# MATH 537, Fall 2020

# Ordinary Differential Equations

Lecture #16: Part II

Chapter 5

Higher-Dimensional Linear Algebra

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# Section 5.2

# A Summary For Section 5.2: Eigenvalues

- Eigenvalue Problems

$$\boxed{AV = \lambda V \quad \Rightarrow \quad (A - \lambda I)V = 0 \quad \Rightarrow \quad |A - \lambda I| = 0}$$

- Linearly Independent eigenvectors associated with real and distinct eigenvalues

$\boxed{det|B| \neq 0}$ $\qquad$ $B$ consists of linearly independent vectors

- Diagonalization

$T = (V_1, V_2, \cdots, V_n),$ $\qquad$ $V_j$ are eigenvectors

$$T^{-1}AT = D$$

# 5.2 Eigenvalue Problem and LI Eigenvectors

**Definition**
A vector $V$ is an *eigenvector* of an $n \times n$ matrix $A$ if $V$ is a nonzero solution to the system of linear equations $(A - \lambda I)V = 0$. The quantity $\lambda$ is called an *eigenvalue* of $A$, and $V$ is an eigenvector associated to $\lambda$.

$$\boxed{AV = \lambda V \quad \Rightarrow \quad (A - \lambda I)V = 0 \quad \Rightarrow \quad |A - \lambda I| = 0}$$

**Proposition.** *Suppose $\lambda_1, \ldots, \lambda_\ell$ are real and distinct eigenvalues for $A$ with associated eigenvectors $V_1, \ldots, V_\ell$. Then the $V_j$ are linearly independent.* ■

$\boxed{V_j \text{ are Linearly Independent}}$

# Sect 5.2 Diagonalization

**Corollary.** *Suppose $A$ is an $n \times n$ matrix with real, distinct eigenvalues. Then there is a matrix $T$ such that*

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \boxed{= D}$$

*where all of the entries off the diagonal are* $0$.

For example,

$T = (V_1, V_2, \cdots, V_n)$, $\quad V_j$ are eigenvectors

# Sect 5.2 Eigenvalues and Eigenvectors

**Example.** Let

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix}.$$

$$AV = \lambda V \qquad A - \lambda I = \begin{pmatrix} 1-\lambda & 2 & -1 \\ 0 & 3-\lambda & -2 \\ 0 & 2 & -2-\lambda \end{pmatrix}$$

$$|A - \lambda I = 0| \qquad (1-\lambda)(\lambda^2 - \lambda - 2) = 0 \qquad \lambda = 2, 1, -1$$

$$\lambda_1 = 2 \qquad \lambda_2 = 1 \qquad \lambda_3 = -1$$

$$V_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad V_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

# Construct $T$ and $D$

Let $D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ $\lambda_j$ are eigenvalues of the matrix A

Let $T = (V_1, \quad V_2, \quad V_3)$, $V_j$ are the eigenvectors corresponding to $\lambda_j$

$\lambda_1 = 2$ $\lambda_2 = 1$ $\lambda_3 = -1$

$V_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ $V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $V_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad T = (V_1, \quad V_2, \quad V_3) = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

# Diagnoalization using $T^{-1}AT$

Compute $AT = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix}\begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

- Pick one and compute
- During the difficult time, the following is added to make our life easier:
  - you will receive additional 10 points for your next homework if your selection matches the one I preselected (to appear later) and your answer is correct.
- Send your results via "chat" (e.g., a21=???)
- You have 2 minutes

- You are a winner if you select $a_{33}$ and compute it correctly.

$$AT = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix}\begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 1 & 0 \\ 4 & 0 & -1 \\ 2 & 0 & -2 \end{pmatrix}$$

# Diagnoalization using $T^{-1}AT$

$$AT = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix}\begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 6 & 1 & 0 \\ 4 & 0 & -1 \\ 2 & 0 & -2 \end{pmatrix}$$

What we have done is called “parallel computing”:

- Compute each of $a_{ij}$ in parallel
- Load imbalance may appear, e.g., $a_{12} = 1 + 2 * 0 * +(-1) * 0$
- The last element determines the timing for the entire task.

