Lecture 3:

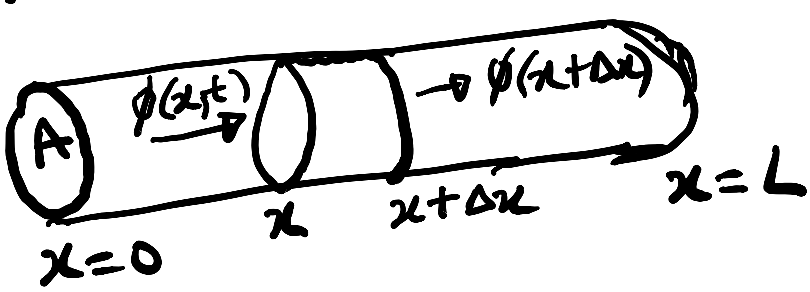
Outline

1) Heat Equation
   - Derivation
   - Temperature & Heat equation

2) Heat equation: Equilibrium temperature distribution
   - Dirichlet
   - Insulated

Derivation of heat Conduction in a One-dimensional Rod

Consider a rod of constant cross-sectional area A of length L



Let $e(x,t)$ represent the thermal energy density.

That is the amount of thermal energy per unit volume

We assume that all thermal quantities are constant across a section; That is the rod is one-dimensional.

A way to accomplish this is to <u>insulate</u> perfectly the lateral surfaces

<u>Heat energy</u>

In the thin slice of the rod, if Δx is very small then heat energy $= e(x,t) A \Delta x$

<u>Conservation of heat energy</u>

rate of change of heat energy in time = heat energy flowing across boundaries per unit time + heat energy generated inside per unit time

For the small slice, the rate of change of heat energy is

$$\frac{\partial}{\partial t}\left[e(x,t)\, A\, \Delta x\right]$$

<u>Heat flux</u>: The amount of thermal energy per unit time flowing to the right per unit surface area is defined as $\phi(x,t)$

If $\phi(x,t) < 0 \Rightarrow$ heat flows to the left.

Heat energy flowing across the boundary of the slice per unit time is

$$\phi(x,t)A - \phi(x+\Delta x,t)A$$

<u>Heat sources</u>

$Q(x,t)$ = heat energy per unit volume generated per unit time.

$Q(x,t)$ is approximately constant in space for a thin slice.

Therefore, the total thermal energy generated per time in the thin slice is approximately:

$$Q(x,t)\, A\, \Delta x$$

Conservation of heat energy on the thin slice is

$$\frac{\partial}{\partial t}\left[e(x,t)\, A\Delta x\right] \approx \phi(x,t)A - \phi(x+\Delta x,t) + Q(x,t)A\Delta x$$

divide through $A\Delta x$ and take the limit as $\Delta x \to 0$

$$\frac{\partial e}{\partial t} = \lim_{\Delta x \to 0} \frac{\phi(x,t) - \phi(x+\Delta t,t)}{\Delta x} + Q(x,t)$$

$$\frac{\partial e}{\partial t} = \frac{-\partial \phi}{\partial x} + Q \qquad (1)$$

Temperature and specific heat

Let $u(x,t)$ — be the temperature of a material

$c(x)$ — be the specific heat of a material

(i.e the heat energy required to raise a unit mass of a material a unit of temperature)

$\rho(x)$ — Mass density (per unit volume)

Based on the above:

thermal energy, $e(x,t) = c(x)\rho(x)u(x,t)$ (2)

Fourier's Law: Heat flows proportional to the negative gradient of the temperature.

Therefore
$$\phi(x,t) = -K_0(x) \frac{\partial u}{\partial x}(x,t)$$
③

From the heat conduction equation ①, we substitute equation ② and ③ to obtain the heat equation:
$$c(x)\rho(x) \frac{\partial u}{\partial t}(x,t) = \frac{\partial}{\partial x}\left(K_0(x) \frac{\partial u}{\partial x}(x,t)\right) + Q(x,t)$$

If $c(x)$, $\rho(x)$, $K_0(x)$ are constant and there is no heat source or sink (i.e. $Q(x,t)=0$)

then the heat equation has the form
$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$
where $k = \frac{K_0}{c\rho}$ is thermal diffusivity

Heat equation

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}, \quad t > 0, \quad x = (0, L)$$

we need initial conditions and boundary conditions.

