Math 532: Homework 10/Midterm Two
Due 12/11/19
Everyone turns in an individual copy.

Book Problems

1. (2pts each) 6.79.2, 6.81.3, 7.86.2
2. (5pts each) 6.81.8, 6.83.6, 7.86.10, 7.88.11, 7.88.12

Beyond the Book Problems

3. (12 pts) Let $h(t)$ and $g(t)$ be continuous functions on the real line, $f, g \in C(\mathbb{R})$, such that

$$\int_{-\infty}^{\infty} |h(t)|^2 \, dt < \infty, \quad \int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty.$$

Then we have the *Cauchy-Schwarz* inequality which says that

$$\int_{-\infty}^{\infty} |g(t)| \, |h(t)| \, dt \leq \left(\int_{-\infty}^{\infty} |h(t)|^2 \, dt \right)^{1/2} \left(\int_{-\infty}^{\infty} |g(t)|^2 \, dt \right)^{1/2}$$

We further choose $h(t)$ to be real for t real, $h \in C^1(\mathbb{R})$, and for $z \in \mathbb{C}$, $h(z)$ is uniformly continuous for $0 \leq \text{Im}(z) \leq \epsilon_0$, with $\epsilon < \epsilon_0$. This means that for z_1, z_2 such that $0 \leq \text{Im}(z_j) \leq \epsilon_0$ we have for all $\tilde{\epsilon} > 0$ there exists a $\delta > 0$ such that

$$|z_1 - z_2| < \delta \rightarrow |h(z_1) - h(z_2)| \leq \tilde{\epsilon},$$

where the bounds hold uniformly for any choices of z_j in the given region.

(a) Using the Cauchy-Schwarz inequality, show that if we define

$$f_+(z) = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{h(t)}{t - z} dt,$$

then $f_+(z)$ is analytic for $z \in \mathbb{C}_+$ where

$$\mathbb{C}_+ = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}.$$

Just assume derivatives pass through integrals without fuss.

(b) In order to understand how $f_+(z)$ behaves on the real line, we, for the sake of the collective sanity of the class, define $f_+(z)$ for $z = x \in \mathbb{R}$ to be

$$f_+(x) = \lim_{\epsilon \to 0} \int_{L_{\epsilon,+}} \frac{h(t)}{t-x} dt,$$

where we define the contour $L_{\epsilon,+}(x)$ to be

$$L_{\epsilon,+}(x) = (-\infty, x-\epsilon] \cup C_{\epsilon,+}(x) \cup [x+\epsilon, \infty),$$

where $C_{\epsilon,+}(x)$ is a half-circle of radius $\epsilon > 0$ centered at x which is above the real-line in $\mathbb{C}$, i.e. for $z \in C_{\epsilon,+}(x)$, $\mathrm{Im}(z) \geq 0$. Using the uniform continuity of $h(z)$ so that $h(x + \epsilon e^{i\theta}) \to h(x)$ as $\epsilon \to 0^+$, show for $z = x \in \mathbb{R}$, that we have that

$$f_+(x) = h(x) + i(\mathcal{H}h)(x),$$

where the *Hilbert Transform* of h, say $(\mathcal{H}h)(x)$, is given by

$$(\mathcal{H}h)(x) = \frac{1}{\pi} \lim_{\epsilon \to 0} \left(\int_{-\infty}^{x-\epsilon} \frac{h(t)}{t-x} dt + \int_{x+\epsilon}^{\infty} \frac{h(t)}{t-x} dt \right).$$

To show this limit exists, choose real values $b > x$ and $c < x$ so that for ϵ sufficiently small, $b > x + \epsilon$ and $c < x - \epsilon$. Then we must have that

$$\int_{-\infty}^{x-\epsilon} \frac{h(t)}{t-x} dt + \int_{x+\epsilon}^{\infty} \frac{h(t)}{t-x} dt =$$
$$\int_{-\infty}^{c} \frac{h(t)}{t-x} dt + \int_{c}^{x-\epsilon} \frac{h(t)}{t-x} dt + \int_{x+\epsilon}^{b} \frac{h(t)}{t-x} dt + \int_{b}^{\infty} \frac{h(t)}{t-x} dt$$

Show that the Cauchy-Schwarz Inequality guarantees the existence of the integrals

$$\int_{-\infty}^{c} \frac{h(t)}{t-x} dt, \quad \int_{b}^{\infty} \frac{h(t)}{t-x} dt$$

As for the remaining pieces, we see that we can write

$$\int_{c}^{x-\epsilon} \frac{h(t)}{t-x} dt + \int_{x+\epsilon}^{b} \frac{h(t)}{t-x} dt = h(x) \left(\int_{c}^{x-\epsilon} \frac{dt}{t-x} + \int_{x+\epsilon}^{b} \frac{dt}{t-x} \right)$$
$$+ \int_{c}^{x-\epsilon} \frac{h(t)-h(x)}{t-x} dt + \int_{x+\epsilon}^{b} \frac{h(t)-h(x)}{t-x} dt$$

Basic integration and the use of the Mean-Value Theorem and the assumption of a continuous derivative give us the limit.

