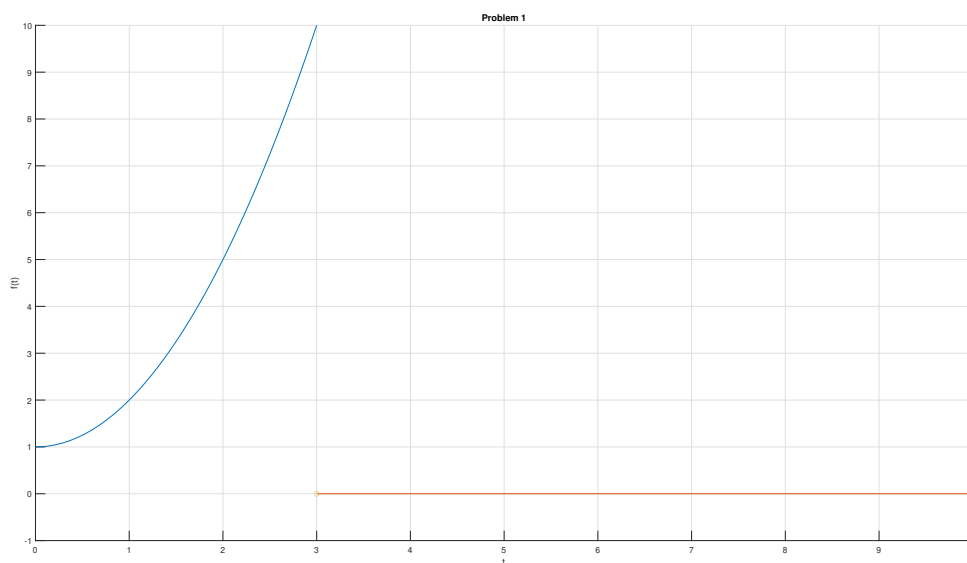


**Quiz 10**  
**Differential Equations**  
**Math 337**  
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**Problem 1:** Consider the function  $f(t)$  defined as follows:

$$f(t) = \begin{cases} t^2 + 1 & 0 \leq t \leq 3 \\ 0 & t > 3 \end{cases}$$

Sketch a graph of this function and write it in terms of the step function,  $u_c(t)$ , which is defined in the lecture notes. Further, write the function with the step function so that every element is readily found in the Laplace table. (Something like  $u_c(t) \sin(t - c)$ .) Finally, find the Laplace Transform of  $f(t)$ ,  $F(s) = \mathcal{L}[f(t)]$ .



$$\begin{aligned} f(t) &= (t^2 + 1)(u_0(t) - u_3(t)) \\ &= (t^2)u_0(t) + u_0(t) - ((t - 3) + 3)^2u_3(t) - u_3(t) \\ &= (t^2)u_0(t) + u_0(t) - (t - 3)^2u_3(t) - 6(t - 3)u_3(t) - 9u_3(t) - u_3(t) \\ &= (t^2)u_0(t) + u_0(t) - (t - 3)^2u_3(t) - 6(t - 3)u_3(t) - 10u_3(t) \end{aligned}$$

$$F(s) = \frac{1}{s^2} + \frac{1}{s} - e^{-3s} \left( \frac{2}{s^3} + \frac{6}{s^2} + \frac{10}{s} \right)$$

**Problem 2:** Solve the following initial value problem with Laplace transforms:

$$y'' + 2y' + 5y = f(t) = \begin{cases} 5 & 0 \leq t \leq 4 \\ -(t-9) & 4 \leq t \leq 9, \\ 0 & t > 9 \end{cases}, \quad y(0) = 1, \quad y'(0) = 4$$

