Homework 5
Ordinary Differential Equations
Math 537
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Problem 1: Consider the following second-order ordinary differential equations (ODEs) for nonlinear pendulum oscillations:

$$\frac{d^2\theta}{dt^2} + \epsilon\frac{d\theta}{dt} + \sin(\theta) = 0. \tag{1.1}$$

Applying Taylor series expansions, Eq. (1.1) can be simplified into one of the following systems:

$$\frac{d^2\theta}{dt^2} + \theta = 0. \tag{1.2}$$

$$\frac{d^2\theta}{dt^2} + \epsilon\frac{d\theta}{dt} + \theta = 0. \tag{1.3}$$

$$\frac{d^2\theta}{dt^2} + \left(\theta - \frac{\theta^3}{6}\right) = 0. \tag{1.4}$$

(a) Perform a linear stability analysis in each of Eqs. (1.2)-(1.4).

(1.2) We can find the eigenvalues from solving its characteristic equation.

$$\lambda^2 + 1 = 0$$

Thus our eigenvalues are $\lambda = \pm i$. This means that we get a **Clockwise Center**.

(1.3) We can find the eigenvalues from solving its characteristic equation.

$$\lambda^2 + \epsilon\lambda + 1 = 0$$

Thus our eigenvalues are $\lambda = \frac{1}{2}\left(-\epsilon \pm \sqrt{\epsilon^2 - 4}\right)$.

For $\epsilon < -2$, we get 2 positive real eigenvalues. This means that we get an **Unstable Source**.

For $\epsilon = -2$, we get $\lambda = 1$. This means that we get an **Unstable Source**.

For $-2 < \epsilon < 0$, we get 2 imaginary eigenvalues with positive real parts. This means that we get an **Unstable Spiral**.

For $\epsilon = 0$, we get $\lambda = \pm i$. This means that we get a **Clockwise Center**.

For $0 < \epsilon < 2$, we get 2 imaginary eigenvalues with negative real parts. This means that we get a **Stable Spiral**.

For $2 < \epsilon$, we get 2 negative real eigenvalues. This means that we get a **Stable Sink**.

For $\epsilon = 2$, we get $\lambda = -1$. This means that we get a **Stable Sink**.

1.4 Let $\phi = \theta'$ and $\phi' = \theta\left(\frac{\theta^2}{6} - 1\right)$.

$$X' = \begin{pmatrix}\phi\\ \phi'\end{pmatrix} = \begin{pmatrix}0 & 1\\ \frac{\theta^2}{6} - 1 & 0\end{pmatrix}\begin{pmatrix}\theta\\ \phi\end{pmatrix} = AX$$

Notice we can find the critical points:

$$X' = \begin{pmatrix}0\\ 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ \frac{\theta^2}{6} - 1 & 0\end{pmatrix}\begin{pmatrix}\theta\\ \phi\end{pmatrix} = AX$$

From this we get the following critical points: $(\theta, \phi) = (\pm\sqrt{6}, 0)$ and $(\theta, \phi) = (0, 0)$.

Now notice the Jacobian:

$$J(\theta, \phi) = \begin{bmatrix}\frac{d\phi}{d\theta} & \frac{d\phi}{d\theta'}\\ \frac{d\phi'}{d\theta} & \frac{d\phi'}{d\theta'}\end{bmatrix} = \begin{bmatrix}0 & 1\\ \frac{\theta^2}{2} - 1 & 0\end{bmatrix}$$

Notice the Jacobians and eigenvalues evaluated at the critical points:

$$J(0,0) = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

From solving $|A - \lambda I| = 0$, we get that $\lambda = \pm i$. Thus we get a **Clockwise Center**.

$$J(\pm\sqrt{6}, 0) = \begin{bmatrix}0 & 1\\ 2 & 0\end{bmatrix}$$

From solving $|A - \lambda I| = 0$, we get that $\lambda = \pm\sqrt{2}$. Thus we get a **Saddle Point**.

(b) Discuss the concept of structural stability using results in (1a).

