MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #10

Chapter 3 Phase Portraits for Planar Systems
Changing Coordinates

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Is Weather Chaotic?

Coexistence of Chaos and Order within a Generalized Lorenz Model

by

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"A Paradigm Shift" in Predictability Study

- *``As with Poincare and Birkhoff, everything centers around periodic solutions" (Lorenz, 1993).*
- After Lorenz (1963, 1972), Prof. Lorenz and chaos researchers focused on the existence of non-periodic solutions and their complexities.
- Based on the concept of attractor coexistence within the original and generalized Lorenz models (Shen, 2019a), we (Shen et al., 2019; 2020) propose a revised view that focus on the duality of chaos and order.

- **Shen, B.-W.***, R. A. Pielke Sr., X. Zeng, J.-J. Baik, S. Faghih-Naini#, J. Cui#, and R. Atlas, 2020: Is Weather Chaotic? Coexistence of Chaos and Order within a Generalized Lorenz Model. Bulletin of American Meteorological Society. Available from ResearchGate: http://doi.org/10.13140/RG.2.2.21811.07204 (Accepted, Sep. 18, 2020)
- **Shen, B.-W.*,** R. A. Pielke Sr., X. Zeng, J.-J. Baik, S. Faghih-Naini, J. Cui, R. Atlas, T.A. Reyes, 2020: Is Weather Chaotic? Coexisting Chaotic and Non-Chaotic Attractors within Lorenz Models. *The 13th Chaos International Conference (CHAOS2020). 9-12 June 2020. (virtual conference).*

Changing Coordinates

Despite differences in the associated phase portraits, we really have dealt with only three type of matrices in these past four sections:

diagonalization
$$\boxed{\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}}, \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

(I) **(II)** **(III)**

- Any 2×2 matrix that is in one of these three forms is said to be in canonical form.
- Given any linear system $X' = AX$, we can always "change coordinates" so that the new system's coefficient matrix is in canonical form

A Summary for the Three Cases

TBD

Goal: Solve the following 2D system

$$\boxed{X' = AX}$$

Compute the eigenvalues and eigenvectors of A,
U_1 and U_2

$$AU_j = \lambda_j U_j, \qquad j = 1,2$$

Construct $T = (V_1, \mathrm{V}_2)$, $B = T^{-1}AT$ and $\mathrm{X} = TY$ using the following

(I) real eigenvalues

$$(V_1, \mathrm{V}_2) = (U_1, U_2)$$

$$B = T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

(II) complex eigenvalues

$$(V_1, \mathrm{V}_2) = (Re(U_1), Im(U_1))$$

$$B = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

(III) repeated eigenvalues

$$V_1 = U_1$$
$$(A - \lambda I)V_2 = V_1$$

$$B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$\boxed{Y' = BY}$$

A Linear Map and Matrix

A linear map (or *linear transformation*) on $\mathbb{R}^2$ is a function $T : \mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

That is, T simply multiplies any vector by the 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We will thus think of the linear map and its matrix as being interchangeable, so that we also write

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Invertible Matrix and Its Inverse Matrix

Proposition. T *he* 2×2 *matrix* T *is invertible if and only if* $\det T \neq 0$. $\square$

the matrix

$$S = \frac{1}{\det T}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

serves as T^{-1} if $\det T \neq 0$. If $\det T = 0$, we know from Chapter 2 that there are

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

A Summary for Real Eigenvalues

Goal: Solve the following 2D system

$$\boxed{X' = AX}$$

Compute the eigenvalues and eigenvectors of A, V_1 and V_2

$$AV_j = \lambda_j V_j, \qquad j = 1,2$$

Construct T using V_1 and V_2

$$\boxed{Y' = DY} \qquad D = T^{-1}AT$$

$$T = (V_1, V_2)$$

$$Y' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} Y$$

$$Y = c_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}$$

$$X = TY$$

$$= (V_1, V_2) \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}$$

$$= e^{\lambda_1 t}\, V_1 + c_2 e^{\lambda_2 t} V_2$$

Solve a Linear System by Changing Coordinates

$$X' = AX$$

Find T $\quad X = TY \qquad$ T: converts Y into X

so that Y is a solution to the following ODE

$$\boxed{Y' = (T^{-1}AT)Y}$$

Verify

$$(TY)' = TY' = T(T^{-1}AT)Y = ATY \qquad TY \text{ is a solution to } X' = AX$$

(T is not a function of time)

- We think of T as a change of coordinates
- The linear map T converts solutions of $Y' = (T^{-1}AT)Y$ to the solutions of $X' = AX$
- Alternatively, T^{-1} takes the solutions of $X' = AX$ to the solutions of $Y' = (T^{-1}AT)Y$.

Linearly Conjugate

Proposition: If L_1 and L_2 are linearly conjugate,

$$L_1(x) = A_1 x \text{ and } L_2(x) = A_2 x$$

Then, A_1 and A_2 have the same eigenvalues.

Definition: Let L_1 and L_2 be linear maps of R^n. Let L_1 and L_2 are linearly conjugate if there is an invertible linear map P such that

$$L_1 = P^{-1} \circ L_2 \circ P \qquad e.g., D = T^{-1} A T$$

How to find P (or T)? Construct T using eigenvectors of A:

$$P = [V_1, V_2 \ldots, V_n], \text{ V}_j \text{ are eigenvectors.}$$

Diagonalization (for Real Eigenvalues)

Example. (Real Eigenvalues) Suppose the matrix A has two real, distinct eigenvalues λ_1 and λ_2 with associated eigenvectors V_1 and V_2. Let T be the matrix with columns V_1 and V_2. Thus, $TE_j = V_j$ for $j = 1, 2$ where the E_j form the standard basis of $\mathbb{R}^2$. Also, $T^{-1}V_j = E_j$. Therefore, we have

$$\begin{aligned}(T^{-1}AT)E_j &= T^{-1}AV_j = T^{-1}(\lambda_j V_j)\\ &= \lambda_j T^{-1}V_j\\ &= \lambda_j E_j.\end{aligned}$$

