Lecture: Separation of Variables (Chapter 2)

Outline:

i) Homogeneous heat equation
- Basic definitions
- Principle of Superposition

ii) Separation of Variables method
- Reduce PDE to ODEs
- Eigenfunctions
- Superposition

iii) Orthogonality and Computer approximation
- Orthogonality of Sine functions
- Heat equation examples
- MATLAB

Homogeneous Heat Equation

Assume a uniform rod of length L, so that K, c, ρ do not vary in x

General heat equation:

PDE: $$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} + \frac{Q(x,t)}{c\rho}, \quad 0<x<L, \; t>0,$$

IC: $u(x,0) = f(x)$, $0 < x < L$

BC: $u(0,t) = T_1(t)$ and $u(L,t) = T_2(t)$

If $Q(x,t) = 0$, then PDE is homogeneous

If $T_1(t) = T_2(t) = 0$, the BC is homogeneous.

Definition: Linearity

An operator $\mathcal{L}$ is linear if and only if $\mathcal{L}[c_1 u_1 + c_2 u_2]$

$$= c_1 \mathcal{L}[u_1] + c_2 \mathcal{L}[u_2]$$

for any two functions u_1 and u_2 and constants c_1 and c_2.

Given the linear operator

$$\frac{\partial}{\partial t} - k \frac{\partial^2}{\partial x^2} = \mathcal{L}$$

The following shows linearity of the heat operator:

$$\mathcal{L}[c_1 u_1 + c_2 u_2] = \frac{\partial}{\partial t} - k \frac{\partial^2}{\partial x^2}(c_1 u_1 + c_2 u_2)$$

$$= c_1 \frac{\partial u_1}{\partial t} + c_2 \frac{\partial u_2}{\partial t} - k c_1 \frac{\partial^2 u_1}{\partial x^2}$$

$$-Kc_2\frac{\partial^2 u_2}{\partial x^2}$$

$$= c_1 \mathcal{L}[u_1] + c_2\mathcal{L}[u_2].$$

Theorem: Principle of Superposition

If u_1 and u_2 satisfy the linear homogeneous equation ($\mathcal{L}[u]=0$), then any arbitrary combination, $c_1u_1 + c_2u_2$, also satisfies the linear homogeneous equation.

Concept of linearity and homogeneity also apply to BCs.

Solving the Homogeneous Heat Equation : Method of Seperation of variables

(1) $\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$, $t > 0$, $0 < x < L$

(2) $u(x,0) = f(x)$, $0 < x < L$

(3) $u(0,t) = 0$ and $u(L,t) = 0$, $t > 0$

Seperation of variable :

Seperate $u(x,t)$ into a product of a function of x and a function of t.

$u(x,t) = \phi(x) G(t)$ (4)

Substitute (4) into (1)

e.g. $u(x,t) = xt$
$u(x,t) = \phi(x) G(t)$

$$\frac{\partial u}{\partial t} = \phi(x) \frac{dG}{dt}$$

$$\frac{\partial^2 u}{\partial x^2} = G(t) \frac{d^2\phi}{dx^2}$$

Note

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(\phi(x) G(t)\right)$$

$$= \phi(x) \frac{dG}{dt}$$

This gives

$$\phi(x) \frac{dG}{dt} = \kappa \, G(t) \frac{d^2\phi}{dx^2}$$

divide through by $\phi(x) G(t) \kappa$

$$\frac{1}{\kappa G} \frac{dG}{dt} = \frac{1}{\phi} \frac{d^2\phi}{dx^2}$$

Since the LHS depends only on the independent variable t and the right hand side (RHS) depends on the independent variable x, these must equal a constant.

Therefore,

$$\frac{1}{kG}\frac{dG}{dt} = \frac{1}{\phi}\frac{d^2\phi}{dx^2} = -\lambda$$

Seperation of variable results in two ODEs

$$\frac{dG}{dt} = -\lambda kG \quad \text{and} \quad \frac{d^2\phi}{dx^2} = -\lambda\phi$$

BC with the seperation assumption gives

$$u(0,t) = G(t)\,\phi(0) = 0$$
$$u(L,t) = G(t)\,\phi(L) = 0$$
$$\Rightarrow \phi(0) = \phi(L) = 0$$

The time dependent ODE is readily solved since it is a first order ODE in time;

$$\frac{dG}{dt} = -\lambda k G$$

$$G(t) = Ce^{-k\lambda t}$$

We now have an expression for $G(t)$.

