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# **MATH 537, Fall 2020**

# **Ordinary Differential Equations**

Lecture #22

Chapter 6  
Quasi-Periodic Motions

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## Section 6.2 Harmonic Oscillators (uncoupled)

s1

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Consider a pair of undamped harmonic oscillators whose equations are

$$x_1'' = -\omega_1^2 x_1$$

$$x_2'' = -\omega_2^2 x_2.$$

$$Y(t) = \begin{pmatrix} x_1(t) \\ y_1(t) \\ x_2(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} a_1 \cos \omega_1 t + b_1 \sin \omega_1 t \\ -a_1 \sin \omega_1 t + b_1 \cos \omega_1 t \\ a_2 \cos \omega_2 t + b_2 \sin \omega_2 t \\ -a_2 \sin \omega_2 t + b_2 \cos \omega_2 t \end{pmatrix}$$

# Polar Coordinates

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$$x'_j = \omega_j y_j$$

$$y'_j = -\omega_j x_j.$$

$$r'_j = 0$$

$$\theta'_j = -\omega_j.$$

$$\theta'_1 = -\omega_1$$

$$\theta'_2 = -\omega_2.$$

It is convenient to think of  $\theta_1$  and  $\theta_2$  as variables in a square of sidelength  $2\pi$  where we glue together the opposite sides  $\theta_j = 0$  and  $\theta_j = 2\pi$  to make the torus. In this square our vector field now has constant slope

$$\frac{\theta'_2}{\theta'_1} = \frac{\omega_2}{\omega_1}.$$

Therefore solutions lie along straight lines with slope  $\omega_2/\omega_1$  in this square. When a solution reaches the edge  $\theta_1 = 2\pi$  (say, at  $\theta_2 = c$ ), it instantly reappears on the edge  $\theta_1 = 0$  with the  $\theta_2$  coordinate given by  $c$ , and then continues onward with slope  $\omega_2/\omega_1$ . A similar identification occurs when the solution meets  $\theta_2 = 2\pi$ .

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# (Uncoupled) Oscillators

$$x_1'' = -\omega_1^2 x_1 \rightarrow \text{periodic}$$

$$x_2'' = -\omega_2^2 x_2 \rightarrow \text{periodic}$$

$$\begin{pmatrix} x_1' \\ y_1' \\ x_2' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_2^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix}$$

$$x_1' = y_1$$

$$y_1' = -\omega_1^2 x_1$$

$$x_2' = y_2$$

$$y_2' = -\omega_2^2 x_2$$

uncoupled

$$A - \lambda I = \begin{pmatrix} -\lambda & 1 & 0 & 0 \\ -\omega_1^2 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & -\omega_2^2 & -\lambda \end{pmatrix}$$

$$|A - \lambda I| = 0 \quad -\lambda \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -\omega_2^2 & -\lambda \end{vmatrix} - \begin{vmatrix} -\omega_1^2 & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -\omega_2^2 & -\lambda \end{vmatrix} = 0$$

$\lambda_{1,2} = \pm i \omega_1$

$\lambda_{3,4} = \pm i \omega_2$

# Changing Coordinates: Construct the Linear Map

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$$\lambda_{1,2} = \pm i \omega_1$$

$$V_1^T = (1, i\omega_1, 0, 0)$$

$$W_1^T = Re(V_1^T) = (1, 0, 0, 0)$$

$$\lambda_{3,4} = \pm i \omega_2$$

$$V_2^T = (0, 0, 1, i\omega_2)$$

$$W_3^T = Re(V_2^T) = (0, 0, 1, 0)$$

$$(W_1 \quad W_2 \quad W_3 \quad W_4)$$

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega_2 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_2^2 & 0 \end{pmatrix}$$

$$T^{-1}AT = \begin{pmatrix} 0 & \omega_1 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 \\ 0 & 0 & -\omega_2 & 0 \end{pmatrix}$$

# Changing Coordinates: Matlab Code

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- `syms w1 w2`
- `T=[1 0 0 0; 0 w1 0 0; 0 0 1 0; 0 0 0 w2]`
- `A=[0 1 0 0; -w1^2 0 0 0; 0 0 0 1; 0 0 -w2^2 0]`
- `inv(T)*A*T`

`ans =`

```
[ 0, w1, 0, 0]
[ -w1, 0, 0, 0]
[ 0, 0, 0, w2]
[ 0, 0, -w2, 0]
```

# Solutions & $\frac{dr_j}{dt} = 0$

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2<sup>nd</sup>-order ODE

$$x_1'' = -\omega_1^2 x_1 \rightarrow \text{periodic} \quad x(t) = a_1 \cos(\omega_1 t) + b_1 \sin(\omega_1 t)$$

2D system

$$x_1' = \omega_1 y_1 \quad x_1(t) = a_1 \cos(\omega_1 t) + b_1 \sin(\omega_1 t)$$

$$y_1' = -\omega_1 x_1 \quad y_1(t) = -a_1 \sin(\omega_1 t) + b_1 \cos(\omega_1 t)$$

$$x_1^2 = a_1^2 \cos^2(\omega_1 t) + 2a_1 b_1 \cos(\omega_1 t) \sin(\omega_1 t) + b_1^2 \sin^2(\omega_1 t)$$

$$y_1^2 = a_1^2 \sin^2(\omega_1 t) - 2a_1 b_1 \cos(\omega_1 t) \sin(\omega_1 t) + b_1^2 \cos^2(\omega_1 t)$$

$$x_1^2 + y_1^2 = a_1^2 (\cos^2(\omega_1 t) + \sin^2(\omega_1 t)) + b_1^2 (\cos^2(\omega_1 t) + \sin^2(\omega_1 t)) = a_1^2 + b_1^2$$

$$x_1^2 + y_1^2 = a_1^2 + b_1^2 = r_1^2 \quad \frac{dr_1}{dt} = 0$$

Similarly,

$$\frac{dr_j}{dt} = 0$$

$$x_2^2 + y_2^2 = a_2^2 + b_2^2 = r_2^2 \quad \frac{dr_2}{dt} = 0$$

# Solutions & $\frac{d\theta_j}{dt} = -\omega_j$

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$$\tan(\theta_j) = \frac{y_j}{x_j}$$

$$\frac{d}{dt} \tan(\theta_j) = \frac{d}{dt} \left( \frac{y_j}{x_j} \right) \quad \frac{d\theta_j}{dt} \sec^2(\theta_j) = \frac{x_j y'_j - y_j x'_j}{x_j^2}$$

(next slide)

$$\frac{x_j y'_j - y_j x'_j}{x_j^2} = \frac{-r_j^2 \omega_j}{(r_j \cos(\theta_j))^2} = \frac{-\omega_j}{(\cos(\theta_j))^2} = -\omega_j \sec^2(\theta_j)$$

$$\frac{d\theta_j}{dt} \sec^2(\theta_j) = -\omega_j \sec^2(\theta_j)$$

$$\frac{d\theta_j}{dt} = -\omega_j$$

$$\theta_j = \theta_{j0} - \omega_j t$$

$$2\pi = |-\omega_j| T$$

$$T = \frac{2\pi}{\omega_j}, \quad period$$

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Show  $x_j y'_j - y_j x'_j = -r_j^2 \omega_j$

$$x_j = a_j \cos(\omega_j t) + b_j \sin(\omega_j t) \quad (a) \quad x'_j = -a_j \omega_j \sin(\omega_j t) + b_j \omega_j \cos(\omega_j t) \quad (c)$$

$$y_j = -a_j \sin(\omega_j t) + b_j \cos(\omega_j t) \quad (b) \quad y'_j = -a_j \omega_j \cos(\omega_j t) - b_j \omega_j \sin(\omega_j t) \quad (d)$$

$$y_j x'_j = (b)(c) = a_j^2 \omega_j \sin^2(\omega_j t) + b_j^2 \omega_j \cos^2(\omega_j t) - 2a_j b_j \omega_j \sin(\omega_j t) \cos(\omega_j t)$$

$$x_j y'_j = (a)(d) = -a_j^2 \omega_j \cos^2(\omega_j t) - b_j^2 \omega_j \sin^2(\omega_j t) - 2a_j b_j \omega_j \sin(\omega_j t) \cos(\omega_j t)$$

$$x_j y'_j - y_j x'_j = (ad) - (bc) = (-a_j^2 \omega_j - b_j^2 \omega_j) = -r_j^2 \omega_j$$

# Periodic Composite Motion

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$$x_1'' = -\omega_1^2 x_1 \rightarrow \text{periodic}$$

$$x_1(t) = a_1 \cos(\omega_1 t) + b_1 \sin(\omega_1 t)$$

$$x_2'' = -\omega_2^2 x_2 \rightarrow \text{periodic}$$

$$x_2(t) = a_2 \cos(\omega_2 t) + b_2 \sin(\omega_2 t)$$

Q: under which condition a composite motion with the two frequencies is periodic?

