Today 11/5

- Turn in HW 4 Thursday - 1.
- Test 2, Thursday 11/21
- HW 5 & HW 6 last 2 HWs
  - each will have new & review problems

— Limits of Functions 3.7.

So far...

① Limits of sequences

② Continuity of functions

Setup: Let $D \subseteq \mathbb{R}$. We say $x_0 \in \mathbb{R}$ is a limit point of D iff

$$\exists \{x_n\} \subseteq D \setminus \{x_0\} \text{ s.t. } \lim_{n\to\infty} x_n = x_0.$$

Examples:

① Consider $(1,3) = D$.

The set $[1,3]$ is all limit points of D

② Consider $D = [2,10) \cup \{11, 12, 13\}$.

The set of all limit points of D is $[2,10]$.

Definition: Suppose $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$.

Suppose x_0 is a limit pt of D. Then

$$\lim_{x \to x_0} f(x) = l \in \mathbb{R}$$

iff

$$\forall \{x_n\} \in D \setminus \{x_0\}, \text{ if } \lim_{n\to\infty} x_n = x_0,$$

$$\text{then } \lim_{n\to\infty} f(x_n) = l.$$

"The limit of f at x_0 is l"

$\boxed{\varepsilon-\delta \text{ Limit Thm.}}$ Suppose $f: D \to \mathbb{R}$ and x_0 is a l.p. of D.

$$\lim_{x \to x_0} f(x) = \ell$$

iff

$\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $\forall x \in D$ if $0 < |x - x_0| < \delta$, then

$$|f(x) - \ell| < \varepsilon.$$

<u>proof:</u> $(\longrightarrow)$ <u>Contrapositive</u>

Suppose $\exists \varepsilon > 0$, $\forall \delta > 0$, $\exists x \in D$ where $0 < |x - x_0| < \delta$ <u>and</u> $|f(x) - \ell| \geq \varepsilon$

(Show $\exists \{x_n\} \subseteq D \setminus \{x_0\}$ s.t. $\lim_{n\to\infty} x_n = x_0$ <u>and</u> $\lim_{n\to\infty} f(x_n) \neq \ell$.)

Let $n \geq 1$. Use $\delta = 1/n$. Then $\exists x_n \in D$ s.t.

$$0 < |x_n - x_0| < \frac{1}{n} \text{ and } |f(x_n) - \ell| \geq \varepsilon.$$

Notice that $\lim_{n\to\infty} x_n = x_0$. <u>and</u> $\lim_{n\to\infty} f(x_n) \neq \ell$.

($\Leftarrow$) Suppose:

$\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $\forall x \in D$ if $0 < |x - x_0| < \delta$,
then $|f(x) - \ell| < \varepsilon$.

Let $\{x_n\} \subseteq D \setminus \{x_0\}$, and suppose $\lim_{n\to\infty} x_n = x_0$.

(Show $\lim_{n\to\infty} f(x_n) = \ell$.).

Let $\varepsilon > 0$. Note $\exists \delta > 0$ s.t. $\forall x \in D$ if $0 < |x - x_0| < \delta$, then
$|f(x) - \ell| < \varepsilon$.

Since $\lim_{n\to\infty} x_n = x_0$, $\exists N$ s.t. $\forall n \geq N$, $|x_n - x_0| < \delta$.

Let $n \geq N$. Then $|x_n - x_0| < \delta$ so $|f(x_n) - \ell| < \varepsilon$. $\blacksquare$

Remarks:

① We could now show: Suppose $f: D \to \mathbb{R}$ and $x_0 \in D$ is a l.p. of D.

f is continuous at x_0 iff $\lim_{x \to x_0} f(x) = f(x_0)$.

② "Limit Laws" they work!

③ We can slightly alter definitions to do:

- one-sided limits
- infinite limits

Examples:

① Use sequential arguing:

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2.$$

(Old way: Limit Laws fail.

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}} = 2 \quad)$$

Note 1 is a limit point for the domain of our function $f: \mathbb{R}\setminus\{1\} \to \mathbb{R}$ by $f(x) = \dfrac{x^2-1}{x-1}$.

Let $\{x_n\} \subseteq \mathbb{R}\setminus\{1\}$. Suppose $\lim\limits_{n\to\infty} x_n = 1$.

Consider $\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} \dfrac{x_n^2-1}{x_n-1} = \lim\limits_{n\to\infty} (x_n+1) = 2$.

Using limit laws of sequences.

② Use the ε-δ limit def to show

$$\lim_{x \to 3} (x^2 + 3x) = 18.$$

Let $\varepsilon > 0$

Let $\delta = \min\{1, \frac{\varepsilon}{10}\}$.

Suppose $x \in \mathbb{R}$ and $0 < |x-3| < \delta$.

Since $\delta \leq 1$, $\quad 2 < x < 4$.

So $\quad 8 < x+6 < 10$.

Also since $\delta \leq \frac{\varepsilon}{10}$, $\quad |x-3| < \frac{\varepsilon}{10}$

So $\quad |x-3| \cdot 10 < \varepsilon$

And $\quad |x-3|(x+6) < |x-3| \cdot 10 < \varepsilon$.

So $\quad |x^2+3x-18| < \varepsilon$. $\square$.

SIDE:

WANT: $|f(x) - \ell| < \varepsilon$

Goal: $0 < |x - x_0| < \delta$.

$|x^2+3x-18| < \varepsilon$.

$|(x-3)(x+6)| < \varepsilon$.

$\boxed{x \text{ is near } 3}$

$2 < x < 4$.

$8 < x+6 < 10$.

$|x-3||x+6| < |x-3|(10) < \varepsilon$.

$|x-3| < \frac{\varepsilon}{10}$