MATH 537, Fall 2020
Ordinary Differential Equations

Lecture #3

Chapter 1 First Order Equations
Constant & Periodic Harvesting and Bifurcations

Instructor: Dr. Bo-Wen Shen*

Department of Mathematics and Statistics
San Diego State University

Review of Important Concepts

1. Bifurcation;
2. Critical points, $f(x_c) = 0$;
3. (equilibrium points = fixed points = critical points)
4. Derivative tests
5. General solution
6. Initial Value Problem (IVP)
7. Particular solution
8. Phase Line
9. Separable ODEs
10. Sink vs. Source
11. Stable vs. Unstable Solutions, $f'(x_c)$.
12. Structurally Stable vs. Unstable (i.e., with bifurcation)

Sect. 1.2: the Logistic Equation

$$\boxed{x' = ax}$$

- linear population model if $a > 0$
- x: population (i.e., assume $\mathrm{x} > 0$).
- $\frac{\mathrm{dx}}{\mathrm{dt}}$: the rate of growth of the population, (called a growth rate, or an exponential growth rate)
- $\frac{\mathrm{dx}}{\mathrm{dt}}$ is proportional to x

$$\boxed{x' = ax\left(1 - \frac{\mathrm{x}}{\mathrm{N}}\right)}$$

- $\frac{\mathrm{dx}}{\mathrm{dt}}$ is proportional to x for small x (and $\mathrm{x} < N$).
- $\frac{\mathrm{dx}}{\mathrm{dt}}$ becomes negative for large x (i.e., $\mathrm{x} > N$).
- N is called carrying capacity.

We choose $\mathrm{N} = 1$ (see Quiz II)

$$\boxed{x' = ax\,(1 - x)}$$

$$\boxed{\equiv f_a(x)}$$

- first order, nonlinear, separable
- autonomous, ($f(x) = ax\,(1 - x)$ is not an explicit function of time).

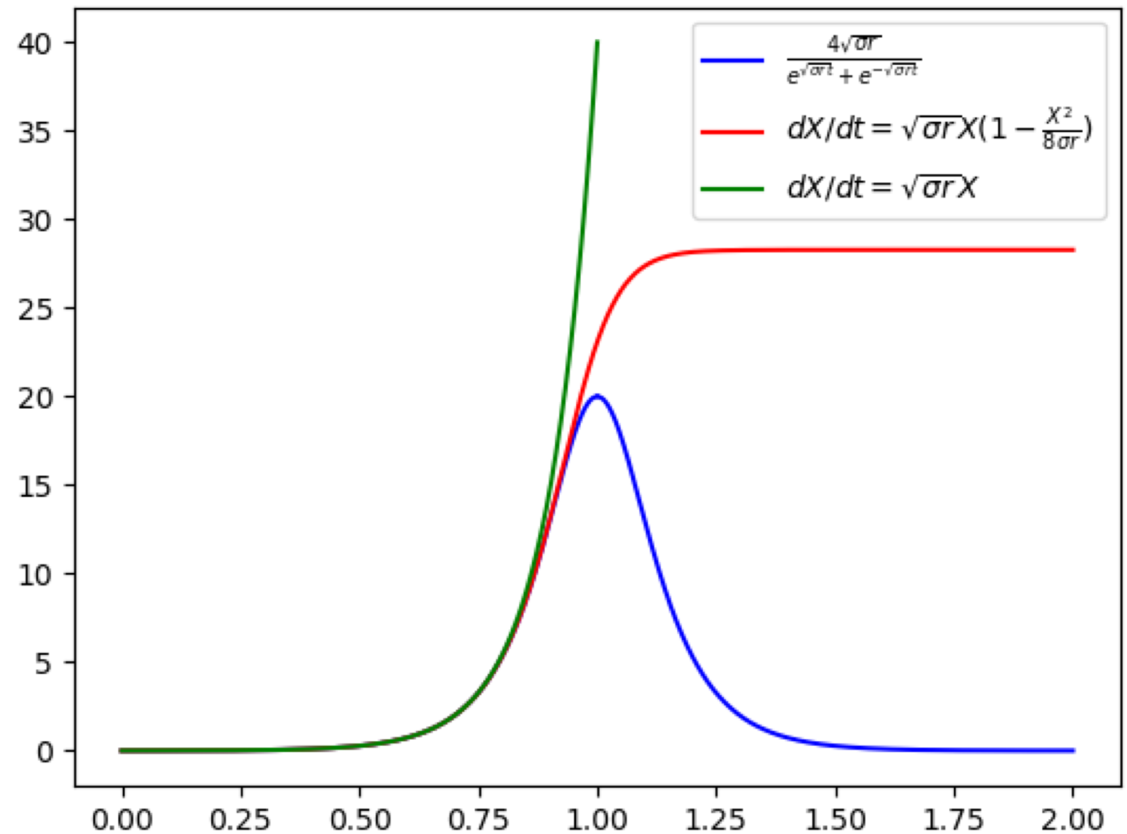
The Role of Nonlinearity

$$\boxed{x' = ax}$$

$$\boxed{x' = ax\,(1 - x)}$$

The role of nonlinearity

→ suppress growth



A Saddle Point

$$x' = x^2 = f(x)$$

Is the critical point a sink or source?

critical points $\boxed{f(x) = 0}$

$$\boxed{x = 0}$$

1st derivative $f'(x) = 2x$

$x = 0$ $\quad f'(0) = 0\ ?$

A saddle point or a half-stable critical point (e.g., Strogatz)

Based on the definition, we can obtain:

$x < 0$	$x > 0$
$\frac{dx}{dt} > 0$	$\frac{dx}{dt} > 0$
positive direction	positive direction

$\mathrm{x} = 0$

A saddle point

Apply a perturbation method

$$x = x_c + \varepsilon(t)$$

$$\boxed{x = 0 + \varepsilon} \qquad \varepsilon' = \varepsilon^2$$

The above discussions can help determine $x = 1$ is a saddle point within $x' = (x-1)^2$.

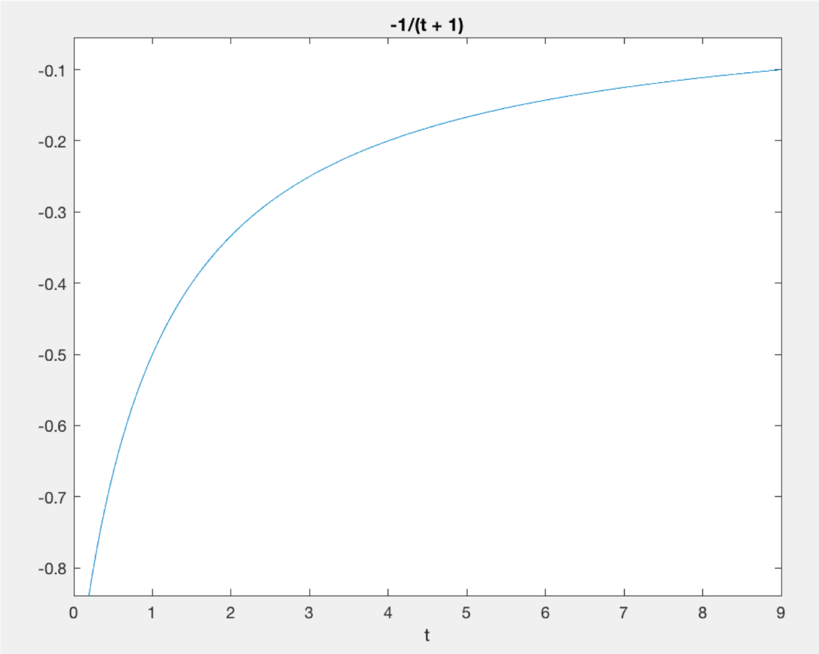
A Saddle Point

```
syms t x0
x0=-1
fun=-x0/(x0*t-1)
ezplot (fun, [0, 9])
```

