MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #24
Chapter 7
Existence and Uniqueness,
Continuous and Sensitive Dependence on ICs,
and Linearization Theorem

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References

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Outline

1. Existence and Uniqueness Theorems
2. Lipschitz Condition
3. Continuous Dependence of Solutions on Initial Conditions (CDIC)
4. Sensitive Dependence of Solutions on Initial Conditions (SDIC)
5. Linearization Theorems & Linearized Systems

Fundamental Concepts

1. **Existence:** Each point in the (t, x)-plane has a solution passing through it. The solution has slope given by the differential equation at that point.
2. **Uniqueness:** Only one solution passes through any particular (t, x).
3. **Continuous dependence:** Solutions through nearby initial conditions remain close over short time intervals. In other words, the flow $F(t, x_0)$ is a continuous function of x_0 as well as t. $|X(t) - Y(t)| < |X_0 - Y_0|e^{K(t-t_0)}$

- *"Sensitive dependence on initial conditions" (SDIC) means that nearby trajectories separate exponentially fast, i.e., the system has a positive Liapunov exponent. (Strogatz, p331)*
- *We may define an infinite invariant set of a map $F: R \to R$ to have sensitive dependence on initial conditions if there exists $\delta > 0$ such that for all $x \in S$ and all neighbourhoods N (however small) of x there exists $y \in N$ and $n > 0$ such that $|F^n(x) - F^n(y)| > \delta$. So neighbouring orbits, however close initially, separate from one another, although each keeps close to the invariant set. (Drazin, p140; Devaney, p49)*

Alligood et al.

Existence Theorem: A Quick Look (1D)

Existence Theorem

Let the right side $\boxed{f(x, y)}$ *of the ODE in the initial value problem*

I.V.P.

(1) $$y' = f(x, y), \qquad y(x_0) = y_0$$

$$y' = \frac{dy}{dx}$$

be continuous at all points (x, y) *in some rectangle*

$$R\colon |x - x_0| < a, \qquad |y - y_0| < b \qquad \text{(Fig. 26)}$$

and **bounded** *in* R*; that is, there is a number* K *such that*

(2) $$\boxed{|f(x, y)| \leqq K} \qquad \textit{for all } (x, y) \textit{ in } R.$$

Then the initial value problem (1) *has at least one solution* $y(x)$*. This solution exists at least for all* x *in the* $\boxed{\textit{subinterval } |x - x_0| < \alpha}$ *of the interval* $|x - x_0| < a$*; here,* α *is the smaller of the two numbers* a *and* b/K*.*

x & f bounded ➔ at least one solution within an subinterval Kreyszig

Uniqueness Theorem: A Quick Look (1D)

Uniqueness Theorem

Let f and its partial derivative $f_y = \partial f/\partial y$ be continuous for all (x, y) in the rectangle R (Fig. 26) and bounded, say,

$$(3) \qquad \text{(a)} \quad |f(x, y)| \leqq K, \qquad \text{(b)} \quad |f_y(x, y)| \leqq M \qquad \textit{for all } (x, y) \textit{ in } R.$$

Then the initial value problem (1) *has at most one solution $y(x)$. Thus, by Theorem* 1, *the problem has precisely one solution. This solution exists at least for all x in that subinterval $|x - x_0| < \alpha$.*

f & f_y bounded➔ at most one solution

M has a special name.

Kreyszig

Existence and Uniqueness: A Simple Statement (1D)

THEOREM 2.2.1 Existence and Uniqueness

Consider the initial-value problem

$$y' = f(x, y) \quad \text{with} \quad y(x_0) = y_0.$$

$$y' = \frac{dy}{dx}$$

If f and $\partial f/\partial y$ are continuous functions on the rectangular region

$$R : a < x < b, \quad c < y < d$$

containing the point (x_0, y_0), then there exists an interval

$$|x - x_0| < h$$

centered at x_0 on which there exists one and only one solution to the differential equation that satisfies the initial condition.

f & $f_y(x, y)$ continuous over an finite rectangular "region" for x and y.

Wirkus and Swift

Existence and Uniqueness: A Simple Statement (1D)

A similar statement:

Existence and Uniqueness of Solution

Theorem 1. Consider the initial value problem

$$\frac{dy}{dx} = f(x, y) \ , \qquad y(x_0) = y_0 \ .$$

If f and $\partial f / \partial y$ are continuous functions in some rectangle

$$R = \{(x, y) \colon a < x < b, c < y < d\}$$

that contains the point (x_0, y_0), then the initial value problem has a unique solution $\phi(x)$ in some interval $x_0 - \delta < x < x_0 + \delta$, where δ is a positive number.

f & $f_y(x, y)$ continuous over an finite rectangular “region” for x and y.

Nagle et al.

