Homework 4
Partial Differential Equations
Math 531
Stephen Giang RedID: 823184070

Exercise 2.5.1a: Solve Laplace's equation inside a rectangle $0 \leq x \leq L, 0 \leq y \leq H$, with the following boundary conditions [*Hint*: Separate variables. If there are two homogeneous boundary conditions in y, let $u(x,y) = h(x)\phi(y)$, and if there are two homogeneous boundary conditions in x, let $u(x,y) = \phi(x)h(y)$.]:

$$\frac{\partial u}{\partial x}(0,y) = 0, \qquad \frac{\partial u}{\partial x}(L,y) = 0, \qquad u(x,0) = 0, \qquad u(x,H) = f(x)$$

Let the following be true:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 \leq x \leq L, \qquad 0 \leq y \leq H$$

with

$$u(x,y) = \phi(x)h(y), \qquad \phi'(0) = 0, \quad \phi'(L) = 0, \quad h(0) = 0$$

Taking Laplace's Equation, we get the following:

$$\phi''(x)h(y) + \phi(x)h(y)'' = 0$$

$$\frac{\phi''(x)}{\phi(x)} = -\frac{h''(y)}{h(y)} = -\lambda$$

From this, we get the following:

$$\phi'' + \lambda\phi = 0, \qquad h'' - \lambda h = 0$$

From this, we can see this is an eigenvalue problem:

(a) ($\lambda = 0$):

$$\phi'' = 0 \quad \rightarrow \quad \phi' = c_1 \quad \rightarrow \quad \phi = c_1 x + c_2$$
$$h'' = 0 \quad \rightarrow \quad h' = d_1 \quad \rightarrow \quad h = d_1 y + d_2$$

Substituting in our BC's, we get:

$$\phi'(0) = \phi'(L) = c_1 = 0 \quad \rightarrow \quad \phi(x) = c_2$$

So now we have our first eigenfunction:

$$\phi(x) = c_2 \text{ with } \lambda = 0$$

Now we can solve for $h(y)$:

$$h(0) = d_2 = 0 \quad \rightarrow \quad h(y) = d_1 y$$

From here, we get the following:

$$u(x, y) = c_2 d_1 y$$

Now we can simply set the following and get the first product solution:

$$\boldsymbol{u_0(x, y) = A_0 y}$$

(b) ($\lambda < 0$):

$$\phi'' - |\lambda|\phi = 0$$

Using the characteristic equation, we get:

$$\phi(x) = c_1 \cosh(\sqrt{|\lambda|}x) + c_2 \sinh(\sqrt{|\lambda|}x) \qquad \phi'(x) = c_1\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}x) + c_2\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}x)$$

Using the BC's, we get:

$$\phi'(0) = c_2\sqrt{|\lambda|} = 0 \quad \rightarrow \quad \sqrt{|\lambda|} > 0 \quad \rightarrow \quad c_2 = 0$$
$$\phi'(L) = c_1\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}L) = 0 \quad \rightarrow \quad \sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}L) \neq 0 \quad \rightarrow \quad c_1 = 0$$
$$\phi(x) = 0$$

Thus we get the following trivial solution:

$$\boldsymbol{u(x, y) = 0}$$

(c) ($\lambda > 0$):

$$\phi'' + \lambda\phi = 0$$

Using the characteristic equation, we get:

$$\phi(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x) \qquad \phi'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

Using the BC's, we get:

$$\phi'(0) = c_2\sqrt{\lambda} = 0 \quad \rightarrow \quad \sqrt{\lambda} > 0 \quad \rightarrow \quad c_2 = 0$$
$$\phi'(L) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0$$

(i) ($c_1 = 0$):

$$\phi(x) = 0$$

Thus we get the following trivial solution:

$$\boldsymbol{u(x,y) = 0}$$

(ii) ($\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0$):

$$\sin(\sqrt{\lambda}L) = 0 \qquad \rightarrow \qquad \sqrt{\lambda}L = n\pi \qquad \rightarrow \qquad \lambda = \frac{n^2\pi^2}{L^2}$$

So now we have our n eigenfunctions:

$$\phi_n(x) = c_1 \cos\left(\frac{n\pi x}{L}\right)$$

we can now substitute our eigenvalues into the other ODE, and we get:

$$h'' - \frac{n^2\pi^2}{L^2}h = 0$$

When solving this ODE, we get the following linear independent solutions:

$$h_n(y) = d_1 \cosh\left(\frac{n\pi y}{L}\right) + d_2 \sinh\left(\frac{n\pi y}{L}\right)$$

If we substitute our BC ($h(0) = 0$) in, we get

$$h_n(0) = 0 = d_1 \qquad \rightarrow \qquad h_n(y) = d_2 \sinh\left(\frac{n\pi y}{L}\right)$$

From here, we get the following n product solutions:

$$\boldsymbol{u_n(x,y) = A_n \cos\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right)}$$

By the Principle of Superposition, we get the following:

$$\begin{aligned} u(x,y) &= u_0(x,y) + u_1(x,y) + ... + u_n(x,y) \\ &= A_0 y + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right) \end{aligned}$$

We can now include our nonhomogeneous solution and get the following:

$$u(x,H) = f(x) = A_0 H + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi H}{L}\right)$$

Using the orthogonality of cosines, we get:

$$A_0 = \frac{1}{LH}\int_0^L f(x)\,dx \qquad A_n = \frac{2}{L\sinh\left(\frac{n\pi H}{L}\right)}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$$

