Note: For full credit you must show intermediate steps in your calculations.

1. (8pts) The lecture notes examined the negative feedback of glucose and insulin. A classic enzymatic negative feedback model satisfies the system:

$$\begin{aligned}\dot{x}_1 &= \frac{3}{1+0.2x_2} - 0.5x_1,\\ \dot{x}_2 &= 5x_1 - x_2,\end{aligned}$$

where x_1 is an enzyme and x_2 is the endproduct. Find the positive equilibrium for this model ($x_1 > 0$ and $x_2 > 0$). Compute the Jacobian Matrix for this system. Evaluate the Jacobian matrix at the equilibrium. Determine the eigenvalues for this model and determine the qualitative behavior of this model near the equilibrium. Sketch a phase portrait for this model for non-negative x_1 and x_2 ($x_1 \geq 0$ and $x_2 \geq 0$). (Slides 25-31)

To find the equilibria, x_{1e} and x_{2e}, we set the derivatives equal to zero, so

$$x_{1e} = \frac{6}{1+0.2x_{2e}} \qquad \text{and} \qquad x_{2e} = 5x_{1e}.$$

Thus, we have:

$$x_{1e} = \frac{6}{1+x_{1e}} \qquad \text{or} \qquad x_{1e}^2 + x_{1e} - 6 = 0,$$

which gives $x_{1e} = -3$ or 2. Thus, $x_{2e} = -15$ or 10. The positive equilibrium is $(x_{1e}, x_{2e}) = (2, 10)$.

The Jacobian matrix satisfies:

$$J(x_1, x_2) = \begin{pmatrix} -0.5 & -\frac{0.6}{(1+0.2x_2)^2} \\ 5 & -1 \end{pmatrix},$$

which when evaluated at $(x_{1e}, x_{2e}) = (2, 10)$ gives:

$$J(2, 10) = \begin{pmatrix} -0.5 & -\frac{0.2}{3} \\ 5 & -1 \end{pmatrix}.$$

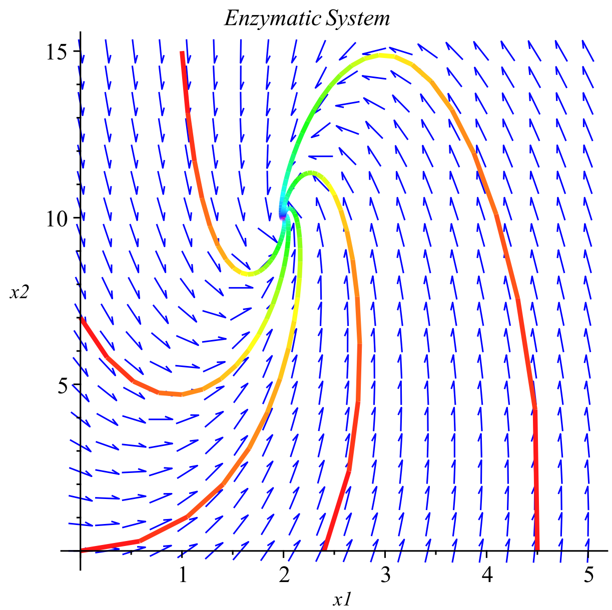
The eigenvalues satisfy:

$$\begin{vmatrix} -0.5-\lambda & \frac{-0.2}{3} \\ 5 & -1-\lambda \end{vmatrix} = (\lambda+1)(\lambda+0.5) + \tfrac{1}{3} = \lambda^2 + \tfrac{3}{2}\lambda + \tfrac{5}{6} = 0.$$

It follows that

$$\lambda = -\tfrac{3}{4} \pm i\tfrac{\sqrt{39}}{12} \approx -0.75 \pm 0.52042i,$$

which implies that $(x_{1e}, x_{2e}) = (2, 10)$ is a stable spiral. Below is a graph showing the phase portrait of this classic enzymatic negative feedback model.



2. (8pts) A very popular ecological model is the predator-prey model (Lotka-Volterra). Consider the system of ODEs:

$$\begin{aligned} \dot{x}_1 &= 0.1x_1 - 0.05x_1x_2, \\ \dot{x}_2 &= 0.001x_1x_2 - 0.04x_2. \end{aligned}$$

Associate each variable with the prey or the predator and explain briefly your reasoning. Find all equilibria for this model. Compute the Jacobian Matrix for this system. Evaluate the Jacobian matrix at all of the equilibria. Determine the eigenvalues at each of the equilibria for this model and determine the qualitative behavior of this model near these equilibria. Sketch a phase portrait for this model for non-negative x_1 and x_2 ($x_1 \geq 0$ and $x_2 \geq 0$).

The prey is given by x_1, while x_2 is the predator. We see that the prey exhibits Malthusian growth in the absence of the predator and loses population in the presence of the predator. The differential equation for the predator shows that the predator population declines without prey, while its population grows in the presence of prey.

As in the previous problem, the equilibria are found by solving the derivatives equal to zero ($\dot{x}_1 = \dot{x}_2 = 0$). We factor and obtain the equations to be solved simultaneously:

$$0.1x_{1e}(1 - 0.5x_{2e}) = 0 \qquad \text{and} \qquad 0.01x_{2e}(0.1x_{1e} - 4) = 0.$$

The first equation gives either $x_{1e} = 0$ or $x_{2e} = 2$. From the second equation, if $x_{1e} = 0$, then $x_{2e} = 0$ (extinction equilibrium). However, if $x_{2e} = 2$, then $x_{1e} = 40$ (coexistence equilibrium).

The Jacobian matrix is given by:

$$J(x_1, x_2) = \begin{pmatrix} 0.1 - 0.05x_2 & -0.05x_1 \\ 0.001x_2 & 0.001x_1 - 0.04 \end{pmatrix}.$$

At the extinction equilibrium $(0, 0)$, we have

$$J(0, 0) = \begin{pmatrix} 0.1 & 0 \\ 0 & -0.04 \end{pmatrix},$$

which has eigenvalues $\lambda_1 = 0.1$ and $\lambda_2 = -0.04$ (diagonal elements), so is a *saddle node*. At the coexistence equilibrium $(40, 2)$, we have

$$J(40, 2) = \begin{pmatrix} 0 & -2 \\ 0.002 & 0 \end{pmatrix}, \qquad \text{with c.e.} \quad \lambda^2 + 0.004 = 0,$$

which has eigenvalues $\lambda = \pm i\sqrt{0.004} = \pm 0.01\sqrt{40}i \approx \pm 0.0632456i$, so is a *center* (neurally stable). Below is a graph showing the phase portrait of this classic enzymatic negative feedback model.