Spring 2021 Math 531 Take-Home Exam 2 **Due Fri April 16, 2021**

I, ________________________________(your name), pledge that this exam is completely my own work, and that I did not take, borrow or steal work from any other person, and that I did not allow any other person to use, have, borrow or steal portions of my work. I understand that if I violate this honesty pledge, I am subject to disciplinary action pursuant to the appropriate sections of the San Diego State University Policies.

1. The Laplace equation

$$\nabla^2 u(x, y, z) = 0,$$

represents the steady-state heat equation without sources. Using circular cylindrical coordinates,

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \\ z &= z, \end{aligned}$$

show that the Laplace's equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

2. Given the heat equation on a radially symmetric disk

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \qquad 0 < r < a, \quad t > 0,$$

with boundary condition

$$u(a, t) = 0, \quad t > 0,$$

and initial condition

$$u(r, 0) = f(r), \quad t > 0,$$

State clearly the implicit boundary conditions. State clearly your Sturm-Liouville problem(s) and any orthogonality relationships. Solve this problem (showing the full Fourier series solution before applying the initial condition), then using orthogonality relative to the initial condition, reduce the Fourier series solution. (Don't try to reduce your integrals in r.)

3. a Solve the heat equation on a disk

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}, \qquad 0 < r < 1, \quad -\pi < \theta < \pi, \quad t > 0,$$

with the boundary condition

$$u(1, \theta, t) = \sin(3\theta),$$

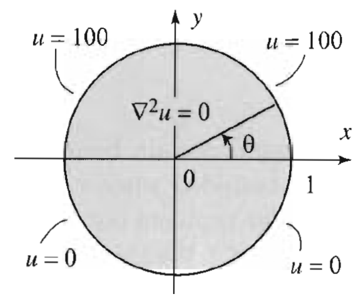
and the initial condition

$$u(r,\theta,0)=0.$$

State clearly the implicit boundary conditions. State clearly your Sturm-Liouville problem(s) and any orthogonality relationships. Solve this problem (showing the full Fourier series solution before applying the initial condition), then using orthogonality relative to the initial condition, reduce the Fourier series solution. (Don't try to reduce your integrals in r.)

b Find the steady-state temperature in the disk.

The disk has a radius of 1. The upper half of the circumference is kept at 100° and the lower half is kept at 0°.



4. Consider heat conduction in a sphere given by:

$$\frac{\partial u}{\partial t}=k\left(\frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial u}{\partial \rho}\right)+\frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right)+\frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}\right),\quad 0<\rho<a,\ -\pi<\theta\le\pi,\ 0\le\phi\le\pi,\ t>0,$$

with the boundary condition

$$\frac{\partial u}{\partial \rho}(a,\theta,\phi,t)=0,$$

and initial conditions:

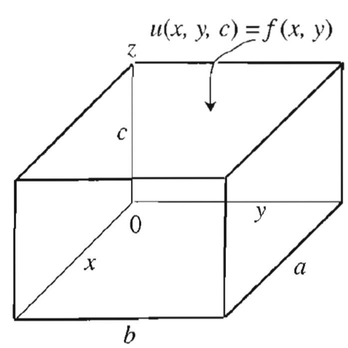
$$u(\rho,\theta,\phi,0)=F(\rho,\phi)\sin(3\theta).$$

Solve this equation noting any other boundary conditions you might need to apply. State clearly your Sturm-Liouville problem(s) and any orthogonality relationships.

5. Find the steady-state temperature in a cube, which satisfies:

$$\nabla^2 u(x,y,z)=0,\qquad 0<x<a,\quad 0<y<b,\quad 0<z<c.$$

The cube is kept at 0°C on all faces except on the upper horizontal face.



6. Solve the heat equation

$$\frac{\partial u}{\partial t} = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2}\right), \qquad 0 < r < 2, \quad 0 < z < 7, \quad t > 0,$$

inside a cylinder subject to the initial condition

$$u(r, z, 0) = 100,$$

if the boundary conditions are :

$$u(2, z, t) = 0, \quad u(r, 0, t) = 0, \quad u(r, 7, t) = 100,$$

7. Find the eigenvalues and eigenfunctions which arise from the Sturm-Liouville problem:

$$x\frac{d}{dx}\left(x\frac{d\phi}{dx}\right) + (\lambda x^2 - 9)\phi = 0, \quad 0 < x < 6,$$

with $\phi'(6) = 0$ and $\phi(x)$ bounded for $x \in [0, 6]$. Clearly state the orthogonality relationship for the eigenfunctions and use the eigenfunctions to find the Fourier expansion for a function $f(x)$. Give an integral expression for the Fourier coefficients. Assume that

$$f(x) = \begin{cases} x, & x < \frac{1}{2}. \\ 1 - x, & x > \frac{1}{2}. \end{cases}$$