Midterm 2
Linear Algebra
Math 524
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Problem 1: Let $T \in \mathcal{L}(\mathbb{F}^3)$ be defined by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$

(a) Find all eigenvalues and eigenspaces of T

Notice: We can define the matrix of T with respect to the basis z_1, z_2, z_3

$$M(T) = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

Because $M(T)$ is an upper triangular matrix, we can see that $\lambda_1 = 0, \lambda_2 = 5$

When $\lambda_1 = 0$

$$M(T) - 0I = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So $\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, for $(M(T) - \lambda I)\vec{x} = \vec{0}$ and $\lambda = 0$, so the eigenspace for this eigenvalue is $\vec{x}$ and the zero vector.

When $\lambda_2 = 5$

$$M(T) - 5I = \begin{pmatrix} -5 & 2 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -5 & 2 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So $\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, for $(M(T) - \lambda I)\vec{x} = \vec{0}$ and $\lambda = 5$, so the eigenspace for this eigenvalue is $\vec{x}$ and the zero vector.

(b) Find a Basis for range(T)

$$\text{range(T)} = \text{span}\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \}$$

(c) Find a Basis for the null(T)

$$\text{null(T)} = \text{span}\{ \begin{pmatrix}1\\0\\0\end{pmatrix} \}$$

(d) Is $\mathbb{F}^3 = \text{null(T)} \oplus \text{range(T)}$?

Because $\text{null(T)} \cap \text{range(T)} = \begin{pmatrix}1\\0\\0\end{pmatrix} \neq \vec{0}$: $\qquad \mathbb{F}^3 \neq \text{null(T)} \oplus \text{range(T)}$

(e) Is $\mathbb{F}^3 = E(\lambda_1, T) \oplus E(\lambda_2, T)$?

Because $E(\lambda_1, T) = span\{ \begin{pmatrix}1\\0\\0\end{pmatrix} \}$ and $E(\lambda_2, T) = span\{ \begin{pmatrix}0\\0\\1\end{pmatrix} \}$, such that $E(\lambda_1, T) + E(\lambda_2, T) = \text{span}\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \}$, $\mathbb{F}^3 \neq E(\lambda_1, T) \oplus E(\lambda_2, T)$

(f) Is T diagonalizable? Why/Why not?

We can define the matrix of T with respect to the basis z_2, z_1, z_3

$$M(T) = \begin{pmatrix}2 & 0 & 0\\0 & 0 & 0\\0 & 0 & 5\end{pmatrix}$$

Because the only non zero entries lie in the diagonal, T is diagonalizable

(g) let $S = T^2$ (T from above)

(i) Find all eigenvalues and eigenspaces of S
Notice:

$$S(z_1, z_2, z_3) = T(T(z_1, z_2, z_3)) = T(2z_2, 0, 5z_3) = (0, 0, 25z_3)$$

We can define the matrix of S with respect to the basis z_1, z_2, z_3

$$M(S) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 25 \end{pmatrix}$$

Because $M(S)$ is an upper triangular matrix, we can see that $\lambda = 25$

$$M(S) - 25I = \begin{pmatrix} -25 & 0 & 0 \\ 0 & -25 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -25 & 0 & 0 \\ 0 & -25 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So $\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, for $(M(S) - \lambda I)\vec{x} = \vec{0}$ and $\lambda = 25$, so the eigenspace for this eigenvalue is $\vec{x}$ and the zero vector.

(ii) Find a Basis for range(S)

$$\text{range(S)} = \text{span}\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \}$$

(iii) Find a Basis for the null(S)

$$\text{null(S)} = \text{span}\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \}$$

(iv) Is $\mathbb{F}^3 = \text{null(S)} \oplus \text{range(S)}$?

$\text{null(S)} + \text{range(S)} = \text{span}\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \}$, and $\text{null(S)} \cap \text{range(S)} = 0$, then $\mathbb{F}^3 = \text{null(S)} \oplus \text{range(S)}$?

(v) Is $\mathbb{F}^3 = E(\lambda, S)$

$$E(\lambda, S) = span\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \} \neq span\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \}, \text{ so } \mathbb{F}^3 \neq E(\lambda, S)$$

(vi) Is S diagonalizable? Why/Why not?

Because M(S) can be written as a matrix with its only non zero entries being on its diagonal, M(S) is diagonalizable.

