Quiz 1
Differential Equations
Math 337
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Problem 1: Consider the initial value problem (IVP):

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Find the general solution to this problem, create a phase portrait, and solve the initial value problem. Describe the qualitative behavior shown in the phase portrait.

Solution 1: Let: $\begin{vmatrix} 0-\lambda & 1 \\ 6 & 1-\lambda \end{vmatrix} = 0$

$$\begin{aligned} (\lambda)(\lambda-1)-6 = \lambda^2-\lambda-6 &= 0 \\ = (\lambda-3)(\lambda+2) &= 0 \\ \lambda &= 3, -2 \end{aligned}$$

Let $\lambda_1 = -2$

$$\begin{pmatrix} 0--2 & 1 \\ 6 & 1--2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Let $\lambda_2 = 3$

$$\begin{pmatrix} 0-3 & 1 \\ 6 & 1-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ 6 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\boldsymbol{\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t}}$$

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 1 & 1 & 3 \\ -2 & 3 & 4 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 3 & 3 & 9 \\ 2 & -3 & -4 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 1 & 1 & 3 \\ 5 & 0 & 5 \end{pmatrix}$$

So $c_1 = 1$ and $c_2 = 2$, thus the solution holds as:

$$\boldsymbol{\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-2t} + 2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t}}$$

Because the eigenvalues have opposite signs, the phase portrait shows a saddle point.

Problem 2: Consider the differential equation:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Find the general solution to this problem and create a phase portrait. Describe the qualitative behavior shown in the phase portrait.

Solution 2: Let: $\begin{vmatrix} 0-\lambda & 1 \\ 0 & 0-\lambda \end{vmatrix} = 0$

$$\lambda^2 = 0$$
$$\lambda = 0 \text{ mult. } 2$$

$$\begin{pmatrix} 0-0 & 1 \\ 0 & 0-0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Notice:

$$\dot{x}_2 = 0 \qquad\qquad \dot{x}_1 = x_2 = C_2$$
$$x_2 = C_2 \qquad\qquad x_1 = C_2 t + C_1$$

$$\boldsymbol{\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} t \\ 1 \end{pmatrix}}$$

Because the eigenvectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} t \\ 1 \end{pmatrix}$, phase portraits are horizontal lines that are parallel to the x_1 axis. The $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ eigenvector shows us that the phase portraits are horizontal lines, and the $\begin{pmatrix} t \\ 1 \end{pmatrix}$ eigenvector shows us that with forward time, the phase portrait moves right, and with backwards time, it moves left.

Problem 3: Consider the differential equation with the parameter α

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \alpha & 2 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Find the general solutions and create phase portraits for the values $\alpha = -6$ and $\alpha = 3$. Describe the qualitative behavior shown in the phase portraits.

Solution 3: $(\alpha = -6)$ Let $\begin{vmatrix} -6-\lambda & 2 \\ -2 & 0-\lambda \end{vmatrix} = 0$

$$\begin{aligned} (\lambda+6)(\lambda)+4 &= 0 \\ \lambda^2+6\lambda+4 &= 0 \\ \lambda &= \frac{-6 \pm \sqrt{36-4(4)}}{2} \\ &= \frac{-6 \pm \sqrt{20}}{2} \\ &= \frac{-6 \pm 2\sqrt{5}}{2} \\ &= -3 \pm \sqrt{5} \end{aligned}$$

Let $\lambda_1 = -3+\sqrt{5}$

$$\begin{pmatrix} -6-(-3+\sqrt{5}) & 2 \\ -2 & 0-(-3+\sqrt{5}) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3-\sqrt{5} & 2 \\ -2 & 3-\sqrt{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3+\sqrt{5} \end{pmatrix}$$

Let $\lambda_2 = -3-\sqrt{5}$

$$\begin{pmatrix} -6-(-3-\sqrt{5}) & 2 \\ -2 & 0-(-3-\sqrt{5}) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3+\sqrt{5} & 2 \\ -2 & 3+\sqrt{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3-\sqrt{5} \end{pmatrix}$$

$$\boldsymbol{\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 3+\sqrt{5} \end{pmatrix} e^{(-3+\sqrt{5})t} + c_2 \begin{pmatrix} 2 \\ 3-\sqrt{5} \end{pmatrix} e^{(-3-\sqrt{5})t}}$$

Because of the negative eigenvalues, the phase portrait has a stable node (sink).