Thus the matrix $T^{-1}AT$ assumes the canonical form

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and the corresponding system is easy to solve. ■

Discussed in the next slide

the Linear Map T & Diagonalization $T^{-1}AT = \mathrm{D}$

construct $T = [V_1, V_2]$ $\quad AV_1 = \lambda_1 V_1 \quad AV_2 = \lambda_2 V_2$

two standard basis vectors $\quad E_1 = \begin{pmatrix}1\\0\end{pmatrix} \quad E_2 = \begin{pmatrix}0\\1\end{pmatrix}$

obtain $\quad TE_1 = V_1 \quad TE_2 = V_2 \quad T^{-1}V_1 = E_1 \quad T^{-1}V_2 = E_2$

consider $\quad \boxed{(T^{-1}AT)E_j} = (T^{-1}A)TE_j = (T^{-1}A)V_j = (T^{-1})\lambda_j V_j$

$$= \lambda_j (T^{-1})V_j \boxed{= \lambda_j E_j}$$

$$j = 1 \quad (T^{-1}AT)E_1 = \lambda_1 E_1 = \begin{pmatrix}\lambda_1\\0\end{pmatrix}$$

$$j = 2 \quad (T^{-1}AT)E_2 = \lambda_2 E_2 = \begin{pmatrix}0\\\lambda_2\end{pmatrix}$$

$$\boxed{T^{-1}AT = \begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix}}$$

diagonalization

Diagonalization $T^{-1}AT = \mathrm{D}$

Construct $T = [V_1, \mathrm{V}_2]$

Show $\boxed{T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}}$

Let $T^{-1}AT = \begin{pmatrix} x & u \\ y & w \end{pmatrix}$ $\quad E_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad E_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Apply $(T^{-1}AT)E_1 = \lambda_1 E_1$

Apply $(T^{-1}AT)E_2 = \lambda_2 E_2$

- Find (x, y) or (u, w)
- Send your results via ”chat”
- You have 3 minutes

Diagonalization $T^{-1}AT = \mathrm{D}$

Construct $T = [V_1, \mathrm{V}_2]$ Show $\boxed{T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}}$

Let $T^{-1}AT = \begin{pmatrix} x & u \\ y & w \end{pmatrix}$ $E_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $E_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Apply $(T^{-1}AT)E_1 = \lambda_1 E_1$ $\boxed{\begin{pmatrix} x & u \\ y & w \end{pmatrix} E_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix}}$ $x = \lambda_1 \,\&\, y = 0$

Apply $(T^{-1}AT)E_2 = \lambda_2 E_2$ $\boxed{\begin{pmatrix} x & u \\ y & w \end{pmatrix} E_2 = \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix}}$ $u = 0 \,\&\, w = \lambda_2$

$$\boxed{T^{-1}AT = \begin{pmatrix} x & u \\ y & w \end{pmatrix} = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix} = D}$$

Example

Example. As a further specific example, suppose

$$A = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}.$$

$$x' = -x$$

$$y' = x - 2y$$

eigenvalue problem $|A - \lambda I| = \begin{vmatrix} -1-\lambda & 0 \\ 1 & -2-\lambda \end{vmatrix} = 0$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\lambda = -1, -2$$

Example 1: Find T by Solving $|A - \lambda I| = 0$

Solve for eigenvectors $\quad AV_0 = \lambda V_0 \quad A = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}$

$$-x_0 = \lambda x_0$$
$$x_0 - 2y_0 = \lambda y_0$$

Consider $\lambda = -1$

$$-x_0 = -x_0$$
$$x_0 - 2y_0 = -y_0$$

$$\boxed{x_0 = y_0}$$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0 \end{pmatrix} = y_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Obtain $\boxed{V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$ as an eigenvector associated with $\lambda = -1$

Similarly, $\boxed{V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$ as an eigenvector associated with $\lambda = -2$

$$\boxed{X = \alpha X_1 + \beta X_2 = \alpha e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta e^{-2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$$X_1 = e^{\lambda_1 t} V_1$$
$$X_2 = e^{\lambda_2 t} V_2$$

Construct $T = [V_1, V_2] = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

Verify $T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$A = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$T^{-1} = \frac{1}{\det(T)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$T^{-1}\mathrm{AT} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Solve $Y' = (T^{-1}AT)Y$ to Obtain Y and X

$$T^{-1}\mathrm{AT} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$Y = \begin{pmatrix} u \\ w \end{pmatrix} \qquad Y' = (T^{-1}AT)Y \qquad \begin{aligned} u' &= -u \\ w' &= -2w \end{aligned}$$

$$E_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$Y = \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} \alpha e^{-t} \\ \beta e^{-2t} \end{pmatrix} = \alpha e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$X = TY = T\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha e^{-t} \\ \beta e^{-2t} \end{pmatrix} = \begin{pmatrix} \alpha e^{-t} \\ \alpha e^{-t} + \beta e^{-2t} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha e^{-t} \\ \alpha e^{-t} \end{pmatrix} + \begin{pmatrix} 0 \\ \beta e^{-2t} \end{pmatrix} \boxed{= \alpha e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha \mathrm{X}_1 + \beta X_2}$$

Compute $Y = T^{-1}X$ to Obtain an Eq. for Y'

$$x' = -x$$

$$y' = x - 2y$$

$$Y = \binom{u}{w} \qquad T^{-1} = \frac{1}{\det(T)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