Thus we get our desired solution:

$$\begin{aligned} \boldsymbol{u(x,y)} &\boldsymbol{= \frac{1}{LH}\int_0^L f(x)\,dx} \\ &\boldsymbol{+ \sum_{n=1}^{\infty}\left[\frac{2}{L\sinh\left(\frac{n\pi H}{L}\right)}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx\right]\cos\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)} \end{aligned}$$

Exercise 2.5.1g: Solve Laplace's equation inside a rectangle $0 \leq x \leq L, 0 \leq y \leq H$, with the following boundary conditions [*Hint*: Separate variables. If there are two homogeneous boundary conditions in y, let $u(x,y) = h(x)\phi(y)$, and if there are two homogeneous boundary conditions in x, let $u(x,y) = \phi(x)h(y)$.]:

$$\frac{\partial u}{\partial x}(0,y) = 0, \qquad \frac{\partial u}{\partial x}(L,y) = 0, \qquad u(x,0) = \begin{cases} 0 & x > L/2 \\ 1 & x < L/2 \end{cases}, \qquad \frac{\partial u}{\partial y}(x,H) = 0$$

Let the following be true:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 \leq x \leq L, \qquad 0 \leq y \leq H$$

with

$$u(x,y) = \phi(x)h(y), \qquad \phi'(0) = 0, \quad \phi'(L) = 0, \quad h'(H) = 0.$$

Also let the following be true:

$$u(x,0) = g(x) = \begin{cases} 0 & x > L/2 \\ 1 & x < L/2 \end{cases}$$

Taking Laplace's Equation, we get the following:

$$\phi''(x)h(y) + \phi(x)h(y)'' = 0$$

$$\frac{\phi''(x)}{\phi(x)} = -\frac{h''(y)}{h(y)} = -\lambda$$

From this, we get the following:

$$\phi'' + \lambda\phi = 0, \qquad h'' - \lambda h = 0$$

From this, we can see this is an eigenvalue problem:

(a) ($\lambda = 0$):

$$\phi'' = 0 \quad \rightarrow \quad \phi' = c_1 \quad \rightarrow \quad \phi = c_1 x + c_2$$
$$h'' = 0 \quad \rightarrow \quad h' = d_1 \quad \rightarrow \quad h = d_1 y + d_2$$

Substituting in our BC's, we get:

$$\phi'(0) = \phi'(L) = c_1 = 0 \quad \rightarrow \quad \phi(x) = c_2$$

So now we have our first eigenfunction:

$$\phi(x) = c_2 \text{ with } \lambda = 0$$

Now we can solve for $h(y)$:

$$h'(H) = d_1 = 0 \quad \rightarrow \quad h(y) = d_2$$

From here, we get our first product solution:

$$\boldsymbol{u_0(x, y) = A_0}$$

(b) ($\lambda < 0$):

$$\phi'' - |\lambda|\phi = 0$$

Using the characteristic equation, we get:

$$\phi(x) = c_1 \cosh(\sqrt{|\lambda|}x) + c_2 \sinh(\sqrt{|\lambda|}x) \qquad \phi'(x) = c_1\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}x) + c_2\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}x)$$

Using the BC's, we get:

$$\phi'(0) = c_2\sqrt{|\lambda|} = 0 \quad \rightarrow \quad \sqrt{|\lambda|} > 0 \quad \rightarrow \quad c_2 = 0$$
$$\phi'(L) = c_1\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}L) = 0 \quad \rightarrow \quad \sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}L) \neq 0 \quad \rightarrow \quad c_1 = 0$$
$$\phi(x) = 0$$

Thus we get the following trivial solution:

$$\boldsymbol{u(x, y) = 0}$$

(c) ($\lambda > 0$):

$$\phi'' + \lambda\phi = 0$$

Using the characteristic equation, we get:

$$\phi(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x) \qquad \phi'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

Using the BC's, we get:

$$\phi'(0) = c_2\sqrt{\lambda} = 0 \quad \rightarrow \quad \sqrt{\lambda} > 0 \quad \rightarrow \quad c_2 = 0$$
$$\phi'(L) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0$$

(i) $(c_1 = 0)$:

$$\phi(x) = 0$$

Thus we get the following trivial solution:

$$\boldsymbol{u(x, y) = 0}$$

(ii) $(\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0)$:

$$\sin(\sqrt{\lambda}L) = 0 \qquad \rightarrow \qquad \sqrt{\lambda}L = n\pi \qquad \rightarrow \qquad \lambda = \frac{n^2\pi^2}{L^2}$$

So now we have our n eigenfunctions:

$$\phi_n(x) = c_1 \cos\left(\frac{n\pi x}{L}\right)$$

we can now substitute our eigenvalues into the other ODE, and we get:

$$h'' - \frac{n^2\pi^2}{L^2}h = 0$$

When solving this ODE, we get the following linear independent solutions:

$$h_n(y) = d_1 \cosh\left(\frac{n\pi(H-y)}{L}\right) + d_2 \sinh\left(\frac{n\pi(H-y)}{L}\right)$$

$$h_n'(y) = d_1 \frac{-n\pi}{L} \sinh\left(\frac{n\pi(H-y)}{L}\right) + d_2 \frac{-n\pi}{L} \cosh\left(\frac{n\pi(H-y)}{L}\right)$$

If we substitute our BC $(h'(H) = 0)$ in, we get

$$h_n(H) = 0 = d_2 \qquad \rightarrow \qquad h_n(y) = d_1 \cosh\left(\frac{n\pi(H-y)}{L}\right)$$

