Final Exam
Partial Differential Equations
Math 531
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Problem 1: Consider the PDE for the heat equation on a semi-infinite domain:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad t > 0, \qquad x > 0,$$

with the following boundary condition and initial condition:

$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad \text{and} \qquad u(x,0) = xe^{-x^2/2},$$

Solve this problem for $u(x,t)$ using Fourier cosine transform. Leave your answer in integral form. That is, do not attempt to evaluate the Fourier integral expression for the solution $u(x,t)$.

Let the following be true:

$$u(x,t) = \phi(x)h(t)$$

So we can substitute this in and get the following equations assuming $\lambda = \omega^2$:

$$h(t) = Ce^{-\omega^2 kt} \qquad \phi(x) = c_1 \cos \omega x + c_2 \sin \omega x$$

Using our boundary conditions, we get:

$$\phi'(x) = -c_1 \sin \omega x + c_2 \omega \cos \omega x \qquad \phi'(0) = 0 = c_2\omega \quad \rightarrow \quad c_2 = 0$$

Thus we get the following for $u(x,t)$:

$$u(x,t) = \int_0^\infty A(\omega) \cos(\omega x)\, e^{-\omega^2 kt}\, d\omega$$

using the initial condition, gives us:

$$u(x,0) = xe^{-x^2/2} = \int_0^\infty A(\omega)\cos(\omega x)\ d\omega$$

with the following coefficients:

$$A(\omega) = \frac{2}{\pi}\int_0^\infty xe^{-x^2/2}\cos(\omega x)\ dx$$

Problem 2: Consider the nonhomogeneous wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad t > 0 \qquad \text{and} \qquad 0 < x < \pi,$$

with one end fixed and the other having a time dependent (sinusoidal) forcing condition:

$$u(0,t) = 0 \qquad \text{and} \qquad u(\pi,t) = A\sin(\omega t).$$

Assume homogeneous initial conditions:

$$u(x,0) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = 0.$$

(a) Find a linear (in x) reference distribution, $r(x,t)$, with $u(x,t) = v(x,t) + r(x,t)$, such that the PDE in $v(x,t)$ has homogeneous boundary conditions. Be sure to note the changes in both the PDE and the initial conditions with this reference function.

Notice the following:

$$\frac{\partial^2 r}{\partial x^2} = 0 \qquad \to \qquad r(0,t) = 0 \quad \text{and} \quad r(\pi,t) = A\sin(\omega t)$$

After solving, we get the following

$$r(x,t) = \frac{A\sin(\omega t)}{\pi} x$$

Thus we get the following:

$$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2} + \frac{A\omega^2 \sin(\omega t) x}{\pi} \quad \text{with} \quad v(0,t) = v(\pi,t) = 0, \quad v(x,0) = 0, \quad \frac{\partial v}{\partial t}(x,0) = \frac{A\omega x}{\pi}$$

(b) The PDE in $v(x,t)$ is nonhomogeneous, but has homogeneous boundary conditions, so apply the method of eigenfunction expansion with $v(t) = \sum_{n=0}^{\infty} a_n(t)\phi_n(x)$ where $\phi_n(x)$ are the appropriate eigenfunctions corresponding to the homogeneous boundary conditions to solve this problem. The PDE in $v(x,t)$ has the form:

$$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2} + Q(x,t), \qquad t > 0 \quad \text{and} \quad 0 < x < \pi.$$

Write the nonhomogeneous function $Q(x,t)$ in an eigenfunction expansion, $Q(x,t) = \sum_{n=0}^{\infty} q_n(t)\phi_n(x)$, and determine the Fourier coefficients, $q_n(t)$, for this function. The problem for $v(x,t)$ will have a second order nonhomogeneous ODE in $a_n(t)$, which may have a messy expression, so can be left in integral form. (Variation of parameters solution) The expressions for the Fourier coefficients from the initial conditions can also be left in integral form, but you do need to write these integrals.

