MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #24
Chapter 7
Existence and Uniqueness,
Continuous and Sensitive Dependence on ICs,
and Linearization Theorem

Instructor: Dr. Bo-Wen Shen*
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San Diego State University

References

- Boyce, W. E. and R. C. DiPrima, 2012: Elementary Differential Equations. [Section 9.3]
- M.W. Hirsch, S. Smale, and R.L. Devaney, 2013: Differential Equations, Dynamical Systems and an Introduction to Chaos, 3rd Ed. Academic Press. ISBN: 0-12-349703-5. [HSD]
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References

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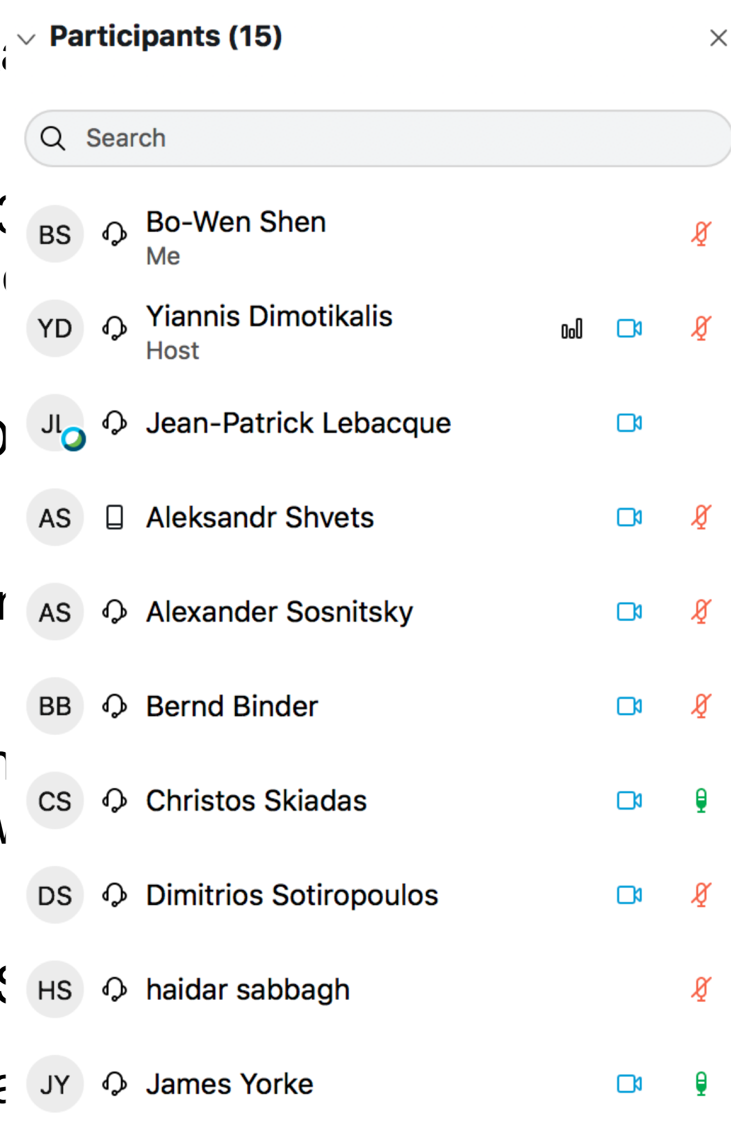

Outline

1. Existence and Uniqueness Theorems
2. Lipschitz Condition
3. Continuous Dependence of Solutions on Initial Conditions (CDIC)
4. Sensitive Dependence of Solutions on Initial Conditions (SDIC)
5. Linearization Theorems & Linearized Systems

Fundamental Concepts

1. **Existence:** Each point in the (t, x)-plane has a solution passing through it. The solution has slope given by the differential equation at that point.
2. **Uniqueness:** Only one solution passes through any particular (t, x).
3. **Continuous dependence:** Solutions through nearby initial conditions remain close over short time intervals. In other words, the flow $F(t, x_0)$ is a continuous function of x_0 as well as t. $|X(t) - Y(t)| < |X_0 - Y_0|e^{K(t-t_0)}$

- *"Sensitive dependence on initial conditions" (SDIC) means that nearby trajectories separate exponentially fast, i.e., the system has a positive Liapunov exponent. (Strogatz, p331)*
- *We may define an infinite invariant set of a map $F\colon R \to R$ to have sensitive dependence on initial conditions if there exists $\delta > 0$ such that for all $x \in S$ and all neighbourhoods N (however small) of x there exists $y \in N$ and $n > 0$ such that $|F^n(x) - F^n(y)| > \delta$. So neighbouring orbits, however close initially, separate from one another, although each keeps close to the invariant set. (Drazin, p140; Devaney, p49)*

Alligood et al.

Existence Theorem: A Quick Look (1D)

Existence Theorem

Let the right side $\boxed{f(x, y)}$ *of the ODE in the initial value problem*

I.V.P.

(1) $$y' = f(x, y), \qquad y(x_0) = y_0$$

$$y' = \frac{dy}{dx}$$

be continuous at all points (x, y) *in some rectangle*

$$R: |x - x_0| < a, \qquad |y - y_0| < b \qquad \text{(Fig. 26)}$$

and **bounded** *in* R*; that is, there is a number* K *such that*

(2) $$\boxed{|f(x, y)| \leqq K} \qquad \textit{for all } (x, y) \textit{ in } R.$$

Then the initial value problem (1) *has at least one solution* $y(x)$*. This solution exists at least for all* x *in the* $\boxed{\textit{subinterval } |x - x_0| < \alpha}$ *of the interval* $|x - x_0| < a$*; here,* α *is the smaller of the two numbers* a *and* b/K*.*

x & f bounded➔ at least one solution within an subinterval Kreyszig

Uniqueness Theorem: A Quick Look (1D)

Uniqueness Theorem

Let f and its partial derivative $f_y = \partial f/\partial y$ be continuous for all (x, y) in the rectangle R (Fig. 26) and bounded, say,

$$(3) \qquad \boxed{\text{(a)} \quad |f(x, y)| \leqq K,} \qquad \boxed{\text{(b)} \quad |f_y(x, y)| \leqq M} \qquad \textit{for all } (x, y) \textit{ in } R.$$

Then the initial value problem (1) has at most one solution $y(x)$. Thus, by Theorem 1, the problem has precisely one solution. This solution exists at least for all x in that subinterval $|x - x_0| < \alpha$.

f & f_y bounded→ at most one solution

M has a special name.

Kreyszig

Existence and Uniqueness: A Simple Statement (1D)

THEOREM 2.2.1 Existence and Uniqueness

Consider the initial-value problem

$$y' = f(x, y) \quad \text{with} \quad y(x_0) = y_0.$$

$$y' = \frac{dy}{dx}$$

If f and $\partial f/\partial y$ are continuous functions on the rectangular region

$$R : a < x < b, \quad c < y < d$$

containing the point (x_0, y_0), then there exists an interval

$$|x - x_0| < h$$

centered at x_0 on which there exists one and only one solution to the differential equation that satisfies the initial condition.

f & $f_y(x, y)$ continuous over an finite rectangular "region" for x and y.

Wirkus and Swift

Existence and Uniqueness: A Simple Statement (1D)

A similar statement:

Existence and Uniqueness of Solution

Theorem 1. Consider the initial value problem

$$\frac{dy}{dx} = f(x, y) \ , \qquad y(x_0) = y_0 \ .$$

If f and $\partial f/\partial y$ are continuous functions in some rectangle

$$R = \{(x, y): a < x < b, c < y < d\}$$

that contains the point (x_0, y_0), then the initial value problem has a unique solution $\phi(x)$ in some interval $x_0 - \delta < x < x_0 + \delta$, where δ is a positive number.

f & $f_y(x, y)$ continuous over an finite rectangular "region" for x and y.

Nagle et al.

Existence and Uniqueness

Namely, if the functions $P(t)$ and $Q(t)$ are continuous in an interval I containing t_0, then there exists a solution to

(1) $$y'(t) + P(t)y(t) = Q(t) \ , \qquad y(t_0) = y_0 \ ,$$

on the *entire* interval I. In the preceding section the best we could say was that if f and $\partial f/\partial y$ are continuous in a rectangle

$$R = \{(t, y): a < t < b, c < y < d\}$$

containing (t_0, y_0), then the solution to

$$y'(t) = f\big(t, y(t)\big) \ , \qquad y(t_0) = y_0 \ ,$$

exists on *some* subinterval $[t_0 - h, t_0 + h] \subset [a, b]$.

For example, the seemingly simple nonlinear problem

(2) $$y'(t) = y^2(t) \ , \qquad y(0) = a(\neq 0) \ ,$$

has the solution $y(t) = 1/(a^{-1} - t)$, which is undefined at $t = 1/a$ even though $f(t, y) = y^2$ is very well behaved.

Nagle et al.
(6th ed)

Important Dates (derived from the first lecture)

Quiz1: Aug 26 (W)
Quiz2: Sep 2 (W)
HW1: Sep 11 (F)
HW2: Sep 25 (F)
Part A: Sep 30 (W)
Part B: Oct 2 (F)

August

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September

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Quiz: Oct 7 (W)
HW3: Oct 16 (F)
HW4: Oct 30 (F)
HW5: Nov 13 (F)
Quiz: Nov 25 (W)
HW6: Dec 4 (F)
Part A: Dec 11 (F)
Part B: Dec 14 (M)

October

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November

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December

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Fall 2020 Academic Calendar: November

November 26–27	Holiday—Thanksgiving recess. No classes will be held Nov. 25, but campus will remain open. Campus closed Nov. 26-27. Faculty/staff holiday.

- 11/23 (Mon.): asynchronous recorded lecture for Quiz 8 of Math537
- 11/25 (Wed.): No classes
- Quiz 8 (20 points), Due 9 am on 11/25 (Wed.)
- recorded lecture: https://bit.ly/3pxcwZ1

November

Su	Mo	Tu	We	Th	Fr	Sa
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https://registrar.sdsu.edu/calendars/academic_calendars/fall_2020_academic_calendar

Quiz-8

https://bit.ly/3pxcwZ1

Quiz-8
Math 537 Ordinary Differential Equations
Due 9:00AM Wednesday, November 25, 2020

Student Name: ______________________________ **ID** ____________

Goal: Understand three types of solutions within the Lorenz model and two kinds of attractor coexistence within the generalized Lorenz model.

Total points: 20

1: [20 points] Based on the recorded video (a link provided during the lecture), please answer the following questions:

(a) [10 points] Briefly discuss three types of solutions within the Lorenz model.

(b) [10 points] Briefly discuss two kinds of attractor coexistence within the generalized Lorenz model.

