MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #25

Limit Cycles, Closed Orbits, and

Poincare-Bendixson Theorem in 2D Systems

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Quora Home Following Answer Spaces Notifications

Comic Book Matchups | DC Comics Superheroes | Hypothetical Superhero Battles | Iron Man (Marvel character) | +7

Iron Man vs. Superman -- who would win?

https://www.quora.com/Who-would-win-if-Superman-fights-against-Thor

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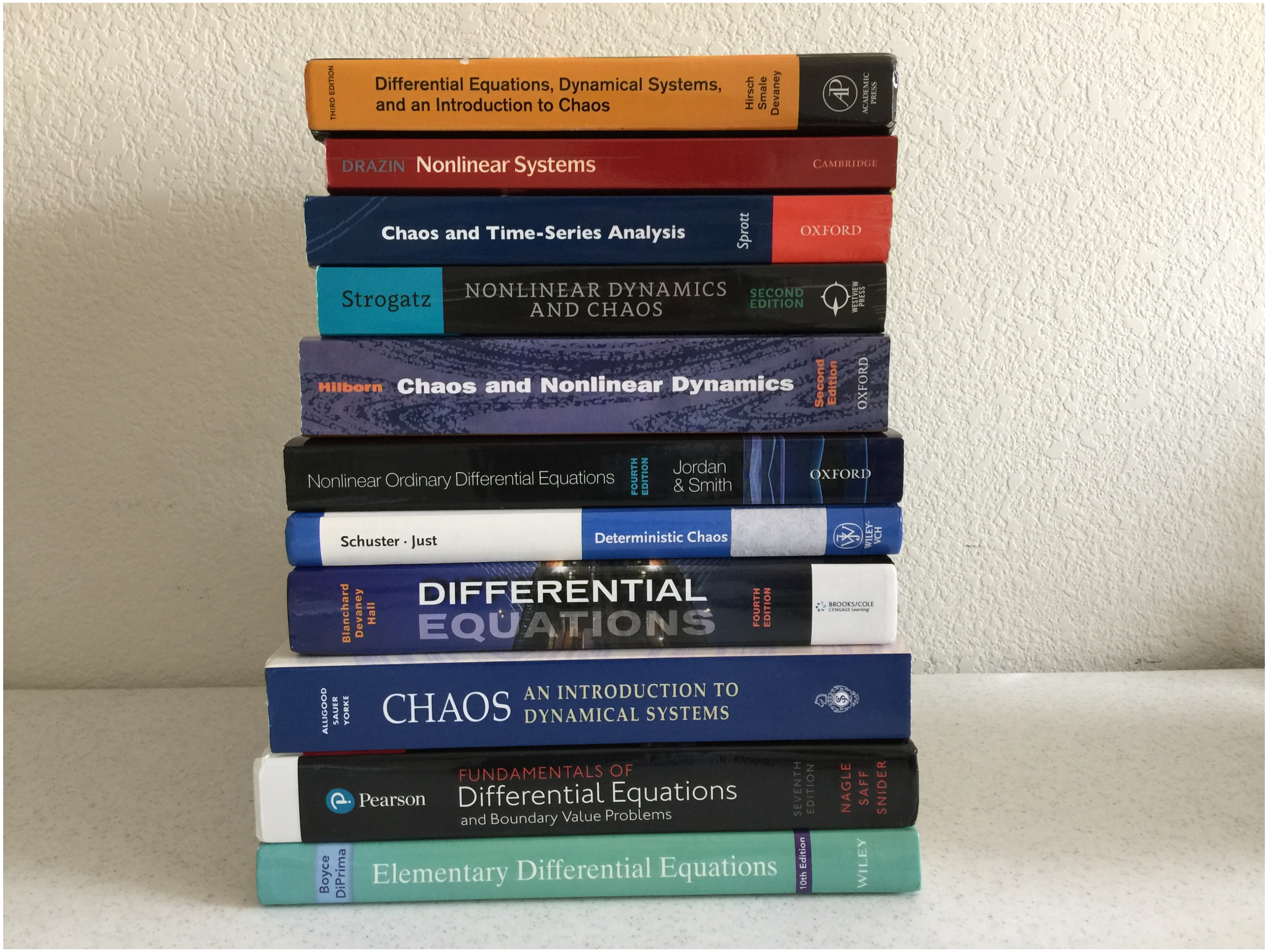
88 Answers

Outline

- Introduction
  - Limit cycle
  - A mini review of vector calculus
  - Gradient system ($\vec{v} = \nabla\phi$) for irrotational flow ($\nabla \times \vec{v} = 0$)
  - Hamiltonian system for incompressible flow ($\nabla \cdot \vec{v} = 0$)
- Energy methods for stability analysis near a critical point
  - The limit cycle of van dan Pol Equation
- Methods for ruling out closed orbits
  - I. Existence of a Laypunov function
  - II. Zero curl ($\nabla \times \vec{v} = 0$)
  - III. Positive ($\nabla \cdot \vec{v} > 0$) or negative ($\nabla \cdot \vec{v} < 0$) divergence
- Poincare-Bendixson Theorem
- Summary
  - Eigenvalue analysis for gradient and Hamiltonian Systems
  - Poincare-Bendixson Theorem

References


THIRD EDITION
Differential Equations, Dynamical Systems, and an Introduction to Chaos
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ACADEMIC PRESS
DRAZIN Nonlinear Systems
CAMBRIDGE
Chaos and Time-Series Analysis
Sprott
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Strogatz
NONLINEAR DYNAMICS AND CHAOS
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Second Edition
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Poincare-Bendixson Theorem

TBD

The Poincare–Bendixson Theorem essentially determines all of the possible limiting behaviors of a planar flow (HSD).

Consider a particular trajectory starting in R. The Poincare-Bendixson Theorem states that there are only two (three) possibilities for that trajectory (Hilborn, p101) (R is a closed bounded subset of the plane):

1. The trajectory approaches a fixed point of the system as $t \to \infty$.
2. The trajectory approaches a limit cycle as $t \to \infty$.
3. The trajectory is a limit cycle.

Hilborn

Poincare-Bendixson Theorem

TBD

- The theorem works only in two dimensions because only in two dimensions does a closed curve separate the space into a region "inside" the curve and a region "outside."
- Thus a trajectory starting inside the limit cycle can never get out and a trajectory starting outside can never get in.
- from the Poincare-Bendixson Theorem we arrive at an important result: Chaotic trajectories (in a bounded system) cannot occur in a state space of two dimensions.
- For systems described by differential equations, we need at least three state-space dimensions for chaos.

Hilborn

Terminology

- Limit cycle: an isolated closed path

Consider $X' = F(X)$, here $X' = (x', y') = \vec{v}$.

- Gradient system ($\vec{v} = \nabla\phi$) for irrotational flow ($\nabla\times\vec{v} = 0$)
  - ✓ ϕ : velocity potential; $\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right)$
  - ✓ The direction of vector filed is perpendicular to the tangent vector of the contour line of ϕ (i.e., a constant value of the potential function).
- Hamiltonian system for incompressible flow ($\nabla\cdot\vec{v} = 0$)
  - ✓ ψ: streamfunction; $\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \left(\frac{\partial\psi}{\partial y}, \frac{-\partial\psi}{\partial x}\right)$
  - ✓ The direction of the flow is parallel to the tangent vector of the contour line of ψ.

Limit Cycles

- An isolated closed path is called a limit cycle: "Isolated" in the sense that there is no other closed path in its immediate neighborhood (Jordan and Smith).

Nagle et al.

A new edition

Limit Cycle

Definition 5. A nontrivial† closed trajectory with at least one other trajectory spiraling into it (as time approaches plus or minus infinity) is called a **limit cycle.**

An old edition

Limit Cycle

Definition 5. A nontrivial† closed trajectory that is isolated is called a **limit cycle.**

ω-limit set and α-limit set

- A point z in R^n is in the ω-limit set $\omega(v_0)$ of the solution curve $\mathrm{F}(t, v_0)$ if there is a sequence of points increasingly far out along the orbit (that is, $t \to \infty$) which converges to z.
- Specifically, z is in $\omega(v_0)$ if there exists an unbounded increasing sequence $\{t_n\}$ of real numbers with $\lim_{n\to\infty} \mathrm{F}(t_n, v_0) = z$.
- A point z in R^n is in the α-limit set $\alpha(v_0)$ if there exists an unbounded decreasing sequence $\{t_n\}$ of real numbers $(t_n \to \infty)$ with $\lim_{n\to\infty} \mathrm{F}(t_n, v_0) = z$.

Alligood et al.

ω-Limit Cycle: Definition

- If γ is an ω-limit cycle, there exists $X \notin \gamma$ such that

$$\lim_{t\to\infty} d(\phi_t(X), \gamma) = 0.$$

- Geometrically this means that some solution $(\phi_t(X))$ spirals toward γ as $t \to \infty$.

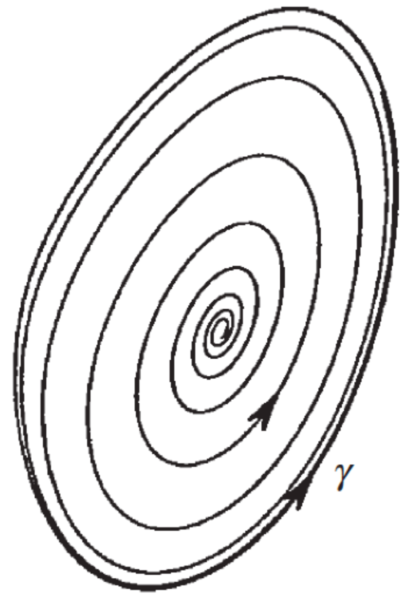


Figure 10.10 A solution spiraling toward a limit cycle.

HSD

Limit Cycles and Hopf Bifurcation: Ex 1

Example. (Hopf Bifurcation) Consider the system

$$x' = ax - y - x(x^2 + y^2)$$
$$y' = x + ay - y(x^2 + y^2).$$

There is an equilibrium point at the origin and the linearized system is

$$X' = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix} X.$$

The eigenvalues are $a \pm i$, so we expect a bifurcation when $a = 0$.

Spiral source, spiral sink, or center

To see what happens as a passes through 0, we change to polar coordinates. The system becomes

$$\begin{aligned} r' &= ar - r^3 \\ \theta' &= 1. \end{aligned}$$

Note that the origin is the only equilibrium point for this system, since $\theta' \neq 0$. For $a < 0$ the origin is a sink since $ar - r^3 < 0$ for all $r > 0$. Thus all solutions tend to the origin in this case. When $a > 0$, the equilibrium becomes a source. So what else happens? When $a > 0$ we have $r' = 0$ if $r = \sqrt{a}$. So the circle of radius $\sqrt{a}$ is a periodic solution with period 2π. We also have $r' > 0$ if $0 < r < \sqrt{a}$, while $r' < 0$ if $r > \sqrt{a}$. Thus, all nonzero solutions spiral toward this circular solution as $t \to \infty$.

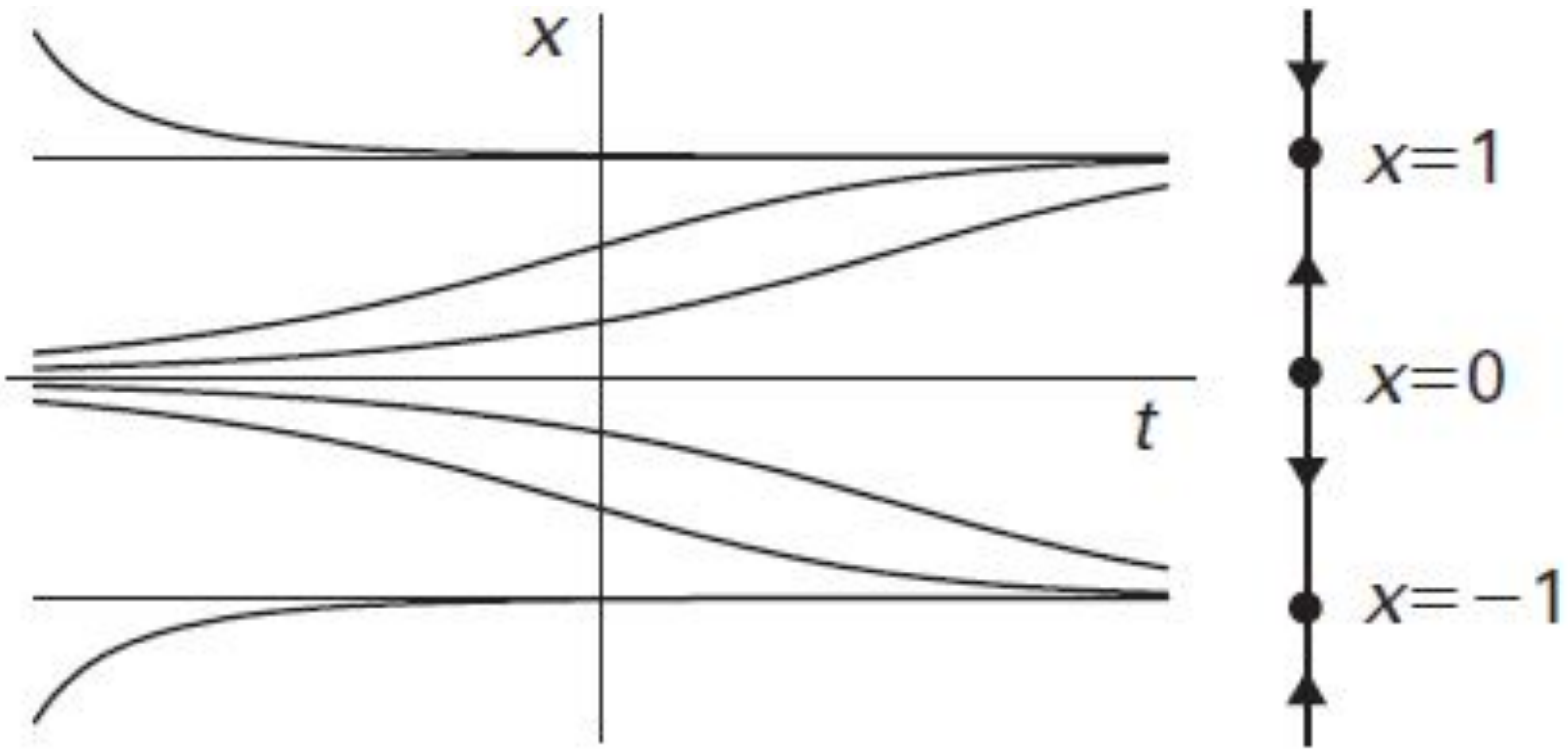
This type of bifurcation is called a *Hopf bifurcation*. Thus at a Hopf

HW (Math537)

2: [35 points] $dx/dt = -(ax + x^3)$ for $x \geq 0$ and $x(t=0) = x_o$. [Hint: set $r = x^2$, solve for r and discuss the results when $a < 0$, $a = 0$ or $0 < a$.]

$$\frac{dx}{dt} = x - x^3$$

"$a = 1$" is different from that in the previous slide



Sink

Source

Sink

A Limit Cycle: Ex 1

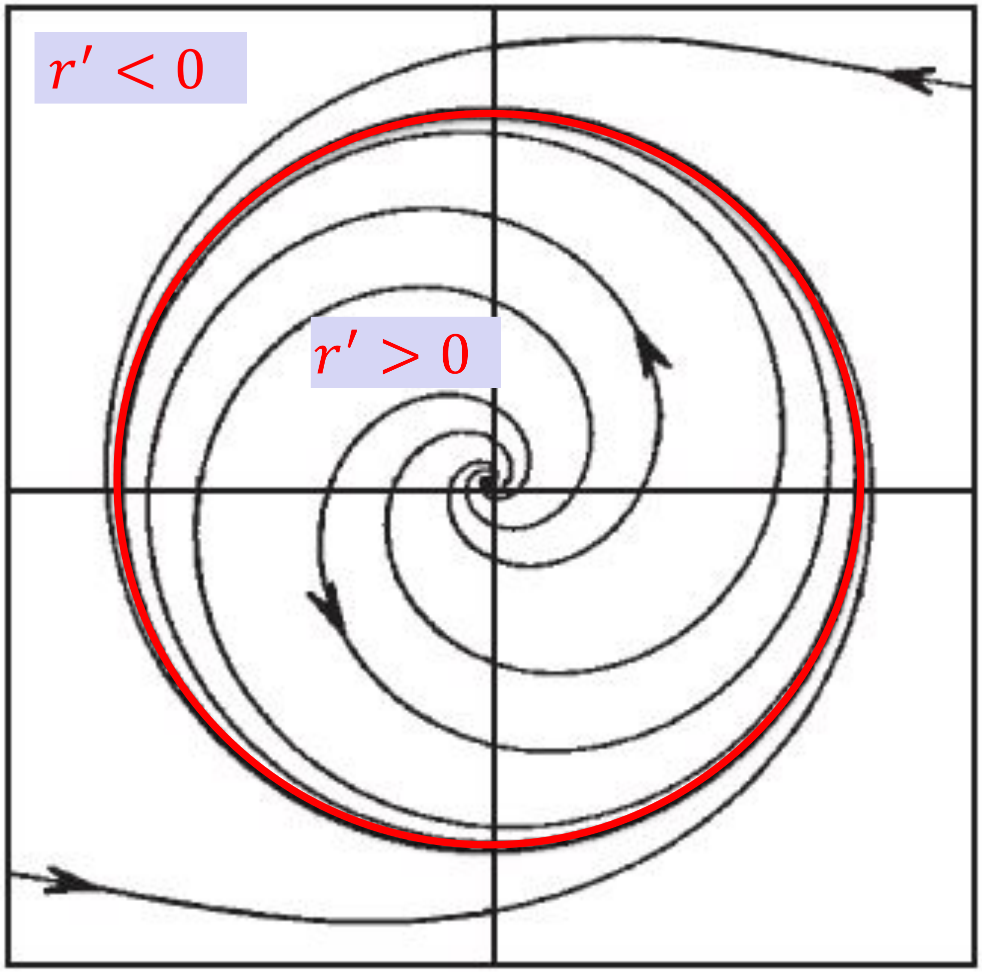


Figure 10.1 The phase plane for $r' = (r - r^3)$, $\theta' = 1$.