Notice the following eigenvalues and eigenfunction:

$$\phi_n(x) = \sin nx \qquad \lambda_n = n^2$$

Now we can get the following solution for $v(t)$:

$$v(t) = \sum_{n=1}^{\infty} a_n(t) \sin nx$$

Now we can resubstitute this into our original equation:

$$\sum_{n=1}^{\infty} \frac{d^2 a_n(t)}{dt^2} \sin nx = -c^2 \sum_{n=1}^{\infty} n^2 a_n(t) \sin nx + \frac{A\omega^2 \sin(\omega t) x}{\pi}$$

Notice we get the following:

$$\sum_{n=1}^{\infty} \left(\frac{d^2 a_n(t)}{dt^2} + c^2 n^2 a_n(t) \right) \sin nx = \frac{A\omega^2 \sin(\omega t) x}{\pi}$$

Solving for a_n when $n \neq 0$ and using the initial condition:

$$\frac{d^2 a_n(t)}{dt^2} + c^2 n^2 a_n(t) = 0 \qquad \rightarrow \qquad a_n(t) = c_2 \sin(cnt)$$

Solving for a_n when $n = 0$:

$$\frac{d^2 a_0(t)}{dt^2} = \frac{A\omega^2 \sin(\omega t) x}{\pi} \qquad \rightarrow \qquad a_0(t) = -\frac{A \sin(\omega t) x}{\pi}$$

Now we can use our initial condition to solve for $a_n(0)$:

$$\frac{\partial v}{\partial t}(x,0) = \sum_{n=1}^{\infty} \frac{da_n(0)}{dt} \sin nx = \frac{A\omega x}{\pi} \qquad \rightarrow \qquad \frac{da_n(0)}{dt} = \frac{2}{\pi} \int_0^{\pi} \left(\frac{A\omega x}{\pi} \right) \sin nx \, dx = c_2(cn)$$

Thus, we get our missing coefficient such that:

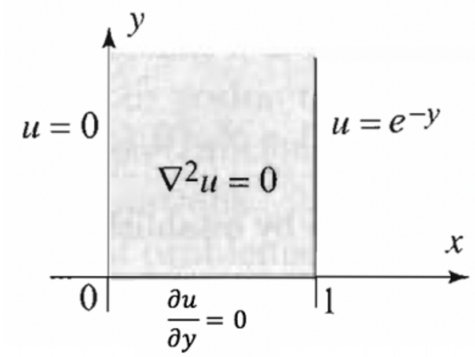
$$c_2 = \frac{2}{\pi(cn)} \int_0^{\pi} \left(\frac{A\omega x}{\pi} \right) \sin nx \, dx$$

Problem 3:

(a) Find the solution for the Laplace equation in a semi-infinite strip;

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad y > 0, \quad 0 < x < 1,$$

with boundary conditions provided in the figure



Notice the following boundary conditions:

$$u(0, y) = 0 \qquad \frac{\partial u}{\partial y}(x, 0) = 0 \qquad u(1, y) = e^{-y}$$

We can let $u(x, y) = f(x)g(y)$ and $\lambda > 0$ to get the following:

$$f''g = -fg'' \qquad \rightarrow \qquad \frac{f''}{f} = -\frac{g''}{g} = -\lambda \qquad f'' + \lambda f = 0 \qquad g'' - \lambda g = 0$$

From our x-dependent ODE and its boundary condition $f(0) = 0$, we get the following:

$$f(x) = c_2 \sin \sqrt{\lambda} x$$

From our y-dependent ODE and its boundary condition $g'(0) = 0$ along with the boundedness condition that $g(y)$ must be finite as $y \rightarrow \infty$, we get the following:

$$g(y) = d_1 e^{-\sqrt{\lambda} y}$$

Thus we get the following for $u(x, y)$:

$$u(x, y) = A(y) \sin \sqrt{\lambda} x e^{-\sqrt{\lambda} y}$$

Using the other boundary condition now gets us:

$$A(y) = \frac{1}{\sin \sqrt{\lambda} e^{(-\sqrt{\lambda}+1)y}}$$

Problem 4: Solve the nonhomogeneous heat equation

$$\frac{\partial u}{\partial t} = \nabla^2 u + \sin(2x)\sin(3y), \qquad 0 < x < \pi, \quad 0 < y < \pi, \quad t > 0,$$

$$u(x,y,0) = \sin(4x)\sin(7y) \qquad u(x,0,t) = u(x,\pi,t) = u(0,y,t) = u(\pi,y,t) = 0$$

