Laplace
Differential Equations
Math 337
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Problem 18: Consider the initial value problem:

$$y' + 3y = \begin{cases} 0 & 0 \leq t < 1 \\ 12 & 1 \leq t < 5 \\ 0 & 5 \leq t < \infty \end{cases}, \qquad y(0) = 8.$$

(a) Take the Laplace transform of both sides of the given differential equation to create the corresponding algebraic equation. Denote the Laplace transform of $y(t)$ by $Y(s)$. Do not move any terms from one side of the equation to the other (until you get to part (b) below).

Using the definition of Laplace Transforms of derivatives:

$$\mathcal{L}(y' + 3y) = sY(s) - y(0) + 3Y(s) = sY(s) - 8 + 3Y(s)$$

We can convert the piece-wise into the following Heaviside function:

$$12(h(t-1) - h(t-5))$$

Using the definition of Laplace Transforms of Heaviside functions:

$$\mathcal{L}(12(h(t-1) - h(t-5))) = \frac{12(e^{-s} - e^{-5s})}{s}$$

Thus we get the following equality:

$$sY(s) - 8 + 3Y(s) = 12(e^{-s} - e^{-5s})$$

(b) Solve your equation for $Y(s)$

Through simple algebra, we get:

$$Y(s) = \frac{12e^{-s}}{s(s+3)} + \frac{12e^{-5s}}{s(s+3)} + \frac{8}{s+3}$$

(c) Take the inverse Laplace transform of both sides of the previous equation to solve for $y(t)$. Use $h(t-a)$ for the Heaviside function shifted a units horizontally. (Class notes have $u_a(t) = h(t-a)$.)

Notice the partial fraction decomposition:

$$\frac{12}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3} = \frac{(A+B)s+3A}{12}$$

So we get $A = 4$ and $B = -4$

$$\frac{12}{s(s+3)} = \frac{4}{s} - \frac{4}{s+3}$$

So now to find $y(t)$, we take the inverse Laplace Transform of each term:

$$\begin{aligned}\mathcal{L}^{-1}(Y(s)) &= \mathcal{L}^{-1}\left(e^{-s}\frac{4}{s}\right) - \mathcal{L}^{-1}\left(e^{-s}\frac{4}{s+3}\right) + \mathcal{L}^{-1}\left(e^{-5s}\frac{4}{s}\right) - \mathcal{L}^{-1}\left(e^{-5s}\frac{4}{s+3}\right) + \mathcal{L}^{-1}\left(\frac{8}{s+3}\right) \\ &= 4h(t-1) - 4h(t-1)e^{-3(t-1)} + 4h(t-5) - 4h(t-5)e^{-3(t-5)} + 8e^{-3t} \\ &= 4h(t-1)(1-e^{-3(t-1)}) + 4h(t-5)(1-e^{-3(t-5)}) + 8e^{-3t}\end{aligned}$$

Problem 23: Consider the initial value problem:

$$y'' + 16y = 64t, \qquad y(0) = 8, \qquad y'(0) = 2.$$

(a) Take the Laplace transform of both sides of the given differential equation to create the corresponding algebraic equation. Denote the Laplace transform of $y(t)$ by $Y(s)$. Do not move any terms from one side of the equation to the other (until you get to part (b) below).

Using the definition of Laplace Transforms of derivatives:

$$\begin{aligned}\mathcal{L}(y'' + 16y) &= s^2Y(s) - sy(0) - y'(0) + 16Y(s) = s^2Y(s) - 8s - 2 + 16Y(s)\\ \mathcal{L}(64t) &= \frac{64}{s^2}\end{aligned}$$

Thus we get the following equality:

$$(s^2 + 16)Y(s) - (8s + 2) = \frac{64}{s^2}$$

(b) Solve your equation for $Y(s)$

Through simple algebra, we get:

$$Y(s) = \frac{64}{s^2(s^2+16)} + \frac{8s}{s^2+16} + \frac{2}{s^2+16}$$

(c) Take the inverse Laplace transform of both sides of the previous equation to solve for $y(t)$.

Notice the partial fraction decomposition:

$$\begin{aligned}\frac{64}{s^2(s^2+16)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2+16}\\ 64 &= As^3 + 16As + Bs^2 + 16B + Cs^3 + Ds^2\\ &= (A + C)s^3 + (B + D)s^2 + 16As + 16B\end{aligned}$$

So we get $B = 4, D = -4, A = C = 0$

$$\frac{64}{s^2(s^2+16)} = \frac{4}{s^2} - \frac{4}{s^2+16}$$

So now to find $y(t)$, we take the inverse Laplace Transform of each term:

$$\begin{aligned}\mathcal{L}^{-1}(Y(s)) &= \mathcal{L}^{-1}\left(\frac{4}{s^2}\right) + \mathcal{L}^{-1}\left(\frac{-4}{s^2+16}\right) + \mathcal{L}^{-1}\left(\frac{8s}{s^2+16}\right) + \mathcal{L}^{-1}\left(\frac{2}{s^2+16}\right)\\ &= \mathcal{L}^{-1}\left(\frac{4}{s^2}\right) - \frac{1}{2}\mathcal{L}^{-1}\left(\frac{4}{s^2+16}\right) + \mathcal{L}^{-1}\left(\frac{8s}{s^2+16}\right)\\ &= 4t - \frac{1}{2}\sin(4t) + 8\cos(4t)\end{aligned}$$

Problem 24: Consider the initial value problem:

$$y'' + 25y = \cos(5t), \qquad y(0) = 6, \qquad y'(0) = 9.$$

(a) Take the Laplace transform of both sides of the given differential equation to create the corresponding algebraic equation. Denote the Laplace transform of $y(t)$ by $Y(s)$. Do not move any terms from one side of the equation to the other (until you get to part (b) below).

