MATH 525
Section 3.4: Extended Codes

October 22, 2020

Let C be an (n, k, d)-linear code. We form a new code C^* by appending a 0 at the end of every codeword of C with even weight, and a 1 at the end of every codeword of C with odd weight. This extra digit is called an overall parity-check digit. C^* is called the the extended code of C.

Example

C	C^*
000000	0000000
100101	1001011
110011	1100110
010110	0101101
011001	0110011
111100	1111000
101010	1010101
001111	0011110

Note that in the new code C^* every word has even weight.

If G is the generator matrix of C, then

$$G^* = [G|b]$$

is the generator matrix of C^*, where the column labeled b is appended so that every row of G^* has even weight. Alternatively, $b = Gj$ where j is the $n \times 1$ column vector of all 1's.

Example

Let

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then

$$G^* = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

If H is a parity-check matrix of C, then

$$H^* = \left[\begin{array}{c|c} H & j \\ \hline \mathbf{0} & 1 \end{array}\right]$$

is a parity-check matrix for C^*. Indeed,

$$G^* \cdot H^* = [G|b] \cdot \left[\begin{array}{c|c} H & j \\ \hline \mathbf{0} & 1 \end{array}\right] = [GH, Gj + b] = [\mathbf{0}, \mathbf{0}].$$

Remarks:

1. Note that if $v \in C$ and v^* is the corresponding codeword in C^*, then:

$$\mathrm{wt}(v^*) = \begin{cases} \mathrm{wt}(v) & \text{if } \mathrm{wt}(v) \text{ is even;} \\ \mathrm{wt}(v) + 1 & \text{if } \mathrm{wt}(v) \text{ is odd.} \end{cases}$$

2. If $d(C) = $ odd, then $d(C^*) = d(C) + 1$;
   if $d(C) = $ even, then $d(C^*) = d(C)$.