**Classwork 8**
**Abstract Algebra**
**Math 320**
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**Problem 1:** Consider the polynomial $f(x) = x^4 + x^3 + x^2 + 1$ in $\mathbb{Z}_3[x]$. Prove that $f(x)$ is irreducible. If you only check for roots, you will receive 0 points, and if you use the Rational Roots Test or Eisenstein, you will receive 0 points

Notice: The possible roots of $f(x)$ are $0, 1, 2$:

$$\begin{aligned} f(0) &= 1 \\ f(1) &= 1 \\ f(2) &= 2 \end{aligned}$$

Thus, there are no roots, meaning the factors must be both degree 2, that is:

$$\begin{aligned} x^4 + x^3 + x^2 + 1 &= (ax^2 + bx + c)(dx^2 + ex + f) \\ &= x^4 + (a + c)x^3 + (ac + b + d)x^2 + (bc + ad)x + bd \end{aligned}$$

Thus we get

$$a + c = 1 \tag{1}$$
$$ac + b + d = 1 \tag{2}$$
$$bc + ad = 1 \tag{3}$$
$$bd = 1 \tag{4}$$

Notice because of (4), we have $b = d = 1$ or $b = d = 2$.

$$\begin{aligned} b = d &= 1 \\ ac + 2 &= 1 \\ ac &= 2 \end{aligned} \qquad\qquad \begin{aligned} b = d &= 2 \\ ac + 1 &= 1 \\ ac &= 0 \end{aligned}$$

If $ac = 2$, then either $a = 1$,$c = 2$ or $a = 2, c = 1$. Either way $a + c = 0 \neq 1$.

If $ac = 0$, then if we let $a = 0$, then $c = 1$. But this contradicts (3), because this is the case when $b = 2$, and $(2)(1) = 2 \neq 1$. The same is true, when we let $c = 0$.

**Problem 2:** Write out the multiplication table for $K = \mathbb{Z}_2[x]/(x^2 + x + 1)$. Use your table to explain why K is a field

| | $0$ | $1$ | $x$ | $x+1$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $1$ | $0$ | $1$ | $x$ | $x+1$ |
| $x$ | $0$ | $x$ | $x+1$ | $1$ |
| $x+1$ | $0$ | $x+1$ | $1$ | $x$ |

A field is defined as a commutative ring with identity such that all its nonzero elements have a multiplicative inverse.

Well we can see that $K$ is a commutative ring as the table is symmetric meaning that $a * b = b * c \in K$.

Also we can see that each row, and each column contain 1, meaning each nonzero element has a multiplicative inverse.

**Problem 3:** Every element of $R = \mathbb{Q}[x]/(x^2 - 2)$ is a congruence class and can be written in the form $[ax + b]$. Determine the rules for multiplication of congruence classes in $R$. That if, if $[ax + b][cx + d] = [rx + s]$, solve for $r$ and $s$ in terms of $a, b, c, d$

Notice: $[x^2] = [2]$

$$\begin{aligned}(ax + b)(cx + d) &= acx^2 + adx + bcx + bd \\ &= 2ac + adx + bcx + bd \\ &= (ad + bc)x + (2ac + bd)\end{aligned}$$

So we get:

$$r = ad + bc \qquad\qquad s = 2ac + bd$$

**Problem 4:** Let $f(x), g(x) \in F[x]$, not both zero. Prove that if there exist $u(x), v(x) \in F[x]$ such that $f(x)u(x) + g(x)v(x) = 1_F$, then $f(x)$ and $g(x)$ are relatively prime.

Let $d(x) = gcd(f(x), g(x))$ and $f(x)u(x) + g(x)v(x) = 1_F$.

So notice now for $a(x), b(x) \in F[x]$:

$$f(x) = a(x)d(x)$$
$$g(x) = b(x)d(x)$$

And now we can replace these values into our beginning equation:

$$f(x)u(x) + g(x)v(x) = 1_F$$
$$d(x)(a(x)u(x) + b(x)v(x)) = 1_F$$

Thus we get $d(x)|1_F$. Because $d(x)|1_F$, degree of $d(x)$ must be 0, such that $d(x) = 1_F$