Exam 2
Partial Differential Equations
Math 531
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Problem 1: The Laplace equation

$$\nabla^2 u(x, y, z) = 0,$$

represents the steady-state heat equation without sources. Using circular cylindrical coordinates,

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

show that the Laplace's equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Notice the following for r:

$$x = r\cos\theta \quad \rightarrow \quad \frac{\partial x}{\partial r} = \cos\theta \qquad y = r\sin\theta \quad \rightarrow \quad \frac{\partial y}{\partial r} = \sin\theta$$

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \\
\frac{\partial^2 u}{\partial r^2} &= \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)\cos\theta + \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial y}\right)\sin\theta \\
&= \left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial u}{\partial x}\frac{\partial y}{\partial r}\right)\cos\theta + \left(\frac{\partial}{\partial x}\frac{\partial u}{\partial y}\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\right)\sin\theta \\
&= \left(\frac{\partial^2 u}{\partial x^2}\cos\theta + \frac{\partial^2 u}{\partial y \partial x}\sin\theta\right)\cos\theta + \left(\frac{\partial^2 u}{\partial x \partial y}\cos\theta + \frac{\partial^2 u}{\partial y^2}\sin\theta\right)\sin\theta \\
&= \frac{\partial^2 u}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 u}{\partial y \partial x}\sin\theta\cos\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta
\end{aligned}$$

Notice the following for θ:

$$x = r\cos\theta \quad \rightarrow \quad \frac{\partial x}{\partial \theta} = -r\sin\theta \qquad y = r\sin\theta \quad \rightarrow \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\begin{aligned}
\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta) \\
\frac{\partial^2 u}{\partial \theta^2} &= \left(\frac{\partial u}{\partial x}(-r\cos\theta) + \frac{\partial u}{\partial y}(-r\sin\theta)\right) + \left(\frac{\partial}{\partial \theta}\frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial}{\partial \theta}\frac{\partial u}{\partial y}(r\cos\theta)\right) \\
&= \left(\frac{\partial u}{\partial x}(-r\cos\theta) + \frac{\partial u}{\partial y}(-r\sin\theta)\right) \\
&\qquad + \left(\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial u}{\partial x}\frac{\partial y}{\partial \theta}\right)(-r\sin\theta) + \left(\frac{\partial}{\partial x}\frac{\partial u}{\partial y}\frac{\partial x}{\partial \theta} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}\right)(r\cos\theta)\right) \\
&= \left(\frac{\partial u}{\partial x}(-r\cos\theta) + \frac{\partial u}{\partial y}(-r\sin\theta)\right) \\
&\qquad + \left(\left(\frac{\partial^2 u}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 u}{\partial y \partial x}(r\cos\theta)\right)(-r\sin\theta) + \left(\frac{\partial^2 u}{\partial x \partial y}(-r\sin\theta) + \frac{\partial^2 u}{\partial y^2}(r\cos\theta)\right)(r\cos\theta)\right) \\
&= \left(\frac{\partial u}{\partial x}(-r\cos\theta) + \frac{\partial u}{\partial y}(-r\sin\theta)\right) + \left(\frac{\partial^2 u}{\partial x^2}(r\sin\theta)^2 + 2\frac{\partial^2 u}{\partial y \partial x}(-r^2\cos\theta\sin\theta) + \frac{\partial^2 u}{\partial y^2}(r\cos\theta)^2\right) \\
&= -r\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right) + r^2\left(\frac{\partial^2 u}{\partial x^2}\sin^2\theta - 2\frac{\partial^2 u}{\partial y \partial x}\cos\theta\sin\theta + \frac{\partial^2 u}{\partial y^2}\cos^2\theta\right)
\end{aligned}$$

Putting it all together, we get:

$$
\begin{aligned}
\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= \frac{-1}{r}\frac{\partial u}{\partial r} + \left(\frac{\partial^2 u}{\partial x^2}\sin^2\theta - 2\frac{\partial^2 u}{\partial y \partial x}\cos\theta\sin\theta + \frac{\partial^2 u}{\partial y^2}\cos^2\theta\right) \\
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= \frac{-1}{r}\frac{\partial u}{\partial r} + \left(\frac{\partial^2 u}{\partial x^2}\sin^2\theta - 2\frac{\partial^2 u}{\partial y \partial x}\cos\theta\sin\theta + \frac{\partial^2 u}{\partial y^2}\cos^2\theta\right) \\
&\qquad + \left(\frac{\partial^2 u}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 u}{\partial y \partial x}\sin\theta\cos\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta\right) \\
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} &= \frac{-1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \\
\nabla^2 u(x,y,z) &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0
\end{aligned}
$$

