MATH 531 PDE

HW 2 due 02/05/2021

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1. **(3 pts)**
Consider the partial differential equation for heat in a one-dimensional rod with temperature $u(x,t)$:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}.$$

Assume initial condition:

$$u(x,0) = f(x)$$

and boundary conditions:

$$u(0,t) = 16 \quad u(3,t) = 0$$

Determine the equilibrium temperature distribution:
$u(x) =$ ______________________

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2. **(3 pts)**
Consider the partial differential equation for heat in a one-dimensional rod with temperature $u(x,t)$:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}.$$

Assume initial condition:

$$u(x,0) = f(x)$$

and boundary conditions:

$$\frac{\partial u}{\partial x}(0,t) = 0 \quad u(2,t) = 14$$

Determine the equilibrium temperature distribution:
$u(x) =$ ______________________

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3. **(3 pts)**
Consider the partial differential equation for heat in a one-dimensional rod with temperature $u(x,t)$:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + Q,$$

with $Q/k = 3$.
Assume initial condition:

$$u(x,0) = f(x)$$

and boundary conditions:

$$u(0,t) = 14 \quad u(1,t) = 3$$

Determine the equilibrium temperature distribution:
$u(x) =$ ______________________

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4. **(4 pts)**
Determine the equilibrium temperature distribution for a one-dimensional rod composed of two different materials in perfect thermal contact at $x = 1$. For $0 < x < 1$, there is one material ($c\rho = 1$, $K_0 = 0.6$) with a constant source ($Q = 2.5$), whereas for the other $1 < x < 2$ there are no sources ($Q = 0$, $c\rho = 1.8$, $K_0 = 1.9$) with $u(0) = 0$ and $u(2) = 0$. (Hint: See Exercise 1.3.2.)

Determine the equilibrium temperature distribution for each segment of the rod:
For $0 \le x \le 1$, $u(x) =$ ______________

For $1 \le x \le 2$, $u(x) =$ ______________

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5. **(5 pts)**
Consider the partial differential equation for heat in a one-dimensional rod with temperature $u(x,t)$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2.$$

Assume initial condition:

$$u(x,0) = 4e^{-x}\sin\left(\frac{\pi x}{3}\right),$$

and boundary conditions:

$$\frac{\partial u}{\partial x}(0,t) = 3 \quad \frac{\partial u}{\partial x}(3,t) = \beta$$

For what values of β are there solutions to this heat equation?

$\beta =$ ______

Determine the equilibrium temperature distribution.
$u(x) =$ ______________________

In your homework assignment write a brief paragraph explaining what is occurring physically to allow the unique equilibrium solution.

6. **(5 pts)**
Consider the partial differential equation for heat in a one-dimensional rod with temperature $u(x,t)$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

Assume initial condition:

$$u(x,0) = 6xe^{-x/1},$$

and boundary conditions:

$$\frac{\partial u}{\partial x}(0,t) = 1 \quad \frac{\partial u}{\partial x}(1,t) = \beta$$

For what values of β are there solutions to this heat equation?

$\beta =$ ______

Determine the equilibrium temperature distribution.
$u(x) =$ ____________________

In your homework assignment write a brief paragraph explaining what is occurring physically to allow the unique equilibrium solution.

7. **(5 pts)**
Consider the partial differential equation for heat in a one-dimensional rod with temperature $u(x,t)$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1.1x - \beta.$$

Assume initial condition:

$$u(x,0) = 0.2x^2 \sin\left(\frac{\pi x}{2}\right),$$

and boundary conditions:

$$\frac{\partial u}{\partial x}(0,t) = 0 \quad \frac{\partial u}{\partial x}(2,t) = 0$$

For what values of β are there solutions to this heat equation?

$\beta =$ ______

Determine the equilibrium temperature distribution.
$u(x) =$ ____________________

In your homework assignment write a brief paragraph explaining what is occurring physically to allow the unique equilibrium solution.

Problem 8 (11pts): Hint: Read chapter 1.1-1.4

1.2.9. Consider a thin one-dimensional rod without sources of thermal energy whose lateral surface area is not insulated.

(a) Assume that the heat energy flowing out of the lateral sides per unit surface area per unit time is $w(x,t)$. Derive the partial differential equation for the temperature $u(x,t)$.

(b) Assume that $w(x,t)$ is proportional to the temperature difference between the rod $u(x,t)$ and a known outside temperature $\gamma(x,t)$. Derive that

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) - \frac{P}{A}[u(x,t) - \gamma(x,t)]h(x), \qquad (1.2.15)$$

where $h(x)$ is a positive x-dependent proportionality, P is the lateral perimeter, and A is the cross-sectional area.

(c) Compare (1.2.15) to the equation for a one-dimensional rod whose lateral surfaces are insulated, but with heat sources.

(d) Specialize (1.2.15) to a rod of circular cross section with constant thermal properties and 0° outside temperature.

*(e) Consider the assumptions in part (d). Suppose that the temperature in the rod is uniform [i.e., $u(x,t) = u(t)$]. Determine $u(t)$ if initially $u(0) = u_0$.

Problem 9 (9pts): Hint: Read chapter 1.1-1.4

1.4.12. Suppose the concentration $u(x,t)$ of a chemical satisfies Fick's law (1.2.13), and the initial concentration is given $u(x,0) = f(x)$. Consider a region $0 < x < L$ in which the flow is specified at both ends $-k\frac{\partial u}{\partial x}(0,t) = \alpha$ and $-k\frac{\partial u}{\partial x}(L,t) = \beta$. Assume α and β are constants.

(a) Express the conservation law for the entire region.

(b) Determine the total amount of chemical in the region as a function of time (using the initial condition).

(c) Under what conditions is there an equilibrium chemical concentration and what is it?