Homework 7.2
Linear Algebra
Math 524
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Problem 7.C.2: Suppose T is a positive operator on V. Suppose $v, w \in V$ are such that

$$Tv = w \qquad \text{and} \qquad Tw = v$$

Prove that $v = w$.

Because T is a positive operator, $\langle Tv, v \rangle \geq 0$. Notice the following:

$$\begin{aligned} \langle T(v-w), (v-w) \rangle &\geq 0 \\ \langle Tv - Tw, v - w \rangle &= -\langle v - w, v - w \rangle \geq 0 \\ &= -||v-w||^2 \geq 0 \end{aligned}$$

Because of the square, it means that $-||v-w||^2 = 0$. Thus $||v-w|| = 0$, so $v = w$

Problem 7.C.4: Suppose $T \in \mathcal{L}(V, W)$. Prove that T^*T is a positive operator on V and TT^* is a positive operator on W

Notice the following:

$$(T^*T)^* = T^*T^{**} = T^*T \qquad\qquad (TT^*)^* = T^{**}T^* = TT^*$$

Because of the following, we can see that both are self-adjoint. So now we can notice the following:

$$\begin{aligned} \langle T^*Tv, v\rangle &= \langle T^2v, v\rangle \\ &= \langle Tv, Tv\rangle \\ &= ||Tv||^2 \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \langle TT^*v, v\rangle &= \langle (T^*)^2v, v\rangle \\ &= \langle T^*v, T^*v\rangle \\ &= ||T^*v||^2 \geq 0 \end{aligned}$$

Thus T^*T and TT^* are positive operators on V and W respectively

Problem 7.C.7: Suppose T is a positive operator on V. Prove that T is invertible if and only if

$$\langle Tv, v\rangle > 0$$

for every $v \in V$, with $v \neq 0$

(=>). Let $\langle Tv, v\rangle > 0$.

If T is not invertible then there exists $v \neq 0 \in V$, such that $Tv = 0$. So we can see that $\langle Tv, v\rangle = 0$. This contradicts that $\langle Tv, v\rangle > 0$, so T has to be invertible

(<=) Let T be invertible.

We can define an operator, $S^2 = T$, where S is the square root operator of T because T is a positive operator. So now we can notice the following

$$\langle Tv, v\rangle = \langle S^2 v, v\rangle = \langle Sv, Sv\rangle = ||Sv||^2 > 0$$

$\square$

Problem 7.D.1: Fix $u, v \in V$ with $u \neq 0$. Define $T \in \mathcal{L}(V)$ by

$$Tv = \langle v, u \rangle x$$

for every $v \in V$. Prove that

$$\sqrt{T^*T}v = \frac{||x||}{||u||}\langle v, u \rangle u$$

By definition of T, we can see the following:

$$T^*Tv = T^*\langle v, u \rangle x = \langle v, u \rangle T^*(x) = \langle v, u \rangle \langle x, x \rangle u = ||x||^2 \langle v, u \rangle u$$

We can see that the map $R \in \mathcal{L}(V)$, with:

$$Rv = \frac{||x||}{||u||}\langle v, u \rangle u$$

is a square root of T^*T. We can also see that $\langle Rv, v \rangle \geq 0$. If we let $e_1, ..., e_n$ be an orthonormal basis of V, then we can write the following:

$$u = a_1e_1 + ... + a_ne_n$$

for some $a_1, ..., a_n$. So now we have:

$$R(e_j) = \frac{||x||}{||u||}\langle e_j, u \rangle u = \frac{||x||}{||u||}\left(a_j\bar{a_1}e_1 + ... + a_j\bar{a_n}e_n\right)$$

Now we can see that $\mathcal{M}(R) = \frac{||x||}{||u||}a_j\bar{a_k}$. Now we can see that $\mathcal{M}(R) = \mathcal{M}(R^*)$. Thus R is the square root of T^*T and self-adjoint. So the result below is true:

$$\sqrt{T^*T}v = \frac{||x||}{||u||}\langle v, u \rangle u$$

Problem 7.D.2: Give an example of $T \in \mathcal{L}(\mathbb{C}^2)$ such that 0 is the only eigenvalue of T and the singular values of T are 5, 0.

Notice the following:

$$\mathcal{M}(T) = \begin{pmatrix} 0 & 0 \\ 5 & 0 \end{pmatrix}$$

Because $\mathcal{M}(T)$ is a triangular matrix, the eigenvalues are its entries in the diagonal, being 0.

Also Notice that to find the singular values:

$$\mathcal{M}(T)\mathcal{M}(T^*) = \begin{pmatrix} 0 & 0 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 25 \end{pmatrix}$$

Because $\mathcal{M}(T^*)\mathcal{M}(T)$ is a triangular matrix, the eigenvalues are its entries in the diagonal, being 0 and 25. By 7.52, the singular values are the non-negative square root values of $\mathcal{M}(T^*)\mathcal{M}(T)$'s eigenvalues, which are 0 and 5

Problem 7.D.5: Suppose $T \in \mathcal{L}(\mathbb{C}^2)$ is defined by $T(x, y) = (-4y, x)$. Find the singular values of T.

Notice we can write $\mathcal{M}(T)$ as a matrix in respect to a basis (x, y):

$$\mathcal{M}(T) = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}$$

Thus $\mathcal{M}(T^*) = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$. Now we can see the following:

$$\mathcal{M}(T)\mathcal{M}(T^*) = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} = \begin{pmatrix} 16 & 0 \\ 0 & 1 \end{pmatrix}$$

The singular values are going to be the square root of the eigenvalues of $\mathcal{M}(T)\mathcal{M}(T^*)$, so they are 4 and 1.