Mid Term Part A
Math 537 Ordinary Differential Equations
Due Sep 30, 2020

Student Name: ______________________ **ID** __________

Rules

A. The exam must be taken completely alone. Showing it or discussing it with anybody is forbidden.

B. Make an effort to make your submission clear and readable. Severe readability issues may be penalized by grade.

C. Please submit your work to Gradescope by 11:59 pm on Sep. 30, 2020.

1: [30 points] The well-known "SIR" epidemic model (Kermack and McKendrick, 1927) consists of three first-order ordinary differential equations (ODEs) for three time dependent variables, S, I, and R, that represent susceptible, infected, and recovered individuals, respectively. In HW2, we have reduced the system of three ODEs into a single ODE with one time dependent variable R, as follows:

$$\frac{dR}{dt} = \nu \left(N - R - S(0)e^{-\frac{\beta}{N\nu}(R(t)-R(0))} \right). \tag{1.1}$$

Here, three parameters, $\beta > 0$, $\nu > 0$, and $N > 0$, represent a transmission rate, a recovery rate, and a fixed population ($N = S + I + R$), respectively. $S(0)$ and $R(0)$ denote the initial values of S and R, respectively. Complete the following problems.

(a) [6 points] Consider the following logistic equation:

$$\frac{df}{dt} = f(1 - f). \tag{1.2}$$

Introduce a new dependent variable (g) to transform Eq. (1.2) into the following ODE:

$$\frac{dg}{dt} = \frac{1}{4} - g^2. \tag{1.3}$$

(b) [8 points] Express the solutions of Eqs. (1.2) and (1.3) in terms of the sigmoid and hyperbolic tangent functions, respectively.

(c) [6 points] Apply a Taylor series expansion with $e^{-x} \approx 1 - x + x^2/2$ to simplify the term $e^{-\frac{\beta}{N\nu}(R(t)-R(0))}$ in Eq. (1.1). Then, perform a (linear) stability analysis.

(d) [10 points] Solve the ODE derived in problem (1c) using a small non-negative $R(0)$.

2: [25 points] A nonlinear, non-dissipative Lorenz model is written as follows:

$$\frac{d^2X}{dt^2} - (\sigma r + C)\, X + \frac{X^3}{2} = 0. \tag{2}$$

Here, we assume that both σ and r are positive, and choose $C = 0$ for convenience. Complete the following problems.

(a) [3 points] Transform the 2nd order ODE in Eq. (2) into a system of the first order ODEs, (i.e., $Y = X'$).

(b) [3 points] Find critical points in the above 2D system in problem (2a).

(c) [6 points] Compute the Jacobian matrix of the above 2D system.

(d) [13 points] Perform a linear stability analysis for all of the critical points.

3: [25 points] Consider the general, linear, 2D system as follows:

$$X' = AX, \tag{3.1}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } X = \begin{pmatrix} x \\ y \end{pmatrix}.$$

By properly choosing a linear map (or linear transformation) T, the above system can be transformed into the system with its matrix in one of the following three forms:

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}. \tag{3.2}$$

(a) [5 points] Discuss the conditions under which the general system in Eq. (3.1) can be transformed into the system with one of of the matrices in Eq. (3.2).

(b) [10 points] Discuss how to construct a linear map to achieve the goals in problem (3a) for all of the three cases. [Hints: construct a 2x2 matrix T that can convert the given linear system into one with a different coefficient matrix that is in canonical form.]

(c) [10 points] Apply $(a, b, c, d) = (-2, 1, -9/4, 1)$ to illustrate the above procedures in problem (3b). [Hint: construct T and compute $T^{-1}AT$.]

4: [20 points] Show off Your Skills and and Knowledge.

(a) [7 points] Design your problem using the skills and knowledge that have been discussed in the textbook or lectures.

(b) [7 points] Discuss why your problem is unique, as compared to the above problems and/or problems in homework (1-2).

(c) [6 points] Solve the problem.