10/1 2.4 Subsequences & Compact Sets.

Examples: (A) Not all sequences have convergent subsequences.

Eg. $a_n = n^2$.

(B) $\{\sin(n)\}_{n=1}^{\infty}$ ~ since this is bounded we will show today a convergent subsequence exists. We just can't construct it.

2.4 Subsequences & sequential compactness

Definition Suppose $\{a_n\} \subseteq \mathbb{R}$ is a sequence.

Let $n_1, n_2, \ldots$ be a strictly increasing sequence of natural numbers.

Then $b_k = a_{n_k}$ define terms of a subsequence of $\{a_n\}_{n=1}^{\infty}$ Usually shorthanded as $\{a_{n_k}\}_{k=1}^{\infty}$

Ex: $\{(-1)^n\}_{n=1}^{\infty} = \{a_n\}$.

$n_1 = 1, \ n_2 = 3, \ n_3 = 5, \ \ldots$

Then $\{a_{n_k}\} = \{(-1)^{2k-1}\}_{k=1}^{\infty}$

Prop: 2.30 Suppose $\{a_n\}$ is a convergent sequence s.t. $\lim_{n\to\infty} a_n = a$. Every subsequence also converges to a.

Proof: Let $\{a_{n_k}\}$ be an arbitrary subsequence.

Show: $\lim_{k\to\infty} a_{n_k} = a$.

$\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t. $\forall k \geq K$, $|a_{n_k} - a| < \varepsilon$.

Let $\varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|a_n - a| < \varepsilon$.

Since $\{n_k\}$ is strictly increasing, $\exists K$ s.t.

$n_K > N$. Let $k \geq K$. Then $n_k > n_K > N$.

Thus $|a_{n_k} - a| < \varepsilon$.

Definition (only useful for the next result!)

Suppose $\{a_n\}$ is a sequence. We say $m \in \mathbb{N}$ is a *peak index* for the sequence iff

$$\forall n \geq m, \quad a_m \geq a_n.$$

Ex $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$: every index $n \geq 1$ is a peak index.

$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\}$ no peak index

$\left\{(-1)^n\right\}_{n=1}^{\infty}$ even n are peak indices

Thm 2.32 Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Then there exists a monotone subsequence $\{a_{n_k}\}_{k=1}^{\infty}$

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proof: There are finitely many or infinitely many peak indices.

case 1: Suppose there are infinitely many.

Let n_1 be the first. Generally n_k be the kth peak index.

Fix $k \in \mathbb{N}^+$.

Notice that n_k is peak, so $\forall n \geq n_k$, $a_{n_k} \geq a_n$.

Since $n_{k+1} > n_k$, $a_{n_{k+1}} \leq a_{n_k}$.

I.e. $\{a_{n_k}\}_{k=1}^{\infty}$ is decreasing.

Case 2: Suppose finitely many peak indices.

Suppose $N \in \mathbb{N}$ is greater than all peak indices.

Let $n_1 = N+1$.

So n_1 is not peak.

So $\exists n_2 > n_1$ s.t. $a_{n_1} < a_{n_2}$.

Letting $k \geq 2$, define $n_{k+1} > n_k$ where $a_{n_k} < a_{n_{k+1}}$.

By construction, $\{a_{n_k}\}_{k=1}^{\infty}$ is increasing.

$\boxtimes$

Thm 2.33 Suppose $\{a_n\}$ is bounded.

Then $\{a_n\}$ has a convergent subsequence.

proof: By Thm 2.32, there exists a monotone $\{a_{n_k}\}_{k=1}^{\infty}$. Since $\{a_n\}$ is bounded $\exists M \in \mathbb{R}$ s.t. $\forall n$, $|a_n| \leq M$. Thus $\forall k$, $|a_{n_k}| \leq M$ and $\{a_{n_k}\}_{k=1}^{\infty}$ is bounded. By the Monotone Convergence Thm, $\lim_{k \to \infty} a_{n_k}$ exists.

Def: Suppose $S \subseteq \mathbb{R}$ and $S \neq \emptyset$.

We say S is <u>sequentially</u> <u>compact</u>

iff

$\forall \{a_n\} \subseteq S, \exists \{a_{n_k}\}$ s.t. $\lim_{k\to\infty} a_{n_k} \in S$.

S is <u>not</u> <u>sequentially</u> <u>compact</u>

$\exists \{a_n\} \subseteq S, \forall \{a_{n_k}\}$ $\lim_{k\to\infty} a_{n_k}$ DNE OR $\lim_{k\to\infty} a_{n_k} \notin S$.

Ex! The set $[5, \infty)$ is <u>not</u> ~~compact~~ <u>sequentially compact</u>

$a_n = 5 + n^2$.

All subsequences are increasing & unbounded, thus do not converge.

Ex 2 Let $S = (0, 1]$.

Let $a_n = \frac{1}{n}$, $n \geq 1$.

Then $\{a_n\}_{n=1}^{\infty} \subseteq S$ and $\lim_{n \to \infty} a_n = 0 \notin S$.

By Prop 2.30, letting $\{a_{n_k}\}$ be a subsequence,

$$\lim_{k \to \infty} a_{n_k} = 0 \notin S.$$

Thm: Bolzano-Weirstrass Let $a < b$. Then $[a,b]$ is sequentially compact.

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proof: Let $\{a_n\} \subseteq [a,b]$.

Since $\{a_n\}$ is bounded, $\exists \{a_{n_k}\}$ s.t. $\lim_{k \to \infty} a_{n_k}$ exists. Since $\{a_{n_k}\} \subseteq [a,b]$ which is closed, $\lim_{k \to \infty} a_{n_k} \in [a,b]$. (Thm 2.22). ☒

Section 2.5 Optional Extension on Compactness & Covering sets.

Definitions Suppose $S \subseteq \mathbb{R}$. Let I_n be open intervals s.t. $S \subseteq \bigcup_{n=1}^{\infty} I_n$.

We say $\{I_n\}_{n=1}^{\infty}$ is an <u>open cover</u> of S.

We say S is <u>compact</u> iff every open cover has a "finite subcover" i.e. $\exists N$ s.t.

$$S \subseteq \bigcup_{n=1}^{N} I_n.$$

Thm 2.LAST Hiene-Borel. Let $S \subseteq \mathbb{R}$.

T.F.A.E.

(1) S compact

(2) S sequentially compact

(3) S closed & bounded.

Test 1 Extra Questions

① Let $\{a_n\}$, $\{b_n\}$ be convergent sequences.

If $\forall n$, $a_n \le b_n$, then $\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$.

② Suppose $\forall n$, $a_n \le c_n \le b_n$. Let $L \in \mathbb{R}$.

If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$, then $\lim_{n\to\infty} c_n = L$.

③ Suppose $\{a_n\} \subseteq \mathbb{R}$. We say $\{a_n\}$ <u>goes to infinity</u> iff

$$\forall M \in \mathbb{R}^+, \exists N, \forall n \ge N, \; a_n > M.$$

<u>Prove:</u> $\{a_n\}$ goes to infinity iff $\lim_{n\to\infty} \frac{1}{a_n} = 0$.

④ Suppose $\{a_n\}$ is increasing and unbounded. Prove $\{a_n\}$ does not converge.