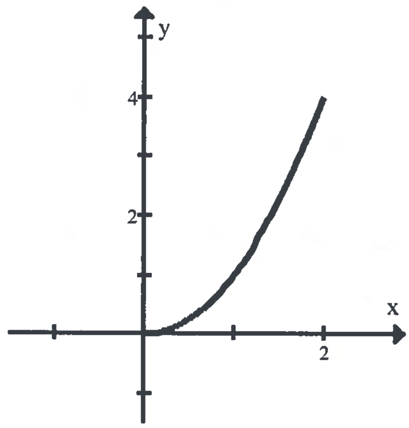
$$S = 2\pi \int_a^b r(x)\sqrt{1+\left(f'(x)\right)^2}\, dx$$

where $r(x)$ is the distance between the graph and the axis.



Example 1:
Find the area of the surface formed by revolving the graph of $y = x^2$ on the interval $[0,2]$ about the x-axis.

Example 2:
Find the area of the surface formed by revolving the graph of $y = x^2$ on the interval $[0,2]$ about the y-axis. (No calculator)



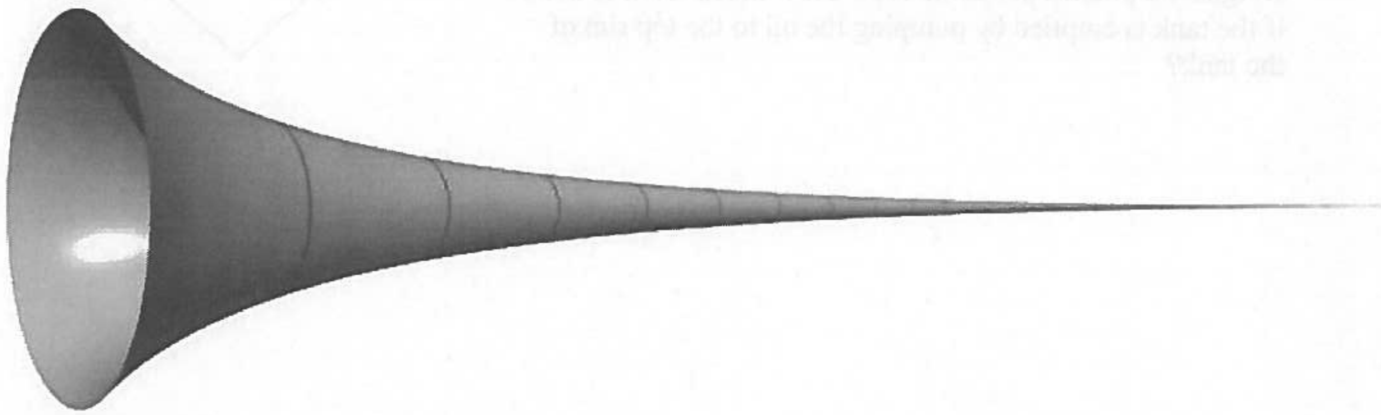
Assignment A-2

1. Find the area of the surface generated by revolving $y = x^3$ on the interval $0 \le x \le 4$ about the x-axis.
2. Find the area of the surface generated by revolving $y = \sqrt{x}$ on the interval $1 \le x \le 4$ about the x-axis.
3. Find the area of the surface generated by revolving $y = x^2 + \dfrac{1}{x^2}$ on the interval $1 \le x \le 5$ about the x-axis.
4. Find the area of the surface generated by revolving $y = 3x$ on the interval $1 \le x \le 10$ about the x-axis.

5. Find the area of the surface generated by revolving $y = x^2 + 2$ on the interval $0 \le x \le 2$ about the y-axis.
6. Find the area of the surface generated by revolving $y = \sin x$ on the interval $0 \le x \le \frac{\pi}{2}$ about the y-axis.
7. Find the area of the surface generated by revolving $y = \ln x$ on the interval $1 \le x \le e$ about the y-axis.
8. A helium gas storage tank with a solid top is formed in the shape of $y = x^4$ revolved about the y-axis. The height of the tank is 16 feet.
   a. What is the surface area of the entire tank?
   b. If the tank is constructed with metal $1/8$ inch $\left(1/96 \text{ feet}\right)$ thick how much metal was used to construct the tank.
   c. If the metal weighs 170 pounds per cubic foot, how much does the empty metal tank weigh?
9. Gabriel's Horn

   A solid is formed by revolving the region below the graph of $y = \frac{1}{x}$ for $x \ge 1$ about the x- axis.
   a. Find the volume of the solid by evaluating an improper integral without using a calculator.
   b. Use a calculator to find the surface area of the solid for :
      i. $1 \le x \le 10$.
      ii. $1 \le x \le 1000$.
      iii. $1 \le x \le 10000000$.
   c. What surface area do you expect for $x \ge 1$?

