The Essentials of Calculus

With Early Transcendentals

Gary L. Taylor & J Michael Shaw

Including all material required for the AP™
Calculus AB Course
in a Sequence Compatible with AP Classroom

Gary L. Taylor
J Michael Shaw

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THE ESSENTIALS OF CALCULUS WITH EARLY TRANSCENDENTALS

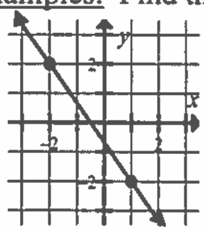
LESSON 0.1 SLOPES, LINES, CALCULATOR REVIEW

The slope of a line is symbolized by the letter "m".

Slope $= m = \dfrac{rise}{run} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\Delta y}{\Delta x}$

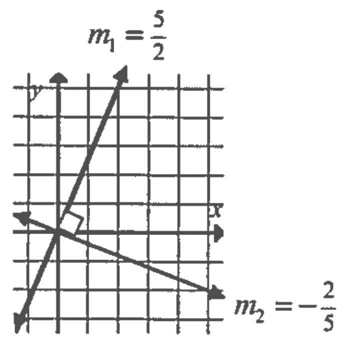
Examples: Find the slopes of the lines containing each pair of points.

1.



2. $(-2,0)$ *and* $(4,2)$ 3. $(3,2)$ *and* $(2,2)$ 4. $(3,2)$ *and* $(3,5)$

Parallel lines have equal slopes ($m_1 = m_2$).
Perpendicular lines have slopes which are opposite reciprocals $\left(m_1 = -\frac{1}{m_2}\right)$.



Equations for lines

point-slope form: $y - y_1 = m(x - x_1)$

slope-intercept form: $y = mx + b$ (where b is the y-intercept)

general form : $Ax + By + C = 0$ (where A, B, and C are integers)

Examples: Find an equation of each line described.

5. a line through (2,3) with slope $m = -3$

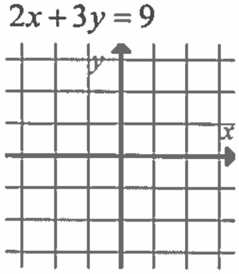
6. a vertical line through $(-1,2)$

7. a line through $(-1,2)$ parallel to the graph of $2x - 5y = 5$ (in slope-intercept form)

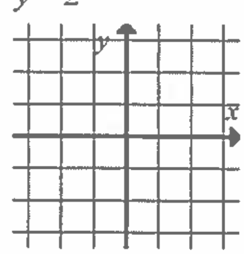
8. a line through $(-1,2)$ perpendicular to the graph of $2x - 5y = 5$ (in general form)

Examples: Draw a graph of each line.

9. $2x + 3y = 9$



10. $y = 2$



Calculator Examples:

11. Find a window to show a complete graph of $y = f(x) = -0.2x^3 - 2.2x^2 + 1.6x + 1$.
    Indicate the scale on the graph or give your window setting.

12. Find the zeros of $y = f(x) = -0.2x^3 - 2.2x^2 + 1.6x + 1$.

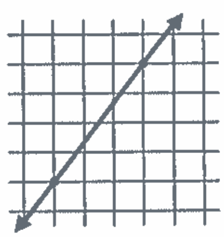
13. Find the points of intersection of $y = -x^3 + 12x^2 + 9x - 3$ and $3x - y + 5 = 0$. Write the equation you are solving.

14. Use a calculator to solve $|x^2 - 5| \geq 4$. Write your answer in both inequality notation and interval notation.

ASSIGNMENT 0.1

Find the slopes of these lines.

1.



2. through $(2,-6)$ *and* $(5,-12)$
3. through $(3,6)$ *and* $(-2,6)$
4. through $(-6,5)$ *and* $(4,3)$

Find an equation for each line.

5. through (1,2) with $m = -2$
6. through (2,0) and (3,1), in slope-intercept form
7. through (1,7) with undefined slope
8. through (1,7) with $m = 0$
9. vertical with x-intercept at 4
10. through $(-1,-3)$ parallel to the graph of $y = 3x - 5$, in general form
11. through (2,3) perpendicular to the graph of $2x - 3y = 7$
12. through $(2,-3)$ perpendicular to the graph of $x = 5$

Graph without using a calculator.

13. $y = -3x + 2$
14. $x = -2$
15. $2x + 5y + 10 = 0$

16. Show work to determine if $(3,5)$, $(7,0)$, and $(-1,11)$ are collinear (lie on the same line).

Use a calculator for problems 17-27. Answers should be accurate to three or more decimal places (rounded or truncated).

17. Find an appropriate window to show a complete graph of $y = x^3 + 4x^2 - 5x$. Your window should show all zeros and all local maximum and minimum points (turn-around points). Draw a window rectangle on your own paper and accurately draw the graph. Indicate the scale on the graph or give the window setting.
18. Find the zeros of $y = f(x) = x^3 + 4x^2 - 5x$. Write the equation you are solving on your paper.
19. Copy and complete the table at the right for this same function from problem 18.

x	$f(x)$
7	
7.1	524.051
7.2	
7.3	
7.4	

Be sure to show three decimal places.

20. Find $f(-2.1576)$ for this same function.
21. Find the x- and y-coordinates of the local maximum and minimum points of $f(x)$.
22. Find the intersection points of the $f(x)$ function and $g(x) = -3x^2 - 5x + 15$. Write the equation you are solving.
23. Solve $x^3 + 4x^2 - 5x = -3x^2 - 5x$.
24. Solve $x^3 + 4x^2 - 5x \le 0$. Write your answer in interval notation. No work is required.
25. Find the points of intersection of the graphs of $x^2 + y = 4$ and $2x - y = 1$. Write the equation you are solving.
26. Find the x-coordinate(s) of the point(s) of intersection of the graphs of $x + y = 7$ and $2x - 3y = -1$. Write the equation you are solving.
27. Solve $\log(2x^2 - 5) = 0$.

28. Show algebraic steps (without a calculator) to find the intersection point(s) of the two functions $f(x) = \begin{cases} x^2 + 2, & x \le 1 \\ 3, & x > 1 \end{cases}$ and $g(x) = -x + 8$.

29. Verify your intersection point(s) from Problem 28 without using a calculator by graphing $f(x)$ and $g(x)$ on the same axes.

LESSON 0.2 FUNCTIONS, INVERSES, GRAPHING ADJUSTMENTS

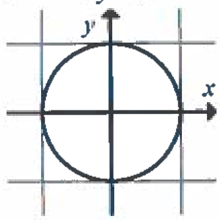
Relation: any set of ordered pairs (any set of points on a graph)

Function: a special type of relation. y is a function of x if for each x-value there is only one y-value. The graph of a function passes the vertical line test. This is written $y = f(x)$.

$$\left.\begin{array}{ll} \text{Domain:} & \text{the set of all } x\text{-values} \\ \text{Range:} & \text{the set of all } y\text{-values} \end{array}\right\} \text{ assuming } y \text{ is a function of } x$$

<u>Examples:</u> Determine whether each is a function of x.

1. $x+y=1$
2. $x^2+y^2=1$



3. $y=-x^2+1$
4. $x+y^2=1$

Given: $f(x)=3x-1$ and $g(x)=x^2$. Find the following.

5. $f(10)=$
6. $g(x+\Delta x)=$
7. $g(f(x))=$
8. $(f\circ g)(x)=$

Determine the domain and range for each function.

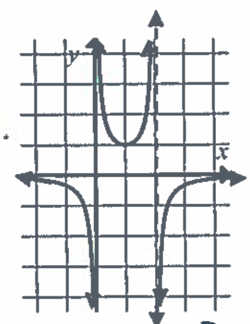
9. $f(x)=\sqrt{x-1}$

Do:
Ra:

10. $g(x)=\dfrac{1}{x-2}$

Do:
Ra:

11.



Do: Ra:

<u>One-to-one Function:</u> a function in which not only is there only one y for each x, but there is also only one x for each y. The graph passes the <u>horizontal line test</u> as well as the vertical line test.

<u>Inverse Function:</u> found by switching x and y and solving for the new y. $f^{-1}(x)$ is the symbol for the inverse of $f(x)$. Only one-to-one functions have inverse functions. Since x and y are switched to produce inverse functions, the domain of f is the range of f^{-1} and vice versa. If (a,b) is in the f function, then (b,a) is in the f^{-1} function.

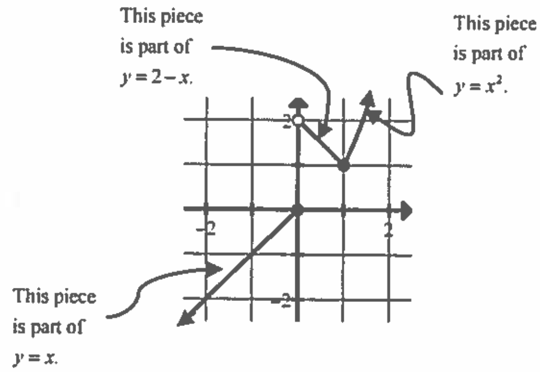
<u>Examples:</u>

12. Which of the relations in Examples 1-4 above is a function with an inverse function?

13. Find the inverse of $f(x)=2x^3-1$.

Piecewise Function: a function defined differently on different pieces of its domain.

Example: $f(x)=\begin{cases} x, & x\le 0 \\ 2-x, & 0<x<1 \\ x^2, & x\ge 1 \end{cases}$

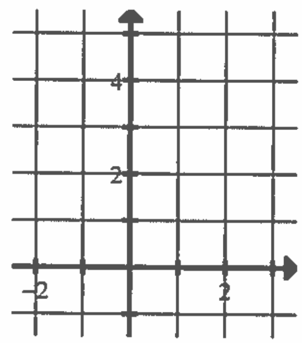


Examples:

14. Graph this piecewise function and give the domain and range.

$f(x)=\begin{cases} |x|, & x<1 \\ x+2, & x\ge 1 \end{cases}$

Do:
Ra:



Zeros: x-values for which y equals zero.
Conventionally, zeros are written as single values (e.g. $x=2$ or $x=5$) while x-intercepts are written as ordered pairs (e.g. (2,0) or (5,0)).

Find the zeros without using a calculator.

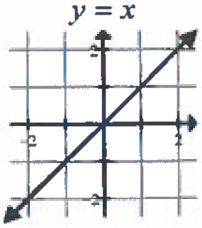
15. $f(x)=x^2-3x-4$

16. $y=\dfrac{x^2-4}{x^2+4}$

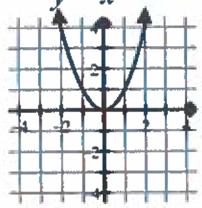
Parent Graphs

These graphs occur so frequently in this course that it would be worth your time to learn (memorize) them.

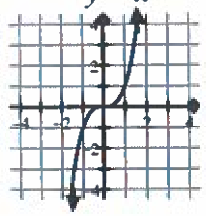
$y = x$



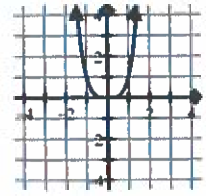
$y = x^2$



$y = x^3$



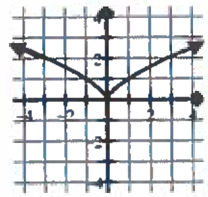
$y = x^4$



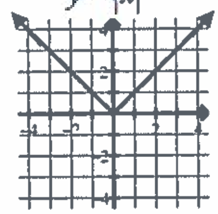
$y = \sqrt{x}$

$y = \sqrt[3]{x}$

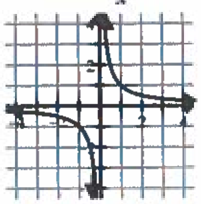
$y = x^{\frac{2}{3}}$



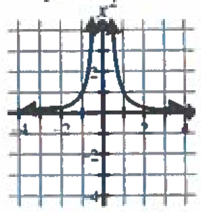
$y = |x|$



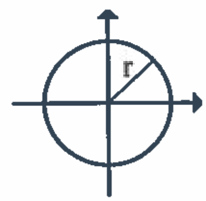
$y = \frac{1}{x}$



$y = \frac{1}{x^2}$



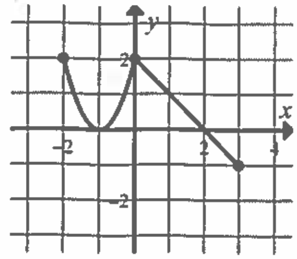
$x^2 + y^2 = r^2$



Graphing Adjustments to $y = f(x)$

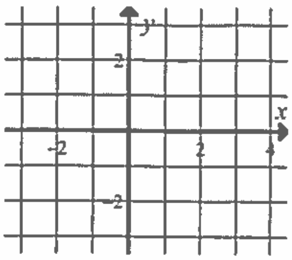
1. $y = -f(x)$ reflect across the x-axis
2. $y = f(-x)$ reflect across the y-axis
3. $y = f(x) + d$ shift up if $d > 0$, shift down if $d < 0$
4. $y = f(x + c)$ shift left if $c > 0$, shift right if $c < 0$
5. $y = a \cdot f(x)$ vertical stretch if $a > 1$, vertical squeeze if $a < 1$ (assumes a is positive, if a is negative a reflection is needed)
6. $y = f(b \cdot x)$ horizontal squeeze if $b > 1$, horizontal stretch if $b < 1$ (assumes b is positive, if b is negative a reflection is needed)
7. $y = |f(x)|$ reflect all points below the x-axis across the x-axis. Leave points above the x-axis alone.
8. $y = f(|x|)$ eliminate completely all points left of the y-axis. Leave points right of the y-axis alone. Replace the left half of the graph with a reflection of the right half. Your graph should then show y-axis symmetry.

Note: Adjustments to functions always produce functions.

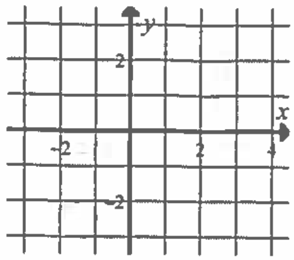


Examples: Use the graph of $y = f(x)$ shown to sketch the following:

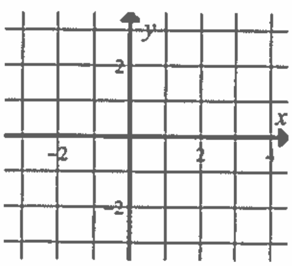
17. $y = f(x+2)$



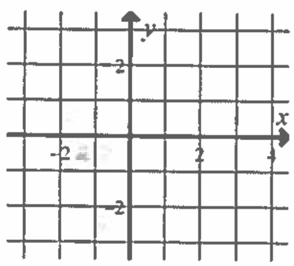
18. $y = -f(x)+2$



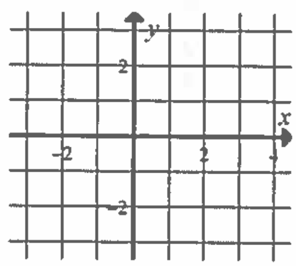
19. $y = \frac{1}{2}f(-x)$



20. $y = |f(2x)|$



21. $y = f(|x|)$



ASSIGNMENT 0.2

1. If $f(x) = 3x-2$, find the following.
   a. $f(0)$ b. $f(-3)$ c. $f(b)$ d. $f(x-1)$

2. If $g(x) = \frac{|x|}{x}$, find the following.
   a. $g(2)$ b. $g(-2)$ c. $g(x^2)$

3. If $f(x) = x^2 - x$, find $\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$.

Without using a calculator, find the domain and range of the given function and draw its graph. When possible make use of the parent graphs in this lesson.

4. $f(x) = \sqrt{x+1}$ 5. $g(x) = x^2+2$ 6. $h(x) = 4-x$

Without using a calculator determine whether y is a function of x.

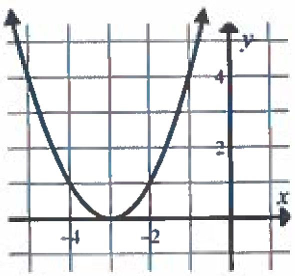
7. $2x+3y=4$ 8. $x^2+y^2=4$

9. Use the <u>parent graph</u> of $y=\sqrt{x}$ to graph the following.

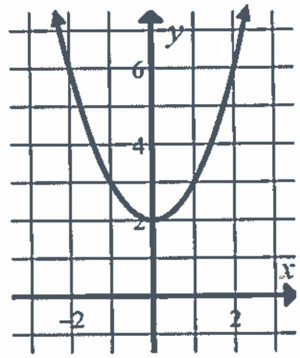
a. $y=\sqrt{x}+2$ b. $y=-\sqrt{x}$ c. $y=\sqrt{x-2}$ d. $y=2\sqrt{x}$

10. Use the parent graph of $y=x^2$ to determine an equation for each graph.

a.



b.



11. If $f(x)=\sqrt{x}$ and $g(x)=x^2-1$, find the following.

a. $f(g(1))$ b. $g(f(1))$ c. $(g\circ f)(x)$

12. If $f(x)=x+1$ and $g(x)=\frac{1}{x}$, find the following.

a. $(f\circ g)(x)$ b. the domain of $(f\circ g)$

c. $(g\circ f)(x)$ d. the domain of $(g\circ f)$

13. Are the two composite functions $(f \circ g)$ and $(g\circ f)$ from problem 12 equal?

14. If $f(x)=2x^2$, $g(x)=x+5$, $h(x)=2x-7$, and $k(x)=3$, find the following.

a. $k(2)$ b. $f(k(x))$ c. $(f\circ f)(x)$

d. $k(g(x))$ e. $(g\circ h)(3)$

Find the inverse function for each of the following showing organized work.

15. $y=2x-1$ 16. $f(x)=\sqrt[3]{x}-1$ 17. $g(x)=x$ 18. $h(x)=\sqrt{x}$

19. Draw a graph of $h(x)$ and $h^{-1}(x)$ from problem 18. Did your answer on problem 18 include the domain restriction needed for $h^{-1}(x)$?

20. If $f(x)=\sqrt{x-2}$, $g(x)=x^2$, and $h(x)=\frac{1}{x^2}$, find the following.

a. $g(f(x))$ b. the domain of $(g\circ f)$

c. $h(f(x))$ d. the domain of $(h\circ f)$

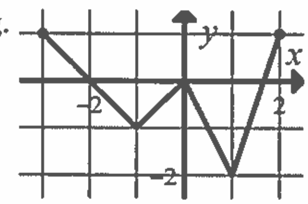
21. Without using a calculator graph this piecewise function.

$$f(x)=\begin{cases} x+2, & x<-2 \\ -x, & -2\le x\le 2 \\ x^2-6, & x>2 \end{cases}$$

Find the zeros of these functions without using a calculator.

22. $f(x)=\dfrac{x^2-3x+2}{x^2-1}$ 23. $g(x)=2x^3-8x$

Use the graph of $y = f(x)$ to draw **accurate** graphs of the following.



24. $y=-f(x)$ 25. $y=f(-x)$ 26. $y=|f(x)|$

27. $y=f(|x|)$ 28. $y=f(x)-1$ 29. $y=f(x-1)$

30. $y=\frac{1}{2}f(x)$ 31. $y=f\left(\frac{1}{2}x\right)$ 32. $y=|f(x)-1|$

Use a calculator for the rest of the assignment. Write answers to three or more decimal place accuracy.

33. Find the zeros of $f(x)=x^3-3x^2-2x+4$.

34. Solve $x^3-2x^2+5=\sqrt{3x+10}$.

Find the domain and the range for each function.

35. $y=2x^2+3x+6$ 36. $y=\dfrac{|x-2|}{x-2}$ 37. $y=\sqrt{7-x^2}$

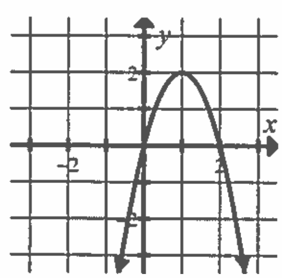
Use graphing adjustments to draw an accurate graph of each of the following without using a calculator. If necessary, use the parent graphs shown on Page 6.

38. $y=|x+2|$ 39. $y=|x|+2$ 40. $y=\dfrac{1}{x-1}$ 41. $y=\sqrt{|x|}$

42. $y=2\sqrt[3]{x}$ 43. $y=-\dfrac{1}{x^2}$ 44. $y=\dfrac{1}{(-x)^2}$ 45. $y=(x+3)^3-2$

46. Write an equation for the curve shown.

47. Check your answer by graphing with a calculator.



LESSON 0.3 INTERCEPTS, SYMMETRY, EVEN/ODD, INTERSECTIONS

x - and y - intercepts

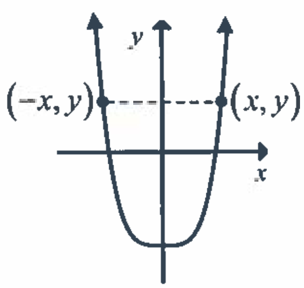
x-intercepts are points where a graph crosses or touches the x-axis. The y-coordinate is zero. To find the x-intercept, let $y = 0$ and solve for x.

y-intercepts are points where a graph crosses or touches the y-axis. The x-coordinate is zero. To find the y-intercept, let $x = 0$ and solve for y.

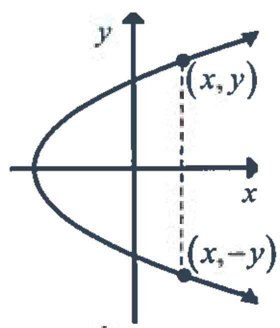
Example 1.

Find the x- and y-intercepts for $y^2 - 3 = x$.

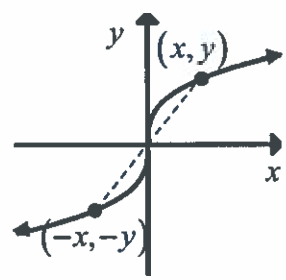
Symmetry



y-axis symmetry
reflection across
the y-axis



x-axis symmetry
reflection across
the x-axis



origin symmetry
reflection through
the origin (0,0)

Graphs can be symmetric to other lines and points. However, we will concentrate on these three.

Formal tests for symmetry:

1. y-axis: replacing x with $-x$ produces an equivalent equation
2. x-axis: replacing y with $-y$ produces an equivalent equation
3. origin: replacing x with $-x$ and y with $-y$ produces an equivalent equation

Informal tests for symmetry:

1. y-axis: substituting a number and its opposite for x give the same y-value
2. x-axis: substituting a number and its opposite for y give the same x-value
3. origin: substituting a number and its opposite for x give opposite y-values

Note: These informal tests are not foolproof. Think about whether other numbers would work the same. If your substitution produces zero, try another number.

<u>Examples:</u> Find the type(s) of symmetry for the graph of:

2. $y = 2x^3 - x$ 3. $y = |x| - 2$ 4. $|y| = x - 2$

Even/Odd Functions

A function is defined to be **even** if $f(-x) = f(x)$ for all x in the domain of $f(x)$. Even functions have graphs with <u>y-axis symmetry</u>. Examples: $y = x^2$, $y = x^4$, $y = x^2 + 3$, $y = x^4 + x^2$

A function is defined to be **odd** if $f(-x) = -f(x)$ for all x in the domain of $f(x)$. Odd functions have graphs with <u>origin symmetry</u>. Examples: $y = x$, $y = x^3$, $y = x^5$, $y = x^5 - x^3$

<u>Examples:</u> Determine whether the following functions are even, odd, or neither.

5. $f(x) = x^3 - x$ 6. $g(x) = x^2 - 4$ 7. $h(x) = x^2 + 2x + 2$

Points of Intersection of Two Graphs <u>(without a calculator)</u>

Method 1. Solve one equation for one variable and substitute into the other equation.

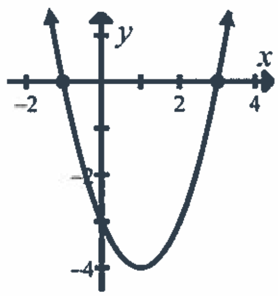
Method 2. Solve both equations for the same variable and set equal.

<u>Example 8.</u> Without using a calculator, find all points of intersection for the graphs of $x - y = 1$ and $x^2 - y = 3$.

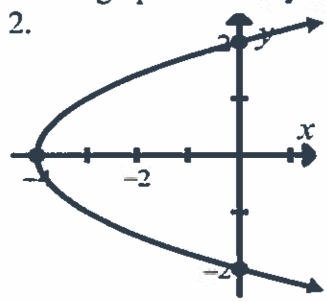
ASSIGNMENT 0.3

Find the x- and y-intercepts for these graphs. Write your answers as ordered pairs.

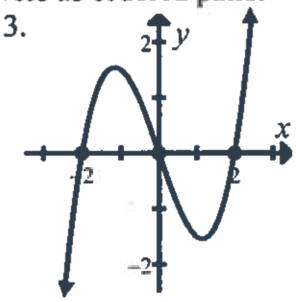
1.



2.



3.



Find the intercepts for the graphs of these equations. Do not use a calculator.

4. $y=3x-2$ 5. $y=x^2-4x+3$ 6. $y=x\sqrt{x^2-9}$

7. $y=\dfrac{x-2}{x+3}$ 8. $xy^2+x^2+4y-4=0$ 9. $y=\sqrt{x^2-9}$

Check for x-axis, y-axis, or origin symmetry. Do not use a calculator.

10. the graph of Problem 1 on this assignment.
11. the graph of Problem 2 on this assignment.
12. the graph of Problem 3 on this assignment.

13. $y=x^2-2$ 14. $y=x^3+x$ 15. $y=\dfrac{x}{x^2+1}$

16. $y^2=x-2$ 17. $y=x^3+3$

18. Which of the graphs in Problems 1-3 represent(s) an odd function?
19. Which of the graphs in Problems 1-3 represent(s) an even function?

Without using a calculator determine whether the following functions are even, odd, or neither.

20. $f(x)=4-x^2$ 21. $g(x)=x(x^2-4)$ 22. $h(x)=x^3-1$

For Problems 23-25 find intercepts, symmetry, and sketch a graph without using a calculator.

23. $y=x+2$ 24. $y=\dfrac{1}{x}$ 25. $y=x^2+3$

Find the points of intersection for the graphs of these equations without using a calculator. **Show algebra steps!**

26. $\begin{cases} y=x^3 \\ y=x \end{cases}$ 27. $\begin{cases} x^2+y^2=25 \\ y-x=1 \end{cases}$

28. Is the point (1,4) on the graph of $2x-3y=10$?
29. If $(2,-1)$ is a point on the graph of $y=kx^3$, find the value of k.

Use a calculator to determine whether the following functions are even, odd, or neither.

30. $f(x)=\sqrt{x^2-x^4}$

31. $g(x)=\left|x^3-x\right|$

32. $h(x)=\sin x^3$

33. $j(x)=\log x^3$

For Problems 34-36 (using a calculator):
a. list the intercepts
b. identify the symmetry
c. draw a window rectangle and sketch the graph

34. $y=-2x^2+2x+1$

35. $3x^2+y^2=4$

36. $x+y^2=4$

Use a calculator on Problems 37-40. Answers should be accurate to three or more decimal places.

37. If $y=x^3+4x^2-5x$, find the value(s) of x when $y=20$. Write the equation you are solving on your paper.

38. Solve $|3x-7|<9$.

39. Draw a window rectangle and sketch a graph of $y=(x-5)^{\frac{2}{3}}$.

40. Draw more than one window rectangle to show all local maximum and minimum points and end behavior of $f(x)=\frac{1}{4}x^4-\frac{19}{6}x^3-\frac{11}{4}x^2+5x$.

Find the slope for each of the following.

41. a line through (2,3) and (2,7)

42. the graph of $2x+3y=8$

43. a line perpendicular to $x=4$

Find an equation for each of the following.

44. a line through (0,2) with slope $m=-\frac{2}{3}$

45. a line through (1,2) perpendicular to the graph of $y=\frac{2}{3}x$

LESSON 0.4 REVIEW OF BASIC TRIGONOMETRY

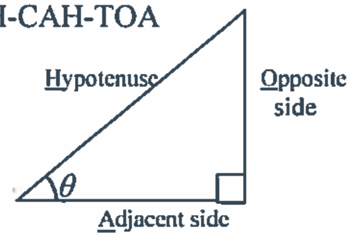
Basic Right Triangle Trigonometry:

The basic right triangle trigonometric ratios are given by SOH-CAH-TOA

$\sin\theta = \dfrac{opp}{hyp}$ (SOH) $\qquad \csc\theta = \dfrac{1}{\sin\theta}$

$\cos\theta = \dfrac{adj}{hyp}$ (CAH) $\qquad \sec\theta = \dfrac{1}{\cos\theta}$

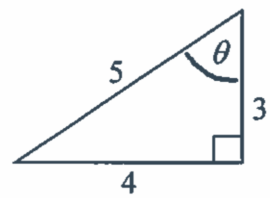
$\tan\theta = \dfrac{opp}{adj}$ (TOA) $\qquad \cot\theta = \dfrac{1}{\tan\theta}$



$0^\circ < \theta < 90^\circ$

When using right triangle trigonometry, angles are usually measured in degrees.

Example 1: Use the triangle at right to find

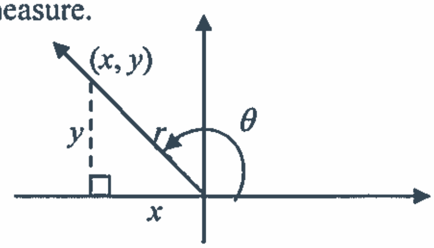


a. $\sin\theta$ b. $\cos\theta$ c. $\tan\theta$ d. $\sec\theta$

Trigonometric Functions Defined as Circular Functions:

Angles in a right triangle must be positive and less than or equal to 90°. A less restrictive way of defining trigonometric (trig) ratios is to use angles which can be any measure.

At right is an angle in standard position. The vertex of the angle is the origin. The initial side of the angle is the positive x-axis. In the figure shown, the terminal side was formed by a counter-clockwise rotation, so the measure of the angle, (θ), is positive Clockwise rotations produce negative angles.



When trig functions are defined using rotations from an initial ray (side) in the coordinate plane, they are called circular functions. In Calculus, angles are usually defined by circular trig functions and are almost always measured in radians. $(2\pi^R = 360^\circ)$

The circular function trig definitions are (see figure):

$\sin\theta = \dfrac{y}{r} \qquad \csc\theta = \dfrac{r}{y}$

$\cos\theta = \dfrac{x}{r} \qquad \sec\theta = \dfrac{r}{x}$

$\tan\theta = \dfrac{y}{x} \qquad \cot\theta = \dfrac{x}{y}$

$$\begin{cases} \theta \text{ is any measure} \\ r = \sqrt{x^2 + y^2} \quad \text{(Positive)} \\ x \text{ and } y \text{ may be } +,\ -,\ \text{or } 0 \end{cases}$$

Example 2: Find $\sin\theta$, $\csc\theta$, and $\cot\theta$, if θ is an angle in standard position whose terminal side passes through the point $(-5, 2)$.

Circular function trigonometry makes use of reference angles in triangles and is really not much different than right triangle trigonometry. Think of it as an extension of right triangle trig.

$30^{\circ} - 60^{\circ} - 90^{\circ}$ and $45^{\circ} - 45^{\circ} - 90^{\circ}$ reference triangles can be used to find trig ratios of angles which are multiples of 30° or 45°.

Example 3: Draw angles in standard position and make "reference triangles" to find:

a. $\cos 210^{\circ}$

b. $\tan 315^{\circ}$

Example 4: Since 2π radians $= 360^{\circ}$, it follows that $\pi^{R} = 180^{\circ}$, and the following common radian measures should be easy to think about in degrees. Convert each common radian measure to degrees.

a. $\frac{\pi}{2} =$

b. $\frac{\pi}{4} =$

c. $\frac{\pi}{3} =$

d. $\frac{\pi}{6} =$

Example 5: Convert from radians to degrees or degrees to radians without using a calculator.

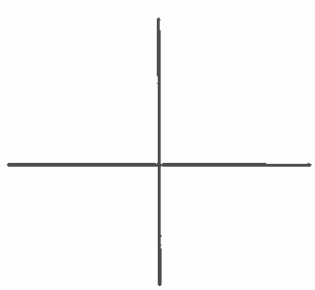
a. $\frac{5\pi}{4} =$

b. $270^{\circ} =$

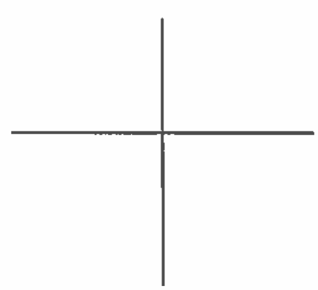
c. $-120^{\circ} =$

Examples: Draw angles in standard position, and make "reference triangles" to find the following without using a calculator:

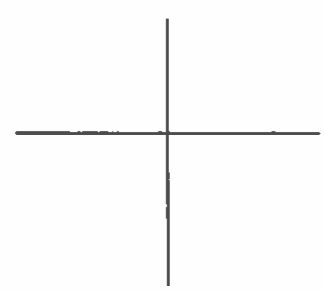
6. $\tan\frac{5\pi}{6}$



7. $\cos\left(\frac{-3\pi}{4}\right)$

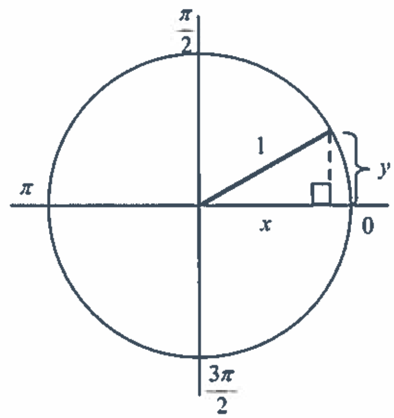


8. $\csc\frac{5\pi}{3}$



A unit circle is created by letting $r = 1$ when dealing with the circular trig functions.

Then, $\sin\theta = y$, $\cos\theta = x$, and $\tan\theta = \dfrac{y}{x}$.



Example 9: Use a unit circle to find:

a. $\sin\frac{\pi}{6}$ b. $\sin 0$ c. $\cos 0$

d. $\sin\frac{\pi}{2}$ e. $\cos\frac{\pi}{2}$ f. $\sin\pi$

g. $\tan\pi$ h. $\sin\frac{3\pi}{2}$ i. $\cos\frac{3\pi}{2}$

j. $\cos(-\pi)$ k. $\tan\left(\frac{-\pi}{2}\right)$

As you can see from Example 9, the unit circle is particularly useful when finding trig ratios for the quadrant separators (since no "reference triangles" can be built for them).

Sine and cosine are the two most important trig functions. The other trig functions can all be built as ratios of the sine and cosine functions.

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\tan\theta}$ $\csc\theta = \dfrac{1}{\sin\theta}$ $\sec\theta = \dfrac{1}{\cos\theta}$

Example 10: If $\sin\theta = \frac{3}{5}$, find the possible values for

a. $\csc\theta$ b. $\cos\theta$ c. $\tan\theta$

Solving trigonometric equations requires you to "**work backwards**" from ratios to angles.

Example 11: Solve the following trig equations without using a calculator. Find all of the solutions in the interval $[0, 2\pi)$.

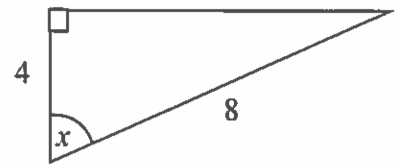
a. $\csc x = \dfrac{-2}{\sqrt{3}}$ b. $\cot\theta = \sqrt{3}$

c. $2\cos^2\theta - 1 = 0$ **Note:** $\cos^2\theta$ means $(\cos\theta)^2$. This is trig symbolism.

For these problems, you must be very careful with your "**SIGNS.**"

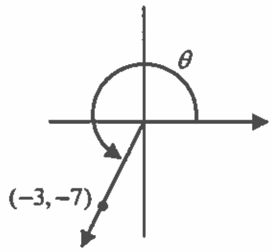
ASSIGNMENT 0.4

1. Use the triangle at right to find the following without using a calculator.
   a. $\cos x$ b. $\sin x$ c. $\tan x$
   d. $\cot x$ e. $\csc x$ f. $\sec x$



Use the following points on the terminal sides of angles, θ, in standard position to find the trig ratios in Problems 2 and 3 without using a calculator.

2.

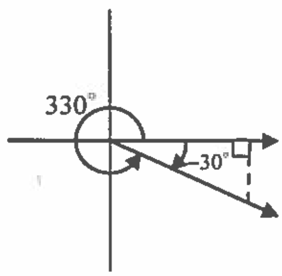


   a. $\tan\theta$ b. $\sin\theta$ c. $\sec\theta$

3. Point: $(5,-2)$
   Sketch a diagram (like the one in Problem 2) containing the given point before you find the trig ratios.
   a. $\cos\theta$ b. $\csc\theta$ c. $\cot\theta$

For Problems 4 and 5, use either a $30^\circ - 60^\circ - 90^\circ$ reference triangle or a $45^\circ - 45^\circ - 90^\circ$ reference triangle to find the trig ratios without using a calculator.

4.



   a. $\cos 330^\circ$ b. $\tan 330^\circ$ c. $\sec(-30^\circ)$

5. Sketch a diagram (like the one in Problem 4) before you find the trig ratios.
   a. $\sin 135^\circ$ b. $\csc(-225^\circ)$ c. $\cot 135^\circ$

6. Convert from radians to degrees <u>or</u> degrees to radians without using a calculator.
   a. $\frac{3\pi}{2}$ b. $\frac{-4\pi}{3}$ c. 225° d. -150°

7. Use a calculator to convert from radians to degrees <u>or</u> degrees to radians.
   a. $\frac{11\pi^R}{15}$ b. -2.5^R c. 312° d. $-8\pi^\circ$

For Problems 8 and 9, find the trig ratios without using a calculator. Sketch diagrams with reference triangles first.

8. a. $\sin\frac{5\pi}{4}$ b. $\cos\frac{5\pi}{4}$ c. $\tan\frac{5\pi}{4}$

9. a. $\cos\frac{2\pi}{3}$ b. $\tan\frac{2\pi}{3}$ c. $\csc\frac{2\pi}{3}$

10. List the quadrant in which θ lies if:

a. $\sin\theta < 0$, but $\tan\theta > 0$

b. $\cos\theta > 0$, but $\cot\theta < 0$

11. If $\cos\theta = \frac{-1}{3}$ and $\sin\theta > 0$, find:

a. $\sin\theta$

b. $\tan\theta$

c. $\sec\theta$

12. Without a calculator, find two values of x, where $0 \le x < 2\pi$ such that:

a. $\tan x = \sqrt{3}$

b. $\csc x = -2$

For Problems 13 and 14, solve for θ, where $0 \le \theta < 2\pi$, without using a calculator.

13. $\sec^2\theta - 4 = 0$

14. $\sin^2\theta = \cos^2\theta$

LESSON 0.5 TRIGONOMETRY WITH A CALCULATOR, GRAPHS OF TRIGONOMETRIC FUNCTIONS

When using a calculator with trig functions, it is important that the calculator is set in the correct mode (radians or degrees). In Calculus, we will deal almost entirely with radian measure.

Example 1: Use a calculator to find:

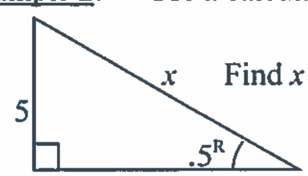
a. $\sin 2$

b. $\tan\left(\frac{-\pi}{5}\right)$

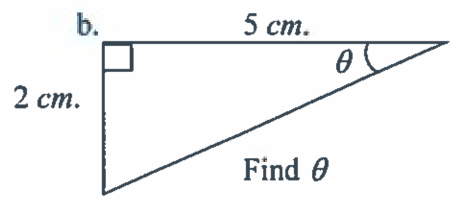
c. $\sec 1.3$

Example 2: Use a calculator to find the missing measure in each triangle.

a.



b.



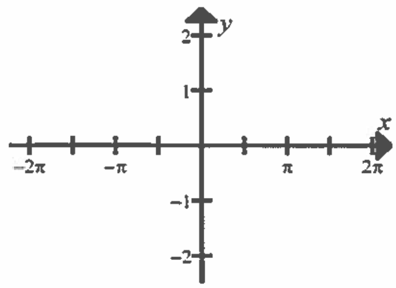
Graphs of Trig Functions:

Trig functions are periodic (their graphs repeat after a certain period or cycle).
The sine, cosine, cosecant, and secant functions all have a period of 2π.
The tangent and cotangent functions have a period of π.

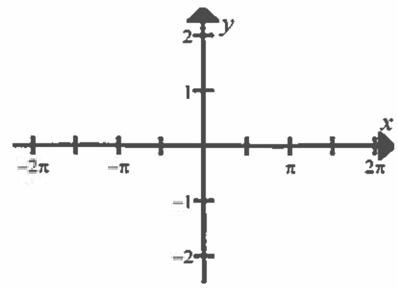
You should be able to easily graph the trig functions by using trig values at $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and by using the fact that the functions are periodic.
(You should also use $x = \pm\frac{\pi}{4}$ for the tangent and cotangent graphs.)

Example 3: Graph each of the following.

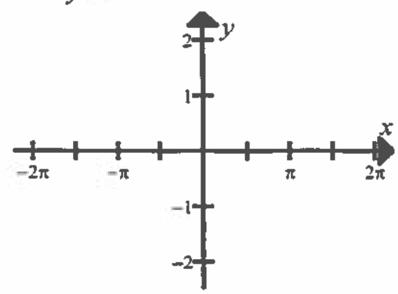
a. $y = \sin x$



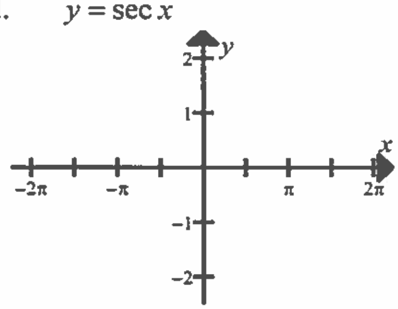
b. $y = \cos x$



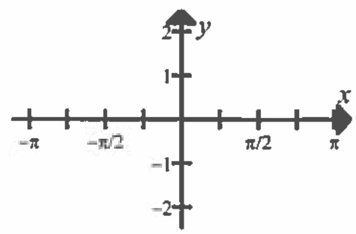
c. $y = \csc x$



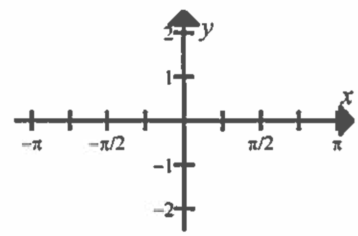
d. $y = \sec x$



e. $y = \tan x$



f. $y = \cot x$



Remember: Each of these last two functions has a period of π.

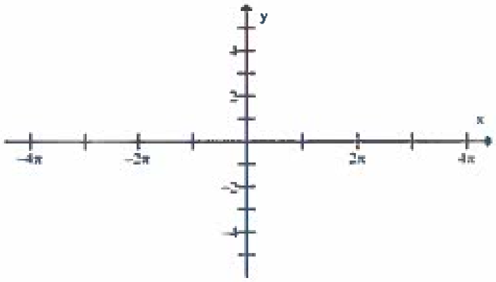
You should be able to use the parent trig graphs to graph functions of the form
$y = a\sin\big(b(x+c)\big)+d$.
The chart below provides an aid, but remember to think of "adjustments to graphs."

FUNCTION	PERIOD	AMPLITUDE	HORIZONTAL SHIFT	VERTICAL SHIFT
$y = a\ \sin\big(b(x+c)\big)+d$	$\frac{2\pi}{\lvert b\rvert}$	$\lvert a\rvert$	$-c$	d
$y = a\ \cos\big(b(x+c)\big)+d$				
$y = a\ \tan\big(b(x+c)\big)+d$	$\frac{\pi}{\lvert b\rvert}$	None	$-c$	d
$y = a\ \cot\big(b(x+c)\big)+d$				
$y = a\ \sec\big(b(x+c)\big)+d$	$\frac{2\pi}{\lvert b\rvert}$	None	$-c$	d
$y = a\ \csc\big(b(x+c)\big)+d$				

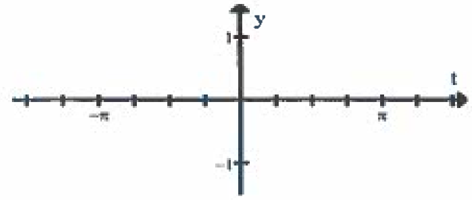
When c is positive, the horizontal shift is to the left. When c is negative, the horizontal shift is to the right. Horizontal shift is often called phase shift for periodic functions.

<u>Example 4</u>: Without using a calculator, sketch two cycles of:

a. $f(x) = -5\cos\left(\frac{x}{2}\right)$



b. $g(t) = \sin\left(2t - \frac{\pi}{2}\right)$



The sine and cosine functions are related to each other by the basic <u>Pythagorean Identity</u>:

$$\sin^2\theta + \cos^2\theta = 1 \quad \text{or} \quad \sin^2 x + \cos^2 x = 1$$

<u>Example 5</u>: Use the Pythagorean Identity to rewrite $2\cos\theta - \sin^2\theta = -2$ in a form which only contains one trig function. Then, without using a calculator, solve for θ on the interval $[0, 2\pi)$.

Use your calculator to verify your solution.

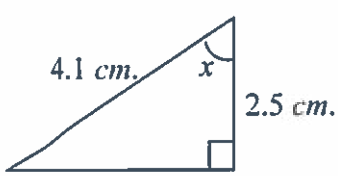
ASSIGNMENT 0.5

1. Use a calculator to find (to 3 or more decimal place accuracy):

   a. $\cos\frac{3\pi}{5}$ b. $\csc(-2.9)$ c. $\tan\left(\frac{5}{8}\right)$ d. $\cot(-1.6\pi)$

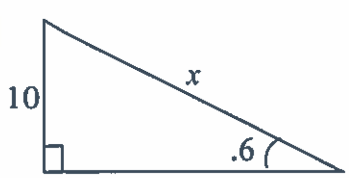
For Problems 2-5, use a calculator to find the values of x in each triangle.

2.

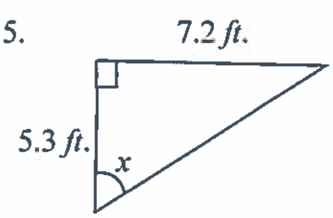


3.

4.



5.



Use a calculator to solve for x on the interval $[0, 2\pi)$ for Problems 6 and 7.

6. $\sin x - 2\cos x = 0$

7. $\tan x = \csc^2 x - 2$

For Problems 8-11, find:

a. the period
b. the amplitude (if there is one)
c. the horizontal shift (<u>phase shift</u>) including direction
d. the vertical shift including direction

8. $f(x) = -4\cos(2x) + 1$

9. $g(\theta) = \tan\left(\pi\left(\theta - \frac{1}{8}\right)\right)$

10. $P(t) = 112\sin(3t + \pi)$

11. $y = 2\sec\left(\dfrac{x}{2}\right) - 12$

Without a calculator, graph each of the functions in Problems 12-15 in a separate coordinate plane. Sketch two complete cycles for each graph.

12. $y = 2\cos x + 3$

13. $f(x) = \left|\tan\left(x - \frac{\pi}{4}\right)\right|$

14. $y = -\frac{1}{2}\sin\dfrac{x}{2}$

15. $g(x) = \csc(-2x)$

List the discontinuities for the functions in Problems 16 and 17. Do not use a calculator.

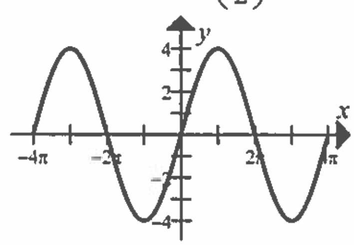
16. $f(x) = \sec(\pi x)$ on $[0, 2]$.

17. $g(x) = \tan\left(\dfrac{x}{2}\right)$ on $[0, 4\pi]$.

18. Find equations in the indicated forms for the graph at right.

    a. $y = a\sin(b(x - c)) + d$

    b. $y = a\cos(b(x - c)) + d$



19. Use a calculator to find $\lim\limits_{x \to 0} \dfrac{\sin(\pi - x)}{\sin(3x)}$.

20. Without a calculator, find all six trig ratios for an angle θ (in standard position) whose terminal side contains the point $(-2, -7)$.

21. Without a calculator, find all six trig ratios for an angle θ whose measure is:

    a. $\frac{5\pi}{4}$ b. $\frac{11\pi}{6}$ c. $\frac{5\pi}{2}$ d. $-\pi$

    First, sketch a reference triangle (if possible).

22. If $\sin\theta = \frac{3}{4}$ and $\cos\theta < 0$, find $\cos\theta$ and $\tan\theta$.

Without a calculator, solve for θ on the interval $[0, 2\pi)$ in Problems 23-25.

23. $\cot\theta = \frac{1}{\sqrt{3}}$

24. $\tan\theta - \sin\theta = 0$

25. $2\sin^2\theta = \cos\theta + 1$

LESSON 0.6 EXPONENTIAL FUNCTIONS

An **exponential function** is a function represented by a constant base with a variable exponent. For example, $f(x)=2^x$, $y=e^x$, and $g(x)=3^{x^2-5}$ are exponential functions.
These basic properties of exponents are used when working with exponential functions.

For a and b positive real numbers and x and y any real numbers:

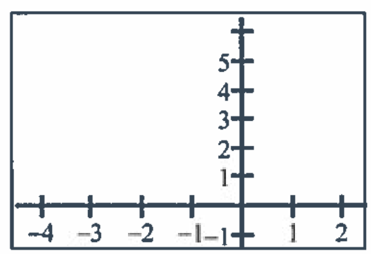
1. $a^0=1$
2. $a^x a^y=a^{x+y}$
3. $\dfrac{a^x}{a^y}=a^{x-y}$
4. $\left(a^x\right)^y=a^{xy}$
5. $(ab)^x=a^x b^x$
6. $\left(\dfrac{a}{b}\right)^x=\dfrac{a^x}{b^x}$
7. $a^{-x}=\dfrac{1}{a^x}$

Note: $(a+b)^x \neq a^x+b^x$

When simplifying, do not leave answers with negative exponents.

Examples: Simplify without using a calculator.

1. $27^{\frac{4}{3}}$
2. $\left(e+\dfrac{1}{e}\right)^0$
3. $\left(\dfrac{e^5\cdot e^{-3}}{e^4}\right)^2$
4. $5^3\cdot 25^{-2}$
5. Solve $9^x=27$ without using a calculator.
6. Use a calculator to carefully graph $y=2^x$, $y=5^x$, and $y=e^x$ in the same coordinate plane. Do you see any similarities in the graphs?



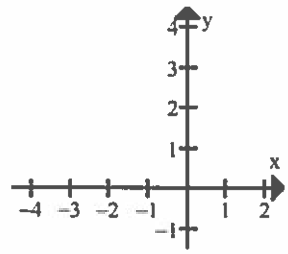
Graphs of Exponential Functions: If $f(x)=a^x$ and $a>1$, then

1. The domain of $f(x)$ is $(-\infty,\infty)$. The range of $f(x)$ is $(0,\infty)$.
2. The graph of $f(x)$ is continuous, increasing, concave upward, and one-to-one (has an inverse function).
3. The x-axis is a horizontal asymptote to the left: $\lim\limits_{x\to-\infty} f(x)=0$. *
   (Also, $\lim\limits_{x\to\infty} f(x)=\infty$) *
4. The y-intercept is (0, 1). Another key point is $(1, a)$.

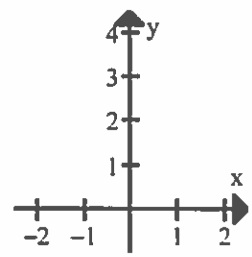
*This notation using limits will be developed completely in the next unit.

The letter e used as a base in Examples 2, 3, and 6, is not an unknown. It is a number called the natural base for exponential functions. It is the most common base in Calculus, because functions with base e are easier to differentiate and integrate than functions with other bases. By definition, $e = \lim_{x \to 0}(1+x)^{\frac{1}{x}}$. To three decimal places, $e \approx 2.718$.

Example 7: Without using a calculator, sketch a graph of $y = e^x$.



Example 8: Using adjustments to the graph from Example 7, graph $f(x) = e^{-x} + 1$ without using a calculator. Write an equation for the graph's asymptote.



ASSIGNMENT 0.6

Simplify without a calculator.

1. $8^{\frac{2}{3}}$	2. $25^{\frac{-3}{2}}$	3. $3^0 - 5^0$	4. $(3-5)^0$
5. $\dfrac{4}{4^3}$	6. $(3^{-2})^{-1}$	7. $(3^{-4})\,(9^3)$	8. $\dfrac{8^2}{4^3}$
9. $\left(\dfrac{2}{e}\right)^{-3}$	10. $\left(\dfrac{-e^2}{e^{-2}}\right)^2$	11. $\left(\dfrac{e^2 \cdot e^{-1}}{e^{-4}}\right)^3$	12. $(e+3)^2$

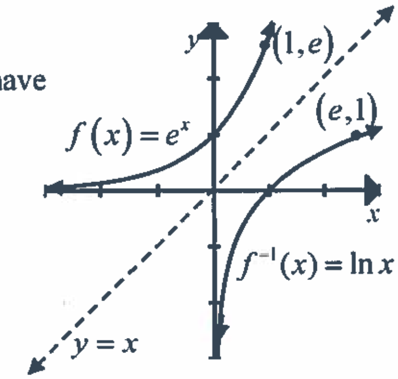
Solve for x without a calculator.

13. $2^x = 16$	14. $3^{2x-3} = 27$	15. $\left(\frac{1}{2}\right)^{2x} = 8$
16. $x^{\frac{4}{3}} = 16$	17. $\left(\dfrac{e^2}{e^x}\right)^3 = e^9$	18. $(5-e)^x = 1$

For Problems 19-24, sketch a graph without using a calculator. List all intercepts, and write an equation for each asymptote. Use a separate coordinate plane for each graph.

19. $y = 2^x$	20. $y = 2^{-x}$	21. $y = -2^x$
22. $y = 2^{x+1}$	23. $y = 2^x - 1$	24. $y = 2^{\lvert x \rvert}$

LESSON 0.7 LOGARITHMIC FUNCTIONS



Since $f(x)=e^x$ is one-to-one (continuous and increasing), it must have an inverse. However, if you switch x and y in the equation $y=e^x$ to get $x=e^y$, you cannot isolate the new y by using algebraic methods. So, we must define $f^{-1}(x)$ for the function $f(x)=e^x$. For $f(x)=e^x$, $f^{-1}(x)$ is called the **natural logarithmic function**, and we write $f^{-1}(x)=\ln x$ (so that $x=e^y$ and $y=\ln x$ must be equivalent). In general, if $f(x)=a^x$ $(a>0)$, then $f^{-1}(x)=\log_a x$ (so that $x=a^y$ and $y=\log_a x$ must be equivalent).

Note: $\log_e x$ is usually written as $\ln x$ and $\log_{10} x$ is usually written simply as $\log x$.

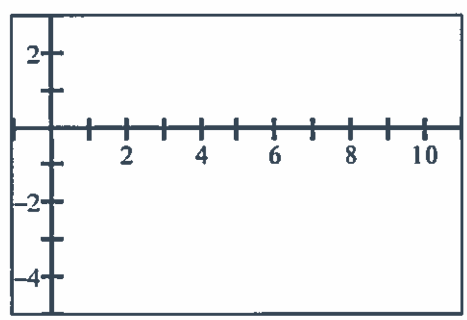
Graphs of Logarithmic Functions: If $f(x)=\log_a x$ and $a>1$, then

1. The domain of $f(x)$ is $(0,\infty)$. The range of $f(x)$ is $(-\infty,\infty)$.
2. The graph of $f(x)$ is continuous, increasing, concave downward, and one-to-one (has an inverse function).
3. The y-axis is a vertical asymptote downward: $\lim\limits_{x\to 0} f(x)=-\infty$ * (Also, $\lim\limits_{x\to\infty} f(x)=\infty$) *
4. The x-intercept is $(1, 0)$. Another key point is $(a, 1)$.

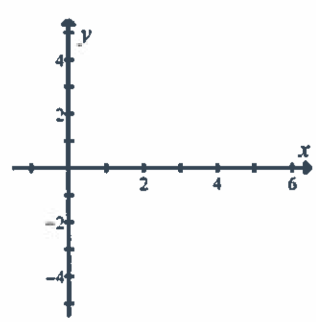
*This notation using limits will be developed completely in the next unit.

Compare these graphical characteristics of $f(x)=\log_a x$ to those of $f(x)=a^x$ from Lesson 0.6 (page 22).

Example 1: Use a calculator to graph $y=\ln x$ and $y=\log x$ in the same coordinate plane. Do you see any similarities in the graphs?



Example 2: Without using a calculator, sketch a graph of $y=|\ln(x-2)|$. Write an equation for the graph's asymptote.



For changing forms of an equation involving exponentials or logarithms, we use the following **Change of Form Definition:**

Exponential form	$\begin{cases} x = e^y \leftrightarrow y = \ln x \\ x = a^y \leftrightarrow y = \log_a x \end{cases}$	Logarithmic form

Example 3: Change the following equations from exponential form to logarithmic form or vice versa.

a. $3^4 = 81$ b. $e^0 = 1$ c. $\log(.1) = -1$

Example 4:

a. Since $e^0 = 1$, $\ln 1 =$ b. Since $e^1 = e$, $\ln e =$

c. Because the natural exponential function and the natural logarithmic function are inverses, $\ln e^n = e^{\ln n} =$

Example 5: Use the inverse idea from Example 4c. to simplify.

a. $\ln e^{\sqrt{2}} =$ b. $e^{\ln(3x)} =$ c. $10^{\log 2} =$ d. $\log_2 2^{x^2} =$

Properties of Logarithms:

1. $\ln(ab) = \ln a + \ln b$
2. $\ln \frac{a}{b} = \ln a - \ln b$
3. $\ln a^n = n \ln a$

These properties work for any bases, but only if $a > 0$ and $b > 0$

Example 6: Expand using Logarithm Properties 1-3 above.

a. $\ln \frac{5}{8}$ b. $\ln \sqrt[3]{x^2+1}$

Example 7: Condense into a single logarithm. ($x > 0$ and $y > 0$)

a. $-3\ln x + 5\ln y$ b. $\frac{1}{2}\ln x + \ln(x+1) - 3\ln y$

Example 8: Solve for x.

a. $y = e^{2x-5} + 6$

b. $\log_2 x - \log_2(x-8) = 3$

Change of Base Formula: $\log_a x = \dfrac{\log_b x}{\log_b a}$

Since the only two logarithmic bases on your calculator are 10 (log key) and e (ln key), you will change bases on your calculator in one of two ways:

$$\log_a x = \frac{\log x}{\log a} \qquad \text{or} \qquad \log_a x = \frac{\ln x}{\ln a}$$

Example 9: Use your calculator to find $\log_7 112$ to 3 or more decimal places.

Example 10:

a. Find an exact value for x, if $3^{x+2} = 6$.

b. Use your calculator to find a decimal value for your answer from Part a. to 3 or more decimal places.

ASSIGNMENT 0.7

Decide whether each statement in Problems 1-8 is true or false for $a > 0$ and $b > 0$. (Check your answers before working on the rest of the assignment.)

1. $\log(a+b) = \log a + \log b$
2. $\ln(a+b) = \ln a \cdot \ln b$
3. $\log a - \log b = \dfrac{\log a}{\log b}$
4. $\log \dfrac{a}{b} = \dfrac{\log a}{\log b}$
5. $(\ln x)^3 = 3\ln x$
6. $\ln x^3 = 3\ln x$
7. $\ln x^2 = 2\ln x$, for all x
8. $\ln x^2 = 2\ln x$, for $x > 0$

For Problems 9-12, change each equation from exponential form to logarithmic form or vice versa.

9. $5^{-3} = \dfrac{1}{125}$
10. $e^x = 17$
11. $\log_3 729 = 6$
12. $\log x = -2$

Simplify each expression in Problems 13-16.

13. $e^{\ln(2x+1)}$

14. $\ln e^{a+b}$

15. $\log_5 5^{\sqrt{p}}$

16. $3^{\log_3 m^2}$

For Problems 17-20, solve for x without using a calculator. Simplify your answers.

17. $\log_2 x = 3$

18. $\ln x = -1$

19. $x^2 - 1 = \log_3 27$

20. $\log_x 64 = 3$

For Problems 21-26, sketch a graph without using a calculator. List all x-intercepts, and write an equation for each asymptote. Use a separate coordinate plane for each graph.

21. $y = \log_2 x$

22. $y = \log_2(x+3)$

23. $y = \log_2(-x)$

24. $y = |\log_2 x|$

25. $y = \log_2 |x|$

In Problems 26 and 27, $f(x)$ is given. Without using a calculator, find $f^{-1}(x)$, and graph both f and f^{-1} in the same coordinate plane.

26. $f(x) = e^{2x}$

27. $f(x) = \ln(x-1)$

Remember that your graphs should be reflections of each other across $y = x$.

Use Properties of Logarithms to expand the expressions in Problems 28-30.
(All variables represent positive quantities.)

28. $\ln \dfrac{a}{bc}$

29. $\log\left(xy^2\right)$

30. $\ln \dfrac{(a+b)^2}{c}$

Use Properties of Logarithms to condense the expressions in Problems 31-33 into single logarithms. (All variables represent positive quantities).

31. $\log x + 2\log y$

32. $3\ln x - \frac{1}{2}\ln y$

33. $\ln a - (2\ln b - \ln c)$

For Problems 34-36, solve for t without using a calculator.

34. $\ln e^{t^2-t} = 6$

35. $e^{2t-1} - 3 = 0$

36. $\log_2 t + \log_2(t+2) = 3$

Use a calculator to solve for x in Problems 37 and 38. (Express answers to 3 or more decimal place accuracy.)

37. $3e^{-x+1} = 5 - x^2$

38. $\ln(.5x) = .2 - e^x$

Find the values of the logarithms in Problems 39 and 40. (Express answers to 3 or more decimal place accuracy.)

39. $\log_3 20$

40. $\log_5(.02)$

ADDITIONAL PRACTICE

Lesson 0.1

Draw accurate graphs for the following without using a calculator.

1. $4x+2y=6$ 2. $y=\dfrac{-x+4}{2}$

3. Find equations for lines passing through $(-1,3)$ with the following characteristics.

a. $m=\dfrac{2}{3}$ b. parallel to $2x+4y=7$

c. passing through the origin d. perpendicular to the x-axis

Use a calculator for Problems 4-6. Remember to show three or more decimal place accuracy for all answers that are not exact.

4. Solve $3x^3-3x+1\le 0$.
5. Solve $|3x+5|>2$.
6. Find the x-value(s) of the point(s) of intersection for the graphs of $x-y^2=-7$ and $2x-3y+12=0$. Write the equation you are solving.

Lesson 0.2

Draw accurate graphs for the following without using a calculator. Use the parent graphs on Page 6 to help you whenever possible.

7. $y=\dfrac{1}{x}+1$ 8. $y=\sqrt[3]{x-2}$ 9. $y=|x^2-2|$ 10. $y=x^{\frac{2}{3}}-1$

11. $y-x^2=0$ 12. $x=y^2$ 13. $y=x^3-1$

14. For which of the relations in Problems 7-13 is y not a function of x?

15. If $f(x)=1-x^2$ and $g(x)=2x+1$, find the following.

a. $f(x)+g(x)$ b. $f(g(x))$ c. $(g\circ f)(2)$

Find the zeros without using a calculator.

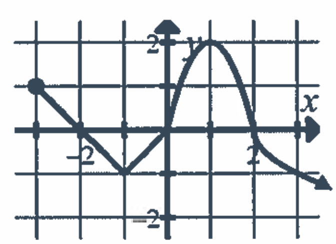
16. $f(x)=x^4-7x^2+12$ 17. $g(x)=\begin{cases}2x-1, & x<0\\ x^2-4, & x\ge 0\end{cases}$

18. Find the inverse function for $f(x)=(x^3-1)^5$.

Use the graph of $y=f(x)$ at the right to draw an accurate graph for each of the following.

19. $y=\frac{1}{2}|f(x)|$ 20. $y=|f(2x)|$

21. $y=f(x-2)-2$



22. Without using a calculator sketch a graph of $g(x)=\begin{cases} x-1, & x\le 0 \\ x^2-1, & 0<x<2 \\ 4, & x\ge 2 \end{cases}$

Lesson 0.3

Without using a calculator, find the point(s) of intersection of the graphs of the following. **Show algebra steps!**

23. $\begin{cases} y=x+5 \\ y=-2x+8 \end{cases}$

24. $\begin{cases} x^2-y^2=9 \\ x^2+y^2=9 \end{cases}$

For each function in Problems 25-27, without using a calculator:

a. find the domain and the range.
b. find the intercepts.
c. discuss the symmetry.
d. tell whether the function is even, odd, or neither.
e. draw an accurate graph.

25. $y=\left|x^2-4\right|$

26. $y=-\sqrt{x+4}$

27. $y=\left|x^3\right|-2$

Use a calculator for Problems 28-30. Remember to show three or more decimal place accuracy for all answers that are not exact.

a. find the domain and the range.
b. find the intercepts.
c. discuss the symmetry.
d. tell whether the function is even, odd, or neither.
e. draw an accurate graph.

28. $y=\dfrac{x}{x^2-4}$

29. $y=-\sqrt{5-2x^2}$

30. $y=3x^2-3x-5$

Are the following functions even, odd, or neither? Do not use a calculator.

31. $g(x)=\dfrac{x}{\left|x^3-x\right|}$

32. $h(x)=\dfrac{x-1}{\left|x^3-x\right|}$

Lesson 0.4

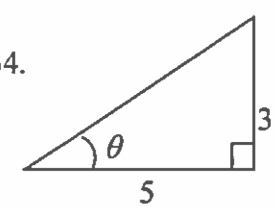
33. Without a calculator, convert from degrees to radians for Part a. and radians to degrees for Part b.

a. 300°

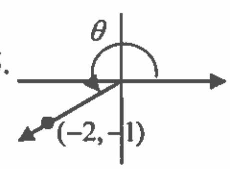
b. $-\frac{5\pi}{2}$

For Problems 34-36, find $\sin\theta$, $\cos\theta$, and $\tan\theta$ without using a calculator.

34.

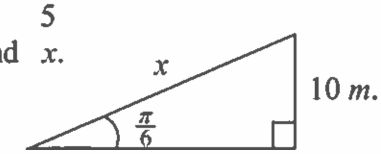


35.



36. The measure of θ is $\frac{8\pi}{3}$.

37. Find x.



Evaluate each of the following without using a calculator.

38. $\sin\pi$ 39. $\cos\frac{3\pi}{2}$ 40. $\tan\frac{4\pi}{3}$ 41. $\sec\frac{5\pi}{4}$ 42. $\csc\frac{7\pi}{6}$ 43. $\cot\frac{-3\pi}{2}$

44. Given $\tan\theta = \frac{-5}{12}$, $\sin\theta > 0$, find $\cos\theta$.

Solve the following equations on the interval $[0, 2\pi)$.

45. $3\tan^2\theta - 1 = 0$ 46. $3\csc^2\theta - 4 = 0$

Lesson 0.5

47. Use a calculator to solve $\tan^2 x = \sin(2x) + 3$ on $[0, \pi)$.

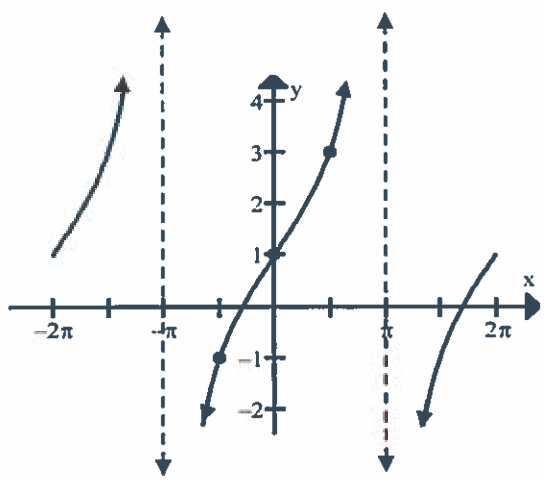
Do not use a calculator for Problems 48-50.

48. Find the amplitude, period, and phase shift for $y = \frac{-3}{4}\cos(3x - \pi)$.

49. Sketch the graph of $y = -\frac{1}{2}\sin\left(x - \frac{\pi}{2}\right)$.

50. Find the discontinuities of $f(x) = \csc(4x)$.

51. Write an equation of the form $y = a\tan\left(b(x - c)\right) + d$ for the graph below.



Lesson 0.6 and 0.7

For Problems 52 and 53, find the inverse of the given function, and then sketch the function and its inverse in the same coordinate plane.

52. $f(x) = \ln(-x)$　　　　53. $g(x) = e^{2x} + 1$

Simplify the expressions in Problems 54 and 55 without using a calculator.

54. $\log_3 \frac{1}{27}$　　　　55. $\ln \frac{e^{10}}{e^3}$

56. Use Log Properties to expand $\ln\left(x\sqrt{y-1}\right)$.

57. Use Log Properties to condense $2\log p - 3\log q - \log r$ into a single logarithm.

Solve for x without a calculator:

58. $e^{2x-3} - 5 = 0$　　　　59. $1 - 3\ln x = -5$

60. Without using a calculator, list the domain, two points that the graph of the function contains, and the asymptote for the graph of the function. Then sketch each graph in the same coordinate plane.
    a. $y = e^x$　　　　b. $y = \ln x$

61. Using adjustments to the graph of $y = e^x$ from Problem 60, sketch $y = e^{-x} - 3$. What is the asymptote for this graph?

62. Find the inverse of $y = e^{-x} - 3$, and sketch its graph in the same coordinate plane that you used for Problem 61. What is the asymptote for this graph?

For Problems 63-67, solve for x without using a calculator. Simplify your answers.

63. $\log_5 x = -2$　　　　64. $e^{\ln(-2x+3)} = 5$　　　　65. $\ln|2x-1| = 0$

66. $\log_3 x = \log_3(2x+1) - \log_3(x+4) + 1$　　　　67. $4^3 = 8^{2x-1}$

UNIT 0 SUMMARY

Slope: $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Equations for lines:

Point-Slope form $y - y_1 = m(x - x_1)$

Slope-Intercept form $y = mx + b$ (where b is the y-intercept)

Domain: all possible x-values
Range: all possible y-values

Inverse functions: found by switching x and y and solving for the new y.

Parent graphs and graphing adjustments: see Page 6

Intercepts:

To find the x-intercept, let $y = 0$ and solve for x.

To find the y-intercept, let $x = 0$ and solve for y.

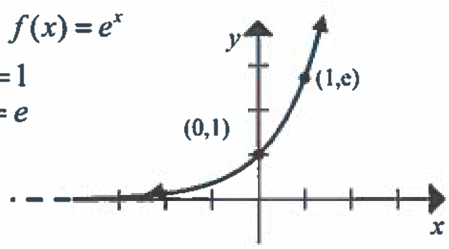
Symmetry:
Informal tests:
1. y-axis: substituting a number and its opposite for x give the same y-value.
2. x-axis: substituting a number and its opposite for y give the same x-value.
3. origin: substituting a number and its opposite for x give opposite y-values.

Even/odd functions:
Even functions have graphs with y-axis symmetry.
Odd functions have graphs with origin symmetry.

Exponential and Logarithmic Graphs:

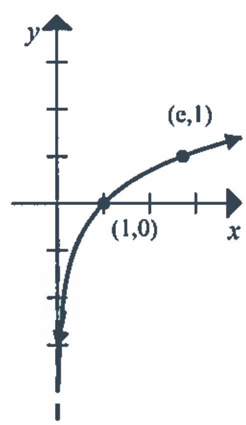
Graph of $f(x) = e^x$

$e^0 = 1$
$e^1 = e$



Graph of $g(x) = \ln x$

$\ln 1 = 0$
$\ln e = 1$



All basic exponential $(f(x) = a^x)$ and logarithmic $(g(x) = \log_a x)$ graphs with $a > 0$ are similar to the graphs shown above.

$f(x) = e^x$ and $g(x) = \ln x$ are inverse functions, so $\ln e^x = e^{\ln x} = x$.

Change of Form Definition: $$\text{Exponential form} \left\{ \begin{array}{l} x = e^y \leftrightarrow y = \ln x \\ x = a^y \leftrightarrow y = \log_a x \end{array} \right\} \text{Logarithmic form}$$

Properties of Logarithms: only true when $a > 0$ and $b > 0$

1. $\ln(ab) = \ln a + \ln b$
2. $\ln \frac{a}{b} = \ln a - \ln b$
3. $\ln a^n = n \ln a$

Change of Base: $\log_a x = \dfrac{\ln x}{\ln a}$

LESSON 1.1 LIMITS , CONTINUITY

Limits

Informally, a **limit is a $y-$value** which a function approaches as x approaches some value.

$\lim_{x\to c} f(x) = L$ means as x approaches c, $f(x)$ approaches the y-value of L.

