MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #10

Chapter 3 Phase Portraits for Planar Systems
Changing Coordinates

Instructor: Dr. Bo-Wen Shen*

Department of Mathematics and Statistics
San Diego State University

Changing Coordinates

Despite differences in the associated phase portraits, we really have dealt with only three type of matrices in these past four sections:

diagonalization
$$\boxed{\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}}, \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

(I) (II) (III)

- Any 2×2 matrix that is in one of these three forms is said to be in canonical form.
- Given any linear system $X' = AX$, we can always "change coordinates" so that the new system's coefficient matrix is in canonical form

A Summary for the Three Cases

TBD

Goal: Solve the following 2D system

$$\boxed{X' = AX}$$

Compute the eigenvalues and eigenvectors of A,

U_1 and U_2

$$AU_j = \lambda_j U_j, \qquad j = 1,2$$

Construct $T = (V_1, V_2)$, $B = T^{-1}AT$ and $Y = TX$ using the following

(I) real eigenvalues

$$(V_1, V_2) = (U_1, U_2)$$

$$B = T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

(II) complex eigenvalues

$$(V_1, V_2) = (Re(U_1), Im(U_1))$$

$$B = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

(III) repeated eigenvalues

$$V_1 = U_1$$
$$(A - \lambda I)V_2 = V_1$$

$$B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$\boxed{Y' = BY}$$

A Linear Map and Matrix

A linear map (or *linear transformation*) on $\mathbb{R}^2$ is a function $T : \mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

That is, T simply multiplies any vector by the 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We will thus think of the linear map and its matrix as being interchangeable, so that we also write

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Invertible Matrix and Its Inverse Matrix

Proposition. T *he* 2×2 *matrix* T *is invertible if and only if* $\det T \neq 0$. $\square$

the matrix

$$S = \frac{1}{\det T} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

serves as T^{-1} if $\det T \neq 0$. If $\det T = 0$, we know from Chapter 2 that there are

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

A Summary for Real Eigenvalues

Goal: Solve the following 2D system

$$\boxed{X' = AX}$$

Compute the eigenvalues and eigenvectors of A, V_1 and V_2

$$AV_j = \lambda_j V_j, \qquad j = 1,2$$

Construct T using V_1 and V_2

$$T = (V_1, \mathrm{V}_2)$$

$$X = TY$$

$$= (V_1, \mathrm{V}_2)\begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}$$

$$= e^{\lambda_1 t}\, V_1 + c_2 e^{\lambda_2 t} V_2$$

$$\boxed{Y' = DY} \qquad D = T^{-1}AT$$

$$Y' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} Y$$

$$Y = c_1 e^{\lambda_1 t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda_2 t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}$$

Solve a Linear System by Changing Coordinates

$$X' = AX$$

Find T $\quad X = TY \quad$ T: converts Y into X

so that Y is a solution to the following ODE

$$\boxed{Y' = (T^{-1}AT)Y}$$

Verify

$$(TY)' = TY' = T(T^{-1}AT)Y = ATY$$

TY is a solution to $X' = AX$

(T is not a function of time)

- We think of T as a change of coordinates
- The linear map T converts solutions of $Y' = (T^{-1}AT)Y$ to the solutions of $X' = AX$
- Alternatively, T^{-1} takes the solutions of $X' = AX$ to the solutions of $Y' = (T^{-1}AT)Y$.

Linearly Conjugate

Proposition: If L_1 and L_2 are linearly conjugate,

$$L_1(x) = A_1 x \text{ and } L_2(x) = A_2 x$$

Then, A_1 and A_2 have the same eigenvalues.

Definition: Let L_1 and L_2 be linear maps of R^n. Let L_1 and L_2 are linearly conjugate if there is an invertible linear map P such that

$$L_1 = P^{-1} \circ L_2 \circ P \qquad e.g., D = T^{-1} A T$$

How to find P (or T)? Construct T using eigenvectors of A:

$$P = [V_1, V_2 \ldots, V_n], \ \mathrm{V}_j \text{ are eigenvectors.}$$

Diagonalization (for Real Eigenvalues)

Example. (Real Eigenvalues) Suppose the matrix A has two real, distinct eigenvalues λ_1 and λ_2 with associated eigenvectors V_1 and V_2. Let T be the matrix with columns V_1 and V_2. Thus, $TE_j = V_j$ for $j = 1, 2$ where the E_j form the standard basis of $\mathbb{R}^2$. Also, $T^{-1}V_j = E_j$. Therefore, we have

$$\begin{aligned}(T^{-1}AT)E_j &= T^{-1}AV_j = T^{-1}(\lambda_j V_j)\\ &= \lambda_j T^{-1}V_j\\ &= \lambda_j E_j.\end{aligned}$$

