Homework 6
Ordinary Differential Equations
Math 537
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Problem 1: Compute the Picard iterations for the initial value problem:

$$\frac{dy}{dt} = ay, \qquad y(t=0) = 1.$$

Notice the following:

Let $u_0(t) = 1, F(y) = ay$

$$\begin{aligned}
u_0(t) &= 1 \\
u_1(t) &= 1 + \int_0^t F(u_0(s))\,ds = 1 + a\int_0^t ds = 1 + at \\
u_2(t) &= 1 + \int_0^t F(u_1(s))\,ds = 1 + a\int_0^t (1+as)\,ds = 1 + at + \frac{a^2t^2}{2} \\
u_3(t) &= 1 + \int_0^t F(u_2(s))\,ds = 1 + a\int_0^t \left(1 + as + \frac{a^2s^2}{2}\right)\,ds \\
&= 1 + at + \frac{(at)^2}{2} + \frac{(at)^3}{6} \\
u_k(t) &= \sum_{n=0}^{k} \frac{(at)^n}{n!}
\end{aligned}$$

$$\textbf{As } \boldsymbol{k \to \infty, u_k(t) = \sum_{n=0}^{k} \frac{(at)^n}{n!} = e^{at}}$$

Problem 2: Consider the following second-order homogeneous nonlinear differential equation:

$$\frac{d^2X}{dt^2} + h\left(X, \frac{dX}{dt}\right) + g\left(X\right) = 0.$$

Let

$$E = \frac{1}{2}\left(\frac{dX}{dt}\right)^2 + \int g\left(X\right)dX.$$

(a) Show that $\frac{dE}{dt} = -h\frac{dX}{dt}$.

Notice the following:

$$\begin{aligned}
\frac{dE}{dt} &= \frac{dX}{dt}\frac{d^2X}{dt^2} + \frac{dX}{dt}\frac{d}{dx}\int g(X) \\
&= \left(\frac{d^2X}{dt^2} + g(X)\right)\frac{dX}{dt} \\
&= -h\frac{dX}{dt}
\end{aligned}$$

(b) Consider the Van der Pol equation:

$$\frac{d^2X}{dt^2} + \mu\left(X^2 - 1\right)\frac{dX}{dt} + X = 0.$$

Discuss the conditions under which $\frac{dE}{dt}$ is positive (and negative)

Notice the following:

$$h(X, \frac{dX}{dt}) = \mu\left(X^2 - 1\right)\frac{dX}{dt} \qquad \frac{dE}{dt} = -h\frac{dX}{dt} = -\mu\left(X^2 - 1\right)\left(\frac{dX}{dt}\right)^2$$

Notice the following:

$$\left(\frac{dX}{dt}\right)^2 \geq 0$$

Let $\mu > 0$, we get the following results:

(a) For $\frac{dE}{dt} < 0$ (negative), $X^2 - 1 > 0$, such that $|X| > 1$.

(b) For $\frac{dE}{dt} > 0$ (positive), $X^2 - 1 < 0$, such that $|X| < 1$.

For $\mu < 0$, we get the following results:

(a) For $\frac{dE}{dt} < 0$ (negative), $X^2 - 1 < 0$, such that $|X| < 1$.

(b) For $\frac{dE}{dt} > 0$ (positive), $X^2 - 1 > 0$, such that $|X| > 1$.

Problem 3: Consider the following second-order differential equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = y,$$

which has an irregular singular point at ∞. Apply the substitution $y = e^{S(x)}$ to show that the leading behavior of $y(x)$ is given by

$$y(x) \sim cx^{-1/4}e^{2x^{1/2}}, x \to +\infty$$

here c is a constant.

Notice the following:

$$y = e^{S(x)} \qquad \frac{dy}{dx} = S'(x)e^{S(x)} \qquad \frac{d^2y}{dx^2} = \left(S''(x) + (S'(x))^2\right)e^{S(x)}$$

We can rewrite it as follows:

$$x\left(S'' + (S')^2\right)e^{S(x)} + S'e^{S(x)} - e^{S(x)} = 0$$
$$xS'' + x(S')^2 + S' - 1 = 0$$

Now we can drop the all the small terms and rewrite the equation:

$$x(S')^2 + S' - 1 \sim 0, \qquad x \to \infty, \tag{1}$$

Now we can see the following:

$$S' \sim \frac{-1 \pm \sqrt{1+4x}}{2x} \sim \pm\frac{1}{\sqrt{x}}, \qquad x \to \infty$$

Thus we get:

$$S(x) = 2\sqrt{x} + C(x)$$

with $C(x) << 2\sqrt{x}, C' << x^{-1/2}, C'' << x^{-3/2}$.

