Today: 10/24. 3.2 & 3.3

Extreme & Intermediate Value Thms.

3.2 **Thm:** Compactness - Continuity Theorem.

Let $S \subseteq \mathbb{R}$ be sequentially compact.

Suppose $f: S \to \mathbb{R}$ is continuous.

Then $f(S) = \operatorname{im}(f) = \{ f(x) \mid x \in S \}$ is sequentially compact.

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proof: Let $\{y_n\}_{n \geq 1}^{\infty} \subseteq f(S)$.

So for each n, $\exists x_n \in S$ s.t. $f(x_n) = y_n$.

So $\{x_n\} \subseteq S$, sequentially compact so $\exists \{x_{n_k}\} \subseteq S$

s.t. $x_0 = \lim_{k \to \infty} x_{n_k} \in S$. Consider $\{y_{n_k}\} = \{f(x_{n_k})\} \subseteq f(S)$.

Since f is continuous at D,

$$\lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} y_{n_k} = f(D) \in f(S). \quad \boxed{\text{E}}$$

<u>Lemma</u>: Suppose $S \subseteq \mathbb{R}$ and S is sequentially compact. Then S is bounded.

<u>proof</u>: Suppose S is unbounded (above, WLOG).

For any $n \in \mathbb{N}$, $\exists\, x_n \in S$ s.t. $x_n > n$.

Let $\{x_{n_k}\}$ be any subsequence.

Let $M \in \mathbb{R}$. $\exists\, n_k \in \mathbb{N}$ s.t. $n_k > M$.

In particular, let $k > M$ and we have

$n_k \geq k > M$. Thus $x_{n_k} > n_k \geq k > M$.

So $\{x_{n_k}\}_{k=1}^{\infty}$ is unbounded. Thus $\{x_{n_k}\}_{k=1}^{\infty}$ does not converge. $\boxed{\text{E}}$

$\Rightarrow \Leftarrow$

Lemma: If $S \subseteq \mathbb{R}$ is sequentially compact, then

$\sup(S) \in S$ and $\inf(S) \in S$.

I.e. $\max(S)$ and $\min(S)$ exist.

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proof: Suppose $S \subseteq \mathbb{R}$ is sequentially compact.

By previous lemma, S is bounded thus

$M := \sup(S)$ exists. Fix $n \geq 1$, note that

$M - \frac{1}{n}$ is not an upper bound for S.

Thus $\exists\, x_n \in S$ s.t. $M - \frac{1}{n} < x_n \leq M$.

Since $\lim_{n\to\infty}(M - \frac{1}{n}) = M = \lim_{n\to\infty} M$, $\lim_{n\to\infty} x_n = M$

by the squeeze thm. Since S is sequentially

compact $\lim_{n\to\infty} x_n = M \in S$.

Extreme Value Theorem: Thm 3.9

Suppose $f : [a,b] \to \mathbb{R}$ is continuous. Then f attains both max & min values.

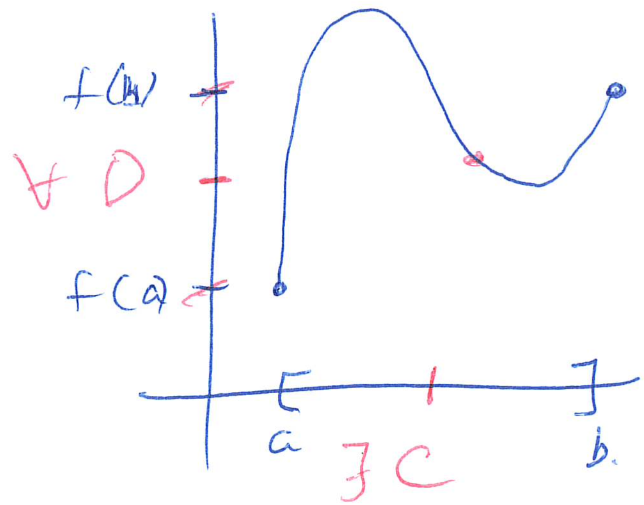
Proof: Since $[a,b]$ is sequentially compact, we know $f([a,b])$ is sequentially compact.

Thus $\exists y \in f([a,b])$ s.t. $\forall z \in f([a,b])$, $z \le y$.

