

B: Sink ($\lambda_1 < \lambda_2 < 0$): move toward (0,0)

$$\begin{aligned}x' &= \lambda_1 x \\y' &= \lambda_2 y\end{aligned}\qquad A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Let $|A - \lambda I| = 0 \Rightarrow \boxed{\lambda = \lambda_{1,2}}$ $V_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Thus, a general solution is written as

$$X(t) = \alpha e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \begin{aligned}x &= \alpha e^{\lambda_1 t} \\y &= \beta e^{\lambda_2 t}\end{aligned}$$

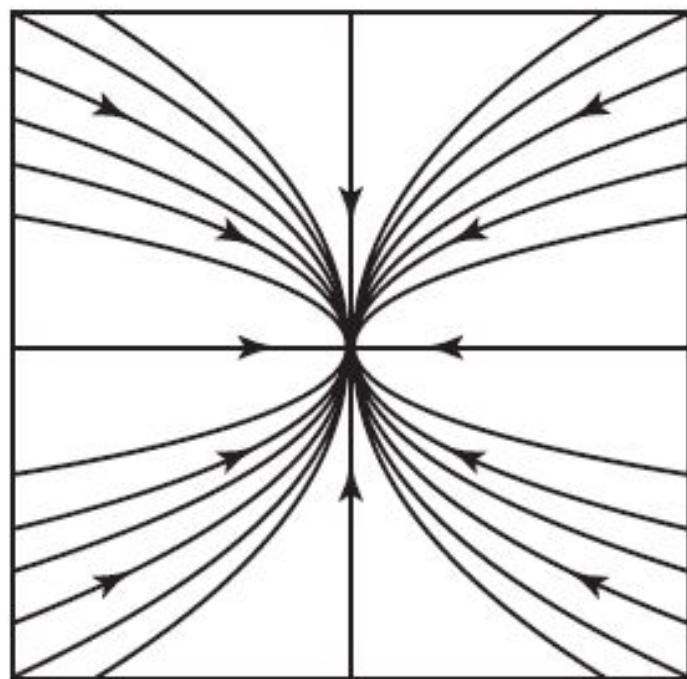
$$\frac{dy}{dx} = \frac{\lambda_2 \beta e^{\lambda_2 t}}{\lambda_1 \alpha e^{\lambda_1 t}} = \frac{\lambda_2 \beta}{\lambda_1 \alpha} e^{(\lambda_2 - \lambda_1)t} \qquad \rightarrow \infty \text{ (or } -\infty \text{) as } t \rightarrow \infty$$
$$\frac{dy}{dt} \qquad (\lambda_2 - \lambda_1) > 0$$

- These solutions tend to the origin (a sink) tangentially to the y axis (i.e., vertical asymptotes)
- x tends to zero much quickly
- λ_1 (λ_2) is referred to as the stronger (weaker) eigenvalue.

B: Sink ($\lambda_1 < \lambda_2 < 0$)

$$\lambda_1 < \lambda_2 < 0$$

Trajectories (time varying)



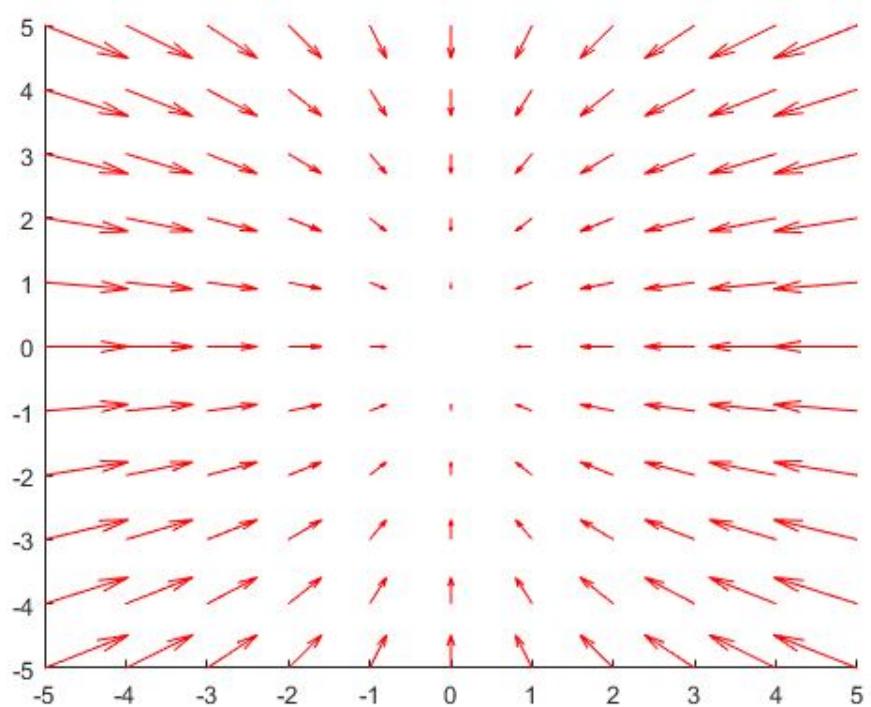
(a)

Figure 3.3 Phase portraits for (a) a sink

$$X' = -2x$$

$$Y' = -y$$

Vector Fields (at a given time)



MATLAB Plot for Figure 3.3a

- These solutions tend to the origin (a sink) tangentially to the y axis (i.e., vertical asymptotes), associated with the weaker eigenvalue (λ_2).

The Euler Method

From Calc II, we have:

$$\frac{dy}{dx} = F(x, y) \quad x_n = x_{n-1} + h \quad h = \Delta x, \text{ an increment}$$

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}) \quad n = 1, 2, 3, \dots$$

Now, we consider the following:

$$\frac{dy}{dt} = F(y) \quad y: \textit{displacement},$$

$$\frac{dy}{dt}: \textit{velocity}$$

Given a $F(y) = -y$, we have

$$\frac{dy}{dt} = -y \quad \frac{y_{n+1} - y_n}{\Delta t} = -y_n \quad y_{n+1} = y_n - \Delta t y_n$$

B: Trajectories (derived from Vector Fields)

$$x' = -2x$$
$$y' = -y$$

$$X' = -2x$$
$$Y' = -y$$

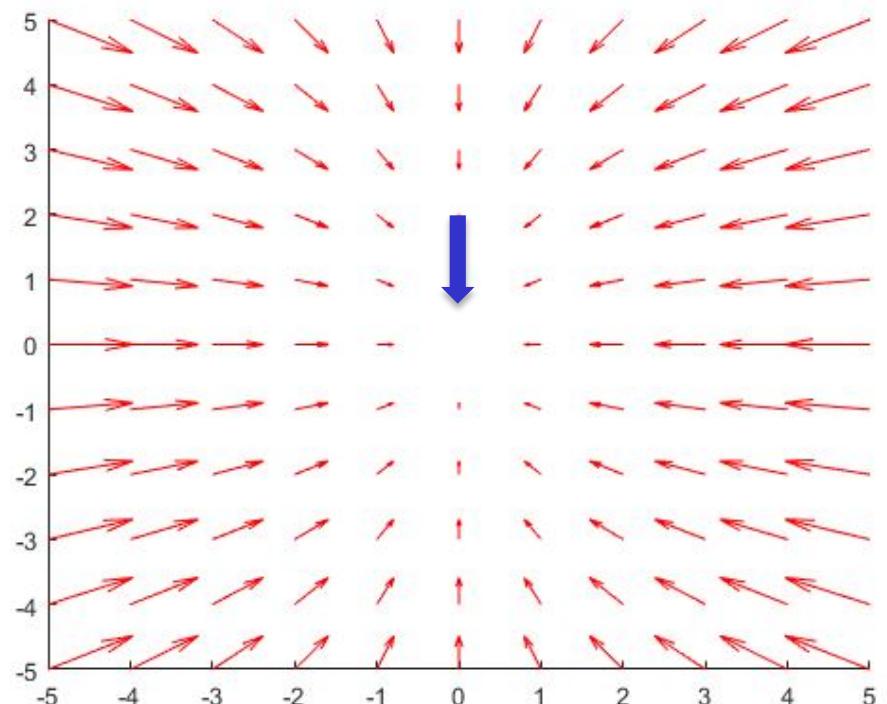
Apply the Euler Method

$$y_{n+1} = y_n - \Delta t y_n \quad \Delta t = 0.5$$

At point B: $(x_n, y_n) = (0, 2)$

$$\begin{aligned} y_{n+1} &= y_n - \Delta t y_n \\ &= 2 - 0.5 * 2 \\ &= 1 < y_n \rightarrow \text{southward} \end{aligned}$$