Examples

limits:

1. $\lim_{x\to -4} f(x) =$
2. $\lim_{x\to -1} f(x) =$
3. $\lim_{x\to 2} f(x) =$
4. $\lim_{x\to 3} f(x) =$
5. $\lim_{x\to 5} f(x) =$

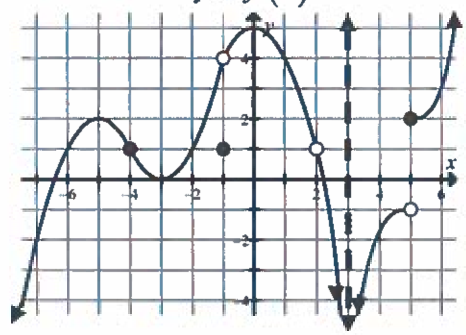
one-sided limits:

6. $\lim_{x\to 5^-} f(x) =$
7. $\lim_{x\to 5^+} f(x) =$

function values:

8. $f(-4) =$
9. $f(-1) =$
10. $f(2) =$
11. $f(3) =$
12. $f(5) =$

$y = f(x)$



Continuity

Informally, a function is continuous where it can be drawn without lifting a pencil. Roughly, continuous means "connected."
Formally, a function is continuous where its limit and function value are the same.

Definition of Continuity

A function f is continuous at an x-value c if and only if $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$.

In this course, we will work with three types of discontinuities: holes, vertical asymptotes, and jumps (breaks). A fourth type of discontinuity is an oscillating discontinuity (these rarely appear). To investigate this fourth type, graph $y = \sin\frac{1}{x}$ on a calculator and look at windows close to $x = 0$.

Example 13. List the x-values of the discontinuities of the function $y = f(x)$ graphed above.

All discontinuities can be classified as removable or nonremovable.
Removable discontinuities occur when the function has a limit (holes in the graph).
Nonremovable discontinuities occur when the limit of the function does not exist (jumps and vertical asymptotes).

Example 14. Which of the discontinuities from Example 13 are removable?

At x-values where a function is continuous, limits can be found by **direct substitution**.

Examples:

15. $\lim_{x\to 3}\left(3x^2+2\right)=$ 16. $\lim_{x\to 1}\frac{x^2+x}{x+1}=$ 17. $\lim_{x\to\frac{\pi}{3}}\cos(2x)=$

For **piecewise functions**, one-sided limit evaluation is often necessary.

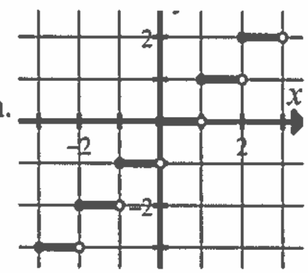
Examples:

18. If $f(x)=\begin{cases}4-x, & x\le 1\\ 4x-x^2, & x>1\end{cases}$, $\lim_{x\to 1}f(x)=$

19. If $g(x)=\begin{cases}3x-x^3, & x\le 1\\ 2x^2-1, & x>1\end{cases}$, $\lim_{x\to 1}g(x)=$

20. For this same g function, $\lim_{x\to -1}g(x)=$

Another function requiring one-sided limit analysis is a step function called the **Greatest Integer Function** also known as the **Floor Function**.
$f(x)=\lfloor x\rfloor=$ the greatest integer less than or equal to x. The graph is shown.

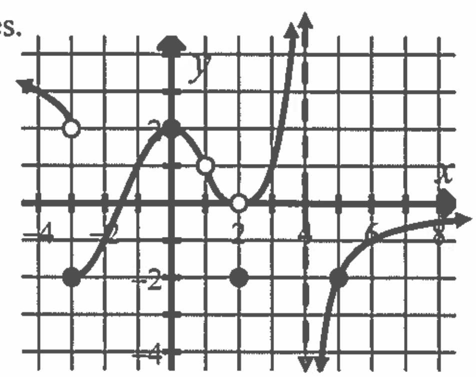


Examples: Find the following limits.

21. $\lim_{x\to\frac{1}{2}}\lfloor x\rfloor=$ 22. $\lim_{x\to 1}\lfloor x\rfloor=$

ASSIGNMENT 1.1

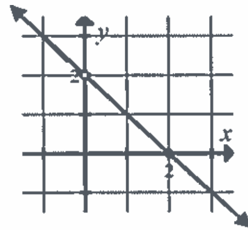
Use the graph of $y=f(x)$ at the right to find these values.



1. $\lim_{x\to 5}f(x)$ 2. $\lim_{x\to -3}f(x)$ 3. $\lim_{x\to -3^-}f(x)$
4. $\lim_{x\to -3^+}f(x)$ 5. $f(-3)$ 6. $\lim_{x\to 4^-}f(x)$
7. $\lim_{x\to 0}f(x)$ 8. $f(0)$ 9. $\lim_{x\to 4}f(x)$
10. $\lim_{x\to 4^+}f(x)$ 11. $f(4)$ 12. $f(2)$
13. $\lim_{x\to 2}f(x)$ 14. $f(1)$ 15. $\lim_{x\to 1}f(x)$
16. List the x-values of all removable discontinuities of $f(x)$.
17. List the x-values of all nonremovable discontinuities of $f(x)$.

Use the graph shown to find each value.

18. a. $\lim\limits_{x\to 0} f(x)$ b. $\lim\limits_{x\to 2} f(x)$
c. $f(0)$ d. removable discontinuities

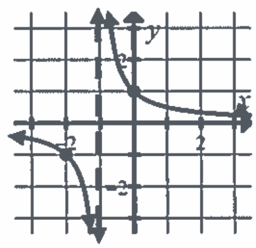


$$f(x)=\frac{-2x^2+4x}{2x}$$

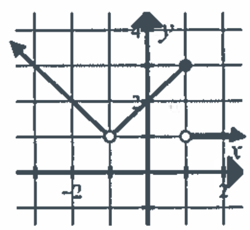
Use the graphs shown to find each value.

19. a. $\lim\limits_{x\to 0} g(x)$ b. $\lim\limits_{x\to -1^-} g(x)$ c. $\lim\limits_{x\to -1^+} g(x)$
d. $\lim\limits_{x\to -1} g(x)$ e. $g(-1)$ f. $g(0)$
g. $\lim\limits_{x\to -2} g(x)$ h. removable discontinuities
i. nonremovable discontinuities

$$g(x)=\frac{1}{x+1}$$



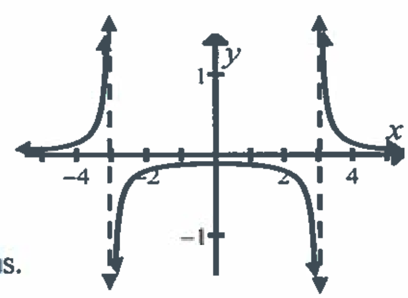
20. a. $h(-1)$ b. $h(1)$
c. $\lim\limits_{x\to -1} h(x)$ d. $\lim\limits_{x\to 1^-} h(x)$
e. $\lim\limits_{x\to 1^+} h(x)$ f. $\lim\limits_{x\to 1} h(x)$
g. removable discontinuities
h. nonremovable discontinuities



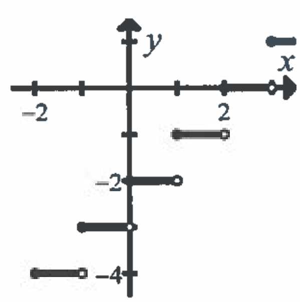
$$h(x)=\begin{cases} -x, & x<-1 \\ x+2, & -1<x\le 1 \\ 1, & x>1 \end{cases}$$

21. a. $\lim\limits_{x\to 3} f(x)$ b. $\lim\limits_{x\to 0} f(x)$
c. $\lim\limits_{x\to -3^-} f(x)$ d. $f(3)$
e. Use interval notation to show where f is continuous.

$$f(x)=\frac{1}{x^2-9}$$

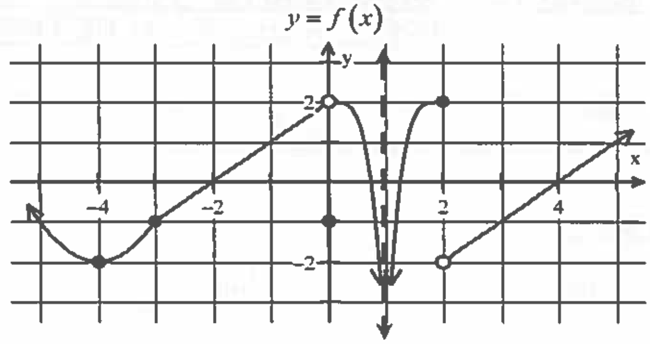


22. a. $g(1)$ b. $\lim\limits_{x\to 1} g(x)$
c. $\lim\limits_{x\to 1^-} g(x)$ d. $\lim\limits_{x\to 1^+} g(x)$
e. removable discontinuities
f. nonremovable discontinuities



$$g(x)=\lfloor x \rfloor - 2$$

23. Identify each x-value at which the function shown appears to be discontinuous and classify each as removable or nonremovable.



Find the indicated limits without using a calculator.

24. $\lim_{x\to 2}(x^2-4)$ 25. $\lim_{x\to -3}(x^2-x)$ 26. $\lim_{x\to 0}\sqrt{x^2+9}$ 27. $\lim_{x\to 3^-}(2x-5)$

28. $\lim_{x\to -1}\dfrac{x}{x^2+1}$ 29. $\lim_{x\to -2}\sqrt[3]{x^2+4}$ 30. $\lim_{x\to 0}\dfrac{x}{x-1}$ 31. $\lim_{x\to 0}(3x-3)^3$

32. $\lim_{x\to \pi}\sin x$ 33. $\lim_{x\to \frac{\pi}{2}}\cos x$ 34. $\lim_{x\to \pi}\tan x$ 35. $\lim_{x\to \frac{\pi}{2}}\cos(2x)$

36. $\lim_{x\to 2}\cos\dfrac{\pi x}{3}$ 37. $\lim_{x\to 3}\sec\dfrac{\pi x}{4}$

Use these piecewise functions to find the following without using a calculator.

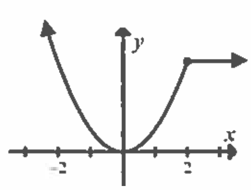
38. $f(x)=\begin{cases} x^2-2, & x\le 0 \\ x+2, & x>0 \end{cases}$ a. $\lim_{x\to 0^-}f(x)$ b. $\lim_{x\to 0^+}f(x)$ c. $\lim_{x\to 0}f(x)$
d. Give the intervals where f is continuous.

39. $g(x)=\begin{cases} 3x^2-x, & x\le 2 \\ 2x+6, & x>2 \end{cases}$ a. $\lim_{x\to 2}g(x)$ b. $\lim_{x\to 0}g(x)$
c. List the interval(s) where g is continuous.

40. Find the interval(s) where $y=\sqrt{x-1}$ is continuous.

41. Find the discontinuities of $f(x)=\dfrac{x}{x(x+1)}$ and classify each of them as removable or nonremovable.

42. $f(x)=\begin{cases} x^2, & x\le 2 \\ a, & x>2 \end{cases}$



If $f(x)$ is a continuous function as shown in the graph, find the value of the unknown a.

LESSON 1.2 MORE LIMITS, MORE CONTINUITY, INTERMEDIATE VALUE THEOREM

If direct substitution does not give an answer to a limit problem because an indeterminate form is obtained (usually $\frac{0}{0}$), use algebraic techniques to change the form of the limit.

Examples:

1. $\lim_{x\to 0}\frac{x}{x(x+1)}$

2. $\lim_{x\to 2}\frac{x-2}{x^2-4}$

3. $\lim_{x\to 1}\frac{x^3-1}{x-1}$

4. $\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}$

5. $\lim_{x\to 2^+}\frac{x}{x-2}$

6. Discuss the continuity of

$$f(x)=\begin{cases}\dfrac{x^2-2x-3}{x-3}, & x\neq 3\\ 5, & x=3\end{cases}$$

7. If $g(x)=\begin{cases}3x^2+a, & x>2\\ x-3, & x\leq 2\end{cases}$ is a continuous function, find the value of a.

8. Use a calculator to fill in the tables to help find these limits if $f(x)=\dfrac{x^3-3x^2+x+2}{x^3-2x^2-x+2}$.

a. $\lim_{x\to 1} f(x)$

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$						

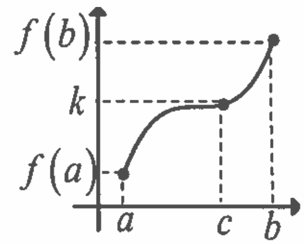
b. $\lim_{x\to 2} f(x)$

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$						

c. Can a table like these be used to find limits with absolute certainty?

Intermediate Value Theorem

If f is continuous on $[a,b]$ and k is any y-value between $f(a)$ and $f(b)$, then there is at least one x-value c between a and b such that $f(c)=k$.



Examples:

9. Does the Intermediate Value Theorem guarantee a c-value on the given interval.

   a. $f(x)=x^2-x,$ $f(c)=12,\ \ [0,5]$

   b. $g(x)=\dfrac{x^2-4}{x-2},$ $g(c)=4,\ \ [0,3]$

10. Find the value of c in Example 9a.

ASSIGNMENT 1.2

Find the indicated limits without using a calculator. **Show steps using correct <u>limit symbolism</u>!**

1. $\lim\limits_{x\to 1}\dfrac{x^2-1}{x-1}$
2. $\lim\limits_{x\to -1}\dfrac{2x^2-x-3}{x+1}$
3. $\lim\limits_{x\to -1}\dfrac{x^2-1}{x-1}$
4. $\lim\limits_{x\to -1}\dfrac{x^3+1}{x+1}$
5. $\lim\limits_{x\to 5^+}\dfrac{x-5}{x^2-25}$
6. $\lim\limits_{x\to -5}\dfrac{x-5}{x^2-25}$
7. $\lim\limits_{x\to 2}\dfrac{2-x}{x^2-4}$
8. $\lim\limits_{x\to 3}\dfrac{x^2-9}{x^2-6x+9}$
9. $\lim\limits_{x\to 2^-}\dfrac{1}{x^2-4}$
10. $\lim\limits_{x\to 1}\dfrac{x}{x^2+1}$
11. $\lim\limits_{x\to 4}\dfrac{\sqrt{x}-2}{x-4}$
12. $\lim\limits_{x\to 3}\dfrac{x-3}{\sqrt{x+1}-2}$
13. $\lim\limits_{x\to 0^-}\dfrac{|x|}{x}$
14. $\lim\limits_{x\to 0}\dfrac{|x|}{x}$
15. $\lim\limits_{x\to 3^+}\lfloor x-1\rfloor$
16. $\lim\limits_{x\to 3^-}\lfloor x-1\rfloor$
17. $\lim\limits_{x\to 3}\lfloor x-1\rfloor$
18. $\lim\limits_{x\to 2}\lfloor x+6\rfloor$
19. $\lim\limits_{x\to 3}\left\lfloor \dfrac{x}{2}\right\rfloor$
20. $\lim\limits_{x\to 5^-}\lfloor 2x-3\rfloor$
21. $\lim\limits_{x\to 7}\csc\dfrac{\pi x}{6}$
22. $\lim\limits_{x\to \pi}\cot\dfrac{x}{6}$
23. $\lim\limits_{x\to 5\pi}\cos\dfrac{x}{3}$
24. $\lim\limits_{x\to 3}\begin{cases}\frac{1}{2}x+1, & x\le 3\\ \dfrac{12-2x}{3}, & x>3\end{cases}$
25. $\lim\limits_{x\to 1}\begin{cases}x^2+1, & x<1\\ x^3+1, & x\ge 1\end{cases}$
26. $\lim\limits_{x\to 2}\begin{cases}x-2, & x\le 0\\ x+2, & x>0\end{cases}$

Use the function $g(x)=\begin{cases}2\sin\dfrac{3x}{2}, & x\le \pi\\ \sec\dfrac{11x}{6}, & x>\pi\end{cases}$ for problems 27-29. Do not use a calculator.

27. $\lim\limits_{x\to \pi^+}g(x)$
28. $\lim\limits_{x\to \pi^-}g(x)$
29. $\lim\limits_{x\to \pi}g(x)$

Use a calculator to find these limits.

30. (a) $\lim\limits_{x\to 1}\dfrac{\sin x}{6x}$ (b) $\lim\limits_{x\to 0}\dfrac{\sin x}{6x}$

31. $\lim\limits_{x\to -2}\dfrac{x^3+2x^2-x-2}{x^3+2x^2+x+2}$

Find all discontinuities for these functions and classify them as removable or nonremovable. Do not use a calculator.

32. $f(x)=\dfrac{2x-3}{x+1}$

33. $f(x)=2x-3$

34. $f(x)=\dfrac{1}{x^2-9}$

35. $f(x)=\dfrac{x}{x^2+x}$

36. $f(x)=\dfrac{x^2-9}{x+3}$

37. $f(x)=\begin{cases} x^2, & x\le 0 \\ x, & x>0 \end{cases}$

38. $f(x)=\begin{cases} x-3, & x\le 1 \\ x, & x>1 \end{cases}$

39. $f(x)=\begin{cases} 2x-5, & x>3 \\ x^2-8, & x<3 \end{cases}$

40. $f(x)=\lfloor x-1\rfloor$

41. $f(x)=\left\lfloor \frac{x}{2}\right\rfloor$

Use a calculator to find all discontinuities for these functions and classify them as removable or nonremovable.

42. $f(x)=\dfrac{10x}{6x^3-31x^2+23x+20}$

43. $f(x)=\dfrac{x}{x^3+4x}$

44. $f(x)=\left\lfloor \frac{x}{4}\right\rfloor$

45. If the function $f(x)=\begin{cases} x^3, & x\le 2 \\ ax^2, & x>2 \end{cases}$ is continuous, find the value of a.

Determine whether the Intermediate Value Theorem would guarantee a c-value on the given interval.

46. $f(x)=x^2+x-1,\quad f(c)=11,\quad [0,5]$

47. $f(x)=\dfrac{x}{x-1},\quad f(c)=1,\quad [0,2]$

48. $f(x)=|x|,\quad f(c)=3,\quad [-4,1]$

49. $f(x)=\begin{cases} x, & x\le 1 \\ 3, & x>1 \end{cases},\quad f(c)=2,\quad [0,4]$

50. Find the c-value in Problem 46.

51. Find the c-value in Problem 48.

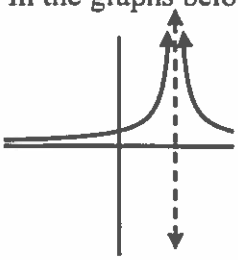
LESSON 1.3 INFINITE LIMITS, LIMITS AT INFINITY

Review:

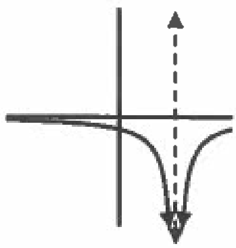
The graph of the function $f(x)=\dfrac{x-1}{(x-1)(x-2)^2(x-4)^3}$ has a **hole** at $x=$ _____,

an **even vertical asymptote** at $x=$ ______,
and an **odd vertical asymptote** at $x=$ ______.

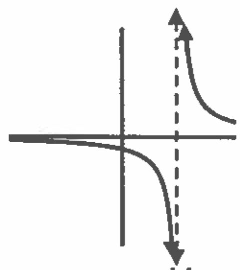
In the graphs below, the vertical asymptotes are labeled odd or even.



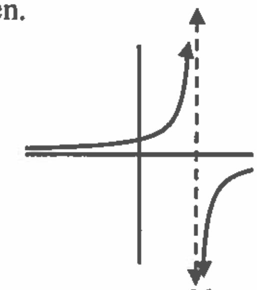
even



even



odd



odd

Infinite Limits

You have seen examples where a limit does not exist at a vertical asymptote. Such non-existent limits can be expressed as infinite limits if the vertical asymptote is even or if you are finding one-sided limits. We will write $\lim\limits_{x\to c} f(x)=\infty$ or $\lim\limits_{x\to c} f(x)=-\infty$.

The examples below make use of your knowledge of even and odd vertical asymptotes as well as holes.

Examples:

1. $\lim\limits_{x\to 2^+} \dfrac{x+3}{x-2} =$

2. $\lim\limits_{x\to 2^-} \dfrac{x+3}{x-2} =$

3. $\lim\limits_{x\to 1^+} \dfrac{x-2}{(x-1)^2} =$

4. $\lim\limits_{x\to 1^-} \dfrac{x-2}{(x-1)^2} =$

5. $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2} =$

Limits at Infinity

If the graph of a function $f(x)$ approaches a horizontal asymptote to the left and/or the right, $f(x)$ is said to have a limit at infinity. If the asymptote is $y = L$ then $\lim\limits_{x\to\infty} f(x) = L$. In other words, limits at infinity give us end behaviors for graphs of functions. For "large" values of x, the highest degree terms in the numerator and denominator dominate the other terms and are the only terms you need to consider.

Review Examples: Find the horizontal asymptotes.

6. $f(x)=\dfrac{5x^4-3x^2+2}{10x^4+3}$ 7. $g(x)=\dfrac{5x^4-3x^2+2}{10x^5+3}$ 8. $h(x)=\dfrac{5x^4-3x^2+2}{10x^3+3}$

Examples: Find the following limits.

9. $\lim\limits_{x\to\infty}\dfrac{5x^4-3x^2+2}{10x^4+3}=$ 10. $\lim\limits_{x\to-\infty}\dfrac{5x^4-3x^2+2}{10x^5+3}=$ 11. $\lim\limits_{x\to\infty}\dfrac{5x^4-3x^2+2}{10x^3+3}=$

12. $\lim\limits_{x\to\infty}\dfrac{(2x+3)(x-1)^2}{(x+2)(3x-1)^2}=$

Note: Make sure you consider highest degree **terms** – not highest degree **factors.**

Rational functions like those above have at most one horizontal asymptote, so the limit is the same whether x approaches ∞ or $-\infty$. However, radical functions frequently have two horizontal asymptotes.

Examples: Find these limits.

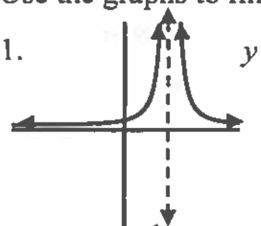
13. $\lim\limits_{x\to\infty}\dfrac{\sqrt{4x^2-3}}{x}=$ 14. $\lim\limits_{x\to-\infty}\dfrac{\sqrt{4x^2-3}}{x}=$

Use the end behavior of $y=e^x$ to find these limits.

15. $\lim\limits_{x\to-\infty} e^x$ 16. $\lim\limits_{x\to\infty} e^x$ 17. $\lim\limits_{x\to\infty}\left(e^{-x}+\dfrac{2x^2-x}{x^2+3}\right)$ 18. $\lim\limits_{x\to\infty}\left(\dfrac{2x^2+4x}{e^x+3x^2}\right)$

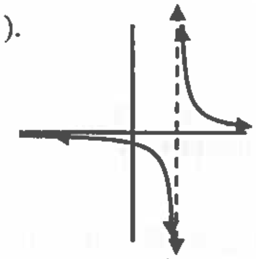
ASSIGNMENT 1.3

Use the graphs to find these limits (answer ∞ or $-\infty$).

1. 

$y=\dfrac{1}{(x-2)^2}$

a. $\displaystyle\lim_{x\to 2^-}\frac{1}{(x-2)^2}$ b. $\displaystyle\lim_{x\to 2^+}\frac{1}{(x-2)^2}$

2. 

$y=\dfrac{1}{x-2}$

a. $\displaystyle\lim_{x\to 2^-}\frac{1}{x-2}$ b. $\displaystyle\lim_{x\to 2^+}\frac{1}{x-2}$

Find the vertical asymptotes without using a calculator, and classify each of them as even or odd.

3. $f(x)=\dfrac{1}{x^2}$ 4. $f(x)=\dfrac{x}{x(x-1)^2}$ 5. $f(x)=\dfrac{x}{x^2-4}$ 6. $f(x)=\dfrac{x}{x^2-x-2}$

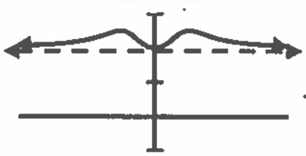
Find these limits without using a calculator. Whenever appropriate answer ∞ or $-\infty$.

7. $\displaystyle\lim_{x\to 3^-}\frac{x}{x-3}$ 8. $\displaystyle\lim_{x\to 3}\frac{x}{x-3}$ 9. $\displaystyle\lim_{x\to 1^+}\frac{x}{x^2-x}$ 10. $\displaystyle\lim_{x\to 0}\frac{x}{x^2-x}$

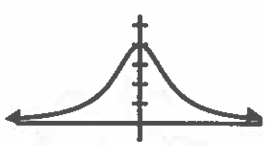
11. $\displaystyle\lim_{x\to 3}\frac{x+3}{x^2-6x+9}$ 12. $\displaystyle\lim_{x\to 3}\frac{x-3}{x^2-6x+9}$ 13. $\displaystyle\lim_{x\to 0}\frac{x^2-2x}{x^3}$ 14. $\displaystyle\lim_{x\to 0^-}\left(\frac{1}{x}-10\right)$

15. $\displaystyle\lim_{x\to \frac{\pi}{2}^+}\frac{3}{\cos x}$ 16. $\displaystyle\lim_{x\to \pi}\frac{x}{\csc x}$

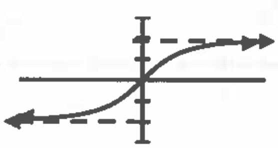
Find these limits without using a calculator.

17. 

$f(x)=\dfrac{2x^4+x^2+2}{x^4+1}$ a. $\displaystyle\lim_{x\to\infty} f(x)$ b. $\displaystyle\lim_{x\to-\infty} f(x)$

18. 

$f(x)=\dfrac{4}{x^2+1}$ a. $\displaystyle\lim_{x\to\infty} f(x)$ b. $\displaystyle\lim_{x\to-\infty} f(x)$

19. 

$f(x)=\dfrac{2x}{\sqrt{x^2+2}}$ a. $\displaystyle\lim_{x\to\infty} f(x)$ b. $\displaystyle\lim_{x\to-\infty} f(x)$

20. $\displaystyle\lim_{x\to\infty}\frac{2x+5}{3x-4}$ 21. $\displaystyle\lim_{x\to-\infty}\frac{1-5x^3}{10x^3-x^2}$ 22. $\displaystyle\lim_{x\to\infty}\frac{x(2x-1)^2}{3x(x-3)^2}$ 23. $\displaystyle\lim_{x\to-\infty}\frac{4x^2+3}{2x}$

24. $\displaystyle\lim_{x\to-\infty}\frac{x}{\sqrt{x^2+x}}$ 25. $\displaystyle\lim_{x\to\infty}\frac{x}{\sqrt{x^2+x}}$ 26. $\displaystyle\lim_{x\to-\infty}\frac{2-x}{\sqrt{x^2-3}}$ 27. $\displaystyle\lim_{x\to-\infty}\frac{2x^2-2}{\sqrt{x^4}}$

28. $\lim\limits_{x\to\infty} \dfrac{\sin x}{x+1}$ 29. $\lim\limits_{x\to-\infty} \left(\dfrac{4e^x+2x}{3x}\right)$ 30. $\lim\limits_{x\to\infty} \left(x^5 e^x+2\right)$

31. Use a calculator to find $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

Without using a calculator, find the discontinuities. Which are removable?

32. $f(x)=\dfrac{x+2}{x^2-4}$ 33. $f(x)=\left\lfloor \frac{1}{3}x \right\rfloor$

34. $f(x)=\begin{cases} x^2, & x\le 2 \\ 2x-2, & x>2 \end{cases}$ 35. $f(x)=\begin{cases} x^2, & x\le 2 \\ 2x, & x>2 \end{cases}$ 36. $f(x)=\dfrac{|x+2|}{x+2}$

37. If $f(x)=\begin{cases} 2ax-6, & x\le 2 \\ x^2+a, & x>2 \end{cases}$ is a continuous function, find the value of a.

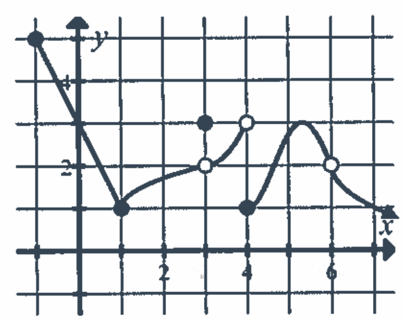
Use a calculator to find all discontinuities.

38. $f(x)=\begin{cases} \dfrac{\cos x-1}{x}, & x<0 \\ 5x, & x>0 \end{cases}$ 39. $f(x)=\dfrac{x^2-4}{x^3-2x^2-2x+4}$

Does the Intermediate Value Theorem guarantee a value of c in the given interval? If so, find the c-value. If not, explain why not.

40. $f(x)=\dfrac{x^2-x}{x}$, $f(c)=-1$ on $[-2,2]$ 41. $f(x)=x^2-x$, $f(c)=-1$ on $[-2,2]$

42. $f(x)=x^2-x$, $f(c)=5$ on $[-2,2]$



Use the graph of $y=f(x)$ at the right for Problems 43-52.

Find these limits.

43. $\lim\limits_{x\to 6} f(x)$ 44. $\lim\limits_{x\to 4^+} f(x)$ 45. $\lim\limits_{x\to 4^-} f(x)$

46. $\lim\limits_{x\to 4} f(x)$ 47. $\lim\limits_{x\to 3} f(x)$ 48. $\lim\limits_{x\to 1} f(x)$

Determine if $f(x)$ is continuous at these x-values. Use correct notation to explain your answer using the definition of continuity from page 34.

49. $x=1$ 50. $x=3$ 51. $x=4$ 52. $x=6$

LESSON 1.4 CURVE SKETCHING

What to watch for:

Domain: possible x-values. Avoid zero denominators and square roots of negative numbers.

Vertical Asymptotes: denominator restrictions from the **reduced** function (write equations in the form $x = a$)

Holes: denominator restrictions from the **original** function which are no longer restricted in the reduced function (plug into the reduced function to find the y-value and write as ordered pairs)

x-intercepts: let $y = 0$, solve for x (write as ordered pairs)

y-intercepts: let $x = 0$, solve for y (write as ordered pairs)

End Behavior: look at highest degree terms in the numerator and the denominator, analyze for "large" positive and negative x-values

Even/Odd: for vertical asymptotes and x-intercepts (or holes on the x-axis)
Note: These come from degrees of factors in the reduced function.

Examples: For Examples 1-3 give the domain, reduce the function, find vertical asymptotes, holes, and end behavior.

1. $f(x) = \dfrac{x+2}{x^2-2x}$

Do:

V.A.:

E.B.:

2. $g(x) = \dfrac{2x^3}{(x+3)^2}$

Do:

V.A.:

E.B.:

3. $h(x) = \dfrac{x^2+2x-8}{x^2-4}$

$h(x) = \dfrac{(x+4)(x-2)}{(x+2)(x-2)}$

Do:

$h_{red}(x) =$

V.A.:

Hole:

E.B.:

Curve Sketching Recipe:

1. Give the domain.
2. Reduce $f(x)$. Oftentimes, you must factor before you can reduce.
3. Find vertical asymptotes and holes.
4. Give x- and y-intercepts.
5. Find the end behavior (horizontal asymptotes or other).
6. (optional) Check for symmetry.
7. (if needed) Find a starting point.
8. Graph.

Examples: Follow the **Curve Sketching Recipe** to graph.

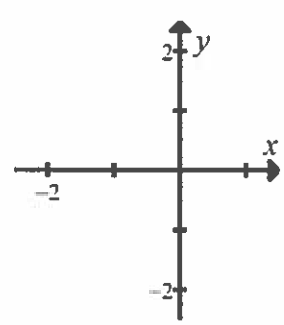
4. $f(x) = x(x-1)(x+2)^2$

Do:

x-int.:

y-int.:

E.B.:



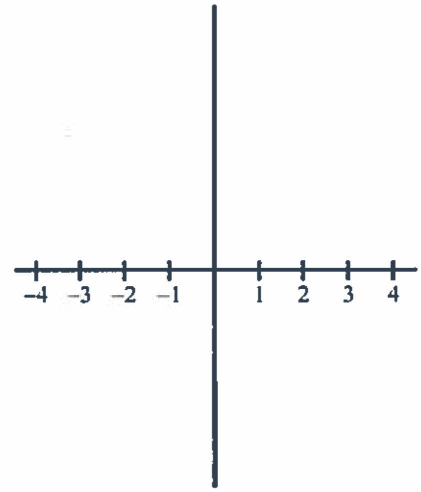
5. $g(x)=\dfrac{x(x-1)^2(x+3)^3}{x^2(x-1)(x-3)^2}$

Do.: $g_{red}(x)=$

V.A.: Holes:

x-int.: y-int.:

E.B.:



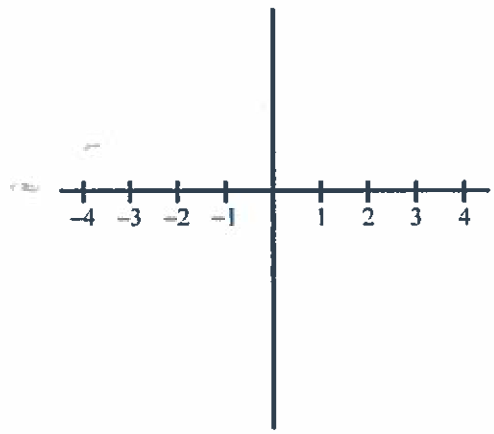
6. $y=\dfrac{x+1}{\sqrt{x^2}}$

Do.: V.A.:

Holes: x-int.:

y-int.: E.B.:

Starting Point:



ASSIGNMENT 1.4

Find the indicated characteristics of these functions without using a calculator.

1. $f(x)=\dfrac{x(x-1)^3}{x^2(x-1)}$, domain, hole, and vertical asymptote

2. $f(x)=\dfrac{-x}{\sqrt{x^2-1}}$, domain and **all** horizontal asymptotes

Follow the **Curve Sketching Recipe** to graph each function without using a calculator.

3. $f(x)=(x+2)(x-1)^2$

4. $f(x)=\dfrac{x-2}{x+2}$

5. $f(x)=\dfrac{(x+2)^2(x-4)}{(x-1)^2}$

6. $f(x)=\dfrac{x(x-1)^3}{x^2(x-1)}$ (see Problem 1)

7. $f(x)=\dfrac{(x-1)^2(x^2+1)}{x^2-1}$

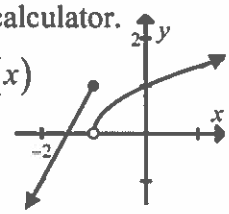
8. $f(x)=\dfrac{1}{\sqrt{x}}$

9. $f(x)=\dfrac{-x}{\sqrt{x^2-1}}$ (see Problem 2)

Find these limits without using a calculator.

10. $\lim\limits_{x\to 2}\dfrac{2-x}{x^2-4}$

11. $y=f(x)$



a. $\lim\limits_{x\to -1^-} f(x)$ b. $\lim\limits_{x\to -1^+} f(x)$ c. $\lim\limits_{x\to -1} f(x)$

12. $\lim\limits_{x\to 4}\dfrac{\sqrt{x}-2}{x-4}$ 13. $\lim\limits_{x\to 3}\dfrac{|x-3|}{x-3}$ 14. $\lim\limits_{x\to 1}\begin{cases} x, & x\le 1 \\ x-1, & x>1\end{cases}$ 15. $\lim\limits_{x\to 0}\begin{cases} x^2-5, & x\le 0 \\ 2x-5, & x>0\end{cases}$

16. $\lim\limits_{x\to 1}\left\lfloor \frac{x}{4}+2\right\rfloor$ 17. $\lim\limits_{x\to 1}\lfloor 4x+2\rfloor$ 18. $\lim\limits_{x\to 2}\dfrac{x}{x-2}$ 19. $\lim\limits_{x\to 2}\dfrac{x^2-2x}{x-2}$

20. $\lim\limits_{x\to 2}\dfrac{x}{(x-2)^2}$ 21. $\lim\limits_{x\to\infty}\dfrac{3x^2-5x}{7x^2+2}$ 22. $\lim\limits_{x\to 2}\left(x^2-9\right)$ 23. $\lim\limits_{x\to -\infty}\dfrac{x}{x^2-3}$

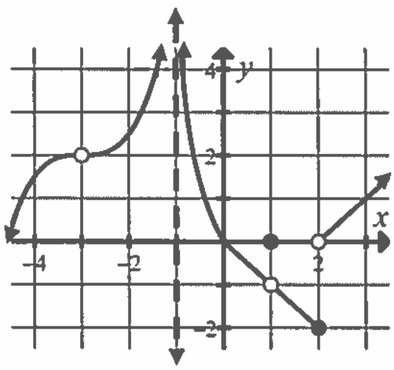
24. $\lim\limits_{x\to\infty}\dfrac{x(3x-5)}{3x+2}$ 25. $\lim\limits_{x\to 0}\dfrac{x}{3x+2}$ 26. $\lim\limits_{x\to 2}\dfrac{3x+5}{\tan\frac{\pi x}{4}}$ 27. $\lim\limits_{x\to -\infty}\left(e^x-\dfrac{3x}{x^2+x}\right)$

Find all discontinuities without using a calculator. Which are removable?

28. $f(x)=\dfrac{x}{x^2-4}$ 29. $f(x)=\dfrac{x-2}{x^2-4}$ 30. $f(x)=\lfloor 2x\rfloor$

31. $f(x)=\begin{cases} -2x+1, & x\le 2 \\ x^2-4x+1, & x>2\end{cases}$

32.



LESSON 1.5 RATE OF CHANGE, SQUEEZE THEOREM, LIMITS OF COMPOSITIONS OF DISCONTINUOUS FUNCTIONS

Rate of Change:

Another meaning for slope is rate of change. In this course there will be two situations where you will use slopes (rates of change).

1. **Average Rate of Change** This is the slope between two points on a graph or a rate of change between two points in time. It is found algebraically using a method from previous courses. $\text{AROC} = m = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$ or $\text{AROC} = \dfrac{\Delta y}{\Delta t} = \dfrac{y_2 - y_1}{t_2 - t_1}$
2. **Instantaneous Rate of Change** This is the slope at a single point on a curve or a rate of change at a single instant in time. It can be approximated using one or more average rates of change or found exactly using a Calculus technique that will be shown in the next unit.

Examples:

1. Given $f(x) = x^3 - 2x^2 - 10$

a. find the average rate of change of $f(x)$ between $x = 2$ and $x = 3$.

b. find the average rate of change of $f(x)$ between $x = 3$ and $x = 4$.

c. Which of these is likely to be a better estimate of the instantaneous rate of change at $x = 2.4$?

2. The data in the table shows the mileage from the start of a four hour car trip recorded at one hour intervals. Assume the car continued in the same straight line.

time in hours	0	1	2	3	4
miles from start	0	55	120	180	250

a. Find the average rate of change (average speed) of the car for the final two hours of the trip.

b. Estimate the instantaneous speed at the 1.5 hour instant.

c. During which hour does the data suggest the car reached the greatest instantaneous speed?

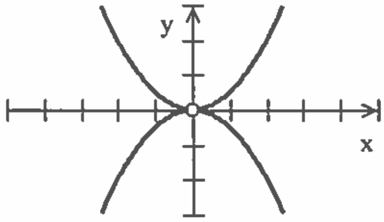
Squeeze Theorem (Sandwich Theorem)

If $f(x) \le g(x) \le h(x)$ for all $x \ne c$ in some interval containing c and if $\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x)$, then $\lim_{x\to c} g(x) = L$.

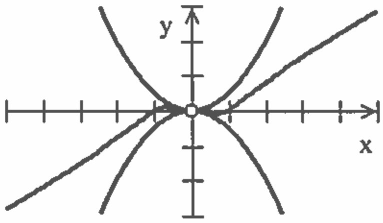
Informally: If a function g is squeezed (sandwiched) between two other functions with the same limit then g also approaches that same limit.

Examples:

3. The graphs of $f(x)=\dfrac{x^3}{2x}$ and $g(x)=\dfrac{-x^3}{2x}$ are shown.
   Find $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to 0} g(x)$.



4. The graph of a third function $k(x)$ is shown along with the two functions from example 3.
   If $g(x)\le k(x)\le f(x)$ find $\lim\limits_{x\to 0} k(x)$. Explain.



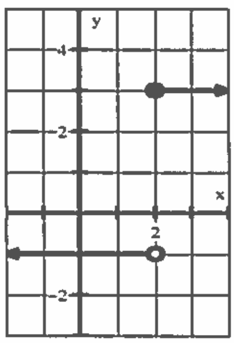
Use the functions graphed to find the following limits.

5. $\lim\limits_{x\to 3}\dfrac{(f(x))^2}{g(x)+1}=$

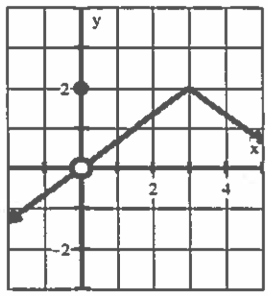
6. $\lim\limits_{x\to 2.5} g(f(x))=$

7. $\lim\limits_{x\to 3} f(g(x))=$

$y=f(x)$

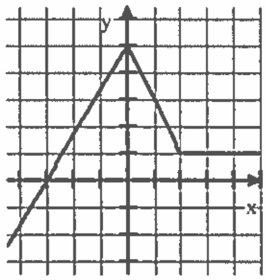


$y=g(x)$



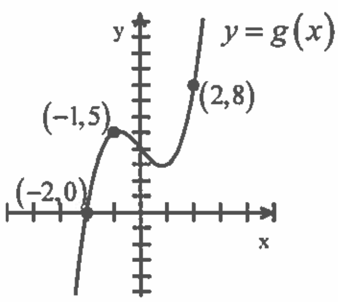
ASSIGNMENT 1.5

1. The function $y=f(x)$ graphed at the right is a piecewise linear function. Find the instantaneous rate of change at each of the following x-values.
   a. $x=-1$
   b. $x=1$
   c. $x=4$



$y=f(x)$

2. The function $y=g(x)$ is graphed at the right.
   a. Find the average rate of change on the interval $[-2,-1]$.
   b. Find the average rate of change on the interval $[-1,2]$.
   c. Which of these is a better approximation for the instantaneous rate of change of $g(x)$ at $x=-1.5$?



3. Approximate the instantaneous rate of change of $y=3e^x+5\sin x$ at $x=3.3$ by finding the average rate of change on the interval $[3,4]$ accurate to three decimal places.

4. The data in the table below gives times and distances for a marathoner at selected points in the race.

time in minutes	0	40	55	95	129
miles from start	0	8	12	20	26

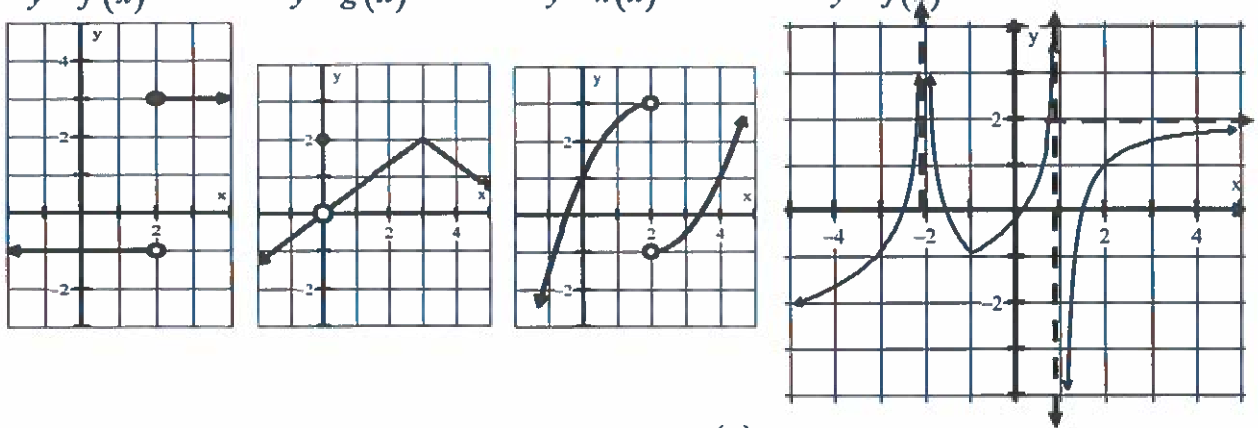
a. Find the runner's average rate of change (speed in miles per minute) for the 26 miles included in the table.
b. Approximate the instantaneous speed at the half-marathon spot (13.1 miles).
c. Which of the intervals shown in the table was the slowest for the runner?

5. If $f(x)=\dfrac{6x-18}{x-3}$ and $g(x)=\dfrac{6\sin\frac{\pi x}{6}}{\cos(x-3)}$ and it is known that $f(x)\le h(x)\le g(x)$ on the interval $[2,4]$ except at $x=3$. Find $\lim\limits_{x\to 3} h(x)$. Explain your reasoning.

6. Given $f(x)=\dfrac{x^2-4}{x+2}$ and $f(x)\le h(x)\le j(x)$ for all x except $x=-2$. If $\lim\limits_{x\to -2} h(x)$ can be found by using the Squeeze Theorem what is $\lim\limits_{x\to -2} j(x)$?

Use the four functions graphed below to find the limits shown or state that the limit does not exist.

$y=f(x)$ $\quad$ $y=g(x)$ $\quad$ $y=h(x)$ $\quad$ $y=j(x)$



7. $\lim\limits_{x\to -2} j(x)$ $\quad$ 8. $\lim\limits_{x\to 1} j(x)$ $\quad$ 9. $\lim\limits_{x\to -1} \dfrac{f(x)-2}{(j(x))^2}$ $\quad$ 10. $\lim\limits_{x\to \infty} h(j(x))$

11. $\lim\limits_{x\to -1} g(f(x)+1)$ $\quad$ 12. $\lim\limits_{x\to 0} f(|x|+2)$ $\quad$ 13. $\lim\limits_{x\to 0}(g(x)\cdot f(x+2))$ $\quad$ 14. $\lim\limits_{x\to -2} j(j(x))$

15. Find the equation of the horizontal asymptote for the function $g(x)=\dfrac{x^3+x}{e^x+x}$ without using a calculator.

16. Find $\lim\limits_{x\to 0}\dfrac{\sin x+2e^x}{\cos x}$ without using a calculator.

17. Find the values of a and b so that $f(x)=\begin{cases} x-1, & x\le -1 \\ ax+b, & -1<x<1 \\ 2x+1, & x\ge 1 \end{cases}$ is continuous.

18. Use a calculator to find this limit $\lim\limits_{x\to 2}\dfrac{|2-x|}{25x-50}$.

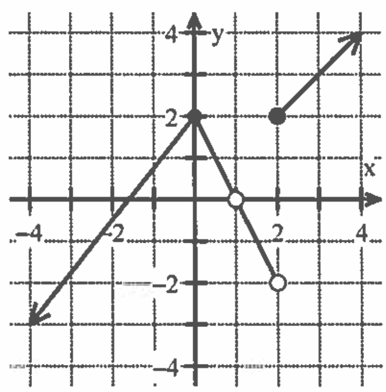
19. Determine whether the Intermediate Value Theorem would guarantee a c-value where $f(c)=6$, for the function $f(x)=\dfrac{x^2+x}{x-1}$ on the interval $\left[\frac{5}{2},4\right]$.

20. If your answer to problem 19 is yes, find the c-value. If your answer is no, try it again.

Find the following limits without using a calculator.

21. $\lim\limits_{x\to -2}(3x-3)$ 22. $\lim\limits_{x\to 2^-}\left\lfloor \frac{x}{2}-4\right\rfloor$ 23. $\lim\limits_{x\to 2}\left\lfloor \frac{x}{2}-4\right\rfloor$ 24. $\lim\limits_{x\to 3^+}\left\lfloor \frac{x}{2}-4\right\rfloor$

Use the graph of $y=f(x)$ for Problems 25-31.
Find the following limits and function values.



25. $\lim\limits_{x\to 2}f(x)$ 26. $f(2)$ 27. $\lim\limits_{x\to 2^-}f(x)$

28. $\lim\limits_{x\to 1}f(x)$ 29. $\lim\limits_{x\to 0}f(x)$

30. List all removable discontinuities of $f(x)$.

31. List all nonremovable discontinuities of $f(x)$.

ASSIGNMENT 1.6 REVIEW

Find the following limits without using a calculator.

1. $\lim\limits_{x\to 2}(5x-3)$ 2. $\lim\limits_{x\to 3}\dfrac{x^2-9}{3-x}$ 3. $\lim\limits_{t\to -3}\dfrac{3+t}{t^2-9}$ 4. $\lim\limits_{x\to -1}\dfrac{x+1}{x^2+2x+1}$

5. $\lim\limits_{t\to 2}\dfrac{t^2-4}{t^2-3t+2}$ 6. $\lim\limits_{x\to 0^+}\left(x+\dfrac{1}{x^3}\right)$ 7. $\lim\limits_{x\to \frac{1}{2}}\dfrac{4x-2}{2x-1}$ 8. $\lim\limits_{x\to 1}\dfrac{x-1}{x^4-1}$

9. $\lim\limits_{x\to 1}\dfrac{x^2-2x+1}{x+1}$ 10. $\lim\limits_{x\to 1}\dfrac{x^2-2x+1}{x-1}$ 11. $\lim\limits_{x\to 2}\dfrac{3x+5}{5x-2}$ 12. $\lim\limits_{x\to 2}\dfrac{\frac{1}{2}-\frac{1}{x}}{x-2}$

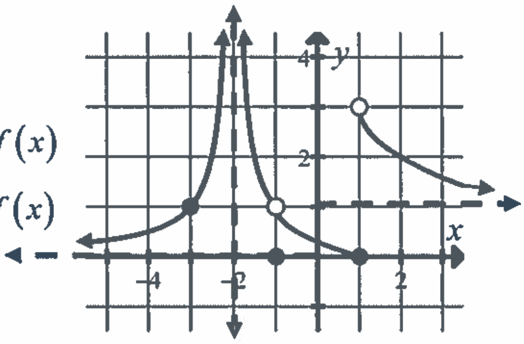
13. $\lim\limits_{x\to\infty}\dfrac{x^2-9}{2x^2+9}$ 14. $\lim\limits_{x\to\infty}\dfrac{(2x-1)^2}{x^2-9}$ 15. $\lim\limits_{x\to -\infty}\dfrac{3x^2-5}{x+1}$ 16. $\lim\limits_{x\to\infty}\dfrac{2x}{(x-3)^2}$

17. $\lim\limits_{x\to 2^-}\dfrac{x}{x^2-4}$ 18. $\lim\limits_{x\to -\infty}\dfrac{-\sqrt{x^2}}{x}$ 19. $\lim\limits_{x\to 2}\left\lfloor \frac{x}{3}+5\right\rfloor$ 20. $\lim\limits_{x\to 3^-}\left\lfloor \frac{x}{3}+5\right\rfloor$

21. $\lim\limits_{x\to 3}\left\lfloor \frac{x}{3}+5\right\rfloor$ 22. $\lim\limits_{x\to \frac{\pi}{2}}\dfrac{3x+5\cos x}{-2\sin x}$ 23. $\lim\limits_{x\to\infty}\dfrac{e^{-2x}+100}{x^2-9}$ 24. $\lim\limits_{x\to \frac{\pi}{4}}\dfrac{\tan x}{\cos x+\sin x}$

Use the graph of $y = f(x)$ for Problems 25-35.

Find the following limits and function values.



25. $\lim_{x\to -1} f(x)$ 26. $f(-1)$ 27. $f(1)$

28. $\lim_{x\to 1} f(x)$ 29. $\lim_{x\to -3} f(x)$ 30. $\lim_{x\to 1^+} f(x)$

31. $\lim_{x\to \infty} f(x)$ 32. $\lim_{x\to -\infty} f(x)$ 33. $\lim_{x\to -2} f(x)$

34. List all removable discontinuities of $f(x)$.

35. List all nonremovable discontinuities of $f(x)$.

36. On what intervals is $y = \dfrac{x}{x^2 - x}$ continuous? Do not use a calculator.

37. On what intervals is $y = \sqrt{x+2}$ continuous? Do not use a calculator.

Use the function $g(x) = \begin{cases} x+2, & x \le 0 \\ x^2+2, & 0 < x < 2 \\ 2, & x \ge 2 \end{cases}$ for Problems 38-44.

38. Sketch a graph of $g(x)$ without using a calculator.

Find the following limits.

39. $\lim_{x\to 0} g(x)$ 40. $\lim_{x\to 2} g(x)$ 41. $\lim_{x\to 2^-} g(x)$ 42. $\lim_{x\to 1} g(x)$ 43. $\lim_{x\to \infty} g(x)$

44. List all discontinuities of $g(x)$.

45. If $f(x) = \begin{cases} x+1, & 1 < x < 3 \\ x^2 + ax + b, & x \le 1 \text{ or } x \ge 3 \end{cases}$ is continuous, find the values of a and b.

46. Find the domain, vertical asymptotes, holes, intercepts, end behavior, and graph for the function $y = \dfrac{x(x-1)}{x^2-1}$.

47. Determine whether the Intermediate Value Theorem would guarantee a c-value on the given interval. If it does, find the c value. If it does not apply, write a sentence explaining why.

a. $f(x) = x^2 - 4x - 8$ $f(c) = 4$ $[-4, -1]$

b. $f(x) = \dfrac{x+3}{x-2}$ $f(c) = 4$ $[0, 5]$

c. $f(x) = \dfrac{x+5}{x-1}$ $f(c) = 3$ $[2, 5]$

48. If $f(x) = 2x^3 - 3x + 2$ approximate the instantaneous rate of change of $f(x)$ at $x = 1$ by finding the average rate of change on the interval $[0, 3]$.

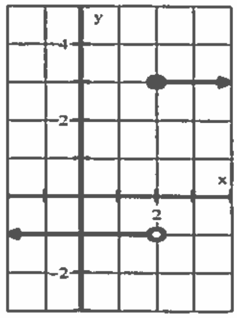
Use the functions graphed below to find the limits shown or state that the limit does not exist.

49. $\lim_{x \to 0} f(h(x))$

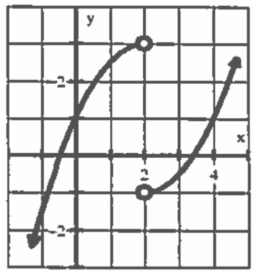
50. $\lim_{x \to 2}\left(\left(f(x)-1\right)^2-6\right)$

51. $\lim_{x \to 2}\left(h(x)+f(x)\right)$

$y=f(x)$



$y=h(x)$



52. If the Squeeze Theorem can be applied and $g(x) \le f(x) \le h(x)$ except at $x=5$ where $g(x)=\dfrac{x^2-25}{x-5}$, find $\lim_{x \to 5} h(x)$ and $\lim_{x \to 5} f(x)$.

53. Find the indicated information for the following rational functions and then graph. Make sure your answers are in the right form and your graphs are consistent with your list.

a. $f(x)=\dfrac{(x-3)(x+2)^2(x-1)^2}{(x+1)(x-3)(x-1)(x-2)^2}$

Domain: $f_{red}=$ Holes:

VA: EB: x-int:

y-int:

b. $p(x)=\dfrac{(4-x)}{\sqrt{x^2-9}}$

Domain: Holes:

VA: EB: x-int:

y-int:

UNIT 1 SUMMARY

Limits:
A limit is a y-value.
Analyze left and/or right behavior.
Use direct substitution.

Discontinuities: holes, vertical asymptotes, and jumps (breaks).
Removable (holes). Nonremovable (jumps and vertical asymptotes).

Limit at infinity: (end behavior)
Consider the highest degree terms in the numerator and denominator.

Curve sketching:
Vertical Asymptotes: denominator restrictions from the **reduced** function (write equations in the form $x = a$)
Holes: denominator restrictions from the **original** function which are no longer restricted in the reduced function (plug into the reduced function to find the y-value and write as ordered pairs)

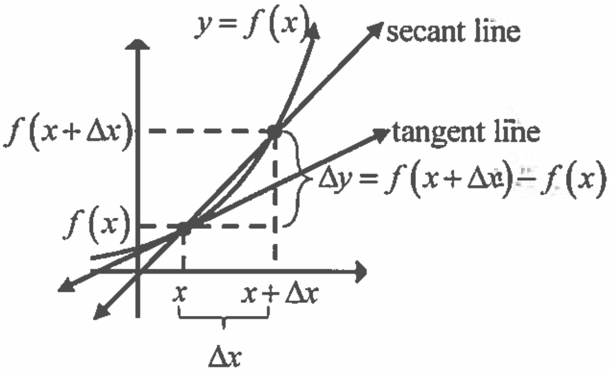
Average Rate of Change: (the slope between two points) $\text{AROC} = \dfrac{y_2 - y_1}{x_2 - x_1}$
Instantaneous Rate of Change: can be approximated with an average rate of change

Squeeze Theorem (Sandwich Theorem)

If $f(x) \le g(x) \le h(x)$ for all $x \neq c$ in some interval containing c and if $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$, then $\lim_{x \to c} g(x) = L$.

LESSON 2.1 LIMIT DEFINITION OF THE DERIVATIVE, ALTERNATE FORM OF THE LIMIT DEFINITION

Instead of approximating instantaneous rates of change as you did in the previous unit the process of differentiation allows us to find them precisely.
Any nonvertical line has the same slope at every point. In Calculus we frequently deal with the slope of a curve. The slope of a curve is defined to be the same as the slope of the curve's tangent line at a given point. To find the slope of a tangent line we use a limit of the slope of a secant line.



$$m_{\sec} = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$m_{\tan} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

The slope of a tangent line is called the **derivative** of the function at a given x-value. The most commonly used symbol for the derivative is $f'(x)$. Here are some other notations you will encounter (assume $y = f(x)$).

$$\boxed{f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx}f(x) = m_{\tan} = \lim_{\Delta x \to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{h \to 0}\frac{f(x+h)-f(x)}{h}}$$

A vertical tangent line has no slope, so a curve has no derivative at any point where it has a vertical tangent line. Differentiation is the process of finding derivatives. If a derivative exists at a point on a curve, the function is said to be differentiable at that point.

Examples:

1. If $f(x) = x^2 + 2$

   a. find $f'(x)$.

   $f'(x) = \lim_{\Delta x \to 0}$ ______________

   b. use your answer from part a. to find $f'(-3)$.

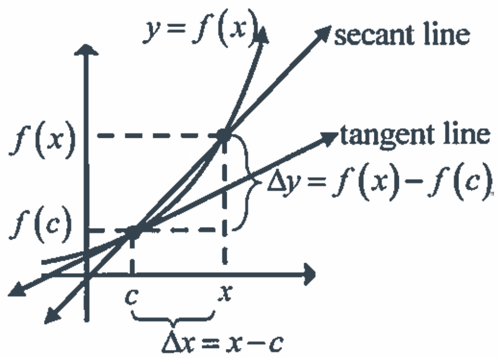
2. If $y=\sqrt{x}$, find y'.

$y'=$

3. Given $y=f(t)=\frac{2}{t}$, find the derivative of y with respect to t.

$\frac{dy}{dt}=f'(t)=$

Alternate Form of the Limit Definition of the Derivative
(Gives the value of the derivative at a single point.)



$$m_{\tan}=\lim_{x\to c} m_{\sec}$$

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

Example 4. If $f(x)=x^3$, use the alternate form of the derivative to find $f'(3)$.

$f'(3)=$

ASSIGNMENT 2.1

Use the limit definition of the derivative to find $f'(x)$ or $f'(t)$.

1. $f(x)=-3x$ 2. $f(x)=x^2-1$ 3. $f(x)=\frac{1}{x-1}$

4. $f(t)=t^3-12t$ 5. $f(x)=3$

Use the alternate form of the limit definition of the derivative to find the indicated derivative.

6. $f(x)=x^2-1$ Find $f'(2)$. 7. $f(x)=x^3-2x^2-1$ Find $f'(2)$.

8. $f(x)=\frac{1}{x}$ Find $f'(3)$. 9. $f(x)=(x-1)^{\frac{2}{3}}$ Find $f'(1)$.

10. If $y=x^2-x$, use the limit definition of the derivative to find y'.
11. If $y=x^3+1$, use the limit definition of the derivative to find $\frac{dy}{dx}$.
12. If $f(x)=2x^2+4$, use the limit definition of the derivative to find $f'(x)$.
    Then find $f'(4)$.
13. If $f(x)=2x^2+4$, use the alternate form of the limit definition of the derivative to find $f'(4)$.

These review problems require arithmetic and algebra skills that will be **absolutely essential** on the next assignment!

Simplify without using a calculator:

14. $\frac{2}{3}-1$ 15. $-\frac{1}{2}-1$ 16. $-\frac{5}{3}-1$ 17. x^{-2} 18. $x^{\frac{1}{2}} \bullet x^2$

19. $\left(x^{\frac{1}{2}}\right)^2$ 20. $\frac{x^9}{x^3}$ 21. Rewrite $\frac{16}{(2x)^3}$ without using a fraction.

22. Rewrite $\sqrt[3]{x^2}$ without using a radical symbol.
23. Rewrite $\frac{2}{\sqrt{x^3}}$ without using a fraction or a radical symbol.

LESSON 2.2 DERIVATIVE RULES (short cuts), TANGENT LINES, NONDIFFERENTIABILITY, RATE OF CHANGE

The expression $\frac{d}{dx}$ means to differentiate with respect to x. The most common derivative $\frac{d}{dx}y$ is usually written as $\frac{dy}{dx}$.

Derivative Rules:

- Power Rule: $\frac{d}{dx}x^n = nx^{n-1}$
- Constant Rule: If c is any constant, $\frac{d}{dx}c = 0$.
- Scalar Multiple Rule: If c is any constant, $\frac{d}{dx}\left(c\,f(x)\right) = c\,f'(x)$.
- Sum Rule: $\frac{d}{dx}\left(f(x)+g(x)\right) = f'(x)+g'(x)$

Examples: Differentiate.

1. $f(x)=x^4$

   $f'(x)=$

2. $y=x^{-\frac{2}{3}}+3$

   $y'=$

3. $h(t)=5-\dfrac{1}{2t^3}$

   $h(t)=$

   $h'(t)=$

4. $f(x)=\dfrac{5}{(2x)^3}$

Higher-Order Derivatives

Since the derivative of a function is another function, we can repeat the differentiation process to find the derivative of a derivative. The result is still another function which could again be differentiated. These derivatives are called higher-order derivatives.

Notation:

First Derivative:	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}f(x)$
Second Derivative:	y''	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx^2}f(x)$
Third Derivative:	y'''	$f'''(x)$	$\frac{d^3y}{dx^3}$	$\frac{d^3}{dx^3}f(x)$
Fourth Derivative:	$y^{(4)}$	$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$\frac{d^4}{dx^4}f(x)$

Example 5. For $f(x)=\dfrac{1}{2\sqrt[3]{x^2}}$, find $f'(1)$ and $f''(-8)$.

Equation of a Tangent Line:

Since the derivative of a function gives us a slope formula for tangent lines to the graph of the function, the derivative can be used to find equations of tangent lines.

Sometimes we will want to find a line perpendicular to the tangent line at the point of tangency. Such a line is called a normal line.

$$m_{\text{normal}} = \frac{-1}{m_{\text{tangent}}}$$

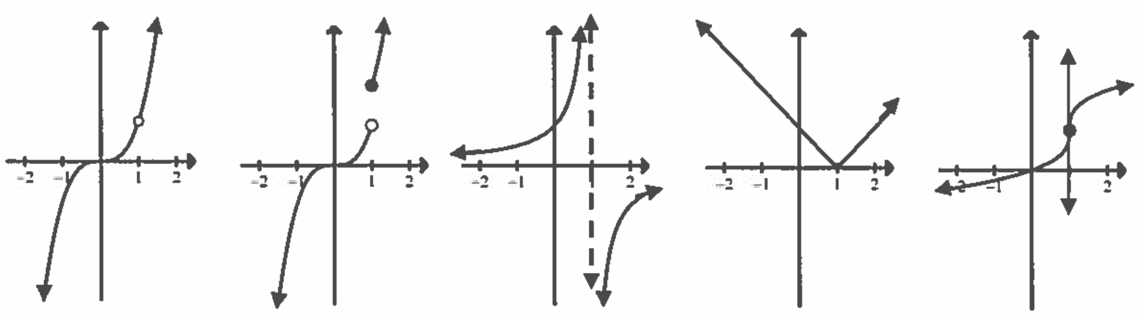
Examples:

6. Find an equation of the line tangent to the graph of $f(x) = 4x^5 - 3x^2 + 5$ at the point (1,6).

7. Find an equation of the normal line to the same curve at the same point.

NONDIFFERENTIABILITY (when a derivative does not exist)

Each of these functions has no derivative when $x = 1$.



hole | jump | vertical asymptote | sharp turn | vertical tangent

These five characteristics destroy differentiability:

1. Holes
2. Jumps (breaks)
3. Vert. Asymptotes

} discontinuities

4. Sharp Turns
5. Vert. Tangent Lines

Note:
If a function is not continuous, it is not differentiable (see the first three figures above).
A function may be continuous and still not be differentiable (see the last two figures above).

<u>Examples:</u> Find the x-values where $f(x)$ is <u>not</u> differentiable. <u>Give a reason for each</u>.

8. $f(x)=|x|$

9. $f(x)=\begin{cases} x^2, & x\le 0 \\ x, & x>0 \end{cases}$

10. $f(x)=\begin{cases} x^2, & x\le 0 \\ x^2+1, & x>0 \end{cases}$

11. $f(x)=\dfrac{x}{x(x-1)}$

12. $f(x)=\sqrt[3]{x}$

<u>Rate of Change:</u>

Remember the method of finding average rate of change

1. **<u>Average Rate of Change</u>** This is the slope between two points. It is found without using a derivative (algebraically). AROC = $m=\dfrac{\Delta y}{\Delta x}=\dfrac{y_2-y_1}{x_2-x_1}$

Using differentiation we can now also find the instantaneous rate of change.

2. **<u>Instantaneous Rate of Change</u>** This is the slope at a single point. It is usually found by using a derivative (calculus). IROC = $m=f'(c)$

<u>Examples</u>:

13. If $f(x)=x^3+2x$, find the average rate of change from $x=10$ to $x=30$.

14. If $f(x)=x^3+2x$, find the instantaneous rate of change when $x=10$.

ASSIGNMENT 2.2 **Do not use a calculator on this entire assignment.**

Find the derivative. Use correct symbolism.

1. $y=2$
2. $f(x)=x^2$
3. $g(x)=x^3+1$
4. $y=t+2$
5. $f(t)=-2t^2-3t+2$
6. $f(x)=-\frac{1}{3}x^2-\frac{2}{5}x+\frac{5}{2}$

Find the value of the derivative of the function at the given point. Show steps with correct symbolism without using a calculator.

7. $f(x)=3x^{-2}$ at $(1,3)$
8. $g(x)=x^2-2x$ at $(2,0)$
9. $h(x)=x^3-1$ at $(1,0)$
10. $f(x)=2-x^3$ at $(2,-6)$

Differentiate each function. Show steps with correct symbolism.

11. $y=\dfrac{1}{x}$
12. $f(x)=x^2-\dfrac{4}{x^2}$
13. $y=(2x-1)^2$
14. $g(x)=x(x^2+1)$
15. $y=\dfrac{\sqrt{x}}{x}$
16. $y=\sqrt[3]{x}+\sqrt{x^3}$
17. $f(t)=\dfrac{t^2-2t}{t}$
18. $f(x)=\dfrac{1}{\sqrt[3]{x^2}}$
19. $y=\dfrac{1}{3x^2}$
20. $y=\dfrac{1}{(3x)^2}$

Find the indicated value or expression. Show steps with correct symbolism.

21. $y=3x^2,\quad y''=?$
22. $f(x)=\sqrt{x}+2,\quad f'(4)=?$
23. $f(t)=2-\dfrac{2}{t},\quad f''(2)=?$
24. $y=x(x-2),\quad \dfrac{d^2y}{dx^2}=?$
25. $f^{(3)}(x)=2x-1,\quad f^{(5)}(3)=?$
26. $\frac{d}{dx}(x^3+5)=?$
27. $\dfrac{d^2}{dx^2}(3x-x^{-1})=?$

Find an equation of a line with the following characteristics.

28. tangent to the graph of $f(x)=x^2-1$ at the point (2,3)
29. tangent to the graph of $f(x)=\frac{2}{x}$ when $x=1$
30. normal to the graph of $f(x)=\frac{2}{x}$ when $x=1$
31. tangent to the graph of $y=x^2-2x+3$ when $x=1$
32. Find the x-values of all points where the graph of $f(x)=3x^3+2x-2$ has a slope of 11.

33. Find the x-values of all points where the graph of $y = x^4 - 3x^2 + 2$ has a horizontal tangent line.

34. Find the point(s) where the graph of $y = \frac{1}{x}$ has a slope of $-\frac{1}{4}$.

35. Find the average rate of change of the function $f(x) = 3x^3 - 4$ between $x = 2$ and $x = 4$.

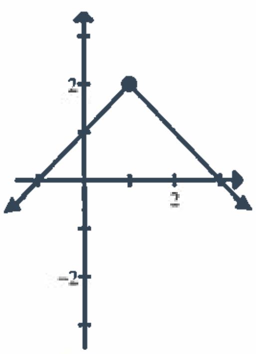
36. Find the instantaneous rate of change of the function $f(x) = 3x^3 - 4$ at $x = 3$.

37. Find the average rate of change of $y = \frac{x}{x+2}$ on the interval $[1,4]$.

38. Find the rate of change of $y = \frac{x^2 - x}{x^2}$ at the point (1,0).

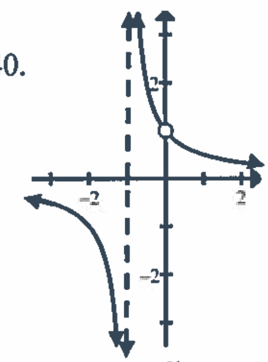
Find the x-values where the function is not differentiable. Give a reason for each value.

39.



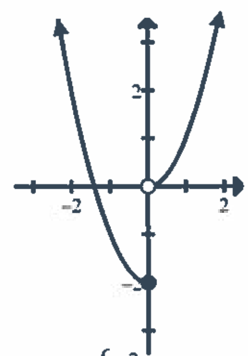
$y = -|x-1| + 2$

40.



$y = \frac{x}{x^2 + x}$

41.



$y = \begin{cases} x^2 - 2, & x \le 0 \\ x^2, & x > 0 \end{cases}$

42. $f(x) = x^{\frac{2}{3}}$

43. $f(x) = 3x^{\frac{1}{3}}$

44. $f(x) = \begin{cases} 3x^2 - 2, & x \le 1 \\ x, & x > 1 \end{cases}$

45. $f(x) = \begin{cases} x^2, & x \le 0 \\ -x^2, & x > 0 \end{cases}$

46. Use the limit definition of the derivative to find $f'(x)$ if $f(x) = 2x^2 - 5$.

47. Use the alternate form of the limit definition of the derivative to find $f'(1)$ if $f(x) = x^2 + 2x$.

48. Find $\lim\limits_{h \to 0} \frac{(x+h)^3 - x^3}{h}$ mentally without showing any steps.

LESSON 2.3 CALCULATOR DIFFERENTIATION, DERIVATIVES OF $\sin x, \cos x, e^x$, and $\ln x$

Calculator Differentiation

Some CAS calculators can find derivatives of functions in symbolic form. This is not one of the uses of calculators that are allowed on the Advanced Placement test and will not be helpful. All graphing calculators can be used to find the value of a derivative at a specific point and this is allowed and required on the Advanced Placement test.
For example a TI84 can find a derivative at a point using nDeriv( ) in the Math menu.
Use a graphing calculator to find the following. As always give answers accurate to three decimal places.

Examples:

1. If $f(x)=x^3+3^x$, find $f'(2)$.

   $f'(2)=$

2. If $g(x)=\ln(x^2-3)$, find $g'(2)$, $g'(4)$, and sketch a graph of $g'(x)$.

   Hint: To save time and avoid confusing parentheses, let $y_1=\ln(x^2-3)$.

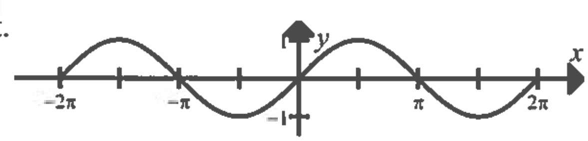
   $g'(2)=$ $g'(4)=$

   To graph $g'(x)$, let $y_2=\left.\frac{d}{dx}(y_1)\right|_{x=x}$.

3. If $f(x)=|x|$, find $f'(0)$.

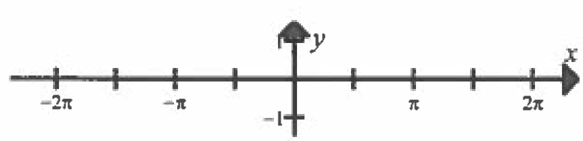
Some calculators are unable to give a correct answer to this last problem. Make sure you understand the limitations of your calculator.

The graph of $f(x)=\sin x$ is shown at the right.



Example 4:
Estimate slopes for the graph of $f(x)=\sin x$ at $x=-2\pi, \frac{-3\pi}{2}, -\pi, \frac{-\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π.

Plot those slopes in the coordinate plane at right, and connect them to make a smooth continuous curve. This is the graph of $f'(x)$.



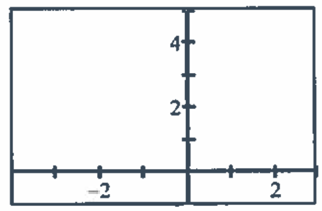
$f'(x)=$

Example 5:
Use your calculator to sketch a graph of the derivative of $y = \cos x$.
Can you identify your result?

