MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #28

Perturbation Theory

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References

- Bender, C. M., and S. A. Orszag, 2010: *Advanced Mathematical Methods for Scientists and Engineers.* Springer-Verlag, 593 pp. ISBN 978-1-4419-3187-0. [BO]

- Boyce W. and R. C. DiPrima, 2012: Elementary Differential Equations. Tenth Edition. John Wiley & Sons, INC. 832 pages ISBN: 978-0-470-45831-0

- Kreyszig, E., 2011: Advanced Engineering Mathematics. 10th edition. John Wiley & Sons, INC. ISBN 978-0-470-45836-5. 1113 pp.

- Haberman, R. 2013: *Applied Partial Differential Equations*, 5th edition, by Publisher: Pearson/Prentice Hall. ISBN-10: 0-321-79705-1. ISBN-13: 978-0321797056.

- Mathews J. and R. L. Walker, 1970: Mathematical Methods of Physics. 2nd edition, 501pp.

Perturbation Theory

Perturbation Theory

- Perturbation theory is a large collection of iterative methods for obtaining approximate solutions to problems involving a small parameter ε.

Three typical steps of perturbative analysis are:

1. Convert the original problem into a perturbation problem by introducing the small parameter ε.
2. Assume an expression for the answer in the form of a perturbation series and compute the coefficients of that series.
3. Recover the answer to the original problem by summing the perturbation series for the appropriate value of ε.

Perturbation Theory: An Example

$$x^3 - (4+\epsilon)x + 2\epsilon = 0 \qquad \epsilon = 0.001$$

$$x(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n \qquad \boxed{x(\epsilon) = a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots}$$

$$(a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots)^3 - (4+\epsilon)(a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots) + 2\epsilon = 0$$

$O(\epsilon^0)$ or $\epsilon = 0$

$$(a_0)^3 - (4)(a_0) = 0$$

$$a_0 = 0, \pm 2$$

Perturbation Theory: An Example

$$x^3 - (4+\epsilon)x + 2\epsilon = 0 \qquad \epsilon = 0.001$$

$$x(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n \qquad \boxed{x(\epsilon) = a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots}$$

$$(a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots)^3 - (4+\epsilon)(a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots) + 2\epsilon = 0$$

$O(\epsilon^1)$

$$\boxed{3a_0^2 a_1\epsilon - (4a_1\epsilon + a_0\epsilon) + 2\epsilon = 0}$$

$a_0 = -2$

$$12a_1 - (4a_1 - 2) + 2 = 0$$

$$8a_1 + 4 = 0$$

$$a_1 = -\frac{1}{2}$$

Perturbation Theory: An Example

$$x^3 - (4+\epsilon)x + 2\epsilon = 0 \qquad \epsilon = 0.001$$

$$x(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n \qquad x(\epsilon) = a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots$$

$$(a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots)^3 - (4+\epsilon)(a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots) + 2\epsilon = 0$$

$O(\epsilon^2)$

$$3a_0a_1^2\epsilon^2 + 3a_0^2a_2\epsilon^2 - (4a_2\epsilon^2 + a_1\epsilon^2) = 0$$

$a_0 = -2$

$$-6a_1^2 + 12a_2 - (4a_2 + a_1) = 0$$

$a_1 = -\frac{1}{2}$

$$8a_2 = 6a_1^2 + a_1 = 1 \qquad a_2 = \frac{1}{8}$$

$$x(\epsilon) = a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots$$

$$\boxed{x(\epsilon) = -2 - \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2 + \cdots}$$

$$\epsilon = 0.001$$

Perturbation Theory

Perturbation theory is a large collection of iterative methods for obtaining approximate solutions to problems involving a small parameter ε. Three typical steps of perturbative analysis are:

1. Convert the original problem into a perturbation problem by introducing the small parameter ε.

$$x^3 - 4.001x + 0.002 = 0. \quad \Rightarrow \quad x^3 - (4 + \varepsilon)x + 2\varepsilon = 0.$$

1. Assume an expression for the answer in the form of a perturbation series and compute the coefficients of that series.

$$\boxed{x(\varepsilon) = \sum_{n=0}^{\infty} a_n \varepsilon^n.} = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \ldots$$

1. Recover the answer to the original problem by summing the perturbation series for the appropriate value of ε.

$$x_1 = -2 - \tfrac{1}{2}\varepsilon + \tfrac{1}{8}\varepsilon^2 + \cdots.$$ If we now set $\varepsilon = 0.001$, we obtain x_1

van der Poll Eq.

The Limit Cycle of van dan Pol Equation: Ex 2

3: [30 points] Consider the following differential equation:

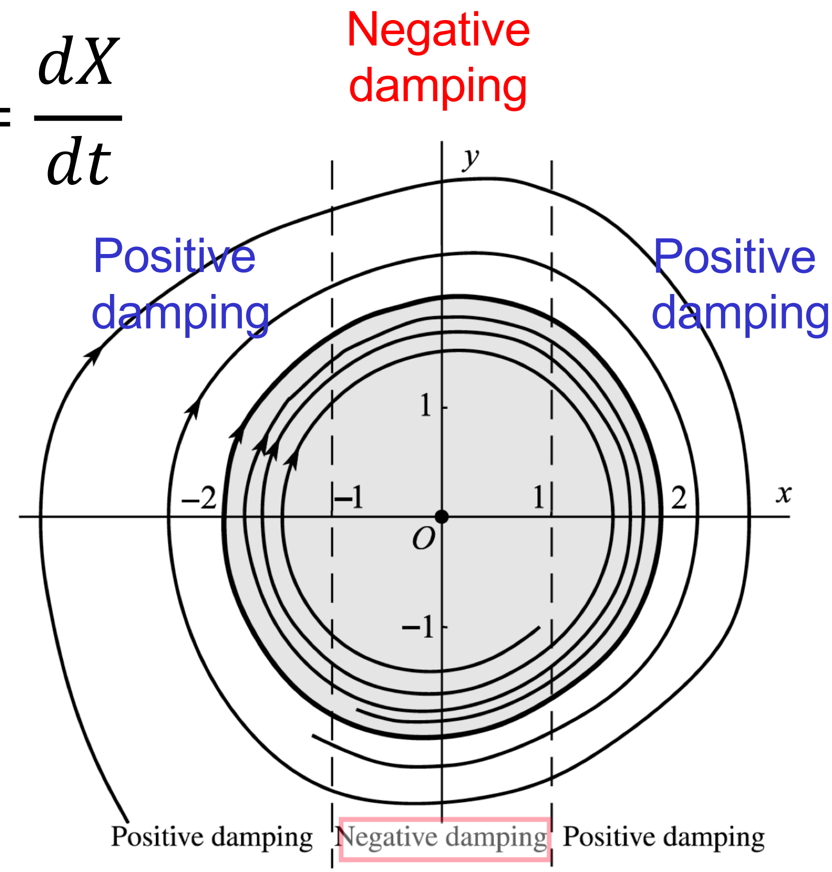
$$\frac{d^2X}{dt^2} + b(X^2-1)\frac{dX}{dt} + X = 0.$$

$$\frac{dE}{dt} = -hX' = -b(X^2-1)v^2; \qquad v = \frac{dX}{dt}$$

Assume b ($b = 0.1$) is positive.

- $\frac{dE}{dt} = 0$ when $x = 1$;
- $\frac{dE}{dt} < 0$ when $|X| > 1$; "*sink*"
- $\frac{dE}{dt} > 0$ when $|X| < 1$; "*source*"

Jordan and Smith (p126)



A Perturbative Analysis of the Solution

Consider the following Van der Pol equation

$$\frac{d^2X}{dt^2} + b(X^2 - 1)\frac{dX}{dt} + X = 0, \tag{1}$$

which can be written as follows:

$$\frac{d^2X}{dt^2} + X = \epsilon\frac{dX}{dt} - \epsilon X^2\frac{dX}{dt}, \tag{2}$$

where $\boxed{\epsilon \text{ is introduced to replace } b}$ in order to perform a perturbative analysis.

We seek a first-order expansion for the solution in the form

Assume

$$X = X_o + \epsilon X_1 + \ldots \tag{3}$$

Plugging the above into Eq. (2), we have

$$\left(X_o'' + \epsilon X_1''\right) + (X_o + \epsilon X_1) = \epsilon\left(\frac{dX_o}{dt} + \epsilon\frac{dX_1}{dt}\right) - \epsilon\left(X_o + \epsilon X_1\right)^2\left(\frac{dX_o}{dt} + \epsilon\frac{dX_1}{dt}\right). \tag{4}$$

A Perturbative Analysis of the Solution

$$\left(\boxed{X_o''} + \epsilon X_1''\right) + \boxed{(X_o} + \epsilon X_1) = \epsilon\left(\frac{dX_o}{dt} + \epsilon\frac{dX_1}{dt}\right) - \epsilon\,(X_o + \epsilon X_1)^2\left(\frac{dX_o}{dt} + \epsilon\frac{dX_1}{dt}\right). \tag{4}$$

Considering terms with ϵ^0, we have

$$\boxed{X_o'' + X_o = 0.} \tag{5}$$

$$\boxed{x(t) \approx a\cos t, \quad y(t) \approx -a\sin t}$$

A Perturbative Analysis of the Solution

$$\left(X_o'' + \boxed{\epsilon X_1''}\right) + (X_o + \boxed{\epsilon X_1}) = \epsilon \left(\boxed{\frac{dX_o}{dt}} + \epsilon \frac{dX_1}{dt}\right) - \epsilon \boxed{(X_o} + \epsilon X_1)^2 \left(\boxed{\frac{dX_o}{dt}} + \epsilon \frac{dX_1}{dt}\right). \tag{4}$$