Structural Stability is when the linear stability can withstand small changes in perturbation without changing. A linear flow is structurally stable if and only if it is hyperbolic, meaning not having real parts zero. In (1a):

For equation (1.2), the linear stability was a center so it is **Structurally Unstable**.

For equation (1.3), the linear stability changed a lot for different values of ϵ. It was **Structurally Stable** as long as $\epsilon \neq 0$.

For equation (1.4), the linear stability was **Structurally Stable** except at the origin where it was **Structurally Unstable**

Problem 2: Consider the following system:

$$\frac{d^2x}{dt^2} - \alpha x = e^{\beta t}. \tag{2.1}$$

Complete the following problems with $(\alpha, \beta) = (1, -1)$ and $(\alpha, \beta) = (-1, -1)$

(a) Solve Eqs. (2.1) for the solutions.

(i) Let $(\alpha, \beta) = (1, -1)$. Notice we can solve for the homogeneous solution of Eq (2.1).

We can find the eigenvalues from solving its characteristic equation.

$$\lambda^2 - 1 = 0 \qquad \lambda = \pm 1$$

So we get the homogeneous solution:

$$x_h = c_1 e^{-t} + c_2 e^{t}$$

Using the Method of Undetermined Coefficients, we can try:

$$x_p = Ae^{-t}t, \qquad x_p' = -Ae^{-t}t + Ae^{-t}, \qquad x_p'' = Ae^{-t}t - 2Ae^{-t}$$

Now we can evaluate y_p into Eq (2.1):

$$Ae^{-t}t - 2Ae^{-t} - Ae^{-t}t = -2Ae^{-t} = e^{-t}$$

Thus we get that

$$A = \frac{-1}{2}$$

This gives us the solution:

$$\boldsymbol{x = c_1 e^{-t} + c_2 e^{t} - \frac{1}{2} t e^{-t}}$$

(ii) Let $(\alpha, \beta) = (-1, -1)$. Notice we can solve for the homogeneous solution of Eq (2.1).

We can find the eigenvalues from solving its characteristic equation.

$$\lambda^2 + 1 = 0 \qquad \lambda = \pm i$$

So we get the homogeneous solution:

$$x_h = c_1 \cos t + c_2 \sin t$$

Using the Method of Undetermined Coefficients, we can try:

$$x_p = Ae^{-t}, \qquad x_p' = -Ae^{-t}, \qquad x_p'' = Ae^{-t}$$

Now we can evaluate y_p into Eq (2.1):

$$Ae^{-t} + Ae^{-t} = 2Ae^{-t} = e^{-t}$$

Thus we get that

$$A = \frac{1}{2}$$

This gives us the solution:

$$\boldsymbol{x = c_1 \cos t + c_2 \sin t + \frac{1}{2} e^{-t}}$$

(b) Convert Eqs. (2.1) into an autonomous linear system which consists of three first-order differential equations.

Notice we can rewrite the equation as the following:

$$\frac{d^2x}{dt^2} = \alpha x + e^{\beta t}$$

Let $u = x'$, $v = e^{\beta t}$ and $w = v' = \beta e^{\beta t}$. From this we get:

$$\frac{dx}{dt} = u \qquad \frac{du}{dt} = \alpha x + v \qquad \frac{dv}{dt} = \beta v$$

Using this we get the system:

$$\boldsymbol{X'} = \begin{pmatrix} \boldsymbol{x'} \\ \boldsymbol{u'} \\ \boldsymbol{v'} \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{1} & \boldsymbol{0} \\ \boldsymbol{\alpha} & \boldsymbol{0} & \boldsymbol{1} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{\beta} \end{pmatrix} \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix} = \boldsymbol{AX}$$

(c) Solve for the eigenvalues and eigenvectors of the autonomous systems in (2b).

