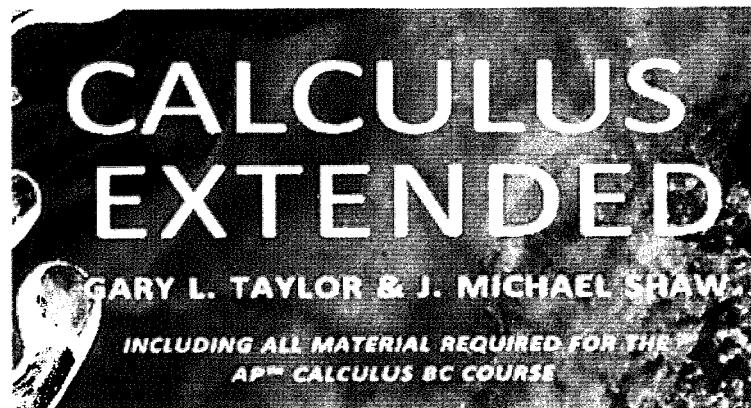


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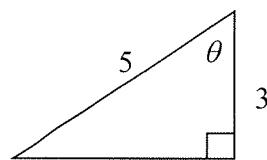
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Lesson 1-1 Trig Review, Limits, Continuity

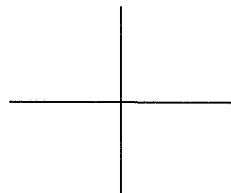
Trig Review

Example 1: Use the triangle at right to find

- a. $\sin \theta$ b. $\tan \theta$ c. $\sec \theta$



Example 2: Find the following, if θ is an angle in standard position whose terminal side passes through the point $(-5, 2)$.



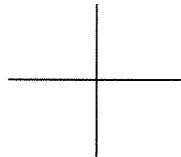
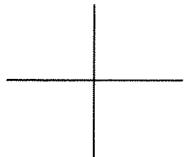
$$\sin \theta =$$

$$\csc \theta =$$

$$\cot \theta =$$

Example 3: Draw angles in standard position and make “reference triangles” to find:

- a. $\cos 210^\circ$ b. $\tan 315^\circ$



Example 4: Since 2π radians $= 360^\circ$, it follows that $\pi^R = 180^\circ$, and the following common radian measures should be easy to think about in degrees. Convert each common radian measure to degrees.

- a. $\frac{\pi}{2} =$ b. $\frac{\pi}{4} =$ c. $\frac{\pi}{3} =$ d. $\frac{\pi}{6} =$

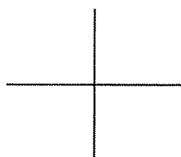
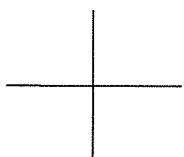
Example 5: Convert from radians to degrees or degrees to radians without using a calculator.

- a. $\frac{5\pi}{4} =$ b. $270^\circ =$ c. $-120^\circ =$

Examples: Draw angles in standard position, and make “reference triangles” to find the following without using a calculator:

6. $\cos\left(\frac{-3\pi}{4}\right) =$

7. $\csc\frac{5\pi}{3} =$



A unit circle is created by letting $r = 1$ when dealing with the circular trig functions.

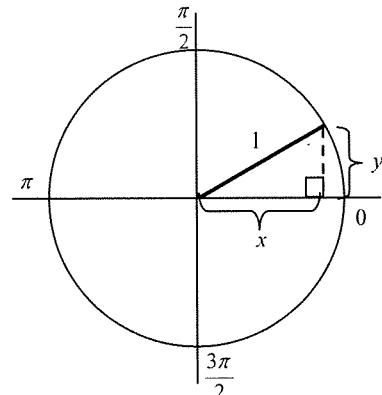
Then, $\sin \theta = y$, $\cos \theta = x$, and $\tan \theta = \frac{y}{x}$.

Example 8: Use a unit circle to find:

a. $\sin \frac{\pi}{6} =$ b. $\sin 0 =$ c. $\cos 0 =$

d. $\sin \frac{\pi}{2} =$ e. $\tan\left(\frac{-\pi}{2}\right) =$ f. $\tan \pi =$

g. $\csc \frac{3\pi}{2} =$ h. $\cos \frac{3\pi}{2} =$



Limits

Informally, a limit is a y-value which a function approaches as x approaches some value.

$\lim_{x \rightarrow c} f(x) = L$ means as x approaches c , $f(x)$ approaches the y -value of L .

Examples

limits:

9. $\lim_{x \rightarrow -4} f(x) =$ 16. $f(-4) =$

10. $\lim_{x \rightarrow -1} f(x) =$ 17. $f(-1) =$

11. $\lim_{x \rightarrow 2} f(x) =$ 18. $f(2) =$

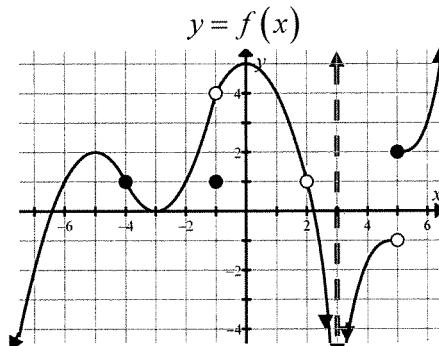
12. $\lim_{x \rightarrow 3} f(x) =$ 19. $f(3) =$

13. $\lim_{x \rightarrow 5} f(x) =$ 20. $f(5) =$

one-sided limits:

14. $\lim_{x \rightarrow 5^-} f(x) =$

15. $\lim_{x \rightarrow 5^+} f(x) =$



Continuity

Informally, a function is continuous where it can be drawn without lifting a pencil. Roughly, continuous means "connected."

Formally, a function is continuous where its limit and function value are the same.

In this course, we will work with three types of discontinuities: holes, vertical asymptotes, and jumps (breaks). A fourth type of discontinuity is an oscillating discontinuity (these rarely appear).

To investigate this fourth type, graph $y = \sin \frac{1}{x}$ on a calculator and look at windows close to $x = 0$.

Example 21. List the x -values of the discontinuities of the function $y = f(x)$ graphed on page 2.

All discontinuities can be classified as removable or nonremovable.

Removable discontinuities occur when the function has a limit (holes in the graph).

Nonremovable discontinuities occur when the limit of the function does not exist (jumps, vertical asymptotes, or oscillations).

Example 22. Which of the discontinuities from Example 21 are removable?

At x -values where a function is continuous, limits can be found by direct substitution.

Examples:

$$23. \lim_{x \rightarrow 3} (3x^2 + 2) =$$

$$24. \lim_{x \rightarrow 1} \frac{x^2 + x}{x + 1} =$$

$$25. \lim_{x \rightarrow \frac{\pi}{3}} \cos(2x) =$$

For piecewise functions, one-sided limit evaluation is often necessary.

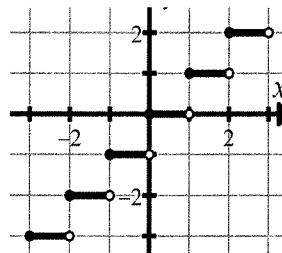
Examples:

$$26. \text{ If } f(x) = \begin{cases} 4-x, & x \leq 1 \\ 4x-x^2, & x > 1 \end{cases}, \quad \lim_{x \rightarrow 1} f(x) =$$

$$27. \text{ If } g(x) = \begin{cases} 3x-x^3, & x \leq 1 \\ 2x^2-1, & x > 1 \end{cases}, \quad \lim_{x \rightarrow 1} g(x) =$$

$$28. \text{ For this same } g \text{ function, } \lim_{x \rightarrow -1} g(x) =$$

Another function requiring one-sided limit analysis is a step function called the Greatest Integer Function. $f(x) = \|x\| =$ the greatest integer less than or equal to x . The graph is shown at the right.



Examples: Find the following limits.

$$29. \lim_{x \rightarrow -\frac{1}{2}} \|x\| =$$

$$30. \lim_{x \rightarrow 1} \|x\| =$$

$$31. \lim_{x \rightarrow 5^-} \|2x-3\| =$$

Assignment 1-1

Use the appearance of the graph shown at the right to find the following limit and function values.

1. $\lim_{x \rightarrow -4} f(x)$

2. $\lim_{x \rightarrow -1} f(x)$

3. $\lim_{x \rightarrow 0} f(x)$

4. $f(0)$

5. $\lim_{x \rightarrow 0^-} f(x)$

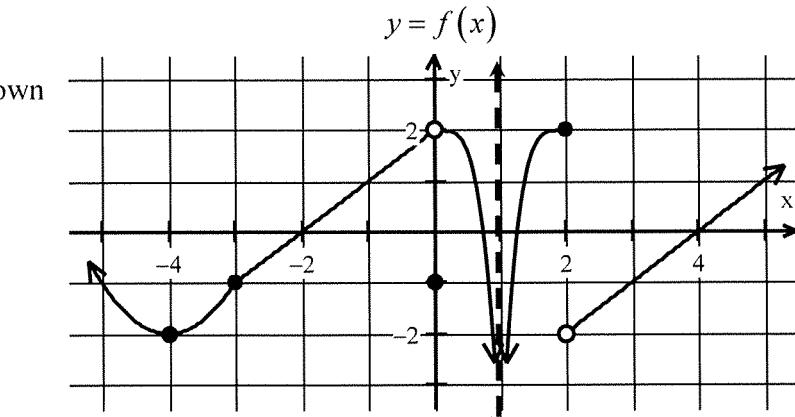
6. $\lim_{x \rightarrow 1} f(x)$

7. $f(1)$

8. $\lim_{x \rightarrow 2} f(x)$

9. $\lim_{x \rightarrow 2^-} f(x)$

10. $\lim_{x \rightarrow 2^+} f(x)$



11. $f(2)$

12. $\lim_{x \rightarrow 4} f(x)$

13. $\lim_{x \rightarrow 4^+} f(x)$

Use the function $g(x) = \begin{cases} 2x - 3, & x \leq 0 \\ -x - 3, & 0 < x \leq 2 \\ 3x, & x > 2 \end{cases}$ for problems 14-20.

14. Sketch an accurate graph without using a calculator.

15. $\lim_{x \rightarrow 0} g(x) =$

16. $\lim_{x \rightarrow 2} g(x) =$

17. $\lim_{x \rightarrow 2^-} g(x) =$

18. $\lim_{x \rightarrow 2^+} g(x) =$

19. $g(2) =$

20. $\lim_{x \rightarrow -2} g(x) =$

Find each of the following limits without using a calculator. Simplify your answers.

21. $\lim_{x \rightarrow 0} (2x - 5)$

22. $\lim_{x \rightarrow -3} (x^2 - 5x + 4)$

23. $\lim_{x \rightarrow 2} \frac{2x - 5}{\sqrt{x + 7}}$

24. $\lim_{x \rightarrow -2} |3x + 5|$

25. $\lim_{x \rightarrow 2} \frac{3x - 6}{\sqrt{x + 6}}$

26. $\lim_{x \rightarrow \pi} \sin x$

27. $\lim_{x \rightarrow \frac{\pi}{2}} \cos x$

28. $\lim_{x \rightarrow \pi} \tan x$

29. $\lim_{x \rightarrow \frac{\pi}{2}} \cos(2x)$

30. $\lim_{x \rightarrow 2} \cos \frac{\pi x}{3}$

31. $\lim_{x \rightarrow 3} \sec \frac{\pi x}{4}$

32. $\lim_{x \rightarrow 7} \csc \frac{\pi x}{6}$

33. $\lim_{x \rightarrow \pi} \cot \frac{x}{6}$

34. $\lim_{x \rightarrow 5\pi} \cos \frac{x}{3}$

Use the function $f(x) = \begin{cases} 6x - 3x^3, & x \leq 2 \\ 4x - x^4, & x > 2 \end{cases}$ for problems 35, 36.

35. $\lim_{x \rightarrow 2^+} f(x)$

36. $\lim_{x \rightarrow 2} f(x)$

Use the function $g(x) = \begin{cases} 2\sin\frac{3x}{2}, & x \leq \pi \\ \sec\frac{11x}{6}, & x > \pi \end{cases}$ for problems 37-39.

37. $\lim_{x \rightarrow \pi^+} g(x)$ 38. $\lim_{x \rightarrow \pi^-} g(x)$ 39. $\lim_{x \rightarrow \pi} g(x)$

Use the functions $g(x) = 3x^2 - 5x$ and $f(x) = \sqrt[3]{3x + 5}$ for problems 40-42.

40. $\lim_{x \rightarrow 2} g(x)$ 41. $\lim_{x \rightarrow 1} f(x)$ 42. $\lim_{x \rightarrow 3} f(g(x))$

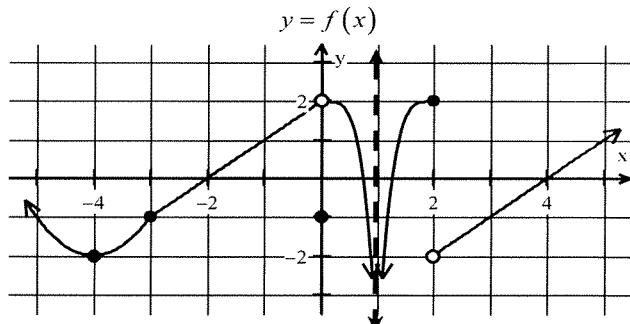
If $\lim_{x \rightarrow 3} h(x) = 5$ and $\lim_{x \rightarrow 3} k(x) = 3$ find the following limits.

43. $\lim_{x \rightarrow 3} (h(x) + k(x))$ 44. $\lim_{x \rightarrow 3} (h(x)k(x))$ 45. $\lim_{x \rightarrow 3} \frac{h(x)}{k(x)}$

The symbol $\| \ |$ is used to represent the Greatest Integer Function in the following problems. Find these limits without using a calculator or state that the limit does not exist.

46. $\lim_{x \rightarrow 3^-} \|x - 1\|$ 47. $\lim_{x \rightarrow 3^+} \|x - 1\|$ 48. $\lim_{x \rightarrow 3} \|x - 1\|$ 49. $\lim_{x \rightarrow 3} \left\| \frac{x}{2} - 1 \right\|$ 50. $\lim_{x \rightarrow 3^-} \|4x - 1\|$

51. Identify each x -value at which the function shown appears to be discontinuous and classify each as removable or nonremovable.



Find all discontinuities for the following functions and classify each as removable or nonremovable. Do not use a calculator.

52. $f(x) = \begin{cases} 3x^3 + 4x, & x \leq -2 \\ x^4 + 16, & x > -2 \end{cases}$ 53. $f(x) = \begin{cases} 3x^3 + 4x, & x \leq 2 \\ x^4 + 16, & x > 2 \end{cases}$

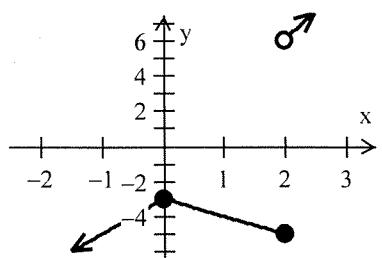
54. $g(x) = \begin{cases} 2\sin\frac{\pi x}{2}, & x \leq 1 \\ \cos\frac{\pi x}{3}, & x > 1 \end{cases}$

55. $g(x) = \begin{cases} \cos x, & x \leq 0 \\ -x + 1, & 0 < x \leq 2 \\ \sin\frac{\pi x}{2}, & x > 2 \end{cases}$

56. $f(x) = \|x + 5\|$

57. $h(x) = \left\| \frac{x}{3} \right\|$

Selected Answers:

1. -2 2. 1 3. 2 5. 2 6. DNE or $-\infty$ 7. DNE 9. 2 10. -2 11. 2 12. 0
 14.  15. -3 16. DNE 17. -5 19. -5 20. -7 21. -5
 22. 28 23. $-\frac{1}{3}$ 24. 1 26. 0 27. 0 29. -1
 30. $-\frac{1}{2}$ 31. $-\sqrt{2}$ 32. -2 34. $\frac{1}{2}$ 35. -8 36. DNE
 38. -2 40. 2 41. 2 42. $\sqrt[3]{41}$ 43. 8 44. 15
 45. $\frac{5}{3}$ 46. 1 48. DNE 50. 10

51. $x = 0$ remov., $x = 1$ nonrem., $x = 2$ nonrem. 52. $x = -2$ nonrem. 53. no discontinuities
 54. $x = 1$ nonrem. 56. nonremovable discontinuity at every integer

Lesson 1-2 More Limits, Continuity, Intermediate Value Theorem, Graphing Adjustments

If direct substitution does not give an answer to a limit problem because an indeterminate form is obtained (usually $\frac{0}{0}$), use algebraic techniques to change the form of the limit.

Examples:

1. $\lim_{x \rightarrow 0} \frac{x}{x(x+1)}$

2. $\lim_{x \rightarrow 2} \frac{x-2}{x^2-4}$

3. $\lim_{x \rightarrow 1} \frac{x^3-1}{x-1}$

4. $\lim_{x \rightarrow 1} \frac{\sqrt{x}-1}{x-1}$

5. $\lim_{x \rightarrow 2^+} \frac{x}{x-2}$

6. Discuss the continuity of

$$f(x) = \begin{cases} \frac{x^2-2x-3}{x-3}, & x \neq 3 \\ 5, & x = 3 \end{cases}$$

7. If $g(x) = \begin{cases} 3x^2 + a, & x > 2 \\ x-3, & x \leq 2 \end{cases}$

is a continuous function,
find the value of a .

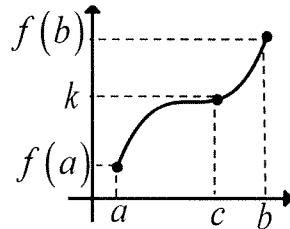
8. Use a calculator to find these limits.

a. $\lim_{x \rightarrow 1} \frac{x^3 - 3x^2 + x + 2}{x^3 - 2x^2 - x + 2}$

b. $\lim_{x \rightarrow 2} \frac{x^3 - 3x^2 + x + 2}{x^3 - 2x^2 - x + 2}$

Intermediate Value Theorem

If f is continuous on $[a, b]$ and k is any y -value between $f(a)$ and $f(b)$, then there is at least one x -value c between a and b such that $f(c) = k$. In other words, f takes on every y -value between $f(a)$ and $f(b)$.



Examples:

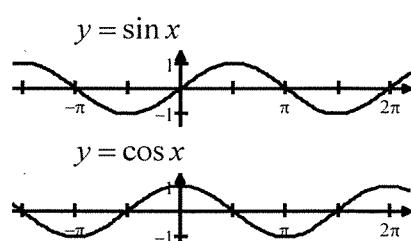
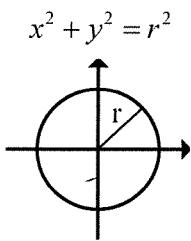
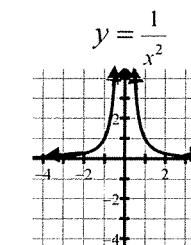
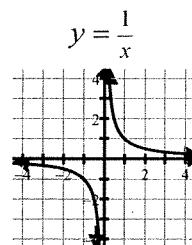
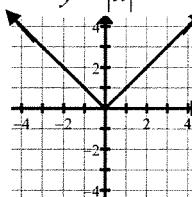
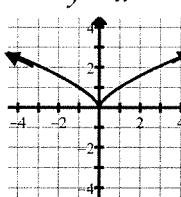
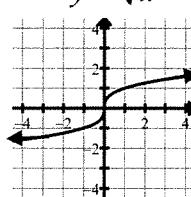
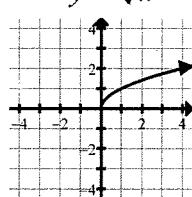
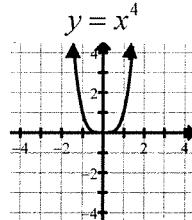
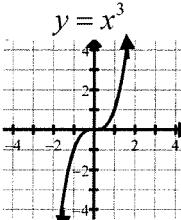
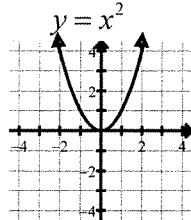
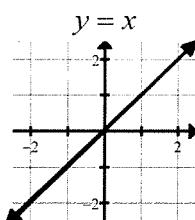
9. Does the Intermediate Value Theorem guarantee a c -value on the given interval.

a. $f(x) = x^2 - x$, \quad b. $g(x) = \frac{x^2 - 4}{x - 2}$,
 $f(c) = 12, \quad [0, 5] \quad g(c) = 4, \quad [0, 3]$

10. Find the value of c in Example 9a.

Parent Graphs

These graphs occur so frequently in this course that it would be worth your time to learn (memorize) them.



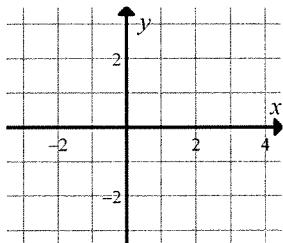
Graphing Adjustments to $y = f(x)$

1. $y = -f(x)$ reflect across the x -axis
2. $y = f(-x)$ reflect across the y -axis
3. $y = f(x) + d$ shift up if $d > 0$, shift down if $d < 0$
4. $y = f(x + c)$ shift left if $c > 0$, shift right if $c < 0$
5. $y = a \cdot f(x)$ vertical stretch if $a > 1$, vertical squeeze if $a < 1$
(assumes a is positive, if a is negative a reflection is needed)
6. $y = f(b \cdot x)$ horizontal squeeze if $b > 1$, horizontal stretch if $b < 1$
(assumes b is positive, if b is negative a reflection is needed)
7. $y = |f(x)|$ reflect all points below the x -axis across the x -axis. Leave points above the x -axis alone.
8. $y = f(|x|)$ eliminate completely all points left of the y -axis. Leave points right of the y -axis alone. Replace the left half of the graph with a reflection of the right half. Your graph should then show y -axis symmetry.

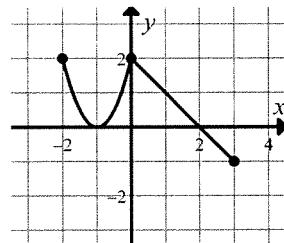
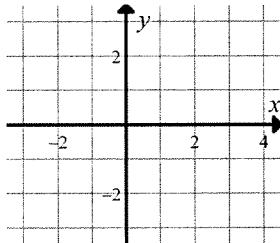
Note: Adjustments to functions always produce functions.

Examples: Use the graph of $y = f(x)$ shown to sketch the following:

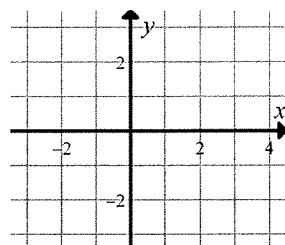
11. $y = f(x + 2)$



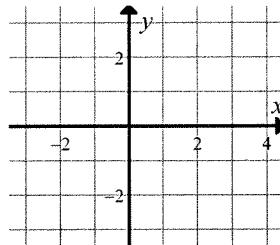
12. $y = -f(x) + 2$



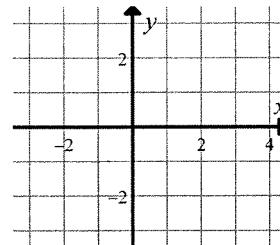
13. $y = \frac{1}{2}f(-x)$



14. $y = |f(2x)|$



15. $y = f(|x|)$



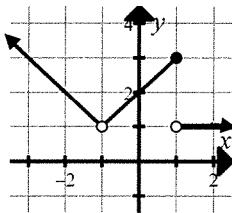
Assignment 1-2Find the indicated limits without using a calculator. **Show steps using correct limit symbolism!**

$$\begin{array}{cccc}
 1. \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} & 2. \lim_{x \rightarrow -1} \frac{2x^2 - x - 3}{x + 1} & 3. \lim_{x \rightarrow -1} \frac{x^2 - 1}{x - 1} & 4. \lim_{x \rightarrow -1} \frac{x^3 + 1}{x + 1} \\
 5. \lim_{x \rightarrow -1} \frac{x}{x^2 + 1} & 6. \lim_{x \rightarrow 5^+} \frac{x - 5}{x^2 - 25} & 7. \lim_{x \rightarrow -5} \frac{x - 5}{x^2 - 25} & 8. \lim_{x \rightarrow 2} \frac{2 - x}{x^2 - 4} \\
 9. \lim_{x \rightarrow 3} \frac{x^2 - 9}{x^2 - 6x + 9} & 10. \lim_{x \rightarrow -2} \sqrt[3]{x^2 + 4} & 11. \lim_{x \rightarrow 0} \frac{x}{x - 1} & 12. \lim_{x \rightarrow 2^-} \frac{1}{x^2 - 4} \\
 13. \lim_{x \rightarrow 1} \frac{x}{x^2 + 1} & 14. \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4} & 15. \lim_{x \rightarrow 3} \frac{x - 3}{\sqrt{x+1} - 2} & 16. \lim_{x \rightarrow 0^+} \frac{|x|}{x} \\
 17. \lim_{x \rightarrow 0} \frac{|x|}{x} & 18. \lim_{x \rightarrow \frac{\pi}{2}} \sin x & 19. \lim_{x \rightarrow \pi} \sec x & 20. \lim_{x \rightarrow \frac{\pi}{2}} \cos(3x) \\
 21. \lim_{x \rightarrow 5} \csc \frac{\pi x}{6} & 22. \lim_{x \rightarrow 3^+} \|x - 1\| & 23. \lim_{x \rightarrow 3^-} \|x - 1\| & 24. \lim_{x \rightarrow 3} \|x - 1\| \\
 25. \lim_{x \rightarrow 2} \|x + 6\| & 26. \lim_{x \rightarrow 3} \left\| \frac{x}{2} \right\| & 27. \lim_{x \rightarrow 5^-} \|2x - 3\| & 28. \lim_{x \rightarrow 5} \frac{\frac{1}{x} - \frac{1}{5}}{x - 5} \\
 29. \lim_{x \rightarrow 3} \begin{cases} \frac{1}{2}x + 1, & x \leq 3 \\ \frac{12 - 2x}{3}, & x > 3 \end{cases} & 30. \lim_{x \rightarrow 1} \begin{cases} x^2 + 1, & x < 1 \\ x^3 + 1, & x \geq 1 \end{cases} & 31. \lim_{x \rightarrow 2} \begin{cases} x - 2, & x \leq 0 \\ x + 2, & x > 0 \end{cases} &
 \end{array}$$

Use a calculator to find these limits.

$$32. \text{(a)} \lim_{x \rightarrow 1} \frac{\sin x}{6x} \quad \text{(b)} \lim_{x \rightarrow 0} \frac{\sin x}{6x} \quad 33. \lim_{x \rightarrow -2} \frac{x^3 + 2x^2 - x - 2}{x^3 + 2x^2 + x + 2} \quad 34. \lim_{x \rightarrow 2} \frac{|2 - x|}{25x - 50}$$

35. a. $h(-1)$ b. $h(1)$
 c. $\lim_{x \rightarrow 1} h(x)$ d. $\lim_{x \rightarrow 1^-} h(x)$
 e. $\lim_{x \rightarrow 1^+} h(x)$ f. $\lim_{x \rightarrow 1} h(x)$
 g. removable discontinuities
 h. nonremovable discontinuities



$$h(x) = \begin{cases} -x, & x < -1 \\ x + 2, & -1 < x \leq 1 \\ 1, & x > 1 \end{cases}$$

Find all discontinuities for these functions and classify them as removable or nonremovable. Do not use a calculator.

$$\begin{array}{cccc}
 36. f(x) = \frac{2x - 3}{x + 1} & 37. f(x) = 2x - 3 & 38. f(x) = \frac{1}{x^2 - 9} & 39. f(x) = \frac{x}{x^2 + x} \\
 40. f(x) = \frac{x^2 - 9}{x + 3} & 41. f(x) = \begin{cases} x^2, & x \leq 0 \\ x, & x > 0 \end{cases} & 42. f(x) = \begin{cases} x - 3, & x \leq 1 \\ x, & x > 1 \end{cases} & \\
 43. f(x) = \begin{cases} 2x - 5, & x > 3 \\ x^2 - 8, & x < 3 \end{cases} & 44. f(x) = \|x - 1\| & 45. f(x) = \left\| \frac{x}{2} \right\| &
 \end{array}$$

Use a calculator to find all discontinuities for these functions and classify them as removable or nonremovable.

46. $f(x) = \frac{10x}{6x^3 - 31x^2 + 23x + 20}$

47. $f(x) = \frac{x}{x^3 + 4x}$

48. $f(x) = \left\| \frac{x}{4} \right\|$

49. If the function $f(x) = \begin{cases} x^3, & x \leq 2 \\ ax^2, & x > 2 \end{cases}$ is continuous, find the value of a .

50. Find the values of a and b so that $f(x) = \begin{cases} x-1, & x \leq -1 \\ ax+b, & -1 < x < 1 \\ 2x+1, & x \geq 1 \end{cases}$ is continuous.

Determine whether the Intermediate Value Theorem would guarantee a c -value on the given interval.

51. $f(x) = x^2 + x - 1, \quad f(c) = 11, \quad [0, 5]$

52. $f(x) = \frac{x}{x-1}, \quad f(c) = 1, \quad [0, 2]$

53. $f(x) = |x|, \quad f(c) = 3, \quad [-4, 1]$

54. $f(x) = \begin{cases} x, & x \leq 1 \\ 3, & x > 1 \end{cases}, \quad f(c) = 2, \quad [0, 4]$

55. $f(x) = \frac{x^2 + x}{x-1}, \quad f(c) = 6, \quad \left[\frac{5}{2}, 4 \right]$

56. Find the c -value in Problem 51.

57. Find the c -value in Problem 53.

58. Find the c -value in Problem 55.

59. Find an equation of the line which intersects the graph of $f(x) = \begin{cases} x^2 + 1, & x < 1 \\ x^3 + 1, & x \geq 1 \end{cases}$ when $x = -2$ and again when $x = 2$.

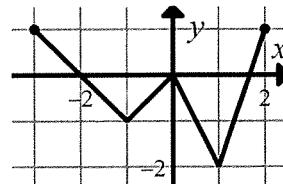
60. Use the parent graph of $y = \sqrt{x}$ to graph the following.

a. $y = \sqrt{x} + 2$ b. $y = -\sqrt{x}$ c. $y = 2\sqrt{x}$

Use the graph of $y = f(x)$ to draw accurate graphs of the following.

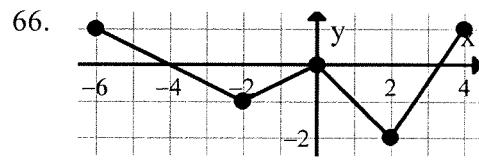
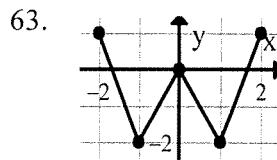
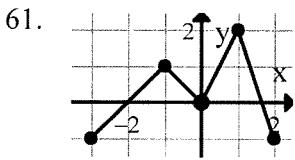
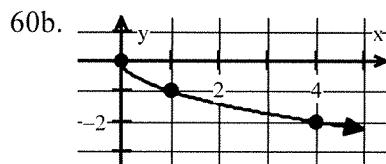
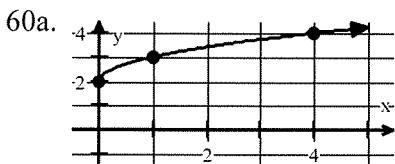
61. $y = -f(x)$ 62. $y = |f(x)|$ 63. $y = f(|x|)$

64. $y = f(x) - 1$ 65. $y = \frac{1}{2}f(x)$ 66. $y = f\left(\frac{1}{2}x\right)$



Selected Answers:

1. 2 2. -5 3. 0 4. 3 6. $\frac{1}{10}$ 7. DNE 8. $-\frac{1}{4}$ 9. DNE 10. 2 11. 0
 12. DNE or $-\infty$ 14. $\frac{1}{4}$ 15. 4 16. -1 18. 1 19. -1 21. 2 22. 2 24. DNE 26. 1
 27. 6 28. $-\frac{1}{25}$ 29. DNE 30. 2 32a. .140 b. .166 or .167 34. DNE 35a. DNE
 35b. 3 c. 1 e. 1 g. $x = -1$ 36. $x = -1$ nonremovable 38. $x = \pm 3$ nonremovable
 39. $x = 0$ removable, $x = -1$ nonremovable 40. $x = -3$ removable
 42. $x = 1$ nonremovable 44. nonremovable discontinuity at every integer
 45. nonremovable discontinuity at every even integer 47. $x = 0$ removable 49. $a = 2$
 50. $a = \frac{5}{2}$, $b = \frac{1}{2}$ 51. yes 52. no 53. yes 54. no 56. $c = 3$ 57. $c = -3$
 59. $y - 5 = x + 2$ or $y - 9 = x - 2$ or $y = x + 7$



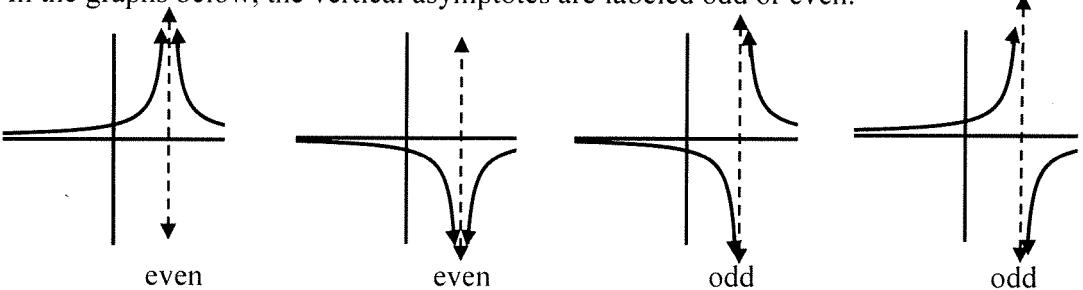
Lesson 1-3 Infinite Limits, Limits at Infinity, Curve Sketching

Review:

The graph of the function $f(x) = \frac{x-1}{(x-1)(x-2)^2(x-4)^3}$ has a **hole** at $x = \underline{\hspace{2cm}}$,

an **even vertical asymptote** at $x = \underline{\hspace{2cm}}$,
 and an **odd vertical asymptote** at $x = \underline{\hspace{2cm}}$.

In the graphs below, the vertical asymptotes are labeled odd or even.



Infinite Limits

You have seen examples where a limit does not exist at a vertical asymptote. Such non-existent limits can be expressed as infinite limits if the vertical asymptote is even or if you are finding one-sided limits. We will write $\lim_{x \rightarrow c} f(x) = \infty$ or $\lim_{x \rightarrow c} f(x) = -\infty$.

The examples below make use of your knowledge of even and odd vertical asymptotes as well as holes.

Examples:

$$1. \lim_{x \rightarrow 2^+} \frac{x+3}{x-2} = \quad 2. \lim_{x \rightarrow 2^-} \frac{x+3}{x-2} = \quad 3. \lim_{x \rightarrow 1^+} \frac{x-2}{(x-1)^2} =$$

$$4. \lim_{x \rightarrow 1^-} \frac{x-2}{(x-1)^2} = \quad 5. \lim_{x \rightarrow 2} \frac{x^2-4}{x-2} =$$

Limits at Infinity

If the graph of a function $f(x)$ approaches a horizontal asymptote to the left and/or the right, $f(x)$ is said to have a limit at infinity. If the asymptote is $y = L$ then $\lim_{x \rightarrow \infty} f(x) = L$. In other words, limits at infinity give us end behaviors for graphs of functions. For “large” values of x , the highest degree terms in the numerator and denominator dominate the other terms and are the only terms you need to consider.

Review Examples: Find the horizontal asymptotes.

$$6. f(x) = \frac{5x^4 - 3x^2 + 2}{10x^4 + 3} \quad 7. g(x) = \frac{5x^4 - 3x^2 + 2}{10x^5 + 3} \quad 8. h(x) = \frac{5x^4 - 3x^2 + 2}{10x^3 + 3}$$

Examples: Find the following limits.

$$9. \lim_{x \rightarrow \infty} \frac{5x^4 - 3x^2 + 2}{10x^4 + 3} = \quad 10. \lim_{x \rightarrow -\infty} \frac{5x^4 - 3x^2 + 2}{10x^5 + 3} = \quad 11. \lim_{x \rightarrow \infty} \frac{5x^4 - 3x^2 + 2}{10x^3 + 3} =$$

$$12. \lim_{x \rightarrow \infty} \frac{(2x+3)(x-1)^2}{(x+2)(3x-1)^2} = \quad \text{Note: Make sure you consider highest degree **terms** – not highest degree **factors**.}$$

Rational functions like those above have at most one horizontal asymptote, so the limit is the same whether x approaches ∞ or $-\infty$. However, radical functions frequently have two horizontal asymptotes.

Examples: Find these limits.

$$13. \lim_{x \rightarrow \infty} \frac{\sqrt{4x^2 - 3}}{x} = \quad 14. \lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2 - 3}}{x} =$$

Curve Sketching

Examples: For Examples 15-17 give the domain, reduce the function, find vertical asymptotes, holes, and end behavior.

15. $f(x) = \frac{x+2}{x^2 - 2x}$

Do:

V.A.:

E.B.:

16. $g(x) = \frac{2x^3}{(x+3)^2}$

Do:

V.A.:

E.B.:

17. $h(x) = \frac{x^2 + 2x - 8}{x^2 - 4}$

$$h(x) = \frac{(x+4)(x-2)}{(x+2)(x-2)}$$

Do:

$$h_{red}(x) =$$

V.A.:

Hole:

E.B.:

Curve Sketching Recipe:

1. Give the domain (watch for denom. restrictions, radical restrictions).
2. Reduce $f(x)$. Oftentimes, you must factor before you can reduce.
3. Find vertical asymptotes (denom. restr. after reducing) and holes (denom. restr. which reduce away).
4. Give x - and y -intercepts.
5. Find the end behavior (horizontal asymptotes or other) using highest degree terms of num. and denom.
6. (if needed) Find a starting point.
7. Graph.

Examples: Follow the **Curve Sketching Recipe** to graph.

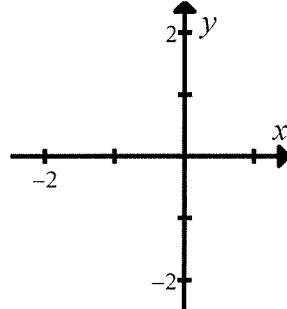
18. $f(x) = x(x-1)(x+2)^2$

Do:

y -int.:

x -int.:

E.B.:



19. $g(x) = \frac{x(x-1)^2(x+3)^3}{x^2(x-1)(x-3)^2}$

Do.:

$$g_{red}(x) =$$

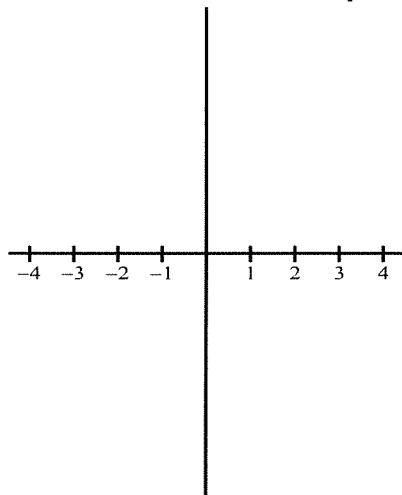
V.A.:

Holes:

x -int.:

y -int.:

E.B.:



20. $y = \frac{x+1}{\sqrt{x^2 - 4}}$

Do.:

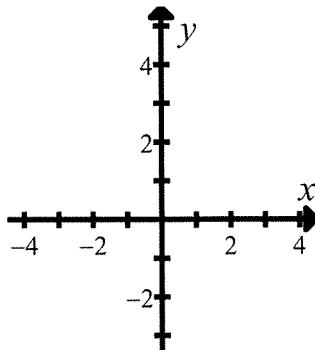
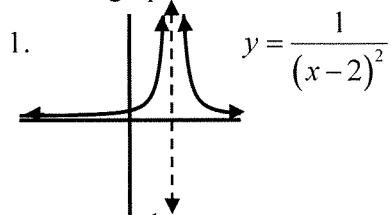
V.A.:

Holes:

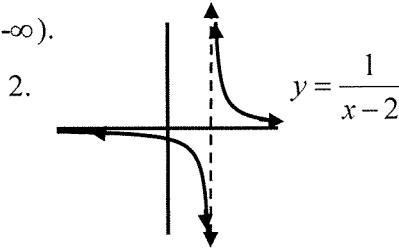
 x -int.: y -int.:

E.B.:

Starting Point:

**Assignment 1-3**Use the graphs to find these limits (answer ∞ or $-\infty$).

a. $\lim_{x \rightarrow 2^-} \frac{1}{(x-2)^2}$ b. $\lim_{x \rightarrow 2^+} \frac{1}{(x-2)^2}$



a. $\lim_{x \rightarrow 2^-} \frac{1}{x-2}$ b. $\lim_{x \rightarrow 2^+} \frac{1}{x-2}$

Find the vertical asymptotes, if any, without using a calculator, and classify each of them as even or odd.

3. $f(x) = \frac{1}{x^2}$ 4. $f(x) = \frac{x}{x(x-1)^2}$ 5. $f(x) = \frac{x}{x^2-4}$ 6. $f(x) = \frac{x}{x^2-x-2}$

7. $g(x) = \frac{x^3-1}{x-1}$ 8. $g(x) = \csc(\pi x)$

Find these limits without using a calculator. Whenever appropriate answer ∞ or $-\infty$.

9. $\lim_{x \rightarrow 3^+} \frac{x}{x-3}$	10. $\lim_{x \rightarrow 3^-} \frac{x}{x-3}$	11. $\lim_{x \rightarrow 1^+} \frac{x}{x^2-x}$	12. $\lim_{x \rightarrow 0^-} \frac{x}{x^2-x}$
13. $\lim_{x \rightarrow 3} \frac{x+3}{x^2-6x+9}$	14. $\lim_{x \rightarrow 3} \frac{x-3}{x^2-6x+9}$	15. $\lim_{x \rightarrow 0} \frac{x^2-2x}{x^3}$	16. $\lim_{x \rightarrow 0^+} \left(\frac{1}{x} - 10 \right)$
17. $\lim_{x \rightarrow \frac{\pi}{2}^+} \frac{3}{\cos x}$	18. $\lim_{x \rightarrow \pi^-} \frac{x}{\csc x}$		

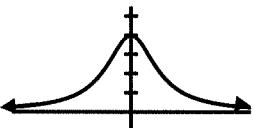
Find these limits without using a calculator.

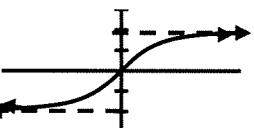
19.

$f(x) = \frac{2x^4 + x^2 + 2}{x^4 + 1}$

a. $\lim_{x \rightarrow \infty} f(x)$ b. $\lim_{x \rightarrow -\infty} f(x)$

Find these limits without using a calculator.

20.  a. $\lim_{x \rightarrow \infty} f(x)$ b. $\lim_{x \rightarrow -\infty} f(x)$

21.  a. $\lim_{x \rightarrow \infty} f(x)$ b. $\lim_{x \rightarrow -\infty} f(x)$

22. $\lim_{x \rightarrow \infty} \frac{2x+5}{3x-4}$ 23. $\lim_{x \rightarrow -\infty} \frac{1-5x^3}{10x^3-x^2}$ 24. $\lim_{x \rightarrow \infty} \frac{x(2x-1)^2}{3x(x-3)^2}$ 25. $\lim_{x \rightarrow -\infty} \frac{4x^2+3}{2x}$

26. $\lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2+x}}$ 27. $\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2+x}}$ 28. $\lim_{x \rightarrow -\infty} \frac{2-x}{\sqrt{x^2-3}}$ 29. $\lim_{x \rightarrow -\infty} \frac{2x^2-2}{\sqrt{x^4}}$

30. $\lim_{x \rightarrow \infty} \frac{\sin x}{x+1}$

Use a calculator to find the following limits.

31. $\lim_{x \rightarrow \infty} (x^5 e^{-x} + 2)$ 32. $\lim_{x \rightarrow -\infty} \frac{|2x+5|}{x-7}$

Follow the **Curve Sketching Recipe** to graph each function without using a calculator. List intercepts, asymptotes, holes, end behavior, etc. Show accurate graphs.

33. $f(x) = (x+2)(x-1)^2$ 34. $f(x) = \frac{x-2}{x+2}$ 35. $f(x) = \frac{x(x-1)^3}{x^2(x-1)}$

36. $f(x) = \frac{1}{\sqrt{x}}$ 37. $f(x) = \frac{-x}{\sqrt{x^2-1}}$

38. If $f(x) = \begin{cases} 2ax-6, & x \leq 2 \\ x^2+a, & x > 2 \end{cases}$ is a continuous function, find the value of a .

Use a calculator to find all discontinuities.

39. $f(x) = \begin{cases} \frac{\cos x - 1}{x}, & x < 0 \\ 5x, & x > 0 \end{cases}$ 40. $f(x) = \frac{x^2-4}{x^3-2x^2-2x+4}$

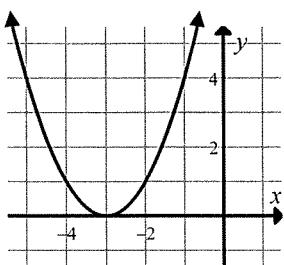
Does the Intermediate Value Theorem guarantee a value of c in the given interval? If so, find the c -value. If not, explain why not.

41. $f(x) = \frac{x^2-x}{x}$, $f(c) = -1$ on $[-2, 2]$ 42. $f(x) = x^2 - x$, $f(c) = -1$ on $[-2, 2]$

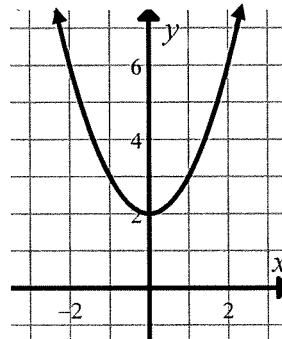
43. $f(x) = x^2 - x$, $f(c) = 5$ on $[-2, 2]$

44. Use the parent graph of $y = x^2$ to determine an equation for each graph.

a.



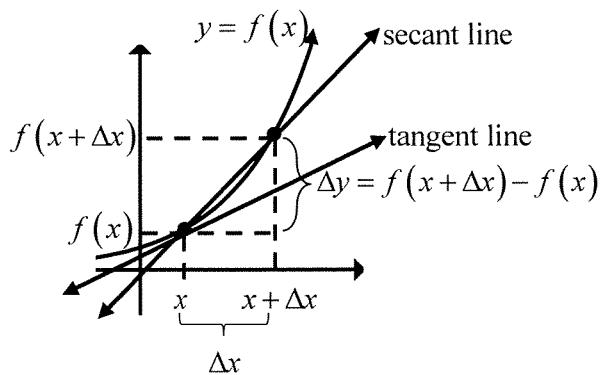
b.


Selected Answers:

3. $x = 0$ (even) 5. $x = \pm 2$ (both odd) 7. none 8. $0, \pm 1, \pm 2, \pm 3, \dots$ 9. $-\infty$ 10. DNE
 11. ∞ 12. -1 14. DNE 15. $-\infty$ 16. $-\infty$ 18. 0 19a. 2 b. 2 21a. 2 b. -2 22. $\frac{2}{3}$
 23. $-\frac{1}{2}$ 25. $-\infty$ or DNE 26. -1 27. 1 30. 0 31. 2
 33. x -int: $(-2, 0)$ (odd) 34. Do: $x \neq -2$ 38. $a = \frac{10}{3}$ 39. $x = 0$
 (1, 0) (even) VA: $x = -2$ (odd) 41. No, f is disc. at $x = 0$
 y-int: $(0, 2)$ x-int: $(2, 0)$ 44a. $y = (x+3)^2$
 EB: like $y = x^3$ y-int: $(0, -1)$
 EB: HA $y = 1$

Lesson 1-4 Limit Definition of the Derivative, Alternate Form, Trig Review

Any nonvertical line has the same slope at every point. In Calculus we frequently deal with the slope of a curve. The slope of a curve is defined to be the same as the slope of the curve's tangent line at a given point. To find the slope of a tangent line we use a limit of the slope of a secant line.



$$m_{\sec} = \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$m_{\tan} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The slope of a tangent line is called the **derivative** of the function at a given x -value. The most commonly used symbol for the derivative is $f'(x)$. Here are some other notations you will encounter (assume $y = f(x)$).

$$f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx} f(x) = m_{\tan} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

A vertical tangent line has no slope, so a curve has no derivative at any point where it has a vertical tangent line. Differentiation is the process of finding derivatives. If a derivative exists at a point on a curve, the function is said to be differentiable at that point.

Examples:

1. If $f(x) = x^2 + 2$

a. find $f'(x)$.

b. use your answer

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

from part a. to find

$$f'(-3).$$

2. If $y = \sqrt{x}$, find y' .

$$y' =$$

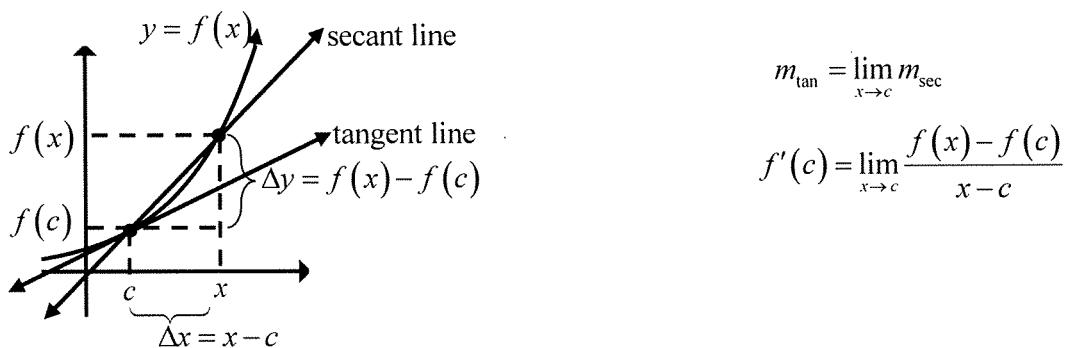
3. Given $y = f(t) = \frac{2}{t}$,

find the derivative of y with respect to t .

$$\frac{dy}{dt} = f'(t) =$$

Alternate Form of the Limit Definition of the Derivative

(Gives the value of the derivative at a single point.)



Example 4. If $f(x) = x^3$, use the alternate form of the derivative to find $f'(3)$.

$$f'(3) =$$

Solving Trig Equations

Example 5: Solve the following trig equations without using a calculator. Find all of the solutions in the interval $[0, 2\pi)$.

a. $\csc x = \frac{-2}{\sqrt{3}}$

b. $\cot \theta = \sqrt{3}$

c. $2\cos^2 \theta - 1 = 0$

d. $\cos^2 \theta - \cos \theta = 0$

Assignment 1-4

Use the limit definition of the derivative to find $f'(x)$ or $f'(t)$. Show correct limit symbolism.

$$1. f(x) = -3x \quad 2. f(x) = x^2 - 1 \quad 3. f(x) = \frac{1}{x-1} \quad 4. f(t) = t^3 - 12t \quad 5. f(x) = 3$$

Use the alternate form of the limit definition of the derivative to find the indicated derivative.

$$6. f(x) = x^2 - 1 \quad \text{Find } f'(2). \quad 7. f(x) = x^3 - 2x^2 - 1 \quad \text{Find } f'(2).$$

$$8. f(x) = \frac{1}{x} \quad \text{Find } f'(3). \quad 9. f(x) = (x-1)^{\frac{2}{3}} \quad \text{Find } f'(1).$$

10. If $y = x^2 - x$, use the limit definition of the derivative to find y' .

11. If $y = x^3 + 1$, use the limit definition of the derivative to find $\frac{dy}{dx}$.

12. If $f(x) = 2x^2 + 4$, use the limit definition of the derivative to find $f'(x)$. Then find $f'(4)$.

13. If $f(x) = 2x^2 + 4$, use the alternate form of the limit definition of the derivative to find $f'(4)$.

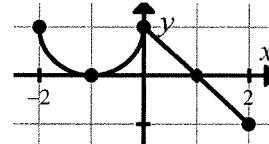
For Problems 14-17, solve for θ , where $0 \leq \theta < 2\pi$, without using a calculator.

$$14. \sec^2 \theta - 4 = 0 \quad 15. \sin^2 \theta = \cos^2 \theta \quad 16. \tan \theta - \sin \theta = 0 \quad 17. 2\sin^2 \theta = \cos \theta + 1$$

18. Use a calculator to solve for x on the interval $[0, 2\pi]$ for $\tan x = \csc^2 x - 2$.

Use the graph of $y = f(x)$ shown to graph the following.

$$19. y = |f(x)| \quad 20. y = f(x-2)+1 \quad 21. y = -2f(x)$$



22. Find the domain, vertical asymptotes, holes, intercepts, end behavior, and graph for the function $y = \frac{x(x-1)}{x^2-1}$.

Use the graph of $y = f(x)$ for Problems 23-32.

Find the following limits and function values.

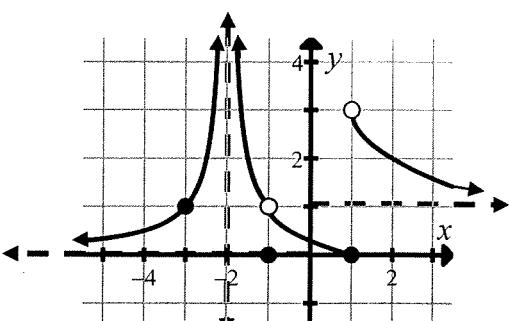
$$23. \lim_{x \rightarrow -1^-} f(x) \quad 24. f(-1) \quad 25. \lim_{x \rightarrow 1^+} f(x)$$

$$26. \lim_{x \rightarrow 1^+} f(x) \quad 27. \lim_{x \rightarrow -3} f(x) \quad 28. \lim_{x \rightarrow \infty} f(x)$$

$$29. \lim_{x \rightarrow -\infty} f(x) \quad 30. \lim_{x \rightarrow -2} f(x)$$

31. List all removable discontinuities of $f(x)$.

32. List all nonremovable discontinuities of $f(x)$.



Find the following limits without using a calculator.

$$\begin{array}{lllll}
 33. \lim_{x \rightarrow 2} (5x - 3) & 34. \lim_{x \rightarrow 3} \frac{x^2 - 9}{3 - x} & 35. \lim_{t \rightarrow -3} \frac{3+t}{t^2 - 9} & 36. \lim_{t \rightarrow 2} \frac{t^2 - 4}{t^2 - 3t + 2} & 37. \lim_{x \rightarrow 0^+} \left(x + \frac{1}{x^3} \right) \\
 38. \lim_{x \rightarrow \frac{1}{2}} \frac{4x - 2}{2x - 1} & 39. \lim_{x \rightarrow 1} \frac{x - 1}{x^4 - 1} & 40. \lim_{x \rightarrow 1} \frac{x^2 - 2x + 1}{x + 1} & 41. \lim_{x \rightarrow 1} \frac{x^2 - 2x + 1}{x - 1} & 42. \lim_{x \rightarrow 2} \frac{3x + 5}{\tan \frac{\pi x}{4}}
 \end{array}$$

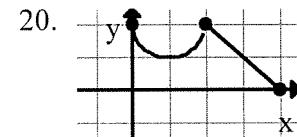
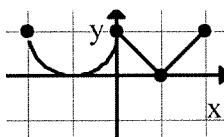
Selected Answers:

1. $f'(x) = -3$ 2. $f'(x) = 2x$ 3. $f'(x) = -\frac{1}{(x-1)^2}$ 4. $f'(t) = 3t^2 - 12$ 5. $f'(x) = 0$

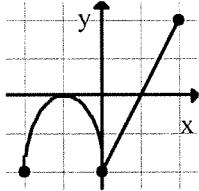
6. $f'(2) = 4$ 7. $f'(2) = 4$ 8. $f'(3) = -\frac{1}{9}$ 9. $f'(1)$ is undefined 10. $y' = 2x - 1$

11. $\frac{dy}{dx} = 3x^2$ 12. $f'(x) = 4x$, $f'(4) = 16$ 13. $f'(4) = 16$ 15. $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

16. $\theta = 0, \pi$ 17. $\theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$



21.



22. Do: $x \neq \pm 1$

$$y_{red} = \frac{x}{x+1}$$

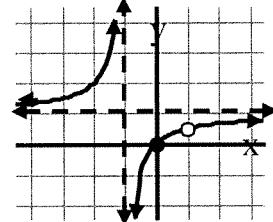
Hole: $\left(1, \frac{1}{2} \right)$

EB: HA $y = 1$

VA: $x = -1$ (odd)

x-int: $(0, 0)$ (odd)

y-int: $(0, 0)$



23. 1

24. 0

25. DNE

26. 3

27. 1

28. 1

29. 0

30. ∞ or DNE

31. $x = -1$

32. $x = -2, 1$

33. 7

34. -6

35. $-\frac{1}{6}$

36. 4

37. ∞ or DNE

38. 2

39. $\frac{1}{4}$

40. 0 41. 0 42. 0

Lesson 1-5 Differentiation Rules (shortcuts), Tangent Lines, Differentiability, Rates of Change

Derivative Rules:

Power Rule: $\frac{d}{dx} x^n = nx^{n-1}$

Constant Rule: If c is any constant, $\frac{d}{dx} c = 0$.

Scalar Multiple Rule: If c is any constant, $\frac{d}{dx}(c f(x)) = c f'(x)$.

Sum Rule: $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$

Examples: Differentiate.

$$1. \quad f(x) = x^4 \quad 2. \quad y = x^{-\frac{2}{3}} + 3 \quad 3. \quad h(t) = 5 - \frac{1}{2t^3} \quad 4. \quad f(x) = \frac{5}{(2x)^3}$$

$$f'(x) = \quad y' = \quad h(t) =$$

$$h'(t) =$$

Higher-Order Derivatives

Since the derivative of a function is another function, we can repeat the differentiation process to find the derivative of a derivative. The result is still another function which could again be differentiated. These derivatives are called higher-order derivatives.

Notation:

<u>First Derivative:</u>	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx} f(x)$
--------------------------	------	---------	-----------------	---------------------

<u>Second Derivative:</u>	y''	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx^2} f(x)$
---------------------------	-------	----------	---------------------	-------------------------

<u>Third Derivative:</u>	y'''	$f'''(x)$	$\frac{d^3y}{dx^3}$	$\frac{d^3}{dx^3} f(x)$
--------------------------	--------	-----------	---------------------	-------------------------

<u>Fourth Derivative:</u>	$y^{(4)}$	$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$\frac{d^4}{dx^4} f(x)$
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Example 5. For $f(x) = \frac{1}{2\sqrt[3]{x^2}}$, find $f'(1)$ and $f''(-8)$.

Equation of a Tangent Line:

Since the derivative of a function gives us a slope formula for tangent lines to the graph of the function, the derivative can be used to find equations of tangent lines.

Sometimes we will want to find a line perpendicular to the tangent line at a certain point. Such a line is called a normal line.

$$m_{\text{normal}} = \frac{-1}{m_{\text{tangent}}}$$

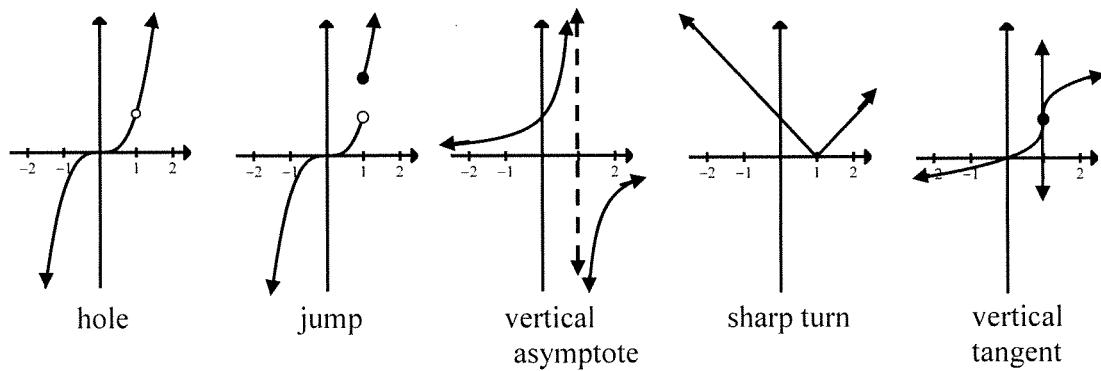
Examples:

6. Find an equation of the line tangent to the graph of $f(x) = 4x^5 - 3x^2 + 5$ at the point (1,6).

7. Find an equation of the normal line to the same curve at the same point.

NONDIFFERENTIABILITY (when a derivative does not exist)

Each of these functions has no derivative when $x = 1$.



These five characteristics destroy differentiability:

- | | | |
|--|--|--|
| 1. Holes
2. Jumps (breaks)
3. Vert. Asymptotes | $\left. \begin{array}{l} \\ \\ \end{array} \right\}$ | 4. Sharp Turns
5. Vert. Tangent Lines |
|--|--|--|

Note:

If a function is not continuous, it is not differentiable (see the first three figures above).

A function may be continuous and still not be differentiable (see the last two figures above).

Examples: Find the x -values where $f(x)$ is not differentiable. Give a reason for each.

$$8. \quad f(x) = |x| \quad 9. \quad f(x) = \begin{cases} x^2, & x \leq 0 \\ x, & x > 0 \end{cases} \quad 10. \quad f(x) = \begin{cases} x^2, & x \leq 0 \\ x^2 + 1, & x > 0 \end{cases}$$

$$11. \quad f(x) = \frac{x}{x(x-1)} \quad 12. \quad f(x) = \sqrt[3]{x}$$

Rate of Change:

Another meaning for slope is rate of change. We now have two ways to find slopes (rates of change).

1. **Average Rate of Change** This is the slope between two points. It is found without using a derivative (algebraically).
$$\text{AROC} = m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$
2. **Instantaneous Rate of Change** This is the slope at a single point. It is usually found by using a derivative (calculus).
$$\text{IROC} = m = f'(c)$$

Examples:

13. If $f(x) = x^3 + 2x$, find the average rate of change from $x = 10$ to $x = 30$.

14. If $f(x) = x^3 + 2x$, find the instantaneous rate of change when $x = 10$.

Assignment 1-5

Find the derivative. Use correct symbolism.

1. $y = 2$	2. $f(x) = x^2$	3. $g(x) = x^3 + 1$
4. $y = t + 2$	5. $f(t) = -2t^2 - 3t + 2$	6. $f(x) = -\frac{1}{3}x^2 - \frac{2}{5}x + \frac{5}{2}$

Find the value of the derivative of the function at the given point. Show steps with correct symbolism.

7. $f(x) = 3x^{-2}$ at $(1, 3)$	8. $g(x) = x^2 - 2x$ at $(2, 0)$
9. $h(x) = x^3 - 1$ at $(1, 1)$	10. $f(x) = 2 - x^3$ at $(2, -6)$

Differentiate each function. Show steps with correct symbolism.

11. $y = \frac{1}{x}$	12. $f(x) = x^2 - \frac{4}{x^2}$	13. $y = (2x - 1)^2$	14. $g(x) = x(x^2 + 1)$
15. $y = \frac{\sqrt{x}}{x}$	16. $y = \sqrt[3]{x} + \sqrt{x^3}$	17. $f(t) = \frac{t^2 - 2t}{t}$	18. $f(x) = \frac{1}{\sqrt[3]{x^2}}$
19. $y = \frac{1}{3x^2}$	20. $y = \frac{1}{(3x)^2}$	21. $f(x) = \frac{x^2 - x - 1}{\sqrt{x}}$	22. $y = (3x^2 - 5)(x + 7)$

Find the indicated value or expression. Show steps with correct symbolism.

23. $y = 3x^2$, $y'' = ?$ 24. $f(x) = \sqrt{x+2}$, $f'(4) = ?$ 25. $f(t) = 2 - \frac{2}{t}$, $f''(2) = ?$

26. $y = x(x-2)$, $\frac{d^2y}{dx^2} = ?$ 27. $f^{(3)}(x) = 2x-1$, $f^{(5)}(3) = ?$ 28. $\frac{d}{dx}(x^3 + 5) = ?$

29. $\frac{d^2}{dx^2}(3x - x^{-1}) = ?$

30. Find the second derivative of $f(x) = \frac{x^2 - 4x - 6}{2x}$.

Find an equation of a line with the following characteristics.

31. tangent to the graph of $f(x) = x^2 - 1$ at the point (2,3)

32. tangent to the graph of $f(x) = \frac{2}{x}$ when $x = 1$

33. normal to the graph of $f(x) = \frac{2}{x}$ when $x = 1$

34. tangent to the graph of $y = x^2 - 2x + 3$ when $x = 1$

35. Find the x -values of all points where the graph of $f(x) = 3x^3 + 2x - 2$ has a slope of 11.

36. Find the x -values of all points where the graph of $y = x^4 - 3x^2 + 2$ has a horizontal tangent line.

37. Find the point(s) where the graph of $y = \frac{1}{x}$ has a slope of $-\frac{1}{4}$.

38. Find the average rate of change of the function $f(x) = 3x^3 - 4$ between $x = 2$ and $x = 4$.

39. Find the instantaneous rate of change of the function $f(x) = 3x^3 - 4$ at $x = 3$.

40. Find the average rate of change of $y = \frac{x}{x+2}$ on the interval $[1, 4]$.

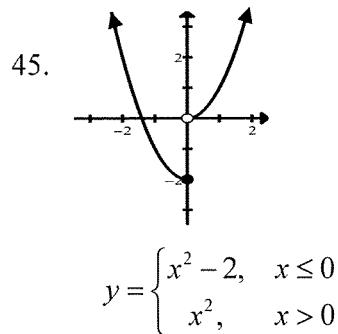
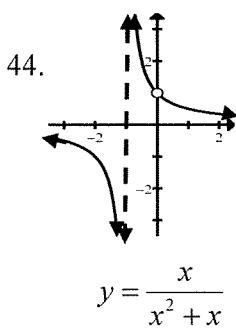
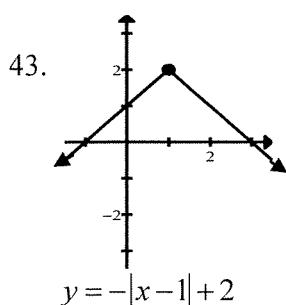
41. Find the rate of change of $y = \frac{x^2 - x}{x^2}$ at the point (1,0).

42. If $f(x) = 2x^3 - 3x + 2$ find:

a. the average rate of change on the interval $[0, 3]$.

b. the instantaneous rate of change at $x = 3$.

Find the x -values where the function is not differentiable. Give a reason for each value.



Find the x -values where the function is not differentiable. Give a reason for each value.

46. $f(x) = x^{\frac{2}{3}}$

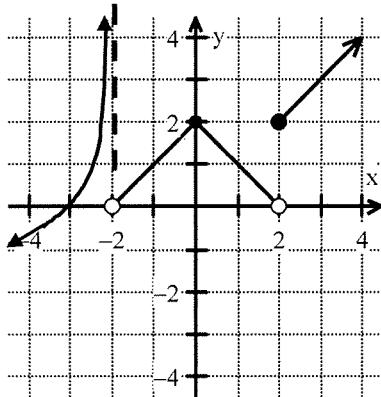
47. $f(x) = 3x^{\frac{1}{3}}$

For each of the following piecewise functions:

- Find any x -values at which the function is discontinuous.
- Differentiate the function.
- Find any x -values at which the function is not differentiable.

48. $f(x) = \begin{cases} 3x^2 - x, & x \leq 1 \\ 5x - 3, & x > 1 \end{cases}$
 49. $f(x) = \begin{cases} 3x^2 - x, & x \leq 1 \\ 5x - 2, & x > 1 \end{cases}$
 50. $f(x) = \begin{cases} 3x^2 - x, & x \leq 1 \\ 4x - 2, & x > 1 \end{cases}$

51. a. Identify any x -values at which the function shown is not continuous.
 b. Identify any x -values at which the function shown is not differentiable.



52. Use the limit definition of the derivative to find $f'(x)$ if $f(x) = 2x^2 - 5$.

53. Use the alternate form of the limit definition of the derivative to find $f'(1)$ if $f(x) = x^2 + 2x$.

54. Find $\lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h}$ mentally without showing any steps.

Selected Answers:

1. $y' = 0$ 3. $g'(x) = 3x^2$ 5. $f'(t) = -4t - 3$ 6. $f'(x) = -\frac{2}{3}x - \frac{2}{5}$ 7. $f'(1) = -6$

9. $h'(1) = 3$ 10. $f'(2) = -12$ 11. $y' = -\frac{1}{x^2}$ 12. $f'(x) = 2x + \frac{8}{x^3}$ 14. $g'(x) = 3x^2 + 1$

15. $y' = -\frac{1}{2}x^{-\frac{3}{2}}$ 17. $f'(t) = 1, t \neq 0$ 19. $y' = -\frac{2}{3x^3}$ 20. $y' = -\frac{2}{9x^3}$

21. $f'(x) = \frac{3}{2}x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}}$ 22. $y' = 9x^2 + 42x - 5$ 24. $f'(4) = \frac{1}{4}$

25. $f''(2) = -\frac{1}{2}$ 27. $f^{(5)}(3) = 0$ 30. $f''(x) = -\frac{6}{x^3}$ 31. $y - 3 = 4(x - 2)$

Selected Answers continued:

33. $y - 2 = \frac{1}{2}(x - 1)$ 34. $y = 2$ 35. $x = \pm 1$ 36. $x = 0, \pm \sqrt{\frac{3}{2}}$ 38. AROC = 84

39. $f'(3) = 81$ 41. $y'(1) = 1$ 42a. AROC = 15 b. $f'(3) = 51$ 43. $x = 1$ (sharp turn)

44. $x = -1$ (vert. asympt.), $x = 0$ (hole) 47. $x = 0$ (vert. tang.)

48a. always continuous b. $f'(x) = \begin{cases} 6x - 1, & x \leq 1 \\ 5, & x > 1 \end{cases}$ c. always differentiable

49a. discontinuous at $x = 1$ b. $f'(x) = \begin{cases} 6x - 1, & x < 1 \\ 5, & x > 1 \end{cases}$ c. not differentiable at $x = 1$

Lesson 1-6 Position → Velocity → Acceleration, Calculator Differentiation

Important Terms

Position Function

gives the location of an object at time t , usually $s(t)$, $x(t)$, or $y(t)$
Velocity Function the rate of change (derivative) of position, usually $v(t)$
 Velocity is positive for upward or rightward motion and
 negative for downward or leftward motion.

Acceleration Function

the rate of change (derivative) of velocity, usually $a(t)$

Initial Position

starting position (at $t = 0$), s_o

Initial Velocity

starting velocity (at $t = 0$), v_o

Speed

the absolute value of velocity

Displacement

the net change in position, (final pos. – original pos.)

Total Distance

total distance traveled by the object in the time interval

(takes into account all direction changes)

Example 1. If $s(t) = t^3 + t$, find $v(t)$ and $a(t)$.

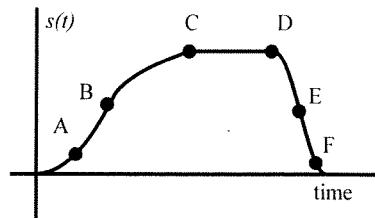
Examples: Use the position function $s(t) = 16t^3 - 36t^2 + 24$ of an object moving on a horizontal line for Examples 2-11. Distance units are measured in feet and time units are measured in seconds.

- | | | |
|--|--|---|
| 2. What is the initial position of the object? | 3. What is the velocity of the object at $t = 1$ second? | 4. What is the speed of the object at $t = 1$ second? |
|--|--|---|

5. What is the acceleration of the object at $t = 1$ second?
6. When is the object at rest?
7. When is the object moving right?
8. When is the object moving left?
9. When is the velocity of the object equal to $54 \frac{\text{ft}}{\text{sec}}$?
10. What is the displacement of the object between $t = 0$ and $t = 2$ seconds?
11. What is the total distance traveled by the object between $t = 0$ and $t = 2$ seconds?

The graph shows the position function of a radio controlled model car. Answer these questions and explain.

12. Was the car going faster at A or at B?
13. When was the car stopped?
14. At which point was the car's velocity the greatest?
15. At which point was the car's speed the greatest?



Vertical Motion Examples:

Suppose $s(t) = -16t^2 + 48t + 160$ gives the position (in feet) above the ground for a ball thrown into the air from the top of a high cliff (where time is measured in seconds).

16. Find the initial velocity.
17. At what time does the ball hit the ground?
18. At what time does the ball reach its maximum height?

Calculator Differentiation

A TI-83 or TI-84 calculator can be used to find the value of a derivative at a specific point using $\frac{dy}{dx}$ in the calculate menu or nDeriv in the math menu. It can also graph the derivative of a given function using nDeriv in the math menu. Since nDeriv works for both of these situations, and in some situations is more accurate, it is the recommended method.

Note: Newer operating systems have a MATHPRINT setting that simplifies this process slightly.

19. If $f(x) = x^3 + 3^x$, find $f'(2)$.

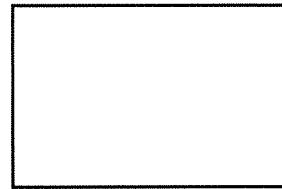
$$f'(2) = \text{nDeriv}(x^3 + 3^x, x, 2) =$$

20. If $g(x) = \ln(x^2 - 3)$, find $g'(2)$, $g'(4)$, and sketch a graph of $g'(x)$.

Hint: To save time and avoid confusing parentheses, let $y_1 = \ln(x^2 - 3)$.

$$g'(2) = \text{nDeriv}(y_1, x, 2) = \quad g'(4) =$$

To graph $g'(x)$, let $y_2 = \text{nDeriv}(y_1, x, x)$.



21. If $f(x) = |x|$, find $f'(0)$.

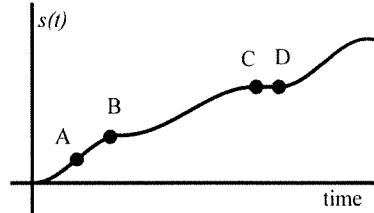
Assignment 1-6

You may use a calculator for these questions.

1. The position, in meters, of a particle moving in a straight line is given by $x(t) = 4t^3 + 6t + 2.5$ (where t is measured in seconds).
 - a. Find the velocity function.
 - b. Find the velocity at time $t = 2$ seconds.
 - c. Find the acceleration function.
 - d. Find the acceleration at time 3 seconds.
 - e. When is the velocity of the particle 18 meters per second?
 - f. Find the velocity when the position of the particle is 25 meters.
 - g. Find the initial position.
 - h. Find the particle's displacement from 0 to 1.5 seconds.
2. A helium balloon rises so that its height (position) is given by $s(t) = t^2 + 3t + 5$ (where height is measured in feet and time is measured in seconds). Assume $t \geq 0$.
 - a. When is the balloon 45 feet high?
 - b. How fast is the balloon rising at time 1 second?
 - c. How fast is the balloon rising at time 4 seconds?
 - d. What is the balloon's velocity when it is 45 feet high?

