Homework 1
Partial Differential Equations
Math 531
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Problem 1: Given the ODE:

$$y'' - y = 0.$$

(a) Show that $y_1(t) = e^t$ and $y_2(t) = e^{-t}$ are solutions to the differential equation. In addition, show that this pair form a linearly independent set using the definition of linear independence.

We can show that the given functions are solutions to the ODE by substituting them into the ODE and the equality remaining true. So notice the following:

$$y_1(t)'' - y_1(t) = \frac{d^2}{dt^2}\left(e^t\right) - e^t = e^t - e^t = 0$$

$$y_2(t)'' - y_2(t) = \frac{d^2}{dt^2}\left(e^{-t}\right) - e^{-t} = e^{-t} - e^{-t} = 0$$

Thus we have shown that $\boldsymbol{y_1(t) = e^t}$ and $\boldsymbol{y_2(t) = e^{-t}}$ are solutions to the given ODE.

Proof. Let the following be true:

$$c_1 e^t + c_2 e^{-t} = 0$$

where c_1 and c_2 are constants. We can substitute $t = 0$ and $t = 1$ to get the following:

$$c_1 + c_2 = 0 \qquad c_1 e + c_2 e^{-1} = 0$$

From the left equation, we get that $c_1 = -c_2$. By substituting this into the right equation we get:

$$-c_2 e + c_2 e^{-1} = 0$$

Solving this we get that $c_2 e^{-1} = c_2 e$. Thus the only way for this to be true is if $c_1 = c_2 = 0$.

Thus proving the set to be linear independent.

$\square$

(b) Also, show that $y_1(t) = \sinh t$ and $y_2(t) = \sinh(1-t)$ are solutions to the differential equation. In addition, show that this pair form another linearly independent set.

We can show that the given functions are solutions to the ODE by substituting them into the ODE and the equality remaining true. So notice the following:

$$y_1(t)'' - y_1(t) = \frac{d^2}{dt^2}(\sinh t) - \sinh t = \sinh t - \sinh t = 0$$

$$y_2(t)'' - y_2(t) = \frac{d^2}{dt^2}(\sinh(1-t)) - \sinh(1-t) = \sinh(1-t) - \sinh(1-t) = 0$$

Thus we have shown that $y_1(t) = \sinh(t)$ and $y_2(t) = \sinh(1-t)$ are solutions to the given ODE.

Proof. Let the following be true:

$$c_1 \sinh(t) + c_2 \sinh(1-t) = 0$$

where c_1 and c_2 are constants. We can substitute $t = 0$ and $t = 1$ to get the following:

$$c_2 \sinh(1) = 0 \qquad c_1 \sinh(1) = 0$$

From this, we get that $c_1 = c_2 = 0$.

Thus proving the set to be linear independent.

$\square$

Problem 2: Consider the following second order linear homogeneous differential equation:

$$y'' - 2ay' + (a^2 + b^2)y = 0,$$

where the parameters are fixed and positive, so assume $a > 0$ and $b > 0$.

(a) Find the general solution to this ordinary differential equation (ODE).

Notice the characteristic equation:

$$\lambda^2 - 2a\lambda + (a^2 + b^2) = 0$$

Using the quadratic formula, we can solve for λ:

$$\lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2 + b^2)}}{2} = \frac{2a \pm \sqrt{-4b^2}}{2} = a \pm bi$$

Because we have $\lambda \in \mathbb{C}$, we get the following general solution:

$$y(t) = Ce^{(a \pm bi)t} = Ce^{at}e^{(\pm i\, bt)} = e^{at}\left(c_1 \cos bt + c_2\, i \sin bt\right)$$

(b) Find the unique solution to the initial value problem (IVP), where the initial conditions for the ODE are:

$$y(0) = y_0 \qquad \text{and} \qquad y'(0) = z_0.$$

Notice the following:

$$y(0) = c_1 = y_0$$

$$y'(t) = e^{at}\left[(c_1 a \cos bt + c_2 a\, i \sin bt) + (-c_1 b \sin bt + c_2 b\, i \cos bt)\right]$$

$$y'(0) = c_1 a + c_2 bi = y_0 a + c_2 bi = z_0 \qquad c_2 = \frac{z_0 - y_0 a}{bi}$$

Thus we get the following unique solution:

$$y(t) = e^{at}\left(y_0 \cos bt + \frac{z_0 - y_0 a}{b} \sin bt\right)$$

(c) Now consider the ODE with boundary conditions:

$$y(0) = A \qquad \text{and} \qquad y(x_0) = B.$$

Give conditions on the boundary condition parameters, A, B, and $x_0 > 0$, such that this boundary value problem (BVP) has:

(i) A unique solution (ii) No solution (iii) Infinitely many solutions

When this BVP has a unique solution, give the solution to the BVP.

(i) Notice the following:

$$y(0) = c_1 = A$$

$$y(x_0) = e^{ax_0}\left(A\cos bx_0 + c_2\, i\sin bx_0\right) = B$$

$$c_2 = \frac{Be^{-ax_0} - A\cos bx_0}{i\,\sin bx_0}$$

Thus we get the unique solution:

$$y(t) = e^{at}\left(A\cos bt + \frac{Be^{-ax_0} - A\cos bx_0}{\sin bx_0}\sin bt\right)$$

(ii) Let $x_0 = c\pi/b$ with c being an even positive integer. Also let $B \neq Ae^{ax_0}$. Now notice the following:

$$y(0) = c_1 = A$$

$$y(x_0) = e^{ax_0}c_1 = B \qquad c_1 = \frac{B}{e^{ax_0}}$$

Because we have a conflict in the value of c_1, thus we get no solution.

(iii) Let $x_0 = c\pi/b$ with c being an even positive integer. Also let $B = Ae^{ax_0}$. Now notice the following:

$$y(0) = c_1 = A$$

$$y(x_0) = e^{ax_0}c_1 = 0 \qquad c_1 = A$$

Thus we get infinite solutions.