MATH 537, Fall 2020
Ordinary Differential Equations

Lecture #29

The WKBJ/LG Approximation

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Final Exam(35%)

- Part A: take-home problems, 15% Dec 11 (F)
- Part B: 8:00-10:00 am, 20% Dec 14 (M)

Students need to submit their Part-B work (to GradeScope) by 10:10 am on Dec 14.

– MWF/MW Classes

Class Meeting Start Time	Examination Date	Exam Hour
0700 MWF	Monday, Dec. 14	0600-0800
0800 MWF	Friday, Dec. 11	0800-1000
0900 MWF	Monday, Dec. 14	0800-1000

https://registrar.sdsu.edu/calendars/final_exam_schedule/fall_2020_final_exam_schedule

MT Part B: 8:00-10:00 AM, December 14 (Monday) (this slide will be available @canvas/supp)

❖ Please read the following instructions carefully.

➢ "conceptual, comprehensive" and "less technical"

1) Students should submit their Part-B work (to GradeScope) by 10:10 am on December 14.
   - ➢ Document any issues (e.g., using screenshots) and report via emails or "chat" as soon as possible.
   - ➢ Submit your work to GradeScope (under "InBox") after 10:10 am on December 14.
2) Enable your camera during the exam time period.
3) The following rules are applied for the Openbook Exam:
   - ➢ Materials are allowed;
   - ➢ However, the exam must be taken completely alone. Showing it or discussing it with anybody is forbidden.
4) Document/Record the time interval for completing each of the **four (or three, TBD)** selected problems; For example, 9:15-9:27 for problem 2 (a deduction of 10 points may be applied for each of the problems which does not include the time interval for completion).

References

- Bender, C. M., and S. A. Orszag, 2010: *Advanced Mathematical Methods for Scientists and Engineers.* Springer-Verlag, 593 pp. ISBN 978-1-4419-3187-0.
- Boyce W. and R. C. DiPrima, 2012: Elementary Differential Equations. Tenth Edition. John Wiley & Sons, INC. 832 pages ISBN: 978-0-470-45831-0
- Gill, A., 1982: Atmosphere-Ocean Dynamics. Academic Press. ISBN 0-12-283520-4. 681 pp.
- Haberman, R. 2013: *Applied Partial Differential Equations*, 5th edition, by Publisher: Pearson/Prentice Hall. ISBN-10: 0-321-79705-1. ISBN-13: 978-0321797056.
- Kreyszig, E., 2011: Advanced Engineering Mathematics. 10th edition. John Wiley & Sons, INC. ISBN 978-0-470-45836-5. 1113 pp.
- Mathews J. and R. L. Walker, 1970: Mathematical Methods of Physics. 2nd edition, 501pp.

Supplemental Materials

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WKBJ-LG.pdf	11:21pm	11:21pm		2.8 MB	
PertubationMethod-2020.pdf	11:29pm	11:29pm		29 KB	

Singular Perturbation Problems: Rapidly Decaying vs. Rapidly Oscillatory

1. A local breakdown occurs where the approximate solution is exponentially increasing or decreasing. This kind of behavior is called dissipative because the rapidly varying component of the solution decays exponentially (dissipates) away from the point of local breakdown.

$$\boxed{\epsilon y'' - y' = 0, \qquad y(0) = 0, \qquad y(1) = 1,}$$

boundary layer techniques

2. A global breakdown is typically associated with rapidly oscillatory, or dispersive, behavior. A dispersive solution is wavelike with very small and slowly changing wavelengths and slowly varying amplitudes as functions of x.

$$\boxed{\epsilon y'' + y = 0, \qquad y(0) = 0, \qquad y(1) = 1,}$$

WKBJ

Singular Perturbation Problems: Dissipative vs. Dispersive

1. A local breakdown occurs where the approximate solution is exponentially increasing or decreasing. This kind of behavior is called dissipative because the rapidly varying component of the solution decays exponentially (dissipates) away from the point of local breakdown.

$$\varepsilon y'' - y' = 0, \qquad y(0) = 0,\ y(1) = 1,$$

$$y(x) = \frac{e^{x/\varepsilon} - 1}{e^{1/\varepsilon} - 1}.$$

boundary layer techniques

2. A global breakdown is typically associated with rapidly oscillatory, or dispersive, behavior. A dispersive solution is wavelike with very small and slowly changing wavelengths and slowly varying amplitudes as functions of x.

$$\varepsilon y'' + y = 0, \qquad y(0) = 0,\ y(1) = 1,$$

$$y(x) = \frac{\sin\,(x/\sqrt{\varepsilon})}{\sin\,(1/\sqrt{\varepsilon})}, \qquad \varepsilon \neq (n\pi)^{-2},$$

WKBJ

Appearance of Rapid Oscillations on a Global Scale

Example 3 *Appearance of rapid variation on a global scale.* In the previous example we saw that the exact solution varies rapidly in the neighborhood of $x = 1$ for small ε and develops a discontinuity there in the limit $\varepsilon \to 0+$. A solution to a boundary-value problem may also develop discontinuities throughout a large region as well as in the neighborhood of a point.

The boundary-value problem $\varepsilon y'' + y = 0$ $[y(0) = 0,\ y(1) = 1]$ is a singular perturbation problem because when $\varepsilon = 0$, the solution to the unperturbed problem, $y = 0$, does not satisfy the boundary condition $y(1) = 1$. The exact solution, when ε is not of the form $(n\pi)^{-2}$ $(n = 0, 1, 2, \ldots)$, is $y(x) = \sin(x/\sqrt{\varepsilon})/\sin(1/\sqrt{\varepsilon})$. Observe that $y(x)$ becomes discontinuous throughout the inter-

$$\epsilon y'' + y = 0, \qquad y(0) = 0, \qquad y(1) = 1,$$

An exact solution $y = \dfrac{\sin(x/\sqrt{\epsilon})}{\sin(1/\sqrt{\epsilon})}$ when $\epsilon \neq \dfrac{1}{(n\pi)^2}$, $\mathrm{n} = 1,2,3\cdots$

$$\epsilon^2 y'' + y = 0, \qquad y(0) = 0, \qquad y(1) = 1,$$

An exact solution $y = \dfrac{\sin(x/\epsilon)}{\sin(1/\epsilon)}$ when $\epsilon \neq \dfrac{1}{(n\pi)}$, $\mathrm{n} = 1,2,3\cdots$

Appearance of Rapid Oscillations on a Global Scale

An exact solution $y = \dfrac{\sin(x/\sqrt{\epsilon})}{\sin(1/\sqrt{\epsilon})}$ when $\epsilon \neq \dfrac{1}{(n\pi)^2}$, $\mathrm{n} = 1,2,3 \cdots$

