MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #20

Chapter 5

Section 5.6 Genericity

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Important Concepts

- Generic: typical, large subset
- Structural Stability: a flow with a center is not structurally stable
- Hyperbolic: $Re(\lambda) \neq 0$
- Open: if X belongs to an open set U, any point sufficiently near to X also belongs to U
- Dense: $\mathrm{U} \in R^n$ is dense if there are points in U arbitrarily close to each point in R^n

Sect. 5.6: Genericity

We have mentioned several times that "most" matrices have distinct eigenvalues. Our goal in this section is to make this precise.

- A system is 'generic' if it does the 'typical' things and avoids the exceptional things that occur with zero probability (Sprott, p81)
- The word "generic" refers to behavior that is typical—behavior that we normally expect to see (Alliwood et al. 1996)
- In measure theory, a generic property is one that holds almost everywhere (Wikipedia)
- Intuitively speaking, a generic property is one that "almost all" matrices have (Hirsch et al. 2013).

Structural Stability

- **Roughly speaking, a dynamical system (vector field or map) is said to be structurally stable if nearby systems have qualitatively the same dynamics (Wiggins, 0000)**
- In applications we require our mathematical models to be robust. By this we mean that **their qualitative properties should not change significantly when the model is subjected to small, allowable perturbations** (Arrowsmith & Place, 1990).
- A dynamical system whose topological properties are shared by all sufficiently close neighboring systems is said to be structurally stable (Arrowsmith & Place, 1990).
- In § 3.1, we show that a linear flow is structurally stable if and only if it is hyperbolic and that hyperbolicity of flow is a generic property of linear transformations on R^n (Arrowsmith & Place, 1990).
- Thus the structurally stable linear flows are characterized by a single, hyperbolic fixed point at the origin and structural stability of flow is a generic property of linear transformations (Arrowsmith & Place, 1990).

An Example: Non-hyperbolic Critical Point

Example 12.0.1. Consider the simple harmonic oscillator

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\omega_0^2 x, \end{aligned} \qquad (x, y) \in \mathbb{R}^2. \tag{12.0.1}$$

We know everything about this system. It has a nonhyperbolic fixed point of $(x, y) = (0, 0)$ surrounded by a one-parameter family of periodic orbits, each having frequency ω_0. The phase portrait of (12.0.1) is shown in Figure 12.0.1 (note: strictly speaking, the phase curves are circles for $\omega_0 = 1$ and ellipses otherwise). Is (12.0.1) stable with respect to perturbations (note: this is a new concept of stability, as opposed to the idea of stability of specific solutions discussed in Chapter 1)? Let us try a few perturbations and see what happens.

$$\boxed{x'' = -\omega_o^2 x}$$

Center

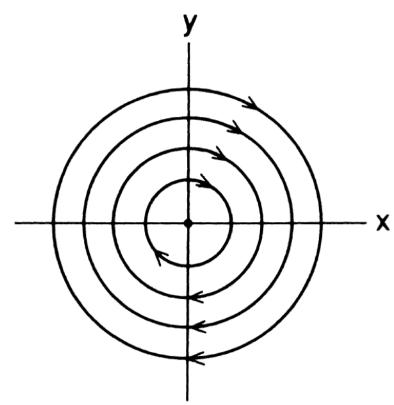


FIGURE 12.0.1.

Wiggins (2003)

