Homework 7
Abstract Algebra
Math 320
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Section 3.2 Problem 9: Show that the set S of matrices of the form $\begin{pmatrix} a & 4b \\ b & a \end{pmatrix}$, with a and b real numbers is a subring of $\mathbb{M}(\mathbb{R})$

Solution. Let A $\in S = \begin{pmatrix} a_1 & 4b_1 \\ b_1 & a_1 \end{pmatrix}$, and B $\in S = \begin{pmatrix} a_2 & 4b_2 \\ b_2 & a_2 \end{pmatrix}$ with a and $b \in \mathbb{R}$

1) Notice: Let $a_1 = 0, b_1 = 0$, so

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0_S \in S$$

2) $A + B = \begin{pmatrix} a_1 & 4b_1 \\ b_1 & a_1 \end{pmatrix} + \begin{pmatrix} a_2 & 4b_2 \\ b_2 & a_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & 4(b_1 + b_2) \\ b_1 + b_2 & a_1 + a_2 \end{pmatrix} \in S$

3) Notice: Let -B = $\begin{pmatrix} -a_2 & -4b_2 \\ -b_2 & -a_2 \end{pmatrix}$, so -B $\in S$

$$A + -B = \begin{pmatrix} a_1 & 4b_1 \\ b_1 & a_1 \end{pmatrix} + \begin{pmatrix} -a_2 & -4b_2 \\ -b_2 & -a_2 \end{pmatrix} = \begin{pmatrix} a_1 - a_2 & 4(b_1 - b_2) \\ b_1 - b_2 & a_1 - a_2 \end{pmatrix} \in S$$

4) $A * B = \begin{pmatrix} a_1 & 4b_1 \\ b_1 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & 4b_2 \\ b_2 & a_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + 4b_1 b_2 & 4(a_1 b_2 + b_1 a_2) \\ b_1 a_2 + a_1 b_2 & 4b_1 b_2 + a_1 a_2 \end{pmatrix}$

$$= \begin{pmatrix} a_1 a_2 + 4b_1 b_2 & 4(a_1 b_2 + b_1 a_2) \\ a_1 b_2 + b_1 a_2 & a_1 a_2 + 4b_1 b_2 \end{pmatrix} \in S$$

Thus S is a subring of $\mathbb{M}(\mathbb{R})$ as it holds closure under subtraction, multiplication and contains 0

□

Section 3.2 Problem 12: Let a and b be elements of a ring R.

(a) Prove that the equation $a + x = b$ has a unique solution in R. (You must prove that there is a solution and that this solution is the only one.)

(b) If R is a ring with identity and a is a unit, prove that the equation $ax = b$ has a unique solution in R.

Solution a). Notice: Because R is a ring, $\forall a \in R, \exists(-a)$ such that $a + -a = 0_R$

$$a + x = b$$
$$x = b - a$$

Thus there exists a solution to the equation, $a + x = b$

Let x_1, x_2 both be solutions to the equation, $a + x = b$

$$a + x_1 = b$$
$$a + x_2 = b$$
$$x_1 = b - a$$
$$x_2 = b - a$$
$$x_1 = x_2$$

Thus there exists a unique solution to the equation, $\boldsymbol{a + x = b}$ □

Solution b). Let R be a ring with identity and a be a unit. So $\exists 1_R$ and $\exists x$ such that $ax = 1_R$

Let x_1, x_2 both be solutions to the equation $ax = b$

$$ax_1 = b$$
$$ax_2 = b$$
$$ax_1 = ax_2$$
$$ax_1 - ax_2 = 0$$
$$a(x_1 - x_2) = 0$$
$$a^{-1}a(x_1 - x_2) = a^{-1}0$$
$$x_1 - x_2 = 0$$
$$x_1 = x_2$$

Thus there exists a unique solution to the equation $\boldsymbol{ax = b}$

□