Quiz 2
Differential Equations
Math 337
Stephen Giang

Problem 1: Use the information in the lecture notes for the salt mixing problem to verify that the Example on Slide 10 satisfies the conditions required to maintain a constant volume in each vessel. Also, show steps (Gaussian elimination, row-reduce echelon, or some other technique from Linear Algebra) to verify that the equilibrium given on Slide 11 follows from the salt mixing model for the Example from Slide 10. (Slides 5-12).

Solution. Constant Volume Conditions:

$$\begin{array}{llc} f_1 + f_2 = f_6 & .20 + .15 = .35 & \checkmark \\ f_1 + f_3 = f_4 & .20 + .25 = .45 & \checkmark \\ f_2 + f_5 = f_3 & .15 + .10 = .25 & \checkmark \\ f_5 + f_6 = f_4 & .10 + .35 = .45 & \checkmark \end{array}$$

$$\begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \end{pmatrix} = \begin{pmatrix} \frac{-.45}{100} & \frac{.25}{100} \\ \\ \frac{.10}{60} & \frac{-.25}{60} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} \frac{.2(7)}{100} \\ \\ \frac{.15(12)}{60} \end{pmatrix} = \begin{pmatrix} -0.0045 & 0.0025 \\ 0.00167 & -0.004617 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} 0.014 \\ 0.03 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.0045 & 0.0025 \\ 0.00167 & -0.004617 \end{pmatrix} \begin{pmatrix} c_{1e} \\ c_{2e} \end{pmatrix} + \begin{pmatrix} 0.014 \\ 0.03 \end{pmatrix}$$

$$\begin{pmatrix} -0.014 \\ -0.03 \end{pmatrix} = \begin{pmatrix} -0.0045 & 0.0025 \\ 0.00167 & -0.004617 \end{pmatrix} \begin{pmatrix} c_{1e} \\ c_{2e} \end{pmatrix}$$

$$\begin{pmatrix} -0.0045 & 0.0025 & -0.014 \\ \frac{.10}{60} & \frac{-.25}{60} & -0.03 \end{pmatrix} \to \begin{pmatrix} -1 & .5556 & -3.1111 \\ 1 & -2.5 & -18 \end{pmatrix} \quad \to \begin{pmatrix} -1 & .5556 & -3.1111 \\ 0 & -1.9444 & -21.1111 \end{pmatrix}$$

$$\begin{pmatrix} -1 & .5556 & -3.1111 \\ 0 & 1 & 10.85714 \end{pmatrix} \to \begin{pmatrix} -1 & .5556 & -3.1111 \\ 0 & -.5556 & -6.0317 \end{pmatrix} \quad \to \begin{pmatrix} 1 & 0 & 9.14286 \\ 0 & 1 & 10.85714 \end{pmatrix}$$

$\square$

Problem 2: The pharmokinetic model is presented on Slides 14-15. It is stated that the *trace* satisfies $tr(\mathbf{A}) < 0$, the *determinant* is $\det|\mathbf{A}| > 0$, and the *discriminant* is $D > 0$. Provide details that verify these conditions, assuming all parameters in the matrix A are positive.

$$\begin{pmatrix} \dot{d}_1 \\ \dot{d}_2 \end{pmatrix} = \begin{pmatrix} -(K_{pb} + K_e) & K_{bp} \\ K_{pb} & -K_{bp} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