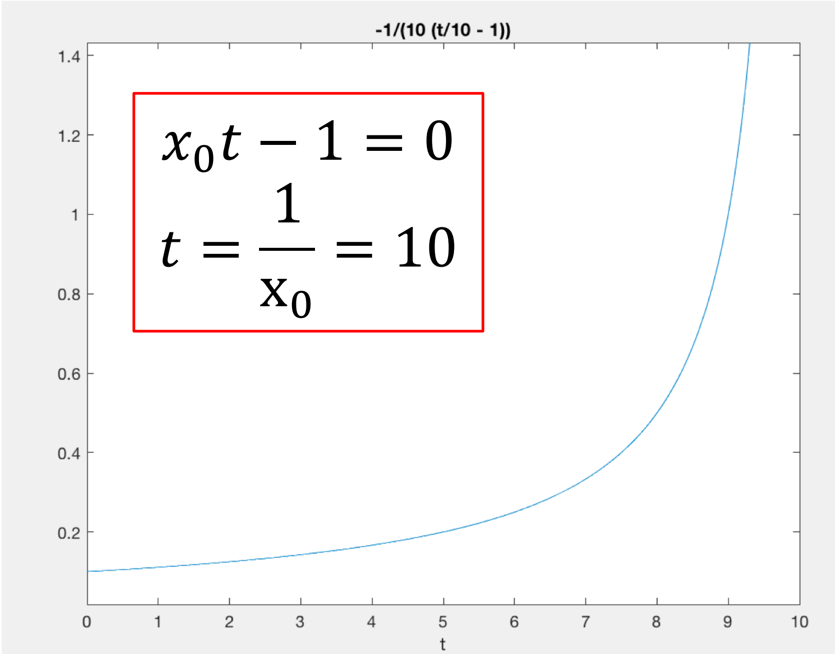
$x' = x^2$

$$\boxed{x = \frac{-x_0}{x_0 t - 1}}$$

```
syms t x0
x0=0.1
fun=-x0/(x0*t-1)
ezplot (fun, [0, 10])
```



$x_0 < x_c = 0,$ move toward the critical point



$x_0 > x_c = 0,$ move away from the critical point (x_0 is close to x_c)

Additional Notes

Supp

The Logistic equation includes the following features:

- For $\mathrm{a} > 0$ the basin of attraction of $x_c = 1$ is $x > 0$, while negative values of x attract to minus infinity.
- In contrast to the logistic map (i.e., difference equation), the logistic equation has no oscillatory nor chaotic solutions.

When both f and f' are zero at the critical point,

- the stability is determined by the sign of the first non-vanishing higher derivatives;
- If that derivative is even (e.g., f''), the point is a saddle point, attracting on one side but repelling on the other.
- If that derivative is odd, it follows the same sign rules as f'.

Bifurcation vs. Saddle Points:

- The former appears in associated with the changes of parameter(s).
- There are various kinds of Bifurcation

Sprott (2003)

Review of Horizontal and Vertical Asymptotes

$$y = f(x)$$

Asymptotes

(i) *Horizontal Asymptotes.* Recall from Section 2.6 that if either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$. If it turns out that $\lim_{x\to\infty} f(x) = \infty$ (or $-\infty$), then we do not have an asymptote to the right, but that is still useful information for sketching the curve.

$y = L$ is a horizontal asymptote if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$

(ii) *Vertical Asymptotes.* Recall from Section 2.2 that the line $x = a$ is a vertical asymptote if at least one of the following statements is true:

1

$$\lim_{x\to a^+} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty$$

$$\lim_{x\to a^+} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty$$

Review of Horizontal and Vertical Asymptotes

V EXAMPLE 1 Use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

$y = 2$ is a horizontal asymptote

$\lim_{x\to\infty} f(x) = 2$ or $\lim\limits_{x\to-\infty} f(x) = 2$

$x = 1$ and $x = -1$ are vertical asymptotes

$(the\ denonimator\ (x^2-1)\ is\ zero)$

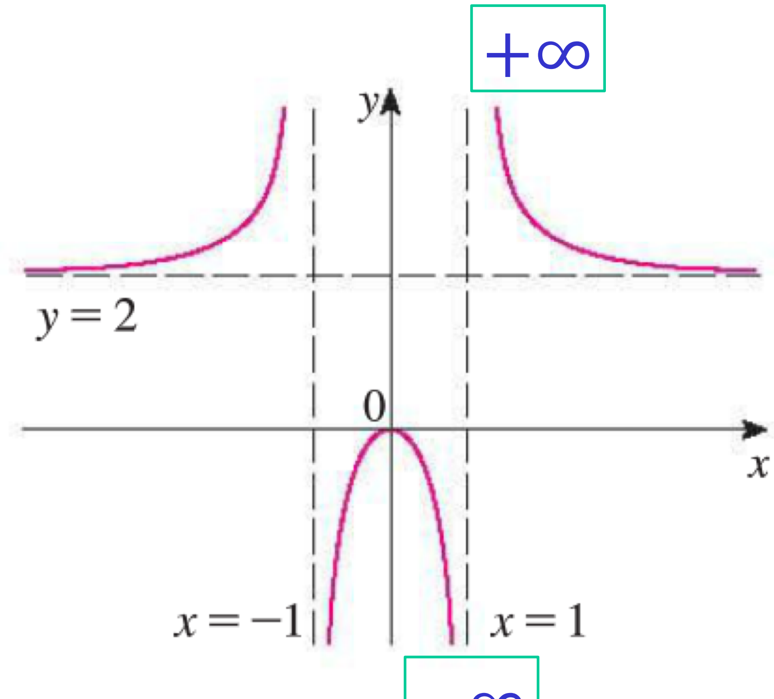


FIGURE 6
Finished sketch of $y = \dfrac{2x^2}{x^2 - 1}$

Horizontal and Vertical Asymptotes

V **EXAMPLE 1** Use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

A. The domain is

$$\{x \mid x^2 - 1 \neq 0\} = \{x \mid x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$$

B. The x- and y-intercepts are both 0.

C. Since $f(-x) = f(x)$, the function f is even. The curve is symmetric about the y-axis.

D.
$$\lim_{x \to \pm\infty} \frac{2x^2}{x^2 - 1} = \lim_{x \to \pm\infty} \frac{2}{1 - 1/x^2} = 2$$

Therefore the line $y = 2$ is a horizontal asymptote.

Since the denominator is 0 when $x = \pm 1$, we compute the following limits:

$$\lim_{x \to 1^+} \frac{2x^2}{x^2 - 1} = \infty \qquad \lim_{x \to 1^-} \frac{2x^2}{x^2 - 1} = -\infty$$

$$\lim_{x \to -1^+} \frac{2x^2}{x^2 - 1} = -\infty \qquad \lim_{x \to -1^-} \frac{2x^2}{x^2 - 1} = \infty$$

Therefore the lines $x = 1$ and $x = -1$ are vertical asymptotes. This information about limits and asymptotes enables us to draw the preliminary sketch in Figure 5, showing the parts of the curve near the asymptotes.

Review of Slant Asymptotes

$$y = f(x)$$

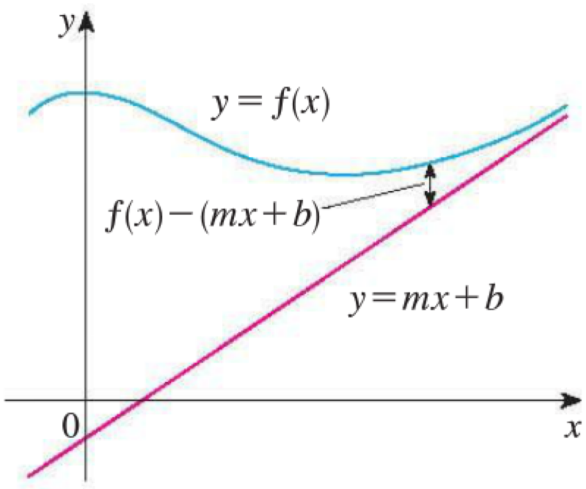
Slant Asymptotes

Some curves have asymptotes that are *oblique*, that is, neither horizontal nor vertical. If

$$\boxed{\lim_{x \to \infty} [f(x) - (mx + b)] = 0}$$

then the line $y = mx + b$ is called a **slant asymptote** because the vertical distance

between the curve $y = f(x)$ and the line $y = mx + b$ approaches 0, as in Figure 12. (A similar situation exists if we let $x \to -\infty$.) For rational functions, slant asymptotes occur when the degree of the numerator is one more than the degree of the denominator. In such a case the equation of the slant asymptote can be found by long division as in the following example.



An Example for a Slant Asymptote

EXAMPLE 6 Sketch the graph of $f(x) = \dfrac{x^3}{x^2 + 1}$.

The domain is $\mathbb{R} = (-\infty, \infty)$.
The x- and y-intercepts are both 0.
Since $f(-x) = -f(x)$, f is odd and its graph is symmetric about the origin.
Since $\boxed{x^2 + 1}$ is never 0, there is no vertical asymptote. Since $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$, there is no horizontal asymptote. But long division gives

denominator

$$f(x) = \frac{x^3}{x^2 + 1} = \boxed{x} - \frac{x}{x^2 + 1}$$

$$f(x) - x = -\frac{x}{x^2 + 1} = -\frac{\dfrac{1}{x}}{1 + \dfrac{1}{x^2}} \to 0 \quad \text{as} \quad x \to \pm\infty$$

So the line $\boxed{y = x}$ is a slant asymptote.

