MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #16

Chapter 5

Higher-Dimensional Linear Algebra

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Chapter 5 vs. Chapter 6

Chapter 5: $AX = b$

Chapter 6: $X' = AX$

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A Brief Review of Linear Algebra

Please review slides @Canvas

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Linear Algebra (LA): Eight Axioms

Definition. *Let V be a set on which addition and scalar multiplication are defined. (That is, if $\mathbf{x}$ and $\mathbf{y}$ are elements of V, and c is a scalar, then $\mathbf{x}+\mathbf{y}$ and $c\mathbf{x}$ are elements of V.) If the following eight axioms are satisfied by all elements $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ of V and all scalars a and b, then V is called a* vector space *and the elements of V are called* vectors. *If these axioms apply to multiplication by real scalars, then V is called a* real vector space. *If the axioms apply to multiplication by complex scalars, then V is a* complex vector space.

(1) *Commutativity of addition:* $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$.

(2) *Associativity of addition:* $(\mathbf{x}+\mathbf{y})+\mathbf{z}=\mathbf{x}+(\mathbf{y}+\mathbf{z})$.

LA: Eight Axioms (.continued)

(3) *Additive identity: There exists a vector, denoted* $\mathbf{0}$, *such that, for every vector* $\mathbf{x}$, $\mathbf{0} + \mathbf{x} = \mathbf{x}$.

(4) *Additive inverses: For every vector* $\mathbf{x}$ *there exists a vector* $(-\mathbf{x})$ *such that* $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$.

(5) *First distributive law:* $a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$.

(6) *Second distributive law:* $(a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$.

(7) *Multiplicative identity:* $1(\mathbf{x}) = \mathbf{x}$.

(8) *Relation to ordinary multiplication:* $(ab)\mathbf{x} = a(b\mathbf{x})$.

Closure (Wikipedia)

- A set is closed under an operation if performance of that operation on members of the set always produces a member of that set.
- For example, the positive integers are closed under addition, but not under subtraction: 1-2 is not a positive integer even though both 1 and 2 are positive integers.
- A set that is closed under an operation or collection of operations is said to satisfy a closure property.

Subset vs. Subspace

Definition. *A set S is* closed under addition *if the sum of any two elements of S is in S, and is* closed under scalar multiplication *if the product of an arbitrary scalar and an arbitrary element of S is in S.*

Frequently we consider a subset W of a vector space V. In this case, addition and scalar multiplication are already defined, and already satisfy the eight axioms. If W is closed under addition and scalar multiplication, then W is a vector space in its own right, and we call W a *subspace* of V.

subspace: closed under (1) addition and (2) scalar multiplication

Subset vs. Subspace: Examples

(1) Let $W = \{\mathbf{x} \in \mathbb{R}^2 | x_1 + x_2 = 0\}$. The sum of any two vectors in W is in W, and any scalar multiple of a vector in W is in W. (Check this!) W is a subspace of the vector space $\mathbb{R}^2$.

(2) More generally, let A be any fixed $m \times n$ matrix and let $W = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} = 0\}$. W is closed under addition and scalar multiplication (why?), so W is a subspace of $\mathbb{R}^n$.

(3) Let $\mathbb{R}[t]$ be the set of all real-valued polynomials in a fixed variable t. Polynomials are continuous functions on $[0, 1]$, so $\mathbb{R}[t]$ is a subset of $C^0[0, 1]$. The sum of polynomials is a polynomial, as is the product of a scalar and a polynomial, so $\mathbb{R}[t]$ is a subspace of $C^0[0, 1]$.

(4) Let $\mathbb{R}_n[t]$ be the set of real polynomials of degree n or less. For $n < m$, $\mathbb{R}_n[t]$ is a subspace of $\mathbb{R}_m[t]$, and $\mathbb{R}_n[t]$ is always a subspace of $\mathbb{R}[t]$.

