Sept 23, 2024

Model solution:

$$\frac{d\vec{x}}{dt} = f(\vec{x}, \vec{p})$$

Solution: $$\int_{\vec{x}_0}^{\vec{x}(t)} \frac{1}{f(\vec{x},\vec{p})}\, d\vec{x} = \int_0^t ds$$

General form of the solution: $\vec{x}(t)$

– obtained by integration.

- possible integration $\Rightarrow$ closed form solution.

Example: Malthusian equation (single variable)

$$\begin{cases} \dfrac{dx}{dt} = rx \\ x(0) = x_0 \end{cases} \longrightarrow r = b - \delta$$

$$\frac{dx}{dt} = rx \Rightarrow \int_{x_0}^{x(t)} \frac{1}{x}\, dx = \int_0^t r\, ds$$

$$\Rightarrow \boxed{x(t) = x_0 e^{rt}}$$

- Doubling time, t_2 ($\delta = 0$):

$$2x_0 = x_0 e^{b t_2}$$

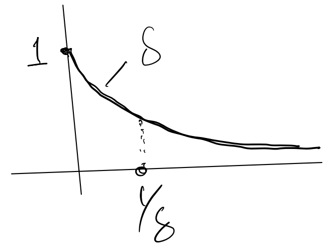
$$\Rightarrow t_2 = \frac{\ln(2)}{b}$$

- Half life, t_h ($b = 0$):

$$\frac{x_0}{2} = x_0 e^{-\delta t_h}$$

$$\Rightarrow \ln(0.5) = -\delta t_h \Rightarrow t_h = \frac{\ln(2)}{\delta}$$

- Average life span, $t_m(d=0)$: $\underline{\underline{\frac{1}{\delta}}}$



Numerical Integration (Approximate solution)

Example: MATLAB (ode23, ode45, ode15s)

Example:

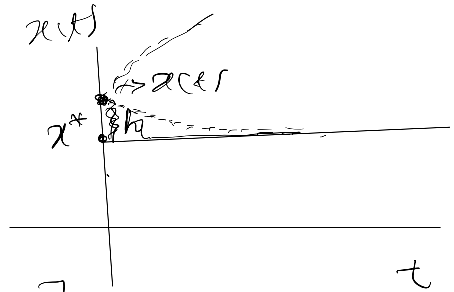
Model:
$$\begin{cases} \dfrac{dx_1}{dt} = x_1(1-x_1-a x_2), & x_1(0)=0.9 \\ \dfrac{dx_2}{dt} = c x_2(1-b x_1 - x_2), & x_2(0)=0.1 \\ a=0.03, \quad b=0.03, \quad c=1 \end{cases}$$

Files:
- Solver2DOdeModel.m
- Deqns2DOdeModel.m

Qualitative Analysis (1-dimension)

- $\dfrac{dx}{dt} = f(x, \vec{p})$ [Model]
- Equilibria: $x(t) = x^*$, for all t (constant)
  - obtained by solving $f(x^*, p) = 0$

- Local stability of x^*:



$h(t) = x(t) - x^*$

$\Rightarrow \dfrac{dh}{dt} = \dfrac{dx}{dt} - 0$

$\Rightarrow \dfrac{dh}{dt} = f(x)$ [$\because \vec{p}$ is constant]

$\quad = f(x^* + h)$

$$\Rightarrow \frac{dh}{dt} = \underbrace{f(x^*)}_{\overset{\shortparallel}{0}} + \left.\frac{df}{dx}\right|_{x^*} h + \cancel{h.o.t}\ 0$$

$$\Rightarrow \frac{dh}{dt} = \underbrace{\left(\left.\frac{df}{dx}\right|_{x^*}\right)}_{r} h$$

Solution $\Rightarrow$ $$h(t) = h_0 e^{\left(\left.\frac{df}{dx}\right|_{x^*}\right)t}$$

$$\xrightarrow{\text{as } t \to \infty} \begin{cases} 0 & \text{if } \left(\left.\frac{df}{dx}\right|_{x^*}\right) < 0 \quad \text{[Asymptotically stable]} \\ \infty & \text{if } \left(\left.\frac{df}{dx}\right|_{x^*}\right) > 0 \quad \text{[Unstable]} \end{cases}$$

- Bifurcation Analysis: Change in parameters may cause change in equilibria and stability. These phenomenon are called bifurcations.
  - For bifurcation analysis, study the equilibria and stability for changing parameters

Example:

1. Malthusian: $\frac{dx}{dt} = \underbrace{rx}_{f(x)}$
2. Logistic: $\frac{dx}{dt} = \underbrace{rx\left(1 - \frac{x}{K}\right)}_{f(x)}$.

- Equilibria:
  1. $f(x) = rx = 0 \Rightarrow x^* = 0$
  2. $f(x) = rx\left(1 - \frac{x}{K}\right) \Rightarrow x^* = 0,\ x^* = K$

- **Stability**:

1. $$\frac{df}{dx} = r$$

$$\underbrace{\left.\frac{df}{dx}\right|_{x^*=0}} = \underset{\sim}{r}$$

$\Rightarrow$ $\underbrace{x^*=0}$ is asymptotically stable if $\underline{r<0}$ and unstable if $r>0$.

2. $$\frac{df}{dx} = \frac{d}{dx}\left[rx\left(1-\frac{x}{K}\right)\right] = \underbrace{r\left(1-\frac{2x}{K}\right)}$$

$\underline{x^*=0}$: $$\left.\frac{df}{dx}\right|_{x^*=0} = r$$

$\Rightarrow$ $x^*=0$ is asymptotically stable if $r<0$ and unstable if $r>0$.

