MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #1b

A Brief Review of Fundamental ODEs

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Outline

- A review of first order ODEs
  - Separable ODEs
  - Linear ODEs
  - Bernoulli Equation
  - Exact ODEs; differential
- A review of second order ODEs
  - Homogeneous, linear ODEs with constant coefficients
  - Euler-Cauchy Equation
- An outlook

(1) Separable ODEs

Many practically useful ODEs can be reduced to the form

(1) $$g(y)\,y' = f(x)$$

Dependent vs.
Independent variables

by purely algebraic manipulations. Then we can integrate on both sides with respect to x, obtaining

(2) $$\int g(y)\,y'\,dx = \int f(x)\,dx + c.$$

On the left we can switch to y as the variable of integration. By calculus, $y'\,dx = dy$, so that

(3) $$\int g(y)\,dy = \int f(x)\,dx + c.$$

If f and g are continuous functions, the integrals in (3) exist, and by evaluating them we obtain a general solution of (1). This method of solving ODEs is called the **method of separating variables**, and (1) is called a **separable equation**, because in (3) the variables are now separated: x appears only on the right and y only on the left.

Separable ODEs: Examples

equations	Initial conditions	solutions	remarks
$y' = 1 + y^2$	$y(0) = \tan(c)$	$y(x) = \tan(x + c)$	$-c - \frac{\pi}{2} < x < -c + \frac{\pi}{2}$
$y' = -2xy$	$y(0) = \mathrm{c}$	$y(x) = \mathrm{ce}^{-\mathrm{x}^2}$	Bell-shaped

(2) Linear First-Order ODEs

A first-order ODE is said to be **linear** if it can be brought into the form

$$y' + P(x)y = Q(x)$$

by algebra, and **nonlinear** if it cannot be brought into this form.

The defining feature of the linear ODE (1) is that it is linear in both the unknown function y and its derivative $y' = dy/dx$, whereas p and r may be ***any*** given functions of x. If in an application the independent variable is time, we write t instead of x.

Homogeneous Linear ODE. $Q(x) = 0$

(3) $$y_h(x) = ce^{-\int P(x)dx} \quad (c = \pm e^{c^*} \quad \text{when} \quad y \gtrless 0);$$

the **integrating factor** $$I(x) = e^{\int P(x)dx} \propto y_h^{-1}$$

Nonhomogeneous Linear ODE.

$$y(x) = \frac{1}{I(x)}\left[\int I(x)\,Q(x)\,dx + C\right]$$

Total output = response to the “forcing” Q + Response to the initial Data

fyi: An Earlier Version

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$I(x) = e^{\int P(x)\,dx}$$

$$\big(I(x)y\big)' = I(x)\,Q(x)$$

$$\boxed{y(x) = \frac{1}{I(x)}\left[\int I(x)\,Q(x)\,dx + C\right]}$$

(3) Bernoulli Equations and Logistic Equations

Linear ODEs: $y' + P(x)y = Q(x)$

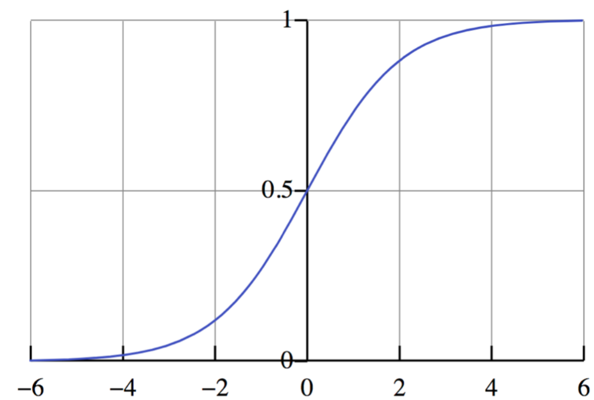
Bernoulli Equations

Equations	New variable	Linear ODE	Solutions
$y' + p(x)y = g(x)y^a$	$u(x) = y^{1-a}$	$u' + (1-a)pu$ $= (1-a)g$	
$y' = Ay - By^2$ (Logistic Eq.)	$u = y^{-1}$	$u' + Au = B$	$y = \dfrac{1}{ce^{-At} + B/A}$ (sigmoid function)