From here, we get the following n product solution:

$$\boldsymbol{u_n(x, y) = A_n \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi(H-y)}{L}\right)}$$

By the Principle of Superposition, we get the following:

$$\begin{aligned} u(x,y) &= u_0(x,y) + u_1(x,y) + ... + u_n(x,y) \\ &= A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi(H-y)}{L}\right) \end{aligned}$$

We can now include our nonhomogeneous solution and get the following:

$$u(x,0) = g(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi H}{L}\right)$$

Using the orthogonality of cosines, we get:

$$A_0 = \frac{1}{L}\int_0^L g(x)\,dx = \frac{1}{L}\left(\int_0^{L/2} 1\,dx + \int_{L/2}^{L} 0\,dx\right) = \frac{1}{L}\left(\frac{L}{2}\right) = \frac{1}{2}$$

$$\begin{aligned} A_n &= \frac{2}{L\cosh\left(\frac{n\pi H}{L}\right)} \int_0^L g(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{2}{L\cosh\left(\frac{n\pi H}{L}\right)} \left(\int_0^{L/2} \cos\left(\frac{n\pi x}{L}\right)\,dx + \int_{L/2}^{L} 0\,dx\right) \\ &= \frac{2}{L\cosh\left(\frac{n\pi H}{L}\right)} \left(\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^{L/2}\right) \\ &= \frac{2}{n\pi\cosh\left(\frac{n\pi H}{L}\right)} \sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

Thus we get our desired solution:

$$\boldsymbol{u(x,y) = \frac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{n\pi\cosh\left(\frac{n\pi H}{L}\right)}\sin\left(\frac{n\pi}{2}\right)\right]\cos\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi(H-y)}{L}\right)}$$

Exercise 2.5.2: Consider $u(x,y)$ satisfying Laplace's equation inside a rectangle ($0 < x < L, 0 < y < H$) subject to the boundary conditions

$$\frac{\partial u}{\partial x}(0,y) = 0, \qquad \frac{\partial u}{\partial y}(x,0) = 0$$

$$\frac{\partial u}{\partial x}(L,y) = 0, \qquad \frac{\partial u}{\partial y}(x,H) = f(x)$$

(a) *Without* solving this problem, briefly explain the physical condition under which there is a solution to this problem.

We know that this rectangle is insulated on 3 sides: $\frac{\partial u}{\partial x}(0,y) = 0$ (no change in x at (0,y), so left side of rectangle is insulated), $\frac{\partial u}{\partial x}(L,y) = 0$ (no change in x at (L,y), so right side of rectangle is insulated), $\frac{\partial u}{\partial y}(x,0) = 0$ (no change in y at (x,0), so bottom side of rectangle is insulated). Thus the only way for this to have a solution is for the top of the rectangle to be insulated:

$$\boldsymbol{\int_0^L \frac{\partial u}{\partial y}(x,H)\,dx = \int_0^L f(x)\,dx = 0}$$

We take the integral from 0 to L to denote that the net change in heat energy between the left and right endpoints is 0.

(b) Solve this problem by the method of separation of variables. Show that the method works only under the condition of part (a). [*Hint*: You may use (5.16) without derivation.]

Let the following be true:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 \le x \le L, \qquad 0 \le y \le H$$

with

$$u(x,y) = \phi(x)h(y), \qquad \phi'(0) = 0, \quad \phi'(L) = 0, \quad h'(0) = 0.$$

Taking Laplace's Equation, we get the following:

$$\phi''(x)h(y) + \phi(x)h(y)'' = 0$$

$$\frac{\phi''(x)}{\phi(x)} = -\frac{h''(y)}{h(y)} = -\lambda$$

From this, we get the following:

$$\phi'' + \lambda\phi = 0, \qquad h'' - \lambda h = 0$$

From this, we can see this is an eigenvalue problem:

(a) ($\lambda = 0$):

$$\phi'' = 0 \quad \rightarrow \quad \phi' = c_1 \quad \rightarrow \quad \phi = c_1 x + c_2$$
$$h'' = 0 \quad \rightarrow \quad h' = d_1 \quad \rightarrow \quad h = d_1 y + d_2$$

Substituting in our BC's, we get:

$$\phi'(0) = \phi'(L) = c_1 = 0 \quad \rightarrow \quad \phi(x) = c_2$$

So now we have our first eigenfunction:

$$\phi(x) = c_2 \text{ with } \lambda = 0$$

Now we can solve for $h(y)$:

$$h'(0) = d_1 = 0 \quad \rightarrow \quad h(y) = d_2$$

From here, we get our first product solution:

$$\boldsymbol{u_0(x, y) = A_0}$$

(b) ($\lambda < 0$):

$$\phi'' - |\lambda|\phi = 0$$

Using the characteristic equation, we get:

$$\phi(x) = c_1 \cosh(\sqrt{|\lambda|}x) + c_2 \sinh(\sqrt{|\lambda|}x) \qquad \phi'(x) = c_1\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}x) + c_2\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}x)$$

Using the BC's, we get:

$$\phi'(0) = c_2\sqrt{|\lambda|} = 0 \quad \rightarrow \quad \sqrt{|\lambda|} > 0 \quad \rightarrow \quad c_2 = 0$$
$$\phi'(L) = c_1\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}L) = 0 \quad \rightarrow \quad \sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}L) \neq 0 \quad \rightarrow \quad c_1 = 0$$
$$\phi(x) = 0$$

