MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #18

Chapter 5

Higher-Dimensional Linear Algebra

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Section 5.4: A Summary

A function $T: R^n \to R^n$ is linear,
if $T(X) = AX$ for some $n \times n$ matrix A.

- T is a linear map or linear transformation.
- $T(\alpha X + \beta Y) = \alpha T(X) + \beta T(Y)$ for any $\alpha, \beta \in R$ and $X, Y \in R^n$.
- T is invertible if the matrix A (associated with T) has an inverse.

The most important types of subspaces are the kernels and ranges of linear maps.

- $Ker\ T$: the set of vectors mapped to $\vec{0}$ by T, i.e., $TX = 0$.
- $Range\ T$: (consists of) all vectors W for which there exists a vector (V) for which, $TV = W$.
- Dim($Ker\ T$)+Dim($Range\ T$)=n.

Notes for “Ranges”

Calc III: Domain vs. Range, $z = f(x, y)$

$W = TV$ $\quad\quad W = TV = xC_1 + yC_2 + zC_3$

$T = [C_1, C_2, C_3]$, consisting of three column vectors

$V = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ appearing as “coefficients”

W is a linear combination of $C_1, C_2, \ldots C_n$

- Dim($Range\ T$) = # of freedom in W = # of “coefficients”
- = # of LI column vectors of T

Notes for Ranges, $TV = W$

The most important types of subspaces are the kernels and ranges of linear maps.

- $Ker\ T$: the set of vectors mapped to $\vec{0}$ by T, i.e., $TX = 0$.
- $Range\ T$: (consists of) all vectors W for which there exists a vector for which, $TV = W$.
- Dim($Ker\ T$)+Dim($Range\ T$) = n.

- Dim($Range\ T$) = # of LI column vectors of the matrix T
- For example, if Dim($Range\ T$) = n, rank of (T) = n;
- Thus, $TX = 0$ has a unique solution,
- indicating Ker T = {0} and Dim (Ker T) = 0

Equivalent Statements for a Non-singular Matrix

Square Matrix

Theorem 1.3. *For $A \in \mathbb{C}^{m \times m}$, the following conditions are equivalent:*

(a) *A has an inverse A^{-1},*

(b) $\boxed{\operatorname{rank}(A) = m,}$

(c) $\boxed{\operatorname{range}(A) = \mathbb{C}^m}$,

(d) $\operatorname{null}(A) = \{0\}$,

(e) 0 *is not an eigenvalue of A,*

(f) 0 *is not a singular value of A,*

(g) $\boxed{\det(A) \neq 0.}$

dim range(A) = m

Ax=0 ➔ x=0; “null space”=”kernel”

Section 5.4

Definition
Let $\mathcal{S}$ be a subspace of $\boxed{\mathbb{R}^n}$. A collection of vectors $\boxed{V_1, \ldots, V_k}$ is a *basis* of $\mathcal{S}$ if the V_j are linearly independent and span $\mathcal{S}$.

Proposition. *Every basis of a subspace $\mathcal{S} \subset \mathbb{R}^n$ has the same number of elements.*

Let $AV = \lambda V \quad \Rightarrow \quad (A - \lambda I)V = 0$

Define $B = A - \lambda I$

Find X for $BX = 0$

Ker B consists of $A's$ eigenvectors

A Proof

Proposition. *Every basis of a subspace $\mathcal{S} \subset \mathbb{R}^n$ has the same number of elements.*

Goal: given k LI vector $V_1, V_2, \dots V_k$;

assume $W_1, W_2, \dots W_{k+l}$ and represent each W_k using $V_1, V_2, \dots V_k$

show that $W_1, W_2, \dots W_{k+l}$ are LD

Observe that the system of k linear equations in the $(k+l)$ unknown given by

$$k \times (k+l) \qquad (k+l) \times 1 \qquad (k+1)\ variables$$

$$k\ EQs \quad \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k+l} \\ a_{21} & a_{22} & \cdots & \\ \vdots & \vdots & & \\ a_{k1} & \vdots & & a_{kk+l} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{k+l} \end{pmatrix} = 0$$

$$\Rightarrow \sum_{j=1}^{k+l} a_{ij} x_j = 0, for\ i = 1 \dots k \qquad (\boldsymbol{A1})$$

A Proof

Observe that the system of k linear equations in the $(k+l)$ unknown variables (e.g., Eq. A1), it always has a non-zero solution. Why?

