Homework 9
Partial Differential Equations
Math 531
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Excersise 8.2.2: Consider the heat equation with time-dependent sources and boundary conditions

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t) \qquad u(x,0) = f(x)$$

Reduce the problem to one with homogeneous boundary conditions if

(b)

$$u(0,t) = A(t) \qquad \text{and} \qquad \frac{\partial u}{\partial x}(L,t) = B(t)$$

Let $u_E(x,t)$ be the linear PDE that satisfies the steady state problem:

$$u_E(0,t) = A(t) \qquad \frac{\partial u_E}{\partial x}(L,t) = B(t) \qquad \to \qquad u_E(x,t) = A(t) + xB(t)$$

We can now define $v(x,t) = u(x,t) - u_E(x,t)$ with the following PDE's:

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t} - \frac{\partial u_E}{\partial t} \quad \to \quad \frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} + \frac{\partial u_E}{\partial t} \qquad\qquad \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u_E}{\partial x^2} = \frac{\partial^2 u}{\partial x^2}$$

Now we can convert our original PDE:

$$\frac{\partial v}{\partial t} + \frac{\partial u_E}{\partial t} = k\frac{\partial^2 v}{\partial x^2} + Q(x,t)$$

Now notice this is a PDE with the following homogeneous boundary conditions:

$$v(0,t) = u(0,t) - u_E(0,t) = 0 \qquad \frac{\partial v}{\partial x}(L,t) = \frac{\partial u}{\partial x}(L,t) - \frac{\partial u_E}{\partial x}(L,t) = 0$$

and the following initial condition:

$$v(x,0) = u(x,0) - u_E(x,0) = f(x) - (A(0) + xB(0))$$

Excersise 8.2.5: Solve the initial value problem for a two-dimensional heat equation inside a circle (of radius a) with time-independent boundary conditions

$$\frac{\partial u}{\partial t} = k\nabla^2 u \qquad u(a,\theta,t) = g(\theta) \qquad u(r,\theta,0) = f(r,\theta)$$

Let $u_E(r,\theta)$ be the equilibrium temperature distribution:

$$\nabla^2 u_E = 0 \qquad \rightarrow \qquad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_E}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_E}{\partial \theta^2} = 0$$

Using separation of variables, we get: $u_E(r,\theta) = F(r)G(\theta)$:

$$\frac{r}{F}\frac{d}{dr}\left(r\frac{dF}{r}\right) + \frac{1}{G}\frac{d^2G}{d\theta^2} = 0 \qquad \rightarrow \qquad \frac{r}{F}\frac{d}{dr}\left(r\frac{dF}{r}\right) = -\frac{1}{G}\frac{d^2G}{d\theta^2} = \lambda$$

Thus we get the following ODE's:

$$r^2F'' + rF' - \lambda F = 0 \qquad G'' + \lambda G = 0$$

Using the fact of the geometry of the circle, and that $F(r)$ is finite at $r = 0$, we get the following eigenvalues and eigenfunction:

$$\lambda = 0 \qquad \rightarrow \qquad G(\theta) = 1 \qquad F(r) = 1$$

$$\lambda = m^2 \qquad \rightarrow \qquad G(\theta) = a\cos m\theta + b\sin m\theta \qquad F(r) = c_1 r^m$$

From this, we get the following:

$$u_E(r,\theta) = \sum_{m=0}^{\infty}\left(A_m\cos m\theta + B_m\sin m\theta\right)r^m$$

Using the boundary conditions, we get:

$$u_E(a,\theta) = A_0 + \sum_{m=1}^{\infty}\left(A_m\cos m\theta + B_m\sin m\theta\right)a^m$$

From here, we get the following coefficients:

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\,d\theta \qquad A_m = \frac{1}{a^m\pi}\int_{-\pi}^{\pi} g(\theta)\cos m\theta\,d\theta \qquad B_m = \frac{1}{a^m\pi}\int_{-\pi}^{\pi} g(\theta)\sin m\theta\,d\theta$$

