Today: 9/3

- Finish "completeness" idea in 1.1 (text).
- Absolute Values & Distances 2.6 (Gilles Notes)

Recall: Suppose $S \subseteq \mathbb{R}$ and $S \neq \emptyset$.

We say $x \in \mathbb{R}$ is an upper bound

iff

$\forall y \in S, \; x \geq y.$

We say $x \in \mathbb{R}$ is the least upper bound (supremum) of S and write ~~$x = \sup S$~~ $x = \sup S$.

iff

1. x is an upper bound of S.
2. $\forall y \in \mathbb{R}$, if y is an upper bound of S, then $x \leq y$.

Remarks

① sup S is either a number or denoted $+\infty$.

$S = [1, \infty)$

S has no upper bound.

we write $\sup S = +\infty$.

② <u>Be careful</u>: sup S may or may not be a member of S.

$$\sup\,(-7,-3) = -3 = \sup\,(-7,-3].$$

"Completeness Axiom"

Suppose $S \subseteq \mathbb{R}$ and $S \neq \phi$.

If S is bounded above, then sup S exists.

Prop 1.3 Let $c > 0$. Then $\exists!\, b \geq 0$ such that
$$b^2 = c.$$

proof: Suppose $S = \{x \in \mathbb{R} \mid x \geq 0 \text{ and } x^2 < c\}$.

1. We will show S is bounded above.

Notice $(c+1)^2 = c^2 + 2c + 1 > 2c > c$.

So $c+1 \notin S$.

Let $x \in S$ (arbitrary). So $x \geq 0$ and $x^2 < c$.

So $x^2 < (c+1)^2$.

Thus $0 < (c+1)^2 - x^2 = (c+1-x)(c+1+x)$.

Since $c+1+x > 0$, we divide to get

$$0 < c+1-x$$

I.e. $x < c+1$. Thus $c+1$ is an upper bound.

By the Completeness Axiom $\exists\, b \in \mathbb{R}$ s.t. $b = \sup(S)$.

2. Suppose $b^2 < c$. (we seek contradiction.)

We will show $\exists\, \varepsilon > 0$ s.t. $(b+\varepsilon)^2 < c$

In this case, $b+\varepsilon \in S$ and $b+\varepsilon > b$.

This <u>contradicts</u> b as an upper bound of S.

Let $\varepsilon = \frac{1}{2} \cdot \min\left\{b, \frac{c-b^2}{3b}\right\}$. So $\varepsilon < b$ and $\varepsilon < \frac{c-b^2}{3b}$.

$$
\begin{aligned}
\text{So } (b+\varepsilon)^2 = b^2 + 2b\varepsilon + \varepsilon^2 &< b^2 + 2b\varepsilon + b\varepsilon \\
&= b^2 + 3b\varepsilon \\
&< b^2 + \cancel{3b} \cdot \frac{c-b^2}{\cancel{3b}} \\
&= c
\end{aligned}
$$

So $(b+\varepsilon)^2 < c$.

So $b^2 \geq c$.

$(b-\varepsilon)^2 > c.$

$b^2 - 2b\varepsilon + \varepsilon^2 > c.$

$b^2 - c > 2b\varepsilon - \varepsilon^2.$

$\frac{b^2 - c}{2b} > \frac{2b\varepsilon}{2b}.$

$\frac{b^2-c}{2b} > \varepsilon.$

$(b-\varepsilon)^2$

$= b^2 - 2b\varepsilon + \varepsilon^2$

$> b^2 - 2b\varepsilon$

$> b^2 - \cancel{2b} \cdot \frac{b^2-c}{\cancel{2b}}$

$= c.$

Scratch!

3. Suppose $b^2 > c$. (We head for contradiction.)

We will show $\exists\, \varepsilon > 0$ s.t. $(b-\varepsilon)^2 > c$.

Just as in part 1, this will show $b-\varepsilon$ is an upper bound for S. But $b-\varepsilon < b$ and this contradicts b as a least upper bound.

Exercise for you!

Let $\varepsilon = \frac{b^2-c}{2b} > 0$.

Then $(b-\varepsilon)^2 = b^2 - 2b\varepsilon + \varepsilon^2 > b^2 - 2b\varepsilon = b^2 - \cancel{2b} \cdot \frac{b^2-c}{\cancel{2b}} = c$.

By this argument and using $b^2 \geq c$ we conclude $b^2 = c$.

Section 2.6 Gilles Notes. Absolute Value and The triangle inequality.

Definition $\forall x \in \mathbb{R}$, $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$.

Remark: We typically ~~think~~ think $\forall x, y \in \mathbb{R}$ $|x-y|$ represents distance between x & y.

Lemma: Suppose $x \in \mathbb{R}$ and $\varepsilon > 0$. We have

$$|x| < \varepsilon$$

iff

$$-\varepsilon < x < \varepsilon.$$

Proof: $(\longrightarrow)$ Suppose $|x| < \varepsilon$. Notice $x \geq 0$ or $x < 0$.

Case 1: Suppose $x \geq 0$. Then $|x| = x < \varepsilon$.

Since $\varepsilon > 0$, $-\varepsilon < 0 \leq x$.

Thus $-\varepsilon < x < \varepsilon$.

Case 2: Suppose $x < 0$. Then $|x| = -x < \varepsilon$.

So $x > -\varepsilon$.

And also $x < 0 < \varepsilon$.

Thus $-\varepsilon < x < \varepsilon$.

$(\longleftarrow)$ Proof as an exercise for you!