(I) Disappearance of Periodic Orbits

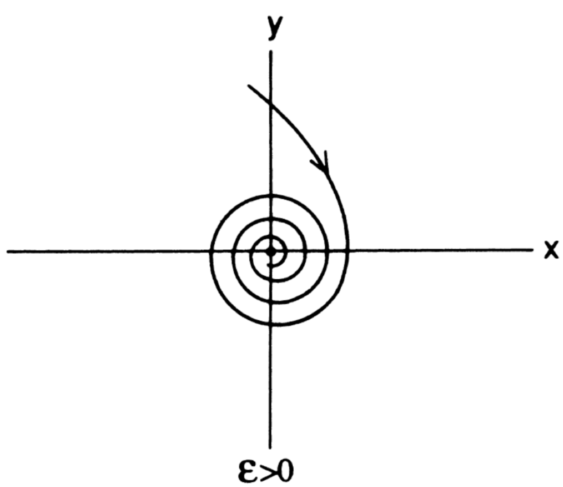
Consider the perturbed system

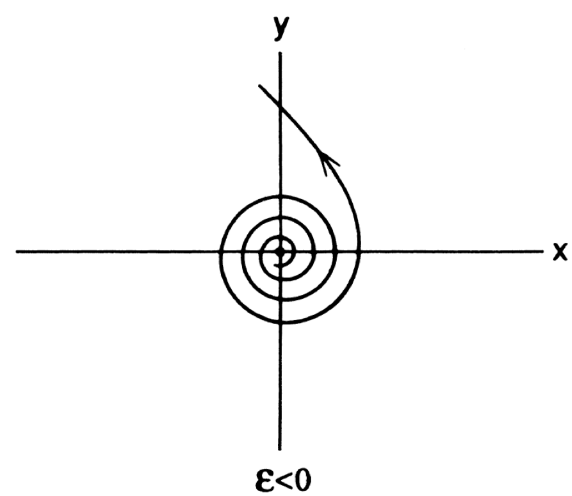
$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= -\omega_0^2 x - \varepsilon y.\end{aligned} \tag{12.0.2}$$

It is easy to see that the origin is a hyperbolic fixed point, a sink for $\varepsilon > 0$ and a source for $\varepsilon < 0$. However, all the periodic orbits are destroyed (use Bendixson's criteria). Thus, this perturbation radically alters the structure of the phase space of (12.0.1); see Figure 12.0.2.

$$\boxed{x'' = -\omega_0^2 x - \epsilon x'}$$

Linear, Dissipative Perturbation





Wiggins (2003)

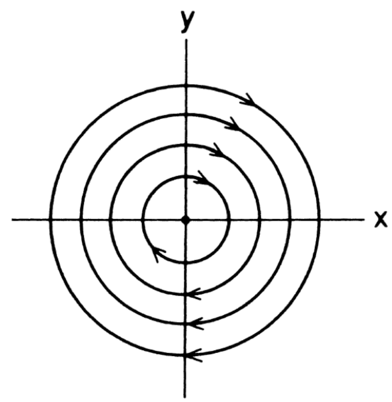
Sink

Source

(I) Structurally Unstable

$$\boxed{x'' = -\omega_o^2 x}$$

Center

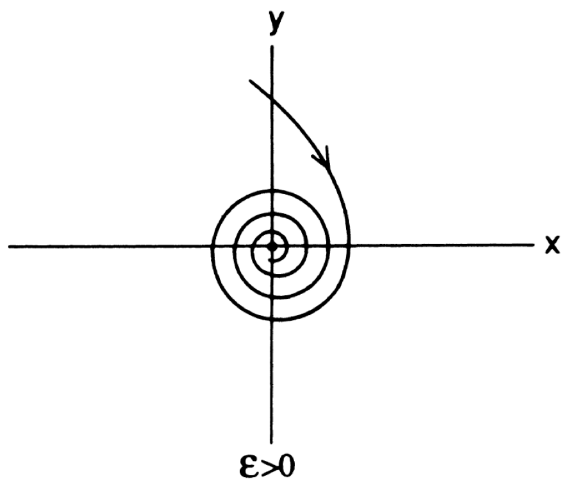


$$\lambda = \pm i\beta$$

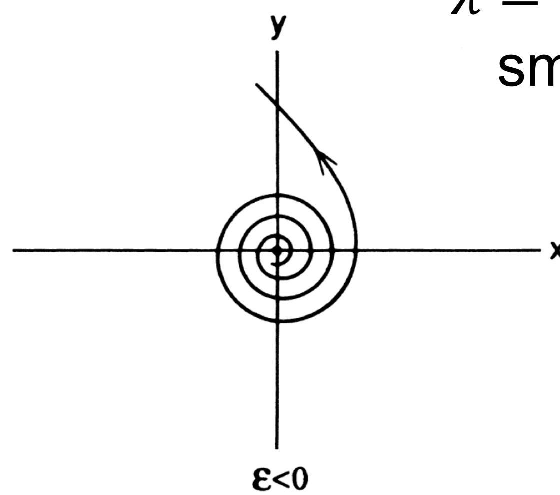
$$\boxed{x'' = -\omega_o^2 x - \epsilon x'}$$

$\lambda = \alpha \pm i\beta,$
small α

Linear, Dissipative Perturbation



Sink



Source

Wiggins (2003)

