Final Part B
Ordinary Differential Equations
Math 537
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Problem 1 [8:40am, 8:53am]: Consider the following 2nd order ODEs:

$$\frac{d^2X}{dt^2} - a^2X = F(t) \tag{1.1}$$

$$\frac{d^2X}{dt^2} + a^2X = F(t) \tag{1.2}$$

Complete the following problems:

(a) Assume that $F = G = 0$ and a is a positive constant. Solve Eqs. (1.1) and (1.2) for general solutions

Notice the characteristic equation for Eq (1.1):

$$\lambda^2 - a^2 = 0, \qquad \lambda = \pm a$$

Thus from these eigenvalues, we get the following solution:

$$X = c_1e^{-a} + c_2e^{a}$$

Notice the characteristic equation for Eq (1.2):

$$\lambda^2 + a^2 = 0, \qquad \lambda = \pm ia$$

Thus from these eigenvalues, we get the following solution:

$$X = c_1\left(\cos(at) + i\sin(at)\right)$$

(b) Find $F(t)$ so that Eq. (1.1) has repeated eigenvalue. Briefly discuss the characteristics of the particular solution. [a is a positive constant.]

(c) Find $G(t)$ so that Eq. (1.1) has repeated eigenvalue. Briefly discuss the characteristics of the particular solution. [a is a positive constant.]

(d) State the conditions under which the solutions in problem (1a) change rapidly within an interval or oscillate rapidly over a global scale.

(e) Assume $F = G = 0$ and $a(t) > 0$ is a function of time. Briefly discuss how to solve both systems.

Problem 2 [9:47am, 9:57am]: Consider the following 2nd order ODE for nonlinear pendulum oscillations (as shown in Figure 1):

$$\frac{d^2\theta}{dt^2} + \epsilon\frac{d\theta}{dt} + \sin\theta = 0 \tag{2.1}$$

Apply the full equation and its simplified versions to discuss the following concepts:

(a) Locally linearized systems near a stable or unstable critical point.

(b) The impact of dissipation on the local, global, and structural stability.

(c) The impact of nonlinearity (e.g., represented by a cubic term) on the local, global, and structural stability.

$$\frac{d^2z}{dt^2} + \theta = 0. \tag{2.2}$$

$$\frac{d^2z}{dt^2} + \epsilon\frac{dz}{dt} + \theta = 0. \tag{2.3}$$

$$\frac{d^2z}{dt^2} + \left(\theta - \frac{\theta^3}{6}\right) = 0. \tag{2.4}$$

(2.2) We can find the eigenvalues from solving its characteristic equation.

$$\lambda^2 + 1 = 0$$

Thus our eigenvalues are $\lambda = \pm i$. This means that we get a **Clockwise Center**.

(2.3) We can find the eigenvalues from solving its characteristic equation.

$$\lambda^2 + \epsilon\lambda + 1 = 0$$

Thus our eigenvalues are $\lambda = \frac{1}{2}\left(-\epsilon \pm \sqrt{\epsilon^2 - 4}\right)$.

For $\epsilon < -2$, we get 2 positive real eigenvalues. This means that we get an **Unstable Source**.

For $\epsilon = -2$, we get $\lambda = 1$. This means that we get an **Unstable Source**.

For $-2 < \epsilon < 0$, we get 2 imaginary eigenvalues with positive real parts. This means that we get an **Unstable Spiral**.

For $\epsilon = 0$, we get $\lambda = \pm i$. This means that we get a **Clockwise Center**.

For $0 < \epsilon < 2$, we get 2 imaginary eigenvalues with negative real parts. This means that we get a **Stable Spiral**.

For $2 < \epsilon$, we get 2 negative real eigenvalues. This means that we get a **Stable Sink**.

For $\epsilon = 2$, we get $\lambda = -1$. This means that we get a **Stable Sink**.