3. A ball rolls on an inclined plane with position function $s(t) = 2t^3 + 3t^2 + 5$ (where position is measured in centimeters and time is measured in seconds).
- Find the ball's velocity at time 2 seconds.
 - When is the velocity of the ball 30 centimeters per second?

4. The graph at the right shows the position function of a car. Answer these questions and explain each answer.
- What was the car's initial position?
 - Was the car going faster at A or at B?
 - Was the car speeding up or slowing down at B?
 - What happened between C and D?

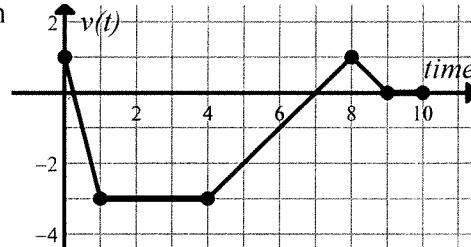


5. A particle moves along a horizontal line with position function $x(t) = t^3 - 3t^2$ (where position is measured in centimeters and time is measured in minutes).
- Find the particle's displacement between $t = 0$ minutes and $t = 5$ minutes.
 - Find the particle's velocity when $t = 4$ minutes.
 - Find the particle's acceleration when $t = 4$ minutes.
 - At what time does the particle change direction?
 - What is the total distance traveled by the particle between 0 and 5 minutes?

$$\text{Average Velocity} = \frac{\text{displacement}}{\text{elapsed time}}$$

$$\text{Average Speed} = \frac{\text{total distance}}{\text{elapsed time}}$$

- Find the particle's average velocity (average rate of change of position) between $t = 0$ and $t = 5$ minutes.
 - Find the particle's average speed between $t = 0$ and $t = 5$ minutes.
6. The graph at the right shows the velocity function of a particle moving horizontally.
- When does the particle move left?
 - When is the particle's acceleration positive?
 - When is the speed greatest?
 - When does the particle stop for more than an instant?



7. The position at time t seconds of a pebble dropped from an initial height of 600 feet is given by $s(t) = -16t^2 + 600$.
- At what time will the pebble hit the ground?
 - What is the pebble's velocity when it hits the ground?
 - What is the pebble's speed when it hits the ground?

Do not use a calculator on problems 8-17.

Find $f'(x)$.

8. $f(x) = 2x - \frac{3}{x^3}$

9. $f(x) = (2x+3)^2$

Evaluate the derivative of $f(x)$ at the indicated point for Problems 10 and 11.

10. $f(x) = 2x\sqrt{x}$ at $(4, 16)$

11. $f(x) = \sqrt[3]{x^2}$ at $(-8, 4)$

12. If $y = x(x-2)$ find $\frac{d^2y}{dx^2}$.

13. Find an equation of a line tangent to the graph of $f(x) = 2x^4 - 3x^3$ when $x = 1$.

14. Find a point on the graph of $f(x) = x^4 + 3$ where a tangent line has a slope of -4 .

15. Use the limit definition of the derivative to find $f'(x)$ if $f(x) = 3x^2 - x$.

16. If $f(x) = x^3 + 5$, find the instantaneous rate of change at $x = 1$.

17. If $f(x) = x^3 + 5$, find the average rate of change between $x = 0$ and $x = 2$.

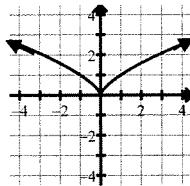
18. If $f(x) = 7^x$, use a calculator to find $f'(3)$.

19. If $g(x) = \sin x^3 + 4x^3$, find $g'(2)$, $g'(-4)$, and $g''(1)$.

20. The graph of $f(x) = x^{\frac{2}{3}}$ is shown.

a. Is f continuous at $x = 0$?

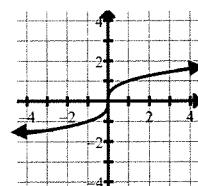
b. Is f differentiable at $x = 0$? If not, why not?



21. The graph of $f(x) = x^{\frac{1}{3}}$ is shown.

a. Is f continuous at $x = 0$?

b. Is f differentiable at $x = 0$? If not, why not?



22. $f(x) = \begin{cases} x^2 + 1, & x \leq 0 \\ x^2 + x + 1, & x > 0 \end{cases}$

a. Is f continuous at $x = 0$?

b. Is f differentiable at $x = 0$?
If not, why not?

23. $f(x) = \begin{cases} x^2 + 1, & x \leq 0 \\ -x^2 + 2, & x > 0 \end{cases}$

a. Is f continuous at $x = 0$?

b. Is f differentiable at $x = 0$?
If not, why not?

24. $f(x) = \begin{cases} x^2 + 1, & x \leq 0 \\ -x^2 + 1, & x > 0 \end{cases}$

- a. Is f continuous at $x = 0$?
- b. Is f differentiable at $x = 0$?
If not, why not?

25. $f(x) = \left\| \frac{x}{2} + 1 \right\|$

- a. Is f continuous at $x = 0$?
- b. Is f differentiable at $x = 0$?
If not, why not?

26. Find the domain, reduced function, vertical asymptote(s), hole(s), intercept(s), and end behavior of the function $f(x) = \frac{(x-1)^2(x^2+1)}{x^2-1}$. Then graph the function.

Evaluate the following limits without using a calculator.

27. $\lim_{x \rightarrow 2} \frac{\frac{1}{2} - \frac{1}{x}}{x-2}$ 28. $\lim_{x \rightarrow \infty} \frac{x^2 - 9}{2x^2 + 9}$ 29. $\lim_{x \rightarrow \infty} \frac{(2x-1)^2}{x^2 - 9}$ 30. $\lim_{x \rightarrow -\infty} \frac{3x^2 - 5}{x+1}$ 31. $\lim_{x \rightarrow -\infty} \frac{-\sqrt{x^2}}{x}$

32. $\lim_{x \rightarrow 2} \left\| \frac{x}{3} + 5 \right\|$ 33. $\lim_{x \rightarrow 3^-} \left\| \frac{x}{3} + 5 \right\|$ 34. $\lim_{x \rightarrow 3} \left\| \frac{x}{3} + 5 \right\|$

35. If $f(x) = \begin{cases} x+1, & 1 < x < 3 \\ x^2 + ax + b, & x \leq 1 \text{ or } x \geq 3 \end{cases}$ is continuous, find the values of a and b .

Selected Answers

1. b. $V(2) = 54 \frac{m}{sec}$ d. $A(3) = 72 \frac{m}{sec^2}$ e. $t = \pm 1 \text{ sec}$ f. $V(1.5) = 33 \frac{m}{sec}$ h. 22.5 m

2. a. $t = 5 \text{ sec}$ b. $V(1) = 5 \frac{ft}{sec}$ d. $V(5) = 13 \frac{ft}{sec}$ 3. b. $t = -2.791, 1.791 \text{ sec}$

4. c. slowing down (the slope of S is decreasing) 5. a. disp. = 50 cm

5. b. $V(4) = 24 \frac{cm}{min}$ d. $t = 0.2 \text{ min}$ e. TD = 58 cm f. $V_{avg} = 10 \frac{cm}{min}$

5. g. avg. speed = $\frac{58}{5} \frac{cm}{min}$ 6. a. $\frac{1}{4} < t < 7$ c. $1 \leq t \leq 4$

7. a. $t = 6.123 \text{ or } 6.124 \text{ sec}$ b. $V(6.123) = -195.959 \frac{ft}{sec}$ 8. $f'(x) = 2 + \frac{9}{x^4}$

9. $f'(x) = 8x + 12$ 10. $f'(4) = 6$ 11. $f'(-8) = -\frac{1}{3}$ 12. $y'' = 2$

13. $y+1 = -1(x-1)$ 14. $(-1, 4)$ 16. $f'(1) = 3$ 17. AROC = 4

18. $f'(3) = 667.447 \text{ or } 667.448$ 20. a. yes b. no, sharp turn 21. a. yes b. no, vert. tan.

22. a. yes b. no, sharp turn 23. a. no b. no, jump 24. a. yes b. yes 25. a. no b. no

26. Do: $x \neq \pm 1$ $f_{red}(x) = \frac{(x-1)(x^2+1)}{x+1}$ VA: $x = -1$ (odd) Hole: $(1, 0)$ (odd)

y -int: $(0, -1)$ EB: like $y = x^2$

27. $\frac{1}{4}$ 28. $\frac{1}{2}$ 29. 4 30. $-\infty$ or DNE 31. 1 32. 5 33. 5 34. DNE

35. $a = -3, b = 4$

CALCULUS EXTENDED UNIT 1 SUMMARY

Limits:

A limit is a y -value.

Analyze left and/or right behavior.

Use direct substitution or use algebra to change form.

Discontinuities: holes, vertical asymptotes, and jumps (breaks).

Removable (holes). Nonremovable (jumps and vertical asymptotes).

Limit at infinity: (end behavior)

Consider the highest degree terms in the numerator and denominator.

$$\text{Limit Definition of the Derivative: } f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\text{Alternate Form of the Limit Definition of the Derivative: } f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

(Gives the value of the derivative at a single point.)

Power Rule: (short-cut derivative)

$$\frac{d}{dx} x^n = nx^{n-1}$$

$$\text{Equation of a Tangent Line: Use the derivative to find m. } y - y_1 = m(x - x_1)$$

Nondifferentiability: (where the derivative does not exist)

1. Discontinuities
2. Sharp Turns
3. Vertical Tangent Lines

$$\text{Average Rate of Change: (the slope between two points) AROC} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Instantaneous Rate of Change: (slope at a single point) IROC} = f'(c)$$

Pos. → Vel. → Acc. (differentiate)

Speed (the absolute value of velocity)

Displacement the net change in position, (final pos. – original pos.)

Total Distance total distance traveled by the object in the time interval
(takes into account all direction changes)

$$\text{Calculator Derivative: } f'(c) = \text{nDeriv}(f(x), x, c)$$

Lesson 2-1 Product Rule, Quotient Rule, Trig Rules

Product Rule: $\frac{d}{dx}(f(x) \cdot g(x)) = f(x)g'(x) + g(x)f'(x)$ or $\boxed{\frac{d}{dx}(f \cdot s) = fs' + sf'}$

Quotient Rule: $\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$ or $\boxed{\frac{d}{dx} \frac{t}{b} = \frac{bt' - tb'}{b^2}}$

Examples: Differentiate.

1. $f(x) = (3x^2 - 2)(2x + 3)$

2. $y = \frac{2x^2 - 4x + 3}{2 - 3x}$

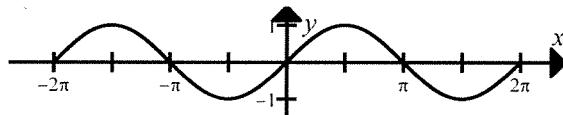
3. $y = \frac{-9}{5x^2}$

4. If $f(2) = 3$ and $f'(2) = -4$
find $g'(2)$ when $g(x) = x^2 f(x)$.

The graph of $f(x) = \sin x$ is shown at right.

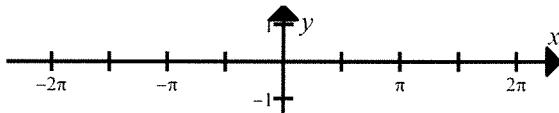
Example 5: Estimate slopes for
the graph of $f(x) = \sin x$ at

$x = -2\pi, \frac{-3\pi}{2}, -\pi, \frac{-\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π .



Plot those slopes in the coordinate plane at right, and connect them to make a smooth continuous curve. This is the graph of $f'(x)$.

$f'(x) =$



Derivatives of the Trigonometric Functions:

1. $\frac{d}{dx} \sin x = \cos x$

2. $\frac{d}{dx} \cos x =$

3. $\frac{d}{dx} \tan x =$

4. $\frac{d}{dx} \cot x = -\csc^2 x$

5. $\frac{d}{dx} \sec x = \sec x \tan x$

6. $\frac{d}{dx} \csc x = -\csc x \cot x$

Examples: Differentiate each of the following.

6. $y = \frac{\sin x}{\cos x}$

7. $f(x) = 2x \cos x$

Example 8. Find an equation of a line tangent to the graph of $y = \frac{\sin \theta}{\theta}$ when $\theta = \pi$.

Assignment 2-1

1. Use the Product Rule to differentiate. Simplify your answer. $f(x) = (x^2 - 2)(4x + 3)$
2. Differentiate without using the Product Rule. $f(x) = (x^2 - 2)(4x + 3)$
3. Use the Quotient Rule to differentiate. $f(x) = \frac{2x + 1}{x^2 + 2}$
4. Differentiate without using the Quotient or Product Rules. $f(x) = \frac{x^2 - 4}{x + 2}$
5. Differentiate without using the Quotient or Product Rules. $g(x) = \frac{2}{5x^2}$

Differentiate by any method you wish.

6. $y = \frac{12x^2 - 4}{4}$
7. $f(t) = \frac{1}{t^2}(t^3 - t^2)$
8. $g(x) = 2(x^2 + 5x - 3)$
9. $f(x) = \frac{2x - 3}{3x - 2}$
10. $y = x^2 \sin x$
11. $y = \sqrt{x}(x+1)$
12. $f(x) = \frac{x^2 - c}{x^2 + c}$, c is a constant

Differentiate by any method you wish.

13. $f(x) = \frac{\cos x}{x^3}$

14. $g(x) = \frac{2x-4}{2\sqrt{x}}$

15. $y = \frac{2(1-\sin x)}{3\cos x}$

Find the indicated derivative value. Simplify.

16. $g(x) = x \cos x$ find $g'(\frac{\pi}{4})$

17. $f(x) = x^2 + 5x - \tan x$ find $f'(0)$

18. $f(x) = \frac{x^2 - 9}{x - 2}$ find $f'(1)$

19. $h(t) = \frac{\sec t}{t^2}$ find $h'(\pi)$

20. Find the second derivative of $y = \frac{4x^3}{3}$.

21. Find an equation of the tangent line to the graph of $f(x) = \frac{x}{x+1}$ at the point $(-2, 2)$.

22. Find the x -coordinate(s) of point(s) at which the graph of $g(x) = (2x-1)(x^2 + 3)$ has a tangent line parallel to the graph of $y = 6x + 1$.

23. Find the x -coordinate(s) of point(s) at which the graph of $f(x) = \frac{x^2}{x+1}$ has a horizontal tangent line.

24. Find the average rate of change of $f(x) = \frac{x}{\sin x}$ on the interval $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$.

25. Find the rate of change of $f(x) = \frac{x}{\sin x}$ when $x = \frac{\pi}{6}$.

26. Given $g(x) = x \cdot f(x)$. If $f(2) = 3$ and $f'(2) = -2$, find $g'(2)$.

27. If $h(x) = \frac{f(x)}{x^2}$, $f(2) = 3$, and $f'(2) = -2$, find $h'(2)$.

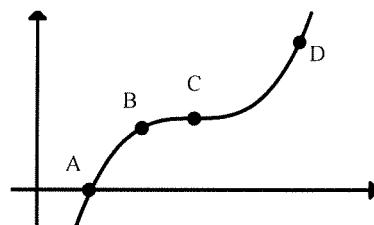
28. If $f(x) = \frac{g(x)}{h(x)}$, $g(3) = 2$, $g'(3) = 3$, $h(3) = 4$, and $h'(3) = 5$, find $f'(3)$.

29. The volume formula of a cube is $V = s^3$. Find the rate of change of the volume with respect to the side length when $s = 4$.

30. A particle moves horizontally according to the equation $s = t^2 - 5t + 4$. When is the particle moving left?

31. Which is greater – the average rate of change between points A and B or the instantaneous rate of change at B?

32. Copy the figure on your own paper and sketch a tangent line which intersects the curve between points C and D whose slope is equal to the average rate of change between C and D.



33. The height in feet of a rock thrown vertically on the moon is given by the equation $h = -\frac{27}{10}t^2 + 27t + 6$ (where time is measured in seconds). When does the rock reach its greatest height? How high is it?
34. If $f(x) = (2x^2 - 1)^3$, use a calculator to graph $f'(x)$. Also find $f'(1)$ and $f'(8)$.
35. If $f(x)$ is an n^{th} degree polynomial, find $f^{(n+1)}(x)$.
36. True or False? If $y = f(x)g(x)$, then $y' = f'(x)g'(x)$.
37. Find the total distance (in meters) traveled between $t = 0$ and $t = 4$ seconds by a particle whose position equation is $s(t) = \frac{1}{4}t^4 - 3t^3 + 4t^2 - 4$.
38. A model rocket is fired straight upward. The engine burns for two seconds. The rocket continues to coast upward then starts to fall. A parachute is released ten seconds after the launch. The graph shows the rocket's velocity.
- What was the rocket's greatest velocity?
 - Estimate the velocity at time 7 sec.
 - Was the rocket moving upward or downward at time 7 sec.?
 - Estimate the rocket's speed when the parachute was released.
 - Estimate when the rocket started to fall.
 - Estimate when the acceleration was the greatest.
 - When was the acceleration constant? Estimate the value of this constant acceleration.
-
39. The position function of a particle moving horizontally along the x -axis is $x(t) = 2t^3 - 3t^2 + 2$.
- Find the velocity function.
 - Find the acceleration function.
 - When is the particle at rest?
 - When is the particle moving left?
 - When is the particle's velocity equal to 12?
 - Find the particle's speed when the acceleration is equal to zero.
 - Find the particle's displacement from $t = -1$ to $t = 3$.
 - Find the total distance traveled by the particle between $t = -1$ and $t = 3$.

Selected Answers:

1. $f'(x) = 12x^2 + 6x - 8$ 3. $f'(x) = \frac{(x^2 + 2)2 - (2x+1)2x}{(x^2 + 2)^2}$ 4. $f'(x) = 1, x \neq -2$

5. $g'(x) = \frac{-4}{5x^3}$ 7. $f'(t) = 1, t \neq 0$ 9. $f'(x) = \frac{5}{(3x-2)^2}$ 10. $y' = x^2 \cos x + 2x \sin x$

11. $y' = \frac{3}{2}x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}$ 13. $f'(x) = \frac{-x^3 \sin x - 3x^2 \cos x}{x^6}$ 14. $g'(x) = \frac{1}{2}x^{-\frac{1}{2}} + x^{-\frac{3}{2}}$

16. $g'\left(\frac{\pi}{4}\right) = \frac{-\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}}$ 17. $f'(0) = 4$ 18. $f'(1) = 6$ 20. $y'' = 8x$

21. $y - 2 = 1(x + 2)$ 22. $x = 0, \frac{1}{3}$ 23. $x = 0, -2$ 24. AROC = $\frac{1}{2}$ 25. $f'\left(\frac{\pi}{6}\right) = \frac{\frac{1}{2} - \frac{\pi}{6}\left(\frac{\sqrt{3}}{2}\right)}{\frac{1}{4}}$

26. $g'(2) = -1$ 27. $-\frac{5}{4}$ 29. $\left.\frac{dv}{ds}\right|_{s=4} = 48$ 30. moving left on $(-\infty, \frac{5}{2})$

33. $t = 5 \text{ sec}, h(5) = 73.5 \text{ ft}$ 35. 0 37. TD = 66.5 m

38b. $v(7) \approx 40 \frac{ft}{sec}$ d. $|v(10)| = 50 \frac{ft}{sec}$ f. just before $t = 2 \text{ sec}$ 39a. $v(t) = 6t^2 - 6t$

39b. $a(t) = 12t - 6$ c. $t = 0, 1$ d. $(0, 1)$ e. $t = 2, -1$ f. $\left|v\left(\frac{1}{2}\right)\right| = \frac{3}{2}$ g. disp. = 32 h. TD = 34

Lesson 2-2 Chain Rule

If $y = f(g(x))$, then $y' =$

or in another form: $\frac{d}{dx}f(u) =$ (where u is a function of x)

Examples: Differentiate each of the following.

1. $y = (x^2 - 3)^2$ 2. $y = \sin(3x)$ 3. $f(x) = \sqrt{(3x^2 - x)^3}$

4. $y = \cos^2 x$

5. $g(x) = \frac{1}{2x+1}$

6. $f(t) = \sin^3(4t^2)$

Examples: Differentiate each of the following.

7. $y = x^2 \sqrt{1-x^2}$
(simplify by factoring)

8. If $p(2)=5$, $q(2)=3$, $q'(2)=2$, $p'(3)=4$,
and $q'(3)=\frac{3}{2}$, find $\frac{d}{dx} p(q(x))$ at $x=2$.

Assignment 2-2

Find the derivative.

1. $y = (3x+5)^3$

2. $f(x) = 3(7x+5)^4$

3. $y = \sqrt{2-3x}$

4. $f(t) = \frac{1}{(1-t)^2}$

5. $y = \sqrt[3]{(x^2+1)^2}$

6. $g(x) = x(2x+3)^3$

7. $y = \frac{1}{\sqrt{x+1}}$

8. $f(x) = \frac{3x-2}{x+1}$

9. $g(x) = \sec(4x)$

10. $y = 4 \tan(2x)$

11. $f(\theta) = \frac{1}{2} \sin^2(3\theta)$

12. $y = \sqrt{\frac{4x^3-2x}{2x}}$

Find an equation of the line tangent to the graph of f at the given point.

13. $f(x) = \sqrt{2x^2+2}$ at $(-1, 2)$

14. $f(x) = \frac{x+4}{x}$ at $(2, 3)$

15. $f(x) = \frac{1}{\sqrt{(9x)^3}}$ at $\left(\frac{1}{4}, \frac{8}{27}\right)$

16. $f(x) = \frac{1}{x^2} + \sqrt{\cos x}$ at $\left(2\pi, \frac{1}{4\pi^2} + 1\right)$

Find the indicated derivatives.

17. $\frac{d}{dx}(2 \sin x - 3)^4$

18. $\frac{d^2}{dt^2}(t^2 - 1)^{\frac{3}{2}}$

19. Find the point(s) at which a line tangent to the graph of $f(x) = (2x-3)^3$ is parallel to the graph of $y = 24x - 7$. You may use a calculator.

20. If $g(x) = (f(x))^3$, $f(1) = 2$, and $f'(1) = 4$, find $g'(1)$.

21. If $f(2) = 3$, $g(2) = 2$, $f'(2) = -1$, $g'(2) = 4$, and $g'(3) = 5$, find:

a. $\frac{d}{dx} g(f(x))$ at $x = 2$

b. $\frac{d}{dx}(g(x)f(x))$ at $x = 2$

c. $\frac{d}{dx} \sqrt{g(x)}$ at $x = 2$

d. $\frac{d}{dx} \frac{g(x)}{f(x)}$ at $x = 2$

22. Find an equation of the line tangent to the graph of $y = 3^x$ when $x = 1.2$. You will need a calculator.

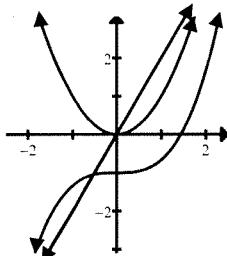
23. A sprinter in a 100 meter dash is clocked every 10 meters as shown in the table.

meters covered	0	10	20	30	40	50	60	70	80	90	100
time in seconds	0	1.4	2.5	3.5	4.5	5.4	6.6	7.6	8.5	9.3	10.2

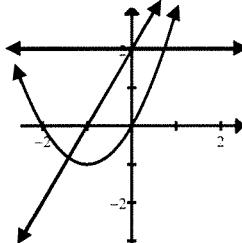
- a. How long does it take the sprinter to finish the race?
 b. What is the sprinter's average speed over the first 50 meters?
 c. What is the sprinter's average speed over the second 50 meters?
 d. What is the sprinter's average speed between 20 and 30 meters?
 e. What is the sprinter's average speed between 30 and 40 meters?
 f. What is the sprinter's approximate speed as he passes the 30 meter mark?
 g. During which ten meter segment of the race is the sprinter running the fastest?
 h. During which portion of the race is the sprinter accelerating the fastest?
24. A particle is moving vertically on the y -axis with position $y(t) = \frac{1}{3}t^3 + t^2 - 3t + 7$. When is the particle moving upward? Do not use a calculator.

The functions f , f' , and f'' are shown on the same graph. Determine which is which.

25.



26.



27. If $f(x) = 5x^2 - 3x + 2$, use the alternate form of the limit definition of the derivative to find $f'(1)$. Show steps with correct limit symbolism.

28. The position function of an object is $s(t) = \frac{2}{3}t^3 - \frac{5}{2}t^2 + 3$. Find the displacement of the object between time $t = 0$ and time $t = 3$. Find the total distance traveled by the object between time $t = 0$ and time $t = 3$:

29. The height (position) in feet of a ball thrown straight down from a tall building is given by $h = -16t^2 - 22t + 220$ (where t is measured in seconds).
- a. What is the ball's initial height?
 b. What is the ball's initial velocity?
 c. What is the speed of the ball after 3 seconds?
 d. How far does the ball travel in the first 3 seconds?

Selected Answers:

1. $y' = 9(3x+5)^2$
3. $y' = -\frac{3}{2}(2-3x)^{-\frac{1}{2}}$
4. $f'(t) = \frac{2}{(1-t)^3}$
6. $g'(x) = 6x(2x+3)^2 + (2x+3)^3$
7. $y' = -\frac{1}{2}(x+1)^{-\frac{3}{2}}$
8. $f'(x) = \frac{5}{(x+1)^2}$
9. $g'(x) = 4\sec(4x)\tan(4x)$
10. $y' = 8\sec^2(2x)$
11. $f'(\theta) = 3\sin(3\theta)\cos(3\theta)$
12. $y' = \frac{2x}{\sqrt{2x^2-1}}$
13. $y-2 = -1(x+1)$
16. $y - \left(\frac{1}{4\pi^2} + 1\right) = -\frac{1}{4\pi^3}(x-2\pi)$
17. $8\cos x(2\sin x-3)^3$
18. $\frac{3(2t^2-1)}{\sqrt{t^2-1}}$
19. $\left(\frac{5}{2}, 8\right), \left(\frac{1}{2}, -8\right)$
20. $g'(1) = 48$
- 21a. -5 b. 10 c. $\sqrt{2}$ d. $\frac{14}{9}$
22. $y-3.737 = 4.105(x-1.2)$ or $y-3.737 = 4.106(x-1.2)$
- 23b. Avg. Speed = $9.259 \frac{m}{sec}$
- e. $10 \frac{m}{sec}$
- g. between 80 and 90 m
24. $t < -3$ or $t > 1$
25. f looks like a cubic, f' looks like a parabola, and f'' looks like a line
28. disp. = -4.5, TD = 5.916 or 5.917
- 29a. $h(0) = 220 \text{ ft}$
- c. $|v(3)| = 118 \frac{ft}{sec}$

Lesson 2-3 Implicit Differentiation

All the derivatives you have done to this point have been of explicit equations. For example

$y = x^2$, $y = \frac{x}{x-1}$, and $y = \sqrt{2x+1}$ all explicitly express y in terms of x .

In this lesson you will be working with implicit equations where the relationship between x and y is only implied. $x^2 + y^2 = 1$, $xy + y^2 = 3$, and $xy = 1$ are all examples of implicit equations.

It is possible to differentiate implicit equations using implicit differentiation.

Procedure:

1. Differentiate both sides with respect to x .
(Remember the y' "hook-on factor" for any term involving y .)
2. Collect all y' (or $\frac{dy}{dx}$) terms on one side of the equation.
3. Factor out y' .
4. Divide to solve for y' .

Warm-up Examples: Differentiate.

1. $y = x$
2. $y = x^2$
3. $y = (2x-1)^2$
4. $y = (f(x))^2$
5. $x = y^2$

Examples:

- Given $x^2 - 2y^3 + 3x = 6$, find y' .
 - Find the slope of the lines tangent to and normal to the graph of $x^2 + 4y^2 = 25$ at $(3,2)$.
 - Given $x^3 - 2xy + y^3 = 5x$, find $\frac{dy}{dx}$ and evaluate at the point $(1,2)$.
 - Given $x^2 + y^2 = 3$, find y'' in terms of y .
 - Given $\cot y = x - y$ find $\frac{dy}{dx}$.

Assignment 2-3

Use implicit differentiation to find y' .

$$1. \ x^2 + y^2 = 4$$

$$2. \quad xy = 7$$

$$3. \ x^2y^3 + y = x^3$$

$$4. \cos x - 2\sin(2y) = 4$$

$$5. \tan(xy) = y^2$$

$$6. \sqrt{x} + \sqrt{y} = 9$$

$$7. \ y^2 = \frac{x}{x+1}$$

Find $\frac{dy}{dx}$ and evaluate the derivative at the given point.

8. $x^3 - xy = 3$ at $(1, -2)$

$$9. \quad y^3 + y^2 + x^2 = 0 \quad \text{at } (2, -2)$$

$$10. \ x^3 + y = 2xy \quad \text{at } (1,1)$$

$$11. \sec(x+y) = x+1 \quad \text{at } (0,0)$$

12. Differentiate implicitly to find the slope of the graph of $x^2 + y^2 = 25$, at the point $(3, -4)$.

13. Use explicit differentiation to find the slope of the graph of $x^2 + y^2 = 25$, at the point $(3, -4)$.

You must first solve for y . If your answer does not match the correct answer to Problem 12, find your mistake.

14. Given $x^2 + xy = 4$, find $\frac{d^2y}{dx^2}$ in terms of x and y .
15. Find an equation of the line tangent to the circle $x^2 + y^2 = 169$ at the point $(-12, 5)$.
16. Find the point(s) at which the graph of $x^2 + 2y^2 - 4y - 6 = 0$ has a horizontal tangent line.
17. Find the point(s) at which the graph of $x^2 + 2y^2 - 4y - 6 = 0$ has a vertical tangent line.
18. The volume formula for a sphere is $V = \frac{4}{3}\pi r^3$. Find the rate of change of the volume with respect to the radius $\left(\frac{dV}{dr}\right)$, when the radius is 4.

Use the figure at the right for Problems 19-26.

19. Find $\lim_{x \rightarrow 1} f(x)$.

20. Find $\lim_{x \rightarrow -3} f(x)$.

21. Find $\lim_{x \rightarrow -\infty} f(x)$.

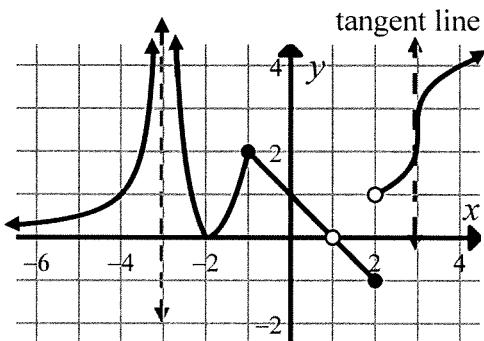
22. Find $\lim_{x \rightarrow 2} f(x)$.

23. Find $\lim_{x \rightarrow 2^+} f(x)$.

24. List the x -values of all discontinuities of $f(x)$.

25. Which of these discontinuities are removable?

26. List the x -values where $f(x)$ is not differentiable.



27. Use a calculator to solve this inequality. $2x^2 - 7x + 4 \leq 0$.

28. An object's velocity in meters per second is $v = 2t^3 - 9t^2 + 12t - 5$. Find the object's speed each time the acceleration is zero.

29. Find $\frac{d}{dx} \left(\frac{x^2 - 2x}{x^2} \right)$.

Use the function $g(x) = \begin{cases} x-1, & x \leq 0 \\ x^2-1, & 0 < x < 2 \\ 4, & x \geq 2 \end{cases}$ for Problems 30-37.

30. Sketch a graph of $g(x)$.

Find the following limits.

31. $\lim_{x \rightarrow 0} g(x)$

32. $\lim_{x \rightarrow 2} g(x)$

33. $\lim_{x \rightarrow 2^-} g(x)$

34. $\lim_{x \rightarrow 1} g(x)$

35. List all discontinuities of $g(x)$.

36. Find $g'(x)$

37. List all x -values at which g is **not** differentiable.

Selected Answers

1. $y' = -\frac{x}{y}$ 2. $y' = -\frac{y}{x}$ 3. $\frac{dy}{dx} = \frac{3x^2 - 2xy^3}{3x^2y^2 + 1}$ 4. $y' = \frac{\sin x}{-4\cos(2y)}$ 6. $y' = -\sqrt{\frac{y}{x}}$

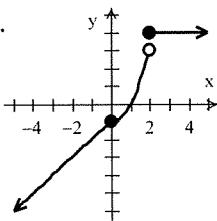
8. $\frac{dy}{dx} = \frac{y - 3x^2}{-x} = \frac{3x^2 - y}{x}$, $\left.\frac{dy}{dx}\right|_{(1,-2)} = 5$ 9. $\frac{dy}{dx} = \frac{-2x}{3y^2 + 2y}$, $\left.\frac{dy}{dx}\right|_{(2,-2)} = -\frac{1}{2}$

10. $y' = \frac{2y - 3x^2}{1 - 2x}$; $y'(1,1) = 1$ 12. $y'(3,-4) = \frac{3}{4}$ 14. $y'' = \frac{2x + 2y}{x^2}$

15. $y - 5 = \frac{12}{5}(x + 12)$ 16. $(0,3), (0,-1)$ 17. $(\pm\sqrt{8}, 1)$ 18. $\frac{dV}{dr}(4) = 64\pi$

27. $.719 \leq x \leq 2.780$ (or 2.781) 28. $|v(1)| = 0 \frac{m}{sec}$, $|v(2)| = 1 \frac{m}{sec}$ 29. $\frac{2}{x^2}$

30. 31. -1 32. DNE 33. 3 34. 0 35. $x = 2$



Lesson 2-4 Related Rates Story Problems

In related rate story problems, the idea is to find a rate of change (with respect to time) of one quantity by using the rate of change (with respect to time) of a related quantity.

Procedure for Related Rate Problems

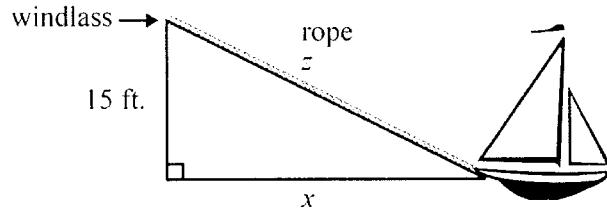
1. Draw a figure (if necessary) and choose variables for all unknowns.
2. Write what is given and what is to be found using your variables and $\frac{d}{dt}$ symbols.
3. Write an equation relating the variables.
 - (a) If a quantity is **changing** it must be represented with a variable letter.
 - (b) If a quantity is **constant** it must be represented with a number value.
 - (c) Look for secondary relationships between quantities to reduce the number of variables.
4. Implicitly differentiate both sides with respect to t .
5. Substitute number values and solve.
6. Write a sentence to explain the meaning of your answer.

Examples:

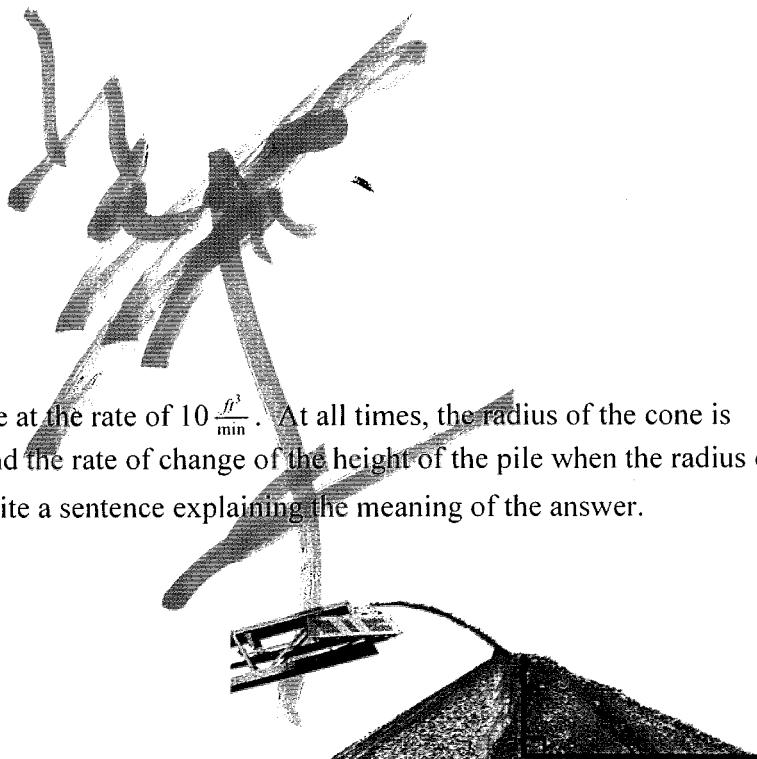
1. In this problem, the first three steps of the procedure are done. We need to complete only the last two steps.

Given: $\frac{dx}{dt} = -6$, $y = 4$, $xy = 12$ Find: $\frac{dy}{dt}$

2. A windlass is used to tow a boat to a dock. The rope is attached to the boat at a point 15 feet below the level of the windlass. If the windlass pulls in rope at the rate of 30 feet per minute, at what rate is the boat approaching the dock when there is 25 feet of rope out? Write a sentence explaining the meaning of the answer.



3. A policeman traveling south toward an intersection spots a speeding car traveling east away from the intersection. When the policeman is .6 mi from the intersection and the car is .8 mi from the intersection, the policeman's radar shows the distance between them is increasing at the rate of 20 mph. If the speed of the police car is 60 mph, what is the speed of the car? Write a sentence explaining the meaning of the answer.



4. Gravel is falling on a conical pile at the rate of $10 \frac{\text{ft}^3}{\text{min}}$. At all times, the radius of the cone is twice the height of the cone. Find the rate of change of the height of the pile when the radius of the pile is 6 ft. $V = \frac{1}{3}\pi r^2 h$ Write a sentence explaining the meaning of the answer.

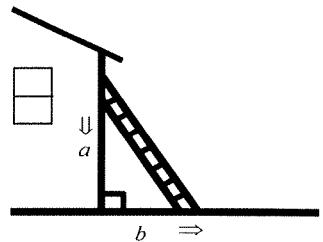


Assignment 2-4

1. Given: $\frac{dx}{dt} = 3$, $x = 4$, $y = x^2 - 2x$ Find: $\frac{dy}{dt}$
2. Given: $\frac{dx}{dt} = 10$, $y = 5$, $x^2 + y^2 = 169$ Find: $\frac{dy}{dt}$

Write sentences explaining the meaning on problems 3-12.

3. The radius of a circle is increasing at the rate of 3 cm/sec . At the instant the radius is 4 cm., find:
- the rate of change of the area of the circle.
 - the rate of change of the circumference of the circle.
4. A large spherical balloon is inflated at the rate of $18 \frac{\text{ft}^3}{\text{min}}$. When the radius is 2 feet, how fast is the radius changing? $V = \frac{4}{3}\pi r^3$
5. All edges of a cube are expanding at the rate of $2 \frac{\text{in}}{\text{sec}}$. At the instant the edges are all 5 inches long find:
- the rate of change of the surface area of the cube.
 - the rate of change of the volume of the cube.
6. A point is moving on the curve $y = \sqrt{x}$ so that the x -coordinate is changing at the rate of 3 cm/min . When the y -coordinate is 25 centimeters, find the rate of change of the y -coordinate.
7. A 10 foot ladder is leaning against a house as shown. The base of the ladder is pulled away from the house at the rate of $1.5 \frac{\text{ft}}{\text{sec}}$. At the instant the base of the ladder is 6 feet from the house, answer these questions.
- Find the rate at which the top of the ladder is moving.
 - Find the rate at which the area of the triangle formed is changing.
 - Is the area from part b increasing or decreasing? Would your answer be the same at all times?
 - Let θ = the angle formed by the ladder and the house, find the rate of change of θ with respect to t .
8. A rectangle is formed whose length is twice the width. It is enlarged to a similar (same shape) rectangle as the width changes at the rate of 4 inches per minute. When the width is 10 inches, how fast is the area of the rectangle changing?
9. Sand is added to a conical pile at the rate of $8 \frac{\text{ft}^3}{\text{min}}$. If the diameter of the cone is twice the height of the cone, find the rate of change of the height of the pile when the volume of the pile is 1125π cubic feet. $V = \frac{1}{3}\pi r^2 h$
10. A bicyclist and a jogger are moving on two perpendicular intersecting streets. The bicyclist is moving north toward the intersection at the rate of 60 feet per second and the jogger is moving west away from the intersection at the rate of 15 feet per second. What is the rate at which the straight line distance between them is changing when the bicyclist is 120 feet from the intersection and the jogger is 50 feet from the intersection?



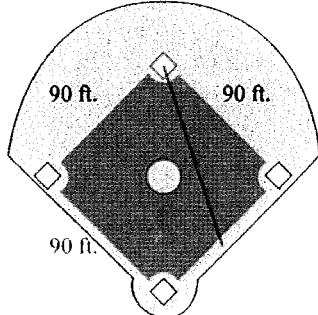
11. At halftime of a football game, two fan volunteers are attached together with a bungee cord and positioned on the sideline at the 50 yard line. One fan runs along the sideline at $4 \frac{yd}{sec}$ and the other runs on the 50 yard line toward midfield at $3 \frac{yd}{sec}$. At what rate is the bungee cord being stretched when the sideline runner reaches the 30 yard line?

12. A base runner who is running at the rate of 25 feet per second is halfway to first base. Find the rate of change of his distance from second base.

Differentiate.

13. $f(x) = \frac{x^3 - 3x}{x^2 - 4}$

14. $y = \sqrt{2x-1} \sqrt[3]{3x+1}$



15. Find an equation of the tangent line to the graph of $y = (3x^2 + 2)^4$ when $x = 0$.

16. If $f(x) = \frac{x^2 + 3}{x}$
- find the average rate of change from $x = 1$ to $x = 3$.
 - find the instantaneous rate of change at $x = 3$.

Answer these same three questions for Problems 17-20.

- What is the derivative of the function from the left at $x = 1$?
- What is the derivative of the function from the right at $x = 1$?
- Is the function differentiable at $x = 1$?

17. $f(x) = \begin{cases} (x-1)^3, & x \leq 1 \\ (x-1)^2, & x > 1 \end{cases}$

18. $f(x) = \begin{cases} \frac{1}{2}x^2 - \frac{1}{2}, & x \leq 1 \\ x-1, & x > 1 \end{cases}$

19. $f(x) = \begin{cases} \frac{1}{2}x^2 - \frac{1}{2}, & x < 1 \\ x-1, & x \geq 1 \end{cases}$

20. $f(x) = \begin{cases} x^2 - 2x, & x \leq 1 \\ x^2 - 2x + 1, & x > 1 \end{cases}$

21. If $f(x) = x^2 - x$, use the limit definition of the derivative to find $f'(x)$.

22. If $y = \frac{1}{x^2} + \frac{1}{2x}$ find $\frac{d^2y}{dx^2}$.

23. Find an equation of the normal line to the graph of $y = \sqrt[3]{(2x-1)^2}$ at the point $(1,1)$.

24. If $r(3) = -2$, $s(3) = 2$, $r'(3) = 3$, $s'(3) = 5$, and $r'(2) = 4$, find $\frac{d}{dt} r(s(t))$ at $t = 3$

Selected Answers

1. $\frac{dy}{dt} = 18$ 2. $\frac{dy}{dt} = \pm 24$

Write sentence answers for 3-12.

3a. $\frac{dA}{dt} = 24\pi \frac{\text{cm}^2}{\text{sec}}$ b. $\frac{dC}{dt} = 6\pi \frac{\text{cm}}{\text{sec}}$ 4. $\frac{dr}{dt} = \frac{9}{8\pi} \frac{\text{ft}}{\text{min}}$ 5a. $\frac{dA}{dt} = 120 \frac{\text{in}^2}{\text{sec}}$ b. $\frac{dV}{dt} = 150 \frac{\text{in}^3}{\text{sec}}$

6. $\frac{dy}{dt} = \frac{3}{50} \frac{\text{cm}}{\text{min}}$ 7a. $\frac{da}{dt} = -\frac{9}{8} \frac{\text{ft}}{\text{sec}}$ b. $\frac{dA}{dt} = \frac{21}{8} \frac{\text{ft}^2}{\text{sec}}$ c. increasing, no d. $\frac{d\theta}{dt} = \frac{3}{16} \frac{\text{radians}}{\text{sec}}$

8. $\frac{dA}{dt} = 160 \frac{\text{in}^2}{\text{min}}$ 9. $\frac{dh}{dt} = \frac{8}{225\pi} \frac{\text{ft}}{\text{min}}$ 10. $-49.615 \frac{\text{ft}}{\text{sec}}$ 11. $5 \frac{\text{yd}}{\text{sec}}$ 12. $-11.180 \frac{\text{ft}}{\text{sec}}$

15. $y = 16$ 16a. AROC = 0 b. $f'(3) = \frac{2}{3}$ 17a. 0 b. 0 c. yes 18a. 1 b. 1 c. yes

22. $y'' = 6x^{-4} + x^{-3}$ 23. $y - 1 = -\frac{3}{4}(x - 1)$ 24. 20

Lesson 2-5 Absolute Extrema and the Mean Value Theorem

Definitions (informal)

- The **absolute maximum** (global maximum) of a function is the y-value that is greater than or equal to all other y-values in the function.
- The **absolute minimum** (global minimum) of a function is the y-value that is less than or equal to all other y-values in the function.
- A **relative maximum** (local maximum) of a function is a y-value that is greater than or equal to all “nearby” y-values in the function.
- A **relative minimum** (local minimum) of a function is a y-value that is less than or equal to all “nearby” y-values in the function.
- **Extrema** (extreme values) are either maximum values (maxima) or minimum values (minima).
- **Critical Numbers** are x-values at which $f(x)$ exists but $f'(x)$ is either zero or undefined.

THE CANDIDATE TEST:

(Procedure for finding absolute (global extrema))

1. Find all critical numbers of the function.
2. Find y-values at each critical number and at each endpoint of the interval.
3. Choose the least and greatest y-values as absolute extrema.

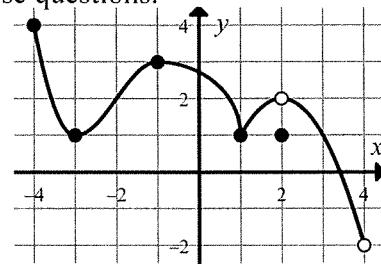
Note:

Absolute extrema can occur either at critical numbers or endpoints.

Relative extrema can occur only at critical numbers. We will not consider endpoint extrema to be relative extrema.

Examples: Use the figure of $y = f(x)$ at the right to answer these questions.

1. What is the absolute maximum of f ?
2. At what x -value does f have an absolute maximum?
3. What is the absolute maximum point on f ?
4. What is the absolute minimum of f ?
5. At what x -value(s) does f have a relative minimum?
6. At what x -value(s) does f have a relative maximum?



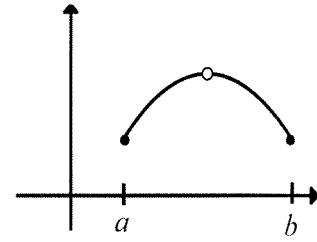
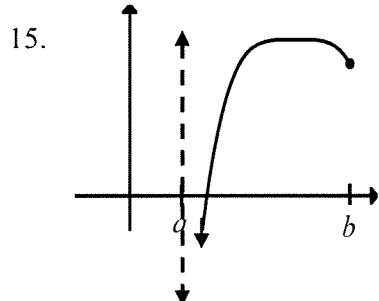
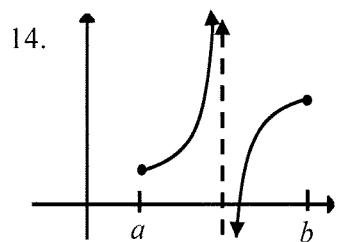
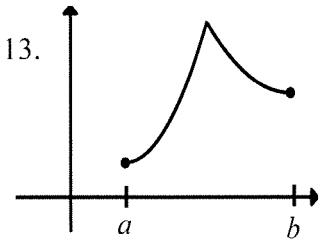
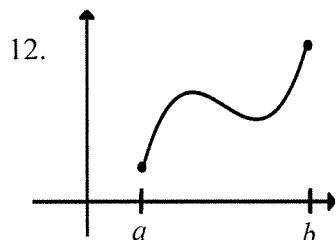
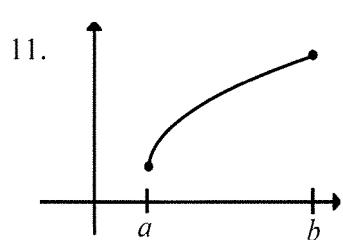
Examples:

7. Find the global extrema of $f(x) = \frac{1}{3}x^3 - 2x^2$ on the interval $[-1, 3]$.
8. Find the absolute maximum and absolute minimum values of $f(x) = |x - 2|$ on the interval $[0, 5]$.
9. Find the extrema of $f(x) = 3x^{\frac{2}{3}} - 2x$ on $[-1, 3]$.
10. Find the global maximum and minimum of $g(x) = 2\sin x - \cos(2x)$ on the interval $[0, 2\pi]$.

Discovering the Mean Value Theorem

For Examples 11-16 draw these lines (if possible).

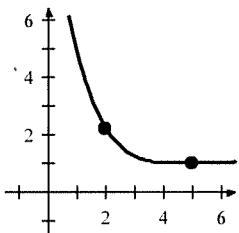
- Draw the secant line between the two points $(a, f(a))$ and $(b, f(b))$.
- Draw all tangent lines parallel to the secant line.



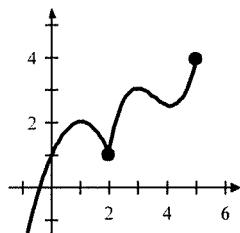
For examples 17-19

- Draw the secant line between the two points $(2, f(2))$ and $(5, f(5))$.
- Draw all tangent lines parallel to the secant line at some point on the interval $(2, 5)$.
- Estimate the value of c where $(c, f(c))$ is a point of tangency.

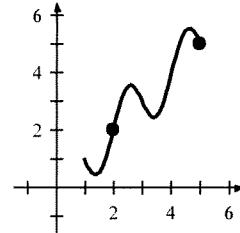
17. $c \approx \underline{\hspace{2cm}}$



18. $c \approx \underline{\hspace{2cm}}$



19. $c \approx \underline{\hspace{2cm}}$

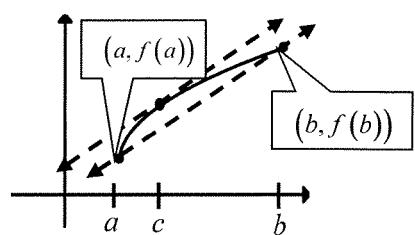


MEAN VALUE THEOREM: If f is continuous on $[a, b]$ and differentiable on (a, b) , then there is a number c in (a, b)

such that
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

tangent slope
(inst. rt. of ch.)

secant slope
(avg. rt. of ch.)



Informally: The Mean Value Theorem states that given the right conditions of continuity and differentiability, there will be at least one tangent line parallel to the secant line.

In still other words: The instantaneous rate of change (slope of tangent) will equal the average rate of change (slope of secant) at least once.

Example 20. Given $f(x) = 3 - \frac{6}{x}$, find all c which satisfy the Mean Value Theorem on the interval $[3, 6]$.

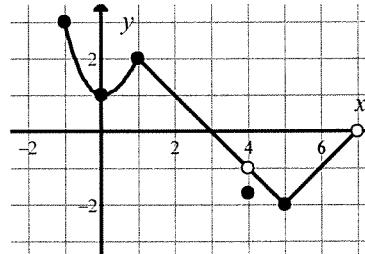
Assignment 2-5

Use the graph of $y = f(x)$ at the right for Problems 1-4.

1. What is the absolute maximum value of $f(x)$?
2. At what point does $f(x)$ reach a global minimum?
3. At what x -value(s) does $f(x)$ have a relative minimum?
4. At what x -value(s) does $f(x)$ have a relative maximum?

Find the critical numbers of these functions.

5. $y = x^2 - 3x$	6. $f(x) = 6x^{\frac{2}{3}} - 2x$	7. $y = -2x + 3$
8. $y = x+2 $	9. $f(x) = \cos^2 x + \sin x$ on $[0, 2\pi]$	



Without using a calculator, find the absolute maximum and absolute minimum values of $f(x)$ on the given interval. Show organized work using the Candidate Test.

10. $f(x) = x^2 - 3x$ on $[-1, 5]$	11. $f(x) = 6x^{\frac{2}{3}} - 2x$ on $[-1, 27]$
12. $f(x) = x^3 + 3x^2$ on $[-3, 1]$	13. $f(x) = x^3 + 3x^2$ on $[-1, \frac{1}{2}]$

Does the Mean Value Theorem apply to the given function on the given interval? If it does, find the c -value. If it does not, explain why not. Do not use a calculator.

14. $f(x) = x $ on $[-1, 3]$	15. $f(x) = x^2 - 2x$ on $[1, 3]$
16. $f(x) = x^2 - 3x + 2$ on $[1, 2]$	17. $f(x) = x^{\frac{2}{3}}$ on $[-2, 2]$

Without using a calculator, find the absolute maximum and absolute minimum values of $f(x)$ on the given interval.

18. $f(x) = |x+2|$ on $[-3, 0]$

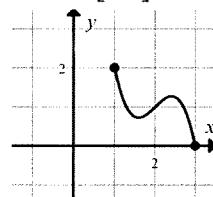
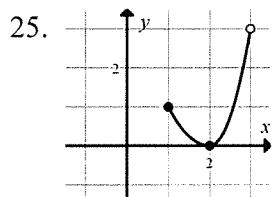
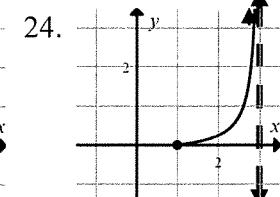
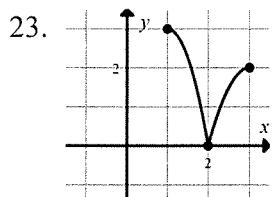
19. $f(x) = 3(x-2)$ on $[-3, 1]$

20. $f(x) = \frac{x^2}{x^2 + 2}$ on $[-1, 2]$

21. $f(x) = \cos^2 x + \sin x$ on $[0, 2\pi]$ see Problem 9.

22. $f(x) = \begin{cases} -x^2 + 2, & x \leq 0 \\ -x + 2, & 0 < x < 6 \\ -4 & x \geq 6 \end{cases}$ on $[-1, 8]$

Find the absolute minimum and maximum values of the functions graphed on $[1, 3]$.



Answer True or False

27. The minimum of a function on a closed interval can occur at two different places.
28. Every continuous function has both a maximum and a minimum on a closed interval.
29. The functions $f(x)$ and $g(x) = f(x) + c$ where c is a constant have the same critical numbers.
30. The functions $f(x)$ and $g(x) = f(x+c)$ where c is a constant have the same critical numbers.

Does the Mean Value Theorem apply to the given function on the given interval? If it does, find the c -value. If it does not, explain why not. Do not use a calculator.

31. $f(x) = x^{\frac{2}{3}}$ on $[0, 1]$

32. $f(x) = \frac{1}{x-4}$ on $[2, 6]$

33. $f(x) = \frac{x^2 - x}{x}$ on $[-1, 1]$

34. $f(x) = x^2 - 2x$ on $[0, 2]$

35. $f(x) = \sin x$ on $[0, \pi]$

36. $f(x) = \tan x$ on $[0, \pi]$

Find the c -value guaranteed by the Mean Value Theorem for the given function on the given interval. You may use a calculator.

37. $f(x) = x^3 - 2x^2 + 3$ on $[-1, 2]$

38. $f(x) = \frac{1}{x-1}$ on $[2, 3]$

39. $f(x) = 2\sin x + \sin(2x)$ on $[0, \pi]$

40. The height, in feet, of an object at time t seconds is given by $h = -16t^2 + 200$.

- a. Find the average velocity of the object during the first 3 seconds.
- b. Use the Mean Value Theorem to find the time at which the object's instantaneous velocity equals this average velocity.

Selected Answers:

1. 3 2. $(5, -2)$ 3. $x = 0, 4, 5$ 4. $x = 1$ 5. $x = \frac{3}{2}$ 6. $x = 0, 8$ 8. $x = -2$
 9. $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$ 10. max. $f = 10$, min. $f = -\frac{9}{4}$ 11. max. $f = 8$, min. $f = 0$
 13. max. $f = 2$, min. $f = 0$ 15. $c = 2$ 16. $c = \frac{3}{2}$ 17. MVT does not apply because f is
 not differentiable at $x = 0$ (sharp turn). 19. max. $f = -3$, min. $f = -15$
 20. max. $f = \frac{2}{3}$, min. $f = 0$ 21. max. $f = \frac{5}{4}$, min. $f = -1$ 23. max. value = 3, min. value = 0
 24. no max. value, min. value = 0 26. max. value = 2, min. value = 0 27. T 29. T
 31. $c = \frac{8}{27}$ 34. $c = 1$ 37. $c = -0.215, 1.548$ or 1.549 38. $c = 2.414$ 39. $c = 1.047$
 40b. $t = \frac{3}{2} \text{ sec}$

Lesson 2-6 Increasing/Decreasing Functions, First Derivative Test**PROCEDURE: (Increasing/ Decreasing and First Derivative Test)**

1. Find **domain** restrictions.
2. Find all **critical numbers** (where $f'(x) = 0$ or $f'(x)$ is undefined – but domain restrictions cannot be critical numbers).
3. Locate critical numbers and domain restrictions on an **f' number line**.
Label critical numbers **CN**.
4. Test the sign of $f'(x)$ in each interval and **label the signs** on the number line.
5. List **increasing/decreasing intervals** and/or identify **relative min/max** x -values. If requested, find y -values or points.

Examples: Find the intervals on which these functions are increasing and decreasing and find all local extrema points.

1. $f(x) = (x^2 - 9)^{\frac{2}{3}}$ 2. $y = x - 2 \sin x$ on $(0, 2\pi)$

3. $f(x) = \frac{x^4 + 3}{3x}$

4. $f(x) = x^3 - 3x^2 + 3x$

The function in Example 4 is a strictly monotonic function. Strictly increasing or strictly decreasing functions are called monotonic.

Example 4 illustrates two **important points**.

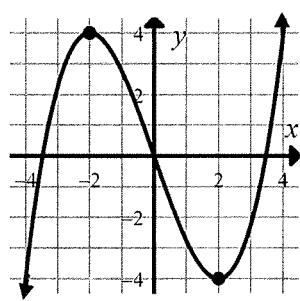
1. Not every critical number produces a relative maximum or minimum.
2. Even though the slope at $x = 1$ is zero, the function is always increasing.

Assignment 2-6

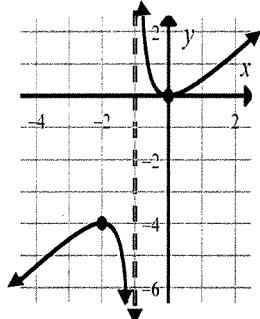
Use the graphs shown with each function to answer these four questions.

- a. On which intervals is f increasing?
- b. On which intervals is f decreasing?
- c. At which point(s) does f have a relative minimum?
- d. At which point(s) does f have a relative maximum?

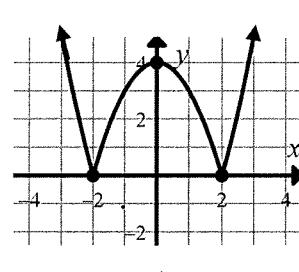
1. $f(x) = \frac{1}{4}x^3 - 3x$



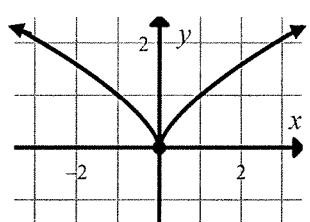
2. $f(x) = \frac{x^2}{x+1}$



3. $f(x) = |x^2 - 4|$

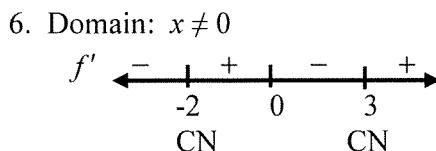
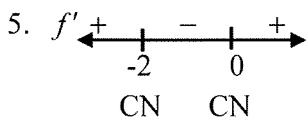


4. $f(x) = x^{\frac{2}{3}}$

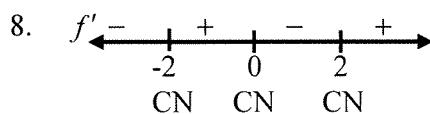
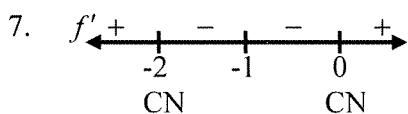


Use the f' number lines shown to answer these four questions.

- On which intervals is f increasing?
- On which intervals is f decreasing?
- At which x -values does f have a relative minimum?
- At which x -values does f have a relative maximum?



Which of the functions in Problems 1-4 would have the following f' number lines?



Without using a calculator, answer these four questions for Problems 9-19. Show organized work and an f' number line to support your answers.

- On which intervals is f increasing?
- On which intervals is f decreasing?
- At which point(s) does f have a relative minimum?
- At which point(s) does f have a relative maximum?

9. $f(x) = x^2 - 8x$

10. $f(x) = -2x^2 + 4x + 6$

11. $f(x) = 2x^3 + 3x^2 - 12x$

12. $f(x) = \frac{3}{5}x^5 - \frac{3}{2}x^3 + 1$

13. $f(x) = x^3 + 1$

14. $f(x) = \frac{x}{x^2 - 9}$

15. $f(x) = (x+1)^{\frac{2}{3}}$

16. $f(x) = \frac{x-1}{x+1}$

17. $f(x) = x^4 - 2x^3$

18. $f(x) = \frac{x}{2} + \cos x$ on $[0, 2\pi]$

19. $f(x) = \sin x + \cos x$ on $(0, 2\pi)$

Determine whether the following functions are strictly monotonic on the interval $(0, \infty)$. Do not use a calculator.

20. $f(x) = x^2$

21. $g(x) = x^{\frac{2}{3}}$

22. $h(x) = \frac{1}{3}x^3 - x$

For Problems 23-25 the height, in feet, of a ball is given by the position function $s(t) = -16t^2 + 64t + 6$. Assume $0 \leq t \leq 4$ seconds.

23. On which interval of time is the ball moving upward?

24. What is the maximum height of the ball?

An object has **increasing velocity** when its acceleration is greater than zero. It has **increasing speed** when the velocity and acceleration have the same sign.

25. For what times is

- the velocity of the ball increasing?
- the speed of the ball increasing?

26. If the function $f(x) = x^2 + ax + b$ has a relative minimum point at $(2, -4)$, solve for a and b .
27. Without using a calculator find the absolute extrema of $f(x) = 4x^3 - 12x + 1$ on the interval $[-1, 3]$.
28. Without using a calculator find the absolute extrema of $f(x) = |x - 2| + 2$ on $[1, 4]$.
29. Find all critical numbers of $f(x) = \|x\|$
30. Can the Mean Value Theorem be used for the function $f(x) = \frac{x}{x-2}$ on the interval $[0, 3]$?
If it can be used, find the c -value. If it cannot, explain why not.
31. Find the c -value guaranteed by the Mean Value Theorem for the function $f(x) = x^3 - x^2 - 2x$ on the interval $[-1, 1]$. You may use a calculator.
32. A point moves along the curve $y = \sqrt{x}$ so that the y -coordinate is increasing at the rate of $4 \frac{\text{cm}}{\text{sec}}$. Find the rate of change of the x -coordinate with respect to time when $y = 2 \text{ cm}$.
33. If $x^2 - xy + y^2 = 5$ find $\frac{dy}{dx}$.
34. Find an equation of the tangent line to the graph of $x^2 + 2y + y^2 = 1$ at the point $(1, -2)$.
35. If $r(3) = -2$, $s(3) = 2$, $r'(3) = 3$, $s'(3) = 5$, and $r'(2) = 4$, find:
 a. $\frac{d}{dt}(r(t) \bullet s(t))$ at $t = 3$ b. $\frac{d}{dt} \frac{r(t)}{s(t)}$ at $t = 3$ c. $\frac{d}{dt} r(s(t))$ at $t = 3$

Selected Answers:

1. a. $(-\infty, -2)$, $(2, \infty)$ b. $(-2, 2)$ c. $(2, -4)$ d. $(-2, 4)$
 3. a. $(-2, 0)$, $(2, \infty)$ b. $(-\infty, -2)$, $(0, 2)$ c. $(\pm 2, 0)$ d. $(0, 4)$
 5. a. $(-\infty, -2)$, $(0, \infty)$ b. $(-2, 0)$ c. $x = 0$ d. $x = -2$
 7. problem 2 9. a. $(4, \infty)$ b. $(-\infty, 4)$ c. $(4, -16)$ d. none
 10. a. $(-\infty, 1)$ b. $(1, \infty)$ c. none d. $(1, 8)$
 12. a. $(-\infty, 0)$, $(1, \infty)$ b. $(0, 1)$ c. $\left(1, \frac{1}{10}\right)$ d. $(0, 1)$
 13. a. $(-\infty, \infty)$ b. none c. none d. none
 14. a. none b. $(-\infty, -3)$, $(-3, 3)$, $(3, \infty)$ c. none d. none
 16. a. $(-\infty, -1)$, $(-1, \infty)$ b. none c. none d. none
 18. a. $\left[0, \frac{\pi}{6}\right), \left(\frac{5\pi}{6}, 2\pi\right)$ b. $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ c. $\left(\frac{5\pi}{6}, \frac{5\pi}{12} - \frac{\sqrt{3}}{2}\right)$ d. $\left(\frac{\pi}{6}, \frac{\pi}{12} + \frac{\sqrt{3}}{2}\right)$
 20. yes 22. no 23. $[0 \text{ sec}, 2 \text{ sec})$ 25. a. never 26. $a = -4, b = 0$
 27. max. $f = 73$, min. $f = -7$ 31. $c = -.333$ 32. $\frac{dx}{dt} = 16 \frac{\text{cm}}{\text{sec}}$ 33. $y' = \frac{-2x+y}{-x+2y}$
 34. $y+2=1(x-1)$ 35. a. -4 b. 4 c. 20

CALCULUS EXTENDED UNIT 2 SUMMARY**Product Rule:**

$$\frac{d}{dx}(f \cdot s) = fs' + sf'$$

Quotient Rule:

$$\frac{d}{dx} \frac{t}{b} = \frac{bt' - tb'}{b^2}$$

Chain Rule:

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

(chain rule version)

Trig Rules: $\frac{d}{dx} \sin x = \cos x$

$$\frac{d}{dx} \sin u = \cos u u'$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} \cos u = -\sin u u'$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \tan u = \sec^2 u u'$$

$$\frac{d}{dx} \cot x = -\csc^2 x$$

$$\frac{d}{dx} \cot u = -\csc^2 u u'$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\frac{d}{dx} \sec u = \sec u \tan u u'$$

$$\frac{d}{dx} \csc x = -\csc x \cot x$$

$$\frac{d}{dx} \csc u = -\csc u \cot u u'$$

CALCULUS EXTENDED UNIT 2 SUMMARY (CONTINUED)

An object has **increasing velocity** when its acceleration is greater than zero.
It has **increasing speed** when the velocity and acceleration have the same sign.

Implicit Differentiation: differentiate both sides with respect to x , remember the y' or $\frac{dy}{dx}$ “chain rule part” and remember to use the Product Rule for any xy term

Related Rate Problems:

Write an equation relating the variables.

- (a) If a quantity is **changing** it must be represented with a variable letter.
- (b) If a quantity is **constant** it must be represented with a number value.
- (c) Look for secondary relationships between quantities to reduce the number of variables.
Implicitly differentiate both sides with respect to t .

Definition of **Critical Numbers**: x -values where f' is zero or undefined (cannot be domain restrictions)

Absolute Extrema: Compare y -values at endpoints and critical numbers.

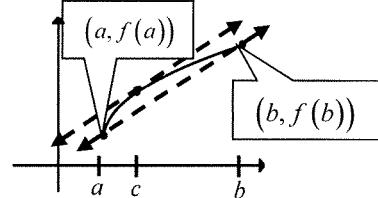
Relative (Local) Extrema:

First Derivative Test: Find critical numbers, and make an f' number line. Domain restrictions must be on all number lines – but cannot be max/min points. An f' number line also gives increasing/decreasing intervals.

Mean Value Theorem: f must be continuous on $[a,b]$ and differentiable on (a,b) .

inst. rt. ch. = avg. rt. ch.
tangent slope = secant slope

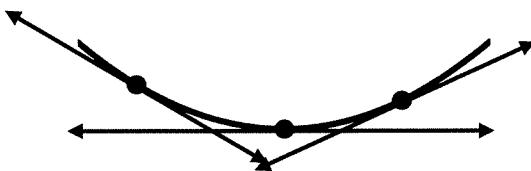
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



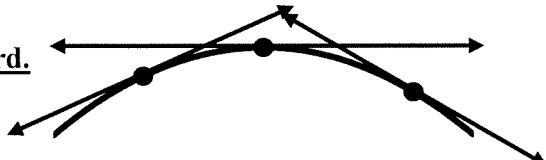
MVT is used to find the c -value.

Lesson 3-1**Concavity, Points of Inflection,
The Second Derivative Test**

A graph with this shape is called **concave upward**.
 The tangent lines lie **below** the graph. The slopes
 of the tangent lines are increasing which means
 $f''(x) \geq 0$.



A graph with this shape is called **concave downward**.
 The tangent lines lie **above** the graph. The slopes
 of the tangent lines are decreasing which means
 $f''(x) \leq 0$.

**Nonmathematical Memory Device:**

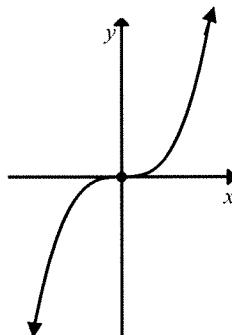
Concave upward \leftrightarrow positive \leftrightarrow smiley face \leftrightarrow

Concave downward \leftrightarrow negative \leftrightarrow frowny face \leftrightarrow

A point on a graph is a **point of inflection** if:

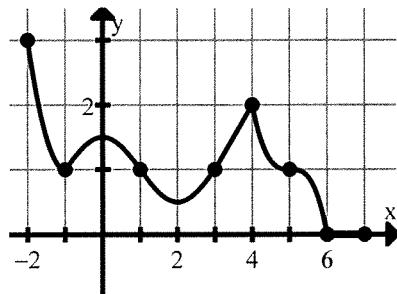
1. the graph has a tangent line at that point **and**
2. the graph changes concavity at that point.

On the graph of $y = x^3$ shown, the point $(0,0)$ is a point of inflection.



Examples: Use the graph at the right to answer these questions. Base your answers on appearances of the graph.

1. On which intervals is the graph concave upward?
2. On which intervals is the graph concave downward?
3. On which intervals does the graph have no concavity?
4. What are the points of inflection?



Analytically we find concavity intervals and points of inflection by using a second derivative number line.

The procedure is parallel to the procedure used in the last lesson to find increasing/decreasing intervals and relative extrema by using a first derivative number line.

Examples:

5. Determine the points of inflection and discuss the concavity for the graph of $f(x) = x^4 + x^3 - 3x^2 + 1$.

6. If $f(x) = \frac{x^2 + 1}{x^2 - 4}$ and $f''(x) = \frac{10(3x^2 + 4)}{(x^2 - 4)^3}$, list the intervals where the graph of f is concave upward, concave downward, and list the points of inflection.

THE SECOND DERIVATIVE TEST FOR RELATIVE EXTREMA

This test does not require a second derivative number line. It does not find points of inflection. It is used to find relative extrema (max/min).

Procedure:

1. Use f' to find critical numbers.
2. Plug critical numbers into f'' and analyze concavity to determine if the function has a relative minimum or maximum.

Note: The Second Derivative Test does not always give an answer (when $f''(x) = 0$). Use it only when the directions require it or when the given information requires it.

Examples:

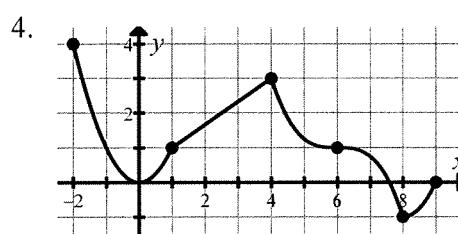
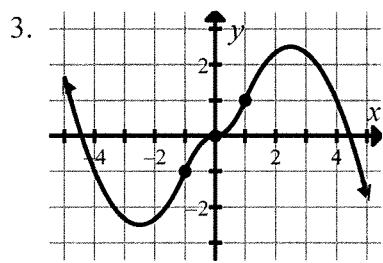
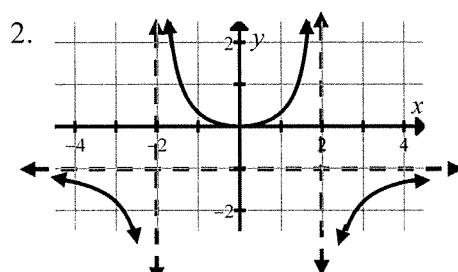
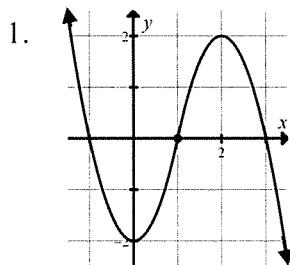
7. Use the Second Derivative Test to find the relative minimum and relative maximum points for the graph of $f(x) = -3x^4 + 6x^2$.

8. $g(x)$ is a function such that $g'(-3)=0$, $g'(-1)=0$, $g'(0)=0$, and $g'(2)=0$. If $g''(-3)=4$, $g''(-1)=-2$, $g''(0)=0$, $g''(1)=5$, and $g''(2)=3$, find the x -values of the relative maximum and relative minimum values when possible.

Assignment 3-1

Use the appearance of these graphs to answer these three questions.

- On which interval(s) is the graph of the function concave upward?
- On which interval(s) is the graph of the function concave downward?
- What are the points of inflection?



Show organized steps and a second derivative number line to answer the same three questions for these functions.

- $g(x) = 6x^2 - 4x^3$
- $f(x) = x^4 - 4x^3 + 2$
- $g(x) = \cos x + \sin x$ on $[0, 2\pi]$
- $f(x) = 3x^5 - 5x^4$
- $f(x) = x^{\frac{2}{3}} - 3$
- $f(x) = -2 \sin x - \frac{1}{2}x^2 + 2x + 1$ on $[0, 2\pi]$

11. $g(x) = x^4 - 4x$

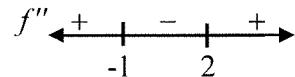
Use the Second Derivative Test to find the relative extrema points (see Example 7 on the previous page).