$$\frac{dx}{dt} = ax - y - x(x^2 + y^2),$$

$$\frac{dy}{dt} = x + ay - y(x^2 + y^2),$$

where $a = 1.0$ and $t \in [0, 10]$.

$$r' = r(1 - r^2),$$
$$a = 1 > 0$$

$$r' = 0 \; when \; r = 1$$

$$r' < 0 \; when \; r > 1$$

$$r' > 0 \; when \; r < 1$$

(A source at the origin)

Limit Cycle

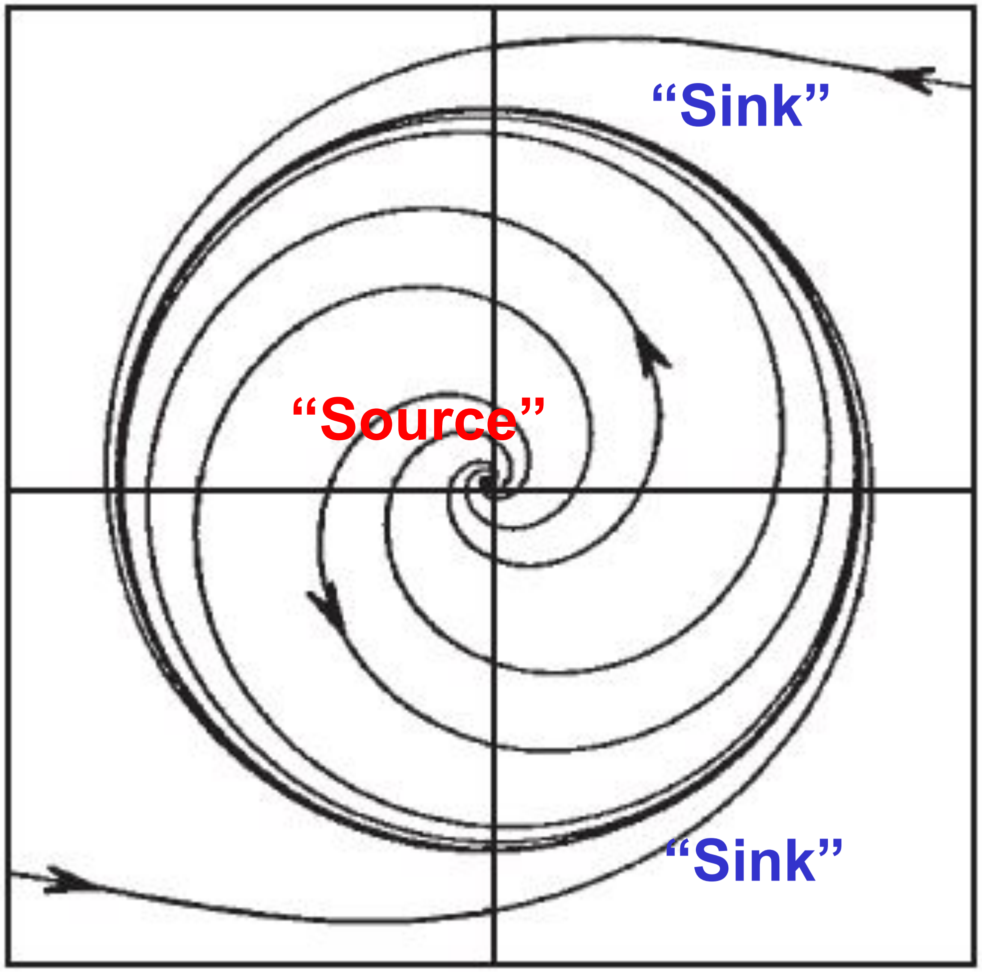


Figure 10.1 The phase plane for $r' = (r - r^3)$, $\theta' = 1$.

Limit Cycles and Hopf Bifurcation

A sink at the origin (for $a < 0$) A source at the origin (for $a > 0$)

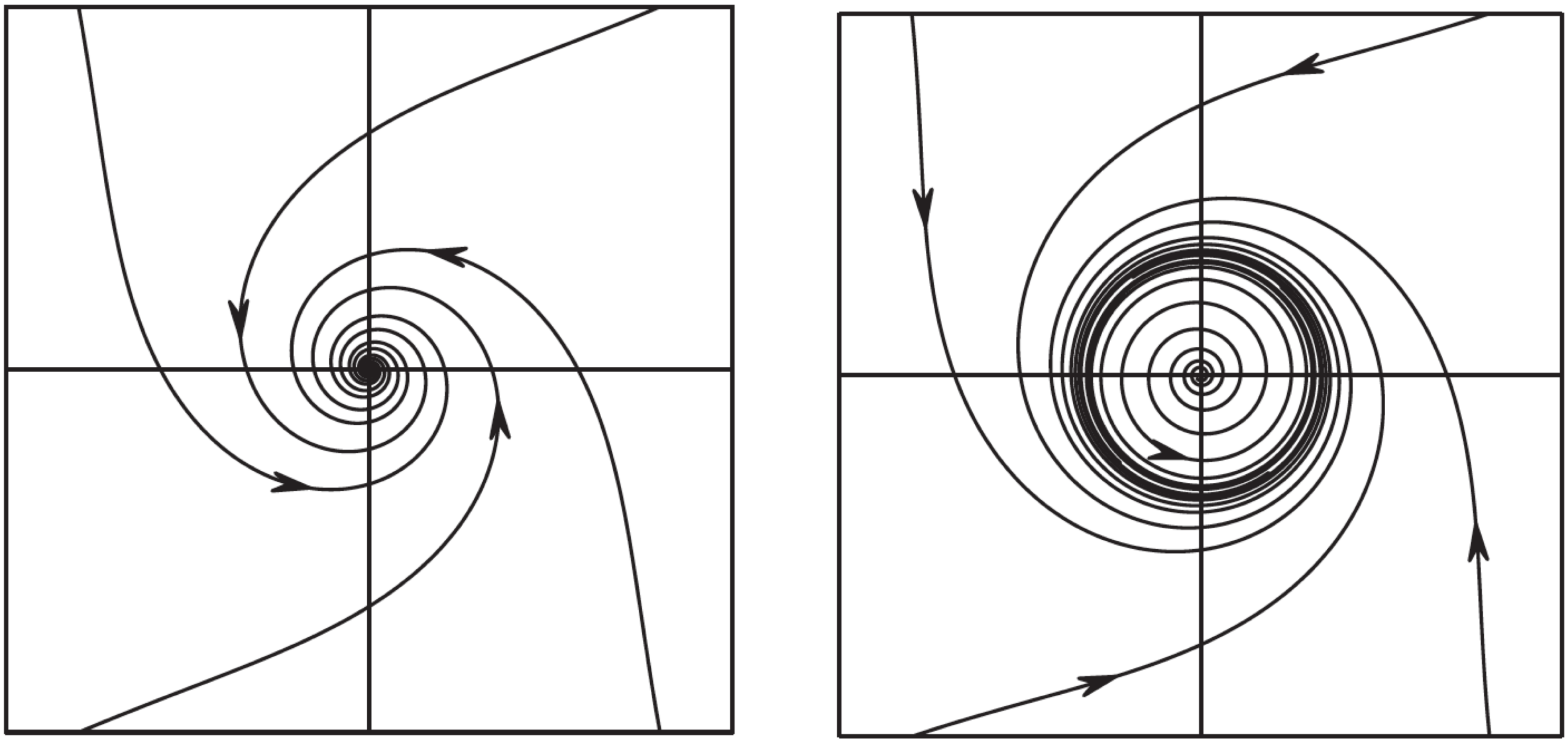
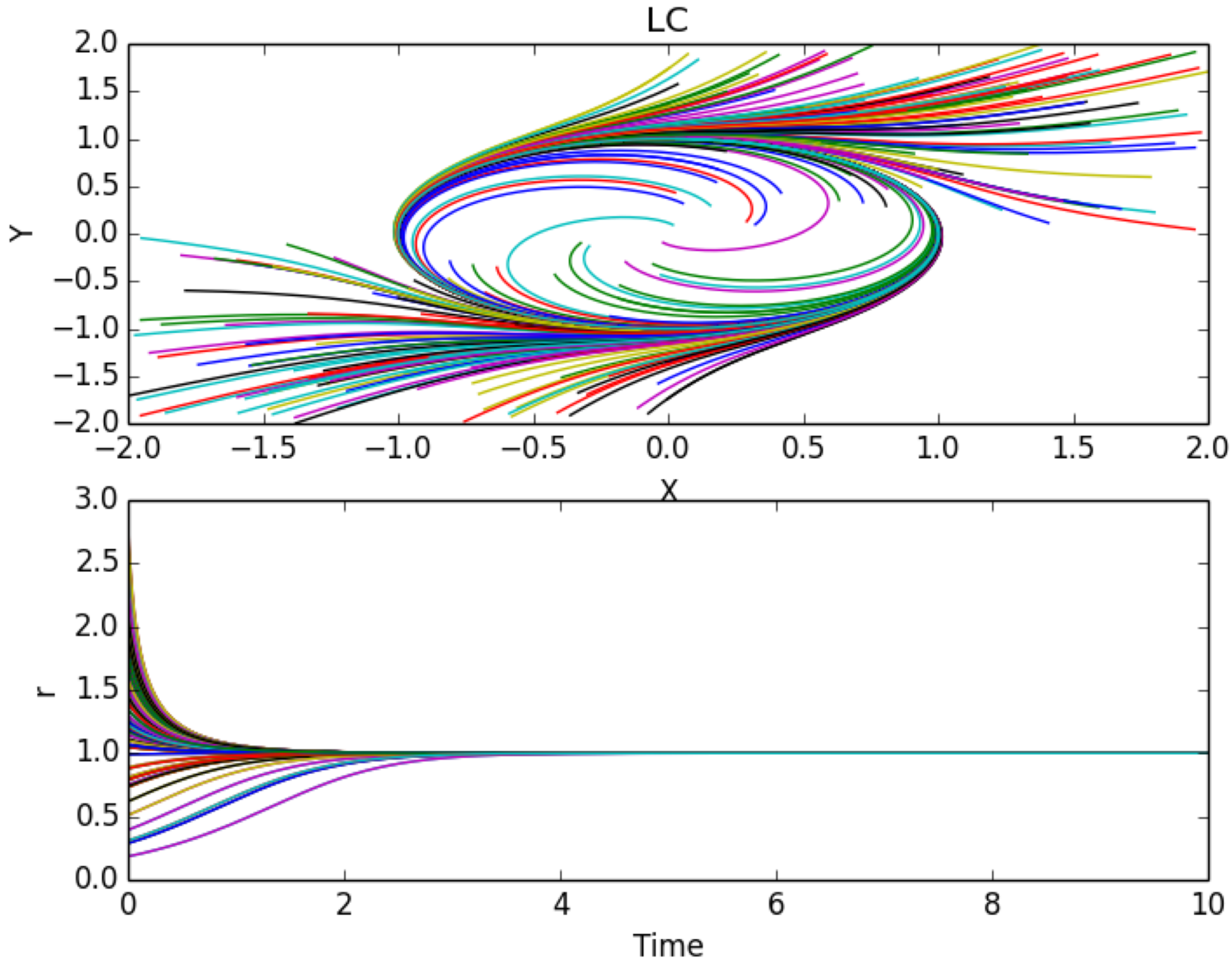


Figure 8.10 Hopf bifurcation for $a < 0$ and $a > 0$.

Limit Cycle: Ex 1 (cont.)

Ensemble Runs



Limit Sets: An Alternative "Ex1"

Exercise 4-32 Consider the system

$$\dot{x} = y + x(1 - x^2 - y^2) \qquad \dot{y} = -x + y(1 - x^2 - y^2) \tag{4-72}$$

Using the polar coordinates

$$r^2 = x^2 + y^2 \qquad \theta = \arctan\frac{y}{x}$$

$$r\dot{r} = x\dot{x} + y\dot{y} \qquad r^2\dot{\theta} = x\dot{y} - y\dot{x}$$

reduce the system to

$$\dot{r} = r(1 - r^2) \qquad \dot{\theta} = -1$$

and verify that the general solution is given by

$$r = \frac{1}{(1 + ce^{-2t})^{1/2}} \qquad \theta = -(t - t_0)$$

With $t_0 = 0$, we have

$$x = \frac{\cos t}{(1 + ce^{-2t})^{1/2}} \qquad y = -\frac{\sin t}{(1 + ce^{-2t})^{1/2}}$$

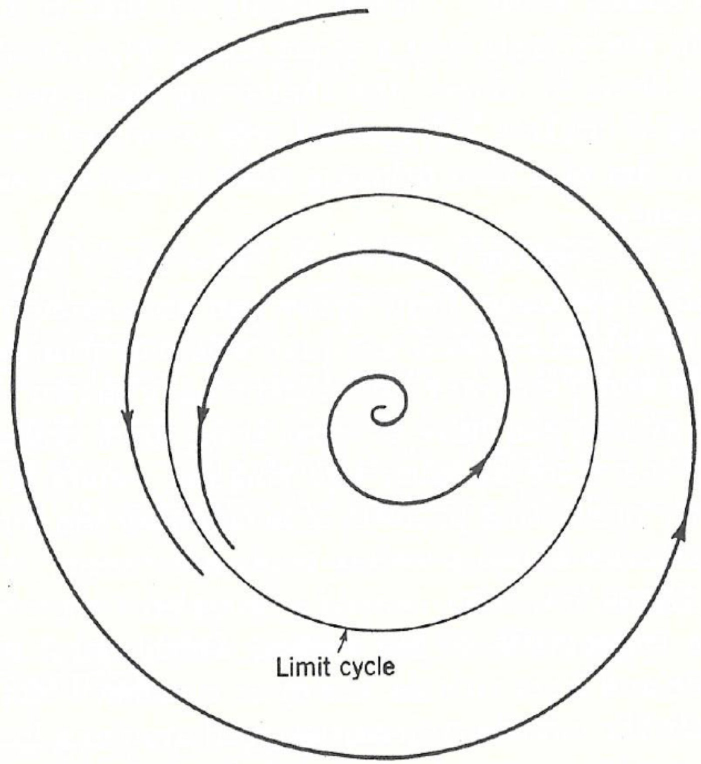


Fig. 4-5

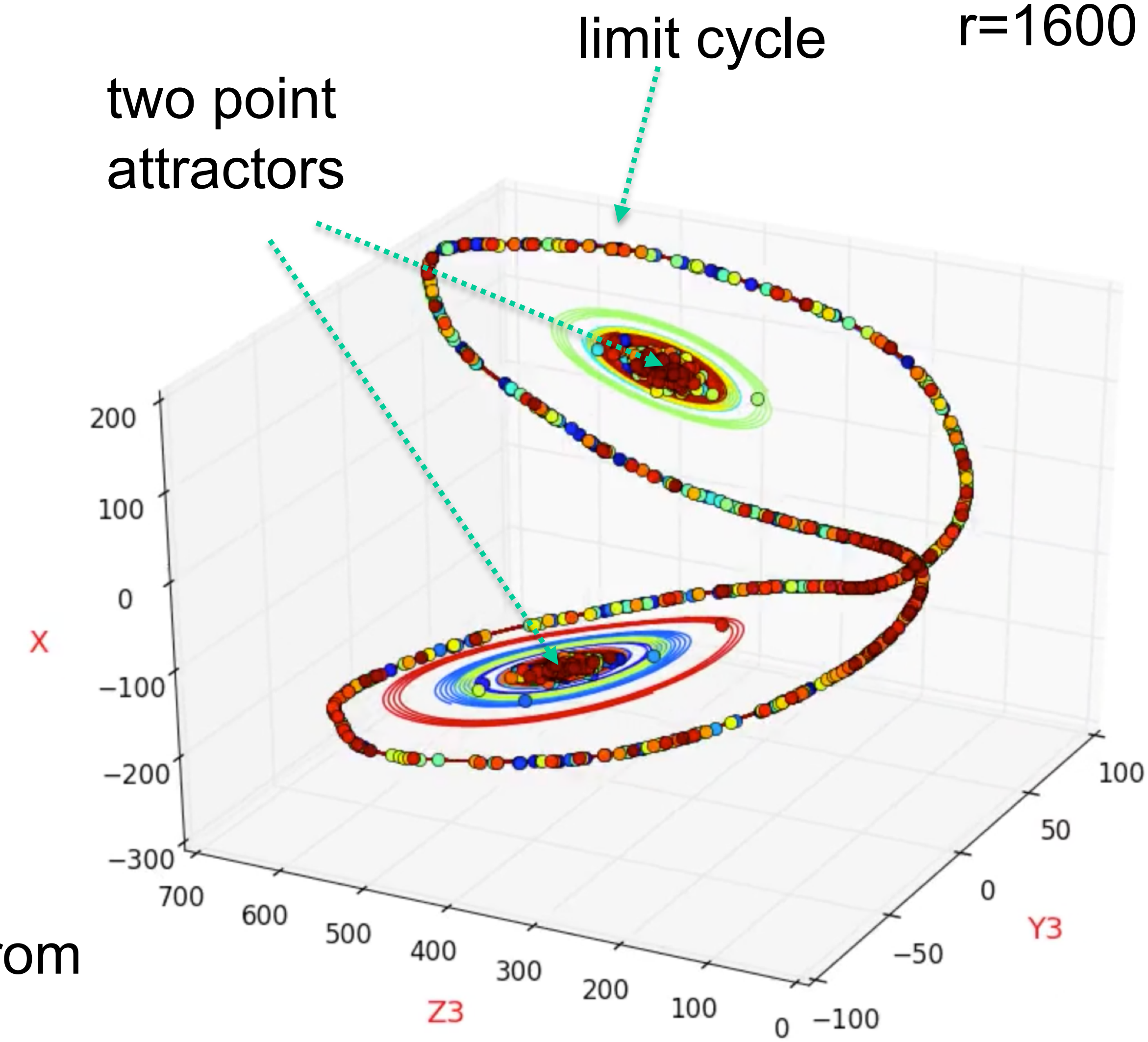
Saaty and Bram (1964)
Boyce and DiPrima

The Second Kind of Attractor Coexistence

- For the 2nd kind of attractor coexistence, a limit cycle (LC) that is an isolated closed orbit coexists with point attractors.

r=1600

- Time evolution of 2,048 orbits in the X-Y_3-Z_3 space using the 9DLM.
- The total simulation time is $\tau = 3.5$.
- Transient orbits are only kept for the last 0.25 time units, i.e. for the time interval of [max (0, T-0.25), T] at a given time T.