Uniqueness (N-dimensional)

Uniqueness: (bounded derivatives)

Continuation of Solution

Theorem 5. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{f}(t, \mathbf{x})$ denote the vector function $\mathbf{f}(t, \mathbf{x}) = \big(f_1(t, x_1, \ldots, x_n), \quad \ldots, \quad f_n(t, x_1, \ldots, x_n)\big)$. Suppose $\mathbf{f}$ and $\partial\mathbf{f}/\partial x_i$, $i = 1, \ldots, n$, are continuous on the strip

$$R = \{(t, \mathbf{x}) \in \mathbf{R}^{n+1}\colon a \le t \le b,\ \mathbf{x} \text{ arbitrary}\}$$

containing the point $(t_0, \mathbf{x}_0)$. Assume further that there exists a positive constant L such that, for $i = 1, \ldots, n$,

$$\boxed{\left|\frac{\partial \mathbf{f}}{\partial x_i}(t, \mathbf{x})\right| \le L} \tag{3}$$

for all $(t, \mathbf{x})$ in R. Then the initial value problem

$$\mathbf{x}'(t) = \mathbf{f}(t, \mathbf{x}) \ , \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \ , \tag{4}$$

has a unique solution on the entire interval $a \le t \le b$.

What's in Eq. (3)? Determinant of the Jacobian matrix

Nagle et al.?

Existence and Uniqueness Theorem for Linear ODEs

High order ODEs

Existence and Uniqueness Theorem for Linear Equations

Theorem 7. Suppose $p_1(t), \ldots, p_n(t)$ and $g(t)$ are continuous on an interval (a, b) containing the point t_0. Then, for every choice of the initial values $y_0, y_1, \ldots, y_{n-1}$, there exists a unique solution on the whole interval (a, b) to the initial value problem

$$y^{(n)}(t) + p_1(t)y^{(n-1)}(t) + \cdots + p_n(t)y(t) = g(t) \; ; \tag{14}$$

$$y(t_0) = y_0 \; , \qquad y'(t_0) = y_1 \; , \qquad \ldots , \qquad y^{(n-1)}(t_0) = y_{n-1} \; .$$

I.V.P.

Nagle et al.

Existence and Uniqueness Theorem for Linear Systems

Existence and Uniqueness Theorem for Linear Systems

Theorem 6. Suppose the $n \times n$ matrix function $\mathbf{A}(t) = \left[a_{ij}(t)\right]$ and the vector function $\mathbf{f}(t)$ are continuous on an open interval (a, b) that contains the point t_0. Then, for any choice of the initial vector $\mathbf{x}_0$, there exists a unique solution on the whole interval (a, b) to the initial value problem

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t) , \qquad \mathbf{x}(t_0) = \mathbf{x}_0 . \tag{12}$$

non-autonomous with forcing term(s)

Nagle et al.

Existence and Uniqueness for Linear Systems

Existence and Uniqueness for X'=A(t)X (linear nonautonomous)

Theorem. *Let $A(t)$ be a continuous family of $n \times n$ matrices defined for $t \in [\alpha, \beta]$. Then the initial value problem*

$$X' = A(t)X,\ X(t_0) = X_0$$

has a unique solution that is defined on the entire interval $[\alpha, \beta]$. ∎

non-autonomous (with no additional forcing term)

HSD

Smoothness of Flows

Smoothness of Flows

HSD

Theorem. (Smoothness of Flows) *Consider the system $X' = F(X)$ where F is C^1. Then the flow $\phi(t, X)$ of this system is a C^1 function; that is, $\partial\phi/\partial t$ and $\partial\phi/\partial X$ exist and are continuous in t and X.* ∎

- For functions for which derivatives of all orders exist and are continuous functions, we will call this type of function a smooth function (e.g., Alligood et al)

- C^k function: A function is C^k if it is k-times differentiable.
- Diffeomorphism: A C^k -diffeomorphism $f: M \to N$ is a mapping f which is 1-1, onto, and has the property that both f and f^{-1} are k-times differentiable.
- Homeomorphism: A homeomorphism is a C^0 diffeomorphism, i.e. a continuous mapping $f: M \to N$ with a continuous inverse.

The Fundamental Local Theorem of ODEs

The Existence and Uniqueness Theorem. *Consider the initial value problem*

$$X' = F(X),\ X(t_0) = X_0,$$

where $X_0 \in \mathbb{R}^n$*. Suppose that* $F : \mathbb{R}^n \to \mathbb{R}^n$ *is* C^1*. Then, first, there exists a solution of this initial value problem, and second, this is the only such solution. More precisely, there exists an* $a > 0$ *and a unique solution,*

$$X : (t_0 - a, t_0 + a) \to \mathbb{R}^n,$$

of this differential equation satisfying the initial condition $X(t_0) = X_0$.

an finite interval

HSD

Picard Iteration for the Proof

Without dwelling on the details here, the proof of this theorem depends on an important technique known as *Picard iteration*. Before moving on, we illustrate how the Picard iteration scheme used in the proof of the theorem works in several special examples. The basic idea behind this iterative process is to construct a sequence of functions that converges to the solution of the differential equation. The sequence of functions $u_k(t)$ is defined inductively by $u_0(t) = x_0$, where x_0 is the given initial condition, and then

$$X' = F(X) \quad X(t=0) = X_0$$

$$\mathrm{u}_0(t) = x_0$$

$$u_{k+1}(t) = x_0 + \int_0^t F\big(u_k(s)\big)ds$$

Example 1 for Picard Iteration

$$u_{k+1}(t) = x_0 + \int_0^t F(u_k(s))ds \qquad \mathrm{u}_0(t) = x_0$$

Example. Consider the simple differential equation $x' = x$. We will produce the solution of this equation satisfying $x(0) = x_0$. We know, of course, that this solution is given by $x(t) = x_0 e^t$. We will construct a sequence of functions $u_k(t)$, one for each k, that converges to the actual solution $x(t)$ as $k \to \infty$.

