MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #14

Review

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Grading Policy

- **Grading Policy:** The final grades for this section will be determined as follows:
  - Homework 30%
  - Quizzes 5%
  - Midterm (Sep. 30 & Oct. 2) 30%
    - Part A: take-home, due 11:59 pm Sep. 30 15%
    - Part B: exam, 9:00-9:50 am Oct. 2 15%
  - Final Exam (Dec. 11&14) 35%
    - Part A: take-home, due 11:59 pm Dec 11 15%
    - Part B: exam, 8:00-10:00 am Dec 14 20%
  - Total----------------------------------- 100%

- **Assignments:** Bi-weekly will be due at 11:50 pm Friday
- Quiz I: Due at 9:00 am Aug 26, 2020
- HW1: Posted on Aug. 28 (Fri); Due on September 11 (Fri).

Mid Term (30%)

- Part A: take-home problems, 15% Sep. 30 (W)

Rules

A. The exam must be taken completely alone. Showing it or discussing it with anybody is forbidden.

B. Make an effort to make your submission clear and readable. Severe readability issues may be penalized by grade.

C. Please submit your work to Gradescope by 11:59 pm on Sep. 30, 2020.

- Part B: 9:00-9:50 am, 15%, Oct 2 (F)

  - Students need to submit their Part-B work (to GradeScope) by 10:00 am on Oct 2.
  - "conceptual, comprehensive" and "less technical"
  - Additional information is provided near the end.

MT2 Part B: 9:00-9:50 AM on Oct. 2, 2020

- The exam must be taken completely alone. Showing it or discussing it with anybody is forbidden.
- There are 5 (or 6, TBD) regular problems. Each problem is worth 25 points. Please complete 4 of them.
- You have 50 minutes (between 9:00 and 9:50 am)
- Submit your work to GradeScope by 10:00 am on Oct. 2, 2020

  document any issues (e.g., using screenshots) and report via emails or chat as soon as possible
- Please keep your Zoom connection and enable your Camera.
- Use “Chat” (under Zoom) or “emails” to ask questions or report problems.

Biweekly Homework (30%)

Due Dates

- ~~HW1: Sep 11 (F)~~
- ~~HW2: Sep 25 (F)~~
- HW3: Oct 16 (F), available on Oct. 2.
- HW4: Oct 30 (F)
- HW5: Nov 13 (F)
- HW6: Dec 4 (F)

- Late submission:
  - submitted within one week after the deadline
  - a deduction of 10 points for late submission

A Balance

Freedom vs. Protection

Flexibility vs. Constraint

➢ Find the Hanger

- ➢ Formulate
- ➢ Solve
- ➢ Interpret

Chap 1:

$$x' = -\beta(x - x_{c+})(x - x_{c-})$$

Stability Analysis

1D, linear	2D, linear
$x' = ax$ $x = ke^{\lambda t}$ $\lambda = \mathrm{a}$	$x' = ax + by$ $y' = cx + dy$ $X = \begin{pmatrix} x \\ y \end{pmatrix}; A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
$\lambda > 0, source$ $\lambda < 0, sink$	$X' = AX$ $AX = \lambda X$

1D, nonlinear	2D, nonlinear
$x' = f(x, a)$	$x' = F(x, y)$ $y' = G(x, y)$ $J = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{X_c}$
$x' \approx f'(x_c)(x - x_c)$ $\lambda = f'(x_c)$	$X' \approx JX$ $JX = \lambda X$

a collection of the first partial derivatives

linearization

One Slide Summary

1st order	2nd order	eigenvalue problem
$y' = \alpha y - \beta y^{2}$ (logistic eq.)	$x'' + \beta x' + \alpha x = 0$	$x' = ax + by$ $y' = cx + dy$
$y' = \alpha y - \beta y^{3}$	$x' = y$ $y' = -\alpha x - \beta y$	$X' = AX$ $AX = \lambda X$ $X = \begin{pmatrix} x \\ y \end{pmatrix}; A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

nonlinear	a system of ODEs	
$\boxed{x'' - \alpha x + \beta x^{3} = 0 \quad \text{(DE-sech)}}$	$x' = y \equiv F$ $y' = \alpha x - \beta x^{3} \equiv G$	$JX = \lambda X$ $J = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{X_c}$
$x'' - \alpha x + \beta x^{2} = 0$ (DE-sech²)	$x' = y \equiv F$ $y' = \alpha x - \beta x^{2} \equiv G$	

Tips

- Study slides for lectures #13 and #14 (review)
- Review HW1 and HW2
- Complete MT Part A soon

HW #1

- Derivative Tests vs. Analysis using a Perturbation Method

- Source, Sink, Saddle
- The Logistic Eq: $x' = ax(1-x)$, separable
- ➔ $x' = -\beta(x - x_{c+})(x - x_{c-})$

HW #2

1. 2nd order ODE vs. a system of first-order ODEs (in a matrix form)
2. Saddle vs. Center for a 2D system
3. Saddle vs. Sink within a Linearized Lorenz model (in preparation for ”understanding” the so-called Lorenz Geometrical model.
4. The SIR model: 3D vs. 1D

MT Part A

1. The SIR model: a simplified version vs. Logistic Eq.
2. A nonlinear, non-dissipative Lorenz model: linearization and linear stability analysis
3. A general linear 2D system:
   - eigenvalue problems
   - changing coordinates
   - canonical form
4. Show off your skills and knowledge

Abbreviations

BVP: boundary value problems

DE: differential equations

EP: equilibrium points

IC: initial conditions

IVP: initial value problems

LI: linearly independent

ODE: ordinary differential equations

PDE: partial differential equations

TBD: to be discussed later

Supp: Supplemental Materials (Optional)

1.1 Important Concepts ($x' = f(x)$)

1. Bifurcation;
2. Critical points, $f(x_c) = 0$;
3. (equilibrium points = fixed points = critical points)
4. Derivative tests
5. General solution
6. Initial Value Problem (IVP)
7. Particular solution
8. Phase Line
9. Separable ODEs
10. Sink vs. Source
11. Stable vs. Unstable Solutions, $f'(x_c) < 0$ vs. $f'(x_c) > 0$
12. Structurally Stable vs. Unstable (i.e., with bifurcation)

1.1 Equilibrium Points & Phase Line

- Given $x' = f(x; a)$, equilibrium points, also known as fixed points or critical points, are defined when $f(x_c) = 0$.

- Example 1: Consider $x' = ax$. $x = 0$ is a critical point.

- We may construct a space using dependent variables as coordinates. Such a space is called a phase space (or state space, e.g., Hilborn 2000).
- A 1-D phase space is called a phase line.
- The phase line: as the solution is a function of time, we may view it as a particle moving along the real line.
- For linear stability analysis of a single first-order ODE (e.g., $x' = f(x; a)$, we analyze the sign of x' near one of the system's critical points.

1.1 Unstable vs. Stable Solutions: $x' = ax$

- The phase line: as the solution is a function of time, we may view it as a particle moving along the real line.

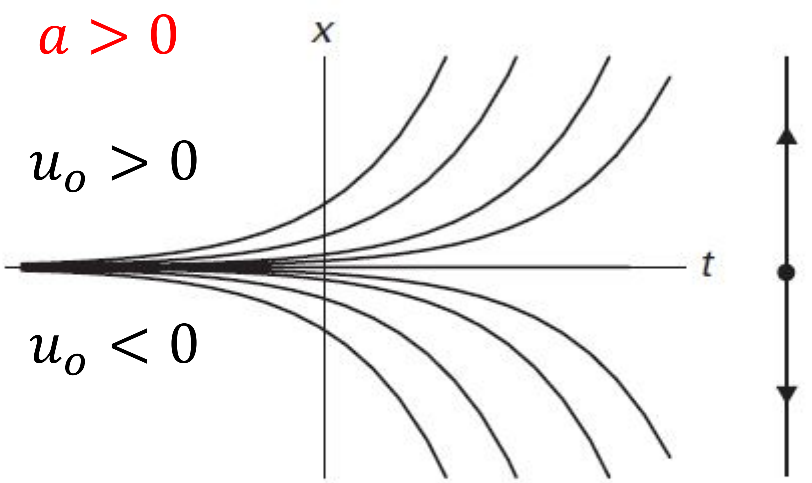


Figure 1.1 The solution graphs and phase line for $x' = ax$ for $a > 0$. Each graph represents a particular solution.

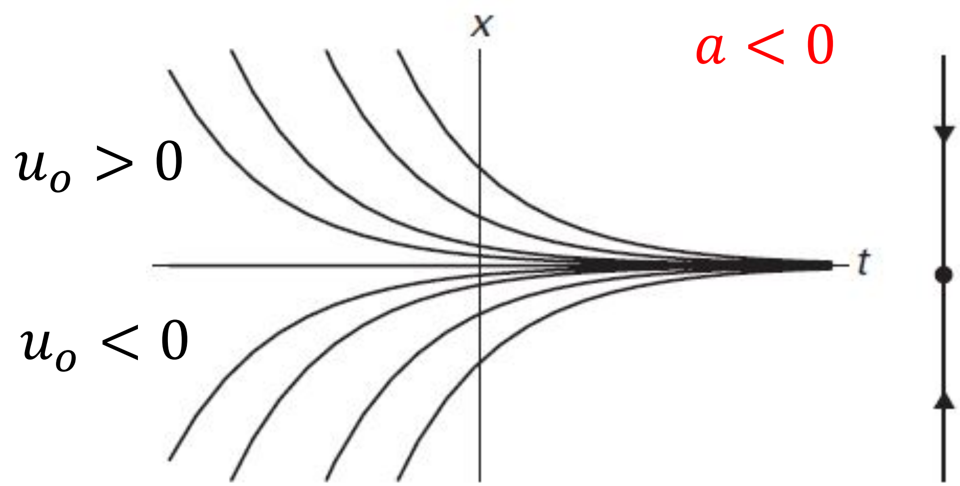


Figure 1.2 The solution graphs and phase line for $x' = ax$ for $a < 0$.

- Unstable solutions with $a > 0$
- moving away from an equilibrim point, $x = 0$.
- $x = 0$ is a source.

- Stable solutions with $a < 0$
- moving toward an equilibrim point, $x = 0$.
- $x = 0$ is a sink.

1.1 Linear (Local) Stability Analysis for 1st Order ODEs

consider a general case

$$\frac{dx}{dt} = f(x)$$

$$\boxed{x' = ax}$$

find critical points

$$f(x_c) = 0$$

linearize $f(x)$ wrt a critical pt

$$\frac{dx}{dt} = f(x) = f(x_c) + f'(x_c)(\mathrm{x} - \mathrm{x_c}) + \cdots$$

(a linearized system)

$$\boxed{\frac{dx}{dt} = f'(x_c)(\mathrm{x} - \mathrm{x_c}) + \cdots}$$

$$\boxed{x' = ax}$$

find solution

$$\mathrm{x} - \mathrm{x_c} = \mathrm{c_0}\exp(f'(x_c)t)$$

$$\boxed{\lambda = f'(x_c)}$$

λ : eigenvalue

stability

the critical point is stable if $f'(x_c) < 0$	a sink
the critical point is unstable if $f'(x_c) > 0$	a source

1.1 Bifurcation of $\frac{dx}{dt} = ax$

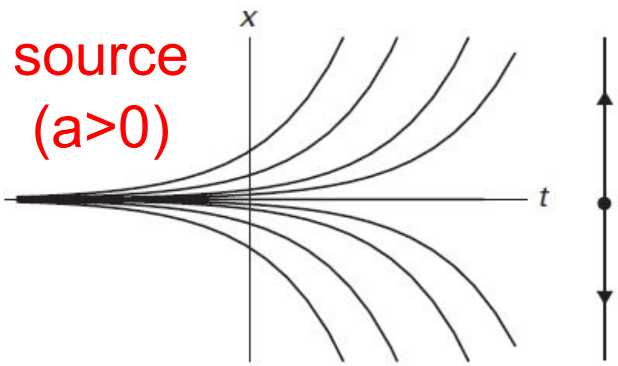


Figure 1.1 The solution graphs and phase line for $x' = ax$ for $a > 0$. Each graph represents a particular solution.

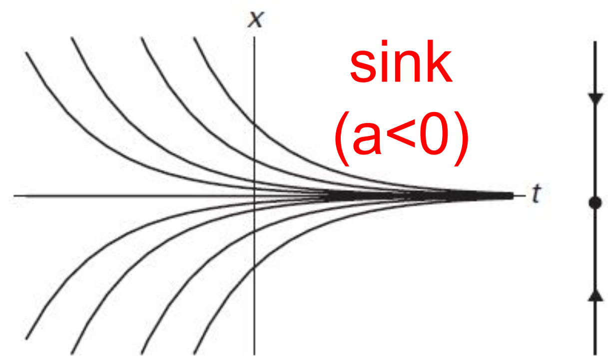


Figure 1.2 The solution graphs and phase line for $x' = ax$ for $a < 0$.

- A bifurcation occurs when there is a "significant" change in the structure of the solutions of the system as "a" varies.
- In the previous example, solutions are unstable for $a > 0$ but stable for $a < 0$. Thus, we have a bifurcation at $a = 0$.
- The simplest types of bifurcations occur when the number of equilibrium solutions changes as "a" varies. For example, within $x' = 1 - ax^2$, there are two critical points for $a > 0$ but no critical points for $a \leq 0$.

Definition: Bifurcation Points

$$\frac{dx}{dt} = f(x, a)$$

critical points

$$f(x, a) = 0$$

bifurcation points

$$\boxed{f(x, a) = 0 \;\&\; f_x(x, a) = 0}$$

example

$$\frac{dx}{dt} = ax$$

critical points

$$ax = 0 \qquad \rightarrow x = 0$$

bifurcation points

$$f_x(x, a) = a \;\rightarrow a = 0$$

Sect. 1.2: the Logistic Equation

$$\boxed{x' = ax}$$

- linear population model if $a > 0$
- x: population (i.e., assume $x > 0$).
- $\frac{dx}{dt}$: the rate of growth of the population, (called a growth rate, or an exponential growth rate)
- $\frac{dx}{dt}$ is proportional to x

$$\boxed{x' = ax\left(1 - \frac{x}{N}\right)}$$

- $\frac{dx}{dt}$ is proportional to x for small x (and $x < N$).
- $\frac{dx}{dt}$ becomes negative for large x (i.e., $x > N$).
- N is called carrying capacity.

We choose $N = 1$ (see Quiz II)

$$\boxed{x' = ax\,(1 - x)}$$

$$\boxed{\equiv f_a(x)}$$

- first order, nonlinear, separable
- autonomous, ($f(x) = ax\,(1 - x)$ is not an explicit function of time).