Thus the matrix $T^{-1}AT$ assumes the canonical form

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and the corresponding system is easy to solve. ■

Discussed in the next slide

the Linear Map T & Diagonalization $T^{-1}AT = \mathrm{D}$

construct $T = [V_1, V_2]$ $\qquad AV_1 = \lambda_1 V_1 \qquad AV_2 = \lambda_2 V_2$

two standard basis vectors $\qquad E_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

obtain $\qquad TE_1 = V_1 \qquad TE_2 = V_2 \qquad T^{-1}V_1 = E_1 \qquad T^{-1}V_2 = E_2$

consider

$$\boxed{(T^{-1}AT)E_j} = (T^{-1}A)TE_j = (T^{-1}A)V_j = (T^{-1})\lambda_j V_j$$

$$= \lambda_j (T^{-1})V_j \boxed{= \lambda_j E_j}$$

$$j = 1 \quad (T^{-1}AT)E_1 = \lambda_1 E_1 = \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix}$$

$$j = 2 \quad (T^{-1}AT)E_2 = \lambda_2 E_2 = \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix}$$

$$\boxed{T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}}$$

diagonalization

Diagonalization $T^{-1}AT = \mathrm{D}$

Construct $T = [V_1, \mathrm{V}_2]$

Show $\boxed{T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}}$

Let $T^{-1}AT = \begin{pmatrix} x & u \\ y & w \end{pmatrix}$ $E_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $E_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Apply $(T^{-1}AT)E_1 = \lambda_1 E_1$

Apply $(T^{-1}AT)E_2 = \lambda_2 E_2$

- Find (x, y) or (u, w)
- Send your results via "chat"
- You have 3 minutes

Diagonalization $T^{-1}AT = \mathrm{D}$

Construct $T = [V_1, \mathrm{V}_2]$ Show $\boxed{T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}}$

Let $T^{-1}AT = \begin{pmatrix} x & u \\ y & w \end{pmatrix}$ $E_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $E_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Apply $(T^{-1}AT)E_1 = \lambda_1 E_1$ $\boxed{\begin{pmatrix} x & u \\ y & w \end{pmatrix} E_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix}}$ $x = \lambda_1 \,\&\, y = 0$

Apply $(T^{-1}AT)E_2 = \lambda_2 E_2$ $\boxed{\begin{pmatrix} x & u \\ y & w \end{pmatrix} E_2 = \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix}}$ $u = 0 \,\&\, w = \lambda_2$

$$\boxed{T^{-1}AT = \begin{pmatrix} x & u \\ y & w \end{pmatrix} = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix} = D}$$

Example

Example. As a further specific example, suppose

$$A = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}.$$

$$x' = -x$$

$$y' = x - 2y$$

eigenvalue problem $|A - \lambda I| = \begin{vmatrix} -1-\lambda & 0 \\ 1 & -2-\lambda \end{vmatrix} = 0$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\lambda = -1, -2$$

Example 1: Find T by Solving $|A - \lambda I| = 0$

Solve for eigenvectors $\quad AV_0 = \lambda V_0 \quad A = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}$

$$-x_0 = \lambda x_0$$
$$x_0 - 2y_0 = \lambda y_0$$

Consider $\lambda = -1$

$$-x_0 = -x_0$$
$$x_0 - 2y_0 = -y_0$$

$$\boxed{x_0 = y_0} \qquad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0 \end{pmatrix} = y_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Obtain $\boxed{V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$ as an eigenvector associated with $\lambda = -1$

Similarly, $\boxed{V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$ as an eigenvector associated with $\lambda = -2$

$$\boxed{X = \alpha X_1 + \beta X_2 = \alpha e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$$X_1 = e^{\lambda_1 t} V_1$$
$$X_2 = e^{\lambda_2 t} V_2$$

