Quiz 8
Differential Equations
Math 337
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Problem 1: The Fourier sine transform is defined by:

$$F(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\sin(\omega x)dx$$

while its inverse transform is given by:

$$f(x) = \int_0^\infty F(\omega)\sin(\omega x)d\omega$$

Consider $F(\omega) = e^{-\beta\omega}, \beta > 0 (\omega \geq 0)$. Find the inverse Fourier sine transform by evaluating:

$$f(x) = \int_0^\infty e^{-\beta\omega}\sin(\omega x)d\omega$$

Show your integration methods (integration by parts) in solving this problem. This result gives you one *transform pair for a Fourier sine transform* table.

Notice the following:

$$f(x) = \int_0^\infty e^{-\beta\omega}\sin(\omega x)d\omega$$

Using integration by parts, let $u = \sin(\omega x), dV = e^{-\beta\omega}$, we get

$$\int_0^\infty e^{-\beta\omega}\sin(\omega x)d\omega = \sin(\omega x)\left(\frac{-e^{-\beta\omega}}{\beta}\right) + \frac{\omega}{\beta}\int_0^\infty e^{-\beta\omega}\cos(\omega x)d\omega$$

Now we use integration by parts again, and let $u = \cos(\omega x), dV = e^{-\beta\omega}$, we get

$$\int_0^\infty e^{-\beta\omega}\cos(\omega x)d\omega = \cos(\omega x)\left(\frac{-e^{-\beta\omega}}{\beta}\right) - \frac{\omega}{\beta}\int_0^\infty e^{-\beta\omega}\sin(\omega x)d\omega$$

After substituting the previous equation into the original we get:

$$\begin{aligned}\int_0^\infty e^{-\beta\omega}\sin(\omega x)d\omega &= \sin(\omega x)\left(\frac{-e^{-\beta\omega}}{\beta}\right) + \frac{\omega}{\beta}\left(\cos(\omega x)\left(\frac{-e^{-\beta\omega}}{\beta}\right) - \frac{\omega}{\beta}\int_0^\infty e^{-\beta\omega}\sin(\omega x)d\omega\right)\\ &= \left(\sin(\omega x) + \frac{\omega}{\beta}\cos(\omega x)\right)\left(\frac{-e^{-\beta\omega}}{\beta}\right) - \frac{\omega^2}{\beta^2}\int_0^\infty e^{-\beta\omega}\sin(\omega x)d\omega\end{aligned}$$

Now we can add the last term on the right side to the left side and get:

$$\frac{\beta^2+\omega^2}{\beta^2}\int_0^\infty e^{-\beta\omega}\sin(\omega x)d\omega = \left(\sin(\omega x) + \frac{\omega}{\beta}\cos(\omega x)\right)\left(\frac{-e^{-\beta\omega}}{\beta}\right)$$

Thus we get the result:

$$f(x) = \int_0^\infty e^{-\beta\omega}\sin(\omega x)d\omega = \lim_{A\to\infty}\left(\sin(\omega x) + \frac{\omega}{\beta}\cos(\omega x)\right)\left(\frac{-\beta e^{-\beta\omega}}{\beta^2+\omega^2}\right)\Bigg|_{\omega=0}^{\omega=A}$$

Problem 2: Use the definition of the Laplace transform to find:

$$\mathcal{L}(\cosh(\beta t)), \qquad s > \beta$$

form the integrals in the definition and solve them. Use the definition of $\cosh(\beta t)$ in terms of the appropriate sum of exponentials to work your integrals. Write your answer with one common denominator

Notice the following:

$$\begin{aligned}
\mathcal{L}(\cosh(\beta t)) &= \int_0^\infty e^{-st}\cosh(\beta t)\,dt \\
&= \int_0^\infty e^{-st}\frac{e^{\beta t}+e^{-\beta t}}{2}\,dt \\
&= \frac{1}{2}\int_0^\infty e^{-(s-\beta)t}\,dt + \frac{1}{2}\int_0^\infty e^{-(s+\beta)t}\,dt \\
&= \lim_{A\to\infty}\left(\frac{1}{-2(s-\beta)}e^{-(s-\beta)t} + \frac{1}{-2(s+\beta)}e^{-(s+\beta)t}\right)\Bigg|_{t=0}^{t=A} \\
&= -\left(\frac{1}{-2(s-\beta)} + \frac{1}{-2(s+\beta)}\right) \\
&= -\left(\frac{s+\beta}{-2(s-\beta)(s+\beta)} + \frac{s-\beta}{-2(s-\beta)(s+\beta)}\right) \\
&= \frac{s}{s^2-\beta^2}
\end{aligned}$$

Problem 3: Use the result in Question 2 to solve the initial value problem with Laplace transforms:

$$y'' - 9y = 0, \qquad y(0) = 6, \qquad y'(0) = 0$$

Thus, your answer should include the cosh function.

Notice the following, and let $\mathcal{L}(y) = Y(s)$:

$$\mathcal{L}\left(y'' - 9y = 0\right) \to s^2 Y(s) - sy(0) - y'(0) - 9Y(s) = 0$$

So through simple algebra:

$$\begin{aligned}(s^2 - 9)Y(s) - 6s &= 0 \\ Y(s) &= \frac{6s}{s^2 - 9} \\ &= 6 * \frac{s}{s^2 - 3^2}\end{aligned}$$

Thus we get the result that

$$\mathcal{L}^{-1}(Y(s)) = y = \mathcal{L}^{-1}\left(6 * \frac{s}{s^2 - 3^2}\right) = 6\cosh(3t)$$

Problem 4: Solve the following initial value problem with *Laplace transforms*:

$$y'' + 2y' + y = 12te^{-t}, \qquad y(0) = 3, \qquad y'(0) = -2$$

Notice the following, and let $\mathcal{L}(y) = Y(s)$, with $(\mathcal{L}(e^{-t}f(t)) = F(s+1))$:

$$\mathcal{L}\left(y'' + 2y' + y = 12te^{-t}\right) \to s^2Y(s) - sy(0) - y'(0) + 2sY(s) - y(0) + Y(s) = \frac{12}{(s+1)^2}$$

So through simple algebra:

$$\begin{aligned}
(s^2 + 2s + 1)Y(s) - (3s + 1) &= \frac{12}{(s+1)^2} \\
Y(s) &= \frac{12}{(s+1)^2(s+1)^2} + \frac{3s+1}{(s+1)^2} \\
&= \frac{12}{(s+1)^4} + \frac{3(s+1)}{(s+1)^2} - \frac{2}{(s+1)^2} \\
&= \frac{12}{(s+1)^4} + \frac{3}{s+1} - \frac{2}{(s+1)^2}
\end{aligned}$$

Thus we get the result that

$$\mathcal{L}^{-1}(Y(s)) = y = \mathcal{L}^{-1}\left(\frac{12}{(s+1)^4} + \frac{3}{s+1} - \frac{2}{(s+1)^2}\right) = (2t^3 + 3 - 2t)e^{-t}$$