$$\text{Let } Y^T = (x_1, y_1, x_2, y_2)$$

- $Y$  is a periodic function with a period  $\tau$  if and only if there exists integers  $m$  and  $n$  such that

$$\omega_1 \tau = m2\pi$$

$$\omega_2 \tau = n2\pi$$

$$\tau = \frac{m2\pi}{\omega_1}$$

$$\tau = \frac{n2\pi}{\omega_2}$$

$$\frac{m2\pi}{\omega_1} = \frac{n2\pi}{\omega_2}$$

$$\frac{\omega_2}{\omega_1} = \frac{n}{m}$$

The ratio must be a rational number

$$x_1(t) = \sin(2 * 2\pi t)$$

$$x_2(t) = \sin(5 * 2\pi t)$$

# Periodicity vs. Quasi-periodicity

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$$\frac{dr_j}{dt} = 0$$

$$\frac{d\theta_j}{dt} = -\omega_j \quad \rightarrow$$

$$\begin{aligned}\theta'_1 &= -\omega_1 \\ \theta'_2 &= -\omega_2\end{aligned}$$

$$\begin{aligned}\theta_1 &= \theta_1(t=0) - \omega_1 t \\ \theta_2 &= \theta_2(t=0) - \omega_2 t\end{aligned}$$

- **Periodic:**  $\frac{\omega_2}{\omega_1}$  is a rational number, e.g.,  $\frac{\omega_2}{\omega_1} = \frac{5}{2}$

Periodic,  $\frac{\omega_2}{\omega_1} = \frac{5}{2}$

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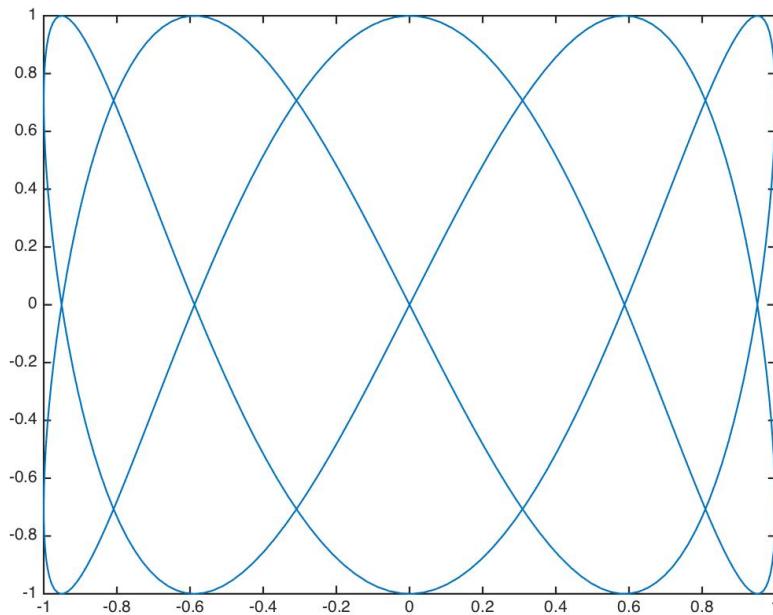
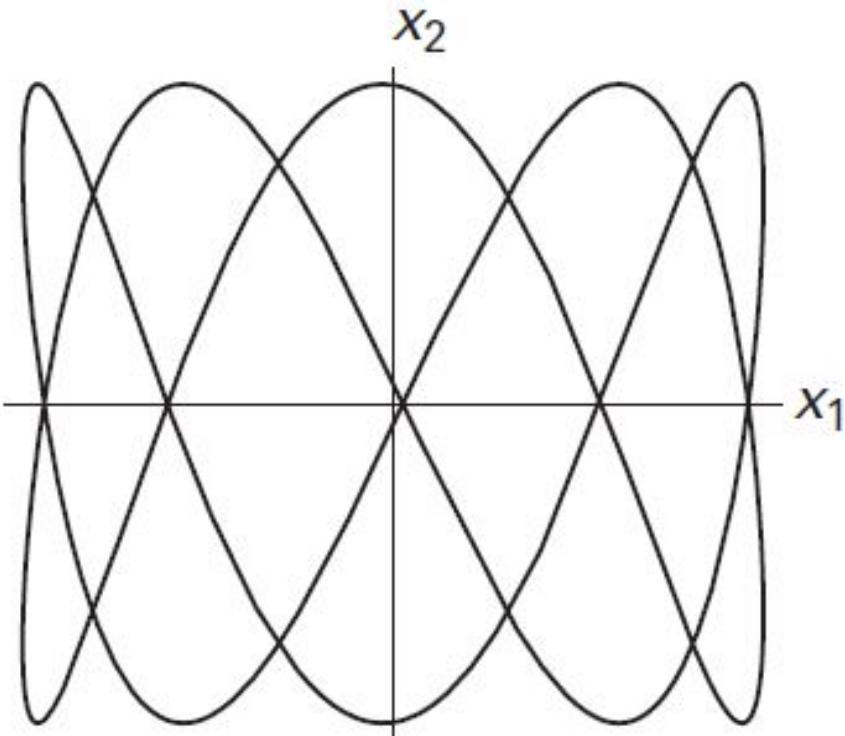


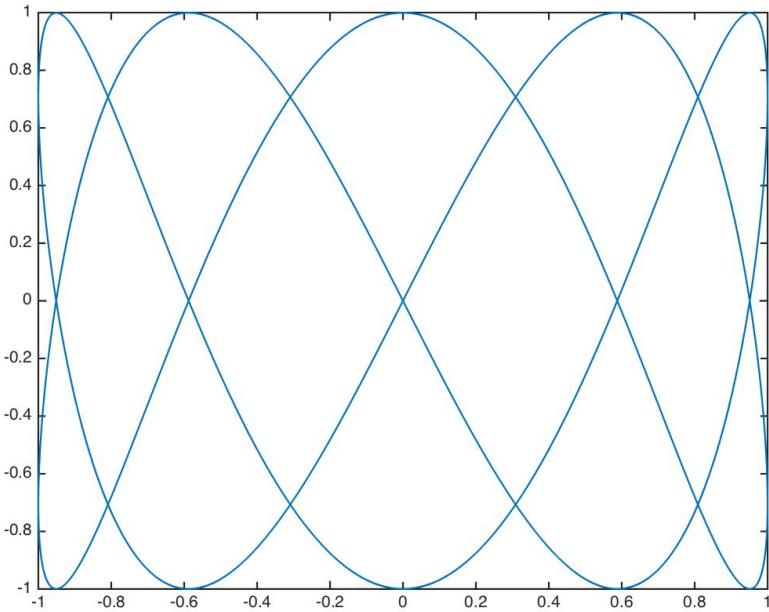
Figure 6.6 A solution with frequency ratio  $5/2$  projected into the  $x_1 x_2$ -plane. Note that  $x_2(t)$  oscillates five times and  $x_1(t)$  only twice before returning to the initial position.

```
time=linspace(0,1, 400) %t=0~1  
x1=sin(2*2*pi*time)  
x2=sin(5*2*pi*time)  
%fig=figure()  
plot(x1, x2)  
saveas(gcf, 'Figure6_6_numerical', 'jpg' )
```

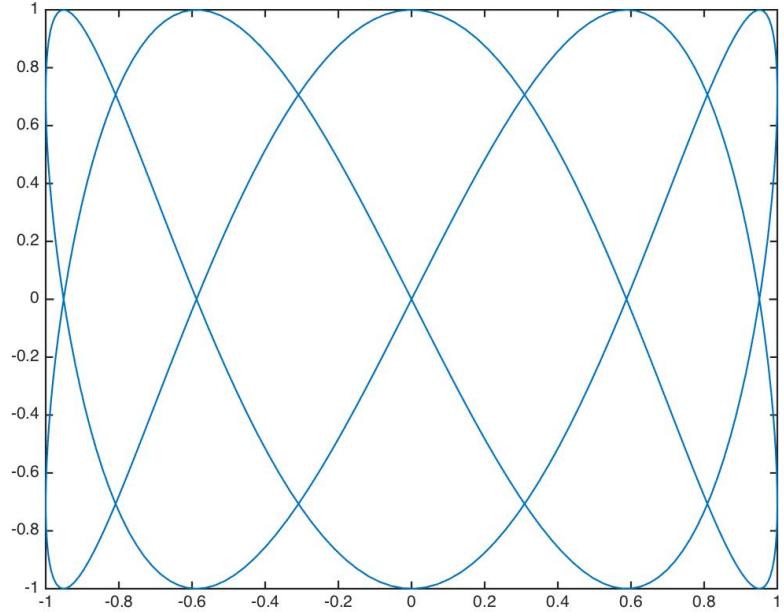
## Section 6.2: Periodic, $\frac{\omega_2}{\omega_1} = \frac{5}{2}$

S3

T=0~4



T=0~1



```
time=linspace(0,4, 1600) %t=0~4  
x1=sin(2*2*pi*time)  
x2=sin(5*2*pi*time)  
%fig=figure()  
plot(x1, x2)  
saveas(gcf, 'Figure6_6_numerical', 'jpg' )
```

```
time=linspace(0,1, 400) %t=0~1  
x1=sin(2*2*pi*time)  
x2=sin(5*2*pi*time)  
%fig=figure()  
plot(x1, x2)  
saveas(gcf, 'Figure6_6_numerical', 'jpg' )
```

By our identification, this solution has now returned to the circle  $\theta_1 = 0$ . The solution may cross the boundary  $\theta_2 = 2\pi$  several times in making this transit, but it does eventually return to  $\theta_1 = 0$ . So we may define the Poincaré map on  $\theta_1 = 0$  by assigning to  $x_0$  on this circle the corresponding coordinate of the point of first return. Suppose that this first return occurs at the point  $\theta_2(\tau)$  where  $\tau$  is the time for which  $\theta_1(\tau) = 2\pi$ . Since  $\theta_1(t) = \theta_1(0) - \omega_1 t$ , we have  $\tau = 2\pi/\omega_1$ . Thus  $\theta_2(\tau) = x_0 + 2\pi(\omega_2/\omega_1)$ .

