Homework 2
Linear Algebra
Math 524
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Section 2.A Problem 8: Prove or give a counterexample: If v_1, v_2, ..., v_m is a linearly independent list of vectors in $\mathbb{V}$ and $\lambda \in \mathbb{F}$ with $\lambda \neq 0$, then λv_1, λv_2, ..., λv_m is linearly independent.

Solution 2.A Problem 8: Let v_1, v_2, ..., v_m be a linearly independent list of vectors in $\mathbb{V}$ and $\lambda \in \mathbb{F}$ with $\lambda \neq 0$.
By the Definition of Linear Independence:

$$0 = \sum_{k=1}^{m} a_k v_k$$

with $a_1, a_2, ..., a_m \in \mathbb{F}$, the following must be true: $a_k = 0$ for $\{k = 0, 1, 2, ..., m\}$

To prove λv_1, λv_2, ..., λv_m to be linearly independent, the following must be true:

$$0 = \sum_{k=1}^{m} a_k \lambda v_k$$

with $a_1, a_2, ..., a_m \in \mathbb{F}$ and $a_k = 0$ for $\{k = 0, 1, 2, ..., m\}$.

$$\sum_{k=1}^{m} a_k \lambda v_k = \lambda \sum_{k=1}^{m} a_k v_k \tag{1}$$

Because the only way for

$$\sum_{k=1}^{m} a_k v_k = 0 \text{ was for } a_k = 0 \text{ for } \{k = 0, 1, 2, ..., m\}.$$

That means the only way for

$$\sum_{k=1}^{m} a_k \lambda v_k = 0 \text{ is if } a_k = 0 \text{ for } \{k = 0, 1, 2, ..., m\} \text{ also.}$$

Thus $\boldsymbol{\lambda v_1}$, $\boldsymbol{\lambda v_2}$, ..., $\boldsymbol{\lambda v_m}$ is linearly independent.

Section 2.A Problem 9: If v_1, v_2, ..., v_m and w_1, w_2, ..., w_m are linearly independent lists of vectors in $\mathbb{V}$, then $v_1 + w_1$, $v_2 + w_2$, ..., $v_m + w_m$ is linearly independent.

Solution 2.A Problem 9: Let v_1, v_2, ..., v_m and w_1, w_2, ..., w_m be linearly independent lists of vectors in $\mathbb{V}$.
By the Definition of Linear Independence,

$$0 = \sum_{k=1}^{m} a_k v_k \text{ and } 0 = \sum_{k=1}^{m} b_k w_k$$

with $a_1, a_2, ..., a_m, b_1, b_2, ..., b_m, \in \mathbb{F}$, the following must be true: $a_k = 0$ and $b_k = 0$ for $\{k = 0, 1, 2, ..., m\}$

$$\sum_{k=1}^{m} c_k(v_k + w_k) = \sum_{k=1}^{m} c_k v_k + \sum_{k=1}^{m} c_k w_k \tag{2}$$

Because the only way for

$$\sum_{k=1}^{m} c_k v_k = 0 \text{ was for } c_k = 0 \text{ for } \{k = 0, 1, 2, ..., m\}.$$

And the only way for

$$\sum_{k=1}^{m} c_k w_k = 0 \text{ was for } c_k = 0 \text{ for } \{k = 0, 1, 2, ..., m\}.$$

That means the only way for

$$\sum_{k=1}^{m} c_k(v_k + w_k) = 0 \text{ is if } c_k = 0 \text{ for } \{k = 0, 1, 2, ..., m\} \text{ also.}$$

Thus $\boldsymbol{v_1 + w_1}$, $\boldsymbol{v_2 + w_2}$, ..., $\boldsymbol{v_m + w_m}$ is linearly independent.