Problem 2: Given the heat equation on a radially symmetric disk

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \qquad 0 < r < a, t > 0,$$

with boundary condition

$$u(a,t) = 0, \qquad t > 0,$$

and initial condition

$$u(r,0) = f(r), \qquad t > 0$$

State clearly the implicit boundary conditions. State clearly your Sturm-Liouville problem(s) and any orthogonality relationships. Solve this problem (showing the full Fourier series solution before applying the initial condition), then using orthogonality relative to the initial condition, reduce the Fourier series solution. (Don't try to reduce your integrals in r.)

Let the following be true:

$$u(r,t) = \phi(r)h(t) \qquad \phi(a) = 0$$

Notice the following from separation of variables:

$$h' + \lambda k h = 0 \qquad \frac{d\phi}{dr} + r\frac{d^2\phi}{dr^2} + \lambda r\phi = 0$$

Solving the time dependent ODE gives us the following:

$$h(t) = Ce^{-\lambda kt}$$

Using a simple scaling transformation of $z = \sqrt{\lambda}r$, we get the following equation fo the spacial ODE:

$$z^2\frac{d^2\phi}{dr^2} + z\frac{d\phi}{dr} + (z^2 - 0^2)\phi = 0$$

We can now see that this is a Bessel Differential Equation of order 0:

$$\phi(r) = c_1 J_0(\sqrt{\lambda}r) + c_2 Y_0(\sqrt{\lambda}r)$$

By the boundedness condition, we get that $|\phi(0)| < \infty$, and that $\lim_{z\to 0} Y_0(z) = \pm\infty$, such that $c_2 = 0$:

$$\phi(r) = c_1 J_0(\sqrt{\lambda}r)$$

Using the boundary condition, we get:

$$f(a) = 0 = c_1 J_0\left(\sqrt{\lambda}a\right) \qquad \to \qquad \lambda_{0n} = \left(\frac{z_{0n}}{a}\right)^2$$

where z_{0n} is the n^{th} zero of J_0.

Now we get the following for u:

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0\left(\sqrt{\lambda_{0n}}r\right) e^{-\lambda_{0n}kt}$$

Now if we use the initial condition, we get the following:

$$u(r,0) = f(r) = \sum_{n=1}^{\infty} A_n J_0\left(\sqrt{\lambda_{0n}}r\right)$$

Now we can solve for the following coefficient:

$$A_n = \frac{\int_0^a f(r) J_0\left(\sqrt{\lambda_{0n}}r\right) r\,dr}{\int_0^a J_0^2\left(\sqrt{\lambda_{0n}}r\right) r\,dr}$$

Problem 3:

(a) Solve the heat equation on a disk

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \qquad 0 < r < 1, \quad -\pi < \theta < \pi, \quad t > 0,$$

with the boundary condition

$$u(1, \theta, t) = \sin 3\theta,$$

and the initial condition

$$u(r, \theta, 0) = 0$$

State clearly the implicit boundary conditions. State clearly your Sturm-Liouville problem(s) and any orthogonality relationships. Solve this problem (showing the full Fourier series solution before applying the initial condition), then using orthogonality relative to the initial condition, reduce the Fourier series solution. (Don't try to reduce your integrals in r.)

Let the following be true:

$$u(r,\theta,t) = f(r)g(\theta)h(t)$$

Using separation of variables, we get the following three equations:

$$h' = -\lambda h \qquad g'' = -\mu g \qquad r\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\lambda r^2 - \mu)f = 0$$

Solving the time dependent ODE, we get:

$$h(t) = Ce^{-\lambda t}$$

Solving the angular ODE, we get the following eigenvalue and eigenfunctions:

$$\mu = m^2 \qquad g(\theta) = a_m \cos m\theta + b_m \sin m\theta$$

Using a simple scaling transformation of $z = \sqrt{\lambda}r$, we get the following equation fo the radial ODE:

$$z^2\frac{d^2f}{dr^2} + z\frac{df}{dr} + (z^2 - m^2)f = 0$$

We can now see that this is a Bessel Differential Equation of order m:

$$f(r) = c_1 J_m(\sqrt{\lambda}r) + c_2 Y_m(\sqrt{\lambda}r)$$

By the boundedness condition, we get that $|f(0)| < \infty$, and that $\lim_{z\to 0} Y_0(z) = \pm\infty$, such that $c_2 = 0$:

$$f(r) = c_1 J_m(\sqrt{\lambda}r)$$

Now we get the following for u:

$$u(r,\theta,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn}J_m\left(\sqrt{\lambda}r\right)\cos m\theta\, e^{-\lambda t} + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{mn}J_m\left(\sqrt{\lambda}r\right)\sin m\theta\, e^{-\lambda t}$$