Logistic activation function

$$S(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{e^x+1}.$$

(Wikipedia)



1=on

0=off

(4) Exact ODEs

$$\boxed{du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.}$$

Consider $\quad Mdx + Ndy$

Seek $\quad Mdx + Ndy = f_x dx + f_y dy = \nabla f \cdot d\vec{r} = df$

$$when\ N_y - M_x = 0,$$

$$i.e., M = f_x \ and\ N = f_y$$

f : potential function

Calc III: Divergence-Curl and Curl-Gradient

Supp

11 Theorem If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\operatorname{div} \operatorname{curl} \mathbf{F} = 0$$

DC

$$div\ H = 0 \Rightarrow H = curl\ F$$

ψ : stream function

3 Theorem If f is a function of three variables that has continuous second-order partial derivatives, then

$$\operatorname{curl}(\nabla f) = \mathbf{0}$$

CG

$$curl\ H = 0 \Rightarrow H = \nabla f$$

Conservative

$$\nabla \times \vec{F} = 0 \Rightarrow \vec{F} = \nabla f$$

f : potential function

Review: 2nd order ODEs

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$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

Two basic facts enable us to solve homogeneous linear equations. The first of these says that if we know two solutions y_1 and y_2 of such an equation, then the **linear combination** $y = c_1y_1 + c_2y_2$ is also a solution.

3 Theorem If $y_1(x)$ and $y_2(x)$ are both solutions of the linear homogeneous equation (2) and c_1 and c_2 are any constants, then the function

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

is also a solution of Equation 2.

Are y_1 and y_2 Linearly Independent?

Linear Dependence and Independence of Solutions

Let the ODE (1) *have continuous coefficients* $p(x)$ *and* $q(x)$ *on an open interval* I. *Then two solutions* y_1 *and* y_2 *of* (1) *on* I *are linearly dependent on* I *if and only if their* "**Wronskian**"

(6)
$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

is 0 *at some* x_0 *in* I. *Furthermore, if* $W = 0$ *at an* $x = x_0$ *in* I, *then* $W = 0$ *on* I; *hence, if there is an* x_1 *in* I *at which* W *is not* 0, *then* y_1, y_2 *are linearly independent on* I.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Two Types of 2nd Order ODEs

- Homogeneous Linear ODEs with Constant Coefficients
  - $ay" + by' + c = 0$ $\quad\quad (P, Q, R \rightarrow a, b, c)$
  - $y = e^{rx}$ ➔ $ar^2 + br + c = 0$
- Euler-Cauchy Equations
  - $x^2y" + axy' + b = 0$
  - $y = x^m$ ➔ $m^2 + (a - 1)m + b = 0$

Key Concepts within Selected 2nd Order ODEs

- Homogenous
- Linear
- Basis functions
- Linearly Independent
- Wronskian
- Superposition Principle

Fundamental Theorem for the Homogeneous Linear ODE (2)

For a homogeneous linear ODE (2), *any linear combination of two solutions on an open interval I is again a solution of* (2) *on I. In particular, for such an equation, sums and constant multiples of solutions are again solutions.*

(I): Homogeneous Linear ODEs with Constant Coefficients

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$$ay'' + by' + cy = 0$$

where a, b, and c are constants and $a \neq 0$.

Summary of Cases I–III

Case	Roots of (2)	Basis of (1)	General Solution of (1)
I	Distinct real λ_1, λ_2	$e^{\lambda_1 x}, e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
II	Real double root $\lambda = -\frac{1}{2}a$	$e^{-ax/2}, xe^{-ax/2}$	$y = (c_1 + c_2 x)e^{-ax/2}$
III	Complex conjugate $\lambda_1 = -\frac{1}{2}a + i\omega$, $\lambda_2 = -\frac{1}{2}a - i\omega$	$e^{-ax/2} \cos \omega x$ $e^{-ax/2} \sin \omega x$	$y = e^{-ax/2}(A \cos \omega x + B \sin \omega x)$