- $f(x) = x^3 - 3x^2 + 6$
- $f(x) = -\frac{1}{4}x^4 + \frac{9}{2}x^2 + 5$
- $g(x) = 2 \sin x + 3$ on $[0, 2\pi]$
- $g(x) = \sqrt{x^2 + 2}$

For these problems, find the x -values of points which must be relative minimum points and the x -values of points which must be relative maximum points.

16. $-3, 1$, and 3 are critical numbers of f and 17. $f'(-2) = f'(0) = f'(4) = 0$

$f''(-3) = -2$, $f''(1) = 0$, and $f''(3) = 2$.



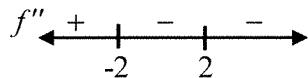
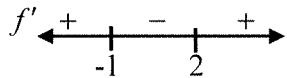
Without using a calculator, find local maximum and minimum points and points of inflection. Then sketch a graph. It is not necessary to find the x -intercepts. Show organized steps with f' and f'' number lines to support your answers.

18. $f(x) = 2x^3 - 3x^2 - 12x + 5$

19. $f(x) = 4x^3 - x^4$

20. Use the following information to sketch a possible graph of f .

f is a continuous function, $f(-1) = 3$, $f(2) = 0$, $f(-2) = -1$, $f'(2)$ does not exist.



21. Use a calculator to determine if the function $f(x) = 4x^3 + \sin(5x)$ is concave upward or downward on an interval including $x = .523$. Be sure your calculator is in radians mode.

22. Find the absolute maximum and absolute minimum for the function $f(x) = x^4 - 2x^2$ on the interval $[-2, 1]$. Do not use a calculator.

23. Find the c -value guaranteed by the Mean Value Theorem for the function $f(x) = \sqrt{x-2}$ on the interval $[2, 6]$.

24. If the graph of $y = ax^3 + bx^2 + cx + d$ has a point of inflection at the point $(0, 2)$ and a relative maximum at the point $(-1, 4)$, find the values of a, b, c , and d .

25. a. Sketch a smooth curve whose slope is always positive and whose slope is increasing.
b. Sketch a smooth curve whose slope is always positive and whose slope is decreasing.
c. Sketch a smooth curve whose slope is always negative and whose slope is increasing.
d. Sketch a smooth curve whose slope is always negative and whose slope is decreasing.

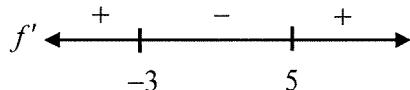
26. Given: $x^2 + xy + y = -3$

- a. Find the value of $\frac{dy}{dx}$ at the point $(1, -2)$. b. Find the value of $\frac{d^2y}{dx^2}$ at $(1, -2)$.

- c. Does the curve have a relative maximum, a relative minimum, or neither at $(1, -2)$.

Use the f' number line shown for Problems 27-29.

Fill in the blank with $>$ or $<$.



27. If $g(x) = f(x) + 2$, then $g'(4) \underline{\hspace{2cm}} 0$.

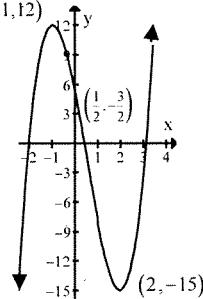
28. If $g(x) = -2f(x)$, then $g'(4) \underline{\hspace{2cm}} 0$.

29. If $g(x) = f(x-10)$, then $g'(4) \underline{\hspace{2cm}} 0$.

30. $x(t) = t^3 - 27t + 50$ is the position function of a particle moving along a horizontal line.
- When is the velocity of the particle increasing?
 - When is the speed of the particle increasing?
 - When is the distance of the particle from the origin increasing?

Selected Answers:

- 1a. $(-\infty, 1)$ b. $(1, \infty)$ c. $(1, 0)$ 2a. $(-2, 2)$ b. $(-\infty, -2), (2, \infty)$ c. none
 4a. $(-2, 1), (4, 6), (8, 9)$ b. $(6, 8)$ c. $(6, 1)$ 5a. $(-\infty, \frac{1}{2})$ b. $(\frac{1}{2}, \infty)$ c. $(\frac{1}{2}, 1)$
 6a. $(-\infty, 0), (2, \infty)$ b. $(0, 2)$ c. $(0, 2), (2, -14)$ 7a. $(\frac{3\pi}{4}, \frac{7\pi}{4})$ b. $[0, \frac{3\pi}{4}), (\frac{7\pi}{4}, 2\pi]$
 7c. $(\frac{3\pi}{4}, 0), (\frac{7\pi}{4}, 0)$ 9a. none b. $(-\infty, 0), (0, \infty)$ c. none 10a. $(\frac{\pi}{6}, \frac{5\pi}{6})$
 10b. $[0, \frac{\pi}{6}), (\frac{5\pi}{6}, 2\pi]$ c. $\left(\frac{\pi}{6}, \frac{-\pi^2}{72} + \frac{\pi}{3}\right), \left(\frac{5\pi}{6}, \frac{-25\pi^2}{72} + \frac{5\pi}{3}\right)$ 12. rel. max. $(0, 6)$ rel. min. $(2, 2)$
 13. rel. max. $(\pm 3, \frac{101}{4})$ rel. min. $(0, 5)$ 14. rel. max. $(\frac{\pi}{2}, 5)$ rel. min. $(\frac{3\pi}{2}, 1)$
 15. rel. min. $(0, \sqrt{2})$ 16. rel. max. at $x = -3$, rel. min. at $x = 3$ 18. $(-1, 12)$
 21. concave down ($f''(.523) = -.013$)
 22. max. $f = 8$, min. $f = -1$ 23. c = 3
 24. $a = 1, b = 0, c = -3, d = 2$
 26a. 0 b. -1 c. relative maximum



Lesson 3-2 Curve Sketching with Extrema and Inflection Points

You used a Precalculus Curve Sketching Recipe in Unit 1. The following incorporates the additional Calculus techniques you have recently learned.

Curve Sketching Recipe:

- Give the domain.
- Reduce $f(x)$.
- Find vertical asymptotes and holes.
- Give x - and y -intercepts.
- Find the end behavior (horizontal asymptotes or other).
- (optional) Check for symmetry.
- Find increasing/decreasing intervals and relative extrema (show an f' number line).
- Find concavity and points of inflection (show an f'' number line).
- Graph.

Examples:

1. (a rational function) $f(x) = \frac{3x-2}{x^2-2x+1}$, $f'(x) = \frac{-3x+1}{(x-1)^3}$, $f''(x) = \frac{6x}{(x-1)^4}$

Do.:

V.A.:

Holes:

x-int.:

y-int.:

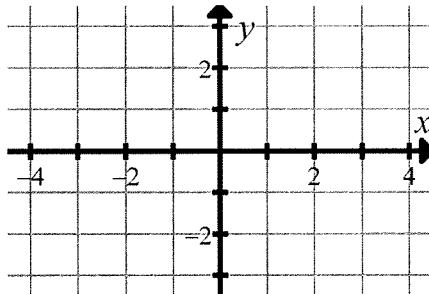
E.B.:

Symmetry:

Rel. Max.:

Rel. Min.:

P.I.:



2. (a radical function) $f(x) = \frac{x}{\sqrt{x^2+2}}$, $f'(x) = \frac{2}{\sqrt{(x^2+2)^3}}$, $f''(x) = \frac{-6x}{\sqrt{(x^2+2)^5}}$

Do.:

V.A.:

Holes:

x-int.:

y-int.:

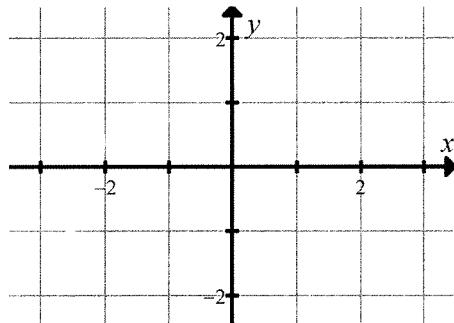
E.B.:

Symmetry:

Rel. Max.:

Rel. Min.:

P.I.:

**Assignment 3-2**

Without using a calculator, find local extrema, points of inflection, and sketch a graph. Show organized steps and justification. It is not necessary to find x-intercepts and there are no domain restrictions or asymptotes. However, an end behavior analysis will be helpful.

1. $y = x^3 - 3x^2 + 5$ 2. $y = 1 - x - x^3$ 3. $y = x^4 - 4x^3 + 16$

Find intercepts and relative extrema and graph these functions without using a calculator.

4. $f(x) = x^2 - 2x - 8$ 5. $g(x) = |x^2 - 2x - 8|$

Find relative extrema, points of inflection, and end behavior and graph without a calculator.

6. $y = \frac{2x^2}{x^2+3}$, $y'' = \frac{-36(x^2-1)}{(x^2+3)^3}$

Find the domain, relative extrema, asymptotes, and end behavior and graph without a calculator.

There are no points of inflection.

7. $f(x) = \frac{x^2+1}{2x}$

Find the domain, reduced function, hole, intercepts, relative extrema, and points of inflection. Then graph without using a calculator.

8. $f(x) = \frac{x^2\sqrt{4-x}}{x}$, $f' = \frac{8-3x}{2\sqrt{4-x}}$, $f'' = \frac{3x-16}{4(4-x)^{\frac{3}{2}}}$, $f\left(\frac{8}{3}\right) = 3.079$

9. Without a calculator, find the domain, x -intercepts, and relative extrema. Then graph $f(x)$.

There are no points of inflection. $f(x) = \sqrt{9-x^2}$

10. Find the domain, x - and y -intercepts, relative extrema, and points of inflection for $y = \sin x + \cos x$ on $[0, 2\pi]$. Then sketch the graph of y without using a calculator.

11. Without using a calculator, graph $g(x) = |x^3 - 3x|$.

12. True or False? If $f'(x) > 0$ for all real x -values, then $\lim_{x \rightarrow \infty} f(x) = \infty$.

Show a graph to illustrate your answer.

Find the discontinuities. Which are removable? Do not use a calculator.

13. $f(x) = \frac{x-2}{x^2-3x+2}$ 14. $f(x) = \begin{cases} x^2-4x, & x \leq 0 \\ x-1, & x > 0 \end{cases}$ 15. $f(x) = \begin{cases} x \\ 3 \end{cases}$

Find these limits without a calculator.

16. $\lim_{x \rightarrow 2} \frac{x-2}{x^2-4}$ 17. $\lim_{x \rightarrow 2} \frac{x+2}{x^2-4}$ 18. $\lim_{x \rightarrow \infty} \frac{x+2}{x^2+4}$

19. Find an equation of the line tangent to the graph of $y = 4^{x-2}$ at the point where $x = 5$. You may use a calculator.

20. a. Find the points at which the graph of $x^2 + 4y + 2y^2 = 6$ has horizontal tangent lines.
 b. Determine whether each of these points is a local minimum or a local maximum.
 (Show organized work using the Second Derivative Test.)

21. Find the points at which the graph of $x^2 + 4y + 2y^2 = 6$ has vertical tangent lines.

22. Given: $f(4) = 2$, $g(4) = 3$, $f'(4) = -1$, and $g'(4) = 5$,

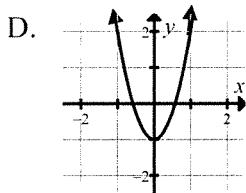
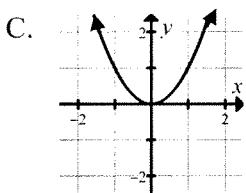
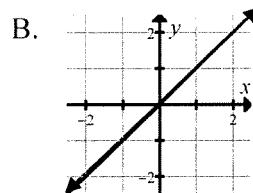
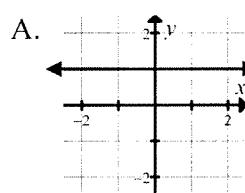
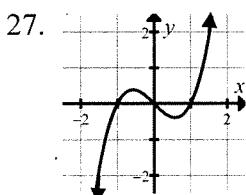
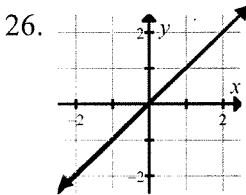
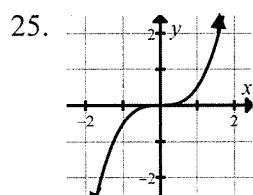
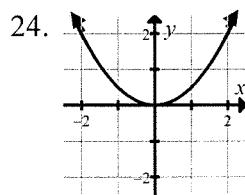
- a. If $h(x) = f(x) \cdot g(x)$, find $h'(4)$.
 b. If $j(x) = (f(x))^3$, find $j'(4)$.

23. The position (in cm) of an object moving on a horizontal line is given by $s(t) = 2t^3 - 3t^2 - t + 8$ (where time is measured in seconds). Answer the following questions.

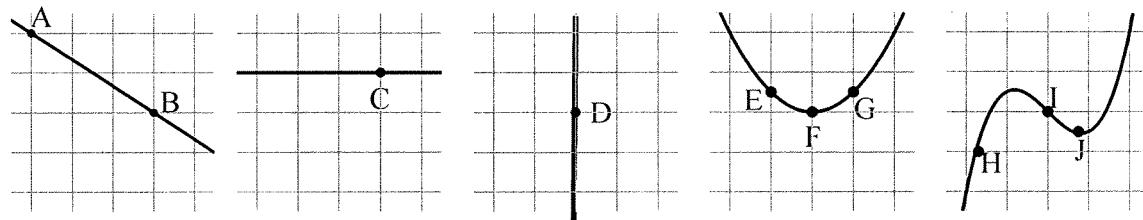
You may use a calculator.

- What is the object's velocity equation?
- What is the object's initial velocity?
- What is the object's acceleration equation?
- What is the object's acceleration at $t = 3$ seconds?
- What is the object's speed at $t = 1$ second?
- When is the object moving left?
- What is the object's displacement between zero and two seconds?
- What is the object's total distance traveled between zero and two seconds?
- When is the object's velocity decreasing?
- When is the object's speed decreasing.

Match the graph of f in the top row with the appropriate graph of f' in the bottom row.

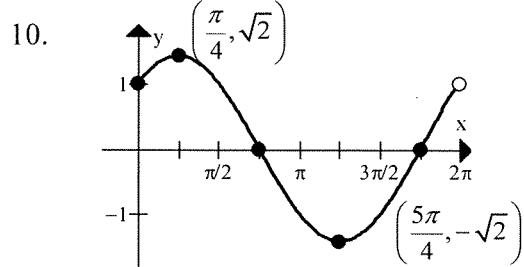
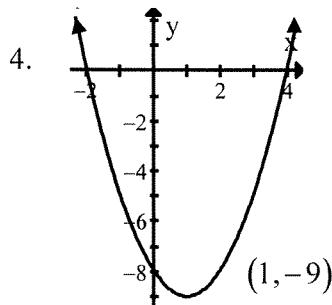


28. Find (or estimate) the slope of each graph at each lettered point.



Selected Answers:

- | | |
|--|---------------------------------|
| 1. rel. max. $(0, 5)$ rel. min. $(2, 1)$ PI $(1, 3)$ | 2. no rel. extrema, PI $(0, 1)$ |
| 3. rel. min. $(3, -11)$ PI $(0, 16), (2, 0)$ | |

More Selected Answers:

6. rel. min. $(0, 0)$, P.I. $(\pm 1, \frac{1}{2})$, E.B. horiz. asym. $y = 2$

7. Do: $x \neq 0$, rel. max. $(-1, -1)$, rel. min. $(1, 1)$,
V. A. $x = 0$ (odd), E. B. like $y = \frac{1}{2}x$

9. Do: $-3 \leq x \leq 3$, x-int. $(\pm 3, 0)$, rel. max. $(0, 3)$

13. $x = 1$ (nonrem.), $x = 2$ (rem.)

14. $x = 0$ (nonrem.) 16. $\frac{1}{4}$ 18. 0

20.a. $(0, -3), (0, 1)$ 21. $(\pm\sqrt{8}, -1)$ 22. a. 7

23. b. $v(0) = -1 \frac{\text{cm}}{\text{sec}}$ d. $a(3) = 30 \frac{\text{cm}}{\text{sec}^2}$

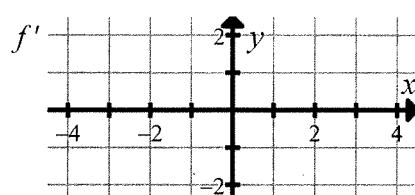
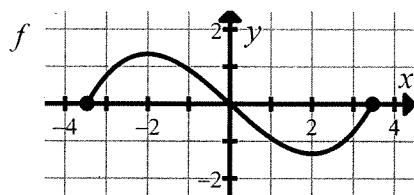
23 f. $-1.145 < t < 1.145$ h. T. D. = 6.151 or 6.152

24. B 28. (A) $-\frac{2}{3}$, (G) 1

Lesson 3-3 Graphing Derivatives and Antiderivatives from Graphs

Derivatives: f graph $\rightarrow f'$ graph (or f' graph $\rightarrow f''$ graph)
Find (or estimate) slopes and plot them as points.

Example: 1. Use the graph of f shown to sketch a graph of f' .

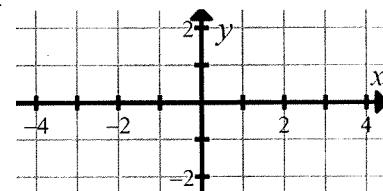
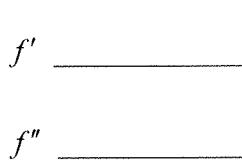
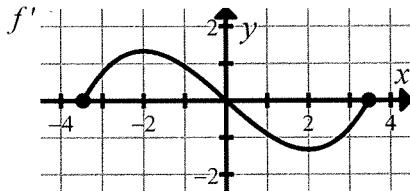


Antiderivatives: f' graph $\rightarrow f$ graph

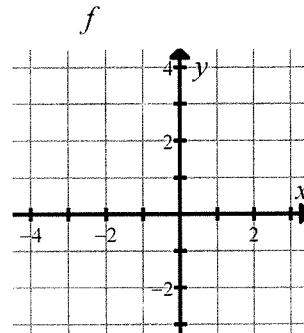
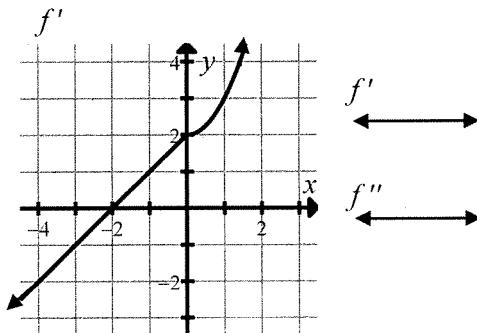
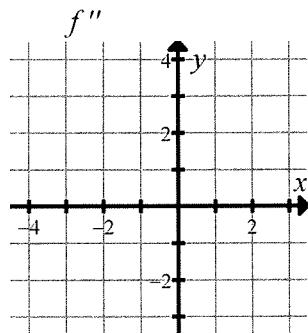
1. Make an f' number line by using the location or position of the points on the f' graph. This does not involve the slopes of f' .
2. Make an f'' number line by using the slopes of the f' graph.
3. Combine information from both number lines to graph f . If no starting point is given, you are free to shift the graph vertically.

Examples:

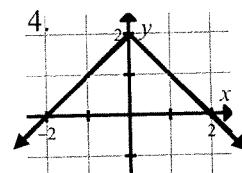
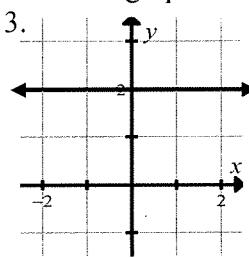
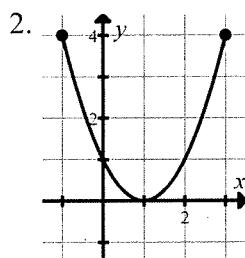
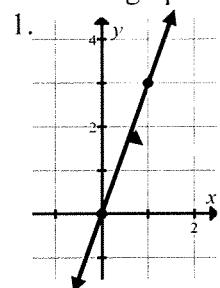
2. Use the graph of f' shown to sketch a graph of f with a starting point of $(0,1)$.



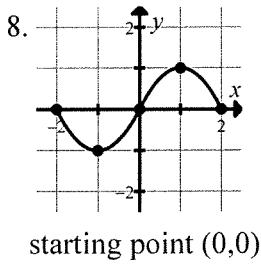
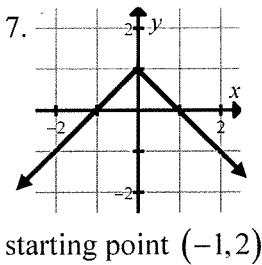
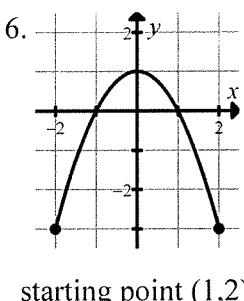
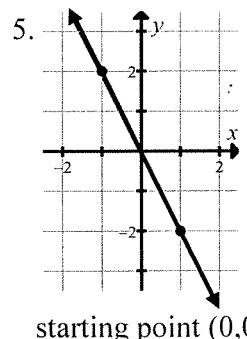
3. Use the graph of f' shown to sketch a graph of f'' and a possible graph of f .

**Assignment 3-3**

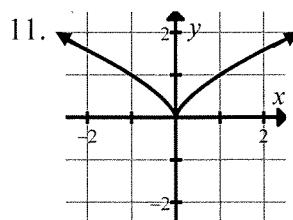
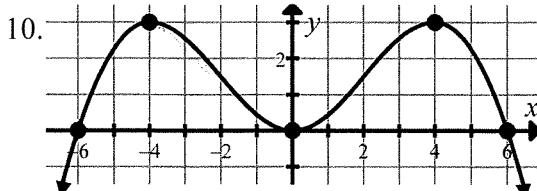
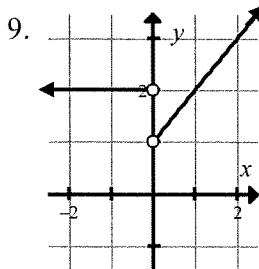
Sketch a graph of the derivative of the function whose graph is shown.



Use the graph of f' shown and the given starting point to graph f (the antiderivative).



Use the graph of f' shown to sketch a graph of f'' and a possible graph of f .



12. Find the vertical asymptotes, end behavior, and relative extrema. Then graph

$$f(x) = \frac{1}{x^2 - 2x - 8} \text{ without using a calculator.}$$

13. If $f(x) = x(x-4)^3$, find relative extrema points and points of inflection. Then graph f without using a calculator. Hint: $f''(x) = 12(x-4)(x-2)$.

14. Use the Second Derivative Test to find the relative extrema points of $f(x) = x^3 - 3x^2 - 5$.

15. Use the following information to sketch a possible graph of f .

$$f(0) = f(4) = 0, \quad f(2) = -2,$$

$f'(x) < 0$ when $x < 2$, $f'(x) > 0$ when $x > 2$, $f'(2)$ does not exist,

$$f''(x) < 0 \text{ when } x \neq 2$$

16. Find the c -value guaranteed by the Mean Value Theorem for $f(x) = x^3 - 2x + 3$ on the interval $[0, 2]$.

17. Find the absolute minimum and absolute maximum of the function $f(x) = x^3 - 12x - 2$ on the interval $[0, 4]$ without using a calculator.

18. Without using a calculator, sketch a graph of $f(x) = |-x^2 - 6x|$.

19. The graph of $y = ax^2 + bx + c$ passes through the point $(1, 6)$ and has a tangent line at $(0, 16)$ which is parallel to the graph of $y = -12x - 2$. Find a , b , and c .

20. If the only critical number of a function $f(x)$ is $x = 3$, $f'(2) = -6$, and $f'(4) = 7$, does f have a local minimum or a local maximum at $x = 3$? Assume f is continuous.

21. If $x = 3$ is a critical number of a function $g(x)$ and $g''(3) = -6$, does g have a relative minimum or a relative maximum at $x = 3$?

True or False?

22. Every fourth degree polynomial has three critical numbers.

23. Every fourth degree polynomial has at most three critical numbers.

24. If a polynomial has two critical numbers, one must be at a relative maximum and the other must be at a relative minimum.

25. If $f'(2) > 0$, then f is increasing at $x = 2$.
 26. If a function f is increasing at $x = 2$, then $f'(2) > 0$.

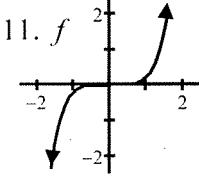
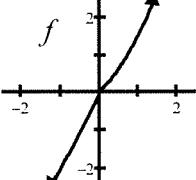
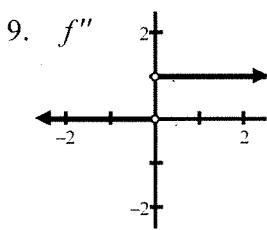
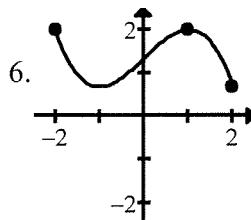
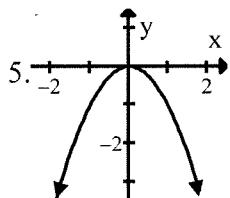
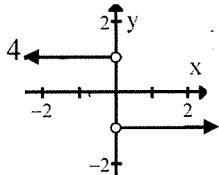
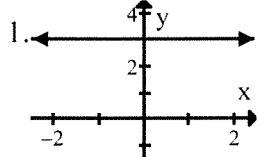
Find y' .

27. $y = x\sqrt{3x^2 + 2}$

28. $x^2 - 2xy + y = 8$

29. Without using a calculator, find the domain, vertical asymptote, hole, x -intercept, and end behavior, and then graph $f(x) = \frac{2x^2 + 2x}{x^2 - x}$.

Selected Answers:



These f graphs could be shifted vertically.

12. V.A.: $x = 4, -2$, H.A.: $y = 0$, Rel. Max.: $(1, -\frac{1}{9})$, no Rel. Min.

13. Rel. Min.: $(1, -27)$, no Rel. Max., P.I.: $(2, -16), (4, 0)$

14. Rel. Max.: $(0, -5)$, Rel. Min.: $(2, -9)$ 16. $c = \sqrt{\frac{4}{3}}$

17. abs. min. $f = -18$, abs. max. $f = 14$ 19. $a = 2, b = -12, c = 16$

27. $y' = 3x^2(3x^2 + 2)^{-\frac{1}{2}} + (3x^2 + 2)^{\frac{1}{2}}$ 28. $y' = \frac{-2x + 2y}{-2x + 1}$

29. x -int.: $(-1, 0)$ (odd), Hole: $(0, -2)$, H. A.: $y = 2$

Lesson 3-4 Optimization Story Problems

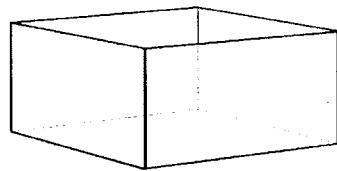
One of the most common applications of Calculus involves determining maximum or minimum values.

Procedure:

1. Choose variables and/or draw a labeled figure.
2. Write a primary equation. Isolate whatever is to be maximized or minimized.
3. Rewrite with only one variable on each side. This may require a secondary equation.
4. Find the domain.
5. Take the derivative, find critical numbers, make a number line, etc.

Examples:

1. A box with no lid is to be made from 48 cm^2 of material. If the box must have a square base, find the dimensions that produce a maximum volume. Explain the meaning of the answer with a complete sentence.

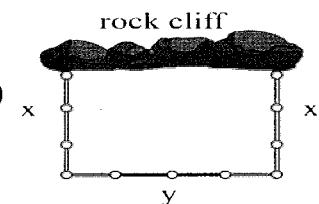


2. The sum of two nonnegative numbers is 30. Find both numbers if the sum of twice the first plus the square of the second is a
- maximum. Explain the meaning of the answer with a complete sentence.
 - minimum. Explain the meaning of the answer with a complete sentence.

Assignment 3-4

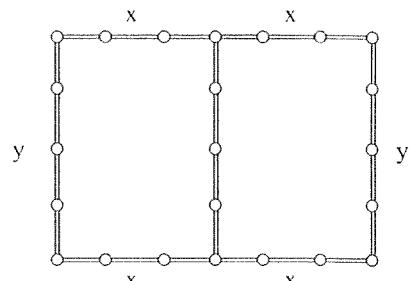
Write a complete sentence explaining the meaning of each answer for problems 1-12.

- The product of two positive numbers is 100. Find the two numbers so that the sum of the numbers is as small as possible.
- The sum of two nonnegative numbers is 25. Find the two numbers so that the sum of the first plus the square of the second is a maximum.
- A rancher plans to fence in three sides of a rectangular pasture, with the fourth side being against a rock cliff. He needs to enclose 320,000 square meters of pasture. What dimensions would require the least amount of fence material?

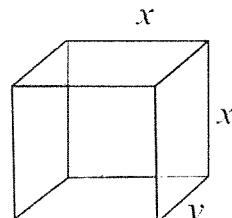


4. The perimeter of a rectangle is 80 feet. Find the length and width so that the rectangle has a maximum area.

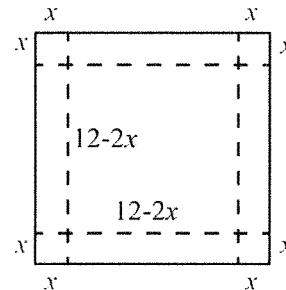
5. Two adjacent rectangular corrals are to be made using 240 feet of fencing. The fence must extend around the outer perimeter and across the middle as shown in the diagram. Find the dimensions so that the total enclosed area is as large as possible.



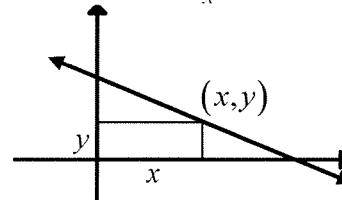
6. A shelter at a bus stop is to be made with three Plexiglas sides and a Plexiglas top. If the volume of the shelter is 486 cubic feet, find the dimensions that require the least amount of Plexiglas.



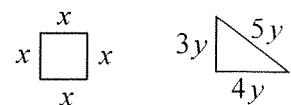
7. A box is made by cutting small squares from each corner of a piece of square material 12 inches on each side and then folding up the flaps. Find the length of each side of the square cutouts that will produce the greatest volume box.



8. A rectangle is positioned with one vertex on the line $y = -\frac{1}{2}x + 3$ in the first quadrant as shown. Find the point so that the rectangle has a maximum area.



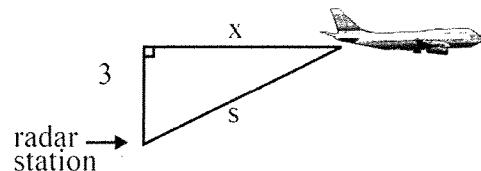
9. 36 total inches of wire is to be used to form the perimeter(s) of a square and/or a 3-4-5 ratio right triangle. Find the dimensions of the figure(s) that enclose both a maximum and minimum total area.



10. A box with an open top has a square base. If the volume of the box is 4000 cubic centimeters, what dimensions minimize the amount of material used?

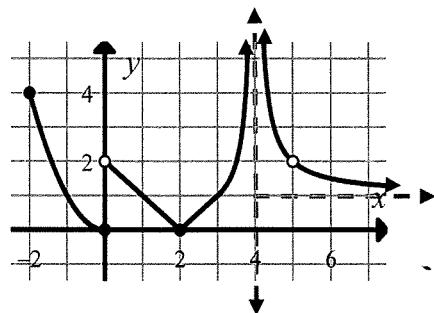
11. The volume formula for a cone is $V = \frac{1}{3}\pi r^2 h$. If $\frac{dr}{dt} = 3 \frac{\text{in}}{\text{min}}$ and $h = 3r$, find $\frac{dV}{dt}$ when $r = 6$ inches.

12. An airplane flying at an altitude of 3 miles flies directly over a radar station. When the plane is 5 miles away from the station, the radar shows the distance s is changing at the rate of 300 miles per hour. What is the plane's speed?

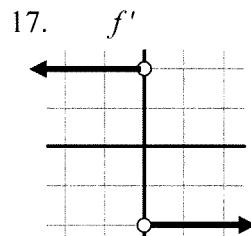
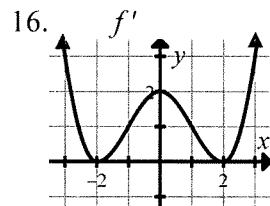
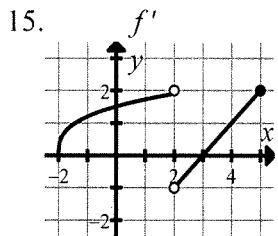
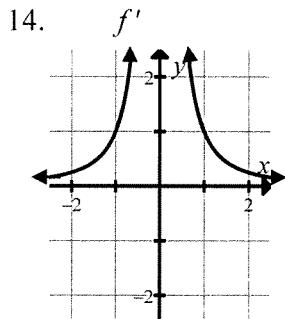


13. Use the graph of $y = f(x)$ at the right for these problems.

- Find $\lim_{x \rightarrow 0} f(x)$.
- Find $\lim_{x \rightarrow 0^+} f(x)$.
- Find $\lim_{x \rightarrow 4} f(x)$.
- Find $\lim_{x \rightarrow \infty} f(x)$.
- Find $\lim_{x \rightarrow 3} f(x)$.
- List the discontinuities of f .
- Which of these discontinuities are removable?
- Find the absolute maximum of $f(x)$ on $[-2, 3]$.
- Find the absolute minimum of $f(x)$ on $[-2, 3]$.
- Find $f'(1)$.
- Find $f''(1)$.
- List all x -values where $f'(x)$ does not exist.
- List all x -values at which $f(x)$ has a local minimum.
- List all x -values at which $f(x)$ has a local maximum.



Use these graphs of f' to graph f'' .



18 and 19. Use the graphs of f' for Problems 14 and 15 above to sketch a possible graph of f .

- Use the graph of f' in Problem 16 to sketch a graph of f with the starting point $(-2, -2)$.
- Use the graph of f' in Problem 17 to sketch a continuous graph of f with the starting point $(0, 2)$.

Selected Answers:

- | | |
|--|---|
| 1. The numbers are both 10. | 2. The first number is 0 and the second is 25. |
| 3. The sides perpendicular to the rock are both 400 m and the other side is 800 m. | |
| 4. The rectangle's length and width are both 20 ft. | |
| 5. The shared side is 40 feet and the other dimension of each corral is 30 feet. | |
| 6. The back of the shelter is a 9 feet by 9 feet square and the other dimension is 6 feet. | |
| 7. The square cutouts are 2 inches on each side. | 8. The vertex is at the point $\left(3, \frac{3}{2}\right)$. |

More Selected Answers:

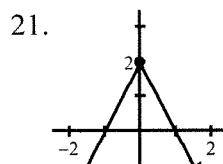
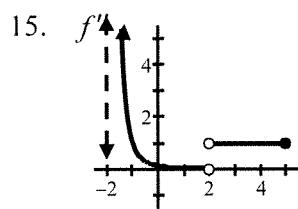
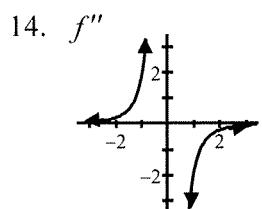
9. The minimum area is enclosed by a square with 3.6 in. sides and a triangle with sides 5.4 in., 7.2 in., and 9 in. The maximum area is enclosed by a square with 9 in. sides and no triangle.

10. The length and width of the box are both 20 cm. and the height is 10 cm.

11. The volume is increasing at the rate of $324\pi \frac{\text{in}^3}{\text{min}}$. 12. The plane's speed is 375 mph.

13a. DNE b. 2 c. DNE or ∞ d. 1 e. 1 f. $x = 0, 4, 5$ g. $x = 5$ h. 4 i. 0 j. -1 k. 0

13l. $x = 0, 2, 4, 5$ (also possibly $x = 3$) m. $x = 0, 2$ n. none



There could be a jump at $x = 0$.

Lesson 3-5 Approximating with the Tangent Line

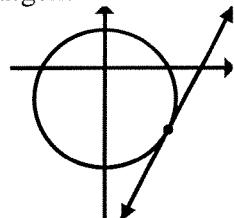
In many instances, finding a value of a function is difficult or impossible. With the use of Calculus techniques, we can approximate the function value by finding a y -value on a tangent line to the function. Since this method involves using a linear function (the tangent line function) at a nearby point, it is sometimes called a local linearization approximation.

Examples:

1. If $(2, -2)$ is a point on the graph of $x^2 + y^2 + 2y = 4$, use the equation of a tangent

line passing through the point $(2, -2)$ to approximate a y -coordinate

(a) when the x -coordinate is 2.1.



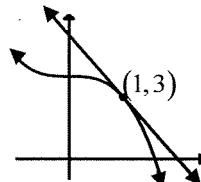
(b) when the x -coordinate is 1.9.

2. If $f(2) = 3$ and $f'(2) = -2$, use local linearization to approximate $f(2.01)$,

Assignment 3-5

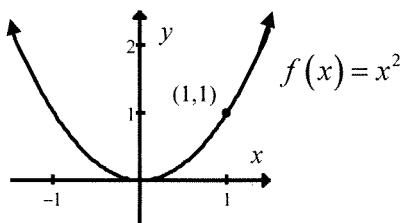
1. a. Write an equation of the tangent line shown.
 b. Use this tangent line equation to approximate $f(1.1)$.

$$f(x) = -x^3 + 4$$

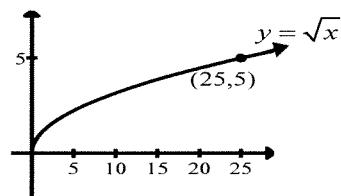


- c. What is the actual value of $f(1.1)$?
 2. Make a large copy of the graph on your own paper.

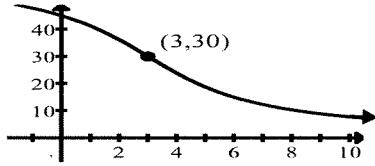
- a. Draw the tangent line at the point $(1, 1)$.
 b. Write an equation of this tangent line.
 c. Label a point on your tangent line with an x -coordinate of .9 as point A.
 d. Use your equation of the tangent line to approximate $f(.9)$ by finding the y -coordinate of your point A..
 e. Label a point B on the parabola with an x -coordinate of .9. What is the actual value of $f(.9)$?
 f. Use the same tangent line to approximate $f(.6)$. How accurate is your approximation?



3. Approximate $\sqrt{26}$ using the equation of a tangent line.
 You must choose your own equation and point. The graph shown should help.



4. The graph of a function $y = f(x)$ is shown. If $f'(3) = -9$, use local linearization to approximate $f(3.1)$.

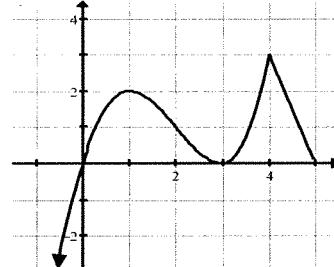


5. Find the actual value of $f(3.1)$ from problem 4 or explain why it cannot be found.
 6. The point $(5, 20)$ is on the curve $y = x\sqrt{x^2 - 9}$. Use a tangent line to approximate the y -coordinate when $x = 5\frac{1}{5}$.
7. The length of one side of a square is found to be 8 in. with a possible measurement error of $\frac{1}{16}$ in.
- a. Instead of using the actual area formula ($A = s^2$), approximate the area of the square using a local linearization of the area formula if the length of the side is really $8\frac{1}{16}$ inches (without using a calculator).
 b. Find the approximate area if the side is actually $7\frac{15}{16}$ inches.
 c. Use your answers from parts a and b to give an approximate range of values for the area of the square.
8. Use a tangent line equation to approximate $f(8.01)$ if $f(x) = \sqrt[3]{x}$ (without using a calculator).

9. The point $(1,2)$ is on the graph of $x^3 + xy + y^4 = 19$. Use the equation of a tangent line to approximate a y -coordinate when $x = 1.1$.
10. Use a calculator to find an actual y -coordinate on the graph of the curve from problem 9 when $x = 1.1$. Show the equation you are solving.
11. Given the function $y = x^3$
- use the equation of a tangent line to approximate $\left(2\frac{1}{6}\right)^3$ without using a calculator.
 - find the actual value of $\left(2\frac{1}{6}\right)^3$?
12. An interstate driver is traveling 420 miles across a state from south to north without stopping. At noon she notices her speed is 60 miles per hour and her position is at interstate mile marker 240. Note: Interstate mile markers increase from south to north.
- Use this data to write a linear function (local linearization) which could be used to estimate her position as a function of time. Assume $t = 0$ at noon.
 - Approximate her position at 2:00 pm.
 - Approximate her position at 10:30 am.
 - What is the domain on which your linear function can be applied?

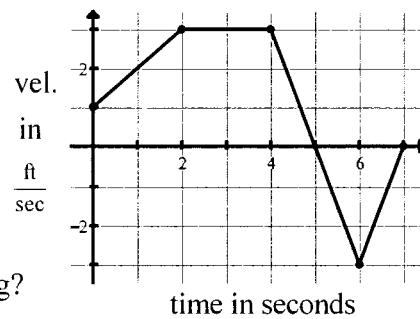
Use the graph of $f'(x)$ shown for problems 13 and 14.

13. Sketch a graph of $f''(x)$.
14. Sketch a graph of $f(x)$ which contains the point $(0,0)$.

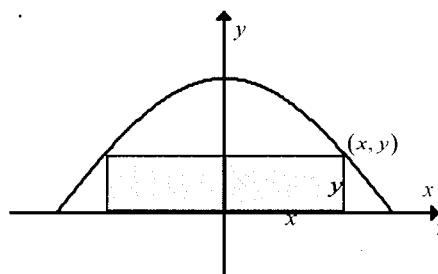


15. Find the absolute extrema of the function $f(x) = -x^2 + 3x$ on the interval $[-1, 3]$.
16. The second of two positive numbers is the reciprocal of the first. Find the two numbers so that their sum is a minimum.
17. The function $f(x) = \frac{4000}{1+100e^{-0.5x}}$ can be used to model how a disease spreads in an isolated population of 4000 people. x represents the time in days since the sickness started and $f(x)$ represents the number of people who have become sick.
- Use a calculator to help answer the questions below.
- How many people have become sick by the tenth day?
 - How fast was the disease spreading on the tenth day?
 - Find the maximum point on f' . What does the x -coordinate represent? What does the y -coordinate represent?
 - How many people have caught the disease when the curve is the steepest?
 - Why would the slope of the curve decrease after a period of time?
 - When was the rate of the spread of the disease increasing the fastest?

18. A rancher plans to fence in three sides of a rectangular pasture with the fourth side being against a rock cliff. If he has 1200 yards of fencing to use, what is the maximum area he can enclose?
-
19. Use the intercepts, vertical asymptotes, relative extrema, and end behavior to graph f , if $f(x) = \frac{x+1}{x^2 - 4x + 4}$ and $f'(x) = \frac{-x-4}{(x-2)^3}$. Do not use a calculator.
20. Find all points of inflection of $f(x) = \frac{1}{4}x^4 - 2x^3 + 2x - 6$.
21. Find all relative extrema points on the graph of $y = \frac{x}{x^3 + 4x}$.
22. Find $\lim_{x \rightarrow \infty} \frac{2x^3 + 5}{10x^3 + 6x - 3}$.
23. Find the c -value guaranteed by the Mean Value Theorem for the function $y = -3x^3 - 2x^2 + 3x - 5$ on the interval $[-2, 0]$. You may use a calculator.
- Use the graph of a velocity function for an object moving horizontally shown at the right for problems 24-29.
24. Find the object's acceleration at time 5 seconds.
25. Find the speed of the object at time 6 seconds.
26. On which interval of time is the object moving right?
27. On which interval(s) of time is the object's velocity increasing?
28. On which interval(s) of time is the object's speed increasing?
29. At what time is the object farthest right?



30. A rectangle is positioned with two points on the curve $y = \cos x$ as shown. Find the point (x, y) so that the area of the rectangle is a maximum. You may use a calculator but only to solve an equation.
-



Selected Answers:

- 1a. $y = -3x + 6$ b. $f(1.1) \approx 2.7$ c. $f(1.1) = 2.669$ 2b. $y = 2x - 1$
 3. $\sqrt{26} \approx 5.1$ 4. $f(3.1) \approx 29.1$ 6. $y \approx 22.05$ 7a. $A \approx 65$ sq. in.
 8. $f(8.01) \approx 2.000$ or 2.001 9. $y \approx 1.984$ or 1.985 11a. $(2\frac{1}{6})^3 \approx 10$ b. $(2\frac{1}{6})^3 = 10.171$
 12a. $s(t) \approx 60t + 240$ c. mile 150 16. Both numbers are 1.
 18. The maximum area is 180,000 yd^2 .
 19. V.A.: $x = 2$ (even), x -int.: $(-1, 0)$ (odd), y -int.: $(0, \frac{1}{4})$, Rel. Min.: $(-4, -\frac{1}{12})$, H.A: $y = 0$
 23. $c = -1.190$ or -1.191 24. $a(5) = -3 \frac{ft}{sec^2}$ 25. $speed(6) = 3 \frac{ft}{sec}$
 27. $[0, 2]$ sec, $(6, 7]$ sec 28. $[0, 2]$ sec, $(5, 6)$ sec 29. 5 seconds 30. $(.860, .652)$

Lesson 3-6 Antiderivatives

Warm-up Examples: Differentiate each of the following.

1. $f(x) = x^3$ 2. $f(x) = x^3 - 10$ 3. $f(x) = x^3 + C$
 where C is any constant (number)

So what should you get when you antidifferentiate (integrate) $f'(x) = 3x^2$? $f(x) = \underline{\hspace{2cm}}$

This problem can be written as $\int 3x^2 dx =$

The symbol \int is called an integral symbol and tells you to integrate (antidifferentiate) the expression which follows it. That expression is called an integrand. dx indicates that you are integrating with respect to the variable x but does not affect the integration process. C is called the constant of integration and must be written as part of your answer when you are antidifferentiating.

Integration Rules:

Power Rule: $\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$

Constant Rule: $\int k dx = kx + C \quad (\text{if } k \text{ is any constant})$

Scalar Multiple Rule: $\int k f(x) dx = k \int f(x) dx \quad (\text{if } k \text{ is any constant})$
 (Constants may be “factored out” of the integral expression.
NEVER “factor out” a variable.)

Sum Rule: $\int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx$

Trig Rules: $\int \cos x dx =$ $\int \sin x dx =$
 $\int \sec^2 x dx =$ $\int \csc^2 x dx =$
 $\int \sec x \tan x dx =$ $\int \csc x \cot x dx =$

Examples: Evaluate (Integrate).

4. $\int x^3 dx$ 5. $\int 2 dx$ 6. $\int (t^4 + 2) dt$

7. $\int (2y^2 + 4y + 1) dy$ 8. $\int \left(\frac{3}{x^2} - \frac{1}{\sqrt{x}} \right) dx$ 9. $\int \frac{\sqrt{x} + 1}{x^2} dx$

10. $\int (2 \cos \theta - 3 \sin \theta) d\theta$ 11. $\int \frac{\cos x}{\sin^2 x} dx$

Sometimes an initial condition is given which makes it possible to solve for C .

Example 12: If $f'(x) = x^{-3}$ and $f(1) = \frac{3}{2}$, find $f(x)$.

Remember: Pos. \rightarrow Vel. \rightarrow Acc. (Differentiate), so Acc. \rightarrow Vel. \rightarrow Pos. (Integrate).

Example 13: The acceleration of a particle at time t is given by $a(t) = 4t - 3$, $v(1) = 6$ and $s(2) = 5$.

a. Find the velocity equation. $v(t) =$

b. Find the position equation. $s(t) =$

Example 14: Given that on earth, the acceleration of an object due to gravity is approximately -32 ft/sec^2 (negative indicates downward), develop

a. the equation for velocity of the object. (v_0 = initial velocity)

$$v(t) =$$

b. the equation for position of the object. (s_0 = initial position)

$$s(t) =$$

Note: The two equations $v(t) = -32t + v_0$ and $s(t) = -16t^2 + v_0 t + s_0$ may be used for any motion affected only by the earth's gravity.

Assignment 3-6

For Problems 1-4, rewrite the integrand and then integrate.

1. $\int \frac{1}{x^3} dx$

2. $\int \sqrt[4]{t} dt$

3. $\int (x+1)(x-2) dx$

4. $\int \frac{2y}{\sqrt{y}} dy$

Evaluate (integrate) each integral in Problems 5-17.

$$\begin{array}{lll} 5. \int (2x^3 - x^2 + 1) dx & 6. \int \frac{1}{3x^2} dx & 7. \int \frac{1}{(3x)^2} dx \\ 9. \int \left(5x^{\frac{1}{4}} - x^{-\frac{2}{3}}\right) dx & 10. \int (3t - 10)^2 dt & 11. \int (2 \sin x + 3 \cos x) dx \\ 13. \int \frac{8x^4 - 2x^2 + 1}{2x^2} dx & 14. \int \frac{2\sqrt{t} - 1}{\sqrt{t}} dt & 15. \int \sqrt{y}(y^2 + 2\sqrt{y}) dy \\ 17. \int \sec x(\sec x + \tan x) dx \end{array}$$

18. If $f'(x) = 3x^2 - 4x + 2$ and $f(1) = -3$, find $f(x)$.

19. The derivative of a function is $\frac{dy}{dt} = \frac{-3}{t^2} + 1$. If the graph of the function contains the point $(3, 10)$, find the equation of the function.

20. a. Find an equation for the family of functions whose derivative is $y' = 3\sqrt{x}$.
 b. Find the particular function from the family in Part a. whose curve passes through the point $(4, 0)$.

21. Find $g(x)$, given that: $g''(x) = 2x - 3$, $g'(0) = -5$, and $g(-1) = 2$.

22. Find $f(x)$, given that: $f''(x) = \sin x$, $f'(0) = 2$, and $f(0) = 2$.

23. Evaluate $\frac{d}{dx} \int (2x - 1)^3 dx$. Hint: This is a derivative of an integral.

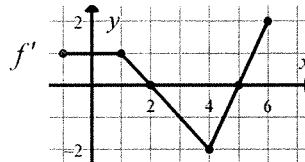
24. Find $f(x)$, given that: $f'(x) = \begin{cases} 2x+8, & x \leq 2 \\ 3x^2, & x > 2 \end{cases}$, $f(1) = 1$.

25. The position of a particle moving on the x -axis is given by $x(t) = t^4 - 4t^3 + 2$.
 a. Find the velocity and acceleration equations for the particle.
 b. Find when the particle is moving left.
 c. Find the velocity when the acceleration is zero.

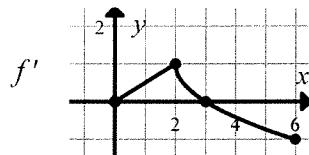
26. The acceleration of an object moving along a horizontal path is given by the equation $a(t) = 6t - 4$. The object's initial velocity is 5, and its initial position is -2.
 a. Find a velocity equation for the object.
 b. Find the velocity of the object when $t = 2$.
 c. Find a position equation for the object.
 d. Find the object's position when $t = 2$.

27. The velocity of an object moving along a vertical path is given by the equation $v(t) = \sqrt{t} + 1$, $t \geq 0$.
- Find an equation for the object's acceleration.
 - Find the acceleration of the object when $t = 9$.
 - The object's position at $t = 9$ is 20. Find an equation for the object's position.
28. A ball is dropped from a bridge which is 160 feet above a river. How long will it take the ball to hit the water?
29. A ball is thrown straight upward from a height of 6 feet with a velocity of 64 feet per second. How high will the ball go?
30. For the first 4 seconds of a race, a sprinter accelerates at a rate of 3 meters per second per second (3 m/sec^2). He then continues to run at the constant speed that he has attained for the rest of the race.
- Write a piecewise function to express the sprinter's velocity $v(t)$ as a function of time.
 - Find $v(2)$, $v(4)$, and $v(6)$.
 - Write a piecewise function to express the sprinter's position $s(t)$ as a function of time.
 - How far does the sprinter run during the first 4 seconds of the race?
 - How long will it take the sprinter to run 100 m?

31. Use the graph of f' shown to graph f'' and a possible graph of f .

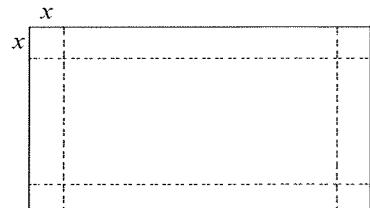


32. Use the graph of f' shown to graph f'' and a graph of f with the starting point $(0, 2)$.



33. Find the cubic function of the form $y = ax^3 + bx^2 + cx + d$ which has a relative maximum point at $(0, 2)$ and a point of inflection at $(-1, -2)$.

34. A box is to be made by cutting small squares from each corner of a 3 ft by 5 ft rectangular piece of material. Find the size of the square cutouts that would produce a box with maximum volume. (Your $V' = 0$ equation will not be factorable. You may use a calculator to solve it.) Show three or more decimal place accuracy.



35. Find the volume of the box in Problem 34. Show 3 or more decimal place accuracy.

36. If $y = \frac{x}{x+5}$, find the equation of the tangent line when $x = -6$ and use it to approximate the y -coordinate when $x = -6\frac{1}{5}$.

37. Without using a calculator, find the domain, the intercepts, the vertical asymptote, the end behavior, the relative extrema, and the points of inflection. Then sketch a graph of $f(x) = \frac{x^3 + 2}{x}$. Hint: $f'(x) = \frac{2x^3 - 2}{x^2}$ and $f''(x) = \frac{2x^3 + 4}{x^3}$.

38. Without using a calculator, find the intercepts, the local extrema, and the points of inflection, and then draw a graph of $f(x) = |-x^3 + 3x^2|$.

39. The point $(1, -2)$ is on the graph of $y^2 - x^2 + 2x = 5$.

a. Find the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(1, -2)$.

b. Does the graph have a local maximum, a local minimum, or neither at this point.

Selected Answers:

1. $-\frac{1}{2x^2} + C$
2. $\frac{4}{5}t^{\frac{5}{4}} + C$
3. $\frac{x^3}{3} - \frac{x^2}{2} - 2x + C$
4. $\frac{4}{3}y^{\frac{3}{2}} + C$
6. $-\frac{1}{3x} + C$
7. $-\frac{1}{9x} + C$
10. $3t^3 - 30t^2 + 100t + C$
11. $-2\cos x + 3\sin x + C$
13. $\frac{4}{3}x^3 - x - \frac{1}{2}x^{-1} + C$
14. $2t - 2t^{\frac{1}{2}} + C$
16. $-\cot \theta - \theta^2 + C$
17. $\tan x + \sec x + C$
18. $f(x) = x^3 - 2x^2 + 2x - 4$
19. $y = \frac{3}{t} + t + 6$
- 20a. $y = 2x^{\frac{3}{2}} + C$
- b. $y = 2x^{\frac{3}{2}} - 16$
21. $g(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 - 5x - \frac{7}{6}$
22. $f(x) = -\sin x + 3x + 2$
24. $f(x) = \begin{cases} x^2 + 8x - 8, & x \leq 2 \\ x^3 + 4, & x > 2 \end{cases}$
- 25a. $v(t) = 4t^3 - 12t^2$, $a(t) = 12t^2 - 24t$
- b. $t < 0, 0 < t < 3$
- c. $v(0) = 0, v(2) = -16$
- 26a. $v(t) = 3t^2 - 4t + 5$
- c. $s(t) = t^3 - 2t^2 + 5t - 2$
- 27c. $s(t) = \frac{2}{3}t^{\frac{3}{2}} + t - 7$
28. $t = \sqrt{10}$ sec
29. 70 feet
- 30a. $v(t) = \begin{cases} 3t, & 0 \leq t \leq 4 \\ 12, & t > 4 \end{cases}$
- c. $s(t) = \begin{cases} \frac{3}{2}t^2, & 0 \leq t \leq 4 \\ 12t - 24, & t > 4 \end{cases}$
- 30e. 10.333 sec
33. $y = -2x^3 - 6x^2 + 2$
34. The sides of the squares are .606 ft or .607 ft.
35. $V = 4.104 \text{ ft}^3$
36. $y \approx 5$
37. Do: $x \neq 0$, V. A: $x = 0$ (odd), x-int.: $(-\sqrt[3]{2}, 0)$, no y-int.,
- E. B: like $y = x^2$, Rel. Min: $(1, 3)$, no Rel. Max, P. I: $(-\sqrt[3]{2}, 0)$
- 39a. $\frac{dy}{dx}(1, -2) = 0$ and $\frac{d^2y}{dx^2}(1, -2) = -\frac{1}{2}$
- b. local maximum by the second derivative test

CALCULUS EXTENDED UNIT 3 SUMMARY

Points of Inflection and Concavity:

Find possible points of inflection (where f'' is zero or undefined), and make an f'' number line. Remember, domain restrictions cannot be points of inflection but must be on all number lines.

Second Derivative Test for Relative Extrema:

This test does not require a second derivative number line. It does not find points of inflection. It is used to find relative extrema (max/min).

Procedure:

1. Use f' to find critical numbers.
2. Plug critical numbers into f'' and analyze concavity to determine if a relative min. or max.

Curve Sketching:

Precalculus: domain, intercepts, vert. asymptotes, holes, end behavior, symmetry

Calculus: f' number line \rightarrow inc./decr. and max./min. f'' number line \rightarrow concavity, pts. of infl.

Graph to Graph:

- $f' \rightarrow f''$ Find slopes on f' and plot them as points on f'' .
- $f' \rightarrow f$ Make an f' number line using the **location** of points on the f' graph.
- Make an f'' number line using the **slope** at points on the f' graph.
- Use both number lines to sketch a graph of f .

Max/Min Applications:

Procedure:

1. Choose variables and/or draw a labeled figure.
2. Write a primary equation. Isolate whatever is to be maximized or minimized.
3. Rewrite with only one variable on each side. This may require a secondary equation.
4. Find the domain.
5. Take the derivative, find critical numbers, make a number line, etc.

Approximations using a tangent line (sometimes called a local linearization approximation):

Find the equation of a tangent line at a convenient point. Plug in a new x -value to find a new y -value on the tangent line which is close to a y -value on the curve.

Integration Rules:

$$\text{Power Rule: } \int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\text{Trig Rules: } \int \cos x dx = \sin x + C \quad \int \sin x dx = -\cos x + C$$

$$\int \sec^2 x dx = \tan x + C \quad \int \csc^2 x dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C \quad \int \csc x \cot x dx = -\csc x + C$$

Remember: Pos. \rightarrow Vel. \rightarrow Acc. (Differentiate), so Acc. \rightarrow Vel. \rightarrow Pos. (Integrate)

Lesson 4-1 Reverse Chain Rule, u -substitution

Warm-up Examples:

Differentiate

$$1. \frac{d}{dx}(1+5x)^4 = \quad 2. \frac{d}{dx}\sin(1+5x) =$$

Now, integrate

$$3. \int 5(1+5x)^3 dx = \quad 4. \int 5\cos(1+5x) dx =$$

Note: You “hooked on” the derivative of the inside function in Examples 1 and 2, so you had to “unhook” the derivative of the inside function in Examples 3 and 4.

Each of the Integration Rules from the last lesson can now be generalized as Reverse Chain Rule integrals.

x form	u form (Reverse Chain Rule)
<u>Power Rule:</u> $\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$	$\int u^n u' dx = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
<u>Trig Rules:</u> $\int \cos x dx = \sin x + C$	$\int \cos u u' dx = \sin u + C$
$\int \sin x dx = -\cos x + C$	$\int \sin u u' dx = -\cos u + C$
$\int \sec^2 x dx = \tan x + C$	$\int \sec^2 u u' dx = \tan u + C$
$\int \csc^2 x dx = -\cot x + C$	$\int \csc^2 u u' dx = -\cot u + C$
$\int \sec x \tan x dx = \sec x + C$	$\int \sec u \tan u u' dx = \sec u + C$
$\int \csc x \cot x dx = -\csc x + C$	$\int \csc u \cot u u' dx = -\csc u + C$

Examples: Integrate.

$$5. \int (3x-1)^{10} dx \quad 6. \int (3t^2 + 2t)(t^3 + t^2) dt \quad 7. \int \frac{6x^2}{\sqrt{4x^3 - 5}} dx$$

$$8. \int (y^3 + 1)^2 dy \quad 9. \int \sin(4x) dx \quad 10. \int 3\theta \cos \theta^2 d\theta$$

$$11. \int \sin^2 x \cos x dx$$

u-Substitution

For more complicated integration problems, simple rules for integration might fail, and you may have to make some type of substitution to be able to integrate. In this course, a common substitution will be to let $u = \text{the radicand}(\sqrt{\text{radicand}})$ part of the expression and to change the variable throughout the integral before integrating. You should use this method of substitution (called u -substitution) only when simpler methods don't work. It should be your last resort.

Procedure for u -substitution: (for $\int __ dx$ problems requiring the method)

1. Let $u = \text{radicand}$ (part inside the $\sqrt{\quad}$ symbol).
2. Solve for x (in terms of u).
3. Differentiate the equation from Step 2.
4. Find dx .
5. Substitute u -expressions for x -expressions in the integral.
Note: Most often, $dx \neq du$. Don't forget to substitute for dx .
6. Integrate.
7. Substitute back, so that your final answer is again in terms of x .

Sometimes it is easier to do Step 3 before Step 2. These two steps are reversible.

Examples: Integrate.

12. $\int x\sqrt{x-1} dx$

13. $\int \frac{2x-1}{\sqrt{2x+3}} dx$

Assignment 4-1

Evaluate (integrate) in Problems 1-8.

1. $\int 3(3x-2)^5 dx$

2. $\int (5t-3)^8 dt$

3. $\int \frac{x^2}{\sqrt{4-x^3}} dx$

4. $\int \sqrt{y^2-3} y dy$

5. $\int \frac{5x^2}{(x^3+2)^6} dx$

6. $\int \frac{-3}{\sqrt{1-v}} dv$

7. $\int (2x^2-3x)^4 (4x-3) dx$

8. $\int \frac{x-1}{(2x^2-4x)^5} dx$

Evaluate (integrate) in Problems 9-20.

$$\begin{array}{lll}
 9. \int \frac{(\sqrt{t}-4)^{10}}{\sqrt{t}} dt & 10. \int \frac{1}{\sqrt[3]{5x}} dx & 11. \int (2u+1)^2 du \\
 13. \int \frac{x^2-3}{x^2} dx & 14. \int \frac{3x^2+x-2}{\sqrt{x}} dx & 15. \int \left(2 + \frac{1}{x}\right)^4 \frac{1}{x^2} dx \\
 17. \int \frac{\cos \frac{1}{x}}{x^2} dx & 18. \int \sec(2x-1)\tan(2x-1)dx & 19. \int \tan^5 \theta \sec^2 \theta d\theta \\
 & & 20. \int \frac{\csc^2 t}{\cot^4 t} dt
 \end{array}$$

Use u -substitution to evaluate in Problems 21-23.

$$\begin{array}{lll}
 21. \int 30x\sqrt{x+1} dx & 22. \int \frac{3x-5}{\sqrt{\frac{1}{2}x-1}} dx & 23. \int (5x-8)\sqrt{1-x} dx \\
 24. \text{If } f''(x) = x^{\frac{-4}{3}}, f'(8) = \frac{3}{2}, \text{ and } f(27) = 5, \text{ find } f(x). \\
 25. \text{The derivative of a function is } \frac{dy}{dx} = 6x\sqrt{x^2-3}. \text{ Find the function if } (2, 5) \text{ is a point on the graph of the function.} \\
 26. \text{If } f'(x) = \cos(3x) \text{ and } f\left(\frac{\pi}{6}\right) = 2, \text{ find } f(x). \\
 27. \text{Evaluate } \frac{d}{dx} \int (x^2 - 3)^4 dx. \\
 28. \text{The velocity of a particle moving along a vertical line is given by the equation } v(t) = \left(\frac{1}{3}t - 1\right)^2. \text{ The particle's position at time zero is 4.}
 \end{array}$$

- a. Find an equation for the particle's acceleration $a(t)$.
- b. Find an equation for the particle's position $y(t)$.
- c. At what time(s) is the particle at rest?
- d. At what time(s) is the particle moving upward?
- e. For what value(s) of t does the particle's speed equal the particle's velocity?
- f. Find the total distance traveled by the particle from $t = 3$ to $t = 9$.
- g. Find the interval(s) of time for which the speed of the particle is increasing

29. Find equations for the lines tangent to and normal to the graph of $y = \sqrt{3x-5}$ when $x = 2$.
30. Find the instantaneous rate of change for $f(t) = \frac{t}{t+1}$ when $t = 1$.
31. Find the average rate of change for $f(t) = \frac{t}{t+1}$ on $[0, 2]$.
32. Which of the rates of change from Problems 30 and 31 represents:
 - a. the slope of a secant line for the graph of $f(t)$?
 - b. the slope of a tangent line for the graph of $f(t)$?
33. Find the value of c in $[0, 2]$ such that $f'(c) =$ the average rate of change of $f(t) = \frac{t}{t+1}$ on $[0, 2]$. It is at this t -location that the slopes of what two lines are the same? (MVT).

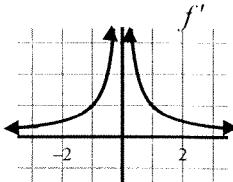
34. Differentiate $2x^2 + y^2 = 4y$ implicitly to find the point(s) where the curve has
 a. horizontal tangents. b. vertical tangents.

35. $(2, 7)$ is a point on the curve of $f(x) = x^3 - 3x + 5$. Use a tangent line to approximate $f(2.1)$.

36. The graph of $f'(x)$ is shown at right.

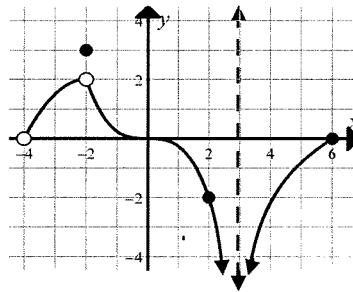
a. Use the given graph to make f' and f'' number lines.

b. Sketch a graph of f which passes through the points $(1, -1)$ and $(-1, 1)$.



37. Use the graph at right to find:

- | | |
|-------------------------------------|------------|
| a. $\lim_{x \rightarrow -2^-} f(x)$ | b. $f(-2)$ |
| c. $\lim_{x \rightarrow 2^+} f(x)$ | d. $f(2)$ |
| e. $\lim_{x \rightarrow 3^-} f(x)$ | f. $f(3)$ |
| g. $\lim_{x \rightarrow -4^+} f(x)$ | h. $f(-4)$ |



38. Use the alternate form of the limit definition of the derivative to find $f'(2)$ for $f(x) = x^2 + 1$.

Selected Answers:

1. $\frac{1}{6}(3x-2)^6 + C$
2. $\frac{1}{45}(5t-3)^9 + C$
3. $-\frac{2}{3}\sqrt{4-x^3} + C$
4. $-\frac{1}{3}(x^3+2)^{-5} + C$
5. $6\sqrt{1-v} + C$
6. $-\frac{1}{16}(2x^2-4x)^{-4} + C$
7. $\frac{2}{11}(\sqrt{t}-4)^{11} + C$
8. $\frac{3}{2\sqrt[3]{5}}x^{\frac{2}{3}} + C$ or $\frac{3}{10}(5x)^{\frac{2}{3}} + C$
9. $\frac{4}{3}u^3 + 2u^2 + u + C$ or $\frac{1}{6}(2u+1)^3 + C$
10. $x + \frac{3}{x} + C$
11. $\frac{6}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + C$
12. $-\frac{1}{5}(2 + \frac{1}{x})^5 + C$
13. $-\cos(\pi\theta) + C$
14. $\frac{1}{2}\sec(2x-1) + C$
15. $\frac{1}{6}\tan^6 \theta + C$
16. $12(x+1)^{\frac{5}{2}} - 20(x+1)^{\frac{3}{2}} + C$
17. $2\left(\frac{1}{2}x-1\right)^{\frac{3}{2}} + 4\left(\frac{1}{2}x-1\right)^{\frac{1}{2}} + C$
18. $2(1-x)^{\frac{5}{2}} + 2(1-x)^{\frac{3}{2}} + C$
19. $-\frac{9}{2}x^{\frac{2}{3}} + 3x - \frac{71}{2}$
20. $y = 2(x^2-3)^{\frac{3}{2}} + 3$
21. $f(x) = \frac{1}{3}\sin(3x) + \frac{5}{3}$
22. $a(t) = \frac{2}{9}t - \frac{2}{3}$
23. $y(t) = \left(\frac{1}{3}t-1\right)^3 + 5$
24. e. at all times
25. f. 8
26. g. $\frac{1}{4}$
27. h. $\frac{1}{3}$
28. i. $c = -1 + \sqrt{3} = .732$ Tangent and secant lines have the same slope.
29. j. $(0, 0), (0, 4)$
30. k. $(\pm\sqrt{2}, 2)$
31. l. $f(2.1) \approx 7.9$
32. m. $f'(2) = 4$

Lesson 4-2 Definite Integrals, Calculator Integration, The Fundamental Theorem of Calculus

A Definite Integral is written with upper and lower limits attached to an integration expression..

The value of a definite integral $\left(\int_a^b f(x) dx \right)$ may be thought of as a “signed area” from the lower limit a (usually a left side boundary) to the upper limit b (usually a right-side boundary), and between the curve of $f(x)$ and the x -axis. The value may be positive, negative, or zero.

Unlike the previous integration process which produced an indefinite integral (an antiderivative) representing a family of curves, a definite integral represents **a number value**.

Calculator Integration: A TI-84 calculator can be used to find the value of a definite integral from a to b by using $\int f(x) dx$ in the calculate menu or fnInt in the math menu. The calculate menu shows a graphical representation of the “signed area” together with the value of the definite integral.

Examples:

Use the calculate menu to evaluate the following definite integrals.

$$1. \int_{-3}^1 (x^3 - 6x) dx \quad 2. \int_{-\sqrt{6}}^{\sqrt{6}} (x^3 - 6x) dx \quad 3. \int_{-5}^5 |x^3 - 6x| dx$$

The math menu only provides the value of the definite integral, but that is usually all that we need. The math menu gives a more accurate answer. fnInt is recommended for all problems from now on.

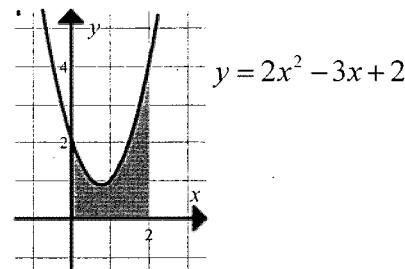
Note: Newer operating systems have a MATHPRINT setting that simplifies this process.

Use the math menu to evaluate:

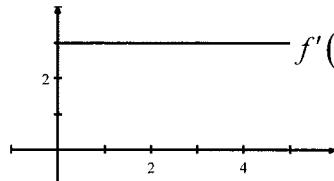
$$4. \int_{-5}^5 |x^3 - 6x| dx = \text{fnInt}(\text{abs}(x^3 - 6x), x, -5, 5)$$

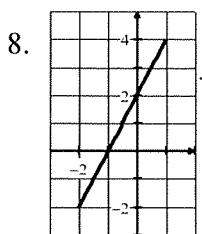
$$5. \text{ Use the idea of “signed area” to evaluate } \int_0^3 |2x - 1| dx \text{ without using a calculator.}$$

6. Set up a definite integral which could be used to find the area of the region bounded by the graph of $y = 2x^2 - 3x + 2$ (shown at right), the x -axis, and the vertical lines $x = 0$ and $x = 2$.



Discovering the Fundamental Theorem of Calculus

7.  a. Find $\int_0^5 f'(x) dx$ b. Write an equation for $f'(x)$ on $[0, 5]$.
c. Find $f(5)$ if $f(1) = 4$. d. Find $f(5) - f(0)$



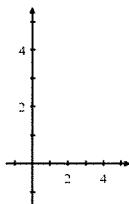
8. a. Find $\int_{-2}^1 f'(x) dx$
 b. Write an equation for $f'(x)$ on $[-2, 1]$.
 c. Find $f(x)$ if $f(1) = 0$.
 d. Find $f(1) - f(-2)$

9. Given $x(t) = -\frac{1}{2}t^2 + 4t - 3$ is the position equation for an object moving on the x -axis.

- a. Find the displacement of the object on the interval $[1, 4]$. b. Find the velocity equation.

$$v(t) =$$

- c. Sketch a graph of $v(t)$.



- d. Find $\int_1^4 v(t) dt$

Notice for each of these, the answers to parts a and d are the same.

If f' is a continuous function on $[a, b]$, then $\int_a^b f'(x) dx = f(x) \Big|_a^b = f(b) - f(a)$

This relationship is known as the **Fundamental Theorem of Calculus**.

Evaluate using the Fundamental Theorem of Calculus without using a calculator.

$$10. \int_0^4 (2\sqrt{y} + 1) dy \quad 11. \int_0^1 (4t + 1)^5 dt \quad 12. \int_1^5 \frac{x}{\sqrt{2x-1}} dx$$

$$13. \int_0^{\frac{\pi}{2}} \cos(2x) dx$$

START PLUS ACCUMULATION METHOD

Since $\int_a^b f'(x) dx = f(b) - f(a)$, it follows that $f(b) = f(a) + \int_a^b f'(x) dx$. This means a

function value at an endpoint can be found as a starting value plus a definite integral.

Examples:

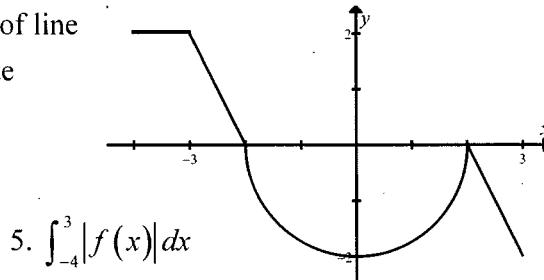
14. If $f'(x) = 3x^2 + 3$ and $f(0) = 4$,
 find $f(2)$ without a calculator.

15. If an object's velocity is $v(t) = 2^t$
 and $s(2) = 8$ find $s(3)$.

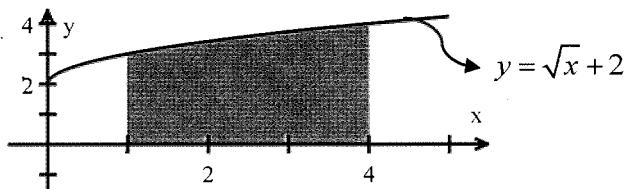
Assignment 4-2

The graph of the function $y = f(x)$ consists of line segments and a semicircle as shown. Evaluate the following using geometry formulas.

1. $\int_{-4}^{-2} f(x) dx$
2. $\int_0^2 f(x) dx$
3. $\int_{-4}^3 f(x) dx$
4. $\int_{-2}^2 (f(x) + 2) dx$



5. $\int_{-4}^3 |f(x)| dx$
6. Show an integral setup and evaluate to find the area shaded in the figure shown without using a calculator.