$\underline{x^*=K}$: $$\left.\frac{df}{dx}\right|_{x^*=K} = \underbrace{r\left(1-\frac{2K}{K}\right)} = -r$$

$\Rightarrow$ $x^*=K$ is asymptotically stable if $r>0$ and unstable if $r<0$.

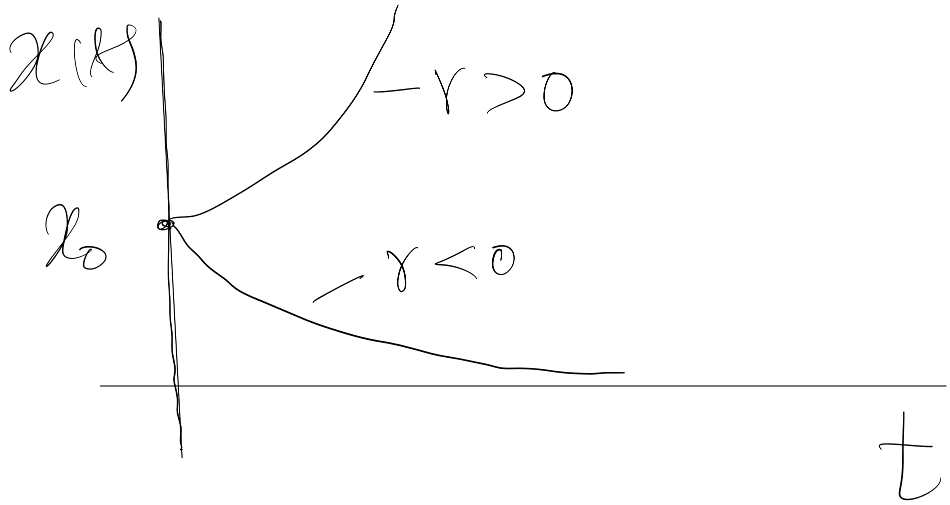
- **Bifurcation**:

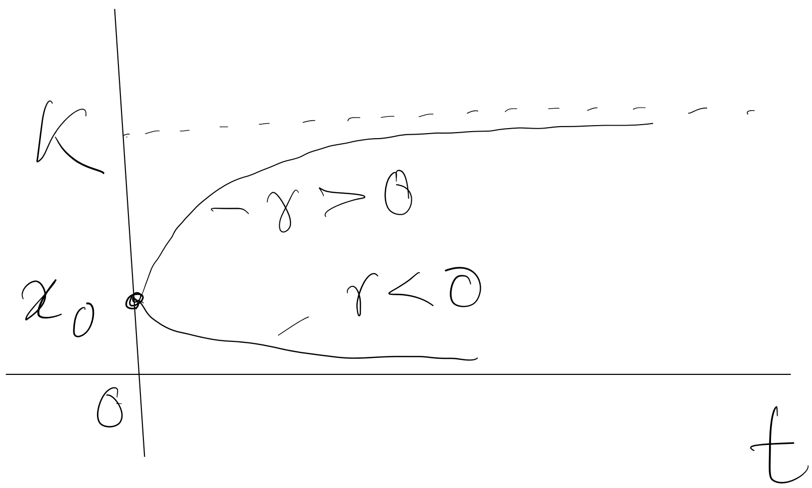
1. Stability changes at $r=0$. Therefore, the bifurcation occurs at the bifurcation point $r=0$.

2. Same as 1. $(r=0)$

Graphical Analysis:

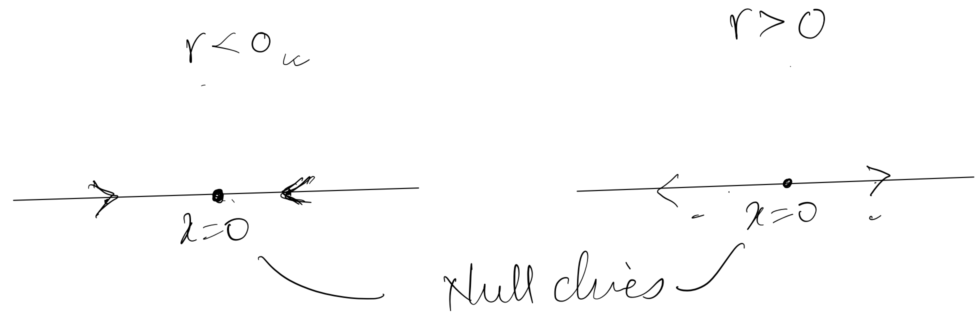
- Solution graph:

1. 

2. 

Phase diagram :

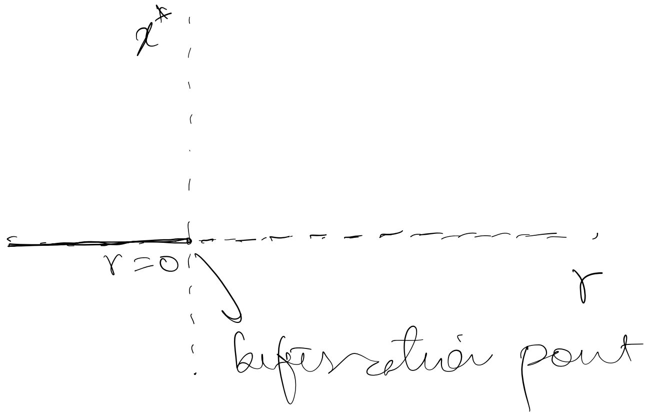
1.



2.

3. Bifurcation diagram

1.



2.