Construct $T = [V_1, V_2] = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

Verify $T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$A = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$T^{-1} = \frac{1}{\det(T)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$T^{-1}\mathrm{AT} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Solve $Y' = (T^{-1}AT)Y$ to Obtain Y and X

$$T^{-1}\mathrm{AT} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$Y = \begin{pmatrix} u \\ w \end{pmatrix} \qquad Y' = (T^{-1}AT)Y \qquad \begin{aligned} u' &= -u \\ w' &= -2w \end{aligned}$$

$$E_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$Y = \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} \alpha e^{-t} \\ \beta e^{-2t} \end{pmatrix} = \alpha e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$X = TY = T\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha e^{-t} \\ \beta e^{-2t} \end{pmatrix} = \begin{pmatrix} \alpha e^{-t} \\ \alpha e^{-t} + \beta e^{-2t} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha e^{-t} \\ \alpha e^{-t} \end{pmatrix} + \begin{pmatrix} 0 \\ \beta e^{-2t} \end{pmatrix} = \alpha e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha \mathrm{X}_1 + \beta X_2$$

Compute $Y = T^{-1}X$ to Obtain an Eq. for Y'

$$x' = -x$$

$$y' = x - 2y$$

$$Y = \begin{pmatrix} u \\ w \end{pmatrix} \qquad T^{-1} = \frac{1}{\det(T)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$Y = \begin{pmatrix} u \\ w \end{pmatrix} = T^{-1}X = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -x + y \end{pmatrix}$$

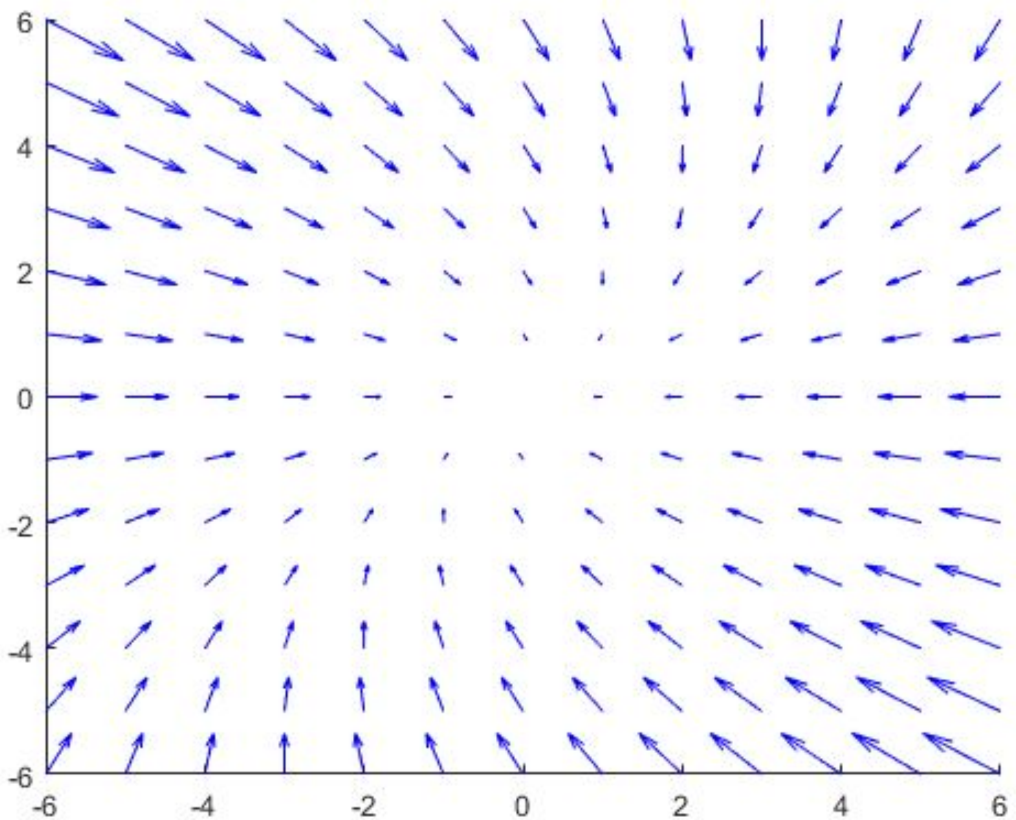
$$u = x$$
$$w = -x + y$$

$$u' = x' = -x = -u$$

$$w' = -x' + y' = x + x - 2y = 2(x - y) = -2w$$

Section 3.4: Changing Coordinates

$X' = 2x + y \qquad Y' = -x + 2y$



MATLAB Plot for Figure 3.6

A Summary for (I) Real Eigenvalues

Goal: Solve the following 2D system

$$X' = AX \qquad X' = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix} X$$

