Homework 6
Partial Differential Equations
Math 533
Stephen Giang RedID: 823184070

Problem 1 Exercise 4.4.1: Consider vibrating strings of uniform density ρ_0 and tension T_0

(a) What are the natural frequencies of a vibrating string of length L fixed at both ends?

Notice the linear combination of all product solutions:

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \sin \frac{n\pi x}{L} \cos \frac{n\pi c x}{L} + B_n \sin \frac{n\pi x}{L} \sin \frac{n\pi c x}{L} \right)$$

Notice that the circular frequency is:

$$\omega = \frac{n\pi c}{L} \qquad c = \sqrt{\frac{T_0}{\rho_0}}$$

which is measured in 2π units of time. The natural frequency is simply this circular frequency in cycles per second:

$$f = \frac{\omega}{2\pi} = \frac{nc}{2L}$$

(b) What are the natural frequencies of a vibrating string of length H, which is fixed at $x = 0$ and "free" at the other end [i.e., $\partial u/\partial x(H,t) = 0$]? Sketch a few modes of vibration as in Fig. 4.1.

Notice the following from the given information:

$$u(x,t) = \phi(x)G(t) \qquad \phi(0) = 0 \qquad \phi'(H) = 0$$

Solving this, we get:

$$\phi = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x \qquad \phi' = -c_1 \sqrt{\lambda} \sin \sqrt{\lambda} x + c_2 \sqrt{\lambda} \cos \sqrt{\lambda} x$$

Notice the following from the boundary conditions:

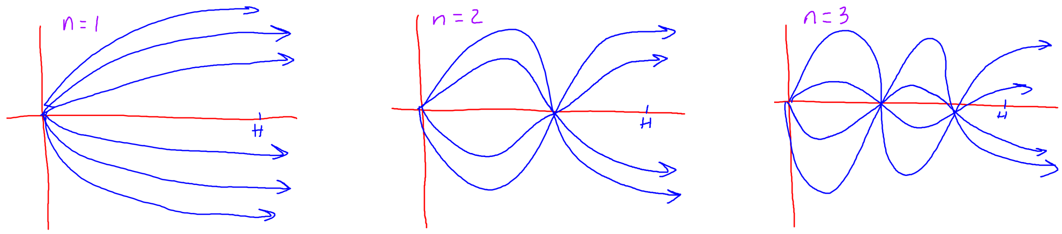
$$\phi(0) = c_1 = 0 \qquad \rightarrow \qquad \phi'(H) = c_2 \sqrt{\lambda} \cos \sqrt{\lambda} H$$

We get the following from the non trivial solution:

$$\lambda = \left(\frac{(2n-1)\pi}{2H} \right)^2$$

From here, we get the circular and natural frequency:

$$\omega = \frac{(2n-1)\pi c}{2H} \qquad f = \frac{\omega}{2\pi} = \frac{(2n-1)c}{4H} \qquad c = \sqrt{\frac{T_0}{\rho_0}}$$



(c) Show that the modes of vibration for the *odd* harmonics (i.e., $n = 1, 3, 5, ...$) of part (a) are identical to modes of part (b) if $H = L/2$. Verify that their natural frequencies are the same. Briefly explain using symmetry arguments.

Notice $u(x,t)$ from part (a) for $n = 2k - 1$:

$$u(x,t) = \sum_{k=1}^{\infty} \left(A_{2k-1} \sin \frac{(2k-1)\pi x}{L} \cos \frac{(2k-1)\pi c x}{L} + B_{2k-1} \sin \frac{(2k-1)\pi x}{L} \sin \frac{(2k-1)\pi c x}{L} \right)$$

Notice $u(x,t)$ from part (b) for $2H = L$:

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_{2n-1} \sin \frac{(2n-1)\pi x}{L} \cos \frac{(2n-1)\pi c x}{L} + B_{2n-1} \sin \frac{(2n-1)\pi x}{L} \sin \frac{(2n-1)\pi c x}{L} \right)$$

