9/10
Section 1.2
(mostly)

Extended Ideas...

Suppose $S \subseteq \mathbb{R}$ and $S \neq \emptyset$.

We have a definition of supremum already.

We can invert it to obtain an infimum definition

We say S is bounded below and $x \in \mathbb{R}$ is a lower bound of S iff $\forall y \in S$, $x \leq y$.

We say $x \in \mathbb{R}$ is an infimum of S (greatest lower bound g.l.b. S) (written $\inf S$)

iff

1. x is a lower bound of S.
2. $\forall y \in \mathbb{R}$, if y is a lower bound of S, $y \leq x$.

We say $c \in S$ is a maximum of S iff c is an upper bound of S.

We say $c \in S$ is a minimum of S iff c is a lower bound of S.

Careful with the membership of c, x above here.

~~Continuing in section 1.2~~

Prop 1.6 $\forall n \in \mathbb{N}, \forall k \in \mathbb{Z}, k \notin (n, n+1)$

proof idea:

1. By construction, there is no natural number between 0 & 1.
   So $\forall k \in \mathbb{Z}, k \notin (0,1)$.
2. Proceed by contradiction.
   Suppose $\exists n \in \mathbb{N}$ and $\exists k \in \mathbb{Z}$ s.t. $k \in (n, n+1)$.
   So $n < k < n+1$
   Thus $0 < k-n < 1$.
   Since $k-n \in \mathbb{Z}$ and $k-n \in (0,1)$ we have a contradiction.

Prop 1.7 (A lot like well-ordering Property)

Suppose $S \subseteq \mathbb{Z}$ with $S \neq \emptyset$.

If S is bounded above, then $\max S$ exists.

proof: Suppose S is bounded above.

By completeness, $\exists a \in \mathbb{R}$ s.t. $a = \sup S$.

By definition, $a-1$ is not an upper bound of S.

Thus $\exists m \in S$ s.t. $a-1 < m$.

Thus $a < m+1$.

~~Let $x \in S$~~ (Goal: show $m = \max S$)

Let $x \in S$.

By def $x \leq a$.

So $x < m+1$. By Prop 1.6 $x \notin (m, m+1)$

Thus $x \leq m$. So $m \in S$ is an upper bound and $m = \max S$.

Dense Sets

We will show $\mathbb{Q} \subseteq \mathbb{R}$ is dense in $\mathbb{R}$.

- everything in $\mathbb{R}$ can be approximated by things in $\mathbb{Q}$.

- We often use "$A \subseteq B$ is dense in B" where B has complicated members and A has simple members to approximate them.

Ex: taylor polynomials converging to smooth functions

__Definition__ Suppose $S \subseteq \mathbb{R}$. We say that S is __dense__ in $\mathbb{R}$ iff $\forall a, b \in \mathbb{R}$ with $a < b$, we have

$$(a, b) \cap S \neq \phi.$$

__Thm 1.8__ $\forall c \in \mathbb{R}$, $\exists! k \in \mathbb{Z}$ s.t. $k \in [c, c+1)$

(exists a unique).

__proof:__ ~~Define S~~ Let $c \in \mathbb{R}$.

Define $S = \{ n \in \mathbb{Z} \mid n < c+1 \}$.

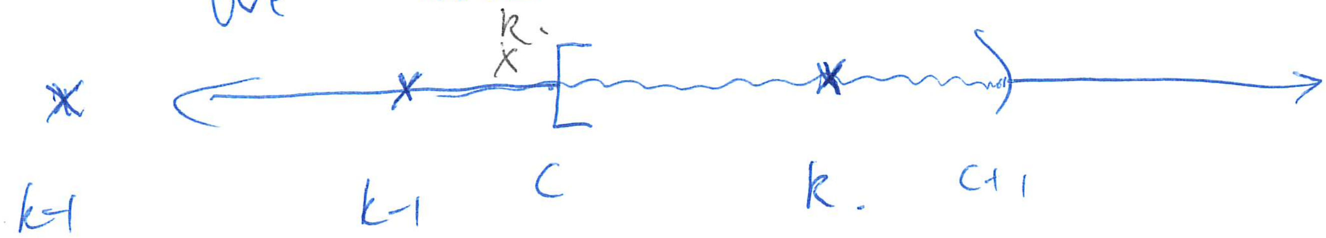
Consider $-(c+1)$ and use Thm 1.5 (Archimedes)

So $\exists N \in \mathbb{N}$ $\quad -(c+1) < N$

So $\quad c+1 > -N \in \mathbb{Z}$.