Thus we get the following trivial solution:

$$\boldsymbol{u(x, y) = 0}$$

(c) ($\lambda > 0$):

$$\phi'' + \lambda\phi = 0$$

Using the characteristic equation, we get:

$$\phi(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x) \qquad \phi'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

Using the BC's, we get:

$$\phi'(0) = c_2\sqrt{\lambda} = 0 \quad \rightarrow \quad \sqrt{\lambda} > 0 \quad \rightarrow \quad c_2 = 0$$
$$\phi'(L) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0$$

(i) ($c_1 = 0$):

$$\phi(x) = 0$$

Thus we get the following trivial solution:

$$\boldsymbol{u(x,y) = 0}$$

(ii) ($\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0$):

$$\sin(\sqrt{\lambda}L) = 0 \qquad \rightarrow \qquad \sqrt{\lambda}L = n\pi \qquad \rightarrow \qquad \lambda = \frac{n^2\pi^2}{L^2}$$

So now we have our n eigenfunctions:

$$\phi_n(x) = c_1 \cos\left(\frac{n\pi x}{L}\right)$$

we can now substitute our eigenvalues into the other ODE, and we get:

$$h'' - \frac{n^2\pi^2}{L^2}h = 0$$

When solving this ODE, we get the following linear independent solutions:

$$h_n(y) = d_1 \cosh\left(\frac{n\pi y}{L}\right) + d_2 \sinh\left(\frac{n\pi y}{L}\right)$$

$$h_n'(y) = d_1\frac{n\pi}{L} \sinh\left(\frac{n\pi y}{L}\right) + d_2\frac{n\pi}{L} \cosh\left(\frac{n\pi y}{L}\right)$$

If we substitute our BC ($h'(0) = 0$) in, we get

$$h_n(0) = 0 = d_2 \qquad \rightarrow \qquad h_n(y) = d_1 \cosh\left(\frac{n\pi y}{L}\right)$$

From here, we get the following n product solutions:

$$\boldsymbol{u_n(x,y) = A_n \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi y}{L}\right)}$$

By the Principle of Superposition, we get the following:

$$\begin{aligned}
u(x,y) &= u_0(x,y) + u_1(x,y) + ... + u_n(x,y) \\
&= A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi y}{L}\right)
\end{aligned}$$

We can now include our nonhomogeneous solution and get the following:

$$\frac{\partial u}{\partial y}(x,H) = f(x) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \left(\frac{n\pi}{L} \sinh\left(\frac{n\pi H}{L}\right)\right)$$

Notice we can verify our assertion of part (a):

$$\begin{aligned}
\int_0^L f(x)\,dx &= \int_0^L \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) \left(\frac{n\pi}{L} \sinh\left(\frac{n\pi H}{L}\right)\right)\,dx \\
&= \sum_{n=1}^{\infty} A_n \left[\int_0^L \cos\left(\frac{n\pi x}{L}\right)\,dx\right] \left(\frac{n\pi}{L} \sinh\left(\frac{n\pi H}{L}\right)\right)\,dx \\
&= \sum_{n=1}^{\infty} A_n \left[\frac{L}{n\pi} \sin\left(\frac{n\pi x}{L}\right)\Big|_0^L\right] \left(\frac{n\pi}{L} \sinh\left(\frac{n\pi H}{L}\right)\right)\,dx \\
&= 0
\end{aligned}$$

Using the orthogonality of cosines, we get:

$$A_n = \frac{2}{n\pi \sinh\left(\frac{n\pi H}{L}\right)} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right)\,dx$$

Thus we get our desired solution:

$$\begin{aligned}
\boldsymbol{u(x,y)} &\boldsymbol{= A_0} \\
&\boldsymbol{+ \sum_{n=1}^{\infty} \left[\frac{2}{n\pi \sinh\left(\frac{n\pi H}{L}\right)} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right)\,dx\right] \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi y}{L}\right)}
\end{aligned}$$

(c) The solution [part (b)] has an arbitrary constant. Determine it by consideration of the time-dependent heat equation subject to the initial condition

$$u(x,y,0) = g(x,y)$$

Because we know that three sides of the rectangle are insulated, we get that the total energy in the rectangle $\mathcal{D}$ must be constant, so we get the following:

$$\begin{aligned}
\int\int_{\mathcal{D}} u(x,y,t)\,dx\,dy &= \int\int_{\mathcal{D}} u(x,y,0)\,dx\,dy \\
&= \int\int_{\mathcal{D}} g(x,y)\,dx\,dy \\
&= \int\int_{\mathcal{D}} u(x,y)\,dx\,dy \\
&= \int\int_{\mathcal{D}} \left(A_0 + \sum_{n=1}^{\infty}\left[\frac{2}{n\pi\sinh\left(\frac{n\pi H}{L}\right)}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx\right]\ldots \right. \\
&\qquad\qquad \left.\cos\left(\frac{n\pi x}{L}\right)\cosh\left(\frac{n\pi y}{L}\right)\right)\,dx\,dy \\
&= A_0\int_0^L\int_0^H\,dxdy + \sum_{n=1}^{\infty}\left[\frac{2}{n\pi\sinh\left(\frac{n\pi H}{L}\right)}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx\right]\ldots \\
&\qquad\qquad \left(\int_0^L \cos\left(\frac{n\pi x}{L}\right)\,dx\right)\left(\int_0^H \cosh\left(\frac{n\pi y}{L}\right)\,dy\right) \\
&= A_0LH + \sum_{n=1}^{\infty}\left[\frac{2}{n\pi\sinh\left(\frac{n\pi H}{L}\right)}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx\right]\ldots \\
&\qquad\qquad \left(\int_0^L \cos\left(\frac{n\pi x}{L}\right)\,dx\right)\left(\int_0^H \cosh\left(\frac{n\pi y}{L}\right)\,dy\right)
\end{aligned}$$