- In the first quadrant, a particle moves westward ($x' < 0$) and southward ($y' < 0$).
- In the third quadrant, a particle moves eastward westward ($x' > 0$) and northward ($y' > 0$).



MATLAB Plot for Figure 3.3a

B: Sink ($\lambda_1 < \lambda_2 < 0$): a general case

Thus, a general solution is written as

$$X(t) = \begin{pmatrix} x \\ y \end{pmatrix} = \alpha e^{\lambda_1 t} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \beta e^{\lambda_2 t} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$x = \alpha e^{\lambda_1 t} u_1 + \beta e^{\lambda_2 t} v_1$$

$$y = \alpha e^{\lambda_1 t} u_2 + \beta e^{\lambda_2 t} v_2$$

$$\frac{dy}{dt} = \frac{\lambda_1 \alpha e^{\lambda_1 t} u_1 + \lambda_2 \beta e^{\lambda_2 t} v_1}{\lambda_1 \alpha e^{\lambda_1 t} u_2 + \lambda_2 \beta e^{\lambda_2 t} v_2} = \frac{\lambda_1 \alpha e^{(\lambda_1 - \lambda_2)t} u_1 + \lambda_2 \beta v_1}{\lambda_1 \alpha e^{(\lambda_1 - \lambda_2)t} u_2 + \lambda_2 \beta v_2} \rightarrow \frac{v_2}{v_1} \text{ as } t \rightarrow \infty$$

$(\lambda_1 - \lambda_2) < 0$



- All solutions (except for those on the straight line corresponding to the stronger eigenvalue) **tend to the origin (a sink) tangentially** to the straight-line solution corresponding to **the weaker eigenvalue (λ_2)**.
- λ_1 (λ_2) is referred to as the stronger (weaker) eigenvalue.

(C): A Source ($0 < \lambda_1 < \lambda_2$)

$$\begin{aligned}x' &= \lambda_1 x \\y' &= \lambda_2 y\end{aligned}$$

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Let $|A - \lambda I| = 0 \Rightarrow$

$$\lambda = \lambda_{1,2}$$

$$V_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus, a general solution is written as

$$X(t) = \alpha X_1 + \beta X_2 = \alpha e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\lambda_2 \beta e^{\lambda_2 t}}{\lambda_1 \alpha e^{\lambda_1 t}} = \frac{\lambda_2 \beta}{\lambda_1 \alpha} e^{(\lambda_2 - \lambda_1)t} \rightarrow \infty \text{ (or } -\infty \text{) as } t \rightarrow -\infty$$

tend away from (0,0)

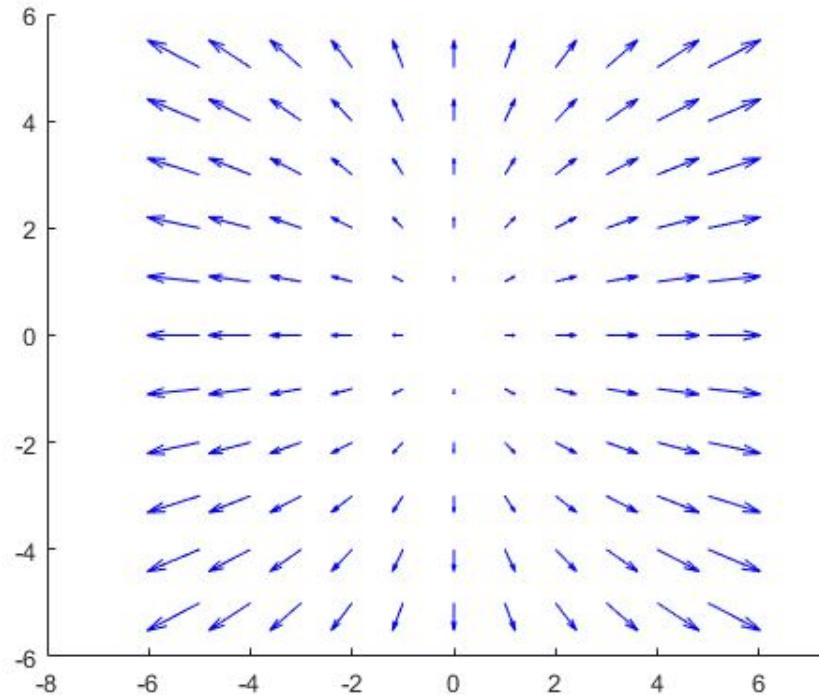
(C): A Source ($0 < \lambda_2 < \lambda_1$)

$$X' = 2x$$

$$Y' = y$$

$$\lambda_1 > 0; \quad \lambda_2 > 0$$

Vector Fields (at a given time)



MATLAB Plot for Figure 3.3b

Trajectories (time varying)

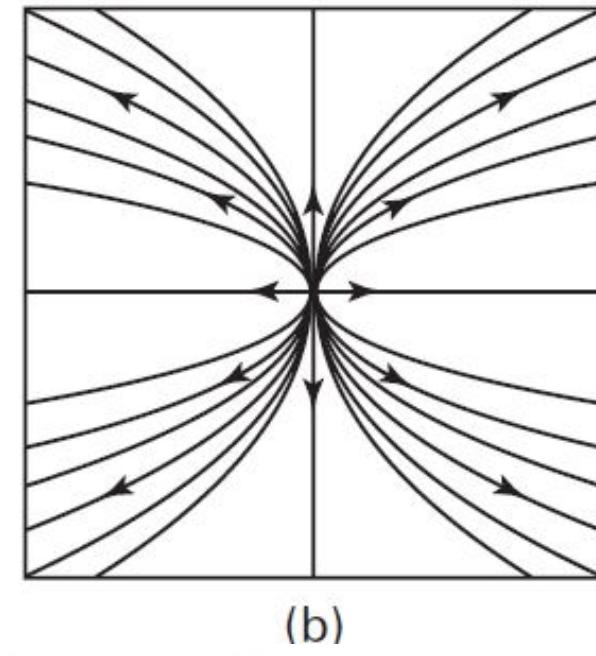


Figure 3.3 Phase portraits for
(b) a source.

The general solution and phase portrait remain the same, except that all solutions now **tend away from (0,0)** along the same paths. See Figure 3.3b.

MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #9

Chapter 3 Phase Portraits for Planar Systems
Complex Eigenvalues
&
Repeated Eigenvalues

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Simple 2D Systems with Complex Eigenvalues

$$x' = ax + by \quad (= P(x, y)) \quad (1)$$

$$y' = cx + dy \quad (= Q(x, y)) \quad (2)$$

$$X' = AX \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

(I) pure imaginary eigenvalues

$$A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$$

$$x' = \cancel{ax} + \beta y \quad (= P(x, y))$$

$$y' = \beta x + \cancel{dy} \quad (= Q(x, y))$$

D. $\lambda_{1,2} = \pm i \beta$: center

(II) complex eigenvalues with $Re \neq 0$

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

$$x' = \alpha x + \beta y \quad (= P(x, y))$$

$$y' = \beta x + \alpha y \quad (= Q(x, y))$$

E. $\lambda_{1,2} = \alpha \pm i \beta$: spiral source or sink

Review: Complex Roots

5

$$ay'' + by' + cy = 0$$

where a , b , and c are constants and $a \neq 0$.

Summary of Cases I–III

Case	Roots of (2)	Basis of (1)	General Solution of (1)
III	Complex conjugate $\lambda_1 = -\frac{1}{2}a + i\omega$, $\lambda_2 = -\frac{1}{2}a - i\omega$	$e^{-ax/2} \cos \omega x$ $e^{-ax/2} \sin \omega x$	$y = e^{-ax/2}(A \cos \omega x + B \sin \omega x)$

Review: 2nd order ODEs with Complex Eigenvalues

$$\lambda_{1,2} = \alpha \pm i\beta$$

$$x = c_1 x_1 + c_2 x_2 = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$= c_1 e^{(\alpha+i\beta)t} + c_2 e^{(\alpha-i\beta)t}$$

$$= e^{\alpha x} (c_1 e^{(i\beta)t} + c_2 e^{(-i\beta)t})$$

$$= e^{\alpha x} (c_1 (\cos(\beta t) + i \sin(\beta t)) + c_2 (\cos(\beta t) - i \sin(\beta t)))$$

$$= e^{\alpha x} ((c_1 + c_2) \cos(\beta t) + i(c_1 - c_2) \sin(\beta t))$$

$$= e^{\alpha x} (A \cos(\beta t) + B \sin(\beta t)) \quad A = c_1 + c_2 \quad B = i(c_1 - c_2)$$

(D) A Center with $\lambda = \pm i\beta$

Consider

$$\begin{aligned}x' &= \beta y \\y' &= -\beta x\end{aligned}$$

$$A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} -\lambda & \beta \\ -\beta & -\lambda \end{pmatrix}$$

Let $|A - \lambda I| = 0 \Rightarrow$

$$\lambda^2 + \beta^2 = 0$$

$$\lambda = \pm i\beta$$

$$AV_0 = \lambda V_0$$

$$\begin{aligned}\beta y_0 &= \lambda x_0 \\-\beta x_0 &= \lambda y_0\end{aligned}$$

$$\lambda = i\beta$$

$$\begin{aligned}\beta y_0 &= i\beta x_0 \\-\beta x_0 &= i\beta y_0\end{aligned}$$

- Find the eigenvector
- Send your results via "chat"
- You have 3 minutes

(D) A Center with $\lambda = \pm i\beta$

Consider

$$\begin{aligned}x' &= \beta y \\y' &= -\beta x\end{aligned}$$

$$A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} -\lambda & \beta \\ -\beta & -\lambda \end{pmatrix}$$

Let $|A - \lambda I| = 0 \Rightarrow$

$$\lambda^2 + \beta^2 = 0$$

$$\lambda = \pm i\beta$$

$$AV_0 = \lambda V_0$$

$$\begin{aligned}\beta y_0 &= \lambda x_0 \\-\beta x_0 &= \lambda y_0\end{aligned}$$

$$\lambda = i\beta$$

$$\begin{aligned}\beta y_0 &= i\beta x_0 \\-\beta x_0 &= i\beta y_0\end{aligned} \quad y_0 = ix_0 \quad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ ix_0 \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Obtain

$$V_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

as an eigenvector associated with $\lambda = i\beta$

Similarly,

$$V_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

as an eigenvector associated with $\lambda = -i\beta$

(D) A Center with $\lambda = \pm i\beta$

Thus, a general solution is written as

$$\begin{aligned} X(t) &= ae^{i\beta t} \begin{pmatrix} 1 \\ i \end{pmatrix} + be^{-i\beta t} \begin{pmatrix} 1 \\ -i \end{pmatrix} \\ &= a(\cos(\beta t) + i\sin(\beta t)) \begin{pmatrix} 1 \\ i \end{pmatrix} + b(\cos(\beta t) - i\sin(\beta t)) \begin{pmatrix} 1 \\ -i \end{pmatrix} \\ &= \begin{pmatrix} (a+b)\cos(\beta t) \\ (-a-b)\sin(\beta t) \end{pmatrix} + i \begin{pmatrix} (a-b)\sin(\beta t) \\ (a-b)\cos(\beta t) \end{pmatrix} \\ &= (a+b) \begin{pmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{pmatrix} + i(a-b) \begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix} \\ &= c_1 X_{re}(t) + c_2 X_{im} \end{aligned}$$

How to quickly find two LI solutions with trig functions?

$$X_{re}(t) = \operatorname{Re} \left(e^{i\beta t} \begin{pmatrix} 1 \\ i \end{pmatrix} \right)$$

$$X_{im}(t) = \operatorname{Im} \left(e^{i\beta t} \begin{pmatrix} 1 \\ i \end{pmatrix} \right)$$

(D) A Center with $\lambda = \pm i\beta$

$$e^{i\beta t} \begin{pmatrix} 1 \\ i \end{pmatrix} = (\cos(\beta t) + i\sin(\beta t)) \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \cos(\beta t) + i\sin(\sin(\beta t)) \\ Ans \end{pmatrix}$$

- Find “Ans”
- Send your results via "chat"
- You have 2 minutes

$$e^{i\beta t} \begin{pmatrix} 1 \\ i \end{pmatrix} = (\cos(\beta t) + i\sin(\beta t)) \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \cos(\beta t) + i\sin(\sin(\beta t)) \\ i \cos(\beta t) - \sin(\beta t) \end{pmatrix}$$

$$Re \left(e^{i\beta t} \begin{pmatrix} 1 \\ i \end{pmatrix} \right) = \begin{pmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{pmatrix} = X_{re}$$

$$Im \left(e^{i\beta t} \begin{pmatrix} 1 \\ i \end{pmatrix} \right) = \begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix} = X_{im}$$

(D) A Center with $\lambda = \pm i\beta$

$$Re \left(e^{i\beta t} \begin{pmatrix} 1 \\ i \end{pmatrix} \right) = Re \left((\cos(\beta t) + i\sin(\beta t)) \begin{pmatrix} 1 \\ i \end{pmatrix} \right) = \begin{pmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{pmatrix} = X_{re}$$

$$Im \left(e^{i\beta t} \begin{pmatrix} 1 \\ i \end{pmatrix} \right) = Im \left((\cos(\beta t) + i\sin(\beta t)) \begin{pmatrix} 1 \\ i \end{pmatrix} \right) = \begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix} = X_{im}$$

Thus, a general solution is written as

$$X(t) = ae^{i\beta t} \begin{pmatrix} 1 \\ i \end{pmatrix} + be^{-i\beta t} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$= c_1 X_{re}(t) + c_2 X_{im}$$

(D) A Center with $\lambda = \pm i\beta$

Thus, a general solution is written as

$$X(t) = ae^{i\beta t} \begin{pmatrix} 1 \\ i \end{pmatrix} + be^{-i\beta t} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$= c_1 X_{re}(t) + c_2 X_{im}$$

$$= c_1 \begin{pmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix}$$

$$x(t) = c_1 \cos(\beta t) + c_2 \sin(\beta t)$$

also obtained by solving a 2nd-order ODE for x

$$y(t) = -c_1 \sin(\beta t) + c_2 \cos(\beta t)$$

Note that $\beta y = x'$

$$X_{re}(t) = \begin{pmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{pmatrix}$$

$$X_{im}(t) = \begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix}$$

Solution for a Center with $\lambda = \pm i\beta$

$$\lambda_{1,2} = \pm i\beta$$

Trajectories (time varying)

