MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #27

Approximate Solution of Linear Differential Equations

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A Note

- You may recall our promise that the style of mathematics would change.
- When we derive a local series about an ordinary point or a regular singular point, we use equal signs because such a series actually converges to a solution of the differential equation.
- We have learned in this section that the local behavior of a solution near an irregular singular point may be relatively, but not approximately, equal to the exact solution because the solution changes so rapidly near such a point.
- We have had to exchange equality and approximate equality for relative (asymptotic) equality.
- When a solution is changing rapidly, an asymptotic relation may be far more informative than an approximate equality.

Leading Behavior vs. Controlling Factor

- We will refer to the first term in a series as the leading behavior of the series.
- The leading behavior is determined by those contributions to $S(x)$ that do not vanish as x approaches the irregular singularity.
- We will also refer to the most rapidly changing component of the leading behavior in the limit x → x_o as the controlling factor.

The following represent the first terms of infinite series…

y_1 $$y(x) \sim c_1 x^{3/4} \boxed{e^{2x^{-1/2}}}, \qquad x \to 0+; \qquad \boxed{(3.4.4a)}$$

(discussed below)

the other solution has the behavior

y_2 $$y(x) \sim \boxed{c_2 x^{3/4} \boxed{e^{-2x^{-1/2}}}}, \qquad x \to 0+. \qquad (3.4.4b)$$

leading behavior (three terms)

controlling factor (exponential)

- Observe that these behaviors all involve exponentials of functions which become singular at the irregular singularity of the differential equation.
- Thus, these two functions have essential singularities at x = 0.

Procedures for Obtaining Asymptotic Solutions

$$y = \boxed{c_1 x^{\frac{3}{4}} e^{\frac{2}{\sqrt{x}}}} \boxed{\sum_{n=0}^{\infty} a_n (x)^{n/2}}$$

leading behavior　　asymptotic series

- First, by means of the exponential substitution $y = \exp(S(x))$ we determined the leading behavior of y(x) as $x \to x_o$.

$$S = 2x^{-1/2} + \frac{3}{4}\ln(x) + d \qquad y_{leading} \sim c_1 x^{3/4} e^{2/\sqrt{x}}$$

leading behavior

- Next, we refined this approximation to y(x) by
  (1) peeling or factoring off the leading behavior and

$$y = y_{leading} * w(x), \qquad w(x) = 1 + \epsilon(x)$$

  (2) expanding what remains as a series of fractional powers of $(x - x_0)$

$$\boxed{w(x) = \sum_{n=0}^{\infty} a_n (x^{\alpha})^n}$$

asymptotic series

Asymptotic Series vs. Convergent Power Series

	convergent power series	divergent asymptotic series
limit to the accuracy	no limit	an upper limit
Improving the accuracy	taking more terms	finding the optimal number of terms

- We can compare this rule with the way we would evaluate the sum of the convergent power series for a fixed value of x. For this series, there is no limit to the accuracy; we can always improve the accuracy by taking more terms in the partial sum.
- However, for a divergent asymptotic series, for each given value of x there is an upper limit to the accuracy and if we take either more or less than the optimal number of terms in the partial sum according to our rule, we usually decrease the accuracy.
- If we are not satisfied with this maximal accuracy, then to improve it we must take x closer to x_o or in the case of the series in (3.5.10) we must take x closer to $+\infty$.)

Power, Frobenius, and Asymptotic Series

Power Series

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Frobenius Series

$$y = x^r \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Asympotic Series

$$w(x) = \sum_{n=0}^{\infty} a_n (x^{\alpha})^n$$

α may be non-integers.

leading behavior

$\sim \exp(S(x))$

$$y = \boxed{c_1 x^{\frac{3}{4}} e^{\frac{2}{\sqrt{x}}}} \sum_{n=0}^{\infty} a_n (x)^{n/2}$$

leading behavior

asymptotic series

An Introduction to Asymptotics

(I) We must introduce two new symbols which express the relative behavior of two functions. The notation

$$f(x) \ll g(x), \qquad x \to x_0,$$

which is read "$f(x)$ is much smaller than $g(x)$ as x tends to x_0," means

much smaller than

$$\boxed{\lim_{x \to x_0} f(x)/g(x) = 0.}$$

(II) Second, the notation

$$f(x) \sim g(x), \qquad x \to x_0,$$

which is read "$f(x)$ is asymptotic to $g(x)$ as x tends to x_0," means that the relative error between f and g goes to zero as $x \to x_0$:

$$f(x) - g(x) \ll g(x), \qquad x \to x_0,$$

or, equivalently,

asymptotic to

$$\boxed{\lim_{x \to x_0} f(x)/g(x) = 1.}$$

Note that if $f(x) \sim g(x)$ $(x \to x_0)$ then $g(x) \sim f(x)$ $(x \to x_0)$.

