Homework 5
Numerical Matrix Analysis
Math 543
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Section 12 Problem 3: The goal of this problem is to explore some properties of random matrices. Your job is to be a laboratory scientist, performing experiments that lead to conjectures and more refined experiments. Do not try to prove anything. Do produce well-designed plots, which are worth a thousand numbers. Define a random matrix to be an mxm matrix whose entries are independent samples from the real normal distribution with mean zero and standard deviation $m^{-1/2}$. (In MATLAB, A = randn(m,m)/sqrt (m).) The factor $\sqrt{m}$ is introduced to make the limiting behavior clean as m $\to \infty$.

(a) What do the eigenvalues of a random matrix look like? What happens, say, if you take 100 random matrices and superimpose all their eigenvalues in a single plot? If you do this for m = 8, 16, 32, 64, ... , what pattern is suggested? How does the spectral radius $\rho(A)$ (Exercise 3.2) behave as m $\to \infty$?

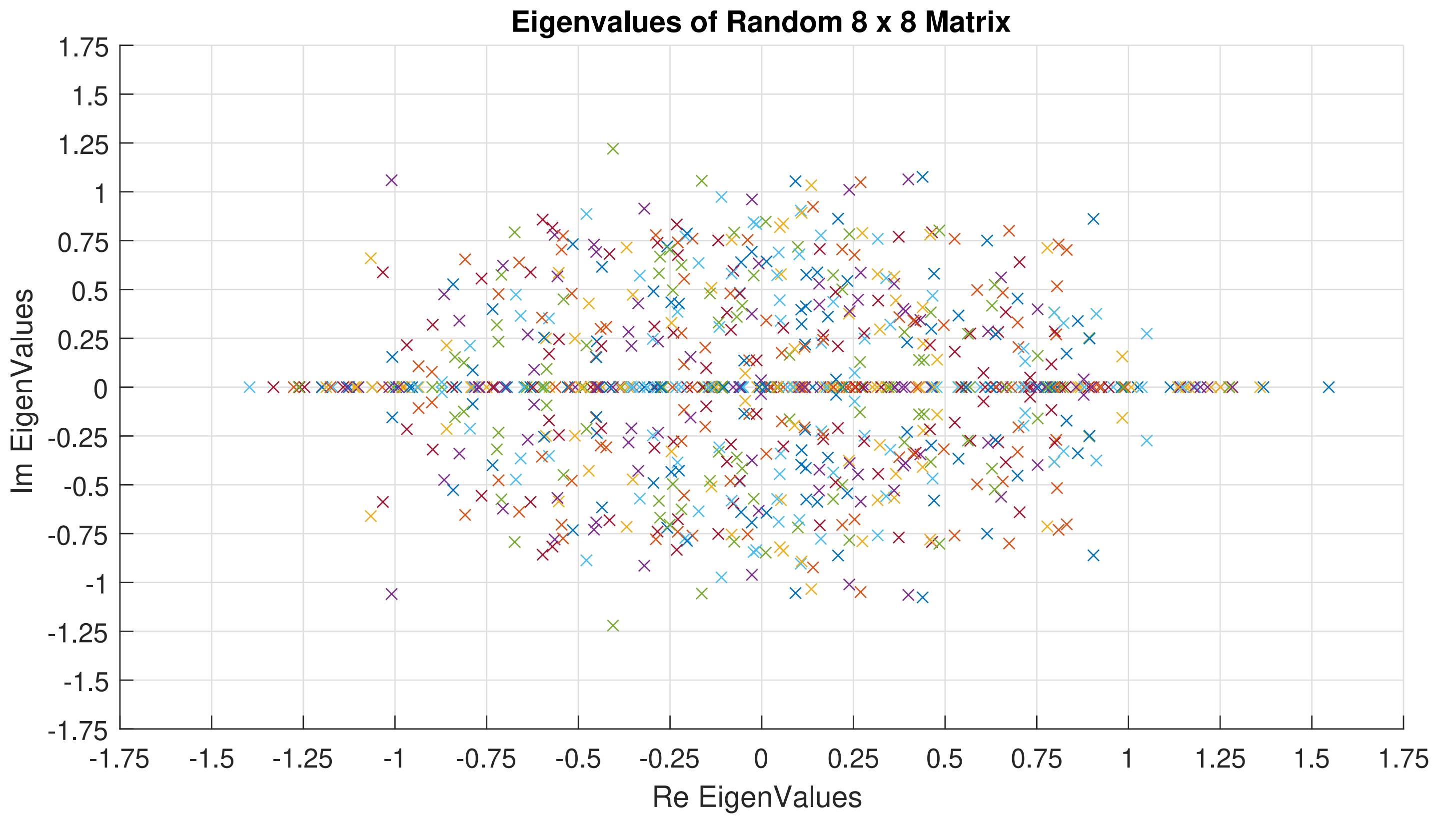
(b) What about norms? How does the 2-norm of a random matrix behave as m $\to \infty$? Of course, we must have $\rho(A) < ||A||$ (Exercise 3.2). Does this inequality appear to approach an equality as m $\to \infty$?

