Homework 3.1
Linear Algebra
Math 524
Stephen Giang

Section 3.A Problem 4: Suppose $T \in \mathcal{L}(V, W)$ and $v_1, ..., v_m$ is a list of vectors in V such that $Tv_1, ..., Tv_m$ is a linearly independent list in W. Prove that $v_1, ..., v_m$ is linearly independent.

Solution 3.A.4. Let $T \in \mathcal{L}(V, W)$ and $v_1, ..., v_m$ is a list of vectors in V such that $Tv_1, ..., Tv_m$ is a linearly independent list in W.

$$\begin{aligned}
&\text{By Definition of Linearly Independent:}\\
&0 = a_1Tv_1 + ... + a_mTv_m \qquad \text{for } \{a_1, ..., a_m\} = 0 \in \mathbb{F}\\
&0 = T(a_1v_1 + ... + a_mv_m)
\end{aligned}$$

Because $\{a_1, ..., a_m\} = 0$, $v_1, ..., v_m$ is linearly independent.

□

Section 3.A Problem 14: Suppose V is finite-dimensional with dim $V \geq 2$. Prove that there exist S,T $\in \mathcal{L}(V, V)$ such that $ST \neq TS$.

Solution 3.A.14. Let V be finite-dimensional with dim $V \geq 2$ and S,T $\in \mathcal{L}(V, V)$. Let $v_1, ..., v_m$ be a basis of V

$$\begin{aligned}
&\text{Let } T(v_1) = v_2, \quad T(v_2) = v_1 \quad T(v_m) = v_m\\
&\text{Let } S(v_1) = v_1, \quad S(v_2) = 2v_2 \quad S(v_m) = mv_1\\
&\text{By Theorem 3.5, there exists a unique linear map for T and S}\\
&ST(v_1) = S(T(v_1)) = S(v_2) = 2v_2\\
&TS(v_1) = T(S(v_1)) = T(v_1) = v_2
\end{aligned}$$

Thus $ST \neq TS$.

□

Section 3.B Problem 5: Give an example of a linear map $T : R^4 \to R^4$ such that range T = null T.

Solution 3.B.5. Let $T(v_1, v_2, v_3, v_4) = (v_3, v_4, 0, 0)$

$$\text{Range(T)} = \{(v_1, v_2, v_3, v_4) \in \mathbb{R}^4 : v_3 = v_4 = 0\} = \text{ null(T)}$$

$\square$

Section 3.B Problem 6: Prove that there does not exist a linear map $T : R^5 \to R^5$ such that range T = null T.

Solution 3.B.6. Let $T : R^5 \to R^5$ and range T = null T

By Theorem 3.22: dim R^5 = dim(null T) + dim(range T)
$\dim(R^5) = 5$
Because null T = range T, dim(null T) = dim(range T)
Thus dim(null T) = dim(range T) = $2.5 \notin \mathbb{Z}$

Thus there does not exist a linear map $T : R^5 \to R^5$ such that range T = null T.

$\square$

Section 3.B Problem 9: Suppose $T \in \mathcal{L}(V, W)$ is injective and $v_1, ..., v_m$ is linearly independent in V. Prove that $Tv_1, ..., Tv_m$ is linearly independent in W.

Solution 3.B.9. Let $T \in \mathcal{L}(V, W)$ be injective and $v_1, ..., v_m$ be linearly independent in V

Because $v_1, ..., v_m$ is linearly independent in V:
$0 = a_1v_1 + ... + a_mv_m \quad$ for $\{a_1, ..., a_m\} = 0 \in \mathbb{F}$
Because T is injective:
$T(0) = T(a_1v_1 + ... + a_mv_m) \quad$ for $\{a_1, ..., a_m\} = 0 \in \mathbb{F}$
$0 = a_1Tv_1 + ... + a_mTv_m \quad$ for $\{a_1, ..., a_m\} = 0 \in \mathbb{F}$

By definition of Linearly Independence, $Tv_1, ..., Tv_m$ is linearly independent in W

$\square$

Section 3.C Problem 2: Suppose $D \in \mathcal{L}(P_3(\mathbf{R}), P_2(\mathbf{R})$ is the differentiation map defined by $Dp = p'$. Find a basis of $P_3(\mathbf{R})$ and a basis of $P_2(\mathbf{R})$ such that the matrix of D with respect to these bases is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Solution 3.C.2. Let $D \in \mathcal{L}(P_3(\mathbf{R}), P_2(\mathbf{R})$ be the differentiation map defined by $Dp = p'$

Basis of $P_3(\mathbf{R})$: $\{1, x, x^2, x^3\}$

Basis of $P_2(\mathbf{R})$: $\{1, 2x, 3x^2\}$

□

Section 3.C Problem 3: Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exist a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 0 except that the entries in row j , column j , equal 1 for $1 \leq j \leq$ dim range T.

Solution 3.C.3. Let V and W be finite-dimensional and $T \in \mathcal{L}(V, W)$. Let $v_1, ..., v_m$ and $Tv_1, ..., Tv_m$ be bases of V and W respectively.

By Definition of the Matrix of a Linear Map:

$$Tv_k = \sum_{j=1}^{m} A_{j,k} T v_j$$

The only way for $\sum_{j=1}^{m} A_{j,k} T v_j = Tv_k$ with $v_1, ..., v_m$ being a basis of V and $Tv_1, ..., Tv_m$ being a basis of W is for $A_{j,k} = 0$ except when $j = k$, $A_{j,k} = 1$, where $A_{j,k}$ are the constants of Tv_k as a linear combination of Tv_j

□