MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #6

Chapter 2 Systems of ODEs
Sections 2.3-2.7

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A Brief Note for Sections 2.1-2.2

- A C^∞ or C^k function: C^k have k continuous derivatives

$$X' = F(X), \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad F(X) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$

- Column vectors, X & $F(X)$
- Vector Fields, $F(X) = \big(f\big(x,y\big), g(x,y)\big)^{\mathrm{T}} = (P,Q)^{\mathrm{T}}$
- Methods for solving a system of 2 (linear) first-order ODEs include:
  - Convert the system of ODEs into a 2nd order ODE and then solve the 2nd order ODE
  - Assume $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{\lambda t}$ and solve for λ.

$f'(x_c) \rightarrow \lambda\ (eigenvalue)$

$$\boxed{x' = ax} \qquad x' = f(x)$$

assume

$$x = ke^{\lambda t}$$

$$\lambda = a = f'(x_c)$$

$\boxed{\text{the solution is stable (unstable) if } \lambda < 0\ (\lambda > 0)}$

consider a general case

$$x' = f(x)$$

linearize $f(x)$ wrt a critical pt

$$x' = f(x) \approx \cancel{f(x_c)} + f'(x_c)(\mathrm{x} - \mathrm{x_c}) + \cdots$$

the critical point is stable if $f'(x_c) < 0$
the critical point is unstable if $f'(x_c) > 0$

assume

$$x - x_c = ke^{\lambda t} \qquad \lambda ke^{\lambda t} \approx f'(x_c)ke^{\lambda t}$$

$$\lambda = f'(x_c)$$

λ : eigenvalue

the crtical point is stable (unstable) if $\lambda < 0\ (\lambda > 0)$

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Lecture #5

A Brief Review of Linear Algebra

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Topics discussed On September 4

- Fundamental Concepts
  - Eight Axioms; Subset vs. Subspace
- Basic Matrix Operations
  - Addition, Subtraction, Scalar Multiplication, Matrix Multiplication
- Matrix Properties
  - Rank, Range, Kernel, and Nullspace
- Main Types of Matrices
  - Matrix Transpose, Identity, and Inverse;
  - Symmetric, Skew-symmetric, and Orthogonal Matrices
- Linear Dependence/Independence and Linear Systems
- Elimination and LU Decomposition
  - Elementary Row Operations, Augmented Matrix, and Row Echelon Form
- Fundamental Theorem for Linear Systems
- Eigenvalue Problem
  - Similarity Transformation and Diagonalization
- Quadratic Forms and Rayleigh Quotient

Online

- Section 2.3: Preliminaries from Algebra
- Section 2.4: Planar Linear System
- Lecture #5 for Eigenvalue Problem

ODEs vs. LA

TBD

High-Order ODEs		System of Algebraic Eqs.
Systems of M Linear ODEs	$X = X_o e^{\lambda t} \Rightarrow$	Linear Algebraic Eq.
	features near X_c $\Leftarrow$	Eigenvalue Problems
Systems of Nonlinear ODEs	$\Rightarrow$ linearization via Jacobian Matrix	Stability and Bifurcation analysis
Data Producer for various types of data, including chaotic, periodic, quasi-periodic solutions, etc.	Time Series of M Variables	• Multivariate Analysis • Spectral analysis • Principle Component Analysis (PCA: for Dimension Reduction) • Single Value Decomposition (SVD)
”Prediction” “Causality”		• Analysis • Prediction with empirical equations

Key Concepts for Eigenvectors

Eigenvectors and generalized eigenvectors: An eigenvector (or right eigenvector) v of an $n\times n$ matrix A is a nonzero vector which satisfies $Av = \lambda v$ or $(A - \lambda I)v = 0$. A generalized (right) eigenvector is defined as $(A - \lambda I)^k\, v = 0$ for some $1 \leq k \leq n$. Namely,

- Eigenvectors: $(A - \lambda I)v = 0$
- Generalized eigenvectors: $(A - \lambda I)^k\, v = 0$ for some $1 \leq k \leq n$.

