Review: Second order differential equations

Examples: Consider an initial value problem (IVP)

$$y'' - y = 0 \; ; \quad y(0) = y_0 \text{ and } y'(0) = y_0$$

This is a second order Linear homogeneous differential equation.

Solve this by attempting the solution

$y(t) = ce^{\lambda t}$, which results in:

$$c\lambda^2 e^{\lambda t} - ce^{\lambda t} = 0$$

$$ce^{\lambda t}(\lambda^2 - 1) = 0$$

This results in the characteristic

equation:

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

This gives the general solution:

$$y(t) = c_1 e^{t} + c_2 e^{-t}$$

From the initial condition

$$c_1 + c_2 = y_0$$

$$c_1 - c_2 = y_{p0}$$

This gives that:

$$c_1 = \frac{y_0 + y_{p0}}{2} \quad \text{and} \quad c_2 = \frac{y_0 - y_{p0}}{2}$$

Thus

$$y(t) = \frac{y_0 + y_{p0}}{2} e^{t} + \frac{y_0 - y_{p0}}{2} e^{-t}$$

$$\equiv y_0 \cosh(t) + y_{p0} \sinh(t)$$

N.B.

$$\frac{y_0}{2} e^{t} + \frac{y_0}{2} e^{-t}$$

$$= y_0 \cosh(t)$$

Example: first order systems of Differential Equations

Consider the ODE

$$y'' - y = 0$$

Let $y_1(t) = y(t)$ and

$$y_2(t) = y'(t) = y_1'(t)$$

So:

$$y_2'(t) = y''(t) = y_1(t)$$

The second order DE can be written as follows (first order system of ODEs)

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

i.e $\dot{y} = Ay$

The characteristic equation of the matrix :

$$\det |A - \lambda I| = \det \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$$

$$= \lambda^2 - 1 = 0$$

Which is the same as for the ODE before.

Once again, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$

Consider the eigenvalue $\lambda_1 = 1$ for the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; the eigenvector is $\xi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Similarly, the eigenvector for $\lambda_2 = -1$ is $\xi_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

It follows that the solution to the system of DEs $\dot{y} = Ay$ is

$$y = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$$

Example: Boundary Value Problems (BVPs)

Consider the BVP

$$y'' - y = 0 \quad ; \quad y(0) = A \ , \ y(1) = B$$

which gives the general solution:

$$y(t) = c_1 e^{t} + c_2 e^{-t}$$

with some algebra the unique solution becomes:

$$y(t) = \frac{B - Ae^{-1}}{e - e^{-1}} e^{t} - \frac{(B - Ae)}{e - e^{-1}} e^{-t}$$

Which gives

$$y(t) = \frac{B}{\sinh(1)} \sinh(t) + \frac{A}{\sinh(1)} \sinh(1-t)$$

i.e

$$y(t) = d_1 \sinh(t) + d_2 \sinh(1-t)$$

where $d_1 := \frac{B}{\sinh(1)}$ and

$$d_2 := \frac{A}{\sinh(1)}$$

Note: $\sinh(t)$ & $\sin(1-t)$ are linear Independent combination of e^t and e^{-t}.

Linear independence

Definition of Linear independence

Let V be a vector space of all real-valued function of a real variable x.

A set of functions,

Note

$x = y$

$z = 2y$

$\Rightarrow x$ is a linear combination of z and vice versa

$z = 2x$

$\{f_i(x)\}_{i=1}^{n}$ is <u>linearly independent</u>

if and only if a linear combination of those functions

$$c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0$$

for all x,

implies that all <u>the</u> constants, $c_i = 0$

<u>Example</u>

Consider the set of functions $\{e^{t}, e^{-t}\}$ and assume that

$$c_1 e^{t} + c_2 e^{-t} = 0 \quad \text{for all } t$$

Solving this equation gives

$$c_1 e^{2t} = -c_2 \quad \text{for all } t$$

which only occurs when $c_1 = 0$ and $c_2 = 0$

Theorem: Existence and Uniqueness of the IVP

(1) $$y' = f(t, y) \text{ with } y(0) = 0$$

If f and $\frac{df}{dy}$ are continuous in a rectangle R where $R: |t| \leq a$; $|y| \leq b$ then there is some interval $|t| \leq h \leq |a|$ in which there exist a Unique Solution $y = \phi(t)$ Of the IVP (1).

This theorem states that assuming the function f is smooth, then the first-order differential equation has a unique solution through a specific initial condition.

Here we are primarily considering $f(t,y)$ linear in y, this theorem is satisfied.

Does this theorem hold for BVPs?

Example: Harmonic Oscillator

Consider the IVP

$$y'' + y = 0, \quad y(0) = A, \quad y'(0) = B$$

The characteristic equation is

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

The general solution:

$$y(t) = c_1 \cos(t) + c_2 \sin(t)$$

The unique solution is

$$y(t) = A\cos(t) + B\sin(t)$$

This is a classic harmonic undamped oscillator

Example Harmonic oscillator

Now consider the BVP:

$$y'' + y = 0 \,, \quad y(0) = A \text{ and } y(1) = B$$

The general solution is

$$y(t) = c_1 \cos(t) + c_2 \sin(t)$$

The BC are easily solved to give:

$$y(t) = A\cos(t) + \frac{B - A\cos(1)}{\sin(1)} \sin(t)$$

This again gives a unique solution

Example (Harmonic Oscillators)

Now consider BVP:

$$y'' + y = 0 \,, \quad y(0) = A \,; \quad y(\pi) = B$$

The general solution:

$$y(t) = c_1 \cos(t) + c_2 \sin(t)$$

The condition $y(0) = A$ implies

$c_1 = A$

However, $y(\pi) = B$ gives

$$y(\pi) = A\cos(\pi) + c_2 \sin(\pi)$$
$$= -A = B$$

This only has a solution if $B = -A$

Furthermore, if $B = -A$, the arbitrary constant c_2 remains undetermined, so takes any value:

- If $B \neq -A$ then <u>no solution exist</u>
- If $B = -A$ then <u>infinitely many solutions exist</u>

and satisfy

$$y(t) = A\cos(t) + c_2 \sin(t)$$

where c_2 is arbitrary.

<u>Theorem Boundary value problem</u>

(General Case)

Consider the second order linear BVP: $y'' + py' + qy = 0$

$y(a) = A$ and $y(b) = B$,

where $p, q, a \neq b, A, B$ are constants

Exactly one of the following conditions holds:

- There is a unique solution to the BVP
- There is no solution to the BVP
- There are infinitely many solutions to the BVP.

The previous example demonstrates this theorem and will be critical to solving many of our PDEs this semester.

$\lambda = \pm i$

$y = c_1 e^{it} + c_2 e^{-it}$

euler