Compute the eigenvalues and eigenvectors of A, V_1 and V_2

$$AV_j = \lambda_j V_j, \qquad j = 1,2$$

Construct T using V_1 and V_2

$$T = (V_1, V_2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$Y' = DY \qquad D = T^{-1}AT$$

$$Y' = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} Y$$

$$X = TY$$

$$= (V_1, V_2)\begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}$$

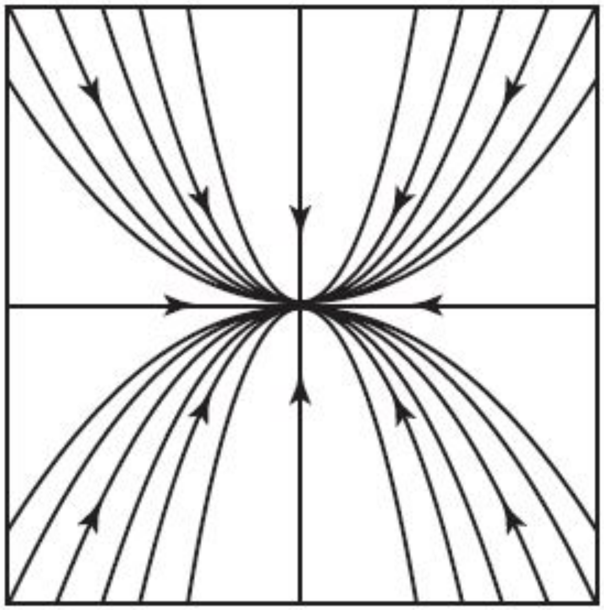
$$= e^{\lambda_1 t}\, V_1 + c_2 e^{\lambda_2 t} V_2$$

$$Y = c_1 e^{\lambda_1 t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda_2 t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

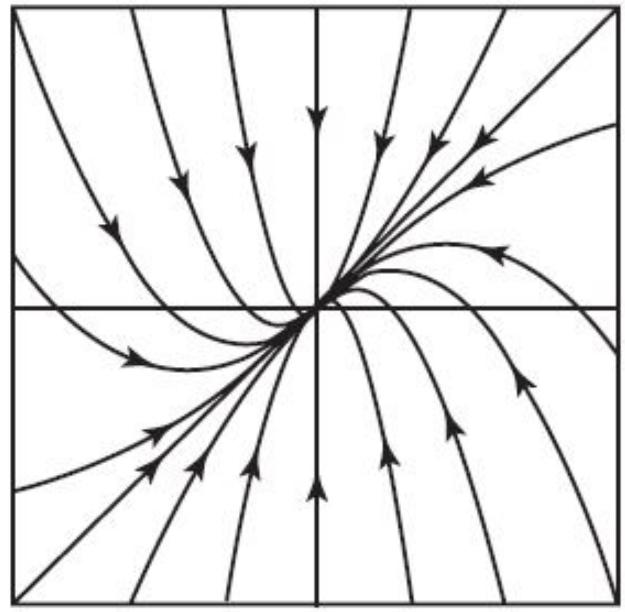
$$= \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}$$

A Summary for (I) Real Eigenvalues

find the solutions to one of the systems in the canonical form Y $\Rightarrow$ $X = TY$



$$T = (V_1, V_2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$



$$Y' = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} Y$$

$$Y' = DY$$

$$AV_j = \lambda_j V_j$$
$$j = 1,2$$

$$X' = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix} X$$

$$X' = AX$$

A Summary for (II) Complex Eigenvalues

Consider $X' = AX$ and A has complex eigenvalues

Goal: Find T so that $Y = TX$ and $Y' = BY$ $\quad B = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$

Let U_j and λ_j represent the eigenvectors and eigenvalues, respectively. Thus, we have $AU_j = \lambda_j U_j$.

without loss of generality, we have $\lambda_1 = \alpha + i\beta$ and $U_1 = V_1 + iV_2$.

V_1 and V_2 are real, $V_1 = \mathrm{Re}(U_1)$ and $V_2 = \mathrm{Im}(U_1)$.

Below, we show that

- V_1 and V_2 are linearly independent
- When $T = (\mathrm{Re}(U_1), \mathrm{Im}(U_1))$, we obtain $B = T^{-1}\mathrm{A}\mathrm{T} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$.

Real Basis Vectors for Complex Eigenvalues

$AU_j = \lambda_j U_j$, $\lambda_1 = \alpha + i\beta$ and $U_1 = V_1 + iV_2$. Show that

- V_1 and V_2 are linearly independent
- In other words, $Re(U_1)$ and $Im(U_1)$ are linearly independent

We assume that V_1 and V_2 are not linearly independent, $V_1 = cV_2$.

