Quiz 3
Ordinary Differential Equations
Math 537
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Goal: Based on Eq. (1.3) of the Mid-Term Part A, we will derive a second order ODE with its solution as a hyperbolic secant squared function (sech^2). The second-order ODE is mathematically identical to the Korteweg-de Vries (KdV) equation in a traveling-wave coordinate. The solution is known as a solitary wave or logistic distribution. It also represents the solution (I) of the simplified SIR model under the condition of "weak outbreak".

Problem 1: In the Mid-term Part A, we have completed the following:

(*) Consider the following logistic equation:

$$\frac{df}{dt} = f(1-f). \tag{MT - 1.2}$$

Introduce a new dependent variable (g) to transform Eq. (MT-1.2) into the following ODE:

$$\frac{dg}{dt} = \frac{1}{4} - g^2. \tag{MT - 1.3}$$

(*) Express the solutions of Eqs. (MT-1.2) and (MT-1.3) in terms of the sigmoid and hyperbolic tangent functions, respectively

Here, by defining $Z = dg/dt$, please derive the following ODE from Eq. (MT-1.3)

$$\frac{d^2Z}{dt^2} - Z + 6Z^2 = 0, \tag{1.1}$$

which can be written using a new time variable (τ) as follows:

$$\frac{d^2Z}{d\tau^2} - \frac{Z}{2} + 3Z^2 = 0. \tag{1.2}$$

Eq. (1.2) is mathematically identical to the KdV equation in the travelingwave coordinate (Shen 2020, IJBC, in press).

Notice the following, when we let $Z = dg/dt$:

$$\begin{aligned}
Z &= \frac{dg}{dt} = \frac{1}{4} - g^2 \\
\frac{dZ}{dt} &= \frac{d^2g}{dt^2} = -2g\frac{dg}{dt} = -2g\left(\frac{1}{4} - g^2\right) = -\frac{g}{2} + 2g^3 \\
\frac{d^2Z}{dt^2} &= \frac{d^3g}{dt^3} = \left(-\frac{1}{2} + 6g^2\right)\frac{dg}{dt} = \left(-\frac{1}{2} + 6g^2\right)\left(\frac{1}{4} - g^2\right) = -\frac{1}{8} + 2g^2 - 6g^4
\end{aligned}$$

Now Notice the following:

$$\begin{aligned}
\frac{d^2Z}{dt^2} - Z + 6Z^2 &= \left(-\frac{1}{8} + 2g^2 - 6g^4\right) - \left(\frac{1}{4} - g^2\right) + 6\left(\frac{1}{4} - g^2\right)^2 \\
&= \left(-\frac{2}{16} + 2g^2 - 6g^4\right) - \left(\frac{4}{16} - g^2\right) + 6\left(\frac{1}{16} - \frac{1}{2}g^2 + g^4\right)^2 \\
&= -\frac{2}{16} + 2g^2 - 6g^4 - \frac{4}{16} + g^2 + \frac{6}{16} - 3g^2 + 6g^4 \\
&= \left(-\frac{2}{16} - \frac{4}{16} + \frac{6}{16}\right) + \left(2g^2 + g^2 - 3g^2\right) + \left(-6g^4 + 6g^4\right) \\
&= 0
\end{aligned}$$

Notice the following with a new time variable τ, with $\frac{dg}{d\tau} = \frac{dg}{dt}\frac{dt}{d\tau}$:

$$\begin{aligned}
Z &= \frac{dg}{dt} = \frac{1}{4} - g^2 \\
\frac{dZ}{d\tau} &= \frac{d^2g}{d\tau\, dt} = -2g\frac{dg}{d\tau} = -2g\frac{dg}{dt}\frac{dt}{d\tau} = -2g\left(\frac{1}{4} - g^2\right)\frac{dt}{d\tau} = \left(-\frac{g}{2} + 2g^3\right)\frac{dt}{d\tau} \\
\frac{d^2Z}{d\tau^2} &= \frac{d^3g}{d\tau^2\, dt} = \left(-\frac{1}{2} + 6g^2\right)\frac{dg}{d\tau}\frac{dt}{d\tau} = \left(-\frac{1}{2} + 6g^2\right)\frac{dg}{dt}\frac{dt}{d\tau}\frac{dt}{d\tau} = \left(-\frac{1}{2} + 6g^2\right)\left(\frac{1}{4} - g^2\right)\left(\frac{dt}{d\tau}\right)^2 \\
&= \left(-\frac{1}{8} + 2g^2 - 6g^4\right)\left(\frac{dt}{d\tau}\right)^2
\end{aligned}$$

Notice that we have the following:

$$\begin{aligned}
\frac{d^2Z}{dt^2} &= Z - 6Z^2 \\
\frac{d^2Z}{d\tau^2} &= \frac{Z}{2} - 3Z^2 \\
&= \frac{1}{2}\left(Z - 6Z^2\right) \\
&= \frac{1}{2}\frac{d^2Z}{dt^2}
\end{aligned}$$

$$\begin{aligned}
\left(-\frac{1}{8} + 2g^2 - 6g^4\right)\left(\frac{dt}{d\tau}\right)^2 &= \frac{1}{2}\left(-\frac{1}{8} + 2g^2 - 6g^4\right) \\
\left(\frac{dt}{d\tau}\right)^2 &= \frac{1}{2} \\
\frac{dt}{d\tau} &= \frac{\pm 1}{\sqrt{2}} \\
\int dt &= \int \frac{\pm 1}{\sqrt{2}} d\tau
\end{aligned}$$

So we get that $t + C = \frac{\pm 1}{\sqrt{2}}\tau$, such that $\tau = \pm\sqrt{2}t + C$