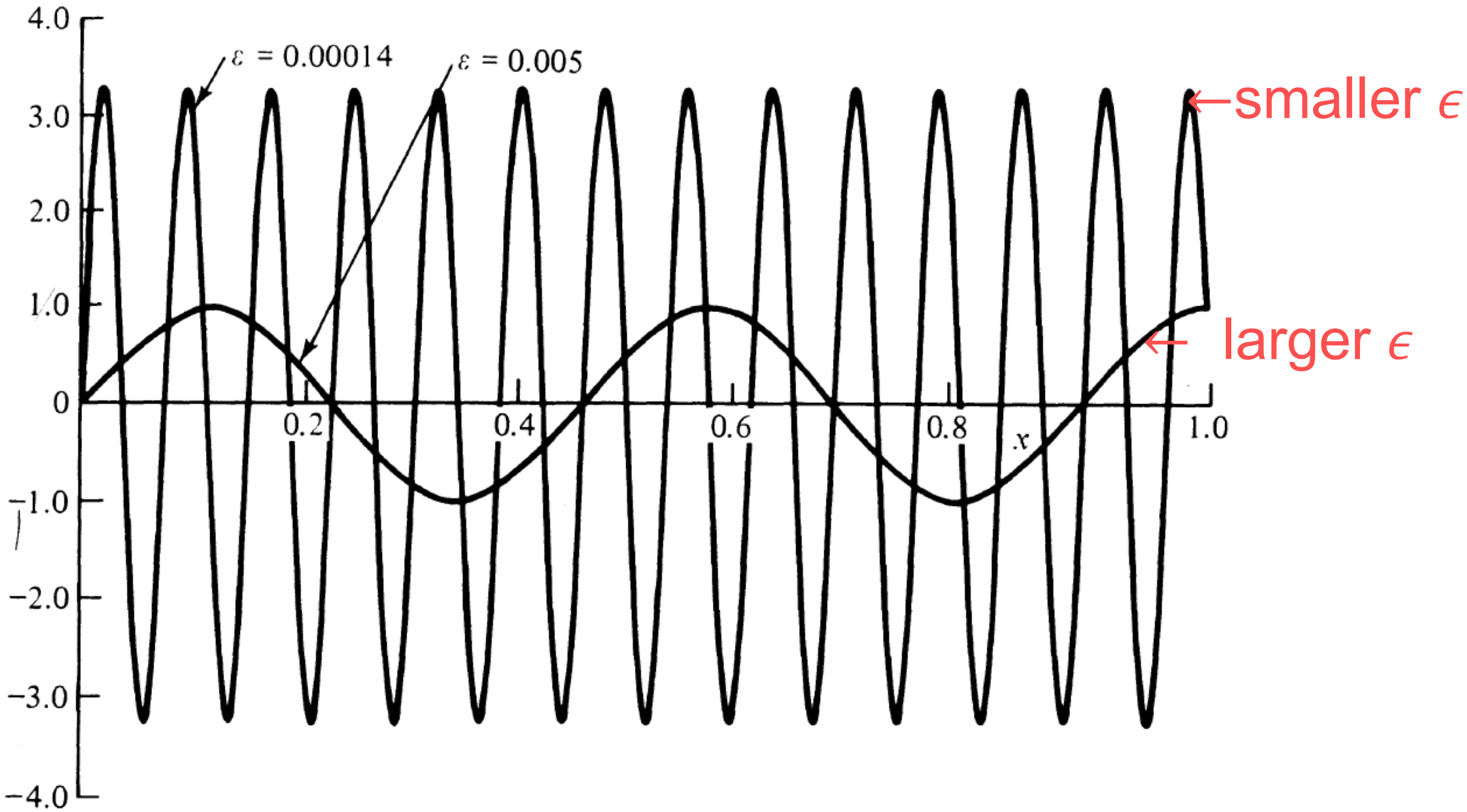


Figure 7.3 A plot of $y(x) = [\sin(x\varepsilon^{-1/2})]/[\sin(\varepsilon^{-1/2})]$ $(0 \le x \le 1)$ for $\varepsilon = 0.005$ and 0.00014. As ε gets smaller the oscillations become more violent; as $\varepsilon \to 0+$, $y(x)$ becomes discontinuous over the entire interval. The WKB approximation is a perturbative method commonly used to describe functions like $y(x)$ which exhibit rapid variation on a global scale.

Why WKBJ?

Boundary-layer techniques are not powerful enough to handle dispersive phenomena. To see why, let us try to solve (10.1.1) using boundary-layer methods. Setting $\varepsilon = 0$ in (10.1.1) gives the outer solution $y_{\text{out}}(x) = 0$, which is obviously a terrible approximation to the actual solution in (10.1.2). The actual solution in Fig. 7.3 looks like a sequence of internal boundary layers with no outer solution at all. Even for this very simple problem, boundary-layer analysis is insufficient.

From our understanding of Chap. 9 we can intuit that it is the absence of a one-derivative term which causes the global breakdown of the solution to (10.1.1). In Sec. 9.6 we showed that internal boundary layers may occur in the solution of $\varepsilon y'' + a(x)y' + b(x)y = 0 \quad [y(0) = A, \quad y(1) = B]$ at isolated points for which $a(x) = 0$. When $a(x) \equiv 0$ on an interval, it is not surprising to find that the solution is rapidly varying on the entire interval. Fortunately, WKB theory provides a simple and general approximation method for linear differential equations which treats dissipative and dispersive phenomena equally well.

WKBJ or LG Approximation

- This method is named after physicists Gregor **Wentzel**, Hendrik Anthony **Kramers**, and Léon **Brillouin**, who all developed it in 1926.
- In 1923, mathematician Harold **Jeffreys** had developed a general method of approximating solutions to linear, second-order differential equations, a class that includes the Schrödinger equation. The Schrödinger equation itself was not developed until two years later.
- Earlier appearances of essentially equivalent methods are: Francesco Carlini in 1817, Joseph **Liouville** (mathematician and engineer) in 1837, George Green (mathematical physicist) in 1837, Lord Rayleigh in 1912 and Richard Gans in 1915. (Wikipedia)

This lecture includes the following discussions from three references:

I. The WKB method in Mathews and Walker (1970)

II. The Liouville-Green Approximation in Gill (1982)

III. The WKB method in Bender and Orszag (2010)

Review: Method of Dominant Balance: An Illustration

$$y'' + p(x)y' + q(x)y = 0$$

$\boxed{y = e^{S(x)}}$ $\qquad y' = S'e^{S(x)}$ $\qquad y'' = S''e^{S(x)} + (S')^2 e^{S(x)}$

$$S''e^{S(x)} + (S')^2 e^{S(x)} + pS'e^{S(x)} + qe^{S(x)} = 0$$

$S'' + (S')^2 + pS' + q = 0$ $\qquad$ divide by $e^{s(x)}$

$\boxed{S'' << (S')^2, \quad \text{as } x \to x_0}$

1. drop (all) terms that are small

$\boxed{(S')^2 \sim -pS' - q, \text{ as } x \to x_0}$

2. replace "=" by "~" and solve the system (note that don't have "0" on the RHS)

Asymptotic differential equations

3. check whether the solution is consistent with the approximation, i.e., whether $S'' << (S')^2$ is valid.

