Math 337 - Elementary Differential Equations

Lecture Notes – Introduction to Differential Equations

Joseph M. Mahaffy,
⟨jmahaffy@sdsu.edu⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://jmahaffy.sdsu.edu

Spring 2020

Outline

Contact Information

Professor Joseph Mahaffy

Office	GMCS-593
Email	jmahaffy@sdsu.edu
Web	http://jmahaffy.sdsu.edu
Phone	(619)594-3743
Office Hours	TBA and by Appointment

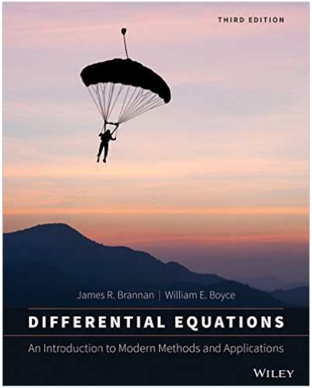
Basic Information: Text/Topics

Text: The text is optional and old editions are fine.

Brannan and Boyce: Differential Equations:
An Introduction to Modern Methods and Applications.

Wiley 2015.

ISBN 978-1-118-53177-8



Lecture Notes available at Bookstore

Basic Information: Text/Topics

1. Introductory Definitions
2. Qualitative Methods and Direction Fields
3. Linear Equations
4. Separable Equations
5. Exact and Bernoulli Equation
6. Existence and Uniquess
7. Numerical Methods
8. 2D Linear Systems
9. Second Order Differential Equations
10. Laplace Transforms
11. Power Series

Other Differential Equation Courses

Differential Equations and Dynamical Systems: Several courses extend the material from this class. Courses from the Nonlinear Dynamical Systems Group.

- **Math 531:** ***Partial Differential Equations***
- **Math 537:** ***Ordinary Differential Equations***
- **Math 538:** ***Discrete Dynamical Systems and Chaos***
- **Math 542:** ***Introduction to Computational Ordinary Differential Equations***

Basic Information: Grading

Approximate Grading

Homework, including WeBWorK	33%
Homework Quizzes	7%
3 Exams	36%
Final	24%

- Homework includes electronic HW with WeBWorK and written problems (often inside WW problems). Critical to **keep up** on HW after each lecture.
- Exams are based heavily on HW problems and examples from lectures.
- Final: Friday, May 8, 13:00 – 15:00

Expectations and Procedures, I

- Most class attendance is OPTIONAL — Homework and announcements will be posted on the class web page. If/when you attend class:
    - Please be on time.
    - Please pay attention.
    - Please turn off mobile phones.
    - Please be courteous to other students and the instructor.
    - Abide by university statutes, and all applicable local, state, and federal laws.

Expectations and Procedures, II

- Please, turn in assignments on time. (The instructor reserves the right not to accept late assignments, and there is a maximum of **2** extensions of WeBWorK during the semester.)
- The instructor will make special arrangements for students with documented learning disabilities and will `try` to make accommodations for other unforeseen circumstances, *e.g.* illness, personal/family crises, etc. in a way that is fair to all students enrolled in the class. ***Please contact the instructor EARLY regarding special circumstances.***
- Students are expected ***and encouraged*** to ask questions in class!
- Students are expected ***and encouraged*** to to make use of office hours! If you cannot make it to the scheduled office hours: contact the instructor to schedule an appointment!

Expectations and Procedures, III

- Missed midterm exams: Don't miss exams! The instructor reserves the right to schedule make-up exams, modify the type and nature of this make-up, and/or base the grade solely on other work (including the final exam).
- Missed final exam: Don't miss the final! Contact the instructor ASAP or a grade of incomplete or F will be assigned.
- ***Academic honesty***: Submit your own work. Any cheating will be reported to University authorities and a **ZERO** will be given for that HW assignment or Exam.

MatLab

- Students can obtain **MatLab** from EDORAS Academic Computing.
- Google **SDSU MatLab** or access **https://edoras.sdsu.edu/ download/matlab.html**.
- **MatLab** and **Maple** can also be accessed in the **Computer Labs GMCS 421**, **422**, **425**, and the Library.
- A discounted student version of **Maple** is available.

Math 337: Formal Prerequisites

`Math 254` or `Math 342A` or `AE 280` (Soon will not be allowed for students with credit in AE 280.)

