9/26 2.3 MCT
2.4 sequential compactness.

## Last Class...

**Thm 2.25** MCT.

Suppose $\{a_n\}$ is monotone.

The sequence $\{a_n\}$ converges iff it is bounded.

When it converges,

if $\{a_n\}$ increasing, $\lim_{n\to\infty} a_n = \sup\{a_n\}$.

if $\{a_n\}$ decreasing, $\lim_{n\to\infty} a_n = \inf\{a_n\}$.

- Caution: Abuse of notation above!

## Thm 2.29 Nested Interval Thm.

Suppose $\{a_n\}$ and $\{b_n\}$ are such that ~~a~~ $\forall n$, $a_n < b_n$.

Define $I_n = [a_n, b_n]$

Suppose $\forall n$, $I_{n+1} \subseteq I_n$. (nested)

Suppose $\lim_{n\to\infty} (b_n - a_n) = 0$.

---

Then $\exists! x$ s.t. $\forall n$, $x \in I_n$ and

$$\lim_{n\to\infty} a_n = x = \lim_{n\to\infty} b_n .$$

---

proof: Let $n \in \mathbb{N}^*$.

Notice that $I_n \subseteq [a_1, b_1]$.

Thus $a_n \le b_1$ and $b_n \ge a_1$

So $\{a_n\}$ is bounded above by $b_1$

and $\{b_n\}$ ~~a~~ is bounded below by $a_1$.

Also since $I_{n+1} \subseteq I_n$, we have

$$a_n \leq a_{n+1} < b_{n+1} \leq b_n .$$

Thus $\{a_n\}$ is increasing & bounded above <u>and</u>

$\{b_n\}$ is decreasing & bounded below.

Thus $\lim_{n\to\infty} \{a_n\} = a$ and $\lim_{n\to\infty} b_n = b$. by MCT.

Also $\lim_{n\to\infty} b_n - a_n = b - a = 0$. So $a = b = x$ in statement.

Suppose for uniqueness, $\forall n \in \mathbb{N}^+$, $y \in I_n$.

Suppose $x \neq y$. WLOG suppose $x < y$.

Notice $\forall n$, $a_n \leq y \leq b_n$. So $y$ is a lower bound for $\{b_n\}$.

By the MCT, $x = \inf\{b_n\}$. (greatest lower bound).

Since $y$ is a lower bound <u>and</u> $y > x$, we contradict

$x = \inf\{b_n\}$. $\Rightarrow\Leftarrow$.

## 2.4 Subsequences & sequential compactness

**Definition** Suppose $\{a_n\} \subseteq \mathbb{R}$ is a sequence.

Let $n_1, n_2, \ldots$ be a strictly increasing sequence of natural numbers.

Then $b_k = a_{n_k}$ define terms of a subsequence of $\{a_n\}$. Usually shorthanded as $\{a_{n_k}\}$.

Ex: $\{(-1)^n\}_{n=1}^{\infty} = \{a_n\}$.

$n_1 = 1, \; n_2 = 3, \; n_3 = 5, \; \ldots$

Then $\{a_{n_k}\} = \{(-1)^{2k-1}\}_{k=1}^{\infty}$

<u>Prop</u>: Suppose $\{a_n\}$ is a convergent sequence. s.t.

$\lim_{n\to\infty} a_n = a$. Every subsequence also converges to $a$.

<u>Proof</u>: Let $\{a_{n_k}\}$ be an arbitrary subsequence.

<u>Show</u>: $\lim_{k\to\infty} a_{n_k} = a$.

$\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t. $\forall k \geq K$, $|a_{n_k} - a| < \varepsilon$.

Let $\varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|a_n - a| < \varepsilon$.

Since $\{n_k\}$ is strictly increasing, $\exists K$ s.t.

$n_K > N$. Let $k \geq K$. Then $n_k > n_K > N$.

Thus $|a_{n_k} - a| < \varepsilon$.