Math 524: Linear Algebra

Notes #7.2 — Operators on Inner Product Spaces

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Spring 2020

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Outline

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Student Learning Targets, and Objectives

Target Positive Operators

Objective Be able to characterize Positive Operators, and in particular construct the Unique Positive Square Root Operator.

Target Isometries

Objective Be able to state the definition of, and characterize Isometries

Target Polar Decomposition

Objective Be able to *abstractly construct** the Polar Decomposition of an Operator, through Identification of the appropriate Isometry and Postive Operator.

Target Singular Value Decomposition

Objective Be able to *abstractly construct** the Singular Value Decomposition of an Operator, by Identifying the Singular Values and Orthonormal Bases.

* Generally practical constructions must be addressed with computational tools from [MATH 543].

Positive Operators

Definition (Positive Operator)

An operator $T \in \mathcal{L}(V)$ is called **positive** if T is self-adjoint and

$$\langle T(v),\, v\rangle \geq 0$$

$\forall v \in V$.

If V is a complex vector space, then the requirement that T is self-adjoint can be dropped from the definition above:

Rewind (Over $\mathbb{C}$, $\langle T(v),\, v\rangle \in \mathbb{R}\ \forall v \in V$ Only for Self-Adjoint Operators [NOTES#7.1])

Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Then T is self-adjoint if and only if

$$\langle T(v),\, v\rangle \in \mathbb{R}$$

$\forall v \in V$.

Positive Operators

Example (Positive Operators)

- If U is a subspace of V, then the orthogonal projections P_U and $P_{U^\perp}$ are positive operators
- If $T \in \mathcal{L}(V)$ is self-adjoint and $b, c \in \mathbb{R}$ are such that $b^2 < 4c$, then $(T^2 + bT + cI)$ is a positive operator, as shown by the proof of [Invertible Quadratic (Operator) Expressions (Notes#7.1)]

Rewind (Invertible Quadratic (Operator) Expressions [Notes#7.1])

Suppose $T \in \mathcal{L}(V)$ is self-adjoint, and $b, c \in \mathbb{R} : b^2 < 4c$, then

$$T^2 + bT + cI$$

is invertible.

Square Root

Definition (Square Root)

An operator R is called a **square root** of an operator T if $R^2 = T$.

Example (Square Root)

If $T \in \mathcal{L}(\mathbb{F}^3)$ is defined by $T(z_1, z_2, z_3) = (z_3, 0, 0)$, then the operator $R \in \mathcal{L}(\mathbb{F}^3)$ defined by $R(z_1, z_2, z_3) = (z_2, z_3, 0)$ is a square root of T:

$$R^2(z_1, z_2, z_3) = R(z_2, z_3, 0) = (z_3, 0, 0) = T(z_1, z_2, z_3)$$

Example (n-th Roots?)

If $T \in \mathcal{L}(\mathbb{F}^{n+1})$ is defined by $T(z_1, \dots, z_{n+1}) = (z_{n+1}, 0, \dots, 0)$, then the operator $R \in \mathcal{L}(\mathbb{F}^{n+1})$ defined by $R(z_1, \dots, z_{n+1}) = (z_2, z_3, \dots, z_{n+1}, 0)$ is an nth root of T:

$$\begin{aligned} R^n(z_1, \dots, z_n) &= R^{n-1}(z_2, z_3, \dots, z_{n+1}, 0) = R^{n-2}(z_3, z_4, \dots, z_{n+1}, 0, 0) \\ &= \dots = (z_{n+1}, 0, \dots, 0) = T(z_1, \dots, z_{n+1}) \end{aligned}$$

“Positive” vs “Non-Negative” vs “Semi-Positive”

Comment (“Positive” vs “Non-Negative” vs “Semi-Positive”)

The positive operators correspond to the numbers $[0, \infty)$, so a more precise terminology would use the term **non-negative** instead of positive.

However, operator-theorists consistently call these the positive operators.

Restricted to the Matrix-Vector “universe” we tend to talk about (strictly) *Positive Definite* and *Positive Semi-Definite* Matrices (“Matrix-Operators,” if you want).

Characterization of Positive Operators

Theorem (Characterization of Positive Operators)

Let $T \in \mathcal{L}(V)$, then the following are equivalent

(a) *T is positive*

(b) *T is self-adjoint and all the eigenvalues of T are non-negative*

(c) *T has a positive square root*

(d) *T has a self-adjoint square root;*

(e) *there exists an operator $R \in \mathcal{L}(V)$ such that $T = R^* R$*

MATRICES: CHOLESKY FACTORIZATION; OR "HERMITIAN LU-FACTORIZATION"

Characterization of Positive Operators

Proof (Characterization of Positive Operators)

(a)$\Rightarrow$(b) T is positive ($\langle T(v), v\rangle \geq 0$, and $T = T^*$); suppose λ is an eigenvalue of T and v the corresponding eigenvector, then

$$0 \leq \langle T(v), v\rangle = \langle \lambda v, v\rangle = \lambda \langle v, v\rangle$$

$\Rightarrow \lambda \in [0, \infty)$ $\qquad \Rightarrow$ (b)

Characterization of Positive Operators

Proof (Characterization of Positive Operators)

(b)$\Rightarrow$(c) T is self-adjoint ($T = T^*$) and $\lambda(T) \in [0, \infty)$. By [Complex Spectral Theorem (Notes#7.1)] or [Real Spectral Theorem (Notes#7.1)], there is an orthonormal basis $v_1, \dots, v_n$ of V consisting of eigenvectors of T; let $\lambda_k : T(v_k) = \lambda_k v_k$; thus $\lambda_k \in [0, \infty)$. Let $R \in \mathcal{L}(V)$ such that

$$R(v_k) = \sqrt{\lambda_k}\, v_k, \;\; k = 1, \dots, n$$

R is a positive operator, and $R^2(v_k) = \lambda_k v_k = T(v_k)$, $k = 1, \dots, n$; *i.e.* $R^2 = T$.

