Homework 4
Algebraic Coding Theory
Math 525
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Problem 1: Consider the subset

$$S = \{11000, 00011, 01110\}$$

of K^5.

(a) Find the code C generated by S (i.e., list all of its codewords).

The code C generated by S is the basis for the code $C = \langle S \rangle$.

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}, \qquad G = REF(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

So the code C generated by S is $\{10101, 01101, 00011\}$.

(b) Find $C^{\perp}$, the dual code of C (i.e., list all of its codewords).

Notice we can write $G' = [I_k|X]$.

$$G' = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}, \qquad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 3 & 5 \end{pmatrix}$$

Now we can write $H' = \left[\frac{X}{I_{n-k}}\right]$

$$H' = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 3 & 5 \end{pmatrix}$$

So we get the following:

$$H = C^{\perp} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Problem 2: Consider the set

$$S = \{110011, 010100, 001101, 100111\}$$

of words in K^6.

(a) Find a generator matrix G, in RREF, for the code $C = \langle S \rangle$. What is dim C?

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \qquad G = RREF(C) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

The dimension of C is the amount of rows of its generated matrix G, $k = 3$

(b) From the matrix G above, find a parity-check matrix H for C

$$H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice the parity check matrix $H = \left[\frac{X}{I_{n-k}}\right]$ where X is the matrix from $G = [I_k|X]$

(c) Use H to determine the distance of C Notice we can write matrix H as the following:

$$H = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice that $h_2 + h_4 = \vec{\mathbf{0}}$, so the distance of C, $d = 2$

Problem 3: Let

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

be a matrix with entries in K^8 and let C be the code generated by it.

(a) Find a systematic encoding matrix G for C.

$$G = RREF(M) = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

(b) Use G to encode the information vector $(u_0, u_1, u_2, u_3) \in K^4$

$$u \cdot G = (u_0, u_1, u_2, u_3) \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$= (u_0, u_1, u_2, u_3, u_0 + u_1 + u_2, u_0 + u_1 + u_3, u_0 + u_2 + u_3, u_1 + u_2 + u_3)$$

(c) Find the dimension of C and $C^\perp$. Find the number of codewords in C and $C^\perp$.

The dimension of C is equal to the number of rows in G. So the dim $C = k = 4$. The number of codewords in $C^\perp$ is $|C| = 2^k = 2^4$

The dimension of $C^\perp$ is equal to $n - k$. So the dim $C^\perp = 8 - 4 = 4$. The number of codewords in $C^\perp$ is $|C^\perp| = 2^{n-k} = 2^4$

(d) Find a parity-check matrix H for C.

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(e) From H, conclude that $C = C^{\perp}$ in this case.

Notice that if we take each column of H, h_1, h_2, h_3, h_4, we get that $h_i^T H = \vec{0}, \forall i \in \{1,2,3,4\}$. By the theorem in 2.7, that makes $h_i \in C$. Now since that the columns of H form a basis of $C^{\perp}$, $C^{\perp} \subseteq C$. But because they are of the same size, $|C^{\perp}| = |C| = 2^4$, thus getting us $C = C^{\perp}$

Problem 4:

2.6.5.a Find a generator matrix in RREF for each of the following codes.

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.6.6.a Find a generator matrix for each of the following codes. Give the dimension of the code.

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}, \qquad dim\, C = 3$$

2.6.10.a.i For each of the following generating matrices, encode the given messages:

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}, u = 100, \qquad uG = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

2.6.13 Find the number of messages which can be sent, and the information rate r, for each of the linear codes in Exercises 2.6.6 and 2.6.7

2.6.6.a $C = \{000000, 001011, 010101, 011110, 100110, 101101, 110011, 111000\}$

Information Rate: $R = \frac{\log_2 2^{k=3}}{n=6} = \frac{1}{2}$. The number of messages that can be sent: $|C| = 8$

Problem 5:

2.7.4.a Find a parity-check matrix from each of the following codes.

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$H' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad \sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \qquad H = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

2.7.5 Find a parity-check matrix for each of the following codes (the generating matrices were constructed in Exercises 2.6.6 and 2.6.7).

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}, \qquad H = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.7.9 In each part, a parity-check matrix for a linear code C is given. Find (i) a generator matrix for $C^{\perp}$; (ii) a generator matrix for C.

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 3 & 2 & 5 & 6 \end{pmatrix}, \qquad H' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G' = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}, \qquad \sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 3 & 2 & 5 & 6 \end{pmatrix}, \qquad G = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

2.7.10 List all the words in the dual code $C^{\perp}$ for the code $C = \{00000, 11111\}$. Then find generating and parity-check matrices for $C^{\perp}$

$C^{\perp}$ consists of $|C^{\perp}| = 2^{k=5}$

2.7.11 For each code C described below, find the dimension of C, the dimension of $C^{\perp}$, the size of generating and parity-check matrices for C and for $C^{\perp}$, the number of words in C and in $C^{\perp}$, and the information rates r of C and $C^{\perp}$

C has length $n = 2^t - 1$ and dimension t.

