Homework 2
Discrete Dynamical Systems and Chaos
Math 538
Stephen Giang RedID: 823184070

Problem T1.3: Solve the inequality $|f(x) - 0| > |x - 0|$, where $f(x) = \frac{3x-x^3}{2}$. This identifies points whose distance from 0 increases on each iteration. Use the result to find a large set of initial conditions that do not converge to any sink of f.

Lets solve the given equality:

$$\left|\frac{3x - x^3}{2}\right| > |x|$$

$$\begin{aligned}
\left(\frac{3x - x^3}{2}\right)^2 &> x^2 \\
\left(3x - x^3\right)^2 &> 4x^2 \\
x^6 - 6x^4 + 9x^2 &> 4x^2 \\
x^6 - 6x^4 + 5x^2 &> 0 \\
x^2(x^4 - 6x^2 + 5) &> 0 \\
x^2(x^2 - 5)(x^2 - 1) &> 0 \\
x^2(x - \sqrt{5})(x + \sqrt{5})(x - 1)(x + 1) &> 0
\end{aligned}$$

Looking at a number line, we can see that the solution is the following:

$$(-\infty, -\sqrt{5}) \cup (-1, 0) \cup (0, 1) \cup (\sqrt{5}, \infty)$$

We can see that for fixed points $x^* = 0, -1, 1$, $x^* = 0$ is a source that makes $f(x)$ converge to the sinks of $x^* = 1$ and $x^* = -1$.

We can see that for all values $|x| > \sqrt{5}$, we get that:

$$|f^{(k+1)}(x)| > |f^{(k)}(x)| \quad \rightarrow \quad \lim_{k\to\infty} |f^k(x)| = \infty$$

which shows that for $|x| > \sqrt{5}$, x does not converge to any sink of f.

Problem T1.4: Let p be a fixed point of a map f. Given some $\epsilon > 0$, find a geometric condition under which all points x in $N_\epsilon(p)$ are in the basin of p. Use cobweb plot analysis to explain your reasoning.

We can informally define basin of p to be the set of initial points that converge into the sink $x^* = p$

Let $\epsilon > 0$ such that all values $N_\epsilon(p) = (p, p+\epsilon)$. We can choose some $0 < \delta < \epsilon$ such that $x = p + \delta \in N_\epsilon(p)$.

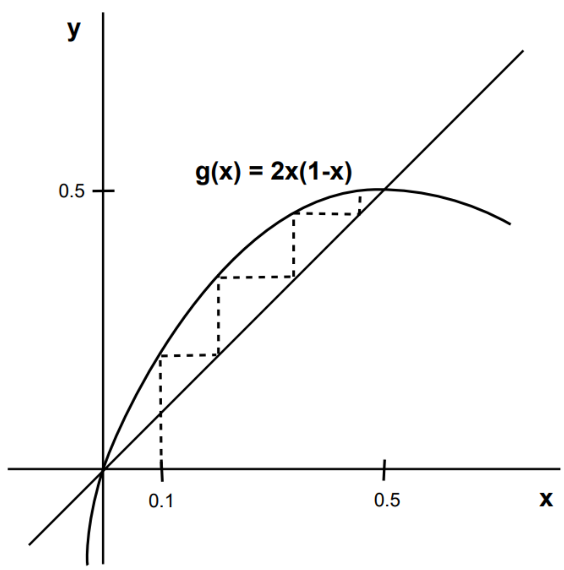
If we evaluate the sink condition of Theorem 1.5, we get:

$$\frac{|f(p+\delta) - f(p)|}{|p+\delta-p|} < 1$$

such that the geometric condition under which all points x in $N_\epsilon(p)$ are in the basin of p is as follows:

$$|f(p+\delta) - f(p)| < \delta < \epsilon$$

Notice for the example: $g(x) = 2x(1-x)$, we get a sink at $x = \frac{1}{2}$ because the $\Delta g(x) > \Delta x$ near $x = \frac{1}{2}$.



Problem 1.1: Let $\ell(x) = ax + b$, where a and b are constants. For which values of a and b does ℓ have an attracting fixed point? A repelling fixed point?