"Dissipations"

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y = 0. \tag{1a}$$

Transform the above equation into one without the first derivative, which is shown as follows

$$\frac{d^2u}{dx^2} + q(x)u = 0, \tag{1b}$$

by making the substitution

$$y(x) = P(x)u(x).$$

$$P(x) = e^{-\int \frac{a(x)}{2} dx}$$

positive dissipation when $a(x) > 0$

negative dissipation when $a(x) < 0$

A Quick Look: $y'' + f(x)y = 0$

$y'' + p(x)y' + q(x)y = 0$

$y = e^{S(x)}$

$(S')^2 \sim -pS' - q$, as $x \to x_0$

Asymptotic differential equations

$\boxed{y'' + f(x)y = 0}$

$p(x) = 0; q = f$

$\boxed{(S')^2 \sim -f, \text{ as } x \to x_0}$

If $S(x) = \lambda x$

$\boxed{(\lambda)^2 \sim -f}$ $\lambda \sim \pm i\sqrt{f}$

$y \sim e^{i\sqrt{f}x} = \cos(\sqrt{f}x + \alpha)$

$\boxed{\text{wavelength, } L = \frac{2\pi}{\sqrt{f}}}$

or, frequency , $\omega = \frac{2\pi}{\sqrt{f}}$

$y'' = Q(x)y$ vs. Schrodinger Eq.

Example 5 *Local behavior of solutions near an irregular singular point of a general nth-order Schrödinger equation.* In this example we derive an extremely simple and important formula for the leading behavior of solutions to the *n*th-order Schrödinger equation

$$\frac{d^n y}{dx^n} = Q(x)y \tag{3.4.27}$$

near an irregular singular point at x_0.

The exponential substitution $y = e^S$ and the asymptotic approximations $d^kS/dx^k \ll (S')^k$ as $x \to x_0$ for $k = 2, 3, \ldots, n$ give the asymptotic differential equation $(S')^n \sim Q(x)$ $(x \to x_0)$. Thus, $S(x) \sim \omega \int^x [Q(t)]^{1/n}\, dt$ $(x \to x_0)$, where ω is an nth root of unity. This result determines the n possible controlling factors of $y(x)$.

The leading behavior of $y(x)$ is found in the usual way (see Prob. 3.27) to be

$$\boxed{y(x) \sim c[Q(x)]^{(1-n)/2n} \exp\left\{\omega \int^x [Q(t)]^{1/n}\, dt\right\}, \qquad x \to x_0.} \tag{3.4.28}$$

If $x_0 \neq \infty$, (3.4.28) is valid if $|(x - x_0)^n Q(x)| \to \infty$ as $x \to x_0$. If $x_0 = \infty$, then (3.4.28) is valid if $|x^n Q(x)| \to \infty$ as $x \to \infty$. This important formula forms the basis of WKB theory and will be rederived perturbatively and in much greater detail in Sec. 10.2. If $Q(x) < 0$, solutions to (3.4.27) oscillate as $x \to \infty$; the nature of asymptotic relations between oscillatory functions is discussed in Sec. 3.7.

(I): WKBJ in Mathews and Walker

1-4 THE WKB METHOD

The WKB method provides approximate solutions of differential equations of the form

$$\frac{d^2y}{dx^2} + f(x)y = 0 \tag{1-85}$$

provided $f(x)$ satisfies certain restrictions discussed below, which may be summarized in the phrase "$f(x)$ is slowly varying." Recall that any linear homogeneous second-order equation may be put in this form by the transformation (1-41). The one-dimensional Schrödinger equation is of this form and the method was developed for quantum-mechanical applications by Wentzel, by Kramers,[9] and by Brillouin, whence the name. The method had been given previously by Jeffreys.[10]

The solutions of Eq. (1-85) with $f(x)$ constant suggest the substitution

$$\boxed{y = e^{i\phi(x)}} \tag{1-86}$$

(I) WKBJ Approximation

TBD

$$\frac{d^2y}{dx^2} + f(x)y = 0 \qquad (1\text{-}85)$$

$y = e^{i\phi(x)}$ $\qquad \phi^2 \sim f$ $\qquad \phi \sim \pm\sqrt{f}$ $\qquad y = e^{S(x)}$

$$y(x) \approx \frac{1}{(f(x))^{1/4}} \left\{ c_+ \exp\left[i \int \sqrt{f(x)}\, dx \right] + c_- \exp\left[-i \int \sqrt{f(x)}\, dx \right] \right\} \qquad (1\text{-}90)$$

The condition of validity (that ϕ'' be "small") is

$$|\phi''| \approx \frac{1}{2} \left| \frac{f'}{\sqrt{f}} \right| \ll |f| \qquad (1\text{-}89)$$

From (1-86) and (1-88) we see that $1/\sqrt{f}$ is roughly $1/(2\pi)$ times one "wavelength" or one "exponential length" of the solution y. Thus the condition of validity of our approximation is simply the intuitively reasonable one that the change in $f(x)$ in one wavelength should be small compared to $|f|$.

(I) The WKBJ Method

$$y'' + f(x)y = 0$$

$\boxed{y = e^{iS(x)}}$ $\quad y' = iS'e^{iS(x)} \quad y'' = iS''e^{S(x)} - (S')^2 e^{S(x)}$

$$iS''e^{S(x)} - (S')^2 e^{S(x)} + f(x)e^{S(x)} = 0$$

$iS'' - (S')^2 \sim -f(x)$ divide by $e^{s(x)}$

$\boxed{S'' << (S')^2, \quad \text{as } x \to x_0}$

1. drop (all) terms that are small

$$-(S')^2 \sim -f(x)$$

$$S' \sim \pm\sqrt{f}$$

$$S \sim \pm \int \sqrt{f}\,dx$$

(I) The WKBJ Method: $S'' << (S')^2$

$\boxed{S'' << (S')^2, \quad \text{as } x \to x_0}$

$$S \sim \pm \int \sqrt{f}\, dx \qquad S' \sim \pm \sqrt{f} \qquad S'' \sim \pm \frac{f'}{2\sqrt{f}}$$

$\boxed{S'' << (S')^2, \quad \text{as } x \to x_0}$

$$\frac{f'}{2\sqrt{f}} \ll f$$

$\boxed{\text{wavelength, } L \sim \frac{2\pi}{\sqrt{f}}}$

$$\frac{f'L}{4\pi} \ll f \qquad f'L \approx \int_0^L f'\, dx$$

accumulated changes in one wavelength

- The changes in $f(x)$ in one wavelength (i.e., $f'(x)L \sim \frac{f'(x)}{\sqrt{f}}$) should be small compared to $f(x)$.
- Stated alternatively, "$f(x)$ is slowly varying".