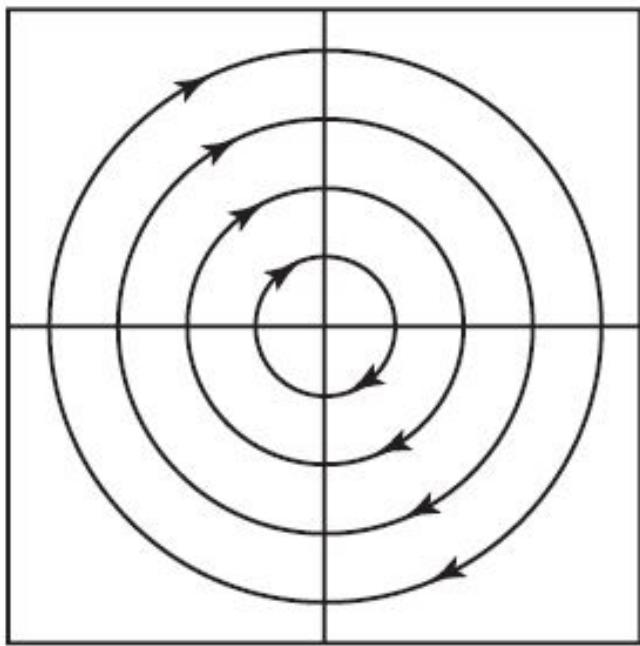
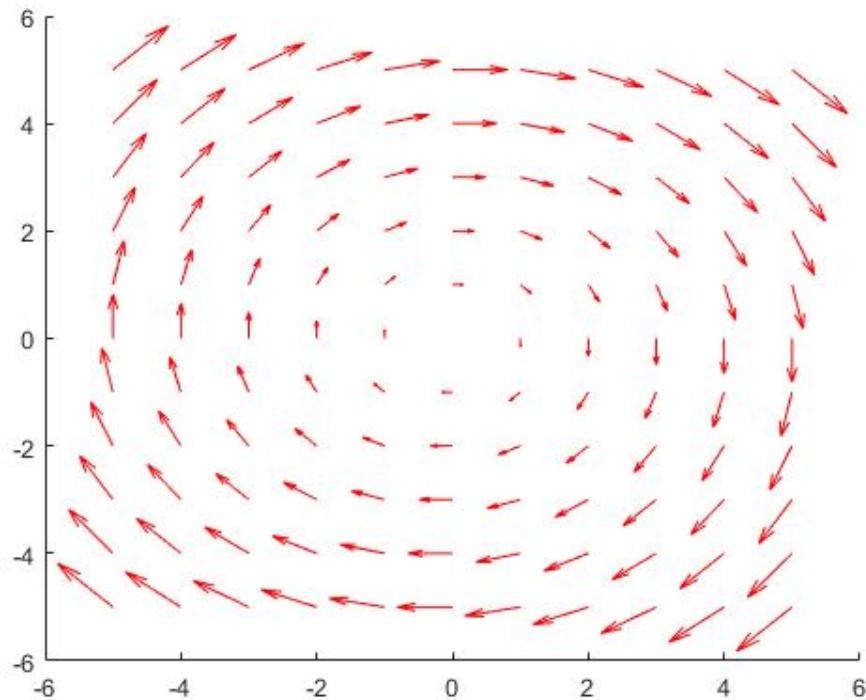


Figure 3.4 Phase portrait for a center.

$$X' = 2y$$

$$Y' = -2x$$

Vector Fields (at a given time)



MATLAB Plot for Figure 3.4

$$\nabla \cdot \vec{F} = P_x + Q_y = 0$$

$$\nabla \times \vec{F} = Q_x - P_y = -2 < 0$$

A Planar System on a Complex Plane

$$\frac{d}{dt}$$

$$\begin{aligned}x' &= y \\y' &= -x,\end{aligned}$$

$$\frac{d}{d\tau}$$

$$\begin{aligned}x' &= \beta y \\y' &= -\beta x,\end{aligned}$$

$$t = \beta\tau$$



$$u = x + iy$$

$$u' = x' + iy'$$

$$= y - ix = -i(x + iy)$$

$$= -iu$$

$$u' = -iu$$

$$u = c_1 e^{-it} = c_1 \cos(t) - i c_1 \sin(t)$$

$$x = c_1 \cos(t)$$

$$y = -c_1 \sin(t)$$

$$w = x - iy$$

- Find the ODE for w
- Send your results via "chat"
- You have 3 (up to 5) minutes

A Planar System on a Complex Plane

$$\frac{d}{dt}$$

$$\begin{aligned}x' &= y \\y' &= -x,\end{aligned}$$

$$\frac{d}{d\tau}$$

$$\begin{aligned}x' &= \beta y \\y' &= -\beta x,\end{aligned}$$

$$t = \beta\tau$$



$$u = x + iy$$

$$u' = x' + iy'$$

$$= y - ix = -i(x + iy)$$

$$= -iu$$

$$u' = -iu$$

$$u = c_1 e^{-it} = c_1 \cos(t) - ic_1 \sin(t)$$

$$x = c_1 \cos(t)$$

$$y = -c_1 \sin(t)$$

$$w = x - iy$$

$$w' = x' - iy'$$

$$= y + ix = i(x - iy)$$

$$= iw$$

$$w = iw$$

$$w = c_2 e^{it} = c_2 \cos(t) + ic_2 \sin(t)$$

$$x = c_2 \cos(t)$$

$$y = -c_2 \sin(t)$$

Simple 2D Systems with Complex Eigenvalues

$$x' = ax + by \quad (= P(x, y)) \quad (1)$$

$$y' = cx + dy \quad (= Q(x, y)) \quad (2)$$

$$X' = AX \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

(I) pure imaginary eigenvalues

$$A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$$

$$x' = \cancel{ax} + \beta y \quad (= P(x, y))$$

$$y' = \beta x + \cancel{dy} \quad (= Q(x, y))$$

D. $\lambda_{1,2} = \pm i \beta$: center

(II) complex eigenvalues with $Re \neq 0$

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

$$x' = \alpha x + \beta y \quad (= P(x, y))$$

$$y' = \beta x + \alpha y \quad (= Q(x, y))$$

E. $\lambda_{1,2} = \alpha \pm i \beta$: spiral source or sink

(E): Spiral Sink or Spiral Source with $\lambda = \alpha \pm i\beta$

Consider $x' = \alpha x + \beta y$ $A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$

$$y' = -\beta x + \alpha y$$

Let $|A - \lambda I| = 0 \Rightarrow \boxed{\lambda = \alpha \pm i\beta}$

$$AV_0 = \lambda V_0 \quad \begin{aligned} \alpha x_0 + \beta y_0 &= \lambda x_0 \\ -\beta x_0 + \alpha y_0 &= \lambda y_0 \end{aligned}$$