$$F(x) = x \qquad \mathrm{u}_0(t) = x_0$$

$$u_1(t) = x_0 + \int_0^t F(u_0(s))ds = x_0 + \int_0^t x_0 ds = x_0 + x_0 t$$

$$u_2(t) = x_0 + \int_0^t F(u_1(s))ds$$

- Find u_2
- Submit your results via "chat"
- You have 3 minutes

Example 1 for Picard Iteration

$$u_{k+1}(t) = x_0 + \int_0^t F\big(u_k(s)\big)ds \qquad \mathrm{u}_0(t) = x_0$$

$$F(x) = x \qquad \mathrm{u}_0(t) = x_0$$

$$u_1(t) = x_0 + \int_0^t F\big(u_0(s)\big)ds = x_0 + \int_0^t x_0 ds = \boxed{x_0 + x_0 t}$$

$$u_2(t) = x_0 + \int_0^t F\big(u_1(s)\big)ds = x_0 + \int_0^t \boxed{(x_0 + x_0 s)} ds = x_0 + x_0 t + \frac{1}{2} x_0 t^2$$

$$u_{k+1}(t) = x_0 \sum_{i=0}^{k+1} \frac{t^i}{i!}$$

As $k \to \infty$, u_k converges to

- "Simplify" u_k as $k \to \infty$ (into an elementary function)
- You have 2 minutes

Example 1 for Picard Iteration

$$u_{k+1}(t) = x_0 + \int_0^t F\big(u_k(s)\big)ds \qquad\qquad \mathrm{u}_0(t) = x_0$$

$$F(x) = x \qquad\qquad \mathrm{u}_0(t) = x_0$$

$$u_1(t) = x_0 + \int_0^t F\big(u_0(s)\big)ds = x_0 + \int_0^t x_0 ds = \boxed{x_0 + x_0 t}$$

$$u_2(t) = x_0 + \int_0^t F\big(u_1(s)\big)ds = x_0 + \int_0^t \boxed{(x_0 + x_0 s)} ds = x_0 + x_0 t + \frac{1}{2} x_0 t^2$$

$$u_{k+1}(t) = x_0 \sum_{i=0}^{k+1} \frac{t^i}{i!}$$

As $k \to \infty$, u_k converges to

$$x_0 \sum_{i=0}^{\infty} \frac{t^i}{i!} = x_o e^t$$

Example 1 for Picard Iteration

$$u_{k+1}(t) = x_0 + \int_0^t F(u_k(s))ds \qquad \mathrm{u}_0(t) = x_0$$

Example. Consider the simple differential equation $x' = x$. We will produce the solution of this equation satisfying $x(0) = x_0$. We know, of course, that this solution is given by $\boxed{x(t) = x_0 e^t}$. We will construct a sequence of functions $u_k(t)$, one for each k, that converges to the actual solution $x(t)$ as $k \to \infty$.

$$F(x) = x \qquad \mathrm{u}_0(t) = x_0$$

$$x_0 \sum_{i=0}^{\infty} \frac{t^i}{i!} = x_o e^t = x(t)$$

Example 2 for Picard Iteration

Example. For an example of Picard iteration applied to a system of differential equations, consider the linear system

$$X' = F(X) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X$$

with initial condition $X(0) = (1, 0)$. As we have seen, the solution of this initial value problem is

$$X(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}.$$

$$U_0(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad F(U_0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U_0(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$U_1(t) = U_0 + \int_0^t F(U_0) ds = U_0 + \int_0^t \begin{pmatrix} 0 \\ -1 \end{pmatrix} ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -t \end{pmatrix} = \begin{pmatrix} 1 \\ -t \end{pmatrix}$$

$$F(U_1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U_1(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -t \end{pmatrix} = \begin{pmatrix} -t \\ -1 \end{pmatrix}$$

$$U_0 = \binom{1}{0} \qquad F(U_1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U_1(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \binom{1}{-t} = \binom{-t}{-1}$$

$$U_2(t) = U_0 + \int_0^t F(U_1)ds = U_0 + \int_0^t \binom{-s}{-1} ds = \binom{1}{0} + \begin{pmatrix} -\frac{1}{2}t^2 \\ -t \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{2}t^2 \\ -t \end{pmatrix}$$

$$F(U_2) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U_2(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 - \frac{1}{2}t^2 \\ -t \end{pmatrix} = \begin{pmatrix} -t \\ -1 + \frac{1}{2}t^2 \end{pmatrix}$$

$$U_3(t) = U_0 + \int_0^t F(U_2)ds = \begin{pmatrix} 1 - t^2/2 \\ -t + t^3/3! \end{pmatrix}$$

$$U_4(t) = \begin{pmatrix} 1 - \frac{t^2}{2} + t^4/4! \\ -t + t^3/3! \end{pmatrix} \qquad U_k(t) = \binom{u_{1k}}{u_{2k}}$$