We let the following be true now:

$$v(r,\theta,t) = u(r,\theta,t) - u_E(r,\theta) \quad \text{with} \quad v(a,\theta,t) = 0 \qquad v(r,\theta,0) = f(r,\theta) - u_E(r,\theta)$$

Similar to previous homeworks, we get the following for $v(r,\theta,t)$:

$$v(r,\theta,t) = \sum_{n=1}^{\infty} A_{0n} J_0(\sqrt{\lambda_{0n}} r) e^{-\lambda_{0n} kt} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_m(\sqrt{\lambda_{mn}} r) \left(A_{mn}\cos m\theta + B_{mn}\sin m\theta\right) e^{-\lambda_{mn} kt}$$

From here, we use our initial conditions:

$$\begin{aligned} v(r,\theta,0) &= f(r,\theta) - u_E(r,\theta) \\ &= \sum_{n=1}^{\infty} A_{0n} J_0(\sqrt{\lambda_{0n}} r) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_m(\sqrt{\lambda_{mn}} r) \left(A_{mn}\cos m\theta + B_{mn}\sin m\theta\right) \end{aligned}$$

From here, we get the following coefficients:

$$\begin{aligned} A_{0n} &= \frac{\int_0^a \int_{-\pi}^{\pi} \left(f(r,\theta) - u_E(r,\theta)\right) J_0\left(\sqrt{\lambda_{0n}} r\right) r\, dr\, d\theta}{2\pi \int_0^a J_0^2\left(\sqrt{\lambda_{0n}} r\right)\, dr} \\ A_{mn} &= \frac{\int_0^a \int_{-\pi}^{\pi} \left(f(r,\theta) - u_E(r,\theta)\right) J_m\left(\sqrt{\lambda_{mn}} r\right) \cos m\theta\, r\, dr\, d\theta}{2\pi \int_0^a J_m^2\left(\sqrt{\lambda_{mn}} r\right)\, dr} \\ B_{mn} &= \frac{\int_0^a \int_{-\pi}^{\pi} \left(f(r,\theta) - u_E(r,\theta)\right) J_m\left(\sqrt{\lambda_{mn}} r\right) \sin m\theta\, r\, dr\, d\theta}{2\pi \int_0^a J_m^2\left(\sqrt{\lambda_{mn}} r\right)\, dr} \end{aligned}$$

Excersise 8.4.2: Use the method of eigenfunction expansions to solve, without reducing to homogeneous boundary conditions,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad u(0,t) = A \qquad u(L,t) = B \qquad u(x,0) = f(x)$$

Using an eigenfunction expansion, we choose get the Strum-Liouville Problem in the spatial domain:

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \quad \text{with} \quad \phi(0) = 0 \quad \phi(L) = 0$$

Notice the eigenvalues and eigenfunctions:

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad \phi_n(x) = \sin\frac{n\pi x}{L}$$

Thus we can let the following be true:

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t)\phi(x) \quad \text{with} \quad u(x,0) = f(x) = B_n(0)\phi(x) \quad \to \quad B_n(0) = \frac{2}{L}\int_0^L f(x)\phi(x)\,dx$$

Notice the following from that and orthogonality of sines:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\frac{dB_n(t)}{dt}\phi(x) = k\frac{\partial^2 u}{\partial x^2} \qquad \frac{dB_n(t)}{dt} = \frac{2k}{L}\int_0^L \frac{\partial^2 u}{\partial x^2}\phi(x)$$

Notice the following from Green's formula:

$$\begin{aligned}
\int_0^L \left(u\frac{d^2\phi_n}{dx^2} - \phi_n\frac{\partial^2 u}{\partial x^2}\right)dx &= u\frac{d\phi_n}{dx} - \phi_n\frac{\partial u}{\partial x}\bigg|_0^L \\
&= \left(u(L,t)\frac{d\phi_n(L)}{dx} - \phi_n(L)\frac{\partial u}{\partial x}(L,t)\right) - \left(u(0,t)\frac{d\phi_n(0)}{dx} - \phi_n(0)\frac{\partial u}{\partial x}(0,t)\right) \\
&= B\frac{d\phi_n(L)}{dx} - A\frac{d\phi_n(0)}{dx} \\
&= \frac{n\pi}{L}\Big((-1)^n B - A\Big)
\end{aligned}$$

