MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #30

A Final Review

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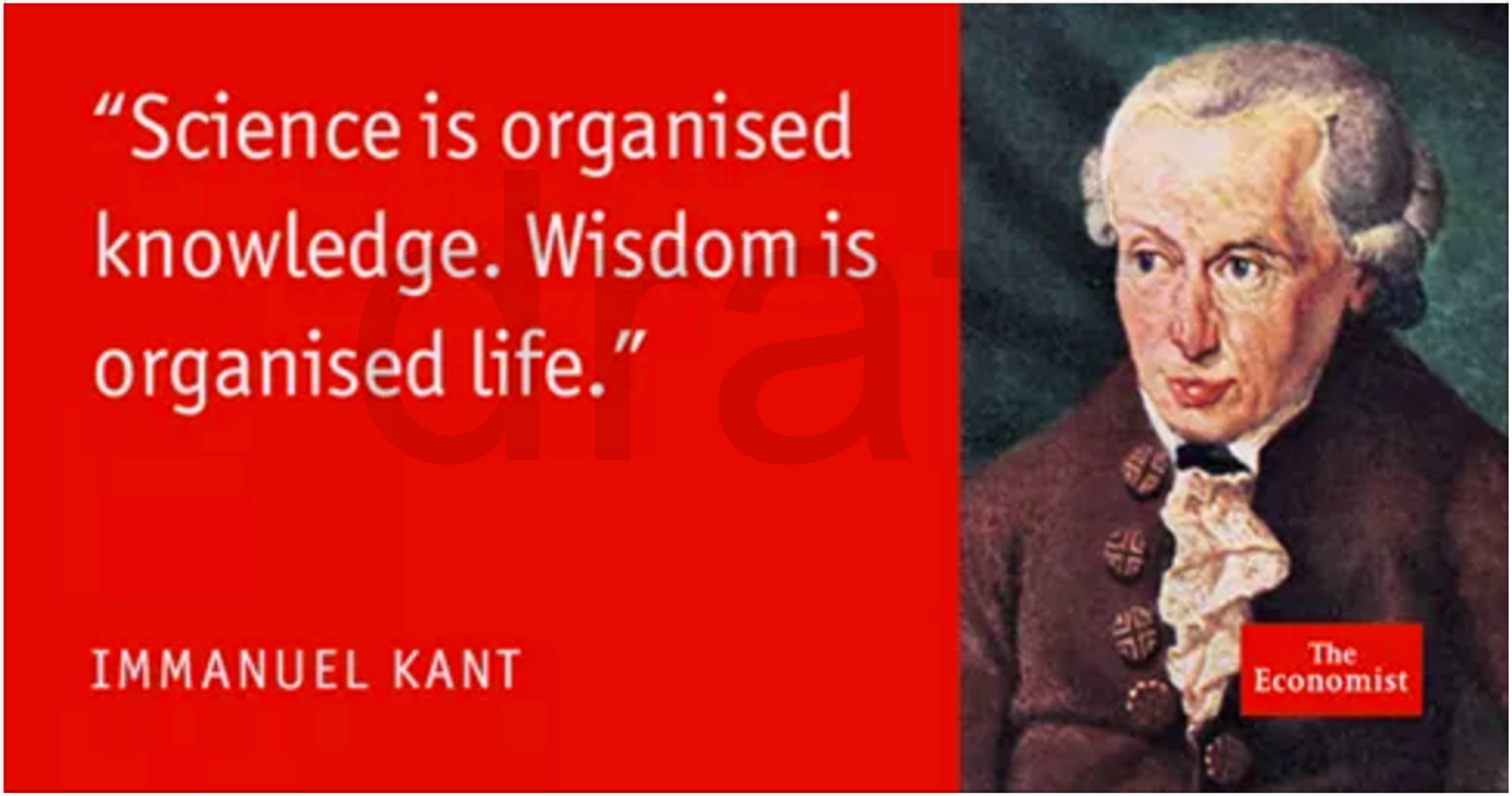
A Personal View on Education

"It is not so very important for a person to learn facts. For that he does not really need a college. He can learn them from books. The value of an education in a liberal arts college is not the learning of many facts, but the training of the mind to think something that cannot be learned from textbooks."

1. Formulate
2. Solve
3. Interpret

The quote comes from Einstein's 1921 response to Thomas Edison's statement that a college education is useless.
http://www.quora.com/Did-Einstein-say-Education-is-not-the-learning-of-facts-but-the-training-of-minds-to-think

Immanuel Kant



One Slide Summary

(A) 1st order	(B) 2nd order	(C) eigenvalue problem
$\boxed{y' = \alpha y - \beta y^{2}}$ (logistic eq.)	$x'' + \beta x' + \alpha x = 0$	$x' = ax + by$ $y' = cx + dy$
$y' = \alpha y - \beta y^3$	$x' = y$ $y' = -\alpha x - \beta y$	$X' = AX$ $AX = \lambda X$ $X = \begin{pmatrix} x \\ y \end{pmatrix}; A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

(D) nonlinear	(E) a system of ODEs	**(F)**
$x'' - \alpha x + \beta x^3 = 0$ (DE-sech)	$x' = y \equiv F$ $y' = \alpha x - \beta x^3 \equiv G$	$JX = \lambda X$ $J = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{X_c}$
$x'' - \alpha x + \beta x^2 = 0$ (DE-sech^2)	$x' = y \equiv F$ $y' = \alpha x - \beta x^2 \equiv G$	

Sect. 1.2: the Logistic Equation

$$\boxed{x' = ax}$$

- linear population model if $a > 0$
- x: population (i.e., assume $\mathrm{x} > 0$).
- $\frac{\mathrm{dx}}{\mathrm{dt}}$: the rate of growth of the population, (called a growth rate, or an exponential growth rate)
- $\frac{\mathrm{dx}}{\mathrm{dt}}$ is proportional to x

$$\boxed{x' = ax\left(1 - \frac{\mathrm{x}}{\mathrm{N}}\right)}$$

- $\frac{\mathrm{dx}}{\mathrm{dt}}$ is proportional to x for small x (and $\mathrm{x} < N$).
- $\frac{\mathrm{dx}}{\mathrm{dt}}$ becomes negative for large x (i.e., $\mathrm{x} > N$).
- N is called carrying capacity.

We choose $\mathrm{N} = 1$ (see Quiz II)

$$\boxed{x' = ax\,(1 - x)}$$

$$\boxed{\equiv f_a(x)}$$

- first order, nonlinear, separable
- autonomous, $(f(x) = ax\,(1 - x)$ is not an explicit function of time).

1.2 Logistic Equation: Solutions

separable ODE

$$\boxed{x' = a(x - x^2)}$$

$$\frac{dx}{x - x^2} = adt$$

the method of partial fractions

$$\frac{dx}{x(1-x)} = adt$$

$$\left(\frac{1}{x} + \frac{1}{1-x}\right) dx = adt$$

$x \in (0,1)$ (see Quiz II)

$$\ln(x) - \ln(1-x) = at + C$$

$$\ln\left(\frac{x}{1-x}\right) = at + C$$

$$\ln\left(\frac{x}{1-x}\right) = at + C$$

$$\left(\frac{x}{1-x}\right) = C_0 e^{at}$$

$$x = C_0 e^{at}(1 - \mathrm{x})$$

$$(1 + C_0 e^{at})x = C_0 e^{at}$$

$$\boxed{x = \frac{C_0 e^{at}}{1 + C_0 e^{at}}}$$

$$x \to 1 \; as \; t \; \to \infty$$

1.2 Analysis of Solutions (sigmoid function)

$$\frac{dx}{dt} = 3(x - x^2),$$

$$1 > x_0 > 0$$

general solution $$x = \frac{C_0 e^{at}}{1+C_0 e^{at}}$$ $x \in (0,1)$

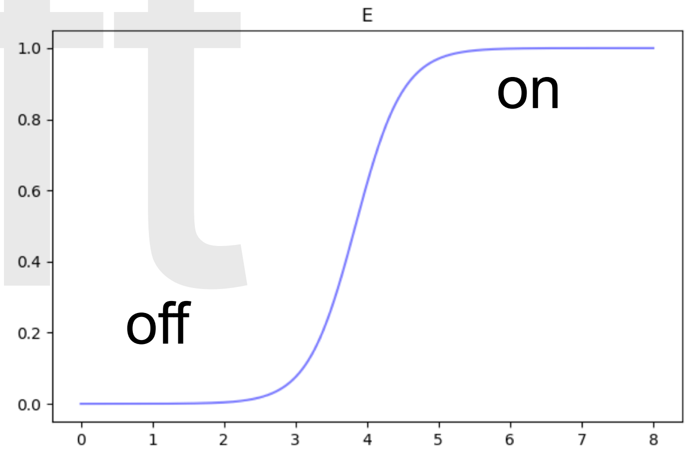
apply an IC $$x(0) = \frac{C_0}{1+C_0} = x_0$$

$$C_0 = \frac{x_0}{1 - x_0}$$

$$\boxed{x = \frac{x_0 e^{at}}{1-x_0+x_0 e^{at}}}$$

$x_0 > 0$

sigmoid function



$x \to 1 \; as \; t \; \to \infty \; (forward \; in \; time)$ on

$x \to 0 \; as \; t \; \to -\infty \; (backward \; in \; time)$ off

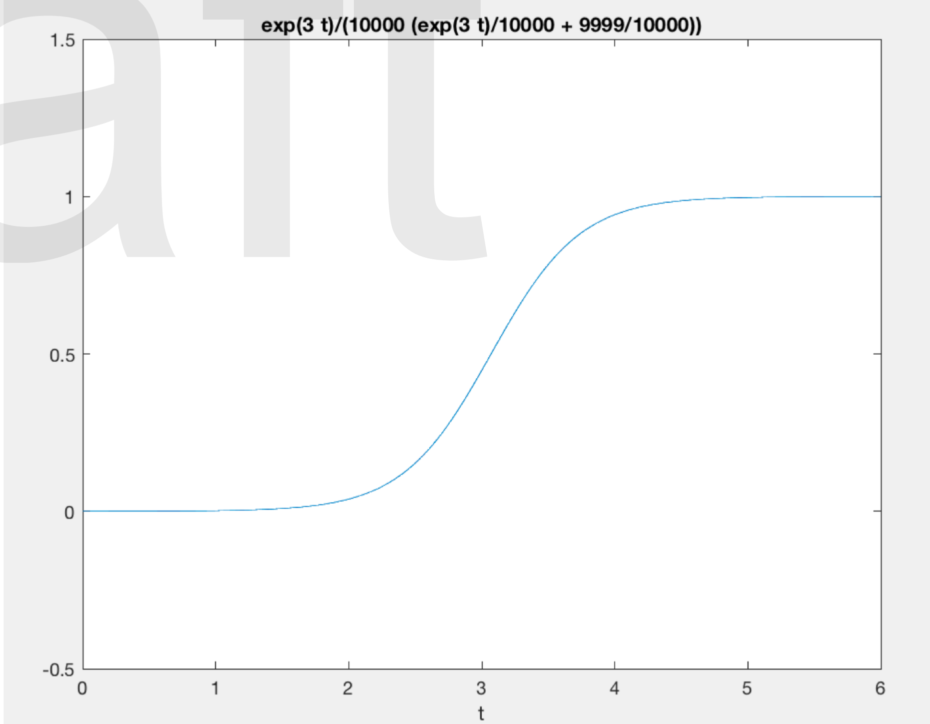
$x \in \; [0, 1]$

1.2 Symbolic Plotting: $0 < x_0 < 1$

```
syms t x0 a
a=3
x0=0.0001
fun=x0*exp(a*t)/(1-x0+x0*exp(a*t))
ezplot (fun, [0, 6, -0.5, 1.5])
```

$$x' = 3(x - x^2)$$

$$\boxed{x = \frac{x_0 e^{at}}{1 - x_0 + x_0 e^{at}}}$$



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$x'' - \alpha x + \beta x^2 = 0$ (DE-sech^2)	$x' = y \equiv F$ $y' = \alpha x - \beta x^2 \equiv G$	

Limit Cycles

- An isolated closed path is called a limit cycle: “Isolated” in the sense that there is no other closed path in its immediate neighborhood (Jordan and Smith).

Nagle et al.

