Oct 09, 2024

Chapter-3: Modeling with Delay Differential Equation.

Motivation:

Recall $\frac{dx}{dt} = f(x)x$

$f(x)$ = growth rate per capita.

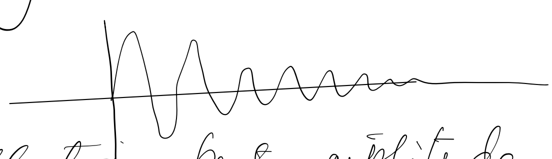
Assumptions behind this model:

- $x(t)$ is a smooth function
- environment is closed.
- environment is spatially homogeneous
- temporal homogeneity of the species/environment

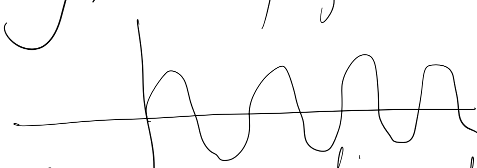
$\Longrightarrow$

- a scalar ODE which is autonomous (the parameters are independent of time)
- the dynamics is determined by the structure of equilibria.
- monotonic convergence is the "generic" behavior.

OBSERVATION: Even for a single species in a very carefully controlled laboratory, the population oscillates.



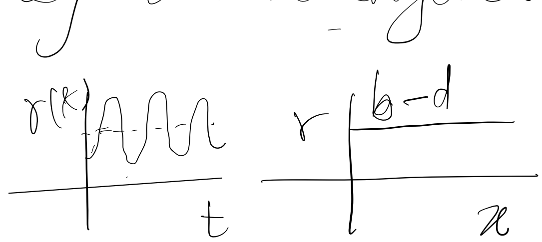
oscillation but amplitude decays



periodic oscillation

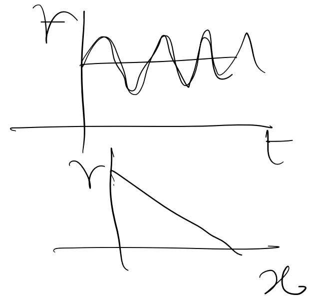
Explaination:

- the oscillation is caused by the heterogeneity (temporal) of environment.



Malthusian $\rightarrow$ $\dfrac{dx}{dt} = r(t)x$

Logistic $\rightarrow$ $\dfrac{dx}{dt} = r(t)\, x\left[1 - \dfrac{x}{K}\right]$



- there are other mechanism of oscillation.

DELAY — the "internal" temporal structure of the species

Recall: $\dfrac{1}{x(t)} \dfrac{dx(t)}{dt} = f(x(t))$

Per capita growth rate = a function of the species at the current time.

Difficulty:

In relatity,

$$\frac{1}{x(t)} \frac{dx(t)}{dt} = -\text{death rate} + \text{birth rate}$$

$$= -\delta(x(t)) + b(\text{past population})$$

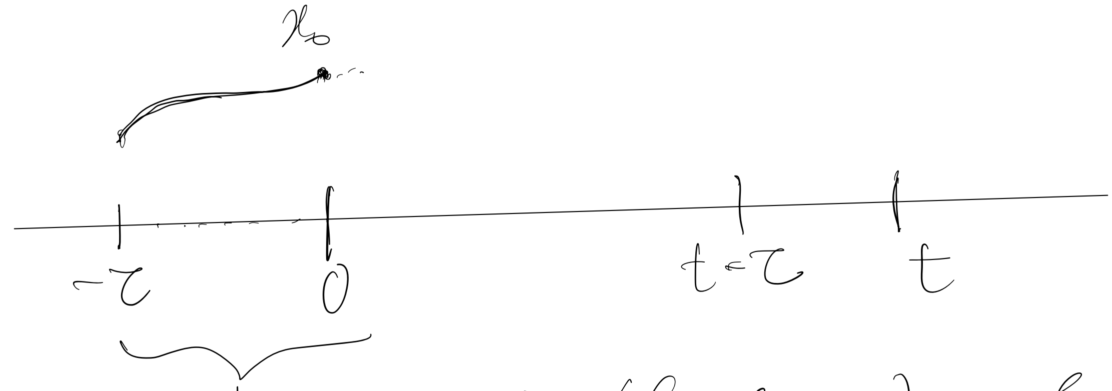
instantaneous   delayed (not instantaneous) per capita growth rate is not necessarily instantaneous but is delays due to

