Quiz 12
Differential Equations
Math 337
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Problem 1: Consider the Taylor series given by the following:

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (x-1)^{2n-1}}{3n-1}$$

Find all values of x where this series converges absolutely, diverges, or converges conditionally. Give the series test that shows the convergence or divergence. What is the radius of convergence for this series about x = 1?

Notice the ratio test:

$$\begin{aligned}\lim_{n\to\infty} \frac{(-1)^{n+1}(x-1)^{2n+1}}{3n+2} * \frac{3n-1}{(-1)^n(x-1)^{2n-1}} &= \lim_{n\to\infty} \frac{-(x-1)^2(3n-1)}{3n+2} \\ &= -(x-1)^2\end{aligned}$$

The series converges absolutely:

$$\begin{aligned}(x-1)^2 &< 1 \\ |x-1| &< 1 \\ 0 < x &< 2\end{aligned}$$

The series diverges:

$$\begin{aligned}(x-1)^2 &> 1 \\ |x-1| &> 1 \\ x < 0 \text{ or } x &> 2\end{aligned}$$

The series converges conditionally at $x = 0$ and $x = 2$.

The radius of convergence for this series about $x = 1$ is $\rho = 1$

Problem 2: Solve the following ODE with a power series method:

$$(4 - x^2)y'' - xy' + 16y = 0$$

Assume a power series solution of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Clearly state the recurrence relation. Determine all the coefficients a_n for $n = 2, ..., 10$ in terms of the two arbitrary constants, a_0 and a_1. Find the two linearly independent solutions, y_1 and y_2, up to and including terms of x^{10}. (You are not expected to find a closed form solution of any infinite series.) Determine all values of x where your solutions y_1 and y_2 converge absolutely.

Notice the following:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\begin{aligned}
(4 - x^2)y'' - xy' + 16y &= \sum_{n=2}^{\infty} 4n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 16 a_n x^n \\
&= \sum_{n=0}^{\infty} 4(n+2)(n+1)a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 16 a_n x^n \\
&= 8a_2 + 24a_3 x + \sum_{n=2}^{\infty} 4(n+2)(n+1)a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n \\
&- a_1 x - \sum_{n=2}^{\infty} n a_n x^n + 16a_0 + 16a_1 x + \sum_{n=2}^{\infty} 16 a_n x^n \\
&= \sum_{n=2}^{\infty} (4(n+2)(n+1)a_{n+2} - (n^2 - 16)a_n)x^n + (24a_3 + 15a_1)x + 8(a_2 + 2a_0) \\
&= 0
\end{aligned}$$

The recurrence relation is as follows:

$$\begin{aligned}
4(n+2)(n+1)a_{n+2} - (n^2 - 16)a_n &= 0 \\
24a_3 + 15a_1 &= 0 \\
a_2 + 2a_0 &= 0
\end{aligned}$$

Notice we get the following from the recurrence relation:

$$\begin{aligned}
a_{n+2} &= \frac{n^2 - 16}{4(n+2)(n+1)} a_n \\
a_2 &= -2a_0 \\
a_3 &= \frac{-5}{8} a_1
\end{aligned}$$

Notice the following coefficients:

$$
\begin{aligned}
a_0 &= a_0 \\
a_1 &= a_1 \\
a_2 &= -2a_0 \\
a_3 &= \frac{-15}{4(3)(2)}a_1 \\
a_4 &= \frac{-1}{4}a_2 = \frac{-1}{4}(-2a_0) = \frac{1}{2}a_0 \\
a_5 &= \frac{-7}{4(5)(4)}a_3 = \frac{-7}{4(5)(4)}\left(\frac{-15}{4(3)(2)}a_1\right) \\
a_6 &= 0a_4 \\
a_7 &= \frac{9}{4(7)(6)}a_5 = \frac{9}{4(7)(6)}\left(\frac{-7}{4(5)(4)}\right)\left(\frac{-15}{4(3)(2)}a_1\right) \\
a_8 &= \frac{5}{56}a_6 = 0 \\
a_9 &= \frac{33}{4(9)(8)}a_7 = \frac{33}{4(9)(8)}\left(\frac{9}{4(7)(6)}\right)\left(\frac{-7}{4(5)(4)}\right)\left(\frac{-15}{4(3)(2)}a_1\right) \\
a_{10} &= \frac{2}{15}a_8 = 0
\end{aligned}
$$

So we get the following:

$$
y(x) = a_0\left(1 - 2x^2 + \frac{1}{2}x^4\right) + a_1\left(x + \sum_{n=1}^{\infty}\frac{[(2n-1)^2-16][(2n-3)^2-16]......[1-16]}{4^n(2n+1)!}x^{2n+1}\right)
$$

So thus we get:

$$
y_1 = 1 - 2x^2 + \frac{1}{2}x^4 \qquad\qquad y_2 = x + \frac{-5}{8}x^3 + \frac{7}{128}x^5 + \frac{3}{1024}x^7 + \frac{33}{98304}x^9
$$

Notice that through the ratio test of

$$
\lim_{n\to\infty}\left|\frac{a_n+2}{a_n}\right|x^2 = \lim_{n\to\infty}\left|\frac{n^2-16}{4(n+2)(n+1)}\right|x^2 = \frac{x^2}{4} < 1
$$

So we get that y_2 converges absolutely for $|x| < 2$ and y_1 converges for all x