Notice we get the following:

$$\int_0^L \cos\left(\frac{n\pi x}{L}\right)\,dx = \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L = 0$$

Thus, we get the following:

$$\int\int_{\mathcal{D}} g(x,y)\,dx\,dy = A_0LH$$

Which gives us our arbitrary constant:

$$\boldsymbol{A_0 = \frac{1}{LH}\int\int_{\mathcal{D}} g(x,y)\,dx\,dy}$$

Exercise 2.5.6b: Solve Laplace's equation inside a semicircle of radius $a(0 < r < a, 0 < \theta < \pi)$ subject to the boundary conditions [Hint: In polar coordinates,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

it is known that if $u(r,\theta) = \phi(\theta)G(r)$, then $\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}$.]:

The diameter is insulated and $u(a,\theta) = g(\theta)$.

Let the following be true:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad 0 < r < a, \qquad 0 < \theta < \pi$$

with

$$u(r,\theta) = \phi(\theta)G(r)$$

Notice the boundary conditions:

$$\phi'(0) = 0 \qquad \phi'(\pi) = 0 \qquad u(a,\theta) = g(\theta)$$

Taking Laplace's Equation, we get the following:

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{\phi''}{\phi} = \lambda$$

From this, we get the following:

$$\phi'' + \lambda\phi = 0 \qquad r\frac{d}{dr}\left(r\frac{dG}{dr}\right) - \lambda G = r^2 G'' + rG' - \lambda G = 0$$

From this, we can see this is an eigenvalue problem:

(a) ($\lambda = 0$):

$$\phi'' = 0 \qquad \rightarrow \qquad \phi' = c_1 \qquad \rightarrow \qquad \phi = c_1\theta + c_2$$

$$r\frac{d}{dr}\left(r\frac{dG}{dr}\right) = 0 \qquad \rightarrow \qquad r\frac{dG}{dr} = d_1 \qquad \rightarrow \qquad G = d_1 \ln r + d_2$$

Substituting in our BC's, we get:

$$\phi'(0) = \phi'(\pi) = c_1 = 0$$

So now we have our first eigenfunction:

$$\phi(\theta) = c_2 \text{ with } \lambda = 0$$

Now we can solve for $G(r)$ using the boundedness condition. This implies that $d_1 = 0$, thus we get:

$$G(r) = d_2$$

From here, we get our first product solution:

$$\boldsymbol{u_0(r,\theta) = A_0}$$

(b) ($\lambda < 0$):

$$\phi'' - |\lambda|\phi = 0$$

Using the characteristic equation, we get:

$$\phi(\theta) = c_1 \cosh(\sqrt{|\lambda|}\theta) + c_2 \sinh(\sqrt{|\lambda|}\theta) \qquad \phi'(\theta) = c_1\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}\theta) + c_2\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}\theta)$$

Using the BC's, we get:

$$\phi'(0) = c_2\sqrt{|\lambda|} = 0 \qquad \rightarrow \qquad c_2 = 0$$

$$\phi'(\pi) = c_1\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}\pi) = 0 \qquad \sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}\pi) \neq 0 \qquad \rightarrow \qquad c_1 = 0$$

Thus we get the following trivial solution:

$$\boldsymbol{u(r,\theta) = 0}$$

(c) ($\lambda > 0$):

$$\phi'' + \lambda\phi = 0$$

Using the characteristic equation, we get:

$$\phi(\theta) = c_1\cos(\sqrt{\lambda}\theta) + c_2\sin(\sqrt{\lambda}\theta) \qquad \phi'(\theta) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\theta) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\theta)$$

Using the BC's, we get:

$$\phi'(0) = c_2\sqrt{\lambda} = 0 \qquad \rightarrow \qquad c_2 = 0$$
$$\phi'(\pi) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)$$

(i) ($c_1 = 0$):

$$\phi(\theta) = 0$$

Thus we get the following trivial solution:

$$\boldsymbol{u(r,\theta) = 0}$$

(ii) ($\sin(\sqrt{\lambda}\pi) = 0$):

$$\sin(\sqrt{\lambda}\pi) = 0 \qquad \rightarrow \qquad \sqrt{\lambda}\pi = n\pi \qquad \rightarrow \qquad \lambda = n^2$$

So now we have our n eigenfunctions:

$$\phi_n(\theta) = c_1\cos(n\theta)$$

we can now substitute our eigenvalues into the other ODE, and we get:

$$r^2G'' + rG' - n^2G = 0$$

Let the following be true:

$$G = cr^{\alpha} \qquad \rightarrow \qquad G' = \alpha cr^{\alpha-1} \qquad \rightarrow \qquad G'' = (\alpha^2 - \alpha)cr^{\alpha-2}$$

$$\begin{aligned} r^2(\alpha^2 - \alpha)cr^{\alpha-2} + r\alpha cr^{\alpha-1} - n^2cr^{\alpha} &= 0 \\ r^{\alpha}c\left(\alpha^2 - \alpha + \alpha - n^2\right) &= 0 \\ \alpha &= \pm n \end{aligned}$$