- The animation is available from https://goo.gl/sMhoUb.

Hopf Bifurcation in Linear Systems

(4) Finally, consider a system without nonlinear terms:

$$\dot{z} = (\alpha + i)z.$$

This system also has a family of periodic orbits of increasing amplitude, but all of them are present at $\alpha = 0$ when the system has a *center* at the origin (see Figure 3.9). It can be said that the limit cycle paraboloid "degenerates"

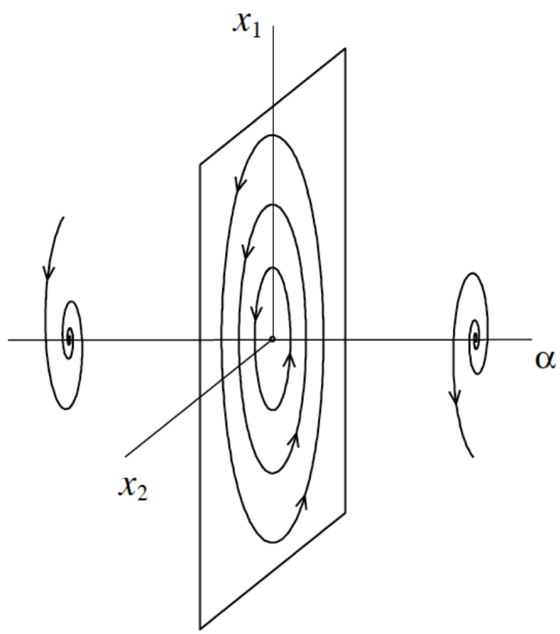


FIGURE 3.9. "Hopf bifurcation" in a linear system.

into the plane $\alpha = 0$ in (x, y, α)-space in this case. This observation makes natural the appearance of small limit cycles in the nonlinear case. ♢

Kuznetsov

A Brief Review on Calc III

Learning Outcomes

Formulas for Grad, Div, Curl, and the Laplacian

	Cartesian (x, y, z) $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors in the directions of increasing x, y, and z. P, Q, and R are the scalar components of $\mathbf{F}(x, y, z)$ in these directions.
Gradient	$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$
Divergence	$\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$
Curl	$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$
Laplacian	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

The Fundamental Theorem of Line Integrals

1. Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ be a vector field whose components are continuous throughout an open connected region D in space. Then there exists a differentiable function f such that

$$\mathbf{F} = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

if and only if for all points A and B in D the value of $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path joining A to B in D.

2. If the integral is independent of the path from A to B, its value is

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

Green's Theorem (Tangential Form)

$$\iint_R \nabla \times \vec{F} \cdot \vec{k}\, dx dy = \oint \vec{F} \cdot d\vec{r}$$

Stokes' Theorem

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

Green's Theorem (Normal Form)

$$\iint_R \nabla \cdot \vec{F}\, dx dy = \oint \vec{F} \cdot \vec{n}\, ds$$

Divergence Theorem

$$\iiint_E \operatorname{div} \mathbf{F}\, dV = \iint \vec{F} \cdot \vec{n}\, dS$$

Curl and Divergence

- $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, a "meta" vector: $F = (P, Q, R)$

- Gradient:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (f_x, f_y, f_z)$$

- Curl ("a Cross product of ∇ and F"):

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (R_y - Q_z, P_z - R_x, Q_x - P_y)$$

- Divergence ("a Dot product of ∇ and F"):

$$\nabla \bullet F = (P_x + Q_y + R_z)$$

Curl and Divergence (2D)

- $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$, a "meta" vector: $F = (P, Q)$

- Gradient: $\nabla f = (f_x, f_y)$

- Curl ("a Cross product of ∇ and F"):

$$\nabla \times F \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = (0, 0, Q_x - P_y)$$

$$\nabla \times \vec{F} \cdot \vec{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

- Divergence ("a Dot product of ∇ and F"):

$$\nabla \cdot F = P_x + Q_y$$

Vector Calculus Identities (2D)

- curl (gradient f)=0

$$\mathrm{f} = \phi$$

$$\nabla\times(\nabla\phi) = 0$$

$$\boxed{\vec{v} = \nabla\phi}$$

$$(u, v) = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right)$$

$$\phi: velocity\ potential$$

$$\boxed{\nabla\times\vec{v} = 0}$$

$$irrotational\ flow$$

- div (curl F)=0

$$F = (P, Q, R)$$

$$\nabla\times F = \left(R_y - Q_z, P_z - R_x, Q_x - P_y\right)$$

$$F = (0, 0, \psi); \quad \nabla\times F = \left(\psi_y, -\psi_x, 0\right)$$

$$\boxed{\vec{v} = \nabla\times F}$$

$$(u, v) = \left(\frac{\partial\psi}{\partial y}, \frac{-\partial\psi}{\partial x}\right)$$

$$\psi: streamfunction$$

$$\boxed{\nabla\cdot\vec{v} = 0}$$

$$incompressible\ flow$$

Path Independence

$$\boxed{\nabla\times\vec{v} = 0}$$

$$irrotational\ flow$$

$$\boxed{\nabla\cdot\vec{v} = 0}$$

$$incompressible\ flow$$

Green's Theorem (Tangential Form)

$$\oint \vec{V}\cdot d\vec{r} = \iint \nabla\times\vec{V}\cdot\vec{k}\,dxdy$$

$$\oint \vec{V}\cdot d\vec{r} = 0$$

$$\oint Pdx + Qdy = 0$$

Green's Theorem (Normal Form)

$$\oint \vec{V}\cdot\vec{n}\,ds = \iint \nabla\cdot\vec{V}dxdy$$

$$\oint \vec{V}\cdot\vec{n}\,ds = 0$$

$$\oint Pdy - Qdx = 0$$

Review: Path Independence

$\vec{F} = (P, Q)$ $\quad \vec{T}$ tangent vector

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot d\vec{r} \text{ (circulation)}$$

$$= \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} \right) dt$$

$$\overset{?}{=} \int_a^b \nabla f \cdot d\vec{r} = \int_a^b \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$\nabla \times \vec{F} = 0$ Irrotational (e.g., A2)

$$\nabla \times \vec{F}\cdot\vec{k} = 0 = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

$$= \int_a^b df = f(b) - f(a)$$ potential function

$\vec{n}$ normal vector

$$\int \vec{F}\cdot\vec{n}\,ds = \int P\,dy - Q\,dx \text{ (flux)}$$

$$= \int_a^b \left(P\frac{dy}{dt} - Q\frac{dx}{dt} \right) dt$$

$$\overset{?}{=} \int_a^b \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy$$

$\nabla \cdot \vec{F} = 0$ incompressible (e.g., A1)

$$\vec{F} = (P, Q) = \left(\frac{\partial \psi}{\partial y}, \frac{-\partial \psi}{\partial x} \right)$$

$$= \int_a^b d\psi = \psi(b) - \psi(a)$$ stream function

Review: Complex Potential

$$\Phi = \phi + i\psi$$

$$\frac{d\Phi}{dz} = \frac{\partial \Phi}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = \phi_x - i\phi_y$$

$$\frac{d\Phi}{dz}dz = \left(\phi_x - i\phi_y\right)\left(dx + idy\right)$$

$$= \left(\phi_x dx + \phi_y dy\right) + i\left(\phi_x dy - \phi_y dx\right)$$

$$= \left(Pdx + Qdy\right) + i\left(Pdy - Qdx\right)$$

$$= \vec{F}\cdot\vec{T} + i\vec{F}\cdot\vec{n}$$

Circulation + i Flux

Cauchy Riemann Equations

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \frac{\partial \phi}{\partial y} = \frac{-\partial \psi}{\partial x}$$

$$\vec{F} = \left(P, Q\right) = \left(\phi_x, \phi_y\right)$$

$$\phi = \ln\sqrt{x^2 + y^2}$$

(A4)

$$\psi = \tan^{-1}\left(\frac{y}{x}\right)$$

(A3)

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Energy Methods for Stability Analysis

$$\frac{d^2x}{dt^2} + h\left(x, \frac{dx}{dt}\right) + g(x) = 0$$ (nonlinear)

multiply both sides by $\frac{dx}{dt}$

$$x' \frac{d^2x}{dt^2} + x'h + x'g(x) = 0$$

$$\underbrace{\frac{1}{2}\frac{d}{dt}(x')^2}_{d(KE)/dt} + x'h + \underbrace{\frac{d}{dt}\int g(x)\,dx}_{d(PE)/dt} = 0$$

$$\frac{dE}{dt} = \frac{d}{dt}(KE + PE) = -x'h$$

Nagle et al.

Energy Methods for Stability Analysis

$$\frac{d^2x}{dt^2} + h\left(x, \frac{dx}{dt}\right) + g(x) = 0$$ (nonlinear)

$F = -\nabla G$

$$G(x) = \int g(x)\, dx, potential\ energy$$

$$E(x, v) = \frac{1}{2}v^2 + G(x), total\ energy$$

$$v = \frac{dx}{dt}$$

$$\frac{dE}{dt} = -\frac{dx}{dt}h = -vh;$$

- $\frac{dE}{dt} = 0$ when $h = 0$; $conservative$
- $\frac{dE}{dt} > 0$ when $vh < 0$
- $\frac{dE}{dt} < 0$ when $vh > 0$

Nagle et al.

Pause: 2nd Order ODEs vs. 1st Order ODEs

2D $$\frac{d^2x}{dt^2} + h\left(x, \frac{dx}{dt}\right) + g(x) = 0$$ (nonlinear)

$$F = -\nabla G$$

- 1D gradient
- $G(x)$, not $G(x, x')$

$$G(x) = \int g(x)\, dx, potential\ energy$$

$$E(x, v) = \frac{1}{2}v^2 + G(x), total\ energy$$

$$v = \frac{dx}{dt}$$

1D $$\frac{dx}{dt} + f = 0$$

Let $f = \frac{dG}{dx}$

$$G(x) = \int f(x)\, dx, potential\ function$$

M-D $$\frac{d\vec{x}}{dt} + \vec{f} = 0$$ $\vec{f} ?= \nabla G$ $(f_1\, f_2, f_3) = (\mathrm{G_x}, \mathrm{G_y}, \mathrm{G_z})$

Strogatz; Nagle et al.

Ex1 with 1st Order ODEs

$$\frac{dx}{dt} + f = 0$$

$$V = G(x) = \int f(x)\, dx, PE$$

Consider the following

$$\frac{dx}{dt} + x = 0$$

$$V = \int f(x)\, dx = \frac{1}{2}x^2 + C$$

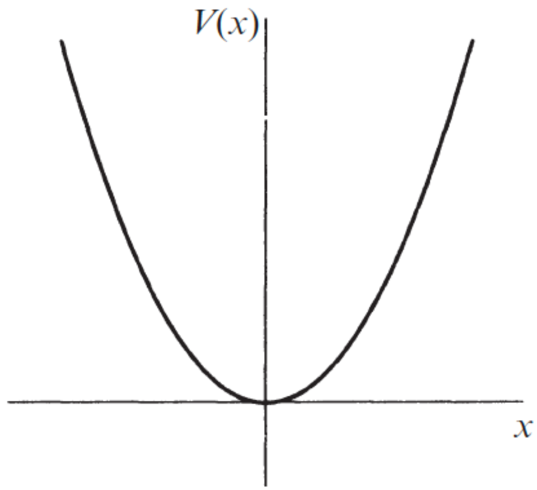


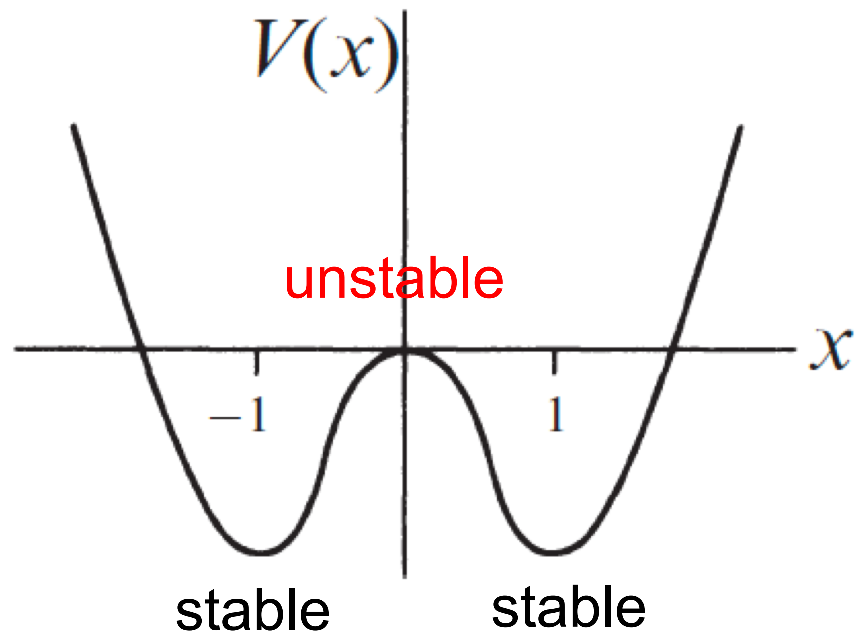
Figure 2.7.2

- $x = 0$ where PE has a minimum is a stable point.
- The above analysis is consistent with the linear stability analysis near a critical point.
- Note that the solution is $x = x_0 e^{-t}$.

Ex2 with 1st Order ODEs

$$\frac{dx}{dt} - x + x^3 = 0$$

$$V = \int f(x)\, dx = \frac{-1}{2} x^2 + \frac{1}{4} x^4 C$$



HW (Math537)

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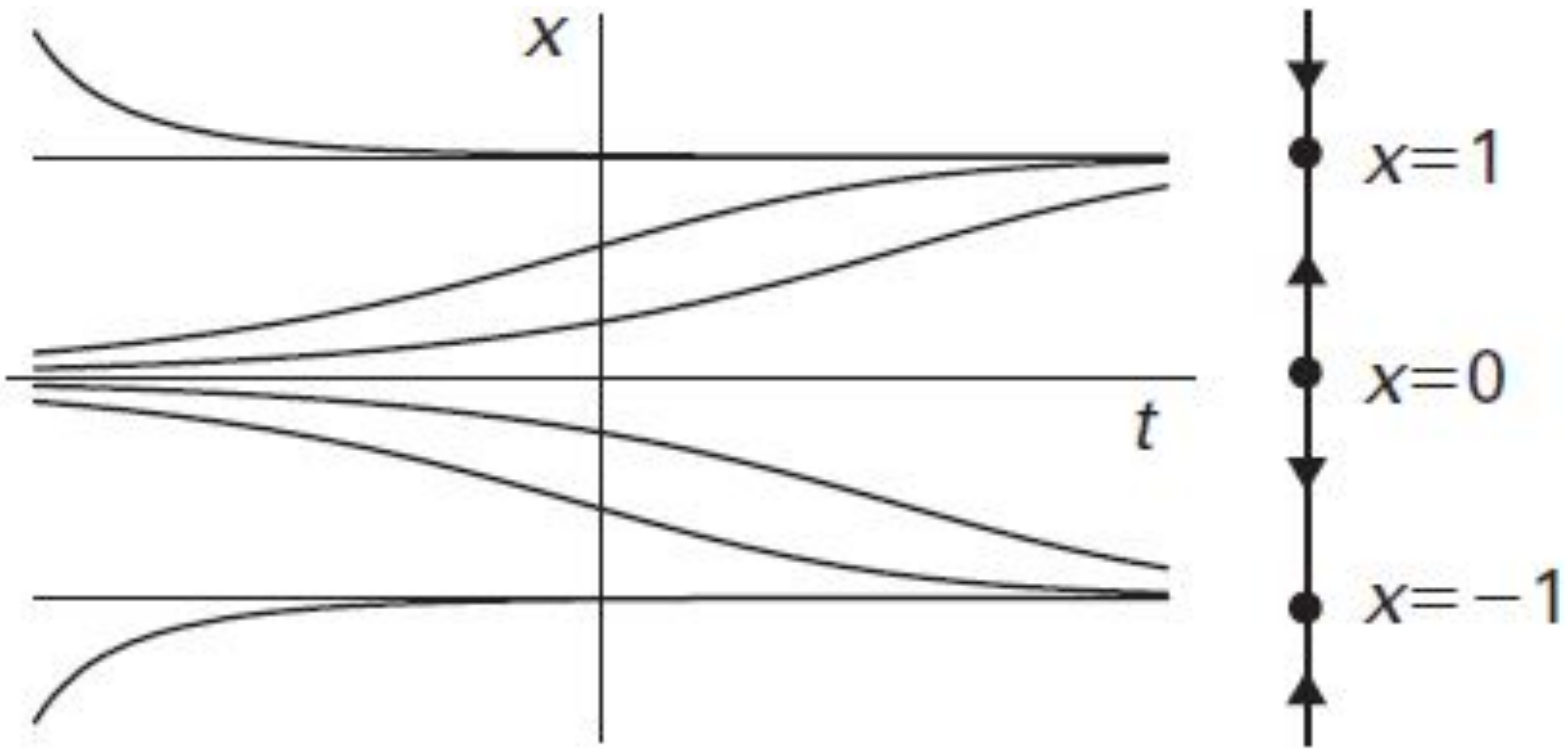
$$\frac{dx}{dt} = x - x^3$$

"$a = 1$" is different from that in the previous slide



Sink

Source

Sink

2^{nd} Order ODEs

2D $$\frac{d^2x}{dt^2} + h\left(x, \frac{dx}{dt}\right) + g(x) = 0$$ (nonlinear)

$F = -\nabla G$

- 1D gradient
- $G(x)$, not $G(x, x')$

$$V = G(x) = \int g(x)\, dx, potential\ energy$$

$$E(x, v) = \frac{1}{2}v^2 + G(x), total\ energy$$

$$v = \frac{dx}{dt}$$

a **minimum of** $\mathcal{V}(x)$ generates a **centre** (stable);
a **maximum** of $\mathcal{V}(x)$ generates a **saddle** (unstable);
a **point of inflection** leads to a **cusp,** as shown in Fig. 1.12(c),

Note that within conservative systems ($h = 0$), $\lambda_1 + \lambda_2 = 0$ which does not allows a pure "sink" or a pure "source".

