HW ①

② (a) typo Change "x" to "n".

Constructing $\mathbb{R}$

- There are several approaches:

There exists a unique, complete, Archimedean-ordered field we call $\mathbb{R}$.

3 Key Parts

1. elements.
2. algebra.

★ 3. geometry. — order & completeness ★

1. <u>elements</u>

- $\mathbb{N}$ to start. , +, ×
- $\mathbb{Z}$ by including additive inverses, −
- $\mathbb{Q}$. by including division, ÷
- <u>need</u> irrational numbers too – where do they come from? "COMPLETENESS"

2. <u>algebra</u> – topic for another course!
   - $\mathbb{R}$ is a <u>field</u>.
   - we will assume this part is understood.

(Gilles Notes <u>2.2</u> / Preliminary Section text)

3. Geometry

Order / Positivity (2.3 Gilles / Preliminaries text)

P1 $\forall a, b \in \mathbb{R}$ if $a > 0$ and $b > 0$, then $ab > 0$ and $a+b > 0$.

P2 $\forall a \in \mathbb{R}$ $a > 0$ xor $-a > 0$ xor $a = 0$

A few corollaries...

(i) $\forall a \neq 0$, $a^2 > 0$.

proof: Let $a \in \mathbb{R}$ and $a \neq 0$.

By P2, $a > 0$ or $-a > 0$.

Case 1: Suppose $a > 0$.

By P1, $a \cdot a = a^2 > 0$.

Case 2: Suppose $-a > 0$.

By P1, $(-a)(-a) = a^2 > 0$.

Done by cases 1 & 2. $\blacksquare$.

Definition we read $a > 0$ as a is *positive*.

When $-a > 0$ we say a is *negative* and write $0 > a$.

Also, saying $a > b$ is equivalent to $a - b > 0$.

(ii) $\forall a, b, c \in \mathbb{R}$,

① if $a > b$ and $c > 0$, then $ac > bc$.

② if $a > b$ and $c < 0$, then $ac < bc$.

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proof: Suppose $a, b, c \in \mathbb{R}$..

For ①, Suppose $a > b$ and $c > 0$.

So $a - b > 0$ and $c > 0$.

By P1, $c(a-b) > 0$

So $ac - bc > 0$

So $ac > bc$ by def.

For ②, Suppose $a > b$ and $c < 0$.

So $a-b > 0$ and $-c > 0$.

By P1, $-c(a-b) > 0$.

$bc - ac > 0$.

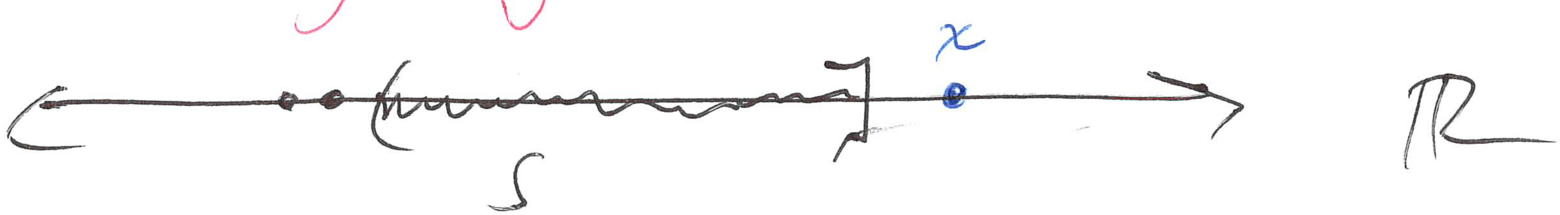
So $bc > ac$. $\boxtimes$.

Completeness

Defs: Suppose that $S \subseteq \mathbb{R}$. We say that $x \in \mathbb{R}$ is an <u>upper bound</u> for S iff $\forall y \in S$, $x \geq y$.

We say S is <u>bounded above</u> iff $\exists x \in \mathbb{R}$ s.t. x is an upper bound for S.

<u>remark</u>: When an upper bound exists for S, there are infinitely many more!



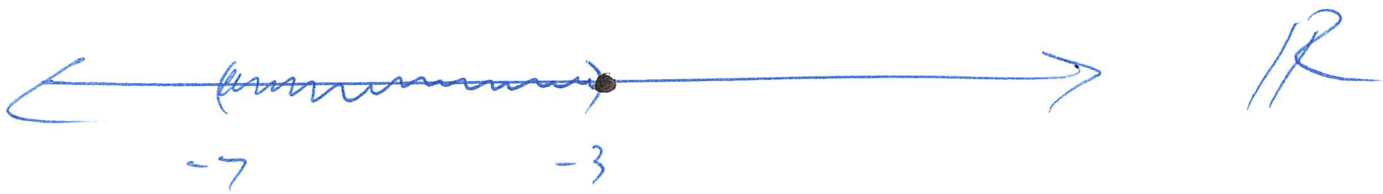
Def: Suppose $S \subseteq \mathbb{R}$ and $S \neq \emptyset$. We say $x \in \mathbb{R}$ is the least upper bound of S (the supremum of S) (notations: $x = \text{l.u.b. } S$ or $x = \sup S$).

iff

1. $\forall y \in S$, $x \geq y$
2. $\forall y \in \mathbb{R}$, if y is an upper bound of S, then $x \leq y$.

__Example:__ Suppose that $S = (-7, -3)$

Then $-3 = \sup S$.



① Notice if we let $y \in S$, then

$$-7 < y < -3$$

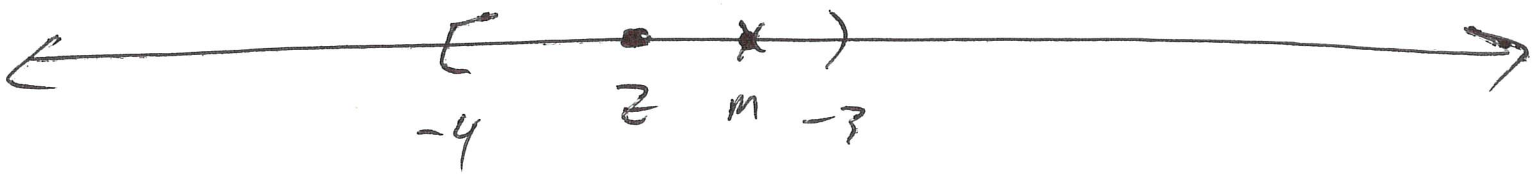
So $y < -3$ implies -3 is an upper bound.

② Suppose z is an upper bound of S.

There are 2 cases, $z < -3$ or $z \geq -3$.

__Case 1:__ suppose $z < -3$. If $z < -4$, then z is not an upper bound for S. So $-4 \leq z < -3$

Consider the midpoint $m = \dfrac{z + (-3)}{2}$.



Since $z < -3$, $z + (-3) < (-3) + (-3)$.

So
$$\frac{z + (-3)}{2} = m < \frac{(-3) + (-3)}{2} = -3.$$

Since $z < -3$, $z + z < z + (-3)$.

So
$$\frac{z+z}{2} = z < \frac{z + (-3)}{2} = m.$$

So $-4 \leq z < m < -3$. So $m \in S'$ and thus z is not an upper bound of S'. ($\Rightarrow\Leftarrow$).

Case 2: Suppose $z \geq -3$.

Then -3 is less or equal to the upper bound z.

Thus $-3 = \sup S'$.