Sect. 2.5: Eigenvalues, Eigenvectors, and Solutions

Definition
A $\boxed{\text{nonzero vector } V_0}$ is called an *eigenvector* of A if $AV_0 = \lambda V_0$ for some λ. The constant λ is called an *eigenvalue* of A.

Theorem. *Suppose that V_0 is an eigenvector for the matrix A with associated eigenvalue λ. Then the function* $\boxed{X(t) = e^{\lambda t} V_0 \textit{ is a solution}}$ *of the system* $X' = AX$. ∎

V_0: eigenvector
λ: eigenvalue

If the system has eigenvectors, $|A - \lambda I| = 0$. characteristic equation

We call $|A - \lambda I|$ the characteristic polynomial.

Examples: Verify $AV_0 = \lambda V_0$

Example. Consider

$$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}.$$

Then A has an eigenvector $V_0 = (3, 1)$ with associated eigenvalue $\boxed{\lambda = 2}$ since

$$AV_0 = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}. = \lambda V_0$$

$$1 \times 3 + 3 \times 1 = 6$$

$$1 \times 3 + (-1) \times 1 = 2$$

Similarly, $V_1 = (1, -1)$ is an eigenvector with associated eigenvalue $\lambda = -2$.

■

General Solutions and Eigenvectors/Eigenvalues

Thus, for the system

$$X' = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} X$$

we now know $\boxed{\text{three solutions:}}$ the equilibrium solution at the origin together with

$\lambda = 2$ $\lambda = -2$

non-trivial solutions $X_1(t) = e^{2t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $X_2(t) = e^{-2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$e^{2t}V_0$ $e^{-2t}V_1$

We will see that we can use these solutions to generate *all* solutions of this system in a moment, but first we address the question of how to find eigenvectors and eigenvalues.

General Solution c $\boxed{X = \alpha X_1 + \beta\, X_2 = \alpha e^{\lambda_0 t} V_0 + \beta e^{\lambda_1 t} V_1}$

Examples: Solve $|A - \lambda I| = 0$

Example. We return to the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \quad for\ X' = AX.$$

define $\quad X = \begin{pmatrix} x \\ y \end{pmatrix}; \; A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$

eigenvalue problem $\quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{vmatrix} = 0$

$$(1-\lambda)(-1-\lambda) - 3 = \lambda^2 - 1 - 3 = 0$$

$$\lambda^2 - 4 = 0$$

$$\lambda = 2, -2$$

Examples: Solve $|A - \lambda I| = 0$

Solve for eigenvectors $AV_0 = \lambda V_0$

$$x_0 + 3y_0 = \lambda x_0$$

$$x_0 - y_0 = \lambda y_0$$

Consider $\lambda = 2$

$$\boxed{\begin{aligned} x_0 + 3y_0 &= 2x_0 \\ x_0 - y_0 &= 2y_0 \end{aligned}} \qquad \boxed{x_0 = 3y_0} \qquad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 3y_0 \\ y_0 \end{pmatrix} = y_0 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Obtain

$\boxed{V_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}}$ as an eigenvector associated with $\lambda = 2$

Similarly,

$\boxed{V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$ as an eigenvector associated with $\lambda = -2$

General Solution $\boxed{X = \alpha X_1 + \beta X_2 = \alpha e^{2t} V_1 + \beta e^{-2t} V_2}$

A Summary

$AX = \gamma$	$X' = AX$
	$(A - \lambda I)V_0 = 0$
$\lvert A \rvert \neq 0$, unique sol $\lvert A \rvert \neq 0$ & γ=0, trivial sol	$\lvert A - \lambda I \rvert \neq 0$, trivial sol
$\lvert A \rvert = 0$ • no solution • Infinitely many solutions	$\lvert A - \lambda I \rvert = 0$ • Infinitely many solutions
	• The above is called an eigenvalue problem • $Let\ \ AV_1 = \lambda_1 V_1;\ AV_2 = \lambda_2 V_2,$ we have a general solution as follows: $X = c_1\, V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t}$
	• 1D $x' = f(x)$ • $x' = f(x) \approx f'(\mathrm{x_c})(\mathrm{x} - \mathrm{x_c})$ • $\lambda = f'(\mathrm{x_c})$

