MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #11

Chapter 4 Classification of Planar Systems

The Trace-Determinant Plane

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Topics

Prerequisites: Math 337 (ODEs) with minimum grade of C

Topics covered in this course

1. First Order Equations
2. Planar Linear Systems
3. Phase Portraits for Planar Systems
4. Classification of Planar Systems
5. Higher Dimensional Linear Algebra
6. Higher Dimensional Linear Systems
7. Nonlinear Systems
8. Asymptotic Series and Local Analysis
9. Perturbation Series
10. Boundary Layer Theory
11. WKB Theory

Hirsch, Smale, and Devaney (**HSD**)

~ 9.5 weeks

"bridge"

Bender and Orszag (**BO**)

~4 weeks

Computing: You are encouraged to apply mathematics software, such as Matlab, R or Python for plotting.

4.1: Classification

For a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad A - \lambda I = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

we know that the eigenvalues are the roots of the characteristic equation, which may be written

$$|A - \lambda I| = 0 \qquad \lambda^2 - (a+d)\lambda + (ad - bc) = 0.$$

Thus the eigenvalues satisfy

$$\lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0$$

$$T = trace\ (A) = tr\ (A) = a + d \qquad D = det\ (A) = ad - bc$$

$$\boxed{\lambda^2 - T\lambda + D = 0}$$

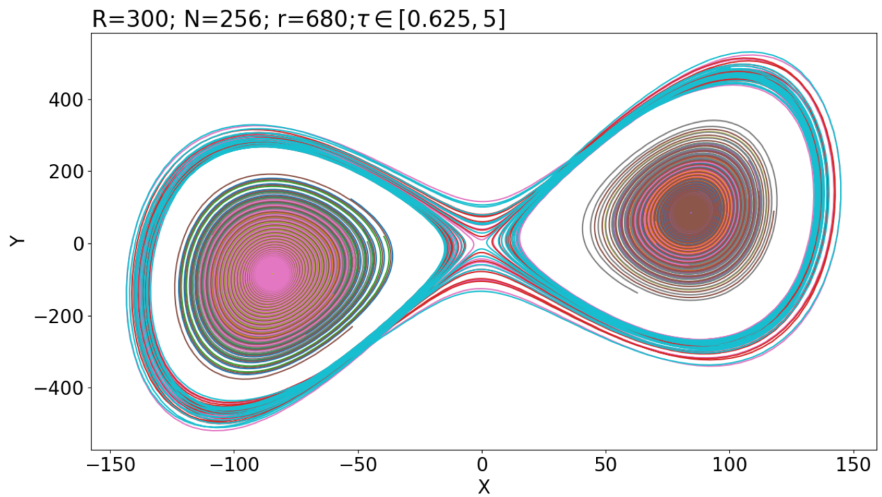
A Summary for Classification

- $Re(\lambda_1) > 0 \,\&\, Re(\lambda_2) > 0:$ Source
- $Re(\lambda_1) < 0 \,\&\, Re(\lambda_2) < 0:$ Sink
- $\lambda_1\lambda_2 < 0:$ Saddle

- $Re(\lambda) = 0 \ \&\ Im(\lambda) \neq 0$: Center
- $Re(\lambda) \neq 0 \ \&\ Im(\lambda) \neq 0$ Spiral sink or source

Pause: Similarity

- In some cases, we obtained realistic 30-day predictions with a predictability of over two weeks.
- We suggested that the entirety of weather possesses a dual nature of chaos and order.
- The above refined view is neither too optimistic nor pessimistic as compared to the Laplacian view of deterministic predictability and the Lorenz view of deterministic chaos.



