MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #21

Chapter 6

Higher Dimensional Linear Systems

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Quiz 4

Goal: (1) Understand the relationship between the Logistic (differential) equation and the Logistic map; (2) Understand how higher derivative tests may help analyze the stability of critical points.

Total points: 50

1: [25 points] Consider the Logistic equation:

$$\frac{dX}{dt} = rX(1-X). \tag{1.1}$$

(a) Assume a time step $\triangle t$ and apply the Euler method to derive a discrete equation where X_{n+1} can be computed from X_n.

(b) Introduce a new variable Y and transform the above discrete equation into the following equation:

$$Y_{n+1} = \rho Y_n(1-Y_n). \tag{1.2}$$

Express Y_n in terms of X_n and find ρ.

Eq. (1.2) is called the Logistic map that possesses chaotic solutions for large values of ρ.

The Logistic Map: Discrete vs. Continuous

$$\frac{dX}{dt} = rX(1-X).$$

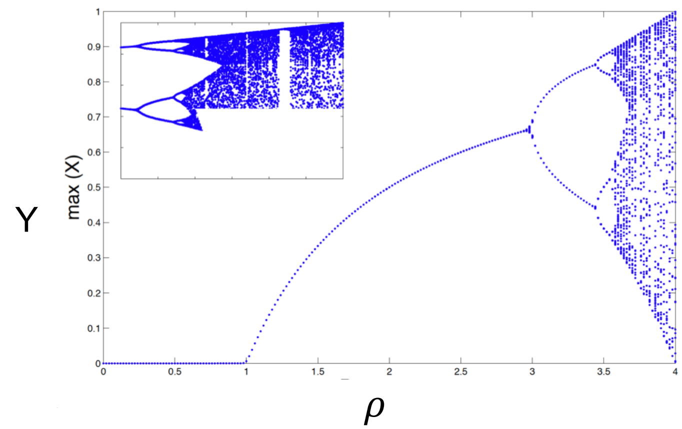
Flow

$$\rho = 1 + r\Delta t \text{ and } X_n = \frac{1+r\Delta t}{r\Delta t} Y_n,$$

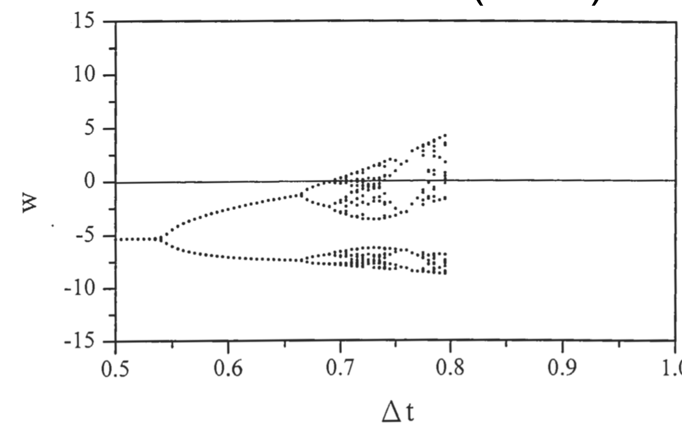
$$Y_{n+1} = \rho Y_n (1 - Y_n)$$

Map

Logistic Map



Shen and Lin (1995)



$$\frac{dW}{dt} = -g + \alpha \mid W \mid (-W), \qquad \alpha = \frac{3}{4}\frac{\rho}{\rho_p}\frac{0.46}{D}$$

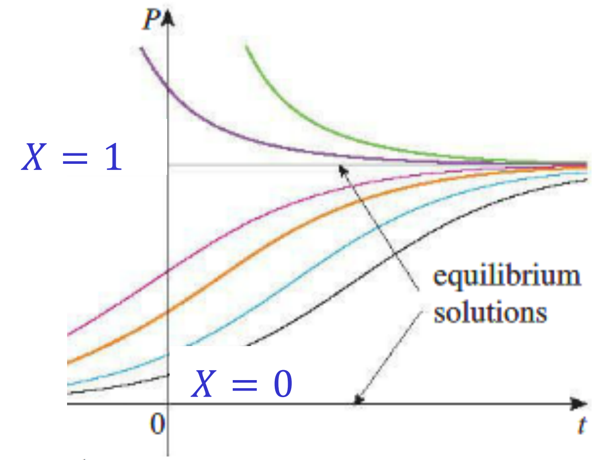
$$\frac{W_{n+1} - W_n}{\Delta t} = -g + \alpha \mid W_n \mid (-W_n)$$

Vertical Fall Through a Fluid: Terminal Velocity. The magnitude of F(W) is proportional to W^2. To ensure that the force remains resistive, we must remember that the sign preceding the F(W).

Computational Chaos: An Illustration

- Logistic Equation (Flow)

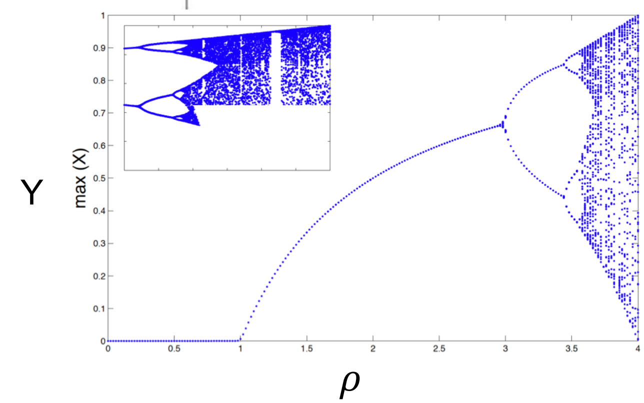
$$\frac{dX}{dt} = rX(1 - X).$$



- Logistic Map

$$Y_{n+1} = \rho Y_n(1 - Y_n)$$

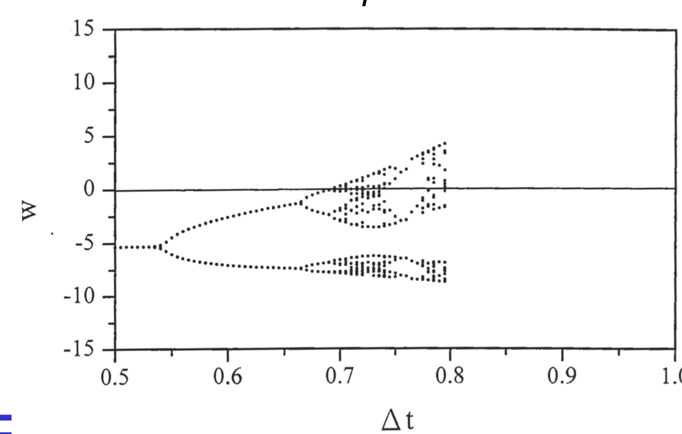
$$\rho = 1 + r\Delta t \text{ and } X_n = \frac{1+r\Delta t}{r\Delta t} Y_n,$$



- Terminal Velocity

$$\frac{dW}{dt} = -g + \alpha \mid W \mid (-W), \qquad \alpha = \frac{3}{4}\frac{\rho}{\rho_p}\frac{0.46}{D}$$

The magnitude of dW/dt is proportional to W^2
(Shen and Lin, 1995)