(5) Instead of considering polynomials with real coefficients, we could consider polynomials with complex coefficients to get examples of complex vector spaces. The space of polynomials with complex coefficients is usually denoted $\mathbb{C}[t]$.

(6) Let $W' = \{\mathbf{x} \in \mathbb{R}^2 | x_1 + x_2 = 1\}$. W' is not a vector space, as the sum of two elements of W', or a scalar multiple of an element of W', is typically not in W'. (Again, check this!)

(7) Let $\mathbb{Z}^2 = \{\mathbf{x} \in \mathbb{R}^2 | x_1 \text{ and } x_2 \text{ are integers } \}$. $\mathbb{Z}^2$ is closed under addition, but not under scalar multiplication. $\mathbb{Z}^2$ is a sub*set* of $\mathbb{R}^2$, but not a sub*space*.

See the next slide

Subset vs. Subspace: Examples (.continued)

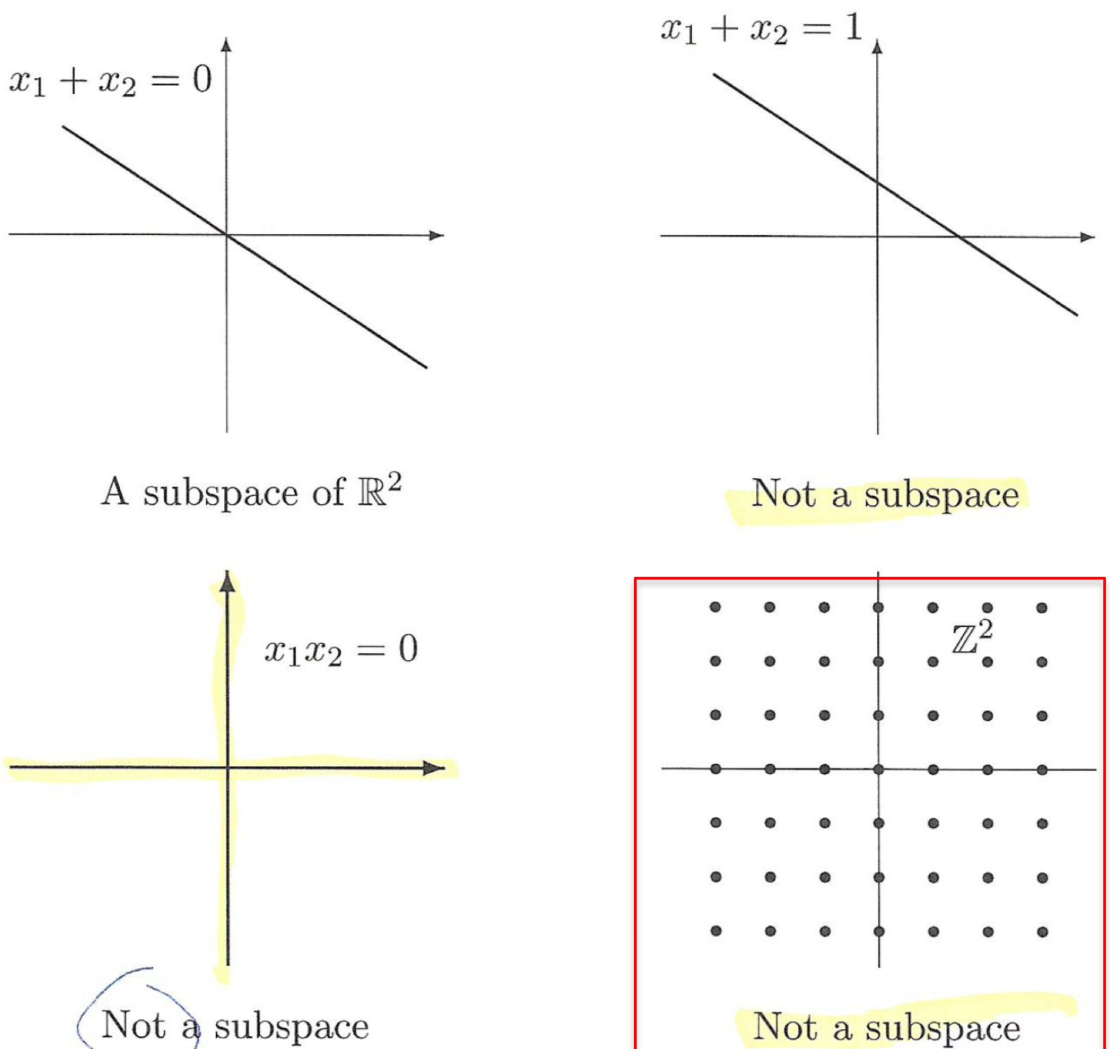

Figure 2.1. Four subsets of $\mathbb{R}^2$

Sect 5.1: Preliminaries from Linear Algebra

Consider a collection of vectors, $V_1, V_2, \ldots, V_k$ in R^n.

- These vectors are linearly independent if, whenever

  $\alpha_1 V_1 + \alpha_2 V_2 \ldots + \alpha_k V_k = 0$ with $\alpha_j \in R$, it follows that each $\alpha_j = 0$.

- The vectors are linearly dependent if some α_j are not zero.

  (at least, one vector can be linearly represented by other vector(s)).

- If $V_1, V_2, \ldots, V_k$ are linearly independent and W is a linear combination

  $$W = \beta_1 V_1 + \beta_2 V_2 \ldots + \beta_k V_k,$$

  then the β_j are unique.

Sect 5.1: Preliminaries from Linear Algebra

More generally, in $\mathbb{R}^n$, a collection of vectors $V_1, \ldots, V_k$ in $\mathbb{R}^n$ is said to be *linearly independent* if, whenever

$$\alpha_1 V_1 + \cdots + \alpha_k V_k = 0$$

