MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #8

Chapter 3 Phase Portraits for Planar Systems

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An Epidemic Model: SIR

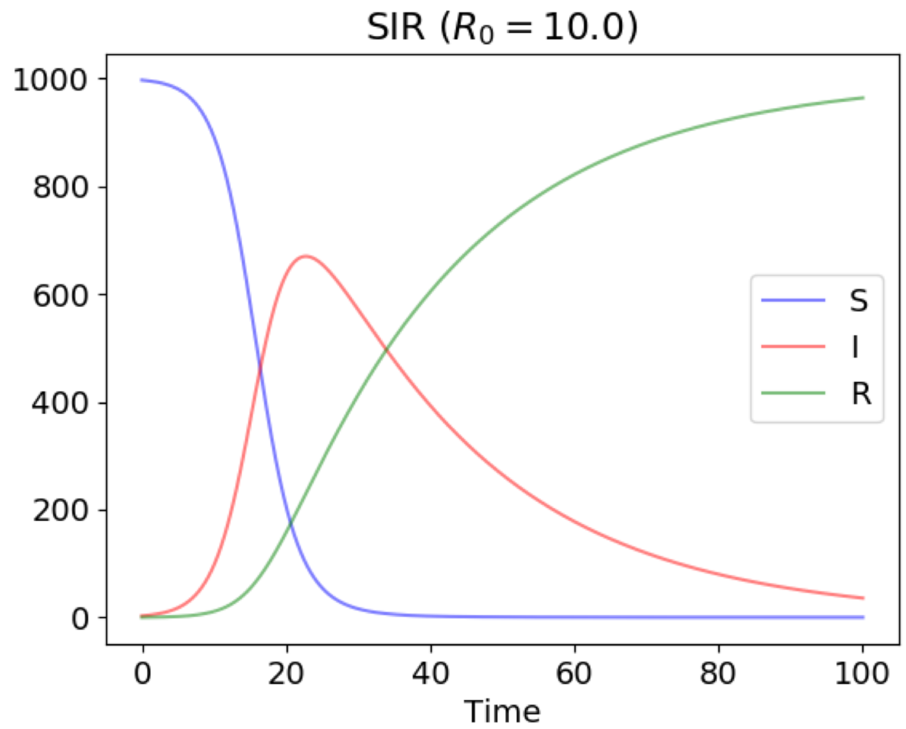
Susceptible | Infectious | Recovered

$$\frac{dS}{dt} = -\frac{\beta}{N} SI,$$

$$\frac{dI}{dt} = \frac{\beta}{N} SI - \nu I,$$

$$\frac{dR}{dt} = \nu I.$$

$$N = S + I + R = constant$$



Reproduction of Wikipedia

$\beta > 0$: infection rate
(transmission rate, or transmission coefficients);

$\nu > 0$: recovery rate;

$R_0 = \frac{\beta}{\nu}$: basic reproduction #, or contact #.

Problem 4: An Epidemic Model

4: [25 points] Consider the following epidemic model (Kermack and McKendrick, 1927), which is called the "SIR" model:

2D

$$\boxed{\begin{aligned} \frac{dS}{dt} &= -\frac{\beta}{N} SI, \\ \frac{dI}{dt} &= \frac{\beta}{N} SI - \nu I, \end{aligned}} \quad \begin{aligned} &= F(S,I) \quad (4.1) \\ &= G(S,I) \quad (4.2) \end{aligned}$$

$$\frac{dR}{dt} = \nu I. \qquad = H(I) \tag{4.3}$$

Here, S, I, and R denote susceptible, infected, and recovered individuals, respectively. Three parameters, $\beta > 0$, $\nu > 0$, and $N > 0$, represent a transmission rate, a recovery rate, and a fixed population ($N = S + I + R$), respectively. Complete the following derivations to convert Eqs. (4.1)-(4.3) into the following equations:

1D with one ODE

$$\boxed{\begin{aligned} S &= S(0) e^{-\frac{\beta}{N\nu}(R(t)-R(0))}, && (4.4) \\ I &= N - S(0) e^{-\frac{\beta}{N\nu}(R(t)-R(0))} - R, && (4.5) \\ \frac{dR}{dt} &= \nu \left(N - R - S(0) e^{-\frac{\beta}{N\nu}(R(t)-R(0))} \right), && (4.6) \end{aligned}}$$

where $S(0)$ and $R(0)$ represent the initial values of S and R, respectively.

Problem 4: An Epidemic Model

(a) Show

$$S + I + R = constant = N \tag{4.7}$$

(i.e., $\frac{d(S+I+R)}{dt} = 0$).

(b) Apply Eqs (4.1) and (4.2) to obtain the following:

$$\frac{S'}{S} = -\frac{\beta}{N\nu} R'.$$

Integrate the above Eq. to obtain Eq. (4.4), yielding $S = S(R)$.

(c) Apply Eqs. (4.4) and (4.7) to find Eq. (4.5) for I, which is a function of R.

(d) Apply the above to obtain Eq. (4.6).

(e) Briefly discuss how to analyze Eq. (4.6) to reveal the characteristics of the solution.

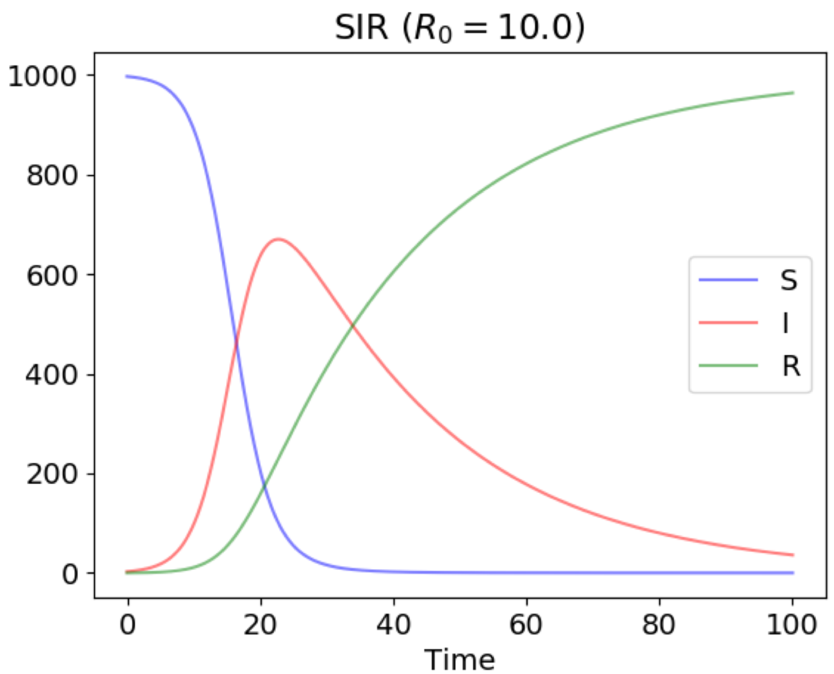
Note that based on Eqs. (4.4)-(4.6), we can obtain the solutions by solving a single first order ODE in Eq. (4.6) for $R(t)$, and then compute $S(t)$ and $R(t)$ using Eqs. (4.4) and (4.5), respectively.

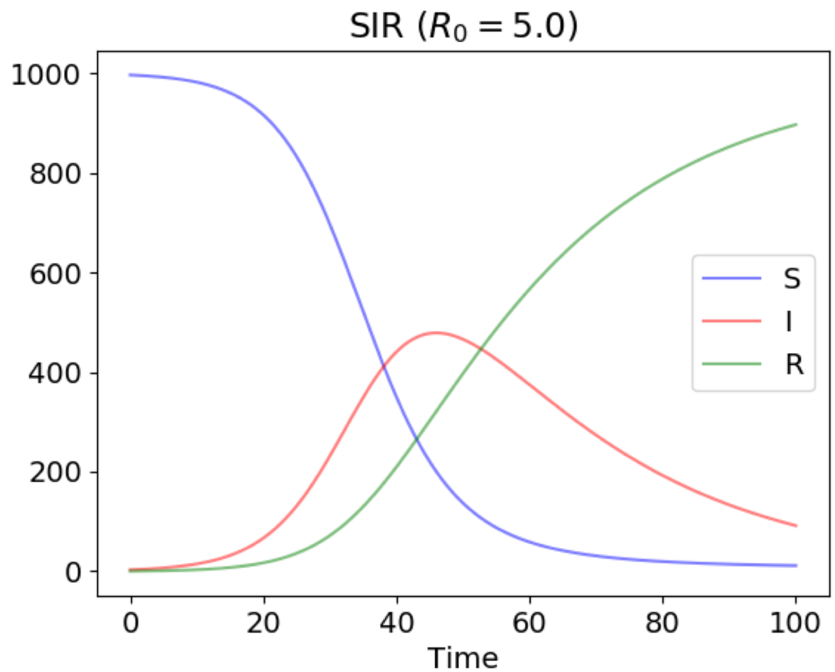
Flattening the Curve: Impact of $\mathrm{R}_0 = \frac{\beta}{\nu}$

$\mathrm{R}_0 = \frac{\beta}{\nu}$: basic reproduction #, or contact #.

$\beta > 0$: infection rate
(transmission rate, or transmission coefficients);

$\nu > 0$: recovery rate;