Thus $S \neq \phi$ and since $S \subseteq \mathbb{Z}$ is bounded above by $c+1$, $\exists k \in S$ s.t. $\max S = k$.

We have



We need to show $k \geq c$ so the picture is sensible.

Suppose (to reach a contradiction) that $k < c$.

Then $k+1 < c+1$. Since $k+1 \in \mathbb{Z}$, $k+1 \in S$.

and $k+1 > k = \max S$ $(\Rightarrow\Leftarrow)$. Thus $k \geq c$.

Thus $k \in [c, c+1)$.

Suppose now $k' \in [c, c+1) \cap \mathbb{Z}$ also. So $c \leq k' < c+1$.

Also $c \leq k < c+1$ so that $-(c+1) < -k \leq -c$.

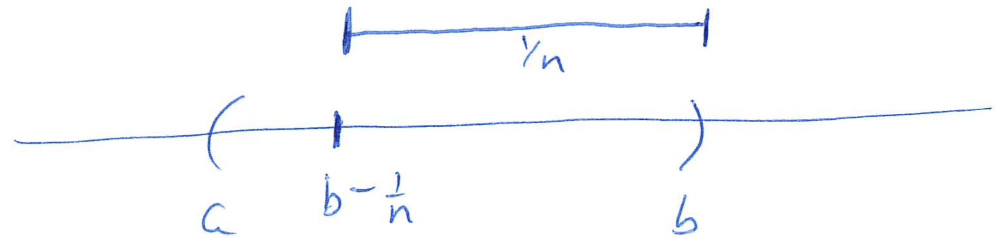
$$-1 < k' - k < 1.$$

Since $k' - k \in \mathbb{Z}$, $k' - k = 0$. ☒

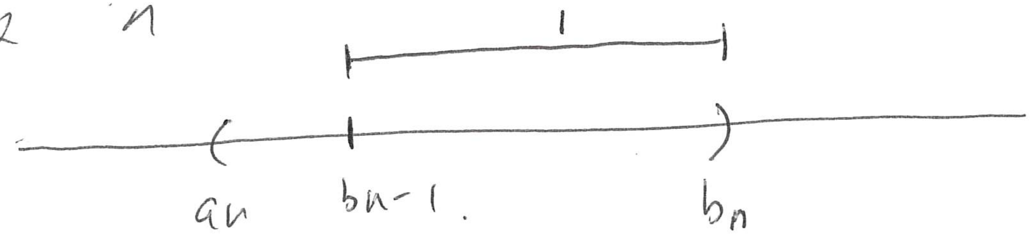
We have uniqueness, $k' = k$.

Thm 1.9 $\mathbb{Q}$ is dense in $\mathbb{R}$.

proof: Suppose $a, b \in \mathbb{R}$ with $a < b$.
Since $0 < b-a$, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < b-a$ by Thm 1.5. Note that $a < b - \frac{1}{n}$. (*)



Multiply by n



Apply Thm 1.8 to $[bn-1, bn)$ to obtain $m \in \mathbb{Z}$
s.t. $bn - 1 \le m < bn$.

Thus $b - \frac{1}{n} \le \frac{m}{n} < b$. By (*) $a < b - \frac{1}{n}$ we have $a < b - \frac{1}{n} \le \frac{m}{n} < b$. I.e. $\frac{m}{n} \in (a, b)$.

Thus $\mathbb{Q}$ is dense in $\mathbb{R}$.

Function Review

Suppose A, B are non-empty sets.

Most generally, a <u>function</u> $f: A \to B$ can be described as $f \subseteq A \times B$.

with properties

① $\forall a \in A, \exists (a,b) \in f$. everywhere defined

② $\forall (a_1, b_1), (a_2, b_2) \in f$, if $a_1 = a_2$, then $b_1 = b_2$.

well-defined.

We call A the <u>domain</u> of f

B is the <u>codomain</u> (range) of f.

The <u>image of f</u>, $im(f) = \{ b \in B \mid \exists a \in A, (a,b) \in f \}$,

$f(a) = b$.

$= \{ f(a) \mid a \in A \}, \subseteq B$.