Notice the following:

$$\begin{aligned} \mathcal{L}[y'' + 2y' + 5y] &= s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + 5Y(s) \\ &= (s^2 + 2s + 5)Y(s) - (s + 6) \end{aligned}$$

$$\begin{aligned} f(t) &= 5(u_0(t) - u_4(t)) - (t-9)(u_4(t) - u_9(t)) \\ &= 5u_0(t) - 5u_4(t) - ((t-4) - 5)u_4(t) + (t-9)u_9(t) \\ &= 5u_0(t) - 5u_4(t) - (t-4)u_4(t) + 5u_4(t) + (t-9)u_9(t) \\ &= 5u_0(t) - (t-4)u_4(t) + (t-9)u_9(t) \\ \mathcal{L}[f(t)] &= \frac{5}{s} - \frac{e^{-4s}}{s^2} + \frac{e^{-9s}}{s^2} \end{aligned}$$

Thus we get the equality:

$$Y(s) = \frac{5}{s(s^2 + 2s + 5)} - \frac{e^{-4s}}{s^2(s^2 + 2s + 5)} + \frac{e^{-9s}}{s^2(s^2 + 2s + 5)} + \frac{s + 6}{s^2 + 2s + 5}$$

Notice the partial fractions decomposition:

$$\begin{aligned} \frac{1}{s(s^2 + 2s + 5)} &= \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 5} \\ 1 &= (A + B)s^2 + (2A + C)s + 5A \end{aligned}$$

So we get  $A = \frac{1}{5}, B = \frac{-1}{5}, C = \frac{-2}{5}$

$$\frac{1}{s(s^2 + 2s + 5)} = \frac{1}{5s} - \frac{s + 2}{5(s^2 + 2s + 5)}$$

Notice the partial fractions decomposition:

$$\begin{aligned} \frac{1}{s^2(s^2 + 2s + 5)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 2s + 5} \\ 1 &= (A + C)s^3 + (2A + B + D)s^2 + (5A + 2B)s + 5B \end{aligned}$$

So we get  $B = \frac{1}{5}, A = \frac{-2}{25}, C = \frac{2}{25}, D = \frac{-1}{25}$

$$\frac{1}{s^2(s^2 + 2s + 5)} = \frac{-2}{25s} + \frac{1}{5s^2} + \frac{2s - 1}{25(s^2 + 2s + 5)}$$

So we can now rewrite  $Y(s)$

$$Y(s) = \left( \frac{1}{s} - \frac{s+1}{(s+1)^2+4} - \frac{1}{(s+1)^2+4} \right) - \frac{e^{-4s}}{25} \left( \frac{-2}{s} + \frac{5}{s^2} + \frac{2(s+1)}{(s+1)^2+4} - \frac{3}{(s+1)^2+4} \right) \\ + \frac{e^{-9s}}{25} \left( \frac{-2}{s} + \frac{5}{s^2} + \frac{2(s+1)}{(s+1)^2+4} - \frac{3}{(s+1)^2+4} \right) + \left( \frac{s+1}{(s+1)^2+4} + \frac{5}{(s+1)^2+4} \right)$$

Now we can take the Laplace inverse:

$$y(t) = \left( 1 - e^{-t} \cos(2t) - \frac{1}{2} e^{-t} \sin(2t) \right) - \frac{u_4(t)}{25} \left( -2 + 5(t-4) + 2e^{-(t-4)} \cos(2(t-4)) - \frac{3}{2} \sin(2(t-4)) \right) \\ + \frac{u_9(t)}{25} \left( -2 + 5(t-9) + 2e^{-(t-9)} \cos(2(t-9)) - \frac{3}{2} \sin(2(t-9)) \right) + \left( \cos(2t) + \frac{5}{2} \sin(2t) \right)$$

**Problem 3:** The limiting solution is:

$$y(t) = 1 - \frac{u_4(t)}{25} \left( -2 + 5(t-4) - \frac{3}{2} \sin(2(t-4)) \right) \\ + \frac{u_9(t)}{25} \left( -2 + 5(t-9) - \frac{3}{2} \sin(2(t-9)) \right) + \left( \cos(2t) + \frac{5}{2} \sin(2t) \right)$$