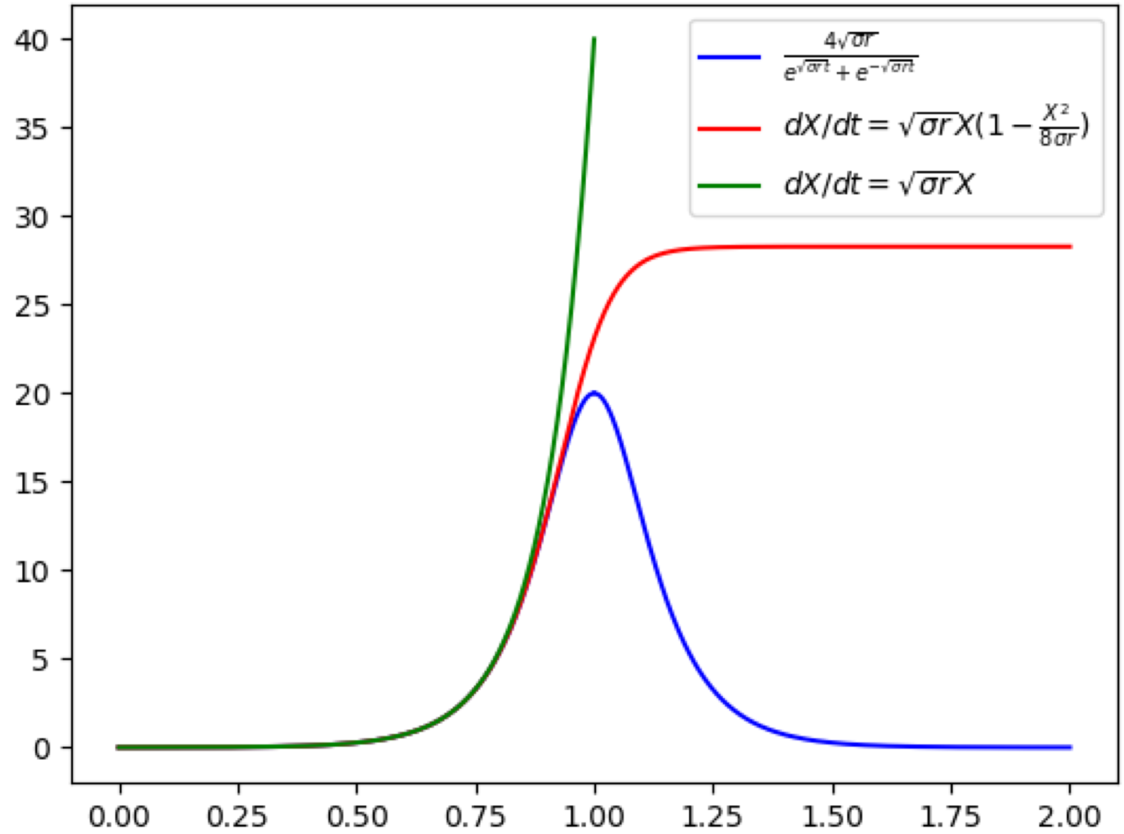
The Role of Nonlinearity

$\boxed{x' = ax}$ exponential growth

$\boxed{x' = ax\,(1-x)}$ logistic growth

The role of nonlinearity

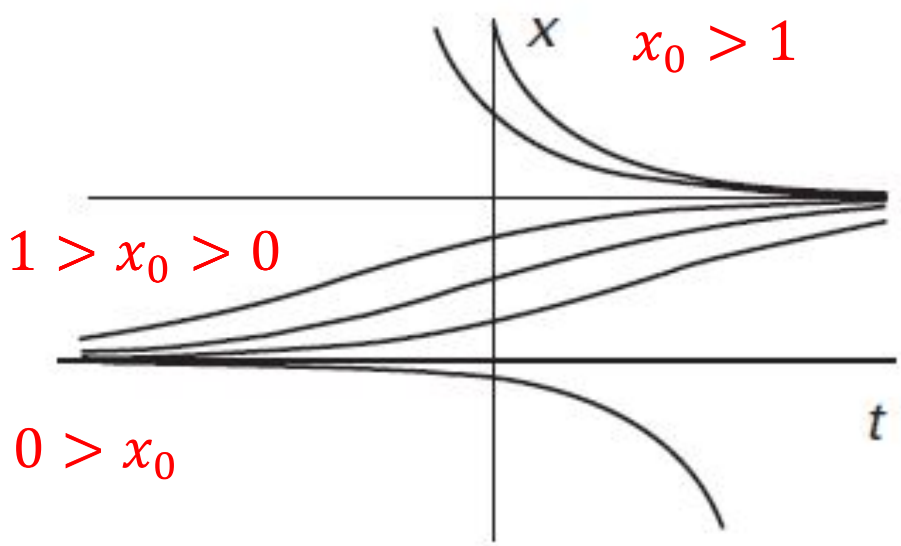
→ suppress growth

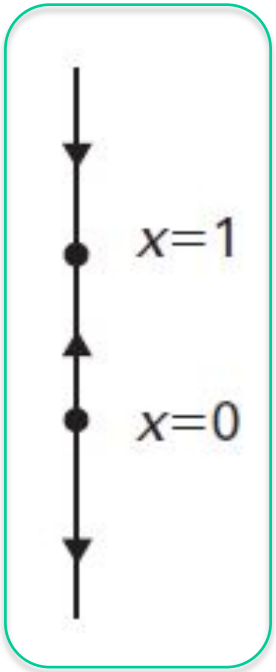


1.2: Analysis of Solutions

$$x' = a(x - x^2)$$

$$x = \frac{x_0 e^{at}}{1 - x_0 + x_0 e^{at}}$$





Sink

Source

Phase line: TBD

1.2 Stability Analysis: Derivative Tests

$$x' = a(x - x^2)$$

$$\boxed{f_a(x) = a(x - x^2)}$$

critical points $\boxed{f_a(x) = 0}$

$$\boxed{x = 0 \; or \; x = 1}$$

1st derivative $f'(x) = a - 2ax$

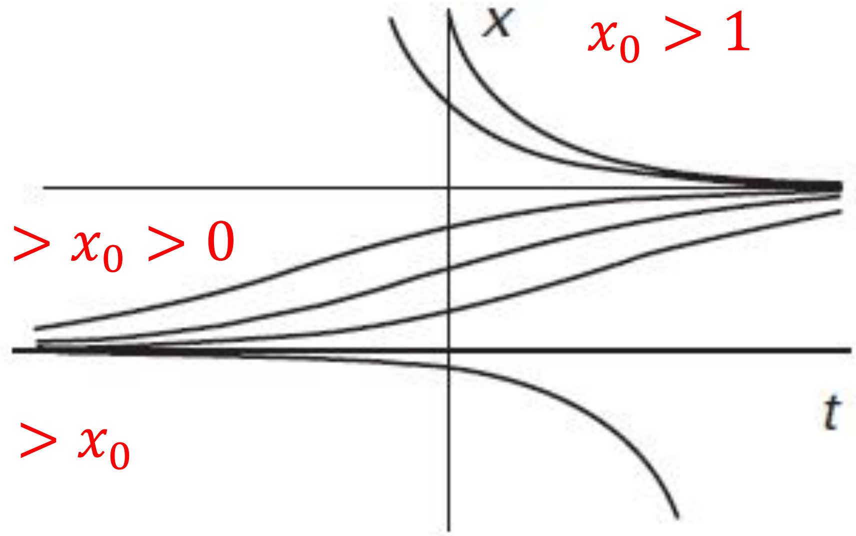
$x = 0$ $\quad f'(0) = a > 0$

unstable

$x = 1$ $\quad f'(1) = -a < 0$

stable

$x_0 > 1$



Sink

$1 > x_0 > 0$

Source

$0 > x_0$

source

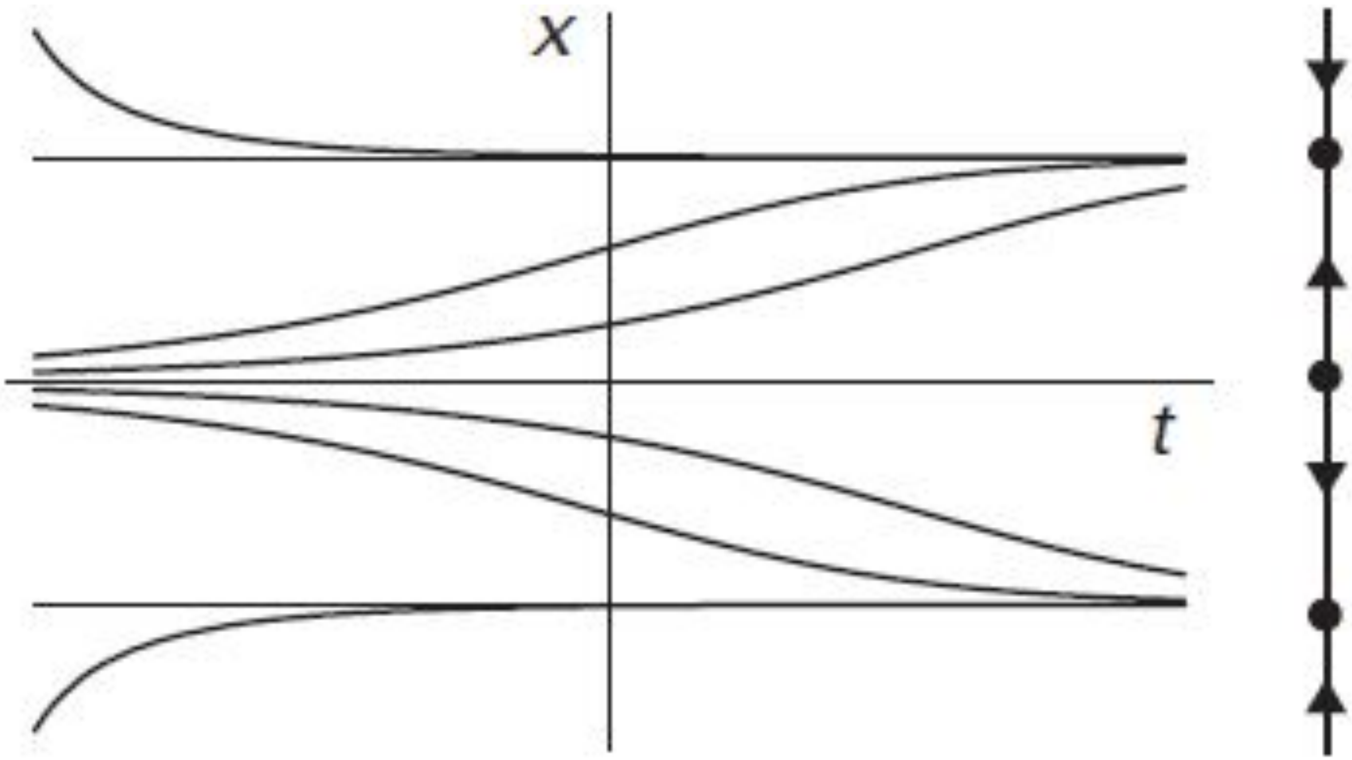
x = 0

sink

x = 1

Section 1.2

$$\frac{dx}{dt} = x - x^3$$



Sink

Source

Sink

A Saddle Point

$$x' = x^2 = f(x)$$

Is the critical point a sink or source?

critical points $\boxed{f(x) = 0}$

$\boxed{x = 0}$

1st derivative $f'(x) = 2x$

$x = 0$ $f'(0) = 0\ ?$

A saddle point or a half-stable critical point (e.g., Strogatz)

Based on the definition, we can obtain:

$x < 0$	$x > 0$
$\frac{dx}{dt} > 0$	$\frac{dx}{dt} > 0$
positive direction	positive direction

$\mathrm{x} = 0$

A saddle point

Apply a perturbation method

$$x = x_c + \varepsilon(t)$$

$\boxed{x = 0 + \varepsilon}$ $\varepsilon' = \varepsilon^2$

The above discussions can help determine $x = 1$ is a saddle point within $x' = (x-1)^2$.

Additional Notes

Supp

The Logistic equation includes the following features:

- For $\mathrm{a} > 0$ the basin of attraction of $x_c = 1$ is $x > 0$, while negative values of x attract to minus infinity.
- In contrast to the logistic map (i.e., difference equation), the logistic equation has no oscillatory nor chaotic solutions.

When both f and f' are zero at the critical point,

- the stability is determined by the sign of the first non-vanishing higher derivatives;
- If that derivative is even (e.g., f''), the point is a saddle point, attracting on one side but repelling on the other.
- If that derivative is odd, it follows the same sign rules as f'.

Bifurcation vs. Saddle Points:

- The former appears in associated with the changes of parameter(s).
- There are various kinds of Bifurcation

Quiz 4 (Due Oct 21)

Sprott (2003)

1.3: Constant Harvesting and Bifurcations

$$x' = x(1-x) - h = f(x,h)$$

$$f(x,h) = 0 \;\&$$
$$f_x(x,h) = 0$$

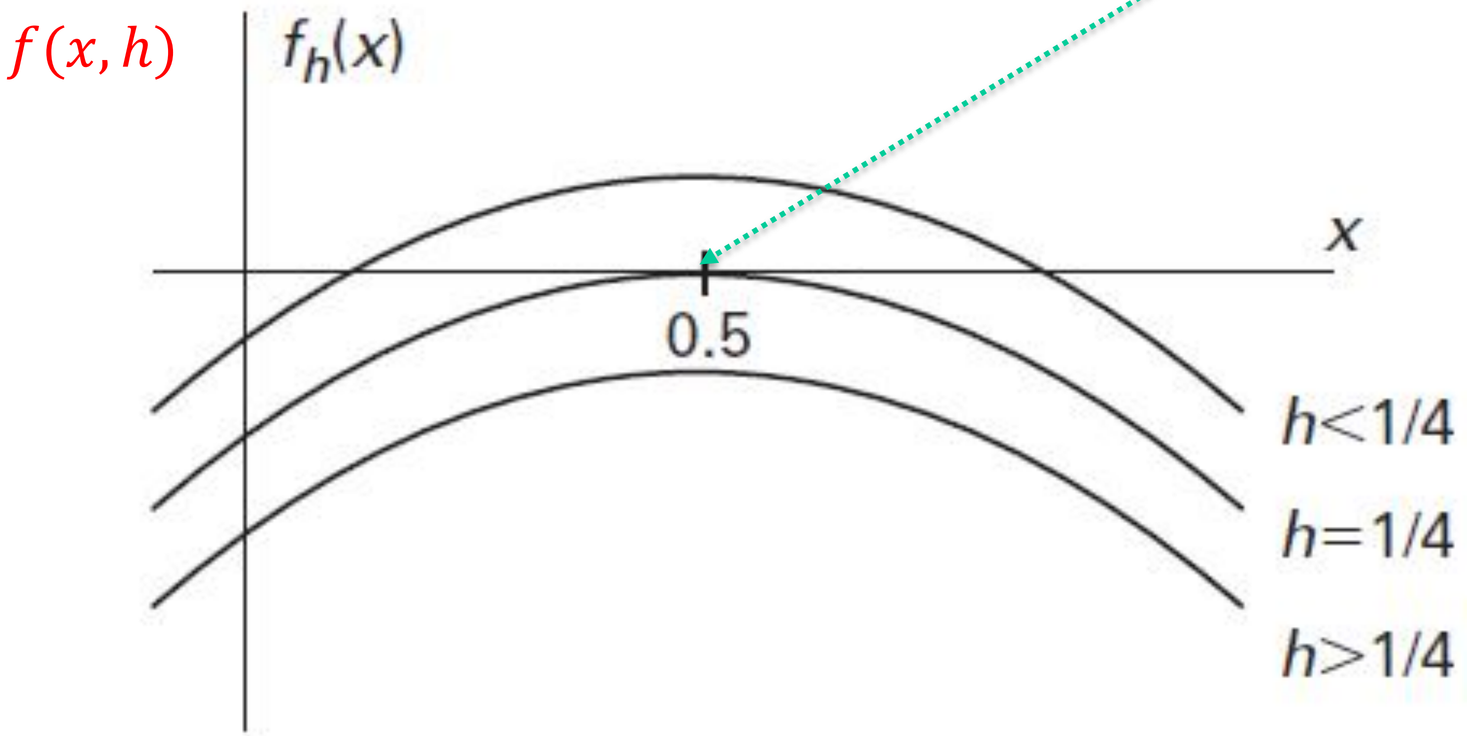


Figure 1.6 The graphs of the function $f_h(x) = x(1-x) - h$.

Bifurcations

- A Bifurcation occurs when there is a “significant” change in the structure of the solutions of the system as system parameter “a” (or “h”) varies.
- The simplest types of bifurcations occur when the number of equilibrium solutions changes as “a” (or “h”) varies.

Bifurcation Point(s)

$$\boxed{x' = x(1-x) - h = f(x,h)}$$

bifurcation points

$$\boxed{f(x,h) = 0 \;\&\; f_x(x,h) = 0}$$

$$f(x,h) = 0 \qquad \boxed{x(1-x) - h = 0} \qquad \boxed{h = 1/4}$$

$$\boxed{x_c = \frac{1 \pm \sqrt{1-4h}}{2}}$$

$$f_x(x,h) = 0 \qquad \boxed{1 - 2x = 0} \qquad \boxed{x = \frac{1}{2}}$$

1.3: Stability Analysis

$$\boxed{x' = x(1-x) - h = f(x,h)} \qquad \boxed{x_c = \frac{1 \pm \sqrt{1-4h}}{2}} \qquad \frac{dx}{dt} = f'(x_c)(\mathrm{x} - \mathrm{x_c}) + \cdots$$

$h > \frac{1}{4}, \rightarrow \textit{no critical points}$ because of $f(x,h) \neq 0$

$$x' = -x^2 + x - h = -\left(x - \frac{1}{2}\right)^2 + \frac{1}{4} - h < 0$$

$$h = \frac{1}{4}, \rightarrow \textit{critical point } x_c = 1/2$$

$$x' = -\left(x - \frac{1}{2}\right)^2 < 0$$

a saddle at $x_c = 1/2$

$$h < \frac{1}{4}, \rightarrow \textit{two critical points}, x_{c1,2} = \frac{1 \pm \sqrt{1-4h}}{2}$$

$$\boxed{f_x(x) = -2\mathrm{x} + 1}$$

$f_x(x_{c1}) < 0$	stable	a sink
$f_x(x_{c2}) > 0$	unstable	a source

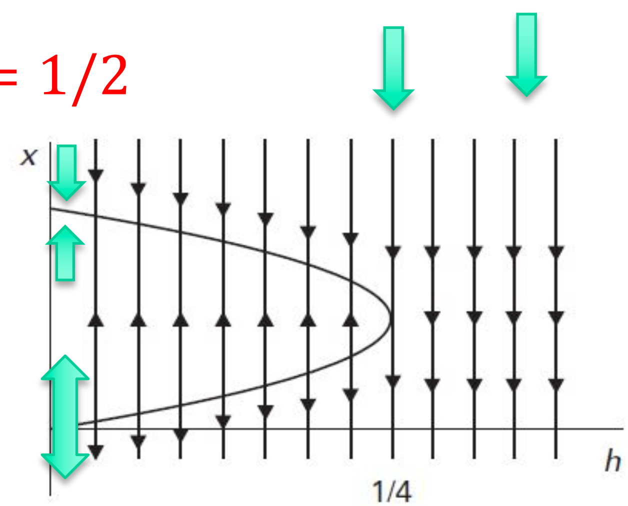


Figure 1.7 The bifurcation diagram for $f_h(x) = x(1-x) - h$.

$$\frac{dx}{dt} = x(1-x) - h$$

$$\boxed{\frac{dx}{dt} = -\beta(\mathrm{x} - \mathrm{x}_{c+})(\mathrm{x} - \mathrm{x}_{c-})}$$