Note: This unbounded shape called Gabriel's Horn has a finite volume but an infinite surface area. This famous mathematical paradox means the horn would be filled with a finite volume of paint but this paint would not be enough to cover the inner surface of the horn since it has an infinite area.



Lesson A-3 Work

Work done by a constant force: $W = FD$

Example 1:

Find the work done in lifting a 40 pound suitcase 2 feet and moving it horizontally 10 feet.

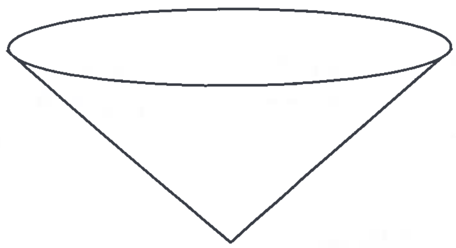
Work done by a variable force or over a variable distance: $W = \int_a^b \textit{force} \bullet \textit{distance}$

Example 2:

A bucket of water weighing 100 newtons (N) is at the bottom of a 20 meter well attached to a rope weighing 0.4 N/m. How much work is done in lifting the bucket to the surface by winding the rope around a windlass?

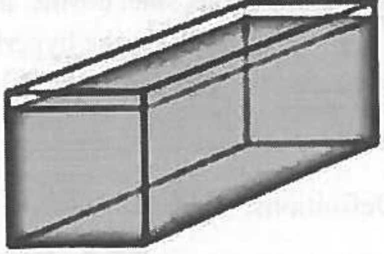
Example 3:

A conical tank with a height of 10 feet and a top diameter of 20 feet is filled to within 1 foot of the top with oil that weighs 50 pounds per cubic foot. How much work is done if the tank is emptied by pumping the oil to the top rim of the tank?



Assignment A-3

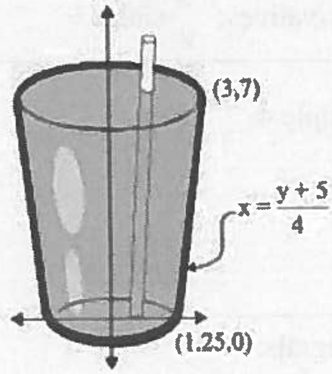
1. Find the work done in lifting a 50 pound concrete block from the ground to a height of 6 feet.
2. A rectangular tank has a base 7 feet by 17 feet and a height of 9 feet. The tank is filled with water which weighs 62.4 pounds per cubic foot to a depth of 8 feet. The water is pumped out over the top edge of the tank.
   a. How much work is done to empty half of the water in the tank?
   b. How much work is done to completely empty the water in the tank?



3. A cylindrical gasoline tank has a height of 13 meters and a diameter of 25 meters. Gasoline weighs 8300 newtons per cubic meter. How much work is done to empty a full tank by pumping the gasoline to the top of the tank.

4. A 20 foot chain is hanging freely from a winch 20 feet above ground level. The chain weighs 4 pounds per foot.
   a. How much work is done to raise the bottom of the chain to a height of 15 feet?
   b. How much work is done to wind up the full 20 feet of chain?
   c. If a 200 pound weight is attached to the end of the chain, how much work is done to raise the chain and weight 5 feet?

5. A drinking glass is filled with water that weighs .58 ounces per cubic inch. An 8 inch straw extends one inch above the 7 inch tall glass. How much work does it take to drink the water through the straw?



Lesson A-4 Hyperbolic Functions

The trig functions sine, cosine, and tangent are often defined using points on the unit circle $x^2 + y^2 = 1$. Similarly the hyperbolic functions can be defined using points on a unit hyperbola $-x^2 + y^2 = 1$. They can also be defined using exponential functions as follows.

Definitions: $\sinh x = \dfrac{e^x - e^{-x}}{2}$	$\cosh x =$	$\tanh x =$

Example 1: Find the value of $\sinh 0$, $\cosh 0$, and $\tanh 0$ without using a calculator.

Example 2. Use a calculator to find $\sinh \pi$.

Example 3: Find $\lim\limits_{x\to\infty} \tanh x$ without using a calculator.