(I-a) The WKBJ Method: Improving $S(x)$

$$S \sim \int \sqrt{f}\,dx + C(x), \qquad C(x) << \int \sqrt{f}\,dx$$

$$S' \sim \sqrt{f} + C'(x)$$

$$S'' \sim \frac{\frac{1}{2} f'(x)}{\sqrt{f}} + C''(x) \qquad iS'' - (S')^2 + f(x) = 0$$

$$i\left(\frac{\frac{1}{2} f'(x)}{\sqrt{f}} + C''(x)\right) - \left(\sqrt{f} + C'(x)\right)^2 \sim -f(x)$$

expand

$$i\left(\frac{\frac{1}{2} f'(x)}{\sqrt{f}} + C''(x)\right) - \left(f(x) + 2\sqrt{f}\,C' + (C')^2\right) \sim -f(x)$$

(I-a) The WKBJ Method: Improving $S(x)$

$$i\left(\frac{\frac{1}{2}f'(x)}{\sqrt{f}}+C''(x)\right)-\left(\cancel{f(x)}+2\sqrt{f}C'+(C')^2\right)\sim -\cancel{f(x)}$$

$$i\left(\boxed{\frac{\frac{1}{2}f'(x)}{\sqrt{f}}}+C''(x)\right)\sim\left(\boxed{2\sqrt{f}C'}+(C')^2\right)$$

$$C'(x)<<\sqrt{f} \qquad C''\ll\frac{\frac{1}{2}f'}{\sqrt{f}}$$

$$i\left(\frac{\frac{1}{2}f'(x)}{\sqrt{f}}\right)\sim\left(2\sqrt{f}C'\right) \qquad C'\sim i\left(\frac{\frac{1}{2}f'(x)}{2f}\right)=\frac{i}{4}\frac{f'(x)}{f}$$

$$C\sim\frac{i}{4}\ln(f(x))$$

(I-a) The WKBJ Method : Improving $S(x)$

$$S \sim \int \sqrt{f}\,dx + C(x), \qquad C \sim \frac{i}{4}\ln(fx)$$

$$S \sim \int \sqrt{f}\,dx + \frac{i}{4}\ln\big(f(x)\big), \qquad \frac{1}{4}\ln(f(x)) << \int \sqrt{f}\,dx$$

$$\boxed{y = e^{iS(x)} \sim \exp\left(i\int \sqrt{f}\,dx - \frac{1}{4}\ln\big(f(x)\big)\right)}$$

$$\boxed{= \exp\left(-\frac{1}{4}\ln\big(f(x)\big)\right)\exp\left(i\int \sqrt{f}\,dx\right)}$$

$$\boxed{= \exp\left(\ln\big(f(x)\big)^{-1/4}\right)\exp\left(i\int \sqrt{f}\,dx\right)}$$

$$\boxed{= \big(f(x)\big)^{-1/4}\exp\left(i\int \sqrt{f}\,dx\right)}$$

$$\boxed{= \frac{1}{\sqrt[4]{f}}\exp\left(i\int \sqrt{f}\,dx\right)}$$

(I-b) The WKBJ Method : Improving $S(x)$

$$y'' + f(x)y = 0$$

$$\boxed{iS'' - (S')^2 \sim -f(x)}$$

$$S' \sim +\sqrt{f} \qquad\qquad S'' \sim \frac{f'}{2\sqrt{f}}$$

Plug S'' (not S') into the above asymptotic equation

$$i\frac{f'}{2\sqrt{f}} - (S')^2 \sim -f(x) \qquad\qquad (S')^2 \sim f(x) + i\frac{f'}{2\sqrt{f}}$$

$$S' \sim \pm\sqrt{f(x) + i\frac{f'}{2\sqrt{f}}} = \pm\sqrt{f(x)}\sqrt{1 + i\frac{f'}{2\sqrt[3]{f}}}$$

$$S' \sim \pm\sqrt{f(x)}\left(1 + \frac{1}{2}i\frac{f'}{2\sqrt[3]{f}} + \cdots\right) \qquad S' \sim \pm\left(\sqrt{f(x)} + \frac{i}{4}\frac{f'}{f}\right)$$

(I-b) The WKBJ Method: Improving $S(x)$

$$S' \sim \pm\left(\sqrt{f(x)} + \frac{i}{4}\frac{f'}{f}\right)$$

$$S' \sim +\left(\sqrt{f(x)} + \frac{i}{4}\frac{f'}{f}\right)$$

$$S \sim +\left(\int \sqrt{f}\,dx + \frac{i}{4}\ln(f(x))\right)$$

$$\boxed{y = e^{iS(x)} \sim \exp\left(i\int \sqrt{f}\,dx - \frac{1}{4}\ln\big(f(x)\big)\right)}$$

$$\boxed{= \frac{1}{\sqrt[4]{f}}\exp\left(i\int \sqrt{f}\,dx\right)}$$

WKBJ vs. Liouville-Green Approximation

$$y'' + f(x)y = 0$$

$$\boxed{y = \frac{1}{\sqrt[4]{f}} \exp\left(i \int \sqrt{f}\, dx\right)}$$

- Wentzel, Kramers, and Brillouin (1926)
- Jeffreys (1923)

(II) The Liouville-Green or WKBJ Approximation

A problem of this type was tackled by Liouville (1837) and Green (1838) and is discussed in textbooks on asymptotic theory such as those of Erdélyi (1956, Chapter 4) and Olver (1974, Chapter 6). The approximate solution is therefore called the Liouville–Green approximation. It was also (and still is) called the WKB or WKBJ approximation, based on the initials of more recent authors, until it was realized that the method was used much earlier by Liouville and Green.

