Note: For full credit you must show intermediate steps in your calculations.

1. (4pts) Consider the initial value problem (IVP):

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Find the general solution to this problem, create a phase portrait, and solve the initial value problem. Describe the *qualitative behavior* shown in the phase portrait. (Slides 19-22)

Solution: The characteristic equation satisfies:

$$\begin{vmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{vmatrix} = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2) = 0.$$

The eigenvalues and associated eigenvectors are:

$$\lambda_1 = 3, \quad \xi_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = -2, \quad \xi_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

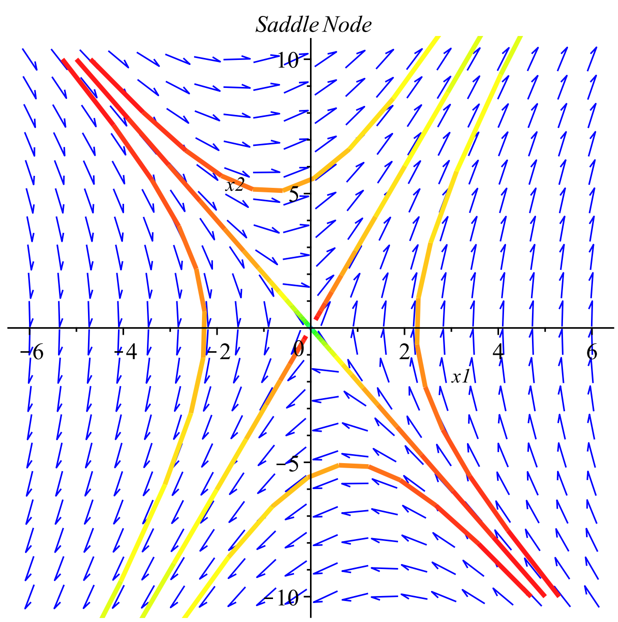
which gives the general solution:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t}.$$

The solution to the IVP is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t}.$$

This problem produces a *saddle node* seen below:



2. (4pts) Consider the differential equation:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Find the general solution to this problem and create a phase portrait. Describe the qualitative behavior shown in the phase portrait. (Slides 44-49)

Solution: The characteristic equation satisfies:

$$\begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 = 0.$$

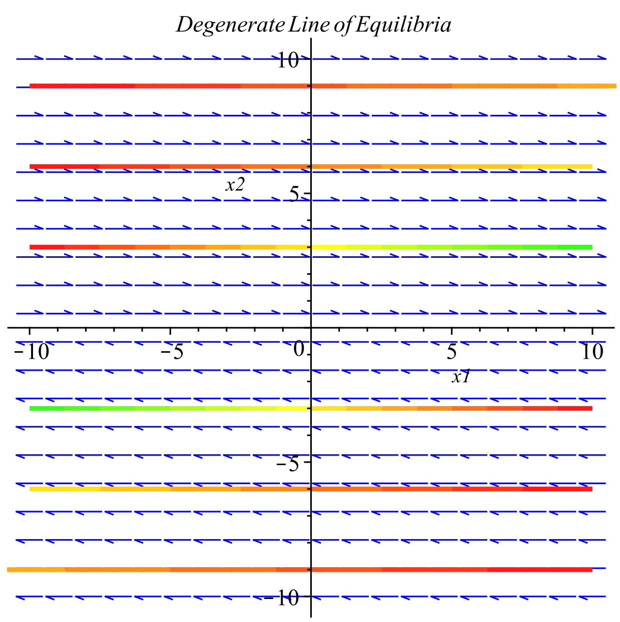
It follows that $\lambda = 0$ is an eigenvalue with algebraic multiplicity of 2, but it only has the eigenvector, $\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so has geometric multiplicity of 1. There are two ways to solve this problem. If one begins by integrating the equation for $\dot{x}_2$, one obtains $x_2(t) = c_2$. This is substituted into the equation for $\dot{x}_1$, and one obtains $x_1(t) = c_2 t + c_1$. Alternately, one can solve the higher nullspace/eigenspace problem:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \xi_2 = \xi_1, \quad \text{which is satisfied by} \quad \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In either case, we find the general solution:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} t \\ 1 \end{pmatrix}.$$

This problem produces the degenerate case with a *line of equilibria* along the x_1-axis seen below:



3. (4pts) Consider the differential equation with the parameter α:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \alpha & 2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Find the general solutions and create phase portraits for the values $\alpha = -6$ and $\alpha = 3$. Describe the qualitative behavior shown in the phase portraits. (Slides 50-54)

Solution: For $\alpha = -6$, the characteristic equation satisfies:

$$\begin{vmatrix} -6-\lambda & 2 \\ -2 & -\lambda \end{vmatrix} = \lambda^2 + 6\lambda + 4 = 0.$$