Example 4 *Asymptotic relations.*

(*a*) $x \ll 1/x$ $(x \to 0)$.
(*b*) $x^{1/2} \ll x^{1/3}$ $(x \to 0+)$, where the limit $x \to 0+$ means that x approaches 0 through positive values only.
(*c*) $(\log x)^5 \ll x^{1/4}$ $(x \to +\infty)$.
(*d*) $x^{1/2} \sim 2$ $(x \to 4)$.
(*e*) $e^x + x \sim e^x$ $(x \to +\infty)$, but note that the difference between the left and right sides of this relation, x, goes to ∞ as $x \to +\infty$. Thus, even if two functions are asymptotic, they may not be approximately equal.
(*f*) $x^2 \not\sim x$ $(x \to 0)$ because x^2 and x approach 0 at different rates. In this case, even though x^2 and x are approximately equal to 0 as $x \to 0$, they are not asymptotic.
(*g*) It is a common mistake to assert that a function is asymptotic to zero. For example, the equation $\boxed{x^3 \sim 0 \ (x \to 0) \text{ is wrong;}}$ by definition, no nonzero function can ever be asymptotic to zero.
(*h*) $x \ll -10$ $(x \to 0+)$, even though the signs are different.

Function vs. Series

No good

(*g*) It is a common mistake to assert that a function is asymptotic to zero. For example, the equation $\boxed{x^3 \sim 0\ (x \to 0) \text{ is wrong}}$; by definition, no nonzero function can ever be asymptotic to zero.

good

$w(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}$ $(x \to +\infty;\ a_0 = 1)$. Substituting this expansion into the differential equation for w gives

$$\boxed{0 \sim} \sum_{n=0}^{\infty} n(n+1)a_n x^{-n} - 2\sum_{n=0}^{\infty} n a_n x^{1-n} + \left(\tfrac{1}{4} - \nu^2\right)\sum_{n=0}^{\infty} a_n x^{-n}, \qquad x \to +\infty.$$

You might think that this asymptotic relation is formulated poorly because in Sec. 3.4 we warned that a function could not be asymptotic to 0. However, by the definition of an asymptotic power series, the function 0 *does* have an asymptotic power series expansion whose coefficients are all 0. Therefore, since the coefficients of any asymptotic power series are unique, we may equate to 0 the coefficients of all powers of $1/x$ in the above relation:

(p. 93)

Asymptotic (Power) Series

Definition The power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is said to be *asymptotic to the function* $y(x)$ *as* $x \to x_0$ and we write $y(x) \sim \sum_{n=0}^{\infty} a_n(x-x_0)^n$ $(x \to x_0)$ if $y(x) - \sum_{n=0}^{N} a_n(x-x_0)^n \ll (x-x_0)^N$ $(x \to x_0)$ for every N.

$$y \sim \sum_{n=0}^{\infty} a_n(x-x_0)^n$$

$$\varepsilon_N = y - \sum_{n=0}^{N} a_n(x-x_0)^n \qquad \left(\varepsilon_N = \sum_{n=N+1}^{\infty} a_n(x-x_0)^n\right)$$

The first term is $(x-x_0)^{N+1}$.

$$\varepsilon_N << (x-x_0)^N$$

Asymptotic (Power) Series

We also encounter asymptotic series in nonintegral powers of $x - x_0$. The series $\sum_{n=0}^{\infty} a_n(x-x_0)^{\alpha n}$ $(\alpha > 0)$ is asymptotic to the function $y(x)$ if $y(x) - \sum_{n=0}^{N} a_n(x-x_0)^{\alpha n} \ll (x-x_0)^{\alpha N}$ $(x \to x_0)$, for every N.

$$y \sim \sum_{n=0}^{\infty} a_n(x-x_0)^{\alpha n} \qquad (\alpha > 0)$$

$$\varepsilon_N = y - \sum_{n=0}^{N} a_n(x-x_0)^{\alpha n} << (x-x_0)^{\alpha N}$$