Thus R is a positive square root of T. $\Rightarrow$ (c)

Characterization of Positive Operators

Proof (Characterization of Positive Operators)

(c)⇒(d) By definition, every positive operator is self-adjoint.

(d)⇒(e) Assume $\exists R \in \mathcal{L}(V)$ so that $R = R^*$ and $R^2 = T$:
Then $T = R^* R$ $\Rightarrow$ (e)

(e)⇒(a) Suppose $\exists R \in \mathcal{L}(V) : T = R^* R$, then $T^* = (R^* R)^* = R^*(R^*)^* = R^* R = T$. (which makes T self-adjoint).

Also,

$$\langle T(v), v \rangle = \langle (R^* R)(v), v \rangle = \langle R(v), R(v) \rangle \geq 0$$

$\forall v \in V$, hence T is positive. $\Rightarrow$ (a)

We now have (a)⇒(b)⇒(c)⇒(d)⇒(e)⇒(a). $\surd$

Uniqueness of the Square Root

Theorem (Each Positive Operator Has Only One Positive Square Root)

Every positive operator on V has a unique positive square root.

Comment ("Positive Operators Act Like Real Numbers")

Each non-negative number has a unique non-negative square root.

Again, positive operators have "real" properties.

Comment (What is Unique?)

A positive operator can have infinitely many square roots; only one of them can be positive.

Uniqueness of the Square Root

Proof (Each Positive Operator Has Only One Positive Square Root)

Suppose $T \in \mathcal{L}(V)$ is positive; let $t \in V$ be an eigenvector, and $\lambda^T \geq 0$: $T(t) = \lambda^T t$.

Let R be a positive square root of T.

NOTE: We show $R(t) = \sqrt{\lambda^T}\, t \Rightarrow$ the action of R on the eigenvectors of T is uniquely determined. Since there is a basis of V consisting of eigenvectors of T [$\mathbb{C}/\mathbb{R}$ SPECTRAL THEOREM (NOTES#7.1)], this implies that R is uniquely determined.

To show that $R(t) = \sqrt{\lambda^T}\, t$, we use the fact that [$\mathbb{C}/\mathbb{R}$ SPECTRAL THEOREM (NOTES#7.1)] guarantees an orthonormal basis $r_1, \ldots, r_n$ of V consisting on eigenvectors of R. Since R is a positive operator $\lambda(R) \geq 0$ $\Rightarrow \exists \lambda_1^R, \ldots, \lambda_n^R \geq 0$ such that $R(r_k) = \lambda_k^R r_k$ for $k = 1, \ldots, n$.

$\rightarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow$

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Uniqueness of the Square Root

Proof (Each Positive Operator Has Only One Positive Square Root)

Since $r_1, \ldots, r_n$ is a basis of V, we can write $t \stackrel{!}{=} (a_1 r_1 + \cdots + a_n r_n)$, for $a_1, \ldots, a_n \in \mathbb{F}$, thus

$$\begin{aligned} R(t) &= a_1 \lambda_1^R r_1 + \cdots + a_n \lambda_n^R r_n \\ R^2(t) &= a_1 (\lambda_1^R)^2 r_1 + \cdots + a_n (\lambda_n^R)^2 r_n \end{aligned}$$

But $R^2 = T$ (by assumption, it is a positive square root of T), and $T(t) = \lambda^T t$; therefore, the above implies

$$a_1 \lambda^T r_1 + \cdots + a_n \lambda^T r_n = a_1 (\lambda_1^R)^2 r_1 + \cdots + a_n (\lambda_n^R)^2 r_n$$

$\Rightarrow a_j(\lambda^T - (\lambda_j^R)^2) = 0$, $j = 1, \ldots, n$ (either $a_j = 0$, or $(\lambda^T - (\lambda_j^R)^2) = 0$).

$$\text{Hence, } t = \sum_{j: a_j \neq 0} a_j r_j \quad \Rightarrow \quad R(t) = \sum_{j: a_j \neq 0} a_j \sqrt{\lambda^T}\, r_j = \sqrt{\lambda^T}\, t,$$

which is what we needed to show. $\surd$

Isometries — Norm-Preserving Operators

Definition (Isometry)

- An operator $S \in \mathcal{L}(V)$ is called an **isometry** if
$$\|S(v)\| = \|v\|$$
$\forall v \in V$.
- "An operator is an isometry if it preserves norms."

Rewind (Orthogonal Transformations [MATH-254 (NOTES#5.3)])

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is called orthogonal if it preserves the length of vectors:
$$\|T(\vec{x})\| = \|\vec{x}\|, \ \forall \vec{x} \in \mathbb{R}^n.$$
If $T(\vec{x}) = A\vec{x}$ is an orthogonal transformation, we say that A is an orthogonal (or *unitary*, when it has complex entries) matrix.

