MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #15

Chapter 4 Classification of Planar Systems
Dynamical Classification

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Section 4.2: Dynamical Classification

- We will consider two systems to be dynamically equivalent if there is a function h that takes one flow to the other.
- We require that this function be a homeomorphism; that is, h is a one-to-one, onto, and continuous function with an inverse that is also continuous.

Homeomorphism

- A continuous deformation between a coffee mug and a donut (torus) illustrating that they are homeomorphic (wikipedia)
- There need not be a continuous deformation for two spaces to be homeomorphic — only a continuous mapping with a continuous inverse function.

Homeomorphism and Conjugacy (TBD)

Suppose I and J are intervals and $f: I \to I$ and $g: J \to J$. We say that f and g are *conjugate* if there is $\boxed{\text{a homeomorphism } h: I \to J}$ such that h satisfies the *conjugacy equation* $h \circ f = g \circ h$. Just as in the case of flows, a conjugacy takes orbits of f to orbits of g. This follows since we have $h(f^n(x)) = g^n(h(x))$ for all $x \in I$, so h takes the nth point on the orbit of x under f to the nth point on the orbit of $h(x)$ under g. Similarly, h^{-1} takes orbits of g to orbits of f. ■

From the point of view of chaotic systems, conjugacies are important since they map one chaotic system to another.

Proposition. *Suppose $f: I \to I$ and $g: J \to J$ are conjugate via h, where both I and J are are closed intervals in $\mathbb{R}$ of finite length. If f is chaotic on I, then g is chaotic on J.*

Mathematical Problems for the Next Century

- Smale, S., 1998: Mathematical Problems for the Next Century. The Mathematical Intelligencer 20, no. 2, pages 7–15. Smale's List of 1998.

V. I. Arnold, on behalf of the International Mathematical Union has written to a number of mathematicians with a suggestion that they describe some great problems for the next century.

Arnold's invitation is inspired in part by Hilbert's list of 1900 (see e.g. (Browder, 1976)) and I have used that list to help design this essay. I have listed 18 problems

Problem 14: Lorenz attractor.

Is the dynamics of the ordinary differential equations of Lorenz (1963), that of the geometric Lorenz attractor of Williams, Guckenheimer and Yorke?

- Problem 14 asks if the dynamics of the original equations is the same as that of the geometric model.
- The most complete positive answer would be to describe a homeomorphism of R^3 to R^3 which would take solutions of the Lorenz equations to solutions of the geometric attractor.

The Lorenz Attractor Exists (Tucker, 2008)

The Lorenz attractor exists

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(Reçu le 15 janvier 1999, accepté après révision le 12 avril 1999)

Abstract. We prove that the Lorenz equations support a strange attractor, as conjectured by Edward Lorenz in 1963. We also prove that the attractor is robust, i.e., it persists under small perturbations of the coefficients in the underlying differential equations. The proof is based on a combination of normal form theory and rigorous numerical computations. © Académie des Sciences/Elsevier, Paris

of unpredictability in many systems. Numerical simulations for an open neighbourhood of the classical parameter values $\sigma = 10$, $\beta = 8/3$ and $\varrho = 28$ suggest that almost all points in phase space tend to a strange attractor $\mathcal{A}$—*the Lorenz attractor*. Based on numerical data, a geometric model describing the dynamics of the flow was introduced by Guckenheimer and Williams (*see* [2], [7]). We prove that this model does indeed give an accurate description of the dynamics of (1).

Tucker (2002) and Immler (2018)

RESEARCH ARTICLE | 28 SEPTEMBER 2020

Is Weather Chaotic? Coexistence of Chaos and Order within a Generalized Lorenz Model

Bo-Wen Shen ; Roger A. Pielke, Sr.; Xubin Zeng; Jong-Jin Baik; Sara Faghih-Naini; Jialin Cui; Robert Atlas

Bull. Amer. Meteor. Soc. 1–28.

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In response to reviewers comments, we have provided more than 100 pages responses, including the following from Tucker (2002) and Immler (2018):

Here, recent mathematical work on the Lorenz model is worth mentioning. To understand how accurately the geometric model of Guckenheimer and Williams describes the dynamics of the strange attractor within the Lorenz model, the 14th mathematical problem of Smale's list (Smale, 1998) looked for a proof (i.e. homeomorphism) for revealing dynamic equivalence between the two models. In 2002, a rigorous proof was provided by Tucker (2002]). Tucker's study suggests that the Lorenz strange attractor is not a numerical artifact (Stewart, 2000). More recently, Immler (2018) completed a dissertation entitled "A Verified ODE Solver and Smale's 14th Problem", turning the numerical portion of Tucker's proof into solid formal foundations.

A Geometric Model for the Lorenz Attractor

1. The following geometric model for the Lorenz attractor was originally proposed by Guckenheimer and Williams (1979).

$$x' = -3x$$
$$y' = 2y$$
$$z' = -z.$$

- Linear,
- Uncoupled,
- Two negative eigenvalues
- **One critical point (a saddle)**
- **No recurrence**

2. Tucker (1999) showed that this model does indeed correspond to the Lorenz system for certain parameters.

3. Stewart (2002) stated *"thanks to Tucker, dynamical systems theorists can at last stop worrying about whether their most potent icon might suddenly fall apart. And Lorenz's original insight, that the strange behavior of his equations was not a numerical artefact, can no longer be disputed."*

- Guckenheimer, J., and Williams, R. F., 1979: Structural stability of Lorenz attractors. Publ. Math. IHES . 50 (1979), 59.
- Tucker, W., 1999: The Lorenz attractor exists. C. R. Acad. Sci. Paris Sér. I Math. 32, 1197.
- Stewart, I., 2000: The Lorenz attractor exists. Nature, vol 406, No 6799, 948-949.

