Homework 5
Partial Differential Equations
Math 531
Stephen Giang RedID: 823184070

Problem 1: Consider the function $f(x) = x^2$.Use MatLab to create the computer graphics to show the following:

- In all graphs include the original function for $x \in [-4, 4]$. (Don't extend to the full interval.)
- Find the Fourier sine series, including the Fourier coefficients, for $f(x)$ for $x \in [0, 3]$

Notice the following Fourier sine series:

$$f(x) = x^2 = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{3}\right)$$

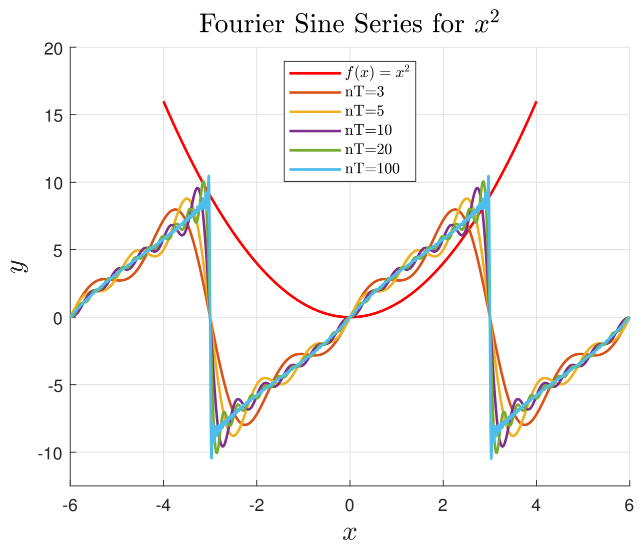
Notice the following:

$$\begin{aligned}
\int x^2 \sin\left(\frac{n\pi x}{3}\right) dx &= \frac{-3x^2}{n\pi} \cos\left(\frac{n\pi x}{3}\right) + \frac{6}{n\pi} \int x \cos\left(\frac{n\pi x}{3}\right) dx \\
&= \frac{-3x^2}{n\pi} \cos\left(\frac{n\pi x}{3}\right) + \frac{18x}{(n\pi)^2} \sin\left(\frac{n\pi x}{3}\right) - \frac{18}{(n\pi)^2} \int \sin\left(\frac{n\pi x}{3}\right) dx \\
&= \frac{-3x^2}{n\pi} \cos\left(\frac{n\pi x}{3}\right) + \frac{18x}{(n\pi)^2} \sin\left(\frac{n\pi x}{3}\right) + \frac{54}{(n\pi)^3} \cos\left(\frac{n\pi x}{3}\right)
\end{aligned}$$

Notice the Fourier coefficient:

$$\begin{aligned}
B_n &= \frac{2}{3} \int_0^3 x^2 \sin\left(\frac{n\pi x}{3}\right) dx \\
&= \frac{2}{3} \left(\frac{-3x^2}{n\pi} \cos\left(\frac{n\pi x}{3}\right) + \frac{18x}{(n\pi)^2} \sin\left(\frac{n\pi x}{3}\right) + \frac{54}{(n\pi)^3} \cos\left(\frac{n\pi x}{3}\right) \right) \Bigg|_0^3 \\
&= \frac{-18}{n\pi} \cos(n\pi) + \frac{36}{(n\pi)^3} \cos(n\pi) - \frac{36}{(n\pi)^3}
\end{aligned}$$