Gill (1982)

(II): The Liouville-Green Approximation

$$\frac{d^2w}{dz^2} + m^2 w = 0 \qquad (A)$$

$$\phi = \int m\, dz \qquad \mathbb{W} = \sqrt{m} w$$

$$\frac{dw}{dz} = \frac{dw}{d\phi}\frac{d\phi}{dz} = \frac{dw}{d\phi} z$$

$$\frac{d^2w}{dz^2} = \left(\frac{d^2w}{d\phi^2} m + \frac{dw}{d\phi}\frac{dm}{d\phi}\right) m$$

$$m\left(\frac{d^2w}{d\phi^2} m + \frac{dw}{d\phi}\frac{dm}{d\phi}\right) + m^2 w = 0 \qquad (B)$$

(II): The Liouville-Green Approximation

$$m\left(m\frac{d^2w}{d\phi^2}+\frac{dm}{d\phi}\frac{dw}{d\phi}\right)+m^2w=0 \qquad (B)$$

$$\mathbb{W}=\sqrt{m}w \qquad \frac{d\mathbb{W}}{d\phi}=\sqrt{m}\frac{dw}{d\phi}+\frac{1}{2}w\frac{\frac{dm}{d\phi}}{\sqrt{m}}$$

$$\frac{d^2\mathbb{W}}{d\phi^2}=\frac{1}{\sqrt{m}}\left(m\frac{d^2w}{d\phi^2}+\frac{dm}{d\phi}\frac{dw}{d\phi}+w\frac{d^2m}{d\phi^2}\right)$$

$$m\frac{d^2w}{d\phi^2}+\frac{dm}{d\phi}\frac{dw}{d\phi}=\sqrt{m}\frac{d^2\mathbb{W}}{d\phi^2}-w\frac{d^2m}{d\phi^2}$$

Plug into Eq. (B)

$$m\left(\sqrt{m}\frac{d^2\mathbb{W}}{d\phi^2}-w\frac{d^2m}{d\phi^2}\right)+m^2w=0 \qquad (C)$$

(II): The Liouville-Green Approximation

$$m\left(\sqrt{m}\frac{d^2\mathbb{W}}{d\phi^2} - w\frac{d^2m}{d\phi^2}\right) + m^2w = 0 \qquad (C)$$

$$\mathbb{W} = \sqrt{m}w$$

$$m\left(\sqrt{m}\frac{d^2\mathbb{W}}{d\phi^2} - \frac{\mathbb{W}}{\sqrt{m}}\frac{d^2m}{d\phi^2}\right) + \frac{m^2\mathbb{W}}{\sqrt{m}} = 0$$

Multiply by $\sqrt{m}$

$$\left(m^2\frac{d^2\mathbb{W}}{d\phi^2} - m\mathbb{W}\frac{d^2m}{d\phi^2}\right) + m^2\mathbb{W} = 0$$

$$\frac{d^2\mathbb{W}}{d\phi^2} + \mathbb{W} - \frac{1}{m}\frac{d^2m}{d\phi^2}\mathbb{W} = 0$$

(II): The Liouville-Green Approximation

$$\frac{d^2\mathbb{W}}{d\phi^2} + \mathbb{W} - \frac{1}{m}\frac{d^2 m}{d\phi^2}\mathbb{W} = 0$$

small

$$\frac{d^2\mathbb{W}}{d\phi^2} + \mathbb{W} = 0$$

$$\boxed{\mathbb{W} \sim \exp(i\phi)}$$

recall

$$\phi = \int m\, dz$$

$$\boxed{\mathbb{W} \sim \exp\left(i\int m\, dz\right)}$$

recall

$$\mathbb{W} = \sqrt{m}w$$

$$w \sim \frac{\mathbb{W}}{\sqrt{m}} = \frac{1}{\sqrt{m}}\exp\left(i\int m\, dz\right)$$

WKBJ vs. Liouville-Green Approximation

$$y'' + f(x)y = 0$$

$$\frac{d^2w}{dz^2} + m^2 w = 0 \qquad (A)$$

$$\boxed{y = \frac{1}{\sqrt[4]{f}} \exp\left(i \int \sqrt{f}\, dx\right)}$$

$$w \sim \boxed{\frac{1}{\sqrt{m}}} \exp\left(i \int m \, dz\right)$$

This becomes part of the coefficient if m is constant.

- Wentzel, Kramers, and Brillouin (1926)
- Jeffreys (1923)

- Liouville (1837)
- Green (1838)

A Note on the WKBJ Approximation

1. The method fails if f(x) changes too rapidly [because the method requires $S'' << (S')^2$, as $x \to x_0$] or if f(x) passes through zero.
2. The latter is a serious difficulty since we often wish to join an oscillatory solution in a region where f(x) > 0 to an "exponential" one in a region where f(x) < 0.
3. The above problem should be solved using the so-called connection formulas relating the constants c_+ and c_- of the WKB solutions on either side of a point where f(x) = 0.

$x \ll x_0, f(x) < 0$:

exponential

$$y(x) \approx \frac{a}{\sqrt[4]{-f(x)}} \exp\left[+\int_x^{x_0} \sqrt{-f(x)}\, dx\right] + \frac{b}{\sqrt[4]{-f(x)}} \exp\left[-\int_x^{x_0} \sqrt{-f(x)}\, dx\right] \quad (1\text{-}91)$$

$x \gg x_0, f(x) > 0$:

oscillatory

$$y(x) \approx \frac{c}{\sqrt[4]{f(x)}} \exp\left[+i\int_{x_0}^{x} \sqrt{f(x)}\, dx\right] + \frac{d}{\sqrt[4]{f(x)}} \exp\left[-i\int_{x_0}^{x} \sqrt{f(x)}\, dx\right] \quad (1\text{-}92)$$

Singular Perturbation Problems: Rapidly Decaying vs. Rapidly Oscillatory

1. A local breakdown occurs where the approximate solution is exponentially increasing or decreasing. This kind of behavior is called dissipative because the rapidly varying component of the solution decays exponentially (dissipates) away from the point of local breakdown.

$$\boxed{\epsilon y'' - y' = 0, \qquad y(0) = 0, \qquad y(1) = 1,}$$

boundary layer techniques

2. A global breakdown is typically associated with rapidly oscillatory, or dispersive, behavior. A dispersive solution is wavelike with very small and slowly changing wavelengths and slowly varying amplitudes as functions of x.

$$\boxed{\epsilon y'' + y = 0, \qquad y(0) = 0, \qquad y(1) = 1,}$$

WKBJ

Singular Perturbation Problems: Dissipative vs. Dispersive

1. A local breakdown occurs where the approximate solution is exponentially increasing or decreasing. This kind of behavior is called dissipative because the rapidly varying component of the solution decays exponentially (dissipates) away from the point of local breakdown.

$$\varepsilon y'' - y' = 0, \qquad y(0) = 0,\ y(1) = 1,$$

boundary layer techniques

$$y(x) = \frac{e^{x/\varepsilon} - 1}{e^{1/\varepsilon} - 1}.$$

2. A global breakdown is typically associated with rapidly oscillatory, or dispersive, behavior. A dispersive solution is wavelike with very small and slowly changing wavelengths and slowly varying amplitudes as functions of x.

$$\varepsilon y'' + y = 0, \qquad y(0) = 0,\ y(1) = 1,$$

WKBJ

$$y(x) = \frac{\sin\,(x/\sqrt{\varepsilon})}{\sin\,(1/\sqrt{\varepsilon})}, \qquad \varepsilon \neq (n\pi)^{-2},$$