A new edition

Limit Cycle

Definition 5. A nontrivial† closed trajectory with at least one other trajectory spiraling into it (as time approaches plus or minus infinity) is called a **limit cycle.**

An old edition

Limit Cycle

Definition 5. A nontrivial† closed trajectory that is isolated is called a **limit cycle.**

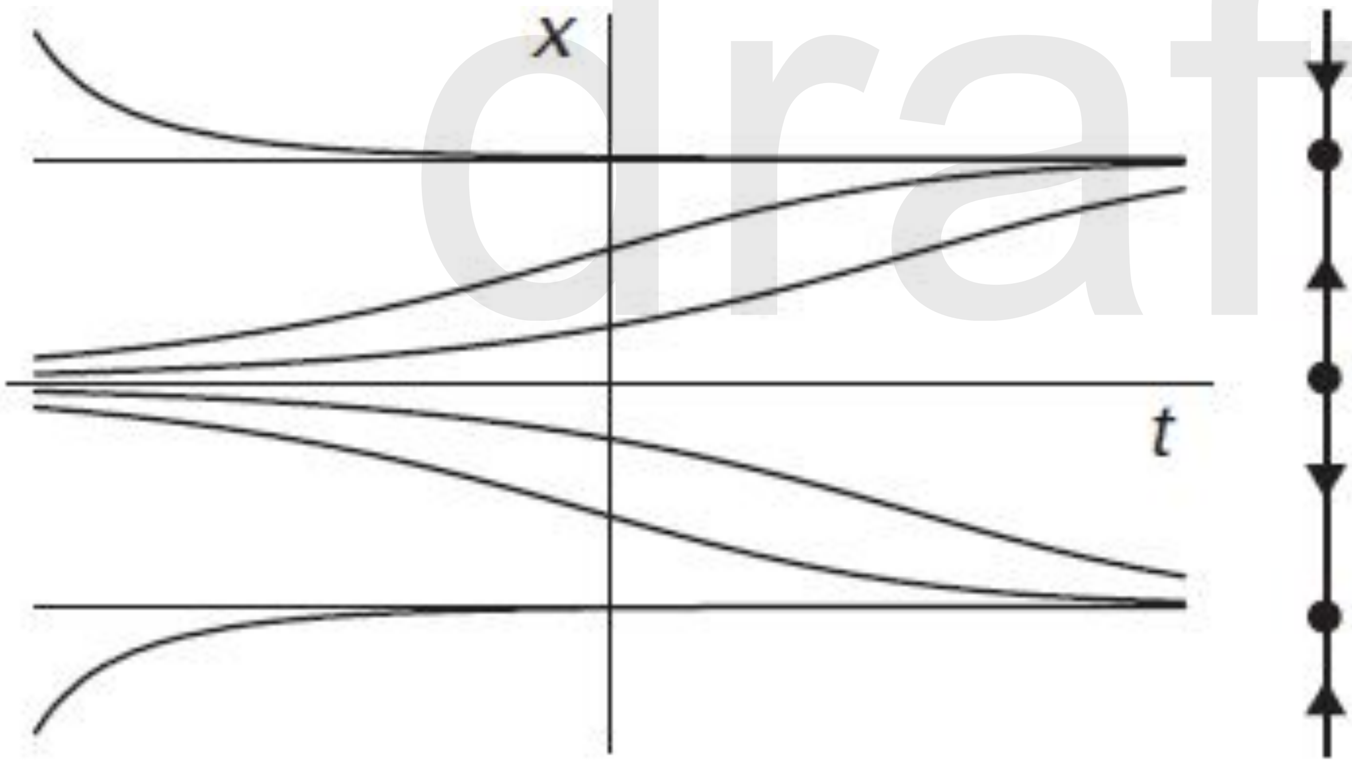
HW #1

2: [35 points] $dx/dt = -(ax + x^3)$ for $x \geq 0$ and $x(t=0) = x_o$. [Hint: set $\boxed{r = x^2,}$ solve for r and discuss the results when $a < 0$, $a = 0$ or $0 < a$.]

$$\frac{dx}{dt} = x - x^3$$

"$a = 1$" is different from that in the previous slide

Logistic eq.



Sink

Source

Sink

A Limit Cycle: Ex 1

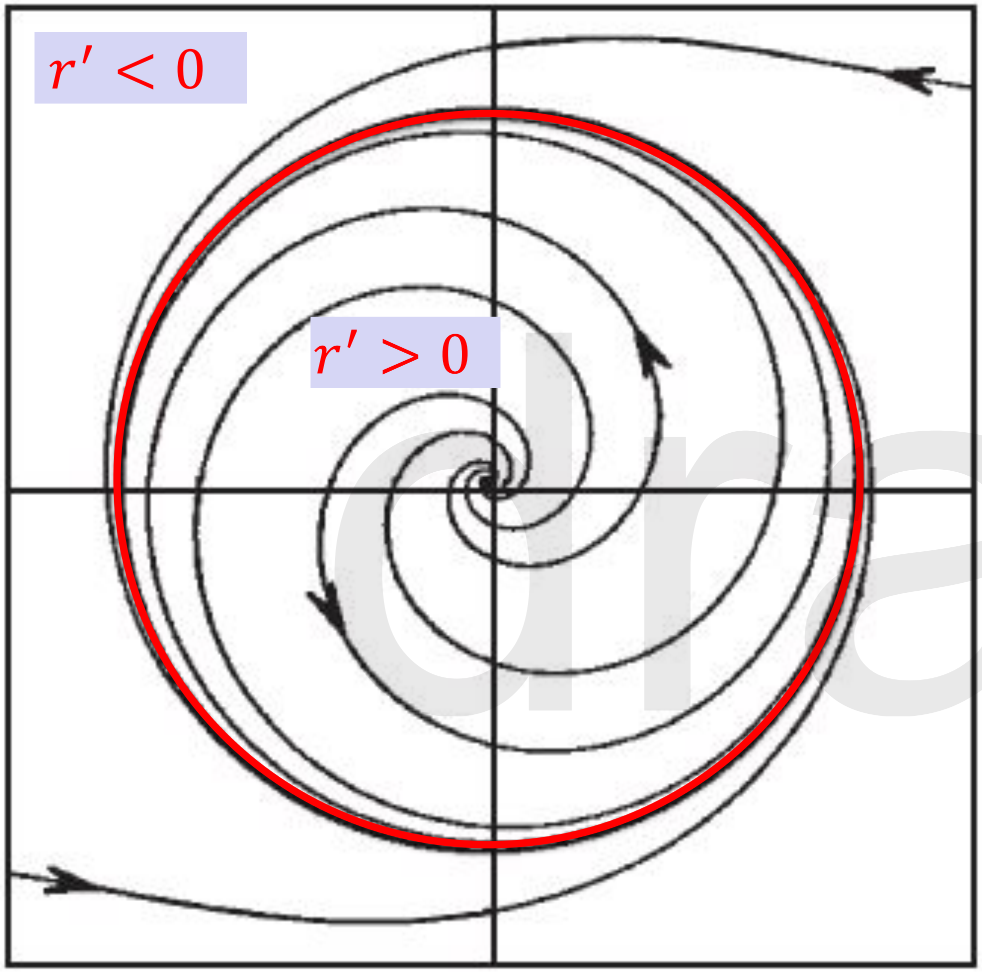


Figure 10.1 The phase plane for $r' = (r - r^3)$, $\theta' = 1$.

$$\frac{dx}{dt} = ax - y - x(x^2 + y^2),$$

$$\frac{dy}{dt} = x + ay - y(x^2 + y^2),$$

where $a = 1.0$ and $t \in [0, 10]$.

$$r' = r(1 - r^2),$$
$$a = 1 > 0$$

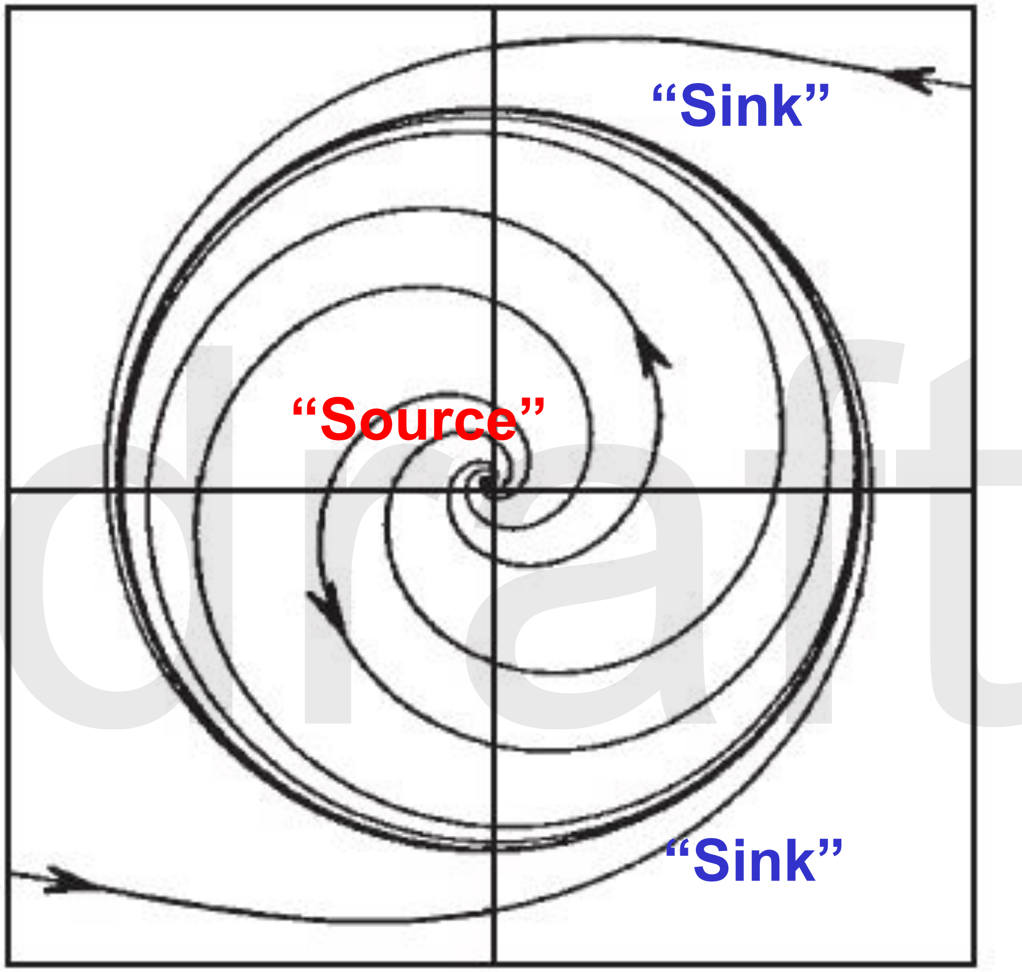
$r' = 0$ *when* $r = 1$

$r' < 0$ *when* $r > 1$

$r' > 0$ *when* $r < 1$

(A source at the origin)

Limit Cycle



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draft

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$x'' - \alpha x + \beta x^{2} = 0$ (DE-sech^2)	$x' = y \equiv F$ $y' = \alpha x - \beta x^2 \equiv G$	

Equations for Homoclinic Orbits

3D-NLM

$$\frac{d^2X}{d\tau^2} + \left(\frac{X^2}{2} - (\sigma r + \frac{C_1}{C_o})\right) X = 0,$$

$$\overline{\mathrm{KE}} + \overline{\mathrm{PE}} = C_o \left(\frac{X^2}{2} - \sigma Z\right) = C_1,$$

$$\overline{\mathrm{KE}} + \overline{\mathrm{APE}} = \frac{C_o}{2} \left(X^2 - \frac{\sigma}{r}(Y^2 + Z^2)\right) = C_2.$$

$C_1 = 0$

$C_2 = 0$

EQ. for Homoclinic orbits

$$\boxed{\frac{d^2X}{d\tau^2} + \left(\frac{X^2}{2} - \sigma r\right) X = 0,}$$

$$\frac{X^2}{2} - \sigma Z = 0,$$

$$X^2 - \frac{\sigma}{r}(Y^2 + Z^2) = 0.$$

- $\overline{KE}, \overline{PE}$, and $\overline{APE}$, represent the domain-averaged kinetic energy, potential energy, and available potential energy, respectively.

- A homoclinic orbit begins and ends at the saddle point.

DE-sech (of the 3D-NLM) vs. NLS Equation

$$\boxed{\frac{d^2X}{d\tau^2} + \left(\frac{X^2}{2} - \sigma r\right) X = 0,} \xrightarrow{\times X'} X'\frac{d^2X}{d\tau^2} + \frac{X^3}{2}X' - \sigma r X X' = 0$$

Integrate with respect to τ

$$\frac{1}{2}(X')^2 + \frac{X^4}{8} - \frac{\sigma r}{2}X^2 = E_1$$

3D-NLM

$$\boxed{\left(\frac{dX}{d\tau}\right)^2 - \sigma r X^2 + \frac{X^4}{4} = 0,}$$

E_1=0

DE-sech

$$\Updownarrow$$

NLS equation for the amplitude of a traveling wave

$$\left(\frac{dh}{dx}\right)^2 + \delta h^2 + \frac{\gamma}{2}h^4 = 0,$$

$$\delta < 0 \; and \;\; \gamma > 0$$

DE-sech^2 of the 3D-NLM (and the KdV Eq)

$$\frac{d^2X}{d\tau^2} + \left(\frac{X^2}{2} - \sigma r\right) X = 0,$$

$$\frac{X^2}{2} - \sigma Z = 0,$$

$$X^2 - \frac{\sigma}{r}(Y^2 + Z^2) = 0.$$

differentiating both sides and making a square

$$\left(\frac{dZ}{d\tau}\right)^2 = \frac{1}{\sigma^2} X^2 \left(\frac{dX}{d\tau}\right)^2.$$

$$\left(\frac{dX}{d\tau}\right)^2 - \sigma r X^2 + \frac{X^4}{4} = 0,$$

DE-sech

$$\left(\frac{dZ}{d\tau}\right)^2 = \frac{1}{\sigma^2}\left(\sigma r X^4 - \frac{X^6}{4}\right)$$

(D) nonlinear
$x'' - \alpha x + \beta x^3 = 0$ (DE-sech)
$x'' - \alpha x + \beta x^2 = 0$ (DE-sech^2)

$$X^2 = 2\sigma Z$$

$$\left(\frac{dZ}{d\tau}\right)^2 = 4\sigma r Z^2 - 2\sigma Z^3.$$

$$\frac{d}{d\tau}$$

$$\boxed{\frac{d^2Z}{d\tau^2} + 3\sigma Z^2 - 4\sigma r Z = 0.}$$

DE-sech^2

Analytical Solutions of Homoclinic Orbits/Solitary Waves

DE-sech
$$\boxed{\left(\frac{dX}{d\tau}\right)^2 - \sigma r X^2 + \frac{X^4}{4} = 0,}$$