- Graph the original function and the Fourier sine series, where you use 3, 5, 10, 20, and 100 terms. Show the graph for $x \in [-6, 6]$.




```matlab
close all; clc; clear;
figure();hold on; grid on;

x = linspace(-4,4,2000);
g = x.^2;
plot(x,g,'r-','LineWidth',1.5)

numTerms = [3,5,10,20,100];
x = linspace(-6,6,2000);
for i = 1 : size(numTerms,2)
  plot(x,diffFTerms(numTerms(i), x),'LineWidth',1.5);
end

xlabel('$x$','FontSize',16,'interpreter','latex');
ylabel('$y$','FontSize',16, 'interpreter','latex');
title('Fourier Sine Series for $x^2$','FontSize',16, 'interpreter','latex');
legend('$f(x) = x^2$','nT=3', 'nT=5', 'nT=10', 'nT=20', 'nT=100', 'interpreter','latex', 'location', 'north' );
xlim([-6,6]);
ylim([-12.5,20]);

print -depsc Prob1Sine.eps

function f = diffFTerms(Nf, x)
b = zeros(1,Nf);
f = 0;
  for n = 1 : Nf
    npi = n * pi;
    b(n) = ( (-18/npi)*cos(npi) ) + ( (36/(npi^3))*cos(npi) ) + ( 36/(npi^3) ) ;
    fn = b(n)*sin((npi*x)/3);
    f=f+fn;
  end
end
```

- Find the Fourier cosine series, including the Fourier coefficients, for f(x) for $x \in [0, 3]$.

  Notice the following Fourier cosine series:

$$f(x) = x^2 = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{3}\right)$$

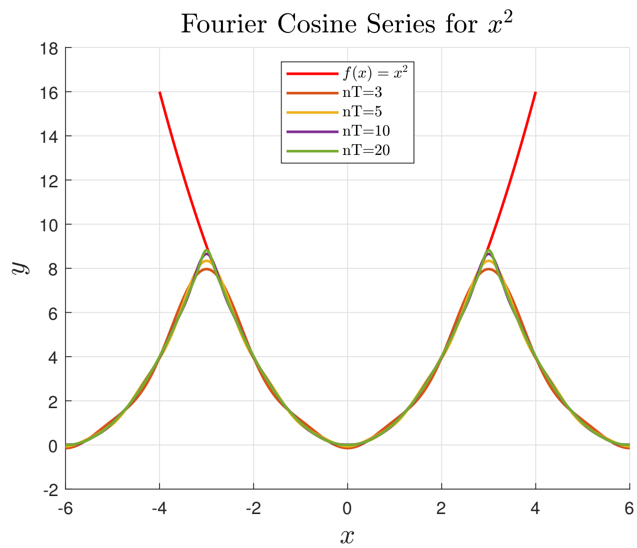
  Notice the following:

$$\begin{aligned}
\int x^2 \cos\left(\frac{n\pi x}{3}\right)\, dx &= \frac{3x^2}{n\pi} \sin\left(\frac{n\pi x}{3}\right) - \frac{6}{n\pi} \int x \sin\left(\frac{n\pi x}{3}\right)\, dx \\
&= \frac{3x^2}{n\pi} \sin\left(\frac{n\pi x}{3}\right) + \frac{18x}{(n\pi)^2} \cos\left(\frac{n\pi x}{3}\right) - \frac{18}{(n\pi)^2} \int \cos\left(\frac{n\pi x}{3}\right)\, dx \\
&= \frac{3x^2}{n\pi} \sin\left(\frac{n\pi x}{3}\right) + \frac{18x}{(n\pi)^2} \cos\left(\frac{n\pi x}{3}\right) - \frac{54}{(n\pi)^3} \sin\left(\frac{n\pi x}{3}\right)
\end{aligned}$$

  Notice the Fourier coefficients:

$$\begin{aligned}
A_0 &= \frac{1}{3} \int_0^3 x^2\, dx = \frac{x^3}{9}\bigg|_0^3 = 3 \\
A_n &= \frac{2}{3} \int_0^3 x^2 \cos\left(\frac{n\pi x}{3}\right)\, dx \\
&= \frac{2}{3} \left(\frac{3x^2}{n\pi} \sin\left(\frac{n\pi x}{3}\right) + \frac{18x}{(n\pi)^2} \cos\left(\frac{n\pi x}{3}\right) - \frac{54}{(n\pi)^3} \sin\left(\frac{n\pi x}{3}\right) \right)\bigg|_0^3 \\
&= \frac{36}{(n\pi)^2} \cos(n\pi)
\end{aligned}$$

