Homework 7
Partial Differential Equations
Math 531
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Excersise 7.3.1d: Consider the heat equation in a two-dimensional rectangular region $0 < x < L, 0 < y < H$,

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

subject to the initial condition

$$u(x, y, 0) = \alpha(x, y).$$

[Hint: You many assume without derivation that product solutions $u(x, y, t) = \phi(x, y)h(t) = f(x)g(y)h(t)$ satisfy $\frac{dh}{dt} = -\lambda kh$, the two-dimensional eigenvalue problem $\nabla^2\phi + \lambda\phi = 0$ with further separation

$$\frac{d^2 f}{dx^2} = -\mu f, \qquad \frac{d^2 g}{dy^2} + (\lambda - \mu)g = 0,$$

or you may use results of the two-dimensional eigenvalue problem.] Solve the initial value problem and analyze the temperature as $t \to \infty$ if the boundary conditions are

$$u(0, y, t) = 0, \qquad \frac{\partial u}{\partial x}(L, y, t) = 0, \qquad \frac{\partial u}{\partial y}(x, 0, t) = 0, \qquad \frac{\partial u}{\partial y}(x, H, t) = 0$$

Let the following be true:

$$u(x,y,t) = f(x)g(y)h(t) \quad \text{with} \quad f(0)=0, \quad f'(L)=0, \quad g'(0)=0, \quad g'(H)=0$$

Also let the following be true throughout the rest of this assignment:

$$n, m, \ell \in \mathbb{Z}^+$$

Notice the following ODE, and different values for μ:

$$\frac{d^2 f}{dx^2} = -\mu f$$

($\mu < 0$):

$$f(x) = c_1 \cosh(\sqrt{\mu}x) + c_2 \sinh(\sqrt{\mu}x), \quad f'(x) = c_1\sqrt{\mu}\sinh(\sqrt{\mu}x) + c_2\sqrt{\mu}\cosh(\sqrt{\mu}x)$$

Using the boundary conditions, we get:

$$f(0) = c_1 = 0 \qquad \rightarrow \qquad f'(L) = c_2\sqrt{\mu}\cosh(\sqrt{\mu}L) \quad \rightarrow \quad c_2 = 0$$

Thus, we get the trivial solution:

$$f(x) = 0 \qquad \rightarrow \qquad u(x,y,t) = 0$$

($\mu = 0$):

$$f''(x) = 0, \quad f'(x) = c_1, \quad f(x) = c_1 x + c_2$$

Using the boundary conditions, we get:

$$f(0) = c_2 = 0 \qquad \rightarrow \qquad f'(L) = c_1 = 0$$

Thus, we get the trivial solution:

$$f(x) = 0 \qquad \rightarrow \qquad u(x,y,t) = 0$$

($\mu > 0$):

$$f(x) = c_1 \cos(\sqrt{\mu}x) + c_2 \sin(\sqrt{\mu}x), \quad f'(x) = -c_1\sqrt{\mu}\sin(\sqrt{\mu}x) + c_2\sqrt{\mu}\cos(\sqrt{\mu}x)$$

Using the boundary conditions, we get:

$$f(0) = c_1 = 0 \qquad \rightarrow \qquad f'(L) = c_2\sqrt{\mu}\cos(\sqrt{\mu}L)$$

If we let $c_2 = 0$, we get the trivial solution. Notice the other condition to allow a non-trivial solution:

$$\cos(\sqrt{\mu}L) = 0 \quad \rightarrow \quad \sqrt{\mu}L = \frac{(2n+1)\pi}{2} \quad \rightarrow \quad \mu = \left(\frac{(2n+1)\pi}{2L}\right)^2$$

Thus, we get this ODE's n eigenfunctions and n eigenvalues:

$$\mu_n = \left(\frac{(2n+1)\pi}{2L}\right)^2 \qquad f_n(x) = \sin\left(\frac{(2n+1)\pi x}{2L}\right)$$

Notice the following ODE:

$$\frac{d^2g}{dy^2} + (\lambda - \mu)g = 0 \qquad \rightarrow \qquad \frac{d^2g}{dy^2} = -(\lambda - \mu)g$$

