Math 337 - Elementary Differential Equations

Lecture Notes – Separable Differential Equations

Joseph M. Mahaffy,
⟨jmahaffy@sdsu.edu⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://jmahaffy.sdsu.edu

Spring 2020

Outline

Introduction

Introduction

- Desiccation of a Cell
- Separable Differential Equation
- Applications
  - Population of Italy

Desiccation of a Cell

Desiccation of a Cell: This example examines water loss through the surface of a cell

- Most cells are primarily water
- The loss of water due to desiccation is primarily through the surface of the cell
- *Surface area* $\propto$ *length*^{2}, while *volume* $\propto$ *length*^{3}
- The rate of change in the *surface area* $\propto$ *volume*$^{2/3}$
- An appropriate model for the desiccation of a cell is

$$\frac{dV}{dt} = -kV^{2/3}$$

  where $V(t)$ is the volume of the cell
- This is a **nonlinear DE**, so other techniques to solve

Separation of Variables

Definition (Separable Differential Equation)

Consider the differential equation

$$\frac{dy}{dt} = f(t, y),$$

and suppose that $f(t, y)$ can be written as the product of a function, $p(t)$, that only depends on t and another function, $q(y)$, that depends only on y. The differential equation

$$\frac{dy}{dt} = f(t, y) = p(t)q(y),$$

is called separable.

Separation of Variables

Theorem (Solution of a Separable Differential Equation)

Consider the ***separable differential equation***

$$\frac{dy}{dt} = p(t)q(y),$$

and assume that $q(y)$ is nonzero for y values of interest. The solution of this differential equation satisfies

$$\int q^{-1}(y)dy = \int p(t)dt.$$

Example of Desiccation of a Cell 1

Desiccation of a Cell: The model satisfies

$$\frac{dV}{dt} = -kV^{2/3}$$

- Suppose that the initial volume of water in the cell is $V(0) = 8 \text{ mm}^3$
- Suppose that 6 hours later the volume of water has decreased to $V(6) = 1 \text{ mm}^3$
- Solve this differential equation
- Find k and graph the solution
- Determine when all of the water has left the cell

Example of Desiccation of a Cell 2

Solution: The model is a separable differential equation

$$\frac{dV}{dt} = -kV^{2/3}$$

- Separate variables to give

$$\int V^{-2/3} dV = -\int k\,dt$$

- Upon integration,

$$3V^{1/3}(t) = -kt + C$$

- Equivalently,

$$V(t) = \left(\frac{-kt + C}{3}\right)^3$$

- The initial condition gives

$$V(0) = 8 = \left(\tfrac{C}{3}\right)^3 \quad \text{or} \quad C = 6$$

Example of Desiccation of a Cell 3

Solution: The model is given by

$$V(t) = \left(\frac{-kt+6}{3}\right)^3$$

- The other condition gives

$$V(6) = 1 = \left(\frac{-6k+6}{3}\right)^3 = (-2k+2)^3$$

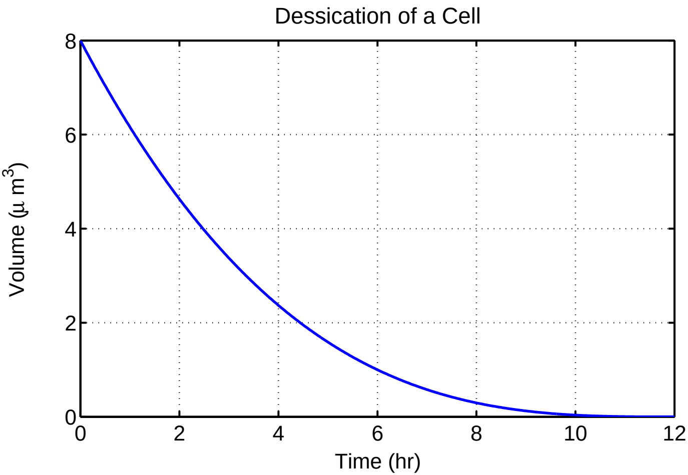
- So $k = \frac{1}{2}$
- The solution to this problem is

$$V(t) = \left(2 - \frac{t}{6}\right)^3$$

- The solution vanishes (all the water evaporates) at $t = 12$

Example of Desiccation of a Cell 4

Graphs of Desiccation of a Cell



Example 1 - Separable Differential Equation 1

Example - Separable Differential Equation Consider the differential equation

$$\frac{dy}{dt} = 2ty^2$$

Solution:

- Separate the variables t and y
  - Put only $2t$ and dt on the right hand side
  - And only y^2 and dy are on the left hand side
- The integral form is

$$\int \frac{dy}{y^2} = \int 2t\,dt$$

Example 1 - Separable Differential Equation 2

Solution (cont) The two integrals are

$$\int \frac{dy}{y^2} = \int 2t\,dt$$

- The two integrals are easily solved

$$-\frac{1}{y} = t^2 + C$$

- **Note** that you only need to put **one arbitrary constant**, despite solving two integrals
- This is easily rearranged to give the solution in explicit form

$$y(t) = -\frac{1}{t^2 + C}$$

General Separable Differential Equation 1

General Separable Differential Equation: An equivalent form is

$$M(t) + N(y)\frac{dy}{dt} = 0 \tag{1}$$

Theorem

Let $H_1(t)$ and $H_2(y)$ be any antiderivatives of M and N with

$$H_1'(t) = M(t) \qquad \text{and} \qquad H_2'(y) = N(y).$$

Then (1) can be written as a total derivative

$$\frac{d}{dt}\left[H_1(t) + H_2(y)\right] = 0,$$

*so the **general solution** of (1) is*

$$H_1(t) + H_2(y) = C.$$

General Separable Differential Equation 2

General Separable Differential Equation: $M(t) + N(y)\frac{dy}{dt} = 0$ with $H_1'(t) = M(t)$ and $H_2'(y) = N(y)$ has an **implicit representation** of the solution

$$H_1(t) + H_2(y) = C.$$

Often this general form does not allow an **explicit solution**

This form of solution should remind you of the **total differential** from Calculus

$$M(t)dt + N(y)dy = 0$$

These result **integral curves** or **level curves** for each C

This form will prove more useful for **Exact DEs**, which arise in potential problems from Physics

Example 2 - Separable Differential Equation 1

Example 2: Consider the initial value problem

$$\frac{dy}{dt} = \frac{4\,\sin(2t)}{y} \quad \text{with} \quad y(0) = 1$$

Skip Example

Solution: Begin by separating the variables, so

$$\int y\,dy = 4\int \sin(2t)dt$$

Solving the integrals gives

$$\frac{y^2}{2} = -2\,\cos(2t) + C$$

Example 2 - Separable Differential Equation 2

Solution (cont) Since

$$\frac{y^2}{2} = -2\,\cos(2t) + C$$

We write

$$y^2(t) = 2C - 4\,\cos(2t) \qquad \text{or} \qquad y(t) = \pm\sqrt{2C - 4\,\cos(2t)}$$

From the initial condition

$$y(0) = 1 = \sqrt{2C - 4\,\cos(0)} = \sqrt{2C - 4}$$

Thus, $2C = 5$, and

$$y(t) = \sqrt{5 - 4\,\cos(2t)}$$

Example 3 - Separable Differential Equation 1

Example 3: Consider the initial value problem

$$\frac{dy}{dt} = -y\frac{(1+2\,t^2)}{t} \quad \text{with} \quad y(1) = 2$$

Skip Example

Solution: Begin by separating the variables, so

$$\int \frac{dy}{y} = -\int \frac{(1+2\,t^2)}{t}dt = -\int \frac{dt}{t} - 2\int t\,dt$$

Solving the integrals gives

$$\ln(y) = -\ln(t) - t^2 + C$$

Example 3 - Separable Differential Equation 2

Solution (cont): Since

$$\ln(y) = -\ln(t) - t^2 + C$$

Exponentiate both sides to give

$$y(t) = e^{-\ln(t) - t^2 + C} = e^{-\ln(t)} e^{-t^2} e^{C} = \frac{A}{t} e^{-t^2}$$

where $A = e^C$
With the initial condition

$$y(1) = 2 = A\,e^{-1} \quad \text{or} \quad A = 2\,e^{1}$$

The solution is

$$y(t) = \frac{2}{t} e^{1-t^2}$$

Modified Malthusian Growth Model 1

Modified Malthusian Growth Model Consider the model

$$\frac{dP}{dt} = (a\,t + b)P \quad \text{with} \quad P(0) = P_0$$

- This equation is linear and separable

$$\int \frac{dP}{P} = \int (a\,t + b)dt$$

- Integrating

$$\ln(P(t)) = \frac{a\,t^2}{2} + b\,t + C$$

- Exponentiating

$$P(t) = e^{\left(\frac{a\,t^2}{2} + b\,t + C\right)}$$

Modified Malthusian Growth Model 2

Modified Malthusian Growth Model: With $P(0) = e^C = P_0$, the model can be written

$$P(t) = P_0 e^{\left(\frac{a\,t^2}{2} + b\,t\right)}$$

- This model has 3 unknowns, P_0, a, and b
- We fit the census data in 1790 and 1990 of 3.93 million and 248.7 million
- Choose the third data value from the census in 1890, where the population is 62.95 million
- Again take t to be the years after 1790, then $P_0 = 3.93$