Derivatives of Two Trigonometric Functions:

$\frac{d}{dx}\sin x = \cos x$ $\frac{d}{dx}\cos x = -\sin x$

Example 6:
Use your calculator to graph $y = e^x$ and its derivative in the same coordinate plane. What do you notice?



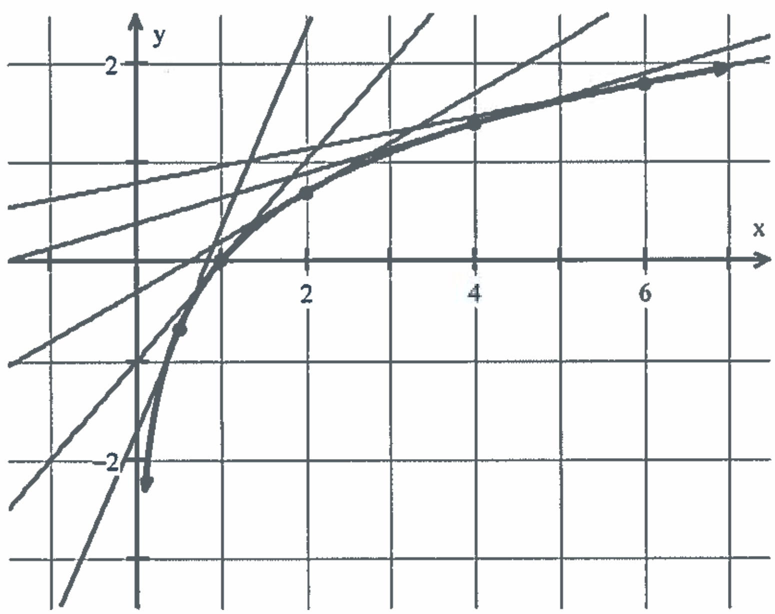
e is the only base for which the basic exponential function and its derivative are the same.

Differentiating the Exponential Function:

$\frac{d}{dx}e^x = e^x$

Example 7:
The function $y = \ln x$ is graphed at the right with some tangents shown.

Fill in the table by estimating slopes at the points indicated with as much precision as possible.



x	$\frac{d}{dx}\ln x$
$\frac{1}{2}$	
1	
2	
4	
6	

Can you make a conclusion about the derivative of the natural logarithm function?

Differentiating the Natural Logarithmic Function:

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Examples:

Differentiate each of the following functions using correct notation without using a calculator.

8. $y = 2e^x - 3\ln x + \sqrt{x}$

9. $g(\theta) = 3\sin\theta - 5\cos\theta$

Examples:

Find the indicated value without using a calculator.

10. If $h(x) = 15\sin(x) + 4e^x$ find $h'(0)$.

11. If $f(t) = 4\ln t - 3e^t$ find $f''(5)$.

Example 12: Find an equation of the line tangent to the graph of $f(x) = -3\cos x$ when $x = \frac{\pi}{4}$ without using a calculator.

ASSIGNMENT 2.3

Use a graphing calculator for problems 1-3.

1. If $f(x) = (2x^2 - 1)^3$, use a calculator to graph $f'(x)$. Also find $f'(1)$ and $f'(8)$.
2. If $f(x) = 7^x$, find $f'(3)$.
3. If $g(x) = \sin x^3 + 4x^3$, find $g'(2)$, $g'(-4)$, and $g''(1)$.

Differentiate each of the following without using a calculator. Show steps and answers using correct notation.

4. $f(x) = 2\sqrt{x} + 3e^x$

5. $y = \frac{2}{x} - \ln x$

6. $g(\theta) = 3\sin(\theta)$

7. $f(x) = \dfrac{3x^2 + 4x - 2x\cos x}{2x}$

8. $y = ex^2 - 2e^x$

9. $f(t) = 3\ln t + \sin(t)$

10. $y = f(x) + g(x)$

Find the indicated value without using a calculator. Show steps and answers using correct notation.

11. $g(x) = x + 2\cos x$ find $g'\left(\frac{\pi}{4}\right)$

12. $f(x) = x^2 + 5x - \ln x$ find $f'(1)$

13. $f(x) = 7e^x + \sin(x)$ find $f'(0)$

14. $h(t) = \dfrac{5t^4 + 9t^3 - 6t^2 e^t}{3t^2}$ find $h''(2)$

15. $f(x) = \pi \sin x - \pi^2 \ln x$ find $f''\left(\frac{\pi}{2}\right)$

16. Find an equation of a line tangent to $f(x) = 6\sqrt[3]{x}$ at $x = 8$ without using a calculator.

17. Find an equation of a line tangent to $g(\theta) = \frac{3}{2} + \sin(\theta)$ at $\theta = \frac{\pi}{6}$ without using a calculator.

18. Find the x-coordinate(s) of point(s) at which the graph of $g(x) = x - e^x$ has a tangent line parallel to the graph of $y = 7$ without using a calculator.

The following two limits can now be found with very little work.

19. $\displaystyle\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$

20. $\displaystyle\lim_{h \to 0} \frac{2e^{x+h} - 2e^x}{h}$

21. Find the domain, vertical asymptotes, holes, intercepts, end behavior, and graph for the function $y = \dfrac{x(x-1)}{x^2 - 1}$ without using a calculator.

22. Use the Limit Definition of the Derivative to find $f'(x)$ if $f(x) = 3x^2 + x$.

23. Use the Alternate Form of the Limit Definition of the Derivative to find $f'(3)$ if $f(x) = 3x^2 - 2$.

24. If $f(x) = 2x^3 - 3x + 2$ find:

    a. the average rate of change on the interval $[0, 3]$.

    b. the instantaneous rate of change at $x = 3$.

Differentiate without using a calculator.

25. $f(x) = 2x(x^2 + 3)$

26. $g(x) = \dfrac{3x^2 - 6x + 9}{3x}$

27. $f(x) = \dfrac{1}{4x^4}$

For each of the following piecewise functions:

a. Find any x-values at which the function is discontinuous.

b. Differentiate the function.

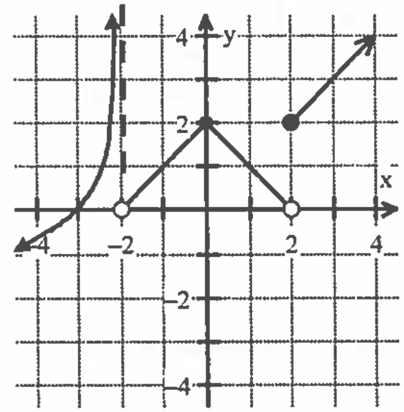
c. Find any x-values at which the function is not differentiable.

28. $f(x) = \begin{cases} 3x^2 - x, & x \le 1 \\ 5x - 3, & x > 1 \end{cases}$

29. $f(x) = \begin{cases} 3x^2 - x, & x \le 1 \\ 5x - 2, & x > 1 \end{cases}$

30. $f(x) = \begin{cases} 3x^2 - x, & x \le 1 \\ 4x - 2, & x > 1 \end{cases}$

31. a. Identify any x-values at which the function shown is not continuous.
    b. Identify any x-values at which the function shown is not differentiable.



LESSON 2.4 POSITION, VELOCITY, ACCELERATION

Important Terms	
Position Function	gives the location of an object at time t, usually $s(t)$, $x(t)$, or $y(t)$
Velocity Function	the rate of change (derivative) of position, usually $v(t)$ Velocity is positive for upward or rightward motion and negative for downward or leftward motion.
Acceleration Function	the rate of change (derivative) of velocity, usually $a(t)$
Initial Position	starting position (at $t=0$), s_o
Initial Velocity	starting velocity (at $t=0$), v_o
Speed	the absolute value of velocity
Displacement	the net change in position, (final pos. – original pos.)
Total Distance	total distance traveled by the object in the time interval (takes into account all direction changes)

Example 1. If $s(t)=t^3+t$, find $v(t)$ and $a(t)$.

Examples: Use the position function $s(t)=16t^3-36t^2+24$ of an object moving on a horizontal line for Examples 2-11. Distance units are measured in feet and time units are measured in seconds.

2. What is the initial position of the object?

3. What is the velocity of the object at $t=1$ second?

4. What is the speed of the object at $t=1$ second?

5. What is the acceleration of the object at $t = 1$ second?

6. When is the object at rest?

7. When is the object moving right?

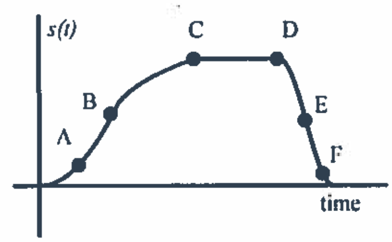
8. When is the object moving left?

9. When is the velocity of the object equal to 54 $^{ft}/_{sec}$?

10. What is the displacement of the object between $t = 0$ and $t = 2$ seconds?

11. What is the total distance traveled by the object between $t = 0$ and $t = 2$ seconds?

The graph models the position function of a radio controlled model car. Answer these questions and explain.



12. Was the car going faster at A or at B?

13. When was the car stopped?

14. At which point was the car's velocity the greatest?

15. At which point was the car's speed the greatest?

Vertical Motion Examples:

Suppose $s(t) = -16t^2 + 48t + 160$ gives the position (in feet) above the ground for a ball thrown into the air from the top of a high cliff (where time is measured in seconds).

16. Find the initial velocity.

17. At what time does the ball hit the ground?

18. At what time does the ball reach its maximum height?

ASSIGNMENT 2.4

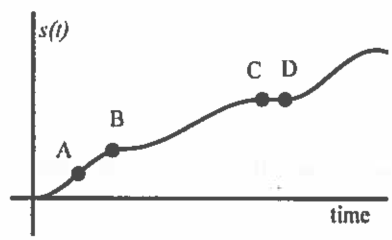
You may use a calculator for these questions.

1. The position, in meters, of a particle moving in a straight line is given by $x(t) = 4t^3 + 6t + 2.5$ (where t is measured in seconds).
   a. Find the velocity function.
   b. Find the velocity at time $t = 2$ seconds.
   c. Find the acceleration function.
   d. Find the acceleration at time 3 seconds.
   e. When is the velocity of the particle 18 meters per second?
   f. Find the velocity when the position of the particle is 25 meters.
   g. Find the initial position.
   h. Find the particle's displacement from 0 to 1.5 sec

2. A helium balloon rises so that its height (position) is given by $s(t) = t^2 + 3t + 5$ (where height is measured in feet and time is measured in seconds). Assume $t \geq 0$.
   a. When is the balloon 45 feet high?
   b. How fast is the balloon rising at time 1 second?
   c. How fast is the balloon rising at time 4 seconds?
   d. What is the balloon's velocity when it is 45 feet high?

3. A ball rolls on an inclined plane with position function $s(t) = 2t^3 + 3t^2 + 5$ (where position is measured in centimeters and time is measured in seconds).
   a. Find the ball's velocity at time 2 seconds.
   b. When is the velocity of the ball 30 centimeters per second?

4. The graph at the right models the position function of a car. Answer these questions and explain each answer.
   a. What was the car's initial position?
   b. Was the car going faster at A or at B?
   c. Was the car speeding up or slowing down at B?
   d. What happened between C and D?

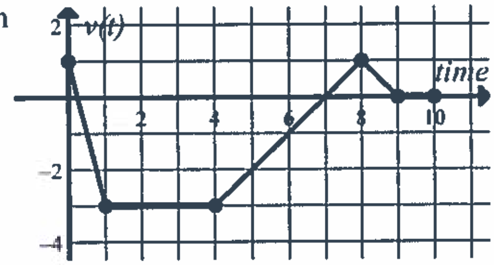


5. A particle moves along a horizontal line with position function $x(t) = t^3 - 3t^2$ (where position is measured in centimeters and time is measured in minutes).
   a. Find the particle's displacement between $t = 0$ minutes and $t = 5$ minutes.
   b. Find the particle's velocity when $t = 4$ minutes.
   c. Find the particle's acceleration when $t = 4$ minutes.
   d. At what time does the particle change direction?
   e. What is the total distance traveled by the particle between 0 and 5 minutes?

Average Velocity $= \dfrac{\text{displacement}}{\text{elapsed time}}$	**Average Speed** $= \dfrac{\text{total distance}}{\text{elapsed time}}$

   f. Find the particle's average velocity (average rate of change of position) between $t = 0$ and $t = 5$ minutes.
   g. Find the particle's average speed between $t = 0$ and $t = 5$ minutes.

6. The graph at the right shows the velocity function of a particle moving horizontally.
   a. When does the particle move left?
   b. When is the particle's acceleration positive?
   c. When is the speed greatest?
   d. When does the particle stop for more than an instant?



7. The position at time t seconds of a pebble dropped from an initial height of 600 feet is given by $s(t) = -16t^2 + 600$.
   a. At what time will the pebble hit the ground?
   b. What is the pebble's velocity when it hits the ground?
   c. What is the pebble's speed when it hits the ground?

Find $f'(x)$ without using a calculator.

8. $f(x) = 2x - \dfrac{3}{x^3}$ 9. $f(x) = (2x+3)^2$ 10. $h(t) = 2\sin t - 3\ln t$ 11. $y = 2e^x + \ln x$

Evaluate the derivative of $f(x)$ at the indicated point without using a calculator.

12. $f(x) = 2x\sqrt{x}$ at $(4,16)$ 13. $f(x) = \sqrt[3]{x^2}$ at $(-8,4)$ 14. $f(x) = e\cos x - 9e^x$ at $x = 0$

15. If $y = x(x-2)$ find $\dfrac{d^2y}{dx^2}$.

16. Find an equation of a line tangent to the graph of $f(x) = 2x^4 - 3x^3$ when $x = 1$ without using a calculator.

17. Find a <u>point</u> on the graph of $f(x) = x^4 + 3$ where a tangent line has a slope of -4 without using a calculator

18. Find the x-value(s) where the function $f(x) = \begin{cases} \dfrac{x}{x^2-1}, & x \le 2 \\ x-1, & x > 2 \end{cases}$ is <u>not</u> differentiable. Give a reason for each x-value.

19. Use the limit definition of the derivative to find $f'(x)$ if $f(x) = 3x^2 - x$.

20. If $f(x) = x^3 + 5$, find the instantaneous rate of change at $x = 1$.

21. If $f(x) = x^3 + 5$, find the average rate of change between $x = 0$ and $x = 2$.

LESSON 2.5 PRODUCT AND QUOTIENT RULES

Product Rule: $\frac{d}{dx}\left(f(x)\bullet g(x)\right)=f(x)g'(x)+g(x)f'(x)$ or $\boxed{\frac{d}{dx}(f\bullet s)=fs'+sf'}$

Quotient Rule: $\frac{d}{dx}\frac{f(x)}{g(x)}=\frac{g(x)f'(x)-f(x)g'(x)}{\left(g(x)\right)^2}$ or $\boxed{\frac{d}{dx}\frac{t}{b}=\frac{bt'-tb'}{b^2}}$

Examples: Differentiate without using a calculator.

1. $f(x)=(3x^2-2)(2x+3)$

2. $y=\dfrac{2x^2-4x+3}{2-3x}$

3. $y=\dfrac{-9}{5x^2}$

4. $g(x)=3e^x\sin(x)$

ASSIGNMENT 2.5

1. Use the Product Rule to differentiate. Simplify your answer. $f(x)=(x^2-2)(4x+3)$
2. Differentiate without using the Product Rule. $f(x)=(x^2-2)(4x+3)$
3. Use the Quotient Rule to differentiate. $f(x)=\dfrac{2x+1}{x^2+2}$
4. Differentiate without using the Quotient or Product Rules. $f(x)=\dfrac{x^2-4}{x+2}$

Differentiate by any method you wish without using a calculator.

5. $y=\dfrac{12x^2-4}{4}$

6. $f(t)=\dfrac{1}{t^2}(t^3-t^2)$

7. $g(x)=2(x^2+5x-3)$

8. $f(x)=\dfrac{2x-3}{3x-2}$

9. $y=\sqrt{x}(x+1)$

10. $f(x)=\dfrac{\cos x}{x^3}$

11. $y=\dfrac{2(1-\sin x)}{3\cos x}$

12. $f(x)=\dfrac{x^2-c}{x^2+c}$, c is a constant

13. Find the second derivative of $y = \frac{4x^3}{3}$.

14 Find an equation of the tangent line to the graph of $f(x) = \frac{x}{x+1}$ at the point $(-2, 2)$.

15. Find the x-coordinate(s) of point(s) at which the graph of $g(x) = (2x-1)(x^2+3)$ has a tangent line parallel to the graph of $y = 6x+1$.

16. Find an equation of the line tangent to the graph of $h(x) = 2x\sin x$ at $x = \frac{\pi}{2}$.

17. Find the average rate of change of $f(x) = \frac{1}{x+1}$ on the interval $[0, 3]$.

18. Find the rate of change of $f(x) = \frac{1}{x+1}$ when $x = 2$.

19. Find the average rate of change of $h(x) = \cos(2x)$ on the interval $\left[-\frac{\pi}{8}, \frac{\pi}{8}\right]$.

20. Given $g(x) = x \bullet f(x)$.
    a. Use the product rule to find $g'(x)$.
    b. If $f(2) = 3$ and $f'(2) = -2$, find $g'(2)$.

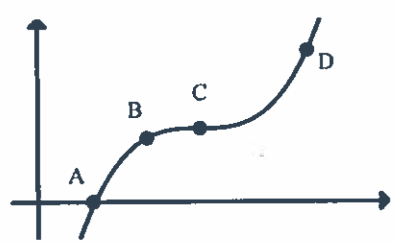
21. If $h(x) = \frac{f(x)}{x^2}$, $f(2) = 3$, and $f'(2) = -2$, find $h'(2)$.

22. The volume formula of a cube is $V = s^3$. Find the rate of change of the volume with respect to the side length when $s = 4$.

23. A particle moves horizontally according to the equation $s = t^2 - 5t + 4$. When is the particle moving left?

24. Which is greater – the average rate of change between points A and B or the instantaneous rate of change at B?

25. Copy the figure on your own paper and sketch a tangent line which intersects the curve between points C and D whose slope is equal to the average rate of change between C and D.



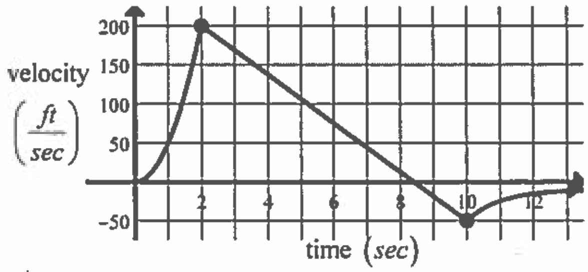
26. The height in feet of a rock thrown vertically on the moon is given by the equation $h = -\frac{27}{10}t^2 + 27t + 6$ (where time is measured in seconds). When does the rock reach its greatest height? How high is it?

27. If $f(x)$ is an n^{th} degree polynomial, find $f^{(n+1)}(x)$.

28. True or False? If $y = f(x)g(x)$, then $y' = f'(x)g'(x)$.

29. Find the total distance (in meters) traveled between $t = 0$ and $t = 4$ seconds by a particle whose position equation is $s(t) = \frac{1}{4}t^4 - 3t^3 + 4t^2 - 4$.

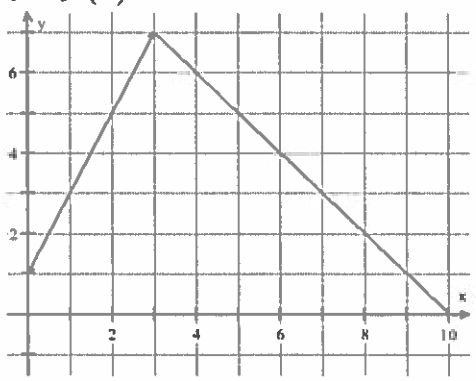
30. A model rocket is fired straight upward. The engine burns for two seconds. The rocket continues to coast upward then starts to fall. A parachute is released ten seconds after the launch. The graph shows the rocket's velocity.



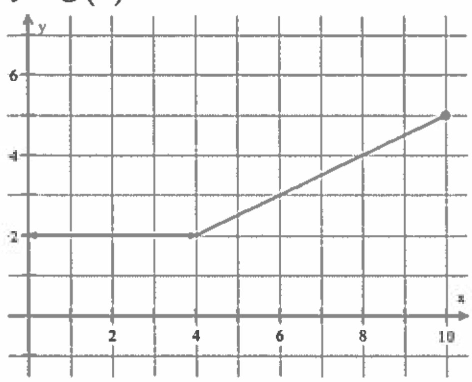
a. What was the rocket's greatest velocity?
b. Estimate the velocity at time 7 *sec.*
c. Was the rocket moving upward or downward at time 7 *sec.*?
d. Estimate the rocket's speed when the parachute was released.
e. Estimate when the rocket started to fall.
f. Estimate when the acceleration was the greatest.
g. When was the acceleration constant? Estimate the value of this constant acceleration.

31. $f(x)$ and $g(x)$ are piecewise linear functions graphed below.

$y = f(x)$



$y = g(x)$



a. If $h(x) = f(x) \cdot g(x)$ find $h'(2)$.

b. If $j(x) = \dfrac{f(x)}{2g(x)}$ find $j'(6)$.

LESSON 2.6 DERIVATIVES OF $\tan x$, $\cot x$, $\sec x$, and $\csc x$

Use the trig identities $\sin^2 x + \cos^2 x =$ $\tan x = \frac{\quad}{\quad}$ $\sec x = \frac{\quad}{\quad}$

and the quotient rule to derive formulas for:

$\frac{d}{dx}\tan x =$ $\frac{d}{dx}\sec x =$

Derivations for derivative formulas for $\cot x$ and $\csc x$ are very similar.

Derivatives of Trig Functions:

$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\cos x = -\sin x$
$\frac{d}{dx}\tan x = \sec^2 x$	$\frac{d}{dx}\cot x = -\csc^2 x$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\frac{d}{dx}\csc x = -\csc x \cot x$

From now on these formulas can be used making it unnecessary to use the quotient rule process for these simple functions. By far the most common errors using them involve incorrect signs. Can you see a quick way to remember which derivatives need a negative sign?

Example1: Find the indicated derivative.

$\frac{d}{dx}(4\cot(x)) =$

Examples: Differentiate the following without using a calculator.

2. $y = e^x \tan(x)$ 3. $f(t) = \frac{\csc t}{t^2}$ 4. $g(\theta) = \sec\theta \bullet \ln\theta$

Example 5: Without a calculator, find the slope of the graph of $y = -3\tan(x)$, where:

a. $x = \frac{\pi}{4}$ b. $x = \frac{-\pi}{3}$

ASSIGNMENT 2.6

Differentiate in Problems 1-12 without using a calculator.

1. $f(x)=x-3\sec x$
2. $y=3\cos t\cot t$
3. $g(x)=\dfrac{\csc x}{x}$
4. $h(\theta)=5\tan\theta-\theta^2+7$
5. $f(x)=\ln x \bullet \cot x$
6. $y=\csc x-\sec x$
7. $f(y)=\dfrac{y^2-\sec y}{y^3}$
8. $h(x)=\sin x+\pi x^2$
9. $f(\theta)=\dfrac{e^\theta}{\sin\theta}$
10. $y=3\sqrt[3]{x}+8$
11. $f(x)=\dfrac{3x^2-2x}{5x^2}$
12. $f(t)=2t(2t-3)^2$

Find the indicated derivative value without using a calculator. Simplify.

13. $g(x)=x\cos x$ find $g'\left(\frac{\pi}{4}\right)$
14. $f(x)=x^2+5x-\tan x$ find $f'(0)$
15. $f(x)=\dfrac{x^2-9}{x-2}$ find $f'(1)$
16. $h(t)=\frac{\sec t}{t^2}$ find $h'(\pi)$

For Problems 17-20, write an equation for a line tangent to each curve at the given point. Do not use a calculator.

17. $f(x)=-\cot(x)$ contains the point $\left(\frac{\pi}{6},-\sqrt{3}\right)$
18. $f(x)=2\tan x$ at the point $\left(-\frac{\pi}{4},-2\right)$.
19. $y=2\sec x$ at the point where $x=\frac{\pi}{4}$
20. $g(x)=3x\tan(x)$ at the point where $x=\frac{\pi}{4}$

21. Use a calculator to write an equation for the line tangent to the graph of $f(x)=\ln(|\cos x|+2)$ at the point where $x=.821$.

22. Find the average rate of change of $f(x)=\dfrac{x}{\sin x}$ on the interval $\left[\frac{\pi}{6},\frac{\pi}{2}\right]$ without using a calculator.

23. Find the rate of change of $f(x)=\dfrac{x}{\sin x}$ when $x=\frac{\pi}{6}$ without using a calculator.

24. Find the x-coordinate(s) of point(s) at which the graph of $f(x)=\dfrac{x^2}{x+1}$ has a horizontal tangent line.

25. If $f(x)=\dfrac{g(x)}{h(x)}$, $g(3)=2$, $g'(3)=3$, $h(3)=4$, and $h'(3)=5$, find $f'(3)$.

Use the function $g(x)=\begin{cases} x-1, & x\le 0 \\ x^2-1, & 0<x<2 \\ 4, & x\ge 2 \end{cases}$ for Problems 26-31.

26. Sketch a graph of $g(x)$.

Find the following limits.

27. $\lim\limits_{x\to 0} g(x)$ 28. $\lim\limits_{x\to 2} g(x)$ 29. $\lim\limits_{x\to 2^-} g(x)$ 30. $\lim\limits_{x\to 1} g(x)$

31. List all discontinuities of $g(x)$.

ASSIGNMENT 2.7 REVIEW

Find the derivatives of these functions.

1. $f(x)=x^4-3x^3+x-2$
2. $f(x)=x^{\frac{1}{3}}-x^{\frac{-1}{3}}$
3. $f(t)=\dfrac{3}{4t^3}$
4. $y=\dfrac{\tan t}{t}$
5. $h(\theta)=2\theta\cos\theta-\sin\theta$
6. $f(x)=\dfrac{-2x^2}{x+1}$
7. $y=\dfrac{x^3-x^2}{x^2}$
8. $f(x)=-3(2-3x)^2$
9. $g(t)=-4\cot t$

10. If $f(x)=x^2+1$, use the limit definition of the derivative to find $f'(x)$.
11. If $f(x)=x^2-2x$, use the alternate form of the limit definition of the derivative to find $f'(3)$.
12. If $y=\dfrac{1}{x^2}+\dfrac{1}{2x}$ find $\dfrac{d^2y}{dx^2}$.
13. If $f(x)=3e^x\sec x$ find an equation of the line tangent to the graph of f when $x=0$.
14. Find an equation of the tangent line to the graph of $f(x)=4\ln x$ when $x=e^2$.
15. Find an equation of the tangent line to the graph of $y=x\sin(x)$ at the point $(\pi,0)$.

16. Use the values given in the table to find the following.

t	$r(t)$	$r'(t)$	$s(t)$	$s'(t)$
3	−2	3	2	5

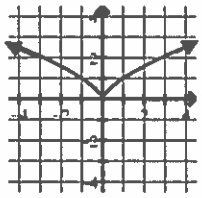
a. $\frac{d}{dt}(2s(t)+4r(t))$ at $t=3$ b. $\frac{d}{dt}(r(t)\bullet s(t))$ at $t=3$ c. $\frac{d}{dt}\dfrac{r(t)}{s(t)}$ at $t=3$

17. The position function of a particle moving horizontally along the x-axis is $x(t)=2t^3-3t^2+2$.
    a. Find the velocity function.
    b. Find the acceleration function.
    c. When is the particle at rest?
    d. When is the particle moving left?
    e. When is the particle's velocity equal to 12 feet per second?
    f. Find the particle's speed when the acceleration is equal to zero.
    g. Find the particle's displacement from $t=-1$ to $t=3$ seconds.
    h. Find the total distance traveled by the particle between $t=-1$ and $t=3$ seconds.

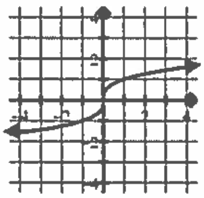
18. $f(x)=4^x-3x$ (Use a calculator for these problems.)
    a. Find $f'(3)$. b. Find $f''(1)$. c. Graph $f'(x)$.

19. If $f(x)=\dfrac{4}{(3x)^2}$, find:
    a. the average rate of change of $f(x)$ from $x=1$ to $x=2$.
    b. the instantaneous rate of change of $f(x)$ at $x=1$.

20. The graph of $f(x)=x^{\frac{2}{3}}$ is shown.
    a. Is f continuous at $x=0$?
    b. Is f differentiable at $x=0$? If not, why not?



21. The graph of $f(x)=x^{\frac{1}{3}}$ is shown.
    a. Is f continuous at $x=0$?
    b. Is f differentiable at $x=0$? If not, why not?



22. $f(x)=\begin{cases} x^2+1, & x\le 0 \\ x^2+x+1, & x>0 \end{cases}$
    a. Is f continuous at $x=0$?
    b. Is f differentiable at $x=0$? If not, why not?

23. $f(x)=\begin{cases} x^2+1, & x\le 0 \\ -x^2+2, & x>0 \end{cases}$
    a. Is f continuous at $x=0$?
    b. Is f differentiable at $x=0$? If not, why not?

24. $f(x)=\begin{cases} x^2+1, & x\le 0 \\ -x^2+1, & x>0 \end{cases}$
    a. Is f continuous at $x=0$?
    b. Is f differentiable at $x=0$? If not, why not?

25. Without using a calculator, find the <u>points</u> where the graph of $f(x)=x^3-3x^2+3$ has horizontal tangents.

UNIT 2 SUMMARY

Limit Definition of the Derivative: $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

Alternate Form of the Limit Definition of the Derivative:
(Gives the value of the derivative at a single point.) $f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$

Power Rule: $\frac{d}{dx}x^n=nx^{n-1}$

Equation of a Tangent Line: Use the derivative to find m. $y-y_1=m(x-x_1)$

Nondifferentiability: (where the derivative does not exist)

1. Holes
2. Jumps (breaks)
3. Vert. Asymptotes
4. Sharp Turns
5. Vertical Tangent Lines

Average Rate of Change: (the slope between two points) AROC = $\frac{y_2-y_1}{x_2-x_1}$

Instantaneous Rate of Change: (slope at a single point) IROC = $f'(c)$

Pos.→Vel.→Acc. (differentiate)
Speed (the absolute value of velocity)
Displacement (the net change in position = final position – original position)
Total Distance (consider all direction changes → make a velocity number line)

Product Rule: $\boxed{\frac{d}{dx}(f\bullet s)=fs'+sf'}$

Quotient Rule: $\boxed{\frac{d}{dx}\frac{t}{b}=\frac{bt'-tb'}{b^2}}$

Calculator Derivative: $f'(c)=\left.\frac{d}{dx}f(x)\right|_{x=c}$ found in the math menu on a TI84

Derivatives of Trig Functions:

$\frac{d}{dx}\sin x=\cos x$

$\frac{d}{dx}\cos x=-\sin x$

$\frac{d}{dx}\tan x=\sec^2 x$

$\frac{d}{dx}\cot x=-\csc^2 x$

$\frac{d}{dx}\sec x=\sec x\tan x$

$\frac{d}{dx}\csc x=-\csc x\cot x$

Other Derivatives:

$\frac{d}{dx}e^x=e^x$

$\frac{d}{dx}\ln x=\frac{1}{x}$

LESSON 3.1 THE CHAIN RULE

Discovery Example: Use the product rule to differentiate.

$\frac{d}{dx}\left(x^3-3\right)^2=\frac{d}{dx}\left(\left(x^3-3\right)\left(x^3-3\right)\right)$

This can be generalized to the following rule.

Chain Rule: (used to differentiate any composition of functions)

$\frac{d}{dx}f\left(g(x)\right)=f'\left(g(x)\right)g'(x)$ or in another form $\frac{d}{dx}f(u)=f'(u)u'$ where u is a function of x.

Examples: Differentiate.

1. $f(x)=\left(3x^3-5x\right)^4$
2. $y=\sin(3x)$
3. $g(x)=\sqrt{\left(2x^2-x\right)^3}$
4. $h(\theta)=3\cos^2\theta$
5. $g(x)=\dfrac{1}{2x+1}$
6. $f(x)=3x^2\sqrt[3]{9-4x^2}$
7. $f(t)=\sin^3\left(4t^2\right)$
8. Given this data find the following:

x	$p(x)$	$q(x)$	$p'(x)$	$q'(x)$
2	5	3	7	6
3	4	$\frac{1}{2}$	8	$\frac{3}{2}$

a. If $f(x)=p(x)\bullet q(x)$ find $f'(2)$.

b. If $h(x)=p\left(q(x)\right)$ find $h'(2)$.

Note: You must quickly learn to distinguish between the Chain Rule and the Product Rule!

ASSIGNMENT 3.1

Find the derivative without using a calculator.

1. $y=(3x+5)^3$
2. $f(x)=3(7x+5)^4$
3. $y=\sqrt{2-3x}$
4. $f(t)=\dfrac{1}{(1-t)^2}$
5. $y=\sqrt[3]{(x^2+1)^2}$
6. $g(x)=x(2x+3)^3$
7. $y=\dfrac{1}{\sqrt{x+1}}$
8. $f(x)=\dfrac{3x-2}{x+1}$
9. $g(x)=\sec(4x)$
10. $y=4\tan(2x)$
11. $f(\theta)=\frac{1}{2}\sin^2(3\theta)$
12. $y=\sqrt{\dfrac{4x^3-2x}{2x}}$

Find an equation of the line tangent to the graph of f at the given point without using a calculator.

13. $f(x)=\sqrt{2x^2+2}$ at $(-1,2)$
14. $f(x)=\dfrac{x+4}{x}$ at $(2,3)$
15. $f(x)=\dfrac{1}{\sqrt{(9x)^3}}$ at $\left(\frac{1}{4},\frac{8}{27}\right)$
16. $f(x)=\dfrac{1}{x^2}+\sqrt{\cos x}$ at $\left(2\pi,\dfrac{1}{4\pi^2}+1\right)$

Find the indicated derivatives.

17. $\frac{d}{dx}(2\sin x-3)^4$
18. $\dfrac{d^2}{dt^2}(t^2-1)^{\frac{3}{2}}$

19. Find the point(s) at which a line tangent to the graph of $f(x)=(2x-3)^3$ is parallel to the graph of $y=24x-7$. You may use a calculator.

20. If $g(x)=(f(x))^3$, $f(1)=2$, and $f'(1)=4$, find $g'(1)$.

21. Given these values

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	3	2	-1	4
3	-2	$\frac{1}{2}$	6	5

find the following derivatives.

a. $\frac{d}{dx}g(f(x))$ at $x=2$

b. $\frac{d}{dx}(g(x)f(x))$ at $x=2$

c. $\frac{d}{dx}\sqrt{g(x)}$ at $x=2$

d. $\frac{d}{dx}\dfrac{g(x)}{f(x)}$ at $x=2$

22. Find an equation of the line tangent to the graph of $y=3^x$ when $x=1.2$. You will need a calculator.

23. If $f(x)=5x^2-3x+2$, use the alternate form of the limit definition of the derivative to find $f'(1)$. Show steps with correct limit symbolism.

24. The position function of an object is $s(t)=\frac{2}{3}t^3-\frac{5}{2}t^2+3$. Find the **displacement** of the object between time $t=0$ and time $t=3$. Find the **total distance** traveled by the object between time $t=0$ and time $t=3$.

LESSON 3.2 CHAIN RULE WITH EXPONENTIALS AND LOGS INCLUDING BASES OTHER THAN *e*

Applying the chain rule to exponential and logarithmic functions gives us the following formulas.

Given u is a function of x.

Since $\frac{d}{dx}e^x = e^x$, it follows that $\frac{d}{dx}e^u = e^u u'$.

Since $\frac{d}{dx}\ln x = \frac{1}{x}$, it follows that $\frac{d}{dx}\ln u = \frac{1}{u}u'$ or more simply $\frac{d}{dx}\ln u = \frac{u'}{u}$.

Using the inverse relationship of logs and exponentials we can derive a formula for differentiating exponentials with bases other than e.

$\frac{d}{dx}a^x = \frac{d}{dx}e^{\left(\ln a^x\right)}$ inverse property of exponentials and logs

$= \frac{d}{dx}e^{(x\ln a)}$ log property

$=$ chain rule derivative

$=$ same log property in reverse

$=$ same inverse property of exponentials and logs in reverse

$\frac{d}{dx}a^x = a^x \ln a$ and the chain rule form $\frac{d}{dx}a^u = a^u u' \ln a$

Using the change of base formula $\log_a x = \dfrac{\ln x}{\ln a}$ we can derive a formula for differentiating logs with bases other than e.

$\frac{d}{dx}\log_a x = \frac{d}{dx}\frac{\ln x}{\ln a}$ change of base formula

$= \frac{1}{\ln a}\frac{d}{dx}\ln x$ since $\ln a$ is a constant

$=$ derivative of $\ln x$

$\frac{d}{dx}\log_a x = \dfrac{1}{x\ln a}$ and the chain rule form $\frac{d}{dx}\log_a u = \dfrac{u'}{u\ln a}$

Examples:

Differentiate each of the following.

1. $f(x) = e^{x^2+3x}$

2. $g(x) = \ln(3x)$ use $\frac{d}{dx}\ln u = \frac{u'}{u}$

3. $g(x) = \ln(3x)$ use $\ln(3x) = \ln 3 + \ln x$

4. $y = \ln\left(t^2 + t\right)$

5. $h(x) = x \ln x$ 6. $g(t) = e^{\frac{-3}{t}}$ 7. $f(v) = 3^{\sqrt{v}}$

Example 8:

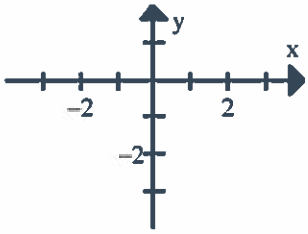
a. Graph $y = \ln|x|$ in the coordinate plane at right.

b. For the right half of the graph the absolute value has no effect, so $\frac{d}{dx}\ln|x| = \frac{1}{x}$ for $x > 0$.

What is $\frac{d}{dx}\ln|x|$ when $x < 0$?

Plot some slopes and think about it.

$\frac{d}{dx}\ln|x| =$ for $x < 0$.



When differentiating the natural log function, $\ln|x|$ and $\ln|u|$ the absolute value can be ignored. In these cases, absolute value may change the domain of the function – but not the derivative.

Example 9:

Find $\frac{d}{dy}\ln|5 - 2y^3|$

When possible, expand logarithmic functions before differentiating them.

Example 10: Differentiate $y = \ln\frac{x\sqrt{2x+1}}{x^2+1}$.

First, rewrite as $y =$

Then, $y' =$

Example 11: If $y = \log_2(x^2 + 1)$, find $y'(2)$

ASSIGNMENT 3.2

Differentiate in Problems 1-17 without using a calculator.

1. $f(x)=e^{5x+1}$
2. $g(y)=2^{-5y}$
3. $h(t)=5^{t^2-2t}$
4. $y=\ln x^5$
5. $y=e^{\sqrt{x}-2}$
6. $y=e^{2\sin x}$
7. $y=\ln\left(x^2-5x\right)$
8. $f(x)=2x^2e^{3x}$
9. $g(x)=\dfrac{e^x}{\sin x^2}$
10. $g(y)=(\ln y)^4$
11. $y=\ln\dfrac{x^2}{x-1}$
12. $h(x)=(e^{-2x}-1)^3$
13. $f(x)=\ln\left|x^3+2x\right|$
14. $f(x)=\log_5\left|x^2-1\right|$
15. $g(t)=\ln\sqrt{t^3-t}$
16. $h(x)=\tan x\ln x$

17. For $y=\dfrac{e^{-3x}}{3}+2e^{2x}$, find $\dfrac{d^2y}{dx^2}$.

18. Write an equation for the line tangent to the graph of $f(x)=e^{x^2-4x+3}$ at the point where $x=1$ without using a calculator.

19. Find an equations for the tangent line to $y=x^2-\ln(x+1)+1$ at the point $(0, 1)$ without using a calculator.

20. The height (position) in feet of a ball thrown straight down from a tall building is given by $h=-16t^2-22t+220$ (where t is measured in seconds).
    a. What is the ball's initial height?
    b. What is the ball's initial velocity?
    c. What is the speed of the ball 3 seconds after it was thrown?
    d. How far does the ball travel in the first 3 seconds?

21. A sprinter in a 100 meter dash is clocked every 10 meters as shown in the table.

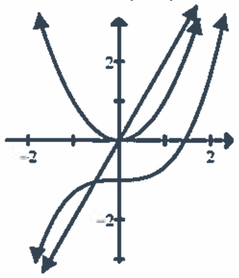
meters covered	0	10	20	30	40	50	60	70	80	90	100
time in seconds	0	1.4	2.5	3.5	4.5	5.4	6.6	7.6	8.5	9.3	10.2

    a. How long does it take the sprinter to finish the race?
    b. What is the sprinter's average speed over the first 50 meters?
    c. What is the sprinter's average speed over the second 50 meters?
    d. What is the sprinter's average speed between 20 and 30 meters?
    e. What is the sprinter's average speed between 30 and 40 meters?
    f. What is the sprinter's approximate speed as he passes the 30 meter mark?
    g. During which ten meter segment of the race is the sprinter running the fastest?
    h. During which portion of the race is the sprinter accelerating the fastest?

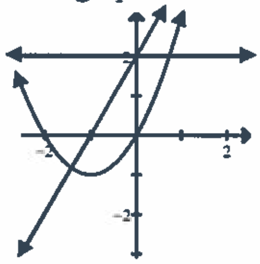
22. A particle is moving vertically on the y-axis with position $y(t)=\frac{1}{3}t^3+t^2-3t+7$. When is the particle moving upward? Do not use a calculator.

The functions f, f', and f'' are shown on the same graph. Determine which is which.

23.



24.



Use the function $g(x)=\begin{cases} 3x-1, & x\le 0 \\ \frac{4}{27}x^3-1, & 0<x<3 \\ 4, & x\ge 3 \end{cases}$ for Problems 25-32.

25. Sketch a graph of $g(x)$.

Find the following limits.

26. $\lim\limits_{x\to 0} g(x)$ 27. $\lim\limits_{x\to 3} g(x)$ 28. $\lim\limits_{x\to 3^-} g(x)$ 29. $\lim\limits_{x\to \frac{3}{\sqrt[3]{4}}} g(x)$

30. List all discontinuities of $g(x)$.

31. Find $g'(x)$

32. List all x-values at which g is **not** differentiable.

<u>LESSON 3.3</u> <u>IMPLICIT DIFFERENTIATION</u>

All the derivatives you have done to this point have been of <u>explicit equations</u>. For example $y=x^2$, $y=\dfrac{x}{x-1}$, and $y=\sqrt{2x+1}$ all <u>explicitly</u> express y in terms of x.

In this lesson you will be working with <u>implicit equations</u> where the relationship between x and y is only <u>implied</u>. $x^2+y^2=1$, $xy+y^2=3$, and $xy=1$ are all examples of <u>implicit</u> equations.

It is possible to differentiate implicit equations using **<u>implicit differentiation</u>**.

<u>Procedure:</u>

1. Differentiate both sides with respect to x.
   (Remember the y' "chain rule factor" for any term involving y.)
2. Collect all y' (or $\frac{dy}{dx}$) terms on one side of the equation.
3. Factor out y'.
4 Divide to solve for y'.

<u>Warm-up Examples:</u> Differentiate.

1. $y=x$ 2. $y=x^2$ 3. $y=(2x-1)^2$ 4. $y=(f(x))^2$ 5. $x=y^2$

Examples:

1. Given $x^2-2y^3+3x=6$, find y'.

2. Find the slope of the line tangent to the graph of $x^2+4y^2=25$ at (3,2).

3. Given $x^3-2xy+y^3=5x$, find $\frac{dy}{dx}$ and evaluate at the point (1,2).

4. Given $x^2+y^2=3$, find y'' in terms of y.

5. Given $\cot y=x-y$ find $\frac{dy}{dx}$.

ASSIGNMENT 3.3

Use implicit differentiation to find y'.

1. $x^2+y^2=4$
2. $xy=7$
3. $\sqrt{x}+\sqrt{y}=9$
4. $\cos x-2\sin(2y)=4$
5. $y^2=\dfrac{x}{x+1}$
6. $x^2y^3+y=x^3$
7. $\tan(xy)=y^2$

Find $\frac{dy}{dx}$ and evaluate the derivative at the given point.

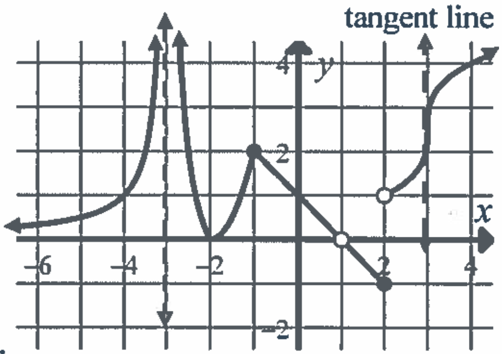
8. $x^3-xy=3$ at $(1,-2)$
9. $e^y+\ln\left(\frac{1}{2}y\right)-x^2=0$ at $(e,2)$
10. $x^3+y=2xy$ at $(1,1)$
11. $\sec(x+y)=x+1$ at $(0,0)$

12. Use implicit differentiation to find y' and evaluate the derivative at the point $(3,-4)$ if $x^2+y^2=25$.
13. Use <u>explicit</u> differentiation (the old way) to find y' and evaluate the derivative at the point $(3,-4)$ if $y=-\sqrt{25-x^2}$.
14. Given $x^2+xy=4$, find $\dfrac{d^2y}{dx^2}$ in terms of x and y.
15. Find an equation of the line tangent to the circle $x^2+y^2=169$ at the point $(-12,5)$.
16. Find the <u>point(s)</u> at which the graph of $x^2+2y^2-4y-6=0$ has a <u>horizontal</u> tangent line.
17. Find the <u>point(s)</u> at which the graph of $x^2+2y^2-4y-6=0$ has a <u>vertical</u> tangent line.

18. If f is an unknown differentiable function of y where $f(3)=0$ and $f'(3)=2$ and the ordered pair (1,3) is a point on the curve $\dfrac{f(y)}{x}=x^3-y^2+8$ find $\left.\frac{dy}{dx}\right|_{(1,3)}$.

19. The volume formula for a sphere is $V=\frac{4}{3}\pi r^3$. Find the rate of change of the volume with respect to the radius $\left(\frac{dV}{dr}\right)$, when the radius is 4.

Use the figure at the right for Problems 20-27.



20. Find $\lim\limits_{x\to 1} f(x)$.
21. Find $\lim\limits_{x\to -3} f(x)$.
22. Find $\lim\limits_{x\to -\infty} f(x)$.
23. Find $\lim\limits_{x\to 2} f(x)$.
24. Find $\lim\limits_{x\to 2^+} f(x)$.
25. List the x-values of all discontinuities of $f(x)$.
26. Which of these discontinuities are removable?
27. List the x-values where $f(x)$ is <u>not</u> differentiable.

28. An object's velocity in meters per second is $v=2t^3-9t^2+12t-5$. Find the object's speed each time the acceleration is zero.

29. Find $\dfrac{d}{dx}\left(\dfrac{x^2-2x}{x^2}\right)$ without using a calculator.

30. If $f(x)=3^{x^2}-\log_3\left(x^2+x\right)$ find $f'(x)$ without using a calculator.

LESSON 3.4 RELATED RATES

In related rate story problems, the idea is to find a rate of change (with respect to time) of one quantity by using the rate of change (with respect to time) of a related quantity.

Procedure for Related Rate Problems

1. Draw a figure (if necessary) and choose variables for all unknowns.
2. Write what is given and what is to be found using your variables and $\frac{d}{dt}$ symbols.
3. Write an equation relating the variables.
   (a) If a quantity is **changing** it must be represented with a variable letter.
   (b) If a quantity is **constant** it must be represented with a number value.
   (c) Look for secondary relationships between quantities to reduce the number of variables.
4. Implicitly differentiate both sides with respect to t.
5. Substitute number values and solve.

Geometry Formulas

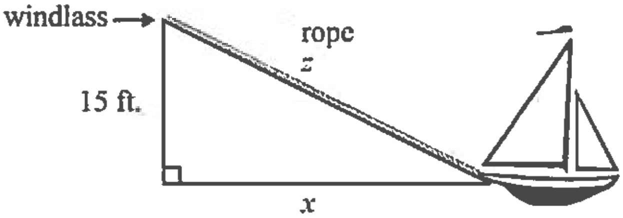
Right Triangle	Circle	Sphere	Cube	Cone	Rectangle
$a^2+b^2=c^2$	$A=\pi r^2$	$V=\frac{4}{3}\pi r^3$	$V=e^3$	$V=\frac{1}{3}\pi r^2 h$	$A=lw$
$A=\frac{1}{2}bh$	$C=2\pi r$	$A=4\pi r^2$	$A=6e^2$		$P=2l+2w$

Examples:

1. In this problems, the first three steps of the procedure are done. We need to complete only the last two steps.

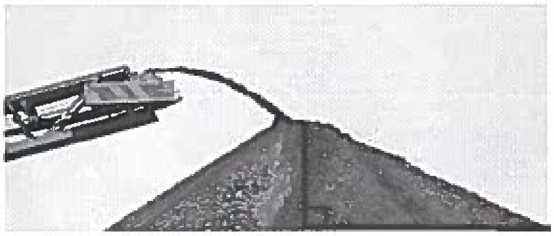
   Given: $\frac{dx}{dt}=-6,\ y=4,\ xy=12$ Find: $\frac{dy}{dt}$

2. A windlass is used to tow a boat to a dock. The rope is attached to the boat at a point 15 feet below the level of the windlass. If the windlass pulls in rope at the rate of 30 feet per minute, at what rate is the boat approaching the dock when there is 25 feet of rope out? Write a sentence explaining the meaning of the answer.



3. A policeman traveling south toward an intersection spots a speeding car traveling east away from the intersection. When the policeman is .6 mi from the intersection and the car is .8 mi from the intersection, the policeman's radar shows the distance between them is increasing at the rate of 20 mph. If the speed of the police car is 60 mph, what is the speed of the car? Write a sentence explaining the meaning of the answer.

4. Gravel is falling on a conical pile at the rate of $10\,\frac{ft^3}{\min}$. At all times, the radius of the cone is twice the height of the cone. Find the rate of change of the height of the pile when the radius of the pile is $6\,ft$. $V=\frac{1}{3}\pi r^2 h$ Write a sentence explaining the meaning of the answer.



ASSIGNMENT 3.4

1. Given: $\frac{dx}{dt}=3,\ x=4,\ y=x^2-2x$ Find: $\frac{dy}{dt}$

2. Given: $\frac{dx}{dt}=10,\ y=5,\ x^2+y^2=169$ Find: $\frac{dy}{dt}$

Write sentences explaining the meaning on problems 3-12.

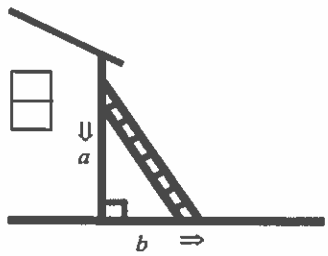
3. The radius of a circle is increasing at the rate of $3\,{}^{cm}\!/_{sec}$. At the instant the radius is 4 *cm*., find:
   a. the rate of change of the area of the circle.
   b. the rate of change of the circumference of the circle.

4. A large spherical balloon is inflated at the rate of $18\frac{ft^3}{min}$. When the radius is 2 feet, how fast is the radius changing? $V = \frac{4}{3}\pi r^3$

5. All edges of a cube are expanding at the rate of $2\frac{in}{sec}$. At the instant the edges are all 5 inches long find:
   a. the rate of change of the surface area of the cube.
   b. the rate of change of the volume of the cube.

6. A point is moving on the curve $y = \sqrt{x}$ so that the x-coordinate is changing at the rate of $3\,{}^{cm}\!/_{min}$. When the y-coordinate is 25 centimeters, find the rate of change of the y-coordinate.

7. A 10 foot ladder is leaning against a house as shown. The base of the ladder is pulled away from the house at the rate of $1.5\ \frac{ft}{sec}$. At the instant the base of the ladder is 6 feet from the house, answer these questions.



   a. Find the rate at which the top of the ladder is moving.
   b. Find the rate at which the area of the triangle formed is changing.
   c. Is the area from part b increasing or decreasing? Would your answer be the same at all times?
   d. Let $\theta =$ the angle formed by the ladder and the house, find the rate of change of θ with respect to t.

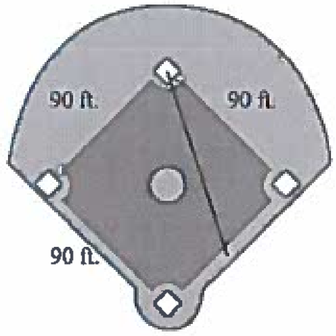
8. A rectangle is formed whose length is twice the width. It is enlarged to a similar (same shape) rectangle as the width changes at the rate of 4 inches per minute. When the width is 10 inches, how fast is the area of the rectangle changing?

9. Sand is added to a conical pile at the rate of $8\frac{ft^3}{\min}$. If the diameter of the cone is twice the height of the cone, find the rate of change of the height of the pile when the volume of the pile is 1125π cubic feet. $V = \frac{1}{3}\pi r^2 h$

10. A bicyclist and a jogger are moving on two perpendicular intersecting streets. The bicyclist is moving north toward the intersection at the rate of 60 feet per second and the jogger is moving west away from the intersection at the rate of 15 feet per second. What is the rate at which the straight line distance between them is changing when the bicyclist is 120 feet from the intersection and the jogger is 50 feet from the intersection?

11. At halftime of a football game, two fan volunteers are attached together with a bungee cord and positioned on the sideline at the 50 yard line. One fan runs along the sideline at $4\ \frac{yd}{sec}$ and the other runs on the 50 yard line toward midfield at $3\ \frac{yd}{sec}$. At what rate is the bungee cord being stretched when the sideline runner reaches the 30 yard line?

12. A base runner who is running at the rate of 25 feet per second is halfway to first base. Find the rate of change of his distance from second base.



Differentiate.

13. $f(x)=e^{x^2-5x}$

14. $y=\ln(2x+5)\sqrt[3]{3x+1}$

15. Find an equation of the tangent line to the graph of $y=\left(3x^2+2\right)^4$ when $x=0$.

16. If $f(x)=\dfrac{x^2+3}{x}$

a. find the average rate of change from $x=1$ to $x=3$.

b. find the instantaneous rate of change at $x=3$.

Answer these same three questions for Problems 17-20.

(a) What is the derivative of the function from the left at $x=1$?
(b) What is the derivative of the function from the right at $x=1$?
(c) Is the function differentiable at $x=1$?

17. $f(x)=\begin{cases}(x-1)^3, & x\le 1\\ (x-1)^2, & x>1\end{cases}$

18. $f(x)=\begin{cases}\frac{1}{2}x^2-\frac{1}{2}, & x\le 1\\ x-1, & x>1\end{cases}$

19. $f(x)=\begin{cases}\frac{1}{2}x^2-\frac{1}{2}, & x<1\\ x-1, & x>1\end{cases}$

20. $f(x)=\begin{cases}x^2-2x, & x\le 1\\ x^2-2x+1, & x>1\end{cases}$

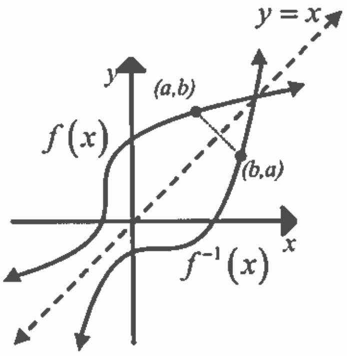
21. If $f(x)=x^2-x$, use the limit definition of the derivative to find $f'(x)$.

22. If $y=\dfrac{1}{x^2}+\sin(4x)$ find $\dfrac{d^2y}{dx^2}$.

23. Find an equation of the tangent line to the graph of $y=\ln|\sec x|$ at the point where $x=\pi$.

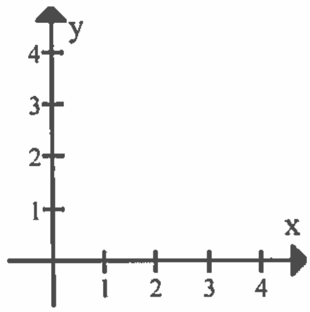
24. If $r(3)=-2$, $s(3)=2$, $r'(3)=3$, $s'(3)=5$, and $r'(2)=4$, find $\frac{d}{dt}r(s(t))$ at $t=3$

LESSON 3.5 DERIVATIVES OF INVERSE FUNCTIONS



At the right are the graphs of a function $f(x)$ and its inverse $f^{-1}(x)$. Remember that if the graph of f contains the point (a, b), then the graph of f^{-1} contains the point (b, a). Also, the graph of f^{-1} is the reflection of the graph of f across the line $y = x$.

From the graphs, do you see a relationship between the slope of the graph of f at (a, b) and the slope of the graph of f^{-1} at (b, a)?



Example 1: Let $f(x) = \sqrt{x}$.

a. Sketch the graph of $f(x)$.

b. Find $f^{-1}(x)$. Hint: You must list a domain restriction.

c. Sketch the graph of $f^{-1}(x)$ in the same coordinate plane as the graph of $f(x)$.

d. Differentiate both $f(x)$ and $f^{-1}(x)$.

e. Find the slope of the graph of $f(x)$ at (4, 2) and the slope of the graph of $f^{-1}(x)$ at (2, 4).

f. What conclusion can you make about these slopes?

Since slope $= m = \dfrac{\Delta y}{\Delta x}$, it should make sense that switching x and y (for inverse functions) should produce reciprocal slopes for inverse functions.

Derivatives of Inverse Functions:

If (a, b) is a point on f, then (b, a) is a point on f^{-1}, and $(f^{-1})'(b) = \dfrac{1}{f'(a)}$

or if f and g are inverse functions, then $g'(x) = \dfrac{1}{f'(g(x))}$.

Derivatives of inverses have reciprocal slopes at "image points" (points reflected across $y = x$).
(a, b) and (b, a) are image points.

Note: When finding derivatives of inverse functions, do not use the same x-value for both f and f^{-1}. This hardly ever works. (It only works when the x- and y-values of the ordered pairs are the same.)

Example 2: Let f and g be inverse functions such that f has the function and derivative values shown in the table.

x	$f(x)$	$f'(x)$
-1	1	$\frac{3}{2}$
0	2	2
1	5	$\frac{1}{2}$

From the given information, find each of the following if possible. The table below is a good way to organize the given information.

a. $g'(1)$ b. $g'(2)$ c. $g'(3)$

d. $g'(0)$ e. $g'(5)$

$f(x)$ points		$g(x)$ points	
(,)	$m=$	(,)	$m=$
(,)	$m=$	(,)	$m=$
(,)	$m=$	(,)	$m=$

Example 3:

If $f(3)=28$ where $f(x)=x^3-2x+7$ find $(f^{-1})'(28)$.

ASSIGNMENT 3.5

1. $f(x)=x^3-1$. Let $g(x)=f^{-1}(x)$.
   a. Find $g(x)$.
   b. Graph $f(x)$ and $g(x)$ in the same coordinate plane.
   c. Find $f'(x)$ and $g'(x)$.
   d. Find $f'(1)$ and $g'(0)$.
   e. What is the relationship between the slopes in Part d?

2. $f(x)=\sqrt{x+1}$. Let $g(x)=f^{-1}(x)$
   a. Find $g(x)$.
   b. Graph $f(x)$ and $g(x)$ in the same coordinate plane.
   c. Find $f'(x)$ and $g'(x)$.
   d. Find $f'(3)$ and $g'(2)$.
   e. What is the relationship between the slopes in Part d?

3. Let f and g be inverse functions such that: $\begin{cases} f(-1)=0,\ f(0)=1, \text{ and } f(1)=3 \\ f'(-1)=\frac{4}{3},\ f'(0)=\frac{1}{5}, \text{ and } f'(1)=2 \end{cases}$

   Find each of the following (if possible).

   a. $g'(-1)$ b. $g'(0)$ c. $g'(1)$ d. $g'(2)$ e. $g'(3)$

4. If $f(2)=3$ and $f'(2)=4$, find $\left(f^{-1}\right)'(3)$.

5. If $(1,2)$ is a point on $f(x)=x^3+2x-1$, find $\left(f^{-1}\right)'(2)$.

6. If $f(x)=x^3-\dfrac{4}{x}$ $(x>0)$, find $\left(f^{-1}\right)'(6)$.

f and g are inverse functions in Problems 7-9. Find g' at the given value.

7. $f(2)=5$
   $f'(2)=\frac{-2}{3}$
   $g'(5)=$

8. $f(x)=x^5+2x^3-1$
   $f(1)=2$
   $g'(2)=$

9. $f(x)=e^{x-2}$
   $g'(1)=$

Differentiate in Problems 10-13 without using a calculator.

10. $g(t)=-4\cot\left(3t^2\right)$
11. $y=\dfrac{\tan t}{t}$
12. $f(x)=e^{\sec x}$
13. $h(\theta)=2\theta\cos\theta-\sin\theta$

Find the indicated derivatives for Problems 14-20 without using a calculator..

14. $y=x^5+\dfrac{1}{x}$, find $\dfrac{d^2y}{dx^2}$
15. $f(x)=\dfrac{x^2}{e^x}$, find $f'(x)$
16. $g(t)=(2t-1)^5$, find $g''(t)$
17. $\frac{d}{dx}\left(6^{2x}-3\right)^4=$
18. $\frac{d}{dx}\ln\left|\sin^3 x\right|=$
19. $\sin\left(t^2\right)=ye^t$, find $\frac{dy}{dt}$
20. $\sin(y-2x)=x^2-10$, find $\frac{dy}{dx}$

21. An object moves along a vertical path with its position at time t (in seconds), according to the equation $y(t)=te^{t+1}$ (where y is measured in centimeters (cm)).
    a. Find the object's position at time $t=-2\,sec.$
    b. Find the equation for the object's velocity.
    c. Find the object's velocity at time $t=-2\,sec.$
    d. Find an equation for the object's acceleration.
    e. Find $a(-2)$.
    f. For what interval(s) of time is the object moving downward?

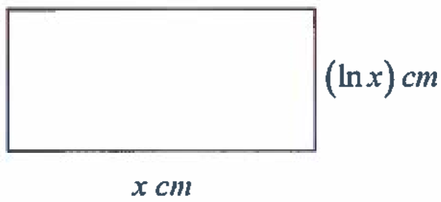
Use a calculator to evaluate problems 22-23.

22. find $f'(1.237)$ for $f(x)=\sqrt{e^x+5}$ 23. find $f''(1.237)$ for $f(x)=\sqrt{e^x+5}$

For Problems 24-27, find each limit (if it exists) without using a calculator.

24. $\lim\limits_{x\to -3} \dfrac{2x-1}{3x+4}$ 25. $\lim\limits_{t\to 2} \dfrac{t^2-4}{t^2-3t+2}$ 26. $\lim\limits_{x\to -1} \dfrac{x+1}{\sqrt{x^2+1}}$ 27. $\lim\limits_{x\to -\infty} \dfrac{x+1}{\sqrt{x^2+1}}$

28. If the area of the rectangle shown is increasing at the rate of $4\ cm^2/_{sec}$, find $\dfrac{dx}{dt}$ when $x=e\ cm$.



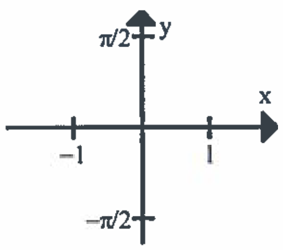
LESSON 3.6 INVERSE TRIGONOMETRIC FUNCTIONS, DIFFERENTIATING INVERSE TRIGONOMETRIC FUNCTIONS

None of the six basic trig functions is one-to-one, so none of them have an inverse function. However, we can use domain restrictions to make the trig functions one-to-one, so that they do have inverses. In this course, we will deal only with the inverse trig functions for the sine, cosine, and tangent functions.

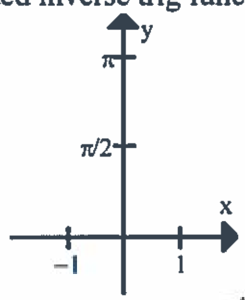
Definition of the Inverse Trig Functions:

Function	Domain (x values)	***Range (y values)**
$y=\arcsin x \leftrightarrow \sin y = x$	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$y=\arccos x \leftrightarrow \cos y = x$	$[-1,1]$	$[0,\pi]$
$y=\arctan x \leftrightarrow \tan y = x$	$(-\infty,\infty)$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

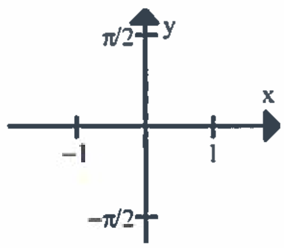
Example 1: Graph the indicated inverse trig functions in the coordinate planes below:



$y=\arcsin x=\sin^{-1} x$

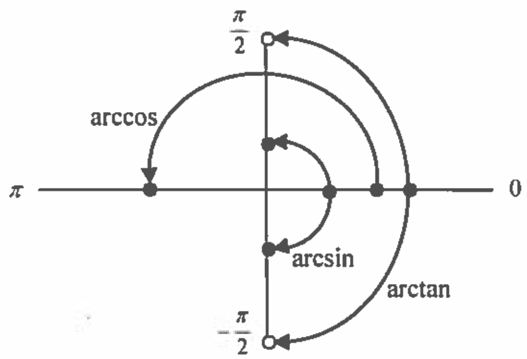


$y=\arccos x=\cos^{-1} x$



$y=\arctan x=\tan^{-1} x$

*A geometric representation of the range values for each inverse trig function is shown in the coordinate plane at right.



Remember that the answer to an inverse trig problem must fall in the correct range and that there is only one correct answer.

Example 2:

a. $\sin\frac{-\pi}{6} =$ b. $\sin\frac{7\pi}{6} =$ c. $\sin\frac{11\pi}{6} =$ d. $\arcsin\left(-\frac{1}{2}\right) =$

Example 3: Evaluate without a calculator.

a. $\arctan 1$ b. $\arccos(-1)$ c. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ d. $\arcsin 2$

Example 4: Use a calculator to evaluate.

a. $\arcsin(0.3)$ b. $\arctan\left(-\frac{5}{2}\right)$

Example 5: Simplify without a calculator.

a. $\sin\left(\arcsin\frac{\sqrt{3}}{2}\right)$ b. $\tan(\arctan 3)$ c. $\arccos\left(\cos\frac{\pi}{3}\right)$ d. $\arcsin\left(\sin\frac{11\pi}{6}\right)$

Example 6: Solve for x. $\arcsin(x^2 - 3) = \frac{\pi}{2}$

<u>Examples:</u> Sketch a right triangle, and evaluate without a calculator.

7. Given $x = \arccos \frac{2}{\sqrt{5}}$, find $\tan x$.

8. Given that $y = \arcsin x$, find $\cos y$.

<u>Example 9:</u> Use your work from Example 8 to find $\frac{d}{dx} \arcsin x$.

<u>Derivatives of the Inverse Trig Functions</u>:

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx} \arcsin u = \frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx} \arccos u = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2} \qquad \frac{d}{dx} \arctan u = \frac{u'}{1+u^2}$$

(where u is a function of x)

<u>Examples:</u> Differentiate.

10. $g(y) = \arctan(2y-1)$

11. $f(x) = \arcsin\sqrt{x}$

12. $h(t) = \cos^{-1}(\ln t)$

ASSIGNMENT 3.6

Evaluate the expressions in Problems 1-4 without using a calculator.

1. $\arcsin\frac{\sqrt{3}}{2}$ 2. $\arctan(-1)$ 3. $\arccos\left(\frac{-1}{2}\right)$ 4. $\tan^{-1}\sqrt{3}$

Use a calculator to evaluate in Problems 5-8.

5. $\arctan(-3)$ 6. $\arccos(.8)$ 7. $\arcsin\left(\frac{-1}{3}\right)$ 8. $\arccos\left(\sqrt{2}-1\right)$

Simplify the expressions in Problems 9-12 without using a calculator.

9. $\cos\left(\arccos\left(\frac{-2}{3}\right)\right)$ 10. $\tan\left(\arctan(2x+3)\right)$

11. $\arcsin\left(\cos\frac{\pi}{2}\right)$ 12. $\arctan\left(\tan\frac{4\pi}{3}\right)$ Be careful!

In Problems 13 and 14, solve for x without using a calculator.

13. $\arctan(3-x)=\frac{-\pi}{4}$ 14. $\arccos\left(x^2-2\right)=\pi$

For Problems 15 and 16, evaluate without using a calculator. First, sketch a triangle for each problem.

15. Find $\cos y$, given that $y=\arcsin\left(\frac{-4}{5}\right)$. 16. Find $\sin x$, given that $x=\arctan(3)$.

Differentiate in Problems 17-28 without using a calculator.

17. $y=2\arctan(3x)$ 18. $f(x)=\arcsin\left(x^2-1\right)$ 19. $g(y)=\arcsin e^{-y}$

20. $h(t)=\arctan t^{\frac{3}{2}}$ 21. $y=x^2\cos^{-1}x$ 22. $f(\theta)=\arctan(\ln\theta)$

23. $y=\cos\left(\ln t^2\right)$ 24. $f(x)=\tan(x)\ln|x-1|$ 25. $g(t)=2\cos^2\sqrt{t}$

26. $f(x)=\dfrac{x^2-3}{\tan x}$ 27. $y=x\sin(-3x)$ 28. $h(y)=\left(\ln(\sec y)\right)^3$

29. For $y=x\sin x$, evaluate $\dfrac{d^2y}{dx^2}$ at $x=\dfrac{\pi}{4}$.

30. For $\sec y-xy=x+2$, find $\frac{dy}{dx}$.

31. Without using a calculator, find the x-values where $y=e^x\sin x$ has horizontal tangents on the interval $[-\pi,\pi]$.

32. Find a and b so that $f(t)$ is <u>differentiable</u> at $t=-1$.

$$f(t)=\begin{cases}at^3+bt^2-2, & t\le -1\\ -bt^2+at-4, & t>-1\end{cases}$$

33. $y(t) = t^3 - t^{-3}$ represents the position of a point on the y-axis at time $t > 0$.
$v(t)$ represents the velocity and $a(t)$ represents the acceleration of the point.

Without a calculator, find

a. $y(2)$ b. $v(2)$ c. $a(2)$

34. Use a calculator to find $\left(g^{-1}\right)'(2)$ for $g(x) = \sqrt{x^3 + 2x + 5}$.

ASSIGNMENT 3.7 REVIEW

Find the derivatives of these functions without using a calculator.

1. $y = \sqrt{x^2 - x}$ 2. $g(x) = x(x-1)^3$ 3. $y = \ln\left(x^2 \sqrt[3]{x+1}\right)$ 4. $g(t) = t^3 e^{-t}$

5. $y = 4\left(\log_6\left(x^2+1\right)\right)^3$ 6. $y = \ln\left(\arctan(2x)\right)$ 7. $f(x) = \ln(x)\arctan(2x)$

8. $g(x) = \left(\ln\left(\arcsin x^2\right)\right)^3$ 9. $x = \cos(\arcsin t)$ 10. $y = \arctan(v-1)^2$

11. Write an equation for the line tangent to $y = e^{-x} - 3$ when $x = 0$.

12. If $x^2 - xy + y^2 = 5$ find $\frac{dy}{dx}$.

13. Find an equation of the tangent line to the graph of $x^2 + 2y + y^2 = 1$ at the point $(1, -2)$.

14. Find an equation of the line tangent to the graph of $y = \sqrt{\cos\left(x - \frac{\pi}{4}\right)}$ when $x = \frac{\pi}{4}$.

15. Use the data given in the table to find the following:

a. $\frac{d}{dx}\left(f(x) \cdot g(x)\right)$ at $x = 1$ b. $\frac{d}{dx}\frac{f(x)}{g(x)}$ at $x = 1$

c. $\frac{d}{dx} f\left(g(x)\right)$ at $x = 1$ d. $\frac{d}{dx} g\left(f(x)\right)$ at $x = 1$

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	2	5	1	3
2	4	−6	−2	7

Find the limits in Problems 16-18 without using a calculator.

16. $\lim\limits_{x\to\infty} \frac{x(2x-3)^2}{x^3+10}$ 17. $\lim\limits_{x\to\infty} \frac{x(2x-3)}{x^3+10}$ 18. $\lim\limits_{x\to 1} \frac{\arctan(x) + \frac{\pi}{4}}{x+1}$

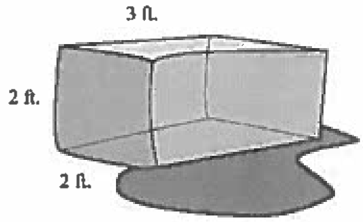
19. For $4\cos x \sin y - \sqrt{6} = 0$, find $\frac{dy}{dx}$ at $\left(\frac{\pi}{6}, \frac{\pi}{4}\right)$.

20. Given $(2, -1)$ is a point on the graph of $f(x) = x^3 + x - 11$, find $\left(f^{-1}\right)'(-1)$.

21. If $g(x)=e^x+3x$, find $\left(g^{-1}\right)'(1)$.

22. A point moves along the curve $y=\sqrt{x}$ so that the y-coordinate is increasing at the rate of $4\,\frac{cm}{sec}$. Find the rate of change of the x-coordinate with respect to time when $y=2\ cm$.

