MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #12

Linearization Theorem

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Outline

1. Existence and Uniqueness Theorems
2. Lipschitz Condition
3. Continuous Dependence of Solutions on Initial Conditions (CDIC)
4. Sensitive Dependence of Solutions on Initial Conditions (SDIC)
5. Linearization Theorems & Linearized Systems

Hyperbolic

Definition
A matrix A is *hyperbolic* if none of its eigenvalues has real part 0. We also say that the system $X' = AX$ is *hyperbolic.*

$$\boxed{Re(\lambda_j) \neq 0}$$

Linearization Theorem

TBD

THEOREM 6.2.3
Let $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ be a nonlinear system of n first-order equations with $\mathbf{x}^*$ as an equilibrium solution and $\mathbf{f}$ a sufficiently smooth vector function. If $\mathrm{Re}(\lambda_i) \neq 0$ for all i, then the predictions given by the linear stability results of Theorem 6.2.2 hold for the equilibrium solution in the nonlinear system.

Hyperbolic

Wirkus and Swift

Conjugacy and Linearization Theorem

TBD

We can now conjugate the flow of a nonlinear system near a hyperbolic equilibrium point that is a sink to the flow of its linearized system. Indeed, the argument used in the second example of the previous section goes over essentially unchanged. In similar fashion, nonlinear systems near a hyperbolic source are also conjugate to the corresponding linearized system.

This result is a special case of the following more general theorem.

The Linearization Theorem. *Suppose the n-dimensional system* $X' = F(X)$ *has an equilibrium point at* X_0 *that is hyperbolic. Then the nonlinear flow is conjugate to the flow of the linearized system in a neighborhood of* X_0. ∎

a.k.a. Hartman–Grobman Theorem

Dynamical equivalence

Hartman–Grobman Theorem

TBD

An underlying theme throughout the first chapter of this book was that the orbit structure near a hyperbolic fixed point was qualitatively the same as the orbit structure given by the associated linearized dynamical system. A theorem proved independently by Hartman [1960] and Grobman [1959] makes this precise. We will describe the situation for vector fields.

Consider a $\mathbf{C}^r$ $(r \geq 1)$ vector field

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \tag{19.12.18}$$

where f is defined on a sufficiently large open set of $\mathbb{R}^n$. Suppose that (19.12.18) has a *hyperbolic* fixed point at $x = x_0$, i.e.,

$$f(x_0) = 0,$$

and $Df(x_0)$ has no eigenvalues on the imaginary axis. Consider the associated linear vector field

$$\dot{\xi} = Df(x_0)\xi, \qquad \xi \in \mathbb{R}^n. \tag{19.12.19}$$

Then we have the following theorem.

Theorem 19.12.6 (Hartman and Grobman) *The flow generated by (19.12.18) is* $\mathbf{C}^0$ conjugate *to the flow generated by (19.12.19) in a neighborhood of the fixed point* $x = x_0$.

Wiggins (2003)

Hartman–Grobman Theorem

TBD

Theorem 1.3.1 (Hartman–Grobman). *If $Df(\bar{x})$ has no zero or purely imaginary eigenvalues then there is a homeomorphism h defined on some neighborhood U of $\bar{x}$ in $\mathbb{R}^n$ locally taking orbits of the nonlinear flow ϕ_t of* (1.3.1), *to those of the linear flow $e^{tDf(\bar{x})}$ of* (1.3.2). *The homeomorphism preserves the sense of orbits and can also be chosen to preserve parametrization by time.*

A more delicate situation in which the nonlinear and linear flows are related via *diffeomorphisms* (Sternberg's theorem) requires certain nonresonance conditions among the eigenvalues of $Df(\bar{x})$. We shall not consider this here, but see the discussion of normal forms in Chapter 3.

When $Df(\bar{x})$ has no eigenvalues with zero real part, $\bar{x}$ is called a *hyperbolic* or *nondegenerate* fixed point and the asymptotic behavior of solutions near it (and hence its stability type) is determined by the linearization. If any one of the eigenvalues has zero real part, then stability cannot be determined by linearization.

