MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #23

Sections 6.3 & 6.4

Higher Dimensional Linear Systems

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Key Concepts

Eigenvectors and generalized eigenvectors: An eigenvector (or right eigenvector) v of an $n \times n$ matrix A is a nonzero vector which satisfies $Av = \lambda v$ or $(A - \lambda I)v = 0$. A generalized (right) eigenvector is defined as $(A - \lambda I)^k v = 0$ for some $1 \leq k \leq n$.

Namely,

1. Eigenvectors: $(A - \lambda I)v = 0$
2. Generalized eigenvectors: $(A - \lambda I)^k v = 0$ for some $1 \leq k \leq n$.

For example, given a matrix with repeated eigenvalue, two independent vectors can be obtained as follows:

- $(A - \lambda I)v_1 = 0$
- $(A - \lambda I)^2 v_2 = 0 \qquad \Rightarrow (A - \lambda I)v_2 = v_1$

In the following, u_j (or U_j) reprent generalized eigenvectors V_{j+1}.
For example, $U_1 = V_2$

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Lecture #5

A Brief Review of Linear Algebra

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Generalized Eigenvector (for Repeated Eigenvalues)

Definition 5.6.1

We say that $\mathbf{u} \neq \mathbf{0}$ is a **generalized eigenvector** of $\mathbf{A}$ associated to the eigenvalue λ if

$$(\mathbf{A} - \lambda\mathbf{I})^k \mathbf{u} = \mathbf{0} \tag{5.78}$$

for some integer $k > 0$. The index of the generalized eigenvector is the smallest k satisfying (5.78).

Here u_j (or U_j) reprent V_{j+1}. For example, $U_1 = V_2$

Let V_1 be the eigenvector of A associated to the eigenvalue λ. Namely, $(A - \lambda I)V_1 = 0$. (a notation used in the HSD)

Consider u_1 to be V_2 and $(A - \lambda I)V_2 = V_1$. Therefore, we have $(A - \lambda I)^2 V_2 = (A - \lambda I)V_1 = 0$. V_2 is a generalized eigenvector.

Wirkus and Swift

Generalized Eigenvspace

- For k > 1, we see that the original eigenvector (v) gives rise to a set of generalized eigenvectors (u_1, $u_2 \ldots u_k$):

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{u}_{k-1} = \mathbf{u}_{k-2},$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{u}_{k-2} = \mathbf{u}_{k-3}, \quad \cdots ,$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{u}_3 = \mathbf{u}_2,$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{u}_2 = \mathbf{u}_1,$$

$$\boxed{(\mathbf{A} - \lambda\mathbf{I})\mathbf{u}_1 = \mathbf{v}.}$$

The set $\{\mathbf{v}, \mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{k-1}\}$ satisfying (5.79) is called a **chain** of generalized eigenvectors. The chain is determined entirely by the choice of $\mathbf{v}$, which is referred to as the **bottom of the chain**. For those that have read Section 5.3, we define

$$\tilde{E}_\lambda = \{v | (\mathbf{A} - \lambda\mathbf{I})^k \mathbf{v} = \mathbf{0} \text{ for some } k\} \qquad (5.80)$$

as the generalized eigenspace of λ. From (5.50), we note that $E_\lambda \subseteq \tilde{E}_\lambda$. If λ is an eigenvalue of $\mathbf{A}$ of multiplicity m, then $\tilde{E}_\lambda$ is a subspace of dimension m.

Construct a Solution

Let v and u_j represent the "regular" and generalized eigenvectors, respectively.

Let $x_0 = e^{\lambda t} v$

apply u_1 to obtain

apply u_2 to obtain

represent a solution

THEOREM 5.6.3

Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for the $k \times k$ matrix $\mathbf{A}$ in which (i) λ is an eigenvalue of multiplicity k with a single eigenvector $\mathbf{v}$ and (ii) $\{\mathbf{v}, \mathbf{u}_1, \cdots, \mathbf{u}_{k-1}\}$ is the corresponding chain of generalized eigenvectors. Set

$$\mathbf{x}_1 = te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{u}_1$$

$$\mathbf{x}_2 = \frac{t^2}{2!}e^{\lambda t}\mathbf{v} + te^{\lambda t}\mathbf{u}_1 + e^{\lambda t}\mathbf{u}_2$$

$$\vdots$$

$$\mathbf{x}_{k-1} = \frac{t^{k-1}}{(k-1)!}e^{\lambda t}\mathbf{v} + \frac{t^{k-2}}{(k-2)!}e^{\lambda t}\mathbf{u}_1 \cdots + te^{\lambda t}\mathbf{u}_{k-2} + e^{\lambda t}\mathbf{u}_{k-1}^{(k)}.$$

