Oct 14, 2024

Logistic Delay Differential Equation:

$$\underbrace{\frac{dx(t)}{dt}}=\underbrace{r\,x(t)\left[1-\frac{x(t-\tau)}{K}\right]}.$$

- Scaling (Dimensional Analysis).

$$x(t)=[x]\,x^*(t),$$
$$t=[t]\,t^*$$

$$\frac{dx(t)}{dt}=[x]\frac{dx^*(t)}{dt}=[x]\cdot\frac{dx^*([t]t^*)}{d([t]t^*)}$$
$$=\frac{[x]}{[t]}\frac{dx^*([t]t^*)}{dt^*}.$$

Assuming $y^*(t^*)=x^*([t]t^*)$,

we get $$\frac{dx(t)}{dt}=\frac{[x]}{[t]}\frac{dy^*(t^*)}{dt^*}.$$

and $$r\,x(t)\left[1-\frac{x(t-\tau)}{K}\right]=r\,x(\underbrace{[t]t^*})\left[1-\frac{x([t]t^*-\tau)}{K}\right]$$
$$=r[x]y^*(t^*)\left[1-\frac{[x]}{K}x^*([t]t^*-\tau)\right]$$

$$\therefore\ \frac{\cancel{[x]}}{[t]}\cdot\frac{dy^*(t^*)}{dt^*}=r\cancel{[x]}y^*(t^*)\left[1-\frac{[x]\,x^*([t]t^*-\tau)}{K}\right]$$

or, $$\frac{dy^*(t^*)}{dt^*}=\underbrace{[t]\cdot r}\,y^*(t^*)\left[1-\frac{[x]\,x^*([t]t^*-\tau)}{K}\right]$$

$$[x]=K,\quad [t]=\frac{1}{r}$$

$$\Rightarrow\ \frac{dy^*(t^*)}{dt^*}=y^*(t^*)\left[1-x^*\left([t]\left(t^*-\frac{\tau}{[t]}\right)\right)\right]$$
$$=y^*(t^*)\left[1-y^*(t^*-\tau r)\right]$$

$$\left[\because\ y^*(t^*)=x^*([t]t^*)\right]$$

$$\Rightarrow \frac{dy^*(t^*)}{dt^*} = y^*(t^*)\left[1 - y^*(t^* - \tau^*)\right]$$

$$\Rightarrow \frac{dy}{dt} = y(t)\left[1 - y(t-\tau)\right]$$

Here τ is the relative size of delay which is very important as it may cause oscillation or because of which carrying capacity is not constant.

$\therefore$ the relative size of the delay is one of the sources for the system to loose its stability at carry capacity $(y=1)$ and for the occurance of non-linear oscillation

Linear stability Analysis:

$$y(t) = 1 + \varepsilon Z(t) \quad , \varepsilon << 1$$

[$\because$ $K \sim 1$ and case to observe is near carrying capacity]

$$\Rightarrow \varepsilon \frac{dZ(t)}{dt} = \left[1 + \varepsilon Z(t)\right]\left[\cancel{1} - \cancel{1} - \varepsilon Z(t-\tau)\right]$$

$$\Rightarrow \frac{dZ(t)}{dt} = -Z(t-\tau) - \varepsilon Z(t) Z(t-\tau)$$

$$\xrightarrow{\varepsilon << 1} \frac{dZ}{dt} = -Z(t-\tau)$$

Let $Z(t) = Z_0 e^{\lambda t}$ (solution)

Then $\frac{d}{dt} Z_0 e^{\lambda t} = -Z_0 e^{\lambda(t-\tau)}$

or, $Z_0 \lambda e^{\lambda t} = -Z_0 e^{\lambda t} \cdot e^{-\lambda\tau}$

$$\Rightarrow \lambda = -e^{-\lambda\tau}$$

$$\Rightarrow \lambda + e^{-\lambda\tau} = 0 \quad \text{[transcendental equation]}$$

This is the characteristic equation.

