Today 11/7 Uniform Continuity 3.4/3.5

Suppose $f: D \to \mathbb{R}$.

"continuous on D"

$\forall x_0 \in D$, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in D$,
if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.

"uniformly continuous on D" (ε-δ version)

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x_0, x \in D$,
if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.

- same δ works for all x_0 in domain

- needed for integral development
- "uniform" idea is generalized in sequences of functions.

(sequential definition) $f: D \to \mathbb{R}$ <u>uniformly continuous</u>

iff

$\forall \{u_n\}, \{v_n\} \subseteq D$, if $\lim_{n\to\infty} (u_n - v_n) = 0$, then $\lim_{n\to\infty} \left(f(u_n) - f(v_n)\right) = 0$.

<u>Example 1:</u>

(A) $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = 3x+1$. is uniformly cont.

<u>proof:</u> Let $\varepsilon > 0$.

Let $\delta = \varepsilon/3 > 0$.

Let $x_0, x \in D = \mathbb{R}$.

Suppose $|x_0 - x| < \delta = \varepsilon/3$.

Then $|3x_0 - 3x| < \varepsilon$

So $|(3x_0+1) - (3x+1)| < \varepsilon$.

And $|f(x_0) - f(x)| < \varepsilon$.

SIDE:

$|f(x_0) - f(x)| < \varepsilon$

<u>Goal:</u> $|x_0 - x| < \underline{\delta ??}$

$|(3x_0+1) - (3x+1)| < \varepsilon$

$|x_0 - x| < \varepsilon/3$.

(b) Let $f(x) = mx + b$ where $m \neq 0$ and $\mathcal{D} = \mathbb{R}$.

Proof f is uniformly continuous.

Proof. Let $\{u_n\}, \{v_n\} \subseteq \mathbb{R}$.

Suppose $\lim_{n\to\infty} (u_n - v_n) = 0$.

Compute $\lim_{n\to\infty} (f(u_n) - f(v_n))$

$$= \lim_{n\to\infty} ((mu_n + b) - (mv_n + b))$$

$$= m \lim_{n\to\infty} (u_n - v_n) = 0 \quad \text{using limit laws.} \quad \boxtimes$$

Thm 3.22 ε-δ definition is equivalent to sequential def for uniform continuity.

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proof: ($\rightarrow$) Suppose $f: D \to \mathbb{R}$ satisfies the ε-δ uniform continuity.

Suppose $\{u_n\}, \{v_n\} \subseteq D$ and $\lim_{n\to\infty}(u_n - v_n) = 0$.

(Prove $\lim_{n\to\infty}(f(u_n) - f(v_n)) = 0$).

Let $\varepsilon > 0$. Let $\delta > 0$ be s.t. $\forall x, x_0 \in D$ if

$|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.

Note $\exists N$ s.t. $\forall n \geq N$, $|u_n - v_n| < \delta$.

Let $n \geq N$. Since $|u_n - v_n| < \delta$, $|f(u_n) - f(v_n)| < \varepsilon$.

So $\lim_{n\to\infty}(f(u_n) - f(v_n)) = 0$.

($\Leftarrow$) Suppose $f: D \to \mathbb{R}$ does not meet ε-δ uniform continuity.

So $\exists \varepsilon > 0$, $\forall \delta > 0$, $\exists x, x_0 \in D$ s.t.

$$|x - x_0| < \delta \text{ and } |f(x) - f(x_0)| \geq \varepsilon.$$

Show: $\exists \{u_n\}, \{v_n\} \subseteq D$ s.t. $\lim_{n\to\infty} (u_n - v_n) = 0$ and $\lim_{n\to\infty} (f(u_n) - f(v_n)) \neq 0$

Let $n \in \mathbb{N}^+$. Consider $\delta = \frac{1}{n}$.

We can choose $u_n, v_n \in D$ s.t.

$$|u_n - v_n| < \delta = \frac{1}{n} \text{ and } |f(u_n) - f(v_n)| \geq \varepsilon.$$

So clearly $\lim_{n\to\infty} (u_n - v_n) = 0$ and $\lim_{n\to\infty} (f(u_n) - f(v_n)) \neq 0$.

Example: $f(x) = x^2$ for $f: \mathbb{R} \to \mathbb{R}$.

f is not uniformly continuous.

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proof: Let $u_n = n$, $v_n = n + \frac{1}{n}$.

Notice $\lim_{n\to\infty} (u_n - v_n) = \lim_{n\to\infty} \frac{1}{n} = 0$.

Notice $\lim_{n\to\infty} (f(u_n) - f(v_n))$

$= \lim_{n\to\infty} \left(n^2 - \left(n^2 + 2 + \frac{1}{n^2}\right)\right)$

$= -2 \neq 0$. $\boxtimes$

Thm 3.17 A continuous function $f : [a,b] \to \mathbb{R}$ is uniformly continuous.

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proof: Contradiction Suppose f is continuous on $[a,b]$ and not uniformly continuous.

$\exists \{u_n\}, \{v_n\} \subseteq [a,b]$ s.t. $\lim_{n\to\infty} [u_n - v_n] = 0$ and $\lim_{n\to\infty} (f(u_n) - f(v_n)) \neq 0$.

By exercise 12 & possibly "passing to a subsequence" $\exists \varepsilon > 0$ s.t. $\forall n$, $|f(u_n) - f(v_n)| \geq \varepsilon$.

Since $[a,b]$ is sequentially compact, $\exists \{u_{n_k}\}$ s.t.

$$\lim_{k\to\infty} u_{n_k} = x_0 \text{ for } x_0 \in [a,b].$$

Since $\lim_{k\to\infty} (u_{n_k} - v_{n_k}) = 0$ and by HW problem 1, on HW #5,

$\lim_{k\to\infty} v_{n_k} = x_0$. Since f is continuous at x_0,

$$\lim_{k\to\infty} (f(u_{n_k}) - f(v_{n_k})) = 0 \quad (\Rightarrow\Leftarrow).$$