Then $e^{\lambda t}\mathbf{v}, \mathbf{x}_1, \cdots, \mathbf{x}_{k-1}$ are linearly independent and the general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ can be written as

$$\mathbf{x} = c_1 e^{\lambda t}\mathbf{v} + c_2\mathbf{x}_1 + c_3\mathbf{x}_2 + \cdots + c_k\mathbf{x}_{k-1}.$$

$$\boxed{x = c_1 x_0 + c_2 x_1 + c_3 x_2 \ldots + c_k x_{k-1}}$$

Revisit: Example in Chapters 3

$$x' = \lambda x + y$$

$$y' = \lambda y.$$

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

$$\alpha e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda t} \begin{pmatrix} t \\ 1 \end{pmatrix}.$$

$$A - \lambda I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$\lambda < 0$

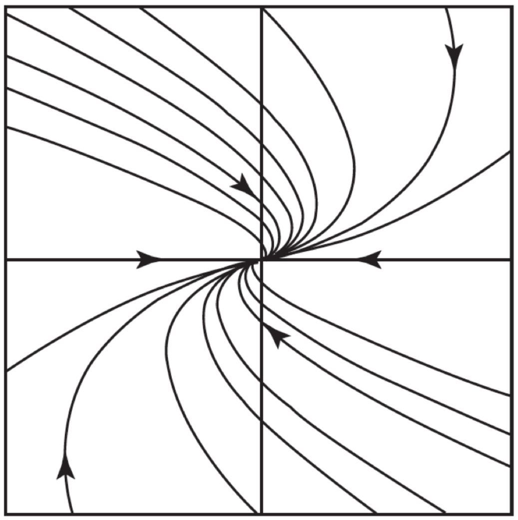

Fig. 3.6 Phase portrait for a system with repeated negative eigenvalues. Solutions tend toward or away from the origin in a direction tangent to the eigenvector (1,0).

Dim R=1 (# of the LI column vector)

Dim K=1 (Dim K = 2 – Dim R) (one “regular” eigenvector) $\quad (A - \lambda I)V_1 = 0$

Revisit: using the formula on slide #9

Solve for V_1 $\quad AV_1 = \lambda V_1 \quad A = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \quad \boxed{\lambda = \lambda_1} \quad V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Solve for U_1 $\quad \boxed{(A - \lambda I)U_1 = V_1}$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} U_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \boxed{y = 1} \quad U_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Obtain X_0 $\quad X_0 = e^{\lambda t} V_1 = e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Construct X_1 $\quad \boxed{X_1 = te^{\lambda t} V_1 + e^{\lambda t} U_1} = e^{\lambda t} \begin{pmatrix} t \\ 0 \end{pmatrix} + e^{\lambda t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{\lambda t} \begin{pmatrix} t \\ 1 \end{pmatrix}$

Obtain X $\quad \boxed{X = c_1 X_0 + c_2 X_1 = c_1 e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda t} \begin{pmatrix} t \\ 1 \end{pmatrix}}$

Goal: Repeated Eigenvalue

TBD

Example. Let

$$X' = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} X.$$

The only eigenvalue for this system is λ, and its only eigenvector is $(1,0,0)$. We

Altogether, we find

$$X(t) = c_1 e^{\lambda t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{\lambda t} \boxed{\begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix}} + c_3 e^{\lambda t} \boxed{\begin{pmatrix} t^2/2 \\ t \\ 1 \end{pmatrix}},$$

which is the general solution. Despite the presence of the polynomial terms

Review: Section 5.5: Repeated Eigenvalues in R^3

- We first consider the case of R^3. If A has repeated eigenvalues in R^3, then all eigenvalues must be real.

