Final Exam Part B
Math 537 Ordinary Differential Equations
8:00-10:00 AM Dec 14, 2020

Student Name: ______________________________ **ID** ____________

A. The exam must be taken completely alone. Showing it or discussing it with anybody is forbidden.

B. Make an effort to make your submission clear and readable. Severe readability issues may be penalized by grade.

C. There are five problems. **Select and complete four of them.**

D. Please submit your work to Gradescope by 10:10 AM on Dec. 14, 2020.

E. Additional instructions are provided on Slides via Zoom during the exam period.

* *"Be kind, for everyone you meet is fighting a hard battle,"* Ian Maclaren.

1: [25 points] Consider the following 2nd order ODEs:

$$\frac{d^2X}{dt^2} - a^2X = F(t), \tag{1.1}$$

$$\frac{d^2X}{dt^2} + a^2X = G(t). \tag{1.2}$$

Complete the following problems:

(a) [5 points] Assume that $F = G = 0$ and a is a positive constant. Solve Eqs. (1.1) and (1.2) for general solutions.

(b) [10 points] Find $F(t)$ so that Eq. (1.1) has repeated eigenvalue. Briefly discuss the characteristics of the particular solution. [a is a positive constant.]

(c) [10 points] Find $G(t)$ so that Eq. (1.2) has repeated eigenvalue. Briefly discuss the characteristics of the particular solution. [a is a positive constant.]

(d) [BP] State the conditions under which the solutions in problem (1a) change rapidly within an interval or oscillate rapidly over a global scale.

(e) [BP] Assume $F = G = 0$ and $a(t) > 0$ is a function of time. Briefly discuss how to solve both systems.

2: [25 points] Consider the following 2nd order ODE for nonlinear pendulum oscillations (as shown in Figure 1):

$$\frac{d^2\theta}{dt^2} + \epsilon\frac{d\theta}{dt} + sin(\theta) = 0. \tag{2.1}$$

Apply the full equation and its simplified versions to discuss the following concepts:

(a) [5 points] Locally linearized systems near a stable or unstable critical point.

(b) [10 points] The impact of dissipation on the local, global, and structural stability.

(c) [10 points] The impact of nonlinearity (e.g., represented by a cubic term) on the local, global, and structural stability.

(*) Please discuss stability using eigenvalues, extrema of potential energy, etc.

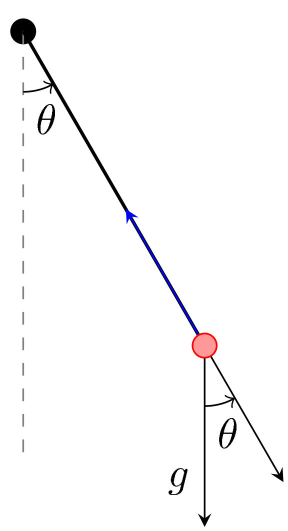


Figure 1: A pendulum consisting of a weightless rod of length L and a bob with a mass of m. The bob and the point of support are marked with a red and black dot, respectively. The parameter "g" denotes the gradational force. The angle θ is measured in the counterclockwise direction. Stable and unstable equilibrium points appear at $\theta = 0$ and $\theta = \pi$, respectively.

3: [25 points] Consider the Lorenz model:

$$\frac{dX}{dt} = -\sigma X + \sigma Y, \tag{3.1}$$

$$\frac{dY}{dt} = -XZ + rX - \alpha Y, \tag{3.2}$$

$$\frac{dZ}{dt} = XY - \beta Z. \tag{3.3}$$

(a) [5 points] Briefly discuss methods for analyzing the above nonlinear system.

(b) [5 points] Compute a Jacobian matrix to obtain a linearized system.

(c) [10 points] Apply a perturbation method to obtain systems of equations for basic state ($O(\epsilon^0)$) and perturbation ($O(\epsilon^1)$) variables.

(d) [5 points] Given $\sigma = 10$, $\alpha = 1$, and $\beta = 8/3$, briefly discuss the characteristics of three types of solutions within different intervals of heating parameters (r).

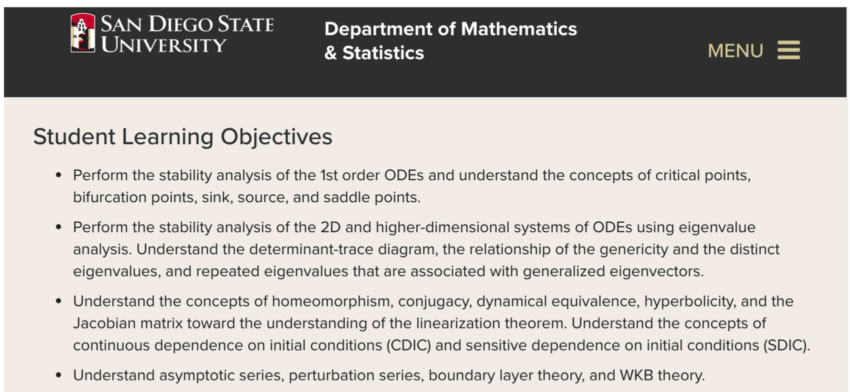
(e) [BP] Find critical points for positive parameters.

