# HW 2

① Suppose $c < x$.

Then $0 < x - c$

So $0 < \frac{x-c}{2}$

So $c < c + \frac{x-c}{2}$ $(\forall)$

Also since $c < x$,

$$c + x < x + x$$

So $\frac{c+x}{2} < x$

Thus $c + \frac{x-c}{2} = \frac{c+x}{2} < x$. $(\forall\forall)$

By $(\forall)$ and $(\forall\forall)$,

$$c < c + \frac{x-c}{2} < x.$$

② Base: Let $n = 1$ and $x \in \mathbb{R}$.

Clearly $|x| = |x|$ and we are done.

Inductive Step: Suppose $N \geq 1$.

Suppose $\forall x_1, \ldots, x_N \in \mathbb{R}$, we have

$$\left| \sum_{i=1}^{N} x_i \right| \leq \sum_{i=1}^{N} |x_i|.$$

Let $x_1, x_2, \ldots, x_{N+1} \in \mathbb{R}$.

Then $\left| \sum_{i=1}^{N+1} x_i \right| = \left| \sum_{i=1}^{N} x_i + x_{N+1} \right|$

$\leq \left| \sum_{i=1}^{N} x_i \right| + |x_{N+1}|$ by $\Delta$-inequality.

$\leq \sum_{i=1}^{N} |x_i| + |x_{N+1}|$ by Inductive Hyp.

Thus $\left| \sum_{i=1}^{N+1} x_i \right| \leq \sum_{i=1}^{N+1} |x_i|$. $\blacksquare$

(3) __Base__: $(1+a)^1 = 1 + a = 1 + (2^1 - 1)a$.

__Inductive Step__: Let $N \geq 1$.

Suppose $(1+a)^N \leq 1 + (2^N - 1)a$.

In the case $a = 0$,

$$(1+a)^{N+1} = 1 = 1 + (2^N - 1)a.$$

Suppose $0 < a \leq 1$. Then $a^2 \leq a$.

So $(1+a)^{N+1} = (1+a)^N (1+a) \leq (1 + (2^N - 1)a)(1+a)$

$= 1 + a + (2^N - 1)a + (2^N - 1)a^2$

$\leq 1 + a + (2^N - 1)a + (2^N - 1)a$.

$= 1 + (2^N + 2^N - 2 + 1)a$.

$= 1 + (2^{N+1} - 1)a$. $\blacksquare$

④ (a) $\forall n \in \mathbb{Z}^+$, $\forall b \in \mathbb{R}^+$,

$$(1+b)^n \geq 1 + nb + \frac{n(n-1)}{2} b^2$$

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proof: Let $n \in \mathbb{Z}^+$, $b \in \mathbb{R}^+$.

Case $n=1$: $(1+b)^1 = 1 + 1 \cdot b + \frac{1(1-1)}{2} b^2 = 1+b$.

Case $n=2$: $(1+b)^2 = 1 + 2b + b^2 = 1 + 2 \cdot b + \frac{2(2-1)}{2} b^2$.

Case $n \geq 3$:

$$(1+b)^n = \sum_{j=0}^{n} \binom{n}{j} 1^{n-j} b^j$$

$$= \binom{n}{0} b^0 + \binom{n}{1} b^1 + \binom{n}{2} b^2 + \sum_{j=3}^{n} \binom{n}{j} b^j$$

$$= 1 + nb + \frac{n(n-1)}{2} b^2 + \sum_{j=3}^{n} \binom{n}{j} b^j$$

$$\geq 1 + nb + \frac{n(n-1)}{2} b^2, \text{ provided}$$

$\sum_{j=3}^{n} \binom{n}{j} b^j \geq 0$. (To see this, let $3 \leq j \leq n$. Then $\binom{n}{j} b^j \geq 0$. Thus $\sum_{j=3}^{n} \binom{n}{j} b^j \geq 0$.)

(b) Let $n \geq 1$. Apply Binomial Theorem to

$$(1+(-1))^n = 0^n = 0.$$

$$0 = (1+(-1))^n = \sum_{j=0}^{n} \binom{n}{j} (1)^{n-j} (-1)^j$$

$$= \sum_{j=0}^{n} \binom{n}{j} (-1)^j$$

$$= \binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \dots + (-1)^n \binom{n}{n}.$$

5 Suppose $S \subseteq \mathbb{R}$ and $S \neq \emptyset$.

($\rightarrow$) Suppose $S$ has a maximum.
Then $\exists c \in S$ s.t. $c$ is an upper bound of $S$.
Suppose $d \in \mathbb{R}$ is any upper bound for $S$.
Since $c \in S$, $c \leq d$. Thus $c = \sup S$.

($\leftarrow$) Suppose $\sup S \in S$.
Then $\sup S$ is an upper bound for $S$ and
$\sup S \in S$. Thus $\sup S = \max S$ and
$S$ has a maximum.

(6) (a) $S' = \{ 1/n \mid n \in \mathbb{Z}^+ \}$.

$\inf S' = 0$

$\min S'$ DNE.

$\sup S' = 1 = \max S'$.

(b) $T = \{ x \in \mathbb{R} \mid x^2 < 2 \}$.

$\inf T = -\sqrt{2}$

$\sup T = \sqrt{2}$

$\max T$ DNE and $\min T$ DNE.