

**Systems 2D**  
**Differential Equations**  
**Math 337**  
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**Problem 11 (c):**

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

**Solution 11 (c):** Let  $\begin{vmatrix} 0 - \lambda & 2 \\ -3 & 5 - \lambda \end{vmatrix} = 0$

$$\begin{aligned} (\lambda)(\lambda - 5) + 6 &= \lambda^2 - 5\lambda + 6 = 0 \\ &= (\lambda - 3)(\lambda - 2) = 0 \\ \lambda &= 3, 2 \end{aligned}$$

Let  $\lambda_1 = 3$

$$\begin{pmatrix} 0 - 3 & 2 \\ -3 & 5 - 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Let  $\lambda_2 = 2$

$$\begin{pmatrix} 0 - 2 & 2 \\ -3 & 5 - 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

Because the eigenvalues are both positive, we have an unstable node. As  $t \rightarrow \infty$ , the phase portrait is going away from the origin, which is why it creates an unstable node.

**Problem 12 (c):**

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -12 & -10 \\ 15 & 13 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

**Solution 12 (c):** Let  $\begin{vmatrix} -12 - \lambda & -10 \\ 15 & 13 - \lambda \end{vmatrix} = 0$

$$\begin{aligned} (\lambda + 12)(\lambda - 13) + 150 &= \lambda^2 - \lambda + 6 = 0 \\ &= (\lambda - 3)(\lambda + 2) = 0 \\ \lambda &= 3, -2 \end{aligned}$$

Let  $\lambda_1 = 3$

$$\begin{pmatrix} -12 - 3 & -10 \\ 15 & 13 - 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -15 & -10 \\ 15 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

Let  $\lambda_2 = -2$

$$\begin{pmatrix} -12 - -2 & -10 \\ 15 & 13 - -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -10 & -10 \\ 15 & 15 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$$

Because the eigenvalues have opposite signs, we have a saddle point. As  $t \rightarrow \infty$ , the phase portrait is going away from the origin along one eigenvector and going towards the origin along the other eigenvector, which is why it creates a saddle point.

**Problem 13 (c):**

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -25 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

**Solution 13 (c):** Let  $\begin{vmatrix} 0 - \lambda & -25 \\ 1 & 0 - \lambda \end{vmatrix} = 0$

$$\begin{aligned} (\lambda)(\lambda) + 25 &= \lambda^2 + 25 = 0 \\ &= \lambda^2 = -25 \\ \lambda &= \pm 5i \end{aligned}$$

Let  $\lambda_1 = 5i$

$$\begin{pmatrix} 0 - 5i & -25 \\ 1 & 0 - 5i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -5i & -25 \\ 1 & -5i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5i \\ 1 \end{pmatrix}$$

$$x_1(t) = \begin{pmatrix} 5i \\ 1 \end{pmatrix} (\cos(5t) + i \sin(5t))$$

$$u(t) + iw(t) = \begin{pmatrix} -5 \sin(5t) \\ \cos(5t) \end{pmatrix} + i \begin{pmatrix} 5 \cos(5t) \\ \sin(5t) \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{pmatrix} = c_1 \begin{pmatrix} -5 \sin(5t) \\ \cos(5t) \end{pmatrix} + c_2 \begin{pmatrix} 5 \cos(5t) \\ \sin(5t) \end{pmatrix}$$

Because the eigenvalues' real part is 0, we have a center or ellipse. As  $t \rightarrow \infty$ , the phase portrait moves in a counter clockwise rotation.

**Problem 14 (c):**

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 6 & -9 \\ 1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

**Solution 14 (c):** Let  $\begin{vmatrix} 6 - \lambda & -9 \\ 1 & 6 - \lambda \end{vmatrix} = 0$

$$(\lambda - 6)(\lambda - 6) + 9 = \lambda^2 - 12\lambda + 45 = 0$$

$$\begin{aligned} \lambda &= \frac{12 \pm \sqrt{144 - 180}}{2} \\ &= 6 \pm 3i \end{aligned}$$

Let  $\lambda_1 = 6 + 3i$

$$\begin{pmatrix} 6 - (6 + 3i) & -9 \\ 1 & 6 - (6 + 3i) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3i & -9 \\ 1 & -3i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3i \\ 1 \end{pmatrix}$$

$$x_1(t) = \begin{pmatrix} 3i \\ 1 \end{pmatrix} (\cos(3t) + i \sin(3t)) e^{6t}$$

