Quiz 5
Ordinary Differential Equations
Math 537
Stephen Giang RedID: 823184070

Problem 1: Consider the following a 3×3 matrices with repeated eigenvalue:

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

(a) Determine the dimensions of the kernel and range for each of the above matrices.

Notice for the case 1 matrix, we have

$$(A - \lambda I) = 0$$

So we get that $\boldsymbol{dim(Ker(T)) = 3, dim(Range(T)) = 0}$

Notice for the case 2 matrix, we have

$$(A - \lambda I)V_1 = 0, \qquad (A - \lambda I)V_2 = V_1$$

So we get that $\boldsymbol{dim(Ker(T)) = 2, dim(Range(T)) = 1}$

Notice for the case 3 matrix, we have

$$(A - \lambda I)V_1 = 0, \qquad (A - \lambda I)V_2 = V_1, \qquad (A - \lambda I)^2 V_3 = V_1$$

So we get that $\boldsymbol{dim(Ker(T)) = 1, dim(Range(T)) = 2}$

(b) Provide examples for each of the above cases.

Notice $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. When we multiply $T^{-1}AT$, where T is the matrix with its columns being the eigenvectors of A. We get an uncoupled matrix, with $\lambda = 2$, such that $T^{-1}AT = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, which is in the form of the case 1 matrix.

Notice $A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ -1 & 1 & 2 \end{pmatrix}$. When we multiply $T^{-1}AT$, where T is the matrix with its columns being the eigenvectors of A. We get a matrix, with $\lambda = 2$, such that $T^{-1}AT = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, which is in the form of the case 2 matrix.

Notice $A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & -1 & 2 \end{pmatrix}$. When we multiply $T^{-1}AT$, where T is the matrix with its columns being the eigenvectors of A. We get a matrix, with $\lambda = 2$, such that $T^{-1}AT = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$, which is in the form of the case 3 matrix.