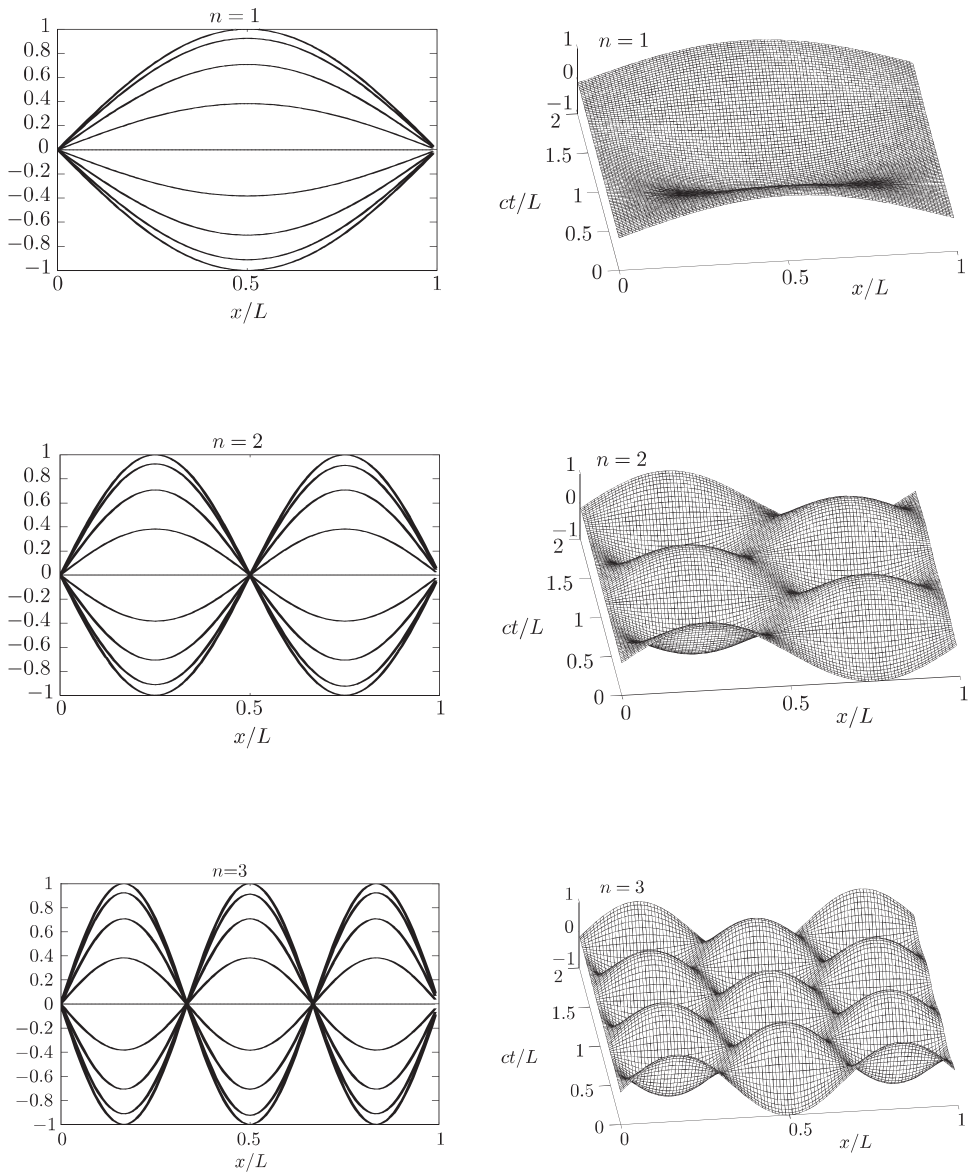


FIGURE 4.1 Normal modes of vibration for a string.