2^{nd} Order ODEs

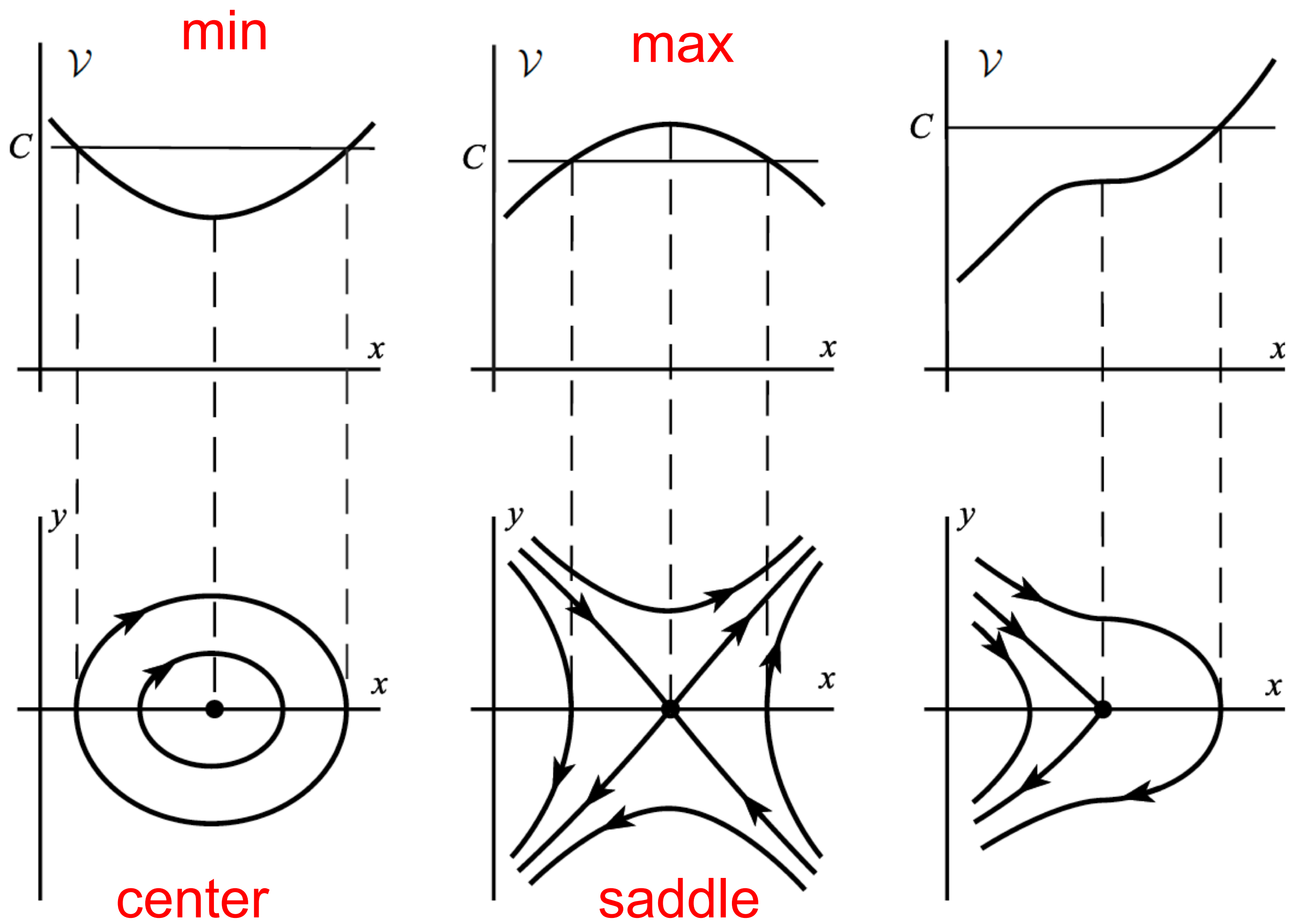


Figure 1.12 Typical phase diagrams arising from the stationary points of the potential energy.

$$\frac{d^2x}{dt^2} + x^3 = 0$$

$$V = G(x) = \int x^3\, dx = \frac{1}{4}x^4 + C$$

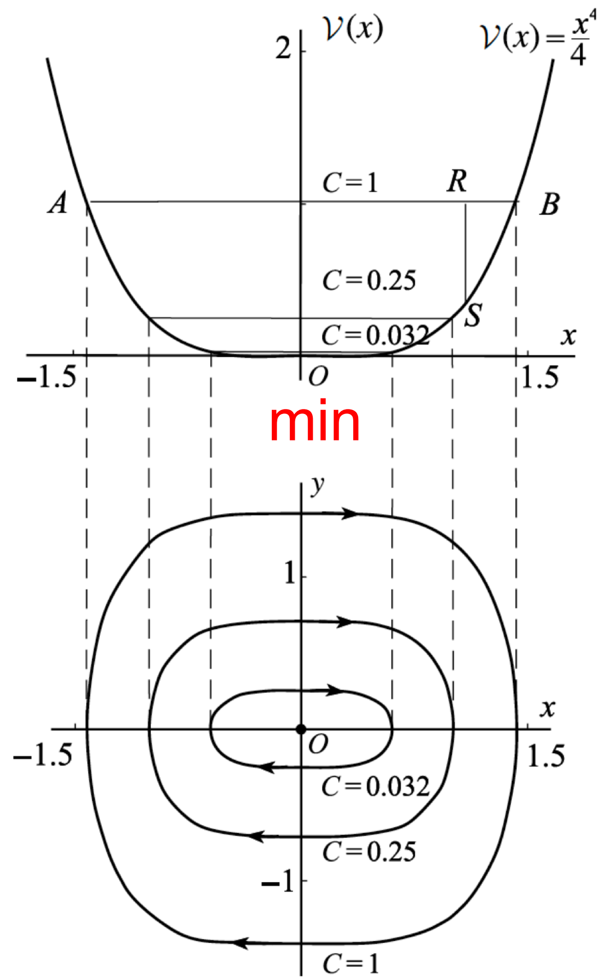


Figure 1.11

center

Example

$$\frac{d^2x}{dt^2} + x - x^3 = 0$$

$$V = \int (x - x^3)\, dx = \frac{1}{2}x^2 - \frac{1}{4}x^4 + C$$

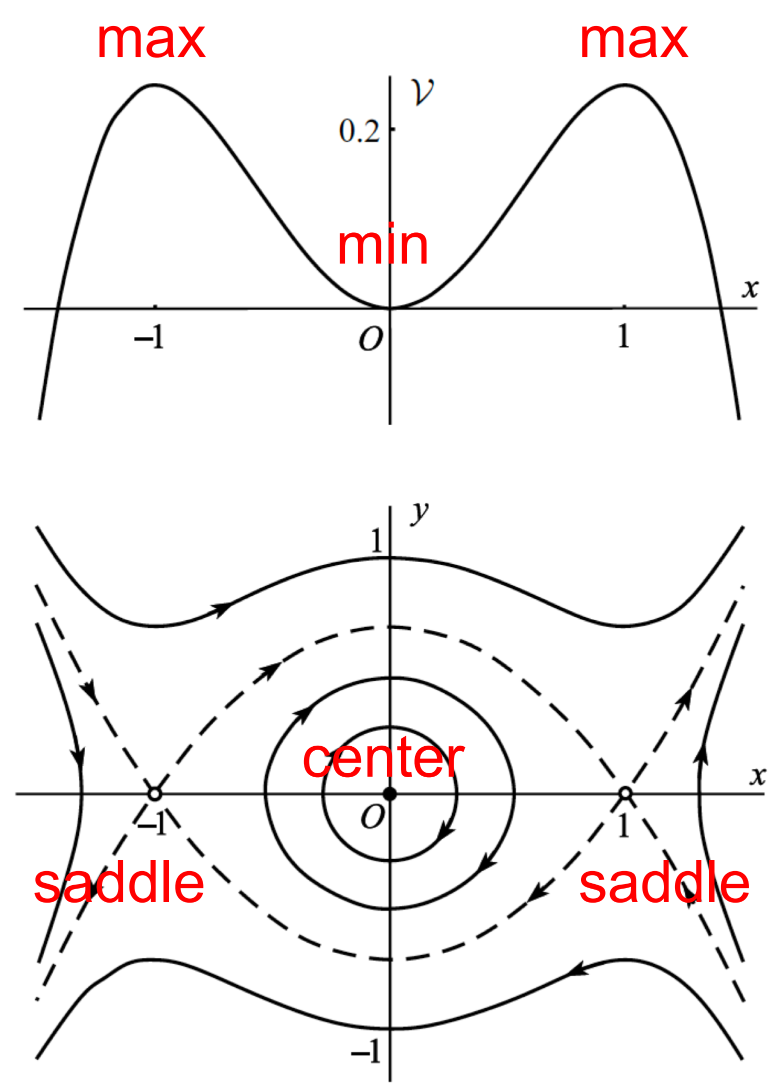


Figure 1.13 The dashed phase paths are separatrices associated with the equilibrium points at $(-1, 0)$ and $(1, 0)$.

HW5

1: [25 points] Consider the following second-order ordinary differential equations (ODEs) for nonlinear pendulum oscillations:

$$\frac{d^2\theta}{dt^2} + \epsilon\frac{d\theta}{dt} + sin(\theta) = 0. \tag{1.1}$$

Applying Taylor series expansions, Eq. (1) can be simplified into one of the following systems:

$$\frac{d^2\theta}{dt^2} + \theta = 0. \tag{1.2}$$

$$\frac{d^2\theta}{dt^2} + \epsilon\frac{d\theta}{dt} + \theta = 0. \tag{1.3}$$

$$\frac{d^2\theta}{dt^2} + \left(\theta - \frac{\theta^3}{6}\right) = 0. \tag{1.4}$$

(a) [21 points] Perform a linear stability analysis in each of Eqs. (1.2)-(1.4).

(b) [4 points] Discuss the concept of structural stability using results in (1a).

HW5

$$\frac{d^2\theta}{dt^2} + \left(\theta - \frac{\theta^3}{6}\right) = 0. \quad (1.4)$$

(a) [21 points] Perform a linear stability analysis in each of Eqs. (1.2)-(1.4).

(b) [4 points] Discuss the concept of structural stability using results in (1a).

(a2) for Eq. (1.4):

(i) three crritical points at $(\theta, z) = (0, 0)$, $(\pm\sqrt{6}, 0)$.

(ii) a center at $(\theta, z) = (0, 0)$ (with $\lambda = \pm i$).

(iii)saddle points at $(\theta, z) = (\pm\sqrt{6}, 0)$ (with $\lambda = \pm\sqrt{2}$).

Energy Methods for Stability Analysis

Consider $h = a(x)\frac{dx}{dt}$ $\qquad \frac{d^2x}{dt^2} + a(x)\frac{dx}{dt} + g(x) = 0$

$$G(x) = \int g(x)\,dx, potential\ energy$$

$$E(x, v) = \frac{1}{2}v^2 + G(x), total\ energy$$

$$\frac{dE}{dt} = -\frac{dx}{dt}h = -a\left(\frac{dx}{dt}\right)^2 = -av^2; \qquad v = \frac{dx}{dt}$$

- $\frac{dE}{dt} = 0$ when $a = 0$; $\qquad$ $e.g., conservative\ when\ a \equiv 0$
- $\frac{dE}{dt} < 0$ when $a > 0$; $\qquad$ positive damping
- $\frac{dE}{dt} > 0$ when $a < 0$; $\qquad$ *negative damping*

Nagle et al.

The Limit Cycle of van dan Pol Equation: Ex 2

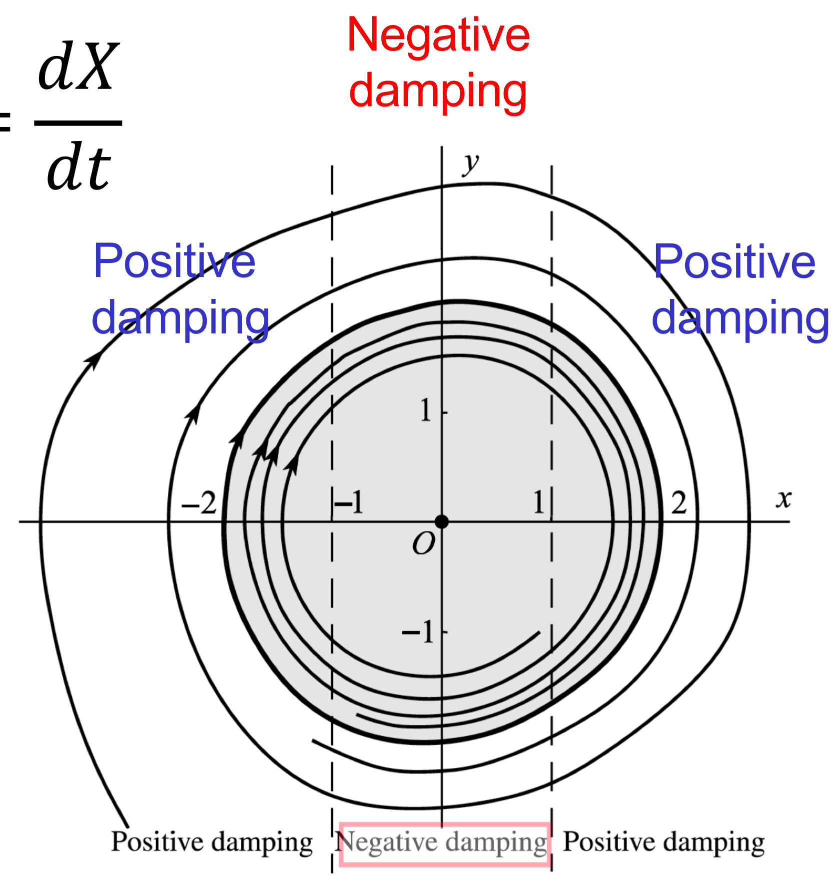
3: [30 points] Consider the following differential equation:

$$\frac{d^2X}{dt^2} + b(X^2 - 1)\frac{dX}{dt} + X = 0.$$

$$\frac{dE}{dt} = -hX' = -b(X^2 - 1)v^2; \qquad v = \frac{dX}{dt}$$

Assume b ($b = 0.1$) is positive.

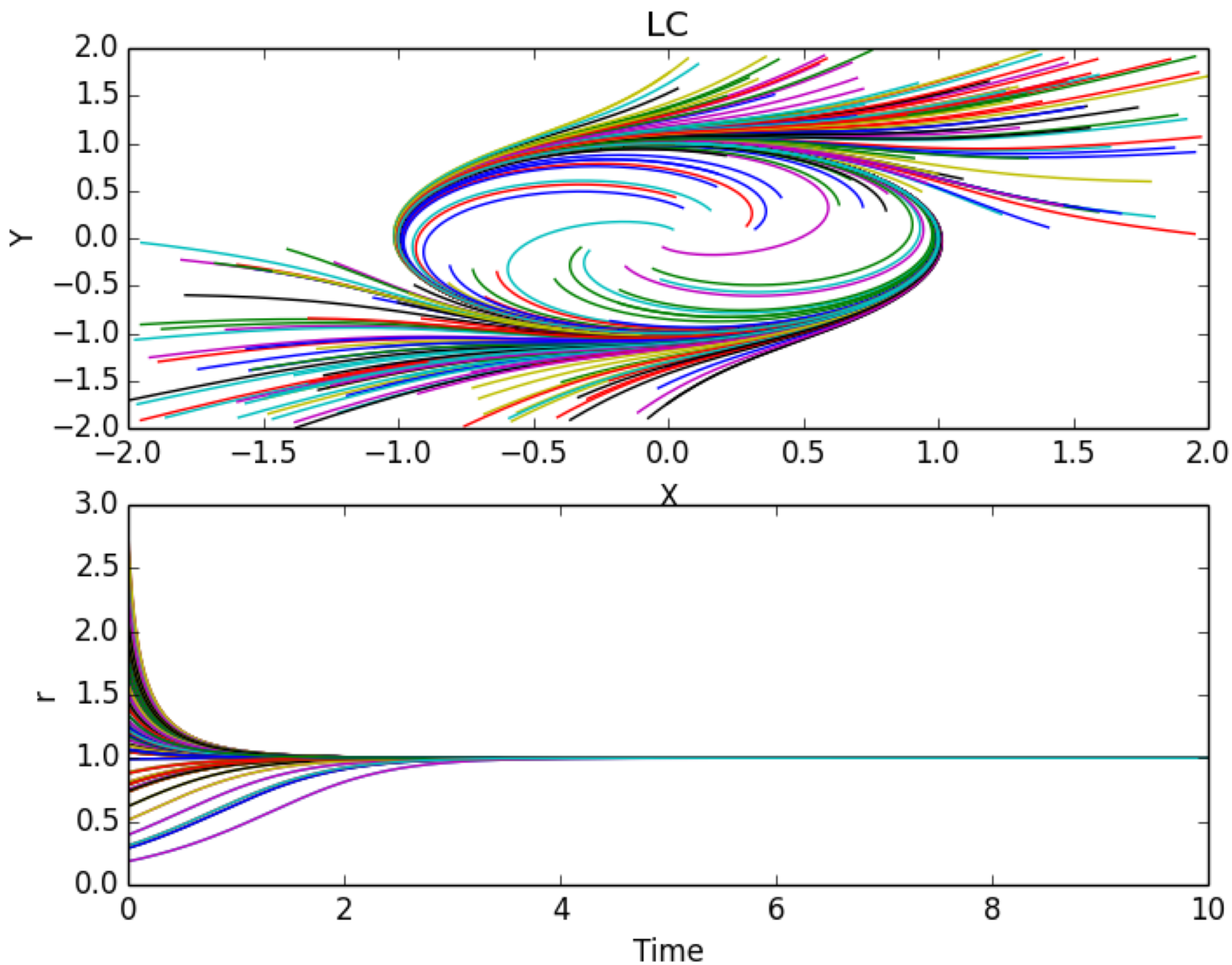
- $\frac{dE}{dt} = 0$ when $x = 1$;
- $\frac{dE}{dt} < 0$ when $|X| > 1$; "*sink*"
- $\frac{dE}{dt} > 0$ when $|X| < 1$; "*source*"



Jordan and Smith (p126)

Review: Limit Cycle in Ex 1

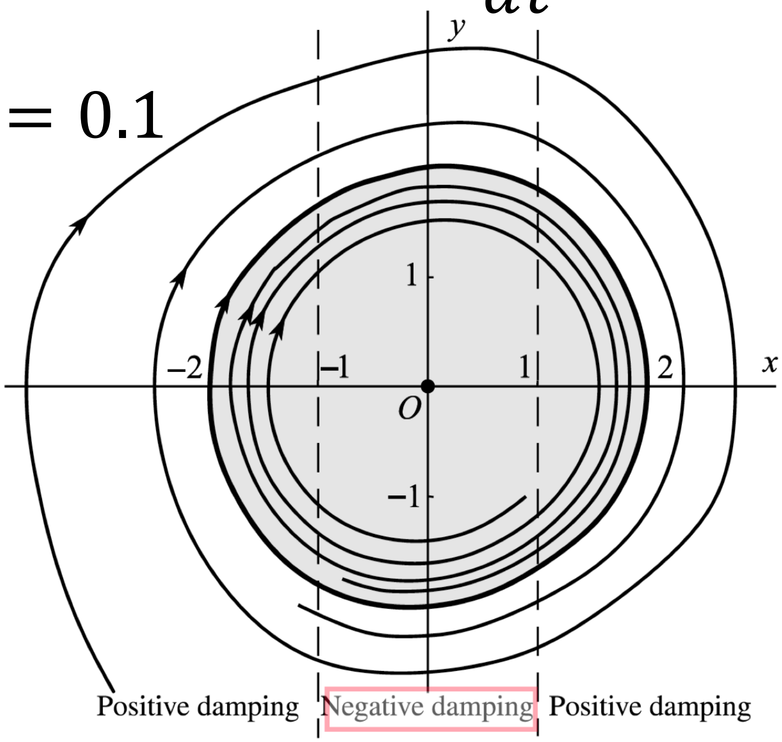
Ensemble Runs



The Limit Cycle of van dan Pol Equation (cont.)

