MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #8

Chapter 3 Phase Portraits for Planar Systems

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Re-Review: Type (I) ODEs: $ax''+bx'+cx=0$

(A) $y = e^{rt}$

$$ar^2 + br + c = 0$$

Characteristic Equation

(B) $$x'' + \frac{b}{a}x' + \frac{c}{a}x = 0$$

let $$x' = y$$

obtain $$y' = -\frac{c}{a}x - \frac{b}{a}y$$

define $$X = \begin{pmatrix} x \\ y \end{pmatrix};\quad A = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}$$

$$X' = AX$$

assume $$X = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{\lambda t};$$

eigenvalue problem $$|A - \lambda I| = \begin{vmatrix} 0-\lambda & 1 \\ -\dfrac{c}{a} & -\dfrac{b}{a} - \lambda \end{vmatrix} = 0$$

$$a\lambda^2 + b\lambda + c = 0$$

Review: A Brief Summary for Type (I) ODEs

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$$ay'' + by' + cy = 0$$

where a, b, and c are constants and $a \neq 0$.

Summary of Cases I–III

what is the most essential part?

Case	Roots of (2)	Basis of (1)	General Solution of (1)
I	Distinct real λ_1, λ_2	$\boxed{e^{\lambda_1 x}, e^{\lambda_2 x}}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

Real Distinct Eigenvalues

Consider

$$x' = \lambda_1 x$$
$$y' = \lambda_2 y$$

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} \lambda_1 - \lambda & 0 \\ 0 & \lambda_2 - \lambda \end{pmatrix}$$

Let $|A - \lambda I| = 0 \Longrightarrow$ $(\lambda_1 - \lambda)(\lambda_2 - \lambda) = 0$ $\boxed{\lambda = \lambda_{1,2}}$

Real distinct eigenvalues include:

- $\lambda_1 < 0 < \lambda_2$ (different signs): saddle
- $\lambda_1 < \lambda_2 < 0$ (both are negative): sink
- $0 < \lambda_1 < \lambda_2$ (both are positive): source

Saddle ($\lambda_1 < 0 < \lambda_2$)

$$x' = \lambda_1 x$$
$$y' = \lambda_2 y$$

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Let $|A - \lambda I| = 0 \Longrightarrow$ $\boxed{\lambda = \lambda_{1,2}}$