Guckenheimer and Holmes (1983)

Hartman–Grobman Theorem

TBD

In mathematics, in the study of dynamical systems, the **Hartman–Grobman theorem** or **linearization theorem** is a theorem about the local behavior of dynamical systems in the neighbourhood of a hyperbolic equilibrium point. It asserts that linearization—a natural simplification of the system—is effective in predicting qualitative patterns of behavior.

The theorem states that the behavior of a dynamical system in a domain near a hyperbolic equilibrium point is qualitatively the same as the behavior of its linearization near this equilibrium point, where hyperbolicity means that no eigenvalue of the linearization has real part equal to zero. Therefore, when dealing with such dynamical systems one can use the simpler linearization of the system to analyze its behavior around equilibria.[1]

(wikipedia)

Hartman–Grobman Theorem

TBD

Main theorem [edit]

Consider a system evolving in time with state $u(t) \in \mathbb{R}^n$ that satisfies the differential equation $du/dt = f(u)$ for some smooth map $f : \mathbb{R}^n \to \mathbb{R}^n$. Suppose the map has a hyperbolic equilibrium state $u^* \in \mathbb{R}^n$: that is, $f(u^*) = 0$ and the Jacobian matrix $A = [\partial f_i / \partial x_j]$ of f at state u^* has no eigenvalue with real part equal to zero. Then there exists a neighborhood N of the equilibrium u^* and a homeomorphism $h : N \to \mathbb{R}^n$, such that $h(u^*) = 0$ and such that in the neighbourhood N the flow of $du/dt = f(u)$ is topologically conjugate by the continuous map $U = h(u)$ to the flow of its linearization $dU/dt = AU$.[2][3][4][5]

Even for infinitely differentiable maps f, the homeomorphism h need not to be smooth, nor even locally Lipschitz. However, it turns out to be Hölder continuous, with an exponent depending on the constant of hyperbolicity of A.[6]

The Hartman–Grobman theorem has been extended to infinite dimensional Banach spaces, non-autonomous systems $du/dt = f(u,t)$ (potentially stochastic), and to cater for the topological differences that occur when there are eigenvalues with zero or near-zero real-part.[7][8][9][10]

(wikipedia)

Hyperbolicity ($Re(\lambda) \neq 0$) and Linearization Theorem

- Thus the dynamics near a hyperbolic fixed point are structurally stable, while the non-hyperbolic fixed is not structurally stable.
- This assures that the local linearization is a valid approximation for hyperbolic fixed points in any number of dimensions.
- (in other words) the validity of local linearization of nonlinear dynamics near an equilibrium is guaranteed only in the generic case of hyperbolic fixed point. (p 200)
- (p 206) For example, a generalized hyperbolic structure, the horseshoe map, enabled Smale to prove that what is called chaos may be structurally stable.
- (p 206) In higher-dimensional phase space, non-hyperbolic structure can be a more serious problem.

Thompson and Stewart, p 198

Hyperbolicity

If $\mathrm{Re}(\lambda) \neq 0$ for both eigenvalues, the fixed point is often called ***hyperbolic***. (This is an unfortunate name—it sounds like it should mean "saddle point"—but it has become standard.) Hyperbolic fixed points are sturdy; their stability type is unaffected by small nonlinear terms. Nonhyperbolic fixed points are the fragile ones.

We've already seen a simple instance of hyperbolicity in the context of vector fields on the line. In Section 2.4 we saw that the stability of a fixed point was accurately predicted by the linearization, *as long as* $f'(x^*) \neq 0$. This condition is the exact analog of $\mathrm{Re}(\lambda) \neq 0$.

