Quiz 3
Differential Equation
Math 337
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Problem 1: The lecture notes examined the negative feedback of glucose and insulin. A classic enzymatic negative feedback model satisfies the system:

$$\begin{aligned}\dot{x}_1 &= \frac{3}{1+0.2x_2} - 0.5x_1\\ \dot{x}_2 &= 5x_1 - x_2\end{aligned}$$

where x_1 is an enzyme and x_2 is the endproduct. Find the positive equilibrium for this model ($x_1 > 0$ and $x_2 > 0$). Compute the Jacobian Matrix for this system. Evaluate the Jacobian matrix at the equilibrium. Determine the eigenvalues for this model and determine the qualitative behavior of this model near the equilibrium. Sketch a phase portrait for this model for non-negative x_1 and x_2 ($x_1 \geq 0$ and $x_2 \geq 0$).

Positive Equilibrium Point (x_1, x_2)**:**

$$\begin{aligned}0 &= \frac{3}{1+0.2x_2} - 0.5x_1\\ 0 &= 5x_1 - x_2\end{aligned}$$

So we can see that $5x_1 = x_2$, so the following holds:

$$\begin{aligned}0 &= \frac{3}{1+0.2x_2} - 0.5x_1\\ &= \frac{3}{1+0.2(5x_1)} - 0.5x_1\\ &= \frac{3}{1+x_1} - 0.5x_1\\ &= 3 - 0.5x_1(1+x_1)\\ &= 0.5x_1^2 + 0.5x_1 - 3\\ &= x_1^2 + x_1 - 6\\ &= (x_1+3)(x_1-2)\end{aligned}$$

So $x_1 = 2$ with $x_2 = 5x_1 = 10$

Jacobian Matrix and EigenValues:

$$J(x_1, x_2) = \begin{pmatrix} \frac{-1}{2} & \frac{-15}{(5+x_2)^2} \\ 5 & -1 \end{pmatrix} \qquad J(2, 10) = \begin{pmatrix} \frac{-1}{2} & \frac{-1}{15} \\ 5 & -1 \end{pmatrix}$$

$$\begin{aligned} \begin{vmatrix} \frac{-1}{2} - \lambda & \frac{-1}{15} \\ 5 & -1 - \lambda \end{vmatrix} &= (\lambda + 1)(\lambda + \frac{1}{2}) + \frac{1}{3} \\ &= \lambda^2 + \frac{3}{2}\lambda + \frac{1}{2} + \frac{1}{3} \\ &= \lambda^2 + \frac{3}{2}\lambda + \frac{5}{6} \\ &= 6\lambda^2 + 9\lambda + 5 = 0 \end{aligned}$$

$$\begin{aligned} \lambda &= \frac{-9 \pm \sqrt{81 - 4(5)(6)}}{12} \\ &= \frac{-9 \pm \sqrt{39}i}{12} \\ &= \frac{-3}{4} \pm \frac{\sqrt{39}i}{12} \end{aligned}$$

Near this equilibrium, because of the negative real parts of the eigenvalues, the phase portrait will be of a **stable focus** around this equilibrium point.

Problem 2: A very popular ecological model is the predator-prey model (Lotka-Volterra). Consider the system of ODEs:

$$\dot{x}_1 = 0.1x_1 - 0.05x_1x_2$$
$$\dot{x}_2 = 0.001x_1x_2 - 0.04x_2$$

Associate each variable with the prey or the predator and explain briefly your reasoning. Find all equilibria for this model. Compute the Jacobian Matrix for this system. Evaluate the Jacobian matrix at all of the equilibria. Determine the eigenvalues at each of the equilibria for this model and determine the qualitative behavior of this model near these equilibria. Sketch a phase portrait for this model for non-negative x_1 and x_2 ($x_1 \geq 0$ and $x_2 \geq 0$).

Variable Reasoning: x_1 is the prey and x_2 is the predator. We know that x_1 is the prey because x_1's population declines with the presence of the other species x_2. And we know that x_2 is the predator because x_2's population grows with the the presence of the other species x_1.