DE-sech^2
$$\boxed{\frac{d^2 Z}{d\tau^2} + 3\sigma Z^2 - 4\sigma r Z = 0.}$$

taking square roots

EQ:
$$\boxed{\frac{dX}{d\tau} = \pm\sqrt{\sigma r X^2 - \frac{X^4}{4}}.}$$

IC:
$$(X, Y, Z) = (2\sqrt{\sigma r}, 0, 2r)$$

Solutions:
$$\boxed{X(\tau) = \frac{4\sqrt{\sigma r}}{e^{\sqrt{\sigma r}\tau} + e^{-\sqrt{\sigma r}\tau}} = 2\sqrt{\sigma r} sech(\sqrt{\sigma r}\tau),}$$

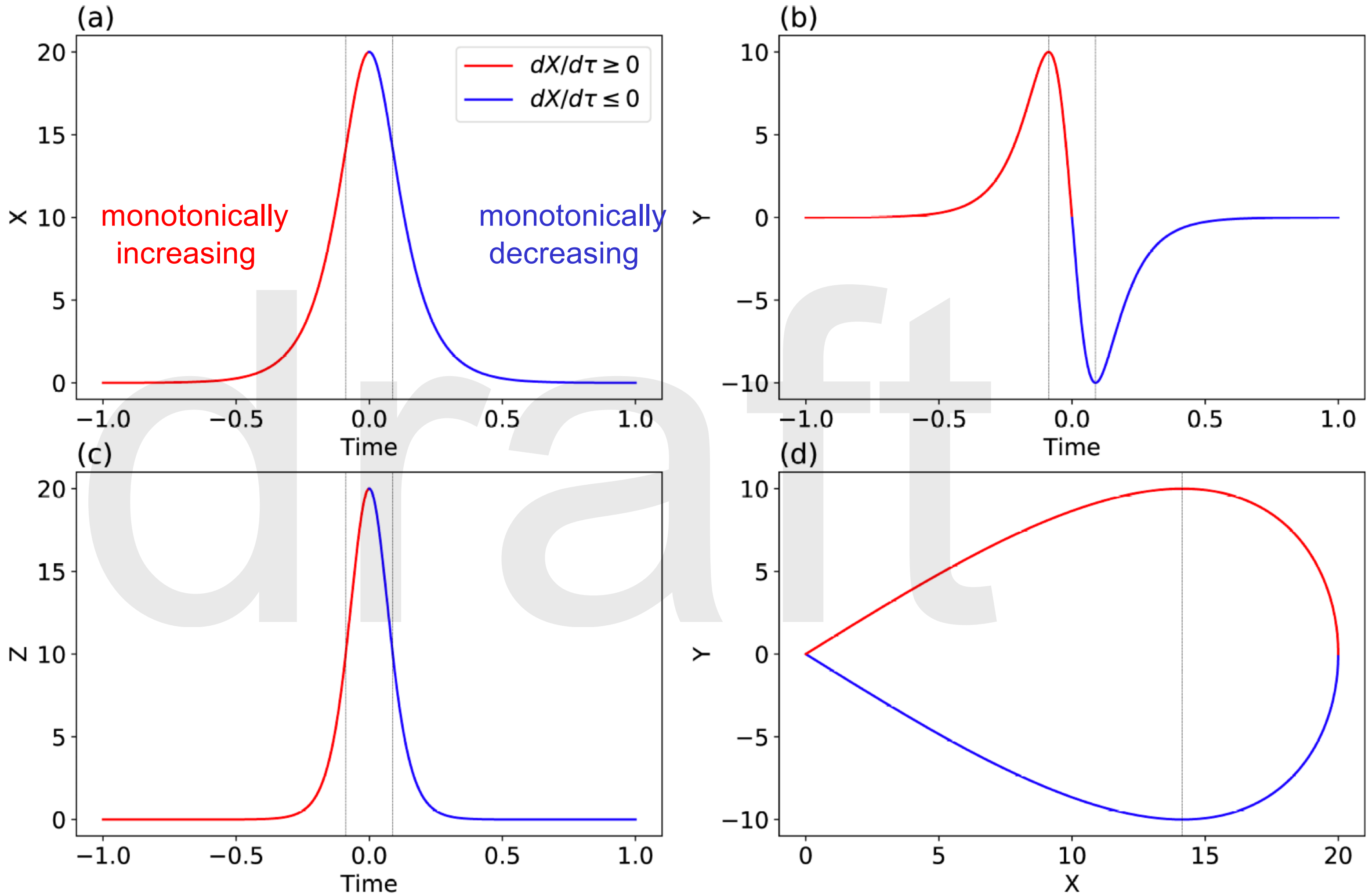
$$Y = \frac{1}{\sigma}\frac{dX}{d\tau} \qquad Y(\tau) = -(2r) tanh(\sqrt{\sigma r}\tau) sech(\sqrt{\sigma r}\tau),$$

$$Z = \frac{X^2}{2\sigma} \qquad \boxed{Z(\tau) = \frac{X^2(\tau)}{2\sigma} = (2r) sech^2(\sqrt{\sigma r}\tau).}$$

Homoclinic Orbits and Solitary Waves

sech:
X-comp
&
NLS

sech^2
Z-comp
&
KdV



Pulse width is determined by $[-\tau_t, \tau_t]$

Vertical lines indicate $\tau = \pm\tau_t$ or $X = \pm X_t$ where $\frac{d^2X}{d\tau^2} = 0$

Mathematical Universalities

3D-NLM	Other Models	Solutions
The Equation for X'' $\frac{\mathrm{d}^2X}{\mathrm{d}\tau^2} - (\sigma r + C_{ic})X + \frac{X^3}{2} = 0$ (Eq. 6)	The invisid Pedlosky Model $\frac{d^2R}{d\tau^2} - (1 + D_0)R + R^3 = 0$	cn and dn
The Equation for X'' (1) the same as the above (Eq. 6), or (2) $\frac{\mathrm{d}^2X}{\mathrm{d}\tau^2} + \epsilon\frac{\mathrm{d}X}{\mathrm{d}\tau} - (\sigma r + C_{ic})X + \frac{X^3}{2} = 0$ (Eq. 4)	The Duffing Equation $\frac{d^2g}{d\tau^2} + \delta\frac{dg}{d\tau} + \alpha g + \beta g^3 = \gamma cos(\omega\tau)$ $\delta = 0$, $\gamma = 0$, $\alpha = -(\sigma r + C_{ic})$, $\beta = 1/2$	cn
The Equation for $(X')^2$ $\left(\frac{dX}{d\tau}\right)^2 - \sigma r X^2 + \frac{X^4}{4} = 0$ (Eq. 9)	The Nonlinear Schrodinger Equation $\left(\frac{dh}{dx}\right)^2 + \delta h^2 + \frac{\gamma}{2}h^4 = 0$ $\delta < 0$, $\gamma > 0$	$sech$
The Equation for Z'' $\frac{d^2Z}{d\zeta^2} + 3Z^2 - 4rZ = 0$ (Eq. 10) $\zeta = \sqrt{\sigma}\tau$	The Korteweg-de Vries Equation $\frac{d^2f}{d\xi^2} + 3f^2 - cf = 0$ $c = 4r$	$sech^2$

Quiz 3: Logistic Eq. vs. the KdV Eq.

(A) 1st order
$y' = \alpha y - \beta y^2$ (logistic eq.)
$y' = \alpha y - \beta y^3$

(D) nonlinear
$x'' - \alpha x + \beta x^3 = 0$ (DE-sech)
$x'' - \alpha x + \beta x^2 = 0$ (DE-sech^2)

1: [30 points] In the Mid-term Part A, we have completed the following:

(*) Consider the following logistic equation:

$$\frac{df}{dt} = f(1-f). \qquad (MT-1.2)$$

Introduce a new dependent variable (g) to transform Eq. (MT-1.2) into the following ODE:

$$\frac{dg}{dt} = \frac{1}{4} - g^2. \qquad (MT-1.3)$$

(*) Express the solutions of Eqs. (MT-1.2) and (MT-1.3) in terms of the sigmoid and hyperbolic tangent functions, respectively.

Here, by defining $Z = dg/dt$, please derive the following ODE from Eq. (MT-1.3):

$$\frac{d^2Z}{dt^2} - Z + 6Z^2 = 0, \qquad (1.1)$$

which can be written using a new time variable (τ) as follows:

$$\frac{d^2Z}{d\tau^2} - Z/2 + 3Z^2 = 0. \qquad (1.2)$$

Eq. (1.2) is mathematically identical to the KdV equation in the traveling-wave coordinate (Shen 2020, IJBC, in press).

Important Concepts and/or ODEs

Asymptotic Matching	Homoclinic Orbits
Asymptotic Series	Jacobian Matrix & Linearization
A System of ODEs	Logistic Eq.
Bifurcation	Lorenz Model
Boundary Layer Theory	Perturbation Theory/Method
Critical Points	Quasi-periodicity
Conjugacy	Sigmoid function
DE-sech (e.g., Duffing Eq. & Nonlinear Schrodinger Eq.)	Solitary Wave
DE-$sech^2$ (e.g., KdV Eq.)	Stability Analysis (source, sink, saddle)
Eigenvalue Problem	Variational Eq.
Homeomorphism	WKB Analysis (oscillatory vs. exponential solutions)

More Concepts

CDIC vs. SDIC

Limit Cycle

Phase space

Method of Dominant Balance

Asymptotic Matching

Asymptotic Series

Boundary Layer Theory/Problem

Method of Dominant Balance

Power, Frobenius, and Asymptotic Series

Power Series

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Frobenius Series

$$y = x^r \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Asympotic Series

$$w(x) = \sum_{n=0}^{\infty} a_n (x^{\alpha})^n$$

α may be non-integers.

leading behavior
$\sim \exp(S(x))$

$$y = \boxed{c_1 x^{\frac{3}{4}} e^{\frac{2}{\sqrt{x}}}} \sum_{n=0}^{\infty} a_n (x)^{n/2}$$

leading behavior | asymptotic series

Convergent vs. Asymptotic vs. Power Series

$$\text{Convergent:} \quad \varepsilon_N(x) = \sum_{n=N+1}^{\infty} a_n (x - x_0)^n \to 0, \qquad N \to \infty;\ x \text{ fixed.}$$

- ϵ_N goes to zero as $N \to \infty$.
- Convergence is an absolute concept.

$$\text{Asymptotic:} \quad \varepsilon_N(x) \ll (x - x_0)^N, \qquad x \to x_0;\ N \text{ fixed.}$$

- ϵ_N goes to zero faster than $(x - x_o)^N$, but needs not to go to zero as $N \to \infty$.
- Asymptoticity is an relative property.

Review: Method of Dominant Balance: An Illustration

$$y'' + p(x)y' + q(x)y = 0$$

$\boxed{y = e^{S(x)}}$ $\qquad y' = S'e^{S(x)} \qquad y'' = S''e^{S(x)} + (S')^2 e^{S(x)}$

$$S''e^{S(x)} + (S')^2 e^{S(x)} + pS'e^{S(x)} + qe^{S(x)} = 0$$

$S'' + (S')^2 + pS' + q = 0$ divide by $e^{S(x)}$

$\boxed{S'' << (S')^2, \quad \text{as } x \to x_0}$

1. drop (all) terms that are small

$\boxed{(S')^2 \sim -pS' - q, \text{ as } x \to x_0}$

2. replace "=" by "~" and solve the system (note that don't have "0" on the RHS)

Asymptotic differential equations

3. check whether the solution is consistent with the approximation, i.e., whether $S'' << (S')^2$ is valid.