Uniqueness (N-dimensional)

Uniqueness: (bounded derivatives)

In this section we show that for *linear* systems in normal form, a solution exists over the entire interval. This fact is a consequence of the following theorem.

Continuation of Solution

Theorem 5. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{f}(t, \mathbf{x})$ denote the vector function $\mathbf{f}(t, \mathbf{x}) = \big(f_1(t, x_1, \ldots, x_n), \quad \ldots, \quad f_n(t, x_1, \ldots, x_n)\big)$. Suppose $\mathbf{f}$ and $\partial \mathbf{f}/\partial x_i$, $i = 1, \ldots, n$, are continuous on the strip

$$R = \{(t, \mathbf{x}) \in \mathbf{R}^{n+1}\colon a \le t \le b, \mathbf{x} \text{ arbitrary}\}$$

containing the point $(t_0, \mathbf{x}_0)$. Assume further that there exists a positive constant L such that, for $i = 1, \ldots, n$,

$$\left| \frac{\partial \mathbf{f}}{\partial x_i}(t, \mathbf{x}) \right| \le L \tag{3}$$

for all $(t, \mathbf{x})$ in R. Then the initial value problem

$$\mathbf{x}'(t) = \mathbf{f}(t, \mathbf{x}) \ , \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \ , \tag{4}$$

has a unique solution on the entire interval $a \le t \le b$.

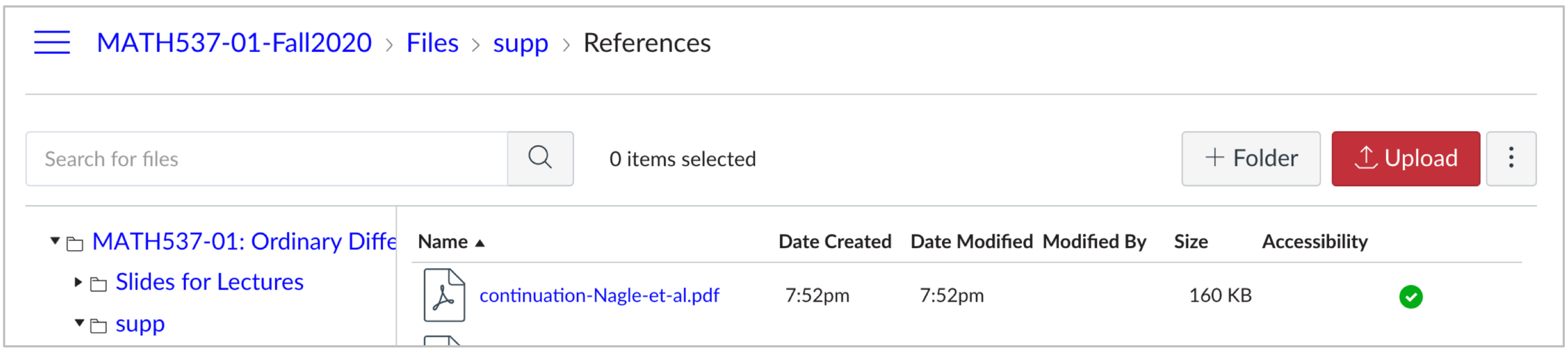
What's in Eq. (3)? Determinant of the Jacobian matrix

Nagle et al. (6th ed, p797)

Continuation of Solutions

Theorem 5 is called a **continuation theorem** because it specifies conditions that guarantee that the solution to initial value problem (4) can be **continued** from the small interval $[t_0 - h, t_0 + h]$ to the entire interval $[a, b]$. As we now show, these conditions are satisfied by linear systems.

Additional materials are available on canvas



Nagle et al.
(6th ed, p797)

An Old Version (presented and posted)

Uniqueness (N-dimensional)

Uniqueness: (bounded derivatives)

Continuation of Solution

Theorem 5. Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{f}(t, \mathbf{x})$ denote the vector function $\mathbf{f}(t, \mathbf{x}) = \left(f_1(t, x_1, \ldots, x_n), \quad \ldots, \quad f_n(t, x_1, \ldots, x_n)\right)$. Suppose $\mathbf{f}$ and $\partial \mathbf{f}/\partial x_i$, $i = 1, \ldots, n$, are continuous on the strip

$$R = \left\{(t, \mathbf{x}) \in \mathbf{R}^{n+1} : a \le t \le b, \mathbf{x} \text{ arbitrary}\right\}$$

containing the point $(t_0, \mathbf{x}_0)$. Assume further that there exists a positive constant L such that, for $i = 1, \ldots, n$,

$$\boxed{\textbf{(3)} \qquad \left|\frac{\partial \mathbf{f}}{\partial x_i}(t, \mathbf{x})\right| \le L}$$

for all $(t, \mathbf{x})$ in R. Then the initial value problem

$$\textbf{(4)} \qquad \mathbf{x}'(t) = \mathbf{f}(t, \mathbf{x}) \ , \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \ ,$$

has a unique solution on the entire interval $a \le t \le b$.

What's in Eq. (3)? Determinant of the Jacobian matrix

Nagle et al.?

A Recorded Lecture for Quiz 8

On the Dual Nature of Chaos and Order in Weather and Climate: New Insights and Opportunities Within a Generalized Lorenz Model

By

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Computational Science Research Center
San Diego State University
06 November 2020

https://bit.ly/3pxcwZ1

On the Dual Nature of Chaos and Order 1 San Diego State University, 06 November 2020

An Updated Information for the Host

I have worked for NASA/GSFC for 15 years since March 1999. I was a NASA AIST PI during 2008-2015. (There are only 4~5 AIST PIs at each of major NASA centers. There are about 8,000 employees at GSFC.)

Here are two links that support the above:
https://bwshen.sdsu.edu/shen_honors_and_awards.html
http://bwshen.sdsu.edu/pdf/Petascale_Computing_2007.pdf

At GSFC, there are basically three types of employees on campus. (a) civil servants; (b) university associates (associated with different universities, including UMCP, UMBC, GMU, etc); (c) company employees (e.g., SAIC).

Some civil servants provide visions and leadership and many others are responsible for administrative tasks (which are not very interesting to some researchers).
In general, major research activities are led and performed by university associates. (Unless my understanding is not correct, all of the employees at NASA/JPL/Caltech belong to university associates, managed by a University, Caltech).

On the Dual Nature of Chaos and Order in Weather

BAMS

RESEARCH ARTICLE | 28 SEPTEMBER 2020

Is Weather Chaotic? Coexistence of Chaos and Order within a Generalized Lorenz Model

Bo-Wen Shen ; Roger A. Pielke, Sr.; Xubin Zeng; Jong-Jin Baik; Sara Faghih-Naini; Jialin Cui; Robert Atlas

Bull. Amer. Meteor. Soc. 1–28.

https://doi.org/10.1175/BAMS-D-19-0165.1

By revealing two kinds of attractor coexistence within Lorenz models, we suggest that the entirety of weather possesses a dual nature of chaos and order with distinct predictability.

”A Paradigm Shift” in Predictability Study

- *``As with Poincare and Birkhoff, everything centers around periodic solutions” (Lorenz, 1993).*

- After Lorenz (1963, 1972), Prof. Lorenz and chaos researchers focused on the existence of non-periodic solutions and their complexities.

- Based on the concept of attractor coexistence within the original and generalized Lorenz models (Shen, 2019a), we (Shen et al., 2020a, b) propose a revised view that focus on the duality of chaos and order.

- An effective detection and classification of chaotic and non-chaotic processes may help extend the lead time of predictions.

A Revised View: a Dual Nature of Chaos and Order

Is Weather Chaotic? Coexistence of Chaos and Order Within a Generalized Lorenz Model

by

Bo-Wen Shen[1*], Roger A. Pielke Sr.[2], Xubin Zeng[3], Jong-Jin Baik[4], Tiffany Reyes[1], Sara Faghih-Naini[5], Robert Atlas[6], and Jialin Cui[1]

[1]San Diego State University, USA
[2]CIRES, University of Colorado at Boulder, USA
[3]The University of Arizona, USA
[4]Seoul National University, South Korea
[5]Friedrich-Alexander University Erlangen-Nuremberg, Germany
[6]AOML, National Oceanic and Atmospheric Administration, USA

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Big Data, Data Assimilation and Uncertainty Quantification

Institut Henri Poincaré (IHP), Paris, France

12-15 November 2019

Institut Henri Poincaré (IHP), Paris, France

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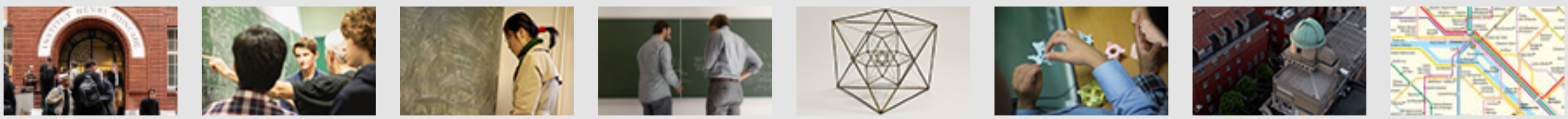

ABOUT IHP

Located in the heart of the 5th arrondissement of Paris, the Henri Poincaré Institute is one of the oldest and most dynamic international structures dedicated to mathematics and theoretical physics.

Director's message

Please here to find the director's message for 2019

Missions

The IHP perpetuates the tradition of diversity and universalism which Poincaré embodies

History

Since its creation in 1928, the IHP has taken an interest in all the disciplinary fields for which mathematics constitute a key element, ranging from general relativity to computer science to mathematical biology...

References

- Boyce, W. E. and R. C. DiPrima, 2012: Elementary Differential Equations. [Section 9.3]
- M.W. Hirsch, S. Smale, and R.L. Devaney, 2013: Differential Equations, Dynamical Systems and an Introduction to Chaos, 3rd Ed. Academic Press. ISBN: 0-12-349703-5. [HSD]
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- Nagle K., E. Staff, D. Snider, 2011: Fundamentals of Differential Equations. Pearson; 8th edition.