- Graph the original function and the Fourier cosine series, where you use 3, 5, 10, and 20 terms. Show the graph for $x \in [-6, 6]$.




```
close all; clc; clear;
figure();hold on; grid on;

x = linspace(-4,4,2000);
g = x.^2;
plot(x,g,'r-','LineWidth',1.5)

numTerms = [3,5,10,20];
x = linspace(-6,6,2000);
for i = 1 : size(numTerms,2)
  plot(x,diffFTerms(numTerms(i), x),'LineWidth',1.5);
end

xlabel('$x$','FontSize',16,'interpreter','latex');
ylabel('$y$','FontSize',16, 'interpreter','latex');
title('Fourier Cosine Series for $x^2$','FontSize',16, 'interpreter','latex');
legend('$f(x) = x^2$','nT=3', 'nT=5', 'nT=10', 'nT=20', 'nT=100', 'interpreter','latex', 'location', 'north' );
xlim([-6,6]);
ylim([-2,18]);

print -depsc Prob1Cosine.eps

function f = diffFTerms(Nf, x)
a0 = 3;
a = zeros(1,Nf);
f = a0;
  for n = 1 : Nf
    npi = n * pi;
    a(n) = (36 / (npi^2))*cos(npi);
    fn = a(n)*cos((npi*x)/3);
    f=f+fn;
  end
end

```

Problem 2: Exercise 3.3.1c:

For the following functions, sketch $f(x)$, the Fourier series of $f(x)$, the Fourier sine series of $f(x)$, and the Fourier cosine series of $f(x)$:

$$f(x) = \begin{cases} x & x < 0 \\ 1+x & x > 0 \end{cases}$$

Notice the Fourier Series:

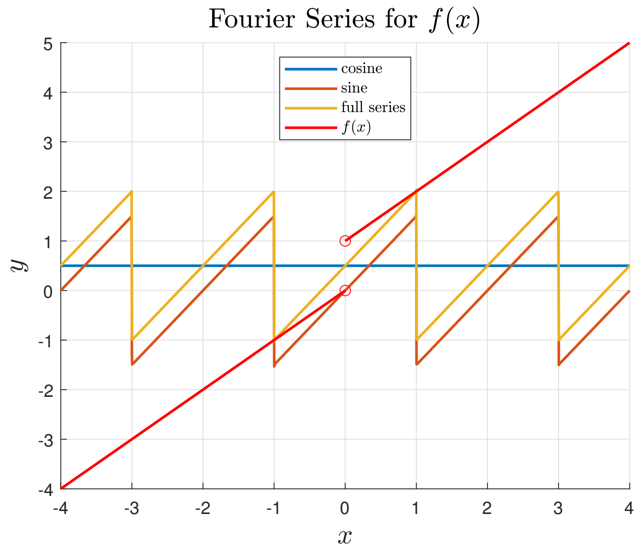
$$\begin{aligned}
f(x) &= A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \\
A_0 &= \frac{1}{2L}\int_{-L}^{0} f(x)\,dx + \frac{1}{2L}\int_{0}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{0} x\,dx + \frac{1}{2L}\int_{0}^{L}(1+x)\,dx \\
&= \frac{x^2}{4L}\bigg|_{-L}^{0} + \left(\frac{x}{2L} + \frac{x^2}{4L}\right)\bigg|_{0}^{L} = \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
A_n &= \frac{1}{L}\int_{-L}^{0} f(x)\cos\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^{0} x\cos\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{0}^{L}(1+x)\cos\frac{n\pi x}{L}\,dx \\
&= \frac{1}{L}\left(\frac{xL}{n\pi}\sin\frac{n\pi x}{L} + \left(\frac{L}{n\pi}\right)^2\cos\frac{n\pi x}{L}\right)\bigg|_{-L}^{0} + \frac{1}{L}\left(\frac{(1+x)L}{n\pi}\sin\frac{n\pi x}{L} + \left(\frac{L}{n\pi}\right)^2\cos\frac{n\pi x}{L}\right)\bigg|_{0}^{L} \\
&= \frac{x}{n\pi}\sin\frac{n\pi x}{L} + \left(\frac{L}{(n\pi)^2}\right)\cos\frac{n\pi x}{L}\bigg|_{-L}^{0} + \frac{(1+x)}{n\pi}\sin\frac{n\pi x}{L} + \left(\frac{L}{(n\pi)^2}\right)\cos\frac{n\pi x}{L}\bigg|_{0}^{L} \\
&= \frac{L}{(n\pi)^2} - \frac{(-1)^n L}{(n\pi)^2} + \frac{(-1)^n L}{(n\pi)^2} - \frac{L}{(n\pi)^2} = 0
\end{aligned}$$