We can see that this is similar to our previous ODE, so we can infer that when $\lambda - \mu \leq 0$, we get the trivial solution, so notice the following: $(\lambda > \mu)$:

$$g(y) = d_1 \cos(\sqrt{\lambda - \mu} y) + d_2 \sin(\sqrt{\lambda - \mu} y) \qquad g'(y) = -d_1\sqrt{\lambda - \mu}\sin(\sqrt{\lambda - \mu} y) + d_2\sqrt{\lambda - \mu}\cos(\sqrt{\lambda - \mu} y)$$

Using the boundary conditions, we get:

$$g'(0) = d_2\sqrt{\lambda - \nu} = 0 \quad \rightarrow \quad d_2 = 0 \qquad \rightarrow \qquad g'(H) = -d_1\sqrt{\lambda - \mu}\sin(\sqrt{\lambda - \mu} H)$$

If we let $d_1 = 0$, we get the trivial solution. Notice the other condition to allow a non-trivial solution:

$$\sin(\sqrt{\lambda - \mu} H) = 0 \quad \rightarrow \quad \sqrt{\lambda - \mu} H = m\pi \quad \rightarrow \quad \lambda - \mu = \left(\frac{m\pi}{H}\right)^2$$

Using the result from the previous ODE, we get this ODE's mn eigenfunctions and mn eigenvalues:

$$\lambda_m - \mu_n = \left(\frac{m\pi}{H}\right)^2 \qquad g_{mn}(y) = \cos\left(\frac{m\pi y}{H}\right)$$

Notice the eigenvalues and eigenfunctions for $\phi(x, y)$:

$$\lambda_{mn} = \left(\frac{m\pi}{H}\right)^2 + \mu_n = \left(\frac{m\pi}{H}\right)^2 + \left(\frac{(2n+1)\pi}{2L}\right)^2 \qquad \phi_{mn}(x, y) = \sin\left(\frac{(2n+1)\pi x}{2L}\right)\cos\left(\frac{m\pi y}{H}\right)$$

Notice the following ODE:

$$\frac{dh}{dt} = -\lambda k h$$

We get the following solution:

$$h_{mn}(t) = Ce^{-\lambda_{mn} kt}$$

From here, we get the following solution for $u(x, y, t)$ using the Principle of Superposition:

$$\boldsymbol{u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{(2n+1)\pi x}{2L}\right)\cos\left(\frac{m\pi y}{H}\right) e^{-\lambda_{mn} kt}}$$

where λ_{mn} are the eigenvalues of the spatial component.

Now we solve for the coefficients using the initial condition:

$$u(x, y, 0) = \alpha(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\left(\frac{(2n+1)\pi x}{2L}\right)\cos\left(\frac{m\pi y}{H}\right)$$

Using the orthogonality of sines and cosines, we get:

$$\boldsymbol{A_{mn} = \frac{4}{LH}\int_0^L \int_0^H \alpha(x, y) \sin\left(\frac{(2n+1)\pi x}{2L}\right)\cos\left(\frac{m\pi y}{H}\right) dy\, dx}$$

Excersise 7.3.2a: Consider the heat equation in a three-dimensional box-shaped region, $0 < x < L, 0 < y < H, 0 < z < W$,

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

subject to the initial condition

$$u(x, y, z, 0) = \alpha(x, y, z)$$

[Hint: You many assume without derivation that the product solutions $u(x, y, z, t) = \phi(x, y, z)h(t)$ satisfy $\frac{dh}{dt} = -\lambda kh$ and satisfy the three-dimensional eigenvalue problem $\nabla^2\phi + \lambda\phi = 0$, or you may use results of the three-dimensional eigenvalue problem.] Solve the initial value problem and analyze the temperature as $t \to \infty$ if the boundary conditions are

$$u(0, y, z, t) = 0, \qquad \frac{\partial u}{\partial y}(x, 0, z, t) = 0, \qquad \frac{\partial u}{\partial z}(x, y, 0, t) = 0,$$

$$u(L, y, z, t) = 0, \qquad \frac{\partial u}{\partial y}(x, H, z, t) = 0, \qquad u(x, y, W, t) = 0$$

Let the following be true:

$$u(x,y,z,t) = \phi(x,y,z)h(t) = f(x)g(y)q(z)h(t)$$

with the following boundary conditions:

$$f(0)=0, \quad f(L)=0, \quad g'(0)=0, \quad g'(H)=0, \quad q'(0)=0, \quad q(W)=0$$