Plenary Talks at the 2020 CMSIM in October

You are also invited to subscribe to attend the web conference and follow the 3 day program along with the important Keynote, Plenary and Invited contributions from:

James A. Yorke, University of Maryland, USA on “The equations of nature and the nature of equations” with Sana Jahedi

Harold M Hastings, Bard College at Simon's Rock, USA on “Vector difference equations, Gerschgorin’s theorem, and design of multi-networks to reduce spread of epidemics” with Tai Young-Taft

Vladimir L. Kalashnikov, Sapienza Universita di Roma, Italy on “Spatiotemporal Turbulence in a Multimode Fiber Laser” with Stefano Wabnitz

Shunji Kawamoto, Osaka Prefecture University, Japan on “Chaos identification of a colliding constraint body on a moving belt”

George Savvidy, National Centre for Scientific Research “Demokritos”, Athens, Greece on “Maximally Chaotic Dynamical Systems”

A. Shvets, National Technical University of Ukraine “Igor Sikorsky Kyiv Polytechnic Institute”, Kyiv, Ukraine on “Generalizing of Attractor Notion for Spherical Pendulum Systems” with S. Donetskyi

Bo-Wen Shen, San Diego State University, USA on “Applying a Kernel PCA Method to Reveal Coexisting Attractors within a Generalized Lorenz Model” with Jialin Cui

Tatiana F. Filippova, Krasovskii Institute of Mathematics and Mechanics, Ekaterinburg, Russia on “Interacting Populations: Dynamics and Viability in Bounded Domains under Uncertainty”

Marek Lampart, VSB-Technical University of Ostrava, IT4Innovations, Czech Republic on “Chaos identification of a colliding constraint body on a moving belt” with Jaroslav Zapomĕl

J.P. Lebacque, UGE (University Gustave Eiffel) COSYS GRETTIA, France on “Chaotic behavior of dynamical systems associated with dynamic traffic assignment in transportation” with M.M. Khoshyaran

Evelina V. Prozorova, St. Petersburg State University, Russia on “Mechanism of Formation of Fluctuation Phenomena”

C H Skiadas, ManLab, TUC, Greece on “How the unsolved problem of finding the Healthy Life Expectancy (HLE) in the far past was resolved: The case of Sweden (1751-2016) with forecasts to 2060 and comparisons with HALE” https://osf.io/preprints/socarxiv/akf8v/

Existence and Uniqueness Theorem for Linear ODEs

High order ODEs

Existence and Uniqueness Theorem for Linear Equations

Theorem 7. Suppose $p_1(t), \ldots, p_n(t)$ and $g(t)$ are continuous on an interval (a, b) containing the point t_0. Then, for every choice of the initial values $y_0, y_1, \ldots, y_{n-1}$, there exists a unique solution on the whole interval (a, b) to the initial value problem

(14)
$$y^{(n)}(t) + p_1(t)y^{(n-1)}(t) + \cdots + p_n(t)y(t) = g(t) \; ;$$
$$y(t_0) = y_0 \; , \qquad y'(t_0) = y_1 \; , \qquad \ldots, \qquad y^{(n-1)}(t_0) = y_{n-1} \; .$$

I.V.P.

Nagle et al.

Existence and Uniqueness Theorem for Linear Systems

Existence and Uniqueness Theorem for Linear Systems

Theorem 6. Suppose the $n \times n$ matrix function $\mathbf{A}(t) = \left[a_{ij}(t)\right]$ and the vector function $\mathbf{f}(t)$ are continuous on an open interval (a, b) that contains the point t_0. Then, for any choice of the initial vector $\mathbf{x}_0$, there exists a unique solution on the whole interval (a, b) to the initial value problem

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t) \;, \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \;. \tag{12}$$

non-autonomous with forcing term(s)

Nagle et al.

Existence and Uniqueness for Linear Systems

Existence and Uniqueness for X'=A(t)X (linear nonautonomous)

Theorem. *Let $A(t)$ be a continuous family of $n \times n$ matrices defined for $t \in [\alpha, \beta]$. Then the initial value problem*

$$X' = A(t)X, \; X(t_0) = X_0$$

has a unique solution that is defined on the entire interval $[\alpha, \beta]$. ∎

non-autonomous (with no additional forcing term)

HSD

Smoothness of Flows

Theorem. (Smoothness of Flows) *Consider the system $X' = F(X)$ where F is C^1. Then the flow $\phi(t, X)$ of this system is a C^1 function; that is, $\partial\phi/\partial t$ and $\partial\phi/\partial X$ exist and are continuous in t and X.* ∎

C^1 ➔ Locally Lipschitz

- For functions for which derivatives of all orders exist and are continuous functions, we will call this type of function a smooth function (e.g., Alligood et al)

- C^k function: A function is C^k if it is k-times differentiable.
- Diffeomorphism: A C^k -diffeomorphism $f: M \to N$ is a mapping f which is 1-1, onto, and has the property that both f and f^{-1} are k-times differentiable.
- Homeomorphism: A homeomorphism is a C^0 diffeomorphism, i.e. a continuous mapping $f: M \to N$ with a continuous inverse.

HSD

The Fundamental Local Theorem of ODEs

The Existence and Uniqueness Theorem. *Consider the initial value problem*

$$X' = F(X), \; X(t_0) = X_0,$$

where $X_0 \in \mathbb{R}^n$. Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is C^1. Then, first, there exists a solution of this initial value problem, and second, this is the only such solution. More precisely, there exists an $a > 0$ and a unique solution,

$$X : (t_0 - a, t_0 + a) \to \mathbb{R}^n,$$

of this differential equation satisfying the initial condition $X(t_0) = X_0$. ∎

an finite interval

C^1 ➔ Locally Lipschitz

HSD

Fixed Point Theorem

Banach Fixed Point Theorem

Theorem 1. Suppose g is a differentiable function on the closed interval I and $g(x)$ lies in I for all x in I. If there exists a constant K such that $|g'(x)| \leq K < 1$ for all x in I, then the equation $x = g(x)$ has a unique solution in I. Moreover, the sequence of **successive approximations** defined by

$$x_{n+1} = g(x_n), \quad n = 0, 1, 2, \ldots, \tag{10}$$

converges to this solution for any choice of the starting value x_0 in I.

MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #24-Supp

Fixed Point Iteration

Available on canvas

Nagle et al.

Picard Iteration

To describe the procedure, we begin by expressing the initial value problem

$$\textbf{(1)} \qquad y'(x) = f\big(x, y(x)\big) \;, \qquad y(x_0) = y_0$$

as an **integral equation.** This is accomplished by integrating both sides of equation (1) from x_0 to x:

$$\textbf{(2)} \qquad \int_{x_0}^{x} y'(t)\,dt = y(x) - y(x_0) = \int_{x_0}^{x} f\big(t, y(t)\big)\,dt \;.$$

Setting $y(x_0) = y_0$ and solving for $y(x)$, we have

$$\textbf{(3)} \qquad y(x) = y_0 + \int_{x_0}^{x} f\big(t, y(t)\big)\,dt \;. \qquad\qquad x = g(x)$$

$$\textbf{(15)} \qquad y_{n+1}(x) := T[y_n](x) = y_0 + \int_{x_0}^{x} f\big(t, y_n(t)\big)\,dt \;, \qquad\qquad x_{n+1} = g(x_n)$$

where we must specify the starting function $y_0(x)$. The simplest choice for $y_0(x)$ is to let $y_0(x) \equiv y_0$. The successive approximations y_n of the solution to the integral equation (3) are called the **Picard iterations.**[†]

Nagle et al.

Picard Iteration for the Proof

Without dwelling on the details here, the proof of this theorem depends on an important technique known as *Picard iteration*. Before moving on, we illustrate how the Picard iteration scheme used in the proof of the theorem works in several special examples. The basic idea behind this iterative process is to construct a sequence of functions that converges to the solution of the differential equation. The sequence of functions $u_k(t)$ is defined inductively by $u_0(t) = x_0$, where x_0 is the given initial condition, and then

$$X' = F(X) \quad X(t=0) = X_0$$

$$\mathrm{u}_0(t) = x_0$$

$$u_{k+1}(t) = x_0 + \int_0^t F\big(u_k(s)\big)ds$$

Example 1 for Picard Iteration

$$u_{k+1}(t) = x_0 + \int_0^t F(u_k(s))ds \qquad u_0(t) = x_0$$

Example. Consider the simple differential equation $x' = x$. We will produce the solution of this equation satisfying $x(0) = x_0$. We know, of course, that this solution is given by $x(t) = x_0 e^t$. We will construct a sequence of functions $u_k(t)$, one for each k, that converges to the actual solution $x(t)$ as $k \to \infty$.

$$F(x) = x \qquad u_0(t) = x_0$$

$$u_1(t) = x_0 + \int_0^t F(u_0(s))ds = x_0 + \int_0^t x_0 ds = x_0 + x_0 t$$

$$u_2(t) = x_0 + \int_0^t F(u_1(s))ds$$

- Find u_2
- Submit your results via “chat”
- You have 3 minutes

Example 1 for Picard Iteration

$$u_{k+1}(t) = x_0 + \int_0^t F(u_k(s))ds \qquad \mathrm{u}_0(t) = x_0$$

$$F(x) = x \qquad \mathrm{u}_0(t) = x_0$$

$$u_1(t) = x_0 + \int_0^t F(u_0(s))ds = x_0 + \int_0^t x_0 ds = \boxed{x_0 + x_0 t}$$

$$u_2(t) = x_0 + \int_0^t F(u_1(s))ds = x_0 + \int_0^t \boxed{(x_0 + x_0 s)} ds = x_0 + x_0 t + \frac{1}{2} x_0 t^2$$

$$u_{k+1}(t) = x_0 \sum_{i=0}^{k+1} \frac{t^i}{i!}$$

As $k \to \infty$, u_k converges to ?

