I, ________________________________(your name), pledge that this exam is completely my own work, and that I did not take, borrow or steal work from any other person, and that I did not allow any other person to use, have, borrow or steal portions of my work. I understand that if I violate this honesty pledge, I am subject to disciplinary action pursuant to the appropriate sections of the San Diego State University Policies.

1. a. Find the eigenvalues and eigenfunctions for the Sturm-Liouville problem

$$\phi'' + \lambda\phi = 0, \qquad 0 < x < 4,$$
$$\text{with B.C.}'\text{s} \quad \phi(0) = 0 \quad \text{and} \quad 2\phi(4) + \phi'(4) = 0.$$

b. Use the eigenfunctions from Part a to represent the function

$$f(x) = \begin{cases} 5, & 0 < x < 2, \\ 0, & 2 \le x < 4. \end{cases}$$

and find the generalized Fourier coefficients.

c. What does the Fourier series converge to at $x = 1$? at $x = 2$? at $x = 3$? at $x = -1$? Does this Fourier series produce a periodic extension for all x? Explain.

d. Use the computer to find the numerical values of the first 50 eigenvalues. (Only write the values for λ_1, λ_2, λ_5, λ_{10}, λ_{25}, and λ_{50}.) Graphically, show $f(x)$ and the approximation using 50 terms in the Fourier series for $x \in [0, 4]$ and $x \in [-4, 8]$. What is the absolute error between your 50 term Fourier series and the value of $f(x)$ at $x = 0.05$, $x = 1$, $x = 2.5$, and $x = 3.75$. Find the maximum actual error between the 50 term approximation and the actual function. (The maximum error is for the Gibb's phenomenon and not the obvious error caused by the jump discontinuity.) Give both the $x_{max} \in (0, 4)$ value and the Fourier series value at x_{max} for this maximum error.

2. Consider the non-homogeneous heat equation given by:

$$\frac{1}{k}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \gamma^2(u - T), \quad 0 < x < L, \quad t > 0.$$

With boundary conditions and initial condition:

$$u(0, t) = T_0, \quad u(L, t) = T_1, \quad \text{and} \quad u(x, 0) = 0.$$

Give a physical interpretation of the partial differential equation, its parameters, and its boundary and initial conditions. Solve the steady state problem.

3. Consider the heat equation given by:

$$\frac{\partial u}{\partial t} = k\nabla^2 u, \quad 0 < x < 2, \quad 0 < y < 3, \quad t > 0.$$

With boundary conditions:

$$\frac{\partial u}{\partial x}(0, y, t) = A(3 - y), \quad \frac{\partial u}{\partial x}(2, y, t) = y^2, \quad \frac{\partial u}{\partial y}(x, 0, t) = 0, \quad \text{and} \quad \frac{\partial u}{\partial y}(x, 3, t) = 0,$$

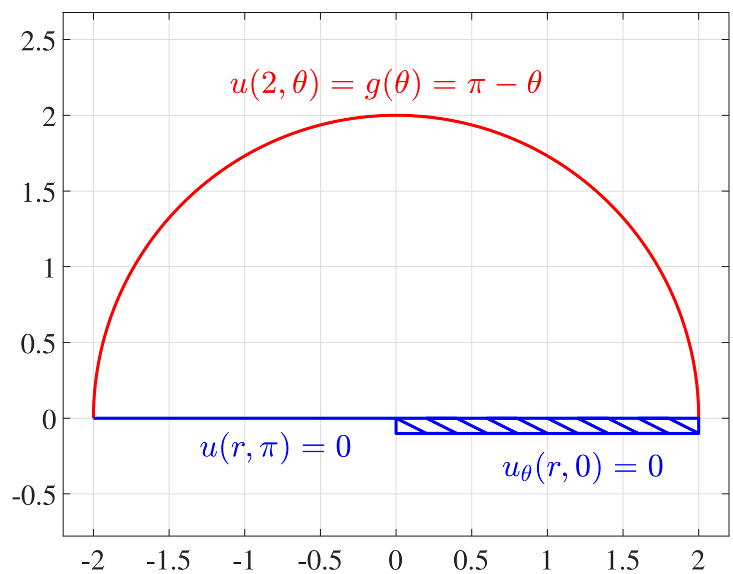
and initial condition:

$$u(x, y, 0) = x(3 - y).$$

Find the condition on A (A constant) that allows the steady state problem to be solvable on the rectangular domain. Solve the steady state problem.

4. a. Find the steady-state temperature distribution for the Figure below (assuming the faces are insulated). The region is a semi-circular region satisfying Laplace's equation, where the edge along the positive x-axis is insulated and the edge along the negative x-axis is fixed at 0. Along the semi-circular edge, we have:

$$u(2, \theta) = g(\theta) = \pi - \theta.$$



b. **Extra-credit**: Use MatLab to create a colored heat map displaying the steady-state temperature distribution in this region. You must include your program.

5. If convection is taken into account, the equation for heat conduction and convection in a one-dimensional rod is given by:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - v_0\frac{\partial u}{\partial x}, \quad 0 < x < L, \quad t > 0.$$

Let $k = 1$, $v_0 = 0.6$, and $L = 5$. Assume the following boundary conditions and initial conditions:

$$u(0, t) = 0, \quad u(L, t) = 0, \quad \text{and} \quad u(x, 0) = f(x).$$

a. Use separation of variables to create two ordinary differential equations.

b. From the spatial ordinary differential equation, create a Sturm-Liouville eigenvalue problem. Identify explicitly the functions $p(x)$, $q(x)$, and $\sigma(x)$. Find the eigenvalues and eigenfunctions for this problem. Explicitly write the orthogonality condition for this problem.

c. Solve the original partial differential equation with its boundary and initial conditions. Write clearly your integral for finding the Fourier coefficients.

6. a. The displacement of a uniform thin beam in a medium that resists motion satisfies the beam equation:

$$\frac{\partial^4 u}{\partial x^4} = -\frac{\partial^2 u}{\partial t^2} - 0.1\frac{\partial u}{\partial t}, \quad 0 < x < 2, \quad t > 0.$$

If the beam is simply supported at the ends, then the boundary conditions are:

$$u(0,t) = 0, \quad u_{xx}(0,t) = 0, \quad u(2,t) = 0, \quad u_{xx}(2,t) = 0.$$

Assume that there is initially no displacement and that an initial velocity, $u_t(x,0) = 1$ is given to the beam. Solve this initial-boundary value problem. You can assume that the eigenvalues are real, but show clearly how you obtain all eigenvalues and eigenfunctions.

b. Use 20 terms in the series solution of $u(x,t)$ and have the computer graph the displacement of the beam at times $t = 0$, 1, 2, 5, 10, and 20.

c. **Extra-credit**: Use MatLab to create a smooth surface showing the time evolution of the beam for $0 \leq x \leq 2$ and $0 \leq t \leq 50$. You must include your program.