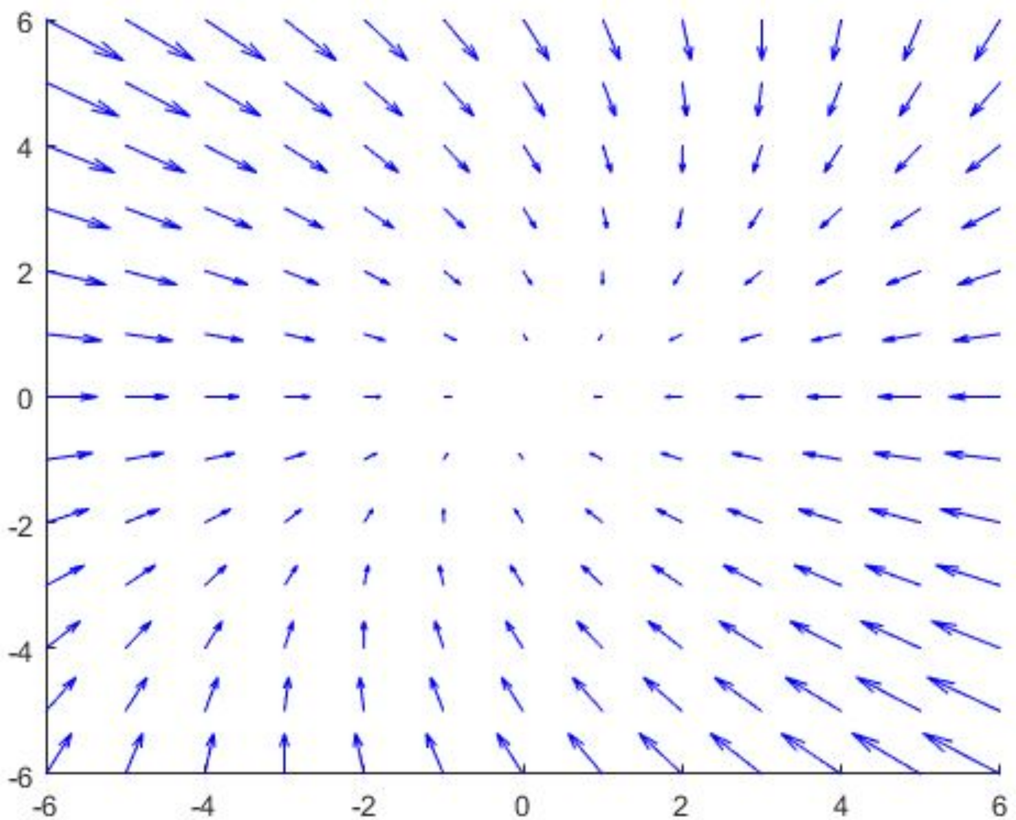
$$Y = \binom{u}{w} = T^{-1}X = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\binom{x}{y} = \binom{x}{-x+y}$$

$$u = x$$
$$w = -x + y$$

$$u' = x' = -x = -u$$

$$w' = -x' + y' = x + x - 2y = 2(x - y) = -2w$$

X' = 2x + y   Y' = -x + 2y



MATLAB Plot for Figure 3.6

$\lambda = -1, -2$

(a sink)

A Summary for (I) Real Eigenvalues

Goal: Solve the following 2D system

$$X' = AX \qquad X' = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix} X$$

Compute the eigenvalues and eigenvectors of A, V_1 and V_2

$$AV_j = \lambda_j V_j, \qquad j = 1,2$$

Construct T using V_1 and V_2

$$T = (V_1, V_2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$Y' = DY \qquad D = T^{-1}AT$$

$$Y' = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} Y$$

$$X = TY$$

$$= (V_1, V_2) \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}$$

$$= e^{\lambda_1 t}\, V_1 + c_2 e^{\lambda_2 t} V_2$$

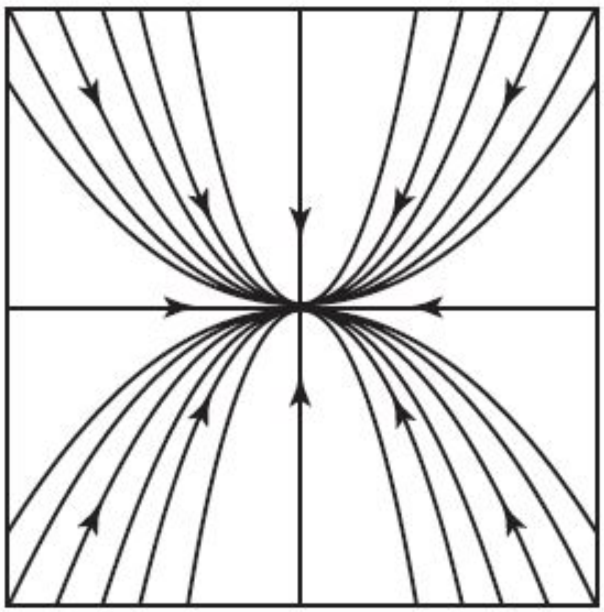
$$Y = c_1 e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}$$