For Problems 7 and 8 sketch a graph for each function, and use the idea of "signed areas" to evaluate these definite integrals using geometry formulas without using a calculator.

- | | |
|--------------------------|----------------------------|
| 7. $f(x) = x - 1$ | 8. $g(x) = 2x + 3$ |
| a. $\int_{-2}^2 f(x) dx$ | b. $\int_{-2}^2 f(x) dx$ |
| a. $\int_{-2}^0 g(x) dx$ | b. $\int_{-2}^0 g(x) dx$ |

Evaluate the definite integrals in Problems 9-22 without using a calculator.

- | | | | |
|--|---|---|---|
| 9. $\int_0^2 (1-2y) dy$ | 10. $\int_0^1 (t^2 - 1)^4 t dt$ | 11. $\int_0^1 x(4x-3)^2 dx$ | 12. $\int_{-2}^{-1} \left(\frac{1-3x^4}{x^2}\right) dx$ |
| 13. $\int_1^4 \frac{2\sqrt{x}-1}{\sqrt{x}} dx$ | 14. $\int_1^8 \left(u^{\frac{2}{3}} + u^{-\frac{1}{3}}\right) du$ | 15. $\int_1^2 \frac{dx}{2\sqrt{3x-2}}$ | 16. $\int_0^\pi \sin x dx$ |
| 17. $\int_0^\pi \cos x dx$ | 18. $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sec^2 \theta d\theta$ | 19. $\int_0^{\frac{\pi}{2}} \sin\left(\frac{2x}{3}\right) dx$ | 20. $\int_0^\pi (3 \sin x + \sin(2x)) dx$ |
| *21. $\int_0^3 \frac{2x}{\sqrt{x+1}} dx$ | *22. $\int_1^2 x \sqrt[3]{x-1} dx$ | ** 23. $\int_0^4 x^2 - 4 dx$ | |

*(Hint: Problems 21 and 22 require u -substitution.)

**(Hint: For Problem 23 sketch a graph and split the integral into two integrals without absolute value.)

Use your calculator to evaluate the definite integrals in Problems 24-26. Express answers to 3 or more decimal place accuracy.

- | | | |
|----------------------------------|----------------------------------|--|
| 24. $\int_1^{12} \frac{1}{x} dx$ | 25. $\int_0^6 \sqrt{y^3 + 1} dy$ | 26. $\int_0^{\frac{\pi}{3}} \sin^2 \theta d\theta$ |
|----------------------------------|----------------------------------|--|

27. Given $f'(x) = |8 \sin(10x)|$,

a. use a calculator to find $\int_0^4 f'(x) dx$. b. if $f(0) = 3$, find $f(4)$.

28. If an object's acceleration is $a(t) = \cos t^2$ and $v(1) = 5 \frac{ft}{sec}$, find $v(3)$.

29. If $f(x)$ is an even function (graph symmetric to the y -axis) and $\int_0^3 f(x) dx = 3$, find

a. $\int_{-3}^0 f(x) dx$ b. $\int_{-3}^0 -f(x) dx$ c. $\int_{-3}^3 f(x) dx$ d. $\int_3^0 f(x) dx$

30. If $g(x)$ is an odd function (graph symmetric to the origin) and $\int_0^3 g(x) dx = -5$,

find (if possible – one cannot be found).

a. $\int_{-3}^0 g(x) dx$ b. $\int_{-3}^3 g(x) dx$ c. $\int_{-3}^3 |g(x)| dx$
 d. $\int_3^3 g(x) dx$ e. $\int_{-3}^0 5g(x) dx$ f. $\int_0^1 g(x) dx + \int_1^3 g(x) dx$

31. For the functions f and g from Problems 29 and 30, find

a. $\int_0^3 (f(x) + g(x)) dx$ b. $\int_{-3}^0 2(g(x) - f(x)) dx$ c. $\int_k^k (f(x) - g(x)) dx$

32. $x(t) = t^3 - 3t^2 + 1$ represents the position equation for a particle moving along the x -axis.

- | | |
|--|--|
| a. Find the velocity equation. | b. Find the acceleration equation. |
| c. Find the velocity at $t = 1$. | d. Find the speed at $t = 1$. |
| e. When is the particle's velocity decreasing? | f. Find the displacement on $[1, 4]$. |
| g. Find the total distance traveled from $t = 1$ to $t = 4$. (Show a velocity number line). | |
| h. Find $\int_1^4 v(t) dt$ without using a calculator. Compare your answer to Part f. | |
| i. Use your calculator to find $\int_1^4 v(t) dt$. Compare your answer to Part g. | |

*You now have two ways to find displacement and total distance. Using definite integrals, **displacement** = $\int_a^b v(t) dt$ and **total distance** = $\int_a^b |v(t)| dt$ on the interval $[a, b]$. Given a choice of methods, always do total distance by evaluating a definite integral on your calculator.

33. Find the area between $f(x) = 2^x$ and the x -axis on the interval $[1, 3]$. Show an integral set up, and evaluate using a calculator.

34. $f(x) = \frac{x}{x^2 + 1}$, $f'(x) = \frac{1 - x^2}{(x^2 + 1)^2}$, and $f''(x) = \frac{2x(x^2 - 3)}{(x^2 + 1)^3}$.

- a. Without using a calculator, list the domain, asymptotes, and intercepts for $f(x)$.
- b. Find the relative extrema points of $f(x)$.
- c. Find the points of inflection of $f(x)$.
- d. Sketch $f(x)$ without using a calculator.

Selected Answers:

$$\begin{array}{llllll}
 1. 3 & 2. -\pi & 4. 8-2\pi & 5. 4+2\pi & 6. \boxed{\left(\frac{16}{3}+8\right)-\left(\frac{2}{3}+2\right)} = \frac{32}{3} & 7a. -4 \quad b. 5 \\
 \\
 8a. 2 & 9. -2 & 10. \frac{1}{10} & 11. \frac{1}{2} & 12. \boxed{(1+1)-\left(\frac{1}{2}+8\right)} = -\frac{13}{2} & \\
 \\
 14. \boxed{\left(\frac{3}{5}\bullet 32 + \frac{3}{2}\bullet 4\right)-\left(\frac{3}{5}+\frac{3}{2}\right)} = \frac{231}{10} & 15. \frac{1}{3} & 16. 2 & 17. 0 & 18. 2\sqrt{3} & 19. \frac{3}{4} \\
 \\
 20. 6 & 21. \boxed{\left(\frac{4}{3}\bullet 8-8\right)-\left(\frac{4}{3}-4\right)} = \frac{16}{3} & 22. \boxed{\frac{3}{7}+\frac{3}{4}} = \frac{33}{28} & & & \\
 \\
 23. \boxed{\left(\frac{-8}{3}+8\right)+\left(\frac{64}{3}-16\right)-\left(\frac{8}{3}-8\right)} = 16 & 24. 2.484 \text{ or } 2.485 & & & & \\
 \\
 27a. 20.533 \text{ or } 20.534 & b. 23.533 \text{ or } 23.534 & 29a. 3 & c. 6 & 30a. 5 & b. 0 & 31a. -2 \\
 32c. -3 & f. 18 & g. 22 & 33. 8.656 & 34a. Do.: all reals, V.A.: none, H.A.: $y=0$, & & \\
 34a. x -int.: $(0,0)$ odd, y -int.: $(0,0)$ & b. rel. min. at $\left(-1, \frac{-1}{2}\right)$, rel. max. at $\left(1, \frac{1}{2}\right)$ & & & & & \\
 34c. P.I. at $(0,0)$, $\left(-\sqrt{3}, \frac{-\sqrt{3}}{4}\right)$, $\left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$ & & & & & &
 \end{array}$$

Lesson 4-3 The Second FTC, Rate Graphs, Average Value of a Function

The following examples serve as an informal guide toward discovering the Second Fundamental Theorem of Calculus.

1. $\int_0^x (2t-3) dt =$

2. $\frac{d}{dx} \int_0^x (2t-3) dt =$

3. $\int_{10}^x f'(t) dt =$

4. $\frac{d}{dx} \int_{10}^x f'(t) dt =$

Second Fundamental Theorem of Calculus:

For any constant a , $\frac{d}{dx} \int_a^x f(t) dt = f(x)$ (if f is continuous from a to x).

Now find:

5. $\int_0^{x^2} f'(t) dt =$

6. $\frac{d}{dx} \int_0^{x^2} f'(t) dt =$

7. $\int_{x^3}^{2x} f'(t) dt =$

8. $\frac{d}{dx} \int_{x^3}^{2x} f'(t) dt =$

Second Fundamental Theorem (Chain Rule Version):

If u and v are functions of x , then $\frac{d}{dx} \int_u^v f(t) dt = f(v)v' - f(u)u'$

(if f is continuous from u to v). Note the “hook-ons” v' and u' .

Examples: Find each of the following without integrating.

9. $\frac{d}{dx} \int_x^0 (2t - 3) dt =$

10. $\frac{d}{dx} \int_2^5 (2t - 3) dt =$

11. $\frac{d}{dx} \int_{-1}^{x^3} (t^2 + 2t) dt =$

12. $\frac{d}{dx} \int_{2x}^{3x} (t^2 + 2t) dt =$

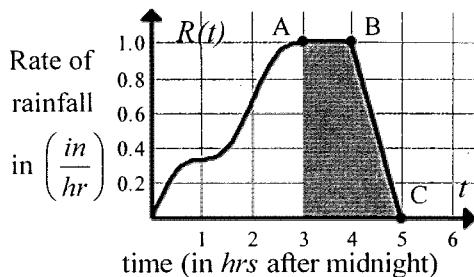
13. $\frac{d}{dx} \int_{f(x)}^{g(x)} (2t - 3) dt =$

14. If $f(x) = \int_0^{3x^2} (1 - t^2)^{10} dt$, then $f'(x) =$

Interpretation of “Rate” Graphs:

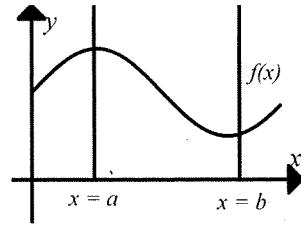
Examples:

The graph at right models the rate of rainfall in inches per hour from midnight until 6:00 A.M. during a tropical rainstorm.



15. Write a complete sentence to explain what Point A on the graph represents. Include numbers and units in your answer.
16. What is the slope of the graph between Points A and B?
17. Write a complete sentence to explain the meaning of your answer to Example 16. Include numbers and units in your answer.
18. What is the slope of the graph between Points B and C?
19. Write a complete sentence to explain the meaning of your answer to Example 18. Include numbers and units in your answer.
20. Find $\int_3^5 R(t) dt$.
21. Write a complete sentence to explain the meaning of your answer to Example 20. Include numbers and units in your answer.
22. Approximate the value of $\int_0^6 R(t) dt$ using geometrical regions. Show computations.
23. Write a complete sentence to explain the meaning of your answer to Example 22. Include numbers and units in your answer.

Discovering the Formula for the Average Value of a Function.
The average value of a function represents its average "height."



Average Value of a Function:

Average value of $f(x)$ on $[a, b] = f(c) =$

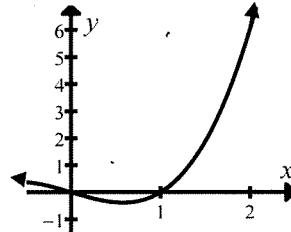
$$f_{\text{avg}} = \frac{\int_a^b f(x) dx}{b-a}$$

"area" of region
"width" of region

or $\frac{1}{b-a} \int_a^b f(x) dx$

$f(x)$ must be continuous on $[a, b]$.

Example 24: Find the average value of $f(x) = x^3 - x$ on the interval $[0, 2]$ without using a calculator.



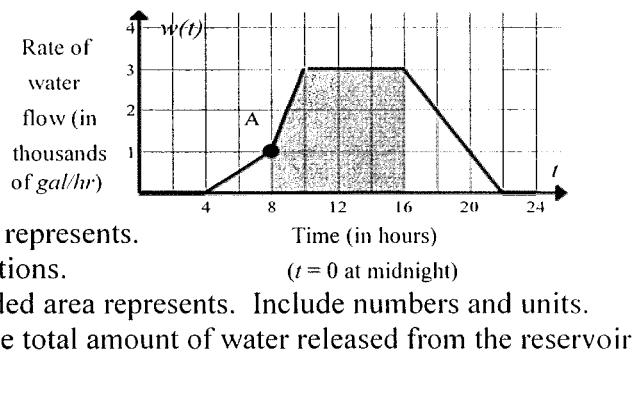
Example 25: Use your calculator to find the value of c in the interval $[0, 2]$ where $f(c) =$ the average value found in Example 24.

Assignment 4-3

Use the Second Fundamental Theorem of Calculus to evaluate in Problems 1-7.

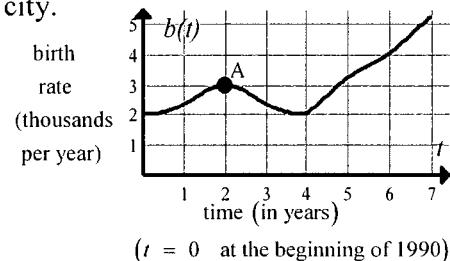
1. $\frac{d}{dx} \int_{-3}^x (t^2 - t + 1) dt$
2. $\frac{d}{dx} \int_2^x \sqrt[3]{t^2 - 5t} dt$
3. $\frac{d}{dt} \int_6^t \frac{2}{x-5} dx, t > 6$
4. $\frac{d}{dt} \int_0^{2t} \sin x dx$
5. $\frac{d}{dx} \int_x^3 (1-t)^4 dt$
6. $\frac{d}{dx} \int_{x^2}^3 (1-t)^4 dt$
7. $\frac{d}{dt} \int_{3t}^{4t} 2^x dx$
8. $\int_1^a t(3t^2 - 1)^5 dt$

9. The graph at the right represents the rate of flow of irrigation water (in thousands of gallons/hour) from a reservoir during a 24 hour period.



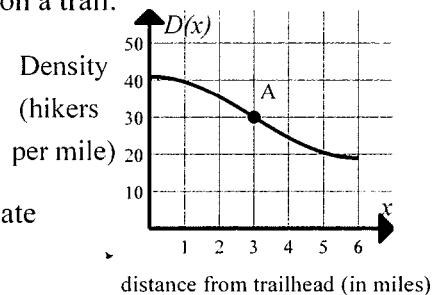
10. The graph at the right models the birth rate in a Utah city.

- a. Tell what Point A represents.
- b. Approximate $\int_2^6 b(t) dt$ using geometrical shapes. Show computations.
- c. Tell what your answer to Part b. represents.
- d. When was the birth rate the lowest, and what was the birth rate at that time?



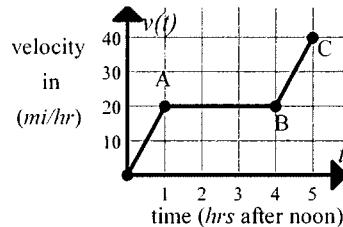
11. The graph at the right represents the density of hikers on a trail.

- a. Tell what Point A represents.
- b. $\int_0^3 D(x) dx = 110$. Write a sentence with numbers and units stating what this represents.
- c. If $D'(3) = -6$, use local linearization to approximate the density of hikers 3.1 miles from the trailhead.



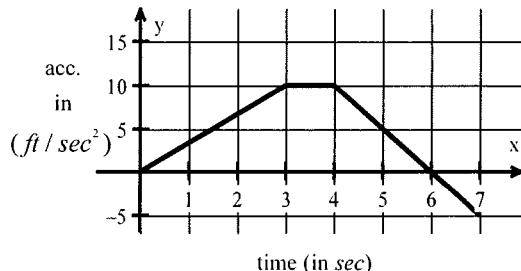
12. The graph at right models velocity.

- a. Tell what Point A represents.
- b. Tell what the slope between B and C represents.
- c. What is the velocity at 12:15? at 5:00?
- d. What is the acceleration at 12:30? at 2:00?
- e. What distance is traveled between noon and 5:00?



13. The graph at right models acceleration

- a. What is the acceleration at $t = 2 \text{ sec}$?
- b. When is the acceleration 10 ft/sec^2 ?
- c. What is the minimum acceleration?
- d. If the initial velocity is zero, what is the velocity at $t = 6 \text{ sec}$?
- e. If the initial velocity is 20 ft/sec , what is the velocity at $t = 6 \text{ sec}$?
- f. If the initial velocity is 20 ft/sec , what is the velocity at $t = 7 \text{ sec}$?



For Problems 14-18, find the average value of each function on the given interval.

14. No calculator.

$$f(x) = x^3 \text{ on } [0, 2]$$

15. No calculator.

$$g(t) = \frac{1}{(t-1)^2} \text{ on } [2, 5]$$

16. No calculator.

$$f(y) = 2y - \sqrt{y} \text{ on } [1, 4]$$

17. Use a calculator.

$$f(t) = \frac{t^2 - 1}{\sqrt{t} + 1} \text{ on } [4, 3.2]$$

18. Use a calculator.

$$h(x) = .5^x \text{ on } [-2, 1]$$

19. Find the exact x -value where the function in Problem 14 equals its average value.

20. Use a calculator to solve for c for the equation $h(c) = h_{avg}$ in Problem 18. Your answer should be expressed to 3 or more decimal place accuracy.

Differentiate in Problems 21-23.

$$21. y = (3x^2 - 5)^8$$

$$22. f(x) = x^2 \sqrt[4]{x^3 - 5}$$

$$23. g(t) = \frac{t^2 - 1}{t^3 + 3}$$

24. Use the limit definition of the derivative to find $f'(x)$, if $f(x) = x^2 - x$.

$$25. \text{ For } f(x) = \begin{cases} 2x^2 - x + 1, & x \leq 1 \\ x^3 + 1, & x > 1 \end{cases} \text{ find:}$$

- a. $f(1)$ b. $\lim_{x \rightarrow 1^-} f(x)$ c. $\lim_{x \rightarrow 1^+} f(x)$ d. $f'(x)$ e. $f'(1)$ if it exists

26. Is the function in Problem 25 continuous on $(-\infty, \infty)$? Why or why not?

27. Is the function in Problem 25 differentiable on $(-\infty, \infty)$? Why or why not?

28. Find equations for lines tangent to the graph of the function in Problem 25 when

- a. $x = 1$ b. $x = 0$ c. $x = 2$

29. Suppose $\int_0^3 f(x) dx = 5$, $\int_3^5 f(x) dx = -2$, and $\int_0^3 g(x) dx = -1$. Find:

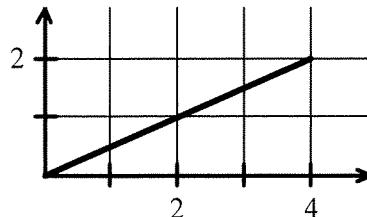
- a. $\int_0^3 (f(x) + g(x)) dx$ b. $\int_0^3 3g(x) dx$ c. $\int_0^3 (2g(x) - 5f(x)) dx$
 d. $\int_0^5 f(x) dx$ e. $\int_5^0 2f(x) dx$ f. $\int_3^3 g(x) dx$

30. For functions f and g in Problem 29, suppose f is even and g is odd. Find:

- a. $\int_{-3}^0 f(x) dx$ b. $\int_{-3}^0 g(x) dx$ c. $\int_{-3}^3 -f(x) dx$ d. $\int_{-3}^3 |g(x)| dx$
 if $x = 0$ is the only zero of $g(x)$

31. Evaluate $\int_1^5 x^2 \sqrt{x-1} dx$ without a calculator.

32. Find $f(4)$ if $f(2) = 3$ and the graph of $y = f'(x)$ is shown.



Selected Answers:

1. $x^2 - x + 1$ 4. $2\sin(2t)$ 6. $-2x(1-x^2)^4$ 8. $\frac{1}{36}(3a^2-1)^6 - \frac{1}{36} \cdot 2^6$

9a. At 8:00 AM water is flowing from the reservoir at the rate of 1000 $\frac{\text{gal}}{\text{hr}}$.9b. 22000 gal e. 33000 gal 10b. approximately 11000 c. Approximately 11,000 babies were born in the Utah city during 1992, 1993, 1994, and 1995. 11c. $D(3.1) \approx 29.4 \frac{\text{hikers}}{\text{mile}}$

12c. $v(.25) = 5 \frac{\text{mi}}{\text{hr}}$, $v(5) = 40 \frac{\text{mi}}{\text{hr}}$ e. 100 miles 13a. $\frac{20}{3} \frac{\text{ft}}{\text{sec}^2}$ d. $35 \frac{\text{ft}}{\text{sec}}$ e. $55 \frac{\text{ft}}{\text{sec}}$ f. $52.5 \frac{\text{ft}}{\text{sec}}$

14. 2 15. $\frac{1}{4}$ 16. $\frac{1}{3} \left(\left(16 - \frac{16}{3} \right) - \left(1 - \frac{2}{3} \right) \right) = \frac{31}{9}$ 17. 1.105 18. 1.683 19. $x = \sqrt[3]{2}$

20. c = -751 22. $f'(x) = x^2 \cdot \frac{1}{4} (x^3 - 5)^{-\frac{3}{4}} \cdot 3x^2 + (x^3 - 5)^{-\frac{1}{4}} \cdot 2x$ 24. $2x - 1$

28a. $y - 2 = 3(x - 1)$ b. $y - 1 = -1(x - 0)$ c. $y - 9 = 12(x - 2)$ 29c..-27 f. 0

30c. -10 31. $\frac{2}{7} \cdot 2^7 + \frac{4}{5} \cdot 2^5 + \frac{2}{3} \cdot 8 = \frac{7088}{105}$ 32. 6

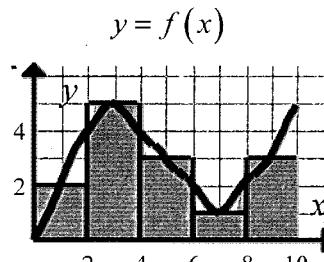
Lesson 4-4 Approximation Using Riemann Sums and Trapezoids

Some functions cannot be integrated, and sometimes you are given data or a graph – but not an actual function. It is still possible to approximate “areas.” One method of approximating a definite integral is to add areas of rectangles. This is called a **Riemann Sum**.

Example 1:

Approximate $\int_0^{10} f(x) dx$ by adding the areas of the five rectangles shown.

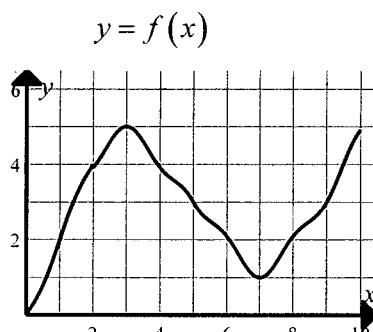
This is a Midpoint Riemann Sum.



x	$f(x)$
0	0
1	2
2	4
3	5
4	4
5	3
6	2
7	1
8	2
9	3
10	5

Example 2:

Approximate $\int_0^{10} f(x) dx$ by using 5 rectangles of equal width ($n = 5$) and an Upper (Circumscribed) Riemann Sum. Draw rectangles on the figure.

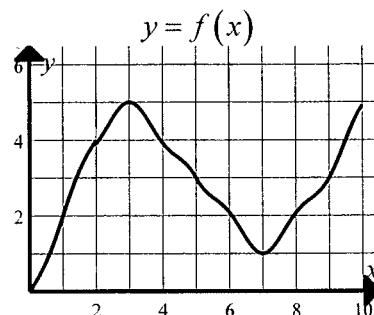


Example 3:

Approximate $\int_0^{10} f(x) dx$ by using a

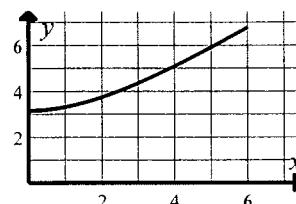
Lower (Inscribed) Riemann Sum with $n = 5$.

Draw rectangles.

Example 4:

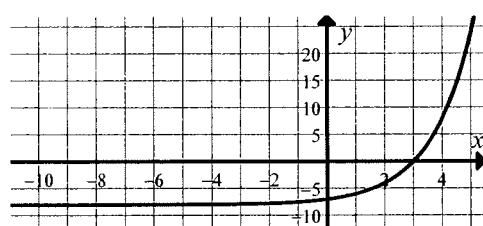
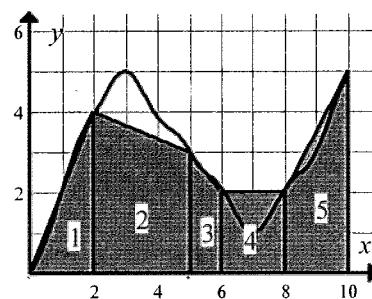
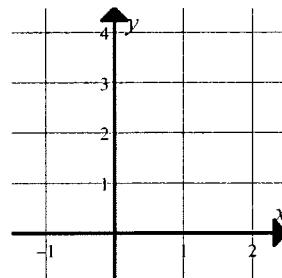
Approximate $\int_0^6 \sqrt{x^2 + 10} dx$ using a

Midpoint Riemann Sum with 3 equal subdivisions.

Example 5: Approximate $\int_{-10}^5 (2^x - 8) dx$

by using five Right hand rectangles whose widths are determined by the intervals separating the following x values:

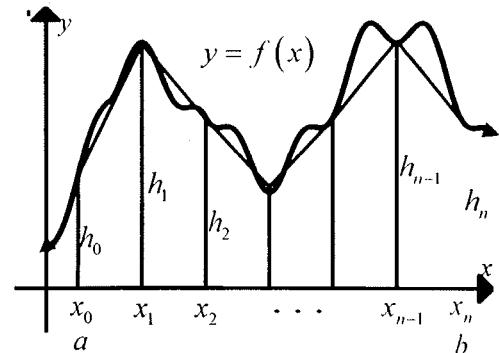
$$x = -10, x = -4, x = 0, x = 2, x = 3, \text{ and } x = 5.$$

Example 6: Approximate $\int_0^{10} f(x) dx$ by adding the areas of the 5 "trapezoids" shown in the graph at the right.Example 7: Use four trapezoids of equal width to find an approximation for the area under the curve $y = x^2$, bounded below by the x -axis, on the left by $x = 0$, and on the right by $x = 2$. Sketch the curve and draw the trapezoids first.

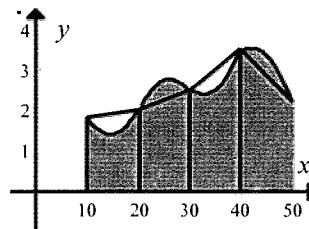
When using trapezoids for approximations, if the subdivisions produce equal widths (see the figure at the right), then

$$\int_a^b f(x) dx \approx \frac{1}{2} w(h_0 + 2h_1 + 2h_2 + \dots + 2h_{n-1} + h_n)$$

This formula is called the **Trapezoidal Rule**.



Example 8: Use the Trapezoidal Rule with four equal subdivisions to approximate the area shown.

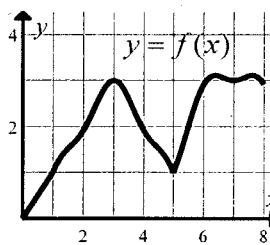


x	y
10	1.8
20	2
30	2.5
40	3.5
50	2.2

Assignment 4-4

Show set ups on all Riemann Sum and Trapezoidal Rule problems.

1. Use the graph of $y = f(x)$ at right and the table shown to approximate $\int_0^8 f(x) dx$ using
 - a Midpoint Riemann Sum with 4 equal subdivisions.
 - a Left hand Riemann Sum with 8 equal subdivisions.
 - the Trapezoidal Rule with $n = 8$.



x	f(x)
0	0
1	1
2	2
3	3
4	2
5	1
6	3
7	3
8	3

2. Use the data in the table below to approximate the area between the graph of $f(t)$ and the t -axis, from $t = 1$ to $t = 13$, using a Midpoint Riemann Sum with 6 rectangles of equal width. Plot the data, sketch a graph, and draw rectangles first.

t	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$f(t)$	0	5	8	11	12	15	17	18	15	13	12	9	6	4

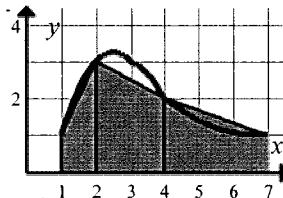
3. Use the data in the table at right to approximate $\int_2^4 f(x) dx$ using a Midpoint Riemann Sum with two equal subintervals.

x	2.0	2.5	3.0	3.5	4.0
y	3	2	4	3	5

For Problems 4-7, use the indicated type of Riemann Sum to approximate the value of each definite integral. Draw an accurate sketch (without using a calculator if possible), and show your rectangles for each problem.

4. $\int_2^8 \frac{1}{x} dx \approx$ Right hand Riemann Sum with 3 equal subdivisions.
 5. $\int_0^{16} (\sqrt{x+1}) dx \approx$ Midpoint Riemann Sum with 4 rectangles of equal width.
 6. $\int_0^4 \left(\frac{1}{2}x^2 - 2x + 4\right) dx \approx$ Lower (Inscribed) Riemann Sum with 4 equal subdivisions.
 7. $\int_0^4 \left(\frac{1}{2}x^2 - 2x + 4\right) dx \approx$ Upper (Circumscribed) Riemann Sum with 4 equal subdivisions.

8. Use the trapezoids shown to approximate $\int_1^7 f(x) dx$.

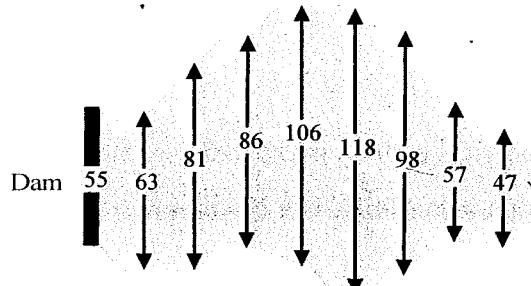


9. Approximate $\int_{-3}^5 \left(\sqrt[3]{x} - \frac{1}{2}\right) dx$ by using 4 regions whose widths are determined by the intervals separating $x = -3, x = 0, x = 1, x = 2$, and $x = 5$ using the following methods.
 a. a Riemann Sum with Left hand rectangles. b. trapezoids.

10. The points shown are from a continuous function f . Use the points in the table and the Trapezoidal Rule with four equal subdivisions to approximate $\int_2^4 f(x) dx$.

x	y
2.00	4.12
2.25	3.76
2.50	3.21
2.75	3.58
3.00	3.94
3.25	4.15
3.50	4.69
3.75	5.44
4.00	7.52

11. A man-made lake has the shape illustrated in the figure with measurements 20 feet apart. Use the Trapezoidal Rule to estimate the surface area of the lake.

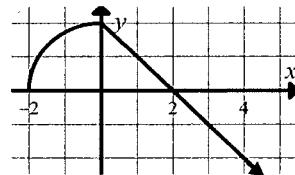


12. Use the Trapezoidal Rule with four equal subdivisions to approximate the value of $\int_0^4 \sqrt{x^2 + 1} dx$. Draw an accurate sketch and show your trapezoids.

Use a calculator for Problems 13,14.

13. For $f(x) = \ln x$, find $f''(2)$
 14. Find the area between $f(x) = e^x$ and the x -axis on the interval $[0, 2]$.
 First show an integral set up.

Use the graph of $y = f(x)$ at right to graph each of the functions in Problems 15-17. Graph each problem on a separate coordinate plane.



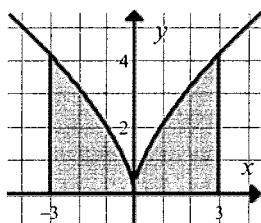
15. $y = -f(x) + 1$ 16. $y = |f(x + 1)|$ 17. $y = 2f(-x)$

Evaluate the expressions in Problems 18-21.

18. $\frac{d}{dx} \int_0^x \sqrt{t} dt$ 19. $\frac{d}{dx} \int_{2x}^0 \sqrt{t} dt$ 20. $\frac{d}{dx} \int_0^{x^2} \sqrt{t} dt$ 21. $\int_0^x \sqrt{t} dt$

For Problems 22,23, set up definite integrals which could be used to find the areas of the regions shown or described. You do not need to evaluate the integrals that you set up.

22. $f(x) = 2x^{\frac{2}{3}}$
 on $[-3, 3]$



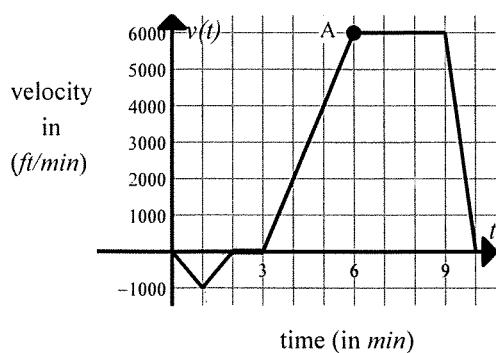
23. Region bounded by
 $g(y) = y^2 + 1$,
 $x = 0$, $y = -2$, and
 $y = 3$

Without a calculator, sketch graphs and use geometry to evaluate Problems 24 and 25.

24. $\int_0^4 |3x - 2| dx$ 25. $\int_{-2}^1 (2 - |x|) dx$

26. A spherical balloon is expanding at the rate of $5 \text{ cm}^3/\text{sec}$. How fast is the diameter of the balloon increasing when its volume is $36\pi \text{ cm}^3$? $\left(V = \frac{4}{3}\pi r^3\right)$

27. The “rate graph” at right represents the velocity of a car during a 10 minute factory test drive along a straight path.
- On what interval(s) of time was the car moving backward (reverse)? forward? at rest?
 - Write a sentence telling what Point A represents.
 - Find the speed of the car at $t = 1$, $t = 2$, and $t = 4$ min.
 - Find the acceleration of the car on the time interval $(3, 6)$.
 - On what time interval(s) is the car’s acceleration the greatest?
 - On what time interval(s) is the absolute value of the car’s acceleration the greatest?
 - On what time interval(s) was the car speeding up (increasing in speed)?
 - $\int_0^{10} v(t) dt$
 - $\int_0^{10} |v(t)| dt$
 - Write a sentence telling what $\int_0^{10} v(t) dt$ represents. Include numbers and units.
 - Write a sentence telling what $\int_0^{10} |v(t)| dt$ represents. Include numbers and units.
 - If the initial position of the car is 5000 feet, what is the position of the car when $t = 10$ min.
28. The product of two positive numbers is 80. Find the numbers so that the sum of the first number and five times the second number is a minimum.
29. If $\int_0^2 f(x) dx = 4$ and f is an odd function, find:
- $\int_{-2}^0 f(x) dx$
 - $\int_{-2}^2 f(x) dx$
 - $\int_0^2 -f(x) dx$
 - $\int_0^2 3f(x) dx$
 - $\int_0^2 (f(x)+1) dx$ (Be careful on this one.)


Selected Answers:

- 1a. $2(1+3+1+3) = 16$ b. $1(0+1+2+3+2+1+3+3) = 15$
- 1c. $\frac{1}{2} \cdot 1(0+2 \cdot 1+2 \cdot 2+2 \cdot 3+2 \cdot 2+2 \cdot 1+2 \cdot 3+2 \cdot 3+3) = 16.5$
2. $2(8+12+17+15+12+6) = 140$ 4. $2\left(\frac{1}{4} + \frac{1}{6} + \frac{1}{8}\right) = \frac{13}{12}$
5. $4(\sqrt{3} + \sqrt{7} + \sqrt{11} + \sqrt{15}) = 46.269$ or 46.270 6. $1(2.5+2+2+2.5) = 9$ 8. $2+5+\frac{9}{2} = \frac{23}{2}$
- 9a. $3\left(\sqrt[3]{-3} - \frac{1}{2}\right) + 1\left(\sqrt[3]{0} - \frac{1}{2}\right) + 1\left(\sqrt[3]{1} - \frac{1}{2}\right) + 3\left(\sqrt[3]{2} - \frac{1}{2}\right) = 3\left(\sqrt[3]{-3} - \frac{1}{2}\right) + 3\left(\sqrt[3]{2} - \frac{1}{2}\right)$

More Selected Answers:

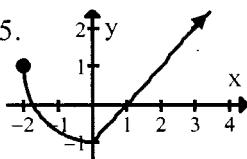
10. $\left[\frac{1}{2} \cdot \frac{1}{2} (4.12 + 2 \cdot 3.21 + 2 \cdot 3.94 + 2 \cdot 4.69 + 7.52) \right] = 8.83$

12. $\left[\frac{1}{2} \cdot 1 (1 + 2 \cdot \sqrt{2} + 2 \cdot \sqrt{5} + 2 \cdot \sqrt{10} + \sqrt{17}) \right] = 9.374$

13. -2.250

14. 6.389

15.



18. \sqrt{x}

19. $-2\sqrt{2x}$

21. $\frac{2}{3}x^{\frac{3}{2}}$

22. $\int_{-3}^3 2x^3 dx$ or $2 \int_0^3 2x^3 dx$

23. $\int_{-2}^3 (y^2 + 1) dy$

24. $\frac{52}{3}$

25. 3.5

26. $\frac{5}{18\pi} \frac{cm}{sec}$

- 27a. reverse $(0,2)$, forward $(3,10)$, at rest $[2,3]$ b. Six minutes after the start of the test
 drive the car's velocity was 6000 feet per minute. d. 2000 ft/min/min f. $(9,10)$ h. $29,000 \text{ ft}$
 i. $31,000 \text{ ft}$ l. $34,000 \text{ ft}$ 28. The first number is 20 and the second is 4. 29a. -4 d. 12

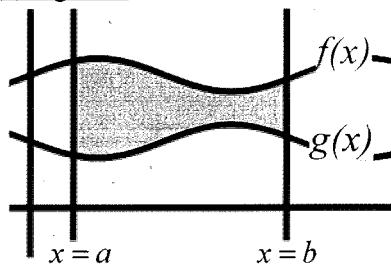
Lesson 4-5 Area between Curves

$\int_a^b f(x) dx$ produces a value ("signed area") which may be positive, negative, or zero.

However, if you are asked to find an actual area, that area cannot be negative.

Area of a Region Between Two Curves

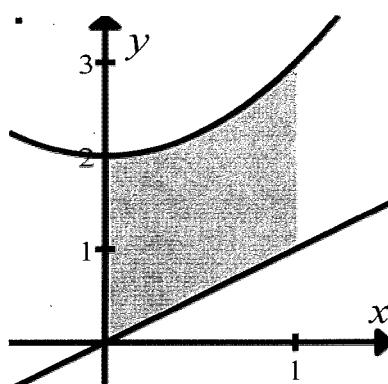
$A = \int_a^b (f(x) - g(x)) dx$ in the figure shown.



For functions of x , $A = \int_a^b (\text{top curve} - \text{bottom curve}) dx$.

For functions of y , $A = \int_a^b (\text{right curve} - \text{left curve}) dy$.

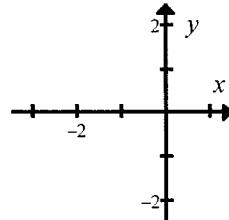
Example 1: Find the area of the region bounded by $y = x^2 + 2$, $y = x$, $x = 0$, and $x = 1$.



Sometimes you have to find where two curves intersect to determine “boundaries” for your region(s). These intersections will provide you with limits of integration for your integral(s). You must show an equation set up, even when using a calculator to find the intersections.

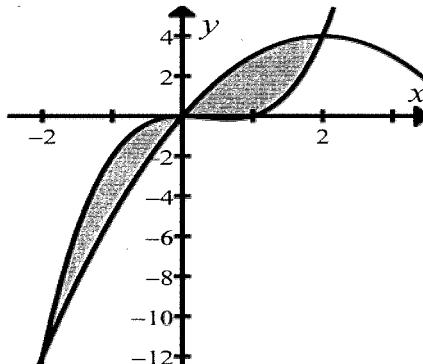
Example 2: (Functions of y)

Find the area of the region bounded by $x = y^2 - 3$ and $y = x + 1$.



Example 3: Find the total area of the regions located between the two curves as shown. You may use a calculator.

$$f(x) = x^3 - x^2 \text{ and } g(x) = -x^2 + 4.1x$$

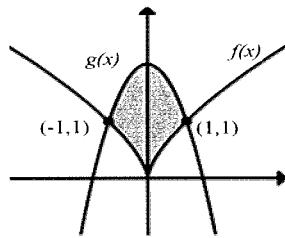


Example 4: Without using a calculator set up an integral for the area of one region between the graphs of $y = \sin \theta$ and $y = \cos \theta$

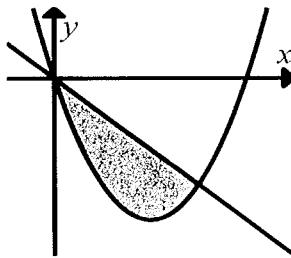
Assignment 4-5

For Problems 1 and 2, set up integrals that could be used to find the areas of the shaded regions. Do not integrate. Show the equation(s) used to find the limits of integration for Problem 2 without using a calculator.

1. $f(x) = x^{\frac{2}{3}}$
 $g(x) = -x^2 + 2$
 $A = \int_{-1}^1$ _____

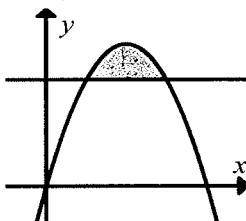


2. $f(x) = x^2 - 4x$
 $g(x) = -x$

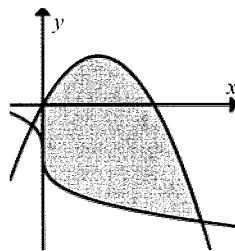


For Problems 3 and 4, show equations used to find the limits of integration, show integral set ups for the areas of the shaded regions, and then find the areas.

3. No calculator.
 $f(x) = 3$
 $g(x) = 4x - x^2$



4. Use a calculator.
 $f(x) = 2x - x^2$
 $g(x) = -\sqrt[3]{x} - 1$



For Problems 5 and 6, sketch regions bounded by the graphs of the given equations, show equations used to find the limits of integration, show integral set ups, and find the areas.

5. Use a calculator.

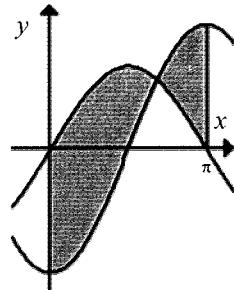
$$y = 3^t$$

$$y = \sqrt{t+2}$$

6. No calculator. Hint: Write as functions of y . (Isolate x)

$$x = y^2 - 3 \quad y = -\frac{1}{2}x$$

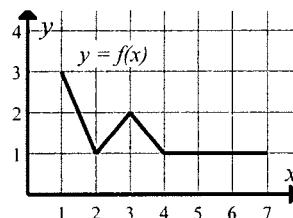
7. Without using absolute value, why can't the area of the region(s) bounded by $y = -4x$ and $y = -x^3$ be written as a single integral using -2 and 2 as limits of integration? (Sketch graphs of $y = -4x$ and $y = -x^3$ in one coordinate plane before you answer this question.) Then use the symmetry of the graphs to write a single integral for the area.
8. Without using a calculator, sketch a graph showing one of the regions bounded between $y = \cos x$ and $y = 2 - \cos x$. Then find the area of the region.



9. Use a calculator to find the shaded area between the curves $f(x) = 2 \sin x$ and $g(x) = -3 \cos x$ on $[0, \pi]$ as shown at right. Show an integral set up and express your final answer to 3 or more decimal place accuracy.

10. The graph of a function f is shown in the figure at right.

a. Evaluate $\int_1^7 f(x) dx$.



- b. Determine the average value of the function on the interval $[1, 7]$.

- c. Determine the answers to Parts a. and b. if the graph is shifted two units upward.

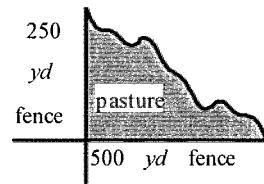
For Problems 11-12, sketch graphs and approximate the values of the intervals using

- a. a Midpoint Riemann Sum. b. the Trapezoidal Rule
where n' = the number of equal subdivisions.

11. $\int_0^8 x^3 dx$
 $n = 4$

12. $\int_0^{12} -\sqrt{x} dx$
 $n = 3$

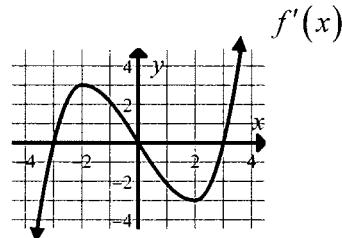
13. A pasture is bounded by a river and two fences as shown in the figure. Approximate the total square yards of pasture area using
a. a Midpoint Riemann sum using five equal subdivisions.
b. the Trapezoidal Rule where $n = 10$.



x	0	50	100	150	200	250	300	350	400	450	500
y	250	210	190	190	170	120	80	60	60	40	0

14. Two men in a search party begin walking from Search Headquarters at the same time. One man walks North at a rate of 4 ft/sec , while the other man walks West at a rate of 3 ft/sec . After both men have walked for one minute, find
a. the distance separating the two men.
b. the rate at which the distance between the two men is changing.

15. Use the graph of f' at right to sketch
a. a graph of f'' .
b. a graph of f which passes through
the origin of the coordinate plane.
Use a separate coordinate plane for each
graph.



16. Use the Trapezoidal Rule with four equal subdivisions to approximate the value of $\int_3^{11} \frac{2}{x-1} dx$.

Evaluate in Problems 17-19.

17. $\frac{d}{dx} \int_0^{2x} \sin t dt$

18. $\frac{d}{dx} \int_{x^2}^5 \sin t dt$

19. $\frac{d}{dt} \int_t^{2t} 2^x dx$

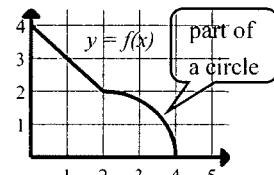
20. Find the value of this definite integral $\int_0^{\frac{\pi}{4}} \tan x dx$.

Evaluate the integrals in Problems 21-24 without using a calculator.

21. $\int_{-2}^2 |x^3| dx$ 22. $\int_1^3 \frac{1}{t^2} dt$ 23. $\int (x^2 - 2x)^5 (x-1) dx$ 24. $\int \sqrt{y} \left(1 - \frac{1}{y}\right) dy$

25. Use geometry to find $\int_0^4 f(x) dx$
for the function shown at right.

$$f(x) = \begin{cases} 4-x, & 0 \leq x \leq 2 \\ \sqrt{4x-x^2}, & 2 < x \leq 4 \end{cases}$$



26. Use the f' and f'' number lines below to sketch a possible graph of a continuous function f .

$$f' \leftarrow \begin{array}{c} - \\ + \\ 0 \\ 2 \end{array} \rightarrow$$

$$f'' \leftarrow \begin{array}{c} + \\ - \\ -2 \\ 0 \end{array} \rightarrow$$

Use a calculator for Problems 27 and 28.

27. Find the average (mean) value of $f(x) = \log x$ on $[1, 4]$.
 28. Find the x -value which produces the average value in Problem 27.

29. Use the acceleration graph at right to find the following for an object moving along a straight path.

- a. $a(1)$ b. $a(6)$ c. $a(9)$ d. $a(12)$

Suppose $v(0) = 2$ m/sec. Find:

- e. $v(3)$ f. $v(7)$ g. $v(10)$ h. $v(12)$

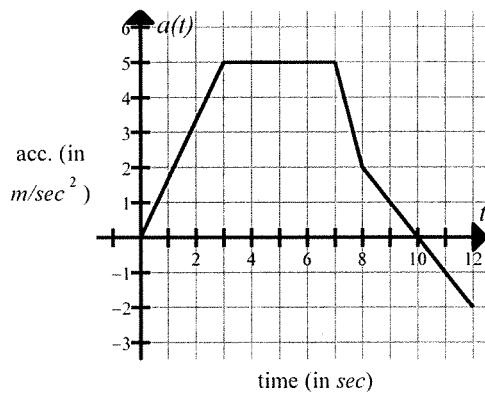
- i. At what time on $[0, 12]$ was the object moving the fastest?

Justify your answer.

- j. At what time on $[0, 12]$ was the object moving the slowest?

Justify your answer.

- k. When was the object slowing down (speed decreasing)?



Selected Answers:

4. 6.356 5. 1.167 or 1.168 6. $\boxed{\left(-1 - \frac{1}{3} + 3\right) - (-9 + 9 - 9)} = \frac{32}{3}$ 8. 4π 9. 7.211

10a. 8 b. $\frac{4}{3}$ 11b. $\boxed{\frac{1}{2} \cdot 2(0 + 2 \cdot 2^3 + 2 \cdot 4^3 + 2 \cdot 6^3 + 8^3)} = 1088$

12a. $\boxed{4(-\sqrt{2} - \sqrt{6} - \sqrt{10})} = -28.103 \text{ or } -28.104$

13b. $\boxed{\frac{1}{2} \cdot 50(250 + 2 \cdot 210 + 2 \cdot 190 + 2 \cdot 190 + 2 \cdot 170 + 2 \cdot 120 + 2 \cdot 80 + 2 \cdot 60 + 2 \cdot 60 + 2 \cdot 40 + 0)} = 62,250 \text{ sq. yds,}$

14a. 300 ft. b. $5 \frac{ft}{sec}$ 17. $2\sin(2x)$ 20. .346 or .347 21. 8 22. $\frac{2}{3}$

23. $\frac{1}{12}(x^2 - 2x)^6 + C$ 24. $\frac{2}{3}y^{\frac{3}{2}} - 2y^{\frac{1}{2}} + C$ 25. $6 + \pi$ 27. .368 28. $x = 2.335 \text{ or } 2.336$

29a. $\frac{5}{3} \frac{m}{sec^2}$ c. $1 \frac{m}{sec^2}$ e. $9.5 \frac{m}{sec}$ f. $29.5 \frac{m}{sec}$ i. 10 sec. j. 0 sec.

Lesson 4-6 Volumes of Solids with Known Cross Sections (including discs and washers)

Area formulas for common cross sections:

Square

Rectangle

Semicircle

Triangle

Equilateral Triangle

$$A = s^2$$

$$A = bh$$

$$A = \frac{1}{2} \pi r^2$$

$$A = \frac{1}{2} bh$$

$$A = \frac{\sqrt{3}}{4} s^2$$

Volumes of Solids with Known Cross Sections

For cross sections perpendicular to the x -axis:

$$V = \int_a^b A dx, \text{ where } A \text{ is a function of } x \text{ and gives the area of a representative cross section.}$$

For cross sections perpendicular to the y -axis:

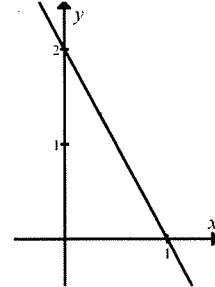
$$V = \int_a^b A dy, \text{ where } A \text{ is a function of } y.$$

Examples:

- Find the volume of the solid whose base is a triangle bounded by $y = -2x + 2$, $x = 0$, and $y = 0$, and whose cross sections are squares which are perpendicular to the x -axis.

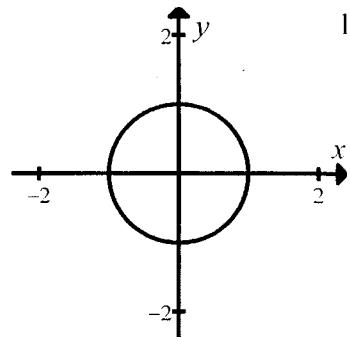
$$V = \int_a^b A dx$$

$$V = \int_0^1 s^2 dx =$$



- Set up (but do not integrate) integrals for the volumes of the solids with the same base as in Example 1, but whose cross sections are semicircles perpendicular to the x -axis.
- Set up (but do not integrate) an integral for the volume of a solid whose base is bounded by $y = -x^2 + 2$ and $y = x$ and whose cross sections are rectangles of height $\frac{1}{4}$ perpendicular to the x -axis.

4. Set up integrals for the volumes of the solids whose base is the circle $x^2 + y^2 = 1$ and whose cross sections are:
- Equilateral triangles perpendicular to the y -axis.

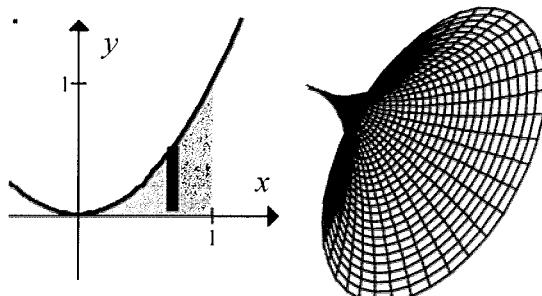


- Rectangles whose heights are three times their bases and whose bases are perpendicular to the y -axis.

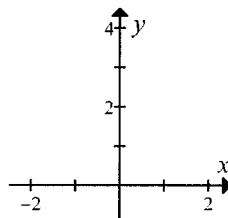
Circular cross sections can be formed by revolving very thin (essentially no width) rectangles about an axis of revolution. These circular cross sections are more commonly called discs.

Disc Formula:	$V = \pi \int_a^b r^2 \ (dx \text{ or } dy)$
----------------------	--

Example 5: Set up an integral for the volume of the solid formed by revolving the region bounded by $y = x^2$, $y = 0$, and $x = 1$ about the x -axis.



Example 6: Find the volume of the solid formed by revolving the region in Quadrant I bounded by $y = x^2$, $x = 0$, and $y = 4$ about the y -axis.



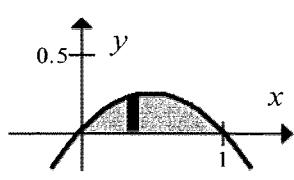
When a region is revolved about a line which is not one of its boundaries, its volume is formed from a sum of volumes of washers (at least some of the discs have holes in them).

Washer Formula: $V = \pi \int_a^b (R^2 - r^2) (dx \text{ or } dy)$

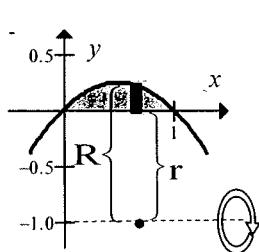
R = Outer radius (from the axis of revolution)
 r = Inner radius (from the axis of revolution)

Example 7: Set up integrals for the volumes of the solids formed by revolving the region bounded by $y = -x^2 + x$ and $y = 0$

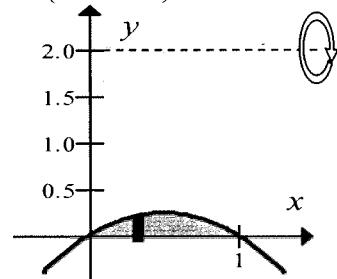
- a. about the x -axis
 (Disks)



- b. about $y = -1$
 (Washers)

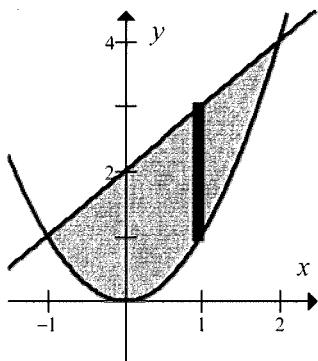


- c. about $y = 2$
 (Washers)

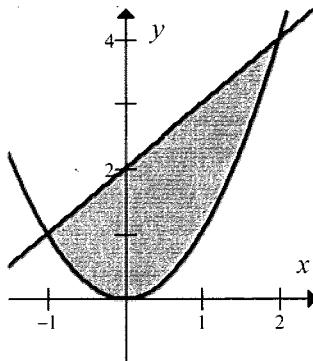


Example 8: Set up integrals for the volumes of the solids formed by revolving the region bounded by $y = x^2$ and $y = x + 2$

- a. about the x -axis



- b. about the line $y = 4$

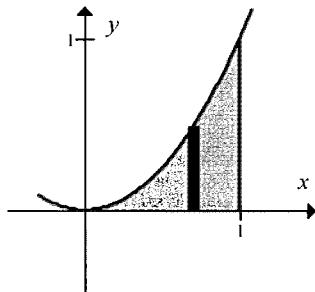


Remember: PerpenDiscular

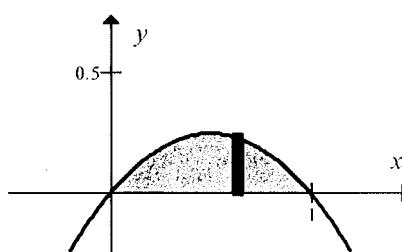
Assignment 4-6

Set up (but do not integrate) integrals for evaluating the volumes of the solids formed by the given cross sections.

1. Base region bounded by $y = x^2$,
 $y = 0$, and $x = 1$ as shown.



2. Base region bounded by $y = -x^2 + x$
and $y = 0$ as shown.



- a. Squares
b. Semicircles
c. Rectangles with height 2

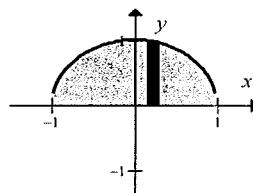
- a. Squares
b. Equilateral triangles
c. Rectangles whose heights are half of their bases

Sketch the graphs of the regions and set up the integrals used for finding the volumes.

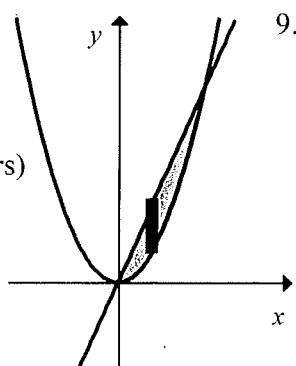
3. The base region is the circle $x^2 + y^2 = 4$ with square cross sections perpendicular to the y -axis.
4. The base region is bounded by $y = 2 - x$ and $y = x^2$ with right triangle cross sections perpendicular to the x -axis. The base of the triangle sits in the region and the height = 3 times the base.
5. Find the volume of a solid whose base is bounded by $y = \sqrt{x}$, $y = 0$, and $x = 4$, and whose cross sections are squares perpendicular to the x -axis.
6. Find the volume of a solid whose base is bounded by $y = 2x$, $y = 0$, and $x = 3$, and whose cross sections are semicircles perpendicular to the x -axis.

Use discs or washers to set up (but not integrate) integrals for finding volumes of the solids described.

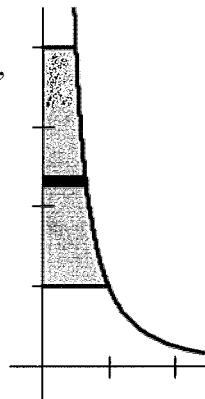
7. Region bounded by $y = \sqrt{1 - x^2}$ and $y = 0$
a. revolved about the x -axis
b. revolved about $y = -1$



8. Region bounded by
 $y = x^2$ and $y = 2x$
a. revolved about the x -axis (Washers)
b. revolved about $y = 5$ (Washers)



9. Region bounded by $y = \frac{1}{x^2}$,
 $x = 0$, $y = 1$, and $y = 4$
a. revolved about the y -axis (Discs)
b. revolved about $x = -1$ (Washers)



Set up (but do not integrate) integrals for finding the volumes of the solids described. Sketch each region and show at least one representative rectangle.

10. Region bounded by

$$y = \frac{1}{x}, y = 0, x = 1, \text{ and } x = 4$$

- a. revolved about the x -axis
- b. revolved about $y = -3$
- c. revolved about $y = 5$

11. Region bounded by

$$y = \sqrt{x}, x = 0, \text{ and } y = 2$$

- a. revolved about the y -axis
- b. revolved about $x = -2$
- c. revolved about $x = 4$

12. Without using a calculator, find the volume of the solid formed when the region bounded by

$$y = \sqrt[3]{x}, x = 0, \text{ and } y = 2$$

- a. the x -axis
- b. the y -axis

13. Sketch the region, and use a calculator to find the volumes of the solids formed when the region bounded by $f(x) = 2^{x-1}$ and $g(x) = 2 - x^2$ is revolved about:

- a. the x -axis
- b. $y = 3$

14. Sketch the region, and set up integrals for the volumes of the solids formed by the given cross sections. The base region is bounded by $f(x) = |x|, y = 0, x = -2, \text{ and } x = 3$.

- a. squares perp. to the x -axis
- b. semicircles perp. to the x -axis

Sketch the regions bounded by the given equations, and find the areas of the regions. Show your integral set ups, and do not use a calculator.

15. $f(x) = 2x^2 + 3x$ and $g(x) = 2$

16. $f(y) = y^2 - 2y$ and $g(y) = 2 - y$

Differentiate these functions.

17. $y = \frac{3x^3 - x + 1}{x^2}$

18. $f(x) = (x - \sqrt{x})^{10}$

19. $g(y) = y^3(2y - 1)^{\frac{2}{3}}$

20. $h(t) = \frac{2t - 1}{3t^2 + 5}$

Antidifferentiate.

21. $y' = \frac{(x-2)^2}{\sqrt{x}}$

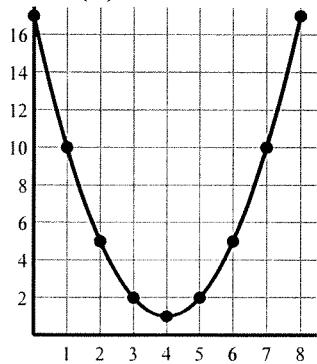
22. $g'(y) = \frac{4y}{\sqrt{y^2 + 3}}$

23. $h'(t) = \frac{t}{\sqrt{t+3}}$

Use Riemann Sums to approximate the values of the definite integrals.

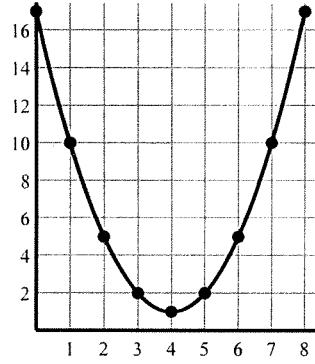
24. $\int_0^8 f(x) dx$. Midpoint R.S. ($n = 4$)

$$f(x) = x^2 - 8x + 17$$



25. $\int_0^8 f(x) dx$. Left hand R.S. ($n = 4$)

$$f(x) = x^2 - 8x + 17$$



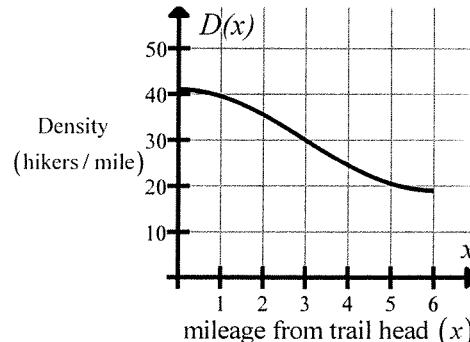
26. $\int_0^9 (x^2 - 8x + 17) dx \approx \underline{\hspace{2cm}}$. Use a Riemann Sum with 3 inscribed rectangles of equal width. (Draw a sketch.) Why is this such a poor approximation?

27. Find $\frac{d}{dx} \int_{x^3}^x \cos(2t+1) dt$ without integrating.

For Problems 28-32, use the graph and table at the right.

28. Show an integral set up for the average density of hikers between mile 1 and mile 5.
 29. Use a Midpoint Riemann Sum with $n = 2$ to approximate this same average density.
 30. Use four equal width trapezoids to approximate this average density on the same interval.
 31. Estimate where on the trail the density of hikers is equal to this same average density.
 32. Write a complete sentence using numbers and units to tell what $\int_1^5 D(x) dx$ represents.

x	1	2	3	4	5
D	40	35	30	25	20



33. If $f'(x) = \sin^2 x$ and $f(0) = 2$, find $f(3)$.

Selected Answers:

1a. $V = \int_0^1 (x^2)^2 dx$ b. $V = \int_0^1 \frac{1}{2} \pi \left(\frac{1}{2} x^2\right)^2 dx$ c. $V = \int_0^1 2x^2 dx$

2a. $V = \int_0^1 (-x^2 + x)^2 dx$ 3. $V = \int_{-2}^2 (2\sqrt{4-y^2})^2 dy$

5. 8 6. $\frac{9}{2}\pi$ 7a. $\pi \int_{-1}^1 (\sqrt{1-x^2})^2 dx$ b. $\pi \int_{-1}^1 \left((1+\sqrt{1-x^2})^2 - 1^2\right) dx$

More Selected Answers:

8a. $\pi \int_0^2 ((2x)^2 - (x^2)^2) dx$ b. $\pi \int_0^2 ((5-x^2)^2 - (5-2x)^2) dx$

9b. $\pi \int_1^4 \left(\left(1 + \frac{1}{\sqrt{y}} \right)^2 - 1^2 \right) dy$ 10b. $\pi \int_1^4 \left(\left(3 + \frac{1}{x} \right)^2 - 3^2 \right) dx$ c. $\pi \int_1^4 \left(5^2 - \left(5 - \frac{1}{x} \right)^2 \right) dx$

11a. $\pi \int_0^2 (y^2)^2 dy$ b. $\pi \int_0^2 ((2+y^2)^2 - 2^2) dy$ c. $\pi \int_0^2 (4^2 - (4-y^2)^2) dy$

12a. $\pi \left(32 - \frac{3}{5} \cdot 8^{\frac{5}{3}} \right)$ b. $\pi \frac{2^7}{7}$ 13a. 16.302 or 16.303 b. 28.688 or 28.689

14b. $\frac{1}{2} \pi \int_{-2}^3 \left(\frac{|x|}{2} \right)^2 dx$ 15. $\left[\left(1 - \frac{1}{12} - \frac{3}{8} \right) - \left(-4 + \frac{16}{3} - 6 \right) \right] = \frac{125}{24}$ 18. $f'(x) = 10(x - \sqrt{x})^9 \left(1 - \frac{1}{2}x^{-\frac{1}{2}} \right)$

19. $g'(y) = \frac{4}{3} y^3 (2y-1)^{-\frac{1}{3}} + 3y^2 (2y-1)^{\frac{2}{3}}$ 21. $y = \frac{2}{5}x^{\frac{5}{2}} - \frac{8}{3}x^{\frac{3}{2}} + 8x^{\frac{1}{2}} + C$

22. $g(y) = 4\sqrt{y^2+3} + C$ 23. $h(t) = \frac{2}{3}(\sqrt{t+3})^3 - 6\sqrt{t+3} + C$ 24. 48

26. 24, This is not accurate because only a small part of the area under the curve is covered by the three rectangles.

27. $\cos(2x+1) - 3x^2 \cos(2x^3+1)$ 28. $D_{avg} = \frac{\int_1^5 D(x) dx}{5-1}$ 29. $D_{avg} \approx 30 \frac{\text{hikers}}{\text{mile}}$

30. $D_{avg} \approx 30 \frac{\text{hikers}}{\text{mile}}$ 31. approx. 3 miles from the trailhead

32. $\int_1^5 D(x) dx$ represents the total number of hikers on the trail at a distance between 1 and 5 miles from the trailhead. 33. 3.569 or 3.570

CALCULUS EXTENDED UNIT 4 SUMMARY**Three Ways to Integrate** (so far):

1. Term by term.
2. Reverse Chain Rule
3. u -Substitution

Calculator Integration:

$$\int_a^b f(x) dx = \text{fnInt}(f(x), x, a, b)$$

Definite Integrals

$$\int_a^b f'(x) dx = f(x) \Big|_a^b = f(b) - f(a) \quad \text{or} \quad f(b) = f(a) + \int_a^b f'(x) dx$$

Second Fundamental Theorem of Calculus:

Basic form: $\frac{d}{dx} \int_a^x f(t) dt = f(x)$ Chain form: $\frac{d}{dx} \int_u^v f(t) dt = f(v)v' - f(u)u'$

where u and v are functions of x
Remember the hook-ons for this form.

Average Value of a Function: $f_{avg} = \frac{\int_a^b f(x) dx}{b-a}$ $\left(\text{avg. height} = \frac{\text{area}}{\text{width}} \right)$

Formula for Riemann Sums: (Rectangles) $\int_a^b f(x) dx \approx w(h_1 + h_2 + h_3 + \dots + h_n)$

Use formulas only when you have a common width. $w = \frac{b-a}{n}$

Trapezoidal Rule: $\int_a^b f(x) dx \approx \frac{1}{2} w(h_0 + 2h_1 + 2h_2 + \dots + 2h_{n-1} + h_n)$

a → v → s: (Integration)

Displacement: (By Integration) $Displacement = \int_a^b v(t) dt$

Total Distance: (By Integration) $Tot. Dist. = \int_a^b |v(t)| dt$

Area Between Two Curves: $A = \int_a^b (f(x) - g(x)) dx$ or $A = \int_a^b (f(y) - g(y)) dy$
 top curve – bottom curve right curve – left curve

Volumes with Known Cross Sections: $V = \int_a^b A dx$ or dy $A = \text{area of cross section}$

Area formulas for common cross sections:

Square	Rectangle	Semicircle	Triangle	Equilateral Triangle
$A = s^2$	$A = bh$	$A = \frac{1}{2} \pi r^2$	$A = \frac{1}{2} bh$	$A = \frac{\sqrt{3}}{4} s^2$

Volumes of Revolution:

(Discs): $V = \pi \int_a^b r^2 dx$ **(Washers):** $V = \pi \int_a^b (R^2 - r^2) dx$

Lesson 5-1 Shell Method Volume, Arc Length

Key: The representative element (rectangle) must be parallel to the axis of revolution. Recall that for discs or washers, the element had to be perpendicular to the axis of revolution..

Remember: PARASHELL vs PERPENDISCLAR

Revolving rectangular elements about a parallel axis produces cylindrical shells (like the wrappings around a toilet paper roll).

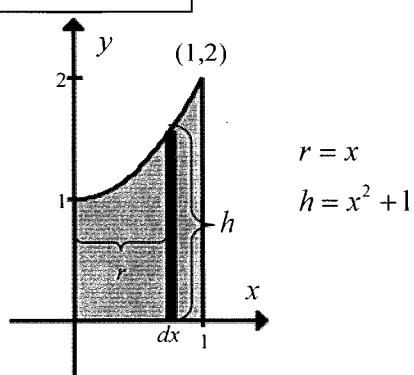
The volume formula for the shell method is:

$$V = 2\pi \int_a^b rh \, dx \text{ (or } dy\text{)} \quad r > 0, h > 0$$

Examples:

- Find the volume of the solid formed by revolving the region bounded by $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about the y -axis.

$$V =$$



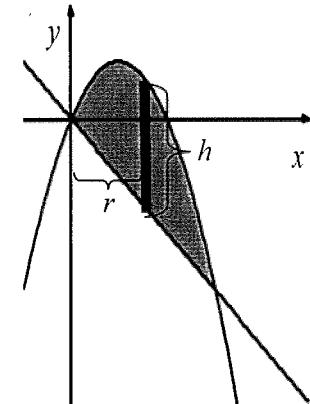
Why would using the disc method for this problem be much harder?

- Set up (but do not integrate) an integral giving the volume of the solid formed by revolving the region bounded by $y = 2x - x^2$ and $y = -x$ about the y -axis.

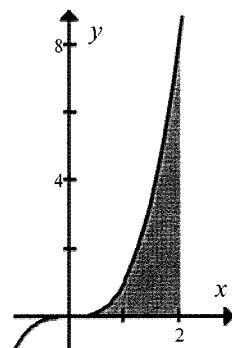
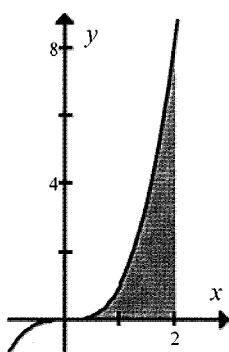
$$r =$$

$$h =$$

$$V =$$



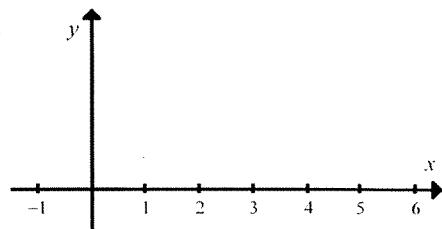
- Use both the shell method and the disc method to find the volume formed by revolving the region bounded by $y = x^3$, $x = 2$, and $y = 0$ about the x -axis.



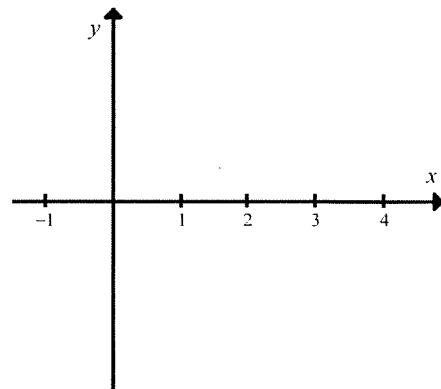
For revolutions about lines other than the x - and y -axes, the formula is still $V = 2\pi \int_a^b rh \, dx$ (or dy), but r is slightly harder to find.
(Remember that both r and h must be nonnegative.)

Example 4: Use the Shell Method to set-up integrals which could be used to find the volumes of the solids formed when the region bounded by $y = \sqrt{x-1}$, $y = 0$, and $x = 5$ is revolved about:

- the y -axis
- $x = -2$
- $x = 5$



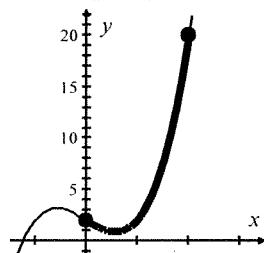
Example 5: Find the volume of the solid formed by revolving the region bounded by $x = y^2$ and $x = 4$ about the line $y = -3$.



Example 6: Set up an integral (but do not integrate) which could be used to find the volume of the solid which would be formed if the region from Example 5 were revolved about the line $y = 3$,

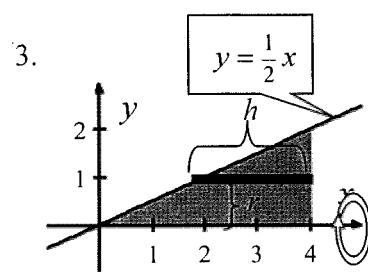
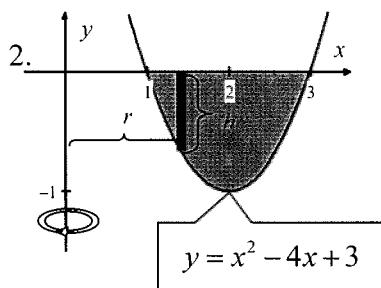
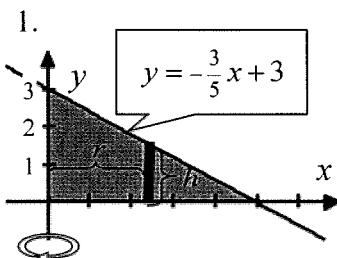
Arc Length Formula: Arc Length = $\int_a^b \sqrt{1 + (f'(x))^2} \, dx$

Example 7: Find the length of the arc of the curve $y = 3x^3 - 3x + 2$ on the interval $[0, 2]$.

