Day 1 — Logic & Proofwriting Overview

– See Gilles Notes – Chapter 1.

Objectives:

- Negating Statements
- Proofwriting form basics.

Notation: Sets, (use capital letters)

$x \in A$

$y \notin A$. membership

$B \subseteq A$ subset

$B \subset A$ proper (not equal to) subset

$\mathbb{N} = \{0, 1, 2, \ldots\}$.

$\mathbb{Z} = \{\ldots, -1, 0, 1, 2, \ldots\}$.

$\mathbb{Q}$ ~ rational numbers.

$\mathbb{R}$ ~ real numbers.

Quantifiers:

$\forall x \in S$ "for all x in the set S ..."

$\exists x \in S$ "there exists an x in S ..."

$\exists! x \in S$ "there exists a unique x in S ..."

Logical Connectives: usually I use words.

$p \wedge q$

$p \vee q$

$p \Rightarrow q$ "if p, then q."

$\overline{p} \equiv \neg p$, "not p"

Basic Logical Equivalences

- verified via truth table in MATH 245.

① $p \Rightarrow q \equiv (\neg q) \Rightarrow (\neg p) \equiv (\neg p) \vee q$

② $\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$

$\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$.

③

$\neg \forall x \in S,\ p(x) \equiv \exists x \in S,\ \neg p(x)$.

$\neg \exists x \in S,\ p(x) \equiv \forall x \in S,\ \neg p(x)$.

Examples: Negate the given statements. Which are true?

(A) $\exists x \in \mathbb{R}, \ x^2 < 0.$ false

$\forall x \in \mathbb{R}, \ x^2 \geq 0.$ true

(B) $\forall x \in \mathbb{R}, \ 2x+1=0 \ \vee \ x<0.$ false

$\exists x \in \mathbb{R}, \ 2x+1 \neq 0 \ \wedge \ x \geq 0.$ true

BASIC Proof Forms.

Statement: $\forall x, P(x)$.

Direct Proof: Assume x is arbitrary.

$\vdots$

Conclude $P(x)$.

$\exists x, P(x)$.

Direct Proof: Present your x.

$\vdots$

Conclude $P(x)$.

Statement: $p \Rightarrow q$.

direct: Assume p.

$\vdots$

Show q.

contrapositive: Suppose $\neg q$.

$\vdots$

Show $\neg p$.

contradiction Suppose p and $\neg q$.

$\vdots$

Conclude FALSE.

<u>Definitions</u> Let $n \in \mathbb{Z}$.

We say n is <u>even</u> iff $\exists k \in \mathbb{Z}$ s.t. $n = 2k$.

We say n is <u>odd</u> iff $\exists k \in \mathbb{Z}$ s.t. $n = 2k+1$.

– Every integer is either even or odd <u>and</u> not both.

<u>Example Proof:</u> $\forall n \in \mathbb{Z}$ if n is odd, then $7n$ is odd.

<u>proof</u>: Let $n \in \mathbb{Z}$.

Suppose n is odd.

So $\exists k \in \mathbb{Z}$ s.t. $n = 2k+1$.

So $7n = 7(2k+1)$

$7n = 14k + 6 + 1$

$7n = 2(7k+3)+1$.

Since $7k+3 \in \mathbb{Z}$, $7n$ is odd.

<u>Converse</u>: Prove $\forall n \in \mathbb{Z}$, if $7n$ is odd, then n is odd.

<u>proof</u>: Let $n \in \mathbb{Z}$ (arbitrary).

Suppose n is even (Proceed by contraposition.)

Then $\exists k \in \mathbb{Z}$ s.t. $n = 2k$.

So $7n = 14k = 2(7k)$.

Since $7k \in \mathbb{Z}$, $7n$ is even.

<u>Remark</u>: We just proved:

$\forall n \in \mathbb{Z}$, n is odd <u>iff</u> $7n$ is odd.

Prove: $\forall \varepsilon \in (0,1)$, $\forall n \in \mathbb{N}$, if $\frac{1-\varepsilon}{\varepsilon} < n$, then $\frac{1}{n+1} < \varepsilon$.

Proof: Suppose $\varepsilon \in (0,1)$ and $n \in \mathbb{N}$.

Suppose $\frac{1-\varepsilon}{\varepsilon} < n$.

Then $1 - \varepsilon < \varepsilon n$

$$1 < (n+1)\varepsilon$$

So $\frac{1}{n+1} < \varepsilon$.

Proof by Induction

Statement: $\forall n \in \mathbb{N},\ p(n)$.

- direct proofs work much of the time
- proof by induction (weak).

① Prove $p(0)$. (BASE CASE)

② Prove: $\forall k \in \mathbb{N},\ p(k) \Rightarrow p(k+1)$.

(Induction Step ~ typically direct).