Homework 2
Math Modeling
Math 636
Stephen Giang RedID: 823184070

Problem 1: The dynamics of the number of photons $n(t)$ in a laser field is given by

$$\frac{dn}{dt} = (GN_0 - k)n - \alpha G n^2$$

where G is the gain coefficient for simulated emission, k is the decay rate due to photon loss by scattering, α is the rate at which atoms drop back to their ground states and in the absence of a laser field, the number of excited atoms is kept fixed at N_0.

(i) Find the equilibrium points of the system and comment on its stability.

$$\begin{aligned} 0 &= (GN_0 - k)n - \alpha G n^2 \\ 0 &= n((GN_0 - k) - \alpha G n) \end{aligned}$$

Notice the equilibrium points of the system:

$$n_1 = 0 \qquad n_2 = \frac{GN_0 - k}{\alpha}$$

Let the following be true:

$$f(n) = (GN_0 - k)n - \alpha G n^2 \qquad \frac{df}{dn} = (GN_0 - k) - 2\alpha G n$$

Such that the stability of each point can be found:

(a) $n = 0$

$$\left.\frac{df}{dn}\right|_{n=0} = GN_0 - k$$

We get a stable point if $GN_0 > k$ and an unstable point if $GN_0 < k$.

(b) $n = \frac{GN_0 - k}{\alpha}$

$$\left.\frac{df}{dn}\right|_{n=\frac{GN_0-k}{\alpha}} = GN_0 - k - 2G(GN_0 - k) = (1 - 2G)(GN_0 - k)$$

We get a stable point if $GN_0 > k$ and $G < \frac{1}{2}$, or $GN_0 < k$ and $G > \frac{1}{2}$.

We get an unstable point if $GN_0 > k$ and $G > \frac{1}{2}$, or $GN_0 < k$ and $G < \frac{1}{2}$.

(ii) Show that the system undergoes a transcritical bifurcation at $N_0 = k/G$.

Notice that when $N_0 = k/G$, we get:

$$n_1 = n_2 = 0$$

Notice that because we get a solution of a multiplicity of 2, the system undergoes a transcritical bifurcation.

Problem 2: The following system of differential equations describe the motions of a certain pendulum:

$$\frac{d\theta}{dt} = y,$$
$$\frac{dy}{dt} = -5\sin\theta - \frac{9}{13}y,$$

where θ is the angle between the rod and the downward vertical direction and $\frac{d\theta}{dt}$ is the speed at which the angle changes. Find all the steady state solutions for the system. Also, identify if the steady state solutions are stable.

$$0 = y$$
$$0 = -5\sin\theta - \frac{9}{13}y$$
$$0 = -5\sin\theta$$
$$\theta = n\pi \qquad n \in \mathbb{Z}$$

We reach a steady state when $\theta = n\pi$ for $n \in \mathbb{Z}$.

Notice the Jacobian of the system:

$$J(\theta, y) = \begin{pmatrix} 0 & 1 \\ -5\cos\theta & -\frac{9}{13} \end{pmatrix}$$

Finding the eigenvalues at the given solutions will show us stability:

$$|J(n\pi, 0) - \lambda I| = \begin{pmatrix} 0 - \lambda & 1 \\ -1^n(5) & \frac{-9}{13} - \lambda \end{pmatrix} = \lambda^2 + \frac{9}{13}\lambda + (-1)^n(5) = 0$$

Thus, we get:

$$\lambda = \frac{-(9/13) \pm \sqrt{(9/13)^2 - 4((-1)^n(5))}}{2} = \frac{-(9/13) \pm \sqrt{(9/13)^2 + ((-1)^{n+1}(20))}}{2}$$

If n is even, we get complex eigenvalues with negative real parts, such that (n, θ) would be stable solutions.

If n is odd, we get real eigenvalues that are postive and negative, such that (n, θ) would be unstable solutions.

Problem 3: A hypothetical reaction in the study of isothermal autocatalytic reactions was considered by Gray and Scott (1985), whose kinetics in dimensionless form are given as follows:

$$\frac{dx}{dt} = a(1-x) - xy^2,$$
$$\frac{dy}{dt} = xy^2 - (a+k)y,$$

where a and k are positive parameters. Show that the saddle node bifurcation occurs at $k = -a \pm \frac{\sqrt{a}}{2}$.

$$0 = a(1-x) - xy^2$$
$$0 = xy^2 - (a+k)y$$
$$0 = y(xy - (a+k))$$

We get the following solutions from $y = 0$:

$$y = 0 \text{ with } x = 1$$

Solving for the other solution we get:

$$y = \frac{a+k}{x}$$
$$0 = a - ax - x\left(\frac{a+k}{x}\right)^2$$
$$0 = a - ax - \frac{(a+k)^2}{x}$$
$$0 = ax - ax^2 - (a+k)^2$$
$$0 = ax^2 - ax + (a+k)^2$$

Such that we get:

$$x = \frac{a \pm \sqrt{a^2 - 4a(a+k)^2}}{2a} \text{ with } y = \frac{a+k}{x}$$

Evaluating at $k = -a \pm \frac{\sqrt{a}}{2}$, we get:

$$x = \frac{a}{2a} = \frac{1}{2} \text{ with } y = \pm\sqrt{a}$$

Notice how the double solution became a single solution of multiplicity 2 at $k = -a \pm \frac{\sqrt{a}}{2}$, which shows the saddle node bifurcation.