Strogatz (2015), p156

Hyperbolicity, Linearization, and Homemorphism

TBD

These ideas also generalize neatly to higher-order systems. A fixed point of an nth-order system is *hyperbolic* if all the eigenvalues of the linearization lie off the imaginary axis, i.e., $\mathrm{Re}(\lambda_i) \neq 0$ for $i = 1, \ldots, n$. The important *Hartman–Grobman theorem* states that the local phase portrait near a hyperbolic fixed point is "topologically equivalent" to the phase portrait of the linearization; in particular, the stability type of the fixed point is faithfully captured by the linearization. Here ***topologically equivalent*** means that there is a ***homeomorphism*** (a continuous deformation with a continuous inverse) that maps one local phase portrait onto the other, such that trajectories map onto trajectories and the sense of time (the direction of the arrows) is preserved.

Strogatz (2015), p156

Hyperbolicity, Linearization, Homemorphism

TBD

Intuitively, two phase portraits are topologically equivalent if one is a distorted version of the other. Bending and warping are allowed, but not ripping, so closed orbits must remain closed, trajectories connecting saddle points must not be broken, etc.

Hyperbolic fixed points also illustrate the important general notion of structural stability. A phase portrait is ***structurally stable*** if its topology cannot be changed by an arbitrarily small perturbation to the vector field. For instance, the phase portrait of a saddle point is structurally stable, but that of a center is not: an arbitrarily small amount of damping converts the center to a spiral.

non-hyperbolic

Strogatz (2015), p156

Linearization: An Illustration

Consider the following system of first-order ODEs

$$x' = F(x, y) \qquad\qquad y' = G(x, y)$$

Find critical points

$$F(x_c, y_c) = 0 \qquad\qquad G(x_c, y_c) = 0$$

Linearize F and G with respect to one of the critical points

$$F(x, y) = F(x_c, y_c) + F_x\,(x_c, y_c)(x - x_c) + F_y(x_c, y_c)(y - y_c)$$

$$G(x, y) = G(x_c, y_c) + G_x\,(x_c, y_c)(x - x_c) + G_y(x_c, y_c)(y - y_c)$$

Express the above in a matrix form

$$\begin{pmatrix} F(x, y) \\ G(x, y) \end{pmatrix} = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{x_c, y_c} \begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix} = J(F, G) \begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix}$$

Linearization: An Illustration

Consider the following system of first-order ODEs

$$\boxed{x' = F(x,y) \qquad\qquad y' = G(x,y)} \tag{1}$$

Express the above in a matrix form with a Jacobian matrix

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{x_c,y_c} \begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix} = J(F,G) \begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix} \tag{2}$$

Define perturbations, u and v, with respect to their critical point

$$u = x - x_c \qquad\qquad v = y - y_c \tag{3}$$

Obtain

$$u' = x' \qquad\qquad v' = y' \tag{4}$$

Plug Eqs. (3) and (4) into Eq. (2), yielding:

$$\boxed{\begin{pmatrix} u' \\ v' \end{pmatrix} = J(F,G) \begin{pmatrix} u \\ v \end{pmatrix}} \qquad\qquad J(F,G) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{x_c,y_c} \tag{5}$$

Linear Stability Analysis

Consider the following system of first-order ODEs

$$\boxed{x' = F(x,y) \qquad\qquad y' = G(x,y)}$$

Find critical points

$$F(x_c, y_c) = 0 \qquad\qquad G(x_c, y_c) = 0$$

Compute the Jacobian matrix and evaluate it at a critical point

$$J(F,G) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{x_c, y_c}$$

Solve an eigenvalue problem:

$$JV = \lambda V \qquad V = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\boxed{|J - \lambda I| = 0}$$

Linear Stability and Jacobian

THEOREM 6.2.2

Let $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ be a nonlinear system of n first-order equations with $\mathbf{x}^*$ as an equilibrium solution and $\mathbf{f}$ a sufficiently smooth vector function. Let $\mathbf{J}$ be the Jacobian (the matrix of partial derivatives) evaluated at this equilibrium solution:

$$\mathbf{J}(\mathbf{x}^*) = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots & \partial f_1/\partial x_n \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \cdots & \partial f_2/\partial x_n \\ \vdots & & \ddots & \\ \partial f_n/\partial x_1 & \partial f_n/\partial x_2 & \cdots & \partial f_n/\partial x_n \end{bmatrix}_{\mathbf{x}=\mathbf{x}^*}. \tag{6.13}$$