The Lorenz Model and the Geometric Model

The Lorenz Model

$$\frac{dX}{d\tau} = \boxed{-\sigma X} + \boxed{\sigma Y},$$

$$\frac{dY}{d\tau} = -XZ + \boxed{rX} - Y,$$

$$\frac{dZ}{d\tau} = XY \boxed{- bZ}.$$

A Geometric Model by

Guckenheimer and Williams (1979)

$$x' = \boxed{-3x}$$

$$y' = \boxed{2y}$$

$$z' = \boxed{-z}.$$

- Missing nonlinear terms (no -XZ and XY)
- Missing recurrence (no complex eigenvalues)

➔ Understanding Butterfly Effects, Searching for Recurrence

Simplified Models

(1) The Lorenz Model	(2) The Geometric Model
$\frac{dX}{d\tau} = -\sigma X + \sigma Y,$ $\frac{dY}{d\tau} = -XZ + rX - Y,$ $\frac{dZ}{d\tau} = XY - bZ.$	$\frac{dX}{d\tau} = -3X,$ $\frac{dY}{d\tau} = 2Y,$ $\frac{dZ}{d\tau} = -Z.$
(3) The Limiting Equations	**(4) The Non-dissipative Lorenz Model**
$\frac{dX}{d\tau} = \sigma Y,$ $\frac{dY}{d\tau} = -XZ,$ $\frac{dZ}{d\tau} = XY.$	$\frac{dX}{d\tau} = \sigma Y,$ $\frac{dY}{d\tau} = -XZ + rX,$ $\frac{dZ}{d\tau} = XY.$ complex eigenvalues

Table S1: The Lorenz model (Lorenz, 1963) and three simplified versions, including the geometric model (Guckenheimer and Williams, 1979), the limiting equations (Sparrow, 1982), and the non-dissipative Lorenz model (e.g., Shen 2018).

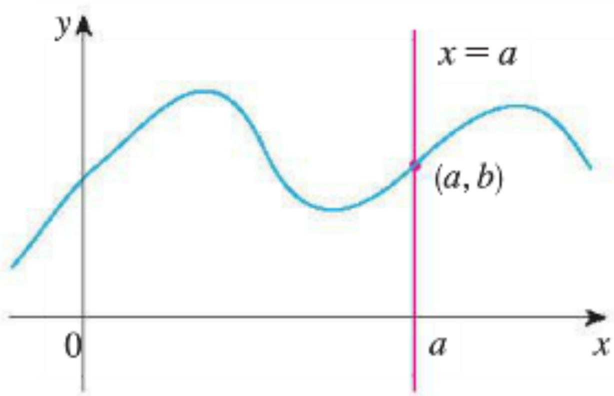
2: [25 points] A nonlinear, non-dissipative Lorenz model is written as follows:

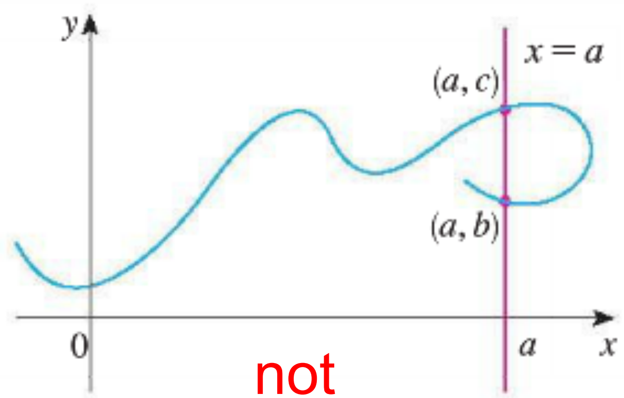
$$\frac{d^2X}{dt^2} - (\sigma r + C)\, X + \frac{X^3}{2} = 0. \qquad (2)$$

A Brief Review: A Function

A **function** f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

The Vertical Line Test A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.





not
a function

- Given an x, there is a y, y=f(x).
- The vertical line test can be applied to verify the above
- A function may have the property of many-to-one, more than one intersections for y=f(x) and a horizontal line, say, y=y1.

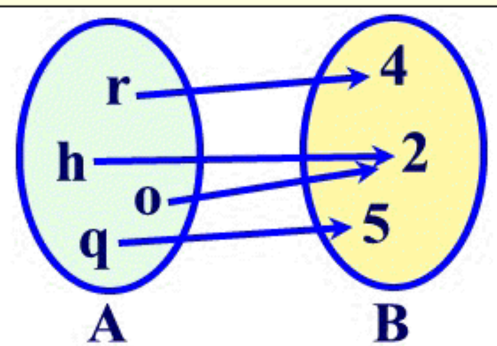
onto (surjective)

Definition: Let A and B be intervals and f: A→B. The function f(x) is onto if for any y in B there is an $x \in A$ such that f(x)=y. [Devaney, Def 2.2]

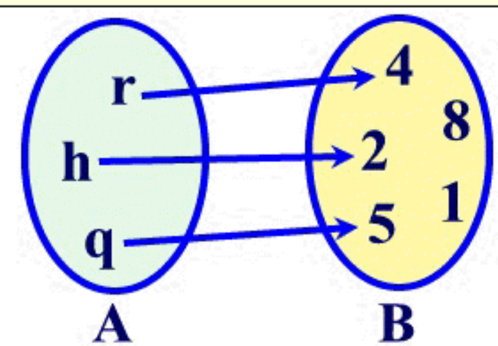
Onto Function

A function f from **A** to **B** is called **onto** if for all b in **B** there is an a in **A** such that $f(a) = b$. All elements in **B** are used.