(II) Nonlinear Perturbation

Nonlinear Perturbation

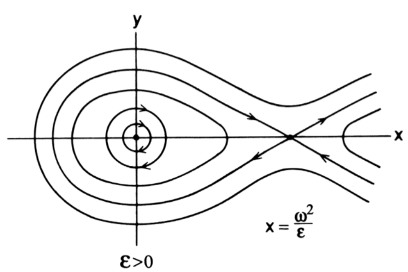
Consider the perturbed system

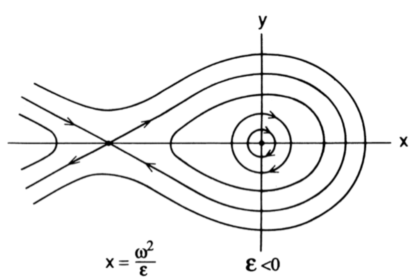
$$\dot{x} = y, \qquad\qquad (12.0.3)$$
$$\dot{y} = -\omega_0^2 x + \boxed{\varepsilon x^2}.$$

The perturbed system now has two fixed points given by

$$(x, y) = (0, 0), \qquad\qquad (12.0.4)$$
$$(x, y) = (\omega_0^2/\varepsilon, 0).$$

The origin is still a center (i.e., unchanged by the perturbation), and the new fixed point is a saddle and far away for ε small.



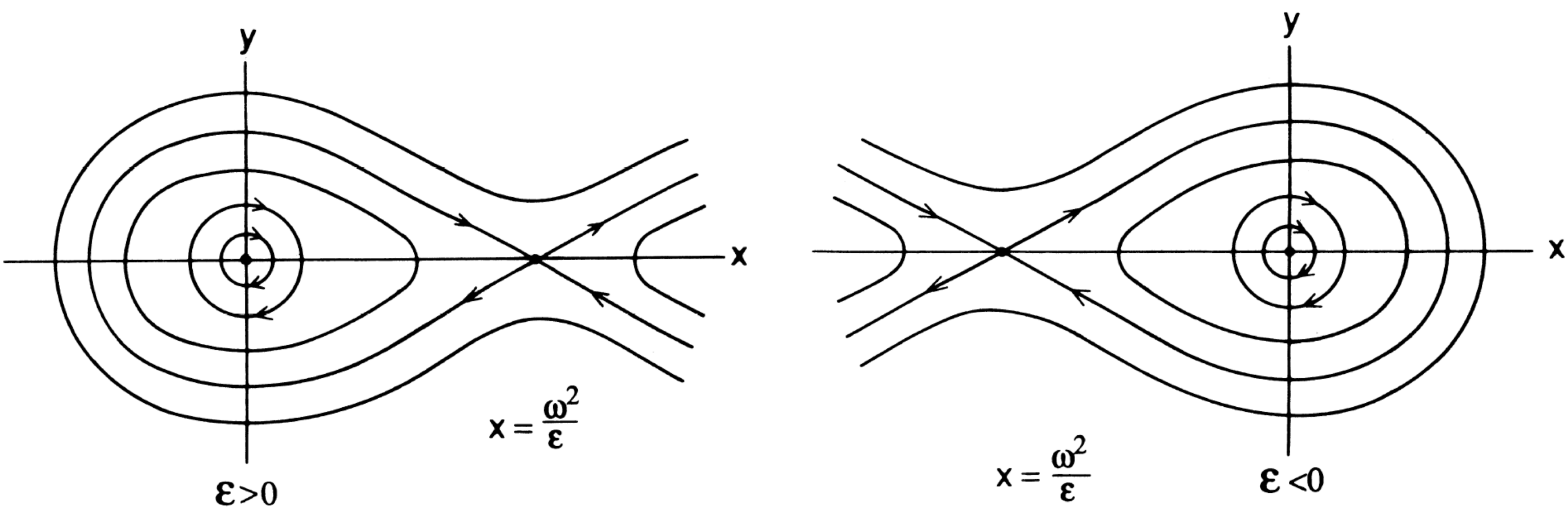


Wiggins (2003)

(II) Nonlinear Perturbation

Center

Center



Wiggins (2003)

(III) Time-Dependent Perturbation

Consider the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\omega_0^2 x + \varepsilon x \cos t. \end{aligned} \tag{12.0.6}$$

This perturbation is of a very different character than the previous two. Writing (12.0.6) as an autonomous system (see Chapter 7)