Maps vs. Flows

Supp

Maps	Flows
Discrete time	*Continuous* time
Variables change *abruptly*	Variables change *smoothly*
Described by *algebraic* equations	Described by *differential* equations
Complicated 1-D dynamics	*Simple* 1-D dynamics
$X_{n+1} = f(X_n)$	$dx/dt = f(x)$
Capital letters	Lower case letters
Example: $X_{n+1} = AX_n$	Example: $dx/dt = ax$
Solution: $X_{n+1} = A^n X_0$	Solution: $x = x_0 e^{at}$
Growth for $A > 1$	Growth for $a > 0$
Decay for $A < 1$	Decay for $a < 0$
Solution is called an *orbit*	Solution is called a *trajectory*
$n \to t \Rightarrow A \to e^a$	$t \to n \Rightarrow a \to \ln(A)$

$r > 1$ and $\lambda > 0$

$$y = y_0 e^{\lambda t}$$

$$y_n = y_0 e^{\lambda n \Delta t}$$

$$\frac{y_{n+1}}{y_n} = e^{\lambda \Delta t}$$

$$r = \frac{y_{n+1}}{y_n} = e^{\lambda \Delta t}$$

$$r > 1 \; when \; \lambda > 0$$

$$r < 1 \; when \; \lambda < 0$$

Review: A Summary for 2D Systems

Goal: Solve the following 2D system

$$\boxed{X' = AX}$$

Compute the eigenvalues and eigenvectors of A,
U_1 and U_2

$$AU_j = \lambda_j U_j, \qquad j = 1,2$$

Construct $T = (V_1, \mathrm{V}_2)$, $B = T^{-1}AT$ and $\mathrm{X} = TY$ using the following

(I) real eigenvalues

$$(V_1, \mathrm{V}_2) = (U_1, U_2)$$

$$B = T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

(II) complex eigenvalues

$$(V_1, \mathrm{V}_2) = (Re(U_1), Im(U_1))$$

$$B = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

(III) repeated eigenvalues

$$V_1 = U_1$$
$$(A - \lambda I)V_2 = V_1$$

$$B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$\boxed{Y' = BY}$$

Review: Saddle, Source and Sink in 2D Systems

$$\lambda^2 - T\lambda + \mathrm{D} = 0$$

$$\lambda_{\pm} = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

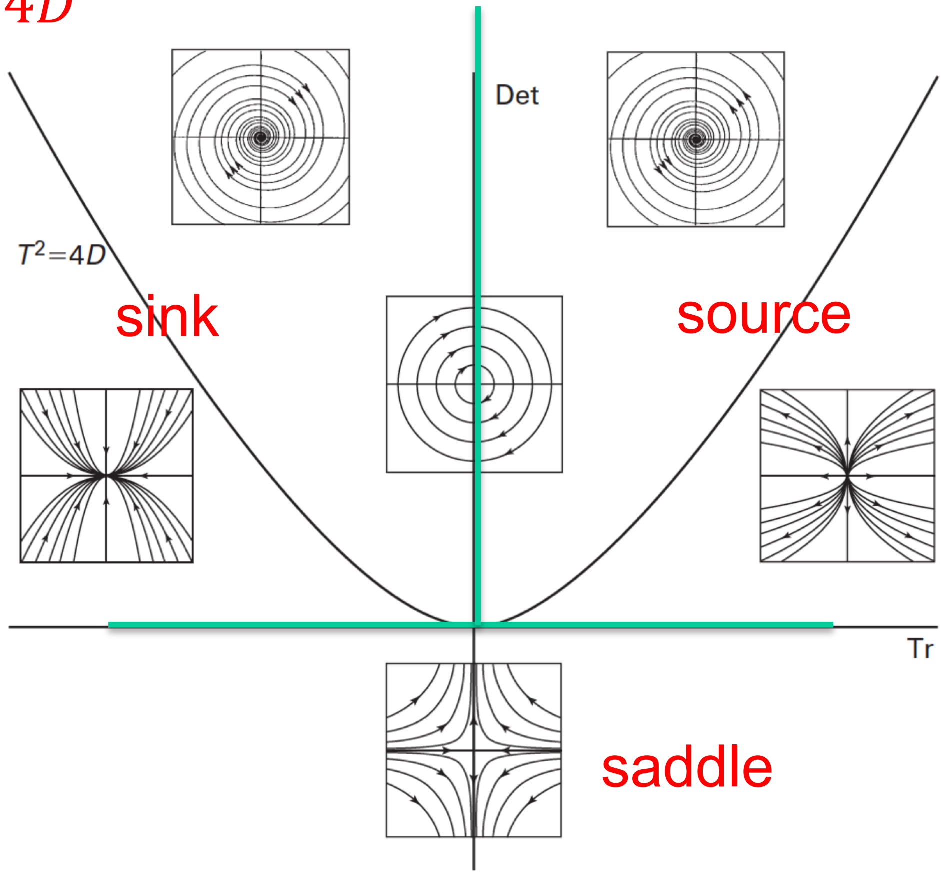
$$\boxed{\lambda_+ + \lambda_- = T = tr}$$

$$(\lambda - \lambda_+)(\lambda - \lambda_-) = 0$$

$$\lambda^2 - (\lambda_+ + \lambda_-)\lambda + \lambda_+\lambda_- = 0$$

$$\boxed{\lambda_+\lambda_- = D = determinat}$$

$T^2 = 4D$



- $D < 0$, λ_+ and λ_- have different signs ➔ saddle
- $D > 0$, λ_+ and λ_- have the same sign ➔ source with $T > 0$
  ➔ sink with $T < 0$

Review: Find Solutions using $T^{-1}AT$ for 2D Systems

Goal: Solve the following 2D system

$$\boxed{X' = AX}$$

Compute the eigenvalues and eigenvectors of A, V_1 and V_2

$$AV_j = \lambda_j V_j, \qquad j = 1,2$$

Construct T using V_1 and V_2

$$T = (V_1, V_2)$$

$$X = TY$$

$$= (V_1, V_2)\begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}$$

$$= e^{\lambda_1 t}\, V_1 + c_2 e^{\lambda_2 t} V_2$$

$$\boxed{Y' = DY} \qquad D = T^{-1}AT$$

$$Y' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} Y$$

$$Y = c_1 e^{\lambda_1 t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda_2 t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}$$

Review: Canonical Form within 2D Systems

Any 2x2 matrix that is in one of the following three forms is said to be in canonical form.

