Quiz 9
Differential Equations
Math 337
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Problem 1: Suppose that $f(t)$ has a *Laplace Transform* $F(s) = \mathcal{L}[f(t)]$, where

$$F(s) = \frac{12}{s^3} + \frac{16}{s^2+9} + \frac{20s}{(s+1)^2+4}$$

Find $f(t)$

To find $f(t)$, we need to take the Laplace inverse of each term:

$$f(t) = 6t^2 + \frac{16\sin(3t)}{3} + 20e^{-t}\cos(2t)$$

Problem 2: Suppose that $f(t)$ has a *Laplace Transform* $F(s) = \mathcal{L}[f(t)]$, where

$$F(s) = \frac{3s^2 + 8s + 80}{s^2(s^2 - 6s + 40)}$$

Find $f(t)$

Notice the following:

$$F(s) = \frac{3}{(s-3)^2 + 31} + \frac{8}{s(s^2 - 6s + 40)} + \frac{80}{s^2(s^2 - 6s + 40)}$$

Notice the partial fractions decomposition:

$$\frac{8}{s(s^2 - 6s + 40)} = \frac{A}{s} + \frac{Bs + C}{s^2 - 6s + 40}$$
$$8 = (A + B)s^2 + (-6A + C)s + 40A$$

So we can see that $A = \frac{1}{5}, B = \frac{-1}{5}, C = \frac{6}{5}$

$$\begin{aligned} \frac{8}{s(s^2 - 6s + 40)} &= \frac{1}{5s} + \frac{-s + 6}{5(s^2 - 6s + 40)} \\ &= \frac{1}{5s} - \frac{s - 3}{5((s-3)^2 + 31)} + \frac{9}{5((s-3)^2 + 31)} \end{aligned}$$

Notice another partial fractions decomposition:

$$\begin{aligned} \frac{80}{s^2(s^2 - 6s + 40)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 - 6s + 40} \\ 80 &= As^3 - 6As^2 + 40As + Bs^2 - 6Bs + 40B + Cs^3 + Ds^2 \\ &= (A + C)s^3 + (-6A + B + D)s^2 + (40A - 6B)s + 40B \end{aligned}$$

So we can see that $B = 2, A = \frac{3}{10}, D = \frac{-2}{10}, C = \frac{-3}{10}$

$$\frac{80}{s^2(s^2 - 6s + 40)} = \frac{3}{10s} + \frac{2}{s^2} - \frac{3(s-3)}{10((s-3)^2 + 31)} - \frac{11}{10((s-3)^2 + 31)}$$

After resubstitution, we get:

$$\begin{aligned} F(s) = &\frac{3}{(s-3)^2 + 31} + \frac{1}{5s} - \frac{s - 3}{5((s-3)^2 + 31)} + \frac{9}{5((s-3)^2 + 31)} \\ &+ \frac{3}{10s} + \frac{2}{s^2} - \frac{3(s-3)}{10((s-3)^2 + 31)} - \frac{11}{10((s-3)^2 + 31)} \end{aligned}$$

After taking the Laplace inverse of each term, we get:

$$\begin{aligned} f(t) &= \frac{3\sin(\sqrt{31}t)e^{3t}}{\sqrt{31}} + \frac{1}{5} - \frac{\cos(\sqrt{31}t)e^{3t}}{5} + \frac{9\sin(\sqrt{31}t)e^{3t}}{5\sqrt{31}} + \frac{3}{10} + 2t \\ &= \frac{3\cos(\sqrt{31}t)e^{3t}}{10} + \frac{11\sin(\sqrt{31}t)e^{3t}}{10\sqrt{31}} \end{aligned}$$

Problem 3: Solve the following initial value problem with *Laplace transforms*:

$$y'' - 4y' + 8y = 20\cos(2t), \qquad y(0) = 5, \qquad y'(0) = 6.$$

So we need to take the Laplace transform of each side:

$$\begin{aligned}
\mathcal{L}[y'' - 4y' + 8y] &= s^2Y(s) - sy(0) - y'(0) - 4sY(s) + 4y(0) + 8Y(s) \\
&= (s^2 - 4s + 8)Y(s) - (5s - 14) \\
\mathcal{L}[20\cos(2t)] &= \frac{20s}{s^2+4}
\end{aligned}$$

So we get the following equality:

$$Y(s) = \frac{20s}{(s^2+4)(s^2-4s+8)} + \frac{5s-14}{s^2-4s+8}$$

Notice the partial fractions decomposition:

$$\begin{aligned}
\frac{20s}{(s^2+4)(s^2-4s+8)} &= \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2-4s+8} \\
20s &= (A+C)s^3 + (-4A+B+D)s^2 + (8A-4B+4C)s + (8B+4D)
\end{aligned}$$

We can solve the following using system of equations:

$$\begin{aligned}
A + C &= 0 \\
-4A + B + D &= 0 \\
8A - 4B + 4C &= 20 \\
8B + 4D &= 0
\end{aligned}$$

Using matrix reduction, we get $A = 1, B = -4, C = -1, D = 8$.

$$\frac{20s}{(s^2+4)(s^2-4s+8)} = \frac{s-4}{s^2+4} - \frac{s-8}{s^2-4s+8}$$

So we get the following after some algebra:

$$\begin{aligned}
Y(s) &= \frac{s-4}{s^2+4} - \frac{s-8}{s^2-4s+8} + \frac{5s-14}{s^2-4s+8} \\
&= \frac{s}{s^2+4} - \frac{4}{s^2+4} - \frac{s-2}{(s-2)^2+4} + \frac{6}{(s-2)^2+4} + \frac{5(s-2)}{(s-2)^2+4} - \frac{4}{(s-2)^2+4} \\
&= \frac{s}{s^2+4} - \frac{4}{s^2+4} + \frac{4(s-2)}{(s-2)^2+4} + \frac{2}{(s-2)^2+4}
\end{aligned}$$

Now we take Laplace inverse of each term and get:

$$y(t) = \cos(2t) - 2\sin(2t) + 4\cos(2t)e^{2t} + \sin(2t)e^{2t}$$