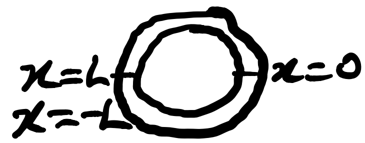
Lecture 7 Heat Conduction in a Ring

The model satisfies the heat equation:



A thin insulated wire that is deformed into a ring

PDE: $u_t = K u_{xx}$, $t>0$, $-L<x<L$

BC: Periodic BC (homogeneous)

$$u(-L,t) = u(L,t)$$

$$u_x(-L,t) = u_x(L,t)$$

IC: $u(x,0) = f(x)$, $-L<x<L$

Solution

Seperation of variables:

$$u(x,t) = \phi(x)\,G(t)$$

gives $\phi G' = K\phi'' G$ or $\dfrac{G'}{KG} = \dfrac{\phi''}{\phi} = -\lambda$

The time varying ODE is

$$G' = -K\lambda G$$

Which has the Solution:

$$G(t) = Ae^{-\kappa\lambda t}$$

The second ODE is

$$\phi'' + \lambda\phi = 0, \quad \phi(-L) = \phi(L)$$
$$\text{and } \phi'(-L) = \phi'(L)$$

Consider three different cases based on the value of λ:

(i) Let $\lambda = 0$, then $\phi'' = 0$ or

$$\phi(x) = C_2 x + C_1$$

Apply the BC: Since $\phi(-L) = \phi(L)$

$$\Rightarrow \phi(-L) - \phi(L) = 0$$
$$\phi(-L) - \phi(L) = -2C_2 L := 0$$

Since $L \neq 0$

then $C_2 = 0$

Also, Apply BC: $\phi'(-L) = \phi'(L)$

we obtain $C_2 = 0$

Which gives no new information, therefore

c_1 is arbitrary.

We have an eigenvalue $\lambda_0 = 0$ with associated eigenfunction

$$\phi_0(x) = 1.$$

Case ii) $\lambda = -\alpha^2 < 0$, $\alpha > 0$

then $\phi'' - \alpha^2 \phi = 0$ so

$$\phi(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$$

Apply the BC: $\phi(-L) = \phi(L)$

$$c_1 \cosh(-\alpha L) + c_2 \sinh(-\alpha L) = c_1 \cosh(\alpha L) + c_2 \sinh(\alpha L)$$

$$\Rightarrow -c_2 \sinh(\alpha L) = c_2 \sinh(\alpha L)$$

Therefore $c_2 = 0$

Applying the BC: $\phi'(-L) = \phi'(L)$

$$c_1 \alpha \sinh(-\alpha L) + c_2 \alpha \cosh(-\alpha L) = c_1 \alpha \sinh(\alpha L) + c_2 \alpha \cosh(\alpha L)$$

So: $2 c_1 \alpha \sinh(\alpha L) = 0$ or $c_1 = 0$

Thus, if $\lambda < 0$, we get the trivial solution $\phi(x) = 0$.

iii) Let $\lambda = \alpha^2 > 0$, $\alpha > 0$

then $\phi'' + \alpha^2 \phi = 0$ and $\boxed{\phi(x) = c_1 \sin(\alpha x) + c_2 \cos(\alpha x)}$

Apply the BC: $\phi(-L) = \phi(L)$ gives

$$c_1 \cos(-\alpha L) + c_2 \sin(-\alpha L) = c_1 \cos(\alpha L) + c_2 \sin(\alpha L)$$

which gives that

$$c_2 \sin(\alpha L) = 0,$$

$c_2 \neq 0$

and $\sin(\alpha L) = 0$

$\alpha L = n\pi$

$\alpha_n = \frac{n\pi}{L}$

and $\lambda_n = \alpha_n^2 = \frac{n^2\pi^2}{L^2}$

are eigenvalues

The BC: $\phi'(-L) = \phi'(L)$

gives

$$-c_1 \alpha \sin(-\alpha L) + c_2 \alpha \cos(-\alpha L) = -c_1 \alpha \sin(\alpha L)$$

$$+ C_2 \alpha \cos(\alpha L)$$

$$2C_1 \alpha \sin(\alpha L) = 0 \ ,$$

$C_1 \neq 0$ and

$$\sin(\alpha L) = 0$$

$$\alpha L = n\pi$$

$$\alpha_n = \frac{n\pi}{L}$$

$$\lambda_n := \alpha_n^2 = \frac{n^2\pi^2}{L^2}$$

are eigenvalues

with corresponding eigenfunction

$$\phi_n(x) = A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \cdots$$

The product solution are:

$$U_0 = A_0 \ ;$$

$$U_n(x,t) = e^{-\frac{k n^2 \pi^2}{L^2} t}\left(A_n \cos\left(\frac{n\pi}{L} x\right) + B_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

The principle of superposition gives the solution:

$$U(x,t) = A_0 + \sum_{n=1}^{\infty} e^{-\frac{kn^2\pi^2}{L^2}t}\left(A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

Apply IC gives

$$(1) \quad U(x,0) = f(x) = A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

Using the orthogonality of the eigenfunctions gives:

