Homework 1
Ordinary Differential Equations
Math 537
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Solve the following problems, discuss results, and perform linear stability analysis near equilibrium points.

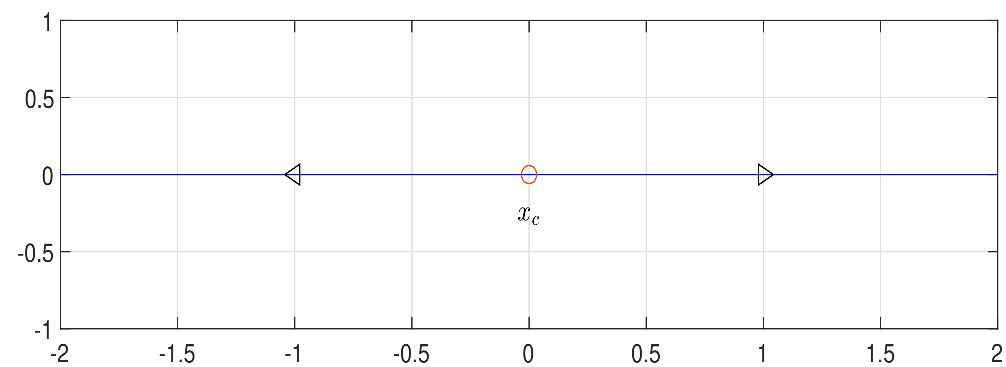
Problem 1:

$$\frac{dx}{dt} = f(x),$$

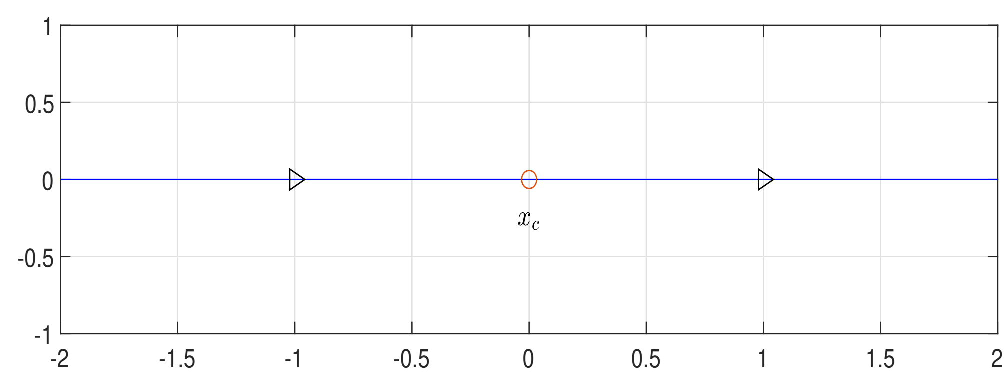
here (i) $f(x) = x$; (ii) $f(x) = x^2$; and (iii) $f(x) = x^3$

(a) Perform (linear) stability analysis.

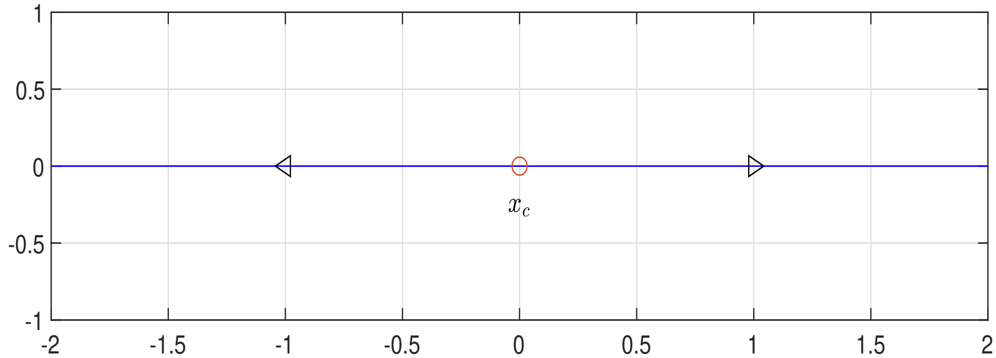
(i) For $x' = f(x) = x$, we get the fixed point $x_c = 0$. We can now see that because $f'(0) = 1 > 0$, the critical point is unstable.
We can see that for $x < 0$, we get $x' < 0$, and for $x > 0$, we get $x' > 0$, thus giving us a source.