Notice the following equality:

$$\int_0^L u\frac{d^2\phi_n}{dx^2}\,dx - \int_0^L \phi_n\frac{\partial^2 u}{\partial x^2}\,dx = \int_0^L\left(u\frac{d^2\phi_n}{dx^2} - \phi_n\frac{\partial^2 u}{\partial x^2}\right)dx$$

$$\int_0^L u\frac{d^2\phi_n}{dx^2}\,dx - \int_0^L\left(u\frac{d^2\phi_n}{dx^2} - \phi_n\frac{\partial^2 u}{\partial x^2}\right)dx = \int_0^L \phi_n\frac{\partial^2 u}{\partial x^2}\,dx$$

Thus we get the following:

$$
\begin{aligned}
\int_0^L \phi_n \frac{\partial^2 u}{\partial x^2}\, dx &= \int_0^L u \frac{d^2 \phi_n}{dx^2}\, dx - \int_0^L \left(u \frac{d^2\phi_n}{dx^2} - \phi_n \frac{\partial^2 u}{\partial x^2} \right)\, dx \\
&= \int_0^L u \frac{d^2\phi_n}{dx^2}\, dx - \frac{n\pi}{L}\Big((-1)^n B - A \Big) \\
&= \int_0^L u\left(-\lambda_n \phi_n\right)\, dx - \frac{n\pi}{L}\Big((-1)^n B - A \Big) \\
&= -\lambda_n \int_0^L \left(\sum_{m=1}^{\infty} B_m(t)\phi_m(x) \right) \phi_n\, dx - \frac{n\pi}{L}\Big((-1)^n B - A \Big) \\
&= -\lambda_n \left(\frac{L}{2} \right) B_n(t) - \frac{n\pi}{L}\Big((-1)^n B - A \Big)
\end{aligned}
$$

From here, we get the following:

$$
\begin{aligned}
\frac{dB_n(t)}{dt} &= \frac{2k}{L} \int_0^L \frac{\partial^2 u}{\partial x^2} \phi(x) \\
&= \frac{2k}{L} \left(-\lambda_n \left(\frac{L}{2} \right) B_n(t) - \frac{n\pi}{L}\Big((-1)^n B - A \Big) \right) \\
&= -\lambda_n k B_n(t) - \frac{2kn\pi}{L^2}\Big((-1)^n B - A \Big)
\end{aligned}
$$

Now we multiply by $e^{-\lambda_n k t}$ and solve for B_n, we get:

$$
B_n(t) = B_n(0) e^{-\lambda_n k t} - \frac{2}{n\pi}\Big((-1)^n B - A \Big) \left(1 - e^{-\lambda_n k t} \right)
$$

Excersise 8.4.3: Consider

$$c(x)\rho(x)\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right] + q(x)u + f(x,t)$$

$$u(x,0) = g(x) \qquad u(0,t) = \alpha(t) \qquad u(L,t) = \beta(t)$$

Assume that the eigenfunction $\phi_n(x)$ of the related homogeneous problem are known.

(a) Solve without reducing to a problem with homogeneous boundary conditions.