- These courses all require **Calculus 151**.
  - Assume good knowledge of ***differentiation*** and ***integration***.
  - Understand series techniques (especially ***Taylor's Theorem***)
  - Recall ***Partial Fractions Decomposition***.
- These courses all have sections on basic **Linear Algebra**.

Introduction

Introduction

- Differential equations frequently arise in modeling situations
- They describe population growth, chemical reactions, heat exchange, motion, and many other applications
- Differential equations are continuous analogs of discrete dynamical systems

Malthusian Growth 1

Discrete Malthusian Growth Model:

- Let the initial population, $P(t_0) = P_0$
- Define $t_n = t_0 + n\Delta t$ and $P_n = P(t_n)$
- Let r be the per capita growth rate per unit time
- The Discrete Malthusian Growth Model satisfies:

$$P_{n+1} = P_n + r\Delta t P_n = (1 + r\Delta t)P_n$$

- New population = Old population + per capita growth rate × length of time × Old population

Malthusian Growth 2

Discrete Malthusian Growth: $P_{n+1} = (1 + r\Delta t)P_n$, so

$$\begin{aligned} P_1 &= (1 + r\Delta t)P_0 \\ P_2 &= (1 + r\Delta t)P_1 = (1 + r\Delta t)^2 P_0 \\ P_3 &= (1 + r\Delta t)P_2 = (1 + r\Delta t)^3 P_0 \\ &\vdots \\ P_n &= (1 + r\Delta t)P_{n-1} = (1 + r\Delta t)^n P_0 \end{aligned}$$

The solution of this discrete model is

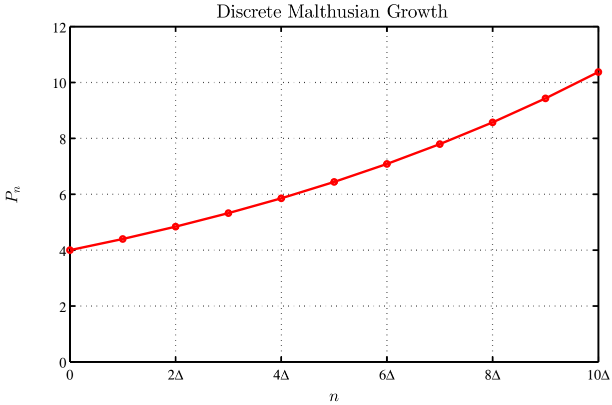
$$P_n = (1 + r\Delta t)^n P_0,$$

which is an exponential growth

Malthusian Growth

Discrete Malthusian Growth:

$$P_{n+1} = (1 + 0.1\Delta t)P_n \qquad P_0 = 4$$



Malthusian Growth

Malthusian Growth: Let $P(t)$ be the population at time $t = t_0 + n\Delta t$ and rearrange the model above

$$\begin{aligned} P_{n+1} - P_n &= r\Delta t P_n \\ P(t+\Delta t) - P(t) &= \Delta t \cdot rP(t) \\ \frac{P(t+\Delta t) - P(t)}{\Delta t} &= rP(t) \end{aligned}$$

Let Δt become very small

$$\lim_{\Delta t \to 0} \frac{P(t+\Delta t) - P(t)}{\Delta t} = \frac{dP(t)}{dt} = rP(t),$$

which is a **Differential Equation**

Malthusian Growth

Solution of Malthusian Growth Model: The Malthusian growth model

$$\frac{dP(t)}{dt} = rP(t)$$

- The rate of change of a population is proportional to the population
- Let c be an arbitrary constant, so try a solution of the form

$$P(t) = ce^{kt}$$

- Differentiating

$$\frac{dP(t)}{dt} = cke^{kt},$$

which if $k = r$ is $rP(t)$, so satisfies the differential equation

Malthusian Growth

Solution of Malthusian Growth Model The Malthusian growth model satisfies

$$P(t) = ce^{rt}$$

- With the initial condition, $P(t_0) = P_0$, then the unique solution is
$$P(t) = P_0 e^{r(t-t_0)}$$
- Malthusian growth is often called exponential growth

