Math336 Final Exam, Dr. Samuel Shen's Class

December 11, 2020, Friday, 10:30AM, due 11:59PM on the second day/Sat/Dec 12

Due to COVID, this is an open-book exam. You can use google to search for any helpful information. You can use a calculator, computer, or experiment. However, you CANNOT ask anyone to help you. Problems #1, 2, 3 and 4 are the hand-written part. Start each problem on a new page. When done, scan all the pages into pdf. Problem #5 is for R programming. Save your R results using `Compile Report...` *function in RStudio to produce a pdf file, or copying-and-pasting your code and output to a WORD file and converting it into pdf. Finally, you combine the hand-written part together with the pdf file for R to form your final single pdf file for submission. Save and submit your R file too. Therefore, you will submit two files to BB: a single pdf file and a single R file.*

Student Name:

Student ID Number:

1. $[16\ points]$

   (a) Figure 1 shows a simple pendulum with mass m , string length l , and the Earth gravitational acceleration g. Use the dimensional analysis method to determine the period τ of the pendulum as a function of m, l, g, determined up to a dimensionless constant α, i.e., $\tau = \alpha m^a l^b g^c$.

   (b) Use the conservation law of energy to find an approximate value of α in Part (a) under the condition of $\sin x \approx x$ when x is close to be zero.

   (c) The string length is increased to $l_2 = 1.022l$ due to expansion in a higher temperature environment. The corresponding period is denoted by τ_2, which is equal to $\tau_2 = k\tau$, where τ is found in Part (a) of this problem. Calculate the value of k?

   (d) Given that the period $\tau = 1.0\ [seconds]$, and $g = 9.79525\ [m/s^2]$, calculate the string length l with unit in centimeters.

   [Requirements: Show your work and steps. You can use a calculator or R to do the calculation. You do not need to submit your R code for this problem even if you use R here.

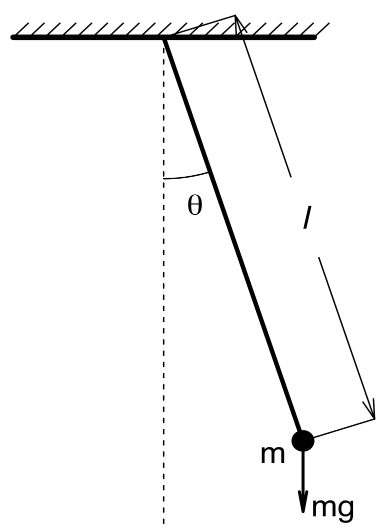

Figure 1 Simple pendulum of mass m and length l under the action of Earth's gravitational force.

2. [14 $points$] The SVD result of a matrix A is below

```
svdA=svd(A)

svdA$d
[1] 3.0 1.0

svdA$u
     [,1] [,2]
[1,]    0    1
[2,]    1    0
svdA$v
     [,1] [,2]
[1,]  0.0    1
[2,]  0.6    0
[3,]  0.8    0
```

(a) [6 points] Write down three matrices U, D, V in the SVD formula $A = UDV'$ where V' denotes the transpose matrix of V.

(b) [8 points] Use the SVD formula $A = UDV'$ to recover the original matrix A by hand calculation for the multiplication of the three matrices.

Requirements: Show your work and steps. You can use a calculator to do your number multiplication, but you still need to show your work. You may use R to verify your solution, but that is not required.

3. [16 points] The Buffon's needle problem: A needle of length l is dropped onto a floor with equally spaced parallel lines, as shown in Figure 2. The distance between each nearby two lines is d.
(a) [8 points] Derive the formula to calculate the probability of the needle crossing, or touching a line when $l < d$, i.e., the case of short needles. Express your formula in terms of l and d.
Requirements: You must draw a diagram of a needle and two lines, clearly mark the needle's position using symbols y and θ for its relevant distance and angle. You must draw a figure on the $\theta - y$ plane to formulate a geometric probability problem. Write down the needle cross condition in terms of y, θ, d and l.
(b) [4 points] Compute the probability of crossing or touching when $l = 0.2$ $[meter]$ and $d = 0.3$ $[meter]$.
(c) [4 points] Given $d = 0.25$ $[meter]$, what is the needle length l so that the the probability of crossing or touching is 0.5?

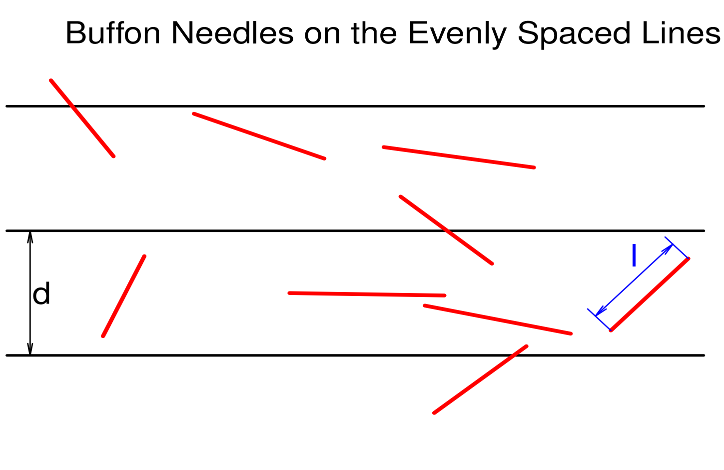

Figure 2 The Buffon's needle problem: a geometric probability example.

