Assignment 2, Math 330

Name:

Please turn this in on Tuesday, September 17. Feel free to work with others on this and other homework assignments.

1. Suppose that $c, x \in \mathbb{R}$ and that $c < x$. Prove that

$$c < c + \frac{x-c}{2} < x$$

2. Use induction to prove this extension of the triangle inequality: $\forall n \in \mathbb{Z}^+, \forall x_1 \dots x_n \in \mathbb{R}$, we have

$$\left| \sum_{i=1}^{n} x_i \right| \leq \sum_{i=1}^{n} |x_i|.$$

3. Suppose that $0 \leq a \leq 1$. Use induction to prove:

$$\forall n \in \mathbb{Z}^+, \quad (1+a)^n \leq 1 + (2^n - 1)a$$

4. The Binomial Theorem is a fact about the expanded version of $(a+b)^n$. In particular, it says:

$$\forall a, b \in \mathbb{R}, \forall n \in \mathbb{Z}^+, \quad (a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j \quad \text{where} \quad \binom{n}{j} = \frac{n!}{j!(n-j)!}.$$

You can lookup a proof of the theorem in (almost) any book on discrete math. Use the theorem to prove the following two statements.

(a) Prove that $\forall n \in \mathbb{Z}^+, \forall b \in \mathbb{R}^+$, we have

$$(1+b)^n \geq 1 + nb + \frac{n(n-1)}{2} b^2$$

(b) Prove that for every integer $n \geq 1$ we have

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n} = 0.$$

5. For a set $S \subseteq \mathbb{R}$, a number $c \in S$ is called a **maximum** of S (and written $c = \max S$) provided c is an upper bound of S. Prove that $\forall S \subseteq \mathbb{R}$ with $S \neq \emptyset$, the set S has a maximum iff S is bounded above and $\sup S \in S$. Give an example of a set that is nonempty, bounded above and that has no maximum.

6. For each of the following, find the maximum, minimum, supremum, and infimum if they are defined. You do not need to justify your answers here.

(a) $\{1/n \mid n \in \mathbb{Z}^+\}$

(b) $\{x \in \mathbb{R} \mid x^2 < 2\}$