A Quick Look: $y'' + f(x)y = 0$

$y'' + p(x)y' + q(x)y = 0$

$y = e^{S(x)}$

$(S')^2 \sim -pS' - q$, as $x \to x_0$

Asymptotic differential equations

$\boxed{y'' + f(x)y = 0}$

$p(x) = 0; q = f$

$\boxed{(S')^2 \sim -f, \text{ as } x \to x_0}$

If $S(x) = \lambda x$

$\boxed{(\lambda)^2 \sim -f}$ $\quad \lambda \sim \pm i\sqrt{f}$

$y \sim e^{i\sqrt{f}x} = \cos(\sqrt{f}x + \alpha)$

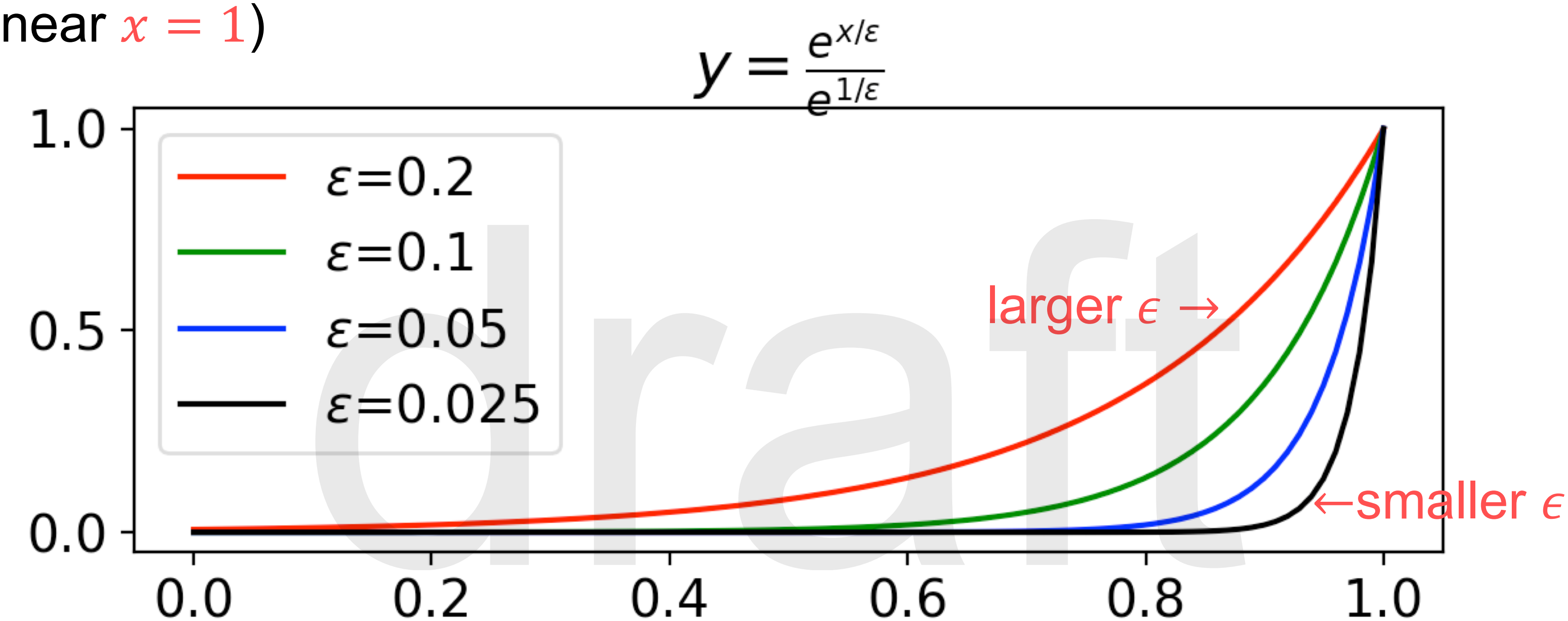
$\boxed{\text{wavelength, } L = \frac{2\pi}{\sqrt{f}}}$

or, frequency , $\omega = \frac{2\pi}{\sqrt{f}}$

Boundary Layer

boundary layer (near $x = 1$)

$$\varepsilon y' = y \qquad y(x=0) = e^{-\epsilon}$$



- For a very small ϵ (e.g., a black curve), solution y is almost a constant except for a very narrow interval near $x = 1$ where a boundary layer is defined.
- Stated alternatively, When ϵ is small, y varies rapidly near $x = 1$; this localized region of rapid variation is called a boundary layer.

Boundary Layer Problem: Matching

$$\boxed{\epsilon y'' + (1+\epsilon)y' + y = 0}$$

$$y(0) = 0 \;\&\; y(1) = 1$$

$$y(x) = \frac{e^{-x} - e^{-x/\epsilon}}{e^{-1} - e^{-1/\epsilon}}$$

$$\boxed{y_{out} = \lim_{\epsilon \to 0} y = \frac{e^{-x}}{e^{-1}} = e^{1-x}}$$

$$x = \epsilon \mathbb{X} \qquad x = O(\epsilon)$$

$$y(x) = \frac{e^{-\epsilon \mathbb{X}} - e^{-\mathbb{X}}}{e^{-1} - e^{-1/\epsilon}}$$

$$\boxed{y_{in} = \lim_{\epsilon \to 0} y = \frac{e^{0} - e^{-\mathbb{X}}}{e^{-1}}}$$

$$= e - e^{1-\mathbb{X}}$$

$$\boxed{\text{TBD: } \mathbb{Y} \sim -e^{1-\mathbb{X}} + e^{1-x}}$$

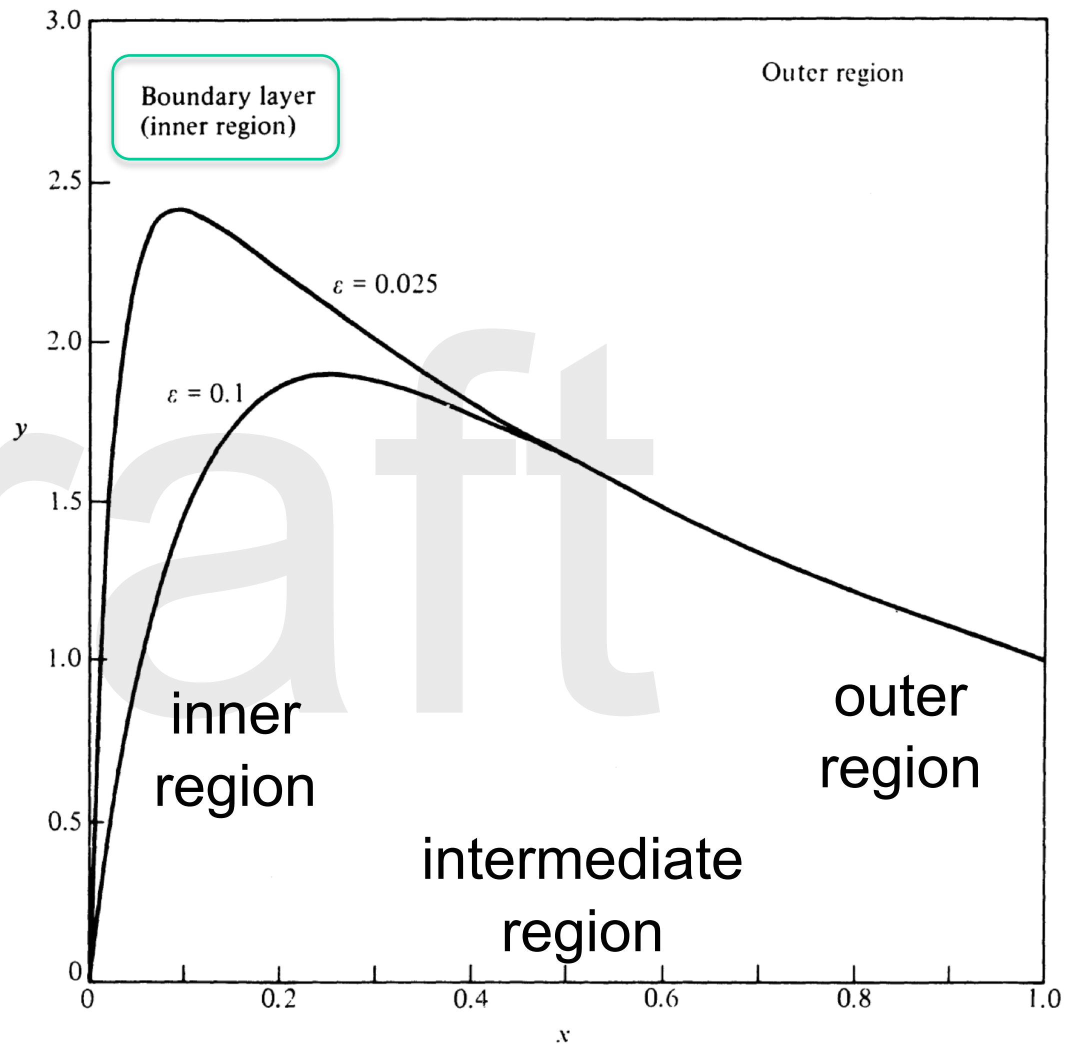


Figure 9.1 A plot of $y(x) = (e^{-x} - e^{-x/\varepsilon})/(e^{-1} - e^{-1/\varepsilon})$ $(0 \le x \le 1)$ for $\varepsilon = 0.1$ and 0.025. Note that $y(x)$ is slowly varying for $\varepsilon \ll x \le 1$ $(\varepsilon \to 0+)$. However, on the interval $0 \le x \le O(\varepsilon)$, $y(x)$ rises abruptly from 0 and becomes discontinuous in the limit $\varepsilon \to 0+$. This narrow and isolated region of rapid change is called a boundary layer.

Matching: Why?

inner region:
2^{nd} order ODE:
1BC

$$\frac{d^2\mathbb{Y}}{d\mathbb{X}^2}+\frac{d\mathbb{Y}}{d\mathbb{X}}+\epsilon\left(\frac{d\mathbb{Y}}{d\mathbb{X}}+\mathbb{Y}\right)=0$$

$$y(0)=0$$

$$x=\epsilon\mathbb{X}$$
(rescaling)

$$\mathbb{Y}(\mathbb{X})=y(x)$$

outer region:
1^{st} order ODE:
1BC

$$\epsilon\left(\frac{d^2y}{dx^2}+\frac{dy}{dx}\right)+\frac{dy}{dx}+y=0$$

$$y(1)=1$$

draft

matching to determine coefficients

$$\boxed{\begin{gathered}\mathbb{Y}\sim A_0\left(1-e^{-\mathbb{X}}\right)\\+\epsilon\left(A_1\left(1-e^{-\mathbb{X}}\right)-A_0\mathbb{X}\right)+\cdots\end{gathered}}$$

$\mathbb{X}\to\infty$
equal

$$\boxed{y_{out}\sim e^{1-x}}$$

Matching (an intermediate component)

$\boxed{\mathbb{Y} \sim -e^{1-\mathbb{X}} + e^{1-x}}$

$\boxed{y_{match} = e^{1-x}}$

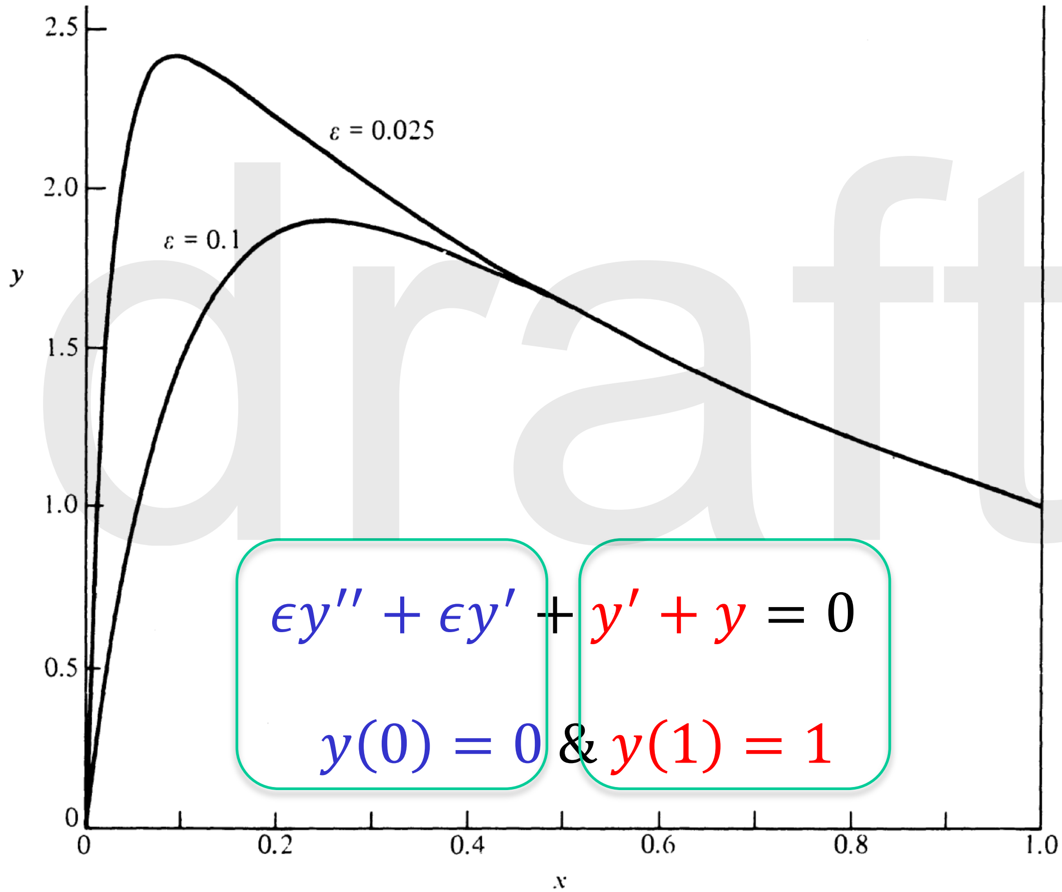
$\boxed{y_{out} \sim e^{1-x}}$

$y(0) = 0$

$y(1) = 1$

$x = \epsilon \mathbb{X}$
(rescaling)

$\mathbb{Y}(\mathbb{X}) = y(x)$



$$\epsilon y'' + \epsilon y' + y' + y = 0$$

$$y(0) = 0 \ \& \ y(1) = 1$$

$\boxed{y_{unif} = y_{in} + y_{out} - y_{match} = -e^{1-\mathbb{X}} + e^{1-x}}$

CDIC vs. SDIC

CDIC vs. SDIC

Table R4: Definitions of CDIC and SDIC

	CDIC	SDIC
Definition	solutions through nearby ICs remain close over short time intervals	*The property for an orbit when most other orbits that pass close to it at some point do not remain close to it as time advances.*
An example	"gradual divergence" or convergence of initial nearby trajectories	"rapid divergence" of initial nearby trajectories (e.g., see TC tracks in Figure R14)

Fundamental Concepts

1. **Existence:** Each point in the (t, x)-plane has a solution passing through it. The solution has slope given by the differential equation at that point.
2. **Uniqueness:** Only one solution passes through any particular (t, x).
3. **Continuous dependence:** Solutions through nearby initial conditions remain close over short time intervals. In other words, the flow $F(t, x_0)$ is a continuous function of x_0 as well as t. $|X(t) - Y(t)| < |X_0 - Y_0| e^{K(t-t_0)}$