First, notice the following:

$$\ell(x) = ax + b \qquad \rightarrow \qquad \ell'(x) = a$$

Consider the following values for a and b:

(a) $(|a| < 1, b \in \mathbb{R})$

by Th. 1.5: ℓ has a stable/attracting fixed point.

(b) $(|a| > 1, b \in \mathbb{R})$

by Th. 1.5: ℓ has an unstable/repelling fixed point.

(c) $(a = -1, b \in \mathbb{R})$

Because $\ell(x) = -x + b$, there exists a period 2 orbit around the fixed point $x^* = b/2$

(d) $(a = 1, b = 0)$

Because $\ell(x) = x$, every point is a fixed point, but none are either a stable/attracting or an unstable/repelling fixed point.

(e) $(a = 1, b \in \mathbb{R}\backslash\{0\})$

Because $\ell(x)||x$, there are no fixed points.

Problem 1.2:

(a) Let $f(x) = x - x^2$. Show that $x = 0$ is a fixed point of f, and describe the dynamical behavior of points near 0.

To show that $x = 0$ is a fixed point, we need to solve the following equality:

$$f(x) = x \quad \rightarrow \quad x - x^2 = x \quad \rightarrow \quad -x^2 = 0 \quad \rightarrow \quad x^* = 0$$

Notice the following behavior:

(i) For $x < 0$, we get that $f(x) < 0$ and:

$$|f^{(k+1)}(x)| > |f^{(k)}(x)| \quad \rightarrow \quad \lim_{k\to\infty} |f^k(x)| = \infty$$

Meaning $x^* = 0$ is being repelling on the interval: $(-\infty, 0)$

(ii) For $0 < x < 1$, we get that $f(x) > 0$ and:

$$|f^{(k+1)}(x)| < |f^{(k)}(x)| \quad \rightarrow \quad \lim_{k\to\infty} |f^k(x)| = 0$$

Meaning $x^* = 0$ is being attracting on the interval: $(0, 1)$

(iii) For $x > 1$, we get that $f(x) < 0$, which then maps to the first case, such that $x^* = 0$ is being repelling on the interval: $(0, \infty)$

(b) Let $g(x) = \tan x, -\pi/2 < x < \pi/2$. Show that $x = 0$ is a fixed point of g, and describe the dynamical behavior of points near 0.

To show that $x = 0$ is a fixed point, we need to solve the following equality:

$$f(x) = x \quad \rightarrow \quad \tan x = x \quad \rightarrow \quad \tan 0 = 0$$

Notice the following behavior:

(i) For $x < 0$, we get that $f(x) < 0$ and:

$$|f^{(k+1)}(x)| > |f^{(k)}(x)| \quad \rightarrow \quad \lim_{k\to\infty} |f^k(x)| = \infty$$

Meaning $x^* = 0$ is being repelling on the interval: $(-\infty, 0)$

(ii) For $x > 0$, we get that $f(x) < 0$ and:

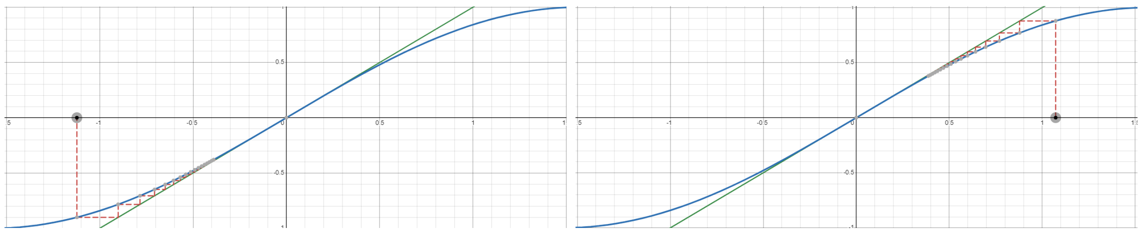
$$|f^{(k+1)}(x)| > |f^{(k)}(x)| \quad \rightarrow \quad \lim_{k\to\infty} |f^k(x)| = \infty$$

Meaning $x^* = 0$ is being repelling on the interval: $(-\infty, 0)$

Thus, giving us that $x^* = 0$ is a source on the domain $-\pi/2 < x < \pi/2$.