Isometries — Norm-Preserving Operators

Example

Suppose $\lambda_1, \dots, \lambda_n$ are scalars with $|\lambda_k| = 1$, and $S \in \mathcal{L}(V)$ satisfies $S(s_j) = \lambda_j s_j$ for some orthonormal basis $s_1, \dots, s_n$ of V.

We demonstrate that S is an isometry.

Let $v \in V$, then

$$\begin{aligned} v &= \langle v, s_1 \rangle s_1 + \cdots + \langle v, s_n \rangle s_n \\ \|v\|^2 &\stackrel{1}{=} |\langle v, s_1 \rangle|^2 + \cdots + |\langle v, s_n \rangle|^2 \end{aligned}$$

$$\begin{aligned} S(v) &= \langle v, s_1 \rangle S(s_1) + \cdots + \langle v, s_n \rangle S(s_n) \\ &= \lambda_1 \langle v, s_1 \rangle s_1 + \cdots + \lambda_n \langle v, s_n \rangle s_n \\ \|S(v)\|^2 &\stackrel{1}{=} |\lambda_1|^2 |\langle v, s_1 \rangle|^2 + \cdots + |\lambda_n|^2 |\langle v, s_n \rangle|^2 \\ &= |\langle v, s_1 \rangle|^2 + \cdots + |\langle v, s_n \rangle|^2 \end{aligned}$$

$\stackrel{1}{=}$ [Writing a Vector as a Linear Combination of Orthonormal Basis (Notes#6)]

Characterization of Isometries

Theorem (Characterization of Isometries)

Suppose $S \in \mathcal{L}(V)$, then the following are equivalent:

(a) *S is an isometry*

(b) $\langle S(u), S(v)\rangle = \langle u, v\rangle \; \forall u, v \in V$

(c) *$S(u_1), \ldots, S(u_n)$ is orthonormal for every orthonormal list of vectors $u_1, \ldots, u_n$ in V*

(d) *there exists an orthonormal list of vectors $u_1, \ldots, u_n$ of V such that $S(u_1), \ldots, S(u_n)$ is orthonormal*

(e) $S^*S = I$

(f) $SS^* = I$

(g) *S^* is an isometry*

(h) *S is invertible and $S^{-1} = S^*$*

Some Help for the Proof

Theorem (The Inner Product on a Real Inner Product Space)

Suppose V is a real inner product space, then

$$\langle u,\, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4}$$

$\forall u, v \in V$.

Theorem (The Inner Product on a Complex Inner Product Space)

Suppose V is a complex inner product space, then

$$\langle u,\, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - i\|u-iv\|^2}{4}$$

$\forall u, v \in V$.

The proofs for these identities are by "direct computation" (very similar to what we did in [NOTES#7.1]). The bottom line is that we can express the inner product in terms of the norm.

Characterization of Isometries

Proof (Characterization of Isometries)

(a)⇒(b) Suppose S is an isometry; the "help theorems" show that inner products can be computed from norms. Since S preserves norms, $\Rightarrow$ S preserves inner products. $\Rightarrow$ (b)

(b)⇒(c) Assume S preserves inner products, let $u_1, \dots, u_n$ be an orthonormal list of vectors in V; $S(u_1), \dots, S(u_n)$ must be an orthonormal list of vectors since $\langle S(u_i), S(u_j) \rangle = \langle u_i, u_j \rangle = \delta_{ij}$. $\Rightarrow$ (c)

(c)⇒(d) $\surd$

Characterization of Isometries

Proof (Characterization of Isometries)

(d)⇒(e) Let $u_1, \dots, u_n$ be an orthonormal basis of V such that $S(u_1), \dots, S(u_n)$ is orthonormal. Thus

$$\langle S^*S(u_j),\, u_k\rangle = \langle S(u_j),\, S(u_k)\rangle = \langle u_j,\, u_k\rangle$$

All $v, w \in V$ can be written as unique linear combinations of $u_1, \dots, u_n$, therefore $\langle S^*S(v),\, w\rangle = \langle v,\, w\rangle \Rightarrow S^*S = I$. $\Rightarrow$ (e)

(e)⇒(f) $S^*S = I. \Rightarrow \{S^*(SS^*) = S^*,\, (SS^*)S = S\} \Rightarrow SS^* = I.$ $\Rightarrow$ (f)

(f)⇒(g) $SS^* = I$, let $v \in V$, then

$$\|S^*(v)\|^2 = \langle S^*(v),\, S^*(v)\rangle = \langle SS^*(v),\, v\rangle = \langle v,\, v\rangle = \|v\|^2$$

$\Rightarrow S^*$ is an isometry. $\Rightarrow$ (g)

Characterization of Isometries

Proof (Characterization of Isometries)

(g)⇒(h) S^* is an isometry. We can apply the previously shown parts of the theorem, in particular (a)⇒(e), and (a)⇒(f) to S^* (with $(S^*)^*$). This gives $S^*S = SS^* = I$, which means that S is invertible, and $S^{-1} = S^*$. $\Rightarrow$ (h)

(h)⇒(a) S is invertible, and $S^{-1} = S^*$; let $v \in V$, then

$$\|S(v)\|^2 = \langle S(v),\, S(v)\rangle = \langle (S^*S)(v),\, v\rangle = \langle v,\, v\rangle = \|v\|^2$$

that is S is an isometry. $\Rightarrow$ (a)

We now have (a)⇒(b)⇒(c)⇒(d)⇒(e)⇒(f)⇒(g)⇒(h)⇒(a). $\surd$

Description of Isometries when $\mathbb{F} = \mathbb{C}$

Theorem (Description of Isometries when $\mathbb{F} = \mathbb{C}$)