Considering terms with ϵ^0, we have

$$X_o'' + X_o = 0. \tag{5}$$

Considering terms with ϵ^1, we have

$$\boxed{X_1'' + X_1} = \boxed{X_o'} - \boxed{X_o^2 X_o'}. \tag{6}$$

X_0 acts as a forcing term

From Eq. (5), we have the solution of X_o as follows:

$$X_o = acos(t + \beta), \tag{7}$$

A Perturbative Analysis of the Solution

$$X_1'' + X_1 = \left(-a + \frac{1}{4}a^3\right) sin(t+\beta) + \frac{1}{4}a^3 sin(3(t+\beta)). \tag{12}$$

Consider $X_1 = u + v$, which satisfy the following:

$$u'' + u = \left(-a + \frac{1}{4}a^3\right) sin(t+\beta), \tag{13}$$

nature freq:
$\lambda = \pm i$

$$v'' + v = \frac{a^3}{4} sin(\boxed{3}(t+\beta)). \tag{14}$$

A particular solution of Eq. (13) is:

$$u_p = \frac{at}{2}\left(1 - \frac{1}{4}a^2\right) cos(t+\beta), \tag{15}$$

repeated eigenvalue

A particular solution of Eq. (14) is:

$$v_p = \frac{1}{32}a^3 sin(3(t+\beta)). \tag{16}$$

distinct eigenvalues

A red circle indicates a nonlinear term.

A Perturbative Analysis of the Solution

From Eqs. (3), (7), (15-16), we have

$$X = acos(t+\beta) + \epsilon\left[\frac{at}{2}\left(1 - \frac{1}{4}a^2\right)cos(t+\beta) + \frac{1}{32}a^3 sin(3(t+\beta))\right] + \cdot \quad (17)$$

As a result of the presence of the mixed-secular term, the above expansion is non-uniform for $t \geq O(\epsilon^{-1})$ because the correction term is the order or larger than the first term. The mixed-secular term in Eq. (17) disappears if

$$a(1 - \frac{1}{4}a^2) = 0, \qquad \boxed{\frac{1}{4}a^2 - 1 = 0} \quad (18)$$

leading to $a = 0$ $a = \pm 2$, the latter of which provides an estimate on the amplitude (a). Therefore, the solution becomes

$$X = 2cos(t+\beta) + \frac{1}{4}\epsilon sin(3(t+\beta)) + \cdots \quad (19)$$

3D-NLM and 5D-NLM

3D Lorenz Model (3DLM)

$$\frac{dX}{d\tau} = \sigma Y - \sigma X = F,$$

$$\frac{dY}{d\tau} = -XZ + rX - Y = G,$$

$$\frac{dZ}{d\tau} = XY - bZ = H.$$

$$X = X_c + \epsilon X'$$
$$Y = Y_c + \epsilon Y'$$
$$Z = Z_c + \epsilon Z'$$

reference (or basic) state

perturbations

From Eqs. (1-3), the Jacobian matrix is written as follows:

$$A_1 = \begin{pmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \\ H_x & H_y & H_z \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - Z_c & -1 & -X_c \\ Y_c & X_c & -b \end{pmatrix}_{(X_c, Y_c, Z_c)} .$$

3DLM: Eqs for the basic/reference state

Plugging the above into Eqs. (1-3), we have

$$\frac{dX_c}{d\tau} + \epsilon\frac{dX'}{d\tau} = \sigma(Y_c + \epsilon Y') - \sigma(X_c + \epsilon X') \quad (4)$$

$$\frac{dY_c}{d\tau} + \epsilon\frac{dY'}{d\tau} = -(X_c Z_c + \epsilon X_c Z' + \epsilon Z_c X' + \epsilon^2 X' Z') + r(X_c + \epsilon X') - (Y_c + \epsilon Y') \quad (5)$$

$$\frac{dZ_c}{d\tau} + \epsilon\frac{dZ'}{d\tau} = (X_c Y_c + \epsilon X_c Y' + \epsilon Y_c X' + \epsilon^2 X' Y') - b(Z_c + \epsilon Z') \quad (6)$$

ε^0:

$$\boxed{\begin{aligned} \frac{dX_c}{d\tau} &= \sigma Y_c - \sigma X_c \\ \frac{dY_c}{d\tau} &= -X_c Z_c + r X_c - Y_c \\ \frac{dZ_c}{d\tau} &= X_c Y_c - b Z_c \end{aligned}}$$

3DLM: Eqs for Perturbations

ε^1:

$$\frac{dX'}{d\tau} = \sigma Y' - \sigma X'$$

$$\frac{dY'}{d\tau} = -Z_c X' + rX' - Y' - X_c Z'$$

$$\frac{dZ'}{d\tau} = X'Y_c + X_c Y' - bZ'$$

$$A_2 = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - Z_c & -1 & -X_c \\ Y_c & X_c & -b) \end{pmatrix}$$

- The above derivations show that the Jacobian matrix A1 in the (nonlinear) 3DLM is the same as the matrix A2.
- In other words, the system with the equations associated with ε^1 represents the locally linearized equations of the 3DLM.

$O(\epsilon)$ for 5D-NLM

$$\frac{dX'}{d\tau} = \sigma Y', \tag{6}$$

$$\frac{dY'}{d\tau} = (r - Z_c)X' - X_c Z' - FN(X'Z'), \tag{7}$$

$$\frac{dZ'}{d\tau} = (Y_c - Y_{1c})\,X' + X_c Y' - X_c Y_1' + FN(X'Y' - X'Y_1'), \tag{8}$$

$$\frac{dY_1'}{d\tau} = (Z_c - 2Z_{1c})X' + X_c Z' - 2X_c Z_1' + FN(X'Z' - 2X'Z_1'), \tag{9}$$

$$\frac{dZ_1'}{d\tau} = 2Y_{1c}X' + 2X_c Y_1' + 2FN(X'Y_1'). \tag{10}$$

$O(\epsilon)$ for 5D-NLM near a Critical Point

setting $Y_c = 0, Y_{1c} = 0, Z_c = r,\ Z_{1c} = \frac{r}{2}$ and $FN = 0$, we obtain:

$$\frac{dX'}{d\tau} = \sigma Y',$$

$$\frac{dY'}{d\tau} = -X_c Z',$$

$$\frac{dZ'}{d\tau} = X_c Y' - X_c Y_1',$$

$$\frac{dY_1'}{d\tau} = X_c Z' - 2X_c Z_1',$$

$$\frac{dZ_1'}{d\tau} = 2X_c Y_1'.$$

$$\frac{d^2Y'}{d\tau^2} = -X_c \dot{Z}' = -X_c(X_c Y' - X_c Y_1') = -X_c^2(Y' - Y_1') \tag{22}$$

$$\frac{d^2Y_1'}{d\tau^2} = X_c \dot{Z}' - 2X_c \dot{Z}_1' = X_c(X_c Y' - X_c Y_1') - 2X_c(2X_c Y_1') = X_c^2 Y' - 5X_c^2 Y_1' \tag{23}$$

Q3 of HW5

3: [40 points] Consider the following coupled harmonic oscillator (as shown in Fig. 1):

$$\frac{d^2x_1}{dt^2} = -k_1x_1 + k_2(x_2 - x_1),$$

$$\frac{d^2x_2}{dt^2} = -k_2(x_2 - x_1).$$

Let $k_1 = 4X_c^2$ and $k_2 = X_c^2$ (and $m_1 = m_2 = 1$).

The same math system with quasi-periodic solutions.

