Quiz 1
Ordinary Differential Equations
Math 537
Stephen Giang

Problem 1: Consider the following first-order ODE's: Please provide one example for each of the below ODEs and discuss the corresponding solutions.

(a) Separable ODEs

$$\frac{dy}{dt} = yt \qquad\qquad \ln(y) = \frac{t^2}{2} + C$$
$$\frac{dy}{y} = t\,dt \qquad\qquad y = Ce^{\frac{t^2}{2}}$$

For Separable ODEs, they will be in the form of $\frac{dy}{dt} = f(t,y) = p(t)q(y)$, with solutions

$$y = g(t)$$

(b) Linear ODEs

$$\frac{dy}{dt} + y = e^t \qquad \frac{d}{dt}(e^t y) = e^{2t} \qquad y = \frac{1}{2}e^t + Ce^{-t}$$
$$e^t\frac{dy}{dt} + e^t y = e^{2t} \qquad e^t y = \frac{1}{2}e^{2t} + C$$

For Linear ODEs, they will be in the form of $\frac{dy}{dt} + p(t)y = g(t)$, with solutions of

$$y(t) = e^{-\int p(t)\,dt}\left[\int e^{\int p(t)\,dt} g(t)\,dt + C\right]$$

(c) Exact ODEs

$$2xy^2 + 2yx^2\frac{dy}{dt} = 0$$

Notice that it is in the form of $M(x,y) + N(x,y)y' = 0$, with $M_y(x,y) = 4xy = N_x(x,y)$, thus giving us the solution ($\phi(x,y)$):

$$\int 2xy^2\,dx = x^2y^2 + h(y)$$
$$\int 2yx^2\,dy = x^2y^2 + g(x)$$
$$\phi(x,y) = x^2y^2 + C$$

For Exact ODEs, they will be in the form of $M(x,y)+N(x,y)y' = 0$, with $M_y(x,y) = N_x(x,y)$, with solutions of

$$\phi(x,y) = \int M(x,y)\,dx = \int N(x,y)\,dy$$

(d) Bernoulli Equations

$$\frac{dy}{dt} + y = y^2$$

Notice we can substitute $u = y^{1-2} = \frac{1}{y}$, $\frac{du}{dt} = \frac{-1}{y^2}\frac{dy}{dt}$ into the equation after multiplying the equation by $\frac{-1}{y^2}$:

$$\frac{du}{dt} - u = -1 \qquad\qquad \ln(u-1) = t + C$$

$$\frac{du}{u-1} = dt \qquad\qquad u = Ce^t + 1$$

Now we need to resubstitute $u = y^{1-2}$, and we get:

$$\frac{1}{y} = Ce^t + 1$$

$$y = \frac{1}{Ce^t + 1}$$

Problem 2: Consider the following homogeneous linear 2nd-order ODEs with constant coefficients:

$$y'' + ay' + by = 0$$

where a and b are constant. Please discuss three types of solutions based on the so-called characteristic equation.

By converting this equation into a system of first order differential equations. We can then find the characteristic equation to be $\lambda^2 + a\lambda + b = 0$. We can find the roots λ_1 and λ_2 from the characteristic equations. Notice the three types of solutions:

(a) Distinct Real Roots with ($\lambda_1 \neq \lambda_2$):

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

(b) Equal Real Roots with ($\lambda_1 = \lambda_2$):

$$y(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_2 t}$$

(c) Roots with ($\lambda_{1,2} = \mu \pm i\nu$):

$$c_1 e^{\mu t} \cos(\nu t) + c_2 e^{\mu t} \sin(\nu t)$$

Problem 3: Consider the following Euler-Cauchy equation:

$$x^2y'' + axy' + by = 0$$

where a and b are constant.

(a) Please discuss three types of solutions: We can substitute $y(x) = x^r$. This will get us its characteristic equation to be $r^2 + (a-1)r + b$ and get 2 solutions r_1 and r_2 with $x > 0$:

(i) Distinct Real Roots with ($r_1 \neq r_2$):

$$y(t) = c_1x^{r_1} + c_2x^{r_2}$$

(ii) Equal Real Roots with ($r_1 = r_2$):

$$y(t) = (c_1 + c_2 \ln x)x^{r_1}$$

(iii) Roots with ($r_{1,2} = \mu \pm i\nu$):

$$y(t) = x^{\mu}\left[c_1 \cos(\nu \ln x) + c_2 \sin(\nu \ln x)\right]$$

(b) Introduce a new independent variable (t), $x = e^t$, to convert the above Euler-Cauchy equation into a second-order ODE with constant coefficients (i.e., in the form of Eq. 2).

Notice the following with $x = e^t$:

$$\begin{aligned}
y' &= \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dt}e^{-t} \\
y'' &= \frac{d^2y}{dx^2} = \left(\frac{d}{dx}\right)\frac{dy}{dx} = \frac{\frac{d}{dt}}{\frac{dx}{dt}}\left(\frac{dy}{dx}\right) \\
&= \frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}} = \frac{e^{-t}\frac{d^2y}{dt^2} - e^{-t}\frac{dy}{dt}}{e^t} \\
&= e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)
\end{aligned}$$

Now we can resubsitute the following into our Cauchy-Euler equation with $x = e^t$:

$$\begin{aligned}
&e^{2t}\left(e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\right) + ae^t\left(\frac{dy}{dt}e^{-t}\right) + by = 0 \\
&\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = 0
\end{aligned}$$

This looks very similar to our characteristic equation as noticed in (a).

Problem 4: Provide a brief summary on what has been completed in this assignment.

In this assignment, we reviewed the techniques to solve first-order ODEs, the different type of solutions from 2nd-order ODE's, the different types of solutions from Cauchy-Euler Equations and how the Cauchy-Euler Equation can be converted into a constant-coefficient 2nd Order ODE.