- "Simplify" u_k as $k \to \infty$ (into an elementary function)
- You have 2 minutes

Example 1 for Picard Iteration

$$u_{k+1}(t) = x_0 + \int_0^t F\big(u_k(s)\big)ds \qquad\qquad \mathrm{u}_0(t) = x_0$$

$$F(x) = x \qquad\qquad \mathrm{u}_0(t) = x_0$$

$$u_1(t) = x_0 + \int_0^t F\big(u_0(s)\big)ds \quad = x_0 + \int_0^t x_0 ds \quad = \boxed{x_0 + x_0 t}$$

$$u_2(t) = x_0 + \int_0^t F\big(u_1(s)\big)ds \quad = x_0 + \int_0^t \boxed{(x_0 + x_0 s)} ds \quad = x_0 + x_0 t + \frac{1}{2} x_0 t^2$$

$$u_{k+1}(t) = x_0 \sum_{i=0}^{k+1} \frac{t^i}{i!}$$

As $k \to \infty$, u_k converges to

$$x_0 \sum_{i=0}^{\infty} \frac{t^i}{i!} = x_o e^t$$

Example 1 for Picard Iteration

$$u_{k+1}(t) = x_0 + \int_0^t F\big(u_k(s)\big)ds \qquad \mathrm{u}_0(t) = x_0$$

Example. Consider the simple differential equation $x' = x$. We will produce the solution of this equation satisfying $x(0) = x_0$. We know, of course, that this solution is given by $\boxed{x(t) = x_0 e^t}$. We will construct a sequence of functions $u_k(t)$, one for each k, that converges to the actual solution $x(t)$ as $k \to \infty$.

$$F(x) = x \qquad \mathrm{u}_0(t) = x_0$$

$$x_0 \sum_{i=0}^{\infty} \frac{t^i}{i!} = \boxed{x_o e^t} = x(t)$$

Example 2 for Picard Iteration

Example. For an example of Picard iteration applied to a system of differential equations, consider the linear system

$$X' = F(X) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X$$

with initial condition $X(0) = (1, 0)$. As we have seen, the solution of this initial value problem is

$$X(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}.$$

$$U_0(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad F(U_0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U_0(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$U_1(t) = U_0 + \int_0^t F(U_0)ds = U_0 + \int_0^t \begin{pmatrix} 0 \\ -1 \end{pmatrix} ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -t \end{pmatrix} = \begin{pmatrix} 1 \\ -t \end{pmatrix}$$

$$F(U_1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U_1(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -t \end{pmatrix} = \begin{pmatrix} -t \\ -1 \end{pmatrix}$$

$$U_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad F(U_1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U_1(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -t \end{pmatrix} = \begin{pmatrix} -t \\ -1 \end{pmatrix}$$

$$U_2(t) = U_0 + \int_0^t F(U_1) ds = U_0 + \int_0^t \begin{pmatrix} -s \\ -1 \end{pmatrix} ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} t^2 \\ -t \end{pmatrix} = \begin{pmatrix} 1 - \frac{1}{2} t^2 \\ -t \end{pmatrix}$$

$$F(U_2) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} U_2(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 - \frac{1}{2} t^2 \\ -t \end{pmatrix} = \begin{pmatrix} -t \\ -1 + \frac{1}{2} t^2 \end{pmatrix}$$

$$U_3(t) = U_0 + \int_0^t F(U_2) ds = \begin{pmatrix} 1 - t^2/2 \\ -t + t^3/3! \end{pmatrix}$$

$$U_4(t) = \begin{pmatrix} 1 - \frac{t^2}{2} + t^4/4! \\ -t + t^3/3! \end{pmatrix} \qquad U_k(t) = \begin{pmatrix} u_{1k} \\ u_{2k} \end{pmatrix}$$

- Please project to "guess" what U_k is when $k \to \infty$?
- (it contains elementary functions.)
- You have 2 minutes

$$U_0 = \begin{pmatrix}1\\0\end{pmatrix} \qquad F(U_1) = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix} U_1(t) = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}1\\-t\end{pmatrix} = \begin{pmatrix}-t\\-1\end{pmatrix}$$

$$U_2(t) = U_0 + \int_0^t F(U_1)ds = U_0 + \int_0^t \begin{pmatrix}-s\\-1\end{pmatrix} ds = \begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}-\frac{1}{2}t^2\\-t\end{pmatrix} = \begin{pmatrix}1-\frac{1}{2}t^2\\-t\end{pmatrix}$$

$$F(U_2) = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix} U_2(t) = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}1-\frac{1}{2}t^2\\-t\end{pmatrix} = \begin{pmatrix}-t\\-1+\frac{1}{2}t^2\end{pmatrix}$$

$$U_3(t) = U_0 + \int_0^t F(U_2)ds = \begin{pmatrix}1-t^2/2\\-t+t^3/3!\end{pmatrix}$$

$$U_4(t) = \begin{pmatrix}1-\frac{t^2}{2}+t^4/4!\\-t+t^3/3!\end{pmatrix}$$

$$U(t) = \begin{pmatrix}\cos(t)\\-\sin(t)\end{pmatrix}$$

Example 2 for Picard Iteration

Example. For an example of Picard iteration applied to a system of differential equations, consider the linear system

$$X' = F(X) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X$$

with initial condition $X(0) = (1,0)$. As we have seen, the solution of this initial value problem is

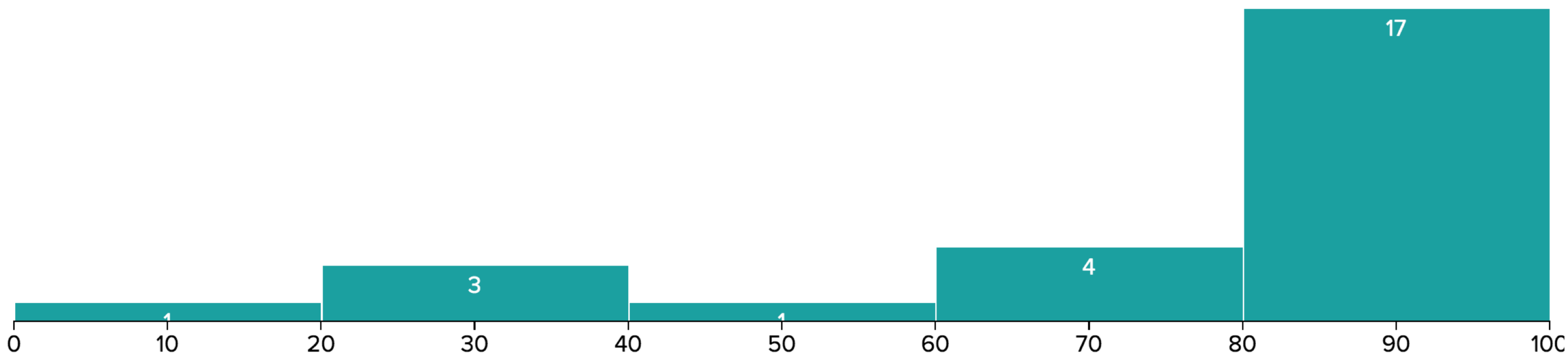
$$\boxed{X(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}.}$$

$$U_{k+1}(t) = U_0 + \int_0^t F(U_k)ds \qquad U(t) = \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix}$$

HW4

Review Grades for **HW #4**

● REGRADE REQUESTS OPEN ● GRADES PUBLISHED



MINIMUM	MEDIAN	MAXIMUM	MEAN	STD DEV
12.0	87.0	100.0	77.92	25.66

Q1

1: [15 points] Consider the Lorenz model:

$$\frac{dX}{dt} = -\sigma X + \sigma Y, \tag{1.1}$$

$$\frac{dY}{dt} = -XZ + rX - Y, \tag{1.2}$$

$$\frac{dZ}{dt} = XY - \beta Z. \tag{1.3}$$

(a) Find the Jacobian matrix at the trivial critical point $(X, Y, Z) = (0, 0, 0)$. [5 points]

(b) Choose $\sigma = 10$. Perform a (linear) stability analysis in r, β-space using the matrix in (a). [10 points]

[Hint: Describe the regions where the Jacobian matrix has real and/or complex eigenvalues.]

MATH537-01-Fall2020 › Announcements › HW #4 + supp for Gauss_Elimination

HW #4 + supp for Gauss_Elimination

Bo-Wen Shen

Oct 17 at 4:48pm

All Sections

Dear Students:

I'd like to draw your attention to the following:

(1) as discussed in HW2, the Lorenz model has non-negative parameters.

This may simplify your work for HW4.

(2) I uploaded a pdf file regarding Gauss_Elimination to canvas/supp for students who need additional help.

Thanks and have a good weekend.

Best,

-Bowen

Q1

(a)

$$J(X,Y,Z) = \begin{pmatrix} -\sigma & \sigma & 0 \\ -Z+r & -1 & -X \\ Y & X & -\beta \end{pmatrix}$$

$$J(0,0,0) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}$$

(b1) $\lambda_3 = -\beta < 0$ (all of the parameters are non-negative.)

(b2a) $\lambda_{1,2} = -\frac{1}{2}(\sigma+1) \pm \frac{1}{2}\sqrt{(\sigma+1)^2 + 4\sigma(r-1)}$

(b2b) $\lambda_{1,2} < 0$ a sink for $0 \leq r < 1$

(b2c) $\lambda_1 < 0$ $\lambda_2 = 0$ for $r = 1$, a stable point,

(b2d) $\lambda_1\lambda_2 < 0$ for $r > 1$, a saddle

Q2

2: [20 points] Consider the non-dissipative Lorenz model:

$$\frac{dX}{dt} = \sigma Y, \tag{2.1}$$

$$\frac{dY}{dt} = -XZ + rX, \tag{2.2}$$

$$\frac{dZ}{dt} = XY. \tag{2.3}$$

(a) Find critical points. [5 points]

(b) Find the Jacobian matrix at critical points(s). [5 points]

(c) Perform a linear stability analysis at each of the critical points. [10 points]

Q2

(a) critical points: $(X,Y,Z)=(0,0,Z_0)$, $(X,Y,Z)=(X_0,0,r)$;

(b)

$$J(X,Y,Z)=\begin{pmatrix} 0 & \sigma & 0 \\ -Z+r & 0 & -X \\ Y & X & 0 \end{pmatrix}$$

(b1)

$$J(0,0,Z_0)=\begin{pmatrix} 0 & \sigma & 0 \\ -Z_0+r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(b2)

$$J(X_0,0,r)=\begin{pmatrix} 0 & \sigma & 0 \\ 0 & 0 & -X_0 \\ 0 & X_0 & 0 \end{pmatrix}$$

Q2

(c1) at $(0, 0, Z_0)$, $\lambda_3 = 0$

(c1a) $\lambda_1\lambda_2 < 0$ for $Z_0 < r$, a saddle

(c1b) $\lambda_{1,2} = \pm i\sqrt{\sigma(Z_0 - r)}$ for $Z_0 > r$, a center

(c1c) $Z_0 = r$, a repeated eigenvalue $\lambda = 0$ of multiplicity 3.

(c2) at $(X_0, 0, r)$, $\lambda_3 = 0$

(c2a) $X_0 \neq 0$, $\lambda_{1,2} = \pm iX_0$, a center

(c2b) $X_0 = 0$, a repeated eigenvalue $\lambda = 0$ of multiplicity 3.

MT Part-A Q2

- What's the full name of the model we just discussed?
- When did you first solve the model?
- You have 1 minute

2: [25 points] A nonlinear, non-dissipative Lorenz model is written as follows:

$$\frac{d^2X}{dt^2} - (\sigma r + C)\,X + \frac{X^3}{2} = 0. \qquad (2)$$

Here, we assume that both σ and r are positive, and choose $C = 0$ for convenience. Complete the following problems.

(a) [3 points] Transform the 2nd order ODE in Eq. (2) into a system of the first order ODEs, (i.e., $Y = X'$).

(b) [3 points] Find critical points in the above 2D system in problem (2a).

(c) [6 points] Compute the Jacobian matrix of the above 2D system.

(d) [13 points] Perform a linear stability analysis for all of the critical points.

