09/04/2024

Traffic Flow (Contd.)

- Formulate the model:

Definition of variable:

Velocity: for a given car in a single lane,

$x(t)$: position (centre of the car)

$\dfrac{dx(t)}{dt}$ = velocity.

Assuming there are multiple cars $C_1, C_2, \dots$

$x_i(t)$: position of C_i

$u_i(t)$: velocity of C_i

$$\Rightarrow \quad u_i(t) = \frac{dx_i(t)}{dt}$$

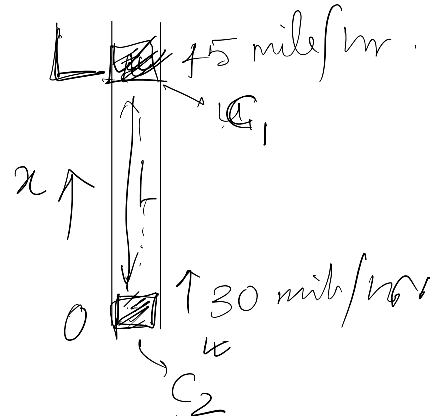
The velocity field: $u(x,t)$ such that

$$u(x_i(t), t) = u_i(t).$$

Given conditions:

C_1: moves at speed 45 mile/hr and initially located at $L>0$.

C_2: moves at speed 30 mile/hr and initially located at 0.

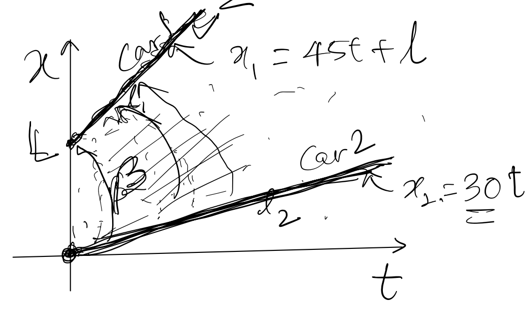


$$\left.\begin{aligned} u_1(t) = \frac{dx_1(t)}{dt} &= 45 \\ x_1(0) &= L \end{aligned}\right\} \text{Initial value problem of ODE.}$$

$$\Rightarrow x_1(t) = 45t + L$$

$$\left.\begin{aligned} u_2(t) = \frac{dx_2(t)}{dt} &= 30 \\ x_2(0) &= 0 \end{aligned}\right\}$$

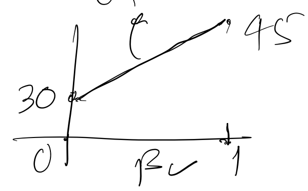
$$\Rightarrow \quad x_2(t) = 30t$$



Goal: Model formulation of $u(x,t)$ such that $u(l_1) = 45$

$u(l_2) = 30$

$u_\beta - 30 = \frac{\cancel{15}\,45-30}{1-0}(\beta-0)$



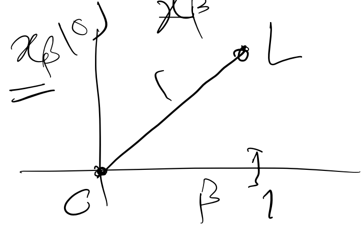
Assume there exists C_β, $\beta \in [0,1]$

$$u_\beta(t) = 30 + 15\beta .$$

$$\frac{dx_\beta(t)}{dt} = u_\beta(t) = 30 + 15\beta .$$

$$\Rightarrow x_\beta(t) = (30+15\beta)t + x_\beta(0) .$$

$x_\beta = \beta L$



$$x_\beta(0) = \beta L .$$

Analysis:

Velocity field

$$x_\beta(t) = (30+15\beta)t + \beta L$$

$$\Rightarrow \beta = \frac{x_\beta(t) - 30t}{15t + L}$$

Now, $u(x_\beta(t), t) = u_\beta(t) = 30 + 15\beta$

$$= 30 + 15 \cdot \frac{x_\beta(t) - 30t}{15t + L}$$

$$\Rightarrow u(x_\beta(t), t) = \frac{15x_\beta(t) + 30L}{15t + L}$$

$$\therefore \boxed{u(x,t) = \frac{15x + 30L}{15t + L}}$$

DIMENSIONAL ANALYSIS

Math: # of basis in vector space.

Physics: An expression for a derived physical quantity in terms of fundamental quantities such that mass (M), length (L) or time (T).

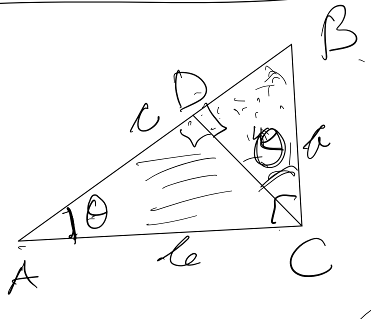
Eg. Area $= L^2$

$$\text{Acceleration} = \frac{[\text{vel}]}{[\text{time}]} = \frac{[\text{dist}]}{[\text{time}][\text{time}]}$$

$$= \underline{LT^{-2}}$$

Application:
- allow consistency in model.
- derivation of some formula (related theorem)
- obtain solution

Example: Pythagorous Theorem



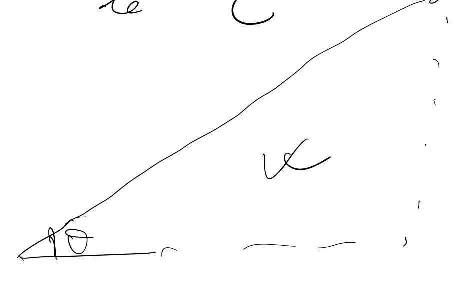
Note:

$$\text{Area}(ABC) = f(c, \theta)$$

θ — nondimensional

$$[c] = L$$

$$[\text{area}] = L^2$$



Dimensional analysis allows

$$\text{area}(ABC) = c^2 \cdot F(\theta)$$

$$\text{area}(ACD) = b^2 F(\theta)$$

$$\text{area}(BCD) = a^2 F(\theta)$$

Note: $F(\theta) = \frac{1}{2}\sin\theta\cos\theta$

area(ABC) = area(ACD) + area(BCD)

$$\Rightarrow \boxed{c^2 = b^2 + a^2}$$

Example:

$$\left.\begin{aligned} \frac{dx}{dt} &= -\lambda\, x(t) \\ x(0) &= x_0 \end{aligned}\right\}$$

x : density of chemical (drug)

(solution $x = ?$)

unit

$[x] = \frac{M}{L^3}$ mg/ml^3

$[t] = T$ hr

$[x_0] = \frac{M}{L^3}$ mg/ml^3

$$\frac{M}{L^3}\cdot\frac{1}{T} = -\lambda\cdot\frac{M}{L^3}$$

$$\Rightarrow \lambda = T^{-1}$$ per h.

Assume $x(t) = f(t, \lambda, x_0)$

Then $[x] = [t^a \lambda^b x_0^c]$

or $$\frac{M}{L^3} = T^a \frac{1}{T^b}\left(\frac{M}{L^3}\right)^c$$

$$\Rightarrow \; M L^{-3} = T^{a-b} \cdot M^{c} \cdot L^{-3c}$$

$$\Rightarrow \; \left.\begin{aligned} a-b &= 0 \\ c &= 1 \\ -3c &= -3 \end{aligned}\right\}$$

$$\therefore \quad \boxed{x = x_0 (\lambda t)^{a}}$$

Intuition: x decreases as t increases

$$a < 0$$

$$(\lambda t)^a \simeq \cancel{e^{-\lambda t}} \; ?$$