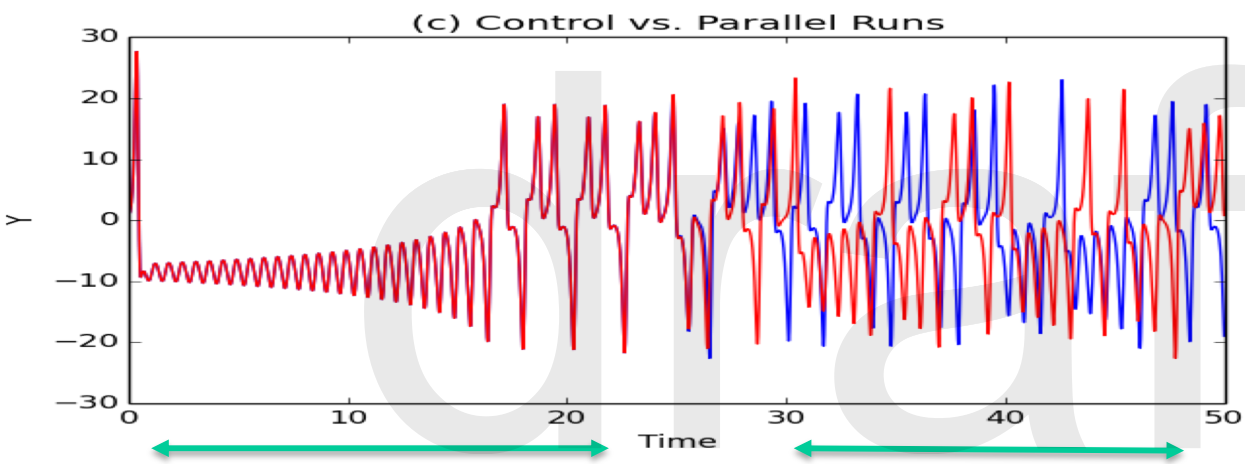
- *"Sensitive dependence on initial conditions" means that nearby trajectories separate exponentially fast, i.e., the system has a positive Liapunov exponent. (Strogatz, p331)*
- *We may define an infinite invariant set of a map $F: R \rightarrow R$ to have sensitive dependence on initial conditions if there exists $\delta > 0$ such that for all $x \in S$ and all neighbourhoods N (however small) of x there exists $y \in N$ and $n > 0$ such that $|F^n(x) - F^n(y)| > \delta$. So neighbouring orbits, however close initially, separate from one another, although each keeps close to the invariant set. (Drazin, p140; Devaney, p49)*

Alligood et al.

CDIC vs. SDIC

1. The butterfly effect of the first kind (BE1):

   Indicating *sensitive dependence on initial conditions* (Lorenz, 1963).



continuous dependence (within a time interval)

sensitive dependence

- control run (blue): $(X, Y, Z) = (0,1,0)$
- parallel run (red): $(X, Y, Z) = (0,1+\epsilon, 0)$, $\epsilon = 1e-10$.

A System of ODEs
&
Eigenvalue Problem

Chapter 3: 2D Linear Systems

$$x' = ax + by \quad (= P(x,y)) \qquad (1)$$

$$y' = cx + dy \quad (= Q(x,y)\ ) \qquad (2)$$

$$X' = AX \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

1. Eigenvalue problem: $|A - \lambda I| = 0$
2. Two linearly independent solutions, $e^{\lambda_1 t} V_1$ and $e^{\lambda_2 t} V_2$
3. Real eigenvalues for a source, sink, or saddle
4. Complex eigenvalues for a center, spiral sink or spiral source
5. Diagonalization
6. Changing coordinates Linearly Conjugate

Eigenvalues and Eigenvectors

$$X' = AX \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\vec{X} = e^{\lambda t}\vec{V} \qquad A\vec{V} = \lambda\vec{V}$$

Definition
A nonzero vector V_0 is called an *eigenvector* of A if $AV_0 = \lambda V_0$ for some λ. The constant λ is called an *eigenvalue* of A.

Theorem. *Suppose that V_0 is an eigenvector for the matrix A with associated eigenvalue λ. Then the function $X(t) = e^{\lambda t}V_0$ is a solution of the system $X' = AX$.* ■

The Trace-Determinant Plane

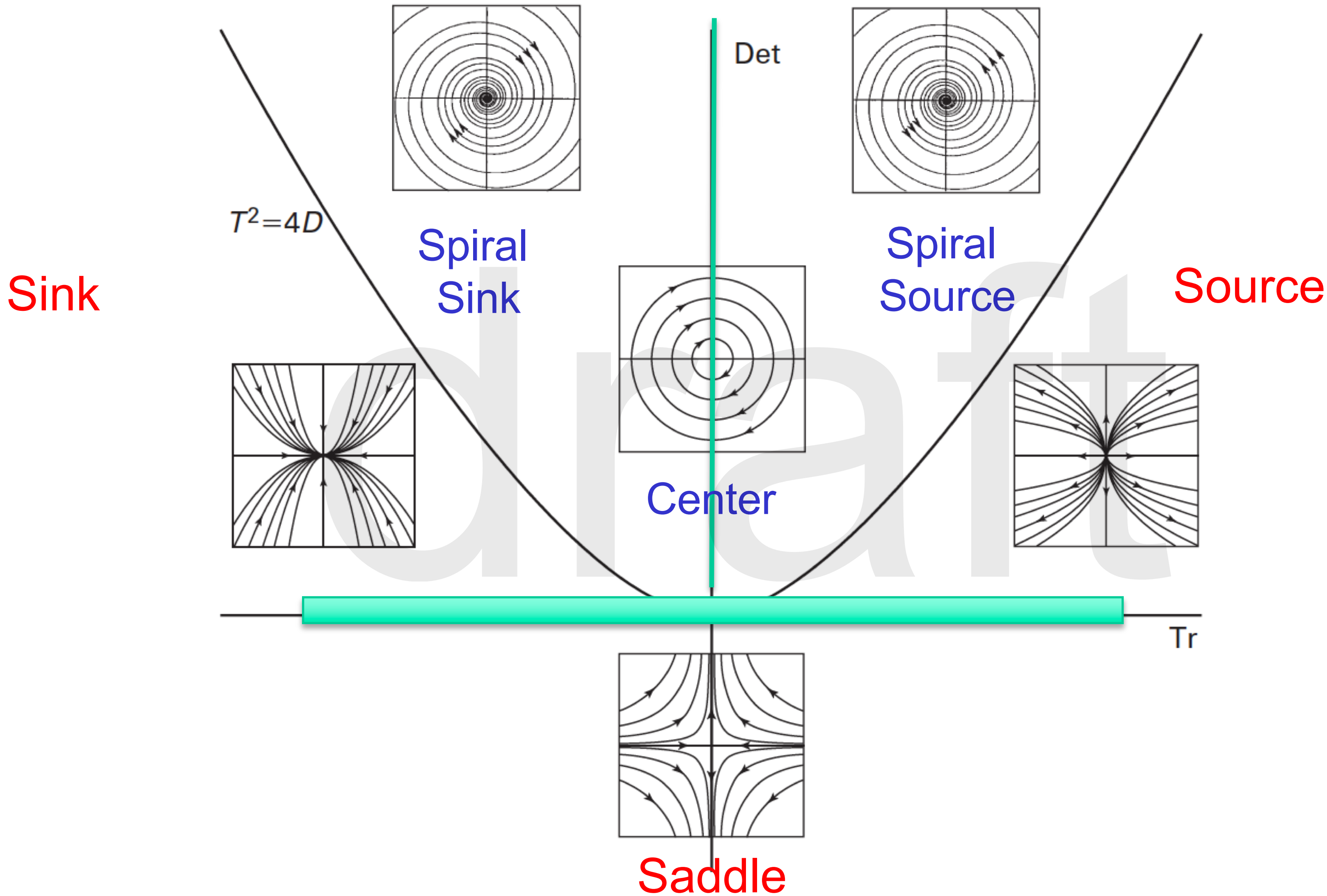


Figure 4.1 The trace-determinant plane. Any resemblance to any of the authors' faces is purely coincidental.

Review: Saddle, Source and Sink in 2D Systems

$$\lambda^2 - T\lambda + D = 0$$

$$\lambda_\pm = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

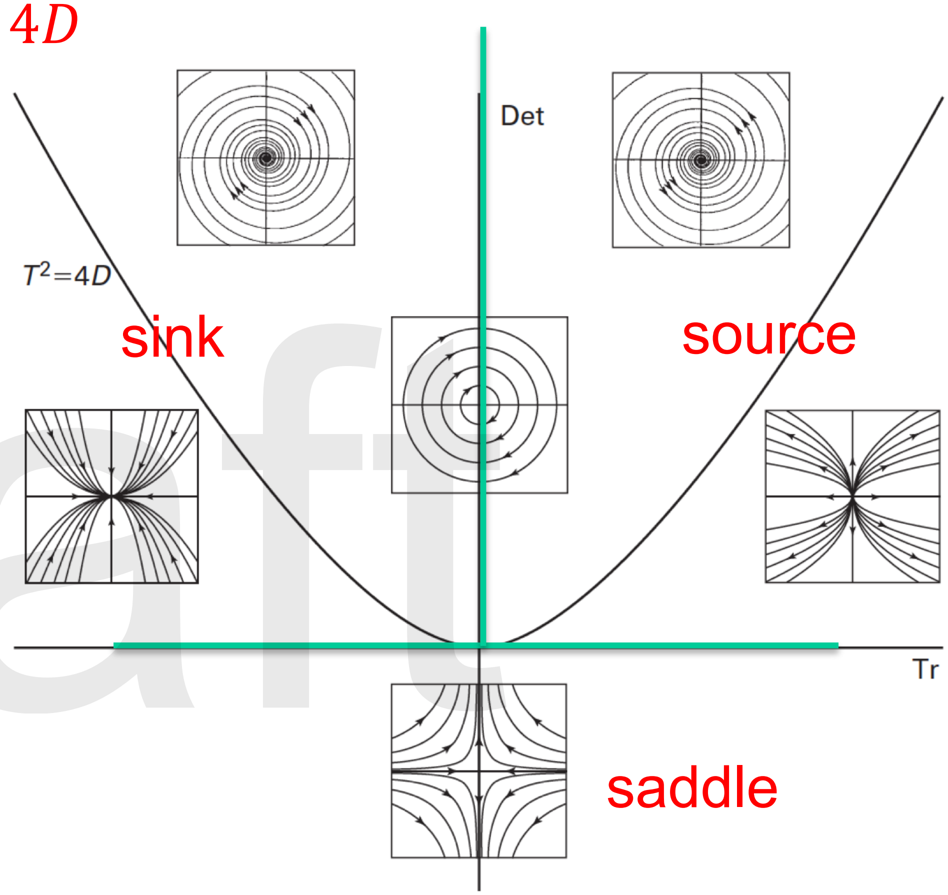
$$\boxed{\lambda_+ + \lambda_- = T = tr}$$

$$(\lambda - \lambda_+)(\lambda - \lambda_-) = 0$$

$$\lambda^2 - (\lambda_+ + \lambda_-)\lambda + \lambda_+\lambda_- = 0$$

$$\boxed{\lambda_+\lambda_- = D = determinat}$$

$T^2 = 4D$



- $D < 0$, λ_+ and λ_- have different signs ➔ saddle
- $D > 0$, λ_+ and λ_- have the same sign ➔ source with $T > 0$
  ➔ sink with $T < 0$

Classification for 3D Systems

- Saddle (three real eigenvalues), $\lambda_{1,2} < 0$ & $\lambda_3 > 0$
- Sink, $\lambda_{1,2,3} < 0$
- Source, $\lambda_{1,2,3} > 0$
- Spiral center, $Re(\lambda_{1,2}) = 0$ & $\lambda_3 < 0$
- Spiral source, $Re(\lambda_{1,2}) > 0$ & $\lambda_3 > 0$
- Spiral sink, $Re(\lambda_{1,2}) < 0$ & $\lambda_3 < 0$
- Spiral saddle (Saddle focus), $Re(\lambda_{1,2}) < 0$ & $\lambda_3 > 0$

- Stable subspace: $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_k$ are negative
- Unstable subspace: $\lambda_{k+1}, \lambda_{k+2}, \lambda_{k+3} \ldots \lambda_n$ are positive.

Spiral Sink/Source and Saddle Focus in the 3D Space

Spiral Sink: $Re(\lambda_{1,2}) < 0$ and $Re(\lambda_3) < 0$

Spiral Source: $Re(\lambda_{1,2}) > 0$ and $Re(\lambda_3) > 0$

Spiral Saddle (Saddle Focus): $Re(\lambda_{1,2}) < 0$ and $Re(\lambda_3) > 0$;

(Ott, p334)

Non-trivial critical points of the Lorenz Model:

$Re(\lambda_{1,2}) > 0$ and $Re(\lambda_3) < 0$, → spiral point

The General Solution

Theorem. *Suppose A has a pair of real eigenvalues $\lambda_1 \neq \lambda_2$ and associated eigenvectors V_1 and V_2. Then the general solution of the linear system $X' = AX$ is given by*

$$X(t) = \alpha e^{\lambda_1 t} V_1 + \beta e^{\lambda_2 t} V_2. \qquad \blacksquare$$

2^{nd} Order ODEs vs. A System of 1^{st} Order ODEs

(A) $ar^2 + br + c = 0$

$y = e^{rt}$

(B) $x'' + \frac{b}{a}x' + \frac{c}{a}x = 0$

let $x' = y$

obtain $y' = -\frac{c}{a}x - \frac{b}{a}y$

define $X = \begin{pmatrix} x \\ y \end{pmatrix}$; $A = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}$

$$X' = AX$$

assume $X = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{\lambda t}$;

eigenvalue problem $|A - \lambda I| = \begin{vmatrix} 0 - \lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a} - \lambda \end{vmatrix} = 0$

Characteristic Equation

$$a\lambda^2 + b\lambda + c = 0$$

Bifurcation,

Bifurcation Points

Critical Points

Critical Points, Equilibrium Points, and Fixed Points

- Given $x' = f(x; a)$, equilibrium points, also known as fixed points or critical points, are defined when $f(x_c) = 0$.