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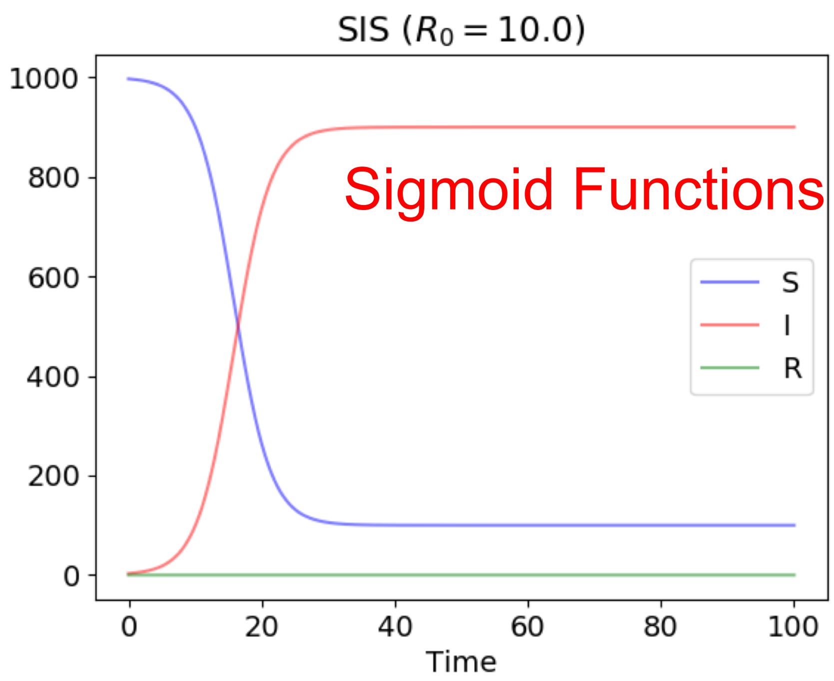
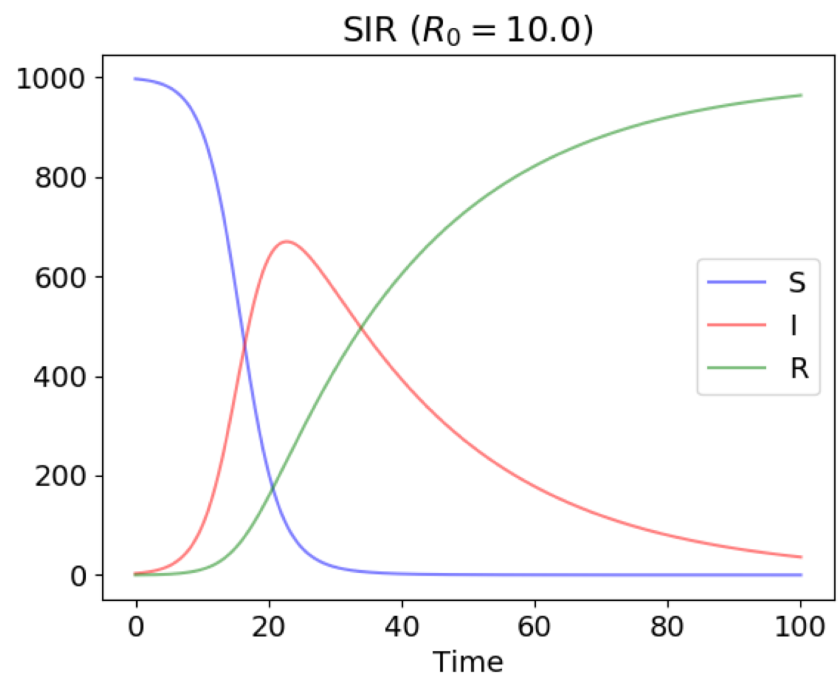
SIS vs. SIR: Sigmoid Functions

$$\frac{dS}{dt} = -\frac{\beta}{N} SI + \nu I,$$
$$\frac{dI}{dt} = \frac{\beta}{N} SI - \nu I,$$

$$\frac{dS}{dt} = -\frac{\beta}{N} SI,$$
$$\frac{dI}{dt} = \frac{\beta}{N} SI - \nu I,$$
$$\frac{dR}{dt} = \nu I.$$





A Summary for Chapter 2

$AX = \gamma$	$X' = AX$
	$(A - \lambda I)V_0 = 0$
$\|A\| \neq 0$, unique sol $\|A\| \neq 0$ & γ=0, trivial sol	$\|A - \lambda I\| \neq 0$, trivial sol
$\|A\| = 0$ • no solution • Infinitely many solutions	$\|A - \lambda I\| = 0$ • Infinitely many solutions
	• The above is called an eigenvalue problem • $Let\ \ AV_1 = \lambda_1 V_1;\ AV_2 = \lambda_2 V_2,$ we have a general solution as follows: $X = c_1\, V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t}$
	• 1D $x' = f(x)$ • $x' = f(x) \approx f'(\mathrm{x_c})(\mathrm{x} - \mathrm{x_c})$ • $\lambda = f'(\mathrm{x_c})$

Chapter 3: 2D Linear Systems

$$x' = ax + by \quad (= P(x, y)) \qquad (1)$$

$$y' = cx + dy \quad (= Q(x, y)\,) \qquad (2)$$

$$X' = AX \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

1. Eigenvalue problem: $|A - \lambda I| = 0$
2. Two linearly independent solutions, $e^{\lambda_1 t} V_1$ and $e^{\lambda_2 t} V_2$
3. Real eigenvalues for a source, sink, or saddle
4. Complex eigenvalues for a center, spiral sink or spiral sink
5. Diagonalization
6. Changing coordinates Linearly Conjugate

Re-Review: Type (I) ODEs: $ax'' + bx' + cx = 0$

(A) $y = e^{rt}$

$$ar^2 + br + c = 0$$

(B)

$$x'' + \frac{b}{a}x' + \frac{c}{a}x = 0$$

let
$$\boxed{\begin{aligned} x' &= y \\ y' &= -\frac{c}{a}x - \frac{b}{a}y \end{aligned}}$$
obtain

define $X = \begin{pmatrix} x \\ y \end{pmatrix}$; $A = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}$

$$X' = AX$$

assume $X = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{\lambda t}$;

eigenvalue problem
$$|A - \lambda I| = \begin{vmatrix} 0 - \lambda & 1 \\ -\dfrac{c}{a} & -\dfrac{b}{a} - \lambda \end{vmatrix} = 0$$

Characteristic Equation

$$a\lambda^2 + b\lambda + c = 0$$

Review: A Brief Summary for Type (I) ODEs

5

$$ay'' + by' + cy = 0$$

where a, b, and c are constants and $a \neq 0$.

Summary of Cases I–III

what is the most essential part?

Case	Roots of (2)	Basis of (1)	General Solution of (1)
I	Distinct real λ_1, λ_2	$\boxed{e^{\lambda_1 x}, e^{\lambda_2 x}}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

Simple 2D Systems with Real Distinct Eigenvalues

$$x' = ax + by \quad (= P(x, y)) \qquad (1)$$

$$y' = cx + dy \quad (= Q(x, y)\ ) \qquad (2)$$

$$X' = AX \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\boxed{\begin{aligned} x' &= \lambda_1 x + \cancel{by} \quad (= P(x, y)) \\ y' &= \cancel{cx} + \lambda_2 y \quad (= Q(x, y)) \end{aligned}}$$

A. $\lambda_1 < 0 < \lambda_2$ (different signs): saddle

B. $\lambda_1 < \lambda_2 < 0$ (both are negative): sink

C. $0 < \lambda_1 < \lambda_2$ (both are positive): source

Simple 2D Systems with Real Distinct Eigenvalues

Consider
$$\boxed{\begin{aligned} x' &= \lambda_1 x \\ y' &= \lambda_2 y \end{aligned}}$$
$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} \lambda_1 - \lambda & 0 \\ 0 & \lambda_2 - \lambda \end{pmatrix}$$

Let $|A - \lambda I| = 0 \Longrightarrow$ $(\lambda_1 - \lambda)(\lambda_2 - \lambda) = 0$ $\boxed{\lambda = \lambda_{1,2}}$

Real distinct eigenvalues include:

A. $\lambda_1 < 0 < \lambda_2$ (different signs): saddle

B. $\lambda_1 < \lambda_2 < 0$ (both are negative): sink

C. $0 < \lambda_1 < \lambda_2$ (both are positive): source

Review: A "Meta" Vector: $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$

- Consider a "meta" vector $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$, a function $f = f(x, y)$ and a vector $\vec{F} = (P(x, y), Q(x, y))$

We can define the following:

$\nabla : nabla$

- Gradient:

$$\nabla f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (f_x, f_y)$$

- Curl (a Cross product of ∇ and $\vec{F}$):

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x, y) & Q(x, y) & 0 \end{vmatrix} = k\left(Q_x - P_y\right)$$

- Divergence (a Dot product of ∇ and $\vec{F}$):

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot (P, Q) = P_x + Q_y$$

Simple 2D Systems with Real Distinct Eigenvalues

Consider
$$\boxed{\begin{aligned} x' &= \lambda_1 x = P \\ y' &= \lambda_2 y = Q \end{aligned}}$$

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad A - \lambda I = \begin{pmatrix} \lambda_1 - \lambda & 0 \\ 0 & \lambda_2 - \lambda \end{pmatrix}$$

Let $|A - \lambda I| = 0 \Longrightarrow$ $(\lambda_1 - \lambda)(\lambda_2 - \lambda) = 0$ $\boxed{\lambda = \lambda_{1,2}}$

Real distinct eigenvalues include: $\boxed{\nabla \cdot \vec{F} = P_x + Q_y = \lambda_1 + \lambda_2}$

A. $\lambda_1 < 0 < \lambda_2$ (different signs): saddle $\nabla \cdot \vec{F}\ undetermined$

B. $\lambda_1 < \lambda_2 < 0$ (both are negative): sink $\nabla \cdot \vec{F} < 0$

C. $0 < \lambda_1 < \lambda_2$ (both are positive): source $\nabla \cdot \vec{F} > 0$

- What's $\nabla \cdot \vec{F}$ for a saddle?
- Send your answer via "chat"

$\nabla \times \vec{F} = 0$

(A) A Saddle ($\lambda_1 < 0 < \lambda_2$)

$$x' = \lambda_1 x$$
$$y' = \lambda_2 y$$

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Let $|A - \lambda I| = 0 \Longrightarrow$ $\boxed{\lambda = \lambda_{1,2}}$

$AX = \lambda X \Longrightarrow$

$$\lambda_1 x = \lambda \mathrm{x}$$
$$\lambda_2 y = \lambda y$$

Consider $\lambda = \lambda_1$

Obtain

$$\lambda_1 x = \lambda_1 \mathrm{x}$$
$$\lambda_2 y = \lambda_1 y$$

$$\boxed{\begin{matrix} x: any \\ y = 0 \end{matrix}}$$

$$V_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Similarly, for $\lambda = \lambda_{2,}$ we have

$$V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

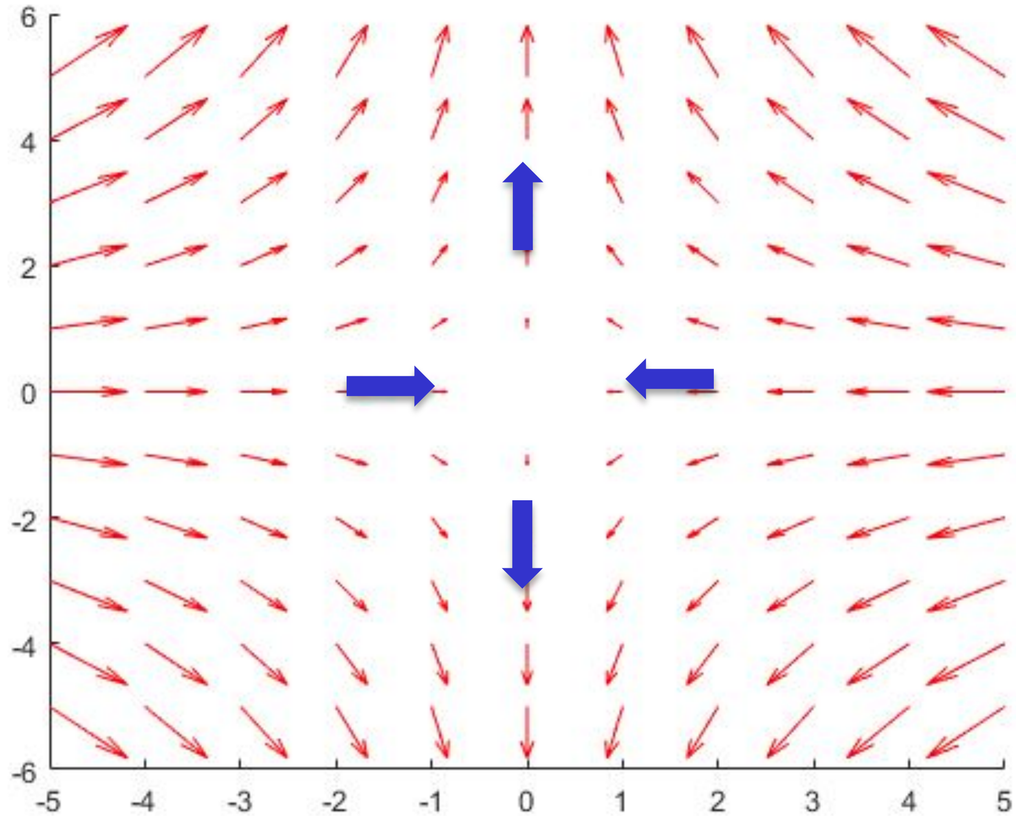
Thus, a general solution is written as

$$X(t) = \alpha X_1 + \beta X_2 = \alpha e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(A) A Saddle ($\lambda_1 < 0 < \lambda_2$)