(2.4) Let $\phi = \theta'$ and $\phi' = \theta\left(\frac{\theta^2}{6} - 1\right)$.

$$X' = \begin{pmatrix} \phi \\ \phi' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{\theta^2}{6} - 1 & 0 \end{pmatrix} \begin{pmatrix} \theta \\ \phi \end{pmatrix} = AX$$

Notice we can find the critical points:

$$X' = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{\theta^2}{6} - 1 & 0 \end{pmatrix} \begin{pmatrix} \theta \\ \phi \end{pmatrix} = AX$$

From this we get the following critical points: $(\theta, \phi) = (\pm\sqrt{6}, 0)$ and $(\theta, \phi) = (0, 0)$.

Now notice the Jacobian:

$$J(\theta, \phi) = \begin{bmatrix} \frac{d\phi}{d\theta} & \frac{d\phi}{d\theta'} \\ \frac{d\phi'}{d\theta} & \frac{d\phi'}{d\theta'} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{\theta^2}{2} - 1 & 0 \end{bmatrix}$$

Notice the Jacobians and eigenvalues evaluated at the critical points:

$$J(0, 0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

From solving $|A - \lambda I| = 0$, we get that $\lambda = \pm i$. Thus we get a **Clockwise Center**.

$$J(\pm\sqrt{6}, 0) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

From solving $|A - \lambda I| = 0$, we get that $\lambda = \pm\sqrt{2}$. Thus we get a **Saddle Point**.

We get that for each point in which we have a saddle point, we get local minima for potential energy, and for centers, we get local maxima for potential energy.

Problem 3 [8:07am, 8:39am][9:30, 9:43am] : Consider the Lorenz model:

$$\frac{dX}{dt} = -\sigma X + \sigma Y \tag{3.1}$$

$$\frac{dY}{dt} = -XZ + rX - \alpha Y \tag{3.2}$$

$$\frac{dZ}{dt} = XY - \beta Z \tag{3.3}$$

(a) Briefly discuss methods for analyzing the above nonlinear system.

We can compute the Jacobian and then evaluate the critical points using the Jacobian. Then we can get the eigenvalues, and solve for the general solution.

(b) Compute a Jacobian matrix to obtain a linearized system.

Let $f_1 = \frac{dX}{dt}$, $f_2 = \frac{dY}{dt}$, $f_3 = \frac{dZ}{dt}$. Notice the following Jacobian:

$$J(X,Y,Z) = \begin{bmatrix} \frac{\partial f_1}{\partial X} & \frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial Z} \\ \frac{\partial f_2}{\partial X} & \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial Z} \\ \frac{\partial f_3}{\partial X} & \frac{\partial f_3}{\partial Y} & \frac{\partial f_3}{\partial Z} \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ -Z+r & -\alpha & -X \\ Y & X & -\beta \end{bmatrix}$$

(c) Apply a perturbation method to obtain systems of equations for basic state $O(\epsilon^0)$ and perturbation $O(\epsilon^1)$ variables

Notice the perturbation steps:

$$\frac{dX}{dt} + \epsilon\frac{d^2X}{dt} = -\sigma(X + \epsilon X') + \sigma(Y + \epsilon Y')$$

$$\frac{dY}{dt} + \epsilon\frac{d^2Y}{dt} = -(X + \epsilon X')(Z + \epsilon Z') + r(X + \epsilon X') - \alpha(Y + \epsilon Y')$$

$$\frac{dZ}{dt} + \epsilon\frac{d^2Z}{dt} = (X + \epsilon X')(Y + \epsilon Y') - \beta(Z + \epsilon Z')$$

Now if we look at the basic state of $O(\epsilon^0)$, we get:

$$\frac{dX}{dt} + \epsilon\frac{d^2X}{dt} = -\sigma X + \sigma Y$$

$$\frac{dY}{dt} + \epsilon\frac{d^2Y}{dt} = -XZ + rX - \alpha Y$$

$$\frac{dZ}{dt} + \epsilon\frac{d^2Z}{dt} = XY - \beta Z$$

Now if we look at the basic state of $O(\epsilon^1)$, we get:

$$\frac{dX}{dt} + \epsilon\frac{d^2X}{dt} = -\sigma X' + \sigma Y'$$

$$\frac{dY}{dt} + \epsilon\frac{d^2Y}{dt} = X'Z + XZ' + rX' - \alpha Y'$$

$$\frac{dZ}{dt} + \epsilon\frac{d^2Z}{dt} = X'Y + XY' - \beta Z'$$

(d) Given $\sigma = 10$, $\alpha = 1$, $\beta = 8/3$, briefly discuss the characteristics of three types of solutions within different intervals of heating parameters (r).