Derivatives: $\dfrac{d}{dx}\sinh x =$	$\dfrac{d}{dx}\cosh x =$	$\dfrac{d}{dx}\tanh x =$

Example 4: $\dfrac{d}{dx}\sinh x^2 =$

Integrals: $\int \sinh x\,dx =$	$\int \cosh x\,dx =$	$\int \operatorname{sech}^2 x\,dx =$

Example 5: $\int \cosh(2x)\sinh^2(2x)\,dx =$

Assignment A-4

1. Find the value of $\sinh 1$, $\cosh 1$, and $\tanh 1$ without using a calculator.
2. Use a calculator to find sinh 4.
3. Use a calculator to find $\tanh(-3)$.

Find the derivative.

4. $f(x)=\ln(\cosh x)$ 5. $g(x)=x\sinh x$ 6. $h(x)=\frac{1}{2}\tanh(4x)-x^3$

7. Without using a calculator find an equation of a line tangent to the graph of $y=e^{\sinh x}$ when $x=0$.

Find the integral.

8. $\int \sinh(2-3x)\,dx$ 9. $\int \frac{\sinh x}{\cosh x}\,dx$ 10. $\int x\,\mathrm{sech}^2(x^2)\,dx$

11. A region is bounded by $y=\sinh x$, $y=0$, and $x=1$.

a. Sketch a graph of the region.
b. Find the area of the region without using a calculator.
c. Find the surface area of the region revolved about the x-axis.
d. Find the surface area of the region revolved about the y-axis.

Lesson A-5 Graphing Points in Three Dimensions (rectangular, cylindrical, spherical)

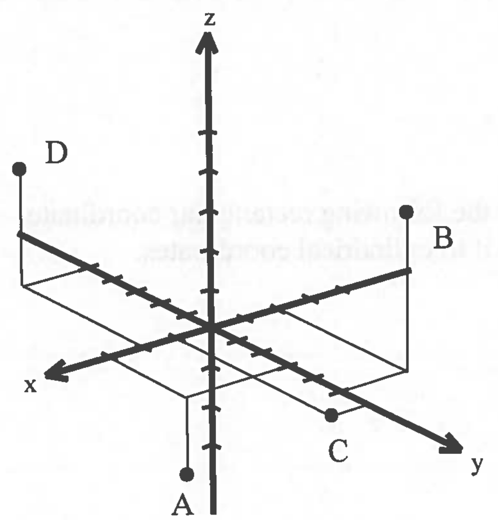
Example 1: Give an ordered triple for each lettered point on this rectangular graph.

A

B

C

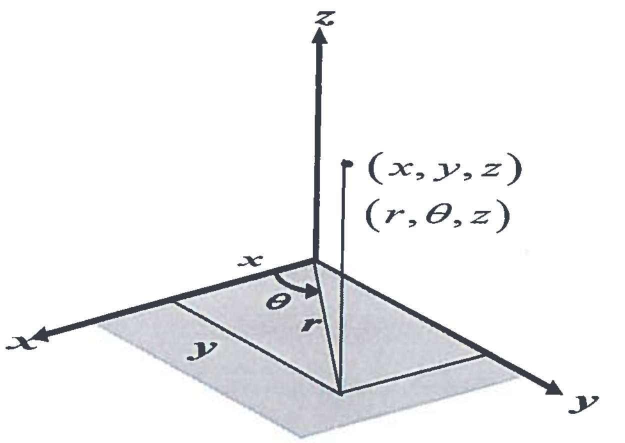
D



Example 2: Plot the following points on the same rectangular system.
(2,3,4), (3,-4,-2), (0,3,2)

Cylindrical Coordinates

Cylindrical coordinates can be thought of as an extension of polar coordinates in the third dimension. For the ordered triple (r,θ,z), r and θ have the same meaning as polar coordinates and z has the same meaning as the z in rectangular three dimensional systems.



Conversion Equations	
$x^2+y^2=$	$x=$
$\tan\theta=$	$y=$
	$z=$

Example 3: Plot the following cylindrical coordinate point and convert it to rectangular coordinates.