(ii) For $x' = f(x) = x^2$, we get the fixed point $x_c = 0$. We notice that because $f'(0) = 0$, the critical point is half-stable.
We can see that for $x < 0$, we get $x' > 0$, and for $x > 0$, we get $x' > 0$, thus giving us a saddle point.



(iii) For $x' = f(x) = x^3$, we get the fixed point $x_c = 0$. We notice that because $f'(0) = 0$, the critical point is half-stable.
We can see that for $x < 0$, we get $x' < 0$, and for $x > 0$, we get $x' > 0$, thus giving us a source.



(b) Find and analyze the corresponding solutions

(i) $\frac{dx}{dt} = f(x) = x$: (Separable)

$$\frac{dx}{dt} = x \qquad \ln x = t + C$$

$$\int \frac{dx}{x} = \int dt \qquad x = Ce^t$$

If we let $x(0) = x_0$, we get the following solution:

$$x = x_0 e^t$$

Now we can see that as $t \to -\infty$, $x \to 0$, and as $t \to \infty$, $x \to \infty$.

(ii) $\frac{dx}{dt} = f(x) = x^2$: (Separable)

$$\frac{dx}{dt} = x^2 \qquad \frac{-1}{x} = t + C$$

$$\int \frac{dx}{x^2} = dt \qquad x = \frac{-1}{t + C}$$

If we let $x(0) = x_0$, we get the following solution:

$$x = \frac{x_0}{-x_0\, t + 1}$$

Now we can see that as $t \to -\infty$, $x \to 0$, and as $t \to \infty$, $x \to 0$.

(iii) $\frac{dx}{dt} = f(x) = x^3$: (Separable)

$$\frac{dx}{dt} = x^3 \qquad \frac{1}{-2x^2} = t + C$$

$$\int \frac{dx}{x^3} = dt \qquad x = \pm\sqrt{\frac{1}{-2t + C}}$$

If we let $x(0) = x_0$, we get the following solution:

$$x = \pm\sqrt{\frac{x_0^2}{-2tx_0^2 + 1}}$$

Now we can see that as $t \to -\infty$, $x \to 0$. Because we cannot have a negative inside the radical, we get undefined values of x, as $t \to \infty$.

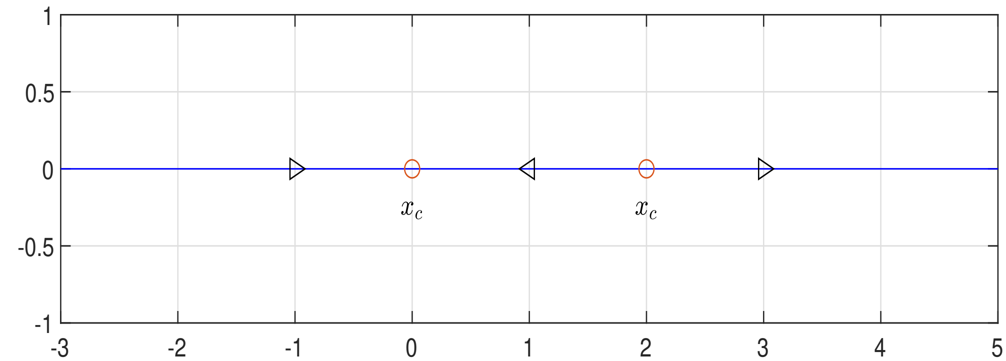
Problem 2:

$$\frac{dx}{dt} = x^2 - 2x$$

Notice we can find fixed points, when $f(x) = x^2 - 2x = 0$. We can see that we get the following fixed points: $x_c = 0, 2$.