1.3: Constant Harvesting and Bifurcations

$$x' = x(1-x) - h = f(x,h)$$

$$f(x,h) = 0 \; \&$$
$$f_x(x,h) = 0$$

$f(x,h)$

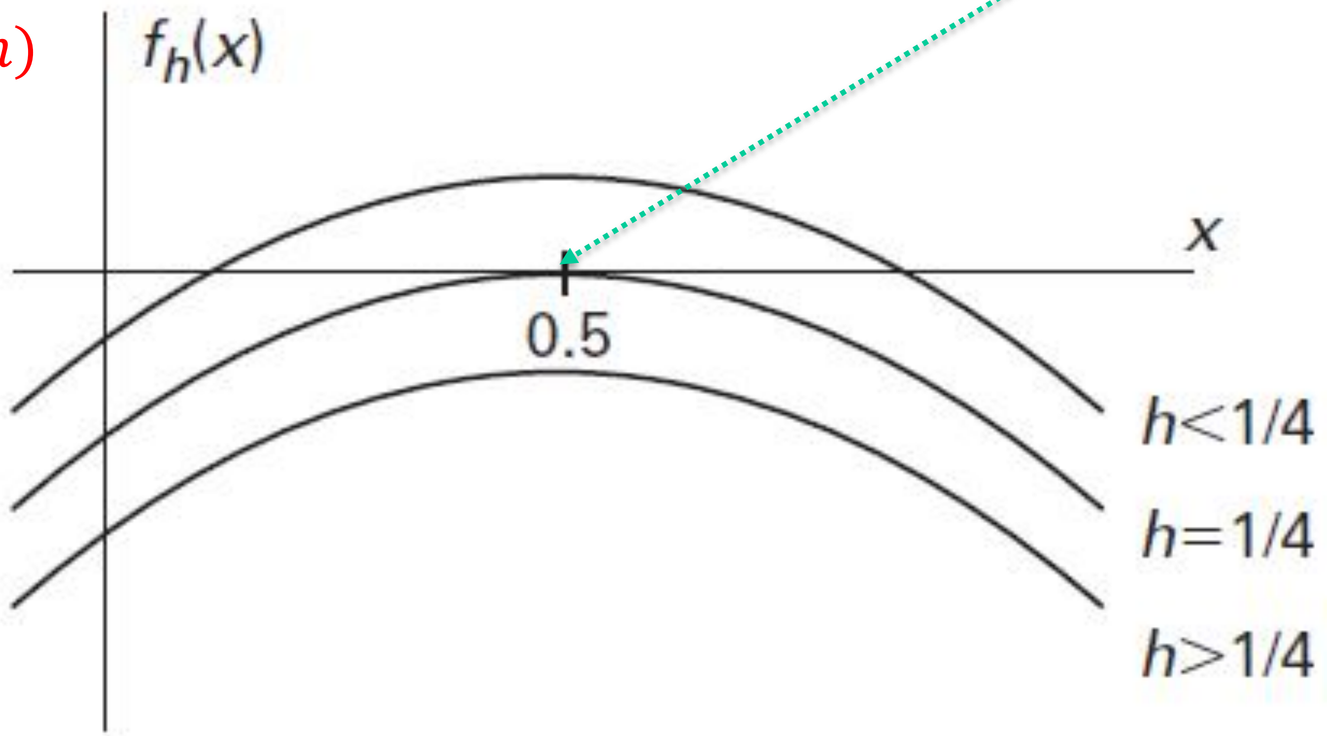


Figure 1.6 The graphs of the function $f_h(x) = x(1-x) - h$.

Bifurcations

- A Bifurcation occurs when there is a “significant” change in the structure of the solutions of the system as system parameter “a” varies.
- The simplest types of bifurcations occur when the number of equilibrium solutions changes as “a” varies.

Bifurcation Point(s)

$$\boxed{x' = x(1-x) - h = f(x,h)}$$

bifurcation points

$$\boxed{f(x,h) = 0 \ \& \ f_x(x,h) = 0}$$

$f(x,h) = 0$ $\quad \boxed{x(1-x) - h = 0}$ $\quad \boxed{h = 1/4}$

$$\boxed{x_c = \frac{1 \pm \sqrt{1-4h}}{2}}$$

$f_x(x,h) = 0$ $\quad \boxed{1 - 2x = 0}$ $\quad \boxed{x = \frac{1}{2}}$

1.3: Stability Analysis

$$\boxed{x' = x(1-x) - h = f(x,h)}$$

$$\boxed{x_c = \frac{1 \pm \sqrt{1-4h}}{2}}$$

$$\frac{dx}{dt} = f'(x_c)(\mathrm{x} - \mathrm{x_c}) + \cdots$$

$h > \frac{1}{4}, \rightarrow$ *no critical points* because of $f(x,h) \neq 0$

$$x' = -x^2 + x - h = -\left(x - \frac{1}{2}\right)^2 + \frac{1}{4} - h < 0$$

$h = \frac{1}{4}, \rightarrow$ *critical point* $x_c = 1/2$

$$x' = -\left(x - \frac{1}{2}\right)^2 < 0$$

a saddle at $x_c = 1/2$

$h < \frac{1}{4}, \rightarrow$ *two critical points*, $x_{c1,2} = \dfrac{1 \pm \sqrt{1-4h}}{2}$

$$\boxed{f_x(x) = -2\mathrm{x} + 1}$$

$f_x(x_{c1}) < 0$ stable a sink

$f_x(x_{c2}) > 0$ unstable a source

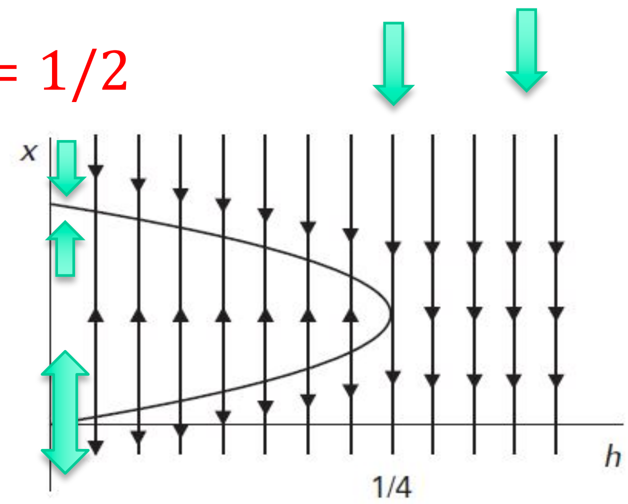


Figure 1.7 The bifurcation diagram for $f_h(x) = x(1-x) - h$.