MT Part-A Q2

(a) $X' = Y$; $Y' = \sigma r X - X^3/2$

(b) $(X, Y) = (0, 0), (\pm\sqrt{2\sigma r}, 0)$

(c)

$$J = \begin{pmatrix} 0 & 1 \\ \sigma r - \frac{3}{2}X^2 & 0 \end{pmatrix}$$

(d1) at $(X, Y) = (0, 0)$, $\lambda_{1,2} = \pm\sqrt{\sigma r}$, a saddle

(d2) at $(X, Y) = (\pm\sqrt{2\sigma r}, 0)$, $\lambda_{1,2} = \pm i\sqrt{2\sigma r}$, centers

Q3

3: [35 points] Consider the following harmonic oscillators:

$$\frac{d^2x_1}{dt^2} = -k_1x_1, \tag{3.1}$$

$$\frac{d^2x_2}{dt^2} = -k_2x_2. \tag{3.2}$$

Let $k_1 = 4\omega_1^2$ and $k_2 = \omega_2^2$.

(a) Convert the above equations into a linear system with four first-order differential equations. Find the matrix A that represents the 4D system. [5 points]

(b) Find the eigenvalues and eigenvectors of A in the 4-D phase space. [15 points]

(c) Find the linear map T using (b) and compute $T^{-1}AT$. [15 points]

Q3

(a)

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -4\omega_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \omega_2^2 & 0 \end{pmatrix}$$

(b) $\lambda_{1,2} = \pm 2i\omega_1$ and $\lambda_{3,4} = \pm i\omega_2$

$$V_1 = \begin{pmatrix} 1 \\ 2i\omega_1 \\ 0 \\ 0 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ i\omega_2 \end{pmatrix}$$

$$T = (Re(V_1), Im(V_1), Re(V_3), Im(V_3)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2\omega_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega_2 \end{pmatrix}$$

Q3

$$T^{-1}AT = \begin{pmatrix} 0 & 2\omega_1 & 0 & 0 \\ -2\omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 \\ 0 & 0 & -\omega_2 & 0 \end{pmatrix}$$

Q4

4: [30 points] Consider the following matrix:

$$A = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

(a) Find the eigenvector(s) and generalized eigenvector(s) associated with the matrix A. [15 points.]

(b) Construct a linear map T using the eigenvector(s) and generalized eigenvector(s) in (a) and compute $T^{-1}AT$. [15 points.]

Q4

(a) $\lambda = 2$ as a repeated eigenvalue.

$$V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$(A - \lambda I)V_2 = V_1;$

$$V_2 = \begin{pmatrix} 1 \\ 1/3 \\ 0 \end{pmatrix}$$

$(A - \lambda I)V_3 = V_2;$

$$V_3 = \begin{pmatrix} 1 \\ 1/3 \\ -1/3 \end{pmatrix}$$

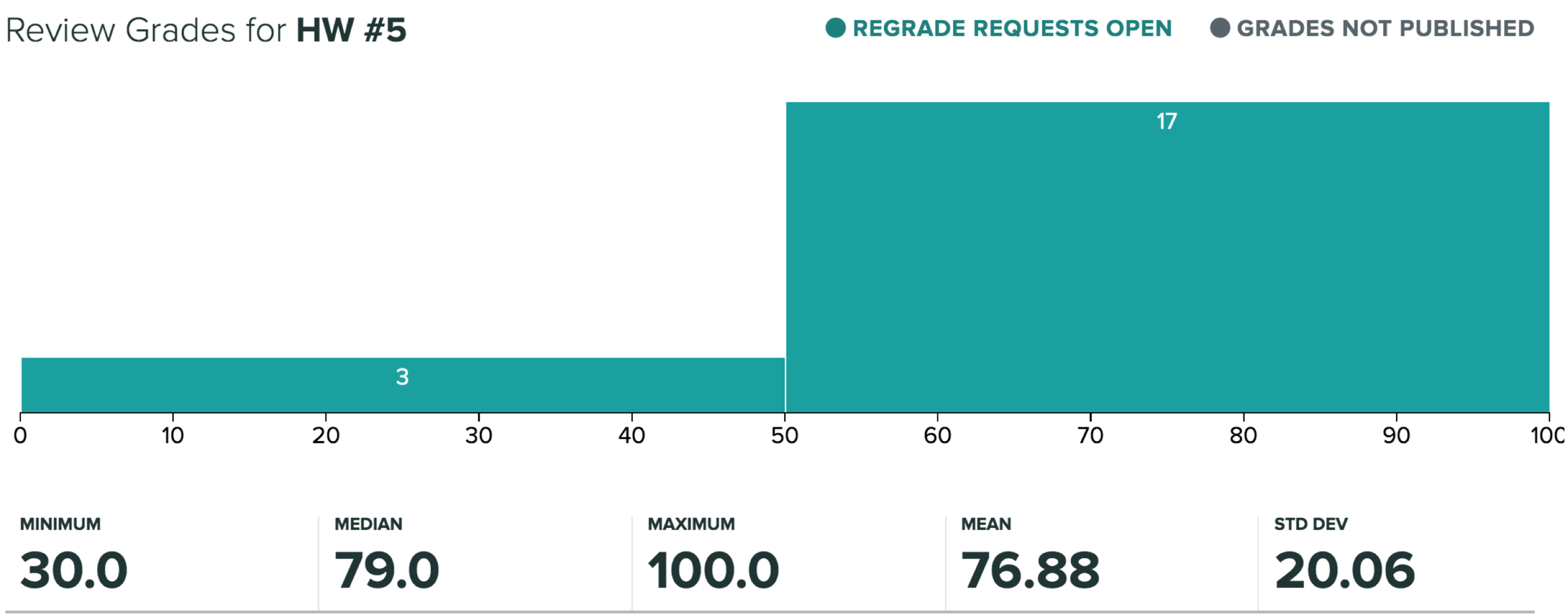
(b1)

$$T = (V_1, V_2, V_3) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1/3 & 1/3 \\ 0 & 0 & -1/3 \end{pmatrix}$$

(b2)

$$T^{-1}AT = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

HW5



Review Grades for **HW #5**
REGRADE REQUESTS OPEN
GRADES NOT PUBLISHED
17
3
0
10
20
30
40
50
60
70
80
90
MINIMUM
30.0
MEDIAN
79.0
MAXIMUM
100.0
MEAN
76.88
STD DEV
20.06

Picard Iteration for the Proof

Without dwelling on the details here, the proof of this theorem depends on an important technique known as *Picard iteration*. Before moving on, we illustrate how the Picard iteration scheme used in the proof of the theorem works in several special examples. The basic idea behind this iterative process is to construct a sequence of functions that converges to the solution of the differential equation. The sequence of functions $u_k(t)$ is defined inductively by $u_0(t) = x_0$, where x_0 is the given initial condition, and then

$$X' = F(X) \quad X(t=0) = X_0$$

$$\mathrm{u}_0(t) = x_0$$

$$u_{k+1}(t) = x_0 + \int_0^t F(u_k(s))ds$$

$$x' = x \quad x(t=0) = x_0$$

$$F(x) = x$$

$$\mathrm{u}_0(t) = x_0$$

$$x_0 \sum_{i=0}^{\infty} \frac{t^i}{i!} = \boxed{x_o e^t} = x(t)$$

The Fundamental Local Theorem for Non-autonomous Eqs.

Theorem. *Let $\mathcal{O} \subset \mathbb{R} \times \mathbb{R}^n$ be open and $F : \mathcal{O} \to \mathbb{R}^n$ a function that is C^1 in X and continuous in t. If $(t_0, X_0) \in \mathcal{O}$, there is an open interval J containing t and a unique solution of $X' = F(t, X)$ defined on J and satisfying $X(t_0) = X_0$.*

C^1 ➔ Locally Lipschitz

A special case is given below.

Corollary. *Let $A(t)$ be a continuous family of $n \times n$ matrices. Let $(t_0, X_0) \in J \times \mathbb{R}^n$. Then the initial value problem*

$$X' = A(t)X, \; X(t_0) = X_0$$

has a unique solution on all of J.

HSD, p401

Outline

1. Existence and Uniqueness Theorems
2. Lipschitz Condition
3. Continuous Dependence of Solutions on Initial Conditions (CDIC)
4. Sensitive Dependence of Solutions on Initial Conditions (SDIC)
5. Linearization Theorems & Linearized Systems

Boundedness and Lipschitz Function

We call the function $F(t, X)$ *Lipschitz in* X if there is a constant $K \geq 0$ such that

$$|F(t, X_1) - F(t, X_2)| \leq K|X_1 - X_2|$$

for all (t, X_1) and (t, X_2) in $\mathcal{O}$. Locally Lipschitz in X is defined analogously.

Definition 5.1 A function $f(t, y)$ is said to satisfy a **Lipschitz condition** in the variable y on a set $D \subset \mathbb{R}^2$ if a constant $L > 0$ exists with

$$|f(t, y_1) - f(t, y_2,)| \leq L|y_1 - y_2|,$$

whenever (t, y_1) and (t, y_2) are in D. The constant L is called a **Lipschitz constant** for f. ■

Partial derivatives are bounded (e.g., in the next slide)

Suppose $f(t, y)$ is defined on a convex set $D \subset \mathbb{R}^2$. If a constant $L > 0$ exists with

$$\left|\frac{\partial f}{\partial y}(t, y)\right| \leq L, \quad \text{for all } (t, y) \in D, \tag{5.1}$$

then f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant L. ■

- A convex set
- Lipschitz condition

Example 1 Show that $f(t, y) = t|y|$ satisfies a Lipschitz condition on the interval $D = \{(t, y) \mid 1 \leq t \leq 2 \text{ and } -3 \leq y \leq 4\}$.