Let the following be true:

$$\sigma = c\rho \qquad u(x,t) = \sum_{n=0}^{\infty} B_n(t)\phi_n(t)$$

Substituting, we get:

$$\sigma \sum_{n=0}^{\infty} B_n'(t)\phi_n(t) = \frac{\partial}{\partial x}\left[K_0(x)\frac{\partial u}{\partial x}\right] + q(x)u + f(x,t)$$

Similar to our last problem, we can use Green's formula to get:

$$\sigma\frac{dB_n(t)}{dt} = -\lambda_n k B_n(t) + f_n(t) - \frac{k\sqrt{\lambda_n}\,((-1)^n\beta(t) - \alpha(t))}{\int_0^L \phi_n^2(x)\sigma\,dx}$$

with the following:

$$f_n(t) = \frac{\int_0^L f(x,t)\phi_n(x)\sigma\,dx}{\int_0^L \phi_n^2(x)\sigma\,dx}$$

Multiplying by $e^{\frac{\lambda_n}{\sigma}t}$ and solving for B_n, we get:

$$B_n(t) = e^{-\frac{\lambda_n}{\sigma}t}\left(\frac{1}{\sigma}\int e^{\frac{\lambda_n}{k}t} f_n(t)\,dt + \frac{\frac{k}{n}\sqrt{\lambda_n}}{\int_0^L \phi_n^2(x)\sigma, dx}\int -e^{\frac{\lambda_n}{\sigma}}\,((-1)^n\beta(t) - \alpha(t))\,dt\right) + B_n(0)e^{-\frac{\lambda_n}{\sigma}t}$$

with the following coefficients found from the initial condition:

$$B_n(0) = \frac{\int_0^L g(x)\phi_n(x)\sigma\,dx}{\int_0^L \phi_n^2(x)\sigma, dx}$$

(b) Solve by first reducing to a problem with homogeneous boundary conditions

Let the following be true:

$$u_E(x,t) = \alpha(t) + \frac{x}{L}\left(\beta(t) - \alpha(t)\right)$$

Now we set the following and substitute:

$$v(x,t) = u(x,t) - u_E(x,t) \qquad c\rho\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial v}{\partial x}\right) + vq(x) + f(x,t)$$

with the following conditions:

$$v(0,t) = v(\pi,t) = 0 \qquad v(x,0) = g(x) - \alpha(0) - \frac{x}{L}\left(\beta(0) - \alpha(0)\right)$$

We now let the following:

$$v(x,t) = \sum_{n=1}^{\infty} B_n(t)\phi_n(x) \qquad \sigma = c\rho$$

Using Green's formula, we get:

$$\sigma\frac{dB_n(t)}{dt} = -\lambda_n B_n(t) + f_n(t)$$

where

$$f_n(t) = \frac{\int_0^L f(x,t)\phi_n(x)\sigma\,dx}{\int_0^L \phi_n^2(x)\sigma\,dx}$$

Multiplying by $e^{\frac{\lambda_n}{\sigma}t}$ and solving for B_n, we get:

$$B_n(t) = e^{-\frac{\lambda_n}{\sigma}t}\frac{1}{\sigma}\int e^{\frac{\lambda_n}{\sigma}t} f_n(t)\,dt + B_n(0)e^{-\frac{\lambda_n}{\sigma}}$$

with the following coefficients found from the initial condition:

$$B_n(0) = \frac{\int_0^L g(x) - \alpha(0) - \frac{x}{L}\left(\beta(0) - \alpha(0)\right)\phi_n(x)\sigma\,dx}{\int_0^L \phi_n^2(x)\sigma\,dx}$$

Excersise 9.2.1: Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q(x,t) \qquad u(x,0) = g(x)$$

In all cases, obtain formulas similar to (2.20) by introducing a Green's function.

Notice equation (2.20):

$$u(x,t) = \int_0^L g(x_0)G(x,t;x_0,0)\,dx_0 + \int_0^L \int_0^t Q(x_0,t_0)G(x,t;x_0,0)\,dt_0\,dx_0 \tag{2.20}$$

(c) Solve using any method if

$$\frac{\partial u}{\partial t}(0,t) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial t}(L,t) = 0$$

(d) Use Green's formula instead of term-by-term differentiation if

$$\frac{\partial u}{\partial x}(0,t) = A(t) \qquad \text{and} \qquad \frac{\partial u}{\partial x}(L,t) = \beta(t)$$