Now we notice the following ODE's:

$$f'' = -\mu f \qquad g'' = -\nu g \qquad q'' + (\lambda - \mu - \nu) = 0 \qquad h' = -\lambda k h$$

To avoid the trivial solution, we will only notice the ODE's when the following is true:

$$\mu > 0 \qquad \nu > 0 \qquad \lambda > \mu + \nu$$

Thus we get the following:

$$f(x) = c_1\cos(\sqrt{\mu}x) + c_2\sin(\sqrt{\mu}x), \quad f'(x) = -c_1\sqrt{\mu}\sin(\sqrt{\mu}x) + c_2\sqrt{\mu}\cos(\sqrt{\mu}x)$$

$$g(y) = d_1\cos(\sqrt{\nu}y) + d_2\sin(\sqrt{\nu}y) \qquad g'(y) = -d_1\sqrt{\nu}\sin(\sqrt{\nu}y) + d_2\sqrt{\nu}\cos(\sqrt{\nu}y)$$

$$q(z) = b_1\cos(\sqrt{\lambda-\mu-\nu}z) + b_2\sin(\sqrt{\lambda-\mu-\nu}z)$$

$$q'(y) = -b_1\sqrt{\lambda-\mu-\nu}\sin(\sqrt{\lambda-\mu-\nu}z) + b_2\sqrt{\lambda-\mu-\nu}\cos(\sqrt{\lambda-\mu-\nu}z)$$

Now we use our boundary conditions, to get the following:

$$f(0) = c_1 = 0 \qquad f(L) = c_2\sin(\sqrt{\mu}L) = 0 \qquad \sqrt{\mu}L = n\pi \qquad \mu = \left(\frac{n\pi}{L}\right)^2$$

$$g'(0) = d_2\sqrt{\nu} = 0 \qquad d_2 = 0 \qquad g'(H) = -d_1\sqrt{\nu}\sin(\sqrt{\nu}H) = 0 \qquad \sqrt{\nu}H = m\pi \qquad \nu = \left(\frac{m\pi}{H}\right)^2$$

$$q'(0) = b_2\sqrt{\lambda-\mu-\nu} = 0 \qquad b_2 = 0$$

$$q(W) = b_1\cos(\sqrt{\lambda-\mu-\nu}W) \qquad \sqrt{\lambda-\mu-\nu}W = \frac{(2\ell+1)\pi}{2} \qquad \lambda_\ell - \mu_n - \nu_m = \left(\frac{(2\ell+1)\pi}{2W}\right)^2$$

So thus we get the following eigenvalues and eigenfunctions:

$$\mu_n = \left(\frac{n\pi}{L}\right)^2 \qquad f_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

$$\nu_m = \left(\frac{m\pi}{H}\right)^2 \qquad g_m(x) = \cos\left(\frac{m\pi y}{H}\right)$$

$$\lambda_\ell - \mu_n - \nu_m = \left(\frac{(2\ell+1)\pi}{2W}\right)^2 \qquad q_\ell(x) = \cos\left(\frac{(2\ell+1)\pi z}{2W}\right)$$

Thus we get the eigenvalues and eigenfunctions for $\phi(x,y,z)$

$$\lambda_{mn\ell} = \mu_n + \nu_m + \left(\frac{(2\ell+1)\pi}{2W}\right)^2 = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 + \left(\frac{(2\ell+1)\pi}{2W}\right)$$

$$\phi(x,y,z) = \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\cos\left(\frac{(2\ell+1)\pi z}{2W}\right)$$

Now we can solve for the time dependent ODE:

$$h' = -\lambda k h \qquad \to \qquad h(t) = Ce^{-\lambda k t}$$

From here, we get the following solution for $u(x, y, z, t)$ using the Principle of Superposition:

$$\boldsymbol{u(x,y,z,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} A_{mn\ell} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{H}\right) \cos\left(\frac{(2\ell+1)\pi z}{2W}\right) e^{-\lambda_{mn\ell} k t}}$$

where $\lambda_{mn\ell}$ are the eigenvalues of the spatial component.