Let $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ be the n (real or complex, possibly repeated) eigenvalues of the Jacobian matrix.

a. If the real part of the eigenvalue $\mathrm{Re}(\lambda_i) < 0$ for all i, then the equilibrium is stable.

b. If the real part of the eigenvalue $\mathrm{Re}(\lambda_i) < 0$ for at least one i and $\mathrm{Re}(\lambda_j) > 0$ for at least one j, then the equilibrium is a saddle.

c. If the real part of the eigenvalue $\mathrm{Re}(\lambda_i) > 0$ for all i, then the equilibrium is unstable.

d. If any of the eigenvalues are complex, then the stable or unstable equilibria is a spiral; if all of the eigenvalues are real, it is a node.

e. If a pair of complex conjugate eigenvalues $\lambda_i, \overline{\lambda_i}$ satisfy $\mathrm{Re}(\lambda_i) = 0$, then the equilibrium is a linear center in the plane containing the corresponding eigenvectors.

Wirkus and Swift

Example

Consider the following system of first-order ODEs

$$x' = ax + by \quad (= F(x,y)) \qquad (1)$$

$$y' = cx + dy \quad (= G(x,y)\,) \qquad (2)$$

$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

Find critical points

$$x_c = 0 \qquad\qquad y_c = 0$$

Compute the Jacobian matrix at a critical point

$$J(F,G) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{x_c, y_c} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Solve an eigenvalue problem:

$$JV = \lambda V \qquad\qquad V = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\boxed{|J - \lambda I| = 0}$$

A Linearized Lorenz Model

$$\frac{dX}{dt} = -\sigma X + \sigma Y,$$
$$\frac{dY}{dt} = rX - Y.$$

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \qquad A = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix}$$

Complete the following to obtain the Jacobian matrix:

$$F(X_c, Y_c) = 0 \implies X_c = Y_c = 0$$

$$J(F,G) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{x_c, y_c} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix}$$

Example: 3-Dimenaional Lorenz Model (3DLM)

1) **r** – Rayleigh number: (Ra/Rc)

   a dimensionless measure of temperature difference between top and bottom surfaces of liquid; proportional to effective force on fluid

2) **σ** – Prandtl number: (ν/κ)

   the ratio of the kinetic viscosity (κ, momentum diffusivity) to thermal diffusivity (ν)

3) **b** – Physical proportion: $(4/(1+a^2))$, b=8/3.

4) **a** – a=l/m, the ratio of the vertical height h of the fluid layer to the horizontal size of the convection rolls. b =8/3.

- l=aπ/H and m=π/H,

$$\frac{dX}{d\tau} = -\sigma X + \sigma Y, \qquad M_1$$

$$\frac{dY}{d\tau} = -XZ + rX - Y, \qquad M_2$$

$$\frac{dZ}{d\tau} = XY - bZ. \qquad M_3$$

-XZ is associated with the J(M1, M3), indicating the impact of the M3 mode. With no –XZ, the above system is reduced to become a system with linear terms only, leading to an unstable solution as r>1.

Three Dimensional Systems

$$\frac{dx}{dt} = F(x, y, z)$$

$$\frac{dy}{dt} = G(x, y, z)$$

$$\frac{dz}{dt} = H(x, y, z)$$

$$J = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} & \frac{\partial H}{\partial z} \end{pmatrix}_{(x_c, y_c, z_c)} = A$$