Solution For each pair of points (t, y_1) and (t, y_2) in D we have

$$|f(t, y_1) - f(t, y_2)| = |t|y_1| - t|y_2|| = |t|\boxed{||y_1| - |y_2||} \leq 2\boxed{|y_1 - y_2|}.$$

Thus f satisfies a Lipschitz condition on D in the variable y with Lipschitz constant 2. The smallest value possible for the Lipschitz constant for this problem is $L = 2$, because, for example,

$$|f(2, 1) - f(2, 0)| = |2 - 0| = 2|1 - 0|. \quad \blacksquare$$

$$(|y_1| - |y_2|)^2 - (|y_1 - y_2|)^2$$

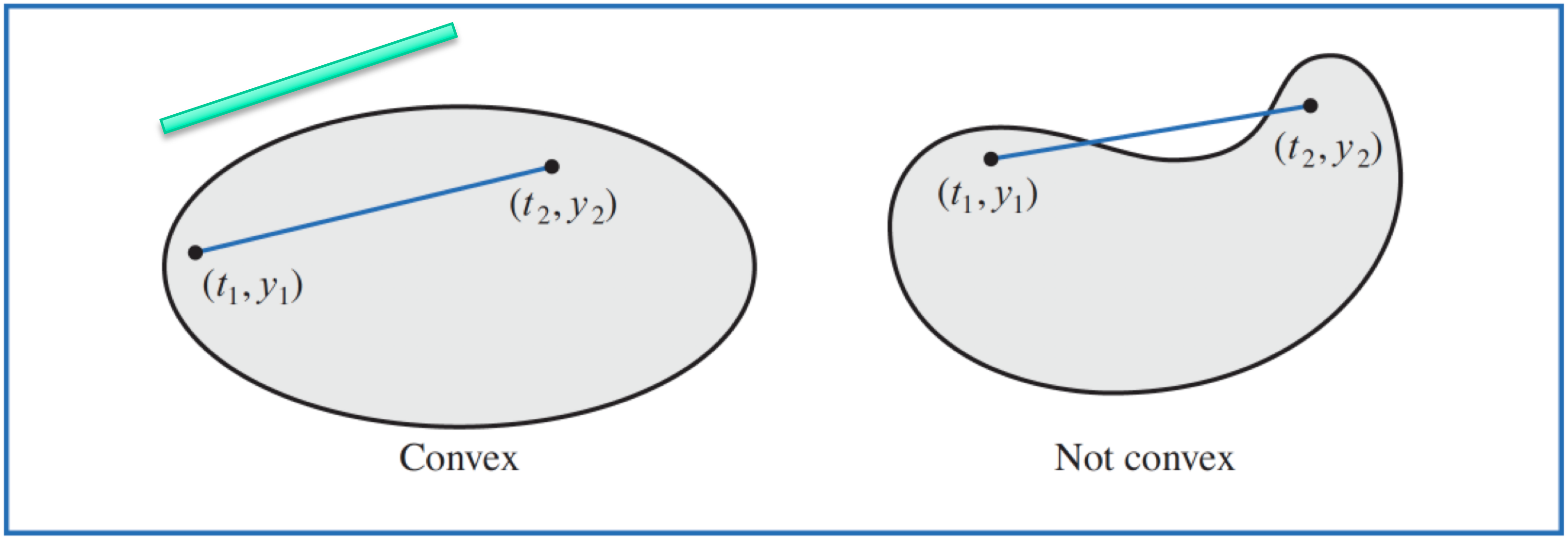
$$= -2|y_1||y_2| - (-2y_1y_2)$$

$$= 2y_1y_2 - 2|y_1||y_2| \leq 0$$

Definition 5.2 A set $D \subset \mathbb{R}^2$ is said to be **convex** if whenever (t_1, y_1) and (t_2, y_2) belong to D, then $((1-\lambda)t_1 + \lambda t_2, (1-\lambda)y_1 + \lambda y_2)$ also belongs to D for every λ in $[0, 1]$. ■

In geometric terms, Definition 5.2 states that a set is convex provided that whenever two points belong to the set, the entire straight-line segment between the points also belongs to the set. (See Figure 5.1.) The sets we consider in this chapter are generally of the form $D = \{(t, y) \mid a \le t \le b \text{ and } -\infty < y < \infty\}$ for some constants a and b. It is easy to verify (see Exercise 7) that these sets are convex.

Figure 5.1



Uniqueness (Burden et al., 2014)

Theorem 5.4 Suppose that $D = \{(t, y) \mid a \le t \le b \text{ and } -\infty < y < \infty\}$ and that $f(t, y)$ is continuous on D. If f satisfies $\boxed{\text{a Lipschitz condition}}$ on D in the variable y, then the initial-value problem

$$y'(t) = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

has $\boxed{\text{a unique solution}}$ $y(t)$ for $a \le t \le b$. ■

Example 2 Use Theorem 5.4 to show that there is a unique solution to the initial-value problem

$$y' = 1 + t\sin(ty), \quad \boxed{0 \le t \le 2,} \quad y(0) = 0.$$

Solution Holding t constant and applying the Mean Value Theorem to the function

$$f(t, y) = 1 + t\sin(ty),$$

we find that when $y_1 < y_2$, a number ξ in (y_1, y_2) exists with

$$\frac{f(t, y_2) - f(t, y_1)}{y_2 - y_1} = \boxed{\frac{\partial}{\partial y} f(t, \xi) = t^2\cos(\xi t).}$$

Thus

$$\boxed{|f(t, y_2) - f(t, y_1)|} = |y_2 - y_1||t^2\cos(\xi t)| \le \boxed{4|y_2 - y_1|,}$$

and f satisfies a Lipschitz condition in the variable y with Lipschitz constant $L = 4$. Additionally, $f(t, y)$ is continuous when $0 \le t \le 2$ and $-\infty < y < \infty$, so Theorem 5.4 implies that a unique solution exists to this initial-value problem.

If you have completed a course in differential equations you might try to find the exact solution to this problem. ∎

Lipschitz Continuity

- f is Lipschitz continuous wrt. y.
- f satisfies a Lipschitz condition.
- f is locally Lipschitz.
- F has Lipschitz constant.

HISTORICAL NOTE

Rudolph Lipschitz (1832 – 1903), inventor of Lipschitz continuity. This concept was used by him to prove the existence and uniqueness of solutions to IVP of ODE in 1876.

If f is **Lipschitz continuous** wrt. y, *i.e.*

$$\|f(t,y) - f(t,y^*)\| \leq L\|y - y^*\|$$

Lipschitz Functions and Lipschitz Constant

Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set. A function $F : \mathcal{O} \to \mathbb{R}^n$ is said to be *Lipschitz* on $\mathcal{O}$ if there exists a constant K such that

$$|F(Y) - F(X)| \leq K|Y - X|$$

for all $X, Y \in \mathcal{O}$. We call *K a Lipschitz constant* for F. More generally, we say that F is *locally Lipschitz* if each point in $\mathcal{O}$ has a neighborhood $\mathcal{O}'$ in $\mathcal{O}$ such that the restriction F to $\mathcal{O}'$ is Lipschitz. The Lipschitz constant of $F|\mathcal{O}'$ may vary with the neighborhoods $\mathcal{O}'$.

Lemma. *Suppose that the function $F : \mathcal{O} \to \mathbb{R}^n$ is C^1. Then F is locally Lipschitz.*

C^1 ➔ Locally Lipschitz

HSD

Outline

1. Existence and Uniqueness Theorems
2. Lipschitz Condition
3. Continuous Dependence of Solutions on Initial Conditions (CDIC)
4. Sensitive Dependence of Solutions on Initial Conditions (SDIC)
5. Linearization Theorems & Linearized Systems

CDIC vs. SDIC

TBD

	CDIC	SDIC
Definition	solutions through nearby ICs remain close over short time intervals	*The property for an orbit when most other orbits that pass close to it at some point do not remain close to it as time advances.*

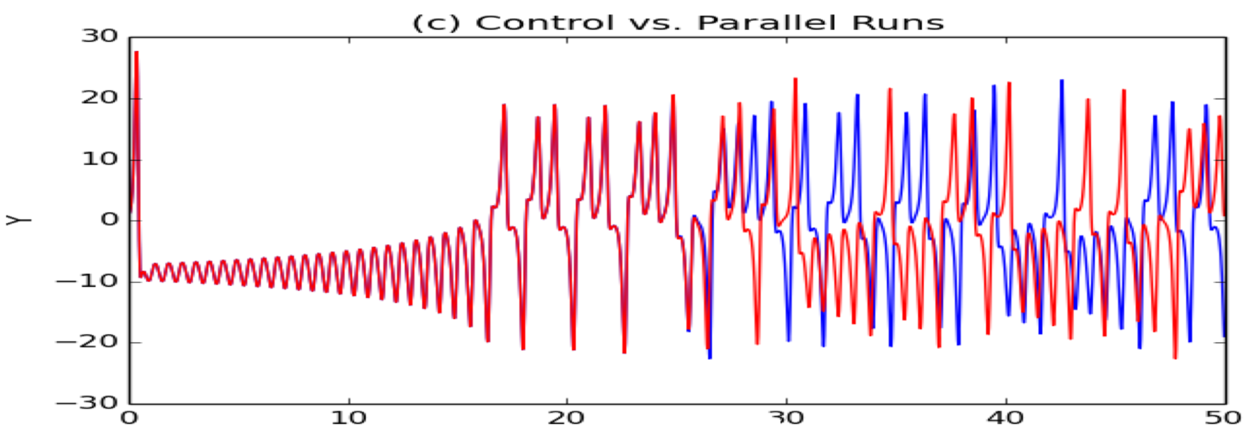
- nearby trajectories diverge no faster than an exponential rate
- nearby trajectories separate exponentially fast

- *Continuous dependence on initial conditions (CDIC)*
- *"Sensitive dependence on initial conditions" (SDIC) means that nearby trajectories separate exponentially fast, i.e., the system has a positive Liapunov exponent. (Strogatz, p331)*
- *We may define an infinite invariant set of a map $F: R \to R$ to have sensitive dependence on initial conditions if there exists $\delta > 0$ such that for all $x \in S$ and all neighbourhoods N (however small) of x there exists $y \in N$ and $n > 0$ such that $|F^n(x) - F^n(y)| > \delta$. So neighbouring orbits, however close initially, separate from one another, although each keeps close to the invariant set. (Drazin, p140; Devaney, p49)*

A Quick Note on CDIC and SDIC

1. The butterfly effect of the first kind (BE1):

   Indicating *sensitive dependence on initial conditions* (Lorenz, 1963).



Time

continuous dependence (within a time interval)

sensitive dependence

- control run (blue): $(X, Y, Z) = (0,1,0)$
- parallel run (red): $(X, Y, Z) = (0,1 + \epsilon, 0)$, $\epsilon = 1e - 10$.