Consider $\lambda = \alpha + i\beta$

$$\begin{aligned} \alpha x_0 + \beta y_0 &= (\alpha + i\beta)x_0 \\ -\beta x_0 + \alpha y_0 &= (\alpha + i\beta)y_0 \end{aligned} \quad y_0 = ix_0 \quad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ ix_0 \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Obtain

$$V_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

as an eigenvector associated with $\lambda = \alpha + i\beta$

(E): Spiral Sink or Spiral Source with $\lambda = \alpha \pm i\beta$

Thus, a general solution is written as

$$X(t) = ae^{\alpha+i\beta t} \begin{pmatrix} 1 \\ i \end{pmatrix} + be^{\alpha-i\beta t} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$
$$= e^{\alpha t} (c_1 X_{re}(t) + c_2 X_{im})$$

$$X_{re}(t) = \begin{pmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{pmatrix} \quad X_{im}(t) = \begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix}$$

$$Re \left(e^{i\beta t} \begin{pmatrix} 1 \\ i \end{pmatrix} \right) = Re \left((\cos(\beta t) + i\sin(\beta t)) \begin{pmatrix} 1 \\ i \end{pmatrix} \right) = \begin{pmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{pmatrix} = X_{re}$$

$$Im \left(e^{i\beta t} \begin{pmatrix} 1 \\ i \end{pmatrix} \right) = Im \left((\cos(\beta t) + i\sin(\beta t)) \begin{pmatrix} 1 \\ i \end{pmatrix} \right) = \begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix} = X_{im}$$

(E): Spiral Sink or Spiral Source with $\lambda = \alpha \pm i\beta$

Thus, a general solution is written as

$$X(t) = ae^{\alpha+i\beta t} \begin{pmatrix} 1 \\ i \end{pmatrix} + be^{\alpha-i\beta t} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$= e^{\alpha t} (c_1 X_{re}(t) + c_2 X_{im})$$

decaying
or growing

oscillatory

$$X_{re}(t) = \begin{pmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{pmatrix}$$

$$X_{im}(t) = \begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix}$$

- $\alpha < 0$ spirals into the origin, a **spiral sink**

oscillatory with time varying radii

- $\alpha > 0$ spirals away the origin, a **spiral source**

A Spiral Sink with $\lambda = \alpha \pm i\beta$: Oscillatory Decay

- red: $e^{\alpha t}, \alpha < 0$
- green: $\sin(\beta t)$
- blue: $e^{\alpha t} \sin(\beta t)$

```
syms t a b
```

```
a=-1
```

```
b=2*pi
```

```
fexp=exp(a*t)
```

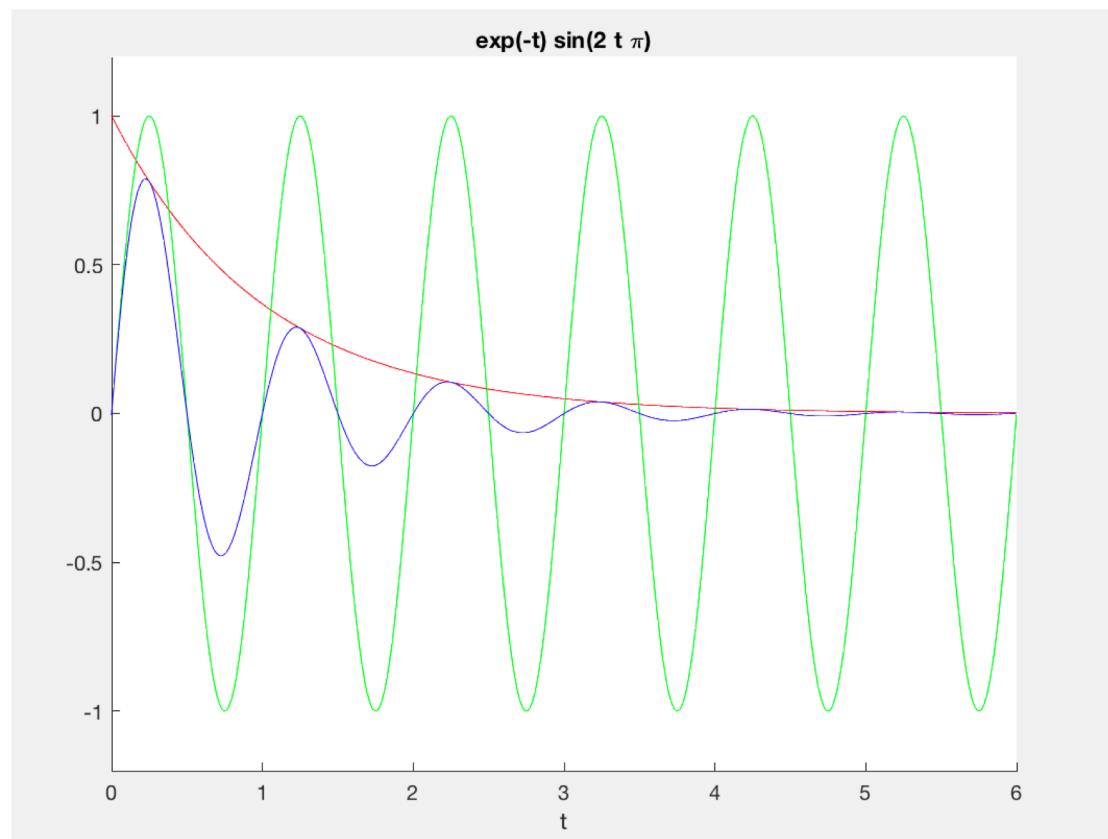
```
fosc=sin(b*t)
```

```
hold on
```

```
ezplot (fexp, [0, 6, -1.2, 1.2])
```

```
ezplot (fosc, [0, 6, -1.2, 1.2])
```

```
ezplot (fexp*fosc, [0, 6, -1.2, 1.2])
```



Codes

```
clear
syms t a b
a=-1
b=2*pi
fexp=exp(a*t)
fosc=sin(b*t)
hold on
ez1=ezplot (fexp, [0, 6, -1.2, 1.2])
ez2=ezplot (fosc, [0, 6, -1.2, 1.2])
ez3=ezplot (fexp*fosc, [0, 6, -1.2, 1.2])
set(ez1,'color',[1 0 0])
set(ez2,'color',[0 1 0])
set(ez3,'color',[0 0 1])
```

