1. (5 pts) Consider the interaction of two species of animals in a habitat. We are told that the change of the populations $x(t)$ and $y(t)$ can be modeled by the equations

$$\frac{dx}{dt} = 0.4x + 0.5y,$$

$$\frac{dy}{dt} = 1.5x - 0.6y.$$

For this system, the smaller eigenvalue is ________ and the larger eigenvalue is ________.

If $y' = Ay$ is a differential equation, how would the solution curves behave?

- A. All of the solution curves would run away from 0. (Unstable node)
- B. All of the solutions curves would converge towards 0. (Stable node)
- C. The solution curves converge to different points.
- D. The solution curves would race towards zero and then veer away towards infinity. (Saddle)

The solution to the above differential equation with initial values $x(0) = 7$, $y(0) = 7$ is

$x(t) =$ ________,

$y(t) =$ ________.

Answer(s) submitted:

- `-11/10`
- `9/10`
- `D`
- `7e^(9t/10)`
- `7e^(9t/10)`

(correct)

Correct Answers:

- `-1.1`
- `0.9`
- `D`
- `0*0.5*exp(-1.1*t)+14*0.5*exp(0.9*t)`
- `0*(-1.1-0.4)*exp(-1.1*t)+14*(0.9-0.4)*exp(0.9*t)`

---

2. (5 pts) Solve the system

$$\frac{dx}{dt} = \begin{bmatrix} -5 & -4 \\ 4 & -5 \end{bmatrix} x$$

with $x(0) = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

Give your solution in real form.

$x_1 =$ ________,

$x_2 =$ ________.

?1. Describe the trajectory.

Answer(s) submitted:

- `-5sin(4t)e^(-5t) + 4cos(4t)e^(-5t)`
- `5cos(4t)e^(-5t) + 4sin(4t)e^(-5t)`
- `Spiral inward counterclockwise`

(correct)

Correct Answers:

- `e^(-5*t) * (4*cos(4*t) - 5*sin(4*t))`
- `e^(-5*t) * (4*sin(4*t) + 5*cos(4*t))`
- `SPIRAL INWARD COUNTERCLOCKWISE`

---

3. (5 pts) Solve the system

$$\frac{dx}{dt} = \begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix} x$$

with $x(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

Give your solution in real form.

$x_1 =$ ________,

$x_2 =$ ________.

?1. Describe the trajectory.

Answer(s) submitted:

- `2cos(t)`
- `2(2cos(t)+sin(t))`
- `Ellipse counterclockwise`

(correct)

Correct Answers:

- `2*(- (2/-1)*sin(1*t) + cos(1*t)) + (4/-1)*(sin(1*t))`
- `2*(-(2*2/-1)*sin(1*t) + 1*sin(1*t)) + 4*(cos(1*t) + (2/-1)`
- `ELLIPSE COUNTERCLOCKWISE`

---

4. (5 pts) Solve the system

$$\frac{dx}{dt} = \begin{bmatrix} 1 & 6 \\ -1 & 6 \end{bmatrix} x$$

with the initial value $x(0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

$$x(t) = \begin{bmatrix} \_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_ \end{bmatrix}.$$

Answer(s) submitted:

- `-3(2)e^(4t) + 2(3)e^(3t)`
- `-3(1)e^(4t) + 2(1)e^(3t)`

(correct)

Correct Answers:

- `2*e^(3*t)*3 + -3*e^(4*t)*2`
- `2*e^(3*t)*1 + -3*e^(4*t)*1`

5. (3 pts) Find the equilibrium solution for

$$x_1'(t) = -7.2 + 1.2x_1 - 0.8x_2$$

$$x_2'(t) = -13.8 + 2.1x_1 - 1.2x_2$$

$$x_1(0) = 12;\ x_2(0) = 8$$

Equilibrium: $x_1^e =$ __________,
$x_2^e =$ __________.

[?] 1. Describe the trajectory.

Answer(s) submitted:

- `10`
- `6`
- `Ellipse counterclockwise`

(correct)

Correct Answers:

- `10`
- `6`
- `ELLIPSE COUNTERCLOCKWISE`

---

6. (3 pts) Find the equilibrium solution for

$$x_1'(t) = -9.7 + 1.2x_1 - 0.5x_2$$

$$x_2'(t) = -9.8 + 1.4x_1 - 0.8x_2$$

$$x_1(0) = 20;\ x_2(0) = 34$$

Equilibrium: $x_1^e =$ __________,
$x_2^e =$ __________.

If $y' = Ay$ is a differential equation, how would the solution curves behave?