$$Y' = x_2$$
$$Y_1' = x_1$$

$O(\epsilon)$ for the 5D-NLM near a critical point

$$\frac{d^2Y'}{d\tau^2} = -k_2(Y' - Y_1') = -X_c^2(Y' - Y_1')$$

$$\frac{d^2Y_1'}{d\tau^2} = -k_1Y_1' - k_2(Y_1' - Y') = X_c^2Y' - 5X_c^2Y_1'$$

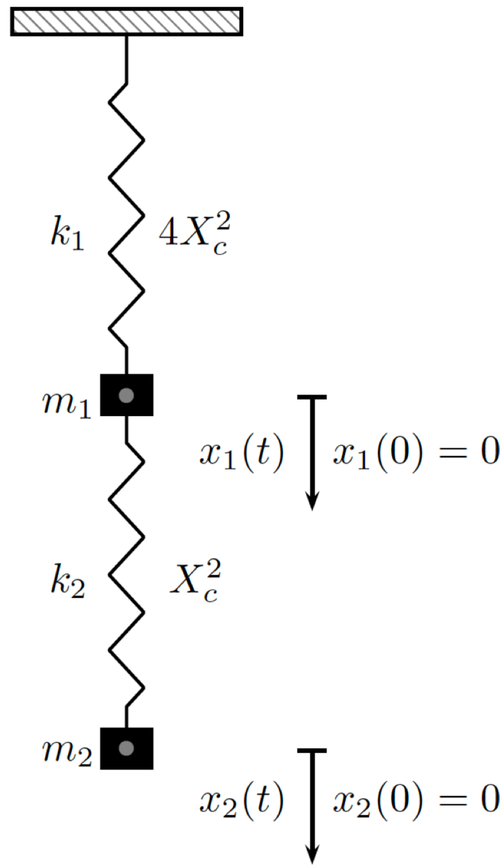


Figure 1: Coupled spring/mass system

Quasi-Periodic Solutions in the 5D-NLM

Courtesy of
Sara Faghih-Naini

Introduction: Quasiperiodic Solutions

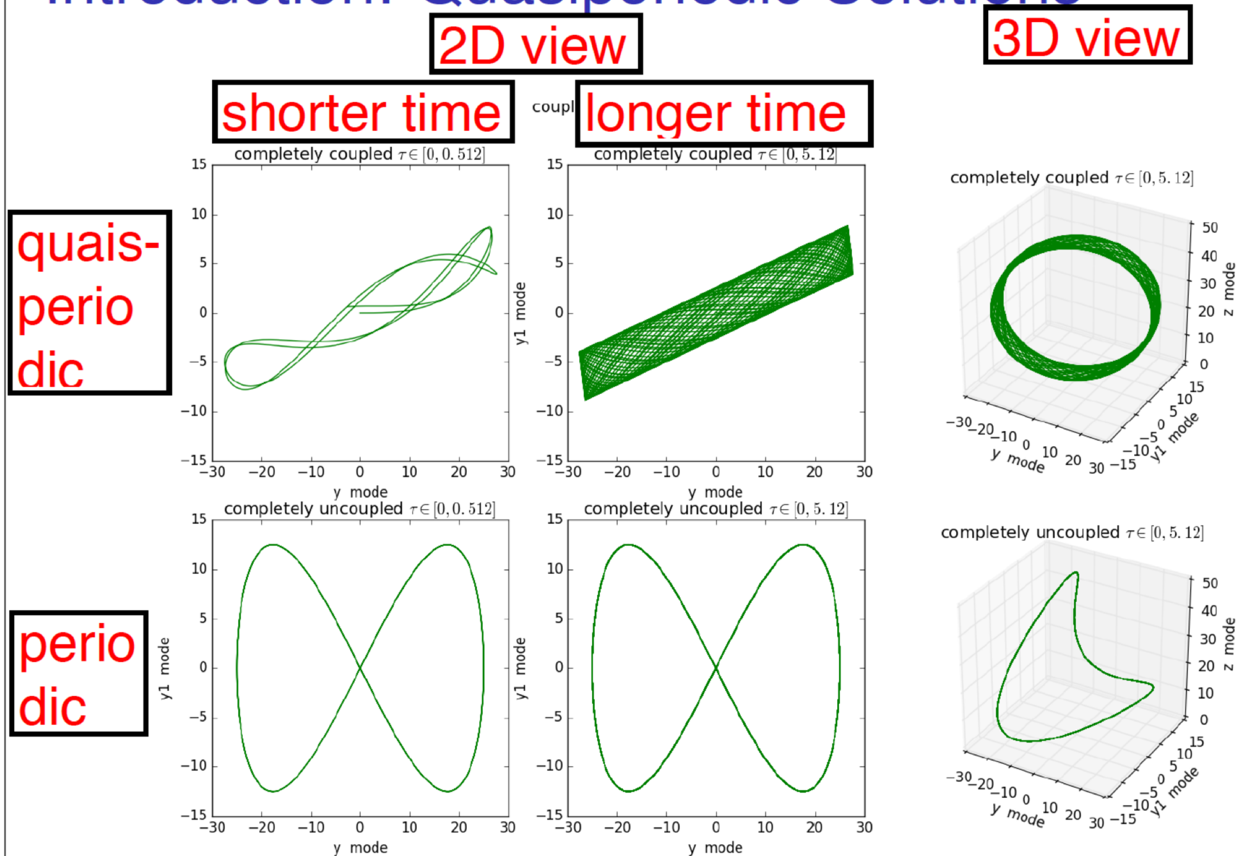


Figure: Quasiperiodic solution of coupled LL 5D NLM with frequency ratio $\frac{1}{2}\left(3-\sqrt{5}\right)$ (top) and periodic solution of uncoupled LL 5D NLM with frequency ratio 2 (bottom) up to time 0.512 (left) and 5.12 (middle)

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Singular Perturbation

Regular Perturbation Problems

- We usually have (or can introduce) a small parameter ϵ, and the solution (for example) $y(x, \epsilon)$ depends on ϵ.
- The solution is said to be a regular perturbation problem if
  - $y(x, \epsilon) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \cdots$
- The first term (called the leading-order term) y_0 is often a well-known solution corresponding to the unperturbed problem $\epsilon = 0$.
- Usually only the first few additional terms are needed.
- The higher-order terms such as $y_1(x)$ can be successively determined by substituting the regular expansion into the original partial differential equation.

Haberman (2013)

A Singular Perturbation Problem

Example 1 *Roots of a polynomial.* How does one determine the approximate roots of

$$\varepsilon^2 x^6 - \varepsilon x^4 - x^3 + 8 = 0? \tag{7.2.1}$$

We may begin by setting $\varepsilon = 0$ to obtain the unperturbed problem $-x^3 + 8 = 0$, which is easily solved:

$$x = 2,\ 2\omega,\ 2\omega^2, \tag{7.2.2}$$

where $\omega = e^{2\pi i/3}$ is a complex root of unity. Note that the unperturbed equation has only three roots while the original equation has six roots. This abrupt change in the character of the solution, namely the disappearance of three roots when $\varepsilon = 0$, implies that (7.2.1) is a singular perturbation problem. Part of the exact solution ceases to exist when $\varepsilon = 0$.

- The unperturbed equation (i.e.,$\epsilon = 0$) has only three roots!

Singular → Regular Perturbation Problem

Example 1 *Roots of a polynomial.* How does one determine the approximate roots of

$$\varepsilon^2 x^6 - \varepsilon x^4 - x^3 + 8 = 0? \tag{7.2.1}$$

We may begin by setting $\varepsilon = 0$ to obtain the unperturbed problem $-x^3 + 8 = 0$, which is

$$\epsilon^2 x^6 - \epsilon y^4 - x^3 + 8 = 0$$ A singular perturbation problem

A scale transformation: $x = \epsilon^{2/3} y$ (rescaling)

$$y^6 - y^3 + 8\epsilon^2 - \epsilon^{\frac{1}{3}} y^4 = 0$$ A regular perturbation problem

Thus, the magnitudes of the three missing roots are $O(\varepsilon^{-2/3})$ as $\varepsilon \to 0$. This result is a clue to the structure of the perturbation series for the missing roots. In particular, it suggests a *scale transformation* for the variable x:

$$x = \varepsilon^{-2/3} y. \tag{7.2.4}$$

Substituting (7.2.4) into (7.2.1) gives

$$y^6 - y^3 + 8\varepsilon^2 - \varepsilon^{1/3} y^4 = 0. \tag{7.2.5}$$

This is now a *regular* perturbation problem for y in the parameter $\varepsilon^{1/3}$ because the unperturbed problem $y^6 - y^3 = 0$ has six roots $y = 1, \omega, \omega^2, 0, 0, 0$. Now, no roots disappear in the limit $\varepsilon^{1/3} \to 0$.

Review: Regular Perturbation Problems

- We usually have (or can introduce) a small parameter ϵ, and the solution (for example) $y(x,\epsilon)$ depends on ϵ.
- The solution is said to be a regular perturbation problem if
  - $y(x,\epsilon) = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \cdots$
- The first term (called the leading-order term) y_0 is often a well-known solution corresponding to the unperturbed problem $\epsilon = 0$.
- Usually only the first few additional terms are needed.
- The higher-order terms such as $y_1(x)$ can be successively determined by substituting the regular expansion into the original partial differential equation.

Haberman (2013)

Singular Perturbation Problems and Methods

- Multiple Scale Methods: (two time scales)
  - $\frac{d^2u}{dt^2} + u = -\epsilon\left(\frac{du}{dt}\right)^3$
  - $T = \epsilon t$ (two time scales, fast variable t, and slow variable T)
  - $\frac{d}{dt} = \frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial T}$
- Boundary Layer Methods: (two regions)
  - $\epsilon\frac{d^2u}{dx^2} - \frac{du}{dx} + 2xu = 0$
  - Outer region (away from the boundary layer)
    - $u(x,\epsilon) = \mathrm{u}_0(x) + \epsilon u_1(x) + \epsilon^2 u_2(x) + \cdots$
    - $O(\epsilon)$: $\frac{du_0}{dx} - 2xu_0 = 0;$ (an unperturbed problem)
  - Inner region
    - $x = x_0 + \epsilon\mathbb{X}$ (rescaling); $u(x) = \mathbb{U}(\mathbb{X})$
    - $\frac{d^2\mathbb{U}}{d\mathbb{X}^2} - \frac{d\mathbb{U}}{d\mathbb{X}} + 2\epsilon(1+\epsilon\mathbb{X})\mathbb{U} = 0$

$$\boxed{\frac{d^2u}{dt^2} + u = -\varepsilon \left(\frac{du}{dt}\right)^3,}$$

The method of multiple scales. The regular perturbation expansion is not valid for large times. However, the regular perturbation expansion suggests that interesting dynamics occur when $\varepsilon t = O(1)$. If we are interested in the solution on that long time scale, then we assume the solution $u(t, T, \varepsilon)$ depends on two variables, a **fast variable** t and a **slow variable**

$$\boxed{T = \varepsilon t,} \tag{14.9.8}$$

$$\boxed{\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T},}$$

Haberman (2013)

$$\boxed{\frac{d^2u}{dt^2}+u} = -\varepsilon\left(\frac{du}{dt}\right)^3,$$

unperturbed equation $\epsilon = 0$

Now the regular expansion, $u(t,T,\varepsilon) = u_0(t,T) + \varepsilon u_1(t,T) + \cdots$, yields

$$O(\varepsilon^0) \quad : \quad \boxed{\frac{\partial^2 u_0}{\partial t^2} + u_0 = 0}$$

the leading-order eq.

$$O(\varepsilon^1) \quad : \quad \frac{\partial^2 u_1}{\partial t^2} + u_1 = -\left(\frac{\partial u_0}{\partial t}\right)^3 - 2\frac{\partial}{\partial T}\left(\frac{\partial u_0}{\partial t}\right). \qquad (14.9.12)$$

Only the order ε term $-2\frac{\partial}{\partial T}\left(\frac{\partial u_0}{\partial t}\right)$ differs from the naive perturbation expansion (14.9.4) and (14.9.5).