with $\alpha_j \in \mathbb{R}$, it follows that each $\alpha_j = 0$. If we can find such $\alpha_1, \ldots, \alpha_k$, not all of which are 0, then the vectors are *linearly dependent*. Note that if $V_1, \ldots, V_k$ are linearly independent and W is the linear combination

$$W = \beta_1 V_1 + \cdots + \beta_k V_k,$$

then the β_j are unique. This follows since, if we could also write

If $V_1, V_2, \ldots V_k$ are LI and $W = \beta_1 V_1 + \beta_2 V_2, \ldots + \beta_k V_k$,

$(\beta_1,\ \beta_2, \ldots \beta_k)$ are unique.

A proof is given below.

LI Vectors and Unique Coef. for Linear Combination

$\boxed{\text{If } V_1, V_2, \dots V_k \text{ are LI and } W = \beta_1 V_1 + \beta_2 V_2, \dots + \beta_k V_k, \; (\beta_1,\ \beta_2, \dots \beta_k) \text{ are unique.}}$

A proof is given below.

Consider W as a linear combination with different coefficients:

$$W = r_1 V_1 + r_2 V_2, \dots + r_k V_k$$

Thus, we have

$$W - W = 0 = (\beta_1 - r_1)V_1 + (\beta_2 - r_2\ )V_2, \dots + (\beta_k - r_k)V_k$$

Since $V_1, V_2, \dots V_k$ are LI, the coefficients are zero in the above Eq.

Thus, we have $\quad 0 = \beta_1 - r_1 = \beta_2 - r_2 \dots = \beta_k - r_k$

$\boxed{(\beta_1,\ \beta_2, \dots \beta_k) \text{ are unique.}}$

Example 1 for LI Vectors

Example. The vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ are clearly linearly independent in $\mathbb{R}^3$. More generally, let E_j be the vector in $\mathbb{R}^n$ whose jth component is 1 and all other components are 0. Then the vectors $E_1, \ldots, E_n$ are linearly independent in $\mathbb{R}^n$. The collection of vectors $E_1, \ldots, E_n$ is called the *standard basis* of $\mathbb{R}^n$. We will discuss the concept of a basis in Section 5.4. ■

