FROM 9.3 FINISHING...

Lemma: Suppose $x \in \mathbb{R}$ and $\varepsilon > 0$. We have

$$|x| < \varepsilon$$

iff

$$-\varepsilon < x < \varepsilon.$$

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proof: ($\Leftarrow$). Suppose $-\varepsilon < x < \varepsilon$.

case 1: Suppose $x \geq 0$. Then $|x| = x < \varepsilon$.

case 2: Suppose $x < 0$. Then $|x| = -x$.

Since $-\varepsilon < x$, we have

$$\varepsilon > -x = |x|.$$

By cases 1 & 2, $|x| < \varepsilon$.

9/5 • HW 2 is posted!!

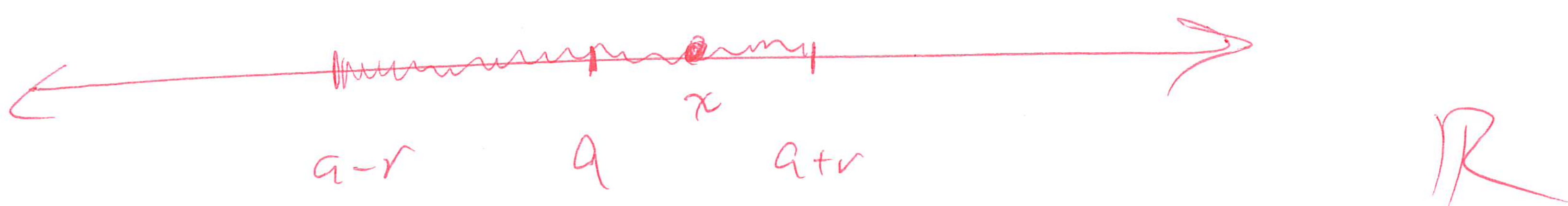
$(\mathbb{Z}^+ = \mathbb{Z} \cap (0,\infty)$, $\mathbb{R}^+ = \mathbb{R} \cap (0,\infty).)$

Today: - More w/ absolute value / Δ-inequality.
(Giller notes 2.6 / text 1.3)

• 1.2 text

Prop 2.6.1 $\forall a \in \mathbb{R}$, $r \in \mathbb{R}^+$, we have

1. $\{x \in \mathbb{R} \mid |x-a| < r\} = (a-r, a+r)$.
2. $\{x \in \mathbb{R} \mid |x-a| \leq r\} = [a-r, a+r]$.



2.6.2 $\forall a, b \in \mathbb{R}$, we have $|ab| = |a| \cdot |b|$.

- You should be able to write out the 4 cases.

Text: Prop 1.12

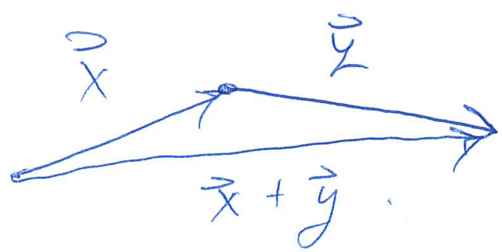
$\forall a \in \mathbb{R}$, $\forall r \in \mathbb{R}^+$, let $x \in \mathbb{R}$. T.F.A.E.

(i) $|x-a| < r$

(ii) $-r < x-a < r$

(iii) $x \in (a-r, r+a)$.

Triangle Inequality:



$\|\vec{x}+\vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|$.

Notes 2.6.3 / Text Thm 1.11

$\forall a, b \in \mathbb{R}$, we have $|a+b| \le |a| + |b|$.

proof: Let $a, b \in \mathbb{R}$

Claim: $-|a| \le a \le |a|$.

Case 1: Suppose $a \ge 0$. $|a| = a \ge a \ge 0 \ge -|a|$.

Case 2: Suppose $a < 0$. $|a| = -a$

So $-|a| = a \le a < 0 \le |a|$.

Similarly, $-|b| \le b \le |b|$.

So $-|a| + (-|b|) \le a+b \le |a| + |b|$.

$-(|a| + |b|) \le a+b \le |a| + |b|$.

By our Lemma, $|a+b| \le |a| + |b|$.