Suppose V is a complex inner product space and $S \in \mathcal{L}(V)$. Then the following are equivalent:

(a) *S is an isometry*

(b) *There is an orthonormal basis of V consisting of eigenvectors of S whose corresponding eigenvalues all have absolute value 1*

Proof (Description of Isometries when $\mathbb{F} = \mathbb{C}$)

The example on slide 16 shows (b)⇒(a). To show (a)⇒(b), we assume S is an isometry and use [Complex Spectral Theorem (Notes#7.1)] to guarantee an orthonormal basis $s_1, \dots, s_n$ of V consisting of eigenvectors of S. Let $\lambda_1, \dots, \lambda_n$ be the corresponding eigenvalues. Then

$$|\lambda_j| = \|\lambda_j s_j\| = \|S(s_j)\| = \|s_j\| = 1,$$

that is $|\lambda_j| = 1$ $j = 1, \dots, n$. $\surd$.

Upcoming: [Description of Isometries when $\mathbb{F} = \mathbb{R}$ (Notes#7.2–Preview)].

"Preview"

Preview (Description of Isometries when $\mathbb{F} = \mathbb{R}$)

Suppose V is a real inner product space and $S \in \mathcal{L}(V)$. Then the following are equivalent:

(a) S is an isometry

(b) There is an orthonormal basis of V with respect to which S has a block-diagonal matrix such that each block on the diagonal is a 1-by-1 matrix containing 1 or -1, or is a 2-by-2 matrix of the form

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \quad \theta \in (0, \pi)$$

$\langle\langle\langle$ Live Math $\rangle\rangle\rangle$

e.g. 7C-$\{1, 6\}$

Live Math :: Covid-19 Version

7C-1: Prove or give a counterexample: If $T \in \mathcal{L}(V)$ is self-adjoint and there exists an orthonormal basis $u_1, \ldots, u_n$ of V such that $\langle T(u_k),\, u_k \rangle \geq 0\ \forall k$, then T is a positive operator.

We have no theorem that helps us, so therefore we suspect the statement is false.

⋇ Constructing a Counter-Example ⋇

Consider $V = \mathbb{R}^2$, with the standard inner product, and standard basis. Let $T(x_1, x_2) = (x_2, x_1)$.

Live Math :: Covid-19 Version

⋇ T is Self-Adjoint: ⋇

$$\begin{aligned}
\langle (x_1, x_2),\ T^*(y_1, y_2) \rangle &= \langle T(x_1, x_2),\ (y_1, y_2) \rangle = \langle (x_2, x_1),\ (y_1, y_2) \rangle \\
&= x_2 y_1 + x_1 y_2 = \langle (x_1, x_2),\ (y_2, y_1) \rangle \\
\hline
T^*(y_1, y_2) &= (y_2, y_1)
\end{aligned}$$

⋇ However, T is not a Positive Operator ⋇

$$\begin{aligned}
\langle T(1,0),\ (1,0) \rangle &= 0 \quad \text{OK} \\
\langle T(0,1),\ (0,1) \rangle &= 0 \quad \text{OK} \\
\langle T(1,-1),\ (1,-1) \rangle &= -2 \quad \text{NOT} \geq 0
\end{aligned}$$

Analogies: $\mathbb{C}$ and $\mathcal{L}(V)$

$\mathbb{C}$	$\mathcal{L}(V)$
z	T
z^*	T^*
$z = \Re(z) \geq 0$ (non-negative)	$\langle T(v), v \rangle \geq 0$ (positive)
$z^* z = \|z\|^2 = 1$ (unit circle)	$T^* T = I$ (isometry)

Any complex $z \in \mathbb{C} \setminus \{0\}$ can be written in the form

$$z = \left(\frac{z}{|z|} \right) |z| = \left(\frac{z}{|z|} \right) \sqrt{z^* z},$$

where, of course

$$w = \left(\frac{z}{|z|} \right) \in \{\text{unit circle}\},$$

Polar Decomposition

Notation ($\sqrt{T}$, The Square Root of T)

If T is a positive operator, then $\sqrt{T}$ is the unique positive square root of T.

T^*T is a positive operator for every $T \in \mathcal{L}(V)$

$$\langle (T^*T)(v),\, v\rangle = \langle T(v),\, T(v)\rangle = \|T(v)\|^2 \geq 0,$$

therefore $\sqrt{T^*T}$ is a always well defined.

Theorem (Polar Decomposition)

Suppose $T \in \mathcal{L}(V)$. Then there exists an isometry $S \in \mathcal{L}(V)$ such that

$$T = S\sqrt{T^*T}$$

Polar Decomposition — Why Should We Care???

The [Polar Decomposition Theorem] shows that we can write any operator on V as the product of an isometry, and a positive operator.

The characterization of the positive operators is given by the [$\mathbb{C}/\mathbb{R}$ Spectral Theorem (Notes#7.1)]; and

- we have characterized the isometries over $\mathbb{C}$ in [Description of Isometries when $\mathbb{F} = \mathbb{C}$]; and
- have "previewed" the characterization over $\mathbb{R}$ [Description of Isometries when $\mathbb{F} = \mathbb{R}$ (Notes#7.2–Preview)].