Singular Perturbation Problems

Rapid changes $\epsilon y'' - y' = 0, \quad y(0) = 0, \quad y(1) = 1,$

Rapid oscillations $\epsilon y'' + y = 0, \quad y(0) = 0, \quad y(1) = 1,$

Singular Perturbation Problems

$$y(x) = exp\left[\frac{1}{\epsilon}\sum_{n=0}^{\infty}\epsilon^n S_n(x)\right]$$

"negative power"

$$= exp\left[\frac{1}{\epsilon}S_0 + S_1 + \epsilon S_2 + \cdots\right]$$

$$S_1 << \frac{1}{\epsilon}S_0$$

$$\epsilon S_2 << S_1$$

$$\epsilon^n S_{n+1} << S_n$$

A Note

- DEs with irregular singularities or regular perturbation problems:

$\boxed{\text{e.g., } y'' + f(x)y = 0}$

- The Exponential Approximation

$$y(x) \sim e^{S(x)}$$

- Singular perturbation problems

$\boxed{\text{e.g. } \epsilon y'' + y = 0}$

- The Exponential Approximation (a.k.a. the WKB approximation)

$$y(x) \sim A(x) e^{S(x)/\epsilon}$$

- Formal WKB Expansion (using an exponential power series)

$$y(x) = exp\left[\frac{1}{\epsilon}\sum_{n=0}^{\infty} \epsilon^n S_n(x)\right] = exp\left[\frac{1}{\epsilon} S_0 + S_1 + \epsilon S_2 + \cdots\right]$$

A Note on the Exponential Approximation

- Dissipative and dispersive phenomena are both characterized by exponential behavior, where the exponent is real in the former case and imaginary in the latter case.
- It is natural to seek an approximate solution of the form

$$y(x) \sim A(x) e^{\frac{S(x)}{\epsilon}}, \qquad \epsilon \to 0+$$

- The phase S(x) is assumed nonconstant and slowly varying in a breakdown region.
- When S is real, there is a boundary layer of thickness δ ;
- When S is imaginary, there is a region of rapid oscillation characterized by waves having wavelength of order δ.
- When S(x) is constant, the behavior of y(x), which is characteristic of an outer solution in boundary-layer theory, is expressed by the slowly varying amplitude function A(x).
- The exponential approximation in (10.1.3) is conventionally known as a WKB approximation.

(III) The WKB method in Bender and Oraszag

$$\epsilon^2 y'' = Q(x)y, \qquad \epsilon \to 0 \; and \; Q(x) \neq 0$$

$$y \sim exp\left[\frac{1}{\epsilon}S(x) + C(x) + \cdots\right]$$

$$y' \sim exp\left[\frac{1}{\epsilon}S(x) + C(x) + \cdots\right]\left(\frac{S'}{\epsilon} + C'\right)$$

$$y'' \sim exp\left[\frac{1}{\epsilon}S(x) + C(x) + \cdots\right]\left(\frac{S'}{\epsilon} + C'\right)^2 + exp\left[\frac{1}{\epsilon}S(x) + C(x) + \cdots\right]\left(\frac{S''}{\epsilon} + C''\right)$$

$$y'' \sim y\left(\frac{S'}{\epsilon} + C'\right)^2 + y\left(\frac{S''}{\epsilon} + C''\right)$$

$$\epsilon^2\left[\left(\frac{S'}{\epsilon} + C'\right)^2 + \left(\frac{S''}{\epsilon} + C''\right)\right] = Q(x)$$

(III) The WKB method in Bender and Oraszag: $Q(x) > 0$

$$\epsilon^2\left[\left(\frac{S'}{\epsilon}+C'\right)^2+\left(\frac{S''}{\epsilon}+C''\right)\right]=Q(x)$$

$$(S')^2+2\epsilon S'C'+\epsilon^2(C')^2+\epsilon S''+\epsilon^2 C''=Q(x)$$

$O(\epsilon^0)$:

$$(S')^2 \sim Q(x) \qquad S' \sim \sqrt{Q(x)} \qquad S'' \sim \frac{Q'}{2\sqrt{Q(x)}}$$

$$S' \sim -\sqrt{Q(x)} \qquad S'' \sim \frac{-Q'}{2\sqrt{Q(x)}}$$

$O(\epsilon^1)$:

$$2S'C' \sim -S'' \qquad C' \sim -\frac{S''}{2S'} \qquad C' \sim -\frac{\frac{Q'}{2\sqrt{Q(x)}}}{2\sqrt{Q(x)}}=\frac{-Q'}{4Q}$$

(III) The WKB method in Bender and Oraszag: $Q(x) > 0$

$$\epsilon^2 y'' = Q(x)y, \qquad \epsilon \to 0 \; and \; Q(x) \neq 0$$

$$y \sim exp\left[\frac{1}{\epsilon}S(x) + C(x) + \cdots\right]$$

$$S' \sim \sqrt{Q(x)} \qquad\qquad S \sim \int \sqrt{Q(x)}dx$$

$$C' \sim -\frac{\frac{Q'}{2\sqrt{Q(x)}}}{2\sqrt{Q(x)}} = \frac{-Q'}{4Q} \qquad\qquad C \sim -\frac{1}{4}\ln(Q)$$

$$y \sim \exp(C(x))exp\left[\frac{1}{\epsilon}S(x)\right]$$

$$y \sim \frac{1}{\sqrt[4]{Q(x)}}exp\left[\frac{1}{\epsilon}\int \sqrt{Q(x)}dx\right]$$

(III) The WKB method in Bender and Oraszag: $Q(x) < 0$

$$\epsilon^2\left[\left(\frac{S'}{\epsilon}+C'\right)^2+\left(\frac{S''}{\epsilon}+C''\right)\right]=Q(x)$$

$$(S')^2+2\epsilon S'C'+\epsilon^2(C')^2+\epsilon S''+\epsilon^2 C''=Q(x)$$

$O(\epsilon^0)$:

$$(S')^2 \sim Q(x) = -P(x) \qquad S' \sim i\sqrt{P(x)} \qquad S'' \sim \frac{iP'}{2\sqrt{P(x)}}$$

$$S' \sim -i\sqrt{P(x)} \qquad S'' \sim \frac{-iP'}{2\sqrt{P(x)}}$$

$O(\epsilon^1)$:

$$2S'C' \sim -S'' \qquad C' \sim -\frac{S''}{2S'} \qquad C' \sim -\frac{\frac{P'}{2\sqrt{P(x)}}}{2\sqrt{P(x)}} = \frac{-P'}{4P}$$