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\omega_0^2 x + \varepsilon x \cos \theta, \\ \dot{\theta} &= 1, \end{aligned} \tag{12.0.7}$$

we see that the time-dependent perturbation has the effect of enlarging the dimension of the system. However, in any case, $(x, y) = (0, 0)$ is still a fixed point of (12.0.6), although it is interpreted as a periodic orbit of (12.0.7). We now ask what the nature of the flow is near $(x, y) = (0, 0)$, which is a difficult question to answer due to the time dependence. Equation (12.0.6) is known as the Mathieu equation, and for $\omega_0 = n/2$, n an integer, it is possible for the system to exhibit parametric resonance resulting in a solution starting near the origin that grows without bound. Thus, the flow of (12.0.7) near the origin differs very much from the flow of (12.0.1) near the origin. For more information on the Mathieu equation see Nayfeh and Mook [1979].

Arrowsmith & Place (1990)

Proposition 3.1.1 *A linear flow or diffeomorphism on* $\mathbb{R}^n$ *is structurally stable in* $L(\mathbb{R}^n)$ *if and only if it is hyperbolic.*

transformations. Let $SF(\mathbb{R}^n) \subseteq L(\mathbb{R}^n)$ denote the set of linear transformations which give rise to structurally stable flows on $\mathbb{R}^n$.

Proposition 3.1.2 *The set* $SF(\mathbb{R}^n)$ *is open and dense in* $L(\mathbb{R}^n)$.

Sect. 5.6: Open

Recall that a set $\mathcal{U} \subset \mathbb{R}^n$ is *open* if whenever $X \in \mathcal{U}$ there is an open ball about X contained in $\mathcal{U}$; that is, for some $a > 0$ (depending on X) the open ball about X of radius a,

$$\{Y \in \mathbb{R}^n \mid |Y - X| < a\},$$

is contained in $\mathcal{U}$. Using geometrical language we say that if X belongs to an open set $\mathcal{U}$, any point sufficiently near to X also belongs to $\mathcal{U}$.

1. Open set: A set U in a metric space is open if for each $x \in U$ there is an $\epsilon > 0$ such that $d(x, y) < \epsilon$ implies $y \in U$.
2. Open set: if X belongs to an open set U, any point sufficiently near to X also belongs to U.

Sect. 5.6: Dense

1. A dense set: $\mathrm{U} \in R^n$ is dense if there are points in U arbitrarily close to each point in R^n.
2. More precisely, if $\mathrm{X} \in R^n$, then for every $\epsilon > 0$ there exists some $\mathrm{Y} \in U$ and $|\mathrm{X} - \mathrm{Y}| < \epsilon$.
3. Equivalently, U is dense in R^n if $\mathrm{V} \cap \mathrm{U}$ is nonempty for every nonempty open set $\mathrm{V} \subset R^n$.

- For example, the rational numbers form a dense subset of R, as do the irrational numbers.

$$\{(x, y) \in \mathbb{R}^2 \mid \text{both } x \text{ and } y \text{ are rational}\}$$

is a dense subset of the plane.

General Mathematical Terms

- Boundary: The boundary ∂A of a closed set A is the set of points of A which are not in the interior of A.
- Closed set: A set A is closed if it contains all its limit points.
- Interior: The interior of a set A, int A, is the largest open set contained in A.
- Limit point: The point x is a limit point of a set A if every neighborhood of x contains a point of $A - \{x\}$.
- Metric space: A metric space A is a set together with a distance function $d: A \times A \rightarrow R$ which satisfies: (1) $d\,(x, y) \geq 0$ with equality if and only if $x = y$; (2) $d\,(x, y)$= $d\,(y, x)$; and (3) $d\,(x, y) + d\,(y, z) \geq d\,(x, z)$(the triangle inequality).
- Neighborhood: A neighborhood of a point x is a set U which contains x in its interior.
- Open set: A set U in a metric space is open if for each $x \in U$ there is an $\epsilon > 0$ such that $d\,(x, y) < \epsilon$ implies $y \in U$.

(Guckenheimer and Holmes, 1983)

Sect. 5.6: Open & Dense

A set is both open and dense:

- Every point in the complement of U can be approximated arbitrarily closely by points of U (since U is dense),
- but no point in U can be approximated arbitrarily closely by points in the complement (because U is open).

➢ An open and dense set is a very fat set, as the following proposition shows (p. 101 of Hirsch et al. 2013).

➢ We therefore think of a subset of R^n as being large if this set contains an open and dense subset (p. 101 of Hirsch et al. 2013).