$AX = \lambda X \Longrightarrow$

$$\lambda_1 x = \lambda \mathrm{x}$$
$$\lambda_2 y = \lambda y$$

Consider $\lambda = \lambda_1$

Obtain
$$\lambda_1 x = \lambda_1 \mathrm{x}$$
$$\lambda_2 y = \lambda_1 y$$

$$\boxed{\begin{matrix} x: any \\ y = 0 \end{matrix}}$$

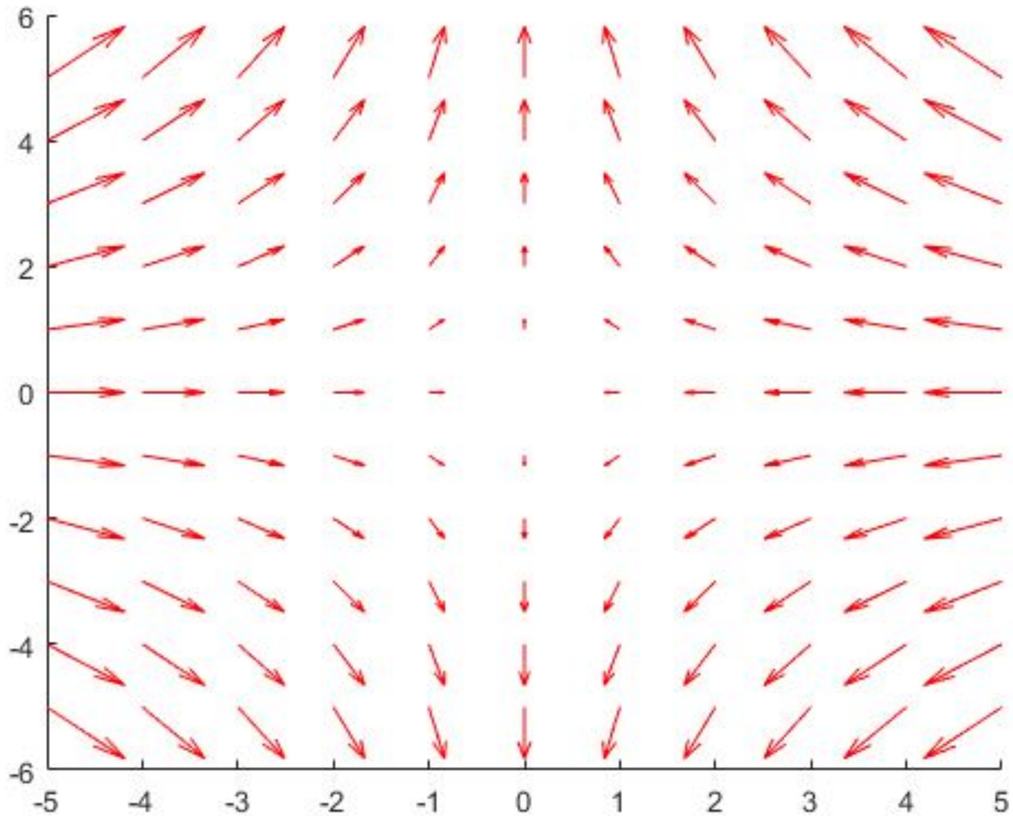
$$V_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Similarly, for $\lambda = \lambda_{2,}$ we have

$$V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus, a general solution is written as

$$X(t) = \alpha X_1 + \beta X_2 = \alpha e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



MATLAB Plot for Figure 3.1

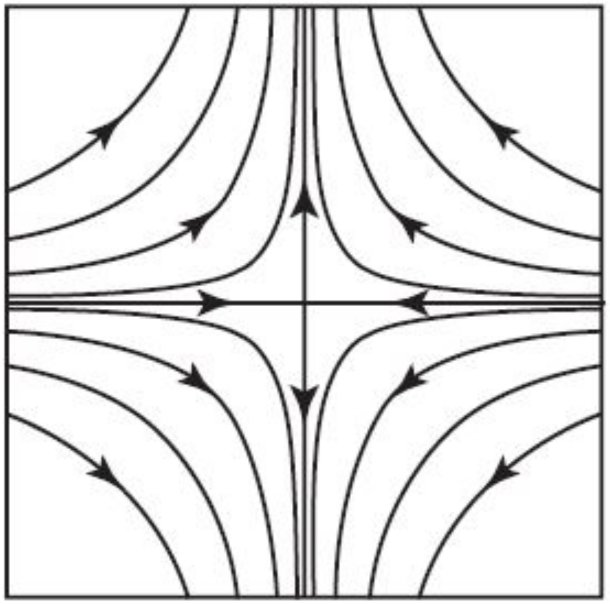


Figure 3.1 Saddle phase portrait for $x' = -x$, $y' = y$.