The leading-order equation is the unperturbed linear oscillator, but we must remember that $\frac{\partial}{\partial t}$ means keeping T fixed. Thus, the general solution of (14.9.11) is

$$\boxed{u_0(t,T) = A(T)e^{it} + A^*(T)e^{-it},} \qquad (14.9.13)$$

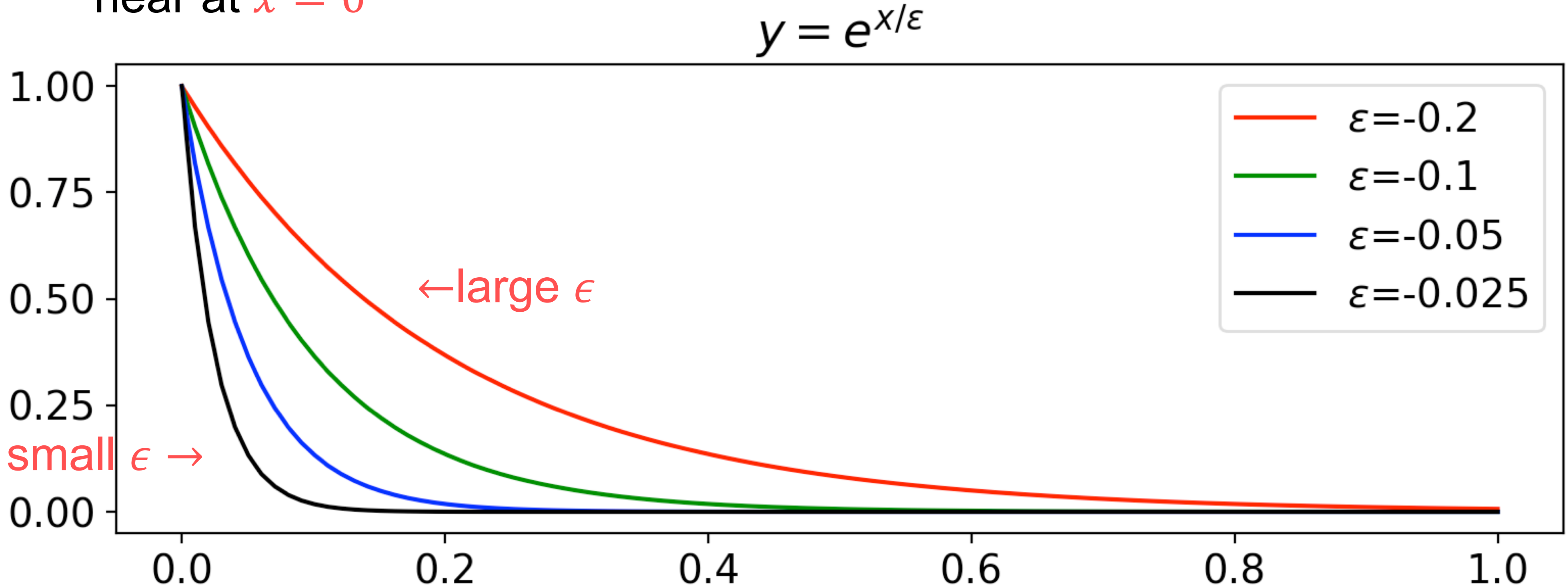
where $A(T)$ is an arbitrary complex function of the slow time variable.

Boundary Layer within a 1st Order ODE

Stiff ODE $\quad y' = \lambda y \quad \lambda < 0 \; and \; |\lambda| \text{ is large}$

boundary layer $\quad \varepsilon y' = y \quad \varepsilon = \frac{1}{\lambda} \quad y(x = 0) = 1$

near at $x = 0$

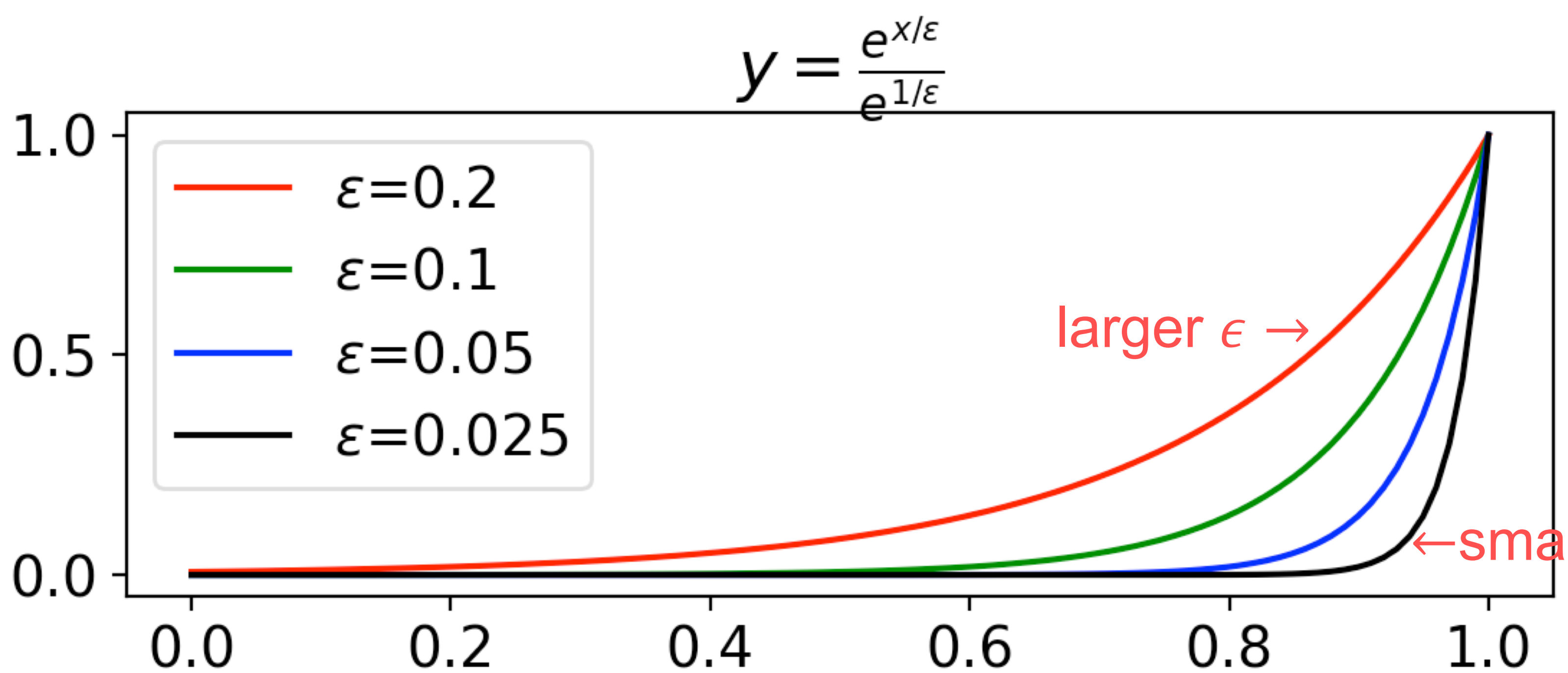


Boundary Layer

boundary layer $\varepsilon y' = y$ $y(x=0) = \frac{1}{e^{\epsilon}}$

near at $x = 1$

$i.e., y(x=1) = 1$

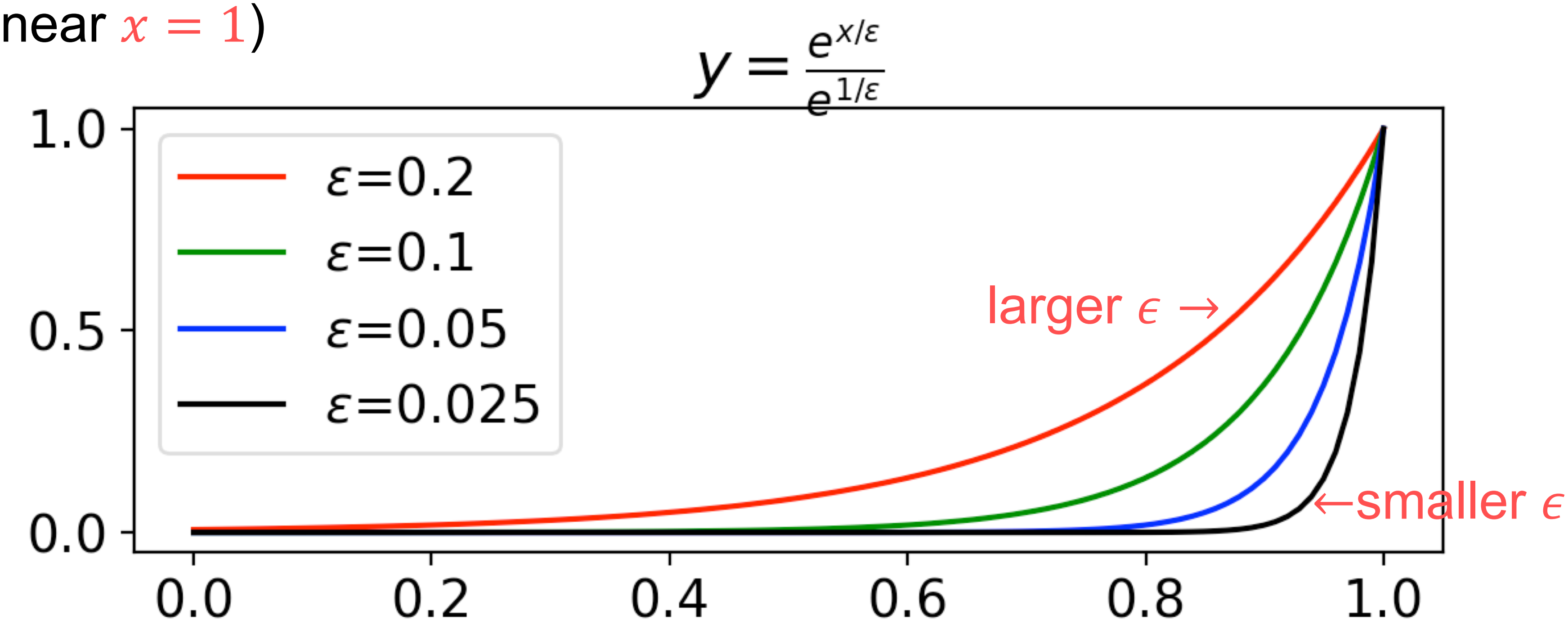


rapid change

Boundary Layer

boundary layer $\varepsilon y' = y$ $y(x=0) = e^{-\epsilon}$

(near $x = 1$)