Now if we use the boundary condition, we get the following:

$$u(1,\theta,t) = \sin 3\theta = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn}J_m\left(\sqrt{\lambda}\right)\cos m\theta\, e^{-\lambda t} + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{mn}J_m\left(\sqrt{\lambda}\right)\sin m\theta\, e^{-\lambda t}$$

We can now see that $m = 3$:

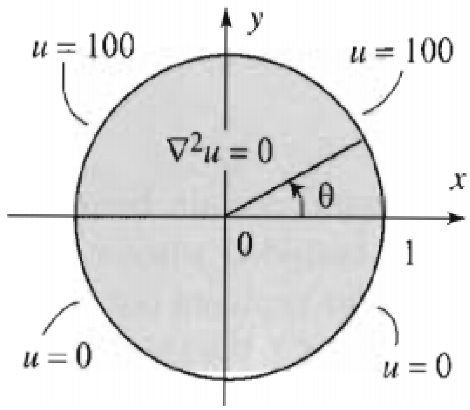
$$1 = \sum_{n=1}^{\infty} A_{3n}J_3\left(\sqrt{\lambda_{3n}}\right)\, e^{-\lambda_{3n}t}$$

Now we can solve for the following coefficient:

$$A_{3n} = \frac{\int_0^a J_3\left(\sqrt{\lambda_{3n}}r\right) r\, dr}{e^{-\lambda_{3n}t}\int_0^a J_3^2\left(\sqrt{\lambda_{3n}}r\right) r\, dr}$$

(b) Find the steady-state temperature in the disk.

The disk has a radius of 1. The upper half of the circumference is kept at 100° and the lower half is kept at 0°



Problem 4: Consider heat conduction in a sphere given by:

$$\frac{\partial u}{\partial t} = k\left(\frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial u}{\partial \phi}\right) + \frac{1}{\rho^2\sin\phi}\frac{\partial^2 u}{\partial \theta^2}\right),$$

$$0 < \rho < a, \quad -\pi < \theta \leq \pi, \quad 0 \leq \phi \leq \pi, \quad t > 0$$

with the boundary condition

$$\frac{\partial u}{\partial \rho}(a, \theta, \phi, t) = 0,$$

and initial conditions:

$$u(\rho, \theta, \phi, 0) = F(\rho, \phi)\sin 3\theta$$

Solve this equation noting any other boundary conditions you might need to apply. State clearly your Sturm-Liouville problem(s) and any orthogonality relationships.

Notice the following separation of variables:

$$u(\rho, \theta, \phi, t) = w(\rho, \theta, \phi)h(t) = f(\rho)q(\theta)g(\phi)h(t)$$

From this, and our original equation, we get the following time equation:

$$h' + \lambda kt = 0$$

Notice the following spatial equations:

$$q'' + \gamma q = 0 \qquad \frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) + \left(\lambda\rho^2 - \mu\right) f = 0 \qquad \frac{d}{d\phi}\left(\sin\phi \frac{dg}{d\theta}\right) + \left(\mu \sin\phi - \frac{m^2}{\sin\phi}\right) g = 0$$

Notice the solution to the time dependent ODE:

$$h(t) = Ce^{-\lambda kt}$$

Notice from the θ dependent ODE, we get the following eigenvalues and eigenfunctions:

$$\gamma = m^2 \qquad q(\theta) = a_m \cos m\theta + b_m \sin m\theta$$

Notice the radial ODE, if we let $\mu = n(n+1)$:

$$\frac{d}{d\rho}\left(\rho^2 \frac{df}{d\rho}\right) + \left(\lambda\rho^2 - n(n+1)\right) f = 0 \qquad \rightarrow \qquad f(\rho) = \rho^{-1/2} J_{n+\frac{1}{2}}\left(\sqrt{\lambda}\rho\right)$$