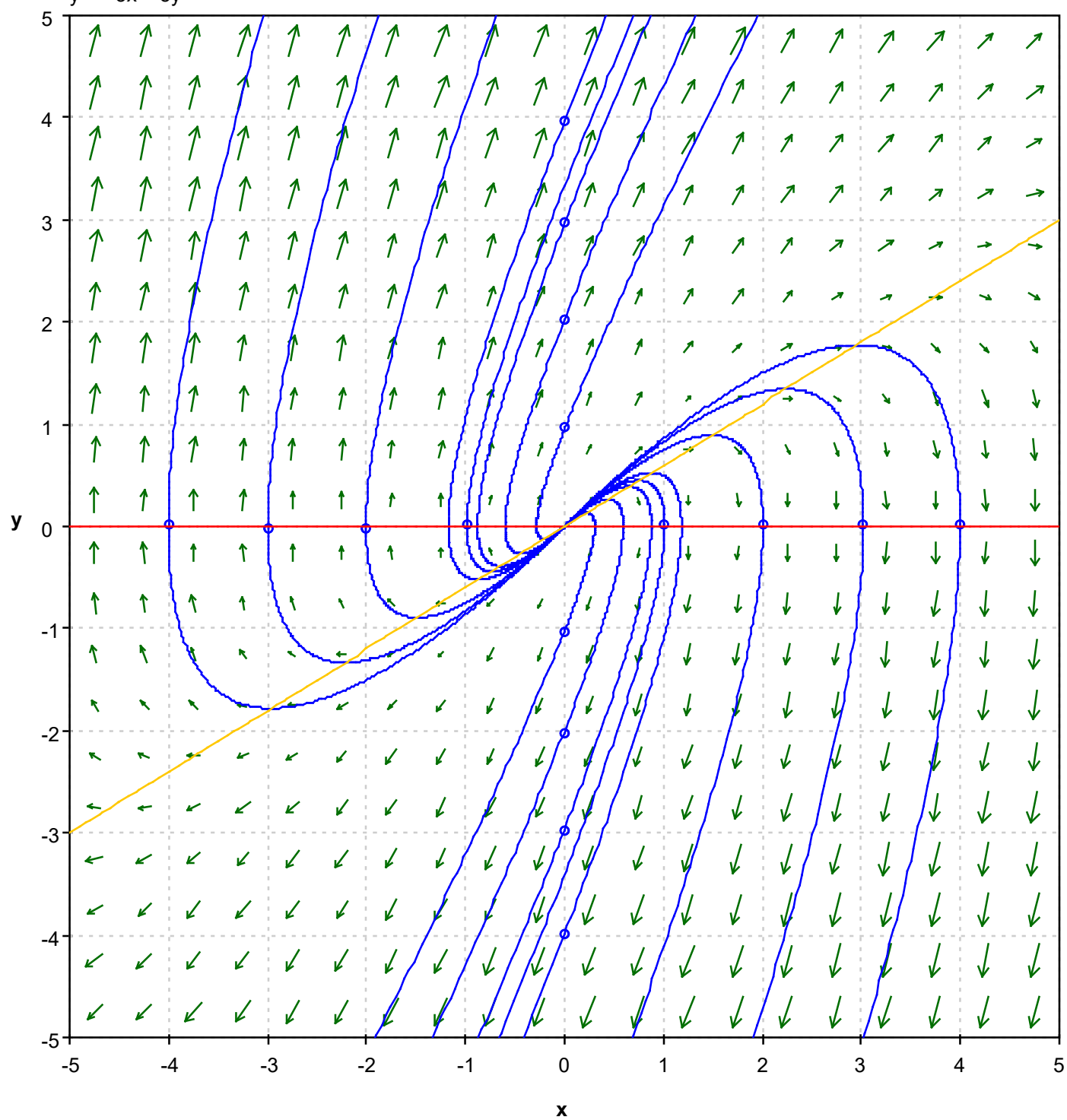
$$u(t) + iw(t) = \begin{pmatrix} -3 \sin(3t) \\ \cos(3t) \end{pmatrix} e^{6t} + i \begin{pmatrix} 3 \cos(3t) \\ \sin(3t) \end{pmatrix} e^{6t}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 \begin{pmatrix} -3 \sin(3t) \\ \cos(3t) \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} 3 \cos(3t) \\ \sin(3t) \end{pmatrix} e^{6t}$$

Because the eigenvalues' real part is positive, we have a spiral source. As  $t \rightarrow \infty$ , the phase portrait is going away from the origin, which is why it creates a spiral source.