in R^3

$$\left.\begin{aligned} E_1 &= \begin{pmatrix}1\\0\\0\end{pmatrix} \\ E_2 &= \begin{pmatrix}0\\1\\0\end{pmatrix} \\ E_3 &= \begin{pmatrix}0\\0\\1\end{pmatrix}\end{aligned}\right\} \text{L.I.}$$

in R^n

$$E_j = \begin{pmatrix}\vdots\\0\\0\\1\\0\\0\\\vdots\end{pmatrix} \quad j^{th} \text{ element} \qquad \text{L.I.}$$

standard basis

Example 1: LI Vectors

If $V_1, V_2, \dots V_n$ are LI and $\beta_1 V_1 + \beta_2 V_2, \dots + \beta_n V_n = 0$,

$\beta_1 = \beta_2 \dots = \beta_n = 0$.

standard basis

$$E_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad E_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

form a linear combination

$$\beta_1 E_1 + \beta_2 E_2 + \beta_3 E_3 = 0$$

plug in Ej

$$\beta_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \beta_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

obtain (3) Eqs.

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = 0$$

solve for β_j

$$\beta_1 = 0 = \beta_3 = \beta_3$$

Note that:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \neq 0$$

Review: Triple Products

Triple Products

The product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ that occurs in Property 5 is called the **scalar triple product** of the vectors **a**, **b**, and **c**. Notice from Equation 12 that we can write the scalar triple product as a determinant:

13
$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

14 The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

determinant = a triple product, e.g., “volume” for 3 LI vectors

Review: Geometric Meaning of Triple Product

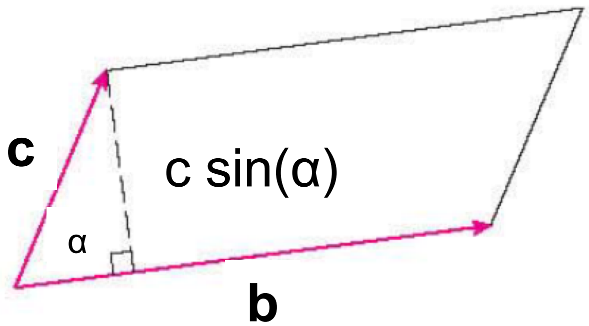

FIGURE 2

h = |a| cos(θ)

height

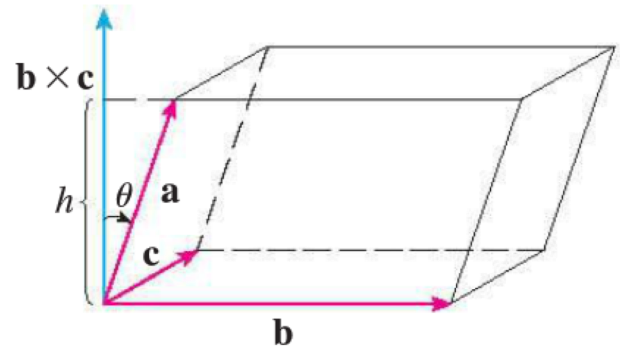

FIGURE 3

If α is the angle between **b** and **c** (0≤α≤π), then
|**b** x **c** | = |**b**| |**c**| sin(α)

area

b x **c** determines a normal vector, **n**, which is perpendicular to the plane.