Section 3.1: Real Distinct Eigen Values

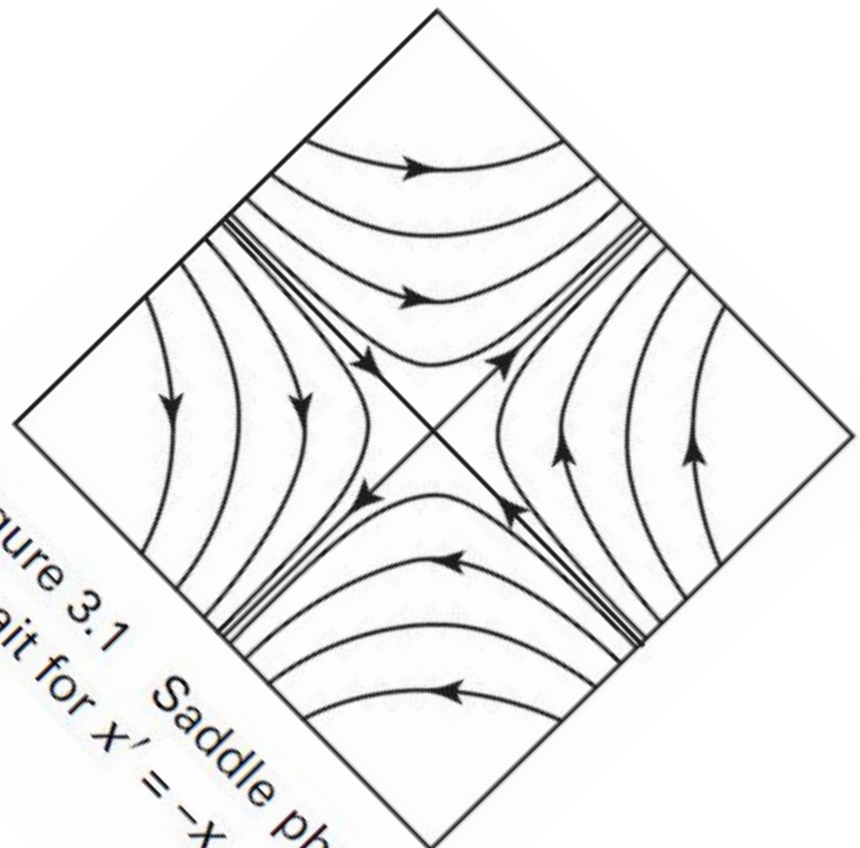


Figure 3.1 Saddle phase portrait for $x' = -x$, $y' = y$.

A saddle point

Examples: Solve $|A - \lambda I| = 0$

Example. We return to the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}.$$

define $\quad X = \begin{pmatrix} x \\ y \end{pmatrix}; \quad A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$

eigenvalue problem $\quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{vmatrix} = 0$

$$\lambda^2 - 4 = 0$$

$$\lambda = 2, -2$$

Examples: Solve $|A - \lambda I| = 0$

Solve for eigenvectors $AV_0 = \lambda V_0$

$$x_0 + 3y_0 = \lambda x_0$$
$$x_0 - y_0 = \lambda y_0$$

Consider $\lambda = 2$

$$\boxed{\begin{aligned} x_0 + 3y_0 &= 2x_0 \\ x_0 - y_0 &= 2y_0 \end{aligned}} \qquad \boxed{x_0 = 3y_0} \qquad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 3y_0 \\ y_0 \end{pmatrix} = y_0 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Obtain $\boxed{V_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}}$ as an eigenvector associated with $\lambda = 2$

Similarly, $\boxed{V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$ as an eigenvector associated with $\lambda = -2$

$$\boxed{X = \alpha X_1 + \beta X_2 = \alpha e^{\lambda_1 t} V_1 + \beta e^{\lambda_2 t} V_2}$$

$$\boxed{X = \alpha e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \beta e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

Saddle ($\lambda_1 < 0 < \lambda_2$)

$$X = \alpha X_1 + \beta X_2 = \alpha e^{2t}\begin{pmatrix}3\\1\end{pmatrix} + \beta e^{-2t}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$X \sim \alpha e^{2t}\begin{pmatrix}3\\1\end{pmatrix} = X_1(t) \ \ as\ t \to \infty\ \&\ \alpha \neq 0$$

$$X \sim \beta e^{-2t}\begin{pmatrix}1\\-1\end{pmatrix} = X_2(t)\ as\ t \to -\infty\ \&\ \beta \neq 0$$