- maturation time
- hatching period
- slow replacement of food supply
- gestation

General Delay Differential Equation (DDE) Model:

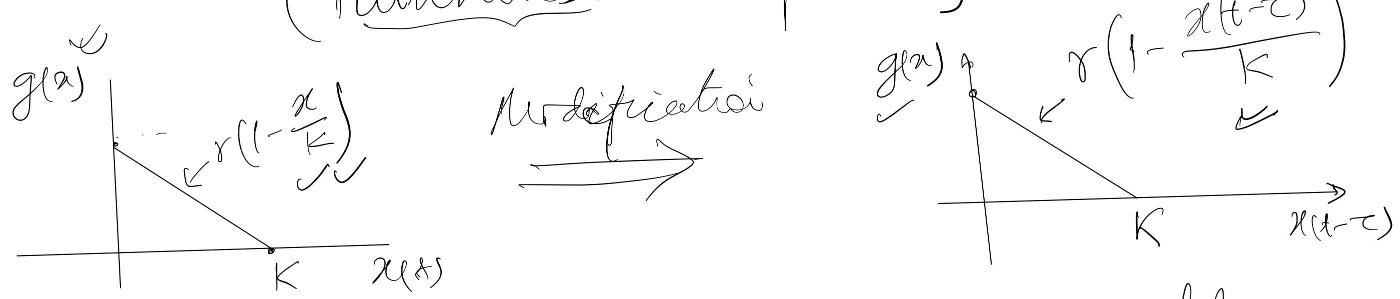
$$\frac{dx(t)}{dt} = f(t, x(t), x(t-\tau)), \text{ where}$$

$\tau > 0$ is delay and $x(t-\tau) = \{x(\tau) : \tau \le t\}$ gives the trajectory of the solution in the past.



initial condition (history), which is function. (Functional Space)

Example: Logistic Delay Model
(Hutchinson's Equation)



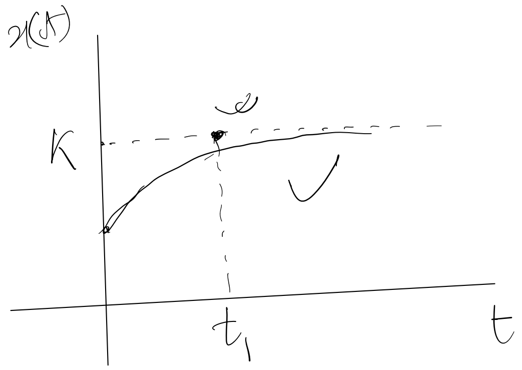
$\tau > 0$ - average delay.

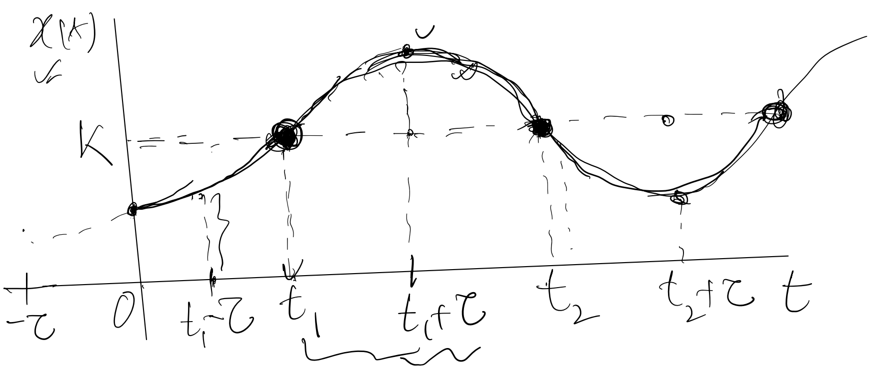
$\therefore$ Logistic delay model

$$\frac{1}{x(t)}\frac{dx(t)}{dt} = r\left[1 - \frac{x(t-\tau)}{K}\right]$$

$$\Rightarrow \boxed{\frac{dx(t)}{dt} = r\,x(t)\left[1 - \frac{x(t-\tau)}{K}\right]}$$