Thus, the [Polar Decomposition Theorem] provides us with a "complete" characterization of all operators in the sense of the [$\mathbb{C}/\mathbb{R}$ Spectral Theorem (Notes#7.1)] and the matching [Description of Isometries when $\mathbb{F} = \mathbb{C}$, or $\mathbb{F} = \mathbb{R}$] results.

I do daresay, this is quite a major result, indeed.

Polar Decomposition

Proof (Polar Decomposition)

Let $v \in V$, then

$$\begin{aligned} \|T(v)\|^2 &= \langle T(v),\, T(v)\rangle &&= \langle (T^*T)(v),\, v\rangle \\ &= \langle (\sqrt{T^*T})(\sqrt{T^*T})(v),\, v\rangle &&= \langle (\sqrt{T^*T})(v),\, (\sqrt{T^*T})(v)\rangle \\ &= \|(\sqrt{T^*T})(v)\|^2 \end{aligned}$$

Thus

$$\|T(v)\| = \|(\sqrt{T^*T})(v)\|, \ \forall v \in V. \tag{PD-1}$$

We define a linear map $S_1 : \mathrm{range}(\sqrt{T^*T}) \mapsto \mathrm{range}(T)$ by

$$S_1(\sqrt{T^*T})(v) = T(v) \tag{PD-2}$$

The goal is to extend S_1 to an isometry $S \in \mathcal{L}(V)$ so that $T = S\sqrt{T^*T}\ldots$

Polar Decomposition

Proof (Polar Decomposition)

First, we make sure S_1 is well defined: let $v_1, v_2 \in V$ such that $\sqrt{T^*T}(v_1) = \sqrt{T^*T}(v_2)$. For (PD-2) to make sense, we need $T(v_1) = T(v_2)$.

$$\begin{aligned} \|T(v_1) - T(v_2)\| &= \|T(v_1 - v_2)\| \overset{\text{(PD-1)}}{=} \|\sqrt{T^*T}(v_1 - v_2)\| \\ &= \|\sqrt{T^*T}(v_1) - \sqrt{T^*T}(v_2)\| = 0 \end{aligned}$$

Hence $T(v_1) = T(v_2)$, and S_1 is well-defined (we leave the verification of the basic linear mapping properties as an "exercise.")

By definition (PD-2) $S_1 : \text{range}(\sqrt{T^*T}) \mapsto \text{range}(T)$; together with (PD-1), we have that

$$\|S_1(u)\| = \|u\|, \; \forall u \in \text{range}(\sqrt{T^*T})$$

Polar Decomposition

Proof (Polar Decomposition)

Now, we extend S_1 to an isometry S on all of V:

By construction S_1 is injective, so the [FUNDAMENTAL THEOREM OF LINEAR MAPS (NOTES#3.1)] gives

$$\dim(\mathrm{range}(\sqrt{T^*T})) = \dim(\mathrm{range}(T))$$

By [DIMENSION OF THE ORTHOGONAL COMPLEMENT (NOTES#6)]

$$\dim(\mathrm{range}(\sqrt{T^*T})^\perp) = \dim(\mathrm{range}(T)^\perp)$$

Let $e_1, \ldots, e_m$ be an orthonormal basis of $(\mathrm{range}(\sqrt{T^*T}))^\perp$, and $f_1, \ldots, f_m$ be an orthonormal basis of $(\mathrm{range}(T))^\perp$.
Both bases have the same length.

Polar Decomposition

Proof (Polar Decomposition)

Now, we define linear map $S_2 : (\mathrm{range}(\sqrt{T^*T}))^\perp \mapsto (\mathrm{range}(T))^\perp$ by

$$S_2(a_1 e_1 + \cdots + a_m e_m) = a_1 f_1 + \cdots + a_m f_m$$

[The Norm of an Orthonormal Linear Combination (Notes#6)] guarantees $\|S_2(w)\| = \|w\|$, $\forall w \in (\mathrm{range}(\sqrt{T^*T}))^\perp$.

Due to [Direct Sum of a Subspace and its Orthogonal Complement (Notes#6)] any $v \in V$ can be uniquely written in the form

$$v = u + w, \quad u \in \mathrm{range}(\sqrt{T^*T}), \; w \in (\mathrm{range}(\sqrt{T^*T}))^\perp \qquad \text{(PD-3)}$$

Polar Decomposition

Proof (Polar Decomposition)

Now, we define $S(v)$ by

$$S(v) = S_1(u) + S_2(w), \quad u \in \mathrm{range}(\sqrt{T^*T}),\ w \in (\mathrm{range}(\sqrt{T^*T}))^{\perp}$$

$\forall v \in V$ we have

$$S(\sqrt{T^*T}(v)) = S_1(\sqrt{T^*T}(v)) = T(v)$$

so $T = S\sqrt{T^*T}$. We must show that S is an isometry; with the decomposition (PD-3) $v = u + w$ $(u \perp w)$, we can use the [Pythagorean Theorem ($\approx$500 BC)]:

$$\begin{aligned} \|S(v)\|^2 &= \|S_1(u) + S_2(w)\|^2 \stackrel{\mathrm{PT}^*}{=} \|S_1(u)\|^2 + \|S_2(w)\|^2 \\ &= \|u\|^2 + \|w\|^2 \stackrel{\mathrm{PT}}{=} \|v\|^2 \end{aligned}$$

$\stackrel{\mathrm{PT}^*}{=}$ holds since $S_1(u) \in (\mathrm{range}(T))$, and $S_2(w) \in (\mathrm{range}(T)^{\perp})$

Polar Decomposition

Comment

When $\mathbb{F} = \mathbb{C}$ let $T = S\sqrt{T^*T}$ be the Polar Decomposition of an operator $T \in \mathcal{L}(V)$, where S is an isometry.