The second ODE is a BVP and is in a class we'll be calling <u>Sturm-Liouville Problems</u>:

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0$$

with $\phi(0) = 0$ and $\phi(L) = 0$

$\phi(x) = 0$ is a trivial solution.

We seek <u>nontrivial solutions</u> to the BVP

From our experience with ODEs, we readily see that we have four cases:

1) $\lambda = 0$ 2) $\lambda < 0$ 3) $\lambda > 0$ 4) λ is complex

We ignore case 4, and later show that Sturm–Liouville problems only have real λ.

Consider Case 1: $\lambda = 0$,

$$\frac{d^2\phi}{dx^2} = 0 \quad \text{with } \phi(0) = 0 \text{ and } \phi(L) = 0$$

General Solution to the BVP is

$$\phi(x) = c_1 x + c_2$$

applying the BC:

$$\phi(0) = c_2 = 0$$

$$\phi(L) = c_1 L = 0$$

or $c_1 = 0$

It follows that when $\lambda = 0$ the

Unique solution to the BVP is the trivial solution $u(x,t)=0$

Consider Case 2 : $\lambda = -\alpha^2 < 0$ with $\alpha > 0$,

$$\frac{d^2\phi}{dx^2} - \alpha^2 \phi = 0$$ with

$\phi(0) = 0$ and

$\phi(L) = 0$

General solution to the BVP is :

$$\phi(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x)$$

Applying the BC :

$$\phi(0) = c_1 = 0$$

$$\phi(L) = c_2 \sinh(\alpha L) = 0$$

Note : $\sinh(x) = 0$ only when $x = 0$

Since $\alpha > 0$ and $L > 0$, this implies that $\sinh(\alpha L) \neq 0$.

Therefore $c_2 = 0$

Therefore, when $\lambda < 0$, the unique solution ~~to~~ the BVP is the trivial solution.

Consider Case 3: $\lambda = \alpha^2 > 0$ with $\alpha > 0$

$$\frac{d^2\phi}{dx^2} + \alpha^2 \phi = 0$$ with $\phi(0) = 0$ and $\phi(L) = 0$

The general solution of the BVP is:

$$\phi(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x)$$

Applying the BC :

$\phi(0) = c_1 = 0$ and

$\phi(L) = c_2 \sin(\alpha L) = 0$

$c_2 = 0$ leads to trivial solutions

$\sin(\alpha L) = 0$ gives non trivial solutions when $\alpha L = n\pi$,

$n = 1, 2, \cdots$ or

$\alpha = \dfrac{n\pi}{L}$ or $\lambda = \alpha^2 = \dfrac{n^2\pi^2}{L^2}$,

$n = 1, 2, \cdots$

Therefore, for $\lambda = \alpha^2 > 0$, the BVP

$$\frac{d^2\phi}{dx^2} + \alpha^2 \phi = 0$$ with

$\phi(0) = \phi(L) = 0$

gives non-trivial solutions:

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right),$$

$n = 1, 2, \cdots$

Called eigenfunctions and the associated eigenvalues

$$\lambda = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, \cdots$$

Recall the t-equation:

$$G(t) = Ce^{-k\lambda t} \text{ becomes}$$

$$G_n(t) = Ce^{-k\frac{n^2\pi^2}{L^2}t}$$

From the separation of variable assumption, we obtain the <u>product solution</u>:

$$u_n(x,t) = G_n(x,t)\,\phi_n(x)$$

$$= B_n e^{-k\frac{n^2\pi^2}{L^2}t} \sin\left(\frac{n\pi x}{L}\right),$$

$$n = 1, 2, \cdots$$

So far, we have satisfied the

B.C. of the PDE. How do we satisfy the IC?

We will show this with some examples.

Example 1

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}$$

B.C.: $u(0,t)=0$ and $u(10,t)=0$

I.C: $u(x,0) = 4\sin\left(\frac{3\pi x}{10}\right)$

Solution

From separation of variables results, we obtain the product solution:

$$u_n(x,t) = B_n e^{-K\frac{n^2\pi^2}{L^2}t}\sin\left(\frac{n\pi x}{L}\right)$$

$n=1,2,\ldots$

$$U_n(x,t) = B_n e^{-\frac{K n^2 \pi^2}{100} t} \sin\left(\frac{n\pi x}{10}\right)$$

$n = 1, 2, \ldots$

Which satisfies the BCs.