The first kind of attractor coexistence

https://pixy.org/src/5/56514.jpg

The Trace-Determinant Plane

$T^2 = 4D$

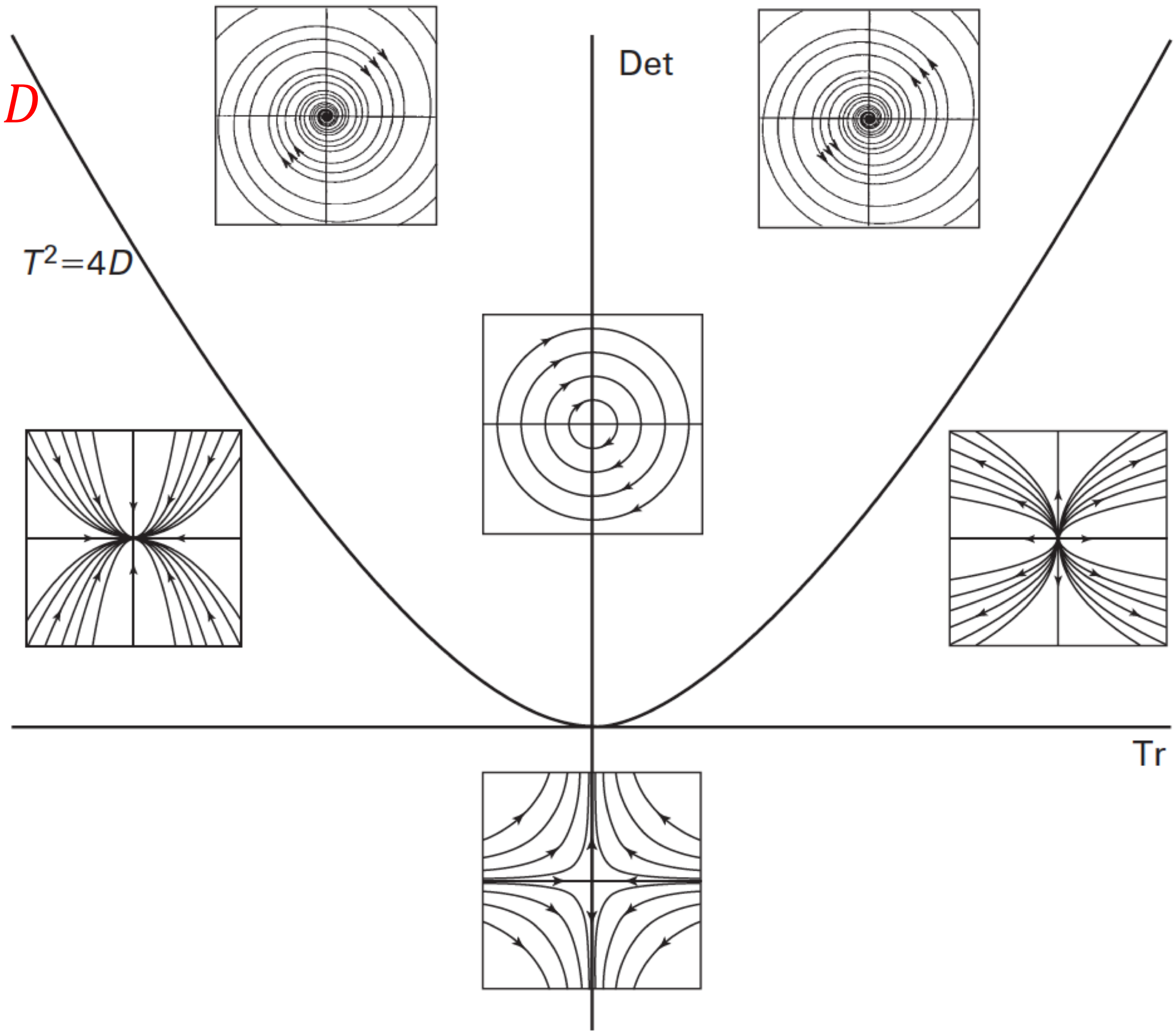


Figure 4.1 The trace-determinant plane. Any resemblance to any of the authors' faces is purely coincidental.