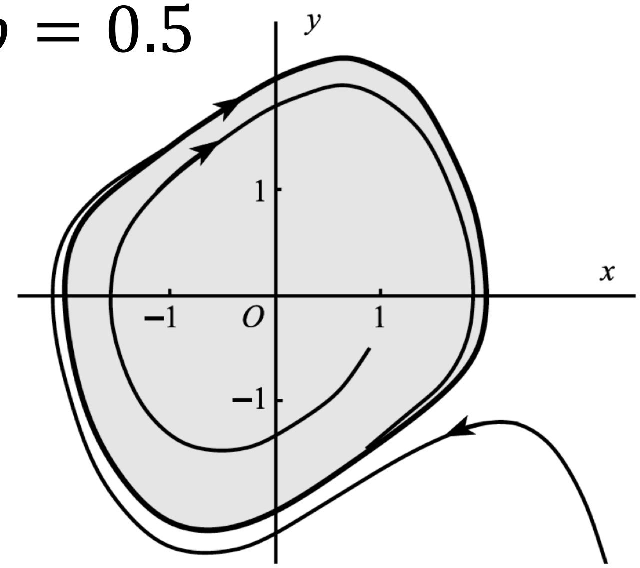
$$\frac{d^2X}{dt^2} + b(X^2 - 1)\frac{dX}{dt} + X = 0$$

$b = 0.1$



Positive damping | Negative damping | Positive damping

$b = 0.5$



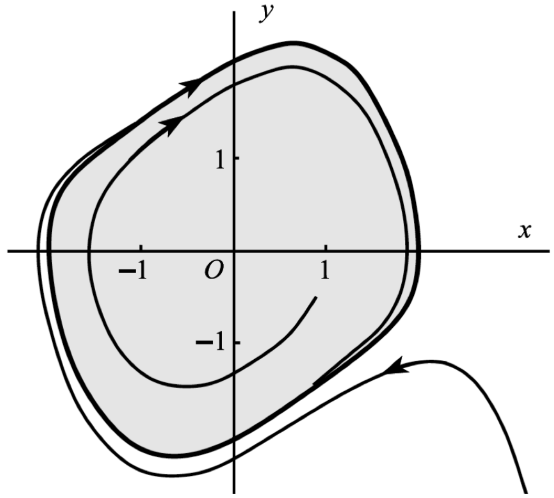
The limit cycle is the outer rim of the shaded region.

Jordan and Smith

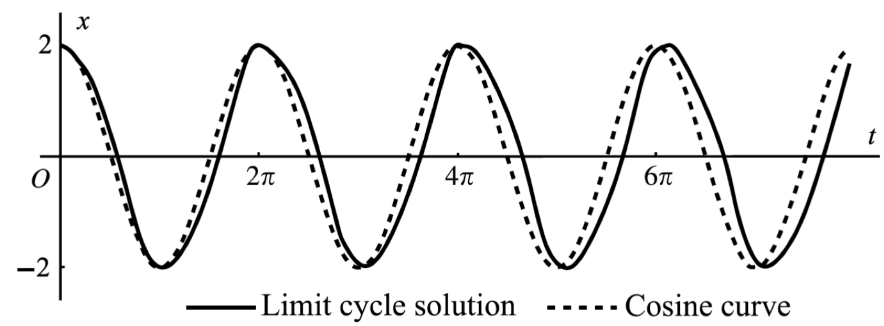
The Limit Cycle of van dan Pol Equation (cont.)

$$b = 0.5$$

(a)



(b)



(Spectral Analysis?)

Jordan and Smith

An Estimate of the Amplitude for a LC

When b is small, $b = \epsilon$, we have $\quad x'' + \epsilon(x^2 - 1)x' + x = 0$

By ignoring the dissipative term, we have

$$\boxed{x(t) \approx a\cos t, \quad y(t) \approx -a\sin t} \qquad y = \frac{dx}{dt}, T \approx 2\pi$$

$$h = \epsilon(x^2 - 1)x'; \qquad \frac{dE}{dt} = -hx' = -\epsilon(x^2 - 1)(x')^2;$$

$$\int_0^{2\pi} \frac{dE}{dt} dt = E(2\pi) - E(0) = 0 \quad \Longrightarrow \int_0^{2\pi} (x^2 - 1)(x')^2 dt = 0$$

$$\cos^2(t) = \frac{1 + \cos(2t)}{2}$$

$$\Longrightarrow \int_0^{2\pi} (a^2 cos^2(t) - 1)a^2 \sin^2(t)\, dt = 0$$

$$\sin^2(t) = \frac{1 - \cos(2t)}{2}$$

$$\int_0^{2\pi} (a^2 \frac{1 + \cos(2t)}{2} - 1) \frac{1 - \cos(2t)}{2} dt = 0 \qquad \int_0^{2\pi} (a^2 \frac{1 - \cos^2(2t)}{4} - \frac{1}{2})\, dt = 0$$

An Estimate of the Amplitude for a LC

When b is small, $b=\epsilon$, we have $x''+\epsilon(x^2-1)x'+x=0$

By ignoring the dissipative term, we have

$$\boxed{x(t)\approx a\cos t,\quad y(t)\approx -a\sin t} \qquad y=\frac{dx}{dt}, T\approx 2\pi$$

$$h=\epsilon(x^2-1)x'; \qquad \frac{dE}{dt}=-hx'=-\epsilon(x^2-1)(x')^2;$$

$$\int_0^{2\pi}\frac{dE}{dt}dt=E(2\pi)-E(0)=0 \quad \Longrightarrow \int_0^{2\pi}(x^2-1)(x')^2dt=0$$

$$\Longrightarrow \int_0^{2\pi}(a^2cos^2(t)-1)a^2\sin^2(t)\,dt=0 \qquad \int_0^{2\pi}(a^2\frac{1-\cos^2(2t)}{4}-\frac{1}{2})\,dt=0$$

$$\Longrightarrow \int_0^{2\pi}(a^2\frac{2-(1+\cos(4t))}{8}-\frac{1}{2})\,dt=0 \qquad \boxed{\frac{1}{4}a^2-1=0}$$

A Perturbative Analysis of the Solution

TBD

Consider the following Van der Pol equation

$$\frac{d^2X}{dt^2} + b(X^2-1)\frac{dX}{dt} + X = 0, \tag{1}$$

which can be written as follows:

$$\frac{d^2X}{dt^2} + X = \epsilon\frac{dX}{dt} - \epsilon X^2\frac{dX}{dt}, \tag{2}$$

where ϵ is introduced to replace b in order to perform a perturbative analysis.

We seek a first-order expansion for the solution in the form

$$X = X_o + \epsilon X_1 + \ldots \tag{3}$$

Plugging the above into Eq. (2), we have

$$\left(X_o'' + \epsilon X_1''\right) + (X_o + \epsilon X_1) = \epsilon\left(\frac{dX_o}{dt} + \epsilon\frac{dX_1}{dt}\right) - \epsilon\,(X_o + \epsilon X_1)^2\left(\frac{dX_o}{dt} + \epsilon\frac{dX_1}{dt}\right). \tag{4}$$

A Perturbative Analysis of the Solution

TBD

$$\boxed{\left(X_o''}+\epsilon X_1''\right)+\boxed{(X_o}+\epsilon X_1)=\epsilon\left(\frac{dX_o}{dt}+\epsilon\frac{dX_1}{dt}\right)-\epsilon\left(X_o+\epsilon X_1\right)^2\left(\frac{dX_o}{dt}+\epsilon\frac{dX_1}{dt}\right). \quad (4)$$

Considering terms with ϵ^0, we have

$$\boxed{X_o''+X_o=0.} \quad (5)$$

$$\boxed{x(t)\approx a\cos t,\qquad y(t)\approx -a\sin t}$$

A Perturbative Analysis of the Solution

TBD

$$\left(X_o'' + \boxed{\epsilon X_1''}\right) + (X_o + \boxed{\epsilon X_1}) = \epsilon\left(\boxed{\frac{dX_o}{dt}} + \epsilon\frac{dX_1}{dt}\right) - \epsilon\left(\boxed{X_o} + \epsilon X_1\right)^2\left(\boxed{\frac{dX_o}{dt}} + \epsilon\frac{dX_1}{dt}\right). \tag{4}$$

Considering terms with ϵ^0, we have

$$X_o'' + X_o = 0. \tag{5}$$

Considering terms with ϵ^1, we have

$$\boxed{X_1'' + X_1} = \boxed{X_o'} - \boxed{X_o^2 X_o'}. \tag{6}$$

X_0 acts as a forcing term

From Eq. (5), we have the solution of X_o as follows:

$$X_o = acos(t + \beta), \tag{7}$$

A Perturbative Analysis of the Solution

TBD

$$X_1'' + X_1 = \left(-a + \frac{1}{4}\,a^3\right) sin(t+\beta) + \frac{1}{4}\,a^3\, sin(3(t+\beta)). \tag{12}$$

Consider $X_1 = u + v$, which satisfy the following:

$$u'' + u = \left(-a + \frac{1}{4}\,a^3\right) sin(t+\beta), \tag{13}$$

$$v'' + v = \frac{a^3}{4} sin(3(t+\beta)). \tag{14}$$

A particular solution of Eq. (13) is:

$$u_p = \frac{at}{2}\left(1 - \frac{1}{4}\,a^2\right) cos(t+\beta), \tag{15}$$

A particular solution of Eq. (14) is:

$$v_p = \frac{1}{32} a^3 sin(3(t+\beta)). \tag{16}$$

A red circle indicates a nonlinear term.

A Perturbative Analysis of the Solution

TBD

From Eqs. (3), (7), (15-16), we have

$$X = acos(t+\beta) + \epsilon \left[\frac{at}{2} \left(1 - \frac{1}{4} a^2 \right) cos(t+\beta) + \frac{1}{32} a^3 sin(3(t+\beta)) \right] + \cdot \quad (17)$$

As a result of the presence of the mixed-secular term, the above expansion is non-uniform for $t \geq O(\epsilon^{-1})$ because the correction term is the order or larger than the first term. The mixed-secular term in Eq. (17) disappears if

$$a(1 - \frac{1}{4}a^2) = 0, \qquad \boxed{\frac{1}{4}a^2 - 1 = 0} \quad (18)$$

leading to $a = 0$ $a = \pm 2$, the latter of which provides an estimate on the amplitude (a). Therefore, the solution becomes

$$X = 2cos(t+\beta) + \frac{1}{4}\epsilon sin(3(t+\beta)) + \cdots \quad (19)$$

The Limit Cycle of Van dan Pol Equation: A Theorem

We conclude this section with a statement of a theorem on **Lienard equations** due to N. Levinson and O. K. Smith,[†] which can be used to show that van der Pol's equation has a unique nonconstant periodic solution (see Problem 24).

Levinson and Smith's Theorem

Theorem 8. Let $f(x)$ and $g(x)$ be continuous functions and let

$$F(x) := \int_0^x f(s)\,ds\ , \qquad G(x) := \int_0^x g(s)\,ds\ .$$

The Lienard equation

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0$$

$$\frac{d^2X}{dt^2} + b(X^2 - 1)\frac{dX}{dt} + X = 0$$

Van der Pol's Eq.

has a unique nonconstant periodic solution whenever all of the following conditions hold:

(a) $f(x)$ is even.

(b) $F(x) < 0$, for $0 < x < a$;
$F(x) > 0$, for $x > a$, for some a.

(c) $F(x) \to +\infty$ as $x \to +\infty$, monotonically for $x > a$.

(d) $g(x)$ is an odd function with $g(x) > 0$ for $x > 0$.

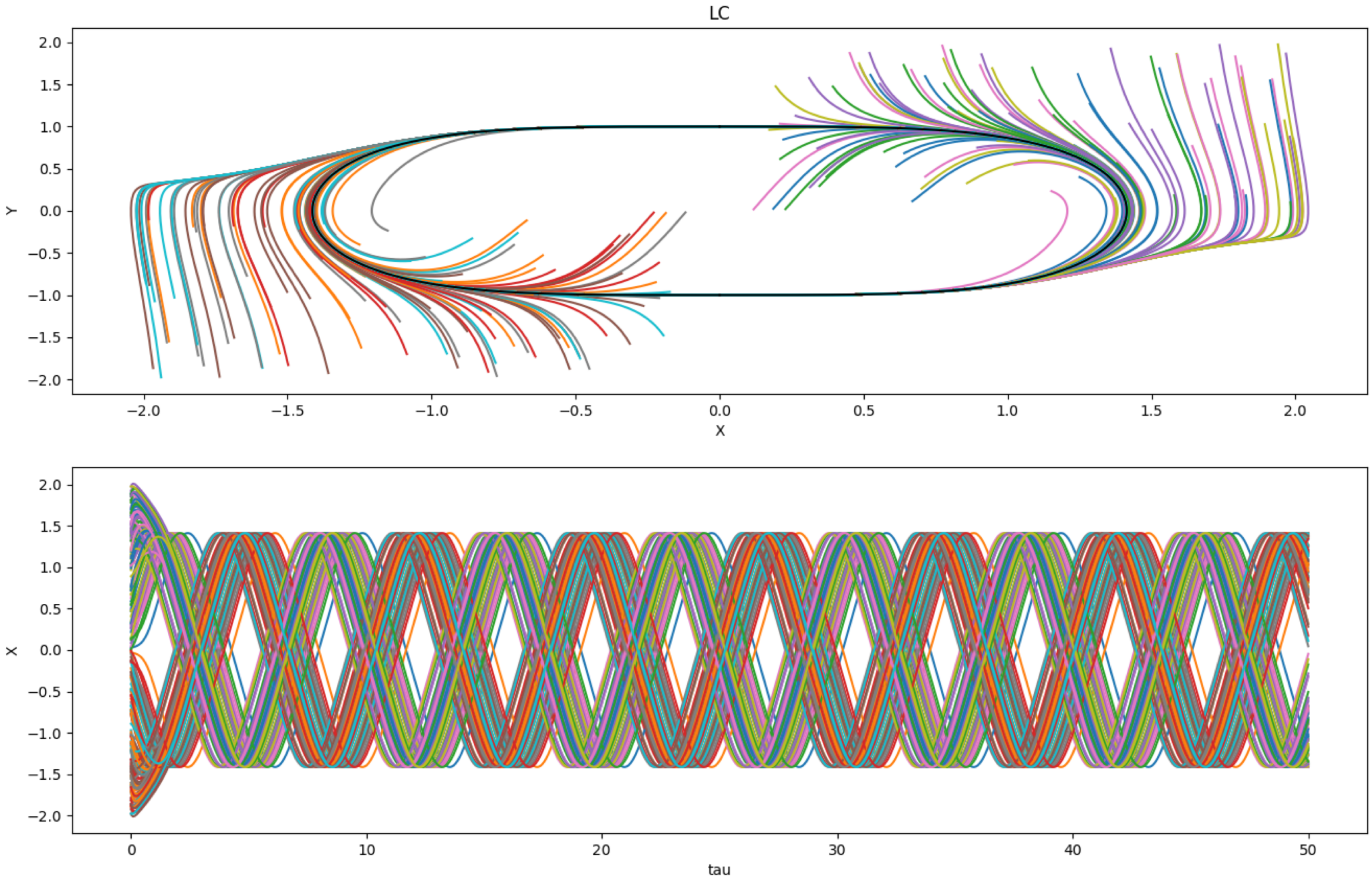
(e) $G(x) \to +\infty$ as $x \to +\infty$.

Nagle et al.