If we substitute this into equation (1) and combine the $C(x)$ terms, we get:

$$xC'' + x(C')^2 + (2\sqrt{x}+1)C' + \frac{1}{2\sqrt{x}} = 0$$

Notice the following from the facts about $C(x)$:

$$1 << 2\sqrt{x}, \qquad xC'' << \frac{1}{2\sqrt{x}}, \qquad x(C')^2 << 2\sqrt{x}C', \qquad x \to \infty$$

Thus we get

$$2\sqrt{x}C' \sim -\frac{1}{2\sqrt{x}}, \qquad C' \sim -\frac{1}{4x}$$

Thus we get that

$$C(x) = \frac{-1}{4}\ln x + d$$

Finally, this leads to the following:

$$S(x) = 2\sqrt{x} + \frac{-1}{4}\ln x + d$$

Notice that this shows the leading behavior:

$$\boldsymbol{y \sim e^{S(x)} \sim e^{2x^{1/2}}e^{-(1/4)\ln x}e^{d} \sim cx^{-1/4}e^{2x^{1/2}}, \qquad x \to \infty}$$

Problem 4: Consider a boundary-layer problem with the following second order linear differential equation:

$$\epsilon \frac{d^2y}{dx^2} + (1+\epsilon)\frac{dy}{dx} + y = 0,$$
$$y(0) = 0 \text{ and } y(1) = 1.$$

(a) Solve for the exact solution.

Notice we can get the characteristic equation:

$$\epsilon\lambda^2 + (1+\epsilon)\lambda + 1 = 0$$

Now notice the lambda values from the quadratic equation:

$$\begin{aligned}
\lambda &= \frac{1}{2\epsilon}\left(-(1+\epsilon) \pm \sqrt{(1+\epsilon)^2 - 4\epsilon}\right) \\
&= \frac{1}{2\epsilon}\left(-(1+\epsilon) \pm \sqrt{\epsilon^2 + 2\epsilon + 1 - 4\epsilon}\right) \\
&= \frac{1}{2\epsilon}\left(-(1+\epsilon) \pm \sqrt{\epsilon^2 - 2\epsilon + 1}\right) \\
&= \frac{1}{2\epsilon}\left(-(1+\epsilon) \pm \sqrt{(\epsilon-1)^2}\right) \\
&= \frac{1}{2\epsilon}\left(-(1+\epsilon) \pm (\epsilon-1)\right) \\
&= \frac{-1}{\epsilon}, \quad -1
\end{aligned}$$

So we get the following general solution:

$$y = c_1 e^{-x} + c_2 e^{-x/\epsilon}$$

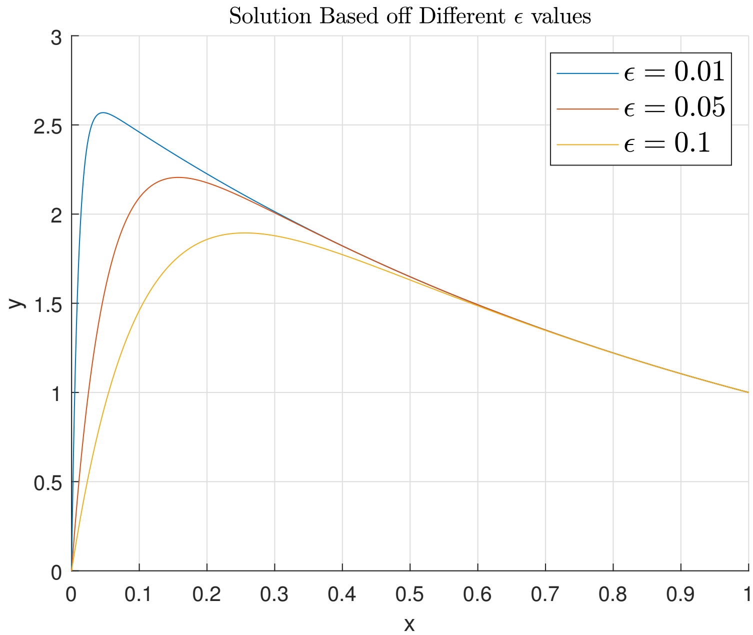
Notice the matrix and its reduced row echelon form found through the Maple Software:

$$\begin{bmatrix} 1 & 1 & 0 \\ e^{-1} & e^{-1/\epsilon} & 1 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & \frac{-1}{e^{-1/\epsilon}-e^{-1}} \\ 0 & 1 & \frac{1}{e^{-1/\epsilon}-e^{-1}} \end{bmatrix}$$

Thus we get the following exact solution:

$$\boldsymbol{y = \frac{e^{-x/\epsilon} - e^{-x}}{e^{-1/\epsilon} - e^{-1}}}$$

(b) Plot the solution for $\epsilon = 0.01, 0.05$, and 0.1.



(c) Determine the inner and outer limit of the solution.

Notice the following for the outer limit:

$$\lim_{\epsilon \to 0} y(x) = \frac{e^{-x}}{e^{-1}} = e^{1-x}$$

Let $x = \epsilon \mathbb{X}$ and notice the following for the inner limit:

$$y = \frac{e^{-\mathbb{X}} - e^{-\epsilon \mathbb{X}}}{e^{-1/\epsilon} - e^{-1}} \qquad \lim_{\epsilon \to 0} y(x) = \frac{e^{-\mathbb{X}} - 1}{-e^{-1}} = e - e^{1-\mathbb{X}}$$

Thus we get the following:

$$\mathbb{Y} \sim e^{1-x} - e^{1-\mathbb{X}}$$