Height = | projection of vector **a** onto **n** | = $\left|\vec{a}\cdot\frac{\vec{n}}{|\vec{n}|}\right| = \frac{\left|\vec{a}\cdot(\vec{b}\times\vec{c})\right|}{\left|\vec{b}\times\vec{c}\right|}$

14 The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$$

height x area

Example 2

Example. The vectors (1,0,0), (1,1,0), and (1,1,1) in $\mathbb{R}^3$ are also linearly independent, for if we have

Start with a linear combination:

$$\alpha_1 \begin{pmatrix}1\\0\\0\end{pmatrix} + \alpha_2 \begin{pmatrix}1\\1\\0\end{pmatrix} + \alpha_3 \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}\boxed{\alpha_1+\alpha_2+\alpha_3}\\ \boxed{\alpha_2+\alpha_3}\\ \boxed{\alpha_3}\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix},$$

Solve the above three equations for the coefficients α_j

$$\alpha_3 = 0 \qquad \alpha_2 = 0 \qquad \alpha_1 = 0$$

$$\begin{vmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 1\end{vmatrix} = 1 \neq 0$$

Example 2

On the other hand, the vectors $(1,1,1)$, $(1,2,3)$, and $(2,3,4)$ are linearly dependent, for we have

$$\boxed{1}\begin{pmatrix}1\\1\\1\end{pmatrix}+\boxed{1}\begin{pmatrix}1\\2\\3\end{pmatrix}-\boxed{1}\begin{pmatrix}2\\3\\4\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

■

Linear combination with non-zero coefficients

$$\begin{vmatrix}1 & 1 & 2\\1 & 2 & 3\\1 & 3 & 4\end{vmatrix}=1\begin{vmatrix}2 & 3\\3 & 4\end{vmatrix}-\begin{vmatrix}1 & 2\\3 & 4\end{vmatrix}+1\begin{vmatrix}1 & 2\\2 & 3\end{vmatrix}$$

$$=-1-(-2)+(-1)=0$$

Example: Chapter 14 of Calc III

$$\boxed{df = f_x dx + f_y dy = \nabla f \cdot d\vec{r}}$$

$$D_u f = (f_x, f_y) \cdot (a, b) = \nabla f \cdot \frac{d\vec{r}}{|d\vec{r}|}$$

$$\boxed{if\ df = 0 \text{ for all } d\vec{r} = (dx, dy)}$$

max and min

$$f_x dx + f_y dy = 0$$

If dx *and* dy *are LI,*

- Find f_x and f_y
- Send your results via "chat"
- You have 2 minutes

Example: Chapter 14 of Calc III

$$\boxed{df = f_x dx + f_y dy = \nabla f \cdot d\vec{r}}$$

$$D_u f = (f_x, f_y) \cdot (a, b) = \nabla f \cdot \frac{d\vec{r}}{|d\vec{r}|}$$

$\boxed{if\ df = 0 \text{ for all } d\vec{r} = (dx, dy)}$ max and min

$f_x dx + f_y dy = 0$ & dx *and* dy *are LI*

coefficients are zero

$$f_x = 0 = f_y$$

$$\boxed{\nabla f = 0}$$

Sect 5.1: Subspace

given $V_1, \ldots, V_k \in \mathbb{R}^n$, the set

$$\mathcal{S} = \{\alpha_1 V_1 + \cdots + \alpha_k V_k \mid \alpha_j \in \mathbb{R}\}$$

is a subspace of $\mathbb{R}^n$. In this case we say that $\mathcal{S}$ is *spanned* by $V_1, \ldots, V_k$. Equivalently, it can be shown (see Exercise 12 at the end of this chapter) that a subspace $\mathcal{S}$ is a nonempty subset of $\mathbb{R}^n$ having the following two properties:

1. If $X, Y \in \mathcal{S}$, then $X + Y \in \mathcal{S}$; addition
2. If $X \in \mathcal{S}$ and $\alpha \in \mathbb{R}$, then $\alpha X \in \mathcal{S}$. scalar multiplication

subspace: closed under (1) addition and (2) scalar multiplication

Example 1: Subspace

Example. Any straight line through the origin in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$, since this line may be written as $\{tV \mid t \in \mathbb{R}\}$ for some nonzero $V \in \mathbb{R}^n$. The single vector V spans this subspace. The plane $\mathcal{P}$ defined by $x + y + z = 0$ in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

$$z = -x - y \qquad W = \begin{pmatrix} x \\ y \\ -(x+y) \end{pmatrix} = x\underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{W_1} + y\underbrace{\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}}_{W_2}$$

