Quiz 4
Differential Equations
Math 337
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Problem 1: Consider the 2^{nd} order linear homogeneous ODE given by:

$$y'' + 6y' + 13y = 0$$

Find two linearly independent solutions, y_1 and y_2, for this ODE and write the general solution to this problem. Show these solutions form a Fundamental set of solutions by computing the Wronskian, $W[y_1, y_2](t)$ and showing it is nonzero for all t.

Notice: To find the eigenvalues, we can write the characteristic equation and solve:

$$\lambda^2 + 6\lambda + 13 = 0$$

Now we solve using the quadratic equation:

$$\begin{aligned}\lambda &= \frac{-6 \pm \sqrt{36 - 4(13)}}{2} \\ &= \frac{-6 \pm 4i}{2} \\ &= -3 \pm 2i\end{aligned}$$

So we get the general solution to this being:

$$c_1 e^{-3t}\cos(2t) + c_2 e^{-3t}\sin(2t)$$

Meaning:

$$y_1 = e^{-3t}\cos(2t) \qquad\qquad y_2 = e^{-3t}\sin(2t)$$

We can prove that these solutions form a Fundamental Set of Solutions by proving that the Wronskian is nonzero for all t.

$$W[y_1, y_2](t) = \begin{vmatrix} e^{-3t}\cos(2t) & e^{-3t}\sin(2t) \\ -3e^{-3t}\cos(2t) - 2e^{3t}\sin(2t) & -3e^{-3t}\sin(2t) + 2e^{-3t}\cos(2t) \end{vmatrix}$$

$$\begin{aligned} &= (e^{-3t}\cos(2t))(-3e^{-3t}\sin(2t) + 2e^{-3t}\cos(2t)) - (e^{-3t}\sin(2t))(-3e^{-3t}\cos(2t) - 2e^{-3t}\sin(2t)) \\ &= -3e^{-6t}\sin(2t)\cos(2t) + 2e^{-6t}\cos^2(2t) + 3e^{-6t}\sin(2t)\cos(2t) + 2e^{-6t}\sin^2(2t) \\ &= 2e^{-6t}(\cos^2(2t) + \sin^2(2t)) \\ &= 2e^{-6t} > 0 \qquad \forall t \end{aligned}$$

Problem 2: Consider the 2^{nd} order linear homogeneous ODE given by:

$$y'' - y' - 2y = 54te^{2t} - 20t$$

Find the general solution to this problem, using the Method of Undetermined Coefficients. You must show your steps for finding the coefficients of the particular solution.

Notice: To find the eigenvalues, we can write the characteristic equation and solve:

$$\begin{aligned} \lambda^2 - \lambda - 2 &= 0 \\ (\lambda - 2)(\lambda + 1) &= 0 \\ \lambda &= 2, -1 \end{aligned}$$

So we get the homogeneous solution to this being:

$$y_h = c_1 e^{2t} + c_2 e^{-t}$$

To get the particular solution, we set the following:

$$\begin{aligned} y_p &= \left(At^2 + Bt\right) e^{2\,t} + Ct + D \\ y_p' &= (2\,At + B)\, e^{2\,t} + 2\,\left(At^2 + Bt\right) e^{2\,t} + C \\ y_p'' &= 2\,Ae^{2\,t} + 4\,(2\,At + B)\, e^{2\,t} + 4\,\left(At^2 + Bt\right) e^{2\,t} \end{aligned}$$

By plugging this into the differential equation, we get:

$$\begin{aligned} y_p'' - y_p' - 2y_p &= 6Ate^{2t} + (2A + 3B)e^{2t} - 2Ct - (C + 2D) \\ &= 54te^{2t} - 20t \end{aligned}$$

To solve, we set the following:

$$\begin{aligned} 6A &= 54 & -2C &= -20 \\ 2A + 3B &= 0 & C + 2D &= 0 \\ A = 9, B &= -6 & C = 10, D &= -5 \end{aligned}$$

Now we have the particular solution:

$$y_p = \left(9t^2 - 6t\right) e^{2\,t} + 10t - 5$$

Now we also have the general solution:

$$y(t) = c_1 e^{2t} + c_2 e^{-t} + \left(9t^2 - 6t\right) e^{2\,t} + 10t - 5$$

Problem 3 (a): An important 2^{nd} order nonlinear homogeneous ODE shown on Slide 7 describes the motion of a pendulum and satisfies:

$$\theta'' + 0.2\theta' + 4.01 \sin\theta = 0$$

where $\theta(t)$ is the angle of the pendulum from the downward vertical. Transform this 2^{nd} order nonlinear ODE into a system of 1^{st} order ODEs by letting $x_1(t) = \theta(t)$ and $x_2(t) = \dot{x}_1(t) = \theta'(t)$. Find all equilibria by letting $\dot{x}_1 = \dot{x}_2 = 0$.