Proposition. *Let $\mathcal{V}_1, \ldots, \mathcal{V}_m$ be open and dense subsets of $\mathbb{R}^n$. Then*

intersection

$$\mathcal{V} = \mathcal{V}_1 \cap \ldots \cap \mathcal{V}_m$$

is also open and dense.

Open & Dense

Here is a simple example of an open and dense subset of $\mathbb{R}^2$:

$$\mathcal{V} = \{(x, y) \in \mathbb{R}^2 \mid xy \neq 1\}.$$

This, of course, is the complement in $\mathbb{R}^2$ of the hyperbola defined by $xy = 1$. Suppose $(x_0, y_0) \in \mathcal{V}$. Then $x_0 y_0 \neq 1$ and if $|x - x_0|$, $|y - y_0|$ are small enough, then $xy \neq 1$; this proves that $\mathcal{V}$ is open. Given any $(x_0, y_0) \in \mathbb{R}^2$, we can find (x, y) as close as we like to (x_0, y_0) with $xy \neq 1$; this proves that $\mathcal{V}$ is dense.

Open & Dense: a Generic Property

Theorem. *The set $\mathcal{M}$ of matrices in $L(\mathbb{R}^n)$ that have n distinct eigenvalues is open and dense in $L(\mathbb{R}^n)$.*

- A property P of matrices is a generic property if the set of matrices having property P contains an open and dense set in $L(R^n)$.
- Intuitively speaking, a generic property is one that "almost all" matrices have.
- Thus, having all distinct eigenvalues is a generic property of $n \times n$ matrices.

A Proof: “dense”

Theorem. *The set $\mathcal{M}$ of matrices in $L(\mathbb{R}^n)$ that have n distinct eigenvalues is open and dense in $L(\mathbb{R}^n)$.*

- Every point in the complement of U can be approximated arbitrarily closely by points of U (since U is dense).

1. Consider the matrix with some repeated eigenvalues

2. Choose distinct values, λ_j, such that $|\lambda_j - \lambda|$ is small

$$(\mathrm{i}) \quad \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 & 1 & & & \\ & \lambda_2 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_j \end{pmatrix}.$$

- $T^1 AT$ has distinct eigenvalues;
- Similarly, we can a new matrix arbitrarily close to $T^1 AT$ with distinct eigenvalues.

A Proof: "dense"

Repeated complex eigenvalue(s)

$$
\text{(ii)} \quad \begin{pmatrix} C_2 & I_2 & & & \\ & C_2 & I_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & I_2 \\ & & & & C_2 \end{pmatrix},
$$

$$
C_2 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
$$

Choose distinct values, α_j, such that $|\alpha_j - \alpha|$ is small

$$
C_{2j} = \begin{pmatrix} \alpha_j & \beta \\ \beta & \alpha_j \end{pmatrix}
$$

- Follow the analysis in the previous slide

A Proof: ”open”

- To prove that M is open, consider the characteristic polynomial of a matrix $\mathrm{A} \in L(R^n)$.
- If we vary the entries of A slightly, then the characteristic polynomial’s coefficients vary only slightly.
- Therefore, the roots of this polynomial in $\mathbb{C}$ move only slightly as well.
- Thus, if we begin with a matrix that has distinct eigenvalues, nearby matrices have this property as well.
- This proves that M is open.

A "typical" $n \times n$ matrix has n distinct eigenvalues

In the sequel, we will denote by $\mathbb{R}^{n\times n}$ the vector space of $n \times n$ matrices with real entries. Similarly, $\mathbb{C}^{n\times n}$ will denote the vector space of $n \times n$ matrices with complex entries. Let

$$U_n = \{A \in \mathbb{C}^{n\times n} : A \text{ has } n \text{ distinct eigenvalues}\}$$

and

$$V_n = \{A \in \mathbb{R}^{n\times n} : A \text{ has } n \text{ distinct (possibly complex) eigenvalues}\}.$$

Theorem 1. *The set U_n is an open and dense subset of $\mathbb{C}^{n\times n}$.*

Theorem 2. *The set V_n is an open and dense subset of $\mathbb{R}^{n\times n}$.*

Closed vs. Open Sets

Theorem 11.5 *In the space of $n \times n$ matrices,*

(a) The set of matrices that have multiple eigenvalues (at least one eigenvalue of multiplicity 2 or more) is closed. repeated eigenvalues

(b) The set of matrices that have distinct eigenvalues is open.

Proof. There two parts.

