Sept 09, 2024

NON-DIMENSIONALIZATION

- Purpose
  - Identify small terms
  - terms are combined during scaling making model simple to analyze.
  - helps for numerical computation.

$$A = \underbrace{\underbrace{B}_{10^{-50}}\ \underbrace{C}_{10^{49}}}_{10^{-1}}$$

- Why/how: size of quantity is measured (scale) by comparison with another variable of the same dimension
  ⇒ smallness is measured by comparison.

- Process:

  (1) If x is a variable and $[x]$ is a scale, then $x = [x]x^*$, and x^* is a dimensionless variables

  (2) Substitute the relations of $x = [x]x^*$ (alike) in the model and divide both sides by proper scale (Important)

  ⇒ all terms involved will be dimensionally homogenous

  [Rescaling: rescale to allow multiple terms of the same highest order (1)].

RULES of THUMB:

1. (always) Make as many non-dimensional constants equal to one as possible.
2. (usually) Make the constants that appear in the initial and boundary conditions equal to one.
3. (usually) If there is a non-dimensional constant, would simplify if we were to set it equal to zero, the problem significantly, allow it to remain free and then see when we can make it small.

Example - 1: Decay of radioactive materials ($x(t)$)

$$\begin{cases} \dfrac{dx}{dt} = -\lambda\, x(t) \\ x(0) = x_0 \end{cases}$$

t = time unit
λ = per unit time
x = amount.

UNIT: $\dfrac{\text{amount}}{\text{time}} = \dfrac{1}{\text{time}} \cdot \text{amount}.$

- $x = [x]\, x^*$
  $t = [t]\, t^*$
- $\dfrac{dx}{dt} = [x] \cdot \dfrac{dx^*}{dt} = [x] \cdot \dfrac{dx^*}{dt^*} \cdot \dfrac{dt^*}{dt} = [x] \cdot \dfrac{dx^*}{dt} \cdot \dfrac{1}{[t]}$
  $= \dfrac{[x]}{[t]} \cdot \dfrac{dx^*}{dt^*}.$

$$\Rightarrow \frac{\cancel{[x]}}{[t]} \cdot \frac{dx^*}{dt^*} = -\lambda \cdot \cancel{[x]}\, x^*$$

$$\Rightarrow \frac{dx^*}{dt^*} = -\lambda [t]\, x^*.$$

$$[x]\, x^*(0) = x_0$$

$$\Rightarrow x^*(0) = \frac{x_0}{[x]}$$

$\cdot\ [x] = x_0, \quad [t] = \frac{1}{\lambda}$ } scaling

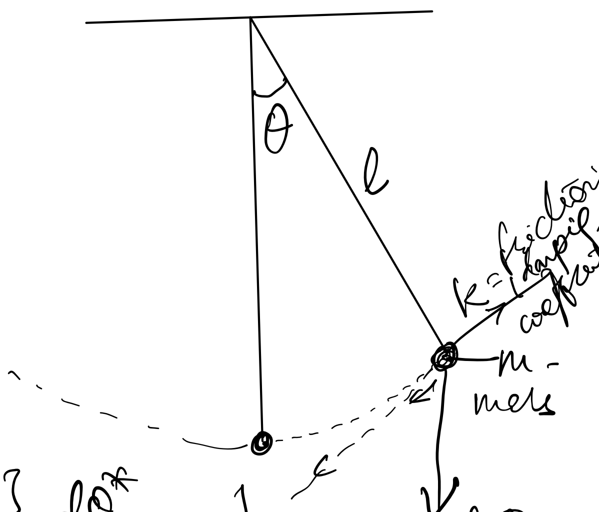
$$\Rightarrow \left.\begin{aligned} \frac{dx^*}{dt^*} &= -x^* \\ x^*(0) &= 1 \end{aligned}\right\}$$ dimensionless equations

$x^*, t^* \to$ dimensionless

solution is $x^*(t^*) = e^{-t^*}$

Example - 2: Damped pendulum

$$\begin{cases} l\dfrac{d^2\theta}{dt^2} + k\dfrac{d\theta}{dt} + g\sin\theta = 0 \\ \theta(0) = \theta_0, \quad \dfrac{d\theta}{dt}(0) = \omega_0 \end{cases}$$



$\cdot\ \theta = [\theta]\theta^*$

$t = [t]t^*$

$$\cdot\ \frac{d\theta}{dt} = [\theta]\cdot\frac{d\theta^*}{dt} = [\theta]\cdot\frac{d\theta^*}{dt^*}\cdot\frac{dt^*}{dt} = \frac{[\theta]}{[t]}\frac{d\theta^*}{dt^*}$$

$$\frac{d^2\theta}{dt^2} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{[\theta]}{[t]}\cdot\frac{d}{dt}\left(\frac{d\theta^*}{dt^*}\right)$$

$$= \frac{[\theta]}{[t]}\cdot\frac{d}{dt^*}\left(\frac{d\theta^*}{dt^*}\right)\cdot\frac{dt^*}{dt}$$

$$= \frac{[\theta]}{[t]}\cdot\frac{d^2\theta^*}{dt^{*2}}\cdot\frac{1}{[t]}$$

$$= \frac{[\theta]}{[t]^2}\cdot\frac{d^2\theta^*}{dt^{*2}}.$$

• Substitute:

$$l\frac{[\theta]}{[t]^2}\cdot\frac{d^2\theta^*}{dt^{*2}}+k\frac{[\theta]}{[t]}\frac{d\theta^*}{dt^*}+g\sin([\theta]\theta^*)=0$$

$$\Rightarrow\quad \frac{d^2\theta^*}{dt^{*2}}+\frac{k[t]}{l}\frac{d\theta^*}{dt^*}+\underbrace{\frac{g[t]^2}{l[\theta]}}\sin([\theta]\theta^*)=0$$

$$\theta^*(0)=\frac{\theta_0}{[\theta]}$$

$$\frac{d\theta}{dt}(0)=\frac{[\theta]}{[t]}\cdot\underbrace{\frac{d\theta^*}{dt^*}(0)}$$

$$\Rightarrow\quad \frac{d\theta^*}{dt^*}(0)=\frac{[t]}{[\theta]}\omega_0.$$

• Scaling: Balance θ and $\frac{d\theta}{dt}$

$$[\theta]=\theta_0 \quad\Longrightarrow\quad \theta^*(0)=1$$

$$\frac{[t]}{[\theta]}\omega_0=1 \quad\Longrightarrow\quad [t]=\frac{\theta_0}{\omega_0}.$$

Then, $$\left.\begin{array}{l}\dfrac{d^2\theta^*}{dt^{*2}}+\alpha\dfrac{d\theta^*}{dt^*}+\beta\sin(\gamma\theta^*)=0\end{array}\right\}$$

where $\alpha=\frac{k[t]}{l}=\frac{k\,\theta_0}{l\,\omega_0}$

$$\beta = \frac{[t]^2 g}{l\theta} = \frac{g\theta_0}{l\omega_0^2}.$$

$$\gamma = \theta_0.$$

$$\left.\begin{aligned} &\theta^*(0) = 1 \\ &\frac{d\theta^*}{dt^*}(0) = 1 \end{aligned}\right\}$$

- Is this scaling reasonable?

= YES, if α, β, γ are of order 1 or smaller.

- NO (inappropriate) if, for example, β, γ are of order 1 but $\alpha \gg 1$ (large), because

$$\left\{\begin{aligned} &\underbrace{\frac{1}{\alpha}}_{\downarrow\, 0}\frac{d^2\theta^*}{dt^{*2}} + \frac{d\theta^*}{dt^*} + \frac{\beta}{\alpha}\underbrace{\sin(\gamma\theta^*)}_{\to 0} = 0 \\ &\Rightarrow 0 \approx \frac{d\theta^*}{dt^*} \sim 0, \text{ which is} \end{aligned}\right.$$

contradiction for the initial stage $\frac{d\theta^*}{dt^*}(0) = 1$.

- Consider $\beta = O(1)$, $\gamma = O(1)$, $\alpha >> 1$.

Rescaling: $t^* = [t^*]\tilde{t}$

$$\frac{d\theta^*}{dt^*} = \frac{d\theta^*}{d\tilde{t}} \cdot \frac{d\tilde{t}}{dt^*} = \frac{d\theta^*}{d\tilde{t}} \cdot \frac{1}{[t^*]} = \frac{1}{[t^*]} \frac{d\theta^*}{d\tilde{t}}$$

$$\frac{d^2\theta^*}{dt^{*2}} = \frac{d}{dt^*}\left(\frac{1}{[t^*]} \cdot \frac{d\theta^*}{d\tilde{t}}\right)$$

$$= \frac{1}{[t^*]} \cdot \frac{d}{d\tilde{t}}\left(\frac{d\theta^*}{d\tilde{t}}\right) \cdot \frac{1}{[t^*]}$$

$$= \frac{1}{[t^*]^2} \cdot \frac{d^2\theta^*}{d\tilde{t}^2}.$$

Then,

$$\frac{1}{[t^*]^2} \frac{d^2\theta^*}{d\tilde{t}^2} + \alpha \cdot \frac{1}{[t^*]} \frac{d\theta^*}{d\tilde{t}} + \beta \cdot \sin(\gamma\theta^*) = 0$$

$$\Rightarrow \frac{d^2\theta^*}{d\tilde{t}^2} + \underline{\alpha \cdot [t^*]} \cdot \frac{d\theta^*}{d\tilde{t}} + \beta [t^*]^2 \sin(\gamma\theta^*) = 0$$

$$\theta^*(0) = 1$$

$$\frac{d\theta^*}{dt^*}(0) = 1 \Rightarrow \frac{d\theta^*}{d\tilde{t}}(0) = [t^*]$$

$$[t^*] = \frac{1}{\alpha}.$$

$$\frac{d^2\theta^*}{d\tilde{t}^2} + \frac{d\theta^*}{d\tilde{t}} + \eta \sin(\gamma\theta^*) = 0$$

where

$$\theta^*(0) = 1$$

$$\frac{d\theta^*}{d\tilde{t}}(0) = \frac{1}{\alpha}, \quad \eta = \frac{\beta}{\alpha^2} \text{ (order less than 1)}$$