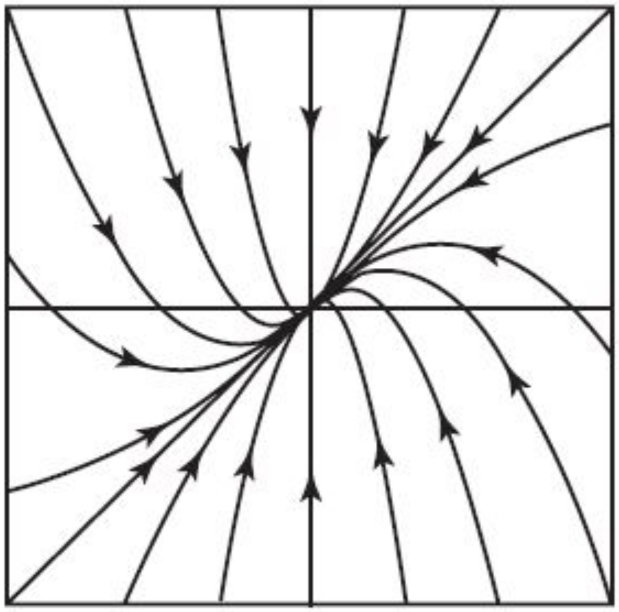
A Summary for (I) Real Eigenvalues

find the solutions to one of the systems in the canonical form

Y $\Rightarrow$ $X = TY$



$T = (V_1, V_2)$
$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$



$$Y' = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} Y$$

$$Y' = DY$$

$$AV_j = \lambda_j V_j$$
$$j = 1,2$$

$$X' = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix} X$$

$$X' = AX$$

A Summary for the Three Cases

Goal: Solve the following 2D system

$$\boxed{X' = AX}$$

Compute the eigenvalues and eigenvectors of A,
U_1 and U_2

$$AU_j = \lambda_j U_j, \qquad j = 1,2$$

Construct $T = (V_1, \mathrm{V}_2)$, $B = T^{-1}AT$ and $\mathrm{X} = TY$ using the following

(I) real eigenvalues

$$(V_1, \mathrm{V}_2) = (U_1, U_2)$$

$$B = T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

(II) complex eigenvalues

$$(V_1, \mathrm{V}_2) = (Re(U_1), Im(U_1))$$

$$B = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

(III) repeated eigenvalues

$$V_1 = U_1$$
$$(A - \lambda I)V_2 = V_1$$

$$B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$\boxed{Y' = BY}$$

A Summary for (II) Complex Eigenvalues

Consider $X' = AX$ and A has complex eigenvalues

Goal: Find T so that $Y = TX$ and $Y' = BY$ $\quad B = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$

Let U_j and λ_j represent the eigenvectors and eigenvalues, respectively. Thus, we have $AU_j = \lambda_j U_j$.

without loss of generality, we have $\lambda_1 = \alpha + i\beta$ and $U_1 = V_1 + iV_2$.

V_1 and V_2 are real, $V_1 = \mathrm{Re}(U_1)$ and $V_2 = \mathrm{Im}(U_1)$.

Below, we show that

- V_1 and V_2 are linearly independent
- When $T = (\mathrm{Re}(U_1), \mathrm{Im}(U_1))$, we obtain $B = T^{-1}\mathrm{A}T = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$.

(II) Complex Eigenvalues

Let U_j and λ_j represent the eigenvectors and eigenvalues, respectively. Thus, we have $AU_j = \lambda_j U_j$.

- Basis "vectors" for changing coordinates (within the phase space). Without loss of generality, we have $\lambda_1 = \alpha + i\beta$ and $U_1 = V_1 + iV_2$.

  V_1 and V_2 are real, $V_1 = \mathrm{Re}(U_1)$ and $V_2 = \mathrm{Im}(U_1)$.

- Basis functions for solutions

  $X_{re} = Re\left(e^{i\lambda_1 t} U_1\right) \Rightarrow trig\ functions,$ functions of time

  $X_{im} = Im\left(e^{i\lambda_1 t} U_1\right) \Rightarrow trig\ functions,$ functions of time

Real Basis Vectors for Complex Eigenvalues

$AU_j = \lambda_j U_j$, $\lambda_1 = \alpha + i\beta$ and $U_1 = V_1 + iV_2$. Show that

- V_1 and V_2 are linearly independent
- In other words, $Re(U_1)$ and $Im(U_1)$ are linearly independent

We assume that V_1 and V_2 are not linearly independent, $V_1 = cV_2$.

Since $AU_1 = \lambda_1 U_1$, we have $A(V_1 + iV_2) = \lambda_1(V_1 + iV_2)$

$$A(cV_2 + iV_2) = (\alpha + i\beta)\,(cV_2 + iV_2)$$

$$(c + i)AV_2 = (\alpha + i\beta)(c + i)V_2\text{c}$$

$$AV_2 = (\alpha + i\beta)V_2$$

This is a contradiction, since the LHS is a complex vector, while the RHS is a real vector.

Section 5.3 Complex Eigenvalues

TBD

- Let U be the eigenvector associated with the complex eigenvalue $\alpha + i\beta$
- Show that $\bar{U}$ is an eigenvector associated with the complex eigenvalue $\alpha - i\beta$
- U and $\bar{U}$ are linearly independent, i.e., $\mathrm{cU} + \mathrm{d}\bar{U} = 0 \Leftrightarrow c = d = 0$
  - c & d are complex numbers.

➢ Note that U and $\bar{U}$ yield the same indepdent real functions, because $\mathrm{Re}(U) = Re(\bar{U})$ and $\mathrm{Im}(U) = -Im(\bar{U})$.

➢ From the first bullet, we have $\mathrm{AU} = (\alpha + i\beta\,)U$.

➢ To prove the statement in the 2nd bullet, we consider $\mathrm{A}\bar{U}$:

$$\mathrm{A}\bar{U} = \overline{AU} = \overline{(\alpha + i\beta\,)U} = (\alpha - i\beta)\bar{U}$$

Properties of $\boldsymbol{Re(U_1)}$ and $\boldsymbol{Im(U_1)}$

$AU_j = \lambda_j U_j$, $\lambda_1 = \alpha + i\beta$ and $U_1 = V_1 + iV_2$. Show that

- $AV_1 = \alpha V_1 - \beta V_2$
- $AV_2 = \alpha V_2 + \beta V_1$
- $AV_j \neq \lambda V_j$

$$A(V_1 + iV_2) = \lambda_1(V_1 + iV_2)$$

$$= (\alpha + i\beta)(V_1 + iV_2)$$

$$= (\alpha V_1 - \beta V_2) + i(\alpha V_2 + \beta V_1)$$

$(T^{-1}\mathrm{AT})E_1$ & $(T^{-1}\mathrm{AT})E_2$

$AU_j = \lambda_j U_j$, $\lambda_1 = \alpha + i\beta$ and $U_1 = V_1 + iV_2$. The previous slide yields