Since $AU_1 = \lambda_1 U_1$, we have $A(V_1 + iV_2) = \lambda_1(V_1 + iV_2)$

$$A(cV_2 + iV_2) = (\alpha + i\beta)\,(cV_2 + iV_2)$$

$$(c + i)AV_2 = (\alpha + i\beta)(c + i)V_2\text{c}$$

$$AV_2 = (\alpha + i\beta)V_2$$

This is a contradiction, since the LHS is a complex vector, while the RHS is a real vector.

Properties of $\boldsymbol{Re(U_1)}$ and $\boldsymbol{Im(U_1)}$

$AU_j = \lambda_j U_j$, $\lambda_1 = \alpha + i\beta$ and $U_1 = V_1 + iV_2$. Show that

- $AV_1 = \alpha V_1 - \beta V_2$
- $AV_2 = \alpha V_2 + \beta V_1$
- $AV_j \neq \lambda V_j$

$$A(V_1 + iV_2) = \lambda_1(V_1 + iV_2)$$

$$= (\alpha + i\beta)(V_1 + iV_2)$$

$$= (\alpha V_1 - \beta V_2) + i(\alpha V_2 + \beta V_1)$$

$(T^{-1}\mathrm{AT})E_1$ & $(T^{-1}\mathrm{AT})E_2$

$AU_j = \lambda_j U_j$, $\lambda_1 = \alpha + i\beta$ and $U_1 = V_1 + iV_2$. The previous slide yields

- $AV_1 = \alpha V_1 - \beta V_2$ and $AV_2 = \alpha V_2 + \beta V_1$

When $T = (\mathrm{Re}(U_1), \mathrm{Im}(U_1))$, Show that

- $TE_j = V_j$ ➔ $T^{-1}V_j = E_j$
- $(T^{-1}\mathrm{AT})E_1 = \alpha E_1 - \beta E_2$
- $(T^{-1}\mathrm{AT})E_2 = \beta E_1 + \alpha E_2$

$$T = (\mathrm{Re}(U_1), \mathrm{Im}(U_1)) = (V_1, V_2)$$

$$TE_1 = (V_1, V_2)\begin{pmatrix}1\\0\end{pmatrix} = V_1 \qquad TE_2 = (V_1, V_2)\begin{pmatrix}0\\1\end{pmatrix} = V_2$$

$$(T^{-1}\mathrm{AT})E_1 = (T^{-1}\mathrm{A})\mathrm{V}_1 = T^{-1}(\alpha V_1 - \beta V_2) = \alpha \mathrm{E}_1 - \beta \mathrm{E}_2$$

$$(T^{-1}\mathrm{AT})E_2 = (T^{-1}\mathrm{A})\mathrm{V}_2 = T^{-1}(\alpha V_2 + \beta V_1) = \alpha \mathrm{E}_2 + \beta \mathrm{E}_1$$

Show $T^{-1}AT = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$

Construct $T = (\mathrm{Re}(U_1), \mathrm{Im}(U_1)) = [V_1, \mathrm{V}_2]$ Show $\boxed{T^{-1}AT = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}}$

Let $M = T^{-1}AT = \begin{pmatrix} x & u \\ y & w \end{pmatrix}$

Apply $ME_1 = \alpha \mathrm{E}_1 - \beta \mathrm{E}_2$

Apply $ME_2 = \alpha \mathrm{E}_2 + \beta \mathrm{E}_1$

- Find (x, y) or (u, w)
- Send your results via ”chat”
- You have 3 minutes

Show $T^{-1}AT = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$

Construct $T = (\mathrm{Re}(U_1), \mathrm{Im}(U_1)) = [V_1, V_2]$ Show $\boxed{T^{-1}AT = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}}$

Let $M = T^{-1}AT = \begin{pmatrix} x & u \\ y & w \end{pmatrix}$

Apply $ME_1 = \alpha E_1 - \beta E_2$ $\boxed{\begin{pmatrix} x & u \\ y & w \end{pmatrix} E_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}}$ $x = \alpha \;\&\; y = -\beta$

Apply $ME_2 = \alpha E_2 + \beta E_1$ $\boxed{\begin{pmatrix} x & u \\ y & w \end{pmatrix} E_2 = \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}}$ $u = \beta \;\&\; w = \alpha$

$$\boxed{T^{-1}AT = \begin{pmatrix} x & u \\ y & w \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}}$$