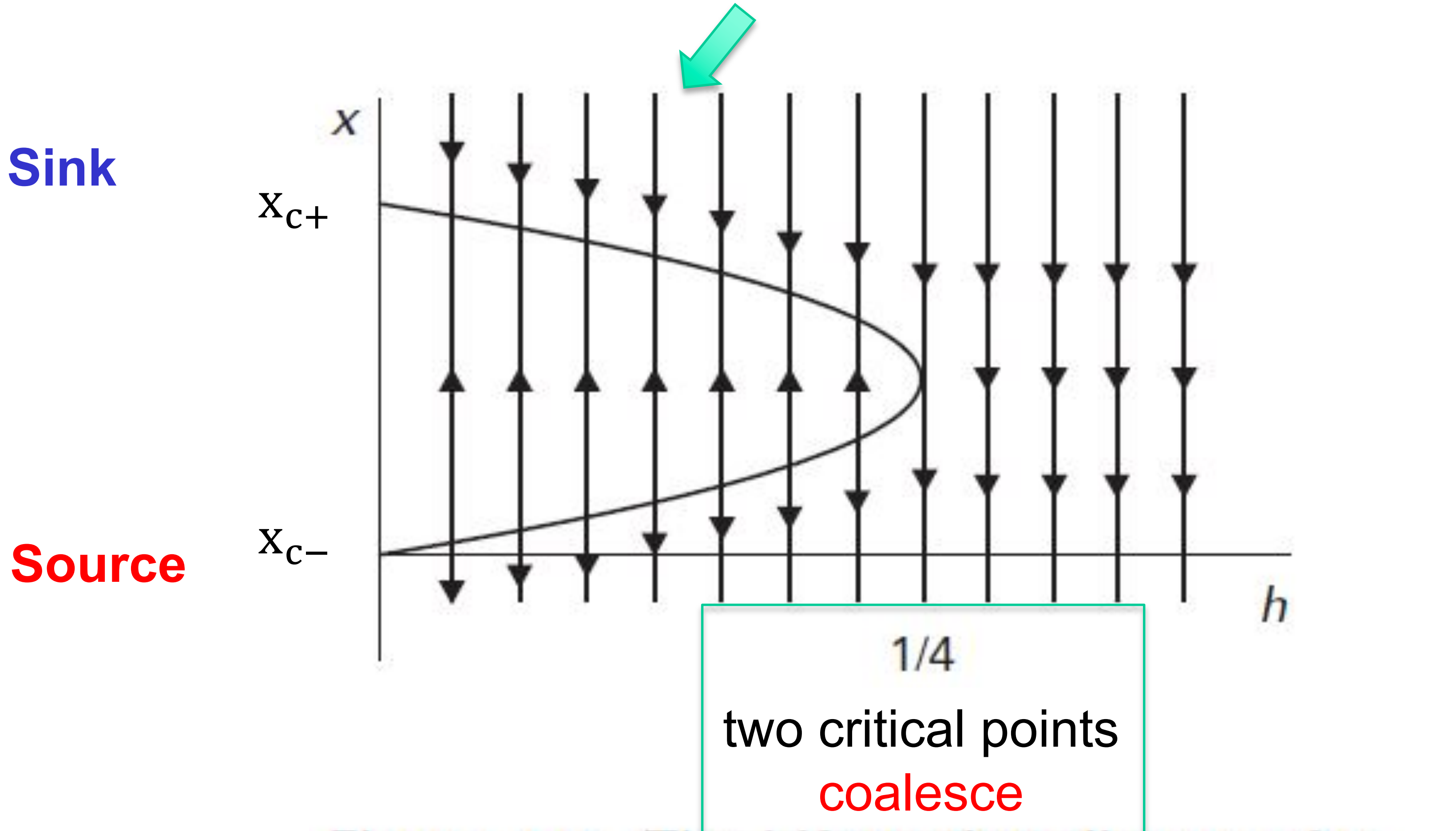
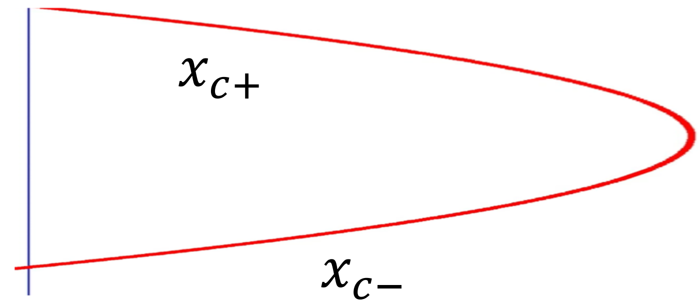


Figure 1.7 The bifurcation diagram for $f_h(x) = x(1-x) - h$.

Stability Analysis: A General Case with Two EPs

$$\frac{dx}{dt} = -\beta(\mathrm{x} - \mathrm{x}_{c+})(\mathrm{x} - \mathrm{x}_{c-}) = f(x)$$



Near x_{c+}

$$f(x) = -\beta(\mathrm{x} - \mathrm{x}_{c+})(\mathrm{x} - \mathrm{x}_{c-})$$

+

$x - x_{c+} < 0$

$$\frac{dx}{dt} > 0$$

positive direction

$x - x_{c+} > 0$

$$\frac{dx}{dt} < 0$$

negative direction

$\mathrm{x} = x_{c+}$

sink

Near x_{c-}

$$f(x) = -\beta(\mathrm{x} - \mathrm{x}_{c+})(\mathrm{x} - \mathrm{x}_{c-})$$

−

$x - x_{c-} < 0$

$$\frac{dx}{dt} < 0$$

negative direction

$x - x_{c-} > 0$

$$\frac{dx}{dt} > 0$$

positive direction

$\mathrm{x} = x_{c-}$

source

1.4: Periodic Harvesting and Periodic Solutions

$x' = x(1-x) - h(1+\sin(2\pi t))$ non-autonomous system

$g(t) = h(1+\sin(2\pi t))$

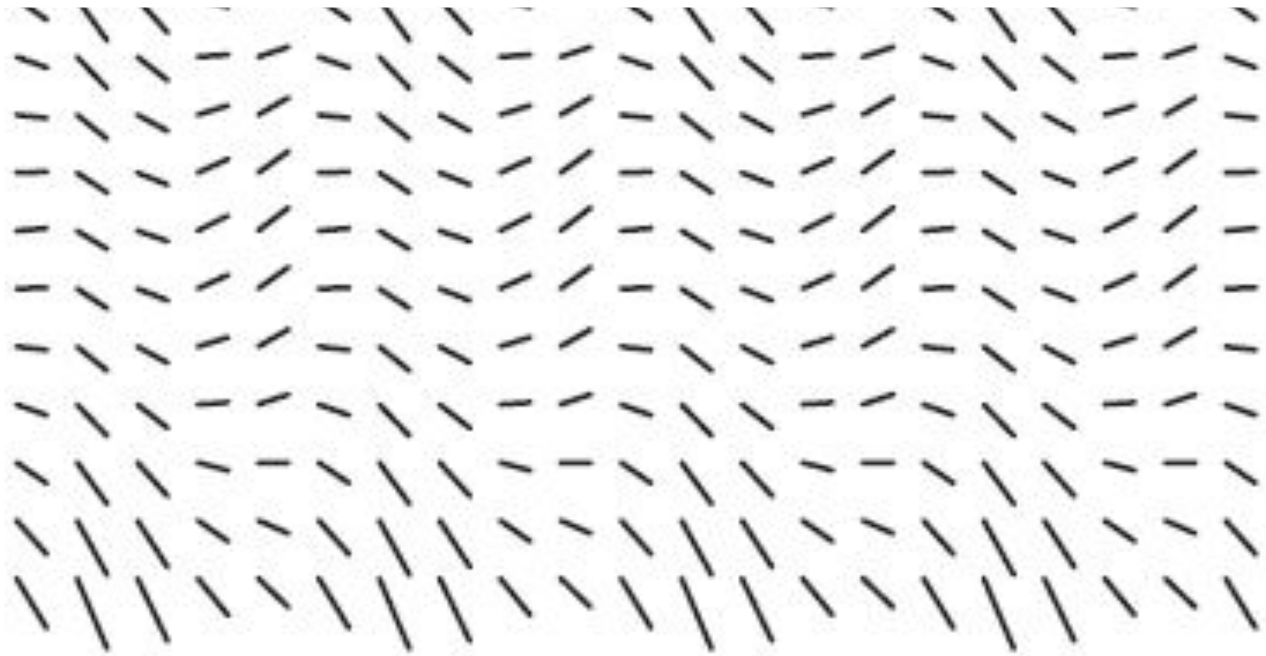


Figure 1.9 The slope field for $f(x) = x(1-x) - h(1+\sin(2\pi t))$.

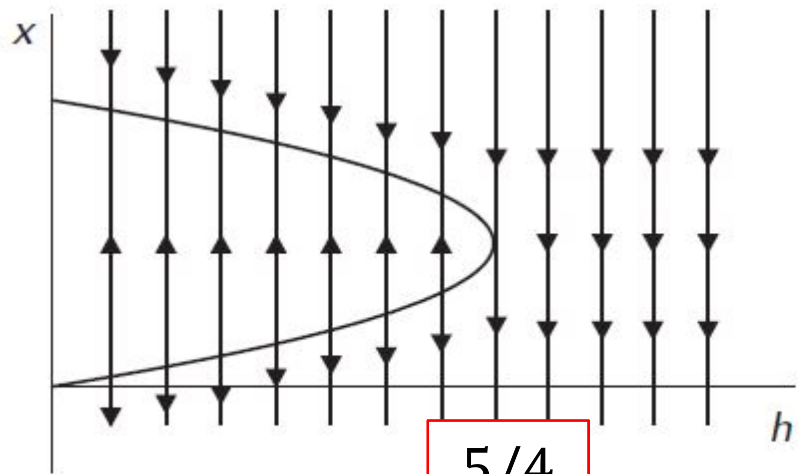
Bifurcation Point(s)

consider $\boxed{x' = 5x(1-x) - h = f(x,h)}$ constant harvesting

bifurcation points $\boxed{f(x,h) = 0 \,\&\, f_x(x,h) = 0}$

$f(x,h) = 0$ $\boxed{5x(1-x) - h = 0}$ $\boxed{h = 5/4}$

$f_x(x,h) = 0$ $\boxed{5 - 10x = 0}$ $\boxed{x = \frac{1}{2}}$

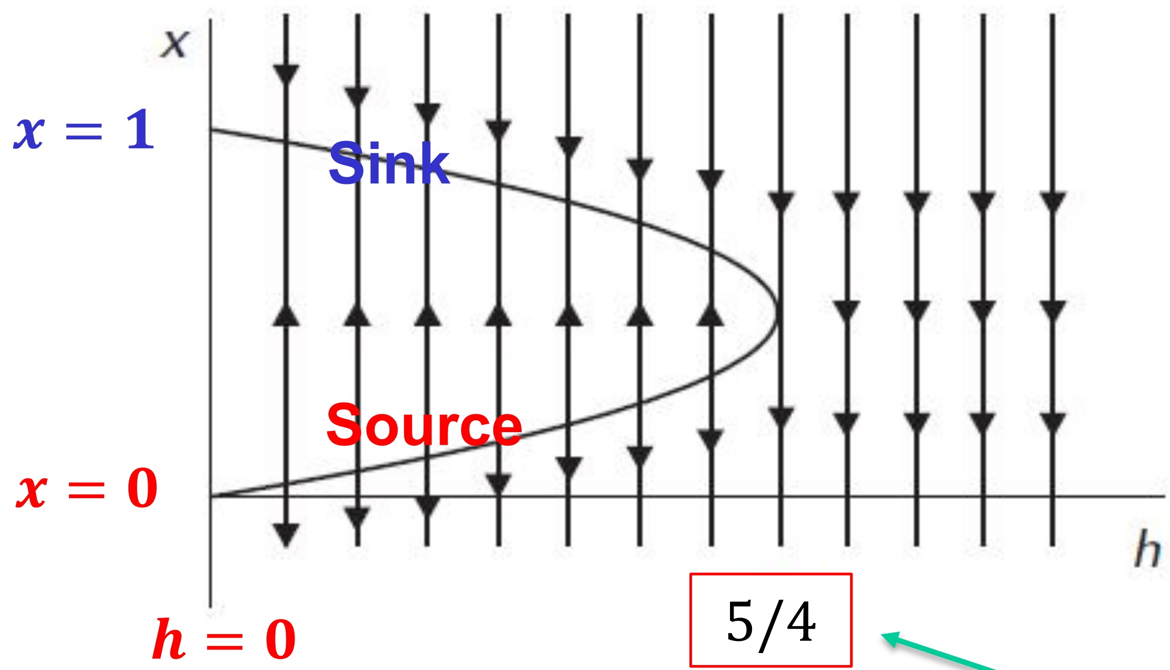


$\boxed{5/4}$

Potential Appearance of Stable and Unstable Points

Consider a time varying "h"

$$x' = 5x(1-x) - g(t) \qquad g(t) = 0.8(1+\sin(2\pi t))$$



For a time varying $g(t)$ that is less than $\frac{5}{4}$,

- stable points may appear between $x = \frac{1}{2}$ and $x = 1$, and
- unstable points may appear between $x = 0$ and $x = \frac{1}{2}$.

Section 1.5: Computing the Poincare Map

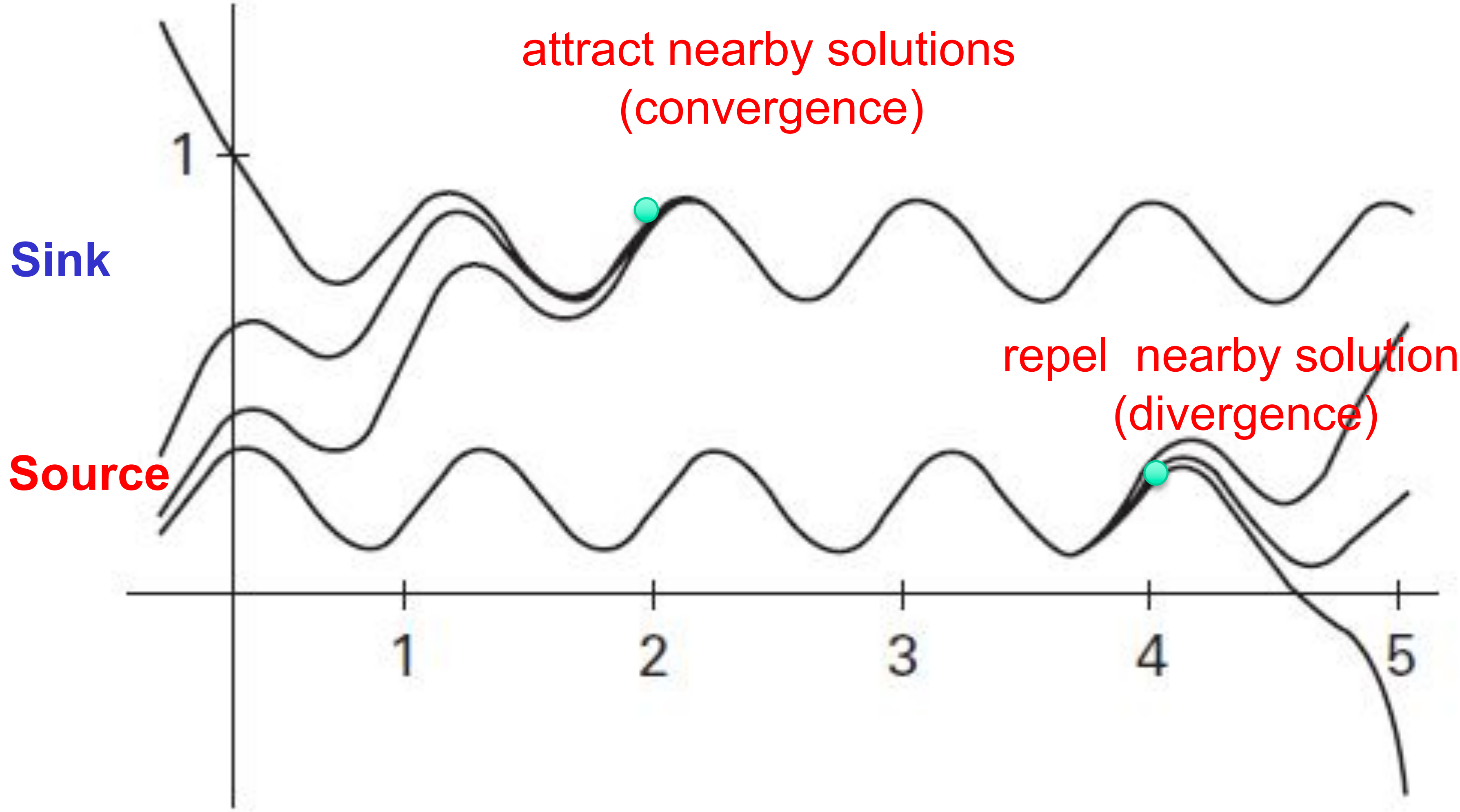
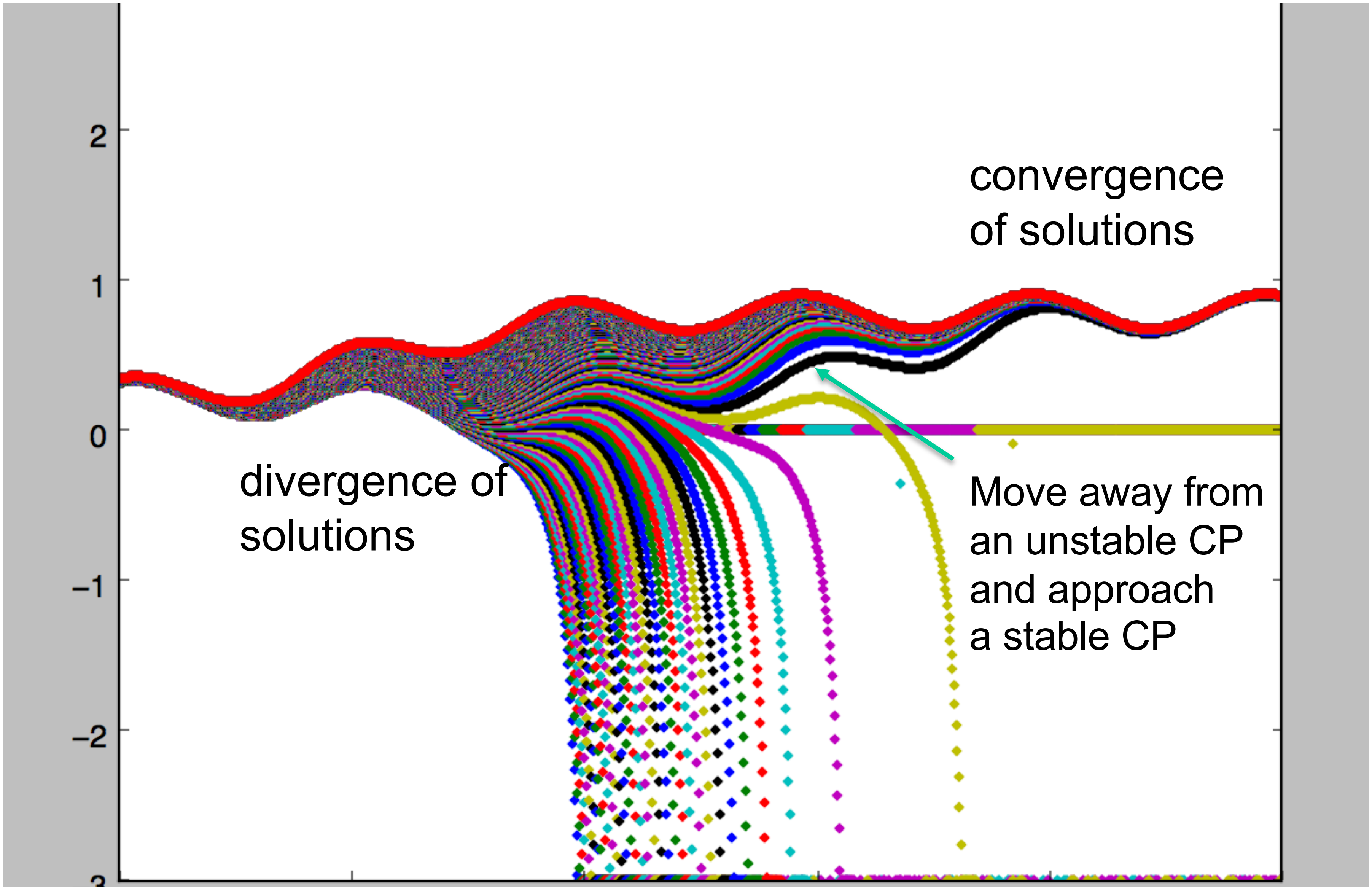
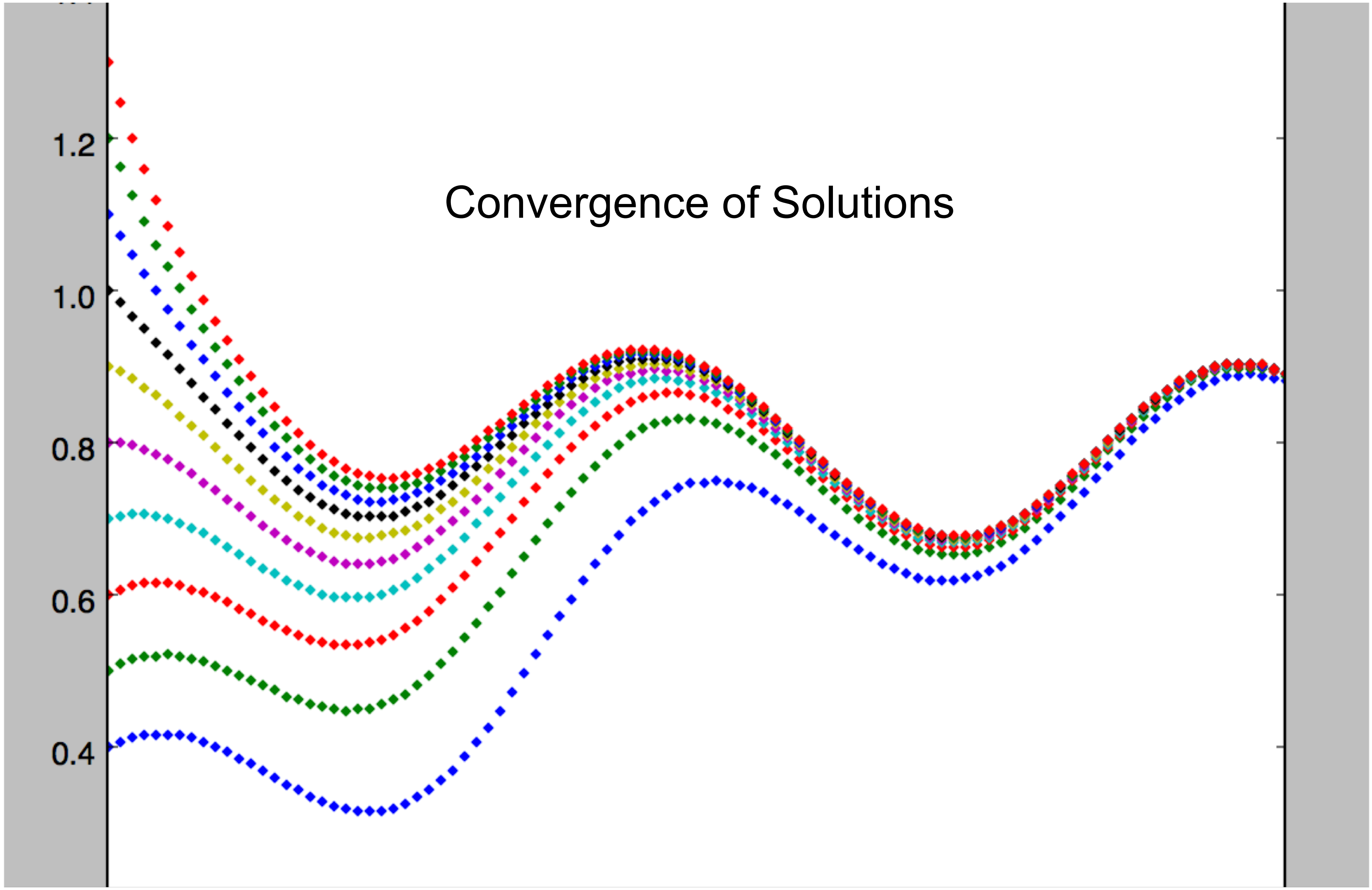