- For a very small ϵ (e.g., a black curve), solution y is almost a constant except for a very narrow interval near $x = 1$ where a boundary layer is defined.
- Stated alternatively, When ϵ is small, y varies rapidly near $x = 1$; this localized region of rapid variation is called a boundary layer.

Singular Perturbation (within a 2nd Order ODE)

Example 2 *Appearance of a boundary layer.* The boundary-value problem

$$\varepsilon y'' - y' = 0, \qquad y(0) = 0,\ y(1) = 1, \tag{7.2.7}$$

is a singular perturbation problem because the associated unperturbed problem

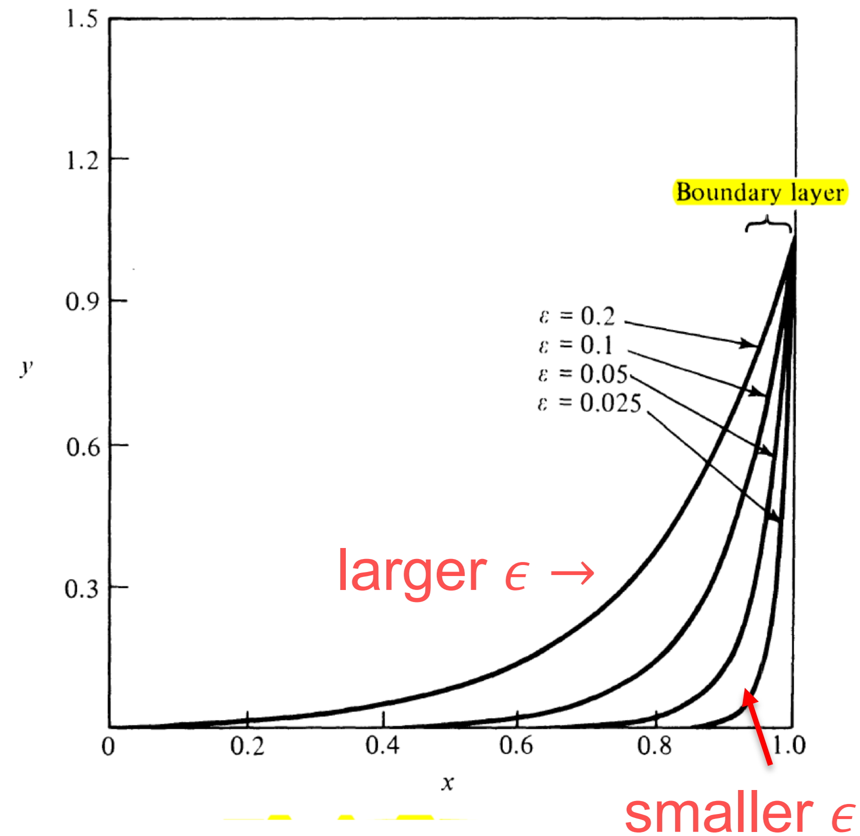
- When $\epsilon = 0$, we have $y' = 0$, yielding no solution that satisfies two boundary conditions (BCs).
- An exact solution is given as follows:

$$y = \frac{e^{x/\epsilon} - 1}{e^{1/\epsilon} - 1}$$

Boundary Layer (*)

$$\epsilon y'' - y' = 0 \quad y(0) = 0 \;\&\; y(1) = 1$$

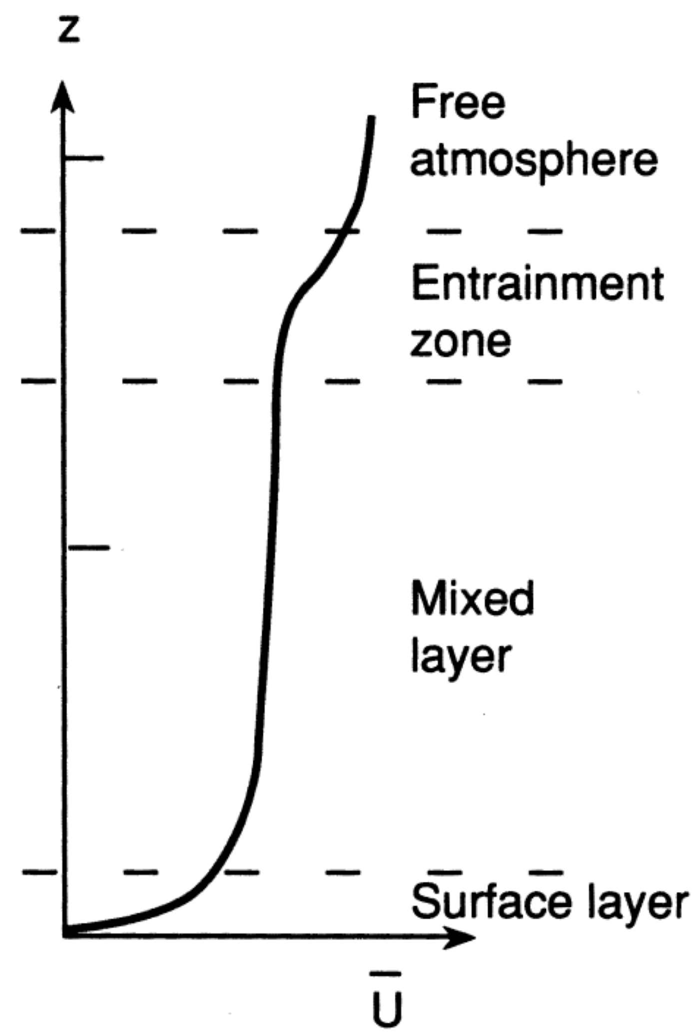
$$y = \frac{e^{x/\epsilon} - 1}{e^{1/\epsilon} - 1}$$



- When ϵ is small, y varies rapidly near $x = 1$; this localized region of rapid variation is called a boundary layer.
- When ϵ is negative, the boundary layer is at $x = 0$ instead of $x = 1$.
- This abrupt jump in the location of the boundary layer as ϵ changes sign reflects the singular nature of the perturbation problem

Atmospheric Surface Layer (Wiki) (*) TBD

- The term boundary layer is used in meteorology and in physical oceanography.
- The atmospheric surface layer is the lowest part of the atmospheric boundary layer (typically the bottom 10% where the log wind profile is valid).
- The surface layer is the layer of a turbulent fluid most affected by interaction with a solid surface or the surface separating a gas and a liquid where the characteristics of the turbulence depend on distance from the interface.
- Surface layers are characterized by large normal gradients of tangential velocity and large concentration gradients of any substances (temperature, moisture, sediments et cetera) transported to or from the interface.



Example

Boundary Layer Problem

$$\boxed{\epsilon y'' + (1+\epsilon)y' + y = 0}$$

$$y(0) = 0 \;\&\; y(1) = 1$$

$$y(x) = \frac{e^{-x} - e^{-x/\epsilon}}{e^{-1} - e^{-1/\epsilon}}$$

$$\boxed{y_{out} = \lim_{\epsilon \to 0} y = \frac{e^{-x}}{e^{-1}} = e^{1-x}}$$

$$x = \epsilon \mathbb{X} \qquad x = O(\epsilon)$$

$$y(x) = \frac{e^{-\epsilon\mathbb{X}} - e^{-\mathbb{X}}}{e^{-1} - e^{-1/\epsilon}}$$

$$\boxed{y_{in} = \lim_{\epsilon \to 0} y = \frac{e^{0} - e^{-\mathbb{X}}}{e^{-1}}}$$