- *Continuous dependence on initial conditions (CDIC)*
- *Sensitive dependence on initial conditions (SDIC)*

CDIC: The Gronwall's Inequality

Gronwall's Inequality. *Let $u : [0, \alpha] \to \mathbb{R}$ be continuous and nonnegative. Suppose $C \geq 0$ and $K \geq 0$ are such that*

$$u(t) \leq C + \int_0^t K u(s)\, ds$$

for all $t \in [0, \alpha]$. Then, for all t in this interval,

$$\boxed{u(t) \leq Ce^{Kt}.}$$

HSD, p. 395

Continuous Dependence on ICs (CDIC)

Theorem 7.16 **Continuous dependence on initial conditions.** *Let* $\mathbf{f}$ *be defined on the open set* U *in* $\mathbb{R}^n$*, and assume that* $\mathbf{f}$ *has Lipschitz constant* L *in the variables* $\mathbf{v}$ *on* U*. Let* $\mathbf{v}(t)$ *and* $\mathbf{w}(t)$ *be solutions of (7.29), and let* $[t_0, t_1]$ *be a subset of the domains of both solutions. Then*

$$|\mathbf{v}(t) - \mathbf{w}(t)| \leq |\mathbf{v}(t_0) - \mathbf{w}(t_0)| e^{L(t-t_0)},$$

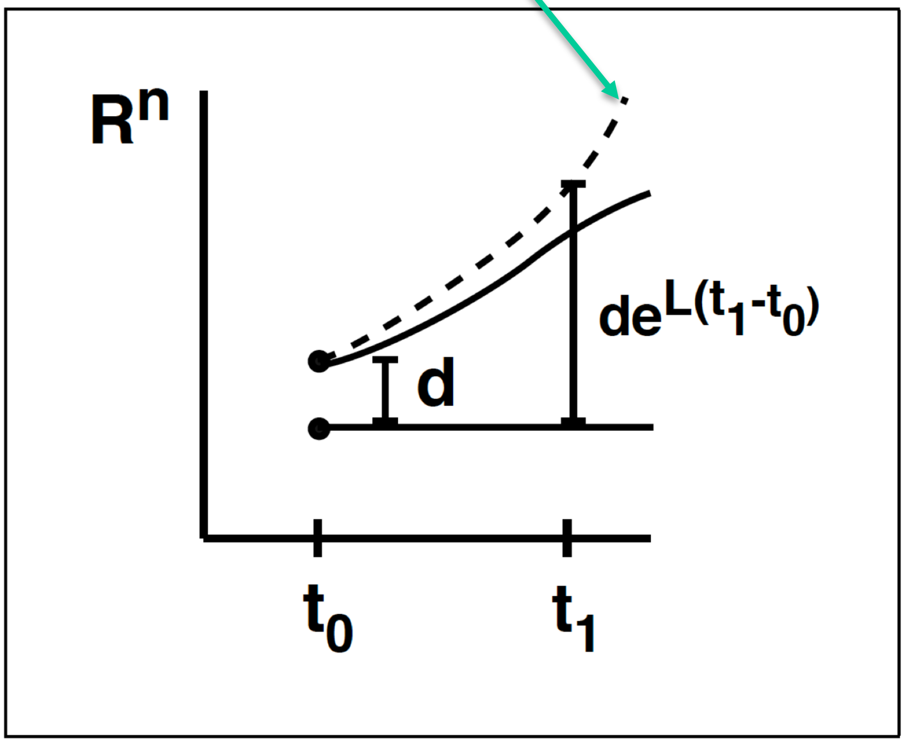
for all t *in* $[t_0, t_1]$.

- diverge no faster than an exponential rate over a (finite) sub-interval.

Alligood et al.

CDIC: The Gronwall Inequality and Lipschitz Constant

exponential function



- diverge no faster than an exponential rate

"Jargon"

L: Lipschitz constant

Figure 7.13 The Gronwall inequality.
Nearby solutions can diverge no faster than an exponential rate determined by the Lipschitz constant of the differential equation.

Alligood et al.

Review: Lipschitz Functions and Lipschitz Constant

Let $\mathcal{O} \subset \mathbb{R}^n$ be an open set. A function $F : \mathcal{O} \to \mathbb{R}^n$ is said to be *Lipschitz* on $\mathcal{O}$ if there exists a constant K such that

$$|F(Y) - F(X)| \leq K|Y - X|$$

for all $X, Y \in \mathcal{O}$. We call *K a Lipschitz constant* for F. More generally, we say that F is *locally Lipschitz* if each point in $\mathcal{O}$ has a neighborhood $\mathcal{O}'$ in $\mathcal{O}$ such that the restriction F to $\mathcal{O}'$ is Lipschitz. The Lipschitz constant of $F|\mathcal{O}'$ may vary with the neighborhoods $\mathcal{O}'$.

Lemma. *Suppose that the function $F : \mathcal{O} \to \mathbb{R}^n$ is C^1. Then F is locally Lipschitz.*

C^1 ➔ Locally Lipschitz

HSD

Continuous Dependence of Solutions on ICs

Theorem. *Consider the differential equation $X' = F(X)$ where $F: \mathbb{R}^n \to \mathbb{R}^n$ is C^1. Suppose that $X(t)$ is a solution of this equation that is defined on the closed interval $[t_0, t_1]$ with $X(t_0) = X_0$. Then there is a neighborhood $U \subset \mathbb{R}^n$ of X_0 and a constant K such that if $Y_0 \in U$, then there is a unique solution $Y(t)$ also defined on $[t_0, t_1]$ with $Y(t_0) = Y_0$. Moreover $Y(t)$ satisfies*

$$|Y(t) - X(t)| \le |Y_0 - X_0| \exp(K(t - t_0))$$

for all $t \in [t_0, t_1]$. K: Lipschitz constant

- X(t) and Y(t) which start out close together remain close together
- Although they may separate from each other, they do so no faster than exponentially (HSD, p. 395)

Corollary. (Continuous Dependence on Initial Conditions) *Let $\phi(t, X)$ be the flow of the system $X' = F(X)$, where F is C^1. Then ϕ is a continuous function of X.* HSD

Continuous Dependence of Solutions on ICs

For the Existence and Uniqueness Theorem to be at all interesting in any physical or even mathematical sense, the result needs to be complemented by the property that the solution $X(t)$ depends continuously on the initial condition $X(0)$. The next theorem gives a precise statement of this property.

Theorem. *Let $\mathcal{O} \subset \mathbb{R}^n$ be open and suppose $F : \mathcal{O} \to \mathbb{R}^n$ has Lipschitz constant K. Let $Y(t)$ and $Z(t)$ be solutions of $X' = F(X)$ that remain in $\mathcal{O}$ and are defined on the interval $[t_0, t_1]$. Then, for all $t \in [t_0, t_1]$, we have*

$$|Y(t) - Z(t)| \leq |Y(t_0) - Z(t_0)| \exp(K(t - t_0)). \qquad \blacksquare$$

- Note that this result says that, if the solutions Y(t) and Z(t) start out close together, then they remain close together for t near t0.
- Although these solutions may separate from each other, they do so no faster than exponentially (HSD, p395).

Continuous Dependence on ICs

Continuous Dependence on Initial Value

Theorem 9. Let f and $\partial f/\partial y$ be continuous functions on $\boxed{\text{an open rectangle}}$

$$R = \{(x, y) : a < x < b, c < y < d\}$$

containing the point (x_0, y_0). Assume that for all $\tilde{y}_0$ sufficiently close to y_0, the solution $\phi(x, \tilde{y}_0)$ to (7) exists on the interval $[x_0 - h, x_0 + h]$ and its graph lies within a fixed closed rectangle $R_0 \subset R$. Then, for $|x - x_0| \le h$,

$$|\phi(x, y_0) - \phi(x, \tilde{y}_0)| \le |y_0 - \tilde{y}_0| e^{Lh} , \tag{8}$$

where L is any positive constant such that $|(\partial f/\partial y)(x, y)| \le L$ for all (x, y) in R_0. Moreover, as $\tilde{y}_0$ approaches y_0, the solution $\phi(x, \tilde{y}_0)$ approaches $\phi(x, y_0)$ uniformly on $[x_0 - h, x_0 + h]$.

Nagle et al.

Continuous Dependence on Parameters

Theorem. (Continuous Dependence on Parameters) *Let $X' = F_a(X)$ be a system of differential equations for which F_a is continuously differentiable in both X and a. Then the flow of this system depends continuously on a as well as X.*

As in the previous case, solutions depend continuously on these parameters provided that the system depends on the parameters in a continuously differentiable fashion. We can see this easily by using a special little trick. Suppose the system

$$X' = F_a(X)$$

depends on the parameter a in a C^1 fashion. Let's consider an "artificially" augmented system of differential equations given by

$$\begin{aligned} x_1' &= f_1(x_1, \ldots, x_n, a) \\ &\vdots \\ x_n' &= f_n(x_1, \ldots, x_n, a) \\ a' &= 0. \end{aligned}$$

This is now an autonomous system of $n+1$ differential equations. Although this expansion of the system may seem trivial, we may now invoke the previous result about continuous dependence of solutions on initial conditions to verify that solutions of the original system depend continuously on a as well.

Continuous Dependence on the RHS

A different kind of continuity is continuity of solutions as functions of the data $F(t, X)$. That is, if $F: \mathcal{O} \to \mathbb{R}^n$ and $G: \mathcal{O} \to \mathbb{R}^n$ are both C^1 in X, and $|F - G|$ is uniformly small, we expect solutions to $X' = F(t, X)$ and $Y' = G(t, Y)$, having the same initial values, to be close. This is true; in fact, we have the following more precise result.

Theorem. *Let $\mathcal{O} \subset \mathbb{R} \times \mathbb{R}^n$ be an open set containing $(0, X_0)$ and suppose that $F, G: \mathcal{O} \to \mathbb{R}^n$ are C^1 in X and continuous in t. Suppose also that for all $(t, X) \in \mathcal{O}$*

$$|F(t,X) - G(t,X)| < \epsilon.$$

Let K be a Lipschitz constant in X for $F(t, X)$. If $X(t)$ and $Y(t)$ are solutions of the equations $X' = F(t, X)$ and $Y' = G(t, Y)$ respectively on some interval J, and $X(0) = X_0 = Y(0)$, then

$$|X(t) - Y(t)| \leq \frac{\epsilon}{K}\left(\exp(K|t|) - 1\right)$$

for all $t \in J$.

$$X' = F(t, X)$$
$$Y' = G(t, Y)$$

$|F - G|$ *is small*

$|X - Y|$ diverges no faster than an exponential growth rate determined by the Lipschitz

HSD, p402

Continuous Dependence on the RHS

Continuous Dependence on $f(x, y)$

Theorem 10. Let f and $\partial f/\partial y$ be continuous functions on an open rectangle $R = \{(x, y): a < x < b, c < y < d\}$ containing the point (x_0, y_0). Let F be continuous on R and assume that

(15) $$|F(x, y) - f(x, y)| \le \varepsilon\ , \qquad \text{for } (x, y) \text{ in } R.$$

Let ϕ be the solution to the initial value problem

(16) $$\frac{dy}{dx} = f(x, y)\ , \qquad y(x_0) = y_0\ ,$$

and let ψ be a solution to

(17) $$\frac{dy}{dx} = F(x, y)\ , \qquad y(x_0) = y_0\ .$$

Assume both solutions $\phi(x)$, $\psi(x)$ exist on $[x_0 - h, x_0 + h]$ and their graphs lie in a closed rectangle $R_0 \subset R$. Then, for $|x - x_0| \le h$,

(18) $$|\phi(x) - \psi(x)| \le \varepsilon h e^{Lh}\ ,$$

where L is any positive constant such that $|(\partial f/\partial y)(x, y)| \le L$ for all (x, y) in R_0.