Limit Cycle: Ex 5

$$\ddot{X} + \left[X^4 + 4\dot{X}^2 - 4\right]\dot{X} + \frac{X^3}{2} = 0$$

$$\dot{X} = Y$$



Shen

Revealing Closed Orbits using Energy Integrals

Supp

$$y_1' = -y_2 - y_2^3$$
$$y_2' = y_1$$

$$y_2'' + y_2 + y_2^3 = 0$$

$$\frac{d^2x}{dt^2} + h\left(x, \frac{dx}{dt}\right) + g(x) = 0$$

$$G(x) = \int g(x)\,dx, PE$$

- A linear analysis shows that the critical point at (0, 0) is a center.
- Below, we show that the critical point is a (nonlinear) center.

$$PE = \int (y_2 + y_2^3) dy_2 = \frac{1}{4}(2y_2^2 + y_2^4)$$

$$KE = \frac{1}{2}(y_2{}')^2 = \frac{1}{2}y_1^2$$

Bender & Orazag

$$\frac{\mathrm{d}}{dt}(KE + PE) = 0 \ (since\ h = 0)$$

$$\boxed{KE + PE = constant = \frac{1}{2}y_1^2 + \frac{1}{4}(2y_2^2 + y_2^4)}$$

A family of concentric closed curves centered at (0, 0)

Outline

- Introduction
  - Limit cycle
  - A mini review of vector calculus
  - Gradient system ($\vec{v} = \nabla\phi$) for irrotational flow ($\nabla\times\vec{v} = 0$)
  - Hamiltonian system for incompressible flow ($\nabla \cdot \vec{v} = 0$)
- Energy methods for stability analysis near a critical point
  - The limit cycle of van dan Pol Equation
- Methods for ruling out closed orbits
  - I. Existence of a Laypunov function
  - II. Zero curl ($\nabla\times\vec{v} = 0$)
  - III. Positive ($\nabla \cdot \vec{v} > 0$) or negative ($\nabla \cdot \vec{v} < 0$) divergence
- Poincare-Bendixson Theorem
- Summary
  - Eigenvalue analysis for gradient and Hamiltonian Systems
  - Poincare-Bendixson Theorem

Methods for Ruling Out Closed Orbits

I. Existence of a Laypunov function

II. Zero curl ($\nabla \times \vec{v} = 0$)

III. Positive ($\nabla \cdot \vec{v} > 0$) or negative ($\nabla \cdot \vec{v} < 0$) divergence
(Divergence is of one sign)

Strogatz

Lyapunov Functions

- The Lyapunov function is constructed as an energy-like function in order to analyze local stability near a critical point.
- Let $E(\vec{X})$ and $\vec{X}_c$ be a Lyapunov function and a critical point.
- The Lyapunov function E: $R^n \to R$ for $\vec{X}_c$ in some neighborhood D of $\vec{X}_c$ has the following properties:

(1) $E(\vec{X}_c) = 0$ and $E(\vec{X}) > 0$ for all $X \neq X_c$ in D, and

(2) $\dot{E}(\vec{X}) \leq 0$ for all X in D,

here, $\dot{E}(\vec{X})$ represents the rate of change of E along a solution trajectory. Therefore, the above properties suggest that the Lyapunov function is positive and decreases along the trajectory. Mathematically, $\dot{E}(\vec{X})$ can be written as follows:

$$\dot{E}(\vec{X}) = \frac{\partial E}{\partial X}\frac{dX}{dt} + \frac{\partial E}{\partial Y}\frac{dY}{dt} + \frac{\partial E}{\partial Z}\frac{dZ}{dt} + \ldots \quad \text{when } \vec{X} = (X, Y, Z, ...), or \qquad (E1)$$

$$\boxed{\frac{dE}{dt} = \nabla E \cdot \left(\frac{dX}{dt}, \frac{dY}{dt}, \frac{dZ}{dt}\right) < 0}$$

An obtuse angle between ∇E and the direction of a flow.

No Closed Orbits When a Lyapunov Function Exist Supp

Liapunov Functions

Even for systems that have nothing to do with mechanics, it is occasionally possible to construct an energy-like function that decreases along trajectories. Such a function is called a Liapunov function. If a Liapunov function exists, then closed orbits are forbidden, by the same reasoning as in Example 7.2.2.

To be more precise, consider a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ with a fixed point at $\mathbf{x}^*$. Suppose that we can find a ***Liapunov function***, i.e., a continuously differentiable, real-valued function $V(\mathbf{x})$ with the following properties:

1. $V(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{x}^*$, and $V(\mathbf{x}^*) = 0$. (We say that V is *positive definite*.)
2. $\dot{V} < 0$ for all $\mathbf{x} \neq \mathbf{x}^*$. (All trajectories flow "downhill" toward $\mathbf{x}^*$.)

Then $\mathbf{x}^*$ is globally asymptotically stable: for all initial conditions, $\mathbf{x}(t) \to \mathbf{x}^*$ as $t \to \infty$. In particular the system has no closed orbits. (For a proof, see Jordan and Smith 1987.)

The intuition is that all trajectories move monotonically down the graph of $V(\mathbf{x})$ toward $\mathbf{x}^*$ (Figure 7.2.1).

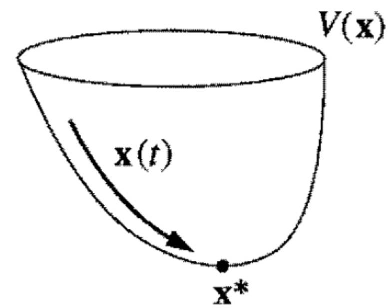


Figure 7.2.1

Large value

small value

- ∇V points in the direction of greatest increase of a function
- $\dfrac{dV}{\mathrm{dt}} = \nabla V \cdot \left(\dfrac{dx}{dt}, \dfrac{dy}{dt}\right)$
- $\dfrac{dV}{\mathrm{dt}} < 0,$ the trajectory moves in the decrease of the function

Strogatz

Strict Lyapunov Functions

Corollary 6. *If L is a strict Liapunov function for a planar system, then there are no limit cycles.* ■

- Liapunov functions not only detect stable equilibria; they can also be used to estimate the size of the basin of attraction of an asymptotically stable equilibrium, as the preceding example shows. (HSD, p199).

HSD

Review: Green's Theorem

Formulas for Grad, Div, Curl, and the Laplacian

	Cartesian (x, y, z) $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors in the directions of increasing x, y, and z. P, Q, and R are the scalar components of $\mathbf{F}(x, y, z)$ in these directions.
Gradient	$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$
Divergence	$\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$
Curl	$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$
Laplacian	$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

The Fundamental Theorem of Line Integrals

1. Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ be a vector field whose components are continuous throughout an open connected region D in space. Then there exists a differentiable function f such that
$$\mathbf{F} = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$
if and only if for all points A and B in D the value of $\int_A^B \mathbf{F} \cdot d\mathbf{r}$ is independent of the path joining A to B in D.
2. If the integral is independent of the path from A to B, its value is
$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

Green's Theorem (Tangential Form)
$$\iint_R \nabla \times \vec{F} \cdot \vec{k}\,dxdy = \oint \vec{F} \cdot d\vec{r}$$

Stokes' Theorem
$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

Green's Theorem (Normal Form)
$$\iint_R \nabla \cdot \vec{F}\,dxdy = \oint \vec{F} \cdot \vec{n}\,ds$$

Divergence Theorem
$$\iiint_E \operatorname{div} \mathbf{F}\,dV = \iint \vec{F} \cdot \vec{n}\,dS$$

Terminology

- Limit cycle: an isolated closed path

Consider $X' = F(X)$, here $X' = (x', y') = \vec{v}$.

- Gradient system ($\vec{v} = \nabla\phi$) for irrotational flow ($\nabla\times\vec{v} = 0$)
  - ✓ ϕ : velocity potential; $\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right)$
  - ✓ The direction of vector filed is perpendicular to the tangent vector of the contour line of ϕ (i.e., a constant value of the potential function).
- Hamiltonian system for incompressible flow ($\nabla\cdot\vec{v} = 0$)
  - ✓ ψ: streamfunction; $\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \left(\frac{\partial\psi}{\partial y}, \frac{-\partial\psi}{\partial x}\right)$
  - ✓ The direction of the flow is parallel to the tangent vector of the contour line of ψ.

Gradient System $\left(\nabla\times\left(\frac{dx}{dt},\frac{dy}{dt}\right)=\mathbf{0}\right)$

- Gradient system ←→ $\left(\frac{dx}{dt},\frac{dy}{dt}\right)=\nabla\phi$
- Gradient system → potential function (ϕ)
- The gradient of the potential function is parallel to the direction of flow (i.e., $\vec{v} \parallel \nabla\phi$)
- The direction of vector filed is in perpendicular to the tangent line of the contour line (i.e., a constant value of the potential function).
- (thus, the flow is across the contour lines of the potential function).
- (Note that the contour line does not represent a streamfuntion)

HSD

Pause: 2^{nd} Order ODEs vs. 1^{st} Order ODEs

2D $$\frac{d^2x}{dt^2} + h\left(x, \frac{dx}{dt}\right) + g(x) = 0$$ (nonlinear)

$$F = -\nabla G$$

- 1D gradient
- $G(x)$, not $G(x, x')$

$$G(x) = \int g(x)\, dx, potential\ energy$$

$$E(x, v) = \frac{1}{2}v^2 + G(x), total\ energy$$

$$v = \frac{dx}{dt}$$

1D $$\frac{dx}{dt} + f = 0$$

Let $f = \frac{dG}{dx}$

$$G(x) = \int f(x)\, dx, potential\ function$$

M-D $$\frac{d\vec{x}}{dt} - \vec{f} = 0 \qquad \vec{f}? = -\nabla G \qquad (f_1\ f_2, f_3) = -(\mathrm{G_x}, \mathrm{G_y}, \mathrm{G_z})$$

Strogatz; Nagle et al.

No Closed Orbits when Curl is Zero

Supp

Gradient Systems

Suppose the system can be written in the form $\dot{\mathbf{x}} = -\nabla V$, for some continuously differentiable, single-valued scalar function $V(\mathbf{x})$. Such a system is called a ***gradient system*** with ***potential function*** V.

Theorem 7.2.1: Closed orbits are impossible in gradient systems.

$$\begin{aligned}\Delta V &= \int_0^T \frac{dV}{dt}\,dt \\ &= \int_0^T (\nabla V \cdot \dot{\mathbf{x}})\,dt \\ &= -\int_0^T \|\dot{\mathbf{x}}\|^2\,dt \\ &< 0\end{aligned}$$

$$\dot{\mathbf{x}} = -\nabla V$$

potential function V.

(unless $\dot{\mathbf{x}} \equiv \mathbf{0}$, in which case the trajectory is a fixed point, not a closed orbit). This contradiction shows that closed orbits can't exist in gradient systems. ■

Strogatz

Additional Details for the Proof

consider $\dot{\mathbf{x}} = -\nabla V$, yielding $\frac{d\vec{r}}{dt} = -\nabla V$

V represents a potential function, determining the potential energy

differential $dV = \nabla V \cdot dr$ $\qquad \frac{dV}{dt} = \nabla V \cdot \frac{d\vec{r}}{dt}$

$$\Delta PE = \Delta V = \int \frac{dV}{dt} dt$$

Check whether ΔPE is zero for one period of time , i.e., $t \in [0, T]$.

$$\Delta PE = \int_0^T \frac{dV}{dt} dt = \int_0^T \nabla V \cdot \frac{d\vec{r}}{dt} dt = \int_0^T -(\nabla V)^2 dt < 0$$

(unless ∇V =0, yielding $\dot{\mathbf{x}} = \mathbf{0}$)

Additional Details for the Proof

Check whether ΔPE is zero for one period of time , i.e., $t \in [0, T]$.

$$\Delta PE = \int_0^T \frac{dV}{dt} dt = \int_0^T \nabla V \cdot \frac{d\vec{r}}{dt} dt = \int_0^T -(\nabla V)^2 dt < 0$$

(unless ∇V =0, yielding $\dot{\mathbf{x}} = \mathbf{0}$)

$$\Delta PE = \int_0^T \nabla V \cdot \frac{d\vec{r}}{dt} dt = \int_A^B \nabla V \cdot d\vec{r} = \int_A^B dV = V(B) - V(A)$$

$\Delta PE \neq 0$ $\quad$ $V(B) \neq V(A)$

- We cannot have a closed orbit and thus cannot apply the Green's theorem (because $d\vec{r} \parallel \nabla V$).

In other words, we cannot select a 2D domain with $d\vec{r} \parallel \nabla V$.

Green's Theorem (Tangential Form) $\oint \vec{V} \cdot d\vec{r} = \iint \nabla \times \vec{V} \cdot \vec{k}\, dxdy$

Gradient System: Eigenvalue Analysis

$$\frac{dX}{dt} = \nabla G \qquad \frac{dx}{dt} = G_x \quad \frac{dy}{dt} = G_y \qquad \frac{d\vec{r}}{dt} = \nabla G$$

$$J(G_x, G_y) = \frac{\partial(G_x, G_y)}{\partial(x, y)} = \begin{bmatrix} G_{xx} & G_{xy} \\ G_{yx} & G_{yy} \end{bmatrix} \qquad A = \begin{bmatrix} G_{xx} & G_{xy} \\ G_{yx} & G_{yy} \end{bmatrix}$$

Matrix A is symmetric

→ eigenvalues are real

→ no closed orbit (or spiral source/sink)

Blandchart et al.

Gradient System $\left(\nabla\times\left(\frac{dx}{dt},\frac{dy}{dt}\right)=\mathbf{0}\right)$

Theorem. (Properties of Gradient Systems) *For the system $X' = -\mathrm{grad}\, V(X)$:*

1. *If c is a regular value of V, then the vector field is perpendicular to the level set $V^{-1}(c)$.*
2. *The critical points of V are the equilibrium points of the system.*
3. *If a critical point is an isolated minimum of V, then this point is an asymptotically stable equilibrium point.* ∎

Vector field in perpendicular to the tangent line of the contour line $(\vec{X} \parallel \nabla V)$

Proposition. *For a gradient system $X' = -\mathrm{grad}\, V(X)$, the linearized system at any equilibrium point has only real eigenvalues.* □

The Jacobian matrix is symmetric → real eigenvalues

→ no closed orbits

HSD

No Closed Orbits when Divergence Is of One Sign

Supp

Bendixson Negative Criterion

Theorem 6. Let $f(x, y)$ and $g(x, y)$ have continuous first partial derivatives in the simply connected domain†† D and assume that

$$f_x(x, y) + g_y(x, y)$$

is of one sign in D. Then there are no (nonconstant) periodic solutions to

$$\frac{dx}{dt} = f(x, y) \ , \tag{4}$$

$$\frac{dy}{dt} = g(x, y) \tag{5}$$

that lie entirely in D.

$$\nabla \cdot (f, g) > 0 \ or < 0$$

Green's Theorem (Normal Form) $\oint \vec{V} \cdot \vec{n}\, ds = \iint \nabla \cdot \vec{V} dx dy$

Nagle et al.

No Closed Orbits when Divergence Is of One Sign

Supp

Theorem 9.7.2 Let the functions F and G have continuous first partial derivatives in a simply connected domain D of the xy-plane. If $F_x + G_y$ has the same sign throughout D, then there is no closed trajectory of the system (15) lying entirely in D.

Green's Theorem (Normal Form) $\oint \vec{V} \cdot \vec{n}\, ds = \iint \nabla \cdot \vec{V}\, dxdy$

$$RHS = \iint \nabla \cdot \vec{V} \text{ dA} \neq 0.$$

$$LHS = \int \vec{V} \cdot d\vec{n} = \int \left(\frac{dx}{dt}, \frac{dy}{dt}\right) \cdot (dy, -dx) = 0.$$

Contradiction

➔no closed orbit

assume that a flow is on a circle

Boyce_n_DiPrima

2D Hamiltonian System:

$$\left(\nabla\cdot\left(\frac{dx}{dt},\frac{dy}{dt}\right)=\mathbf{0}\right)$$

(zero divergence)

2D Hamiltonian System

Supp

By analogy with the form of Hamilton's canonical equations in mechanics, a system

$$\dot{x} = X(x, y), \qquad \dot{y} = Y(x, y) \tag{2.69}$$

is called a **Hamiltonian system** if there exists a function $H(x, y)$ such that

$$X = \frac{\partial H}{\partial y} \quad \text{and} \quad Y = \frac{\partial H}{\partial x}. \tag{2.70}$$

Then H is called the **Hamiltonian function** for the system. A necessary and sufficient condition for (2.69) to be Hamiltonian is that

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0. \tag{2.71}$$

$$\left(\nabla \cdot \left(\frac{\mathrm{dx}}{\mathrm{dt}}, \frac{\mathrm{dy}}{\mathrm{dt}}\right) = 0\right)$$

Hamiltonian System: Conservative $\left(\nabla \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = 0\right)$ Supp

Proposition. *For a Hamiltonian system in $\mathbb{R}^2$, H is constant along every solution curve.* □

Proposition. *Suppose (x_0, y_0) is an equilibrium point for a planar Hamiltonian system. Then the eigenvalues of the linearized system are either $\pm\lambda$ or $\pm i\lambda$ where $\lambda \in \mathbb{R}$.* □

$$x' = \frac{\partial H}{\partial y}(x, y)$$

$$y' = -\frac{\partial H}{\partial x}(x, y)$$

$$\dot{H} = \frac{\partial H}{\partial x}x' + \frac{\partial H}{\partial y}y'$$

$$= \frac{\partial H}{\partial x}\frac{\partial H}{\partial y} + \frac{\partial H}{\partial y}\left(-\frac{\partial H}{\partial x}\right) = 0.$$

$$J = \begin{bmatrix} H_{yx} & H_{yy} \\ -H_{xx} & -H_{xy} \end{bmatrix}$$

- The direction of the flow (dx/dt, dy/dt) and the gradient of H are orthogonal.
- The direction of the flow is parallel to the tangent vector of the contour line of H.
- H is called the Hamiltonian function, and is a streamfunction.
- $a_{11} + a_{22} = 0 \Rightarrow \lambda_1 + \lambda_2 = 0 \Rightarrow saddle\ point\ (witn\ no\ closed\ orbit)\ or\ center$

Hamiltonian System: An Example

Supp

Example. Consider the system

$$x' = y$$
$$y' = -x^3 + x.$$

Alternatively, the system may be written as

$$\boxed{x'' - x + x^3 = 0}$$

A Hamiltonian function is

$$H(x, y) = \frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2} + \frac{1}{4}.$$

Total energy

The constant value 1/4 is irrelevant here; we choose it so that H has minimum value 0, which occurs at $(\pm 1, 0)$, as is easily checked. The only other equilibrium point lies at the origin. The linearized system is

One Saddle
Two Centers

$$X' = \begin{pmatrix} 0 & 1 \\ 1 - 3x^2 & 0 \end{pmatrix} X.$$

At $(0,0)$, this system has eigenvalues ± 1, so we have a saddle. At $(\pm 1, 0)$, the eigenvalues are $\pm\sqrt{2}i$, so we have a center, at least for the linearized system.