When solving this ODE, we get the following linear independent solutions:

$$G(r) = d_1r^{-n} + d_2r^{n}$$

ow we can solve for $G(r)$ using boundedness condition. This implies that $d_1 = 0$, thus we get:

$$G_n(r) = d_2r^n$$

From here, we get the following n product solutions:

$$\boldsymbol{u_n(r,\theta) = A_nr^n\cos(n\theta)}$$

By the Principle of Superposition, we get the following:

$$\begin{aligned} u(r,\theta) &= u_0(r,\theta) + u_1(r,\theta) + ... + u_n(r,\theta) \\ &= A_0 + \sum_{n=1}^{\infty} A_n r^n \cos(n\theta) \end{aligned}$$

We can now include our nonhomogeneous solution and get the following:

$$u(a,\theta) = g(\theta) = A_0 + \sum_{n=1}^{\infty} A_n \, a^n \cos(n\theta)$$

Using the orthogonality of cosines, we get:

$$A_0 = \frac{1}{\pi}\int_0^{\pi} g(\theta)\, d\theta \qquad A_n = \frac{2}{a^n \pi}\int_0^{\pi} g(\theta)\cos(n\theta)\, d\theta$$

Thus we get our desired solution:

$$\boldsymbol{u(x,y) = \frac{1}{\pi}\int_0^{\pi} g(\theta)\, d\theta + \sum_{n=1}^{\infty}\left[\frac{2}{a^n\pi}\int_0^{\pi} g(\theta)\cos(n\theta)\, d\theta\right] r^n \cos(n\theta)}$$

Exercise 2.5.8b: Solve Laplace's equation inside a circular annulus ($a < r < b$) subject to the boundary conditions [Hint: In polar coordinates,

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

it is known that if $u(r,\theta) = \phi(\theta)G(r)$, then $\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2}$.]:

$$\frac{\partial u}{\partial r}(a,\theta) = 0, \qquad u(b,\theta) = g(\theta)$$

Let the following be true:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad 0 < r < a, \qquad 0 < \theta < \pi$$

with

$$u(r,\theta) = \phi(\theta)G(r)$$

Notice the boundary conditions:

$$\phi(-\pi) = \phi(\pi) \qquad \phi'(-\pi) = \phi'(\pi) \qquad u'(a,\theta) = 0 \qquad u(b,\theta) = g(\theta)$$

Taking Laplace's Equation, we get the following:

$$\frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{\phi''}{\phi} = \lambda$$

From this, we get the following:

$$\phi'' + \lambda\phi = 0 \qquad r\frac{d}{dr}\left(r\frac{dG}{dr}\right) - \lambda G = r^2G'' + rG' - \lambda G = 0$$

From this, we can see this is an eigenvalue problem:

(a) ($\lambda = 0$):

$$\phi'' = 0 \qquad \rightarrow \qquad \phi' = c_1 \qquad \rightarrow \qquad \phi = c_1\theta + c_2$$

$$r\frac{d}{dr}\left(r\frac{dG}{dr}\right) = 0 \qquad \rightarrow \qquad r\frac{dG}{dr} = d_1 \qquad \rightarrow \qquad G = d_1 \ln r + d_2$$

Substituting in our BC's, we get:

$$\phi(-\pi) = -c_1\pi + c_2 = c_1\pi + c_2 = \phi(\pi) \qquad \rightarrow \qquad c_1 = 0$$

$$\phi'(-\pi) = c_1 = 0 = \phi'(\pi)$$

So now we have our first eigenfunction:

$$\phi(\theta) = c_2 \text{ with } \lambda = 0$$

From here, we do not have any conditions on $G(r)$, so we get our first product solution:

$$\boldsymbol{u_0(r,\theta) = c_2 G(r) = A_0 \ln r + B_0}$$

(b) ($\lambda < 0$):

$$\phi'' - |\lambda|\phi = 0$$

Using the characteristic equation, we get:

$$\phi(\theta) = c_1 \cosh(\sqrt{|\lambda|}\theta) + c_2 \sinh(\sqrt{|\lambda|}\theta) \qquad \phi'(\theta) = c_1\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}\theta) + c_2\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}\theta)$$

Using the BC's, we get:

$$\phi(-\pi) = c_1 \cosh(\sqrt{|\lambda|}\pi) - c_2 \sinh(\sqrt{|\lambda|}\pi) = c_1 \cosh(\sqrt{|\lambda|}\pi) + c_2 \sinh(\sqrt{|\lambda|}\pi) = \phi(\pi)$$

$$c_2 = 0$$

$$\phi'(-\pi) = -c_1\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}\pi) = c_1\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}\pi) = \phi'(\pi)$$

$$c_1 = 0$$

Thus we get the following trivial solution:

$$\boldsymbol{u(r,\theta) = 0}$$

(c) ($\lambda > 0$):

$$\phi'' + \lambda\phi = 0$$

Using the characteristic equation, we get:

$$\phi(\theta) = c_1\cos(\sqrt{\lambda}\theta) + c_2\sin(\sqrt{\lambda}\theta) \qquad \phi'(\theta) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\theta) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\theta)$$

Using the BC's, we get:

$$\phi(-\pi) = c_1\cos(\sqrt{\lambda}\pi) - c_2\sin(\sqrt{\lambda}\pi) = c_1\cos(\sqrt{\lambda}\pi) + c_2\sin(\sqrt{\lambda}\pi) = \phi(\pi)$$

$$\phi'(-\pi) = c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\pi) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) = \phi'(\pi)$$