Problem 2 Exercise 4.4.9: From (4.1), derive conservation of energy for a vibrating string,

$$\frac{dE}{dt} = c^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\bigg|_0^L, \tag{4.15}$$

where the total energy E is the sum of the kinetic energy, defined by $\int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx$, and the potential energy, defined by $\int_0^L \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx$

Notice equation (4.1) as stated in the problem given:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Notice the equation for total energy, calculated from the sum of kinetic and potential energy.

$$\begin{aligned} E &= KE + PE \\ &= \int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx + \int_0^L \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx \end{aligned}$$

Now notice, when we take the derivative of the total energy:

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt}\int_0^L \frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{d}{dt}\int_0^L \frac{c^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx \\ &= \int_0^L \frac{1}{2}\frac{d}{dt}\left(\left(\frac{\partial u}{\partial t}\right)^2\right) dx + \frac{d}{dt}\int_0^L \frac{c^2}{2}\frac{d}{dt}\left(\left(\frac{\partial u}{\partial x}\right)^2\right) dx \\ &= \int_0^L \frac{\partial u}{\partial t}\left(\frac{\partial^2 u}{\partial t^2}\right) dx + \int_0^L c^2 \frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial t \partial x}\right) dx \\ &= \int_0^L \frac{\partial u}{\partial t}\left(c^2 \frac{\partial^2 u}{\partial x^2}\right) dx + \int_0^L c^2 \frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial t \partial x}\right) dx \\ &= c^2 \int_0^L \frac{\partial u}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right) + \frac{\partial u}{\partial x}\left(\frac{\partial u}{\partial t \partial x}\right) dx \\ &= c^2 \int_0^L \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\right) dx \\ &= c^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\bigg|_0^L \end{aligned}$$

Problem 3 Exercise 4.4.10: What happens to the total energy E of a vibrating string (see Exercise 4.9) Using the result from part (a):

$$\frac{dE}{dt} = c^2 \frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\bigg|_0^L = c^2 \left(\frac{\partial u(L,t)}{\partial x}\frac{\partial u(L,t)}{\partial t} - \frac{\partial u(0,t)}{\partial x}\frac{\partial u(0,t)}{\partial t} \right)$$

(a) If $u(0,T) = 0$ and $u(L,t) = 0$?

With our given boundary conditions, we get the following:

$$\frac{\partial u(0,t)}{\partial t} = 0 \qquad \frac{\partial u(L,t)}{\partial t} = 0$$

Thus, we get with those boundary conditions, we get an insulated system with no change in total energy:

$$\frac{dE}{dt} = 0$$

(b) If $\frac{\partial u}{\partial x}(0,t) = 0$ and $u(L,t) = 0$?

With our given boundary conditions, we get the following:

$$\frac{\partial u(L,t)}{\partial t} = 0$$

Thus, we get with those boundary conditions, we get an insulated system with no change in total energy:

$$\frac{dE}{dt} = 0$$

(c) If $u(0,t) = 0$ and $\frac{\partial u}{\partial x}(L,t) = -\gamma u(L,t)$ with $\gamma > 0$?

With our given boundary conditions, we get the following:

$$\frac{\partial u(0,t)}{\partial t} = 0$$

Thus, we get with those boundary conditions, we get an non-insulated system with a loss in total energy:

$$\frac{dE}{dt} = c^2 \left(-\gamma u(L,t) \frac{\partial u(L,t)}{\partial t} \right) = -c^2 \gamma u(L,t) \frac{\partial u(L,t)}{\partial t} < 0$$

(d) If $\gamma < 0$ in part (c)?

If $\gamma < 0$ in part (c), we would get that $\frac{dE}{dt} > 0$, which would lead into a gain in total energy.

Problem 4 Exercise 5.3.2: Consider

$$\rho \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + \alpha u + \beta \frac{\partial u}{\partial t}$$

(a) Give a brief physical interpretation. What signs must α and β have to be physical?

(b) Allow ρ, α, and β to be functions of x. Show that separation of variables works only if $\beta = c\rho$, where c is a constant.

