Homework 6
Linear Algebra
Math 524
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Section 6.A Problem 1: Show that the function that takes $((x_1, x_2), (y_1, y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2$ to $|x_1 y_1| + |x_2 y_2|$ is not an inner product on $\mathbb{R}^2$

Notice:

$$\begin{aligned} < (1,1) + (-1,-1), (1,1) > &=< (1,1), (1,1) > + < (-1,-1), (1,1) > \\ &= 2 + 2 = 4 \neq 0 =< (0,0), (1,1) > \end{aligned}$$

Because the function does not hold additivity in the first slot, it is not an inner product on $\mathbb{R}^2$

Section 6.A Problem 2: Show that the function that takes $((x_1, x_2, x_3), (y_1, y_2, y_3)) \in \mathbb{R}^3 \times \mathbb{R}^3$ to $x_1 y_1 + x_3 y_3$ is not an inner product on $\mathbb{R}^3$

Notice with $(x_2 \neq 0, y_2 \neq 0)$

$$< (0, x_2, 0), (0, y_2, 0) >= 0$$

Because the function does not hold for definiteness, it is not an inner product on $\mathbb{R}^3$

Section 6.A Problem 4: Suppose V is a real inner product space

(a) Show that $< u+v, u-v >= ||u||^2 - ||v||^2$ for every $u, v \in V$

$$\begin{aligned} < u+v, u-v > &=< u,u > + < u,-v > + < v,u > + < v,-v > \\ &=< u,u > +- < u,v > + < u,v > - < v,v > \\ &= ||u||^2 - ||v||^2 \end{aligned}$$

(b) Show that if $u, v \in V$ have the same norm, then $u+v$ is orthogonal to $u-v$

Let $||u|| = ||v||$

$$\begin{aligned} ||u||^2 &= ||v||^2 \\ < u+v, u-v > &= ||u||^2 - ||v||^2 = 0 \end{aligned}$$

Because $< u+v, u-v >= 0$, then $u+v$ is orthogonal to $u-v$

(c) Use part (b) to show that the diagonals of a rhombus are perpendicular to each other

Because the diagonals of a rhombus with 2 sides being u and the other 2 being v can be written as $u+v$ and $u-v$, part (b) shows us that they are orthogonal, or perpendicular.

Section 6.A Problem 5: Suppose $T \in \mathcal{L}(V)$ is such that $||Tv|| \leq ||v||$ for every $v \in V$. Prove that $T - \sqrt{2}I$ is invertible.

Notice: For T not to be invertible, then $\det(T - \lambda I) = 0$. To find the eigenvalues, λ, then

$$Tv = \lambda v$$

$$\begin{aligned} ||Tv|| &= \sqrt{< Tv, Tv >} \\ &= \sqrt{< \lambda v, \lambda v >} \\ &= \sqrt{\lambda^2}\sqrt{< v, v >} \\ &= |\lambda| ||v|| \\ &\leq ||v|| \end{aligned}$$

$$||Tv|| = |\lambda| ||v|| \leq ||v||$$

Thus $-1 \leq \lambda \leq 1$. Because $\sqrt{2}$ is not within the interval, it is also not an eigenvalue. So $\det(T - \sqrt{2}I) \neq 0$, thus $T - \sqrt{2}I$ is invertible.

Section 6.B Problem 1: Suppose $\theta \in R$. Show that $(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)$ and $(\cos\theta, \sin\theta), (\sin\theta, -\cos\theta)$ are orthonormal bases of $\mathbb{R}^2$.

$$\begin{aligned}
||(\cos\theta, \sin\theta)|| &=< (\cos\theta, \sin\theta), (\cos\theta, \sin\theta) > \\
&= \sqrt{\cos^2\theta + \sin^2\theta} \\
&= \sqrt{1} = 1 \\
||(-\sin\theta, \cos\theta)|| &=< (-\sin\theta, \cos\theta), (-\sin\theta, \cos\theta) > \\
&= \sqrt{\sin^2\theta + \cos^2\theta} \\
&= \sqrt{1} = 1 \\
||(\sin\theta, -\cos\theta)|| &=< (\sin\theta, -\cos\theta), (\sin\theta, -\cos\theta) > \\
&= \sqrt{\sin^2\theta + \cos^2\theta} \\
&= \sqrt{1} = 1
\end{aligned}$$

$$\begin{aligned}
< (\cos\theta, \sin\theta), (-\sin\theta, \cos\theta) > &= \sqrt{-\cos\theta\sin\theta + (\sin\theta\cos\theta} = 0 \\
< (\cos\theta, \sin\theta), (\sin\theta, -\cos\theta) > &= \sqrt{\cos\theta\sin\theta - \sin\theta\cos\theta} = 0
\end{aligned}$$

Because the 2 lists are orthonormal and their length, 2, is equal to $\mathbb{R}^2$'s dimension, then both lists are orthonormal bases.