Type I: Characteristic Equations

5 $$ay'' + by' + cy = 0$$ $y = e^{rx}$

where a, b, and c are constants and $a \neq 0$.

is equal to 0. We know that the exponential function $y = e^{rx}$ (where r is a constant) has the property that its derivative is a constant multiple of itself: $y' = re^{rx}$. Furthermore, $y'' = r^2e^{rx}$. If we substitute these expressions into Equation 5, we see that $y = e^{rx}$ is a solution if

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

or
$$(ar^2 + br + c)e^{rx} = 0$$

But e^{rx} is never 0. Thus $y = e^{rx}$ is a solution of Equation 5 if r is a root of the equation

6 $$ar^2 + br + c = 0$$

Equation 6 is called the **auxiliary equation** (or **characteristic equation**) of the differential equation $ay'' + by' + cy = 0$. Notice that it is an algebraic equation that is obtained from the differential equation by replacing y'' by r^2, y' by r, and y by 1.

Case I: Two Distinct Real Roots

$$ar^2 + br + c = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Distinct real roots c

CASE I $b^2 - 4ac > 0$

In this case the roots r_1 and r_2 of the auxiliary equation are real and distinct, so $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are two linearly independent solutions of Equation 5. (Note that $e^{r_2 x}$ is not a constant multiple of $e^{r_1 x}$.) Therefore, by Theorem 4, we have the following fact.

8 If the roots r_1 and r_2 of the auxiliary equation $ar^2 + br + c = 0$ are real and unequal, then the general solution of $ay'' + by' + cy = 0$ is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

y_1 and y_2 are LI.

EXAMPLE 1 Solve the equation $y'' + y' - 6y = 0$.

Case II: Double Root

Double root

CASE II $b^2 - 4ac = 0$

10 If the auxiliary equation $ar^2 + br + c = 0$ has only one real root r, then the general solution of $ay'' + by' + cy = 0$ is

$$y = c_1 e^{rx} + c_2 x e^{rx}$$

How to find the 2nd solution as a basis fucntion?

The method of variation of parameters:

$$y_2 = u(x) y_1$$

$$u'' = 0$$

$$u = x \qquad\qquad y_2 = x y_1 = x e^{r} x$$

Case III: Complex Roots

Complex roots

CASE III $b^2 - 4ac < 0$

11 If the roots of the auxiliary equation $ar^2 + br + c = 0$ are the complex numbers $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, then the general solution of $ay'' + by' + cy = 0$ is

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

Review: A Brief Summary for Type (I) ODEs

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$$\boxed{ay'' + by' + cy = 0}$$

where a, b, and c are constants and $a \neq 0$.

Summary of Cases I–III

what is the most tricky part?

Case	Roots of (2)	Basis of (1)	General Solution of (1)
I	Distinct real λ_1, λ_2	$e^{\lambda_1 x}, e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
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III	Complex conjugate $\lambda_1 = -\frac{1}{2}a + i\omega$, $\lambda_2 = -\frac{1}{2}a - i\omega$	$e^{-ax/2} \cos \omega x$ $e^{-ax/2} \sin \omega x$	$y = e^{-ax/2}(A \cos \omega x + B \sin \omega x)$

(Real) Double Root

$$\lim_{r \to r_1} \frac{e^{rx} - e^{r_1 x}}{r - r_1}$$

L'Hospital's Rule $\quad \dfrac{\partial}{\partial r} \qquad = \lim_{r \to r_1} \dfrac{x e^{rx}}{1}$

$$= x e^{r_1 x}$$

Are y_1 and y_2 Linearly Independent?