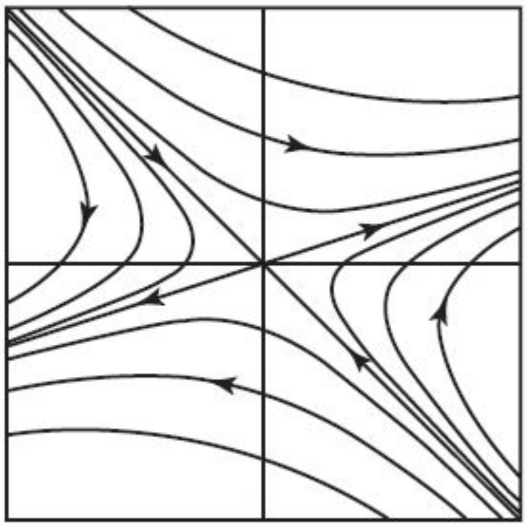
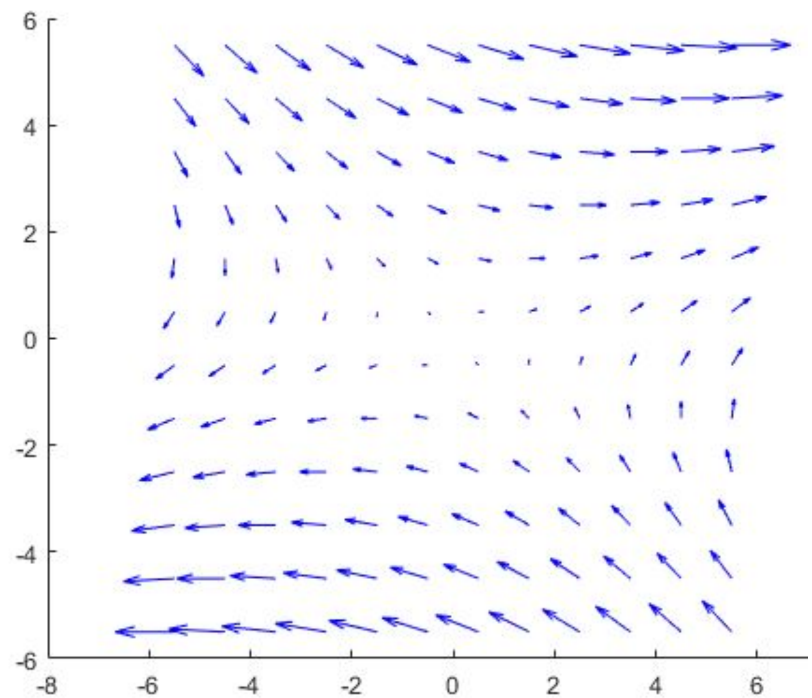


Figure 3.2 Saddle phase portrait for $x' = x + 3y$, $y' = x - y$.



MATLAB

Trajectory, Orbit, and Path: Slope

$$x' = ax + by \quad (= P(x, y)) \qquad (1)$$

$$y' = cx + dy \quad (= Q(x, y)\ ) \qquad (2)$$

From (1-2) we see that the slope of a path passing through a point A: (X,Y) is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{Q(x, y)}{P(x, y)} \qquad (3)$$

- Note that (3) gives no information about the orientation of a path.
- Note further that we must have $P(x, y) \neq 0$ at A.
- If $P(x, y) = 0$ but $\mathrm{Q}(x, y) \neq 0$ at A, we can take $dx/dy = P(x, y)/Q(x, y)$ instead of (3) and conclude from $\frac{dx}{dy} = 0$ that the tangent of C at A is vertical.
- However, what can we do if both P and Q are zero at some point?

Sink ($\lambda_1 < \lambda_2 < 0$): move toward (0,0)

$$x' = \lambda_1 x \qquad A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$
$$y' = \lambda_2 y$$

Let $|A - \lambda I| = 0 \Longrightarrow$ $\boxed{\lambda = \lambda_{1,2}}$ $V_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Thus, a general solution is written as

$$X(t) = \alpha e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$x = \alpha e^{\lambda_1 t}$$

$$y = \beta e^{\lambda_2 t}$$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\lambda_2 \beta e^{\lambda_2 t}}{\lambda_1 \alpha e^{\lambda_1 t}} = \frac{\lambda_2 \beta}{\lambda_1 \alpha} e^{(\lambda_2 - \lambda_1)t} \qquad \to \infty \ as \ t \to \infty$$

- These solutions tend to the origin (a sink) tangentially to the y axis.
- x tends to zero much quickly
- λ_1 (λ_2) is referred to as the stronger (weaker) eigenvalue.

