HW 1 Remarks:

- Looked at ① a, b ② ③ b, c, d

② (b) $\forall n \in \mathbb{N}$, if n is odd, then 2n is even and n^2 is even.

$$p \to q \equiv (\neg p) \vee q$$

$$\neg(p \to q) \equiv \neg((\neg p) \vee q) \equiv p \wedge \neg q$$

$\exists n \in \mathbb{N}$, n is odd AND (2n is odd or n^2 is odd).

(3) (d) Induction $\forall n \in \mathbb{N}, P(n)$.

statement true/false.

CAN'T THEN SAY $P(n) = \frac{n(n+1)(2n+1)}{6}$

looks like a number.

~~Induction Example~~: Let $a, b \in \mathbb{R}\setminus\{0\}$. $\forall n \geq 1$,

$$a^n - b^n = (a-b) \sum_{k=0}^{n-1} a^{n-1-k} b^k.$$

Proof: Notice that if $a = b$, $a^n = b^n$. So

$$a^n - b^n = 0 = (a-b) \sum_{k=0}^{n-1} a^{n-1-k} b^k.$$

Let's now suppose $a \neq b$.

We will show $\forall n \geq 1$, $\sum_{k=0}^{n-1} a^{n-1-k} b^k = \frac{a^n - b^n}{a-b}$.

BASE CASE: $n = 1$. $\sum_{k=0}^{0} a^{-k} b^k = 1 = \frac{a-b}{a-b}$.

Inductive Step: Suppose $\sum_{k=0}^{N-1} a^{N-1-k} b^k = \frac{a^N - b^N}{a-b}$ for some $N \geq 1$.

(Show: $\sum_{k=0}^{N} a^{N-k} b^k = \frac{a^{N+1} - b^{N+1}}{a-b}$)

($\forall N \geq 1,\ P(N) \to P(N+1)$)

$$\sum_{k=0}^{N} a^{N-k} b^k = \sum_{k=0}^{N-1} a^{N-k} b^k + b^N$$

$$= a \sum_{k=0}^{N-1} a^{N-k-1} b^k + b^N$$

$$= a \cdot \frac{a^N - b^N}{a-b} + b^N \cdot \frac{a-b}{a-b}$$

$$= \frac{a^{N+1} - \cancel{ab^N}}{a-b} + \frac{\cancel{b^N a} - b^{N+1}}{a-b}$$

$$= \frac{a^{N+1} - b^{N+1}}{a-b}. \quad \boxed{}$$

Section 2.1 text ~ Beginning of C.7 in Gillés Notes.

Def: A sequence is a real-valued function whose domain is a subset of $\mathbb{N}$.

- domain $D \subseteq \mathbb{N}$.
- codomain is $\mathbb{R}$.

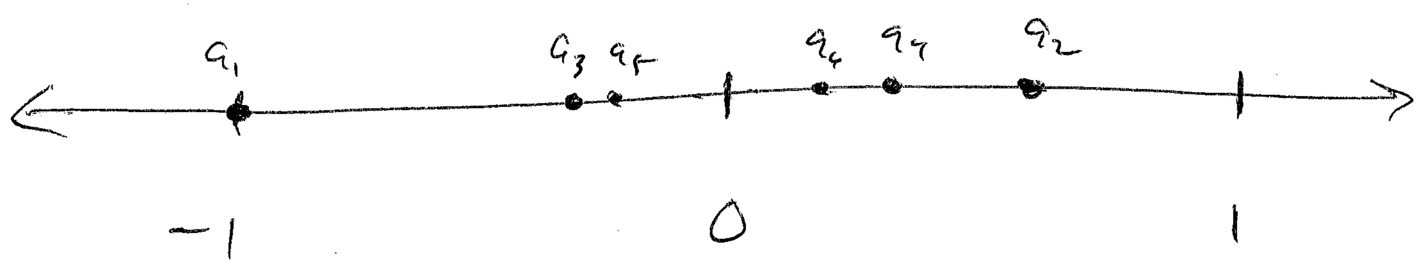
Notations: $a_n = \dfrac{n}{n-4}$, $n \geq 5$.