Linear Dependence and Independence of Solutions

Let the ODE (1) *have continuous coefficients* $p(x)$ *and* $q(x)$ *on an open interval* I. *Then two solutions* y_1 *and* y_2 *of* (1) *on* I *are linearly dependent on* I *if and only if their "***Wronskian***"*

(6) $$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

is 0 *at some* x_0 *in* I. *Furthermore, if* $W = 0$ *at an* $x = x_0$ *in* I, *then* $W = 0$ *on* I; *hence, if there is an* x_1 *in* I *at which* W *is not* 0, *then* y_1, y_2 *are linearly independent on* I.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Type (II): Euler–Cauchy Equations

$$x^2y'' + axy' + by = 0$$

$$y = x^m, \qquad y' = mx^{m-1}, \qquad y'' = m(m-1)x^{m-2}$$

$$m^2 + (a-1)m + b = 0.$$

- Case #1 of two distinct roots, m_1 and m_2;

$$y = c_1x^{m_1} + c_2x^{m_2}$$

- Case #2 of one real repeated root, m;

$$y = c_1\boxed{x^m \ln(x)} + c_2x^m$$

- Case #3 of complex roots, $\alpha \pm \beta i$.

$$y = c_1x^{\alpha}\cos(\beta\ln(x)) + c_2x^{\alpha}\sin(\beta\ln(x))$$

Real Double Root

$$\lim_{r \to r_1} \frac{e^{rx} - e^{r_1 x}}{r - r_1} \qquad \lim_{m \to m_1} \frac{x^m - x^{m_1}}{m - m_1}$$

L'Hospital's Rule $\frac{\partial}{\partial r}$

$$= \lim_{r \to r_1} \frac{x e^{rx}}{1} \qquad \frac{\partial}{\partial m} \quad \lim_{m \to m_1} \frac{x^m \ln(x)}{1}$$

$$= x e^{r_1 x} \qquad x^{m_1} \ln(\mathrm{x})$$

Additional Detail for the Previous Slide

$$Let\ A = x^m$$

$$Goal:\ \frac{\partial A}{\partial m}$$

From the first Eq., we have

$$\ln(A) = m ln(x)$$

$$\frac{\partial}{\partial m}$$

$$\frac{\frac{\partial A}{\partial m}}{A} = ln(x)$$

$$\frac{\partial A}{\partial m} = A ln(x) = x^m \ln(x)$$

Relation between Two Types of 2nd Order ODEs

- Homogeneous Linear ODEs with Constant Coefficients
  - $ay" + by' + c = 0$
  - $y = e^{rx}$

- Euler-Cauchy Equations
  - $x^2y" + axy' + b = 0$
  - $y = x^m$ ➔ $\mathrm{x} = e^t$ ➔ $\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = 0$

(Constant Coefficients)

An Outlook ➔ ODEs in M537

- Homogeneous Linear ODEs with Constant Coefficients
  - $ay" + by' + c = 0$
  - $y = e^{rx}$ ➔ $y = e^{s(x)}$

    W.K.B. method for ODEs with non-constant coefficients

- Euler-Cauchy Equations
  - $x^2y" + axy' + b = 0$
  - $y = x^m$ ➔ $y = x^m \sum a_n x^n$ ➔ asymptotic series

    Frobenious method

An Outlook: A System of First Order ODEs

For example, consider a second-order constant coefficient equation of the form

$$x'' + ax' + bx = 0.$$

If we let $y = x'$, then we may rewrite this equation as a system of first-order equations

$$\begin{aligned} x' &= y \\ y' &= -bx - ay. \end{aligned}$$

An Outlook: Planar Linear System

We now further restrict our attention to the most important class of planar systems of differential equations, namely, linear systems. In the autonomous case, these systems assume the simple form

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}$$

where a, b, c, and d are constants. We may abbreviate this system by using the *coefficient matrix* A where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then the linear system may be written as

$$X' = AX.$$

An Outlook with Type (I) ODEs: $ax'' + bx' + cx = 0$

(A) $y = e^{rt}$

$$ar^2 + br + c = 0$$

Characteristic Equation

(B) $$x'' + \frac{b}{a}x' + \frac{c}{a}x = 0$$

let $x' = y$

obtain $y' = -\frac{c}{a}x - \frac{b}{a}y$

define $X = \begin{pmatrix} x \\ y \end{pmatrix}$; $A = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}$

$$X' = AX$$

assume $X = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{\lambda t}$;

eigenvalue problem $|A - \lambda I| = \begin{vmatrix} 0 - \lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a} - \lambda \end{vmatrix} = 0$

$$a\lambda^2 + b\lambda + c = 0$$