Homework 3
Ordinary Differential Equations
Math 537
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Problem 1: Consider the following system:

$$X' = AX, \tag{1.1}$$

where

$$A = \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix} \text{ and } X = \begin{pmatrix} x \\ y \end{pmatrix}$$

(a) Solve for eigenvalue(s) and eigenvector(s).

Notice we can get the characteristic equation from $A - \lambda I$:

$$\begin{aligned} (\lambda + 5)(\lambda + 2) - 4 &= 0 \\ \lambda^2 + 7\lambda + 10 - 4 &= 0 \\ \lambda + 7\lambda + 6 &= 0 \\ (\lambda + 6)(\lambda + 1) &= 0 \\ \lambda &= -6, -1 \end{aligned}$$

Notice the eigenvectors found from $A - \lambda I$ with $\boldsymbol{\lambda_1 = -6}$ and $\boldsymbol{\lambda_2 = -1}$:

$$\begin{pmatrix} -5-\lambda_1 & 2 \\ 2 & -2-\lambda_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad v_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mathbf{2} \\ \mathbf{-1} \end{pmatrix}$$

$$\begin{pmatrix} -5-\lambda_2 & 2 \\ 2 & -2-\lambda_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \end{pmatrix}$$

(b) Construct T using the results from problem (1a) and calculate $T^{-1}AT$

Notice that the eigenvalues were real and different. So we can construct T from the eigenvectors, such that:

$$\boldsymbol{T} = \begin{pmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{-1} & \mathbf{2} \end{pmatrix}$$

Notice $T^{-1}AT$:

$$\begin{aligned} T^{-1}AT &= \frac{1}{5} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{-6} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix} \end{aligned}$$

(c) Let $X = TY$. Show

$$Y' = (T^{-1}AT)Y, \tag{1.2}$$

Here Y is a column vector and its transpose is defined as $Y^T = (u, w)$.

Notice the following:

$$\begin{pmatrix} u' \\ w' \end{pmatrix} = \begin{pmatrix} -6 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} -6u \\ -w \end{pmatrix}$$

Because we have that $\boldsymbol{u' = \lambda_1 u}$ and $\boldsymbol{w' = \lambda_2 w}$, we have shown the above statement to be true.

(d) Solve Eq. (1.2) for Y.

We can see the eigenvalues because $T^{-1}AT$ is an upper triangular matrix. So we get that $\lambda_1 = -6$ and $\lambda_2 = -1$. We can also easily see the eigenvectors being $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

So we get

$$\boldsymbol{Y = Ae^{-6t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + Be^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

(e) Find the solution X to Eq. (1.1).

$$\begin{aligned} X = TY &= \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} Ae^{-6t} & 0 \\ 0 & Be^{-t} \end{pmatrix} = \begin{pmatrix} 2Ae^{-6t} & Be^{-t} \\ -Ae^{-6t} & 2Be^{-t} \end{pmatrix} \\ &= \boldsymbol{Ae^{-6t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + Be^{-t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}} \end{aligned}$$

Problem 2: Consider the following set of differential equations:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -\omega^2 x - by$$

here both b and ω are real.

(a) Find the conditions under which the system is hyperbolic.

Notice we can rewrite the system as the following:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Notice we can get the characteristic equation from $A - \lambda I$:

$$\lambda(\lambda + b) + \omega^2 = 0$$
$$\lambda^2 + b\lambda + \omega^2 = 0$$

Notice the eigenvalues from the quadratic formula:

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4\omega^2}}{2}$$
$$\lambda_2 = \frac{-b - \sqrt{b^2 - 4\omega^2}}{2}$$

A system is hyperbolic if its matrix A does not have any eigenvalues with real parts 0. In this case, we get eigenvalues with real parts 0 if $\boldsymbol{b = 0}$ or $\boldsymbol{\omega = 0}$, where the 'or' is an inclusive 'or'. So as long as the system does not have these conditions, the system is hyperbolic

(b) Discuss whether the system has a saddle point.

A saddle point occurs when the eigenvalues are real and have opposite signs. To meet the real parameter, we have that $b^2 - 4\omega^2 \geq 0$. From that, we also know that $0 \leq \sqrt{b^2 - 4w^2} \leq b$. From this we have the following:

$$\frac{-b}{2} \leq \lambda_1 \leq 0, \qquad -b \leq \lambda_2 \leq \frac{-b}{2}$$

Because we see that λ_1 and λ_2 never have opposite signs, **the system does not have a saddle point**.

Problem 3: Consider the following two differential equations

$$x'' + ax' + bx = 0$$
$$x'' + cx' + dx = 0$$

Show that the two systems are topologically conjugate when a, b, c and d are positive.

Proof. Notice we can rewrite the systems as the following when we let $y = x'$ with a, b, c and d being positive:

$$X' = \begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} = Ax \tag{3.1}$$

$$X' = \begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -d & -c \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} = Bx \tag{3.2}$$

Notice we can get the characteristic equation of Eq (3.1) from $A - \lambda I$:

$$\lambda(\lambda + a) + b = 0$$
$$\lambda^2 + a\lambda + b = 0$$

Notice the eigenvalues from the quadratic formula:

$$\lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2}, \qquad \lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

Notice the three cases:

(1) $a^2 - 4b > 0$, We get that $0 < \sqrt{a^2 - 4b} < a$

$$\frac{-a}{2} < \lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} < 0$$
$$-a < \lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2} < \frac{-a}{2}$$

(2) $a^2 - 4b = 0$, We get that $\sqrt{a^2 - 4b} = 0$

$$\lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} = \frac{-a}{2}$$
$$\lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2} = \frac{-a}{2}$$

(3) $a^2 - 4b < 0$, We get that $\sqrt{a^2 - 4b} < 0$

$$\lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} = \frac{-a}{2} + i\sqrt{|a^2 - 4b|}$$
$$\lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2} = \frac{-a}{2} - i\sqrt{|a^2 - 4b|}$$

Notice that in all three cases, we get that both eigenvalues do not have real parts 0 and have all negative real parts. Without loss of generality, we can say the same for Eq (3.2). So we get that A and B are hyperbolic. **Finally by the Theorem in Lecture 15, the two systems are conjugate as they both have the same number of eigenvalues (2) with negative real parts.** □