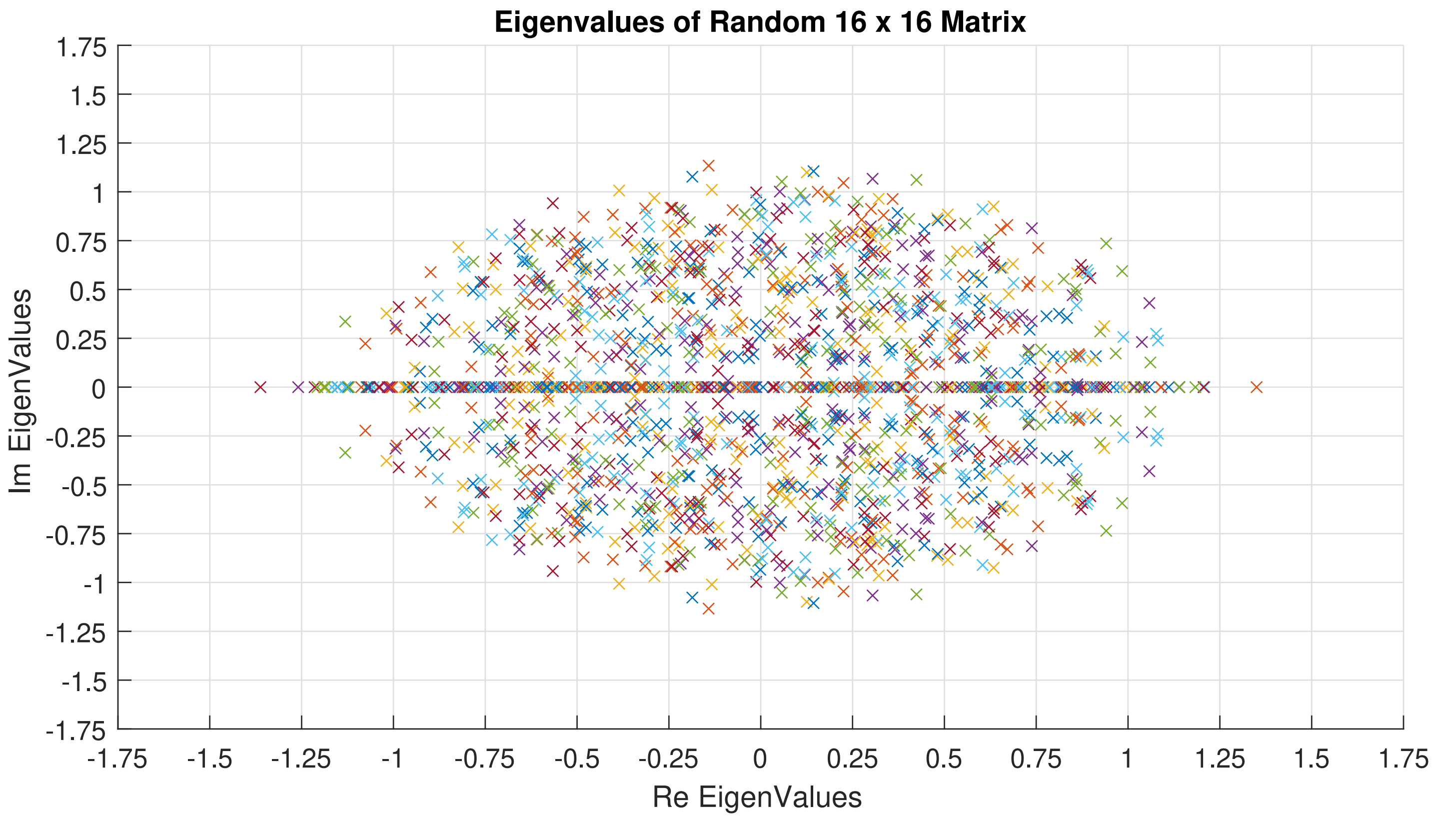
(c) What about condition numbers—or more simply, the smallest singular value σ_{min} ? Even for fixed m this question is interesting. What proportions of random matrices in $\mathbb{R}^{m\times m}$ seem to have $\sigma_{min} < 2^{-1}, 4^{-1}, 8^{-1}$, ...? In other words, what does the tail of the probability distribution of smallest singular values look like? How does the scale of all this change with m ?