If $x_0 = \infty$ the corresponding definition is $y(x) \sim \sum_{n=0}^{\infty} a_n x^{-\alpha n}$ $(x \to \infty)$ if $y(x) - \sum_{n=0}^{N} a_n x^{-\alpha n} \ll x^{-\alpha N}$ $(x \to \infty)$ for every N.

$$y \sim \sum_{n=0}^{\infty} a_n(x-x_0)^{-\alpha n} \qquad (\alpha > 0, x \to \infty)$$

$$\varepsilon_N = y - \sum_{n=0}^{N} a_n(x-x_0)^{-\alpha n} << (x-x_0)^{-\alpha N} \ (x \to \infty)$$

Asymptotic Series: Generalizations of Taylor Series

Example 1 *Taylor series as asymptotic series.* If the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges for $|x-x_0|<R$ to the function $f(x)$, then the series is also asymptotic to $f(x)$ as $x \to x_0$: $f(x) \sim \sum_{n=0}^{\infty} a_n(x-x_0)^n$ $(x \to x_0)$. Since $a_n = f^{(n)}(x_0)/n!$, repeated application of l'Hôpital's rule gives

$$\lim_{x\to x_0} \frac{f(x) - \sum_{n=0}^{N} a_n(x-x_0)^n}{(x-x_0)^{N+1}} = a_{N+1}.$$

Thus, $\varepsilon_N(x) = f(x) - \sum_{n=0}^{N} a_n(x-x_0)^n \ll (x-x_0)^N$ $(x \to x_0)$. We conclude that asymptotic series are generalizations of Taylor series because they include Taylor series as special cases.

$$f(x) \sim \sum_{n=0}^{\infty} a_n(x-x_0)^n \qquad a_n = \frac{f^{(n)}(x_0)}{n!}$$

Taylor Series

$$\left(\varepsilon_N = \sum_{n=N+1}^{\infty} a_n(x-x_0)^n\right) \qquad \lim_{x\to x_0} \frac{\varepsilon_N}{(x-x_0)^{N+1}} = a_{N+1}$$

$$\varepsilon_N = a_{N+1}(x-x_0)^{N+1} + a_{N+2}(x-x_0)^{N+2} + \cdots << (x-x_0)^N$$

Asymptotic Series

Asymptotic Series vs. Convergent Power Series

$$\text{Convergent:} \quad \varepsilon_N(x) = \sum_{n=N+1}^{\infty} a_n(x - x_0)^n \to 0, \qquad N \to \infty;\ x \text{ fixed.}$$

remainder

error terms

$$\text{Asymptotic:} \quad \varepsilon_N(x) \ll (x - x_0)^N, \qquad x \to x_0;\ N \text{ fixed.}$$

the next term

Convergent vs. Asymptotic vs. Power Series

$$\text{Convergent:} \quad \varepsilon_N(x) = \sum_{n=N+1}^{\infty} a_n (x - x_0)^n \to 0, \qquad N \to \infty;\ x \text{ fixed.}$$

- ϵ_N goes to zero as $N \to \infty$.
- Convergence is an absolute concept.

$$\text{Asymptotic:} \quad \varepsilon_N(x) \ll (x - x_0)^N, \qquad x \to x_0;\ N \text{ fixed.}$$

- ϵ_N goes to zero faster than $(x - x_o)^N$, but needs not to go to zero as $N \to \infty$.
- Asymptoticity is an relative property.

Properties of Asymptotic Series

- Nonuniqueness

  The "sum" of a divergent power series is not uniquely determined.

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \Rightarrow \quad f(x) + \underbrace{e^{-(x-x_0)^{-2}}}_{\text{subdominant}} \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n \; (x \to x_0)$$

- Uniqueness

  There is only one asymptotic power series for each function.

  $f(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n \Rightarrow$ The expansion coefficients are unique.

- Equating coefficients in asymptotic series
- Arithmetical operations on asymptotic series

  Arithmetical operations may be performed term by term on asymptotic series.

- Integration of asymptotic series

  Any asymptotic series can be integrated term by term if f(x) is integrable near X_o.

- Differentiation of asymptotic series

  Asymptotic series cannot in general be differentiated term by term.