$$|\vec{V}_1 \times \vec{V}_2| = |\vec{V}_1||\vec{V}_2| sin(\theta)$$

- If V_1 and V_2 are linearly dependent, $c_1 V_1 + c_2 V_2 = 0$ for non-zero $c_1$and c_2, yielding $V_1 = \alpha V_2$ with $\alpha \in R$ and, thus, $|\vec{V}_1 \times \vec{V}_2| = 0$.
- $|\vec{W}_1 \times \vec{W}_2| = |(1, \quad 1, \quad 1)| \neq 0, W_1$ and W_2 are linearly independent.
- Two LI vectors W_1 and W_2 span a "subspace", R^2 (TBD, below)

Example 1: Subspace

Example. Any straight line through the origin in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$, since this line may be written as $\{tV \mid t \in \mathbb{R}\}$ for some nonzero $V \in \mathbb{R}^n$. The single vector V spans this subspace. The plane $\mathcal{P}$ defined by $x + y + z = 0$ in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

$$W = x\begin{pmatrix}1\\0\\-1\end{pmatrix} + y\begin{pmatrix}0\\0\\-1\end{pmatrix}$$

Chose $V_1 = x_1\begin{pmatrix}1\\0\\-1\end{pmatrix} + y_1\begin{pmatrix}0\\0\\-1\end{pmatrix}$ and $V_2 = x_2\begin{pmatrix}1\\0\\-1\end{pmatrix} + y_2\begin{pmatrix}0\\0\\-1\end{pmatrix}$

Obtain $V_1 + V_2 = (x_1+x_2)\begin{pmatrix}1\\0\\-1\end{pmatrix} + (y_1+y_2)\begin{pmatrix}0\\0\\-1\end{pmatrix}$ closed under (1) addition

$$\alpha V_1 = \alpha x_1\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

closed under (2) scalar multiplication

Sect 5.1: Subspace (TBD in Sect. 5.4)

Example. Any straight line through the origin in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$, since this line may be written as $\{tV \mid t \in \mathbb{R}\}$ for some nonzero $V \in \mathbb{R}^n$. The single vector V spans this subspace. The plane $\mathcal{P}$ defined by $\boxed{x + y + z = 0}$ in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$. Indeed, any vector V in $\mathcal{P}$ may be written in the form $(x, y, -x - y)$ or

$$V = x \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

which shows that the vectors $(1, 0, -1)$ and $(0, 1, -1)$ span $\mathcal{P}$. ■

How about x + y + z = c ? (TBD)

Sect 5.1: Preliminaries from Linear Algebra

In linear algebra, one often encounters rectangular $n \times m$ matrices, but in differential equations, most often these matrices are square ($n \times n$). Consequently we will assume that all matrices in this chapter are $n \times n$. We write such a matrix

A system of DEs

➔ A square matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \vdots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix}$$

$$\sum_{j=1}^{n} a_{1j} x_j$$

$$\sum_{j=1}^{n} a_{nj} x_j$$

more compactly as $A = [a_{ij}]$.

For $X = (x_1, \dots, x_n) \in \mathbb{R}^n$, we define the product AX to be the vector

$$AX = \begin{pmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{nj} x_j \end{pmatrix},$$

Sect 5.1: Linearity Properties

Matrix sums are defined in the obvious way. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $n \times n$ matrices, then we define $A + B = C$ where $C = [a_{ij} + b_{ij}]$. Matrix arithmetic has some obvious linearity properties:

1. $A(k_1 X_1 + k_2 X_2) = k_1 A X_1 + k_2 A X_2$ where $k_j \in \mathbb{R}$, $X_j \in \mathbb{R}^n$;
2. $A + B = B + A$;
3. $(A + B) + C = A + (B + C)$.