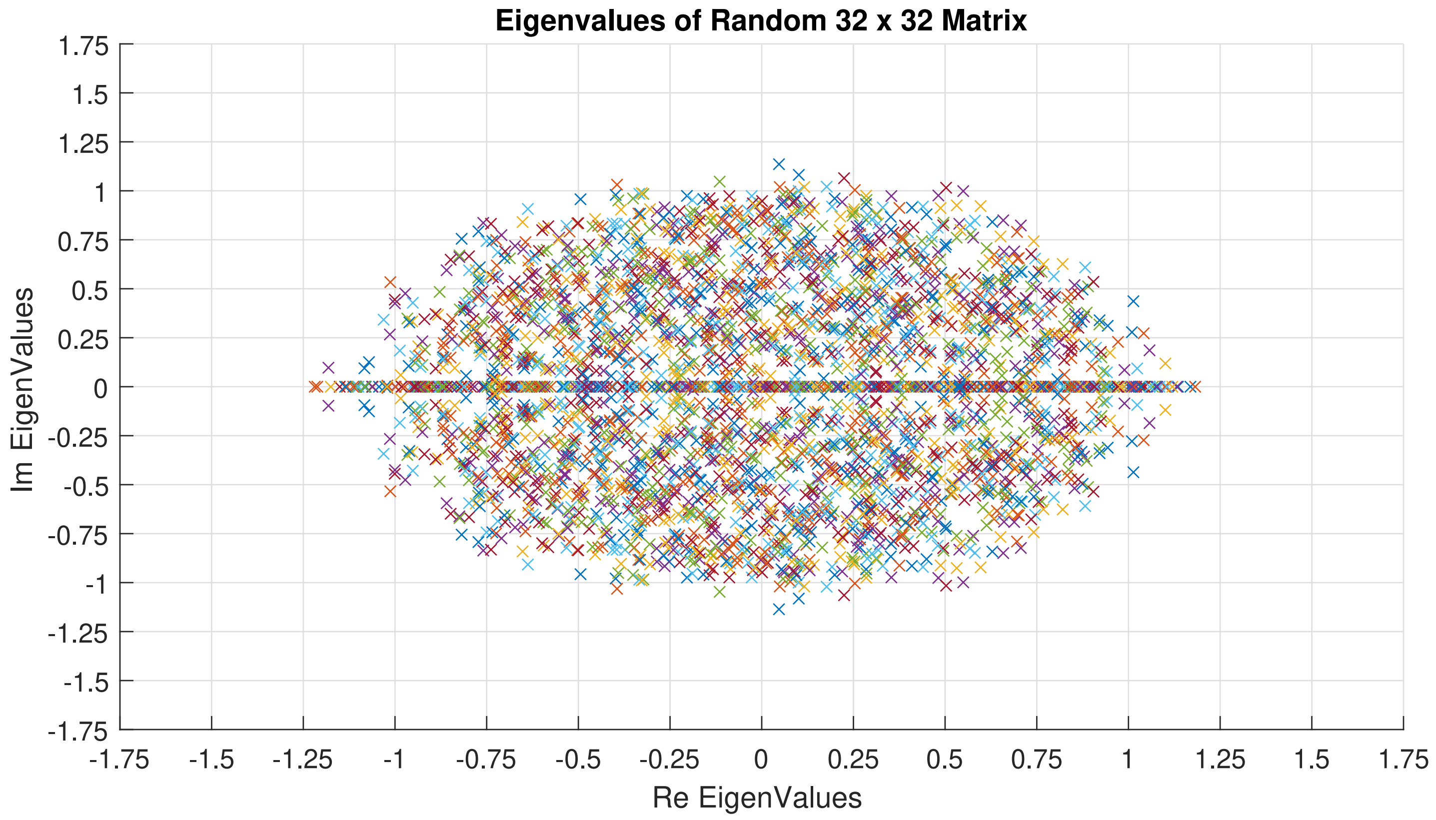
(a) The eigenvalues of random matrices take the shape of circles. As m $\to \infty$, the spectral radius, $\rho(A)$ approaches 1.