Modified Malthusian Growth Model 2

Population Model for U. S. The nonautonomous model is

$$P(t) = 3.93\, e^{\left(\frac{a\,t^2}{2} + b\,t\right)}$$

- Use the census data in 1890 and 1990 to find a and b
- The model gives

$$\begin{aligned} P(100) &= 62.95 = 3.93\, e^{5000a+100b} \\ P(200) &= 248.7 = 3.93\, e^{20000a+200b} \end{aligned}$$

- Taking logarithms, we have the linear equations

$$\begin{aligned} 5000\,a + 100\,b &= \ln\left(\frac{62.95}{3.93}\right) = 2.7737 \\ 20,000\,a + 200\,b &= \ln\left(\frac{248.7}{3.93}\right) = 4.1476 \end{aligned}$$

Modified Malthusian Growth Model

Population Model for U. S. Solving the linear equations

$$\begin{aligned} 5000\,a + 100\,b &= \ln\left(\tfrac{62.95}{3.93}\right) = 2.7737 \\ 20,000\,a + 200\,b &= \ln\left(\tfrac{248.7}{3.93}\right) = 4.1476 \end{aligned}$$

- Multiply the first equation by -2 and add to the second

$$10,000\,a = -2(2.7737) + 4.1476 = -1.3998$$

- Thus, $a = -0.00013998$, which is substituted into the first equation
- It follows that

$$100\,b = 5000(0.00013998) + 2.7737 = 3.473$$

- Solution is $a = -0.00013998$ and $b = 0.03473$

Modified Malthusian Growth Model

Population Models for U. S. The Malthusian growth model fitting the census data at 1790 and 1990 is

$$P(t) = 3.93\, e^{0.02074t}$$

The nonautonomous model fitting the census data at 1790, 1890, and 1990 is

$$P(t) = 3.93\, e^{0.03474t - 0.00006999t^2}$$

Model	1900	2000	2010
U. S. Census Data	76.21	281.4	308.7
Malthusian Growth	38.48	306.1	376.7
Nonautonomous	76.95	264.4	277.0

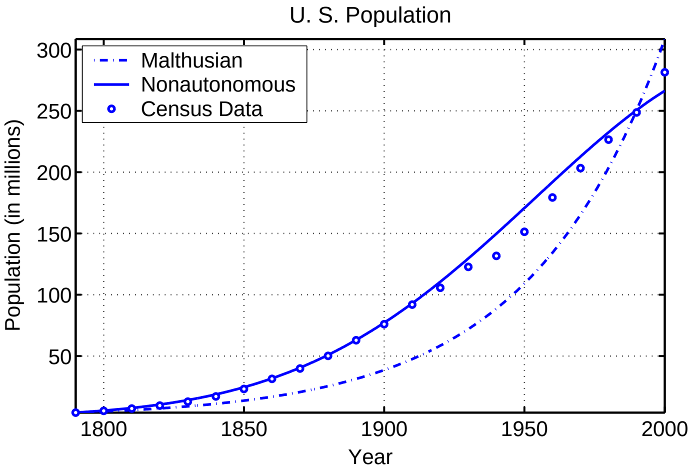
Modified Malthusian Growth Model

Population Models for U. S. The models use limited data for prediction

- For 1900
    - The **Malthusian growth model** is too low by 49.5%
    - The **nonautonomous growth model** is too high by 0.97%
    - The nonautonomous growth model fits quite well
- For 2000 and 2010
    - The **Malthusian growth model** is too high by 8.8% and 22%
    - The **nonautonomous growth model** is too low by 6.0% and 10.3%
    - Neither model fits the census data very well
    - The nonautonomous though fitting better misses the recent higher growth from immigration

Modified Malthusian Growth Model

Graphs of Population Models for U. S.