Now we solve for the coefficients using the initial condition:

$$u(x,y,z,0) = \alpha(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} A_{mn\ell} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{H}\right) \cos\left(\frac{(2\ell+1)\pi z}{2W}\right)$$

Using the orthogonality of sines and cosines, we get:

$$\boldsymbol{A_{mn} = \frac{8}{LHW} \int_0^L \int_0^H \int_0^W \alpha(x,y) \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{H}\right) \cos\left(\frac{(2\ell+1)\pi z}{2W}\right) dz\, dy\, dx}$$

Excersise 7.3.4a: Consider the wave equation for a vibrating rectangular membrane $(0 < x < L, 0 < y < H)$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

subject to the initial conditions

$$u(x, y, 0) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x, y, 0) = \alpha(x, y)$$

[Hint: You many assume without derivation that the product solutions $u(x, y, t) = \phi(x, y)h(t)$ satisfy $\frac{d^2h}{dt^2} = -\lambda c^2 h$ and the two-dimensional eigenvalue problem $\nabla^2\phi + \lambda\phi = 0$, and you may use results of the two-dimensional eigenvalue problem.]

Solve the initial value problem if

$$u(0, y, t) = 0, \qquad u(L, y, t) = 0, \qquad \frac{\partial u}{\partial y}(x, 0, t) = 0, \qquad \frac{\partial u}{\partial y}(x, H, t) = 0$$

Let the following be true:

$$u(x, y, t) = \phi(x, y)h(t) = f(x)g(y)h(t)$$

with the following boundary conditions:

$$f(0) = 0, \quad f(L) = 0, \quad g'(0) = 0, \quad g'(H) = 0$$

Now we notice the following ODE's:

$$f'' = -\mu f \qquad g'' + (\lambda - \mu)g = 0 \qquad h'' = -\lambda c^2 h$$

To avoid the trivial solution, we will only notice the ODE's when the following is true:

$$\mu > 0 \qquad \lambda > \mu$$

Thus we get the following:

$$f(x) = c_1 \cos(\sqrt{\mu}x) + c_2 \sin(\sqrt{\mu}x), \quad f'(x) = -c_1\sqrt{\mu}\sin(\sqrt{\mu}x) + c_2\sqrt{\mu}\cos(\sqrt{\mu}x)$$

$$g(y) = d_1 \cos(\sqrt{\lambda - \mu}y) + d_2 \sin(\sqrt{\lambda - \mu}y) \qquad g'(y) = -d_1\sqrt{\lambda - \mu}\sin(\sqrt{\lambda - \mu}y) + d_2\sqrt{\lambda - \mu}\cos(\sqrt{\lambda - \mu}y)$$

Now we use out boundary conditions to get the following:

$$f(0) = c_1 = 0 \qquad f(L) = c_2 \sin(\sqrt{\mu}L) = 0 \qquad \sqrt{\mu}L = n\pi \qquad \mu = \left(\frac{n\pi}{L}\right)^2$$

$$g'(0) = d_2\sqrt{\lambda - \mu} \qquad d_2 = 0 \qquad g'(H) = -d_1\sqrt{\lambda - \mu}\sin(\sqrt{\lambda - \mu}H) \qquad \sqrt{\lambda - \mu}H = m\pi \qquad \lambda - \mu = \left(\frac{m\pi}{H}\right)^2$$

So thus we get the following eigenvalues and eigenfunctions:

$$\mu_n = \left(\frac{n\pi}{L}\right)^2 \qquad f_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

$$\lambda_m - \mu_n = \left(\frac{m\pi}{H}\right)^2 \qquad g_m(y) = \cos\left(\frac{m\pi y}{H}\right)$$

Thus we get the eigenvalues and eigenfunctions for $\phi(x, y)$:

$$\lambda_{mn} = \left(\frac{m\pi}{H}\right)^2 + \mu_n = \left(\frac{m\pi}{H}\right)^2 + \left(\frac{n\pi}{L}\right)^2 \qquad \phi(x, y) = \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)$$

Now we can solve for the time dependent ODE:

$$h'' = -\lambda c^2 h$$

Notice that we solved for $\lambda = \lambda_{mn} > 0$:

$$h(t) = b_1 \cos(\sqrt{-\lambda c^2}t) + b_2 \sin(\sqrt{-\lambda c^2}t) \quad \rightarrow \quad h'(t) = -b_1\sqrt{-\lambda c^2}\sin(\sqrt{-\lambda c^2}t) + b_2\sqrt{-\lambda c^2}\cos(\sqrt{-\lambda c^2}t)$$