Figure 1.11 Several solutions of $x' = 5x(1 - x) - 0.8(1 + \sin(2\pi t))$.

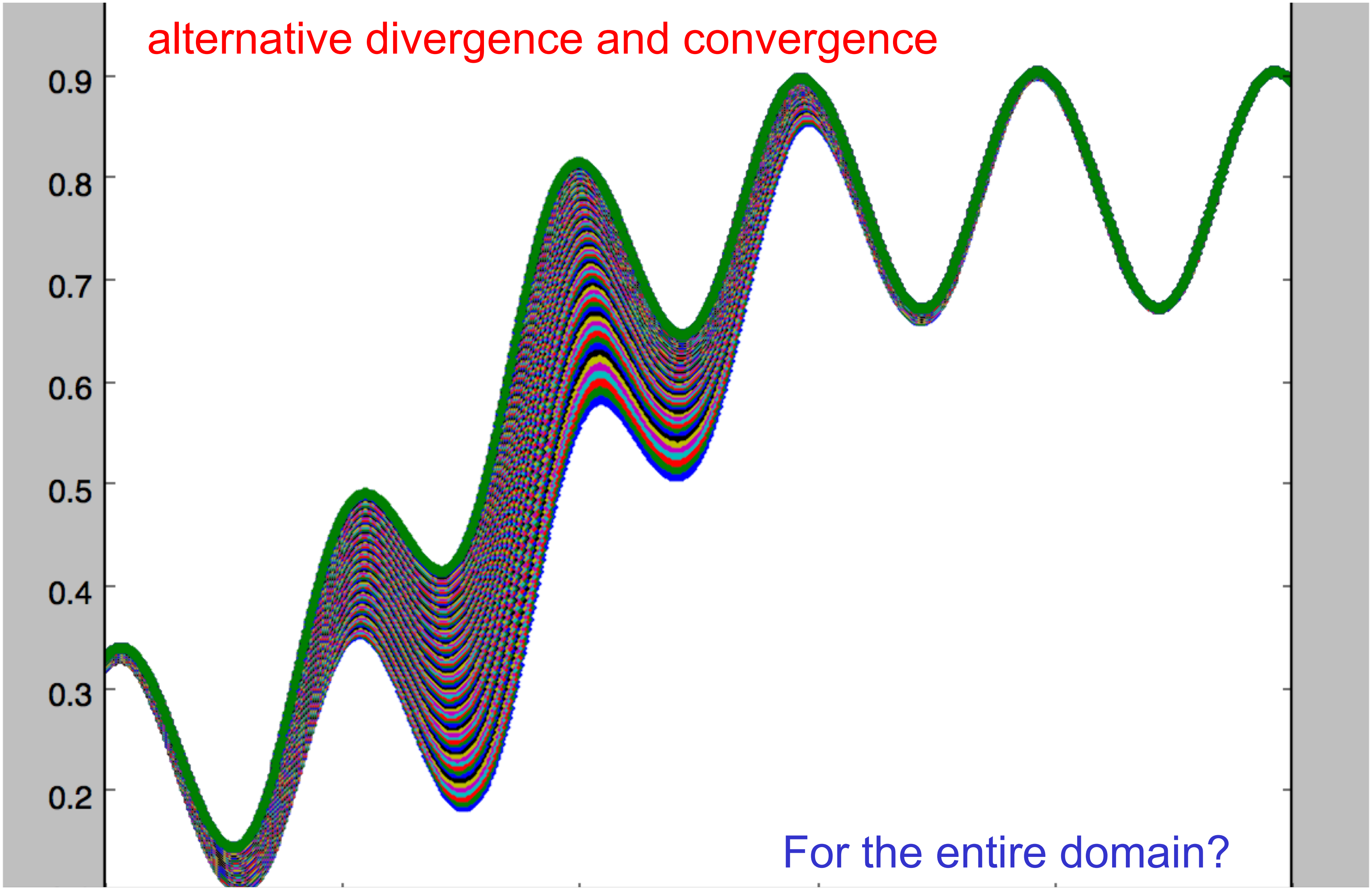
$$x' = 5x(1-x) - 0.8(1+\sin(2\pi t))$$



$$x' = 5x(1-x) - 0.8(1+\sin(2\pi t))$$



$$x' = 5x(1-x) - 0.8(1+\sin(2\pi t))$$



Key ODEs in Chapter 1

$$\frac{dx}{dt} = ax$$

bifurcation at $a = 0$

$$x = x_0 e^{at}$$

$$\boxed{\frac{dx}{dt} = ax\,(1 - x)}$$

bifurcation at $a = 0$

(the Logistic Eq)

$$x = \frac{x_0 e^{at}}{1 - x_0 + x_0 e^{at}}$$

(sigmoid function)

$$\boxed{\frac{dx}{dt} = x(1 - x) - h}$$

bifurcation at $\mathrm{h} = 1/4$

(the Logistic Eq with constant harvesting)

$$\boxed{\frac{dx}{dt} = -\beta(\mathrm{x} - \mathrm{x}_{c+})(\mathrm{x} - \mathrm{x}_{c-})\ for\ h < 1/4}$$

$$\boxed{\frac{dx}{dt} = x(1 - x) - h(1 + \sin(2\pi t))}$$

periodic forcing,
non-autonomous system

(the Logistic Eq with periodic harvesting)

Fundamental Bifurcations

Supp

type		
saddle-node	$\frac{dx}{dt} = a - x^2$	$f_x(x_c, a) = 0$
transcritical	$\frac{dx}{dt} = ax - x^2$	$f_x(x_c, a) = 0$
pitchfork	$\frac{dx}{dt} = ax - x^3$	$f_x(x_c, a) = 0$
Hopf		$Re(\lambda) = 0$
Homoclinic		

Chapter 2. Systems of ODEs in Matrix Form

Our system may then be written more concisely as

$$X' = F(t, X),$$

- The system of equations is called autonomous if none of the f_j depends on t, so the system becomes $X' = F(X)$.
- In analogy with first-order differential equations, a vector X_c for which $F(X_c)$ is called an equilibrium point for the system. An equilibrium point corresponds to a time-independent solution $X(t) = X_c$ of the system as before.

2.2 A System of 1st Order ODEs

$$\boxed{\begin{aligned} x' &= y \\ y' &= -x \end{aligned}}$$

How to solve?

For now, let's transform the above system into a single ODE

$$\boxed{x'' = y' = -x}$$

Assume

$$x = ke^{\lambda t}$$

$$\lambda = \pm i$$

$$x = c_1 \cos(t) \qquad x = c_2\, sin(t)$$

How to obtain y?

$$\boxed{x' = y}$$

$$y = -c_1 \sin(t) \qquad y = c_2 \cos(t)$$

2.2: Alternative Method

$$\boxed{\begin{aligned} x' &= y \\ y' &= -x \end{aligned}}$$ ODEs

How to solve?

Previously, we assume $x = ke^{\lambda t}$

Now, we asume $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{\lambda t}$
$$x = x_0 e^{\lambda t}$$
$$y = y_0 e^{\lambda t}$$

Plug into the above Eq.
$$\boxed{\begin{aligned} \lambda x_0 &= y_0 \quad (1) \\ \lambda y_0 &= -x_0 \quad (2) \end{aligned}}$$ Algebraic Eq.

$\lambda \times (1) + (2)$
$$\lambda^2 x_0 = -x_0$$

$$\lambda = \pm i$$

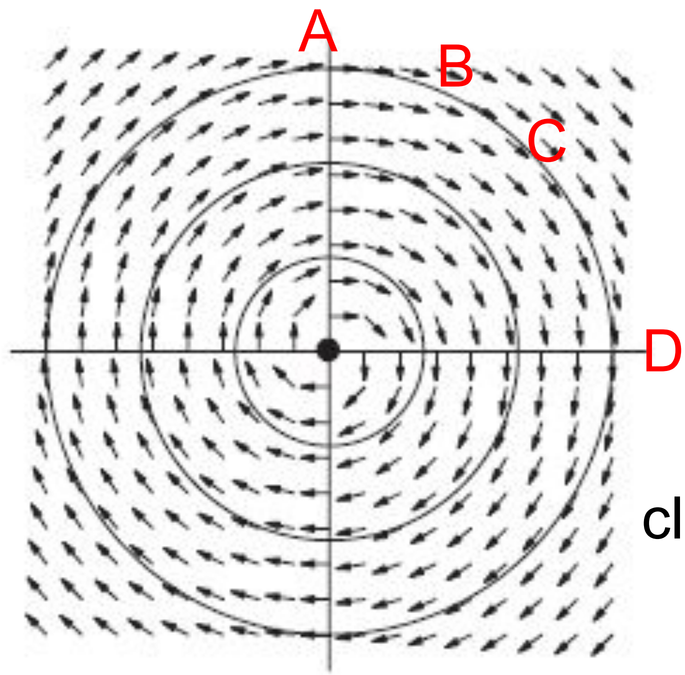
2.2 Vector Fields

$$\boxed{\begin{aligned} x' &= y \\ y' &= -x \end{aligned}}$$

Verify whether the following are the solutions to the above system

$$x = a\sin(t) \qquad y = a\,cos(t)$$

	t	$(\sin(t), cos(t))$
A	0	$(0, 1)$
B	$\pi/6$	$(1/2, \sqrt{3}/2)$
C	$\pi/4$	$(\sqrt{2}/2, \sqrt{2}/2)$
D	$\pi/2$	$(1, 0)$



clockwise

2.2 Planar Systems: Vector Field

$$F(X) = (y, -x) = (P, Q)$$

	(x_i^*, y_i^*)	$\vec{F}$
A	$(0.5, 0)$	$(0, -0.5)$
B	$(0, 0.5)$	$(0.5, 0)$
C	$(-0.5, 0)$	$(0, 0.5)$
D	$(0, -0.5)$	$(-0.5, 0)$

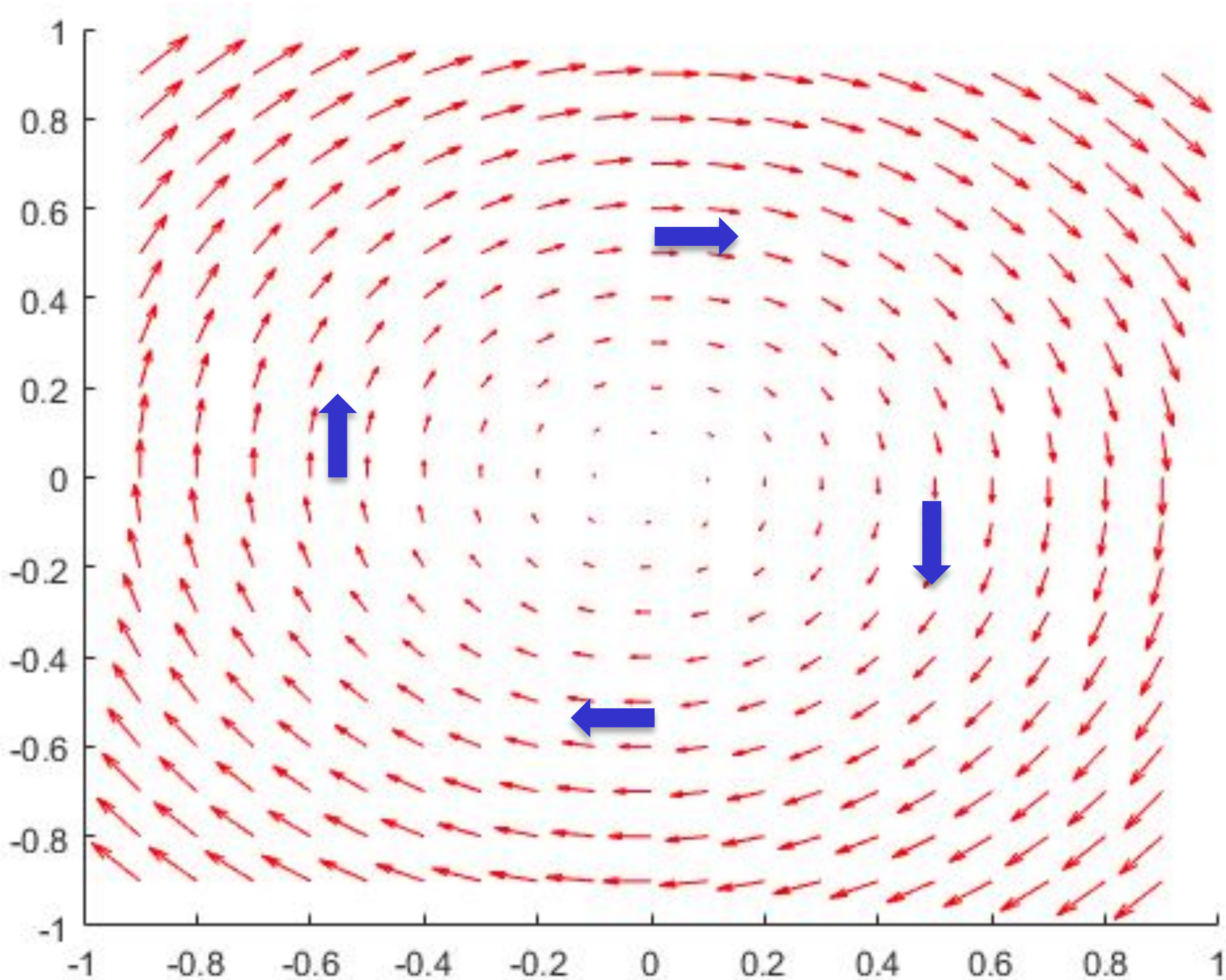
$\nabla \times F = -2$, *clockwise*

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$$\nabla \cdot F = 0$$

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

MATLAB Plot (Direction Field) for Figure 2.1



Stability Analysis

1D, linear	2D, linear
$x' = ax$ $x = ke^{\lambda t}$ $\lambda = \mathrm{a}$	$x' = ax + by$ $y' = cx + dy$ $X = \begin{pmatrix} x \\ y \end{pmatrix}; A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
$\lambda > 0, source$ $\lambda < 0, sink$	$X' = AX$ $AX = \lambda X$