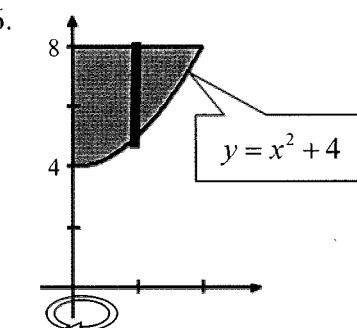
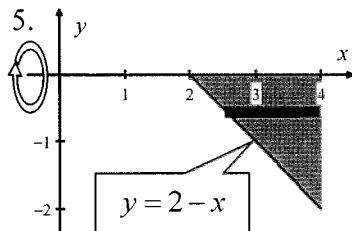
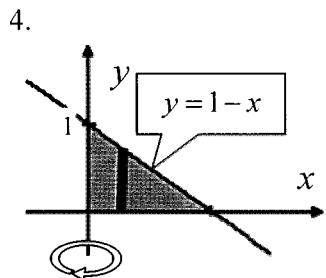


Assignment 5-1:

Give the radius and the height of the shell formed when the representative element is revolved about the indicated axis.



For problems 4-7, set up integrals (but do not integrate) which could be used to find the volumes formed by revolving the plane regions described or shown about the indicated axes. Use the shell method.



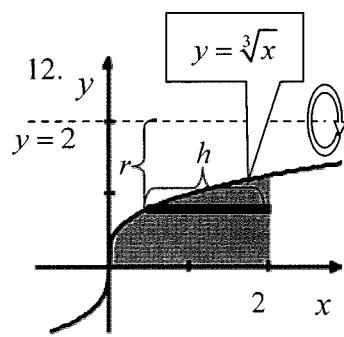
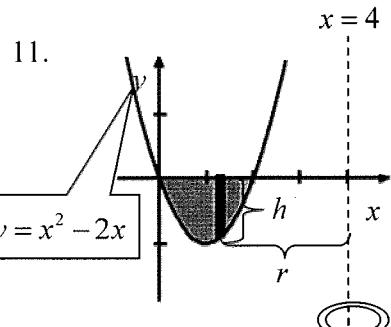
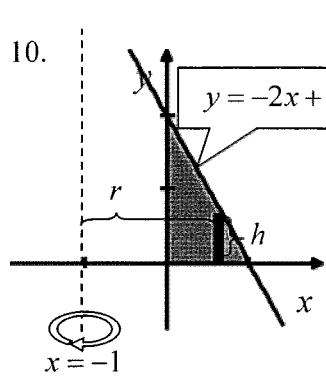
7. Region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 5$ revolved about the x -axis.

Use discs, washers, or shells to find the volumes indicated in problems 8 and 9 without a calculator.

8. Region bounded by $y = x$ and $y = x^3$ in Quadrant I revolved about the x -axis.

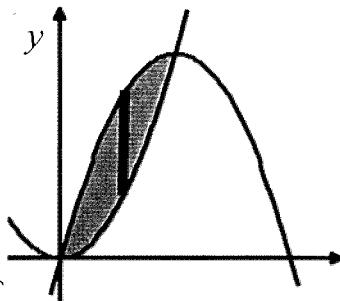
9. Region bounded by $y = \sqrt{x} + 2$, $x = 0$, $y = 0$, and $x = 4$ revolved about the y -axis.

Give the radius and the height of the shell formed when the representative element is revolved about the indicated line.



For problems 13-15, set up integrals (but do not integrate) which could be used to find the volumes formed by revolving the plane region bounded by $y = x^2$ and $y = 4x - x^2$ about the indicated axis of revolution. Use **any method**.

13. about the x -axis
 14. about $x = 4$
 15. about $x = -1$



For problems 16 and 17 draw accurate sketches and set up integrals (but do not integrate) which could be used to find the volumes formed by revolving the plane region bounded by $x = y - y^2$ and $x = 0$ about the indicated axes of revolution. Use **any method**.

Evaluate the integrals in Problems 22-25.

$$22. \int \frac{(x-1)^2}{x^{\frac{3}{2}}} dx \quad 23. \int \frac{x}{\sqrt{4-3x^2}} dx \quad 24. \int_{-4}^{-1} |2x+6| dx \quad 25. \int t\sqrt{t+1} dt$$

26. Set up (but do not integrate) an integral which could be used to find the area bounded by the graphs of $f(x) = -x^2 + 1$ and $g(x) = x^2 - 2x - 3$. Show the equation(s) used to find the limits of integration.

27 Sketch the region bounded by $y = \sqrt{x - 2}$, $x = 6$, and $y = 0$, and find the area of the region.

28. Using the region from Problem 27 as the base, find the volume of the solid formed by using square cross sections perpendicular to the x -axis.

29. The base of a solid is the region bounded by $y = \frac{1}{\sqrt{x}}$, $x = 1$, $x = 4$, and $y = -3$. The solid is formed by using rectangular cross sections whose bases lie in the region and are perpendicular to the x -axis, and whose heights are three times their bases. Set up an integral which represents the volume of the solid, and then use a calculator to find the volume.
30. $f(x) = 2\sqrt{x} - x$. Without using a calculator, find:
- the domain of $f(x)$.
 - the x -intercept(s) for the graph of $f(x)$.
 - the maximum and minimum function values of $f(x)$.
 - the range of $f(x)$.
31. Show that the graph of $f(x) = 2\sqrt{x} - x$ has no points of inflection by building an f'' number line which indicates the concavity for the graph of f .
32. Use the results from Problems 30 and 31 to sketch a graph of $f(x) = 2\sqrt{x} - x$ without using a calculator.
33. Find $\int_0^4 (2\sqrt{x} - x) dx$ without using a calculator.
34. Find the average value of the function $f(x) = 2\sqrt{x} - x$ on $[0, 4]$.
35. Use a calculator to find the value of c where $f(c) = f_{avg}$ for Problem 34.
36. Find the instantaneous rate of change of $f(x) = 2\sqrt{x} - x$ at $x = 2$.
37. Find the average rate of change of $f(x) = 2\sqrt{x} - x$ on $[0, 4]$.
38. Is $f(x) = 2\sqrt{x} - x$ continuous on $[0, 4]$ and differentiable on $(0, 4)$?
39. If your answer to Problem 38 was yes, then the MVT guarantees that $f'(c) = \frac{f(b) - f(a)}{b - a}$ = (the answer to Problem 37) for some c in $(0, 4)$. Find c .
- If your answer to Problem 38 was no, then do it again.

Selected Answers:

- $r = x$, $h = -\frac{3}{5}x + 3$
- $r = x$, $h = -x^2 + 4x - 3$
- $r = y$, $h = 4 - 2y$
- $V = 2\pi \int_0^1 x(1-x) dx$
- $V = 2\pi \int_{-2}^0 -y(2+y) dy$
- $V = 2\pi \int_0^{\sqrt{5}} y(5-y^2) dy$
- $\frac{4\pi}{21}$
- $\frac{288\pi}{5}$
- $r = 1+x$, $h = -2x+2$
- $r = 4-x$, $h = -x^2+2x$
- $r = 2-y$, $h = 2-y^3$
- $V = \pi \int_0^2 \left((4x-x^2)^2 - x^4 \right) dx$
- $V = 2\pi \int_0^2 (1+x)(4x-2x^2) dx$
- $V = 2\pi \int_0^1 (2-y)(y-y^2) dy$
- $V = 2\pi \int_0^1 (3+y)(y-y^2) dy$

More Selected Answers:

18a. $V = \pi \int_0^6 \left(\left(2 + \frac{1}{2}x \right)^2 - 4 \right) dx$ 19a. 6.902 or 6.903 b. 9.852 20. 39.141

22. $\frac{2}{3}x^{\frac{3}{2}} - 4x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} + C$ 23. $-\frac{1}{3}(4 - 3x^2)^{\frac{1}{2}} + C$ 24. 5

25. $\frac{2}{5}(\sqrt{t+1})^5 - \frac{2}{3}(\sqrt{t+1})^3 + C$ 26. $A = \int_{-1}^2 ((-x^2 + 1) - (x^2 - 2x - 3)) dx$ 27. $A = \frac{16}{3}$

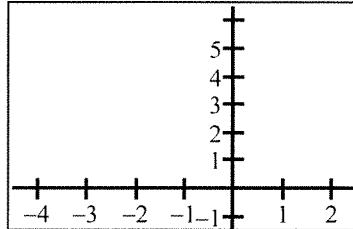
28. V = 8 29. 121.158 or 121.159 30b. x-int.: (0, 0), (4, 0) d. range: $y \leq 1$

33. $\left[\left(\frac{4}{3} \cdot 8 - \frac{1}{2} \cdot 16 \right) - 0 \right] = \frac{8}{3}$ 34. $\frac{2}{3}$ 35. c = .178 or .179, 2.488 37. 0 39. c = 1

Lesson 5-2 Derivatives and Integrals of Exponential Functions

An **exponential function** is a function represented by a constant base with a variable exponent. For example, $f(x) = 2^x$, $y = e^x$, and $g(x) = 3^{x^2-5}$ are exponential functions.

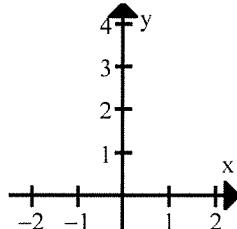
Example 1: Use a calculator to carefully graph $y = 2^x$, $y = 5^x$, and $y = e^x$ in the same coordinate plane. Do you see any similarities in the graphs?



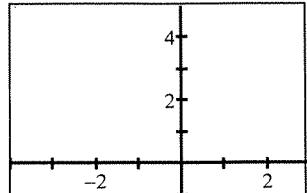
Graphs of Exponential Functions: If $f(x) = a^x$ and $a > 1$, then

1. The domain of $f(x)$ is $(-\infty, \infty)$.
The range of $f(x)$ is $(0, \infty)$.
2. The graph of $f(x)$ is continuous, increasing, concave upward, and one-to-one (has an inverse function).
3. The x -axis is a horizontal asymptote to the left: $\lim_{x \rightarrow -\infty} f(x) = 0$
(Also, $\lim_{x \rightarrow \infty} f(x) = \infty$)
4. The y -intercept is $(0, 1)$.
Another key point is $(1, a)$.

Example 2: Using adjustments to the graph from Example 1, graph $f(x) = e^{-x} + 1$ without using a calculator. Write an equation for the graph's asymptote.



Use your calculator to graph $y = e^x$ and its derivative in the same coordinate plane. What do you notice?



e is the only base for which the basic exponential function and its derivative are the same. For other bases, a logarithmic “hook-on” is required.

Differentiating Exponential Functions:

$$\frac{d}{dx} e^x = e^x$$

$\frac{d}{dx} e^u = e^u u'$ (Chain rule form), where u is a function of x .
(u' is the “hook-on” factor)

$$\frac{d}{dx} a^x = a^x \ln a$$

$\frac{d}{dx} a^u = a^u u' \ln a$ (u' and $\ln a$ are the “hook-on factors”)

Examples: Differentiate

3. $y = e^{x^2 - 3x}$

4. $g(t) = e^{\frac{-t^3}{t}}$

5. $f(v) = 3^{\sqrt{v}}$

6. Find the relative extrema of $f(x) = -xe^{-2x^2}$. List as points and do not use a calculator.

Integrating Exponential Functions:

$$\int e^x dx = e^x + C$$

$$\int e^u u' dx = e^u + C \quad (\text{Reverse Chain Rule form})$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int a^u u' dx = \frac{a^u}{\ln a} + C$$

Examples: Find

7. $\int_0^{\frac{1}{2}} 4^{x^2-1} x \, dx$

8. $\int \frac{e^{\frac{2}{x}}}{3x^2} \, dx$

9. $\int \frac{e^x - e^{-x}}{(e^x + e^{-x})^4} \, dx$

10. $\int e^x \cos e^x \, dx$

Assignment 5-2

For Problems 1-4, sketch a graph without using a calculator. List all intercepts, and write an equation for each asymptote. Use a separate coordinate plane for each graph.

1. $y = 2^x$

2. $y = 2^{-x}$

3. $y = 2^{x+1}$

4. $y = 2^x - 1$

5. Sketch the graph of $y = e^{-x}$, and draw tangent lines at $(0, 1)$ and $(-1, e)$.

6. Find the slope, and write an equation for each tangent line in Problem 5.

a. $(0, 1)$

b. $(-1, e)$

Differentiate in Problems 7-18.

7. $f(x) = e^{5x+1}$

8. $y = e^{\frac{2}{x}}$

9. $g(y) = 2^{-5y}$

10. $h(t) = 5^{t^2-2t}$

11. $y = e^{\sqrt{x}-2}$

12. $f(x) = e^x \sin x$

13. $y = \cos(e^x + x^2)$

14. $f(x) = 2x^2 e^{3x}$

15. $g(x) = \frac{e^x}{x^2 - 1}$

16. $h(x) = (e^{-2x} - 1)^3$

17. $y = \frac{2^{3t}}{t}$

18. $f(\theta) = 3^\theta \sec \theta$

Find $\frac{dy}{dx}$ for Problems 19-21.

19. $ye^x + xe^y = 10x$

20. $\tan(e^x + y) = x^3$

21. $e^{xy} - x^2 = 0$

For Problems 22-35, evaluate each integral.

22. $\int e^{5x} \, dx$

23. $\int \frac{e^u}{u^2} \, du$

24. $\int \frac{e^{\sqrt{y}}}{\sqrt{y}} \, dy$

25. $\int \frac{e^{\tan x}}{\cos^2 x} \, dx$

26. $\int (2x-1)(10^{x^2-x}) \, dx$

27. $\int (e^{-3y} - 2)^5 e^{-3y} \, dy$

28. $\int \frac{e^{-x}}{\sqrt{2-e^{-x}}} \, dx$

29. $\int e^{bx^2-a} x \, dx$
where a and b are constants

30. $\int \frac{e^x - e^{-x}}{(e^x + e^{-x})^3} \, dx$

31. $\int \frac{xe^{2x} - e^x}{e^{2x}} \, dx$

32. $\int_{-1}^1 e^{1-2x} \, dx$

33. $\int_0^2 (e^{\frac{x}{2}} - x) \, dx$

34. $\int_0^4 2^{3t} \, dt$

35. $\int_0^{\frac{1}{2}} e^{\sin(\pi\theta)} \cos(\pi\theta) \, d\theta$

36. Write an equation for the line tangent to the graph of $f(x) = e^{x^2-4x+3}$ at the point where $x = 1$.
37. Write an equation for the line tangent to the graph of $f(x) = 3^{x+2}$ at the point where $x = 1$.
38. Find the second derivative of $f(x) = (1+2x)e^{-2x}$.
39. Find the domain, x - and y -intercepts, extrema, and points of inflection for the graph $y = xe^x - e^x$. Then sketch its graph without using a calculator.
40. If $f''(x) = e^x + e^{-x}$, $f'(0) = 3$, and $f(0) = 4$, find $f(x)$.
41. Find the area bounded by the graphs of $y = 5^x$, $y = 0$, $x = 0$, and $x = 1$ without using a calculator.
42. Approximate $\int_0^2 xe^x dx$ using
 - a Left Hand Riemann Sum with four equal subdivisions.
 - the Trapezoidal Rule with four equal subdivisions.

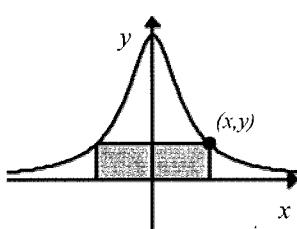
Set up (but do not integrate) integrals for computing the volumes of the solids described in Problems 43-46, where R is the region bounded by $y = \ln(x-1)$, $y = 0$, and $x = 4$. You may use a calculator to graph.

43. The solid is formed by revolving region R about the x -axis.
44. The solid is formed by revolving region R about $y = -2$.
45. The solid is formed by revolving region R about the y -axis.
46. The volume is formed by semicircular cross sections in Region R , whose bases are perpendicular to the x -axis.

47. Find the average value of $y = \frac{1}{(4-x)^4}$ on $[2, 3]$.

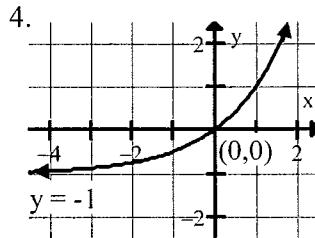
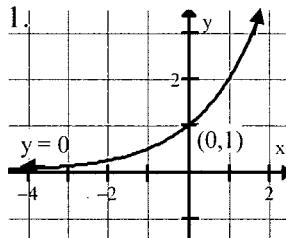
48. A rectangle is inscribed under the curve $y = \frac{4}{1+x^2}$ and is bounded below by the x -axis as shown in the figure at the right.

- Find the point (x, y) as shown in the figure that produces the rectangle having the most area.
- Find the area of the rectangle.



49. Find the length of the curve on the graph of $y = 3x^2 + 5x - 1$ between the x -intercepts.

Selected Answers:

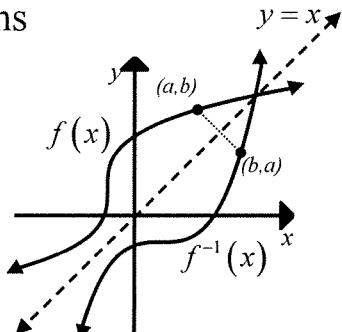


More Selected Answers:

- 6a. $y - 1 = -1(x - 0)$ b. $y - e = -e(x + 1)$ 7. $f'(x) = 5e^{5x+1}$ 8. $y' = -2x^{-2}e^{\frac{2}{x}}$
 10. $h'(t) = 5^{t^2-2t}(2t-2)\ln 5$ 11. $y' = \frac{e^{\sqrt{x}-2}}{2\sqrt{x}}$ 13. $y' = -\sin(e^x+x^2)(e^x+2x)$
 14. $f'(x) = 6x^2e^{3x} + 4xe^{3x}$ 15. $g'(x) = \frac{(x^2-1)e^x - 2xe^x}{(x^2-1)^2}$ 16. $h'(x) = -6e^{-2x}(e^{-2x}-1)^2$
 18. $f'(\theta) = 3^\theta \sec \theta \tan \theta + \sec \theta 3^\theta \ln 3$ 19. $y' = \frac{10 - ye^x - e^y}{e^x + xe^y}$ 21. $y' = \frac{2x - ye^{xy}}{xe^{xy}}$
 22. $\frac{1}{5}e^{5x} + C$ 23. $-\frac{1}{2}e^{\frac{2}{u}} + C$ 25. $e^{\tan x} + C$ 26. $\frac{10^{x^3-x}}{\ln 10} + C$ 27. $-\frac{1}{18}(e^{-3y}-2)^6 + C$
 28. $2\sqrt{2-e^{-x}} + C$ 29. $\frac{1}{2b}e^{bx^2-a} + C$ 31. $\frac{x^2}{2} + e^{-x} + C$ 32. $-\frac{1}{2}e^{-1} + \frac{1}{2}e^3$ 33. $2e-4$
 34. $\frac{2^{12}-1}{3\ln 2}$ 35. $\frac{e-1}{\pi}$ 36. $y - 1 = -2(x - 1)$ 37. $y - 27 = 27 \ln 3(x - 1)$
 38. $f''(x) = 8xe^{-2x} - 4e^{-2x}$ 39. Rel. Min. at $(0, -1)$, P.I. at $(-1, -\frac{2}{e})$
 40. $f(x) = e^x + e^{-x} + 3x + 2$ 41. $\frac{4}{\ln 5}$ 42a. $\frac{1}{2}\left(0 + \frac{1}{2}e^{\frac{1}{2}} + e + \frac{3}{2}e^{\frac{3}{2}}\right)$
 44. $V = \pi \int_2^4 \left((\ln(x-1)+2)^2 - 2^2 \right) dx$ 46. $V = \int_2^4 \frac{1}{2} \pi \left(\left(\frac{\ln(x-1)}{2} \right)^2 \right) dx$ 47. $\frac{7}{24}$
 48a. $(1, 2)$ b. $A = 4$ 49. 6.666 or 6.667

Lesson 5-3 Derivatives of Inverse Functions

At right are the graphs of a function $f(x)$ and its inverse $f^{-1}(x)$. Remember that if the graph of f contains the point (a, b) , then the graph of f^{-1} contains the point (b, a) . Also, the graph of f^{-1} is the reflection of the graph of f across the line $y = x$.



From the graphs above, do you see a relationship between the slope of the graph of f at (a, b) and the slope of the graph of f^{-1} at (b, a) ?

Example 1: Let $f(x) = \sqrt{x}$.

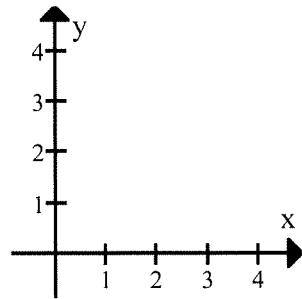
- Sketch the graph of $f(x)$.
- Find $f^{-1}(x)$. Hint: You must list a domain restriction.

- Sketch the graph of $f^{-1}(x)$ in the same coordinate plane.
- Differentiate both $f(x)$ and $f^{-1}(x)$.

- Find the slope of the graph of $f(x)$ at $(4, 2)$ and the slope of the graph of $f^{-1}(x)$ at $(2, 4)$.

- What conclusion can you make about these slopes?

Since slope $= m = \frac{\Delta y}{\Delta x}$, it should make sense that switching x and y (for inverse functions) should produce reciprocal slopes for inverse functions.



Derivatives of Inverse Functions:

If (a, b) is a point on f , then (b, a) is a point on f^{-1} , and $(f^{-1})'(b) = \frac{1}{f'(a)}$

or if f and g are inverse functions, then $g'(x) = \frac{1}{f'(g(x))}$.

Derivatives of inverses have reciprocal slopes at “image points” (points reflected across $y = x$). (a, b) and (b, a) are image points.

Example 2: Let f and g be inverse functions such that:

$$\begin{array}{lll} f(-1) = 1 & f(0) = 2 & f(1) = 5 \\ f'(-1) = \frac{3}{2} & f'(0) = 2 & f'(1) = \frac{1}{2} \end{array}$$

From the given information, find each of the following if possible.

Hint: Make a table or chart to organize your data.

- $g'(1)$
- $g'(2)$
- $g'(3)$
- $g'(0)$
- $g'(5)$

If a function f has an inverse function f^{-1} , then f is one-to-one and must be either strictly increasing or strictly decreasing (strictly monotonic) on its entire domain.

We can use f' to find out where f is increasing and where f is decreasing.

Example 3:

- a. Use $f'(x)$ to show that $f(x) = 6x - x^3$ is not one-to-one on its entire domain.

- b. Find the largest interval containing $x = 0$ for which f is one-to-one.

- c. Find the largest interval containing $x = -2$ for which f has an inverse function.

Assignment 5-3

1. $f(x) = x^3 - 1$. Let $g(x) = f^{-1}(x)$.
 - a. Find $g(x)$.
 - b. Graph $f(x)$ and $g(x)$ in the same coordinate plane.
 - c. Find $f'(x)$ and $g'(x)$.
 - d. Find $f'(1)$ and $g'(0)$. (The points $(1, 0)$ on f and $(0, 1)$ on g are image points).
 - e. What is the relationship between the slopes in Part d?

For Problems 2 and 3, let $g(x)$ be the inverse of $f(x)$. Repeat the five parts in Problem 1 for the given function $f(x)$ and the given point on f . (Use the given point on f when finding f' . You will need to use the image point when finding g' .)

2. $f(x) = 3x - 1$
Point : $(1, 2)$

3. $f(x) = \sqrt{x+1}$
Point: $(3, 2)$

4. Let f and g be inverse functions such that: $\begin{cases} f(-1) = 0, f(0) = 1, \text{ and } f(1) = 3 \\ f'(-1) = \frac{4}{3}, f'(0) = \frac{1}{5}, \text{ and } f'(1) = 2 \end{cases}$

Find each of the following (if possible).

- a. $g'(-1)$
- b. $g'(0)$
- c. $g'(1)$
- d. $g'(2)$
- e. $g'(3)$
5. If $f(2) = 3$ and $f'(2) = 4$, find $(f^{-1})'(3)$.
6. If $(1, 2)$ is a point on $f(x) = x^3 + 2x - 1$, find $(f^{-1})'(2)$.
7. If $f(x) = x^3 - \frac{4}{x}$ ($x > 0$), find $(f^{-1})'(6)$.

f and g are inverse functions in Problems 8-10. Find g' at the given value.

8. $f(2) = 5$	9. $f(x) = x^5 + 2x^3 - 1$	10. $f(x) = e^{x-2}$
$f'(2) = \frac{-2}{3}$	$f(1) = 2$	$g'(1) =$
$g'(5) =$	$g'(2) =$	

For Problems 11-13, use $f'(x)$ to find the largest interval on which $f(x)$ has an inverse function.

11. $f(x) = x^2 - 3x + 10$ (Interval contains $x = 1$)	12. $f(x) = 12x - x^3$ (Interval contains $x = 3$)	13. $f(x) = x^3 - 3x^2 + 3x$
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Differentiate in Problems 14-17.

14. $y = x^5 + \frac{1}{x}$	15. $\frac{d}{dx} (6^{2x} - 3)^4 =$	16. $f(x) = \frac{x^2}{e^x}$	17. $g(t) = (2t - 1)^5$
$\frac{d^2y}{dx^2} =$		$f'(x) =$	$g''(t) =$

Evaluate the integrals in Problems 18-21 without using a calculator.

18. $\int_{-2}^2 \left \frac{3}{4} x^5 \right dx$	19. $\int_{-1}^0 \frac{(e^{-t} + 2)^3}{e^t} dt$	20. $\int \frac{y-1}{3\sqrt[4]{y}} dy$	21. $\int (\sqrt{x} + 1)^2 dx$
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22. Sketch the region bounded by $x = y^2 - 4$ and $x = y - 2$. Then set up an integral for the area, and find the area without using a calculator.
23. Set up (but do not integrate) an integral for the volume of the solid formed by revolving the region from Problem 22 about $x = -4$.
24. An object moves along a vertical path with its position at time t (in seconds), according to the equation $y(t) = te^{t+1}$ (where y is measured in centimeters (cm)).
- Find the object's position at time $t = -2$ sec.
 - Find the equation for the object's velocity.
 - Find the object's velocity at time $t = -2$ sec.
 - Find an equation for the object's acceleration.
 - Find $a(-2)$.
 - For what interval(s) of time is the object moving downward?
 - Find the object's minimum position. Justify your answer.
 - Use a calculator to find the total distance traveled by the object on $[-2, 2]$.

Use a calculator to evaluate problems 25-27.

25. $f'(1.237)$ for $f(x) = \sqrt{e^x + 5}$	26. $f''(1.237)$ for $f(x) = \sqrt{e^x + 5}$	27. $\int_0^{1.237} \sqrt{e^x + 5} dx$
28. Use the Trapezoidal Rule (with $n = 4$) to approximate $\int_0^2 \frac{1}{1+x^3} dx$.		

For Problems 29-32, find each limit (if it exists).

29. $\lim_{x \rightarrow -3} \frac{2x-1}{3x+4}$

30. $\lim_{t \rightarrow 2} \frac{t^2 - 4}{t^2 - 3t + 2}$

31. $\lim_{x \rightarrow -1} \frac{x+1}{\sqrt{x^2+1}}$

32. $\lim_{x \rightarrow -\infty} \frac{x+1}{\sqrt{x^2+1}}$

33. If $f(\pi) = 5$ and $f'(x) = \sin(3x)$, find $f(2\pi)$.

For Problems 34-37, list the x -locations of the discontinuities for each function. Label each discontinuity as removable or non-removable.

34. $f(x) = \begin{cases} x \\ 2 \end{cases}$

35. $g(x) = \frac{x+2}{x^2+x-6}$

36. $h(t) = \frac{\sqrt{t}}{t^2-9}$

37. $y = \begin{cases} \frac{x-2}{x^2+x-6}, & x \neq 2 \\ 0, & x = 2 \end{cases}$

38. Fill in the blank with a number to make

$$y = \begin{cases} \frac{x-2}{x^2+x-6}, & x \neq 2 \\ \underline{\hspace{2cm}}, & x = 2 \end{cases}$$

continuous at $x = 2$

39. Find the values of a and b which make

$$f(x) = \begin{cases} \frac{1}{2}x + b, & x \leq -2 \text{ or } x \geq 1 \\ ax^2, & -2 < x < 1 \end{cases}$$

continuous.

40. Find the values of a and b which make

$$g(x) = \begin{cases} ax^3 - bx + 7, & x \leq 1 \\ -bx^2 + 3bx, & x > 1 \end{cases}$$

continuous and differentiable.

For Problems 41-45, let R be the region bounded by $y = x\sqrt{x+4}$ and the x -axis.

41. Find the area of Region R . (Show an integral set-up first, and do not use a calculator.)
 42. Find the volume of the solid formed by revolving region R about the x -axis. (Show an integral set-up, and then use a calculator to integrate.)
 43. Set up (but do not integrate) an integral for finding the volume of the solid formed by revolving region R about the line $y = 2$.
 44. Set up (but do not integrate) an integral for finding the volume of the solid formed by revolving region R about the y -axis..
 45. Set up (but do not integrate) an integral for finding the volume of the solid formed by equilateral triangle cross sections with bases perpendicular to the x -axis.

Remember: $A = \frac{\sqrt{3}}{4}s^2$

Selected Answers:

1a. $g(x) = \sqrt[3]{x+1}$ c. $f'(x) = 3x^2$, $g'(x) = \frac{1}{3}(x+1)^{-\frac{2}{3}}$ d. $f'(1) = 3$, $g'(0) = \frac{1}{3}$

1e They are reciprocals. 2a. $g(x) = \frac{1}{3}x + \frac{1}{3}$ 3a. $g(x) = x^2 - 1$, $x \geq 0$

3c. $f'(x) = \frac{1}{2\sqrt{x+1}}$, $g'(x) = 2x$ 4a. not possible b. $\frac{3}{4}$ c. 5 5. $\frac{1}{4}$ 6. $\frac{1}{5}$

7. $\frac{1}{13}$ 9. $\frac{1}{11}$ 10. 1 11. $(-\infty, \frac{3}{2}]$ 12. $[2, \infty)$ 14. $\frac{d^2y}{dx^2} = 20x^3 + 2x^{-3}$

15. $8(6^{2x} - 3)^3 6^{2x} \ln 6$ 16.
$$\boxed{f'(x) = \frac{e^x \bullet 2x - x^2 e^x}{(e^x)^2}} = \frac{2x - x^2}{e^x}$$
 18. 16

19. $-\frac{3^4}{4} + \frac{(e+2)^4}{4}$ 20. $\frac{4}{21}y^{\frac{7}{4}} - \frac{4}{9}y^{\frac{3}{4}} + C$ 21. $\frac{1}{2}x^2 + \frac{4}{3}x^{\frac{3}{2}} + x + C$

22. $A = \left[\left(2 - \frac{8}{3} + 4 \right) - \left(\frac{1}{2} + \frac{1}{3} - 2 \right) \right] = \frac{9}{2}$ 24a. $y(-2) = -\frac{2}{e}$ cm b. $v(t) = te^{t+1} + e^{t+1}$

24e. $a(-2) = 0 \frac{\text{cm}}{\text{sec}^2}$ g. -1 cm h. 41.435 cm 26. .471 or .472

28.
$$\boxed{\frac{1}{4} \left(1 + 2 \bullet \frac{1}{1+.5^3} + 2 \bullet \frac{1}{2} + 2 \bullet \frac{1}{1+1.5^3} + \frac{1}{9} \right)} = 1.086 \text{ or } 1.087$$
 30. 4 32. -1

34. $x = \text{any even integer}$ (nonremovable) 35. $x = 2, -3$ (nonrem.) 36. $t = 3$ (nonrem.)

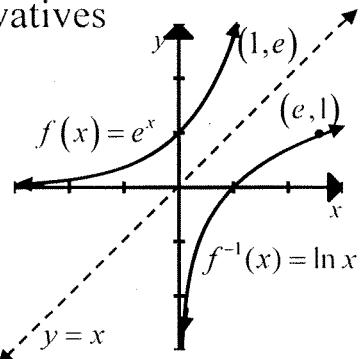
37. $x = 2$ (rem.), $x = -3$ (nonrem.) 38. $\frac{1}{5}$ 39. $a = -\frac{1}{2}$, $b = -1$ 40. $a = 2$, $b = 3$

41. $-\frac{2}{5} \bullet 32 + \frac{8}{3} \bullet 8$ 42. 67.020 or 67.021 45. $V = \int_{-4}^0 \frac{\sqrt{3}}{4} \left(-x\sqrt{x+4} \right)^2 dx$

Lesson 5-4 Logarithmic Functions, Log Derivatives

The inverse of $f(x) = e^x$ is $f^{-1}(x) = \ln x$

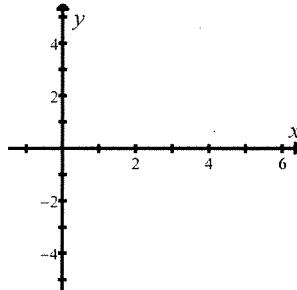
Note: $\log_e x$ is usually written as $\ln x$ and $\log_{10} x$ is usually written simply as $\log x$.



Graphs of Logarithmic Functions: If $f(x) = \log_a x$ and $a > 1$, then

1. The domain of $f(x)$ is $(0, \infty)$.
The range of $f(x)$ is $(-\infty, \infty)$.
2. The graph of $f(x)$ is continuous, increasing, concave downward, and one-to-one (has an inverse function).
3. The y -axis is a vertical asymptote downward: $\lim_{x \rightarrow 0} f(x) = -\infty$
(Also, $\lim_{x \rightarrow \infty} f(x) = \infty$)
4. The x -intercept is $(1, 0)$.
Another key point is $(a, 1)$.

Example 1: Without using a calculator, sketch a graph of $y = |\ln(x-2)|$. Write an equation for the graph's asymptote.



Change of Form Definition:

Exponential form	$\left\{ \begin{array}{l} x = e^y \leftrightarrow y = \ln x \\ x = a^y \leftrightarrow y = \log_a x \end{array} \right\}$	Logarithmic form
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Example 2: Change the following equations from exponential form to log form or vice versa.

a. $3^4 = 81$ b. $\log(1) = -1$

Example 3:

a. Since $e^0 = 1$, $\ln 1 =$ b. Since $e^1 = e$, $\ln e =$ c. $\ln e^n = e^{\ln n} =$

Example 4: Use the inverse idea from Example 3c. to simplify.

a. $\ln e^{\sqrt{2}} =$ b. $e^{\ln(3x)} =$ c. $10^{\log 2} =$ d. $\log_2 2^{x^2} =$

Properties of Logarithms:

1. $\ln(ab) = \ln a + \ln b$ These properties work for any bases,
2. $\ln \frac{a}{b} = \ln a - \ln b$ but only if $a > 0$ and $b > 0$
3. $\ln a^n = n \ln a$

Example 6: Expand using Logarithm Properties 1-3 above.

a. $\ln \frac{5}{8} =$ b. $\ln \sqrt[3]{x^2 + 1} =$

Example 7: Condense into a single logarithm. ($x > 0$ and $y > 0$)

a. $-3 \ln x + 5 \ln y$ b. $\frac{1}{2} \ln x + \ln(x+1) - 3 \ln y$

Example 8: Solve for x .

a. $y = e^{2x-5} + 6$ b. $\log_2 x - \log_2(x-8) = 3$

Change of Base Formula: $\log_a x = \frac{\log_b x}{\log_b a}$

Since the only two logarithmic bases on your calculator are 10 (log key) and e (ln key), you will change bases on your calculator in one of two ways:

$$\log_a x = \frac{\log x}{\log a} \quad \text{or} \quad \log_a x = \frac{\ln x}{\ln a}$$

Example 9: Use your calculator to find $\log_7 112$ to 3 or more decimal places.

Example 10:

- a. Find an exact value for x , if $3^{x+2} = 6$.
- b. Use your calculator to find a decimal value for your answer from Part a.

Differentiating Logarithmic Functions:

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

$$\frac{d}{dx} \ln u = \frac{1}{u} u' = \frac{u'}{u} \quad (\text{Chain rule form})$$

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$$

$$\frac{d}{dx} \log_a u = \frac{u'}{u \ln a} \quad (\text{Chain rule form})$$

Examples: Differentiate.

11. $y = \ln(5x)$

12. $f(t) = \ln(3t^2 - t)$

13. $h(x) = x \ln x$

When differentiating the natural log function, $\frac{d}{dx} \ln|x| = \frac{d}{dx} \ln x$ and $\frac{d}{dx} \ln|u| = \frac{d}{dx} \ln u$.

In these cases, absolute value may change the domain of the function – but not the derivative.

Example 14:

Find $\frac{d}{dy} \ln|5 - 2y^3|$

Example 15:

Differentiate $y = \ln \frac{x\sqrt{2x+1}}{x^2 + 1}$.

Example 16: If $y = \log_2(x^2 + 1)$, find $y'(2)$

So far, we have not encountered any functions having both a variable base and a variable exponent (for example, $y = x^x$, $y = (x^2 + 1)^{x-1}$, $y = x^{\ln x}$). Such functions are neither power functions nor exponential functions. They can only be differentiated using a process called **logarithmic differentiation**.

Procedure for Logarithmic Differentiation: $y = \text{variable base}^{\text{variable exponent}}$

1. Take the natural log (**ln**) of both sides of the equation.
2. Simplify the right hand side of the equation (Log Property 3).
3. Differentiate both sides.
4. Solve for y' .
5. Substitute for y if necessary.

Example 17: Differentiate $y = (x^2 + 1)^{x-1}$. Express your answer in terms of x .

Assignment 5-4

For Problems 1-2, change each equation from exponential form to logarithmic form or vice versa.

1. $e^x = 17$

2. $\log x = -2$

Simplify each expression in Problems 3-4.

3. $\ln e^{a+b}$

4. $3^{\log_3 m^2}$

For Problems 5-6, solve for x without using a calculator. Simplify your answers.

5. $\log_2 x = 3$

6. $\log_x 64 = 3$

For Problems 7-10, sketch a graph without using a calculator. List all x -intercepts, and write an equation for each asymptote. Use a separate coordinate plane for each graph.

7. $y = \log_2 x$

8. $y = \log_2(x+3)$

9. $y = |\log_2 x|$

10. $y = \log_2 |x|$

Use Properties of Logarithms to expand the expressions in Problems 11-12.
(All variables represent positive quantities.)

11. $\ln \frac{a}{bc}$

12. $\log(xy^2)$

Use Properties of Logarithms to condense the expressions in Problems 12-14 into single logarithms.
(All variables represent positive quantities).

13. $3 \ln x - \frac{1}{2} \ln y$

14. $\ln a - (2 \ln b - \ln c)$

For Problems 15-16, solve for t without using a calculator.

15. $\ln e^{t^2-t} = 6$

16. $e^{2t-1} - 3 = 0$

17. Use a calculator to solve for x in $3e^{-x+1} = 5 - x^2$.

18. Find the value of $\log_3 20$. (Express answers to 3 or more decimal place accuracy.)

19. Use an f' number line to determine whether or not $f(x) = x^3 - 9x^2 + 27x - 36$ is strictly monotonic (strictly increasing or strictly decreasing).

Differentiate in Problems 20-34.

20. $y = \ln x^5$

21. $y = \ln(x^2 - 5x)$

22. $f(x) = \ln|x^3 + 2x|$

23. $g(y) = (\ln y)^4$

24. $g(t) = \ln \sqrt{t^3 - t}$

25. $f(x) = \log_5 |x^2 - 1|$

26. $h(x) = (x+1) \ln x$

27. $y = \ln \frac{x^2}{x-1}$

28. $f(t) = \frac{t}{\ln|t|}$

29. $y = \ln(\ln(x-1))$

30. $f(x) = \ln(\cot x)$

31. $f(x) = e^{3x^2-1}$

32. $y = 5^{3x^2-1}$

33. $y = x^x$

34. $y = (2x-1)^{\ln x}$

35. Let $f(x) = \ln \sqrt{x}$.

- Sketch the graph of $f(x) = \ln \sqrt{x}$ without using a calculator.
- On your graph, draw a tangent line at $(1, 0)$.
- Write an equation for the tangent line that you drew.

In Problems 36 and 37, find $\frac{dy}{dx}$ by using implicit differentiation.

36. $y^2 = x \ln y$

37. $x - \ln(xy) = 10$

38. For $y = \frac{e^{-3x}}{3} + 2e^{2x}$, find $\frac{d^2y}{dx^2}$.

39. Find equations for the tangent and normal lines to $x^2 - \ln(x+1) = y^2 - 1$ at the point $(0, 1)$.

40. Find the domain, extrema, and points of inflection for the graph of $y = x - \ln x$. Then sketch the graph without using a calculator.

Evaluate the integrals in Problems 41-42.

41. $\int \frac{1+2e^x+3e^{2x}}{e^{2x}} dx$

42. $\int e^{(2t+4)^2} (t+2) dt$

43. If $f(x) = 2x^5 + x^3 + 1$, find $(f^{-1})'(-2)$.

44. If $f(x) = \sqrt{x-4}$, find $(f^{-1})'(2)$.

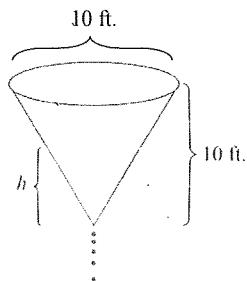
45. If $f(4) = 7$ and $f'(4) = -3$, use the equation of a tangent line to approximate $f\left(4\frac{1}{3}\right)$.

46. A conical tank, as shown at right, has a hole in its bottom and is leaking water at the rate of 1 cubic foot per minute. Find the rate of change in the height, h , of the water in the tank when

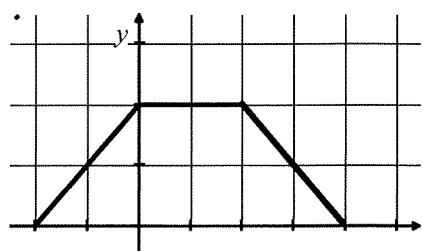
$$h = 4 \text{ ft? } V = \frac{1}{3}\pi r^2 h$$

Write appropriate units for your answer.

Hint: Find a relationship between r and h . $r = \underline{\hspace{2cm}} h$



47. Use the graph of f' shown at right to graph f'' and a possible graph of f .



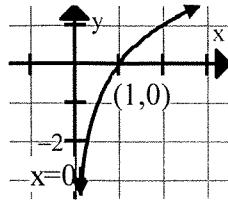
Selected Answers:

1. $\ln 17 = x$

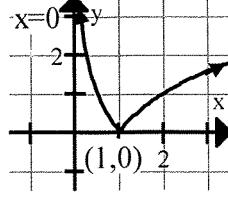
3. $a+b$

5. $x = 8$

7.



9.



11. $\ln a - \ln b - \ln c$

13. $\ln \frac{x^3}{\sqrt{y}}$

14. $\ln \frac{ac}{b^2}$

15. $t = -2, 3$

16. $t = \frac{\ln 3 + 1}{2}$

18. 2.726 or 2.727

19. monotonic increasing

20. $y' = \frac{5}{x}$

22. $f'(x) = \frac{3x^2 + 2}{x^3 + 2x}$

24. $g'(t) = \frac{1}{2} \cdot \frac{3t^2 - 1}{t^3 - t}$

26. $h'(x) = (x+1)\frac{1}{x} + \ln x$

27. $y' = \frac{2}{x} - \frac{1}{x-1}$

28. $f'(t) = \frac{\ln|t| - 1}{(\ln|t|)^2}$

29. $y' = \frac{1}{(x-1)\ln(x-1)}$

30. $f'(x) = \frac{-\csc^2 x}{\cot x}$

31. $f'(x) = e^{3x^2-1} (6x)$

33. $y' = (1 + \ln x)x^x$

34. $y' = \left(\ln x \frac{2}{2x-1} + \ln(2x-1) \frac{1}{x} \right) y$

35c. $y = \frac{1}{2}(x-1)$

36. $y' = \frac{\ln y}{2y - \frac{x}{y}} = \frac{y \ln y}{2y^2 - x}$

37. $y' = \frac{xy - y}{x} = y - \frac{y}{x}$

38. $\frac{d^2y}{dx^2} = 3e^{-3x} + 8e^{2x}$

40. Do.: $x > 0$, Rel. Min.: $(1, 1)$, no P.I.

41. $-\frac{1}{2}e^{-2x} - 2e^{-x} + 3x + C$

42. $\frac{1}{8}e^{(2t+4)^2} + C$

43. $\frac{1}{13}$

45. $f\left(4 \frac{1}{3}\right) \approx 6$

46. $\frac{dh}{dt} = -\frac{1}{4\pi} \frac{ft}{min}$

Lesson 5-5 Integration Involving the Natural Log Function

Differentiation and integration are inverse operations.

So if $\frac{d}{dx} \ln|x| = \frac{1}{x}$, then $\int \frac{1}{x} dx = \ln|x| + C$, and if $\frac{d}{dx} \ln|u| = \frac{u'}{u}$, then $\int \frac{u'}{u} dx = \ln|u| + C$.

Log Rules:	$\int \frac{1}{x} dx = \ln x + C$	and	$\int \frac{u'}{u} dx = \ln u + C$
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Note: Although it is true that both $\frac{d}{dx} \ln x = \frac{1}{x}$ and $\frac{d}{dx} \ln|x| = \frac{1}{x}$, $\int \frac{1}{x} dx = \ln|x|$ only. **Why?**

Examples: Integrate

1. $\int \frac{-3}{x} dx$

2. $\int \frac{P}{P^2 + 1} dP$

3. $\int \frac{9t^2 - 6t}{t^3 - t^2} dt$

4. $\int \frac{\sec^2 x}{\tan x} dx$

When integrating a fraction where the degree of the numerator \geq the degree of the denominator, you will have to use long division (or creative thinking) to “split the fraction.”

Example 5: Integrate $\int \frac{x^2 - 4x + 2}{x^2 + 2} dx$

Example 6: Integrate $\int \frac{\ln x}{x} dx$

Example 7: Integrate $\int \frac{1}{x(2 - \ln x)^3} dx$

Example 8: Integrate $\int \frac{1}{\sqrt{x} - 1} dx$

Example 9: Integrate $\int \cot x dx$

Example 10: Integrate $\int \tan x dx$

$\text{So, } \int \tan x dx = -\ln \cos x + C$ $\int \tan u u' dx = -\ln \cos u + C$	$\int \cot x dx = \ln \sin x + C$ $\int \cot u u' dx = \ln \sin u + C$
--	--

Example 11: Integrate $\int \tan(2x) dx$

Example 12: Integrate $\int \tan^2 x dx$

Assignment 5-5

Evaluate each integral in Problems 1-17 without using a calculator.

1. $\int_1^e \frac{2}{x} dx$

2. $\int_1^3 \frac{4}{2t-1} dt$

3. $\int_{-1}^1 4^{x+1} dx$

4. $\int_0^2 e^{-2x} dx$

5. $\int e^{x^2-1} x dx$

6. $\int \frac{x}{x^2-1} dx$

7. $\int \frac{x^2-1}{x} dx$

8. $\int \frac{4y-6}{y^2-3y+2} dy$

9. $\int \frac{x+1}{x-1} dx$

10. $\int \frac{-1}{(x+1)^3} dx$

11. $\int \frac{3u}{\sqrt[3]{u^2+1}} du$

12. $\int \frac{\sqrt{\ln x}}{x} dx$

13. $\int \frac{2}{x(1+\ln x)^5} dx$

14. $\int \frac{\cos y}{\sin y - 2} dy$

15. $\int \frac{\cot^2 x - 1}{\cot x} dx$

16. $\int \tan^2(3x) dx$

17. $\int \frac{4}{2+\sqrt{x}} dx$

Differentiate in Problems 18-23.

18. $f(x) = \ln(x-1)^{\frac{2}{3}}$

19. $y = \frac{t^2}{\ln t}$

20. $g(y) = \ln|1-\ln y|$

21. $y = x^{2x+1}$

22. $\frac{d}{dx} \ln|\sin^3 x| =$

23. $x^2 + 2 \ln y = y$ (solve for $\frac{dy}{dx}$)

24. List the domain and range for each of the following without using a calculator.

a. $y = \ln(x-2)$

b. $y = \ln((x-2)^2)$

25. For the function $g(x) = \frac{x \ln x}{e^x}$, use a calculator to find each of the following:

- the average rate of change of $g(x)$ on the interval $[1, 4]$.
- the average value of $g(x)$ on the interval $[1, 4]$.
- the instantaneous rate of change of $g(x)$ at $x=1$.
- the x -value(s) where $g(x)$ has a horizontal tangent.

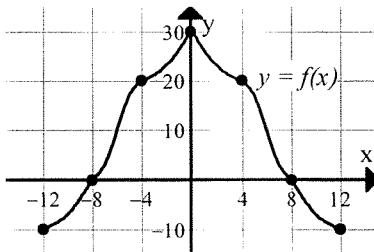
26. Set up one or more integrals which could be used to find the total area of the region(s) bounded by $f(x) = x^2$ and $g(x) = 3x - 3x^2 + x^3$. You may use a calculator to graph the equations, and you do not need to actually integrate.

27. Use the alternate form of the limit definition of the derivative to find $f'(3)$, if $f(t) = 2t^2 - 3$.

28. The graph of $f(x)$ is shown at right.

Approximate $\int_{-12}^{12} f(x) dx$

- a. by using a trapezoidal approximation with equal subintervals of width $\Delta x = 4$.
 - b. by using midpoint rectangles of equal width, where $n = 3$.



29. Region R is bounded by $y = e^x$, $y = 1$, $x = 1$, and $x = 3$.

- a. Without using a calculator sketch Region R .

Set up (but do not integrate) integrals which could be used to find the following.

- b. the area of Region R .
 - c. the volume of the solid formed by revolving Region R about the x -axis.
 - d. the volume of the solid formed by square cross sections in Region R which are perpendicular to the x -axis.
 - e. the volume of the solid formed by revolving Region R about the line $y = 1$.
 - f. the volume of the solid formed by revolving Region R about the line $x = 1$.
 - g. the perimeter of region R .

30. Find a , b , and c for $f(x) = ax^2 + bx + c$, such that $f(1) = 10$, and $f(x)$ has a relative minimum at $(-1, 2)$.

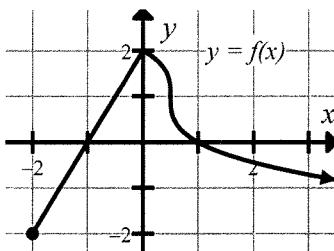
31. If $f(x) = (x-1)^{\frac{2}{3}}$, the Mean Value Theorem does not apply to which interval?
a. $[0, 2]$ b. $[1, 9]$ Why?

32. For Problem 31, use the interval on which the MVT does apply to find the

c -value(s) where $f'(c) = \frac{f(b)-f(a)}{b-a}$

Use the graph of $y = f(x)$ shown at right to graph Problems 33-36 without using a calculator.

33. $y = f(-x) + 1$ 34. $y = f(|2x|)$
 35. $y = -\frac{1}{2}f(x)$ 36. $y = |f(x+2)|$

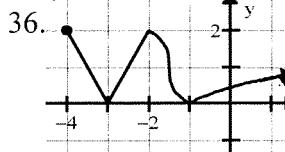
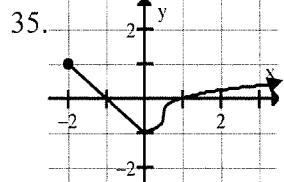
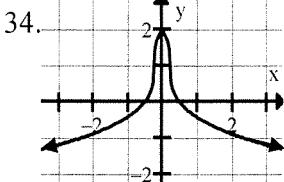
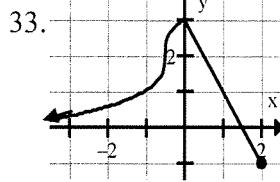


Selected Answers:

1. 2 2. $2\ln 5$ 3. $\frac{15}{\ln 4}$ 4. $-\frac{e^{-4}}{2} + \frac{1}{2}$ 6. $\frac{1}{2}\ln|x^2 - 1| + C$ 7. $\frac{1}{2}x^2 - \ln|x| + C$
 8. $2\ln|y^2 - 3y + 2| + C$ 9. $x + 2\ln|x - 1| + C$ 11. $\frac{9}{4}(u^2 + 1)^{\frac{2}{3}} + C$ 13. $-\frac{1}{2(1 + \ln x)^4} + C$
 14. $\ln|\sin y - 2| + C$ 15. $\ln|\sin x| + \ln|\cos x| + C$ 17. $8\sqrt{x} - 16\ln(\sqrt{x} + 2) + C$
 18. $f'(x) = \frac{2}{3} \cdot \frac{1}{x-1}$ 20. $g'(y) = \frac{-\frac{1}{y}}{1 - \ln y} = -\frac{1}{y(1 - \ln y)}$ 21. $y' = \left(\frac{2x+1}{x} + 2\ln x\right)y$
 23. $y' = \frac{2xy}{y-2}$ 24a. Do.: $x > 2$, Ra.: all reals
 25a. .033 or .034 b. .146 c. .367 or .368 d. .259, 2.239 or 2.240
 28a. $\boxed{\frac{1}{2} \cdot 4(-10 + 2 \cdot 0 + 2 \cdot 20 + 2 \cdot 30 + 2 \cdot 20 + 2 \cdot 0 - 10)} = 240$

29b. $A = \int_1^3 (e^x - 1) dx$ c. $V = \pi \int_1^3 ((e^x)^2 - 1^2) dx$ d. $V = \int_1^3 (e^x - 1)^2 dx$

29e. $V = \pi \int_1^3 (e^x - 1)^2 dx$ 30. $a = 2$, $b = 4$, $c = 4$



Lesson 5-6 Solving Differential Equations, Exponential Growth and Decay

Differential Equations are equations with derivatives in them. In this course, you will only learn how to solve the simplest type of differential equations, in which you can separate variables. You may be asked to find a general solution of the differential equation (which gives you a family of curves) or a particular solution (which gives you a single curve).

Procedure for Solving Differential Equations

1. Rewrite y' as $\frac{dy}{dx}$ (if necessary).
 2. Multiply both sides of the equation by dx (if necessary).
 3. Separate variables. (This is the most crucial step.)
 4. Integrate both sides of the equation. (Remember to add C to one side.)
 5. Solve for y (if necessary).
 6. Use an initial condition to solve for C (if an initial condition is given).
- Steps 5 and 6 are interchangeable.

Example 1: Find a general solution of $x + 2yy' = 0$

Write your solution to Example 1 as a pair of possible functions (in the form $y = f(x)$) for the particular solutions to the differential equation. $y =$ or $y =$

Example 2: Find an equation of a function which contains the point $(0, -3)$, and whose slope is $\frac{xe^{x^2}}{y}$ for each point (x, y) on the curve.

Example 3:

a. Find a general solution of $y - 2 = x \frac{dy}{dx}$.

b. Find a particular solution of $y - 2 = x \frac{dy}{dx}$ if $y(1) = \frac{1}{2}$.

EXPONENTIAL GROWTH AND DECAY

Mathematical models in which the rate of change of a variable is proportional to the variable itself are common in both the business and scientific worlds.

Suppose that the rate of change of y (with respect to time) is proportional to y itself.

$$\frac{dy}{dt} = k \cdot y$$

Rate of change of y with respect to time	=	constant of proportionality	•	amount of substance y present at time t (y is a function of t)
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Example 4: Separate variables and solve the differential equation above.

The equation from Example 4 is called the **Basic Law of Exponential Growth or Decay**:

$$y = Ce^{kt}$$

Constants:

- C is the initial value
(the amount of substance present at time $t = 0$)
- k is the constant of proportionality
($k > 0$ for growth and $k < 0$ for decay)

Variables:

- t is the variable for time
- y is the amount of substance present at time t . (y is a function of t .)

Example 5: What is the rate of growth of the population in a city whose population triples every 100 years? Assume that the population growth can be modeled by the Basic Law of Exponential Growth, and express your answer as a percent (rounded to the nearest hundredth of a percent).

Example 6: Let y represent the mass, in pounds, of a radioactive element whose half-life is 4000 years. If there are 200 pounds of the element in an inactive mine, how much will still remain in 1000 years? Express your answer to 3 or more decimal place accuracy.

Assignment 5-6

For Problems 1-4, find a general solution of each differential equation.

*Solve for y in Problems 2 and 4.

$$1. \quad y' = \frac{x^2 - 1}{2y^2 + 3} \quad *2. \quad e^x y \frac{dy}{dx} = 1 \quad 3. \quad 2xy' = y + 1 \quad *4. \quad (x - 2) \frac{dy}{dx} = 2y$$

For Problems 5-6, find a particular solution of the differential equation with the given initial condition. (Remember to write your solutions in the form $y = f(x)$.)

- $$5. \quad \frac{dy}{dx} = \frac{-2x}{y} \text{ and } y(2) = -4 \qquad \qquad \qquad 6. \quad y = -3x \frac{dy}{dx} \text{ and } y(1) = e$$
7. Find an equation of a function which contains the point $(-2, 1)$ and whose slope is $\frac{x}{2y}$ for each point on the graph of the function.
8. \$1000 is placed into a certificate of deposit (CD) in which interest is compounded continuously at a rate of $5\frac{1}{2}\%$ per year (actual rate of return will be higher due to compounding of interest). Use your calculator and the formula $A = Pe^{rt}$ to find:
- the amount that the CD would be worth in 1 year. 5 years. 10 years.
 - the time it would take the CD to be worth \$1,200.
 - the time it would take the CD to double in value.
9. Suppose 200 bacteria are introduced into a culture to study their rate of growth. Two days later, the culture is found to contain 300 bacteria. Assuming the rate of growth is proportional to the number of bacteria present ($y = Ce^{kt}$), how many bacteria will be present in 3 more days (5 days after the start)?
10. Find the half-life of a radioactive isotope if 4.92 grams out of an initial 5 grams of the isotope remain after 10 years.

11. An isotope of carbon (C^{14}) is used for estimating how long ago certain living organisms were on earth. (The method is called carbon dating.) The half-life of C^{14} is approximately 5730 years. If the skull of an ancient primate contains 10% (.1) of the C^{14} present in the skull of a modern primate of a similar species, estimate how long ago the ancient primate lived (to the nearest thousand years).
12. A worker at a hazardous waste plant was accidentally exposed to toxic chemicals which were absorbed into his bloodstream. Upon feeling ill, the worker went to a hospital and had some blood drawn for testing. The concentration of chemical in the drawn blood was found to be $.0158 \text{ mg/ml}$. Expensive medication was administered to counter the effects of the chemical in the blood, but the doctor on duty knew that the concentration of the chemical in the bloodstream would have to decrease gradually over time according to the Basic Law of Exponential Decay ($y = Ce^{kt}$). Medication would have to be administered every hour until the concentration was below $.0050 \text{ mg/ml}$. Two hours later, blood was again drawn, and it was found to contain a chemical concentration of $.0126 \text{ mg/ml}$. The doctor asked a lab technician to do the following. (You do the same):
- Write the particular solution for exponential decay for the chemical in the patient's blood. (Let $t = 0$ represent the time that blood was first drawn.)
 - Sketch a graph of the function from Part a.
 - Find out how long it will be before the patient can be taken off medication.
 - When the patient has only a negligible amount of chemical in his bloodstream (less than $.0001 \text{ mg/ml}$), he can be released from the hospital. Find out how long the patient has to be hospitalized (from the time he first came to the hospital and had his blood drawn).
 - Occasionally, patients exposed to this chemical suffer damage to their central nervous systems. A maximum concentration of $.020 \text{ mg/ml}$ requires a follow-up examination. The doctor estimated that the maximum concentration of the chemical in the worker's bloodstream occurred 1 hour after exposure. The patient estimated 1 hour after exposure would have been about 3 hours prior to his blood being drawn for the first time. Should the patient be asked to return for a follow-up exam? Why or why not?
 - Find the half-life for the chemical in the bloodstream for the patient.

Simplify in Problems 13 and 14.

13. $\ln \frac{e^2}{e^{\sqrt{x}}}$

14. $e^{\ln a - \ln b}$

Differentiate in Problems 15-17.

15. $y = 3^{2t-1} t^2$

16. $f(y) = \frac{e^{\sqrt{y}}}{y^2}$

17. $f(x) = e^x \ln x$

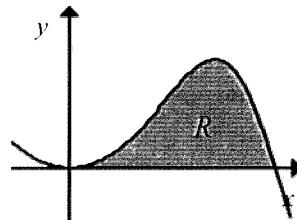
Antidifferentiate in Problems 18-20.

18. $y' = \frac{e^{-x}}{1+e^{-x}}$

19. $g'(x) = \frac{2x-4}{x}$

20. $y' = \frac{(\ln t)^3}{t}$

21. Find the area of the region bounded by $y = \left(\frac{1}{2}\right)^x$, $y = 0$, $x = -2$, and $x = 0$ without using a calculator.
22. Without using a calculator, find the volume of the solid formed by revolving the region bounded by $y = \sqrt{\frac{x}{x^2 + 2}}$, $y = 0$, $x = 0$, and $x = 2$, about the x -axis.
23. Without using a calculator, find the volume of the solid formed by square cross sections perpendicular to the y -axis, whose base is the region bounded by $xy = 4$, $x = 0$, $y = 1$, and $y = 4$.
24. Region R is shown bounded by the graph of $y = \sin x^2$ and the x -axis. Without using a calculator find the volume of the solid formed when region R is revolved about the y -axis.



Use a calculator for Problems 25-31.

25. Find the perimeter of region R from Problem 24.
26. Find the average value of $y = e^{-t^2}$ on $[-1, 1]$. Show an integral set-up.

For Problems 27-30, $f(x) = 3x^2 + \ln|x|$.

27. Find $f'(4)$ 28. Find $\int_1^7 f(x) dx$ 29. Solve $3x^2 + \ln|x| = 0$

30. f is discontinuous when $x = 0$. Is the discontinuity a hole, an asymptote, or a jump?

31. For a particle moving along a straight path with velocity $v(t) = e^{-1.3t} - t \ln(.37t)$, $t > 0$, use your calculator to find:
- the time when the particle is at rest.
 - the speed of the particle at time $t = 4$.
 - the acceleration of the particle at time $t = 5$.
 - the total distance traveled by the particle on the interval $[1, 5]$.
32. Convert $e^{2x-3} = 7$ from exponential form to logarithmic form. Then solve for x .
33. Use log properties to condense the equation. Then solve for x .
- $$3\log_2 x + \log_2(x+1) - 2\log_2 x = 1$$
34. f and g are inverse functions. The graph of g passes through the points $(-1, 2)$, and $(2, -1)$. $f'(-1) = -2$ and $f'(2) = -1$. Find:
- $g'(-1)$
 - $g'(2)$

Selected Answers:

1. $\frac{2}{3}y^3 + 3y = \frac{1}{3}x^3 - x + C$ 2. $y = \sqrt{-2e^{-x} + C}$ or $y = -\sqrt{-2e^{-x} + C}$

3. $\ln|y+1| = \frac{1}{2}\ln|x| + C$ or $\ln|y+1| = \frac{1}{2}\ln|2x| + C$ 4. $y = C(x-2)^2$

5. $y = -\sqrt{-2x^2 + 24}$ 6. $y = e x^{-\frac{1}{3}}$ 7. $y = \sqrt{\frac{1}{2}x^2 - 1}$

8a. $A(1) = \$1056.54$, $A(5) = \$1316.53$, $A(10) = \$1733.25$ c. 12.602 or 12.603 yrs

9. 551 bacteria 10. 429.741 or 429.742 yrs 11. 19,000 yrs

14. $\frac{a}{b}$ 16. $f'(y) = \frac{y^2 e^{\sqrt{y}} \cdot \frac{1}{2} y^{-\frac{1}{2}} - e^{\sqrt{y}} \cdot 2y}{y^4}$ 17. $f'(x) = \frac{e^x}{x} + e^x \ln x$

18. $y = -\ln(1 + e^{-x}) + C$ 21. $\frac{3}{\ln 2}$ 22. $\frac{1}{2}\pi \ln 3$ 23. 12 26. .746 or .747

28. 349.621 29. $x = \pm .488$ or $\pm .489$ 31a. $t = 2.731$ c. $a(5) = -1.617$

31d. $TD = 4.510$ or 4.511 32. $x = \frac{\ln 7 + 3}{2}$ 33. $x = 1$ 34b. $-\frac{1}{2}$

CALCULUS EXTENDED UNIT 5 SUMMARY**Shell Method Volume Formula:**

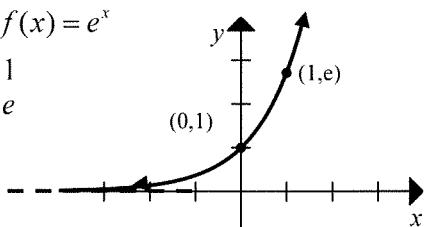
$$V = 2\pi \int_a^b r h \, dx \text{ (or } dy\text{)} \quad r > 0, h > 0$$

Arc Length Formula:

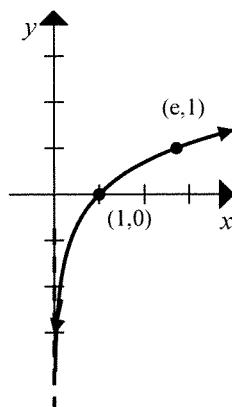
$$\text{Arc Length} = \int_a^b \sqrt{1 + (f'(x))^2} \, dx$$

Exponential and Logarithmic Graphs:Graph of $f(x) = e^x$

$$\begin{aligned} e^0 &= 1 \\ e^1 &= e \end{aligned}$$

Graph of $g(x) = \ln x$

$$\begin{aligned} \ln 1 &= 0 \\ \ln e &= 1 \end{aligned}$$



All basic exponential ($f(x) = a^x$) and logarithmic ($g(x) = \log_a x$) graphs with $a > 0$ are similar to the graphs shown above.

CALCULUS EXTENDED UNIT 5 SUMMARY (CONTINUED)

$f(x) = e^x$ and $g(x) = \ln x$ are inverse functions, so $\ln e^x = e^{\ln x} = x$.

Change of Form Definition:

Exponential form $\left\{ \begin{array}{l} x = e^y \Leftrightarrow y = \ln x \\ x = a^y \Leftrightarrow y = \log_a x \end{array} \right.$	Logarithmic form
---	------------------

Properties of Logarithms: only true when $a > 0$ and $b > 0$

$$1. \quad \ln(ab) = \ln a + \ln b \quad 2. \quad \ln \frac{a}{b} = \ln a - \ln b \quad 3. \quad \ln a^n = n \ln a$$

Change of Base: $\log_a x = \frac{\ln x}{\ln a}$

Derivatives of Inverse Functions:

If f and g are inverse functions, then $f'(a) = \frac{1}{g'(b)}$ where (a, b) is a point on the graph of f and (b, a) is the “image point” on the graph of g .

Procedure for Logarithmic Differentiation: $y = \text{variable base}^{\text{variable exponent}}$

1. Take the natural log (**In**) of both sides of the equation.
2. Simplify the right hand side of the equation (Log Property 3).
3. Differentiate both sides.
4. Solve for y' .
5. Substitute for y if necessary.

Differential Equations: (Equations involving derivatives.)

Solve by separating variables and integrating both sides of the equation.

Procedure for Solving Differential Equations

1. Rewrite y' as $\frac{dy}{dx}$ (if necessary).
 2. Multiply both sides of the equation by dx (if necessary).
 3. Separate variables. (This is the most crucial step.)
 4. Integrate both sides of the equation. (Remember to add C to one side.)
 5. Solve for y (if necessary).
 6. Use an initial condition to solve for C (if an initial condition is given).
- Steps 5 and 6 are interchangeable.

Exponential Growth and Decay: $y = Ce^{kt}$

CALCULUS EXTENDED UNIT 5 SUMMARY (CONTINUED)**Derivatives**

	<i>x</i> form	<i>u</i> form (chain rule)
Exponential Rule	1. $\frac{d}{dx} e^x = e^x$ 2. $\frac{d}{dx} a^x = a^x \ln a$	$\frac{d}{dx} e^u = e^u u'$ $\frac{d}{dx} a^u = a^u u' \ln a$
Log Rule	3. $\frac{d}{dx} \ln x = \frac{1}{x}$ 4. $\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$	$\frac{d}{dx} \ln u = \frac{u'}{u}$ $\frac{d}{dx} \log_a u = \frac{u'}{u \ln a}$

Integrals

	<i>x</i> form	<i>u</i> form (reverse chain rule)
Exponential Rule	1. $\int e^x dx = e^x + C$ 2. $\int a^x dx = \frac{a^x}{\ln a} + C$	$\int e^u u' dx = e^u + C$ $\int a^u u' dx = \frac{a^u}{\ln a} + C$
Log Rule	$3. \int \frac{1}{x} dx = \ln x + C$	$\int \frac{u'}{u} dx = \ln u + C$
Trig Rules	4. $\int \tan x dx = -\ln \cos x + C$ 5. $\int \cot x dx = \ln \sin x + C$	$\int \tan u u' dx = -\ln \cos u + C$ $\int \cot u u' dx = \ln \sin u + C$

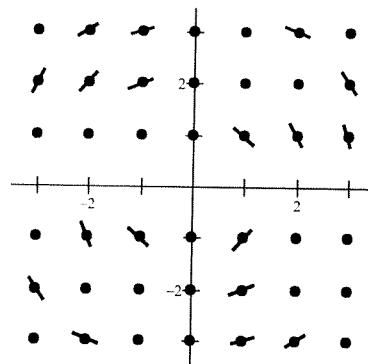
Lesson 6-1 Slope Fields, Euler's Method

A **slope field** is a graphical representation of a set of slopes obtained from a differential equation. Remember that a differential equation involves a derivative. That derivative represents the slopes for a function. Even if you cannot separate variables and integrate, you can still use a differential equation to plot the slopes for a function.

Example 1: Find the slopes given by the differential equation $\frac{dy}{dx} = \frac{-x}{y}$ at the following points:
 a. (3, 2) b. (-1, 3) c. (-2, -1) d. (2, -2)

Why can't you find slopes when $y = 0$?

Example 2: Find and plot the slopes given by $\frac{dy}{dx} = \frac{-x}{y}$ for each remaining marked point (dot) in the coordinate plane at the right.



Example 3: In Example 2, you made what is known as a slope field. Starting at the point (0, 1), follow the flow of the slopes to sketch the solution curve containing (0, 1). Your graph should be "parallel" to the slope lines and be like an "average of slopes" whenever it goes between lines. Your solution curve must represent a function whose domain is the largest possible open interval containing the given point. Sketch a solution curve passing through (-1, 1) and one passing through (0, -3).

Note: The most common student error in sketching a particular solution to a differential equation is to extend the sketch too far and create a graph which is not a function.

Example 4: Solve the differential equation $\frac{dy}{dx} = \frac{-x}{y}$.

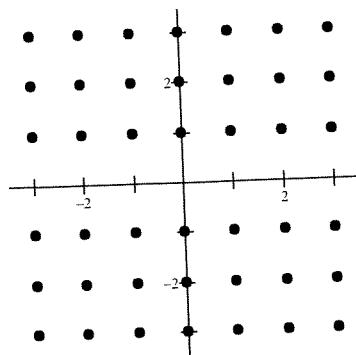
Note: Solving for y yields $y = \underline{\hspace{2cm}}$ or $y = \underline{\hspace{2cm}}$.

Find the particular solution for this differential equation whose graph passes through the point (0, 1).

Find the particular solution whose graph passes through the point (0, -3).

Example 5: For the differential equation $y' = \frac{1}{y}$

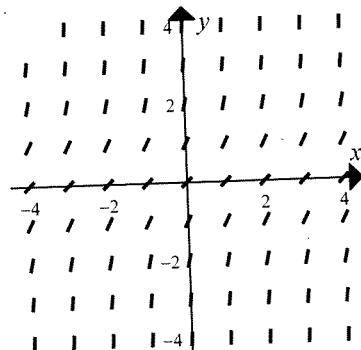
- Draw the slope field in the dot coordinate plane at the right.
- Graph the particular solutions passing through the points $(-2, -1)$ and $(2, 2)$ as functions of x .
- Solve the differential equation.



- Write as functions the particular solutions for the differential equation whose graphs pass through $(-2, -1)$ and $(2, 2)$.

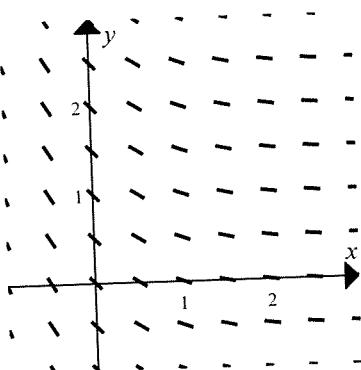
Example 6: Which of the differential equations below matches the slope field shown at the right?

- $y' = x$
- $y' = y$
- $y' = x - y$
- $y' = 1 + y^2$
- $y' = 1 + x^2$



Example 7: The slope field for a certain differential equation is shown at the right. Which of the following could be a specific solution to the differential equation?

- $y = e^x$
- $y = e^{-x}$
- $y = -e^x$
- $y = -\ln x$
- $y = \ln x$

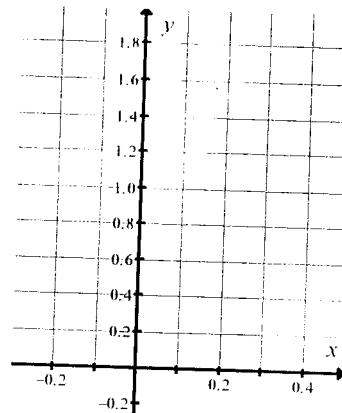


Euler's Method This is a more precise method of graphing an approximate solution to a differential equation.