Using the definition of Laplace Transforms of derivatives:

$$\begin{aligned}\mathcal{L}(y'' + 25y) &= s^2Y(s) - sy(0) - y'(0) + 25Y(s) = s^2Y(s) - 6s - 9 + 25Y(s)\\ \mathcal{L}(\cos(5t)) &= \frac{s}{s^2+25}\end{aligned}$$

Thus we get the following equality:

$$(s^2+25)Y(s) - (6s+9) = \frac{s}{s^2+25}$$

(b) Solve your equation for $Y(s)$

Through simple algebra, we get:

$$Y(s) = \frac{s}{(s^2+25)^2} + \frac{6s}{s^2+25} + \frac{9}{s^2+25}$$

(c) So now to find $y(t)$, we take the inverse Laplace Transform of each term:

$$\mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{s}{(s^2+25)^2}\right) + \mathcal{L}^{-1}\left(\frac{6s}{s^2+25}\right) + \mathcal{L}^{-1}\left(\frac{9}{s^2+25}\right)$$

Notice the following ($\mathcal{L}^{-1}(-F'(s)) = t\mathcal{L}(F(s))$):

$$\begin{aligned}\frac{d}{dx}\left(\frac{-1}{2(s^2+25)}\right) &= \frac{s}{(s^2+25)^2}\\ \mathcal{L}^{-1}\left(\frac{s}{(s^2+25)^2}\right) = -t\mathcal{L}\left(\frac{-1}{2(s^2+25)}\right) &= \frac{t}{10}\sin(5t)\end{aligned}$$

Thus we get the equality:

$$y(t) = \frac{t}{10}\sin(5t) + 6\cos(5t) + \frac{9}{5}\sin(5t)$$

Problem 25: Consider the initial value problem:

$$y'' + 16y = \begin{cases} t & 0 \le t < 3 \\ 0 & 3 \le t < \infty \end{cases}, \qquad y(0) = 0, \qquad y'(0) = 0.$$

(a) Take the Laplace transform of both sides of the given differential equation to create the corresponding algebraic equation. Denote the Laplace transform of $y(t)$ by $Y(s)$. Do not move any terms from one side of the equation to the other (until you get to part (b) below).

Using the definition of Laplace Transforms of derivatives:

$$\mathcal{L}(y'' + 16y) = s^2Y(s) - sy(0) - y''(0) + 16Y(s) = s^2Y(s) + 16Y(s)$$

We can convert the piece-wise into the following Heaviside function:

$$t(h(t) - h(t-3))$$

Using the definition of Laplace Transforms of Heaviside functions:

$$\mathcal{L}(t(h(t) - h(t-3))) = \frac{1}{s^2} - \frac{e^{-3s}}{s^2} - \frac{3e^{-3s}}{s}$$

Thus we get the following equality:

$$(s^2 + 16)Y(s) = \frac{1}{s^2} - \frac{e^{-3s}}{s^2} - \frac{3e^{-3s}}{s}$$

(b) Solve your equation for $Y(s)$

Through simple algebra, we get:

$$Y(s) = \frac{1}{s^2(s^2+16)} - \frac{e^{-3s}}{s^2(s^2+16)} - \frac{3e^{-3s}}{s(s^2+16)}$$

(c) Take the inverse Laplace transform of both sides of the previous equation to solve for $y(t)$. Use $h(t-a)$ for the Heaviside function shifted a units horizontally. (Class notes have $u_a(t) = h(t-a)$.)

Notice the partial fraction decomposition:

$$\begin{aligned}\frac{1}{s^2(s^2+16)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+16} \\ 1 &= As^3 + 16As + Bs^2 + 16B + Cs^3 + Ds^2 \\ &= (A+C)s^3 + (B+D)s^2 + 16As + 16B\end{aligned}$$

So we get $B = \frac{1}{16}, D = -\frac{1}{16}, A = C = 0$

$$\frac{1}{s^2(s^2+16)} = \frac{1}{16s^2} - \frac{1}{16(s^2+16)}$$

Notice the another partial fraction decomposition:

$$\begin{aligned}\frac{3}{s(s^2+16)} &= \frac{A}{s} + \frac{Bs+C}{s^2+16} \\ 3 &= As^2 + 16A + Bs^2 + Cs\end{aligned}$$

So we get $A = \frac{3}{16}, B = \frac{-3}{16}, C = 0$

$$\frac{3}{s(s^2+16)} = \frac{3}{16s} + \frac{-3s}{16(s^2+16)}$$

So now to find $y(t)$, we take the inverse Laplace Transform of each term:

$$\begin{aligned}\mathcal{L}^{-1}(Y(s)) &= \mathcal{L}^{-1}\left(\frac{1}{16s^2}\right) - \mathcal{L}^{-1}\left(\frac{1}{16(s^2+16)}\right) - \mathcal{L}^{-1}\left(\frac{e^{-3s}}{16s^2}\right) + \mathcal{L}^{-1}\left(\frac{e^{-3s}}{16(s^2+16)}\right) \\ &\quad - \mathcal{L}\left(\frac{3e^{-3s}}{16s}\right) + \mathcal{L}\left(\frac{3se^{-3s}}{16(s^2+16)}\right) \\ &= \frac{t}{16} - \frac{\sin(4t)}{64} + \frac{h(t-3)\sin(4(t-3))}{64} - \frac{h(t-3)(t-3)}{16} \\ &\quad - \frac{3h(t-3)}{16} + \frac{3h(t-3)\cos(4(t-3))}{16}\end{aligned}$$