From here, we get the following solution for $u(x,y,t)$ using the Principle of Superposition:

$$\boldsymbol{u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\cos(\sqrt{-\lambda c^2}t)}$$
$$\boldsymbol{+B_{mn}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\sin(\sqrt{-\lambda c^2}t)}$$

$$\boldsymbol{\frac{\partial u}{\partial t}(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} -A_{mn}\sqrt{-\lambda c^2}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\sin(\sqrt{-\lambda c^2}t)}$$
$$\boldsymbol{+B_{mn}\sqrt{-\lambda c^2}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\cos(\sqrt{-\lambda c^2}t)}$$

where λ_{mn} are the eigenvalues of the spatial component.

Now we solve for the coefficients using the initial conditions:

$$\boldsymbol{u(x,y,0) = 0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)}$$
$$\boldsymbol{A_{mn} = 0}$$

$$\boldsymbol{\frac{\partial u}{\partial t}(x,y,0) = \alpha(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\sqrt{-\lambda c^2}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)}$$
$$\boldsymbol{B_{mn} = \frac{4}{LH}\int_0^L\int_0^H \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)\,dy\,dx}$$

Excersise 7.3.5: Consider

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - k\frac{\partial u}{\partial t} \qquad \text{with} \qquad k > 0.$$

(a) Give a *brief* physical interpretation of this equation.

This is a 2 dimensional vibrating membrane that is being damped over time.

(b) Suppose that $u(x, y, t) = f(x)g(y)h(t)$. What ordinary differential equations are satisfied by f, g, and h?

Notice we can rearrange the given ODE:

$$\begin{aligned} fgh'' &= c^2 f''gh + c^2 fg''h - kfgh' \\ fg(h'' - kh') &= h(c^2 f''g + c^2 fg'') \\ \frac{1}{h}(h'' - kh') &= c^2\frac{f''}{f} + c^2\frac{g''}{g} = -\lambda \end{aligned}$$

Thus we get the following ODE's:

$$\boldsymbol{h'' - kh' + \lambda h = 0 \qquad f'' = -\mu f \qquad g'' + \left(\frac{\lambda}{c^2} + \mu\right) = 0}$$

Excersise 7.5.1: The vertical displacement of a nonuniform membrane satisfies

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),$$

where c depends on x and y. Suppose that $u = 0$ on the boundary of an irregularly shaped membrane.

(a) Show that the time variable can be separated by assuming that

$$u(x, y, t) = \phi(x, y)h(t).$$

Show that $\phi(x, y)$ satisfies the eigenvalue problem

$$\nabla^2\phi + \lambda\sigma(x, y)\phi = 0 \qquad \text{with} \qquad \phi = 0 \qquad \text{on the boundary}$$

What is $\sigma(x, y)$?

Notice we can rearrange the given ODE:

$$\begin{aligned} \phi h'' &= c^2 \left(\frac{\partial^2 \phi}{\partial x^2} h + \frac{\partial^2 \phi}{\partial y^2} h \right) \\ \frac{h''}{h} &= \frac{c^2}{\phi} \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) \\ \frac{h''}{h} &= \frac{c^2}{\phi}\nabla^2\phi = -\lambda \end{aligned}$$

From here we get the following:

$$\boldsymbol{\nabla^2\phi(x, y) + \frac{\lambda\phi(x, y)}{c^2(x, y)} = \nabla^2\phi(x, y) + \lambda\sigma(x, y)\phi(x, y) = 0 \qquad \sigma = \frac{1}{c^2(x, y)}}$$

(b) If the eigenvalues are known (and $\lambda > 0$), determine the frequencies of vibration.

Notice if we know the eigenvalues, we can solve the time dependent ODE:

$$h'' + \lambda_n h = 0 \qquad h(t) = c_1 \cos(\sqrt{\lambda_n} t) + c_2 \sin(\sqrt{\lambda_n} t)$$

Thus we get that the frequencies of vibration is:

$$\boldsymbol{\sqrt{\lambda_n}}$$