Today 10/17

3.1 – Algebra of Continuous Functions

3.5 ~ ε-δ criterion for continuity.

Thm 3.4 Suppose that $D \subseteq \mathbb{R}$ and $f, g : D \to \mathbb{R}$.
Suppose that $x_0 \in D$ and f & g are continuous at x_0.

(a) $f + g$

(b) fg ARE ALL CONTINUOUS AT x_0

(c) f/g, provided $g(x_0) \neq 0$.

proof: Suppose $\{x_n\} \subseteq D$ and $\lim_{n\to\infty} x_n = x_0$.
Notice by our continuity assumptions,
$$\lim_{n\to\infty} f(x_n) = f(x_0) \quad \text{and} \quad \lim_{n\to\infty} g(x_n) = g(x_0).$$

By Limit Laws of 2.1,

$$\lim_{n\to\infty} \left(f(x_n) + g(x_n)\right) = f(x_0) + g(x_0)$$

$$\lim_{n\to\infty} \left(f(x_n) g(x_n)\right) = f(x_0) g(x_0)$$

and, if $g(x_0) \neq 0$, $\displaystyle\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \frac{f(x_0)}{g(x_0)}$. ☒

Look over 3.1 ① a, b, c (not d).

<u>Cor 3.5</u> Suppose that $p(x)$ and $q(x)$ are polynomials. <u>Think</u>: $p, q : \mathbb{R} \to \mathbb{R}$

Then the rational function $\frac{p}{q}$ is continuous on the domain $D = \{x \in \mathbb{R} \mid q(x) \neq 0\}$.

<u>proof</u>: Since polynomial functions are continuous, this follows from 3.4 directly.

Thm 3.6 Continuity of Compositions

Definition: Suppose $f : D \to \mathbb{R}$.

We define the image of f $\quad im(f) = f(D)$

$$im(f) = \{ f(x) \mid x \in D \} = f(D)$$

(all the y values that are attained).

Suppose $f : D \to \mathbb{R}$ and that $g : \tilde{D} \to \mathbb{R}$ where $im(f) \subseteq \tilde{D}$.

(Note $g \circ f : D \to \mathbb{R}$)

Suppose f is continuous at x_0 and g is continuous at $f(x_0)$. Then $g \circ f$ is continuous at x_0.

proof: Suppose $\{x_n\} \subseteq D$ and $\lim_{n\to\infty} x_n = x_0$.

By continuity of f at x_0, we know $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Note that $\{f(x_n)\} \subseteq \operatorname{im}(f) \subseteq \tilde{D}$.

Since g is continuous at $f(x_0)$ and $\lim_{n\to\infty} f(x_n) = f(x_0)$, we know

$$\lim_{n\to\infty} (g\circ f)(x_n) = \lim_{n\to\infty} g(f(x_n)) = g(f(x_0)) = (g\circ f)(x_0).$$

$\boxed{\text{R}}$.

3.5 ε-δ criterion for continuity.

Def: Suppose that $f: D \to \mathbb{R}$, and $x_0 \in D$.
We say f meets the ε-δ criterion at x_0
iff

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in D$, if $|x - x_0| < \delta$,
then $|f(x) - f(x_0)| < \varepsilon$.

Remark: This is an interchangable def for continuity at x_0.

Thm 3.20 Suppose $f: D \to \mathbb{R}$ and $x_0 \in D$.
f is continuous at x_0
iff
f meets the ε-δ criterion ~~fo~~ at x_0.

Proof: $(\Longleftarrow)$ Suppose f does not meet ε-δ criterion at x_0.

$\exists \varepsilon > 0, \forall \delta > 0, \exists x \in D$ where $|x - x_0| < \delta$ and $|f(x) - f(x_0)| \geq \varepsilon$.

(Show: f is not continuous at x_0.)

Let $n \geq 1$. $\exists x_n \in D$ s.t. $|x_n - x_0| < \frac{1}{n}$ and $|f(x_n) - f(x_0)| \geq \varepsilon$.

So $\{x_n\}$ form a sequence s.t.

$\{x_n\} \subseteq D$, $\lim_{n\to\infty} x_n = x_0$. (Comparison Lemma)

but $\lim_{n\to\infty} f(x_n) \neq f(x_0)$.

$(\Leftarrow)$ Suppose f meets the ε-δ criterion at x_0.

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in D$ if $|x - x_0| < \delta$, then

$$|f(x) - f(x_0)| < \varepsilon.$$

Suppose $\{x_n\} \subseteq D$ and $\lim_{n\to\infty} x_n = x_0$.

(Show: $\lim_{n\to\infty} f(x_n) = f(x_0)$).

Let $\varepsilon > 0$. Then $\exists \delta > 0$ s.t. $\forall x \in D$

if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.

Since $\lim_{n\to\infty} x_n = x_0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$|x_n - x_0| < \delta.$$

Let $n \geq N$. Since $|x_n - x_0| < \delta$, $|f(x_n) - f(x_0)| < \varepsilon$.

HW

① Suppose $f: \mathbb{R} \to \mathbb{R}$ and $f(5) = 1/2$.

If f is continuous at 5, then $\exists \varepsilon > 0$ s.t. $\forall x \in (5-\varepsilon, 5+\varepsilon)$, we have $f(x) > 0$.

More generally, If $f(x_0) > 0$ and f is continuous at x_0, then $\exists \varepsilon > 0$ s.t. $\forall x \in (x_0 - \varepsilon, x_0 + \varepsilon)$, $f(x) > 0$.