Part a. Let $A_1, A_2, \ldots$ be a sequence of matrices which have multiple eigenvalues. Suppose the sequence converges to A. By the continuous dependence of eigenvalues, A cannot have distinct eigenvalues since if A had distinct eigenvalues, we could find small nonintersecting disks, say, of radius ϵ, about them. But then, for some δ, if $\|A_k - A\|_\infty < \delta$, the eigenvalues of A_k would have to be within ϵ of those of A. Thus, the set of matrices that have multiple eigenvalues is closed.

Part b. Left as exercise. ■

Classification: Saddle, Source and Sink

$$\lambda^2 - T\lambda + \mathrm{D} = 0$$

$$\lambda_{\pm} = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

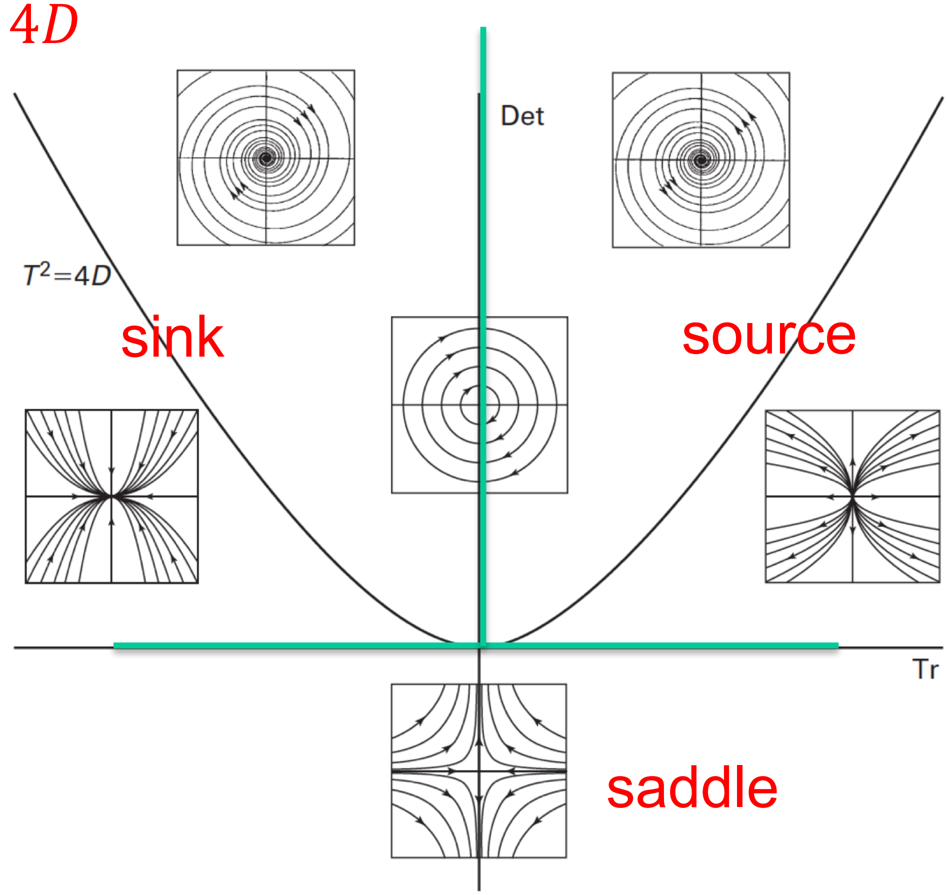
$$\boxed{\lambda_+ + \lambda_- = T = tr}$$

$$(\lambda - \lambda_+)(\lambda - \lambda_-) = 0$$

$$\lambda^2 - (\lambda_+ + \lambda_-)\lambda + \lambda_+\lambda_- = 0$$

$$\boxed{\lambda_+\lambda_- = D = determinat}$$

$T^2 = 4D$



- $D < 0$, λ_+ and λ_- have different signs ➔ saddle
- $D > 0$, λ_+ and λ_- have the same sign ➔ source with $T > 0$