Section 1.3: Constant Harvesting and Bifurcations

$$\frac{dx}{dt} = x(1 - x) - h$$

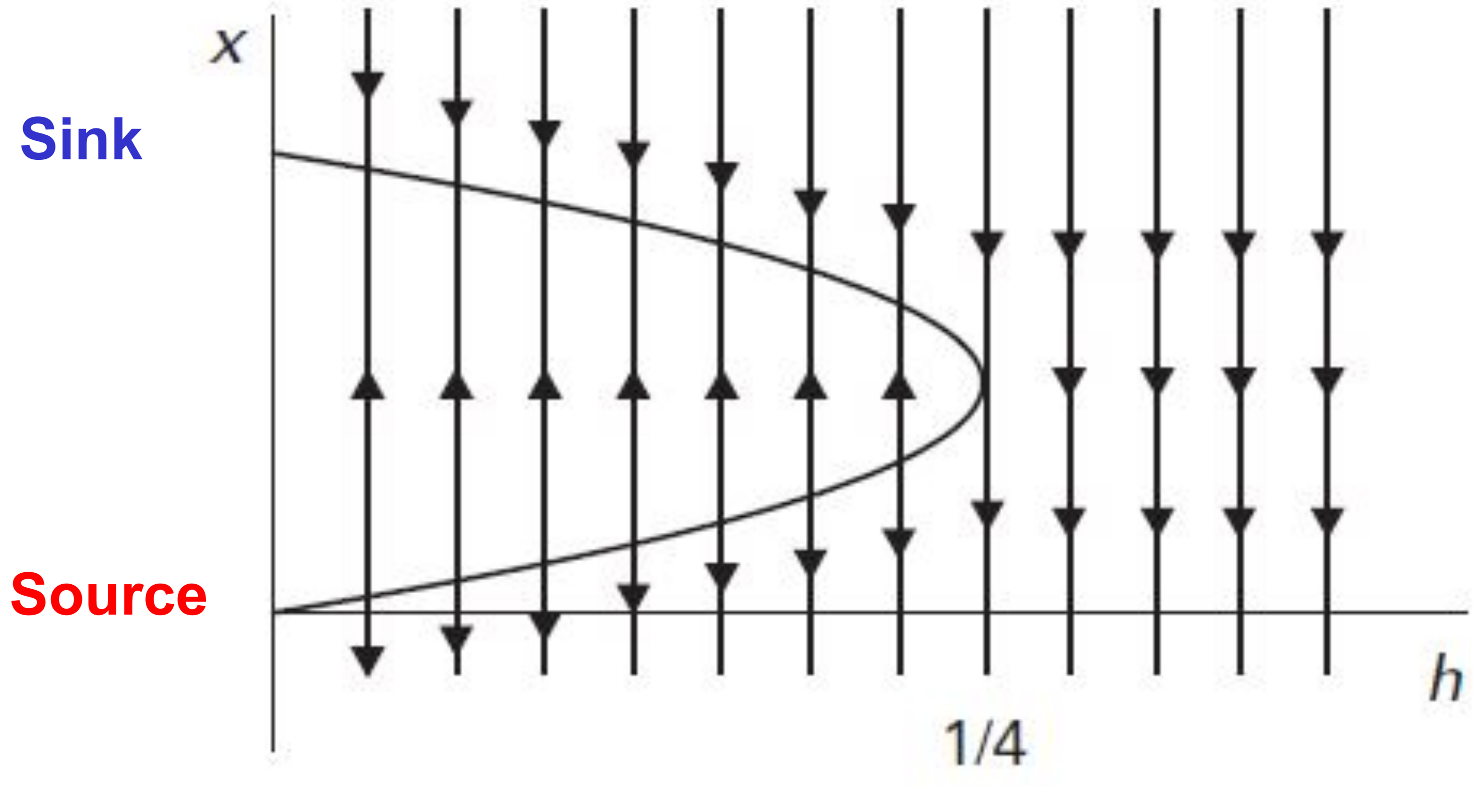


Figure 1.7 The bifurcation diagram for $f_h(x) = x(1 - x) - h$.

1.3 Another Example of a Bifurcation

$$\boxed{x' = x^2 - ax}$$

bifurcation points

$$\boxed{f(x,a) = 0 \;\&\; f_x(x,a) = 0}$$

$$f(x,a) = 0 \qquad \boxed{x^2 - ax = 0} \qquad \boxed{a = 0}$$

$$f_x(x,a) = 0 \qquad \boxed{2x - a = 0} \qquad \boxed{x = \frac{a}{2}}$$

1.3 Another Example of a Bifurcation

$\boxed{x' = x^2 - ax = f(x, a)}$ $\boxed{f_x = 2x - a}$

bifurcation points $f(x, a) = 0$ & $f_x(x, a) = 0$

$\boxed{a = 0}$ $\boxed{x = 0}$

$critical\ points\ x_{c1,2} = a, or\ 0$

$a > 0$:

$x_{c1} = a$: $f_x(a) > 0, source$

$x_{c2} = 0$: $f_x(0) < 0, sink$

$a < 0$:

$x_{c1} = a$: $f_x(a) < 0, sink$

$x_{c2} = 0$: $f_x(0) > 0, source$

$a = 0$: $f_x(0) = 0 \Rightarrow x' = x^2 \geq 0$

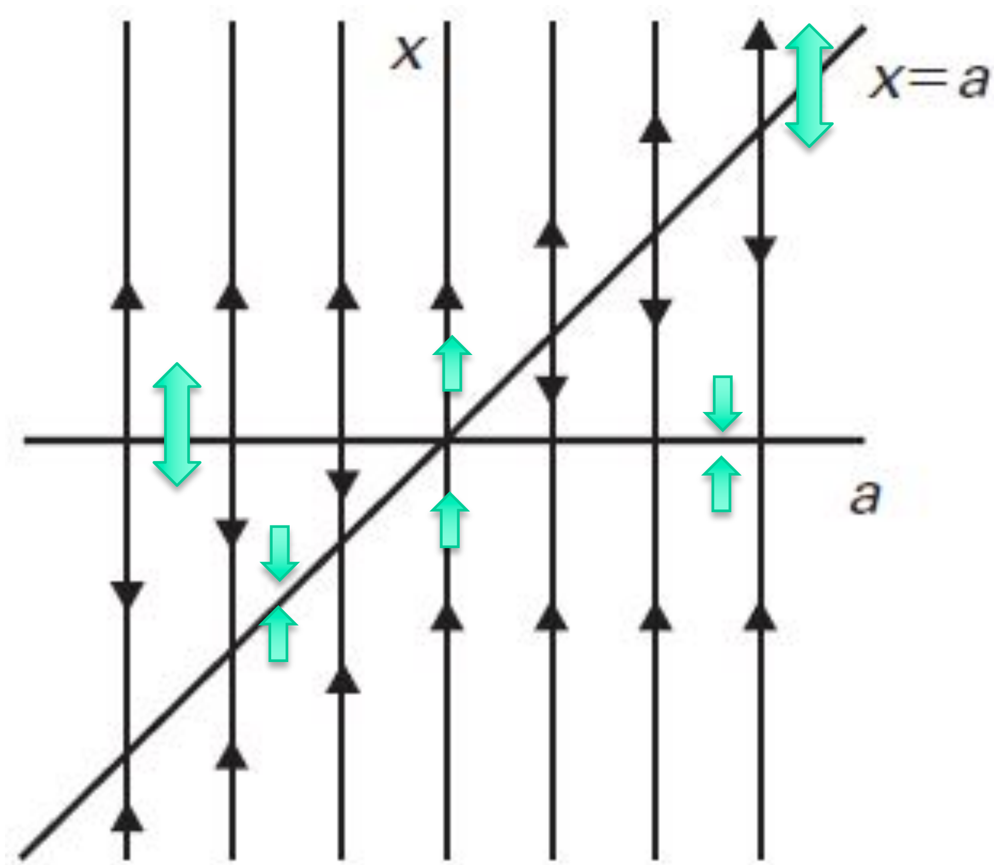
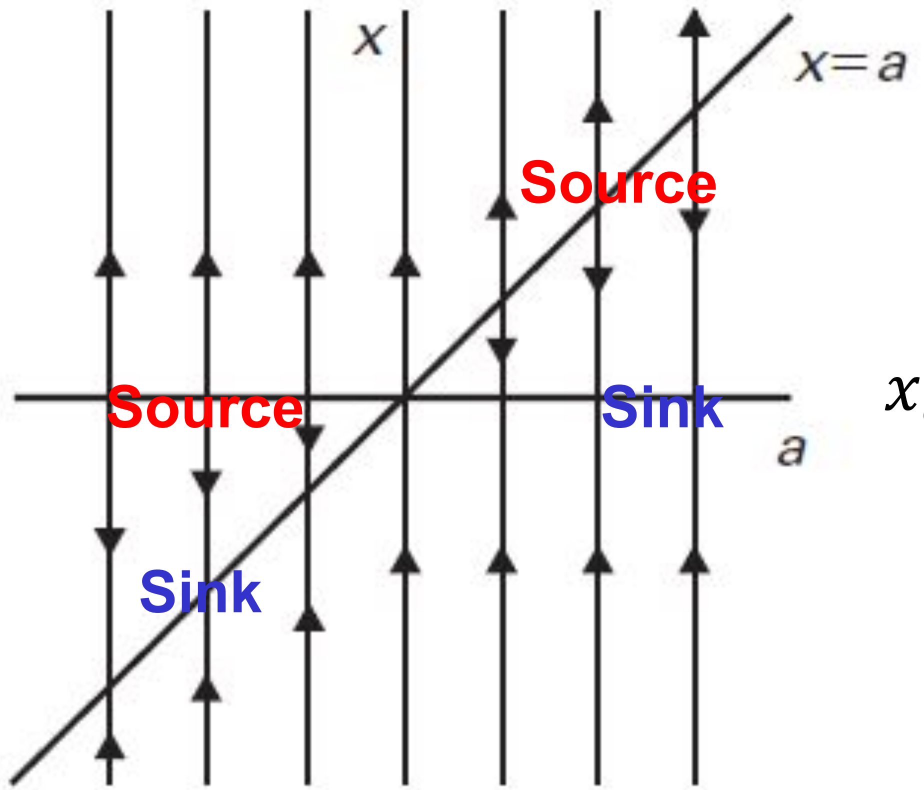


Figure 1.8 The bifurcation diagram for $x' = x^2 - ax$.

Section 1.3: Constant Harvesting and Bifurcations

$$\frac{dx}{dt} = x(x - a)$$

$x_c = a$



$x_c = 0$

Figure 1.8 The bifurcation diagram for $x' = x^2 - ax$.