- Using row reduction, we may first solve for one unknow in terms of others.
- Then, we obtain a system of $(k-1)$ equations in $(k+l-1)$ unknowns
- ➔ a system of $(k-2)$ equations in $(k+l-2)$ unknowns
- ....
- ➔ a system of $(k-(k-1))$ equations in $(k+l-(k-1))$ unknowns
- namely, 1 equation for $(l+1)$ unknows, e.g., $\beta_l z_l + \beta_{l+1} z_{l+1} = 0$

A Proof

From Eq. (A1), we have

$$\sum_{j=1}^{k+l} \alpha_{ij} \boxed{x_j} = 0, for\ i = 1 \dots k \qquad \textbf{(A1)}$$

Observe that the system of k linear equations in the $(k+l)$ unknown variables (e.g., Eq. A1), it always has a non-zero solution.

Since Eq. (A)1 has a non-zero solution $\boxed{, c_1, c_2, \dots c_{k+l},}$ we have

$$\boxed{\sum_{j=1}^{k+l} \alpha_{ij}\, c_j = 0, for\ i = 1 \dots k \qquad (A2)}$$

A Proof

Suppose that $V_1, V_2, \dots V_k$ is a basis for the subspace S

Suppose that $W_1, W_2, \dots W_{k+l}$ is also basis for the subspace S with $l > 0$

Then, each W_k can be represented by a linear combination of using $V_1, V_2, \dots V_k$, so we have constants, α_{ji}, such that

$$W_j = \sum_{i=1}^{k} \alpha_{ji} V_i, \qquad for\ j = 1 \dots (k+l) \qquad \textbf{(B)}$$

From Eq. (A2) with a non-zero solution, $c_1, c_2, \dots c_{k+l}$, we have

$$\sum_{j=1}^{k+l} \alpha_{ij}\, c_j = 0, for\ i = 1 \dots k$$

$$transpose \Longrightarrow \sum_{j=1}^{k+l} c_j \alpha_{ji} = 0, for\ i = 1 \dots k \qquad \textbf{(C)}$$

A Proof

$$W_j = \sum_{i=1}^{k} \alpha_{ji} V_i, for\ j = 1 \ldots (k+l) \quad (\boldsymbol{B})$$

$$\sum_{j=1}^{k+l} c_j \alpha_{ji} = 0, for\ i = 1 \ldots k \quad (\boldsymbol{C})$$

$$consider \sum_{j=1}^{k+l} c_j W_j = \sum_{j=1}^{k+l} c_j \left(\sum_{i=1}^{k} \alpha_{ji} V_i \right) = \sum_{i=1}^{k} \left(\sum_{j=1}^{k+l} c_j \alpha_{ji} \right) V_i = \sum_{i=1}^{k} (0) V_i = 0$$

$$\sum_{j=1}^{k+l} c_j W_j = 0 \quad \text{for non-zero } c_1, c_2, \ldots c_{k+l}$$

$W_1, W_2, \ldots W_{k+l}$ are LD (linearly dependent)

A Summary for the Proof

Proposition. *Every basis of a subspace $\mathcal{S} \subset \mathbb{R}^n$ has the same number of elements.*

Goal: given k LI vector $V_1, V_2, \ldots V_k$;

assume $W_1, W_2, \ldots W_{k+l}$ and represent each W_k using $V_1, V_2, \ldots V_k$

show that $W_1, W_2, \ldots W_{k+l}$ are LD

Review: Subset vs. Subspace

from Lecture #5

Definition. *A set S is* closed under addition *if the sum of any two elements of S is in S, and is* closed under scalar multiplication *if the product of an arbitrary scalar and an arbitrary element of S is in S.*

Frequently we consider a subset W of a vector space V. In this case, addition and scalar multiplication are already defined, and already satisfy the eight axioms. If W is closed under addition and scalar multiplication, then W is a vector space in its own right, and we call W a *subspace* of V.

subspace: closed under (1) addition and (2) scalar multiplication

Review: Closure (Wikipedia)

- A set is closed under an operation if performance of that operation on members of the set always produces a member of that set.
- For example, the positive integers are closed under addition, but not under subtraction: 1-2 is not a positive integer even though both 1 and 2 are positive integers.
- A set that is closed under an operation or collection of operations is said to satisfy a closure property.