(i) ($c_1 = c_2 = 0$):

$$\phi(\theta) = 0$$

Thus we get the following trivial solution:

$$\boldsymbol{u(r,\theta) = 0}$$

(ii) ($\sin(\sqrt{\lambda}\pi) = 0$):

$$\sin(\sqrt{\lambda}\pi) = 0 \qquad \rightarrow \qquad \sqrt{\lambda}\pi = n\pi \qquad \rightarrow \qquad \lambda = n^2$$

So now we have our n eigenfunctions:

$$\phi_n(\theta) = c_1\cos(n\theta) + c_2\sin(n\theta)$$

we can now substitute our eigenvalues into the other ODE, and we get:

$$r^2G'' + rG' - n^2G = 0$$

Let the following be true:

$$G = cr^{\alpha} \qquad \rightarrow \qquad G' = \alpha cr^{\alpha-1} \qquad \rightarrow \qquad G'' = (\alpha^2 - \alpha)cr^{\alpha-2}$$

$$\begin{aligned} r^2(\alpha^2 - \alpha)cr^{\alpha-2} + r\alpha cr^{\alpha-1} - n^2cr^{\alpha} &= 0 \\ r^{\alpha}c\left(\alpha^2 - \alpha + \alpha - n^2\right) &= 0 \\ \alpha &= \pm n \end{aligned}$$

When solving this ODE, we get the following linear independent solutions:

$$G(r) = d_1r^{-n} + d_2r^n$$

From here, we do not have any conditions on $G(r)$, so we get our first product solution:

$$G_n(r) = d_1r^{-n} + d_2r^n$$

From here, we get the following n product solutions:

$$\boldsymbol{u_n(r,\theta) = \left(A_nr^{-n} + B_nr^n\right)\cos(n\theta) + \left(C_nr^{-n} + D_nr^n\right)\sin(n\theta)}$$

By the Principle of Superposition, we get the following:

$$\begin{aligned}u(r,\theta) &= u_0(r,\theta) + u_1(r,\theta) + ... + u_n(r,\theta) \\ &= A_0 \ln\, r + B_0 + \sum_{n=1}^{\infty} \left[\left(A_n r^{-n} + B_n r^n \right) \cos(n\theta) + \left(C_n r^{-n} + D_n r^n \right) \sin(n\theta) \right]\end{aligned}$$

Now we can take the partial derivative in respect to r, so we can use our homogeneous BC:

$$\frac{\partial}{\partial r} u(r,\theta) = \frac{A_0}{r} + \sum_{n=1}^{\infty} \left[\left(-n\, A_n r^{-n-1} + n\, B_n r^{n-1} \right) \cos(n\theta) + \left(-n\, C_n r^{-n-1} + n\, D_n r^{n-1} \right) \sin(n\theta) \right]$$

$$\frac{\partial}{\partial r} u(a,\theta) = \frac{A_0}{a} + \sum_{n=1}^{\infty} \left[\left(-n\, A_n a^{-n-1} + n\, B_n a^{n-1} \right) \cos(n\theta) + \left(-n\, C_n a^{-n-1} + n\, D_n a^{n-1} \right) \sin(n\theta) \right] = 0$$

Because $\cos(n\theta)$ and $\sin(n\theta)$ oscillate, it is impossible to solve this inequality unless we set their coefficients to 0. Then we get the following:

$$A_0 = 0$$

$$nA_n a^{-n-1} = nB_n a^{n-1} \quad \rightarrow \quad A_n = B_n a^{2n} \qquad nC_n a^{-n-1} = nD_n a^{n-1} \quad \rightarrow \quad C_n = D_n a^{2n}$$

Now we can include our nonhomogeneous BC:

$$\begin{aligned}u(b,\theta) = g(\theta) &= A_0 \ln\, b + B_0 + \sum_{n=1}^{\infty} \left[\left(A_n b^{-n} + B_n b^n \right) \cos(n\theta) + \left(C_n b^{-n} + D_n b^n \right) \sin(n\theta) \right] \\ &= B_0 + \sum_{n=1}^{\infty} \left[\left(B_n a^{2n} b^{-n} + B_n b^n \right) \cos(n\theta) + \left(D_n a^{2n} b^{-n} + D_n b^n \right) \sin(n\theta) \right] \\ &= B_0 + \sum_{n=1}^{\infty} B_n \left(a^{2n} b^{-n} + b^n \right) \cos(n\theta) + \sum_{n=1}^{\infty} D_n \left(a^{2n} b^{-n} + b^n \right) \sin(n\theta)\end{aligned}$$

Using the orthogonality of cosines and sines, we get:

$$B_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta)\, d\theta$$

$$B_n = \frac{1}{\pi(a^{2n} b^{-n} + b^n)} \int_{-\pi}^{\pi} g(\theta) \cos(n\theta)\, d\theta \qquad D_n = \frac{1}{\pi(a^{2n} b^{-n} + b^n)} \int_{-\pi}^{\pi} g(\theta) \sin(n\theta)\, d\theta$$

Thus we get our desired solution:

$$\boldsymbol{\begin{aligned}u(r,\theta) = &\frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta)\, d\theta \\ &+ \sum_{n=1}^{\infty} \left(\frac{1}{\pi(a^{2n} b^{-n} + b^n)} \int_{-\pi}^{\pi} g(\theta) \cos(n\theta)\, d\theta \right) \left(a^{2n} b^{-n} + b^n \right) \cos(n\theta) \\ &+ \sum_{n=1}^{\infty} \left(\frac{1}{\pi(a^{2n} b^{-n} + b^n)} \int_{-\pi}^{\pi} g(\theta) \sin(n\theta)\, d\theta \right) \left(a^{2n} b^{-n} + b^n \right) \sin(n\theta)\end{aligned}}$$