23. A block of ice is exposed to heat in such a way that the block maintains a similar shape as it melts. The block of ice is initially 2 feet wide, 2 feet high, and 3 feet long, as shown at right. If the rate of change in the width of the ice is $-\frac{1}{3}$ ft/hr, find:



   a. the rate of change in the volume of the block of ice when the width is 1 ft.
   b. the amount of time it will take for the block of ice to completely melt.

24. Find $\left(f^{-1}\right)'(5)$ if $f(x)=x^3-4x^2+3x-7$ on the interval $[3,\infty)$.

For Problems 25-27, find the indicated derivative.

25. $y=e^{x^2-1}$

$\frac{d^2y}{dx^2}=$

26. $\frac{d}{dx}\left(\frac{\ln\left|x^2-1\right|}{x}\right)=$

27. $y^3-x=y\ln x$

$\frac{dy}{dx}=$

28. f and g are inverse functions. The graph of g passes through the points $(-1,2)$, and $(2,-1)$. $f'(-1)=-2$ and $f'(2)=-1$. Find:

   a. $g'(-1)$
   b. $g'(2)$

If possible, differentiate each function in Problems 29-32. Assume C and c are constants. One example is not possible with techniques you have learned.
<u>Note</u> the differences in each problem.

29. $y=C^c$

30. $y=x^c$

31. $y=c^x$

32. $y=x^x$

UNIT 3 SUMMARY

Chain Rule: $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

Differentiation Rules:

$\frac{d}{dx} e^x = e^x$ $\quad$ $\frac{d}{dx} a^x = a^x \ln a$ $\quad$ $\frac{d}{dx} \ln x = \frac{1}{x}$ $\quad$ $\frac{d}{dx} \ln|x| = \frac{1}{x}$

$\frac{d}{dx} e^u = e^u u'$ $\quad$ $\frac{d}{dx} a^u = a^u u' \ln a$ $\quad$ $\frac{d}{dx} \ln u = \frac{u'}{u}$ $\quad$ $\frac{d}{dx} \ln|u| = \frac{u'}{u}$

$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$ $\quad$ $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$ $\quad$ $\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}}$ $\quad$ $\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$

$\frac{d}{dx} \log_a u = \frac{u'}{u \ln a}$ $\quad$ $\frac{d}{dx} \arcsin u = \frac{u'}{\sqrt{1-u^2}}$ $\quad$ $\frac{d}{dx} \arccos u = \frac{-u'}{\sqrt{1-u^2}}$ $\quad$ $\frac{d}{dx} \arctan u = \frac{u'}{1+u^2}$

Implicit Differentiation: differentiate both sides with respect to x, remember the y' "chain rule factor" and remember to use the Product Rule for xy terms

Related Rates: (story problems)

Procedure for Related Rate Problems

1. Draw a figure (if necessary) and choose variables for all unknowns.
2. Write what is given and what is to be found using your variables and $\frac{d}{dt}$ symbols.
3. Write an equation relating the variables.
   (a) If a quantity is **changing** it must be represented with a variable letter.
   (b) If a quantity is **constant** it must be represented with a number value.
   (c) Look for secondary relationships between quantities to reduce the number of variables.
4. Implicitly differentiate both sides with respect to t.
5. Substitute number values and solve.

Geometry Formulas

Right Triangle	Circle	Sphere	Cube	Cone	Rectangle
$a^2+b^2=c^2$	$A=\pi r^2$	$V=\frac{4}{3}\pi r^3$	$V=e^3$	$V=\frac{1}{3}\pi r^2 h$	$A=lw$
$A=\frac{1}{2}bh$	$C=2\pi r$	$A=4\pi r^2$	$A=6e^2$		$P=2l+2w$

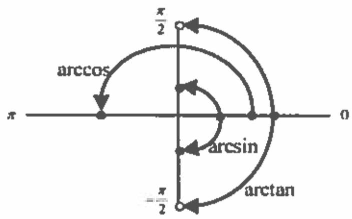
Derivatives of Inverse Functions:

If f and g are inverse functions, then $f'(a)=\dfrac{1}{g'(b)}$ where (a,b) is a point on the graph of f and (b,a) is the "image point" on the graph of g.

Definition of the Inverse Trig Functions:

Function	Domain (x values)	*Range (y values)
$y=\arcsin x \leftrightarrow \sin y = x$	$[-1,1]$	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$y=\arccos x \leftrightarrow \cos y = x$	$[-1,1]$	$[0,\pi]$
$y=\arctan x \leftrightarrow \tan y = x$	$(-\infty,\infty)$	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

*A geometric representation of the range values for each inverse trig function is shown in the coordinate plane at right.

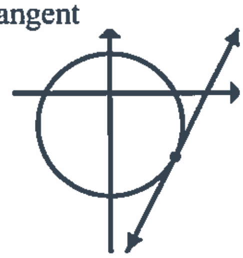


LESSON 4.1 APPROXIMATING WITH THE TANGENT LINE APPLICATIONS OF RATES OF CHANGE

In many instances, finding a value of a function is difficult or impossible. With the use of Calculus techniques, we can approximate the function value by finding a y-value on a tangent line to the function. Since this method involves using a linear function (the tangent line function) at a nearby point, it is sometimes called a local linearization approximation.

Examples:

1. If $(2,-2)$ is a point on the graph of $x^2+y^2+2y=4$, use the equation of a tangent line passing through the point $(2,-2)$ to approximate a y-coordinate



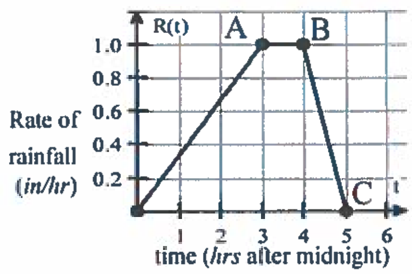
   (a) when the x-coordinate is 2.1.

   (b) when the x-coordinate is 1.9.

2. If $f(2)=3$ and $f'(2)=-2$, use local linearization to approximate $f(2.01)$,

Examples:

The graph at right models the rate of rainfall in inches per hour from midnight until 6:00 A.M. during a tropical rainstorm.



3. Write a complete sentence to explain what Point A on the graph represents. Include numbers and units in your answer.

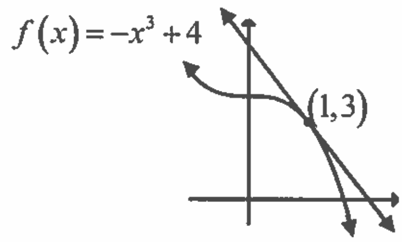
4. What is the slope of the graph between Points A and B?

5. Write a complete sentence to explain the meaning of your answer to Example 4.

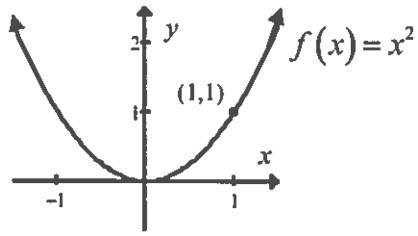
6. What is the slope of the graph between Points B and C?

7. Write a complete sentence to explain the meaning of your answer to Example 6.

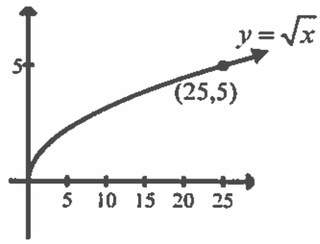
ASSIGNMENT 4.1



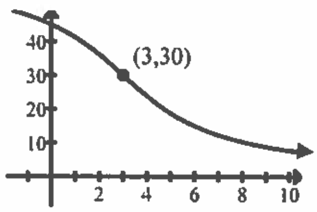
1. a. Write an equation of the tangent line shown.
   b. Use this tangent line equation to approximate $f(1.1)$.
   c. What is the actual value of $f(1.1)$?



2. Make a large copy of the graph on your own paper.
   a. Draw the tangent line at the point (1,1).
   b. Write an equation of this tangent line.
   c. Label a point on your tangent line with an x-coordinate of .9 as point A.
   d. Use your equation of the tangent line to approximate $f(.9)$ by finding the y-coordinate of your point A.
   e. Label a point B on the parabola with an x-coordinate of .9. What is the actual value of $f(.9)$?
   f. Use the same tangent line to approximate $f(.6)$. How accurate is your approximation?



3. Approximate $\sqrt{26}$ using the equation of a tangent line. You must choose your own equation and point. The graph shown should help.



4. The graph of a function $y=f(x)$ is shown. If $f'(3)=-9$, use local linearization to approximate $f(3.1)$.

5. Find the actual value of $f(3.1)$ from problem 4 or explain why it cannot be found.

6. The point (5,20) is on the curve $y=x\sqrt{x^2-9}$. Use a tangent line to approximate the y-coordinate when $x=5\frac{1}{5}$.

7. The length of one side of a square is found to be 8 inches with a possible measurement error of $\frac{1}{16}$ inch.
   a. Instead of using the actual area formula ($A=s^2$), approximate the area of the square using a local linearization of the area formula if the length of the side is really $8\frac{1}{16}$ inches (without using a calculator).
   b. Find the approximate area if the side is actually $7\frac{15}{16}$ inches.
   c. Use your answers from parts a and b to give an approximate range of values for the area of the square.

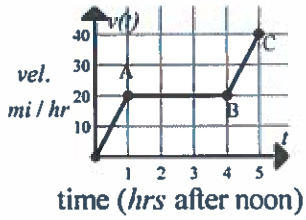
8. Use a tangent line equation to approximate $f(8.01)$ if $f(x)=\sqrt[3]{x}$ (without using a calculator).

9. The point (1,2) is on the graph of $x^3+xy+y^4=19$. Use the equation of a tangent line to approximate a y-coordinate when $x=1.1$.

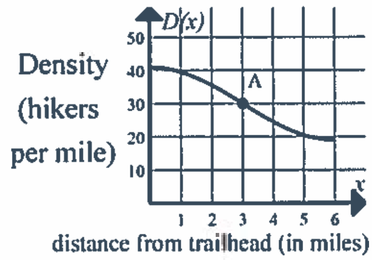
10. Use a calculator to find an actual y-coordinate on the graph of the curve from problem 9 when $x=1.1$. Show the equation you are solving.

11. Given the function $y=x^3$
    a. use the equation of a tangent line to approximate $\left(2\frac{1}{6}\right)^3$ without using a calculator.
    b. find the actual value of $\left(2\frac{1}{6}\right)^3$?

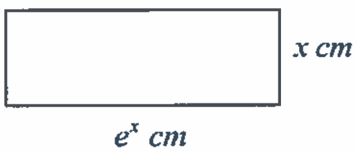
12. The graph at the right models the velocity of a car.
    a. Tell what Point A represents.
    b. Find the slope between B and C and tell what it represents.
    c. What is the velocity at 12:30 pm? at 4:15 pm?
    d. What is the acceleration at 12:30 pm? at 2:00 pm?



13. The graph at the right represents the density of hikers on a trail.
    a. Tell what Point A represents including numbers and units.
    b. If $D'(3)=-6$, tell what this means in the context of the problem using numbers and units.
    c. If $D'(3)=-6$, use local linearization to approximate the density of hikers 3.1 miles from the trailhead.



14. If the area of the rectangle shown is increasing at the rate of $3\ {cm^2}/{sec}$, find $\frac{dx}{dt}$ when $x=2\ cm$.



15. Find the average rate of change of $f(x)=\sin x+\cos x$ on the interval $\left[\frac{\pi}{4},\frac{5\pi}{4}\right]$.

Differentiate in Problems 16-24 without using a calculator .

16. $y=2\arctan\left(e^{2x}\right)$

17. $f(x)=\ln\left|\arcsin x\right|$

18. $y=\dfrac{t^2}{\ln t}$

19. $y=3^{2t-1}t^2$

20. $f(y)=\dfrac{e^{\sqrt{y}}}{y^2}$

21. $f(x)=e^x\ln x$

22. $f(x)=\ln(x-1)^{\frac{2}{3}}$

23. $g(y)=\ln\left|(1-\ln y)\right|$

24. $\ln y=(2x+1)\ln x$, $\frac{dy}{dx}=?$

25. Find these limits without using a calculator.

a. $\lim\limits_{x\to 3}\dfrac{x-3}{x^2-9}$ b. $\lim\limits_{x\to 3}\dfrac{x+3}{x^2-9}$ c. $\lim\limits_{x\to\infty}\dfrac{3x^3+x-3}{2x^3-9}$ d. $\lim\limits_{x\to\infty}\dfrac{3x^3+x-3}{2x^5-9}$

LESSON 4.2 L'HOSPITAL'S RULE

Some limits cannot be found using algebraic methods. If direct substitution produces one of these two indeterminate forms $\left(\frac{0}{0} \text{ or } \frac{\pm\infty}{\pm\infty}\right)$, then a rule known as **L'Hospital's Rule** may help you find the limit.

L'Hospital's Rule

If $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$ or if both of these limits are $\pm\infty$, then $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$

To use L'Hospital's Rule, you need the limit of an expression written in fractional form.

Examples: Evaluate.

1. $\lim_{x\to\infty}\frac{x}{e^x}$

2. $\lim_{x\to\infty}\frac{e^x}{x}$

3. $\lim_{x\to 0}\frac{x}{e^x}$

4. $\lim_{x\to 0}\frac{3-3e^{3x}}{x}$

5. $\lim_{x\to -1}\frac{2x^2-2}{x+1}$ (w/o LR)

6. $\lim_{x\to -1}\frac{2x^2-2}{x+1}$

Do not use L'Hospital's Rule just because a problem "looks like" a candidate for the rule. Example 3 is not a candidate for L'Hospital's Rule, because direct substitution does not produce $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.

If using L'Hospital's Rule leaves you with the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then you can use the rule again. It is a process which can be repeated as many times as necessary. Just remember to use direct substitution at each step to make sure the rule can be used (check for $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ form).

Examples: Evaluate.

7. $\lim_{x\to-\infty} \frac{x^2}{e^{-x}}$

8. $\lim_{x\to 1} \frac{\ln x - x + 1}{x^2 - 2x + 1}$

ASSIGNMENT 4.2

For Problems 1-18, find the indicated limits without using a calculator.
Hint: Not all problems will require the use of L'Hospital's Rule.

1. $\lim_{x\to 2} \frac{x^2-4}{x+2}$

2. $\lim_{x\to 1} \frac{x^2+2x-3}{x-1}$

3. $\lim_{x\to \infty} \frac{e^x}{x^2}$

4. $\lim_{x\to 0} \frac{x}{x-(1-e^x)}$

5. $\lim_{x\to 0^+} \frac{|x|}{x}$

6. $\lim_{x\to 0^-} \frac{|x|}{x}$

7. $\lim_{x\to 0} \frac{|x|}{x}$

8. $\lim_{x\to 2} \frac{x^2-4}{\ln(x-1)}$

9. $\lim_{x\to 0} \frac{2(e^x-1)}{x^2}$

10. $\lim_{x\to \infty} \frac{2x^5-x^2}{3x^5+x^4-5x}$

11. $\lim_{x\to \infty} \frac{2x^4-x^2}{3x^5+x^4-5x}$

12. $\lim_{x\to \infty} \frac{3x^5+x^4-5x}{2x^4-x^2}$

13. $\lim_{x\to \infty} \frac{\sqrt{x^2+5}}{x}$

14. $\lim_{x\to -\infty} \frac{\sqrt{x^2+5}}{x}$

15. $\lim_{x\to 5} \frac{2x-10}{5x}$

16. $\lim_{x\to 0} \frac{\tan x}{x \sec x}$

17. Find $\lim_{\theta\to \frac{\pi}{2}} \frac{-2\cos\theta}{e^{\theta-\frac{\pi}{2}}-1}$

18. $\lim_{x\to 1} \frac{\arctan(x)-\frac{\pi}{4}}{x-1}$

Use a calculator to find the limits in Problems 19-21.

19. $\lim\limits_{x\to\infty} x^{\frac{1}{x}}$ 20. $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ 21. $\lim\limits_{x\to 1.5}\frac{2x^3-3x^2-8x+12}{6x^3-25x^2+34x-15}$

Differentiate without using a calculator.

22. $g(t)=\sin(\arctan t)$ 23. $x^2+2\ln y=y$ $\frac{dy}{dx}=?$ 24. $y=x\sqrt{3x^2+2}$

25. $y=\frac{3x-1}{x^2+2}$ 26. $x^2-2xy+y=8$ $\frac{dy}{dx}=?$

27. Given: $f(4)=2,\ g(4)=3,\ f'(4)=-1,\ f'(3)=-2,$ and $g'(4)=5$,

a. If $h(x)=f(x)\bullet g(x)$, find $h'(4)$. b. If $j(x)=(f(x))^3$, find $j'(4)$.

c. If $k(x)=f(g(x))$, find $k'(4)$.

28. Given: $f(x)=\begin{cases}2x+a, & x\le 2\\ x-b, & 2<x<3\\ x^2-1, & x\ge 3\end{cases}$ is a continuous function, find: a and b.

29. Find the point where the tangent line to the graph of $f(x)=3x^2-2x+5$ is parallel to the graph of $y=10x-3$.

30. The position (in *cm*) of an object moving on a horizontal line is given by $s(t)=2t^3-3t^2-t+8$ (where time is measured in seconds). Answer the following questions. You may use a calculator.

a. What is the object's velocity equation?
b. What is the object's initial velocity?
c. What is the object's acceleration equation?
d. What is the object's acceleration at $t=3$ seconds?
e. What is the object's speed at $t=1$ second?
f. When is the object moving left?
g. What is the object's displacement between zero and two seconds?
h. What is the object's total distance traveled between zero and two seconds?

An object has **increasing velocity** when its acceleration is greater than zero. It has **increasing speed** when the velocity and acceleration have the same sign.

i. When is the object's velocity decreasing?
j. When is the object's speed decreasing.

31. A particle is moving on the curve $4x^2+16y^2=100$. When the particle is in the fourth quadrant with an x-coordinate of 3 cm, the y-coordinate is decreasing at the rate of $\frac{1}{2}\frac{\text{cm}}{\text{sec}}$. Find the rate of change of the x-coordinate and tell what your answer means about the motion of the particle.

LESSON 4.3 ABSOLUTE EXTREMA AND THE MEAN VALUE THEOREM

Definitions (informal)
- The **absolute maximum** (global maximum) of a function is the y-value that is greater than or equal to all other y-values in the function.
- The **absolute minimum** (global minimum) of a function is the y-value that is less than or equal to all other y-values in the function.
- A **relative maximum** (local maximum) of a function is a y-value that is greater than or equal to all "nearby" y-values in the function.
- A **relative minimum** (local minimum) of a function is a y-value that is less than or equal to all "nearby" y-values in the function.
- **Extrema** (extreme values) are either maximum values (maxima) or minimum values (minima).
- **Critical Numbers** are x-values at which $f(x)$ exists but $f'(x)$ is either zero or undefined.

Extreme Value Theorem (EVT)

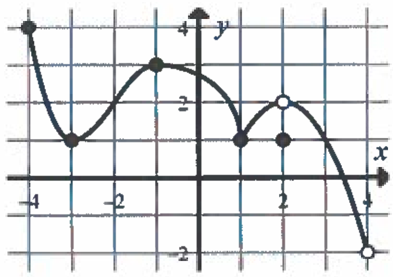
If f is continuous on $[a,b]$ then f has both an absolute (global) minimum and an absolute (global) maximum on the interval.

In practice, the standard method of finding these max/min values is by the **candidate test**.

THE CANDIDATE TEST
PROCEDURE FOR FINDING ABSOLUTE (GLOBAL) EXTREMA:
1. Find all critical numbers of the function.
2. Find y-values at each critical number **and** at each endpoint of the interval.
3. Choose the least and greatest y-values as absolute extrema.

Note:
Absolute extrema can occur either at critical numbers or endpoints.
Relative extrema can occur only at critical numbers. We will not consider endpoint extrema to be relative extrema although some textbooks allow this.

Examples: Use the figure of $y = f(x)$ at the right to answer these questions.



1. What is the absolute maximum of f?
2. At what x-value does f have an absolute maximum?
3. What is the absolute maximum point on f?
4. What is the absolute minimum of f?
5. At what x-value(s) does f have a relative minimum?
6. At what x-value(s) does f have a relative maximum?

Examples:

7. Find the global extrema of $f(x)=\frac{1}{3}x^3-2x^2$ on the interval $[-1,3]$.

8. Find the absolute maximum and minimum values of $f(x)=|x-2|$ on the interval $[0,5]$.

9. Find the extrema of $f(x)=3x^{\frac{2}{3}}-2x$ on $[-1,3]$.

10. Find the global maximum and minimum of $g(x)=2\sin x-\cos(2x)$ on the interval $[0,2\pi]$.

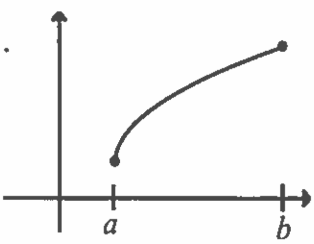
Discovering the Mean Value Theorem

For Examples 11-16 draw these lines (if possible).

(a) Draw the secant line between the two points $(a,f(a))$ and $(b,f(b))$.

(b) Draw all tangent lines parallel to the secant line.

11.



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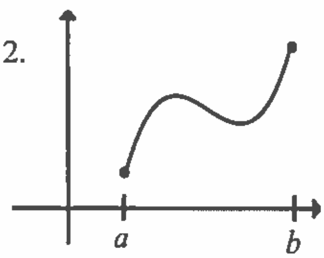

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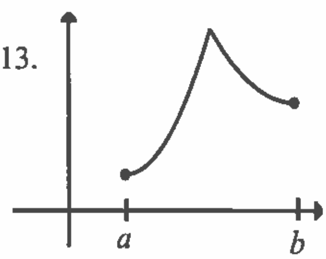
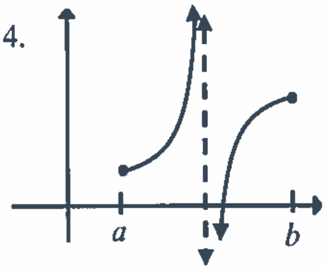
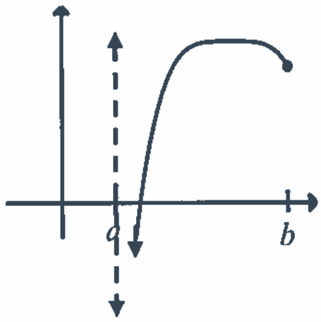


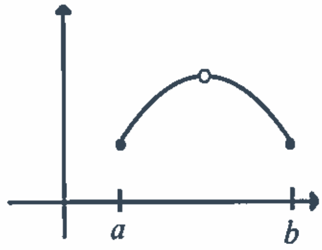
14.



15.



16.



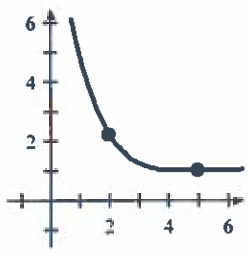
For examples 17-19

(a) Draw the secant line between the two points $(2, f(2))$ and $(5, f(5))$.

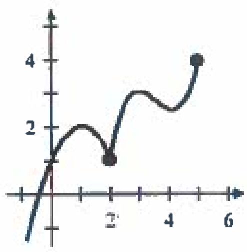
(b) Draw all tangent lines parallel to the secant line at some point on the interval (2,5).

(c) Estimate the value of c where $(c, f(c))$ is a point of tangency.

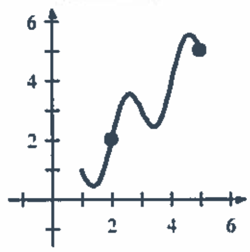
17. $c \approx$ ______



18. $c \approx$ ______



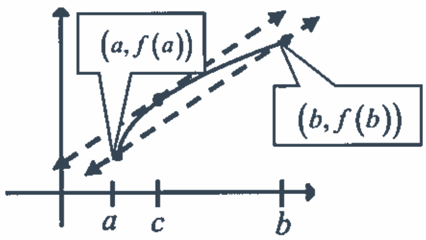
19. $c \approx$ ______



<u>MEAN VALUE THEOREM</u>: If f is continuous on $[a,b]$ and differentiable on (a,b), then there is a number c in (a,b) such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

tangent slope
(inst. rt. of ch.)

secant slope
(avg. rt. of ch.)



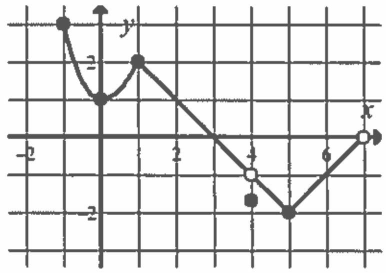
<u>Informally</u>: The Mean Value Theorem states that given the right conditions of continuity and differentiability, there will be at least one tangent line parallel to the secant line.

<u>In still other words</u>: The instantaneous rate of change (slope of tangent) will equal the average rate of change (slope of secant) at least once.

<u>Example 20.</u> Given $f(x) = 3 - \frac{6}{x}$, find all c which satisfy the Mean Value Theorem on the interval $[3,6]$.

ASSIGNMENT 4.3

Use the graph of $y = f(x)$ at the right for Problems 1-4.



1. What is the absolute maximum value of $f(x)$?
2. At what point does $f(x)$ reach a global minimum?
3. At what x-value(s) does $f(x)$ have a relative minimum?
4. At what x-value(s) does $f(x)$ have a relative maximum?

Find the critical numbers of these functions without using a calculator.

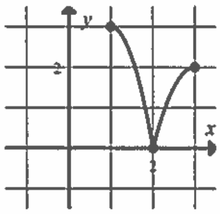
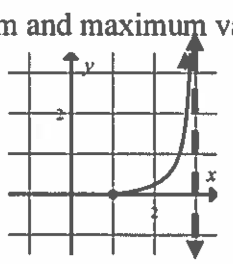
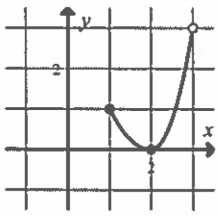
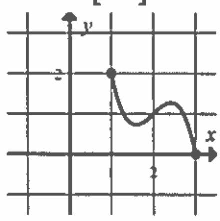
5. $y = x^2 - 3x$ 6. $f(x) = 6x^{\frac{2}{3}} - 2x$ 7. $y = -2x + 3$

8. $y = |x+2|$ 9. $f(x) = \cos^2 x + \sin x$ on $[0, 2\pi]$

Without using a calculator, find the absolute maximum and absolute minimum values of $f(x)$ on the given interval.

10. $f(x) = x^2 - 3x$ on $[-1,5]$
11. $f(x) = 6x^{\frac{2}{3}} - 2x$ on $[-1,27]$
12. $f(x) = x^3 + 3x^2$ on $[-3,1]$
13. $f(x) = x^3 + 3x^2$ on $\left[-1, \frac{1}{2}\right]$
14. $f(x) = |x+2|$ on $[-3,0]$
15. $f(x) = 3(x-2)$ on $[-3,1]$
16. $f(x) = \dfrac{x^2}{x^2+2}$ on $[-1,2]$
17. $f(x) = \cos^2 x + \sin x$ on $[0, 2\pi]$ see Problem 9.
18. $f(x) = \begin{cases} -x^2 + 2, & x \le 0 \\ -x + 2, & 0 < x < 6 \\ -4 & x \ge 6 \end{cases}$ on $[-1,8]$

Find the absolute minimum and maximum values of the functions graphed on $[1,3]$.

19. 
20. 
21. 
22. 

Does the Mean Value Theorem apply to the given function on the given interval? If it does, find the c-value. If it does not, explain why not. Do not use a calculator.

23. $f(x) = |x|$ on $[-1,3]$
24. $f(x) = x^2 - 2x$ on $[1,3]$
25. $f(x) = x^2 - 3x + 2$ on $[1,2]$
26. $f(x) = x^{\frac{2}{3}}$ on $[-2,2]$
27. $f(x) = x^{\frac{2}{3}}$ on $[0,1]$
28. $f(x) = \dfrac{1}{x-4}$ on $[2,6]$

29. $f(x)=\dfrac{x^2-x}{x}$ on $[-1,1]$

30. $f(x)=x^2-2x$ on $[0,2]$

31. $f(x)=\sin x$ on $[0,\pi]$

32. $f(x)=\tan x$ on $[0,\pi]$

Find the c-value guaranteed by the Mean Value Theorem for the given function on the given interval. You may use a calculator.

33. $f(x)=x^3-2x^2+3$ on $[-1,2]$

34. $f(x)=\dfrac{1}{x-1}$ on $[2,3]$

35. $f(x)=2\sin x+\sin(2x)$ on $[0,\pi]$

36. The height, in feet, of an object at time t seconds is given by $h=-16t^2+200$.
   a. Find the average velocity of the object during the first 3 seconds.
   b. Use the Mean Value Theorem to find the time at which the object's instantaneous velocity equals this average velocity.

Find the following limits.

37. $\lim\limits_{t\to\frac{\pi}{4}}\dfrac{\cos t-\sin t}{2-2\tan t}$

38. $\lim\limits_{x\to-\infty} x^2e^x$

LESSON 4.4 INCREASING/DECREASING FUNCTIONS, FIRST DERIVATIVE TEST FOR RELATIVE EXTREMA

PROCEDURE: **(Increasing/ Decreasing and First Derivative Test)**
1. Find **domain** restrictions.
2. Find all **critical numbers** (where $f'(x)=0$ or $f'(x)$ is undefined – but domain restrictions cannot be critical numbers).
3. Locate critical numbers and domain restrictions on an f' **number line.** Label critical numbers CN.
4. Test the sign of $f'(x)$ in each interval and **label the signs** on the number line.
5. List **increasing/decreasing intervals** and/or identify **relative min/max** x-values. If requested, find y-values or points.

Increasing/Decreasing Intervals

A frequent area of confusion for students is the question of whether to include endpoints on intervals. Some textbooks would say the function $y=x^2$ is decreasing on $(-\infty,0)$ and increasing on $(0,\infty)$. Other texts say it is decreasing on $(-\infty,0]$ and increasing on $[0,\infty)$. Using closed intervals is not as intuitive but it is more correct. The concept of increasing/decreasing is not based on a single point but instead is about an interval. The precalculus definition of increasing is that as x-values increase, y-values also increase and if this is true on an open interval it is also true on the closed interval so long as there is continuity. The College Board is well aware of this discrepancy among textbooks and has tried to make sure that students were not penalized. On a free response question, readers have been instructed to count either answer correct. They will never list both open and closed options as multiple choice responses. It is more likely that multiple choice answers will be closed intervals.

The question about whether to include zero when listing where $y = x^3$ is increasing is a completely separate question. All texts would say it is increasing on $(-\infty, \infty)$ even though the derivative is zero at $x = 0$. Any other answer would not receive credit on an AP test.

<u>Examples:</u> Find the intervals on which these functions are increasing and decreasing and find all local extrema <u>points</u>.

1. $f(x) = (x^2 - 9)^{\frac{2}{3}}$
2. $y = x - 2\sin x$ on $(0, 2\pi)$
3. $f(x) = \dfrac{x^4 + 3}{3x}$

4. $f(x) = -xe^{-2x^2}$
5. $f(x) = x^3 - 3x^2 + 3x$

The function in Example 5 is a <u>strictly monotonic</u> function. Strictly increasing or strictly decreasing functions are called monotonic.

Example 5 illustrates two **important points.**

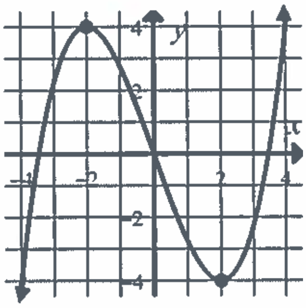
1. Not every critical number produces a relative maximum or minimum.
2. Even though the slope at $x = 1$ is zero, the function is always increasing.

ASSIGNMENT 4.4

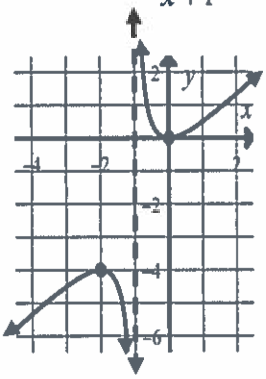
Use the graphs shown with each function to answer these four questions.

a. On which intervals is f increasing?
b. On which intervals is f decreasing?
c. At which point(s) does f have a relative minimum?
d. At which point(s) does f have a relative maximum?

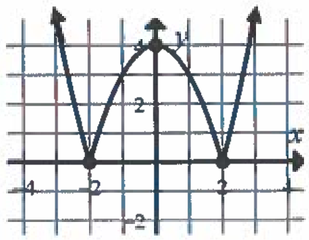
1. $f(x)=\frac{1}{4}x^3-3x$



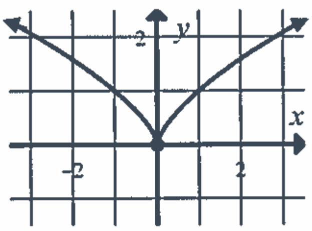
2. $f(x)=\dfrac{x^2}{x+1}$



3. $f(x)=\left|x^2-4\right|$



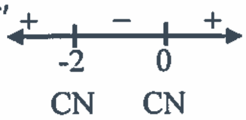
4. $f(x)=x^{\frac{2}{3}}$



Use the f' number lines (sign charts) shown to answer these four questions.

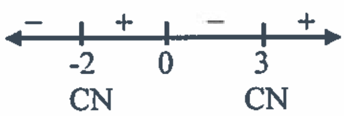
a. On which intervals is f increasing?
b. On which intervals is f decreasing?
c. At which x-values does f have a relative minimum?
d. At which x-values does f have a relative maximum?

5. f'



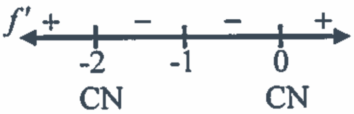
6. Domain: $x \neq 0$

f'

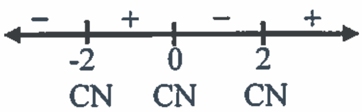


Which of the functions in Problems 1-4 would have the following f' number lines?

7. f'



8. f'



Without using a calculator, answer these four questions for Problems 9-19. Show organized work and an f' number line to support your answers.

a. On which intervals is f increasing?
b. On which intervals is f decreasing?
c. At which point(s) does f have a relative minimum?
d. At which point(s) does f have a relative maximum?

9. $f(x)=x^2-8x$

10. $f(x)=-2x^2+4x+6$

11. $f(x)=2x^3+3x^2-12x$

12. $f(x)=\frac{3}{5}x^{\frac{5}{3}}-\frac{3}{2}x^{\frac{2}{3}}+1$

13. $f(x)=x^3+1$

14. $f(x)=\dfrac{x}{x^2-9}$

15. $f(x)=(x+1)^{\frac{2}{3}}$

16. $f(x)=\dfrac{x-1}{x+1}$

17. $f(x)=x^4-2x^3$

18. $f(x)=\dfrac{x}{2}+\cos x$ on $[0,2\pi)$

19. $f(x)=\sin x+\cos x$ on $(0,2\pi)$

20. If $f'(x)=5x^3-2\sqrt{x+5}+\sin(x^2)$, use your calculator to answer these four questions.
    a. On which intervals is f increasing?
    b. On which intervals is f decreasing?
    c. At which x-values does f have a relative minimum?
    d. At which x-values does f have a relative maximum?

Determine whether the following functions are strictly monotonic on the interval $(0,\infty)$. Do not use a calculator.

21. $f(x)=x^2$

22. $g(x)=x^{\frac{2}{3}}$

23. $h(x)=\frac{1}{3}x^3-x$

For Problems 24-26 the height, in feet, of a ball is given by the position function $s(t)=-16t^2+64t+6$. Assume $0\le t\le 4$ seconds. Do not use a calculator.

24. On which interval of time is the ball moving upward?
25. What is the maximum height of the ball?
26. For what times is
    a. the velocity of the ball increasing?
    b. the speed of the ball increasing?

27. If the function $f(x)=x^2+ax+b$ has a relative minimum point at $(2,-4)$, solve for a and b.

28. Without a calculator find the absolute extrema of $f(x)=4x^3-12x+1$ on the interval $[-1,3]$.

29. Without using a calculator find the absolute extrema of $f(x)=|x-2|+2$ on $[1,4]$.

30. Find all critical numbers of $f(x)=\lfloor x\rfloor$.

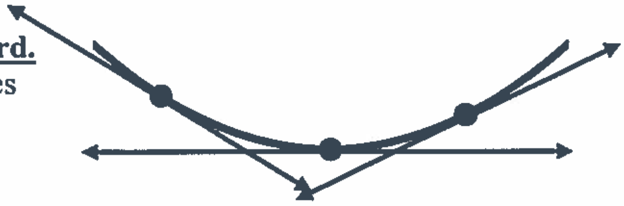
31. Can the Mean Value Theorem be used for the function $f(x)=\dfrac{x}{x-2}$ on the interval $[0,3]$? If it can be used, find the c-value. If it cannot, explain why not.

32. Find the c-value guaranteed by the Mean Value Theorem for the function $f(x)=x^3-x^2-2x$ on the interval $[-1,1]$. You may use a calculator.

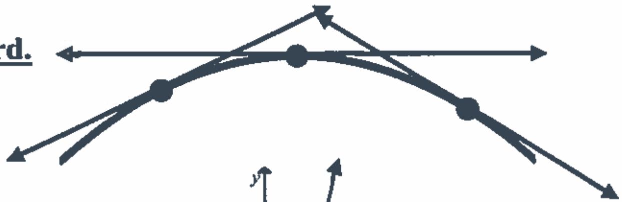
33. Find $\displaystyle\lim_{\theta\to\frac{\pi}{2}}\frac{\sin(2\theta)}{\cos\theta}$.

LESSON 4.5 CONCAVITY AND POINTS OF INFLECTION, THE SECOND DERIVATIVE TEST FOR RELATIVE EXTREMA

A graph with this shape is called **concave upward.** The tangent lines lie **below** the graph. The slopes of the tangent lines are increasing which means $f''(x) \geq 0$.

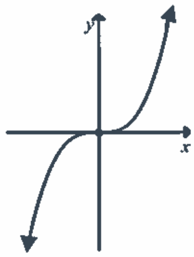


A graph with this shape is called **concave downward.** The tangent lines lie **above** the graph. The slopes of the tangent lines are decreasing which means $f''(x) \leq 0$.



Nonmathematical Memory Device:
Concave upward ↔ positive ↔ smiley face ↔ 

Concave downward ↔ negative ↔ frowny face ↔ 

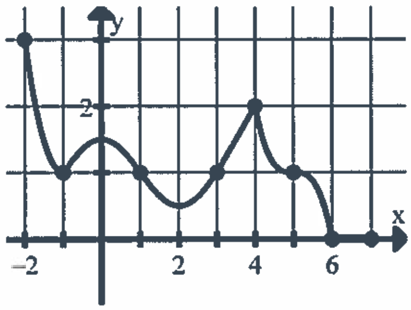


On the graph of $y = x^3$ shown, the point (0,0) is a point of inflection.

A point on a graph is a **point of inflection** if:
1. the graph has a tangent line at that point **and**
2. the graph changes concavity at that point.

Note: Some textbooks use other definitions for a point of inflection that do not require the existence of a tangent line. The AP™ test is carefully written so that this is not an issue.

Examples: Use the graph at the right to answer these questions. Base your answers on appearances of the graph.



1. On which intervals is the graph concave upward?

2. On which intervals is the graph concave downward?

3. On which intervals does the graph have no concavity?

4. What are the points of inflection?

Analytically we find concavity intervals and points of inflection by using a second derivative number line.
The procedure is parallel to the procedure used in the last lesson to find increasing/decreasing intervals and relative extrema by using a first derivative number line.

Most textbooks use open intervals for concavity. However, there is no universal definition of concavity so there is some chance for ambiguity. A few texts define concavity in terms of where the first derivative is increasing/decreasing so the same issue arises about including/excluding endpoints.

Examples:

5. Determine the points of inflection and discuss the concavity for the graph of $f(x)=x^4+x^3-3x^2+1$.

6. If $f(x)=\dfrac{x^2+1}{x^2-4}$ and $f''(x)=\dfrac{10(3x^2+4)}{(x^2-4)^3}$, list the intervals where the graph of f is concave upward, concave downward, and list the points of inflection.

THE SECOND DERIVATIVE TEST FOR RELATIVE EXTREMA

This test does not require a second derivative number line. It does not find points of inflection. It is used to find relative extrema (max/min).

Procedure:

1. Use f' to find critical numbers.
2. Plug critical numbers into f'' and analyze concavity to determine if the function has a relative minimum or maximum.

Note: The Second Derivative Test does not always give an answer (when $f''(x)=0$). Use it only when the directions require it or when the given information requires it.

Examples:

7. Use the Second Derivative Test to find the relative minimum and relative maximum points for the graph of $f(x)=-3x^4+6x^2$.

8. $g(x)$ is a function with some derivative values shown in the table. find the x-values of the relative maximum and relative minimum points when possible.

x	$g'(x)$	$g''(x)$
-3	0	4
-1	0	-1
0	0	0
1	3	5
2	0	3

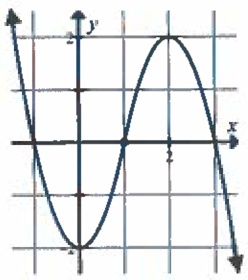
ASSIGNMENT 4.5

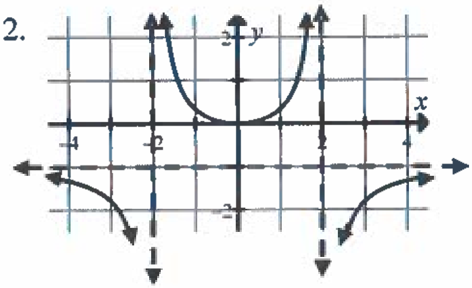
Use the appearance of these graphs to answer these three questions.

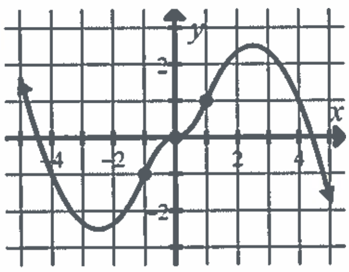
a. On which interval(s) is the graph of the function concave upward?

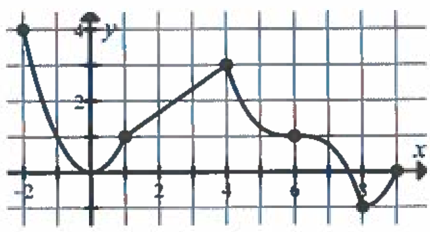
b. On which interval(s) is the graph of the function concave downward?

c. What are the points of inflection?

1. 

2. 

3. 

4. 

Show organized steps and an f'' number line to answer the same three questions for these functions without using a calculator.

5. $f(x)=x^4-4x^3+2$ 6. $g(x)=\cos x+\sin x$ on $[0,2\pi]$ 7. $f(x)=3x^5-5x^4$

8. $f(x)=x^{\frac{2}{3}}-3$ 9. $f(x)=-2\sin x-\frac{1}{2}x^2+2x+1$ on $[0,2\pi]$

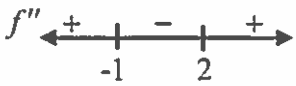
Use the Second Derivative Test to find the relative extrema points (see Example 7 on the previous page).

10. $f(x)=x^3-3x^2+6$ 11. $f(x)=-\frac{1}{4}x^4+\frac{9}{2}x^2+5$ 12. $g(x)=2\sin x+3$ on $[0,2\pi]$

For problems 13 and 14, find the x-values of relative minimum points and the x-values of relative maximum points.

13. -3, 1, and 3 are critical numbers of f and $f''(-3)=-2$, $f''(1)=0$, and $f''(3)=2$.

14. $f'(-2)=f'(0)=f'(4)=0$



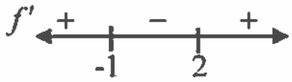
15. Find the relative extrema points and points of inflection for the graph of $y=xe^x-e^x$ without using a calculator.

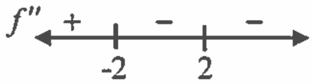
Without using a calculator, find local maximum and minimum points and points of inflection. Then sketch a graph. It is not necessary to find the x-intercepts. Show organized steps with f' and f'' number lines to support your answers.

16. $f(x)=2x^3-3x^2-12x+5$

17. $f(x)=4x^3-x^4$

18. Use the following information to sketch a possible graph of f. f is a continuous function, $f(-1)=3$, $f(2)=0$, $f(-2)=-1$; $f'(2)$ does not exist.





19. Use a calculator to determine if the function $f(x)=4x^3+\sin(5x)$ is concave upward or downward on an interval including $x=.523$. Be sure to use radians mode.

20. Find the <u>absolute</u> maximum and <u>absolute</u> minimum for the function $f(x)=x^4-2x^2$ on the interval $[-2,1]$. Do not use a calculator.

21. Find the c-value guaranteed by the Mean Value Theorem for the function $f(x)=\sqrt{x-2}$ on the interval $[2,6]$.

22. If the graph of $y=ax^3+bx^2+cx+d$ has a point of inflection at the point (0,2) and a relative maximum at the point $(-1,4)$, find the values of a, b, c, and d.

23. a. Sketch a smooth curve whose slope is always positive and whose slope is increasing.
    b. Sketch a smooth curve whose slope is always positive and whose slope is decreasing.
    c. Sketch a smooth curve whose slope is always negative and whose slope is increasing.
    d. Sketch a smooth curve whose slope is always negative and whose slope is decreasing.

Use the f' number line shown for Problems 24-26.
Fill in the blank with > or <.

24. If $g(x)=f(x)+2$, then $g'(4)$____ 0.

25. If $g(x)=-2f(x)$, then $g'(4)$____ 0.

26. If $g(x)=f(x-10)$, then $g'(4)$____ 0.

ASSIGNMENT 4.6 REVIEW

1. If $y=\dfrac{x}{x+5}$, find the equation of the tangent line when $x=-6$ and use it to approximate the y-coordinate when $x=-6\frac{1}{5}$.

2. Use the data in the table shown to find a linear approximation of $g(4.2)$.

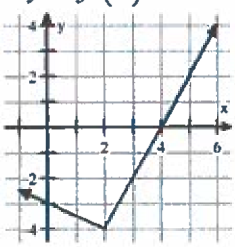
x	$g(x)$	$g'(x)$
4	3	−2

3. An interstate driver is traveling 420 miles across a state from south to north without stopping. At noon she notices her speed is 60 miles per hour and her position is at interstate mile marker 240. Note: Interstate mile markers increase from south to north.
   a. Use this data to write a linear function (local linearization) which could be used to estimate her position as a function of time. Assume $t=0$ at noon.
   b. Approximate her position at 2:00 pm.
   c. Approximate her position at 10:30 am.
   d. What is the domain on which your linear function can be applied?

Find the following limits without using a calculator.

4. $\displaystyle\lim_{\theta\to\frac{\pi}{2}}\frac{2-2\sin\theta}{\theta-\frac{\pi}{2}}$
5. $\displaystyle\lim_{\theta\to\frac{\pi}{2}}\frac{2-\cos\theta}{\theta}$
6. $\displaystyle\lim_{x\to\infty}\frac{e^x-3}{x^2+x+2}$
7. $\displaystyle\lim_{x\to\infty}\frac{\ln(x+1)}{x+2}$
8. $\displaystyle\lim_{x\to\infty}\frac{5x^4-3x^2+5}{x^4+3x-7}$
9. $\displaystyle\lim_{x\to 2}\frac{3e^{x-2}-3}{4x-8}$

$y=f(x)$



$g(x)$ and $g'(x)$ are both continuous

x	$g(x)$	$g'(x)$
4	3	−2

10. Using the graph and table above, find the following limit. $\displaystyle\lim_{x\to 4}\frac{f(x)}{2g(x)-6}$

11. Find the absolute extrema of $f(x)=-\frac{1}{3}x^3+x$ on the interval $[-2,3]$ without using a calculator.

12. Find the absolute extrema of $g(x)=\frac{3}{2}x^{\frac{2}{3}}-x$ on the interval $[-8,1]$ without using a calculator.

Can the Mean Value Theorem be applied for these functions on the given interval? If it applies, find the c-value. If it does not apply, explain why not.

13. $f(x)=\dfrac{1}{x^2}$ on $[-1,2]$
14. $f(x)=\dfrac{x}{x-3}$ on $[4,6]$
15. $f(x)=|x-3|+2$ on $[2,5]$

16. Without using a calculator, find the x-values of all local extrema and points of inflection for the following functions. Show all number lines and analysis.
   a. $y=x^3-3x^2-9x+7$
   b. $y=3x^5-5x^3$

17. Given: $x^2+xy+y=-3$ Do not use a calculator on this problem.
    a. Find the value of $\frac{dy}{dx}$ at the point $(1,-2)$.
    b. Find the value of $\frac{d^2y}{dx^2}$ at $(1,-2)$.
    c. Use the equation of a tangent line to approximate a y-value when $x=0.9$.
    d. Is your approximation in part c an underestimate or an overestimate of the exact value of y? Explain.
    e. Does the curve have a relative maximum, a relative minimum, or neither at $(1,-2)$? Explain.

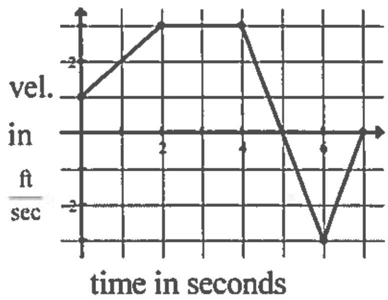
18. Find the maximum and minimum <u>points</u> on the graph of $9x^2+4y^2-54x+8y+49=0$.

19. Find a and b so that the graph of $y=ax^2+bx+3$ has a relative minimum at (2,1).

20. Given $f'(x)=\ln\dfrac{x+1}{x+2}-\cos\left(0.3x^2+5\right)$ and $f''(x)=\dfrac{1}{x+1}-\dfrac{1}{x+2}+0.6x\sin\left(0.3x^2+5\right)$, use a calculator to find the following using the domain $(-1,4)$:
    a. On what interval(s) within the domain is f increasing?
    b. At what x-value(s) within the domain does f have local extrema?
    c. At what x-value(s) within the domain does f have points of inflection?

Use the graph of a velocity function model for an object moving horizontally shown at the right for problems 21-26.

21. Find the object's acceleration at time 5 seconds.
22. Find the speed of the object at time 6 seconds.
23. On which interval of time is the object moving right?
24. On which interval(s) of time is the object's velocity increasing?
25. On which interval(s) of time is the object's speed increasing?
26. At what time is the object farthest right?



27. $x(t)=t^3-27t+50$ is the position function of a particle moving along a horizontal line.
    a. When is the velocity of the particle increasing?
    b. When is the speed of the particle increasing?
    c. What is the displacement of the particle on the interval $[0,4]$?
    d. What is the total distance traveled by the particle on the interval $[0,4]$?

28. A triangular lawn is being mowed so that the base of the triangle of unmown grass is decreasing at the rate of 2 feet per minute. If the altitude of the triangle is always three times the base, how is the area changing when the base is 100 feet?

UNIT 4 SUMMARY

Approximations using a tangent line:
Find the equation of a tangent line at a convenient point. Plug in a new x-value to find a new y-value on the tangent line which is close to a y-value on the curve.

L'Hospital's Rule

If $\lim_{x\to c} f(x)=0$ and $\lim_{x\to c} g(x)=0$ or if both of these limits are $\pm\infty$, then $\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$.

Definition of **Critical Numbers**: x-values where f' is zero or undefined (cannot be domain restrictions)

Absolute Extrema: Compare y-values at endpoints and critical numbers (**Candidate Test**).

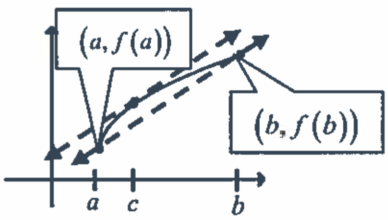
Relative (Local) Extrema:
First Derivative Test (best way): Find critical numbers, and make an f' number line. Domain restrictions must be on all number lines – but cannot be max/min points. An f' number line also gives increasing/decreasing intervals.
Second Derivative Test (Use only when you have to.): Use f' to find the critical numbers – then plug them into f'', and use concavity to see if they are at a maximum or a minimum.

Points of Inflection and Concavity: Find possible points of inflection (where f'' is zero or undefined), and make an f'' number line. Remember, domain restrictions cannot be points of inflection but must be on all number lines.

Mean Value Theorem: f must be continuous on $[a,b]$ and differentiable on (a,b).

inst. rt. ch. = avg. rt. ch.
IROC = AROC
tangent slope = secant slope

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$



MVT is used to find the c-value.

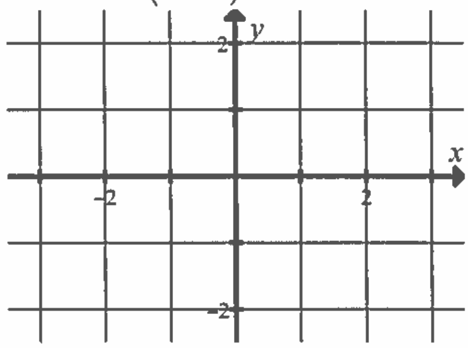
LESSON 5.1 CURVE SKETCHING WITH EXTREMA AND INFLECTION POINTS

Curve Sketching Recipe:

1. Give the domain.
2. Reduce $f(x)$.
3. Find vertical asymptotes and holes.
4. Give x- and y-intercepts.
5. Find the end behavior (horizontal asymptotes or other).
6. Find increasing/decreasing intervals and relative extrema points (show an f' number line).
7. Find concavity and points of inflection (show an f'' number line).
8. Graph.

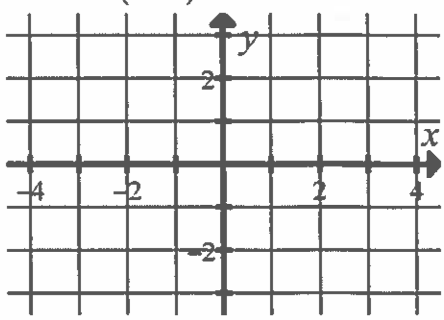
Examples:

1. (a rational function) $f(x)=\dfrac{2x^3}{x^2+1}$, $f'(x)=\dfrac{2x^2(x^2+3)}{(x^2+1)^2}$, $f''(x)=\dfrac{-4x(x^2-3)}{(x^2+1)^3}$



Do.: V.A.:
Holes: x-int.:
y-int.: E.B.:
Rel. Max. Pts.: Rel. Min. Pts.:
P.I.:

2. (a rational function) $f(x)=\dfrac{3x-2}{x^2-2x+1}$, $f'(x)=\dfrac{-3x+1}{(x-1)^3}$, $f''(x)=\dfrac{6x}{(x-1)^4}$



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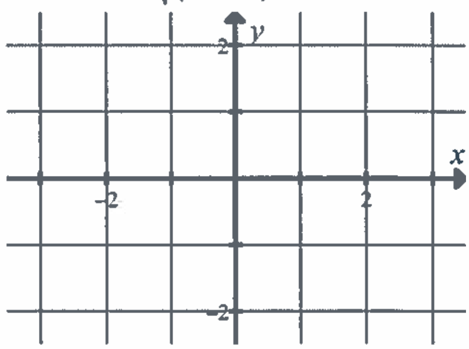
3. (a radical function) $f(x)=\dfrac{x}{\sqrt{x^2+2}}$, $f'(x)=\dfrac{2}{\sqrt{\left(x^2+2\right)^3}}$, $f''(x)=\dfrac{-6x}{\sqrt{\left(x^2+2\right)^5}}$



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ASSIGNMENT 5.1

Without using a calculator, find local extrema points, points of inflection, and sketch a graph. Show organized steps and justification. It is not necessary to find x-intercepts and there are no domain restrictions or asymptotes. However, an end behavior analysis will be helpful.

1. $y=x^3-3x^2+5$ 2. $y=1-x-x^3$ 3. $y=x^4-4x^3+16$

Find intercepts and relative extrema points and graph these functions without using a calculator.

4. $f(x)=x^2-2x-8$ 5. $g(x)=\left|x^2-2x-8\right|$

Find relative extrema points, points of inflection, and end behavior and graph without a calculator.

6. $y=\dfrac{2x^2}{x^2+3}$, $y''=\dfrac{-36\left(x^2-1\right)}{\left(x^2+3\right)^3}$

Find the domain, relative extrema points, asymptotes, and end behavior and graph without a calculator. There are no points of inflection.

7. $f(x)=\dfrac{x^2+1}{2x}$

Find the domain, reduced function, hole, intercepts, relative extrema points, and points of inflection. Then graph without using a calculator.

8. $f(x)=\dfrac{x^2\sqrt{4-x}}{x}$, $f'=\dfrac{8-3x}{2\sqrt{4-x}}$, $f''=\dfrac{3x-16}{4(4-x)^{\frac{3}{2}}}$, $f\left(\frac{8}{3}\right)=3.079$

9. Without a calculator, find the <u>domain</u>, x-intercepts, and relative extrema points. Then graph $f(x)$. There are no points of inflection. $f(x)=\sqrt{9-x^2}$

10. Find the x- and y-intercepts, relative extrema points, and points of inflection for $y=\sin x+\cos x$ on $[0,2\pi)$. Then sketch the graph of y without using a calculator.

11. Find the domain, x- and y-intercepts, local extrema points, and points of inflection for the graph of $y = xe^x - e^x$. Then sketch its graph without using a calculator.

12. Find the domain, local extrema points, and points of inflection for the graph of $y = x - \ln x$. Then sketch the graph without using a calculator.

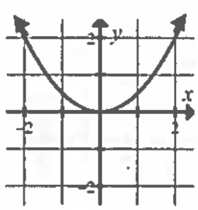
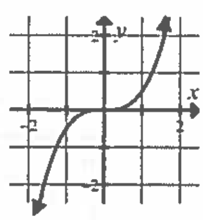
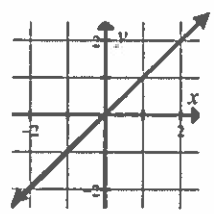
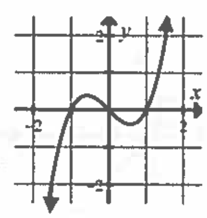
13. True or False? If $f'(x) > 0$ for all real x-values, then $\lim_{x \to \infty} f(x) = \infty$.
Show a graph to illustrate your answer.

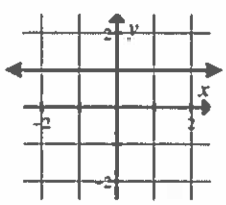
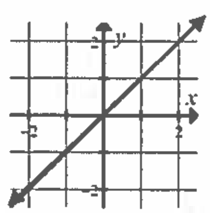
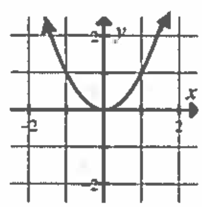
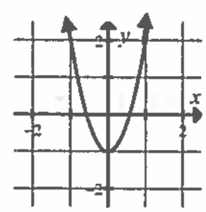
14. Find an equation of the line tangent to the graph of $g(x) = \dfrac{4^{x-2} + \ln x^2}{\sin(x-3)}$ at the point where $x = 5$. You may use a calculator.

15. a. Find the points at which the graph of $x^2 + 4y + 2y^2 = 6$ has horizontal tangent lines.
b. Determine whether each of these points is at a local minimum or a local maximum. (Show organized work using the Second Derivative Test.)

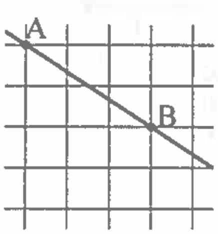
16. Find the points at which the graph of $x^2 + 4y + 2y^2 = 6$ has vertical tangent lines.

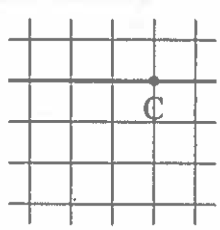
Match the graph of f in the top row with the appropriate graph of f' in the bottom row.

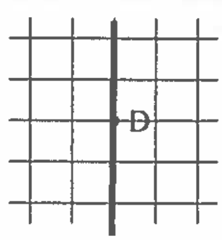
17. 
18. 
19. 
20. 

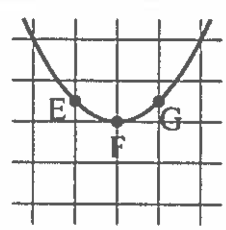
A. 
B. 
C. 
D. 

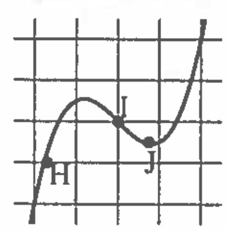
21. Find (or estimate) the slope of each graph at each lettered point.









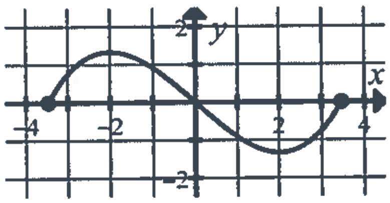


LESSON 5.2 GRAPHING DERIVATIVES AND ANTIDERIVATIVES FROM GRAPHS

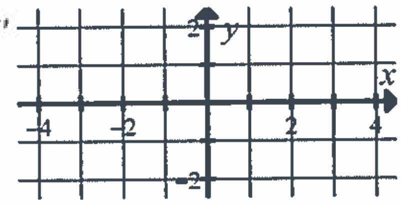
Derivatives: f graph $\rightarrow f'$ graph (or f' graph $\rightarrow f''$ graph)
Find (or estimate) slopes and plot them as points.

Example: 1. Use the graph of f shown to sketch a graph of f'.

f



f'



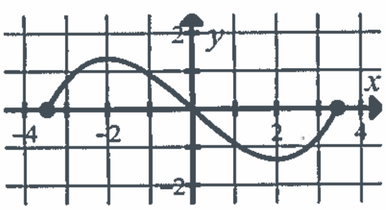
Antiderivatives: f' graph $\rightarrow f$ graph

1. Make an f' number line by using the location or position of the points on the f' graph. This does not involve the slopes of f'.
2. Make an f'' number line by using the slopes of the f' graph.
3. Combine information from both number lines to graph f. If no starting point is given, you are free to shift the graph vertically.

Examples:

2. Use the graph of f' shown to sketch a graph of f with a starting point of (0,1).

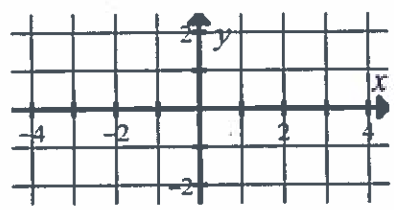
f'



f' ____________

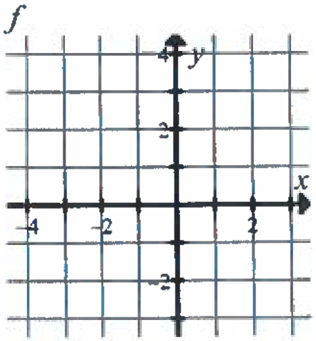
f'' ____________

f



3. Use the graph of f' shown to sketch a graph of f'' and a possible graph of f.

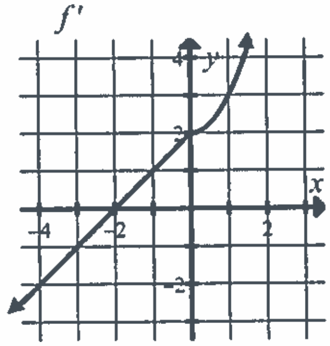
f



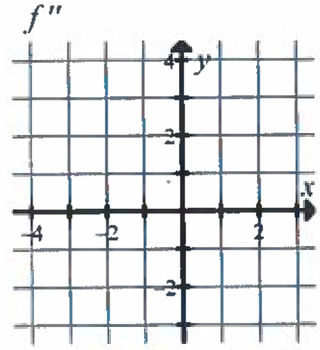
f' $\longleftrightarrow$

f'' $\longleftrightarrow$

f'

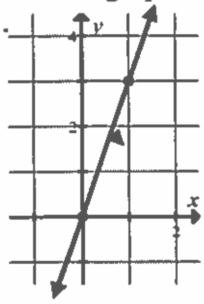


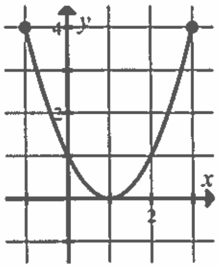
f''

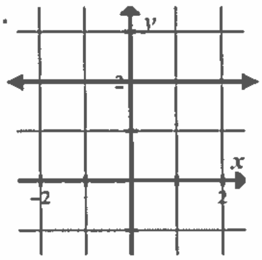


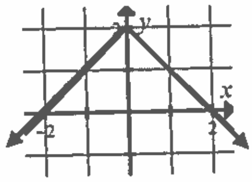
ASSIGNMENT 5.2

Sketch a graph of the derivative of the function whose graph is shown.

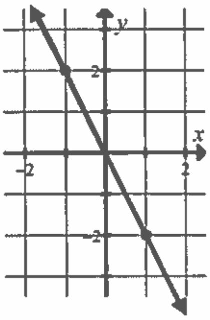
1.


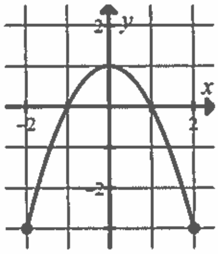
2.


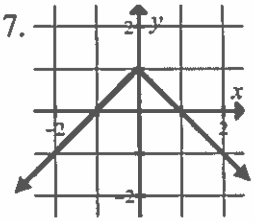
3.


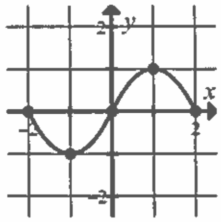
4.


Use the graph of f' shown and the given starting point to graph f (the antiderivative).

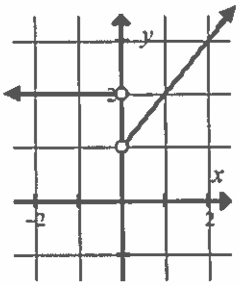
5.

starting point (0,0)

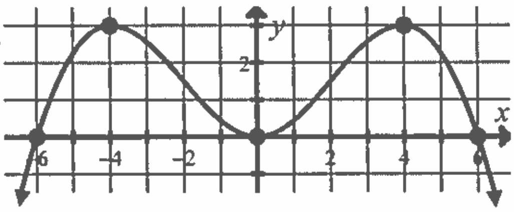
6.

starting point (1,2)

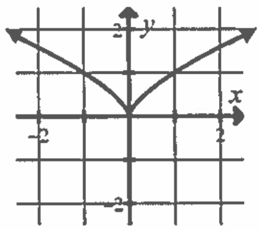
7.

starting point $(-1,2)$

8.

starting point (0,0)

Use the graph of f' shown to sketch a graph of f'' and a possible graph of f.

9.


10.


11.


12. Find the vertical asymptotes, end behavior, and relative extrema points. Then graph $f(x)=\dfrac{1}{x^2-2x-8}$ without using a calculator.

13. If $f(x)=x(x-4)^3$, find relative extrema points and points of inflection. Then graph f without using a calculator. Hint: $f''(x)=12(x-4)(x-2)$.

14. Use the <u>Second Derivative Test</u> to find the relative extrema points of $f(x)=x^3-3x^2-5$.

15. Use the following information to sketch a possible graph of f.
$f(0)=f(4)=0,\ f(2)=-2,$
$f'(x)<0$ when $x<2,\ f'(x)>0$ when $x>2,\ f'(2)$ does not exist,
$f''(x)<0$ when $x\neq 2$

16. Find the c-value guaranteed by the Mean Value Theorem for $f(x)=x^3-2x+3$ on the interval $[0,2]$.

17. Find the absolute minimum and absolute maximum of the function $f(x)=x^3-12x-2$ on the interval $[0,4]$ without using a calculator.

18. Without using a calculator, sketch a graph of $f(x)=\left|-x^2-6x\right|$.

19. The graph of $y=ax^2+bx+c$ passes through the point $(1,6)$ and has a tangent line at $(0,16)$ which is parallel to the graph of $y=-12x-2$. Find a, b, and c.

20. If the only critical number of a function $f(x)$ is $x=3$, $f'(2)=-6$, and $f'(4)=7$, does f have a local minimum or a local maximum at $x=3$? Assume f is continuous.

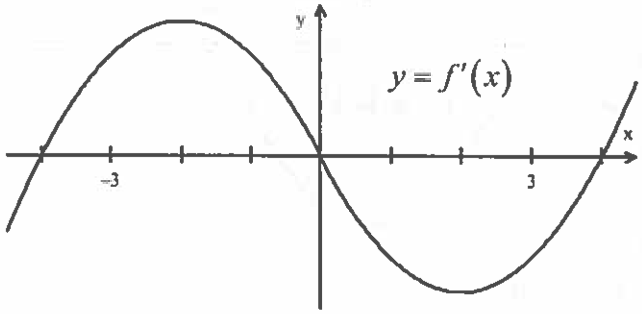
21. If $x=3$ is a critical number of a function $g(x)$ and $g''(3)=-6$, does g have a relative minimum or a relative maximum at $x=3$?

True or False?
22. Every fourth degree polynomial has three critical numbers.
23. Every fourth degree polynomial has at most three critical numbers.
24. If a polynomial has two critical numbers, one must be at a relative maximum and the other must be at a relative minimum.
25. If $f'(2)>0$, then f is increasing on some interval containing $x=2$.
26. If a function f is increasing on an interval containing $x=2$, then $f'(2)>0$.

LESSON 5.3 USING GRAPHS OF THE FIRST DERIVATIVE WITH JUSTIFICATION

An extremely common type of Free Response Question on the AP™ Calculus test is one where a graph of a first derivative is given and students are asked make conclusions about the original function with justification. It is very important that students reference the given graph by name in their justifications. Typically, these justifications require a short sentence. It is unnecessary and unwise to write more than one sentence of justification. If a student justifies correctly but continues by writing a false statement the justification point is lost.
Many students find it helpful to draw first or second derivative number lines (sign charts). These number lines do **not** count as justification but may help the student answer the question correctly which is, of course, the first priority.

The graph below is a graph of a differentiable function $f'(x)$ the derivative of a function $y = f(x)$. This graph of $f'(x)$ has horizontal tangents only at $x = \pm 2$ and its domain is $(-\infty, \infty)$.



Examples:

1. On what **open** interval(s) is f increasing? Justify your answer.

2. On what interval(s) is f decreasing? Justify your answer.

3. At what x-value(s) does f have a relative maximum. Justify your answer.

4. At what x-value(s) does f have a local minimum. Justify your answer.

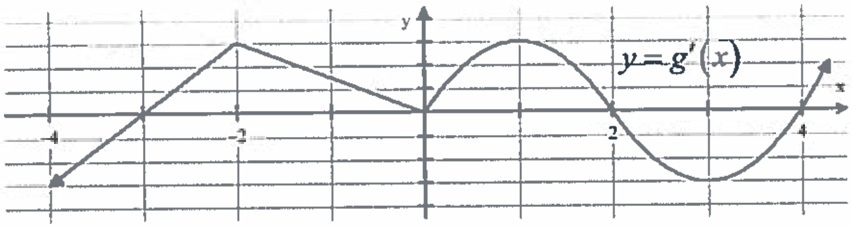
5. On what interval(s) is f concave down? Justify your answer.

6. At what x-value(s) does f have a point of inflection. Justify your answer.

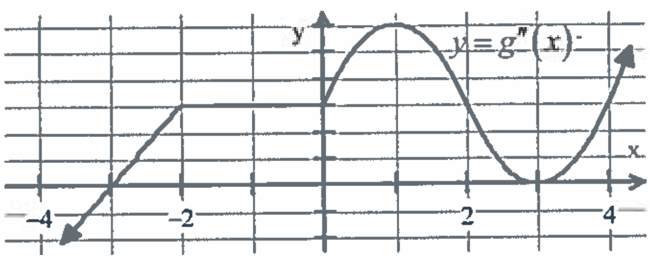
7. On what interval(s) is f both decreasing and concave upward? Justify your answer.

ASSIGNMENT 5.3

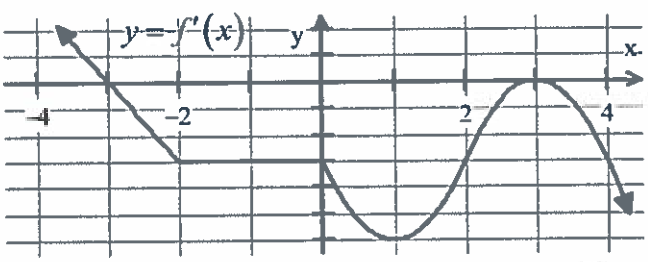
Use the graph shown in each problem to answer the questions and justify your answers. Assume there is no hidden behavior.

1. 

   a. On what interval(s) is g decreasing? Justify.
   b. On what interval(s) is g increasing? Justify.
   c. At what x-value(s) does g have a relative minimum? Justify.
   d. On what interval(s) is g concave upward? Justify.
   e. At what x-value(s) does g have a point of inflection. Justify.