  ➔ sink with $T < 0$

Classification: Saddle, Source and Sink

$$\lambda^2 - T\lambda + \mathrm{D} = 0$$

$$\lambda_{\pm} = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

$$\boxed{\lambda_+ + \lambda_- = T = tr}$$

$$\boxed{\lambda_+ \times \lambda_- = D}$$

- $D < 0$, ➔ saddle
- $D > 0$, ➔ source with $T > 0$
  ➔ sink with $T < 0$

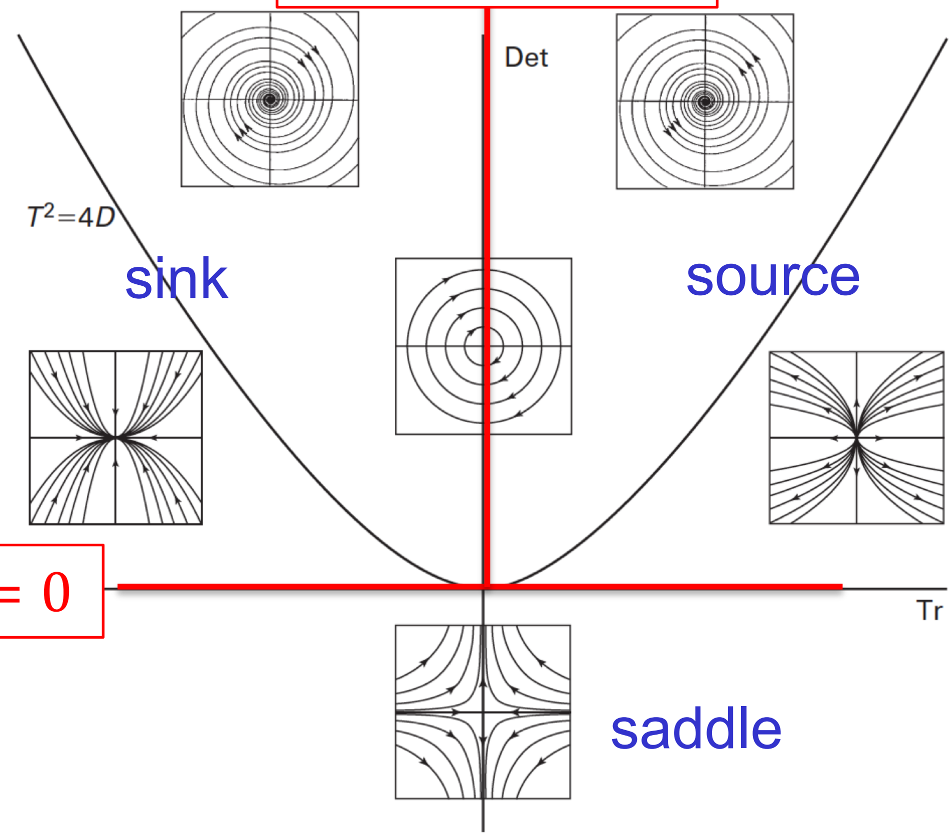
which set is generic?

$$\boxed{Re(\lambda) \neq 0}$$

$T^2 = 4D$

$\boxed{Center, Re = 0}$

$\boxed{D = 0}$



Important Concepts

- Generic: typical, large subset
- Structural Stability: a flow with a center is not structurally stable
- Hyperbolic: $Re(\lambda) \neq 0$
- Open: if X belongs to an open set U, any point sufficiently near to X also belongs to U
- Dense: $\mathrm{U} \in R^n$ is dense if there are points in U arbitrarily close to each point in R^n

Open and Dense (Wiggins, 2003)

Generic Property: A property of a map (resp. vector field) is said to be C^k *generic* if the set of maps (resp. vector fields) possessing that property contains a residual subset in the C^k topology.

Residual Set: Let X be a topological space, and let U be a subset of X. U is called a *residual set* if it is the intersection of a countable number of sets each of which are open and dense in X. If every residual set in X is itself dense in X, then X is called a *Baire space*.

Wiggins (2003)

Wiggins (2003)

Definition 12.1.1 (Structural Stability) *Consider a map $f \in \mathrm{Diff}^r(M, M)$ (resp. a $\mathbf{C}^r$ vector field in $\mathbf{C}^r(M, M)$); then f is said to be* structurally stable *if there exists a neighborhood $\mathcal{N}$ of f in the $\mathbf{C}^k$ topology such that f is $\mathbf{C}^0$ conjugate (resp. $\mathbf{C}^0$ equivalent) to every map (resp. vector field) in $\mathcal{N}$.*

Definition 12.1.2 (Residual Set) *Let X be a topological space, and let U be a subset of X. U is called a* residual set *if it contains the intersection of a countable number of sets, each of which are open and dense in X. If every residual set in X is itself dense in X, then X is called a* Baire space.