Population of Italy 1

Population of Italy: For the last few decades, Italy has had its growth rate decline to where the country does not even have enough births (or immigration) to replace the number of deaths in the country

- The population of Italy was 47.1 million in 1950, 53.7 million in 1970, and 56.8 million in 1990
- Use the data in 1950 and 1990 to find a Malthusian growth model for Italy's population
- Consider the nonautonomous Malthusian growth model given by the differential equation

$$\frac{dP}{dt} = (a\,t + b)P \quad \text{with} \quad P(0) = 47.1$$

with t in years after 1950

- Solve this differential equation
- Find the constants a and b from the data

Population of Italy

Population of Italy (cont):

- If the population of Italy was 50.2 million in 1960 and 57.6 million in 2000, then use each of these models to estimate the populations and determine the error between the models and the actual census values
- Graph the solutions of the two models and the data points from 1950 to 2000
- Find when Italy's population levels off and begins to decline according to the nonautonomous Malthusian growth model

Solution: The Malthusian growth model satisfies

$$\frac{dP}{dt} = rP \qquad \text{with} \qquad P(0) = 47.1$$

Population of Italy

Solution (cont): The solution of the Malthusian growth model is

$$P(t) = 47.1\, e^{rt}$$

- In 1990 the population was 56.8 million, so

$$P(40) = 47.1\, e^{40r} = 56.8$$

- Thus,

$$e^{40r} = \frac{56.8}{47.1} \quad \text{or} \quad r = \frac{1}{40} \ln\left(\frac{56.8}{47.1}\right) = 0.004682$$

- The Malthusian growth model for Italy is

$$P(t) = 47.1\, e^{0.004682\, t}$$

Population of Italy

Solution (cont): The nonautonomous Malthusian growth model is

$$\frac{dP}{dt} = (a\,t + b)P \qquad \text{with} \qquad P(0) = 47.1$$

- Separating variables

$$\int \frac{dP}{P} = \int (a\,t + b)dt$$

- Thus,

$$\ln(P(t)) = \frac{at^2}{2} + bt + c$$

- Exponentiating

$$P(t) = e^{\frac{at^2}{2}+bt+c} = e^c e^{\frac{at^2}{2}+bt}$$

Population of Italy

Solution (cont): The initial condition gives

$$P(0) = 47.1 = e^c$$

- The solution can be written

$$P(t) = 47.1\, e^{\frac{at^2}{2} + bt}$$

- The logarithmic form satisfies

$$\frac{at^2}{2} + bt = \ln(P(t)) - \ln(47.1)$$

- The data from 1970 and 1990 give

$$\begin{aligned} 200\,a + 20\,b &= \ln(53.7) - \ln(47.1) = 0.13114 \\ 800\,a + 40\,b &= \ln(56.8) - \ln(47.1) = 0.18726 \end{aligned}$$

Population of Italy

6

Solution (cont): The equations in a and b are linear equations

- Multiply the first equation by -2 and add it to the second

$$\begin{aligned} -2(200\,a + 20\,b) &= -2(0.13114) \\ 800\,a + 40\,b &= 0.18726 \\ 400\,a &= -0.07502 \end{aligned}$$

- It follows that $a = -0.00018755$
- From either equation above $b = 0.0084325$
- The solution becomes

$$P(t) = 47.1\,e^{0.0084325\,t - 0.00009378\,t^2}$$

Population of Italy

Solution (cont): The two models are given by

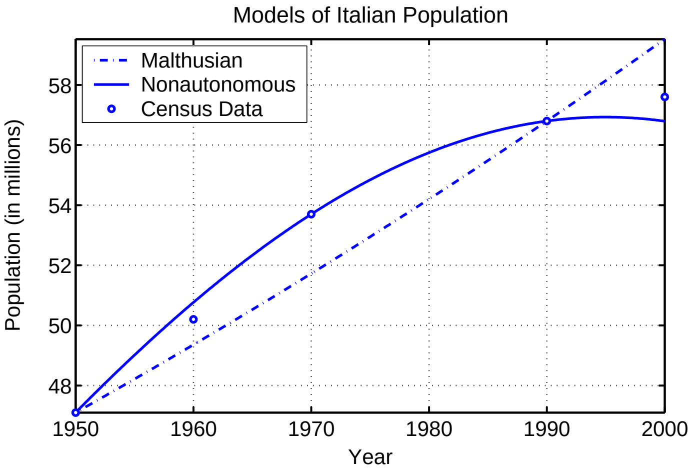
$$P(t) = 47.1\, e^{0.004682\, t} \quad \text{and} \quad P(t) = 47.1\, e^{0.0084325\, t - 0.00009378\, t^2}$$

Below is a Table comparing the models at 1960 and 2000

Model	1960	% Error	2000	% Error
Italy Census Data	50.2	–	57.6	–
Malthusian	49.4	−1.7%	59.5	3.3%
Nonautonomous	50.8	1.1%	56.8	−1.4%