The Trace-Determinant Plane

$T^2 = 4D$

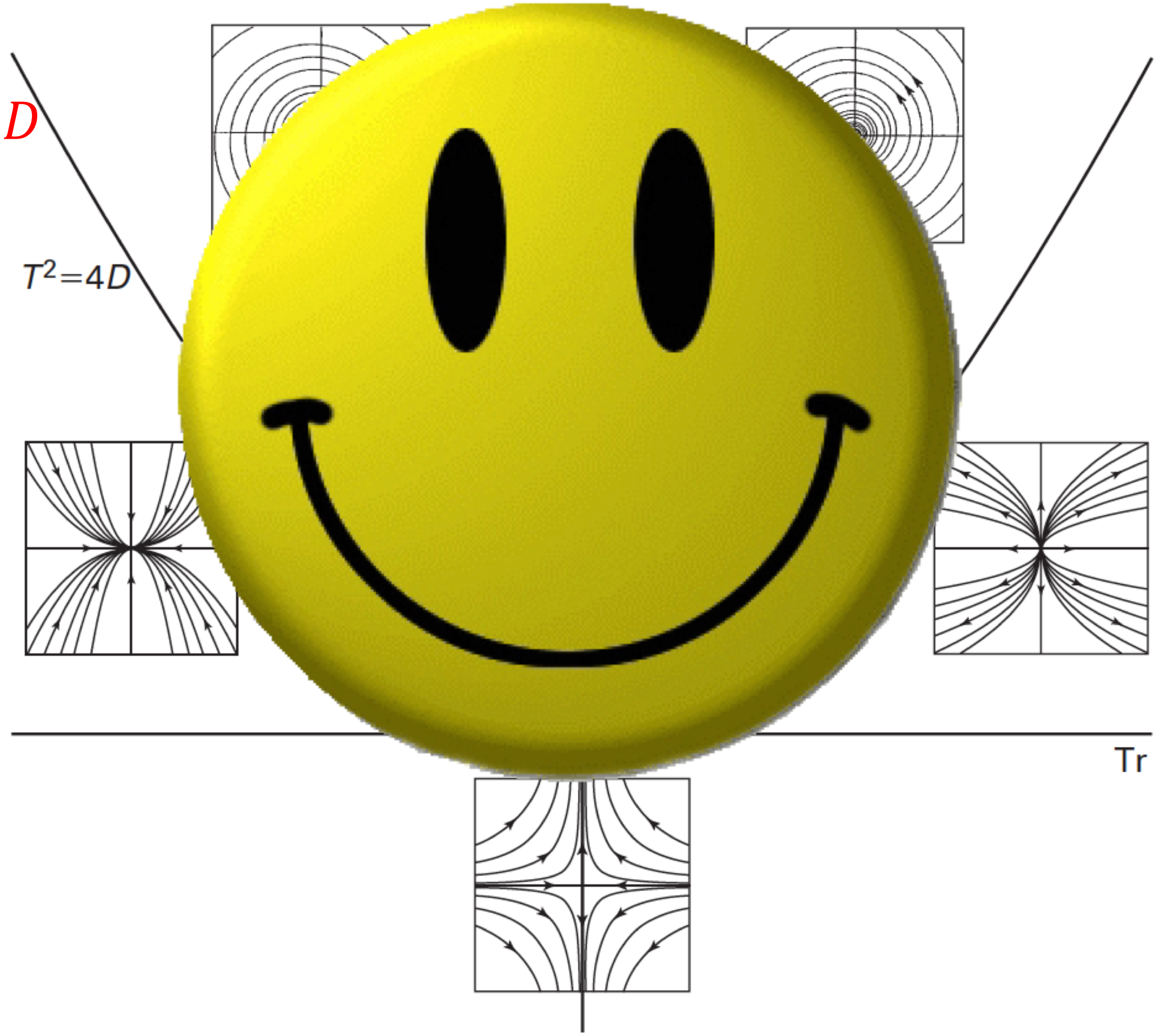


Figure 4.1 The trace-determinant plane. Any resemblance to any of the authors' faces is purely coincidental.

Classification: Smiling Curve $T^2 - 4D = 0$

$$\lambda^2 - T\lambda + \mathrm{D} = 0$$

$$\lambda_\pm = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

$$f(T, D) = T^2 - 4D$$

Define a smiling curve:

$$T^2 = 4D$$

Sample points

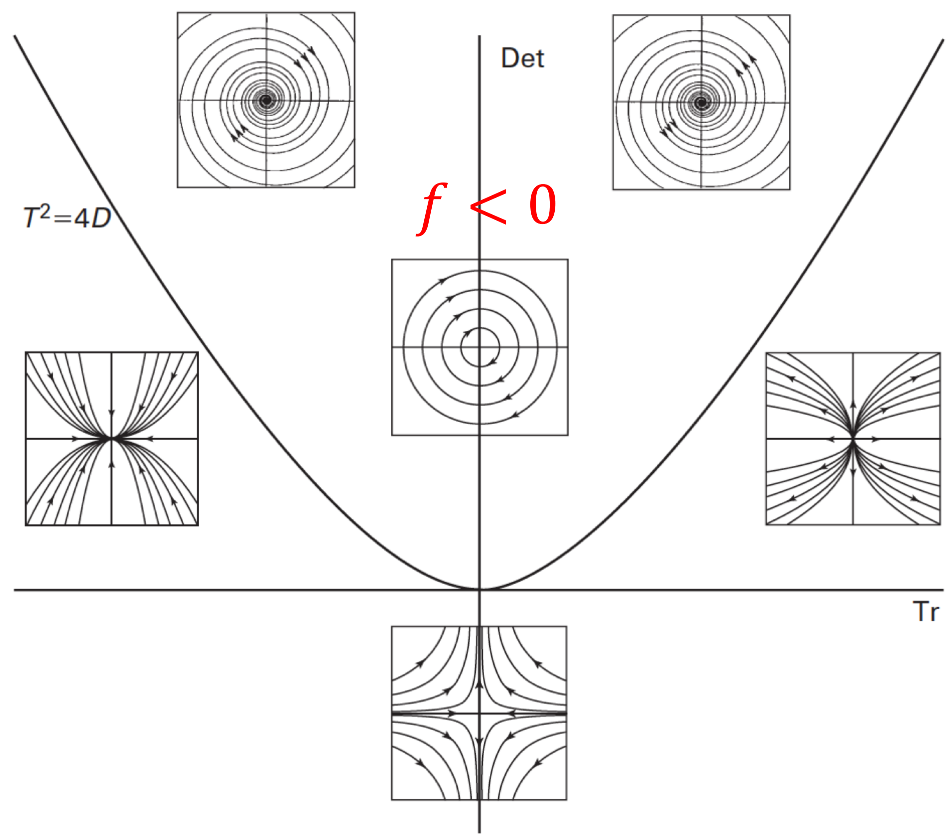
$$f(0,1) = -4 < 0$$

complex eigenvalues

$$f(0,-1) = 4 > 0$$

real eigenvalues

$T^2 = 4D$



$f < 0$

$f > 0$

Classification: Saddle, Source and Sink

$$\lambda^2 - T\lambda + \mathrm{D} = 0$$

$$\lambda_\pm = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

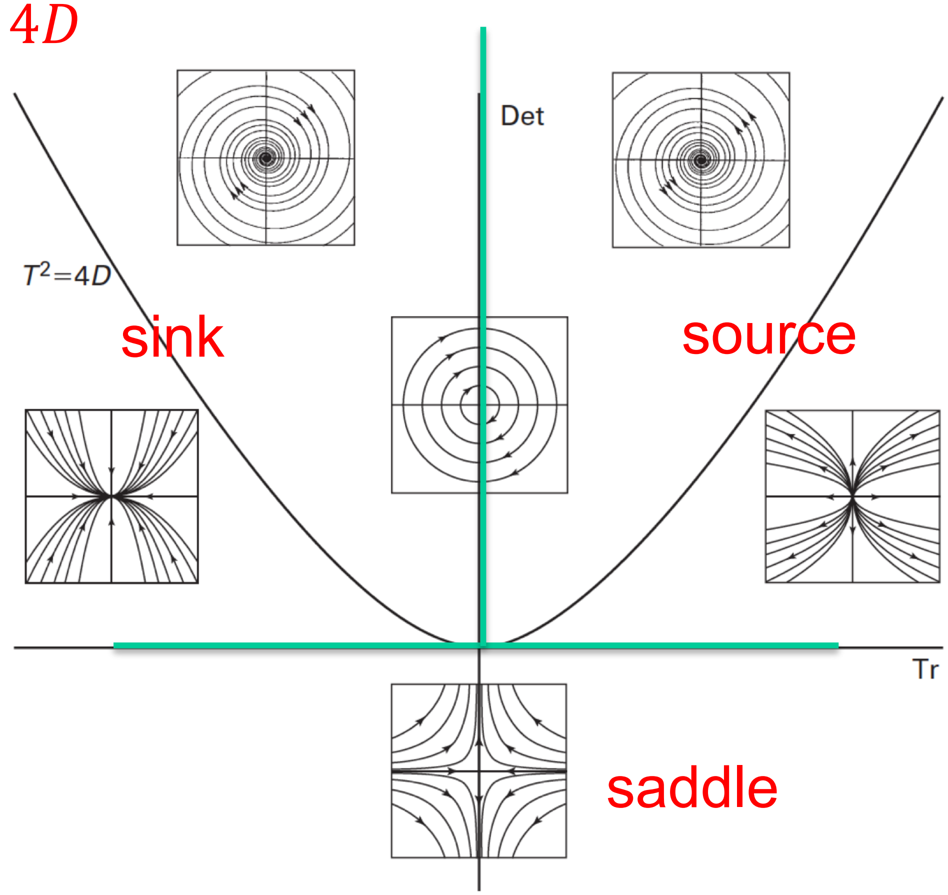
$$\boxed{\lambda_+ + \lambda_- = T = tr}$$

