Homework 3
Abstract Algebra
Math 320
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Section 1.2 Problem 34a: Prove that $(a,b)|(a+b,a-b)$

Solution Problem 34a: Let $d_1 = (a,b)$ and $d_2 = (a+b, a-b)$ $\quad a,b,d_1,d_2,q_1,q_2 \in \mathbb{Z}$

By Corollary 1.3, if $c|(a+b)$ and $c|(a-b)$, then $c|d_2$ $\quad c \in \mathbb{Z}$

$$a = d_1 q_1 \tag{1}$$
$$b = d_1 q_2 \tag{2}$$
$$a+b = d_1(q_1+q_2) \tag{3}$$
$$a-b = d_1(q_1-q_2) \tag{4}$$
$$d_1|(a+b) \text{ and } d_1|(a-b) \tag{5}$$
$$d_1|d_2 = (a,b)|(a+b,a-b) \tag{6}$$

By proving that $d_1|a$ and $d_1|b$, we can prove it divides its sum and difference, thus $\boldsymbol{(a,b)|(a+b,a-b)}$

Section 1.2 Problem 34b: Prove that if a is odd and b is even, then $(a, b) = (a+b, a-b)$

Solution Problem 34b: Let $a = 2q + 1$ and $b = 2k$, $d_1 = (a, b)$ and $d_2 = (a + b, a - b)$ $a, b, d_1, d_2, q, k, r, r_1, s, s_1, c_1, c_2 \in \mathbb{Z}$.

$$\begin{aligned} a + b &= 2(q + k) + 1 && (7) \\ &= 2c_1 + 1 && (8) \\ a - b &= 2(q - k) + 1 && (9) \\ &= 2c_2 + 1 && (10) \end{aligned}$$

Because $a + b$ and $a - b$ is odd, then d_2 is odd.

$$\begin{aligned} a + b &= d_2 r && (11) \\ a - b &= d_2 s && (12) \\ (a + b) + (a - b) &= 2a && (13) \\ &= d_2 r + d_2 s && (14) \\ &= d_2(r + s) && (15) \\ (a + b) - (a - b) &= 2b && (16) \\ &= d_2 r - d_2 s && (17) \\ &= d_2(r - s) && (18) \end{aligned}$$

Because $d_2|2a$ and $d_2|2b$, and because $(d_2, 2) = 1$, $d_2|a$ and $d_2|b$, so $d_2|d_1$.

$$\begin{aligned} a &= d_1 r_1 && (19) \\ b &= d_1 s_1 && (20) \\ a + b &= d_1(r_1 + s_1) && (21) \\ a - b &= d_1(r_1 - s_1) && (22) \end{aligned}$$

So $d_1|d_2$. Because $d_1|d_2$ and $d_2|d_1$, $\boldsymbol{d_1 = d_2}$

Section 1.3 Problem 7: If a, b, c are integers and p is a prime that divides both a and $a + bc$, prove that $p|b$ or $p|c$.

Solution Problem 7: Let $p|a$ and $p|a + bc$ $\quad a, b, c, p, q_1, q_2 \in \mathbb{Z}$

$$\begin{aligned} a &= pq_1 && (23) \\ a + bc &= pq_1 + bc && (24) \\ &= pq_2 && (25) \end{aligned}$$

To have $a + bc = pq_2$, bc needs to be divisible by p. And by prime factorization, **only $\boldsymbol{b}$ or $\boldsymbol{c}$ needs to be divisible by $\boldsymbol{p}$, for their product to be divisible by $\boldsymbol{p}$.**

Section 1.3 Problem 16: Prove that $(a, b) = 1$ if and only if there is no prime p such that $p|a$ and $p|b$.

Solution Problem 16: (=>) Let $(a, b) = 1$ $\quad$ a,$b \in \mathbb{Z}$

$$\text{By Prime Factorization: } a = 1 * \prod_{i=0}^{m} p_i \qquad m = \min\{\text{Amount of Primes for a}\} \tag{26}$$

$$\text{By Prime Factorization: } b = 1 * \prod_{j=0}^{n} p_j \qquad n = \min\{\text{Amount of Primes for b}\} \tag{27}$$

$$\tag{28}$$

If p_i for any $i \in \mathbb{Z}$ equals p_j for any $j \in \mathbb{Z}$, then $p_i = p_j$ would be the GCF, thus contradicting $(a, b) = 1$. **So there are no primes $\boldsymbol{p}$ that would divide both $\boldsymbol{a}$ and $\boldsymbol{b}$.**

Solution Problem 16: (<=) Let there be no prime p such that $p|a$ and $p|b$.

Because of prime factorization, all integers can be written as product of primes. If a and b, do not share any divisor p that is prime, then they have no common divisors greater than 1, **thus (a,b) = 1**.

Section 1.3 Problem 27: If $p > 3$ is prime, prove that $p^2 + 2$ is composite. [Hint: Consider the possible remainders when p is divided by 3.]

Solution Problem 27: Let p be prime, and $p > 3$ $\quad$ p,q,$k \in \mathbb{Z}$ Case 1: (p = 3k+1)

$$\begin{aligned} p^2 + 2 &= (3k + 1)^2 + 2 && (29)\\ &= 9k^2 + 6k + 1 + 2 && (30)\\ &= 3q && (31) \end{aligned}$$

Case 2: (p = 3k+ 2)

$$\begin{aligned} p^2 + 2 &= (3k + 2)^2 + 2 && (32)\\ &= 9k^2 + 12k + 6 && (33)\\ &= 3q && (34) \end{aligned}$$

Because $p \neq 3, p^2{+}2 \neq 3$. Because $p^2{+}2 \neq 3$, and it is divisible by 3, then $\boldsymbol{p^2 + 2}$ is composite. (Note: $p \neq 3k$, because then p would not be prime)