Optimal Asymptotic Approximation

- We look over the individual terms in the asymptotic series; typically the terms get successively smaller for a while, but eventually, because the series is known to diverge, they get larger and larger and tend to infinity.
- For every given value of x we locate the smallest term (i.e., the smallest contribution).
- We then add all the preceding terms in the asymptotic series up to but not including the smallest term. This finite sum of terms usually gives the best estimate of the function because the next term, which approximates the error, is the smallest term in the series.

Example 3 *Numerical evaluation of the asymptotic series for $I_\nu(x)$.* In the last example we obtained an asymptotic expansion for $I_\nu(x)$ valid as $x \to +\infty$. Let us use this expansion to compute $I_5(x)$ for various values of x. Setting $c_1 = (2\pi)^{-1/2}$ and $\nu = 5$ in (3.5.8*a*) and (3.5.9*a*), we obtain the expansion for $I_5(x)$:

$$I_5(x) \sim (2\pi)^{-1/2} e^x x^{-1/2} \left[1 - \frac{(100-1)}{1!\,8x} + \frac{(100-1)(100-9)}{2!\,(8x)^2} - \cdots\right], \qquad x \to +\infty. \tag{3.5.10}$$

Optimal Asymptotic Approximation

- Dependence of optimal approximation on x and N.

partial sums as optimal asymptotic approximations

N	x = 3.0	4.0	5.0	6.0	7.0
0	2.30324 (−1)	1.99471 (−1)	1.78412 (−1)	1.62868 (−1)	1.50786 (−1)
2	1.08147 (0)	4.59816 (−1)	2.39128 (−1)	1.45372 (−1)	1.00804 (−1)
4	2.01953 (−1)	4.74361 (−2)	2.52641 (−2)	2.35810 (−2)	2.61284 (−2)
6	2.11127 (−2)	1.14538 (−2)	1.49262 (−2)	1.98392 (−2)	2.45412 (−2)
7	1.16597 (−2)	1.03611 (−2)	1.47212 (−2)	1.97870 (−2)	2.45248 (−2)
8	5.50542 (−3)	9.82749 (−3)	1.46411 (−2)	1.97700 (−2)	2.45202 (−2)
9	1.20401 (−4)	9.47732 (−3)	1.45991 (−2)	1.97626 (−2)	2.45184 (−2)
10	−5.73580 (−3)	9.19172 (−3)	1.45717 (−2)	1.97585 (−2)	2.45176 (−2)
11	−1.33001 (−2)	8.91505 (−3)	1.45504 (−2)	1.97559 (−2)	2.45172 (−2)
12	−2.45677 (−2)	8.60595 (−3)	1.45314 (−2)	1.97540 (−2)	2.45169 (−2)
13	−4.35276 (−2)	8.21586 (−3)	1.45122 (−2)	1.97523 (−2)	2.45167 (−2)
14	−7.90210 (−2)	7.66817 (−3)	1.44907 (−2)	1.97508 (−2)	2.45166 (−2)
15	−1.52078 (−1)	6.82267 (−3)	1.44641 (−2)	1.97492 (−2)	2.45164 (−2)
20	−1.31437 (1)	−3.61663 (−2)	1.39178 (−2)	1.97329 (−2)	2.45155 (−2)
35	−3.12759 (10)	−1.24079 (6)	−4.90286 (2)	−8.13340 (−1)	2.06197 (−2)
	Exact value of $e^{-x}I_5(x)$				
	4.54090 (−3)	9.24435 (−3)	1.45403 (−2)	1.97519 (−2)	2.45164 (−2)
	Relative error in optimal asymptotic approximation, %				
	21.0	0.57	0.069	0.0024	0.000071

exact solutions

relative error

Table 3.1 Asymptotic approximations to $e^{-x}I_5(x)$ for five values of x using the series in (3.5.10)

Entries in the columns are the partial sums truncated after the x^{-N} term. Underlined partial sums are optimal asymptotic approximations. Notice that even when $x = 7$ the leading term in the asymptotic expansion gives a very poor approximation while the optimal asymptotic truncation is very accurate. The number in parentheses is the power of 10 multiplying the entry.

