Homework 5
Linear Algebra
Math 524
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Section 5.A Problem 4: Suppose that $T \in \mathcal{L}(V)$ and $U_1, ..., U_m$ are subspaces of V invariant under T. Prove that $U_1 + ... + U_m$ is invariant under T.

Solution. Let $T \in \mathcal{L}(V)$ and $U_1, ..., U_m$ be subspaces of V invariant under T.
Let $u \in U_1 + ... + U_m$ and $u_1 \in U_1, ..., u_m \in U_m$ with $u = u_1 + ... + u_m$

$$\begin{aligned} &\text{Notice: } Tu_1 \in U_1, ..., Tu_m \in U_m \\ &Tu = Tu_1 + ... + Tu_m \in U_1 + ... + U_m \end{aligned}$$

Because $Tu \in U_1 + ... + U_m$, this proves that $U_1 + ... + U_m$ is invariant under T.

□

Section 5.A Problem 8: Define $T \in \mathcal{L}(\mathbb{F}^2)$ by $T(w, z) = (z, w)$. Find all the eigenvalues and eigenvectors of T.

Solution. Let $T \in \mathcal{L}(\mathbb{F}^2)$ with $T(w, z) = (z, w)$.

$$\begin{aligned} &\text{Notice:} \qquad Tw = z \qquad Tz = w \\ &Tw = z = \lambda w \qquad Tz = w = \lambda z \\ &\text{Through Subsitution: } z = \lambda^2 z \text{ or } w = \lambda^2 w \\ &\lambda^2 = 1 \\ &\lambda = \pm 1 \end{aligned}$$

For $\lambda = 1$, we need to find a v_1 such that $Tv_1 = v_1$. For $\lambda = -1$, we need to find a v_2 such that $Tv_2 = -v_2$.

$$\begin{aligned} T(w, z) = (z, w) &= 1(w, z) \qquad\qquad & T(w, z) = (z, w) &= -1(w, z) \\ z &= w & z &= -w \end{aligned}$$

The eigenvector corresponding to $\lambda = 1$ would be a vector whose components would be equal to each other such as (w, w) with $w \in \mathbb{F}$. The eigenvector corresponding to $\lambda = -1$ would be a vector whose components would be opposite to each other such as $(w, -w)$ with $w \in \mathbb{F}$.
□

Section 5.A Problem 10 (a): Define $T \in \mathcal{L}(\mathbb{F}^n)$ by $T(x_1, x_2, ..., x_n) = (x_1, 2x_2, ..., nx_n)$. Find all the eigenvalues and eigenvectors of T.

Solution. Let $T \in \mathcal{L}(\mathbb{F}^n)$ with $T(x_1, x_2, ..., x_n) = (x_1, 2x_2, ..., nx_n)$.

$$\begin{aligned}&\text{Notice: } Tx_j = jx_j \qquad 1 \leq j \leq n\\ &jx_j = \lambda x_j\\ &j = \lambda\end{aligned}$$

For $\lambda = j$, we need to find a $(x_1, x_2, ..., x_n)$ such that $T(x_1, x_2, ..., x_n) = j(x_1, x_2, ..., x_n)$

Notice: if we set j = 2: $T(x_1, x_2, ..., x_n) = (x_1, 2x_2, ..., nx_n) = (2x_1, 2x_2, ..., 2x_n)$
So: $x_1 = 2x_1 \qquad 2x_2 = 2x_2 \qquad nx_n = 2x_n$
$0 = 2x_1 - x_1 = x_1 \qquad x_2 = x_2 \qquad 0 = (2 - n)x_n$
So: $x_1 = 0, x_3 = 0, ..., x_n = 0$
Eigenvector for $\lambda = j = 2$: $(0, x_2, 0, ..., 0)$

The eigenvectors that correspond to $\lambda = j$ are $(0, 0, ..., x_j, ..., 0, 0)$ for all $1 \leq j \leq n$.