We can now see that because $f'(0) = -2 < 0$, the critical point is stable.
We can see that for $x < 0$, we get $x' > 0$, and for $0 < x < 2$, we get $x' < 0$, thus giving us a sink.

We can now see that because $f'(2) = 2 > 0$, the critical point is unstable.
We can see that for $0 < x < 2$, we get $x' < 0$, and for $x > 2$, we get $x' > 0$, thus giving us a source.



Notice the solution for $\frac{dx}{dt} = x^2 - 2x$: (Bernoulli's with $\mu = \frac{1}{x}$ and $\frac{d\mu}{dt} = \frac{-1}{x^2}\frac{dx}{dt}$)

$$\frac{dx}{dt} + 2x = x^2 \qquad\qquad \int \frac{d\mu}{2\mu - 1} = \int dt$$

$$\frac{d\mu}{dt} - 2\mu = -1 \qquad\qquad \mu = \frac{Ce^{2t} + 1}{2}$$

Thus we get the following solution after resubstitution after solving for C with $x(0) = x_0$:

$$x = \frac{2x_0}{(2 - x_0)e^{2t} + x_0}$$

Now we can see that as $t \to -\infty$, $x \to 2$, and as $t \to \infty$, $x \to 0$.

Problem 3:

$$\frac{dx}{dt} = -(\alpha x + x^3)$$

for $x \geq 0$ and $x(t = 0) = x_0$.
[Hint: set $r = x^2$, solve for r and discuss the results when $\alpha < 0, \alpha = 0$, or $\alpha > 0$]

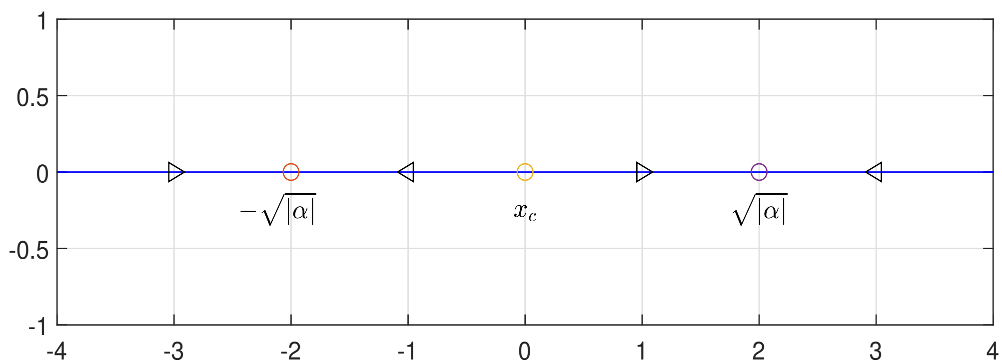
Notice we can find fixed points, when $f(x) = -(\alpha x + x^3) = 0$. We can see that we get the following fixed points: $x_c = 0, \pm\sqrt{-\alpha}$.

(a) For $\alpha < 0$, we get three fixed points $x_c = 0, \pm\sqrt{|\alpha|}$:

We can see that because $f'(-\sqrt{|\alpha|}) = 2\alpha < 0$, the critical point is stable.
We can see that for $x < -\sqrt{|\alpha|}$, we get that $x' > 0$, and for $-\sqrt{|\alpha|} < x < 0$, we get that $x' < 0$, thus giving us a sink.