In the supercomputing world, we may additionally perform the following:

- Decompose data (domain or tasks) into smaller pieces of data;
- Assign sub-tasks (e.g., broadcast data) to different CPUs;
- Gather all of sub-tasks and put them together

# Ch 5. Medium-grained Loop Parallelism

Supp

P1

```
for i in range (0, N/2):
    a[i]=c*b[i]
```

P2

```
for i in range (N/2, N):
    a[i]=c*b[i]
```

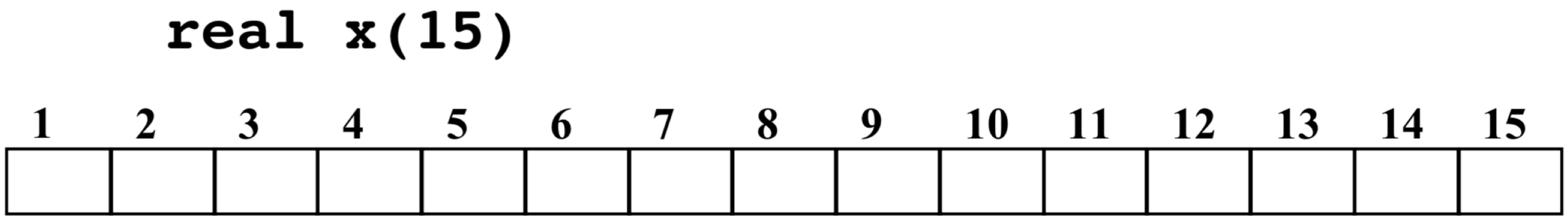
```
N=1000
N2=N/2
a=np.zeros(N)
b=np.linspace(0, N-1, N)
c=5

for i in range (0, N):
    a[i]=c*b[i]
```

```
x1 = np.linspace(0, 10, N, endpoint=True)
x2 = np.linspace(0, 10, N, endpoint=False)
```

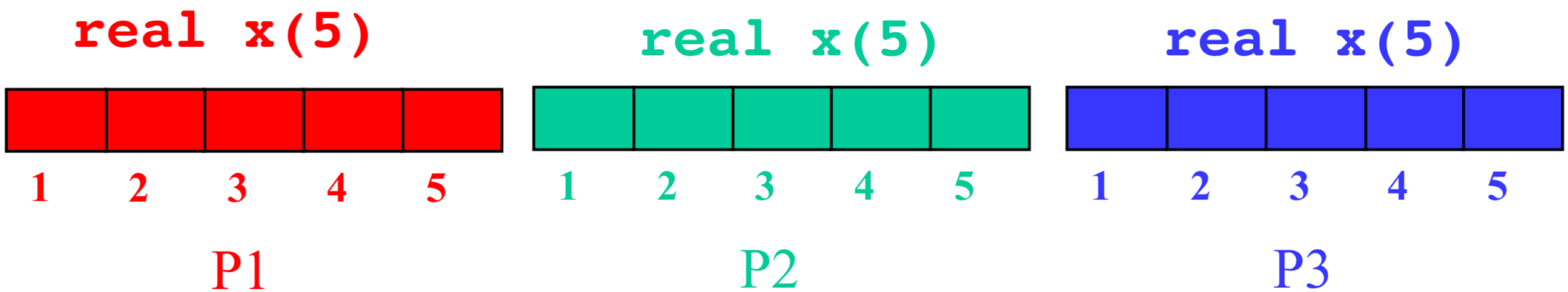
```
xsum = 0.0
 do i=1,15
   xsum =  xsum + x(i)
enddo
```

```
real x(15)
```

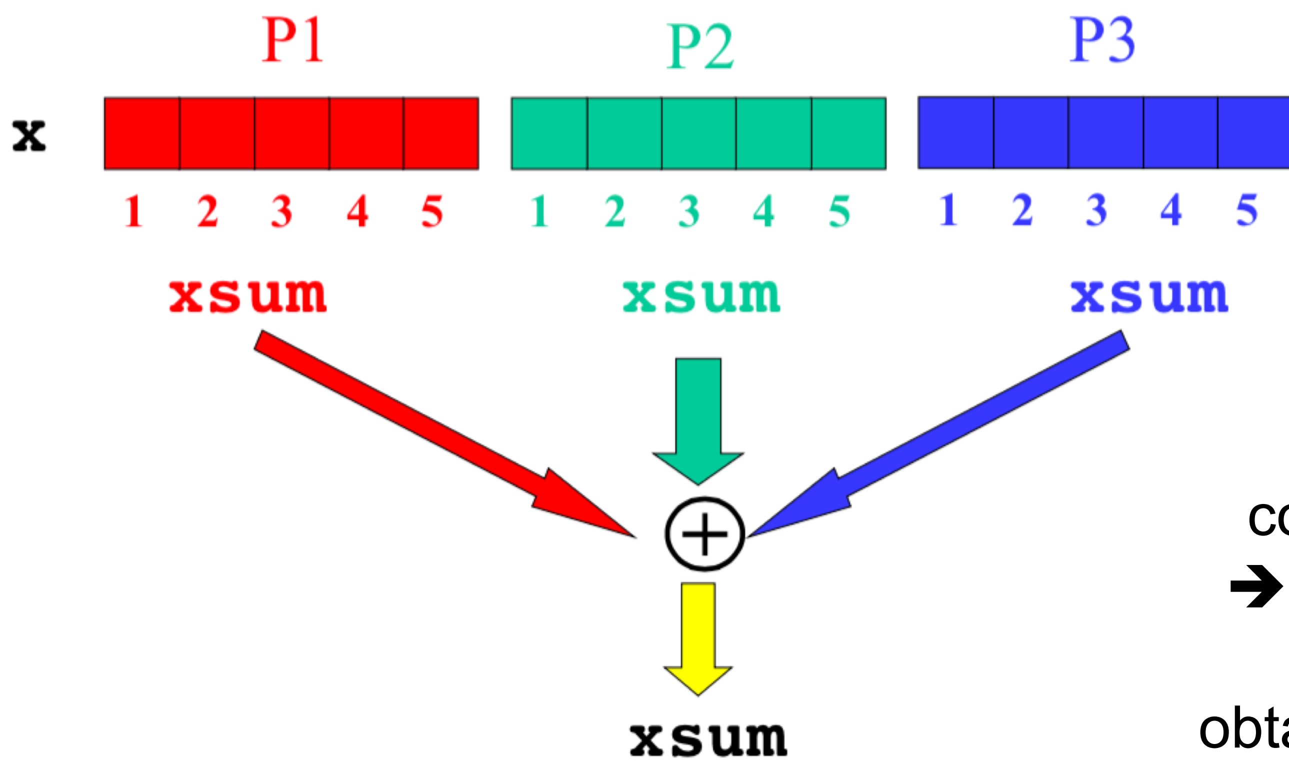
| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| “Global” | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| “Local” | 1 | 2 | 3 | 4 | 5 | 1 | 2 | 3 | 4 | 5 | 1 | 2 | 3 | 4 | 5 |
| | | | P1 | | | | | P2 | | | | | P3 | | |