1D, nonlinear	2D, nonlinear
$x' = f(x, a)$	$x' = F(x, y)$ $y' = G(x, y)$ $J = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{X_c}$
$x' \approx f'(x_c)(x - x_c)$ $\lambda = f'(x_c)$	$X' \approx JX$ $JX = \lambda X$

a collection of the first partial derivatives

linearization

ODEs vs. LA

TBD

High-Order ODEs		System of Algebraic Eqs.
Systems of M Linear ODEs	$X = X_o e^{\lambda t} \Rightarrow$	Linear Algebraic Eq.
	features near X_c $\Leftarrow$	Eigenvalue Problems
Systems of Nonlinear ODEs	$\Rightarrow$ linearization via Jacobian Matrix	Stability and Bifurcation analysis
Data Producer for various types of data, including chaotic, periodic, quasi-periodic solutions, etc.	Time Series of M Variables	• Multivariate Analysis • Spectral analysis • Principle Component Analysis (PCA: for Dimension Reduction) • Single Value Decomposition (SVD)
”Prediction” “Causality”		• Analysis • Prediction with empirical equations

Key Concepts for Eigenvectors

Eigenvectors and generalized eigenvectors: An eigenvector (or right eigenvector) v of an $n{\times}n$ matrix A is a nonzero vector which satisfies $Av = \lambda v$ or $(A - \lambda I)v = 0$. A generalized (right) eigenvector is defined as $(A - \lambda I)^k\, v = 0$ for some $1 \le k \le n$. Namely,

- Eigenvectors: $(A - \lambda I)v = 0$
- Generalized eigenvectors: $(A - \lambda I)^k\, v = 0$ for some $1 \le k \le n$.

For example, given a matrix with repeated eigenvalue, two independent vectors can be obtained as follows:

- $(A - \lambda I)v_1 = 0$
- $(A - \lambda I)^2\, v_2 = 0 \qquad \Rightarrow (A - \lambda I)v_2 = v_1$

Section 2.3: Preliminaries from Algebra

We now further restrict our attention to the most important class of planar systems of differential equations, namely, linear systems. In the autonomous case, these systems assume the simple form

$$x' = ax + by$$
$$y' = cx + dy$$

where a, b, c, and d are constants. We may abbreviate this system by using the *coefficient matrix* A where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then the linear system may be written as

$$X' = AX.$$

Trajectory, Orbit, and Path

$$x' = ax + by \quad (= P(x, y)) \qquad (1)$$

$$y' = cx + dy \quad (= Q(x, y)\ ) \qquad (2)$$

- A solution x(t), y(t) of (1) represents a curve C in the xy-plane (or a point as a degenerate case). This curve is called a solution curve or path (sometimes a trajectory or orbit) of (I).
- The sense of increasing t is called the positive sense on C and can be marked by an arrow head. This defines an orientation on C.
- If t is time and C the path of a moving body, the positive sense is the sense in which the body moves along C as time progresses.
- The present xy-plane is often called the phase plane of (1-2).

Section 2.3: A Summary

Consider the following two lines in R^2

$$ax + by = \alpha$$
$$cx + dy = \beta,$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$|A| = 0 \Longrightarrow$ two lines are parallel or overlap

no solution or
infinitely many solutions

$|A| \neq 0 \Longleftrightarrow$ two lines intersect

unique solution

$\vec{n}_1$ and $\vec{n}_2$ and are LI,
to be discussed below.

Critical Points with $AX = 0$

Consider the following system of ODEs

$$x' = ax + by$$
$$y' = cx + dy$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then the linear system may be written as

$$X' = AX, \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

The critical points of the above system are defined as $X' = 0$, yielding

$$\boxed{AX = 0}$$

Special Linear Systems ($AX = 0$ for Critical Points)

Consider a special case of the following two lines in R^2

$$ax + by = \alpha, \qquad ax + by = 0,$$
$$cx + dy = \beta, \qquad cx + dy = 0,$$

$$\boxed{AX = 0, \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}}$$

which represent two lines passing through the origin. As a result, (0, 0) is a solution. We have the following two scenarios:

- $|A| = 0 \Longrightarrow$ two lines overlap — infinitely many solutions
- $|A| \neq 0 \Longleftrightarrow$ two lines intersect — unique solution of (0,0)

$$\boxed{\text{If the system has non-trivial critical points, } |A| = 0.}$$

Eigenvalue Problem

Definition
A nonzero vector V_0 is called an *eigenvector* of A if $AV_0 = \lambda V_0$ for some λ. The constant λ is called an *eigenvalue* of A.

Eigenvalue problem $\quad (A - \lambda I)V_0 = 0$

V_0: eigenvector
λ: eigenvalue

- $|A - \lambda I| = 0 \Longrightarrow$ infinitely many solutions
- $|A - \lambda I| \neq 0 \Longleftrightarrow$ unique solution of (0,0)

If the system has non-trivial solutions, $|A - \lambda I| = 0$.

$\boxed{\text{If the system has eigenvectors, } |A - \lambda I| = 0.}$

A Summary

$AX = \gamma$	$X' = AX$
	$(A - \lambda I)V_0 = 0$
$\lvert A \rvert \neq 0,$ unique sol $\lvert A \rvert \neq 0$ & γ=0, trivial sol	$\lvert A - \lambda I \rvert \neq 0,$ trivial sol
$\lvert A \rvert = 0$ • no solution • Infinitely many solutions	$\lvert A - \lambda I \rvert = 0$ • Infinitely many solutions
$\lvert A \rvert = 0$ for non-trivial EPs (because of γ=0)	• The above is called an eigenvalue problem • $Let\ AV_1 = \lambda_1 V_1;\ AV_2 = \lambda_2 V_2,$ we have a general solution as follows: $X = c_1 V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t}$
	• 1D $x' = f(x)$ • $x' = f(x) \approx f'(\mathrm{x_c})(\mathrm{x} - \mathrm{x_c})$ • $\lambda = f'(\mathrm{x_c})$

Eigenvalue Problem: Example

ODEs

$$\boxed{\begin{aligned} x' &= y \\ y' &= -x \end{aligned}}$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

We assume

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{\lambda t}$$

$$\begin{aligned} x &= x_0 e^{\lambda t} \\ y &= y_0 e^{\lambda t} \end{aligned}$$

Plug into the above Eq.

$$\boxed{\begin{aligned} \lambda x_0 &= y_0 \\ \lambda y_0 &= -x_0 \end{aligned}}$$

Algebraic Eq.

Express the above in a matrix form

$$\lambda \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = A \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$A \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \lambda \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 0$$

$$(A - \lambda I)V_0 = 0 \qquad V_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

V_0: eigenvector
λ: eigenvalue

Eigenvalue Problem (cont.)

Consider $(A - \lambda I)V_0 = 0$ with $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $V_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

V_0: eigenvector
λ: eigenvalue

Compute $A - \lambda I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix}$

If the system has eigenvectors, $|A - \lambda I| = 0.$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \qquad \boxed{\lambda = \pm i}$$

Consider $\lambda = i$

$$\boxed{\begin{aligned} \lambda x_0 &= y_0 \\ \lambda y_0 &= -x_0 \end{aligned}} \qquad \boxed{\begin{aligned} i x_0 &= y_0 \\ i y_0 &= -x_0 \end{aligned}}$$

("overlap") $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ i x_0 \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ i \end{pmatrix}$

Obtain $\boxed{V_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}}$ as an eigenvector associated with $\lambda = i$

Similarly, we have $\boxed{V_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}}$ as an eigenvector associated with $\lambda = -i$

Eigenvectors as "basis vectors"

Theorem. *Suppose A has a pair of real eigenvalues $\lambda_1 \neq \lambda_2$ and associated eigenvectors V_1 and V_2. Then the general solution of the linear system $X' = AX$ is given by*

$$X(t) = \alpha e^{\lambda_1 t} V_1 + \beta e^{\lambda_2 t} V_2. \quad \blacksquare$$

Chapter 3: 2D Linear Systems

$$x' = ax + by \quad (= P(x, y)) \qquad (1)$$

$$y' = cx + dy \quad (= Q(x, y)\ ) \qquad (2)$$

$$X' = AX \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

1. Eigenvalue problem: $|A - \lambda I| = 0$
2. Two linearly independent solutions, $e^{\lambda_1 t} V_1$ and $e^{\lambda_2 t} V_2$
3. Real eigenvalues for a source, sink, or saddle
4. Complex eigenvalues for a center, spiral sink or spiral source
5. Diagonalization
6. Changing coordinates Linearly Conjugate

Simple 2D Systems with Real Distinct Eigenvalues

Consider $\boxed{\begin{aligned} x' &= \lambda_1 x \\ y' &= \lambda_2 y \end{aligned}}$ $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} \lambda_1 - \lambda & 0 \\ 0 & \lambda_2 - \lambda \end{pmatrix}$$

Let $|A - \lambda I| = 0 \Longrightarrow$ $(\lambda_1 - \lambda)(\lambda_2 - \lambda) = 0$ $\boxed{\lambda = \lambda_{1,2}}$

Real distinct eigenvalues include:

A. $\lambda_1 < 0 < \lambda_2$ (different signs): saddle

B. $\lambda_1 < \lambda_2 < 0$ (both are negative): sink

C. $0 < \lambda_1 < \lambda_2$ (both are positive): source

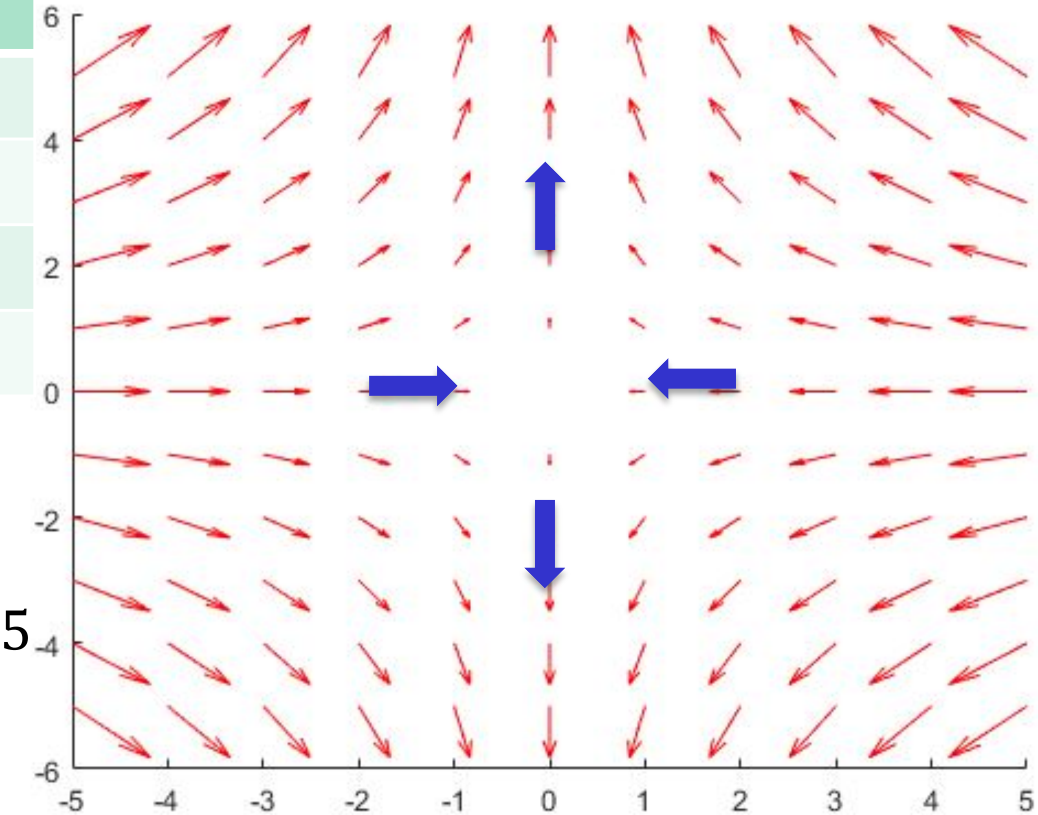
(A) A Saddle ($\lambda_1 < 0 < \lambda_2$)

$x' = -x$

$y' = y$

$F(X) = (-x, y) = (P, Q)$

	(x_i^*, y_i^*)	$\vec{F}$
A	$(2, 0)$	$(-2, 0)$
B	$(0, 2)$	$(0, 2)$
C	$(-2,0)$	$(2,0)$
D	$(0, -2)$	$(0, -2)$



MATLAB Plot for Figure 3.1

Apply the Euler Method

$$\frac{x_{n+1} - x_n}{dt} = -x_n \qquad dt = 0.5$$

At point A: $(x_n, y_n) = (2,0)$

$$x_{n+1} = x_n - dt\ x_n$$

$$= 2 - 0.5 * 2$$

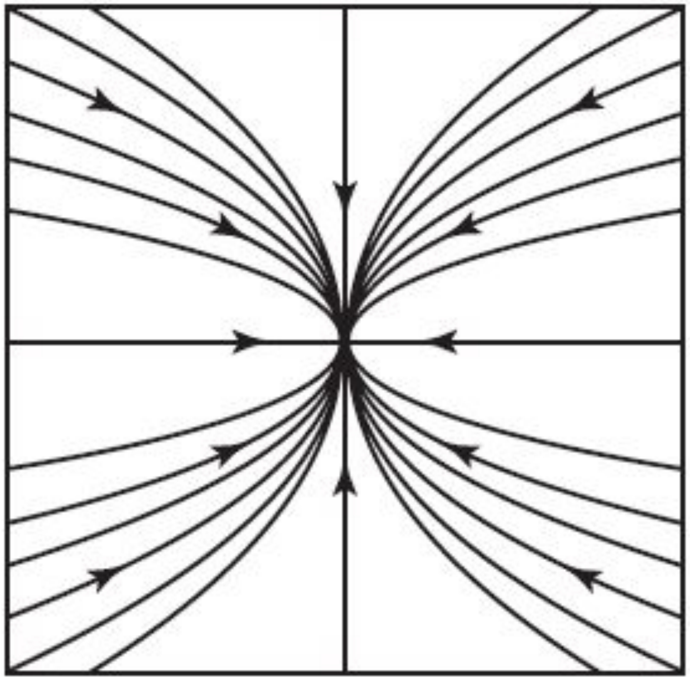
$$= 1 < x_n$$

move westward

B: Sink ($\lambda_1 < \lambda_2 < 0$)

$\lambda_1 < \lambda_2 < 0$

Trajectories (time varying)

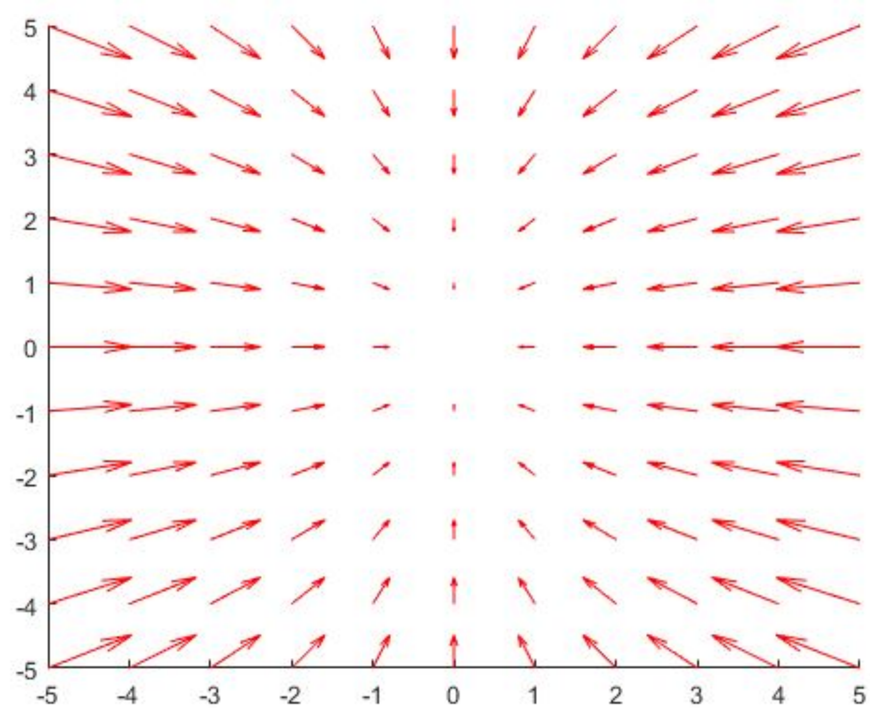


(a)

Figure 3.3 Phase portraits for (a) a sink

X' = -2x Y' = -y

Vector Fields (at a given time)



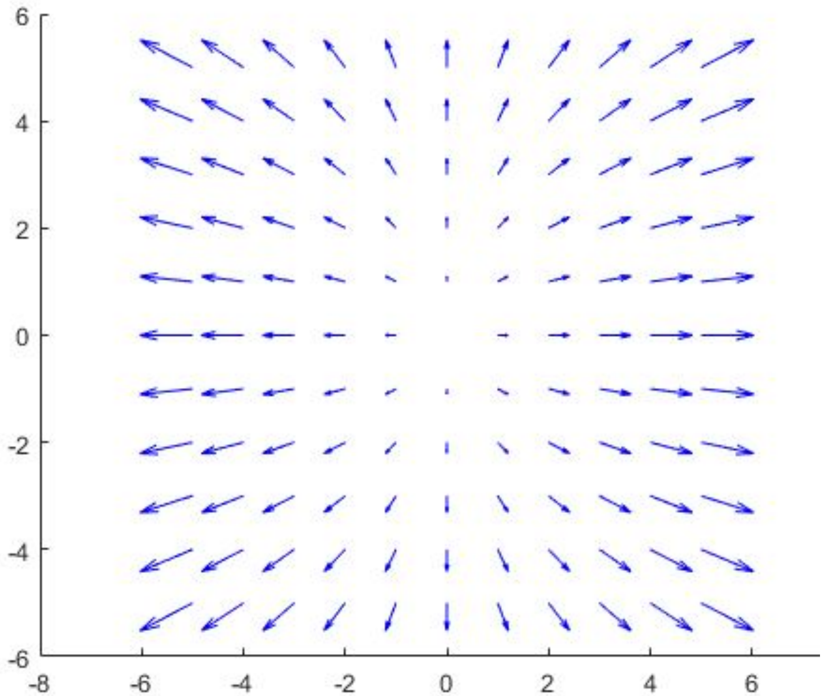
MATLAB Plot for Figure 3.3a

- These solutions tend to the origin (a sink) tangentially to the y axis (i.e., vertical asymptotes), associated with the weaker eigenvalue (λ_2).

