Quiz 2
Ordinary Differential Equations
Math 537
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Problem 1: Introduce a new time variable τ to convert the following ODE:

$$\frac{dy}{dt} = \sigma y$$

into

$$\frac{dy}{d\tau} = y$$

Notice the following:

$$\frac{dy}{dt} = \frac{\frac{dy}{d\tau}}{\frac{dt}{d\tau}} = \sigma y$$
$$\frac{dy}{d\tau} = \frac{dt}{d\tau}\sigma y = y$$

From this, we can conclude that

$$\frac{dt}{d\tau}\sigma = 1, \qquad \frac{dt}{d\tau} = \frac{1}{\sigma}$$

So it is now clear that if we introduce a variable $\tau = t\sigma$, we can convert the original equation into the equation we want.

Problem 2: Consider the following Logistic equation

$$\frac{dy}{dt} = \alpha y - \beta y^2$$

Convert the above ODE into the following ODE

$$\frac{dz}{d\tau} = z - z^2$$

by introducing a new time variable τ and a new time-dependent variable z. Find τ and z.

Notice the following:

$$\frac{dy}{dt} = \alpha y - \beta y^2 = \alpha y \left(1 - \frac{\beta}{\alpha} y\right)$$

Let $z = \frac{\beta}{\alpha} y$. This means that $y = \frac{\alpha}{\beta} z$, $\frac{dy}{dz} = \frac{\alpha}{\beta}$. Let $\frac{d\tau}{dt} = \alpha$

$$\begin{aligned}
\frac{dy}{dt} &= \frac{\frac{dy}{dz}}{\frac{dt}{dz}} = \frac{dy}{dz}\frac{dz}{dt} \\
&= \frac{dy}{dz}\frac{\frac{dz}{d\tau}}{\frac{dt}{d\tau}} = \frac{dy}{dz}\frac{dz}{d\tau}\frac{d\tau}{dt} \\
&= \frac{\alpha}{\beta}\frac{dz}{d\tau}\alpha = \frac{\alpha^2}{\beta}\frac{dz}{d\tau} \\
\alpha y \left(1 - \frac{\beta}{\alpha} y\right) &= \frac{\alpha^2}{\beta} z(1-z) = \frac{\alpha^2}{\beta}\frac{dz}{d\tau} \\
\frac{dz}{d\tau} &= z(1-z) = z - z^2
\end{aligned}$$

Now we can solve for τ and z:

$$\frac{dz}{d\tau} - z = -z^2$$

Let $\mu(\tau) = \frac{1}{z}$ and $\frac{d\mu}{d\tau} = \frac{-1}{z^2}\frac{dz}{d\tau}$, and multiply both sides of the equation by $\frac{-1}{z^2}$:

$$\begin{aligned} \frac{-1}{z^2}\frac{dz}{d\tau} + \frac{1}{z} &= 1 \\ \frac{d\mu}{d\tau} + \mu &= 1 \end{aligned}$$

This now becomes a separable equation and we get:

$$\begin{aligned} \frac{d\mu}{1-\mu} &= d\tau \\ -\ln(1-\mu) &= \tau + C \\ \mu &= Ce^{-\tau} + 1 \\ \frac{1}{z} &= Ce^{-\tau} + 1 \\ z &= \frac{1}{Ce^{-\tau} + 1} \end{aligned}$$

We can solve for C by finding $z(0) = z_0 = \frac{1}{C+1} \iff C = \frac{1}{z_0} - 1$. Furthermore, we get:

$$z = \frac{1}{\left(\frac{1}{z_0} - 1\right)e^{-\tau} + 1}$$

We also find τ by integrating our let statement, and get $\tau = \alpha t$

Problem 3: Consider the improper integral

$$\int_{-1}^{1} \frac{1}{x}\,dx \tag{3}$$

(a) Verify whether the following derivations are correct. The above has the following two parts:

$$\int_{-1}^{0} \frac{1}{x}\,dx + \int_{0}^{1} \frac{1}{x}\,dx$$

By introducing $y = -x$ for the first part, we have

$$\int_{1}^{0} \frac{1}{y}\,dy + \int_{0}^{1} \frac{1}{x}\,dx$$

which becomes

$$-\int_{0}^{1} \frac{1}{y}\,dy + \int_{0}^{1} \frac{1}{x}\,dx = 0.$$

We can verify the first step by noticing:

$$\int_{a}^{b} f(x)\,dx = \int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx \qquad \text{(by Prop. of Integrals)}$$

We can verify the second step by noticing:

$$\begin{aligned}
y = -x, &\qquad x = -y \\
dy = -dx, &\qquad dx = -dy \\
\int_{x=-1}^{x=0} \frac{1}{x}\,dx &= \int_{-y=-1}^{-y=0} \frac{1}{-y}(-1)\,dy && \text{(by Subsitution)} \\
&= \int_{1}^{0} \frac{1}{y}\,dy && \text{(Simplify)}
\end{aligned}$$