Proposition. *Suppose A is a 3×3 matrix for which λ is the only eigenvalue. Then we may find a change of coordinates T such that $T^{-1}AT$ assumes one of the following three forms:*

$$\text{(i)} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad \text{(ii)} \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad \text{(iii)} \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

(uncoupled)

$$(A - \lambda I)V = 0$$

Dim K=3
Dim R=0

$$(A - \lambda I)V_1 = 0$$

$$(A - \lambda I)V_2 = V_1$$

$$\boxed{(A - \lambda I)^2 V_2 = 0}$$

Dim K=2, V_1 & V_3
Dim R=1

$$(A - \lambda I)V_1 = 0$$

$$(A - \lambda I)V_2 = V_1$$

$$(A - \lambda I)^2 V_3 = V_1$$

Dim K=1
Dim R=2

Type (iii): Repeated Eigenvalues in R^3

$$\text{(iii)} \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

$$(A-\lambda I)V_1 = 0$$

$$(A-\lambda I)V_2 = V_1$$

$$(A-\lambda I)^2 V_3 = V_1$$

Dim K=1
Dim R=2

one "regular" eigenvector

We start from the following:

$$A - \lambda I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Two LI column vector ➔ Dim (range) = 2

➔ Dim (kernel) = 1

Find V_1

$$A = \begin{pmatrix} \lambda_0 & 1 & 0 \\ 0 & \lambda_0 & 1 \\ 0 & 0 & \lambda_0 \end{pmatrix}$$

$$\boxed{A - \lambda I = \begin{pmatrix} \lambda_0 - \lambda & 1 & 0 \\ 0 & \lambda_0 - \lambda & 1 \\ 0 & 0 & \lambda_0 - \lambda \end{pmatrix}}$$

$\lambda = \lambda_0$

$$A - \lambda I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$V_1 = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$(A - \lambda I)V_1 = 0$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0$$

$$\begin{pmatrix} \beta \\ \gamma \\ 0 \end{pmatrix} = 0$$

$$V_1 = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Find (the 1st) Generalized Eigenvector U_1

$\boxed{(A-\lambda I)V_2 = V_1}$ $\quad A-\lambda I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad U_1 = V_2 = \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \beta_1 \\ \gamma_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$V_2 = \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ 1 \\ 0 \end{pmatrix} \qquad U_1 = V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Here u_j (or U_j) reprent V_{j+1}. For example, $U_1 = V_2$.

Find (the 2nd) Generalized Eigenvector U_2

$$\boxed{(A-\lambda I)V_3 = V_2} \quad A-\lambda I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad U_1 = V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad V_3 = \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix} = V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \beta_2 \\ \gamma_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \alpha_2 \\ 0 \\ 1 \end{pmatrix} \qquad U_2 = V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Here u_j (or U_j) reprent V_{j+1}. For example, $U_1 = V_2$.

Verification (Double Check)

$$\boxed{(A-\lambda I)^2 V_3 = V_1} \qquad A-\lambda I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad V_3 = \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix}$$

$$(A-\lambda I)^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boxed{(A-\lambda I)^2 V_3 = V_1} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \gamma_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \alpha_2 \\ \beta_2 \\ 1 \end{pmatrix} \qquad V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Construct a Solution using Slide #9

compute X_0 $\quad X_0 = e^{\lambda t} V_1$

compute X_1

$$X_1 = te^{\lambda t} V_1 + e^{\lambda t} U_1 = te^{\lambda t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + e^{\lambda t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

compute X_2

$$X_2 = \frac{t^2}{2} e^{\lambda t} V_1 + te^{\lambda t} U_1 + e^{\lambda t} U_2$$

$$= \frac{t^2}{2} e^{\lambda t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + te^{\lambda t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + e^{\lambda t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Obtain X $\quad \boxed{X = c_1 X_0 + c_2 X_1 + c_3 X_2}$

$$X = c_1 e^{\lambda t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{\lambda t} \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} + c_3 e^{\lambda t} \begin{pmatrix} \frac{t^2}{2} \\ t \\ 0 \end{pmatrix}$$

Section 6.3: Repeated Eigenvalues

$$X(t) = c_1 e^{\lambda t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{\lambda t} \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} + c_3 e^{\lambda t} \begin{pmatrix} t^2/2 \\ t \\ 1 \end{pmatrix}$$