Notes 2.6.4 Reverse Triangle Inequality

$\forall a,b \in \mathbb{R}$, $||a|-|b|| \leq |a-b|$.

proof: ~~Notice~~ Let $a,b \in \mathbb{R}$.

Note $|a| = |a-b+b| \leq |a-b| + |b|$ by 2.6.3

Thus $|a|-|b| \leq |a-b|$. $(\star\star)$

Similarly $|b| = |b-a+a| \leq |b-a| + |a|$.

and so $|b|-|a| \leq |a-b|$.

Thus $-(|b|-|a|) = |a|-|b| \geq -|a-b|$. $(\star)$

So by $(\star)$ and $(\star\star)$ we have

$$-|a-b| \leq |a|-|b| \leq |a-b|.$$

By our Lemma, $||a|-|b|| \leq |a-b|$.

Example Problems: Proofs involving inequalities.

1. Build new from what you know in proof.
   Build from what you want on the side.
2. Estimate on the correct side PRACTICE.

Cauchy's Inequality: $\forall a, b \in \mathbb{R}$ $\quad ab \leq \frac{1}{2}(a^2+b^2)$.

proof: Let $a, b \in \mathbb{R}$.

We know $0 \leq (a-b)^2$

So $0 \leq a^2 - 2ab + b^2$

So $2ab \leq a^2 + b^2$

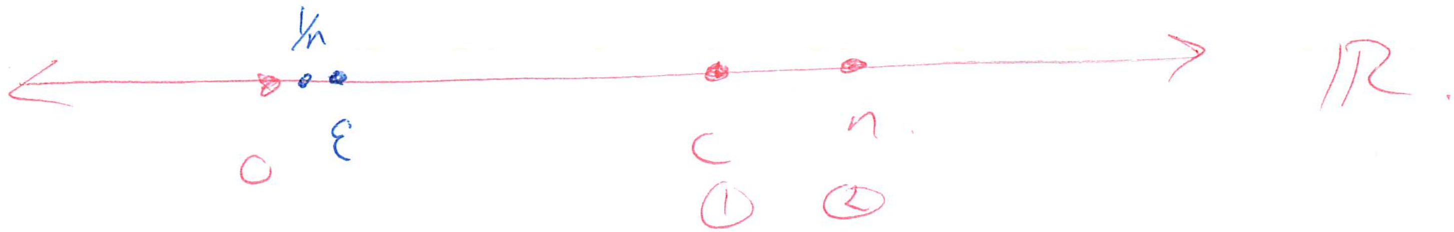
So $ab \leq \frac{1}{2}(a^2+b^2)$.

Text 1.2 "Distribution of $\mathbb{Z}$ & $\mathbb{Q}$ in $\mathbb{R}$".

Theorem 1.5 The Archimedian Property.

(i) $\forall c \in \mathbb{R}^+, \exists n \in \mathbb{N}$ s.t. $n > c$.

(ii) $\forall \varepsilon \in \mathbb{R}^+, \exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon$.



Proof: (ii) Follows directly from (i).

For (i), proceed by contradiction

Suppose $\exists c \in \mathbb{R}^+, \forall n \in \mathbb{N}$ $n \leq c$. So c is an upper bound for $\mathbb{N}$.

So $\mathbb{N}$ are bounded above. By completeness, $\exists b \in \mathbb{R}$ s.t.

$\sup \mathbb{N} = b$.

So $b - \frac{1}{2}$ is not an upper bound for $\mathbb{N}$.

Thus $\exists N \in \mathbb{N}$ s.t. $N > b - \frac{1}{2}$.

So $N+1 > b - \frac{1}{2} + 1 > b$.

Since $N+1 \in \mathbb{N}$, b is not an upper bound for $\mathbb{N}$ ($\Rightarrow\Leftarrow$).

For (ii), let $\varepsilon > 0$.

Then $\frac{1}{\varepsilon} > 0$.

By (i) $\exists n \in \mathbb{N}$ s.t. $n > \frac{1}{\varepsilon}$.

So $n\varepsilon > 1$

So $\varepsilon > 1/n$.