Note that $\lambda + e^{-\lambda\tau} = 0$ has no real solution as $g(\lambda) = \lambda + e^{-\lambda\tau}$ has absolute minimum of $\underline{1}$ at $\lambda = 0$

$$\left[\because g'(\lambda) = 1 - \tau e^{-\lambda\tau} = 0 \Rightarrow \frac{1}{\tau} = e^{-\lambda\tau} \Rightarrow \lambda = -\frac{1}{\tau}\ln\tau \Rightarrow \text{absolute min of } 1 \text{ at } \lambda = 0\right]$$

Let's seek for complex solution $\lambda = a + ib$

$$a + ib + e^{-\tau(a+ib)} = 0$$

$$\Rightarrow a + ib + e^{-a\tau} \cdot e^{-ib\tau} = 0$$

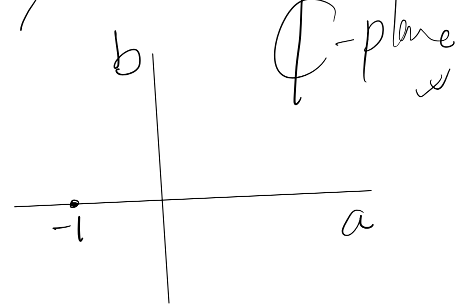
$$\Rightarrow a + ib + e^{-a\tau}(\cos b\tau - i\sin b\tau) = 0$$

$$\Rightarrow (a + e^{-a\tau}\cos b\tau) + i(b - e^{-a\tau}\sin b\tau) = 0$$

Equating the real and imaginary parts, we get

$$\begin{cases} a + e^{-a\tau}\cos b\tau = 0 \\ b - e^{-a\tau}\sin b\tau = 0 \end{cases}$$

$$\Rightarrow \begin{cases} e^{-a\tau}\cos b\tau = -a \\ e^{-a\tau}\sin b\tau = b \end{cases}$$



- $\tau = 0$ (no delay)

$a = -1$, $b = 0$

$z(t) = z_0 e^{-t} \longrightarrow 0$ as $t \to \infty$

$\Rightarrow y(t) \longrightarrow 1$ as $t \to \infty$

$\therefore$ $y^* = 1$ (carrying capacity) is asymptotically stable.

- $\tau \ll 1$ (small delay)

$\Rightarrow a < 0$, $0 < b \ll 1$

$\Rightarrow e^{\lambda t} = e^{at} \cdot e^{ibt} \longrightarrow 0$ as $t \to \infty$

$$Z(t) = Z_0 e^{\lambda t} = Z_0 \underbrace{e^{at}} (\cos bt + i \sin bt)$$

$$\longrightarrow 0 \quad \text{as } t \to \infty$$

i.e. $y(t) \longrightarrow 1$ as $t \to \infty$.

$\Rightarrow y^*(t) = 1$ (carrying capacity) is asymptotically stable.

$\Rightarrow$ No oscillation can be observed in a not too large delay.

. large τ:

Q: (Is there a value of τ) what would be the critical value of τ when one obtains periodic solution?

Assume that $\underline{\underline{\hat{\tau}}}$ is the critical value, $a = 0$ for $\tau = \underline{\underline{\hat{\tau}}}$.

$$\Rightarrow \begin{cases} \cos b\hat{\tau} = 0 \\ \sin b\hat{\tau} = b \end{cases}$$

$$\Rightarrow \begin{cases} b\hat{\tau} = \frac{\pi}{2} \quad (+2k\pi) \\ \cos^2 b\hat{\tau} + \sin^2 b\hat{\tau} = b^2 \Rightarrow b = 1 \text{ (positive value)} \end{cases}$$

$$\Rightarrow \hat{\tau} = \frac{\pi}{2}$$

At critical value $\hat{\tau} = \frac{\pi}{2}$,

$$Z(t) = Z_0 e^{it} = \underline{Z_0}(\underline{\cos t} + i \underline{\sin t}).$$

In this case, $\frac{dZ(t)}{dt} = -Z(t-\tau)$ has a solution $Z(t) = \cos t + i \sin t$, periodic with period 2π.

For the original logistic equation, the

oscillation takes place at $\tau = \frac{\hat{\tau}}{r} = \frac{\pi}{2r}$.

$$\frac{\text{Period}}{\text{delay}} = \frac{2\pi}{\pi/2r} = 4r$$

$\therefore$ Period $= 4r$ delay.