- $AV_1 = \alpha V_1 - \beta V_2$ and $AV_2 = \alpha V_2 + \beta V_1$

When $T = (\mathrm{Re}(U_1), \mathrm{Im}(U_1))$, Show that

- $TE_j = V_j \rightarrow T^{-1}V_j = E_j$
- $(T^{-1}\mathrm{AT})E_1 = \alpha E_1 - \beta E_2$
- $(T^{-1}\mathrm{AT})E_2 = \beta E_1 + \alpha E_2$

$$T = (\mathrm{Re}(U_1), \mathrm{Im}(U_1)) = (V_1, V_2)$$

$$TE_1 = (V_1, V_2)\begin{pmatrix}1\\0\end{pmatrix} = V_1 \qquad TE_2 = (V_1, V_2)\begin{pmatrix}0\\1\end{pmatrix} = V_2$$

$$(T^{-1}\mathrm{AT})E_1 = (T^{-1}\mathrm{A})\mathrm{V}_1 = T^{-1}(\alpha V_1 - \beta V_2) = \alpha \mathrm{E}_1 - \beta \mathrm{E}_2$$

$$(T^{-1}\mathrm{AT})E_2 = (T^{-1}\mathrm{A})\mathrm{V}_2 = T^{-1}(\alpha V_2 + \beta V_1) = \alpha \mathrm{E}_2 + \beta \mathrm{E}_1$$

Show $T^{-1}AT = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$

Construct $T = (\mathrm{Re}(U_1), \mathrm{Im}(U_1)) = [V_1, V_2]$ Show $\boxed{T^{-1}AT = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}}$

Let $M = T^{-1}AT = \begin{pmatrix} x & u \\ y & w \end{pmatrix}$

Apply $ME_1 = \alpha \mathrm{E}_1 - \beta \mathrm{E}_2$

Apply $ME_2 = \alpha \mathrm{E}_2 + \beta \mathrm{E}_1$

- Find (x, y) or (u, w)
- Send your results via ”chat”
- You have 3 minutes

Show $T^{-1}AT = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$

Construct $T = (\mathrm{Re}(U_1), \mathrm{Im}(U_1)) = [V_1, V_2]$ Show $\boxed{T^{-1}AT = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}}$

Let $M = T^{-1}AT = \begin{pmatrix} x & u \\ y & w \end{pmatrix}$

Apply $ME_1 = \alpha E_1 - \beta E_2$ $\boxed{\begin{pmatrix} x & u \\ y & w \end{pmatrix} E_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}}$ $x = \alpha \;\&\; y = -\beta$

Apply $ME_2 = \alpha E_2 + \beta E_1$ $\boxed{\begin{pmatrix} x & u \\ y & w \end{pmatrix} E_2 = \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}}$ $u = \beta \;\&\; w = \alpha$

$$\boxed{T^{-1}AT = \begin{pmatrix} x & u \\ y & w \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}}$$

Example 2

Example. (Another Harmonic Oscillator) Consider the second-order equation

$$x'' + 4x = 0.$$

This corresponds to an undamped harmonic oscillator with mass 1 and spring constant 4. As a system, we have

$$X' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} X = AX.$$

$$x' = y$$
$$y' = -4x$$

$$A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$$

$$AU_j = \lambda_j U_j \,,$$
$$U_1 = V_1 + iV_2$$

Let $|A - \lambda I| = 0 \Longrightarrow$ $\begin{vmatrix} -\lambda & 1 \\ -4 & -\lambda \end{vmatrix} = 0$ $\boxed{\lambda^2 + 4 = 0}$

$$\boxed{\lambda = \pm 2i}$$

Find Basis Vectors

$$AX = \lambda X \Longrightarrow \quad \begin{aligned} y &= \lambda x \\ -4x &= \lambda y \end{aligned}$$

Consider $\lambda = 2i$

$$\begin{aligned} y &= 2ix \\ -4x &= 2iy \end{aligned} \qquad y = 2ix \qquad U_1 = \binom{x}{y} = x\binom{1}{2i}$$

$$[V_1, V_2] = (\mathrm{Re}(U_1), \mathrm{Im}(U_1))$$

Consider $\lambda = -2i$

$$\begin{aligned} y &= -2ix \\ -4x &= -2iy \end{aligned}$$

- Find V_1 and V_2
- Send your results via ”chat”
- You have 3 minutes

Find Basis Vectors

$$AX = \lambda X \Longrightarrow \quad \begin{aligned} y &= \lambda x \\ -4x &= \lambda y \end{aligned}$$

Consider $\lambda = 2i$

$$\begin{aligned} y &= 2ix \\ -4x &= 2iy \end{aligned} \qquad y = 2ix \qquad U_1 = \begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ 2i \end{pmatrix}$$

$$[V_1, V_2] = (\mathrm{Re}(U_1), \mathrm{Im}(U_1)) \qquad V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

Consider $\lambda = -2i$

$$\begin{aligned} y &= -2ix \\ -4x &= -2iy \end{aligned} \qquad y = -2ix \qquad U_2 = \begin{pmatrix} x \\ y \end{pmatrix} = x \begin{pmatrix} 1 \\ -2i \end{pmatrix}$$

$$[V_1, V_2] = (\mathrm{Re}(U_2), \mathrm{Im}(U_2)) \qquad V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