4. [18 $points$]

(a) [8 points] Derive a formula for the monthly mortgage payment x, expressed in terms of the principal amount P, monthly interest rate r, and total number of months of the loan n. Show your work and the detailed steps. The answer for this problem (a) is a formula.

(b) [4 points] Given the following data: The principal amount (i.e., the total loan) is $P = \$400,000$, the annual interest rate is 3.0% (converted into the monthly rate 0.25%), and the loan is to be paid off in 30 years (equivalent to 360 months). Use the above derived formula and the data to compute the monthly mortgage payment x by a calculator or R. The answer should be an amount of money per month. You do not need to submit the R code for this problem.

(c) [3 points] If the annual rate is reduced to 2.95% in the above data, what is the monthly mortgage payment?

(d) [3 points] If the principal is increased to $P = \$470,000$, the annual rate is 2.875%, and the loan period is still 30 years, what is the the monthly mortgage payment now?

5. [36 $points$] **The R programming part**

(i) [8 points]

Use R to solve the following linear equations for x_1, x_2, x_3, x_4:

$$\begin{cases} -x_1 + 2.9x_2 + \quad x_3 - \quad x_4 = & 1 \\ -2.5x_1 - 1.9x_2 + \quad x_3 \quad\quad\quad = & 2.1 \\ 2.1x_1 - 3.8x_2 - \quad 4x_3 - \quad 3x_4 = & 0 \\ x_1 - \quad x_2 - 3.1x_3 - 8.6x_4 = & 2.5 \end{cases}$$

Copy your R solution result to your R code as comment lines after #.

(ii) [8 points]

Write an R code of Monte Carlo simulation to approximately evaluate the volume of a ball of radius equal to 1.0 in 6-dimensional space. Please use at least 100,000 points.

(iii) [8 points]

Use Monte Carlo method to approximately evaluate the following integral

$$\int_1^2 \frac{1 - x^2 + 9\cos x}{x(1 + x + \sin x)}\, dx \tag{0.1}$$

Please use at least 100,000 points.

(iv) [12 points] Figure 3 shows the history of the global average December mean temperature anomalies. Use R and the dataset `EarthTemperatureData.tex` or `EarthTemperatureData.csv` downloadable from BB's Assignment/Final Exam block to plot a similar figure but for **November** and with the following requirements.

(a) Replace "Samuel Shen" and "December" in the main title by your name and November.

(b) Change the curve's color from black to orange and use `lwd=3`.

(c) Compute the linear trend of the November temperature anomalies for the period from 1901 to 2000 using R command `lm( )`. Please note that this is NOT the entire data time period of 1850-2015.
Hint: See Fig. 3.6 in the textbook for reference.

(d) Plot the trend line from 1901 to 2000 in the blue color. The blue trend line must be limited within the time period of 1901-2000, not the entire data time period of 1850-2015. Use `lwd=3` for the trend line.

(e) Change the text "December trend = 0.52 deg C/century" to "1901-2000 November trend = ?? deg C per century", and use the trend calculated from Step (c) in the position "??".

(f) Save your plot as a png file. If your `Compile Report ...` is successful, you do not need to do this figure saving step.

(g) Find the hottest and coldest November temperature anomalies. Which years did they occur?

(h) Save all your work as a pdf file and combine all your work for this exam into a single pdf file.

(i) Submit both pdf and R files into BB.

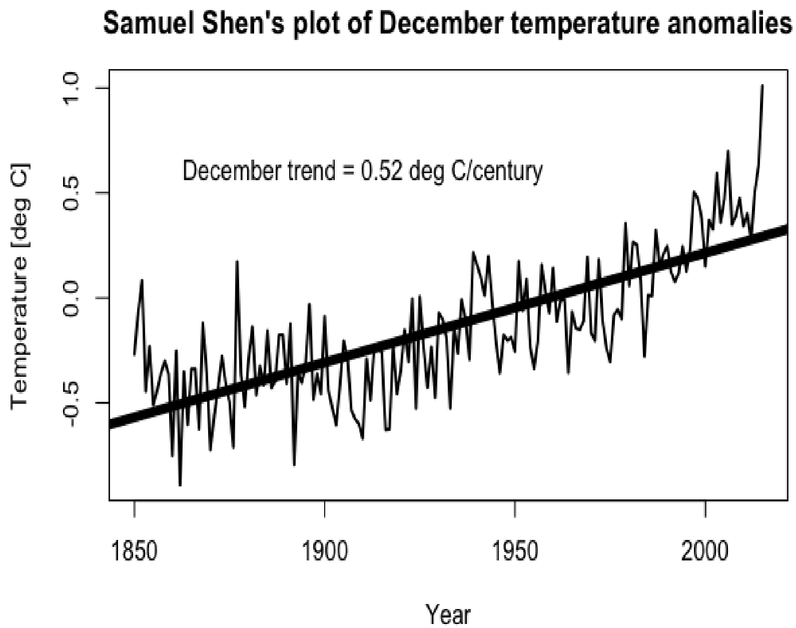

Figure 3 Global average December mean global average surface air temperature anomalies from 1850-2015.