Let $u(x,t) = \phi(x)G(t)$, so we can get the following:

$$\begin{aligned} \rho\phi(x)G''(t) &= T_0\phi''(x)G(t) + \alpha\phi(x)G(t) + \beta\phi(x)G'(t) \\ \rho\phi(x)(G''(t)) &= G(t)\left(T_0\phi''(x) + \alpha\phi(x) + \beta\phi(x)\frac{G'(t)}{G(t)}\right) \\ \frac{G''}{G} &= \frac{T_0}{\rho}\frac{\phi''}{\phi} + \frac{\alpha}{\rho} + \frac{\beta}{\rho}\frac{G'}{G} \\ \frac{G''}{G} - \frac{\beta}{\rho}\frac{G'}{G} &= \frac{T_0}{\rho}\frac{\phi''}{\phi} + \frac{\alpha}{\rho} \end{aligned}$$

From here we can see, we can no longer separate these two equations. With β on the same side of the equation as $G(t)$, the only way to separate the variables is for $\beta = c\rho$, giving us the following:

$$\frac{G''}{G} - c\frac{G'}{G} = \frac{T_0}{\rho}\frac{\phi''}{\phi} + \frac{\alpha}{\rho} = -\lambda$$

(c) If $\beta = c\rho$, show that the spatial equation is a Sturm–Liouville differential equation. Solve the time equation.

Notice the spatial equation:

$$\frac{1}{\rho}\left(T_0\frac{\phi''}{\phi} + \alpha\right) = -\lambda \qquad \rightarrow \qquad T_0\phi'' + (\lambda\rho + \alpha)\phi = 0$$

Notice that this is a Sturm–Liouville differential equation with

$$p(x) = T_0, \qquad \sigma(x) = \rho(x), \qquad q(x) = \alpha(x)$$

Notice the time equation:

$$\frac{G''}{G} - c\frac{G'}{G} = -\lambda \qquad \rightarrow \qquad G'' - cG' + \lambda G = 0$$

Notice from the characteristic equation, we get:

$$r = \frac{c \pm \sqrt{c^2 - 4\lambda}}{2}$$

From here, we can see the following solutions:

(a) Let $c^2 = 4\lambda$: we get the following solution:

$$G = (c_1 + c_2\, t)\, e^{ct/2}$$

(b) Let $c^2 < 4\lambda$: we get the following solution:

$$G = e^{c/2}\left(c_1 \cos\left(\frac{\sqrt{|c^2 - 4\lambda|}}{2}\, t\right) + c_2 \sin\left(\frac{\sqrt{|c^2 - 4\lambda|}}{2}\, t\right)\right)$$

(c) Let $c^2 > 4\lambda$: we get the following solution:

$$G = c_1 e^{\frac{c+\sqrt{c^2-4\lambda}}{2}} + c_2 e^{\frac{c-\sqrt{c^2-4\lambda}}{2}}$$

Problem 5 Exercise 5.3.3: Consider the non-Sturm–Liouville differential equation

$$\frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + [\lambda\beta(x) + \gamma(x)]\,\phi = 0.$$

Multiply this equation by $H(x)$. Determine $H(x)$ such that the equation may be reduced to the standard Sturm–Liouville form:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + [\lambda\sigma(x) + q(x)]\,\phi = 0.$$

Given $\alpha(x), \beta(x)$, and $\gamma(x)$, what are $p(x), \sigma(x)$, and $q(x)$?

Notice the result by multiplying the equation by $H(x)$:

$$H(x)\frac{d^2\phi}{dx^2} + H(x)\alpha(x)\frac{d\phi}{dx} + [\lambda H(x)\beta(x) + H(x)\gamma(x)]\,\phi = 0.$$

Now we can notice that if we let the following be true, we can reduce the non-Sturm–Liouville differential equation into the standard Sturm–Liouville form

$$p(x) = H(x) \qquad p'(x) = H'(x) = H(x)\alpha(x) \qquad \sigma(x) = H(x)\beta(x) \qquad q(x) = H(x)\gamma(x)$$

Now we can write the first two terms into the derivative of $H(x)\phi'(x)$

$$\frac{d}{dx}\left(H(x)\frac{d\phi}{dx}\right) + [\lambda H(x)\beta(x) + H(x)\gamma(x)]\,\phi = 0.$$

Now we simply substitute to get the final result:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + [\lambda\sigma(x) + q(x)]\,\phi = 0.$$

Now, given $\alpha(x), \beta(x), \gamma(x)$, we can calculate $H(x)$:

$$H'(x) = H(x)\alpha(x) \qquad \rightarrow \qquad H(x) = Ce^{\int \alpha(x)\,dx}$$