Example 8. Use Euler's method to construct an approximate solution for the differential equation $\frac{dy}{dx} = y$. Start at the point $(0,1)$ and use step size $\Delta x = .1$

x	y	$\frac{dy}{dx}$	$\Delta y = (\text{slope}) \Delta x$

$$y(.3) \approx$$



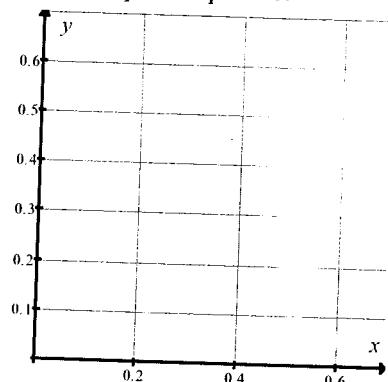
Example 9. Solve $\frac{dy}{dx} = y$ algebraically. Fill in the table with the actual values of y .

x	y
0	1
.1	
.2	
.3	

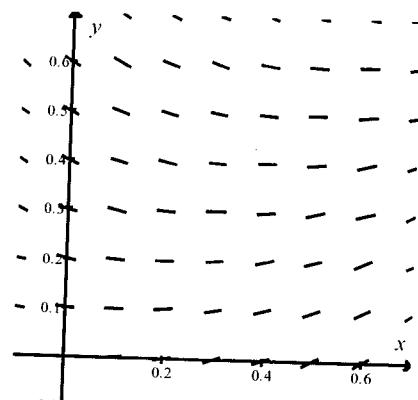
Example 10. Use Euler's Method to approximate the particular solution of the diff. eq. $y' = x - y$ passing through the point $(0, 0.5)$. Let $\Delta x = .2$ and do three steps ($n = 3$). Graph the points.

x	y	$\frac{dy}{dx}$	$\Delta y = (\text{slope}) \Delta x$

$$y(.6) \approx$$

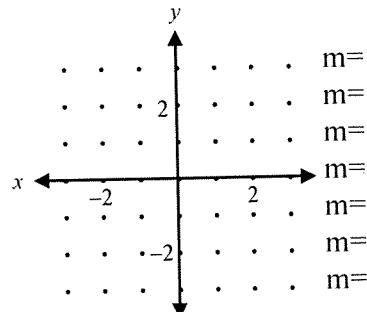
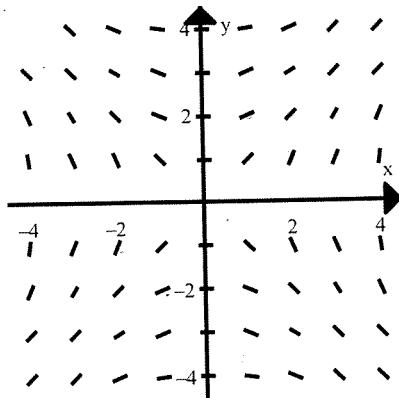


Example 11. Sketch a particular solution of the diff. eq. $y' = x - y$ passing through the point $(0, 0.5)$ using the slope field given. Do the two graphs coincide?



Assignment 6-1

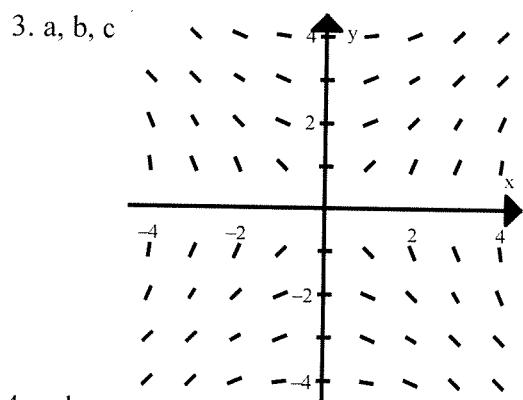
1. Find the slopes given by the differential equation $y' = \frac{x^2}{y-2}$ at each of the following points:
- (0,0)
 - (1,1)
 - (-2,4)
 - (4,-2)
 - (-3,-3)
 - (5,12)
2. For the differential equation in Problem 1, why are there no slopes when $y=2$?
3. The slope field for $y' = \frac{x}{y}$ is shown at the right.
- Plot the following points on the slope field:
 - (1, 2)
 - (3, 1)
 - (0, 3)
 - (0, -2)
 - (-2, -1)
 - Plot a separate solution curve through each of the points from Part a. Remember that the curves have to be functions.
 - What would a solution curve containing (2, 2) look like?
 - Solve the differential equation $y' = \frac{x}{y}$.
4. For the differential equation $\frac{dy}{dx} = y$
- Draw the slope field for the differential equation.
 - Graph the particular solutions passing through the points (0,1) and (0,-1).
 - Solve the differential equation, and find the particular solutions that contain the points (0, 1) and (0, -1).
5. For the differential equation $\frac{dy}{dx} = \frac{x}{2}$
- Draw the slope field for the differential equation.
 - Graph the particular solution passing through the point (-1,1).
 - Solve the differential equation, and find the particular solution that contains the point (-1,1).
6. Repeat the three parts of Problem 5 for the differential equation $y' = \frac{1}{2y}$. For this problem, draw your graph as and write your solution as a function of x .
7. Repeat the three parts of Problem 5 for the differential equation $y'y^2 = 1$.
8. Repeat the three parts of Problem 5 for the differential equation $\frac{dy}{dx} = 2x(y-1)$, but use the origin (instead of (-1,1)) for Parts b. and c.



Assignment 6-1 tear-out page

Name _____ Per _____

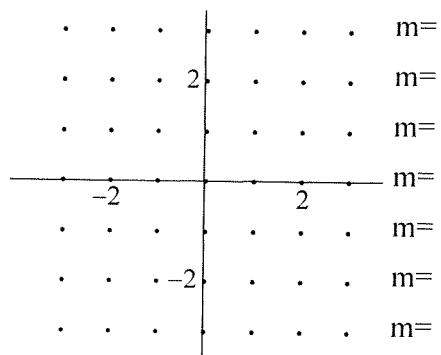
1. a. ____ b. ____ c. ____ d. ____ e. ____ f. ____ 2.



3. d. solve $y' = \frac{x}{y}$

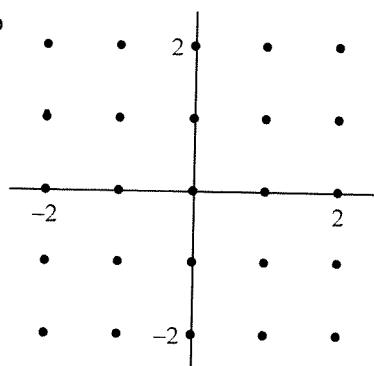
4. a, b

4. c.



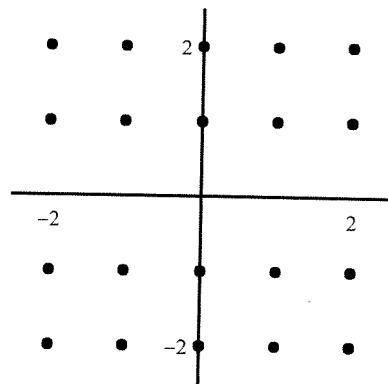
5. a, b

5. c.

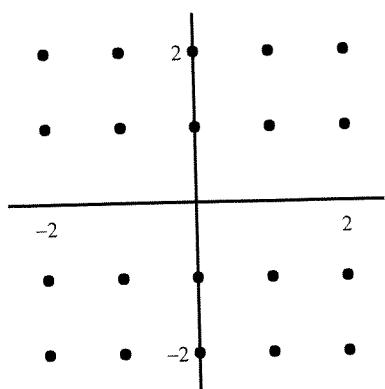


6. a, b

6. c.

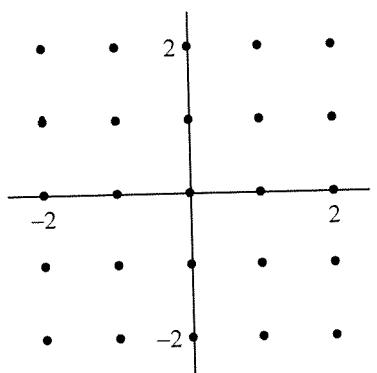


7. a, b



7. c.

8. a, b

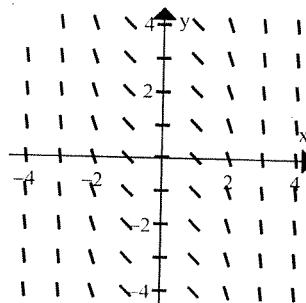


8. c.

9. ____ 10. ____

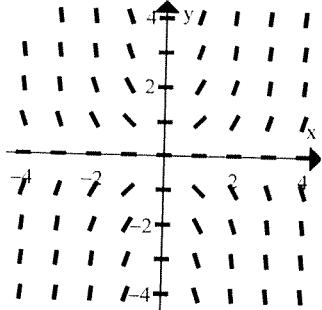
9. The slope field for a certain differential equation is shown at the right. Which of the following could be a specific solution to that differential equation.

a. $\frac{1}{3}x^3 + y = 2$ b. $x^2 + y^2 = 4$
 c. $x^2 - y^2 = 4$ d. $y = \frac{4}{x}$ e. $\frac{y}{x} = 4$



10. Which of the differential equations below matches the slope field shown at the right?

a. $\frac{dy}{dx} = x - y$ b. $\frac{dy}{dx} = y - x$
 c. $\frac{dy}{dx} = \frac{x}{y}$ d. $\frac{dy}{dx} = \frac{y}{x}$ e. $\frac{dy}{dx} = xy$



The differential equation $x^3y'' = 4x^3y$ cannot be solved using the method of separation of variables. Use substitution to determine if the function given is a solution of this differential equation.

11. $y = \sin(2x)$

12. $y = e^{2x}$

13. **Without using a calculator**, use Euler's Method with a step size of 0.1 to approximate $f(3)$ if $f(0) = 3$ and $f'(x) = x + y$.

14. **Without using a calculator**, use Euler's Method with 3 steps each with a size of $\frac{1}{2}$ to approximate a y-value if $y(0) = 2$ and $y' = 2x - 3y$.

15. **Using a calculator**, if $y(1) = 2$ and $y' = e^{xy}$ use 4 steps of Euler's Method to approximate $y(0.8)$.

16. If $f'(t) = \lim_{h \rightarrow 0} \frac{t+h+\sqrt{t+h}-(t+\sqrt{t})}{h}$, find $f(t)$

Find the x -value(s) where each of the functions in Problems 17-21 is not differentiable. Give a reason why each function is not differentiable for those values of x .

17. $f(x) = |x^2 - 9|$

18. $g(x) = \|2(x-1)\|$

19. $p(x) = \frac{x^2(x-2)}{x(x+1)}$

20. $q(x) = x - x^{\frac{1}{3}}$

21. $h(x) = \begin{cases} 2x+1, & x \leq 1 \\ \frac{1}{2}x^2+x+\frac{3}{2}, & x > 1 \end{cases}$

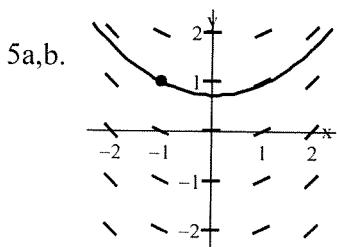
22. If a particle moves along the curve $y = x^{\frac{2}{3}}$, such that $\frac{dx}{dt} = 3$ for all x , find:
- $\frac{dy}{dt}$ when $x = -1$
 - $\frac{dy}{dt}$ when $x = 8$
 - $\lim_{x \rightarrow \infty} \frac{dy}{dt}$
 - $\lim_{x \rightarrow 0^-} \frac{dy}{dt}$
23. a. Use a tangent line to the graph of $y = x^{\frac{2}{3}}$ to approximate $(8.1)^{\frac{2}{3}}$.
 b. Why could a tangent line to $y = x^{\frac{2}{3}}$ at $x = 0$ not be used to approximate $(.1)^{\frac{2}{3}}$?
24. Without using a calculator, find the domain, x - and y -intercepts, asymptotes, and holes for $y = \frac{2x^3 - 2x}{x^3 - 4x}$. (Remember to factor, reduce, and use the reduced function for everything except domain restrictions.) Then, sketch the graph of $y = \frac{2x^3 - 2x}{x^3 - 4x}$ without using a calculator. Check your graph with a calculator.
25. A region in Quadrant 1 is enclosed by the graphs of $y = x$, $y = \cos x$, and the y -axis. Sketch the region, draw a representative rectangle, set up an integral, and find the volume of the solid for each revolution described. Use a calculator, and make sure that it is radian mode.
 - about the x -axis.
 - about $y = -1$
 - about the y -axis
 - about $x = -1$
- Evaluate in Problems 26-31 without using a calculator.
26. $\int_0^2 \frac{1}{5-2t} dt$
27. $\int_2^3 \frac{y+1}{y-1} dy$
28. $\int \frac{x^2}{x^3 + 10} dx$
29. $\int \frac{du}{u(\ln u)^3}$
30. $\int 5^{2x} dx$
31. $\int 5^2 dx$
32. Sketch a graph of a function $f(x)$ having the following characteristics:
 f is continuous. $f(-2) = f(0) = 0$.
 $f'(0)$ is undefined. $f'(x) > 0$ for $x < -1$ and for $x > 0$. $f'(x) < 0$ on $(-1, 0)$.
 $f''(x) < 0$ for $x < 0$, and $f''(x) > 0$ for $x > 0$
33. Find the equation of a tangent line to $x^2 + 3yx = y^2 + 3$ at $(1, 2)$.
34. A radioactive element has a half-life of 1000 years. How much of 200 grams of the element will remain after 750 years?

Selected Answers

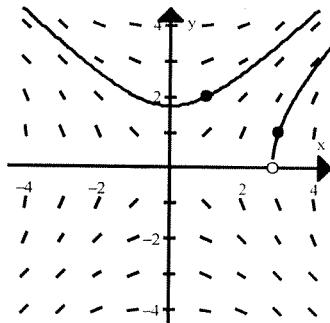
1a. 0 c. 2 e. $-\frac{9}{5}$

3d. $y = \sqrt{x^2 + C}$ or $y = -\sqrt{x^2 + C}$

4c. $y = e^x$ for $(0, 1)$



c. $y = \frac{1}{4}x^2 + \frac{3}{4}$



6c. $y = \sqrt{x+2}, x > -2$

7c. $y = \sqrt[3]{3x+4}, x > -\frac{4}{3}$

8c. $y = -e^{x^2} + 1$

11. not a solution 13. $f(0.3) \approx 4.024$ 14. $y\left(\frac{3}{2}\right) \approx \frac{1}{2}$ 15. $y(0.8) \approx 1.078$ or 1.079

17. $x = \pm 3$ (sharp turns) 19. $x = 0$ (hole), $x = -1$ (VA) 20. $x = 0$ (vertical tangent)

22a. -2 c. 0 23a. $4\frac{1}{30}$ 24. Do: $x \neq 0, \pm 2$, $y_{red} = \frac{2(x-1)(x+1)}{(x-2)(x+2)}$, VA: $x = \pm 2$ (odd),

Hole: $\left(0, \frac{1}{2}\right)$, x-int: $(\pm 1, 0)$ (odd), no y-int., HA: $y = 2$ 25a. 1.520 b. 4.036 or 4.037

25c. .643 d. 3.159 or 3.160 26. $\ln \sqrt{5}$ 27. $1 + 2 \ln 2$ 29. $-\frac{1}{2}(\ln u)^{-2} + C$

31. $25x + C$ 33. $y - 2 = 8(x - 1)$ 34. 118.920 or 118.921 grams

Lesson 6-2. Inverse Trig Functions

Definition of the Inverse Trig Functions:

Function

Domain (x values)

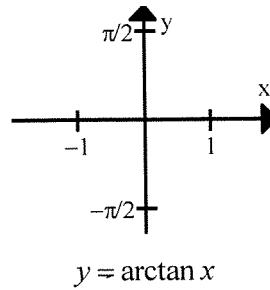
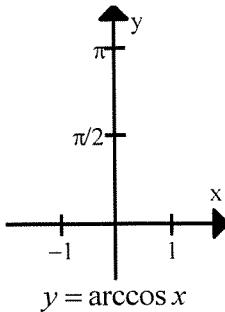
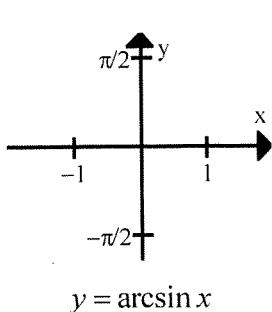
*Range (y values)

$$y = \arcsin x \leftrightarrow \sin y = x \quad [-1, 1] \quad \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

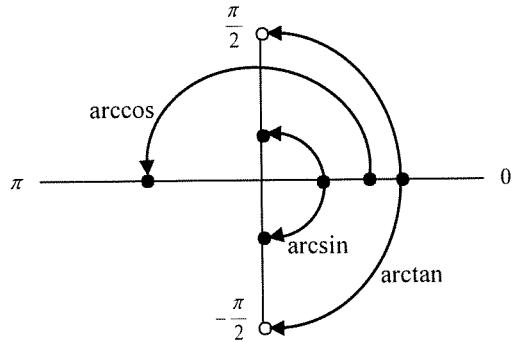
$$y = \arccos x \leftrightarrow \cos y = x \quad [-1, 1] \quad [0, \pi]$$

$$y = \arctan x \leftrightarrow \tan y = x \quad (-\infty, \infty) \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Example 1: Graph the indicated inverse trig functions in the coordinate planes below:



*A geometric representation of the range values for each inverse trig function is shown in the coordinate plane at right.



Example 2: Evaluate without a calculator.

a. $\arctan 1$

b. $\cos^{-1}(-1)$

c. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

d. $\arcsin 2$

Example 3: Solve for x . $\arcsin(x^2 - 3) = \frac{\pi}{2}$

Examples: Sketch a right triangle, and evaluate without a calculator.

4. Find $\tan x$, given that $x = \arccos \frac{2}{\sqrt{5}}$ 5. Find $\cos y$, given that $y = \arcsin x$

Example 6: Use your answer from Example 5 to find $\frac{d}{dx} \arcsin x$.

Derivatives of the Inverse Trig Functions:

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arcsin u = \frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arccos u = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$$

$$\frac{d}{dx} \arctan u = \frac{u'}{1+u^2}$$

(where u is a function of x)Examples: Differentiate.

7. $g(y) = \arctan(2y-1)$

8. $f(x) = \arcsin \sqrt{x}$

9. $h(t) = \arccos(\ln t)$

10. $y = \arcsin \frac{x}{2}$ simplify

Integration of Inverse Trig Functions

1. $\int \frac{u'}{\sqrt{a^2 - u^2}} dx = \arcsin \frac{u}{a} + C$

2. $\int \frac{u'}{a^2 + u^2} dx = \frac{1}{a} \arctan \frac{u}{a} + C$

Examples: Integrate.

11. $\int \frac{1}{\sqrt{4-x^2}} dx$

12. $\int \frac{dx}{4x^2 + 25}$

13. $\int \frac{8}{3+4x^2} dx$

14. $\int \frac{8x}{3+4x^2} dx$

15. $\int \frac{8x^2}{3+4x^2} dx$

16. $\int \frac{x+4}{\sqrt{4-x^2}} dx$

Assignment 6-2

Evaluate the expressions in Problems 1-4 without using a calculator.

1. $\arcsin \frac{\sqrt{3}}{2}$ 2. $\arctan(-1)$ 3. $\arccos\left(\frac{-1}{2}\right)$ 4. $\arctan \sqrt{3}$

In Problems 5 and 6, solve for x without using a calculator.

5. $\arctan(3-x) = \frac{-\pi}{4}$ 6. $\arccos(x^2 - 2) = \pi$

For Problems 7 and 8, evaluate without using a calculator. First, sketch a triangle for each problem.

7. Find $\cos y$, given that $y = \arcsin\left(\frac{-4}{5}\right)$. 8. Find $\sin x$, given that $x = \arctan(3)$.

Differentiate in Problems 9-14. Do not use a calculator.

9. $y = 2 \arctan(3x)$ 10. $f(x) = \arcsin(x^2 - 1)$ 11. $g(y) = \arcsin e^{-y}$
12. $h(t) = \arctan t^{\frac{3}{2}}$ 13. $y = x^2 \arccos x$ 14. $f(\theta) = \arctan(\ln \theta)$

Evaluate the integrals in Problems 15-26. Do not use a calculator.

15. $\int_0^{\frac{1}{4}} \frac{1}{\sqrt{1-4x^2}} dx$ 16. $\int_0^{\frac{5}{3}} \frac{2}{9x^2 + 25} dx$ 17. $\int_0^1 \frac{x^2}{x+1} dx$ 18. $\int \frac{8}{2+(2t+1)^2} dt$
19. $\int \frac{w^2}{\sqrt{4-w^6}} dw$ 20. $\int \frac{dx}{x\sqrt{16-(\ln x)^2}}$ 21. $\int \frac{\sqrt{\arctan \theta}}{1+\theta^2} d\theta$ 22. $\int \frac{5x}{x^2+1} dx$
23. $\int \frac{5x^2}{x^2+1} dx$ 24. $\int \frac{\cos t}{\sqrt{25-\sin^2 t}} dt$ 25. $\int \frac{e^{-2v}}{3+e^{-4v}} dv$ 26. $\int \frac{2x}{\sqrt[3]{3x+1}} dx$

Do not use a calculator on problems 27-32.

27. Find the average value of $y = \frac{6}{1+(2x)^2}$ on the interval $\left[0, \frac{1}{2}\right]$.

28. Find the area of the region bounded by $y = \frac{3}{\sqrt{9-x^2}}$, $y = 0$, $x = 0$, and $x = 1.5$.

29. For $4 \cos x \sin y - \sqrt{6} = 0$, find $\frac{dy}{dx}$ at $\left(\frac{\pi}{6}, \frac{\pi}{4}\right)$.

Integrate in Problems 30-32.

30. $\int \frac{\sec^2(2x)}{\sqrt{\tan(2x)}} dx$ 31. $\int \frac{e^t \cos e^t}{\sin e^t} dt$ 32. $\int \cot^2(3x) dx$ Hint: You need to make a trig substitution before integrating.

33. Use a calculator to find $(g^{-1})'(2)$ for $g(x) = \sqrt{x^3 + 2x + 5}$.

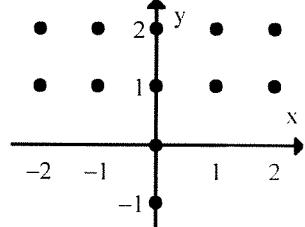
34. Consider the differential equation given by $\frac{dy}{dx} = \frac{xy}{2}$.

- a. Sketch a slope field on your own paper for the given differential equation at the twelve indicated points, and sketch a particular solution containing the point $(0, 1)$.

- b. Use Euler's Method starting at the point $(0, 1)$ with step size $\Delta x = 1$ to find an approximate y -value when $x = 2$.

- c. Solve the differential equation $\frac{dy}{dx} = \frac{xy}{2}$ and find the particular solution containing the point $(0, 1)$. Give your answer in the form $y = f(x)$.

- d. Use your solution to find the exact y -value when $x = 2$.



35. The number of rabbits on an island during a measured five year period of time was found to model the function $N(t) = 1000 \sin(2t) - 300 \cos(t) + 3000$ on the interval $[0, 5]$ years.

- a. Use a calculator to find $N(2)$ and $N(4)$. Specify units and round to the nearest whole number.

- b. Find $N'(t)$ without using a calculator.

- c. Use a calculator to find $N'(1)$ and $N'(3)$. Specify units and round to the nearest whole number.

- d. At what time(s) during the five years of population measurement was the rabbit population the greatest? Use an N' number "line" (actually, a segment) to justify your answer.

- e. What was the greatest number of rabbits on the island during the five years which were measured?

- f. Does $N(t)$ go through at least one full period (cycle) during the five years?

- g. What is the period of $N(t)$?

36. In 1990, the population of a city was 123,580. In 2000, the city's population was 152,918. Assuming that the population is increasing at a rate proportional to the existing population, use your calculator to estimate the city's population in 2025. Express your answer to the nearest person.

Selected Answers:

- $$1. \frac{\pi}{3} \quad 2. -\frac{\pi}{4} \quad 3. \frac{2\pi}{3} \quad 5. x=4 \quad 6. x=\pm 1 \quad 7. \cos y = \frac{3}{5} \quad 8. \sin x = \frac{3}{\sqrt{10}}$$
- $$9. y' = \frac{6}{1+9x^2} \quad 11. g'(y) = \frac{-e^{-y}}{\sqrt{1-(e^{-y})^2}} \quad 12. h'(t) = \frac{3\sqrt{t}}{2(1+t^3)} \quad 13. \frac{-x^2}{\sqrt{1-x^2}} + 2x \arccos x$$
- $$15. \frac{\pi}{12} \quad 16. \frac{\pi}{30} \quad 17. -\frac{1}{2} + \ln 2 \quad 18. \frac{4}{\sqrt{2}} \arctan \frac{2t+1}{\sqrt{2}} + C \quad 20. \arcsin \frac{\ln x}{4} + C$$
- $$21. \frac{2}{3} (\arctan \theta)^{\frac{3}{2}} + C \quad 23. 5x - 5 \arctan x + C \quad 25. -\frac{1}{2\sqrt{3}} \arctan \frac{e^{-2v}}{\sqrt{3}} + C$$
- $$26. \frac{2}{15} \left(\sqrt[3]{3x+1} \right)^5 - \frac{1}{3} \left(\sqrt[3]{3x+1} \right)^2 + C \quad 27. \frac{3\pi}{2} \quad 28. \frac{\pi}{2} \quad 29. \frac{1}{\sqrt{3}} \quad 30. \sqrt{\tan(2x)} + C$$
- $$31. \ln |\sin e'| + C \quad 32. -\frac{1}{3} \cot(3x) - x + C \quad 33. 1.528 \text{ or } 1.529 \quad 34c. y = e^{\frac{1}{4}x^2}$$
- $$35b. N'(t) = 2000 \cos(2t) + 300 \sin t \quad d. 3.876 \text{ or } 3.877 \text{ years} \quad e. 4217 \text{ rabbits} \quad g. 2\pi \text{ years}$$
- $$36. 260,456 \text{ people}$$

Lesson 6-3**Integration by Parts**

Integration by parts is a method of integration used mainly for products of algebraic and transcendental functions (such as $\int xe^x dx$) or products of two transcendental functions (such as $\int e^x \sin x dx$).

Development of the formula for integration by parts: If u and v are both functions of x , then

$$\frac{d}{dx}(uv) =$$

Formula for integration by parts:

$$\int uv' dx = uv - \int vu' dx \quad \text{or} \quad \int u dv = uv - \int v du$$

Strategy: Let u be the part whose derivative is “simpler” (or at least no more complicated) than u itself. Let dv be the more complicated part (or the part which can easily be integrated). Also, remember that you typically have only two choices. If one choice doesn’t work, try the other.

Example 1. $\int xe^x dx$

Let $u =$

$du =$

Let $dv =$

$v =$

Example 2. $\int x \sin(3x) dx$

Example 3. $\int \arcsin x dx$

$\int xe^x dx =$

$\int x \sin(3x) dx =$

$\int \arcsin x dx =$

Example 4. $\int x^2 \sin(2x) dx$

Example 5. $\int_1^e x^2 \ln x dx$

Example 6. Complete

the square to find

$$\int \frac{1}{x^2 + 4x + 8} dx .$$

Assignment 6-3:

Integrate without using a calculator. Integration by parts will be used on most, but not all problems.

1. $\int x \sin x \, dx$

2. $\int x \cos(2x) \, dx$

3. $\int 4x e^{2x} \, dx$

4. $\int x^2 e^{x^3} \, dx$

5. $\int \frac{x}{e^x} \, dx$

6. $\int \ln x \, dx$

7. $\int \frac{\ln x}{x^2} \, dx$

8. $\int \frac{(\ln x)^2}{x} \, dx$

9. $\int \arctan x \, dx$

10. $\int x^2 \cos x \, dx$

11. $\int x^2 e^{2x} \, dx$

12. $\int_0^1 \frac{x}{1+x^2} \, dx$

13. $\int_{-3}^{-1} \frac{1}{\sqrt{7-x^2-6x}} \, dx$

14. $\int_0^1 x e^{3x} \, dx$

15. $\int_1^{e^2} x \ln x \, dx$

16. $\int x \arctan x \, dx^*$

Differentiate.

17. $y = 2 \arctan(e^{2x})$

18. $f(x) = \ln|\arcsin x|$

19. $g(t) = \sin(\arctan t)$

* $\int \frac{x^2}{1+x^2} \, dx$ can be integrated using long division

Evaluate the expressions. Remember not to use a calculator.

20. $\arccos \frac{1}{2}$

21. $\arcsin\left(\frac{-\sqrt{3}}{2}\right)$

22. $\arctan\left(\frac{-1}{\sqrt{3}}\right)$

23. Sketch a triangle to find $\tan y$, given that $y = \arcsin \frac{2}{3}$.

24. $\arcsin(x^2 - 1) = \frac{\pi}{6}$. Solve for x .

Let R be the region bounded by $y = e^{\frac{-x}{2}}$, $y = 0$, $x = 0$, and $x = k$, ($k > 0$).

25. Sketch Region R , set up an integral for its area, and find the area (in terms of k).

26. Find the volume (in terms of k) of the solid formed by revolving Region R about the x -axis.

27. Find the volume (in terms of k) of the solid formed by revolving Region R about the y -axis.

28. If $3x^2 - 2y^2 = 4xy + 20$, find $\frac{dy}{dx}$.

29. Find the point(s) at which the graph from Problem 28 has vertical tangents.

30. Find equations of lines tangent and normal to the graph from Problem 28 at the point $\left(-\sqrt{\frac{20}{3}}, 0\right)$.

31. Find a general solution of $\sqrt[3]{y} y' - \sqrt[3]{x} = 0$.

32. Find the particular solution of $2xyy' = \ln x$, if the graph of the particular solution contains the point $(e, 1)$. Make sure that your answer expresses y as a function of x . ($y = f(x)$)

33. Use Euler's method with three steps to approximate $y(0.6)$ if $y(0) = 2$ on the solution of the differential equation $y' = x + y$. Do **not** use a calculator.

Selected Answers:

1. $-x \cos x + \sin x + C$ 2. $\frac{1}{2}x \sin(2x) + \frac{1}{4}\cos(2x) + C$ 3. $2xe^{2x} - e^{2x} + C$
 5. $-xe^{-x} - e^{-x} + C$ 6. $x \ln x - x + C$ 7. $-\frac{1}{x} \ln x - \frac{1}{x} + C$ 8. $\frac{1}{3}(\ln x)^3 + C$
 9. $x \arctan x - \frac{1}{2} \ln(1+x^2) + C$ 10. $x^2 \sin x + 2x \cos x - 2 \sin x + C$
 11. $\frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$ 12. $\frac{1}{2} \ln 2$ 13. $\frac{\pi}{6}$ 14. $\frac{2}{9}e^3 + \frac{1}{9}$
 15. $\frac{3}{4}e^4 + \frac{1}{4}$ 16. $\frac{1}{2}x^2 \arctan x - \frac{1}{2}x + \frac{1}{2} \arctan x + C$ 17. $y' = \frac{4e^{2x}}{1+e^{4x}}$
 18. $f'(x) = \frac{1}{\arcsin x \sqrt{1-x^2}}$ 20. $\frac{\pi}{3}$ 22. $-\frac{\pi}{6}$ 23. $\frac{2}{\sqrt{5}}$ 24. $x = \pm \sqrt{\frac{3}{2}}$
 25. $A = -2e^{-\frac{k}{2}} + 2$ 26. $V = -\pi e^{-k} + \pi$ 28. $y' = \frac{3x-2y}{2y+2x}$ 29. $(2, -2), (-2, 2)$
 30. tangent: $y = \frac{3}{2}\left(x + \sqrt{\frac{20}{3}}\right)$ normal: $y = -\frac{2}{3}\left(x + \sqrt{\frac{20}{3}}\right)$ 31. $y^{\frac{4}{3}} = x^{\frac{4}{3}} + C$
 32. $y = \sqrt{\frac{1}{2}(\ln x)^2 + \frac{1}{2}}$ 31. $y^{\frac{4}{3}} = x^{\frac{4}{3}} + C$ 32. $y = \sqrt{\frac{1}{2}(\ln x)^2 + \frac{1}{2}}$ 33. 3.584

Lesson 6-4 Partial Fractions, Mixed Integration

We have often simplified an expression like $\frac{1}{x-4} - \frac{1}{x-3}$ by getting a common denominator and combining the two fractions into one. By a reverse process we can sometimes split a single fraction in two to make integration easier.

Example 1:

$$\int \frac{1}{x^2 - 7x + 12} dx$$

Example 2:

$$\int \frac{5x-3}{x^2 - 2x - 3} dx$$

Example 3:

$$\int \frac{2x-2}{x^2-2x-3} dx$$

Example 4:

$$\int \frac{x^3-x+2}{x^2+x-2} dx$$

Example 5: Integrate these four “look-alike” integrals. Although they have similar appearances, they will require you to use three completely different integration formulas.

a. $\int \frac{dx}{1+x^2}$

b. $\int \frac{dx}{1+x}$

c. $\int \frac{x dx}{1+x^2}$

d. $\int \frac{dx}{(1+x)^2}$

Example 6: Integrate the following two “look-alikes.”

a. $\int \frac{dx}{x \ln x}$

b. $\int \frac{\ln x}{x} dx$

Assignment 6-4

Integrate without using a calculator.

1. $\int \frac{1}{x^2-1} dx$

2. $\int \frac{3}{x^2-x-2} dx$

3. $\int \frac{5x-2}{2x^2-x-1} dx$

4. $\int \frac{2x^2+2x-2}{x^3-x} dx$

5. $\int 3x \ln x dx$

6. $\int x^2 \sin(3x) dx$

7. $\int \frac{2x-\sqrt{x}+3}{\sqrt{x}} dx$

8. $\int (2x+1)^6 dx$

9. $\int (3t^2-1)^2 dt$

10. $\int \frac{\sqrt{\ln y}}{y} dy$

11. $\int \frac{\sec^2 \theta}{1+\tan \theta} d\theta$

12. $\int \frac{\sec^2 \theta}{1+\tan^2 \theta} d\theta$

13. $\int \frac{\sec^2 \theta}{(1+\tan \theta)^2} d\theta$

14. $\int \frac{\tan \theta \sec^2 \theta}{1+\tan^2 \theta} d\theta$

15. $\int \frac{2x^2-4}{x+1} dx$

16. $\int e^{\cos(3u)} \sin(3u) du$

17. $\int x^2 (\sin x^3)^{\frac{3}{2}} \cos x^3 dx$

18. $\int \frac{e^{5x}-e^x+2}{e^{2x}} dx$

19. $\int \frac{4x}{\sqrt{1-x^4}} dx$

20. $\int \frac{x+5}{x^2+16} dx$

21. $\int \frac{1}{t^2-10t+32} dt$

22. $\int \frac{2x+12}{x^2+4x} dx$

23. $\int_0^\pi 2x \cos x dx$

Differentiate:

24. $f(x) = \frac{x^2 - 3}{\tan x}$

25. $h(y) = (\ln(\sec y))^3$

26. $x = \cos(\arcsin t)$

27. $y = \arctan(v-1)^2$

28. For $y = x \sin x$, evaluate $\frac{d^2y}{dx^2}$ at $x = \frac{\pi}{4}$ without a calculator.

29. Without using a calculator, find the x -values where $y = e^x \sin x$ has horizontal tangents on the interval $[-\pi, \pi]$.

30. Without using a calculator, find the area of the region bounded by

$$y = \frac{2x}{x^4 + 1}, y = 0, \text{ and } x = 1.$$

31. a. Find a general solution of the differential equation $y = \frac{\cos(-x)}{y'}$.

- b. Write an equation for the particular solution containing the point $(\frac{\pi}{2}, -3)$.

32. Find the slopes of the solution curves for $\frac{dy}{dx} = -xy$ at the following points:

a. $(0, 0)$

b. $(0, 1)$

c. $(1, -1)$

d. $(1, 1)$

e. $(-1, -2)$

33. a. Sketch the slope field for $\frac{dy}{dx} = -xy$, using $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$.

Plot your slopes on dot graph paper.

- b. Sketch the particular solution which contains the point $(0, -2)$.

34. Find the particular solution of differential equation $\frac{dy}{dx} = -xy$ which contains the point $(0, -2)$. (Solve for y , and compare the graph of the equation to your sketch from Problem 33b).

Selected Answers:

1. $\frac{1}{2} \ln|x-1| - \frac{1}{2} \ln|x+1| + C$

2. $\ln\left|\frac{x-2}{x+1}\right| + C$

3. $\frac{3}{2} \ln|2x+1| + \ln|x-1| + C$

4. $2 \ln|x| + \ln|x-1| - \ln|x+1| + C$

5. $\frac{3}{2}x^2 \ln x - \frac{3}{4}x^2 + C$

6. $-\frac{1}{3}x^2 \cos(3x) + \frac{2}{9}x \sin(3x) + \frac{2}{27} \cos(3x) + C$

7. $\frac{4}{3}x^{\frac{3}{2}} - x + 6x^{\frac{1}{2}} + C$

8. $\frac{1}{14}(2x+1)^7 + C$

9. $\frac{9}{5}t^5 - 2t^3 + t + C$

10. $\frac{2}{3}(\ln y)^{\frac{3}{2}} + C$

12. $\theta + C$

13. $-(1+\tan \theta)^{-1} + C$

14. $-\ln|\cos \theta| + C$ or $\frac{1}{2} \ln(1 + \tan^2 \theta) + C$

15. $x^2 - 2x - 2 \ln|x+1| + C$

More Selected Answers:

17. $\frac{2}{15}(\sin x^3)^{\frac{5}{2}} + C$ 18. $\frac{1}{3}e^{3x} + e^{-x} - e^{-2x} + C$ 19. $2\arcsin x^2 + C$

20. $\frac{1}{2}\ln(x^2 + 16) + \frac{5}{4}\arctan\frac{x}{4} + C$ 22. $3\ln|x| - \ln|x+4| + C$ 23. -4

25. $h'(y) = 3(\ln(\sec y))^2 \tan y$ 26. $x' = -\frac{t}{\sqrt{1-t^2}}$ 27. $y' = \frac{2(v-1)}{1+(v-1)^4}$

28. $y''\left(\frac{\pi}{4}\right) = -\frac{\pi}{4\sqrt{2}} + \frac{2}{\sqrt{2}}$ 29. $x = -\frac{\pi}{4}, \frac{3\pi}{4}$ 30. $\frac{\pi}{4}$

31a. $y = \sqrt{-2\sin(-x) + C}$ or $y = -\sqrt{-2\sin(-x) + C}$ b. $y = -\sqrt{-2\sin(-x) + 7}$

32a. 0 b. 0 c. 1 d. -1 e. -2 34. $y = -2e^{-\frac{1}{2}x^2}$

Lesson 6-5 Logistic Equations

Exponential growth modeled by $y = Ce^{kt}$ assumes unlimited growth and is unrealistic for most population growth. More typically the growth rate decreases as the population grows and there is a maximum population M called the carrying capacity. This is modeled by the **logistic** differential equation $\frac{dP}{dt} = kP(M - P)$.

The solution equation is of the form $P = \frac{M}{1 + Ce^{-kt}}$. **Note:** Unlike in the exponential growth equation, C is **not** the initial amount.

Example: A national park is capable of supporting no more than 100 grizzly bears. We model the equation with a logistic differential equation with $k = 0.001$.

a. Write the differential equation.

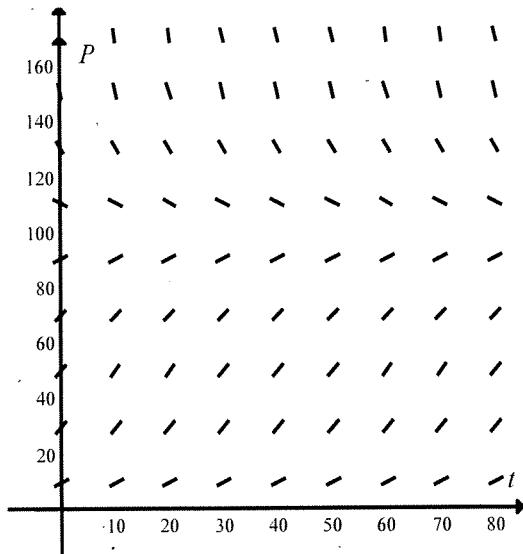
b. The slope field for this differential equation is shown.

Where does there appear to be a horizontal asymptote?

What happens if the starting point is above this asymptote?

What happens if the starting point is below this asymptote?

c. If the park begins with ten bears, sketch a graph of $P(t)$ on the slope field.



d. Solve the differential equation to find $P(t)$ with this initial condition.

e. Instead of solving the differential equation, use the general form

of a logistic equation $P = \frac{M}{1+Ce^{-kMt}}$ to find the same solution.

f. Find $\lim_{t \rightarrow \infty} P(t)$

g. When will the bear population reach 50?

h. When is the bear population growing the fastest.

Assignment 6-5

1. Logistic growth has $k = 0.00025$, $M = 200$, and $P(0) = 10$.

a. Write a differential equation for the population.

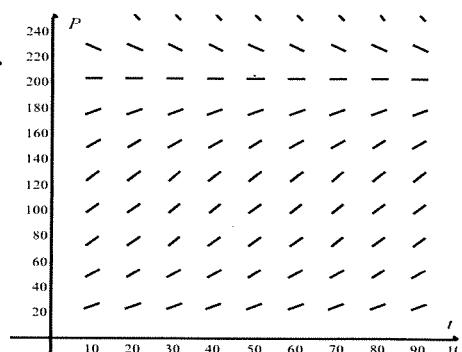
b. Sketch the population function on the slope field.

c. Find a formula for P .

Show all steps using partial fractions.

d. When is the population growing the fastest?

2. If $\frac{dP}{dt} = .04P - .0004P^2$ find k and the carrying capacity.



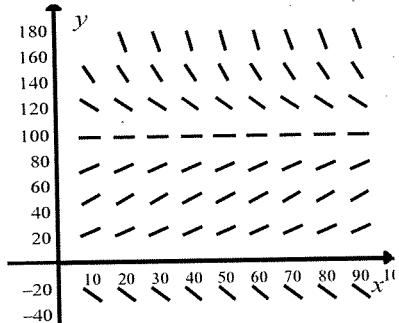
Match each of these differential equations with one of slope fields shown.

3. $\frac{dy}{dx} = .065y$

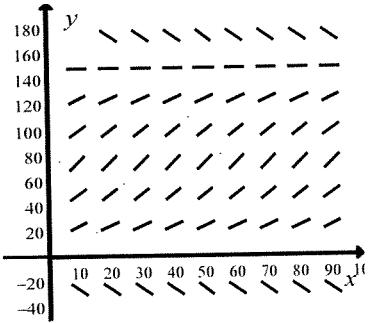
4. $\frac{dy}{dx} = .0006y(100 - y)$

5. $\frac{dy}{dx} = .06y\left(1 - \frac{y}{150}\right)$

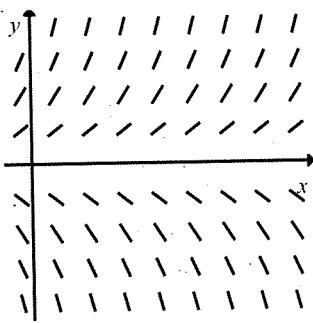
A.



B.



C.



6. Given the logistic equation $P(t) = \frac{2000}{1 + 19e^{-0.6t}}$:

- a. find the carrying capacity.
- b. find the value of k .
- c. find the initial population.
- d. find the time at which the population reaches 500.
- e. give the logistic differential equation.

7. Given the logistic differential equation $\frac{dP}{dt} = .03P(100 - P)$:

- a. find the value of k .
- b. find the carrying capacity.
- c. find the value of P when $\frac{dP}{dt}$ is the greatest.

8. Given the logistic differential equation $\frac{dy}{dt} = 2y\left(1 - \frac{y}{50}\right)$:

- a. find the value of k .
- b. find the value of M .
- c. give the logistic equation if $y(0) = 10$.

9. A 200 gallon tank can support no more than 150 guppies. Six guppies are introduced into the tank. Assume that the rate of growth of the population is $\frac{dP}{dt} = .0015P(150 - P)$, where time t is measured in weeks.

- a. Find a formula for the guppy population in terms of t .
- b. How long will it take for the guppy population to be 100? 125?

10. The amount of food placed daily into a biology lab enclosure can support no more than 200 fruit flies. A biologist releases 25 flies into the enclosure. Four days later she counts 94 flies.
- a. Give the logistic equation.
 - b. Find the number of flies on the 7th day.
 - c. When will there be 175 flies?
 - d. Find the logistic differential equation.
 - e. Find the population and the time at which the growth is the fastest.
 - f. Starting with 94 flies on day 4 and a step size of 1 day, use Euler's Method to approximate the number of flies on the 7th day.

Evaluate these integrals without a calculator.

11. $\int_0^{\frac{\pi}{3}} \sec^2 \theta d\theta$
12. $\int_0^1 \frac{36dx}{(2x+1)^3}$
13. $\int \frac{\sqrt{x}}{x-1} dx$ (hint: let $u = \sqrt{x}$)
14. $\int_0^{\pi} e^{\tan x} \sec^2 x dx$

15. $\int \frac{\cos x}{2-\sin x} dx$
16. $\int \frac{t}{t^2+5} dt$
17. $\int \frac{\tan(\ln y)}{y} dy$
18. $\int \frac{dx}{x \ln x}$

19. $\int x^2 \cos x dx$
20. $\int \frac{1}{2} \sin \sqrt{x} dx$

Find the indicated limits without a calculator.

21. $\lim_{x \rightarrow 2^-} \frac{|x-2|}{x-2}$
22. $\lim_{x \rightarrow 2^+} \frac{|x-2|}{x-2}$
23. $\lim_{x \rightarrow 1} \frac{x^2+x-2}{x-1}$
24. $\lim_{x \rightarrow -1} \frac{\sqrt{x+2}-1}{x+1}$

25. For $f(x) = \begin{cases} x^2 - 3, & x < 1 \\ 4, & x = 1 \\ -2x, & 1 < x < 2 \\ -x^3 + 8, & x \geq 2 \end{cases}$ Find a. $\lim_{x \rightarrow 1} f(x) = ?$ b. $\lim_{x \rightarrow 2} f(x) = ?$

26. $\frac{d}{dt} \int_{-t}^t xe^x dx = ?$
27. $\frac{d}{dx} \int_3^{x^2} (1 - \sin t) dt = ?$
28. $\frac{d}{dx} \left(\sqrt[4]{x} - \frac{x-2}{x^3} \right) = ?$

29. Find a and b so that $f(t)$ is differentiable at $t = -1$.

$$f(t) = \begin{cases} at^3 + bt^2 - 2, & t \leq -1 \\ -bt^2 + at - 4, & t > -1 \end{cases}$$

30. $y(t) = t^3 - t^{-3}$ represents the position of a point on the y -axis at time $t > 0$.
 $v(t)$ represents the velocity and $a(t)$ represents the acceleration of the point.

Without a calculator, find

- a. $y(2)$
- b. $v(2)$
- c. $a(2)$

Find the indicated derivatives.

31. $y = \cos(\ln t^2)$
32. $f(x) = \tan(x) \ln|x-1|$
33. $\frac{d}{dt} \left(\frac{\sin^2 t}{t^2} \right) = ?$

$y' = ?$ $f'(x) = ?$

SELECTED ANSWERS

- 1a. $\frac{dp}{dt} = .00025P(200 - P)$ c. $P(t) = \frac{200}{1 + 19e^{-0.05t}}$ 2. $k = .0004, M = 100$
 6a. 2000 b. .0003 c. 100 d. 3.076 e. $\frac{dp}{dt} = .0003P(2000 - P)$ 8c. $y = \frac{50}{1 + 4e^{-2t}}$
 9a. $P(t) = \frac{150}{1 + 24e^{-0.225t}}$ 10a. $P(t) = \frac{200}{1 + 7e^{-0.456t}}$ b. 155 flies c. 8.526 days
 10d. $\frac{dp}{dt} = .002P(200 - P)$ 11. $\sqrt{3}$ 12. 8 13. $2\sqrt{x} + \ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| + C$ 14. $e-1$
 16. $\frac{1}{2}\ln(t^2 + 5) + C$ 18. $\ln|\ln x| + C$ 19. $x^2 \sin x + 2x \cos x - 2 \sin x + C$
 20. $-\sqrt{x} \cos \sqrt{x} + \sin \sqrt{x} + C$ 21. -1 22. DNE 23. 3 25a. -2 26. $te^t - te^{-t}$
 28. $\frac{1}{4}x^{-\frac{3}{4}} + 2x^{-3} - 6x^{-4}$ 29. $a = -2, b = -1$ 30b. $12\frac{3}{16}$ c. $12 - 12 \cdot 2^{-5} = 11\frac{5}{8}$
 31. $y' = \frac{-2 \sin(\ln t^2)}{t}$ 33. $\frac{t^2 \cdot 2 \sin t \cdot \cos t - \sin^2 t \cdot 2t}{t^4}$

Lesson 6-6 L'Hôpital's Rule

Some limits cannot be found using algebraic methods. If direct substitution produces one of these two indeterminate forms $\left(\frac{0}{0} \text{ or } \frac{\pm\infty}{\pm\infty}\right)$, then a rule known as **L'Hopital's Rule** may help you find the limit.

L' Hopital's Rule

If $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$

To use L' Hopital's Rule, you must have the limit of an expression which is written in fractional form.

Examples: Evaluate.

$$1. \lim_{x \rightarrow \infty} \frac{x}{e^x} \quad 2. \lim_{x \rightarrow \infty} \frac{e^x}{x} \quad 3. \lim_{x \rightarrow 0} \frac{x}{e^x}$$

4. $\lim_{x \rightarrow 0^-} \frac{e^x}{x}$

5. $\lim_{x \rightarrow 0} \frac{3 - 3e^{3x}}{x}$

6. $\lim_{x \rightarrow -\infty} x^2 e^x$

7. $\lim_{x \rightarrow 1} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$

8. $\lim_{x \rightarrow -1} \frac{2x^2 - 2}{x+1}$ (w/o LR)

9. $\lim_{x \rightarrow -1} \frac{2x^2 - 2}{x+1}$

The following limits are not $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ forms. Identify the form and tell which are indeterminate.

10. $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x$

11. $\lim_{x \rightarrow 0^+} x^x$

12. $\lim_{x \rightarrow \infty} x^x$

13. $\lim_{x \rightarrow 0^+} (\sin x)^{\frac{1}{x}}$

Find the following limits.

14. $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x$

15. $\lim_{x \rightarrow \infty} x^x$

16. Sometimes limits can be evaluated quickly using the concept of relative growth rates. Rank the following in order of rate of growth as x approaches infinity from slowest to fastest.

$$y = x^2, \quad y = 1, \quad y = x, \quad y = \ln x, \quad y = x^{10}, \quad y = e^x$$

Use the concept of relative growth rates to evaluate the following when possible.

17. $\lim_{x \rightarrow \infty} \frac{\ln x}{e^x + x^3}$

18. $\lim_{x \rightarrow 0} \frac{e^x}{\ln x}$

Assignment 6-6

Find the indicated limits without using a calculator.

Hint: Not all problems will require the use of L' Hopital's Rule.

1. $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x + 2}$

2. $\lim_{x \rightarrow 1} \frac{x^2 + 2x - 3}{x - 1}$

3. $\lim_{x \rightarrow 0} \frac{e^x - x - 1}{x^2}$

4. $\lim_{x \rightarrow 0} \frac{x}{x - (1 - e^x)}$

5. $\lim_{x \rightarrow 0^+} \frac{|x|}{x}$

6. $\lim_{x \rightarrow 0^-} \frac{|x|}{x}$

7. $\lim_{x \rightarrow 0} \frac{|x|}{x}$

8. $\lim_{x \rightarrow 2} \frac{x^2 - 4}{\ln(x - 1)}$

9. $\lim_{x \rightarrow 0} \frac{2(e^x - 1)}{x^2}$

10. $\lim_{x \rightarrow \infty} \frac{2x^5 - x^2}{3x^5 + x^4 - 5x}$

11. $\lim_{x \rightarrow \infty} \frac{2x^4 - x^2}{3x^5 + x^4 - 5x}$

12. $\lim_{x \rightarrow \infty} \frac{e^x}{x^2}$

13. $\lim_{x \rightarrow \infty} \frac{3x^5 + x^4 - 5x}{2x^4 - x^2}$

14. $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 5}}{x}$

15. $\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 5}}{x}$

16. $\lim_{x \rightarrow 5} \frac{2x - 10}{5x}$

17. $\lim_{x \rightarrow 0} \frac{\tan x}{x \sec x}$

18. $\lim_{t \rightarrow \frac{\pi}{4}} \frac{\cos t - \sin t}{2 - 2 \tan t}$

19. $\lim_{\theta \rightarrow \frac{\pi}{2}} \frac{\sin(2\theta)}{\cos \theta}$

20. $\lim_{x \rightarrow 1} \frac{\arctan(x) - \frac{\pi}{4}}{x - 1}$

21. $\lim_{x \rightarrow 0^+} (e^x + x)^{\frac{3}{x}}$

22. $\lim_{x \rightarrow 0^+} (1+x)^{\frac{1}{x}}$

23. $\lim_{x \rightarrow \infty} (2+x)^{\frac{1}{x}}$

24. $\lim_{x \rightarrow 1^+} (\ln x)^{1-x}$

25. Use the concept of relative growth rate to evaluate the following.

a. $\lim_{x \rightarrow \infty} \frac{e^x - x}{x^2}$

b. $\lim_{x \rightarrow \infty} \frac{\ln x}{x}$

26. Use a calculator to find $\lim_{x \rightarrow 1.5} \frac{2x^3 - 3x^2 - 8x + 12}{6x^3 - 25x^2 + 34x - 15}$

Antidifferentiate

27. $f'(x) = e^{x^2+5}x$

28. $x'(t) = \frac{2t-1}{e^{t^2-t}}$

Evaluate without using a calculator.

29. $\int_2^e \frac{\sqrt[3]{\ln x}}{x} dx$

30. $\int_0^1 3^{-x} dx$

31. $\int \frac{t-2}{t^2-4t-5} dt$

32. $\int \frac{1}{\sqrt{-t^2+4t+12}} dt$

33. What interest rate would be required for a savings account to triple in value every 20 years, if interest is continuously compounded? ($A = Pe^{rt}$) Use your calculator, and express your answer to the nearest hundredth of a percent.

34. A population of rabbits is given by the formula $P(t) = \frac{1000}{1+e^{4.8-7t}}$ where time is measured in months.

- Find: a. the value of k . b. the carrying capacity.
c. the initial number of rabbits. d. the time at which the population is growing the fastest.

35. A wild animal preserve can support no more than 250 lowland gorillas. Twenty eight gorillas were known to be in the reserve in 1970. Assume that the rate of growth of the population is $\frac{dP}{dt} = .0004P(250 - P)$ where t is measured in years. Find a formula for the gorilla population.

When will the population reach 249 gorillas?

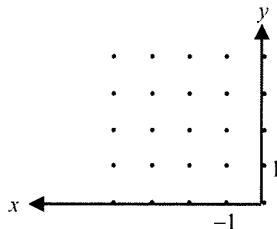
Use the differential equation $y' - 2y = xy$ for Problems 36-41.

36. Solve for y' and find slopes for the solution curve(s) to the differential equation at:

- a. $(0, 2)$ b. $(-2, 0)$ c. $(-1, 1)$
d. $(-4, 1)$ e. $(k, 0)$ where k is a constant

37. Sketch a slope field on your own paper for the equation
(in Quadrant 2 of a coordinate plane - like the one shown
at the right).

38. Sketch a solution curve containing the point $(0, 2)$.



39. Use Euler's method with the same starting point and $\Delta x = -\frac{1}{3}$ to approximate $y(-1)$.

40. Solve the differential equation, and find the particular solution containing the point $(0, 2)$. Use your calculator to graph the particular solution, and compare it to your graph from Problem 38.

41. Find the exact value of $y(-1)$.

42. Let R be the region enclosed by the graphs of $y = \cos(x^3)$ and $y = x^2$. Set up and evaluate an integral that gives the area of R . (Show the equation used to find the limits of integration. Make sure that your calculator is in radian mode.)
43. A particle moves along the x -axis so that, at any time $t \geq 0$, its velocity is given by $v(t) = \ln(t+1) - 2t + 1$. Find the total distance traveled by the particle from $t = 0$ to $t = 2$. (Show an integral set up and an answer.)
44. Integrate $\int \frac{3-3x}{9-x^2} dx$.

Selected Answers:

1. 0 2. 4 3. $\frac{1}{2}$ 4. $\frac{1}{2}$ 6. -1 8. 4 10. $\frac{2}{3}$ 11. 0 12. ∞ or DNE
 14. 1 16. 0 17. 1 18. $\frac{1}{2\sqrt{2}}$ 19. 2 20. $\frac{1}{2}$ 21. e^6 22. e 23. 1
 24. 1 27. $f(x) = \frac{e^{x^2+5}}{2} + C$ 28. $x(t) = -\frac{1}{e^{t^2-t}} + C$ 29. $\frac{3}{4} - \frac{3}{4}(\ln 2)^{\frac{4}{3}}$
 30. $-\frac{1}{3\ln 3} + \frac{1}{\ln 3}$ 31. $\frac{1}{2} \ln|t^2 - 4t - 5| + C$ 32. $\arcsin \frac{t-2}{4} + C$ 33. 5.49%
 34c. ≈ 8 rabbits d. 6.857 months 35. $P(t) = \frac{250}{1 + 7.929e^{-0.1t}}$, about year 2046
 36a. 4 c. 1 e. 0 40. $y = 2e^{\frac{1}{2}x^2+2x}$ 42. 1.248 or 1.249 43. 1.540
 44. $2 \ln|3+x| + \ln|3-x| + C$

CALCULUS EXTENDED UNIT 6 SUMMARY

Euler's Method: This is a method of finding an approximate y -value on a solution to a differential equation.

x	y	$\frac{dy}{dx}$	$\Delta y = (\text{slope}) \Delta x$

Definition of the Inverse Trig Functions:

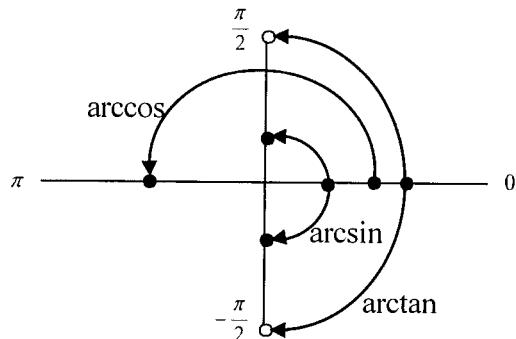
Function	Domain (x values)	*Range (y values)
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$$\underline{y = \arcsin x \leftrightarrow \sin y = x} \quad [-1,1] \quad \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$$

$$\underline{y = \arccos x \leftrightarrow \cos y = x} \quad [-1,1] \quad [0, \pi]$$

$$\underline{y = \arctan x \leftrightarrow \tan y = x} \quad (-\infty, \infty) \quad \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$$

*A geometric representation of the range values for each inverse trig function is shown in the coordinate plane at right.



Derivatives of the Inverse Trig Functions:

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arcsin u = \frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arccos u = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$$

$$\frac{d}{dx} \arctan u = \frac{u'}{1+u^2}$$

(where u is a function of x)

Integration of Inverse Trig Functions

$$1. \int \frac{u'}{\sqrt{a^2 - u^2}} dx = \arcsin \frac{u}{a} + C$$

$$2. \int \frac{u'}{a^2 + u^2} dx = \frac{1}{a} \arctan \frac{u}{a} + C$$

Formula for Integration by Parts:

$$\int u dv = uv - \int v du$$

Partial Fractions: Example: Write fractions like $\frac{2}{(x-3)(x-4)}$ as $\frac{A}{(x-3)} + \frac{B}{(x-4)}$ then find A and B in order to integrate.

$$\text{Logistic Equations: } (M \text{ is the carrying capacity}) \quad \frac{dP}{dt} = kP(M-P) \quad P = \frac{M}{1+Ce^{-kMt}}$$

L' Hopital's Rule

If $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$

To use L' Hopital's Rule, you must have the limit of an expression which is written in fractional form. For limits in other indeterminate forms such as 1^∞ , ∞^0 , or 0^0 use logarithms to rewrite them in fraction form.

Lesson 7-1 Improper Integrals

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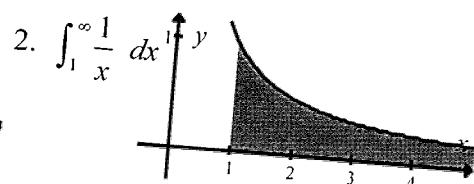
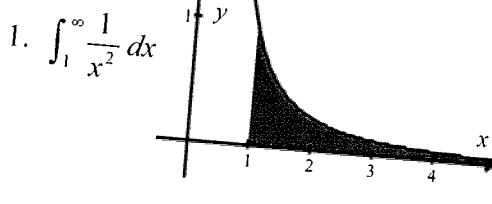
An integral is called **improper** if

1. one or both limits of integration are infinite
2. the function has an infinite discontinuity (a vertical asymptote) at or between the limits

Explain why each of the following is improper.

$$1. \int_1^{\infty} \frac{1}{x} dx \quad 2. \int_{-\infty}^{\infty} \frac{1}{x^2+1} dx \quad 3. \int_1^5 \frac{1}{\sqrt{x-1}} dx \quad 4. \int_{-2}^2 \frac{1}{(x+1)^2} dx$$

Evaluate the following improper integrals. Identify those which diverge.



3. $\int_0^{\infty} \cos x dx$

4. $\int_1^{\infty} xe^{-x} dx$

5. $\int_{-1}^2 \frac{dx}{x^3}$

Assignment 7-1

Which of the integrals in Problems 1-4 is/are improper. For any which are improper identify why the integral is improper. **Do not evaluate the integrals.**

$$1. \int_0^9 \frac{x+3}{\sqrt[3]{x}} dx \quad 2. \int_0^1 \frac{x^3}{3x-2} dx \quad 3. \int_0^4 \frac{1}{x^2-4x-5} dx \quad 4. \int_0^\infty \frac{x}{e^x} dx$$

Evaluate these improper integrals or show that the integral diverges without using a calculator. Show correct limit symbolism.

$$\begin{array}{llll} 5. \int_0^9 \frac{1}{\sqrt{x}} dx & 6. \int_2^{11} \frac{1}{\sqrt{(x-2)^3}} dx & 7. \int_0^\infty \frac{4}{e^x} dx & 8. \int_{-\infty}^0 e^{3x} dx \\ 9. \int_1^\infty \frac{3}{x^4} dx & 10. \int_1^\infty \frac{1}{\sqrt[3]{x}} dx & 11. \int_{-\infty}^0 xe^x dx & 12. \int_0^\infty \frac{x}{e^x} dx \\ 13. \int_e^\infty \frac{1}{x(\ln x)^4} dx & 14. \int_0^\infty \frac{e^x}{e^x + 3} dx & 15. \int_0^\infty \sin x dx & 16. \int_0^1 x \ln x dx \\ 17. \int_0^3 \frac{6}{x^2 - 9} dx & 18. \int_0^9 \frac{1}{\sqrt[3]{x-1}} dx & 19. \int_0^4 \frac{1}{(x-1)^2} dx & \end{array}$$

20. A region R satisfies the inequalities $y \leq \frac{1}{x^2}$, $y \geq 0$, $x \geq 1$.

- a. Sketch the region.
- b. Find the area of the region if possible.
- c. Find the volume of the solid formed by revolving the region about the x -axis if possible.
- d. Find the volume of the solid formed by revolving the region about the y -axis if possible.

21. Sketch a possible graph for $f(x)$, given the following characteristics:

$$\begin{aligned} f(0) &= 1, & f(1) \text{ does not exist,} & f(2) = 2 \\ f'(x) &< 0 \text{ for } x < 1 \text{ and } 1 < x < 2, & f'(x) > 0 \text{ for } x > 2 \\ f''(x) &< 0 \text{ for } x < 1, & f''(x) > 0 \text{ for } x > 1 \end{aligned}$$

22. Find:

- a. the instantaneous rate of change of $f(x) = -2x + \ln x$ at $x = 1$ and at $x = 2$.
- b. the average rate of change of $f(x) = -2x + \ln x$ on the interval $[1, 2]$.

23. Find the value of c where $f'(c) = \frac{f(b) - f(a)}{b - a}$, for $f(x) = -2x + \ln x$ on $[1, 2]$.

24. Find the average value of $f(x) = -2x + \ln x$ on $[1, 2]$.

25. Without using a calculator, find the maximum value of $f(x) = -2x + \ln x$.

26. Use the equation of a tangent line at $x = 1$ to approximate $f(.9)$ on the graph $f(x) = -2x + \ln x$.

27. Approximate $\int_1^4 (-2x + \ln x) dx$ by using a Trapezoidal Rule approximation with 3 equal subdivisions.

Simplify the expressions in Problems 28-30 without using a calculator.

28. $\frac{d}{dt} \int_1^t (-2x + \ln x) dx$ 29. $\frac{d}{dx} \int_{x^2}^5 \sin(2t) dt$ 30. $\int_0^1 xe^{x^2} dx$

For Problems 31-33, write an equation for a line tangent to each curve at the given point. Do not use a calculator.

31. $f(x) = -\cos(2x)$ 32. $g(x) = \tan(x^2 - 1)$ 33. $y = 2 \sin(-x)$
 contains the point $\left(\frac{\pi}{6}, -\frac{1}{2}\right)$ at the point where $x = 1$ at the point where $x = \frac{\pi}{4}$

34. Use a calculator to write equations for the lines tangent to and normal to the graph of $f(x) = \ln(|\cos x| + 2)$ at the point where $x = .822$.

Selected Answers:

1. improper (V.A. at $x = 0$)
2. improper (V.A. at $x = \frac{2}{3}$)
3. not improper
4. improper (infinite limit)
5. 6
6. diverges
7. 4
8. $\frac{1}{3}$
9. 1
11. -1
12. 1
13. $\frac{1}{3}$
14. diverges
16. $-\frac{1}{4}$
17. diverges
18. $\frac{9}{2}$
19. diverges
- 22a. $f'(1) = -1$, $f'(2) = -\frac{3}{2}$
23. $c = \frac{1}{\ln 2} = 1.442$ or 1.443
24. -2.613 or -2.614
25. $-1 + \ln \frac{1}{2}$
26. $f(.9) \approx -1.9$
27. $\left[\frac{1}{2} \cdot 1 \left[(-2 + \ln 1) + 2(-4 + \ln 2) + 2(-6 + \ln 3) + (-8 + \ln 4) \right] \right] = -12.515$
29. $-2x \sin(2x^2)$
30. $\frac{1}{2}e - \frac{1}{2} = .859$
31. $y + \frac{1}{2} = \sqrt{3} \left(x - \frac{\pi}{6} \right)$
32. $y = 2(x - 1)$
33. $y + \sqrt{2} = -\sqrt{2} \left(x - \frac{\pi}{4} \right)$
34. Tangent: $y - .986 = -.273(x - .822)$
 Normal: $y - .986 = 3.659(x - .822)$ or $y - .986 = 3.660(x - .822)$

Lesson 7-2 Sequences

Factorial Definition: $n! = n(n-1)(n-2)(n-3)\dots 1$

Examples;

1. Simplify $\frac{(n+2)!}{n!} =$

A **sequence** is an ordered list of numbers.

Examples:

2. Write out the first five terms of the

$$\text{sequence } \{a_n\} \text{ if } a_n = \frac{n}{n+1}.$$

3. Write out the first five terms of the recursively defined sequence.

$$a_1 = 5, \quad a_{k+1} = 3a_k + 4$$

4. Write a recursive definition for the sequence $2, -6, 18, -54, 162, \dots$

5. Write an expression for the n th term of the sequence.

Convergence or Divergence of a Sequence

If $\{a_n\}$ is a sequence and $\lim_{n \rightarrow \infty} a_n = L$ then L is the limit of the sequence and it converges to L .

Example:

6. Find the limit of the sequence $\{b_n\} = \left\{ \frac{n}{1-2n} \right\}$.

If $\lim_{n \rightarrow \infty} a_n$ does not exist, then the sequence $\{a_n\}$ does not have a limit and $\{a_n\}$ diverges.

Examples: Determine if these sequences converge or diverge and find the limit if possible.

$$7. \{a_n\} = \left\{ 3 + (-1)^n \right\}$$

$$8. \{a_n\} = \left\{ \frac{\ln(n^2)}{n} \right\}$$

A sequence is monotonic if it is nonincreasing or nondecreasing.

A sequence is bounded if every term is greater than or equal to some number N (the lower bound) and also every term is less than or equal to some number M (the upper bound).

Examples: Determine whether these sequences are monotonic and bounded.

$$9. \{a_n\} = \left\{ 3 + (-1)^n \right\}$$

$$10. \{b_n\} = \left\{ \frac{2n}{1+n} \right\}$$

11. $\{a_n\} = \left\{ \frac{1}{n} \right\}$

12. $\{c_n\} = \left\{ \frac{n^2}{1+n} \right\}$

Assignment 7-2

Without using a calculator, write the first five terms of the sequence with the given n^{th} term. Assume $n = 1, 2, 3, \dots$

1. $a_n = \frac{2^n}{n!}$

2. $a_n = \left(-\frac{1}{3} \right)^n$

3. $a_n = \cos\left(\frac{n\pi}{2}\right)$

Write the first five terms of the recursive sequence.

4. $a_1 = 2, a_{n+1} = 3(a_n + 2)$

5. $a_1 = 0, a_{n+1} = \frac{\pi}{2} \left(\sin\left(a_n + \frac{\pi}{2}\right) \right)$

Write a recursive definition of the sequence.

6. $4, 7, 10, 13, \dots$

7. $5, 10, 20, 40, \dots$

8. $5, -\frac{5}{2}, \frac{5}{4}, -\frac{5}{8}, \dots$

Simplify without using a calculator.

9. $\frac{7!}{10!}$

10. $\frac{(2n+1)!}{(2n-1)!}$

Find the limit of each sequence or state that the sequence diverges.

11. $a_n = \frac{n^2}{3n^2 - 5}$

12. $a_n = \frac{\ln n^2}{3n}$

13. $a_n = \cos \frac{1}{n}$

14. $a_n = (-1)^n \frac{n^2}{n^2 + 2}$

15. $a_n = \frac{3n}{\sqrt{n^2 - 5}}$

16. $a_n = \frac{\cos n}{n}$

17. $a_n = (-1)^n \frac{n}{n^2 + 2}$

18. $a_n = \frac{(n+1)!}{n!}$

Write an expression for the n^{th} term of each sequence. Assume $n = 1, 2, 3, \dots$

19. $-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{16}, \dots$

20. $\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \dots$

21. $\frac{2}{1}, \frac{4}{3}, \frac{8}{7}, \frac{16}{15}, \dots$

22. $\frac{3}{1}, \frac{3}{2}, \frac{3}{6}, \frac{3}{24}, \dots$

23. $\frac{1}{2}, \frac{x}{6}, \frac{x^2}{24}, \frac{x^3}{120}, \dots$

24. $-1, 1, 3, 5, \dots$

25. $\frac{1}{1}, \frac{4}{3}, \frac{9}{9}, \frac{16}{27}, \dots$

Determine whether each sequence with the given n^{th} term is monotonic and/or bounded.

26. $a_n = (-1)^n \frac{1}{n^2}$

27. $a_n = \frac{n^3 \cos(n\pi)}{n^2 + 1}$

28. $a_n = \left(\frac{1}{2} \right)^n$

29. $a_n = \left(\frac{3}{2} \right)^n$

Integrate:

30. $\int_0^{\infty} e^{-x} dx$

31. $\int_0^6 \frac{4}{\sqrt{6-x}} dx$

32. $\int_0^{\infty} \frac{e^x}{1+e^x} dx$

33. Find $g(x)$, given that: $g''(x) = 2x - 3$, $g'(0) = -5$, and $g(-1) = 2$.
34. The acceleration of an object moving along a horizontal path is given by the equation $a(t) = 6t - 4$. The object's initial velocity is 5, and its initial position is -2.
- Find a velocity equation for the object.
 - Find the velocity of the object when $t = 2$.
 - Find a position equation for the object.
 - Find the object's position when $t = 2$.
35. A ball is dropped from a bridge which is 160 feet above a river. How long will it take the ball to hit the water? Use the equation $s(t) = -16t^2 + v_o t + s_o$.
36. Find an equation of a line tangent to the curve $y = x^{\frac{2}{3}}$ which is parallel to the line $2x - 6y = 5$.
37. Find the cubic function of the form $y = ax^3 + bx^2 + cx + d$ which has a relative maximum point at $(0, 2)$ and a point of inflection at $(-1, -2)$.
38. List the domain, vertical asymptote(s), hole(s), x - and y -intercepts, and end behavior for the graph of $y = \frac{x^2}{x^2 - 1}$. Then sketch the graph without using a calculator.
39. Find the average value of $f(x) = \frac{1}{\sqrt{2-x}}$ on the interval $[-2, 1]$.
40. Sketch the region bounded by $x = y^2 - y$ and $x = 0$, and find the area of the region. (Show an integral set-up first.)
41. For $f(x) = x^3$, find the value of c in $[-2, 0]$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.
- Differentiate:
42. $y = 3\cos(t-2)$
43. $f(x) = x - \sin(2x)$
44. $g(x) = \cos^2(x^2 + 1)$
45. $h(\theta) = \tan(3\theta) - \theta^2$
46. $f(x) = \ln(\cot x)$
47. $y = \csc(2x) - \sec(x^2)$
48. $f(y) = \frac{y^2 - \sec y}{y^3}$
49. $g(t) = t^3 \tan^3(t^3)$

Selected Answers:

1. $2, 2, \frac{4}{3}, \frac{2}{3}, \frac{4}{15}$

2. $-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, -\frac{1}{243}$

4. 2, 12, 42, 132, 402

6. $a_1 = 4, a_{n+1} = a_n + 3$

9. $\frac{1}{720}$

11. $\frac{1}{3}$

13. 1

14. The sequence diverges.

More Selected Answers:

16. 0 17. 0 19. $a_n = \frac{(-1)^n}{n^2}$ 22. $a_n = \frac{3}{n!}$ 25. $a_n = \frac{n^2}{3^{n-1}}$
 26. not monotonic, bounded 28. monotonic, bounded 30. 1 31. $8\sqrt{6}$ 32. diverges
 33. $g(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 - 5x - \frac{7}{6}$ 34a. $v(t) = 3t^2 - 4t + 5$ c. $s(t) = t^3 - 2t^2 + 5t - 2$
 35. $t = \sqrt{10}$ sec 36. $y - 4 = \frac{1}{3}(x - 8)$ 37. $y = -2x^3 - 6x^2 + 2$
 38. Do.: $x \neq \pm 1$, V.A.: $x = \pm 1$ (odd), no hole, x -int.: $(0, 0)$ (even), y -int.: $(0, 0)$,
 E.B.: H.A. $y = 1$
 39. $f_{avg} = \frac{2}{3}$ 40. $A = \frac{1}{6}$ 41. $c = -\sqrt{\frac{4}{3}}$ 42. $y' = -3 \sin(t - 2)$
 43. $f'(x) = 1 - 2 \cos(2x)$ 44. $g'(x) = -4x \cos(x^2 + 1) \sin(x^2 + 1)$ 46. $f'(x) = \frac{-\csc^2 x}{\cot x}$
 47. $y' = -2 \csc(2x) \cot(2x) - 2x \sec x^2 \tan x^2$ 48. $f'(y) = \frac{y^3(2y - \sec y \tan y) - (y^2 - \sec y)3y^2}{y^6}$

Lesson 7-3 Convergent/Divergent Series, Geometric Series, n^{th} Term Test**Series:**

Recall from the last lesson that a **sequence** is an **ordered list** of numbers. In this lesson we will work with a **series** which is a **sum** of numbers.