Let the following be true:

$$u(x,y,t) = f(x)g(y)h(t)$$

Using separation of variables, we get:

$$f'' = -\lambda f \qquad g'' = -\mu g \qquad h' + (\lambda+\mu)h = 0$$

Notice the following for the x dependent equation, we get:

$$f_(x) = a_1 \cos\sqrt{\lambda}x + a_2 \sin\sqrt{\lambda}x$$

Now we apply the following boundary conditions, $f(0) = f(\pi) = 0$:

$$f(0) = a_1 = 0 \qquad f(\pi) = a_2 \sin\sqrt{\lambda}\pi \qquad \lambda_m = m^2$$

Notice the following for the y dependent equation, we get:

$$g(y) = b_1 \cos\sqrt{\mu}y + b_2 \sin\sqrt{\mu}y$$

Now we apply the following boundary conditions, $g(0) = g(\pi) = 0$:

$$g(0) = b_1 = 0 \qquad g(\pi) = b_2 \sin\sqrt{\mu}\pi \qquad \mu_n = n^2$$

Notice the following for the t dependent equation, we get:

$$h(t) = Ce^{-(\lambda+\mu)t}$$

Thus we get the following:

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin(mx)\sin(ny)e^{-(\lambda+\mu)t}$$

Now notice the resubstitution into our original equation:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left((m^2+n^2)-(\lambda+\mu)\right)A_{mn}e^{-(\lambda+\mu)t}\sin(mx)\sin(ny) = \sin(2x)\sin(3y)$$

Thus, we get the following for $m \neq 2$ and $n \neq 3$:

$$A_{mn} = 0$$

Thus we get the following for $u(x,y,t)$:

$$u(x,y,t) = A_{(2)(3)}\sin(2x)\sin(3y)e^{-(\lambda+\mu)t}$$

Now we can apply our initial condition and get:

$$u(x,y,0) = \sin(4x)\sin(7y) = A_{(2)(3)}\sin(2x)\sin(3y)$$

Thus we get the following for $A_{(2)(3)}$:

$$A_{(2)(3)} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi}\sin(4x)\sin(7y)\sin(2x)\sin(3y)\,dx\,dy$$

Problem 5: Solve the nonhomogeneous partial differential equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + e^{-2t}\sin(5x), \qquad t > 0 \quad \text{and} \quad 0 < x < \pi,$$

with initial and boundary conditions:

$$u(x,0) = 0, \qquad u(0,t) = 1 \quad \text{and} \quad u(\pi,t) = 0$$

We can begin by solving for the steady state problem ($t \to \infty$):

$$\frac{d^2 u_E}{dx^2} = 0 \quad \text{with} \quad u_E(0) = 1 \qquad u_E(\pi) = 0$$

Solving this, we get the following:

$$u_E(x) = \frac{-x}{\pi} + 1$$

Now we can resubstitute the following:

$$v(x,t) = u(x,t) - u_E(x) \qquad \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + e^{-2t}\sin(5x) \qquad v(0,t) = 0 \quad v(\pi) = 0 \quad v(x,0) = \frac{x}{\pi} - 1$$

We now solved for $v(x,t)$'s eigenfunction and its solution:

$$\phi_n(x) = \sin nx \qquad v(t) = \sum_{n=1}^{\infty} a_n(t)\sin nx$$

Now we can resubstitute this into our original equation:

$$\sum_{n=1}^{\infty} \frac{da_n(t)}{dt}\sin nx = -\sum_{n=1}^{\infty} n^2 a_n(t)\sin nx + e^{-2t}\sin(5x)$$