Sect 5.1: Matrix Multiplication

The product of the $n \times n$ matrices A and B is defined to be the $n \times n$ matrix $AB = [c_{ij}]$ where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj},$$

so that c_{ij} is the dot product of the ith row of A with the jth column of B. We

if A, B, and C are $n \times n$ matrices, then

1. $\boxed{(AB)C = A(BC);}$
2. $A(B + C) = AB + AC;$
3. $(A + B)C = AC + BC;$
4. $k(AB) = (kA)B = A(kB)$ for any $k \in \mathbb{R}$.

The associative property of multiplication of matrices
or simply the associate law

Sect 5.1: Matrix Multiplication

$AB \neq BA$

For example

$$AB = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

whereas

$$BA = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

$When\ AB = AC,$ it is not necessary to have $B = C$.

Sect 5.1: Inverse Matrix A^{-1}: $AA^{-1} = \mathrm{I}$

The $n \times n$ matrix A is *invertible* if there exists an $n \times n$ matrix C for which $AC = CA = I$, where I is the $n \times n$ identity matrix that has 1s along the diagonal and 0s elsewhere. The matrix C is called the *inverse* of A. Note that if A has an inverse, then this inverse is unique. For if $AB = BA = I$ as well, then

$$C = CI = C(AB) = (CA)B = IB = B.$$

The inverse of A is denoted by A^{-1}.

Sect 5.1: A Unique Solution for an Invertible Matrix

If A is invertible, then the vector equation $AX = V$ has a unique solution for any $V \in \mathbb{R}^n$. Indeed, $\boxed{A^{-1}V \text{ is one solution.}}$ Moreover, it is the only one, for if Y is another solution, then we have

$$Y = (A^{-1}A)Y = A^{-1}(AY) = A^{-1}V.$$

$A^{-1}A = I$ $\quad$ $AY = V$ $\quad$ $A^{-1}V$ is a sol.

Namely,

$$AX = V \quad \Longrightarrow \quad X = A^{-1}V$$

Recall (from LA):

The system AX has a unique solution if and only if the reduced row echelon form of the matrix A is the identify matrix I.

Sect 5.1: Elementary Row Operations

The reduced row echelon form of the matrix A is obtained by applying to A a sequence of elementary row operations of the form:

1. Add k times row i of A to row j;
2. Interchange row i and j;
3. Multiply row i by $k \neq 0$.

Note that these elementary row operations correspond exactly to the operations that are used to solve linear systems of algebraic equations:

1. Add k times equation i to equation j;
2. Interchange equations i and j;
3. Multiply equation i by $k \neq 0$.

Sect 5.1: Key Points

Proposition. *Let A be an $n \times n$ matrix. Then the system of algebraic equations $AX = V$ has a unique solution for any $V \in \mathbb{R}^n$ if and only if A is invertible.* ∎

Proposition. *The matrix A is invertible if and only if the columns of A form a linearly independent set of vectors.*

A unique solution for $AX = V$ $\Leftrightarrow$ A^{-1} exists $\Leftrightarrow$ column vectors are LI

A Invertible Matrix and its LI Column Vectors

Proposition. *The matrix A is invertible if and only if the columns of A form a linearly independent set of vectors.*

A proof

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & \cdots & \\ \vdots & \vdots & & \\ \vdots & \vdots & & \end{pmatrix} = [V_1 \quad V_2 \quad \cdots \quad V_n]$$

$$V_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \end{pmatrix} \qquad V_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}$$

$$AE_j = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & \cdots & \\ \vdots & \vdots & & \\ \vdots & \vdots & & \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix} = V_j$$

A Invertible Matrix and its LI Column Vectors: a proof

Proposition. *The matrix A is invertible if and only if the columns of A form a linearly independent set of vectors.*

Assume V_j are not LI, we may have real numbers, $\alpha_1, \alpha_2, \dots, \alpha_n$ not all zero, such at

$$\sum_{j=1}^{n} \alpha_j V_j = 0 \qquad AE_j = V_j \qquad \sum_{j=1}^{n} \alpha_j AE_j = 0$$

$$A\left(\sum_{j=1}^{n} \alpha_j E_j\right) = 0$$

- The equation $AX = 0$ has two solutions, i.e., the 0 vector and the non-zero $\alpha_1, \alpha_2, \dots, \alpha_n$.
- This contradicts the previous proposition (e.g., a unique solution for an invertible matrix).