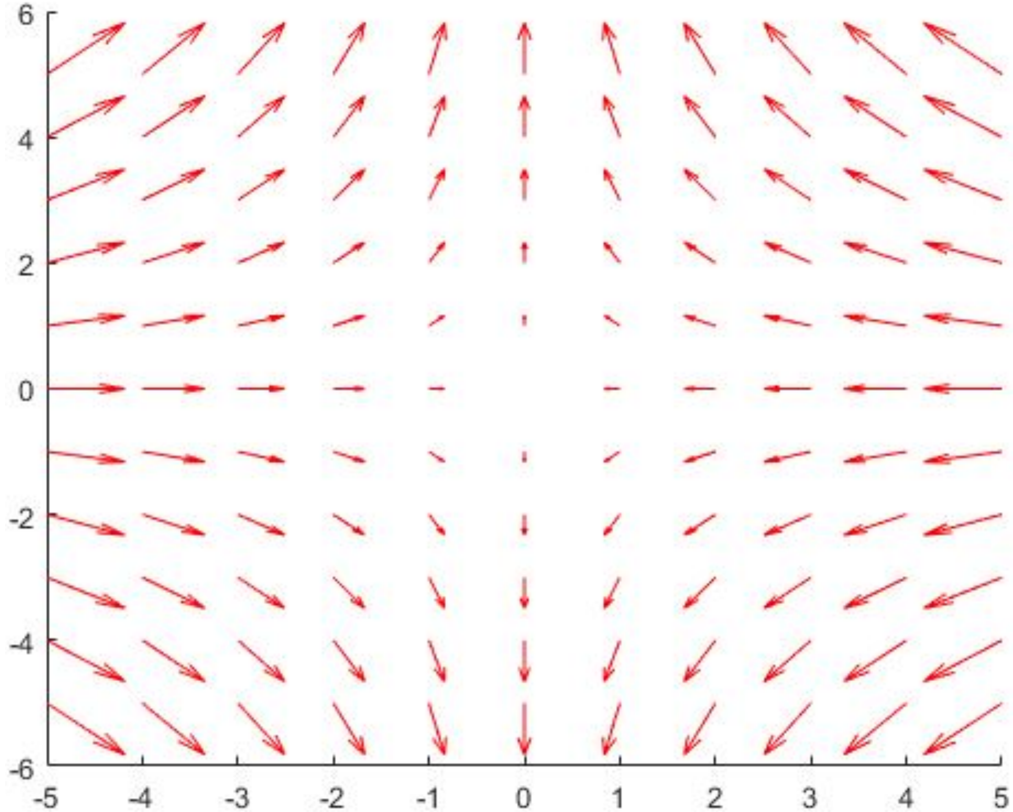
$$F(X) = (-x, y) = (P, Q)$$

	(x_i^*, y_i^*)	$\vec{F}$
A	$(2, 0)$	$(-2, 0)$
B	$(0, 2)$	$(0, 2)$
C	$(-2,0)$	$(2,0)$
D	$(0, -2)$	$(0, -2)$



MATLAB Plot for Figure 3.1

(A) A Saddle ($\lambda_1 < 0 < \lambda_2$)



MATLAB Plot for Figure 3.1

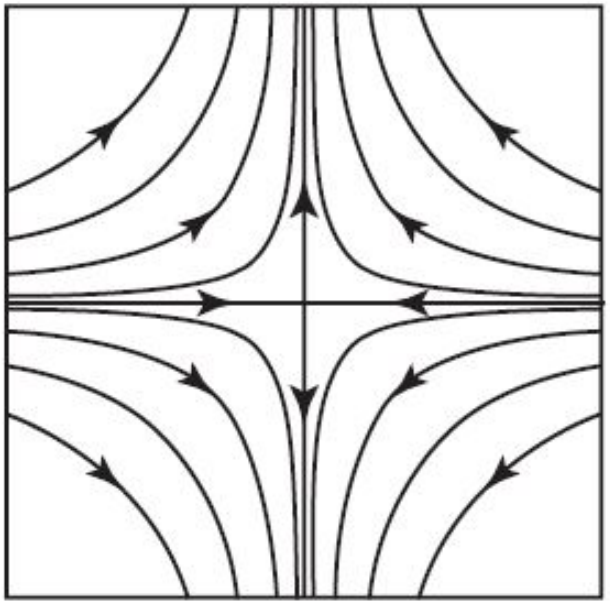
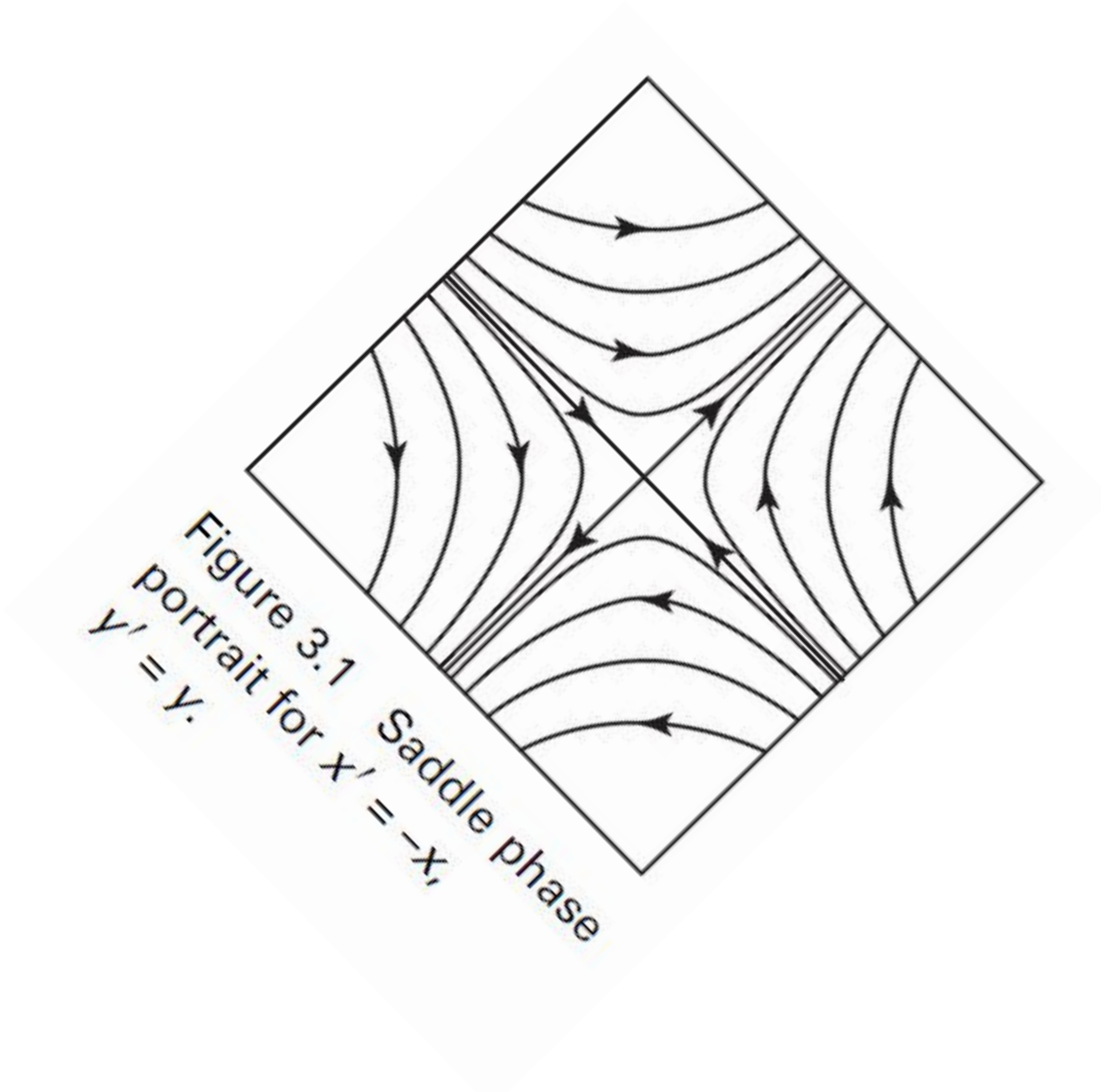


Figure 3.1 Saddle phase portrait for $x' = -x$, $y' = y$.