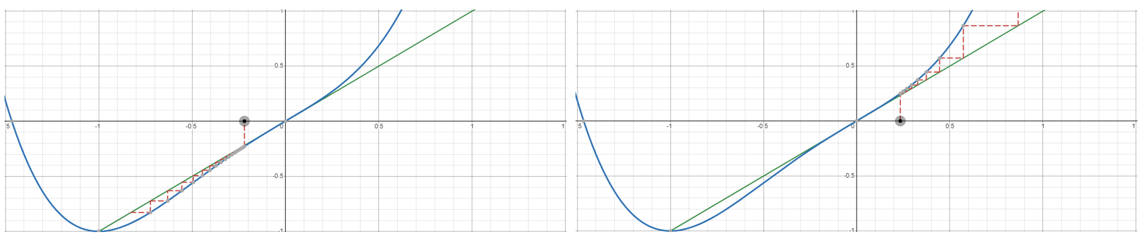
(c) Give an example of a function h for which $h'(0) = 1$ and $x = 0$ is an attracting fixed point.

Notice the following stable attracting fixed point at $x = 0$ of $h(x) = \sin x$



(d) Give an example of a function h for which $h'(0) = 1$ and $x = 0$ is a repelling fixed point.

Notice the following stable attracting fixed point at $x = 0$ of $h(x) = x^4 + x^3 + x$



Problem 1.3: Let $f(x) = x^3 + x$. Find all fixed points of f and decide whether they are sinks or sources. You will have to work without Theorem 1.5, which does not apply.

To find a fixed point, we set the following to be true and solve:

$$f(x) = x \qquad \rightarrow \qquad x^3 + x = x \qquad \rightarrow \qquad x^3 = 0 \qquad \rightarrow \qquad x^* = 0$$

Notice the following inequality:

$$|x^3 + x - 0| = |x(x^2+1)| = |x|(x^2+1) > |x| \quad \rightarrow \quad (x^2+1) > 1$$

By definition of a source, we can see that the distance between $f(x)$ and 0 is always greater than the distance between x and 0 such that:

$$\lim_{k\to\infty} |f^k(x)| = \infty$$

Problem (EXTRA): State and Prove a nonlinear version of the Stability Theorem (Theorem 1.5) when linear stability fails (i.e., $|f'(x^*)| = 1$).

Let $x = x^*$ be a fixed point such that $f(x^*) = x^*$. Let $|f'(x^*)| = 1$. Let there also exist a $\epsilon > 0$ such that $x^* \in N_\epsilon(x^*)$.

We can take any function and expand it to its taylor series:

$$f(x^* + \epsilon) = f(x^*) + f'(x^*)\epsilon + \frac{f''(x^*)}{2}\epsilon^2 + ... + \frac{f^{(n)}(x^*)}{n!}\epsilon^n$$

Moving some terms to the other side and setting $f'(x^*) = 1$, we get:

$$f(x^* + \epsilon) - f(x^*) = \epsilon + \frac{f''(x^*)}{2}\epsilon^2 + ... + \frac{f^{(n)}(x^*)}{n!}\epsilon^n = \epsilon + \sum_{n=2}^{\infty} \frac{f^{(n)}(x^*)}{n!}\epsilon^n$$

Taking the absolute value of both sides and dividing by ϵ, we get the result from Theorem 1.5:

$$\frac{|f(x^* + \epsilon) - f(x^*)|}{\epsilon} = \frac{1}{\epsilon}\left|\epsilon + \sum_{n=2}^{\infty} \frac{f^{(n)}(x^*)}{n!}\epsilon^n\right| = \left|1 + \sum_{n=2}^{\infty} \frac{f^{(n)}(x^*)}{n!}\epsilon^{n-1}\right|$$

Looking at Theorem 1.5, we get the following conclusion:

(a) x^* is a sink:
$$\left|1 + \sum_{n=2}^{\infty} \frac{f^{(n)}(x^*)}{n!}\epsilon^{n-1}\right| < 1$$

(b) x^* is a source:
$$\left|1 + \sum_{n=2}^{\infty} \frac{f^{(n)}(x^*)}{n!}\epsilon^{n-1}\right| > 1$$