General Solutions

$$X_{re} = Re\left(e^{i2t}\begin{pmatrix}1\\2i\end{pmatrix}\right) = Re\left((\cos(2t) + isin(2t))\begin{pmatrix}1\\2i\end{pmatrix}\right) = \begin{pmatrix}\cos(2t))\\-2sin(2t)\end{pmatrix}$$

$$X_{im} = \mathrm{Im}\left(e^{i2t}\begin{pmatrix}1\\2i\end{pmatrix}\right) = Re\left((\cos(2t) + isin(2t))\begin{pmatrix}1\\2i\end{pmatrix}\right) = \begin{pmatrix}\sin(2t)\\2cos(2t)\end{pmatrix}$$

$$X(t) = c_1 X_{re}(t) + c_2 X_{im}$$

$$x = c_1\cos(2t) + c_2\sin(2t)$$

$$y = -2c_1\sin(2t) + 2c_2\cos(2t)$$

$$x^2 + (\frac{y}{2})^2 = c_1^2 + c_2^2$$

Recall

$$x' = y$$
$$y' = -4x$$

$$xx' + \frac{1}{4}yy' = 0$$

$$\frac{1}{2}(x^2 + \frac{1}{4}y^2)' = 0$$

$$x^2 + \frac{1}{4}y^2 = C$$

ellipse

Illustrate $T^{-1}AT = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$

$$A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \qquad T = [V_1, V_2] = (\mathrm{Re}(U_1), \mathrm{Im}(U_1))$$

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$T^{-1} = \frac{1}{\det(T)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$T^{-1}\mathrm{A}\mathrm{T} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

Compute $Y = T^{-1}X$ to Obtain an Eq. for Y'

$$x' = y$$
$$y' = -4x$$

$$Y = \begin{pmatrix} u \\ w \end{pmatrix}$$

$$T^{-1} = \frac{1}{\det(T)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$$

$$Y = \begin{pmatrix} u \\ w \end{pmatrix} = T^{-1}X = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y/2 \end{pmatrix}$$

$$u = x; \quad w = \frac{y}{2}$$

$$u' = x' = y = 2w$$

$$w' = \frac{y'}{2} = -2x = -2u$$

$$u = c_1 \cos(2t) + c_2 \sin(2t)$$

$$w = -c_1 \sin(2t) + c_2 \cos(2t)$$

$$u^2 + w^2 = c_1^2 + c_2^2$$

$$uu' + ww' = 0$$

$$\frac{1}{2}(u^2 + w^2)' = 0$$

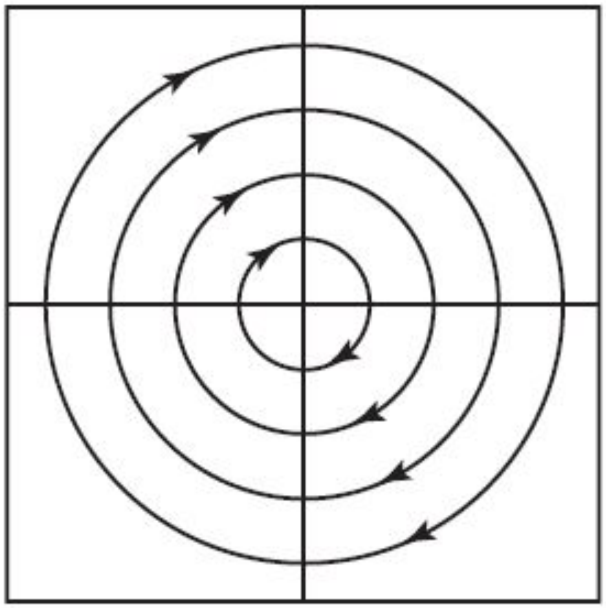
$$u^2 + w^2 = C$$

circle

Changing Coordinates: (II) Complex Eigenvalues)

Y

(circular)



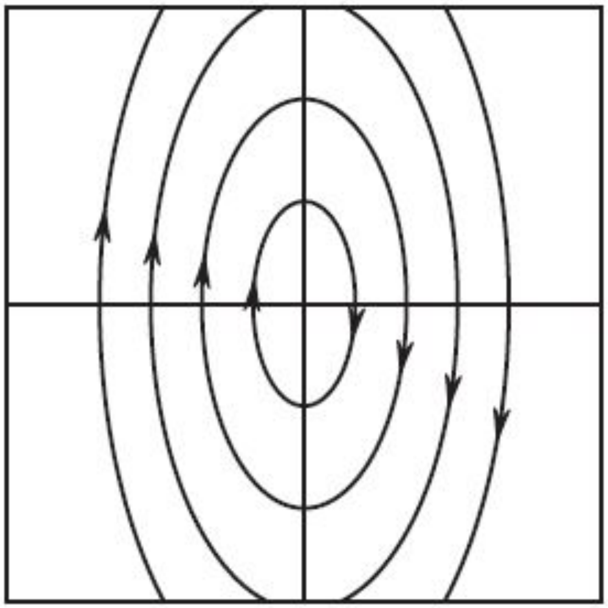
$$AU_j = \lambda_j U_j \,,$$
$$U_1 = V_1 + iV_2$$

$$T = (\mathrm{Re}(U_1), \mathrm{Im}(U_1))$$
$$= (V_1 \; V_2)$$

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$X = TY$

(ellipse)



$$Y' = (T^{-1}\mathrm{AT})Y$$

$$T^{-1}\mathrm{AT} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

$$X' = AX$$

$$A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$$

A Summary for Repeated Eigenvalues

Construct T as follows:

- $AV = \lambda V$
- $(A - \lambda I)V_2 = V$.
- $T = [V, \mathrm{V}_2]$, which leads to $T^{-1}\mathrm{AT} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

Changing Coordinates: (III) Repeated Eigenvalues)

$AV = \lambda V$, λ represents a repeated eigenvalue. Assume two vectors W and V are linearly independent. Show that

- when $AW = c_1 W + c_2 V$, $c_1 = \lambda$.