$\left(4,\frac{\pi}{2},-2\right)$

Example 4: Convert the following cylindrical coordinate point to rectangular coordinates without graphing. $\left(3,\frac{\pi}{6},4\right)$

Example 5: Plot the following rectangular coordinate point and convert it to cylindrical coordinates.
cylindrical

$(-4,0,2)$

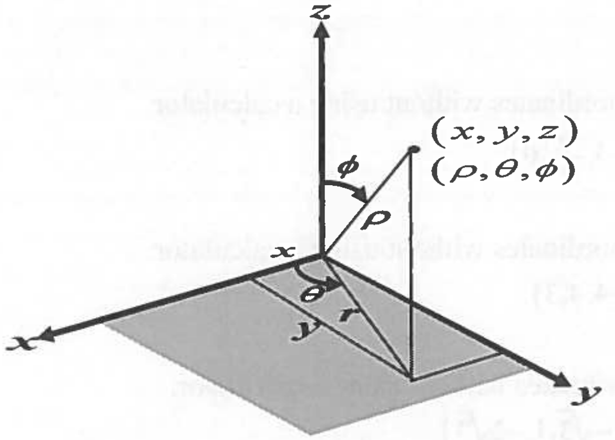
Example 6: Convert the following rectangular coordinate point to

coordinates without graphing.

$\left(-3\sqrt{2},3\sqrt{2},-4\right)$

Spherical Coordinates

Spherical coordinates are given as ordered triples (ρ,θ,ϕ) where ρ is the distance from the origin, θ is the same angle used in cylindrical and polar coordinates, and ϕ is the angle measured from the positive z-axis.



Conversion Equations

$\rho^2 = x^2 + y^2 + z^2$ $\quad x = \rho\sin\phi\cos\theta$

$\tan\theta = \dfrac{y}{x}$ $\quad y = \rho\sin\phi\sin\theta$

$\quad z = \rho\cos\phi$

$\cos\phi = \dfrac{z}{\rho}$

Example 7: Plot the following rectangular coordinate point and convert it to spherical coordinates.
spherical
$(0,-4,0)$

Example 8: Convert the following rectangular coordinate point to

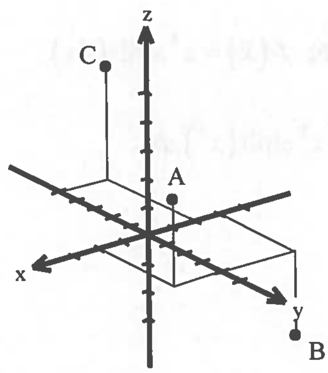
coordinates without graphing.
$\left(\sqrt{3},1,2\sqrt{3}\right)$

Example 9: Plot the following spherical coordinate point and convert it to rectangular coordinates.
rectangular
$\left(6,\frac{3\pi}{2},\frac{\pi}{2}\right)$

Example 10: Convert the following spherical coordinate point to
coordinates without graphing.
$\left(5,\frac{\pi}{4},\frac{3\pi}{4}\right)$

Assignment A-5

1. Find the Cartesian (rectangular) coordinates for each labeled point.



2. Plot these points on the same coordinate system:
   A $(2,4,-3)$, B $(-2,-3,4)$, C $(3,-4,0)$

3. Plot these points on the same coordinate system:
   D $(3,0,3)$, E $(0,-4,4)$, F $(-4,3,5)$

Convert these cylindrical coordinates to rectangular coordinates without using a calculator.

4. $(4,0,10)$
5. $\left(6,\frac{\pi}{3},-5\right)$
6. $\left(-3,\frac{7\pi}{6},0\right)$

Convert these rectangular coordinates to cylindrical coordinates without using a calculator.

7. $(0,3,-2)$
8. $\left(2,2\sqrt{3},5\right)$
9. $(-4,4,3)$

Convert these rectangular coordinates to spherical coordinates without using a calculator.

10. $(3,0,0)$
11. $\left(-1,\sqrt{3},2\right)$
12. $\left(-\sqrt{3},1,-2\sqrt{3}\right)$

Convert these spherical coordinates to rectangular coordinates without using a calculator.

13. $\left(3,\frac{\pi}{4},\frac{\pi}{6}\right)$
14. $(6,2,0)$
15. $\left(8,\pi,\frac{\pi}{4}\right)$

Review:

16. Find the surface area of the solid generated when the region bounded by $y=4\sqrt{x}$, $y=0$, and $x=4$ is revolved about:
    a. the x-axis.   b. the y-axis.

17. A well has a 3 foot diameter and is 200 feet deep. If the well is full of water weighing 62.4 pounds per cubic foot to within 50 feet of ground level, how much work is required to pump the well dry?

18. A chain weighing 60 newtons per meter extends 50 meters up from an attached 250 newton weight. How much work is required to wind up the entire chain and lift the weight the full 50 meters?

19. Differentiate $f(x)=x^2\sinh(4x)$.

20. Integrate $\int x^2\sinh\left(x^3\right)dx$.

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