- A. All of the solution curves would run away from the equilibrium point. (Unstable node)
- B. The solution curves would race towards the equilibrium point and then veer away towards infinity. (Saddle)
- C. The solution curves converge to different points.
- D. All of the solutions curves would converge towards the equilibrium point. (Stable node)

Answer(s) submitted:

- `11`
- `7`
- `B`

(correct)

Correct Answers:

- `11`
- `7`
- `B`

---

7. (5 pts)

(1) Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 8 & -6 \\ 0 & 2 \end{bmatrix}$.

$\lambda_1 =$ ___, $\vec{v}_1 = \begin{bmatrix} \text{---} \\ \text{---} \end{bmatrix}$, and $\lambda_2 =$ ___, $\vec{v}_2 = \begin{bmatrix} \text{---} \\ \text{---} \end{bmatrix}$

(2) Solve the system of differential equations $\vec{x}' = \begin{bmatrix} 8 & -6 \\ 0 & 2 \end{bmatrix} \vec{x}$ satisfying the initial conditions $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$.

$x_1(t) =$ __________________
$x_2(t) =$ __________________

Answer(s) submitted:

- `8`
- `-3(1)e^(8t)+-1(1)e^(2t)`
- `-3(0)e^(8t)+-1(1)e^(2t)`

(correct)

Correct Answers:

- `2; -1; -1; 8; 1; 0`
- `1 * -1 e^(2 t) + -3 * 1 e^(8 t)`
- `1 * -1 e^(2 t) + -3 * 0 e^(8 t)`

---

8. (5 pts) Consider the linear system

$$\vec{y}' = \begin{bmatrix} 6 & 4 \\ -12 & -8 \end{bmatrix} \vec{y}.$$

(1) Find the eigenvalues and eigenvectors for the coefficient matrix.

$\lambda_1 =$ ___, $\vec{v}_1 = \begin{bmatrix} \text{---} \\ \text{---} \end{bmatrix}$, and $\lambda_2 =$ ___, $\vec{v}_2 = \begin{bmatrix} \text{---} \\ \text{---} \end{bmatrix}$

(2) For each eigenpair in the previous part, form a solution of $\vec{y}' = A\vec{y}$. Use t as the independent variable in your answers.

$\vec{y}_1(t) = \begin{bmatrix} \text{-----} \\ \text{-----} \end{bmatrix}$ and $\vec{y}_2(t) = \begin{bmatrix} \text{-----} \\ \text{-----} \end{bmatrix}$

(3) Does the set of solutions you found form a fundamental set (i.e., linearly independent set) of solutions? [?]

Answer(s) submitted:

- `0`
- `-2`
- `Yes, it is a fundamental set`

(correct)
Correct Answers:

- `0; 2; -3; -2; -1; 2`
- `2*e^(0*t); -3*e^(0*t); -1*e^(-2*t); 2*e^(-2*t)`
- `Yes, it is a fundamental set`

---

9. (4 pts)

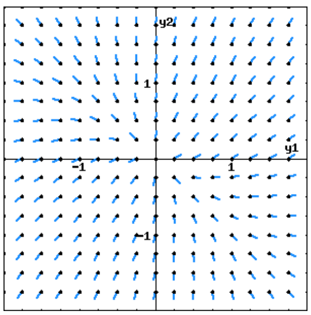
Match each linear system with one of the phase plane direction fields. (The blue lines are the arrow shafts, and the black dots are the arrow tips.)

$\boxed{?}$1. $\vec{y}' = \frac{1}{2}\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}\vec{y}$

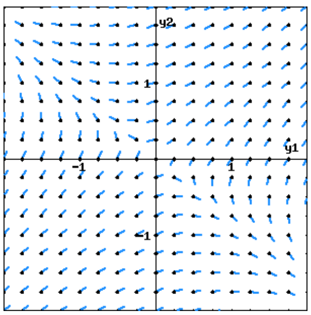
$\boxed{?}$2. $\vec{y}' = \frac{-1}{3}\begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix}\vec{y}$

$\boxed{?}$3. $\vec{y}' = \begin{bmatrix} 1 & -3 \\ 0 & -2 \end{bmatrix}\vec{y}$

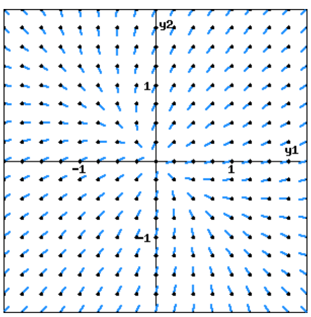
$\boxed{?}$4. $\vec{y}' = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\vec{y}$



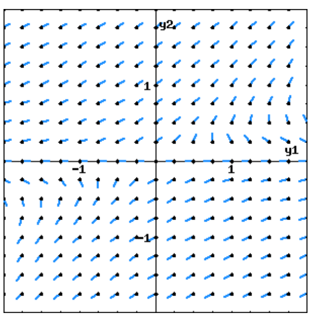
A



B



C



D

Answer(s) submitted:

- `B`
- `A`
- `D`
- `C`

(correct)
Correct Answers:

- `B`
- `A`
- `D`
- `C`

---

10. (3 pts)

Suppose

$$\vec{y}(t) = c_1 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\vec{y}(1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

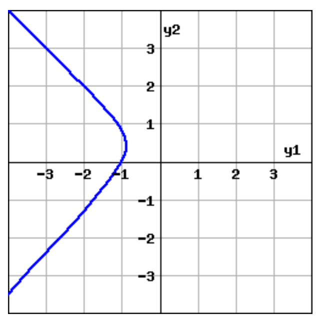
(a) Find c_1 and c_2.

$c_1 =$ __________
$c_2 =$ __________

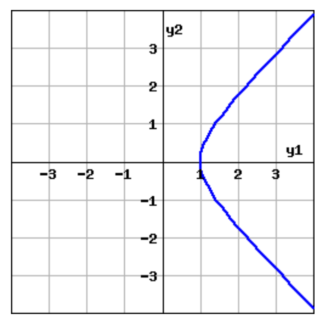
(b) Sketch the phase plane trajectory that satisfies the given initial condition. Which graph most closely resembles the graph you drew? $\boxed{?}$

(c) What is the approximate direction of travel for the solution curve, as t increases from $-\infty$ to $+\infty$?

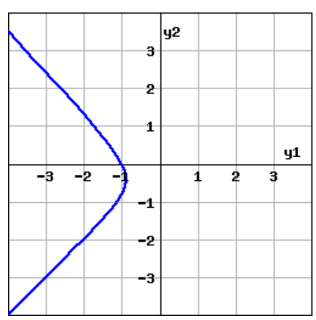
- A. along the line $y = -x$ toward the origin and then along the line $y = x$ away from the origin
- B. along the line $y = x$ toward the origin and then along the line $y = -x$ away from the origin
- C. none of the above



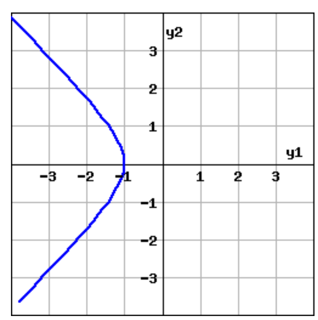
A



B



C



D

Answer(s) submitted:

- `-e/2`
- `-1/(2e)`
- `D`
- `A`

(correct)
Correct Answers:

- `-1.35914091422952`
- `-0.183939720585721`
- `D`
- `A`

---

11. (4 pts)

(1) Find the most general real-valued solution to the linear system of differential equations $\vec{x}' = \begin{bmatrix} 0 & 2 \\ -3 & 5 \end{bmatrix} \vec{x}$.

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \end{bmatrix} + c_2 \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \end{bmatrix}$$

(2) In the phase plane, this system is best described as a

- spiral sink
- saddle
- sink / stable node
- source / unstable node
- center point / ellipses
- spiral source
- none of these

(3) In your written HW, show your calculations for obtaining the eigenvalues and eigenvectors. Sketch a graph of the phase portrait, showing several typical solutions. (The sketch can be hand drawn or computer generated.) Explain why you observe the behavior sketched in a brief paragraph.

Answer(s) submitted:

- `2e^(3t)`
- `source / unstable node`

(correct)
Correct Answers:

- `-2*e^(3*t); -3*e^(3*t); 1*e^(2*t); 1*e^(2*t)`
- `source / unstable node`

---

12. (4 pts)

(1) Find the most general real-valued solution to the linear system of differential equations $\vec{x}' = \begin{bmatrix} -12 & -10 \\ 15 & 13 \end{bmatrix} \vec{x}$.

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \end{bmatrix} + c_2 \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \end{bmatrix}$$

(2) In the phase plane, this system is best described as a

- sink / stable node
- saddle
- source / unstable node
- center point / ellipses
- spiral source
- spiral sink
- none of these

(3) In your written HW, show your calculations for obtaining the eigenvalues and eigenvectors. Sketch a graph of the phase portrait, showing several typical solutions. (The sketch can be hand drawn or computer generated.) Explain why you observe the behavior sketched in a brief paragraph.

Answer(s) submitted:

- `2e^(3t)`
- `saddle`

(correct)
Correct Answers:

- `-2*e^(3*t); 3*e^(3*t); -1*e^(-2*t); 1*e^(-2*t)`
- `saddle`

---

13. (4 pts)

(1) Find the most general real-valued solution to the linear system of differential equations $\vec{x}' = \begin{bmatrix} 0 & -25 \\ 1 & 0 \end{bmatrix} \vec{x}$.