$$\begin{aligned}
B_n &= \frac{1}{L}\int_{-L}^{0} f(x)\sin\frac{n\pi x}{L}\,dx + \frac{1}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{-L}^{0} x\sin\frac{n\pi x}{L} + \frac{1}{L}\int_{0}^{L}(1+x)\sin\frac{n\pi x}{L} \\
&= \frac{1}{L}\left(\frac{-xL}{n\pi}\cos\frac{n\pi x}{L} + \left(\frac{L}{n\pi}\right)^2\sin\frac{n\pi x}{L}\right)\bigg|_{-L}^{0} + \frac{1}{L}\left(\frac{-(1+x)L}{n\pi}\cos\frac{n\pi x}{L} + \left(\frac{L}{n\pi}\right)^2\sin\frac{n\pi x}{L}\right)\bigg|_{0}^{L} \\
&= \frac{-x}{n\pi}\cos\frac{n\pi x}{L} + \left(\frac{L}{(n\pi)^2}\right)\sin\frac{n\pi x}{L}\bigg|_{-L}^{0} + \frac{-(1+x)}{n\pi}\cos\frac{n\pi x}{L} + \left(\frac{L}{(n\pi)^2}\right)\sin\frac{n\pi x}{L}\bigg|_{0}^{L} \\
&= \frac{-L(-1)^n}{n\pi} + \frac{-(1+L)(-1)^n}{n\pi} = \frac{-(1+2L)(-1)^n}{n\pi}
\end{aligned}$$

Notice the Fourier sine and cosine series of $f(x)$ respectively:

$$f(x) \sim \sum_{n=1}^{\infty} \frac{-2L(-1)^n}{(n\pi)^2}\sin\frac{n\pi x}{L} \qquad f(x) \sim 1 + \frac{L}{2} + \sum_{n=1}^{\infty} \frac{2L((-1)^n - 1)}{(n\pi)^2}\cos\frac{n\pi x}{L}$$




```
close all; clc; clear;
figure();hold on; grid on;

L = 1;

x = linspace(-4,4,2000);
for i = 1 : 3
  plot(x,diffFTerms(x, L, i),'LineWidth',1.5);
end

x = linspace(-4,0,2000);
g = x;
plot(x,g,'r-','LineWidth',1.5)
plot(0,0,'ro')
x = linspace(0,4,2000);
g = 1+x;
plot(x,g,'r-','LineWidth',1.5)
plot(0,1,'ro')

xlabel('$x$','FontSize',16,'interpreter','latex');
ylabel('$y$','FontSize',16, 'interpreter','latex');
title('Fourier Series for $f(x)$','FontSize',16, 'interpreter','latex');
legend('cosine', 'sine', 'full series', '$f(x)$', 'interpreter','latex', 'location
    ', 'north' );

print -depsc Prob2.eps

function f = diffFTerms(x, L, num)
  Nf = 1000;
  a0 = (1/2);
  a = zeros(1,Nf);
  b = zeros(1,Nf);
  if (num == 2), f = 0;
  else, f = a0;
  end
  for n = 1 : Nf
    npi = n * pi;
    a(n) = 0;
```

```
    b(n) = (-(1 + (2*L))*((-1)^n)) / (npi);
    if (num == 1), fn = a(n)*cos((npi*x)/L);
    elseif (num == 2), fn = b(n)*sin((npi*x)/L);
    elseif (num == 3), fn = a(n)*cos((npi*x)/L) + b(n)*sin((npi*x)/L);
    end
    f=f+fn;
  end
end