(A) A Saddle ($\lambda_1 < 0 < \lambda_2$)



A saddle point

4 Fun: Solve $|A - \lambda I| = 0$

Example. We return to the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}.$$

define $$X = \begin{pmatrix} x \\ y \end{pmatrix}; \; A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$$

- Find the eigenvalues of the above systems
- Send your results via ”chat”
- You have 5 minutes

(A) A Saddle: Solve $|A - \lambda I| = 0$

Example. We return to the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}.$$

define $\quad X = \begin{pmatrix} x \\ y \end{pmatrix}; \; A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$

eigenvalue problem $\quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{vmatrix} = 0$

$$\lambda^2 - 4 = 0$$

$$\lambda = 2, -2$$

(A) A Saddle: Solve $|A - \lambda I| = 0$

Solve for eigenvectors $\quad AV_0 = \lambda V_0$

$$x_0 + 3y_0 = \lambda x_0$$
$$x_0 - y_0 = \lambda y_0$$

Consider $\lambda = 2$

$$\boxed{\begin{aligned} x_0 + 3y_0 &= 2x_0 \\ x_0 - y_0 &= 2y_0 \end{aligned}} \qquad \boxed{x_0 = 3y_0} \qquad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 3y_0 \\ y_0 \end{pmatrix} = y_0 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Obtain $\boxed{V_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}}$ as an eigenvector associated with $\lambda = 2$

Similarly, Consider $\lambda = -2$

$$\boxed{\begin{aligned} x_0 + 3y_0 &= -2x_0 \\ x_0 - y_0 &= -2y_0 \end{aligned}}$$

- Find the eigenvector
- Send your results via "chat"
- You have 3 minutes

(A) A Saddle: Solve $|A - \lambda I| = 0$

Solve for eigenvectors $\quad AV_0 = \lambda V_0$

$$x_0 + 3y_0 = \lambda x_0$$
$$x_0 - y_0 = \lambda y_0$$

Consider $\lambda = 2$

$$\boxed{\begin{aligned} x_0 + 3y_0 &= 2x_0 \\ x_0 - y_0 &= 2y_0 \end{aligned}} \qquad \boxed{x_0 = 3y_0} \qquad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 3y_0 \\ y_0 \end{pmatrix} = y_0 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Obtain $\boxed{V_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}}$ as an eigenvector associated with $\lambda = 2$

Similarly, $\boxed{V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$ as an eigenvector associated with $\lambda = -2$

$$\boxed{X = \alpha X_1 + \beta X_2 = \alpha e^{\lambda_1 t} V_1 + \beta e^{\lambda_2 t} V_2}$$

$$\boxed{X = \alpha e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \beta e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

(A) A Saddle ($\lambda_1 < 0 < \lambda_2$)

$$X = \alpha X_1 + \beta X_2 = \alpha e^{2t}\begin{pmatrix}3\\1\end{pmatrix} + \beta e^{-2t}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$X \sim \alpha e^{2t}\begin{pmatrix}3\\1\end{pmatrix} = X_1(t) \;\; as\ t \to \infty\ \&\ \alpha \neq 0$$

exponential growth

$$X \sim \beta e^{-2t}\begin{pmatrix}1\\-1\end{pmatrix} = X_2(t)\ as\ t \to -\infty\ \&\ \beta \neq 0$$

exponential decay

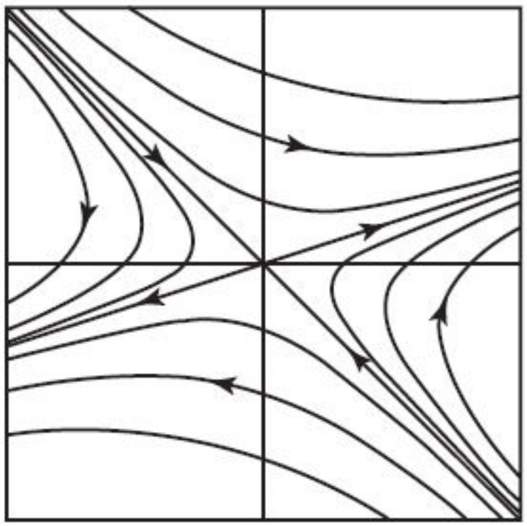
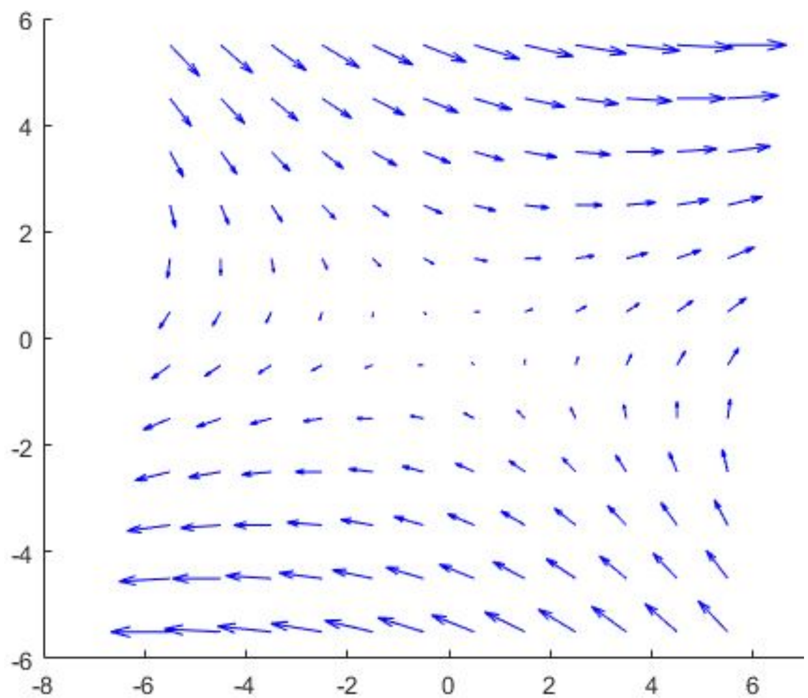


Figure 3.2 Saddle phase portrait for $x' = x + 3y$, $y' = x - y$.



MATLAB

Trajectory, Orbit, and Path: Slope

$$x' = ax + by \quad (= P(x,y)) \qquad (1)$$

$$y' = cx + dy \quad (= Q(x,y)\ ) \qquad (2)$$

From (1-2) we see that the slope of a path passing through a point A: (X,Y) is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{Q(x,y)}{P(x,y)} \qquad (3)$$

- Note that (3) gives no information about the orientation of a path.
- Note further that we must have $P(x,y) \neq 0$ at A.
- If $P(x,y) = 0$ but $\mathrm{Q}(x,y) \neq 0$ at A, we can take $dx/dy = P(x,y)/Q(x,y)$ instead of (3) and conclude from $\frac{dx}{dy} = 0$ that the tangent of C at A is vertical.
- However, what can we do if both P and Q are zero at some point?

B: Sink ($\lambda_1 < \lambda_2 < 0$): move toward (0,0)

$$x' = \lambda_1 x \qquad\qquad A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$
$$y' = \lambda_2 y$$

Let $|A - \lambda I| = 0 \Longrightarrow$ $\boxed{\lambda = \lambda_{1,2}}$ $\quad V_1 = \binom{x}{y} = \binom{1}{0} \qquad V_2 = \binom{0}{1}$

Thus, a general solution is written as

$$X(t) = \alpha e^{\lambda_1 t} \binom{1}{0} + \beta e^{\lambda_2 t} \binom{0}{1} \qquad \begin{aligned} x &= \alpha e^{\lambda_1 t} \\ y &= \beta e^{\lambda_2 t} \end{aligned}$$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\lambda_2 \beta e^{\lambda_2 t}}{\lambda_1 \alpha e^{\lambda_1 t}} = \frac{\lambda_2 \beta}{\lambda_1 \alpha} e^{(\lambda_2 - \lambda_1) t} \qquad \rightarrow \infty \ (or\ -\infty)\ as\ t \rightarrow \infty$$

$$(\lambda_2 - \lambda_1) > 0$$

- These solutions tend to the origin (a sink) tangentially to the y axis (i.e., vertical asymptotes)
- x tends to zero much quickly
- λ_1 (λ_2) is referred to as the stronger (weaker) eigenvalue.

B: Sink $(\lambda_1 < \lambda_2 < 0)$

$\lambda_1 < \lambda_2 < 0$

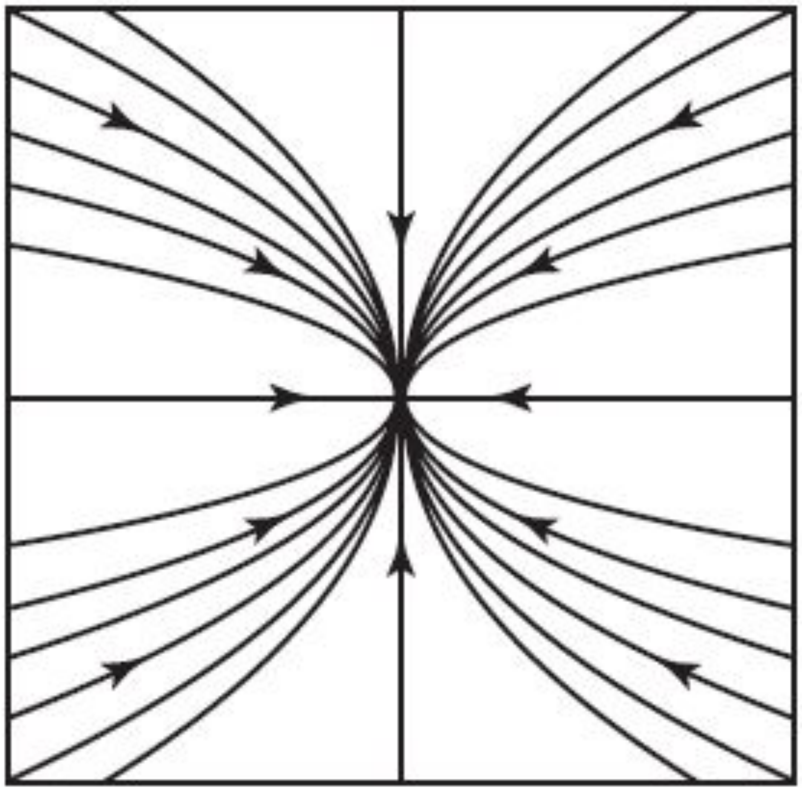
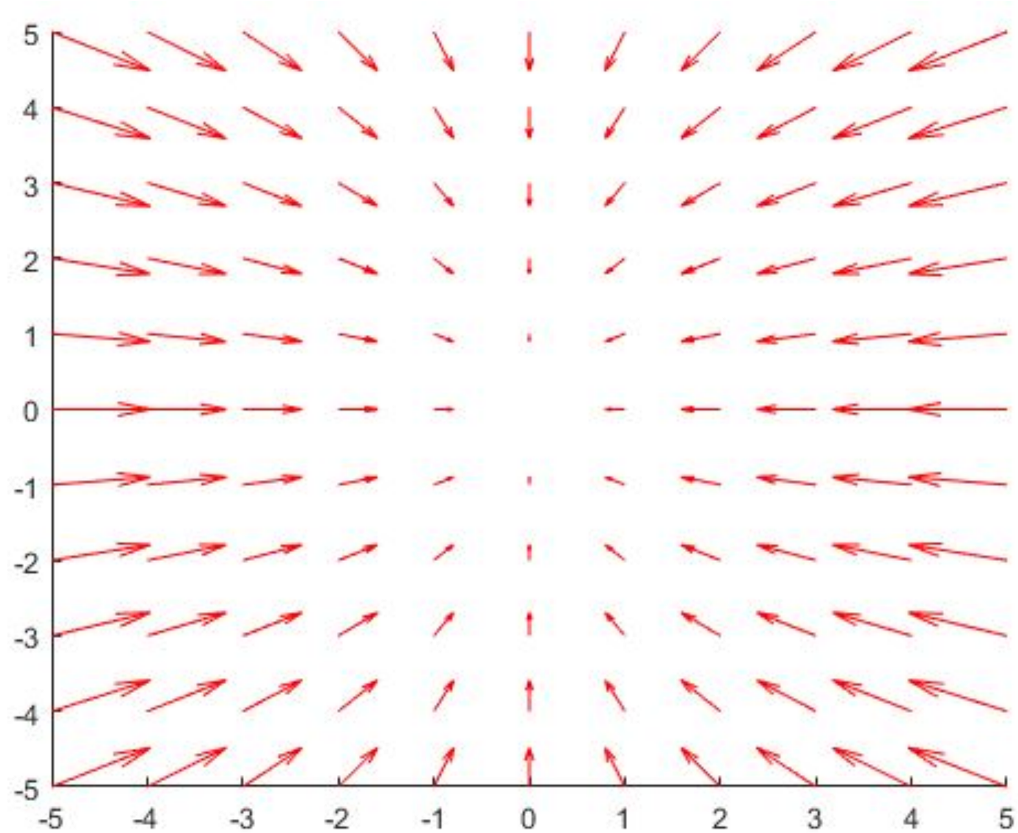