Then

(1) there is an orthonormal basis, $\mathfrak{B}_1(V)$, of V with respect to which S has a diagonal matrix, and

(2) there is an orthonormal basis, $\mathfrak{B}_2(V)$, of V with respect to which $\sqrt{T^*T}$ has a diagonal matrix.

WARNING: Usually, there does **not** exist an orthonormal basis that diagonalizes $\mathcal{M}(S)$, and $\mathcal{M}(\sqrt{T^*T})$ at the same time.

Singular Value Decomposition

So far, we have used the eigenvalues (and eigenvectors) to describe the properties of operators.

Rewind (Eigenspace, $E(\lambda, T)$)

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The **Eigenspace** of T corresponding to λ denoted $E(\lambda, T)$ is defined to be

$$E(\lambda, T) = \mathrm{null}(T - \lambda I)$$

$E(\lambda, T)$ is the set of all eigenvectors of T corresponding to λ, along with the 0 vector.

We are particularly interested in (obsessed with?) scenarios where we can find orthonormal bases; this is the focus of [SCHUR'S THEOREM (NOTES#6)], [COMPLEX SPECTRAL THEOREM (NOTES#7.1)], and [REAL SPECTRAL THEOREM (NOTES#7.1)]

In [POLAR DECOMPOSITION THEOREM] we needed (in general) 2 orthonormal bases to perform the decomposition. The Singular Value Decomposition is an "alternate" way to leverage the use of 2 bases.

Singular Value Decomposition

Definition (Singular Values, σ)

Suppose $T \in \mathcal{L}(V)$. The **singular values** of T are the eigenvalues, in this context denoted σ_i, of $\sqrt{T^*T}$, with each eigenvalue repeated $\dim(E(\sigma_i, \sqrt{T^*T})$ times.

In applications, and algorithms, it is customary to sort the singular values in descending order, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$.

The singular values of T are all non-negative, because they are the eigenvalues of the positive operator $\sqrt{T^*T}$.

Singular Value Decomposition

Example ($T(z_1, z_2, z_3, z_4) = (0, 3z_1, 2z_2, -3z_4)$)

Let $T \in \mathcal{L}(\mathbb{F}^4)$ be defined by

$$T(z_1, z_2, z_3, z_4) = (0, 3z_1, 2z_2, -3z_4)$$

we find the singular values.

(1) First we find the eigenvalues, $\lambda(T)$; consider:

$$\lambda(z_1, z_2, z_3, z_4) = (0, 3z_1, 2z_2, -3z_4)$$

the only solutions are $\lambda \in \{0, -3\}$, and the eigenspaces are given by

$$\begin{cases} E(\lambda = 0, T) & = & \mathrm{span}((0,0,1,0)) \\ E(\lambda = -3, T) & = & \mathrm{span}((0,0,0,1)) \end{cases}$$

Since $\dim(E(0,T)) + \dim(E(-3,T)) = 2 < 4 = \dim(\mathbb{F}^4)$ we cannot fully diagonalize the operator.

$\mathbb{F}^4 \neq E(-3,T) \oplus E(0,T) \Rightarrow$ No Diagonalization.

Singular Value Decomposition

Example ($T(z_1, z_2, z_3, z_4) = (0, 3z_1, 2z_2, -3z_4)$)

(2) Next, we find the adjoint T^*; T^*T, and $\sqrt{T^*T}$:

$$\begin{aligned}
\langle z,\ T^*(w)\rangle &= \langle T(z),\ w\rangle = \langle (0, 3z_1, 2z_2, -3z_4),\ (w_1, w_2, w_3, w_4)\rangle \\
&= 3z_1w_2 + 2z_2w_3 - 3z_4w_4 \\
&= \langle (z_1, z_2, z_3, z_4),\ (3w_2, 2w_3, 0, -3w_4)\rangle
\end{aligned}$$

$$\begin{aligned}
T^*(w) &= (3w_2, 2w_3, 0, -3w_4) \\
T^*T(z) &= T^*(0, 3z_1, 2z_2, -3z_4) = (9z_1, 4z_2, 0, 9z_4) \\
\sqrt{T^*T}(z) &= (3z_1, 2z_2, 0, 3z_4)
\end{aligned}$$

$$\begin{aligned}
\lambda(T^*) &= \{-3, 0\} \\
\lambda(T^*T) &= \{9, 4, 0\} \\
\lambda(\sqrt{T^*T}) &= \{3, 2, 0\}
\end{aligned}$$

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Singular Value Decomposition

Example ($T(z_1, z_2, z_3, z_4) = (0, 3z_1, 2z_2, -3z_4)$)

(3) We need the eigenspaces of $\sqrt{T^*T}$:

$$\begin{aligned} E(0; \sqrt{T^*T}) &= \operatorname{span}((0,0,1,0)) \\ E(2; \sqrt{T^*T}) &= \operatorname{span}((0,1,0,0)) \\ E(3; \sqrt{T^*T}) &= \operatorname{span}((1,0,0,0),(0,0,0,1)) \end{aligned}$$

Thus, the singular values are $\sigma(T) = \{3, 3, 2, 0\}$.

