Oct 16, 2024

Review:

HW - 1 : Q1

h - height

a - accelation due to gravity

m - mass

V_0 = initial velocity.

$$h = a^x m^y V_0^z$$

$$[L]^1 = \left[\frac{L}{T^2}\right]^x \cdot [M]^y \left[\frac{L}{T}\right]^z = L^{x+z} M^y \cdot T^{-2x-z}$$

$$\Rightarrow \left.\begin{aligned} x+z &= 1 \\ y &= 0 \\ -2x-z &= 0 \end{aligned}\right\} \quad x=-1,\ y=0,\ z=2.$$

$$\therefore\ h \sim a^{-1} m^0 \cdot V_0^2$$

$$\Rightarrow h \sim V_0^2/a.$$

HW-1; Q 2.1:

$$\rightarrow \frac{dy}{dt} = ry\left(1-\frac{y}{K}\right)$$

UNIT of r: $\frac{1}{time}$

$$y = [y]\,y^*$$
$$t = [t]\,t^*$$

$$\frac{dy}{dt} = \frac{\cancel{[y]}}{[t]}\frac{dy^*}{dt^*} = r\cancel{[y]}y^*\left[1-\frac{[y]y^*}{K}\right]$$

$$\frac{dy^*}{dt} = [t]\,r\,y^*\left[1-\frac{[y]\,y^*}{K}\right]$$

$$[t] = \frac{1}{r}, \quad [y] = K$$

$$\frac{dy^*}{dt} = y^*\left(1-y^*\right)$$

HW-1 Q 2.2:

$$\frac{dy}{dt} = sy\underbrace{(a-y)}_{\text{same}}(y-b) \rightarrow \frac{[M]}{[T]} = \frac{1}{[T][M]^2}[M][M][M]$$

$$y = [y]\,y^*, \quad t = [t]\,t^*$$

$$\frac{dy}{dt} = \frac{\cancel{[y]}}{[t]}\frac{dy^*}{dt^*} = s\cancel{[y]}y^*\left(a-[y]y^*\right)\left([y]y^*-b\right)$$

$$\Rightarrow \frac{dy^*}{dt^*} = s[t]\,y^*\left(a-[y]\,y^*\right)\left([y]\,y^*-b\right)$$

$$[y] = a$$

$$\frac{dy^*}{dt^*} = s[t]\, y^* \left(a - a y^*\right)\left(a y^* - b\right)$$

$$= s[t]\, y^* \cdot a \left(1 - y^*\right) \cdot a \left(y^* - \frac{b}{a}\right)$$

$$= s[t] \cdot a^2\, y^* \left(1 - y^*\right)\left(y^* - \alpha\right),$$

$$\alpha = \frac{b}{a}.$$

$$[t] = \frac{1}{s a^2} \qquad \frac{1}{T a^2} \cdot a^2 = [T] \quad [t] = \frac{1}{s}$$

$$\Rightarrow \frac{dy^*}{dt^*} = y^*\left(1 - y^*\right)\left(y^* - \alpha\right)$$

HW 1 - Q3:

$$\frac{dN}{dt} = r_B N\left(1 - \frac{N}{K_B}\right) - B\frac{N^2}{A^2 + N} \quad (1)$$

$$\Downarrow$$

$$\frac{du}{d\tau} = ru\left(1 - \frac{u}{q}\right) - \frac{u^2}{1 + u^2}$$

$$N = [N] N^*, \quad t = [t] t^*$$

Eq (1) $\Longrightarrow$

$$\frac{dN^*}{dt^*} = r_B [t] N^* \left[1 - \frac{[N][N^*]}{K_B}\right] - \frac{B[t][N] N^{*2}}{A^2\left(1 + \frac{[N]^2 N^{*2}}{A^2}\right)}$$

$$[N] = A, \quad [t] = \frac{A^{\cancel{2}}}{B \cancel{A}} = \frac{A}{B}.$$

$$\frac{dN^*}{dt^*} = \underbrace{r_B \cdot \frac{A}{B}}_{r} \cdot N^* \left[1 - \frac{A N^*}{K_B}\right] - \frac{N^{*2}}{1 + N^{*2}}$$

$$N^* = u, \quad r = r_B \frac{A}{B}, \quad q = \frac{K_B}{A}$$

$$\Rightarrow \frac{du}{dt} = r u \left(1 - \frac{u}{q}\right) - \frac{u^2}{1+u^2}$$