Thus $\exists x^* \in [a,b]$ s.t. $y = f(x^*)$ and f attains a max value. $\blacksquare$.

3.3 Intermediate Value Theorem:

Picture:



Suppose $f: [a,b] \to \mathbb{R}$ is continuous.

If D is strictly between $f(a)$ & $f(b)$, then

$\exists\, C$ s.t. $a < C < b$ and $f(C) = D$.

proof: "BISECTION METHOD"

Suppose D is strictly between $f(a)$ & $f(b)$.

W.L.O.G. assume $f(a) < D < f(b)$.

Let $a_1 = a$ and $b_1 = b$. Consider $m_1 = \frac{a+b}{2}$.

Case 1: Suppose $f(m_1) = D$. done.

case 2: Suppose $f(m_1) < D$.

Let $a_2 = m_1$ and $b_2 = b_1$.

Then $f(a_2) < D < f(b_2)$

and $b_2 - a_2 = \frac{b_1 - a_1}{2}$. and $(a_2, b_2) \subseteq (a_1, b_1)$.

~~By the same process, given~~

Case 3: Suppose $f(m_1) > D$. Then let $a_2 = a_1$ and $b_2 = m_1$

Then $f(a_2) < D < f(b_2)$ and $b_2 - a_2 = \frac{b_1 - a_1}{2}$ and $(a_2, b_2) \subseteq (a_1, b_1)$.

Consider the n^{th} step, $f(a_n) < D < f(b_n)$.

Construct m_n and redefine $(a_{n+1}, b_{n+1}) \subseteq (a_n, b_n)$

and $b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}$ and $f(a_{n+1}) < D < f(b_{n+1})$.

in the same way as case $n=1$.

We (you) can show by induction that

$$b_n - a_n = \frac{b-a}{2^{n-1}}.$$

Thus $\lim_{n\to\infty}(b_n - a_n) = \lim_{n\to\infty} 2(b-a)\left(\frac{1}{2}\right)^n = 0.$

Thus $\exists c \in \mathbb{R}$ s.t.

$\forall i$, $c \in (a_i, b_i)$ and $c = \lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i$.

Since f is continuous at c,

$$f(c) = \lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n).$$

Since $\forall n$, $f(a_n) < D$ and $f(b_n) > D$,

$$\lim_{n\to\infty} f(a_n) \le D \quad \text{and} \quad \lim_{n\to\infty} f(b_n) \ge D.$$

Thus $f(c) \le D$ and $f(c) \ge D$.

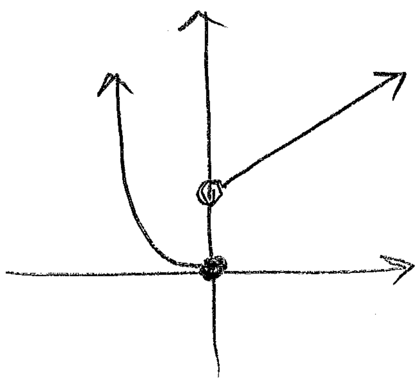
So $\boxed{f(c) = D.}$

TMT

Today: 10/29. Continuity Examples

3.1 (3) Let
$$f(x) = \begin{cases} x^2 & \text{if } x \le 0 \\ x+1 & \text{if } x > 0. \end{cases}$$

At what points is f continuous? JUSTIFY.



$\boxed{\text{Discontinuous at } x=0}$

$\exists \{x_n\} \subseteq \mathbb{R}$ s.t. $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} f(x_n) \ne f(0)$

Let $x_n = \frac{1}{n}$ for $n \ge 1$.

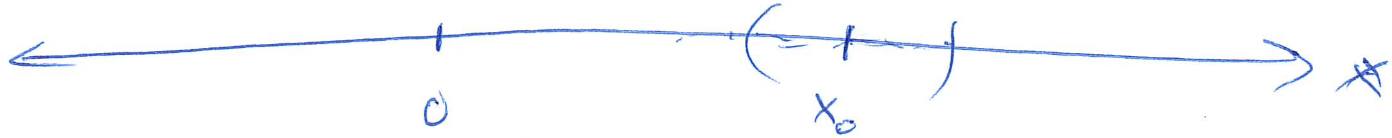
Then $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty}\left(\frac{1}{n}+1\right)$
$$= 1 \ne 0.$$

Continuous $\forall x_0 \in \mathbb{R}$ where $x_0 \neq 0$.

