Classwork 7
Abstract Algebra
Math 320
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Problem 1: Determine if the following polynomials are irreducible. Justify your answers.

(a) $x^6 + 30x^5 - 15x^3 + 6x - 120$

By Eisenstein's Criterion, 3 is a prime that does not divide the coefficient to x^6, but does divide all other coefficients. As well, $3^2 = 9$, does not divide the constant term -120, thus (a) is irreducible.

(b) $7x^3 - 36x^2 - x + 11$

We can test reducibility with the Rational Root Test, and by having no roots, proves it is irreducible by corollary 4.19.

$\frac{r}{s} = \pm 1, \pm 11, \pm 7, \pm \frac{11}{7}$. Let $f(x) = 7x^3 - 36x^2 - x + 11$

$$\begin{aligned} f(1) &= -19 & f(-1) &= -31 \\ f(11) &= 4961 & f(-11) &= -13651 \\ f(7) &= 641 & f(-7) &= -4147 \\ f\left(\frac{11}{7}\right) &= \frac{-2563}{49} & f\left(\frac{11}{7}\right) &= \frac{-5071}{49} \end{aligned}$$

Because there does not exist $f(\frac{r}{s}) = 0$, (b) is irreducible.

(c) $x^4 + 14x^3 + 9x^2 - x + 3$

We can test reducibility with the Rational Root Test, and by having roots, proves it is reducible by corollary 4.19.

Notice if we let $f(x) = x^4 + 14x^3 + 9x^2 - x + 3$, $f(-1) = 0$, thus -1 is a root. Thus we can write $f(x) = (x + 1)g(x)$, for some $g(x)$. Thus (c) is reducible.

Problem 2: Use Eisenstein's Criterion to show that $f(x) = x^4 + 1$ is irreducible in $\mathbb{Q}[x]$ by replacing x with $x + 1$. You may assume the following fact: If $g(x) = f(x + 1)$ is irreducible, then so is $f(x)$.

Notice $g(x) = f(x + 1)$:

$$(x + 1)^4 + 1 = x^4 + 4x^3 + 6x^2 + 4x + 2$$

By Eisenstein's Criterion, 2 is a prime that does not divide the coefficient to x^4, but does divide all other coefficients. As well, $2^2 = 4$, does not divide the constant term 2, thus $f(x)$ is irreducible.

Problem 3: Prove that $x^3 + nx + 2$ is irreducible over $\mathbb{Q}[x]$ for all integers $n \neq 1, -3, -5$.

We can test reducibility with the Rational Root Test, and by having no roots, proves it is irreducible by corollary 4.19.

$\frac{r}{s} = \pm 1, \pm 2$. Let $f(x) = x^3 + nx + 2$

$$f(1) = n + 3 \qquad f(-1) = -(n - 1)$$
$$f(2) = 2(n + 5) \qquad f(-2) = -2(n + 3)$$

Because for $n \neq 1, -3, -5$, we can see that there does not exists an x, such that $f(x) = 0$, thus (3) is irreducible.