## Parallel Code



```
xsum = 0.0
 do i=1,15
    xsum = xsum + x(i)
enddo
```

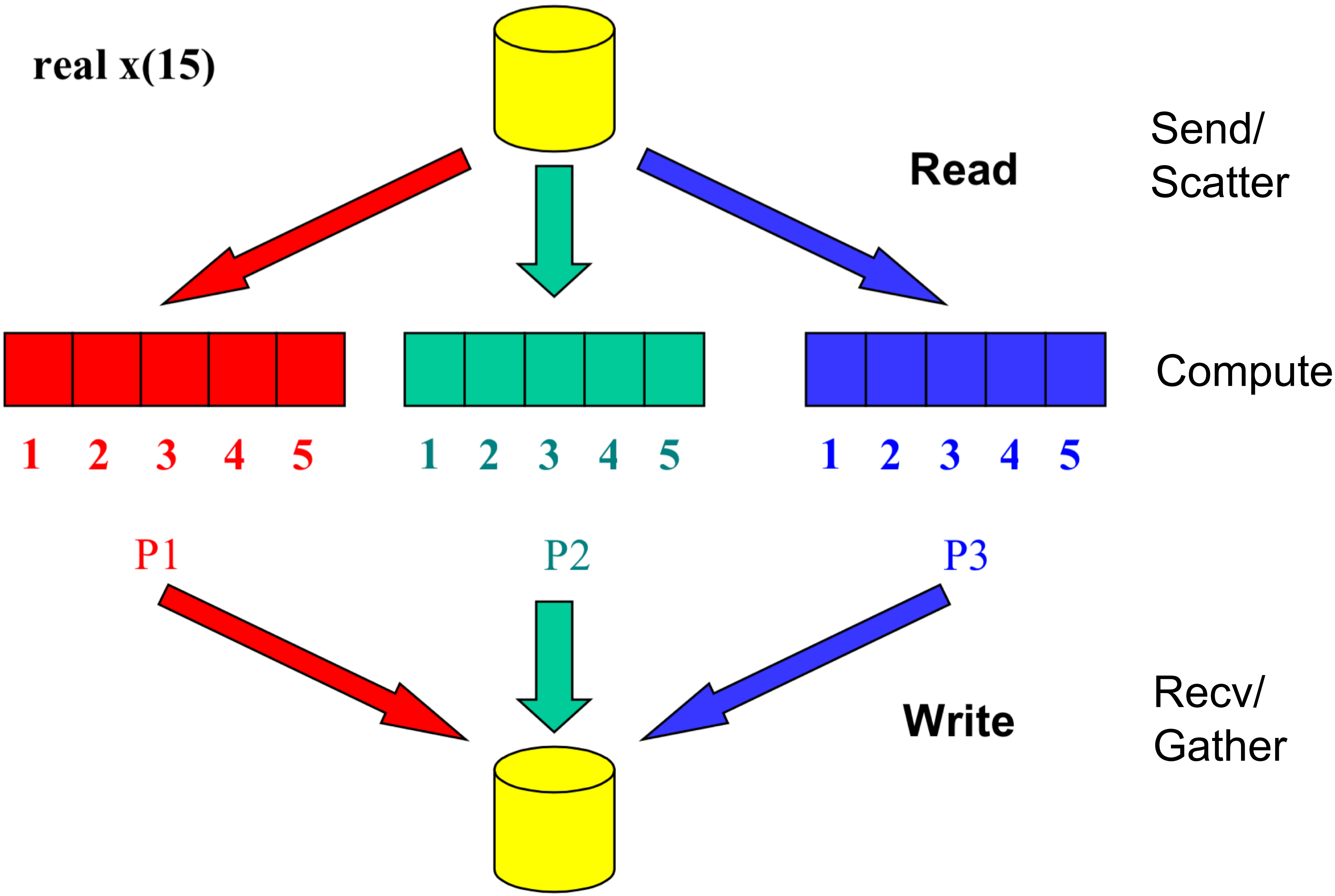



domain decomposition

compute local sums

collect local sums
➔ synchronization

obtain the global sum


real x(15)
Read
Send/
Scatter
Compute
1 2 3 4 5