Section 2.A Problem 11: Suppose v_1, v_2, ..., v_m is linearly independent in $\mathbb{V}$ and $w \in \mathbb{V}$. Show that v_1, v_2, ..., v_m, w is linearly independent if and only if

$$w \notin \operatorname{span}(v_1, v_2, ..., v_m).$$

Solution 2.A Problem 11: (=>)
Let v_1, v_2, ..., v_m, w and v_1, v_2, ..., v_m be linearly independent, and suppose $w \in \operatorname{span}(v_1, v_2, ..., v_m)$. Let $a_i \in \mathbb{R}$ with $i \in \mathbb{Z}^+$

Because $w \in \operatorname{span}(v_1, v_2, ..., v_m)$.,

$$w = a_1v_1 + a_2v_2 + ... + a_mv_m \tag{3}$$
$$0 = a_1v_1 + a_2v_2 + ... + a_mv_m + -w \tag{4}$$

Because we can write 0 as a linear combination of w and vectors: v_i for $i \in \mathbb{Z}^+$, the coefficients are not all 0 as the coefficient to w is -1, thus this contradicts that v_1, v_2, ..., v_m, w is linearly independent. So $w \notin \operatorname{span}(v_1, v_2, ..., v_m)$.

Solution 2.A Problem 11: (<=) Let $w \notin \operatorname{span}(v_1, v_2, ..., v_m)$ and v_1, v_2, ..., v_m, w be linearly dependent, but v_1, v_2, ..., v_m be linearly independent.

Because v_1, v_2, ..., v_m, w is linearly dependent, v_j can be written as a linear combination of v_1, v_2, ..., v_{j-1}. But because v_1, v_2, ..., v_m is linearly independent, there does not exist a v_j such that it is a linear combination of v_1, v_2, ..., v_{j-1}. Now because $w \notin \operatorname{span}(v_1, v_2, ..., v_m)$ and $v_j \notin \operatorname{span}(v_1, v_2, ..., v_{j-1})$, v_1, v_2, ..., v_m, w has to be linearly independent.

Section 2.B Problem 3 (a): Let U be the subspace of $\mathbb{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}$$

Find a basis of U

Solution 2.B Problem 3 (a): Let $u = (3x_2, x_2, 7x_4, x_4, x_5) \in U$ Let $c_i \in \mathbb{R}$ for $i \in \mathbb{Z}^+$

Let vectors $v_1 = (3, 1, 0, 0, 0) \in U$ and $v_2 = (0, 0, 7, 1, 1) \in U$ as they satisfy the parameters of u.

$$0 = c_1v_1 + c_2v_2$$

They are also Linearly Independent of each other as the only way for their sum to be 0, is for their coefficients to also be 0.

$$u = c_1v_1 + c_2v_2$$

$\forall u$ can be written as a combination of v_1 and v_2, so it spans U. **Thus $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ make a basis for U**

Section 2.B Problem 3 (b): Extend the basis in part (a) to basis in $\mathbb{R}^5$.

Solution 2.B Problem 3 (b): Let $v = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$. Let $c_i \in \mathbb{R}$ for $i \in \mathbb{Z}^+$

Because $u = (3x_2, x_2, 7x_4, x_4, x_5)$, x_2, x_4, x_5 are independent of each other, but $x_1 = 3x_2$ and $x_3 = 7x_4$, are dependent of the other elements. To allow the basis to span $\mathbb{R}^5$, we have to add in $v_3 = (1, 0, 0, 0, 0)$ and $v_4 = (0, 0, 1, 0, 0)$, so we can make any vector in $\mathbb{R}^5$.
Any vector, $v \in \mathbb{R}^5$ can be written as

$$v = c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4$$

The vectors v_1, v_2, v_3, v_4 are also linearly independent of each other as the only way for

$$0 = c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4$$

is for all $c_i = 0$ for $i = \{1, 2, 3, 4\}$. **Thus this is a basis of $\mathbb{R}^5$**

Section 2.B Problem 3 (c): Find a subspace W of $\mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$.

Solution 2.B Problem 3 (c): Let W = span(v_3, v_4). Because all vectors of $\mathbb{R}^5$ can be written as a linear combination of $v_1, v_2 \in U$ and $v_3, v_4 \in W$, $\boldsymbol{U \oplus W}$.