Example: Malthusian Growth 1

Example: Malthusian Growth Consider the Malthusian growth model

$$\frac{dP(t)}{dt} = 0.02\,P(t) \quad \text{with} \quad P(0) = 100$$

Skip Example

- Find the solution
- Determine how long it takes for this population to double

Example: Malthusian Growth 2

Solution: The solution is given by

$$P(t) = 100\, e^{0.02t}$$

Since $P(0) = 100$, satisfying the initial condition, then by computing

$$\frac{dP}{dt} = 0.02(100\, e^{0.02t}) = 0.02\, P(t),$$

we find that this solution satisfies the differential equation

The population doubles when

$$200 = 100\, e^{0.02t}$$

$$0.02t = \ln(2) \qquad \text{or} \qquad t = 50 \ln(2) \approx 34.66$$

What is a Differential Equation?

What is a Differential Equation?

Definition (Differential Equation)

An equation that contains derivatives of one or more unknown functions with respect to one or more independent variables is said to be a **differential equation**.

- The classical example is Newton's Law of motion
  - The mass of an object times its acceleration is equal to the sum of the forces acting on that object
  - Acceleration is the first derivative of velocity or the second derivative of position
- In biology, a differential equation describes a growth rate, a reaction rate, or the change in some physiological state

Types of Differential Equations

- This course considers **Ordinary Differential Equations**, where the **unknown function and its derivatives** depend on a single **independent variable**
- Mathematical physics often needs **Partial Differential Equations**, where the **unknown function and its derivatives** depend on two or more **independent variables**
  - **Example: Heat Equation**

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2}$$

- This course also examines some **Systems of Ordinary Differential Equations**, where there are several interacting **unknown functions and their derivatives** each depending on a single **independent variable**

Classification

Definition (Order)

The **order** of a **differential equation** matches the order of the highest derivative that appears in the equation.

Definition (Linear Differential Equation)

An n^{th} order ordinary differential equation $F(t, y, y', ..., y^{(n)}) = 0$ is said to be **linear** if it can be written in the form

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + ... + a_n(t)y = g(t).$$

The functions a_0, a_1, ... a_n, called the **coefficients** of the equation, can depend at most on the independent variable t. This equation is said to be **homogeneous** if the function $g(t)$ is zero for all t. Otherwise, the equation is **nonhomogeneous**.

Applications of Differential Equations 1

Radioactive Decay: Let $R(t)$ be the amount of a radioactive substance

- Radioactive elements transition through decay into another state at a rate proportional to the amount of radioactive material present
- The differential equation is

$$\frac{dR(t)}{dt} = -k\,R(t) \quad \text{with} \quad R(0) = R_0$$

- This is a **first order**, **linear**, **homogeneous differential equation**
- Like the Malthusian growth model, this has an exponential solution

$$R(t) = R_0 e^{-kt}$$

Applications of Differential Equations 2

Harmonic Oscillator: A Hooke's law spring exerts a force that is proportional to the displacement of the spring

- Newton's law of motion: Mass times the acceleration equals the force acting on the mass
- The simplest spring-mass problem is

$$my'' = -cy \qquad \text{or} \qquad y'' + k^2 y = 0$$

- This is a **second order**, **linear**, **homogeneous differential equation**
- The general solution is

$$y(t) = c_1 \cos(kt) + c_2 \sin(kt),$$

where c_1 and c_2 are arbitrary constants

Applications of Differential Equations 3

Swinging Pendulum: A pendulum is a mass attached at one point so that it swings freely under the influence of gravity

- Newton's law of motion (ignoring resistance) gives the differential equation

$$my'' + g\sin(y) = 0$$

- The variable y is the angle of the pendulum, m is the mass of the bob of the pendulum, and g is the gravitational constant
- This is a **second order**, **nonlinear**, **homogeneous differential equation**
- This problem does not have an easily expressible solution

Applications of Differential Equations

Logistic Growth: Most populations are limited by food, space, or waste build-up, thus, cannot continue to grow according to Malthusian growth

- The Logistic growth model has a Malthusian growth term and a term limiting growth due to crowding
- The differential equation is

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{M}\right)$$

- P is the population, r is the Malthusian rate of growth, and M is the carrying capacity of the population
- This is a **first order**, **nonlinear**, **homogeneous differential equation**
- We solve this problem later in the semester