HSD

Hamiltonian System: An Example

Supp

equilibrium point lies at the origin. The linearized system is

One Saddle
Two Centers

$$X' = \begin{pmatrix} 0 & 1 \\ 1 - 3x^2 & 0 \end{pmatrix} X.$$

At (0,0), this system has eigenvalues ± 1, so we have a saddle. At $(\pm 1, 0)$, the eigenvalues are $\pm\sqrt{2}i$, so we have a center, at least for the linearized system.

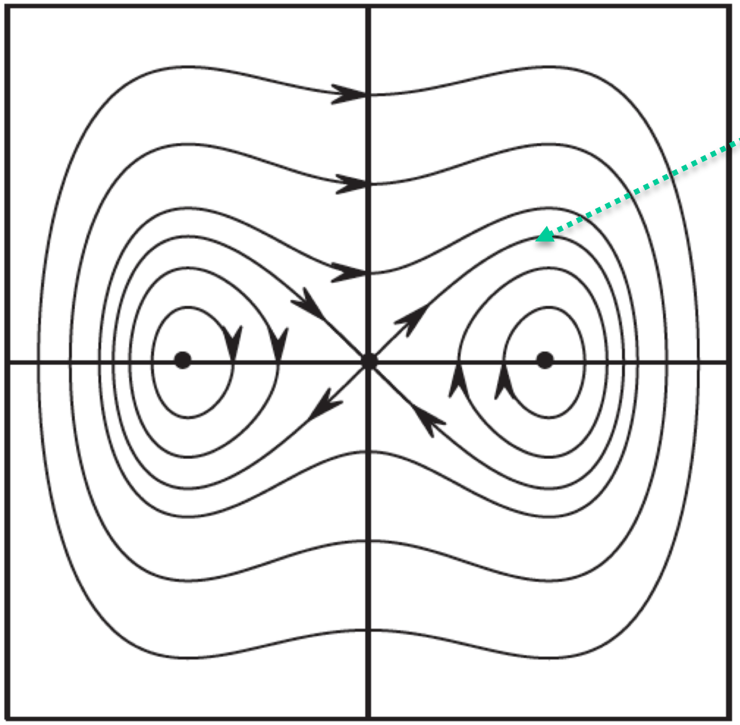
Plotting the level curves of H and adding the directions at nonequilibrium points yields the phase portrait shown in Figure 9.11. Note that the equilibrium points at $(\pm 1, 0)$ remain centers for the nonlinear system. Also note that the stable and unstable curves at the origin match up exactly. That is, we have solutions that tend to $(0,0)$ in both forward and backward time. Such solutions are known as *homoclinic solutions* or *homoclinic orbits*. ∎

HSD

Hamiltonian System: An Example (cont.)

Supp

$$\boxed{x'' - x + x^3 = 0}$$



Homoclinic orbits

Center --- Saddle --- Center

Figure 9.11 Phase portrait for $x' = y,\ y' = -x^3 + x.$

- H is called the Hamiltonian function, and is a streamfunction.
- The tangent vector of the contour line of H represents the direction of the flow.

HSD

MT Part A

2: [25 points] A nonlinear, non-dissipative Lorenz model is written as follows:

$$\frac{d^2X}{dt^2} - (\sigma r + C)\,X + \frac{X^3}{2} = 0. \tag{2}$$

Here, we assume that both σ and r are positive, and choose $C = 0$ for convenience. Complete the following problems.

(a) [3 points] Transform the 2nd order ODE in Eq. (2) into a system of the first order ODEs, (i.e., $Y = X'$).

(b) [3 points] Find critical points in the above 2D system in problem (2a).

(c) [6 points] Compute the Jacobian matrix of the above 2D system.

(d) [13 points] Perform a linear stability analysis for all of the critical points.

MEDIAN	MAXIMUM	MEAN
76.5	100.0	71.88

Methods for Ruling Out Closed Orbits

I. Existence of a Laypunov function

II. Zero curl ($\nabla \times \vec{v} = 0$)

III. Positive ($\nabla \cdot \vec{v} > 0$) or negative ($\nabla \cdot \vec{v} < 0$) divergence
(Divergence is of one sign)

Strogatz

Outline

- Introduction
  - Limit cycle
  - A mini review of vector calculus
  - Gradient system ($\vec{v} = \nabla\phi$) for irrotational flow ($\nabla\times\vec{v} = 0$)
  - Hamiltonian system for incompressible flow ($\nabla \cdot \vec{v} = 0$)
- Energy methods for stability analysis near a critical point
  - The limit cycle of van dan Pol Equation
- Methods for ruling out closed orbits
  - I. Existence of a Laypunov function
  - II. Zero curl ($\nabla\times\vec{v} = 0$)
  - III. Positive ($\nabla \cdot \vec{v} > 0$) or negative ($\nabla \cdot \vec{v} < 0$) divergence
- Poincare-Bendixson Theorem
- Summary
  - Eigenvalue analysis for gradient and Hamiltonian Systems
  - Poincare-Bendixson Theorem

Poincare-Bendixson Theorem

The Poincare–Bendixson Theorem essentially determines all of the possible limiting behaviors of a planar flow (HSD).

Consider a particular trajectory starting in R. The Poincare-Bendixson Theorem states that there are only two (three) possibilities for that trajectory (Hilborn, p101) (R is a closed bounded subset of the plane):

1. The trajectory approaches a fixed point of the system as $t \to \infty$.

2. The trajectory approaches a limit cycle as $t \to \infty$.

3. The trajectory is a limit cycle.

Hilborn

A Limit Cycle and Critical Points

Limit Cycles Surround Critical Points

Theorem 5. A limit cycle in the plane must enclose at least one critical point. Moreover, any critical point enclosed by a limit cycle cannot be a saddle point.

Nagle et al.

Theorem 9.7.1 Let the functions F and G have continuous first partial derivatives in a domain D of the xy-plane. A closed trajectory of the system (15) must necessarily enclose at least one critical (equilibrium) point. If it encloses only one critical point, the critical point cannot be a saddle point.

Boyce and DiPrima

Poincare-Bendixson Theorem

Poincaré–Bendixson Theorem

Theorem 7. Let f and g have continuous first partial derivatives on the closed bounded region R and assume the system

$$\frac{dx}{dt} = f(x, y) \ , \quad \frac{dy}{dt} = g(x, y)$$

has no critical points in R. If a solution $x = \phi(t)$, $y = \psi(t)$ to the system exists for all $t \geq t_0$ and its trajectory $\Gamma(t) := \big(\phi(t), \psi(t)\big)$ remains inside R for $t \geq t_0$, then either Γ is a limit cycle or it spirals toward a limit cycle in R. In either case, the system has a nonconstant periodic solution.

Please note the following properties within the Poincare-Bendixson Theorem:

- R has a hole in it (Nagle et al., p765)
- R cannot be simply connected; it must have a hole (Boycy-DiPrima, p569)
- [In the example by Strogatz (p204), a ring-shaped region is used.]
- [BWS: it excludes a starting point at a (stable) critical point.]

Nagle et al.

Find Region R where Periodic Solution exists

Supp

Example 2 Show that the equation

$$\text{(10)} \qquad \frac{d^2x}{dt^2} + \boxed{\left[4x^2 + \left(\frac{dx}{dt}\right)^2 - 4\right]\frac{dx}{dt}} + x^3 = 0$$

has a nonconstant periodic solution.

$$PE = G(x) = \int x^3 dx = \frac{1}{4}x^4$$

$$KE = \frac{1}{2}(x')^2 = \frac{1}{2}y^2 \qquad y = x'$$

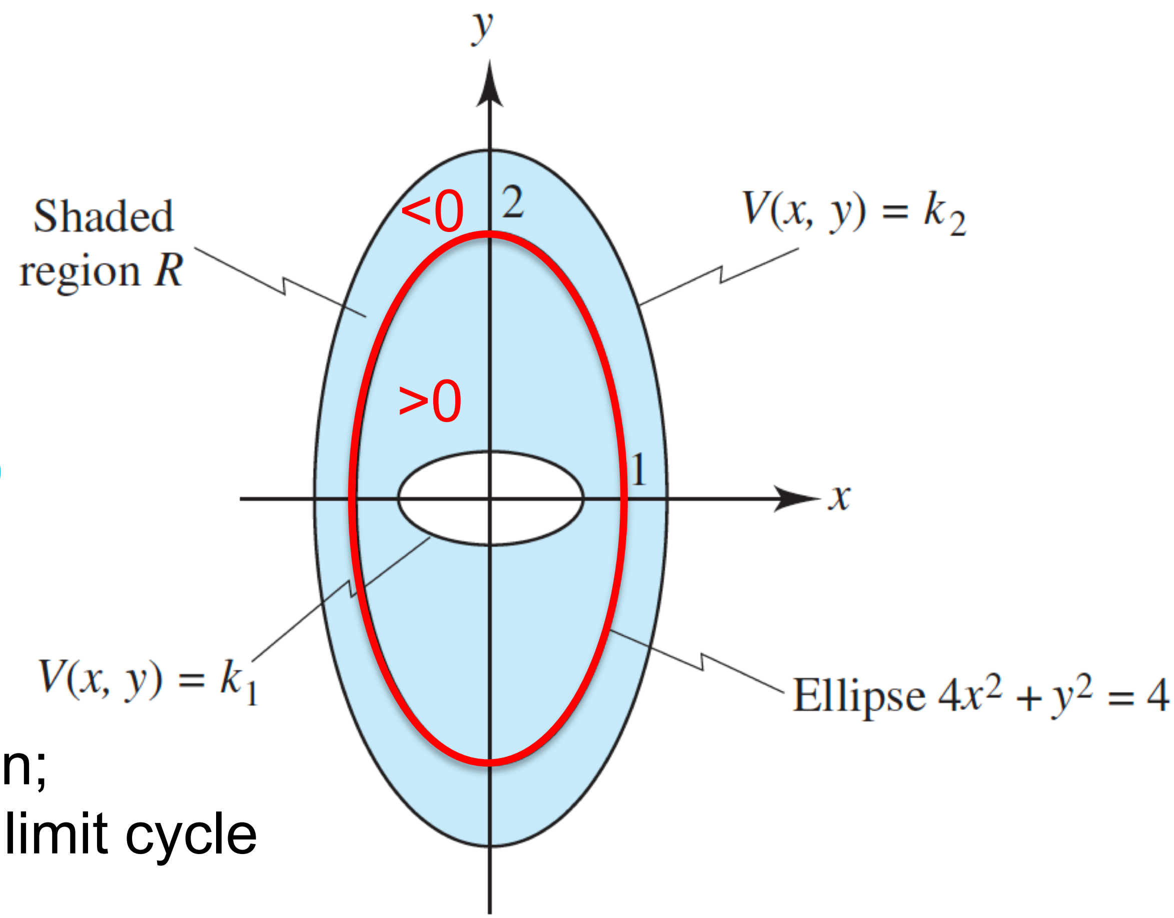
$$\frac{d(KE+PE)}{dt} = -\frac{dx}{dt}h = -x'[4x^2 + (x')^2 - 4]x' = -(y')^2[4x^2 + y^2 - 4]$$

$$\frac{d(KE+PE)}{dt} = \begin{cases} \le 0, & for\ 4x^2 + y^2 \ge 4 \\ \ge 0 & for\ 4x^2 + y^2 \le 4 \end{cases}$$

$$\frac{d(KE+PE)}{dt} = \begin{cases} \leq 0, & for\ 4x^2+y^2 \geq 4 \\ \geq 0 & for\ 4x^2+y^2 \leq 4 \end{cases}$$

red: $4x^2+y^2=4$

light blue: shaded region R
(where periodic solution exists)



Notes:
(1) R does not contain the origin;
(2) The ellipse in red is not the limit cycle

Figure 12.32 Region R where periodic solution exists

Poincare-Bendixson Theorem

Theorem 9.7.3 **(Poincaré–Bendixson[11] Theorem)**

Let the functions F and G have continuous first partial derivatives in a domain D of the xy-plane. Let D_1 be a bounded subdomain in D, and let R be the region that consists of D_1 plus its boundary (all points of R are in D). Suppose that R contains no critical point of the system (15). If there exists a constant t_0 such that $x = \phi(t)$, $y = \psi(t)$ is a solution of the system (15) that exists and stays in R for all $t \geq t_0$, then either $x = \phi(t)$, $y = \psi(t)$ is a periodic solution (closed trajectory), or $x = \phi(t)$, $y = \psi(t)$ spirals toward a closed trajectory as $t \to \infty$. In either case, the system (15) has a periodic solution in R.

The same as that in Nagle et al.

Boyce_n_DiPrima

Poincare-Bendixson Theorem

7.3 Poincaré–Bendixson Theorem

Now that we know how to rule out closed orbits, we turn to the opposite task: finding methods to *establish that closed orbits exist* in particular systems. The following theorem is one of the few results in this direction. It is also one of the key theoretical results in nonlinear dynamics, because it implies that chaos can't occur in the phase plane, as discussed briefly at the end of this section.

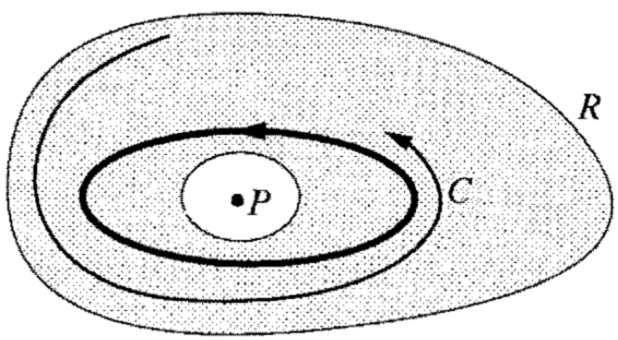
Poincaré–Bendixson Theorem: Suppose that:

(1) R is a closed, bounded subset of the plane;

(2) $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is a continuously differentiable vector field on an open set containing R;

(3) R does not contain any fixed points; and

(4) There exists a trajectory C that is "confined" in R, in the sense that it starts in R and stays in R for all future time (Figure 7.3.1).



Then either C is a closed orbit, or it spirals toward a closed orbit as $t \to \infty$. In either case, R *contains a closed orbit* (shown as a heavy curve in Figure 7.3.1).

The same as that in Nagle et al.

Strogatz

Poincare-Bendixson Theorem (.continued)

In Figure 7.3.1, we have drawn R as a ring-shaped region because any closed orbit must encircle a fixed point (P in Figure 7.3.1) and no fixed points are allowed in R.

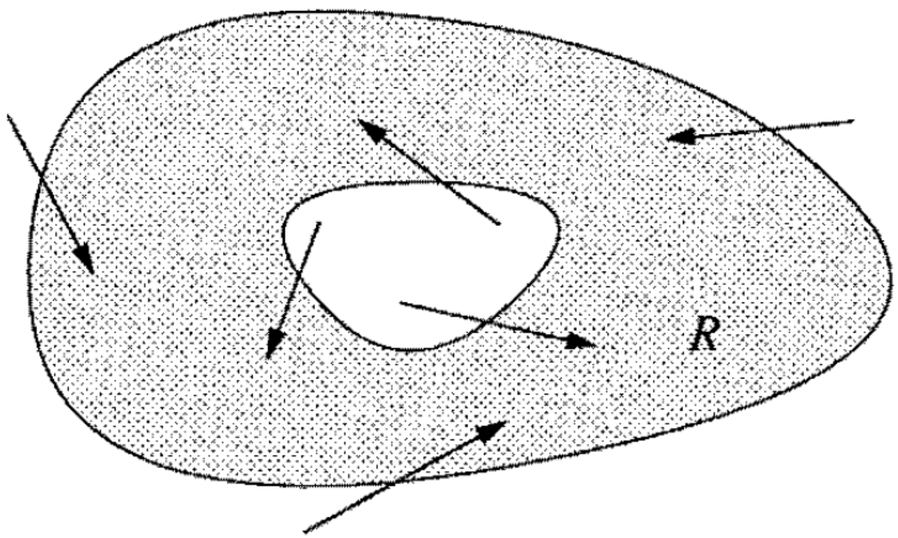


Figure 7.3.2

When applying the Poincaré–Bendixson theorem, it's easy to satisfy conditions (1)–(3); condition (4) is the tough one. How can we be sure that a confined trajectory C exists? The standard trick is to construct a ***trapping region*** R, i.e., a closed connected set such that the vector field points "inward" everywhere on the boundary of R (Figure 7.3.2). Then *all* trajectories in R are confined. If we can also arrange that there are no fixed points in R, then the Poincaré–Bendixson theorem ensures that R contains a closed orbit.

Strogatz

Poincare-Bendixson Theorem

Theorem. (Poincaré–Bendixson) *Suppose that Ω is a nonempty, closed, and bounded limit set of a planar system of differential equations that contains no equilibrium point. Then Ω is a closed orbit.*

- If γ is an ω-limit cycle, there exists $X \notin \gamma$ such that

$$\lim_{t\to\infty} d(\phi_t(X), \gamma) = 0.$$

- Geometrically this means that some solution spirals toward γ as $t \to \infty$.

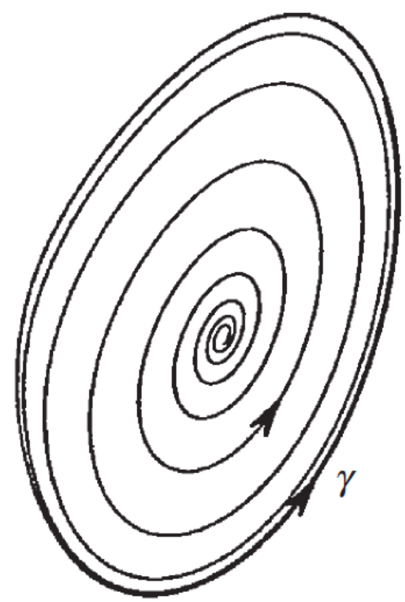


Figure 10.10 A solution spiraling toward a limit cycle.

