Homework 6
Numerical Matrix Analysis
Math 543
Stephen Giang

Problem TB-18.1:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \\ 1 & 1.0001 \end{bmatrix}, \qquad b = \begin{bmatrix} 2 \\ 0.0001 \\ 4.0001 \end{bmatrix}$$

(a) What are the matrices A^+ and P for this example? Give exact answers.

$$\begin{aligned}
A^+ = (A^*A)^{-1}A^* &= \left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.0001 & 1.0001 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \\ 1 & 1.0001 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.0001 & 1.0001 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 3.0002 \\ 3.0002 & 3.00040002 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.0001 & 1.0001 \end{bmatrix} \\
&= \frac{1}{2 * 10^{-8}} \begin{bmatrix} 3.00040002 & -3.0002 \\ -3.0002 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.0001 & 1.0001 \end{bmatrix} \\
&= 50,000,000 \begin{bmatrix} .00020002 & -.0001 & -.0001 \\ -.0002 & .0001 & .0001 \end{bmatrix} \\
&= \begin{bmatrix} 10,001 & -5,000 & -5,000 \\ -10,000 & 5,000 & 5,000 \end{bmatrix} \\
P = AA^+ &= \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \\ 1 & 1.0001 \end{bmatrix} \begin{bmatrix} 10,001 & -5,000 & -5,000 \\ -10,000 & 5,000 & 5,000 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}
\end{aligned}$$

(b) Find the exact solutions x and $y = Ax$ to the least squares problem $Ax \approx b$

$$\begin{aligned}
A^+Ax = x = A^+b &= \begin{bmatrix} 10,001 & -5,000 & -5,000 \\ -10,000 & 5,000 & 5,000 \end{bmatrix} \begin{bmatrix} 2 \\ 0.0001 \\ 4.0001 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
y = Ax = AA^+b = Pb &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 2 \\ 0.0001 \\ 4.0001 \end{bmatrix} = \begin{bmatrix} 2 \\ 2.0001 \\ 2.0001 \end{bmatrix}
\end{aligned}$$

(c) What are $\kappa(A), \theta$, and η? From here on, numerical answers are acceptable.

$$\kappa(A) = ||A||||A^+|| \approx 42429.2354161703$$
$$\theta = \cos^{-1}\frac{||y||}{||b||} \approx 0.684702873261185^R$$
$$\eta = \frac{||A||||x||}{||y||} \approx 1.000000000555537$$

(d) What are the four condition numbers of Theorem 18.1?

	y	x
b	$\frac{1}{\cos\theta}$	$\frac{\kappa(A)}{\eta\cos\theta}$
A	$\frac{\kappa(A)}{\cos\theta}$	$\kappa(A) + \frac{\kappa(A)^2\tan\theta}{\eta}$

=

	y	x
b	1.290977236078942	54775.17703608065
A	54775.17706651028	1469883252.863362

(e) Give examples of perturbations δb and δA that approximately attain these four condition numbers.

Let $\delta b = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$, with $a \in \mathbb{R}$, Now notice that $P\delta b = \delta b$, and $y = Pb$

$$\begin{aligned}
\kappa_{b\mapsto y} &= \frac{||P(b+\delta b) - Pb||}{||y||} \Big/ \frac{||\delta b||}{||b||} \\
&= \frac{||P\delta b = \delta b||}{||y||} * \frac{||b||}{||\delta b||} \\
&= \frac{||b||}{||y||} \\
&= 1.290977236078942
\end{aligned}$$

Let $\delta b = \begin{bmatrix} 1 \\ -0.5 \\ -0.5 \end{bmatrix}$, so $||A^+\delta b|| = 21213.91056005508 = ||A^+||||\delta b||$, and $x = A^+b$.

$$\kappa_{b\mapsto x} = \frac{||A^+\delta b||}{||A^+b||} \Big/ \frac{||\delta b||}{||b||} = \frac{21213.91056005508}{1.414213562373095} * \frac{4.472225399060293}{1.224744871391589} = 54775.17703608065$$

Let $\delta A = \begin{bmatrix} 10^{-12} & -10^{-12} \\ -10^{-12} & 10^{-12} \\ 10^{-12} & -10^{-12} \end{bmatrix}$, so $\tilde{A} = A + \delta A$, and $\tilde{y} = \tilde{A}(\tilde{A}^*\tilde{A})^{-1}\tilde{A}^*b = \begin{bmatrix} 2.000000080004251 \\ 2.000099959999875 \\ 2.000099960003150 \end{bmatrix}$

$$\kappa_{A \mapsto y} = \frac{||\tilde{y} - y||}{||y||} \Big/ \frac{||\delta A||}{||A||} = \frac{9.798024764246804 * 10^{-8}}{3.464217086160112} * \frac{2.449571394482489}{2.449489742783178 * 10^{-12}}$$
$$= 28284.46119148417$$

Let the above be true, so $\tilde{x} = \tilde{A}^+ b = \begin{bmatrix} 1.001200092248810 \\ 0.998799987752254 \end{bmatrix}$.

$$\kappa_{A \mapsto x} = \frac{||\tilde{x} - x||}{||x||} \Big/ \frac{||\delta A||}{||A||} = \frac{0.001697102831166}{1.414213562373095} * \frac{2.449571394482489}{2.449489742783178 * 10^{-12}}$$
$$= 1200072922.383391$$

Problem PB-14.1: We could use these matrices (A_k) to least-squares-fit polynomials (of matching degree k) to some data-set with 101 measurements. Is it necessarily better to have more model parameters (i.e. fitting a higher degree polynomial)? — Discuss.

Based on my observations, it is not necessarily better to have more model parameters as the Vandermonde Matrix is very ill-conditioned. So as k increases, (the degree of the polynomial gets higher), we reach larger and larger condition numbers showing us that the matrix is ill conditioned, meaning we lose a lot of accuracy when trying to use it. As you can see from the plot, you get insanely high condition numbers such as 10^{27}

```
A = [1 1;1 1.0001; 1 1.0001 ];
b = [2; 0.0001; 4.0001];

Ap = (transpose(A)*A)\transpose(A);

P = A*Ap;
x = Ap*b;
y = P*b;

%-----------------------------------------------------------------

da = [10^(-12) -10^(-12); -10^(-12) 10^(-12); 10^(-12) -10^(-12)];
At = A + da;

Atp = (transpose(At)*At)\transpose(At);
Pt = At*Atp;
xt = Atp*b;
yt = Pt*b;

( norm(yt - y) / norm(y) ) / (norm(da) / norm(A)) %#ok
( norm(xt - x) / norm(x) ) / (norm(da) / norm(A)) %#ok
```

```
clear
figure(1)
clf
hold off

grid on
hold on

x = linspace(0,1,101);
x = transpose(x);

bigK = 1000;
c = zeros(bigK,1);

for k = 0 : bigK
    A = x .^ 0;
    for i = 1 : k
        A = horzcat(A,x.^i);
    end
    c(1 + k,1) = log10(cond(A,2));
end

k = 0 : bigK;
plot(k,c,'r')

title('Condition Numbers for Vandermonde Matrix, $A_k$','interpreter','latex');
xlabel('k values');
ylabel('$log_{10} \kappa (A_k)$','interpreter','latex')
xticks(0:25:bigK);
yticks(0:1:30);
```


Condition Numbers for Vandermonde Matrix, A_k
$log_{10}\kappa(A_k)$
k values
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
0
25
50
75
100
125
150
175
200
225
250
275
300
325
350
375
400
425
450
475
500
525
550
575
600
625
650
675
700
725
750
775
800
825
850
875
900
925
950
975
1000