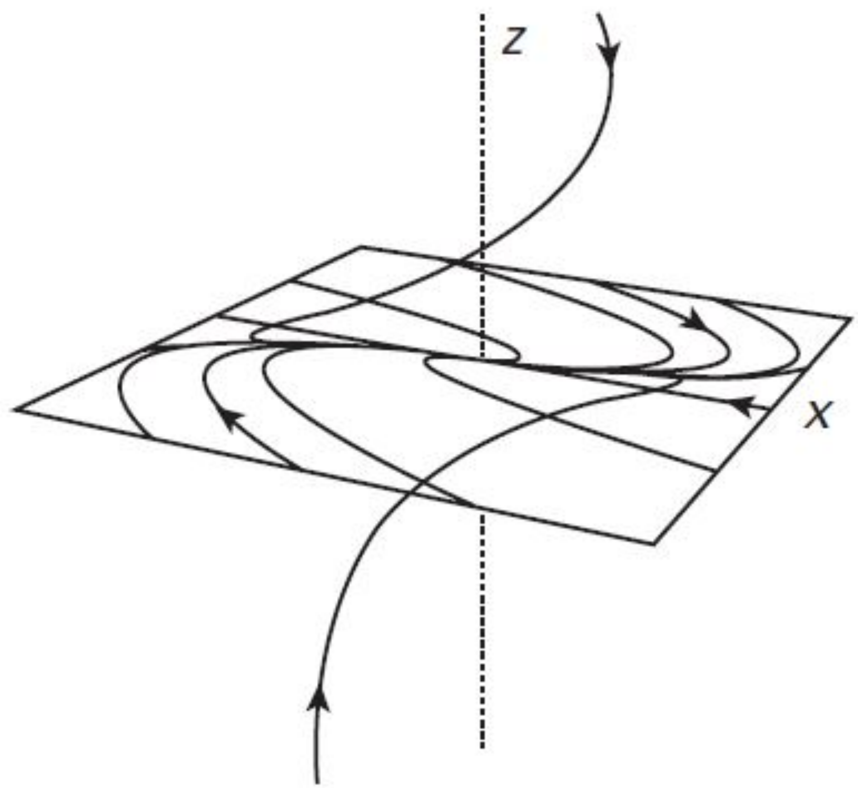

Figure 6.9 The phase portrait for repeated real eigenvalues.

Some representative solutions when $\lambda < 0$ are shown in Figure 6.9. Note that there is only one straight-line solution for this system; this solution lies on the x-axis. Also, the xy–plane is invariant and solutions there behave exactly as in the planar repeated eigenvalue case.

Repeated Eigenvalue

Example. Let

$$X' = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} X.$$

The only eigenvalue for this system is λ, and its only eigenvector is $(1, 0, 0)$. We

Altogether, we find

$$X(t) = c_1 e^{\lambda t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{\lambda t} \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} + c_3 e^{\lambda t} \begin{pmatrix} t^2/2 \\ t \\ 1 \end{pmatrix},$$

which is the general solution. Despite the presence of the polynomial terms

In Relation to Non-autonomous Systems

A high dimensional autonomous system

$$\frac{du}{dt} = \alpha u + \boxed{v}$$

A low dimensional system + a forcing term

$$\frac{du}{dt} = \alpha u + \boxed{\sin(\beta t)}$$

- Construct an ODE with $\mathrm{v} = \sin(\beta t)$ as a solution
- Submit your results via "chat"
- You have 3 minutes

In Relation to Non-autonomous Systems

A high dimensional autonomous system

$$\frac{du}{dt} = \alpha u + \boxed{v}$$

A low dimensional system + a forcing term

$$\frac{du}{dt} = \alpha u + \boxed{\sin(\beta t)}$$

$$\frac{d^2 v}{dt^2} = -\beta^2 v$$

$$\frac{dv}{dt} = \beta w;$$

$$\frac{dw}{dt} = -\beta v$$

In Relation to Non-autonomous Systems

A high dimensional autonomous system

$$\frac{du}{dt} = \alpha u + \boxed{v}$$

$$\frac{dv}{dt} = \beta w;$$

$$\frac{dw}{dt} = -\beta v$$

$$v(0) = 0; w(0) = -\beta \qquad \lambda = \alpha, \pm i\beta$$

A low dimensional system + a forcing term

$$\frac{du}{dt} = \alpha u + \boxed{\sin(\beta t)}$$

In Relation to Non-autonomous Systems

A high dimensional autonomous system

$$\frac{du}{dt} = \alpha u + \boxed{v}$$

A low dimensional system + a forcing term

$$\frac{du}{dt} = \alpha u \boxed{+ e^{\beta t}}$$

- Construct an ODE with $\mathrm{v} = e^{\beta}$ as a solution
- Submit your results via “chat”
- You have 1 minute

In Relation to Non-autonomous Systems

A high dimensional autonomous system

$$\frac{du}{dt} = \alpha u + \boxed{v}$$

$$\frac{dv}{dt} = \beta v;$$

$$\Longleftrightarrow$$

A low dimensional system + a forcing term

$$\frac{du}{dt} = \alpha u \boxed{+ e^{\beta t}}$$