Boundary Conditions:

- Dirichlet or prescribed

  e.g $u(0,t) = U_0(t)$

- Neumann (e.g insulated BC)

  e.g $\frac{\partial u}{\partial x}(0,t) = 0$

- Neumann: (e.g prescribed flux)

  e.g. $-K_0 \frac{\partial u}{\partial x}(0,t) = \phi(t)$

- Robin or mixed BC

  e.g Newton Cooling

Note:

$\frac{\partial u}{\partial x} = u_x$

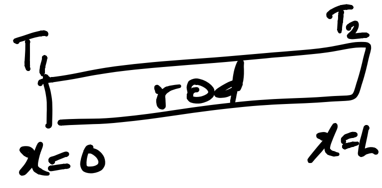
$$K_0 u_x(0,t) = H(u(0,t) - u_E(t))$$

Heat equation: Equilibrium temperature distribution with Dirichlet B.C.

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad u = \text{temperature}$$

with initial conditions and Dirichlet boundary conditions

$$u(x,0) = f(x), \quad u(0,t) = T_1(t)$$
$$u(L,t) = T_2(t)$$



Suppose that the Boundary Conditions

are constant.

$$T_1(t) = T_1 \quad \text{and} \quad T_2(t) = T_2$$

Examine the steady state or equilibrium solution; which implies that

$$\frac{\partial u}{\partial t} = 0 \,, \quad \text{so} \quad u(x,t) = u(x)$$

The equilibrium heat equation (ODE) problem reduces to

$$\frac{d^2u}{dx^2} \quad \text{with } u(0) = T_1 \text{ and } u(L) = T_2$$

$c_2 = T_1$

$T_2 = c_1 L + T_1$

$c_1 = \dfrac{T_2 - T_1}{L}$

The solution of this ODE is

$$u(x) = c_1 x + c_2$$

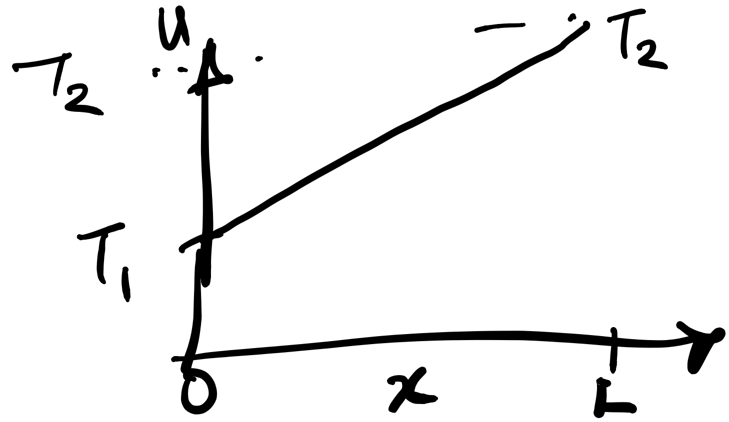
Apply B.Cs we obtain that

$$c_2 = T_1 \quad \text{and} \quad c_1 = \frac{(T_1 - T_2)}{L}$$

$$\boxed{u(x) = -\frac{(T_1 - T_2)}{L}x + T_1}$$

<u>equilibrium</u> solution for <u>the</u> heat equation with fixed temperature at each end (i.e Dirichlet BC)

Thus, <u>the</u> temperature equilibrate to a linear function connecting the two end temperatures



<u>Heat equation</u> : Equilibrium distribution of temperature with <u>insulated</u> <u>BC</u>

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad (4)$$

Note
IC = initial Condition

BC = boundary Condition

IC: $u(x,0) = f(x)$

BC: $u_x(0,t)=0$ and $u_x(L,t)=0$

As before, the equilibrium Problem is $\frac{d^2u}{dt^2}=0$ with $u'(0)=0$ and $u'(L)=0$

The solution of the ODE is

$$u(x) = c_1 x + c_2$$

But $u'(x) = c_1$

$\Rightarrow u'(0) = 0 = c_1 \quad \therefore c_1 = 0$

The boundary condition gives no information on c_2

Therefore the ODE has the solution $\boxed{u(x) = c_2}$

So what is c_2?

Since the lateral sides and the ends are insolated then the thermal energy is conserved.

i.e $$\frac{d}{dt}\int_0^L c\rho\, u(x)\, dx = -K_0 \frac{\partial u}{\partial x}(0,t) + K_0 \frac{\partial u}{\partial x}(L,t) = 0$$

Obtained by integrating equation 4 from $x=0$ to $x=L$

The initial thermal energy is

$$c\rho \int_0^L f(x)\, dx = c\rho \int_0^L u(x)\, dx = c\rho \int_0^L c_2\, dx$$
$$= c\rho L c_2$$

It follows that

$$u(x) = c_2 = \frac{1}{L}\int_0^L f(x)\,dx$$

i.e the average of the initial temperature distribution.