Figure 3.3 Phase portraits for (a) a sink

$X' = -2x$ $Y' = -y$



MATLAB Plot for Figure 3.3a

B: Sink ($\lambda_1 < \lambda_2 < 0$): a general case

Thus, a general solution is written as

$$X(t) = \binom{x}{y} = \alpha e^{\lambda_1 t} \binom{u_1}{u_2} + \beta e^{\lambda_2 t} \binom{v_1}{v_2}$$

$$x = \alpha e^{\lambda_1 t} u_1 + \beta e^{\lambda_2 t} v_1$$

$$y = \alpha e^{\lambda_1 t} u_2 + \beta e^{\lambda_2 t} v_2$$

$(\lambda_1 - \lambda_2) < 0$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\lambda_1 \alpha e^{\lambda_1 t} u_1 + \lambda_2 \beta e^{\lambda_2 t} v_1}{\lambda_1 \alpha e^{\lambda_1 t} u_2 + \lambda_2 \beta e^{\lambda_2 t} v_2} = \frac{\lambda_1 \alpha e^{(\lambda_1 - \lambda_2)t} u_1 + \lambda_2 \beta v_1}{\lambda_1 \alpha e^{(\lambda_1 - \lambda_2)t} u_2 + \lambda_2 \beta v_2} \rightarrow \frac{v_2}{v_1} \quad as\ t \rightarrow \infty$$

- All solutions (except for those on the straight line corresponding the stronger eigenvalue) tend to the origin (a sink) tangentially to the straight-line solution corresponding to the weaker eigenvalue (λ_2).
- λ_1 (λ_2) is referred to as the stronger (weaker) eigenvalue.

(C): A Source $(0 < \lambda_1 < \lambda_2)$

$$x' = \lambda_1 x$$
$$y' = \lambda_2 y$$

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Let $|A - \lambda I| = 0 \Longrightarrow$ $\boxed{\lambda = \lambda_{1,2}}$

$$V_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

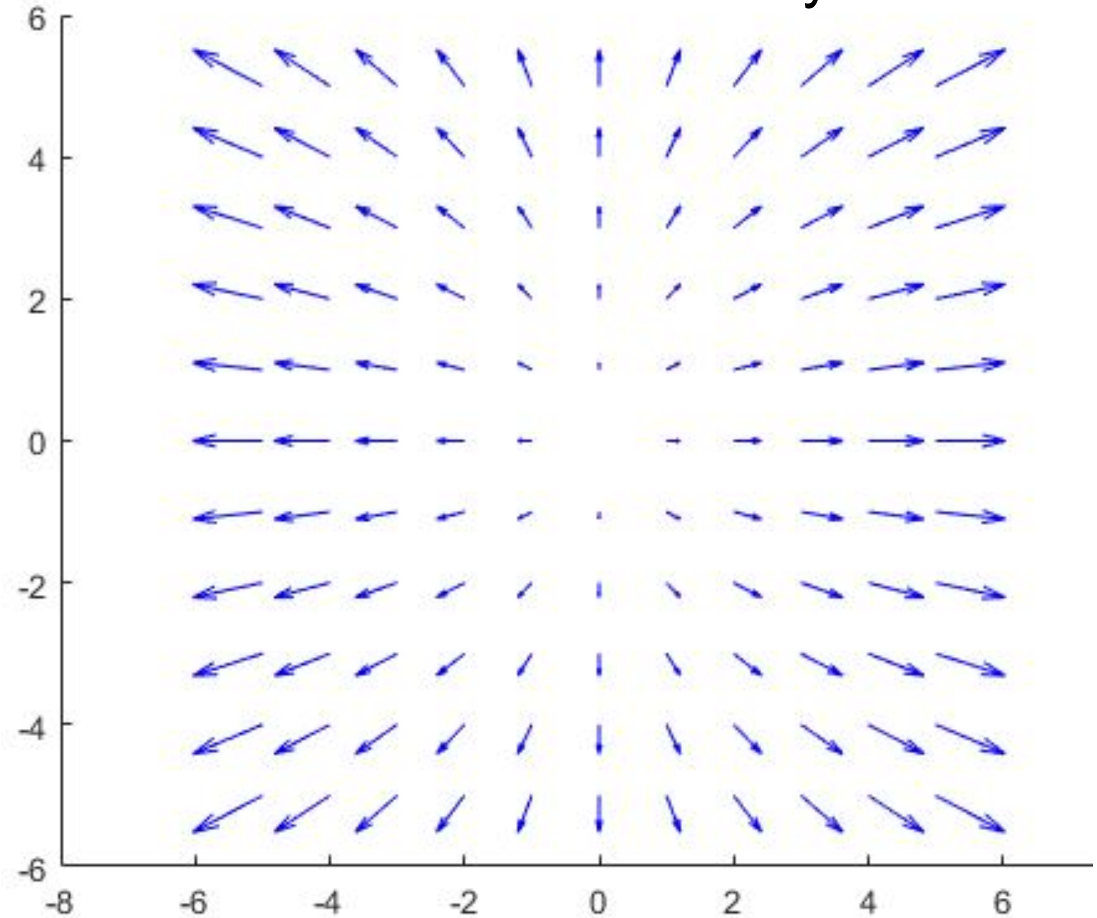
Thus, a general solution is written as

$$X(t) = \alpha X_1 + \beta X_2 = \alpha e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(C): A Source $(0 < \lambda_1 < \lambda_2)$

$\lambda_1 > 0$; $\lambda_2 > 0$

$X' = 2x$ $Y' = y$



MATLAB Plot for Figure 3.3b

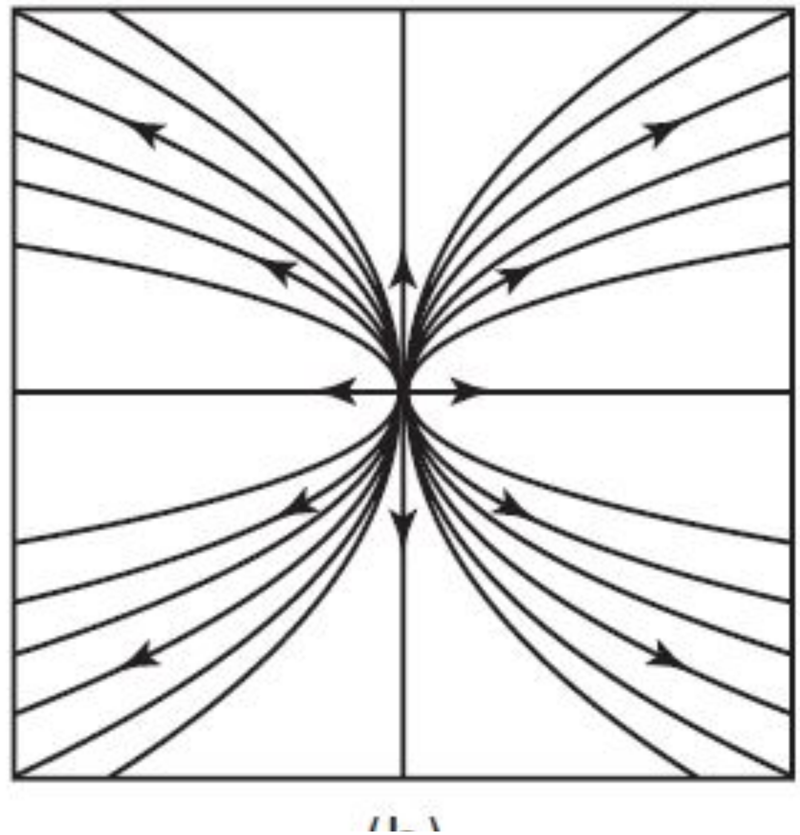


Figure 3.3 Phase portraits for (b) a source.

Simple 2D Systems with Complex Eigenvalues

$$x' = ax + by \quad (= P(x, y)) \qquad (1)$$

$$y' = cx + dy \quad (= Q(x, y)\ ) \qquad (2)$$

$$X' = AX \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\boxed{A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}}$$

$$\boxed{A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}}$$

$$\boxed{\begin{aligned} x' &= \cancel{ax} + \beta y \quad (= P(x, y)) \\ y' &= \beta x + \cancel{dy} \quad (= Q(x, y)) \end{aligned}}$$

$$\boxed{\begin{aligned} x' &= \alpha x + \beta y \quad (= P(x, y)) \\ y' &= \beta x + \alpha y \quad (= Q(x, y)) \end{aligned}}$$

D. $\lambda_{1,2} = \pm i\,\beta$: center

E. $\lambda_{1,2} = \alpha \pm i\,\beta$: spiral source or sink

Review: Complex Roots

5

$$ay'' + by' + cy = 0$$

where a, b, and c are constants and $a \neq 0$.

Summary of Cases I–III

what is the most essential part?

