Test 1 Material?

- definition of <u>density</u>: what is it?

Suppose $S \subseteq \mathbb{R}$. We say S is <u>dense</u> in $\mathbb{R}$

iff

$\forall a, b \in \mathbb{R}$ with $a < b$, $\exists x \in S$ s.t.

$$a < x < b.$$

– for all open intervals (no matter how small or where) we can find an element of S in the interval.

iff

$\forall x \in \mathbb{R}$, $\exists \{a_n\} \subseteq S$ s.t. $\lim_{n\to\infty} a_n = x$.

– every number in $\mathbb{R}$ can be approximated by number in S.

Test I Material Update

Cut off material at the end of 2.4 (text).

Definitions: subsequence, sequentially compact

$S \subseteq \mathbb{R}$ is sequentially compact iff $\forall \{a_n\} \subseteq S$, $\exists \{a_{n_k}\}$ subsequence such that $\lim_{k\to\infty} a_{n_k} \in S$.

2.4 **Results:**

Prop 2.30
If $\{a_n\}$ converges, then every subsequence converges to that same limit

Thm 2.33 All bounded sequences have convergent subsequences.

Thm 2.34 All intervals $[a,b]$ are sequentially compact for $a<b$ in $\mathbb{R}$.

Sequence of number topics...

1. 9.1 Cauchy sequences.
2. limsup & liminf <u>or</u> $\overline{\lim}$ $\underline{\lim}$

<u>Definition</u> Suppose $\{a_n\} \subseteq \mathbb{R}$. We say the sequence is <u>Cauchy</u> iff

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n,m \geq N \text{ we have } |a_n - a_m| < \varepsilon.$$

- this <u>does</u> <u>not</u> prove convergence yet.
- you can investigate / test without a candidate limit value.

Thm 9.2 All convergent sequences are Cauchy.

$$\forall \{a_n\} \subseteq \mathbb{R} \text{ convergent}, \ \forall \varepsilon > 0, \ \exists N \in \mathbb{N} \text{ s.t. } \forall n, m \geq N, \ |a_n - a_m| < \varepsilon.$$

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proof: Let $\{a_n\} \subseteq \mathbb{R}$ be convergent.

Let $\varepsilon > 0$.

Let $a = \lim_{n \to \infty} a_n$. Then $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$

$$|a_n - a| < \frac{\varepsilon}{2}.$$

Let $m, n \geq N$.

$$|a_m - a_n| = |a_m - a + a - a_n| \leq |a_m - a| + |a - a_n|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \square$$

Prop 9.3 All Cauchy sequences are bounded.

$\forall \{a_n\} \subseteq \mathbb{R}$, if $\{a_n\}$ is Cauchy, then $\exists M \in \mathbb{R}^+$ s.t.

$\forall n$, $|a_n| \leq M$.

proof: Let $\{a_n\} \subseteq \mathbb{R}$ and suppose it is Cauchy.

So $\exists N$ s.t. $\forall m, n \geq N$, $|a_m - a_n| < 1$.

So $\forall n \geq N$ we have $|a_n - a_N| < 1$.

By reverse Δ-inequality $|a_n| - |a_N| < 1$ whenever $n \geq N$.

Thus $\forall n \geq N$, $|a_n| < 1 + |a_N|$.

Let $M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a_N| + 1\}$,

and this M works by construction.

$\blacksquare$

Prop 9.4 Suppose $\{a_n\} \subseteq \mathbb{R}$.

The sequence converges iff the sequence is Cauchy.

proof: ($\rightarrow$) Done by 9.2.

($\leftarrow$) Suppose the sequence is Cauchy.

By 9.3, $\{a_n\}$ is bounded.

By 2.32, $\{a_n\}$ has a monotone subsequence $\{a_{n_k}\}$.

Now $\{a_{n_k}\}$ is bounded & monotone, thus

$\lim_{k\to\infty} a_{n_k} = a$ exists.

Let $\varepsilon > 0$. $\exists N$ st. $\forall n, m \geq N$

$$|a_n - a_m| < \varepsilon/2.$$

Also $\exists K$ st. $\forall k \geq K$

$$|a_{n_k} - a| < \varepsilon/2.$$

Let $n \geq N$. Choose k st. $n_k > N$ and $k > K$.

$$|a_n - a| = |a_n - a_{n_k} + a_{n_k} - a| \leq |a_n - a_{n_k}| + |a_{n_k} - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\blacksquare$