Modified Malthusian Growth Model

Graphs of Population Models for Italy



Population of Italy

Solution (cont): The nonautonomous model is

$$\frac{dP}{dt} = (0.0084325 - 0.00018755\,t)P(t)$$

- The population growth slows to zero, so the population levels off, when $\frac{dP}{dt} = 0$
- This occurs when

$$0.0084325 - 0.00018755\,t = 0 \quad \text{or} \quad t = 44.96 \text{ years}$$

- The nonautonomous Malthusian growth model predicts that Italy's population leveled off in 1995 (45 years after 1950)
- Data indicates that 2000 was the peak of Italy's population, so the model does reasonably well

Population of Study 1

Two Models of Population: A population of animals is studied over a period of days

Model 1: Linear time-varying model:

$$\frac{dP}{dt} = (0.05 - 0.00088\,t)P, \qquad P(0) = 64$$

Model 2: Nonlinear time-varying model:

$$\frac{dP}{dt} = (0.3\,e^{-0.01t} - 0.1)P^{2/3}, \qquad P(0) = 64$$

- Give the modeling differences and interpretations
- Solve each differential equation
- Find when the models predict a maximum and find the maximum population

Population of Study

Model 1: Linear time-varying model:

$$\frac{dP}{dt} = (0.05 - 0.00088\,t)P, \qquad P(0) = 64$$

Model 2: Nonlinear time-varying model:

$$\frac{dP}{dt} = (0.3\,e^{-0.01t} - 0.1)P^{2/3}, \qquad P(0) = 64$$

- The first model uses a basic Malthusian growth
- Model 1 has the growth rate dropping linearly in time
- The second model assumes the population is depending on absorption through surface area
- Model 2 has the growth rate dropping exponentially in time to a basic linear decay term

Population of Study

Model 1: Consider:

$$\frac{dP}{dt} = (0.05 - 0.00088\,t)P, \qquad P(0) = 64$$

and rewrite as

$$\frac{dP}{dt} + (0.00088\,t - 0.05)P = 0$$

Integrating factor is

$$\mu(t) = e^{\int (0.00088\,t - 0.05)\,dt} = e^{(0.00044\,t^2 - 0.05t)},$$

so

$$\frac{d}{dt}\left(e^{(0.00044\,t^2 - 0.05t)}P(t)\right) = 0$$

Solution is

$$e^{(0.00044\,t^2 - 0.05t)}P(t) = C \qquad \text{or} \qquad P(t) = C\,e^{(0.05t - 0.00044\,t^2)}$$

Population of Study

Model 1 solution: From before

$$P(t) = C\,e^{(0.05t - 0.00044\,t^2)} = 64\,e^{(0.05t - 0.00044\,t^2)}$$

with initial condition.

Maximum population occurs when $\frac{dP}{dt} = 0$, so

$$0.05 - 0.00088\,t_m = 0 \qquad \text{or} \qquad t_m = 56.82$$

The maximum population is

$$P(t_m) = 64\,e^{(0.05t_m - 0.00044\,t_m^2)} = 264.9$$

Population of Study

Model 2: Consider **Separable DE**:

$$\frac{dP}{dt} = (0.3\,e^{-0.01t} - 0.1)P^{2/3}, \qquad P(0) = 64$$

and rewrite as

$$\int P^{-2/3}dP = \int (0.3\,e^{-0.01t} - 0.1)dt$$

Integrating gives

$$3P^{1/3}(t) = -30e^{-0.01t} - 0.1t + C$$

so

$$P(t) = \left(\frac{C}{3} - 10e^{-0.01t} - \frac{t}{30}\right)^3$$

Since $P(0) = 64 = \left(\frac{C}{3} - 10\right)$ or $\frac{C}{3} = 14$, the solution is,

$$P(t) = \left(14 - 10e^{-0.01t} - \frac{t}{30}\right)^3$$

Population of Study

Model 2 solution: From before

$$P(t) = \left(14 - 10e^{-0.01t} - \frac{t}{30}\right)^3$$

with initial condition.

Maximum population occurs when $\frac{dP}{dt} = 0$, so

$$0.3\,e^{-0.01t_m} - 0.1 = 0 \qquad \text{or} \qquad t_m = 100\ln(3) = 109.86$$

The maximum population is

$$P(t_m) = \left(14 - 10e^{-0.01t_m} - \frac{t_m}{30}\right)^3 = 343.7$$

Population of Study

Graphs of the Population Models