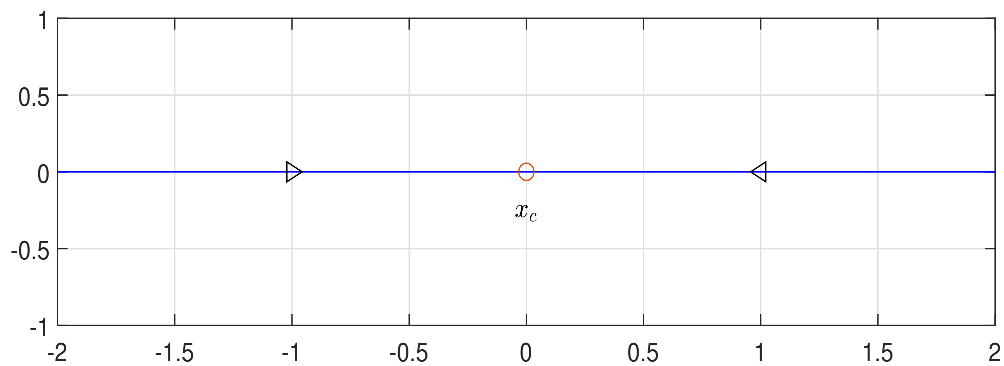
We can see that because $f'(0) = -\alpha > 0$, the critical point is unstable.
We can see that for $-\sqrt{|\alpha|} < x < 0$, we get that $x' < 0$, and for $0 < x < \sqrt{|\alpha|}$, we get that $x' > 0$, thus giving us a source.

We can see that because $f'(\sqrt{|\alpha|}) = 2\alpha < 0$, the critical point is stable.
We can see that for $0 < x < \sqrt{|\alpha|}$, we get that $x' > 0$, and for $x > \sqrt{|\alpha|}$, we get that $x' < 0$, thus giving us a sink.



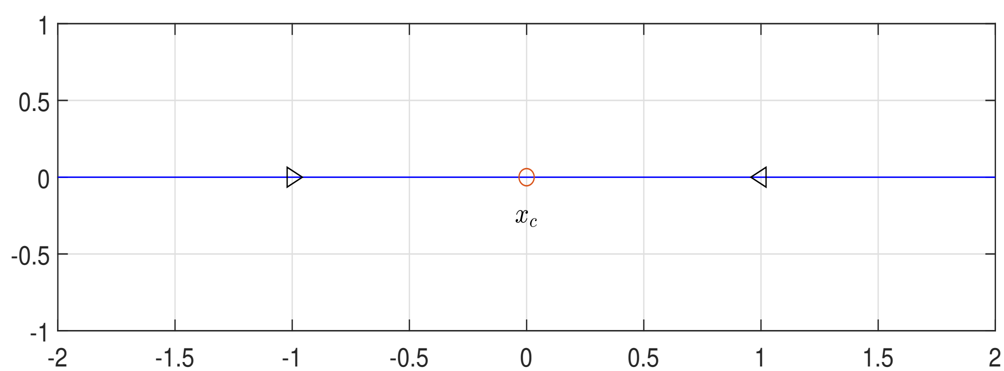
(b) For $\alpha = 0$, we get one fixed point $x_c = 0$:

We can see that because $f'(0) = 0$, the critical point is half-stable.
We can see that for $x < 0$, we get that $x' > 0$, and for $x > 0$, we get that $x' < 0$, thus giving us a sink.



(c) For $\alpha > 0$, we get one fixed point $x_c = 0$, we ignore the fixed point $x_c = i\sqrt{\alpha}$ because it is non-real:

We can see that because $f'(0) = -\alpha < 0$, the critical point is stable.
We can see that for $x < 0$, we get that $x' > 0$, and for $x > 0$, we get that $x' < 0$, thus giving us a sink.



(d) Notice that at $x_c = 0$, the fixed point changed from source to a sink as α changed. For $\alpha < 0$, $x_c = 0$ was a source, and for $\alpha \geq 0$, $x_c = 0$ was a sink. Also notice that $\forall x_c \in \{-\sqrt{|\alpha|}, 0, \sqrt{|\alpha|}\}, f'(x_c) = C\alpha$ with C representing some constant. Thus we have a bifurcation at $\alpha = 0$

