Note: For full credit you must show intermediate steps in your calculations.

1. (4pts) Use the information in the lecture notes for the salt mixing problem to verify that the Example on Slide 10 satisfies the conditions required to maintain a constant volume in each vessel. Also, show steps (Gaussian elimination, row-reduce echelon, or some other technique from Linear Algebra) to verify that the equilibrium given on Slide 11 follows from the salt mixing model for the Example from Slide 10. (Slides 5-12)

To maintain constant volumes, $V_1 = 100l$ and $V_2 = 60l$, we need the following conditions to hold:

$$f_1 + f_2 = f_6, \qquad f_1 + f_3 = f_4, \qquad f_2 + f_5 = f_3, \qquad f_5 + f_6 = f_4.$$

From the information on Slide 10, we have:

$$\begin{aligned} f_1 + f_2 &= 0.2 + 0.15 = f_6 = 0.35, & \qquad f_1 + f_3 &= 0.2 + 0.25 = f_4 = 0.45, \\ f_2 + f_5 &= 0.15 + 0.1 = f_3 = 0.25, & \qquad f_5 + f_6 &= 0.1 + 0.35 = f_4 = 0.45, \end{aligned}$$

which shows the balance of flows in and out of the two vessels. The equilibrium satisfies $\dot{c}_1 = \dot{c}_2 = 0$, so we are solving the matrix equation:

$$\begin{pmatrix} 0.0045 & -0.0025 \\ -0.00167 & 0.004167 \end{pmatrix} \begin{pmatrix} c_{1e} \\ c_{2e} \end{pmatrix} = \begin{pmatrix} 0.014 \\ 0.03 \end{pmatrix}.$$

If we multiply the first row by 2000 and the second row by 1200, then we have

$$\left[\begin{array}{cc|c} 9 & -5 & 28 \\ -2 & 5 & 36 \end{array}\right] (R_2 + \tfrac{2}{9}R_1) \to \left[\begin{array}{cc|c} 9 & -5 & 28 \\ 0 & \frac{35}{9} & \frac{380}{9} \end{array}\right]$$

Back substitution gives $c_{2e} = \frac{380}{35} = 10.85714$, so $c_{1e} = \frac{28}{9} + \frac{380}{63} = 9.14286$, which is the result on Slide 11.

2. (4pts) The pharmokinetic model is presented on Slides 14-15. It is stated that the *trace* satisfies $tr(\mathbf{A}) < 0$, the *determinant* is $\det|\mathbf{A}| > 0$, and the *discriminant* is $D > 0$. Provide details that verify these conditions, assuming all parameters in the matrix $\mathbf{A}$ are positive.

The matrix for the pharmokinetic model is:

$$\mathbf{A} = \begin{pmatrix} -(K_{pb} + K_e) & K_{bp} \\ K_{pb} & -K_{bp} \end{pmatrix}.$$

It follows that the $tr(\mathbf{A}) - (K_{pb} + K_e) - K_{bp} = -(K_{pb} + K_e + K_{bp}) < 0$. Also, $\det|\mathbf{A}| = (K_{pb} + K_e)K_{bp} - K_{pb}K_{bp} = K_eK_{bp} > 0$. Finally, the discriminant satisfies:

$$\begin{aligned} D &= (K_{pb} + K_{bp} + K_e)^2 - 4K_{bp}K_e \\ &= K_{pb}^2 + K_{bp}^2 + K_e^2 + 2(K_{pb}K_{bp} + K_{pb}K_e + K_{bp}K_e) - 4K_{bp}K_e \\ &= K_{pb}^2 + 2(K_{pb}K_{bp} + K_{pb}K_e) + (K_{bp} - K_e)^2 \ > \ 0. \end{aligned}$$

From the Stability Diagram of the Linear System B lecture, this shows that this model always has a *stable node* at $d_1 = d_2 = 0$.