- Please project to "guess" what U_k is when $k \to \infty$?
- (it contains elementary functions.)
- You have 2 minutes

$$U_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad F(U_1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U_1(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -t \end{pmatrix} = \begin{pmatrix} -t \\ -1 \end{pmatrix}$$

$$U_2(t) = U_0 + \int_0^t F(U_1) ds = U_0 + \int_0^t \begin{pmatrix} -s \\ -1 \end{pmatrix} ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}t^2 \\ -t \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{2}t^2 \\ -t \end{pmatrix}$$

$$F(U_2) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U_2(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 - \frac{1}{2}t^2 \\ -t \end{pmatrix} = \begin{pmatrix} -t \\ -1 + \frac{1}{2}t^2 \end{pmatrix}$$

$$U_3(t) = U_0 + \int_0^t F(U_2) ds = \begin{pmatrix} 1 - t^2/2 \\ -t + t^3/3! \end{pmatrix}$$

$$U_4(t) = \begin{pmatrix} 1 - \frac{t^2}{2} + t^4/4! \\ -t + t^3/3! \end{pmatrix} \qquad U(t) = \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}$$

Example 2 for Picard Iteration

Example. For an example of Picard iteration applied to a system of differential equations, consider the linear system

$$X' = F(X) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X$$

with initial condition $X(0) = (1, 0)$. As we have seen, the solution of this initial value problem is

$$\boxed{X(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}.}$$

$$U_{k+1}(t) = U_0 + \int_0^t F(U_k)ds \qquad U(t) = \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}$$

The Fundamental Local Theorem for Non-autonomous Eqs.

Theorem. *Let $\mathcal{O} \subset \mathbb{R} \times \mathbb{R}^n$ be open and $F: \mathcal{O} \to \mathbb{R}^n$ a function that is C^1 in X and continuous in t. If $(t_0, X_0) \in \mathcal{O}$, there is an open interval J containing t and a unique solution of $X' = F(t, X)$ defined on J and satisfying $X(t_0) = X_0$.* ∎

A special case is given below.

Corollary. *Let $A(t)$ be a continuous family of $n \times n$ matrices. Let $(t_0, X_0) \in J \times \mathbb{R}^n$. Then the initial value problem*

$$X' = A(t)X, \; X(t_0) = X_0$$

has a unique solution on all of J. ∎

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1. Existence and Uniqueness Theorems
2. Lipschitz Condition
3. Continuous Dependence of Solutions on Initial Conditions (CDIC)
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Boundedness and Lipschitz Function

We call the function $F(t, X)$ *Lipschitz in* X if there is a constant $K \geq 0$ such that

$$|F(t, X_1) - F(t, X_2)| \leq K|X_1 - X_2|$$

for all (t, X_1) and (t, X_2) in $\mathcal{O}$. Locally Lipschitz in X is defined analogously.

Definition 5.1 A function $f(t, y)$ is said to satisfy a **Lipschitz condition** in the variable y on a set $D \subset \mathbb{R}^2$ if a constant $L > 0$ exists with

$$|f(t, y_1) - f(t, y_2,)| \leq L|y_1 - y_2|,$$

whenever (t, y_1) and (t, y_2) are in D. The constant L is called a **Lipschitz constant** for f. ■

Partial derivatives are bounded (e.g., in the next slide)

Suppose $f(t, y)$ is defined on $\boxed{\text{a convex set } D} \subset \mathbb{R}^2$. If a constant $L > 0$ exists with

$$\left|\frac{\partial f}{\partial y}(t, y)\right| \le L, \quad \text{for all } (t, y) \in D, \tag{5.1}$$

then f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L. ■

- A convex set
- Lipschitz condition

Example 1 Show that $f(t, y) = t|y|$ satisfies a Lipschitz condition on the interval $D = \{(t, y) \mid 1 \leq t \leq 2 \text{ and } -3 \leq y \leq 4\}$.

Solution For each pair of points (t, y_1) and (t, y_2) in D we have

$$|f(t, y_1) - f(t, y_2)| = |t|y_1| - t|y_2|| = |t|\boxed{||y_1| - |y_2||} \leq 2\boxed{|y_1 - y_2|}.$$

Thus f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant 2. The smallest value possible for the Lipschitz constant for this problem is $L = 2$, because, for example,