```

Problem 3: Exercise 3.3.14:

(a) Consider a function $f(x)$ that is even around $x = L/2$. Show that the odd coefficients (n odd) of the Fourier cosine series of $f(x)$ on $0 \le x \le L$ are zero.

Notice the Fourier cosine series of $f(x)$:

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

We disregard A_0 because we are focusing on odd values of n.

Notice the following:

$$\cos\frac{n\pi(L/2)}{L} = \cos\frac{n\pi}{2} = 0 \text{ for } n = 2k+1 \quad k = 1, 3, 5...$$

Notice the following:

$$\begin{aligned}
\cos\left(\frac{n\pi(\frac{L}{2}+x)}{L}\right) &= \cos\left(\frac{n\pi(\frac{L}{2})}{L} + \frac{n\pi x}{L}\right) \\
&= \cos\left(\frac{n\pi(\frac{L}{2})}{L}\right)\cos\left(\frac{n\pi x}{L}\right) - \sin\left(\frac{n\pi(\frac{L}{2})}{L}\right)\sin\left(\frac{n\pi x}{L}\right) \\
&= -\sin\left(\frac{n\pi(\frac{L}{2})}{L}\right)\sin\left(\frac{n\pi x}{L}\right) \\
&= -\left(\cos\left(\frac{n\pi(\frac{L}{2})}{L}\right)\cos\left(\frac{n\pi x}{L}\right) + \sin\left(\frac{n\pi(\frac{L}{2})}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\right) \\
&= -\cos\left(\frac{n\pi(\frac{L}{2})}{L} - \frac{n\pi x}{L}\right) \\
&= -\cos\left(\frac{n\pi(\frac{L}{2}-x)}{L}\right)
\end{aligned}$$

So we get that $\cos\frac{n\pi x}{L}$ is odd about $L/2$ with n being odd as well:

Now because $f(x)$ is even about $x = L/2$ and $\cos\frac{n\pi x}{L}$ is odd about $x = L/2$, we get that their product is odd about $x = L/2$, this means we get the following:

$$\frac{2}{L}\int_0^{L/2} f(x)\cos\frac{n\pi x}{L}\,dx = -\frac{2}{L}\int_{L/2}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

Thus, we get that for n being odd, the coefficients A_n is zero:

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = 0$$

(b) Explain the result of part (a) by considering a Fourier cosine series of $f(x)$ on the interval $0 \leq x \leq L/2$.

Notice the Fourier cosine series of $f(x)$:

$$A_n = \frac{2}{L/2} \int_0^{L/2} f(x) \cos \frac{n\pi x}{L/2} \, dx$$

We disregard A_0 because we are focusing on odd values of n.