Example 2

Example. (Another Harmonic Oscillator) Consider the second-order equation

$$x'' + 4x = 0.$$

This corresponds to an undamped harmonic oscillator with mass 1 and spring constant 4. As a system, we have

$$X' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} X = AX.$$

$$x' = y$$
$$y' = -4x$$

$$A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$$

$$AU_j = \lambda_j U_j \,,$$
$$U_1 = V_1 + iV_2$$

Let $|A - \lambda I| = 0 \Longrightarrow \begin{vmatrix} -\lambda & 1 \\ -4 & -\lambda \end{vmatrix} = 0$ $\boxed{\lambda^2 + 4 = 0}$

$$\boxed{\lambda = \pm 2i}$$

Find Basis Vectors

$AX = \lambda X \Longrightarrow$

$$\begin{aligned} y &= \lambda x \\ -4x &= \lambda y \end{aligned}$$

Consider $\lambda = 2i$

$$\begin{aligned} y &= 2ix \\ -4x &= 2iy \end{aligned}$$

$$y = 2ix$$

$$U_1 = \begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 2i \end{pmatrix}$$

$$[V_1, \mathrm{V}_2] = (\mathrm{Re}(U_1), \mathrm{Im}(U_1)) \qquad V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

Consider $\lambda = -2i$

$$\begin{aligned} y &= -2ix \\ -4x &= -2iy \end{aligned}$$

- Find V_1 and V_2
- Send your results via "chat"
- You have 3 minutes

Find Basis Vectors

$$AX = \lambda X \Longrightarrow \quad \begin{aligned} y &= \lambda x \\ -4x &= \lambda y \end{aligned}$$

Consider $\lambda = 2i$

$$\begin{aligned} y &= 2ix \\ -4x &= 2iy \end{aligned} \qquad y = 2ix \qquad U_1 = \begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 2i \end{pmatrix}$$

$$[V_1, V_2] = (\mathrm{Re}(U_1), \mathrm{Im}(U_1)) \qquad V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

Consider $\lambda = -2i$

$$\begin{aligned} y &= -2ix \\ -4x &= -2iy \end{aligned} \qquad y = -2ix \qquad U_2 = \begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ -2i \end{pmatrix}$$

$$[V_1, V_2] = (\mathrm{Re}(U_2), \mathrm{Im}(U_2)) \qquad V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

General Solutions

$$X_{re} = Re\left(e^{i2t}\begin{pmatrix}1\\2i\end{pmatrix}\right) = Re\left(\left(\cos(2t) + isin(2t)\right)\begin{pmatrix}1\\2i\end{pmatrix}\right) = \begin{pmatrix}\cos(2t))\\-2sin(2t)\end{pmatrix}$$

$$X_{im} = \mathrm{Im}\left(e^{i2t}\begin{pmatrix}1\\2i\end{pmatrix}\right) = Re\left(\left(\cos(2t) + isin(2t)\right)\begin{pmatrix}1\\2i\end{pmatrix}\right) = \begin{pmatrix}\sin(2t)\\2cos(2t)\end{pmatrix}$$

$$X(t) = c_1 X_{re}(t) + c_2 X_{im}$$

$$x = c_1\cos(2t) + c_2\sin(2t)$$

$$y = -2c_1\sin(2t) + 2c_2\cos(2t)$$

$$x^2 + (\frac{y}{2})^2 = c_1^2 + c_2^2$$

$$x' = y$$
$$y' = -4x$$

$$xx' + \frac{1}{4}yy' = 0$$

$$\frac{1}{2}(x^2 + \frac{1}{4}y^2)' = 0$$

$$x^2 + \frac{1}{4}y^2 = C$$

ellipse

Illustrate $T^{-1}AT = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$

$$A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \qquad T = [V_1, V_2] = (\mathrm{Re}(U_1), \mathrm{Im}(U_1))$$

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$T^{-1} = \frac{1}{\det(T)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$T^{-1}\mathrm{AT} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

Compute $Y = T^{-1}X$ to Obtain an Eq. for Y'

$$x' = y$$
$$y' = -4x$$

$$Y = \begin{pmatrix} u \\ w \end{pmatrix}$$

$$T^{-1} = \frac{1}{\det(T)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$Y = \begin{pmatrix} u \\ w \end{pmatrix} = T^{-1}X = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y/2 \end{pmatrix}$$