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

real distinct eigenvalues (saddle, sink and source)

$$X(t) = c_1 e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\mu t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

complex eigenvalues $\alpha \pm i\beta$

$$X(t) = c_1 e^{\alpha t} \begin{pmatrix} \cos \beta t \\ -\sin \beta t \end{pmatrix} + c_2 e^{\alpha t} \begin{pmatrix} \sin \beta t \\ \cos \beta t \end{pmatrix}$$

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

repeated eigenvalues

$$X(t) = c_1 e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda t} \begin{pmatrix} t \\ 1 \end{pmatrix}$$

Section 6.1: Distinct Eigenvalues with 3D Systems TBD

(I)

$$X' = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix} X.$$

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$$X(t) = c_1 e^{2t} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Unstable subspace stable subspace

$$\lambda_1 = 2; \lambda_2 = 1; \lambda_3 = -1$$

saddle

(II)

$$X' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} X$$

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$$Y(t) = x_0 \begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix} + y_0 \begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix} + z_0 e^{-t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda = -1; \lambda = \pm i$$

spiral center

(III)

$$A = \begin{pmatrix} -0.1 & 0 & 1 \\ -1 & 1 & -1.1 \\ -1 & 0 & -0.1 \end{pmatrix}.$$

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$$\lambda_{1,2} = -0.1 \pm i$$

$$\lambda_3 = 1$$

$$Y' = \begin{pmatrix} -0.1 & 1 & 0 \\ -1 & -0.1 & 0 \\ 0 & 0 & 1 \end{pmatrix} Y.$$

spiral saddle (saddle focus)

Classification for 3D Systems

TBD

- Sink, $\lambda_{1,2,3} < 0$
- Source, $\lambda_{1,2,3} > 0$
- Spiral center, $Re(\lambda_{1,2}) = 0$ & $\lambda_3 < 0$
- Spiral source, $Re(\lambda_{1,2}) > 0$ & $\lambda_3 > 0$
- Spiral sink, $Re(\lambda_{1,2}) < 0$ & $\lambda_3 < 0$
- Saddle focus (spiral saddle), $Re(\lambda_{1,2}) < 0$ & $\lambda_3 > 0$
- Saddle (three real eigenvalues), $\lambda_{1,2} < 0$ & $\lambda_3 > 0$

- Stable subspace: $\lambda_1, \lambda_2, \lambda_3 \dots \lambda_k$ are negative
- Unstable subspace: $\lambda_{k+1}, \lambda_{k+2}, \lambda_{k+3} \dots \lambda_n$ are positive.

(I) Real Eigenvalues within 3D Systems

Goal: Solve the following 3D system

$$\boxed{X' = AX}$$

$$\boxed{Y' = DY} \qquad D = T^{-1}AT$$

Compute the eigenvalues and eigenvectors of A, V_1, V_2 and V_3

$$AV_j = \lambda_j V_j, \qquad j = 1,2,3$$

$$Y' = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} Y$$

Uncoupled

$$Y = c_1 e^{\lambda_1 t}\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2 e^{\lambda_2 t}\begin{pmatrix}0\\1\\0\end{pmatrix} + +c_3 e^{\lambda_3 t}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

Construct T using V_1, V_2 and V_3

$$T = (V_1, V_2, V_3)$$

$$X = TY$$

$$= (V_1, V_2, V_3)\begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ c_3 e^{\lambda_3 t} \end{pmatrix}$$

$$Y = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ c_3 e^{\lambda_3 t} \end{pmatrix}$$

$$\boxed{X = e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2 + c_3 e^{\lambda_3 t} V_3 = \sum_{j=1}^{3} c_j V_j e^{\lambda_j t}}$$

Additional Details

$$Y' = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} Y$$

$$\boxed{\begin{aligned} y_1' &= \lambda_1 y_1 \\ {y_2}' &= \lambda_2 y_2 \\ {y_3}' &= \lambda_3 y_3 \end{aligned}}$$

Uncoupled

$$\boxed{\begin{aligned} y_1 &= c_1 e^{\lambda_1 t} \\ y_2 &= c_2 e^{\lambda_2 t} \\ y_3 &= c_3 e^{\lambda_3 t} \end{aligned}}$$

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ c_3 e^{\lambda_3 t} \end{pmatrix}$$

$$Y = c_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + + c_3 e^{\lambda_3 t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Example

(I)

$$X' = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix} X.$$

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$$X(t) = c_1 e^{2t} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Unstable subspace stable subspace

saddle

Review (from Sect. 5.2)

Example. Let

$$X' = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix} X. \Rightarrow A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix}.$$

$$AV = \lambda V \qquad A - \lambda I = \begin{pmatrix} 1-\lambda & 2 & -1 \\ 0 & 3-\lambda & -2 \\ 0 & 2 & -2-\lambda \end{pmatrix}$$

$$|A - \lambda I = 0| \qquad (1-\lambda)(\lambda^2 - \lambda - 2) = 0 \qquad \lambda = 2, 1, -1$$

$$\lambda_1 = 2 \qquad \lambda_2 = 1 \qquad \lambda_3 = -1$$

$$V_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad V_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Example: Solutions

(I)

$$X' = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix} X.$$

$$\lambda_1 = 2 \qquad \lambda_2 = 1 \qquad \lambda_3 = -1$$

$$V_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad V_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$X(t) = \boxed{c_1 e^{2t} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}} + \boxed{c_2 e^{t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} + \boxed{c_3 e^{-t} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}}$$

unstable subspace

stable subspace

saddle

(I) $$X(t) = c_1 e^{2t}\begin{pmatrix}3\\2\\1\end{pmatrix} + c_2 e^{t}\begin{pmatrix}1\\0\\0\end{pmatrix} + c_3 e^{-t}\boxed{\begin{pmatrix}0\\1\\2\end{pmatrix}}$$

$\lambda_3 < 0$

$\lambda_{1,2} > 0$

in canonical form

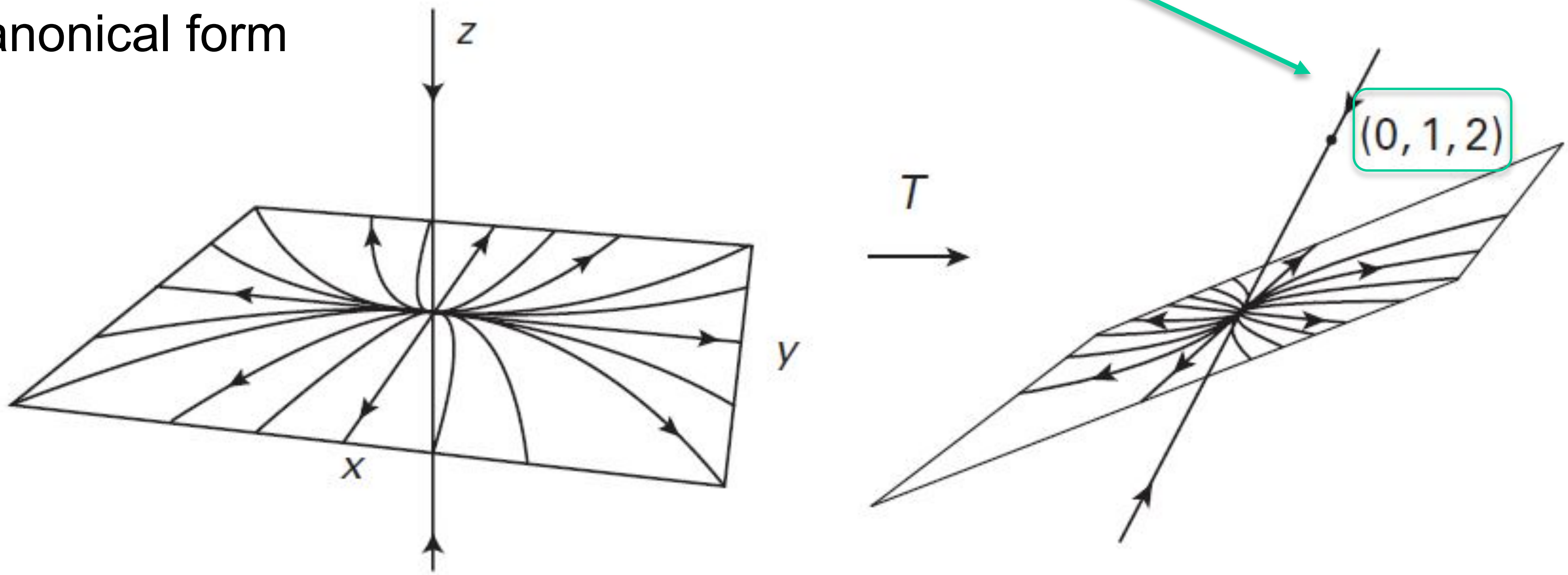


Figure 6.1 The stable and unstable subspaces of a saddle in dimension 3. On the left, the system is in canonical form.