A Spiral Sink with $\lambda = \alpha \pm i\beta$

$$X(t) = e^{\alpha t} (c_1 X_{re}(t) + c_2 X_{im})$$

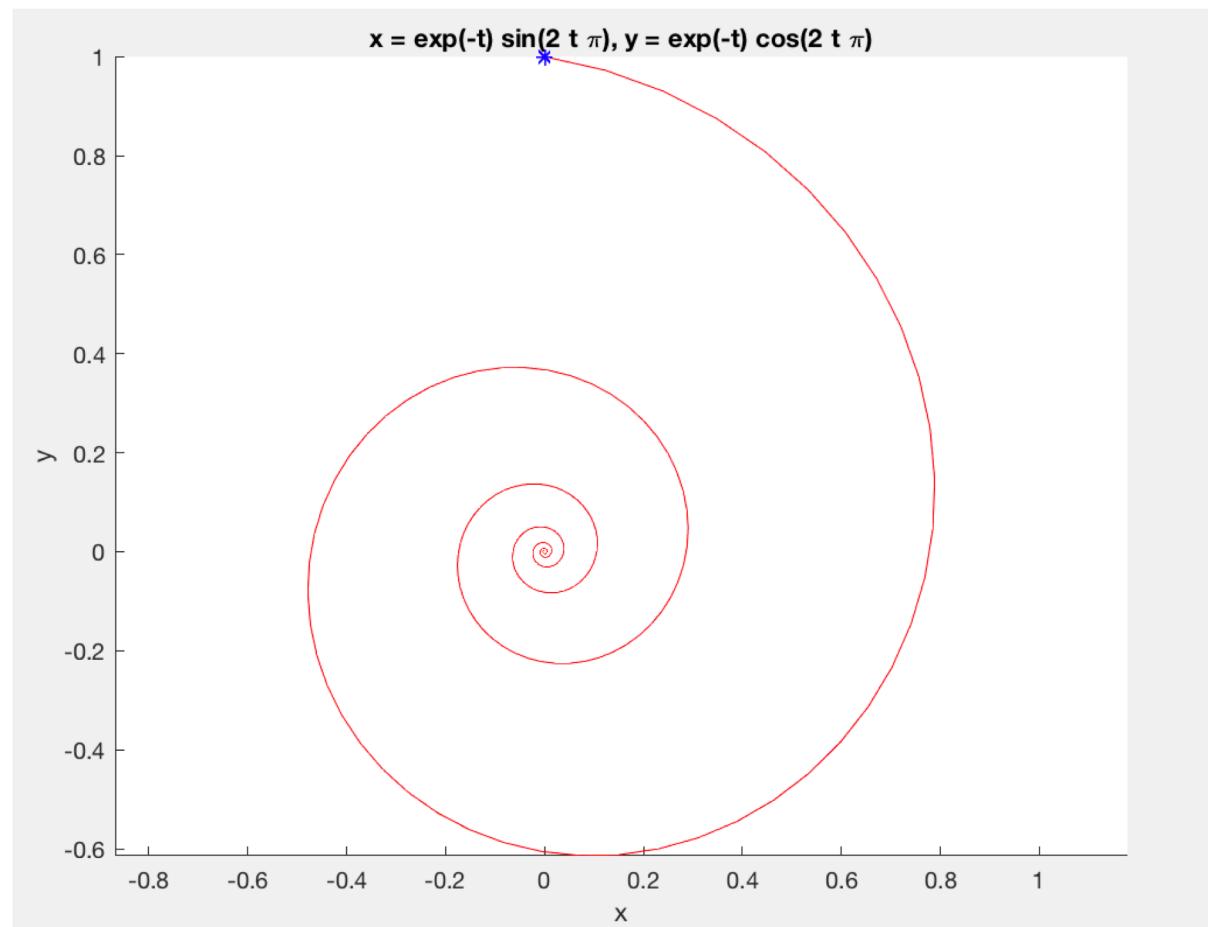
$$X_{im}(t) = \begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix}$$

$$c_1 = 0 ; c_2 = 1$$

$$x(t) = e^{\alpha t} \sin(\beta t)$$
$$y(t) = e^{\alpha t} \cos(\beta t)$$

```
clear
syms t a b x y
a=-1
b=2*pi
x=exp(a*t)*sin(b*t)
y=exp(a*t)*cos(b*t)

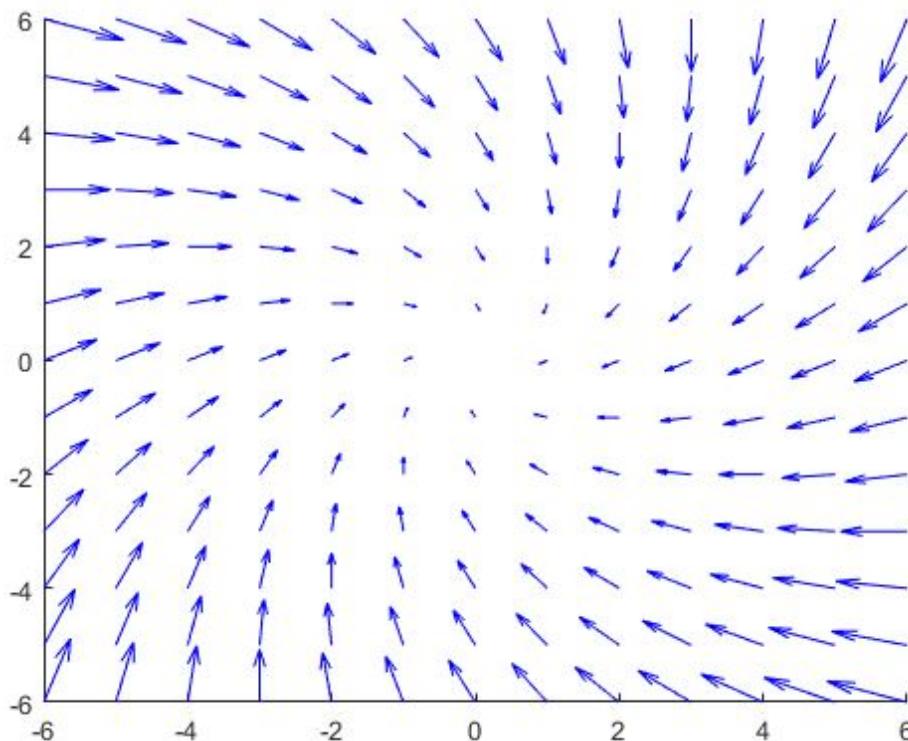
x0=0
y0=1
hold on
ez1=ezplot(x,y, [0, 6])
plot(x0, y0, 'b*')
set(ez1,'color',[1 0 0])
```



A Spiral Sink with $\lambda = \alpha \pm i\beta$

$$X' = -2x + y \quad Y' = -x - 2y$$

Vector Fields (at a given time)



MATLAB Plot for Figure 3.5a

$$\lambda_{1,2} = \alpha \pm i\beta \quad \alpha = -2 < 0$$

Trajectories (time varying)

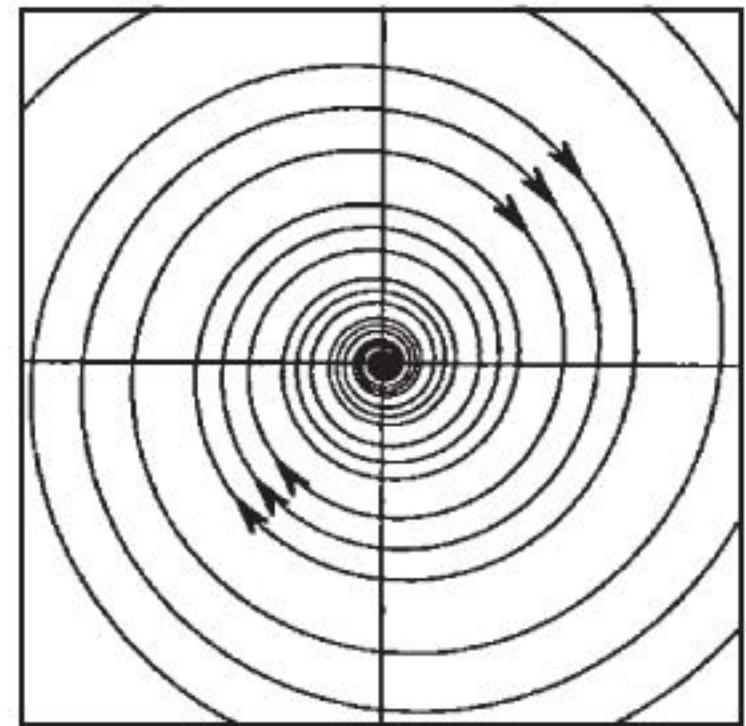


Figure 3.5a Phase portrait for spiral sink

A Spiral Source with $\lambda = \alpha \pm i\beta$

$$\lambda_{1,2} = \alpha \pm i\beta \quad \alpha = 2 > 0$$

$$X' = 2x + y$$

$$Y' = -x + 2y$$

Trajectories (time varying)