Notice we get the following:

$$\sum_{n=1}^{\infty}\left(\frac{da_n(t)}{dt} + n^2 a_n(t)\right)\sin nx = e^{-2t}\sin(5x)$$

Solving for a_n when $n \neq 5$:

$$\frac{da_n(t)}{dt} + n^2 a_n(t) = 0 \qquad \to \qquad a_n(t) = a_n(0)e^{-n^2 t}$$

Solving for a_n when $n = 5$:

$$\frac{da_5(t)}{dt} + 25a_5(t) = e^{-2t} \qquad \to \qquad a_5(t) = \frac{e^{-2t}}{23} + \left(a_5(0) - \frac{1}{23}\right)e^{-25t}$$

Now we can use our initial condition to solve for $a_n(0)$:

$$v(x,0) = \sum_{n=1}^{\infty} a_n(0)\sin nx = \frac{x}{\pi} - 1 \qquad \to \qquad a_n(0) = \frac{2}{\pi}\int_0^{\pi}\left(\frac{x}{\pi} - 1\right)\sin nx\,dx = \frac{-2}{n\pi}$$

Thus, we get the solution:

$$u(x,t) = v(x,t) + u_E(x) = \sum_{n=1}^{\infty} a_n(t)\sin nx + \frac{-x}{\pi} + 1$$

Problem 6: Solve the nonhomogeneous two-dimensional heat equation with circularly symmetric time independent sources, boundary conditions, and initial conditions (inside a circle)

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + Q(r)$$

with

$$u(r,0) = f(r) \quad \text{and} \quad u(a,t) = T.$$

We can begin by solving for the steady state problem ($t \to \infty$):

$$\frac{k}{r}\frac{d}{dr}\left(r\frac{du_E}{dr}\right) + Q(r) = 0 \qquad \to \qquad \frac{d}{dr}\left(r\frac{du_E}{dr}\right) = \frac{-rQ(r)}{k} \qquad \to \qquad \frac{du_E}{dr} = \frac{-1}{kr}\int rQ(r)\,dr$$

Now we can solve for u_E:

$$\int_a^r \frac{du_E}{dr} = u_E(r) - u_E(a) = \int_a^r \frac{-1}{kr}\int rQ(r)\,dr\,dr \qquad u_E(r) = \int_a^r \frac{-1}{kr}\int rQ(r)\,dr\,dr + T$$

Now we can resubstitute the following:

$$v(r,t) = u(r,t) - u_E(r) \qquad \frac{\partial v}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r} - r\frac{\partial u_E}{\partial r}\right) + Q(r) \qquad \to \qquad \frac{\partial v}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)$$

with the following conditions:

$$v(r,0) = f(r) - u_E(r) \qquad v(a,t) = 0$$

We now solved for $v(r,t)$'s eigenfunction and eigenvalues and got the following:

$$\phi_n(r) = c_1 J_0\left(\sqrt{\lambda_n} r\right) \qquad \to \qquad \lambda_n = \left(\frac{z_n}{a}\right)^2$$

We now solved for $v(r,t)$ time dependent equation and got the following:

$$h(t) = e^{-\lambda_n k t}$$

Thus we get the following for $v(r,t)$:

$$v(r,t) = \sum_{n=1}^{\infty} A_n J_0\left(\sqrt{\lambda_n} t\right) e^{-\lambda_n k t}$$

We now solve the following coefficients, using the initial condition:

$$A_n = \frac{\int_0^a \left(f(r) - u_E(r)\right) J_0\left(\sqrt{\lambda_n} t\right) r\,dr}{\int_0^a J_0^2\left(\sqrt{\lambda_n} t\right) r\,dr}$$

Thus we get the solution:

$$u(r,t) = v(r,t) + u_E(r) = \sum_{n=1}^{\infty} A_n J_0\left(\sqrt{\lambda_n} t\right) e^{-\lambda_n k t} + \int_a^r \frac{-1}{kr}\int rQ(r)\,dr\,dr + T$$