Let $x_0 \in \mathbb{R}$ and suppose $x_0 > 0$.

Show $\forall \{x_n\} \subseteq \mathbb{R}$, if $\lim_{n\to\infty} x_n = x_0$, then $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Suppose $\{x_n\} \subseteq \mathbb{R}$ and $\lim_{n\to\infty} x_n = x_0$.



By our Boundedness Lemma, $\exists \beta > 0$ and $N \in \mathbb{N}$ s.t.

$$\forall n \geq N, \quad |x_n| \geq \beta > 0.$$

Let $n \geq N$. Then $f(x_n) = 1 + x_n$.

Note $f(x_0) = 1 + x_0$.

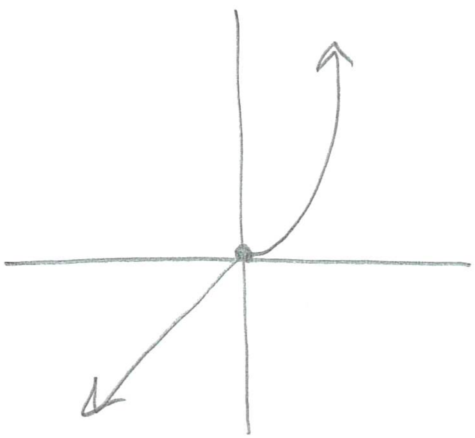
So $$|f(x_n) - f(x_0)| = |x_n - x_0|.$$

So by the comparison Lemma, and since $\lim_{n\to\infty} x_n = x_0$,

$$\lim_{n\to\infty} f(x_n) = f(x_0).$$

(5) Define $f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ x & \text{if } x < 0. \end{cases}$

Prove that f is continuous at $x_0 = 0$.



Show: $\forall \{x_n\} \subseteq \mathbb{R}$, if $\lim_{n\to\infty} x_n = 0$, then $\lim_{n\to\infty} f(x_n) = 0$.

Let $\{x_n\} \subseteq \mathbb{R}$ and suppose $\lim_{n\to\infty} x_n = 0$.

By our boundedness Lemma, $\exists N$ st $\forall n \geq N$, $x_n < 1$.

Thus if $n \geq N$, $f(x_n) = x_n$ or x_n^2.

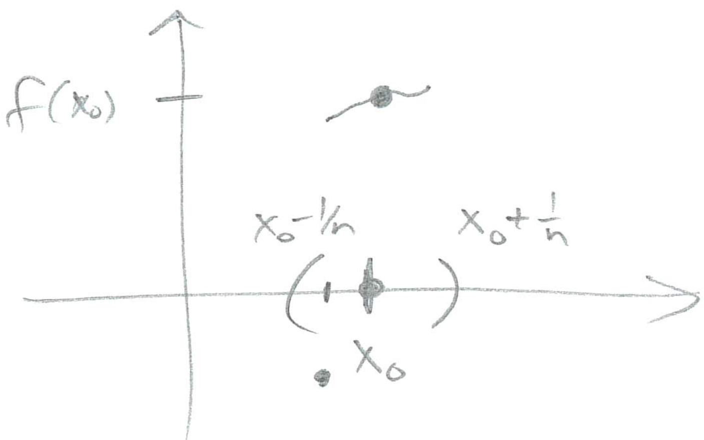
In each case, since $x_n^2 \leq x_n$, we have $|f(x_n)| \leq |x_n|$.

By Comparison Lemma, $\lim_{n\to\infty} f(x_n) = 0$.

(9) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at x_0 and that $f(x_0) > 0$. Prove $\exists n \in \mathbb{N}$ where $\forall x \in I := (x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$, $f(x) > 0$.

proof: Suppose not.

$\forall n \in \mathbb{N}$, $\exists x \in I = (x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$ where $f(x_n) \le 0$.



Let $n \geq 1$.

Choose $x_n \in (x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$

st. $f(x_n) \le 0$.

Consider $\{x_n\}$.

Note that $|x_n - x_0| < \frac{1}{n}$.

So by comparison, $\lim_{n \to \infty} x_n = x_0$.

But $|f(x_0) - f(x_n)| \geq f(x_0) > 0$.

This shows $\{f(x_n)\}$ does not converge to $f(x_0)$.