Review: of Sect 5.1

from Lecture #5

given $V_1, \ldots, V_k \in \mathbb{R}^n$, the set

$$\mathcal{S} = \{\alpha_1 V_1 + \cdots + \alpha_k V_k \mid \alpha_j \in \mathbb{R}\}$$

is a subspace of $\mathbb{R}^n$. In this case we say that $\mathcal{S}$ is *spanned* by $V_1, \ldots, V_k$. Equivalently, it can be shown (see Exercise 12 at the end of this chapter) that a subspace $\mathcal{S}$ is a nonempty subset of $\mathbb{R}^n$ having the following two properties:

1. If $X, Y \in \mathcal{S}$, then $X + Y \in \mathcal{S}$;
2. If $X \in \mathcal{S}$ and $\alpha \in \mathbb{R}$, then $\alpha X \in \mathcal{S}$.

subspace: closed under (1) addition and (2) scalar multiplication

Review: Subset vs. Subspace

(1) Let $W = \{\mathbf{x} \in \mathbb{R}^2 | x_1 + x_2 = 0\}$. The sum of any two vectors in W is in W, and any scalar multiple of a vector in W is in W. (Check this!) W is a subspace of the vector space $\mathbb{R}^2$.

(2) More generally, let A be any fixed $m \times n$ matrix and let $W = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} = 0\}$. W is closed under addition and scalar multiplication (why?), so W is a subspace of $\mathbb{R}^n$.

(3) Let $\mathbb{R}[t]$ be the set of all real-valued polynomials in a fixed variable t. Polynomials are continuous functions on $[0, 1]$, so $\mathbb{R}[t]$ is a subset of $C^0[0, 1]$. The sum of polynomials is a polynomial, as is the product of a scalar and a polynomial, so $\mathbb{R}[t]$ is a subspace of $C^0[0, 1]$.

(4) Let $\mathbb{R}_n[t]$ be the set of real polynomials of degree n or less. For $n < m$, $\mathbb{R}_n[t]$ is a subspace of $\mathbb{R}_m[t]$, and $\mathbb{R}_n[t]$ is always a subspace of $\mathbb{R}[t]$.

(5) Instead of considering polynomials with real coefficients, we could consider polynomials with complex coefficients to get examples of complex vector spaces. The space of polynomials with complex coefficients is usually denoted $\mathbb{C}[t]$.

(6) Let $W' = \{\mathbf{x} \in \mathbb{R}^2 | x_1 + x_2 = 1\}$. W' is not a vector space, as the sum of two elements of W', or a scalar multiple of an element of W', is typically not in W'. (Again, check this!)

(7) Let $\mathbb{Z}^2 = \{\mathbf{x} \in \mathbb{R}^2 | x_1 \text{ and } x_2 \text{ are integers }\}$. $\mathbb{Z}^2$ is closed under addition, but not under scalar multiplication. $\mathbb{Z}^2$ is a sub*set* of $\mathbb{R}^2$, but not a sub*space*.

See the next slide

Review: Subset vs. Subspace (.continued)