Such functions are referred to as **surjective**.



"Onto"
(all elements in B are used)



NOT "Onto"
(the 8 and 1 in Set B are not used)

one-to-one “function” (Injective)

1 Definition A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

[Devaney, Def 2.1]

Horizontal Line Test A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Increasing or decreasing functions are the only type of continuous one-to-one functions of a real variable.

not one-to-one

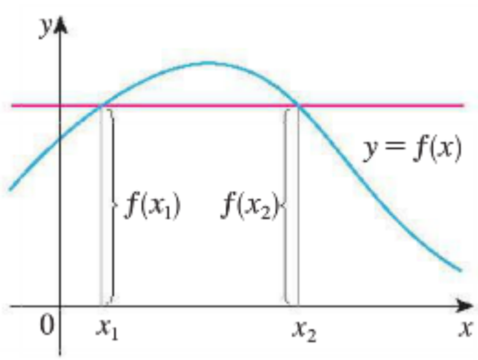


FIGURE 2
This function is not one-to-one because $f(x_1) = f(x_2)$.

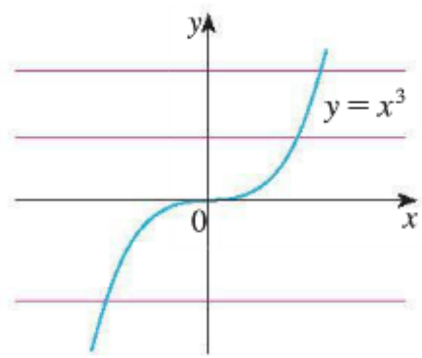
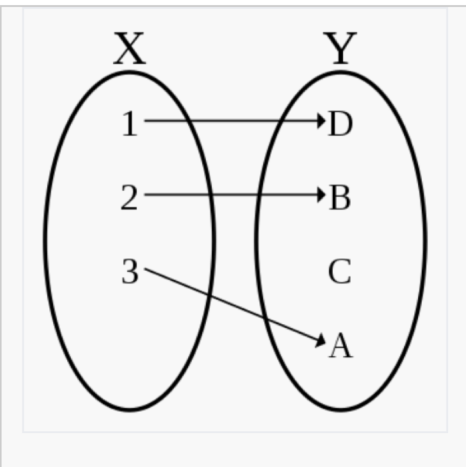


FIGURE 3
$f(x) = x^3$ is one-to-one.

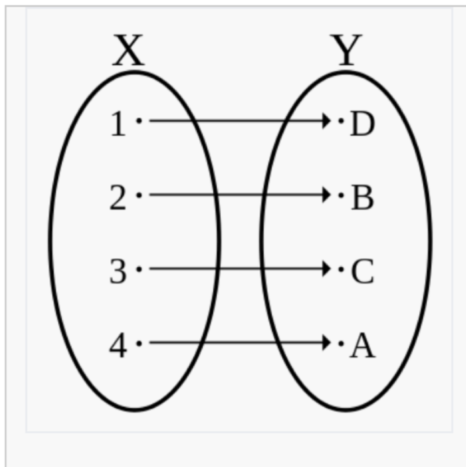
bijective function (one-to-one + onto)

one-to-one functions vs. bijective functions

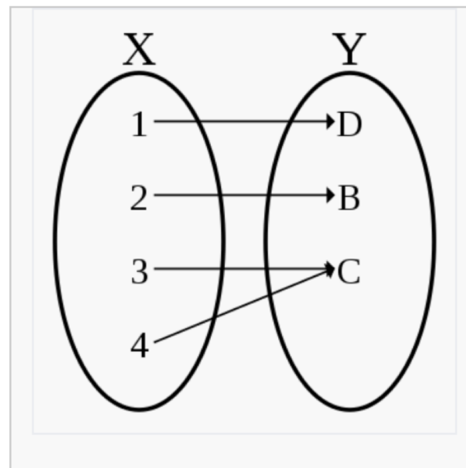
- The term one-to-one function must not be confused with one-to-one correspondence (a.k.a. bijective function, one-to-one + onto), which uniquely maps all elements in both domain and codomain to each other (see figures).



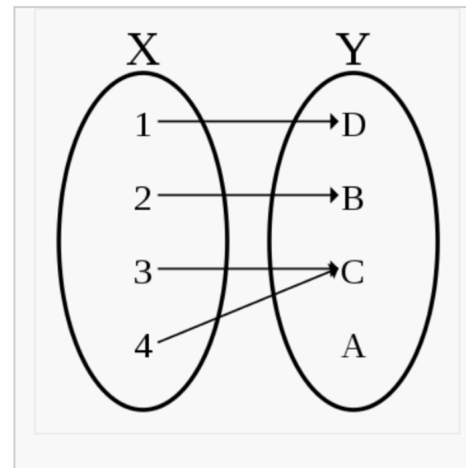
An injective non-surjective function (injection, not a bijection)



An injective surjective function (bijection)



A non-injective surjective function (surjection, not a bijection)



A non-injective non-surjective function (also not a bijection)

Inverse Function

2 Definition Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

$$\text{domain of } f^{-1} = \text{range of } f$$

$$\text{range of } f^{-1} = \text{domain of } f$$

Homeomorphism

Let f: R→ R be a function. We say that $f(\mathrm{x})$ is of class C^r on I if $f^{(r)}(\mathrm{x})$, the r^{th}derivative, exists and is continuous at all x. The function $f(\mathrm{x})$ is C^∞ , if all derivatives exist and are continuous.