$$|f(2, 1) - f(2, 0)| = |2 - 0| = 2|1 - 0|.$$

■

$$(|y_1| - |y_2|)^2 - (|y_1 - y_2|)^2$$

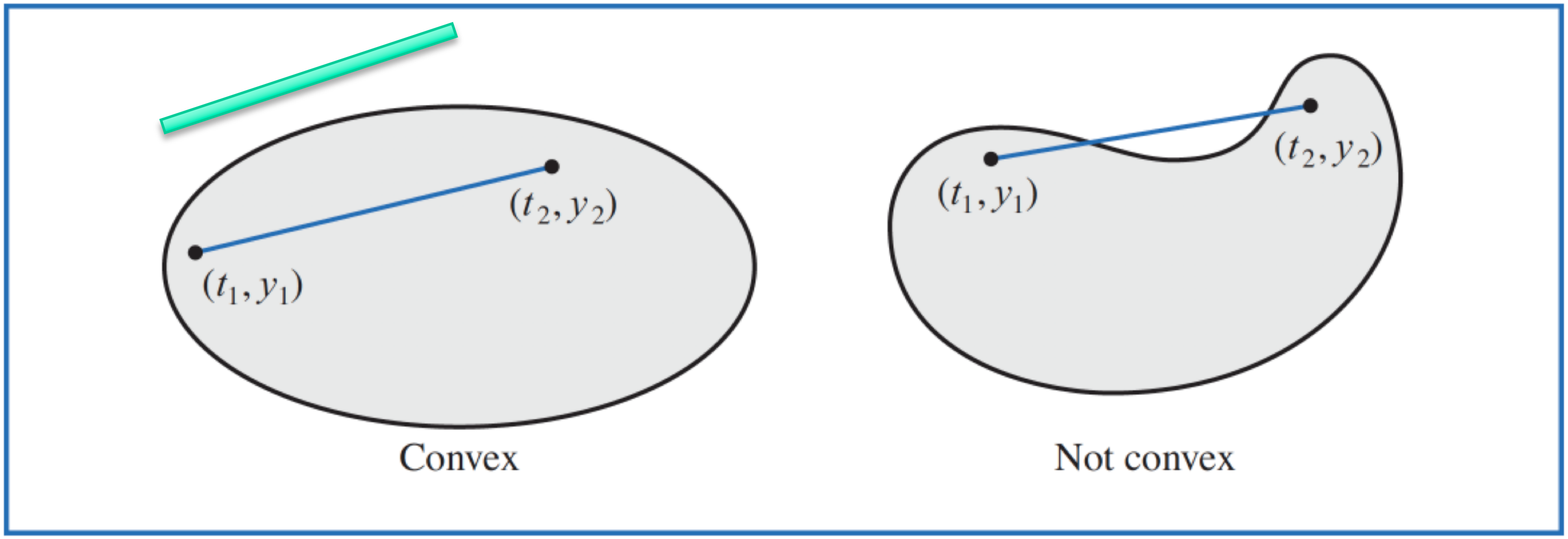
$$= -2|y_1||y_2| - (-2y_1 y_2)$$

$$= 2y_1 y_2 - 2|y_1||y_2| \leq 0$$

Definition 5.2 A set $D \subset \mathbb{R}^2$ is said to be **convex** if whenever (t_1, y_1) and (t_2, y_2) belong to D, then $((1-\lambda)t_1 + \lambda t_2, (1-\lambda)y_1 + \lambda y_2)$ also belongs to D for every λ in $[0, 1]$. ■

In geometric terms, Definition 5.2 states that a set is convex provided that whenever two points belong to the set, the entire straight-line segment between the points also belongs to the set. (See Figure 5.1.) The sets we consider in this chapter are generally of the form $D = \{(t, y) \mid a \le t \le b \text{ and } -\infty < y < \infty\}$ for some constants a and b. It is easy to verify (see Exercise 7) that these sets are convex.

Figure 5.1



Uniqueness (Burden et al., 2014)

Theorem 5.4 Suppose that $D = \{(t, y) \mid a \leq t \leq b \text{ and } -\infty < y < \infty\}$ and that $f(t, y)$ is continuous on D. If f satisfies a Lipschitz condition on D in the variable y, then the initial-value problem

$$y'(t) = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

has a unique solution $y(t)$ for $a \leq t \leq b$. ■

Example 2 Use Theorem 5.4 to show that there is a unique solution to the initial-value problem

$$y' = 1 + t\sin(ty), \quad \boxed{0 \le t \le 2,} \quad y(0) = 0.$$

Solution Holding t constant and applying the Mean Value Theorem to the function

$$f(t, y) = 1 + t\sin(ty),$$

we find that when $y_1 < y_2$, a number ξ in (y_1, y_2) exists with

$$\frac{f(t, y_2) - f(t, y_1)}{y_2 - y_1} = \boxed{\frac{\partial}{\partial y} f(t, \xi) = t^2\cos(\xi t).}$$

Thus

$$|f(t, y_2) - f(t, y_1)| = |y_2 - y_1||t^2\cos(\xi t)| \le 4|y_2 - y_1|,$$

and f satisfies a Lipschitz condition in the variable y with Lipschitz constant $L = 4$. Additionally, $f(t, y)$ is continuous when $0 \le t \le 2$ and $-\infty < y < \infty$, so Theorem 5.4 implies that a unique solution exists to this initial-value problem.

If you have completed a course in differential equations you might try to find the exact solution to this problem. ■

Lipschitz Functions and Lipschitz Constant

Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set. A function $F : \mathcal{O} \to \mathbb{R}^n$ is said to be *Lipschitz* on $\mathcal{O}$ if there exists a constant K such that

$$|F(Y) - F(X)| \leq K|Y - X|$$

for all $X, Y \in \mathcal{O}$. We call K a *Lipschitz constant* for F. More generally, we say that F is *locally Lipschitz* if each point in $\mathcal{O}$ has a neighborhood $\mathcal{O}'$ in $\mathcal{O}$ such that the restriction F to $\mathcal{O}'$ is Lipschitz. The Lipschitz constant of $F|\mathcal{O}'$ may vary with the neighborhoods $\mathcal{O}'$.