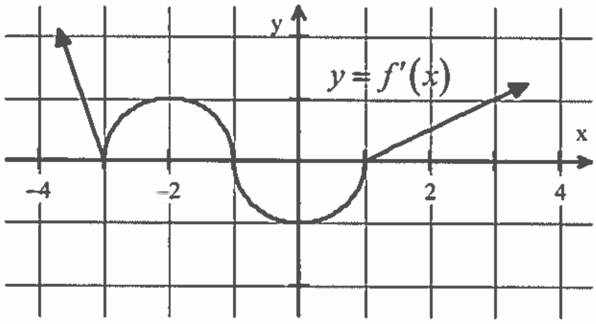
2. 

   a. On what interval(s) is g concave upward? Justify.
   b. At what x-value(s) does g have a point of inflection. Justify.

3. 

   a. On what interval(s) is f decreasing? Justify.
   b. At what x-value does f have a local minimum. Justify.
   c. On what interval(s) is f concave downward? Justify.
   d. At what x-value(s) does f have a point of inflection. Justify.
   e. On what interval(s) is f both increasing and concave upward? Justify.

4. $y=f'(x)$ graphed below is a piecewise function consisting of two linear pieces and two semicircles as shown.



a. On what interval(s) is f decreasing? Justify.
b. On what interval(s) is f concave upward? Justify.
c. At what x-value(s) does f have a point of inflection. Justify.
d. At what x-value(s) does f have a relative minimum? Justify.
e. Find $f''(-3)$ or explain why it does not exist.
f. Find $f''(-2)$ or explain why it does not exist.
g. Find $f''(-1)$ or explain why it does not exist.
h. Find $\lim\limits_{x\to 3}\dfrac{3x-9}{f'(x)-1}$.

Find the discontinuities. Which are removable? Do not use a calculator.

5. $f(x)=\dfrac{x-2}{x^2-3x+2}$ 6. $f(x)=\begin{cases} x^2-4x, & x\le 0 \\ x-1, & x>0 \end{cases}$ 7. $f(x)=\left\lfloor \frac{x}{3} \right\rfloor$

Find these limits without a calculator.

8. $\lim\limits_{x\to 2}\dfrac{x-2}{x^2-4}$ 9. $\lim\limits_{x\to 2}\dfrac{x+2}{x^2-4}$ 10. $\lim\limits_{x\to \infty}\dfrac{x+2}{x^2+4}$

11. Find the absolute extrema of the function $f(x)=-x^2+3x$ on the interval $[-1,3]$.

12. The function $f(x)=\dfrac{4000}{1+100e^{-0.5x}}$ can be used to model how a disease spreads in an isolated population of 4000 people. x represents the time in days since the sickness started and $f(x)$ represents the number of people who have become sick.
Use a calculator to help answer the questions below.
a. How many people have become sick by the tenth day?
b. How fast was the disease spreading on the tenth day?
c. Use the maximum function on your calculator to find the maximum point on f'. What does the x-coordinate represent? What does the y-coordinate represent? **Note:** Finding a maximum or minimum with a calculator will not be allowed on the AP™ Calculus test.
d. How many people have caught the disease when the curve is the steepest?
e. Why would the slope of the curve decrease after a period of time?
f. When was the rate of the spread of the disease increasing the fastest?

13. Without using a calculator, find vertical asymptotes, relative extrema points, and end behavior, and then sketch a graph of $f(x)=\frac{x^2+2}{x^2-9}$.

14. Use the intercepts, vertical asymptotes, relative extrema points, and end behavior to graph f, if $f(x)=\frac{x+1}{x^2-4x+4}$ and $f'(x)=\frac{-x-4}{(x-2)^3}$. Do not use a calculator.

15. Find all points of inflection of $f(x)=\frac{1}{4}x^4-2x^3+2x-6$ without using a calculator.

16. Find all relative extrema <u>points</u> on the graph of $y=\frac{x}{x^3+4x}$ without using a calculator.

17. Find the c-value guaranteed by the Mean Value Theorem for the function $y=-3x^3-2x^2+3x-5$ on the interval $[-2,0]$. You may use a calculator.

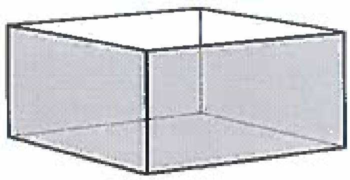
<u>LESSON 5.4</u> <u>MAX/MIN APPLICATIONS (OPTIMIZATION)</u>

One of the most common applications of Calculus involves determining maximum or minimum values.

<u>Procedure:</u>
1. Choose variables and/or draw a labeled figure.
2. Write a primary equation. Isolate whatever is to be maximized or minimized.
3. Rewrite with only one variable on each side. This may require a secondary equation.
4. Find the domain.
5. Take the derivative, find critical numbers, make a number line, etc.

<u>Examples:</u> **Answer with a complete sentence.**

1. A box with no lid is to be made from 48 cm^2 of material. If the box must have a square base, find the dimensions that produce a maximum volume.



2. The product of two positive numbers is 288. Find the two numbers so that the sum of twice the first plus the second is as small as possible.

ASSIGNMENT 5.4

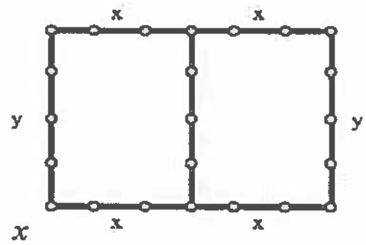
Write sentence answers on 1-6.

1. The product of two positive numbers is 100. Find the two numbers so that the sum of the numbers is as small as possible.

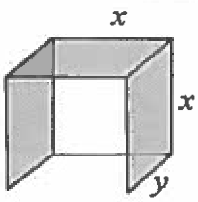
2. The area of a rectangle is 81 cm^2. Find the length and width so that the rectangle has a minimum perimeter.

3. The perimeter of a rectangle is 80 feet. Find the length and width so that the rectangle has a maximum area.

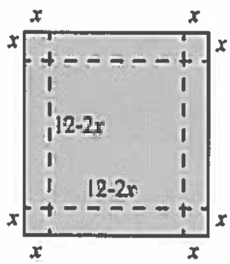
4. Two adjacent rectangular corrals are to be made using 240 feet of fencing. The fence must extend around the outer perimeter and across the middle as shown in the diagram. Find the dimensions so that the total enclosed area is as large as possible.



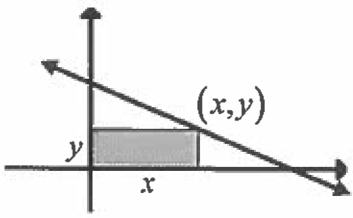
5. A shelter at a bus stop is to be made with three Plexiglas sides and a Plexiglas top. If the volume of the shelter is 486 cubic feet, find the dimensions that require the least amount of Plexiglas.



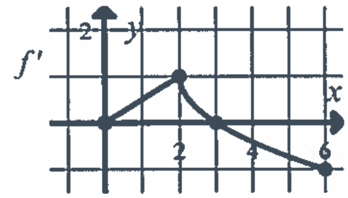
6. A box is made by cutting small squares from each corner of a piece of square material 12 inches on each side and then folding up the flaps. Find the side of the square cutouts that will produce the greatest volume box.



7. A rectangle is positioned with one vertex on the line $y=-\frac{1}{2}x+3$ as shown. Find the point so that the rectangle has a maximum area.

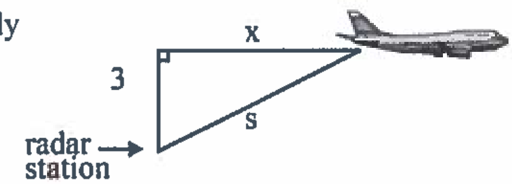


8. Use the graph of f' shown to graph f'' and a graph of f with the starting point $(0,2)$.

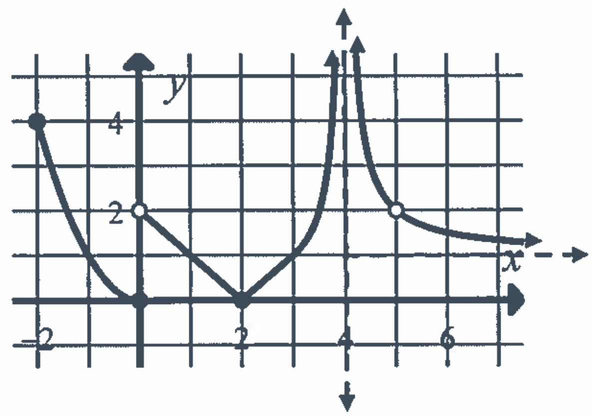


9. The volume formula for a cone is $V = \frac{1}{3}\pi r^2 h$. If $\frac{dr}{dt} = 3\,\frac{in}{min}$ and $h = 3r$, find $\frac{dV}{dt}$ when $r = 6$ inches.

10. An airplane flying at an altitude of 3 miles flies directly over a radar station. When the plane is 5 miles away from the station, the radar shows the distance s is changing at the rate of 300 miles per hour. What is the plane's speed?



11. Use the graph of $y = f(x)$ at the right for these problems.



a. Find $\lim\limits_{x\to 0} f(x)$.

b. Find $\lim\limits_{x\to 0^+} f(x)$.

c. Find $\lim\limits_{x\to 4} f(x)$.

d. Find $\lim\limits_{x\to \infty} f(x)$.

e. Find $\lim\limits_{x\to 3} f(x)$.

f. List the discontinuities of f.

g. Which of these discontinuities are removable?

h. Find the absolute maximum of $f(x)$ on $[-2,3]$.

i. Find the absolute minimum of $f(x)$ on $[-2,3]$.

j. Find $f'(1)$.

k. Find $f''(1)$.

l. List all x-values where $f'(x)$ does not exist.

m. List all x-values at which $f(x)$ has a local minimum.

n. List all x-values at which $f(x)$ has a local maximum.

LESSON 5.5 MORE MAX/MIN APPLICATIONS

Example: The sum of two nonnegative numbers is 30. Find both numbers if the sum of twice the first plus the square of the second is a maximum.

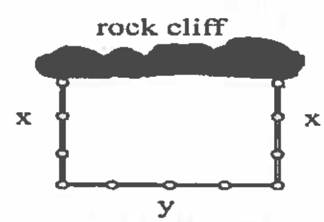
ASSIGNMENT 5.5

1. The product of two positive numbers is 300. Find the two numbers so that the sum of the first plus three times the second is as small as possible.

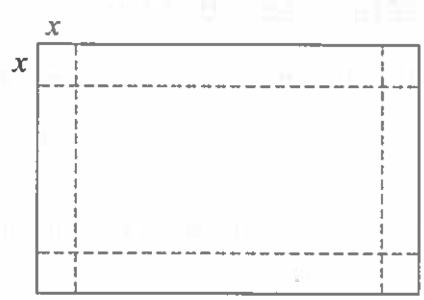
2. The sum of two nonnegative numbers is 25. Find the two numbers so that the sum of the first plus the square of the second is a minimum.

3. The sum of two nonnegative numbers is 25. Find the two numbers so that the sum of the first plus the square of the second is a maximum.

4. A rancher plans to fence in three sides of a rectangular pasture – with the fourth side being against a rock cliff. He needs to enclose 320,000 square meters of pasture. What dimensions would require the least amount of fence material.

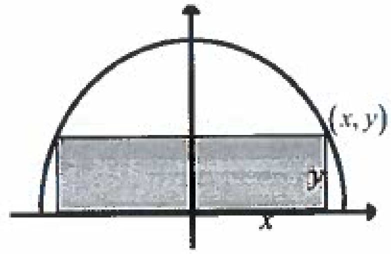


5. A box is to be made by cutting small squares from each corner of a 3 *ft* by 5 *ft* rectangular piece of material. Find the size of the square cutouts that would produce a box with maximum volume. (Your $V' = 0$ equation will not be factorable. You may use a calculator to solve it.) Show three or more decimal place accuracy.



6. Find the volume of the box in Problem 5. Show 3 or more decimal place accuracy.

7. A rectangle is positioned with two points on the semicircle $y=\sqrt{36-x^2}$ as shown. Find the point (x,y) so that the area of the rectangle is a maximum.



8. Find the area of the rectangle in Problem 7.

9. A box with an open top has a square base. If the volume of the box is 4000 cubic centimeters, what dimensions minimize the amount of material used?

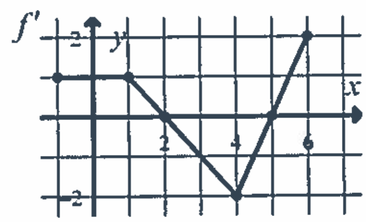
10. Find the relative extrema points and points of inflection and graph f, if $f(x)=-x^4+4x^3-16x+2$ and $f'(x)=-4(x+1)(x-2)^2$.

11. Find the relative extrema points and concavity and graph $g(x)=3x^{\frac{2}{3}}-2x$.

Find the derivative.

12. $y=\dfrac{1}{(x^2-2x)^3}$ 13. $f(x)=\sqrt{x^3-3x+2}$ 14. $g(x)=\dfrac{x+1}{2x+3}$

15. Use the graph of f' shown to graph f'' and a possible graph of f.



ASSIGNMENT 5.6 REVIEW

1. Find the absolute maximum and minimum values of $y=3x^4+4x^3-12x^2+5$ on the interval $\left[-\frac{5}{4},\frac{5}{4}\right]$. Find the critical numbers without using a calculator.

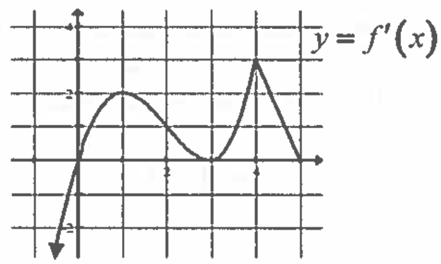
2. Without using a calculator, find the domain, the intercepts, the vertical asymptote, the end behavior, the relative extrema points, and the points of inflection. Then sketch a graph of $f(x)=\dfrac{x^3+2}{x}$. Hint: $f'(x)=\dfrac{2x^3-2}{x^2}$ and $f''(x)=\dfrac{2x^3+4}{x^3}$.

3. Without using a calculator, find the intercepts, the local extrema points, and the points of inflection, and then draw a graph of $f(x)=\left|-x^3+3x^2\right|$. Hint: First graph the same function without the absolute value.

4. If $f(x)=x\sqrt{16-x^2}$, $f'(x)=\dfrac{16-2x^2}{\sqrt{16-x^2}}$, and $f''(x)=\dfrac{2x^3-48x}{\left(16-x^2\right)^{\frac{3}{2}}}$, find the <u>domain</u>, intercepts, relative extrema points, and points of inflection, and then sketch a graph of the function f without using a calculator.

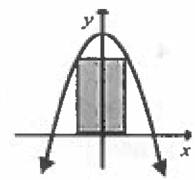
5. Find the intercepts, relative extrema points, and points of inflection, and then graph $f(x)=x^3-3x$ without using a calculator.

Use the graph of $f'(x)$ shown for problems 6 and 7.



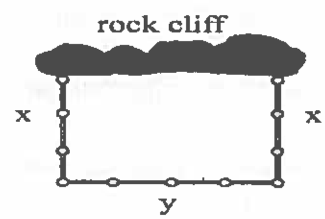
6. Sketch a graph of $f''(x)$.
7. Sketch a graph of $f(x)$ which contains the point $(0,0)$.

8. Find the dimensions of the rectangle with maximum area inscribed under the curve $y=-x^2+12$ as shown.

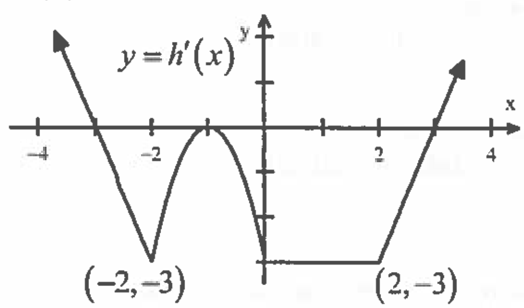


9. The second of two positive numbers is the reciprocal of the first. Find the two numbers so that their sum is a minimum.

10. A rancher plans to fence in three sides of a rectangular pasture with the fourth side being against a rock cliff. If he has 1200 yards of fencing to use, what is the maximum area he can enclose?



11. The graph of $y=h'(x)$ is shown with no hidden behavior.



a. On what interval(s) is h decreasing? Justify your answer.
b. At what x-value(s) does h have a relative minimum? Justify your answer.
c. On what interval(s) is h concave upward? Justify your answer.
d. At what x-value(s) does h have a point of inflection. Justify your answer.
e. On what interval(s) is h increasing and concave downward? Justify your answer.
f. On what interval(s) is h decreasing and linear? Justify your answer.
g. If $f(x)=3\left(h'(x)\right)^2+4x^3$ find $f'(-2.5)$. You may use a calculator.
h Find $\lim\limits_{x\to-3}\dfrac{h'(x)}{x^2-9}$.

12. If $x = 3$ is a critical number of a function f, and $f''(3) < 0$, does f have a local maximum, a local minimum, or neither at $x = 3$?

13. What is the maximum height (in feet) reached by a ball thrown upward, if the ball's height is given by the position equation $s(t) = -16t^2 + 64t + 6$? Do **not** use a calculator.

14. Find a, b, and c for $f(x) = ax^2 + bx + c$, such that $f(1) = 10$, and $f(x)$ has a relative minimum at $(-1, 2)$.

Determine if the Mean Value Theorem can be applied to $f(x)$ on the given interval. If it can be applied, find the c-value. If it cannot be applied, explain why not. You may use a calculator. Answer with three or more decimal place accuracy.

15. $f(x) = x\sqrt{2x-1}$ on $[1,5]$

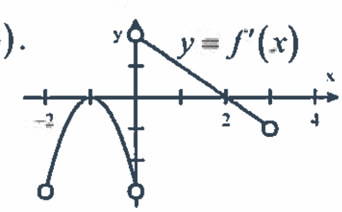
16. $f(x) = \begin{cases} x^2, & x \le 0 \\ x, & x > 0 \end{cases}$ on $[-1,3]$

17. Use the implicit relation $2xy + y^4 = x^2 + 1$ containing the point (2,1) on the graph for the following. Do not use a calculator.
    a. Find $\frac{dy}{dx}$.
    b. Is the curve increasing or decreasing on an interval containing the point (2,1)?
    c. Use the equation of a tangent line to approximate the y-coordinate when $x = 1.9$.
    d. Is the curve concave upward or downward on an interval containing the point (2,1)?
    e. Is your approximation in part c an underestimate or an overestimate. Explain.

18. Given $g'(x) = 2x^3 - 0.1e^x + x$. You may use a calculator.
    a. On what interval(s) is g increasing?
    b. At what x-value(s) does g have a relative minimum?
    c. Find $g''(x)$.
    d. On what interval(s) is g concave upward?
    e. At what x-value(s) does g have a point of inflection?

19. Given $f'(3) = 0$ and $f''(3) = -3$.
    a. Does f have a relative maximum, a relative minimum, or neither at $x = 3$?
    b. If f has only one critical point, what can be said about the absolute extrema of f on $[0,5]$?

20. Use the graph of $y = f'(x)$ shown, given f is continuous on $(-2,3)$.
    a. On what interval(s) is f decreasing? Justify.
    b. At what x-value(s) does f have a relative minimum? Justify.
    c. On what interval(s) is f concave upward? Justify.
    d. At what x-value(s) does f have a point of inflection. Justify.



21. A point moves along the curve $y = \sqrt[3]{x}$ so that the y-coordinate is increasing at the rate of two units per second. At what rate is the x-coordinate changing when $x = 8$ units?

22. Find $\left(f^{-1}\right)'(5)$ if $f(x) = x^3 - 4x^2 + 3x - 7$ on the interval $[3, \infty)$.

UNIT 5 SUMMARY

Curve Sketching:

Precalculus: domain, intercepts, vert. asymptotes, holes, end behavior, symmetry
Calculus: f' number line $\rightarrow$ inc./decr. and max./min.
f'' number line $\rightarrow$ concavity, pts. of infl.

Graph to Graph:

$f' \rightarrow f''$ Find slopes on f' and plot them as points on f''.
$f' \rightarrow f$ Make an f' number line using the **location** of points on the f' graph.
Make an f'' number line using the **slope** at points on the f' graph.
Use both number lines to sketch a graph of f.

Max/Min Applications:

Procedure:
1. Choose variables and/or draw a labeled figure.
2. Write a primary equation. Isolate whatever is to be maximized or minimized.
3. Rewrite with only one variable on each side. This may require a secondary equation.
4. Find the domain.
5. Take the derivative, find critical numbers, make a number line, etc.

LESSON 6.1 ANTIDIFFERENTIATION, INDEFINITE INTEGRALS

Warm-up Examples: Differentiate each of the following.

1. $f(x) = x^3$

2. $f(x) = x^3 - 10$

3. $f(x) = x^3 + C$
where C is any constant (number)

So what should you get when you antidifferentiate $f'(x) = 3x^2$? $f(x) =$ ______

This problem can be written as $\int 3x^2 dx =$

The symbol $\int$ is called an integral symbol and tells you to integrate (antidifferentiate) the expression which follows it. That expression is called an integrand. dx indicates that you are integrating with respect to the variable x but does not affect the integration process. C is called the constant of integration and must be written as part of your answer when you are antidifferentiating.

Integration Rules:

Power Rule: $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C, \quad n \neq -1$

Constant Rule: If k is any constant, $\int k\,dx = kx + C$

Scalar Multiple Rule: If k is any constant, $\int k f(x)\,dx = k\int f(x)\,dx$
(Constants may be "factored out" of the integral expression. **NEVER** "factor out" a variable.)

Sum Rule: $\int \big(f(x) + g(x)\big)\,dx = \int f(x)\,dx + \int g(x)\,dx$

Examples: Evaluate (Integrate).

4. $\int x^3 dx$

5. $\int 2\,dx$

6. $\int (t^4 + 2)\,dt$

7. $\int (2y^2 + 4y + 1)\,dy$

8. $\int \left(\dfrac{3}{x^2} - \dfrac{1}{\sqrt{x}}\right) dx$

9. $\int \dfrac{\sqrt{x}+1}{x^2}\,dx$

Note: Put $+C$ when you integrate, but never when you differentiate.

Sometimes an initial condition is given which makes it possible to solve for C.

Example 10: If $f'(x) = x^{-3}$ and $f(1) = \frac{3}{2}$, find $f(x)$.

Example 11: Evaluate $\frac{d}{dx}\int\left(5\sqrt{x} - \frac{1}{x^2}\right)dx$

If we know the acceleration equation for an object, and if we are given initial conditions for the object's velocity and position, integration allows us to find the velocity and position equations for the object.

Remember: Pos. $\rightarrow$ Vel. $\rightarrow$ Acc. (Differentiate), so Acc. $\rightarrow$ Vel. $\rightarrow$ Pos. (Integrate).

Example 12: The acceleration of a particle at time t is given by $a(t) = 4t - 3$.
$v(1) = 6$ and $s(2) = 5$.

a. Find the velocity equation. $v(t) =$

b. Find the position equation. $s(t) =$

Example 13: Given that on earth, the acceleration of an object due to gravity is approximately $-32\ ft/sec^2$ (negative indicates downward), develop

a. the equation for the velocity of the object.
$v_o =$ initial velocity

$v(t) =$

b. the equation for the position of the object.
$s_o =$ initial position

$s(t) =$

Note: The two equations $v(t) = -32t + v_0$ and $s(t) = -16t^2 + v_0 t + s_0$ may be used for any motion affected only by the earth's gravity.

ASSIGNMENT 6.1

For Problems 1-4, rewrite the integrand and then integrate.

1. $\int \frac{1}{x^3}\,dx$ 2. $\int \sqrt[4]{t}\,dt$ 3. $\int (x+1)(x-2)\,dx$ 4. $\int \frac{2y}{\sqrt{y}}\,dy$

Evaluate (integrate) each integral in Problems 5-13.

5. $\int (2x^3 - x^2 + 1)\,dx$ 6. $\int \frac{1}{3x^2}\,dx$ 7. $\int \frac{1}{(3x)^2}\,dx$

8. $\int \left(3 - \sqrt[5]{y^2}\right) dy$ 9. $\int \left(5x^{\frac{1}{4}} - x^{\frac{-2}{3}}\right) dx$ 10. $\int (3t-10)^2\,dt$

11. $\int \frac{8x^4 - 2x^2 + 1}{2x^2}\,dx$ 12. $\int \frac{2\sqrt{t} - 1}{\sqrt{t}}\,dt$ 13. $\int \sqrt{y}\left(y^2 + 2\sqrt{y}\right) dy$

14. If $f'(x) = 3x^2 - 4x + 2$ and $f(1) = -3$, find $f(x)$.

15. The derivative of a function is $\frac{dy}{dt} = \frac{-3}{t^2} + 1$. If the graph of the function contains the point (3, 10), find the equation of the function.

16. a. Find an equation for the family of functions whose derivative is $y' = 3\sqrt{x}$.
    b. Find the particular function from the family in Part a. whose curve passes through the point (4, 0).

17. Find $g(x)$, given that: $g''(x) = 2x - 3$, $g'(0) = -5$, and $g(-1) = 2$.

18. Evaluate $\frac{d}{dx}\int (2x-1)^3\,dx$. Hint: This is a derivative of an integral.

19. The acceleration of an object moving along a horizontal path is given by the equation $a(t) = 6t - 4$. The object's initial velocity is 5, and its initial position is -2.
    a. Find a velocity equation for the object.
    b. Find the velocity of the object when $t = 2$.
    c. Find a position equation for the object.
    d. Find the object's position when $t = 2$.

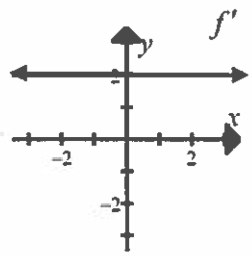
20. The velocity of an object moving along a vertical path is given by the equation $v(t)=\sqrt{t}+1, \quad t\ge 0$.
    a. Find an equation for the object's acceleration.
    b. Find the acceleration of the object when $t=9$.
    c. The object's position at $t=9$ is 20. Find an equation for the object's position.

21. A ball is dropped from a bridge which is 160 feet above a river. How long will it take the ball to hit the water? Use the equation $s(t)=-16t^2+v_o t+s_o$.

22. For the first 4 seconds of a race, a sprinter accelerates at a rate of 3 meters per second per second (3 m/sec^2). He then continues to run at the constant speed that he has attained for the rest of the race.
    a. Write a piecewise function to express the sprinter's velocity $v(t)$ as a function of time.
    b. Find $v(2)$, $v(4)$, and $v(6)$.
    c. Write a piecewise function to express the sprinter's position $s(t)$ as a function of time.
    d. How far does the sprinter run during the first 4 seconds of the race?
    e. How long will it take the sprinter to run 100 m?

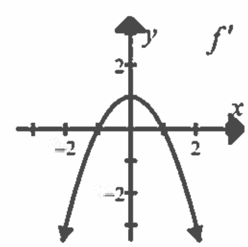
For Problems 23 and 24, the graph of the derivative (f') of a function is given. Sketch a possible graph of the function f.

23.



f contains the point (0, 1)

24.



25. List the domain, vertical asymptote(s), hole(s), x- and y-intercepts, end behavior, and type(s) of symmetry for the graph of $y=\dfrac{x^2}{x^2-1}$. Then sketch the graph without using a calculator.

26. Find an equation of a line tangent to the curve $y=x^{\frac{2}{3}}$ which is parallel to the line $2x-6y=5$.

27. Find the cubic function of the form $y=ax^3+bx^2+cx+d$ which has a relative maximum point at (0, 2) and a point of inflection at $(-1,-2)$.

28. Find each of the following derivatives. It is very important that you know each of these without referring to notes.

a. $\frac{d}{dx}e^x=?$ b. $\frac{d}{dx}\ln x=?$ c. $\frac{d}{dx}\sin x=?$ d. $\frac{d}{dx}\cos x=?$ e. $\frac{d}{dx}\tan x=?$

f. $\frac{d}{dx}\cot x=?$ g. $\frac{d}{dx}\sec x=?$ h. $\frac{d}{dx}\csc x=?$ i. $\frac{d}{dx}a^x=?$ j. $\frac{d}{dx}\log_a x=?$

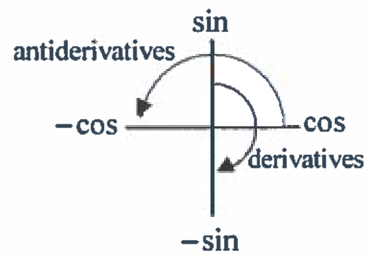
LESSON 6.2 ANTIDIFFERENTIATION OF EXPONENTIALS AND TRIG FUNCTIONS

The derivative rules from previous units can all be reversed to create the following corresponding antiderivative rules (integration rules).

$\int e^x dx = e^x + C$	$\int a^x dx = \dfrac{a^x}{\ln a} + C$	$\int \cos x\, dx = \sin x + C$
$\int \sin x\, dx = -\cos x + C$	$\int \sec^2 x\, dx = \tan x + C$	$\int \csc^2 x\, dx = -\cot x + C$
$\int \sec x \tan x\, dx = \sec x + C$	$\int \csc x \cot x\, dx = -\csc x + C$	

The most common errors made with the trig functions are sign errors.

The simple diagram at the right, reminiscent of the unit circle, may aid in remembering rules for these two functions which are used far more than any other trig functions. Rotate clockwise for derivatives. Rotate counterclockwise (anticlockwise) for antiderivatives.



Examples: Find the following antiderivatives.

1. $\int -3\sin x\, dx$ 2. $\int \left(e^x - 2^x\right) dx$ 3. $\int \left(5\sqrt[3]{t^2} + 3\cos t - 1\right) dt$ 4. $\int \dfrac{\sec y \tan y}{4} dy$

5. If $f'(x) = (\csc x)^2$ and $f\left(\frac{5\pi}{4}\right) = 3$, find $f(x)$.

ASSIGNMENT 6.2

Integrate as indicated without using a calculator.

1. $\int (\sec^2\theta - 2)\, d\theta$
2. $\int \left(\theta^2 + 6\sec\theta\tan\theta\right) d\theta$
3. $\int \dfrac{\cos x}{5} dx$
4. $\int \dfrac{1}{\sin^2 y} dy$
5. $\int \left(\dfrac{2}{e^{-x}} + \sqrt{x}\right) dx$
6. $\int \dfrac{4x^4 - xe^x}{x} dx$
7. $\int (\sin y - 2\cos y)\, dy$
8. $\int \left(6^x - \dfrac{2}{x^4}\right) dx$
9. $\int \dfrac{3 + \cos^2 x}{\cos^2 x} dx$
10. $\int \dfrac{1}{\sin x \tan x} dx$
11. $\int \sqrt{9e^{2x}}\, dx$
12. $\int \left(\dfrac{5^{3x}}{5^{2x}}\right) dx$

13. The velocity of an object moving along a horizontal path is given by the equation $v(t) = 4\sin t + 3t^2$.

    a. Find an equation for the object's acceleration.
    b. Find an equation for the object's position if the initial position is 3.
    c. Find the object's position at $t = \pi$.

For Problems 14-17, write an equation for a line tangent to each curve at the given point. Do not use a calculator.

14. $f(x) = -\cos(2x)$

    Contains the point $\left(\frac{\pi}{6}, -\frac{1}{2}\right)$

15. $g(x) = \tan\left(x^2 - 1\right)$

    at the point where $x = 1$

16. $y = 2\sin(-x)$

    at the point where $x = \frac{\pi}{4}$

17. $f(x) = 4e^x$

    at the point where $x = 0$

18. Use a calculator to write an equation for the line tangent to the graph of $f(x) = \ln\left(|\cos x| + 2\right)$ at the point where $x = .821$.

Find the limits in Problems 19-21 without using a calculator.

19. $\lim\limits_{x \to \frac{\pi}{2}} \left(\dfrac{\cos x}{x - \frac{\pi}{2}}\right)$

20. $\lim\limits_{x \to 0} \dfrac{\sin(2x)}{\sin(-5x)}$

21. $\lim\limits_{\theta \to \frac{\pi}{2}} \dfrac{-2\cos\theta}{e^{\theta - \frac{\pi}{2}} - 1}$

Find derivatives for Problems 22-27 without using a calculator.

22. $y = \dfrac{\sqrt{2t - 3}}{t}$

23. $f(x) = e^{\sec x}$

24. $h(\theta) = 2\theta\cos\theta - \sin(5\theta)$

25. $g(t) = -4\cot\left(3t^2\right)$

26. $\frac{d}{dx}\ln\left|\sin^3 x\right| =$

27. $\sin(y - 2x) = x^2 - 10$

    $\frac{dy}{dx} =$

28. Find the instantaneous rate of change for $f(t) = \dfrac{t}{t+1}$ when $t = 1$.

29. Find the average rate of change for $f(t) = \dfrac{t}{t+1}$ on [0, 2].

30. Which of the rates of change from Problems 28 and 29 represents:

    a. the slope of a secant line for the graph of $f(t)$?
    b. the slope of a tangent line for the graph of $f(t)$?

31. Find the value of c in [0, 2] such that $f'(c) =$ the average rate of change of $f(t) = \dfrac{t}{t+1}$ on [0, 2]. It is at this t-location that the slopes of what two lines are the same? (MVT).

LESSON 6.3 REVERSE CHAIN RULE, u-SUBSTITUTION

Warm-up Examples:
Differentiate

1. $\frac{d}{dx}(1+5x)^4 =$

2. $\frac{d}{dx}\sin(1+5x) =$

Now, integrate

3. $\int 5(1+5x)^3\,dx =$

4. $\int 5\cos(1+5x)\,dx =$

Note: You must insert the chain rule factor, the derivative of the inside function in Examples 1 and 2, so you had to delete the derivative of the inside function in Examples 3 and 4.

Each of the Integration Rules from the last lesson can now be generalized as Reverse Chain Rule integrals.

	x form	**u form** (Reverse Chain Rule)
Power Rule:	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$	$\int u^n u'\,dx = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
Exponential Rules:	$\int e^x\,dx = e^x + C$	$\int e^u u'\,dx = e^u + C$
	$\int a^x\,dx = \frac{a^x}{\ln a} + C$	$\int a^u u'\,dx = \frac{a^u}{\ln a} + C$
Trig Rules:	$\int \cos x\,dx = \sin x + C$	$\int \cos u\; u'\,dx = \sin u + C$
	$\int \sin x\,dx = -\cos x + C$	$\int \sin u\; u'\,dx = -\cos u + C$
	$\int \sec^2 x\,dx = \tan x + C$	$\int \sec^2 u\; u'\,dx = \tan u + C$
	$\int \csc^2 x\,dx = -\cot x + C$	$\int \csc^2 u\; u'\,dx = -\cot u + C$
	$\int \sec x \tan x\,dx = \sec x + C$	$\int \sec u \tan u\; u'\,dx = \sec u + C$
	$\int \csc x \cot x\,dx = -\csc x + C$	$\int \csc u \cot u\; u'\,dx = -\csc u + C$

Examples: Integrate.

5. $\int (3x-1)^{10}\,dx$

6. $\int (3t^2+2t)(t^3+t^2)\,dt$

7. $\int \frac{6x^2}{\sqrt{4x^3-5}}\,dx$

8. $\int (y^3+1)^2\,dy$

9. $\int \sin(4x)\,dx$

10. $\int 3\theta\cos\theta^2\,d\theta$

11. $\int \sin^2 x\cos x\,dx$

u-Substitution

For more complicated integration problems, simple rules for integration might fail, and you may have to make some type of substitution to be able to integrate. In this course, a common substitution will be to let $u =$ the radicand $\left(\sqrt{\text{radicand}}\right)$ part of the expression and to change the variable throughout the integral before integrating. You should use this method of substitution (called u-substitution) only when simpler methods don't work. It should be your last resort.

Procedure for u-substitution: (for $\int \_\_\_\_ dx$ problems requiring the method)

- Let $u =$ radicand, inside the $\sqrt{\ }$ symbol, or some other "inside" function.
- Solve for x (in terms of u).
- Differentiate the equation from Step 2.
- Find dx.
- Substitute u-expressions for x-expressions in the integral.
  Note: Most often, $dx \neq du$. Don't forget to substitute for dx.
- Integrate.
- Substitute back, so that your final answer is again in terms of x.

Sometimes it is easier to do Step 3 before Step 2. These two steps are reversible.

Examples: Integrate.

12. $\int x\sqrt{x-1}\,dx$

13. $\int \frac{2x-1}{(2x+3)^9}\,dx$

ASSIGNMENT 6.3

Evaluate (integrate) in Problems 1-8.

1. $\int 3(3x-2)^5\,dx$
2. $\int (5t-3)^8\,dt$
3. $\int \frac{x^2}{\sqrt{4-x^3}}\,dx$
4. $\int \sqrt{y^2-3}\;y\,dy$
5. $\int \frac{5x^2}{(x^3+2)^6}\,dx$
6. $\int \frac{-3}{\sqrt{1-v}}\,dv$
7. $\int (2x^2-3x)^4(4x-3)\,dx$
8. $\int \frac{x-1}{(2x^2-4x)^5}\,dx$

Evaluate (integrate) in Problems 9-20.

9. $\int \frac{\left(\sqrt{t}-4\right)^{10}}{\sqrt{t}}\,dt$ 10. $\int \frac{1}{\sqrt[3]{5x}}\,dx$ 11. $\int (2u+1)^2\,du$ 12. $\int y^3 4^{3y^4}\,dy$

13. $\int \frac{x^2-3}{x^2}\,dx$ 14. $\int \frac{3x^2+x-2}{\sqrt{x}}\,dx$ 15. $\int \left(2+\frac{1}{x}\right)^4 \frac{1}{x^2}\,dx$ 16. $\int \pi \sin(\pi\theta)\,d\theta$

17. $\int \frac{e^{\frac{1}{x}}}{x^2}\,dx$ 18. $\int \sec(2x-1)\tan(2x-1)\,dx$ 19. $\int \tan^5\theta \sec^2\theta\,d\theta$ 20. $\int \frac{\csc^2 t}{\cot^4 t}\,dt$

Use u-substitution to evaluate in Problems 21-23.

21. $\int 30x\sqrt{x+1}\,dx$ 22. $\int \frac{3x-5}{\sqrt{\frac{1}{2}x-1}}\,dx$ 23. $\int (5x-8)(1-x)^{11}\,dx$

24. If $f''(x)=x^{\frac{-4}{3}}$, $f'(8)=\frac{3}{2}$, and $f(27)=5$, find $f(x)$.

25. The derivative of a function is $\frac{dy}{dx}=6x\sqrt{x^2-3}$. Find the function if $(2,5)$ is a point on the graph of the function.

26. If $f'(x)=\cos(3x)$ and $f\left(\frac{\pi}{6}\right)=2$, find $f(x)$.

27. Evaluate $\frac{d}{dx}\int (x^2-3)^4\,dx$.

28. The velocity of a particle moving along a vertical line is given by the equation $v(t)=\left(\frac{1}{3}t-1\right)^2$. The particle's position at time zero is 4.

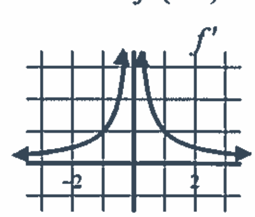
a. Find an equation for the particle's acceleration $a(t)$.
b. Find an equation for the particle's position $y(t)$.
c. At what time(s) is the particle at rest?
d. At what time(s) is the particle moving upward?
e. For what value(s) of t does the particle's speed equal the particle's velocity?
f. Find the total distance traveled by the particle from $t=3$ to $t=9$.
g. Find the interval(s) of time for which the speed of the particle is increasing

29. Find an equation for the line tangent to the graph of $y=\sqrt{3x-5}$ when $x=2$.

30. Differentiate $2x^2+y^2=4y$ implicitly to find the point(s) where the curve has
a. horizontal tangents. b. vertical tangents.

31. (2, 7) is a point on the curve of $f(x)=x^3-3x+5$. Use a tangent line to approximate $f(2.1)$.

32. The graph of $f'(x)$ is shown at right.
a. Use the given graph to make f' and f'' number lines.
b. Sketch a graph of f which passes through the points $(1,-1)$ and $(-1, 1)$.



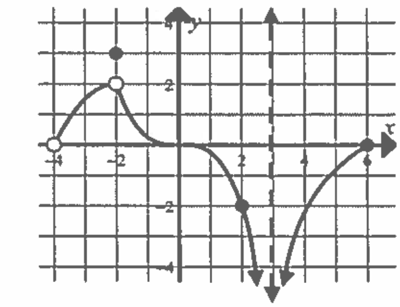
33. Use the graph at right to find:

a. $\lim_{x \to -2} f(x)$ b. $f(-2)$

c. $\lim_{x \to 2} f(x)$ d. $f(2)$

e. $\lim_{x \to 3} f(x)$ f. $f(3)$

g. $\lim_{x \to -4^+} f(x)$ h. $f(-4)$



34. Use the alternate form of the limit definition of the derivative to find $f'(2)$ for $f(x) = x^2 + 1$.

LESSON 6.4 DEFINITE INTEGRALS, CALCULATOR INTEGRATION, THE FUNDAMENTAL THEOREM OF CALCULUS

A Definite Integral is written with upper and lower limits attached to an integration expression. The value of a definite integral $\left(\int_a^b f(x)\,dx\right)$ may be thought of as a "signed area" from the lower limit a (usually a left side boundary) to the upper limit b (usually a right-side boundary), and between the curve of $f(x)$ and the x-axis. The value may be positive, negative, or zero. Unlike the previous integration process which produced an indefinite integral (an antiderivative) representing a family of curves, a definite integral represents **a number value**.

Calculator Integration: A TI-84 calculator can be used to find the value of a definite integral from a to b by using $\int f(x)\,dx$ in the calculate menu or fnInt in the math menu. The calculate menu shows a graphical representation of the "signed area" together with the value of the definite integral.

Examples:
Use the calculate menu to evaluate the following definite integrals.

1. $\int_{-3}^{1} (x^3 - 6x)\,dx$ 2. $\int_{-\sqrt{6}}^{\sqrt{6}} (x^3 - 6x)\,dx$ 3. $\int_{-5}^{5} |x^3 - 6x|\,dx$

The math menu only provides the value of the definite integral, but that is usually all that we need. The math menu gives a more accurate answer. fnInt is recommended for all problems from now on. Note: Newer operating systems have a MATHPRINT setting that simplifies this process.

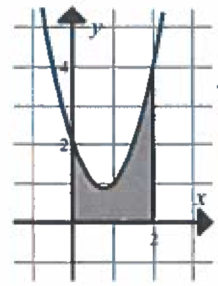
Use the math menu to evaluate:

4. $\int_{-5}^{5} |x^3 - 6x|\,dx =$ fnInt(abs($x^3 - 6x$), x, -5, 5)

This is the syntax for a TI83.

5. Use the idea of "signed area" to evaluate $\int_0^3 |2x-1|\,dx$ without using a calculator.

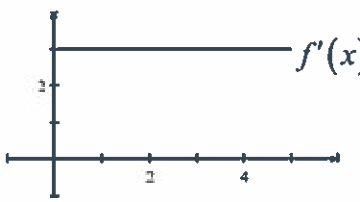
6. Set up a definite integral which could be used to find the area of the region bounded by the graph of $y = 2x^2 - 3x + 2$ (shown at right), the x-axis, and the vertical lines $x = 0$ and $x = 2$.



$y = 2x^2 - 3x + 2$

Discovering the Fundamental Theorem of Calculus

7.

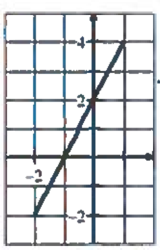


$f'(x)$

a. Find $\int_0^5 f'(x)\,dx$ b. Write an equation for $f'(x)$ on $[0,5]$.

c. Find $f(x)$ if $f(1) = 4$. d. Find $f(5) - f(0)$

8.



$f'(x)$

a. Find $\int_{-2}^1 f'(x)\,dx$ b. Write an equation for $f'(x)$ on $[-2,1]$.

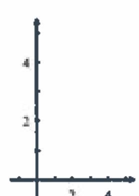
c. Find $f(x)$ if $f(1) = 0$. d. Find $f(1) - f(-2)$

9. Given $x(t) = -\frac{1}{2}t^2 + 4t - 3$ is the position equation for an object moving on the x-axis.

a. Find the displacement of the object on the interval $[1,4]$. b. Find the velocity equation.

$v(t) =$

c. Sketch a graph of $v(t)$.



d. Find $\int_1^4 v(t)\,dt$

Notice for each of these, the answers to parts a and d are the same.

If f' is a continuous function on $[a, b]$, then $\int_a^b f'(x)\,dx = f(x)\Big|_a^b = f(b) - f(a)$

This relationship is known as the **Fundamental Theorem of Calculus**.

Evaluate using the Fundamental Theorem of Calculus without using a calculator.

10. $\int_0^4 \left(2\sqrt{y}+1\right) dy$

11. $\int_0^1 (4t+1)^5\, dt$

12. $\int_1^5 \frac{x}{\sqrt{2x-1}}\, dx$

13. $\int_0^{\frac{\pi}{2}} \cos(2x)\, dx$

START PLUS ACCUMULATION METHOD

Since $\int_a^b f'(x)\,dx = f(b) - f(a)$, it follows that $\boxed{f(b) = f(a) + \int_a^b f'(x)\,dx\ .}$

This means a function value at an endpoint can be found as a starting value plus a definite integral. Although this is only a slight variation of the Fundamental Theorem of Calculus, it gives us a different way to approach problems. This is an extremely common AP™ Calculus type problem.

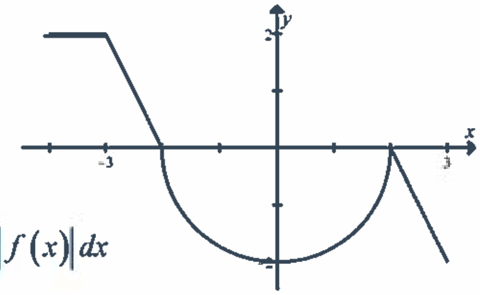
Examples:

14. If $f'(x) = 3x^2 + 3$ and $f(0) = 4$, find $f(2)$ without a calculator.

15. If an object's velocity is $v(t) = 2^{3t^2-5}$ and $s(1) = 8$ find $s(2)$.

ASSIGNMENT 6.4

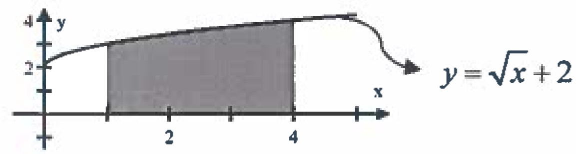
The graph of the function $y = f(x)$ consists of line segments and a semicircle as shown. Evaluate the following using geometry formulas.



1. $\int_{-4}^{-2} f(x)\,dx$ 2. $\int_{0}^{2} f(x)\,dx$

3. $\int_{-4}^{3} f(x)\,dx$ 4. $\int_{-2}^{2} \left(f(x)+2\right)dx$ 5. $\int_{-4}^{3} \left|f(x)\right| dx$

6. Show an integral setup and evaluate to find the area shaded in the figure shown without using a calculator.



For Problems 7 and 8 sketch a graph for each function, and use the idea of "signed areas" to evaluate these definite integrals using geometry formulas without using a calculator.

7. $f(x) = x-1$

a. $\int_{-2}^{2} f(x)\,dx$ b. $\int_{-2}^{2} |f(x)|\,dx$

8. $g(x) = 2x+3$

a. $\int_{-2}^{0} g(x)\,dx$ b. $\int_{-2}^{0} |g(x)|\,dx$

Evaluate the definite integrals in Problems 9-22 without using a calculator.

9. $\int_{0}^{2} (1-2y)\,dy$ 10. $\int_{0}^{1} (t^2-1)^4\, t\,dt$ 11. $\int_{0}^{1} x(4x-3)^2\,dx$ 12. $\int_{-2}^{-1} \left(\frac{1-3x^4}{x^2}\right)dx$

13. $\int_{1}^{4} \frac{2\sqrt{x}-1}{\sqrt{x}}\,dx$ 14. $\int_{1}^{8} \left(u^{\frac{2}{3}}+u^{\frac{-1}{3}}\right)du$ 15. $\int_{1}^{2} \frac{dx}{2\sqrt{3x-2}}$ 16. $\int_{0}^{\pi} \sin x\,dx$

17. $\int_{0}^{\pi} \cos x\,dx$ 18. $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sec^2\theta\,d\theta$ 19. $\int_{0}^{4} xe^{3x^2}\,dx$ 20. $\int_{1}^{3} 3^{6x-1}\,dx$

21. $\int_{0}^{\frac{\pi}{2}} \sin\left(\frac{2x}{3}\right)dx$ 22. $\int_{0}^{\pi} \left(3\sin x+\sin(2x)\right)dx$ ** 23. $\int_{0}^{4} \left|x^2-4\right|dx$

*24. $\int_{0}^{1} \frac{2x}{(x+1)^6}\,dx$ * 25. $\int_{1}^{2} x\,\sqrt[3]{x-1}\,dx$

*(Hint: Problems 24 and 25 require u-substitution.)

**(Hint: For Problem 23 sketch a graph and split the integral into two integrals without absolute value.)

Use your calculator to evaluate the definite integrals in Problems 26-28. Express answers to 3 or more decimal place accuracy.

26. $\int_{1}^{12} \frac{1}{x}\,dx$ 27. $\int_{0}^{6} \sqrt{y^3+1}\,dy$ 28. $\int_{0}^{\frac{\pi}{3}} \sin^2\theta\,d\theta$

29. Given $f'(x) = \left|8\sin(10x)\right|$,

a. use a calculator to find $\int_{0}^{4} f'(x)\,dx$. b. if $f(0)=3$, find $f(4)$.

30. If an object's acceleration is $a(t)=\cos t^2$ and $v(1)=5\frac{\text{ft}}{\text{sec}}$, find $v(3)$.

31. $x(t)=t^3-3t^2+1$ represents the position equation for a particle moving along the x-axis.
   a. Find the velocity equation.
   b. Find the acceleration equation.
   c. Find the velocity at $t=1$.
   d. Find the speed at $t=1$.
   e. When is the particle's velocity decreasing?
   f. Find the displacement on $[1, 4]$.
   g. Find the total distance traveled from $t=1$ to $t=4$. (Show a velocity number line).
   h. Find $\int_1^4 v(t)\,dt$ without using a calculator. Compare your answer to Part f.
   i. Use your calculator to find $\int_1^4 |v(t)|\,dt$. Compare your answer to Part g.

*You now have two ways to find <u>displacement</u> and <u>total distance</u>. Using definite integrals, **displacement** $=\int_a^b v(t)\,dt$ and **total distance** $=\int_a^b |v(t)|\,dt$ on the interval $[a, b]$. Given a choice of methods, always do total distance by evaluating a definite integral on your calculator.

32. Find the area between $f(x)=\ln(2x+5)$ and the x-axis on the interval $[1, 3]$. Show an integral set up, and evaluate using a calculator.

33. If $\int_0^4 f(x)\,dx=3$ and $\int_0^4 g(x)\,dx=-2$ find $\int_0^4 (4f(x)-3g(x))\,dx$.

34. If $\int_0^4 f(x)\,dx=3$ and $\int_0^4 g(x)\,dx=-2$ find $\int_4^0 f(x)\,dx-\int_0^4 g(x)\,dx$.

LESSON 6.5 THE SECOND FUNDAMENTAL THEOREM OF CALCULUS INTEGRATION INVOLVING THE NATURAL LOG FUNCTION

The following examples serve as an informal guide toward discovering the Second Fundamental Theorem of Calculus.

1. $\int_{10}^{x} f'(t)\,dt=$

2. $\frac{d}{dx}\int_{10}^{x} f'(t)\,dt=$

<u>Second Fundamental Theorem of Calculus</u>:

For any constant a, $\frac{d}{dx}\int_a^x f(t)\,dt=f(x)$ (if f is continuous from a to x).

Now find:

3. $\int_0^{x^2} f'(t)\,dt =$

4. $\frac{d}{dx}\int_0^{x^2} f'(t)\,dt =$

5. $\int_{x^3}^{2x} f'(t)\,dt =$

6. $\frac{d}{dx}\int_{x^3}^{2x} f'(t)\,dt =$

<u>Second Fundamental Theorem (Chain Rule Version)</u>:
If u and v are functions of x, then $\frac{d}{dx}\int_u^v f(t)\,dt = f(v)\,v' - f(u)\,u'$
(if f is continuous from u to v). Note the "<u>chain rule factors</u>" v' and u'.

<u>Examples</u>: Find each of the following <u>without integrating</u>.

7. $\frac{d}{dx}\int_x^0 (2t-3)\,dt =$

8. $\frac{d}{dx}\int_2^5 (2t-3)\,dt =$

9. $\frac{d}{dx}\int_{-1}^{x^3} (t^2+2t)\,dt =$

10. If $f(x)=\int_0^{3x^2} (1-t^2)^{10}\,dt$, then $f'(x)=$

Differentiation and integration are inverse operations.
So if $\frac{d}{dx}\ln|x| = \frac{1}{x}$, then $\int \frac{1}{x}\,dx = \ln|x| + C$, and if $\frac{d}{dx}\ln|u| = \frac{u'}{u}$, then $\int \frac{u'}{u}\,dx = \ln|u| + C$.

<u>Log Rules</u>: $\int \frac{1}{x}\,dx = \ln|x| + C$ and $\int \frac{u'}{u}\,dx = \ln|u| + C$

Note: Although it is true that both $\frac{d}{dx}\ln x = \frac{1}{x}$ and $\frac{d}{dx}\ln|x| = \frac{1}{x}$, $\int \frac{1}{x}\,dx = \ln|x|$ only. **Why?**

<u>Examples</u>: Integrate

11. $\int \frac{-3}{x}\,dx$

12. $\int \frac{P}{P^2+1}\,dP$

13. $\int \frac{\sec^2 x}{\tan x}\,dx$

<u>Example 14</u>: Integrate $\int \frac{\ln x}{x}\,dx$

<u>Example 15</u>: Integrate $\int \frac{1}{x(2-\ln x)^3}\,dx$

Examples: Rewrite as a fraction using a trig identity.

16. Integrate $\int \cot x\, dx$

17. Integrate $\int \tan x\, dx$

So,

$$\int \tan x\, dx = -\ln|\cos x| + C \qquad \int \cot x\, dx = \ln|\sin x| + C$$

$$\int \tan u\, u'dx = -\ln|\cos u| + C \qquad \int \cot u\, u'dx = \ln|\sin u| + C$$

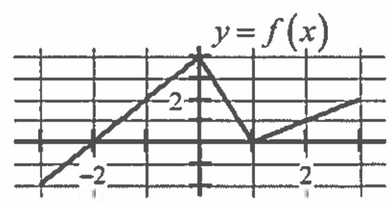
Example 18: Integrate $\int \tan(2x)\, dx$

ASSIGNMENT 6.5

Use the Second Fundamental Theorem of Calculus to evaluate in Problems 1-6.

1. $\frac{d}{dx}\int_{-3}^{x}(t^2 - t + 1)\, dt$
2. $\frac{d}{dx}\int_{2}^{x}\sqrt[3]{t^2 - 5t}\, dt$
3. $\frac{d}{dt}\int_{6}^{t}\frac{2}{x-5}\, dx,\ t > 6$
4. $\frac{d}{dx}\int_{x}^{3}(1-t)^4\, dt$
5. $\frac{d}{dx}\int_{x^2}^{3}(1-t)^4\, dt$
6. $\frac{d}{dt}\int_{3t}^{4t}2^x\, dx$
7. $\int_{1}^{a} t(3t^2 - 1)^5\, dt$

8. If $g(x) = \int_0^{2x} f(t)\, dt$ find each of the following:



a. $g'(x)$ b. $g'(0)$

c. x-value(s) where g has a relative minimum. Justify.

d. x-value(s) where g has a point of inflection. Justify.

e. $g''\left(\frac{1}{4}\right)$

Evaluate each integral in Problems 9-22 without using a calculator.

9. $\int_1^e \frac{2}{x}\, dx$
10. $\int_1^3 \frac{4}{2t-1}\, dt$
11. $\int e^{x^2-1} x\, dx$
12. $\int \frac{x}{x^2-1}\, dx$
13. $\int \frac{x^2-1}{x}\, dx$
14. $\int \frac{4y-6}{y^2-3y+2}\, dy$
15. $\int \frac{-1}{(x+1)^3}\, dx$
16. $\int \frac{3u}{\sqrt[3]{u^2+1}}\, du$
17. $\int \frac{\sqrt{\ln x}}{x}\, dx$
18. $\int \frac{2}{x(1+\ln x)^5}\, dx$
19. $\int \frac{\cos y}{\sin y - 2}\, dy$
20. $\int \frac{e^{2y}}{e^{2y}-2}\, dy$
21. $\int \cot(5x)\, dx$
22. $\int y \tan(y^2)\, dy$

23. Use the substitution $u = f(x)$ to rewrite $\int_1^3 f'(x)\sin(f(x))\,dx$ as an integral with respect to u if $f(1) = 4$ and $f(3) = 9$. Do not integrate.

24. If $g'(x) = \ln\left|\cos x^2\right|$ and $g(4.23) = 5.192$ find $g(2.159)$.

Differentiate in Problems 25-29 without using a calculator.

25. $y = \dfrac{t}{\ln t}$

26. $f(y) = \arctan(3y)$

27. $y = 3^{2x+1}$

28. $\dfrac{d}{dx}\ln\left|\sin^3 x\right| =$

29. $y = 4\arcsin\left(x^2\right)$

30. Use the alternate form of the limit definition of the derivative to find $f'(3)$, if $f(t) = 2t^2 - 3$.

31. Find a, b, and c for $f(x) = ax^2 + bx + c$, such that $f(1) = 10$, and $f(x)$ has a relative minimum at $(-1, 2)$.

32. If $f(x) = (x-1)^{\frac{2}{3}}$, the Mean Value Theorem does not apply to which interval?
 a. $[0, 2]$ b. $[1, 9]$ Why?

33. For Problem 32, use the interval on which the MVT does apply to find the c-value(s) where $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Evaluate the following inverse trig functions:

34. $\tan^{-1}\left(\dfrac{-1}{\sqrt{3}}\right) =$

35. $\arcsin(1) =$

36. $\sin^{-1}\left(-\dfrac{\sqrt{2}}{2}\right) =$

37. $\arctan 1 =$

LESSON 6.6 INTEGRATION INVOLVING INVERSE TRIG FUNCTIONS ADVANCED INTEGRATION TECHNIQUES

When integrating a fraction where the degree of the numerator ≥ the degree of the denominator, you will have to use long division (or creative thinking) to "split the fraction."

Example 1: $\displaystyle\int \frac{x^2 - 4x + 2}{x^2 + 2}\,dx$

Example 2: $\displaystyle\int \frac{1}{\sqrt{x} - 1}\,dx$

Since $\frac{d}{dx}\arcsin u = \dfrac{u'}{\sqrt{1-u^2}}$ and $\frac{d}{dx}\arctan u = \dfrac{u'}{1+u^2}$, it follows that

$$\int \frac{u'}{\sqrt{1-u^2}}\,dx = \arcsin u + C \quad \text{and} \quad \int \frac{u'}{1+u^2}\,dx = \arctan u + C \quad (\text{where } u \text{ is a function of } x).$$

Extending these integration rules gives us these more general **integration rules**.

1. $\displaystyle\int \frac{u'}{\sqrt{a^2-u^2}}\,dx = \arcsin\frac{u}{a} + C$ 2. $\displaystyle\int \frac{u'}{a^2+u^2}\,dx = \frac{1}{a}\arctan\frac{u}{a} + C$

Note: Since $\frac{d}{dx}\arcsin x$ and $\frac{d}{dx}\arccos x$ differ only in sign, it is not necessary to have a third integration rule which integrates into $\arccos x$.

Warm-up Example: Differentiate $y = \arcsin\dfrac{x}{2}$.

Examples: Integrate.

3. $\displaystyle\int \frac{1}{\sqrt{4-x^2}}\,dx$ 4. $\displaystyle\int \frac{dx}{4x^2+25}$ 5. $\displaystyle\int_0^{\frac{1}{2}} \frac{8}{3+4x^2}dx$

6. $\displaystyle\int \frac{8x}{3+4x^2}\,dx$ 7. $\displaystyle\int \frac{8x^2}{3+4x^2}\,dx$ 8. $\displaystyle\int \frac{x+4}{\sqrt{4-x^2}}\,dx$

Example 9. Complete the square to find $\displaystyle\int \frac{1}{x^2+4x+8}\,dx$.

Example 10: $\int \frac{3-x}{\sqrt{1-x^2}}\,dx$

Example 11: $\int \frac{1}{e^x+1}\,dx$

ASSIGNMENT 6.6 Do not use a calculator on this assignment.

Evaluate the integrals in Problems 1-6.

1. $\int_0^{\frac{1}{4}} \frac{1}{\sqrt{1-4x^2}}\,dx$
2. $\int_0^{\frac{5}{3}} \frac{2}{9x^2+25}\,dx$
3. $\int_0^1 \frac{x^2}{x+1}\,dx$
4. $\int \frac{8}{2+(2t+1)^2}\,dt$
5. $\int \frac{w^2}{\sqrt{4-w^6}}\,dw$
6. $\int \frac{dx}{x\sqrt{16-(\ln x)^2}}$

Complete the square to evaluate Problems 7 and 8.

7. $\int \frac{1}{t^2-10t+32}\,dt$
8. $\int_{-3}^{-1} \frac{1}{\sqrt{7-x^2-6x}}\,dx$

Evaluate the integrals in Problems 9-16.

9. $\int \frac{\sqrt{\arctan\theta}}{1+\theta^2}\,d\theta$
10. $\int \frac{5x}{x^2+1}\,dx$
11. $\int \frac{5x^2}{x^2+1}\,dx$
12. $\int \frac{2x^2-4}{x+1}\,dx$
13. $\int \frac{2}{1-e^{2y}}\,dy$
    Hint: See Example 11
14. $\int \frac{3-4t}{t^2+9}\,dt$
    Hint: Split
15. $\int \frac{e^{-2v}}{3+e^{-4v}}\,dv$
16. $\int \frac{2x}{\sqrt[3]{3x+1}}\,dx$ **Hint:** Think about the method of last resort.

Simplify the expressions in Problems 17-19.

17. $\frac{d}{dt}\int_1^t (-2x+\ln x)\,dx$
18. $\frac{d}{dx}\int_{x^2}^5 \sin(2t)\,dt$
19. $\int_0^1 xe^{x^2}\,dx$

Differentiate.

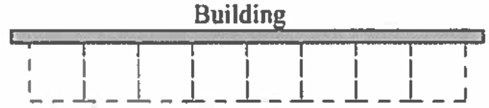
20. $g(y)=\ln|(1-\ln y)|$
21. $x^2+2\ln y=y$ (solve for $\frac{dy}{dx}$)

22. Sketch a possible graph for $f(x)$, given the following characteristics:

$f(0)=1$, $f(1)$ does not exist, $f(2)=2$
$f'(x)<0$ for $x<1$ and $1<x<2$, $f'(x)>0$ for $x>2$
$f''(x)<0$ for $x<1$, $f''(x)>0$ for $x>1$

23. A small dog kennel with 8 individual rectangular holding pens of equal size is to be constructed using 144 *ft* of chain link fencing material. One side of the kennel is to be placed against a building and requires no fencing, as shown in the figure below.

a. Find the dimensions (for each holding pen) that produce a maximum area for each pen.
b. What is that maximum area for <u>each</u> holding pen?



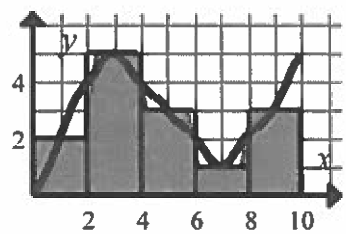
<u>Lesson 6.7</u> <u>APPROXIMATING DEFINITE INTEGRALS WITH RIEMANN SUMS</u>

Some functions cannot be integrated, and sometimes you are given data or a graph – but not an actual function. It is still possible to <u>approximate</u> "areas." One method of approximating a definite integral is to add areas of rectangles. This is called a **<u>Riemann Sum</u>**.

<u>Example 1</u>:
Approximate $\int_0^{10} f(x)\,dx$ by adding the areas of the five rectangles shown.

This is a <u>Midpoint Riemann Sum.</u>

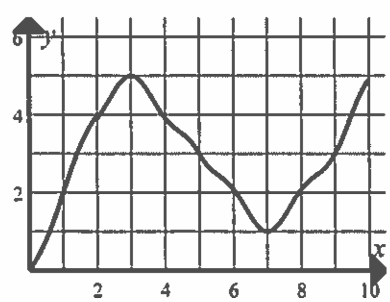
$y=f(x)$



x	$f(x)$
0	0
1	2
2	4
3	5
4	4
5	3
6	2
7	1
8	2
9	3
10	5

<u>Example 2:</u>
Approximate $\int_0^{10} f(x)\,dx$ by using 5 rectangles of equal width $(n=5)$ and a <u>Left Riemann Sum</u>.
Draw rectangles on the figure.

$y=f(x)$



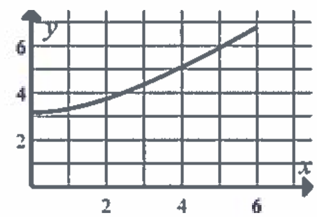
Example 3:

Approximate $\int_0^{10} f(x)\,dx$ by using a Right Riemann Sum with four subdivisions using the data in the table.

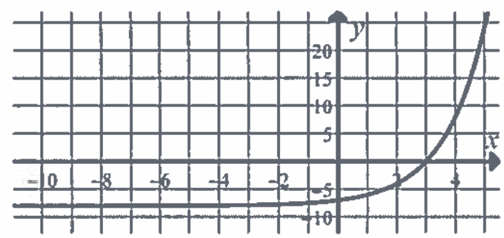
x	0	2	5	9	10
$f(x)$	3	8	2	-1	0

Example 4:

Approximate $\int_0^6 \sqrt{x^2+10}\,dx$ using a Midpoint Riemann Sum with 3 equal subdivisions.

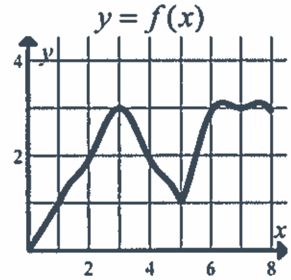


Example 5: Approximate $\int_{-10}^{5} (2^x - 8)\,dx$ by using five Right hand rectangles whose widths are determined by the intervals separating the following x values: $x=-10$, $x=-4$, $x=0$, $x=2$, $x=3$, and $x=5$.



ASSIGNMENT 6.7 Show set ups on all Riemann Sum problems.

1. Use the graph of $y = f(x)$ at right to approximate $\int_0^8 f(x)\,dx$ using
   a. a Midpoint Riemann Sum with 4 equal subdivisions.
   b. a Left hand Riemann Sum with 8 equal subdivisions.
   c. a Right hand Riemann Sum with 8 equal subdivisions.



x	$f(x)$
0	0
1	1
2	2
3	3
4	2
5	1
6	3
7	3
8	3

2. Use the data in the table below to approximate the area between the graph of $f(t)$ and the t-axis, from $t=1$ to $t=13$, using a Midpoint Riemann Sum with 6 rectangles of equal width. Plot the data, sketch a graph, and draw rectangles first.

t	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$f(t)$	0	5	8	11	12	15	17	18	15	13	12	9	6	4

3. Use the data in the table at right to approximate $\int_2^4 f(x)\,dx$ using a Midpoint Riemann Sum with two equal subintervals.

x	2.0	2.5	3.0	3.5	4.0
$f(x)$	3	2	4	3	5

4. Approximate $\int_0^{16}\left(\sqrt{x}+1\right)dx$ using a Left Riemann Sum with 4 rectangles of equal width. Draw an accurate sketch (without using a calculator if possible), and show your rectangles.

5. Is you answer from Problem 4 an underestimate or an overestimate of the actual value of the integral? What characteristic of the square root function makes your answer true?

6. Would a Right Riemann Sum approximation of an integral of an increasing function be an underestimate or an overestimate?

7. Approximate $\int_{-3}^{5}\left(\sqrt[3]{x}-\frac{1}{2}\right)dx$ by using 4 Left rectangles whose widths are determined by the intervals separating $x=-3$, $x=0$, $x=1$, $x=2$, and $x=5$.

8. Approximate $\int_2^8 \frac{1}{x}\,dx$ using a Right Riemann Sum with 3 equal subdivisions.

9. Approximate $\int_0^{10} f(x)\,dx$ by using a Midpoint Riemann Sum with two unequal subdivisions using the data in the table.

x	0	2	4	7	10
$f(x)$	7	-2	0	4	10

Use a calculator for Problems 10-12.

10. Evaluate $\int_{-4}^{5}\left(3x^3-4\right)^{10}dx$

11. For $f(x)=\sin^2\left(3x^2\right)$, find $f'(1.63)$

12. Find the area between $f(x)=\ln\left(\sin(x)+1\right)$ and the x-axis on the interval $[0,\pi]$. First show an integral set up.

Evaluate the expressions in Problems 13-16 without using a calculator.

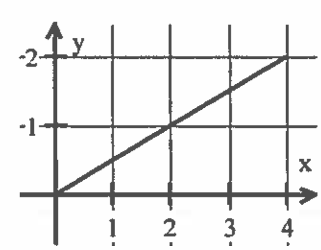
13. $\frac{d}{dx}\int_0^x \sqrt{t}\,dt$

14. $\frac{d}{dx}\int_{2x}^0 \sqrt{t}\,dt$

15. $\frac{d}{dx}\int_0^{x^2} \sqrt{t}\,dt$

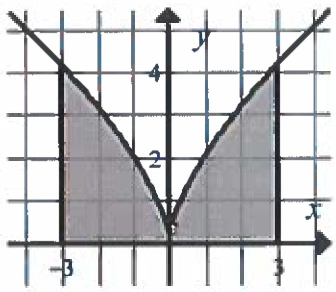
16. $\int_0^x \sqrt{t}\,dt$

17. Find $f(4)$ if $f(2)=3$ and the graph of $y=f'(x)$ is shown.



For Problems 18-20, set up definite integrals which could be used to find the areas of the regions shown or described. Attempt to draw the graphs for Problems 19 and 20 without using a calculator. You do not need to evaluate the integrals that you set up.

18. $f(x)=2x^{\frac{2}{3}}$ on $[-3,3]$



19. Region bounded by

$y=\dfrac{x}{x^2+1}$,

$x=0$, $x=4$, and

$y=0$

20. Region bounded by

$g(y)=y^2+1$,

$x=0$, $y=-2$, and

$y=3$

Without a calculator, sketch graphs and use geometry to evaluate Problems 21and 22.

21. $\int_0^4 |3x-2|\,dx$

22. $\int_{-2}^{1}(2-|x|)\,dx$

Evaluate the following without using a calculator.

23. $\displaystyle\int_{-1}^{0}\frac{1}{\sqrt{3-x^2-2x}}\,dx$ Hint: Comp. Sq.

24. $\displaystyle\int\frac{4x}{\sqrt{1-x^4}}\,dx$

25. $\displaystyle\int\frac{x+5}{x^2+16}\,dx$

26. $\displaystyle\int\frac{\sqrt{x-1}}{x}\,dx$ Hint: Let $u=\sqrt{x-1}$

27. $\displaystyle\int\frac{e^t\cos e^t}{\sin e^t}\,dt$

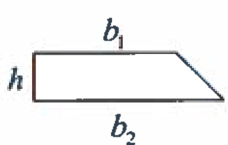
28. $\displaystyle\int\frac{y}{y+2}\,dy$

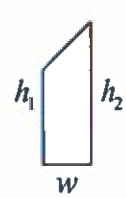
29. A spherical balloon is expanding at the rate of $5\ cm^3/sec$. How fast is the diameter of the balloon increasing when its volume is $36\pi\ cm^3$? $\left(V=\frac{4}{3}\pi r^3\right)$

LESSON 6.8 TRAPEZOIDAL APPROXIMATION, SUMMATION NOTATION

For most functions, using trapezoids to approximate "areas" is more accurate than using rectangles.

The area formula for a single trapezoid is $A=\frac{1}{2}h(b_1+b_2)$. Figure (below left).

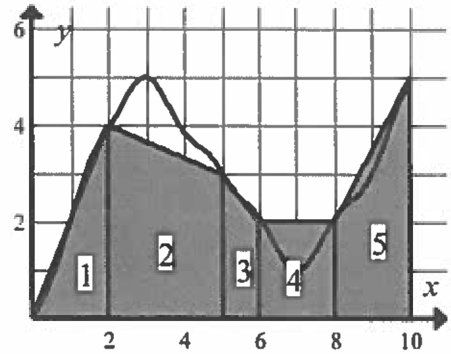




Since trapezoids used in the approximations of "areas" are usually positioned vertically, we will write the formula as $A=\frac{1}{2}w(h_1+h_2)$. Figure (above right).

To approximate the value of a definite integral using trapezoids, use the same strategy as you used for Riemann Sums – but add the "areas" of trapezoids instead of rectangles.

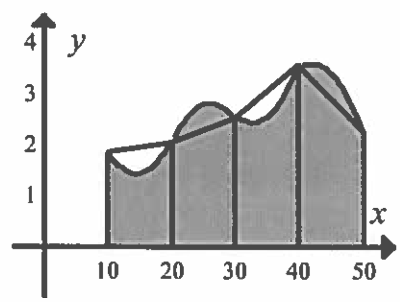
Example 1: Approximate $\int_0^{10} f(x)\,dx$ by adding the areas of the 5 "trapezoids" shown in the graph at the right.