Let f: A→ B. The function $f(\mathrm{x})$ is a homeomorphism if $f(\mathrm{x})$ is one-to-one, onto, and continuous, and $f^{-1}(\mathrm{x})$ is continuous. [Devaney, Def 2.3]

C^0

For example, tan(x) is a homeomorphism between (-π/2, π/2) and R. Thus, we say the open internal (-π/2, π/2) is homeomorphic to R.

Diffeomorphism (Supp)

Let f: A→ B. The function $f(\mathrm{x})$ is a C^r-diffeomorphism if $f(\mathrm{x})$ is a C^r–homeomorphism such that $f^{-1}(\mathrm{x})$ is also C^r. [a differentiable homeomorphism with differentiable inverse] [Devaney, Def 2.4]

For example,

- $\tan(\mathrm{x})$ is a C^∞ diffeomorphism from (-π/2, π/2) to R,
- whereas $f(\mathrm{x}) = x^3$ is a homeomorphism, which is not diffeomorphism since as $f^{-1}(\mathrm{x}) = x^{1/3}$, its derivative $(f^{-1})'(x = 0)$ does not exist.

Review: Isomorphism and homomorphism (wiki) (Supp)

- For most algebraic structures, including groups and rings, a homomorphism is an isomorphism if and only if it is bijective (i.e., onto and one-to-one)
- In topology, where the morphisms are continuous functions, isomorphisms are also called homeomorphisms or bicontinuous functions.
- In mathematical analysis, where the morphisms are differentiable functions, isomorphisms are also called diffeomorphisms.

Guckenheimer and Holmes (1983)

- C^k function: A function is C^k if it is k-times differentiable.
- Diffeomorphism: A C^k-diffeomorphism $f: M \to N$ is a mapping f which is 1-1, onto, and has the property that both f and f^{-1} are k-times differentiable.
- Homeomorphism: A homeomorphism is a C^0 diffeomorphism, i.e. a continuous mapping $f: M \to N$ with a continuous inverse.

Composition

Definition Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(g(x))$ are defined. Figure 11 shows how to picture $f \circ g$ in terms of machines.

EXAMPLE 6 If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

SOLUTION We have

$$f \circ g = f\big(g(x)\big)$$ replace x by g in f

$$g \circ f = g\big(f(x)\big)$$ replace x by f in g

- Find the $f \circ g$ and $g \circ f$
- Send your results via "chat"
- You have 5 minutes

Composition

Definition Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

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EXAMPLE 6 If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

SOLUTION We have

$$(f \circ g)(x) = f(g(x)) = f(x - 3) = (x - 3)^2$$

replace x by g in f

$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 - 3$$

replace x by f in g

Composition vs. Matrix Multiplication

$$L(x) = Ax,$$

which shows that linear maps (L) and matrices (A) are intimately related. The matrix A is called the matrix representation of L.

- Composition of maps is most important in dynamical systems.
- For linear maps, composition is intimately related to matrix multiplication, as shown below.

Proposition: Let L and P be linear maps with matrix representations A and B, respectively. Then,

$$P \circ L(v) = (B \cdot A)v \text{ for all } v \in R^n$$

Linearly Conjugate

Proposition: If L_1 and L_2 are linearly conjugate,

$$L_1(x) = A_1 x \text{ and } L_2(x) = A_2 x$$

Then, A_1 and A_2 have the same eigenvalues.

Definition: Let L_1 and L_2 be linear maps of R^n. Let L_1 and L_2 are linearly conjugate if there is an invertible linear map P such that

$$L_1 = P^{-1} \circ L_2 \circ P \qquad e.g., D = T^{-1}AT$$

How to find P (or T)? Construct T using eigenvectors of A:

$$P = [V_1, V_2 ..., V_n], \text{ V}_j \text{ are eigenvectors.}$$

Similarity and Similarity Transformation (Supp)

Similar Matrices. Similarity Transformation

An $n \times n$ matrix $\hat{\mathbf{A}}$ is called **similar** to an $n \times n$ matrix $\mathbf{A}$ if

$$\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} \tag{4}$$

for some (nonsingular!) $n \times n$ matrix $\mathbf{P}$. This transformation, which gives $\hat{\mathbf{A}}$ from $\mathbf{A}$, is called a **similarity transformation**.

Eigenvalues and Eigenvectors of Similar Matrices

If $\hat{\mathbf{A}}$ is similar to $\mathbf{A}$, then $\hat{\mathbf{A}}$ has the same eigenvalues as $\mathbf{A}$.

Furthermore, if $\mathbf{x}$ is an eigenvector of $\mathbf{A}$, then $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ is an eigenvector of $\hat{\mathbf{A}}$ corresponding to the same eigenvalue.

A and $\hat{A}$ are similar when they have the same eigenvalues.