Solution. Let $K_{bp}, K_{pb}, K_e > 0$

$$\begin{aligned}
\text{Trace(A)} &= -(K_{pb} + K_e) + -K_{bp} = -(K_{pb} + K_e + K_{bp}) < 0 \\
\\
\det|A| &= K_{pb}K_{bp} + K_{bp}K_e - K_{pb}K_{bp} = K_{bp}K_e > 0 \\
\\
\text{D} &= (-K_{pb} - K_e + K_{bp})^2 + 4K_{bp}K_{pb} > 0 \\
&= K_{pb}^2 + K_e^2 + K_{bp}^2 - 2K_{bp}K_e + 2K_{pb}K_e - 2K_{bp}K_{pb} + 4K_{bp}K_{pb} \\
&= K_{pb}^2 + K_e^2 + K_{bp}^2 - 2K_{bp}K_e + 2K_{pb}K_e + 2K_{bp}K_{pb} \\
&= K_{pb}^2 + K_e^2 + K_{bp}^2 + 2K_{bp}K_e + 2K_{pb}K_e + 2K_{bp}K_{pb} - 4K_{bp}K_e \\
&= (K_{pb} + K_e + K_{bp})^2 - 4K_{bp}K_e > 0
\end{aligned}$$

$\square$

Problem 3: Consider the pharmokinetic model:

$$\begin{pmatrix} \dot{d}_1 \\ \dot{d}_2 \end{pmatrix} = \begin{pmatrix} -(K_{pb}+K_e) & K_{bp} \\ K_{pb} & -K_{bp} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

where $K_{pb} = 2, K_{bp} = 5$, and $K_e = 0.5$. Assume the initial conditions: $\begin{pmatrix} d_1(0) \\ d_2(0) \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$
Find the Solution to this initial value problem. State clearly the eigenvalues and eigenvectors.

Solution.

$$\begin{pmatrix} \dot{d}_1 \\ \dot{d}_2 \end{pmatrix} = \begin{pmatrix} -(2+0.5) & 5 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} -2.5 & 5 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

$$\begin{aligned} \begin{vmatrix} -2.5-\lambda & 5 \\ 2 & -5-\lambda \end{vmatrix} &= (\lambda+2.5)(\lambda+5) - 10 \\ &= \lambda^2 + 7.5\lambda + 12.5 - 10 \\ &= \lambda^2 + 7.5\lambda + 2.5 = 0 \\ \lambda &= \frac{-7.5 \pm \sqrt{56.25 - 10}}{2} \\ &= \frac{-7.5 \pm \sqrt{46.25}}{2} \\ &= -0.3496, -7.1504 \end{aligned}$$

Let $\lambda_1 = -0.3496$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2.5-(-0.3496) & 5 \\ 2 & -5-(-0.3496) \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} -2.1504 & 5 \\ 2 & -4.6504 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2.1504 \end{pmatrix}$$

Let $\lambda_2 = -7.1504$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2.5-(-7.1504) & 5 \\ 2 & -5-(-7.1504) \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 4.6504 & 5 \\ 2 & 2.1504 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 5 \\ -4.6504 \end{pmatrix}$$

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = c_1 \begin{pmatrix} 5 \\ 2.1504 \end{pmatrix} e^{-0.3496t} + c_2 \begin{pmatrix} 5 \\ -4.6504 \end{pmatrix} e^{-7.1504}$$

$$\begin{pmatrix} d_1(0) \\ d_2(0) \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 2.1504 & -4.6504 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 5 & 10 \\ 2.1504 & -4.6504 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 2.1504 & -4.6504 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -2.1504 & -2.1504 & -4.3008 \\ 2.1504 & -4.6504 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & -6.8008 & -4.3008 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & .6324 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1.3676 \\ 0 & 1 & .6324 \end{pmatrix}$$

$$\boldsymbol{\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = 1.3676 \begin{pmatrix} 5 \\ 2.1504 \end{pmatrix} e^{-0.3496t} + .6324 \begin{pmatrix} 5 \\ -4.6504 \end{pmatrix} e^{-7.1504}}$$

$\square$

Problem 4: Consider the pharmokinetic model in the previous problem (same parameters). Create a phase portrait and describe the qualitative behavior. On your phase portrait include the specific trajectory for the initial value problem above. (Slides 18-22).

The phase portrait shows of a stable node as both eigenvalues are negative. This means as $t \rightarrow \infty$, the phase portraits approaches the origin