Problem 2: Consider $(x_1, ..., x_n) \in \mathbb{R}^n$, where $x_\ell > 0, \forall \ell \in \{1, ..., n\}$; find a lower bound for

$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} \frac{1}{x_k}\right)$$

Notice the following:
(n = 1)

$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} \frac{1}{x_k}\right) = \left(\sum_{k=1}^{1} x_k\right)\left(\sum_{k=1}^{1} \frac{1}{x_k}\right) = \frac{x_1}{x_1} = 1$$

(n = 2)

$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} \frac{1}{x_k}\right) = \left(\sum_{k=1}^{2} x_k\right)\left(\sum_{k=1}^{2} \frac{1}{x_k}\right) = (x_1 + x_2)\left(\frac{1}{x_1} + \frac{1}{x_2}\right) = 1 + 1 + \frac{x_1}{x_2} + \frac{x_2}{x_1}$$

(n = 3)

$$\begin{aligned}\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} \frac{1}{x_k}\right) = \left(\sum_{k=1}^{3} x_k\right)\left(\sum_{k=1}^{3} \frac{1}{x_k}\right) &= (x_1 + x_2 + x_3)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right) \\ &= 1 + 1 + 1 + \frac{x_1}{x_2} + \frac{x_1}{x_3} + \frac{x_2}{x_1} + \frac{x_2}{x_3} + \frac{x_3}{x_1} + \frac{x_3}{x_2}\end{aligned}$$

Thus the following is true:

$$\left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} \frac{1}{x_k}\right) = n + \sum_{k=1}^{n}\left(x_k \sum_{j=1, j\neq k}^{n} \frac{1}{x_j}\right)$$

Thus the lower bound is n and also the infinum, no matter the value of n,

$$n \leq \left(\sum_{k=1}^{n} x_k\right)\left(\sum_{k=1}^{n} \frac{1}{x_k}\right) = n + \sum_{k=1}^{n}\left(x_k \sum_{j=1, j\neq k}^{n} \frac{1}{x_j}\right)$$

with

$$\sum_{k=1}^{n}\left(x_k \sum_{j=1, j\neq k}^{n} \frac{1}{x_j}\right) > 0, \text{ as } x_\ell > 0$$

Note: Messy Notation:

$$\sum_{k=1}^{n}\left(x_k \sum_{j=1, j\neq k}^{n} \frac{1}{x_j}\right) = \left(\frac{x_1}{x_2} + ... + \frac{x_1}{x_n}\right) + \left(\frac{x_2}{x_1} + \frac{x_2}{x_3} + ... + \frac{x_2}{x_n}\right) + ... + \left(\frac{x_n}{x_1} + ... \frac{x_n}{x_{n-1}}\right)$$

Problem 3: Consider the inner product $< p, q >= \int_0^1 p(x)q(x)dx$ for $p, q \in \mathcal{P}(\mathbb{R})$. On $\mathcal{P}_2(\mathbb{R})$ our friends Gram & Schmidt kindly provide an orthonormal basis:

$$\{ \quad u_1(x) = 1 \quad u_2(x) = \sqrt{3}(-1 + 2x) \quad u_3(x) = \sqrt{5}(1 - 6x + 6x^2) \quad \}$$

Find a polynomial $q \in \mathcal{P}_2(\mathbb{R})$ so that $\forall p \in \mathcal{P}_2(\mathbb{R})$:

$$p\left(\frac{1}{2}\right) = \int_0^1 p(x)q(x)dx$$

Let $\phi(p(x)) =< p, q >= \int_0^1 p(x)q(x)dx = p\left(\frac{1}{2}\right)$

By (6.43) in the textbook,

$$\begin{aligned}
q(x) &= \phi(u_1(x))u_1(x) + \phi(u_2(x))u_2(x) + \phi(u_3(x))u_3(x) \\
&= u_1\left(\frac{1}{2}\right)u_1(x) + u_2\left(\frac{1}{2}\right)u_2(x) + u_3\left(\frac{1}{2}\right)u_3(x) \\
&= 1 + 0(\sqrt{3}(-1 + 2x)) + \frac{-\sqrt{5}}{2}(\sqrt{5}(1 - 6x + 6x^2)) \\
&= 1 + \frac{-5}{2}(1 - 6x + 6x^2)
\end{aligned}$$