4.4.1: uniform density ρ_0 and tension T_0.

a) Natural frequencies of length L fixed at both ends?

Circular frequency (# of oscillations in 2π units time) $= \dfrac{n\pi c}{L}$

$$c = \sqrt{\frac{T_0}{\rho_0}}$$

b) Natural frequencies of length H, fixed at $x=0 \rightarrow \boxed{\phi(0)=0}$

free at $x=H \rightarrow \dfrac{\partial u}{\partial x}(H,t) = 0$

Let $u(x,t) = \phi(x)h(t)$

Then BC: $\dfrac{\partial(\phi(x)h(t))}{\partial x}(H,t) = 0 \rightarrow h(t)\boxed{\phi'(H)=0}$

ODE: $\dfrac{d^2\phi}{dx^2} = -\lambda\phi \rightarrow \phi(x) = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x$

$\phi'(x) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}x + c_2\sqrt{\lambda}\cos\sqrt{\lambda}x$

Using BC: $\phi(0) = 0 = c_1 \cdot 1 + c_2 \cdot 0 \rightarrow \boxed{c_1 = 0}$

Thus $\phi(x) = c_2\sin\sqrt{\lambda}x$

Using other BC: $\phi'(H) = c_2\sqrt{\lambda}\cos\sqrt{\lambda}H \overset{=0}{\rightarrow} c_2 \neq 0, \sqrt{\lambda} \neq 0$, so $\cos\sqrt{\lambda}H = 0$

$\downarrow$

$\cos\dfrac{n\pi H}{H} H$

4.4.9

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad c^2 = \frac{T_0}{\rho_0}$$

① Total Energy = KE + PE

$$E(t) = \int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx + \int_0^L \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx$$

② Differentiate WRT t.

$$\frac{dE}{dt} = \frac{d}{dt}\int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{d}{dt}\int_0^L \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx$$

$$= \frac{1}{2}\int_0^L \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{c^2}{2}\int_0^L \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 dx$$

$$= \cancel{\frac{1}{2}}\int_0^L \cancel{2}\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} dx + \frac{c^2}{\cancel{2}}\int_0^L \cancel{2}\frac{\partial u}{\partial x}\cdot\frac{\partial^2 u}{\partial x \partial t} dx$$

substitution

$$= \int_0^L \frac{\partial u}{\partial t}\cdot\boxed{c^2\frac{\partial^2 u}{\partial x^2}} dx + c^2\int_0^L \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t} dx$$

$$= c^2\int_0^L \underset{v}{\frac{\partial u}{\partial t}}\,\underset{du}{\frac{\partial^2 u}{\partial x^2}} + \underset{u}{\frac{\partial u}{\partial x}}\,\underset{dv}{\frac{\partial^2 u}{\partial x \partial t}}\, dx$$

$$= c^2\int_0^L \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial t}\right] dx$$

$$= c^2\left(\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial t}\right)\Big|_0^L \quad ✓$$

$\boxed{4.4.10}$ $$\frac{dE}{dt} = c^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\Big|_0^L$$

a) $u(0,t) = 0 = u(L,t) \rightarrow$ fixed at both ends

$$\frac{dE}{dt} = c^2\left[\frac{\partial(u(L,t))}{\partial x}\cdot\cancel{\frac{\partial(u(L,t))}{\partial t}}^{\to 0} - \frac{\partial(u(0,t))}{\partial x}\cdot\cancel{\frac{\partial(u(0,t))}{\partial t}}^{\to 0}\right]$$

$$= c^2[0] = 0 \rightarrow \text{energy conserved!}$$

b) $\frac{\partial u}{\partial x}(0,t) = 0$ and $u(L,t) = 0 \rightarrow \frac{\partial u}{\partial t}(L,t) = 0$

$$\frac{dE}{dt} = 0$$

c) $\underbrace{u(0,t) = 0}$ and $\frac{\partial u}{\partial x}(L,t) = -\gamma u(L,t)$, $\gamma > 0$?

$$\frac{\partial u}{\partial t}(0,t) = 0 \rightarrow \text{energy decreases.}$$

$$\frac{dE}{dt} = c^2\left[-\gamma u(L,t)\overbrace{\frac{\partial u}{\partial t}(L,t)}^{\text{chain rule}} - 0\right]$$

$$\int \frac{dE}{dt} = \int_0^t -c^2\gamma\left[u(L,t)\frac{\partial u}{\partial t}(L,t)\right]$$

$$E(t) = -c^2\gamma\cdot\frac{1}{2}(u(L,t))^2\Big|_0^t$$

$$= -\frac{c^2\gamma}{2}\left[u(L,t)^2 - u(0,t)^2\right]$$

d) $\frac{dE}{dt} = +$ so energy increases