Definition 12.1.3 (Generic Property) *A property of a map (resp. vector field) is said to be $\mathbf{C}^k$* generic *if the set of maps (resp. vector fields) possessing that property contains a residual subset in the $\mathbf{C}^k$ topology.*

Examples (Wiggins, 2003)

Example 12.1.1 (Examples of Structurally Stable and Generic Properties).

- Hyperbolic fixed points and periodic orbits are structurally stable and generic.
- The transversal intersection (see Section 12.2) of the stable and unstable manifolds of hyperbolic fixed points and periodic orbits is structurally stable and generic.

The above periodic orbits are stable limit cycles.
The 2^{nd} bullet does not include homoclinic orbits.

Wiggins (2003)

Theorem 12.1.4 (Peixoto's Theorem) *A* $\mathbf{C}^r$ *vector field on a compact boundaryless two-dimensional manifold* M *is structurally stable if and only if*

i) *the number of fixed points and periodic orbits is finite and each is hyperbolic;*

ii) *there are no orbits connecting saddle points;*

iii) *the nonwandering set consists of fixed points and periodic orbits.*

Moreover, if M *is orientable, then the set of such vector fields is open and dense in* $\mathbf{C}^r(M, M)$ *(note: this is stronger than generic).*

Arrowsmith & Place (1990)

The above findings led to two conjectures about flows on manifolds of dimension $n \geqslant 2$:

(i) structural stability is a generic property of such flows;

(ii) the structurally stable flows are characterised in the same way as flows on two-manifolds.

Neither of these conjectures is correct. A counterexample to the first was given by Smale (1966). We do not discuss this example here; the interested reader can

Arrowsmith & Place (1990)

Proposition 3.1.1 *A linear flow or diffeomorphism on $\mathbb{R}^n$ is structurally stable in $L(\mathbb{R}^n)$ if and only if it is hyperbolic.*

transformations. Let $SF(\mathbb{R}^n) \subseteq L(\mathbb{R}^n)$ denote the set of linear transformations which give rise to structurally stable flows on $\mathbb{R}^n$.

Proposition 3.1.2 *The set $SF(\mathbb{R}^n)$ is open and dense in $L(\mathbb{R}^n)$.*

Arrowsmith & Place (1990)

Definition 3.3.1 *A vector field* $\mathbf{X}$ *in* $\mathrm{Vec}^1(D^2)$ *is said to be* structurally stable *if there exists a neighbourhood,* N, *of* $\mathbf{X}$ *in* $\mathrm{Vec}^1(D^2)$ *such that the flow of every* $\mathbf{Y}$ *in* N *is topologically equivalent to that of* $\mathbf{X}$ *on* D^2.

Theorem 3.3.1 (Peixoto) *Let* $\mathbf{X}$ *belong to* $\mathrm{Vec}^1_{\mathrm{in}}(D^2)$. *Then* $\mathbf{X}$ *is structurally stable if and only if its flow satisfies:*

(i) *all fixed points are hyperbolic;*
(ii) *all closed orbits are hyperbolic;*
(iii) *there are no orbits connecting saddle points.*

Notice that items (i) and (ii) in Theorem 3.3.1 simply ensure local structural stability of the fixed points and closed orbits in the flow of $\mathbf{X}$. It is really only item (iii) that involves a global property of the flow.

Theorem 3.3.2 *The subset of vector fields in* $\mathrm{Vec}^1_{\mathrm{in}}(D^2)$ *that are structurally stable is open and dense in* $\mathrm{Vec}^1_{\mathrm{in}}(D^2)$.

Arrowsmith & Place (1990)

Theorem 3.3.3 (Peixoto) *A vector field in* $\mathrm{Vec}^1(M)$ *is structurally stable if and only if its flow satisfies:*

(i) *all fixed points are hyperbolic;*
(ii) *all closed orbits are hyperbolic;*
(iii) *there are no orbits connecting saddle points;*
(iv) *the non-wandering set consists only of fixed points and periodic orbits.*

Moreover, if M *is orientable the set of structurally stable* C^1*-vector fields forms an open dense subset of* $\mathrm{Vec}^1(M)$.

Example 3.3.1 Show that the vector field, **X**, of the differential equation

$$\dot{x} = 2x - x^2, \qquad \dot{y} = -y + xy \tag{3.3.2}$$

is *not* structurally stable on any compact subset of the plane with the line segment joining the singular points of **X** in its interior.