Note: The area formula for a trapezoid also works for a triangle (either $h_1 = 0$ or $h_2 = 0$) or a rectangle $(h_1 = h_2)$.

If you forget the formula for a trapezoid, you can always draw your "areas" as rectangles and triangles (and not even use trapezoids). You could also average the values from a Left and Right Riemann Sum. In any case, make certain to show a clear set-up.

Example 2: Use these trapezoids with four equal subdivisions to approximate the area shown.



x	y
10	1.8
20	2
30	2.5
40	3.5
50	2.2

As the number of subdivisions increases, the accuracy of a Riemann Sum approximation improves. To achieve perfect accuracy we need to approach infinitely many subdivisions. This is the **limit definition of a definite integral.**

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n}\left(f(a+k\Delta x)\Delta x\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\Bigg(\underbrace{f\left(a+k\frac{b-a}{n}\right)}_{\text{right-hand heights}}\overbrace{\frac{b-a}{n}}^{\text{width}}\Bigg)$$

where n is the number of subdivisions.

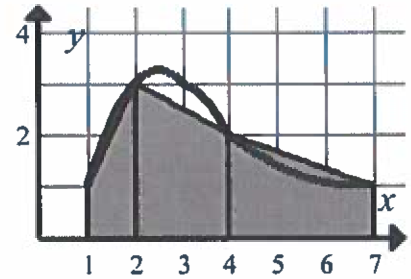
Example 3: Write $\int_1^4 x^5\,dx$ as an infinite Right Riemann Sum.

Example 4: Write $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sin\left(\frac{\pi}{2}+k\bullet\frac{\pi}{n}\right)\frac{\pi}{n}\right)$ as a definite integral.

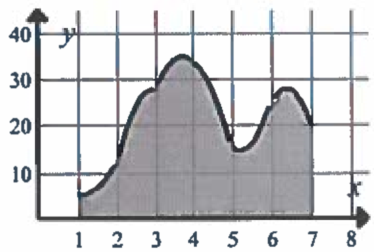
Note: We will focus on Right Riemann Sums for these. A Left Sum would be the same except the summation would be from $k = 0$ to n-1.

ASSIGNMENT 6.8 Show set ups on Problems 1-9.

1. Use the trapezoids shown to approximate $\int_1^7 f(x)\,dx$.



2. The graph in the figure below was recorded by an instrument used to measure a physical quantity. Approximate the area of the shaded region by using six trapezoids of equal width.



x	y
1	5
2	12
3	28
4	34
5	15
6	25
7	20

3. Approximate $\int_{-3}^{5}\left(\sqrt[3]{x}-\frac{1}{2}\right)dx$ by using 4 trapezoids whose widths are determined by the intervals separating $x=-3$, $x=0$, $x=1$, $x=2$, and $x=5$.

4. The points shown are from a continuous function f. Use the points in the table to approximate $\int_2^4 f(x)\,dx$ using two trapezoids of equal width.

x	y
2.00	4.12
2.25	3.76
2.50	3.21
2.75	3.58
3.00	3.94
3.25	4.15
3.50	4.69
3.75	5.44
4.00	7.52

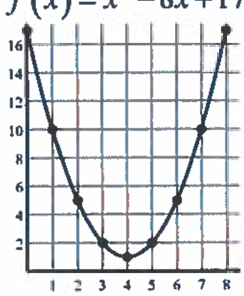
5. Use trapezoids with four equal subdivisions to approximate $\int_{-2}^{2}\sqrt{x^4+1}\,dx$.

6. Look at a graph of the square root function from Problem 5 with a calculator. Is the trapezoidal approximation an underestimate or an overestimate? What characteristic of the graph determines this?

Use Riemann Sums to approximate the values of the definite integrals in Problems 7-9.

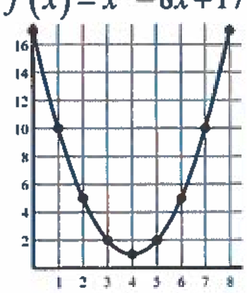
7. $\int_0^8 f(x)\,dx$. Midpoint R.S. with 4 equal subdivisions

$f(x)=x^2-8x+17$



8. $\int_0^8 f(x)\,dx$. Left hand R.S. with 2 equal subdivisions

$f(x)=x^2-8x+17$



Write each Riemann Sum as a definite integral and each definite integral as a right Riemann Sum. Do not evaluate.

9. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left(f\left(2+k\bullet\frac{3}{n}\right)\frac{3}{n}\right)$ 10. $\int_1^5 f(x)\,dx$ 11. $\int_2^4 \sin x\,dx$ 12. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left(\cos\left(0+\frac{k\pi}{n}\right)\frac{\pi}{n}\right)$

13. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left(\sqrt{3+\frac{2k}{n}}\bullet\frac{2}{n}\right)$ 14. $\int_4^5 (x+1)^2\,dx$ 15. $\int_0^5 \left(x^2+1\right)dx$

Evaluate in Problems 16-18.

16. $\frac{d}{dx}\int_0^{2x}\sin^4 t\,dt$ 17. $\frac{d}{dx}\int_{x^2}^{5}\sin\left(t^2\right)dt$ 18. $\frac{d}{dt}\int_t^{2t}2^{x^2}\,dx$

19. Use a calculator to find $\int_{\frac{\pi}{2}}^{\pi}\sin^3(3x-1)\,dx$. (You must be in radian mode.)

Evaluate the integrals in Problems 15-18 without using a calculator.

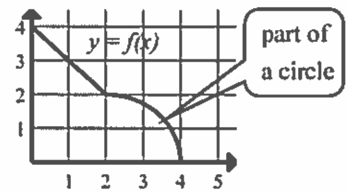
20. $\int_{-2}^{2}\left|x^3\right|dx$ 21. $\int_1^3 \frac{1}{t^2}\,dt$ 22. $\int \left(x^2-2x\right)^5 (x-1)\,dx$

23. $\int \sqrt{y}\left(1-\frac{1}{y}\right)dy$ 24. $\int \frac{8}{2t^2-8t+58}\,dt$ 25. $\int \frac{2\,dx}{9+x^2}$

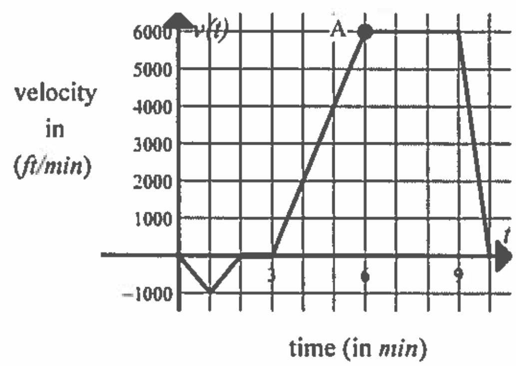
26. $\int \frac{e^{5x}-e^{x}+2}{e^{2x}}\,dx$ 27. $\int \frac{\cos t}{\sqrt{25-\sin^2 t}}\,dt$ 28. $\int_2^3 \frac{y+1}{y-1}\,dy$

29. Use geometry to find $\int_0^4 f(x)\,dx$ for the function shown at right.

$$f(x)=\begin{cases}4-x, & 0\le x\le 2\\ \sqrt{4x-x^2}, & 2<x\le 4\end{cases}$$



30. The "rate graph" at right represents the velocity of a car during a 10 minute factory test drive along a straight path.



a. On what interval(s) of time was the car moving backward (reverse)? forward? at rest?
b. Write a sentence telling what Point A represents.
c. Find the speed of the car at $t=1$, $t=2$, and $t=4$ min.
d. Find the acceleration of the car on the time interval (3, 6).
e. On what time interval(s) is the car's acceleration the greatest?
f. On what time interval(s) is the absolute value of the car's acceleration the greatest?
g. On what time interval(s) was the car speeding up (increasing in speed)?
h. Find $\int_0^{10} v(t)\,dt$ i. Find $\int_0^{10}\left|v(t)\right|dt$
j. Write a sentence telling what $\int_0^{10} v(t)\,dt$ represents. Include numbers and units.
k. Write a sentence telling what $\int_0^{10}\left|v(t)\right|dt$ represents. Include numbers and units.

ASSIGNMENT 6.9 REVIEW

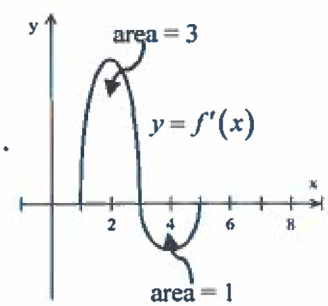
Integrate each of the following without using a calculator.

1. $\int 3x(2x^2+1)^{\frac{1}{3}}\,dx$
2. $\int t\sqrt{2t+1}\,dt$
3. $\int \dfrac{3x^3-2x^2+9}{x^2}\,dx$
4. $\int 5^2\,dx$
5. $\int \dfrac{3}{x^2-6x+18}\,dx$
6. $\int \dfrac{\cos y}{\sin y-2}\,dy$
7. $\int \left(\theta^2+\sec(\theta-1)\tan(\theta-1)\right)d\theta$
8. $\int \dfrac{4}{2+\sqrt{x}}\,dx$
9. $\int \dfrac{e^{\tan x-1}}{\cos^2 x}\,dx$
10. $\int \sin(e^{-t})e^{-t}\,dt$

Evaluate the following without a calculator.

11. $\int_{-1}^{4} \sqrt[4]{3x+4}\,dx$
12. $\int_{0}^{2} \dfrac{1}{5-2t}\,dt$
13. $\int_{2}^{e} \dfrac{\sqrt[3]{\ln x}}{x}\,dx$
14. $\int_{0}^{1} 3^{-x}\,dx$
15. $\int_{0}^{\frac{\pi}{4}} \sin(2x)\,dx$
16. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 3\csc t\cot t\,dt$
17. $\int_{\sqrt{3}}^{2} \dfrac{1}{\sqrt{1-\frac{1}{4}x^2}}\,dx$

18. If $f'(x)=\sin x^3$ and $f(1.2)=6.25$, find $f(3.6)$.



The graph shown is a graph of $y=f'(x)$. The two enclosed regions have areas of 3 and 1 as shown. Use this figure for problems 19-21.

19. If $f(1)=4$, find $f(5)$.
20. If $f(3)=4$, find $f(1)$.
21. If $f(5)=4$, find $f(1)$.

22. Find $\frac{d}{dx}\int_{x^3}^{x} \cos^2(2t+1)\,dt$.

Antidifferentiate in Problems 23 and 24.

23. $f'(x)=e^{x^2+5}x$
24. $x'(t)=\dfrac{2t-1}{e^{t^2-t}}$

25. Use the data in the table shown to approximate $\int_{2}^{14} f(x)\,dx$ with four subdivisions using:
a. a right Riemann Sum. b. a left Riemann Sum. c. trapezoids.

x	2	7	9	10	14
$f(x)$	0	3	8	2	−2

Write each Riemann Sum as a definite integral and each definite integral as a right Riemann Sum. Do not evaluate.

26. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left(\left(3\left(\frac{k}{n}+4\right)+2\right)\frac{1}{n}\right)$
27. $\int_{2}^{7}\left(2x^2+5x\right)dx$
28. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left(\left(\cos\left(\frac{k\pi}{3n}\right)\right)\frac{\pi}{3n}\right)$

29. **Sketch graphs and show shaded areas representing the values of the following .**

I. $\int_{-2}^{2}\left|x^3+x\right|dx$ II. $\int_{-2}^{2}\left|x^2+5x+6\right|dx$ III. $\int_{-2}^{2}\left|x^2+5x-6\right|dx$ IV. $\int_{-2}^{2}\left|x+1\right|dx$

Match each of the integrals to one of the descriptions below.

a. The integral can be evaluated geometrically using areas of triangles, so that no actual integration is necessary.
b. Absolute value is not even necessary for the given limits of integration.
c. Use of symmetry for the graph allows the problem to be done using only one integral that does not involve absolute value.
d. The integral can only be done by using more than one integral. That is, the problem must be split into two or more integrals to eliminate the absolute value.

30. a. Set up integrals that do not involve absolute value which could be used to integrate the integrals shown in Problem 29 I, II, and III.
b. Evaluate the integral in Problem 29 IV using areas of triangles.

Use a calculator for Problems 31-34. For Problems 31-34, $f(x)=3x^2+\ln|x|$.

31. Find $f'(4)$

32. Find $\int_{1}^{7} f(x)\,dx$

33. Solve $3x^2+\ln|x|=0$

34. f is discontinuous when $x=0$. Is the discontinuity a hole, an asymptote, or a jump?

UNIT 6 SUMMARY

Indefinite Integrals $\int f'(x)\,dx = f(x) + C$ You might have an initial condition and be able to solve for C.

Definite Integrals (Fundamental Theorem of Calculus) $\int_a^b f'(x)\,dx = f(x)\Big|_a^b = f(b) - f(a)$

Start Plus Accumulation $f(b) = f(a) + \int_a^b f'(x)\,dx$

Second Fundamental Theorem (Chain Rule Version):

If u and v are functions of x, then $\frac{d}{dx}\int_u^v f(t)\,dt = f(v)\,v' - f(u)\,u'$

(if f is continuous from u to v). Note the "chain rule factors" v' and u'.

Reverse Chain Rule for Integrals

$\int f'(u)u'\,dx = f(u) + C$ (Where u is a function of x)

Five Ways to Integrate

1. Term by term. 2. Reverse Chain Rule 3. u-Substitution
4. Long Division (numerator degree $\geq$ denominator degree)
5. Complete the square in denominator

Procedure for u-substitution:

1. Let $u =$ radicand (part inside the $\sqrt{\ }$ symbol).
2. Solve for x.
3. Differentiate the equation from Step 2.
4. Find dx.
5. Substitute u-expressions for x-expressions in the integral.
6. Integrate.
7. Substitute back, so that your final answer is again in terms of x.

Calculator Integration Math 9 on a TI84

Integrals involving absolute value: draw a graph, use geometry.

Definite Integral $\leftrightarrow$ Infinite Riemann Sum

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n}\left(f(a+k\Delta x)\Delta x\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\Bigg(\underbrace{f\left(a+k\frac{b-a}{n}\right)}_{\text{right-hand heights}}\overbrace{\frac{b-a}{n}}^{\text{width}}\Bigg)$$

where n is the number of subdivisions.

	***x* form**	***u* form** (Reverse Chain Rule)
Power Rule:	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$	$\int u^n u'dx = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
Exponential Rules:	$\int e^x dx = e^x + C$	$\int e^u u'dx = e^u + C$
	$\int a^x dx = \frac{a^x}{\ln a} + C$	$\int a^u u'dx = \frac{a^u}{\ln a} + C$
Trig Rules:	$\int \cos x\,dx = \sin x + C$	$\int \cos u\,u'\,dx = \sin u + C$
	$\int \sin x\,dx = -\cos x + C$	$\int \sin u\,u'\,dx = -\cos u + C$
	$\int \sec^2 x\,dx = \tan x + C$	$\int \sec^2 u\,u'\,dx = \tan u + C$
	$\int \csc^2 x\,dx = -\cot x + C$	$\int \csc^2 u\,u'\,dx = -\cot u + C$
	$\int \sec x \tan x\,dx = \sec x + C$	$\int \sec u \tan u\,u'\,dx = \sec u + C$
	$\int \csc x \cot x\,dx = -\csc x + C$	$\int \csc u \cot u\,u'\,dx = -\csc u + C$
Log Rules:	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$	$\int \frac{u'}{u}\,dx = \ln\lvert u\rvert + C$
Inverse Trig	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$	$\int \frac{u'}{\sqrt{a^2-u^2}}\,dx = \arcsin\frac{u}{a} + C$
	$\int \frac{1}{1+x^2}\,dx = \arctan x + C$	$\int \frac{u'}{a^2+u^2}\,dx = \frac{1}{a}\arctan\frac{u}{a} + C$

LESSON 7.1 SOLVING DIFFERENTIAL EQUATIONS, VERIFYING SOLUTIONS

Differential Equations are equations with derivatives in them. In this course, you will only learn how to solve the simplest type of differential equations, in which you can separate variables. You may be asked to find a general solution of the differential equation (which gives you a family of curves) or a particular solution (which gives you a single curve).

Procedure for Solving Differential Equations

1. Rewrite y' as $\frac{dy}{dx}$ (if necessary).
2. Multiply both sides of the equation by dx (if necessary).
3. Separate variables. (This is the most crucial step.)
4. Integrate both sides of the equation. (Remember to add C to one side.)
5. Solve for y (if necessary).
6. Use an initial condition to solve for C (if an initial condition is given).

Steps 5 and 6 are interchangeable.

Example 1: Find a general solution of $x+2yy'=0$

First, rewrite as $x+2y\frac{dy}{dx}=0$. Then,

Write your solution to Example 1 as a pair of possible functions (in the form $y=f(x)$) for the particular solutions to the differential equation. $y=$ or $y=$

Example 2: Find an equation of a function which contains the point $(0,-3)$, and whose slope is $\frac{xe^{x^2}}{y}$ for each point (x, y) on the curve.

Example 3:

a. Find a general solution of $y-2=x\frac{dy}{dx}$.

b. Find a particular solution of $y-2=x\frac{dy}{dx}$ if $y(1)=\frac{1}{2}$.

An algebra equation like $x^3+x^2+4=0$ cannot be solved using techniques you have learned without a calculator. However, you should be able to answer the following question.

Example 4: (algebra warm-up) Is $x=-2$ a solution of $x^3+x^2+4=0$?

Example 5: Is $y=\frac{1}{2}e^x+e^{-x}$ a solution of the differential equation $y'=e^x-y$?

ASSIGNMENT 7.1

For Problems 1-4, find a general solution of each differential equation.

1. $y'=\dfrac{x^2-1}{2y^2+3}$
2. $e^x y\frac{dy}{dx}=1$ Solve for y.
3. $2xy'=y+1$
4. $(x-2)\frac{dy}{dx}=2y$ Solve for y.

For Problems 5-7, find a particular solution of the differential equation with the given initial condition. (Remember to write your solutions in the form $y=f(x)$.)

5. $\frac{dy}{dx}=\dfrac{-2x}{y}$ and $y(2)=-4$
6. $y=-3x\frac{dy}{dx}$ and $y(1)=e$
7. $\frac{dy}{dt}=ky$ (where k is some constant), and $y(0)=100$. Write y as a function of k and t.

8. Find an equation of a function which contains the point $(-2,1)$ and whose slope is $\dfrac{x}{2y}$ for each point on the graph of the function.

9. Find the solution in the form $y=f(x)$ of the differential equation $\frac{x}{2}\frac{dy}{dx}-1=y^2$ if $f(1)=1$.

Determine whether each of the following is a solution of the differential equation $y''-9y=0$. Show organized work.

10. $y=\sin(3x)$ 11. $y=e^{3x}$ 12. $y=\cos(3x)$ 13. $y=e^{-3x}$

Differentiate in Problems 14-16.

14. $y=3^{2t-1}t^2$ 15. $f(y)=\dfrac{e^{\sqrt{y}}}{y^2}$ 16. $f(x)=e^x\ln x$

Antidifferentiate in Problems 17-19.

17. $y'=\dfrac{e^{-x}}{1+e^{-x}}$ 18. $g'(x)=\dfrac{2x-4}{x}$ 19. $y'=\dfrac{(\ln t)^3}{t}$

20. Find the area of the region bounded by $y=\left(\frac{1}{2}\right)^x$, $y=0$, $x=-2$, and $x=0$ without using a calculator.

21. Use a calculator to find the area between the curve $y=|2\cos x+\cos(2x)|$ and the x-axis, from $x=0$ to $x=\pi$. Show an integral set up and an answer.

22. For a particle moving along a straight path with velocity $v(t)=e^{-1.3t}-t\ln(.37t)$, $t>0$, use your calculator to find:
   a. the time when the particle is at rest.
   b. the speed of the particle at time $t=4$.
   c. the acceleration of the particle at time $t=5$.
   d. the total distance traveled by the particle on the interval $[1, 5]$.

23. If a particle moves along the curve $y=x^{\frac{2}{3}}$, such that $\frac{dx}{dt}=3$ for all x, find:
   a. $\frac{dy}{dt}$ when $x=-1$ b. $\frac{dy}{dt}$ when $x=8$
   c. $\lim\limits_{x\to\infty}\frac{dy}{dt}$ d. $\lim\limits_{x\to 0^-}\frac{dy}{dt}$

24. a. Use a tangent line to the graph of $y=x^{\frac{2}{3}}$ to approximate $(8.1)^{\frac{2}{3}}$.
   b. Why could a tangent line to $y=x^{\frac{2}{3}}$ at $x=0$ <u>not</u> be used to approximate $(.1)^{\frac{2}{3}}$?

25. f and g are inverse functions. The graph of g passes through the points $(-1,2)$, and $(2,-1)$ $f'(-1)=-2$ and $f'(2)=-1$. Find:
   a. $g'(-1)$ b. $g'(2)$

LESSON 7.2 EXPONENTIAL GROWTH AND DECAY AND OTHER MODELS

Mathematical models in which the rate of change of a variable is proportional to the variable itself are common in both the business and scientific worlds.

Suppose that the rate of change of y (with respect to time) is proportional to y itself.

$\frac{dy}{dt}$	$=$	k	$\bullet$	y
Rate of change of y with respect to time	$=$	constant of proportionality	$\bullet$	amount of substance y present at time t (y is a function of t)

Example 1: Separate variables and solve the differential equation above.

The equation from Example 1 is called the **Basic Law of Exponential Growth or Decay**:

$$y = Ce^{kt}$$

Constants:

C is the initial value (the amount of substance present at time $t = 0$)

k is the constant of proportionality ($k > 0$ for growth and $k < 0$ for decay)

Variables:

t is the variable for time

y is the amount of substance present at time t. (y is a function of t.)

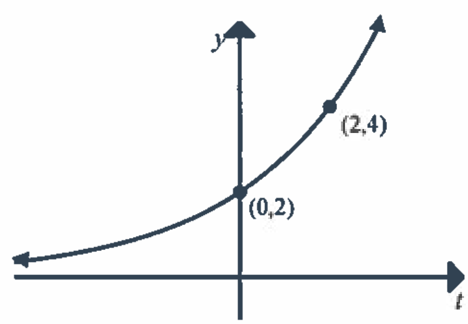
Example 2: What is the rate of growth of the population in a city whose population triples every 100 years? Assume that the population growth can be modeled by the Basic Law of Exponential Growth, and express your answer as a percent (rounded to the nearest hundredth of a percent).

Example 3: Let y represent the mass, in pounds, of a radioactive element whose half-life is 4000 years. If there are 200 pounds of the element in an inactive mine, how much will still remain in 1000 years? Express your answer to 3 or more decimal place accuracy.

Example 4: Newton's Law of Cooling states that the rate of change in the temperature of an object is proportional to the difference between the temperature of the object and the temperature of its surroundings. Suppose a metal figurine, heated to $150°$ F, is brought into a room having a constant temperature of $72°$ F. If the figurine cools from $150°$ to $120°$ in 15 minutes, how long will it take for the figurine to reach a temperature of $100°$ F? Let $y=$ temperature, and express your answers to the nearest minute.

ASSIGNMENT 7.2

1. Find the particular equation of the form $y = Ce^{kt}$ which represents the exponential growth graph shown at right. You must first solve for C. Then, you must solve for k. Finally, you can write the equation in the form $y = Ce^{kt}$ using your values for C and k.



2. \$1000 is placed into a certificate of deposit (CD) in which interest is compounded continuously at a rate of $5\frac{1}{2}\%$ per year (actual rate of return will be higher due to compounding of interest). Use your calculator and the formula $A = Pe^{rt}$ to find:

   a. the amount that the CD would be worth in 1 year. 5 years. 10 years.
   b. the time it would take the CD to be worth \$1,200.
   c. the time it would take the CD to double in value.

3. Suppose 200 bacteria are introduced into a culture to study their rate of growth. Two days later, the culture is found to contain 300 bacteria. Assuming the rate of growth is proportional to the number of bacteria present ($y=Ce^{kt}$), how many bacteria will be present in 3 more days (5 days after the start)?

4. Find the half-life of a radioactive isotope if 4.92 grams out of an initial 5 grams of the isotope remain after 10 years.

5. An isotope of carbon (C^{14}) is used for estimating how long ago certain living organisms were on earth. (The method is called carbon dating.) The half-life of C^{14} is approximately 5730 years. If the skull of an ancient primate contains 10% (.1) of the C^{14} present in the skull of a modern primate of a similar species, estimate how long ago the ancient primate lived (to the nearest thousand years).

6. A worker at a hazardous waste plant was accidentally exposed to toxic chemicals which were absorbed into his bloodstream. Upon feeling ill, the worker went to a hospital and had some blood drawn for testing. The concentration of chemical in the drawn blood was found to be .0158 *mg/ml*. Expensive medication was administered to counter the effects of the chemical in the blood, but the doctor on duty knew that the concentration of the chemical in the bloodstream would have to decrease gradually over time according to the Basic Law of Exponential Decay ($y=Ce^{kt}$). Medication would have to be administered every hour until the concentration was below .0050 *mg/ml*. Two hours later, blood was again drawn, and it was found to contain a chemical concentration of .0126 *mg/ml*. The doctor asked a lab technician to do the following. (You do the same):

   a. Write the particular solution for exponential decay for the chemical in the patient's blood. (Let $t=0$ represent the time that blood was first drawn.)

   b. Sketch a graph of the function from Part a.

   c. Find out how long it will be before the patient can be taken off medication.

   d. When the patient has only a negligible amount of chemical in his bloodstream (less than .0001 *mg/ml*), he can be released from the hospital. Find out how long the patient has to be hospitalized (from the time he first came to the hospital and had his blood drawn).

   e. Occasionally, patients exposed to this chemical suffer damage to their central nervous systems. A maximum concentration of .020 *mg/ml* requires a follow-up examination. The doctor estimated that the maximum concentration of the chemical in the worker's bloodstream occurred 1 hour after exposure. The patient estimated 1 hour after exposure would have been about 3 hours prior to his blood being drawn for the first time. Should the patient be asked to return for a follow-up exam? Why or why not?

   f. Find the half-life for the chemical in the bloodstream for the patient.

7. The rate of change in the temperature, y, of an object in a room with a constant temperature of $70°$ F is proportional to the difference between y and 70, (that is, $y-70$). (This is Newton's Law of Cooling.) If the object cools from $100°$ F to $80°$F in 20 minutes, how long would it take the object to cool from $100°$ F to $75°$ F?

Evaluate in Problems 8-14 without using a calculator.

8. $\int_0^1 \frac{3y^3+y^2+3y+2}{y^2+1}\,dy$ 9. $\int_0^1 \left(x^2-5\right)^2\,dx$ 10. $\int_0^{\frac{2\pi}{3}} \tan\left(\frac{x}{2}\right)dx$ 11. $\int \frac{x^2}{x^3+10}\,dx$

12. $\int \frac{du}{u(\ln u)^3}$ 13. $\int 5^{2x}\,dx$ 14. $\int \frac{\cos\left(\frac{1}{x}\right)}{x^2}\,dx$

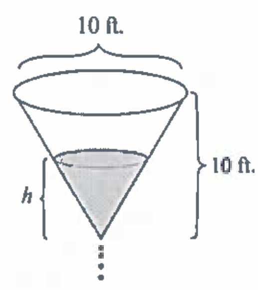
15. Find a general solution of $\sqrt[3]{y}\,y' - \sqrt[3]{x} = 0$.

16. Find the particular solution of $2xyy' = \ln x$, if the graph of the particular solution contains the point $(e,1)$. Make sure that your answer expresses y as a function of x. $(y = f(x))$

17. Show that $y = \frac{1}{2}\sin x - \frac{1}{2}\cos x + e^x$ is a solution of the differential equation $y' - y = \cos x$.

18. A conical tank, as shown at right, has a hole in its bottom and is leaking water at the rate of 1 cubic foot per minute. Find the rate of change in the height, h, of the water in the tank when $h = 4\,ft$? $V = \frac{1}{3}\pi r^2 h$

Write appropriate units for your answer.
Hint: Find a relationship between r and h. $r = \_\_\_h$



Find the indicated limits in Problems 19-21.

19. $\lim\limits_{x \to a} \frac{x-a}{a^2-x^2}$ 20. $\lim\limits_{x \to \infty} \frac{(3x-2)(x+1)^2}{(2x+1)^2(x-5)}$ 21. $\lim\limits_{x \to 0} \frac{1-e^x}{\sin(4x)}$

22. If $f'(t) = \lim\limits_{h \to 0} \frac{t+h+\sqrt{t+h}-(t+\sqrt{t})}{h}$, find $f(t)$.

LESSON 7.3 SLOPE FIELDS

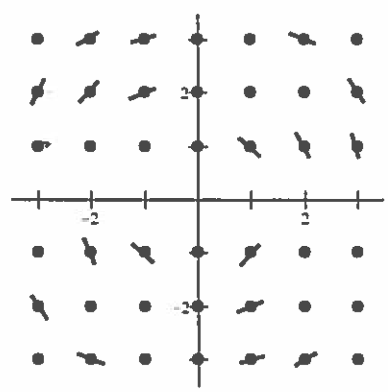
A **slope field** is a graphical representation of a set of slopes obtained from a differential equation. Remember that a differential equation involves a derivative. That derivative represents the slopes for a function. In Lesson 7.1, you learned to solve differential equations by separating variables. Even if you cannot separate variables and integrate, you can still use a differential equation to plot the slopes for a function.

Example 1: Find the slopes given by the differential equation $\frac{dy}{dx}=\frac{-x}{y}$ at the following points:

a. $(3, 2)$ b. $(-1, 3)$ c. $(-2,-1)$ d. $(2,-2)$

Why can't you find slopes when $y=0$?

Example 2: Find and plot the slopes given by $\frac{dy}{dx}=\frac{-x}{y}$ for each remaining marked point (dot) in the coordinate plane at the right.



Example 3: In Example 2, you made what is known as a slope field. Starting at the point $(0, 1)$, follow the flow of the slopes to sketch the solution curve containing $(0, 1)$. Your graph should be "parallel" to the slope lines and be like an "average of slopes" whenever it goes between lines. Your solution curve must represent a function whose domain is the largest possible open interval containing the given point. Sketch a solution curve passing through $(-1, 1)$ and one passing through $(0,-3)$. What type of graph does this differential equation seem to be producing?

Note: The most common student error in sketching a particular solution to a differential equation is to extend the sketch too far and create a graph which is not a function. It is important to set appropriate "boundaries" for your sketch. Why is the x-axis a "boundary" for the differential equation from Examples 1 and 2?

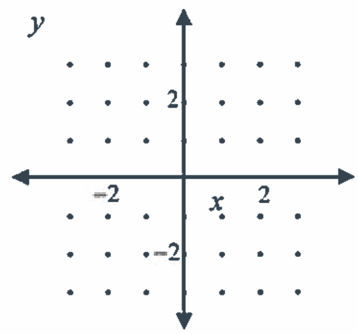
Example 4: Solve the differential equation $\frac{dy}{dx} = \frac{-x}{y}$.

Note: Solving for y yields $y =$ ________________ or $y =$ ________________ .

Find the particular solution for the differential equation $\frac{dy}{dx} = \frac{-x}{y}$ whose graph passes through the point $(0,1)$.

Find the particular solution whose graph passes through the point $(0,-3)$.

Example 5: For the differential equation $y' = \frac{1}{y}$



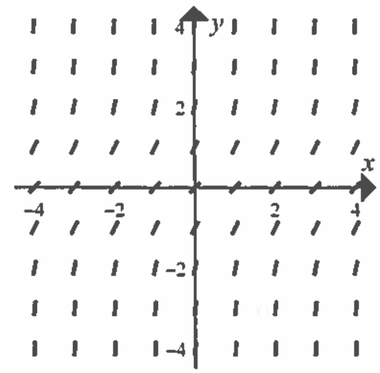
a. Draw the slope field in the dot coordinate plane at right.

b. Graph the particular solutions passing through the points $(-2,-1)$ and $(2,2)$ as functions of x.

c. Solve the differential equation.

d. Write as functions the particular solutions for the differential equation whose graphs pass through $(-2,-1)$ and $(2,2)$.

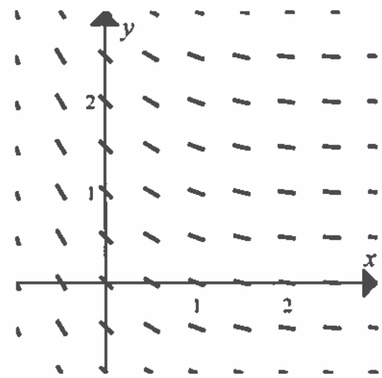
Example 6: Which of the differential equations below matches the slope field shown at right?



a. $y' = x$ b. $y' = y$ c. $y' = x - y$

d. $y' = 1 + y^2$ e. $y' = 1 + x^2$

Example 7: The slope field for a certain differential equation is shown at the right. Which of the following could be a specific solution to the differential equation?



a. $y = e^x$ b. $y = e^{-x}$ c. $y = -e^x$

d. $y = -\ln x$ e. $y = \ln x$

Using Lessons 7.3 and 7.1, you should be able to show graphically a particular solution of a differential equation, and confirm that solution by solving the differential equation (if it is possible to do so).

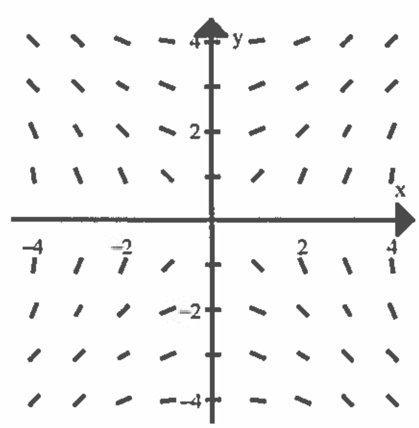
ASSIGNMENT 7.3 A tear-out sheet is provided on page 181 for your work.

1. Find the slopes given by the differential equation $y' = \dfrac{x^2}{y-2}$ at each of the following points:

   a. $(0,0)$ b. $(1,1)$ c. $(-2,4)$ d. $(4,-2)$ e. $(-3,-3)$ f. $(5,12)$

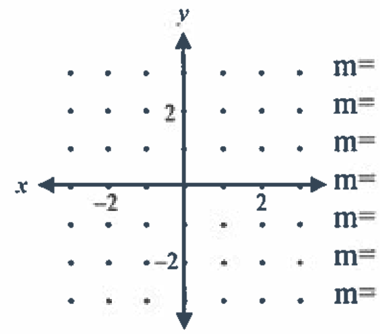
2. For the differential equation in Problem 1, why are there no slopes when $y = 2$?

3. The slope field for $y' = \dfrac{x}{y}$ is shown at right.



   a. Plot the following points on the slope field:
      i. $(1, 2)$ ii. $(3, 1)$ iii. $(0, 3)$
      iv. $(0, -2)$ v. $(-2, -1)$
   b. Plot a separate solution curve through each of the points from Part a. Remember that the curves have to be functions.
   c. What would a solution curve containing $(2, 2)$ look like?
   d. Solve the differential equation $y' = \dfrac{x}{y}$.

4. For the differential equation $\frac{dy}{dx} = y$



   a. Draw the slope field for the differential equation.
   b. Graph the particular solutions passing through the points $(0,1)$ and $(0,-1)$.
   c. Solve the differential equation, <u>and</u> find the particular solutions that contain the points $(0, 1)$ and $(0,-1)$.

5. For the differential equation $\frac{dy}{dx} = \frac{x}{2}$

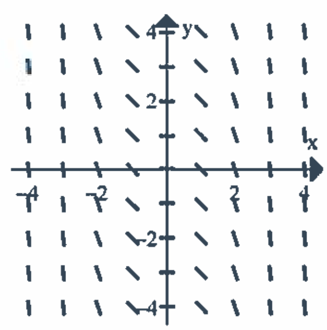
   a. Draw the slope field for the differential equation.
   b. Graph the particular solution passing through the point $(-1, 1)$.
   c. Solve the differential equation, and find the particular solution that contains the point $(-1, 1)$.

6. Repeat the three parts of Problem 5 for the differential equation $y' = \frac{1}{2y}$. For this problem, draw your graph as and write your solution as a <u>function of x</u>.

7. Repeat the three parts of Problem 5 for the differential equation $y'y^2 = 1$.

8. Repeat the three parts of Problem 5 for the differential equation $\frac{dy}{dx} = 2x(y-1)$, but use the <u>origin</u> (instead of $(-1,1)$) for Parts b. and c.

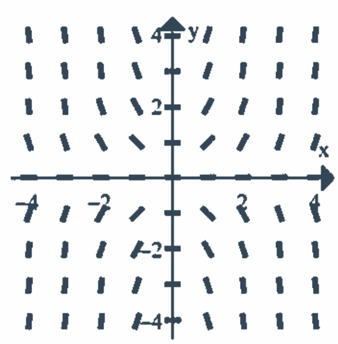
9. The slope field for a certain differential equation is shown at right. Which of the following could be a specific solution to that differential equation.



   a. $\frac{1}{3}x^3 + y = 2$ b. $x^2 + y^2 = 4$

   c. $x^2 - y^2 = 4$ d. $y = \frac{4}{x}$ e. $\frac{y}{x} = 4$

10. Which of the differential equations below matches the slope field shown at right?



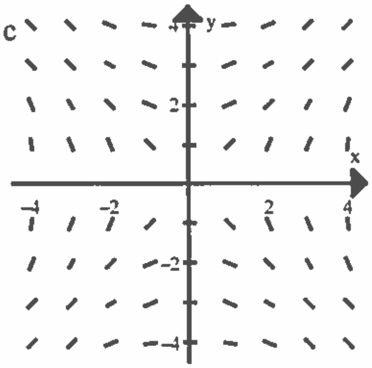
   a. $\frac{dy}{dx} = x - y$ b. $\frac{dy}{dx} = y - x$

   c. $\frac{dy}{dx} = \frac{x}{y}$ d. $\frac{dy}{dx} = \frac{y}{x}$ e. $\frac{dy}{dx} = xy$

Assignment 7.3 Tear-out Sheet Name ____________________

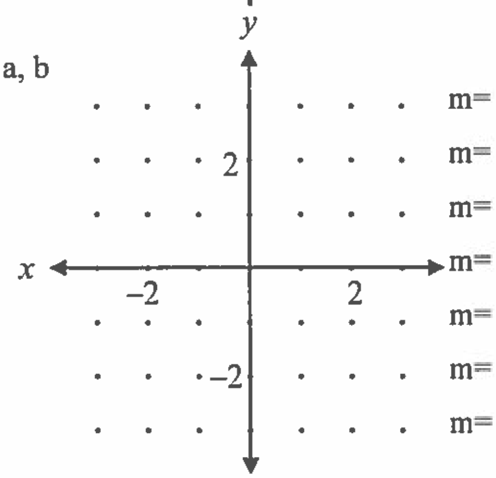
1. a. ____ b. ____ c. ____ d. ____ e. ____ f. ____ 2.

3. a, b, c



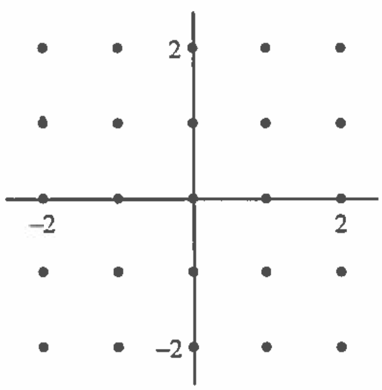
3. d. solve $y' = \frac{x}{y}$

4. a, b



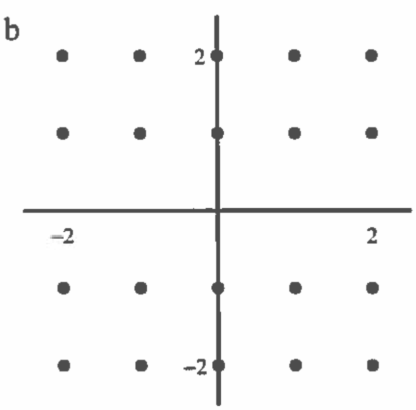
4. c.

5. a, b



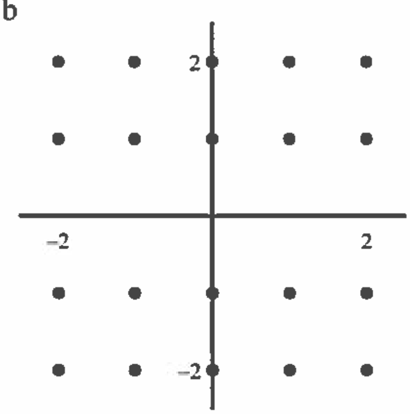
5. c.

6. a, b



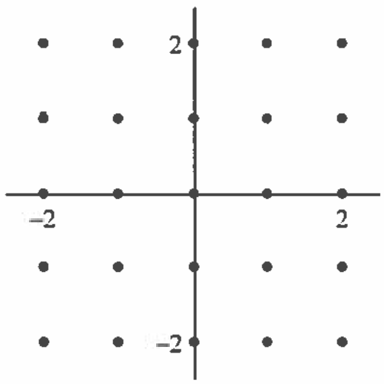
6. c.

7. a, b



7. c.

8. a, b



8. c.

9. ____ 10. ____

11. Use the acceleration graph at right to find the following for an object moving along a straight path.

    a. $a(1)$ b. $a(6)$ c. $a(9)$ d. $a(12)$

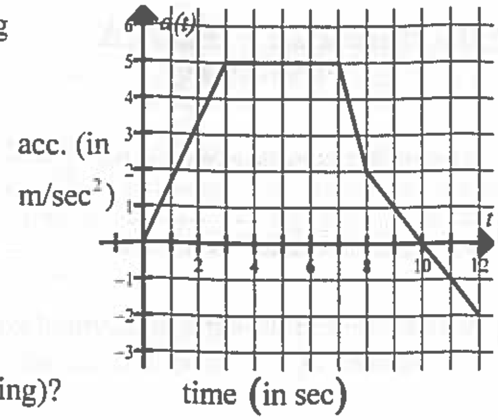
    Suppose $v(0)=2$ *m/sec.* Find:

    e. $v(3)$ f. $v(7)$ g. $v(10)$ h. $v(12)$

    i. At what time on [0,12] was the object moving the fastest? Justify your answer.

    j. At what time on [0,12] was the object moving the slowest? Justify your answer.

    k. When was the object slowing down (speed decreasing)?



Find the x-value(s) where each of the functions in Problems 12-15 is not differentiable. Give a reason why each function is not differentiable for those values of x. Do not use a calculator.

12. $f(x)=\left|x^2-9\right|$

13. $p(x)=\dfrac{x^2(x-2)}{x(x+1)}$

14. $q(x)=x-x^{\frac{1}{3}}$

15. $h(x)=\begin{cases} 2x+1, & x\le 1 \\ \frac{1}{2}x^2+x+\frac{3}{2}, & x>1 \end{cases}$

16. The product of two positive numbers is 80. Find the numbers so that the sum of the first number and five times the second number is a minimum. Do not use a calculator.

17. Use the f' and f'' number lines below to sketch a possible graph of a <u>continuous</u> function f.

f': $-$ on $(-\infty,0)$, $+$ on $(0,2)$, $-$ on $(2,\infty)$; marks at 0, 2

f'': $+$ on $(-\infty,-2)$, $-$ on $(-2,0)$, $-$ on $(0,\infty)$; marks at -2, 0

Evaluate the integrals in Problems 18-24 without using a calculator.

18. $\displaystyle\int \frac{x+1}{x-1}\,dx$

19. $\displaystyle\int \frac{t-2}{t^2-4t-5}\,dt$

20. $\displaystyle\int (\sec^2\theta-2)\,d\theta$

21. $\displaystyle\int \frac{\cos y}{\sin^3 y}\,dy$

22. $\displaystyle\int \sec^5 x\,\tan x\,dx$

23. $\displaystyle\int \frac{\csc^2(\pi x)}{\cot(\pi x)}\,dx$

24. $\displaystyle\int \frac{\cot^2 x-1}{\cot x}\,dx$

The velocity of a moving object is given by $v(t)=\dfrac{1}{4}t^3-2t+1$.

25. If the position at $t=2$ is given by $x(2)=3$, find $x(t)$.

26. Find the total distance traveled by the object on the interval $[0,4]$.

27. $f(x)=\dfrac{x}{x^2+1}$, $f'(x)=\dfrac{1-x^2}{\left(x^2+1\right)^2}$, and $f''(x)=\dfrac{2x\left(x^2-3\right)}{\left(x^2+1\right)^3}$.

    a. Without using a calculator, list the domain, asymptotes, and intercepts for $f(x)$.

    b. Find the relative extrema points of $f(x)$.

    c. Find the points of inflection of $f(x)$.

    d. Sketch $f(x)$ without using a calculator.

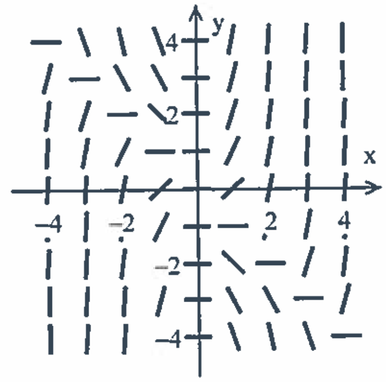
ASSIGNMENT 7.4 REVIEW

1. Find the particular solution of $4y' = 3x^2 + 2x$, if $y(2) = 5$.

2. Find the particular solution of $\frac{dy}{dx} = \frac{x}{y}$, if $y(5) = 3$.

3. Find the particular solution of $\frac{y'}{3x^2} = y - 1$, if $y(0) = 5$.

4. Which of the following differential equations is represented by the slope field shown?



A. $\frac{dy}{dx} = x(x-y)$ B. $\frac{dy}{dx} = x(x+y)$

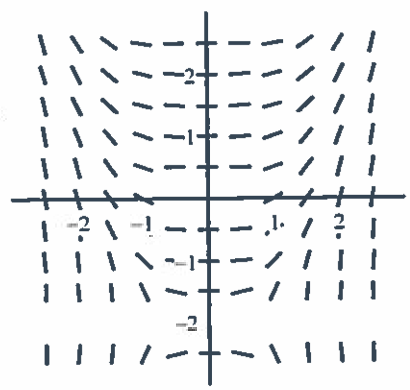
C. $\frac{dy}{dx} = y(x+y)$ D. $\frac{dy}{dx} = \frac{x}{(x+y)}$

E. $\frac{dy}{dx} = xy$

5. Sketch a slope field for the differential equation $\frac{dy}{dx} = x + y$. Use $[-2, 2]$ for both your x-interval and your y-interval.

6. For the differential equation from Problem 5, sketch a particular solution which passes through the point $(1, -1)$.

7. Which of these differential equations corresponds to the slope field shown?



A. $\frac{dy}{dx} = \frac{x^2}{y+2}$ B. $\frac{dy}{dx} = \frac{x^2}{y-2}$

C. $\frac{dy}{dx} = \frac{x^3}{y-2}$ D. $\frac{dy}{dx} = \frac{x^3}{y+2}$

8. If $(2, 6)$ is a point on the solution curve of the differential equation $y + xy' = 5$, determine the concavity of the solution curve at that point.

9. Solve the differential equation $\frac{dH}{dt} = \frac{t^2}{H^2}$ to find an expression for $H(t)$ if $H(-2) = 3$.

10. Which of the following is a solution of the differential equation $y' + y = e^x$?

A. $y = e^x + 2e^{-x}$ B. $y = \frac{1}{2}e^x - 2e^{-x}$ C. $y = e^x - 3e^{-x}$ D. $y = \frac{1}{2}e^{-x} + 3e^x$

11. The rate of coal production, $R(t)$, from a coal mine was 2,000,000 tons per year at the beginning of the year 2000. By the beginning of the year 2006, the coal production had decreased to 1,200,000 tons per year. The mine will be profitable until less than 200,000 tons of coal are produced in a given year. Assuming that the decline in the amount of coal mined per year closely models the equation for exponential decay $\left(y = Ce^{kt}\right)$, find the following:

 a. the particular equation for the amount of coal mined per year. (Let $t = 0$ for the beginning of the year 2000.)
 b. the half-life for the production of coal (from your model in Part a.).
 c. the year when it will no longer be profitable to mine coal.

12. Write a sentence telling what $\int_0^6 R(t)\,dt$ (from Problem 11) represents.

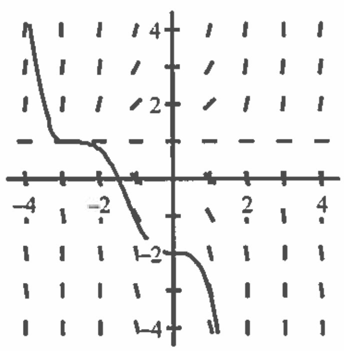
13. Approximate the value of the integral from Problem 12 using the table at the right. (R values are to the nearest thousand tons per year.)

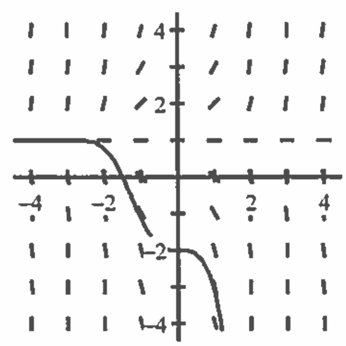
 a. Use a Midpoint Riemann Sum with 3 equal subdivisions.
 b. Use a Trapezoidal approximation with widths of 1,2, and 3 in that order.

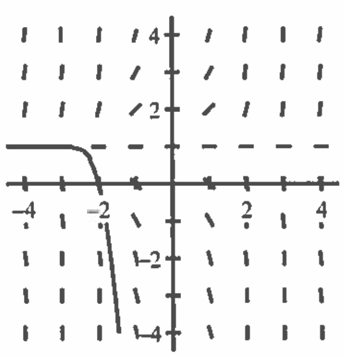
t	$R(t)$
0	2000
1	1837
2	1687
3	1549
4	1423
5	1307
6	1200

14. Use your calculator to find the actual value of $\int_0^6 R(t)\,dt$ from Problem 12 to the nearest thousand tons.

15. Which of the following shows a correct solution curve containing the point $(0,-2)$ for the slope field represented by the differential equation $\dfrac{dy}{dx} = x^2(y-1)$?

A.


B.


C.


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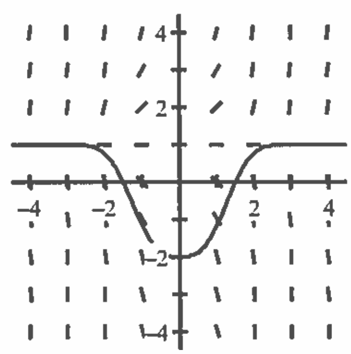
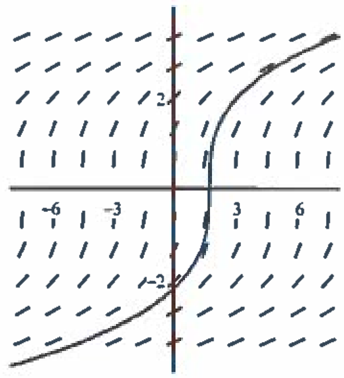
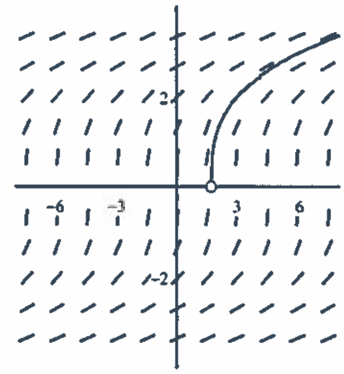


16. Which of the following shows a correct solution curve containing the point $(3,2)$ for the slope field represented by the differential equation $\frac{dy}{dx}=\frac{2}{y^2}$?

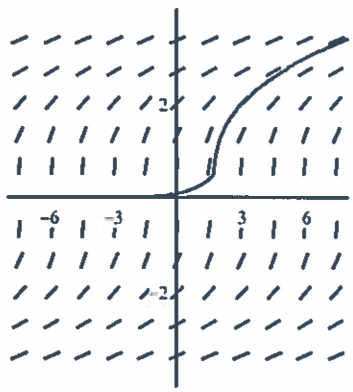
A.



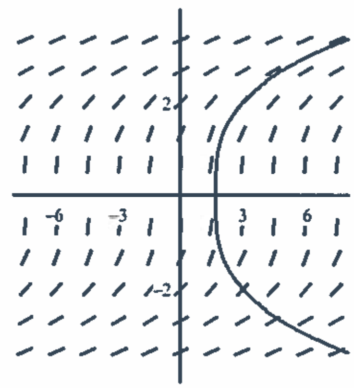
B.



C.



D.



Find the following limits.

17. $\lim\limits_{\theta\to\frac{\pi}{2}} \frac{2-\cos\theta}{\theta}$

18. $\lim\limits_{x\to 2} \frac{3e^{x-2}}{4x-8}$

19. $\lim\limits_{x\to\infty} \frac{e^x-3}{x^2+x+2}$

20. $\lim\limits_{x\to\infty} \frac{\ln(x+1)}{x+2}$

21. $\lim\limits_{x\to\infty} \frac{5x^4-3x^2+5}{x^4+3x-7}$

22. $\frac{d}{dx}\int_{2x}^{3x} \sin t^2 dt = ?$

23. If $f(2)=-15$ and $f'(x)=\ln(x+2)+e^x$ find $f(6)$. You may use a calculator.

24. Without using a calculator, find the x-values of all extrema and points of inflection for the following functions. Show all number lines and analysis.
   a. $y=x^3-3x^2-9x+7$ b. $y=3x^5-5x^3$

25. If you build 3 identical adjacent rectangular pens with 600 feet of fence, what dimensions of the total enclosure will maximize the total area? Show the equations you use and proper calculus steps.

26. Find y' if $3xy - y^2 = 2y - 4x^3$.

27. Integrate the following:

a. $\int \frac{e^x}{e^x + 5} dx$ b. $\int \frac{\ln x + 3}{x} dx$ c. $\int \frac{x^2 - 2x - 1}{x - 1} dx$

28. In 1990, the population of a city was 123,580. In 2000, the city's population was 152,918. Assuming that the population is increasing at a rate proportional to the existing population, use your calculator to estimate the city's population in 2025. Express your answer to the nearest person.

29. A radioactive element has a half-life of 1000 years. How much of 200 grams of the element will remain after 750 years?

UNIT 7 SUMMARY

Differential Equations: (Equations involving derivatives.)

Procedure for Solving Differential Equations

1. Rewrite y' as $\frac{dy}{dx}$ (if necessary).
2. Multiply both sides of the equation by dx (if necessary).
3. Separate variables. (This is the most crucial step.)
4. Integrate both sides of the equation. (Remember to add C to one side.)
5. Solve for y (if necessary).
6. Use an initial condition to solve for C (if an initial condition is given).

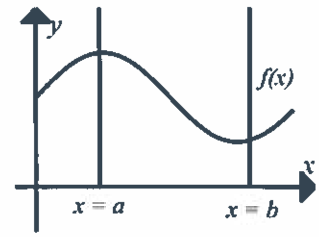
Steps 5 and 6 are interchangeable.

Exponential Growth and Decay: $y = Ce^{kt}$

Slope Fields: Be able to:

Sketch a slope field	given:	a differential equation.
Sketch a solution curve	given:	a differential equation and a starting point.
Match a slope field	with:	a differential equation.
Match a slope field	with:	the solution of a differential equation.

LESSON 8.1 AVERAGE VALUE OF A FUNCTION ACCUMULATION OF RATE FUNCTIONS



Discovering the Formula for the Average Value of a Function.
The average value of a function represents its average "height."

1. Draw a horizontal segment from $x=a$ to $x=b$ in the figure at right which could represent the average height of $f(x)$ on $[a,b]$.

2. Find the area of the rectangle formed.

$$A = \underbrace{\qquad}_{\text{width}} \bullet \; height$$

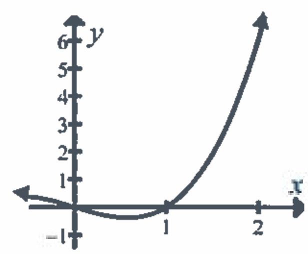
3. $\int_a^b f(x)\,dx = (b-a) \bullet height$ Now solve for the average height.

Average Value of a Function:

Average value of $f(x)$ on $[a, b]$ = $f(c)$ = $\boxed{f_{avg} = \dfrac{\overbrace{\int_a^b f(x)\,dx}^{\text{"area" of region}}}{\underbrace{b-a}_{\text{"width" of region}}}}$ or $\dfrac{1}{b-a}\int_a^b f(x)\,dx$

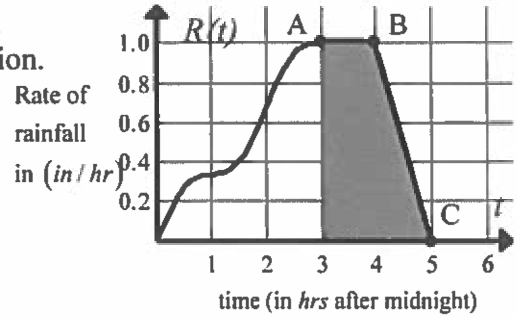
$f(x)$ must be continuous on $[a, b]$.

Example 4: Find the average value of $f(x) = x^3 - x$ on the interval [0, 2] without using a calculator.



Example 5: Use your calculator to find the value of c in the interval [0, 2] where $f(c)$ = the average value found in Example 4.

Accumulation of "Rate Functions":
In Lesson 4.1 you interpreted rate function models. We can now revisit that topic and include the concept of accumulation.



Examples:
The graph at right models the rate of rainfall in inches per hour from midnight until 6:00 A.M. during a tropical rainstorm.

6. Write a complete sentence to explain what Point A on the graph represents. Include numbers and units in your answer to this Example and Examples 8, 10, and 12.

7. What is the slope of the graph between Points B and C?

8. Write a complete sentence to explain the meaning of your answer to Example 7.

9. Find $\int_3^5 R(t)\,dt$.

10. Write a complete sentence to explain the meaning of your answer to Example 9.

11. Approximate the value of $\int_0^6 R(t)\,dt$ using geometrical regions. Show computations.

12. Write a complete sentence to explain the meaning of your answer to Example 11.

ASSIGNMENT 8.1

For Problems 1-5, find the average value of each function on the given interval.

1. $f(x)=x^3$ on $[0,2]$ No calculator.

2. $g(t)=\dfrac{1}{(t-1)^2}$ on $[2,5]$ No calculator.

3. $f(y)=2y-\sqrt{y}$ on $[1,4]$ No calculator.

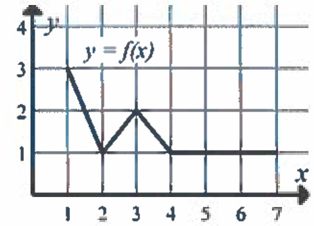
4. $f(t)=\dfrac{t^2-1}{\sqrt{t}+1}$ on $[.4,3.2]$ Use a calculator.

5. $h(x)=.5^x$ on $[-2,1]$ Use a calculator.

6. Find the exact x-value where the function in Problem 1 equals its average value.

7. Use a calculator to solve for c for the equation $h(c)=h_{avg}$ in Problem 5. Your answer should be expressed to 3 or more decimal place accuracy.

8. The graph of a function f is shown in the figure at right.

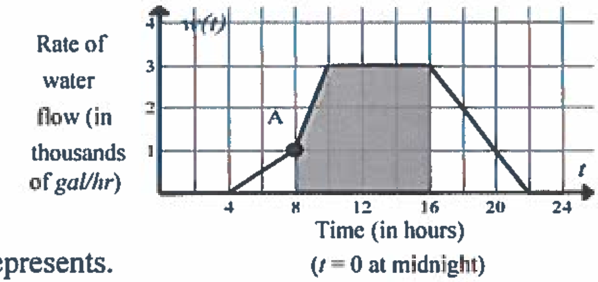


a. Evaluate $\int_1^7 f(x)\,dx$.

b. Determine the average value of the function on the interval $[1,7]$.

c. Determine the answers to Parts a. and b. if the graph is shifted two units upward.

9. The graph at the right represents the rate of flow of irrigation water (in thousands of gallons/hour) from a reservoir during a 24 hour period.



a. Write a sentence telling what Point A represents.

b. Find the shaded area. Show computations.

c. Write a sentence telling what the shaded area represents. Include numbers and units.

d. Set up an integral which represents the total amount of water released from the reservoir during the day shown.

e. Find the value of the integral in Part d.

f. Find the average rate of water flow during the 24 hours. Label units.

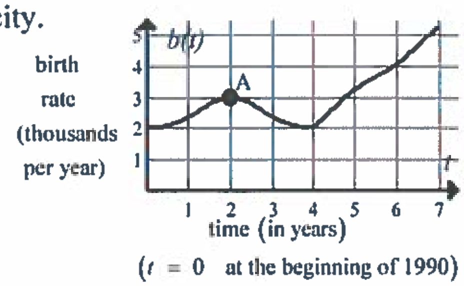
10. The graph at the right models the birth rate in a Utah city.

a. Tell what Point A represents.

b. Approximate $\int_2^6 b(t)\,dt$ using geometrical shapes. Show computations.

c. Tell what your answer to Part b. represents.

d. When was the birth rate the lowest, and what was the birth rate at that time?

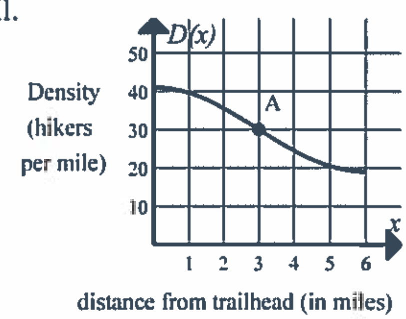


11. The graph at the right represents the density of hikers on a trail.

a. Tell what Point A represents.

b. $\int_0^3 D(x)\,dx = 110$. Write a sentence with numbers and units stating what this represents.

c. If $D'(3) = -6$, use <u>local linearization</u> to approximate the density of hikers 3.1 miles from the trailhead.



12. The graph at right models velocity.

a. Tell what Point B represents.

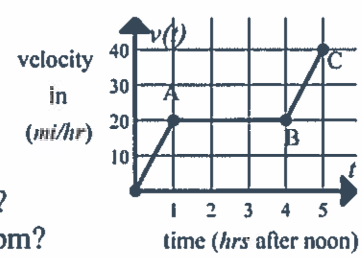
b. Tell what the slope between B and C represents.

c. What is the velocity at 4:30 pm?

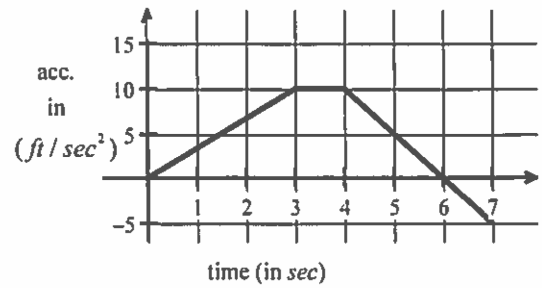
d. What is the acceleration at 4:30 pm?

e. What distance is traveled between noon and 5:00 pm?

f. What is the average velocity between noon and 5:00 pm?



13. The graph at right models acceleration
    a. What is the acceleration at $t = 2\ sec$?
    b. When is the acceleration $10\ ft/sec^2$?
    c. What is the minimum acceleration?
    d. If the initial velocity is zero, what is the velocity at $t = 6\ sec$?
    e. If the initial velocity is $20\ ft/sec$, what is the velocity at $t = 6\ sec$?
    f. If the initial velocity is $20\ ft/sec$, what is the velocity at $t = 7\ sec$?



14. The rate at which oil is leaking from an old storage tank is modeled by the function $L(t) = 5.2e^{0.3t} - 4t$ where t is measured in hours after midnight. At the same time oil is being added to the tank at the rate of $A(t) = \left|56\sin\left(\dfrac{3t^2+5}{6}\right)\right|$ where both functions are measured in liters per hour. At midnight there were 380 liters of oil in the tank.
    a. How much oil has leaked from the tank between midnight and 6 am?
    b. How much oil is left in the tank at 6 am?

For Problems 15 and 16, sketch graphs and <u>approximate</u> the values of the intervals using
a. a Midpoint Riemann Sum. b. a Trapezoidal approximation
where $n =$ the number of equal subdivisions.

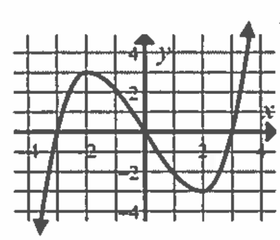
15. $\int_0^8 x^3\,dx$
    $n = 4$

16. $\int_0^{12} -\sqrt{x}\,dx$
    $n = 3$

17. Two men in a search party begin walking from Search Headquarters at the same time. One man walks North at a rate of $4\ ft/sec$, while the other man walks West at a rate of $3\ ft/sec$. After both men have walked for one minute, find
    a. the distance separating the two men.
    b. the rate at which the distance between the two men is changing.

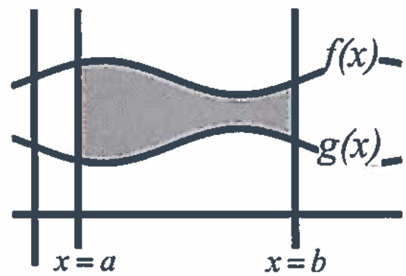
18. Use the graph of f' at right to sketch
    a. a graph of f''.
    b. a graph of f which passes through the origin of the coordinate plane.
    Use a separate coordinate plane for each graph.

$f'(x)$



LESSON 8.2 AREA BETWEEN CURVES

$\int_a^b f(x)\,dx$ produces a value ("signed area") which may be positive, negative, or zero.
However, if you are asked to find an actual area, that area cannot be negative.



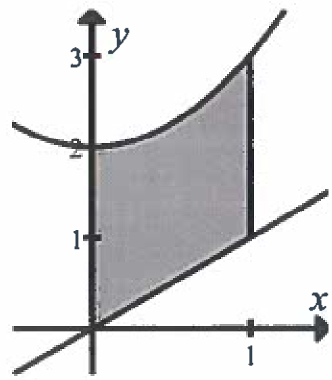
Area of a Region Between Two Curves

$A=\int_a^b (f(x)-g(x))\,dx$ if $f(x)$ and $g(x)$ are continuous and $f(x)\ge g(x)$ on $[a,b]$.

For functions of x, $A=\int_a^b (\text{top curve} - \text{bottom curve})\,dx$.

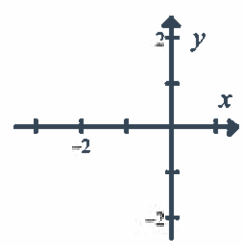
For functions of y, $A=\int_a^b (\text{right curve} - \text{left curve})\,dy$.

Example 1: Find the area of the region bounded by $y=x^2+2$, $y=x$, $x=0$, and $x=1$.

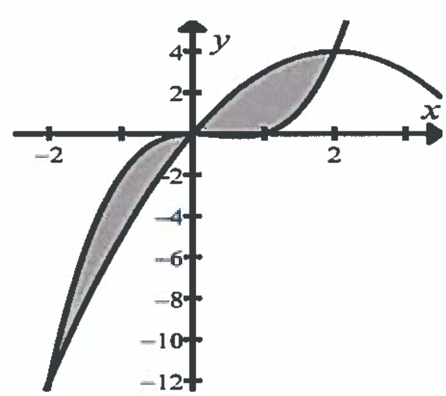


Sometimes you have to find where two curves intersect to determine "boundaries" for your region(s). These intersections will provide you with limits of integration for your integral(s). You must show an equation set up, even when using a calculator to find the intersections.

Example 2: (Functions of y)
Find the area of the region bounded by $x=y^2-3$ and $y=x+1$.



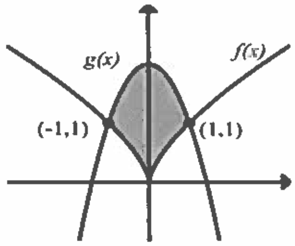
Example 3: Set up integrals for the total area of the regions located between the two curves as shown. You may use a calculator.
$f(x)=x^3-x^2$ and $g(x)=-x^2+4.1x$



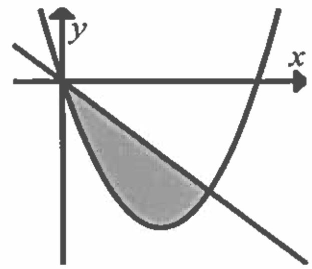
ASSIGNMENT 8.2

For Problems 1 and 2, set up integrals that could be used to find the areas of the shaded regions. Do not integrate. Show the equation(s) used to find the limits of integration for Problem 2 without using a calculator.

1. $f(x) = x^{\frac{2}{3}}$
   $g(x) = -x^2 + 2$
   $A = \int_{-1}^{1}$ ______

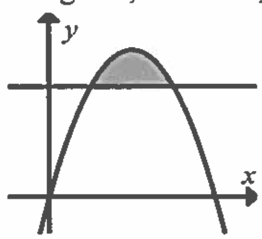


2. $f(x) = x^2 - 4x$
   $g(x) = -x$

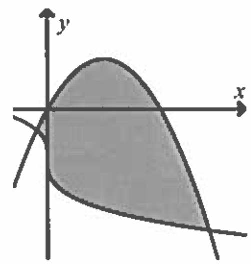


For Problems 3 and 4, show equations used to find the limits of integration, show integral set ups for the areas of the shaded regions, and then find the areas.

3. No calculator.
   $f(x) = 3$
   $g(x) = 4x - x^2$



4. Use a calculator.
   $f(x) = 2x - x^2$
   $g(x) = -\sqrt[3]{x} - 1$



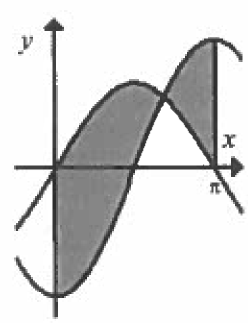
For Problems 5 and 6, sketch regions bounded by the graphs of the given equations, show equations used to find the limits of integration, show integral set ups, and find the areas.

5. Use a calculator.
   $y = 3^t$
   $y = \sqrt{t+2}$

6. No calculator. Hint: Write as functions of y. (Isolate x)
   $x = y^2 - 3$ $\qquad y = -\frac{1}{2}x$

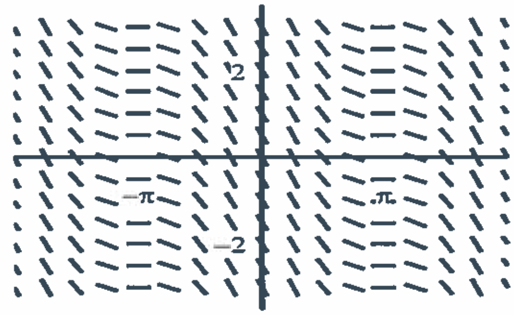
7. a. Sketch graphs of $y = -4x$ and $y = -x^3$ in one coordinate plane.
   b. Explain why the area bounded between the curves cannot be written as a single integral using -2 and 2 as limits of integration.
   c. Use the symmetry of the graphs to write a single integral for the area.

8. Use a calculator to find the shaded area between the curves $f(x) = 2\sin x$ and $g(x) = -3\cos x$ on $[0, \pi]$ as shown at right. Show an integral set up and express your final answer to 3 or more decimal place accuracy. Make sure your calculator is in radian mode.



9. Solve the differential equation $y' = -3x^2y^2$ to find the particular solution passing through the point $(1,9)$. Write your answer in the form $y = f(x)$.

10. Which of these differential equations corresponds to the slope field shown?



A. $\dfrac{dy}{dx} = -\sin x$ B. $\dfrac{dy}{dx} = \cos x + 1$

C. $\dfrac{dy}{dx} = \sin x$ D. $\dfrac{dy}{dx} = -\cos x - 1$

11. Find:
   a. the instantaneous rate of change of $f(x) = -2x + \ln x$ at $x = 1$ and at $x = 2$.
   b. the average rate of change of $f(x) = -2x + \ln x$ on the interval $[1, 2]$.

12. Find the value of c where $f'(c) = \dfrac{f(b) - f(a)}{b - a}$, for $f(x) = -2x + \ln x$ on $[1, 2]$.

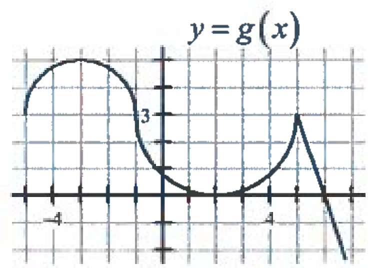
13. Find the average value of $f(x) = -2x + \ln x$ on $[1, 2]$.

14. Without using a calculator, find the maximum value of $f(x) = -2x + \ln x$.

15. Use the equation of a tangent line at $x = 1$ to approximate $f(.9)$ on the graph of $f(x) = -2x + \ln x$.

16. Approximate $\int_1^4 (-2x + \ln x)\,dx$ by using a Trapezoidal approximation with 3 equal subdivisions.

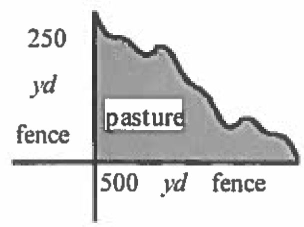
17. The graph at the right is of $y = g(x)$ consisting of two semicircles and a segment as shown. If $f(x) = \int_{-1}^{x} g(t)\,dt$ find each of the following:



   a. an expression for $f'(x)$.
   b. $f'(-1)$.
   c. the interval(s) on which f is increasing. Justify.
   d. the x-value where f has a local maximum. Justify.
   e. the x-value(s) where $f''(x)$ is undefined. Explain.
   f. the x-values of the point(s) of inflection of f. Justify.
   g. the exact value of $f(6)$.
   h. the exact value of $f(-5)$.