Therefore, the Poincaré map on the circle may be written as

$$f(x_0) = x_0 + 2\pi(\omega_2/\omega_1) \bmod 2\pi,$$

where  $x_0 = \theta_2(0)$  is our initial  $\theta_2$  coordinate on the circle. See [Figure 6.8](#). Thus the Poincaré map on the circle is just the function that rotates points on the circle by angle  $2\pi(\omega_2/\omega_1)$ . Since  $\omega_2/\omega_1$  is irrational, this function is called an *irrational rotation* of the circle.

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### Definition

The set of points  $x_0, x_1 = f(x_0), x_2 = f(f(x_0)), \dots, x_n = f(x_{n-1})$  is called the *orbit* of  $x_0$  under iteration of  $f$ .

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The orbit of  $x_0$  tracks how our solution successively crosses  $\theta_1 = 2\pi$  as time increases.

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## Definition

The set of points  $x_0, x_1 = f(x_0), x_2 = f(f(x_0)), \dots, x_n = f(x_{n-1})$  is called the *orbit* of  $x_0$  under iteration of  $f$ .

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**Proposition.** Suppose  $\omega_2/\omega_1$  is irrational. Then the orbit of any initial point  $x_0$  on the circle  $\theta_1 = 0$  is dense in the circle.

# Periodicity vs. Quasi-periodicity

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$$\frac{dr_j}{dt} = 0$$

$$\frac{d\theta_j}{dt} = -\omega_j \quad \rightarrow$$

$$\begin{aligned}\theta'_1 &= -\omega_1 \\ \theta'_2 &= -\omega_2\end{aligned}$$

$$\begin{aligned}\theta_1 &= \theta_1(t=0) - \omega_1 t \\ \theta_2 &= \theta_2(t=0) - \omega_2 t\end{aligned}$$

- **Periodic:**  $\frac{\omega_2}{\omega_1}$  is a rational number, e.g.,  $\frac{\omega_2}{\omega_1} = \frac{5}{2}$
- **Quasi-periodic:**  $\frac{\omega_2}{\omega_1}$  is not a rational number, e.g.,  $\frac{\omega_2}{\omega_1} = \sqrt{2}$

# “Dense” Quasi-periodic Solutions

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$w_1/w_2$   
is irrational



incommensurate  
frequencies

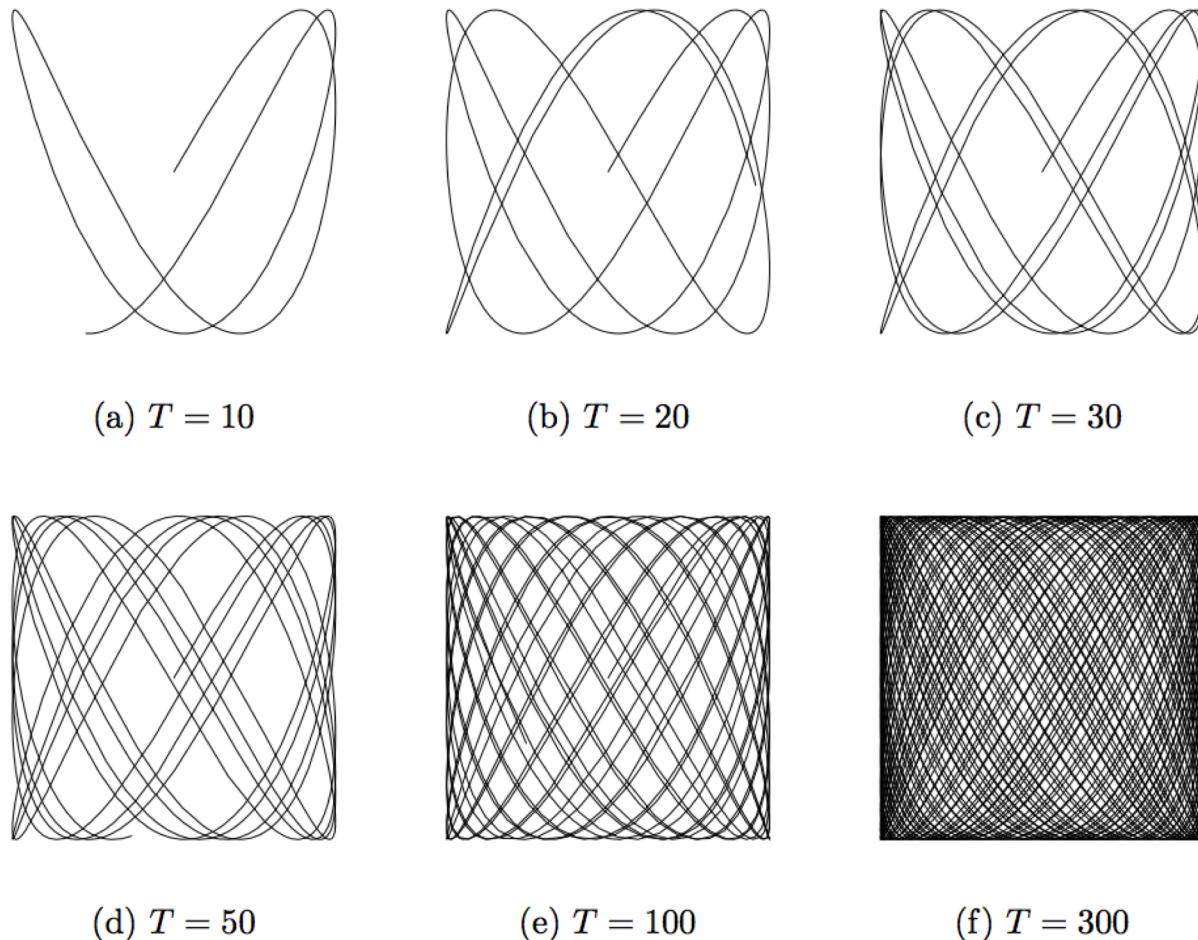
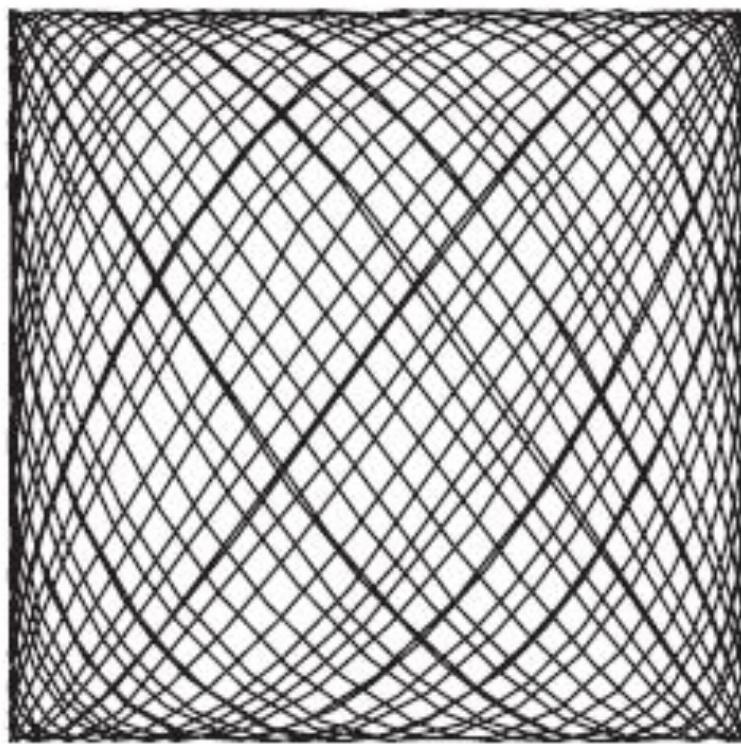