Comment ($T(z_1, z_2, z_3, z_4) = (0, 3z_1, 2z_2, -3z_4)$)

Note that $\lambda(T) = \{0, -3\}$ did not "capture" the 2, but $\sigma(T) = \{3, 3, 2, 0\}$ did.

Singular Value Decomposition

Comment ($T(z_1, z_2, z_3, z_4) = (0, 3z_1, 2z_2, -3z_4)$)

$$\mathcal{M}(T, \{e_i\}) = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}, \quad \mathcal{M}(T^*, \{e_i\}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix},$$

$$\mathcal{M}(\sqrt{T^*T})^2 = \mathcal{M}(T^*)\mathcal{M}(T) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}^2$$

Using [Eigenvalues and Determinants: The Characteristic Equation (Math-254, Notes#7.2)], we can get

$$p_{\mathcal{M}(T)}(\lambda) = \lambda^3(\lambda + 3), \quad p_{\mathcal{M}(\sqrt{T^*T})}(\lambda) = \lambda(\lambda - 2)(\lambda - 3)^2$$

Singular Value Decomposition

Each $T \in \mathcal{L}(V)$ has $\dim(V)$ singular values; this follows from [$\mathbb{C}/\mathbb{R}$ SPECTRAL THEOREM (NOTES#7.1)], and [CONDITIONS EQUIVALENT TO DIAGONALIZABILITY (NOTES#5)] applied to the positive ($\Rightarrow$ self-adjoint) operator $\sqrt{T^*T}$.

The next statement gives a characterization $\forall T \in \mathcal{L}(V)$ in terms of the singular values, and two orthonormal bases of V.

Singular Value Decomposition

Theorem (Singular Value Decomposition)

Suppose $T \in \mathcal{L}(V)$ has singular values $\sigma_1, \ldots, \sigma_n$. Then there exists orthonormal bases $v_1, \ldots, v_n$, and $u_1, \ldots, u_n$ of V such that

$$T(w) = \sigma_1 \langle w,\, v_1 \rangle u_1 + \cdots + \sigma_n \langle w,\, v_n \rangle u_n$$

$\forall w \in V$.

Comment (The Fundamental Theorem of Data Science)

If you want to be Buzzword Compliant, you could call this the Fundamental Theorem of Data Science

Singular Value Decomposition

Proof (Singular Value Decomposition)

By the [$\mathbb{C}/\mathbb{R}$ SPECTRAL THEOREM (NOTES#7.1)], we can find an orthonormal basis $v_1, \ldots, v_n$ of V such that $\sqrt{T^*T}(v_k) = \sigma_k v_k$, $k = 1, \ldots, n$. Hence due to [WRITING A VECTOR AS A LINEAR COMBINATION OF ORTHONORMAL BASIS (NOTES#6)] $\forall w \in V$

$$\begin{aligned} w &= \langle w,\, v_1\rangle v_1 + \cdots + \langle w,\, v_n\rangle v_n \\ \sqrt{T^*T}(w) &= \sqrt{T^*T}\left(\langle w,\, v_1\rangle v_1 + \cdots + \langle w,\, v_n\rangle v_n\right) \\ &= \sigma_1\langle w,\, v_1\rangle v_1 + \cdots + \sigma_n\langle w,\, v_n\rangle v_n \end{aligned}$$

By [POLAR DECOMPOSITION], $\exists$ an isometry $S \in \mathcal{L}(V)$ such that $T = S\sqrt{T^*T}$; thus

$$T(w) = \sigma_1\langle w,\, v_1\rangle S(v_1) + \cdots + \sigma_n\langle w,\, v_n\rangle S(v_n)$$

Let $u_k = S(v_k)$, $k = 1, \ldots, n$, then $u_1, \ldots, u_n$ is an orthonormal basis [CHARACTERIZATION OF ISOMETRIES]; and we have

$$T(w) = \sigma_1\langle w,\, v_1\rangle u_1 + \cdots + \sigma_n\langle w,\, v_n\rangle u_n$$

$\forall w \in V$. $\surd$

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Singular Value Decomposition

Comment (Singular Value Decomposition and Polar Decomposition)

When considering linear maps $T \in \mathcal{L}(V, W)$, we considered

$$\mathcal{M}(T; \mathfrak{B}(V); \mathfrak{B}(W));$$

in the operator setting $(W = V)$ $T \in \mathcal{L}(V)$ we usually consider

$$\mathcal{M}(T; \mathfrak{B}(V)),$$

making the basis $\mathfrak{B}(V)$ play both the input/domain and output/range roles.

In the Polar Decomposition setting, where $T = S\sqrt{T^*T}$, we may consider two bases for V, $\mathfrak{B}_1(V)$, and $\mathfrak{B}_2(V)$, so that

$$\mathcal{M}(S; \mathfrak{B}_1(V)), \text{ and } \mathcal{M}(\sqrt{T^*T}; \mathfrak{B}_2(V))$$

both are diagonal matrices.

