Quiz 10
Differential Equations
Math 337
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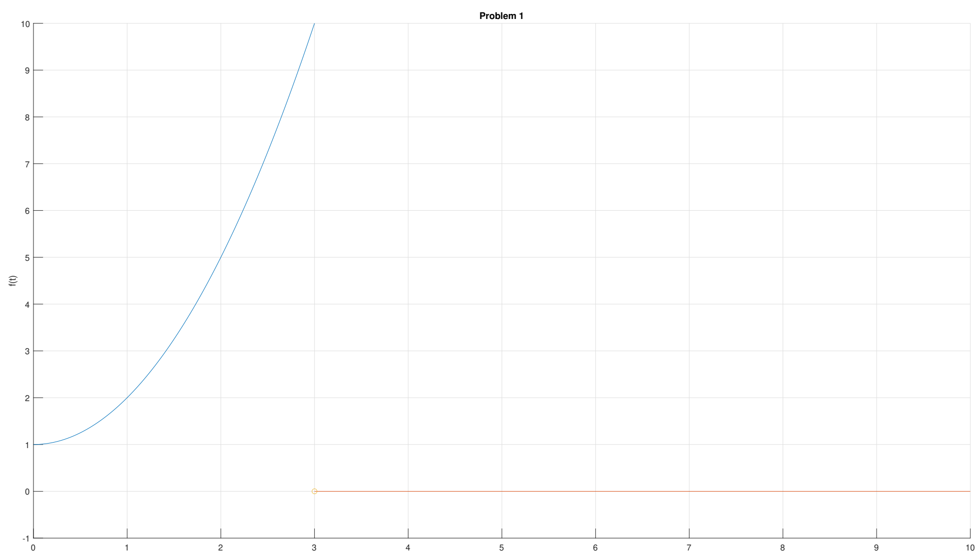
Problem 1: Consider the function $f(t)$ defined as follows:

$$f(t) = \begin{cases} t^2 + 1 & 0 \leq t \leq 3 \\ 0 & t > 3 \end{cases}$$

Sketch a graph of this function and write it in terms of the step function, $u_c(t)$, which is defined in the lecture notes. Further, write the function with the step function so that every element is readily found in the Laplace table. (Something like $u_c(t)\sin(t-c)$.) Finally, find the Laplace Transform of $f(t), F(s) = \mathcal{L}[f(t)]$.



$$\begin{aligned} f(t) &= (t^2+1)(u_0(t) - u_3(t)) \\ &= (t^2)u_0(t) + u_0(t) - ((t-3)+3)^2 u_3(t) - u_3(t) \\ &= (t^2)u_0(t) + u_0(t) - (t-3)^2 u_3(t) - 6(t-3)u_3(t) - 9u_3(t) - u_3(t) \\ &= (t^2)u_0(t) + u_0(t) - (t-3)^2 u_3(t) - 6(t-3)u_3(t) - 10u_3(t) \end{aligned}$$

$$F(s) = \frac{1}{s^2} + \frac{1}{s} - e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{10}{s}\right)$$

Problem 2: Solve the following initial value problem with Laplace transforms:

$$y'' + 2y' + 5y = f(t) = \begin{cases} 5 & 0 \le t \le 4 \\ -(t-9) & 4 \le t \le 9 \\ 0 & t > 9 \end{cases}, \qquad y(0) = 1, \qquad y'(0) = 4$$

Notice the following:

$$\begin{aligned} \mathcal{L}[y'' + 2y' + 5y] &= s^2 Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + 5Y(s) \\ &= (s^2 + 2s + 5)Y(s) - (s + 6) \end{aligned}$$

$$\begin{aligned} f(t) &= 5(u_0(t) - u_4(t)) - (t-9)(u_4(t) - u_9(t)) \\ &= 5u_0(t) - 5u_4(t) - ((t-4) - 5)u_4(t) + (t-9)u_9(t) \\ &= 5u_0(t) - 5u_4(t) - (t-4)u_4(t) + 5u_4(t) + (t-9)u_9(t) \\ &= 5u_0(t) - (t-4)u_4(t) + (t-9)u_9(t) \\ \mathcal{L}[f(t)] &= \frac{5}{s} - \frac{e^{-4s}}{s^2} + \frac{e^{-9s}}{s^2} \end{aligned}$$

Thus we get the equality:

$$Y(s) = \frac{5}{s(s^2+2s+5)} - \frac{e^{-4s}}{s^2(s^2+2s+5)} + \frac{e^{-9s}}{s^2(s^2+2s+5)} + \frac{s+6}{s^2+2s+5}$$

Notice the partial fractions decomposition:

$$\begin{aligned} \frac{1}{s(s^2+2s+5)} &= \frac{A}{s} + \frac{Bs+C}{s^2+2s+5} \\ 1 &= (A+B)s^2 + (2A+C)s + 5A \end{aligned}$$

So we get $A = \frac{1}{5}, B = \frac{-1}{5}, C = \frac{-2}{5}$

$$\frac{1}{s(s^2+2s+5)} = \frac{1}{5s} - \frac{s+2}{5(s^2+2s+5)}$$

Notice the partial fractions decomposition:

$$\begin{aligned} \frac{1}{s^2(s^2+2s+5)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+2s+5} \\ 1 &= (A+C)s^3 + (2A+B+D)s^2 + (5A+2B)s + 5B \end{aligned}$$

So we get $B = \frac{1}{5}, A = \frac{-2}{25}, C = \frac{2}{25}, \frac{-1}{25}$

$$\frac{1}{s^2(s^2+2s+5)} = \frac{-2}{25s} + \frac{1}{5s^2} + \frac{2s-1}{25(s^2+2s+5)}$$

So we can now rewrite $Y(s)$

$$\begin{aligned}Y(s) = &\left(\frac{1}{s} - \frac{s+1}{(s+1)^2+4} - \frac{1}{(s+1)^2+4}\right) - \frac{e^{-4s}}{25}\left(\frac{-2}{s} + \frac{5}{s^2} + \frac{2(s+1)}{(s+1)^2+4} - \frac{3}{(s+1)^2+4}\right)\\ &+ \frac{e^{-9s}}{25}\left(\frac{-2}{s} + \frac{5}{s^2} + \frac{2(s+1)}{(s+1)^2+4} - \frac{3}{(s+1)^2+4}\right) + \left(\frac{s+1}{(s+1)^2+4} + \frac{5}{(s+1)^2+4}\right)\end{aligned}$$

Now we can take the Laplace inverse:

$$\begin{aligned}y(t) = &\left(1 - e^{-t}\cos(2t) - \frac{1}{2}e^{-t}\sin(2t)\right) - \frac{u_4(t)}{25}\left(-2 + 5(t-4) + 2e^{-(t-4)}\cos(2(t-4)) - \frac{3}{2}\sin(2(t-4))\right)\\ &+ \frac{u_9(t)}{25}\left(-2 + 5(t-9) + 2e^{-(t-9)}\cos(2(t-9)) - \frac{3}{2}\sin(2(t-9))\right) + \left(\cos(2t) + \frac{5}{2}\sin(2t)\right)\end{aligned}$$

Problem 3: The limiting solution is:

$$\begin{aligned}y(t) = 1 &- \frac{u_4(t)}{25}\left(-2 + 5(t-4) - \frac{3}{2}\sin(2(t-4))\right)\\ &+ \frac{u_9(t)}{25}\left(-2 + 5(t-9) - \frac{3}{2}\sin(2(t-9))\right) + \left(\cos(2t) + \frac{5}{2}\sin(2t)\right)\end{aligned}$$