name: Solutions

Exam 1, Math 330

Each problem is worth 8 points.

1. (a) Complete the definition:
   We say the sequence $\{a_n\}_{n=1}^{\infty}$ **converges** to $a \in \mathbb{R}$ provided...

$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N},$
if $n \geq N$, then $|a_n - a| < \varepsilon$.

(b) Prove that the sequence converges using the $N - \epsilon$ definition.

$$a_n = \frac{4}{n+3}$$

Scratch

$\left| \frac{4}{n+3} - 0 \right| < \varepsilon.$

$\frac{4}{n+3} < \varepsilon$

$\frac{4}{\varepsilon} < n+3$

$\frac{4}{\varepsilon} - 3 < n$

Let $\varepsilon > 0$.
Suppose $N \in \mathbb{N}$ where $N > \frac{4}{\varepsilon} - 3$.
Let $n \in \mathbb{N}$ and suppose $n \geq N$.

Then $n > \frac{4}{\varepsilon} - 3$

So $n+3 > \frac{4}{\varepsilon}$

So $\varepsilon > \frac{4}{n+3}$

So $\varepsilon > \left| \frac{4}{n+3} - 0 \right|$. $\blacksquare$

Thus $\lim_{n \to \infty} \frac{4}{n+3} = 0$.

2. Suppose that $0 < x < 1$. For $n \in \mathbb{N}$, define $A_n = x^n + x^2$.

(a) Prove that the sequence $\{A_n\}$ is bounded. Let $n \in \mathbb{N}$.

Since $0 < x < 1$, we have $0 < x^n \leq x < 1$.

Thus $x^2 < x^n + x^2 < 1 + x^2$

So $\{A_n\} = \{x^n + x^2\}$ is bounded.

(b) Is the sequence $\{A_n\}$ increasing, $\boxed{\text{decreasing}}$ or neither? Justify your answer.

Let $n \in \mathbb{N}$. Since $0 < x < 1$

$0 < x^{n+1} < x^n$

Thus $x^{n+1} + x^2 < x^n + x^2$

I.e. $\forall n \in \mathbb{N}$, $A_{n+1} < A_n$.

(c) true or false. The sequence $\{A_n\}$ converges.

(By the MCT).

3. Complete the definitions.

(a) The set $K \subseteq \mathbb{R}$ is **closed** provided...

$\forall \{a_n\} \subseteq K$, if $\{a_n\}$ converges, then $\lim_{n\to\infty} a_n \in K$.

(b) Suppose $K \subseteq \mathbb{R}$. We say that $a \in \mathbb{R}$ is the **supremum** of K provided...

1. $a$ is an upper bound for $K$.
   I.e. $\forall x \in K$, $a \geq x$.
2. $a$ is least upper bound.
   I.e. $\forall y \in \mathbb{R}$, if $y$ is an upper bound of $K$, then $x \leq y$.

(c) Give an example of a set $K$ that is closed and has $\sup K = 10$.

$K = [0, 10]$.

5. For each problem, circle T for true or F for false.

T F Every sequence has a convergent subsequence.

E.g. $a_n = n$

T F Every sequence in the set $(0,1)$ has a convergent, monotone subsequence.

$(0,1)$ is bounded.

T F The number 2 is an upper bound for the set $\{x \mid x^2 < x\}$

$\sup\{x \mid x^2 < x\} = 1 < 2.$

T F Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences. If the sequence $\{a_n b_n\}$ converges, then the sequences $\{a_n\}$ and $\{b_n\}$ converge.

For $n \geq 1$, $a_n = n$, $b_n = \frac{1}{n}$.

T F $\forall x \in \mathbb{R}$, if $|x| > 1$, then $|3x^2 - 4x^4| \leq 7|x|^4$

$\Delta$- inequality

T F For every bounded set $S$, we have $\max S = \sup S$.

$S = (0,1)$ MAX $S$ DNE.
SUP $S = 1$.

T F A set $S \subseteq \mathbb{R}$ is dense in $\mathbb{R}$ iff $\forall x \in \mathbb{R}$, $\exists \{a_n\} \subseteq S$ such that $\lim_{n\to\infty} a_n = x$.

sequential density thm.

T F The set $[0, \infty)$ is sequentially compact.

~ unbounded

~ $a_n = n$

then $\{a_n\}$ has no

convergent subsequence.