Thus we get the following for $p(x), \sigma(x)$, and $q(x)$:

$$p(x) = Ce^{\int \alpha(x)\,dx} \qquad \sigma(x) = Ce^{\int \alpha(x)\,dx}\beta(x) \qquad q(x) = Ce^{\int \alpha(x)\,dx}\gamma(x)$$

Problem 6 Exercise 5.3.9: Consider the eigenvalue problem

$$x^2\frac{d^2\phi}{dx^2}+x\frac{d\phi}{dx}+\lambda\phi=0 \text{ with } \phi(1)=0 \text{ and } \phi(b)=0. \tag{3.10}$$

(a) Show that multiplying by $1/x$ puts this in the Sturm–Liouville form. (This multiplicative factor is derived in Exercise 3.3.)

Notice the Sturm–Liouville form:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right]+\left[\lambda\sigma(x)+q(x)\right]\phi=0.$$

Now we multiply the equation by $1/x$:

$$x\frac{d^2\phi}{dx^2}+\frac{d\phi}{dx}+\lambda\frac{1}{x}\phi=0$$

From here, we can write this in the Sturm–Liouville form:

$$\frac{d}{dx}\left[x\frac{d\phi}{dx}\right]+\left[\lambda\frac{1}{x}+0\right]\phi=0 \qquad \rightarrow \qquad \frac{d}{dx}\left[x\frac{d\phi}{dx}\right]+\frac{\lambda}{x}\phi=0$$

In this case, we have the following:

$$p(x)=x \qquad \sigma(x)=\frac{1}{x} \qquad q(x)=0$$

(b) Show that $\lambda \geq 0$.

Notice, we can use the Rayleigh quotient to find λ:

$$\lambda=\frac{-p(x)\phi(x)\phi'(x)\Big|_a^b+\int_a^b\left[p(x)\left(\frac{d\phi}{dx}\right)^2-q(x)\phi^2(x)\right]dx}{\int_a^b\phi^2(x)\sigma(x)\,dx}$$

Substituting our parameters and using our BC's, we get:

$$\lambda=\int_1^b x\left(\frac{d\phi}{dx}\right)^2dx\Big/\int_1^b\frac{\phi^2(x)}{x}\,dx$$

Notice that in the interval $x=[1,b], x>0$, and at all values of x, $(\phi''(x))^2\geq 0$ and $\phi^2(x)\geq 0$. Thus we get the following:

$$\lambda=\int_1^b x\left(\frac{d\phi}{dx}\right)^2dx\Big/\int_1^b\frac{\phi^2(x)}{x}\,dx\geq 0$$

(c) Since (3.10) is an equidimensional equation, determine all positive eigenvalues. Is $\lambda = 0$ an eigenvalue? Show that there is an infinite number of eigenvalues with a smallest but no largest.

Notice that (3.10) is a Cauchy-Euler equidimensional equation, which means that the solution is in the form of $\phi = x^r$. We substitute this into (3.10), and get the following:

$$(r^2 - r)x^r + rx^r + \lambda x^r = x^r\left[r^2 + \lambda\right] = 0 \qquad \rightarrow \qquad r = \pm i\sqrt{\lambda}$$

Notice the solution for $\lambda = 0$:

$$\phi(x) = c_1 + c_2 \ln x$$

Using our boundary conditions, we get:

$$\phi(1) = c_1 = 0 \qquad \phi(b) = c_2 \ln b \qquad c_2 = 0$$

So notice that we get the trivial solution, meaning that $\lambda = 0$ is not an eigenvalue:

$$\phi(x) = 0$$

Now notice the following solution for $\lambda > 0$:

$$\phi(x) = c_1 \cos\left(\sqrt{\lambda} \ln x\right) + c_2 \sin\left(\sqrt{\lambda} \ln x\right)$$

Using our boundary conditions, we get:

$$\phi(1) = c_1 = 0 \qquad \phi(b) = c_2 \sin\left(\sqrt{\lambda} \ln b\right) = 0$$

If $c_2 = 0$, then we would get the trivial solution:

$$\phi(x) = 0$$

If $\sin\left(\sqrt{\lambda} \ln b\right) = 0$, we get the following:

$$\sqrt{\lambda_n} \ln b = n\pi \qquad \rightarrow \qquad \lambda_n = \frac{n^2\pi^2}{(\ln b)^2}$$