Review (Sect 5.2): Construct the Linear Map T

$$V_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad V_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \qquad T = (V_1, \quad V_2, \quad V_3) = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

$$T^{-1}AT = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$\lambda_1 = 2$$

$$\lambda_2 = 1$$

$$\lambda_3 = -1$$

Section 6.1: Sink

$$\lambda_{1,2,3} < 0$$

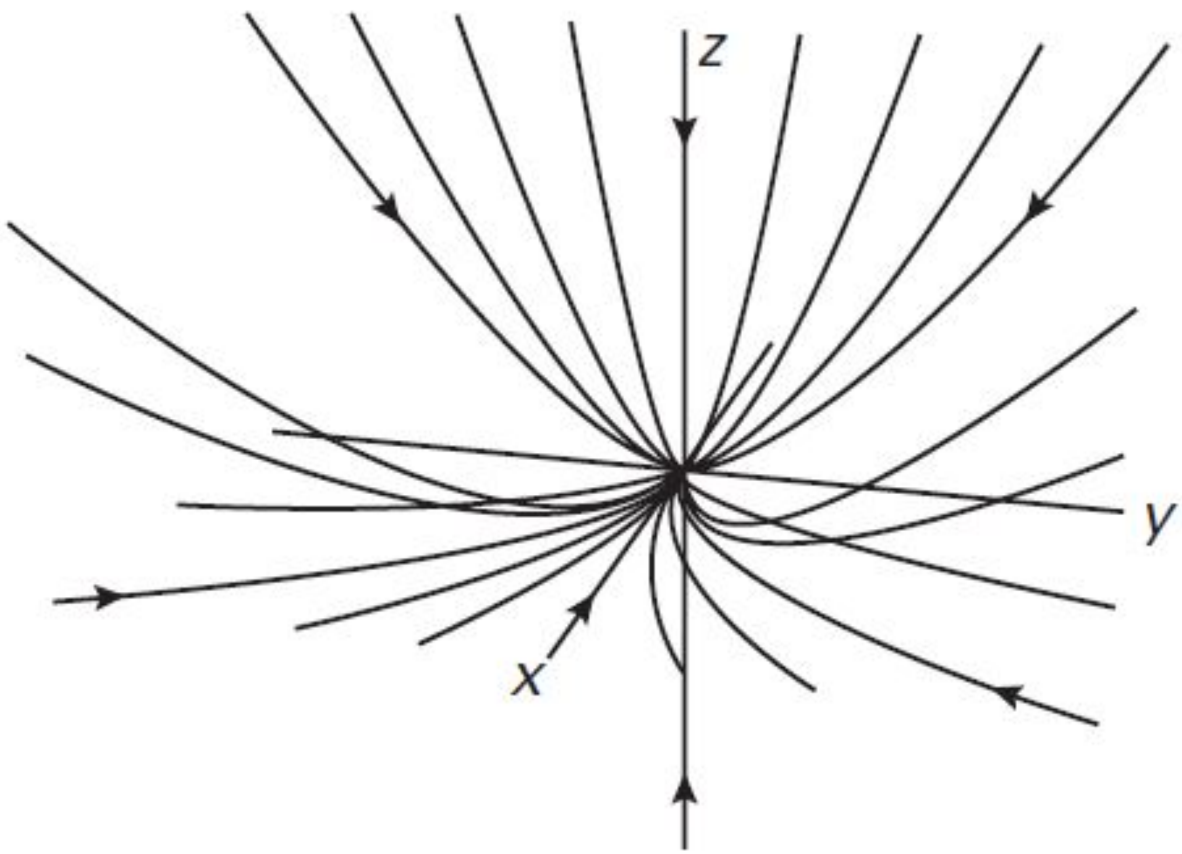


Figure 6.2 A sink in three dimensions.

(II) Complex Eigenvalues: $\alpha \pm i\beta$

Now suppose that the $n \times n$ matrix A has n distinct eigenvalues, of which k_1 are real and $2k_2$ are nonreal, so that $n = k_1 + 2k_2$. Then, as in Chapter 5, we may change coordinates so that the system assumes the form

$$x_j' = \lambda_j x_j$$
$$u_\ell' = \alpha_\ell u_\ell + \beta_\ell v_\ell$$
$$v_\ell' = -\beta_\ell u_\ell + \alpha_\ell v_\ell$$

for $j = 1, \ldots, \boxed{k_1}$ and $\ell = 1, \ldots, \boxed{k_2}$. As in Chapter 3, we therefore have solutions of the form

$\boxed{k_1 = 1;\, k_2 = 1}$

$$x_j(t) = c_j e^{\lambda_j t}$$
$$u_\ell(t) = p_\ell e^{\alpha_\ell t} \cos \beta_\ell t + q_\ell e^{\alpha_\ell t} \sin \beta_\ell t$$
$$v_\ell(t) = -p_\ell e^{\alpha_\ell t} \sin \beta_\ell t + q_\ell e^{\alpha_\ell t} \cos \beta_\ell t.$$

See the next slide for a simpler version

Complex Eigenvalues: $\alpha \pm i\beta$

$$x_1' = \lambda_1 x_1$$

$$\boxed{\begin{aligned} x_2' &= \alpha x_2 + \beta x_3 \\ x_3' &= -\beta x_2 + \alpha x_3 \end{aligned}}$$

coupled

$$\boxed{\begin{pmatrix} x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}}$$

$$x_1(t) = c_1 e^{\lambda_1 t}$$

$$x_2(t) = e^{\alpha t}(p \cos(\beta t) + q \sin(\beta t))$$

$$x_3(t) = e^{\alpha t}(-p \sin(\beta t) + q \cos(\beta t))$$

(II-A) Complex Eigenvalues: $\pm i\beta$

(II)

$$X' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} X$$

$$Y(t) = x_0 \begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix} + y_0 \begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix} + z_0 e^{-t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

spiral center

$$A - \lambda I = \begin{pmatrix} -\lambda & 1 & 0 \\ -1 & -\lambda & 0 \\ 0 & 0 & -1-\lambda \end{pmatrix}$$

$$|A - \lambda I = 0| \qquad (-\lambda)(-\lambda)(-1-\lambda) - 1(-1)(-1-\lambda) = 0$$

$$(\lambda + 1)(\lambda^2 + 1) = 0 \qquad \lambda = -1; \lambda = \pm i$$

Eigenvectors

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$\lambda = -1; \lambda = \pm i$