- Example 1: Consider $x' = ax$. $x = 0$ is a critical point.

- Example 2: Consider $x' = x - x^{\mathbf{2}}$ (i.e., the Logistic Equation). $f(x_c) = 0$ leads to $x - x^{\mathbf{2}} = 0$. Thus, $x = 0$ and $x = 1$ are critical points.

- Example 3: Similarly, within $x' = x - x^{\mathbf{3}}$, three critical points are $x = 0$, $x = 1$ and $x = -1$.

1.1 Bifurcation of $\frac{dx}{dt} = ax$

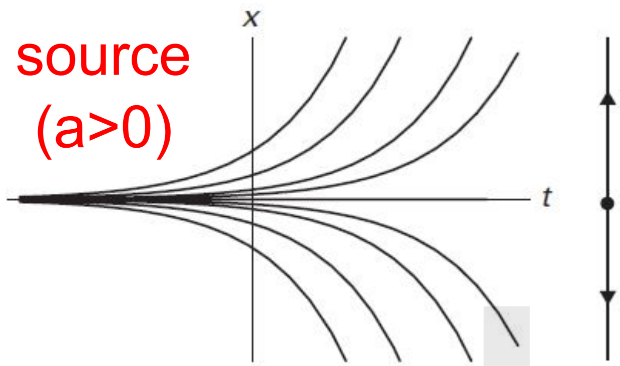


Figure 1.1 The solution graphs and phase line for $x' = ax$ for $a > 0$. Each graph represents a particular solution.

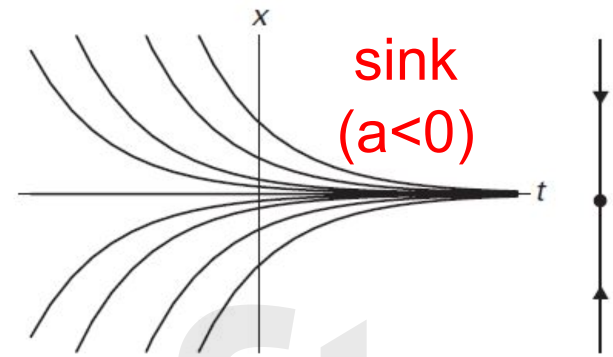


Figure 1.2 The solution graphs and phase line for $x' = ax$ for $a < 0$.

- A bifurcation occurs when there is a “significant” change in the structure of the solutions of the system as “a” varies.
- In the previous example, solutions are unstable for $a > 0$ but stable for $a < 0$. Thus, we have a bifurcation at $a = 0$.
- The simplest types of bifurcations occur when the number of equilibrium solutions changes as “a” varies. For example, within $x' = 1 - ax^2$, there are two critical points for $a > 0$ but no critical points for $a \leq 0$.

Definition: Bifurcation Points

$$\frac{dx}{dt} = f(x, a)$$

critical points $\quad f(x, a) = 0$

bifurcation points $\quad \boxed{f(x, a) = 0 \;\&\; f_x(x, a) = 0}$

example $\quad \dfrac{dx}{dt} = ax$

critical points $\quad ax = 0 \quad \rightarrow x = 0$

bifurcation points $\quad f_x(x, a) = a \rightarrow a = 0$

Conjugacy

Homeomorphism

Linearization Theorem

Hyperbolic

Definition
A matrix A is *hyperbolic* if none of its eigenvalues has real part 0. We also say that the system $X' = AX$ is *hyperbolic.*

$$\boxed{Re(\lambda_j) \neq 0}$$

Conjugacy and Linearization Theorem

We can now conjugate the flow of a nonlinear system near a hyperbolic equilibrium point that is a sink to the flow of its linearized system. Indeed, the argument used in the second example of the previous section goes over essentially unchanged. In similar fashion, nonlinear systems near a hyperbolic source are also conjugate to the corresponding linearized system.

This result is a special case of the following more general theorem.

The Linearization Theorem. *Suppose the n-dimensional system $X' = F(X)$ has an equilibrium point at X_0 that is hyperbolic. Then the nonlinear flow is conjugate to the flow of the linearized system in a neighborhood of X_0.* ■

a.k.a. Hartman–Grobman Theorem

Dynamical equivalence

Hyperbolicity, Linearization, and Homemorphism

TBD

These ideas also generalize neatly to higher-order systems. A fixed point of an *n*th-order system is *hyperbolic* if all the eigenvalues of the linearization lie off the imaginary axis, i.e., $\mathrm{Re}(\lambda_i) \neq 0$ for $i = 1, \ldots, n$. The important *Hartman–Grobman theorem* states that the local phase portrait near a hyperbolic fixed point is "topologically equivalent" to the phase portrait of the linearization; in particular, the stability type of the fixed point is faithfully captured by the linearization. Here ***topologically equivalent*** means that there is a ***homeomorphism*** (a continuous deformation with a continuous inverse) that maps one local phase portrait onto the other, such that trajectories map onto trajectories and the sense of time (the direction of the arrows) is preserved.

Strogatz (2015), p156

Dynamical Equivalence

Definition 3.10 Flow Equivalence

Two flows are conjugate (equivalent) if there exists a one to one map g between corresponding orbits or

$$\phi_t^2 \; o \; g = g \; o \; \phi_t^1$$

The flows are

1. **linearly conjugate** *if* g *is a linear map, then* $g \in C^\infty$, Same eigenvalues
2. **differentiably conjugate** *if* g *is a diffeomorphism,* $g \in C^k$, $k \geq 1$,
3. **topologically conjugate** *if* g *is a homoeomorphism* $g \in C^0$.

Same # of eigenvalues with negative real parts (for 2x2 hyperbolic systems)

Topological Conjugacy in Hyperbolic Systems

Definition
A matrix A is *hyperbolic* if none of its eigenvalues has real part 0. We also say that the system $X' = AX$ is *hyperbolic*.

Theorem. *Suppose that the 2×2 matrices A_1 and A_2 are hyperbolic. Then the linear systems $X' = A_i X$ are conjugate if and only if each matrix has the same number of eigenvalues with negative real part.* ∎

1. One eigenvalue is positive and the other is negative ($\lambda_1 < 0 < \lambda_2$); saddle
2. Both eigenvalues have negative real parts; sink or spiral sink
3. Both eigenvalues have positive real parts. source or spiral source

draft

A System of Nonlinear ODEs

Locally Linearized Systems
&
Jacobian Matrix

Example

Consider the following system of first-order ODEs

$$x' = ax + by \quad (= F(x, y)) \qquad (1)$$

$$y' = cx + dy \quad (= G(x, y)\ ) \qquad (2)$$

$$X = \begin{pmatrix} x \\ y \end{pmatrix}$$

Find critical points

$$x_c = 0 \qquad\qquad y_c = 0$$

Compute the Jacobian matrix at a critical point

$$J(F, G) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{x_c, y_c} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \text{“}A\text{”}$$

Local Linear Stability Analysis

Consider the following system of first-order ODEs

$$\boxed{x' = F(x,y) \qquad\qquad y' = G(x,y)}$$

Find critical points

$$F(x_c, y_c) = 0 \qquad\qquad G(x_c, y_c) = 0$$

Compute the Jacobian matrix and evaluate it at a critical point

$$J(F,G) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{x_c, y_c}$$

Solve an eigenvalue problem:

$$JV = \lambda V \qquad V = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\boxed{|J - \lambda I| = 0}$$

A Locally Linear System

$$\frac{dx}{dt} = F(x,y,z)$$

$$\frac{dy}{dt} = G(x,y,z)$$

$$\frac{dz}{dt} = H(x,y,z)$$

$$J = \left. \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} & \frac{\partial H}{\partial z} \end{pmatrix} \right|_{(x_c, y_c, z_c)} = \boxed{A}$$

$$A\vec{V} = \lambda\vec{V}; \quad \vec{V} = \begin{pmatrix} x_{eigen} \\ y_{eigen} \\ z_{eigen} \end{pmatrix}$$

$$F(x_c, y_c, z_c) = 0$$

$$G(x_c, y_c, z_c) = 0$$

$$H(x_c, y_c, z_c) = 0$$

Determine the Jacobian using Matlab

```
syms x y z r  theta real

x=r * cos(theta)
y=r * sin(theta)
vec1=[x, y]
vec2=[r, theta]
ja=jacobian(vec1, vec2)
jadet=abs(det(ja))
simplify(jadet)

➔
ans =
abs(r)
```

$$|J| = \mathrm{r}$$

```
syms x y z rho phi theta real

assume (phi > 0)

x=rho * sin(phi) * cos(theta)
y=rho * sin(phi) * sin(theta)
z=rho * cos(phi)
vec1=[x, y, z]
vec2=[rho, theta, phi]
ja=jacobian(vec1, vec2)
jadet=abs(det(ja))
simplify(jadet)
➔
ans =
rho^2*abs(sin(phi))
```

$$|J| = \rho^2 \sin(\phi)$$

Perturbation

Perturbation Theory

➢ Perturbation theory is a large collection of iterative methods for obtaining approximate solutions to problems involving a small parameter ε.

Three typical steps of perturbative analysis are:

1. Convert the original problem into a perturbation problem by introducing the small parameter ε.
2. Assume an expression for the answer in the form of a perturbation series and compute the coefficients of that series.
3. Recover the answer to the original problem by summing the perturbation series for the appropriate value of ε.

$$x(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n$$

Perturbation Theory: An Example

Example 1 *Roots of a cubic polynomial.* Let us find approximations to the roots of

$$x^3 - 4.001x + 0.002 = 0. \tag{7.1.1}$$

(1) By introducing a parameter, ε, we convert the original problem to become:

$$\boxed{x^3 - (4 + \varepsilon)x + 2\varepsilon = 0.} \tag{7.1.2}$$

When $\varepsilon = 0.001$, the original equation (7.1.1) is reproduced.

Perturbation Theory

Perturbation theory is a large collection of iterative methods for obtaining approximate solutions to problems involving a small parameter ε. Three typical steps of perturbative analysis are:

1. Convert the original problem into a perturbation problem by introducing the small parameter ε.

$$x^3 - 4.001x + 0.002 = 0. \quad \Rightarrow \quad x^3 - (4+\varepsilon)x + 2\varepsilon = 0.$$

1. Assume an expression for the answer in the form of a perturbation series and compute the coefficients of that series.

$$\boxed{x(\varepsilon) = \sum_{n=0}^{\infty} a_n \varepsilon^n.} = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \ldots$$

1. Recover the answer to the original problem by summing the perturbation series for the appropriate value of ε.

$x_1 = -2 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + \cdots.$ If we now set $\varepsilon = 0.001$, we obtain x_1

A Perturbation Method for Stability Analysis

$$\frac{dx}{dt} = x - x^3 = F(x)$$

Find critical points $\quad x_c = x_c^3$

$$x = x_c + \varepsilon$$

Plug the above into the DE

$$\frac{d\varepsilon}{dt} = A\varepsilon^0 + B\varepsilon^1 + \cdots$$

$$\frac{d\varepsilon}{dt} = x_c + \varepsilon - (x_c^3 + 3\varepsilon x_c^2 + 3\,\varepsilon^2 x_c + \varepsilon^3) \approx \varepsilon(1 - 3x_c^2)$$

- $A = 0$ and $B = 1 - 3x_c^2$
- What's the relation between B and F?
- You have 1 minute

$$B = (1 - 3x_c^2) = F'(x_c) = \lambda$$

draft

A Perturbation Method for Stability Analysis

$$\frac{dx}{dt} = x - x^3 = F(x)$$

$$\boxed{x = x_c + \varepsilon x_1}$$

$O(\varepsilon^0)$

$$\frac{dx_c}{dt} = x_c - x_c^3$$

$O(\varepsilon^1)$

$$\frac{d\varepsilon x_1}{dt} = \varepsilon x_1 - 3x_c^2 \varepsilon x_1 = \varepsilon x_1 (1 - 3x_c^2)$$

$$(1 - 3x_c^2) = \lambda$$

$$\frac{dx}{dt} = x - x^3 = F(x)$$

$$\boxed{x = x_c + \varepsilon}$$

$$\frac{d\varepsilon}{dt} = x_c + \varepsilon - (x_c^3 + 3\varepsilon x_c^2 + 3\,\varepsilon^2 x_c + \varepsilon^3)$$
$$\approx \varepsilon(1 - 3x_c^2)$$

$$(1 - 3x_c^2) = \lambda$$

draft

Phase Line and Phase Space

1.1 Phase Space and Phase Line

- We may construct a space using dependent variables as coordinates. Such a space is called a phase space (or state space, e.g., Hilborn 2000).
- Extended Phase Space: The cartesian product of the phase space with the independent variable (which is often referred to as "time").
- A 1-D phase space is called a phase line.
- For linear stability analysis of a single first-order ODE (e.g., $x' = f(x; a)$), we analyze the sign of x' near one of the system's critical points.