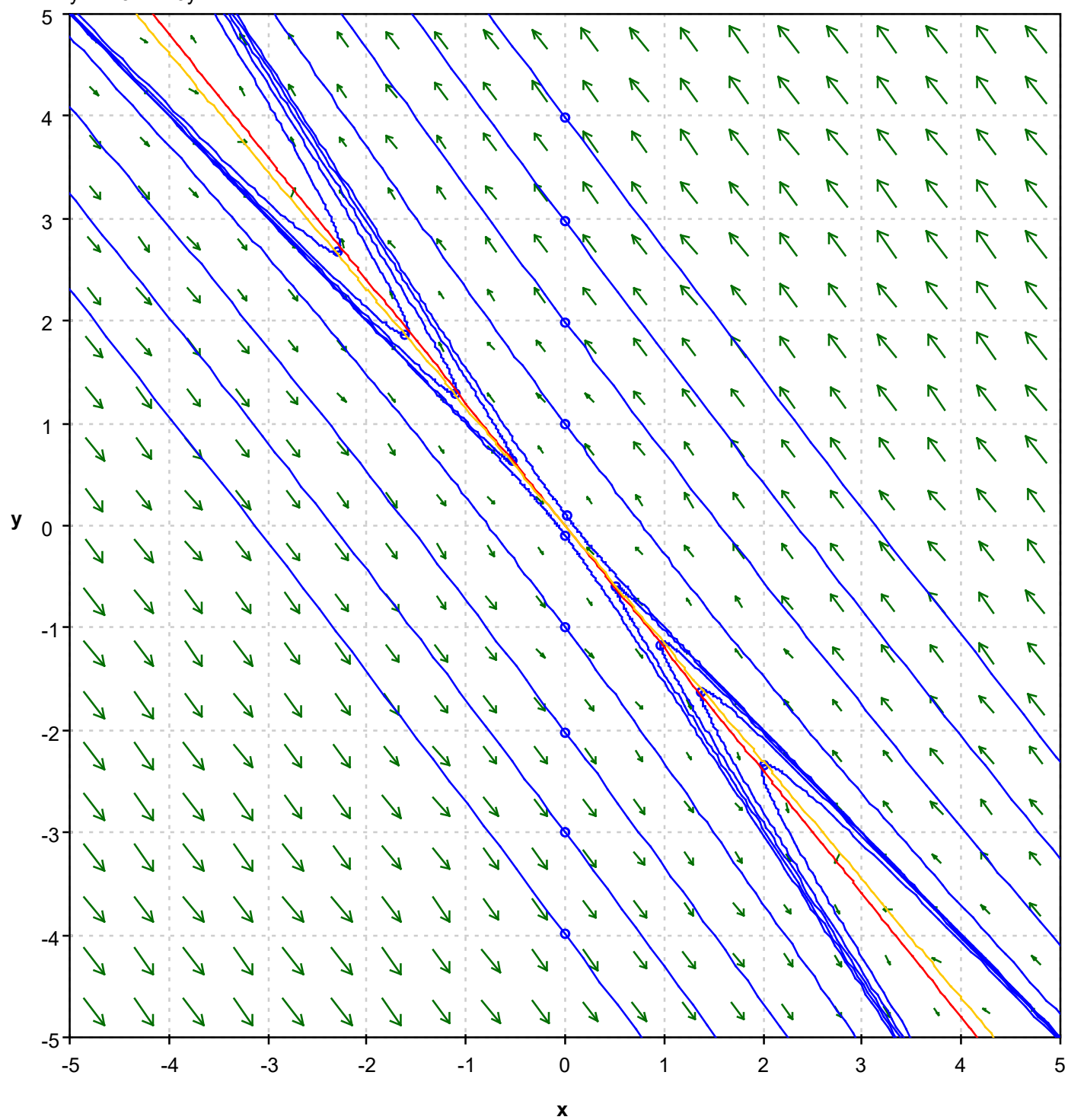
$$x' = 2y$$

$$y' = -3x + 5y$$



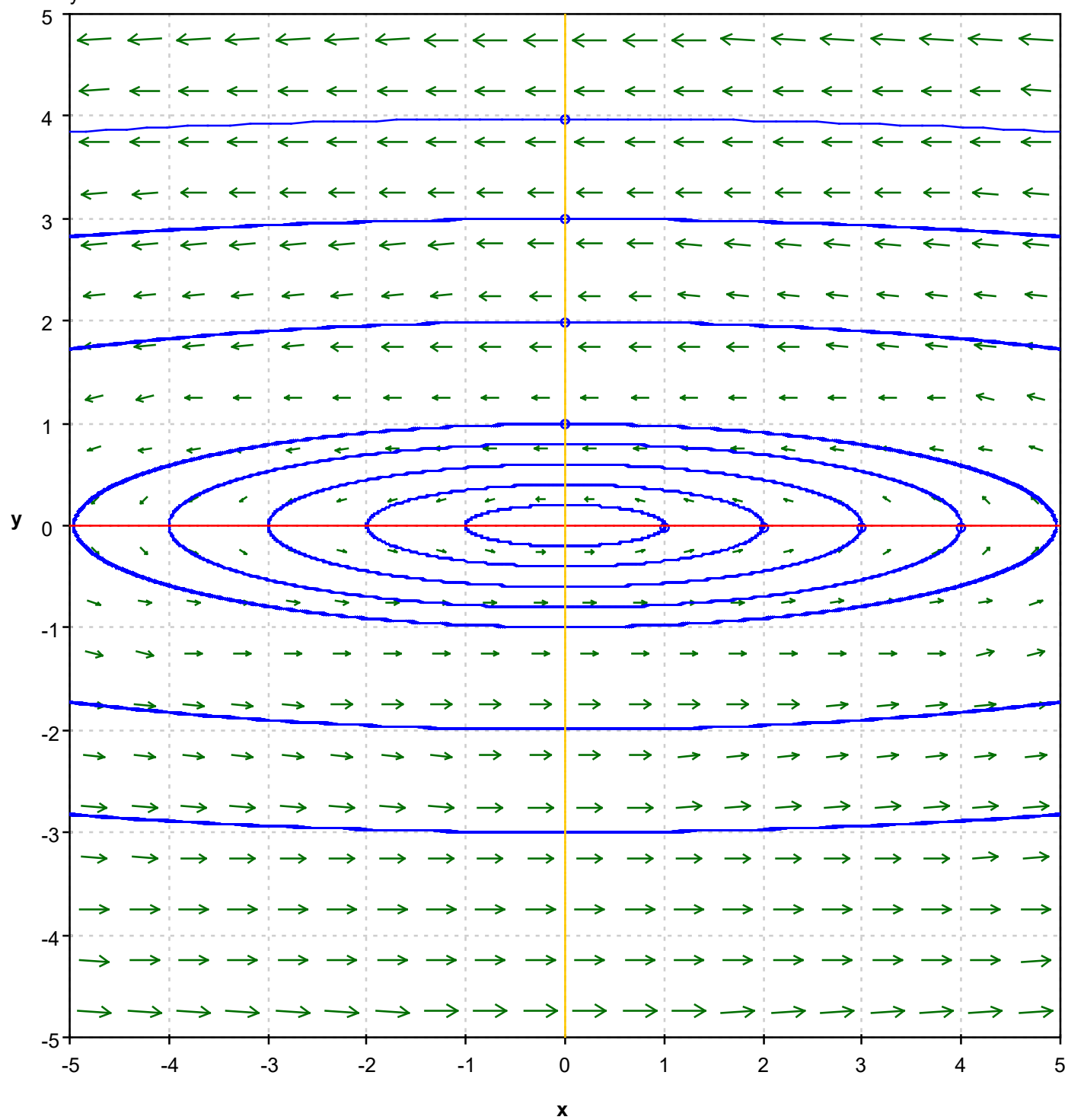
$$x' = -12x - 10y$$

$$y' = 15x + 13y$$



$$x' = -25y$$

$$y' = x$$



$$x' = 6x - 9y$$

$$y' = x + 6y$$