Dynamical Equivalence

(Supp)

L_1 and L_2 are linearly conjugate if $L_1 = P^{-1} \circ L_2 \circ P$

Definition 3.10 Flow Equivalence

Two flows are conjugate (equivalent) if there exists a one to one map g between corresponding orbits or

$$\phi_t^2 \; o \; g = g \; o \; \phi_t^1$$

The flows are

1. **linearly conjugate** *if g is a linear map, then* $g \in C^\infty$,
2. **differentiably conjugate** *if g is a diffeomorphism,* $g \in C^k,\ k \geq 1$,
3. **topologically conjugate** *if g is a homoeomorphism* $g \in \boxed{C^0}$.

Lemma 3.3 *Two linear systems are linearly conjugate if and only if they have the same eigenvalues with the same algebraic and geometric multiplicity.*

Topological Conjugacy

- We will consider two systems to be dynamically equivalent if there is a function h that takes one flow to the other.
- We require that this function be a homeomorphism; that is, h is a one-to-one, onto, and continuous function with an inverse that is also continuous.

Definition
Suppose $X' = AX$ and $X' = BX$ have flows ϕ^A and ϕ^B. These two systems are (topologically) *conjugate* if there exists a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies

$$\phi^B(t, h(X_0)) = h(\phi^A(t, X_0)).$$

The homeomorphism h is called a *conjugacy*. Thus a conjugacy takes the solution curves of $X' = AX$ to those of $X' = BX$.

Topological Conjugacy: $g \circ h = h \circ f$

Definition 7.4: Let f: A→A and g: B→ B be two maps. f and g are said to be topologically conjugate if there exists a homeomorphism h: A→ B such that, $\boxed{g \circ h = h \circ f}$

The homeomorphism h is called a topological conjugacy. [Devaney, Def 7.4]

Mappings which are topologically conjugate are completely equivalent in terms of their dynamics. For example, if f is topologically conjugate to g via h, and p is a fixed point for f, then h(p) is fixed for g. Indeed, h(p)=hf(p)=gh(p).

$$D = T^{-1}AT$$

$$f = h^{-1} \circ g \circ h$$

$$L_1 = P^{-1} \circ L_2 \circ P$$

Topological Conjugacy: $g \circ h = h \circ f$

- We will consider two systems to be dynamically equivalent if there is a function h that takes one flow to the other.
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Definition
Suppose $X' = AX$ and $X' = BX$ have flows ϕ^A and ϕ^B. These two systems are (topologically) *conjugate* if there exists a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies

$$\phi^B(t, h(X_0)) = h(\phi^A(t, X_0)).$$

$$g \circ h = h \circ f$$

$$g = \phi^B; f = \phi^A$$

The homeomorphism h is called a *conjugacy*. Thus a conjugacy takes the solution curves of $X' = AX$ to those of $X' = BX$.

Topological conjugacy

From Wikipedia, the free encyclopedia

In mathematics, two functions are said to be **topologically conjugate** to one another if there exists a homeomorphism that will conjugate the one into the other. Topological conjugacy is important in the study of iterated functions and more generally dynamical systems, since, if the dynamics of one iterated function can be solved, then those for any topologically conjugate function follow trivially.

To illustrate this directly: suppose that f and g are iterated functions, and there exists an h such that

$$g = h^{-1} \circ f \circ h,$$

$$f = h^{-1} \circ g \circ h$$

so that f and g are topologically conjugate. Then of course one must have

$$g^n = h^{-1} \circ f^n \circ h,$$

and so the iterated systems are conjugate as well. Here, $\circ$ denotes function composition.

Example: Topological Conjugacy ($g \circ h = h \circ f$)

Example. For the one-dimensional linear differential equations

$$x' = \lambda_1 x \quad \text{and} \quad x' = \lambda_2 x$$

we have the flows

$$\phi^j(t, x_0) = x_0 e^{\lambda_j t}$$

$0<\lambda_1*\lambda_2$

for $j = 1, 2$. Suppose that λ_1 and λ_2 are nonzero and have the same sign.

Goal:

- We claim that h is a conjugacy between $x'=\lambda_1 x$ and $x'=\lambda_2 x$ if $\lambda_1*\lambda_2>0$.
- Source vs. source
- Sink vs. sink

Example: Topological Conjugacy ($g \circ h = h \circ f$)