partial sums as optimal asymptotic approximations

	x				
N	3.0	4.0	5.0	6.0	7.0
0	2.30324 (−1)	1.99471 (−1)	1.78412 (−1)	1.62868 (−1)	1.50786 (−1)
2	1.08147 (0)	4.59816 (−1)	2.39128 (−1)	1.45372 (−1)	1.00804 (−1)
4	2.01953 (−1)	4.74361 (−2)	2.52641 (−2)	2.35810 (−2)	2.61284 (−2)
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15	−1.52078 (−1)	6.82267 (−3)	1.44641 (−2)	1.97492 (−2)	2.45164 (−2)
20	−1.31437 (1)	−3.61663 (−2)	1.39178 (−2)	1.97329 (−2)	2.45155 (−2)
35	−3.12759 (10)	−1.24079 (6)	−4.90286 (2)	−8.13340 (−1)	2.06197 (−2)
	Exact value of $e^{-x}I_5(x)$				
	4.54090 (−3)	9.24435 (−3)	1.45403 (−2)	1.97519 (−2)	2.45164 (−2)
	Relative error in optimal asymptotic approximation, %				
	21.0	0.57	0.069	0.0024	0.000071

exact solutions

Accuracy -→

Asymptotic Approximation to exp(-x)I_5(x)

- Dependence of optimal approximation on x and N.