Notice the following:

$$\cos \frac{n\pi(L/4)}{L/2} = \cos \frac{n\pi}{2} = 0 \text{ for } n = 2k+1 \quad k = 1, 3, 5...$$

Notice the following:

$$\begin{aligned}
\cos\left(\frac{n\pi(\frac{L}{4}+x)}{L}\right) &= \cos\left(\frac{n\pi(\frac{L}{4})}{L} + \frac{n\pi x}{L}\right) \\
&= \cos\left(\frac{n\pi(\frac{L}{4})}{L}\right) \cos\left(\frac{n\pi x}{L}\right) - \sin\left(\frac{n\pi(\frac{L}{4})}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \\
&= -\sin\left(\frac{n\pi(\frac{L}{4})}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \\
&= -\left(\cos\left(\frac{n\pi(\frac{L}{4})}{L}\right) \cos\left(\frac{n\pi x}{L}\right) + \sin\left(\frac{n\pi(\frac{L}{4})}{L}\right) \sin\left(\frac{n\pi x}{L}\right)\right) \\
&= -\cos\left(\frac{n\pi(\frac{L}{4})}{L} - \frac{n\pi x}{L}\right) \\
&= -\cos\left(\frac{n\pi(\frac{L}{4}-x)}{L}\right)
\end{aligned}$$

So we get that $\cos \frac{n\pi x}{L/2}$ is odd about $L/4$ with n being odd as well:

Now because $f(x)$ is even about $x = L/2$, that means that $\int_0^L f(x)\,dx = 2\int_0^{L/2} f(x)\,dx$ and $\cos \frac{n\pi x}{L/2}$ is odd about $x = L/4$, we get that their product is odd about $x = L/2$, this means we get the following:

$$\frac{2}{L/2} \int_0^{L/4} f(x) \cos \frac{n\pi x}{L/2} \, dx = -\frac{2}{L/2} \int_{L/4}^{L/2} f(x) \cos \frac{n\pi x}{L/2} \, dx$$

Thus, we get that for n being odd, the coefficients A_n is zero:

$$A_n = \frac{2}{L/2} \int_0^{L/2} f(x) \cos \frac{n\pi x}{L/2} \, dx = 0$$

Problem 4: Exercise 3.4.6

There are some things wrong in the following demonstration. Find the mistakes and correct them. In this exercise we attempt to obtain the Fourier cosine coefficients of e^x:

$$e^x = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}. \tag{4.22}$$

Differentiating yields

$$e^x = -\sum_{n=1}^{\infty} \frac{n\pi}{L} A_n \sin\frac{n\pi x}{L},$$

the Fourier sine series of e^x. Differentiating again yields

$$e^x = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 A_n \cos\frac{n\pi x}{L}. \tag{4.23}$$

Since Equations (4.22) and (4.23) give the Fourier cosine series of e^x, they must be identical. Thus,

$$\left.\begin{aligned} A_0 &= 0 \\ A_n &= 0 \end{aligned}\right\} \text{(obviously wrong!)}.$$

By correcting the mistakes, you should be able to obtain A_0 and A_n *without* using the typical technique, that is, $A_n = 2/L \int_0^L e^x \cos n\pi x/L \, dx$.

Notice we cannot differentiate e^x Fourier sine Series as it is not continuous:

$$f(0) = e^0 = 1 \neq -\sum_{n=1}^{\infty} \frac{n\pi}{L} A_n \sin 0 = 0 \qquad f(L) = e^L \neq -\sum_{n=1}^{\infty} \frac{n\pi}{L} A_n \sin n\pi = 0$$

Because we cannot differentiate the Fourier sine Series term by term, we get the following:

$$\begin{aligned} f'(x) &\sim \frac{f(L) - f(0)}{L} + \sum_{n=1}^{\infty} \left(\frac{n\pi}{L} B_n + \frac{2((-1)^n f(L) - f(0))}{L} \right) \cos\frac{n\pi x}{L} \\ e^x &\sim \frac{e^L - 1}{L} + \sum_{n=1}^{\infty} \left(\frac{n\pi}{L} \left(-\frac{n\pi}{L} A_n \right) + \frac{2((-1)^n e^L - 1)}{L} \right) \cos\frac{n\pi x}{L} \end{aligned}$$