18. If $h'(x) = e^{3x^2-4}$ and $h(7) = 9$, find $h(3)$.

19. A pasture is bounded by a river and two fences as shown in the figure. Approximate the total square yards of pasture area using
    a. a Right Riemann sum using five equal subdivisions.
    b. a Trapezoidal approximation with four unequal widths of 50, 100, 150, and 200 yards in that order.



x	0	50	100	150	200	250	300	350	400	450	500
y	250	210	190	190	170	120	80	60	60	40	0

Differentiate the functions in Problems 20 and 21.

20. $y=\dfrac{3x^3-x+1}{x^2}$

21. $f(x)=\left(x-\sqrt{x}\right)^{10}$

Antidifferentiate in Problems 22-24.

22. $y'=\dfrac{(x-2)^2}{\sqrt{x}}$

23. $f'(x)=x(x^2+3)^5$

24. $g'(x)=\dfrac{x}{\sqrt{4-3x^2}}$

Find the indicated limits for Problems 25 and 26.

25. $\lim\limits_{x\to 2^-}\dfrac{|x-2|}{x-2}$

26. $\lim\limits_{x\to 2}\dfrac{|x-2|}{x-2}$

27. Find $\frac{d}{dx}\int_{x^3}^{x}\cos(2t+1)\,dt$ without integrating.

LESSON 8.3 VOLUMES OF SOLIDS WITH KNOWN CROSS SECTIONS

In Geometry, you learned formulas for finding volumes of common three-dimensional solids (cubes, spheres, cones, rectangular prisms, and perhaps others).

Calculus allows us to find volumes of solids whose bases are two-dimensional regions within an x-y coordinate system, and whose heights are formed by cross sections (most often squares, rectangles, semicircles, or triangles) which essentially "stick out from the base" to form the third dimension of the object.

Area formulas for common cross sections:				
Square	Rectangle	Semicircle	Triangle	Equilateral Triangle
$A=s^2$	$A=bh$	$A=\frac{1}{2}\pi r^2$	$A=\frac{1}{2}bh$	$A=\frac{\sqrt{3}}{4}s^2$

Volumes of Solids with Known Cross Sections

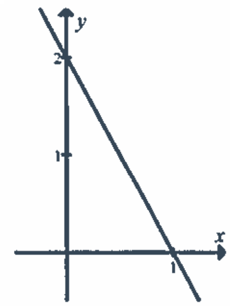
For cross sections perpendicular to the x-axis:

$V=\int_a^b A\,dx$, where A is a function of x and gives the area of a representative cross section.

For cross sections perpendicular to the y-axis:

$V=\int_a^b A\,dy$, where A is a function of y.

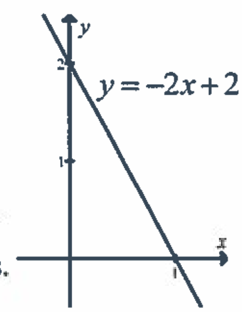
Examples:



1. Find the volume of the solid whose base is a triangle bounded by $y=-2x+2$, $x=0$, and $y=0$, and whose cross sections are squares which are perpendicular to the x-axis.

$$V=\int_a^b A\,dx$$

$$V=\int_0^1 s^2\,dx=$$

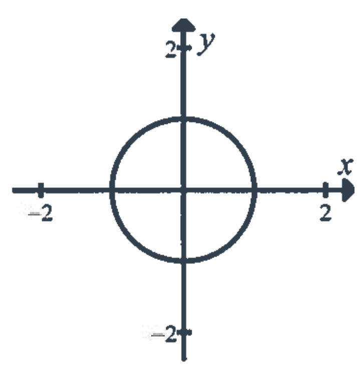


2. Set up (but do not integrate) integrals for the volumes of the solids with the same base as in Example 1, but whose cross sections are:

   a. Semicircles perpendicular to the x-axis.

   b. Rectangles of height $\frac{1}{4}$ which are perpendicular to the y-axis.

3. Set up (but do not integrate) an integral for the volume of a solid whose base is bounded by $y=-x^2+2$ and $y=x$ and whose cross sections are squares perpendicular to the x-axis.



4. Set up integrals for the volumes of the solids whose base is the circle $x^2+y^2=1$ and whose cross sections are:

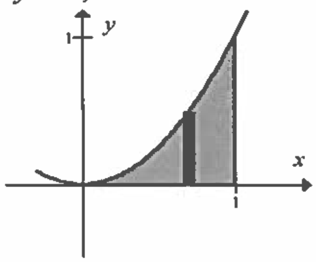
   a. Equilateral triangles perpendicular to the y-axis.

   b. Rectangles whose heights are three times their bases and whose bases are perpendicular to the y-axis.

ASSIGNMENT 8.3

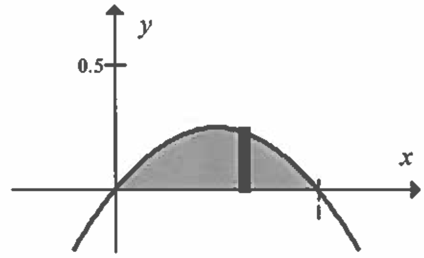
For Problems 1 and 2 set up (but do not integrate) integrals for evaluating the volumes of the solids formed by the given cross sections.

1. Base region bounded by $y = x^2$, $y = 0$, and $x = 1$ as shown.



   a. Squares
   b. Semicircles
   c. Rectangles with height 2

2. Base region bounded by $y = -x^2 + x$ and $y = 0$ as shown.



   a. Squares
   b. Equilateral triangles
   c. Rectangles whose heights are half of their bases

For Problems 3 and 4 sketch a graph of the base region and set up (but do not integrate) integrals for evaluating the volumes of the solids formed by the given cross sections.

3. Base region is the circle $x^2 + y^2 = 4$
   a. Squares perpendicular to the y-axis
   b. Semicircles perpendicular to the y-axis.

4. Base region bounded by $y = 2 - x$ and $y = x^2$
   a. Squares perpendicular to the x-axis
   b. Right triangles perpendicular to the x-axis. The base of the triangle sits in the region and the height $= 3$ times the base.

For Problems 5 and 6 sketch a graph of the base region and find the volumes of the solids formed by the given cross sections.

5. Find the volume of a solid whose base is bounded by $y = \sqrt{x}$, $y = 0$, and $x = 4$, and whose cross sections are squares perpendicular to the x-axis.

6. Find the volume of a solid whose base is bounded by $y = 2x$, $y = 0$, and $x = 3$, and whose cross sections are semicircles perpendicular to the x-axis.

For Problems 7 and 8, sketch the regions bounded by the given equations, and find the areas of the regions. Show your integral set ups, and do not use a calculator.

7. $f(x) = 2x^2 + 3x$
   $g(x) = 2$

8. $f(y) = y^2 - 2y$
   $g(y) = 2 - y$

9. Without using a calculator, find the volume of the solid formed by square cross sections perpendicular to the y-axis, whose base is the region bounded by $xy = 4$, $x = 0$, $y = 1$, and $y = 4$.

10. Find the average value of $f(x) = xe^{x^2}$ on the interval $[0,2]$ without using a calculator.

Use a calculator for Problems 11 and 12.

11. Find the average (mean) value of $f(x) = \log(x^2 + 5)$ on [1, 4].

12. Find the x-value on the interval [1, 4] which produces the average value in Problem 11.

Differentiate the functions in Problems 13 and 14.

13. $g(y) = y^3(2y-1)^{\frac{2}{3}}$

14. $h(t) = \dfrac{2t-1}{3t^2+5}$

Antidifferentiate in Problems 15 and 16.

15. $g'(y) = \dfrac{4y}{\sqrt{y^2+3}}$

16. $h'(t) = \dfrac{t}{\sqrt{t+3}}$ (Be careful on this one.)

Find the indicated limits for Problems 17 and 18.

17. $\lim\limits_{x\to 1} \dfrac{x^2+x-2}{x-1}$

18. $\lim\limits_{x\to -1} \dfrac{\sqrt{x+2}-1}{x+1}$

19. For $f(x) = \begin{cases} x^2-3, & x<1 \\ 4, & x=1 \\ -2x, & 1<x<2 \\ -x^3+8, & x\ge 2 \end{cases}$

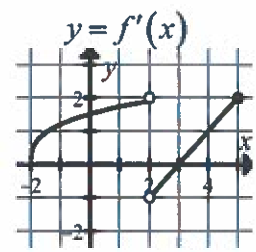
a. $\lim\limits_{x\to 1} f(x) =$

b. $\lim\limits_{x\to 2} f(x) =$

20. Use the graph of $y = f'(x)$ shown at the right to sketch:

a. a graph of $f''(x)$.

b. a graph of $f(x)$ that is continuous on the interval $[-2,5]$ and which contains the point $(2,1)$.



21. Find the equation of a tangent line to $x^2 + 3yx = y^2 + 3$ at $(1,2)$.

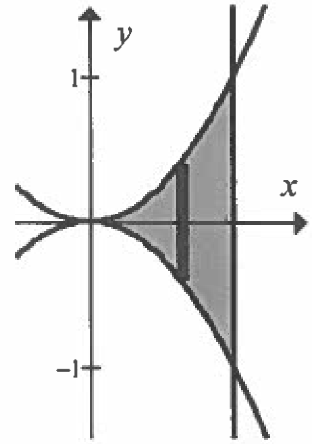
22. Sketch a graph of a function $f(x)$ having the following characteristics:
f is continuous. $f(-2) = f(0) = 0$.
$f'(0)$ is undefined. $f'(x) > 0$ for $x < -1$ and for $x > 0$. $f'(x) < 0$ on $(-1,0)$.
$f''(x) < 0$ for $x < 0$, and $f''(x) > 0$ for $x > 0$.

LESSON 8.4 VOLUMES OF SOLIDS OF REVOLUTION: DISCS AND WASHERS

If, for a given base, you create semicircular cross sections "sticking toward you" and "away from you," you have created circular cross sections. Since the formula for the area of a circle is $A = \pi r^2$, the formula for a volume with circular cross sections is:

$V = \int_a^b \pi r^2\,dx$ or $V = \pi \int_a^b r^2\,dx$ perpendicular to x-axis

$V = \pi \int_a^b r^2\,dy$ perpendicular to y-axis



Warm-up Example: Use the region bounded by $y = x^2$, $y = -x^2$, $x = 0$ and $x = 1$ as shown in the graph at right to create a volume by using circular cross sections perpendicular to the x-axis.

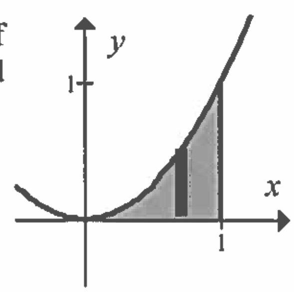
Circular cross sections can also be formed by revolving very thin (essentially no width) rectangles about an axis of revolution. These circular cross sections are more commonly called discs.

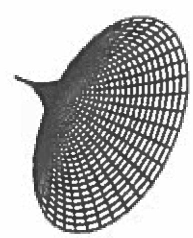
A volume formed by revolving a region about a line that does not pass through the interior of the region is called a solid of revolution. The line is called the axis of revolution.

If a region is bounded by the axis of revolution, the volume of the solid of revolution is a sum of the volumes of essentially an infinite number of cylindrical discs.

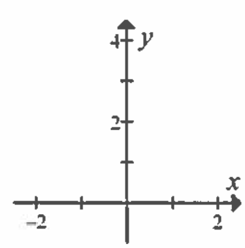
Disc Formula:	$V = \pi \int_a^b r^2\ (dx \text{ or } dy)$

Example 1: Set up an integral for the volume of the solid formed by revolving the region bounded by $y = x^2$, $y = 0$, and $x = 1$ about the x-axis.





Example 2: Find the volume of the solid formed by revolving the region in Quadrant I bounded by $y = x^2$, $x = 0$, and $y = 4$ about the y-axis.

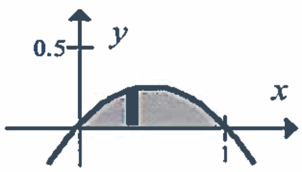


When a region is revolved about a line which is not one of its boundaries, its volume is formed from a sum of volumes of <u>washers</u> (at least some of the discs have holes in them).

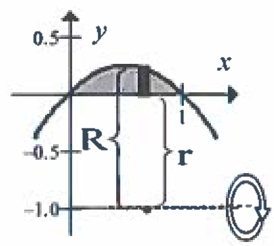
<u>Washer Formula</u>: $V = \pi \int_a^b (R^2 - r^2)\ (dx \text{ or } dy)$	
R = Outer radius (from the axis of revolution)	
r = Inner radius (from the axis of revolution)	

<u>Example 3</u>: <u>Set up</u> integrals for the volumes of the solids formed by revolving the region bounded by $y = -x^2 + x$ and $y = 0$

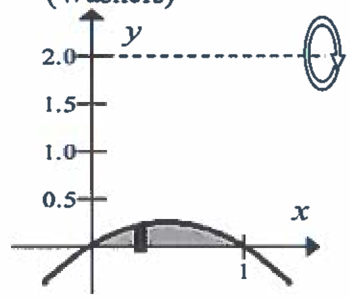
a. about the x-axis (Discs)



b. about $y = -1$ (Washers)

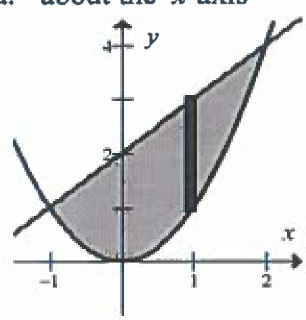


c. about $y = 2$ (Washers)

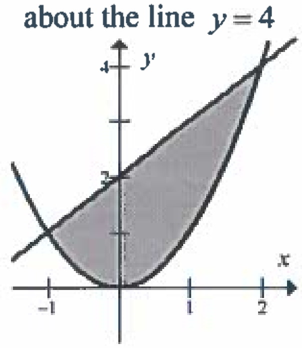


<u>Example 4</u>: <u>Set up</u> integrals for the volumes of the solids formed by revolving the region bounded by $y = x^2$ and $y = x + 2$

a. about the x-axis



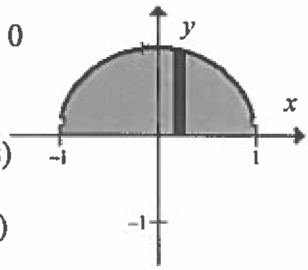
b. about the line $y = 4$



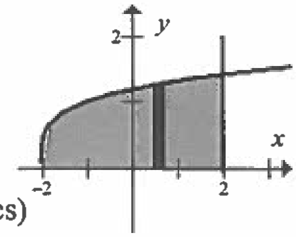
ASSIGNMENT 8.4

For Problems 1-4, use discs or washers to set up (but not integrate) integrals for finding volumes of the solids described. Re-sketch each figure on your assignment sheet, show at least one representative rectangle, and label the radii for each figure.

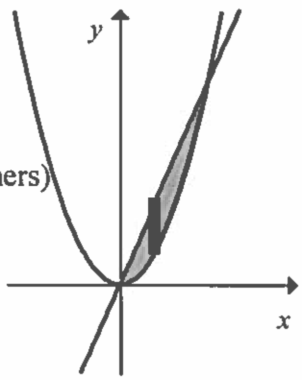
1. Region bounded by $y=\sqrt{1-x^2}$ and $y=0$
   a. revolved about the x-axis (Discs)
   b. revolved about $y=-1$ (Washers)



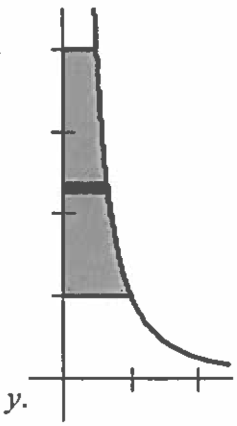
2. Region bounded by $y=\sqrt[4]{x+2}$, $y=0$, and $x=2$
   a. revolved about the x-axis (Discs)
   b. revolved about $y=2$ (Washers)



3. Region bounded by $y=x^2$ and $y=2x$
   a. revolved about the x-axis (Washers)
   b. revolved about $y=5$ (Washers)



4. Region bounded by $y=\dfrac{1}{x^2}$, $x=0$, $y=1$, and $y=4$
   a. revolved about the y-axis (Discs)
   b. revolved about $x=-1$ (Washers)
   Hint: You must rewrite $y=\dfrac{1}{x^2}$ as a function of y.



For Problems 5-7, set up (but do not integrate) integrals for finding the volumes of the solids described. Sketch each region and show at least one representative rectangle.

5. Region bounded by $y=\dfrac{1}{x}$, $y=0$, $x=1$, and $x=4$
   a. revolved about the x-axis
   b. revolved about $y=-3$
   c. revolved about $y=5$

6. Region bounded by $y=\sqrt{x}$, $x=0$, and $y=2$
   a. revolved about the y-axis
   b. revolved about $x=-2$
   c. revolved about $x=4$

7. Region bounded by $y=x^2-2$ and $y=-1$
   a. revolved about the x-axis
   b. revolved about $y=1$

8. Without using a calculator, find the volume of the solid formed when the region bounded by $y=\sqrt[3]{x}$, $x=0$, and $y=2$ is revolved about:
   a. the x-axis
   b. the y-axis

9. Sketch the region, and use a calculator to <u>find</u> the volumes of the solids formed when the region bounded by $f(x)=2^{x-1}$ and $g(x)=2-x^2$ is revolved about:
   a. the x-axis
   b. $y=3$

10. Sketch the region, and <u>set up</u> integrals for the volumes of the solids formed by the given cross sections. The base region is bounded by $f(x)=|x|$, $y=0$, $x=-2$, and $x=3$.
    a. squares perpendicular to the x-axis
    b. semicircles perpendicular to x-axis

11. $f(x)=2\sqrt{x}-x$. Without using a calculator, find:
    a. the domain of $f(x)$.
    b. the x-intercept(s) for the graph of $f(x)$.
    c. the maximum and minimum <u>function values</u> of $f(x)$.
    d. the range of $f(x)$.

12. <u>Show</u> that the graph of $f(x)=2\sqrt{x}-x$ has no points of inflection by building an f'' number line which indicates the concavity for the graph of f.

13. Use the results from Problems 11 and 12 to sketch a graph of $f(x)=2\sqrt{x}-x$ without using a calculator.

14. Find $\int_0^4 (2\sqrt{x}-x)\,dx$ without using a calculator.

15. Find the <u>average value</u> of the function $f(x)=2\sqrt{x}-x$ on $[0,4]$.

16. Use a calculator to find the value of c where $f(c)=f_{avg}$ for Problem 15.

17. Find the instantaneous rate of change of $f(x)=2\sqrt{x}-x$ at $x=2$.

18. Find the average rate of change of $f(x)=2\sqrt{x}-x$ on $[0,4]$.

19. Is $f(x)=2\sqrt{x}-x$ continuous on [0, 4] and differentiable on (0, 4)?

20. If your answer to Problem 19 was yes, then the MVT guarantees that
    $f'(c)=\dfrac{f(b)-f(a)}{b-a}$ (the answer to Problem 18) for some c in (0, 4). Find c.
    If your answer to Problem 19 was no, then do it again.

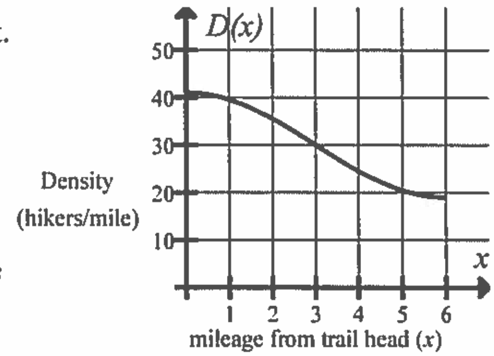
21. A region in Quadrant 1 is enclosed by the graphs of $y=x$, $y=\cos x$, and the y-axis. Sketch the region, draw a representative rectangle, set up an integral, and find the volume of the solid for each revolution described. Use a calculator, and make sure that it is radian mode.
    a. about the x-axis.
    b. about $y=-1$

22. Find the <u>point(s)</u> where the line(s) tangent to the graph of $f(x)=\frac{1}{3}x^3-x^2+x+3$ is/are parallel to the graph of $y-x=5$.

ASSIGNMENT 8.5 REVIEW

For Problems 1-5, use the graph and table at the right.

1. Show an integral set up for the average density of hikers between mile 1 and mile 5.
2. Use a Midpoint Riemann Sum with $n = 2$ to approximate this same average density.
3. Use four equal width trapezoids to approximate this average density on the same interval.
4. Estimate where on the trail the density of hikers is equal to this same average density.
5. Write a complete sentence using numbers and units to tell what $\int_1^5 D(x)\,dx$ represents.



x	1	2	3	4	5
D	40	35	30	25	20

6. Set up (but do not integrate) an integral which could be used to find the area bounded by the graphs of $f(x) = -x^2 + 1$ and $g(x) = x^2 - 2x - 3$. Show the equation(s) used to find the limits of integration.

7. Sketch the region bounded by $y = \sqrt{x-2}$, $x = 6$, and $y = 0$, and find the area of the region.

8. Using the region from Problem 7 as the base, find the volume of the solid formed by using square cross sections perpendicular to the x-axis.

9. Set up (but do not integrate) integrals which could be used to find volumes of the following solids.
   a. Solid formed by revolving the region from Problem 7 about the x-axis.
   b. Solid formed by revolving the region from Problem 7 about the line $y = 4$.

10. The base of a solid is the region bounded by $y = \dfrac{1}{\sqrt{x}}$, $x = 1$, $x = 4$, and $y = -3$. The solid is formed by using rectangular cross sections whose bases lie in the region and are perpendicular to the x-axis, and whose heights are three times their bases. Set up an integral which represents the volume of the solid, and then use a calculator to find the volume of the solid.

11. $v(t) = -\sqrt{t+4}$ represents the velocity equation of an object moving along a vertical path for $t \geq -4$. Let $a(t)$ represent the acceleration and $y(t)$ represent the position of the object at time t. Find:
   a. an equation for the acceleration of the object at time t. $a(t) =$
   b. an equation for the position of the object at time t if $y(0) = -\frac{10}{3}$. $y(t) =$
   c. $y(5)$ d. $v(5)$ e. $a(5)$
   f. the speed of the object at time $t = 5$.

12. $v(t)=\dfrac{t-1}{t^2+1}$ is the velocity equation for an object moving along a horizontal path when $t\ge 0$.
<u>Use a calculator</u> to find:
   a. the velocity of the object at $t=2.3$
   b. the acceleration of the object at $t=2.3$
   c. the <u>displacement</u> of the object from $t=0$ to $t=3$.
   d. the <u>total distance</u> traveled by the object from $t=0$ to $t=3$.

   Show integral set ups for Parts c. and d.

13. If the graph of $f(x)$ is concave upward on (a,b), does a Trapezoidal approximation give a value larger or smaller than the actual value of $\int_a^b f(x)\,dx$?

14. If $f''(x)<0$ on (a,b), does a Trapezoidal approximation give a value larger or smaller than the actual value of $\int_a^b f(x)\,dx$?

15. If $f''(x)=0$ on (a,b), what must be true about the value found using a Trapezoidal approximation and the actual value of $\int_a^b f(x)\,dx$?

For Problems 16-18, list the x-locations of the discontinuities for each function. Label each discontinuity as <u>removable</u> or <u>non-removable</u>.

16. $g(x)=\dfrac{x+2}{x^2+x-6}$

17. $h(t)=\dfrac{\sqrt{t}}{t^2-9}$

18. $y=\begin{cases}\dfrac{x-2}{x^2+x-6}, & x\ne 2\\ 0, & x=2\end{cases}$

19. Fill in the blank with a number to make the function continuous at $x=2$.

$$y=\begin{cases}\dfrac{x-2}{x^2+x-6}, & x\ne 2\\ \underline{\qquad\qquad}, & x=2\end{cases}$$

20. Find the values of a and b which make the function continuous.

$$f(x)=\begin{cases}\frac{1}{2}x+b, & x\le -2 \text{ or } x\ge 1\\ ax^2, & -2<x<1\end{cases}$$

21. Find the values of a and b which make

$$g(x)=\begin{cases}ax^3-bx+7, & x\le 1\\ -bx^2+3bx, & x>1\end{cases}$$

continuous and differentiable.

22. If $y=\dfrac{3}{x}-2\sqrt{x}$, find $\dfrac{d^2y}{dx^2}$.

23. If $3x^2-2y^2=4xy+20$, find $\frac{dy}{dx}$.

24. Find the <u>point(s)</u> at which the graph from Problem 23 has <u>vertical</u> tangents.

25. Find an equation of the line tangent to the graph from Problem 23 at the point $\left(-\sqrt{\frac{20}{3}},\, 0\right)$.

UNIT 8 SUMMARY

Area Between Two Curves: $A = \int_a^b \left(f(x) - g(x)\right) dx$ or $A = \int_a^b \left(f(y) - g(y)\right) dy$

top curve – bottom curve   right curve – left curve

Volumes with Known Cross Sections: $V = \int_a^b A\, dx$

A = area of cross section

Volumes of Revolution (Discs): $V = \pi \int_a^b r^2\, dx$

Volumes of Revolution (Washers): $V = \pi \int_a^b \left(R^2 - r^2\right) dx$

Average Value of a Function: $f_{avg} = \dfrac{\int_a^b f(x)\, dx}{b - a}$

LIST OF DIFFERENTIATION FORMULAS

Power Rule

$\frac{d}{dx} x^n = nx^{n-1}$

General Power Rule

$\frac{d}{dx} u^n = n\,u^{n-1}\,u'$ (where u is a function of x)

Chain Rule

$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

Product Rule: $\frac{d}{dx}(f \bullet s) = fs' + sf'$

Quotient Rule: $\frac{d}{dx}\frac{t}{b} = \frac{bt' - tb'}{b^2}$

Exponential and Logarithmic Rules

$\frac{d}{dx} e^x = e^x$

$\frac{d}{dx} a^x = a^x \ln a$

$\frac{d}{dx} \ln x = \frac{1}{x}$

$\frac{d}{dx} \ln|x| = \frac{1}{x}$

$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$

$\frac{d}{dx} e^u = e^u\, u'$

$\frac{d}{dx} a^u = a^u\, u' \ln a$

$\frac{d}{dx} \ln u = \frac{u'}{u}$

$\frac{d}{dx} \ln|u| = \frac{u'}{u}$

$\frac{d}{dx} \log_a u = \frac{u'}{u \ln a}$

Trigonometric and Inverse Trigonometric Rules

$\frac{d}{dx} \sin x = \cos x$

$\frac{d}{dx} \cos x = -\sin x$

$\frac{d}{dx} \tan x = \sec^2 x$

$\frac{d}{dx} \cot x = -\csc^2 x$

$\frac{d}{dx} \sec x = \sec x \tan x$

$\frac{d}{dx} \csc x = -\csc x \cot x$

$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx} \arccos x = \frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$

$\frac{d}{dx} \sin u = \cos u\, u'$

$\frac{d}{dx} \cos u = -\sin u\, u'$

$\frac{d}{dx} \tan u = \sec^2 u\, u'$

$\frac{d}{dx} \cot u = -\csc^2 u\, u'$

$\frac{d}{dx} \sec u = \sec u \tan u\, u'$

$\frac{d}{dx} \csc u = -\csc u \cot u\, u'$

$\frac{d}{dx} \arcsin u = \frac{u'}{\sqrt{1-u^2}}$

$\frac{d}{dx} \arccos u = \frac{-u'}{\sqrt{1-u^2}}$

$\frac{d}{dx} \arctan u = \frac{u'}{1+u^2}$

LIST OF INTEGRATION FORMULAS

Power Rule for Integrals

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

General (u) Forms (Where u is a function of x)

$$\int u^n\,u'\,dx = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$$

Reverse Chain Rule

$$\int f'(g(x))\,g'(x)\,dx = f(g(x)) + C$$

Exponential and Logarithmic Integrals

$$\int e^x\,dx = e^x + C \qquad\qquad \int e^u\,u'\,dx = e^u + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C \qquad\qquad \int a^u\,u'\,dx = \frac{a^u}{\ln a} + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C \qquad\qquad \int \frac{u'}{u}\,dx = \ln|u| + C$$

Trigonometric Integrals

$$\int \cos x\,dx = \sin x + C \qquad\qquad \int \cos u\,u'dx = \sin u + C$$

$$\int \sin x\,dx = -\cos x + C \qquad\qquad \int \sin u\,u'dx = -\cos u + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad\qquad \int \sec^2 u\,u'dx = \tan u + C$$

$$\int \csc^2 x\,dx = -\cot x + C \qquad\qquad \int \csc^2 u\,u'dx = -\cot u + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad\qquad \int \sec u \tan u\,u'dx = \sec u + C$$

$$\int \csc x \cot x\,dx = -\csc x + C \qquad\qquad \int \csc u \cot u\,u'dx = -\csc u + C$$

$$\int \tan x\,dx = -\ln|\cos x| + C \qquad\qquad \int \tan u\,u'dx = -\ln|\cos u| + C$$

$$\int \cot x\,dx = \ln|\sin x| + C \qquad\qquad \int \cot u\,u'dx = \ln|\sin u| + C$$

$$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C \qquad\qquad \int \tan^2 u\,u'dx = \tan u - x + C$$

Integrals Involving Inverse Trig Functions

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

$$\int \frac{u'}{\sqrt{1-u^2}}\,dx = \arcsin u + C$$

$$\int \frac{u'}{\sqrt{a^2-u^2}}\,dx = \arcsin \frac{u}{a} + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

$$\int \frac{u'}{1+u^2}\,dx = \arctan u + C$$

$$\int \frac{u'}{a^2+u^2}\,dx = \frac{1}{a}\arctan \frac{u}{a} + C$$

APPENDIX A

LESSON A-1 VOLUMES OF REVOLUTION - SHELL METHOD

PART 1 Revolutions about either the y-axis or x-axis

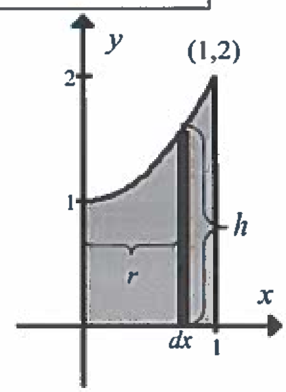
Key: The representative element (rectangle) must be parallel to the axis of revolution. Recall that for discs or washers, the element had to be perpendicular to the axis of revolution.
Remember: **PARASHELL** vs **PERPENDISCULAR**

Revolving rectangular elements about a parallel axis produces cylindrical shells (like the wrappings around a toilet paper roll).

The volume formula for the shell method is: $\boxed{V = 2\pi\int_a^b rh\,dx \text{ (or } dy)}$ $r > 0,\ h > 0$

Examples:

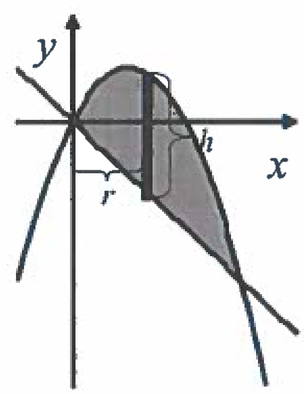
1. Find the volume of the solid formed by revolving the region bounded by $y = x^2 + 1$, $y = 0$, $x = 0$, and $x = 1$ about the y-axis.



$r = x$
$h = x^2 + 1$

V =

Why would using the disc method for this problem be much harder?

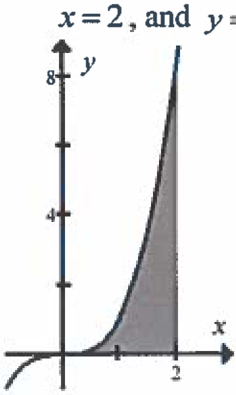


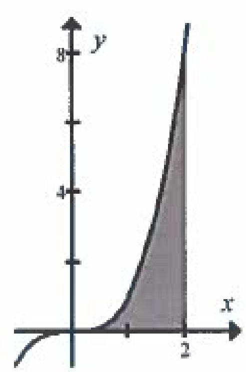
2. Set up (but do not integrate) an integral giving the volume of the solid formed by revolving the region bounded by $y = 2x - x^2$ and $y = -x$ about the y-axis.

$r =$
$h =$

V =

3. Use both the shell method and the disc method to find the volume formed by revolving the region bounded by $y = x^3$, $x = 2$, and $y = 0$ about the x-axis.

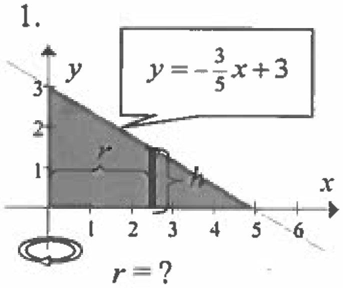




ASSIGNMENT A-1

Give the radius and the height of the shell formed when the representative element is revolved about the indicated axis.

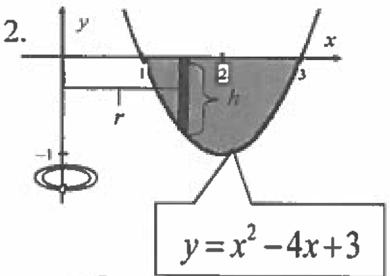
1.



$r=?$

$h=$

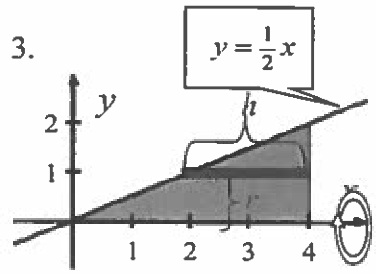
2.



$r=?$

$h=$

3.

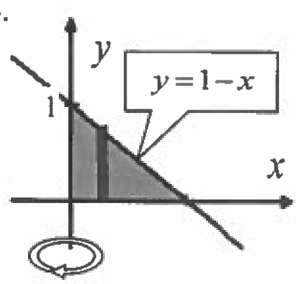


$r=?$

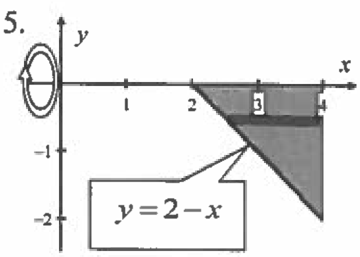
$h=$

For problems 4-9, **set up** integrals (but do not integrate) which could be used to find the volumes formed by revolving the plane regions described or shown about the indicated axes. Use the **shell method**.

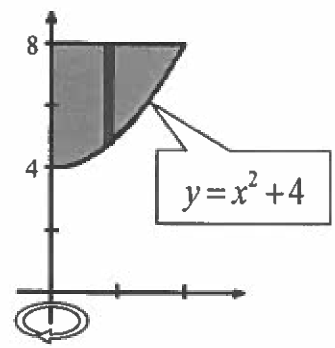
4.



5.



6.



7. Region bounded by $y=x^2$, $y=0$, and $x=2$ revolved about the y-axis.

8. Region bounded by $y=2x$, $y=4$, and $x=0$ revolved about the y-axis.

9. Region bounded by $y=\sqrt{x}$, $y=0$, and $x=5$ revolved about the x-axis.

Use **discs, washers, or shells** to **find the volumes** indicated in problems 10 and 11.

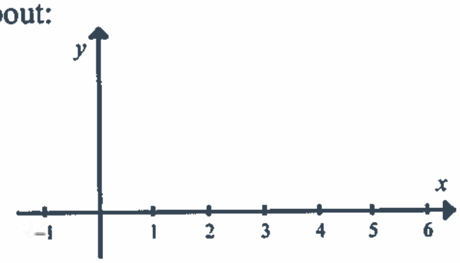
10. Region bounded by $y=x$ and $y=x^3$ in Quadrant I revolved about the x-axis.

11. Region bounded by $y=\sqrt{x}+2$, $x=0$, $y=0$, and $x=4$ revolved about the y-axis.

LESSON A-2 **VOLUMES OF REVOLUTION - SHELL METHOD**
PART 2 Revolutions about other lines

For revolutions about lines other than the x- and y-axes, the formula is still $V = 2\pi\int_a^b rh\,dx$ (or dy), but instead of $r = x$ or $r = y$ (or perhaps $r = -x$ or $r = -y$), r is slightly harder to find. (Remember that both r and h must be nonnegative.)

Example 1. Use the Shell Method to set-up integrals which could be used to find the volumes of the solids formed when the region bounded by $y = \sqrt{x-1}$, $y = 0$, and $x = 5$ is revolved about:

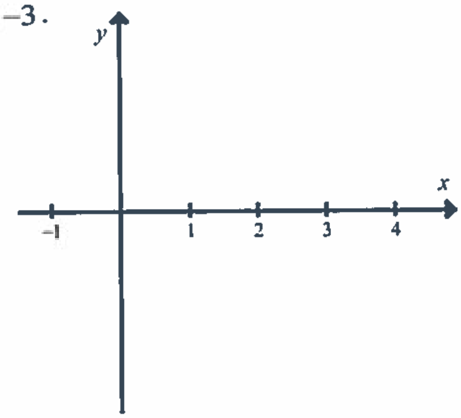


a. the y-axis

b. $x = -2$

c. $x = 5$

Example 2. Find the volume of the solid formed by revolving the region bounded by $x = y^2$ and $x = 4$ about the line $y = -3$.

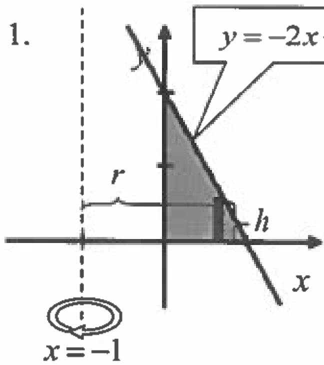


Example 3. Set up an integral (but do not integrate) which could be used to find the volume of the solid which would be formed if the region from Example 2 were revolved about the line $y = 3$.

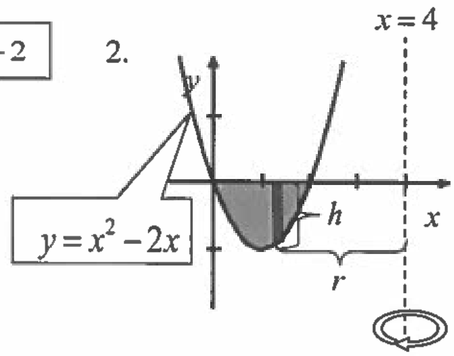
ASSIGNMENT A-2

Give the radius and the height of the shell formed when the representative element is revolved about the indicated line.

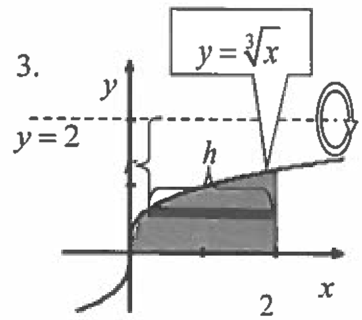
1.



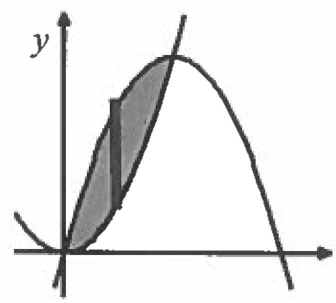
2.



3.



For problems 4-6, **set up** integrals (but do not integrate) which could be used to find the volumes formed by revolving the plane region bounded by $y=x^2$ and $y=4x-x^2$ about the indicated axis of revolution. Use **any method**.



4. about the x-axis

5. about $x=4$

6. about $x=-1$

For problems 7and 8 draw accurate sketches and **set up** integrals (but do not integrate) which could be used to find the volumes formed by revolving the plane region bounded by $x=y-y^2$ and $x=0$ about the indicated axes of revolution. Use **any method**.

7. about $y=2$

8. about $y=-3$

9. **Set up** integrals (but do not integrate) which could be used to find the volumes formed by revolving the plane region bounded by $y=\frac{1}{2}x$, $y=0$, and $x=6$ about the line $y=-2$. Use the indicated method.
   a. washers
   b. shells

10. **Find** the volumes of the solids formed by revolving the region bounded by $y=\dfrac{1}{x^2}$, $y=0$, $x=1$, and $x=3$ about the indicated axes.
   a. about the y-axis
   b. about the line $x=4$

LESSON A-3 INTEGRATION BY PARTS PART 1

Integration by parts is a method of integration used mainly for products of algebraic and transcendental functions (such as $\int xe^x\,dx$) or products of two transcendental functions (such as $\int e^x \sin x\,dx$).

Development of the formula for integration by parts: If u and v are both functions of x, then

$\frac{d}{dx}(uv) =$

Formula for integration by parts:

$$\int uv'\,dx = uv - \int vu'\,dx \quad \text{or} \quad \int u\,dv = uv - \int v\,du$$

Strategy: Let u be the part whose derivative is "simpler" (or at least no more complicated) than u itself. Let dv be the more complicated part (or the part which can easily be integrated). Also, remember that you typically have only two choices. If one choice doesn't work, try the other.

Example 1. $\int xe^x\,dx$

Let $u =$

$du =$

Let $dv =$

$v =$

$\int xe^x\,dx =$

Example 2. $\int x\sin(3x)\,dx$

$\int x\sin(3x)\,dx =$

Example 3. $\int \arcsin x\,dx =$

ASSIGNMENT A-3

Hint: Two of the problems in this assignment do not involve integration by parts. One of these is in the form $\int e^u u'\,dx$ and the other is in the form $\int u^n u'\,dx$ (in disguise). See if you can find them and remember how to do them.

1. $\int x\sin x\,dx$ 2. $\int x\cos(2x)\,dx$ 3. $\int 4xe^{2x}\,dx$ 4. $\int x^2 e^{x^3}\,dx$ 5. $\int \frac{x}{e^x}\,dx$

6. $\int 3x\ln x\,dx$ 7. $\int \ln x\,dx$ 8. $\int \frac{\ln x}{x^2}\,dx$ 9. $\int \frac{(\ln x)^2}{x}\,dx$ 10. $\int \arctan x\,dx$

LESSON A-4 INTEGRATION BY PARTS PART 2

Repeated integration by parts and integration by parts involving definite integrals

Example 1. $\int x^2 \sin(2x)\,dx$

Example 2. $\int_1^e x^2 \ln x\,dx$

ASSIGNMENT A-4

Hint: Two of the problems in this assignment do not involve integration by parts. See if you can find them and integrate them using an appropriate method.

1. $\int x^2 \cos x\,dx$ 2. $\int x^2 \sin(3x)\,dx$ 3. $\int x^2 e^{2x}\,dx$ 4. $\int x\arctan x\,dx$

5. $\int_0^1 \frac{x}{1+x^2}\,dx$ 6. $\int_1^2 xe^{x^2}\,dx$ 7. $\int_0^1 xe^{3x}\,dx$

8. $\int_1^{e^2} x\ln x\,dx$ 9. $\int_0^{\pi} 2x\cos x\,dx$

> $\int \frac{x^2}{1+x^2}\,dx$ can be integrated using long division

THE ESSENTIALS OF CALCULUS WITH EARLY TRANSCENDENTALS

ANSWERS TO SELECTED EXERCISES

Assignment 0.1 page 2

1. $m=\frac{4}{3}$ 3. $m=0$ 6. $y=x-2$ 9. $x=4$ 10. $-3x+y=0$

11. $y-3=-\frac{3}{2}(x-2)$ 16. not collinear 18. $x^3+4x^2-5x=0 \rightarrow x=-5,0,1$

19.

x	$f(x)$
7	504
7.1	524.051
7.2	544.608
7.3	565.677
7.4	587.264

22. $x^3+4x^2-5x=-3x^2-5x+15$
$(1.341, 2.899)$ or $(1.341, 2.900)$
$(-1.678, 14.937)$ or $(-1.679, 14.938)$
$(-6.662, -84.837)$ or $(-6.662, -84.838)$

24. $(-\infty,-5],[0,1]$ 27. $x=\pm 1.732$

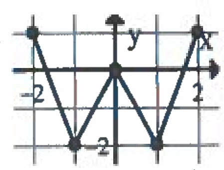
28. (-3,11), (5,3)

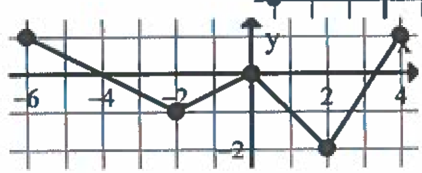
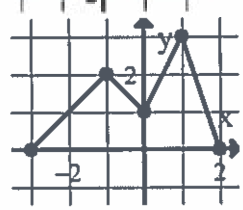
Assignment 0.2 page 7

1. b. -11 d. $3x-5$ 3. $2x+\Delta x-1$ 4. Do: $x\geq -1$, Ra: $y\geq 0$

8. not a function (circle) 10. a. $y=(x+3)^2$ 11. c. $x-1,\ x\geq 0$

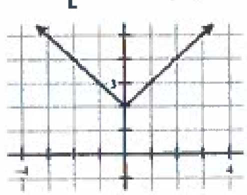
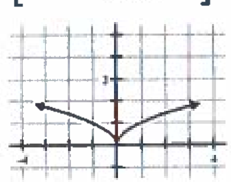
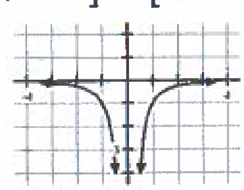
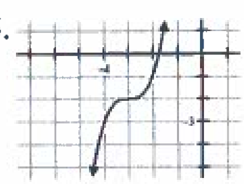
14. c. $8x^4$ e. 4 15. $y=\frac{1}{2}x+\frac{1}{2}$ 20. a. $x-2$ b. $x\geq 2$ c. $\dfrac{1}{x-2}$ d. $x>2$

23. $x=0,\pm 2$ 24.  27. 

31.  32. 

34. $x=-.890$ or $-.891$ 35. Do: all reals, Ra: $y\geq 4.875$

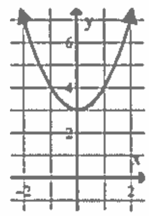
37. Do: $[-2.645, 2.645]$ or $[-2.646, 2.646]$ Ra: $[0, 2.645]$ or $[0, 2.646]$

39.  41.  43.  45. 

Assignment 0.3 page 12

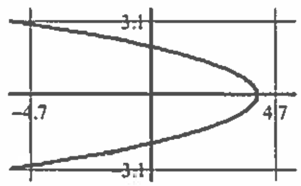
6. x-int. $(\pm 3,0)$, no y-int. 8. x-int. $(\pm 2,0)$, y-int. $(0,1)$ 12. origin 16. x-axis

25. no x-int.,
y-int. $(0,3)$,
y-axis symm.



26. $(0,0), (1,1), (-1,-1)$
27. $(-4,-3), (3,4)$
29. $k=-\frac{1}{8}$

36. x-int. $(4,0)$,
y-int. $(0,\pm 2)$,
x-axis symm.



38. $-.666<x<5.333$ or $-.667<x<5.333$
45. $y-2=\frac{-3}{2}(x-1)$

Assignment 0.4 page 17

1. a. $\frac{1}{2}$ b. $\frac{\sqrt{3}}{2}$ d. $\frac{1}{\sqrt{3}}$ f. 2 2. a. $\frac{7}{3}$ c. $-\frac{\sqrt{58}}{3}$ 3. a. $\frac{5}{\sqrt{29}}$ c. $-\frac{5}{2}$ 4. a. $\frac{\sqrt{3}}{2}$ c. $\frac{2}{\sqrt{3}}$

5. a. $\frac{1}{\sqrt{2}}$ c. -1 6. a. 270° c. $\frac{5\pi}{4}$ 7. a. 132° c. 5.445 8. a. $-\frac{1}{\sqrt{2}}$ c. 1

9. a. $-\frac{1}{2}$ c. $\frac{2}{\sqrt{3}}$ 10. a. Quadrant 3 11. a. $\frac{\sqrt{8}}{3}$ or $\frac{2\sqrt{2}}{3}$ c. -3 12. a. $x=\frac{\pi}{3},\frac{4\pi}{3}$

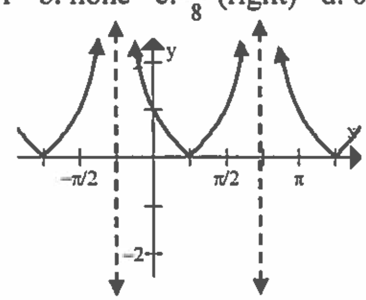
14. $\theta=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$

Assignment 0.5 page 20

1. a. $-.309$ b. -4.179 or -4.180 d. .324 or .325 2. $x=.915$ 3. $x=3.410$ *in.*
5. $x=.936$ 7. $x=.646$ or .647, 3.788 8. a. π b. 4 c. 0 d. 1 (upward)

9. a. 1 b. none c. $\frac{1}{8}$ (right) d. 0 10. a. $\frac{2\pi}{3}$ b. 112 c. $-\frac{\pi}{3}$ (left $\frac{\pi}{3}$) d. 0

13.



16. $x=\frac{1}{2},\frac{3}{2}$ 17. $x=\pi,3\pi$

18. a. $y=4\sin\left(\frac{1}{2}(x-0)\right)+0$ (others are possible)

20. $\sin\theta=-\frac{7}{\sqrt{53}}$, $\tan\theta=\frac{7}{2}$ 21. a. $\sin\theta=-\frac{1}{\sqrt{2}}$, $\tan\theta=1$, $\sec\theta=-\sqrt{2}$

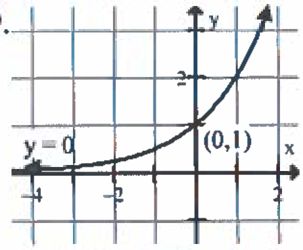
21. b. $\sin\theta=-\frac{1}{2}$, $\tan\theta=-\frac{1}{\sqrt{3}}$, $\sec\theta=\frac{2}{\sqrt{3}}$ c. $\sin\theta=1$, $\cos\theta=0$, $\tan\theta$ is undef.

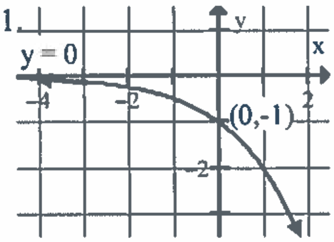
22. $\cos\theta=-\frac{\sqrt{7}}{4}$, $\tan\theta=-\frac{3}{\sqrt{7}}$ 23. $x=\frac{\pi}{3},\frac{4\pi}{3}$ 24. $\theta=0$, π 25. $\theta=\frac{\pi}{3}, \pi, \frac{5\pi}{3}$

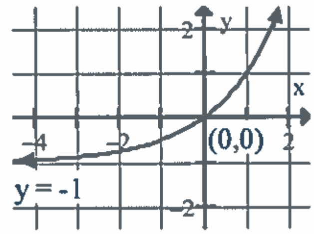
Assignment 0.6 page 23

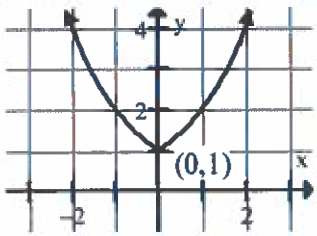
1. 4 3. 0 5. $\frac{1}{16}$ 7. 9 9. $\frac{e^3}{8}$ 11. e^{15} 13. $x=4$ 14. $x=3$ 15. $x=-\frac{3}{2}$

16. $x=\pm 8$ 17. $x=-1$

19. 

21. 

23. 

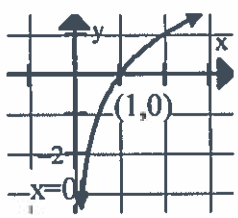
24. 

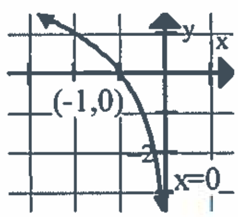
Assignment 0.7 page 26

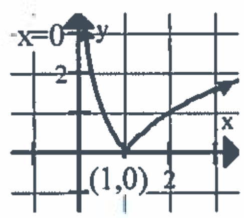
1. false 2. false 3. false 4. false 5. false 6. true 7. false 8. true

9. $\log_5 \frac{1}{125} = -3$ 11. $3^6 = 729$ 13. $2x+1$ 15. $\sqrt{p}$ 17. x = 8 18. $x = \frac{1}{e}$

19. $x = \pm 2$

21. 

23. 

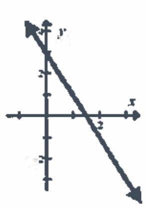
24. 

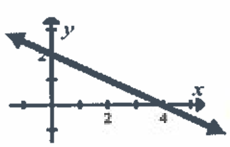
26. $f^{-1}(x) = \frac{1}{2}\ln x$ 27. $f^{-1}(x) = e^x + 1$ 28. $\ln a - \ln b - \ln c$ 30. $2\ln(a+b) - \ln c$

31. $\log(xy^2)$ 33. $\ln \frac{ac}{b^2}$ 34. $t = -2, 3$ 35. $t = \frac{\ln 3 + 1}{2}$ 36. $t = 2$ (-4 is extraneous)

38. $x = .482$ or $.483$ 39. 2.726 or 2.727

Additional Practice 0.1 page 28

1. 

2. 

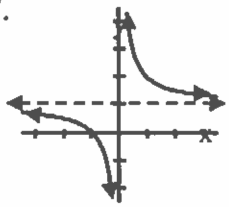
3. a. $y-3=\frac{2}{3}(x+1)$ b. $y-3=\frac{-1}{2}(x+1)$

3. c. $y=-3x$ d. $x=-1$

4. $(-\infty, -1.137]$, $[.394, .742]$ or $[.395, .742]$

5. $x<-2.333$ or $x>-1$ 6. $x=-3$, -6.75

Additional Practice 0.2 page 28

7.


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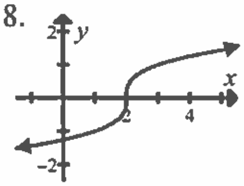

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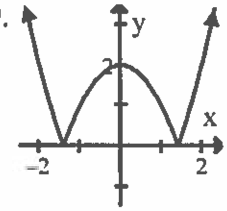

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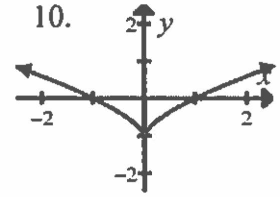

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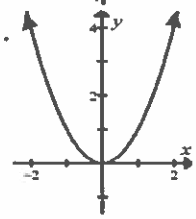

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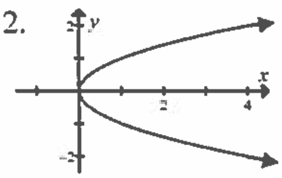
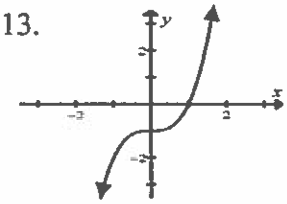


13.


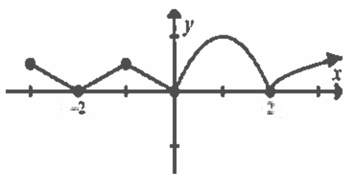
14. Problem 12

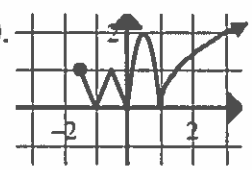
15. a. $-x^2+2x+2$ b. $-4x^2-4x$ c. -5

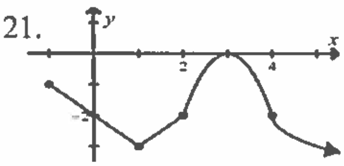
16. $x=\pm 2, \pm\sqrt{3}$

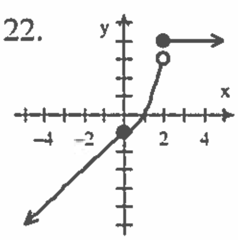
17. $x=2$

18. $f^{-1}(x)=\sqrt[3]{\sqrt[5]{x}+1}$

19.


20.


21.


22.


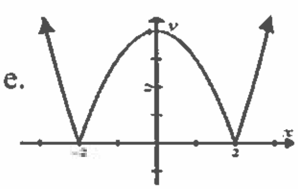
Additional Practice 0.3 page 29

23. $(1,6)$ 24. $(\pm 3,0)$

25. a.Do: all reals Ra: $y \geq 0$

b. x-int: $(\pm 2,0)$ y-int: (0 , 4)

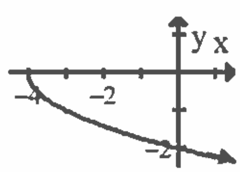
c. y-axis symmetry d. even

e.


26. a. Do: $x \geq -4$, Ra: $y \leq 0$

b. x-int. $(-4,0)$, y-int. $(0,-2)$

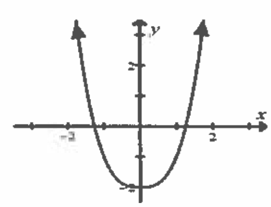
c. no symmetry d. neither

e.


27. a. Do: all reals Ra: $y \geq -2$

b. x-int: $(\pm\sqrt[4]{2},0)$ y-int: (0 , -2)

c. y-axis symmetry d. even

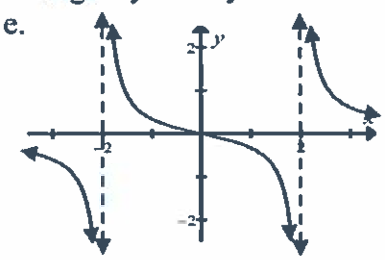
e.


28. a. Do: $x \neq \pm 2$ Ra: all reals

b. x-int: (0,0) y-int: (0 , 0)

c. origin symmetry d. odd

e.

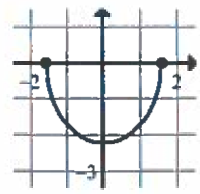


29. a. Do: $[-1.581, 1.581]$, Ra: $[-2.236, 0]$

b. x-int. $(\pm 1.581, 0)$, y-int. $(0, -2.236)$

c. y-axis symm. d. even

e.



30. a. Do: all reals

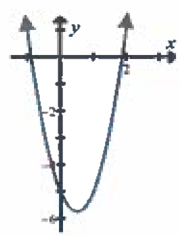
Ra: $y \geq -5.750$

b. x-int: $(-.884, 0)$, (1.884 , 0)

y-int: (0 , -5)

c. no symmetry d. neither

e.



31. odd 32. Neither

Additional Practice 0.4 page 29

33. a. $\frac{5\pi}{3}$ b. $-450°$ 34. $\sin\theta = \frac{3}{\sqrt{34}}$, $\cos\theta = \frac{5}{\sqrt{34}}$, $\tan\theta = \frac{3}{5}$

35. $\sin\theta = -\frac{1}{\sqrt{5}}$, $\cos\theta = -\frac{2}{\sqrt{5}}$, $\tan\theta = \frac{1}{2}$

36. $\sin\theta = \frac{\sqrt{3}}{2}$, $\cos\theta = -\frac{1}{2}$, $\tan\theta = -\sqrt{3}$ 37. $x = 20$ 38. 0 39. 0 40. $\sqrt{3}$ 41. $-\sqrt{2}$ 42. -2 43. 0 44. $-\frac{12}{13}$ 45. $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

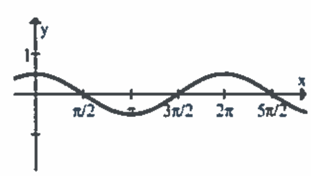
46. $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

Additional Practice 0.5 page30

47. x = 1.097, 2.179

48. amp. $= \frac{3}{4}$, per. $= \frac{2\pi}{3}$, ph. sh. $= \frac{\pi}{3}$ (right)

49.

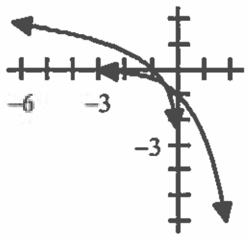


50. disc. at $x = 0, \pm\frac{\pi}{4}, \pm\frac{\pi}{2}, \ldots$

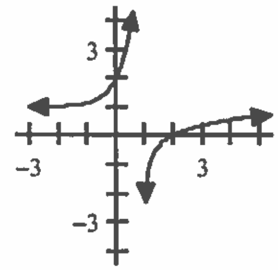
51. $y = 2\tan\left(\frac{1}{2}(x-0)\right)+1$

Additional Practice 0.6 and 0.7 page 30

52. $f^{-1}(x) = -e^x$



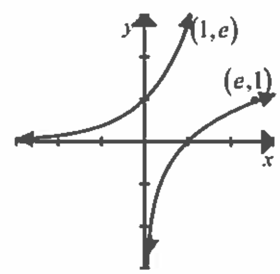
53. $g^{-1}(x) = \frac{1}{2}\ln(x-1)$



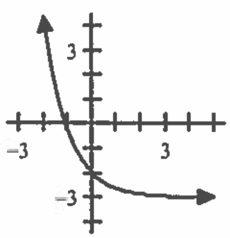
54. -3 55. 7 56. $\ln x + \frac{1}{2}\ln(y-1)$ 57. $\log\frac{p^2}{q^3 r}$ 58. $x = \frac{\ln 5 + 3}{2}$ 59. $x = e^2$

60. a. Do: all reals, points $(0,1)$, $(1,e)$, HA: $y=0$

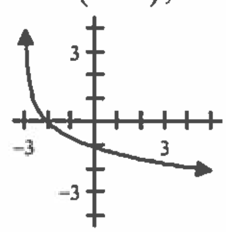
b. Do: $x>0$, points $(1,0)$, $(e,1)$, VA: $x=0$



61. HA: $y=-3$



62. $y = -\ln(x+3)$, VA: $x=-3$



63. $x = \frac{1}{25}$ 64. $x=-1$ 65. $x=0,1$ 66. $x=3$ 67. $\frac{3}{2}$

Assignment 1.1 page 35

1. -2 2. DNE 5. -2 6. ∞ or DNE 10. $-\infty$ or DNE 13. 0
16. $x=1,2$ 17. $x=-3,4$ 18. b. 0 d. $x=0$ 19. b. $-\infty$ or DNE g. -1 i. $x=-1$
20. d. 3 e. 1 21. c. ∞ or DNE e. $(-\infty,-3)$, $(-3,3)$, $(3,\infty)$ 22. b. DNE
22. c. -2 f. every integer 27. 1 28. $-\frac{1}{2}$ 30. 0 32. 0 35. -1 37. $-\sqrt{2}$
38. a. -2 b. 2 c. DNE d. $(-\infty,0)$, $(0,\infty)$ 40. $x \ge 1$

Assignment 1.2 page 39

1. 2 2. -5 4. 3 5. $\frac{1}{10}$ 9. $-\infty$ or DNE 11. $\frac{1}{4}$ 12. 4 13. -1 15. 2
16. 1 19. 1 20. 6 21. -2 23. $\frac{1}{2}$ 25. 2 27. $\frac{2}{\sqrt{3}}$ 30. b. .166 or .167

35. $x=0$ (removable), $x=-1$ (nonrem.) 36. $x=-3$ (removable) 39. $x=3$ (rem.)
41. every even integer (nonrem.) 44. every integral multiple of 4 (nonremovable)
45. $a=2$ 50. $c=3$ 51. $c=-3$

Assignment 1.3 page 43

3. $x=0$ (even) 5. $x=\pm 2$ (both odd) 7. $-\infty$ 8. DNE 9. ∞ 10. -1
12. DNE 13. $-\infty$ 15. $-\infty$ 17. a. 2 b. 2 20. $\frac{2}{3}$ 22. $\frac{4}{3}$ 23. $-\infty$ or DNE
24. -1 27. 2 29. $\frac{2}{3}$ 31. 2.718 32. $x=2$ (nonremovable), $x=-2$ (removable)
34. $x=2$ (nonremovable) 36. $x=-2$ (nonremovable) 37. $a=\frac{10}{3}$
39. $x=\pm 1.414$, 2 41. no, -1 is not between $f(-2)=6$ and $f(2)=2$ 45. 3
46. DNE 49. yes because $f(1)=\lim\limits_{x\to 1} f(x)$ 50. no because $f(3)=3\neq \lim\limits_{x\to 3} f(x)=2$

Assignment 1.4 page 46

1. Do: $x\neq 0,1$
   Hole: $(1,0)$ (even)
   VA: $x=0$ (odd)

2. Do: $(-\infty,-1),(1,\infty)$
   HA: $y=\pm 1$

3. x-int: $(-2,0)$ (odd)
   $(1,0)$ (even)
   y-int: $(0,2)$
   EB: like $y=x^3$

4. Do: $x\neq -2$
   VA: $x=-2$ (odd)
   x-int: $(2,0)$
   y-int: $(0,-1)$
   EB: HA $y=1$

7. Do: $x\neq \pm 1$
   $$f_{red}(x)=\frac{(x-1)(x^2+1)}{x+1}$$
   VA: $x=-1$ (odd) y-int: $(0,-1)$
   Hole: $(1,0)$ (odd) EB: like $y=x^2$

10. $-\frac{1}{4}$ 12. $\frac{1}{4}$ 15. -5 17. DNE 19. 2 20. ∞ or DNE 21. $\frac{3}{7}$ 25. 0
26. 0 29. $x=2$ (removable), $x=-2$ (nonremovable)
30. nonremovable discontinuity at every integral multiple of $\frac{1}{2}$

Assignment 1.5 page 49

1. a. $\frac{5}{3}$ b. -2 c. 0 2. a. 5 3. 99.048 4. a. $\frac{26}{129}\frac{\text{mi}}{\text{min}}$ 5. 6 6. -4 8. DNE
9. -3 11. 2 13. 0 16. 2 17. $a=\frac{5}{2}, b=\frac{1}{2}$ 20. 3 22. -4 24. -3
25. DNE 27. -2 29. 2 30. $x=1$ 31. $x=2$

Assignment 1.6 page 51

1. 7 2. –6 3. $-\frac{1}{6}$ 4. DNE 5. 4 6. ∞ or DNE 7. 2 8. $\frac{1}{4}$ 9. 0

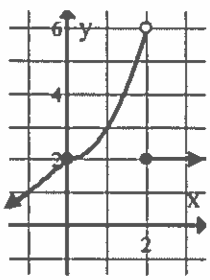
10. 0 11. $\frac{11}{8}$ 12. $\frac{1}{4}$ 13. $\frac{1}{2}$ 14. 4 15. $-\infty$ or DNE 16. 0

17. $-\infty$ or DNE 18. 1 19. 5 20. 5 21. DNE 22. $-\frac{3\pi}{4}$ 23. 0

24. $\frac{\sqrt{2}}{2}$ 25. 1 26. 0 27. 0 28. DNE 29. 1 30. 3 31. 1 32. 0

33. ∞ or DNE 34. $x=-1$ 35. $x=-2,1$ 36. $(-\infty,0),(0,1),(1,\infty)$ 37. $x\geq -2$

38.



39. 2 40. DNE 41. 6 42. 3 43. 2 44. $x=2$

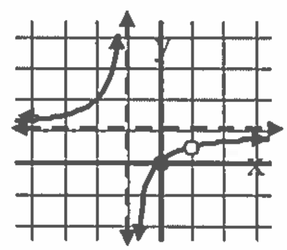
45. $a=-3,\ b=4$

46. Do: $x\neq \pm 1$ VA: $x=-1$ (odd)

$y_{red}=\dfrac{x}{x+1}$ x-int: $(0,0)$ (odd)

Hole: $\left(1,\frac{1}{2}\right)$ y-int: $(0,0)$

EB: HA $y=1$



47. a. Yes, c = -2, b. No, c. Yes, c = 4 48. 15 49. -1 50. -2 51. 2

52. both limits are 10

53. a. D: $x \neq -1, 1, 2, 3$ $f_{red}=\dfrac{(x+2)^2(x-1)}{(x+1)(x-2)^2}$ Holes: (3, 12.5), (1,0) odd

VA: $x = -1$ odd, $x = 2$ even EB: $y = 1$ x-int (-2,0) even y-int: (0,-1)

53. b. D: $(-\infty,-3), (3,\infty)$ VA: $x = 3, x = -3$ EB: $y=-1,\ y=1$ x-int: (4,0)

Assignment 2.1 page 57

1. $f'(x)=-3$ 2. $f'(x)=2x$ 3. $f'(x)=-\dfrac{1}{(x-1)^2}$ 4. $f'(t)=3t^2-12$

5. $f'(x)=0$ 6. $f'(2)=4$ 7. $f'(2)=4$ 8. $f'(3)=-\frac{1}{9}$ 9. $f'(1)$ is undefined

10. $y'=2x-1$ 11. $\frac{dy}{dx}=3x^2$ 12. $f'(x)=4x$, $f'(4)=16$ 13. $f'(4)=16$

15. $-\frac{3}{2}$ 18. $x^{\frac{5}{2}}$ 21. $2x^{-3}$ 22. $x^{\frac{2}{3}}$

Assignment 2.2 page 61

1. $y'=0$ 3. $g'(x)=3x^2$ 5. $f'(t)=-4t-3$ 6. $f'(x)=-\frac{2}{3}x-\frac{2}{5}$ 7. $f'(1)=-6$

9. $h'(1)=3$ 10. $f'(2)=-12$ 11. $y'=-\dfrac{1}{x^2}$ 12. $f'(x)=2x+\dfrac{8}{x^3}$

14. $g'(x)=3x^2+1$ 15. $y'=-\frac{1}{2}x^{-\frac{3}{2}}$ 17. $f'(t)=1,\ t\neq 0$ 19. $y'=-\dfrac{2}{3x^3}$

20. $y'=-\dfrac{2}{9x^3}$ 22. $f'(4)=\dfrac{1}{4}$ 23. $f''(2)=-\dfrac{1}{2}$ 25. $f^{(5)}(3)=0$

28. $y-3=4(x-2)$ 30. $y-2=\frac{1}{2}(x-1)$ 31. $y=2$ 32. $x=\pm 1$ 33. $x=0,\pm\sqrt{\frac{3}{2}}$

35. ARC = 84 36. $f'(3)=81$ 38. $y'(1)=1$ 39. $x=1$ (sharp turn)

40. $x=-1$ (vert. asymp.), $x=0$ (hole) 43. $x=0$ (vert. tang.) 44. $x=1$ (sharp turn)

Assignment 2.3 page 65

2. $f'(3)=$ 667.447 or 667.448 4. $f'(x)=x^{-\frac{1}{2}}+3e^x$ 6. $g'(\theta)=3\cos\theta$

7. $f'(x)=\frac{3}{2}+\sin x$ 9. $f'(t)=\frac{3}{t}+\cos t$ 11. $g'\left(\frac{\pi}{4}\right)=1-\sqrt{2}$ 12. $f'(1)=6$

14. $h''(2)=\frac{10}{3}-2e^2$ 15. $f''\left(\dfrac{\pi}{2}\right)=4-\pi$ 16. $y-12=\frac{1}{2}(x-8)$ 18. $x=0$

19. $\cos x$ 22. $f'(x)=6x+1$ (using the limit definition)

23. $f'(3)=18$ (using the alternate form) 25. $f'(x)=6x^2+6$ 27. $f'(x)=-x^{-5}$

29. a. discontinuous at $x=1$ b. $f'(x)=\begin{cases}6x-1, & x<1\\ 5, & x>1\end{cases}$ c. not differentiable at $x=1$

31. a. $x=\pm 2$

Assignment 2.4 page 69

1. b. $V(2)=54\frac{m}{sec}$ d. $A(3)=72\frac{m}{sec^2}$ e. $t=\pm 1$ sec f. $V(1.5)=33\frac{m}{sec}$ h. 22.5 m

2. a. $t=5$ sec b. $V(1)=5\frac{ft}{sec}$ d. $V(5)=13\frac{ft}{sec}$ 3. b. $t=-2.791,\ 1.791$ sec

4. c. slowing down (the slope of S is decreasing) 5. a. disp. = 50 cm

5. b. $V(4)=24\frac{cm}{min}$ d. t = 0,2 min e. TD = 58 cm f. $V_{avg}=10\frac{cm}{min}$

5. g. avg. speed $=\frac{58}{5}\frac{cm}{min}$ 6. a. $\frac{1}{4}<t<7$ c. $1\le t\le 4$

7. a. $t=6.123$ or 6.124 sec b. $V(6.123)=-195.959\frac{ft}{sec}$ 8. $f'(x)=2+\dfrac{9}{x^4}$

9. $f'(x)=8x+12$ 10. $h'(t)=2\cos t-\frac{3}{t}$ 12. $f'(4)=6$ 13. $f'(-8)=-\frac{1}{3}$

15. $y''=2$ 16. $y+1=-1(x-1)$ 17. $(-1,4)$ 20. $f'(1)=3$ 21. AROC = 4

Assignment 2.5 page 71

1. $f'(x)=12x^2+6x-8$ 3. $f'(x)=\dfrac{(x^2+2)2-(2x+1)2x}{(x^2+2)^2}$ 4. $f'(x)=1,\ x\neq -2$

6. $f'(t)=1,\ t\neq 0$ 8. $f'(x)=\dfrac{5}{(3x-2)^2}$ 9. $y'=\frac{3}{2}x^{\frac{1}{2}}+\frac{1}{2}x^{-\frac{1}{2}}$

10. $f'(x)=\dfrac{-x\sin x-3\cos x}{x^4}$ 13. $y''=8x$ 14. $y-2=1(x+2)$ 15. $x=0,\ \frac{1}{3}$

17. AROC $=-\frac{1}{4}$ 19. AROC $=0$ 20. a. $g'(x)=x\bullet f'(x)+f(x)$ 21. $-\frac{5}{4}$

22. $\frac{dv}{ds}(4)=48$ 23. moving left on $\left(-\infty,\frac{5}{2}\right)$ 26. $t=5\ sec,\ h(5)=73.5\ ft$ 27. 0

29. TD = 66.5 m 30. b. $V(7)\approx 40\frac{ft}{sec}$ d. $|V(10)|=50\frac{ft}{sec}$ f. just before $t=2$ sec

31. a. $h'(2)=4$

Assignment 2.6 page 75

1. $f'(x)=1-3\sec x\tan x$ 3. $g'(x)=\dfrac{-x\csc x\cot x-\csc x}{x^2}$ 4. $h'(\theta)=5\sec^2\theta-2\theta$

7. $f'(y)=\boxed{\dfrac{y^3(2y-\sec y\tan y)-3y^2(y^2-\sec y)}{y^6}\text{ free response answer}}=\boxed{\dfrac{-y^2-y\sec y\tan y+3\sec y}{y^4}\text{ multiple choice answer}}$

9. $f'(\theta)=\dfrac{e^\theta\sin\theta-e^\theta\cos\theta}{\sin^2\theta}$ 11. $f'(x)=\dfrac{2}{5x^2}$ 12. $f'(t)=24t^2-48t+18$

13. $g'\left(\frac{\pi}{4}\right)=\dfrac{-\pi\sqrt{2}+4\sqrt{2}}{8}$ 14. $f'(0)=4$ 15. $f'(1)=6$ 16. $h'(\pi)=\dfrac{2}{\pi^3}$

17. $y+\sqrt{3}=4\left(x-\frac{\pi}{6}\right)$ 18. $y+2=4\left(x+\frac{\pi}{4}\right)$ 20. $y-\frac{3\pi}{4}=\left(\frac{3\pi}{2}+3\right)\left(x-\frac{\pi}{4}\right)$

22. AROC $=\frac{1}{2}$ 23. $f'\left(\frac{\pi}{6}\right)=\dfrac{\frac{1}{2}-\frac{\pi}{6}\frac{\sqrt{3}}{2}}{\frac{1}{4}}=\dfrac{6-\pi\sqrt{3}}{3}$ 24. $x=-2,\ 0$ 27. -1

29. 3

Assignment 2.7 page 76

1. $f'(x)=4x^3-9x^2+1$ 2. $f'(x)=\frac{1}{3}x^{-\frac{2}{3}}+\frac{1}{3}x^{-\frac{4}{3}}$ 3. $f'(t)=-\frac{9}{4t^4}$

4. $y'=\frac{t\sec^2 t-\tan t}{t^2}$ 5. $h'(\theta)=-2\theta\sin\theta+\cos\theta$ 6. $f'(x)=\frac{(x+1)(-4x)-(-2x^2)}{(x+1)^2}$

7. $y'=1,\ x\neq 0$ 8. $f'(x)=36-54x$ 9. $g'(t)=4\csc^2 t$

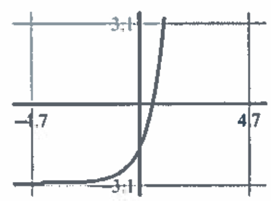
10. $f'(x)=2x$ (using the limit definition) 11. $f'(3)=4$ (using the alternate form)

12. $y''=6x^{-4}+x^{-3}$ 13. $y-3 = 3x$ 14. $y-8=\frac{4}{e^2}(x-e^2)$ 15. $y=-\pi(x-\pi)$

16. a. 22 b. −4 c. 4 17. a. $v(t)=6t^2-6t$ b. $a(t)=12t-6$ c. $t=0,1$

17. d. $0<t<1$ e. $t=2,-1$ f. $\left|v\left(\frac{1}{2}\right)\right|=\frac{3}{2}$ g. disp. = 32 h. TD = 34

18. a. 85.722 or 85.723 b. 7.687 c.



