Today: 10/15 Continuity chapter 3

3.1 Sequential Def of Continuity & Basic Results.

Setup: Suppose $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$.

"real-valued function of a real variable"

Intuition "continuous function"

~ "close in domain" implies "close in range".

~ map compact sets to compact sets

~ map connected sets to connected sets

Definition: Suppose $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$.

Let $x_0 \in D$. We say <u>f is continuous at x_0</u> iff.

$$\forall \{x_n\}_{n=1}^{\infty} \subseteq D, \text{ if } \lim_{n\to\infty} x_n = x_0, \text{ then } \lim_{n\to\infty} f(x_n) = f(x_0)$$

Remark: $\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right)$

We say <u>f is continuous</u> iff

$$\forall x_0 \in D, \ f \text{ is continuous at } x_0.$$

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f is <u>not</u> <u>continuous</u> at $x_0 \in D$.
iff

$$\exists \{x_n\}_{n=1}^{\infty} \subseteq D \text{ s.t. } \lim_{n\to\infty} x_n = x_0. \text{ and } \text{“}\lim_{n\to\infty} f(x_n) \neq f(x_0)\text{”}$$

Example 3.1 Suppose $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x^2 - 2x + 4.$$

The function f is continuous.

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proof: Let $x_0 \in \mathbb{R}$.

Suppose $\{x_n\} \subseteq \mathbb{R}$ and $\lim_{n\to\infty} x_n = x_0$.

Notice $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} (x_n^2 - 2x_n + 4)$.

$$= \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} x_n - 2 \lim_{n\to\infty} x_n + 4$$

(since the limits of the pieces exist)

Limit Laws from 2.1

$$= x_0^2 - 2x_0 + 4$$

$$= f(x_0).$$

$\blacksquare$

Remark: From section 2.1 we know:

Suppose $p(x)$ is a polynomial.
Let $a \in \mathbb{R}$ and $\{a_n\} \subseteq \mathbb{R}$ s.t. $\lim_{n\to\infty} a_n = a$.

Then $\lim_{n\to\infty} p(a_n) = p(a)$.

- All polynomial functions are continuous.

(You can prove this with induction).

Example 3.2 Suppose $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & x \geq 0 \\ 2, & x < 0. \end{cases}$$

Summary: (i) f is not continuous at 0

(ii) f is continuous elsewhere.

proof: (i) Prove: $\exists \{x_n\} \subseteq \mathbb{R}$ s.t. $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} f(x_n) \neq f(0)$.

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Let $x_n = -\frac{1}{n}$ for $n \geq 1$.

Then $\lim_{n\to\infty} x_n = 0$.

But $f(x_n) = 2$ since $x_n < 0$ for all $n \geq 1$.

So $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 2 = 2 \neq 1 = f(0)$.

(ii) Let $x_0 \neq 0$. Show: f is continuous at x_0.

Case 1: Let $x_0 > 0$.

Suppose $\{x_n\} \subseteq \mathbb{R}$ and $\lim_{n\to\infty} x_n = x_0$.

Note that $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|x_n - x_0| < \frac{x_0}{2}$.

So $\forall n \geq N$, $-\frac{x_0}{2} < x_n - x_0 < \frac{x_0}{2}$.

So $\forall n \geq N$, $\frac{x_0}{2} < x_n < \frac{3x_0}{2}$.

(Show: $\lim_{n\to\infty} f(x_n) = f(x_0) = 1$).

Let $\varepsilon > 0$. Let $n \geq N$. Then

$$|f(x_n) - f(x_0)| = |1-1| = 0 < \varepsilon.$$

(Since $x_n > 0$).

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Example 8.3: Dirichlet Function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}. \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

f is discontinuous everywhere.

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proof: Let $x_0 \in \mathbb{R}$.

Case 1: Suppose $x_0 \in \mathbb{R} \setminus \mathbb{Q}$. Note $f(x_0) = 0$.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\exists \{x_n\} \subseteq \mathbb{Q}$ s.t.

$\lim_{n\to\infty} x_n = x_0$. Note for all n, $f(x_n) = 1$.

Thus $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 1 = 1 \neq 0$.

Case 2: Similar argument.

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