Using the boundary condition $f(a) = 0$, we get:

$$\lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2$$

Notice the ϕ dependent ODE, if we let $x = \cos\theta$, we get the following:

$$\frac{d}{dx}\left((1-x^2)\frac{dg}{dx}\right) + \left(n(n+1) - \frac{m^2}{1-x^2}\right) g = 0 \quad \rightarrow \quad g(x) = P_n^m\left(\cos\theta\right)$$

Thus, we get the following for u:

$$u(\rho,\theta,\phi,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn}\rho^{-1/2}J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{mn}}\rho\right)P_n^m\left(\cos\theta\right)\cos m\theta\, e^{-\lambda kt}$$
$$+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{mn}\rho^{-1/2}J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{mn}}\rho\right)P_n^m\left(\cos\theta\right)\sin m\theta\, e^{-\lambda kt}$$

Using our initial condition, we get:

$$u(\rho,\theta,\phi,0) = F(\rho,\phi)\sin 3\theta = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn}\rho^{-1/2}J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{mn}}\rho\right)P_n^m\left(\cos\theta\right)\cos m\theta$$
$$+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{mn}\rho^{-1/2}J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{mn}}\rho\right)P_n^m\left(\cos\theta\right)\sin m\theta$$
$$= \sum_{n=1}^{\infty} B_{3n}\rho^{-1/2}J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{3n}}\rho\right)P_n^3\left(\cos\theta\right)\sin 3\theta$$

Thus, we get the following:

$$F(\rho,\phi) = \sum_{n=1}^{\infty} B_{3n}\rho^{-1/2}J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{3n}}\rho\right)P_n^3\left(\cos\theta\right)$$

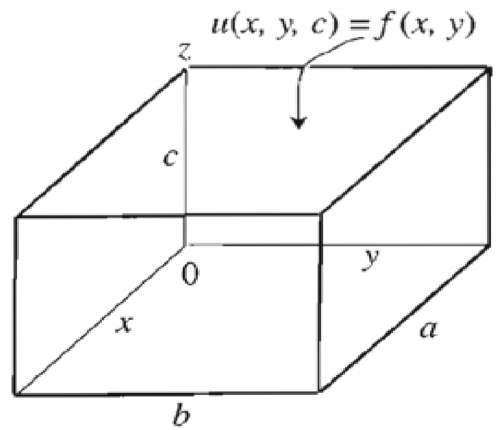
and the following coefficients:

$$B_{3n} = \frac{\int_0^a F(\rho,\phi)J_{n+\frac{1}{2}}\left(\sqrt{\lambda_{3n}}r\right)r\,dr}{\rho^{-1/2}P_n^3(\cos\theta)\int_0^a J^2_{n+\frac{1}{2}}\left(\sqrt{\lambda_{3n}}r\right)r\,dr}$$

Problem 5: Find the steady-state temperature in a cube, which satisfies:

$$\nabla^2 u(x,y,z) = 0, \qquad 0 < x < a, \quad 0 < y < b, \quad 0 < z < c.$$

The cube is kept at 0°C on all faces except on the upper horizontal face.



Notice the following:

$$\nabla^2 u(x,y,z) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Now we let the following be true:

$$u(x,y,z) = f_1(x)g(y)h(z)$$

And we get the following from separation of variables:

$$f_1'' = -\lambda f_1 \qquad g'' = -\mu g \qquad h'' - (\lambda + \mu)h = 0$$

Notice the following for the x dependent equation, we get:

$$f_1(x) = a_1 \cos\sqrt{\lambda}x + a_2 \sin\sqrt{\lambda}x$$

Now we apply the following boundary conditions, $f_1(0) = f_1(a) = 0$:

$$f_1(0) = a_1 = 0 \qquad f_1(a) = a_2 \sin\sqrt{\lambda}a \qquad \lambda_n = \left(\frac{n\pi}{a}\right)^2$$

Notice the following for the y dependent equation, we get:

$$g(y) = b_1 \cos\sqrt{\mu}y + b_2 \sin\sqrt{\mu}y$$

Now we apply the following boundary conditions, $g(0) = g(b) = 0$:

$$g(0) = b_1 = 0 \qquad f(b) = b_2 \sin\sqrt{\mu}b \qquad \mu_m = \left(\frac{m\pi}{b}\right)^2$$