Sink ($\lambda_1 < \lambda_2 < 0$)

$\lambda_1 < \lambda_2 < 0$

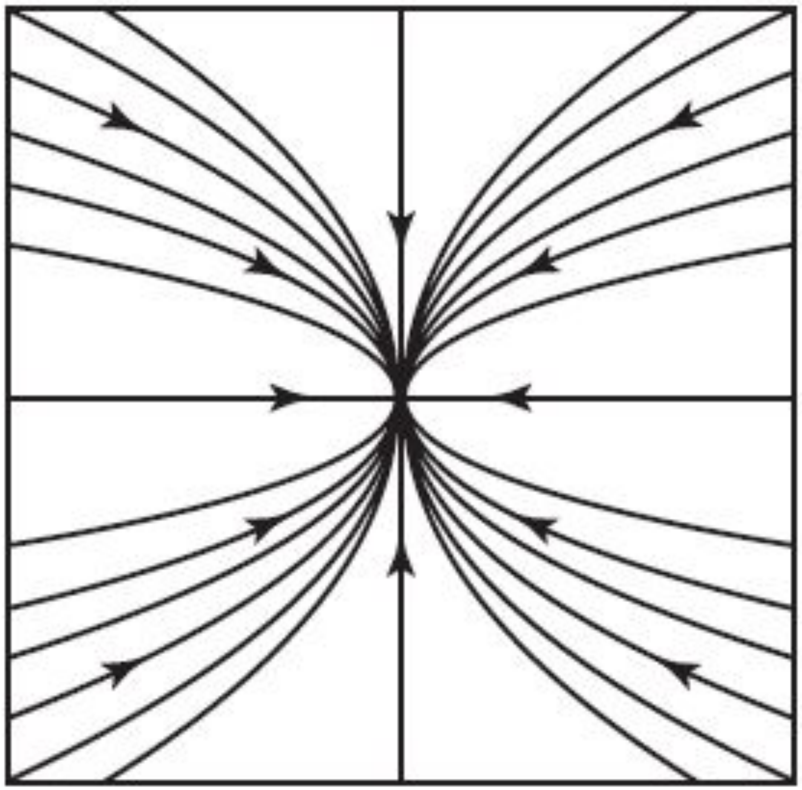
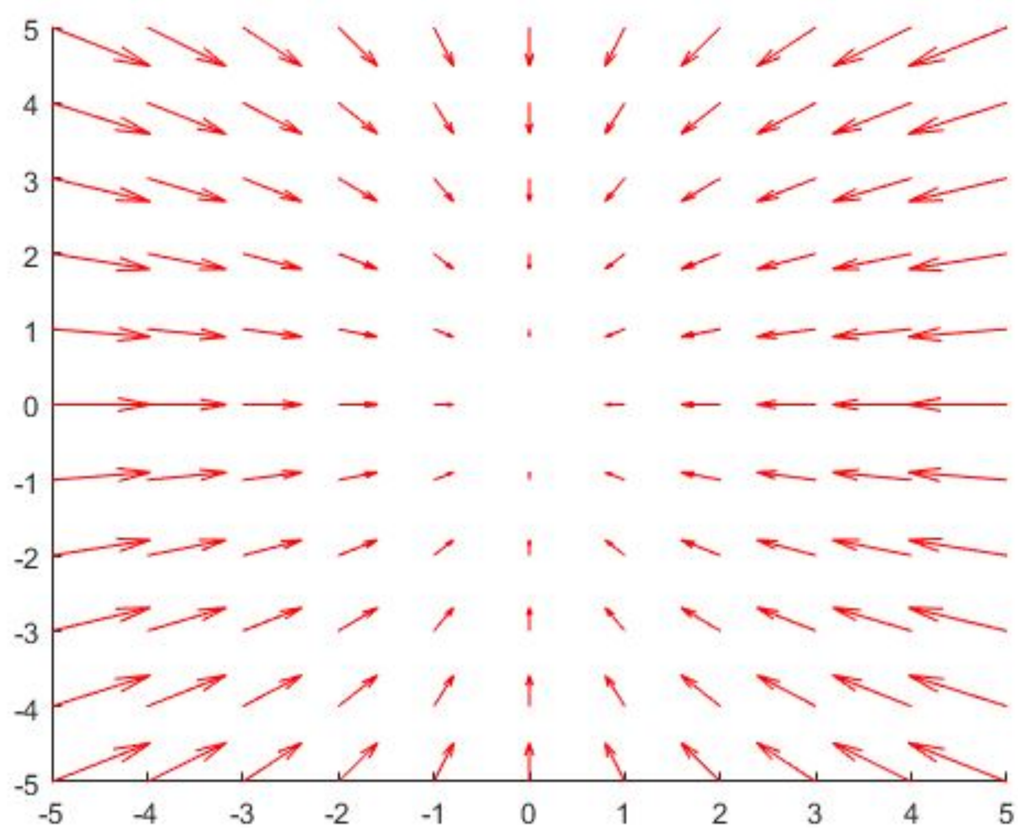


