Homework 3
Partial Differential Equations
Math 531
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Problem 2.2.4: In this exercise we derive superposition principles for nonhomogeneous problems.

(a) Consider $L(u) = f$. If u_p is a particular solution, $L(u_p) = f$, and if u_1 and u_2 are homogeneous solutions, $L(u_i) = 0$, show that $u = u_p + c_1u_1 + c_2u_2$ is another particular solution.

Proof. Let u_p be a particular solution if $L(u_p) = f$, and let u_1 and u_2 be homogeneous solutions if $L(u_i) = 0$. Let $u = u_p + c_1u_1 + c_2u_2$ such that we get the following by the definition of the Linear Operator:

$$L(u) = L(u_p) + c_1L(u_1) + c_2L(u_2) = f$$

Thus, because $\boldsymbol{L(u = u_p + c_1u_1 + c_2u_2) = f}$, then $\boldsymbol{u = u_p + c_1u_1 + c_2u_2}$ is a particular solution from the given statement.

□

(b) If $L(u) = f_1 + f_2$, where u_{pi} is a particular solution corresponding to f_i, what is a particular solution for $f_1 + f_2$

Proof. Let $L(u_{p1}) = f_1, L(u_{p2}) = f_2$, and let $L(u) = f_1 + f_2$ with u being the particular solution.

Notice the following:

$$f_1 + f_2 = L(u_{p1}) + L(u_{p2}) = L(u_{p1} + u_{p2})$$

Thus we get that $\boldsymbol{u = u_{p1} + u_{p2}}$ is a particular solution for $\boldsymbol{f_1 + f_2}$

□

Problem 2.3.1: For the following partial differential equations, what ordinary differential equations are implied by the method of separation of variables?

(b) Let the following be true:

$$u(x,t) = \phi(x)G(t)$$
$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - v_0\frac{\partial u}{\partial x}$$

Now notice the following:

$$\phi(x)\frac{dG}{dt} = k\frac{d^2\phi}{dx^2}G(t) - v_0\frac{d\phi}{dx}G(t)$$
$$\frac{1}{G}\frac{dG}{dt} = \frac{1}{\phi}\left(k\frac{d^2\phi}{dx^2} - v_0\frac{d\phi}{dx}\right) = -\lambda$$

Thus we get the following ODE's:

$$\boldsymbol{\frac{dG}{dt} = -\lambda G \qquad \text{and} \qquad k\frac{d^2\phi}{dx^2} - v_0\frac{d\phi}{dx} + \lambda\phi = 0}$$

(c) Let the following be true:

$$u(x,t) = \phi(x)G(y)$$
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Now notice the following:

$$\frac{d^2\phi}{dx^2}G(y) + \frac{d^2G}{dy^2}\phi(x) = 0$$
$$\frac{d^2\phi}{dx^2}G(y) = -\frac{d^2G}{dy^2}\phi(x)$$
$$\frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\frac{1}{G}\frac{d^2G}{dy^2} = -\lambda$$

Thus we get the following ODE's:

$$\boldsymbol{\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \qquad \text{and} \qquad \frac{d^2G}{dy^2} - \lambda G = 0}$$

(f) Let the following be true:

$$u(x,t) = \phi(x)G(t)$$
$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

Now notice the following:

$$\frac{d^2G}{dt^2}\phi(x) = c^2\frac{d^2\phi}{dx^2}G(t)$$
$$\frac{1}{G}\frac{d^2G}{dt^2} = \frac{c^2}{\phi}\frac{d^2\phi}{dx^2} = -\lambda$$

Thus we get the following ODE's:

$$\boldsymbol{\frac{d^2G}{dt^2} + \lambda G = 0 \qquad \text{and} \qquad \frac{d^2\phi}{dx^2} + \frac{\phi\lambda}{c^2} = 0}$$

Problem 2.3.8: Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \alpha u$$

This corresponds to a one-dimensional rod either with heat loss through the lateral sides with outside temperature $0°$ ($\alpha > 0$) or with insulated lateral sides with a heat sink proportional to the temperature. Suppose that the boundary conditions are

$$u(0,t) = 0 \qquad \text{and} \qquad u(L,t) = 0$$

(a) What are the possible equilibrium temperature distributions if $\alpha > 0$?

Notice that the equilibrium temperature distributions occur as $t \to \infty$, such that we get the following:

$$\frac{\partial u}{\partial t} = 0 = k\frac{\partial^2 u}{\partial x^2} - \alpha u$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\alpha u}{k} = 0$$

From here notice the characteristic equation:

$$\lambda^2 - \frac{\alpha}{k} = 0 \qquad \to \qquad \lambda = \pm\sqrt{\frac{\alpha}{k}}$$

Notice that $\alpha > 0$ and $k > 0$, so we get 2 real roots such that we get the following:

$$u(x) = c_1 e^{\sqrt{\frac{\alpha}{k}}x} + c_2 e^{-\sqrt{\frac{\alpha}{k}}x}$$

From here, we can use the given boundary conditions:

$$c_1 + c_2 = 0 \quad \to \quad c_2 = -c_1$$

$$c_1 e^{\sqrt{\frac{\alpha}{k}}L} - c_1 e^{-\sqrt{\frac{\alpha}{k}}L} = 0 \quad \to \quad c_1 e^{\sqrt{\frac{\alpha}{k}}L} = c_1 e^{-\sqrt{\frac{\alpha}{k}}L}$$