Notice the following for the z dependent equation, we get:

$$h(z) = c_1 \cosh(\sqrt{\lambda + \mu}z) + c_2 \sinh(\sqrt{\lambda + \mu}z)$$

Now we use our boundary condition, $h(0) = 0$, to get $c_1 = 0$:

$$h(z) = c_2 \sinh(\sqrt{\lambda + \mu}z)$$

Thus we get the following for u:

$$u(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{n\pi x}{a} \sin\frac{m\pi x}{b} \sinh(\sqrt{\lambda_n + \mu_m} z)$$

Using our other boundary condition, we get:

$$u(x,y,c) = f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{n\pi x}{a} \sin\frac{m\pi x}{b} \sinh(\sqrt{\lambda_n + \mu_m} c)$$

From here, we can solve for the following coefficient:

$$A_{mn} = \frac{4}{ab\sinh(\sqrt{\lambda_n + \mu_m} c)} \int_0^a \int_0^b \sin\frac{n\pi x}{a} \sin\frac{m\pi x}{b} f(x,y)\, dy\, dx$$

Problem 6: Solve the heat equation

$$\frac{\partial u}{\partial t} = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2}\right), \qquad 0 < r < 2, \quad 0 < z < 7, \quad t > 0,$$

inside a cylinder subject to the initial condition

$$u(r, z, 0) = 100,$$

if the boundary conditions are:

$$u(2, z, t) = 0, \quad u(r, 0, t) = 0, \quad u(r, 7, t) = 100$$

We can separate the original equation and get the following:

$$u(r,z,t) = f(r)g(z)h(t)$$

From here, we get the following, from separating the variables:

$$h' = -\lambda k t \qquad g'' = -\gamma g \qquad r\left(\frac{d}{dr}\left(r\frac{df}{dr}\right)\right) + (\lambda - \gamma)\, r^2 f$$

Solving the time equation, gives us the following:

$$h(t) = Ce^{-\lambda k t}$$

Solving the height equation gives us the following eigenvalues and eigenfunctions:

$$\gamma = m^2 \qquad g(z) = a_m \cos mz + b_m \sin mz$$

Using the homogeneous boundary condition, we get $a_m = 0$, such that:

$$g(z) = b_m \sin mz$$

Solving the radial equation, and using the boundedness condition, gives us the following:

$$f(r) = c_1 J(\sqrt{\lambda} r)$$

Using the boundary condition, we get:

$$\lambda_{mn} = \left(\frac{z_{mn}}{2}\right)^2$$

Thus we get the following for u:

$$u(r,z,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{mn} J(\sqrt{\lambda_{mn}} r) e^{-\lambda_{mn} k t} \sin mz$$

Now we include our nonhomogeneous boundary equations:

$$u(r,7,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{mn} J(\sqrt{\lambda_{mn}} r) e^{-\lambda_{mn} k t} \sin 7m = 100$$

$$u(r,z,0) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{mn} J(\sqrt{\lambda_{mn}} r) \sin mz = 100$$

Problem 7: Find the eigenvalues and eigenfunctions which arise from the Sturm-Liouville problem:

$$x\frac{d}{dx}\left(x\frac{d\phi}{dx}\right)+\left(\lambda x^2-9\right)\phi=0, \qquad 0<x<6,$$

with $\phi'(6)=0$ and $\phi(x)$ bounded for $x\in[0,6]$. Clearly state the orthogonality relationship for the eigenfunctions and use the eigenfunctions to find the Fourier expansion for a function $f(x)$. Give an integral expression for the Fourier coefficients. Assume that

$$f(x)=\begin{cases} x, & x<\frac{1}{2}, \\ 1-x, & x>\frac{1}{2} \end{cases}$$

We can see that $m = 3$ here, such that we get the following for ϕ:

$$\phi(x) = c_1 J_3(\sqrt{\lambda}x) + c_2 Y_3(\sqrt{\lambda}x)$$

The boundedness condition tells us that $c_2 = 0$, such that:

$$\phi(x) = c_1 J_3(\sqrt{\lambda}x)$$

We can also use our boundary condition to get:

$$\phi'(x) = c_1\sqrt{\lambda}J_3(\sqrt{\lambda}x) \qquad \rightarrow \qquad \phi'(6) = 0 = c_1\sqrt{\lambda}J_3(6\sqrt{\lambda}) \qquad \rightarrow \qquad \lambda = \left(\frac{z_{3n}}{6}\right)^2$$