12/5 FINAL EXAM SUGGESTIONS.

- Test 1 & Test 2 Study guides definitions
- Review Test 1 Solutions
- Review HW solutions
- Do HW 6 / Know def $f'(x_0)$

Test 2 Scale:

29↑ A

24 B

19 C

4.1

- Derivative Definition Problems.

Suppose $f: D \to \mathbb{R}$ and x_0 has a neighborhood in D.
We say f is differentiable at x_0 (and use $f'(x_0)$)

iff

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \text{ exists.}$$

Recall: $f: D \to \mathbb{R}$ and x_0 is a limit point of D. We ~~say~~ write $\lim_{x \to x_0} f(x) = L \in \mathbb{R}$

iff

$\forall \{x_n\} \subseteq D \setminus \{x_0\}$, if $\lim_{n\to\infty} x_n = x_0$, then $\lim_{n\to\infty} f(x_n) = L$

(16) <u>Prove</u>: Suppose f is differentiable at $x_0 = 0$.

$$\lim_{x \to 0} \frac{f(x^2) - f(0)}{x} = 0.$$

<u>Note</u>: $\displaystyle \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = f'(0)$. exists.

Suppose $\{x_n\}$ is such that $\lim_{n \to \infty} x_n = 0$ and $\{x_n\}$ is contained in the neighborhood of x_0:

~~Notice: $\lim_{n \to \infty} f(x_n) - f(0)$~~

Consider $\displaystyle \lim_{n \to \infty} \frac{f(x_n^2) - f(0)}{x_n} = \lim_{n \to \infty} x_n \cdot \frac{f(x_n^2) - f(0)}{x_n^2}$.

Noting $\lim_{n \to \infty} x_n^2 = 0$, we have $\displaystyle \lim_{n \to \infty} \frac{f(x_n^2) - f(0)}{x_n^2} = f'(0)$.

Thus $\displaystyle\lim_{n\to\infty} \frac{f(x_n^2)-f(0)}{x_n} = \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} \frac{f(x_n^2)-f(0)}{x_n^2}\right)$

$$= 0 \cdot f'(0) = 0 ,$$

Since $\{x_n\}$ was arbitrary,

$$\lim_{x\to 0} \frac{f(x^2)-f(0)}{x} = 0 .$$

4.1

(15) Suppose f is differentiable at x_0.

Show $\displaystyle\lim_{x\to x_0} \frac{x f(x_0) - x_0 f(x)}{x - x_0} = f(x_0) - x_0 f'(x_0)$.

$$\lim_{x\to x_0} \frac{x f(x_0) - x_0 f(x_0) + x_0 f(x_0) - x_0 f(x)}{x - x_0}$$

Add zero

$$= \lim_{x\to x_0}\left(\cancel{\frac{x - x_0}{x - x_0}}\, f(x_0) + x_0 \left(\frac{f(x_0) - f(x)}{x - x_0} \right)\right) = f(x_0) - x_0 f'(x_0)$$

4.3

(4) For $c>0$, prove the equation does not have 2 solutions in $0<x<1$.

$$x^3-3x+c=0.$$

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proof: Let $c>0$. Suppose the equation has 2 solutions in $0<x<1$.

Define $f:(0,1)\to\mathbb{R}$ by $f(x)=x^3-3x+c$.

So $\exists\, x_1, x_2 \in (0,1)$ s.t. $x_1<x_2$ and $f(x_1)=0=f(x_2)$.

Notice f is continuous on $[x_1,x_2]\subseteq(0,1)$ and differentiable on (x_1,x_2).

So $\exists\, x_3\in(x_1,x_2)$ s.t. $f'(x_3)=0$.

But $f'(x_3)=3(x_3^2-1)<0$. $(\Rightarrow\Leftarrow)$.

⑤ Prove the following has exactly 1 solution.

$$x^5 + 5x + 1 = 0 \quad \text{for} \quad -1 < x < 0.$$

proof: ① Show existence of a solution.

Let $f : [-1, 0] \to \mathbb{R}$ by $f(x) = x^5 + 5x + 1$.

Notice f is ~~diff~~ continuous on $[-1, 0]$,

$f(-1) = -5$ and $f(0) = 1$.

The IVT says $\exists\, x_0 \in (-1, 0)$ s.t. $f(x_0) = 0$ since

$-5 < 0 < 1$.

② Uniqueness: Suppose not. Then $\exists\, x_1, x_2 \in (-1, 0)$ s.t.

$x_1 < x_2$ and $f(x_1) = 0 = f(x_2)$.

Follow last example ...

4.3

(3) For $a, b, c, d \in \mathbb{R}$, define $\mathcal{O} = \{x \mid cx+d \neq 0\}$.

Define $f(x) = \dfrac{ax+b}{cx+d}$ for all $x \in \mathcal{O}$.

Show if f is not constant, then it fails to have any local maximizers or minimizers.

proof sketch:
- Suppose f not constant and suppose $x_0 \in \mathcal{O}$ is a maximizer W.L.O.G.

Goal: $\Rightarrow\!\Leftarrow$.

- f is differentiable on $\mathcal{O}$.
- 4.19 says $f'(x_0) = 0$.
- Compute & show $f'(x_0) \neq 0$.