(III) The WKB method in Bender and Oraszag: $\boldsymbol{Q(x) < 0}$

$$\epsilon^2 y'' = Q(x)y, \qquad \epsilon \to 0 \; and \; Q(x) \neq 0 \qquad Q(x) = -P(x)$$

$$y \sim exp\left[\frac{1}{\epsilon} S(x) + C(x) + \cdots\right]$$

$$S' \sim \sqrt{Q(x)} = i\sqrt{P(x)} \qquad S \sim \int i\sqrt{P(x)}dx$$

$$C' \sim -\frac{\dfrac{Q'}{2\sqrt{Q(x)}}}{2\sqrt{Q(x)}} = \frac{-P'}{4P} \qquad C \sim -\frac{1}{4}\ln(P)$$

$$y \sim \exp(C(x))exp\left[\frac{1}{\epsilon} S(x)\right]$$

$$y \sim \frac{1}{\sqrt[4]{P(x)}} exp\left[\frac{i}{\epsilon}\int \sqrt{P(x)}dx\right]$$

$Q(x) > 0$ vs. $Q(x) < 0$

$$\epsilon^2 y'' = Q(x)y, \qquad \epsilon \to 0 \; and \; Q(x) \neq 0$$

$Q(x) > 0$

$Q(x) < 0$

$(S')^2 \sim Q(x)$

$(S')^2 \sim Q(x) = -P(x)$

$$y \sim \frac{1}{\sqrt[4]{Q(x)}} exp\left[\frac{1}{\epsilon}\int \sqrt{Q(x)}dx\right]$$

$$y \sim \frac{1}{\sqrt[4]{P(x)}} exp\left[\frac{i}{\epsilon}\int \sqrt{P(x)}dx\right]$$

exponential

oscillatory

positive LE
(averaged divergence)

duality

recurrence

WKBJ vs. Liouville-Green

$$y'' + f(x)y = 0$$

$$y = \frac{1}{\sqrt[4]{f}} \exp\left(i \int \sqrt{f}\, dx\right)$$

$$f(x) = m^2$$

$$\frac{d^2 w}{dz^2} + m^2 w = 0 \qquad (A)$$

$$w \sim \boxed{\frac{1}{\sqrt{m}}} \exp\left(i \int m\, dz\right)$$

This becomes part of the coefficient if m is constant.

$$f(x) = -\frac{Q(x)}{\epsilon} = \frac{P(x)}{\epsilon}$$

$$\epsilon^2 y'' = Q(x) y, \qquad Q(x) = -P(x) < 0$$

$$y \sim \frac{1}{\sqrt[4]{P(x)}} exp\left[\frac{i}{\epsilon} \int \sqrt{P(x)} dx\right]$$

Solving the Airy Eq. using the WKBJ Method

Example 1 *Behavior of Airy functions as $x \to +\infty$.* The Airy equation $y'' = xy$ is a Schrödinger equation with $Q(x) = x$ and $\varepsilon = 1$. Thus, from (10.1.11), (10.1.12), and (10.1.14) we have $S_0 = \pm\frac{2}{3}x^{3/2}$, $S_1 = -\frac{1}{4}\ln x$, $S_2 = \pm\frac{5}{48}x^{-3/2}$. We observe that even when $\varepsilon = 1$, the asymptotic inequalities $\varepsilon S_2 \ll S_1 \ll S_0/\varepsilon$, $\varepsilon S_2 \ll 1$ $(x \to +\infty)$ hold. We conclude that for fixed ε the physical-optics approximation is valid as $x \to +\infty$. Indeed, we have just rederived the leading behaviors of

solutions to the Airy equation as $x \to +\infty$ as well as the first correction to the leading behaviors [see (3.5.21)]:

$$y(x) \sim c_\pm x^{-1/4} e^{\pm 2x^{3/2}/3}\left(1 \pm \tfrac{5}{48}x^{-3/2}\right),$$

where $c_\pm$ is a constant. Note that the rapidly varying exponential factors $e^{\pm 2x^{3/2}/3}$ come from the geometrical-optics approximation.

$$\epsilon^2 y'' = Q(x)y$$

$$y'' = xy \qquad \epsilon = 1; \;\; Q(x) = 1 \qquad y = exp\left[\frac{1}{\epsilon}S_0 + S_1 + \epsilon S_2 + \cdots\right]$$

$$S_0 = \pm\frac{2}{3}x^{3/2} \qquad S_1 = -\frac{1}{4}\ln(x) \qquad S_2 = \pm\frac{5}{48}x^{-\frac{-3}{2}} \qquad \epsilon S_2 \ll 1$$

$$y \sim c_\pm \exp(S_0)\exp(S_1) \qquad y \sim c_\pm \exp\left(\pm\frac{2}{3}x^{3/2}\right)x^{-1/4}$$

WKBJ vs. Liouville-Green

$$y'' + f(x)y = 0$$

$$y = \frac{1}{\sqrt[4]{f}} \exp\left(i \int \sqrt{f}\, dx\right)$$

$$f(x) = m^2$$

$$\frac{d^2w}{dz^2} + m^2 w = 0 \qquad (A)$$

$$w \sim \boxed{\frac{1}{\sqrt{m}}} \exp\left(i \int m\, dz\right)$$

This becomes part of the coefficient if m is constant.

$$f(x) = -\frac{Q(x)}{\epsilon} = \frac{P(x)}{\epsilon}$$

$$\epsilon^2 y'' = Q(x) y, \qquad Q(x) = -P(x) < 0$$

$$y \sim \frac{1}{\sqrt[4]{P(x)}} exp\left[\frac{i}{\epsilon} \int \sqrt{P(x)} dx\right]$$