$|C| = 2^t$, $R(C) = \frac{\log_2 2^t}{2^t - 1}$

$C^{\perp}$ has dimension of $n - k = 2^t - 1 - t$ and $|C^{\perp}| = 2^{2^t - 1 - t}$, $R(C^{\perp}) = \frac{\log_2 2^{2^t - 1 - t}}{2^t - 1 - t} = 1$

Problem 6:

2.8.11 Find a generator matrix G in standard form for a code equivalent to the code with given generator matrix G.

$$G = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}, \qquad RREF(G) = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, G' = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

2.8.12 Find a generator matrix G' in standard form for a code C' equivalent to the code C with given parity check matrix H.

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 2 & 4 & 5 & 6 \end{pmatrix}, \qquad H' = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G' = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

2.8.14 (a) yes

(b) yes

(c) no

Problem 7:

2.9.4 Find the distance of the linear code C with each of the given parity-check matrices. Use Theorem 2.9.1 and then check your answer by finding $wt(v)$ for each v in C.

$$H = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Distance of C is 4, Notice $r_2 + r_3 + r_4 + r_5 = \vec{0}$.

2.9.5 Find, by Theorem 2.9.1, the distance of the linear code with the given generator matrix.

$$G = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 4 & 7 & 2 & 5 & 6 & 3 & 8 & 9 \end{pmatrix}, \qquad G' = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$H' = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 4 & 7 & 2 & 5 & 6 & 3 & 8 & 9 \end{pmatrix}, H = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Distance of C is 2.

Problem 8:

2.10.6 List the cosets of each of the following linear codes.

$$C = \{0000, 1001, 0101, 1100\}$$

$$\begin{aligned}
&0000 ::: 0000, 1001, 0101, 1100\\
&0001 ::: 0001, 1000, 0100, 1101\\
&0010 ::: 0010, 1011, 0111, 1110\\
&0011 ::: 0011, 1010, 0110, 1111\\
&0100 ::: 0100, 1101, 0001, 1000\\
&0101 ::: 0101, 1100, 0000, 1001\\
&0110 ::: 0110, 1111, 0011, 1010\\
&0111 ::: 0111, 1110, 0010, 1011\\
&1000 ::: 1000, 0001, 1101, 0100\\
&1001 ::: 1001, 0000, 1100, 0101\\
&1010 ::: 1010, 0011, 1111, 0110\\
&1011 ::: 1011, 0010, 1110, 0111\\
&1100 ::: 1100, 0101, 1001, 0000\\
&1101 ::: 1101, 0100, 1000, 0001\\
&1110 ::: 1110, 0111, 1011, 0010\\
&1111 ::: 1111, 0110, 1010, 0011
\end{aligned}$$

2.10.7 List the cosets of each of the linear codes having the given generator matrix.

$$G = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

2.10.8 List the cosets of the code having the given parity-check matrix.

$$H = \begin{bmatrix} 10 \\ 11 \\ 10 \\ 01 \end{bmatrix}$$

Problem 9:

2.11.8 Construct an SDA assuming IMLD for each of the codes in Exercise 2.10.6.

$$C = \{0000, 1001, 0101, 1100\}$$

Notice the parity check matrix, H:

$$H = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Coset Leader u	Syndrome uH
0000	00
*	01
0010	10
*	11

2.11.9 Construct an SDA assuming IMLD for each of the codes in Exercise 2.10.7.

$$G = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

2.11.10 Construct an SDA assuming IMLD for each of the codes in Exercise 2.10.8.

$$H = \begin{bmatrix} 10 \\ 11 \\ 10 \\ 01 \end{bmatrix}$$

Problem 10:

2.11.16 Again refer to Example 2.11.13 with $w = 110000$ received. Find all the codewords in C closest to w

2.11.19 For each of the following codes, use the SDA to decode the given re-ceived words. (The SDA's for these codes were constructed in Exercises 2.11.8 and 2.11.9.)

$$C = \{0000, 1001, 0101, 1100\}, w = 1110$$

Notice:

$$wH = 10, u = 0010, \qquad v = w + u = 1110 + 0010 = 1100$$

That refers to the coset leader: $u =$

2.11.20 Let C be the code with the parity-check matrix

$$H = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.11.21 Let C be the code of length 7 which has as a parity-check matrix the 7×3 matrix H whose rows are all nonzero words of length 3.

(a) Construct an SDA for C.

(b) Decode 1010101

Problem 11:

2.12.2 Calculate $\theta_p(C)$ for each of the codes in Exercises 2.10.6, 2.10.7, 2.10.8.