Figure 3.3 Phase portraits for (a) a sink

$X' = -2x \qquad Y' = -y$



MATLAB Plot for Figure 3.3a

Sink ($\lambda_1 < \lambda_2 < 0$): a general case

Thus, a general solution is written as

$$X(t) = \binom{x}{y} = \alpha e^{\lambda_1 t}\binom{u_1}{u_2} + \beta e^{\lambda_2 t}\binom{v_1}{v_2}$$

$$x = \alpha e^{\lambda_1 t}u_1 + \beta e^{\lambda_2 t}v_1$$

$$y = \alpha e^{\lambda_1 t}u_2 + \beta e^{\lambda_2 t}v_2$$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\lambda_1 \alpha e^{\lambda_1 t}u_1 + \lambda_2 \beta e^{\lambda_2 t}v_1}{\lambda_1 \alpha e^{\lambda_1 t}u_2 + \lambda_2 \beta e^{\lambda_2 t}v_2} = \frac{\lambda_1 \alpha e^{(\lambda_1-\lambda_2)t}u_1 + \lambda_2 \beta v_1}{\lambda_1 \alpha e^{(\lambda_1-\lambda_2)t}u_2 + \lambda_2 \beta v_2} \quad \rightarrow \frac{v_2}{v_1} \quad as\ t \rightarrow \infty$$

- All solutions (except for those on the straight line corresponding the stronger eigenvalue) tend to the origin (a sink) tangentially to the straight-line solution corresponding to the weaker eigenvalue.
- λ_1 (λ_2) is referred to as the stronger (weaker) eigenvalue.

Source $(0 < \lambda_1 < \lambda_2)$

$$x' = \lambda_1 x$$
$$y' = \lambda_2 y$$

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Let $|A - \lambda I| = 0 \Longrightarrow$ $\boxed{\lambda = \lambda_{1,2}}$

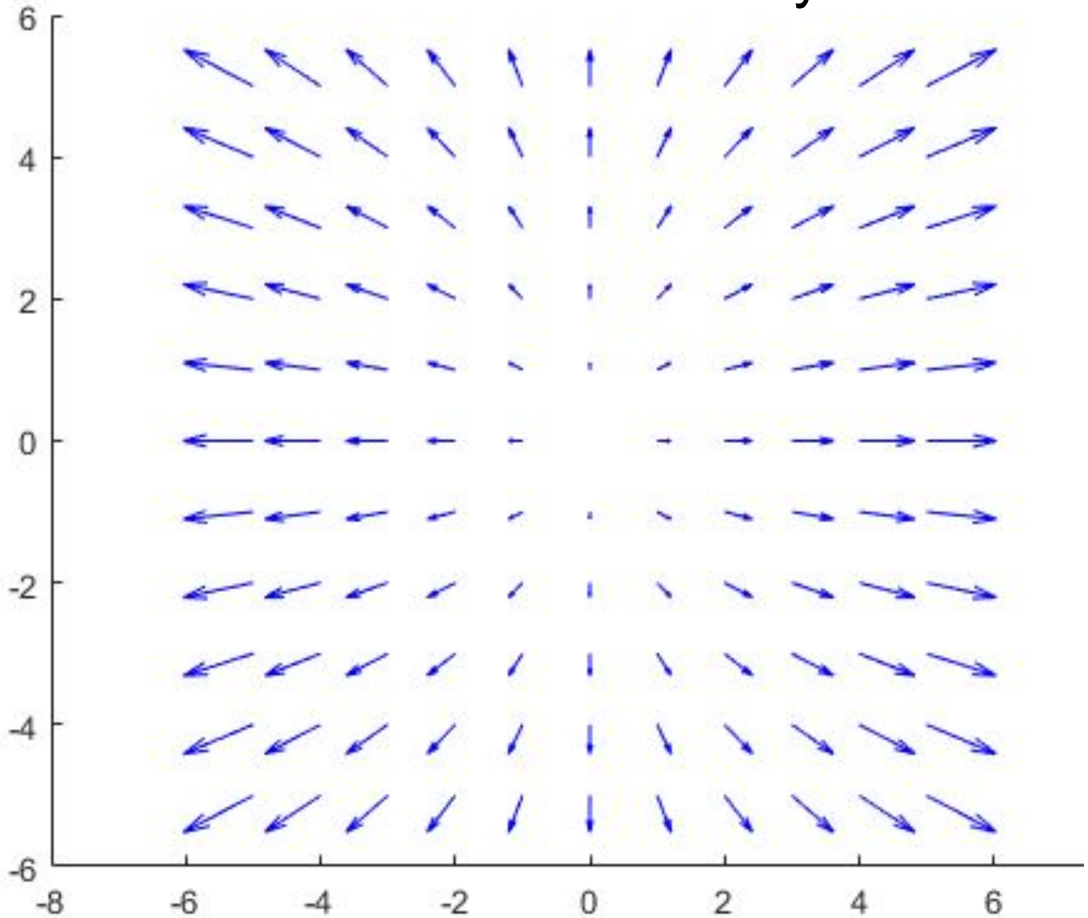
$$V_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus, a general solution is written as