$$(\lambda - \lambda_+)(\lambda - \lambda_-) = 0$$

$$\lambda^2 - (\lambda_+ + \lambda_-)\lambda + \lambda_+\lambda_- = 0$$

$$\boxed{\lambda_+\lambda_- = D = determinat}$$

$T^2 = 4D$



- $D < 0$, λ_+ and λ_- have different signs ➔ saddle
- $D > 0$, λ_+ and λ_- have the same sign ➔ source with $T > 0$

  ➔ sink with $T < 0$

(A) Complex Eigenvalues with $T^2 - 4D < 0$

$$\lambda_{\pm} = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

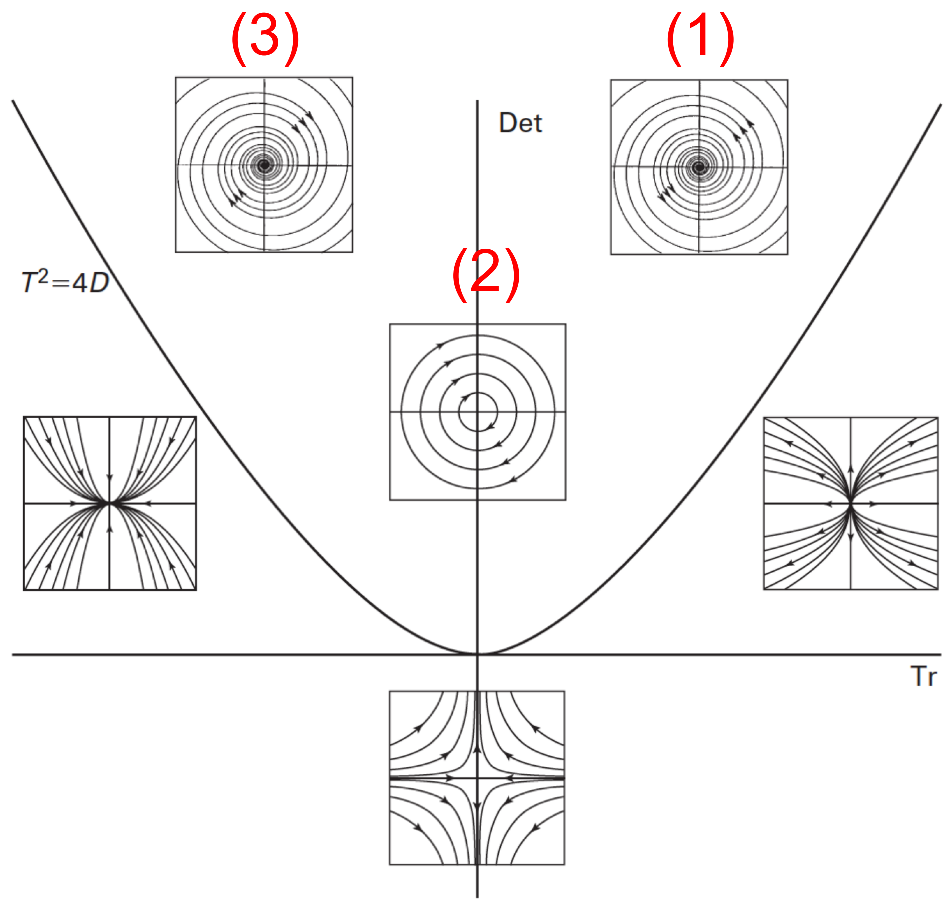
(A) $T^2 - 4\mathrm{D} < 0$ (e.g., larger D)

complex eigenvalues

$T > 0$, spiral source (1)

$T = 0$, center (2)

$T < 0$, spiral sink (3)