$$= e - e^{1-\mathbb{X}}$$

$$\boxed{\text{TBD: } \mathbb{Y} \sim -e^{1-\mathbb{X}} + e^{1-x}}$$

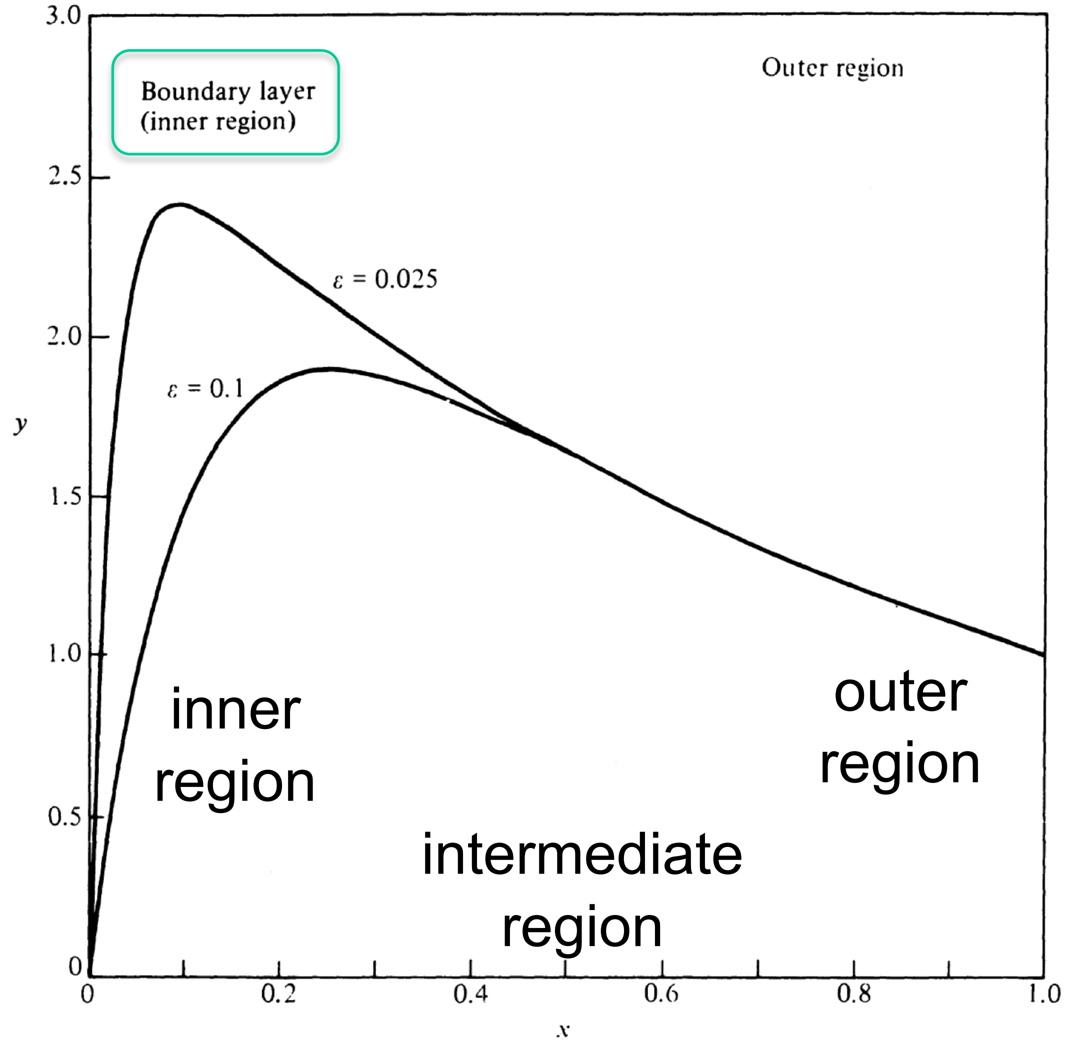


Figure 9.1 A plot of $y(x) = (e^{-x} - e^{-x/\varepsilon})/(e^{-1} - e^{-1/\varepsilon})$ $(0 \le x \le 1)$ for $\varepsilon = 0.1$ and 0.025. Note that $y(x)$ is slowly varying for $\varepsilon \ll x \le 1$ $(\varepsilon \to 0+)$. However, on the interval $0 \le x \le O(\varepsilon)$, $y(x)$ rises abruptly from 0 and becomes discontinuous in the limit $\varepsilon \to 0+$. This narrow and isolated region of rapid change is called a boundary layer.

Boundary Layer Problem

$$\epsilon y'' + (1+\epsilon)y' + y = 0$$

$$y(0) = 0 \ \& \ y(1) = 1$$

$$y(x) = \frac{e^{-x} - e^{-x/\epsilon}}{e^{-1} - e^{-1/\epsilon}}$$

- The outer limit of the solution is obtained by a fixed x outside the boundary layer, $\delta \le x \le 1$, and allowing $\epsilon \to 0^+$.
- The inner limit of the solution in which $\epsilon \to 0^+$ with $x \in O(\epsilon)$ is obtained by introducing $x = \epsilon\mathbb{X}$ and having $\epsilon \to 0^+$ and finite $\mathbb{X}$

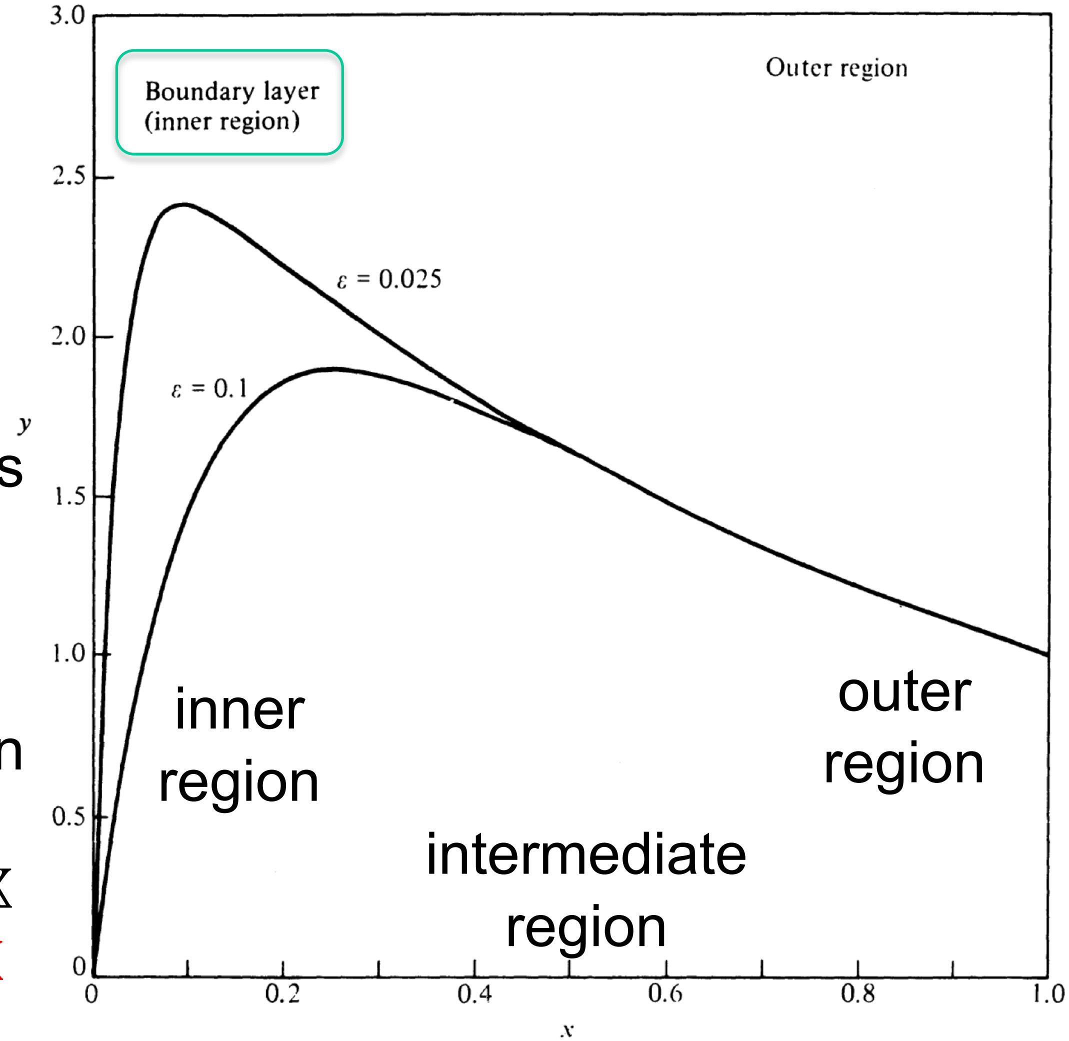


Figure 9.1 A plot of $y(x) = (e^{-x} - e^{-x/\varepsilon})/(e^{-1} - e^{-1/\varepsilon})$ $(0 \le x \le 1)$ for $\varepsilon = 0.1$ and 0.025. Note that $y(x)$ is slowly varying for $\varepsilon \ll x \le 1$ $(\varepsilon \to 0+)$. However, on the interval $0 \le x \le O(\varepsilon)$, $y(x)$ rises abruptly from 0 and becomes discontinuous in the limit $\varepsilon \to 0+$. This narrow and isolated region of rapid change is called a boundary layer.