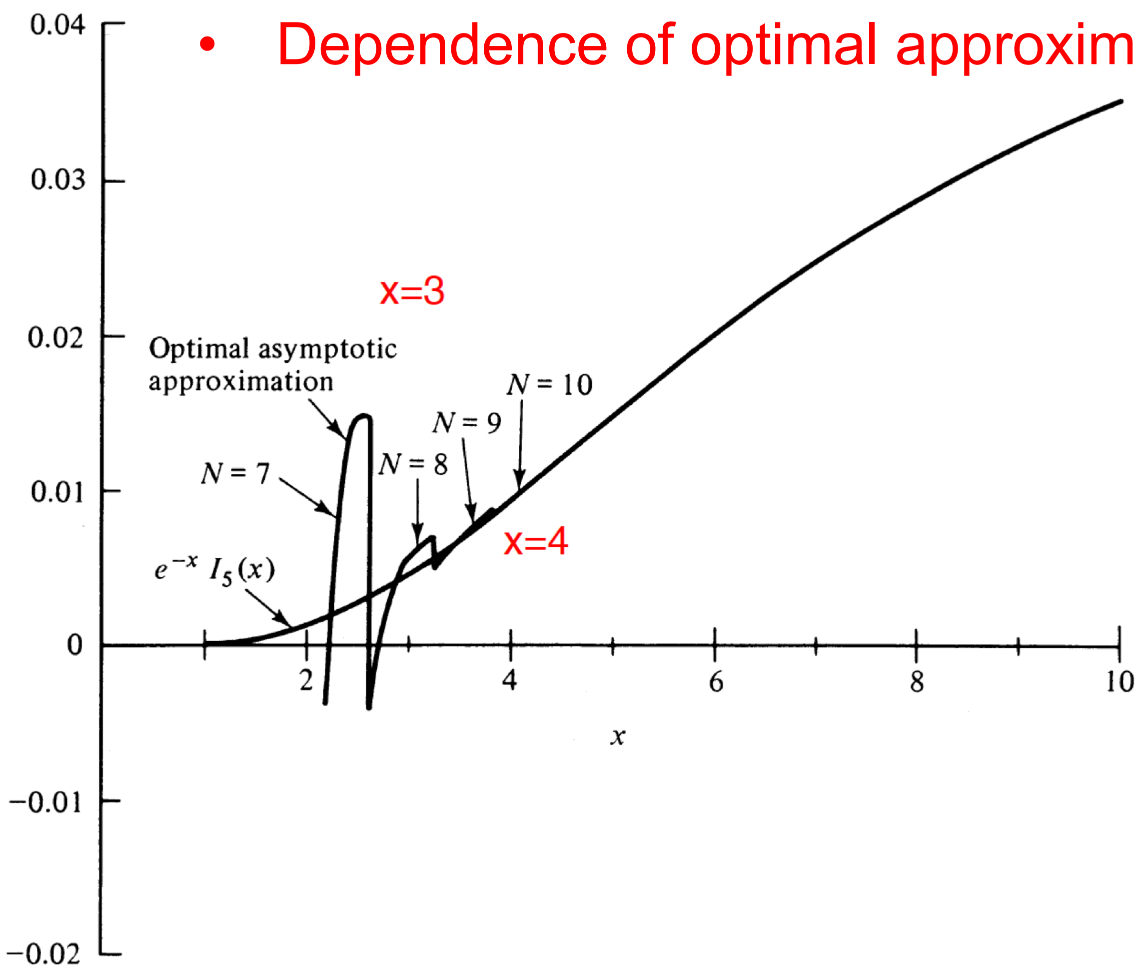
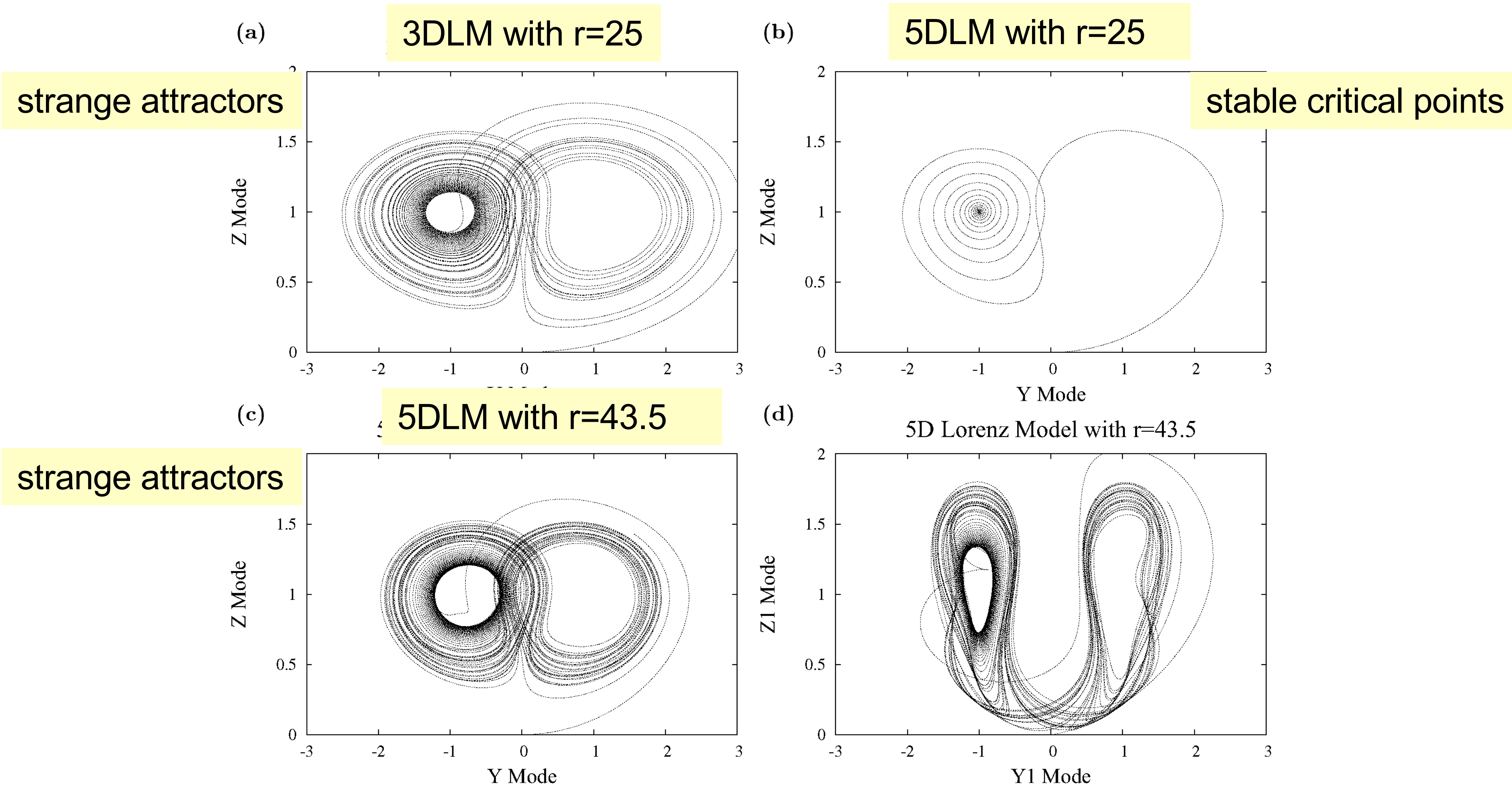


Figure 3.5 A plot of the optimal asymptotic approximation to $e^{-x}I_5(x)$ for $2 \le x \le 10$. For comparison, the exact numerical value of $e^{-x}I_5(x)$ is also shown for $0 \le x \le 10$. These two curves are indistinguishable when $x > 4$. The discontinuities in the optimal asymptotic approximation occur when the optimal number of terms increases by one. Each segment of the optimal asymptotic approximation is labeled by a number N which is the highest power of $1/x$ in the optimal truncation. [Note that we have chosen to plot $e^{-x}I_5(x)$ instead of $I_5(x)$ itself because $I_5(x)$ rapidly runs off scale as x increases.]