Airy Equation $(y'' = xy)$: WKB Analysis

$$y'' = xy$$

connecting
formula

$$Q = x$$

- $x > 0, Q > 0,$ exponential
- $x < 0, Q < 0,$ oscillatory

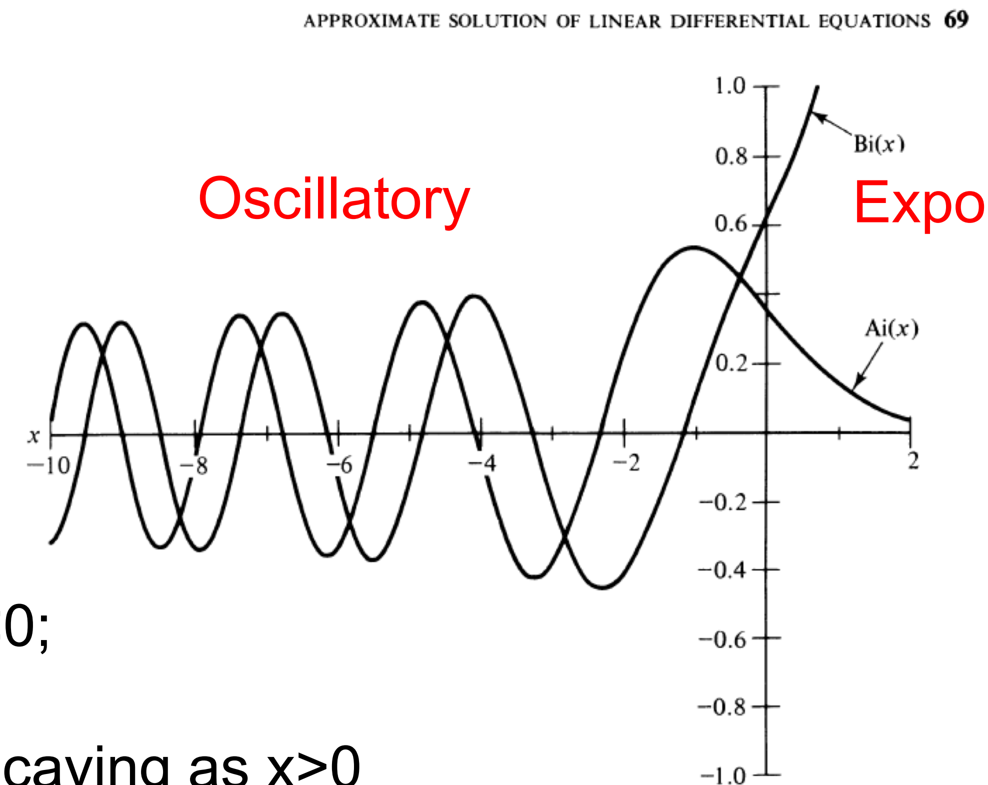
- x=0 is a turning point (where $y'' = 0$)

- Oscillatory solutions as x<0;

- Exponential growing or decaying as x>0

Oscillatory

Exponential

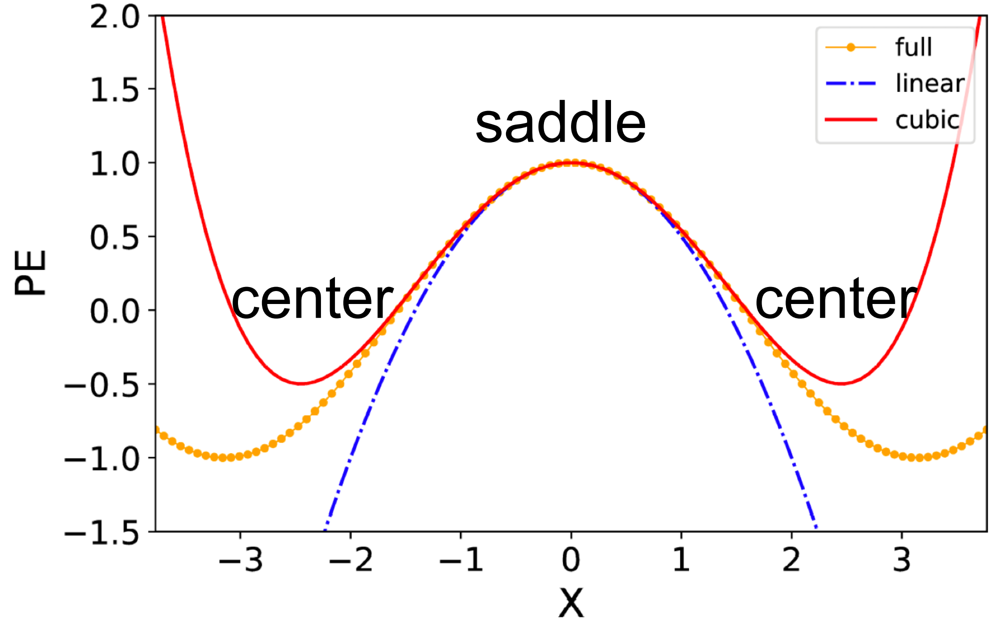


$$X'' - X + \frac{X^3}{6} = 0$$

WBKJ analysis

$$X'' = \left(1 - \frac{X^2}{6}\right) X$$

$$Q = \left(1 - \frac{X^2}{6}\right)$$



exponential

$$1 > \frac{X^2}{6}, \quad \sqrt{6} > X > -\sqrt{6}, \qquad Q(X) > 0, \ \text{exponential}$$

$$1 < \frac{X^2}{6}, \quad |X| > \sqrt{6} \qquad Q(X) < 0, \ \text{oscillatory}$$

A Solution using the WKBJ Method

Example 5 *Local behavior of solutions near an irregular singular point of a general nth-order Schrödinger equation.* In this example we derive an extremely simple and important formula for the leading behavior of solutions to the nth-order Schrödinger equation

$$\frac{d^n y}{dx^n} = Q(x)y \tag{3.4.27}$$

near an irregular singular point at x_0.

The exponential substitution $y = e^S$ and the asymptotic approximations $d^kS/dx^k \ll (S')^k$ as $x \to x_0$ for $k = 2, 3, \ldots, n$ give the asymptotic differential equation $(S')^n \sim Q(x)$ $(x \to x_0)$. Thus, $S(x) \sim \omega \int^x [Q(t)]^{1/n}\, dt$ $(x \to x_0)$, where ω is an nth root of unity. This result determines the n possible controlling factors of $y(x)$.

The leading behavior of $y(x)$ is found in the usual way (see Prob. 3.27) to be

$$\boxed{y(x) \sim c[Q(x)]^{(1-n)/2n} \exp\left\{\omega \int^x [Q(t)]^{1/n}\, dt\right\}, \qquad x \to x_0.} \tag{3.4.28}$$

If $x_0 \neq \infty$, (3.4.28) is valid if $|(x - x_0)^n Q(x)| \to \infty$ as $x \to x_0$. If $x_0 = \infty$, then (3.4.28) is valid if $|x^n Q(x)| \to \infty$ as $x \to \infty$. This important formula forms the basis of WKB theory and will be rederived perturbatively and in much greater detail in Sec. 10.2. If $Q(x) < 0$, solutions to (3.4.27) oscillate as $x \to \infty$; the nature of asymptotic relations between oscillatory functions is discussed in Sec. 3.7.

A Solution using the WKBJ Method (Haberman 2013)

Precise asymptotic techniques[6] beyond the scope of this text determine the slowly varying amplitude. It is known that two independent solutions of the differential equation can be approximated accurately (if λ is large) by

$$\boxed{\phi(x) \approx (\sigma p)^{-1/4} \exp\left[\pm i\lambda^{1/2} \int^{x} \left(\frac{\sigma}{p}\right)^{1/2} dx_0\right],} \tag{5.9.8}$$

where sines and cosines may be used instead. A rough sketch of these solutions

[6]These results can be derived by various ways. such as the W.K.B.(J.) method (which should be called the Liouville-Green method) or the method of multiple scales. References for these asymptotic techniques include books by Bender and Orszag [1999], Kevorkian and Cole [1996], and Nayfeh [2002].