Notice that $L \neq 0$ and $\alpha \neq 0$ and $k \neq 0$. This means that $\sqrt{\frac{\alpha}{k}}L \neq 0$. **Thus we get that the only way for the equality to be true is for $c_1 = 0 = -c_2$. So we get the trivial solution:**.

$$\boldsymbol{u(x) = 0}$$

(b) Solve the time-dependent problem $[u(x,0) = f(x)]$ if $\alpha > 0$. Analyze the temperature for large time ($t \to \infty$) and compare to part (a).

Let the following be true:

$$u(x,t) = \phi(x)G(t)$$
$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \alpha u$$

Now notice the following:

$$\begin{aligned}\frac{dG}{dt}\phi &= k\frac{d^2\phi}{dx^2}G - \alpha\phi G \\ &= kG\left(\frac{d^2\phi}{dx^2} - \frac{\alpha\phi}{k}\right) \\ \frac{1}{kG}\frac{dG}{dt} + \frac{\alpha}{k} &= \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda\end{aligned}$$

From this, we get the following ODE's and its corresponding solutions:

$$\begin{aligned}\frac{1}{kG}\frac{dG}{dt} + \frac{\alpha}{k} &= -\lambda \\ \frac{dG}{G} &= (-\lambda k - \alpha)\,dt\end{aligned} \qquad\qquad \begin{aligned}\frac{1}{\phi}\frac{d^2\phi}{dx^2} &= -\lambda \\ \frac{d^2\phi}{dx^2} + \lambda\phi &= 0\end{aligned}$$

Notice the following cases:

(i) $\lambda = 0$:

$$G(t) = c_1 e^{-\alpha t}$$
$$\phi(x) = d_1 x + d_2$$

From here, we can use the given boundary conditions in $\phi(x)$:

$$\phi(0) = d_2 = 0 \qquad \to \qquad \phi(L) = d_1 L = 0$$

Thus, we get the trivial solution:

$$\boldsymbol{\phi(x) = 0}$$

(ii) $\lambda < 0$:

$$G(t) = c_1 e^{-(\lambda k + \alpha)t}$$
$$\phi(x) = d_1 e^{\sqrt{\lambda}x} + d_2 e^{-\sqrt{\lambda}x}$$

From here, we can use the given boundary conditions in $\phi(x)$:

$$\phi(0) = 0 = d_1 + d_2 \quad \to \quad d_2 = -d_1$$
$$\phi(L) = d_1 e^{\sqrt{\lambda}L} - d_1 e^{-\sqrt{\lambda}L} = 0 \quad \to \quad d_1 e^{\sqrt{\lambda}L} = d_1 e^{-\sqrt{\lambda}L}$$

Because we know $L \neq 0$ and $\lambda \neq 0$, we get that $d_1 = d_2 = 0$, and get the trivial solution:

$$\boldsymbol{\phi(x) = 0}$$

(iii) $\lambda > 0$

$$G(t) = c_1 e^{-(\lambda k+\alpha)t}$$

$$\phi(x) = d_1 \cos(\sqrt{\lambda}x) + d_2 \sin(\sqrt{\lambda}x)$$

From here, we can use the given boundary conditions in $\phi(x)$:

$$d_1 = 0 \qquad \rightarrow \qquad d_2 \sin(\sqrt{\lambda}L) = 0$$

From this, we notice the nontrivial solution, and get the following:

$$\sin(\sqrt{\lambda}L) = 0 \qquad \rightarrow \qquad \sqrt{\lambda} = \frac{n\pi}{L} \quad \rightarrow \quad \lambda = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2...$$

Thus, we get the following solution:

$$u(x,t) = G(t)\phi(x) = B_n e^{-\left(\left(\frac{n\pi}{L}\right)^2 k+\alpha\right)t} \sin\left(\frac{n\pi}{L}x\right)$$

We can now generalize this solution for all values of n, such that:

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\left(\left(\frac{n\pi}{L}\right)^2 k+\alpha\right)t} \sin\left(\frac{n\pi}{L}x\right)$$

Now we can use our initial condition:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

From, here notice by the orthogonality of sines, we get:

$$\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right)\, dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi x}{L}\right)\, dx = B_m \left(\frac{L}{2}\right)$$

From here, we get:

$$B_m = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right)\, dx \qquad \rightarrow \qquad B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx$$

Thus we get the following solution:

$$\boldsymbol{\frac{2}{L}\sum_{n=1}^{\infty}\left[\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx\right] e^{-\left(\left(\frac{n\pi}{L}\right)^2 k+\alpha\right)t} \sin\left(\frac{n\pi}{L}x\right)}$$

From here, also notice that:

$$\boldsymbol{\lim_{t\to\infty} u(x,t) = 0}$$

which does agree with our answer for part (a)