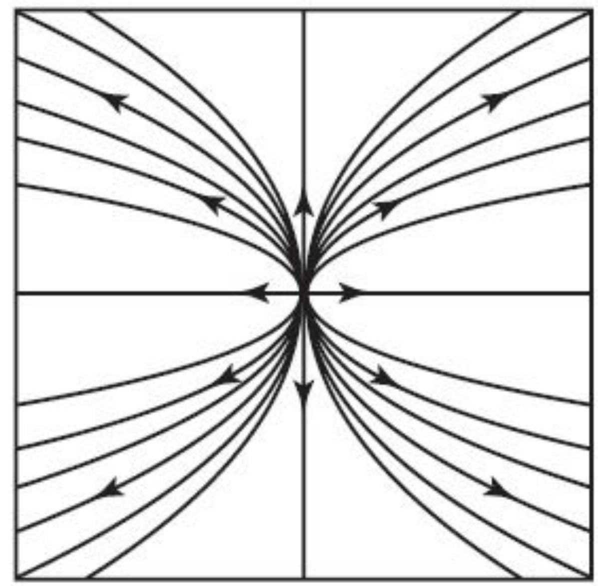
(C): A Source $(0 < \lambda_2 < \lambda_1)$

$X' = 2x$ $\qquad$ $Y' = y$ $\qquad$ $\lambda_1 > 0; \quad \lambda_2 > 0$

Vector Fields (at a given time) $\qquad$ Trajectories (time varying)



MATLAB Plot for Figure 3.3b



(b)

Figure 3.3 Phase portraits for (b) a source.

The general solution and phase portrait remain the same, except that all solutions now tend away from (0,0) along the same paths. See Figure 3.3b.

Simple 2D Systems with Complex Eigenvalues

$$x' = ax + by \quad (= P(x,y)) \qquad (1)$$

$$y' = cx + dy \quad (= Q(x,y)\ ) \qquad (2)$$

$$X' = AX \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

(I) pure imaginary eigenvalues

$$\boxed{A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}}$$

$$\boxed{\begin{aligned} x' &= \cancel{ax} + \beta y \quad (= P(x,y)) \\ y' &= \beta x + \cancel{dy} \quad (= Q(x,y)) \end{aligned}}$$

$\boxed{\lambda_{1,2} = \pm i\,\beta\text{:}\ \ \text{center}}$

(II) complex eigenvalues with $Re \neq 0$

$$\boxed{A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}}$$

$$\boxed{\begin{aligned} x' &= \alpha x + \beta y \quad (= P(x,y)) \\ y' &= \beta x + \alpha y \quad (= Q(x,y)) \end{aligned}}$$

$\lambda_{1,2} = \alpha \pm i\,\beta$:

spiral source or sink

(D) A Center with $\lambda = \pm i\beta$

$$Re\left(e^{i\beta t}\begin{pmatrix}1\\ i\end{pmatrix}\right) = Re\left((\cos(\beta t) + isin(\beta t))\begin{pmatrix}1\\ i\end{pmatrix}\right) = \begin{pmatrix}\cos(\beta t)\\ -sin(\beta t)\end{pmatrix} = X_{re}$$

$$\mathrm{Im}\left(e^{i\beta t}\begin{pmatrix}1\\ i\end{pmatrix}\right) = Im\left((\cos(\beta t) + isin(\beta t))\begin{pmatrix}1\\ i\end{pmatrix}\right) = \begin{pmatrix}\sin(\beta t)\\ cos(\beta t)\end{pmatrix} = X_{im}$$

Thus, a general solution is written as

$$X(t) = a\boxed{e^{i\beta t}\begin{pmatrix}1\\ i\end{pmatrix}} + be^{-i\beta t}\begin{pmatrix}1\\ -i\end{pmatrix}$$

$$= \mathrm{c}_1\mathrm{X}_{\mathrm{re}}(\mathrm{t}) + c_2X_{im}$$

Solution for a Center with $\lambda = \pm i\beta$

$\lambda_{1,2} = \pm\, i\, \beta$

$X' = 2y$ $\quad Y' = -2x$

Trajectories (time varying)

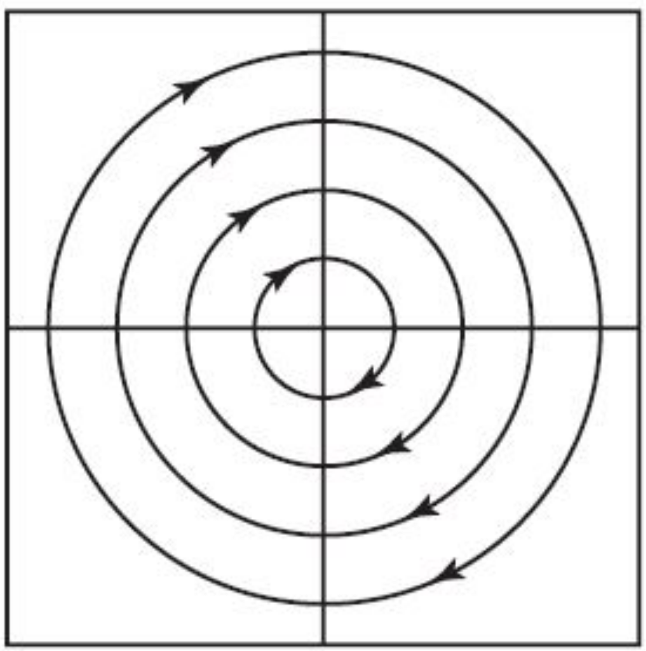
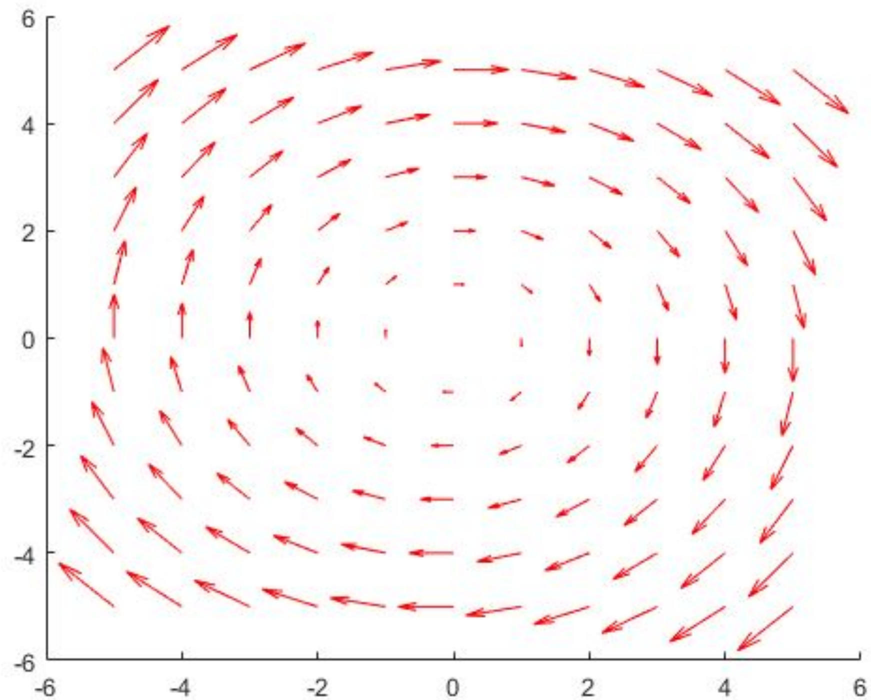


Figure 3.4 Phase portrait for a center.

Vector Fields (at a given time)



MATLAB Plot for Figure 3.4

$$\nabla \cdot \vec{F} = P_x + Q_y = 0$$

$$\nabla \times \vec{F} = Q_x - P_y = -2 < 0$$

Solution for a Center with $\lambda = \pm i\beta$

$$\boxed{\begin{aligned} x' &= y = P \\ y' &= -x = Q \end{aligned}}$$

$$x = c_1\cos(t) + c_2\sin(t)$$
$$y = -c_1\sin(t) + c_2\cos(t)$$

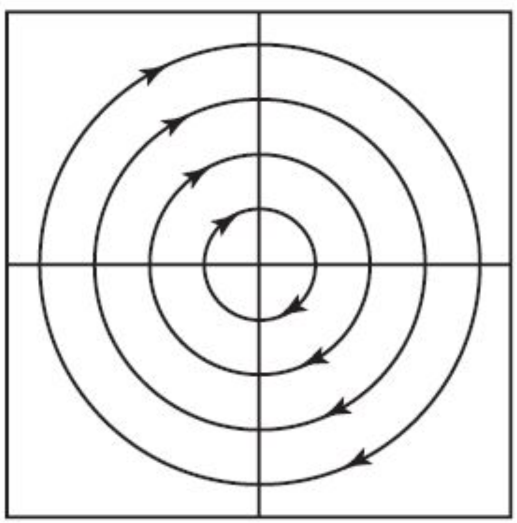
$$\nabla \cdot \vec{F} = P_x + Q_y = 0$$
$$\nabla \times \vec{F} = Q_x - P_y = -2$$

$$x^2 = (c_1\cos(t) + c_2\sin(t))^2 = c_1^2\cos^2(t) + 2c_1c_2\cos(t)\sin(t) + c_2^2\sin^2(t)$$

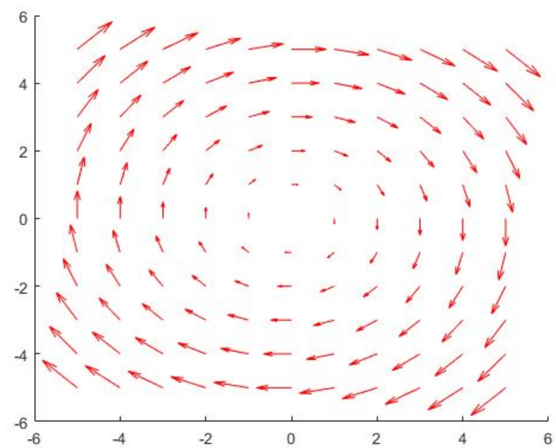
$$y^2 = (-c_1\cos(t) + c_2\sin(t))^2 = c_1^2\sin^2(t) - 2c_1c_2\cos(t)\sin(t) + c_2^2\cos^2(t)$$

$$x^2 + y^2 = c_1^2(\cos^2(t) + \sin^2(t)) + c_2^2(\cos^2(t) + \sin^2(t)) = c_1^2 + c_2^2$$

concentric circles



$$\vec{F} = (P, Q) = (y, -x)$$



A Planar System on a Complex Plane

$$\frac{d}{dt} \qquad\qquad \frac{d}{d\tau}$$

$$\boxed{\begin{aligned} x' &= y \\ y' &= -x, \end{aligned}} \qquad t = \beta\tau \qquad \longleftarrow \qquad \begin{aligned} x' &= \beta y \\ y' &= -\beta x, \end{aligned}$$

$$u = x + iy$$

$$\begin{aligned} u' &= x' + iy' \\ &= y - ix = -i(x + iy) \\ &= -iu \end{aligned}$$

$$\boxed{u' = -iu} \qquad c = c_1 + ic_2$$

$$u = ce^{-it} = c\cos(\mathrm{t}) - ic\sin(\mathrm{t})$$

$$x = \mathrm{c}_1\cos(\mathrm{t}) + \mathrm{c}_2\sin(t)$$

$$y = -\mathrm{c}_1\sin(\mathrm{t}) + \mathrm{c}_2\cos(t)$$

$$w = x - iy$$

$$\begin{aligned} w' &= x' - iy' \\ &= y + ix = i(x - iy) \\ &= iw \end{aligned}$$

$$\boxed{w' = iw} \qquad \mathrm{d} = c_1 - ic_2$$

$$\mathrm{w} = d\,e^{it} = d\cos(\mathrm{t}) + id\ \sin(\mathrm{t})$$

$$x = \mathrm{c}_1\cos(\mathrm{t}) + \mathrm{c}_2\sin(t)$$

$$y = -\mathrm{c}_1\sin(\mathrm{t}) + \mathrm{c}_2\cos(t)$$

Complex Eigenvalues: $\lambda_{1,2} = \alpha \pm i\,\beta$

$$x' = ax + by \quad (= P(x,y)) \qquad (1)$$

$$y' = cx + dy \quad (= Q(x,y)\,) \qquad (2)$$

$$X' = AX \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

(I) pure imaginary eigenvalues

$$A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$$

$$x' = \cancel{ax} + \beta y \quad (= P(x,y))$$

$$y' = \beta x + \cancel{dy} \quad (= Q(x,y))$$

$\lambda_{1,2} = \pm i\,\beta$: center

(II) complex eigenvalues with $Re \neq 0$

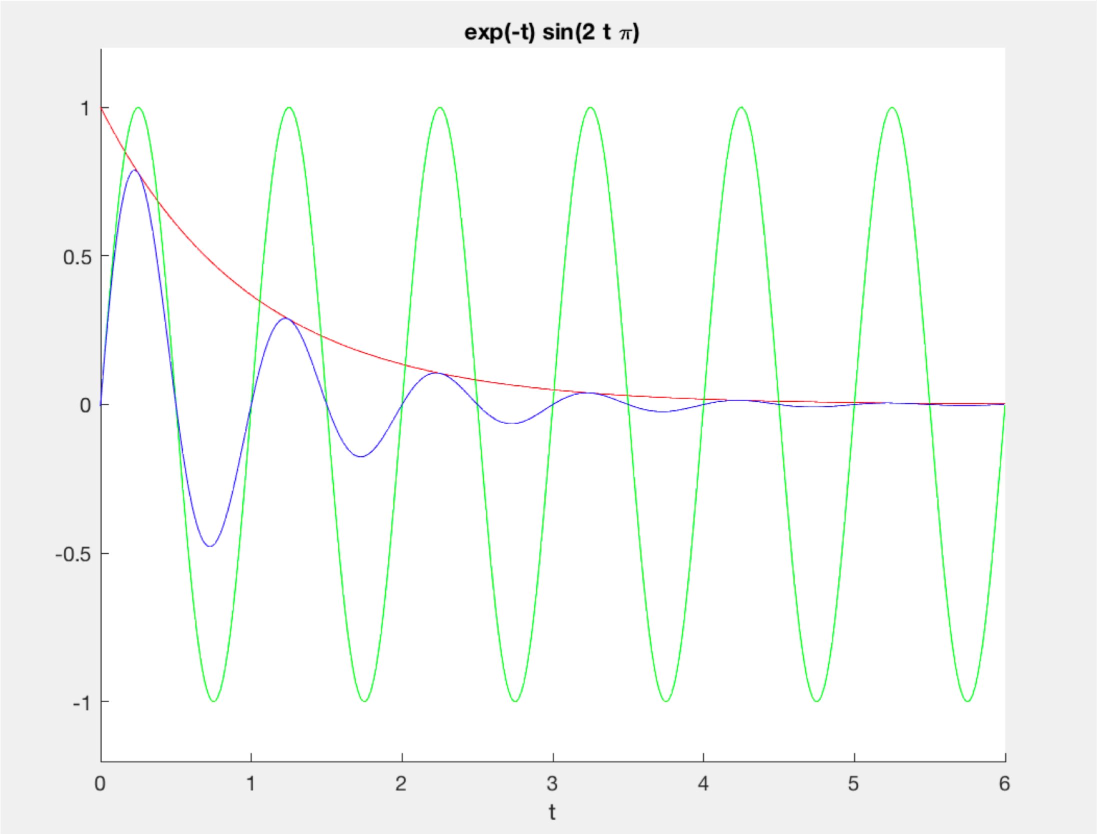
$$\boxed{A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}}$$

$$\boxed{\begin{aligned} x' &= \alpha x + \beta y \quad (= P(x,y)) \\ y' &= \beta x + \alpha y \quad (= Q(x,y)) \end{aligned}}$$