$\exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists n \geq N$ with $|f(x_n) - f(x_0)| \geq \varepsilon$

- $\varepsilon = f(x_0)$
- Fix N. Use $n = N+1$.
- Then $|f(x_n) - f(x_0)| \geq f(x_0) = \varepsilon$.

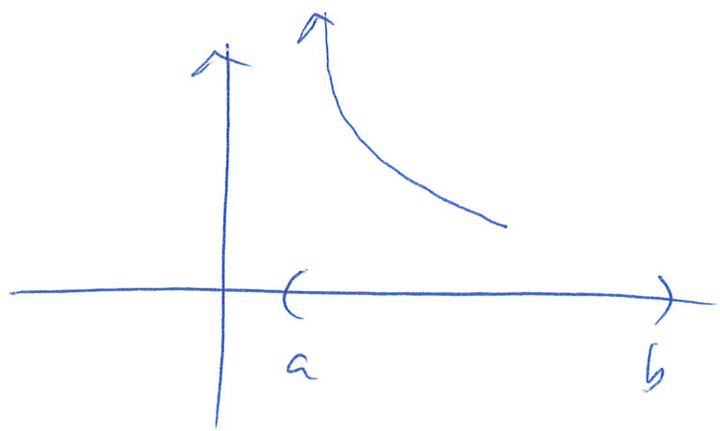
This contradicts continuity assumption at x_0.

3.2 Extreme Value Theorem.

Let $f : [a,b] \to \mathbb{R}$ be continuous.
Then f attains a max & min value on $[a,b]$.

(?) Consider functions $f : (a,b) \to \mathbb{R}$.

(a) Show $\exists f$ s.t. f is unbounded above & continuous.



$$f(x) = \frac{1}{x-a}.$$

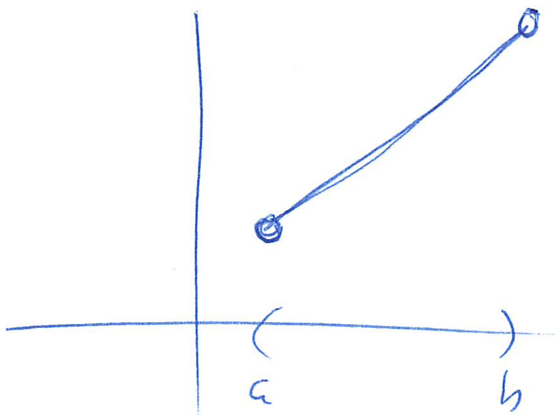
- continuous on (a,b)
- unbounded.

$$x_n = a + \frac{1}{n}.$$

$$f(x_n) = n.$$

(b) Show $\exists f$ s.t. f is bounded but does not attain a max.

Let $f:(a,b) \to \mathbb{R}$ by $f(x) = 2x+1$.