1.4: Periodic Harvesting and Periodic Solutions

$x' = x(1-x) - h(1 + \sin(2\pi t))$ non-autonomous system

$g(t) = h(1 + \sin(2\pi t))$

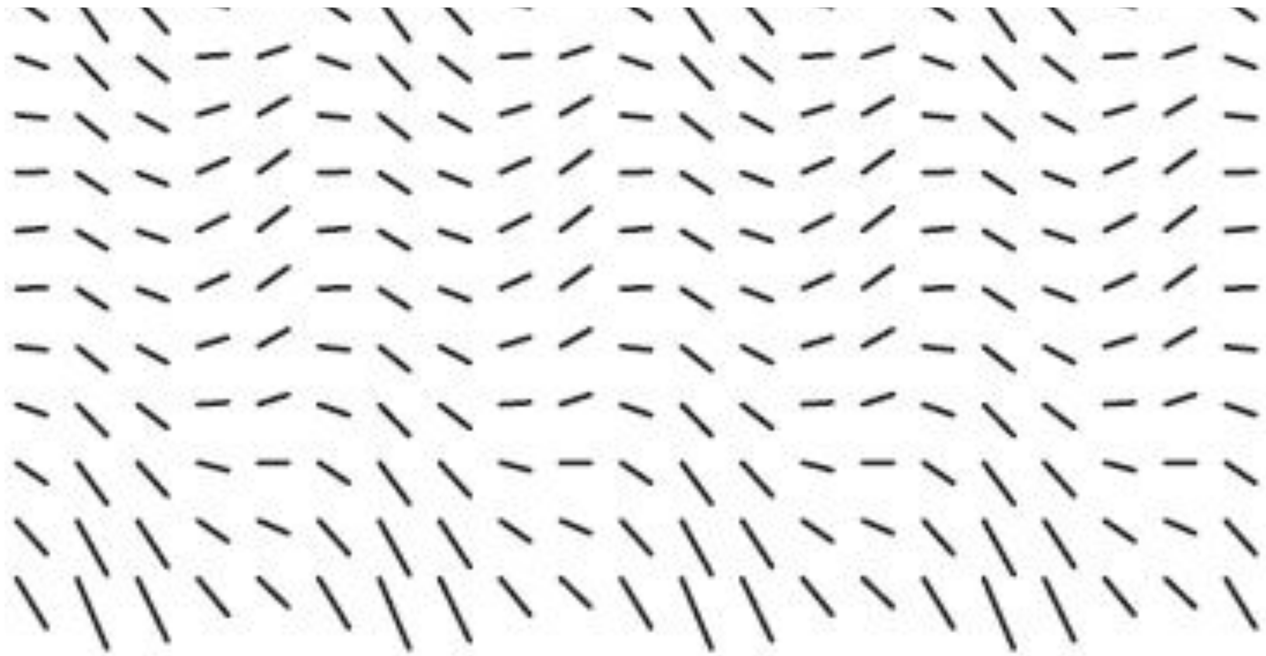


Figure 1.9 The slope field for $f(x) = x(1-x) - h(1 + \sin(2\pi t))$.

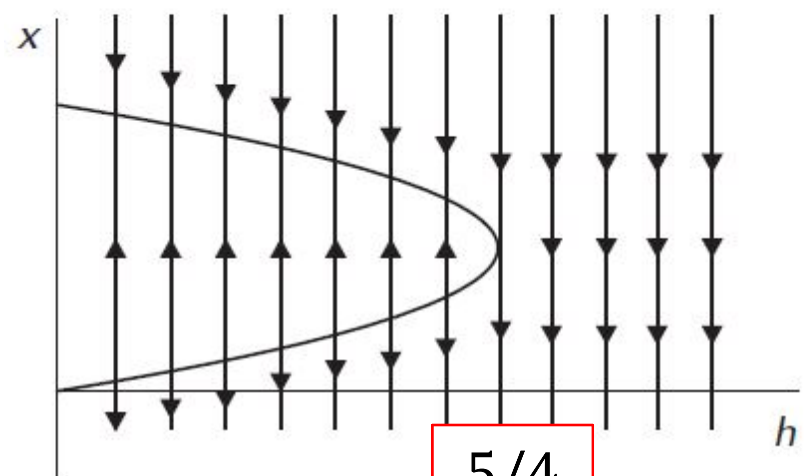
Bifurcation Point(s)

consider $\boxed{x' = 5x(1-x) - h = f(x,h)}$ constant harvesting

bifurcation points $\boxed{f(x,h) = 0 \;\&\; f_x(x,h) = 0}$

$f(x,h) = 0$ $\quad \boxed{5x(1-x) - h = 0}$ $\quad \boxed{h = 5/4}$

$f_x(x,h) = 0$ $\quad \boxed{5 - 10x = 0}$ $\quad \boxed{x = \frac{1}{2}}$

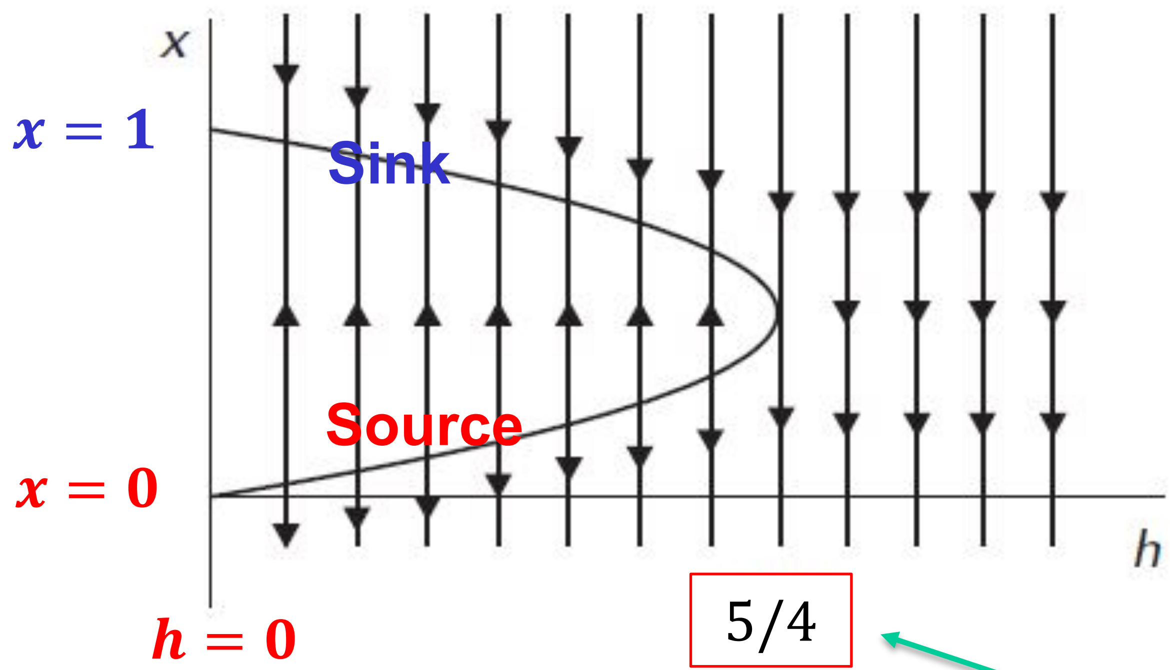


$\boxed{5/4}$

Potential Appearance of Stable and Unstable Points

Consider a time varying "h"