$$X(t) = \alpha X_1 + \beta X_2 = \alpha e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Source ($0 < \lambda_1 < \lambda_2$)

$X' = 2x$ $\quad$ $Y' = y$



MATLAB Plot for Figure 3.3b

$\lambda_1>0; \quad \lambda_2>0$

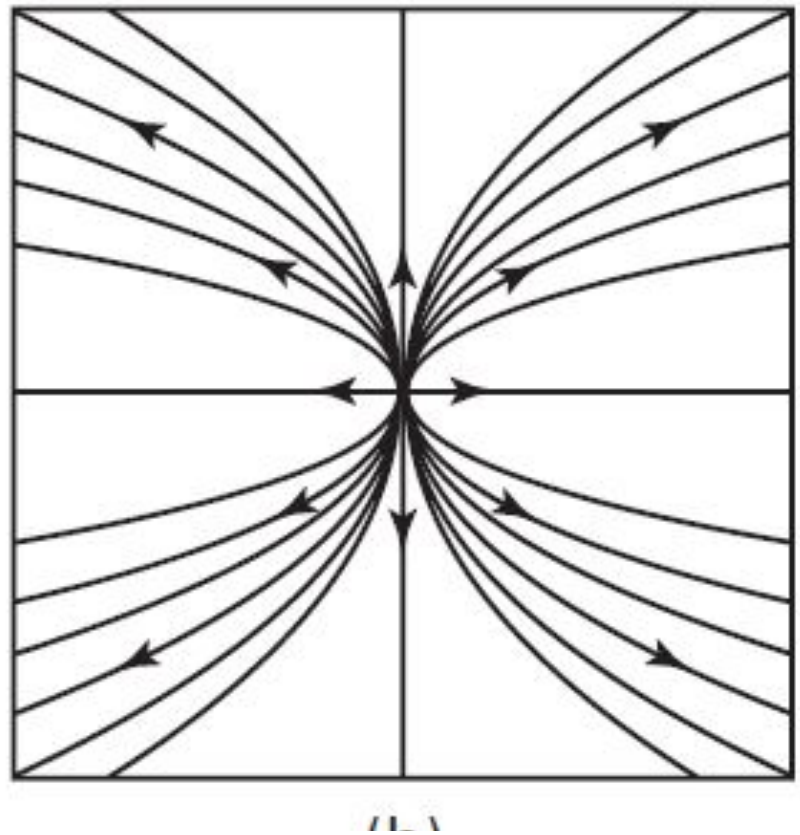


Figure 3.3 Phase portraits for (b) a source.