Now we set our result equal to our original Fourier cosine series:

$$A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} = \frac{e^L - 1}{L} + \sum_{n=1}^{\infty} \left(\frac{n\pi}{L} \left(-\frac{n\pi}{L} A_n \right) + \frac{2((-1)^n e^L - 1)}{L} \right) \cos\frac{n\pi x}{L}$$

Now we get the following results:

$$A_0 = \frac{e^L - 1}{L} \qquad A_n = -\frac{n^2\pi^2}{L^2} A_n + \frac{2((-1)^n e^L - 1)}{L} \quad \to \quad A_n = \frac{2L((-1)^n e^L - 1)}{L^2 + n^2\pi^2}$$

Problem 5: Exercise 3.4.11:

Consider the *nonhomogeneous* heat equation (with a steady heat source):

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + g(x)$$

Solve this equation with the initial condition

$$u(x,0) = f(x)$$

and the boundary conditions

$$u(0,t) = 0 \text{ and } u(L,t) = 0.$$

Assume that a continuous solution exists (with continuous derivatives). [*Hints*: Expand the solution as a Fourier sine series (i.e., use the method of eigenfunction expansion). Expand $g(x)$ as a Fourier sine series. Solve for the Fourier sine series of the solution. Justify all differentiations with respect to x.]

Notice that we can get the solution, u to be in the form of a Fourier sine series:

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\frac{n\pi}{L}$$

We make our coefficient dependent on t, because we need u to be dependent on x and t, and the Fourier sine series is dependent on x.

We can get $g(x)$ as another Fourier sine series:

$$g(x) = \sum_{n=1}^{\infty} G_n \sin\frac{n\pi}{L}$$

Now we get the following:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \frac{dB_n}{dt}\sin\frac{n\pi}{L} \qquad \frac{\partial^2 u}{\partial x^2} = -\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2 B_n(t)\sin\frac{n\pi}{L}$$

Now we resubstitute the following into our original equation:

$$\sum_{n=1}^{\infty}\frac{dB_n}{dt}\sin\frac{n\pi}{L} + k\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2 B_n(t)\sin\frac{n\pi}{L} = \sum_{n=1}^{\infty}\left(\frac{dB_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 B_n(t)\right)\sin\frac{n\pi}{L} = \sum_{n=1}^{\infty} G_n \sin\frac{n\pi}{L}$$

So now we get the following:

$$\frac{dB_n}{dt} + k\left(\frac{n\pi}{L}\right)^2 B_n(t) = G_n(x)$$

Notice, we can solve this first order linear nonhomogeneous equation using an integrating factor:

$$\begin{aligned}
e^{k\left(\frac{n\pi}{L}\right)^2 t} B_n' + k\left(\frac{n\pi}{L}\right)^2 e^{k\left(\frac{n\pi}{L}\right)^2 t} B_n &= e^{k\left(\frac{n\pi}{L}\right)^2 t} G_n(x) \\
e^{k\left(\frac{n\pi}{L}\right)^2 t} B_n &= G(x) \int e^{k\left(\frac{n\pi}{L}\right)^2 t}\, dt \\
B_n &= \frac{G(x)L^2}{k(n\pi)^2} + C_n e^{-k\left(\frac{n\pi}{L}\right)^2 t}
\end{aligned}$$

We can now solve for C_n:

$$B_n(0) = \frac{G(x)L^2}{k(n\pi)^2} + C_n \qquad \rightarrow \qquad C_n = B_n(0) - \frac{G(x)L^2}{k(n\pi)^2}$$

Using the nonhomogeneous boundary condition, we get $B_n(0)$:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n(0) \sin\frac{n\pi}{L} \quad \rightarrow \quad B_n(0) = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}\, dx$$

Thus we get the following solution:

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{G(x)L^2}{k(n\pi)^2} + \left(\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}\, dx - \frac{G(x)L^2}{k(n\pi)^2} \right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \right) \sin\frac{n\pi}{L}$$