Lemma. *Suppose that the function $F : \mathcal{O} \to \mathbb{R}^n$ is C^1. Then F is locally Lipschitz.*

C^1 ➔ Locally Lipschitz

HSD

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CDIC vs. SDIC

TBD

	CDIC	SDIC
Definition	solutions through nearby ICs remain close over short time intervals	*The property for an orbit when most other orbits that pass close to it at some point do not remain close to it as time advances.*

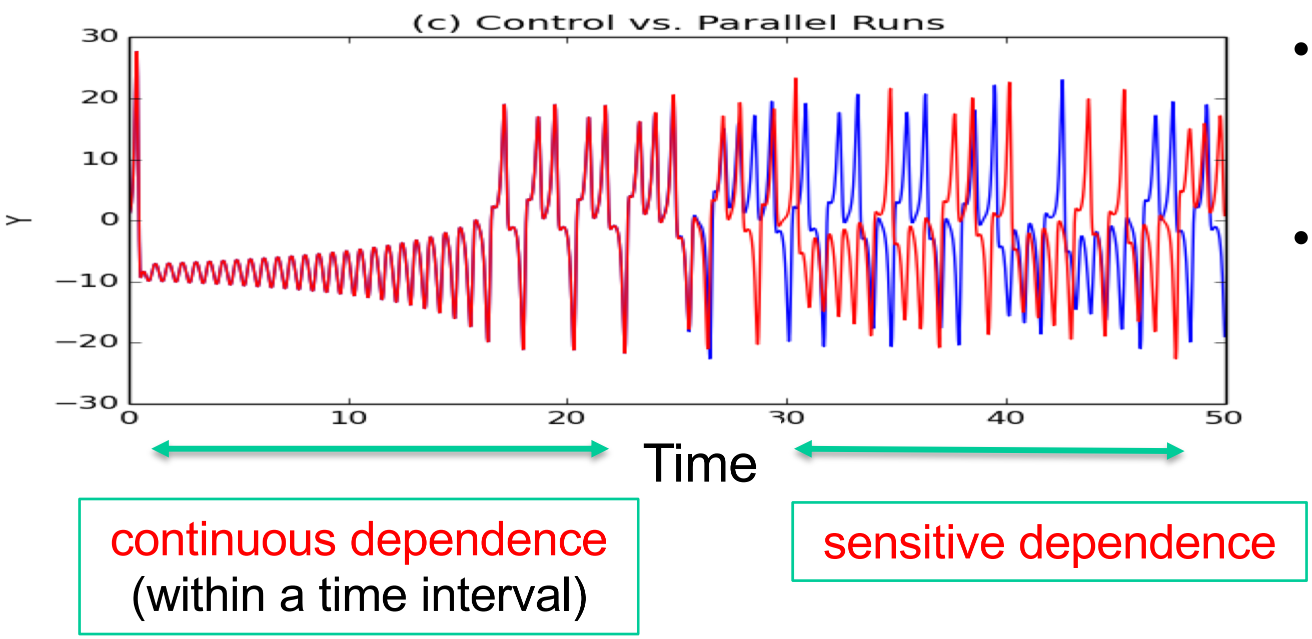
- nearby trajectories diverge no faster than an exponential rate
- nearby trajectories separate exponentially fast

- *Continuous dependence on initial conditions (CDIC)*
- *"Sensitive dependence on initial conditions" (SDIC) means that nearby trajectories separate exponentially fast, i.e., the system has a positive Liapunov exponent. (Strogatz, p331)*
- *We may define an infinite invariant set of a map $F: R \to R$ to have sensitive dependence on initial conditions if there exists $\delta > 0$ such that for all $x \in S$ and all neighbourhoods N (however small) of x there exists $y \in N$ and $n > 0$ such that $|F^n(x) - F^n(y)| > \delta$. So neighbouring orbits, however close initially, separate from one another, although each keeps close to the invariant set. (Drazin, p140; Devaney, p49)*

A Quick Note on CDIC and SDIC

1. The butterfly effect of the first kind (BE1):

   Indicating <u>*sensitive dependence on initial conditions*</u> (Lorenz, 1963).



- control run (blue): $(X, Y, Z) = (0,1,0)$
- parallel run (red): $(X, Y, Z) = (0,1+\epsilon, 0)$, $\epsilon = 1e-10$.

- *Continuous dependence on initial conditions (CDIC)*
- *Sensitive dependence on initial conditions (SDIC)*

CDIC: The Gronwall's Inequality

Gronwall's Inequality. *Let* $u : [0, \alpha] \to \mathbb{R}$ *be continuous and nonnegative. Suppose* $C \geq 0$ *and* $K \geq 0$ *are such that*

$$u(t) \leq C + \int_0^t Ku(s)\, ds$$

for all $t \in [0, \alpha]$. *Then, for all* t *in this interval,*

$$\boxed{u(t) \leq Ce^{Kt}.}$$

Continuous Dependence on ICs (CDIC)

Theorem 7.16 **Continuous dependence on initial conditions.** *Let* $\mathbf{f}$ *be defined on the open set* U *in* $\mathbb{R}^n$*, and assume that* $\mathbf{f}$ *has Lipschitz constant* L *in the variables* $\mathbf{v}$ *on* U*. Let* $\mathbf{v}(t)$ *and* $\mathbf{w}(t)$ *be solutions of (7.29), and let* $[t_0, t_1]$ *be a subset of the domains of both solutions. Then*

$$|\mathbf{v}(t) - \mathbf{w}(t)| \leq |\mathbf{v}(t_0) - \mathbf{w}(t_0)|e^{L(t-t_0)},$$

for all t *in* $[t_0, t_1]$.