For example, given a matrix with repeated eigenvalue, two independent vectors can be obtained as follows:

- $(A - \lambda I)v_1 = 0$
- $(A - \lambda I)^2\, v_2 = 0 \quad \Rightarrow (A - \lambda I)v_2 = v_1$

Section 2.3: Preliminaries from Algebra

We now further restrict our attention to the most important class of planar systems of differential equations, namely, linear systems. In the autonomous case, these systems assume the simple form

$$x' = ax + by$$

$$y' = cx + dy$$

where a, b, c, and d are constants. We may abbreviate this system by using the *coefficient matrix* A where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then the linear system may be written as

$$X' = AX.$$

In a Matrix Form

Consider the following two ODEs:

$$x' = ax + by$$
$$y' = cx + dy$$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\qquad X = \begin{pmatrix} x \\ y \end{pmatrix}$

$$AX = \begin{pmatrix} \boxed{a \quad b} \\ c \quad d \end{pmatrix} \boxed{\begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} ax + by \\ \end{pmatrix}$$

$$AX = \begin{pmatrix} a \quad b \\ \boxed{c \quad d} \end{pmatrix} \boxed{\begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} \\ cx + dy \end{pmatrix}$$

Thus, $AX = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$

$$X' = AX$$

also see Lecture #5

Trajectory, Orbit, and Path

$$x' = ax + by \quad (= P(x, y)) \qquad (1)$$

$$y' = cx + dy \quad (= Q(x, y)\ ) \qquad (2)$$

- A solution x(t), y(t) of (1) represents a curve C in the xy-plane (or a point as a degenerate case). This curve is called a solution curve or path (sometimes a trajectory or orbit) of (I).
- The sense of increasing t is called the positive sense on C and can be marked by an arrow head. This defines an orientation on C.
- If t is time and C the path of a moving body, the positive sense is the sense in which the body moves along C as time progresses.
- The present xy-plane is often called the phase plane of (1-2).

Trajectory, Orbit, and Path: Slope

$$x' = ax + by \quad (= P(x,y)) \qquad (1)$$

$$y' = cx + dy \quad (= Q(x,y)\ ) \qquad (2)$$

From (1-2) we see that the slope of a path passing through a point A: (X,Y) is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{Q(x,y)}{P(x,y)} \qquad (3)$$

- Note that (3) gives no information about the orientation of a path.
- Note further that we must have $P(x,y) \neq 0$ at A.
- If $P(x,y) = 0$ but $\mathrm{Q}(x,y) \neq 0$ at A, we can take $dx/dy = P(x,y)/Q(x,y)$ instead of (3) and conclude from $\frac{dx}{dy} = 0$ that the tangent of C at A is vertical.
- However, what can we do if both P and Q are zero at some point?

Analysis near a Critical Point

TBD

We continue our discussion of homogeneous linear systems with constant coefficients (1). Let us review where we are. From Sec. 4.3 we have

$$(1) \qquad \mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mathbf{y}, \qquad \text{in components,} \qquad \begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 \\ y_2' &= a_{21}y_1 + a_{22}y_2. \end{aligned}$$

From the examples in the last section, we have seen that we can obtain an overview of families of solution curves if we represent them parametrically as $\mathbf{y}(t) = [y_1(t) \quad y_2(t)]^{\mathsf{T}}$ and graph them as curves in the $y_1 y_2$-plane, called the **phase plane**. Such a curve is called a **trajectory** of (1), and their totality is known as the **phase portrait** of (1).

Our examples in the last section show that the general form of the phase portrait is determined to a large extent by the type of **critical point** of the system (1) defined as a point at which dy_2/dy_1 becomes undetermined, 0/0; here [see (9) in Sec. 4.3]

$$\textbf{(3)} \qquad \frac{dy_2}{dy_1} = \frac{y_2'\,dt}{y_1'\,dt} = \frac{a_{21}y_1 + a_{22}y_2}{a_{11}y_1 + a_{12}y_2}.$$

a need of analytical solutions near the critical point

We also recall from Sec. 4.3 that there are various types of critical points.

