11/19 — HW5 solutions posted at 7pm

T/F $\forall f: \mathbb{N} \to \mathbb{R}$, f is continuous.

$\forall x_0 \in \mathbb{N}$, $\forall \{x_n\} \subseteq \mathbb{N}$, if $\lim_{n\to\infty} x_n = x_0$, then $\lim_{n\to\infty} f(x_n) = f(x_0)$.

T/F $\forall f: [0,1) \to \mathbb{R}$, if f cont, then f uniformly continuous.

$$f(x) = \frac{1}{x-1}$$

sequential $f: D \to \mathbb{R}$ uniformly conti

iff

$\forall \{u_n\}, \{v_n\} \subseteq D$, if $\lim_{n\to\infty} (u_n - v_n) = 0$, then $\lim_{n\to\infty} (f(u_n) - f(v_n)) = 0$.

not u. cont.

iff

$\exists \{u_n\}, \{v_n\} \subseteq D$ $\lim_{n\to\infty} (u_n - v_n) = 0$ and $\lim_{n\to\infty} (f(u_n) - f(v_n)) \neq 0$.

$f(x) = \frac{1}{x-1}$

$f: [0,1) \to \mathbb{R}$

Let $u_n = 1 - \frac{1}{n}$, $v_n = 1 - \frac{1}{n^2}$.

$\lim_{n\to\infty} (u_n - v_n) = 1 - 1 = 0$.

$f(u_n) = \frac{1}{1-\frac{1}{n}-1} = -n$ $f(v_n) = \frac{1}{1-\frac{1}{n^2}-1} = -n^2$.

$\lim_{n\to\infty} (f(u_n) - f(v_n)) = \lim_{n\to\infty} (-n + n^2) \neq 0$.

Remarks on limits of functions: Suppose $g, f : D \to \mathbb{R}$.

① If $x_0 \in D$ is a limit point, then f continuous at x_0 is equivalent to $\lim_{x \to x_0} f(x) = f(x_0)$.

② "Limit Laws" Suppose x_0 is a limit point of D and $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = K$.

(a) $\lim_{x \to x_0} \left(f(x) + c\, g(x) \right) = L + cK$.

(b) $\lim_{x \to x_0} f(x) g(x) = LK$

(c) If $K \neq 0$, $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{L}{K}$.

Assuming $x_0 \in D$ has a neighborhood in D implies x_0 is a limit point of D too.

Prop 4.5 Let I be a neighborhood of x_0 and suppose $f: I \to \mathbb{R}$ is differentiable at x_0. Then f is continuous at x_0.

---

proof: Using limit remark ①, consider $\lim_{x \to x_0} f(x) - f(x_0)$.

Since $x \neq x_0$, we can say

$$\lim_{x \to x_0} (f(x) - f(x_0)) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0)$$

$$= f'(x_0) \cdot 0 = 0 \text{ by remark ②.}$$

Prop 4.6 Derivative rules work!

Suppose I is a neighborhood of x_0 and $f, g : I \to \mathbb{R}$ are differentiable at x_0.

(i) $\forall c \in \mathbb{R}$, $(f + cg)'(x_0) = f'(x_0) + c g'(x_0)$. "Linearity"

(ii) $(fg)'(x_0) = f(x_0) g'(x_0) + f'(x_0) g(x_0)$. "Prod"

(iii) Suppose $g(x) \neq 0$ for all $x \in I$.

Then $$\left(\frac{1}{g}\right)'(x_0) = \frac{-g'(x_0)}{(g(x_0))^2}.$$

(iv) Suppose $g(x) \neq 0$ for all $x \in I$.

Then $$\left(\frac{f}{g}\right)'(x_0) = \frac{g(x_0) f'(x_0) - f(x_0) g'(x_0)}{(g(x_0))^2}.$$

proofs are limit computations

proofs:

(i) Let $c \in \mathbb{R}$.

Compute $\displaystyle\lim_{x \to x_0} \frac{(f+cg)(x) - (f+cg)(x_0)}{x - x_0}$

$$= \lim_{x \to x_0} \frac{f(x) + cg(x) - f(x_0) - cg(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} + c \cdot \frac{g(x) - g(x_0)}{x - x_0} \right)$$

$= f'(x_0) + c\,g'(x_0)$ by limit laws in remark ②.

(iii) $$\lim_{x\to x_0} \frac{\left(\frac{1}{g}\right)(x) - \left(\frac{1}{g}\right)(x_0)}{x - x_0}$$

$$= \lim_{x\to x_0} \frac{\frac{1}{g(x)} - \frac{1}{g(x_0)}}{x - x_0} = \lim_{x\to x_0} \frac{\frac{g(x_0) - g(x)}{g(x)g(x_0)}}{x - x_0}$$

$$= \lim_{x\to x_0} \frac{1}{g(x)g(x_0)} \cdot \frac{g(x_0) - g(x)}{x - x_0}$$

$$= \lim_{x\to x_0} \frac{-1}{g(x)g(x_0)} \cdot \frac{g(x) - g(x_0)}{x - x_0}.$$

$$= -\frac{1}{(g(x_0))^2} \cdot g'(x_0)$$ by

1. differentiability of g at x_0
2. continuity of g at x_0
3. limit laws.