□

Section 5.B Problem 1 (a): Suppose $T \in \mathcal{L}(V)$ and there exists a positive integer n such that $T^n = 0$. Prove that $I - T$ is invertible and that

$$(I - T)^{-1} = I + T + ... + T^{n-1}$$

Solution. Let $T \in \mathcal{L}(V)$ and there exists a positive integer n such that $T^n = 0$. Notice: A is invertible iff $AA^{-1} = A^{-1}A = I$

$$\begin{aligned}(I - T)(I + T + ... + T^{n-1}) &= (I + T + ... + T^{n-1}) - (T + T^2 + ... + T^{n-1} + T^n)\\ &= I - T^n = I - 0 = I\end{aligned}$$

This proves that $(I - T)$ is invertible and its inverse is $(I + T + ... + T^{n-1})$

□

Section 5.B Problem 7: Suppose that $T \in \mathcal{L}(V)$. Prove that 9 is an eigenvalue of T^2 if and only if 3 or -3 is an eigenvalue of T.

Solution. Let $T \in \mathcal{L}(V)$. Let λ represent the eigenvalue of T.

$$\begin{aligned}\text{Let } \lambda_1 &= 3 & \text{Let } \lambda_2 &= -3\\ Tv &= 3v & Tv &= -3v\\ T^2v = T(Tv) = T(3v) = 3(3v) &= 9v & T^2v = T(Tv) = T(-3v) = -3(-3v) &= 9v\end{aligned}$$

This proves that the eigenvalue of T^2 is 9 when the eigenvalues of T is 3 or -3.

□

Section 5.B Problem 14: Give an example of an operator whose matrix with respect to some basis contains only 0's on the diagonal, but the operator is invertible

Solution. Let $T \in \mathcal{L}(V)$ and v_1, v_2, v_3 is the basis of V such that $Tv_j = v_1 + v_2 + v_3 - v_j$ for $1 \leq j \leq 3$

$$\mathcal{M}(T) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad \mathcal{M}^{-1}(T) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

T proves there exists an operator whose matrix with respect to some basis contains only 0's on the diagonal. Because $\mathcal{M}^{-1}(T)$ exists proves that the matrix is invertible.

□

Section 5.B Problem 15: Give an example of an operator whose matrix with respect to some basis contains only nonzero numbers on the diagonal, but the operator is not invertible

Solution. Let $T \in \mathcal{L}(V)$ and v_1, v_2 is the basis of V such that for $a \neq 0, b \neq 0 \in \mathbb{F}$, $Tv_j = av_1 + bv_2$ for j = {1, 2}

$$\mathcal{M}(T) = \begin{pmatrix} a & a \\ b & b \end{pmatrix}$$

Because $\mathcal{M}^{-1}(T)$ does not exists proves that the matrix is not invertible, while having its diagonal consist on nonzero numbers. □

Section 5.C Problem 1: Suppose $T \in \mathcal{L}(V)$ is diagonalizable. Prove that $V = \text{null } T \oplus$ range T.

Solution. Let $T \in \mathcal{L}(V)$ be diagonalizable and $v_1, ..., v_n, u_1, ..., u_m$ be a basis of V.

(1) Because T is diagonalizable, each element of the basis of V is an eigenvector, such that $Tv_j = \lambda v_j$ and $Tu_k = 0u_k$.

$$\begin{aligned} &\text{range T} = \text{span}(\lambda v_1, ..., \lambda v_n) \\ &\text{null T} = \text{span}(u_1, ..., u_m) \end{aligned}$$

(2) Let $w \in \text{null } T \cap$ range T.

$$\begin{aligned} w &= a_1 v_1 + ... a_n v_n \\ &= b_1 u_1 + ... + b_m u_m \\ a_1 v_1 + &... a_n v_n - (b_1 u_1 + ... + b_m u_m) = 0 \end{aligned}$$

Because of (1), $V = \text{null } T +$ range T. Because $v_1, ..., v_n, u_1, ..., u_m$ is a basis of V, the constants have to be 0, which means w = 0. This proves that null $T \cap$ range $T = \{0\}$. Which proves that $V = \text{null } T \oplus$ range T.

□

Section 5.C Problem 2: Prove the converse of the statement in the exercise above or give a counterexample to the converse.

Converse: If $V = \text{null } T \oplus \text{ range } T$, then $T \in \mathcal{L}(V)$ is diagonalizable.

Solution. Let $T \in \mathcal{L}(\mathbb{R}^2)$ such that $T(w, z) = (-z, w)$ with $w, z \in \mathbb{R}$.

Notice: null T = span((0,0)) and range T = span(-z,w), $\mathbb{R}^2 = \text{null } T \oplus \text{ range } T$
The eigenvalues of T are, however, $\pm i \notin \mathbb{R}$. Because there does not exist any eigenvalues in $\mathbb{R}$, T is not diagonalizable. So the converse is false! $\square$