10/8

Suppose $\lim_{n\to\infty}(a_n+b_n)$ exists and $\lim_{n\to\infty} a_n$ exists. Then $\lim_{n\to\infty} b_n$ exists.

TRUE

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left((a_n+b_n)+(-1)a_n\right) = \lim_{n\to\infty}(a_n+b_n) - \lim_{n\to\infty} a_n.$$

~~Thes~~ The set $\mathbb{Q}-\mathbb{N}$ (rational but not natural) is dense in $\mathbb{R}$. TRUE

If $\{a_n\} \subseteq \mathbb{Q}$ converges, then $\lim_{n\to\infty} a_n \in \mathbb{Q}$ FALSE

$$\sqrt{2} = \sup\{x \mid x\in\mathbb{Q},\ x^2<2\} = \sup(S)$$

From this, $\exists\ \{x_n\} \subseteq S$ s.t. $\lim_{n\to\infty} x_n = \sqrt{2}$.

Prove $(-\infty, 0]$ is closed.

proof: Let $\{a_n\} \subseteq (-\infty, 0]$. and suppose it converges.

By Boundedness Lemma, $\exists M \in \mathbb{R}^+$ s.t. $\forall n$

$$|a_n| \leq M \text{ and } |a| = \left|\lim_{n\to\infty} a_n\right| \leq M.$$

So $\{a_n\} \subseteq [-M, 0]$.

Since $[-M, 0]$ is closed, $a \in [-M, 0] \subseteq (-\infty, 0]$.

$\boxtimes$

$S \subseteq \mathbb{R}$ closed

iff

$\forall \{a_n\} \subseteq S$, if $\lim_{n\to\infty} a_n = a$ exists, then $a \in S$

Def: $\{a_n\}$ goes to infinity iff $\forall M \in \mathbb{R}^+$, $\exists N$ s.t. $\forall n \geq N$, $a_n > M$.

Prob. Suppose $a_n > 0$ for all n.

$\{a_n\}$ goes to infinity iff $\lim_{n\to\infty} \frac{1}{a_n} = 0$.

proof: $(\rightarrow)$ Suppose $\{a_n\}$ goes to infinity.

Let $\varepsilon > 0$

So $\exists N$ s.t. $\forall n \geq N$, $a_n > \frac{1}{\varepsilon}$.

Let $n \geq N$. $a_n > \frac{1}{\varepsilon}$

So $\varepsilon > \left| \frac{1}{a_n} - 0 \right|$. ∎

$(\leftarrow)$ Suppose $\lim_{n\to\infty} \frac{1}{a_n} = 0$.

Let $M \in \mathbb{R}^+$. Then $\exists N$ s.t. $\forall n \geq N$, $\frac{1}{a_n} < \frac{1}{M}$.

Let $n \geq N$. Then $\frac{1}{a_n} < \frac{1}{M}$. So $M < a_n$.

∎

Today: Bonus Material!! limsup $= \overline{\lim}$

Def: Suppose $\{x_n\}$ is a sequence. We say $x \in \mathbb{R}$ is a clusterpoint of $\{x_n\}$ iff $\forall n \in \mathbb{N}$, $\forall \varepsilon > 0$, $\exists k > n$ s.t. $|x_k - x| < \varepsilon$.

Thm: Suppose $\{x_n\}$ is a sequence

x is a cluster point iff $\exists \{x_{n_k}\}$ s.t. $\lim_{k \to \infty} x_{n_k} = x$.

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proof: $(\longrightarrow)$ Suppose x is a cluster point.

Let $k = 1$. Since x is a c.p., $\exists n_1 > 1$ s.t. $|x_{n_1} - x| < 1$.

Let $k = 2$. $\exists n_2 > n_1$ s.t.

$$|x_{n_2} - x| < \frac{1}{2}$$

Inductively, $|x_{n_k} - x| < \frac{1}{k}$. By comparison, $\lim_{k \to \infty} x_{n_k} = x$.

($\Leftarrow$) Suppose $\exists \{x_{n_k}\}$ s.t. $\lim_{k\to\infty} x_{n_k} = x$.

Let $n \in \mathbb{N}$ and $\varepsilon > 0$.

$\exists K$ s.t. $\forall k \geq K$, $|x_{n_k} - x| < \varepsilon$.

Choose k s.t. $n_k > n$ and $k \geq K$.

Then $|x_{n_k} - x| < \varepsilon$. $\boxtimes$.

Def: Suppose $\{x_n\}$ is a sequence.

Let $S = \{x \mid x \text{ is a cluster pt for } \{x_n\}\}$,

If $\sup S$ exists, we define $\limsup x_n = \overline{\lim_{n\to\infty}}\, x_n = \sup(S)$.

Thm: Suppose $\{x_n\}$ is bounded.

Then $\limsup x_n = \inf\{ \sup\{x_k \mid k \geq n\} \mid n \geq 1\}$.

proof: Let $s = \inf\{ \sup\{x_k \mid k \geq n\} \mid n \geq 1\}$.

1. Show s is a cluster pt.

Let $\varepsilon > 0$, $m \in \mathbb{N}$.

Then $\exists n > m$ s.t. $s \leq \sup\{x_k \mid k \geq n\} < s + \varepsilon$.

So $\exists k \geq n > m$ s.t.

$$s - \varepsilon < x_k < s + \varepsilon.$$

Thus $|s - x_k| < \varepsilon$. So s is a cluster pt.

2. Suppose $x \in \mathbb{R}$ is any other cluster pt for $\{x_n\}$.

Let $\varepsilon > 0$. For each $n \in \mathbb{N}$, $\exists m > n$

s.t. $x - \varepsilon < x_m < x + \varepsilon$

So $x - \varepsilon < \sup\{x_k \mid k \geq n\}$, $\forall n \geq 1$.

~~Thus~~

Thus $x - \varepsilon \leq \inf\{\sup\{x_k \mid k \geq n\} \mid n \geq 1\}$.

So $x - \varepsilon \leq s$.

Since $\varepsilon > 0$ arbitrary, we have $x \leq s$.

Thus by 1, 2, s is the largest cluster pt.