Example. For the one-dimensional linear differential equations

$$x' = \lambda_1 x \quad \text{and} \quad x' = \lambda_2 x$$

we have the flows

$$\phi^j(t, x_0) = x_0 e^{\lambda_j t}$$

$0<\lambda_1*\lambda_2$

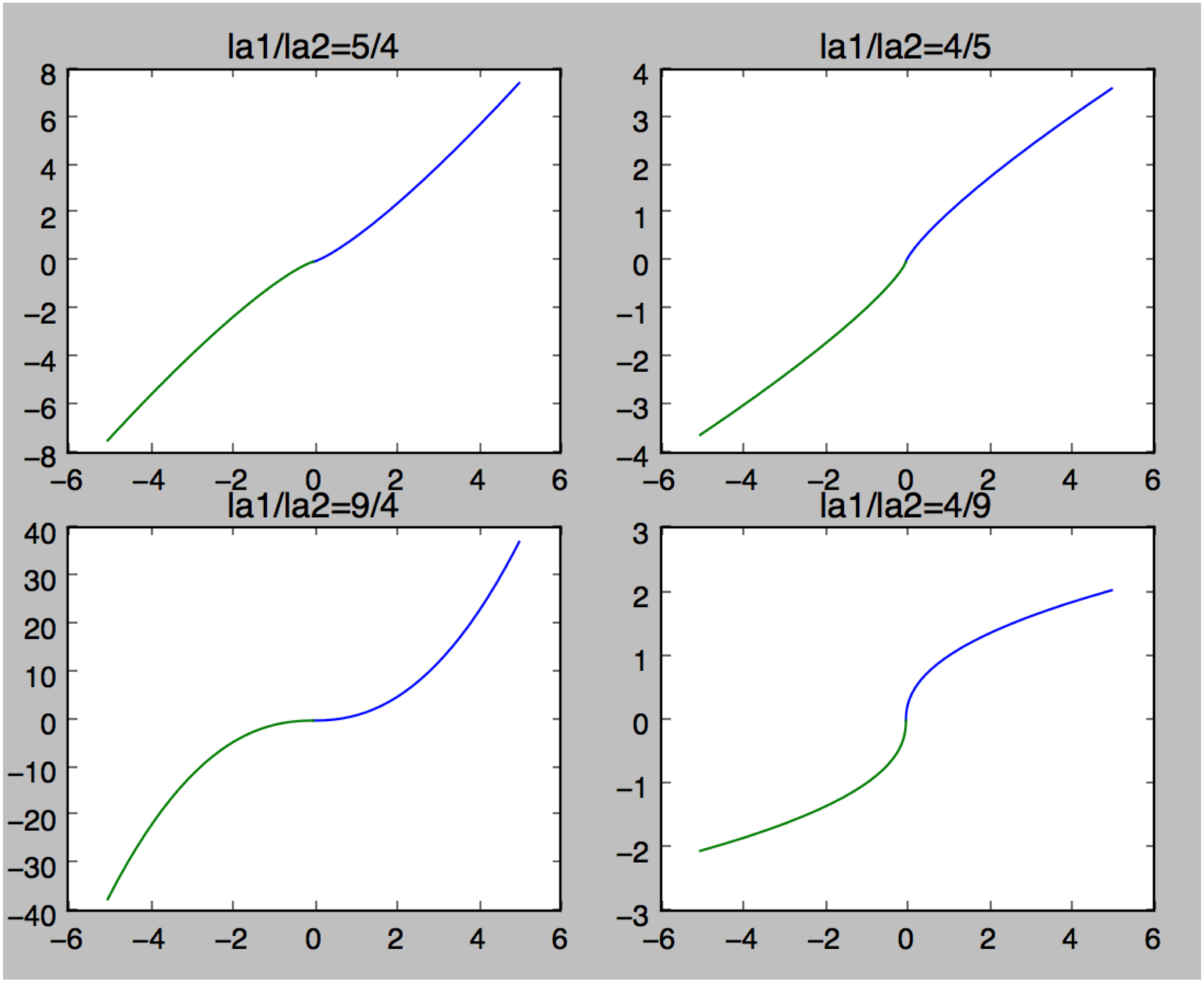
for $j = 1, 2$. Suppose that λ_1 and λ_2 are nonzero and have the same sign.

$$h(x) = \begin{cases} x^{\lambda_2/\lambda_1} & \text{if } x \geq 0 \\ -|x|^{\lambda_2/\lambda_1} & \text{if } x < 0 \end{cases}$$

Goal:

We claim that h is a conjugacy between $x'=\lambda_1 x$ and $x'=\lambda_2 x$ if $\lambda_1*\lambda_2>0$.

A plot for h(x)



A Proof: homeomorphism

$$h(x) = \begin{cases} x^{\lambda_2/\lambda_1}, & x \geq 0 \\ -|x|^{\lambda_2/\lambda_1}, & x < 0 \end{cases}$$

$x \geq 0$

$$ln\left(x^{\lambda_2/\lambda_1}\right) = \frac{\lambda_2}{\lambda_1}\ln(x) \qquad x^{\lambda_2/\lambda_1} = exp\left(\frac{\lambda_2}{\lambda_1}\ln(x)\right)$$

h is a homeomorphism of the real positive axis

$x < 0$

$$y = -x > 0 \qquad ln\left(y^{\lambda_2/\lambda_1}\right) = \frac{\lambda_2}{\lambda_1}\ln(y) \qquad y^{\lambda_2/\lambda_1} = exp\left(\frac{\lambda_2}{\lambda_1}\ln(y)\right)$$

A Proof: $\boldsymbol{g \circ h = h \circ f}$

$$x' = \lambda_1 x, \qquad x = x_0 e^{\lambda_1 t}, \qquad f(t, x_0) = x_0 e^{\lambda_1 t}$$

$$x' = \lambda_2 x, \qquad x = x_0 e^{\lambda_2 t}, \qquad g(t, x_0) = x_0 e^{\lambda_2 t}$$

$x \geq 0$

$$h(x) = x^{\lambda_2/\lambda_1},$$

replace x by f in h

$$\boldsymbol{h \circ f} = h\big(f(t, x_0)\big) = \left(x_0 e^{\lambda_1 t}\right)^{\lambda_2/\lambda_1} = (x_0)^{\lambda_2/\lambda_1} e^{\lambda_2 t}$$

replace x_0 by $h(x_0)$ in g

$$\boldsymbol{g \circ h} = g(t, h(x_0)) = (x_0)^{\lambda_2/\lambda_1} e^{\lambda_2 t}$$

$$h\big(f(t, x_0)\big) = g(t, h(x_0)) \qquad \boldsymbol{g \circ h = h \circ f}$$