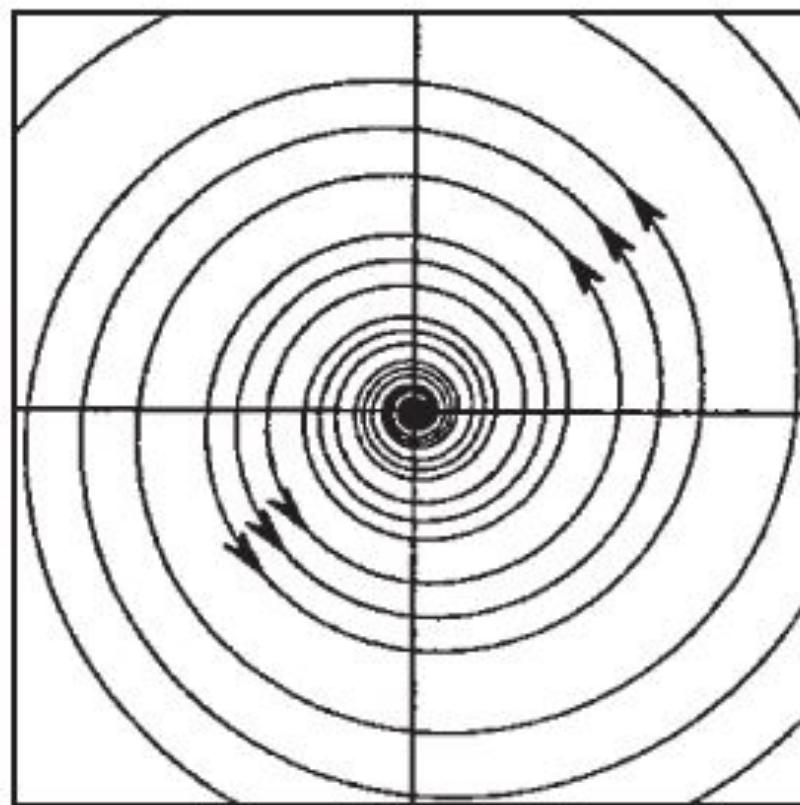


Figure 3.5b Phase portrait for spiral source

Is Weather Chaotic?

Coexistence of Chaos and Order within a Generalized Lorenz Model

by

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"A Paradigm Shift" in Predictability Study

- ``As with *Poincare* and *Birkhoff*, everything centers around *periodic solutions*'' (Lorenz, 1993).
- After Lorenz (1963, 1972), Prof. *Lorenz* and chaos advocates focused on the existence of *non-periodic solutions* and their complexities.
- Based on the concept of *attractor coexistence* within the original and generalized Lorenz models (Shen, 2019a), we (Shen et al., 2019) propose a revised view that focus on *the duality of chaos and order*.

Review: Double Root

5

$$ay'' + by' + cy = 0$$

where a , b , and c are constants and $a \neq 0$.

Summary of Cases I–III

Case	Roots of (2)	Basis of (1)	General Solution of (1)
II	Real double root $\lambda = -\frac{1}{2}a$	$e^{-ax/2}, xe^{-ax/2}$	$y = (c_1 + c_2x)e^{-ax/2}$

Sec 3.3 Repeated Eigenvalues

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\begin{aligned} x' &= \lambda_1 x \\ y' &= \lambda_1 y \end{aligned}$$

Uncoupled System

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$\begin{aligned} x' &= \lambda_1 x + y \\ y' &= \lambda_1 y \end{aligned}$$

Coupled System

x : responder
 y : driver

Sec 3.3 Repeated Eigenvalues: (I)

Consider

$$\begin{aligned}x' &= \lambda_1 x \\y' &= \lambda_1 y\end{aligned}\quad A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} \quad \text{Uncoupled}$$

Let $|A - \lambda I| = 0 \Rightarrow (\lambda - \lambda_1)^2 = 0$

$$\boxed{\lambda = \lambda_1}$$

$$AV_0 = \lambda V_0$$

$$\lambda_1 x_0 = \lambda x_0$$

$$\lambda_1 y_0 = \lambda y_0$$

$$\boxed{\lambda = \lambda_1}$$

$$\lambda_1 x_0 = \lambda_1 x_0 \quad \text{Any number}$$

$$\lambda_1 y_0 = \lambda_1 y_0 \quad \text{Any number}$$

$$V_0 = \begin{pmatrix} x \\ y \end{pmatrix}, \text{ any } x \text{ and } y \quad \text{any straight line through } (0,0)$$

Alternatively, $x = c_1 e^{\lambda_1 t}$ $y = c_2 e^{\lambda_1 t}$

Sec 3.3 Repeated Eigenvalues: (II)

Consider

$$\begin{aligned}x' &= \lambda_1 x + y \\y' &= \lambda_1 y\end{aligned}\quad A = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}$$

coupled

Let $|A - \lambda I| = 0 \Rightarrow (\lambda - \lambda_1)^2 = 0$

$$\boxed{\lambda = \lambda_1}$$

$$\begin{aligned}AV_0 &= \lambda V_0 \\ \lambda_1 x_0 + y_0 &= \lambda x_0 \\ \lambda_1 y_0 &= \lambda y_0\end{aligned}$$

$$\boxed{\lambda = \lambda_1}$$

$$\begin{aligned}y_0 &= 0 \\ \lambda_1 y_0 &= \lambda_1 y_0\end{aligned}\quad V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

How to find V_2 ?

- (1) Apply the concept of generalized eigenvalue problem
- (2) Solve the ODE for the driver that is uncoupled with the ODE for the responder; then solve the ODE for the responder
- (3) Transform the above system to a 2nd order ODE

Definition 5.6.1

We say that $\mathbf{u} \neq \mathbf{0}$ is a **generalized eigenvector** of \mathbf{A} associated to the eigenvalue λ if

$$(\mathbf{A} - \lambda \mathbf{I})^k \mathbf{u} = \mathbf{0} \quad (5.78)$$

for some integer $k > 0$. The index of the generalized eigenvector is the smallest k satisfying (5.78).

Here u may be V_2, V_3, \dots, V_k etc.

Let V_1 be the eigenvector of \mathbf{A} associated to the eigenvalue λ .
Namely, $(\mathbf{A} - \lambda \mathbf{I})V_1 = \mathbf{0}$. (a notation used in the HSD)

Consider u to be V_2 and $(\mathbf{A} - \lambda \mathbf{I})V_2 = V_1$. Therefore, we have
 $(\mathbf{A} - \lambda \mathbf{I})^2 V_2 = (\mathbf{A} - \lambda \mathbf{I})V_1 = \mathbf{0}$. V_2 is a generalized eigenvector.

Wirkus and Swift

Generalized Eigenspace

TBD

For $k > 1$, we see that the original eigenvector (\mathbf{v}) gives rise to a set of generalized eigenvectors ($\mathbf{u}_1, \mathbf{u}_2 \dots \mathbf{u}_k$):

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{u}_{k-1} = \mathbf{u}_{k-2},$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{u}_{k-2} = \mathbf{u}_{k-3}, \quad \dots,$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{u}_3 = \mathbf{u}_2,$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{u}_2 = \mathbf{u}_1,$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{u}_1 = \mathbf{v}.$$

The set $\{\mathbf{v}, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{k-1}\}$ satisfying (5.79) is called a **chain** of generalized eigenvectors. The chain is determined entirely by the choice of \mathbf{v} , which is referred to as the **bottom of the chain**. For those that have read Section 5.3, we define

$$\tilde{E}_\lambda = \{v | (\mathbf{A} - \lambda \mathbf{I})^k \mathbf{v} = \mathbf{0} \text{ for some } k\} \quad (5.80)$$

as the **generalized eigenspace** of λ . From (5.50), we note that $E_\lambda \subseteq \tilde{E}_\lambda$. If λ is an eigenvalue of \mathbf{A} of multiplicity m , then \tilde{E}_λ is a subspace of dimension m .

