Homework 1
Abstract Algebra
Math 320
Stephen Giang

Section 1.1 Problem 9: Prove that the cube of any integer a has to be exactly one of these forms: $9k$ or $9k+1$ or $9k+8$ for some integer k.

Solution:
Let $a \in \mathbb{Z}$. By the Division Algorithm, Let $a = 3q + r, \quad 0 \leq r < 3$.
Remark: Integers are closed under Multiplication and Addition.

Case ($r = 0$):

$$\begin{aligned} a &= 3q && (1)\\ a^3 &= (3q)^3 && (2)\\ &= 9(3q^3) && (3)\\ &= \boldsymbol{9k}, \quad k \in \mathbb{Z} && (4)\\ & && (5) \end{aligned}$$

Case ($r = 1$):

$$\begin{aligned} a &= 3q+1 && (6)\\ a^3 &= (3q+1)^3 && (7)\\ &= (3q)^3 + 3(3q)^2 + 3(3q) + 1 && (8)\\ &= 9(3q^3 + 3(q^2) + q) + 1 && (9)\\ &= \boldsymbol{9k+1}, \quad k \in \mathbb{Z} && (10)\\ & && (11) \end{aligned}$$

Case ($r = 2$):

$$\begin{aligned} a &= 3q+2 && (12)\\ a^3 &= (3q+2)^3 && (13)\\ &= (3q)^3 + 3(3q)^2(2) + 3(3q)(2^2) + 2^3 && (14)\\ &= 9(3q^3 + 3(q^2)(2) + (q)(2^2)) + 2^3 && (15)\\ &= \boldsymbol{9k+8}, \quad k \in \mathbb{Z} && (16)\\ & && (17) \end{aligned}$$

Thus: $\forall a \in \mathbb{Z}, a^3$ can be written in the form: $9k$ or $9k+1$ or $9k+8$ for some integer k.

Section 1.1 Problem 10: Let n be a positive integer. Prove that a and c leave the same remainder when divided by n if and only if $a - c = nk$ for some integer k

Solution: ($\rightarrow$)
Let $a - c = nk, \quad$ with $a, c, k, q_1, q_2, r_1, r_2 \in \mathbb{Z}$

$$\begin{aligned} a &= nq_1 + r_1 && (18) \\ c &= nq_2 + r_2 && (19) \\ a - c &= (nq_1 + r_1) - (nq_2 + r_2) && (20) \\ &= n(q_1 - q_2) + (r_1 - r_2) && (21) \\ &= nk && (22) \end{aligned}$$

Remark: $0 \leq (r_1 - r_2) < n$. To have $a - c = nk$, $(r_1 - r_2) = cn$ for some $c \in \mathbb{Z}$, or have $(r_1 - r_2)$ be a multiple of n. The only multiple of n on the interval [0,n) is 0. So...

$$\begin{aligned} r_1 - r_2 &= 0 && (23) \\ \boldsymbol{r_1} &= \boldsymbol{r_2} && (24) \end{aligned}$$

Thus a, c share the same remainder when divided by n

Solution: ($\leftarrow$)
Let $a, c, k, q_1, q_2, r \in \mathbb{Z}$
Let a and c leave the same remainder when divided by n

$$\begin{aligned} a &= nq_1 + r && (25) \\ c &= nq_2 + r && (26) \\ a - c &= nq_1 + r - (nq_2 + r) && (27) \\ &= nq_1 + r - nq_2 - r && (28) \\ &= nq_1 - nq_2 && (29) \\ &= n(q_1 - q_2) && (30) \\ &= nk && (31) \\ \boldsymbol{a - c} &= \boldsymbol{nk} && (32) \end{aligned}$$

Section 1.2 Problem 4:
a) If $a|b$ and $a|c$, prove that $a|(b+c)$.
b) If $a|b$ and $a|c$, prove that $a|(br+ct)$ for any r, t $\in \mathbb{Z}$.

Solution (a):
Let $a|b$ and $a|c$ for some $a, b, c, x_1, x_2 \in \mathbb{Z}$

$$\begin{aligned} b &= ax_1 && (33) \\ c &= ax_2 && (34) \\ b+c &= ax_1 + ax_2 && (35) \\ &= a(x_1+x_2) && (36) \\ &= ax_3 && (37) \\ & && (38) \\ \boldsymbol{a|(b+c)} & && (39) \end{aligned}$$

Solution (b):
Let $a|b$ and $a|c$ for some $a, b, c, x_1, x_2, r, t \in \mathbb{Z}$

$$\begin{aligned} b &= ax_1 && (40) \\ c &= ax_2 && (41) \\ br+ct &= ax_1r + ax_2t && (42) \\ &= a(x_1r+x_2t) && (43) \\ &= ax_3 && (44) \\ \boldsymbol{a|(br+ct)} & && (45) \end{aligned}$$

Section 1.2 Problem 5: If a and b are nonzero integers such that $a|b$ and $b|a$, prove that $a = \pm b$.

Solution:
Let $a, b \in \mathbb{Z} \setminus \{0\}$ such that $a|b$ and $b|a$, $\quad a, b, x_1, x_2 \in \mathbb{Z}$

$$b = ax_1 \tag{46}$$
$$a = bx_2 \tag{47}$$
$$b = (bx_2)x_1 \tag{48}$$
$$1 = x_2x_1 \tag{49}$$
$$x_2 = 1 \text{ and } x_1 = 1 \tag{50}$$
$$\text{or} \tag{51}$$
$$x_2 = -1 \text{ and } x_1 = -1 \tag{52}$$
$$a = b \text{ or } -b \tag{53}$$
$$\boldsymbol{a} = \pm\boldsymbol{b} \tag{54}$$