Sect 5.1: Determinant

Definition

The *determinant* of $A = [a_{ij}]$ is defined inductively by

$$\det A = \sum_{k=1}^{n} (-1)^{1+k} a_{1k} \det A_{1k}.$$

Example. From the definition we compute

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 1 \det \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 2 \det \begin{pmatrix} 4 & 6 \\ 7 & 9 \end{pmatrix} + 3 \det \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$$

$$= -3 + 12 - 9 = 0.$$

Sect 5.1: Determinant

Example. Expanding the matrix in the previous example along the second and third rows yields the same result:

$$
\begin{aligned}
\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} &= -4\det\begin{pmatrix} 2 & 3 \\ 8 & 9 \end{pmatrix} + 5\det\begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix} - 6\det\begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix} \\
&= 24 - 60 + 36 = 0 \\
&= 7\det\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} - 8\det\begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix} + 9\det\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} \\
&= -21 + 48 - 27 = 0.
\end{aligned}
$$

Sect 5.1: Determinant

Proposition. *If A is an upper or lower triangular $n \times n$ matrix, then* $\det A$ *is the product of the entries along the diagonal. That is,* $\det[a_{ij}] = a_{11} \dots a_{nn}$. ■

Proposition. *Let A and B be $n \times n$ matrices.*

1. *Suppose the matrix B is obtained by adding a multiple of one row of A to another row of A. Then* $\det B = \det A$.
2. *Suppose B is obtained by interchanging two rows of A. Then* $\det B = -\det A$.
3. *Suppose B is obtained by multiplying each element of a row of A by k. Then* $\det B = k \det A$.

Corollary. (Invertibility Criterion) *The matrix A is invertible if and only if* $\det A \neq 0$.

Proposition. $\det(AB) = (\det A)(\det B)$.

Equivalent Statements for a Non-singular Matrix

Square Matrix

Theorem 1.3. *For $A \in \mathbb{C}^{m\times m}$, the following conditions are equivalent:*

(a) *A has an inverse A^{-1},*

$\boxed{\text{(b) } \operatorname{rank}(A) = m,}$

(c) $\operatorname{range}(A) = \mathbb{C}^m$,

(d) $\operatorname{null}(A) = \{0\}$,

(e) *0 is not an eigenvalue of A,*

(f) *0 is not a singular value of A,*

$\boxed{\text{(g) } \det(A) \neq 0.}$

dim range(A) = m

Ax=0 ➔ x=0; “null space”=”kernel”

Equivalent Statements for a Non-singular Matrix

Square Matrix

Theorem 2.5. *Let A be an $n \times n$ matrix. Then the following statements are equivalent:*

(1) *The columns of A are linearly independent.*

(2) *The columns of A span $\mathbb{R}^n$.*

(3) *The columns of A form a basis for $\mathbb{R}^n$.*

(4) *The equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$.*

(5) *A is an invertible matrix.*

(6) *The determinant of A is nonzero.*

(7) *A is row equivalent to the identity matrix.*

Equivalent Statements for a Singular Matrix

Square Matrix

Matrices and Systems of Equations

Theorem 1. Let $\mathbf{A}$ be an $n \times n$ matrix. The following statements are equivalent:

(a) $\mathbf{A}$ is singular (does not have an inverse).

(b) The determinant of $\mathbf{A}$ is zero.

(c) $\mathbf{A}\mathbf{x} = \mathbf{0}$ has nontrivial solutions ($\mathbf{x} \neq \mathbf{0}$).

(d) The columns (rows) of $\mathbf{A}$ form a linearly dependent set.