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0, & n \neq m \\ L, & n = m \neq 0 \\ 2L, & n = m = 0 \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0, & n \neq m \\ L, & n = m \neq 0 \end{cases}$$

Using the orthogonality properties, we obtain from (1) that

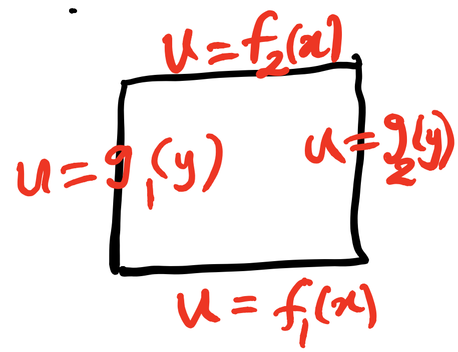
$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

PDE On Rectangular Domains

Laplace's equation in a rectangle with Dirichlet boundary conditions:

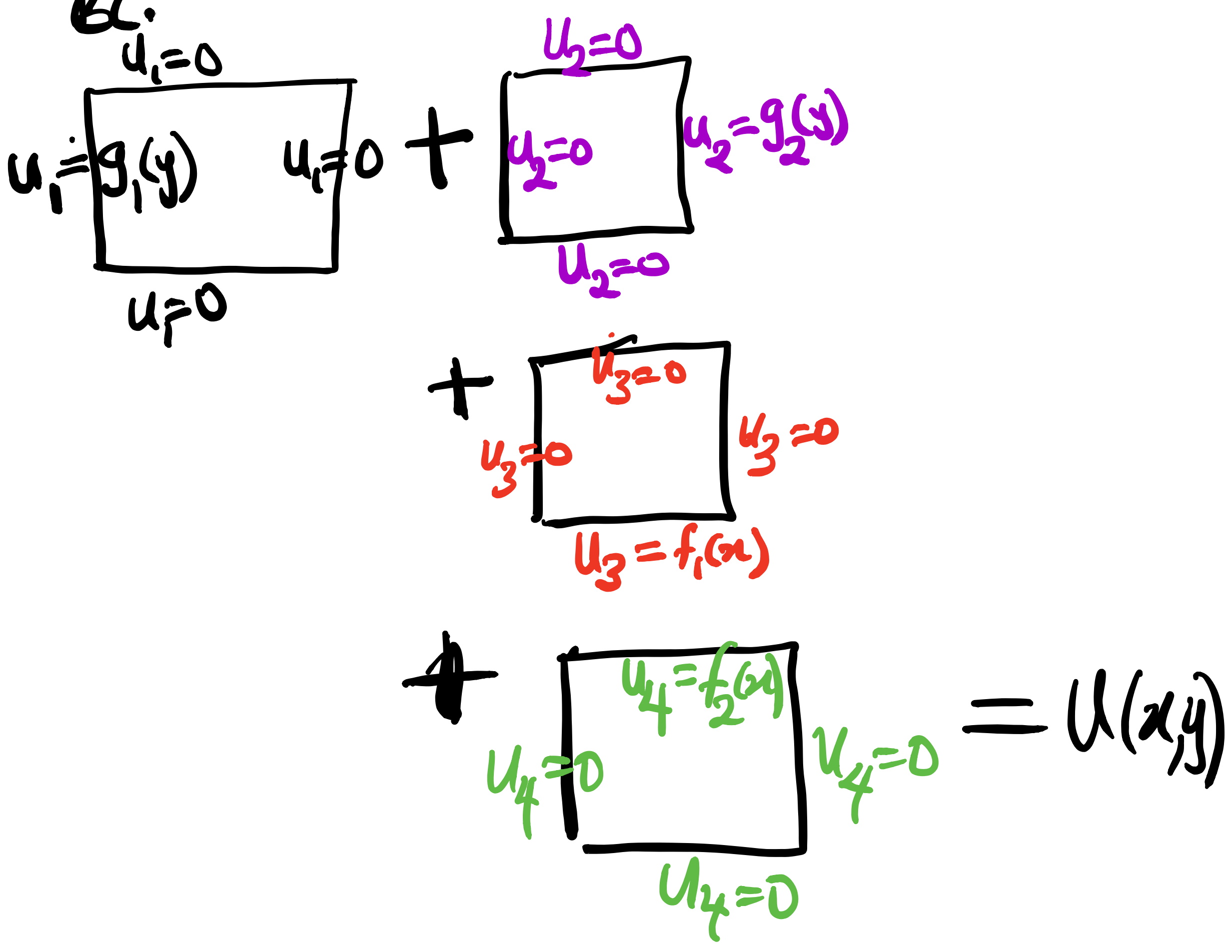


1 $$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad 0<x<L,\ 0<y<H$$

$$u(0,y)=g_1(y)$$

$$u(L,y) = g_2(y)$$
$$u(x,0) = f_1(x)$$
$$u(x,H) = f_2(x)$$

By principle of superposition, we can split problem into four similar problems each with only one non-homogeneous BC.