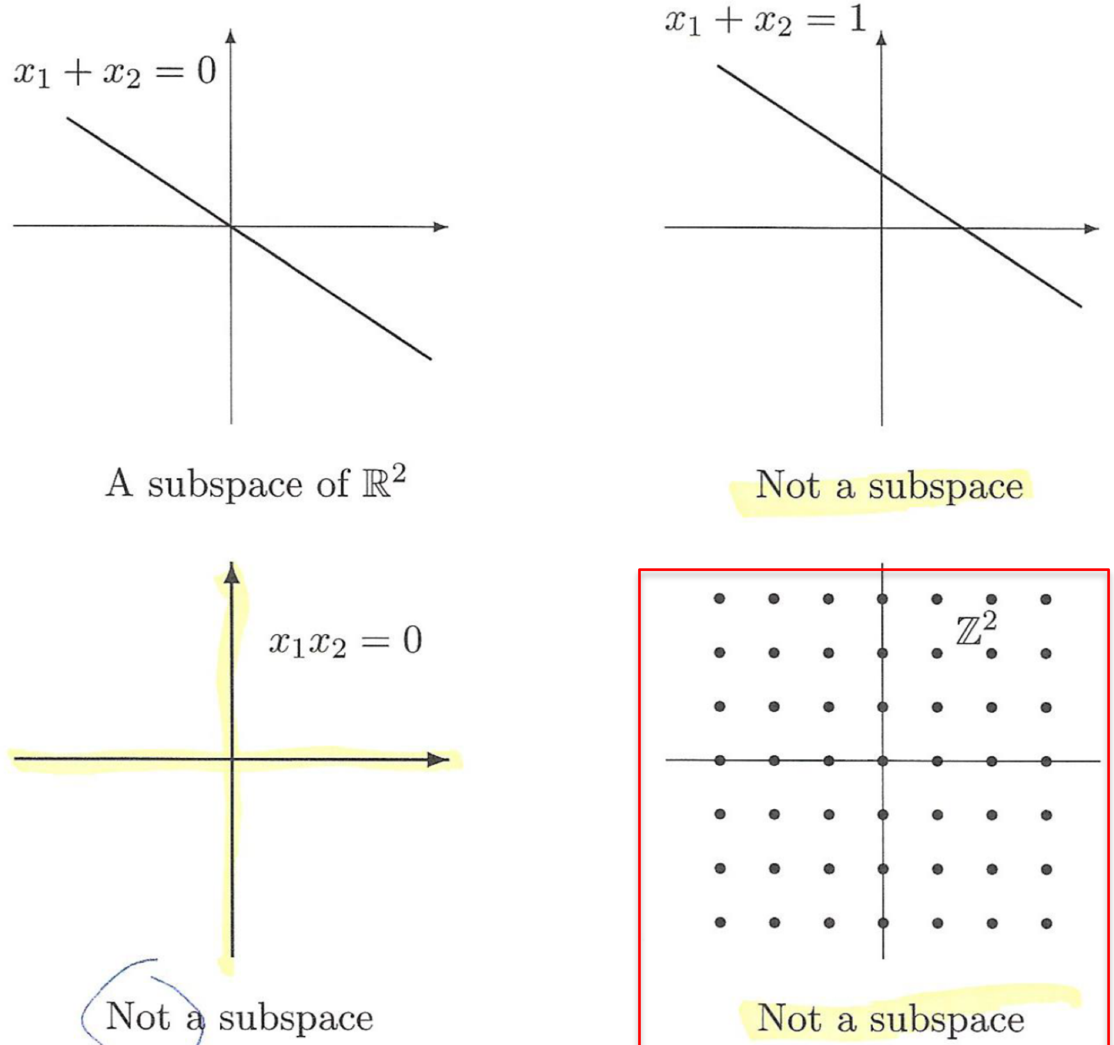

Figure 2.1. Four subsets of $\mathbb{R}^2$

Not closed under scalar multiplication

Review: Example 1: Subspace

Example. Any straight line through the origin in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$, since this line may be written as $\{tV \mid t \in \mathbb{R}\}$ for some nonzero $V \in \mathbb{R}^n$. The single vector V spans this subspace. The plane $\mathcal{P}$ defined by $x + y + z = 0$ in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

$$z = -x - y \qquad W = \begin{pmatrix} x \\ y \\ -(x+y) \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$W_1 \qquad W_2$$

$$|\vec{V}_1 \times \vec{V}_2| = |\vec{V}_1||\vec{V}_2| sin(\theta)$$

- If V_1 and V_2 are linearly dependent, $c_1V_1 + c_2V_2 = 0$ for non-zero $c_1$and c_2, yielding $V_1 = \alpha V_2$ with $\alpha \in R$ and, thus, $|\vec{V}_1 \times \vec{V}_2| = 0$.
- $|\vec{W}_1 \times \vec{W}_2| = |(1, \quad 1, \quad 1)| \neq 0, W_1$ and W_2 are linearly independent.
- Two LI vectors W_1 and W_2 span a subspace, R^2.

Review: Example 1: Subspace

Example. Any straight line through the origin in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$, since this line may be written as $\{tV \mid t \in \mathbb{R}\}$ for some nonzero $V \in \mathbb{R}^n$. The single vector V spans this subspace. The plane $\mathcal{P}$ defined by $x + y + z = 0$ in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

$$W = x\underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{W_1} + y\underbrace{\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}}_{W_2}$$

Chose $V_1 = x_1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + y_1\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

and $V_2 = x_2\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + y_2\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

Obtain $V_1 + V_2 = (x_1+x_2)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + (y_1+y_2)\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

$$\alpha V_1 = \alpha x_1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \alpha y_1\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

1. Is the set closed under addition?
2. Is the set closed under scalar multiplication?

- You have 2 minutes

Review: Example 1: Subspace

Example. Any straight line through the origin in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$, since this line may be written as $\{tV \mid t \in \mathbb{R}\}$ for some nonzero $V \in \mathbb{R}^n$. The single vector V spans this subspace. The plane $\mathcal{P}$ defined by $x + y + z = 0$ in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

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and $V_2 = x_2\begin{pmatrix}1\\0\\-1\end{pmatrix} + y_2\begin{pmatrix}0\\0\\-1\end{pmatrix}$

Obtain $V_1 + V_2 = (x_1+x_2)\begin{pmatrix}1\\0\\-1\end{pmatrix} + (y_1+y_2)\begin{pmatrix}0\\0\\-1\end{pmatrix}$ closed under (1) addition

$\alpha V_1 = \alpha x_1\begin{pmatrix}1\\0\\-1\end{pmatrix} + \alpha y_1\begin{pmatrix}0\\0\\-1\end{pmatrix}$ closed under (2) scalar multiplication

Example 2: Subspace?