Applications of Differential Equations

The van der Pol Oscillator: In electrical circuits, diodes show a rapid rise in current, leveling of the current, then a steep decline

- Biological applications include a similar approximation for nerve impulses
- The van der Pol Oscillator satisfies the differential equation

$$v'' + a(v^2 - 1)v' + v = b$$

- v is the voltage of the system, and a and b are constants
- This is a **second order**, **nonlinear**, **nonhomogeneous differential equation**
- This problem does not have an easily expressible solution, but shows interesting oscillations

Applications of Differential Equations 6

Lotka-Volterra – Predator and Prey Model: Model for studying the dynamics of predator and prey interacting populations

- Model for the population dynamics when one predator species and one prey species are tightly interconnected in an ecosystem
- System of differential equations

$$\begin{aligned} x' &= a\,x - b\,xy \\ y' &= -c\,y + d\,xy \end{aligned}$$

- x is the prey species, and y is the predator species
- This is a **system of first order**, **nonlinear**, **homogeneous differential equations**
- No explicit solution, but we'll study its behavior

Applications of Differential Equations

Forced Spring-Mass Problem with Damping: An extension of the spring-mass problem that includes viscous-damping caused by resistance to the motion and an external forcing function that is applied to the mass

- The model is given by

$$my'' + cy' + ky = F(t)$$

- y is the position of the mass, m is the mass of the object, c is the damping coefficient, k is the spring constant, $F(t)$ is an externally applied force
- This is a **second order**, **linear**, **nonhomogeneous differential equation**
- We'll learn techniques for solving this

Damped Spring-Mass Problem 1

Damped Spring-Mass Problem: Assume a mass attached to a spring with resistance satisfies the second order linear differential equation

$$y''(t) + 2y'(t) + 5y(t) = 0$$

Skip Example

Show that one solution to this differential equation is

$$y_1(t) = 2\,e^{-t}\sin(2t)$$

Damped Spring-Mass Problem 2

Solution: Damped spring-mass problem

- The 1^{st} derivative of $y_1(t) = 2\,e^{-t}\sin(2t)$

$$y_1'(t) = 2e^{-t}(2\cos(2t)) - 2e^{-t}\sin(2t) = 2e^{-t}(2\cos(2t) - \sin(2t))$$

- The 2^{nd} derivative of $y_1(t) = 2\,e^{-t}\sin(2t)$

$$\begin{aligned} y_1''(t) &= 2e^{-t}(-4\sin(2t) - 2\cos(2t)) - 2e^{-t}(2\cos(2t) - \sin(2t)) \\ &= -2e^{-t}(4\cos(2t) + 3\sin(2t)) \end{aligned}$$

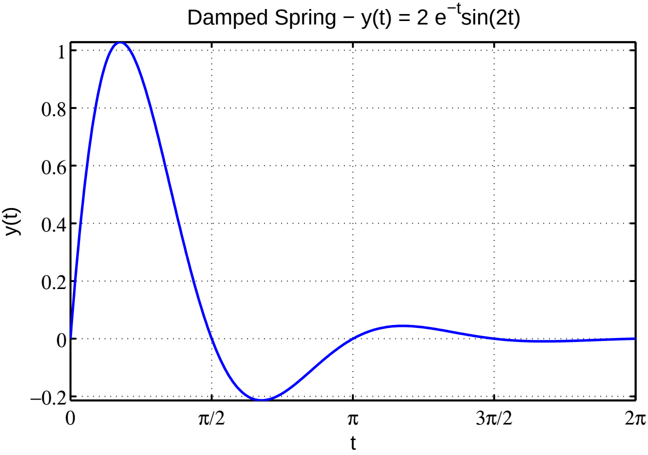
- Substitute into the spring-mass problem

$$\begin{aligned} y_1'' + 2y_1' + 5y &= -2e^{-t}(4\cos(2t) + 3\sin(2t)) \\ &\quad +2(2e^{-t}(2\cos(2t) - \sin(2t))) + 5(2e^{-t}\sin(2t)) \\ &= 0 \end{aligned}$$

It is often **easy** to check that a solution satisfies a differential equation.

