Today 9/17 Convergence Facts 2.1 text

Def. We say $\lim_{n\to\infty} a_n = a$

iff

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$,

if $n \geq N$, then $|a_n - a| < \varepsilon$.

Example: Prove $\lim_{n\to\infty} \left(\frac{2}{\sqrt{n}} + \frac{1}{n} + 3\right) = 3$

Scratch Want: $\left|\frac{2}{\sqrt{n}} + \frac{1}{n} + \cancel{3} - \cancel{3}\right| < \varepsilon$. Get $n > ??$

$$\left|\frac{2\sqrt{n}+1}{n}\right| < \varepsilon$$

$$\frac{9}{\varepsilon^2} < n$$

$$\left|\frac{2\sqrt{n}+1}{n}\right| \leq \left|\frac{2\sqrt{n}+\sqrt{n}}{n}\right| = \left|\frac{3}{\sqrt{n}}\right| < \varepsilon$$

$$\frac{3}{\varepsilon} < \sqrt{n}.$$

Proof: Let $\varepsilon > 0$.

Let $N \in \mathbb{N}$ s.t. $N \geq \frac{9}{\varepsilon^2}$.

Let $n \in \mathbb{N}$ and suppose $n > N \geq \frac{9}{\varepsilon^2}$.

Since $n > \frac{9}{\varepsilon^2} > 0$, we know $\sqrt{n} > \frac{3}{\varepsilon}$.

So $$\varepsilon > \frac{3}{\sqrt{n}} \geq \frac{2\sqrt{n}+1}{n}$$

So $$\varepsilon > \left| \frac{2}{\sqrt{n}} + \frac{1}{n} + 3 - 3 \right|.$$

$\boxtimes$

Lemma 2.9   Comparison Lemma

Suppose $\lim_{n\to\infty} a_n = a$. (Known, simple)

If $\exists C > 0$, $N_1 \in \mathbb{N}$ s.t. $\forall n \geq N_1$

$$|b_n - b| \leq C\,|a_n - a|,$$

then $\lim_{n\to\infty} b_n = b$. (more complex, we estimated).

<u>proof</u>: Suppose $\exists C > 0$, $N_1 \in \mathbb{N}$ s.t. $\forall n \geq N_1$

$$|b_n - b| \leq C\,|a_n - a|.$$

Let $\varepsilon > 0$.

Since $\lim_{n\to\infty} a_n = a$, $\exists N_2 \in \mathbb{N}$ s.t. $\forall n > N_2$,

$$|a_n - a| < \frac{\varepsilon}{C}$$

Let $N = \max\{N_1, N_2\}$.

Suppose $n \in \mathbb{N}$ and $n \geq N$.

Then $|b_n - b| \leq C|a_n - a|$ (since $n \geq N_1$).

$< C\left(\frac{\varepsilon}{C}\right)$ (since $n \geq N_2$).

Thus $|b_n - b| < \varepsilon$. ∎

Boundedness Lemma (Thm 2.18)

Suppose $\lim_{n \to \infty} a_n = a$.

① $\exists M > 0$ s.t. $\forall n$, $|a_n| < M$ and $|a| < M$.

② If $a \neq 0$, then $\exists \beta > 0$ and $N \in \mathbb{N}$ such that $\forall n \geq N$ we have $|a_n| > \beta$ and $|a| > \beta$.

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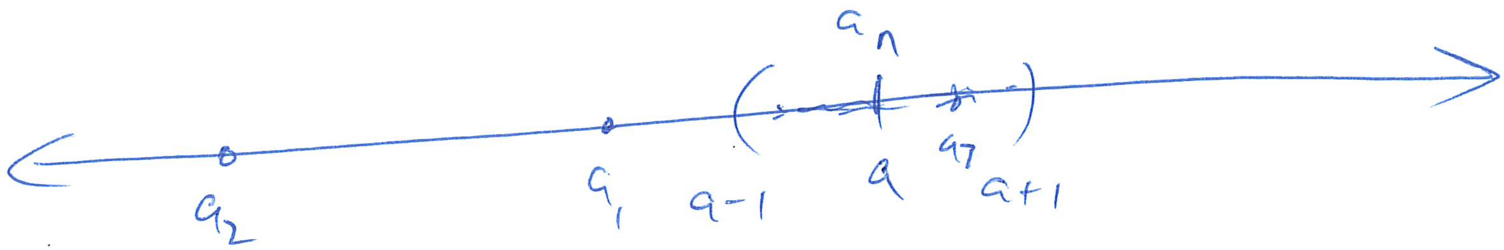
(1)



N_1

$|a_1|, |a_2|, \ldots, |a_{N_1-1}|$ $\qquad$ $|a_n - a| < 1$

For $n \geq N_1$, $\qquad$ $|a_n| - |a| <$

$|a_n| < |a| + 1.$

~~Let M~~ **Proof:** Since $\lim_{n\to\infty} a_n = a$, $\exists N_1 \in \mathbb{N}$ s.t. $\forall n \geq N_1$,

$$|a_n - a| < 1.$$

Notice that for any $n \geq N_1$ we have

$$|a_n| - |a| \leq |a_n - a| < 1$$

and so $\quad |a_n| < 1 + |a|$. when $n \geq N_1$.

Let $M = \max\{|a_1|, \dots, |a_{N_1-1}|, |a|+1\}$.

By the construction of N_1, M the result follows.

(2) Since $\lim_{n\to\infty} a_n = a$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$

we have $|a_n - a| < \frac{|a|}{2}$.

(We are showing $\forall n \geq N$, $|a_n| > \frac{|a|}{2} = \beta$).

So

$$-\frac{|a|}{2} < a_n - a < \frac{|a|}{2}$$

So

$$a - \frac{|a|}{2} < a_n < a + \frac{|a|}{2}.$$

Case 1: Suppose $a > 0$.

Then $a - \frac{|a|}{2} = \frac{a}{2} < a_n$.

Also note $\frac{a}{2} < |a_n|$ (since $a > 0$).

<u>Case 2:</u> Suppose $a < 0$.

Then $a_n < a + \frac{|a|}{2}$.

So $a_n < a - \frac{a}{2}$

So $a_n < \frac{a}{2} < 0$.

So $-a_n > \frac{|a|}{2}$

So $|a_n| > \frac{|a|}{2}$.

<u>By cases 1 & 2</u>, $\forall n \geq N$, $|a_n| > \frac{|a|}{2} = \beta$.

Properties of Limits

Linearity (Thm 2.10, Lemma 2.11, Prop 2.16)

Suppose $a_n \to a$ and $b_n \to b$. Suppose $K \in \mathbb{R}$.

Then $\lim_{n\to\infty} (a_n + K b_n) = a + Kb$.

proof: Let $\varepsilon > 0$.

Then $\exists N_1$ s.t. $\forall n \geq N_1$, $|a_n - a| < \frac{\varepsilon}{2}$.

If $K=0$, the fact is trivial. Suppose $K \neq 0$.

Also $\exists N_2$ s.t. $\forall n \geq N_2$, $|b_n - b| < \frac{\varepsilon}{2|K|}$.

Let $N = \max\{N_1, N_2\}$. Suppose $n \geq N$.

So $|a_n + Kb_n - (a+Kb)| \leq |a_n - a| + |K|\,|b_n - b|$

Since $n \geq N_1$ and $n \geq N_2$ $\quad < \frac{\varepsilon}{2} + \cancel{|K|}\frac{\varepsilon}{\cancel{|K|}\cdot 2} = \varepsilon$. $\boxed{\text{Ed}}$.