A Proof: $\boldsymbol{g \circ h = h \circ f}$

$$x' = \lambda_1 x, \qquad x = x_0 e^{\lambda_1 t}, \qquad f(t, x_0) = x_0 e^{\lambda_1 t}$$

$$x' = \lambda_2 x, \qquad x = x_0 e^{\lambda_2 t}, \qquad g(t, x_0) = x_0 e^{\lambda_2 t}$$

$x < 0$

$$h(x) = -|x|^{\lambda_2/\lambda_1} = -(-x)^{\lambda_2/\lambda_1}$$

$$\boldsymbol{h \circ f} = h\big(f(t, x_0)\big) = -\left(-x_0 e^{\lambda_1 t}\right)^{\lambda_2/\lambda_1} = -(-x_0)^{\lambda_2/\lambda_1} e^{\lambda_2 t}$$

replace x by f in h

$$\boldsymbol{g \circ h} = g(t, h(x_0)) = -(-x_0)^{\lambda_2/\lambda_1} e^{\lambda_2 t}$$

replace x_0 by $h(x_0)$ in g

$$h\big(f(t, x_0)\big) = g(t, h(x_0)) \qquad \boldsymbol{g \circ h = h \circ f}$$

Remarks: 1D Systems: x^{λ_2/λ_1}

There are several things to note here.

- First, λ_1 and λ_2 must have the same sign, because otherwise we would have $|h(0)| = \infty$, in which case h is not a homeomorphism. This agrees with our notion of dynamical equivalence: If λ_1 and λ_2 have the same sign, then their solutions behave similarly as either both tend to the origin or both tend away from the origin.

- Also, note that if $\lambda_2 < \lambda_1$, then h is not differentiable at the origin, whereas if $\lambda_2 > \lambda_1$ then $h^{-1}(x) = (x)^{\lambda_1/\lambda_2}$ is not differentiable at the origin. This is the reason why we require h to be only a homeomorphism and not a diffeomorphism (a differentiable homeomorphism with differentiable inverse): If we assume differentiability, then we must have $\lambda_1 = \lambda_2$, which does not yield a very interesting notion of "equivalence."

- There are three conjugacy "classes": the sinks, the sources, and the special "in-between" case, $x' = 0$, where all solutions are constants. (for $x' = ax$).
  - $\lambda > 0, source$
  - $\lambda < 0, sink$
  - $\lambda = 0,$ constant

Hyperbolic

Definition
A matrix A is *hyperbolic* if none of its eigenvalues has real part 0. We also say that the system $X' = AX$ is *hyperbolic.*

Theorem. *Suppose that the* 2×2 *matrices* A_1 *and* A_2 *are hyperbolic. Then the linear systems* $X' = A_i X$ *are conjugate if and only if each matrix has the same number of eigenvalues with negative real part.* ■

Hyperbolic: $\boldsymbol{Re(\lambda) \neq 0}$

Thus two hyperbolic matrices yield conjugate linear systems if both sets of eigenvalues fall into the same category below:

1. One eigenvalue is positive and the other is negative ($\lambda_1<0<\lambda_2$);
2. Both eigenvalues have negative real parts; $(Re(\lambda_1) < 0, Re(\lambda_2) < 0)$
3. Both eigenvalues have positive real parts. $(Re(\lambda_1) > 0, Re(\lambda_2) > 0)$

Before proving this, note that this theorem implies that a system with a spiral sink is conjugate to a system with a (real) sink. Of course! Even though their phase portraits look very different, it is nevertheless the case that all solutions of both systems share the same fate: They tend to the origin as $t \to \infty$.

Note: For a 2x2 matrix, if eigenvalues (λ_1, λ_2) are complex, then $\lambda_1 = \overline{\lambda_2}$.

$$\left(\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2)\right)$$

Hyperbolicity: $Re(\lambda) \neq 0$

If $\text{Re}(\lambda) \neq 0$ for both eigenvalues, the fixed point is often called ***hyperbolic***. (This is an unfortunate name—it sounds like it should mean "saddle point"—but it has become standard.) Hyperbolic fixed points are sturdy; their stability type is unaffected by small nonlinear terms. Nonhyperbolic fixed points are the fragile ones.

We've already seen a simple instance of hyperbolicity in the context of vector fields on the line. In Section 2.4 we saw that the stability of a fixed point was accurately predicted by the linearization, *as long as* $f'(x^*) \neq 0$. This condition is the exact analog of $\text{Re}(\lambda) \neq 0$.

Strogatz (2015), p156

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Review: Hyperbolicity ($Re(\lambda) \neq 0$)

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1. One eigenvalue is positive and the other is negative ($\lambda_1 < 0 < \lambda_2$);
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$(\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2))$