$$u = x; \quad w = \frac{y}{2}$$

$$u' = x' = y = 2w$$
$$w' = \frac{y'}{2} = -2x = -2u$$

$$u = c_1 \cos(2t) + c_2 \sin(2t)$$
$$w = -c_1 \sin(2t) + c_2 \cos(2t)$$

$$u^2 + w^2 = c_1^2 + c_2^2$$

$$uu' + ww' = 0$$
$$\frac{1}{2}(u^2 + w^2)' = 0$$

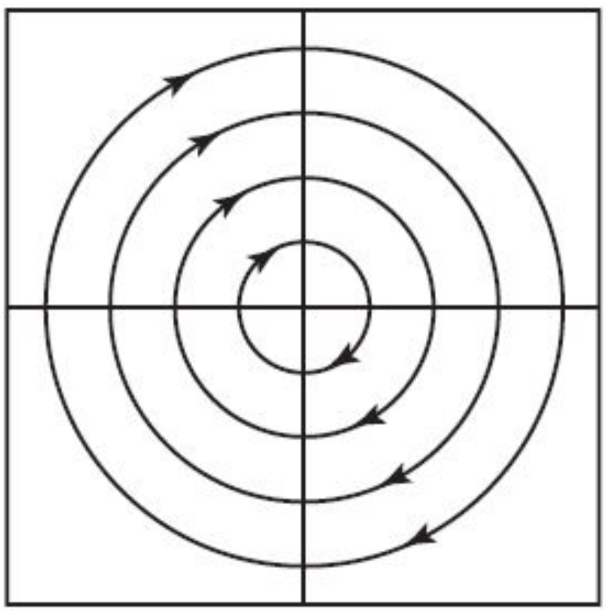
$$u^2 + w^2 = C$$

circle

Changing Coordinates: (II) Complex Eigenvalues)

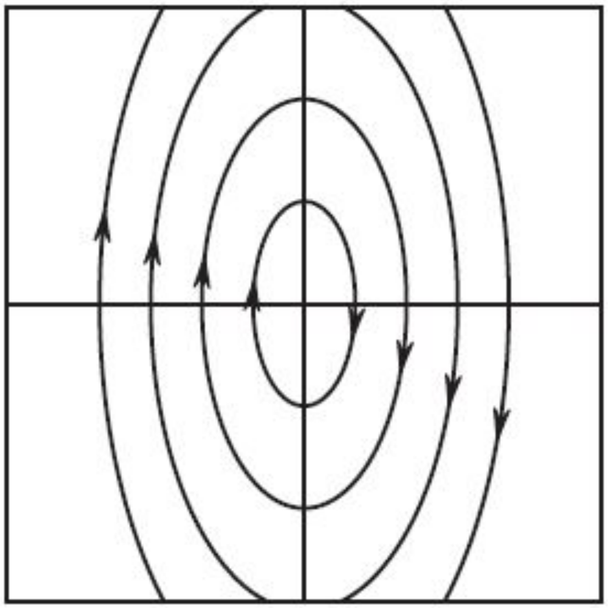
Y

(circular)



$X = TY$

(ellipse)



$$AU_j = \lambda_j U_j\ ,$$
$$U_1 = V_1 + iV_2$$

$$T = (\mathrm{Re}(U_1), \mathrm{Im}(U_1))$$
$$= (V_1\ V_2)$$

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$Y' = (T^{-1}\mathrm{AT})Y$$

$$T^{-1}\mathrm{AT} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

$$X' = AX$$

$$A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$$

A Summary for Repeated Eigenvalues

Construct T as follows:

- $AV = \lambda V$
- $(A - \lambda I)V_2 = V$.
- $T = [V, \mathrm{V}_2]$, which leads to $T^{-1}\mathrm{AT} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

Changing Coordinates: (III) Repeated Eigenvalues)

$AV = \lambda V$, λ represents a repeated eigenvalue. Assume two vectors W and V are linearly independent. Show that

- when $AW = c_1 W + c_2 V$, $c_1 = \lambda$.

Assume $c_1 \neq \lambda$. Consider the vector $W + \left(\frac{c_2}{c_1 - \lambda}\right)$ V.

$$A\left(W + \left(\frac{c_2}{c_1 - \lambda}\right)V\right) = AW + \left(\frac{c_2}{c_1 - \lambda}\right)AV$$

$$c_1 W + c_2 V + \left(\frac{c_2}{c_1 - \lambda}\right)\lambda V \quad = c_1 W + c_2\left(1 + \frac{\lambda}{c_1 - \lambda}\right)V$$

$$= c_1 W + c_2\left(\frac{c_1}{c_1 - \lambda}\right)V \quad = c_1\left(W + \left(\frac{c_2}{c_1 - \lambda}\right)V\right)$$

c_1 is the 2nd eigenvalue different from λ. This is a contradiction. Thus, $c_1 = \lambda$.

