Hw 5 Solutions

① Suppose $\lim_{n\to\infty} x_n = x_0$.

Let $\varepsilon > 0$. Choose $N_1, N_2 \in \mathbb{N}$ s.t. $\forall n \geq N_1$, we have $|x_n - x_0| < \varepsilon/2$ and $\forall n \geq N_2$, we have $|y_n - x_n| < \varepsilon/2$.

Let $N = \max\{N_1, N_2\}$. Let $n \geq N$.

Then
$$\begin{aligned} |y_n - x_0| &= |y_n - x_n + x_n - x_0| \\ &\leq |y_n - x_n| + |x_n - x_0| \\ &< \tfrac{\varepsilon}{2} + \varepsilon/2 = \varepsilon \quad . \blacksquare \end{aligned}$$

② (a) Since S is bounded above $\exists x_0 \in \mathbb{R}$ s.t. $x_0 = \sup S$ by the completeness axiom.

Let $n \geq 1$. Note $x_0 - \frac{1}{n}$ is not an upper bound for S. Choose $x_n \in S$ s.t.

$$x_0 - \frac{1}{n} < x_n.$$

Thus for all n, $x_0 - \frac{1}{n} < x_n \leq x_0 < x_0 + \frac{1}{n}$

I.e., $\forall n$, $|x_n - x_0| < \frac{1}{n}$, and

$\lim_{n\to\infty} x_n = x_0$ by the comparison lemma.

(b) Again $\sup S$ exists by completeness.

Case 1: Suppose $\sup S \in S$.

Then $\sup S = \max S$.

case 2: Suppose $\sup S \notin S$.

Then the sequence $\{x_n\} \subseteq S$

from part (a) exists and $S' = S' \setminus \{\sup S\}$.

I.e. $\sup S$ is a limit point of S.

In each case, $\sup S$ is a maximum of S or a limit point of S.

(c) (i) $S' = [0,1]$

(ii) $S' = [0,1)$

(iii) $S' = \{0,1\}$.

③ $\exists \{a_n\}, \{b_n\} \subseteq \mathbb{R}$ s.t.

$\{a_n b_n\}$ converges and

($\{a_n\}$ does not converge or $\{b_n\}$ does not.)

proof: $\forall n \geq 1$ let $a_n = \frac{1}{n}$ and $b_n = n$.

Then $\{a_n b_n\} = \{1\}$ and $\{b_n\}$ does not converge.

(4) Scratch: $\left|\frac{4n^2+n}{n^2+3n} - 4\right| < \varepsilon$

$$\left|\frac{-11n}{n^2+3n}\right| < \varepsilon.$$

$$\frac{11}{n+3} < \varepsilon$$

$$\frac{11}{\varepsilon} - 3 < n.$$

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proof: Let $\varepsilon > 0$.
Let $N \in \mathbb{N}$ s.t. $N > \frac{11}{\varepsilon}$.

Let $n \geq N$.
Then $\frac{11}{\varepsilon} - 3 < n$.

So $\frac{11}{\varepsilon} < n+3$

So $\frac{11}{n+3} < \varepsilon$.

Thus $\left|\frac{4n^2+n}{n^2+3n} - 4\right| < \varepsilon$. $\blacksquare$

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(5)(a) Scratch

$$|x^2 - 4x + 4| < \varepsilon$$

$$|(x-2)(x-2)| < \varepsilon$$

$$|x-2|(0.1) < \varepsilon.$$

$$|x-2| < \frac{\varepsilon}{0.1}$$

* $1.9 < x < 2.1$
$-.1 < x - 2 < .1$

(a)
proof: Let $\varepsilon > 0$.

Let $\delta = \min\{0.1, \frac{\varepsilon}{0.1}\}$.

Suppose $x \in \mathbb{R}\setminus\{2\}$ and $|x-2| < \delta$.

Then $|x-2| < 0.1$ and $|x-2| < \frac{\varepsilon}{0.1}$

Thus $|(x-2)(x-2)| < (0.1)\left(\frac{\varepsilon}{0.1}\right)$

So $|x^2 - 4x + 4| < \varepsilon$. ☒

(b) scratch

$\left|\frac{2}{x} - \frac{2}{3}\right| < \varepsilon$.

$\left|\frac{6-2x}{3x}\right| < \varepsilon$.

$\left|\frac{3-x}{3x}\right| < \frac{\varepsilon}{2}$.

$\left|\frac{3-x}{3x}\right| < \frac{|3-x|}{6} < \varepsilon/2$ So $|3-x| < 3\varepsilon$.

$2 < x < 4$
$6 < 3x < 12$.

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proof: Let $\varepsilon > 0$. Let $\delta = \min\{1, 3\varepsilon\}$.

Suppose $x \in \mathbb{R}\setminus\{0,3\}$ and $|x-3| < \delta$.

Then $2 < x < 4$ and $|x-3| < 3\varepsilon$.

So $6 < 3x < 12$ and $\left|\frac{2x-6}{6}\right| < \varepsilon$.

So $\left|\frac{2x-6}{3x}\right| < \varepsilon$.

Thus $\left|\frac{2}{x} - \frac{2}{3}\right| < \varepsilon$. ☒

⑥ Scratch WANT: $\frac{3}{(x-2)^2} > M$

$$\sqrt{\frac{3}{M}} > |x-2|$$

proof: Let $M \in \mathbb{N}^+$. Let $\delta = \sqrt{\frac{3}{M}}$. Suppose $0 < |x-2| < \delta$.

Then $|x-2| < \sqrt{\frac{3}{M}}$.

So $0 < (x-2)^2 < \frac{3}{M}$

So $M < \frac{3}{(x-2)^2} = f(x)$ ☒.

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⑦ proof: Let $\varepsilon > 0$. Let $\delta = 2\varepsilon > 0$.

Suppose $x, x_0 \in [1, \infty)$ and $|x - x_0| < \delta$.

Then $|x - x_0| < 2\varepsilon$ and $\sqrt{x} + \sqrt{x_0} \geq 2$.

Notice $|\sqrt{x} - \sqrt{x_0}| = \left| \frac{x - x_0}{\sqrt{x} + \sqrt{x_0}} \right| \leq \left| \frac{x - x_0}{2} \right| < \frac{2\varepsilon}{2} = \varepsilon$

☒.