1 2 3 4 5
1 2 3 4 5
P1
P2
P3
Write
Recv/
Gather

# SPMD: Single Program Multiple Data

Supp

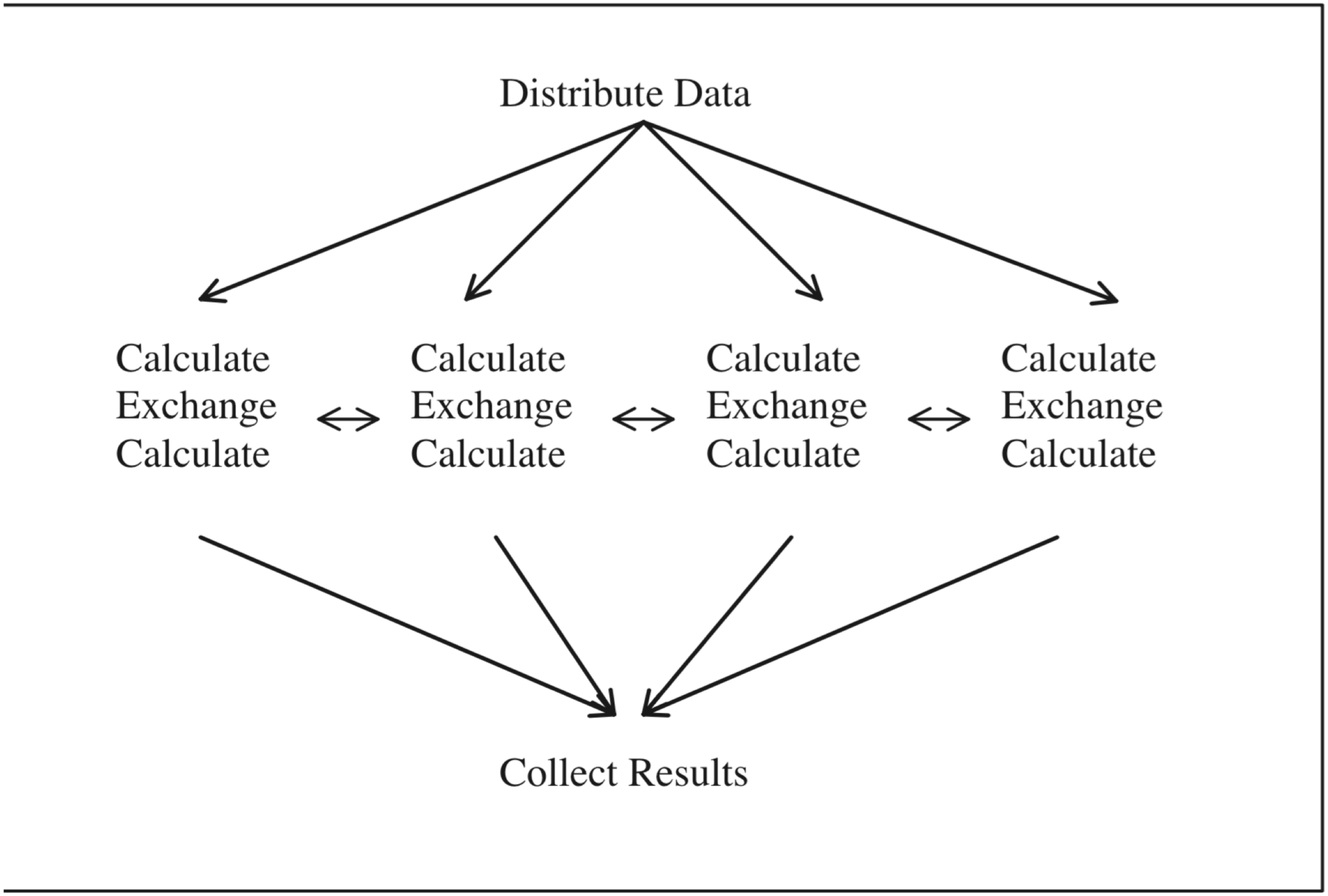



**Figure 1.5** Basic structure of a SPMD program.

# A Minimal Set for MPI

A minimal set of routines that most parallel codes run with are:

- MPI_INIT:

  Initialization. MPI spawns an identical copy of my_proc when mpirun -np N my_proc is issued.
- MPI_COMM_SIZE:

  returns the size or number (i.e., N) of processes in the application.
- MPI_COMM_RANK:

  returns the rank (“id”, 0~N-1) of the calling process.
- MPI_SEND
- MPI_RECV
- MPI_WAIT
- MPI_FINALIZE:

  terminates all MPI processing

# A Minimal Set for MPI in Python

A minimal set of routines that most parallel codes run with are:

| | Fortran | MPI4PY | |
|---|---|---|---|
| | MPI_INIT | ---------- | comm = MPI.COMM_WORLD |
| | MPI_COMM_SIZE | comm.Get_size() | |
| | MPI_COMM_RANK | comm.Get_rank() | |
| | MPI_SEND | comm.send(…) | comm.Send(…) |
| | MPI_RECV | comm.recv(…) | comm.Recv(…) |
| | MPI_WAIT | obj.wait() | |
| | MPI_FINALIZE | ----------- | |

# Diagnoalization using $T^{-1}AT$

$$AT = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 1 & 0 \\ 4 & 0 & -1 \\ 2 & 0 & -2 \end{pmatrix}$$

$$TD = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 1 & 0 \\ 4 & 0 & -1 \\ 2 & 0 & -2 \end{pmatrix}$$

$$TD = AT$$

$$T^{-1}TD = T^{-1}AT$$

$$ID = T^{-1}AT$$

$$D = T^{-1}AT$$

# The Double Zero Eigenvalue

Supp

- A pdf file (from Guckenheimer & Holmes, 1983) is available @canvas/supp.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# Section 5.3 Complex Eigenvalues

$$\text{if } AV = (\alpha + i\beta)V, \text{ we have } A\bar{V} = (\alpha - i\beta)\bar{V}$$

- Let $V$ be the eigenvector associated with the complex eigenvalue $\alpha + i\beta$
- Show that $\bar{V}$ is an eigenvector associated with the complex eigenvalue $\alpha - i\beta$
- $V$ and $\bar{V}$ are linearly independent, i.e., $\mathrm{c}V + \mathrm{d}\bar{V} = 0 \Leftrightarrow c = d = 0$
  - $c$ & $d$ are complex numbers.

➢ Note that $\mathrm{V}$ and $\bar{V}$ yield the same indepdent real functions, because $\mathrm{Re}(V) = Re(\bar{V})$ and $\mathrm{Im}(V) = -Im(\bar{V})$.

➢ From the first bullet, we have $\mathrm{A}V = (\alpha + i\beta\,)V$.

➢ To prove the statement in the 2nd bullet, we consider $\mathrm{A}\bar{V}$:

$$\mathrm{A}\bar{V} = \overline{AV} = \overline{(\alpha + i\beta\,)V} = (\alpha - i\beta)\bar{V}$$

# $Re(V)$ and $Im(V)$ Are Linearly Independent



$V = u + i\,w$ Show that $u$ and $w$ are LI

$\boxed{\text{Assume } w = cu,\ c \in R}$

$\lambda = \alpha + i\beta$

$$AV = A(u + iw) = A(u + icu) \quad (1)$$

$$\lambda V = (\alpha + i\beta)(u + icu) = (\alpha u - c\beta u) + i(\beta u + c\alpha u) \quad (2)$$

Eq. (1) = (2):

real part: $Au = \alpha u - c\beta u$ (3)

imaginary part: $Acu = \beta u + c\alpha u$ (4)

(3)*c = (4),

$$-c^2\beta u = \beta u \qquad \boxed{c^2 = -1, \qquad c = \pm i}$$

contradiction

# Construct a Linear Map, $T$, using Real and Imaginary Parts

Assume $A$ to be a $(2n\times 2n)$ matrix that has the following eigenvalues and eigenvectors:

- $\alpha_j \pm \beta_j$ and $V_j, \bar{V}_j$, $j = 1,2$;
- $AV_j = (\alpha_j + i\beta)V_j$ and $A\bar{V}_j = (\alpha_j - i\beta)\bar{V}_j$