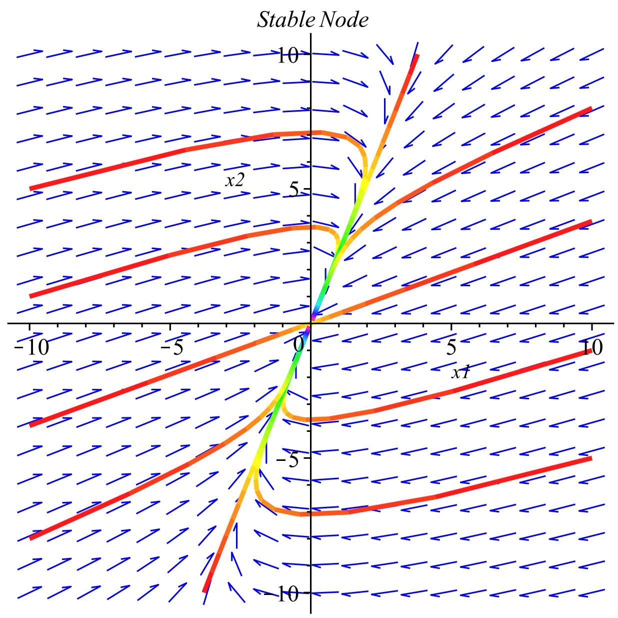
The eigenvalues and associated eigenvectors are:

$$\lambda_1 = -3 + \sqrt{5}, \quad \xi_1 = \begin{pmatrix} 2 \\ 3+\sqrt{5} \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = -3 - \sqrt{5}, \quad \xi_2 = \begin{pmatrix} 2 \\ 3-\sqrt{5} \end{pmatrix},$$

which gives the general solution:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 3+\sqrt{5} \end{pmatrix} e^{(-3+\sqrt{5})t} + c_2 \begin{pmatrix} 2 \\ 3-\sqrt{5} \end{pmatrix} e^{(-3-\sqrt{5})t}.$$

For $\alpha = -6$, the phase portrait produces a *stable node* seen below:



For $\alpha = 3$, the characteristic equation satisfies:

$$\begin{vmatrix} 3-\lambda & 2 \\ -2 & -\lambda \end{vmatrix} = \lambda^2 - 3\lambda + 4 = 0, \qquad \text{so} \qquad \lambda = \frac{3}{2} \pm i\frac{\sqrt{7}}{2}.$$

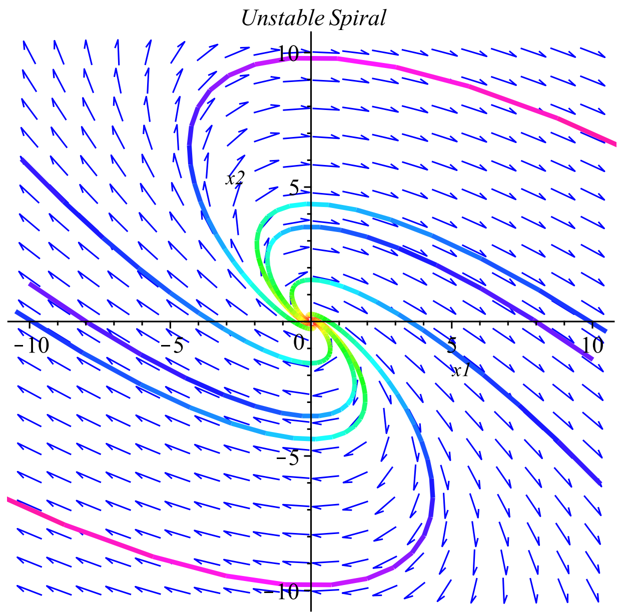
For $\lambda_1 = \frac{3}{2} + i\frac{\sqrt{7}}{2}$, we have the associated eigenvector:

$$\xi_1 = \begin{pmatrix} 2 \\ -\frac{3}{2} + i\frac{\sqrt{7}}{2} \end{pmatrix},$$

which gives the general solution:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{\frac{3t}{2}} \left[c_1 \begin{pmatrix} 2\cos\left(\frac{\sqrt{7}}{2}t\right) \\ -\frac{3}{2}\cos\left(\frac{\sqrt{7}}{2}t\right) - \frac{\sqrt{7}}{2}\sin\left(\frac{\sqrt{7}}{2}t\right) \end{pmatrix} + c_2 \begin{pmatrix} 2\sin\left(\frac{\sqrt{7}}{2}t\right) \\ -\frac{3}{2}\sin\left(\frac{\sqrt{7}}{2}t\right) + \frac{\sqrt{7}}{2}\cos\left(\frac{\sqrt{7}}{2}t\right) \end{pmatrix} \right].$$

For $\alpha = 3$, the phase portrait produces a *unstable spiral* seen below:



4. (4pts) Consider the differential equations $\dot{\mathbf{x}} = J_i\mathbf{x}$, where J_i is each of the following matrices:

$$J_1 = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 5 & 3 \\ -2 & 2 \end{pmatrix}, \qquad J_3 = \begin{pmatrix} 1 & -3 \\ 2 & -5 \end{pmatrix}, \qquad J_4 = \begin{pmatrix} 3 & -2 \\ 6 & -3 \end{pmatrix},$$

Use the diagram on Slide 55 to classify the qualitative behavior for these differential equations ($J_i, i = 1, 2, 3, 4$) without solving the equations.

Solution: The matrix J_1 has $tr(J_1) = -1$, $\det|J_1| = -2$, and $D = 9$, which according to the stability diagram gives the qualitative behavior of $\dot{\mathbf{x}} = J_1\mathbf{x}$ as a *saddle node*.
The matrix J_2 has $tr(J_2) = 7$, $\det|J_2| = 16$, and $D = -15$, which according to the stability diagram gives the qualitative behavior of $\dot{\mathbf{x}} = J_2\mathbf{x}$ as an *unstable spiral*.
The matrix J_3 has $tr(J_3) = -4$, $\det|J_3| = 1$, and $D = 12$, which according to the stability diagram gives the qualitative behavior of $\dot{\mathbf{x}} = J_3\mathbf{x}$ as a *stable node*.
The matrix J_4 has $tr(J_4) = 0$, $\det|J_4| = 3$, and $D = -12$, which according to the stability diagram gives the qualitative behavior of $\dot{\mathbf{x}} = J_4\mathbf{x}$ as a *center*.