Notice the Jacobian, its critical points, and the Jacobian evaluated at these critical points, with the given values:

$$J(X,Y,Z) = \begin{bmatrix} -10 & 10 & 0 \\ -Z+r & -1 & -X \\ Y & X & -8/3 \end{bmatrix}$$

We can find the critical points from the following:

$$\begin{bmatrix} \frac{dX}{dt} \\ \frac{dY}{dt} \\ \frac{dZ}{dt} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ r & -1 & -X \\ 0 & X & -8/3 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

We can see that we get the following critical point: $\mathbf{(0,0,0)}$.

Notice the Jacobian at the critical point:

$$J(0,0,0) = \begin{bmatrix} -10 & 10 & 0 \\ r & -1 & 0 \\ 0 & 0 & -8/3 \end{bmatrix}$$

Now we can find the characteristic equation by solving $|J - \lambda I| = 0$

$$\begin{aligned} (-10-\lambda)(-1-\lambda)(-8/3-\lambda) - 10(r)(-8/3-\lambda) &= 0 \\ -\lambda^3 - \frac{41\,\lambda^2}{3} - \frac{118\,\lambda}{3} - \frac{80}{3} + \frac{80r}{3} + 10\lambda &= 0 \\ \lambda^3 + \frac{41\,\lambda^2}{3} + \frac{88\lambda}{3} + \frac{80}{3}(1-r) &= 0 \end{aligned}$$

(e) Find critical points for positive parameters.

(f) Find critical points within the non-dissipative system (ie, no σX in Eq (3.1) and $\alpha = \beta = 0$)

Problem 5 [8:54am, 9:24am]: Consider a composite motion with the following harmonic oscillators:

$$\frac{d^2x_1}{dt^2} = -\omega_1^2 x_1, \tag{5.1}$$

$$\frac{d^2x_2}{dt^2} = -\omega_2^2 x_2, \tag{5.2}$$

(a) Discuss the condition under which the composite motion with the two frequencies is periodic or quasi-periodic.

From this system, we get 2 solutions:

$$x_1(t) = a_1\cos(\omega_1 t) + b_1\sin(\omega_1 t), \qquad x_2(t) = a_2\cos(\omega_2 t) + b_2\sin(\omega_2 t)$$

To find the conditions in which the frequencies are periodic or quasi-periodic, we consider the following:

$$x_1(t) = a_1\cos(\omega_1 t), \qquad \omega_1(t+T) = \omega_1 t + 2m\pi, \qquad T = \frac{2m\pi}{\omega_1}$$

$$x_2(t) = a_2\cos(\omega_2 t), \qquad \omega_2(t+T) = \omega_2 t + 2n\pi, \qquad T = \frac{2n\pi}{\omega_2}$$

So we get the following

$$\text{Periodic if: } \frac{\omega_1}{\omega_2} = \frac{m}{n} \in \mathbb{Q}, \text{ and Quasi-Periodic if: } \frac{\omega_1}{\omega_2} = \frac{m}{n} \notin \mathbb{Q}$$

(b) Compute and generate a plot (e.g., Figure 4) to illustrate either a periodic or quasi-periodic composite solution (with two frequencies).

```
time = linspace(0,1,1e6);
w1 = 1; w2 = 4;
x1 = cos(2*pi*time*w1);
x2 = cos(2*pi*time*w2);

figure(),
plot(x1,x2); grid on;
title('Periodic')
```