Homework 4
Abstract Algebra
Math 320
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Section 2.1 Problem 14 (a): Prove or disprove: If $ab \equiv 0$ (mod n), then $a \equiv 0$ (mod n) or $b \equiv 0$ (mod n).

Solution 14a.

Disprove :
Let $a = 4, b = 3$
$(4)(3) \equiv 0$ (mod 12)
But $4 \not\equiv 0$ (mod 12) and $3 \not\equiv 0$ (mod 12)

□

Section 2.1 Problem 14 (b): Do part (a) when n is prime

Solution 14b. Let $ab \equiv 0$ (mod n) and $a, b, n, q \in \mathbb{Z}$

$n|(ab - 0)$
$n|ab$
By Theorem 1.5: $\boldsymbol{a \equiv 0}$ **(mod n) or** $\boldsymbol{b \equiv 0}$ **(mod n)**

□

Section 2.1 Problem 15: If $(a, n) = 1$, prove that there is an integer b such that $ab \equiv 1$ (mod n).

Solution 15. Let $(a, n) = 1$

$$ab + n(v) = 1$$
$$ab - 1 = n(-v)$$
$$n|(ab - 1)$$
$$\mathbf{ab} \equiv \boldsymbol{n} \textbf{ (mod n)}$$

□

Section 2.1 Problem 20 (a): Prove or disprove: If $a^2 \equiv b^2 \pmod{n}$, then $a \equiv b \pmod{n}$ or $a \equiv -b \pmod{n}$.

Solution 20a.

$$\begin{aligned}
&5^2 \equiv 1^2 \pmod{24} \\
&\mathit{Disprove}: \\
&5 \not\equiv 1 \pmod{24} \\
&5 \not\equiv -1 \pmod{24}
\end{aligned}$$

□

Section 2.1 Problem 20 (b): Do part (a) when n is prime.

Solution 20b. Let n be prime and $a^2 \equiv b^2 \pmod{n}$

$$\begin{aligned}
&n|(a^2 - b^2) \\
&n|(a+b)(a-b) \\
&\text{By Thm 1.5: } n|(a-(-b)) \text{ or } n|(a-b) \\
&\text{Thus } a \equiv b \pmod{n} \text{ or } a \equiv -b \pmod{n}
\end{aligned}$$

□

Section 2.1 Problem 21 (a): Show that $10^n \equiv 1 \pmod 9$ for every positive n.

Solution 21a. let $n \in \mathbb{Z}^+$

$$\begin{aligned} 10^n - 1 &= (10-1)(10^0 + 10^1 + 10^2 + ... + 10^{n-1}) \\ &= 9(10^0 + 10^1 + 10^2 + ... + 10^{n-1}) \\ & 9|(10^n - 1) \\ & \mathbf{10^n \equiv 1 \pmod 9} \end{aligned}$$

□

Section 2.1 Problem 21 (b): Prove that every positive integer is congruent to the sum of its digits (mod 9) [for example, $38 \equiv 11 \pmod 9$].

Solution 21b. Notice: $\forall n \in \mathbb{Z}$, n $= 10^0 a_0 + 10^1 a_1 + 10^2 a_2 + ... + 10^n a_n$, where a_i are the digits of n with $i \in \mathbb{Z}_{\geq 0}$.

$$\begin{aligned} n &= (1)a_0 + (9+1)a_1 + (99+1)a_2 + ... + (999...99+1)a_n \\ &= 9a_1 + 99a_2 + ... + 999...99a_n + (a_0 + a_1 + a_2 + ... + a_n) \\ & n - (a_0 + a_1 + a_2 + ... + a_n) = 9(a_1 + 11a_2 + ... + 111...11a_n) \\ & 9|(n - (a_0 + a_1 + a_2 + ... + a_n)) \\ & n \equiv (a_0 + a_1 + a_2 + ... + a_n) \pmod 9 \end{aligned}$$

So any integer n is congruent to the sum of it digits (mod 9)

□