9/24 • Assignment 3 posted. Due 10/3
• Test 1 10/10

Today:
~ 2.2 Closed set.
~ 2.3 Monotone Convergence & consequences.

Def: Suppose $S \subseteq \mathbb{R}$. We say S is closed

iff

$\forall \{a_n\} \subseteq S$, if $\{a_n\}$ converges, then $\lim_{n\to\infty} a_n \in S$.

S not closed

iff

$\exists \{a_n\} \subseteq S$ s.t. $\{a_n\}$ converges and $\lim_{n\to\infty} a_n \notin S$.

Examples: $(0,1)$ not closed.

Notice $\left\{\frac{1}{n+1}\right\}_{n=1}^{\infty}$ converges, $\left\{\frac{1}{n+1}\right\}_{n=1}^{\infty} \subseteq (0,1)$.

and $\lim_{n\to\infty} \frac{1}{n+1} = 0 \notin (0,1)$.

Lemma 2.21 Suppose $\lim_{n\to\infty} d_n = d$. and $\forall n \in \mathbb{N}$, $d_n \geq 0$.

Then $d \geq 0$ also.

proof: Suppose $d < 0$.

Let $\varepsilon = -\frac{d}{2} > 0$. Then $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$|d_n - d| < -\frac{d}{2}.$$

Thus $\frac{d}{2} < d_N - d < -\frac{d}{2}$.

So $d_N < \frac{d}{2} < 0 \Rightarrow\Leftarrow$ $\blacksquare$

Thm 2.22 Suppose $c < d$.

Let $\{a_n\} \subseteq [c,d]$. If $\{a_n\}$ converges, then

$\lim_{n\to\infty} a_n \in [c,d]$.

(I.e. $[c,d]$ is closed)

proof: Suppose $\{a_n\}$ converges.

Define $c_n := a_n - c$.

Notice $\forall n$, $c_n \geq 0$.

Since $\{a_n\}$ converges and $\{c\}$ converges,

$\{c_n\}$ converges.

Thus $\lim_{n\to\infty} c_n = \left(\lim_{n\to\infty} a_n\right) - c \geq 0$. (by 2.21)

Thus $\lim_{n\to\infty} a_n \geq c$.

Defining $d_n := d - a_n$, we can show $d \geq \lim_{n\to\infty} a_n$.

Thus $\lim_{n\to\infty} a_n \in [c,d]$. $\boxtimes$

Set Notation : Arbitrary unions / intersections.

Suppose I is an indexing set.

Suppose for each $j \in I$, A_j is a set.

$$\bigcup_{j \in I} A_j = \{x \mid \exists j \in I \text{ where } x \in A_j\}.$$

$$\bigcap_{j \in I} A_j = \{x \mid \forall j \in I, \ x \in A_j\}.$$

2.3 Monotone Sequences.

Definitions: Suppose $\{a_n\}$ is a sequence.

We say $\{a_n\}$ is monotone increasing iff $\forall n \in \mathbb{N}$, $a_{n+1} \geq a_n$.

monotone decreasing $a_{n+1} \leq a_n$.

Remark: We include the prefix "strictly" when the inequalities are strict.

~~Remark:~~

Thm 2.25 Let $\{a_n\}$ be monotone.

The sequence converges iff it is bounded.

Moreover, if $\{a_n\}$ converges, then

$$\lim_{n\to\infty} a_n = \sup\{a_n\} \text{ when } \{a_n\} \text{ increasing}$$

$$\lim_{n\to\infty} a_n = \inf\{a_n\} \text{ when } \{a_n\} \text{ decreases.}$$

---

proof: ($\longrightarrow$) Suppose $\{a_n\}$ converges.

We know $\{a_n\}$ is bounded by earlier Lemma.

($\longleftarrow$) Suppose $\{a_n\}$ is bounded and monotone increasing.

Let $A = \sup\{a_n\}$, which exists by Completeness Axiom.

Let $\varepsilon > 0$.

Since $A-\varepsilon < A$ is not an upper bound, $\exists N$ s.t.

$A-\varepsilon < a_N$. ~~Note that~~ Since $\{a_n\}$ increases,

$\forall n \geq N, \quad a_n \geq a_N > A-\varepsilon.$

Note $\forall n, \quad a_n \leq A < A+\varepsilon.$

Thus $\forall n \geq N, \quad A-\varepsilon < a_n < A+\varepsilon$

So $\quad -\varepsilon < a_n - A < \varepsilon.$

So $\quad |a_n - A| < \varepsilon.$ $\blacksquare$

Ex 2.26 Suppose $\forall n \geq 1$, $S_n := \sum_{k=1}^{n} \frac{1}{k} \cdot \frac{1}{2^k}$.

Then $\{S_n\}$ converges.

---

1. Show the sequence increases.

Let $n \geq 1$.

Notice $S_{n+1} = \sum_{k=1}^{n+1} \frac{1}{k} \cdot \frac{1}{2^k} = \frac{1}{n+1} \cdot \frac{1}{2^{n+1}} + \sum_{k=1}^{n} \frac{1}{k} \cdot \frac{1}{2^k}$.

So $S_{n+1} > S_n$ since $\frac{1}{n+1} \cdot \frac{1}{2^{n+1}} > 0$.

2. Show the sequence is bounded.

Let $k \geq 1$ and $n \geq 1$.

Then $\frac{1}{k} \cdot \frac{1}{2^k} \leq \frac{1}{2^k}$.

Then $S_n = \sum_{k=1}^{n} \frac{1}{k} \cdot \frac{1}{2^k} \leq \sum_{k=1}^{n} \frac{1}{2^k} = \frac{\frac{1}{2} - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} < \frac{\frac{1}{2}}{\frac{1}{2}} = 1$

$\boxed{\text{By 1 \& 2, } \{S_n\} \text{ converges.}}$

Proposition 2.28 Let $0 < c < 1$.

Then $\lim\limits_{n\to\infty} c^n = 0$.

proof: Since $0 < c < 1$, if $n \geq 1$, then

$$c \cdot c^n < c^n \cdot 1.$$

Thus $c^{n+1} < c^n$

So $\{c_n\}$ is monotone decreasing.

Since $0 < c^n \leq c < 1$ $\{c^n\}$ is bounded.

So $\inf\{c^n\} = \lim\limits_{n\to\infty} c^n =: l \geq 0$

Suppose $l > 0$. For all n, $l \leq c^{n+1}$.

So for all n, $\frac{l}{c} \leq c^n$.

So $\frac{l}{c}$ is a lower bound for $\{c^n\}$. And $\frac{l}{c} > l$. $\Rightarrow\Leftarrow$