An infinite series can be represented as $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots$

Example 1:

What happens as more and more terms of a series like $\sum_{n=0}^{\infty} (2n+1) = 1 + 3 + 5 + 7 + \dots$ are added?

When this happens the series is called **divergent**.

Example 2:

What happens as more and more terms of $\sum_{n=1}^{\infty} \frac{3}{10^n} =$ are added?

This is an example of a **convergent** geometric series.

Geometric Series:

If consecutive terms in a series have a common ratio r , the series is called a **geometric series**.

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots$$

is the general form of a geometric series.

first term!

If $|r| \geq 1$ the geometric series **diverges**. If $|r| < 1$ the geometric series **converges** and $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$.

Examples. Determine if these series converge or diverge and, if possible, find the sum of the series.

3. $\sum_{n=0}^{\infty} \frac{3}{2^n}$

4. $\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$

5. $\sum_{n=1}^{\infty} 4\left(-\frac{1}{2}\right)^n$

Example 6. Find the fraction form of the repeating decimal $.0\overline{8}$ using a geometric series.

In general, convergence of a series is less simple than convergence of a sequence,

The sequence $\{a_n\} = \left\{1 + \frac{1}{n}\right\}$ converges because $\lim_{n \rightarrow \infty} a_n = 1$

However the series $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) = (1+1) + \left(1 + \frac{1}{2}\right) + \left(1 + \frac{1}{3}\right) + \cdots$ does **not** converge.

A series **cannot** converge unless the terms approach a limit of **zero**.

n^{th} Term Test for Divergence:

If $\lim_{n \rightarrow \infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges. This test is inconclusive if $\lim_{n \rightarrow \infty} a_n = 0$.

Example 7. Show that $\sum_{n=1}^{\infty} \frac{n!}{2n!+1}$ diverges.

Assignment 7-3

Determine whether each of the following infinite series converges or diverges. Show justification and name the test being used. In addition find the sum of the series, if possible.

1. $\sum_{n=0}^{\infty} 5\left(\frac{2}{3}\right)^n$

2. $\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^n$

3. $\sum_{n=0}^{\infty} 5\left(\frac{3}{2}\right)^n$

4. $\sum_{n=1}^{\infty} \frac{n^2 + 2}{n^2}$

5. $\sum_{n=2}^{\infty} \frac{n^2}{\ln n}$

6. $5 + \frac{5}{2} + \frac{5}{4} + \frac{5}{8} + \dots$

7. $3 + \frac{9}{2} + \frac{27}{4} + \frac{81}{8} + \dots$

8. $1 + 0.2 + 0.04 + 0.008 + \dots$

9. $\sum_{n=1}^{\infty} \frac{n}{\sin n}$

10. $\sum_{n=1}^{\infty} \frac{3^n + 2}{3^{n+2}}$

11. $\sum_{n=0}^{\infty} \frac{e^n}{\pi^{n+1}}$

12. $\frac{1}{2} + \frac{2}{4} + \frac{6}{8} + \frac{24}{16} + \frac{120}{32} + \dots$

13. $\sum_{n=1}^{\infty} (\sin e^{10})^n$

14. $\sum_{n=1}^{\infty} \frac{(-5)^n}{6}$

15. $\sum_{n=1}^{\infty} \left(-\frac{5}{6}\right)^n$

16. $18 - 12 + 8 - \frac{16}{3} + \frac{32}{9} - \dots$

17. $\sum_{n=1}^{\infty} \frac{2n+3}{3n+2}$

18. $\sum_{n=0}^{\infty} \frac{n!}{e^n}$

19. $\sum_{n=0}^{\infty} \frac{4}{3^n}$

20. $\sum_{n=1}^{\infty} 4^{-n}$

21. Given the series $\sum_{n=1}^{\infty} \frac{n^2 + 2}{n^3}$:

a. Find $\lim_{n \rightarrow \infty} a_n$.

b. Explain why the *nth* Term Test **cannot** be used to conclude the series converges.

Evaluate these integrals:

22. $\int_0^{\infty} xe^{-x} dx$

23. $\int_0^3 \frac{1}{x-1} dx$

Determine the convergence/divergence of the following sequences:

24. $\left\{ \left(1 + \frac{2}{n}\right)^n \right\}$

25. $\left\{ \frac{(n+1)!}{n!} \right\}$

Write an expression for the *n*th term of the following sequences: (assume $n = 1, 2, 3, \dots$)

26. $3, 7, 11, 15, \dots$

27. $1, \frac{-1}{2}, \frac{1}{6}, \frac{-1}{24}, \frac{1}{120}, \dots$

28. $2, 6, 18, 54, \dots$

Determine whether the following sequences are monotonic and/or bounded: (assume $n = 1, 2, 3, \dots$)

29. $\left\{ 4 - \frac{1}{n} \right\}$

30. $\left\{ \cos \frac{n\pi}{2} \right\}$

31. $\left\{ \left(\frac{3}{2}\right)^n \right\}$

32. Find the domain, *x*- and *y*-intercepts, relative extrema, and points of inflection for $y = \sin x + \cos x$ on $[0, 2\pi]$. Then sketch the graph of *y* without using a calculator.

Differentiate implicitly to find $\frac{dy}{dx}$.

33. $\cos(y-x) = x^3 + 2$

34. $2xy = \tan y^2$

Find the limits without using a calculator.

35. $\lim_{x \rightarrow \frac{\pi}{2}} \left(\frac{\cos x}{x - \frac{\pi}{2}} \right)$

36. $\lim_{x \rightarrow 0} \frac{\sin(2x)}{\sin(-5x)}$

37. $\lim_{x \rightarrow 0} \frac{\sin(3x)}{x}$

38. $\lim_{t \rightarrow 0} 2t \sec t$

39. Use a calculator to find the area between the curve $y = |2 \cos x + \cos(2x)|$ and the x -axis, from $x = 0$ to $x = \pi$. Show an integral set up and an answer.

40. Use a calculator to find the area of the region in the first quadrant bounded by $y = 3 \cos(2x)$, $y = 3x$, and $x = 0$. Show an integral set up and an answer.

For Problems 41-44, set up (but do not integrate) integrals for finding the volumes of the solids described.

41. the solid is formed by revolving the region from Problem 40 about the x -axis.

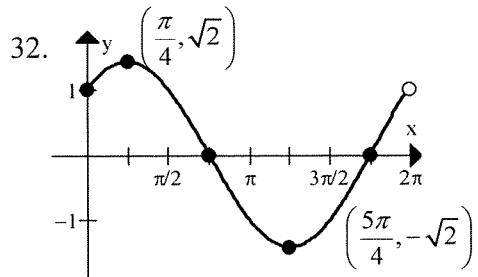
42. the solid is formed by revolving the region from Problem 40 about $y = 4$.

43. the solid is formed by revolving the region from Problem 40 about $x = -1$.

44. the solid is formed by using rectangular cross sections whose bases are in the region from Problem 40 and are perpendicular to the x -axis. The heights of the rectangles are always half of their bases.

Selected Answers:

1. converges by GST, Sum = 15
2. conv. by GST, Sum = 10
3. div. by GST or nTT
5. div. by nTT
6. conv. by GST, Sum = 10
7. div. by GST
8. conv. by GST, Sum = 1.25
9. div. by nTT
11. conv. by GST, Sum = $\frac{1}{\pi - e}$
13. conv. by GST, Sum = -.407 or -.408
14. div. by GST or nTT
15. conv. by GST, Sum = $-\frac{5}{11}$
17. div. by nTT
18. div. by nTT
20. conv. by GST, Sum = $\frac{1}{3}$
22. 1
23. diverges
24. converges to a limit of e^2
25. diverges
26. $a_n = 4n-1$, $n=1, 2, 3, \dots$
28. $a_n = 2(3)^{n-1}$, $n=1, 2, 3, \dots$
29. monotonic increasing, bounded between 3 and 4
30. not monotonic, bounded between -1 and 1



33. $y' = \frac{3x^2}{-\sin(y-x)} + 1 = \frac{3x^2 - \sin(y-x)}{-\sin(y-x)}$

34. $y' = \frac{-2y}{2x - 2y \sec^2 y^2}$

35. -1

37. 3

More Selected Answers:

39. 4.403 or 4.404

40. .888

42. $\pi \int_0^{.5149} \left((4 - 3x)^2 - (4 - 3\cos(2x))^2 \right) dx$

44. $\int_0^{.5149} \frac{1}{2} (3\cos(2x) - 3x)^2 dx$

Lesson 7-4**Power Series, Geometric Power Series,
Integration and Differentiation of Power Series**

A series with variable terms like $1 + x + x^2 + x^3 + \dots + x^n + \dots$ is called a **power series**.

Note that this series is a **geometric power series**. If it converges, what must be true about the variable x ?

For these x -values $1 + x + x^2 + x^3 + \dots + x^n + \dots =$

This means for these x -values, the function $f(x) = \frac{1}{1-x}$ can be written as $f(x) = \sum_{n=0}^{\infty} x^n$.

Examples: Find a power series for each of the following functions. Show four terms and the general term. Also give the series using sigma notation and give the interval of convergence.

1. $f(x) = \frac{1}{1+x}$

2. $g(x) = \frac{3}{1-2x}$

3. $h(x) = \frac{1}{3x}$

Power Series by Substitution:

Examples:

4. Using the power series from Example 1, make a new power series for $f(x^2)$.

5. Using the power series from Example 2, make a new power series for $g(\sqrt{x})$.

Power Series by Differentiation:

Examples:

6. Use the power series for $f(x) = \frac{1}{1-x}$ on the previous page to write a power series for $f'(x)$.

7. If $j(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!} + \dots$ find $j'(x)$.

Can you identify the function $j(x)$?

Power Series by integration:

8. Since $\frac{1}{1+t} = 1 - t + t^2 - t^3 + \dots + (-1)^n t^n + \dots$ (see Example 1)

$\int_0^x \frac{1}{1+t} dt = \int_0^x (1 - t + t^2 - t^3 + \dots + (-1)^n t^n + \dots) dt$ Now integrate both sides.

9. Use the result of Example 8 to write a power series for $f(x) = \ln(x)$.

10. Now write a power series for $g(x) = x^2 \ln x$.

Assignment 7-4:

Find a geometric power series for each of the following functions. Show four terms and the general term. Also give the series using sigma notation and give the interval of convergence.

1. $\frac{1}{1-3x}$

2. $\frac{2}{1-x^3}$

3. $\frac{x}{1+x}$

4. $\frac{1}{1+(-x-3)}$

5. $\frac{3}{4x}$

6. $\frac{1}{2-2x}$

Find a function for each of the following geometric power series. Also give the interval of convergence.

7. $\sum_{n=0}^{\infty} (2x)^n$

8. $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n x^n$

9. $\sum_{n=0}^{\infty} 4(x-1)^n$

10. $\sum_{n=1}^{\infty} (x^2)^n$

11. $\sum_{n=0}^{\infty} (\sin x)^n$

12. Use the result of Example 7 on the previous page to write a power series for $f(x) = e^{x^2}$. Show four terms and the general term.

13. Use the result of Example 7 on the previous page to write a power series for $g(x) = xe^x$. Show four terms and the general term.

14. Find a geometric power series for $g(x) = \frac{1}{1+x}$. Show four terms and the general term.

15. Use the answer to Problem 14 to find a power series for $\frac{1}{1+x^2}$.

16. Use integration to find a power series for $\arctan x$.

Use the function $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$ to find the following. Answer using sigma notation.

17. $f(-x)$

18. $f'(x)$

19. $\int_0^x f(t) dt$

Use the function $g(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$ to find the following.

Show four terms and the general term.

20. $g(\sqrt{x})$

21. $g'(x)$

22. $\int_0^x g(t) dt$

23. Use the power series for $f(x) = \ln(x)$ in Example 9 on the previous page to find a simplified answer for $f(1)$.

Determine whether the following series converge or diverge. Find the sum when possible.

24. $\sum_{n=0}^{\infty} \left(\frac{e}{\pi}\right)^n$

25. $\sum_{n=0}^{\infty} \left(\frac{\pi}{e}\right)^n$

26. $\sum_{n=0}^{\infty} \frac{2n+1}{n+1}$

27. $\sum_{n=2}^{\infty} \frac{n!}{(n-2)!}$

Determine the convergence/divergence of the following sequences:

28. $\left\{ \frac{2n-1}{n+1} \right\}$

29. $\left\{ e^n \right\}$

Determine whether the following sequences are monotonic and/or bounded: (assume $n = 1, 2, 3, \dots$)

30. $\left\{ 4 + \frac{1}{n} \right\}$

31. $\left\{ \frac{n!}{e^n} \right\}$

32. Evaluate the following improper integral $\int_0^1 \frac{1}{\sqrt{3-x^2-2x}} dx$.

Integrate the following:

33. $\int t^2 \tan(t^3) dt$
 34. $\int \frac{2x}{9+x^2} dx$
 35. $\int \frac{2}{9+x^2} dx$
 36. $\int \frac{(e^{2y}-1)^2}{e^y} dy$

37. $\int \frac{dx}{x\sqrt{\ln x}}$
 38. $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$
 39. $\int_1^2 x \sqrt[3]{x-1} dx$

Selected Answers:

1. $1 + 3x + 9x^2 + 27x^3 + \dots + (3x)^n + \dots = \sum_{n=0}^{\infty} (3x)^n ; \left(-\frac{1}{3}, \frac{1}{3} \right)$

2. $2 + 2x^3 + 2x^6 + 2x^9 + \dots + 2(x^3)^n + \dots = \sum_{n=0}^{\infty} 2(x^3)^n ; (-1, 1)$

3. $x - x^2 + x^3 - x^4 + \dots + (-1)^n x^{n+1} + \dots = \sum_{n=0}^{\infty} (-1)^n x^{n+1} = \sum_{n=1}^{\infty} (-1)^{n+1} x^n ; (-1, 1)$

5. $3 + 3(1-4x) + 3(1-4x)^2 + 3(1-4x)^3 + \dots + 3(1-4x)^n + \dots = \sum_{n=0}^{\infty} 3(1-4x)^n ; \left(0, \frac{1}{2} \right)$

6. $\frac{1}{2} + \frac{1}{2}x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \dots + \frac{1}{2}x^n + \dots = \sum_{n=0}^{\infty} \frac{1}{2}x^n ; (-1, 1)$

7. $\frac{1}{1-2x} ; \left(-\frac{1}{2}, \frac{1}{2} \right)$
 8. $\frac{1}{1+\frac{1}{2}x} = \frac{2}{2+x} ; (-2, 2)$
 9. $\frac{4}{1-(x-1)} = \frac{4}{2-x} ; (0, 2)$

10. $\frac{x^2}{1-x^2} ; (-1, 1)$
 12. $e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots + \frac{x^{2n}}{n!} + \dots$

14. $1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots$
 15. $1 - x^2 + x^4 - x^6 + \dots + (-1)^n x^{2n} + \dots$

16. $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + (-1)^n \frac{x^{2n+1}}{2n+1} + \dots$
 17. $f(-x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$

18. $f'(x) = \sum_{n=1}^{\infty} x^{n-1}$
 19. $\int_0^t f(x) dx = \sum_{n=1}^{\infty} \frac{t^{n+1}}{n(n+1)}$

More Selected Answers:

20.
$$g(\sqrt{x}) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \dots + \frac{(-1)^n x^n}{(2n)!} + \dots$$

21.
$$g'(x) = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \dots + \frac{(-1)^n x^{2n-1}}{(2n-1)!} + \dots$$
 24. converge to $\frac{1}{1 - \frac{e}{\pi}}$
 25. diverge

27. diverge 28. converge 29. diverge

30. monotonic decreasing, bounded between 4 and 5

32. $\frac{\pi}{3}$

33. $-\frac{1}{3} \ln |\cos t^3| + C$

34. $\ln(9+x^2) + C$

35. $\frac{2}{3} \arctan \frac{x}{3} + C$

36. $\frac{1}{3} e^{3y} - 2e^y - e^{-y} + C$

37. $2(\ln x)^{\frac{1}{2}} + C$

38. $\ln |e^x - e^{-x}| + C$

39. $\frac{33}{28}$

Lesson 7-5 Taylor Series

In this section you will be finding polynomial functions that can be used to approximate transcendental functions. If $P(x)$ is a polynomial function used to approximate some other function $f(x)$, they must contain the same point with some x -value c . That means $P(c) = f(c)$. To be a better approximation they should have the same slope at that point. This means $P'(c) = f'(c)$. For even greater accuracy, $P''(c) = f''(c)$ and so on. Putting this together gives us the **Taylor Polynomial Expansion**: If $f(x)$ has derivatives of all orders it can be approximated by the polynomial function shown.

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

This is called an n^{th} degree or n^{th} order Taylor Polynomial centered at c or expanded about c .

When the center is at $c = 0$ the Taylor polynomial is called a **Maclaurin Polynomial** which can be written as :

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

Example 1. Use the definition of a Maclaurin Polynomial to find the fifth degree Maclaurin Polynomial for $f(x) = e^x$.

$f(x) =$	$f(0) =$
$f'(x) =$	$f'(0) =$
$f''(x) =$	$f''(0) =$
$f'''(x) =$	$f'''(0) =$
$f^{(4)}(x) =$	$f^{(4)}(0) =$
$f^{(5)}(x) =$	$f^{(5)}(0) =$

$$P_5(x) =$$

This polynomial is a good approximation for $f(x) = e^x$. By extending the pattern into an infinite series it becomes exactly correct instead of an approximation.

$$f(x) = e^x =$$

The general form of Taylor and Maclaurin Polynomials can be extended to Taylor and Maclaurin Series.

Taylor Series (provided $f(x)$ has derivatives of all orders)

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$

Maclaurin Series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

These formulas allow us to form a power series for functions that cannot be written as geometric power series.

Example 2. Use the definition of a Taylor Polynomial to find the fourth order Taylor Polynomial and the Taylor Series for $f(x) = \cos x$ centered at $c = \pi$.

$$f(x) = \quad f(\pi) =$$

$$f'(x) = \quad f'(\pi) =$$

$$f''(x) = \quad f''(\pi) =$$

$$f'''(x) = \quad f'''(\pi) =$$

$$f^{(4)}(x) = \quad f^{(4)}(\pi) =$$

$$P_4(x) =$$

$$f(x) =$$

Example 3. Use your Taylor Polynomial from example 2 to approximate $\cos 3$.

$$\cos 3 \approx$$

Example 4. Use the definition of a Maclaurin Series to find the Maclaurin series for $f(x) = \sin x$.

$$f(x) = \quad f(0) =$$

$$f'(x) = \quad f'(0) =$$

$$f''(x) = \quad f''(0) =$$

$$f'''(x) = \quad f'''(0) =$$

$$f^{(4)}(x) = \quad f^{(4)}(0) =$$

$$f^{(5)}(x) = \quad f^{(5)}(0) =$$

$$f(x) = \sin x =$$

Example 6. Find a power series for $f(x)$ centered at $c = 1$ if $f(1) = 2$ and $f^{(n)}(1) = n!$.

Assignment 7-5:

1. Use the definition to find a fourth degree Maclaurin Polynomial for $f(x) = \frac{1}{e^x}$.
2. Use the polynomial from Problem 1 to approximate $\frac{1}{\sqrt{e}}$.
3. Use the definition to find a fourth degree Maclaurin Polynomial for $f(x) = e^{3x}$.
4. Use the definition to find a fifth degree Maclaurin Polynomial for $g(x) = xe^x$.
5. Use the definition to find a third degree Taylor Polynomial centered at $c = 1$ for $f(x) = \sqrt[3]{x}$.
6. Use the definition to find a fourth degree Taylor Polynomial centered at $c = 1$ for $h(x) = \ln x$.
7. Use the polynomial from Problem 6 to approximate $\ln 1.3$.
8. Use the definition to find a Taylor Series centered at $c = \frac{\pi}{4}$ for $f(x) = \sin x$.
9. Use the definition to find a Taylor Series centered at $c = 0$ for $f(x) = \cos(2x)$. Show four terms (Zero terms don't count.) and a general term.
10. Use the definition to write a Maclaurin Series for $f(x) = \frac{1}{1+x}$. Show four terms and the general term.
11. Write a geometric series expansion for $f(x) = \frac{1}{1+x}$. Also give the interval of convergence.
12. Write four terms and the general term of the Taylor series expansion of $f(x) = \frac{1}{x-1}$ about $x = 2$.
13. Use the series from Problem 12 to find four terms and the general term of the series expansion about $x = 2$ for $\ln|x-1|$.
14. The Taylor Series of a function about $x = 3$ is given by

$$f(x) = 1 + \frac{3(x-3)}{1!} + \frac{5(x-3)^2}{2!} + \frac{7(x-3)^3}{3!} + \dots + \frac{(2n+1)(x-3)^n}{n!} + \dots$$
What is the value of $f'''(3)$?
15. Let $f(x)$ be a function such that $f(0) = 2$, $f'(x) = 3f(x)$, and the n^{th} derivative of f is given by $f^{(n)}(x) = 3f^{(n-1)}(x)$.
 - (a) Give the first four terms and the general term of the Taylor Series for f centered at $x = 0$.
 - (b) Find $f(x)$ by solving the differential equation $f'(x) = 3f(x)$ (that is $y' = 3y$) with the initial condition $f(0) = 2$.

16. Let f be the function defined by the power series $f(x) = 2 + 2x + 2x^2 + 2x^3 + \dots + 2x^n + \dots$

If $g'(x) = f(x)$ and $g(0) = 2$, then $g(x) = ?$ Show four terms and the general term.

17. The Taylor series for a function f about $x = 0$ is $2 + \frac{4}{3}x + \frac{8}{9}x^2 + \frac{16}{27}x^3 + \dots + \frac{2^{n+1}x^n}{3^n} + \dots$ for $-1 < x < 1$. Calculator Allowed.

(a) Write the first four nonzero terms and the general term for f' , the derivative of f .

(b) Using the appropriate second-degree Taylor polynomial approximate $f(0.2)$ and $f'(0.2)$.

(c) Use the values found in part (b) to approximate the equation of the tangent line to f at $x = 0.2$.

18. The Maclaurin series for $f(x)$ is given by $1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \dots + \frac{x^n}{(n+1)!} + \dots$

(a) Find $f'(0)$.

(b) Find $f^{(15)}(0)$.

19. Find a geometric power series for $f(x) = \frac{x}{1+4x^2}$. Show four terms and the general term.

Also give the series using sigma notation and give the interval of convergence.

20. Find a function for the geometric power series $f(x) = \sum_{n=0}^{\infty} 4(3x)^n$. Also give the interval of convergence.

Determine whether the following series converge or diverge. Find the sum when possible.

21. $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$

22. $\sum_{n=0}^{\infty} (1.3)^n$

23. $\sum_{n=0}^{\infty} \frac{n^2 - 1}{n+1}$

24. The derivative of a function is $\frac{dy}{dt} = \frac{-3}{t^2} + 1$. If the graph of the function contains the point $(3, 10)$, find the equation of the function.

25. a. Find an equation for the family of functions whose derivative is $y' = 3\sqrt{x}$.

b. Find the particular function from the family in Part a. whose curve passes through the point $(4, 0)$.

Evaluate the following definite integrals without a calculator.

26. $\int_1^4 \frac{2\sqrt{x}-1}{\sqrt{x}} dx$

27. $\int_1^2 \frac{dx}{2\sqrt{3x-2}}$

28. $\int_0^3 \frac{2x}{\sqrt{x+1}} dx$

Differentiate.

29. $h(x) = \sin(\pi x) + \pi x^2$

30. $f(\theta) = \frac{-2\theta}{\sin \theta}$

31. $f(x) = \tan(\ln(x^2 - 2x))$

32. $h(x) = e^{\sin x \cos x}$

31. $f(x) = \tan(\ln(x^2 - 2x))$

32. $h(x) = e^{\sin x \cos x}$

33. Sketch graphs and show shaded areas representing the values of the following definite integrals.

I. $\int_{-2}^2 |x^3 + x| dx$ II. $\int_{-2}^2 |x^2 + 5x + 6| dx$ III. $\int_{-2}^2 |x^2 + 5x - 6| dx$ IV. $\int_{-2}^2 |x + 1| dx$

Match each of the integrals to one of the descriptions below.

- a. The integral can be evaluated geometrically using areas of triangles, so that no actual integration is necessary.
- b. Absolute value is not even necessary for the given limits of integration.
- c. Use of symmetry for the graph allows the problem to be done using only one integral that does not involve absolute value.
- d. The integral can only be done by using more than one integral. That is, the problem must be split into two or more integrals to eliminate the absolute value.

34. a. Set up integrals that do not involve absolute value which could be used to integrate the integrals shown in Problem 33 I, II, and III.

b. Evaluate the integral in Problem 33 IV using areas of triangles.

Selected Answers:

1. $f(x) \approx 1 - x + \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4$
2. .60677
3. $f(x) \approx 1 + 3x + \frac{9}{2}x^2 + \frac{27}{3!}x^3 + \frac{81}{4!}x^4$
4. $g(x) \approx x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \frac{1}{24}x^5$
5. $f(x) \approx 1 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2 + \frac{10}{27 \cdot 3!}(x-1)^3$
7. .261975
8. $f(x) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \left(x - \frac{\pi}{4} \right) - \frac{1}{\sqrt{2}} \cdot \frac{1}{2!} \left(x - \frac{\pi}{4} \right)^2 - \frac{1}{\sqrt{2}} \cdot \frac{1}{3!} \left(x - \frac{\pi}{4} \right)^3 + \frac{1}{\sqrt{2}} \cdot \frac{1}{4!} \left(x - \frac{\pi}{4} \right)^4 + \dots$
10. $f(x) = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots$
12. $f(x) = 1 - (x-2) + (x-2)^2 - (x-2)^3 + \dots + (-1)^n (x-2)^n + \dots$
13. $\ln|x-1| = (x-2) - \frac{1}{2}(x-2)^2 + \frac{1}{3}(x-2)^3 - \frac{1}{4}(x-2)^4 + \dots + (-1)^n \frac{1}{n+1}(x-2)^{n+1} + \dots$
14. $f'''(3) = 7$
- 15a. $f(x) = 2 + 6x + 9x^2 + 9x^3 + \dots + \frac{2(3^n)}{n!} x^n + \dots$
- b. $f(x) = 2e^{3x}$
16. $g(x) = 2 + \left(2x + x^2 + \frac{2}{3}x^3 + \dots + \frac{2}{n+1}x^{n+1} + \dots \right)$
- 18a. $f'(0) = \frac{1}{2}$
- b. $f^{(15)}(0) = \frac{1}{16}$
19. $f(x) = x - 4x^3 + 16x^5 - 64x^7 + \dots + (-1)^n 4^n x^{2n+1} + \dots = \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n+1}, \quad -\frac{1}{2} < x < \frac{1}{2}$
21. converges to 3
23. diverges
24. $y = \frac{3}{t} + t + 6$
- 25 a. $y = 2x^{\frac{3}{2}} + C$
- b. $y = 2x^{\frac{3}{2}} - 16$
27. $\frac{1}{3}$
28. $\left[\left(\frac{4}{3} \cdot 8 - 8 \right) - \left(\frac{4}{3} - 4 \right) \right] = \frac{16}{3}$
29. $h'(x) = \pi \cos(\pi x) + 2\pi x$
30. $f'(\theta) = \frac{-2 \sin \theta + 2\theta \cos \theta}{\sin^2 \theta}$
31. $f'(x) = \sec^2(\ln(x^2 - 2x)) \frac{2x-2}{x^2-2x}$
32. $h'(x) = e^{\sin x \cos x} (-\sin^2 x + \cos^2 x)$

Lesson 7-6 Elementary Series, Alternating Series

In this section you will be using the four elementary power series. You are expected to know them from memory.

$$\sin x =$$

$$\cos x =$$

$$e^x =$$

$$\ln x =$$

Creating new power series.

Examples: Using these elementary series, find a power series for each of the following functions. Show four terms and the general term.

$$1. \sin x^2 =$$

$$2. \cos \sqrt{x} =$$

$$3. xe^x =$$

$$4. \frac{1}{x} =$$

$$5. \int_1' \ln x \, dx =$$

An **Alternating Series** is a series whose terms alternate between positive and negative. Three of the four elementary series are alternating series.

Alternating Series Test for Convergence:

Let $a_n > 0$. The alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ and $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converge if

1. $a_{n+1} \leq a_n$ for all n after a certain n (terms never increase in absolute value) and
2. $\lim_{n \rightarrow \infty} a_n = 0$

Examples: Determine the convergence or divergence.

6. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ 7. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+1}{n}$

We now have three tests for Convergence/Divergence.

n^{th} Term Test for Divergence: If $\lim_{n \rightarrow \infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

This test is inconclusive if $\lim_{n \rightarrow \infty} a_n = 0$. (cannot be used to show conv.)

Geom. Series Test: $|r| \geq 1 \rightarrow \text{diverges}$, $|r| < 1 \rightarrow \text{converges}$ and $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$.

Alternating Series Test for Convergence: converges if

1. $a_{n+1} \leq a_n$ for all n after a certain n (terms never increase in absolute value)
2. $\lim_{n \rightarrow \infty} a_n = 0$

Assignment 7-6

Use an elementary series to give a series for each of the following functions. Show four terms and the general term.

1. $f(x) = e^{-4x}$ 2. $g(x) = \cos(3x)$ 3. $f(x) = 2 \sin x^2$ 4. $h(x) = (x-1) \ln x$

Write each of the following series as a function using elementary functions.

5. $x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \dots$	6. $1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \dots$
7. $\frac{x-1}{x} - \frac{(x-1)^2}{2x} + \frac{(x-1)^3}{3x} - \frac{(x-1)^4}{4x} + \dots$	8. $1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \dots$

Find the value of each of the following using an elementary function.

9. $1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \dots$	10. $(e-1) - \frac{(e-1)^2}{2} + \frac{(e-1)^3}{3} - \frac{(e-1)^4}{4} + \dots$
11. $5 + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \dots$	12. $-\frac{2^3}{3!} + \frac{2^5}{5!} - \frac{2^7}{7!} + \frac{2^9}{9!} - \dots$

Determine the convergence or divergence of each series. Show justification and name the test used. If possible, find the sum of the series.

13. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

14. $\sum_{n=0}^{\infty} \frac{(-1)^n \sqrt{n}}{n^2 + 1}$

15. $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{n^3 + 2}$

16. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{\ln n}$

17. $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$

18. $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$

19. $\sum_{n=1}^{\infty} \frac{4n+2}{5n-1}$

20. $\sum_{n=1}^{\infty} 2\pi^{-n}$

21. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{5n^2 - 1}$

22. $\sum_{n=0}^{\infty} \frac{(-2)^n}{5^{n+1}}$

23. $1 - \frac{\left(\frac{\pi}{4}\right)^2}{2!} + \frac{\left(\frac{\pi}{4}\right)^4}{4!} - \frac{\left(\frac{\pi}{4}\right)^6}{6!} + \dots$

Given $f(x) = 1 - \frac{4}{3}(x-2)^2 + \frac{16}{5}(x-2)^4 - \frac{2^6}{7}(x-2)^6 + \dots$ is a Taylor Series expansion for $f(x)$ find:

24. a general term for the series.

25. the center of the series.

26. $f(2)$

27. $f'(2)$

28. $f''(2)$

29. $f^{(11)}(2)$

30. $f^{(12)}(2)$

31. $f'(x)$

32. Is the point $(2,1)$ on this same function a local minimum, a local maximum, or neither. Justify.

Write an expression for the n th term of these sequences. Assume $n = 1, 2, 3, \dots$

33. $5, \frac{5}{2}, \frac{5}{6}, \frac{5}{24}, \frac{5}{120}, \dots$

34. $\frac{1}{4}, \frac{1}{7}, \frac{1}{10}, \frac{1}{13}, \dots$

Determine if the following sequences converge or diverge.

35. $\frac{5}{4}, \frac{8}{7}, \frac{11}{10}, \frac{14}{13}, \dots$

36. $a_n = \frac{n^3}{n^2 + 2}$

37. Find the value of $2 - \frac{2}{3} + \frac{2}{9} - \frac{2}{27} + \dots$

38. Find a third degree Taylor Polynomial for $f(x) = \tan x$ centered at $c = \frac{\pi}{4}$.

39. Find a series for the function $f(x) = \cos x$ centered at $c = \frac{3\pi}{4}$. Show four terms.

40. Write the following function as a geometric series showing four terms and the general term and find the interval of convergence. $f(x) = \frac{3}{1+2x}$

41. Find a function for the series $g(x) = 1 - \frac{3}{4}x + \frac{9}{16}x^2 - \frac{27}{64}x^3 + \dots$ and determine its domain.

Give a series representation for these functions. Show four terms and a general term.

42. $f(x) = \int_0^x \frac{\sin t}{t} dt$

43. $g(x) = \int_0^x \frac{\cos t - 1}{t} dt$

Evaluate these integrals.

44. $\int_0^{\infty} \frac{e^{-\frac{1}{x}}}{x^2} dx$

45. $\int_0^3 \frac{x-28}{x^2-x-6} dx$

Selected Answers:

1. $1 - 4x + 8x^2 - \frac{32}{3}x^3 + \dots + \frac{(-1)^n 4^n x^n}{n!} + \dots$

2. $1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \frac{(3x)^6}{6!} + \dots + \frac{(-1)^n 3^{2n} x^{2n}}{(2n)!} + \dots$

4. $(x-1)^2 - \frac{(x-1)^3}{2} + \frac{(x-1)^4}{3} - \frac{(x-1)^5}{4} + \dots + \frac{(-1)^{n+1} (x-1)^{n+1}}{n} + \dots$

5. $x \cos x$ 6. e^{-x^2} 7. $\frac{\ln x}{x}$ 9. -1 10. 1 11. e^4 12. -1.091

 13. converge by AST 14. converge by AST 15. diverge by nTT
 17. converge by AST 18. converge by GST, Sum = 3 19. diverge by nTT

20. converge by GST, Sum = $\frac{2}{\pi-1}$ 21. converge by AST

22. converge by GST, Sum = $\frac{1}{7}$ 23. converge by AST, Sum = $\frac{1}{\sqrt{2}}$

25. $c = 2$ 27. 0 28. $-\frac{8}{3}$ 29. 0

31. $f'(x) = -\frac{8}{3}(x-2) + \frac{64}{5}(x-2)^3 - \frac{2^6 \cdot 6}{7}(x-2)^5 + \dots$ 33. $\frac{5}{n!}$ 35. converges to 1

36. diverges 37. $\frac{3}{2}$ 38. $\tan x \approx 1 + 2\left(x - \frac{\pi}{4}\right) + \frac{4}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{16}{3!}\left(x - \frac{\pi}{4}\right)^3$

40. $f(x) = 3 - 6x + 12x^2 - 24x^3 + \dots + (-1)^n 3 \cdot 2^n x^n + \dots$, IOC: $-\frac{1}{2} < x < \frac{1}{2}$

41. $g(x) = \frac{1}{1 + \frac{3}{4}x}$, Do: $-\frac{4}{3} < x < \frac{4}{3}$

42. $g(x) = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} + \dots$ 44. 1 45. diverges

CALCULUS EXTENDED UNIT 7 SUMMARY

Improper Integrals

1. one or both limits of integration are infinite
2. the function has an infinite discontinuity (a vertical asymptote) at or between the limits

Convergence or Divergence of a Sequence

If $\{a_n\}$ is a sequence and $\lim_{n \rightarrow \infty} a_n = L$ then L is the limit of the sequence and the sequence converges to L .

If $\lim_{n \rightarrow \infty} a_n$ does not exist, then the sequence $\{a_n\}$ does not have a limit and $\{a_n\}$ diverges.

A sequence is monotonic if it is nonincreasing or nondecreasing.

A sequence is bounded if every term is greater than or equal to some number N (the lower bound) and also every term is less than or equal to some number M (the upper bound).

Tests for Convergence (of a series)

n^{th} Term Test for Divergence: If $\lim_{n \rightarrow \infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges. (cannot be used to show conv.)

Geom. Series Test: $|r| \geq 1 \rightarrow \text{diverges}$, $|r| < 1 \rightarrow \text{converges}$ and $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$.

Alternating Series Test for Convergence: converges if

1. $a_{n+1} \leq a_n$ for all n after a certain n (terms never increase in absolute value)
2. $\lim_{n \rightarrow \infty} a_n = 0$

Taylor Series Expansion:

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \dots$$

Maclaurin Series :

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

Elementary Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$$

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \dots + \frac{(-1)^{n+1} (x-1)^n}{n} + \dots$$

Lesson 8-1 Error Approximations

Alternating Series Remainder:

For a convergent alternating series when approximating the sum of a series by using only the first n terms, the error will be less than or equal to the absolute value of the $(n+1)^{\text{st}}$ term (this is the next term or the first unused term).

Example 1. Approximate the sum of $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n!}$ by using the first 6 terms.

Example 2. Find the upper bound for the remainder for the approximation from Example 1.

Example 3. Find upper and lower bounds for the actual sum of the series in Example 1.

Example 4. Approximate $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$ with an error of less than .001.

Example 5. Use an elementary series to find the actual value of the series in Example 4.

This remainder can be written as

$$\boxed{\text{Alternating Series Remainder} \leq \left| \frac{f^{(n+1)}(c)(x-c)^{n+1}}{(n+1)!} \right|}$$

First unused term!

If a nonalternating series is approximated the method is slightly different and slightly harder. It is called the **Lagrange Remainder or Taylor's Theorem Remainder**.

$$\boxed{\text{Lagrange Remainder} \leq \left| \frac{f^{(n+1)}(z)(x-c)^{n+1}}{(n+1)!} \right|}$$

Where z is the x -value between x and c inclusive which makes $|f^{(n+1)}(z)|$ a maximum.

As in an alternating series remainder the $(n+1)^{\text{st}}$ term of the Taylor series is used however, the $(n+1)^{\text{st}}$ derivative factor is carefully chosen.

Choose a value of z which makes the $f^{(n+1)}(z)$ factor a maximum. This may be at the center, at the x -value where f is to be evaluated, or you may know the maximum value in advance (sine and cosine functions have a maximum value of 1).

Example 6. Estimate e^2 using a Maclaurin polynomial of degree 10 for e^x .

Example 7. Use the Lagrange form of the remainder (error) to estimate the accuracy of using this partial sum.

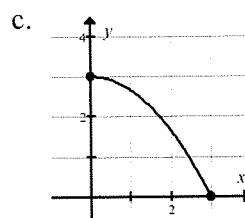
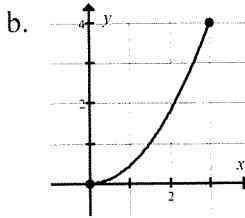
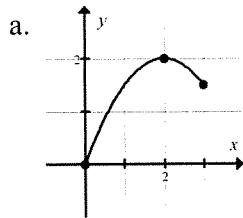
Example 8. If $f^{(5)}(x) = 700 \sin x$ and if $x = .7$ is in the convergence interval for the power series of f centered at $x = 0$, find an upper limit for the error when the fourth-degree Taylor polynomial is used to approximate $f(.7)$.

Example 9. If $f^{(6)}(x)$ is a decreasing function, find the error bound when a 5th degree Taylor polynomial centered at $x = 4$ is used to approximate $f(4.1)$. Assume the series converges for $x = 4.1$.

Assignment 8-1

1. Approximate the sum of the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ with an error less than or equal to 0.001.
2. If the first four terms are used to approximate the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n^3 - 1}$ find an upper bound for the remainder.
3. Approximate e^{-1} with a sixth degree Maclaurin Polynomial and find an upper limit of the Alternating Series Remainder.

4. How many terms of a Maclaurin Polynomial are needed to approximate $\sin 1$ with an error of less than 0.001?
5. How many terms of a Maclaurin Polynomial are needed to approximate $\sin 2$ with an error of less than 0.001?
6. If a Taylor Polynomial centered at 1 is used to approximate $\ln 2$ with an error of less than 0.001, how many terms are needed?
7. If $|f^{(4)}(x)| \leq 4$ find the Lagrange error bound if a third degree Taylor Polynomial centered at $x=1$ is used to approximate $f(2)$. Assume the series converges for $x=2$.
8. If $P_3(2)=5$ for the function from problem 7, find the range of possible values for $f(2)$.
9. If $f^{(6)}(x)=200\sin x$ and $x=.5$ is in the interval of convergence of the power series for f , then find the error when a fifth-degree Taylor polynomial, centered at $x=0$ is used to approximate $f(.5)$.
10. If a sixth degree Taylor Polynomial centered at $x=0$ is used to approximate $f(3)$, find the Lagrange error bound for each of the following if the graph shown is a portion of the graph of $f^{(7)}(x)$. Assume the series converges for $x=3$.



11. Assuming the function from problem 10 is represented by an alternating series, which of the three answers would be the same using an alternating series error bound?
12. The function $f(x)=e^{-2x}$ is approximated by the polynomial $f(x) \approx 1 - 2x + 2x^2 - \frac{4}{3}x^3$. For what x -values will this approximation have an error of less than 0.001?
13. For $f(x)=\ln x$, $c=1$:
- Write a Taylor Polynomial $P_4(x)$.
 - Write a power series for $f(x)$ using Σ notation.
 - Approximate $f(1.3)$ using $P_4(1.3)$.
 - Find the actual value of $f(1.3)$.
 - Find the Lagrange error (remainder) bound, $R_4(1.3)$.
 - Find the number of terms from the Taylor Polynomial needed to approximate $f(1.3)$ with an error (remainder) less than .001.
14. Find an upper limit for the error when the Taylor polynomial $T(x)=x-\frac{x^3}{3!}$ is used to approximate $f(x)=\sin x$ at $x=0.5$.

15. Let $f(x)$ be a function whose Taylor series converges for all x . If $|f^{(n)}(x)| < 1$ what is the minimum number of terms of the Taylor series, centered at $x=1$, necessary to approximate $f(1.2)$ with an error less than 0.00001? Assume the series has no zero terms.

16. (calculator allowed)

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
2	3	4	5	6	7

Let h be a function having derivatives of all orders for $x > 0$, selected values of h and its first four derivatives are indicated in the table above.

- Write the first-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.
- Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.
- Assuming the fourth derivative of h is an increasing function, use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with error less than 3×10^{-5} .

17. If $y = \frac{3}{x} - 2\sqrt{x}$, find $\frac{d^2y}{dx^2}$.

18. Find the point(s) where the line(s) tangent to the graph of $f(x) = \frac{1}{3}x^3 - x^2 + x + 3$ is/are parallel to the graph of $y = x + 5$.

Integrate:

19. $\int \frac{3-4t}{t^2+9} dt$

20. $\int \frac{\arcsin \frac{x}{2}}{\sqrt{4-x^2}} dx$

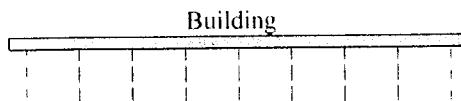
21. $\int \frac{\sqrt{x-1}}{x} dx$ (hint: let $u = \sqrt{x-1}$)

22. Find a general solution of the differential equation $t \frac{dy}{dt} - 2y = \frac{dy}{dt}$. Solve for y .

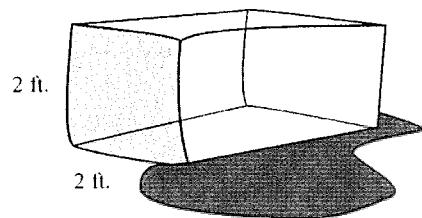
23. The rate of growth of bacteria in a culture is proportional to the number of bacteria present at any time t . If there were 2000 bacteria present 3 days after the introduction of bacteria into the culture, and 5000 present 2 days later, find:
- the growth rate for the bacteria in the culture.
 - the number of bacteria initially introduced into the culture. (Round to the nearest whole number.)
 - the estimated number of bacteria in the culture 12 days after the bacteria were initially introduced. (Round to the nearest hundred.)

24. A small dog kennel with 8 individual rectangular holding pens of equal size is to be constructed using 144 ft of chain link fencing material. One side of the kennel is to be placed against a building and requires no fencing, as shown in the figure below.

- Find the dimensions (for each holding pen) that produce a maximum area for each pen.
- What is that maximum area for each holding pen?



25. A block of ice is exposed to heat in such a way that the block maintains a similar shape as it melts. The block of ice is initially 2 feet wide, 2 feet high, and 3 feet long, as shown at right. If the change in the width of the ice is $-\frac{1}{3}$ ft/hr, find:



- the rate of change in the volume of the block of ice when the width is 1 ft.
- the amount of time it will take for the block of ice to completely melt.

26. Find $\lim_{\theta \rightarrow \frac{\pi}{2}} \frac{-2 \cos \theta}{e^{\theta - \frac{\pi}{2}} - 1}$

Selected Answers:

- .902
- $R \leq \frac{1}{249}$
- .368, $R \leq \frac{1}{5040}$
- three terms
- five terms
- 1000 terms
- $R \leq \frac{1}{6}$
- $4\frac{5}{6} \leq f(2) \leq 5\frac{1}{6}$
- $R \leq .0043$
- a. $R \leq .867$ or $.868$
- b. $R \leq 1.735$ or 1.736
- c. $R \leq 1.301$ or 1.302
11. c
12. $-.197$ or $-.196 < x < .196$ or $.197$
- 13a. $P_4 = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$, b. $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} (x-1)^n}{n}$,
- c. $f(1.3) \approx P_4(1.3) = .261975$,
- d. $f(1.3) = \ln(1.3) = .262364$,
- e. $R_4(1.3) \leq .000486$,
- f. four terms
14. Alt Series Error $\leq .00026$ or Lagrange Error $\leq .0026$
15. five terms
- 16 (a) $h(x) \approx 3 + 4(x-2)$, $h(1.9) \approx 2.6$,
- (b) $h(x) \approx 3 + 4(x-2) + \frac{5}{2}(x-2)^2 + (x-2)^3$, $h(1.9) \approx 2.624$
- (c) $R \leq .000029 < .00003$
17. $y'' = 6x^{-3} + \frac{1}{2}x^{-\frac{3}{2}}$
18. $(0, 3)$, $\left(2, \frac{11}{3}\right)$
19. $\arctan \frac{t}{3} - 2 \ln(t^2 + 9) + C$
20. $\frac{1}{2} \left(\arcsin \frac{x}{2} \right)^2 + C$
21. $2\sqrt{x-1} - 2 \arctan \sqrt{x-1} + C$
22. $y = C(t-1)^2$

More Selected Answers:23a. $k = .458$ b. 505 or 506 bacteria c. 123,500 bacteria24a. 8 ft. by 9 ft. b. 72 ft^2 25a. $-\frac{3}{2} \frac{\text{ft}^3}{\text{hr}}$ b. 6 hours 26. 2**Lesson 8-2 Integral Test and p -Series****Integral Test:**

If the function $f(x)$ is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$ then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x) dx$ either both converge or both diverge.

Note: $\sum_{n=1}^{\infty} a_n \neq \int_1^{\infty} f(x) dx$

Examples: Use the integral test to determine convergence or divergence.

$$1. \sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

$$2. \sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Examples: Use the integral test to determine convergence or divergence of these series.

$$3. \sum_{n=1}^{\infty} \frac{1}{n}$$

$$4. \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

$$5. \sum_{n=1}^{\infty} \frac{1}{n^2}$$

***p*-Series and Harmonic Series:**

If p is a positive constant then $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots$ is called a ***p*-series**.

The last three examples are all *p*-series. Each of them could have been done using the following test.

***p*-series Test**

If $p > 1$ then the *p*-series **converges**. If $0 < p \leq 1$ then the *p*-series **diverges**.

The **harmonic series** is the *p*-series in which $p = 1$. $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$ (Example 3 above)

Examples: Use the *p*-series test to determine convergence or divergence of these series.

6. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \dots$

7. $\sum_{n=1}^{\infty} n^3 \sqrt[3]{n^{-11}}$

Assignment 8-2

Use the Integral Test to show convergence or divergence.

1. $\sum_{n=1}^{\infty} \frac{1}{n+2}$

2. $\sum_{n=1}^{\infty} \frac{1}{e^n}$

3. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+2)}$

4. $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \dots$

5. $\frac{\ln 2}{4} + \frac{\ln 3}{9} + \frac{\ln 4}{16} + \frac{\ln 5}{25} + \dots$

Explain why the Integral Test cannot be used for these series.

6. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+2}$

7. $\sum_{n=1}^{\infty} \frac{1}{n-2}$

8. $\sum_{n=1}^{\infty} \frac{n^2}{n+2}$

Use the *p*-series Test to show convergence or divergence.

9. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$

10. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \dots$

11. $\sum_{n=1}^{\infty} \frac{1}{n^e}$

Determine the convergence or divergence by any method.

12. $\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^{\ln n}$

13. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^5}}$

14. $\sum_{n=1}^{\infty} \left(\frac{5}{4}\right)^n$

15. $\sum_{n=1}^{\infty} \frac{2n}{\sqrt{n^2+2}}$

16. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

17. $\sum_{n=1}^{\infty} \left(\frac{1}{n^2} + \frac{1}{n^3}\right)$

18. $\sum_{n=1}^{\infty} \frac{e^n}{3^{n+1}}$

19. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$

20. $\sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{\sqrt[3]{n}}$

21. If $f(x) = \frac{1}{1+x}$

- Write a fourth degree Maclaurin Polynomial. Hint: It is **not** necessary to find any derivatives.
- Write a power series for the function using \sum notation.
- Approximate $f(0.2)$ using the polynomial from part a.
- Find the Alternating Series error (remainder) bound for this polynomial.
- Find the actual value of $f(0.2)$.
- Find the actual error in the approximation from part c.
- Find the number of terms from the Maclaurin Polynomial needed to approximate $f(0.2)$ with an error (remainder) less than .001.

22. Use one of the elementary functions to find the **simplified** value of the following series.

$$(e-1) - \frac{(e-1)^2}{2} + \frac{(e-1)^3}{3} - \frac{(e-1)^4}{4} + \frac{(e-1)^5}{5} + \dots$$

23. Write the first four nonzero terms and the general term of the Taylor series for $g(x) = \frac{3x^2}{1+x^3}$ about $x = 0$.

24. Find the interval of convergence of the series in Problem 23.

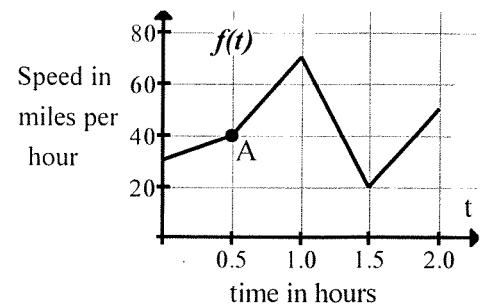
25. Use integration and your series from Problem 23 to find four terms of a series for $f(x) = \ln(1+x^3)$.

26. Use your series from Problem 25 to find an approximation for $\ln \frac{9}{8}$ with an error of less than 0.001.

27. Let $f(x)$ be a function whose Taylor series converges for all x . If $|f^{(n)}(x)| < 4$ what is the minimum number of terms of the Taylor series, centered at $x = 2$, necessary to approximate $f(2.3)$ with an error less than 0.001? Assume there are no zero terms.

28. The graph at the right shows a speed function for a car trip.

- Write a sentence telling what the coordinates of point A represent in words.
(Include numbers and units.)
- Using the graph shown, if $\int_0^2 f(t) dt = 80$, write a sentence telling what this means about the car trip in words. (Include numbers and units.)



Find the indicated antiderivatives.

29. $y' = \frac{\ln(x) - x^2}{x}$, $y = ?$

30. $f'(t) = e^{3t^2+3t}(2t+1)$, $f(t) = ?$

Differentiate.

31. $y = \ln(\arctan(2x))$

32. $f(x) = \ln(x) \arctan(2x)$

33. $g(x) = (\ln(\arcsin x^2))^3$

34. For $\sin(t^2) = ye^t$, find $\frac{dy}{dt}$

Without a calculator, find the volumes of the solids formed by revolving the regions described in Problems 35 and 36 about the indicated axis.

35. The region is bounded by $f(x) = \sqrt{\sin x}$ and $y = 0$ on $[0, \pi]$, about the x -axis

36. The region is bounded by $y = \sin x$, $y = 0$, $x = \frac{\pi}{3}$, and $x = \frac{2\pi}{3}$, about the y -axis.

37. Use the Trapezoidal Rule to approximate $\int_0^{\frac{\pi}{4}} e^{\tan x} dx$ using three equal subdivisions.

Selected Answers:

1. diverges 2. converges 3. diverges 5. converges

6. The series is not always positive. 8. The series is not decreasing. 9. diverges by p-sT

10. converges by p-sT 12. diverges (harmonic series) 13. converges by p-sT

14. diverges by GST 16. converges by IT 18. converges by GST

19. converges by AST 21a. $f(x) \approx 1 - x + x^2 - x^3 + x^4$ b. $f(x) = \sum_{n=0}^{\infty} (-1)^n x^n$

21c. $f(.2) \approx .8336$ d. $ASR \leq .00032$ e. $f(.2) = .8333\dots$ f. error = .0002666\dots g. 5 terms

23. $f(x) = 3x^2 - 3x^5 + 3x^8 - 3x^{11} + \dots + (-1)^{n+1} 3x^{3n-1} + \dots$

25. $\ln(1+x^3) = x^3 - \frac{1}{2}x^6 + \frac{1}{3}x^9 - \frac{1}{4}x^{12} + \dots$ 26. .1171 27. 5 terms

29. $y = \frac{1}{2}(\ln x)^2 - \frac{1}{2}x^2 + C$ 31. $y' = \frac{2}{(1+4x^2)\arctan(2x)}$

32. $f'(x) = \ln x \frac{2}{1+4x^2} + \arctan(2x) \frac{1}{x}$ 33. $g'(x) = 3(\ln(\arcsin x^2))^2 \frac{2x}{\arcsin x^2 \sqrt{1-x^4}}$

34. $y' = \frac{2t \cos t^2 - ye^t}{e^t}$ 35. 2π 37. $\boxed{\frac{1}{2} \cdot \frac{\pi}{12} \left(e^{\tan 0} + 2e^{\tan \frac{\pi}{12}} + 2e^{\tan \frac{\pi}{6}} + e^{\tan \frac{\pi}{4}} \right)} = 1.295$

Lesson 8-3 Direct Comparison Test, Limit Comparison Test

Direct Comparison Test: Let $0 < a_n \leq b_n$ for all n after a certain n .

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Informally:

1. If the “larger” series converges, then the “smaller” series must also converge.
2. If the “smaller” series diverges, then the “larger” series must also diverge.

Examples: Determine the convergence or divergence of the following.

$$1. \sum_{n=1}^{\infty} \frac{1}{1+2^n}$$

$$2. \sum_{n=3}^{\infty} \frac{1}{n-2}$$

$$3. \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$$

Limit Comparison Test: If $a_n > 0$, $b_n > 0$, and $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$

where L is finite and positive, then both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or they both diverge.

This limit comparison works well when comparing a messy algebraic series to an easier p-series. To choose the p -series, disregard all but the highest powers of n in the numerator and denominator.

Examples: Determine the convergence or divergence of the following.

$$4. \sum_{n=2}^{\infty} \frac{1}{3n^2 - 4}$$

$$5. \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+7}$$

$$6. \sum_{n=1}^{\infty} \frac{n}{4n^3 + n^2 + 5}$$

Assignment 8-3

Use the Direct Comparison Test to determine convergence or divergence.

$$\begin{array}{lll} 1. \sum_{n=1}^{\infty} \frac{1}{n^2 + 4} & 2. \sum_{n=5}^{\infty} \frac{1}{n-4} & 3. \sum_{n=2}^{\infty} \frac{2}{\sqrt{n}-1} \\ 5. \sum_{n=1}^{\infty} \frac{3^{n+1}}{4^n + 2} & 6. \sum_{n=1}^{\infty} \frac{\ln n}{n+2} & 7. \sum_{n=1}^{\infty} \frac{1}{n!} \end{array}$$

Use the Limit Comparison Test to determine convergence or divergence.

$$\begin{array}{lll} 8. \sum_{n=1}^{\infty} \frac{n}{n^2 + 4} & 9. \sum_{n=1}^{\infty} \frac{2}{\sqrt{n^2 + 2}} & 10. \sum_{n=1}^{\infty} \frac{2n^2 - 3n + 2}{3n^5 + n - 4} \\ 12. \sum_{n=1}^{\infty} \frac{4}{2n + \sqrt{n^2 + 3}} & 13. \sum_{n=1}^{\infty} \sin \frac{1}{n} & 11. \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2 + 3}} \end{array}$$

14. Use the integral test to determine the convergence of $\sum_{n=1}^{\infty} \frac{n}{(n^2 + 1)^2}$.

15. Use the p -series test to determine the convergence of $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n}$.

Determine the convergence of each of the following by any convergence test.

Do not use the same test twice.

$$16. \sum_{n=0}^{\infty} 4 \left(\frac{1}{6}\right)^n \quad 17. \sum_{n=1}^{\infty} \frac{1}{3^n - 4} \quad 18. \sum_{n=1}^{\infty} \frac{1}{3^n + 4} \quad 19. \sum_{n=1}^{\infty} \frac{n}{3n + 4} \quad 20. \sum_{n=2}^{\infty} (-1)^n \frac{5}{n-1}$$

21. Given $f(x) = e^{-x}$:

- a. Write a fourth degree Taylor Polynomial centered at $c = 1$.
- b. Write a power series for $f(x)$ using \sum notation.
- c. Approximate $f(1.1)$ using the Taylor polynomial from part a.
- d. Find the actual value of $f(1.1)$.
- e. Find the upper limit for the error (remainder) for your approximation in part c.
- f. Find the number of terms from the Taylor Polynomial needed to approximate $f(1.1)$ with an error (remainder) less than .001.

22. Let f be the function given by $f(x) = \sin\left(3x + \frac{\pi}{4}\right)$.

- a. Find $T(x)$ the third-degree Taylor polynomial for f about $x = 0$.
- b. Find the coefficient of x^{18} in the Taylor series for f about $x = 0$.
- c. Use the Lagrange error bound to show that $|f(0.1) - T(0.1)| < 0.0003$.
- d. Write a third-degree Taylor polynomial for $f'(x)$ about $x = 0$.

23. Let f be the function defined by $f(x) = \frac{1}{x-2}$. Without using a calculator:

- Write the first four terms and the general term of the Taylor series expansion of $f(x)$ about $x=3$.
- Use the result of Part (a) to find the first four terms and the general term of the series expansion about $x=3$ for $\ln|x-2|$.
- Use the series in Part (b) to find an approximation for $\ln 1.5$ with an error of less than 0.02.

Integrate.

24. $\int \frac{\cos y}{\sin^3 y} dy$

25. $\int \sec^5 x \tan x dx$

26. $\int \frac{\csc^2(\pi x)}{\cot(\pi x)} dx$

27. $\int \frac{\cos y}{\sin y - 2} dy$

28. $\int \frac{\cot^2 x - 1}{\cot x} dx$

29. $\int \frac{e^{\tan x - 1}}{\cos^2 x} dx$

Selected Answers:

1. conv. 2. div. 3. div. 4. conv. 5. conv. 6. div. 7. conv. 8. div.

9. div. 10. conv. 11. conv. 12. div. 13. div. 14. conv. 15. div.

16. conv. GST 17. conv. LCT 18. Conv. DCT 19. div. nTT

20. conv. AST 21a. $f(x) \approx P_4(x) = \frac{1}{e} - \frac{1}{e}(x-1) + \frac{1}{e} \frac{(x-1)^2}{2!} - \frac{1}{e} \frac{(x-1)^3}{3!} + \frac{1}{e} \frac{(x-1)^4}{4!}$

21b. $f(x) = \sum_{n=0}^{\infty} \frac{1}{e} \frac{(-1)^n (x-1)^n}{n!}$ c. .3328711139 d. $f(1.1) = e^{-1.1} = .3328710837$

e. error $\leq 3.06566201 \times 10^{-8}$ f. 3 terms

22a. $f(x) \approx \frac{1}{\sqrt{2}} + \frac{3}{\sqrt{2}}x - \frac{9}{2\sqrt{2}}x^2 - \frac{27}{6\sqrt{2}}x^3$ b. $\frac{-3^{18}}{18!\sqrt{2}}$

22d. $f'(x) \approx \frac{3}{\sqrt{2}} - \frac{9}{\sqrt{2}}x - \frac{27}{2\sqrt{2}}x^2 + \frac{81}{6\sqrt{2}}x^3$

23a. $f(x) = 1 - (x-3) + (x-3)^2 - (x-3)^3 + \dots + (-1)^n (x-3)^n + \dots$

23b. $\ln|x-2| = (x-3) - \frac{(x-3)^2}{2} + \frac{(x-3)^3}{3} - \frac{(x-3)^4}{4} + \dots + \frac{(-1)^n (x-3)^{n+1}}{n+1} + \dots$

23c. $\ln 1.5 \approx .41666$ 24. $-\frac{1}{2}(\sin y)^{-2} + C$ 25. $\frac{1}{5}(\sec x)^5 + C$

27. $\ln|\sin y - 2| + C$ 28. $\ln|\sin x| + \ln|\cos x| + C$ 29. $e^{\tan x - 1} + C$

Lesson 8-4 Ratio Test

Ratio Test: (useful for series involving factorials or exponentials)

1. $\sum a_n$ converges if $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$
2. $\sum a_n$ diverges if $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$ or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$
3. The Ratio Test is inconclusive if $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$

Examples: Determine the convergence or divergence.

$$1. \sum_{n=0}^{\infty} \frac{n!}{3^n}$$

$$2. \sum_{n=1}^{\infty} \frac{3^{n+1}}{4^n n^2}$$

$$3. \sum_{n=2}^{\infty} \frac{(-1)^n \sqrt{n}}{n-1}$$

$$4. \sum_{n=1}^{\infty} \frac{(2n+1)!!}{3^n (2n-1)n!} = \sum_{n=1}^{\infty} \frac{(2n+1)(2n-1)(2n-3)\cdots 5 \cdot 3 \cdot 1}{3^n (2n-1)n!}$$

Handling Mixed Problems:

1. Does the n^{th} term approach zero? If not, the series diverges (n^{th} term test).
2. Is the series a special type: geometric (r^n) , p -series $\left(\frac{1}{n^p}\right)$, or alternating?

$f(x)$ can be integrated

factorials or exponentials

3. Can you use the Integral Test or the Ratio Test
4. Can you compare to a special type using DCT or LCT?

compare to larger or smaller series

messy algebraic

Convergence/Divergence Tests

n^{th} term test	div. if $\lim_{n \rightarrow \infty} a_n \neq 0$ (cannot be used to show convergence)
Geom. series test	$\sum_{n=0}^{\infty} ar^n \quad r < 1 \rightarrow \text{conv.}, \quad r \geq 1 \rightarrow \text{div.}, \quad S = \frac{a}{1-r}$
p -series	$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad p > 1 \rightarrow \text{conv.}, \quad p \leq 1 \rightarrow \text{div.}$
Alternating series	decr. terms and $\lim_{n \rightarrow \infty} a_n = 0 \rightarrow \text{conv.}$
Integral test	$a_n = f(n) \quad \sum_{n=1}^{\infty} a_n \text{ conv. if } \int_1^{\infty} f(x) dx \text{ conv.}, \quad \sum_{n=1}^{\infty} a_n \text{ div. if } \int_1^{\infty} f(x) dx \text{ div.}$
Ratio test	$\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right < 1 \rightarrow \text{conv.}, \quad \lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right > 1 \rightarrow \text{div.}, \quad (\text{inconclusive if } \lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right = 1)$ (works well for factorials and exponentials)
Direct Comparison	a series with terms smaller than a known convergent series also converges a series with terms larger than a known divergent series also diverges
Limit Comparison	if $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ is finite and positive both series converge or both diverge (use with "messy" algebraic series, usually compared to a p -series)

Mixed Examples: Determine the convergence or divergence.

5.
$$\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$$

6.
$$\sum_{n=1}^{\infty} n^2 e^{-n^3}$$

7.
$$\sum_{n=1}^{\infty} \left(\frac{e}{3} \right)^n$$

8.
$$\sum_{n=1}^{\infty} \frac{1}{4n+5}$$

9.
$$\sum_{n=1}^{\infty} (-1)^n \frac{3}{n^2}$$

10.
$$\frac{1}{10} + \frac{1 \cdot 2}{10^2} + \frac{1 \cdot 2 \cdot 3}{10^3} + \frac{1 \cdot 2 \cdot 3 \cdot 4}{10^4} + \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{10^5} + \dots$$

Assignment 8-4

Use the ratio test to determine convergence or divergence if possible.

1. $\sum_{n=1}^{\infty} \frac{n!}{n^3}$

2. $\sum_{n=0}^{\infty} n\left(\frac{2}{3}\right)^n$

3. $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

4. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$

5. $\sum_{n=1}^{\infty} \frac{(2n)!}{n3^n}$

6. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

7. $\sum_{n=0}^{\infty} \frac{(2n+1)!!}{n!} = \sum_{n=1}^{\infty} \frac{(2n+1)(2n-1)(2n-3)\dots 3 \cdot 1}{n(n-1)(n-2)\dots 2 \cdot 1}$

Determine convergence or divergence using any test.

8. $\sum_{n=1}^{\infty} (-1)^n \frac{3}{n}$

9. $\sum_{n=1}^{\infty} \frac{3}{n}$

10. $\sum_{n=0}^{\infty} \frac{1}{3^n}$

11. $\sum_{n=1}^{\infty} \frac{4}{n\sqrt{n}}$

12. $\sum_{n=1}^{\infty} (-1)^n \frac{3n}{n+1}$

13. $\sum_{n=1}^{\infty} \frac{3n+2}{n^2 + 2n - 4}$

14. $\sum_{n=2}^{\infty} \frac{2^n}{\ln n}$

15. $\sum_{n=1}^{\infty} \frac{|\cos n|}{4^n}$

16. $\sum_{n=1}^{\infty} 4 \left(\frac{5^n}{3^{n+1}} \right)$

17. $\sum_{n=3}^{\infty} \frac{(n-2)3^n}{n!}$

18. Which of the following series is/are equivalent to $\sum_{n=1}^{\infty} \frac{2n}{n+1}$?

- a. $\sum_{n=0}^{\infty} \frac{2(n+1)}{n+2}$ b. $\sum_{n=0}^{\infty} \frac{2n}{n+1}$ c. $1 + \sum_{n=2}^{\infty} \frac{2n}{n+1}$ d. $\sum_{n=1}^{\infty} \left(2 - \frac{2}{n+1} \right)$ e. $\frac{7}{3} + \sum_{n=3}^{\infty} \frac{2n}{n+1}$

19. The function f has derivatives of all orders for all real numbers x . Given $f(3) = -2$,

$$f'(3) = 4, \quad f''(3) = 2, \quad \text{and} \quad f'''(3) = -9.$$

a. Write the third degree Taylor polynomial for f about $x = 3$.

b. Use your part a answer to approximate $f(2.5)$.

c. Given $|f^{(4)}(x)| \leq 3$ use the Lagrange error bound on the approximation to $f(2.5)$ found in part b to find a range of possible values for $f(2.5)$.

d. Write the fourth degree Taylor polynomial for $g(x) = f(x^2 + 3)$ about $x = 0$.

e. Use your answer to part d to determine if g has a local minimum or a local maximum at $x = 0$.

20. Given the series $f(x) = 3 - 15x + 75x^2 - 375x^3 + \dots$ (calculator allowed)

a. Write an expression for $f(x)$ using sigma notation.

b. Find the interval of convergence of the geometric series.

c. Find the exact value of $f(.1)$.

d. Can the exact value of $f(.3)$ be found? If it can, find it. If not, explain why not.

e. Approximate $f(.1)$ using the first four terms of the series.

f. Find the alternating series error bound for this approximation.

g. Find the actual error for this approximation.

21. Calculator Allowed

The table shown is a record of the velocity of a car traveling in a straight line. At time 35 seconds the car attains its absolute maximum velocity and at time 45 seconds the car has a relative minimum velocity.

There are no other local or absolute extrema for the car's velocity between 0 and 50 seconds.

- During what intervals of time is the acceleration of the car positive?
- Find the average acceleration of the car over the interval $0 \leq t \leq 50$ seconds. Label units.
- Find an approximation for the acceleration of the car at $t = 32.5$ seconds. Show the computations you used to arrive at your answer and label units.
- Approximate $\int_0^{50} v(t) dt$ with a Riemann sum, using midpoints of five equal subintervals.
- Using correct units, explain the meaning of your answer to part d.

time (seconds)	$v(t)$ (feet per second)
0	0
5	15
10	21
15	29
20	51
25	65
30	75
35	80
40	75
45	55
50	70

Differentiate.

22. $y = \csc \frac{1}{x} - 3\sqrt{x}$

23. $P(t) = \sqrt{\cos t - \sin t}$

24. $g(x) = \ln |\sec(3x)|$

25. $y = e^{\sin x} e^{\cos x}$

26. $f''(x) = 10x^{-\frac{1}{3}}$, $f'(8) = 50$, and $f(1) = 30$. Find an equation for the curve $f(x) =$

Selected Answers:

- ∞ , div.
- $\frac{2}{3}$, conv.
- 3, div.
- 0, conv.
- ∞ , div.
- 1, inconclusive
- 2, div.
- conv AST
- conv. GST
- conv. pST
- div. LCT
- div. nTT
- div. GST
- conv. RT
- $f(x) \approx -2 + 4(x-3) + (x-3)^2 - \frac{3}{2}(x-3)^3$
- $f(2.5) \approx -3.5625$
- $g(x) \approx -2 + 4x^2 + x^4$
- $\sum_{n=0}^{\infty} 3(-5x)^n$
- $-0.2 < x < 0.2$
- 1.875
- 1.4 ft/sec²
- The car traveled a total distance of approximately 2440 feet between 0 and 50 seconds.
- $P'(t) = \frac{-\sin t - \cos t}{2\sqrt{\cos t - \sin t}}$
- $g'(x) = 3\tan(3x)$
- $y' = e^{\sin x} e^{\cos x} (\cos x - \sin x)$
- $f(x) = 9x^{\frac{5}{3}} - 10x + 31$

Lesson 8-5 Interval of Convergence (by Ratio Test)

The **radius of convergence** of a power series is the distance from the center c at which the series will converge. The **interval of convergence** is the range of x -values within which the series will converge.

Examples: Find the interval of convergence and the radius of convergence of the following power series.

$$1. \sum_{n=1}^{\infty} \frac{2^n x^n}{n}$$

$$2. \sum_{n=0}^{\infty} \frac{(x+1)^n}{2^n}$$

$$3. \sum_{n=0}^{\infty} n! x^n$$

$$4. \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Note: In the last unit we found intervals of convergence for Geometric Series. It was not necessary to check endpoints because Geometric Series diverge when $r = 1$. However the ratio test is inconclusive when the ratio is 1. Therefore when using the Ratio Test to find intervals of convergence it is necessary to check endpoints using one of the other tests (although it is not necessary to check when asked to find only the radius of convergence).

Assignment 8-5

Find the radius of convergence.

1. $\sum_{n=0}^{\infty} \frac{(2x)^n}{n+1}$

2. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n^2}$

Find the interval of convergence. Remember to check endpoints when using the Ratio Test.

3. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$

4. $\sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$

5. $\sum_{n=1}^{\infty} \frac{(-1)^n (x+3)^n}{n(n+1)}$

6. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

7. $\sum_{n=0}^{\infty} (2n)! x^{2n}$

8. $\sum_{n=0}^{\infty} \frac{x^n}{4^{n+1}}$

9. $\sum_{n=1}^{\infty} \frac{(-1)^n (x-3)^n}{n 3^n}$

10. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^n}{n+1}$

11. $\sum_{n=0}^{\infty} \frac{(-1)^n (x-5)^n}{2^{n+1}}$

12. $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2n}$

13. $\sum_{n=0}^{\infty} \frac{(n+1)! x^n}{n!}$

14. $\sum_{n=1}^{\infty} \frac{n+1}{n} (6x)^n$

15. Given $f(x) = \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \frac{x^3}{5!} + \dots + \frac{x^n}{(n+2)!} + \dots$

- a. For what values of x does the given series converge? Show work.
 - b. Let $g(x) = x^2 f(x)$. Write the Maclaurin series for $g(x)$, showing the first four terms and the general term.
 - c. Write $g(x)$ in terms of an elementary function.
 - d. Write $f(x)$ in terms of the same elementary function.
16. Let f be the function given by $f(x) = e^{-x^2}$.
- a. Find the first four terms and the general term of the power series for $f(x)$ about $x=0$.
 - b. Find the interval of convergence of this power series for $f(x)$.
 - c. Use the first four nonzero terms of $f(x)$ to approximate $f(0.8)$.
 - d. Show this approximation has an error less than 0.007.

17. Let f be the function given by $f(t) = \frac{3}{1+t^2}$.

- a. Find the first four nonzero terms and the general term for the power series expansion of $f(t)$ about $t=0$.
- b. Given $g(x) = \int_0^x f(t) dt$, find the first four nonzero terms and the general term for the power series expansion of $g(x)$ about $x=0$.
- c. Find the interval of convergence of the power series for $g(x)$.

Selected Answers:

1. $\frac{1}{2}$ 2. 1 3. $(-1,1]$ 4. $(-3,3)$ 5. $[-4,-2]$ 6. $(-\infty,\infty)$ 8. $(-4,4)$

9. $(0,6]$ 10. $(-1,1]$ 11. $(3,7)$ 12. $[-1,1]$ 13. $(-1,1)$

15a. $(-\infty,\infty)$ b. $g(x) = \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^{n+2}}{(n+2)!} + \cdots$ c. $g(x) = e^x - 1 - x$

16a. $e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots$ b. $(-\infty,\infty)$ c. .5211 d. .00699 < .007

17a. $f(t) = 3 - 3t^2 + 3t^4 - 3t^6 + \cdots + 3(-1)^n t^{2n} + \cdots$

b. $g(x) = 3x - x^3 + \frac{3x^5}{5} - \frac{3x^7}{7} + \cdots + \frac{3(-1)^n x^{2n+1}}{2n+1} + \cdots$ c. $-1 \leq x \leq 1$

Lesson 8-6 Absolute vs Conditional Convergence, Review

Many convergent series have negative terms, alternating or some other pattern. Taking the absolute value of each term of a convergent series with some negative terms makes the new positive series **less** likely to converge since the sum will be greater without negative terms. If the new positive series is still convergent the original series is called **absolutely convergent**. If, on the other hand, the new positive series diverges the original convergent series with negative terms is called **conditionally convergent**.

If a series converges after taking the absolute value of its terms it is guaranteed to also converge with no absolute value.