By inspection, we solve the IC's by taking $n=3$ and $B_n = 4$

This gives the solution:

$$\boxed{u(x,t) = 4 e^{-\frac{9K\pi^2 t}{100}} \sin\left(\frac{3\pi x}{10}\right)}$$

Example 2: Vary the IC and consider the heat equation

PDE: $\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$

BC: $u(0,t) = 0 = u(5,t)$

IC: $u(x,0) = 3\sin\left(\frac{3\pi x}{5}\right) + 7\sin(\pi x)$

With the principle of superposition, we can add our <u>product solutions</u>:

$$u_3(x,t) + u_5(x,t)$$

By inspection, we satisfy the ICs by taking $B_3 = 3$ and $B_5 = 7$

This gives the solution to this example as:

$$u(x,t) = 3e^{-\frac{9k\pi^2 t}{25}} \sin\left(\frac{3\pi x}{5}\right) + 7e^{-k\pi^2 t} \sin(\pi x)$$

Extended Superposition principle

If $u_1, u_2, u_3, \ldots, u_m$ are solutions of a linear homogeneous PDE, ...

then any linear combination is a solution

i.e $c_1u_1 + c_2u_2 + c_3u_3 + \dots + c_m u_m = \sum_{n=1}^{m} c_n u_n$

it follows that for homogeneous heat problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{with } u(0,t)=0 \text{ and } u(L,t)=0$$

we can write a solution of the form

$$u(x,t) = \sum_{n=1}^{m} B_n e^{-\frac{kn^2\pi^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right)$$

N.B: This solution satisfies any IC, where $u(x,0) = \sum_{n=1}^{M} B_n \sin\left(\frac{n\pi x}{L}\right) \equiv f(x)$

That is, any IC that is a finite sum of sine functions.

Arbitrary ICs

What if $f(x)$ is not a finite linear combination of sine functions?

Solution:

Then, we can use Fourier Series.

Note

- Any function with reasonable restriction can be approximated by a linear combination of $\sin\left(\frac{n\pi x}{L}\right)$

- As $n \to \infty$, with some restrictions the RHS converges to $f(x)$

- It remains to find the constants B_n such that

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Orthogonality of Sine

Assume $m \neq n$ integer

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = 0$$

When $m = n$, then

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{L}{2}$$

Therefore,

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0, & m \neq n \\ \frac{L}{2}, & m = n. \end{cases}$$

Whenever $\int_0^L A(x)B(x)\,dx = 0$

we say the functions $A(x)$ and $B(x)$ are <u>orthogonal</u> over <u>the</u> interval $[0, L]$

finding B_n

Consider the expression

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

multiply both sides by $\sin\left(\frac{m\pi x}{L}\right)$

and integrate over $x = [0, L)$

to use the orthogonality of sine functions:

$$\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx = \int_0^L \left(\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)\right) dx$$

Assuming we can interchange the integration and summation

$$\int_0^L f(x) \sin\left(\frac{m\pi x}{L}\right) dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$$

$$= B_m \frac{L}{2}$$

By orthogonality of sin functions.

Therefore, $B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

<u>Example</u>: Consider the equation:

PDE: $u_t = k\,u_{xx}$, $t>0$, $0<x<L$

BC: $u(0,t)=0$, $u(L,t)=0$, $t>0$

IC: $u(x,0)=100$, $0<x<L$

from before the solution satisfies

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\frac{kn^2\pi^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right)$$

The Fourier coefficients are given by:

$$B_n = \frac{2}{L}\int_0^L 100 \sin\left(\frac{n\pi x}{L}\right) dx$$

$$B_n = \frac{200}{L}\left(\frac{-L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_0^L$$

$$B_n = \frac{200}{n\pi}\left(1 - \cos(n\pi)\right)$$

$$= \begin{cases} \frac{400}{n\pi} & , \ n \text{ odd} \\ 0 & , \ n \text{ even} \end{cases}$$

Thus the solution satisfies:

$$\boxed{u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(1-(-1)^n)}{n} e^{-\frac{k n^2 \pi^2 t}{L^2}} \sin\left(\frac{n\pi x}{L}\right)}$$

Because of the exponential decay term, the solution rapidly approaches $u(x,t) \approx \frac{400}{\pi} e^{-\frac{k\pi^2 t}{L^2}} \sin\left(\frac{\pi x}{L}\right)$