Solution 3: $(\alpha = 3)$ Let $\begin{vmatrix} 3-\lambda & 2 \\ -2 & 0-\lambda \end{vmatrix} = 0$

$$\begin{aligned}(\lambda-3)(\lambda)+4&=0\\ \lambda^2-3\lambda+4&=0\end{aligned}$$

$$\begin{aligned}\lambda &= \frac{3\pm\sqrt{9-4(4)}}{2}\\ &= \frac{3\pm\sqrt{9-16}}{2}\\ &= \frac{3\pm\sqrt{7}i}{2}\end{aligned}$$

Let $\lambda_1 = \frac{3+\sqrt{7}i}{2}$

$$\begin{pmatrix} 3-(\frac{3+\sqrt{7}i}{2}) & 2 \\ -2 & 0-(\frac{3+\sqrt{7}i}{2}) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6-(3+\sqrt{7}i) & 4 \\ -4 & 0-(3+\sqrt{7}i) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} 3-\sqrt{7}i & 4 \\ -4 & -3-\sqrt{7}i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -3+\sqrt{7}i \end{pmatrix}$$

$$x_1(t) = \begin{pmatrix} 4 \\ -3+\sqrt{7}i \end{pmatrix} e^{\frac{3}{2}t}\left(\cos\left(\frac{\sqrt{7}}{2}t\right) + i\sin\left(\frac{\sqrt{7}}{2}t\right)\right)$$

$$u(t)+iw(t) = \begin{pmatrix} 4\cos\left(\frac{\sqrt{7}}{2}t\right) \\ -3\cos\left(\frac{\sqrt{7}}{2}t\right) - \sqrt{7}\sin\left(\frac{\sqrt{7}}{2}t\right) \end{pmatrix} + i\begin{pmatrix} 4\sin\left(\frac{\sqrt{7}}{2}t\right) \\ \sqrt{7}\cos\left(\frac{\sqrt{7}}{2}t\right) - 3\sin\left(\frac{\sqrt{7}}{2}t\right) \end{pmatrix}$$

$$\boldsymbol{\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1\begin{pmatrix} 4\cos\left(\frac{\sqrt{7}}{2}t\right) \\ -3\cos\left(\frac{\sqrt{7}}{2}t\right) - \sqrt{7}\sin\left(\frac{\sqrt{7}}{2}t\right) \end{pmatrix} e^{\frac{3}{2}t} + c_2\begin{pmatrix} 4\sin\left(\frac{\sqrt{7}}{2}t\right) \\ \sqrt{7}\cos\left(\frac{\sqrt{7}}{2}t\right) - 3\sin\left(\frac{\sqrt{7}}{2}t\right) \end{pmatrix} e^{\frac{3}{2}t}}$$

Because of the imaginary eigenvalues, with the real part being positive, the phase portrait has an unstable focus

Problem 4: Consider the differential equations $\dot{\boldsymbol{x}} = J_i\mathbf{x}$, where J_i is each of the following matrices:

$$J_1 = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 5 & 3 \\ -2 & 2 \end{pmatrix} \qquad J_3 = \begin{pmatrix} 1 & -3 \\ 2 & -5 \end{pmatrix} \qquad J_4 = \begin{pmatrix} 3 & -2 \\ 6 & -3 \end{pmatrix}$$

Use the diagram on Slide 55 to classify the qualitative behavior for these differential equations (J_i, $i = 1, 2, 3, 4$) without solving the equations.

Solution 4:

$$\begin{aligned} \text{For } J_1, \text{ the eigenvalues are:} \\ (\lambda - 2)(\lambda + 3) + 4 = 0 \\ \lambda^2 + \lambda - 2 = 0 \\ \lambda = 2, -1 \end{aligned} \qquad \begin{aligned} \text{For } J_1, \text{ the Discriminant is:} \\ D_1 = (2 - -3)^2 + 4(4)(-1) \\ D_1 > 0 \end{aligned}$$

$$\begin{aligned} \text{For } J_2, \text{ the eigenvalues are:} \\ (\lambda - 5)(\lambda - 2) + 6 = 0 \\ \lambda^2 - 7\lambda + 16 = 0 \\ \lambda = \frac{7 \pm \sqrt{15}i}{2} \end{aligned} \qquad \begin{aligned} \text{For } J_2, \text{ the Discriminant is:} \\ D_2 = (5 - 2)^2 + 4(3)(-2) \\ D_2 < 0 \end{aligned}$$

$$\begin{aligned} \text{For } J_3, \text{ the eigenvalues are:} \\ (\lambda - 1)(\lambda + 5) + 6 = 0 \\ \lambda^2 + 4\lambda + 1 = 0 \\ \lambda = \frac{-4 \pm \sqrt{12}}{2} \end{aligned} \qquad \begin{aligned} \text{For } J_3, \text{ the Discriminant is:} \\ D_3 = (1 - -5)^2 + 4(-3)(2) \\ D_3 > 0 \end{aligned}$$

$$\begin{aligned} \text{For } J_4, \text{ the eigenvalues are:} \\ (\lambda - 3)(\lambda + 3) + 12 = 0 \\ \lambda^2 + 3 = 0 \\ \lambda = \pm\sqrt{3}i \end{aligned} \qquad \begin{aligned} \text{For } J_4, \text{ the Discriminant is:} \\ D_4 = (3 - -3)^2 + 4(-2)(6) \\ D_4 < 0 \end{aligned}$$

By the Diagram the following is true:

1. J_1's Phase Portrait is a Saddle Point
2. J_2's Phase Portrait is an Unstable Focus
3. J_3's Phase Portrait is a Stable Node
4. J_4's Phase Portrait is a Center