Equilibria (x_1, x_2):

$$0 = x_1(0.1 - 0.05x_2)$$
$$0 = x_2(0.001x_1 - 0.04)$$

$$\begin{array}{rrrr} x_1 = 0 & x_2 = 0 & 0.001x_1 - 0.04 = 0 & 0.1 - 0.05x_2 = 0 \\ -0.04x_2 = 0 & 0.1x_1 = 0 & x_1 = 0.04/0.001 = 40 & x_2 = 0.1/0.05 = 2 \\ x_2 = 0 & x_1 = 0 & x_2 = 2 & x_1 = 40 \end{array}$$

So the equilibria holds as $(0, 0)$ and $(40, 2)$

Jacobian Matrix and EigenValues:

$$J(x_1, x_2) = \begin{pmatrix} 0.1 - 0.05x_2 & -0.05x_1 \\ 0.001x_2 & 0.001x_1 - 0.04 \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 0.1 & 0 \\ 0 & -0.04 \end{pmatrix} \qquad \begin{vmatrix} 0.1 - \lambda & 0 \\ 0 & -0.04 - \lambda \end{vmatrix} = (\lambda - 0.1)(\lambda + 0.04) = 0$$

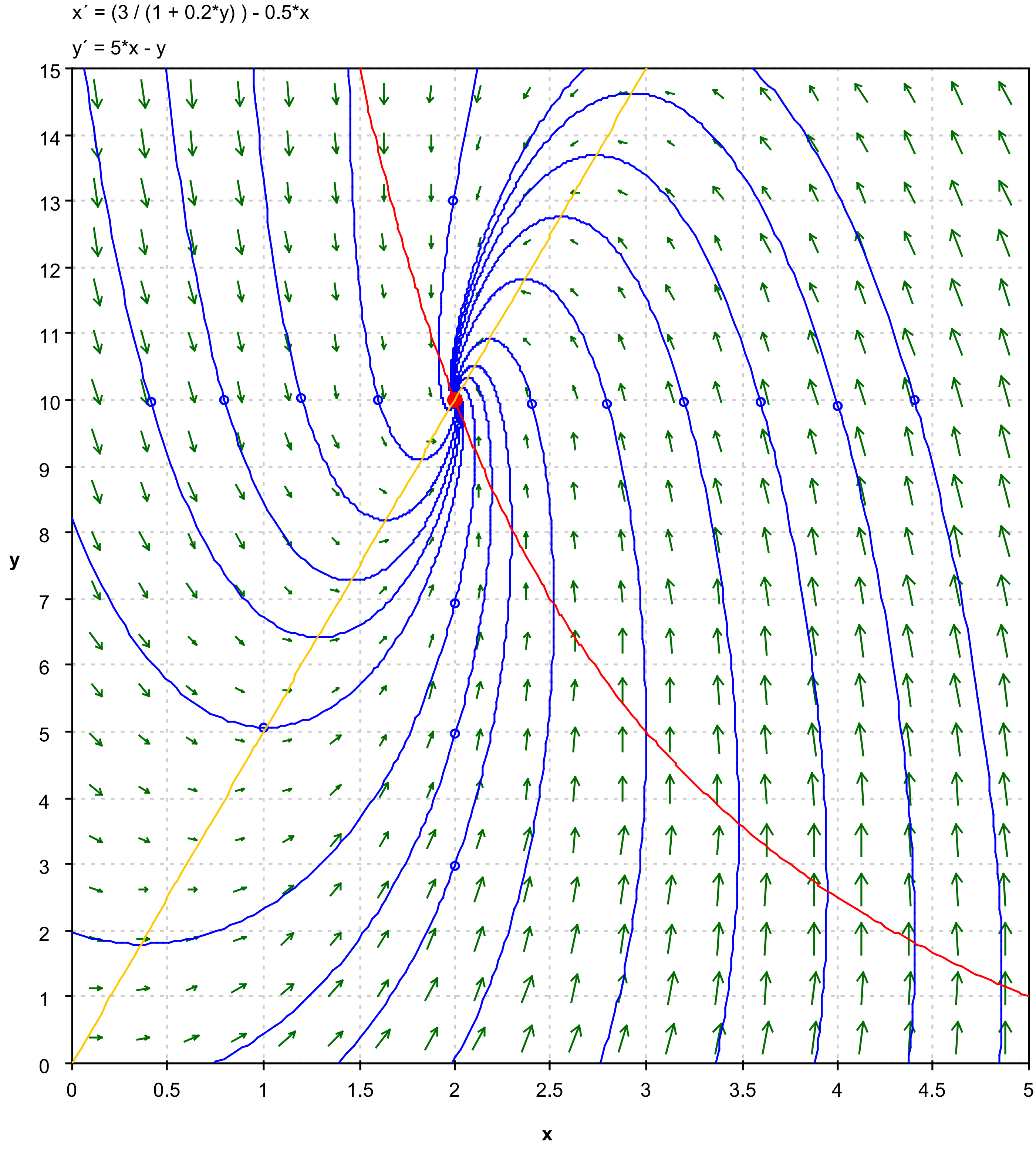
$$\lambda_1 = 0.1, \lambda_2 = -0.04$$