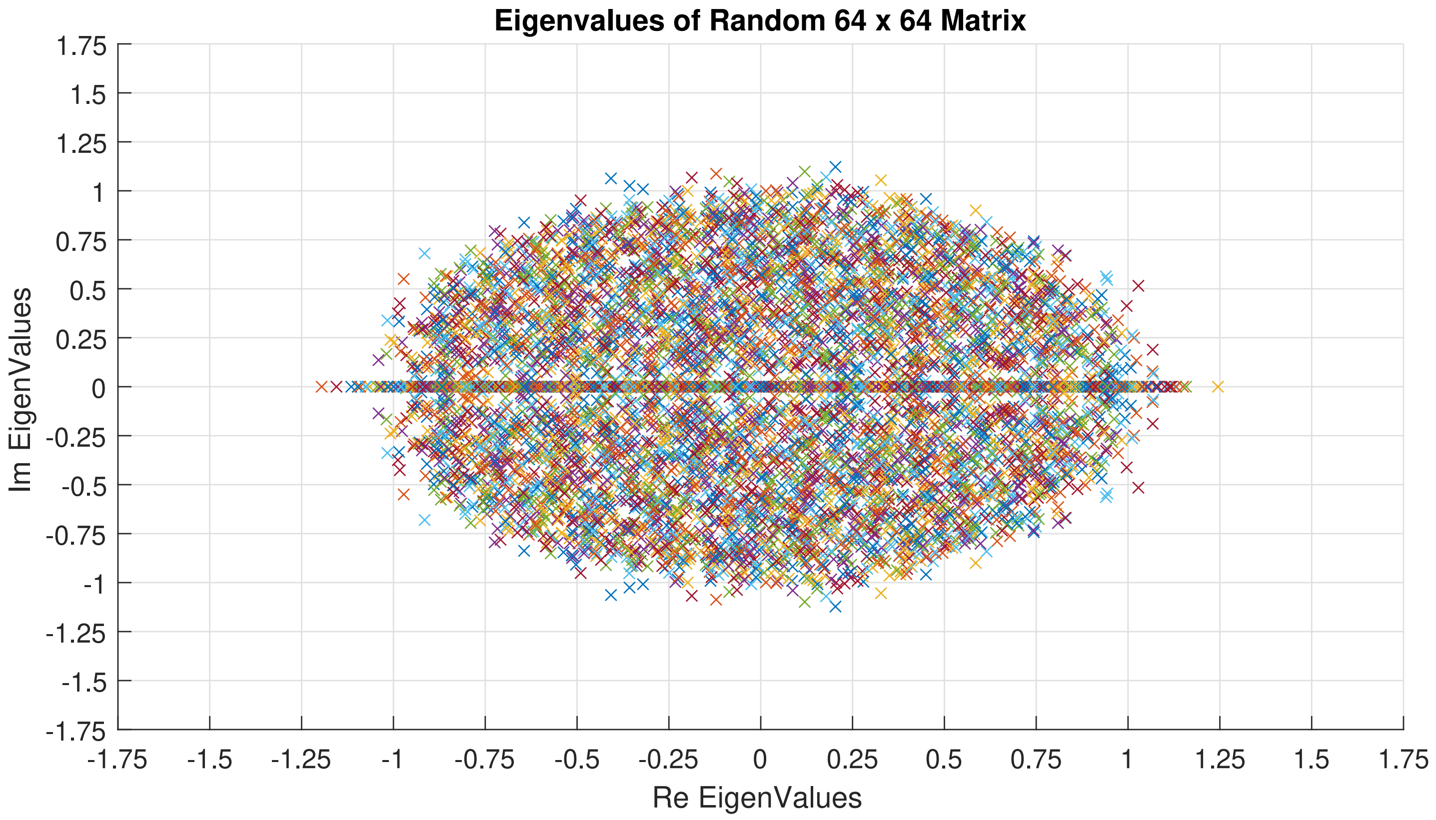
(b) As m $\to \infty$, the 2-norm of the random matrices approach 2. The inequality $\rho(A) <$ $||A||$, remains true as m $\to \infty$. The inequality does not approach an equality as m approaches infinity.