$\lambda_{1,2} = \alpha \pm i\,\beta$:
spiral source or sink

A Spiral Sink with $\lambda = \alpha \pm i\beta$: Oscillatory Decay

- red: $e^{\alpha t}, \alpha < 0$
- green: $\sin(\beta t)$
- blue: $e^{\alpha t}\sin(\beta t)$



Time

```
syms t a b
a=-1
b=2*pi
fexp=exp(a*t)
fosc=sin(b*t)
hold on
ezplot (fexp, [0, 6, -1.2, 1.2])
ezplot (fosc, [0, 6, -1.2, 1.2])
ezplot (fexp*fosc, [0, 6, -1.2, 1.2])
```

A Spiral Sink with $\lambda = \alpha \pm i\beta$

$$X(t) = e^{\alpha t}(c_1 X_{re}(t) + c_2 X_{im})$$

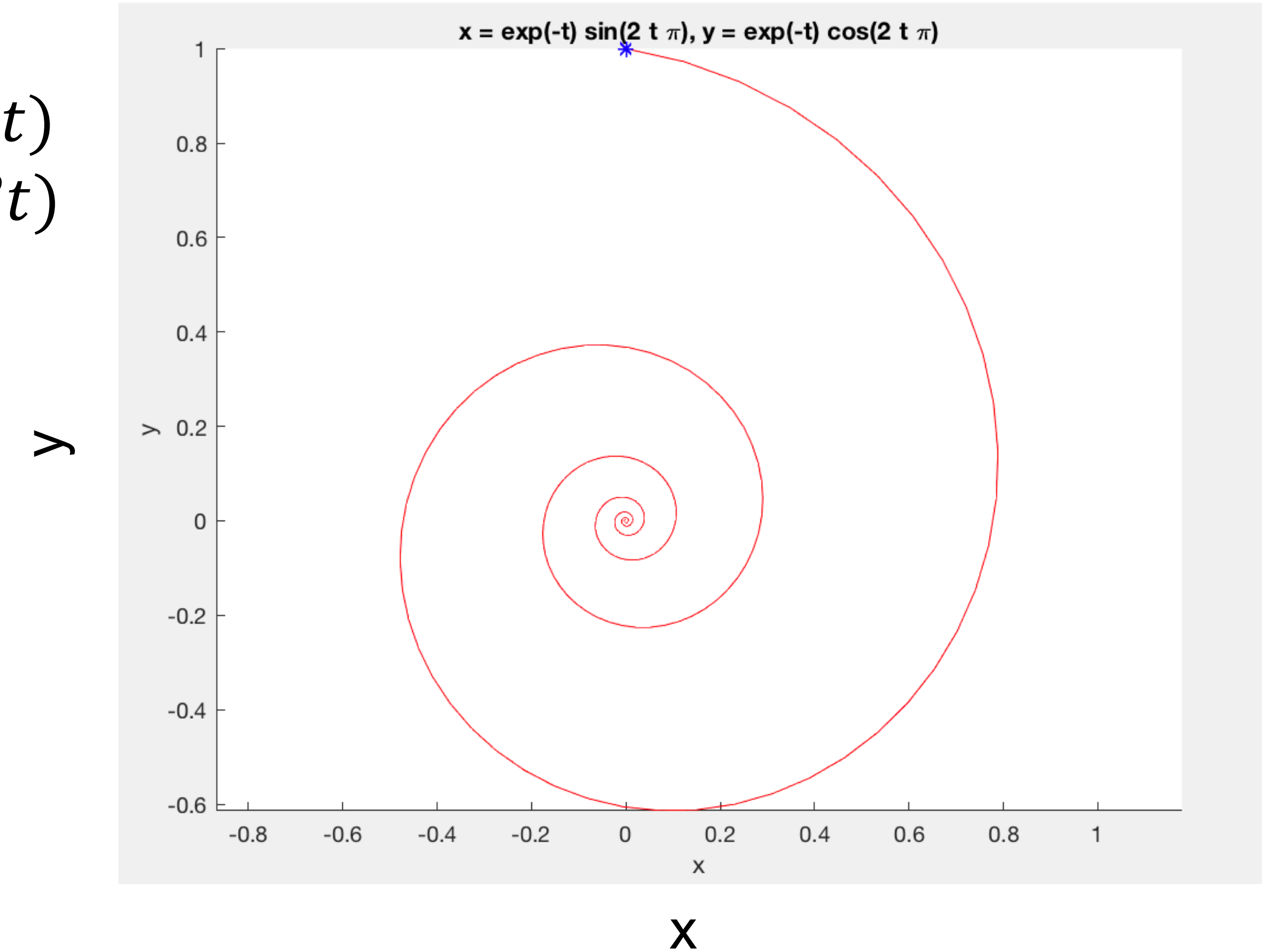
$$X_{im}(t) = \begin{pmatrix} \sin(\beta t) \\ cos(\beta t) \end{pmatrix}$$

$$c_1 = 0\,;\, c_2 = 1$$

$$x(t) = e^{\alpha t}\sin(\beta t)$$
$$y(t) = e^{\alpha t}\cos(\beta t)$$

```
clear
syms t a b x y
a=-1
b=2*pi
x=exp(a*t)*sin(b*t)
y=exp(a*t)*cos(b*t)

x0=0
y0=1
hold on
ez1=ezplot (x,y, [0, 6])
plot(x0, y0, 'b*')
set(ez1,'color',[1 0 0])
```



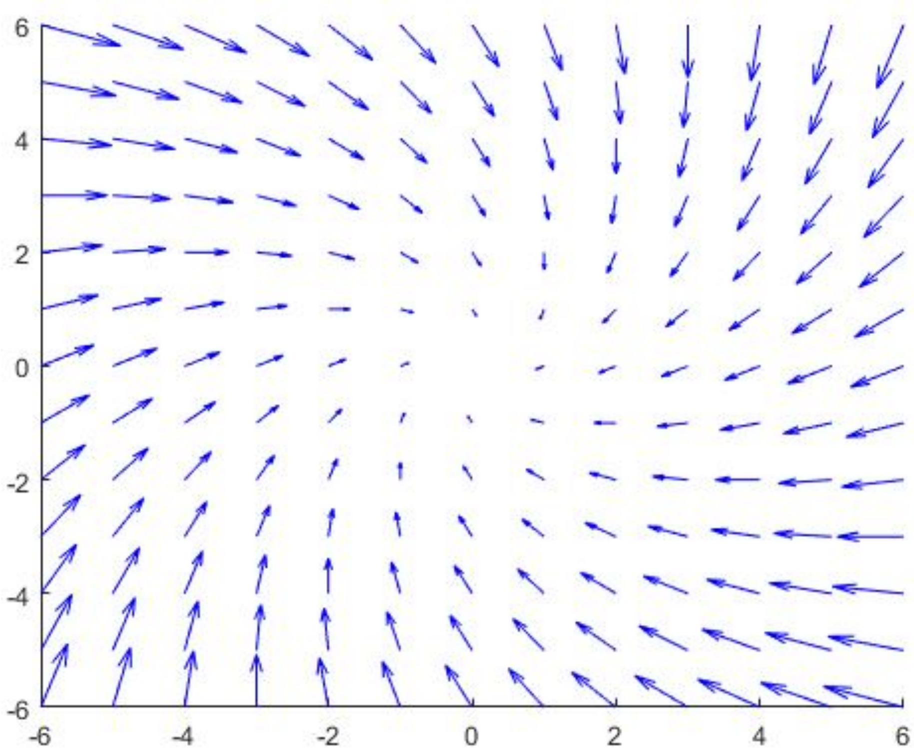
A Spiral Sink with $\lambda = \alpha \pm i\beta$ and $\alpha < 0$

X' = -2x + y   Y' = -x – 2y

$\lambda_{1,2} = \alpha \pm i\,\beta$ $\alpha=-2<0$

Vector Fields (at a given time)

Trajectories (time varying)



MATLAB Plot for Figure 3.5a

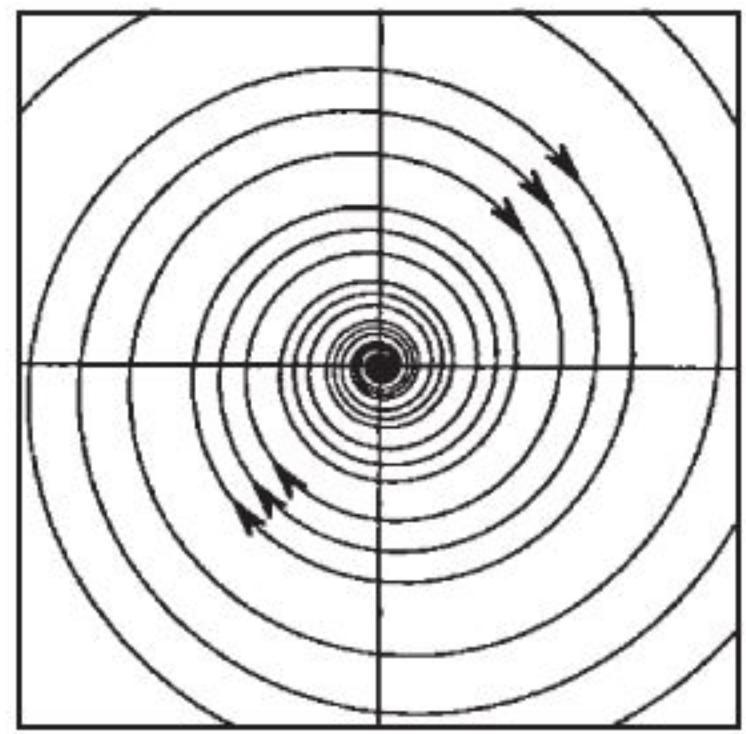


Figure 3.5a Phase portrait for spiral sink

Note: Changing Coordinates for Complex Eigenvalues

Consider $X' = AX$ and A has complex eigenvalues

Goal: Find T so that $Y = TX$ and $Y' = BY$

$$B = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

Let U_j and λ_j represent the eigenvectors and eigenvalues, respectively. Thus, we have $AU_j = \lambda_j U_j$.

without loss of generality, we have $\lambda_1 = \alpha + i\beta$ and $U_1 = V_1 + iV_2$.

V_1 and V_2 are real, $V_1 = \mathrm{Re}(U_1)$ and $V_2 = \mathrm{Im}(U_1)$.

Below, we show that

- V_1 and V_2 are linearly independent
- When $T = (\mathrm{Re}(U_1), \mathrm{Im}(U_1))$, we obtain $B = T^{-1}\mathrm{A}\mathrm{T} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$.

Sec 3.3 Repeated Eigenvalues

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$x' = \lambda_1 x$$
$$y' = \lambda_1 y$$

Uncoupled System

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$x' = \lambda_1 x + y$$
$$y' = \lambda_1 y$$

Coupled System

x: responder

y: driver

A Summary: Repeated Eigenvalues

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$x' = \lambda_1 x + y$$
$$y' = \lambda_1 y$$

Coupled System

y: driver

x: responder

$$\boxed{X = \alpha \overset{X_1}{X_1} + \beta X_2 = \alpha e^{\lambda t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda t}\begin{pmatrix} t \\ 1 \end{pmatrix}}$$

X_1 X_2

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$x' = \lambda_1 x$$
$$y' = \lambda_1 y$$

Uncoupled System

$$x = c_1 e^{\lambda_1 t}$$

$$y = c_2 e^{\lambda_1 t}$$

How to find V_2?

(1) Apply the concept of generalized eigenvalue problem

(2) Solve the ODE for the driver that is uncoupled with the ODE for the responder; then solve the ODE for the responder

(3) Transform the above system to a 2nd order ODE

(1) Solutions using the Concept of Generalized Eigenvector

Solve for V_1 $\quad (A-\lambda I)V_1 = 0 \quad A = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \quad \boxed{\lambda = \lambda_1} \quad V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Solve for V_2 $\quad \boxed{(A-\lambda I)V_2 = V_1}$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \boxed{y = 1} \quad V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Obtain X_1 $\quad X_1 = e^{\lambda t} V_1 = e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Construct X_2 $\quad \boxed{X_2 = t e^{\lambda t} V_1 + e^{\lambda t} V_2} = e^{\lambda t} \begin{pmatrix} t \\ 0 \end{pmatrix} + e^{\lambda t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{\lambda t} \begin{pmatrix} t \\ 1 \end{pmatrix}$

Obtain X $\quad \boxed{X = \alpha X_1 + \beta X_2 = \alpha e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda t} \begin{pmatrix} t \\ 1 \end{pmatrix}}$

(2) Solve the ODEs for the Driver and Responders

Consider
$$x' = \lambda_1 x + y$$
$$y' = \lambda_1 y$$
uncoupled

Start with the uncoupled equation
$$y' = \lambda_1 y$$

$$\boxed{y = \beta e^{\lambda_1 t}}$$

The 1st eq becomes:
$$x' - \lambda_1 x = \beta e^{\lambda_1 t}$$
$$x' + P(t)x = Q(t)$$
$$I = e^{-\lambda_1 t}$$

- Forced problem
- Non-autonomous system
- The system's "frequency" is the same as the "frequency" of the forcing.

$$\boxed{x = \frac{1}{I}\left[\int IQdt + C\right] = \frac{1}{I}(\beta \mathrm{t} + \mathrm{C}) = Ce^{\lambda_1 t} + \beta \mathrm{t} e^{\lambda_1 t}}$$

(3) Convert and Solve a 2nd Order ODE

Consider

$$x' = \lambda_1 x + y$$
$$y' = \lambda_1 y$$

uncoupled

Transform into a 2nd order ODE

$$x'' = \lambda_1 x' + y'$$

$$x'' = \lambda_1 x' + \lambda_1 y \qquad x'' = \lambda_1 x' + \lambda_1 (x' - \lambda_1 x)$$

$$\boxed{x'' - 2\lambda_1 x' + \lambda_1^2 x = 0}$$

Assume $\boxed{x = ke^{\lambda t}}$

Obtain $(\lambda - \lambda_1)^2 = 0$ $\boxed{\lambda = \lambda_1}$

$$\boxed{x_1 = e^{\lambda_1 t}} \qquad \boxed{x_2 = te^{\lambda_1 t} = \lim_{\lambda \to \lambda_1} \frac{e^{\lambda t} - e^{\lambda_1 t}}{\lambda - \lambda_1}}$$

Changing Coordinates

Despite differences in the associated phase portraits, we really have dealt with only three type of matrices in these past four sections:

diagonalization

$$\boxed{\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}}, \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

(I) (II) (III)

- Any 2×2 matrix that is in one of these three forms is said to be in canonical form.
- Given any linear system $X' = AX$, we can always "change coordinates" so that the new system's coefficient matrix is in canonical form

A Summary for the Three Cases

Goal: Solve the following 2D system

$$\boxed{X' = AX}$$

Compute the eigenvalues and eigenvectors of A,
U_1 and U_2

$$AU_j = \lambda_j U_j, \qquad j = 1,2$$

Construct $T = (V_1, V_2)$, $B = T^{-1}AT$ and $X = TY$ using the following

(I) real eigenvalues

$$(V_1, V_2) = (U_1, U_2)$$

$$B = T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

(II) complex eigenvalues

$$(V_1, V_2) = (Re(U_1), Im(U_1))$$

$$B = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

(III) repeated eigenvalues

$$V_1 = U_1$$

$$(A - \lambda I)V_2 = V_1$$

$$B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$\boxed{Y' = BY}$$

(I) Real Eigenvalues

Goal: Solve the following 2D system

$$\boxed{X' = AX}$$

Compute the eigenvalues and eigenvectors of A, V_1 and V_2

$$AV_j = \lambda_j V_j, \qquad j = 1,2$$

Construct T using V_1 and V_2 $\Rightarrow$ $\boxed{Y' = DY}$ $\quad D = T^{-1}AT$

$$T = (V_1, \mathrm{V}_2)$$

$$Y' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} Y$$

$$Y = c_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}$$

$$X = TY$$

$$= (V_1, \mathrm{V}_2) \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}$$

$$= e^{\lambda_1 t}\, V_1 + c_2 e^{\lambda_2 t} V_2$$

(II) Complex Eigenvalues

Let U_j and λ_j represent the eigenvectors and eigenvalues, respectively. Thus, we have $AU_j = \lambda_j U_j$.

- Basis "vectors" for changing coordinates (within the phase space). Without loss of generality, we have $\lambda_1 = \alpha + i\beta$ and $U_1 = V_1 + iV_2$.

  V_1 and V_2 are real, $V_1 = \mathrm{Re}(U_1)$ and $V_2 = \mathrm{Im}(U_1)$.