Review: Determinant

- In linear algebra, the determinant is a useful value that can be computed from the elements of a square matrix. The determinant of a matrix A is denoted det(A), det A, or |A|.
- In the case of a 2 × 2 matrix, the specific formula for the determinant is simply the upper left element times the lower right element, minus the product of the other two elements.

The determinant of a 2 × 2 matrix is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

- For a 3x3 matrix A, we have the following for its determinant |A|.

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= aei + bfg + cdh - ceg - bdi - afh.$$

Geometric Meaning of 2nd-Order Determinant

Consider the following two planes in R^3 (or two lines in R^2)

$$ax + by = \alpha$$
$$cx + dy = \beta,$$

which lead to the following normal vectors

$$\vec{n}_1 = (a, b, 0)$$
$$\vec{n}_2 = (c, d, 0),$$

respectively. Consider a cross product of the two normal vectors:

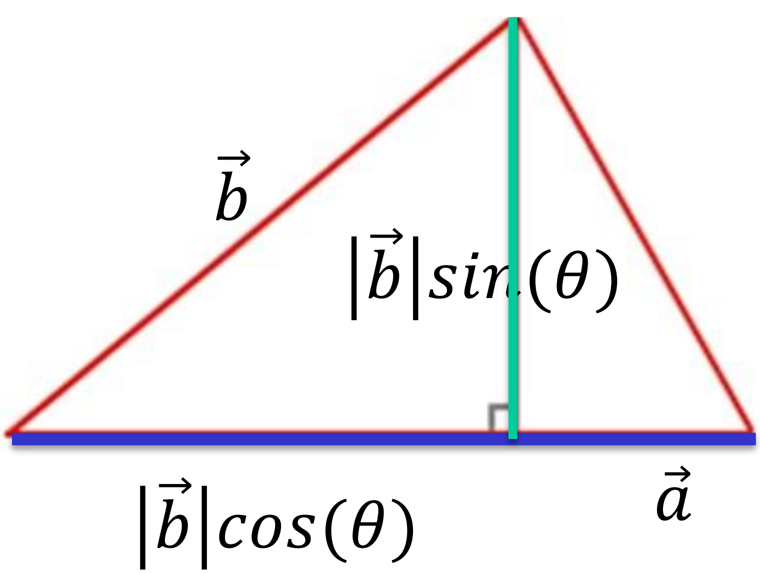
$$|\vec{n}_1 \times \vec{n}_2| = \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & 0 \\ c & d & 0 \end{vmatrix} \right\| = \left|\vec{k}(ad - bc)\right| = \|A\| \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$|\vec{n}_1 \times \vec{n}_2| = |\vec{n}_1||\vec{n}_2| sin(\theta)$$

$overlap\ or\ are\ parallel\ when\ \theta = 0\ with\ \ |A| = 0$

Review of Inner and Cross Products

- Inner Product and Projection
- Cross Production and Height



$$projection = |\vec{b}|cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

$$height = |\vec{b}|sin(\theta) = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|}$$

$$area = 0.5\, width * height$$

$$= 0.5|\vec{a}||\vec{b}|sin(\theta) = 0.5|\vec{a} \times \vec{b}|$$

When $|A| = 0$

Consider the following two planes in R^3 (or two lines in R^2)

$$ax + by = \alpha$$
$$cx + dy = \beta,$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

➔ $\text{overlap or are parallel when } \theta = 0 \text{ with } |A| = 0$

The above is associated with one of the following cases:

(I) no solutions

two lines are paralel

$$3x + 2y = 6$$
$$3x + 2y = 12$$

(II) infinitely many solutions

two lines overlap

$$3x + 2y = 6$$
$$6x + 4y = 12$$

Section 2.3: A Summary

Consider the following two lines in R^2

$$ax + by = \alpha$$
$$cx + dy = \beta,$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$|A| = 0 \Longrightarrow$ two lines are parallel or overlap — no solution or infinitely many solutions

$|A| \neq 0 \Longleftrightarrow$ two lines intersect — unique solution

$\vec{n}_1$ *and* $\vec{n}_2$ and are LI, to be discussed below.