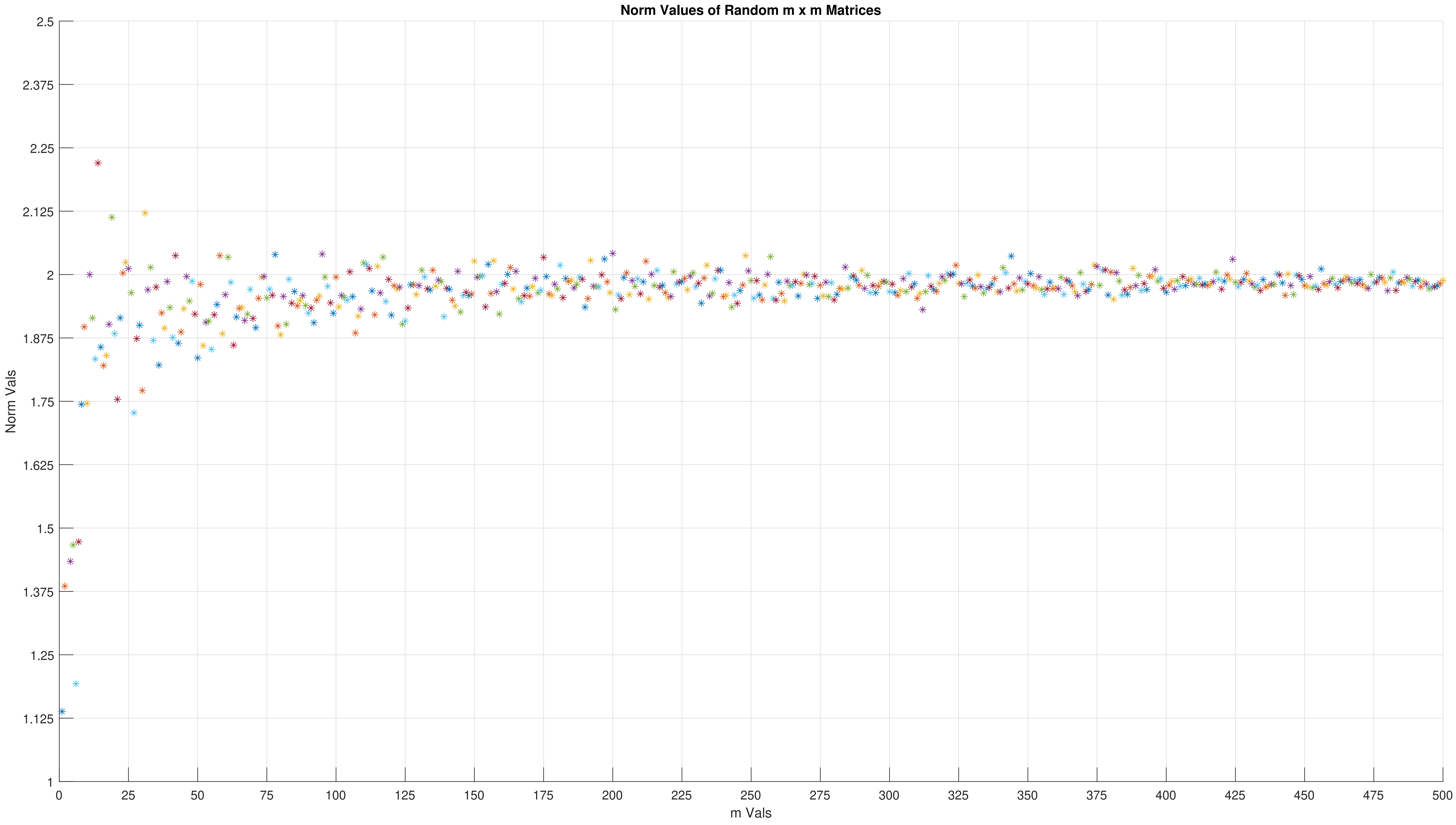
(c) As m $\to \infty$, the proportion between $\delta_{min} < 2^{-1}, 4^{-1}, 8^{-1}$ and the number of random matrices approach 100 %. The proportion between $\delta_{min} < 2^{-1}$ and the number of iterations approaches 100% the fastest as m increases.