Singular Value Decomposition

Comment (Singular Value Decomposition)

Now, in the Singular Value Decomposition we use one basis $\mathfrak{B}_1(V)$ for the input/domain side, and another $\mathfrak{B}_2(V)$ for the output/range side, so that

$$\mathcal{M}(T; \mathfrak{B}_1(V), \mathfrak{B}_2(V)) = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \sigma_n \end{bmatrix} = \mathrm{diag}(\sigma_1, \dots, \sigma_n)$$

Every $T \in \mathcal{L}(V)$ has orthonormal bases $\mathfrak{B}_1(V) = (v_1, \dots, v_n)$ and $\mathfrak{B}_2(V) = (u_1, \dots, u_n)$ so that

$$\mathcal{M}(T; \mathfrak{B}_1(V), \mathfrak{B}_2(V)) = \mathrm{diag}(\sigma_1, \dots, \sigma_n)$$

Singular Value Decomposition

The following result is useful when developing strategies for finding singular values:

Theorem (Singular Values Without Taking Square Root of an Operator)

Suppose $T \in \mathcal{L}(V)$. Then the singular values of T are the nonnegative square roots of the eigenvalues of T^*T, with each eigenvalue σ repeated $\dim(E(\sigma, T^*T))$ times.

Proof (Singular Values Without Taking Square Root of an Operator)

The [$\mathbb{C}/\mathbb{R}$ SPECTRAL THEOREM (NOTES#7.1)] implies that there is an orthonormal basis $v_1, \dots, v_n$ and nonnegative numbers $\sigma_1, \dots, \sigma_n$ such that $T^*T(v_i) = \sigma_i v_i$, $j = 1, \dots, n$. As we have done previously, defining $\sqrt{T^*T}(v_i) = \sqrt{\sigma_i}\, v_i$ gives the desired result.

$\langle\langle\langle$ Live Math $\rangle\rangle\rangle$

e.g. 7D-{4, **6**, 7, 10}

Live Math :: Covid-19 Version

7D-6: Find the singular values of the differentiation operator $D \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$ defined by $Dp = p'$, where the inner product on $\mathcal{P}_2(\mathbb{R})$ is the "Legendre Inner Product", $\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,dx$.

⋇ Reference Orthonormal Basis ⋇

In [NOTES#6] we derived an orthonormal basis for $\mathcal{P}_2(\mathbb{R})$ with this particular inner product:

$$u_0 = \sqrt{\frac{1}{2}}, \quad u_1 = \sqrt{\frac{3}{2}}x, \quad u_2 = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$$

⋇ Matrix wrt. the Standard Basis $\{1, x, x^2\}$ ⋇

The matrix of D with respect to the Standard Basis of $\mathcal{P}_2(\mathbb{R})$ is

$$\mathcal{M}(D, \{1, x, x^2\}) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

which shows that the only Eigenvalue of D is 0 by [Determination of Eigenvalues from Upper-Triangular Matrix (Notes#5)].

⋇ Toward Singular Values... ⋇

However, $\mathcal{M}(D, \{1, x, x^2\})$ cannot be used to compute the singular values since $\{1, x, x^2\}$ is not an orthonormal basis.

⁂ $\mathcal{M}(T, \{u_0(x),\, u_1(x),\, u_2(x)\})$ ⁂

Getting the coefficients for the matrix with respect to the reference orthonormal basis is a little messy, but not too bad:

$$
\begin{aligned}
D\left(\sqrt{\frac{1}{2}}\right) &= 0 \\
D\left(\sqrt{\frac{3}{2}}x\right) &= \sqrt{\frac{3}{2}} &&= \sqrt{3}\cdot\sqrt{\frac{1}{2}} \\
D\left(\sqrt{\frac{45}{8}}\left(x^2-\frac{1}{3}\right)\right) &= \sqrt{\frac{45}{2}}x &&= \sqrt{15}\cdot\sqrt{\frac{3}{2}}x
\end{aligned}
$$

Live Math :: Covid-19 Version

$$\mathcal{M}(T,\ \{u_0(x),\ u_1(x),\ u_2(x)\}) = \begin{bmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{15} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{M}(T^*,\ \{u_0(x),\ u_1(x),\ u_2(x)\}) = \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & \sqrt{15} & 0 \end{bmatrix}$$

$$\mathcal{M}(T^*T) = \begin{bmatrix} 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & \sqrt{15} & 0 \end{bmatrix} \begin{bmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{15} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

Hence, by [Singular Values Without Taking Square Root of an Operator], we have

$$\sigma(T) = \left\{ \sqrt{15},\ \sqrt{3},\ 0 \right\}$$

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Suggested Problems

7.C—1, **2**, **4**, 6, **7**

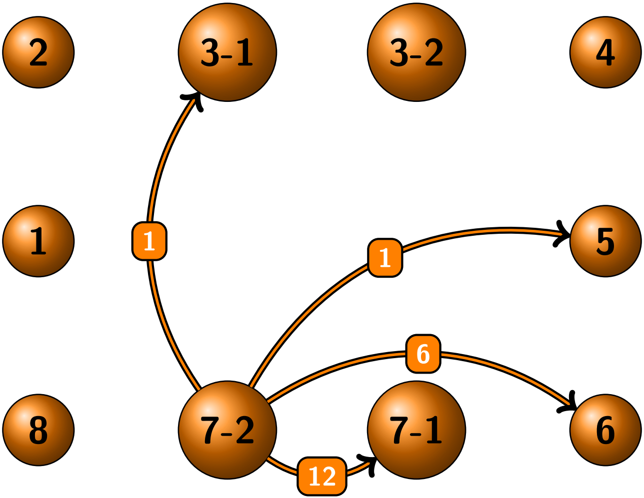
7.D—**1**, **2**, 4, **5**, 6, 7, 10

Assigned Homework — HW#7.2, Due 5/4/2020, 4:00am, Upload to Gradescope

7.C—2, 4, 7

7.D—1, 2, 5

Explicit References to Previous Theorems or Definitions (with count)



Explicit References to Previous Theorems or Definitions