Figure 1.2: Trajectory of the mechanical system at different times.

$$\text{Quasiperiodic: } \frac{\omega_2}{\omega_1} = \sqrt{2}$$

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$T=0 \sim 50\pi$



$T=0 \sim 100\pi$

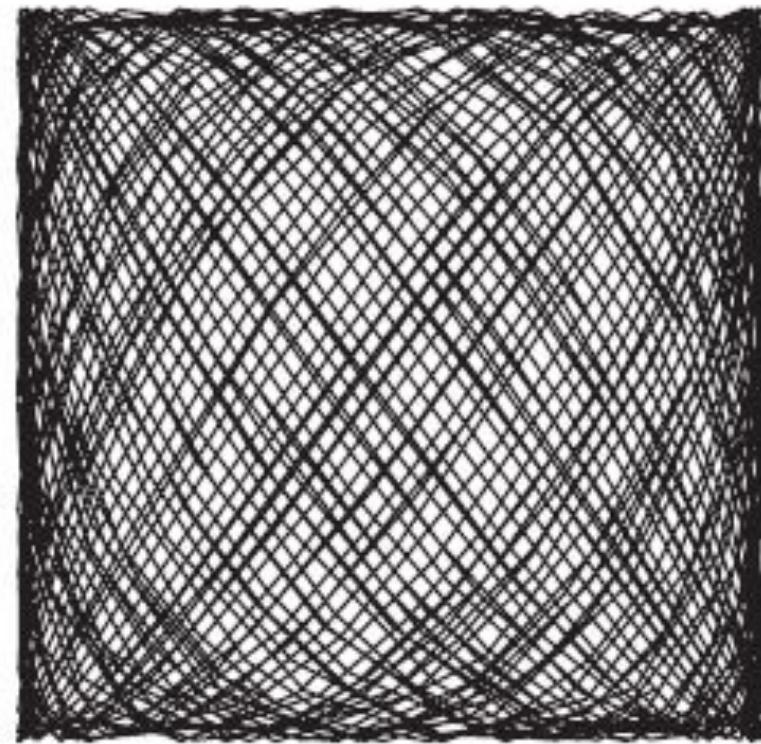


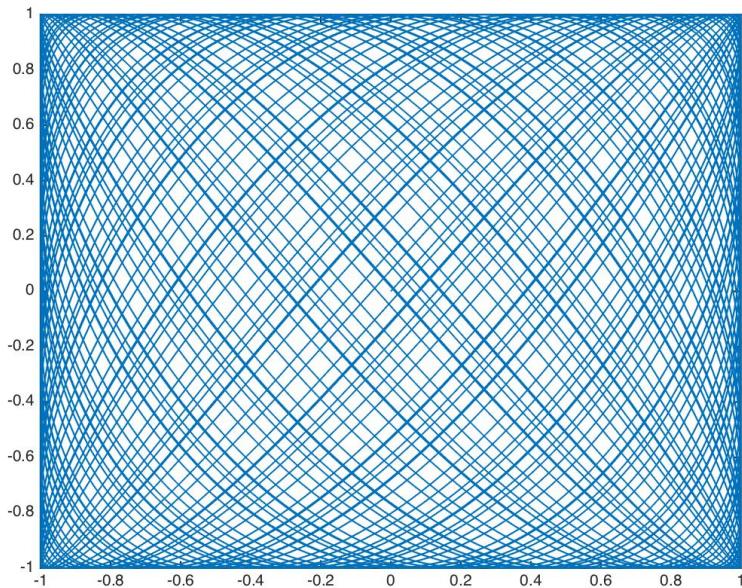
Figure 6.7 A solution with frequency ratio  $\sqrt{2}$  projected into the  $x_1 x_2$ -plane, the left curve computed up to time  $50\pi$ , the right to time  $100\pi$ .

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$$\text{Quasiperiodic: } \frac{\omega_2}{\omega_1} = \sqrt{2}$$

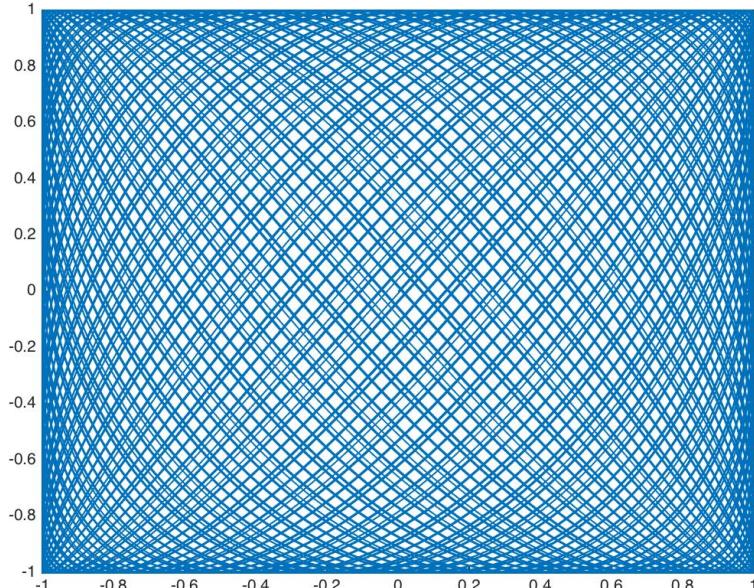
S4

T=0~50



```
time=linspace(0,50, 50000)
x1=sin(2*pi*time)
x2=sin(sqrt(2)*2*pi*time)
%fig=figure()
plot(x1, x2)
```

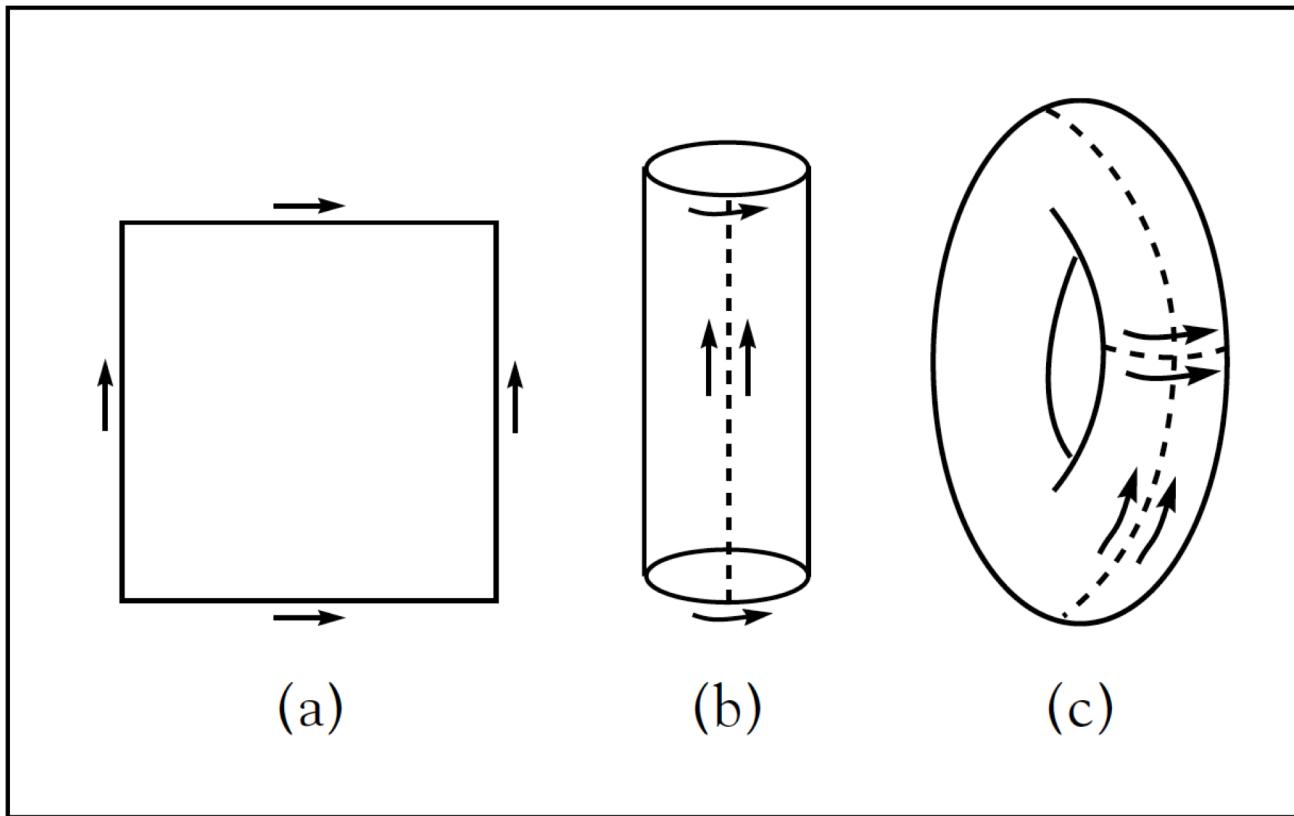
T=0~100



```
time=linspace(0,100, 100000)
x1=sin(2*pi*time)
x2=sin(sqrt(2)*2*pi*time)
%fig=figure()
plot(x1, x2)
```