$AX = \lambda X \Longrightarrow$

$$\boxed{\begin{aligned} y &= \lambda \mathrm{x} \\ -x &= \lambda y \end{aligned}}$$

$$-z = \lambda z$$

coupled

Consider $\lambda = -1$ $\quad x = y = 0$ & z: any $\quad V_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Consider $\lambda = i$

$$\begin{aligned} y &= \mathrm{ix} \\ -x &= iy \\ -z &= iz \end{aligned} \qquad \begin{aligned} y &= \mathrm{ix} \\ z &= 0 \end{aligned} \qquad V_2 = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}$$

Consider $\lambda = -i$

$$\begin{aligned} y &= -\mathrm{ix} \\ -x &= -iy \\ -z &= -iz \end{aligned} \qquad \begin{aligned} y &= -\mathrm{ix} \\ z &= 0 \end{aligned} \qquad V_3 = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

Solutions using Eigenvalues and Eigenvectors

$$\lambda = -1 \qquad \lambda = i \qquad \lambda = -i$$

$$V_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \qquad V_3 = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

$$X = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2 + +c_3 e^{\lambda_3 t} V_3$$

$$X = c_1 e^{t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{it} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} + c_3 e^{-it} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

$$X = c_1 e^{t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_4 \begin{pmatrix} \cos(t) \\ -\sin(t) \\ 0 \end{pmatrix} + c_5 \begin{pmatrix} \sin(t) \\ \cos(t) \\ 0 \end{pmatrix}$$

Solution Pattern: Phase Portrait

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 e^t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_4 \begin{pmatrix} \cos(t) \\ -\sin(t) \\ 0 \end{pmatrix} + c_5 \begin{pmatrix} \sin(t) \\ \cos(t) \\ 0 \end{pmatrix}$$

$$x = c_4 \cos(t) + c_5 \sin(t)$$

$$y = -c_4 \sin(t) + c_5 \cos(t)$$

$$z = c_1 e^t$$

$$\boxed{x^2 + y^2 = c_4^2 + c_5^2}$$

circle

Construct the Linear Map T

$$\lambda = -1 \qquad \lambda = i = i\beta \qquad \lambda = -i = -i\beta$$

$$V_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \qquad V_3 = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

$$T = (V_1, \quad Re(V_2), \quad Im(V_2)) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$T^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\boxed{T^{-1}AT = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & \beta \\ 0 & -\beta & 0 \end{pmatrix}}$$

The Phase Portrait for a Spiral Center

(II) $Re(\lambda_{2,3}) = 0$

$Re(\lambda_{2,3}) = 0$

$\lambda_1 < 0$

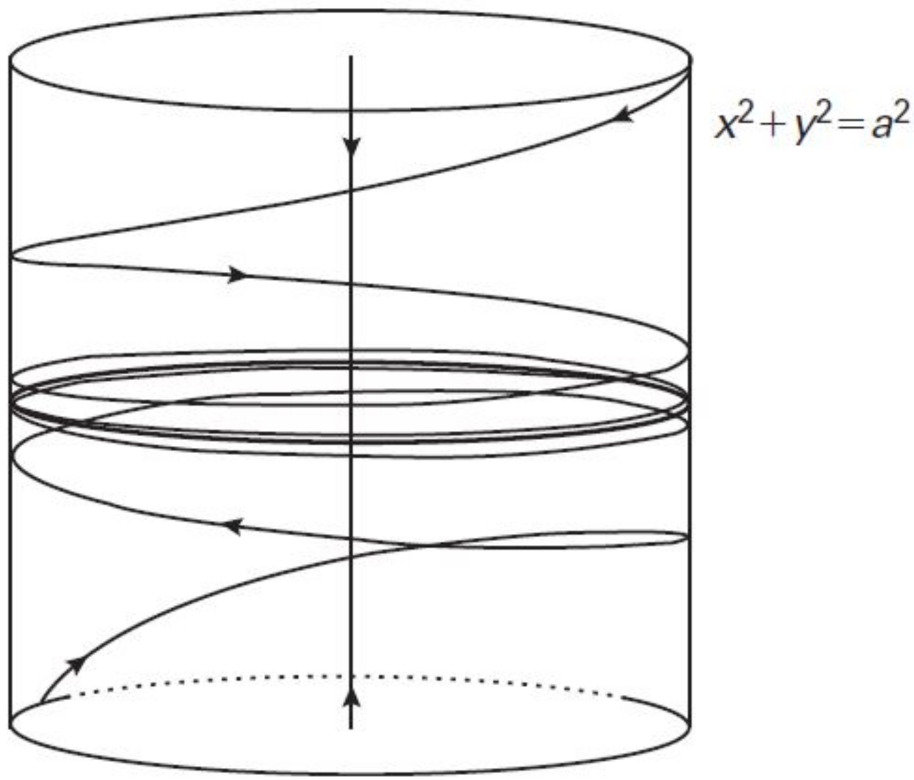


Figure 6.3 The phase portrait for a spiral center.

(II-B) Complex Eigenvalues: $\alpha \pm i\beta$

$$\boxed{Y' = \begin{pmatrix} -0.1 & 1 & 0 \\ -1 & -0.1 & 0 \\ 0 & 0 & 1 \end{pmatrix} Y.}$$

$$A = \begin{pmatrix} -0.1 & 0 & 1 \\ -1 & 1 & -1.1 \\ -1 & 0 & -0.1 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} -0.1-\lambda & 0 & 0 \\ -1 & 1-\lambda & -1.1 \\ 0 & 0 & -0.1-\lambda \end{pmatrix}$$

$$|A - \lambda I| = 0 \qquad (\lambda - 1)((\lambda + 0.1)^2 + 1) = 0 \qquad \lambda_3 = 1; \;\; \lambda_{1,2} = -0.1 \pm i$$

Eigenvectors

$$A = \begin{pmatrix} -0.1 & 0 & 1 \\ -1 & 1 & -1.1 \\ -1 & 0 & -0.1 \end{pmatrix}$$

$$AX = \lambda X \Longrightarrow \qquad \begin{aligned} -0.1x + z &= \lambda \mathrm{x} \\ -x + y - 1.1z &= \lambda y \\ -x - 0.1z &= \lambda z \end{aligned}$$

$\lambda_3 = 1$

$$\begin{aligned} -0.1x + z &= \mathrm{x} \\ -x + y - 1.1z &= y \\ -x - 0.1z &= z \end{aligned} \qquad \begin{aligned} x &= -1.1z \\ z &= 1.1x \\ x &= -1.1z \end{aligned} \qquad x = z = 0 \qquad V_3 = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} = y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$\lambda_1 = -0.1 + i$

$$\begin{aligned} -0.1x + z &= (-0.1 + \mathrm{i})\mathrm{x} \\ -x + y - 1.1z &= (-0.1 + \mathrm{i})y \\ -x - 0.1z &= (-0.1 + \mathrm{i})z \end{aligned} \qquad \begin{aligned} y &= ix \\ z &= ix \end{aligned} \qquad V_1 = \begin{pmatrix} -i \\ 1 \\ 1 \end{pmatrix}$$