Consider $x + y + z = 1$

$$z = 1 - x - y \qquad W = \begin{pmatrix} x \\ y \\ 1-(x+y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + x\underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{W_1} + y\underbrace{\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}}_{W_2}$$

Chose $V_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + x_1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + y_1\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

and $V_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + x_2\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + y_2\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

- Is the set closed under addition?
- You have 1 minute

Obtain $V_1 + V_2 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + (x_1+x_2)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + (y_1+y_2)\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

Example 2: Subspace? (No)

Consider $x+y+z=1$

$$z = 1 - x - y \qquad W = \begin{pmatrix} x \\ y \\ 1-(x+y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + x \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{W_1} + y \underbrace{\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}}_{W_2}$$

Chose $V_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + x_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + y_1 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

and $V_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + y_2 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

Obtain $V_1 + V_2 = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + (x_1+x_2) \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + (y_1+y_2) \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ not closed under (1) addition

Review: Subset vs. Subspace (.continued)

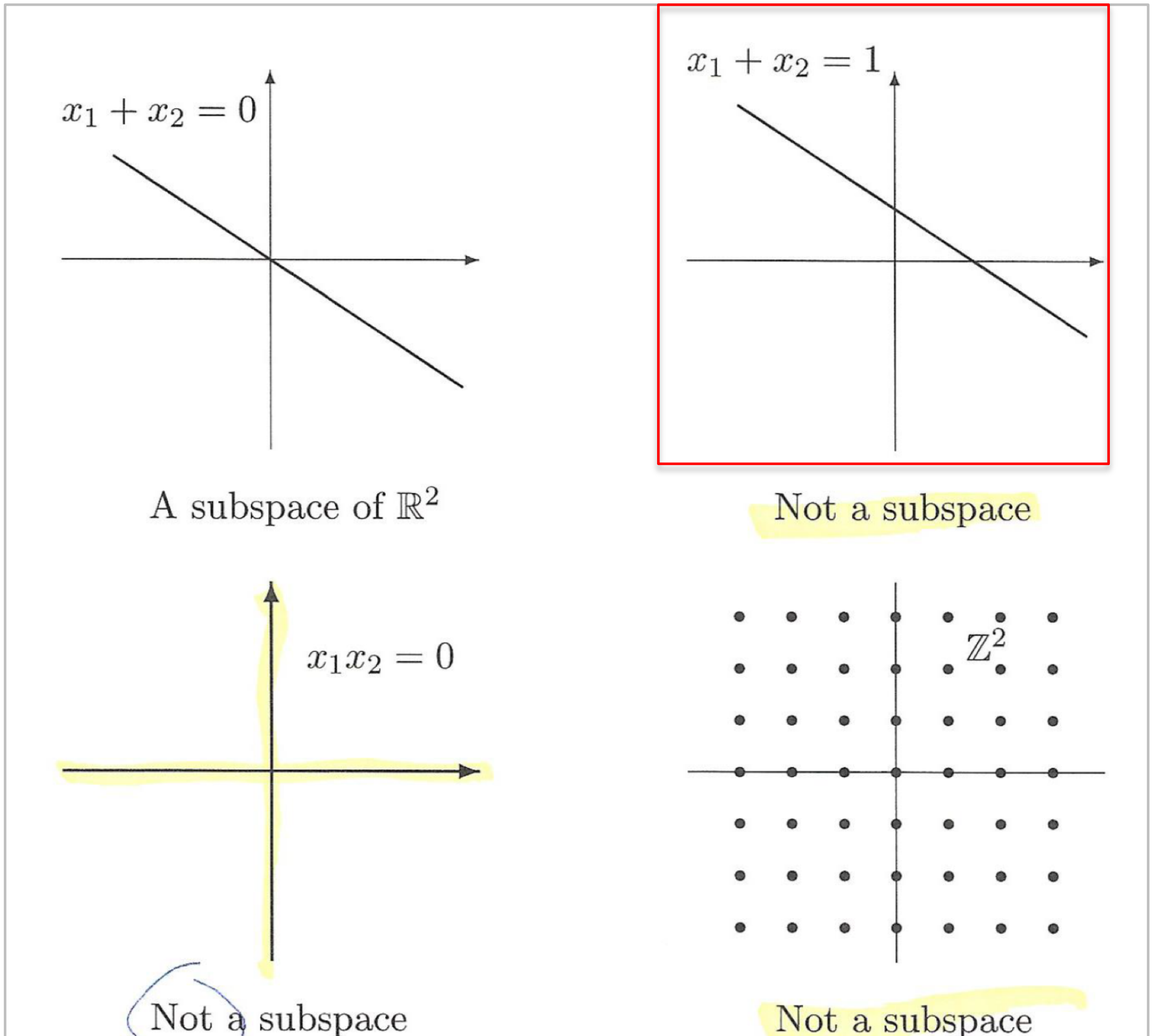

Figure 2.1. Four subsets of $\mathbb{R}^2$

Not closed under addition

Dimension, Ker, and Range

- We may define the dimension of a subspace S as the number of vectors that form any basis for S.
- In particular, R^n is a subspace of itself, and its dimension is clearly n.
- The set consisting of only the 0 vector is also a subspace, and we define its dimension to be zero.
- We write dim S for the dimension of the subspace S.
- We define the kernel of T, denoted Ker T, to be the set of vectors mapped to 0 by T (i.e., $TX = 0$)
- The range of T consists of all vectors W for which there exists a vector V for which $TV = W$.

  [This, of course, is a familiar concept from calculus. The difference here is that the range of T is always a subspace of R^n.]

Dimension, Ker, and Range

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Definition
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Proposition. *Every basis of a subspace $\mathcal{S} \subset \mathbb{R}^n$ has the same number of elements.*

