Midterm - In Class
Partial Differential Equations
Math 531
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Problem 1: Given the following partial differential equation:

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \qquad 0 < x < a, \qquad 0 < y < b, \qquad t > 0$$

with boundary conditions

$$u(0, y, t) = 0, \qquad u(a, y, t) = 0 \qquad 0 < y < b, \qquad t > 0$$

$$u(x, 0, t) = 0, \qquad u(x, b, t) = 0 \qquad 0 < x < a, \qquad t > 0$$

and initial conditions

$$u(x, y, 0) = f(x, y), \qquad 0 < x < a, \qquad 0 < y < b$$

(a) If we separate the variables by letting $u = \phi(x)g(y)h(t)$, we obtain the following three ODEs:

$$\phi'' + \mu^2\phi = 0, \qquad \phi(0) = 0, \qquad \phi(a) = 0,$$

$$g'' + \nu^2 g = 0, \qquad g(0) = 0, \qquad g(b) = 0,$$

$$h' + c^2(\mu^2 + \nu^2)h = 0$$

Solve the three ODEs and obtain the product solution for u.

(i) Notice the first ODE:

$$\phi'' + \mu^2\phi = 0, \qquad \phi(0) = 0, \qquad \phi(a) = 0,$$

Using the characteristic equation, we get:

$$\phi(x) = c_1\cos(\mu x) + c_2\sin(\mu x)$$

Using the boundary conditions, we get:

$$\phi(0) = c_1 = 0 \qquad \phi(a) = c_2\sin(\mu a) = 0$$

If we assume, that $c_2 = 0$, we get the trivial solution:

$$\phi(x) = 0 \qquad \rightarrow \qquad u(x, y, t) = 0$$

If we assume that $c_2 \neq 0$, we get the following n eigenvalues:

$$\sin(\mu a) = 0 \qquad \rightarrow \qquad \mu a = n\pi \qquad \rightarrow \qquad \mu_n = \frac{n\pi}{a}$$

Thus we would get the following n eigenfunctions:

$$\phi_n(x) = B_n\sin\left(\frac{n\pi}{a}x\right)$$

(ii) Notice the second ODE:

$$g'' + \nu^2 g = 0, \qquad g(0) = 0, \qquad g(b) = 0,$$

Using the characteristic equation, we get:

$$g(x) = c_1 \cos(\nu y) + c_2 \sin(\nu y)$$

Using the boundary conditions, we get:

$$g(0) = c_1 = 0 \qquad g(b) = c_2 \sin(\nu b) = 0$$

If we assume, that $c_2 = 0$, we get the trivial solution:

$$g(y) = 0 \qquad \to \qquad u(x, y, t) = 0$$

If we assume that $c_2 \neq 0$, we get the following n eigenvalues:

$$\sin(\nu b) = 0 \qquad \to \qquad \nu b = n\pi \qquad \to \qquad \nu_n = \frac{n\pi}{b}$$

Thus we would get the following n eigenfunctions:

$$g_n(x) = B_n \sin\left(\frac{n\pi}{b} y\right)$$

(iii) Notice the third ODE:

$$h' + c^2(\mu^2 + \nu^2)h = 0$$

Using some simple algebra, we get:

$$\frac{h'}{h} = -c^2(\mu^2 + \nu^2) \qquad \to \qquad \ln(h) = -c^2(\mu^2 + \nu^2)t + C$$

Thus we get the following for h, and we can resubstitute the following μ and ν values:

$$h(t) = Ce^{-c^2(\mu^2+\nu^2)t} \qquad \to \qquad h_n(t) = C_n e^{-c^2\left(\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{a}\right)^2\right)t}$$

Thus we get the following for u_n:

$$u_n(x, y, t) = B_n \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) e^{-c^2\left(\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{a}\right)^2\right)t}$$

Thus we get the following product solution for u:

$$u(x, y, t) = u_1 + u_2 + \cdots \quad \to \quad u(x, y, t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) e^{-c^2\left(\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{a}\right)^2\right)t}$$

Now we can include our initial condition, and get the following:

$$u(x, y, 0) = f(x, y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right)$$

Now we can solve for our coefficients:

$$B_n = \frac{4}{ab} \int_0^a \int_0^b f(x, y) \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) \, dy \, dx$$

(b) Show that if we assume $u = \phi(x)g(y)h(t)$, then the separation of variables method yields:

$$\phi'' + \mu^2\phi = 0, \qquad \phi(0) = 0, \qquad \phi(a) = 0,$$

$$g'' + \nu^2 g = 0, \qquad g(0) = 0, \qquad g(b) = 0,$$

$$h' + c^2(\mu^2 + \nu^2)h = 0$$

Notice the following substitution of u into our original equation:

$$\phi(x)g(y)h'(t) = c^2\phi''(x)g(y)h(t) + c^2\phi(x)g''(y)h(t)$$

$$\frac{h'(t)}{h(t)} = c^2\frac{\phi''(x)}{\phi(x)} + c^2\frac{g''(y)}{g(y)}$$

Notice that because this is a steady state problem, we get:

$$c^2\frac{\phi''(x)}{\phi(x)} + c^2\frac{g''(y)}{g(y)} = 0$$

From here we get:

$$c^2\frac{\phi''(x)}{\phi(x)} = -c^2\frac{g''(y)}{g(y)} = -c^2\mu^2$$

Solving this, we get the first ODE:

$$\phi''(x) + \mu^2\phi(x) = 0$$

If we let $\nu^2 = -\mu^2$, we get the second ODE:

$$g''(y) - \mu^2 g(y) = 0 \qquad \rightarrow \qquad g''(y) + \nu^2 g(y)$$

Then we get the following:

$$\frac{\phi''(x)}{\phi(x)} = -\mu^2 \qquad \frac{g''(y)}{g(y)} = -\nu^2$$

From here, we get:

$$h'(t) = c^2\left(\frac{\phi''(x)}{\phi(x)} + \frac{g''(y)}{g(y)}\right)h(t) \qquad \rightarrow \qquad h'(t) - c^2\left(-\mu^2 + -\nu^2\right)h(t) = 0$$

Thus we get the the third ODE:

$$h'(t) + c^2\left(\mu^2 + \nu^2\right)h(t) = 0$$