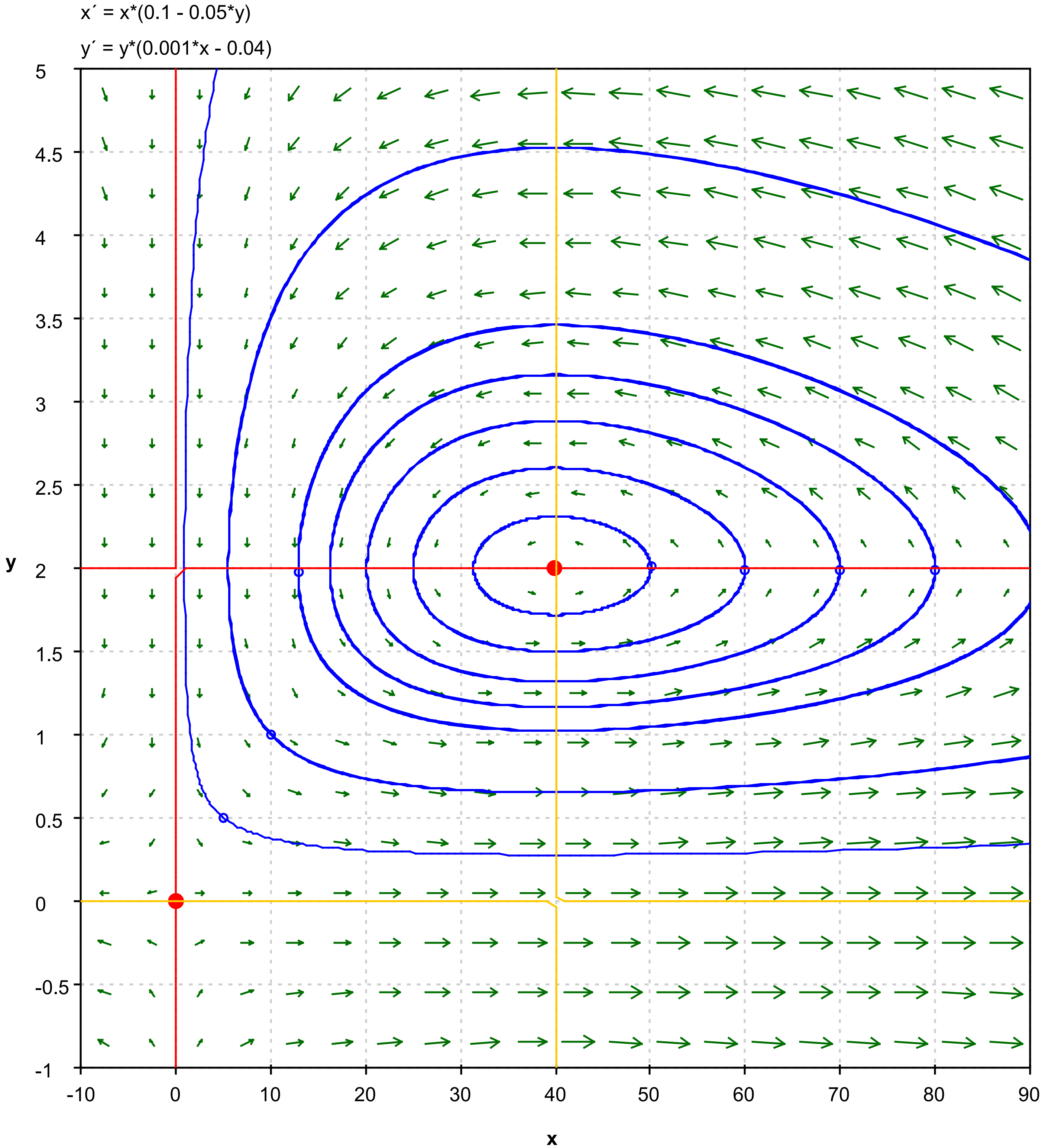
Near this equilibrium, because of the opposite signs of the eigenvalues, the phase portrait will be of a saddle point.

$$J(40,2) = \begin{pmatrix} 0 & -2 \\ 0.002 & 0 \end{pmatrix} \qquad \begin{vmatrix} -\lambda & -2 \\ 0.002 & -\lambda \end{vmatrix} = (\lambda)(\lambda) + .004 = 0$$

$$\lambda = \pm \frac{1}{5\sqrt{10}} i$$

Near this equilibrium, because of the real parts of the eigenvalues being 0, the phase portrait will be of a counter clockwise center around this equilibrium point.


x´ = (3 / (1 + 0.2*y)) - 0.5*x
y´ = 5*x - y
y
x
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0



x´ = x*(0.1 - 0.05*y)
y´ = y*(0.001*x - 0.04)
y
x
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-10
0
10
20
30
40
50
60
70
80
90