Solutions

Exam 2, Math 330

1. (a) Suppose $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. Suppose x_0 is a limit point of D and $L \in \mathbb{R}$. Complete the $\epsilon - \delta$ definition: We say the **limit of f as x approaches x_0 is L** and write $\lim_{x \to x_0} f(x) = L$ iff...

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in D$,

if $0 < |x - x_0| < \delta$, then $|f(x) - L| < \varepsilon$.

(b) Use the $\epsilon - \delta$ definition to prove that $\lim_{x \to 3} (2x^2 - 7x) = -3$

Let $\varepsilon > 0$.

Let $\delta = \min\{\frac{\varepsilon}{5.2}, 0.1\} > 0$.

Suppose $x \in \mathbb{R}$ is such that

$|x - 3| < \delta$.

Then

$|x - 3| < 0.1$

So $0 < 2x - 1 < 5.2$

Also

$|x - 3| < \frac{\varepsilon}{5.2}$

So $|(2x-1)(x-3)| < 5.2\,|x-3| < \varepsilon$.

I.e. $|2x^2 - 7x - (-3)| < \varepsilon$. ∎

SIDE:

WANT:

$|2x^2 - 7x + 3| < \varepsilon$

$|(2x-1)(x-3)| < \varepsilon$

$2.9 < x < 3.1$

$4.8 < 2x - 1 < 5.2$

$|(2x-1)(x-3)| < 5.2\,|x-3| < \varepsilon$.

$|x - 3| < \frac{\varepsilon}{5.2}$

2. We say $f : D \to \mathbb{R}$ is **uniformly continuous on** D iff
$\forall \{u_n\}, \{v_n\} \subseteq D$, if $\lim_{n\to\infty} (u_n - v_n) = 0$, then $\lim_{n\to\infty} (f(u_n) - f(v_n)) = 0$.

   (a) Negate the above definition so that we see what $f : D \to \mathbb{R}$ is **not uniformly continuous on** D means.

   $\exists \{u_n\}, \{v_n\} \subseteq D$ such that

   $$\lim_{n\to\infty} (u_n - v_n) = 0 \text{ and } \lim_{n\to\infty} (f(u_n) - f(v_n)) \neq 0.$$

   (b) Consider $g : (0,2) \to \mathbb{R}$ by $g(x) = \dfrac{1}{x}$. Prove that g is not uniformly continuous on (0,2).

   Let $u_n = \frac{2}{n}$ and $v_n = \frac{1}{n}$ for $n \geq 1$.

   Then $\{u_n\}, \{v_n\} \subseteq (0,2)$ and

   $$\lim_{n\to\infty} \left(\frac{2}{n} - \frac{1}{n}\right) = 0 - 0 = 0.$$

   But $$\lim_{n\to\infty} \left(g(u_n) - g(v_n)\right) = \lim_{n\to\infty} \left(\frac{n}{2} - n\right)$$

   $$= \lim_{n\to\infty} \left(-\frac{n}{2}\right) \neq 0.$$

3. Choose **one** of the following two to prove. **Do not submit answers for both.**

   (a) Suppose $f : [1,3] \to \mathbb{R}$ and $g : [1,3] \to \mathbb{R}$ are continuous functions on $[1,3]$. Define $h : [1,3] \to \mathbb{R}$ by $h(x) = f(x)g(x)$. Suppose that $f(1) = -5 = g(1)$ and that $f(3) = 10 = g(3)$.
   Prove $\exists x_0 \in (1,3)$ such that $h(x_0) = 0$.

   (b) Suppose that $g : [1,3] \to \mathbb{R}$ is a continuous function on $[1,3]$ such that $\forall x \in [1,3], g(x) > 0$. Define $h : [1,3] \to \mathbb{R}$ by $h(x) = \dfrac{1}{g(x)}$.
   Prove $\exists M \in \mathbb{R}$ such that $\forall x \in [1,3]$, we have $h(x) < M$.

(a) Notice that $f(1) = -5$ and $f(3) = 10$.
Since f is continuous on $[1,3]$ and $-5 < 0 < 10$, the IVT says $\exists x_0 \in (1,3)$ s.t.

$$f(x_0) = 0.$$

Then $h(x_0) = f(x_0)\,g(x_0) = 0 \cdot g(x_0) = 0$. $\boxed{4}$

(b) Since g is continuous on $[1,3]$ and $g(x) \neq 0$ on $[1,3]$, $h(x) = \frac{1}{g(x)}$ is continuous on $[1,3]$. By EVT, h attains a max on $[1,3]$ and $\exists x_0 \in [1,3]$ s.t. $\forall x \in [1,3]$, $f(x) \leq f(x_0)$. Use $M = f(x_0) + 1$.

$\boxed{4}$

4. (a) Complete the definition: we say that $x_0 \in \mathbb{R}$ is a **limit point of** $D \subseteq \mathbb{R}$ iff ...

$\exists \{x_n\} \subseteq D \setminus \{x_0\}$ such that $\lim_{n \to \infty} x_n = x_0$.

i. Give an example of a set C with no limit points. (No justification needed.)

$C = \{1, 3\}$

ii. Give an example of a set D with a limit point x_0 where $x_0 \notin D$. (No justification needed.)

$D = [1, 3)$ 3 is a limit point not in the set D

(b) Suppose $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. Complete the sequential definition: we say f is **continous at** $x_0 \in D$ iff...

$\forall \{x_n\} \subseteq D$, if $\lim_{n\to\infty} x_n = x_0$, then $\lim_{n\to\infty} f(x_n) = f(x_0)$.

(c) Use the sequential definition of continuity and limit laws to prove that $f : [2, 5] \to \mathbb{R}$ by $f(x) = \dfrac{3x}{x^2 + 1}$ is continuous at $x_0 = 4$.

Let $\{x_n\} \subseteq [2, 5]$ and suppose $\lim_{n\to\infty} x_n = 4$.

Then $\lim_{n\to\infty} 3x_n = 12$ and $\lim_{n\to\infty} (x_n^2 + 1) = 17$.

Thus $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \dfrac{3x_n}{x_n^2+1} = \dfrac{12}{17} = f(4)$.

$\boxed{4}$

5. For each problem, circle T for true or F for false.

T F For every set $D \in \mathbb{R}$, all points of D are limit points.

T F The function $f : [2, 4] \to \mathbb{R}$ by $f(x) = \dfrac{x-3}{x+1}$ is uniformly continuous.

T F All functions $f : [0, 1] \to \mathbb{R}$ such that $f([0, 1])$ is bounded are continuous.

T F Suppose $f : D \to \mathbb{R}$ is not continuous at $x_0 \in D$. Then $\forall \delta > 0, \exists \epsilon > 0$ such that $\exists x \in D$ where $0 < |x - x_0| < \delta$ and $|f(x) - f(x_0)| < \epsilon$.

T F The function $f : (0, 1] \to \mathbb{R}$ by $f(x) = 2x + 1$ attains a maximum value.

T F There exists a solution to the equation $4x^3 + 3x^2 - 4 = 0$ on the interval $(0, \infty)$.

T F Every function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x)$ is a degree four polynomial has a solution to $f(x) = 0$ for some $x \in \mathbb{R}$.

T F Suppose $a < b$. Every function $f : [a, b] \to \mathbb{R}$ is uniformly continuous.