Homework 7.1
Linear Algebra
Math 524
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Problem 7.A.1: Suppose n is a positive integer. Define $T \in \mathcal{L}(\mathbb{F}^n)$ by

$$T(z_1, ..., z_n) = (0, z_1, ..., z_{n-1})$$

Find a formula for $T^*(z_1, ..., z_n)$.

Notice the following:

$$\begin{aligned}
\langle T(x_1, ..., x_n), (z_1, ..., z_n)\rangle &= \langle (0, x_1, ..., x_{n-1}), (z_1, ..., z_n)\rangle \\
&= x_1 z_2 + ... + x_{n-1} z_n + x_n(0) \\
&= \langle (x_1, ..., x_{n-1}, x_n), (z_2, ..., z_n, 0)\rangle \\
&= \langle (x_1, ..., x_n), T^*(z_1, ..., z_n)\rangle
\end{aligned}$$

Thus $T^*(z_1, ..., z_n) = (z_2, ..., z_{n-1}, 0)$

Problem 7.A.4: Suppose $T \in \mathcal{L}(V, W)$. Prove that

(a) T is injective if and only if T^* is surjective.

(=>). Let T be injective.

Because T is injective, $\operatorname{null} T$ =(rangeT^*)$^\perp$ $= \{0\}$. Because (rangeT^*)$^\perp$ $= \{0\}$, that means (rangeT^*)$= W$, thus proving that T^* is surjective.

(<=) Let T^* be surjective.

Because T^* is surjective, range$T^* = W$, such that (rangeT^*)$^\perp = \operatorname{null} T = \{0\}$. Because null$T = \{0\}$, T is injective.

$\square$

(b) T is surjective if and only if T^* is injective.

(=>). Let T be surjective.

Because T is surjective, range$T = W$ and (range$T$)$^\perp$ $=$ nullT^* $= \{0\}$. Because null$T^* = \{0\}$. Thus T^* is injective.

Let T^* be injective.

Because T^* is injective, $\operatorname{null} T^*$ =(range$T$)$^\perp$ $= \{0\}$. Because (range$T$)$^\perp$ $= \{0\}$, that means (range$T$)$= W$, thus proving that T is surjective. $\square$

Problem 7.A.6: Make $\mathcal{P}_2(\mathbb{R})$ into an inner product space by defining

$$\langle p, q\rangle = \int_0^1 p(x)q(x)$$

Define $T \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$ by $T(a_0 + a_1x + a_2x^2) = a_1x$

(a) Show that T is not self-adjoint.

Notice that self adjoint means $\langle Tp, q\rangle = \langle p, Tq\rangle$.

Let $p = p_0 + p_1x + p_2x^2$ and $q = q_0 + q_1x + q_2x^2$

$$\begin{aligned}\langle Tp, q\rangle &= \langle p_1x, q_0 + q_1x + q_2x^2\rangle \\ &= \int_0^1 p_1q_0x + p_1q_1x^2 + p_1q_2x^3 \\ &= p_1\left(\frac{q_0x^2}{2} + \frac{q_1x^3}{3} + \frac{q_2x^4}{4}\right)\Bigg|_0^1 \\ &= p_1\left(\frac{q_0}{2} + \frac{q_1}{3} + \frac{q_2}{4}\right)\end{aligned}$$

$$\begin{aligned}\langle p, Tq\rangle &= \langle p_0 + p_1x + p_2x^2, q_1x\rangle \\ &= \int_0^1 q_1p_0x + q_1p_1x^2 + q_1p_2x^3 \\ &= q_1\left(\frac{p_0x^2}{2} + \frac{p_1x^3}{3} + \frac{p_2x^4}{4}\right)\Bigg|_0^1 \\ &= q_1\left(\frac{p_0}{2} + \frac{p_1}{3} + \frac{p_2}{4}\right)\end{aligned}$$

As long as $p_0, q_0 \neq 0$, and $q_i \neq p_i$ for $i = \{0, 1, 2\}$, T is not self-adjoint.

(b) The matrix of T with respect to the basis $(1, x, x^2)$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This matrix equals its conjugate transpose, even though T is not self-adjoint. Explain why this is not a contradiction.

This is an assumption that the basis $(1, x, x^2)$ is an orthonormal basis. Notice:

$$\begin{aligned}\langle 1 + x + x^2, 1 + x + x^2\rangle &= \int_0^1 x^4 + 2x^3 + 3x^2 + 2x + 1 \\ &= \frac{x^5}{5} + \frac{x^4}{2} + x^3 + x^2 + x\Bigg|_0^1 \\ &= 3.7 \neq 0\end{aligned}$$

Because the inner product with itself is not 0, it is not an orthonormal basis, thus allowing the matrix to be equal with its conjugate transpose.

Problem 7.A.14: Suppose T is a normal operator on V. Suppose also that $v, w \in V$ satisfy the equations

$$||v|| = ||w|| = 2 \qquad\qquad Tv = 3v \qquad\qquad Tw = 4w$$

Show that $||T(v+w)|| = 10$

Because $||Tv|| \neq ||Tw||$, we know that v, w are orthogonal, or form a right triangle, in which we can use the Pythagorean theorem:

$$\begin{aligned}
\sqrt{||T(v+w)^2||} &= \sqrt{||Tv + Tw||^2} \\
&= \sqrt{||3v + 4w||^2} \\
&= \sqrt{||3v||^2 + ||4w||^2} \\
&= \sqrt{9||v||^2 + 16||w||^2} \\
&= \sqrt{9(4) + 16(4)} = 10
\end{aligned}$$

Problem 7.B.2: Suppose that T is a self-adjoint operator on a finite-dimensional inner product space and that 2 and 3 are the only eigenvalues of T. Prove that $T^2 - 5T + 6I = 0$.

Because T is self-adjoint, there exists an orthonormal basis $e_1, ..., e_n$ that consists of eigenvectors such that for any $i \in \{1, ..., n\}$, $(T - \lambda I)e_i = 0$, for all eigenvalues, λ , of T.

$$(T^2 - 5T + 6I)e_i = (T - 2I)(T - 3I)e_i = 0$$

Because $e_i \neq 0$, then $T^2 - 5T + 6I = 0$.

Problem 7.B.6: Prove that a normal operator on a complex inner product space is self-adjoint if and only if all its eigenvalues are real.

(=>). Let T be a complex inner product space that is self-adjoint.

By 7.13, every eigenvalue of a self-adjoint operator is real.

(<=) Let all the eigenvalues of some operator, T, be real.

Thus there exists a matrix in respect to its basis, $\mathcal{M}(T)$, with its eigenvalues on its diagonals. Thus $\mathcal{M}(T)$ equals its transpose, thus T is self-adjoint. $\square$