Spring 2021 **Math 531 Take-Home Final** **Due Monday, May 10, 2021 by 3 PM**

1. Consider the PDE for the heat equation on a semi-infinite domain:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad t > 0, \quad x > 0,$$

with the following boundary condition and initial condition:

$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad \text{and} \qquad u(x,0) = xe^{-x^2/2},$$

Solve this problem for $u(x,t)$ using Fourier cosine transform.
Leave your answer in integral form. That is, do not attempt to evaluate the Fourier integral expression for the solution $u(x,t)$.

2. Consider the nonhomogeneous wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \qquad t > 0 \quad \text{and} \quad 0 < x < \pi,$$

with one end fixed and the other having a time dependent (sinusoidal) forcing condition:

$$u(0,t) = 0 \qquad \text{and} \qquad u(\pi,t) = A\sin(\omega t).$$

Assume homogeneous initial conditions:

$$u(x,0) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

a. Find a *linear (in x) reference distribution*, $r(x,t)$, with $u(x,t) = v(x,t) + r(x,t)$, such that the PDE in $v(x,t)$ has homogeneous boundary conditions. Be sure to note the changes in both the PDE and the initial conditions with this reference function.

b. The PDE in $v(x,t)$ is nonhomogeneous, but has homogeneous boundary conditions, so apply the *method of eigenfunction expansion* with $v(x,t) = \sum_{n=0}^{\infty} a_n(t)\phi_n(x)$, where $\phi_n(x)$ are the appropriate eigenfunctions corresponding to the homogeneous boundary conditions to solve this problem. The PDE in $v(x,t)$ has the form:

$$\frac{\partial^2 v}{\partial t^2} = c^2\frac{\partial^2 v}{\partial x^2} + Q(x,t), \qquad t > 0 \quad \text{and} \quad 0 < x < \pi.$$

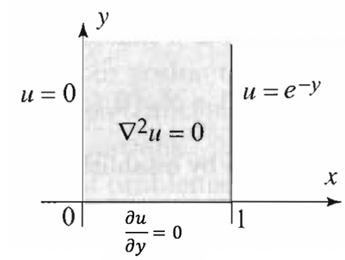
Write the nonhomogeneous function $Q(x,t)$ in an eigenfunction expansion,
$Q(x,t) = \sum_{n=0}^{\infty} q_n(t)\phi_n(x)$, and determine the Fourier coefficients, $q_n(t)$, for this function. The problem for $v(x,t)$ will have a second order nonhomogeneous ODE in $a_n(t)$, which may have a messy expression, so can be left in integral form. (Variation of parameters solution) The expressions for the Fourier coefficients from the initial conditions can also be left in integral form, but you do need to write these integrals.

c. For certain values of c, there are unbounded solutions for $v(x,t)$. Find these values of c and explain why the solution becomes unbounded. (Resonance in the system)

3a. Find the solution for the Laplace equation in a semi-infinite strip;

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad y > 0, \quad 0 < x < 1,$$

with boundary conditions provided in the figure.



3b. Use your solution to create a 3D plot of $u(x,y)$ with $y \in [0,1]$. Your program should integrate at least $\omega = [0,50]$ for Fourier transforms. Be sure to include your program.

4. Solve the nonhomogeneous heat equation

$$\frac{\partial u}{\partial t} = \nabla^2 u + \sin(2x)\sin(3y), \qquad 0 < x < \pi, \quad 0 < y < \pi, \quad t > 0,$$
$$u(x,y,0) = \sin(4x)\sin(7y)$$
$$u(x,0,t) = u(x,\pi,t) = u(0,y,t) = u(\pi,y,t) = 0$$

5. Solve the nonhomogeneous partial differential equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + e^{-2t}\sin(5x), \qquad t > 0 \quad \text{and} \quad 0 < x < \pi,$$

with initial and boundary conditions:

$$u(x,0) = 0, \quad u(0,t) = 1 \quad \text{and} \quad u(\pi,t) = 0.$$

6. Solve the nonhomogeneous two-dimensional heat equation with circularly symmetric time independent sources, boundary conditions, and initial conditions (inside a circle)

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + Q(r)$$

with

$$u(r,0) = f(r) \quad \text{and} \quad u(a,t) = T.$$