Dependence on Mode Truncation in Nonlinear Models

Dependence on Mode Truncation in Nonlinear Models



Shen, B.-W., **2014a:** Nonlinear Feedback in a Five-dimensional Lorenz Model. J. of Atmos. Sci., 71, 1701–1723. doi: http://dx.doi.org/10.1175/JAS-D-13-0223.1

Dependence on Mode Truncation in Lorenz Models

Table 1: Fourier modes selected to construct the 3DLM and higher-order LMs, which is from Table 1 of Roy and Musielak (2007c). The critical values of the normalized Raleigh parameter, shown in red, are derived from Table 2 of Roy and Musielak (2007c).

Table 1
Fourier modes selected to construct the original 3D Lorenz system and other generalized Lorenz systems

Model	Circulation modes	Temperature modes	Temperature modes with $m = 0$		References
3D	$\Psi_1(1,1)$	$\Theta_2(1,1)$	$\Theta_2(0,2)$	rc~24.75	Lorenz [1]
5D	$\Psi_1(1,1)$ $\Psi_1(2,1)$	$\Theta_2(1,1)$ $\Theta_2(2,1)$	$\Theta_2(0,2)$	rc~22.50	Paper II
6D	$\Psi_1(1,1)$ $\Psi_1(2,1)$ $\Psi_1(1,2)$	$\Theta_2(1,1)$ $\Theta_2(2,1)$	$\Theta_2(0,2)$	n/a	Humi [9]
6D	$\Psi_1(1,1)$ $\Psi_1(1,3)$	$\Theta_2(1,1)$ $\Theta_2(1,3)$	$\Theta_2(0,2)$ $\Theta_2(0,4)$	rc~40.15	Kennamer [10]
8D	$\Psi_1(1,1)$ $\Psi_1(2,1)$ $\Psi_1(1,2)$	$\Theta_2(1,1)$ $\Theta_2(2,1)$ $\Theta_2(1,2)$	$\Theta_2(0,2)$ $\Theta_2(0,4)$	rc~35.60	This Paper
9D	$\Psi_1(1,1)$ $\Psi_1(1,2)$ $\Psi_1(1,3)$	$\Theta_2(1,1)$ $\Theta_2(1,2)$ $\Theta_2(1,3)$	$\Theta_2(0,2)$ $\Theta_2(0,4)$ $\Theta_2(0,6)$	rc~40.50	Paper I

Dependence on Mode Truncation in Lorenz Models

model	Ψ	Θ	Θ	rc	References
5DLM	$\psi_1(1,1)$	$\Theta_2(1,1)$, $\Theta_2(1,3)$	$\Theta_2(0,2)$, $\Theta_2(0,4)$	42.9	Shen (2014)
6DLM	$\psi_1(1,1)$, $\psi_1(1,3)$,	$\Theta_2(1,1)$ $\Theta_2(1,3)$	$\Theta_2(0,2)$ $\Theta_2(0,4)$	41.1	Shen(2015)
7DLM	$\psi_1(1,1)$	$\Theta_2(1,1)$, $\Theta_2(1,3)$, $\Theta_2(1,5)$	$\Theta_2(0,2)$, $\Theta_2(0,4)$, $\Theta_2(0,6)$,	~116.9	Shen (2016)
8DLM	$\psi_1(1,1)$, $\psi_1(1,3)$	$\Theta_2(1,1)$, $\Theta_2(1,3)$, $\Theta_2(1,5)$	$\Theta_2(0,2)$, $\Theta_2(0,4)$, $\Theta_2(0,6)$	~103.4	(Shen (2017)
9DLM	$\psi_1(1,1)$, $\psi_1(1,3)$, $\psi_1(1,5)$	$\Theta_2(1,1)$, $\Theta_2(1,3)$, $\Theta_2(1,5)$	$\Theta_2(0,2)$, $\Theta_2(0,4)$, $\Theta_2(0,6)$	~102.9	(Shen (2017)
14DLM	$\psi_1(1,1)$, $\psi_1(1,3)$, $\psi_1(2,2)$, $\psi_1(2,4)$, $\psi_1(3,1)$, $\psi_1(3,3)$	$\Theta_2(1,1)$, $\Theta_2(1,3)$, $\Theta_2(2,2)$, $\Theta_2(2,4)$, $\Theta_2(3,1)$, $\Theta_2(3,3)$	$\Theta_2(0,2)$, $\Theta_2(0,4)$	rc~43	Curry (1978)
10EQs	$\psi_1(1,1)$ $\psi_1(2,2)$	$\Theta_2(1,1)$, $\Theta_2(2,2)$	$\Theta_2(0,2)$, $\Theta_2(0,4)$	n/a	Lucarini and K. Fraedrich (2009)