Proposition. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Then* Ker T *and* Range T *are both subspaces of $\mathbb{R}^n$. Moreover,*

$$\dim \operatorname{Ker} T + \dim \operatorname{Range} T = n.$$

Notes for Ranges, $TV = W$

The most important types of subspaces are the kernels and ranges of linear maps.

- $Ker\ T$: the set of vectors mapped to $\vec{0}$ by T, i.e., $TX = 0$.
- $Range\ T$: (consists of) all vectors W for which there exists a vector for which, $TV = W$.
- Dim($Ker\ T$)+Dim($Range\ T$) = n.

- Dim($Range\ T$) = # of LI column vectors of the matrix T

- For example, if Dim($Range\ T$) = n, rank of (T) = n;
- Thus, $TX = 0$ has a unique solution,
- indicating Ker T = {0} and Dim (Ker T) = 0

Example 3: (a) Range with $TV = W$

Example. Consider the linear map

$$T(X) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} X.$$

If $V^T = (x, y, z)$

(A)

$\boxed{TV = W}$ $\quad W = TV = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z \\ 0 \end{pmatrix} = y\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Linearly Independent

- Vectors W are represented by E_1 and E_2.
- Range T is the set of vectors, W, in the form of $(\beta, \gamma, 0)$.
- Dim (range) = 2 = # of LI column vectors of the matrix T

Example 3: (a) Range with $TV = W$

(B)

$$T = \begin{array}{c} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{array}{ccc} U_1 & U_2 & U_3 \end{array} \end{array}$$

- $\det(T) = 0 \Rightarrow$ Matrix T is singular.
- Two of three column vectors, U_2 and U_3, are linearly independent.
- Dim (range) = # of linearly independence column vectors

Dim (range) = 2

Example 3: (b) Kernel with $TX = 0$

Example. Consider the linear map

(C)

$$T(X) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} X.$$

Kernel $\boxed{TX = 0}$

Let X = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\quad TX = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z \\ 0 \end{pmatrix}$

$\boxed{TX = 0}$ $\quad \begin{pmatrix} y \\ z \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad y = 0 \;\&\; z = 0$

Thus, we have X = $\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$ Dim (kernel) = 1

Example 4: Det, Kernel, Range

Example. Let

$$T(X) = AX = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} X.$$

$$Det \begin{pmatrix} \boxed{1 \quad 2 \quad 3} \\ 4 \quad 5 \quad 6 \\ 7 \quad 8 \quad 9 \end{pmatrix} = \; 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1(45 - 48) - 2(36 - 42) + 3(32 - 35)$$

$$= -3 + 12 - 9 = 0$$

$$det(A) = 0 \Rightarrow volume = 0, not\ linearly\ indepenent$$

Elementary Row Operations

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} X = 0$$

$$\begin{aligned} x + 2y + 3z &= 0 \\ 4x + 5y + 6z &= 0 \\ 7x + 8y + 9z &= 0 \end{aligned}$$

$$\begin{aligned} x + 2y + 3z &= 0 \quad (1) \\ x + \frac{5}{4} y + \frac{3}{2} z &= 0 \quad (2) \\ x + \frac{8}{7} y + \frac{9}{7} z &= 0 \quad (3) \end{aligned}$$