$$x' = 5x(1-x) - g(t) \qquad g(t) = 0.8(1+\sin(2\pi t))$$



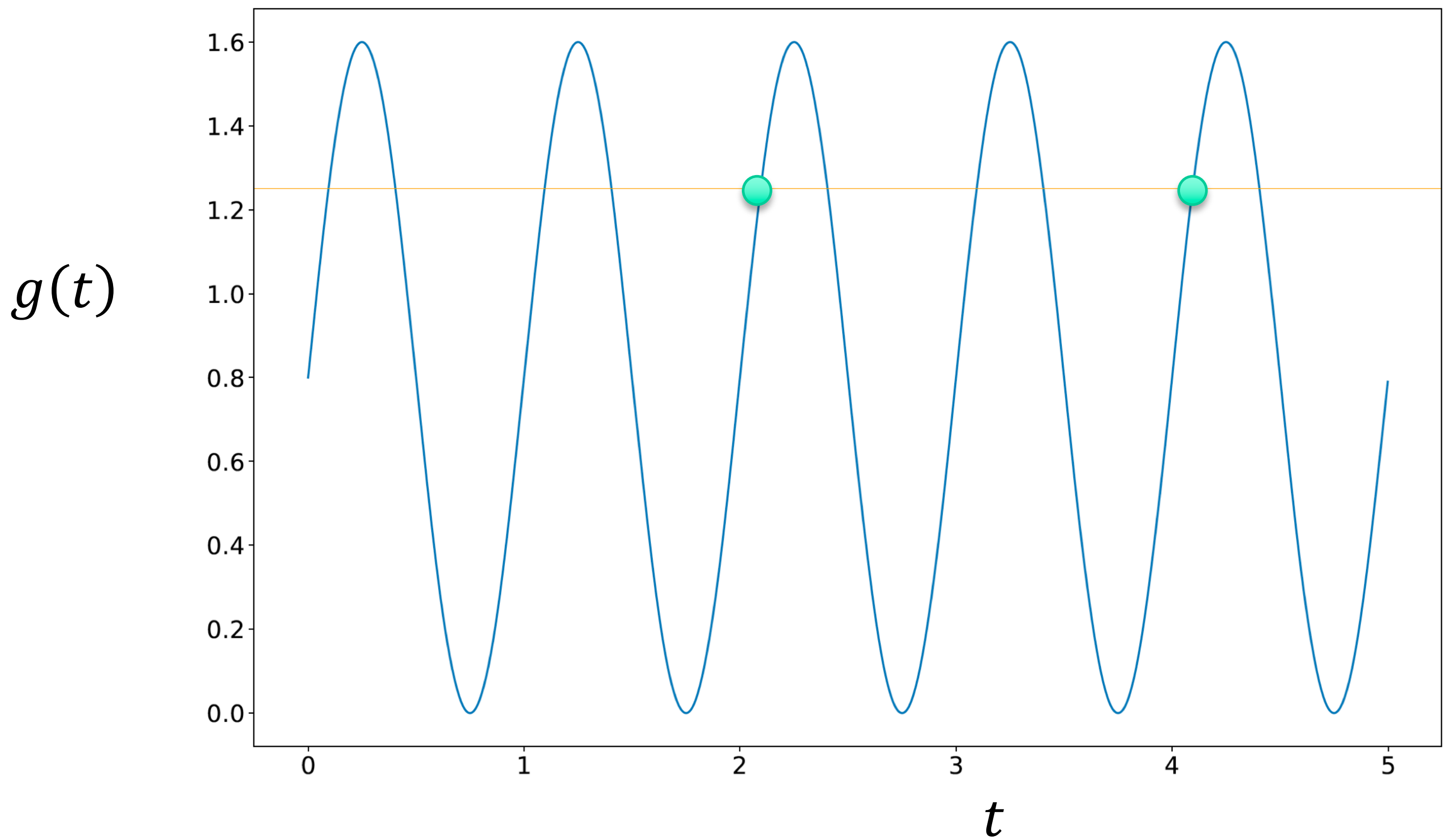
For a time varying g(t) that is less than $\frac{5}{4}$,

- stable points may appear between $x = \frac{1}{2}$ and $x = 1$, and
- unstable points may appear between $x = 0$ and $x = \frac{1}{2}$.

1.4: Periodic Harvesting (Forcing)

$$x' = 5x(1-x) - 0.8(1+\sin(2\pi t))$$

$$g(t) = 0.8(1+\sin(2\pi t))$$



“”no critical point”

$$\frac{5}{4} = \text{“}h\text{”}$$

”two critical points”

1.4: Periodic Harvesting and Periodic Solutions

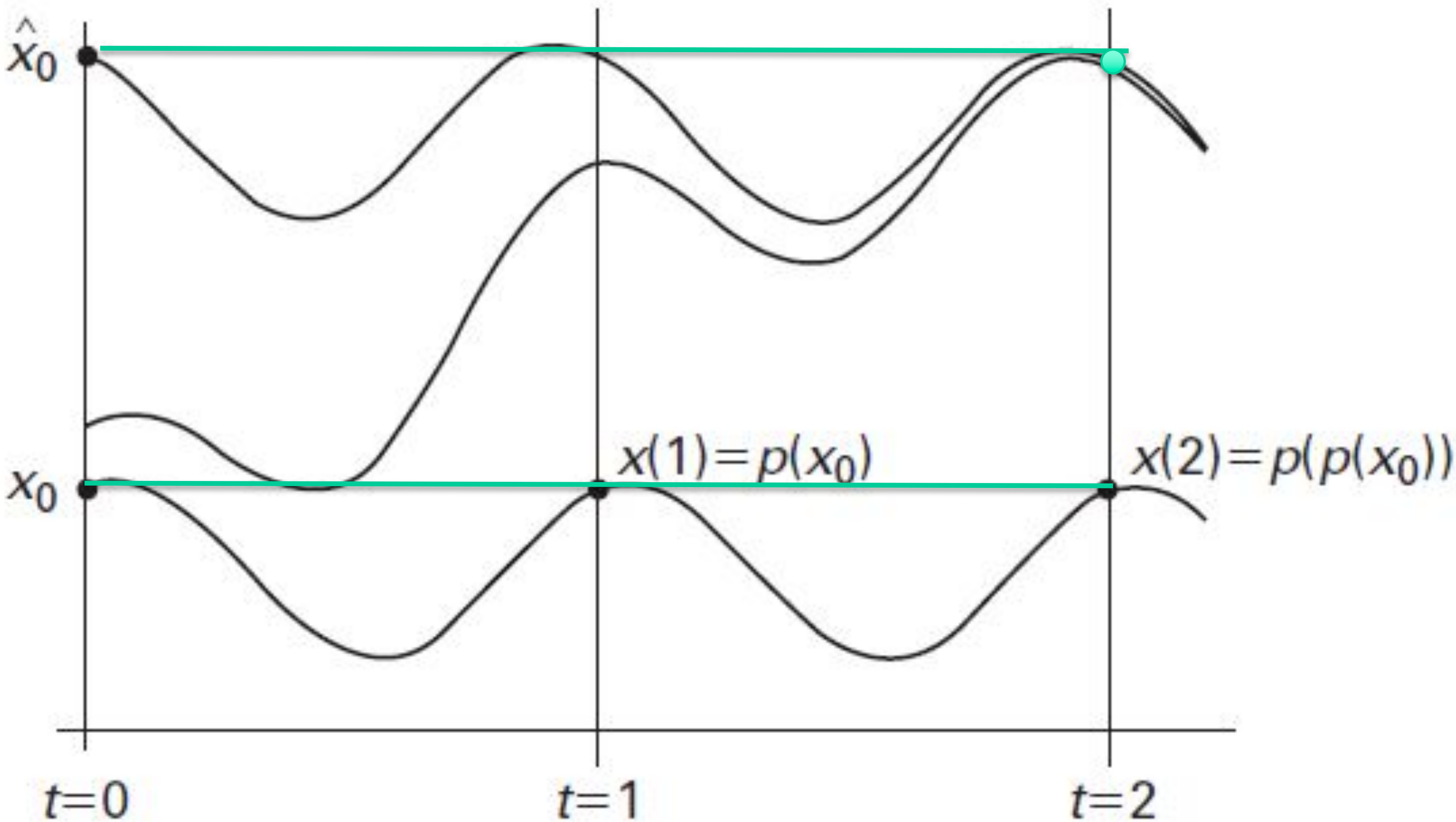


Figure 1.10 The Poincaré map for $x' = 5x(1 - x) - 0.8(1 + \sin(2\pi t))$.

