SOLUTIONS AND HINTS TO PROBLEM SET 7

Problem 1.

3.5.1: From fact (3), we know that a parity-check matrix for C_{24} is

$$H = \begin{bmatrix} I \\ B \end{bmatrix}$$

where B is the matrix shown on page 77. Now, let

$$v = (1, 1, 1, \ldots, 1)$$

be the all-one word of length 24. By direct inspection, one can verify that $vH = 0$ where 0 is the all-zero word (or vector). Hence $v \in C_{24}$.

If C_{24} had a codeword, say, c, of weight 20, then the codeword $v + c$ would have weight 4, a contradiction (recall that C_{24} has no codewords of weight 4, see pp. 78–79).

3.5.2: One parity-check matrix for C_{24} is

$$H = \begin{bmatrix} I \\ B \end{bmatrix}.$$

Then $[B \,|\, I] \cdot H = B \cdot I + I \cdot B = \mathbf{0}$. Since the rank of $[B \,|\, I]$ equals 12, it follows that $[B \,|\, I]$ is also a generator matrix for C_{24}.

3.5.3: A parity-check matrix for C_{24} is

$$H = \begin{bmatrix} I \\ B \end{bmatrix}.$$

Hence, a generator matrix for $C_{24}^{\perp}$ is $[I \,|\, B^T] = [I \,|\, B]$, which, by definition, is also a generator matrix for C_{24}. This shows that the two codes, C_{24} and $C_{24}^{\perp}$, have the same generator matrix, that is, $C_{24} = C_{24}^{\perp}$.

Problem 2.

3.6.5(d): The parity-check matrix H is the one from the previous problem. The received vector is

$$r = [1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 0].$$

The syndrome of r is $s_1 = rH = [1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0]$.

Note that $\mathrm{wt}(s_1) = 4 > 3$. We also have $s_2 = sB = [0, 1, 0, 1, 1, 0, 1, 0, 0, 0, 0, 0]$, that is, $\mathrm{wt}(s_2) = 4 > 3$. By direct inspection, one can verify that

$$\mathrm{wt}(s_1 + b_i) > 2 \quad \text{and} \quad \mathrm{wt}(s_2 + b_i) > 2$$

where b_i denotes the ith row of B, for $1 \le i \le 12$. From these results and Algorithm 3.6.1, it follows that the weight of error pattern is beyond the error-correction capability of C_{24}, namely, 3. Hence, the decoder will not decode r; it will request retransmission.

3.6.6(b): Since $\mathrm{wt}(s_2) = 3$, then (according to Algorithm 3.6.1) the most likely error pattern is

$$u = [0, s_2] = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 0, 0].$$

Problem 3.

3.7.5: The reliability of IMLD for C_{23}, namely, $\theta_p(C_{23})$ can be found via the formula on p. 63. To use it, we need to know the number of coset leaders of weight 0, 1, 2, and 3:

- Clearly, there is exactly 1 coset leader of weight 0.
- There are $\binom{23}{1} = 23$ coset leaders of weight 1.
- There are $\binom{23}{2} = 253$ coset leaders of weight 2.
- There are $\binom{23}{3} = 1771$ coset leaders of weight 3.

The total is $2048 = 2^{11}$, so there are no other coset leaders. Also, by Theorem 3.2.8, all the above coset leaders are unique. Therefore,

$$\theta_p(C_{23}) = p^{23} + 23p^{22}(1-p) + 253p^{21}(1-p)^2 + 1771p^{20}(1-p)^3.$$

3.7.7: The Golay code C_{23} is a linear code of minimum distance equal to 7. Hence the minimum weight of its nonzero codewords is equal to 7 and C_{23} is a 3-error-correcting code. We also showed that C_{23} is a perfect code.

From the above, it follows that *each word of length 23 and weight 4 is at distance 3 from exactly one codeword in C. Moreover, this codeword has weight equal to 7.* Note that there are $\binom{23}{4} = 8855$ words of length 23 having weight equal to 4. Now, for any codeword $c \in C_{23}$ such that $\mathrm{wt}(c) = 7$, the number of words that have weight 4 and are at distance 3 from c is $\binom{7}{3} = 35$. In view of all of this, the number of words of weight 7 in C_{23} is equal to $\frac{8855}{35} = 253$.

3.7.8: There are $\binom{24}{5} = 42504$ words of length 24 having weight equal to 5. Each of these words is at distance 3 from exactly one codeword $v \in C_{24}$ such that $\mathrm{wt}(v) = 8$ (this is the hint of the problem and it is justified below). For each such v, the number of words that have weight equal to 5 and are at distance 3 from v is $\binom{8}{3} = 506$. In view of the above, the number of words of weight 8 in C_{24} is equal to $\frac{42504}{506} = 759$.

Now we will justify the hint: Let $w \in K^{24}$ with $\mathrm{wt}(w) = 5$. Puncture any coordinate of w that contains a 0 and denote the obtained word by $\widehat{w}$. Thus, $\widehat{w} \in K^{23}$ and $\mathrm{wt}(\widehat{w}) = 5$. Since C_{23} is perfect, $\widehat{w}$ is at distance 2 or 3 from exactly one codeword $c \in C_{23}$. Moreover, either $\mathrm{wt}(c) = 7$ or $\mathrm{wt}(c) = 8$. From this, it follows that the original $w \in K^{24}$ is at distance 3 from exactly one codeword in C_{24} of weight equal to 8.