Examples: Determine whether each of the following series is absolutely convergent, conditionally convergent, or divergent.

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

2. $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$

3. $\sum_{n=1}^{\infty} \frac{e^n}{n^e}$

Assignment 8-6

Determine whether each series is absolutely convergent, conditionally convergent, or divergent.

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

2. $\sum_{n=0}^{\infty} \frac{(-1)^n n^2}{n^3 - 4}$

3. $\sum_{n=1}^{\infty} \frac{(-1)^n (n-1)}{n}$

4. $\sum_{n=1}^{\infty} \frac{(\sin n)^3}{n^3}$

Determine the convergence or divergence of the series. Whenever possible, find the sum of the series.

5. $\sum_{n=1}^{\infty} 5 \frac{2^n}{3^{n+2}}$

6. $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n \ln n}$

7. $\sum_{n=0}^{\infty} 3 \left(\frac{\sqrt{5}}{2} \right)^n$

8. $\sum_{n=0}^{\infty} \frac{3^n}{n!}$

9. $\sum_{n=1}^{\infty} \frac{\sqrt{\ln n}}{n}$

10. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n^4}}$

11. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + 3n}}$

12. $\sum_{n=3}^{\infty} \frac{n+1}{n(n-2)}$

13. $\sum_{n=0}^{\infty} \frac{1}{4^n - 3}$

14. $\sum_{n=1}^{\infty} \frac{e^n}{n!}$

15. $\sum_{n=1}^{\infty} \frac{3^n}{n^3}$

16. $\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{(2n+1)!}$

17. $1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \dots + \frac{1}{n!} + \dots$

18. Find a third degree Maclaurin Polynomial for $f(x) = e^{-2x}$.

19. Find a third degree Taylor Polynomial centered at $c = \frac{-3\pi}{4}$ for $f(x) = \tan x$.

20. Use a Taylor Polynomial to approximate $\sin(0.5)$ accurate to the third decimal place (error less than 0.001).

21. Use a Taylor Polynomial to approximate $e^{-0.2}$ accurate to the third decimal place.

22. Find the largest possible value of $|f(1.9) - p(1.9)|$ if $p(x)$ is a second degree Taylor polynomial of $f(x)$ centered at $x = 2$ and $-9 \leq f'''(x) \leq 5$.

23. Show four terms and a general term of a Taylor series centered at $c = -1$ for the function

$$g(x) = \frac{1}{x}.$$

24. Show four terms and a general term of a Taylor series centered at $c = 0$ for the function $f(x) = 2^x$.

25. Show four terms of a Taylor series centered at $c = 0$ for the function $f(x) = \frac{1}{(x+1)^4}$.

Find the interval of convergence for each series.

26. $\sum_{n=0}^{\infty} \frac{(-1)^n (x-3)^n}{(n+1)^2}$

27. $\sum_{n=0}^{\infty} \left(\frac{2x}{5} \right)^n$

28. $\sum_{n=1}^{\infty} \frac{2^n (x-3)^n}{n}$

29. $\sum_{n=0}^{\infty} n! (x-5)^n$

30. Find the series corresponding to the integral $\int_0^x \frac{\sin t}{t} dt$. Show four terms and a general term.

31. Use an appropriate series to approximate $\int_0^1 \sin \sqrt{x} dx$ within 0.001.

32. Do not use a calculator. $\int_0^\pi \left(x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \dots \right) dx = ?$

33. Given $f(x) = \cos x^2$. Do not use a calculator.

a. Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

b. Using this series find $\lim_{x \rightarrow 0} \frac{1 - \frac{1}{2}x^4 - f(x)}{x^8}$.

c. Write the first four nonzero terms of the Taylor series for $\int_0^x \cos t^2 dt$ about $x = 0$.

d. Approximate $\int_0^1 \cos t^2 dt$ using the first two terms of your answer to part c.

e. Show that the approximation found in part d differs from the actual value of $\int_0^1 \cos t^2 dt$ by less than $\frac{1}{200}$.

Answers:

1. converges absolutely 2. converges conditionally 3. diverges
 4. converges absolutely 5. converges, GST; Sum = $\frac{10}{9}$ 6. diverges, nTT 7. div. GST

8. conv. RT 9. div. IT or DCT 10. div. p-sT 11. conv. DCT (or LCT)
 12. div. DCT (or LCT) 13. conv. LCT 14. conv. RT 15. div. nTT (or RT)

16. conv. AST; Sum = 0.141 17. conv. RT (or DCT); Sum = $e - 1$

18. $f(x) \approx 1 - 2x + 2x^2 - \frac{4}{3}x^3$ 19. $f(x) \approx 1 + 2\left(x + \frac{3\pi}{4}\right) + 2\left(x + \frac{3\pi}{4}\right)^2 + \frac{8}{3}\left(x + \frac{3\pi}{4}\right)^3$

20. 0.479 21. 0.818 or 0.819 22. 0.0015

23. $f(x) = -1 - (x+1) - (x+1)^2 - (x+1)^3 - \dots - (x+1)^n - \dots$

24. $f(x) = 1 + (\ln 2)x + \frac{(\ln 2)^2}{2!}x^2 + \frac{(\ln 2)^3}{3!}x^3 + \dots + \frac{(\ln 2)^n}{n!}x^n + \dots$

25. $f(x) = 1 - 4x + 10x^2 - 20x^3 + \dots$ 26. $[2, 4]$ 27. $\left(-\frac{5}{2}, \frac{5}{2}\right)$ 28. $\left[\frac{5}{2}, \frac{7}{2}\right]$ 29. Conv.

only when $x = 5$. 30. $x - \frac{x^3}{3!3} + \frac{x^5}{5!5} - \frac{x^7}{7!7} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!(2n+1)} + \dots$

31. $\approx \frac{2}{3} - \frac{1}{15} + \frac{1}{420} = .602$ 32. π 33a. $f(x) = 1 - \frac{x^4}{2} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \dots + \frac{(-1)^n x^{4n}}{(2n)!} + \dots$

33b. $-\frac{1}{24}$ c. $x - \frac{1}{10}x^5 + \frac{1}{4!9}x^9 - \frac{1}{6!13}x^{13} + \dots$ d. $\frac{9}{10}$

33e. alternating series error $< \frac{1}{216} < \frac{1}{200}$

CALCULUS EXTENDED UNIT 8 SUMMARY

Alternating Series Remainder \leq the first unused term

$$\text{Lagrange Remainder} \leq \frac{|f^{(n+1)}(z)(x-c)^{n+1}|}{(n+1)!}$$

Where z is the x -value between x and c inclusive which makes $|f^{(n+1)}(z)|$ a maximum.

If $\sum_{n=1}^{\infty} |a_n|$ converges, $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, $\sum_{n=1}^{\infty} a_n$ converges conditionally.

Handling Mixed Problems:

1. Does the n^{th} term approach zero? If not, the series diverges (n^{th} term test).
2. Is the series a special type: geometric (r^n) , p -series $\left(\frac{1}{n^p}\right)$, or alternating?

$f(x)$ can be integrated

factorials or exponentials

3. Can you use the Integral Test or the Ratio Test
4. Can you compare to a special type using DCT or LCT?

compare to larger or smaller series

messy algebraic

Convergence/Divergence Tests

n^{th} term test	div. if $\lim_{n \rightarrow \infty} a_n \neq 0$ (cannot be used to show convergence)
Geom. series test	$\sum_{n=0}^{\infty} ar^n \quad r < 1 \rightarrow \text{conv.}, \quad r \geq 1 \rightarrow \text{div.}, \quad S = \frac{a}{1-r}$
p -series	$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad p > 1 \rightarrow \text{conv.}, \quad p \leq 1 \rightarrow \text{div.}$
Alternating series	decr. terms and $\lim_{n \rightarrow \infty} a_n = 0 \rightarrow \text{conv.}$
Integral test	$a_n = f(n) \quad \sum_{n=1}^{\infty} a_n \text{ conv. if } \int_1^{\infty} f(x) dx \text{ conv.}, \quad \sum_{n=1}^{\infty} a_n \text{ div. if } \int_1^{\infty} f(x) dx \text{ div.}$
Ratio test	$\lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right < 1 \rightarrow \text{conv.}, \quad \lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right > 1 \rightarrow \text{div.}, \quad (\text{inconclusive if } \lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right = 1)$ (works well for factorials and exponentials)
Direct Comparison	a series with terms smaller than a known convergent series also converges a series with terms larger than a known divergent series also diverges
Limit Comparison	if $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ is finite and positive both series converge or both diverge (use with "messy" algebraic series, usually compared to a p -series)

Lesson 9-1 Parametric Equations

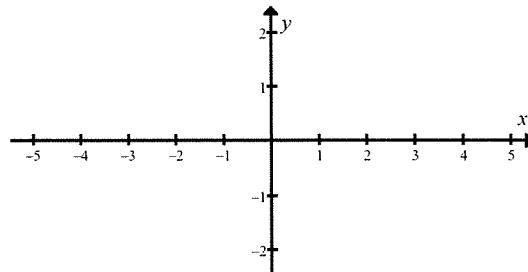
Example 1. Plot points to sketch the curve described by the parametric equations. Mark the orientation on the curve.

$$x = t^2 - 5$$

$$y = \frac{t}{2}$$

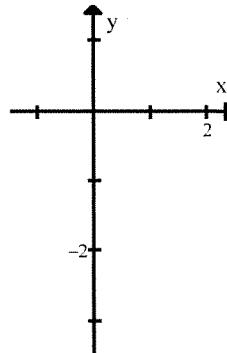
$$-3 \leq t \leq 2$$

t						
x						
y						



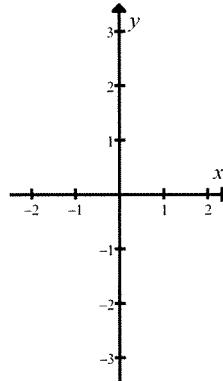
Example 2. Change the following to rectangular form by eliminating the parameter. Then graph.

$$x = \frac{1}{\sqrt{t+1}} \text{ and } y = \frac{t}{t+1}, \quad t > -1$$



Example 3. Eliminate the parameter to sketch the curve.

$$x = \cos \theta \text{ and } y = 3 \sin \theta, \quad 0 \leq \theta \leq 2\pi$$



Even though equations are given in terms of a parameter, it is possible to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ by differentiating and then dividing.

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ and $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$
--

Example 4. If $x = \cos t$ and $y = 3 \sin t$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Example 5. Find the slope and concavity of $x = \sqrt{t}$ and $y = \frac{1}{4}t^2 - 1$, $t \geq 0$ at the point $(2, 3)$.

Example 6. Write an equation of a tangent line to the curve defined by $x = t - 1$ and $y = \frac{1}{t} + 1$ at the point when $t = 1$.

Arc Length: If a curve is smooth and does not intersect itself the length of an arc is given by

$$\text{arc length} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Example 7. Using the parametric equations from example 6, find the arc length on the interval $1 \leq t \leq 3$.

Assignment 9-1:

Using these parametric equations, eliminate the parameter to write the corresponding rectangular equation. Sketch the curve indicating the orientation without using a calculator.

1. $x = 2t - 3$, $y = \frac{2}{3}t + 4$ 2. $x = t^3$, $y = t^2$ 3. $x = \sqrt{t}$, $y = 4 - t$ 4. $x = t^4$, $y = 4 \ln t$

5. Use a calculator set in parametric mode to graph the curve represented by the parametric equations $x = -3 + 4 \cos \theta$ and $y = 1 + 2 \sin \theta$. Then eliminate the parameter.

6. Given the parametric equations $x = -2t + 1$ and $y = t^3 + 3$:

a. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

b. Find an equation of the tangent line when $t = 1$.

c. Use concavity to determine if the tangent line is above the curve or below the curve.

7. Given the parametric equations $x = 3 \cos \theta$ and $y = 3 \sin \theta$:

a. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

b. Find an equation of the tangent line when $\theta = \frac{\pi}{4}$.

c. Use concavity to determine if the tangent line is above the curve or below the curve.

8. Use a calculator to graph the curve represented by the parametric equations $x = 3 \sin(2t)$ and $y = 2 \sin t$. The curve crosses itself at the point $(0,0)$. Find equations of all tangent lines at that point.

Without using a calculator, find all points at which each curve has horizontal and vertical tangents.

9. $x = 2t + 1$, $y = t^2$ 10. $x = t^2 + 1$, $y = t^2 + 4t$

11. $x = t^2 - t + 3$, $y = 4t^3 - 12t$ 12. $x = \tan \theta$, $y = \sec \theta$

Given the parametric equations $x = -3t - 5$ and $y = t^3 - 12t + 3$ (without using a calculator):

13. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t .

14. Use the second derivative test to determine if the curve has a local maximum, a local minimum, or neither when $t = 2$.

15. Use the second derivative test to determine if the curve has a local maximum, a local minimum, or neither at the point $(1, 19)$.

16. Use the second derivative test to determine if the curve has a local maximum, a local minimum, or neither at the point $(-8, -8)$.

Show an integral setup and find the length of each arc on the given interval.

17. $x = 3t - t^2$, $y = 4t^{\frac{3}{2}}$ $1 \leq t \leq 2$ 18. $x = t + \cos t$, $y = t - \sin t$ $0 \leq t \leq \pi$

19. $x = \arccos t$, $y = \ln \sqrt{1+t^2}$ $0 \leq t \leq \frac{1}{2}$

20. Find the length of the arc between the two y -intercepts of $x = t^2 - 1$ and $y = 2t$.

21. Differentiate $\sin(y - 2x) = x^2 - 10$ to find $\frac{dy}{dx}$.

Integrate each of the following:

22. $\int \frac{(\sqrt{t} - 4)^{10}}{\sqrt{t}} dt$

23. $\int y^3 \left(y - \frac{1}{y} \right) dy$

24. $\int \frac{3x^2 + x - 2}{\sqrt{x}} dx$

25. $\int \frac{6 - 4x}{\sqrt{2x-1}} dx$

26. $\int_0^{\frac{2\pi}{3}} \tan\left(\frac{x}{2}\right) dx$

27. $\int \frac{3}{x^2 - 6x + 18} dx$

28. $\int \sin(e^{-t}) e^{-t} dt$

29. $\int \frac{x-1}{(2x^2 - 4x)^5} dx$

30. $\int_{-2}^{-1} \left(\frac{1-3x^4}{x^2} \right) dx$

31. If $f''(x) = x^{-\frac{4}{3}}$, $f'(8) = \frac{3}{2}$, and $f(27) = 5$, find $f(x)$.

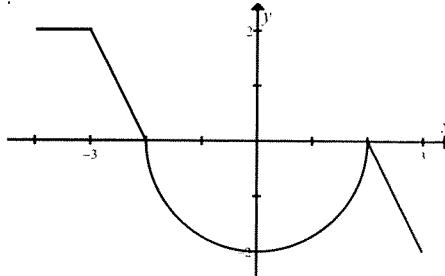
32. Find the domain, vertical asymptote(s), hole(s) in the graph, x - and y -intercepts,

and end behavior for $y = \frac{4x^2 - 16}{x^4 - 16}$. Then sketch its graph without using a calculator.

The graph of the function $y = f(x)$ consists of line segments and a semicircle as shown. Evaluate the following using geometry formulas.

33. $\int_{-4}^3 f(x) dx$

34. $\int_{-2}^2 (f(x) + 2) dx$



Selected Answers:

1. $y = \frac{1}{3}x + 5$ 2. $y = x^{\frac{2}{3}}$ 4. $y = \ln x$ 5. $\left(\frac{x+3}{4}\right)^2 + \left(\frac{y-1}{2}\right)^2 = 1$

6a. $\frac{dy}{dx} = -\frac{3}{2}t^2$, $\frac{d^2y}{dx^2} = \frac{3}{2}t$ b. $y - 4 = -\frac{3}{2}(x+1)$ c. below

7a. $\frac{dy}{dx} = -\cot \theta$, $\frac{d^2y}{dx^2} = -\frac{1}{3}\csc^3 \theta$ b. $y - \frac{3}{\sqrt{2}} = -\left(x - \frac{3}{\sqrt{2}}\right)$ c. above 8. $y = \pm \frac{1}{3}x$

9. H.T: (1,0) V.T: none

10. H.T: (5, -4) V.T: (1,0)

11. H.T: (3, -8), (5, 8) V.T: $\left(\frac{11}{4}, -\frac{11}{2}\right)$

12. H.T: $(0, \pm 1)$ V.T: none

13. $\frac{dy}{dx} = -t^2 + 4$, $\frac{d^2y}{dx^2} = \frac{2}{3}t$

14. min.

15. max.

17. 7.336 or 7.337

18. 3.678

19. .538

20. 4.591

22. $\frac{2}{11}(\sqrt{t} - 4)^{11} + C$

24. $\frac{6}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + C$

More Selected Answers:

25. $4\sqrt{2x-1} - \frac{2}{3}(\sqrt{2x-1})^3 + C$ or $(6-4x)\sqrt{2x-1} + \frac{4}{3}(2x-1)^{\frac{3}{2}} + C$

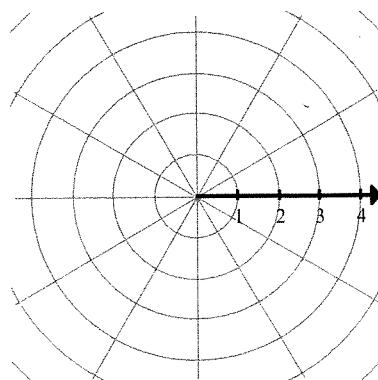
26. $-2 \ln \frac{1}{2} = \ln 4$ 27. $\arctan \frac{x-3}{3} + C$ 29. $-\frac{1}{16}(2x^2 - 4x)^{-4} + C$

30. $(1+1) - \left(\frac{1}{2} + 8\right) = -\frac{13}{2}$ 31. $-\frac{9}{2}x^{\frac{2}{3}} + 3x - \frac{71}{2}$ 34. $8 - 2\pi$

Lesson 9-2 Polar Graphs**Plotting points in polar form**

1. Use the polar grid to plot these polar points.

- A $\left(3, \frac{\pi}{6}\right)$ B $(4, \pi)$
 C $\left(3, -\frac{5\pi}{6}\right)$ D $\left(-2, \frac{3\pi}{2}\right)$

**Point conversions**

2. Write the rectangular point
- $(-1, \sqrt{3})$
- in polar form such that:

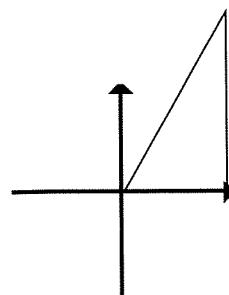
- a. $r > 0, \theta > 0$ b. $r > 0, \theta < 0$ c. $r < 0, \theta > 0$ d. $r < 0, \theta < 0$

3. Change the point
- $(2, \pi)$
- to rectangular form.

4. Change the point
- $(-3, 3)$
- to polar form.

Conversion Equations

$$\begin{aligned} x^2 + y^2 &= & x &= \\ \tan \theta &= & y &= \end{aligned}$$



Equation conversions (rectangular to polar)

5. $y = 4$

6. $3x - y + 2 = 0$

7. $x^2 + y^2 - 2x = 0$

Equation conversions (polar to rectangular)

8. $r = -2$

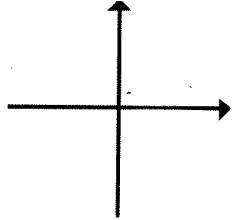
9. $r = 3 \cos \theta$

10. $r = 2 \csc \theta$

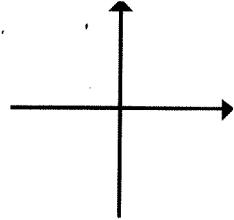
Sketching polar graphs (use a calculator on those in **bold**)

11. Circles

a. $r = 2 \cos \theta$

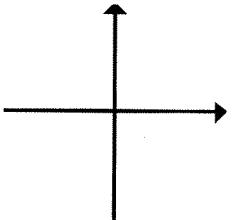


b. $r = 5 \sin \theta$

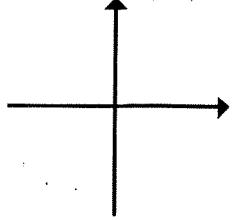


12. Rose petal curves

a. $r = 3 \cos(2\theta)$

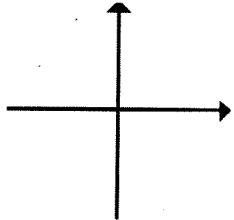


b. $r = 4 \sin(3\theta)$

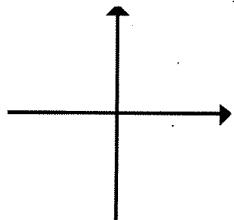


13. Limaçons

a. $r = 2 - 3 \cos \theta$

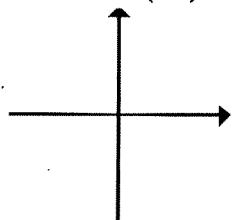


b. $r = 3 + 3 \sin \theta$



14. Lemniscate

$r^2 = 9 \sin(2\theta)$



Sketching polar graphs:

Circles: $r = d \cos \theta$ (x-axis symmetry) $r = d \sin \theta$ (y-axis symmetry) } d is the diameter

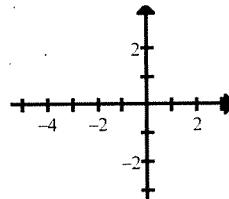
Rose petal curves: $r = a \cos(n\theta)$ (x-axis symmetry) $r = a \sin(n\theta)$ (y-axis symmetry) } a is the maximum r ,
 n petals if n is odd,
 $2n$ petals if n is even

Limaçons: $r = a \pm b \cos \theta$ (x-axis symmetry) } may have inner loop
 $r = a \pm b \sin \theta$ (y-axis symmetry) } may or may not include the pole

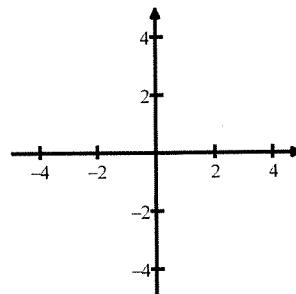
Tangent lines

15. Find an equation of the tangent line to the graph of $r = 2(1 - \sin \theta)$ at the point $(2, 0)$.

16. Find the points at which the graph of $r = 2 - 2 \cos \theta$ has horizontal tangents.

**Tangent lines at the pole**

17. Find the equations of the lines tangent to $r = 4 \sin(3\theta)$ at the pole.



Assignment 9-2

Without using a calculator, accurately plot each of the following polar coordinate points on a separate graph. Give the rectangular coordinates of the point.

1. $\left(5, \frac{\pi}{2}\right)$

2. $\left(-3\sqrt{2}, \frac{3\pi}{4}\right)$

3. $\left(4, -\frac{\pi}{3}\right)$

4. $\left(-1, \frac{7\pi}{6}\right)$

Without using a calculator, plot each of the following rectangular points and give two sets of polar coordinates for $0 \leq \theta < 2\pi$.

5. $(5, -5)$

6. $(-1, \sqrt{3})$

7. $(-5, 0)$

8. $(-5, -5\sqrt{3})$

9. Use a calculator to give rectangular coordinates for the polar point $(-7.2, 4.5)$.

10. Use a calculator to give two sets of polar coordinates ($0 \leq \theta < 2\pi$) for the rectangular point $(-2, 5)$.

Match each of the following equations with one of the descriptions given without using a calculator.

11. $r = 3\sin(2\theta)$

12. $r = 4\cos\theta$

13. $r\cos\theta = 4$

14. $r = 4 + 2\cos\theta$

a. a circle with y -axis symmetry

b. a four petal rose

c. a vertical line

d. a limaçon with y -axis symmetry

e. a circle with x -axis symmetry

f. a horizontal line.

g. a limaçon with x -axis symmetry

Convert the following rectangular equations to polar (solve for r) and sketch the graph.

15. $y = 5$

16. $2x + y + 5 = 0$

17. $y^2 = 2x$

Convert the following polar equations to rectangular and sketch the graph.

18. $r = 5$

19. $\theta = \frac{3\pi}{4}$

20. $r = 3\sec\theta$

Use a calculator to graph. Determine if the interval $0 \leq \theta < 2\pi$ produces a complete graph.

21. $r = 5\sin\left(\frac{3\theta}{2}\right)$

22. $r = 5 - 6\cos\theta$

23. $r = \theta$

24. Without using a calculator graph $r = 2 + 4\sin\theta$. Find an equation of the line tangent to the curve at the point $(2, 0)$.

25. Without using a calculator graph $r = 4\cos\theta$. Find an equation of the line tangent to the curve at the polar point $\left(2\sqrt{3}, \frac{\pi}{6}\right)$.

26. Without using a calculator graph $r = 1 - \sin\theta$. Find the points at which the curve has horizontal tangent lines.

Without using a calculator graph each equation and find equations of each tangent at the pole.

27. $r = 5\sin\theta$

28. $r = 3 - 3\cos\theta$

29. $r = 4\cos(3\theta)$

30. Find the length of the arc of the curve defined by $x = \ln t$ and $y = t + 1$ on the interval $1 \leq t \leq 6$.
 31. Find an equation of the line tangent to the curve defined by $x = \ln t$ and $y = t + 1$ when $t = 1$.
 32. Is the curve defined by $x = \ln t$ and $y = t + 1$ concave upward or downward when $t = 2$?
 33. Give a rectangular equation for the curve defined by $x = \ln t$ and $y = t + 1$.

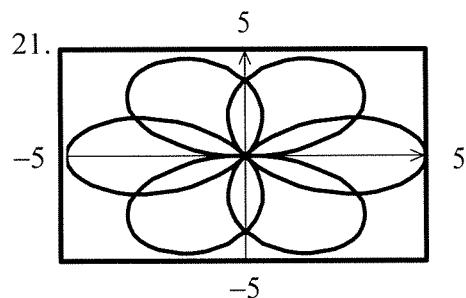
Selected Answers:

show graphs for 1-8 1. $(0,5)$ 2. $(3,-3)$ 3. $(2, -2\sqrt{3})$ 5. $(5\sqrt{2}, \frac{7\pi}{4}), (-5\sqrt{2}, \frac{3\pi}{4})$

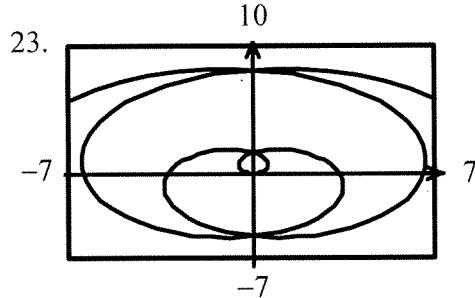
7. $(5, \pi), (-5, 0)$ 8. $(10, \frac{4\pi}{3}), (-10, \frac{\pi}{3})$ 9. $(1.517 \text{ or } 1.518, 7.038)$.

11. b 13. c 15. $r = 5 \csc \theta$ 16. $r = \frac{-5}{2 \cos \theta + \sin \theta}$ 17. $r = \frac{2 \cos \theta}{\sin^2 \theta}$

19. $y = -x$ 20. $x = 3$



$0 \leq \theta < 4\pi$ shows a complete graph.



No interval shows a complete graph.

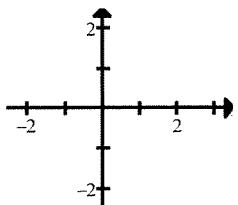
24. $y = \frac{1}{2}(x - 2)$ 25. $y - \sqrt{3} = -\frac{1}{\sqrt{3}}(x - 3)$ 26. $(2, \frac{3\pi}{2}), (\frac{1}{2}, \frac{\pi}{6}), (\frac{1}{2}, \frac{5\pi}{6})$

27. $\theta = 0$ 29. $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$ 30. 5.384 31. $y = x + 2$ 33. $y = e^x + 1$

Lesson 9-3 Polar Area

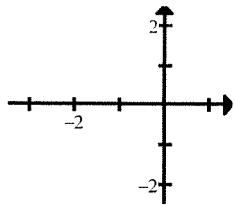
$$\text{Polar area} = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$$

Example 1. Find the area of one petal of the curve $r = 3 \cos(3\theta)$.



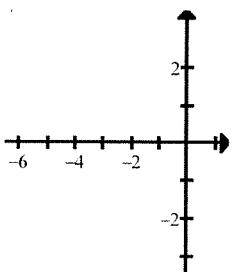
Intersections of Polar Graphs

Example 2. Find the points of intersection of the graphs of $r = 1 - 2 \cos \theta$ and $r = 1$.

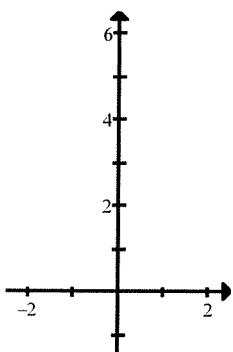


Area between two curves

Example 3. Find the area of the region common to the two regions bounded by $r = -6 \cos \theta$ and $r = 2 - 2 \cos \theta$.



Example 4. Find the area between the loops of $r = 2(1 + 2 \sin \theta)$.



Assignment 9-3

Without using a calculator, graph the following polar curves and find the points of intersection.

1. $r = 1 + \sin \theta$ and $r = 1 - \sin \theta$ 2. $r = 1 + \sin \theta$ and $r = 1 - \cos \theta$

3. Use a calculator to graph the following curves. Then find the points of intersection.

$r = 6 - 8 \sin \theta$ and $r = 2$

Graph the following polar curves without using a calculator. Set up a definite integral for the area of the indicated region. Use a calculator to evaluate the integral.

- | | |
|--|---|
| 4. the interior of $r = 1 - \cos \theta$ | 5. one petal of $r = 4 \sin(3\theta)$ |
| 6. one petal of $r = 3 \cos(2\theta)$ | 7. the common interior of $r = 3 - 2 \cos \theta$ and
$r = -3 + 2 \cos \theta$ |

Use a calculator to graph the following curves. Set up a definite integral for the area of the indicated region. Use a calculator to evaluate the integral.

- | | |
|---|--|
| 8. between the loops of $r = 1 + 2 \sin \theta$ | 9. inside $r = 3 \cos \theta$ and outside $r = 2 - \cos \theta$ |
| 10. common interior of $r = 3$ and
$r = 6 \sin(2\theta)$ | 11. region bounded by $r = \theta + \sin(3\theta)$ and
the x -axis for $0 \leq \theta \leq \pi$ |
12. Given the parametric equations $x = 4t - 1$ and $y = 8t - 4$, eliminate the parameter to write the corresponding rectangular equation. Sketch the curve indicating the orientation without using a calculator.
13. Without using a calculator given the parametric equations $x = 3t + 5$ and $y = 8t^2 + 4$, find an equation of the line tangent to the curve when $x = 2$.
14. Without using a calculator given the parametric equations $x = 4 \cos \theta$ and $y = 8 \sin \theta$, determine the concavity on an interval containing $\theta = \frac{7\pi}{6}$.
15. Given the parametric equations $x = 2 + 2 \cos \theta$ and $y = 1 + \sin \theta$, show work to determine the points of horizontal and vertical tangency. Graph with a calculator to see if your answers appear correct.
16. Without a calculator convert the polar point $\left(3, \frac{3\pi}{2}\right)$ to rectangular form.
17. Without a calculator convert the polar point $\left(4, \frac{2\pi}{3}\right)$ to rectangular form.
18. Without a calculator convert the rectangular point $(-5, -5)$ to polar form. Give two answers such that $0 \leq \theta < 2\pi$.
19. Using a calculator convert the rectangular point $(-1.372, 5.164)$ to polar form. Give two answers such that $0 \leq \theta < 2\pi$.

20. Give the rectangular form of the polar equation $r = 2 \sin \theta$.
21. Given the polar equation $r = 1 - 2 \sin \theta$:
- Use a calculator to sketch a graph.
 - Without using a calculator, find equations of tangents at the pole.
 - Without using a calculator, find an equation of the line tangent to the curve at the point (1,0).
 - Find the points at which the curve has vertical tangents. You may use a calculator.

Selected Answers:

- | | | | | |
|---------------------------------------|--|----------------------------------|--|----------|
| 1. $(1,0), (1,\pi), (0,0)$ | 2. $\left(1+\frac{1}{\sqrt{2}}, \frac{3\pi}{4}\right), \left(1-\frac{1}{\sqrt{2}}, \frac{7\pi}{4}\right), (0,0)$ | 4. 4.712 | | |
| 5. 4.188 or 4.189 | 6. 3.534 | 7. 10.557 or 10.558 | 8. 8.337 or 8.338 | 9. 5.196 |
| 10. 22.110 or 22.111 | 11. 7.000 | 12. $y = 2x - 2$ | 13. $y - 12 = -\frac{16}{3}(x - 2)$ | |
| 15. V.T: $(4,1), (0,1)$ | H.T: $(2,2), (2,0)$ | 17. $(-2, 2\sqrt{3})$ | 18. $(5\sqrt{2}, \frac{5\pi}{4}), (-5\sqrt{2}, \frac{\pi}{4})$ | |
| 19. $(5.343, 1.830), (-5.343, 4.972)$ | 21b. $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$ | c. $y - 0 = -\frac{1}{2}(x - 1)$ | | |

Lesson 9-4 Polar Arc Length, Vector Definitions**Arc Length**

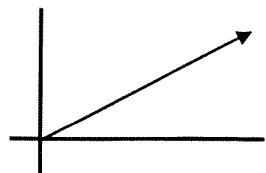
Example 1. Find the length of the arc from $\theta = 0$ to $\theta = 2\pi$ for the curve $r = 2 - 2 \cos \theta$.

Arc Length

$$= \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Vectors (definitions):

Vector-



Magnitude-

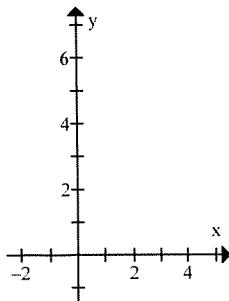
Direction-

Equivalent Vectors-

Component form-

Example 2. The initial point of a vector is $(-1, 3)$ and its terminal point is $(2, 7)$

- a. Graph the vector. b. Graph the vector in standard position on the same axes.



- c. Give the component form of the vector.
d. Find the magnitude of the vector.
e. Find the direction of the vector.

Example 3. Find the direction of a vector given by $\langle -3, 5 \rangle$.

Example 4. If the magnitude of a vector v is $\|v\| = 6$ and its direction is $\theta = \frac{2\pi}{3}$, write the vector in component form.

Assignment 9-4

1. Given the polar curve $r = 4 \sin \theta$.

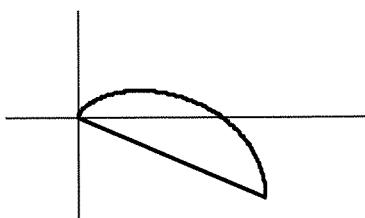
- a. Graph without using a calculator.
- b. Find the circumference using a geometry formula.
- c. Find the circumference showing a polar arc length integral setup and integrate without using a calculator.

2. Graph $r = 4 \cos(2\theta)$ without a calculator. Then use a calculator to find the length of the arc forming one petal.

3. Use a calculator to graph $r = e^{\frac{\theta}{2}}$ on the interval $0 \leq \theta \leq \frac{3\pi}{2}$ and find the length of the curve.

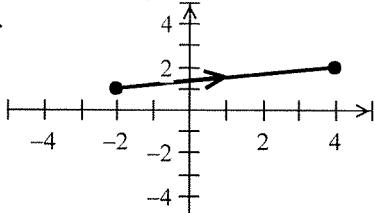
4. The region shown is bounded by the polar curve $r = 1 - \sin \theta$ and the line $\theta = -\frac{\pi}{6}$.

- a. Find the area of the region.
- b. Find the perimeter of the region.

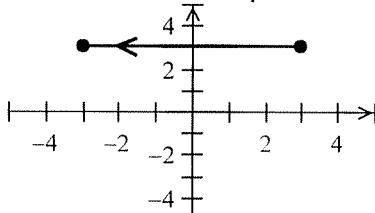


Find the component form of the vector and sketch it with the initial point at the origin.

5.



6.



7. Find the component form of the u and v vectors whose initial and terminal points are given.

Show that u and v are equivalent. $u: (3, -2), (5, 2)$ $v: (-1, -4), (1, 0)$

8. The initial and terminal points of a vector are $(-1, 3)$ and $(3, 6)$.

- Sketch the vector.
- Write the component form.
- Sketch the vector with the initial point at the origin.

9. If the initial point of vector v is $(5, -2)$ and $v = \langle -2, 4 \rangle$, find the terminal point.

10. Find the magnitude of the vector $v = \langle -4, 3 \rangle$.

Find the component form of each vector given the magnitude and the direction without using a calculator.

11. $\|v\| = 5, \theta = 0$.

12. $\|v\| = 6, \theta = \frac{4\pi}{3}$.

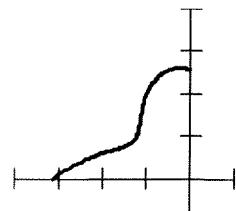
13. $\|v\| = 4, \theta = \frac{7\pi}{4}$.

14. Use a calculator to find the magnitude and the direction of the vector $v = \langle -8, -13 \rangle$.

15. The graph at the right shows the polar curve $r = \theta - \sin(3\theta)$ on the

interval $\frac{\pi}{2} \leq \theta \leq \pi$.

- Find the area of the region bounded by the curve, the x -axis, and the y -axis.
- Find $\frac{dr}{d\theta}$ at $\theta = \frac{3\pi}{4}$ without using a calculator.
- Use your answer to part b to determine if r is increasing or decreasing on an interval containing $\theta = \frac{3\pi}{4}$.
- Find the value of θ on $\frac{\pi}{2} \leq \theta \leq \pi$ at which the curve is closest to the pole.
- Find the x -coordinate of the point on the curve when $\theta = \frac{3\pi}{4}$.
- Find $\frac{dx}{d\theta}$ at $\theta = \frac{3\pi}{4}$ using a calculator.
- Use your answer to part f to determine if x is increasing or decreasing on an interval containing $\theta = \frac{3\pi}{4}$.



16. $v(t) = -\sqrt{t+4}$ represents the velocity equation of an object moving along a vertical path for $t \geq -4$. Let $a(t)$ represent the acceleration and $y(t)$ represent the position of the object at time t . Find:

- an equation for the acceleration of the object at time t .
- an equation for the position of the object at time t if the object's initial position is $-\frac{10}{3}$.
- $y(5)$
- $v(5)$
- $a(5)$
- the speed of the object at time $t=5$.

17. $v(t) = \frac{t-1}{t^2+1}$ is the velocity equation for an object moving along a horizontal path when $t \geq 0$.

Use a calculator to find:

- the velocity of the object at $t=2.3$
- the acceleration of the object at $t=2.3$
- the displacement from $t=0$ to $t=3$.
- the total distance traveled from $t=0$ to $t=3$.

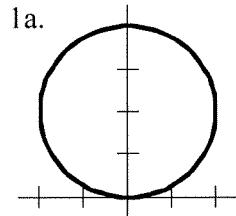
18. If the acceleration of an object is given by $a(t) = e^{\sin t}$ and at time $t=3$ seconds the velocity of the object is 15 feet per second, find the object's velocity at $t=10$ seconds.

19. If the graph of $f(x)$ is concave upward on (a,b) , does the Trapezoidal Rule give a value larger or smaller than the actual value of $\int_a^b f(x) dx$?

20. If $f''(x) < 0$ on (a,b) , does the Trapezoidal Rule give a value larger or smaller than the actual value of $\int_a^b f(x) dx$?

21. If $f''(x) = 0$ on (a,b) , what must be true about the value found using the Trapezoidal Rule and the actual value of $\int_a^b f(x) dx$?

Selected Answers:

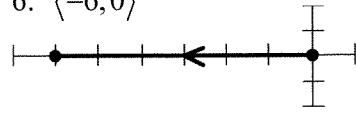


1a. b. 4π 2. 9.688 3. 21.356
4a. .596 or .597 b. 3.499 or 3.500

5. $\langle 6, 1 \rangle$



6. $\langle -6, 0 \rangle$



7. $u = \langle 2, 4 \rangle = v$ 8b. $\langle 4, 3 \rangle$ 9. $(3, 2)$ 10. $\|v\| = 5$ 11. $v = \langle 5, 0 \rangle$
 12. $v = \langle -3, -3\sqrt{3} \rangle$ 13. $v = \langle 2\sqrt{2}, -2\sqrt{2} \rangle$ 14. $\|v\| = 15.264$, $\theta = 4.160$ or 4.161
 15a. 3.756 b. $1 - \frac{3}{\sqrt{2}}$ d. 2.504 or 2.505 e. -1.166 f. -.373

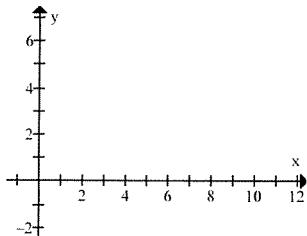
More Selected Answers

16. a. $a(t) = -\frac{1}{2}(t+4)^{-\frac{1}{2}}$ b. $y(t) = -\frac{2}{3}(t+4)^{\frac{3}{2}} + 2$ c. -16 d. -3 e. $-\frac{1}{6}$ f. 3
 17a. $v(2.3) = .206$ or $.207$ b. $a(2.3) = .007$ or $.008$ c. disp. = $-.097$ or $-.098$
 17d. T. D. = $.779$ or $.780$ or $.781$ 18. 23.547 19. larger 20. smaller
 21. They are equal.

Lesson 9-5 Calculus of Vector-Valued Functions**Position vector-** **Speed-****Velocity vector-** **Distance traveled-****Acceleration vector-** **“at rest”**

Example 1. Given a position vector $\langle 3t^2, t^3 - 3t^2 + 4 \rangle$ for a particle moving in the xy -plane find the following.

- a. graph the path of the particle on the interval $0 \leq t \leq 2$ b. the velocity vector at time $t = 1$ c. the speed of the particle at time $t = 1$



- d. the distance traveled between $t = 0$ and $t = 3$ e. the time(s) when the particle is at rest f. the acceleration vector at time $t = 2$
- g. the direction of the particle at time $t = 1$ and when $t = 2$

Example 2. A particle is moving in the xy -plane with acceleration vector $a(t) = \langle -2\cos t, -3\sin t \rangle$.

At time $t = 0$ its velocity vector is $v(0) = \langle 0, 3 \rangle$ and its position vector is $s(0) = \left\langle 2, \frac{\sqrt{2}}{2} \right\rangle$. Find :

a. the velocity vector when $t = \frac{\pi}{4}$

b. the position vector when $t = \frac{\pi}{4}$

c. the speed when $t = \frac{\pi}{4}$

d. the time(s) when the
particle is at rest

e. the direction of the
particle when $t = \frac{\pi}{4}$

f. the distance traveled between $t = 0$ and $t = 2$

Assignment 9-5

1. The position of a particle in the xy -plane is given by $x = 4t^2$ and $y = \sqrt{t}$. Find the following:

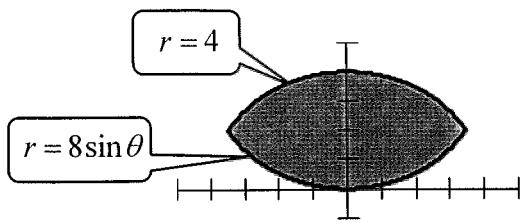
- a. the velocity vector at $t = 4$
- b. the acceleration vector at $t = 4$
- c. the speed of the particle at $t = 4$
- d. the distance the particle moves between $t = 0$ and $t = 4$
- e. the direction of the particle at $t = 4$

2. The position of a particle is given by $x = t^2$ and $y = t^3$. Find the following:

- a. the speed of the particle at $t = 2$
- b. the direction of the particle at $t = 2$
- c. the distance the particle moves between $t = 1$ and $t = 4$
- d. the velocity vector at $t = 3$
- e. the acceleration vector at $t = 4$

3. A particle moves in the xy -plane so that at time t its velocity vector is $v(t) = \langle t^3, \cos(\pi t) \rangle$ and the particle's position vector at time $t = 0$ is $\langle 2, 1 \rangle$.
- What is the position vector of the particle when $t = 2$? Do not use a calculator.
 - What is the acceleration vector of the particle when $t = 2$? Do not use a calculator.
 - What is the direction of the particle when $t = 1$?
 - What is the distance the particle travels between $t = 0$ and $t = 2$?
 - When is the particle at rest?
 - What is the speed of the particle when $t = 2$?
4. A particle moves on the xy -plane so that at time t its coordinates are $x = t^3 + t$ and $y = t^5 - 2t^2$. Find its velocity vector at time $t = 2$.
5. A calculator is allowed on this problem.
The position of an object moving on a curve is $(x(t), y(t))$ at time t . Given $\frac{dx}{dt} = \sqrt{\frac{t}{2+t}}$ and $\frac{dy}{dt} = \cos(t^2 - 1)$. At time $t = 1$, the position of the object is $(2, 4)$.
- Find the position of the object at time $t = 3$.
 - Find the speed of the object at time $t = 3$.
 - Find the total distance traveled by the object over the interval $1 \leq t \leq 3$.
 - Find an equation of the line tangent to the curve at time $t = 3$.
 - Find the acceleration vector at time $t = 3$.
6. Given a parametric curve defined by $x = e^t$ and $y = t + 1$.
- Find the length of the arc of the curve on the interval $1 \leq t \leq 6$.
 - Find an equation of the line tangent to the curve when $t = 1$.
 - Is the curve concave upward or downward when $t = 2$?
 - Give a rectangular equation for the curve.
 - Show an integral setup with respect to the variable x that gives the length of the same arc as that in part a.
7. Find all points of horizontal and vertical tangency to the curve $x = 2 - 2\cos\theta$, $y = 2\sin(2\theta)$.
8. Find the tangents at the pole for $r = 1 + 2\cos\theta$.
9. Find the area common to the interiors of $r = 4\cos\theta$ and $r = 2$.
10. Given the function $f(x) = \ln(x+1)$:
- Write a power series for f showing four terms and the general term.
 - Find the interval of convergence of this power series.
 - Approximate $\ln(1.2)$ by using a fourth degree Taylor polynomial of f .
 - Using your answer for part c and the alternating series remainder, give an upper and lower limit for the actual value of $\ln(1.2)$

11. The figure at the right shows a shaded region bounded by the polar curves $r = 4$ and $r = 8\sin\theta$.



- Find the area of the shaded region.
- Find the perimeter of the shaded region.
- Convert the two polar equations to rectangular form.
- Set up an integral with respect to the variable x and find the area of the shaded region.

Selected Answers:

- 1a. $v(4) = \left\langle 32, \frac{1}{4} \right\rangle$ b. $a(4) = \left\langle 8, -\frac{1}{32} \right\rangle$ c. $\|v(4)\| = 32.001$ d. dist. = 64.413 e. $\theta = .031$
 2a. $\|v(2)\| = 12.649$ b. $\theta = 1.107$ c. dist. = 64.949 d. $v(3) = \langle 6, 27 \rangle$ e. $a(4) = \langle 2, 24 \rangle$
 3a. $s(2) = \langle 6, 1 \rangle$ b. $a(2) = \langle 12, 0 \rangle$ c. $\theta = .418$ d. dist. = 4.567 or 4.568 e. never
 3f. $\|v(2)\| = 8.062$ 4. $\langle 13, 72 \rangle$ 5a. $s(3) = \langle 3.394, 4.280 \text{ or } 4.281 \rangle$ b. $\|v(3)\| = .788$
 5c. dist. = 1.960 or 1.961 d. $y - 4.280 = -.187(x - 3.394)$ or $y - 4.281 = -.188(x - 3.394)$
 5e. $a(3) = \langle .051 \text{ or } .052, -5.936 \rangle$ 6a. 400.891 b. $y - 2 = \frac{1}{e}(x - e)$ c. concave down
 6d. $y = \ln x + 1$ 7. Horiz. $(2 \pm \sqrt{2}, \pm 2)$, Vert. $(0, 0), (4, 0)$ 8. $\theta = \frac{2\pi}{3}$, $\theta = \frac{4\pi}{3}$
 9. 4.913 10a. $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{(-1)^{n+1} x^n}{n} + \dots$ b. $(-1, 1]$ c. .18227
 11a. 19.653 or 19.654 b. 16.755 c. $x^2 + y^2 = 16$, $x^2 + y^2 = 8y$ d. 19.653 or 19.654

Assignment 9-6 Review

- Given the parametric equations $x = t + 3$, $y = t^2$, eliminate the parameter to write the corresponding rectangular equation. Sketch the curve indicating the orientation without using a calculator.
- Given the parametric equations $x = \frac{1}{2}t^2$, $y = \frac{1}{3}t^3 - \frac{1}{2}t^2$:
 - Find the point(s) at which the curve has a horizontal tangent.
 - Use the second derivative test to determine if the curve has a local maximum, a local minimum, or neither at the point found in part a.
 - Find an equation of the tangent line when $t = 6$.
- Given the parametric equations $x = 3\cos\theta$, $y = 2\sin\theta$, find the length of the arc on the interval $0 \leq \theta \leq \frac{\pi}{2}$,
- Convert the polar equation $r = 4\sin\theta$ to rectangular form. Sketch a graph without using a calculator.

5. Convert the polar equation $r = -3 - 3\cos\theta$ to rectangular form. Sketch a graph without using a calculator.
6. Graph $r = 2 - 3\cos\theta$ without using a calculator.
7. Without using a calculator, find an equation of the line tangent to the graph of $r = 2 - 3\cos\theta$ at $\theta = \frac{\pi}{2}$.
8. Find all points of intersection of $r = 2 - 3\cos\theta$ and $r = -5\cos\theta$.
9. Use a calculator to graph $r^2 = 10\sin(2\theta)$ and $r = \sqrt{5}$ and find the area common to the two interiors.
10. Set up an integral for the length of the arc in the first quadrant on the curve $r^2 = 10\sin(2\theta)$. Why is this integral improper?
11. The position of a particle moving in the xy -plane at any time t , $0 \leq t < 2\pi$, is given by the parametric equations $x = 4\cos(2t)$ and $y = 4\sin t$. Do not use a calculator.
- Find the velocity vector for the particle at any time t .
 - Find the velocity vector for the particle at time $t = \pi$.
 - Find the acceleration vector for the particle at time $t = \pi$.
 - Find the speed when $y = -4$.
 - For what values of t is the particle at rest?
 - Find the direction of the particle when $t = \frac{\pi}{2}$.
 - Set up an integral for the distance traveled on the interval $1 \leq t \leq 5$.
12. A particle moves along the graph of $y = \sin x$. If the x -component of acceleration is always 3 and at time $t = 0$, the position of the particle is the point $(\pi, 0)$ and the velocity vector of the particle is $\langle 2, -2 \rangle$. Without using a calculator. Find the x - and y -coordinates of the position of the particle in terms of t .
13. Calculator Allowed
- A particle moving along a curve in the xy -plane has position $(x(t), y(t))$ with $\frac{dx}{dt} = \sin t^2$ and $\frac{dy}{dt} = e^{t^2}$. The position of the particle is $(-1, 4)$ at time $t = 3$.
- Find the acceleration vector at time $t = 2$.
 - Find the position point of the particle at time $t = 0$.
 - At what time does the speed of the particle reach 15 when $t > 0$.
14. Calculator Allowed
- Let g be a function that has derivatives of all orders for all real numbers. Assume $g(0) = 2$, $g'(0) = -1$, $g''(0) = 4$, and $g'''(0) = 3$.
- Write the third-degree Taylor polynomial for g about $x = 0$.
 - Write the fourth-degree Taylor polynomial for $g(x^2)$.
 - Write the third-degree Taylor polynomial for h , where $h(x) = \int_0^x g(t) dt$, about $x = 0$.

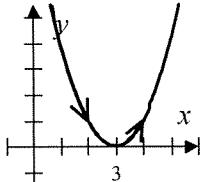
15. Euler's Formula

- Use the elementary series for e^x to write at least six terms of the series for $e^{i\pi}$.
- Simplify your answer using the relationship for powers of i . ($i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, etc.)
- Separate the terms of your simplified series into two infinite series (one with the odd power terms and one with the even power terms). Use elementary series for sine and cosine to evaluate.

Note: This result is sometimes written as the equation $e^{i\pi} + 1 = 0$ called Euler's Formula. This mathematically elegant equation contains the five most "important" numbers in mathematics.

Answers:

1. $y = (x-3)^2$

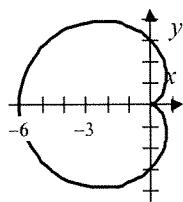


2a. $\left(\frac{1}{2}, -\frac{1}{6}\right)$

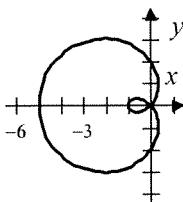
b. local minimum

c. $y - 54 = 5(x - 18)$

5.



6.

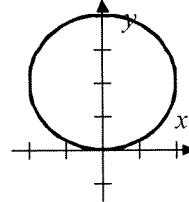


3. 3.966

4. $x^2 + y^2 = 4y$

7. $y - 2 = -\frac{3}{2}(x - 0)$

8. $(5, \pi), (0, 0)$



9. 6.575 or 6.576

10. $\int_0^{\frac{\pi}{2}} \sqrt{10 \sin(2\theta) + \left(\frac{10 \cos(2\theta)}{\sqrt{10 \sin(2\theta)}} \right)^2} d\theta$

This integral is improper since $\frac{dr}{d\theta}$ is undefined at both endpoints.

11a. $v(t) = \langle -8 \sin(2t), 4 \cos t \rangle$ b. $v(\pi) = \langle 0, -4 \rangle$ c. $a(\pi) = \langle -16, 0 \rangle$ d. $\|v\left(\frac{3\pi}{2}\right)\| = 0$

11e. $t = \frac{\pi}{2}, \frac{3\pi}{2}$ f. $\theta = \frac{3\pi}{4}$ g. $\int_1^5 \sqrt{(-8 \sin(2t))^2 + (4 \cos t)^2} dt$

12. $x(t) = \frac{3}{2}t^2 + 2t + \pi$, $y(t) = \sin\left(\frac{3}{2}t^2 + 2t + \pi\right)$ 13a. $a(2) = \langle -2.614 \text{ or } -2.615, 218.393 \rangle$

13b. $(-1.773 \text{ or } -1.774, -1440.545)$ c. $t = 1.645$

14a. $g(x) \approx 2 - x + 2x^2 + \frac{1}{2}x^3$ b. $g(x^2) \approx 2 - x^2 + 2x^4$ c. $h(x) \approx 2x - \frac{1}{2}x^2 + \frac{2}{3}x^3$

15a. $e^{i\pi} = 1 + (i\pi) + \frac{(i\pi)^2}{2!} + \frac{(i\pi)^3}{3!} + \frac{(i\pi)^4}{4!} + \frac{(i\pi)^5}{5!} + \frac{(i\pi)^6}{6!} + \frac{(i\pi)^7}{7!} + \dots$

b. $e^{i\pi} = 1 + (i\pi) - \frac{\pi^2}{2!} - \frac{i\pi^3}{3!} + \frac{\pi^4}{4!} + \frac{i\pi^5}{5!} - \frac{\pi^6}{6!} - \frac{i\pi^7}{7!} + \dots$

c. -1

CALCULUS EXTENDED UNIT 9 SUMMARY

Parametric derivatives: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$

$$\text{Arc Length} = \int_a^b \sqrt{1 + (f'(x))^2} dx = \underbrace{\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt}_{\text{parametric}} = \underbrace{\int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta}_{\text{polar}}$$

Polar Conversions: $r^2 = x^2 + y^2$, $x = r \cos \theta$, $y = r \sin \theta$, $\theta = \arctan \frac{y}{x}$

Sketching polar graphs:

$$\text{Circles: } \begin{aligned} r &= d \cos \theta && (\text{x-axis symmetry}) \\ r &= d \sin \theta && (\text{y-axis symmetry}) \end{aligned} \quad \left. \right\} d \text{ is the diameter}$$

$$\text{Rose petal curves: } \left. \begin{array}{l} r = a \cos(n\theta) \text{ (x-axis symmetry)} \\ r = a \sin(n\theta) \text{ (y-axis symmetry)} \end{array} \right\} \begin{array}{l} a \text{ is the maximum } r, \\ n \text{ petals if } n \text{ is odd,} \\ 2n \text{ petals if } n \text{ is even} \end{array}$$

$$\begin{array}{lll} \text{Limaçons: } & r = a \pm b \cos \theta & (\text{x-axis symmetry}) \\ & r = a \pm b \sin \theta & (\text{y-axis symmetry}) \end{array} \left. \begin{array}{l} \text{may have inner loop} \\ \text{may or may not include the pole} \end{array} \right\}$$

$$\text{Polar Area} = \frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 d\theta$$

Tangent lines at the pole: let $r = 0$

Vectors:

$$\text{Magnitude} = \sqrt{x^2 + y^2}$$

$$\text{Direction} = \theta = \arctan \frac{y}{x}$$

$$\text{Speed} = |v(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$\text{Total Dist.} = \int_{t_1}^{t_2} |v(t)| dt = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Lesson A-1 Area of a Surface of Revolution

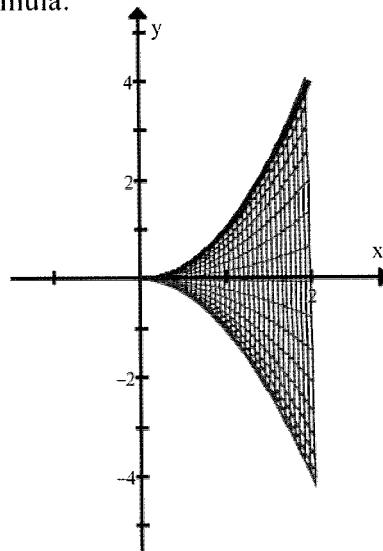
The area of a surface of revolution can be found by using the formula:

$$S = 2\pi \int_a^b r(x) \sqrt{1 + (f'(x))^2} dx$$

where $r(x)$ is the distance between the graph and the axis.

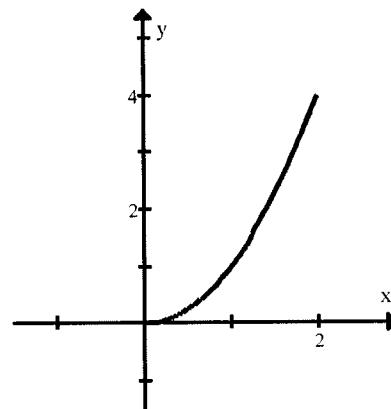
Example 1:

Find the area of the surface formed by revolving the graph of $y = x^2$ on the interval $[0, 2]$ about the x -axis.



Example 2:

Find the area of the surface formed by revolving the graph of $y = x^2$ on the interval $[0, 2]$ about the y -axis. (No calculator)

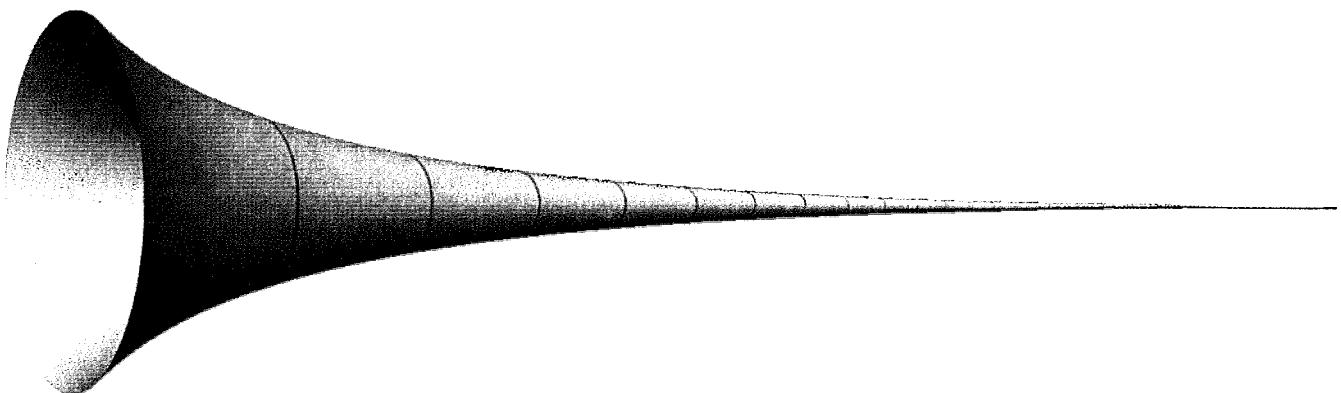


Assignment A-1:

1. Find the area of the surface generated by revolving $y = x^3$ on the interval $0 \leq x \leq 4$ about the x -axis.
2. Find the area of the surface generated by revolving $y = \sqrt{x}$ on the interval $1 \leq x \leq 4$ about the x -axis.
3. Find the area of the surface generated by revolving $y = x^2 + \frac{1}{x^2}$ on the interval $1 \leq x \leq 5$ about the x -axis.
4. Find the area of the surface generated by revolving $y = 3x$ on the interval $1 \leq x \leq 10$ about the x -axis.

5. Find the area of the surface generated by revolving $y = x^2 + 2$ on the interval $0 \leq x \leq 2$ about the y -axis.
6. Find the area of the surface generated by revolving $y = \sin x$ on the interval $0 \leq x \leq \frac{\pi}{2}$ about the y -axis.
7. Find the area of the surface generated by revolving $y = \ln x$ on the interval $1 \leq x \leq e$ about the y -axis.
8. A helium gas storage tank with a solid top is formed in the shape of $y = x^4$ revolved about the y -axis. The height of the tank is 16 feet.
- What is the surface area of the entire tank?
 - If the tank is constructed with metal $\frac{1}{8}$ inch ($\frac{1}{96}$ feet) thick how much metal was used to construct the tank.
 - If the metal weighs 170 pounds per cubic foot, how much does the empty metal tank weigh?
9. Gabriel's Horn
- A solid is formed by revolving the region below the graph of $y = \frac{1}{x}$ for $x \geq 1$ about the x -axis.
- Find the volume of the solid by evaluating an improper integral without using a calculator.
 - Use a calculator to find the surface area of the solid for :
 - $1 \leq x \leq 10$.
 - $1 \leq x \leq 1000$.
 - $1 \leq x \leq 10000000$.
 - What surface area do you expect for $x \geq 1$?

Note: This unbounded shape called Gabriel's Horn has a finite volume but an infinite surface area. This famous mathematical paradox means the horn would be filled with a finite volume of paint but this paint would not be enough to cover the inner surface of the horn since it has an infinite area.



Lesson A-2 Work

Work done by a constant force: $W = FD$

Example 1:

Find the work done in lifting a 40 pound suitcase 2 feet and moving it horizontally 10 feet.

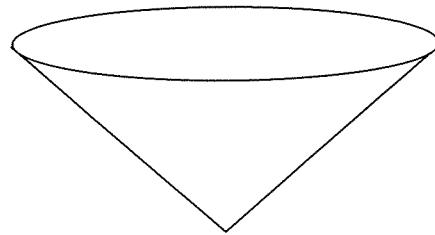
Work done by a variable force or over a variable distance: $W = \int_a^b \text{force} \cdot \text{distance}$

Example 2:

A bucket of water weighing 100 newtons (N) is at the bottom of a 20 meter well attached to a rope weighing 0.4 N/m. How much work is done in lifting the bucket to the surface by winding the rope around a windlass?

Example 3:

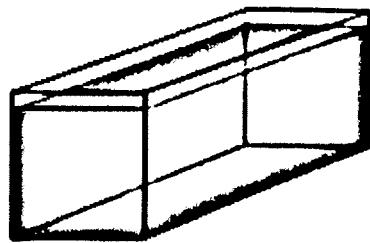
A conical tank with a height of 10 feet and a top diameter of 20 feet is filled to within 1 foot of the top with oil that weighs 50 pounds per cubic foot. How much work is done if the tank is emptied by pumping the oil to the top rim of the tank?



Assignment A-2:

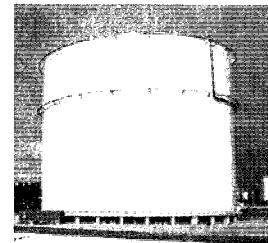
1. Find the work done in lifting a 50 pound concrete block from the ground to a height of 6 feet.

2. A rectangular tank has a base 7 feet by 17 feet and a height of 9 feet. The tank is filled with water which weighs 62.4 pounds per cubic foot to a depth of 8 feet. The water is pumped out over the top edge of the tank.



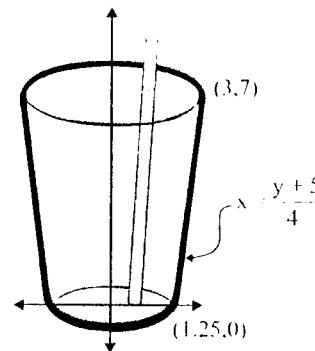
- How much work is done to empty half of the water in the tank?
- How much work is done to completely empty the water in the tank?

3. A cylindrical gasoline tank has a height of 13 meters and a diameter of 25 meters. Gasoline weighs 8300 newtons per cubic meter. How much work is done to empty a full tank by pumping the gasoline to the top of the tank.



4. A 20 foot chain is hanging freely from a winch 20 feet above ground level. The chain weighs 4 pounds per foot.
- How much work is done to raise the bottom of the chain to a height of 15 feet?
 - How much work is done to wind up the full 20 feet of chain?
 - If a 200 pound weight is attached to the end of the chain, how much work is done to raise the chain and weight 5 feet?

5. A drinking glass is filled with water that weighs .58 ounces per cubic inch. An 8 inch straw extends one inch above the 7 inch tall glass. How much work does it take to drink the water through the straw?



Lesson A-3 Hyperbolic Functions

The trig functions sine, cosine, and tangent are often defined using points on the unit circle $x^2 + y^2 = 1$. Similarly the hyperbolic functions can be defined using points on a unit hyperbola $-x^2 + y^2 = 1$. They can also be defined using exponential functions as follows.

Definitions: $\sinh x = \frac{e^x - e^{-x}}{2}$	$\cosh x =$	$\tanh x =$
---	-------------	-------------

Example 1: Find the value of $\sinh 0$, $\cosh 0$, and $\tanh 0$ without using a calculator.

Example 2. Use a calculator to find $\sinh \pi$.

Example 3: Find $\lim_{x \rightarrow \infty} \tanh x$ without using a calculator.

Derivatives: $\frac{d}{dx} \sinh x =$

$\frac{d}{dx} \cosh x =$

$\frac{d}{dx} \tanh x =$

Example 4: $\frac{d}{dx} \sinh x^2 =$

Integrals: $\int \sinh x \, dx =$

$\int \cosh x \, dx =$

$\int \operatorname{sech}^2 x \, dx =$

Example 5: $\int \cosh(2x) \sinh^2(2x) \, dx =$

Assignment A-3:

1. Find the value of $\sinh 1$, $\cosh 1$, and $\tanh 1$ without using a calculator.
2. Use a calculator to find $\sinh 4$.
3. Use a calculator to find $\tanh(-3)$.

Find the derivative.

4. $f(x) = \ln(\cosh x)$ 5. $g(x) = x \sinh x$ 6. $h(x) = \frac{1}{2} \tanh(4x) - x^3$

7. Without using a calculator find an equation of a line tangent to the graph of $y = e^{\sinh x}$ when $x = 0$.

Find the integral.

8. $\int \sinh(2-3x) \, dx$ 9. $\int \frac{\sinh x}{\cosh x} \, dx$ 10. $\int x \operatorname{sech}^2(x^2) \, dx$

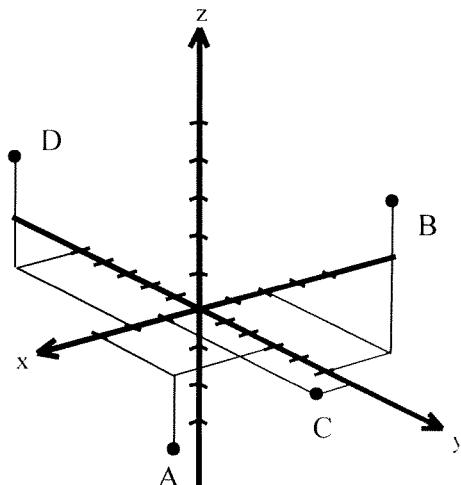
11. A region is bounded by $y = \sinh x$, $y = 0$, and $x = 1$.

- a. Sketch a graph of the region.
- b. Find the area of the region without using a calculator.
- c. Find the surface area of the region revolved about the x -axis.
- d. Find the surface area of the region revolved about the y -axis.

Lesson A-4 Graphing Points in Three Dimensions (rectangular, cylindrical, spherical)

Example 1: Give an ordered triple for each lettered point on this rectangular graph.

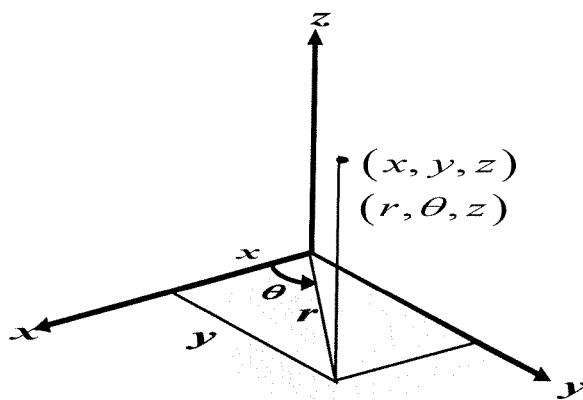
A
B
C
D



Example 2: Plot the following points on the same rectangular system.
(2,3,4), (3,-4,-2), (0,3,2)

Cylindrical Coordinates

Cylindrical coordinates can be thought of as an extension of polar coordinates in the third dimension. For the ordered triple (r, θ, z) , r and θ have the same meaning as polar coordinates and z has the same meaning as the z in rectangular three dimensional systems.



Conversion Equations

$$x^2 + y^2 = r^2 \quad x = r \cos \theta$$

$$\tan \theta = \frac{y}{x} \quad y = r \sin \theta$$

$$z = z$$

Example 3: Plot the following cylindrical coordinate point and convert it to rectangular coordinates.

$$\left(4, \frac{\pi}{2}, -2\right)$$

Example 5: Plot the following rectangular coordinate point and convert it to cylindrical coordinates.

$$(-4, 0, 2)$$

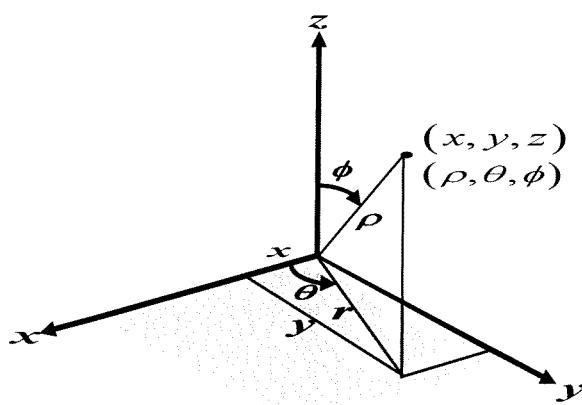
Example 4: Convert the following cylindrical coordinate point to rectangular coordinates without graphing. $\left(3, \frac{\pi}{6}, 4\right)$

Example 6: Convert the following rectangular coordinate point to cylindrical coordinates without graphing.

$$\left(-3\sqrt{2}, 3\sqrt{2}, -4\right)$$

Spherical Coordinates

Spherical coordinates are given as ordered triples (ρ, θ, ϕ) where ρ is the distance from the origin, θ is the same angle used in cylindrical and polar coordinates, and ϕ is the angle measured from the positive z -axis.



Conversion Equations

$\rho^2 = x^2 + y^2 + z^2$	$x = \rho \sin \phi \cos \theta$
$\tan \theta = \frac{y}{x}$	$y = \rho \sin \phi \sin \theta$
$\cos \phi = \frac{z}{\rho}$	$z = \rho \cos \phi$

Example 7: Plot the following rectangular coordinate point and convert it to spherical coordinates.

$$(0, -4, 0)$$

Example 8: Convert the following rectangular coordinate point to spherical coordinates without graphing.

$$(\sqrt{3}, 1, 2\sqrt{3})$$

Example 9: Plot the following spherical coordinate point and convert it to rectangular coordinates.

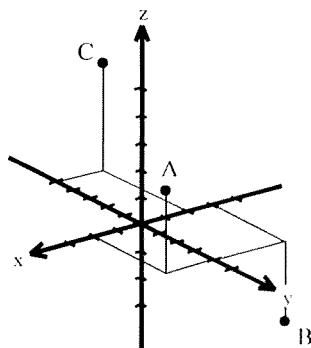
$$\left(6, \frac{3\pi}{2}, \frac{\pi}{2}\right)$$

Example 10: Convert the following spherical coordinate point to rectangular coordinates without graphing.

$$\left(5, \frac{\pi}{4}, \frac{3\pi}{4}\right)$$

Assignment A-4:

1. Find the Cartesian (rectangular) coordinates for each labeled point.



2. Plot these points on the same coordinate system:

$$A (2, 4, -3), \quad B (-2, -3, 4), \quad C (3, -4, 0)$$

3. Plot these points on the same coordinate system:

$$D (3, 0, 3), \quad E (0, -4, 4), \quad F (-4, 3, 5)$$

Convert these cylindrical coordinates to rectangular coordinates without using a calculator.

$$4. (4, 0, 10) \qquad 5. \left(6, \frac{\pi}{3}, -5\right) \qquad 6. \left(-3, \frac{7\pi}{6}, 0\right)$$

Convert these rectangular coordinates to cylindrical coordinates without using a calculator.

$$7. (0, 3, -2) \qquad 8. (2, 2\sqrt{3}, 5) \qquad 9. (-4, 4, 3)$$

Convert these rectangular coordinates to spherical coordinates without using a calculator.

$$10. (3, 0, 0) \qquad 11. (-1, \sqrt{3}, 2) \qquad 12. (-\sqrt{3}, 1, -2\sqrt{3})$$

Convert these spherical coordinates to rectangular coordinates without using a calculator.

$$13. \left(3, \frac{\pi}{4}, \frac{\pi}{6}\right) \qquad 14. (6, 2, 0) \qquad 15. \left(8, \pi, \frac{\pi}{4}\right)$$

16. Find the surface area of the solid generated when the region bounded by $y = 4\sqrt{x}$, $y = 0$, and $x = 4$ is revolved about:

- a. the x -axis. b. the y -axis.

17. A well has a 3 foot diameter and is 200 feet deep. If the well is full of water weighing 62.4 pounds per cubic foot to within 50 feet of ground level, how much work is required to pump the well dry?
18. A chain weighing 60 newtons per meter extends 50 meters up from an attached 250 newton weight. How much work is required to wind up the entire chain and lift the weight the full 50 meters?
19. Differentiate $f(x) = x^2 \sinh(4x)$.
20. Integrate $\int x^2 \sinh(x^3) dx$.

CALCULUS EXTENDED INDEX

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