Case	Roots of (2)	Basis of (1)	General Solution of (1)
III	Complex conjugate $\lambda_1 = -\frac{1}{2}a + i\omega$, $\lambda_2 = -\frac{1}{2}a - i\omega$	$e^{-ax/2}\cos\omega x$ $e^{-ax/2}\sin\omega x$	$y = e^{-ax/2}(A\cos\omega x + B\sin\omega x)$

Review: 2nd order ODEs with Complex Eigenvalues

$$\lambda_{1,2} = \alpha \pm i\beta$$

$$x = c_1 x_1 + c_2\, x_2 = c_1\, e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$= c_1\, e^{(\alpha+i\beta)t} + c_2 e^{(\alpha-i\beta)t}$$

$$= e^{\alpha x}(c_1\, e^{(i\beta)t} + c_2 e^{(-i\beta)t})$$

$$= e^{\alpha x}(c_1\, (\cos(\beta t) + isin(\beta t)) + c_2(\cos(\beta t) - isin(\beta t))$$

$$= e^{\alpha x}((c_1 + c_2)\cos(\beta t) + i(c_1 - c_2)sin(\beta t))$$

$$= e^{\alpha x}(\mathrm{A}\cos(\beta t) + Bsin(\beta t))$$

$$A = c_1 + c_2 \qquad B = i(c_1 - c_2)$$

(D) A Center with $\lambda = \pm i\beta$

Consider $\begin{aligned} x' &= \beta y \\ y' &= -\beta x \end{aligned}$ $\boxed{A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}}$ $A - \lambda I = \begin{pmatrix} -\lambda & \beta \\ -\beta & -\lambda \end{pmatrix}$

Let $|A - \lambda I| = 0 \Longrightarrow$ $\lambda^2 + \beta^2 = 0$ $\boxed{\lambda = \pm i\beta}$

$AV_0 = \lambda V_0$

$$\begin{aligned} \beta y_0 &= \lambda x_0 \\ -\beta x_0 &= \lambda y_0 \end{aligned}$$

$\boxed{\lambda = i\beta}$

$$\begin{aligned} \beta y_0 &= i\beta x_0 \\ -\beta x_0 &= i\beta y_0 \end{aligned}$$

- Find the eigenvector
- Send your results via ”chat”
- You have 3 minutes

(D) A Center with $\lambda = \pm i\beta$

Consider $\begin{aligned} x' &= \beta y \\ y' &= -\beta x \end{aligned}$ $\boxed{A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}}$ $A - \lambda I = \begin{pmatrix} -\lambda & \beta \\ -\beta & -\lambda \end{pmatrix}$

Let $|A - \lambda I| = 0 \Longrightarrow$ $\lambda^2 + \beta^2 = 0$ $\boxed{\lambda = \pm i\beta}$

$AV_0 = \lambda V_0$ $\begin{aligned} \beta y_0 &= \lambda x_0 \\ -\beta x_0 &= \lambda y_0 \end{aligned}$

$\boxed{\lambda = i\beta}$ $\begin{aligned} \beta y_0 &= i\beta x_0 \\ -\beta x_0 &= i\beta y_0 \end{aligned}$ $y_0 = ix_0$ $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ ix_0 \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ i \end{pmatrix}$

Obtain $\boxed{V_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}}$ as an eigenvector associated with $\lambda = i\beta$

Similarly, $\boxed{V_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}}$ as an eigenvector associated with $\lambda = -i\beta$

(D) A Center with $\lambda = \pm i\beta$

Thus, a general solution is written as

$$X(t) = ae^{i\beta t}\begin{pmatrix}1\\ i\end{pmatrix} + be^{-i\beta t}\begin{pmatrix}1\\ -i\end{pmatrix}$$

$$= a\big(\cos(\beta t) + i sin(\beta t)\big)\begin{pmatrix}1\\ i\end{pmatrix} + b\big(\cos(\beta t) - i sin(\beta t)\big)\begin{pmatrix}1\\ -i\end{pmatrix}$$

$$= \begin{pmatrix}(\mathrm{a}+\mathrm{b})\cos(\beta t)\\ (-a-b)sin(\beta t)\end{pmatrix} + i\begin{pmatrix}(a-b)sin(\beta t)\\ (a-b)\cos(\beta t)\end{pmatrix}$$

$$= (\mathrm{a}+\mathrm{b})\begin{pmatrix}\cos(\beta t)\\ -sin(\beta t)\end{pmatrix} + i(a-b)\begin{pmatrix}sin(\beta t)\\ \cos(\beta t)\end{pmatrix}$$

$$\boxed{= \mathrm{c}_1\mathrm{X}_{\mathrm{re}}(\mathrm{t}) + c_2X_{im}}$$

$$\boxed{\mathrm{X}_{\mathrm{re}}(\mathrm{t}) = Re\left(e^{i\beta t}\begin{pmatrix}1\\ i\end{pmatrix}\right) = \begin{pmatrix}\cos(\beta t)\\ -sin(\beta t)\end{pmatrix}}$$

$$\boxed{\mathrm{X}_{im}(\mathrm{t}) = Im\left(e^{i\beta t}\begin{pmatrix}1\\ i\end{pmatrix}\right) = \begin{pmatrix}\sin(\beta t)\\ cos(\beta t)\end{pmatrix}}$$

(D) A Center with $\lambda = \pm i\beta$

Thus, a general solution is written as

$$X(t) = a\boxed{e^{i\beta t}\begin{pmatrix}1\\ i\end{pmatrix}} + be^{-i\beta t}\begin{pmatrix}1\\ -i\end{pmatrix}$$

$$= \mathrm{c}_1\mathrm{X}_{\mathrm{re}}(\mathrm{t}) + c_2X_{im}$$

$$\mathrm{X}_{\mathrm{re}}(\mathrm{t}) = \begin{pmatrix}\cos(\beta t)\\ -sin(\beta t)\end{pmatrix} \quad \mathrm{X}_{im}(\mathrm{t}) = \begin{pmatrix}\sin(\beta t)\\ cos(\beta t)\end{pmatrix}$$

$$Re\left(e^{i\beta t}\begin{pmatrix}1\\ i\end{pmatrix}\right) = Re\left((\cos(\beta t) + isin(\beta t))\begin{pmatrix}1\\ i\end{pmatrix}\right) = \begin{pmatrix}\cos(\beta t)\\ -sin(\beta t)\end{pmatrix} = X_{re}$$

$$\mathrm{Im}\left(e^{i\beta t}\begin{pmatrix}1\\ i\end{pmatrix}\right) = Im\left((\cos(\beta t) + isin(\beta t))\begin{pmatrix}1\\ i\end{pmatrix}\right) = \begin{pmatrix}\sin(\beta t)\\ cos(\beta t)\end{pmatrix} = X_{im}$$

(D) A Center with $\lambda = \pm i\beta$

Thus, a general solution is written as

$$X(t) = \boxed{a e^{i\beta t} \begin{pmatrix} 1 \\ i \end{pmatrix}} + b e^{-i\beta t} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$X_{re}(t) = \begin{pmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{pmatrix}$$

$$X_{im}(t) = \begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix}$$

$$= c_1 X_{re}(t) + c_2 X_{im}$$

$$= c_1 \begin{pmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(\beta t) \\ \cos(\beta t) \end{pmatrix}$$

$$\boxed{x(t) = c_1 \cos(\beta t) + c_2 \sin(\beta t)}$$

also Obtained by solving a 2nd-order ODE for x

$$\boxed{y(t) = -c_1 \sin(\beta t) + c_2 \cos(\beta t)}$$

Note that $\beta y = x'$

Solution for a Center with $\lambda = \pm i\beta$

$\lambda_{1,2} = \pm\, i\, \beta$

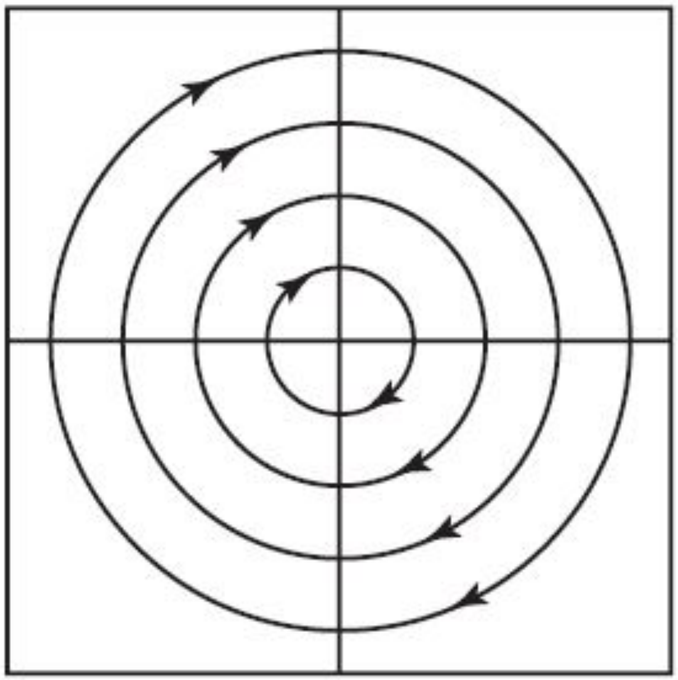
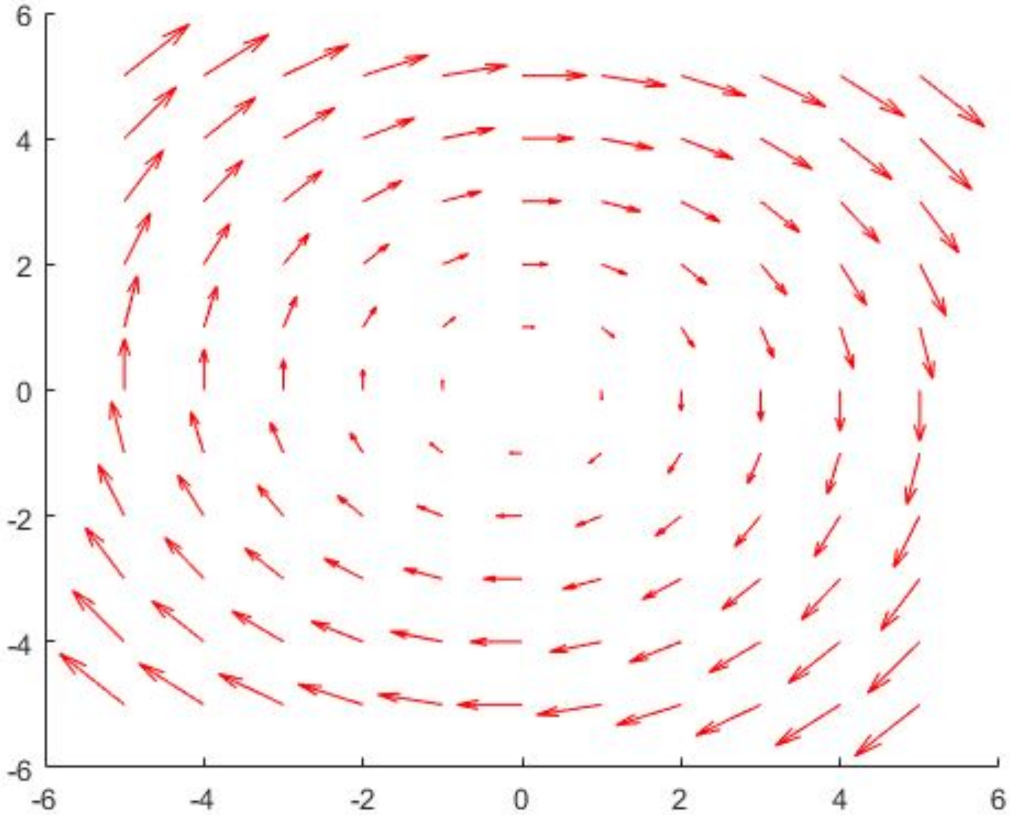


Figure 3.4 Phase portrait for a center.