19. a. AROC = $-\frac{1}{3}$ b. $f'(1)=-\frac{8}{9}$ 20. a. yes b. no, sharp turn

21. a. yes b. no, vertical tangent 22. a. yes b. no, sharp turn 23. a. no b. no, jump

24. a. yes b. yes 25. $(0,3)$, $(2,-1)$

Assignment 3.1 page 80

1. $y'=9(3x+5)^2$ 3. $y'=-\frac{3}{2}(2-3x)^{-\frac{1}{2}}$ 4. $f'(t)=\frac{2}{(1-t)^3}$

6. $g'(x)=6x(2x+3)^2+(2x+3)^3$ 7. $y'=-\frac{1}{2}(x+1)^{-\frac{3}{2}}$ 8. $f'(x)=\frac{5}{(x+1)^2}$

9. $g'(x)=4\sec(4x)\tan(4x)$ 11. $f'(\theta)=3\sin(3\theta)\cos(3\theta)$ 12. $y'=2x(2x^2-1)^{-\frac{1}{2}}$

13. $y-2=-1(x+1)$ 15. $y-\frac{8}{27}=-\frac{16}{9}\left(x-\frac{1}{4}\right)$ 16. $y-\left(\frac{1}{4\pi^2}+1\right)=-\frac{1}{4\pi^3}(x-2\pi)$

17. $4(2\sin x-3)^3(2\cos x)$ 19. $\left(\frac{5}{2},8\right),\left(\frac{1}{2},-8\right)$ 20. $g'(1)=48$ 21. a. −5 c. $\sqrt{2}$

22. $y-3.737=4.105(x-1.2)$ or $y-3.737=4.106(x-1.2)$

24. disp. = −4.5, TD = 5.916 or 5.917

Assignment 3.2 page 83

1. $f'(x)=5e^{5x+1}$ 3. $h'(t)=5^{t^2-2t}(2t-2)\ln 5$ 4. $y'=\frac{5}{x}$ 6. $y'=2e^{2\sin x}\cos x$

8. $f'(x)=6x^2e^{3x}+4xe^{3x}$ 11. $y'=\dfrac{2}{x}-\dfrac{1}{x-1}$ 13. $f'(x)=\dfrac{3x^2+2}{x^3+2x}$

15. $g'(t)=\frac{1}{2}\bullet\dfrac{3t^2-1}{t^3-t}$ 16. $h'(x)=\dfrac{\tan x}{x}+\ln x\sec^2 x$ 17. $\frac{d^2y}{dx^2}=3e^{-3x}+8e^{2x}$

18. $y-1=-2(x-1)$ 20. a. $h(0)=220\ ft$ c. $|V(3)|=118\frac{ft}{sec}$

21. b. Avg. Speed $=9.259\ \frac{m}{sec}$ e. $10\frac{m}{sec}$ g. between 80 and 90 m

22. $t<-3$ or $t>1$ 23. f is a cubic, f' is a parabola, and f'' is a line

26. -1 28. 3 29. 0

Assignment 3.3 page 85

1. $y'=-\dfrac{x}{y}$ 2. $y'=-\dfrac{y}{x}$ 3. $y'=-\sqrt{\dfrac{y}{x}}$ 4. $\frac{dy}{dx}=-\dfrac{\sin x}{4\cos(2y)}$

6. $y'=\dfrac{3x^2-2xy^3}{3x^2y^2+1}$ 7. $\frac{dy}{dx}=\dfrac{-y\sec^2(xy)}{x\sec^2(xy)-2y}$

8. $\dfrac{dy}{dx}=\dfrac{y-3x^2}{-x}=\dfrac{3x^2-y}{x}$; $\left.\dfrac{dy}{dx}\right|_{(1,-2)}=5$ 9. $\dfrac{dy}{dx}=\dfrac{2x}{e^y+\frac{1}{y}}=\dfrac{2xy}{ye^y+1}$; $\left.\dfrac{dy}{dx}\right|_{(e,2)}=\dfrac{4e}{2e^2+1}$

10. $y'=\dfrac{2y-3x^2}{1-2x}$; $y'(1,1)=1$ 11. $y'=\dfrac{1}{\sec(x+y)\tan(x+y)}-1$; $y'|_{(0,0)}$ does not exist

12. $y'=-\dfrac{x}{y}$; $y'(3,-4)=\dfrac{3}{4}$ 14. $y''=\dfrac{2x+2y}{x^2}$ 15. $y-5=\frac{12}{5}(x+12)$

16. $(0,3)$, $(0,-1)$ 17. $(\pm\sqrt{8},1)$ 18. $\left.\frac{dy}{dx}\right|_{(1,3)}=\frac{3}{8}$ 19. $\frac{dV}{dr}(4)=64\pi$

20. 0 22. 0 23. DNE 25. $x=-3,1,2$ 28. $|V(1)|=0\frac{m}{sec}$, $|V(2)|=1\frac{m}{sec}$

29. $\dfrac{2}{x^2}$ 30. $f'(x)=3^{x^2}2x\ln 3-\dfrac{2x+1}{\ln 3(x^2+x)}$

Assignment 3.4 page 88

1. $\frac{dy}{dt}=18$ 2. $\frac{dy}{dt}=\pm 24$

Write sentence answers for 3-12.

3. a. $\frac{dA}{dt}=24\pi\frac{\text{cm}^2}{\text{sec}}$ b. $\frac{dC}{dt}=6\pi\frac{\text{cm}}{\text{sec}}$ 4. $\frac{dr}{dt}=\frac{9}{8\pi}\frac{\text{ft}}{\text{min}}$ 5. a. $\frac{dA}{dt}=120\frac{in^2}{sec}$

5. b. $\frac{dV}{dt}=150\frac{in^3}{sec}$ 6. $\frac{dy}{dt}=\frac{3}{50}\frac{cm}{min}$ 7. a. $\frac{da}{dt}=-\frac{9}{8}\frac{ft}{sec}$ b. $\frac{dA}{dt}=\frac{21}{8}\frac{ft^2}{sec}$

7. c. increasing, no d. $\frac{d\theta}{dt}=\frac{3}{16}\frac{\text{radians}}{\text{sec}}$ 8. $\frac{dA}{dt}=160\frac{in^2}{min}$ 9. $\frac{dh}{dt}=\frac{8}{225\pi}\frac{ft}{min}$

10. $-49.615\frac{ft}{sec}$ 11. $5\frac{yd}{sec}$ 12. $-11.180\frac{ft}{sec}$ 14. $y'=\frac{\ln(2x+5)}{(3x+1)^{\frac{2}{3}}}+\frac{2\sqrt[3]{3x+1}}{2x+5}$

15. $y=16$ 16. a. AROC = 0 b. $f'(3)=\frac{2}{3}$ 17. a. 0 b. 0 c. yes

18. a. 1 b. 1 c. yes 20. a. 0 b. 0 c. no 22. $y''=6x^{-4}-16\sin(4x)$ 24. 20

Assignment 3.5 page 92

1. a. $g(x)=\sqrt[3]{x+1}$ c. $f'(x)=3x^2$, $g'(x)=\frac{1}{3}(x+1)^{-\frac{2}{3}}$ d. $f'(1)=3$, $g'(0)=\frac{1}{3}$

1. e. They are reciprocals. 2. a $g(x)=x^2-1,\ x\geq 0$ 2. c. $f'(x)=\frac{1}{2\sqrt{x+1}}$, $g'(x)=2x$

3. a. not possible b. $\frac{3}{4}$ c. 5 4. $\frac{1}{4}$ 5. $\frac{1}{5}$ 6. $\frac{1}{13}$ 8. $\frac{1}{11}$ 9. 1

10. $g'(t)=24t\csc^2(3t^2)$ 12. $f'(x)=e^{\sec x}\sec x\tan x$

14. $\frac{d^2y}{dx^2}=20x^3+2x^{-3}$ 15. $\boxed{f'(x)=\frac{e^x\bullet 2x-x^2e^x}{(e^x)^2}}=\frac{2x-x^2}{e^x}$ 17. $8(6^{2x}-3)^3 6^{2x}\ln 6$

18. $3\frac{\cos x}{\sin x}=3\cot x$ 19. $\frac{dy}{dt}=\frac{2t\cos(t^2)-ye^t}{e^t}$ 20. $y'=\frac{2x}{\cos(y-2x)}+2$

21. a. $y(-2)=-\frac{2}{e}\text{cm}$ b. $v(t)=te^{t+1}+e^{t+1}$ e. $a(-2)=0\frac{cm}{sec^2}$ 23. .471 or .472

25. 4 27. -1 28. $2\,{}^{cm}/_{sec}$

Assignment 3.6 page 97

1. $\frac{\pi}{3}$ 2. $-\frac{\pi}{4}$ 3. $\frac{2\pi}{3}$ 5. -1.249 7. $-.339$ or $-.340$ 9. $-\frac{2}{3}$ 11. 0 12. $\frac{\pi}{3}$

13. $x=4$ 14. $x=\pm 1$ 15. $\cos y=\frac{3}{5}$ 16. $\sin x=\frac{3}{\sqrt{10}}$ 17. $y'=\dfrac{6}{1+9x^2}$

19. $g'(y)=\dfrac{-e^{-y}}{\sqrt{1-\left(e^{-y}\right)^2}}$ 20. $h'(t)=\dfrac{3\sqrt{t}}{2\left(1+t^3\right)}$ 21. $\dfrac{-x^2}{\sqrt{1-x^2}}+2x\arccos x$

23. $y'=\dfrac{-2\sin\left(\ln t^2\right)}{t}$ 25. $g'(t)=\dfrac{-2\cos\sqrt{t}\sin\sqrt{t}}{\sqrt{t}}$ 27. $y'=-3x\cos(-3x)+\sin(-3x)$

28. $h'(y)=3\left(\ln(\sec y)\right)^2\tan y$ 29. $y''\left(\frac{\pi}{4}\right)=-\frac{\pi}{4\sqrt{2}}+\frac{2}{\sqrt{2}}$ 31. $x=-\frac{\pi}{4},\frac{3\pi}{4}$

32. $a=-2,\ b=-1$ 33. b. $12\frac{3}{16}$ c. $\boxed{12-12\bullet 2^{-5}}=11\frac{5}{8}$ 34. 1.528 or 1.529

Assignment 3.7 page 98

1. $y'=\frac{1}{2}\left(x^2-x\right)^{-\frac{1}{2}}(2x-1)$ 2. $g'(x)=3x(x-1)^2+(x-1)^3$ 3. $y'=\dfrac{2}{x}+\dfrac{1}{3(x+1)}$

4. $g'(t)=-t^3e^{-t}+3t^2e^{-t}$ 5. $\frac{dy}{dx}=12\left(\log_6\left(x^2+1\right)\right)^2\dfrac{2x}{\left(x^2+1\right)\ln 6}$ 6. $y'=\dfrac{2}{\left(1+4x^2\right)\arctan(2x)}$

7. $f'(x)=\ln x\bullet\dfrac{2}{1+4x^2}+\arctan(2x)\bullet\dfrac{1}{x}$ 8. $g'(x)=3\left(\ln\left(\arcsin x^2\right)\right)^2\bullet\dfrac{2x}{\sqrt{1-x^4}\arcsin x^2}$

9. $x'=\dfrac{-t}{\sqrt{1-t^2}}$ 10. $y'=\dfrac{2(v-1)}{1+(v-1)^4}$ 11. $y=-x-2$ 12. $y'=\dfrac{-2x+y}{-x+2y}$

13. $y+2=1(x-1)$ 14. $y=1$ 15. a. 11 b. -1 c. 15 d. 35 16. 4 17. 0

18. $\frac{\pi}{4}$ 19. $\frac{1}{\sqrt{3}}$ 20. $\frac{1}{13}$ 21. $\frac{1}{4}$ 22. $\dfrac{dx}{dt}=16\dfrac{cm}{sec}$ 23. a. $-\frac{3}{2}\frac{ft^3}{hr}$ b. 6 hours

24. $\frac{1}{19}$ 25. $y''=2e^{x^2-1}+4x^2e^{x^2-1}$ 26. $\dfrac{\frac{2x^2}{x^2-1}-\ln\left|x^2-1\right|}{x^2}$ 27. $y'=\dfrac{\frac{y}{x}+1}{3y^2-\ln x}=\dfrac{y+x}{3xy^2-x\ln x}$

28. a. -1 b. $-\frac{1}{2}$ 29. $y'=0$ 30. $y'=cx^{c-1}$ 31. $y'=c^x\ln c$

32. not possible with current techniques

Assignment 4.1 page 103

1. a. $y=-3x+6$ b. $f(1.1)\approx 2.7$ c. $f(1.1)=2.669$ 2. b. $y=2x-1$
3. $\sqrt{26}\approx 5.1$ 4. $f(3.1)\approx 29.1$ 6. $y\approx 22.05$ 7. a. A $\approx$ 65 sq. in.
8. $f(8.01)\approx 2\frac{1}{1200}$ 9. $y\approx 1.984$ or 1.985 11. a. $\left(2\frac{1}{6}\right)^3\approx 10$
12. a. At 1 pm the velocity of the car is 20 miles per hour. b. 20, Between 4 and 5 pm the car is accelerating at the rate of 20 miles per hour per hour. c. 10 miles per hour, 25 miles per hour d. $20\frac{mi}{hr^2}$, $0\frac{mi}{hr^2}$ 13. a. Three miles from the trailhead the density of hikers is 30 hikers per mile. b. Three miles from the trailhead the density of hikers is decreasing at the rate of 6 hikers per mile per mile. c. 29.4 hikers per mile 14. $\frac{1}{e^2}\frac{\text{cm}}{\text{sec}}$
15. AROC $=\frac{-2\sqrt{2}}{\pi}$ 16. $y'=\frac{4e^{2x}}{1+e^{4x}}$ 17. $f'(x)=\frac{1}{\arcsin x\,\sqrt{1-x^2}}$
20. $f'(y)=\frac{y^2e^{\sqrt{y}}\bullet\frac{1}{2}y^{-\frac{1}{2}}-e^{\sqrt{y}}\bullet 2y}{y^4}$ 21. $f'(x)=\frac{e^x}{x}+e^x\ln x$
22. $f'(x)=\frac{2}{3}\bullet\frac{1}{x-1}$ 23. $g'(y)=\frac{-\frac{1}{y}}{1-\ln y}=-\frac{1}{y(1-\ln y)}$
24. $y'=\left(\frac{2x+1}{x}+2\ln x\right)y$ 25. a. $\frac{1}{6}$ b. DNE c. $\frac{3}{2}$ d. 0

Assignment 4.2 page 106

1. 0 2. 4 3. ∞ or DNE 4. $\frac{1}{2}$ 6. -1 8. 4 10. $\frac{2}{3}$ 11. 0 13. 1
15. 0 16. 1 17. 2 18. $\frac{1}{2}$ 19. 1.000 23. $y'=\frac{2xy}{y-2}$
24. $y'=\frac{3x^2}{\sqrt{3x^2+2}}+\sqrt{3x^2+2}$ 25. $y'=\frac{-3x^2+2x+6}{\left(x^2+2\right)^2}$ 26. $\frac{dy}{dx}=\frac{-2x+2y}{-2x+1}$
27. a. 7 b. -12 c. -10 28. $a=3,\ b=-5$ 29. $(2,13)$ 30. a. $v(t)=6t^2-6t-1$
30. b. $v(0)=-1\frac{\text{cm}}{\text{sec}}$ c. $a(t)=12t-6$ d. $a(3)=30\frac{\text{cm}}{\text{sec}^2}$ e. $|v(1)|=1\frac{\text{cm}}{\text{sec}}$
30. f. $(-.145\text{sec},1.145\text{sec})$ g. 2 cm h. 6.151 or 6.152 cm i. $t<\frac{1}{2}$sec
30. j. $(-\infty,-.145\text{sec})$, $\left(\frac{1}{2}\text{sec},1.145\text{sec}\right)$ 31. $\frac{dx}{dt}=-\frac{4}{3}\frac{\text{cm}}{\text{sec}}$ This means the particle is moving leftward at the rate of $\frac{4}{3}\frac{\text{cm}}{\text{sec}}$.

Assignment 4.3 page 111

1. 3   3. $x=0,4,5$ 5. $x=\frac{3}{2}$ 6. $x=0,8$ 8. $x=-2$ 9. $x=\frac{\pi}{6},\frac{\pi}{2},\frac{5\pi}{6},\frac{3\pi}{2}$

10. max. $f=10$, min. $f=-\frac{9}{4}$ 11. max. $f=8$, min. $f=0$

13. max. $f=2$, min. $f=0$ 15. max. $f=-3$, min. $f=-15$

16. max. $f=\frac{2}{3}$, min. $f=0$ 17. max. $f=\frac{5}{4}$, min. $f=-1$

19. max. value = 3, min. value = 0   21. no max. value, min. value = 0

24. c = 2   25. $c=\frac{3}{2}$ 26. MVT does not apply. f is not diff. at $x=0$ (sharp turn)

27. $c=\frac{8}{27}$ 30. c = 1   31. $c=\frac{\pi}{2}$ 33. $c=-.215, 1.548$ or 1.549 34. c = 2.414

36. b. $t=\frac{3}{2}sec$ 37. $\frac{\sqrt{2}}{4}$

Assignment 4.4 page 114

1. a. $(-\infty,-2],[2,\infty)$ b. $[-2,2]$ c. $(2,-4)$ d. $(-2,4)$

3. a. $[-2,0],[2,\infty)$ b. $(-\infty,-2],[0,2]$ c. $(\pm 2,0)$ d. $(0,4)$

5. a. $(-\infty,-2],[0,\infty)$ b. $[-2,0]$ c. $x=0$ d. $x=-2$

7. problem 2   9. a. $[4,\infty)$ b. $(-\infty,4]$ c. $(4,-16)$ d. none

10. a. $(-\infty,1]$ b. $[1,\infty)$ c. none d. $(1,8)$

12. a. $(-\infty,0],[1,\infty)$ b. $[0,1]$ c. $\left(1,\frac{1}{10}\right)$ d. $(0,1)$

13. a. $(-\infty,\infty)$ b. none c. none d. none

14. a. none b. $(-\infty,-3),(-3,3),(3,\infty)$ c. none d. none

16. a. $(-\infty,-1),(-1,\infty)$ b. none c. none d. none

18. a. $\left[0,\frac{\pi}{6}\right],\left[\frac{5\pi}{6},2\pi\right)$ b. $\left[\frac{\pi}{6},\frac{5\pi}{6}\right]$ c. $\left(\frac{5\pi}{6},\frac{5\pi}{12}-\frac{\sqrt{3}}{2}\right)$ d. $\left(\frac{\pi}{6},\frac{\pi}{12}+\frac{\sqrt{3}}{2}\right)$

20. a. $[.936,\infty)$ b. $(-\infty,.936]$ c. $x=.936$ d. none   21. yes   23. no

24. $[0\,\text{sec},2\,\text{sec})$ 26. a. never   27. $a=-4$, $b=0$

28. max. $f=73$, min. $f=-7$ 32. $c=-.333$ 33. 2

Assignment 4.5 page 118

1. a. $(-\infty,1)$ b. $(1,\infty)$ c. $(1,0)$ 2. a. $(-2,2)$ b. $(-\infty,-2),(2,\infty)$ c. none

4. a. $(-2,1),(4,6),(8,9)$ b. $(6,8)$ c. $(6,1)$

5. a. $(-\infty,0),(2,\infty)$ b. $(0,2)$ c. $(0,2),(2,-14)$

6. a. $\left(\frac{3\pi}{4},\frac{7\pi}{4}\right)$ b. $\left[0,\frac{3\pi}{4}\right),\left(\frac{7\pi}{4},2\pi\right]$ c. $\left(\frac{3\pi}{4},0\right),\left(\frac{7\pi}{4},0\right)$

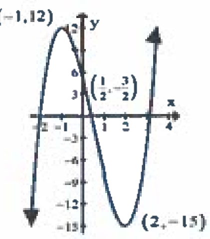
8. a. none b. $(-\infty,0),(0,\infty)$ c. none

9. a. $\left(\frac{\pi}{6},\frac{5\pi}{6}\right)$ b. $\left[0,\frac{\pi}{6}\right),\left(\frac{5\pi}{6},2\pi\right]$ c. $\left(\frac{\pi}{6},\frac{-\pi^2}{72}+\frac{\pi}{3}\right),\left(\frac{5\pi}{6},\frac{-25\pi^2}{72}+\frac{5\pi}{3}\right)$

10. rel. max. (0,6) rel. min. (2,2) 11. rel. max. $\left(\pm 3,\frac{101}{4}\right)$ rel. min. (0,5)

12. rel. max. $\left(\frac{\pi}{2},5\right)$ rel. min. $\left(\frac{3\pi}{2},1\right)$ 13. rel. max. at $x=-3$, rel. min. at $x=3$

15. rel. min. at $(0,-1)$, PI at $\left(-1,\frac{-2}{e}\right)$

16. 

19. concave down ($f''(.523)=-.013$)

20. max. $f=8$, min. $f=-1$ 21. c = 3

22. $a=1, b=0, c=-3, d=2$

24. <

Assignment 4.6 page 120

1. $y\approx 5$ 2. $y\approx 2.6$ 3. a. $s(t)\approx 60t+240$ b. mile 360 c. mile 150 d. $[-4,3]$ hours

4. 0 5. $\frac{4}{\pi}$ 6. ∞ 7. 0 8. 5 9. $\frac{3}{4}$ 10. $-\frac{1}{2}$

11. Abs. Max. $f=\frac{2}{3}$, Abs. Min. $f=-6$ 12. Abs. Max. $g=14$, Abs. Min. $g=0$

13. MVT does not apply. (V.A. at $x=0$) 14. $c=3+\sqrt{3}=4.732$

15. MVT does not apply. (Sharp turn at $x=3$)

16. a. rel. max. at $x=-1$, rel. min. at $x=3$, P.I. at $x=1$

16. b. rel. max. at $x=-1$, rel. min. at $x=1$, P.I. at $x=0,\pm\frac{1}{\sqrt{2}}$

17.a. 0 b. -1 c. $y\approx -2$ d. The approximation is an overestimate since the second derivative is negative. e. relative maximum by the second derivative test since the first derivative is zero and the second derivative is negative.

18. Min. point: $(3,-4)$, Max. point: $(3,2)$ 19. $a=\frac{1}{2}, b=-2$

20. a. $[3.198,4)$ or $[3.199,4)$ b. local min at $x=3.198$ or 3.199

20. c. PI at $x=.483, 2.012, 3.849$ 21. $a(5)=-3\frac{ft}{sec^2}$ 22. $speed(6)=3\frac{ft}{sec}$

23. $[0,5)$ seconds 24. $[0,2)$sec, $(6,7]$sec 25. $[0,2)$sec, $(5,6)$sec 26. 5 seconds

27. a. $t>0$ b. $-3<t<0$, $t>3$ c. -44 d. 64

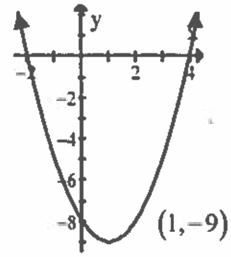
28. The area is decreasing at the rate of $600\frac{ft^2}{min}$.

Assignment 5.1 page 124

1. rel. max. pt. $(0,5)$ rel. min. pt. $(2,1)$ PI $(1,3)$

2. no rel. extrema, PI $(0,1)$

3. rel. min. pt. $(3,-11)$ PI $(0,16),(2,0)$

4. 

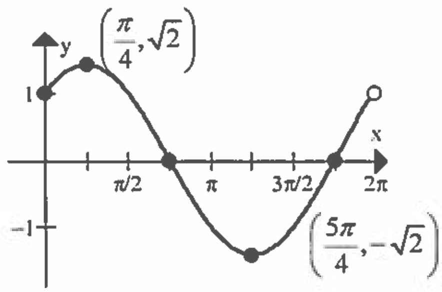
6. rel. min. pt. $(0,0)$, P.I. $\left(\pm 1,\frac{1}{2}\right)$,

E.B. horiz. asymp. y = 2

7. Do: $x \neq 0$, rel. max. pt. $(-1,-1)$, rel. min. pt. $(1,1)$,

V. A. x = 0 (odd), E. B. like $y=\frac{1}{2}x$

9. Do: $-3 \leq x \leq 3$, x-int. $(\pm 3,0)$, rel. max. pt. $(0,3)$

10. 

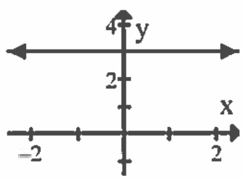
11. Rel. Min. at $(0,-1)$, P.I. at $\left(-1,-\frac{2}{e}\right)$

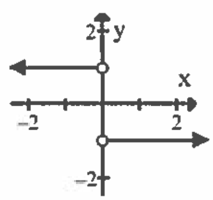
12. Do.: $x>0$, Rel. Min. pt.: $(1,1)$, no P.I.

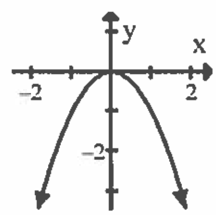
15.a. $(0,-3),(0,1)$ 16. $\left(\pm\sqrt{8},-1\right)$

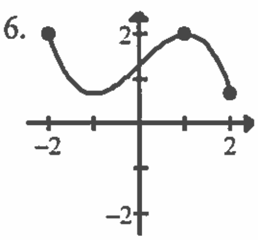
17. B 21. (A) $-\frac{2}{3}$, (G) 1

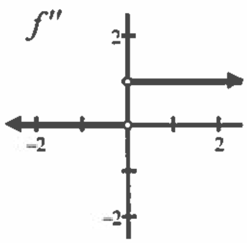
Assignment 5.2 page 127

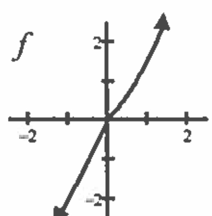
1. 

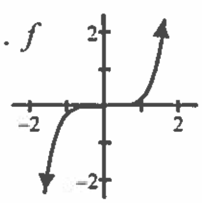
4. 

5. 

6. 

9. f'' 

f 

11. f 

These f graphs could be shifted vertically.

12. V.A.: $x=4,-2$, H.A.: $y=0$, Rel. Max. pt.: $\left(1,-\frac{1}{9}\right)$, no Rel. Min.

13. Rel. Min. pt.: $(1,-27)$, no Rel. Max., P.I.: $(2,-16)$, $(4,0)$

14. Rel. Max. pt.: $(0,-5)$, Rel. Min. pt.: $(2,-9)$

16. $c=\sqrt{\frac{4}{3}}$

17. abs. min. f = −18, abs. max. f = 14

19. $a=2,\ b=-12,\ c=16$

Assignment 5.3 page 130

1. a. $(-\infty,-3]$ and $[2,4]$ because $g' \le 0$ b. $[-3,2]$ and $[4,\infty)$ because $g' \ge 0$
   c. $x=-3,\ 4$ because g' changes from negative to positive
   d. $(-\infty,-2)$, $(0,1)$, and $(3,\infty)$ because g' is increasing
   e. $x=-2,0,1,3$ because g' changes between increasing and decreasing
2. a. $(-3,\infty)$ because $g'' \ge 0$ b. $x=-3$ because g'' changes sign
3. a. $[-3,\infty)$ because $f' \le 0$ d. $x=1,\ 3$ because f' has relative extrema
   e. none because f' is never positive and increasing
4. f. 0 g. DNE because f' has a vertical tangent h. 6
5. $x=1$ nonremovable, $x=2$ removable 8. $\frac{1}{4}$ 9. DNE 10. 0
11. max $f=\frac{9}{4}$, min f = -4
12. a. 2389 or 2390 people b. $481.009\,\frac{people}{day}$ c. (9.210 , 500.000)
13. V. A.: $x=\pm 3$ (odd), Rel. Max. pt.: $\left(0,-\frac{2}{9}\right)$, no Rel. Min., H. A.: $y=1$
14. V.A.: $x=2$ (even), x-int.: $(-1,0)$ (odd), y-int.: $\left(0,\frac{1}{4}\right)$, Rel. Min. pt.: $\left(-4,-\frac{1}{12}\right)$, H.A: $y=0$
17. $c=-1.190$ or -1.191

Assignment 5.4 page 133

1. The numbers are both 10. 2. The rectangle's length and width are both 9 *cm*.
3. The rectangle's length and width are both 20 *ft*. 4. $x=30\,ft.$, $y=40\,ft.$
5. $x=9\,ft.$, $y=6\,ft.$ 6. $x=2\,in.$ 7. $\left(3,\frac{3}{2}\right)$ 9. $\frac{dV}{dt}=324\pi\frac{in^3}{min}$ 10. 375 *mph*
11. f. $x=0,4,5$ l. $x=0,2,4,5$

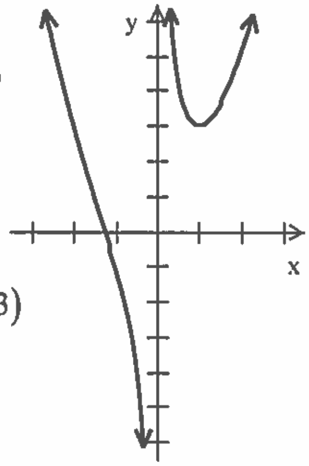
Assignment 5.5 page 135

1. The first number is 30 and the second is 10. 2. The first number is 24.5 and the second is .5. 3. The first number is 0 and the second is 25. 4. $x=400\,m$, $y=800\,m$
5. $x=.606\,ft$ or $.607\,ft$ 6. $V=4.104\,ft^3$ 7. $\left(\sqrt{18},\sqrt{18}\right)$ 8. 36
9. length = 20 *cm*, width = 20 *cm*, height = 10 *cm* 10. Rel. Max. pt.: $(-1,13)$,
10. no Rel. Min., P. I.: $(0,2)$, $(2,-14)$ 11. Rel. Max. pt.: $(1,1)$, Rel. Min. pt.: $(0,0)$,
11. Concave Down: $(-\infty,0)$, $(0,\infty)$ 14. $g'(x)=\dfrac{1}{(2x+3)^2}$

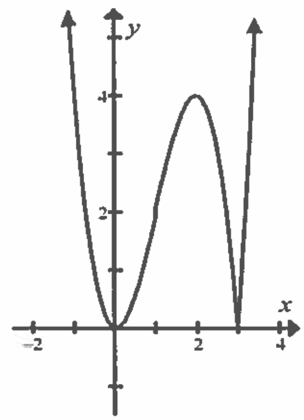
Assignment 5.6 page 136

1. abs. max. value of y is 5, Abs. min. value of y is -14.238

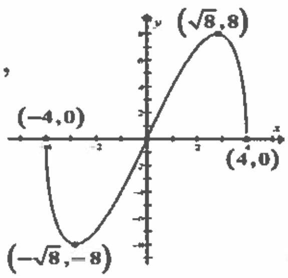
2. Do.: $x \neq 0$
   V. A.: $x = 0$ (odd)
   x-int.: $\left(-\sqrt[3]{2},0\right)$
   no y-int.
   E. B.: like $y = x^2$
   Rel. Min. pt.: $(1,3)$
   no Rel. Max.
   P. I.: $\left(-\sqrt[3]{2},0\right)$



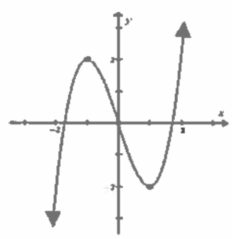
3. x-int: $(0,0)$ $(3,0)$
   y-int: $(0,0)$
   rel. max. pt.: $(2,4)$
   rel. min. pt.: $(0,0)$
   and $(3,0)$
   P. I.: $(1,2)$



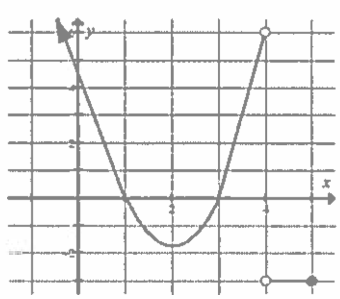
4. Do.: $-4 \le x \le 4$,
   x-int.: $(0,0)$(odd), $(\pm 4,0)$,
   y-int.: $(0,0)$,
   Rel. Max. pt.: $\left(\sqrt{8},8\right)$,
   Rel. Min. pt.: $\left(-\sqrt{8},-8\right)$,
   P.I.: $(0,0)$



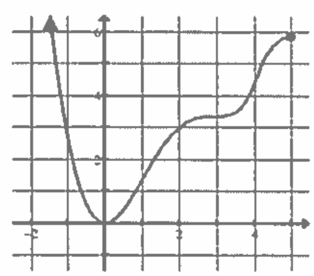
5. x-int.: $(0,0)$, $\left(\pm\sqrt{3},0\right)$(all odd),
   y-int.: $(0,0)$, Rel. Max. pt.: $(-1,2)$,
   Rel. Min. pt.: $(1,-2)$, P.I.: $(0,0)$



6.



7.



8. 4 by 8 9. Both numbers are 1. 10. 180,000 yd^2

11. a. $[-3,3]$ because $h' \le 0$ b. x = 3 because h' changes from negative to positive.
    c. $(-2,-1)$ and $(2,\infty)$ because h' is increasing
    d. $x=-2,-1$ because h' changes between increasing and decreasing
    e. $(-\infty,-3)$ because h' is positive and decreasing
    f. $[0,2]$ because h' is negative and horizontal   g. 102   h. $\frac{1}{2}$
12. local maximum by the second derivative test   13. 70 feet   14. $a=2, b=4, c=4$
15. $c=2.877$ 16. MVT does not apply. (sharp turn at $x=0$)
17. a. $\dfrac{dy}{dx}=\dfrac{x-y}{x+2y^3}$ b. increasing   c. $y\approx\frac{39}{40}$ d. upward   e. underestimate since $y''\big|_{(2,1)}>0$
18. a. $[.108,9.869]$ or $[.109,9.869]$ b. $x=.108$ or $.109$ c. $g''(x)=6x^2-0.1e^x+1$
    d. $(-\infty,8.338)$ or $(-\infty,8.339)$ e. $x=8.338$ or 8.339
19. a. rel. max.   b. The abs. max. value of f is at $x=3$. The abs. min. value of f is at either $x=0$ or $x=5$.
20. a. $(-2,0]$, $[2,3)$ because $f'(x)\le 0$
    b. $x=0$ because $f'(x)$ changes from negative to positive
    c. $(-2,-1)$ because $f'(x)$ is increasing
    d. $x=-1$ because $f'(x)$ has a relative maximum (because $f'(x)$ changes from inc. to dec)
21. $\frac{dx}{dt}=24$ *units per sec.*   22. $\frac{1}{19}$

Assignment 6.1 page 142

1. $-\dfrac{1}{2x^2}+C$ 2. $\frac{4}{5}t^{\frac{5}{4}}+C$ 3. $\dfrac{x^3}{3}-\dfrac{x^2}{2}-2x+C$ 4. $\frac{4}{3}y^{\frac{3}{2}}+C$ 6. $-\dfrac{1}{3x}+C$

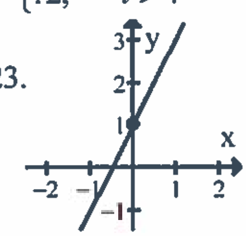
7. $-\dfrac{1}{9x}+C$ 10. $3t^3-30t^2+100t+C$ 11. $\frac{4}{3}x^3-x-\frac{1}{2}x^{-1}+C$ 12. $2t-2t^{\frac{1}{2}}+C$

14. $f(x)=x^3-2x^2+2x-4$ 15. $y=\dfrac{3}{t}+t+6$ 16. a. $y=2x^{\frac{3}{2}}+C$ b. $y=2x^{\frac{3}{2}}-16$

17. $g(x)=\frac{1}{3}x^3-\frac{3}{2}x^2-5x-\frac{7}{6}$ 19. a. $v(t)=3t^2-4t+5$ c. $s(t)=t^3-2t^2+5t-2$

20. c. $s(t)=\frac{2}{3}t^{\frac{3}{2}}+t-7$ 21. $t=\sqrt{10}$ sec   22. a. $v(t)=\begin{cases}3t, & 0\le t\le 4\\ 12, & t>4\end{cases}$

22. c. $s(t)=\begin{cases}\frac{3}{2}t^2, & 0\le t\le 4\\ 12t-24, & t>4\end{cases}$ e. 10.333 *sec*   23.



25. Do.: $x \neq \pm 1$, V.A.: $x = \pm 1$ (odd), no hole, x-int.: $(0,0)$ (even), y-int.: $(0,0)$, E.B.: H.A. $y = 1$

26. $y-4=\frac{1}{3}(x-8)$ 27. $y=-2x^3-6x^2+2$

Assignment 6.2 page 144

1. $\tan\theta-2\theta+C$ 3. $\frac{1}{5}\sin x+C$ 5. $2e^x+\frac{2}{3}x^{\frac{3}{2}}+C$ 6. x^4-e^x+C

8. $\frac{6^x}{\ln 6}+\frac{2}{3}x^{-3}+C$ 9. $3\tan x+x+C$ 10. $-\csc x+C$ 12. $\frac{5^x}{\ln 5}+C$

13. a. $a(t)=4\cos t+6t$ b. $x(t)=-4\cos t+t^3+7$ 14. $y+\frac{1}{2}=\sqrt{3}\left(x-\frac{\pi}{6}\right)$

15. $y=2(x-1)$ 16. $y+\sqrt{2}=-\sqrt{2}\left(x-\frac{\pi}{4}\right)$ 17. $y=4x+4$

18. $y-.986=-.272(x-.821)$ or $y-.986=-.273(x-.821)$ 19. -1 21. 2

23. $f'(x)=e^{\sec x}\sec x\tan x$ 24. $h'(\theta)=-2\theta\sin\theta+2\cos\theta-5\cos(5\theta)$

25. $g'(t)=24t\csc^2(3t^2)$ 26. $\frac{3\cos x}{\sin x}$ 28. $\frac{1}{4}$ 29. $\frac{1}{3}$

31. $c=-1+\sqrt{3}=.732$ Tangent and secant lines have the same slope.

Assignment 6.3 page 147

1. $\frac{1}{6}(3x-2)^6+C$ 2. $\frac{1}{45}(5t-3)^9+C$ 3. $-\frac{2}{3}\sqrt{4-x^3}+C$ 5. $-\frac{1}{3}\left(x^3+2\right)^{-5}+C$

6. $6\sqrt{1-v}+C$ 8. $-\frac{1}{16}\left(2x^2-4x\right)^{-4}+C$ 9. $\frac{2}{11}\left(\sqrt{t}-4\right)^{11}+C$

10. $\frac{3}{2\sqrt[3]{5}}x^{\frac{2}{3}}+C$ or $\frac{3}{10}(5x)^{\frac{2}{3}}+C$ 11. $\frac{4}{3}u^3+2u^2+u+C$ or $\frac{1}{6}(2u+1)^3+C$

13. $x+\frac{3}{x}+C$ 14. $\frac{6}{5}x^{\frac{5}{2}}+\frac{2}{3}x^{\frac{3}{2}}-4x^{\frac{1}{2}}+C$ 16. $-\cos(\pi\theta)+C$

18. $\frac{1}{2}\sec(2x-1)+C$ 20. $\frac{1}{3}(\cot t)^{-3}+C$ 21. $12(x+1)^{\frac{5}{2}}-20(x+1)^{\frac{3}{2}}+C$

22. $8\left(\frac{1}{2}x-1\right)^{\frac{3}{2}}+4\left(\frac{1}{2}x-1\right)^{\frac{1}{2}}+C$ 23. $\frac{1}{4}(1-x)^{12}+\frac{5}{13}(1-x)^{13}+C$

24. $-\frac{9}{2}x^{\frac{2}{3}}+3x-\frac{71}{2}$ 25. $y=2\left(x^2-3\right)^{\frac{3}{2}}+3$ 26. $f(x)=\frac{1}{3}\sin(3x)+\frac{5}{3}$

28. a. $a(t)=\frac{2}{9}t-\frac{2}{3}$ b. $y(t)=\left(\frac{1}{3}t-1\right)^3+5$ e. at all times f. 8

30. a. $(0,0)$, $(0,4)$ b. $\left(\pm\sqrt{2},2\right)$ 31. $f(2.1)\approx 7.9$ 34. $f'(2)=4$

Assignment 6.4 page 152

1. 3 2. $-\pi$ 4. $8-2\pi$ 5. $4+2\pi$ 6. $\boxed{\left(\frac{16}{3}+8\right)-\left(\frac{2}{3}+2\right)}=\frac{32}{3}$

7. a. -4 b. 5 8. a. 2 9. -2 10. $\frac{1}{10}$ 11. $\frac{1}{2}$ 12. $\boxed{(1+1)-\left(\frac{1}{2}+8\right)}=-\frac{13}{2}$

14. $\boxed{\left(\frac{3}{5}\bullet 32+\frac{3}{2}\bullet 4\right)-\left(\frac{3}{5}+\frac{3}{2}\right)}=\frac{231}{10}$ 15. $\frac{1}{3}$ 16. 2 17. 0 18. $2\sqrt{3}$

19. $\frac{1}{6}e^{48}-\frac{1}{6}$ 20. $\dfrac{3^{17}-3^5}{6\ln 3}$ 21. $\frac{3}{4}$ 22. 6 23. $\boxed{\left(\frac{-8}{3}+8\right)+\left(\left(\frac{64}{3}-16\right)-\left(\frac{8}{3}-8\right)\right)}=16$

24. $\boxed{\left(-\frac{1}{2}\bullet 2^{-4}+\frac{2}{5}\bullet 2^{-5}\right)-\left(-\frac{1}{2}+\frac{2}{5}\right)}=\frac{13}{160}$ 25. $\boxed{\frac{3}{7}+\frac{3}{4}}=\frac{33}{28}$ 26. 2.484 or 2.485

29. a. 20.533 or 20.534 b. 23.533 or 23.534 31. c. -3 f. 18 g. 22
33. 18 34. -1

Assignment 6.5 page 155

1. x^2-x+1 4. $-(1-x)^4$ 5. $-2x\left(1-x^2\right)^4$ 7. $\frac{1}{36}\left(3a^2-1\right)^6-\frac{1}{36}\cdot 2^6$

8. a. $g'(x)=2f(2x)$ c. g has a rel. min. at $x=-1$ because $f(2x)$ changes from neg. to pos.

8. e. -16 9. 2 10. $2\ln 5$ 12. $\frac{1}{2}\ln\left|x^2-1\right|+C$ 13. $\frac{1}{2}x^2-\ln|x|+C$

14. $2\ln\left|y^2-3y+2\right|+C$ 16. $\frac{9}{4}\left(u^2+1\right)^{\frac{2}{3}}+C$ 18. $-\dfrac{1}{2(1+\ln x)^4}+C$

19. $\ln|\sin y-2|+C$ 21. $\frac{1}{5}\ln|\sin(5x)|+C$ 24. 6.732 25. $y'=\dfrac{\ln t-1}{(\ln t)^2}$

26. $f'(y)=\dfrac{3}{1+9y^2}$ 28. $3\cot x$ 30. 12 31. $a=2,\ b=4,\ c=4$

33. $c=\frac{91}{27}=3.370$ 34. $-\frac{\pi}{6}$ 35. $\frac{\pi}{2}$ 36. $-\frac{\pi}{4}$ 37. $\frac{\pi}{4}$

Assignment 6.6 page 158

1. $\frac{\pi}{12}$ 2. $\frac{\pi}{30}$ 3. $-\frac{1}{2}+\ln 2$ 4. $\frac{4}{\sqrt{2}}\arctan\frac{2t+1}{\sqrt{2}}+C$ 6. $\sin^{-1}\left(\dfrac{\ln x}{4}\right)+C$

7. $\dfrac{1}{\sqrt{7}}\tan^{-1}\left(\dfrac{t-5}{\sqrt{7}}\right)+C$ 8. $\frac{\pi}{6}$ 9. $\frac{2}{3}(\arctan\theta)^{\frac{3}{2}}+C$ 11. $5x-5\arctan x+C$

12. $x^2-2x-2\ln|x+1|+C$ 13. $-\ln\left|e^{-2y}-1\right|+C$ or $2y-\ln\left|1-e^{2y}\right|+C$

14. $\arctan\frac{t}{3}-2\ln\left(t^2+9\right)+C$ 15. $-\frac{1}{2\sqrt{3}}\arctan\frac{e^{-2v}}{\sqrt{3}}+C$

16. $\frac{2}{15}\left(\sqrt[3]{3x+1}\right)^5-\frac{1}{3}\left(\sqrt[3]{3x+1}\right)^2+C$ 18. $-2x\sin\left(2x^2\right)$ 19. $\frac{1}{2}e-\frac{1}{2}=.859$

20. $g'(y)=\dfrac{-\frac{1}{y}}{1-\ln y}=-\dfrac{1}{y(1-\ln y)}$ 21. $y'=\dfrac{2xy}{y-2}$ 23. a. 8 *ft.* by 9 *ft.* b. 72 ft^2

Assignment 6.7 page 160

1. a. $\boxed{2(1+3+1+3)}=16$ b. $\boxed{1(0+1+2+3+2+1+3+3)}=15$

1. c. $\boxed{1(1+2+3+2+1+3+3+3)}=18$ 2. $\boxed{2(8+12+17+15+12+6)}=140$

4. $\boxed{4\bullet 1+4\bullet\sqrt{5}+4\bullet 3+4\bullet\sqrt{13}}$

7. $\boxed{3\left(\sqrt[3]{-3}-\frac{1}{2}\right)+1\left(\sqrt[3]{0}-\frac{1}{2}\right)+1\left(\sqrt[3]{1}-\frac{1}{2}\right)+3\left(\sqrt[3]{2}-\frac{1}{2}\right)}=3\left(\sqrt[3]{-3}-\frac{1}{2}\right)+3\left(\sqrt[3]{2}-\frac{1}{2}\right)$

9. $4(-2)+6\bullet 4=16$ 12. 1.486 13. $\sqrt{x}$ 14. $-2\sqrt{2x}$ 16. $\frac{2}{3}x^{\frac{3}{2}}$

18. $\int_{-3}^{3}2x^{\frac{2}{3}}\,dx$ or $2\int_{0}^{3}2x^{\frac{2}{3}}\,dx$ 20. $\int_{-2}^{3}(y^2+1)\,dy$ 21. $\frac{52}{3}$ 22. 3.5

23. $\frac{\pi}{6}$ 24. $2\sin^{-1}x^2+C$ 25. $\frac{1}{2}\ln(x^2+16)+\frac{5}{4}\arctan\frac{x}{4}+C$

26. $2\sqrt{x-1}-2\arctan\sqrt{x-1}+C$ 28. $y-2\ln|y+2|+C$ 29. $\frac{5}{18\pi}\frac{cm}{sec}$

Assignment 6.8 page 164

1. $2+5+\frac{9}{2}$ 2. $\boxed{\frac{1}{2}\bullet 1(5+2\bullet 12+2\bullet 28+2\bullet 34+2\bullet 15+2\bullet 25+20)}=126.5$

3. $\frac{3}{2}\left(\sqrt[3]{-3}-\frac{1}{2}-\frac{1}{2}\right)+\frac{1}{2}\left(-\frac{1}{2}+1-\frac{1}{2}\right)+\frac{1}{2}\left(1-\frac{1}{2}+\sqrt[3]{2}-\frac{1}{2}\right)+\frac{3}{2}\left(\sqrt[3]{2}-\frac{1}{2}+\sqrt[3]{5}-\frac{1}{2}\right)$

4. 9.76 7. 48 9. $\int_{2}^{5}f(x)\,dx$ (Other answers are possible if the function is shifted horizontally) 10. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left(f\left(1+k\bullet\frac{4}{n}\right)\frac{4}{n}\right)$ 11. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left(\sin\left(2+\frac{2k}{n}\right)\frac{2}{n}\right)$

13. $\int_{3}^{5}\sqrt{x}\,dx$ 15. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\left(\left(\left(\frac{5k}{n}\right)^2+1\right)\frac{5}{n}\right)$ 16. $2\sin^4(2x)$ 20. 8 21. $\frac{2}{3}$

22. $\frac{1}{12}(x^2-2x)^6+C$ 23. $\frac{2}{3}y^{\frac{3}{2}}-2y^{\frac{1}{2}}+C$ 24. $\frac{4}{5}\tan^{-1}\dfrac{t-2}{5}+C$

26. $\frac{1}{3}e^{3x}+e^{-x}-e^{-2x}+C$ 27. $\arcsin\dfrac{\sin t}{5}+C$ 28. $1+2\ln 2$ 29. $6+\pi$

30. a. reverse $(0,2)$, forward $(3,10)$, at rest $[2,3]$ b. Six minutes after the start of the test drive the car's velocity was 6000 feet per minute. d. 2000 *ft/min/min* f. $(9,10)$ h. 29,000 *ft* i. 31,000 *ft*

Assignment 6.9 page 166

1. $\frac{9}{16}\left(2x^2+1\right)^{\frac{4}{3}}+C$ 2. $\frac{1}{10}(2t+1)^{\frac{5}{2}}-\frac{1}{6}(2t+1)^{\frac{3}{2}}+C$ 3. $\frac{3}{2}x^2-2x-\frac{9}{x}+C$

4. $25x+C$ 5. $\tan^{-1}\left(\frac{x-3}{3}\right)+C$ 6. $\ln\left|\sin y-2\right|+C$ 7. $\frac{1}{3}\theta^3+\sec(\theta-1)+C$

8. $8\sqrt{x}-16\ln\left(\sqrt{x}+2\right)+C$ 9. $e^{\tan x-1}+C$ 10. $\cos e^{-t}+C$ 11. $\boxed{\frac{4}{15}\bullet 16^{\frac{5}{4}}-\frac{4}{15}}=\frac{124}{15}$

12. $\ln\sqrt{5}$ 13. $\frac{3}{4}-\frac{3}{4}(\ln 2)^{\frac{4}{3}}$ 14. $-\frac{1}{3\ln 3}+\frac{1}{\ln 3}$ 15. $\frac{1}{2}$ 16. $6-2\sqrt{3}$ 17. $\frac{\pi}{3}$

18. 6.294 or 6.295 19. 6 20. 1 21. 2 22. $\cos^2(2x+1)-3x^2\cos^2\left(2x^3+1\right)$

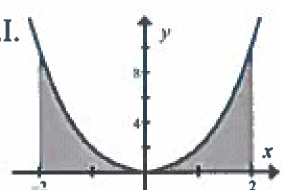
23. $f(x)=\frac{e^{x^2+5}}{2}+C$ 24. $x(t)=-\frac{1}{e^{t^2-t}}+C$ 25.a. $\boxed{5\bullet 3+2\bullet 8+1\bullet 2+4(-2)}=25$

25.b. $\boxed{5\bullet 0+2\bullet 3+1\bullet 8+4\bullet 2}=22$

25.c. $\boxed{\frac{1}{2}\bullet 5(0+3)+\frac{1}{2}\bullet 2(3+8)+\frac{1}{2}\bullet 1(8+2)+\frac{1}{2}\bullet 4(2-2)}=\frac{47}{2}$ 26. $\int_4^5(3x+2)\,dx$

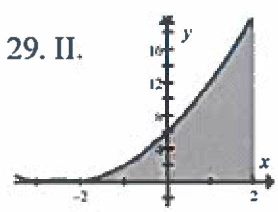
27. $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\left(2\left(2+\frac{5k}{n}\right)^2+5\left(2+\frac{5k}{n}\right)\right)\frac{5}{n}\right)$ 28. $\int_0^{\frac{\pi}{3}}\cos x\,dx$

29.I.



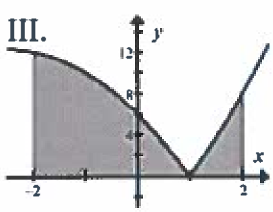
matches with c

29. II.



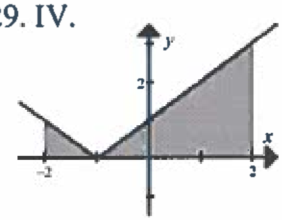
matches with b

29. III.



matches with d

29. IV.



matches with a

30.a. I $\int_{-2}^{2}\left|x^3+x\right|dx=2\int_0^2\left(x^3+x\right)dx$ II $\int_{-2}^{2}\left|x^2+5x+6\right|dx=\int_{-2}^{2}\left(x^2+5x+6\right)dx$

III $\int_{-2}^{2}\left|x^2+5x-6\right|dx=\int_{-2}^{1}\left(-x^2-5x+6\right)dx+\int_1^2\left(x^2+5x-6\right)dx$

30.b. IV $\int_{-2}^{2}\left|x+1\right|dx=5$

31. $f'(4)=24.250$ 32. 349.621 33. $x=\pm.488$ or $\pm.489$ 34. an asymptote

Assignment 7.1 page 171

1. $\frac{2}{3}y^3+3y=\frac{1}{3}x^3-x+C$ 2. $y=\sqrt{-2e^{-x}+C}$ or $y=-\sqrt{-2e^{-x}+C}$

3. $\ln|y+1|=\frac{1}{2}\ln|x|+C$ or $\ln|y+1|=\frac{1}{2}\ln|2x|+C$ 4. $y=C(x-2)^2$

5. $y=-\sqrt{-2x^2+24}$ 6. $y=ex^{-\frac{1}{3}}$ 7. $y=100e^{kt}$ 8. $y=\sqrt{\frac{1}{2}x^2-1}$

9. $y=\tan\left(2\ln x+\frac{\pi}{4}\right)$ 10. not a solution 15. $f'(y)=\dfrac{y^2e^{\sqrt{y}}\bullet\frac{1}{2}y^{-\frac{1}{2}}-e^{\sqrt{y}}\bullet 2y}{y^4}$

16. $f'(x)=\dfrac{e^x}{x}+e^x\ln x$ 17. $y=-\ln\left(1+e^{-x}\right)+C$ 20. $\frac{3}{\ln 2}$ 21. 4.403 or 4.404

22. a. $t=2.731$ c. $a(5)=-1.617$ d. $TD=4.510$ or 4.511 23. a. -2 c. 0

24. a. $4\frac{1}{30}$ 25. b. $-\frac{1}{2}$

Assignment 7.2 page 176

1. $y=2e^{.346t}$ or $y=2e^{.347t}$ or $y=2e^{\left(\frac{1}{2}\ln 2\right)t}$ 2. a. $A(1)=\$1056.54$, $A(5)=\$1316.53$

2. a. $A(10)=\$1733.25$ c. 12.602 or 12.603 *yrs* 3. 551 bacteria

4. 429.741 or 429.742 *yrs* 5. 19,000 *yrs* 7. 32.618 or 32.619 *min* 8. $\frac{5}{2}+\frac{\pi}{4}$

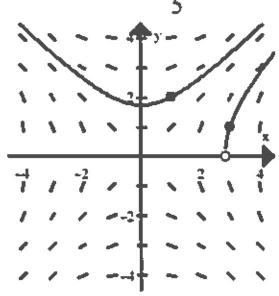
9. $\frac{1}{5}-\frac{10}{3}+25=\frac{328}{15}$ 10. $-2\ln\frac{1}{2}=\ln 4$ 12. $-\frac{1}{2}(\ln u)^{-2}+C$ 14. $-\sin\frac{1}{x}+C$

15. $y^{\frac{4}{3}}=x^{\frac{4}{3}}+C$ 16. $y=\sqrt{\frac{1}{2}(\ln x)^2+\frac{1}{2}}$ 18. $\frac{dh}{dt}=-\frac{1}{4\pi}\frac{ft}{min}$ 19. $-\frac{1}{2a}$

20. $\frac{3}{4}$ 21. $-\frac{1}{4}$

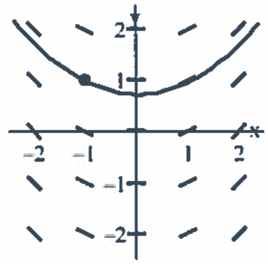
Assignment 7.3 page179

1. a. 0 c. 2 e. $-\frac{9}{5}$

3. b. i,ii.  d. $y=\sqrt{x^2+C}$ or $y=-\sqrt{x^2+C}$

4. c. $y=e^x$ for $(0,1)$

5. a,b.



c. $y=\frac{1}{4}x^2+\frac{3}{4}$

6. c. $y=\sqrt{x+2}$ 7. c. $y=\sqrt[3]{3x+4}$ 8. c. $y=-e^{x^2}+1$

11. a. $\frac{5}{3}\frac{m}{sec^2}$ c. $1\frac{m}{sec^2}$ e. $9.5\frac{m}{sec}$ f. $29.5\frac{m}{sec}$ i. $10\ sec.$ j. $0\ sec.$

12. $x=\pm 3$ (sharp turns) 13. $x=0$ (hole), $x=-1$ (VA) 14. $x=0$ (vertical tangent)

16. The first number is 20 and the second is 4. 18. $x+2\ln|x-1|+C$

20. $\tan\theta-2\theta+C$ 21. $-\frac{1}{2}(\sin y)^{-2}+C$ 22. $\frac{1}{5}(\sec x)^5+C$

24. $\ln|\sin x|+\ln|\cos x|+C$ 25. $x(t)=\frac{1}{16}t^4-t^2+t+4$ 26. 7.128 or 7.129

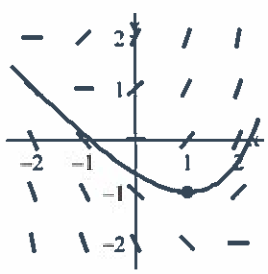
27. a. Do.: all reals, V.A.: none, H.A.: y = 0, x-int.: (0,0) odd, y-int.: (0,0)

b. rel. min. pt. $\left(-1,-\frac{1}{2}\right)$, rel. max. pt. $\left(1,\frac{1}{2}\right)$ c. P.I. $(0,0)$, $\left(\sqrt{3},\frac{\sqrt{3}}{4}\right)$, $\left(-\sqrt{3},\frac{-\sqrt{3}}{4}\right)$

Assignment 7.4 page 184

1. $y=\frac{1}{4}x^3+\frac{1}{4}x^2+2$ 2. $y=\sqrt{x^2-16}$ 3. $y=4e^{x^3}+1$ 4. B

5,6.



7. D 8. Concave upward since $y''|_{(2,6)}=\frac{1}{2}>0$

9. $H(t)=\sqrt[3]{t^3+35}$ 10. B

11. a. $R(t)=2{,}000{,}000e^{-.085t}$ b. 8.141 yrs c. 27.045 *yrs* (early in 2027)

12. $\int_0^6 R(t)\,dt$ gives the total amount of coal (in tons) produced by the mine from the beginning of 2000 through 2005.

13. a. 9,386,000 tons b. 9,428,000 tons 14. 9,397,000 tons 15. B 16. B

17. $\frac{4}{\pi}$ 18. DNE 19. DNE or ∞ 20. 0 21. 5 22. $3\sin(9x^2)-2\sin(4x^2)$

23. 388.130 24. a. max at x = -1, min at x = 3, PI at x = 1

24. b. max at x = -1, min at x = 1, PI at x = 0, $\frac{-1}{\sqrt{2}},\frac{1}{\sqrt{2}}$ 25. Length 150, width 75

26. $y' = \dfrac{12x^2+3y}{2y-3x+2}$ 27. a. $\ln\left(e^x+5\right)+C$ b. $\dfrac{(\ln x+3)^2}{2}+C$

27. c. $\dfrac{x^2}{2}-x-2\ln|x-1|+C$ 28. 260,456 people 29. 118.920 or 118.921 g

Assignment 8.1 page 189

1. 2 2. $\frac{1}{4}$ 3. $\boxed{\frac{1}{3}\left(\left(16-\frac{16}{3}\right)-\left(1-\frac{2}{3}\right)\right)} = \frac{31}{9}$ 4. 1.105 5. 1.683

6. $x=\sqrt[3]{2}$ 7. c = –.751 8. a. 8 b. $\frac{4}{3}$

9. a. At 8:00 AM water is flowing from the reservoir at the rate of 1000 $\frac{gal}{hr}$.
b. 22000 *gal* e. 33000 *gal*

10. b. approximately 11000 c. Approximately 11,000 babies were born in the Utah city during 1992, 1993, 1994, and 1995.

11. c. $D(3.1) \approx 29.4\frac{hikers}{mile}$ 12. c. $v(4.5)=30\frac{mi}{hr}$ e. 100 miles f. $v_{avg}=20\frac{mi}{hr}$

13. a. $\frac{20}{3}\frac{ft}{sec^2}$ d. $35\frac{ft}{sec}$ e. $55\frac{ft}{sec}$ f. $52.5\frac{ft}{sec}$

14. a. 15.527 liters b. 595.531 or 595.532 liters

15. b. $\boxed{\frac{1}{2}\bullet 2\left(0+2^3\right)+\frac{1}{2}\bullet 2\left(2^3+4^3\right)+\frac{1}{2}\bullet 2\left(4^3+6^3\right)+\frac{1}{2}\bullet 2\left(6^3+8^3\right)} = 1088$

16. a. $\boxed{4\left(-\sqrt{2}-\sqrt{6}-\sqrt{10}\right)} = -28.103$ or -28.104 17. a. 300 ft. b. $5\frac{ft}{sec}$

Assignment 8.2 page 193

2. area $=\int_0^3 \left(g(x)-f(x)\right)dx$ 3. $\frac{4}{3}$ 4. 6.356 5. 1.167 or 1.168

6. $\boxed{\left(-1-\frac{1}{3}+3\right)-\left(-9+9-9\right)} = \frac{32}{3}$ 8. 7.211 9. $f(x)=\dfrac{1}{x^3-\frac{8}{9}}=\dfrac{x^2}{9x^3-8}$

10. D 11. a. $f'(1)=-1$, $f'(2)=-\frac{3}{2}$ 12. $c=\frac{1}{\ln 2}=1.442$ or 1.443

13. –2.613 or -2.614 14. $-1+\ln\frac{1}{2}$ 15. $f(.9)\approx -1.9$

16. $\boxed{\frac{1}{2}\bullet 1\left[(-2+\ln 1)+2(-4+\ln 2)+2(-6+\ln 3)+(-8+\ln 4)\right]} = -12.515$

17. a. $f'(x)=g(x)$ c. f is increasing on the interval $[-5,6]$ because $g\geq 0$.
f. f has points of inflection at $x=-3,2,5$ because g changes between incr. and decr.
g. $\frac{39}{2}-\frac{9}{2}\pi$

19. b. $\boxed{\frac{1}{2}\bullet 50(250+210)+\frac{1}{2}\bullet 100(210+190)+\frac{1}{2}\bullet 150(190+80)+\frac{1}{2}\bullet 200(80+0)}=59750$

21. $f'(x)=10\left(x-\sqrt{x}\right)^9\left(1-\frac{1}{2}x^{-\frac{1}{2}}\right)$ 22. $y=\frac{2}{5}x^{\frac{5}{2}}-\frac{8}{3}x^{\frac{3}{2}}+8x^{\frac{1}{2}}+C$

23. $f(x)=\frac{1}{12}\left(x^2+3\right)^6+C$ 24. $-\frac{1}{3}\sqrt{4-3x^2}+C$ 27. $\cos(2x+1)-3x^2\cos\left(2x^3+1\right)$

Assignment 8.3 page 197

1. a. $V=\int_0^1\left(x^2\right)^2dx$ b. $V=\int_0^1\frac{1}{2}\pi\left(\frac{1}{2}x^2\right)^2dx$ c. $V=\int_0^1 2x^2\,dx$

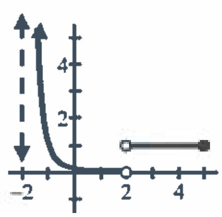
2. a. $V=\int_0^1\left(-x^2+x\right)^2dx$ 3. a. $V=\int_{-2}^2\left(2\sqrt{4-y^2}\right)^2dy$ 4. a. $V=\int_{-2}^1\left(2-x-x^2\right)^2dx$

5. 8 6. $\frac{9}{2}\pi$ 7. $\boxed{\left(1-\frac{1}{12}-\frac{3}{8}\right)-\left(-4+\frac{16}{3}-6\right)}=\frac{125}{24}$ 9. 12 10. $\frac{e^4-1}{4}$

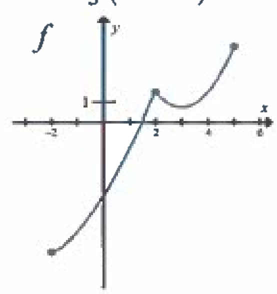
11. 1.049 or 1.050 12. 2.492 13. $g'(y)=\frac{4}{3}y^3(2y-1)^{-\frac{1}{3}}+3y^2(2y-1)^{\frac{2}{3}}$

15. $g(y)=4\sqrt{y^2+3}+C$ 16. $h(t)=\frac{2}{3}\left(\sqrt{t+3}\right)^3-6\sqrt{t+3}+C$ 17. 3 18. $\frac{1}{2}$

20. a. f''



b. f



21. $y-2=8(x-1)$

Assignment 8.4 page 199

1. a. $\pi\int_{-1}^1\left(\sqrt{1-x^2}\right)^2dx$ b. $\pi\int_{-1}^1\left(\left(1+\sqrt{1-x^2}\right)^2-1^2\right)dx$ 2. b. $\pi\int_{-2}^2\left(2^2-\left(2-\sqrt[4]{x+2}\right)^2\right)dx$

3. a. $\pi\int_0^2\left((2x)^2-\left(x^2\right)^2\right)dx$ b. $\pi\int_0^2\left(\left(5-x^2\right)^2-(5-2x)^2\right)dx$

4. b. $\pi\int_1^4\left(\left(1+\frac{1}{\sqrt{y}}\right)^2-1^2\right)dy$ 5. b. $\pi\int_1^4\left(\left(3+\frac{1}{x}\right)^2-3^2\right)dx$ c. $\pi\int_1^4\left(5^2-\left(5-\frac{1}{x}\right)^2\right)dx$

6. a. $\pi\int_0^2\left(y^2\right)^2dy$ b. $\pi\int_0^2\left(\left(2+y^2\right)^2-2^2\right)dy$ c. $\pi\int_0^2\left(4^2-\left(4-y^2\right)^2\right)dy$

7. a. $\pi\int_{-1}^1\left(\left(-\left(x^2-2\right)\right)^2-1^2\right)dx$ b. $\pi\int_{-1}^1\left(\left(1-\left(x^2-2\right)\right)^2-2^2\right)dx$

8. a. $\pi\left(32-\frac{3}{5}\bullet 8^{\frac{5}{3}}\right)$ b. $\pi\frac{2^7}{7}$ 9. a. 16.302 or 16.303 b. 28.688 or 28.689

10. b. $\frac{1}{2}\pi\int_{-2}^{3}\left(\frac{|x|}{2}\right)^2 dx$ 11. b. x-int.: $(0,0)$, $(4,0)$ d. range: $y\leq 1$

14. $\boxed{\left(\frac{4}{3}\bullet 8-\frac{1}{2}\bullet 16\right)-0}=\frac{8}{3}$ 15. $\frac{2}{3}$ 16. $c=.178$ or $.179$, 2.488 18. 0 20. c = 1

21. a. 1.520 b. 4.036 or 4.037 22. $(0,3)$ or $\left(2,\frac{11}{3}\right)$

Assignment 8.5 page 203

1. $D_{avg}=\dfrac{\int_1^5 D(x)\,dx}{5-1}$ 2. $D_{avg}\approx 30\ \frac{hikers}{mile}$ 3. $D_{avg}\approx 30\ \frac{hikers}{mile}$

4. approx. 3 miles from the trailhead 5. $\int_1^5 D(x)\,dx$ represents the total number of hikers on the trail at a distance between 1 and 5 miles from the trailhead.

6. $A=\int_{-1}^{2}\left(\left(-x^2+1\right)-\left(x^2-2x-3\right)\right)dx$ 7. $A=\frac{16}{3}$ 8. V = 8

9. a. $V=\pi\int_2^6\left(\sqrt{x-2}\right)^2 dx$ b. $V=\pi\int_2^6\left(4^2-\left(4-\sqrt{x-2}\right)^2\right)dx$

10. $V=\int_1^4\left(\frac{1}{\sqrt{x}}+3\right)\bullet 3\left(\frac{1}{\sqrt{x}}+3\right)dx=121.158$ or 121.159

11. a. $a(t)=-\frac{1}{2}(t+4)^{-\frac{1}{2}}$ b. $y(t)=-\frac{2}{3}(t+4)^{\frac{3}{2}}+2$ c. -16 d. -3 e. $-\frac{1}{6}$ f. 3

12. a. $v(2.3)=.206$ or $.207$ b. $a(2.3)=.007$ or $.008$ c. disp. = $-.097$ or $-.098$
d. T. D. = .779 or .780 or .781

13. larger 14. smaller 15. They are equal. 16. $x=2,-3$ (nonrem.)

17. $t=3$ (nonrem.) 18. $x=2$ (rem.), $x=-3$ (nonrem.) 19. $\frac{1}{5}$ 20. $a=-\frac{1}{2}$, $b=-1$

21. $a=2$, $b=3$ 22. $y''=6x^{-3}+\frac{1}{2}x^{-\frac{3}{2}}$ 23. $y'=\dfrac{3x-2y}{2y+2x}$ 24. $(2,-2)$, $(-2,2)$

25. $y=\frac{3}{2}\left(x+\sqrt{\frac{20}{3}}\right)$

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