Ex

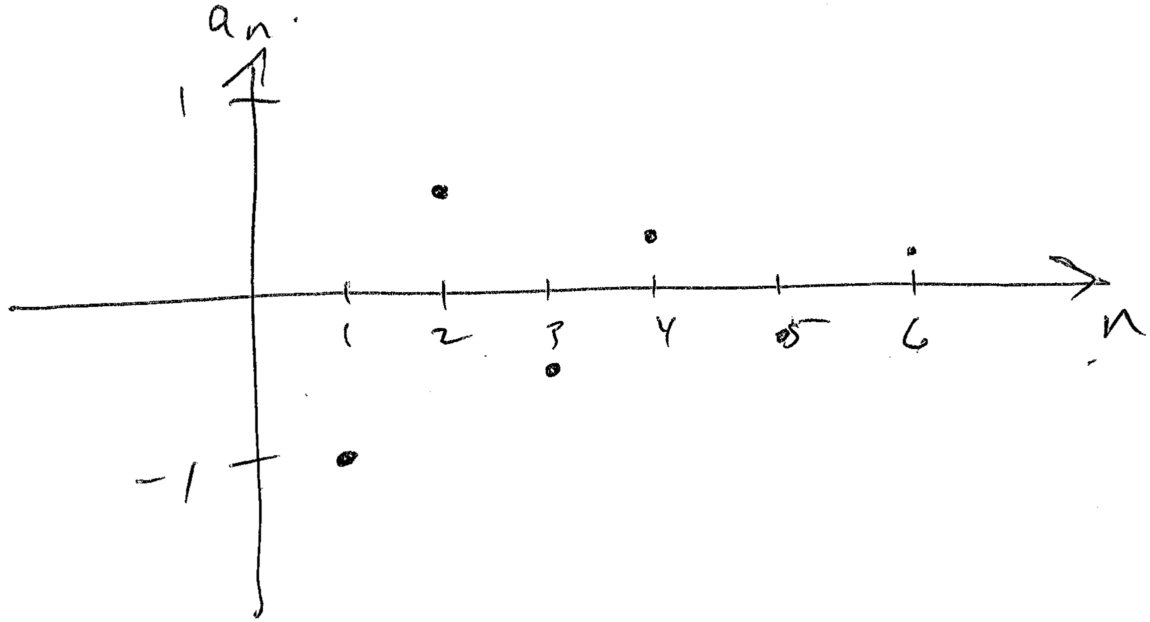
$$\{a_n\}_{n=5}^{\infty} \; , \; \left\{\frac{n}{n-4}\right\}_{n=5}^{\infty}$$

Examples:

$$\left\{ \frac{(-1)^n}{n} \right\}_{n=1}^{\infty} = \left\{ -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \dots \right\}.$$



This is a picture of the image of the sequence



the graph of the sequence.

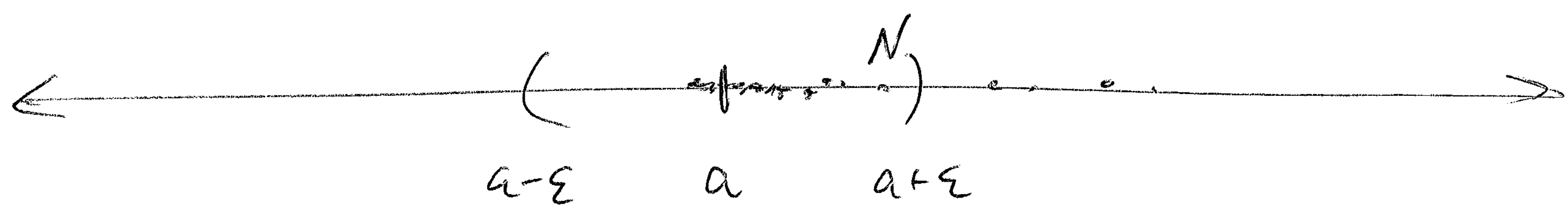
Convergence Definition. Given a sequence $\{a_n\}$,

what does it mean for the *limit* to be a?