$X' = 2y \qquad Y' = -2x$



MATLAB Plot for Figure 3.4

(E): Spiral Sink or Spiral Source with $\lambda = \alpha \pm i\beta$

Consider $$x' = \alpha x + \beta y$$ $$y' = -\beta x + \alpha y$$ $$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

Let $|A - \lambda I| = 0 \Longrightarrow$ $\boxed{\lambda = \alpha \pm i\beta}$

$$AV_0 = \lambda V_0$$

$$\alpha x_0 + \beta y_0 = \lambda x_0$$
$$-\beta x_0 + \alpha y_0 = \lambda y_0$$

Consider $\lambda = \alpha + i\beta$

$$\alpha x_0 + \beta y_0 = (\alpha + i\beta) x_0$$
$$-\beta x_0 + \alpha y_0 = (\alpha + i\beta) y_0$$

$$y_0 = i x_0 \qquad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ i x_0 \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Obtain

$\boxed{V_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}}$ as an eigenvector associated with $\lambda = \alpha + i\beta$

(E): Spiral Sink or Spiral Source with $\lambda = \alpha \pm i\beta$

Thus, a general solution is written as

$$X(t) = ae^{\alpha+i\beta t}\begin{pmatrix}1\\ i\end{pmatrix} + be^{\alpha-i\beta t}\begin{pmatrix}1\\ -i\end{pmatrix}$$

$$= e^{\alpha t}(\mathrm{c}_1\mathrm{X}_{\mathrm{re}}(\mathrm{t}) + c_2 X_{im})$$

$$\mathrm{X}_{\mathrm{re}}(\mathrm{t}) = \begin{pmatrix}\cos(\beta t)\\ -sin(\beta t)\end{pmatrix} \quad \mathrm{X}_{im}(\mathrm{t}) = \begin{pmatrix}\sin(\beta t)\\ cos(\beta t)\end{pmatrix}$$

$$Re\left(e^{i\beta t}\begin{pmatrix}1\\ i\end{pmatrix}\right) = Re\left((\cos(\beta t) + isin(\beta t))\begin{pmatrix}1\\ i\end{pmatrix}\right) = \begin{pmatrix}\cos(\beta t)\\ -sin(\beta t)\end{pmatrix} = X_{re}$$

$$\mathrm{Im}\left(e^{i\beta t}\begin{pmatrix}1\\ i\end{pmatrix}\right) = Im\left((\cos(\beta t) + isin(\beta t))\begin{pmatrix}1\\ i\end{pmatrix}\right) = \begin{pmatrix}\sin(\beta t)\\ cos(\beta t)\end{pmatrix} = X_{im}$$

(E): Spiral Sink or Spiral Source with $\lambda = \alpha \pm i\beta$

Thus, a general solution is written as

$$X(t) = ae^{\alpha+i\beta t}\begin{pmatrix}1\\ i\end{pmatrix} + be^{\alpha-i\beta t}\begin{pmatrix}1\\ -i\end{pmatrix}$$

$$\mathrm{X_{re}(t)} = \begin{pmatrix}\cos(\beta t)\\ -sin(\beta t)\end{pmatrix}$$

$$= e^{\alpha t}(\mathrm{c_1 X_{re}(t)} + c_2 X_{im})$$

$$\mathrm{X}_{im}(\mathrm{t}) = \begin{pmatrix}\sin(\beta t)\\ cos(\beta t)\end{pmatrix}$$

decaying or growing

oscillatory

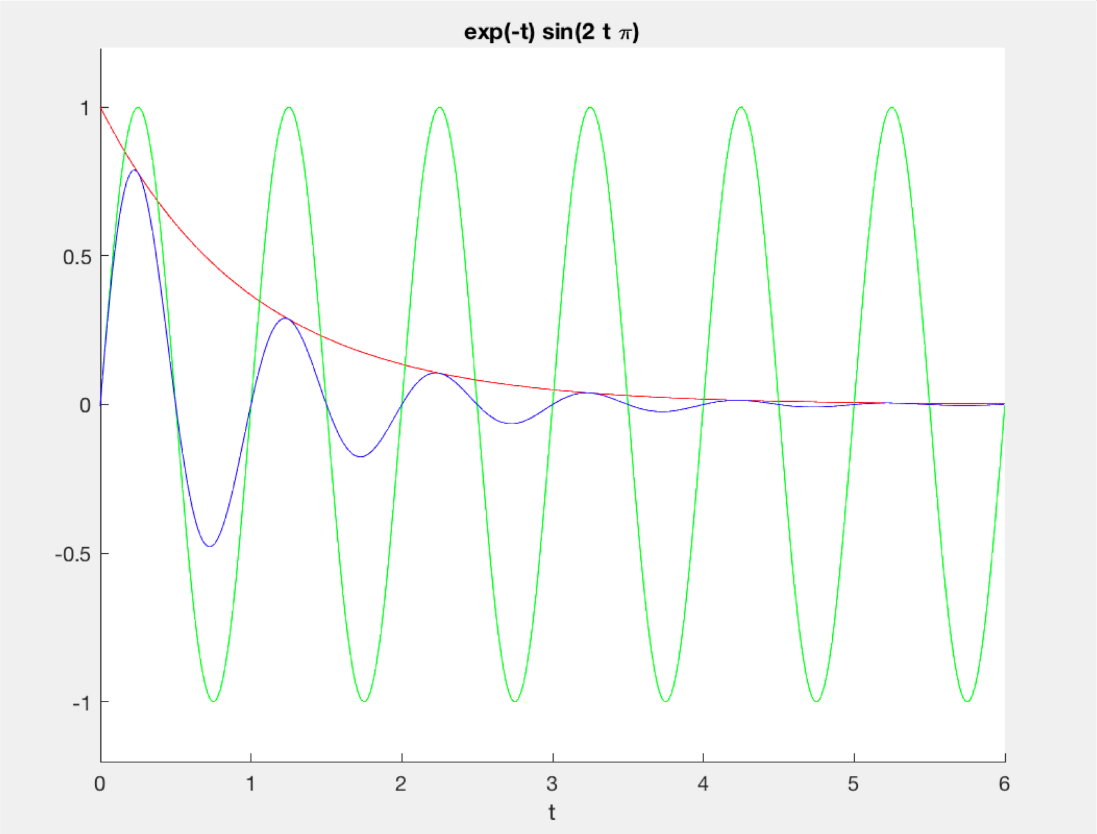
- $\alpha < 0$ spirals into the origin, a spiral sink

oscillatory with time varying radii

- $\alpha > 0$ spirals away the origin, a spiral source

A Spiral Sink with $\lambda = \alpha \pm i\beta$: Oscillatory Decay

- red: $e^{\alpha t}$
- green: $\sin(\beta t)$
- blue: $e^{\alpha t}\sin(\beta t)$




```
syms t a b
a=-1
b=2*pi
fexp=exp(a*t)
fosc=sin(b*t)
hold on
ezplot (fexp, [0, 6, -1.2, 1.2])
ezplot (fosc, [0, 6, -1.2, 1.2])
ezplot (fexp*fosc, [0, 6, -1.2, 1.2])
```

Codes

```
clear
syms t a b
a=-1
b=2*pi
fexp=exp(a*t)
fosc=sin(b*t)
hold on
ez1=ezplot (fexp, [0, 6, -1.2, 1.2])
ez2=ezplot (fosc, [0, 6, -1.2, 1.2])
ez3=ezplot (fexp*fosc, [0, 6, -1.2, 1.2])
set(ez1,'color',[1 0 0])
set(ez2,'color’,[0 1 0])
set(ez3,'color’,[0 0 1])
```

A Spiral Sink with $\lambda = \alpha \pm i\beta$

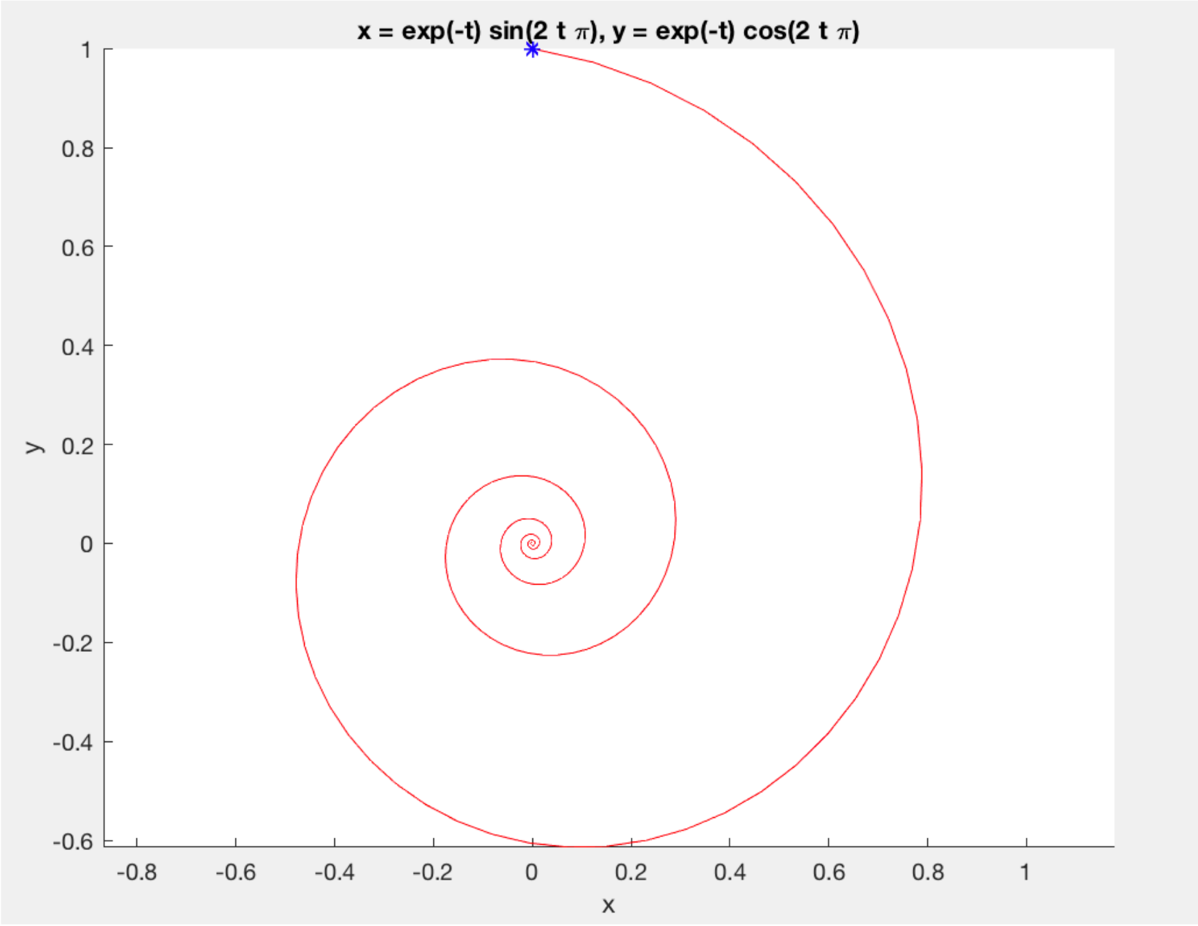
$$X(t) = e^{\alpha t}(c_1 X_{re}(t) + c_2 X_{im})$$