Let $\alpha \neq 0$, and let $x = \sqrt{r}$ with $\frac{dx}{dt} = \frac{1}{2\sqrt{r}}\frac{dr}{dt}$. (Note that $x \neq -\sqrt{r}$, because we have that $x \geq 0$)

$$\frac{dx}{dt} = -(\alpha x + x^3) \qquad \frac{dr}{dt} = -2r(\alpha + r)$$
$$\frac{1}{2\sqrt{r}}\frac{dr}{dt} = -\sqrt{r}(\alpha + r) \qquad \frac{dr}{dt} + 2\alpha r = -2r^2$$

Now we can solve using Bernoulli's with $\mu = \frac{1}{r}$ and $\frac{d\mu}{dt} = \frac{-1}{r^2}\frac{dr}{dt}$

$$\frac{d\mu}{dt} - 2\alpha\mu = 2 \qquad \mu = \frac{Ce^{2\alpha t} - 1}{\alpha}$$
$$\int \frac{d\mu}{1 + \alpha\mu} = \int 2\,dt \qquad r = \frac{\alpha}{Ce^{2\alpha t} - 1}$$

Now we resubstitute $r = x^2$ and solve for C:

$$x = \sqrt{\frac{\alpha}{Ce^{2\alpha t} - 1}} \qquad C = \frac{\alpha + x_0^2}{x_0^2}$$
$$x(0) = x_0 = \sqrt{\frac{\alpha}{C - 1}} \qquad = \frac{\alpha}{x_0^2} + 1$$

Thus, we get the following:

$$x = \sqrt{\frac{\alpha x_0^2}{(\alpha + x_0^2)e^{2\alpha t} - x_0^2}}$$

For $\alpha < 0$, we can see that as $t \to -\infty$, $x \to 0$, and as $t \to \infty$, $x \to \sqrt{|\alpha|}$.

For $\alpha > 0$, we can see that as $t \to -\infty$, we get undefined values of x, and as $t \to \infty$, $x \to 0$.

Let $\alpha = 0$, we get the following equation to solve:

$$\frac{dx}{dt} = -x^3 \qquad\qquad \frac{1}{-2x^2} = -t + C$$

$$\int \frac{dx}{x^3} = \int -\, dt \qquad\qquad x = \sqrt{\frac{1}{2t + C}}$$

If we let $x(0) = x_0$, we get the following solution:

$$x = \sqrt{\frac{x_0^2}{2tx_0^2 + 1}}$$

For $\alpha = 0$, we can see that as $t \to -\infty$, we get undefined values of x, and as $t \to \infty$, $x \to 0$.

Problem 4: Analyze the following ODE with $\beta > 0$:

$$\frac{dx}{dt} = \beta x(1-x) - h$$

for all values of the parameter $h > 0$

Let the following be true:

$$f(x) = \beta x(1-x) - h = -\beta x^2 + \beta x - h$$

Notice we get the derivative $f'(x)$ as the following:

$$f'(x) = -2\beta x + \beta$$

We can find the fixed points x_c using the quadratic formula:

$$x_c = \frac{-\beta \pm \sqrt{\beta^2 - 4\beta h}}{-2\beta}$$

Notice the following:

$$f'\left(\frac{-\beta + \sqrt{\beta^2 - 4\beta h}}{-2\beta}\right) = \sqrt{\beta^2 - 4\beta h} > 0$$
$$f'\left(\frac{-\beta - \sqrt{\beta^2 - 4\beta h}}{-2\beta}\right) = -\sqrt{\beta^2 - 4\beta h} < 0$$

So we get that for all values, $\beta > 4h$, we get an unstable fixed point at $x_c = \frac{-\beta + \sqrt{\beta^2 - 4\beta h}}{-2\beta}$, and a stable fixed point at $x_c = \frac{-\beta - \sqrt{\beta^2 - 4\beta h}}{-2\beta}$

For values $\beta = 4h$, we get that the two earlier fixed points are the same, and get a half-stable fixed point, $x_c = \frac{1}{2}$

Lastly, for values $\beta < 4h$, we get no real fixed points.