Math 532: Midterm 1
Due 10/16/19
Everyone turns in an individual copy. Work alone!!

1. (20 pts) Let
$$f(z) = z + \frac{a^2}{z}, \ a > 0.$$
   - (4 pts) Describe the domain of analyticity of $f(z)$.
   - (6 pts) Find the imaginary part of $f(z)$, i.e. $\text{Im}\{f(z)\}$ in both Cartesian and Polar Coordinates.
   - (10 pts) We call the level sets of $\text{Im}\{f(z)\}$, where
   $$\text{Im}\{f(z)\} = c, \ c \in \mathbb{R},$$
   streamlines. Letting $z = re^{i\theta}$, show that the streamlines for $f(z)$ are found via the polar equation
   $$r(\theta) = \frac{c}{2\sin(\theta)} + \sqrt{a^2 + \frac{c^2}{4\sin^2(\theta)}}.$$
   Sketch the streamlines for $r \geq a$. If this were a fluid flow, describe what object the fluid is flowing around.

2. (10 pts) Using only the Cauchy–Riemann equations, show that if $f(z)$ is analytic on a given domain D and $|f(z)| = \tilde{c} > 0$, then $f(z) = c \in \mathbb{C}$.

3. (5 pts) Let u be a harmonic function on the closed disc $\bar{D}_R((x, y))$. Given that for $0 \leq r \leq R$ that the function
$$u_a(x, y) = \int_0^{2\pi} u\left(x + r\cos(\theta), y + r\sin(\theta)\right) d\theta$$
is independent of r, using L'Hopital's rule and the Fundamental Theorem of Calculus (instead of a bound, hint hint), show that
$$u(x, y) = \frac{1}{\pi R^2} \int_0^R \int_0^{2\pi} ru\left(x + r\cos(\theta), y + r\sin(\theta)\right) d\theta dr.$$

4. (10 pts) Let $f(z)$ be analytic on the bounded domain D. Show that either $|f(z)|$ is constant on $\bar{D}$ or it attains its maximum on ∂D. If you use another result to prove this, clearly state the result. Yes, you can cite the homework. *Hint*: So keep in mind that if

$$f(z) = u(x, y) + iv(x, y)$$

and from analyticity of f we know u and v are harmonic, then we know on a closed disc $\bar{D}_R((x, y)) \subset D$ that

$$\begin{aligned} u(x,y) =& \frac{1}{\pi R^2} \iint_{\bar{D}_R((x,y))} u(\tilde{x}, \tilde{y}) d\tilde{x} d\tilde{y}, \\ v(x,y) =& \frac{1}{\pi R^2} \iint_{\bar{D}_R((x,y))} v(\tilde{x}, \tilde{y}) d\tilde{x} d\tilde{y} \end{aligned}$$

Thus we get

$$u(x, y) + iv(x, y) = \frac{1}{\pi R^2} \iint_{\bar{D}_R((x,y))} \left(u(\tilde{x}, \tilde{y}) + iv(\tilde{x}, \tilde{y}) \right) d\tilde{x} d\tilde{y},$$

and so with a slight abuse of notation we get

$$f(x + iy) = \frac{1}{\pi R^2} \iint_{\bar{D}_R((x,y))} f(\tilde{x} + i\tilde{y}) d\tilde{x} d\tilde{y}.$$

Then we use our favorite inequality involving integrals which says that

$$|f(x + iy)| \leq \frac{1}{\pi R^2} \iint_{\bar{D}_R((x,y))} |f(\tilde{x} + i\tilde{y})| \, d\tilde{x} d\tilde{y}.$$

From there you should be able to use a very similar proof to the one used in the homework to show the result.

5. (10 pts) Suppose $g(x, y)$ is harmonic on some domain D, i.e.

$$\Delta g = \partial_{xx} g + \partial_{yy} g = 0.$$

Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function for $z \in D$. Introducing the change of variables

$$u = u(x, y), \;\; v = v(x, y)$$

by letting $g = g(u(x, y), v(x, y))$, we get via the Chain-Rule the formula

$$\partial_x g = u_x \partial_u g + v_x \partial_v g.$$

- Using the Multiplication-Rule and the Chain-Rule, show that

$$\partial_{xx} g = u_{xx} \partial_u g + u_x^2 \partial_{uu} g + 2u_x v_x \partial_{uv} g + v_{xx} \partial_v g + v_x^2 \partial_{vv} g.$$

  Hint: You need to see that

$$\partial_u g = g_u(u(x, y), v(x, y))$$

  so that

$$\partial_x \partial_u g = u_x g_{uu} + v_x g_{uv}.$$

- Derive a similar formula for $\partial_{yy} g$.
- Using your results from above and the Cauchy–Riemann equations, show that

$$\partial_{xx} g + \partial_{yy} g = \left| f'(z) \right|^2 (\partial_{uu} g + \partial_{vv} g)$$

- Thus, if in x and y, $\Delta g = 0$, what is Δg in the transformed variables u and v? From this result, what can we say about changing the coordinates of harmonic functions via analytic maps $f(z)$?