Notice we can find the eigenvalues from solving $|A - \lambda I| = 0$

$$\begin{aligned} \lambda^2(\beta - \lambda) - \alpha(\beta - \lambda) &= 0 \\ -\lambda^3 + \beta\lambda^2 + \alpha\lambda - \alpha\beta &= 0 \\ \lambda^3 - \beta\lambda^2 - \alpha\lambda + \alpha\beta &= 0 \end{aligned} \qquad \begin{aligned} \lambda^2(\lambda - \beta) - \alpha(\lambda - \beta) &= 0 \\ (\lambda^2 - \alpha)(\lambda - \beta) &= 0 \\ \lambda &= \pm\sqrt{\alpha}, \beta \end{aligned}$$

For $(\alpha, \beta) = (1, -1)$, we get $\lambda = 1$ and $\lambda = -1$ with multiplicity 2.

Notice for $\lambda_1 = 1$, we get the eigenvector: $v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Notice for $\lambda_2 = -1$, we get the eigenvector: $v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

For $\lambda_3 = -1$, we can solve for v_3 by solving $|A - \lambda_3|v_3 = v_2$, such that $v_3 = \begin{pmatrix} 1/2 \\ 1/2 \\ -2 \end{pmatrix}$

For $(\alpha, \beta) = (-1, -1)$, we get $\lambda = \pm i$ and $\lambda = -1$.

Notice for $\lambda_1 = i$, we get the eigenvector: $v_1 = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$.

Notice for $\lambda_2 = -i$, we get the eigenvector, v_2, is just the conjugate of v_1, such that : $v_2 = \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}$.

For $\lambda_3 = -1$, we get the eigenvector: $v_3 = \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix}$

(d) Compare the results in (2a) and (2c)

For $(\alpha, \beta) = (1, -1)$, we get the following solution:

$$\begin{pmatrix} x \\ u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-t} + c_3 \begin{pmatrix} 1/2 \\ 1/2 \\ -2 \end{pmatrix} te^{-t}$$

Notice the solution for x:

$$\boldsymbol{x = c_1 e^t + c_2 e^{-t} + c_3 \frac{1}{2} t e^t}$$

This is the same as part (a) except this has an extra constant c_3.

For $(\alpha, \beta) = (-1, -1)$, we get the following solution:

$$\begin{pmatrix} x \\ u \\ v \end{pmatrix} = c_1 \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix} e^{-t}$$

Notice the solution for x:

$$\boldsymbol{x = -c_1 \sin t + c_2 \cos t + c_3 \frac{1}{2} e^{-t}}$$

This is the same as part (a) except this has an extra constant c_3.

Problem 3: Consider the following coupled harmonic oscillator (as shown in Fig. 1):

$$\frac{d^2x_1}{dt^2} = -k_1x_1 + k_2(x_2 - x_1), \tag{3.1}$$

$$\frac{d^2x_2}{dt^2} = -k_2(x_2 - x_1). \tag{3.2}$$

Let $k_1 = 4X_c^2$ and $k_2 = X_c^2$ (and $m_1 = m_2 = 1$).

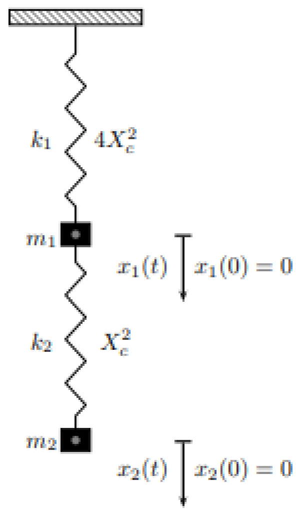

Figure 1: Coupled spring/mass system

(a) Convert the above equations into a linear system with first order differential equations.

Let $y_1 = x_1'$ and $y_2 = x_2'$.

$$\boldsymbol{X'} = \begin{pmatrix} \boldsymbol{y_1} \\ \boldsymbol{y_1'} \\ \boldsymbol{y_2} \\ \boldsymbol{y_2'} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{-5X_c^2} & \mathbf{0} & \boldsymbol{X_c^2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \boldsymbol{X_c^2} & \mathbf{0} & \boldsymbol{-X_c^2} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{x_1} \\ \boldsymbol{y_1} \\ \boldsymbol{x_2} \\ \boldsymbol{y_2} \end{pmatrix} = \boldsymbol{AX}$$

