Math 320 April 30, 2020

Last time: Introduced Ideals

Def: A subset I of a ring R an <u>ideal</u> if

(1) $0_R \in I$

(2) I closed under subtraction

(3) If $r \in R$ and $s \in I$, then

$$rs \in I \text{ and } sr \in I$$

(Note: subsets I that satisfy these 3 conditions are sometimes called 2-sided ideals)

To show a subset I is an ideal, just go through these three steps.
Note that for step 3, r is an <u>arbitrary</u> element of R, and s is an element of I.

Examples:

In $\mathbb{Z}$, let $n \in \mathbb{Z}$ and denote the set of multiples of n by $n\mathbb{Z}$ (so $2\mathbb{Z}$ = even integers)

Then $n\mathbb{Z}$ is an ideal in $\mathbb{Z}$ for all $n \geq 1$.

Show $4\mathbb{Z}$ is an ideal.

(1) $0 = 4 \cdot 0$, so $0 \in 4\mathbb{Z}$. ✓

(2) Let $a, b \in 4\mathbb{Z}$, (i.e. a, b are multiples of 4), so $a = 4k$ and $b = 4m$ f.s. $k, m \in \mathbb{Z}$.

To show closure under subtraction, show $a - b \in 4\mathbb{Z}$:

$$a - b = 4k - 4m = 4(k - m) \in 4\mathbb{Z} \checkmark$$

(3) Let $r \in \mathbb{Z}$ and $s \in 4\mathbb{Z}$, so $s = 4c$ for some $c \in \mathbb{Z}$.

Want to show $rs, sr \in 4\mathbb{Z}$

$$s \cdot r = (4c)(r) = 4(cr) \in 4\mathbb{Z} \checkmark$$

Since $\mathbb{Z}$ is commutative, $sr = rs$.

Thus, $4\mathbb{Z}$ is an ideal in $\mathbb{Z}$.

Another Example:

I = nonunits in $\mathbb{Z}_8$, so

$I = \{0, 2, 4, 6\}$

This is an ideal in $\mathbb{Z}_8$.

- First, notice $0 \in I$.
- Closure under subtraction:
  - $4 - 2 = 2 \in I$
  - $2 - 4 = -2 = 6 \in I$
  - $6 - 2 = 4 \in I$
  - $2 - 6 = -4 = 4 \in I$
  - $6 - 4 = 2 \in I$
  - $4 - 6 = -2 = 6 \in I$.
- Next, show absorption property

Ex: $5 \in \mathbb{Z}_8$, $6 \in I$

$$5 \cdot 6 = 30 = 6 \in I.$$

Note: if $r \in \mathbb{Z}_8$, and $s \in I$, then s is even (considered as an integer)

Then, rs is even, so $(rs, 8) \geq 2$.

Therefore, rs is not a unit, since $(rs, 8) \neq 1$.

Another example:

Let S be the ring:

$$S = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

Some elements are $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 3 & \pi \\ 0 & 3 \end{pmatrix}$, etc

$$J = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathbb{R} \right\}$$

Note: $J \subset S$. Show J is an ideal in S:

(1) Show $0_S \in J$.

2x2 zero matrix

for $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in J$, set $b = 0$ to get

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0_S \in J \checkmark$$

(2) Closure under subtraction:

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \in J:$$

$$\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b-c \\ 0 & 0 \end{pmatrix} \in J$$

(3) Let $A \in S$, $B \in J$

Since $A \in S$, $A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ f.s. $a, b \in \mathbb{R}$

Since $B \in J$, $B = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ f.s. $x \in \mathbb{R}$.

We're not sure if S is commutative so show both $AB, BA \in J$:

$$AB = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ax \\ 0 & 0 \end{pmatrix} \in J$$

$$BA = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & ax \\ 0 & 0 \end{pmatrix} \in J$$

Thus, J is an ideal in S.

Finitely generated Ideals:

Thm 6.2 : Let R be a commutative ring, $c \in R$, and let

R has identity

$$I = \underbrace{\{rc : r \in R\}}_{\text{"multiples of } c\text{"}}$$

Then I is an ideal.

Pf: (1) set $r = 0_R$ to get

$$0_R \cdot c = 0_R \Rightarrow 0_R \in I \checkmark$$

(2) Let $x, y \in I$, so $x = r_1 c$, $y = r_2 c$ f.s. $r_1, r_2 \in R$. Then,

$$x - y = r_1 c - r_2 c = \underbrace{(r_1 - r_2)}_{\in R} c \in I \checkmark$$

(3) Let $z \in R$, $s \in I$, so $s = r \cdot c$ f.s. $r \in R$. Then,

$\cdot\ zs = z(rc) = (zr)\cdot c \in I$

$\cdot\ sz = (rc)\cdot z = z(rc) = (zr)c \in I.$

Therefore, I is an ideal. $\blacksquare$

We denote this ideal by (c), and call (c) the principal ideal generated by c.

Ex: Consider $x^2-2 \in \mathbb{Q}[x]$, then

$$(x^2-2) = \text{multiples of } x^2-2$$
$$= \{ f(x)\cdot(x^2-2) : f(x) \in \mathbb{Q}[x] \}$$

Thm 6.3: Let R be a commutative ring with identity, and let

$$c_1, c_2, \ldots, c_n \in R.$$

Then, the set

$$I = \{ r_1 c_1 + r_2 c_2 + \cdots + r_n c_n : r_i \in R \}$$

is an ideal in R

We denote this ideal by

$$(c_1, c_2, c_3, ---, c_n)$$

and call this the ideal generated by $c_1, c_2, ---, c_n$

Note: $(c_1, c_2, c_3, ---, c_n) =$ "linear combos of $c_1, c_2, ..., c_n$"

Ex: $(2, 3, 7) \subset \mathbb{Z}$

$$(2,3,7) = \{ 2a + 3b + 7c : a, b, c \in \mathbb{Z} \}$$