Homework 3.2
Linear Algebra
Math 524
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Section 3.D Problem 2: Suppose V is finite-dimensional and dim $V > 1$. Prove that the set of noninvertible operators on V is not a subspace of $\mathcal{L}(V)$.

Solution 3.D.2. Let V be finite-dimensional with dim $V = n, \quad n \in \mathbb{Z}^{>1}$.
Let $\{v_1, ..., v_n\}$ be a basis of V. Let $S, T \in \mathcal{L}(V)$

$$\begin{array}{ccc} Sv_1 = v_1 & Sv_2 = 0 & Sv_k = 0 \\ Tv_1 = 0 & Tv_2 = v_2 & Tv_k = v_k \end{array}$$

So S, T are both noninvertible operators on V, such that S is not injective, and T is not surjective

$$(S+T)v_1 = Sv_1 + Tv_1 = v_1$$
$$(S+T)v_k = Sv_k + Tv_k = v_k$$

Because (S+T) is injective and surjective, (S+T) is invertible, so S,T is not closed under addition, thus they are not a subspace of $\mathcal{L}(V)$

□

Section 3.D Problem 3: Suppose V is finite-dimensional, U is a subspace of V, and $S \in \mathcal{L}(V)$. Prove there exists an invertible operator $T \in \mathcal{L}(V)$ such that $Tu = Su$ for every $u \in U$ if and only if S is injective.

Solution 3.D.3 . Let V be finite-dimensional, $U \subseteq V$, and $S \in \mathcal{L}(V)$.
(=>) Let $T \in \mathcal{L}(V)$ be invertible, and $Tu = Su$, $\forall u \in U$.

Because T is invertible, T is injective, such that Tu = 0, u is exclusively 0. And because Tu = Su = 0 only when u is exclusively 0, S is injective as well.

(<=) Let S be injective, and $\{u_1, ..., u_m, v_1, ..., v_n\}$ be an extended Basis of V from U.

Because S is injective, we can have $\{Su_1, ..., Su_m, w_1, ..., w_n\}$ be an extended Basis of V from S. Let T be defined as:

$$Tu_k = Su_k \quad 1 \leq k \leq m$$
$$Tv_j = w_j \quad 1 \leq j \leq n$$

Because T maps the entire basis $\{u_1, ..., u_m, v_1, ..., v_n\}$ to another basis entirely, $\{Su_1, ..., Su_m, w_1, ..., w_n\}$, T is invertible $\in \mathcal{L}(V)$.

□

Section 3.E Problem 2: Suppose $V_1, ..., V_m$ are vector spaces such that $V_1 \times \cdots \times V_m$ is finite dimensional. Prove that V_j is finite-dimensional for each $j = 1, ..., m$

Solution 3.E.2. Let $V_1, ..., V_m$ be vector spaces such that $V_1 \times \cdots \times V_m$ is finite dimensional.

By Theorem 3.76, Dim $(V_1 \times \cdots \times V_m) = \sum_{k=1}^{m}$Dim (V_k).

If any V_k was not finite dimensional, then the dimension of V_k would be infinite, such that the sum with the other dimensions would not be finite, as integers are closed under addition.

□

Section 3.E Problem 4: Suppose $V_1, ..., V_m$ are vector spaces.
Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are isomorphic vector spaces.

Solution 3.E.4. Let $V_1, ..., V_m$ be vector spaces.

Let $f \in \mathcal{L}(V_1 \times \cdots \times V_m, W)$, with $f_i : V_i \to W$,

$$f_i(v_i) = f(0, 0, 0, v_i, 0, 0, 0) \in \mathcal{L}(V_i, W) \quad 1 \leq i \leq m$$

Let $\Gamma : \mathcal{L}(V_1 \times \cdots \times V_m, W) \to \mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ such that

$$\Gamma(f) = (f_1, ..., f_m)$$

Let $\Gamma^{-1} : \mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W) \to \mathcal{L}(V_1 \times \cdots \times V_m, W)$ such that

$$\Gamma^{-1}(f_1, ..., f_m) = (f_1(v_1), ..., f_m(v_m))$$

Because Γ is linear and has an inverse, $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are isomorphic vector spaces.

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