Find the 2nd Basis Vector

Based on the above discussions, when two vectors W and V are linearly independent, we have:

- $AW = \lambda W + c_2 V$ and
- $\boxed{(A-\lambda I)V_2 = V}$ for $\mathrm{W} = \mathrm{c}_2\mathrm{V}_2$.
- Here, V_2 represent a 2nd basis vector

Consider $W = c_2 V_2$, we have

$$c_2 A V_2 = \lambda c_2 V_2 + c_2 V$$

$$A V_2 = \lambda V_2 + V$$

$$(A-\lambda I)V_2 = V$$

$$(A-\lambda I)^2 V_2 = (A-\lambda I)V = 0 \quad \text{because of } AV = \lambda V$$

Example 3: Find Two Basis Vectors

Solve for eigenvalues $AV_0 = \lambda V_0$ $A = \begin{pmatrix} 2 & 1 \\ -\frac{1}{4} & 1 \end{pmatrix}$

$$2x_0 + y_0 = \lambda x_0$$
$$\frac{-x_0}{4} + y_0 = \lambda y_0$$

$$\lambda = \frac{3}{2}$$

Consider $\lambda = \frac{3}{2}$

$$2x_0 + y_0 = \frac{3}{2}x_0$$

$$\frac{-x_0}{4} + y_0 = \frac{3}{2}y_0$$

$$\boxed{\frac{-x_0}{2} = y_0}$$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ \frac{-x_0}{2} \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}$$

Obtain $\boxed{V = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}}$ as an eigenvector associated with $\lambda = \frac{3}{2}$

For V_2, we solve

$$(A - \lambda I)V_2 = V$$

$$\begin{pmatrix} \frac{1}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix} V_2 = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}$$

Example 3: Find V_2 (conti.)

For V_2, we solve

$$(A-\lambda I)V_2 = V. \qquad \begin{pmatrix} \frac{1}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix} V_2 = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \qquad \text{Let } V_2 = \binom{u}{w}$$

$$\frac{1}{2}u + w = 1$$

$$\frac{-1}{4}u - \frac{1}{2}w = -\frac{1}{2}$$

$$\frac{1}{2}u + w = 1$$

$$V_2 = \binom{u}{w} = \binom{1}{1/2}$$

Make sure that V and V2 are linearly independent

Construct T and Compute $T^{-1}\mathrm{AT}$

$$A = \begin{pmatrix} 2 & 1 \\ -1/4 & 1 \end{pmatrix} \qquad T = [V, \mathrm{V}_2] \qquad V = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 1 \\ -1/2 & 1/2 \end{pmatrix} \qquad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$T^{-1} = \frac{1}{\det(T)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1/2 & -1 \\ 1/2 & 1 \end{pmatrix}$$

$$T^{-1}\mathrm{AT} = \begin{pmatrix} 1/2 & -1 \\ 1/2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1/4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1/2 & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 5/4 & -1/2 \\ 3/4 & 3/2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 6/4 & 1 \\ 0 & 6/4 \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad \lambda = \frac{3}{2}$$

A Summary for Repeated Eigenvalues

Construct T as follows:

- $AV = \lambda V$
- $(A - \lambda I)V_2 = V$.
- $T = [V, V_2]$, which leads to $T^{-1}AT = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

$$Y' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} Y \qquad\qquad X' = AX$$

Solve the above for Y and compute $X = TY$

For example,

$$Y' = \begin{pmatrix} 3/2 & 1 \\ 0 & 3/2 \end{pmatrix} Y \qquad\qquad X' = \begin{pmatrix} 2 & 1 \\ -1/4 & 1 \end{pmatrix} X$$

Upper triangle

A Summary for the Three Cases

Goal: Solve the following 2D system

$$\boxed{X' = AX}$$

Compute the eigenvalues and eigenvectors of A,
U_1 and U_2

$$AU_j = \lambda_j U_j, \qquad j = 1,2$$

Construct $T = (V_1, V_2)$, $B = T^{-1}AT$ and $Y = TX$ using the following

(I) real eigenvalues

$$(V_1, V_2) = (U_1, U_2)$$

$$B = T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

(II) complex eigenvalues

$$(V_1, V_2) = (Re(U_1), Im(U_1))$$

$$B = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

(III) repeated eigenvalues

$$V_1 = U_1$$
$$(A - \lambda I)V_2 = V_1$$

$$B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$\boxed{Y' = BY}$$