We can verify the third step by noticing:

$$\begin{aligned}
\int_{1}^{0} \frac{1}{y}\,dy &= -\int_{0}^{1} \frac{1}{y}\,dy && \text{(by the F.T.C)} \\
\lim_{a\to 0}\left[-\int_{a}^{1} \frac{1}{y}\,dy + \int_{a}^{1} \frac{1}{x}\right] &= \lim_{a\to 0}\left[-\ln(y)\big|_{a}^{1} + \ln(x)\big|_{a}^{1}\right] && \text{(Use Limits for Improper Int)} \\
&= \lim_{a\to 0}\left[-\ln(1) + \ln(a) + \ln(1) - \ln(a)\right] = 0 && \text{(Evaluate)}
\end{aligned}$$

(b) Represent Eq. (3) as follows:

$$\lim_{\epsilon\to 0}\int_{-1}^{-\epsilon}\frac{1}{x}\,dx+\lim_{\epsilon\to 0}\int_{\epsilon}^{1}\frac{1}{x}\,dx$$

Complete the above integrals.

Notice we can represent Eq. (3) as follows:

$$\int_{-1}^{1}\frac{1}{x}\,dx=\int_{-1}^{0}\frac{1}{x}\,dx+\int_{0}^{1}\frac{1}{x}\,dx=\lim_{\epsilon\to 0}\int_{-1}^{-\epsilon}\frac{1}{x}\,dx+\lim_{\epsilon\to 0}\int_{\epsilon}^{1}\frac{1}{x}\,dx$$

Notice the following:

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_{-1}^{-\epsilon}\frac{1}{x}\,dx &= \lim_{\epsilon\to 0}\int_{-y=-1}^{-y=-\epsilon}\frac{1}{-y}(-1)\,dy \qquad\qquad \text{(Subsitute } x=-y)\\
&= \lim_{\epsilon\to 0}\int_{1}^{\epsilon}\frac{1}{y}\,dy\\
&= \lim_{\epsilon\to 0}-\int_{\epsilon}^{1}\frac{1}{y}\,dy\\
\lim_{\epsilon\to 0}\int_{-1}^{-\epsilon}\frac{1}{x}\,dx+\lim_{\epsilon\to 0}\int_{\epsilon}^{1}\frac{1}{x}\,dx &= \lim_{\epsilon\to 0}-\int_{\epsilon}^{1}\frac{1}{y}\,dy+\lim_{\epsilon\to 0}\int_{\epsilon}^{1}\frac{1}{x}\,dx\\
&= \lim_{\epsilon\to 0}\left[-\ln(1)+\ln(\epsilon)+\ln(1)-\ln(\epsilon)\right]=0
\end{aligned}$$

(c) Represent Eq. (3) as follows:

$$\lim_{\epsilon\to 0}\int_{-1}^{-2\epsilon}\frac{1}{x}\,dx+\lim_{\epsilon\to 0}\int_{\epsilon}^{1}\frac{1}{x}\,dx$$

Complete the above integrals.

Notice we can represent Eq. (3) as follows:

$$\int_{-1}^{1}\frac{1}{x}\,dx=\int_{-1}^{0}\frac{1}{x}\,dx+\int_{0}^{1}\frac{1}{x}\,dx=\lim_{\epsilon\to 0}\int_{-1}^{-2\epsilon}\frac{1}{x}\,dx+\lim_{\epsilon\to 0}\int_{\epsilon}^{1}\frac{1}{x}\,dx$$

Following the same steps as (3b), we can skip to the following step:

$$\begin{aligned}
\lim_{\epsilon\to 0}\int_{-1}^{-2\epsilon}\frac{1}{x}\,dx+\lim_{\epsilon\to 0}\int_{\epsilon}^{1}\frac{1}{x}\,dx &= \lim_{\epsilon\to 0}-\int_{2\epsilon}^{1}\frac{1}{y}\,dy+\lim_{\epsilon\to 0}\int_{\epsilon}^{1}\frac{1}{x}\,dx\\
&= \lim_{\epsilon\to 0}\left[-\ln(1)+\ln(2\epsilon)+\ln(1)-\ln(\epsilon)\right]=\ln(2)
\end{aligned}$$

(d) Compare the answers in (b) and (c) to provide justifications to your analysis in (a).

I was able to get the same result in (b) and (a). The answers both equal 0, which makes sense graphically because the original equation is symmetric about the origin $(0,0)$, meaning the integral on each side of its vertical asymptote would cancel each other. In (c), because the middle bounds were not the same, one being ϵ and the other being 2ϵ, this results in a different answer. Because ϵ only approaches 0 and not actually equals 0, we cannot say that $2\epsilon=\epsilon$, so (c) cannot be the same result as (b) and (a).