(B) Real Eigenvalues with $T^2 - 4D > 0$

$$\lambda_\pm = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

(B) $T^2 - 4\text{D} > 0$

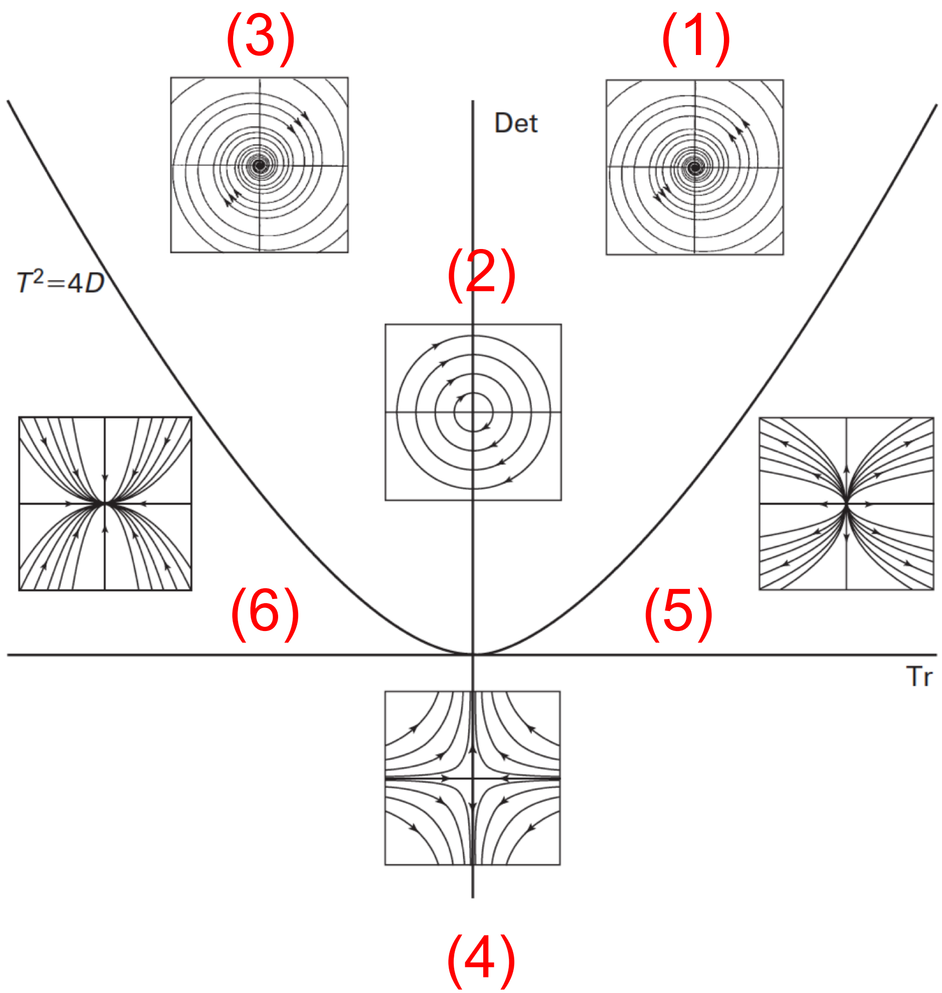
(e.g., small or negative D)

real eigenvalues

$D < 0$, ➔ $\lambda_+ \lambda_- < 0$, saddle (4)

$\text{D} > 0$ & $T > 0$, source (5)

$D > 0$ & $T < 0$, sink (6)