Eq (1)

Eq (1) – Eq. (2)

Eq (1) – Eq. (3)

$$\begin{aligned} x + 2y + 3z &= 0 \\ +\frac{3}{4} y + \frac{3}{2} z &= 0 \\ +\frac{6}{7} y + \frac{12}{7} z &= 0 \end{aligned}$$

$$\begin{aligned} x + 2y + 3z &= 0 \quad (5) \\ y + 2z &= 0 \quad (6) \\ y + 2z &= 0 \quad (7) \end{aligned}$$

Eq (1) – 2*Eq. (2)

Eq (7) – Eq. (6)

$$\begin{aligned} x - z &= 0 \\ y + 2z &= 0 \\ 0 &= 0 \end{aligned}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} X = 0$$

Example 4: Det, Kernel ($AX = 0$), Range

Example. Let

$$T(X) = AX = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} X.$$

$AX = 0 \Rightarrow$ Previously, we obtained $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} X = 0$

$$\begin{aligned} x - z &= 0 \\ y + 2z &= 0 \end{aligned} \qquad \begin{aligned} x &= z \\ y &= -2z \end{aligned}$$

$$\mathrm{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ -2z \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Dim (kernel) = 1

Example 4: Det, Kernel, Range ($TV = W$)

Example. Let

$$T(X) = AX = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} X.$$

$$TV = W \Rightarrow \quad A_1 V = W_1 \qquad A_1 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$ (simplified)

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} V = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \qquad V = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \begin{aligned} x - z &= u_1 \\ y + 2z &= u_2 \\ 0 &= u_3 \end{aligned}$$

Dim (range) = 2

Previously, we obtained

$$det(A) = 0 \Rightarrow volume = 0, not\ linearly\ indepenent$$

Now, consider the reduced matrix

$$\begin{array}{c} A_1 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \\ \quad\quad U_1 \quad U_2 \quad U_3 \end{array}$$

- U_1 and U_2 are LI
- $U_3 = (-1)U_1 + 2U_2$

# of LI column vectors = 2

Dim (range) = 2

Section 5.4: KerT and RangeT

Proposition. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Then* Ker T *and* Range T *are both subspaces of* $\mathbb{R}^n$. *Moreover,*

$$\dim \operatorname{Ker} T + \dim \operatorname{Range} T = n.$$

A proof

(A) Assume $Ker\ T = \{0\}$

(B) Assume $Ker\ T \neq \{0\}$

A Proof: $Ker\ T = \{0\}$

(A) Assume $Ker\ T = \{0\}$

Let $E_1, E_2, \dots E_n$ be the standard basis of R^n

Goal (1): we show that $TE_1, TE_2, \dots TE_n$ are LI.

$TX = 0$ &
$X \in Ker = \{0\}$
$\Rightarrow$
$X = 0$ is
a unique solution
$\Rightarrow$
$|T| \neq 0$

Assume that $TE_1, TE_2, \dots TE_n$ are not LI, we may find $\alpha_1, \alpha_2, \dots \alpha_n$, not all zero, such that

$$\sum_{j=1}^{n} \alpha_j (TE_j) = 0 \quad \Rightarrow T \sum_{j=1}^{n} \alpha_j (E_j) = 0 \quad \Rightarrow T\left(\sum_{j=1}^{n} \alpha_j E_j\right) = 0$$

$Let\ X = \sum_{j=1}^{n} \alpha_j (E_j)$ X is a solution of $TX = 0$

Since $Ker\ T = \{0\}$, $X = 0$ is a unique solution, $\sum_{j=1}^{n} \alpha_j (E_j) = X = 0$

Since $E_1, E_2, \dots E_n$ are LI, each $\alpha_j = 0$. This is a contradiction.

A Proof: $Ker\ T = \{0\}$ (conti.)

Previously, we showed that $TE_1, TE_2, \ldots TE_n$ are LI.

Given a vector $W \in R^n$, we have $W = \sum_{j=1}^{n} \beta_j (TE_j)$, **for some** $\beta_1, \beta_2, \ldots \beta_n$

$$W = T\sum_{j=1}^{n} \beta_j (E_j) = TV, \qquad V = \sum_{j=1}^{n} \beta_j (E_j) \qquad (n\ E_j\ are\ LI)$$

Since $W \in R^n$ and V exists such that $W = TV$, the above shows that $\mathrm{Range}\ \ T = R^n$.

