Homework 10
Abstract Algebra
Math 320
Stephen Giang

Problem 4.5 - 1(a): Use the Rational Root Test to write each polynomial as a product of irreducible polynomials in $\mathbb{Q}[x]$:

$$-x^4 + x^3 + x^2 + x + 2$$

The Rational Root Test says that the possible roots of this equation are $\pm 1, \pm 2$. If we let $f(x) = -x^4 + x^3 + x^2 + x + 2$, we notice the following:

$$f(1) = 4 \qquad f(-1) = 0$$
$$f(2) = 0 \qquad f(-2) = -20$$

So we know now that $(x+1)$ and $(x-2)$ are factors of $f(x)$. After long division, we can see:

$$f(x) = -x^4 + x^3 + x^2 + x + 2 = (x+1)(x-2)(-x^2-1)$$

We also know that $(-x^2 - 1)$ is also irreducible, as its factors can only be of degree one, meaning that if it is irreducible, then it has no roots. This is true as its roots are $\pm i \notin \mathbb{Q}$. Thus we are done.

Problem 4.5 - 4(b): Show that each polynomial is irreducible in $\mathbb{Q}[x]$, as in Example 3.

$$x^4 - 2x^2 + 8x + 1$$

We can see through the Rational Root Test, that the only possible roots would be ± 1. By evaluating it at these values, we can see that the equation does not have any roots. Thus the only factors out of $f(x) = x^4 - 2x^2 + 8x + 1$ are of degree 2, such that for some $a, b, c, d \in \mathbb{Z}$:

$$\begin{aligned} f(x) = x^4 - 2x^2 + 8x + 1 &= (x^2 + ax + b)(x^2 + cx + d) \\ &= x^4 + (a + c)x^3 + (ac + b + d)x^2 + (bc + ad)x + bd \end{aligned}$$

Now we just need to solve for a, b, c, d

$$a + c = 0 \tag{1}$$
$$ac + b + d = -2 \tag{2}$$
$$bc + ad = 8 \tag{3}$$
$$bd = 1 \tag{4}$$

So we can see that $a = -c$ from (1). We can also see that the only choices for b, d is $b = d = 1$ or $b = d = -1$ from (4). After evaluating this into (3), we get $c(b - d) = -8$. Because $b = d$, then the following is impossible as $b - d = 0$, and anything times 0 is 0. Thus we have proved that there does not exist a factorization in $\mathbb{Z}[x]$, and hence also in $\mathbb{Q}[x]$.

Problem 4.5 - 5: Use Eisenstein's Criterion to show that each polynomial is irreducible in $\mathbb{Q}[x]$.

(a) $x^5 - 4x + 22$.

By Eisenstein's Criterion, we can choose a prime number $p = 2$. Because 2 does not divide the coefficient of x^5, 1 , but does divide the other coefficients, -4 and 22, as well as $p^2 = 4$ also does not divide the constant term, 22 , (a) is irreducible.

$$2 \not| 1 \qquad 2|\{-4, 22\} \qquad 4 \not| 22$$

(b) $-7x^4 + 25x^2 - 15x + 10$.

By Eisenstein's Criterion, we can choose a prime number $p = 5$. Because 5 does not divide the coefficient of $-7x^4$, -7 , but does divide the other coefficients, {25, -15, and 10}, as well as $p^2 = 25$ also does not divide the constant term, 10 , (b) is irreducible.

$$5 \not| -7 \qquad 5|\{25, -15, 10\} \qquad 25 \not| 10$$

(c) $5x^{11} - 6x^4 + 12x^3 + 36x - 6$

By Eisenstein's Criterion, we can choose a prime number $p = 3$. Because 3 does not divide the coefficient of $5x^{11}$, 5 , but does divide the other coefficients, {-6, -12, 36, and -6}, as well as $p^2 = 9$ also does not divide the constant term, -6 , (c) is irreducible.

$$3 \not| 5 \qquad 3|\{-6, -12, 36, -6\} \qquad 9 \not| -6$$