(B) Repeated Eigenvalue with $T^2 - 4D = 0$

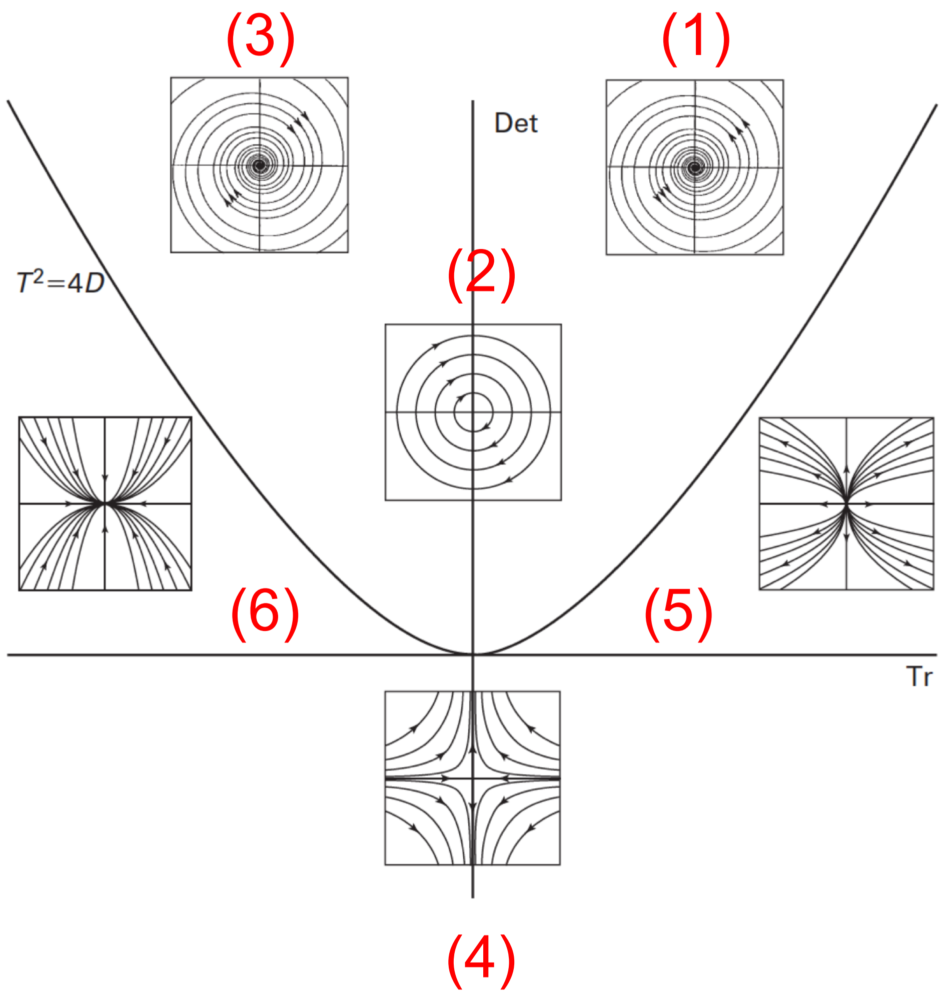
$$\lambda_{\pm} = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

(C) $T^2 - 4\mathrm{D} = 0$

repeated eigenvalue

$T > 0$, source

$T < 0$, sink



Stability Properties of Linear Systems

TABLE 9.1.1 Stability Properties of Linear Systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with $\det(\mathbf{A} - r\mathbf{I}) = 0$ and $\det \mathbf{A} \neq 0$

	Eigenvalues		Type of Critical Point	Stability
source	$r_1 > r_2 > 0$		Node	Unstable
sink	$r_1 < r_2 < 0$		Node	Asymptotically stable
saddle	$r_2 < 0 < r_1$		Saddle point	Unstable
source	$r_1 = r_2 > 0$	repeated	Proper or improper node	Unstable
sink	$r_1 = r_2 < 0$		Proper or improper node	Asymptotically stable
spiral	$r_1, r_2 = \lambda \pm i\mu$		Spiral point	
	$\lambda > 0$			Unstable
	$\lambda < 0$			Asymptotically stable
center	$r_1 = i\mu, r_2 = -i\mu$		Center	Stable

Boyce and Diprima, 2012: Elementary Differential Equations, Tenth Edition. Wiley, 2012.

Remarks

- First, the trace-determinant plane is a two-dimensional representation of what is really a four-dimensional space, since 2x2 matrices are determined by four parameters, the entries of the matrix. Thus there are infinitely many different matrices corresponding to each point in the TD–plane. While all of these matrices share the same eigenvalue configuration, there may be subtle differences in the phase portraits, such as the direction of rotation for centers and spiral sinks and sources, or the possibility of one or two independent eigenvectors in the repeated eigenvalue case.

- We also think of the trace-determinant plane as the analog of the bifurcation diagram for planar linear systems. A one-parameter family of linear systems corresponds to a curve in the TD–plane. When this curve crosses the T-axis, the positive D-axis, or the parabola $T^2 - 4D = 0$, the phase portrait of the linear system undergoes a bifurcation: A major change occurs in the geometry of the phase portrait.

- Finally, note that we may obtain quite a bit of information about the system from D and T without ever computing the eigenvalues. For example, if $D < 0$, we know that we have a saddle at the origin. Similarly, if both D and T are positive, then we have a source at the origin.

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