Lecture 6 Heat equation with insulated BCs

The heat equation with insulated BCs satisfies the following:

PDE $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}$, $t>0$, $0<x<L$ (1)

IC $u(x,0) = f(x)$, $0<x<L$ (2)

BC: $u_x(0,t) = 0$ and $u_x(L,t) = 0$ (3)

Solution

Seperation of variables : $u(x,t) = \phi(x)G(t)$ (4)

Substitute (4) into (1) to obtain

$$\phi G' = K\phi'' G$$

Or

$$\frac{G'}{KG} = \frac{\phi''}{\phi} = -\lambda$$

This gives two ODEs,

the first is: $G' = -\lambda K G$ (5)

with General solution:

$$G(t) = Ae^{-k\lambda t}$$

The second ODE is the Sturm-Liouville / BVP in space is:

$$\phi'' = -\lambda\phi \quad (6)$$

with $\phi'(0) = 0$ and $\phi'(L) = 0$

We consider 3 cases depending on λ.

Case 1: Let $\lambda = 0$ then $\phi'' = 0$
or $\phi(x) = c_2 x + c_1$

Applying the BCs $\phi'(0) = 0$, $\phi'(L) = 0$

$$\phi'(0) = \phi'(L) = c_2 = 0$$

However, c_1 is arbitrary, so we have an eigenvalue $\lambda_0 = 0$ with eigenfunction

$$\phi(x) = c_1$$

$$\phi_0(x) = 1 \text{ Since } c_1 \text{ is arbitrary}$$

<u>Case 2</u> Let $\lambda = -\alpha^2 < 0$, $\alpha > 0$,

Then $\phi'' - \alpha^2 \phi = 0$, so

$$\phi(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$$

Apply B.Cs $\phi'(0) = c_1 \alpha \sinh(0) + c_2 \alpha \cosh(0) = 0$

$$= c_2 \alpha = 0$$

Since $\alpha > 0$

Then $c_2 = 0$

$$\phi'(L) = c_1 \alpha \sinh(\alpha L) = 0$$

$\alpha > 0$

$\sinh(\alpha L) \neq 0$

$$c_1 = 0$$

Only the trivial solution $\phi(x,t) = 0$

Satisfies the BCs.

<u>Case 3</u> Let $\lambda = \alpha^2$, $\alpha > 0$

We obtain the Sturm-Liouville problem:

$$\phi'' + \alpha^2 \phi = 0$$

Applying the BCs, we obtain the following:

$$\phi'(x) = -c_1 \alpha \sin(\alpha x) + c_2 \alpha \cos(\alpha x)$$

$$\phi'(0) = c_2 \alpha = 0$$

Therefore $c_2 = 0$

$$\phi'(L) = -c_1 \alpha \sin(\alpha L) = 0$$

To avoid trivial solutions, we set $\sin(\alpha L) = 0$ therefore

$$\alpha L = n\pi \quad , n = 1, 2, \cdots$$

$$\alpha_n = \frac{n\pi}{L} \; ; \; \lambda_n := \alpha_n^2 = \frac{n^2\pi^2}{L^2}$$

$n = 1, 2, \cdots$

$$\phi_n(x) = c_1 \cos(\alpha_n x)$$

$$:= c_1 \cos\left(\frac{n\pi}{L} x\right)$$

The product solutions are

$$u(x,t) = \phi(x)\,G(t)$$

$$u_0(x,t) = \phi_0 G_0 := A_0$$

$$u_n(x,t) = \phi_n G_n \quad n = 1, 2, 3, \dots$$

$$u_n(x,t) = A_n \cos\left(\frac{n\pi x}{L}\right) \cdot e^{-k\frac{n^2\pi^2}{L^2}t}$$

The principle of superposition gives the solution

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\frac{kn^2\pi^2 t}{L^2}} \cos\left(\frac{n\pi x}{L}\right)$$

To find the coefficients A_0 and A_n, we apply the IC (equation 2)

$$f(x) = u(x,0)$$

$$= A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \quad (7)$$

Note: The eigenfunctions $\phi_i(x)$, $i=1,2,\cdots$ are mutually orthogonal, which allows finding the Fourier coefficient for any initial condition, $f(x)$.

First multiply by $\phi_0 = 1$ and integrate over $[0,L]$, which by orthogonality with $\phi_n(x)$, $n=1,2,3\cdots$ gives:

$$\int_0^L f(x)\,dx = \int_0^L A_0\,dx \quad \text{or} \quad A_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

Next we multiply by $\phi_m(x)$ and integrate over $x \in [0,L]$,

So

$$\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty} A_n \int_0^L \left(\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\right)dx$$

$$= A_m\left(\frac{L}{2}\right)$$

Therefore $A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

We have now found A_0 and A_n

Therefore the Solution of the one-dimensional heat equation with insulated boundaries is given by:

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-k\frac{n^2\pi^2}{L^2}t}\cos\left(\frac{n\pi x}{L}\right)$$

with

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx \quad \text{and}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

The <u>Steady State</u> (i.e. equilibrium) Solution is obtained by examining the $\lim_{t \to \infty} u(x,t)$.

That is $\lim_{t \to \infty} u(x,t) = A_0$

$$:= \frac{1}{L}\int_0^L f(x)\,dx$$

Which is the average temperature From the IC.