Notice that λ increases on the interval $n = [1, \infty)$, where $n \in \mathbb{Z}$, which means that there exists a smallest λ but not a largest. The smallest being:

$$\lambda_1 = \frac{\pi^2}{(\ln b)^2}$$

(d) The eigenfunctions are orthogonal with what weight according to Sturm–Liouville theory? Verify the orthogonality using properties of integrals

Notice the following with $u = \ln x$:

$$\int_1^b \frac{1}{x}\phi_m\phi_n \, dx = \int_1^b \frac{1}{x} \sin\frac{m\pi \ln x}{\ln b} \sin\frac{n\pi \ln x}{\ln b} \, dx = \int_0^{\ln b} \sin\frac{m\pi u}{\ln b} \sin\frac{n\pi u}{\ln b} \, du = \frac{\ln b}{2}$$

Problem 7 Exercise 5.4.5: Consider

$$\rho \frac{\partial u^2}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + \alpha u,$$

where $\rho(x) > 0, \alpha(x) < 0$, and T_0 is constant, subject to

$$u(0,t) = 0 \qquad\qquad u(x,0) = f(x)$$

$$u(L,t) = 0 \qquad\qquad \frac{\partial u}{\partial t}(x,0) = g(x)$$

Assume that the appropriate eigenfunctions are known. Solve the initial value problem.

Let the following be true:

$$u(x,t) = \phi(x)G(t) \quad \phi(0) = 0 \quad \phi(L) = 0 \quad G(0) = f(x) \quad G'(0) = g(x)$$

We can now substitute this into our original equation,

$$\rho\phi(x)G''(t) = T_0\phi''(x)G(t) + \alpha\phi(x)G(t) \qquad \to \qquad \frac{G''}{G} = \frac{T_0\phi''}{\rho\phi} + \frac{\alpha}{\rho} = -\lambda$$

This gives us two ODE's:

$$G'' + \lambda G = 0 \qquad T_0\phi'' + (\alpha + \rho\lambda)\phi = 0$$

Notice this is in the form of the Strum-Liouville form with the following:

$$p(x) = T_0 \qquad q(x) = \alpha(x) \qquad \sigma(x) = \rho(x)$$

Notice the Rayleigh Quotient:

$$\lambda = \int_0^L \left(T_0(\phi')^2 - \alpha\phi^2\right)\, dx \Big/ \int_0^L \phi^2 \rho \, dx$$

We can see that $\lambda > 0$, with this.

We can now solve the time equation to get:

$$G = c_1 \cos\sqrt{\lambda}t + c_2 \sin\sqrt{\lambda}x \qquad G' = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}t + c_2\sqrt{\lambda}\cos\sqrt{\lambda}t$$

So we get the following for $u(x,t)$:

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\sqrt{\lambda_n}t + B_n \sin\sqrt{\lambda_n}t\right)\phi_n(x)$$

We can solve for the following coefficients:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \phi_n(x)$$

From here we get the following:

$$\int_0^L \rho(x) f(x) \phi_n(x)\, dx = \sum_{n=1}^{\infty} A_n \int_0^L \rho(x) \phi_n^2(x)\, dx \qquad A_n = \int_0^L \rho(x) f(x) \phi_n(x)\, dx \Bigg/ \int_0^L \rho(x) \phi_n^2(x)\, dx$$

We can solve for the other coefficient:

$$\frac{\partial}{\partial t} u(x,0) = g(x) = \sum_{n=1}^{\infty} B_n \sqrt{\lambda_n} \phi_n(x)$$

From here we get the following:

$$\int_0^L \rho(x) g(x) \phi_n(x)\, dx = \sum_{n=1}^{\infty} \sqrt{\lambda_n} B_n \int_0^L \rho(x) \phi_n^2(x)\, dx \qquad B_n = \int_0^L \rho(x) g(x) \phi_n(x)\, dx \Bigg/ \sqrt{\lambda_n} \int_0^L \rho(x) \phi_n^2(x)\, dx$$