Damped Spring-Mass Problem 3

Graph of Damped Oscillator



Initial Value Problem

Definition (Initial Value Problem)

An initial value problem for an n^{th} order differential equation

$$y^{(n)} = f(t, y, y', y'', ..., y^{(n-1)})$$

on an interval I consists of this differential equation together with n initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_1, \quad ..., \quad y^{(n-1)}(t_0) = y_{n-1}$$

prescribed at a point $t_0 \in I$, where y_0, y_1, ..., y_{n-1} are given constants.

Under reasonable conditions the solution of an **Initial Value Problem** has a unique solution.

Evaporation Example 1

Evaporation Example: Animals lose moisture proportional to their surface area

Skip Example

- If $V(t)$ is the volume of water in the animal, then the moisture loss satisfies the differential equation

$$\frac{dV}{dt} = -0.03\, V^{2/3}, \qquad V(0) = 8 \text{ cm}^3$$

- The initial amount of water is 8 cm^3 with t in days
- Verify the solution is

$$V(t) = (2 - 0.01t)^3$$

- Determine when the animal becomes totally desiccated according to this model
- Graph the solution

Evaporation Example 2

Solution: Show $V(t) = (2 - 0.01t)^3$ satisfies

$$\frac{dV}{dt} = -0.03\,V^{2/3}, \qquad V(0) = 8 \text{ cm}^3$$

- $V(0) = (2 - 0.01(0))^3 = 8$, so satisfies the initial condition
- Differentiate $V(t)$,

$$\frac{dV}{dt} = 3(2 - 0.01t)^2(-0.01) = -0.03(2 - 0.01t)^2$$

- But $V^{2/3}(t) = (2 - 0.01t)^2$, so

$$\frac{dV}{dt} = -0.03\,V^{2/3}$$

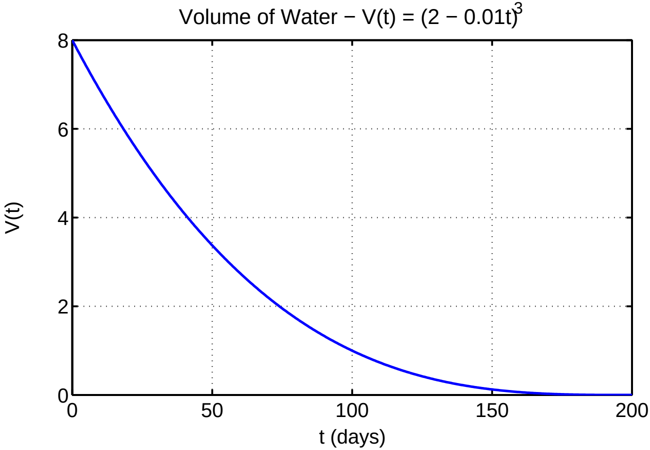
Evaporation Example

Solution (cont): Find the time of total desiccation

- Must solve
$$V(t) = (2 - 0.01t)^3 = 0$$
- Thus,
$$2 - 0.01t = 0 \qquad \text{or} \qquad t = 200$$
- It takes 200 days for complete desiccation

Evaporation Example

Graph of Desiccation



Nonautonomous Example 1

Nonautonomous Example: Consider the nonautonomous differential equation with initial condition (**Initial Value Problem**):

$$\frac{dy}{dt} = -ty^2, \qquad y(0) = 2$$

- Show that the solution to this differential equation, including the initial condition, is