- we need to *get* arbitrarily close & *stay* arbitrarily close

We say $\{a_n\}$ *converges* to a (write $\lim_{n\to\infty} a_n = a$)
iff

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall n \in \mathbb{N},\ \text{if } n \geq N,\ |a_n - a| < \varepsilon.$$



Prop 2.6 The sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to 0.

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proof: Let $\varepsilon > 0$.

By the Archimedian Property Thm 1.5 (b),

$\exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$.

Let $n \geq N$.

So $1 \geq \frac{N}{n}$

So $\frac{1}{N} \geq \frac{1}{n}$.

Thus $\frac{1}{n} \leq \frac{1}{N} < \varepsilon$.

I.e. $\left|\frac{1}{n} - 0\right| < \varepsilon$.

Remark: If a sequence converges to a, then the limit is unique.

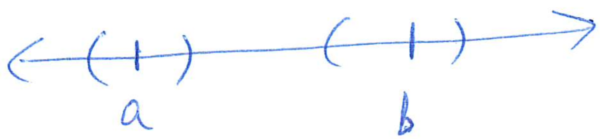
Suppose $\{a_n\}$ is a sequence. Let $a, b \in \mathbb{R}$.

If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} a_n = b$, then $a = b$.

proof: Suppose not. That is, suppose $\lim_{n\to\infty} a_n = a$

and $\lim_{n\to\infty} a_n = b$

and $a \neq b$.



Using $\varepsilon = \frac{|b-a|}{2}$, $\exists N_1, N_2 \in \mathbb{N}$ s.t. $\forall n \geq N_1$ and $n \geq N_2$

$$|a_n - a| < \varepsilon \quad \text{and} \quad |a_n - b| < \varepsilon.$$

Let $n > N_1$ and $n > N_2$.

Then
$$|a-b| = |a - a_n + a_n - b| \leq |a - a_n| + |a_n - b| < \varepsilon + \varepsilon = |b-a|.$$

$(\Rightarrow\Leftarrow)$.

Example: Prove $\lim_{n\to\infty} \frac{n}{n-4} = 1$.

Proof: Let $\varepsilon > 0$

Let $N = \frac{4}{\varepsilon} + 4$.

Suppose $n \in \mathbb{N}$ and $n > N$.

So $n > \frac{4}{\varepsilon} + 4$.

So $n - 4 > \frac{4}{\varepsilon}$

So $\frac{1}{n-4} < \frac{\varepsilon}{4}$

So $\frac{4}{n-4} < \varepsilon$

So $\left| \frac{n}{n-4} - 1 \right| < \varepsilon$. $\boxtimes$

DUMP

WANT: $\left| \frac{n}{n-4} - 1 \right| < \varepsilon$.

AND FIND $n \geq$??

$$\frac{n}{n-4} - 1 = \frac{n}{n-4} - \frac{n-4}{n-4}$$

$$= \frac{4}{n-4} < \varepsilon$$

$$\frac{1}{n-4} < \frac{\varepsilon}{4}$$

$$n-4 > \frac{4}{\varepsilon}$$

$$n > \frac{4}{\varepsilon} + 4$$

N.

Prop 2.7 $\{(-1)^n\}$ does not converge.

Suppose it does converge to $a \in \mathbb{R}$.

Let $\varepsilon = \frac{1}{2}$. $\exists N \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$ if $n \geq N$, $|a_n - a| < \frac{1}{2}$.

Let $n > N$.

If n is even,

$$|a_n - a| < \frac{1}{2}$$

$$|a - 1| < \frac{1}{2}$$

$$-\frac{1}{2} < a - 1 < \frac{1}{2}$$

$$\underline{\frac{1}{2} < a < \frac{3}{2}}$$

If n is odd.

$$|a_n - a| < \frac{1}{2}$$

$$|a + 1| < \frac{1}{2}$$

$$-\frac{1}{2} < a + 1 < \frac{1}{2}$$

$$\underline{-\frac{3}{2} < a < -\frac{1}{2}}.$$

This is a contradiction.