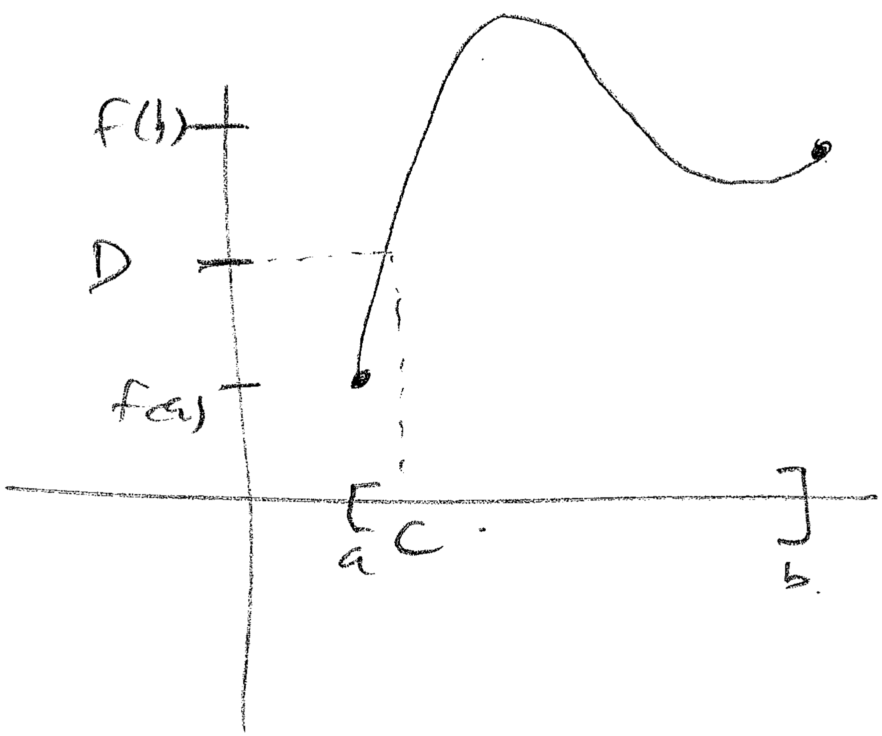


im(f) is bounded <u>but</u>

im(f) has no max/min.

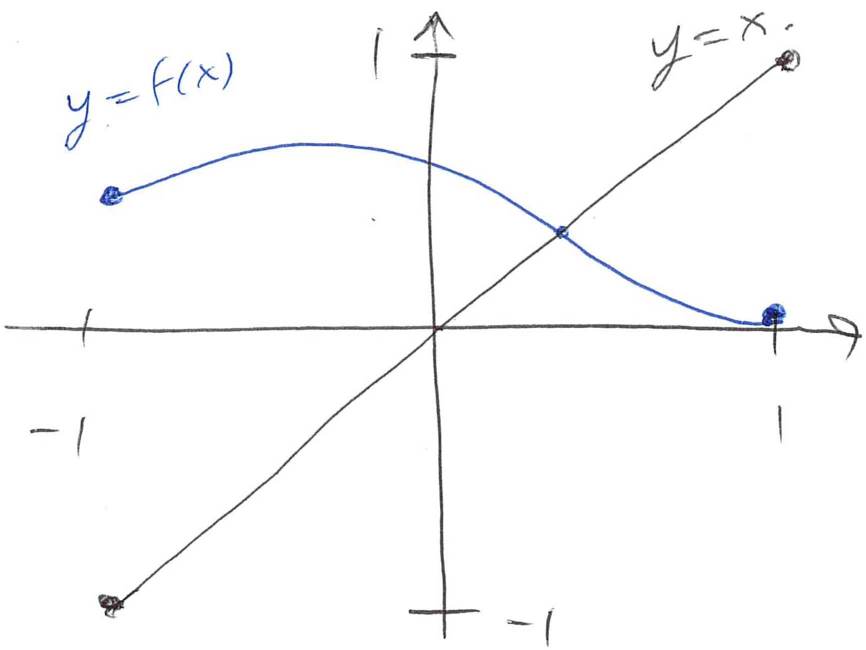
3.3 Intermediate Value Theorem:

If $f:[a,b]\to\mathbb{R}$ is continuous and D is strictly between $f(a)$ & $f(b)$, then $\exists\, c\in\mathbb{R}$ s.t. $a<c<b$ and $f(c)=D$.



Def: Suppose $f: D\to\mathbb{R}$. We say $x=a$ is a *fixed point* iff $f(a)=a$.

④ Suppose $f : [-1, 1] \to [-1, 1]$ and f is continuous.
Then f has a fixed point.



Consider $g : [-1, 1] \to \mathbb{R}$
where $g(x) = f(x) - x$.

Notice $f(-1) \geq -1$.

So $f(-1) - (-1) \geq 0$.

If $f(-1) = -1$ done.

Suppose not, $f(-1) - (-1) > 0$.

Similarly $f(1) \leq 1$

So $f(1) - 1 \leq 0$

If $f(1) = 1$. done.

Suppose not, $f(1) - 1 < 0$.

Thus g is continuous.
Also $g(-1) > 0$ and $g(1) < 0$.
By I.V.T. $\exists\, c \in (-1, 1)$ s.t.

$$g(c) = 0.$$

I.e. $f(c) - c = 0$
And c is a fixed pt of f.

P. 3

(5) Suppose $h, g : [a,b] \to \mathbb{R}$ are continuous.

Suppose $h(a) \leq g(a)$. and $h(b) \geq g(b)$.

Then prove $\exists\, x_0 \in [a,b]$ s.t. $h(x_0) = g(x_0)$.

1. DRAW A PICTURE.

2. Setup a "difference function"

   via (4).

3. Apply IVT.