Quiz 4
Ordinary Differential Equations
Math 537
Stephen Giang RedID: 823184070

Problem 1: Consider the Logistic equation:

$$\frac{dX}{dt} = rX(1-X). \tag{1.1}$$

(a) Assume a time step Δt and apply the Euler method to derive a discrete equation where X_{n+1} can be computed from X_n.

$$\boldsymbol{X_{n+1} = X_n + \Delta t\ (rX_n(1-X_n))}$$

(b) Introduce a new variable Y and transform the above discrete equation into the following equation:

$$Y_{n+1} = \rho Y_n(1-Y_n). \tag{1.2}$$

Express Y_n in terms of X_n and find ρ.

Notice the following:

$$\begin{aligned}
X_{n+1} &= (1+\Delta t\, r)X_n - \Delta t\, r X_n^2 \\
&= (1+\Delta t\, r)X_n \left(1 - \frac{\Delta t\, r}{1+\Delta t\, r} X_n\right) \\
&= \frac{(1+\Delta t\, r)^2}{\Delta t\, r} \frac{\Delta t\, r}{1+\Delta t\, r} X_n \left(1 - \frac{\Delta t\, r}{1+\Delta t\, r} X_n\right)
\end{aligned}$$

Let the following be true:

$$Y_n = \frac{\Delta t\, r}{1+\Delta t\, r} X_n, \qquad \text{such that} \qquad X_{n+1} = \frac{1+\Delta t\, r}{\Delta t\, r} Y_{n+1}$$

So we get:

$$\begin{aligned}
\frac{1+\Delta t\, r}{\Delta t\, r} Y_{n+1} &= \frac{(1+\Delta t\, r)^2}{\Delta t\, r} Y_n\,(1-Y_n) \\
Y_{n+1} &= (1+\Delta t\, r) Y_n (1-Y_n)
\end{aligned}$$

Thus we get the final solutions being:

$$\boldsymbol{Y_n = \frac{\Delta t\, r}{1+\Delta t\, r} X_n \qquad \rho = 1 + \Delta t\, r}$$

Eq. (1.2) is called the Logistic map that possesses chaotic solutions for large values of ρ

Problem 2: Consider the general first-order ODE:

$$x' = f(x). \tag{2}$$

When both f and f' are zero at the critical point, the stability is determined by the sign of the first non-vanishing higher derivatives. Apply Taylor series expansions and provide simple functions $f(x)$ to illustrate the following:

(a) If the first non-vanishing higher derivative is even (e.g., f''), the point is a saddle point, attracting on one side but repelling on the other.

(b) If that derivative is odd, it follows the same sign rules as f'.

Notice the following:

$$x' = f(x) = f(x_c) + f'(x_c)(x - x_c) + f''(x_c)\frac{(x-x_c)^2}{2} + f'''(x_c)\frac{(x-x_c)^3}{6} + ...$$

Notice the following examples:

(a) $f(x) = x^2$, $x_c = 0$

$$x' = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{6} + \cdots + = x^2$$

Notice for $x_c < 0$, $x' > 0$ and $x_c > 0$, $x' > 0$. Notice for $x_c < 0$, the phase portrait is attracted to the critical point. Notice for $x_c > 0$, the phase portrait is repelled to the critical point. Thus we get a saddle point for $x' = x^2$, where the first non-vanishing derivative is even (2^{nd}).

Notice for all functions with the first non-vanishing higher derivative being even, we get:

$$x' = f^{(2n)}(x_c)\frac{(x-x_c)^{2n}}{(2n)!} + \cdots, \text{ where } f^{(2n)}(x_c) = C, \text{ and } n \in \mathbb{Z}\backslash\{0,1\}$$

Now we can see that the exponent on $x - x_c$ being an even number, means that for all values of $x \neq x_c$, we get that $\frac{(x-x_c)^{2n}}{(2n)!} > 0$. Thus we get both sides of the critical point being the same sign, meaning attracting one way and repelling the other, which is known as a saddle point.

(b) $f(x) = x^3$, $x_c = 0$

$$x' = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{6} + \cdots + = x^3$$

Notice for $x_c < 0$, $x' < 0$ and $x_c > 0$, $x' > 0$, we get a source. For $f'''(x_c) > 0$, we get an unstable source, and for $f'''(x_c) < 0$, we get a stable sink. This follows the same sign rules as f'.

Notice for all functions with the first non-vanishing higher derivative being odd, we get:

$$x' = f^{(2n+1)}(x_c)\frac{(x-x_c)^{2n+1}}{(2n+1)!} + \cdots, \text{ where } f^{(2n+1)}(x_c) = C, \text{ and } n \in \mathbb{Z}\backslash\{0,1\}$$

Now we can see that the exponent on $x - x_c$ being an odd number, means that for values of $x < x_c$, we get that $\frac{(x-x_c)^{2n+1}}{(2n+1)!} < 0$ and for values of $x > x_c$, we get that $\frac{(x-x_c)^{2n+1}}{(2n+1)!} > 0$. Thus we get each side being opposite signs of each other. If $f^{(2n+1)}(x_c) < 0$, we get for $x < x_c$, $x' > 0$ and for $x > x_c$, $x' < 0$. We get the reverse for $f^{(2n+1)}(x_c) > 0$. Thus this follows the same sign rules as f'