Outer Region

$$\epsilon y'' + (1+\epsilon)y' + y = 0 \qquad y(0) = 0 \;\&\; y(1) = 1$$

$$y_{out} \sim \sum_{n=0}^{\infty} y_n \epsilon^n = y_0 + \epsilon y_1 + \epsilon^2 y_2$$

$$y(1) = 1 \Rightarrow \qquad y_0(1) = 1, \quad y_1(1) = 0, \qquad y_2(1) = 0$$

$$y' + y = -\epsilon y'' - \epsilon y'$$

$$O(\epsilon^0) \qquad y_0' + y_0 = 0 \qquad y_0(1) = 1$$

$$O(\epsilon^n) \qquad y_{n+1}' + y_n = -y_n'' - y_{n-1}' \qquad y_n(1) = 0, n \geq 1$$

Outer Region: y_0 and y_1

$O(\epsilon^0)$ $\quad y_0' + y_0 = 0 \qquad y_0(1) = 1 \qquad$ (1)

1st order ODE + one BC

$O(\epsilon^n)$ $\quad y_{n+1}' + y_n = -y_n'' - y_{n-1}' \qquad y_n(1) = 0, n \geq 1 \qquad$ (2)

From Eq. (1), we obtain

$$\boxed{y_0 = e^{1-x}}$$

From Eq. (2) with $n = 1$, since the RHS is

$$-y_0'' - y_0' = 0$$

Eq. (2) becomes

$$y_1' + y_1 = 0 \qquad y_1(1) = 0$$

From the above Eq. and BC, we obtain

$y_1 = ce^{-x} \qquad c = 0$ (with the BC) $\qquad \boxed{y_1 = 0}$

Outer Region: $y_2 \cdots$ and y_{out}

$O(\epsilon^0)$ $\quad y_0' + y_0 = 0 \qquad y_0(1) = 1 \qquad$ (1)

$O(\epsilon^n)$ $\quad y_{n+1}' + y_n = -y_n'' - y_{n-1}' \qquad y_n(1) = 0, n \geq 1 \qquad$ (2)

From Eq. (1), we obtain

$$\boxed{y_0 = e^{1-x}}$$

From the above Eq. and BC, we obtain

$$\boxed{y_1 = 0}$$

Eq. (2) becomes

$$y_2' + y_2 = 0 \qquad y_2(1) = 0$$

From the above Eq. and BC, we obtain

$$y_2 = 0 \rightarrow \boxed{y_n = 0}$$

$$\boxed{y_{out} \sim y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots = e^{1-x}}$$

Inner Region (with "rescaling")

$$\boxed{\epsilon\frac{d^2y}{dx^2}+(1+\epsilon)\frac{dy}{dx}+y=0}$$

$$\boxed{y(0)=0\ \&\ y(1)=1}$$

$$x=\epsilon\mathbb{X} \qquad \frac{d}{dx}=\frac{1}{\epsilon}\frac{d}{d\mathbb{X}} \qquad \frac{d^2}{dx^2}=\frac{1}{\epsilon^2}\frac{d^2}{d\mathbb{X}^2}$$

Let $\mathbb{Y}(\mathbb{X})=y(x)$

$$\epsilon\frac{1}{\epsilon^2}\frac{d^2\mathbb{Y}}{d\mathbb{X}^2}+(1+\epsilon)\frac{1}{\epsilon}\frac{d\mathbb{Y}}{d\mathbb{X}}+\mathbb{Y}=0$$

$$\boxed{\frac{1}{\epsilon}\frac{d^2\mathbb{Y}}{d\mathbb{X}^2}+\frac{1}{\epsilon}\frac{d\mathbb{Y}}{d\mathbb{X}}+\frac{d\mathbb{Y}}{d\mathbb{X}}+\mathbb{Y}=0} \qquad \frac{d^2\mathbb{Y}}{d\mathbb{X}^2}+\frac{d\mathbb{Y}}{d\mathbb{X}}+\epsilon\left(\frac{d\mathbb{Y}}{d\mathbb{X}}+\mathbb{Y}\right)=0$$

By comparison, the original ODE is written as follows:

$$\epsilon\left(\frac{d^2y}{dx^2}+\frac{dy}{dx}\right)+\frac{dy}{dx}+y=0$$

Inner Region (with “rescaling”)

$$\boxed{\frac{1}{\epsilon}\frac{d^2\mathbb{Y}}{d\mathbb{X}^2}+\frac{1}{\epsilon}\frac{d\mathbb{Y}}{d\mathbb{X}}+\frac{d\mathbb{Y}}{d\mathbb{X}}+\mathbb{Y}=0}$$

$$\mathbb{Y}\sim\sum_{n=0}^{\infty}\mathbb{Y}_n\epsilon^n=\mathbb{Y}_0+\epsilon\mathbb{Y}_1+\epsilon^2\mathbb{Y}_2+\cdots$$

$O(\epsilon^{-1})$
$$\frac{d^2\mathbb{Y}_0}{d\mathbb{X}^2}+\frac{d\mathbb{Y}_0}{d\mathbb{X}}=0$$

2nd order ODE + one BC

$O(\epsilon^{0})$
$$\frac{d^2\mathbb{Y}_1}{d\mathbb{X}^2}+\frac{d\mathbb{Y}_1}{d\mathbb{X}}+\frac{d\mathbb{Y}_0}{d\mathbb{X}}+\mathbb{Y}_0=0$$

$O(\epsilon^{n})$
$$\frac{d^2\mathbb{Y}_n}{d\mathbb{X}^2}+\frac{d\mathbb{Y}_n}{d\mathbb{X}}+\frac{d\mathbb{Y}_{n-1}}{d\mathbb{X}}+\mathbb{Y}_{n-1}=0,\qquad n\geq 1$$

Inner Region (with "rescaling")

$$\frac{d^2\mathbb{Y}_0}{d\mathbb{X}^2} + \frac{d\mathbb{Y}_0}{d\mathbb{X}} = 0 \qquad \mathbb{Y}(\mathbb{X}) = y(x) \qquad \boxed{y(0) = 0 \;\&\; y(1) = 1}$$

$$\mathbb{Y}_0 = \mathrm{c}_1 + \mathrm{c}_2 e^{-\mathbb{X}}$$

$$\mathbb{Y}_0(0) = 0 \rightarrow \qquad \mathrm{c}_1 = -\mathrm{c}_2 = \mathrm{A}_0$$

$$\boxed{\mathbb{Y}_0 = A_0(1 - e^{-\mathbb{X}})}$$

$$\frac{d^2\mathbb{Y}_1}{d\mathbb{X}^2} + \frac{d\mathbb{Y}_1}{d\mathbb{X}} + \frac{d\mathbb{Y}_0}{d\mathbb{X}} + \mathbb{Y}_0 = 0 \qquad \frac{d^2\mathbb{Y}_1}{d\mathbb{X}^2} + \frac{d\mathbb{Y}_1}{d\mathbb{X}} = -\frac{d\mathbb{Y}_0}{d\mathbb{X}} - \mathbb{Y}_0 = -A_0$$

$$\mathbb{Y}_1 = \mathbb{Y}_{1h} + \mathbb{Y}_{1p} \qquad \mathbb{Y}_{1h} = \mathrm{d}_1 + \mathrm{d}_2 e^{-\mathbb{X}} \qquad \mathbb{Y}_{1p} = -A_0\mathbb{X}$$

$$\mathbb{Y}_1(0) = 0 \rightarrow \qquad \mathrm{d}_1 = -\mathrm{d}_2 = \mathrm{A}_1$$

$$\boxed{\mathbb{Y}_1 = A_1\left(1 - e^{-\mathbb{X}}\right) - A_0\mathbb{X}}$$

$$\boxed{\mathbb{Y} \sim A_0\left(1 - e^{-\mathbb{X}}\right) + \epsilon\left(A_1\left(1 - e^{-\mathbb{X}}\right) - A_0\mathbb{X}\right) + \cdots}$$

Matching: Why?

inner region:
2^{nd} order ODE:
1BC

outer region:
1^{st} order ODE:
1BC

$$\frac{d^2\mathbb{Y}}{d\mathbb{X}^2} + \frac{d\mathbb{Y}}{d\mathbb{X}} + \epsilon\left(\frac{d\mathbb{Y}}{d\mathbb{X}} + \mathbb{Y}\right) = 0$$

$$y(0) = 0$$

$$\epsilon\left(\frac{d^2y}{dx^2} + \frac{dy}{dx}\right) + \frac{dy}{dx} + y = 0$$

$$y(1) = 1$$

$$x = \epsilon\mathbb{X}$$

(rescaling)

$$\mathbb{Y}(\mathbb{X}) = y(x)$$

matching to determine coefficients

$$\boxed{\begin{aligned}&\mathbb{Y} \sim A_0\left(1 - e^{-\mathbb{X}}\right)\\ &+ \epsilon\left(A_1\left(1 - e^{-\mathbb{X}}\right) - A_0\mathbb{X}\right) + \cdots\end{aligned}}$$

$\mathbb{X} \rightarrow \infty$

equal

$$\boxed{y_{out} \sim e^{1-x}}$$

Matching (to determine the coefficients)

$$\frac{d^2\mathbb{Y}}{d\mathbb{X}^2}+\frac{d\mathbb{Y}}{d\mathbb{X}}+\epsilon\left(\frac{d\mathbb{Y}}{d\mathbb{X}}+\mathbb{Y}\right)=0$$