# Construct a Torus

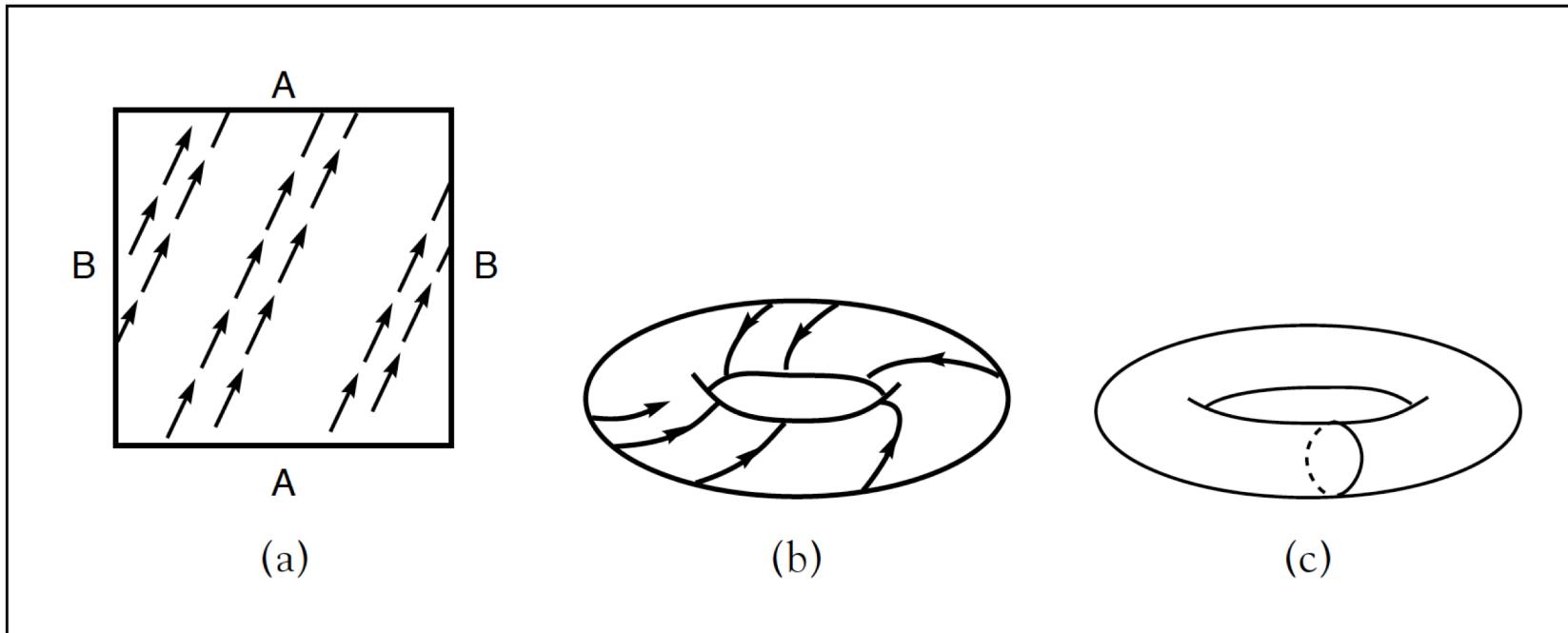
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**Figure 2.28 Construction of a torus in two easy steps.**

(a) Begin with unit square. (b) Identify (glue together) vertical sides. (c) Identify horizontal sides.

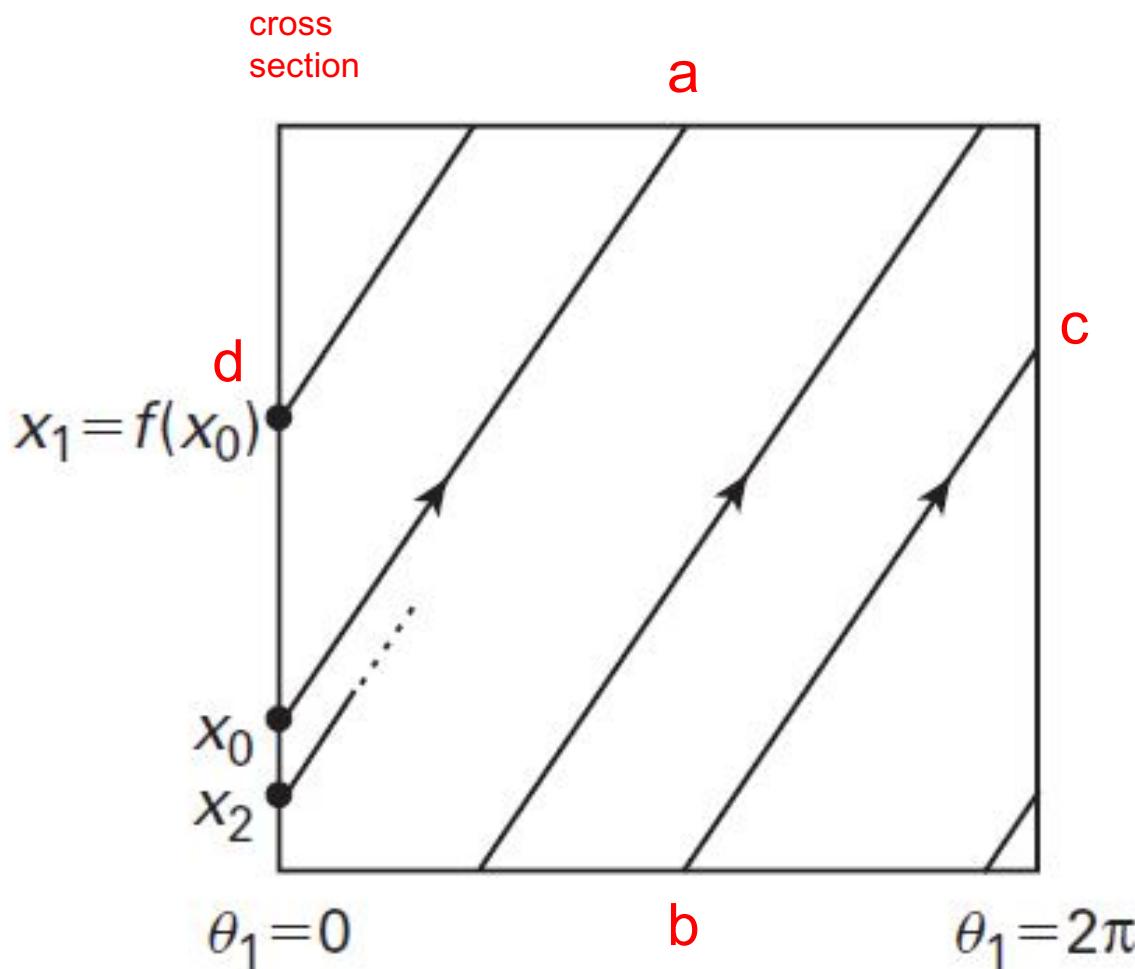
# Construct a Torus



**Figure 8.9 A dense orbit on the torus.**

- (a) The vector field has the same irrational slope at each point in the unit square.
- (b) The square is made into a torus by gluing together the top and bottom (marked A) to form a cylinder and then gluing together the ends (marked B). Each orbit winds densely around the torus. For each point  $\mathbf{u}$  of the torus,  $\mathbf{u}$  belongs to  $\omega(\mathbf{u})$  and  $\alpha(\mathbf{u})$ .
- (c) There is no analogue of the Poincaré-Bendixson Theorem on the torus because there is no Jordan Curve Theorem on the torus. A simple closed curve that does not divide the torus into two parts is shown.