Assume $c_1 \neq \lambda$. Consider the vector $W + \left(\frac{c_2}{c_1 - \lambda}\right) V$.

$$A\left(W + \left(\frac{c_2}{c_1-\lambda}\right)V\right) = AW + \left(\frac{c_2}{c_1-\lambda}\right)AV$$

$$c_1 W + c_2 V + \left(\frac{c_2}{c_1-\lambda}\right)\lambda V \qquad = c_1 W + c_2\left(1 + \frac{\lambda}{c_1-\lambda}\right)V$$

$$= c_1 W + c_2\left(\frac{c_1}{c_1-\lambda}\right)V \qquad = c_1\left(W + \left(\frac{c_2}{c_1-\lambda}\right)V\right)$$

c_1 is the 2nd eigenvalue different from λ. This is a contradiction. Thus, $c_1 = \lambda$.

Find the 2nd Basis Vector

Based on the above discussions, when two vectors W and V are linearly independent, we have:

- $AW = \lambda W + c_2 V$ and
- $\boxed{(A - \lambda I)V_2 = V}$ for $W = c_2 V_2$.
- Here, V_2 represent a 2nd basis vector

Consider $W = c_2 V_2$, we have

$$c_2 A V_2 = \lambda c_2 V_2 + c_2 V$$

$$A V_2 = \lambda V_2 + V$$

$$(A - \lambda I)V_2 = V$$

$$(A - \lambda I)^2 V_2 = (A - \lambda I)V = 0$$ because of $AV = \lambda V$

Example 3: Find Two Basis Vectors

Solve for eigenvalues $AV_0 = \lambda V_0$ $A = \begin{pmatrix} 2 & 1 \\ -\frac{1}{4} & 1 \end{pmatrix}$

$$2x_0 + y_0 = \lambda x_0$$
$$\frac{-x_0}{4} + y_0 = \lambda y_0$$

$$\lambda = \frac{3}{2}$$

Consider $\lambda = \frac{3}{2}$

$$2x_0 + y_0 = \frac{3}{2}x_0$$
$$\frac{-x_0}{4} + y_0 = \frac{3}{2}y_0$$

$$\boxed{\frac{-x_0}{2} = y_0}$$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ \frac{-x_0}{2} \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}$$

Obtain $\boxed{V = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}}$ as an eigenvector associated with $\lambda = \frac{3}{2}$

For V_2, we solve

$$(A - \lambda I)V_2 = V$$

$$\begin{pmatrix} \frac{1}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix} V_2 = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}$$

Example 3: Find V_2 (conti.)

For V_2, we solve

$$(A-\lambda I)V_2 = V. \qquad \begin{pmatrix} \frac{1}{2} & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix} V_2 = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \qquad \text{Let } V_2 = \binom{u}{w}$$

$$\frac{1}{2}u + w = 1$$

$$\frac{-1}{4}u - \frac{1}{2}w = -\frac{1}{2}$$

$$\frac{1}{2}u + w = 1$$

$$V_2 = \binom{u}{w} = \binom{1}{1/2}$$

Make sure that V and V2 are linearly independent

Construct T and Compute $T^{-1}\mathrm{AT}$

$$A = \begin{pmatrix} 2 & 1 \\ -1/4 & 1 \end{pmatrix} \qquad T = [V, \mathrm{V}_2] \qquad V = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 1 \\ -1/2 & 1/2 \end{pmatrix} \qquad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$T^{-1} = \frac{1}{\det(T)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1/2 & -1 \\ 1/2 & 1 \end{pmatrix}$$

$$T^{-1}\mathrm{AT} = \begin{pmatrix} 1/2 & -1 \\ 1/2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1/4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1/2 & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 5/4 & -1/2 \\ 3/4 & 3/2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 6/4 & 1 \\ 0 & 6/4 \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad \lambda = \frac{3}{2}$$

A Summary for Repeated Eigenvalues

Construct T as follows:

- $AV = \lambda V$
- $(A - \lambda I)V_2 = V$.
- $T = [V, \mathrm{V}_2]$, which leads to $T^{-1}\mathrm{AT} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

$$Y' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} Y \qquad X' = AX$$

Solve the above for Y and compute $\mathrm{X} = \mathrm{T}Y$

For example,

$$Y' = \begin{pmatrix} 3/2 & 1 \\ 0 & 3/2 \end{pmatrix} Y \qquad X' = \begin{pmatrix} 2 & 1 \\ -1/4 & 1 \end{pmatrix} X$$

Upper triangle

A Summary for the Three Cases

Goal: Solve the following 2D system

$$\boxed{X' = AX}$$

Compute the eigenvalues and eigenvectors of A,
U_1 and U_2

$$AU_j = \lambda_j U_j, \qquad j = 1,2$$

Construct $T = (V_1, V_2)$, $B = T^{-1}AT$ and $X = TY$ using the following

(I) real eigenvalues

$$(V_1, V_2) = (U_1, U_2)$$

$$B = T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

(II) complex eigenvalues

$$(V_1, V_2) = (Re(U_1), Im(U_1))$$

$$B = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

(III) repeated eigenvalues

$$V_1 = U_1$$

$$(A - \lambda I)V_2 = V_1$$

$$B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$\boxed{Y' = BY}$$

(II) Complex Eigenvalues

Let U_j and λ_j represent the eigenvectors and eigenvalues, respectively. Thus, we have $AU_j = \lambda_j U_j$.

- Basis "vectors" for changing coordinates (within the phase space). Without loss of generality, we have $\lambda_1 = \alpha + i\beta$ and $U_1 = V_1 + iV_2$.

  V_1 and V_2 are real, $V_1 = \mathrm{Re}(U_1)$ and $V_2 = \mathrm{Im}(U_1)$.