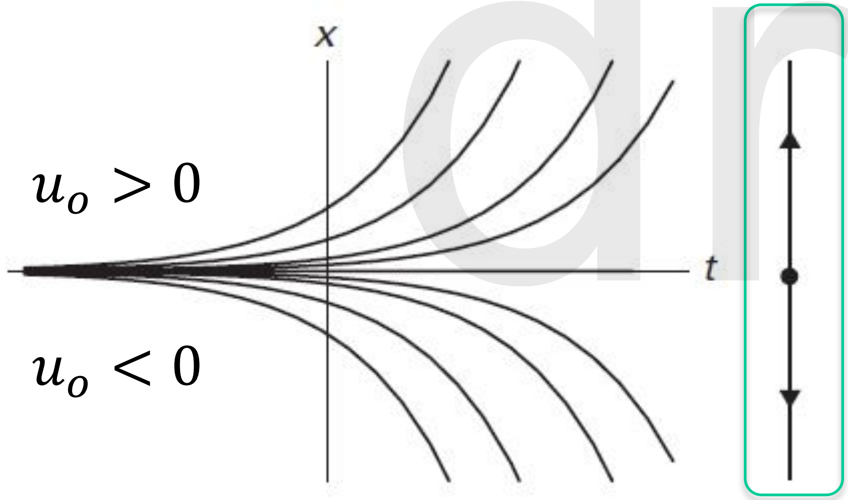
1.1 Phase Line

The phase line:

- as the solution is a function of time, we may view it as a particle moving along the real line, which is called a phase line.
- A line represents intervals of the domain of the derivatives. An interval over which the derivative is positive has an arrow pointing in the positive direction along the line.

$u_o > 0$

$u_o < 0$



$x > 0$

$x' > 0$

$x = 0$

$x < 0$

$x' < 0$

Figure 1.1 The solution graphs and phase line for $x' = ax$ for $a > 0$. Each graph represents a particular solution.

$$\boxed{\begin{array}{c} x' = ax \\ a > 0 \end{array}}$$

$x' < 0$ $\qquad$ $x' > 0$

$x = 0$ $\qquad$ source

Quasi-Periodic

Periodic Composite Motion (with 2 frequencies)

$x_1'' = -\omega_1^2 x_1 \rightarrow periodic$ $\quad x_1(t) = a_1 \cos(\omega_1 t) + \mathrm{b}_1 \sin(\omega_1 t)$

$x_2'' = -\omega_2^2 x_2 \rightarrow periodic$ $\quad x_2(t) = a_2 \cos(\omega_2 t) + \mathrm{b}_2 \sin(\omega_2 t)$

Q: under which condition a composite motion with the two frequencies is periodic?

Based on what we discuss, we can simply consider the following:

$$x_1(t) = a_1 \cos(\omega_1 t) \qquad \omega_1(t+T) = \omega_1 t + 2m\pi \qquad T = \frac{2m\pi}{\omega_1}$$

$$x_2(t) = a_2 \cos(\omega_2 t) \qquad \omega_2(t+T) = \omega_2 t + 2n\pi \qquad T = \frac{2n\pi}{\omega_2}$$

$$\frac{2m\pi}{\omega_1} = \frac{2n\pi}{\omega_2} \qquad \boxed{\frac{\omega_2}{\omega_1} = \frac{n}{m}\text{: a rational number}}$$

Periodicity vs. Quasi-periodicity

$$\frac{dr_j}{dt} = 0$$

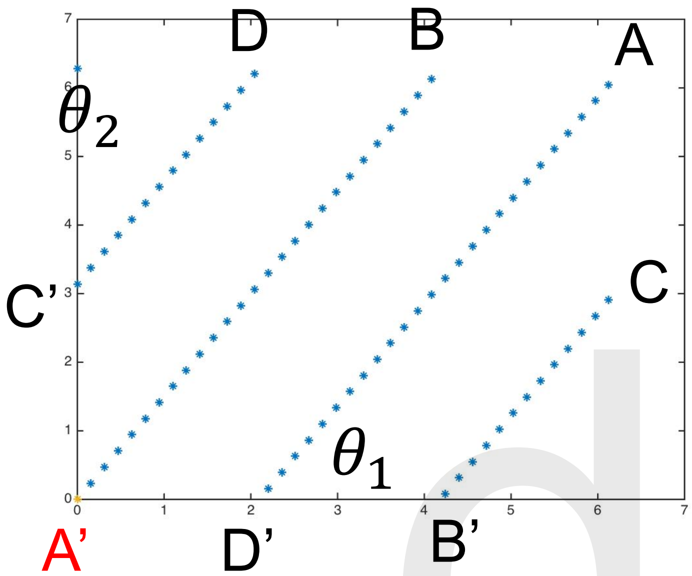
$$\frac{d\theta_j}{dt} = -\omega_j \quad \Rightarrow \quad \begin{aligned} \theta_1' &= -\omega_1 \\ \theta_2' &= -\omega_2 \end{aligned} \qquad \begin{aligned} \theta_1 &= \theta_1(t=0) - \omega_1 t \\ \theta_2 &= \theta_2(t=0) - \omega_2 t \end{aligned}$$

(a parameterization for (“x”, “y”); what’s a direction vector?)

draft

- Periodic: $\frac{\omega_2}{\omega_1}$ is a rational number, e.g., $\frac{\omega_2}{\omega_1} = \frac{3}{2}$

$Periodic: \dfrac{\omega_2}{\omega_1} = \dfrac{3}{2}$



$$\theta_1 = \theta_{1i} + \omega_1 t \quad (1)$$

$$\theta_2 = \theta_{2i} + \omega_2 t \quad (2)$$

Say, $\omega_1 \rightarrow -\omega_1$

$$(\theta_1, \theta_2) = (\theta_{1i}, \theta_{2i}) + t(\omega_1, \omega_2)$$

What we call the above in calc III?

What we call (ω_1, ω_2) in calc III?

at A': $(\theta_{1i}, \theta_{2i}) = (0{,}0)$ Let $\theta_2 = 2\pi$ From (2), $\Delta t_1 = 2\pi/3$

plugging $\Delta t_1 = 2\pi/3$ into (1), we obtain $\theta_1 = 4\pi/3$

at B': $(\theta_{1i}, \theta_{2i}) = (4\pi/3{,}0)$ Let $\theta_1 = 2\pi$ ➔ $\Delta t_2 = \pi/3$; $\theta_2 = \pi$

at C': $(\theta_{1i}, \theta_{2i}) = (0, \pi)$ Let $\theta_2 = 2\pi$ ➔ $\Delta t_3 = \pi/3$; $\theta_1 = 2\pi/3$

at D': $(\theta_{1i}, \theta_{2i}) = (2\pi/3{,}0)$ Let $\theta_1 = 2\pi$ ➔ $\Delta t_4 = 2\pi/3$; $\theta_2 = 2\pi$

$T = 2\pi$

Periodicity vs. Quasi-periodicity

$$\frac{dr_j}{dt} = 0$$

$$\frac{d\theta_j}{dt} = -\omega_j \quad \Rightarrow \quad \begin{aligned} \theta_1' &= -\omega_1 \\ \theta_2' &= -\omega_2 \end{aligned} \qquad \begin{aligned} \theta_1 &= \theta_1(t=0) - \omega_1 t \\ \theta_2 &= \theta_2(t=0) - \omega_2 t \end{aligned}$$

draft

- Periodic: $\frac{\omega_2}{\omega_1}$ is a rational number, e.g., $\frac{\omega_2}{\omega_1} = \frac{5}{2}$
- Quasi-periodic: $\frac{\omega_2}{\omega_1}$ is not a rational number, e.g., $\frac{\omega_2}{\omega_1} = \sqrt{2}$

"Dense" Quasi-periodic Solutions

w1/w2
is irrational

➔
incommensurate
frequencies

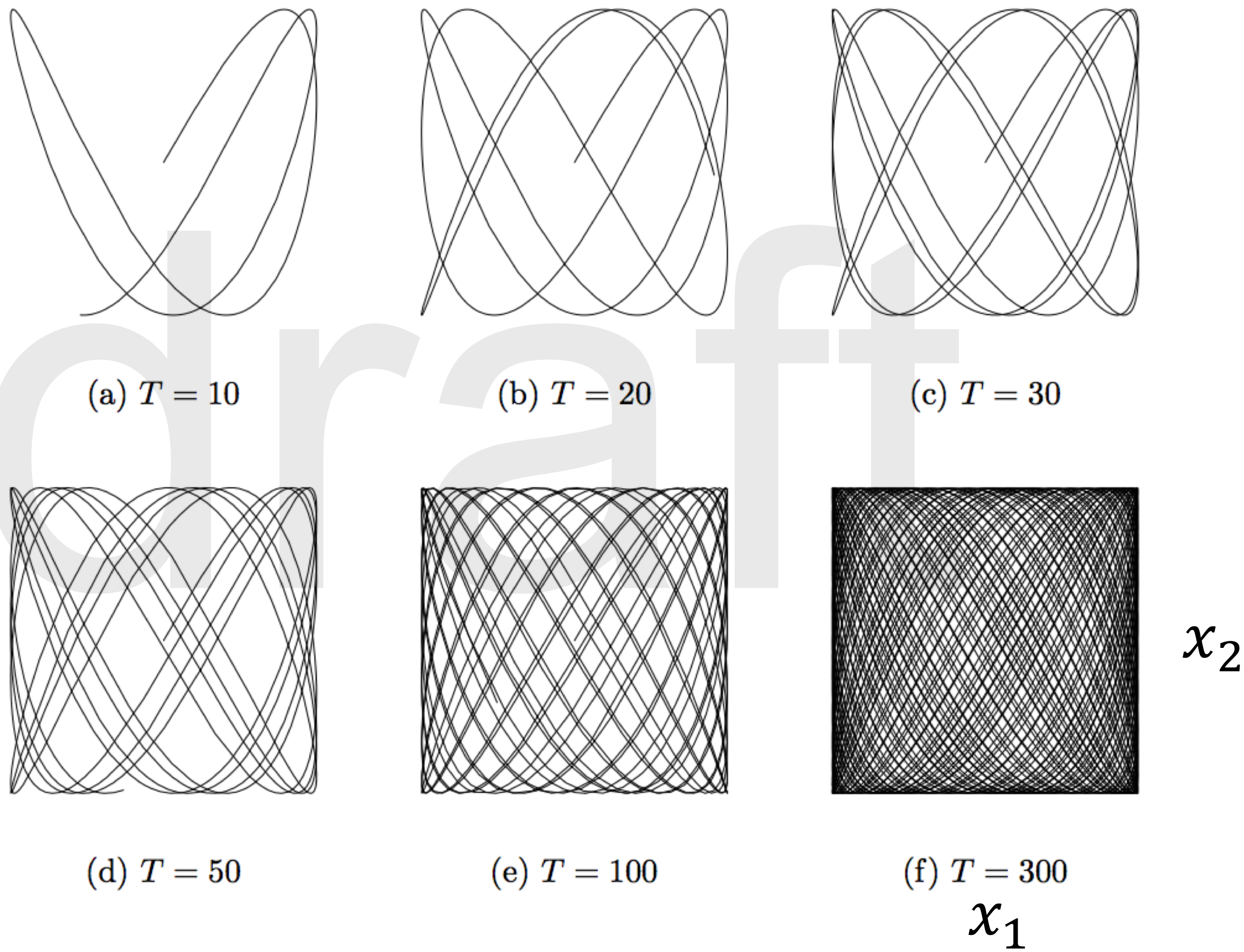
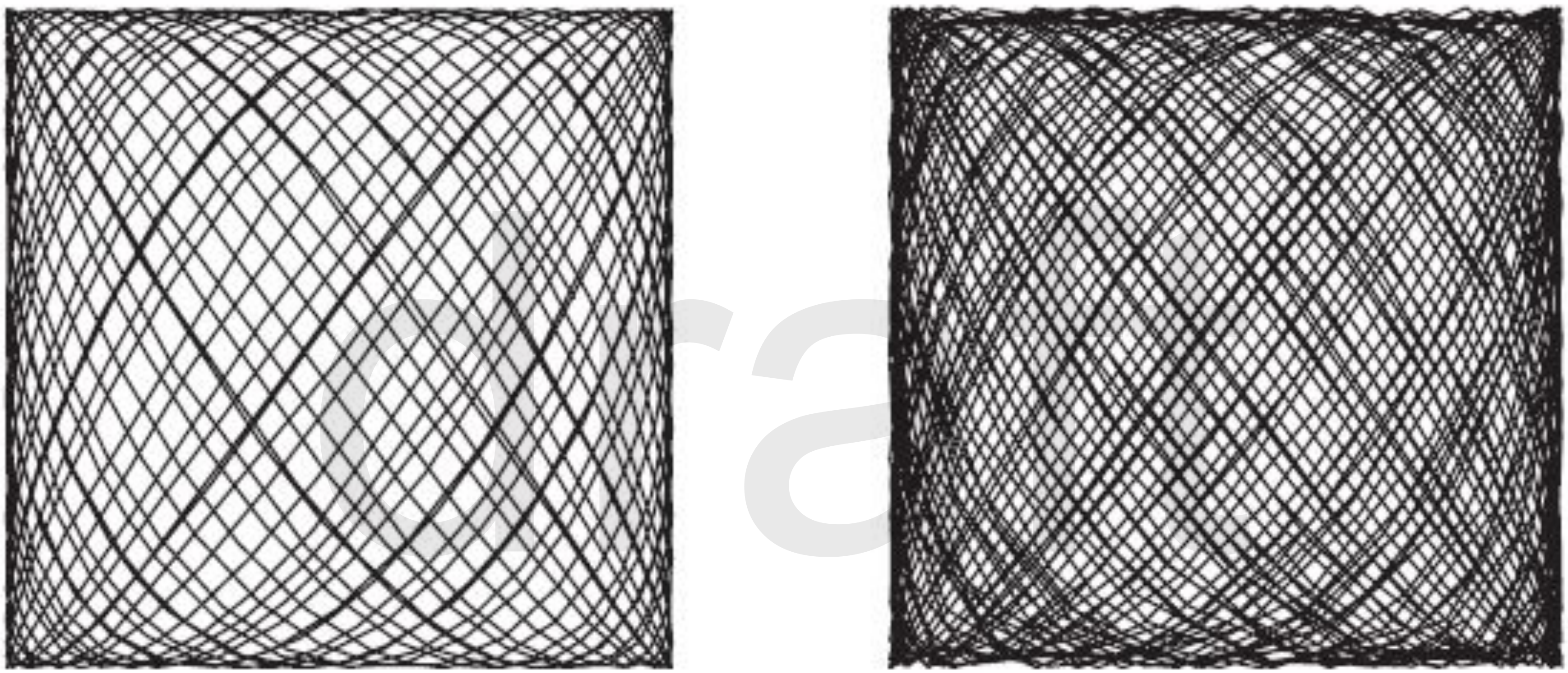


Figure 1.2: Trajectory of the mechanical system at different times.