(f) [BP] Find critical points within the non-dissipative system (i.e., no σX in Eq. (3.1) and $\alpha = \beta = 0$).

4: [25 points + bonus] Based on Student Learning Objectives (SLOs in Figure 2) and your work for Math537, please complete the following:

(a) [25 points] Create your own QuadChart (as shown in Figure 3);

(b) [5 bonus points] Post your QuadChart to any social medias or web sites, and attach a screenshot of the posted work into your Part-B work.



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Student Learning Objectives

- Perform the stability analysis of the 1st order ODEs and understand the concepts of critical points, bifurcation points, sink, source, and saddle points.
- Perform the stability analysis of the 2D and higher-dimensional systems of ODEs using eigenvalue analysis. Understand the determinant-trace diagram, the relationship of the genericity and the distinct eigenvalues, and repeated eigenvalues that are associated with generalized eigenvectors.
- Understand the concepts of homeomorphism, conjugacy, dynamical equivalence, hyperbolicity, and the Jacobian matrix toward the understanding of the linearization theorem. Understand the concepts of continuous dependence on initial conditions (CDIC) and sensitive dependence on initial conditions (SDIC).
- Understand asymptotic series, perturbation series, boundary layer theory, and WKB theory.

Figure 2: Student Learning Objectives for Math537.
`https://math.sdsu.edu/courses/syllabi_math/math537`

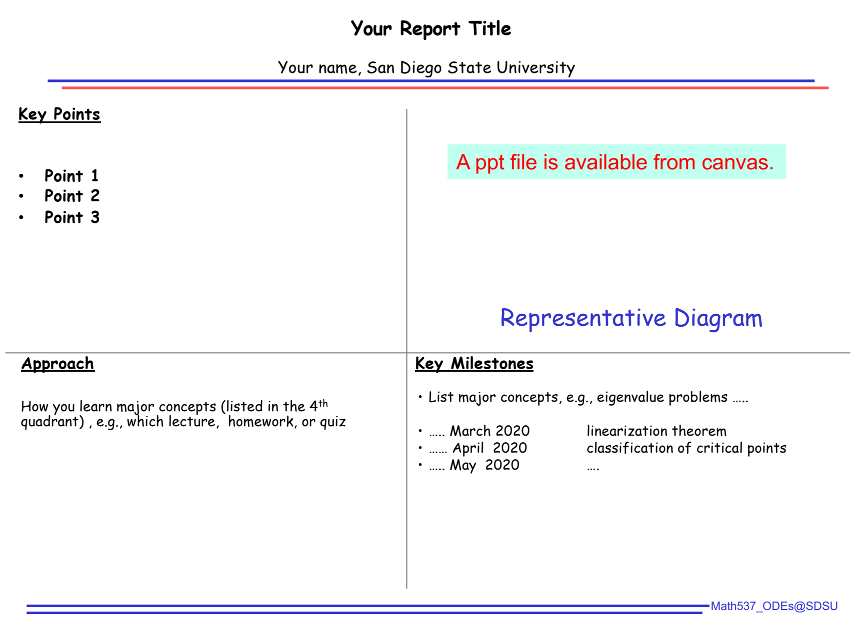

Figure 3: A template for QuadChart

5: [25 points] Consider a composite motion with the following harmonic oscillators:

$$\frac{d^2x_1}{dt^2} = -\omega_1^2 x_1, \tag{5.1}$$

$$\frac{d^2x_2}{dt^2} = -\omega_2^2 x_2. \tag{5.2}$$

(a) [5 points] Discuss the condition under which the composite motion with the two frequencies is periodic or quasi-periodic.

(b) [20 points] Compute and generate a plot (e.g., Figure 4) to illustrate either a periodic or quasi-periodic composite solution (with two frequencies).

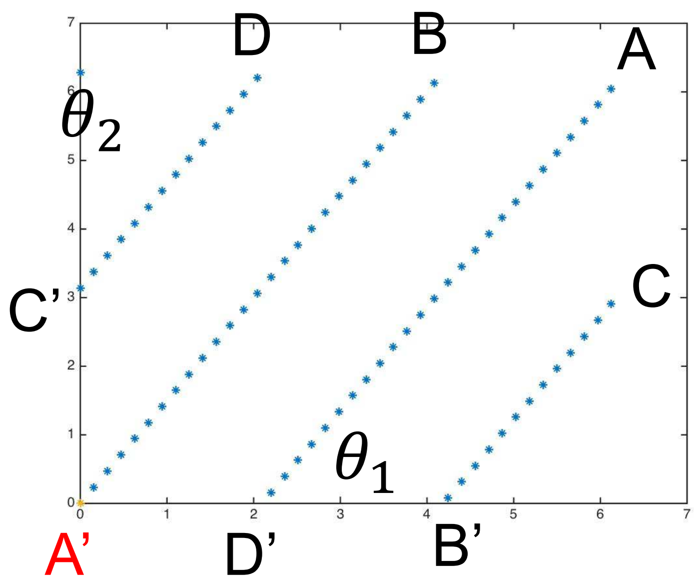

Figure 4: A solution with rational frequency ratio in the $\theta_1 - \theta_2$ plane.