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \end{bmatrix} + c_2 \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \end{bmatrix}$$

(2) In the phase plane, this system is best described as a

- spiral source
- saddle
- sink / stable node
- center point / ellipses
- source / unstable node
- spiral sink
- none of these

(3) In your written HW, show your calculations for obtaining the eigenvalues and eigenvectors. Sketch a graph of the phase portrait, showing several typical solutions. (The sketch can be hand drawn or computer generated.) Explain why you observe the behavior sketched in a brief paragraph.

Answer(s) submitted:

- `-5sin(5t)`
- `center point / ellipses`

(correct)

Correct Answers:

- `5*cos(5*t); sin(5*t); 5*sin(5*t); -[cos(5*t)]`
- `center point / ellipses`

---

14. (4 pts)

(1) Find the most general real-valued solution to the linear system of differential equations $\vec{x}' = \begin{bmatrix} 6 & -9 \\ 1 & 6 \end{bmatrix} \vec{x}$.

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \end{bmatrix} + c_2 \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \end{bmatrix}$$

(2) In the phase plane, this system is best described as a

- spiral source
- center point / ellipses
- source / unstable node
- sink / stable node
- saddle
- spiral sink
- none of these

(3) In your written HW, show your calculations for obtaining the eigenvalues and eigenvectors. Sketch a graph of the phase portrait, showing several typical solutions. (The sketch can be hand drawn or computer generated.) Explain why you observe the behavior sketched in a brief paragraph.

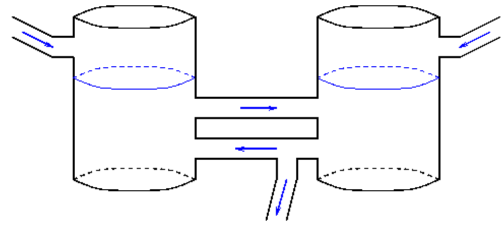
Answer(s) submitted:

- `-3sin(3t)e^(6t)`
- `spiral source`

(correct)

Correct Answers:

- `3*cos(3*t)*e^(6*t); sin(3*t)*e^(6*t); 3*sin(3*t)*e^(6*t); -[cos(3*t)]*e^(6*t)`
- `spiral source`



15. (8 pts)

Consider two interconnected tanks as shown in the figure above. Tank 1 initial contains 100 L (liters) of water and 395 g of salt, while tank 2 initially contains 20 L of water and 480 g of salt. Water containing 50 g/L of salt is poured into tank1 at a rate of 4 L/min while the mixture flowing into tank 2 contains a salt concentration of 15 g/L of salt and is flowing at the rate of 3 L/min. The two connecting tubes have a flow rate of 6.5 L/min from tank 1 to tank 2; and of 2.5 L/min from tank 2 back to tank 1. Tank 2 is drained at the rate of 7 L/min.

You may assume that the solutions in each tank are thoroughly mixed so that the concentration of the mixture leaving any tank along any of the tubes has the same concentration of salt as the tank as a whole. (This is not completely realistic, but as in real physics, we are going to work with the approximate, rather than exact description. The 'real' equations of physics are often too complicated to even write down precisely, much less solve.)

How does the water in each tank change over time?

Let $p(t)$ and $q(t)$ be the amount of salt in g at time t in tanks 1 and 2 respectively. Write differential equations for p and q. (As usual, use the symbols p and q rather than $p(t)$ and $q(t)$.)

$p' =$ __________

$q' =$ __________

Give the initial values:

$$\begin{bmatrix} \text{p(0)} \\ \text{q(0)} \end{bmatrix} = \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \end{bmatrix}.$$

Show the equation that needs to be solved to find a constant solution to the differential equation:

$$\begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \end{bmatrix} = \begin{bmatrix} \_\_\_\_ & \_\_\_\_ \\ \_\_\_\_ & \_\_\_\_ \end{bmatrix} \begin{bmatrix} \text{p} \\ \text{q} \end{bmatrix}.$$

A constant solution is obtained if $p(t) =$ ____ for all time t and $q(t) =$ ____ for all time t.

Answer(s) submitted:

- `4(50) + 2.5(q/20) - 6.5(p/100)`
- `3(15) + 6.5(p/100) - 9.5(q/20)`
- `395`
- `480`
- `-4(50)`
- `-3(15)`
- `-6.5/100`

- `2.5/20`
- `6.5/100`
- `-9.5/20`
- `4423.076923`
- `700`

(correct)
Correct Answers:

- `50*4 - (6.5/100)p + (2.5/20)q`
- `15*3 - ( 2.5/20 + 7/20 )q + ( 6.5/100 )p`
- `395`
- `480`
- `-200`
- `-45`
- `-0.065`
- `0.125`
- `0.065`
- `-0.475`
- `4423.07692307692`
- `700`