MATH 693A Advanced Numerical Methods: Computational Optimization
Homework #1

Due in Canvas, September 16
Dr. Uduak George, Fall 2024

`Problem 1 [65pts] (NW2nd-3.1):`
Program the steepest descent and Newton algorithms using the backtracking line search. Use them to minimize the Rosenbrock function

$$f(\bar{\mathbf{x}}) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$

Set the initial step length $\alpha_0 = 1$ and report the step length used by each method at each iteration. First try the initial point $\bar{\mathbf{x}}_0^T = [1.2,\ 1.2]$ and then the more difficult point $\bar{\mathbf{x}}_0^T = [-1.2,\ 1]$.

Suggested values: $\overline{\alpha} = 1$, $\rho = \frac{1}{2}$, $c = 10^{-4}$.

a. Stop when: $\|\nabla f(\vec{x}_k)\| < 10^{-8}$.

   You should hand in (i) your code (ii) the first 6 and last 6 values of $\vec{x}_k$ obtained from your program for steepest descent and Newton algorithms and (iii) determine the minimizer of the Rosenbrock function $x*$.

b. Repeat (a.) above but stop when $|f(\vec{x}_k)| < 10^{-8}$. Compare your results with those from (a.) and discuss your observation with regards to number of iterations required in order to achieve convergence.

`Problem 2 [10pts]:`
Using the $\vec{x}_k$ values you obtained in Problem 1:
(i) Plot the value of objective function $f(\vec{x}_k)$ against the iteration number for the steepest descent algorithm.
(ii) Plot the value of objective function $f(\vec{x}_k)$ against the iteration number for the Newton algorithms.
(iii) Compare the graph obtained in (i) with the one obtained in (ii). What can you infer about the convergence of the steepest descent and Newton algorithm.

`Problem 3 [10pts]:`
Let

$$\begin{aligned}
f(x,y) &= 5 - 5x - 2y + 2x^2 + 5xy + 6y^2, \\
g(x,y) &= \frac{(x^2 - 0.5) + (y^2 - 3) + (x^2 - 1)(y^2 - 4)}{(x^2 + y^2 + 1)^2} \\
h(x,y) &= \frac{(x^2 - 0.25) + (y^2 - 3) + (x^2 - 0.25)(y^2 - 4)}{(x^2 + y^2 + 1)^2}
\end{aligned}$$

[a.] Determine if the function $f(x,y)$ is convex.
[b.] Create a contour plot and a surface plot for $f(x,y)$, $g(x,y)$ and $h(x,y)$ using a programming language of your choice. Use $x = [-3,3]$ and $y = [-3,3]$

`Problem 4 [5 pts]:`
(i) Show that the sequence $x_k = 1 + (0.5)^{2^k}$ is Q-quadratically convergent.

(ii) Does the sequence $x_k = 1/k!$ converge Q-superlinearly? or Q-quadratically?

`Problem 5 [5 pts]:`
Consider the one-dimensional function

$$f(z) = \begin{cases} (x-1)^2 + 2, & -1 \le x \le 1, \\ 2, & 1 \le x \le 2, \\ -(x-2)^2 + 2, & 2 \le x \le 2.5, \\ (x-3)^2 + 1.5, & 2.5 \le x \le 4, \\ -(x-5)^2 + 3.5, & 4 \le x \le 6, \\ -2x + 14.5, & 6 \le x \le 6.5, \\ 2x - 11.5, & 6.5 \le x \le 8, \end{cases}$$

defined over the interval $[-1, 8]$. (i) Graph the function. (ii) Identify the strict global maximum point. (iii) Identify the local maximum and the strict local minimum points.

`Problem 6 [5 pts]:` Determine if any of the following matrices are positive definite.

$$A = \begin{pmatrix} 4 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} -4 & 1 & 1 \\ 1 & -4 & 1 \\ 1 & 1 & -4 \end{pmatrix}.$$