Construct the Linear Map

$$\lambda = -0.1 + i \qquad \lambda = -0.1 - i \qquad \lambda = 1$$

$$V_1 = \begin{pmatrix} -i \\ 1 \\ 1 \end{pmatrix} \qquad V_2 = \overline{V_1} = \begin{pmatrix} i \\ 1 \\ 1 \end{pmatrix} \qquad V_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$T = (Re(V_1), \quad Im(V_1), \quad V_3) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$T^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \qquad A = \begin{pmatrix} -0.1 & 0 & 1 \\ -1 & 1 & -1.1 \\ -1 & 0 & -0.1 \end{pmatrix}$$

$$\boxed{T^{-1}AT = \begin{pmatrix} -0.1 & 1 & 0 \\ -1.0 & -0.1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}}$$

Solution of Y (to the System in the Canonical Form)

$$\text{“}D\text{”} = T^{-1}AT = \begin{pmatrix} -0.1 & 1 & 0 \\ -1 & -0.1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Y' = DY$$

$$y_1' = \alpha y_1 + \beta y_2$$

$$y_2' = -\beta y_1 + \alpha y_2$$

coupled

$$\boxed{\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}}$$

$$y_3' = \lambda_3 y_x$$

$$y_1(t) = e^{\alpha t}(p\cos(\beta t) + q\sin(\beta t)) = e^{-0.1t}(p\cos(t) + q\sin(t))$$

$$y_2(t) = e^{\alpha t}(q\cos(\beta t) - \mathrm{p}\sin(\beta t)) = e^{-0.1t}(-p\,sin(t) + q\cos(t))$$

$$y_3(t) = d_3 e^{\lambda_3 t} = d_3 e^t$$

A Spiral Saddle in Canonical Form

S5

(II) $Y' = \begin{pmatrix} -0.1 & 1 & 0 \\ -1 & -0.1 & 0 \\ 0 & 0 & 1 \end{pmatrix} Y.$

$\lambda_{1,2} = -0.1 \pm i$

$\lambda_3 = 1$

$Re(\lambda_{1,2}) < 0$

$\lambda_3 > 0$

Saddle focus
(Ott, p333/334)

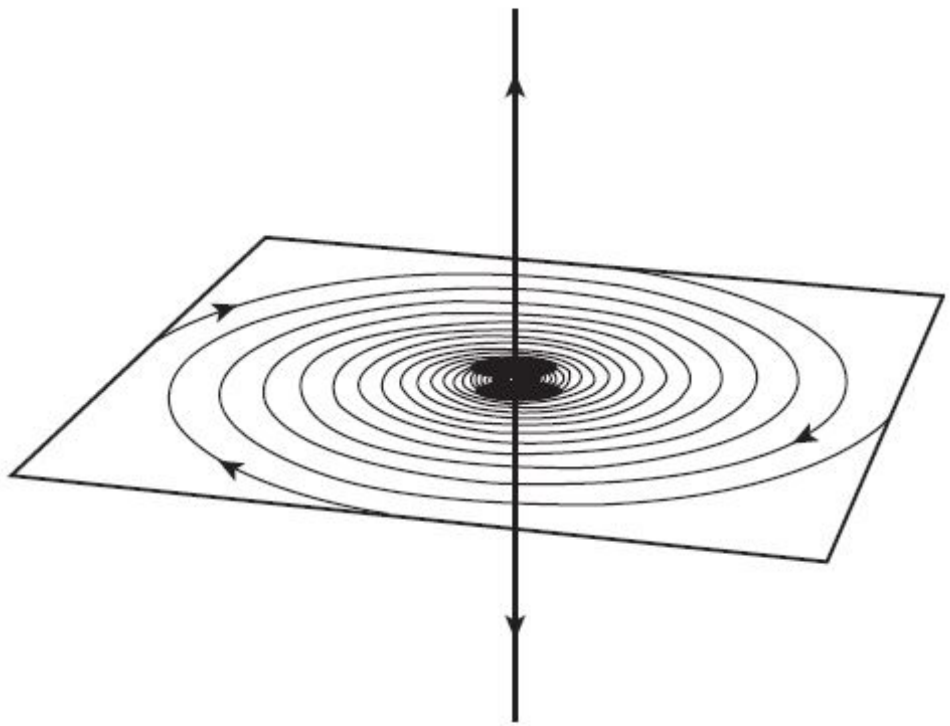


Figure 6.4 A spiral saddle in canonical form.

Compute $\mathbf{X} = \mathbf{T}\boldsymbol{Y}$ (Method I)

$$T = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$y_1(t) = e^{\alpha t}(p\cos(\beta t) + q\sin(\beta t)) = e^{-0.1t}(p\cos(t) + q\sin(t))$$

$$y_2(t) = e^{\alpha t}(q\cos(\beta t) - \mathrm{p}\sin(\beta t)) = e^{-0.1t}(-p\,sin(t) + q\cos(t))$$

$$y_3(t) = d_3 e^{\lambda_3 t} = d_3 e^t$$

$$X = TY = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -y_2 \\ y_1 + y_3 \\ y_1 \end{pmatrix}$$

$$x_1 = -y_2 = e^{-0.1t}(p\sin(t) - q\cos(t))$$

$$x_2 = y_1 + y_3 = e^{-0.1t}(p\cos(t) + q\sin(t)) + d_3 e^t$$

$$x_3 = y_1 = e^{-0.1t}(p\cos(t) + q\sin(t))$$

$p = c_1; q = -c_2$
see details below

Solution: Method (II)

$$\lambda_1 = -0.1 + i \qquad \lambda_2 = -0.1 - i \qquad \lambda_3 = 1$$

$$V_1 = \begin{pmatrix} -i \\ 1 \\ 1 \end{pmatrix} \qquad V_2 = \begin{pmatrix} i \\ 1 \\ 1 \end{pmatrix} \qquad V_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\boxed{X = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2 + +c_3 e^{\lambda_3 t} V_3}$$

$$X = d_1 e^{(-0.1+i)t} \begin{pmatrix} -i \\ 1 \\ 1 \end{pmatrix} + d_2 e^{(-0.1-i)t} \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} + d_3 e^{t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= d_1 e^{-0.1t}(\cos(t) + i\sin(t)) \begin{pmatrix} -i \\ 1 \\ 1 \end{pmatrix} + d_2 e^{-0.1t}(\cos(t) - i\sin(t)) \begin{pmatrix} i \\ 1 \\ 1 \end{pmatrix} + d_3 e^{t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Solution: Method (II)