. The heuristic argument for oscillation:





At some instant t_1, $x(t_1) = K$ and $x(t) < K$ for $t < t_1$.

$$\left.\frac{dx}{dt}\right|_{t=t_1} = r\,x(t_1)\left[1 - \frac{x(t_1-\tau)}{K}\right] > 0 \text{ at } t = t_1$$

$\Rightarrow$ $x(t)$ at $t = t_1$ is still increasing and remains increasing for all $t \in (t_1, t_1+\tau)$

At $t = \underbrace{t_1+\tau}$, $x(t-\tau) = x(t_1) = K$ and $\frac{dx(t)}{dt} = 0$

For $t_1+\tau < t < t_2$, where t_2 is the first instance where $t_2 > t_1$ and $x(t_2) = K$, $t-\tau > t_1$ and $x(t-\tau) > K$ which implies

$$\left.\frac{dx}{dt}\right|_{t=t_2} = r\, x(t_2)\left[1 - \frac{x(t_2-\tau)}{K}\right] < 0 \text{ and}$$

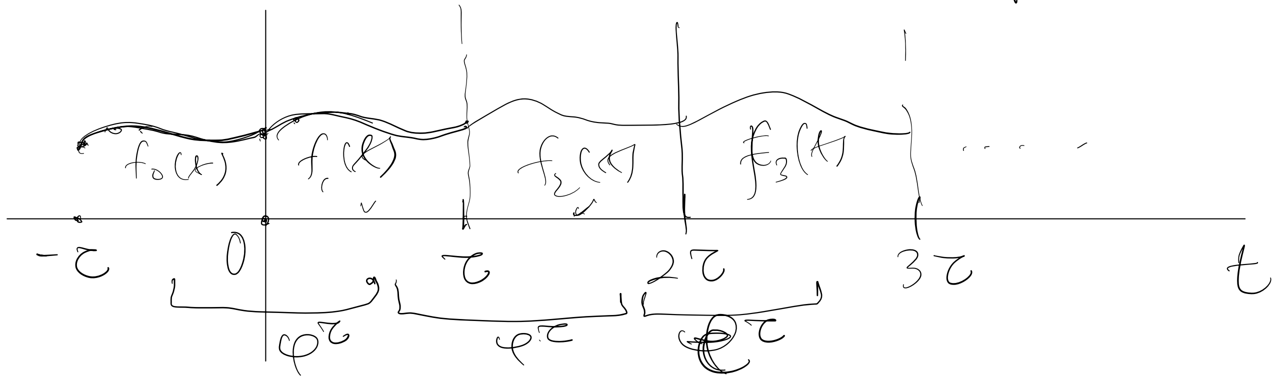
$x(t)$ decreases until $t = t_2+\tau$ since then $\frac{dx}{dt} = 0$ again becomes

$x(t_2+\tau-\tau) = x(t_2) = K$. There is a possibility of periodic oscillatory solution with period 4τ.

Solution of DDE:

$$\begin{cases} \dfrac{dx}{dt} = f\big(x(t), x(t-\tau)\big), & \tau > 0 \\ x(t) = x_0(t), & t \in [-\tau, 0] \end{cases}$$

(history function)



Stepwise method:

For given $f_{i-1}(x(t), x(t-\tau))$, $t \in [t_{i-1}, t_i]$

$$\int_{x(t_{i-1})}^{x(t)} dx(t) = \int_{x(t_{i-1})}^{x(t)} f_{i-1}(x(t), x(t-\tau))\,dt,$$

$t \in [t_i, t_{i+1}]$

Numerical Solution:

Example: Logistic Delay

Matlab files: dde.m

$\Rightarrow$ Delay can cause

- distabilizing effect
- oscillatory pattern formation