Section 6.B Problem 3: Suppose $T \in \mathcal{L}(\mathbb{R}^3)$ has an upper-triangular matrix with respect to the basis $(1,0,0),(1,1,1),(1,1,2)$. Find an orthonormal basis of $\mathbb{R}^3$ with respect to which T has an upper-triangular matrix.

Because we have an upper-triangular matrix with respect to some basis, there exists an orthonormal basis e_1, e_2, e_3 that can be calculated from the Gram-Schmidt Procedure, to which T has an upper triangular matrix.

Let $v_1 = (1,0,0), v_2 = (1,1,1), v_3 = (1,1,2)$

$$e_1 = \frac{v_1}{||v_1||} = \frac{(1,0,0)}{1} = (1,0,0)$$

$$e_2 = \frac{v_2 - < v_2, e_1 > e_1}{||v_2 - < v_2, e_1 > e_1||} = \frac{(1,1,1) - (1,0,0)}{||(1,1,1) - (1,0,0)||} = \frac{(0,1,1)}{\sqrt{2}} = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$$

$$e_3 = \frac{v_3 - < v_3, e_1 > e_1 - < v_3, e_2 > e_2}{||v_3 - < v_3, e_1 > e_1 - < v_3, e_2 > e_2||}$$

$$= \frac{(1,1,2) - (1,0,0) - \frac{3}{\sqrt{2}}(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})}{||1,1,2) - (1,0,0) - \frac{3}{\sqrt{2}}(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})||} = \frac{(0,1,2) - (0, \frac{3}{2}, \frac{3}{2})}{||(0,1,2) - (0, \frac{3}{2}, \frac{3}{2})||}$$

$$= \sqrt{2}(0, \frac{-1}{2}, \frac{1}{2}) = (0, \frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$$

Thus e_1, e_2, e_3 is an orthonormal basis of $\mathbb{R}^3$ with respect to which T has an upper-triangular matrix.

Section 6.B Problem 9: What happens if the Gram–Schmidt Procedure is applied to a list of vectors that is not linearly independent?

Let $v_1, ..., v_j$ be a linearly dependent list of vectors and $v_1, ..., v_{j-1}$ be a linearly independent list such that

$$v_j \in \text{span}(v_1, ..., v_{j-1})$$

Because $v_1, ..., v_{j-1}$ is linearly independent, it can be used to turn into an orthonormal list $e_1, ..., e_{j-1}$ and by 6.30, $v_j =< v_j, e_1 > e_1 + ...+ < v_j, e_{j-1} > e_{j-1}$. If we try to apply Gram-Schmidt on v_j, you would get a denominator of 0, Thus Gram-Schmidt doesn't work on linearly dependent lists.

Section 6.C Problem 5: Suppose V is finite-dimensional and U is a subspace of V. Show that $P_{U^\perp} = I - P_U$, where I is the identity operator of V.

By 6.47, $V = U \oplus U^\perp$. This means that $\forall v \in V, v = u + w$, with $u \in U$ and $w \in U^\perp$. By definition of $P_U, P_U(v) = w$. And by definition of $P_{U^\perp}, P_{U^\perp}(v) = u$.

$$P_{U^\perp}(v) = u = u + w - w = (u + w) - w = I(v) - P_U(v) = (I - P_U)(v)$$

Thus $P_{U^\perp} = I - P_U$

Section 6.C Problem 11: In $\mathbb{R}^4$, Let

$$U = \text{span}\left((1, 1, 0, 0), (1, 1, 1, 2)\right)$$

Find $u \in U$ such that $||u - (1, 2, 3, 4)||$ is as small as possible.

Using Gram-Schmidt, we can find an orthonormal basis

$$\begin{aligned} e_1 &= \frac{(1,1,0,0)}{\sqrt{2}} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0) \\ e_2 &= \frac{(1,1,1,2)- < (1,1,1,2), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0) > (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)}{||(1,1,1,2)- < (1,1,1,2), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0) > (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)||} \\ &= \frac{(0,0,1,2)}{\sqrt{5}} = (0, 0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}) \end{aligned}$$

$P_U(1, 2, 3, 4) =< (1, 2, 3, 4), e_1 > e_1+ < (1, 2, 3, 4), e_2 > e_2$ is the closest $u \in U$ to $(1, 2, 3, 4)$ by 6.56. Which equals

$$(\frac{3}{2}, \frac{3}{2}, \frac{11}{5}, \frac{22}{5})$$