4. (13 pts) Let A be an $n \times n$ matrix with complex entries. We define its *eigenvalues* to be those values $\lambda \in \mathbb{C}$ such that the matrix $A - \lambda I$, where I is the $n \times n$ identity matrix, is not invertible. If a square matrix is not invertible, this means it must have at least one vector in its null space, i.e. there must be a $\mathbf{v}$ such that

$$(A - \lambda I)\,\mathbf{v} = 0,$$

or

$$A\mathbf{v} = \lambda \mathbf{v}.$$

We correspondingly call these vectors $\mathbf{v}$ *eigenvectors*.

We find eigenvalues as roots of the degree n *characteristic polynomial* $p(\lambda) = \det(A - \lambda I)$, where $\det(\cdot)$ denotes the determinant. By factoring across roots, we have that

$$p(\lambda) = (-1)^n \left(\lambda - \lambda_1\right)^{m_{a,1}} \left(\lambda - \lambda_2\right)^{m_{a,2}} \cdots \left(\lambda - \lambda_k\right)^{m_{a,k}},$$

where each λ_j is an eigenvalue, and $m_{a,j}$ is the number of times the eigenvalue is repeated as a root of $p(\lambda)$. Note, we have that

$$\sum_{l=1}^{k} m_{a,l} = n.$$

If for each eigenvalue λ_j, we can find $m_{a,j}$ linearly independent eigenvectors, say $\mathbf{v}_{j,l}$ so that

$$A\mathbf{v}_{j,l} = \lambda_j \mathbf{v}_{j,l}, \; l = 1, \cdots, m_{a,j},$$

then we say A is *diagonalizable* which means we can factor it into the following form

$$A = V \Lambda V^{-1},$$

where V is an $n \times n$ matrix whose columns are all of the eigenvectors arranged so that

$$V = \begin{pmatrix} \mathbf{v}_{1,1} & \cdots & \mathbf{v}_{1,m_{a,1}} & \mathbf{v}_{2,1} & \cdots & \mathbf{v}_{2,m_{a,2}} & \cdots & \mathbf{v}_{j,1} & \cdots & \mathbf{v}_{j,m_{a,j}} \end{pmatrix}$$

and Λ is a diagonal matrix whose entries are the eigenvalues arranged so that

$$\Lambda = \begin{pmatrix} \lambda_1 I_{m_{a,1}} & & & \\ & \lambda_2 I_{m_{a,2}} & & \\ & & \ddots & \\ & & & \lambda_j I_{m_{a,j}} \end{pmatrix},$$

where each matrix $I_{m_{a,j}}$ is the $m_{a,j} \times m_{a,j}$ identity matrix. Finally, corresponding to A, we define the *resolvent matrix* $R(z; A)$ for $z \in \mathbb{C}$ where

$$R(z; A) = (zI - A)^{-1}.$$

Thus, we see that $R(z; A)$ is well defined for all $z \neq \lambda_j$.

(a) Show that if $A = V\Lambda V^{-1}$ then $R(z; A) = V\left(zI - \Lambda\right)^{-1}V^{-1}$. Note, you will need to use $zI = zVV^{-1}$ and $(BC)^{-1} = C^{-1}B^{-1}$.

(b) Show that every entry of $R(z; A)$ is analytic in z except at the eigenvalues of A. Note,

$$(zI - \Lambda)^{-1} = \begin{pmatrix} (z-\lambda_1)^{-1}I_{m_{a,1}} & & & \\ & (z-\lambda_2)^{-1}I_{m_{a,2}} & & \\ & & \ddots & \\ & & & (z-\lambda_j)^{-1}I_{m_{a,j}} \end{pmatrix}.$$

(c) For each eigenvalue λ_j, let $C_\epsilon(\lambda_j)$ be a circular contour of radius ϵ centered at λ_j with ϵ chosen small enough so that all of the contours are disjoint. Define the $n \times n$ matrix $\mathbb{P}_j$ so that

$$\mathbb{P}_j = \frac{1}{2\pi i}\oint_{C_\epsilon(\lambda_j)} R(z; A)dz.$$

Show that

i. $\mathbb{P}_j\mathbb{P}_j = \mathbb{P}_j$

ii. $\mathbb{P}_j\mathbb{P}_l = 0,\ j \neq l$

iii. For any vector $\mathbf{w} \in \mathbb{C}^n$, we have $A\mathbb{P}_j\mathbf{w} = \lambda_j\mathbb{P}_j\mathbf{w}$.

iv. $\sum_{j=1}^{k} \mathbb{P}_j = I$.

v. For any vector $\mathbf{w} \in \mathbb{C}^n$, $A\mathbf{w} = \sum_{j=1}^{k} \lambda_j\mathbb{P}_j\mathbf{w}$.