Section 1.5: Computing the Poincare Map

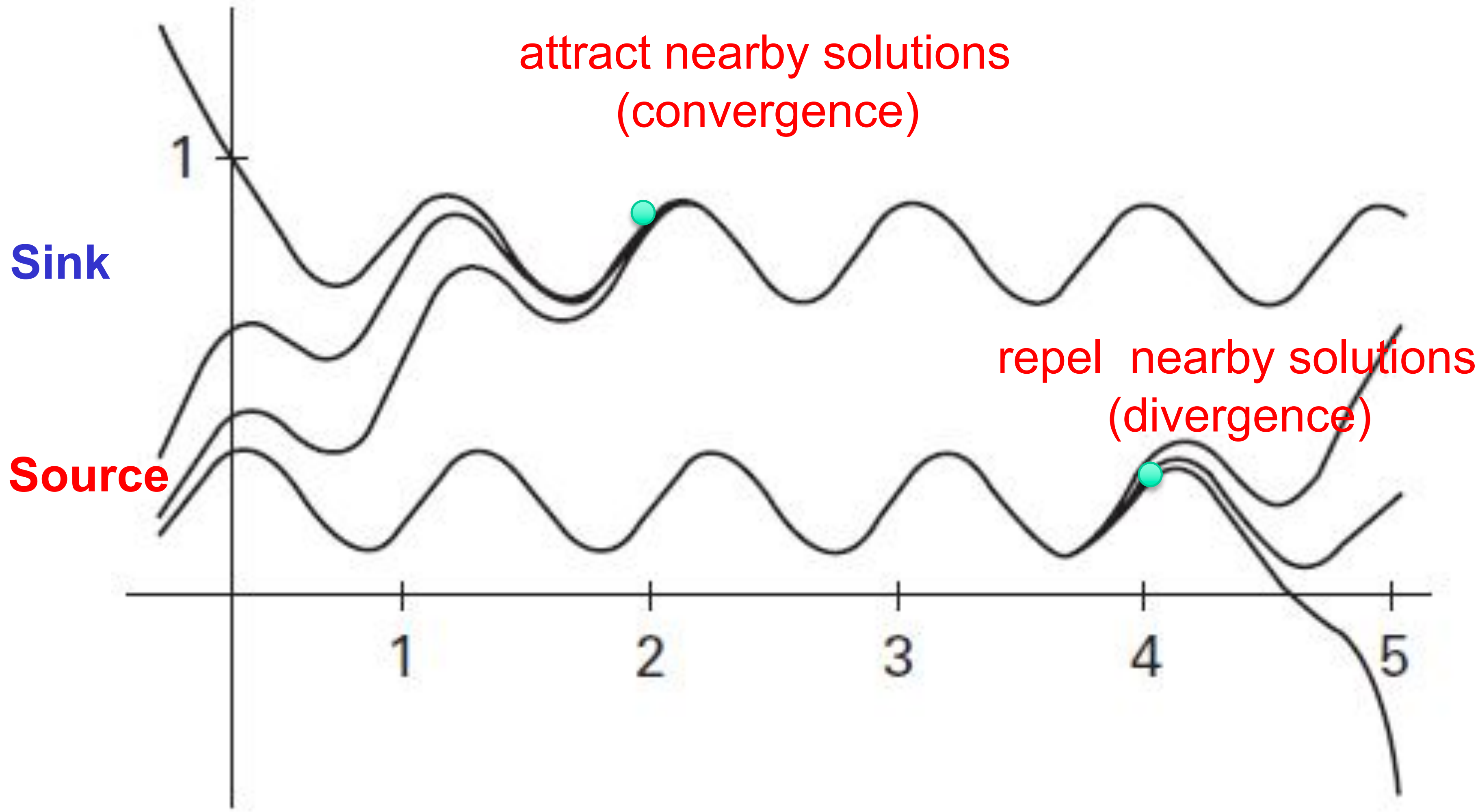
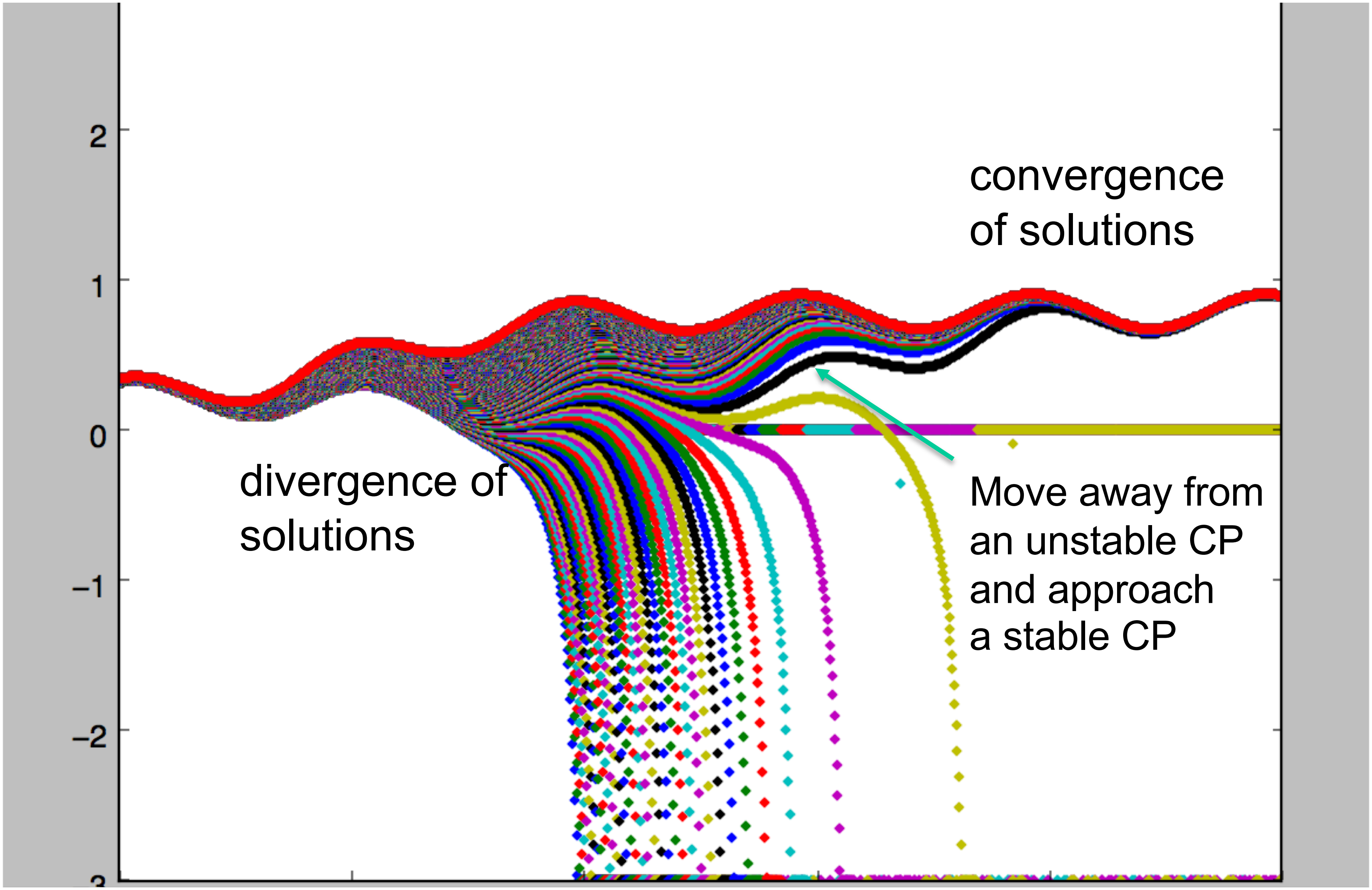
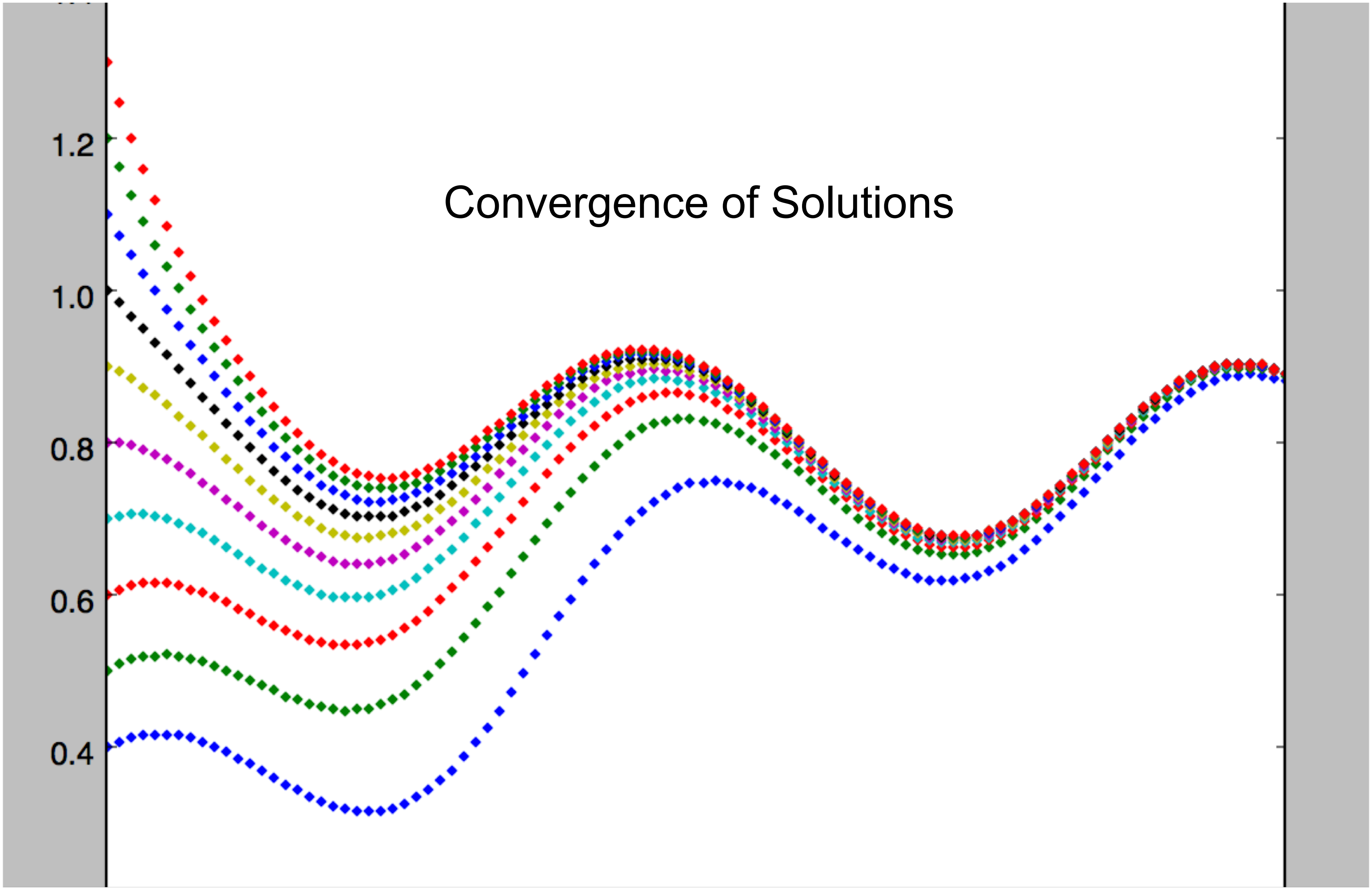