THEOREM 5.6.3

Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for the $k \times k$ matrix \mathbf{A} in which (i) λ is an eigenvalue of **multiplicity k** with a single eigenvector \mathbf{v} and (ii) $\{\mathbf{v}, \mathbf{u}_1, \dots, \mathbf{u}_{k-1}\}$ is the corresponding chain of generalized eigenvectors. Set

$$\begin{aligned}\mathbf{x}_1 &= te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{u}_1 \\ \mathbf{x}_2 &= \frac{t^2}{2!}e^{\lambda t}\mathbf{v} + te^{\lambda t}\mathbf{u}_1 + e^{\lambda t}\mathbf{u}_2 \\ &\vdots \\ \mathbf{x}_{k-1} &= \frac{t^{k-1}}{(k-1)!}e^{\lambda t}\mathbf{v} + \frac{t^{k-2}}{(k-2)!}e^{\lambda t}\mathbf{u}_1 + \dots + te^{\lambda t}\mathbf{u}_{k-2} + e^{\lambda t}\mathbf{u}_{k-1}^{(k)}.\end{aligned}$$

Then $e^{\lambda t}\mathbf{v}, \mathbf{x}_1, \dots, \mathbf{x}_{k-1}$ are linearly independent and the general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ can be written as

$$\mathbf{x} = c_1 e^{\lambda t} \mathbf{v} + c_2 \mathbf{x}_1 + c_3 \mathbf{x}_2 + \dots + c_k \mathbf{x}_{k-1}.$$

(1) Solutions using the Concept of Generalized Eigenvector

Solve for V_1 $(A - \lambda I)V_1 = 0$ $A = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}$ $\lambda = \lambda_1$ $V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Solve for V_2 $(A - \lambda I)V_2 = V_1$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad y = 1 \quad V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Obtain X_1 $X_1 = e^{\lambda t}V_1 = e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Construct X_2 $X_2 = te^{\lambda t}V_1 + e^{\lambda t}V_2 = e^{\lambda t} \begin{pmatrix} t \\ 0 \end{pmatrix} + e^{\lambda t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{\lambda t} \begin{pmatrix} t \\ 1 \end{pmatrix}$

Obtain X $X = \alpha X_1 + \beta X_2 = \alpha e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda t} \begin{pmatrix} t \\ 1 \end{pmatrix}$

(2) Solve the ODEs for the Driver and Responders

Consider

$$\begin{aligned}x' &= \lambda_1 x + y \\y' &= \lambda_1 y\end{aligned}$$

uncoupled

Start with the uncoupled equation

$$y' = \lambda_1 y$$

$$y = \beta e^{\lambda_1 t}$$

The 1st eq becomes:

$$x' - \lambda_1 x = \beta e^{\lambda_1 t}$$

$$x' + P(t)x = Q(t)$$

$$I = e^{-\lambda_1 t}$$

- Forced problem
- Non-autonomous system
- The system's “frequency” is the same as the “frequency” of the forcing.

$$x = \frac{1}{I} \left[\int IQ dt + C \right] = \frac{1}{I} (\beta t + C) = Ce^{\lambda_1 t} + \beta te^{\lambda_1 t}$$

(2) Solve the ODEs for the Driver and Responders

$$x = \frac{1}{I} \left[\int IQdt + C \right] = \frac{1}{I} (\beta t + C) = Ce^{\lambda_1 t} + \beta te^{\lambda_1 t}$$

$$y = \beta e^{\lambda_1 t}$$

$$X = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ce^{\lambda_1 t} + \beta te^{\lambda_1 t} \\ \beta e^{\lambda_1 t} \end{pmatrix} = Ce^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_1 t} \begin{pmatrix} t \\ 1 \end{pmatrix}$$

$$\lambda_1 < 0$$

$$\lim_{t \rightarrow \infty} te^{\lambda_1 t} = \lim_{t \rightarrow \infty} \frac{t}{e^{-\lambda_1 t}} = \lim_{t \rightarrow \infty} \frac{1}{(-\lambda_1)e^{-\lambda_1 t}} = 0$$

Low-D Non-autonomous \rightarrow High-D autonomous

Consider

$$x' - \lambda_1 x = \beta e^{\lambda_1 t}$$

Low dimensional,
Non-autonomous

Introduce a
new variable to
represent the
(time varying)
forcing term

$$y = \beta e^{\lambda_1 t}$$

$$x' = \lambda_1 x + y$$

$$y' = \lambda_1 y$$

$$\begin{aligned} x' &= \lambda_1 x + y \\ y' &= \lambda_1 y \end{aligned}$$

High dimensional,
autonomous

(3) Convert and Solve a 2nd Order ODE

Consider

$$\begin{aligned}x' &= \lambda_1 x + y \\y' &= \lambda_1 y\end{aligned}\quad \text{uncoupled}$$

Transform into
a 2nd order ODE

$$x'' = \lambda_1 x' + y'$$

$$x'' = \lambda_1 x' + \lambda_1 y \quad x'' = \lambda_1 x' + \lambda_1(x' - \lambda_1 x)$$

$$x'' - 2\lambda_1 x' + \lambda_1^2 x = 0$$

Assume

$$x = k e^{\lambda t}$$

Obtain

$$(\lambda - \lambda_1)^2 = 0 \quad \boxed{\lambda = \lambda_1}$$

$$\boxed{x_1 = e^{\lambda_1 t}}$$

$$\boxed{x_2 = t e^{\lambda_1 t} = \lim_{\lambda \rightarrow \lambda_1} \frac{e^{\lambda t} - e^{\lambda_1 t}}{\lambda - \lambda_1}}$$

Repeated Negative Eigenvalue

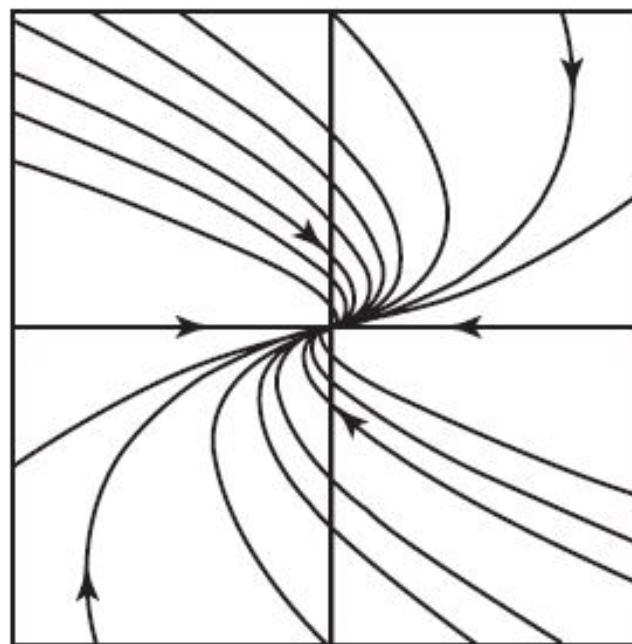


Figure 3.6 Phase portrait for a system with repeated negative eigenvalues.

A Summary: Repeated Eigenvalues

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$\begin{aligned} x' &= \lambda_1 x + y \\ y' &= \lambda_1 y \end{aligned}$$

Coupled System

y : driver

x : responder

$$\begin{matrix} V_1 & V_2 \end{matrix}$$

$$X = \alpha X_1 + \beta X_2 = \alpha e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda t} \begin{pmatrix} t \\ 1 \end{pmatrix}$$

How to find V_2 ?

- (1) Apply the concept of generalized eigenvalue problem
- (2) Solve the ODE for the driver that is uncoupled with the ODE for the responder; then solve the ODE for the responder
- (3) Transform the above system to a 2nd order ODE

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\begin{aligned} x' &= \lambda_1 x \\ y' &= \lambda_1 y \end{aligned}$$

Uncoupled System

$$x = c_1 e^{\lambda_1 t}$$

$$y = c_2 e^{\lambda_1 t}$$