## Quasiperiodic: dense



"a" and "b" are the same

"c" and "d" are the same

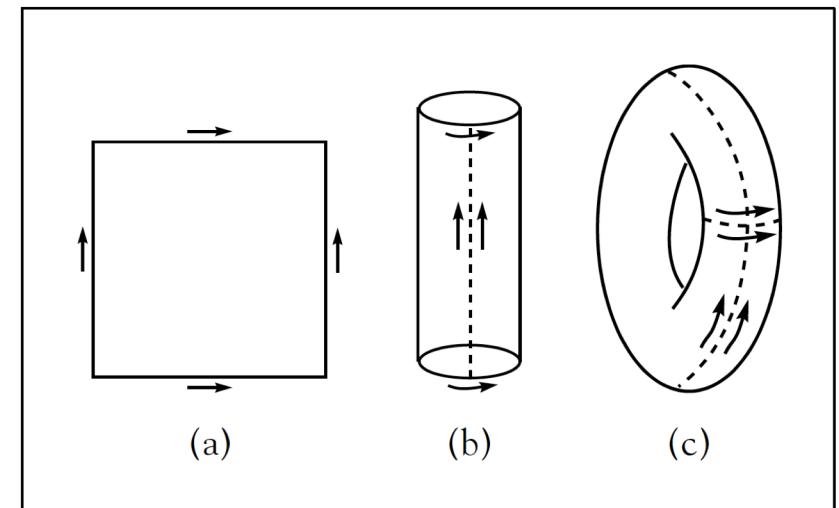
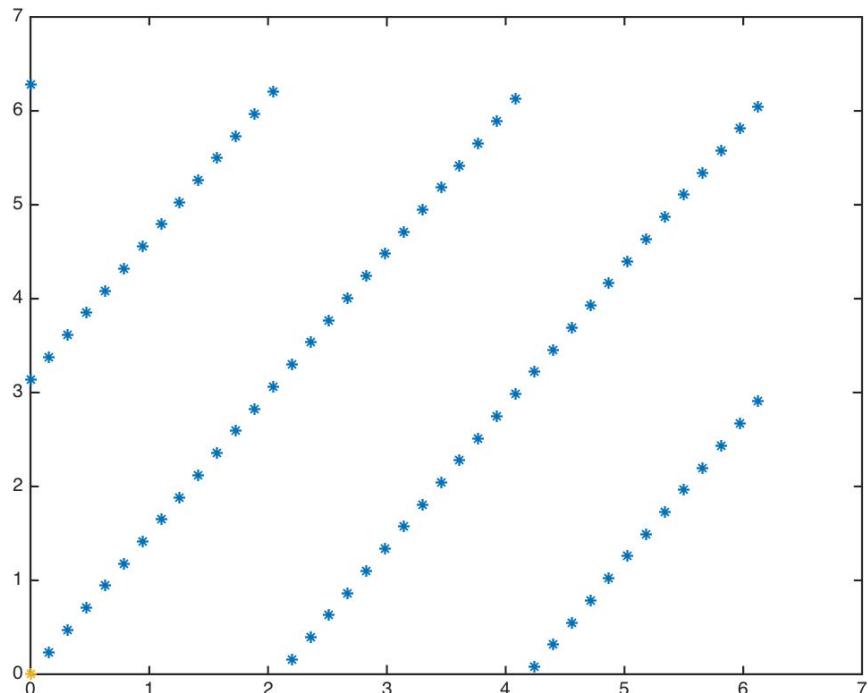


Figure 6.8 The Poincaré map on the circle  $\theta_1 = 0$  in the  $\theta_1 \theta_2$  torus.

$$\text{Periodic: } \frac{\omega_2}{\omega_1} = \frac{3}{2}$$


---

$T=0 \sim 12\pi$



NumPTs=241

theta1=zeros(NumPTs)

theta2=zeros(NumPTs)

Ending\_time=12\*pi

dt=Ending\_time/(NumPTs-1)

tau=0.0

omega1=1.

omega2=1.5

i=1

while i <= NumPTs-1

tau=tau+dt

theta1(i+1)=theta1(i) + omega1\*dt

theta2(i+1)=theta2(i) + omega2\*dt

if (theta1(i+1) >= 2\*pi)

theta1(i+1)=theta1(i+1)-2\*pi

end

if (theta2(i+1) >= 2\*pi)

theta2(i+1)=theta2(i+1)-2\*pi

end

i=i+1

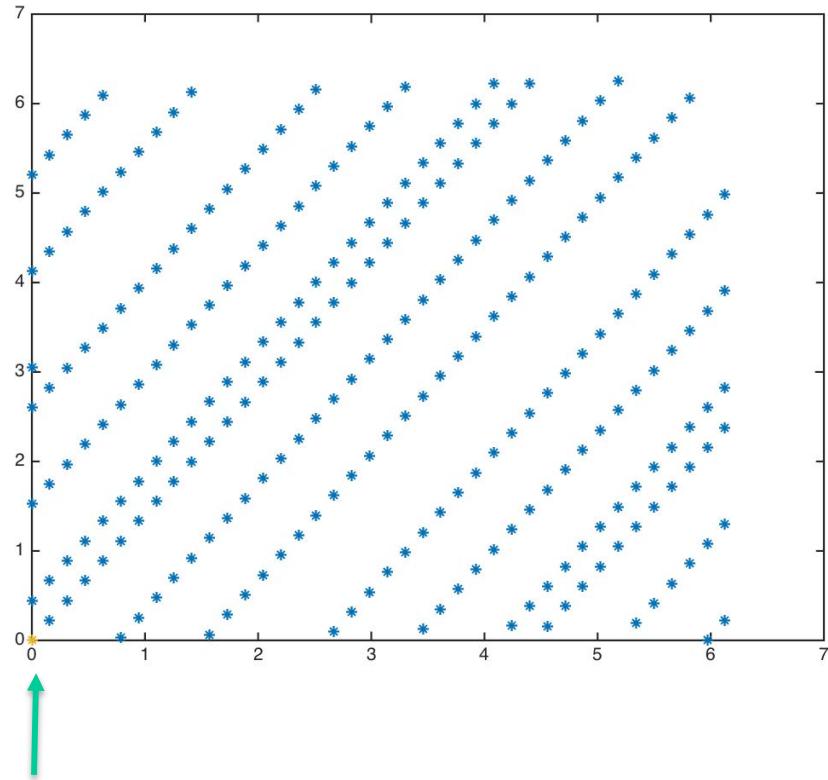
end

plot(theta1, theta2, '\*')

$$\text{Quasiperiodic: } \frac{\omega_2}{\omega_1} = \sqrt{2}$$

S7

$T=0 \sim 12\pi$



NumPTs=241

theta1=zeros(NumPTs)

theta2=zeros(NumPTs)

Ending\_time=12\*pi

dt=Ending\_time/(NumPTs-1)

tau=0.0

omega1=1

omega2=sqrt(2)

i=1

while i <= NumPTs-1

tau=tau+dt

theta1(i+1)=theta1(i) + omega1\*dt

theta2(i+1)=theta2(i) + omega2\*dt

if (theta1(i+1) >= 2\*pi)

theta1(i+1)=theta1(i+1)-2\*pi

end

if (theta2(i+1) >= 2\*pi)

theta2(i+1)=theta2(i+1)-2\*pi

end

i=i+1

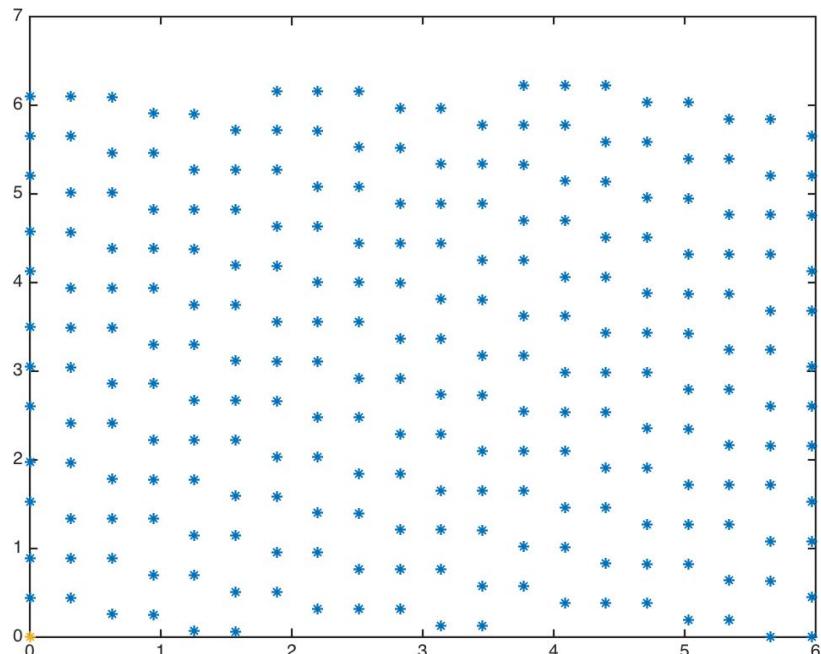
end

plot(theta1, theta2, '\*')

# Quasiperiodic: $\frac{\omega_2}{\omega_1} = \sqrt{2}$ , longer time

S8

$T=0 \sim 24\pi$



NumPTs=241

theta1=zeros(NumPTs)

theta2=zeros(NumPTs)

Ending\_time=24\*pi

dt=Ending\_time/(NumPTs-1)

tau=0.0

omega1=1

omega2=sqrt(2)

i=1

while i <= NumPTs-1

tau=tau+dt

theta1(i+1)=theta1(i) + omega1\*dt

theta2(i+1)=theta2(i) + omega2\*dt

if (theta1(i+1) >= 2\*pi)

theta1(i+1)=theta1(i+1)-2\*pi

end

if (theta2(i+1) >= 2\*pi)

theta2(i+1)=theta2(i+1)-2\*pi

end

i=i+1

end

plot(theta1, theta2, '\*')

## Section 6.2: Quasi-Periodic

Supp

Consider two frequencies,  $\omega$  and  $\Omega$ .