$$= d_1 e^{-0.1t}(\cos(t) + isin(t))\begin{pmatrix} -i \\ 1 \\ 1 \end{pmatrix} + d_2 e^{-0.1t}(\cos(t) - isin(t))\begin{pmatrix} i \\ 1 \\ 1 \end{pmatrix} + d_3 e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= e^{-0.1t}\begin{pmatrix} d_1 \sin(t) + d_2 \sin(t) \\ d_1 \cos(t) + d_2 \cos(t) \\ \textit{the same} \end{pmatrix} + ie^{-0.1t}\begin{pmatrix} -d_1 \cos(t) + d_2 \cos(t) \\ d_1 \sin(t) - d_2 \sin(t) \\ \textit{the same} \end{pmatrix} + d_3 e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= e^{-0.1t}\begin{pmatrix} (d_1+d_2) \sin(t) \\ (d_1+d_2) \cos(t) \\ \textit{the same} \end{pmatrix} + ie^{-0.1t}\begin{pmatrix} -(d_1 - d_2) \cos(t) \\ (d_1-d_2) \sin(t) \\ \textit{the same} \end{pmatrix} + d_3 e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= (d_1+d_2)e^{-0.1t}\begin{pmatrix} \sin(t) \\ \cos(t) \\ \cos(t) \end{pmatrix} - i(d_1 - d_2)e^{-0.1t}\begin{pmatrix} \cos(t) \\ -\sin(t) \\ -\sin(t) \end{pmatrix} + d_3 e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= c_1 e^{-0.1t}\begin{pmatrix} \sin(t) \\ \cos(t) \\ \cos(t) \end{pmatrix} + c_2 e^{-0.1t}\begin{pmatrix} \cos(t) \\ -\sin(t) \\ -\sin(t) \end{pmatrix} + d_3 e^t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$c_1 = d_1 + d_2$$

$$c_2 = -i(d_1 - d_2)$$

Solution Pattern

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 e^{-0.1t} \begin{pmatrix} \sin(t) \\ \cos(t) \\ \cos(t) \end{pmatrix} + c_2 e^{-0.1t} \begin{pmatrix} \cos(t) \\ -\sin(t) \\ -\sin(t) \end{pmatrix} + d_3 e^{t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$x = e^{-0.1t}(c_1 \sin(t) + c_2 \cos(t))$$

$$y = e^{-0.1t}(c_1 \cos(t) - c_2 \sin(t)) + d_3 e^{t}$$

$$\boxed{p = c_1; q = -c_2}$$

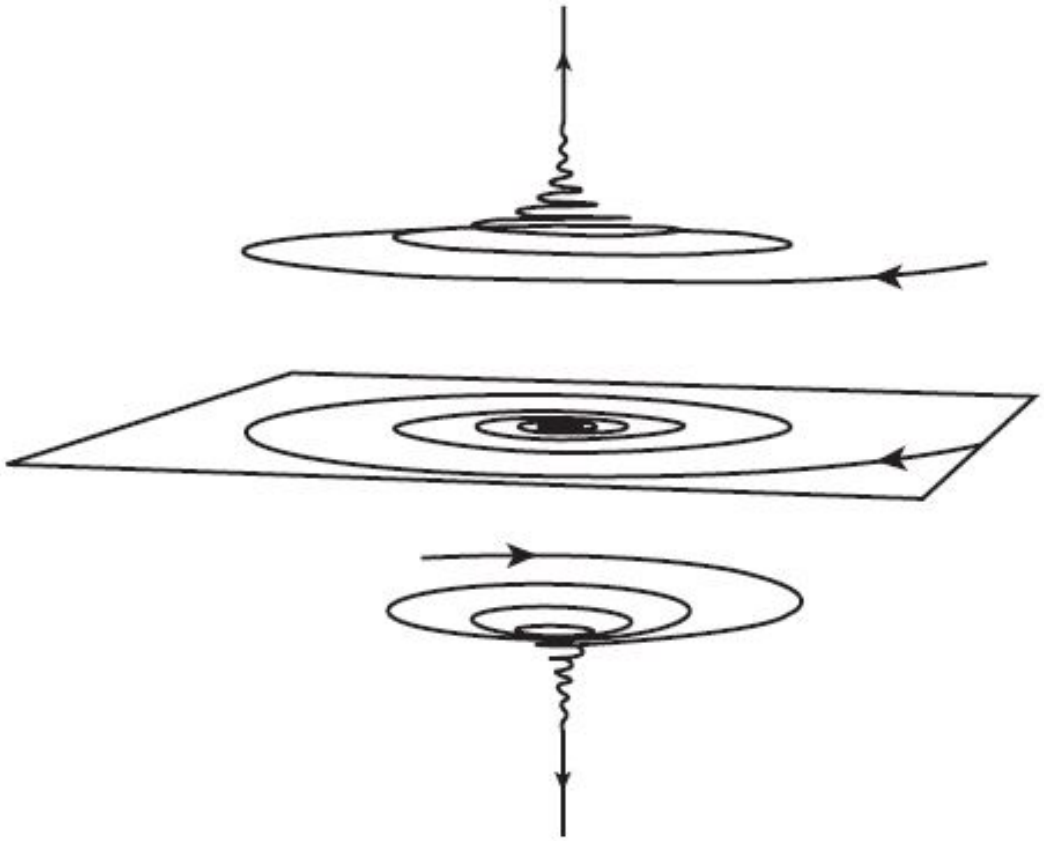
$$z = e^{-0.1t}(c_1 \cos(t) - c_2 \sin(t))$$

$$\boxed{x^2 + z^2 = e^{-0.2t}\left(c_4^2 + c_5^2\right)}$$

Section 6.1: Saddle Focus

$$Re(\lambda_{1,2}) < 0$$

$$\lambda_3 > 0$$



Saddle focus
(Ott, p333/334)

Figure 6.5 Typical solutions of the spiral saddle tend to spiral toward the unstable line.

Section 6.1: Spiral Sink and Source

$$\lambda_3 < 0$$

$$Re(\lambda_{1,2}) < 0$$

$$\lambda_3 > 0$$

$$Re(\lambda_{1,2}) > 0$$

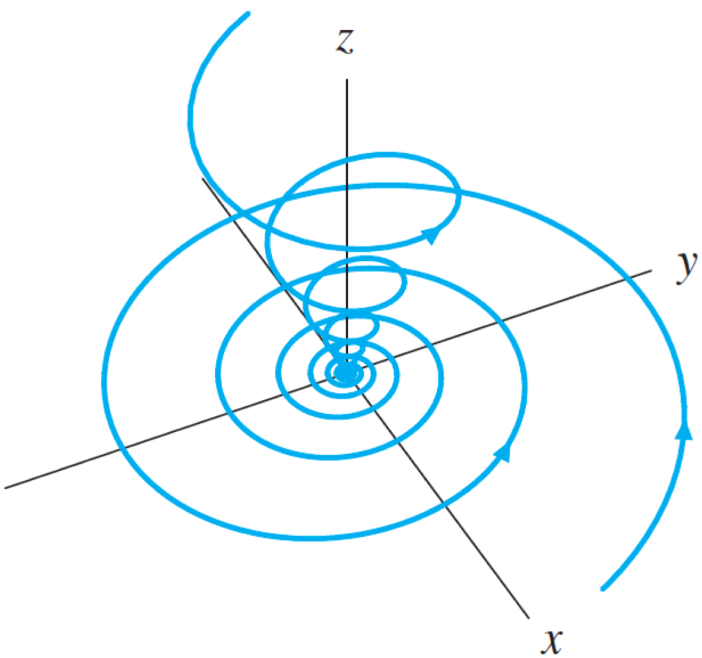


Figure 3.61
Example phase space for spiral sink.