$Quasiperiodic: \dfrac{\omega_2}{\omega_1} = \sqrt{2}$

T=0~50π T=0~100π



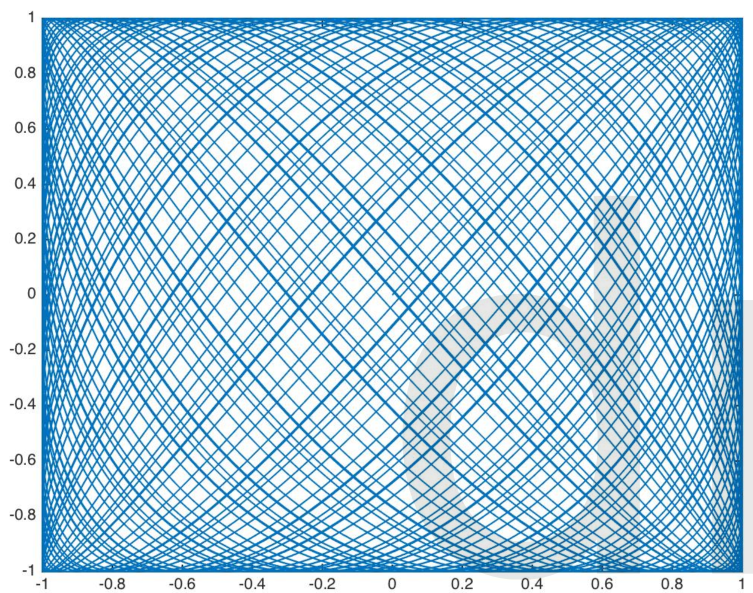
x_2

x_1

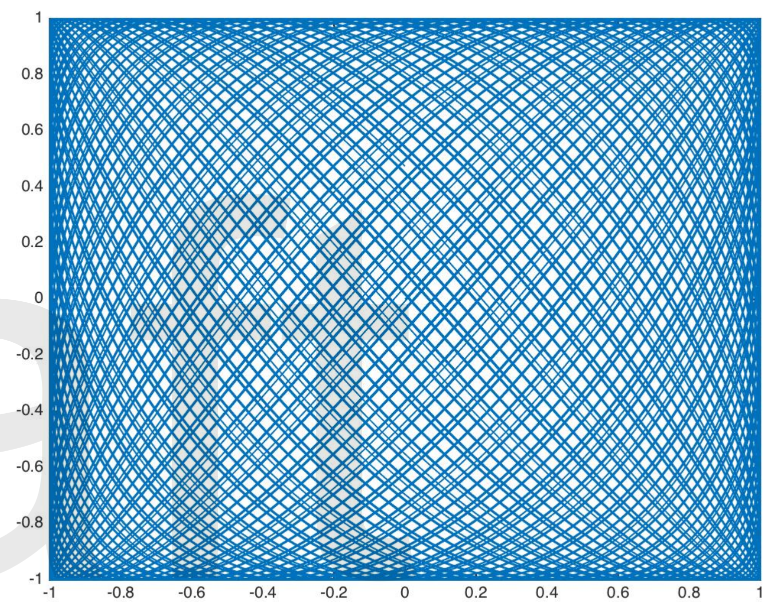
Figure 6.7 A solution with frequency ratio $\sqrt{2}$ projected into the $x_1 x_2$–plane, the left curve computed up to time 50π, the right to time 100π.

$Quasiperiodic: \frac{\omega_2}{\omega_1} = \sqrt{2}$ **S4**

T=0~50



T=0~100



x_2

x_1

```
time=linspace(0,50, 50000)
x1=sin(2*pi*time)
x2=sin(sqrt(2)*2*pi*time)
%fig=figure()
plot(x1, x2)
```

```
time=linspace(0,100, 100000)
x1=sin(2*pi*time)
x2=sin(sqrt(2)*2*pi*time)
%fig=figure()
plot(x1, x2)
```

$O(\epsilon)$ for 5D-NLM

$$\frac{dX'}{d\tau} = \sigma Y', \tag{6}$$

$$\frac{dY'}{d\tau} = (r - Z_c)X' - X_c Z' - FN(X'Z'), \tag{7}$$

$$\frac{dZ'}{d\tau} = (Y_c - Y_{1c})\, X' + X_c Y' - X_c Y_1' + FN(X'Y' - X'Y_1'), \tag{8}$$

$$\frac{dY_1'}{d\tau} = (Z_c - 2Z_{1c})X' + X_c Z' - 2X_c Z_1' + FN(X'Z' - 2X'Z_1'), \tag{9}$$

$$\frac{dZ_1'}{d\tau} = 2Y_{1c}X' + 2X_c Y_1' + 2FN(X'Y_1'). \tag{10}$$

$O(\epsilon)$ for 5D-NLM near a Critical Point

setting $Y_c = 0, Y_{1c} = 0, Z_c = r,\ Z_{1c} = \frac{r}{2}$ and $FN = 0$, we obtain:

$$\frac{dX'}{d\tau} = \sigma Y',$$

$$\frac{dY'}{d\tau} = -X_c Z',$$

$$\frac{dZ'}{d\tau} = X_c Y' - X_c Y_1',$$

$$\frac{dY_1'}{d\tau} = X_c Z' - 2X_c Z_1',$$

$$\frac{dZ_1'}{d\tau} = 2X_c Y_1'.$$

$$\frac{d^2Y'}{d\tau^2} = -X_c\dot{Z}' = -X_c(X_cY' - X_cY_1') = -X_c^2(Y' - Y_1') \tag{22}$$

$$\frac{d^2Y_1'}{d\tau^2} = X_c\dot{Z}' - 2X_c\dot{Z}_1' = X_c(X_cY' - X_cY_1') - 2X_c(2X_cY_1') = X_c^2Y' - 5X_c^2Y_1' \tag{23}$$

Q3 of HW5

3: [40 points] Consider the following coupled harmonic oscillator (as shown in Fig. 1):

$$\frac{d^2x_1}{dt^2} = -k_1x_1 + k_2(x_2 - x_1),$$

$$\frac{d^2x_2}{dt^2} = -k_2(x_2 - x_1).$$

Let $k_1 = 4X_c^2$ and $k_2 = X_c^2$ (and $m_1 = m_2 = 1$).

The same math system with quasi-periodic solutions.

$Y' = x_2$

$Y_1' = x_1$

$O(\epsilon)$ for the 5D-NLM near a critical point

$$\frac{d^2Y'}{d\tau^2} = -k_2(Y' - Y_1') = -X_c^2(Y' - Y_1')$$

$$\frac{d^2Y_1'}{d\tau^2} = -k_1Y_1' - k_2(Y_1' - Y') = X_c^2Y' - 5X_c^2Y_1'$$

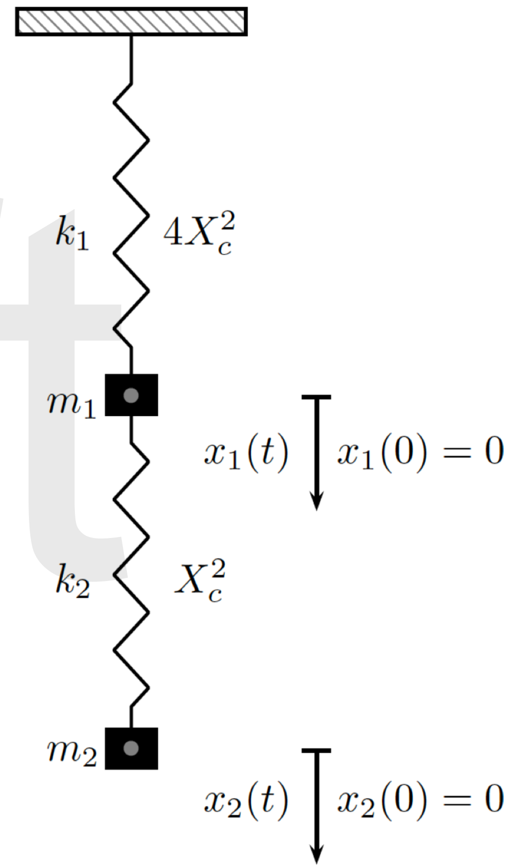


Figure 1: Coupled spring/mass system

Quasi-Periodic Solutions in the 5D-NLM

Supp

Courtesy of
Sara Faghih-Naini

Introduction: Quasiperiodic Solutions

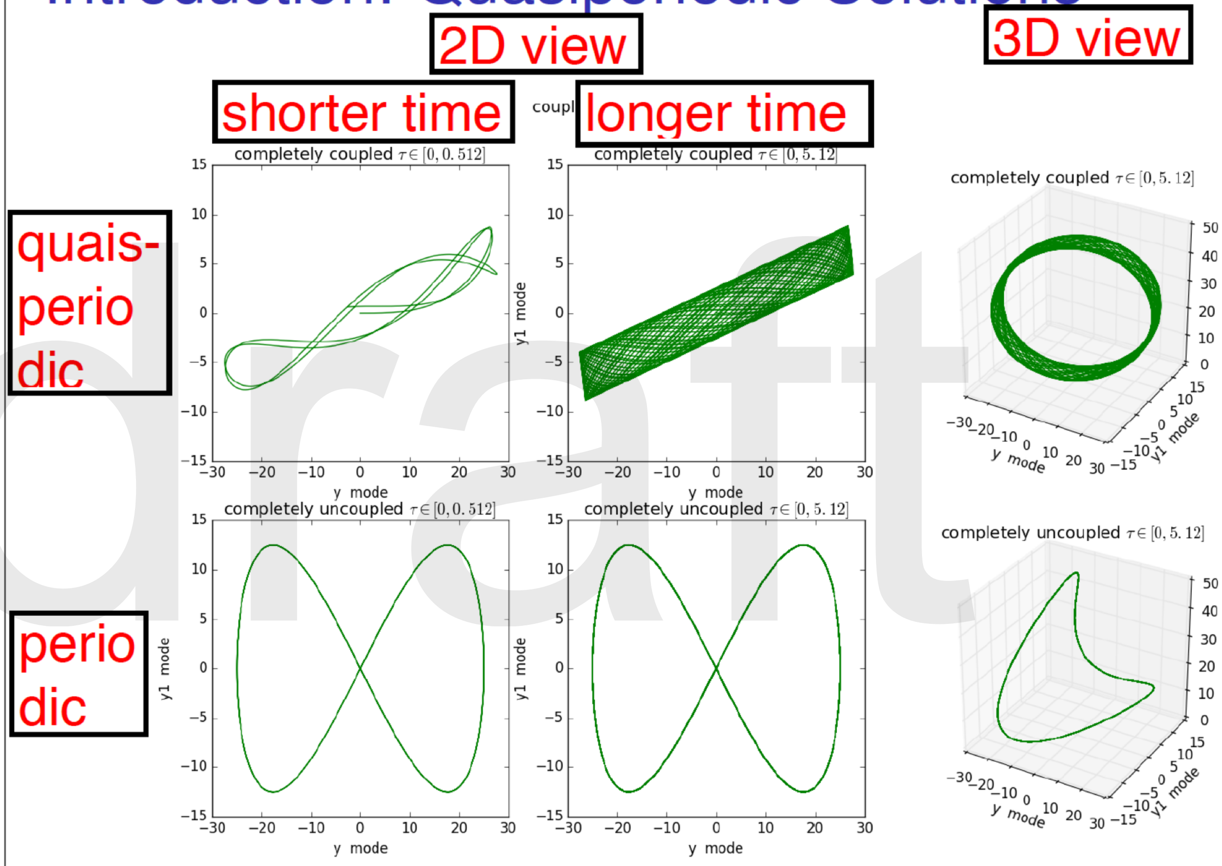


Figure: Quasiperiodic solution of coupled LL 5D NLM with frequency ratio $\frac{1}{2}\left(3-\sqrt{5}\right)$ (top) and periodic solution of uncoupled LL 5D NLM with frequency ratio 2 (bottom) up to time 0.512 (left) and 5.12 (middle).

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