2.6 Solving Linear Systems

Eigenvectors as "basis vectors"

Theorem. *Suppose A has a pair of real eigenvalues $\lambda_1 \neq \lambda_2$ and associated eigenvectors V_1 and V_2. Then the general solution of the linear system $X' = AX$ is given by*

$$X(t) = \alpha e^{\lambda_1 t} V_1 + \beta e^{\lambda_2 t} V_2. \qquad \blacksquare$$

Examples

Example. Consider the second-order differential equation:

$$x'' + 3x' + 2x = 0.$$

let $x' = y$

obtain $y' = -2x - 3y$

define $X = \begin{pmatrix} x \\ y \end{pmatrix}$; $A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$ $\quad X' = AX$

eigenvalue problem $|A - \lambda I| = \begin{vmatrix} 0-\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = 0$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\lambda = -1, -2$$

$$AV_0 = \lambda V_0; \qquad (A - \lambda I)V_0 = 0$$

Solve for eigenvectors $\quad AV_0 = \lambda V_0$

$$y_0 = \lambda x_0$$
$$-2x_0 - 3y_0 = \lambda y_0$$

Consider $\lambda = -1$

$$\boxed{\begin{aligned} y_0 &= -x_0 \\ -2x_0 &= 2y_0 \end{aligned}}$$

("overlap")

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ -x_0 \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Obtain

$\boxed{V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$ as an eigenvector associated with $\lambda = -1$

Similarly,

$\boxed{V_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$ as an eigenvector associated with $\lambda = -2$

Examples

Thus, one eigenvector associated with the eigenvalue -1 is $(1,-1)$. In similar fashion we compute that an eigenvector associated with the eigenvalue -2 is $(1,-2)$. Note that these two eigenvectors are linearly independent. Therefore, by the previous theorem, the general solution of this system is

$$\boxed{X = \alpha e^{\lambda_1 t} V_1 + \beta e^{\lambda_2 t} V_2}$$

$$X(t) = \alpha e^{-t} \begin{pmatrix} \boxed{1} \\ \boxed{-1} \end{pmatrix} + \beta e^{-2t} \begin{pmatrix} \boxed{1} \\ \boxed{-2} \end{pmatrix}.$$

That is, the position of the mass is given by the first component of the solution,

$$\boxed{x(t) = \alpha e^{-t} + \beta e^{-2t},}$$

and the velocity is given by the second component,

$$\boxed{y(t) = x'(t)} = -\alpha e^{-t} - 2\beta e^{-2t}. \qquad \blacksquare$$

Eigenvectors as “Basis Vectors”

THEOREM 5.5.2

Consider the 2×2 system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ whose coefficient matrix has eigenvalues $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$, with α, β real numbers. Choose one of the eigenvectors and write it as $\mathbf{v}_j = \mathbf{a} + i\mathbf{b}$, where $\mathbf{a}, \mathbf{b}$ are real-valued vectors. Then

$$\begin{aligned} \mathbf{x}_1 &= e^{\alpha t}\left[\mathbf{a}(\cos\beta t) - \mathbf{b}(\sin\beta t)\right] \\ \mathbf{x}_2 &= e^{\alpha t}\left[\mathbf{a}(\sin\beta t) + \mathbf{b}(\cos\beta t)\right] \end{aligned} \tag{5.75}$$

are two linearly independent solutions, defined for $-\infty < t < \infty$. The general solution is a linear combination of these two:

$$\mathbf{x} = e^{\alpha t}\left\{c_1\left[\mathbf{a}(\cos\beta t) - \mathbf{b}(\sin\beta t)\right] + c_2\left[\mathbf{a}(\sin\beta t) + \mathbf{b}(\cos\beta t)\right]\right\} \tag{5.76}$$

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Sect. 2.7: The Linearity Principle

Theorem. *Let $X' = AX$ be a planar system. Suppose that $Y_1(t)$ and $Y_2(t)$ are solutions of this system, and that the vectors $Y_1(0)$ and $Y_2(0)$ are linearly independent. Then*

$$X(t) = \alpha Y_1(t) + \beta Y_2(t)$$

is the unique solution of this system that satisfies $X(0) = \alpha Y_1(0) + \beta Y_2(0)$. ■

$$X' = \alpha Y_1' + \beta Y_2' = \alpha A Y_1 + \beta A Y_2 = A(\alpha Y_1 + \beta Y_2) = AX$$