HW Q4.1

$$\ddot{\theta} = \alpha \dot{\theta} + \beta \sin \gamma \theta = 0,$$

$$\theta(0) = 0$$

$$\dot{\theta}(0) = \underline{1}.$$

$$\alpha, \gamma = O(1), \quad \beta >> 1$$

$$t = [t] t^*$$

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{1}{[t]} \frac{d\theta}{dt^*}, \quad \ddot{\theta} = \frac{1}{[t]^2} \frac{d^2\theta}{dt^{*2}}.$$

$$\frac{1}{[t]^2} \cdot \frac{d^2\theta}{dt^{*2}} = \alpha \frac{1}{[t]} \frac{d\theta}{dt^*} + \beta \cdot \sin(\gamma \theta)$$

$$\Rightarrow \frac{d^2\theta}{dt^{*2}} = \underline{\alpha [t]} \frac{d\theta}{dt^*} + \beta \cdot [t]^2 \sin(\gamma \theta)$$

$$\dot{\theta}(0) = \frac{1}{[t]}\frac{d\theta}{dt^*}(0) = 1$$

$$\frac{d\theta}{dt^*}(0) = [t]\cdot 1$$

$$[t] = \frac{1}{\sqrt{\beta}}.$$

$$\Rightarrow \frac{d^2\theta}{dt^{*2}} = \frac{\alpha}{\sqrt{\beta}}\cdot\frac{d\theta}{dt^*} + \sin(\gamma\theta)$$

$$\theta(0) = 0 \qquad \eta = \frac{\alpha}{\sqrt{\beta}} \text{ - order of less than 1.}$$

$$\frac{d\theta}{dt^*}(0) = \frac{1}{\sqrt{\beta}}$$

order of less than 1

HW Q 4.2 $\longrightarrow$ Similar.

HW - ~~B~~ Q 4.3.

$$\alpha \sim \beta\gamma \sim \frac{1}{\gamma} >> 1$$

$$t = [t]t^*, \quad \theta = [\theta]\theta^*,$$

Eq. $\Longrightarrow$ $$\left\{ \frac{d^2\theta^*}{dt^{*2}} + \alpha[t]\frac{d\theta^*}{dt^*} + \beta\frac{[t]^2}{[\theta]}\sin(\gamma\theta^*[\theta]) = 0 \right.$$

$$\begin{cases} \theta^*(0) = \frac{1}{[\theta]}\theta(0) = 0 \\ \frac{d\theta^*}{dt^*} = \frac{1}{[\theta]}[t] \cdot \frac{d\theta}{dt}(0) = \frac{[t]}{[\theta]} \end{cases}$$

$$[\theta] = \frac{1}{\gamma} \qquad [t] = \frac{1}{\alpha}$$

$$\Rightarrow \quad \frac{d^2\theta^*}{dt^{*2}} + \frac{d\theta^*}{dt^*} + \frac{\beta\gamma}{\alpha^2} \cdot \sin(\theta^*) = 0$$

$$\eta = \frac{\beta\gamma}{\alpha^2} \sim \underbrace{\frac{\beta\gamma}{\alpha}}_{O(1)} \cdot \frac{1}{\alpha} << O(1)$$

HW- Q5.1

$$\frac{dy}{dt} = D\frac{d^2y}{dx^2} + \gamma y^3$$

$$\frac{dy}{dt} \sim \frac{[M]}{[T]}$$

$$D\frac{d^2y}{dx^2} = D \cdot \frac{[M]}{[L]^2} \sim \frac{M}{[T]}$$

$$\Rightarrow D \sim \frac{[L]^2}{[T]}$$

$$\gamma y^3 = \gamma [M]^3 \sim \frac{[M]}{[T]}$$

$\Rightarrow \gamma \sim \frac{1}{[M]^2[T]}$

HW. Q5.2 $\longrightarrow$ just regular non-dimen. scale ✓

$\Downarrow$

$$\frac{\partial y^*}{\partial t^*} = \underbrace{\alpha}_{\text{diffusion}} \frac{\partial^2 y^*}{\partial x^{*2}} + \underbrace{\beta}_{\text{other term}} y^{*3}$$

$$\alpha = \frac{D[t]}{[x]^2}, \quad \beta = \gamma[t]y_0^2.$$

For large diffusion relatively,

$$\alpha >> \beta$$

$$\frac{D[t]}{[x]^2} >> \gamma[t]y_0^2$$

$$\Rightarrow \underbrace{\frac{D}{\gamma[x]^2 y_0^2}}_{\text{const}} >> 1$$

$$\Rightarrow \boxed{[x]} << 1 \quad \left(\text{i.e. } \sqrt{\frac{\gamma y_0^2}{D}}\right)$$

For small diffusion relatively

$$\alpha << \beta$$

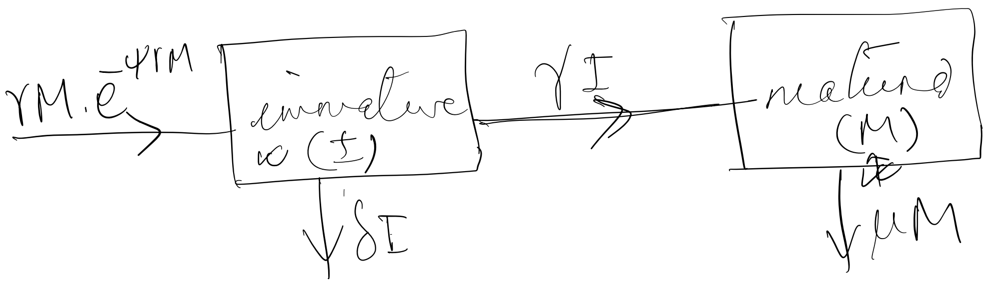
So, $\frac{D}{\gamma [x]^2 y_0^2} << 1$

$\Rightarrow [x] >> 1$

HW-1 Q 7 → straight forward.