Dependence on Mode Truncation in the GLM

model	r_c	heating terms	solutions	references
3DLM	24.74	rX	steady, chaotic, or LC	Lorenz (1963)
3D-NLM	n/a	rX	periodic	Shen (2018)
5DLM	42.9	rX	steady, chaotic, or LC/LT	Shen (2014a,2015a,b)
5D-NLM	n/a	rX	quasi-periodic	Faghih-Naini and Shen (2018)
6DLM	41.1	rX, rX_1	steady or chaotic	Shen (2015a,b)
7DLM	116.9	rX	steady, chaotic or LC/LT	Shen (2016, 2017)
7D-NLM	n/a	rX	quasi-periodic	Shen and Faghih-Naini (2017)
8DLM	103.4	rX, rX_1	steady or chaotic	Shen (2017)
9DLM	102.9	rX, rX_1, rX_2	steady or chaotic	Shen (2017)
9DLMr	679.8	rX	steady, chaotic, or LC/LT	Shen (2019a)

rc: a critical value of the Raleigh parameter for the onset of chaos; LC: limit cycle; LT: limit torus

Aggregated Negative Feedback

Method of Dominant Balance: An Illustration

$$y'' + p(x)y' + q(x)y = 0$$

$\boxed{y = e^{S(x)}}$ $\qquad y' = S'e^{S(x)}$ $\qquad y'' = S''e^{S(x)} + (S')^2 e^{S(x)}$

$$S''e^{S(x)} + (S')^2 e^{S(x)} + pS'e^{S(x)} + qe^{S(x)} = 0$$

$S'' + (S')^2 + pS' + q = 0$ divide by $e^{s(x)}$

$\boxed{S'' << (S')^2, \quad \text{as } x \to x_0}$

1. drop (all) terms that are small

$\boxed{(S')^2 \sim -pS' - q, \text{ as } x \to x_0}$

2. replace "=" by "~" and solve the system (note that don't have "0" on the RHS)
3. check whether the solution is consistent with the approximation, i.e., whether $S'' << (S')^2$ is valid.

Method of Dominant Balance: How

The method of dominant balance is used to identify those terms in an equation that may be neglected in an asymptotic limit, (i.e., S''). The technique consist of three steps:

1. We drop all terms that appear small and replace the exact equation by an asymptotic relation.
2. We replace the asymptotic relation with an equation by exchanging the ~ sign for an = sign and solve the resulting equation exactly (the solution to this equation automatically satisfies the asymptotic relation although it is certainly not the only function that does so).
3. We check that the solution we have obtained is consistent with the approximation made in (1). If it is consistent, we must still show that the equation for the function obtained by factoring off the dominant balance solution from the exact solution itself has a solution that varies less rapidly than the dominant balance solution. When this happens, we conclude that the controlling factor (and not the leading behavior, i.e., the first term) obtained from the dominant balance relation is the same as that of the exact solution.

``stoppage criteria'' : The leading behavior of y(x) is determined by just those contributions to S(x) that do not vanish as x approaches the irregular singularity.

Exceptions for the Assumption of $S'' << (S')^2$

(D) **3.32** The differential equation $y'' + x^{-2}e^{1/x}\sin(e^{1/x})y = 0$ has an irregular singular point at $x = 0$. Show that if we make the exponential substitution $y = e^S$, it is *not* correct to assume $S'' \ll (S')^2$ as $x \to 0+$. What is the leading behavior of $y(x)$?