That is :

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0$$

$$u_1(0,y) = g_1(y) \qquad (2)$$

$$u_1(L,y) = 0$$

$$u_1(x,0) = 0$$

$$u_1(x,H) = 0$$

$\boxed{\text{Note}}$

$$\boxed{\begin{aligned} u_1(x,H) &= X(x)Y(H) \\ &= 0 \\ \Rightarrow Y(H) &= 0 \end{aligned}}$$

we use seperation of variables

Let $u_1(x,y) = X(x)\,Y(y)$

Then the Laplace equation can be written as

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda \qquad (3)$$

$$X'' = \lambda x \qquad (4)$$

$$Y'' = -\lambda Y \qquad (5)$$

$$Y(0) = 0 \quad ; \quad Y(H) = 0 \quad (6)$$

Then, the Y equation (5) and the boundary condition (6) have solution

$$Y = c \sin\left(\frac{n\pi y}{H}\right) \quad (7)$$

for $\lambda = \frac{n^2\pi^2}{H^2} \leftarrow (8)$

Where $n = 1, 2, \ldots$

For the λ in (8), we find the general solution to (4)

$$X(x) = E e^{\frac{n\pi x}{H}} + F e^{-\frac{n\pi x}{H}}$$

or

$$X(x) = a_1 \cosh\left(\frac{n\pi}{H}(x - L)\right) + a_2 \sinh\left(\frac{n\pi}{H}(x - L)\right)$$

The shift in x by L is selected

to Satisfy the boundary Condition at $x=L$ Conveniently. We Impose that $X(L)=0$, which Implies that $a_1=0$. Thus

$$X(x) = a_2 \sinh\left(\frac{n\pi}{H}(x-L)\right)$$

Combining the product solutions:

$$u_1(x,y) = a_2 \sinh\left(\frac{n\pi}{H}(x-L)\right)\sin\left(\frac{n\pi y}{H}\right)$$

By Principle of Superposition:

$$u_1(x,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi}{H}(x-L)\right)\sin\left(\frac{n\pi y}{H}\right)$$

We choose a_n Such that u Satisfies the non-homogeneous BC.

$$u(0,y) = g_1(y)$$

$$g_1(y) := \sum_{n=1}^{\infty} a_n \sinh\left(\frac{-n\pi L}{H}\right)\sin\left(\frac{n\pi y}{H}\right)$$

That is, we multiply both sides by $\sin\left(\frac{n\pi y}{H}\right)$

and use the orthogonality condition

By orthogonality condition of $\sin\left(\frac{n\pi y}{H}\right)$ for y between 0 and H, we obtain

$$a_n \sinh\left(\frac{-n\pi L}{H}\right) = \frac{2}{H}\int_0^H g_1(y)\sin\left(\frac{n\pi y}{H}\right)dy$$

Since $\sinh\left(\frac{-n\pi L}{H}\right)$ is never zero, we can divide by it to obtain that.

$$a_n = \frac{2\int_0^H g_1(y)\sin\left(\frac{n\pi y}{H}\right)dy}{H\sinh\left(\frac{-n\pi L}{H}\right)}$$

We now have a solution for equation 2 :

$$u_1(x,y) = \sum_{n=1}^{\infty} \frac{2}{H\sinh\left(\frac{-n\pi L}{H}\right)} \left\{ \int_0^H g_1(\eta) \sin\left(\frac{n\pi\eta}{H}\right) d\eta \right\} *$$

$$\sinh\left(\frac{n\pi}{H}(x-L)\right) \sin\left(\frac{n\pi y}{H}\right)$$

$$U(x,y) = U_1(x,y) + U_2(x,y) + U_3(x,y) + U_4(x,y)$$

Note: functions are Cosh and Sinh

$$\cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$\sinh x = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

Laplace equation inside a circular disk

Consider a thin circular disk of radius a, (with constant thermal properties). Find the steady state temperature distribution

Laplace equation in polar coordinate:

$$\text{PDE}: \nabla^2 u := \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Derive Laplace equation in polar coordinate:

use the fact that

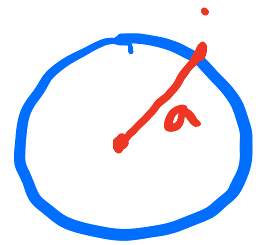
$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y}$$ in

Cartesian Coordinate

$$r^2 = x^2 + y^2; \quad x = r\cos(\theta), \quad y = r\sin(\theta)$$

and chain rule

BC: $u(a,\theta) = f(\theta)$



Periodic BCs and homogeneous:

$$u(r,-\pi) = u(r,\pi)$$

$$u_\theta(r,-\pi) = u_\theta(r,\pi)$$

We Implicit BC: $|u(0,\theta)| < \infty$

that is the solution is bounded at $r=0$.

We have $0 \le r \le a$, and $-\pi \le \theta \le \pi$.

Mathematically, we need conditions at the endpoints of the coordinate system

$r=0$, $r=a$, $\theta=-\pi$ and $\theta=\pi$.

Here only $r=a$ corresponds to a physical boundary.

Separation of variable:

$$u(r,\theta) = \phi(\theta)\, G(r)$$