(b) Find the eigenvalues and eigenvectors

Notice we can find the eigenvalues from solving $|A - \lambda I| = 0$

$$\begin{aligned} \lambda^2(\lambda^2 + X_c^2) + (5X_c^2)(\lambda^2 + X_c^2) - X_c^4 &= 0 \\ (\lambda^2 + 5X_c^2)(\lambda^2 + X_c^2) - X_c^4 &= 0 \\ \lambda^4 + 6\lambda^2 X_c^2 + 4X_c^4 &= 0 \end{aligned}$$

From this we get $\lambda_{1,2} = \pm X_c\, i\, \sqrt{3 - \sqrt{5}}$ and $\lambda_{3,4} = \pm X_c\, i\, \sqrt{3 + \sqrt{5}}$

Notice the eigenvector for $\lambda_1 = X_c\, i\sqrt{3-\sqrt{5}}$:

$$\begin{pmatrix} -X_c\, i\sqrt{3-\sqrt{5}} & 1 & 0 & 0 \\ -5X_c^2 & -X_c\, i\sqrt{3-\sqrt{5}} & X_c^2 & 0 \\ 0 & 0 & -X_c\, i\sqrt{3-\sqrt{5}} & 1 \\ X_c^2 & 0 & -X_c^2 & -X_c\, i\sqrt{3-\sqrt{5}} \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \\ v_{14} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= rref \begin{pmatrix} -X_c\, i\sqrt{3-\sqrt{5}} & 1 & 0 & 0 & 0 \\ -5X_c^2 & -X_c\, i\sqrt{3-\sqrt{5}} & X_c^2 & 0 & 0 \\ 0 & 0 & -X_c\, i\sqrt{3-\sqrt{5}} & 1 & 0 \\ X_c^2 & 0 & -X_c^2 & -X_c\, i\sqrt{3-\sqrt{5}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & \frac{\sqrt{2}\, i}{4X_c}\left(3-\sqrt{5}\right) & 0 \\ 0 & 1 & 0 & 2-\sqrt{5} & 0 \\ 0 & 0 & 1 & \frac{\sqrt{2}\, i}{4X_c}\left(1+\sqrt{5}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From here, we can let $v_{14} = 1$, such that we get the eigenvector:

$$v_1 = \begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \\ v_{14} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}\, i}{4X_c}\left(3-\sqrt{5}\right) \\ \sqrt{5}-2 \\ -\frac{\sqrt{2}\, i}{4X_c}\left(1+\sqrt{5}\right) \\ 1 \end{pmatrix}$$

We can now see that $\lambda_2 = -X_c\, i\sqrt{3-\sqrt{5}}$ is the imaginary conjugate of λ_1 so we get its eigenvector v_2 is the imaginary conjugate of v_1:

$$v_2 = \begin{pmatrix} v_{21} \\ v_{22} \\ v_{23} \\ v_{24} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}\, i}{4X_c}\left(3-\sqrt{5}\right) \\ \sqrt{5}-2 \\ \frac{\sqrt{2}\, i}{4X_c}\left(1+\sqrt{5}\right) \\ 1 \end{pmatrix}$$

Notice the eigenvector for $\lambda_3 = X_c\, i\, \sqrt{3+\sqrt{5}}$:

$$\begin{pmatrix} -X_c\, i\, \sqrt{3+\sqrt{5}} & 1 & 0 & 0 \\ -5X_c^2 & -X_c\, i\, \sqrt{3+\sqrt{5}} & X_c^2 & 0 \\ 0 & 0 & -X_c\, i\, \sqrt{3+\sqrt{5}} & 1 \\ X_c^2 & 0 & -X_c^2 & -X_c\, i\, \sqrt{3+\sqrt{5}} \end{pmatrix} \begin{pmatrix} v_{31} \\ v_{32} \\ v_{33} \\ v_{34} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= rref \begin{pmatrix} -X_c\, i\, \sqrt{3+\sqrt{5}} & 1 & 0 & 0 & 0 \\ -5X_c^2 & -X_c\, i\, \sqrt{3+\sqrt{5}} & X_c^2 & 0 & 0 \\ 0 & 0 & -X_c\, i\, \sqrt{3+\sqrt{5}} & 1 & 0 \\ X_c^2 & 0 & -X_c^2 & -X_c\, i\, \sqrt{3+\sqrt{5}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & -\frac{\sqrt{2}\, i}{4X_c}\left(3+\sqrt{5}\right) & 0 \\ 0 & 1 & 0 & 2+\sqrt{5} & 0 \\ 0 & 0 & 1 & -\frac{\sqrt{2}\, i}{4X_c}\left(1-\sqrt{5}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From here, we can let $v_{34} = 1$, such that we get the eigenvector:

$$v_3 = \begin{pmatrix} v_{31} \\ v_{32} \\ v_{33} \\ v_{34} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}\, i}{4X_c}\left(3+\sqrt{5}\right) \\ -\left(2+\sqrt{5}\right) \\ \frac{\sqrt{2}\, i}{4X_c}\left(1-\sqrt{5}\right) \\ 1 \end{pmatrix}$$

We can now see that $\lambda_4 = -X_c\, i\, \sqrt{3+\sqrt{5}}$ is the imaginary conjugate of λ_3 so we get its eigenvector v_4 is the imaginary conjugate of v_3:

$$v_4 = \begin{pmatrix} v_{41} \\ v_{42} \\ v_{43} \\ v_{44} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}\, i}{4X_c}\left(3+\sqrt{5}\right) \\ -\left(2+\sqrt{5}\right) \\ -\frac{\sqrt{2}\, i}{4X_c}\left(1-\sqrt{5}\right) \\ 1 \end{pmatrix}$$

(c) Find the general solutions.

$$\begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} \frac{\sqrt{2}}{4X_c}\left(3-\sqrt{5}\right)\sin\left(X_c\sqrt{3-\sqrt{5}}\,t\right) \\ \left(\sqrt{5}-2\right)\cos\left(X_c\sqrt{3-\sqrt{5}}\,t\right) \\ \frac{\sqrt{2}}{4X_c}\left(1+\sqrt{5}\right)\sin\left(X_c\sqrt{3-\sqrt{5}}\,t\right) \\ \cos\left(X_c\sqrt{3-\sqrt{5}}\,t\right) \end{pmatrix} + c_2 \begin{pmatrix} -\frac{\sqrt{2}}{4X_c}\left(3-\sqrt{5}\right)\cos\left(X_c\sqrt{3-\sqrt{5}}\,t\right) \\ \left(\sqrt{5}-2\right)\sin\left(X_c\sqrt{3-\sqrt{5}}\,t\right) \\ -\frac{\sqrt{2}}{4X_c}\left(1+\sqrt{5}\right)\cos\left(X_c\sqrt{3-\sqrt{5}}\,t\right) \\ \sin\left(X_c\sqrt{3-\sqrt{5}}\,t\right) \end{pmatrix}$$

$$+c_3 \begin{pmatrix} -\frac{\sqrt{2}}{4X_c}\left(3+\sqrt{5}\right)\sin\left(X_c\sqrt{3+\sqrt{5}}\,t\right) \\ -\left(2+\sqrt{5}\right)\cos\left(X_c\sqrt{3+\sqrt{5}}\,t\right) \\ -\frac{\sqrt{2}}{4X_c}\left(1-\sqrt{5}\right)\sin\left(X_c\sqrt{3+\sqrt{5}}\,t\right) \\ \cos\left(X_c\sqrt{3+\sqrt{5}}\,t\right) \end{pmatrix} + c_4 \begin{pmatrix} \frac{\sqrt{2}}{4X_c}\left(3+\sqrt{5}\right)\cos\left(X_c\sqrt{3+\sqrt{5}}\,t\right) \\ -\left(2+\sqrt{5}\right)\sin\left(X_c\sqrt{3+\sqrt{5}}\,t\right) \\ \frac{\sqrt{2}}{4X_c}\left(1-\sqrt{5}\right)\cos\left(X_c\sqrt{3+\sqrt{5}}\,t\right) \\ \sin\left(X_c\sqrt{3+\sqrt{5}}\,t\right) \end{pmatrix}$$