Thus, both $Ker\ T$ and $\mathrm{Range}\ T$ are subspaces of R^n and we have $\mathrm{dime}\ Ker\ T = 0$ and $\mathrm{dime}\ Range\ T = n$.

A Proof: $Ker\ T \neq \{0\}$

(B) Assume $Ker\ T \neq \{0\}$

0 is not a unique solution for $Ker\ T$.

- We may find a nonzero vector $V_1 \in Ker\ T, TV_1 = 0$. Clearly, we have $T(\alpha V_1) = 0$ for any $\alpha \in R$, so all vectors of the form αV_1 lie in $Ker\ T$.
- If $Ker\ T$ has contain additional vectors, choose one and call it V_2, $TV_2 = 0$. Additionally, we have $T(\alpha_1 V_1 + \alpha_2 V_2) = \alpha_1 TV_1 + \alpha_2 TV_2 = 0$.
- We can continue the above to obtain a set of LI vectors that span $Ker\ T$, showing that $Ker\ T$ is a subspace.
- Now, assume $V_1, V_2, \ldots V_k$ form a basis of $Ker\ T$ where $0 < k < n$.
- Choose vectors $W_{k+1}, W_{k+2}, \ldots W_n$ so that $V_1, V_2, \ldots V_k$, $W_{k+1}, W_{k+2}, \ldots W_n$ for a basis of R^n.
- Let $Z_j = TW_j$ for each j. The vectors Z_j are LD if we had

$$\alpha_{k+1} Z_{k+1} + \alpha_{k+2} Z_{k+2} \ldots + \alpha_n Z_n = 0$$

A Proof: $Ker\ T \neq \{0\}$

(B) Assume $Ker\ T \neq \{0\}$ (.continued)

- Let $Z_j = TW_j$ for each j. Assume that the vectors Z_j are LD if we have

$$\alpha_{k+1} Z_{k+1} + \alpha_{k+2} Z_{k+2} \ldots + \alpha_n Z_n = 0, \text{ yielding}$$

$$\alpha_{k+1} TW_{k+1} + \alpha_{k+2} TW_{k+2} \ldots + \alpha_n TW_n = 0, \text{ yielding}$$

$$T\left(\alpha_{k+1} W_{k+1} + \alpha_{k+2} W_{k+2} \ldots + \alpha_n W_n\right) = 0$$

$$\left(\alpha_{k+1} W_{k+1} + \alpha_{k+2} W_{k+2} \ldots + \alpha_n W_n\right) \in Ker\ T$$

  - The above is impossible, since we cannot write any W_j as a linear combination of the V_j.
  - Thus, Dim (Ker T) + Dim (Range T) = n

Section 5.4: KerT and RangeT

Proposition. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Then* Ker T *and* Range T *are both subspaces of* $\mathbb{R}^n$. *Moreover,*

$$\dim \operatorname{Ker} T + \dim \operatorname{Range} T = n.$$

Corollary. *If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map with* $\dim \operatorname{Ker} T = 0$, *then T is invertible.* ■

Section 5.4: KerT and RangeT

- We remark that it is easy to find a set of vectors that spans Range *T; simply* take the set of vectors that comprise the columns of the matrix associated to *T.*

  ➔ *Dim (range T) = # of LI column vectors of T*

- *This works since the ith column vector of this matrix is the image of the* standard basis vector E_j *under T.*

$$T = [U_1, U_2, \ldots U_j \ \ldots, U_n], \qquad U_j = TE_j$$

- *In particular, if these column vectors are* linearly independent, then *T is invertible and* Ker *T = {0}; there is a unique solution to the* equation $T(X) = V$ *for every* $V \in R^n$.

Section 5.4: A Summary

A function $T: R^n \to R^n$ is linear,
if $T(X) = AX$ for some $n \times n$ matrix A.

- T is a linear map or linear transformation.
- $T(\alpha X + \beta Y) = \alpha T(X) + \beta T(Y)$ for any $\alpha, \beta \in R$ and $X, Y \in R^n$.
- T is invertible if the matrix A (associated with T) has an inverse.

The most important types of subspaces are the kernels and ranges of linear maps.

- $Ker\ T$: the set of vectors mapped to $\vec{0}$ by T, i.e., $TX = 0$.
- $Range\ T$: (consists of) all vectors W for which there exists a vector (V) for which, $TV = W$.
- Dim($Ker\ T$)+Dim($Range\ T$)=n.