MATH 525
Sections 1.12 and 1.10

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Section 1.12 - Error-correcting codes

Definition

A code C corrects the error pattern e if for all $v \in C$,

$$d(v+e, v) < d(v+e, u), \ \forall\, u \in C, u \neq v.$$

Example

$C = \{000, 111\}$ corrects the error patterns $100, 010, 001, 000$, but not $110, 101, 011, 111$.

Theorem

A code of distance d will correct all error patterns of weight $\leq \lfloor \frac{d-1}{2} \rfloor$. Moreover, there exists at least one error pattern of weight $1 + \lfloor \frac{d-1}{2} \rfloor$ which C will not correct.

Example

Let C be a code of distance d. In this example, we will find an error pattern of weight $1 + \lfloor \frac{d-1}{2} \rfloor$ that C does not correct:

Suppose $d = 5$ and let u and v be codewords in C such that:

$$\begin{array}{rcccccccccccc} u & = & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ v & = & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ \hline u+v & = & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}$$

Let $e' = u + v$ and form e from e' be changing $\lfloor \frac{d-1}{2} \rfloor = 2$ of its ones into zeroes. Thus, $e = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$. It follows that $\mathrm{wt}(e) = \lfloor \frac{d+1}{2} \rfloor = 3$,

$$d(v + e, v) \geq d(v + e, u),$$

and C does not correct e.

Definition

A block code C is said to be a *t-error-correcting code* if C corrects all error patterns of weight up to t, but it does not correct at least one error pattern of weight $t + 1$.

Remark: The IMLD table can be used to determine the error patterns that a code C will correct: In each error-pattern column of the IMLD table, every error pattern appears exactly once. An asterisk is placed beside an error pattern e in the column corresponding to a codeword v precisely when v is sent, e occurs, and C correctly decodes the received word $w = v + e$ into v. Thus,

a particular error pattern e is corrected by C if and only if an asterisk is placed beside e in every column of the IMLD table.

Section 1.10 - Reliability of MLD

$\theta_p(C, v)$ = probability that if v is sent over a BSC of reliability p, then IMLD will correctly conclude that v was sent.

To evaluate $\theta_p(C, v)$, we construct the set $L(v)$ which consists of all words in K^n that are closer to v than to any other word in C. It follows that

$$\theta_p(C, v) = \sum_{w \in L(v)} \phi_p(v, w).$$

Example

Let $C = \{000, 111\}$. Calculate $\theta_p(C, 000)$. Start out by writing

$$L(000) = \{000, 100, 010, 001\}.$$

Remark: $L(v)$ can be found from the IMLD table:

$$L(v) = \{w \text{ is in the first column of the IMLD table } | \, w \text{ is decoded into } v\}.$$

(see example on the next page.)

IMLD table for Example 1.9.4 - p. 15 of the textbook.

received word	error patterns			most likely codeword
w	$0000 + w$	$1010 + w$	$0111 + w$	v
0000	0000	1010	0111	0000
0001	0001	1011	0110	0000
0010	0010	1000	0101	−−
0011	0011	1001	0100	0111
0100	0100	1110	0011	0000
0101	0101	1111	0010	0111
0110	0110	1100	0001	0111
0111	0111	1101	0000	0111
1000	1000	0010	1111	−−
1001	1001	0011	1110	−−
1010	1010	0000	1101	1010
1011	1011	0001	1100	1010
1100	1100	0110	1011	−−
1101	1101	0111	1010	0111
1110	1110	0100	1001	1010
1111	1111	0101	1000	0111

$L(0000) = \{0000, 0001, 0100\}$.