Quiz 11
Differential Equations
Math 337
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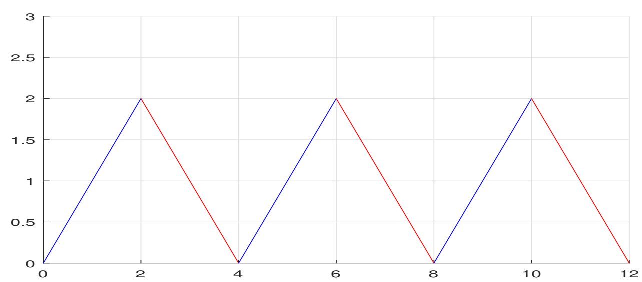
Problem 1: Consider the periodic function $f(t)$ defined as follows:

$$f(t) = \begin{cases} t, & 0 \le t < 2 \\ 4-t, & 2 \le t < 4 \end{cases}, \qquad \text{with} \qquad f(t+4) = f(t)$$

Sketch a graph of this function for $t \in [0, 12]$. Write this function as a window function, $f_4(t)$, (Slide 22) using step functions. Use our Theorem (Slide 23) to obtain the Laplace Transform, $\mathcal{L}[f(t)] = F(s)$. This expression simplifies by dividing out a common factor in the numerator and denominator. Follow the example in the Lecture Slides to express the resulting rational expression in terms of a geometric series.



$$\begin{aligned}
f_4(t) &= t[u_0(t) - u_2(t)] + (4-t)[u_2(t) - u_4(t)] \\
&= tu_0(t) - tu_2(t) + 4u_2(t) - tu_2(t) + (t-4)u_4(t) \\
&= t - 2(t-2)u_2(t) + (t-4)u_4(t)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}[f(t)] &= \frac{1}{1-e^{-4s}}\mathcal{L}[f_4(t)] \\
&= \frac{1}{1-e^{-4s}}\left(\frac{1}{s^2} - \frac{2e^{-2s}}{s^2} + \frac{e^{-4s}}{s^2}\right) \\
&= \frac{1}{1-e^{-4s}}\left(\frac{e^{-4s} - 2e^{-2s} + 1}{s^2}\right) \\
&= \frac{1}{(1-e^{-2s})(1+e^{-2s})}\left(\frac{(1-e^{-2s})^2}{s^2}\right) \\
&= \frac{1-e^{-2s}}{s^2}\left(\frac{1}{1+e^{-2s}}\right) \\
&= \frac{1-e^{-2s}}{s^2}\left(1 - e^{-2s} + e^{-4s} + \ldots + (-1)^n e^{-2ns}\right) \\
&= \frac{1-e^{-2s}}{s^2}\sum_{n=0}^{\infty}(-1)^n e^{-2ns}
\end{aligned}$$

Problem 2: Solve the following initial value problem with Laplace transforms:

$$y' + 4y = f(t), \qquad y(0) = 2$$

where $f(t)$ is the periodic function given in the previous problem above. Show all the steps needed to find $\mathcal{L}[y(t)] = Y(s)$, then show the necessary partial fractions decomposition (PFD) required to make your elements appear in the Laplace table. Finally, invert Y (s) to find your solution

Notice the following:

$$\begin{aligned} \mathcal{L}[y' + 4y] = sY(s) - y(0) + 4Y(s) &= \frac{1 - e^{-2s}}{s^2} \sum_{n=0}^{\infty} (-1)^n e^{-2ns} \\ = (s+4)Y(s) - 2 &= \frac{1 - e^{-2s}}{s^2} \sum_{n=0}^{\infty} (-1)^n e^{-2ns} \end{aligned}$$

Notice the partial fractions decomposition:

$$\begin{aligned} \frac{1}{s^2(s+4)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+4} \\ 1 &= (A+C)s^2 + (4A+B)s + 4B \end{aligned}$$

So we get $B = \frac{1}{4}$, $A = \frac{-1}{16}$, and $C = \frac{1}{16}$

$$\begin{aligned} Y(s) &= \frac{2}{s+4} + \left(\frac{-1}{16s} + \frac{1}{4s^2} + \frac{1}{16(s+4)} \right) \left(1 - e^{-2s}\right) \sum_{n=0}^{\infty} (-1)^n e^{-2ns} \\ &= \frac{2}{s+4} + \left(\frac{-1}{16s} + \frac{1}{4s^2} + \frac{1}{16(s+4)} \right) \left(1 - e^{-2s}\right) \sum_{n=0}^{\infty} (-1)^n e^{-2ns} \\ y(t) &= 2e^{-4t} + \left(\frac{-1}{16} + \frac{t}{4} + \frac{e^{-4t}}{16} \right) (1 - u_2(t)) + \sum_{n=1}^{\infty} (-1)^n u_k(t) \left(\frac{-1}{16} + \frac{t}{4} + \frac{e^{-4t}}{16} \right) \end{aligned}$$

Problem 3: Solve the following initial value problem with Laplace transforms:

$$y'' + 4y' + 5y = \frac{2t}{\pi}(\delta(t-\pi) - \delta(t-2\pi)), \qquad y(0) = 0, \quad y'(0) = 2$$

Use the Laplace table to find your solution. Use the computer to create a graph of your solution for $t \in [0, 15]$. What is the limiting solution for large t?

Notice the following:

$$\begin{aligned}
\mathcal{L}\left[\frac{2t}{\pi}(\delta(t-\pi) - \delta(t-2\pi))\right] &= 2\mathcal{L}\left[\frac{t}{\pi}\delta(t-\pi)\right] - 4\mathcal{L}\left[\frac{t}{2\pi}\delta(t-2\pi)\right] \\
&= 2e^{-\pi s} - 4e^{-2\pi s}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}\left[y'' + 4y' + 5y\right] &= s^2Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 5Y(s) \\
&= (s^2 + 4s + 5)Y(s) - 2
\end{aligned}$$

$$Y(s) = \frac{2}{(s+2)^2+1} + \frac{2e^{-\pi s}}{(s+2)^2+1} - \frac{4e^{-2\pi s}}{(s+2)^2+1}$$

$$y(t) = 2e^{-2t}\sin(t) + 2u_\pi(t)e^{-2(t-\pi)}\sin(t-\pi) - 4u_{2\pi}(t)e^{-2(t-2\pi)}\sin(t-2\pi)$$

The limiting solution:

$$\lim_{t\to\infty} y(t) = 0$$