Define the following

$$\boxed{W_{2j-1} = \frac{1}{2}(V_j + \bar{V}_j) = Re(V_j)} \qquad \boxed{W_{2j} = \frac{-i}{2}(V_j - \bar{V}_j) = Im(V_j)}$$

- Let $\lambda_j = \alpha_j + \beta_j$ which of the following is correct:
  - (1) $AW_j = \lambda_j W_j$;
  - (2) $AW_j = Re(\lambda_j)\, W_j$
  - (3) all of the above
  - (4) none of the above
- Send your results via ”chat”
- You have 60 seconds

# Construct a Linear Map, $T$, using Real and Imaginary Parts

Assume $A$ to be a $(2n\times 2n)$ matrix that has the following eigenvalues and eigenvectors:

- $\alpha_j \pm \beta_j$ and $V_j, \bar{V}_j$, $j = 1,2$;
- $AV_j = (\alpha_j + i\beta)V_j$ and $A\bar{V}_j = (\alpha_j - i\beta)\bar{V}_j$

Define the following

$$\boxed{W_{2j-1} = \frac{1}{2}\left(V_j + \bar{V}_j\right) = Re\left(V_j\right)} \qquad \boxed{W_{2j} = \frac{-i}{2}\left(V_j - \bar{V}_j\right) = Im\left(V_j\right)}$$

Show that (TBD in the next slide)

$$AW_{2j-1} = \alpha_j W_{2j-1} - \beta_j W_{2j} \qquad AW_{2j} = \beta_j W_{2j-1} + \alpha_j W_{2j}$$

Construct $T$ as follows:

$$T = [W_1, W_2, \cdots W_{2j-1}, W_{2j} \cdots W_{2n-1}, W_{2n}]$$

Obtain $Y' = BY, X = TY$, and $B$ is defined as follows:

$$B = T^{-1}AT$$

# Find $AW_{2j-1}$ and $AW_{2j}$

$$W_{2j-1} = \frac{1}{2}(V_j + \bar{V}_j) = Re(V_j) \qquad W_{2j} = \frac{-i}{2}(V_j - \bar{V}_j) = Im(V_j)$$

Show that $AW_{2j-1} = \alpha_j W_{2j-1} - \beta_j W_{2j}$ $\qquad AW_{2j} = \beta_j W_{2j-1} + \alpha_j W_{2j}$

$$AW_{2j-1} = \frac{1}{2}(AV_j + A\bar{V}_j) = \frac{1}{2}\left((\alpha_j + i\beta)V_j + (\alpha_j - i\beta)\bar{V}_j\right)$$

$$= \frac{1}{2}\left(\alpha_j(V_j + \bar{V}_j) + i\beta_j(V_j - \bar{V}_j)\right) = \alpha_j W_{2j-1} - \beta_j W_{2j}$$

$$AW_{2j} = \frac{-i}{2}(AV_j - A\bar{V}_j) = \frac{-i}{2}\left((\alpha_j + i\beta)V_j - (\alpha_j - i\beta)\bar{V}_j\right)$$

$$= \frac{-i}{2}\left(\alpha_j(V_j - \bar{V}_j) + i\beta_j(V_j + \bar{V}_j)\right) = \beta_j W_{2j-1} + \alpha_j W_{2j}$$

## $W_1, W_2, \cdots W_{2j-1}, W_{2j} \cdots W_{2n-1}, W_{2n}$ are LI

$$W_{2j-1} = \frac{1}{2}(V_j + \bar{V}_j) = Re(V_j)$$

$$W_{2j} = \frac{-i}{2}(V_j - \bar{V}_j) = Im(V_j)$$

**Proposition.** *The vectors $W_1, \ldots, W_{2n}$ are linearly independent.*

Form a linear combination:

$$\sum_{j=1}^{n}(c_j W_{2j-1} + d_j W_{2j}) = 0$$

Assume that they are LD with some non-zero $c_j, d_j \in R$

Plug the Eqs. on the top into the above Eq.:

$$\frac{1}{2}\sum_{j=1}^{n}(c_j - i d_j)V_j + (c_j + i d_j)\bar{V}_j = 0$$

Since the $V_j$ and the $\bar{V}_j$ are LI, we have

$$(c_j - i d_j) = 0 = (c_j + i d_j)$$

$\Rightarrow c_j = d_j = 0$ contradiction

# Construct a Linear Map, $T$

$$\boxed{AW_{2j-1} = \alpha_j W_{2j-1} - \beta_j W_{2j}} \qquad AW_{2j} = \beta_j W_{2j-1} + \alpha_j W_{2j}$$

Construct $\quad T = [W_1, W_2, \cdots W_{2j-1}, W_{2j} \cdots W_{2n-1}, W_{2n}]$