3. (4pts) Consider the pharmokinetic model:

$$\begin{pmatrix} \dot{d}_1 \\ \dot{d}_2 \end{pmatrix} = \begin{pmatrix} -(K_{pb}+K_e) & K_{bp} \\ K_{pb} & -K_{bp} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix},$$

where $K_{pb} = 2$, $K_{bp} = 5$, and $K_e = 0.5$. Assume the initial condition:

$$\begin{pmatrix} d_1(0) \\ d_2(0) \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \end{pmatrix}$$

Find the solution to this initial value problem. State clearly the eigenvalues and eigenvectors. (Slides 18-22)

With the given parameters, the pharmokinetic model becomes:

$$\begin{pmatrix} \dot{d}_1 \\ \dot{d}_2 \end{pmatrix} = \begin{pmatrix} -2.5 & 5 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}.$$

The characteristic equation satisfies:

$$\begin{vmatrix} -2.5-\lambda & 5 \\ 2 & -5-\lambda \end{vmatrix} = \lambda^2 + 7.5\lambda + 2.5 = 0, \quad \text{so} \quad \lambda = \frac{-7.5 \pm \sqrt{46.25}}{2} \approx -7.1504, -0.34963.$$

The associated eigenvectors are for:

$$\lambda_1 = -7.1504, \quad \xi_1 = \begin{pmatrix} -1.0752 \\ 1 \end{pmatrix}, \qquad \text{and} \qquad \lambda_2 = -0.34963, \quad \xi_1 = \begin{pmatrix} 2.3252 \\ 1 \end{pmatrix},$$

which gives the general solution:

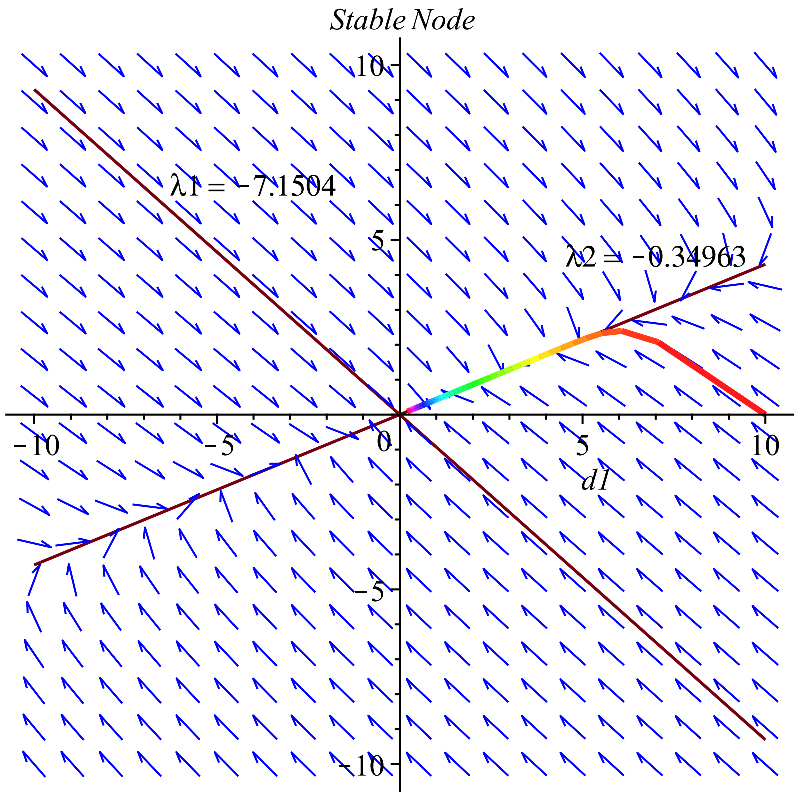
$$\begin{pmatrix} d_1(t) \\ d_2(t) \end{pmatrix} = c_1 \begin{pmatrix} -1.0752 \\ 1 \end{pmatrix} e^{-7.1504t} + c_2 \begin{pmatrix} 2.3252 \\ 1 \end{pmatrix} e^{-0.34963t}.$$

The solution to the IVP (with some help from Maple) is

$$\begin{pmatrix} d_1(t) \\ d_2(t) \end{pmatrix} = \begin{pmatrix} 6.8380e^{-0.34963t} + 3.1620e^{-7.1504t} \\ 2.9409e^{-0.34963t} - 2.9409e^{-7.1504t} \end{pmatrix}.$$

4. (4pts) Consider the pharmokinetic model in the previous problem (same parameters). Create a phase portrait and describe the qualitative behavior. On your phase portrait include the specific trajectory for the initial value problem above. (Slides 18-22)

As noted in Problem 2, the qualitative behavior is a *stable node*with all solutions approaching the origin in time moving first rapidly in the λ_1 direction before slowly moving in the direction of the eigenvector for λ_1. Below is the phase portrait of the pharmokinetic model of Problem 3, showing the eigenvectors and the solution to the IVP.



Stable Node
λ1 = −7.1504
λ2 = −0.34963
d1
-10
-5
0
5
10