$$X_{im}(t) = \begin{pmatrix} \sin(\beta t) \\ cos(\beta t) \end{pmatrix}$$

$$c_1 = 0\,;\, c_2 = 1$$

$$x(t) = e^{\alpha t}\sin(\beta t)$$
$$y(t) = e^{\alpha t}\cos(\beta t)$$

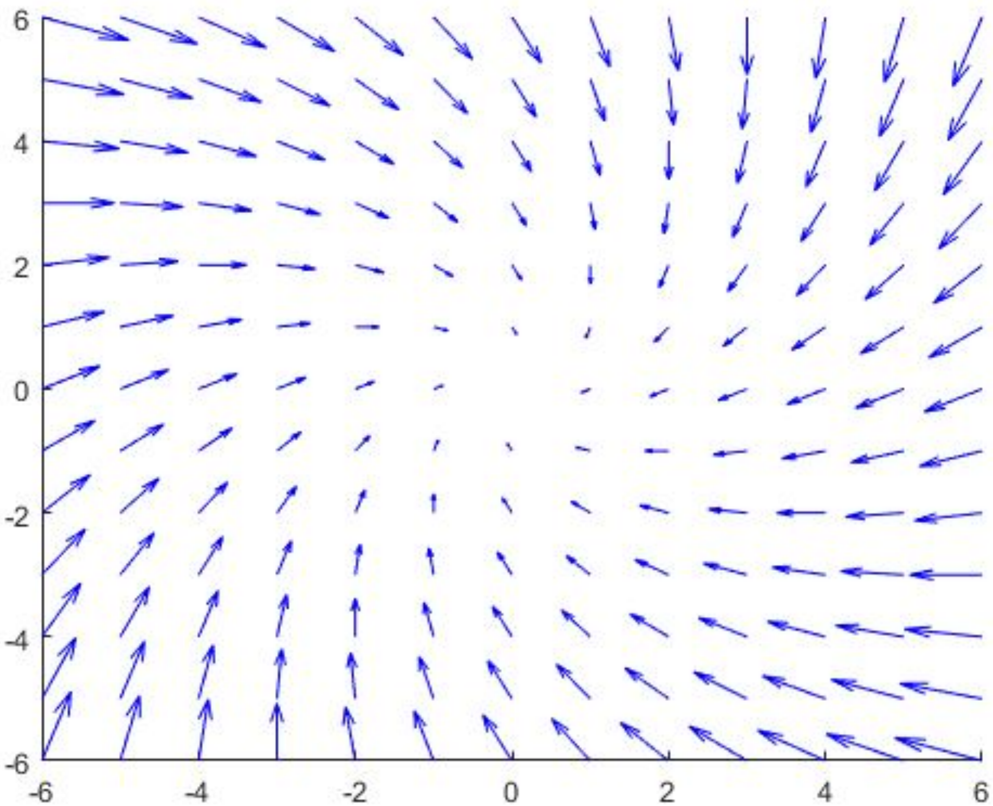
```
clear
syms t a b x y
a=-1
b=2*pi
x=exp(a*t)*sin(b*t)
y=exp(a*t)*cos(b*t)

x0=0
y0=1
hold on
ez1=ezplot (x,y, [0, 6])
plot(x0, y0, 'b*')
set(ez1,'color',[1 0 0])
```



A Spiral Sink with $\lambda = \alpha \pm i\beta$

$\lambda_{1,2} = \alpha \pm i\,\beta \qquad \alpha = -2 < 0$

$X' = -2x + y \qquad Y' = -x - 2y$



MATLAB Plot for Figure 3.5a

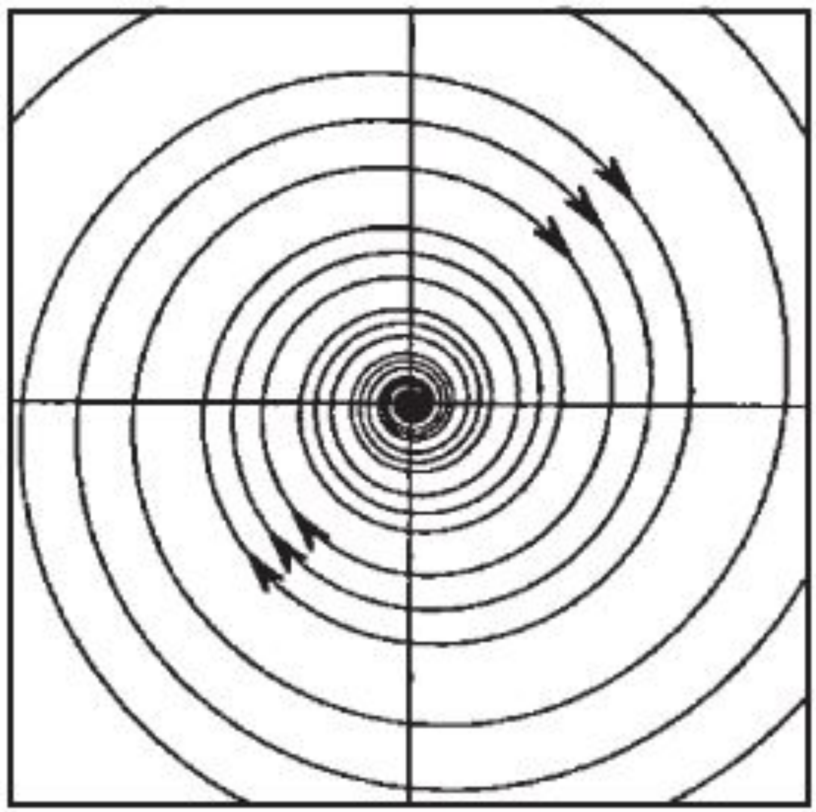


Figure 3.5a Phase portrait for spiral sink

A Spiral Source with $\lambda = \alpha \pm i\beta$

$\lambda_{1,2} = \alpha \pm i\,\beta$ $\alpha = 2 > 0$

X' = 2x + y Y' = -x + 2y

counter-clockwise

clockwise

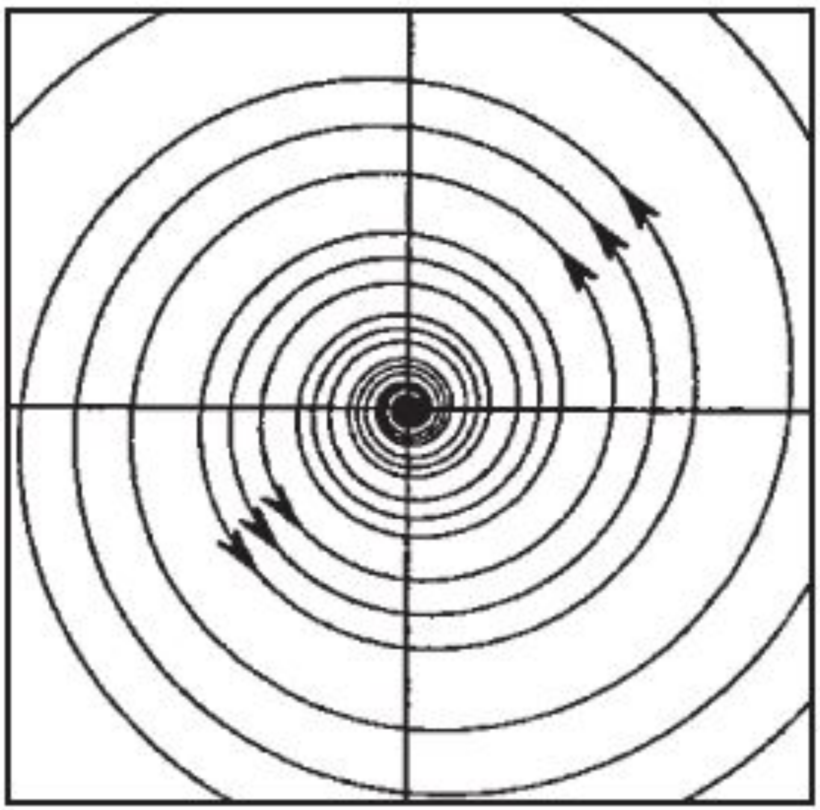
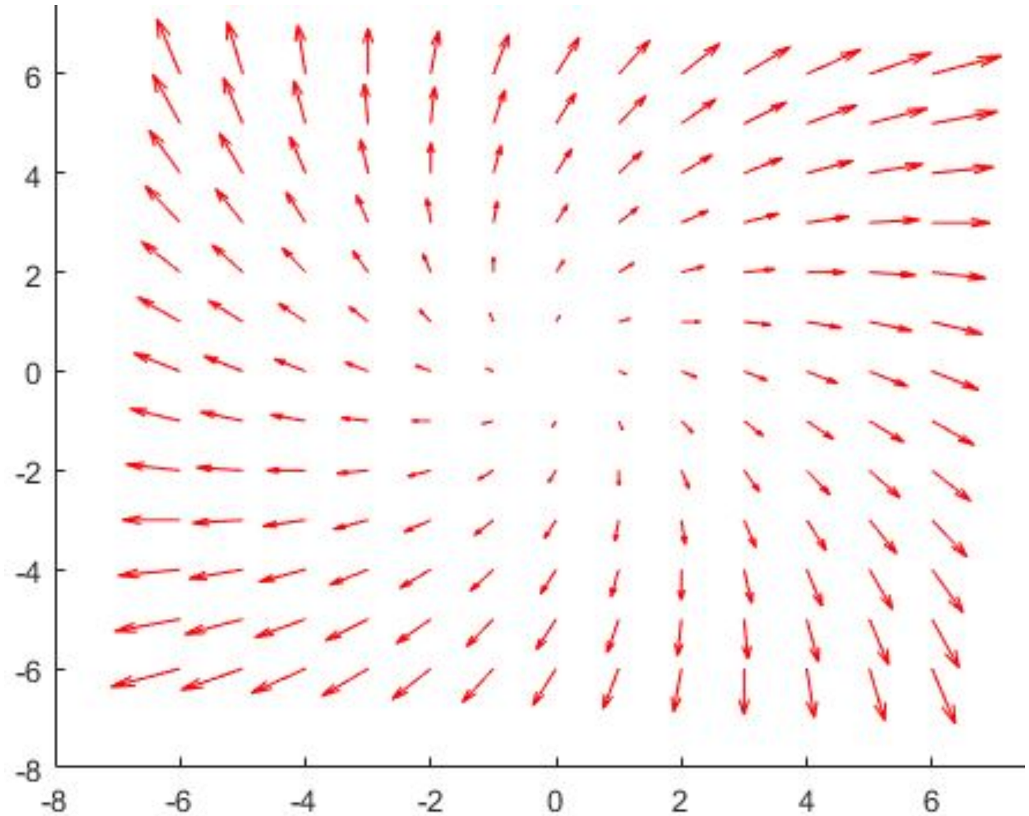


Figure 3.5b Phase portrait for spiral source



MATLAB Plot for Figure 3.5b