HW-2

HW2-Q6.1:



$$\frac{dI}{dt} = rMe^{-\psi r M} - \delta I - \gamma I$$

$$\frac{dM}{dt} = \gamma I - \mu M.$$

Q6.2 with control

$$\frac{dI}{dt} = (1-\theta) r M e^{-\psi r M (1-\theta)} - \delta I - \gamma I$$

$$\frac{dM}{dt} = \gamma I - \mu M. = G$$

Equilibrium:

$$\left.\begin{aligned} \frac{dI}{dt} &= 0 \\ \frac{dM}{dt} &= 0 \end{aligned}\right\} \checkmark$$

$\Longrightarrow$ (1) $I = 0,\ M = 0$ $E_0 = (0, 0)$

(2) $E^* = \begin{cases} I^* = -\ln\left(\frac{(\delta+\gamma)\left(\frac{\mu}{\gamma}\right)}{r(1-\theta)}\right)\frac{\mu}{\varphi\gamma}\left(\frac{1}{r(1-\theta)}\right) \\ M^* = -\ln\left(\frac{(\delta+\gamma)\left(\frac{\mu}{\gamma}\right)}{r(1-\theta)}\right)\left(\frac{1}{\varphi r(1-\theta)}\right) \end{cases}$ $\checkmark$

Stability:

Jacobi

$$J = \begin{bmatrix} \frac{\partial F}{\partial I} & \frac{\partial F}{\partial M} \\ \frac{\partial G}{\partial I} & \frac{\partial G}{\partial M} \end{bmatrix}$$

$$J = \begin{bmatrix} -\delta-\gamma & e^{-\varphi r(1-\theta)M}\left[(1-\theta)r - M\varphi r^2(1-\theta)^2\right] \\ \gamma & -\mu \end{bmatrix}$$

$\checkmark$

At $E_0 = (0,0)$, $I_S = 0$, $M = 0$

$$J\Big|_{E_0} = \begin{bmatrix} -\delta-\gamma & (1-\theta)r \\ \gamma & -\mu \end{bmatrix}$$

$\text{tr}\, J = -\delta - \gamma - \mu < 0$

E_0 is stable $\det J > 0$
unstable $\det J < 0$.

$\det J = \mu(\delta+\gamma) - \gamma r(1-\theta) > 0$ for stable E_0 (Extinction)

~~For~~

$$\theta > \underbrace{1 - \frac{\mu(\delta+\gamma)}{\gamma r}}_{\text{unstable}}$$

~~E~~ otherwise unstable,

Same thing for I^*, M^* =

HW-2 Q4.5.

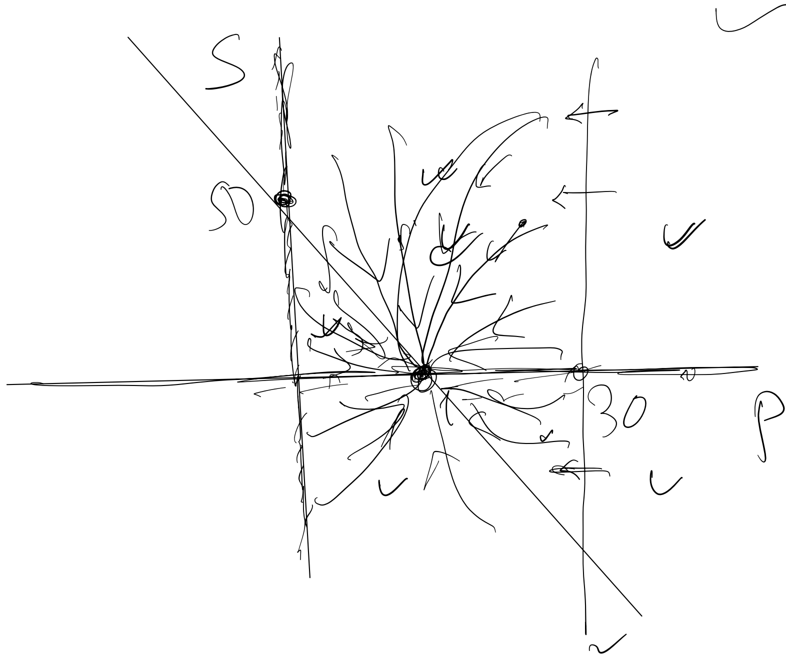
$$\frac{dP}{dt} = P\underbrace{(0.5 - 0.5P - 0.01S)}$$
$$\frac{dS}{dt} = S\underbrace{(-0.3 + 0.01P)}$$
$= 0$

$\frac{dP}{dt} = 0 \Rightarrow \underline{P = 0}$, $\underline{0.5 - 0.5P - 0.01S = 0}$

$\underline{S = 50 - 50P}$

$\frac{dS}{dt} = 0 \Rightarrow \underline{S = 0}$, $-0.3 + 0.01P = 0$

$\Rightarrow \underline{P = 30.}$



HW2-Q 5

Bifurcation diagram