Obtain $\quad TE_j = W_j, \quad j = 1 \sim 2n, \qquad \boxed{T^{-1}W_j = E_j,} \quad j = 1 \sim 2n,$

$$for\ 2j-1, \quad (T^{-1}AT)E_{2j-1} = T^{-1}ATE_{2j-1} = T^{-1}AW_{2j-1} = T^{-1}\boxed{AW_{2j-1}}$$

$$= T^{-1}\left(\alpha_j W_{2j-1} - \beta_j W_{2j}\right)$$

$$= \left(\alpha_j \boxed{T^{-1}W_{2j-1}} - \beta_j T^{-1}W_{2j}\right)$$

$$= \left(\alpha_j E_{2j-1} - \beta_j E_{2j}\right)$$

$$for\ 2j, \quad (T^{-1}AT)E_{2j} = \left(\beta_j E_{2j-1} + \alpha_j E_{2j}\right)$$

# Construct a Linear Map, $T$

Previously, we obtained

$$\boxed{(T^{-1}AT)E_{2j-1} = \left(\alpha_j E_{2j-1} - \beta_j E_{2j}\right)}$$

$$\boxed{(T^{-1}AT)E_{2j} = \left(\beta_j E_{2j-1} + \alpha_j E_{2j}\right)}$$

j=1, we have

$$(T^{-1}AT)E_1 = (\alpha_1 E_1 - \beta_1 E_2) \quad = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} - \beta_1 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ -\beta_1 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

$$(T^{-1}AT)E_2 = (\beta_1 E_1 + \alpha_1 E_2) \quad = \beta_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

# Construct a Linear Map, $T$

Previously, we obtained

$$(T^{-1}AT)E_1 = (\alpha_1 E_1 - \beta_1 E_2) = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} - \beta_1 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ -\beta_1 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

$$(T^{-1}AT)E_2 = (\beta_1 E_1 + \alpha_1 E_2) = \beta_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

Now, we put them together

$$(T^{-1}AT)[E_1, E_2, \cdots E_{2n-1}, E_{2n}] = \begin{pmatrix} \alpha_1 & \beta_1 & & \\ -\beta_1 & \alpha_1 & & \\ & & & \\ & & & \end{pmatrix}$$

# Construct a Linear Map, $T$

Previously, we obtained

$$(T^{-1}AT)\underbrace{[E_1, E_2, \cdots E_{2n-1}, E_{2n}]}_{\textit{Identity matrix } I} = \begin{pmatrix} \alpha_1 & \beta_1 & \\ -\beta_1 & \alpha_1 & \\ & & \\ & & D_j \end{pmatrix}$$

Thus, we have

$$T^{-1}AT = \begin{array}{c} \begin{pmatrix} \boxed{\begin{matrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{matrix}} & & \\ & & \\ & \boxed{\begin{matrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{matrix}} & \\ & & \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \leftarrow 2j-1 \\ \leftarrow 2j \\ \\ \end{array}$$

$$D_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{pmatrix} \quad D_j = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix}$$

# A Linear Map $T$ & $T^{-1}AT$

Construct $T$ as follows:

$$T = [W_1, W_2, \cdots W_{2j-1}, W_{2j} \cdots W_{2n-1}, W_{2n}]$$

Obtain $Y' = BY, X = TY$, and $B$ is defined as follows:

$$B = T^{-1}AT = \begin{pmatrix} D_1 & & \\ & & \\ & & D_n \end{pmatrix} \qquad D_n = \begin{pmatrix} \alpha_n & \beta_n \\ -\beta_n & \alpha_n \end{pmatrix}$$

# Example: Section 6.2 (uncoupled) Oscillators

Consider a pair of undamped harmonic oscillators whose equations are

$$x_1'' = -\omega_1^2 x_1$$
$$x_2'' = -\omega_2^2 x_2.$$

$$Y(t) = \begin{pmatrix} x_1(t) \\ y_1(t) \\ x_2(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} a_1 \cos \omega_1 t + b_1 \sin \omega_1 t \\ -a_1 \sin \omega_1 t + b_1 \cos \omega_1 t \\ a_2 \cos \omega_2 t + b_2 \sin \omega_2 t \\ -a_2 \sin \omega_2 t + b_2 \cos \omega_2 t \end{pmatrix}$$

- Solutions will be discussed in Chapter 6
- Eigenvalues and eigenvectors are discussed below.