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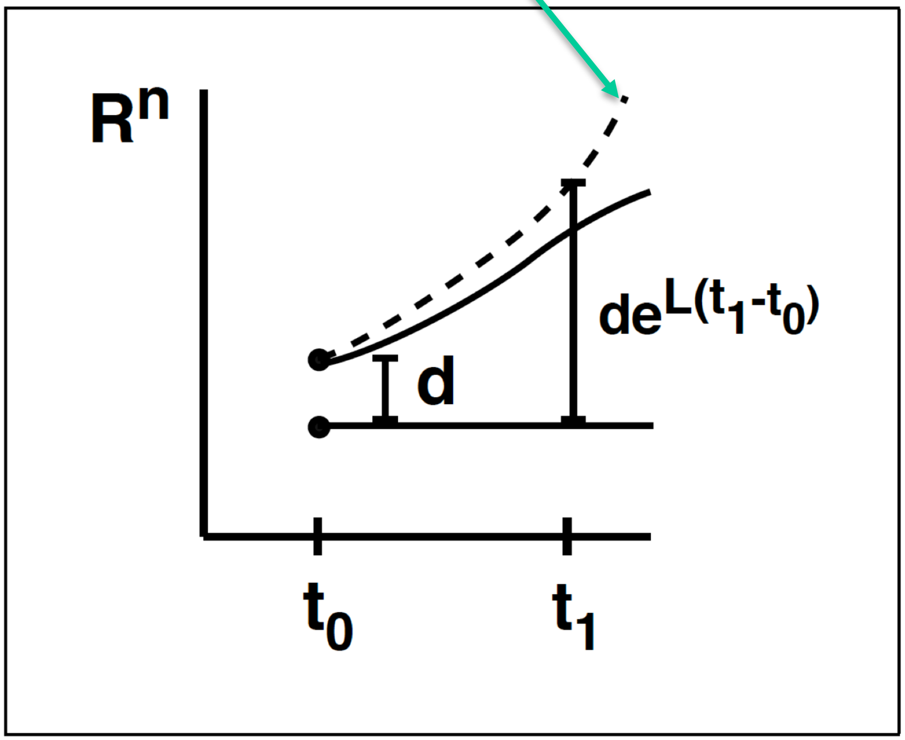
CDIC: The Gronwall Inequality and Lipschitz Constant

exponential function



- diverge no faster than an exponential rate

"Jargon"

L: Lipschitz constant

Figure 7.13 The Gronwall inequality.
Nearby solutions can diverge no faster than an exponential rate determined by the Lipschitz constant of the differential equation.

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Continuous Dependence of Solutions on ICs

Theorem. *Consider the differential equation $X' = F(X)$ where $F : \mathbb{R}^n \to \mathbb{R}^n$ is C^1. Suppose that $X(t)$ is a solution of this equation that is defined on the closed interval $[t_0, t_1]$ with $X(t_0) = X_0$. Then there is a neighborhood $U \subset \mathbb{R}^n$ of X_0 and a constant K such that if $Y_0 \in U$, then there is a unique solution $Y(t)$ also defined on $[t_0, t_1]$ with $Y(t_0) = Y_0$. Moreover $Y(t)$ satisfies*

$$|Y(t) - X(t)| \leq |Y_0 - X_0| \exp(K(t - t_0))$$

for all $t \in [t_0, t_1]$. K: Lipschitz constant ∎

- X(t) and Y(t) which start out close together remain close together
- Although they may separate from each other, they do so no faster than exponentially.

Corollary. (Continuous Dependence on Initial Conditions) *Let $\phi(t, X)$ be the flow of the system $X' = F(X)$, where F is C^1. Then ϕ is a continuous function of X.* HSD ∎

Continuous Dependence of Solutions on ICs

For the Existence and Uniqueness Theorem to be at all interesting in any physical or even mathematical sense, the result needs to be complemented by the property that the solution $X(t)$ depends continuously on the initial condition $X(0)$. The next theorem gives a precise statement of this property.

Theorem. *Let $\mathcal{O} \subset \mathbb{R}^n$ be open and suppose $F : \mathcal{O} \to \mathbb{R}^n$ has Lipschitz constant K. Let $Y(t)$ and $Z(t)$ be solutions of $X' = F(X)$ that remain in $\mathcal{O}$ and are defined on the interval $[t_0, t_1]$. Then, for all $t \in [t_0, t_1]$, we have*

$$|Y(t) - Z(t)| \le |Y(t_0) - Z(t_0)| \exp(K(t - t_0)).$$

- Note that this result says that, if the solutions Y(t) and Z(t) start out close together, then they remain close together for t near t0.
- Although these solutions may separate from each other, they do so no faster than exponentially.