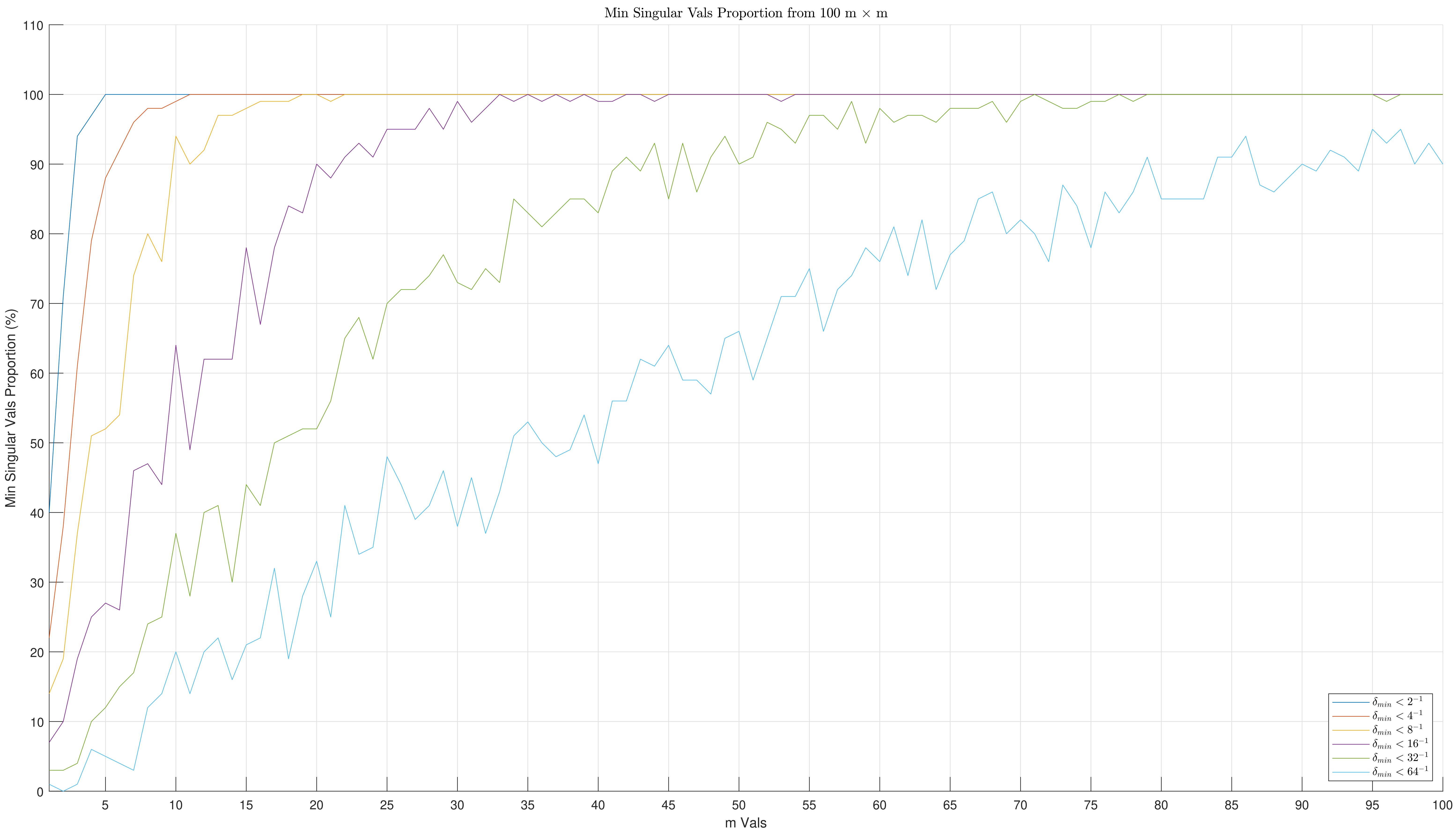

Eigenvalues of Random 8 x 8 Matrix
Im EigenValues
Re EigenValues


Eigenvalues of Random 16 x 16 Matrix
Im EigenValues
Re EigenValues


Eigenvalues of Random 32 x 32 Matrix
Im EigenValues
Re EigenValues


Eigenvalues of Random 64 x 64 Matrix
Im EigenValues
Re EigenValues


Norm Values of Random m x m Matrices
Norm Vals
m Vals
2.5
2.375
2.25
2.125
2
1.875
1.75
1.625
1.5
1.375
1.25
1.125
1
0
25
50
75
100
125
150
175
200
225
250
275
300
325
350
375
400
425
450
475
500



Min Singular Vals Proportion from 100 m × m
Min Singular Vals Proportion (%)
m Vals
0
10
20
30
40
50
60
70
80
90
100
110
5
10
15
20
25
30
35
40
45
50
55
60
65
70
75
80
85
90
95
100
$\delta_{min} < 2^{-1}$
$\delta_{min} < 4^{-1}$
$\delta_{min} < 8^{-1}$
$\delta_{min} < 16^{-1}$
$\delta_{min} < 32^{-1}$
$\delta_{min} < 64^{-1}$