- Basis functions for solutions

  $X_{re} = Re\left(e^{i\lambda_1 t} U_1\right) \Rightarrow trig\ functions,$ functions of time

  $X_{im} = Im\left(e^{i\lambda_1 t} U_1\right) \Rightarrow trig\ functions,$ functions of time

(III) Repeated Eigenvalues

Construct T as follows:

- $AV = \lambda V$
- $(A - \lambda I)V_2 = V$.
- $T = [V, \mathrm{V}_2]$, which leads to $T^{-1}\mathrm{AT} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

$$Y' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} Y \qquad\qquad X' = AX$$

Solve the above for Y and compute $\mathrm{X} = \mathrm{T}Y$

For example,

$$Y' = \begin{pmatrix} 3/2 & 1 \\ 0 & 3/2 \end{pmatrix} Y \qquad\qquad X' = \begin{pmatrix} 2 & 1 \\ -1/4 & 1 \end{pmatrix} X$$

Upper triangle

4.1: Classification

For a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad A - \lambda I = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

we know that the eigenvalues are the roots of the characteristic equation, which may be written

$$|A - \lambda I| = 0 \qquad \lambda^2 - (a+d)\lambda + (ad - bc) = 0.$$

Thus the eigenvalues satisfy

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0$$

$$T = trace\ (A) = tr\ (A) = a + d \qquad D = det\ (A) = ad - bc$$

$$\boxed{\lambda^2 - T\lambda + \mathrm{D} = 0}$$

A Summary for Classification

- $Re(\lambda_1) > 0 \;\&\; Re(\lambda_2) > 0:$ Source
- $Re(\lambda_1) < 0 \;\&\; Re(\lambda_2) < 0:$ Sink
- $\lambda_1\lambda_2 < 0:$ Saddle

- $Re(\lambda) = 0 \;\&\; Im(\lambda) \neq 0:$ Center
- $Re(\lambda) \neq 0 \;\&\; Im(\lambda) \neq 0$ Spiral sink or source

The Trace-Determinant Plane

$T^2 = 4D$

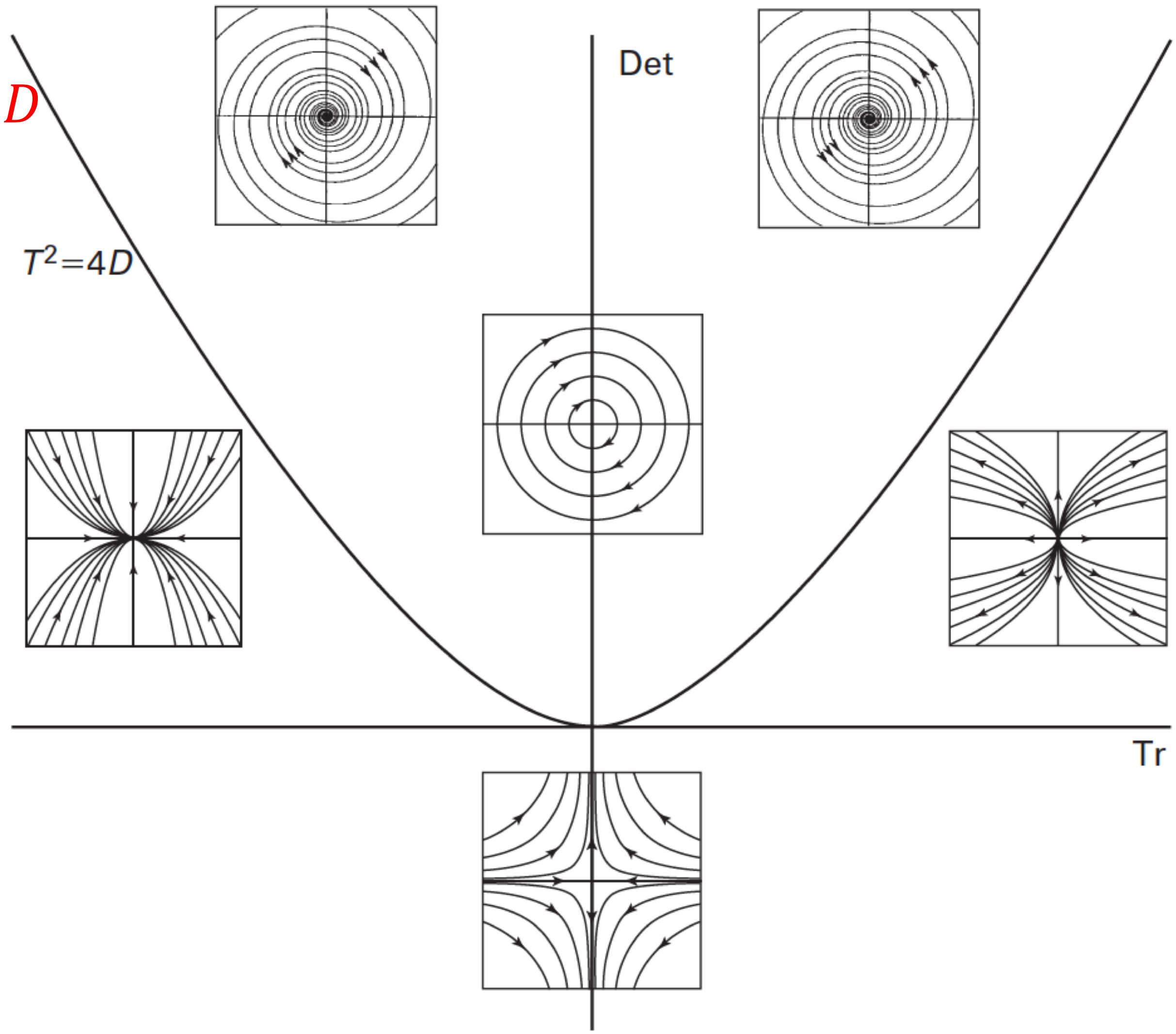


Figure 4.1 The trace-determinant plane. Any resemblance to any of the authors' faces is purely coincidental.

The Trace-Determinant Plane

$T^2 = 4D$

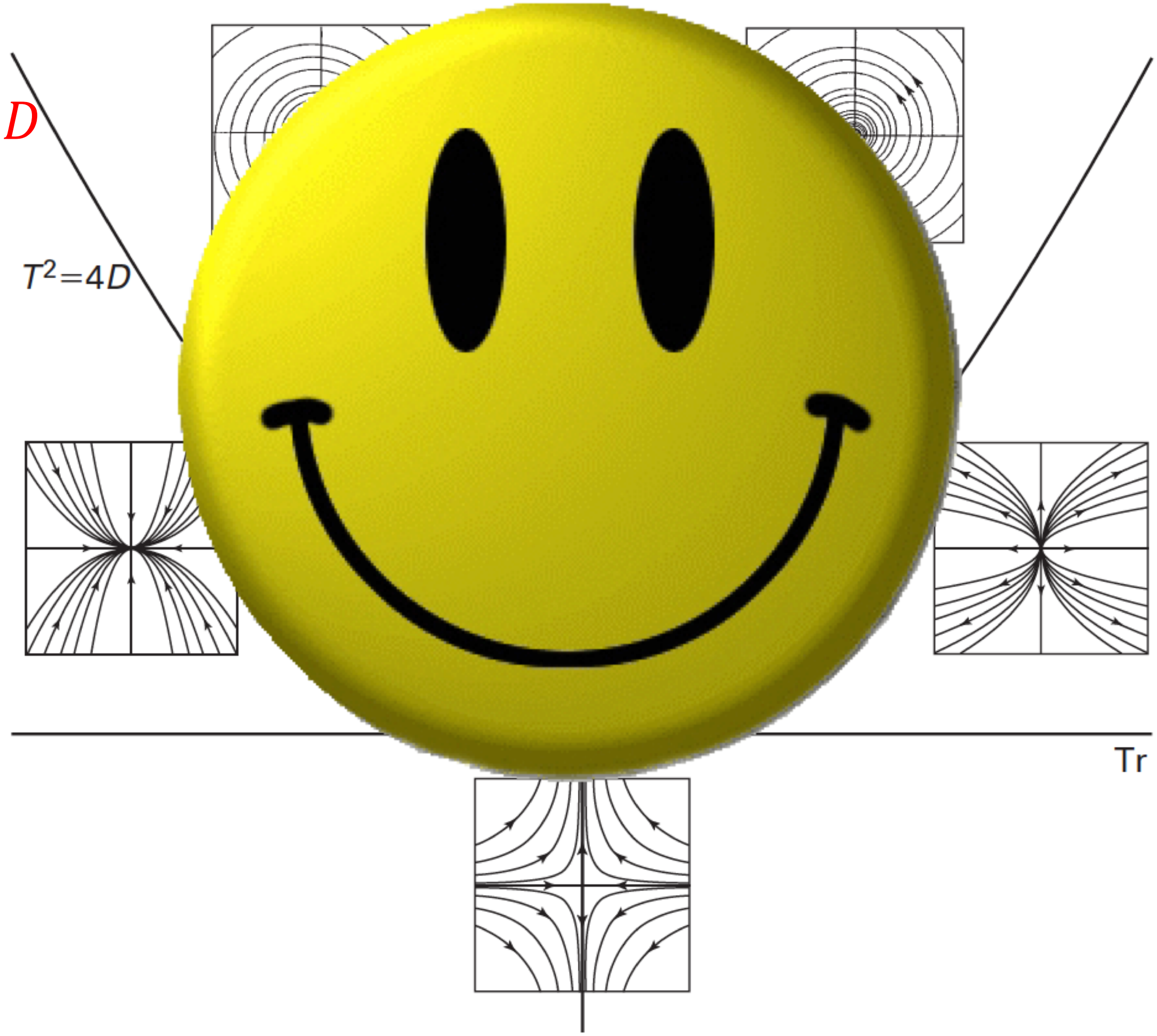


Figure 4.1 The trace-determinant plane. Any resemblance to any of the authors' faces is purely coincidental.

Classification: Smiling Curve $T^2 - 4D = 0$

$$\lambda^2 - T\lambda + \mathrm{D} = 0$$

$$\lambda_{\pm} = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

$$f(T, D) = T^2 - 4D$$

Define a smiling curve:

$$T^2 = 4D$$

Sample points

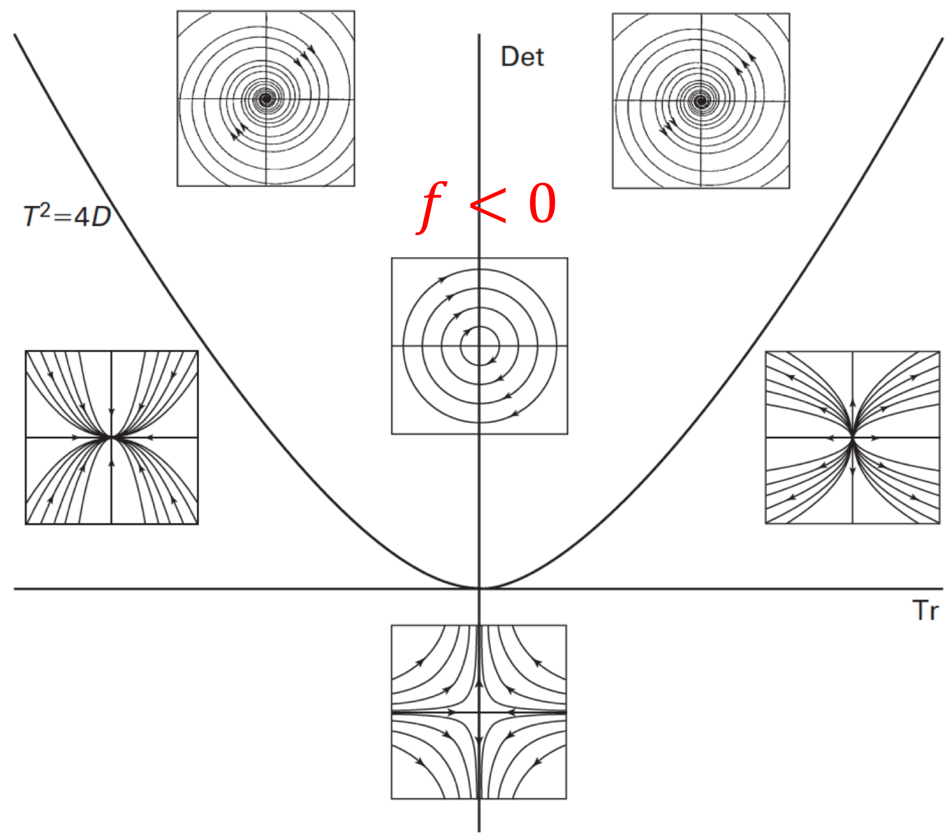
$$f(0,1) = -4 < 0$$

complex eigenvalues

$$f(0,-1) = 4 > 0$$

real eigenvalues

$T^2 = 4D$



$f > 0$

Classification: Saddle, Source and Sink

$$\lambda^2 - T\lambda + \mathrm{D} = 0$$

$$\lambda_\pm = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

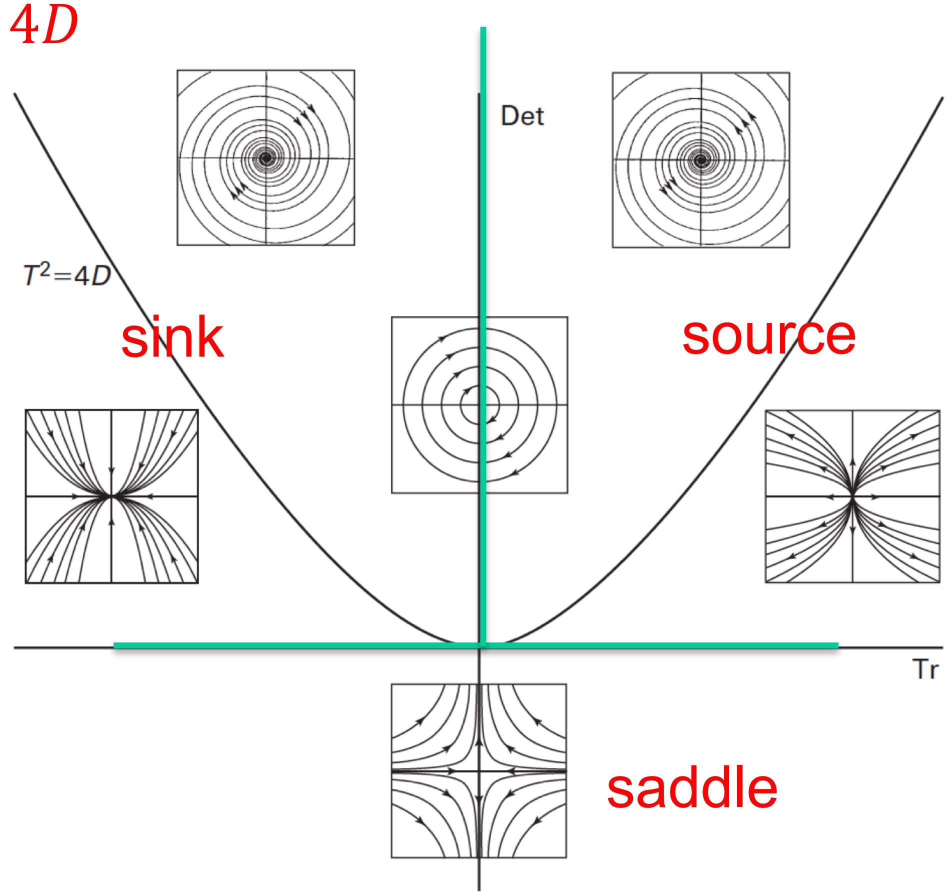
$$\boxed{\lambda_+ + \lambda_- = T = tr}$$

$$(\lambda - \lambda_+)(\lambda - \lambda_-) = 0$$

$$\lambda^2 - (\lambda_+ + \lambda_-)\lambda + \lambda_+\lambda_- = 0$$

$$\boxed{\lambda_+\lambda_- = D = determinat}$$

$T^2 = 4D$



- $D < 0$, λ_+ and λ_- have different signs ➔ saddle
- $D > 0$, λ_+ and λ_- have the same sign ➔ source with $T > 0$

  ➔ sink with $T < 0$

Stability Properties of Linear Systems

TABLE 9.1.1 Stability Properties of Linear Systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with $\det(\mathbf{A} - r\mathbf{I}) = 0$ and $\det \mathbf{A} \neq 0$

	Eigenvalues		Type of Critical Point	Stability
source	$r_1 > r_2 > 0$		Node	Unstable
sink	$r_1 < r_2 < 0$		Node	Asymptotically stable
saddle	$r_2 < 0 < r_1$		Saddle point	Unstable
source	$r_1 = r_2 > 0$	repeated	Proper or improper node	Unstable
sink	$r_1 = r_2 < 0$		Proper or improper node	Asymptotically stable
spiral	$r_1, r_2 = \lambda \pm i\mu$		Spiral point	
	$\lambda > 0$			Unstable
	$\lambda < 0$			Asymptotically stable
center	$r_1 = i\mu, r_2 = -i\mu$		Center	Stable

Boyce and Diprima, 2012: Elementary Differential Equations, Tenth Edition. Wiley, 2012.

Linear Stability Analysis for Nonlinear Systems

Consider the following system of first-order ODEs

$$\boxed{x' = F(x,y) \qquad\qquad y' = G(x,y)}$$

Find critical points

$$F(x_c, y_c) = 0 \qquad\qquad G(x_c, y_c) = 0$$

Compute the Jacobian matrix and evaluate it at a critical point

$$J(F,G) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{x_c, y_c}$$

Solve an eigenvalue problem:

$$JV = \lambda V \qquad V = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\boxed{|J - \lambda I| = 0}$$

Example: A Linearized Lorenz Model

$$\frac{dX}{dt} = -\sigma X + \sigma Y,$$
$$\frac{dY}{dt} = rX - Y.$$

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \qquad A = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix}$$

Complete the following to obtain the Jacobian matrix:

$$F(X_c, Y_c) = 0 \implies X_c = Y_c = 0$$

$$J(F,G) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{x_c, y_c} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix}$$

MT2 Part B: 9:00-9:50 AM on Oct. 2, 2020

- The exam must be taken completely alone. Showing it or discussing it with anybody is forbidden.
- There are 5 (or 6, TBD) regular problems. Each problem is worth 25 points. Please complete 4 of them.
- You have 50 minutes (between 9:00 and 9:50 am)
- Submit your work to GradeScope by 10:00 am on Oct. 2, 2020

  document any issues (e.g., using screenshots) and report via emails or chat as soon as possible

- Please keep your Zoom connection and enable your Camera.
- Use "Chat" (under Zoom) or "emails" to ask questions or report problems.