- $T_1, 2\pi/\omega$ , is the time required to transit the torus the short way;
- $T_2, 2\pi/\Omega$ , is the time required to transit the torus the long way.

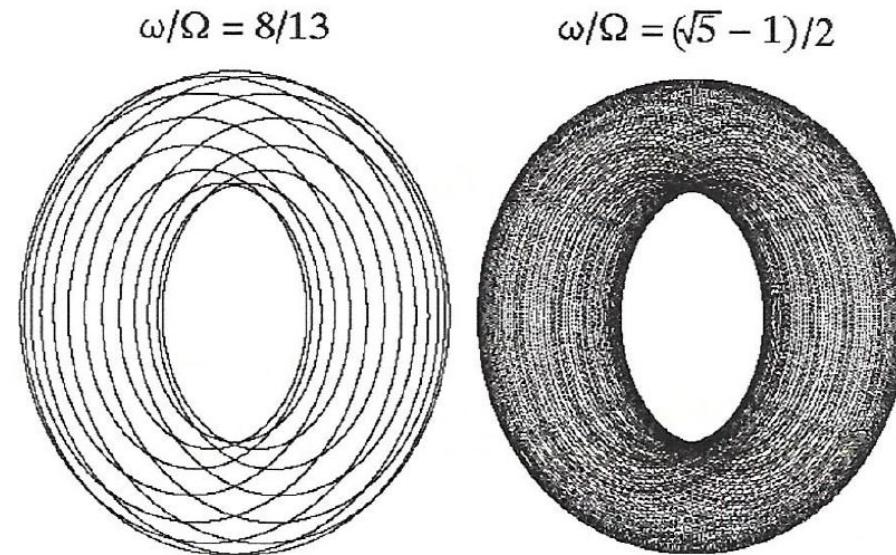


Fig. 3.5 Invariant torus for commensurate and incommensurate frequencies.

- The ratio  $\omega/\Omega$ , called **winding number** (or rotation number), is the number of times the trajectory transits the short way around for each time the long way.
- If the ratio is **rational**, (the frequencies are commensurate), the motion is **periodic** with a period equal to the smallest common multiple of the two periods. This condition is variously called **frequency locking**, phase locking, mode locking, frequency pulling etc.
- If the ratio is **irrational**, (the frequencies are incommensurate), the trajectory fills the whole toroidal surface without ever intersecting itself and the motion is **quasiperiodic**.

**There is no common multiple of the frequencies**, and the period is infinite.

## Section 6.2: Harmonic Oscillators

Supp

The second possibility is that the slope is *irrational* (Figure 8.6.6). Then the flow is said to be *quasiperiodic*. Every trajectory winds around endlessly on the torus, never intersecting itself and yet never quite closing.

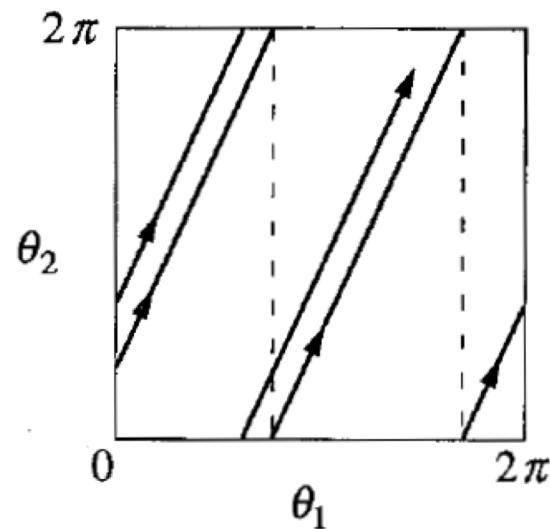


Figure 8.6.6

How can we be sure the trajectories never close? Any closed trajectory necessarily makes an integer number of revolutions in both  $\theta_1$  and  $\theta_2$ ; hence the slope would have to be rational, contrary to assumption.

Furthermore, when the slope is irrational, each trajectory is *dense* on the torus: in other words, each trajectory comes arbitrarily close to any given point on the torus. This is *not* to say that the trajectory passes through each point; it just comes arbitrarily close (Exercise 8.6.3).

Quasiperiodicity is significant because it is a new type of long-term behavior. Unlike the earlier entries (fixed point, closed orbit, homoclinic and heteroclinic orbits and cycles), quasiperiodicity occurs only on the torus.

# Quasi-Periodicity

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- In the quasi-periodic case the motion, strictly speaking, **never exactly repeats itself** (hence, the modifier quasi), but the motion is not chaotic;
- it is composed of two (or more) periodic components, whose presence could be made known by measuring the frequency spectrum (Fourier power spectrum) of the motion.
- We should point out that detecting the difference between quasi-periodic motion and motion with a rational ratio of frequencies, when the integers are large, is a delicate question.
- Whether a given experiment can distinguish the two cases **depends on the resolution** of the experimental equipment. As we shall see later, the behavior of the system can switch abruptly back and forth between the two cases as **a parameter of the system is varied**.
- The important point is that the attractor for the system is a two-dimensional surface of the torus for quasi-periodic behavior.

Hilborn (2000, p135)

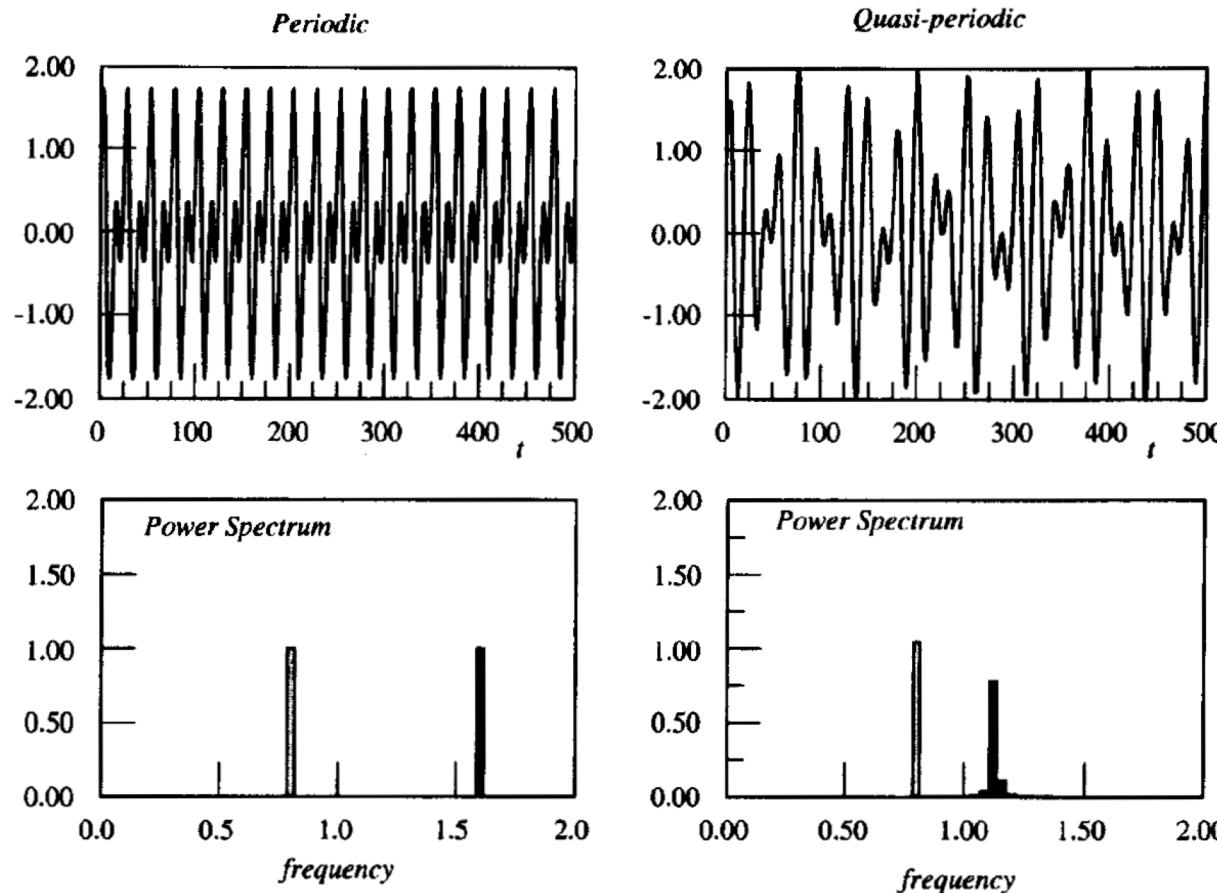
# Quasi-Periodicity

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- If the ratio is irrational, then we say that the system's behavior is quasi-periodic (The terms **conditionally periodic, almost periodic** are sometimes used in place of quasi-periodic.)
- We must ask, therefore, how do we know in practice **whether two frequencies  $f_1$  and  $f_2$  are commensurate or incommensurate?** The problem is that any actual measurement of the frequencies has some finite **precision**.
- Similarly, any numerical calculation, say on a computer, has only finite arithmetical precision: any number used by the computer is effectively a rational number.
- All we can say is that to within the precision of our measurements or **within the precision of our numerical calculations, a given frequency ratio is equal to a particular irrational number** or a particular rational number, which is close to that irrational number.
- Beyond that we cannot say whether the ratio is "really" rational or irrational.