$$x = x_c + \boxed{x_1}$$

$$y = y_c + \boxed{y_1}$$

$$z = z_c + \boxed{z_1}$$

$$F(x_c, y_c, z_c) = 0$$

$$G(x_c, y_c, z_c) = 0$$

$$H(x_c, y_c, z_c) = 0$$

A Locally Linear System

$$\begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dy_1}{dt} \\ \dfrac{dz_1}{dt} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} & \dfrac{\partial F}{\partial z} \\ \dfrac{\partial G}{\partial x} & \dfrac{\partial G}{\partial y} & \dfrac{\partial G}{\partial z} \\ \dfrac{\partial H}{\partial x} & \dfrac{\partial H}{\partial y} & \dfrac{\partial H}{\partial z} \end{pmatrix}_{(x_c, y_c, z_c)} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$$\frac{d\vec{U}}{dt} = \boxed{A}\vec{U}; \quad \vec{U} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \qquad assume \quad \vec{U} = e^{\lambda t} \begin{pmatrix} x_{eigen} \\ y_{eigen} \\ z_{eigen} \end{pmatrix} = e^{\lambda t}\vec{V}$$

A Locally Linear System

$$\frac{dx}{dt} = F(x,y,z)$$

$$\frac{dy}{dt} = G(x,y,z)$$

$$\frac{dz}{dt} = H(x,y,z)$$

$$J = \left(\begin{array}{ccc} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} & \frac{\partial H}{\partial z} \end{array} \right)_{(x_c, y_c, z_c)} = \boxed{A}$$

$$A\vec{V} = \lambda \vec{V}; \quad \vec{V} = \left(\begin{array}{c} x_{eigen} \\ y_{eigen} \\ z_{eigen} \end{array} \right)$$

$$F(x_c, y_c, z_c) = 0$$

$$G(x_c, y_c, z_c) = 0$$

$$H(x_c, y_c, z_c) = 0$$

Example: Jacobian Matrix for the 3DLM

$$\frac{dX}{d\tau} = -\sigma X + \sigma Y, \qquad F(X_c, Y_c, Z_c) = 0 \;\Rightarrow\; X_c = Y_c$$

$$\frac{dY}{d\tau} = -XZ + rX - Y, \qquad G(X_c, Y_c, Z_c) = 0 \;\Rightarrow\; Z_c = r - 1$$

$$\frac{dZ}{d\tau} = XY - bZ. \qquad H(X_c, Y_c, Z_c) = 0 \;\Rightarrow\; X_c = \pm\sqrt{b(r-1)}$$

$$J(F,G,H) = \begin{pmatrix} \frac{\partial(1)}{\partial x} & \frac{\partial(1)}{\partial y} & \frac{\partial(1)}{\partial z} \\ \frac{\partial(2)}{\partial x} & \frac{\partial(2)}{\partial y} & \frac{\partial(2)}{\partial z} \\ \frac{\partial(3)}{\partial x} & \frac{\partial(3)}{\partial y} & \frac{\partial(3)}{\partial z} \end{pmatrix}_{(x_c,\, y_c,\, z_c)} = \begin{pmatrix} -\sigma, & \sigma, & 0 \\ r - Z_c & -1 & -X_c \\ Y_c & X_c & -b \end{pmatrix}$$

Determine the Jacobian using Matlab

```
syms x y z r  theta real

x=r * cos(theta)
y=r * sin(theta)
vec1=[x, y]
vec2=[r, theta]
ja=jacobian(vec1, vec2)
jadet=abs(det(ja))
simplify(jadet)
```

➔

```
ans =
abs(r)
```

$$|J| = \mathrm{r}$$

```
syms x y z rho phi theta real

assume (phi > 0)

x=rho * sin(phi) * cos(theta)
y=rho * sin(phi) * sin(theta)
z=rho * cos(phi)
vec1=[x, y, z]
vec2=[rho, theta, phi]
ja=jacobian(vec1, vec2)
jadet=abs(det(ja))
simplify(jadet)
```

➔

```
ans =
rho^2*abs(sin(phi))
```

$$|J| = \rho^2 \sin(\phi)$$

Perturbation Theory

Perturbation theory is a large collection of iterative methods for obtaining approximate solutions to problems involving a small parameter ε.