Section 2.B Problem 5: Prove or disprove: There exists a basis p_0, p_1, p_2, p_3 of $\mathbb{P}_3(\mathrm{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2.

Solution 2.B Problem 5: Let $p_0 = 1$, $p_1 = x$, $p_2 = x^3 + x^2$, $p_3 = x^3 - x^2$.

To write 0 as a linear combination of p_0, p_1, p_2, p_3, the coefficients would all have to be 0, so p_0, p_1, p_2, p_3 are linearly independent of each other.

We can also write any polynomial as a linear combination of p_0, p_1, p_2, p_3 as $p_3 + p_2$ will get you a polynomial of degree 3. $p_2 - p_3$ will get you a polynomial of degree 2. p_1 will get you a polynomial of degree 1, and p_0 will get you a polynomial of degree 0. So all polynomials are in the span(p_0, p_1, p_2, p_3).

Thus there exists a basis p_0, p_1, p_2, p_3 of $\mathbb{P}_3(\mathrm{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 have degree 2

Section 2.C Problem 5 (a): Let $U = \{p \in \mathbb{P}_4(\mathbb{R}) : p''(6) = 0\}$.
Find the Basis of U

Solution 2.C Problem 5 (a): Let $p(x) \in \mathbb{P}_4(\mathbb{R})$ and $a, b, c, d \in \mathbb{R}$

$$p(x) = a(x-6)^4 + b(x-6)^3 + c(x-6) + d \tag{5}$$
$$p'(x) = 4a(x-6)^3 + 3b(x-6)^2 + c \tag{6}$$
$$p''(x) = 12a(x-6)^2 + 6b(x-6) \tag{7}$$

Because $\forall p(x)$ with $p''(6) = 0, p(x)$ can be written as a linear combination of $\{(x-6)^4, (x-6)^3, (x-6), 1\}$. Thus all $p(x) \in \operatorname{span}(\{(x-6)^4, (x-6)^3, (x-6), 1\})$.

Also because $\{(x-6)^4, (x-6)^3, (x-6), 1\}$ is all of different degrees, they are linearly independent of each other. **Thus $\boldsymbol{\{(x-6)^4, (x-6)^3, (x-6), 1\}}$ is a basis of U.**

Section 2.C Problem 5 (b): Extend the basis in part (a) to a basis of $\mathbb{P}_4(\mathbb{R})$

Solution 2.C Problem 5 (b): Because all the leading terms in each element of my previous basis are of degree, $4, 3, 1, 0$. All I need to do is add in $\{x^2\}$. This will allow my basis to span $\mathbb{P}_4(\mathbb{R})$ and still be linearly independent of each other

Section 2.C Problem 5 (c): Find a subspace W of $\mathbb{P}_4(\mathbb{R})$ such that $\mathbb{P}_4(\mathbb{R}) = U \oplus W$

Solution 2.C Problem 5 (c): Let W $= \{cx^2 : c \in \mathbb{R}\}$. This will allow $U \oplus W$ to make up the entire set of $\mathbb{P}_4(\mathbb{R})$.

Section 2.C Problem 9: Suppose $v_1, ..., v_m$ is linearly independent in V and $w \in V$. Prove that

$$\dim \text{span } (v_1 + w, ..., v_m + w) \geq m - 1$$

Solution 2.C Problem 9:
Notice the following:

$$v_2 - v_1 = (v_2 + w) - (v_1 + w)$$

Thus $v_i - v_1 \in \text{span}(v_1 + w, ..., v_m + w)$ for $2 \leq i \leq m$. Because $v_1, ..., v_m$ is linearly independent, $v_2 - v_1, ..., v_m - v_1$ is also linearly independent. So we can now extended this to a basis in V by Thm 2.33, such that

$$\dim \text{span } (v_1 + w, ..., v_m + w) \geq m - 1$$