A Summary

$AX = \gamma$	$X' = AX$
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	• 1D $x' = f(x)$ • $x' = f(x) \approx f'(\mathrm{x_c})(\mathrm{x} - \mathrm{x_c})$ • $\lambda = f'(\mathrm{x_c})$

Problem 1: Linear Pendulum Oscillations

HW2
Math 537 Ordinary Differential Equations
Due Sep 25, 2020

1: [25 points] Consider the following second-order ordinary differential equations (ODEs) for linear pendulum oscillations:

$$\frac{d^2x}{dt^2} + c\frac{dx}{dt} + Kx = 0, \tag{1}$$

which is a linearized version of the nonlinear system:

$$\frac{d^2x}{dt^2} + c\frac{dx}{dt} + Ksin(x) = 0.$$

Assume $c = 5$ and $K = 4$.

(a) Solve Eq. (1) for solutions.

(b) Convert Eq. (1) into a system of first-order ODEs by introducing $y = dx/dt$. Solve the system of the first-order ODEs.

Problem 2: A System of the First-order ODEs

2: [25 points] Consider the following system of linear ODEs:

$$\frac{dx}{dt} = \alpha y, \tag{2a}$$

$$\frac{dy}{dt} = -\beta x. \tag{2b}$$

Discuss the region in the $\alpha\beta$-plane where this system has different types of eigenvalues.

Problem 3: Linearized Lorenz Model

3: [25 points] Consider the following linearized Lorenz model (Lorenz, 1963):

$$\frac{dX}{dt} = -\sigma X + \sigma Y, \qquad (3a)$$

$$\frac{dY}{dt} = rX - Y. \qquad (3b)$$

Perform a stability analysis for $\sigma > 0$ (i.e., discuss the cases with $r > 1$, $r = 1$, and $r < 1$, respectively.)

Problem 4: An Epidemic Model

4: [25 points] Consider the following epidemic model (Kermack and McKendrick, 1927), which is called the "SIR" model:

$$\frac{dS}{dt} = -\frac{\beta}{N} SI, \tag{4.1}$$

$$\frac{dI}{dt} = \frac{\beta}{N} SI - \nu I, \tag{4.2}$$

$$\frac{dR}{dt} = \nu I. \tag{4.3}$$

Here, S, I, and R denote susceptible, infected, and recovered individuals, respectively. Three parameters, $\beta > 0$, $\nu > 0$, and $N > 0$, represent a transmission rate, a recovery rate, and a fixed population ($N = S + I + R$), respectively. Complete the following derivations to convert Eqs. (4.1)-(4.3) into the following equations:

$$S = S(0) e^{-\frac{\beta}{N\nu}(R(t) - R(0))}, \tag{4.4}$$

$$I = N - S(0) e^{-\frac{\beta}{N\nu}(R(t) - R(0))} - R, \tag{4.5}$$

$$\frac{dR}{dt} = \nu \left(N - R - S(0) e^{-\frac{\beta}{N\nu}(R(t) - R(0))} \right), \tag{4.6}$$

where $S(0)$ and $R(0)$ represent the initial values of S and R, respectively.

Problem 4: An Epidemic Model

(a) Show

$$S + I + R = constant = N \tag{4.7}$$

(i.e., $\frac{d(S+I+R)}{dt} = 0$).

(b) Apply Eqs (4.1) and (4.2) to obtain the following:

$$\frac{S'}{S} = -\frac{\beta}{N\nu} R'.$$

Integrate the above Eq. to obtain Eq. (4.4), yielding $S = S(R)$.

(c) Apply Eqs. (4.4) and (4.7) to find Eq. (4.5) for I, which is a function of R.

(d) Apply the above to obtain Eq. (4.6).

(e) Briefly discuss how to analyze Eq. (4.6) to reveal the characteristics of the solution.

Note that based on Eqs. (4.4)-(4.6), we can obtain the solutions by solving a single first order ODE in Eq. (4.6) for $R(t)$, and then compute $S(t)$ and $R(t)$ using Eqs. (4.4) and (4.5), respectively.