Continuous Dependence on ICs

Continuous Dependence on Initial Value

Theorem 9. Let f and $\partial f/\partial y$ be continuous functions on an open rectangle

$$R = \{(x, y): a < x < b, c < y < d\}$$

containing the point (x_0, y_0). Assume that for all $\tilde{y}_0$ sufficiently close to y_0, the solution $\phi(x, \tilde{y}_0)$ to (7) exists on the interval $[x_0 - h, x_0 + h]$ and its graph lies within a fixed closed rectangle $R_0 \subset R$. Then, for $|x - x_0| \leq h$,

$$|\phi(x, y_0) - \phi(x, \tilde{y}_0)| \leq |y_0 - \tilde{y}_0| e^{Lh} , \tag{8}$$

where L is any positive constant such that $|(\partial f/\partial y)(x, y)| \leq L$ for all (x, y) in R_0. Moreover, as $\tilde{y}_0$ approaches y_0, the solution $\phi(x, \tilde{y}_0)$ approaches $\phi(x, y_0)$ uniformly on $[x_0 - h, x_0 + h]$.

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Continuous Dependence on Parameters

Theorem. (Continuous Dependence on Parameters) *Let $X' = F_a(X)$ be a system of differential equations for which F_a is continuously differentiable in both X and a. Then the flow of this system depends continuously on a as well as X.*

As in the previous case, solutions depend continuously on these parameters provided that the system depends on the parameters in a continuously differentiable fashion. We can see this easily by using a special little trick. Suppose the system

$$X' = F_a(X)$$

depends on the parameter a in a C^1 fashion. Let's consider an "artificially" augmented system of differential equations given by

$$\begin{aligned}
x_1' &= f_1(x_1, \ldots, x_n, a) \\
&\vdots \\
x_n' &= f_n(x_1, \ldots, x_n, a) \\
a' &= 0.
\end{aligned}$$

This is now an autonomous system of $n+1$ differential equations. Although this expansion of the system may seem trivial, we may now invoke the previous result about continuous dependence of solutions on initial conditions to verify that solutions of the original system depend continuously on a as well.

Continuous Dependence on the RHS

A different kind of continuity is continuity of solutions as functions of the data $F(t, X)$. That is, if $F : \mathcal{O} \to \mathbb{R}^n$ and $G : \mathcal{O} \to \mathbb{R}^n$ are both C^1 in X, and $|F - G|$ is uniformly small, we expect solutions to $X' = F(t, X)$ and $Y' = G(t, Y)$, having the same initial values, to be close. This is true; in fact, we have the following more precise result.

Theorem. *Let $\mathcal{O} \subset \mathbb{R} \times \mathbb{R}^n$ be an open set containing $(0, X_0)$ and suppose that $F, G : \mathcal{O} \to \mathbb{R}^n$ are C^1 in X and continuous in t. Suppose also that for all $(t, X) \in \mathcal{O}$*

$$|F(t,X) - G(t,X)| < \epsilon.$$

Let K be a Lipschitz constant in X for $F(t, X)$. If $X(t)$ and $Y(t)$ are solutions of the equations $X' = F(t, X)$ and $Y' = G(t, Y)$ respectively on some interval J, and $X(0) = X_0 = Y(0)$, then

$$|X(t) - Y(t)| \leq \frac{\epsilon}{K}\left(\exp(K|t|) - 1\right)$$

for all $t \in J$.

$$X' = F(t, X)$$
$$Y' = G(t, Y)$$

$|F - G|$ *is small*

$|X - Y|$ diverges no faster than an exponential growth rate determined by the Lipschitz

HSD, p402

Continuous Dependence on the RHS

Continuous Dependence on f(x, y)

Theorem 10. Let f and $\partial f/\partial y$ be continuous functions on an open rectangle $R = \{(x, y): a < x < b, c < y < d\}$ containing the point (x_0, y_0). Let F be continuous on R and assume that

(15) $$|F(x, y) - f(x, y)| \le \varepsilon \ , \qquad \text{for } (x, y) \text{ in } R.$$

Let ϕ be the solution to the initial value problem

(16) $$\frac{dy}{dx} = f(x, y) \ , \qquad y(x_0) = y_0 \ ,$$

and let ψ be a solution to

(17) $$\frac{dy}{dx} = F(x, y) \ , \qquad y(x_0) = y_0 \ .$$

Assume both solutions $\phi(x)$, $\psi(x)$ exist on $[x_0 - h, x_0 + h]$ and their graphs lie in a closed rectangle $R_0 \subset R$. Then, for $|x - x_0| \le h$,

(18) $$|\phi(x) - \psi(x)| \le \varepsilon h e^{Lh} \ ,$$

where L is any positive constant such that $|(\partial f/\partial y)(x, y)| \le L$ for all (x, y) in R_0.

In particular, as F approaches f uniformly on R—that is, as $\varepsilon \to 0^+$ in (15)—the solution $\psi(x)$ approaches $\phi(x)$ uniformly on $[x_0 - h, x_0 + h]$.

Nagle et al.