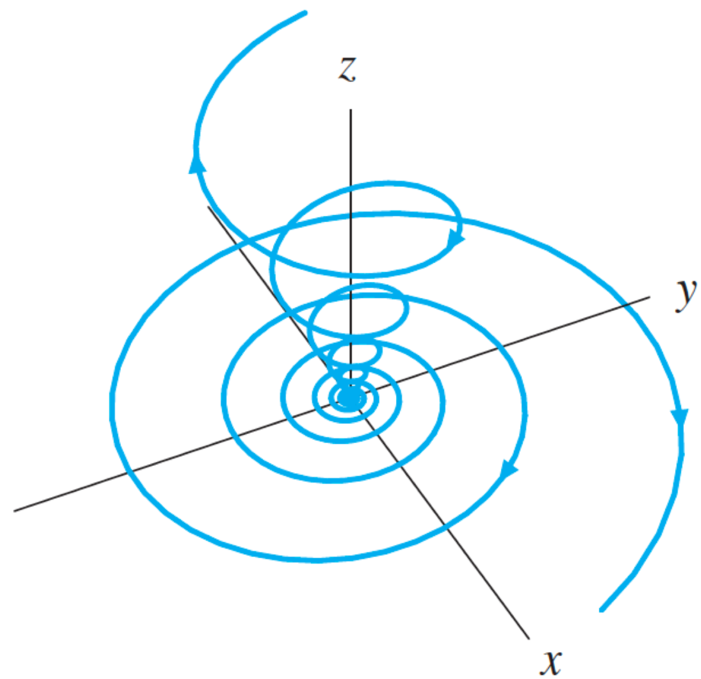


Figure 3.62
Example phase space for spiral source.

Section 6.1: Saddle

$\lambda_3 > 0$

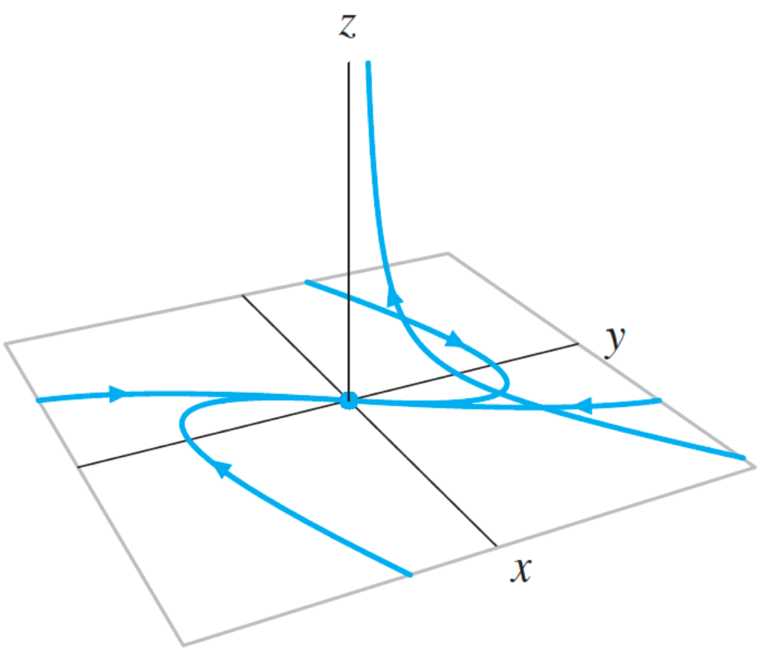
$\lambda_{1,2} < 0$

three real

$\lambda_3 > 0$

$Re(\lambda_{1,2}) < 0$

Saddle focus (Ott, p333/334)



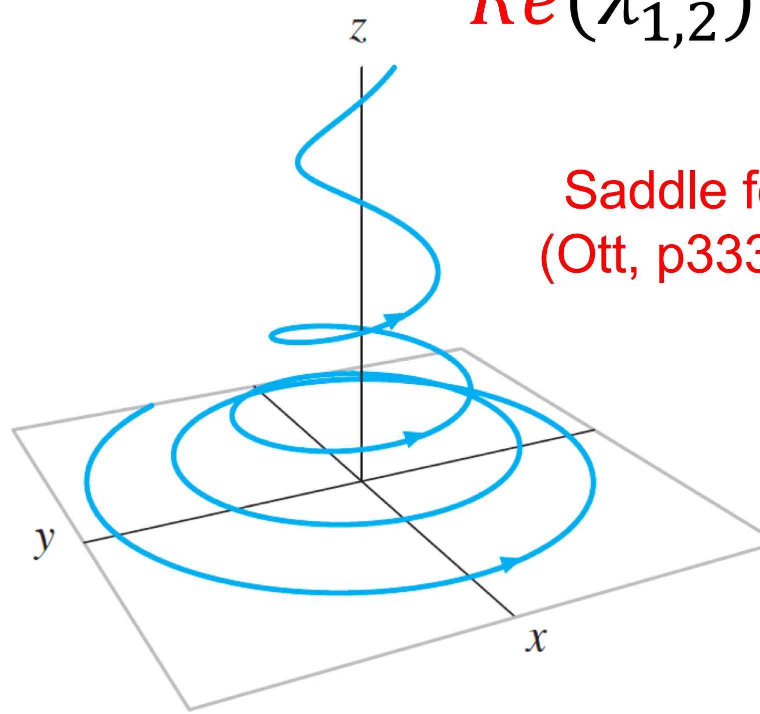


Figure 3.63
Example of a saddle with one positive and two negative eigenvalues.

Figure 3.64
Example of a saddle with one real eigenvalue and a complex conjugate pair of eigenvalues.

Complex Eigenvalues / Eigenvectors in Higher-D Systems

Theorem. *Consider the system $X' = AX$ where A has distinct eigenvalues $\lambda_1, \ldots, \lambda_{k_1} \in \mathbb{R}$ and $\alpha_1 + i\beta_1, \ldots, \alpha_{k_2} + i\beta_{k_2} \in \mathbb{C}$. Let T be the matrix that puts A in the canonical form*

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_{k_1} & & & \\ & & & B_1 & & \\ & & & & \ddots & \\ & & & & & B_{k_2} \end{pmatrix}$$

where

$$B_j = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix}.$$

Then the general solution of $X' = AX$ is $TY(t)$ where

$$Y(t) = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ \vdots \\ c_k e^{\lambda_{k_1} t} \\ a_1 e^{\alpha_1 t} \cos \beta_1 t + b_1 e^{\alpha_1 t} \sin \beta_1 t \\ -a_1 e^{\alpha_1 t} \sin \beta_1 t + b_1 e^{\alpha_1 t} \cos \beta_1 t \\ \vdots \\ a_{k_2} e^{\alpha_{k_2} t} \cos \beta_{k_2} t + b_{k_2} e^{\alpha_{k_2} t} \sin \beta_{k_2} t \\ -a_{k_2} e^{\alpha_{k_2} t} \sin \beta_{k_2} t + b_{k_2} e^{\alpha_{k_2} t} \cos \beta_{k_2} t \end{pmatrix}$$