Three typical steps of perturbative analysis are:

1. Convert the original problem into a perturbation problem by introducing the small parameter ε.
2. Assume an expression for the answer in the form of a perturbation series and compute the coefficients of that series.
3. Recover the answer to the original problem by summing the perturbation series for the appropriate value of ε.

3D Lorenz Model (3DLM)

$$\frac{dX}{d\tau} = \sigma Y - \sigma X = F,$$

$$\frac{dY}{d\tau} = -XZ + rX - Y = G,$$

$$\frac{dZ}{d\tau} = XY - bZ = H.$$

$$X = X_c + \epsilon X'$$
$$Y = Y_c + \epsilon Y'$$
$$Z = Z_c + \epsilon Z'$$

reference (or basic) state

perturbations

From Eqs. (1-3), the Jacobian matrix is written as follows:

$$A_1 = \begin{pmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \\ H_x & H_y & H_z \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - Z_c & -1 & -X_c \\ Y_c & X_c & -b \end{pmatrix}_{(X_c, Y_c, Z_c)} .$$

3DLM: Eqs for the Basic/Reference State (ε^0)

Plugging the above into Eqs. (1-3), we have

$$\frac{dX_c}{d\tau} + \epsilon\frac{dX'}{d\tau} = \sigma(Y_c + \epsilon Y') - \sigma(X_c + \epsilon X') \tag{4}$$

$$\frac{dY_c}{d\tau} + \epsilon\frac{dY'}{d\tau} = -(X_cZ_c + \epsilon X_cZ' + \epsilon Z_cX' + \epsilon^2 X'Z') + r(X_c + \epsilon X') - (Y_c + \epsilon Y') \tag{5}$$

$$\frac{dZ_c}{d\tau} + \epsilon\frac{dZ'}{d\tau} = (X_cY_c + \epsilon X_cY' + \epsilon Y_cX' + \epsilon^2 X'Y') - b(Z_c + \epsilon Z') \tag{6}$$

ε^0:

$$\cancel{\frac{dX_c}{d\tau}} = \sigma Y_c - \sigma X_c$$

$$\cancel{\frac{dY_c}{d\tau}} = -X_cZ_c + rX_c - Y_c$$

$$\cancel{\frac{dZ_c}{d\tau}} = X_cY_c - bZ_c$$

3DLM: Eqs for Perturbations (ε^1)

ε^1:

$$\frac{dX'}{d\tau} = \sigma Y' - \sigma X'$$

$$\frac{dY'}{d\tau} = -Z_c X' + rX' - Y' - X_c Z'$$

$$\frac{dZ'}{d\tau} = X' Y_c + X_c Y' - bZ'$$

$$A_2 = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - Z_c & -1 & -X_c \\ Y_c & X_c & -b) \end{pmatrix}$$

- The above derivations show that the Jacobian matrix A1 in the (nonlinear) 3DLM is the same as the matrix A2.
- In other words, the system with the equations associated with ε^1 represents the locally linearized equations of the 3DLM.

A Perturbation Method

For the “linear” case (FN=0), we have

$$\frac{dX'}{d\tau} = -\sigma X' + \sigma Y',$$

$$\frac{dY'}{d\tau} = (r - Z_c)X' - Y' - X_c Z'$$

$$\frac{dZ'}{d\tau} = Y_c X' + X_c Y' - bZ'$$

$$X = X_c + X'$$

$$Y = Y_c + Y'$$

$$Z = Z_c + Z'$$

ODE Solver

and

$$\begin{pmatrix} \dfrac{dX'}{d\tau} \\ \dfrac{dY'}{d\tau} \\ \dfrac{dZ'}{d\tau} \end{pmatrix} = \begin{pmatrix} -\sigma, & \sigma, & 0 \\ r - Z_c & -1 & -X_c \\ Y_c & X_c & -b \end{pmatrix} \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix}$$

Eigenvalue Analysis