Section 2.3 Linearly Independent

Let V and W be vectors in the plane. We say that V and W are *linearly independent* if V and W do not lie along the same straight line through the origin. The vectors V and W are *linearly dependent* if either V or W is the zero vector or if both lie on the same line through the origin.

Proposition. *Suppose $V = (v_1, v_2)$ and $W = (w_1, w_2)$. Then V and W are linearly independent if and only if*

$$\det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} \neq 0.$$

For a proof, see Exercise 11 at the end of this chapter. ■

$det \neq 0 \Leftrightarrow$ linearly independent $(i.e., \theta \neq 0)$

Basis Vectors

Whenever we have a pair of linearly independent vectors V and W, we may always write any vector $Z \in \mathbb{R}^2$ in a unique way as a *linear combination* of V and W. That is, we may always find a pair of real numbers α and β such that

$$Z = \alpha V + \beta W.$$

Moreover, α and β are unique.

The linearly independent vectors V and W are said to define a *basis* for $\mathbb{R}^2$.

Unique α and β: A Proof

Consider two linearly independent vectors, $V = (v_1, v_2)$ and W $= (w_1, w_2)$

We may express any vector $Z \in R^2$ as follows:

$$Z = \alpha V + \beta W.$$

Here, unique α and β are determined as follows. Suppose Z $= (z_1, z_2)$. We have the following equations:

$$z_1 = \alpha v_1 + \beta w_1$$
$$z_2 = \alpha v_2 + \beta w_2$$

where the v_i, w_i and z_i are known. The system has a unique solution (α, β) since

$$V\ and\ W\ are\ LI. \Rightarrow \quad \det\begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} \neq 0.$$

interaction ➔
a unique solution

Examples

Example. The unit vectors $E_1 = (1,0)$ and $E_2 = (0,1)$ obviously form a basis called the $\boxed{\textit{standard basis}}$ of $\mathbb{R}^2$. The coordinates of Z in this basis are just the "usual" Cartesian coordinates (x, y) of Z. ■

$$det = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \neq 0$$

E_1 and E_2 are linearly Independent

Examples

Example. The vectors $V_1 = (1,1)$ and $V_2 = (-1,1)$ also form a basis of $\mathbb{R}^2$. Relative to this basis, the coordinates of E_1 are $(1/2, -1/2)$ and those of E_2 are $(1/2, 1/2)$

$$det = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \neq 0$$

V_1 and V_2 are linearly Independent

$$E_1 = \boxed{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \alpha V_1 + \beta V_2 = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} \alpha - \beta \\ \alpha + \beta \end{pmatrix}}$$

$$\alpha - \beta = 1$$
$$\alpha + \beta = 0$$

$$\alpha = \frac{1}{2}, \beta = -\frac{1}{2}$$

Similarly,

$$E_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = aV_1 + bV_2 = a \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} a - b \\ a + b \end{pmatrix}$$

$$a = \frac{1}{2}, b = \frac{1}{2}$$

Examples

Example. The vectors $V_1 = (1, 1)$ and $V_2 = (-1, -1)$ do not form a basis of $\mathbb{R}^2$ since these vectors are collinear. Any linear combination of these vectors is of the form

$$\alpha V_1 + \beta V_2 = \begin{pmatrix} \alpha - \beta \\ \alpha - \beta \end{pmatrix}, \qquad y = x$$

which yields only vectors on the straight line through the origin, that is, V_1 and V_2. ■

$$det = \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} = 0$$

V_1 and V_2 are *linearly dependent*

Sect. 2.4 Planar Linear System

Consider the following system of ODEs

$$x' = ax + by$$
$$y' = cx + dy$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then the linear system may be written as

$$X' = AX, \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

The critical points of the above system are defined as $X' = 0$, yielding

$$\boxed{AX = 0}$$

Review: Sect. 2.3

Consider the following two lines in R^2

$$ax + by = \alpha$$
$$cx + dy = \beta,$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$|A| = 0 \Longrightarrow$ two lines are parallel or overlap — no solution or infinitely many solutions

$|A| \neq 0 \Longleftrightarrow$ two lines intersect — unique solution

Special Linear Systems (for Critical Points)