So we can now transform this 2^{nd} order nonlinear ODE into a system of 1^{st} order ODEs

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -4.01\sin(x_1) - 0.2x_2
\end{aligned}$$

We will now set $\dot{x}_1 = \dot{x}_2 = 0$ to find all the equilibria:

$$\begin{aligned}
0 &= x_2 \\
0 &= -4.01\sin(x_1) - 0.2x_2
\end{aligned}$$

Solving the system of 1^{st} order ODEs, we get:

$$\begin{aligned}
0 &= x_2 \\
0 &= -4.01\sin(x_1) - 0.2(0) \\
0 &= -4.01\sin(x_1) \\
0 &= \sin(x_1) \\
n\pi &= x_1 \qquad \forall n \in \mathbb{Z}
\end{aligned}$$

Thus the equilibria is as follows:

$$(n\pi, 0) \qquad \forall n \in \mathbb{Z}$$

Problem 3 (b): Take the nonlinear system of 1^{st} order ODEs found in Part a and determine the Jacobian matrix, $J(x_1, x_2)$, for this system. One equilibrium is $[x_{1e}, x_{2e}]^T = [0, 0]^T$, so compute $J(0, 0)$. Find the eigenvalues for $J(0, 0)$ and use this information to determine the qualitative behavior (e.g., stable node, center, etc.) near this equilibrium, as we did in the previous section. Another equilibrium is $[x_{1e}, x_{2e}]^T = [\pi, 0]^T$, so compute $J(\pi, 0)$. Find the eigenvalues for $J(\pi, 0)$ and use this information to determine the qualitative behavior near this equilibrium.

Using the system of 1^{st} order ODEs, the Jacobian is as follows:

$$J(x_1, x_2) = \begin{pmatrix} 0 & 1 \\ -4.01\cos(x_1) & -0.2 \end{pmatrix}$$

One equilibrium is $[x_{1e}, x_{2e}]^T = [0, 0]^T$, so the Jacobian at that point is:

$$J(0, 0) = \begin{pmatrix} 0 & 1 \\ -4.01 & -0.2 \end{pmatrix}$$

Eigenvalues at this point can be found by taking the determinant of Jacobian:

$$\begin{aligned} \begin{vmatrix} 0-\lambda & 1 \\ -4.01 & -0.2-\lambda \end{vmatrix} &= \lambda(\lambda + 0.2) + 4.01 \\ &= \lambda^2 + 0.2\lambda + 4.01 \\ \lambda &= \frac{-0.2 \pm \sqrt{0.04 - 4(4.01)}}{2} \\ &= \frac{-0.2 \pm \sqrt{-16}}{2} \\ &= -0.1 \pm 2i \end{aligned}$$

These eigenvalues show us that the qualitative behavior near $(0, 0)$ is a **stable focus**.

One equilibrium is $[x_{1e}, x_{2e}]^T = [\pi, 0]^T$, so the Jacobian at that point is:

$$J(\pi, 0) = \begin{pmatrix} 0 & 1 \\ 4.01 & -0.2 \end{pmatrix}$$

Eigenvalues at this point can be found by taking the determinant of Jacobian:

$$\begin{aligned} \begin{vmatrix} 0-\lambda & 1 \\ 4.01 & -0.2-\lambda \end{vmatrix} &= \lambda(\lambda + 0.2) - 4.01 \\ &= \lambda^2 + 0.2\lambda - 4.01 \\ \lambda &= \frac{-0.2 \pm \sqrt{0.04 + 4(4.01)}}{2} \\ &= \frac{-0.2 \pm \sqrt{16.08}}{2} \\ &= -0.1 \pm 2.005 \end{aligned}$$

These eigenvalues show us that the qualitative behavior near $(\pi, 0)$ is a **saddle point**.