Exercise 2.5.15b: Solve Laplace's equation inside a semi-infinite strip ($0 < x < \infty, 0 < y < H$) subject to the boundary conditions [Hint: In Cartesian coordinates, $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, inside a semi infinite strip ($0 \le y \le H$ and $0 \le x \le \infty$), it is known that if $u(x,y) = F(x)G(y)$, then $\frac{1}{F}\frac{d^2F}{dx^2} = -\frac{1}{G}\frac{d^2G}{dy^2}$.]:

$$u(x,0) = 0 \qquad u(x,H) = 0, \qquad u(0,y) = f(y)$$

Let the following be true:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad 0 \le x \le \infty, \qquad 0 \le y \le H$$

with

$$u(x,y) = h(x)\phi(y), \qquad \phi(0) = 0, \quad \phi(H) = 0, \quad h(0) = f(y)$$

Taking Laplace's Equation, we get the following:

$$\phi''(y)h(x) + \phi(y)h(x)'' = 0$$

$$\frac{\phi''(y)}{\phi(y)} = -\frac{h''(x)}{h(x)} = -\lambda$$

From this, we get the following:

$$\phi'' + \lambda\phi = 0, \qquad h'' - \lambda h = 0$$

From this, we can see this is an eigenvalue problem:

(a) ($\lambda = 0$):

$$\phi'' = 0 \qquad \rightarrow \qquad \phi' = c_1 \qquad \rightarrow \qquad \phi = c_1 y + c_2$$

Substituting in our BC's, we get:

$$\phi(0) = c_2 = 0 = c_1 H + c_2 = \phi(H)$$

So now we have our first eigenfunction:

$$\phi(y) = 0 \text{ with } \lambda = 0$$

From here, we get the following trivial solution:

$$\boldsymbol{u(x, y) = 0}$$

(b) ($\lambda < 0$):

$$\phi'' - |\lambda|\phi = 0$$

Using the characteristic equation, we get:

$$\phi(y) = c_1 \cosh(\sqrt{|\lambda|}y) + c_2 \sinh(\sqrt{|\lambda|}y)$$

Using the BC's, we get:

$$\phi(0) = c_1 = 0 \qquad \phi(H) = c_2 \sinh(\sqrt{|\lambda|}H) = 0 \qquad \rightarrow \qquad c_2 = 0$$

Thus we get the following trivial solution:

$$\boldsymbol{u(x, y) = 0}$$

(c) ($\lambda > 0$):

$$\phi'' + \lambda\phi = 0$$

Using the characteristic equation, we get:

$$\phi(y) = c_1 \cos(\sqrt{\lambda}y) + c_2 \sin(\sqrt{\lambda}y)$$

Using the BC's, we get:

$$\phi(0) = c_1 = 0 \qquad \phi(H) = c_2 \sin(\sqrt{\lambda}H) = 0$$

(i) ($c_2 = 0$):

$$\phi(x) = 0$$

Thus we get the following trivial solution:

$$\boldsymbol{u(x, y) = 0}$$

(ii) $(\sqrt{\lambda}\sin(\sqrt{\lambda}H) = 0)$:

$$\sin(\sqrt{\lambda}H) = 0 \qquad \rightarrow \qquad \sqrt{\lambda}H = n\pi \qquad \rightarrow \qquad \lambda = \frac{n^2\pi^2}{H^2}$$

So now we have our n eigenfunctions:

$$\phi_n(y) = c_2 \sin\left(\frac{n\pi y}{H}\right)$$

we can now substitute our eigenvalues into the other ODE, and we get:

$$h'' - \frac{n^2\pi^2}{H^2}h = 0$$

When solving this ODE, we get the following linear independent solutions:

$$h_n(x) = d_1 e^{-\frac{n\pi x}{H}} + d_2 e^{\frac{n\pi x}{H}}$$

By the boundedness condition, we set $d_2 = 0$, so that as x grows, the function is still bounded.

$$h_n(x) = d_1 e^{-\frac{n\pi x}{H}}$$

From here, we get the following n product solutions:

$$\boldsymbol{u_n(x, y) = B_n \sin\left(\frac{n\pi y}{H}\right) e^{-\frac{n\pi x}{H}}}$$

By the Principle of Superposition, we get the following:

$$\begin{aligned} u(x, y) &= u_0(x, y) + u_1(x, y) + ... + u_n(x, y) \\ &= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi y}{H}\right) e^{-\frac{n\pi x}{H}} \end{aligned}$$

We can now include our nonhomogeneous solution and get the following:

$$u(0, y) = f(y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi y}{H}\right)$$

Using the orthogonality of sines, we get:

$$B_n = \frac{2}{H}\int_0^H f(y) \sin\left(\frac{n\pi y}{H}\right)\, dy$$

Thus we get our desired solution:

$$\boldsymbol{u(x, y) = \sum_{n=1}^{\infty} \left[\frac{2}{H}\int_0^H f(y) \sin\left(\frac{n\pi y}{H}\right)\, dy\right] \sin\left(\frac{n\pi y}{H}\right) e^{-\frac{n\pi x}{H}}}$$