Figure 4.1: The trace-determinant plane

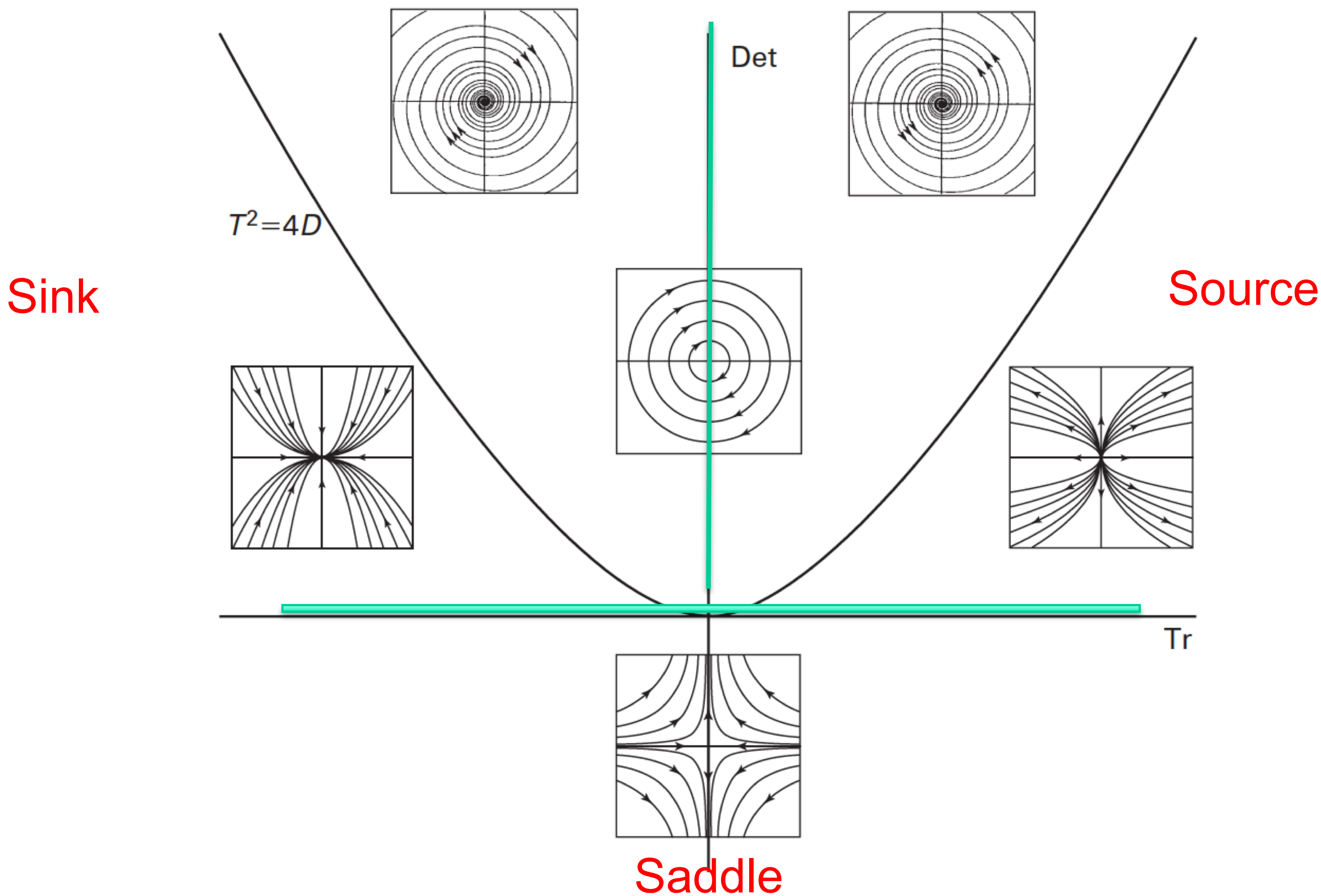

Figure 4.1 The trace-determinant plane. Any resemblance to any of the authors' faces is purely coincidental.

Case 1: $\lambda_j < 0 < \mu_j$ (saddle)

Suppose we have two linear systems $X' = A_i X$ for $i = 1, 2$ such that each A_i has eigenvalues $\lambda_i < 0 < \mu_i$. Thus each system has a saddle at the origin. This is the easy case. As we saw earlier, the real differential equations $x' = \lambda_i x$ have conjugate flows via the homeomorphism

$$h_1(x) = \begin{cases} x^{\lambda_2/\lambda_1} & \text{if } x \geq 0 \\ -|x|^{\lambda_2/\lambda_1} & \text{if } x < 0 \end{cases}.$$

Similarly, the equations $y' = \mu_i y$ have conjugate flows via an analogous function h_2. Now define

$$H(x, y) = (h_1(x), h_2(y)).$$

Then one checks immediately that H provides a conjugacy between these two systems.

$$\begin{array}{lcccc} & h_1(x) & & & h_2(y) \\ A_1: & \boxed{\lambda_1} & < 0 < & & \boxed{\mu_1} \\ A_2: & \boxed{\lambda_2} & < 0 < & & \boxed{\mu_2} \end{array}$$

$$H(x, y) = (h_1(x), h_2(y))$$

Case 2: Two Negative $Re(\lambda_j)$ (Spiral Sink vs. Sink)

Consider the system $X' = AX$ where A is in canonical form with eigenvalues that have negative real parts. We further assume that the matrix A is not in the form

$$\cancel{\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}}$$

with $\lambda < 0$. Thus, in canonical form, A assumes one of the two forms

$$A: \qquad (a) \quad \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \qquad (b) \quad \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

with $\alpha, \lambda, \mu < 0$. We will show that, in either (a) or (b), the system is conjugate to $X' = BX$ where

$$B: \qquad B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It then follows that any two systems of this form are conjugate.

Two systems with A (spiral sink) and B (sink) are conjugate.

Case 3

Finally, suppose that

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

with $\lambda < 0$. The associated vector field need not point inside the unit circle in this case. However, if we let

$$T = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix},$$

then the vector field given by

$$Y' = (T^{-1}AT)Y$$