# Eigenvalues for Oscillators

$$x_1'' = -\omega_1^2 x_1 \rightarrow periodic$$

$$x_2'' = -\omega_2^2 x_2 \rightarrow periodic$$

$$x_1{}' = y_1$$

$$y_1{}' = -\omega_1^2 x_1$$

$$x_2{}' = y_2$$

$$y_2{}' = -\omega_2^2 x_2$$

uncoupled

$$\begin{pmatrix} x_1' \\ y_1' \\ x_2' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_1^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} -\lambda & 1 & 0 & 0 \\ -\omega_1^2 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & -\omega_1^2 & -\lambda \end{pmatrix}$$

$$|A - \lambda I| = 0 \qquad -\lambda \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -\omega_1^2 & -\lambda \end{vmatrix} - \begin{vmatrix} -\omega_1^2 & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -\omega_1^2 & -\lambda \end{vmatrix} = 0$$

$$\lambda_{1,2} = \pm i\, \omega_1$$

$$\lambda_{3,4} = \pm i\, \omega_2$$

# Eigenvectors for Oscillators

$$AV = \lambda V \Rightarrow$$

$$y_1 = \lambda x_1 \qquad y_2 = \lambda x_2$$

$$-\omega_1^2 x_1 = \lambda y_1 \qquad -\omega_2^2 x_2 = \lambda y_2$$

$$\lambda_{1,2} = \pm i\,\omega_1$$

$$\lambda_{3,4} = \pm i\,\omega_2$$

$$\lambda_1 = i\omega_1$$

$$y_1 = i\omega_1\,x_1$$

$$-\omega_1^2 x_1 = i\omega_1 y_1$$

$$\Rightarrow y_1 = i\omega_1\,x_1$$

$$x_1 = x_1$$

$$y_1 = i\omega_1\,x_1$$

$$y_2 = i\omega_1 x_2$$

$$-\omega_2^2 x_2 = i\omega_1 y_2$$

$$\Rightarrow -\omega_1^2 x_2 = -\omega_2^2 \mathrm{x}_2$$

$$x_2 = 0$$

$$y_2 = 0$$

$$\omega_1 \neq \omega_2$$

- Find the eigenvectors U1 and U2 associated with $\lambda_1 = i\omega_1$ & $\lambda_2 = -i\omega_1$
- Send your results via "chat"
- You have 2 minutes

$$U_1 = V_1 = \begin{pmatrix} 1 \\ i\omega_1 \\ 0 \\ 0 \end{pmatrix} \quad U_2 = \bar{V}_1 = \begin{pmatrix} 1 \\ -i\omega_1 \\ 0 \\ 0 \end{pmatrix}$$

# Changing Coordinates: Construct the Linear Map

$\lambda_{1,2} = \pm i\,\omega_1$ $\quad V_1^T = (1, i\omega_1, 0, 0)$

$$W_1^T = Re(V_1^T) = (1, 0, 0, 0)$$

$$W_2^T = Im(V_1^T) = (0, \omega_1\,, 0, 0)$$

$\lambda_{3,4} = \pm i\,\omega_2$ $\quad V_2^T = (0,0,1, i\omega_2)$

$$W_3^T = Re(V_2^T) = (0, 0, 1, 0)$$

$$W_4^T = Im(V_2^T) = (0,0, 0, \omega_2)$$

$$T = \begin{array}{c} (W_1 \quad W_2 \quad W_3 \quad W_4) \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega_2 \end{pmatrix} \end{array} \qquad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_1^2 & 0 \end{pmatrix}$$

$$T^{-1}AT = \begin{pmatrix} 0 & \omega_1 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 \\ 0 & 0 & -\omega_2 & 0 \end{pmatrix}$$

# Changing Coordinates: Matlab Code

- syms w1 w2
- T=[1 0 0 0 ; 0 w1 0 0 ; 0 0 1 0; 0 0 0 w2]
- A=[0 1 0 0 ; -w1^2 0 0 0 ; 0 0 0 1 ; 0 0 -w2^2 0]
- inv(T)*A*T

```
ans =


[   0, w1,   0,  0]
[ -w1,  0,   0,  0]
[   0,  0,   0, w2]
[   0,  0, -w2,  0]
```

# A Summary with Complex Eigenvalues

Assume $A$ to be a $(2n \times 2n)$ matrix that has the following eigenvalues and eigenvectors:

- $\alpha_j \pm \beta_j$ and $V_j, \bar{V}_j$, $j = 1,2$;

Define the following

$$W_{2j-1} = \frac{1}{2}\left(V_j + \bar{V}_j\right) = Re\left(V_j\right) \qquad W_{2j} = \frac{-i}{2}\left(V_j - \bar{V}_j\right) = Im\left(V_j\right)$$

Construct $T$ as follows:

$$T = [W_1, W_2, \cdots W_{2j-1}, W_{2j} \cdots W_{2n-1}, W_{2n}]$$

Obtain $Y' = BY, X = TY$, and $B$ is defined as follows:

$$B = T^{-1}AT = \begin{pmatrix} D_1 & & \\ & \ddots & \\ & & D_n \end{pmatrix} \qquad D_n = \begin{pmatrix} \alpha_n & \beta_n \\ -\beta_n & \alpha_n \end{pmatrix}$$