Problem 4: The favorite food of the tiger shark is the sea turtle. A two-species prey-predator model is given by

$$\frac{dP}{dt} = P(a - bP - cS),$$
$$\frac{dS}{dt} = S(-k + \lambda P),$$

where P is the sea turtle, S is the shark and $a, b, c, k, \lambda > 0$.

(i) Let $b = 0$ and the value of k is increased. Ecologically, what is the interpretation of increasing k and what is its effect on the non-zero equilibrium populations of sea turtles and sharks?

If we increase k, we can see a decrease in the rate of shark population over time. We can also see that the turtle population increases as seen in part (ii)(a).

(ii) Obtain all the equilibrium solutions for $b = 0$ and $b \neq 0$.

(a) $b = 0$

$$0 = P(a - cS),$$
$$0 = S(-k + \lambda P),$$

Such that we get:

$$P = 0 \text{ with } S = 0 \qquad S = \frac{a}{c} \text{ with } P = \frac{k}{\lambda}$$

(b) $b \neq 0$

$$0 = P(a - bP - cS),$$
$$0 = S(-k + \lambda P),$$

One solution, we get is:

$$P = 0 \text{ with } S = 0$$

Notice to find the other solutions:

$$\begin{aligned}
0 &= a - bP - cS \\
S &= \frac{a - bP}{c} \\
0 &= \frac{-k(a - bP)}{c} + \frac{\lambda P(a - bP)}{c} \\
&= -k(a - bP) + \lambda P(a - bP) \\
&= -ka + kbP + a\lambda P - b\lambda P^2 \\
&= ka - (kbP + a\lambda P) + b\lambda P^2
\end{aligned}$$

Such that we get the other solutions to be:

$$S = \frac{a - bP}{c} \text{ with } P = \frac{(a\lambda + bk) \pm \sqrt{(a\lambda + bk)^2 - 4abk\lambda}}{2b\lambda}$$

(iii) Obtain the linearized system of differential equations about the equilibrium point $P^* = \frac{k}{\lambda}$ and $S^* = \frac{a}{c} - \frac{bk}{c\lambda} > 0$, which can be put in the form

$$\frac{dP_1}{dt} = \frac{k}{\lambda}(-bP_1 - cS_1),$$
$$\frac{dS_1}{dt} = \lambda P_1(\frac{a}{c} - \frac{bk}{c\lambda}).$$

(iv) Obtain the condition(s) for which the linearized system is stable.

(v) Draw the solution curves in the phase plane with $a = 0.5, b = 0.5, c = 0.01, k = 0.3, \lambda = 0.01$. What do you expect to happen to the dynamics of the model if $c = 0$?

Problem 5: The spruce budworm model

$$\frac{dN(t)}{dt} = r_B N(t)\left[1 - \frac{N(t)}{K_B}\right] - B\frac{N(t)^2}{A^2 + N(t)^2}$$

can be reduced to the following scaled equation (see HW-1):

$$\frac{du}{d\tau} = ru\left(1 - \frac{u}{q}\right) - \frac{u^2}{1+u^2}$$

Perform the stability analysis and the bifurcation analysis with the parameter r fixed and the parameter q as a bifurcation parameter. Also, plot the bifurcation diagram.

Notice the equilibrium points:

$$\begin{aligned}
0 &= ru\left(1 - \frac{u}{q}\right) - \frac{u^2}{1+u^2} \\
0 &= u\left(r\left(1 - \frac{u}{q}\right) - \frac{u}{1+u^2}\right) \\
0 &= u \\
0 &= r\left(1 - \frac{u}{q}\right) - \frac{u}{1+u^2} \\
0 &= r(1+u^2)\left(1 - \frac{u}{q}\right) - u \\
0 &= (r + ru^2)\left(1 - \frac{u}{q}\right) - u \\
0 &= r - \frac{r}{q}u + ru^2 - \frac{r}{q}u^3 - u \\
0 &= r - \left(\frac{r}{q} + 1\right)u + ru^2 - \frac{r}{q}u^3 \\
0 &= \frac{r}{q}u^3 - ru^2 + \left(\frac{r}{q} + 1\right)u - r
\end{aligned}$$

Problem 6: Matured insects lay eggs with per capita rate of r, which survive and hatch to immature population with survival rate $e^{-\psi x}$, where x is a number of eggs. The immature insects become matured with per capita maturation rate γ. Assume that δ and μ are per capita mortality rate of immature and mature insect populations, respectively.

(i) Develop a patchy model with two patches, one representing immature insects and another representing mature insects.

(ii) Consider a control mechanism which results in the reduction of the egg laying rate, i.e., $r \to (1-\theta)r$ with the control level θ. Perform bifurcation analysis of the model to identify the level of control mechanism for extinction and for persistence of the insect population.