Hilborn (2000, p211)

# Spectra of Periodic and Quasi-Periodic Motions



The second frequency is incommensurate with the first

**Fig. 6.1.** On the left is the time evolution of a system with two frequencies. Here  $f_1 = 2 f_2$ . On the right is the time evolution when the two frequencies are incommensurate  $f_2 = \sqrt{2} f_1$ . The behavior on the right looks quite irregular, but the power spectrum shown in the lower part of the figure indicates that only two frequencies (with different amplitudes) are contributing to the behavior. The crucial point is that in the case on the right has two incommensurate frequencies. The widths of the power spectrum “peaks” are due to the relatively short time interval of data used in the analysis.

Hilborn (2000, p212)

## Review: Harmonic Oscillators (uncoupled)

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Consider a pair of undamped harmonic oscillators whose equations are

$$x_1'' = -\omega_1^2 x_1$$

$$x_2'' = -\omega_2^2 x_2.$$

$$Y(t) = \begin{pmatrix} x_1(t) \\ y_1(t) \\ x_2(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} a_1 \cos \omega_1 t + b_1 \sin \omega_1 t \\ -a_1 \sin \omega_1 t + b_1 \cos \omega_1 t \\ a_2 \cos \omega_2 t + b_2 \sin \omega_2 t \\ -a_2 \sin \omega_2 t + b_2 \cos \omega_2 t \end{pmatrix}$$

# A Coupled Oscillator: Example 1

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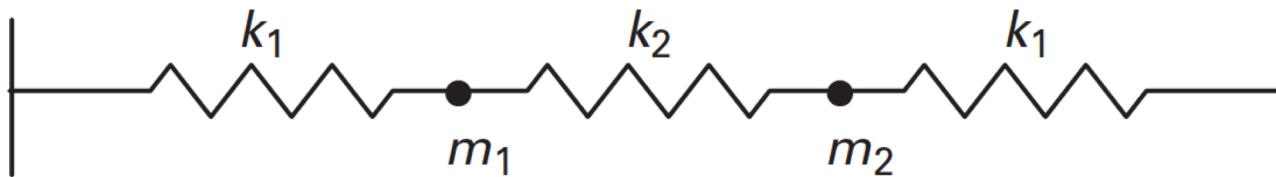


Figure 6.10 A coupled oscillator.

and  $m_2$  attached to springs and walls as shown in Figure 6.10. The springs connecting  $m_j$  to the walls both have spring constants  $k_1$ , while the spring connecting  $m_1$  and  $m_2$  has spring constant  $k_2$ . This coupling means that the motion of either mass affects the behavior of the other.

Let  $x_j$  denote the displacement of each mass from its rest position, and assume that both masses are equal to 1. The differential equations for these coupled oscillators are then given by

$$\begin{aligned}x_1'' &= -(k_1 + k_2)x_1 + k_2x_2 \\x_2'' &= k_2x_1 - (k_1 + k_2)x_2.\end{aligned}$$

## A Coupled Oscillator (.continued)

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These equations are derived as follows. If  $m_1$  is moved to the right ( $x_1 > 0$ ), the left spring is stretched and exerts a restorative force on  $m_1$  given by  $-k_1 x_1$ . Meanwhile, the central spring is compressed, so it exerts a restorative force on  $m_1$  given by  $-k_2 x_1$ . If the right spring is stretched, then the central spring is compressed and exerts a restorative force on  $m_1$  given by  $k_2 x_2$  (since  $x_2 < 0$ ). The forces on  $m_2$  are similar.

- (a) Write these equations as a first-order linear system.
- (b) Determine the eigenvalues and eigenvectors of the corresponding matrix.
- (c) Find the general solution.
- (d) Let  $\omega_1 = \sqrt{k_1}$  and  $\omega_2 = \sqrt{k_1 + 2k_2}$ . What can be said about the periodicity of solutions relative to the  $\omega_j$ ? Prove this.



# A Coupled Oscillator: Example 2

Kreyszig

$$\frac{d^2y_1}{dt^2} = -k_1 y_1 + k_2(y_2 - y_1),$$

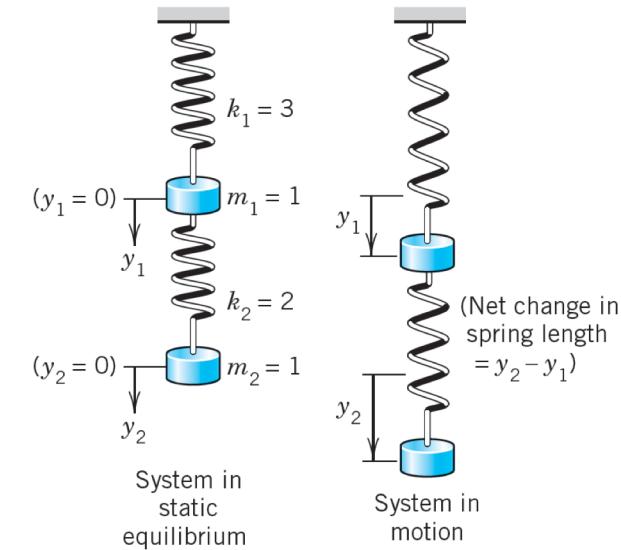
$$\frac{d^2y_2}{dt^2} = -k_2(y_2 - y_1).$$

$$k_1 = 3; k_2 = 2$$

## Vibrating System of Two Masses on Two Springs (Fig. 161)

Mass-spring systems involving several masses and springs can be treated as eigenvalue problems. For instance, the mechanical system in Fig. 161 is governed by the system of ODEs

$$(6) \quad \begin{aligned} y_1'' &= -3y_1 - 2(y_1 - y_2) = -5y_1 + 2y_2 \\ y_2'' &= -2(y_2 - y_1) = 2y_1 - 2y_2 \end{aligned}$$



**Fig. 161.** Masses on springs in Example 4

$$= \mathbf{A}\mathbf{y} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

where  $y_1$  and  $y_2$  are the displacements of the masses from rest, as shown in the figure, and primes denote derivatives with respect to time  $t$ .

# Solve for the Solutions

$$\begin{aligned}y_1'' &= -3y_1 - 2(y_1 - y_2) = -5y_1 + 2y_2 \\y_2'' &= \quad\quad\quad -2(y_2 - y_1) = \quad 2y_1 - 2y_2\end{aligned}$$

$$= \mathbf{A}\mathbf{y} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

$$Y' = AY; \quad Y = \begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix} e^{\omega t};$$

$$\omega^2 \begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix} e^{\omega t} = A \begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix} e^{\omega t}; \quad (A - \omega^2 I) \begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix} = 0;$$

$\begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  is a solution. To obtain non-trivial solutions, we require

$|A - \omega^2 I| = 0$ , which is the same as the following:

$|A - \lambda I| = 0$  if we introduce  $\lambda = \omega^2$  for convenience.

The above indicates that (1) Let  $A$  represent a 2x2 matrix for a system of “two” second-order ODEs (for two oscillators in this case; no first derivatives in the above system); (2) the eigenvalues of the matrix  $A$  represent  $\omega^2$ . Here,  $\omega$  may represent the frequency of the oscillators.



## Section 6.2: Harmonic Oscillators(sup)

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### EXAMPLE 5.5

Let

$$f(r, \theta) = (r^2, \theta + q),$$

where  $r$  and  $\theta$  are polar coordinates, and  $q$  is an angle irrationally related to  $2\pi$ . The orbits of all points inside the unit circle converge to the origin, which is a fixed point. The unit circle is invariant under  $f$ , as points on it rotate through the angle  $q$ . Outside the unit circle, orbits are unbounded. The quasiperiodic orbits at  $r = 1$  have Lyapunov exponents  $\ln 2$  and  $0$ . Although they are not asymptotically periodic and they do exhibit sensitive dependence (due to a positive Lyapunov exponent in the  $r$  direction), we do not want the definition of “chaotic orbit” to apply to these orbits since their motion is quite predictable.