Figure 1.11 Several solutions of $x' = 5x(1 - x) - 0.8(1 + \sin(2\pi t))$.

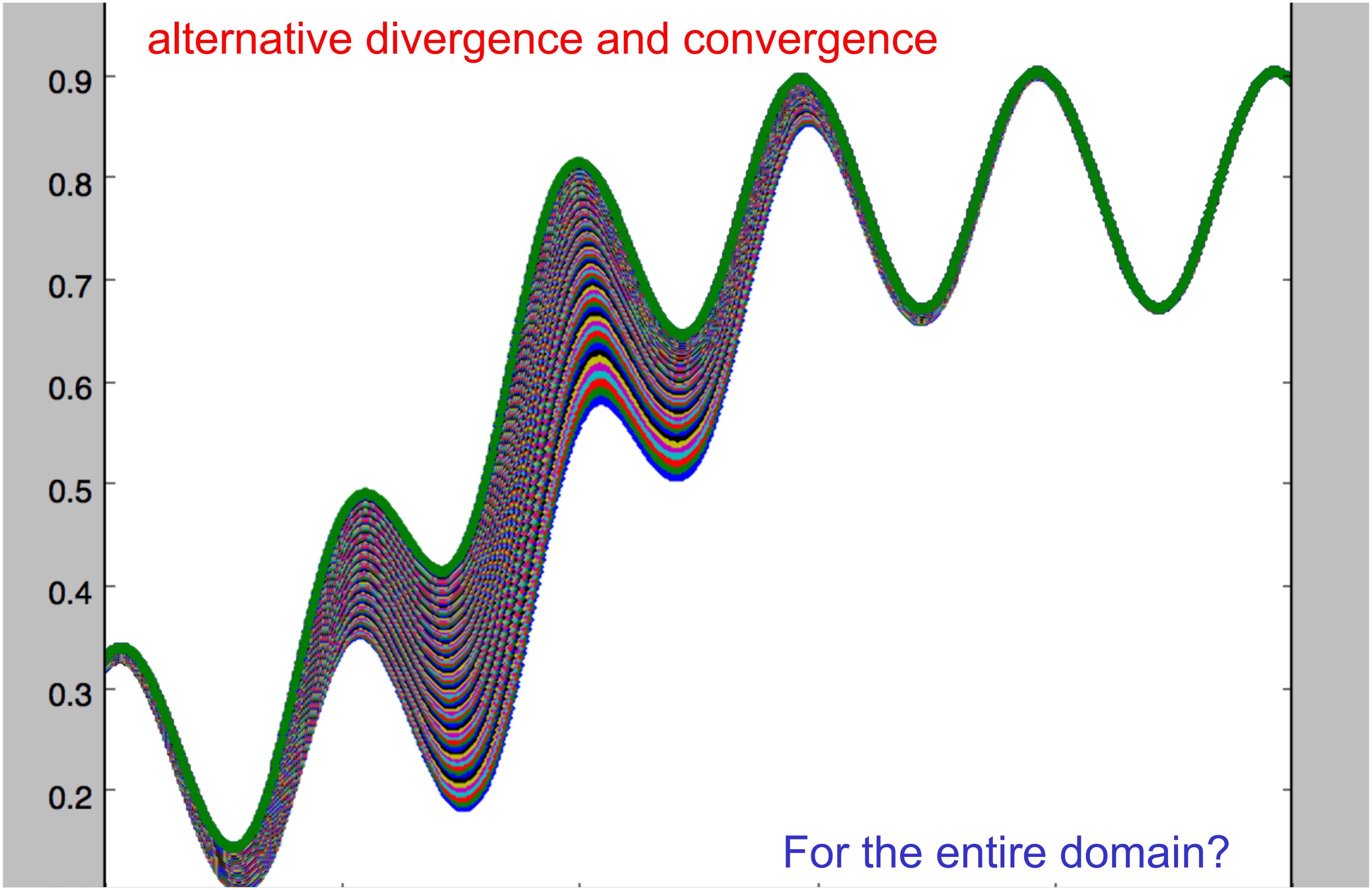
$$x' = 5x(1-x) - 0.8(1+\sin(2\pi t))$$



$$x' = 5x(1-x) - 0.8(1+\sin(2\pi t))$$



$$x' = 5x(1-x) - 0.8(1+\sin(2\pi t))$$



Key ODEs in Chapter 1

$$\frac{dx}{dt} = ax$$

bifurcation at $a = 0$

$$x = x_0 e^{at}$$

$$\boxed{\frac{dx}{dt} = ax\,(1 - x)}$$

bifurcation at $a = 0$

(the Logistic Eq)

$$x = \frac{x_0 e^{at}}{1 - x_0 + x_0 e^{at}}$$

(sigmoid function)

$$\boxed{\frac{dx}{dt} = x(1 - x) - h}$$

bifurcation at $\mathrm{h} = 1/4$

(the Logistic Eq with constant harvesting)

$$\boxed{\frac{dx}{dt} = x(1 - x) - h(1 + \sin(2\pi t))}$$

periodic forcing,
non-autonomous system

(the Logistic Eq with periodic harvesting)

Important Concepts

1. Bifurcation & Bifurcation points
2. Critical points, $f(x_c) = 0$;
3. (equilibrium points = fixed points = critical points)
4. Derivative tests & Perturbation method
5. General solution
6. Initial Value Problem (IVP)
7. Particular solution
8. Phase Line
9. Separable ODEs
10. Sink, Source, an Saddle
11. Stable vs. Unstable Solutions, $f'(x_c)$.
12. Structurally Stable vs. Unstable (i.e., with bifurcation)

Fundamental Bifurcations

Supp

type		
saddle-node	$$\frac{dx}{dt} = a - x^2$$	$f_x(x_c, a) = 0$
transcritical	$$\frac{dx}{dt} = ax - x^2$$	$f_x(x_c, a) = 0$
pitchfork	$$\frac{dx}{dt} = ax - x^3$$	$f_x(x_c, a) = 0$
Hopf		$Re(\lambda) = 0$
Homoclinic		

Terminology

- We will study equations of the following form:

$$x' = f(x, t; a) \qquad \text{(ordinary differential equation)}$$

and

$$x \to g(x; a), \qquad \text{(difference equation)}$$

with $x \in U \subset R^n$, $t \in R^1$, and $\mathrm{a} \in R^p$. We refer to $x, t,$ and a ad dependent variables, independent variables and parameter.

- By a solution of the above differential equation, we mean a map, x, from some interval, $I \in R^1$ into R^n, written as follows:

$$x: I \to R^n$$

$$t \to x(t).$$

- System with $f = f(x; a)$ that is not a function of time is referred to as autonomous systems.

Notation

- To emphasize the dependence of solutions on initial values, x_0, we denote the corresponding solution by $\phi(t, x_0)$.
- This function, $\phi: R \times R \to R$, is called the flow associated with the differential equations.
- If we hold the variable x_0 fixed, the function

$$\mathrm{t} \to \phi(t, x_0)$$

is just an alternative expression of the solution of the differential equations satisfying the ICs.

- Alternatively, the solution can be expressed as $\phi(t, x_0)$ or $\phi_t(x_0)$.