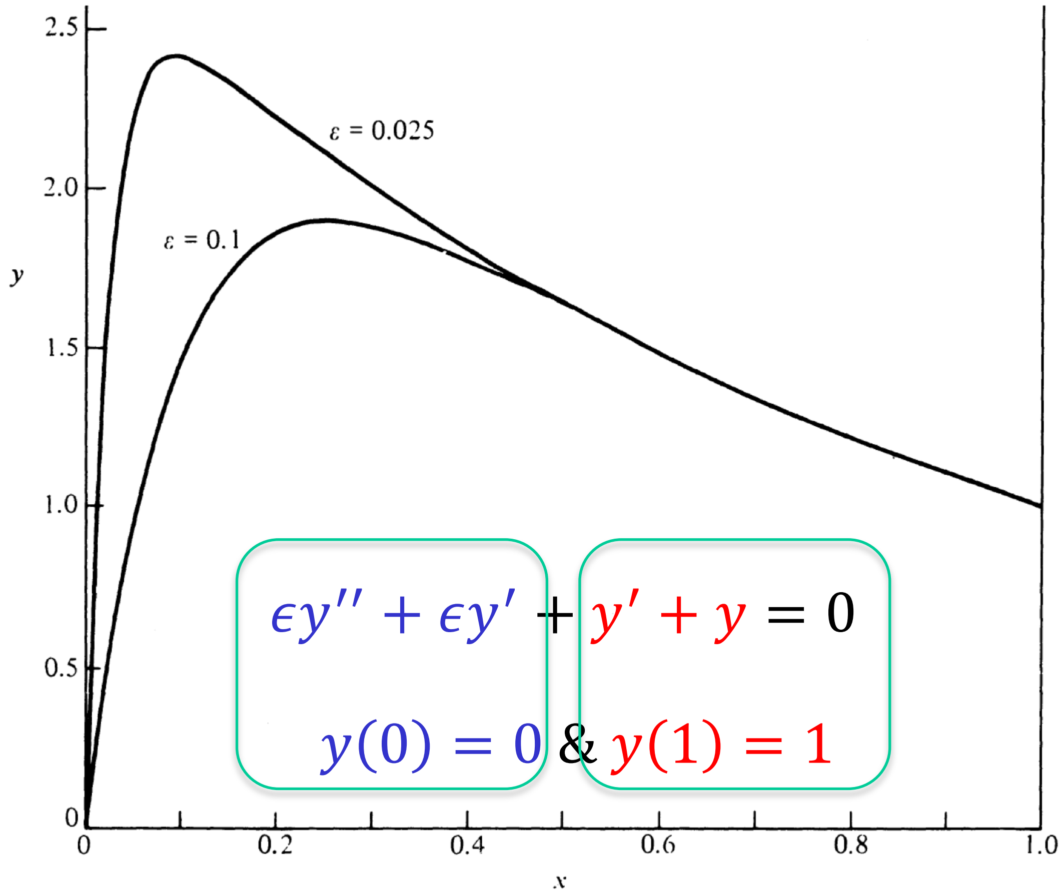
$$\epsilon\left(\frac{d^2y}{dx^2}+\frac{dy}{dx}\right)+\frac{dy}{dx}+y=0$$

$y(0)=0$

$y(1)=1$

$x=\epsilon\mathbb{X}$
(rescaling)

$\mathbb{Y}(\mathbb{X})=y(x)$



$$\epsilon y''+\epsilon y'+y'+y=0$$

$$y(0)=0\ \&\ y(1)=1$$

$$\mathbb{Y}\sim A_0\left(1-e^{-\mathbb{X}}\right)+\epsilon\left(A_1\left(1-e^{-\mathbb{X}}\right)-A_0\mathbb{X}\right)+\cdots$$

$\mathbb{X}\to\infty$

equal

$$y_{out}\sim e^{1-x}$$

Matching (to determine the coefficients)

Inner region

$$\mathbb{Y} \sim \mathbb{Y}_0 + \epsilon \mathbb{Y}_1 + \epsilon^2 \mathbb{Y}_2 + \cdots \quad \boxed{\mathbb{Y}_0 = A_0(1 - e^{-\mathbb{X}})} \quad \boxed{\mathbb{Y}_1 = A_1\left(1 - e^{-\mathbb{X}}\right) - A_0 \mathbb{X}}$$

$$\boxed{\mathbb{Y} \sim A_0\left(1 - e^{-\mathbb{X}}\right) + \epsilon\left(A_1\left(1 - e^{-\mathbb{X}}\right) - A_0 \mathbb{X}\right) + \cdots}$$

Outer region

$$\boxed{y_{out} \sim e^{1-x}} \qquad x = \epsilon \mathbb{X} \qquad y_{out} \sim e\, e^{-x} = e e^{-\epsilon \mathbb{X}} \qquad \boxed{\mathbb{Y}_1 = -e\mathbb{X}}$$

$$\boxed{y_{out} \sim e\, e^{-x} = e\left(1 - \epsilon \mathbb{X} + \frac{(\epsilon \mathbb{X})^2}{2!} + \cdots\right) = e - \epsilon e \mathbb{X} + \cdots}$$

$O(\epsilon^0)$ $\quad A_0 = e$ as $\mathbb{X} \to \infty$

$O(\epsilon^1)$ $\quad \left(A_1\left(1 - e^{-\mathbb{X}}\right) - A_0 \mathbb{X}\right) \sim -e\mathbb{X}$ as $\mathbb{X} \to \infty$ $\qquad A_1 = 0$

Note: $\mathbb{Y}_1$ has a different form from that in Textbook, so A_1 is different. However, $\mathbb{Y}_1 = -e\mathbb{X}$ is correct.

Matching (to determine the coefficients)

$$\mathbb{Y} \sim \mathbb{Y}_0 + \epsilon\mathbb{Y}_1 + \epsilon^2\mathbb{Y}_2 + \cdots$$

$$\boxed{\mathbb{Y}_0 = e\left(1 - e^{-\mathbb{X}}\right) = e - e^{1-\mathbb{X}} \qquad \text{(after being matched)}}$$

$$\boxed{\mathbb{Y}_1 = -e\mathbb{X} \qquad \text{(after being matched)}}$$

$$\boxed{\mathbb{Y}_n = e\frac{(-1)^n}{n!}\mathbb{X}^n \qquad \text{(after being matched)}}$$

$$\mathbb{Y} \sim e - e^{1-\mathbb{X}} + \epsilon(-e\mathbb{X}) + \epsilon^n e\frac{(-1)^n}{n!}\mathbb{X}^n + \cdots$$

$$\mathbb{Y} \sim -e^{1-\mathbb{X}} + e\left(1 - \epsilon\mathbb{X} + \cdots \epsilon^n e\frac{(-1)^n}{n!}\mathbb{X}^n + \cdots\right) = -e^{1-\mathbb{X}} + e\mathrm{e}^{-\epsilon\mathbb{X}}$$

$$= -e^{1-\mathbb{X}} + \mathrm{e}^{1-\epsilon\mathbb{X}} \boxed{= -e^{1-\mathbb{X}} + \mathrm{e}^{1-x}}$$

Matching (an intermediate component)

$\boxed{\mathbb{Y} \sim -e^{1-\mathbb{X}} + e^{1-x}}$ $\quad \boxed{y_{match} = e^{1-x}}$ $\quad \boxed{y_{out} \sim e^{1-x}}$

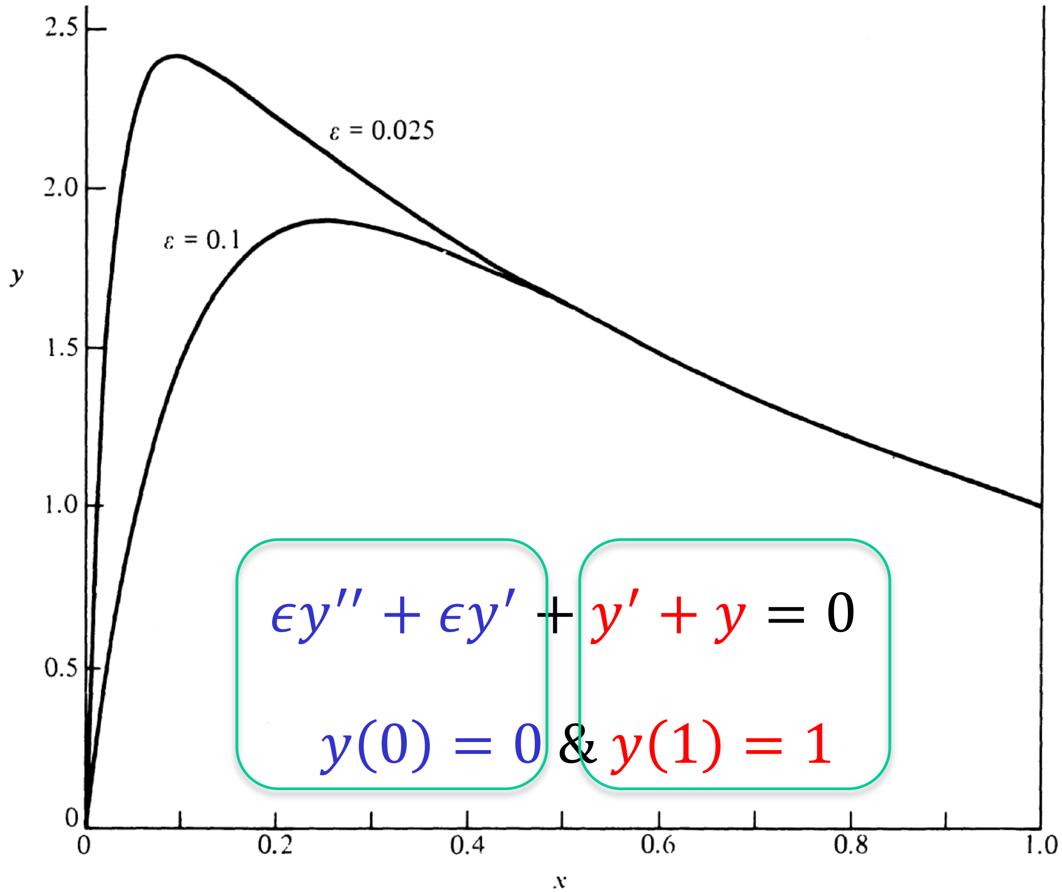
$y(0) = 0$

$y(1) = 1$

$x = \epsilon \mathbb{X}$

(rescaling)

$\mathbb{Y}(\mathbb{X}) = y(x)$



$$\epsilon y'' + \epsilon y' + y' + y = 0$$

$$y(0) = 0 \;\&\; y(1) = 1$$

$$\boxed{y_{unif} = y_{in} + y_{out} - y_{match} = -e^{1-\mathbb{X}} + e^{1-x}}$$

WKBJ