- Basis functions for solutions

  $X_{re} = Re\left(e^{i\lambda_1 t} U_1\right) \Rightarrow trig\ functions,$ functions of time

  $X_{im} = Im\left(e^{i\lambda_1 t} U_1\right) \Rightarrow trig\ functions,$ functions of time

A Summary for Chapter 3: 2D Linear Systems

$$x' = ax + by \quad (= P(x,y)) \qquad (1)$$

$$y' = cx + dy \quad (= Q(x,y)\,) \qquad (2)$$

$$X' = AX \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

1. Eigenvalue problem: $|A - \lambda I| = 0$
2. Two linearly independent solutions, $e^{\lambda_1 t}V_1$ and $e^{\lambda_2 t}V_2$
3. Real eigenvalues for a source, sink, or saddle
4. Complex eigenvalues for a center, spiral sink or spiral sink
5. Diagonalization
6. Changing coordinates Linearly Conjugate

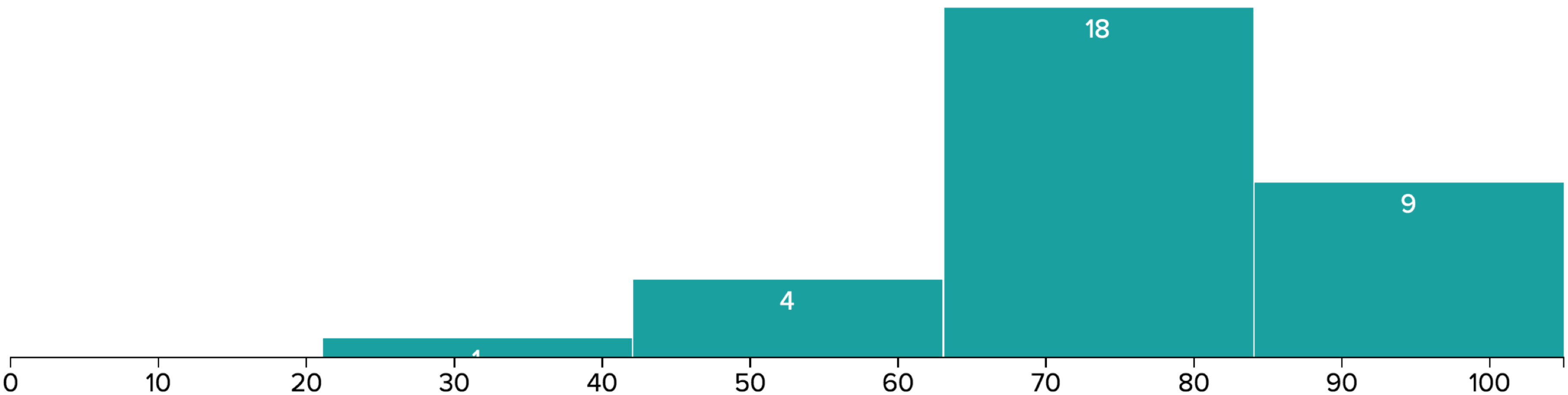
HW #1

- Derivative Tests vs. Analysis using a Perturbation Method
  - See my whiteboard for additional information

- Source, Sink, Saddle
- The Logistic Eq: $x' = ax(1-x)$, separable
- ➔ $x' = \beta(x - x_{c+})(x - x_{c-})$

- Association of Uploaded Pages and Specific Problems

HW #1



MINIMUM	MEDIAN	MAXIMUM	MEAN	STD DEV
33.0	78.0	105.0	75.38	14.73

1: [20+5 points]

$$\frac{dx}{dt} = f(x),$$

here (i) $f(x) = x$; (ii) $f(x) = x^2$; and (iii) $f(x) = x^3$.

(a) Perform (linear) stability analysis.

(b) Find and analyze the corresponding solutions.

(i-a) one unstable critical point at $x = 0$;

(ii-a) one saddle point at $x = 0$;

(iii-a) one unstable critical point at $x = 0$ (hint: apply a perturbation method)

(i-b) sol: $x(t) = x_o e^t$.

(ii-b) sol: $x(t) = \frac{x_o}{1 - x_o t}$.

(iii-b) sol: $x(t) = \frac{x_o}{\sqrt{1 - 2x_o^2 t}}$

A Saddle Point

$$x' = x^2 = f(x)$$

Is the critical point a sink or source?

critical points $\boxed{f(x) = 0}$

$$\boxed{x = 0}$$

1st derivative $f'(x) = 2x$

$x = 0$ $\quad f'(0) = 0\ ?$

A saddle point or a half-stable critical point (e.g., Strogatz)

Based on the definition, we can obtain:

$x < 0$	$x > 0$
$\dfrac{dx}{dt} > 0$	$\dfrac{dx}{dt} > 0$
positive direction	positive direction

$\mathrm{x} = 0$

A saddle point

Apply a perturbation method

$$x = x_c + \varepsilon(t)$$

$\boxed{x = 0 + \varepsilon}$ $\qquad \varepsilon' = \varepsilon^2$

The above discussions can help determine $x = 1$ is a saddle point within $x' = (x-1)^2$.

$x' = x^3$

$$x' = x^3 = f(x)$$

Is the critical point a sink or source?

critical points $\boxed{f(x) = 0}$

$\boxed{x = 0}$

1st derivative $f'(x) = 3x^2$

$x = 0$ $\quad f'(0) = 0\ ?$

Based on the definition, we can obtain:

$x < 0$	$x > 0$
$\dfrac{dx}{dt} < 0$	$\dfrac{dx}{dt} > 0$
negative direction	positive direction

$\mathrm{x} = 0$

a source

A Perturbation Method $x' = x^3$

$$x' = x^3 = f(x)$$

Is the critical point a sink or source?

critical points $\boxed{f(x) = 0}$

$\boxed{x = 0}$

1st derivative $f'(x) = 3x^2$

$x = 0$ $\quad f'(0) = 0\ ?$

Apply a perturbation method

$$x = x_c + \varepsilon(t)$$

$\boxed{x = 0 + \varepsilon}$ $\quad \varepsilon' = \varepsilon^3$

The above discussions help determine $x = 1$ is an unstable point within $x' = (x-1)^3$.

$$x = 1 + \varepsilon(t)$$

$$\varepsilon' = \varepsilon^3$$