Consider a special case of the following two lines in R^2

$$ax + by = \alpha, \qquad ax + by = 0,$$
$$cx + dy = \beta, \qquad cx + dy = 0,$$

$$\boxed{AX = 0, \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}}$$

which represent two lines passing through the origin. As a result, (0, 0) is a solution. We have the following two scenarios:

- $|A| = 0 \Longrightarrow$ two lines overlap — infinitely many solutions
- $|A| \neq 0 \Longleftrightarrow$ two lines intersect — unique solution of (0,0)

$$\boxed{\text{If the system has non-trivial solutions, } |A| = 0.}$$

Sect. 2.4 Planar Linear System

Consider the following system of ODEs

$$X' = AX, \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Proposition. *The planar linear system $X' = AX$ has*

1. *A unique $\boxed{\text{equilibrium point}}$ $(0,0)$ if det $A \neq 0$.*
2. *A straight line of equilibrium points if det $A = 0$ (and A is not the 0 matrix).* ∎

two lines intersect

two lines overlap

Review of 2.2: Alternative Method

$$\boxed{\begin{aligned} x' &= y \\ y' &= -x \end{aligned}}$$ ODEs

How to solve?

Previously, we assume $x = ke^{\lambda t}$

Now, we assume $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{\lambda t}$

$$x = x_0 e^{\lambda t}$$
$$y = y_0 e^{\lambda t}$$

Plug into the above Eq. $\boxed{\begin{aligned} \lambda x_0 &= y_0 \quad (1) \\ \lambda y_0 &= -x_0 \quad (2) \end{aligned}}$ Algebraic Eq.

$\lambda \times (1) + (2)$ $\qquad \lambda^2 x_0 = -x_0$

$$\lambda = \pm i$$

Introduction to Eigenvalue Problem

ODEs

$$\boxed{\begin{aligned} x' &= y \\ y' &= -x \end{aligned}}$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

We assume

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{\lambda t}$$

$$\begin{aligned} x &= x_0 e^{\lambda t} \\ y &= y_0 e^{\lambda t} \end{aligned}$$

Plug into the above Eq.

$$\boxed{\begin{aligned} \lambda x_0 &= y_0 \\ \lambda y_0 &= -x_0 \end{aligned}}$$

Algebraic Eq.

Express the above in a matrix form

$$\lambda \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = A \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$A \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \lambda \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = 0$$

$$(A - \lambda I)V_0 = 0 \qquad V_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

V_0: eigenvector
λ: eigenvalue

Eigenvalue Problem

Definition
A nonzero vector V_0 is called an *eigenvector* of A if $AV_0 = \lambda V_0$ for some λ. The constant λ is called an *eigenvalue* of A.

Eigenvalue problem $\quad (A - \lambda I)V_0 = 0 \quad$ V_0: eigenvector
λ: eigenvalue

- $|A - \lambda I| = 0 \Longrightarrow$ infinitely many solutions
- $|A - \lambda I| \neq 0 \Longleftrightarrow$ unique solution of (0,0)

If the system has non-trivial solutions, $|A - \lambda I| = 0$.

If the system has eigenvectors, $|A - \lambda I| = 0$.

Introduction to Eigenvalue Problem (cont.)

Consider $(A-\lambda I)V_0 = 0$ with $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $V_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

V_0: eigenvector
λ: eigenvalue

Compute $A - \lambda I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix}$

If the system has eigenvectors, $|A - \lambda I| = 0$.

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \qquad \boxed{\lambda = \pm i}$$

Consider $\lambda = i$

$$\boxed{\begin{aligned} \lambda x_0 &= y_0 \\ \lambda y_0 &= -x_0 \end{aligned}} \qquad \boxed{\begin{aligned} i x_0 &= y_0 \\ i y_0 &= -x_0 \end{aligned}}$$

("overlap") $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ i x_0 \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ i \end{pmatrix}$

Obtain $\boxed{V_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}}$ as an eigenvector associated with $\lambda = i$

Similarly, we have $\boxed{V_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}}$ as an eigenvector associated with $\lambda = -i$