The same as that in Nagle et al.

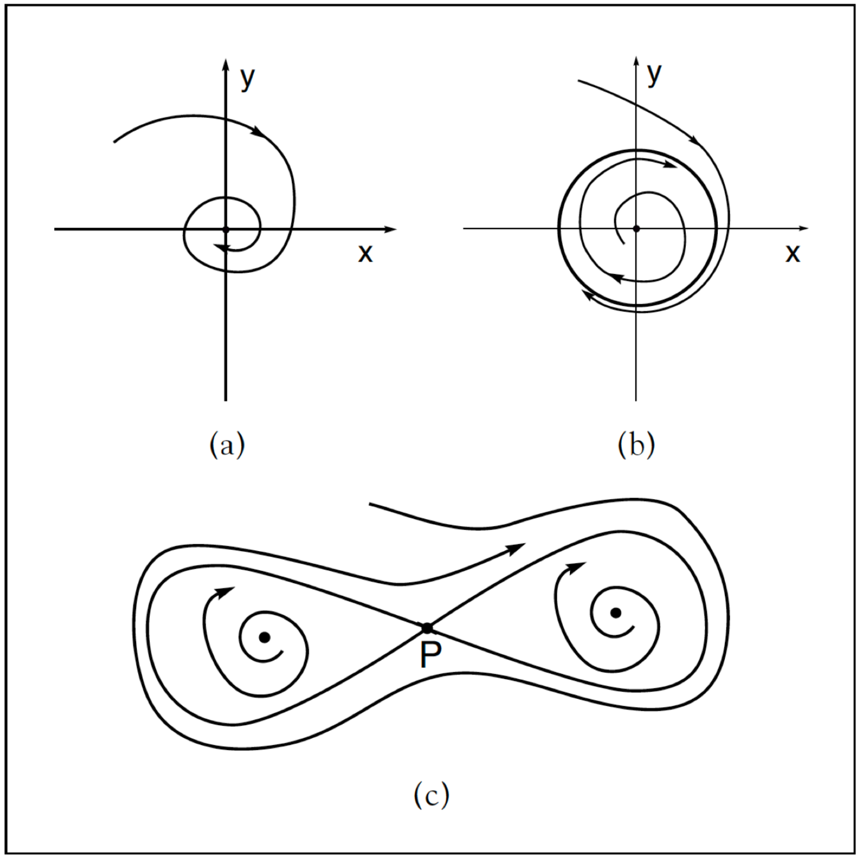
HSD

Three Cases of Poincare-Bendixson Theorem

Steady state
(a point)

Limit Cycle

2D!



Limit Cycle, enclosing three critical points (BWS)

Figure 8.4 Planar limit sets.
The three pictures illustrate the three cases of the Poincaré-Bendixson Theorem. (a) The limit set is one point, the origin. (b) The limit set of each spiraling trajectory is a circle, which is a periodic orbit. (c) The limit set of the outermost trajectory is a figure eight. This limit set must have an equilibrium point P at the vertex of the "eight". It consists of two connecting arcs plus the equilibrium. Trajectories on the connecting arcs tend to P as $t \to \infty$ and as $t \to -\infty$.

Alligood et al.

Poincare-Bendixson Theorem

The Poincaré–Bendixson theorem. If D is a closed bounded region of the (x, y)-plane, and a solution of system (1) for well-behaved $\mathbf{F}$ is such that $\mathbf{x}(t) \in \mathrm{D}$ for all $t \geqslant 0$, then the orbit either is a closed path, approaches a closed path as $t \to \infty$, or approaches an equilibrium point.

1. The trajectory approaches a fixed point of the system as $t \to \infty$.
2. The trajectory approaches a limit cycle as $t \to \infty$.
3. The trajectory is a limit cycle.

Drazin

Poincare-Bendixson Theorem

- The Poincare–Bendixson Theorem essentially determines all of the possible limiting behaviors of a planar flow (HSD).

- Consider a particular trajectory starting in R. The Poincare-Bendixson Theorem states that there are only two (three) possibilities for that trajectory (Hilborn, p101) (R is a closed bounded subset of the plane):

  1. The trajectory approaches a fixed point of the system as $t \to \infty$.

  2. The trajectory approaches a limit cycle as $t \to \infty$.

  3. The trajectory is a limit cycle.

Hilborn

Poincare-Bendixson Theorem

Summarizing: The trajectory depends sensitively on the initial conditions; it is chaotic; it is attracted to a bounded region in phase space; and [according to eq. (6.4)] the volume of this region contracts to zero. This means that the flow of the three-dimensional Lorenz system generates a set of points whose dimension is less than three; i. e., its volume in three-dimensional space is zero. At first sight, one might think of the next lower integer dimension, two. However, this is forbidden by the *Poincaré—Bendixson* theorem which states that there is no chaotic flow in a bounded region in two-dimensional space. We refer, e. g., to the monograph by Hirsch and Smale (1965) for a rigorous proof of this theorem. However, Fig. 68 makes it plausible that both the continuity of the flow lines and the fact that a line divides a plane into two parts restrict the trajectories in two dimensions so strongly that the only possible attractors for a bounded region are limit cycles or fixed points. The solution to this problem is that the set of points to which the trajectory in the Lorenz system is attracted, the so-called Lorenz attractor, has a Hausdorff dimension which is noninteger and lies between two and three (the precise value is $D = 2.06$). This leads, in a natural way, to the concept of a strange attractor which appears in a large variety of physical, nonlinear systems.

Schuster and Just

An Exception: 2D Chaotic Dixon System

The systems in this chapter were inspired by the work of Dixon *et al.* (1993) in which they transformed a set of three ODEs introduced by Cummings *et al.* (1992) to model the dynamical behavior of the magnetic field of a neutron star. The transformation reduced the system to a two-dimensional flow that nevertheless preserves the chaotic behavior in apparent violation of the Poincaré–Bendixson theorem (Hirsch *et al.*, 2004), which states that the attractor for any smooth two-dimensional bounded continuous-time autonomous system is either a stable equilibrium or a limit cycle.

The system derived by Dixon *et al.* (1993) is given by

$$\begin{aligned} \dot{x} &= \frac{xy}{x^2+y^2} - \alpha x \\ \dot{y} &= \frac{y^2}{x^2+y^2} - \beta y + \beta - 1 \end{aligned} \tag{5.1}$$

and is singular at the origin ($x = y = 0$) and thus does not satisfy the smoothness condition required for the Poincaré–Bendixson theorem to apply. All orbits are attracted to the singularity in finite time, and as a result

Sprott, 2010, Elegant Chaos

An Exception: 2D Chaotic Dixon System

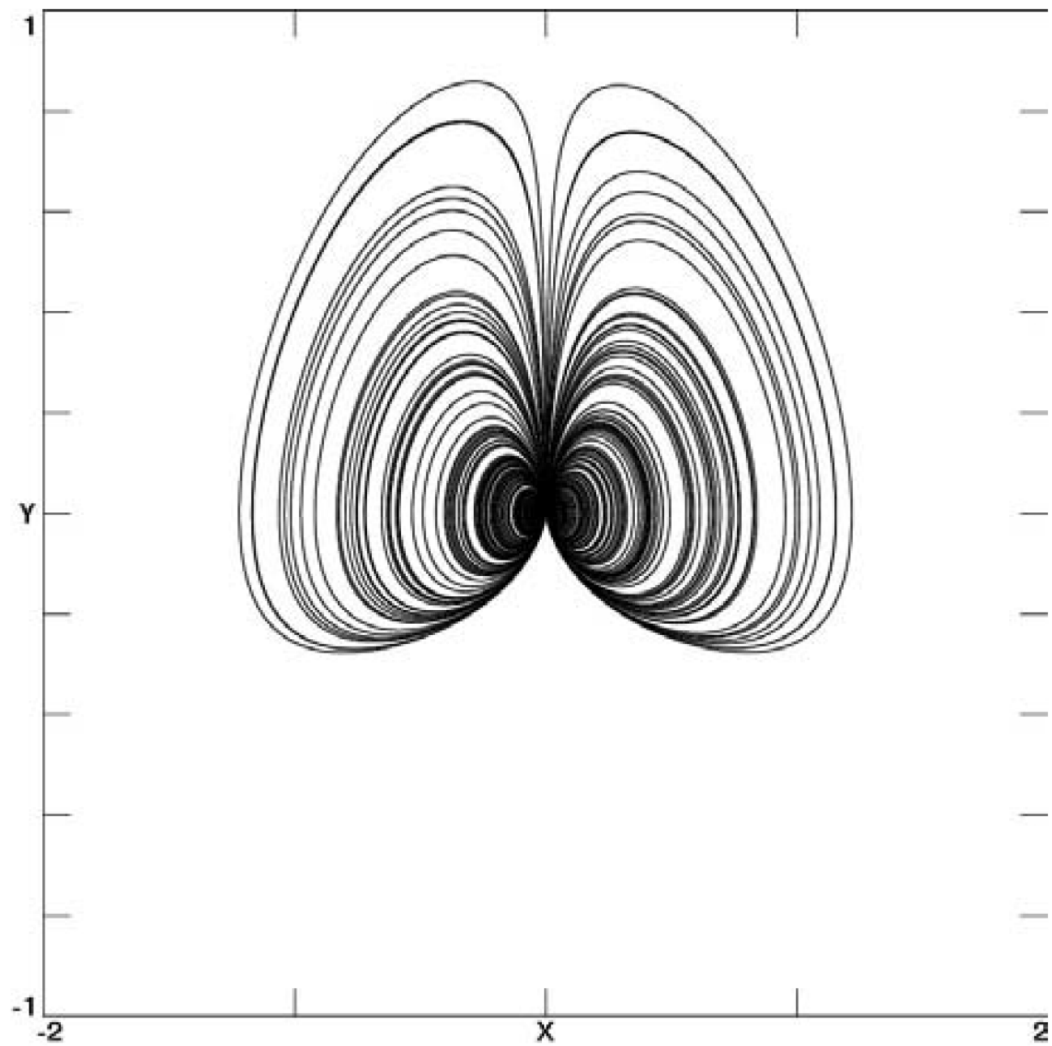


Fig. 5.1 State space plot for the Dixon system in Eq. (5.1) with $(\alpha, \beta) = (0, 0.7)$ for $(x_0, y_0) = (1, 0)$.

An Exception: 2D Chaotic Dixon System

they are sensitive to even the smallest nonzero perturbation, including one arising from the numerical method. The system displays what Dixon *et al.* (1993) call 'S-chaos' (singularity-chaos) for a range of parameters including $\alpha = 0$ and $\beta = 0.7$. The state space plot for this case is shown in Fig. 5.1. A longer calculation of the trajectory would show that it densely fills a region of the xy-plane, and hence it is two-dimensional. Since the resulting plot does not have fractal structure, it is not a proper strange attractor, and the dynamics perhaps should not be considered truly chaotic as pointed out by Alvarez-Ramirez, *et al.* (2005). For the same reason, Lyapunov exponents are not quoted in this chapter because they are not well-defined, and their calculation in the vicinity of the singularity is problematic.

Terminology

- Limit cycle: an isolated closed path

Consider $X' = F(X)$, here $X' = (x', y') = \vec{v}$.

- Gradient system ($\vec{v} = \nabla\phi$) for irrotational flow ($\nabla \times \vec{v} = 0$)
  - ✓ ϕ : velocity potential; $\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right)$
  - ✓ The direction of vector filed is perpendicular to the tangent vector of the contour line of ϕ (i.e., a constant value of the potential function).
- Hamiltonian system for incompressible flow ($\nabla \cdot \vec{v} = 0$)
  - ✓ ψ: streamfunction; $\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \left(\frac{\partial\psi}{\partial y}, \frac{-\partial\psi}{\partial x}\right)$
  - ✓ The direction of the flow is parallel to the tangent vector of the contour line of ψ.

Summary: Eigenvalue Analysis (2D Systems)

- curl (gradient ϕ)=0

$$Gradient\ System$$

$$irrotational\ flow$$

$$\phi: velocity\ potential$$

$$\vec{v} = \nabla\phi = (\phi_x, \phi_y)$$

$$J(\phi_x, \phi_y) = \begin{bmatrix} \phi_{xx} & \phi_{xy} \\ \phi_{yx} & \phi_{yy} \end{bmatrix}$$

- J is a symmetric matrix.
- Its eigenvalues are real.
- There are no center or spiral source/sink

- div (curl F)=0

$$Hamiltonian\ System$$

$$incompressible\ flow$$

$$\psi: streamfunction$$

$$F_o = (0,0,\psi)$$

$$\vec{v} = \nabla \times F_o = (\psi_y, -\psi_x)$$

$$J(\psi_y, -\psi_x) = \begin{bmatrix} \psi_{yx} & \psi_{yy} \\ -\psi_{xx} & -\psi_{xy} \end{bmatrix}$$

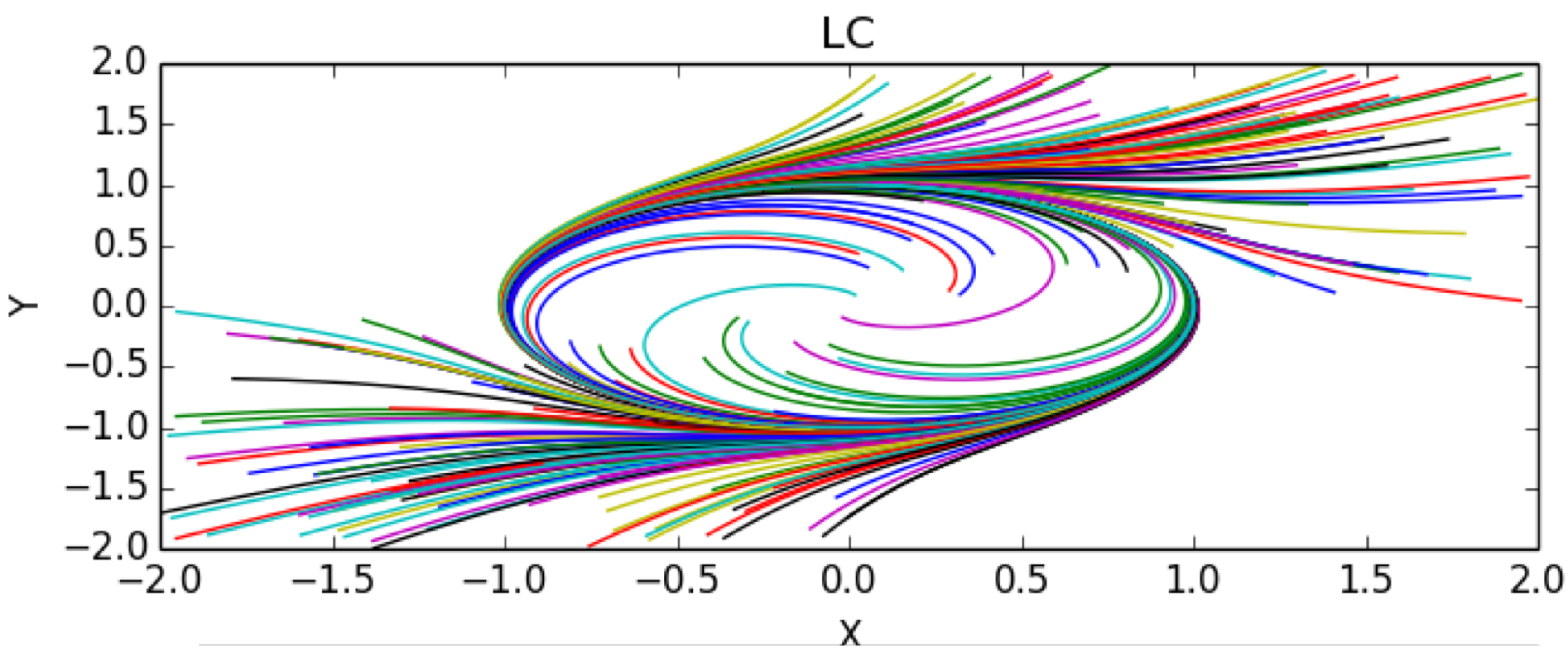
- $Trace = \lambda_1 + \lambda_2 = 0.$
- $Det = \lambda_1 \lambda_2$
- $\lambda_{1,2} = \pm\, \alpha$ (saddle) when Det<0.
- $\lambda_{1,2} = \pm$ iβ (center) when Det>0.

Limit Cycle

- An isolated closed path is called a limit cycle

Limit Cycle

Definition 5. A nontrivial† closed trajectory with at least one other trajectory spiraling into it (as time approaches plus or minus infinity) is called a **limit cycle.**



Poincare-Bendixson Theorem

- The Poincare–Bendixson Theorem essentially determines all of the possible limiting behaviors of a planar flow (HSD).

- Consider a particular trajectory starting in R. The Poincare-Bendixson Theorem states that there are only two (three) possibilities for that trajectory (Hilborn, p101) (R is a closed bounded subset of the plane):

  1. The trajectory approaches a fixed point of the system as $t \to \infty$.
  2. The trajectory approaches a limit cycle as $t \to \infty$.
  3. The trajectory is a limit cycle.

Hilborn

Summary

- Limit cycle: An isolated closed path is called a limit cycle
- Gradient system ($\vec{v} = \nabla\phi$) for irrotational flow ($\nabla\times\vec{v} = 0$)
- Hamiltonian system for incompressible flow ($\nabla \cdot \vec{v} = 0$)
- No closed orbits when the system meets one of the following conditions:
  I. existence of a Laypunov function;
  II. zero curl ($\nabla\times\vec{v} = 0$)
  III. positive ($\nabla \cdot \vec{v} > 0$) or negative ($\nabla \cdot \vec{v} < 0$) divergence
- Poincare-Bendixson Theorem (2D, bounded, autonomous)
  - There are only two (three) possibilities for that trajectory
  - The trajectory approaches a fixed point of the system as $t \to \infty$.
  - The trajectory approaches a limit cycle as $t \to \infty$.
  - The trajectory is the limit cycle.
  - Chaotic trajectories (in a bounded system) cannot occur in a state space of two dimensions (within differential equations).