$$y(t) = \frac{2}{t^2 + 1}$$

- Graph of the solution

Nonautonomous Example 2

Solution: Consider the solution

$$y(t) = \frac{2}{t^2+1} = 2(t^2+1)^{-1}$$

- The initial condition is

$$y(0) = \frac{2}{0^2+1} = 2$$

- Differentiate $y(t)$,

$$\frac{dy}{dt} = -2(t^2+1)^{-2}(2t) = -4t(t^2+1)^{-2}$$

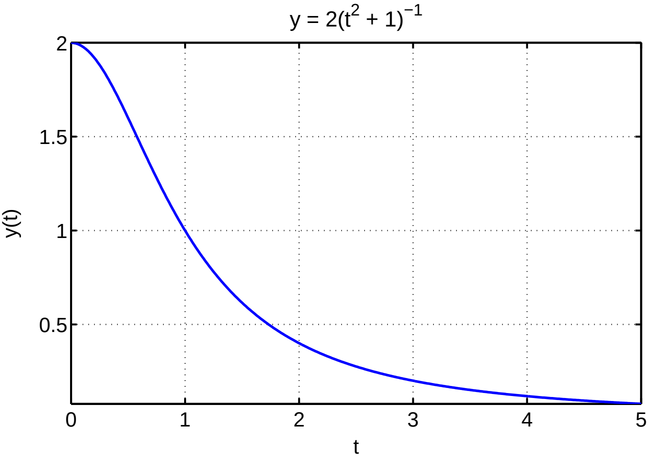
- However,

$$-ty^2 = -t(2(t^2+1)^{-1})^2 = -4t(t^2+1)^{-2}$$

- Thus, the differential equation is satisfied

Nonautonomous Example

Solution of Nonautonomous Differentiation Equation



Introduction to Maple 1

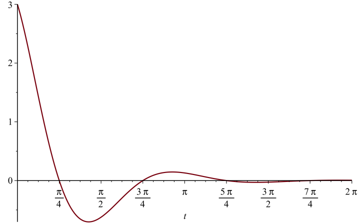
Introduction to Maple: A Symbolic Math Program

We enter a function $y(t) = 3e^{-t}\cos(2t)$,

$y := t \to 3 \cdot \exp(-t) \cdot \cos(2 \cdot t);$

The arrow is $-$ and $>$ and multiplication is $*$. To plot this function

$plot(y(t), t = 0..2 \cdot \text{Pi});$



Introduction to Maple 2

We have the function: $y(t) = 3e^{-t}\cos(2t)$,

This can be differentiated (and stored in variable dy) by typing

$dy := diff(y(t), t);$

Maple gives:

$$dy := -3e^{-t}\cos(2t) - 6e^{-t}\sin(2t)$$

The absolute minimum and a relative maximum are found with Maple:

$tmin := fsolve(dy = 0, t = 1..2); \quad y(tmin);$
$tmax := fsolve(dy = 0, t = 2.5..3.5); \quad y(tmax);$

The result was an **absolute minimum** at $(1.33897, -0.703328)$.
The result was a **relative maximum** at $(2.90977, 0.1462075)$.

Introduction to Maple

With $y(t) = 3e^{-t}\cos(2t)$, we can solve

$$\int 3e^{-t}\cos(2t)dt \qquad \text{and} \qquad \int_0^5 3e^{-t}\cos(2t)dt$$

These can be integrated by typing

$int(y(t), t); \quad int(y(t), t = 0..5); \quad evalf(\%);$

For the indefinite integral, Maple gives:

$$-\frac{3}{5}e^{-t}\cos(2t) + \frac{6}{5}e^{-t}\sin(2t)$$

For the definite integral, Maple gives:

$$\frac{3}{5} - \frac{3}{5}e^{-5}\cos(10) + \frac{6}{5}e^{-5}\sin(10) \quad = \quad 0.59899347$$

Introduction to Maple

Show $y(t) = 3e^{-t}\cos(2t)$ is a solution of the differential equation

$$y'' + 2y' + 5y = 0.$$

The function and derivatives are entered by

$y := t \rightarrow 3 \cdot \exp(-t) \cdot \cos(2 \cdot t);$
$dy := \mathit{diff}(y(t), t);$
$sdy := \mathit{diff}(y(t), t\$2);$

If we type

$sdy + 2 \cdot dy + 5 \cdot y(t);$

Maple gives **0**, which verifies this is a solution.

Introduction to Maple

Maple finds the general solution to the differential equation

$de := diff(Y(t), t\$2) + 2 \cdot diff(Y(t), t) + 5 \cdot Y(t) = 0;$
$dsolve(de, Y(t));$

Maple produces

$$Y(t) = C_1 e^{-t} \sin(2t) + C_2 e^{-t} \cos(2t)$$

To solve an initial value problem, say $Y(0) = 2$ and $Y'(0) = -1$, enter

$dsolve(\{de, Y(0) = 2, D(Y)(0) = -1\}, Y(t));$

Maple produces

$$Y(t) = \frac{1}{2} e^{-t} \sin(2t) + 2e^{-t} \cos(2t),$$

which is made into a useable function by typing

$Y := unapply(rhs(\%), t);$