Math 320 List of Definitions and Theorems, March 12

Definition 1: A function $f : A \to B$ is called **one-to-one** or **injective** if f satisfies the following property: for all $x, y \in A$, if $x \neq y$, then $f(x) \neq f(y)$.

Here is an alternate, but equivalent, definition: If $f(x) = f(y)$, then $x = y$.

Definition 2: A function $f : A \to B$ is called **onto** or **surjective** if f satisfies the following property: for all $y \in B$, there exists $x \in A$ such that $f(x) = y$.

Definition 3: A function $f : A \to B$ is called **bijective** or a **one-to-one correspondence** if f is *both* injective *and* surjective.

Definition 4: A ring R is **isomorphic** to a ring S, which we denote by $R \cong S$, if there exists a function $f : R \to S$, which we call an **isomorphism** such that

(1) f is injective;

(2) f is surjective;

(3) (**Homomorphism property**) $f(a+b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for all $a, b \in R$.

Alternatively, we can say that $f : R \to S$ is an isomorphism if f is a bijection and satisfies the homomorphism property.

Theorem 1: Let R, S, T be rings. Then,

(a) If $R \cong S$, then $S \cong R$,

(b) If $R \cong S$ and $S \cong T$, then $R \cong T$.

Definition 5: Let R and S be rings. A function $f : R \to S$ is called a **homomorphism** if $f(a+b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for all $a, b \in R$.

Theorem 3.10: Let $f : R \to S$ be a homomorphism of rings. Then

(1) $f(0_R) = 0_S$.

(2) $f(-a) = -f(a)$ for every $a \in R$.

(3) $f(a-b) = f(a) - f(b)$ for all $a, b \in R$

If R is a ring with identity and f is surjective, then

(4) S is a ring with identity $1_S = f(1_R)$.

(5) Whenever u is a unit in R, then $f(u)$ is a unit in S and $f(u)^{-1} = f(u^{-1})$.

Corollary 3.11: If $f : R \to S$ is a homomorphism of rings, then the image of f is a subring of S.