In particular, as F approaches f uniformly on R—that is, as $\varepsilon \to 0^+$ in (15)—the solution $\psi(x)$ approaches $\phi(x)$ uniformly on $[x_0 - h, x_0 + h]$.

Nagle et al.

Outline

1. Existence and Uniqueness Theorems
2. Lipschitz Condition
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4. Sensitive Dependence of Solutions on Initial Conditions (SDIC)
5. Linearization Theorems & Linearized Systems

CDIC vs. SDIC

TBD

	CDIC	SDIC
Definition	solutions through nearby ICs remain close over short time intervals	*The property for an orbit when most other orbits that pass close to it at some point do not remain close to it as time advances.*

- nearby trajectories diverge no faster than an exponential rate
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- *Continuous dependence on initial conditions (CDIC)*
- *"Sensitive dependence on initial conditions" (SDIC) means that nearby trajectories separate exponentially fast, i.e., the system has a positive Liapunov exponent. (Strogatz, p331)*
- *We may define an infinite invariant set of a map $F: R \rightarrow R$ to have sensitive dependence on initial conditions if there exists $\delta > 0$ such that for all $x \in S$ and all neighbourhoods N (however small) of x there exists $y \in N$ and $n > 0$ such that $|F^n(x) - F^n(y)| > \delta$. So neighbouring orbits, however close initially, separate from one another, although each keeps close to the invariant set. (Drazin, p140; Devaney, p49)*

Sensitive Dependence on ICs

Chaos is *aperiodic long-term behavior* in a *deterministic* system that exhibits *sensitive dependence on initial conditions*.

1. "Aperiodic long-term behavior" means that there are trajectories which do not settle down to fixed points, periodic orbits, or quasiperiodic orbits as $t \to \infty$. For practical reasons, we should require that such trajectories are not too rare. For instance, we could insist that there be an open set of initial conditions leading to aperiodic trajectories, or perhaps that such trajectories should occur with nonzero probability, given a random initial condition.
2. "Deterministic" means that the system has no random or noisy inputs or parameters. The irregular behavior arises from the system's nonlinearity, rather than from noisy driving forces.
3. "Sensitive dependence on initial conditions" means that nearby trajectories separate exponentially fast, i.e., the system has a positive Liapunov exponent.

Strogatz (2015)

Sensitive Dependence on ICs

Supp

Definition 8.2. $f: J \to J$ has sensitive dependence on initial conditions if there exists $\delta > 0$ such that, for any $x \in J$ and any neighborhood N of x, there exists $y \in N$ and $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$.

Intuitively, a map possesses sensitive dependence on initial conditions if there exist points arbitrarily close to x which eventually separate from x by at least δ under iteration of f. We emphasize that not *all* points near x need eventually separate from x under iteration, but there must be at least one such point in every neighborhood of x. If a map possesses sensitive dependence on initial conditions, then for all practical purposes, the dynamics of the map defy numerical computation. Small errors in computation which are introduced by round-off may become magnified upon iteration. The results of numerical computation of an orbit, no matter how accurate, may bear no resemblance whatsoever with the real orbit.

(Devaney, p49)

Notations for Sensitive Dependence on ICs

We consider $\mathbf{C}^r$ $(r \geq 1)$ autonomous vector fields and maps on $\mathbb{R}^n$ denoted as follows

$$\text{vector field} \qquad \dot{x} = f(x), \qquad (30.0.1)$$

$$\text{map} \qquad x \mapsto g(x). \qquad (30.0.2)$$

We denote the flow generated by (30.0.1) by $\phi(t, x)$ and we assume that it exists for all $t > 0$. We assume that $\Lambda \subset \mathbb{R}^n$ is a compact set invariant under $\phi(t, x)$ (resp. $g(x)$), i.e., $\phi(t, \Lambda) \subset \Lambda$ for all $t \in \mathbb{R}$ (resp. $g^n(\Lambda) \subset \Lambda$ for all $n \in \mathbb{Z}$, except that if g is not invertible, we must take $n \geq 0$). Then we have the following definitions.

(Wiggins, p735)

Sensitive Dependence on ICs

Definition 30.0.1 (Sensitive Dependence on Initial Conditions) *The flow $\phi(t,x)$ (resp. $g(x)$) is said to have* sensitive dependence on initial conditions *on Λ if there exists $\varepsilon > 0$ such that, for any $x \in \Lambda$ and any neighborhood U of x, there exists $y \in U$ and $t > 0$ (resp. $n > 0$) such that $|\phi(t,x) - \phi(t,y)| > \varepsilon$ (resp. $|g^n(x) - g^n(y)| > \varepsilon$).*

Roughly speaking, Definition 30.0.1 says that for any point $x \in \Lambda$, there is (at least) one point arbitrarily close to Λ that diverges from x. Some authors require the rate of divergence to be exponential; for reasons to be explained later, we will not do so. As we will see in the examples, taken by itself sensitive dependence on initial conditions is a fairly common feature in many dynamical systems.

(Wiggins, p735)

Chaos: SDIC + Bounded

Supp

1.5 Sensitive dependence on initial conditions

A defining attribute of an attractor on which the dynamics is *chaotic* is that it displays exponentially sensitive dependence on initial conditions. Consider two nearby initial conditions $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0) = \mathbf{x}_0 + \Delta(0)$, and imagine that they are evolved forward in time by a continuous time dynamical system yielding orbits $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ as shown in Figure 1.14. At time t, the separation between the two orbits is $\Delta(t) = \mathbf{x}_2(t) - \mathbf{x}_1(t)$. If, in the limit $|\Delta(0)| \to 0$, and large t, orbits remain bounded and the difference between the solutions $|\Delta(t)|$ grows exponentially for typical orientation of the vector $\Delta(0)$ (i.e., $|\Delta(t)|/|\Delta(0)| \sim \exp(ht)$, $h > 0$), then we say that the system displays sensitive dependence on initial conditions and

Ott

Chaos: SDIC + Bounded (conti.)

Supp

is chaotic. By bounded solutions, we mean that there is some ball in phase space, $|\mathbf{x}| < R < \infty$, which solutions never leave.[15] (Thus, if the motion is on an attractor, then the attractor lies in $|\mathbf{x}| < R$.) The reason we have imposed the restriction that orbits remain bounded is that, if orbits go to infinity, it is relatively simple for their distances to diverge exponentially. An example is the single, autonomous, linear, first-order differential equation $\mathrm{d}x/\mathrm{d}t = x$. This yields $\mathrm{d}[x_2(t) - x_1(t)]/\mathrm{d}t = [x_2(t) - x_1(t)]$ and hence $\Delta(t) \sim \exp(t)$. Our requirement of bounded solutions eliminates such trivial cases.[16] For the case of the driven damped pendulum

Ott

Chaos: SDIC + Bounded (conti.)

Supp

ates such trivial cases.[16] For the case of the driven damped pendulum equation, we defined three phase space variables, one of which was $x^{(3)} = 2\pi f t$. As defined, $x^{(3)}$ is unbounded since it is proportional to t. The reason we can speak of the driven damped pendulum as being chaotic is that, as previously mentioned, $x^{(3)}$ only occurs as the argument of a sine, and hence it (as well as $x^{(2)} = \theta$) can be regarded as an angle. Thus, the phase space coordinates can be taken as $x^{(1)}, \bar{x}^{(2)}, \bar{x}^{(3)}$, where $\bar{x}^{(2,3)} \equiv x^{(2,3)}$ modulo 2π. Since the variables $\bar{x}^{(2)}$ and $\bar{x}^{(3)}$ lie between 0 and 2π, they are necessarily bounded.

Ott

Continuous vs. Sensitive Dependence on ICs

Curves can now be constructed that follow the slope field and which, by the second concept, do not cross. The third concept is referred to as "continuous dependence of solutions on initial conditions". All solutions of sufficiently smooth differential equations exhibit the property of continuous dependence, which is a consequence of continuity of the slope field. Theorems pertaining to existence, uniqueness, and continuous dependence are discussed in Section 7.4.

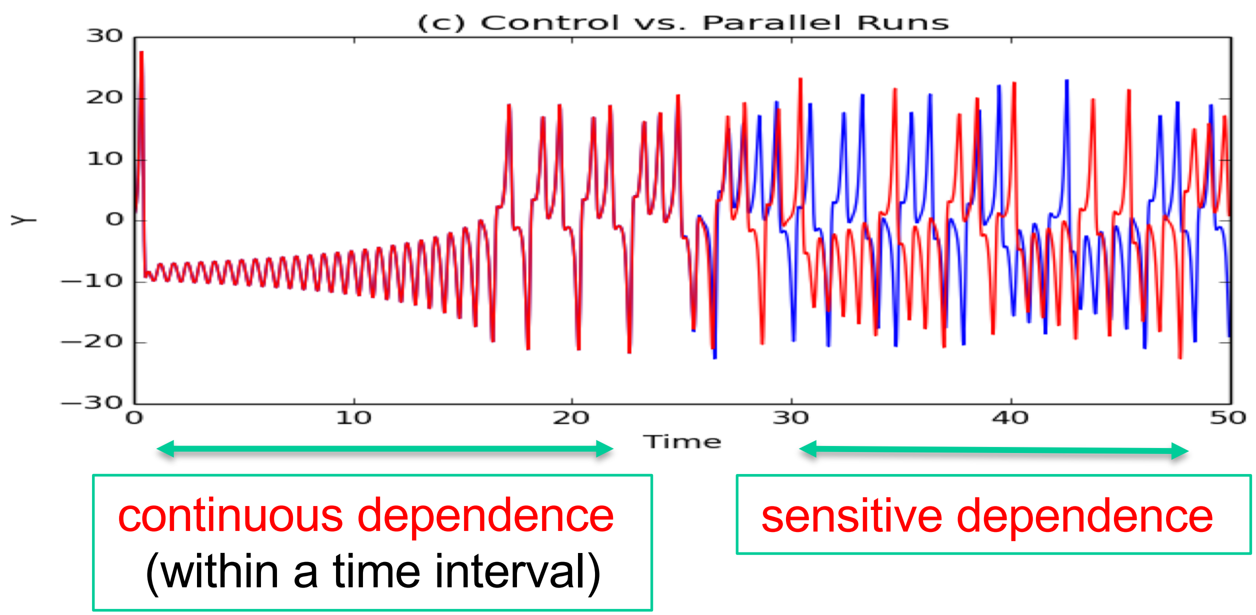
The concept of continuous dependence should not be confused with "sensitive dependence" on initial conditions. This latter property describes the behavior of unstable orbits over longer time intervals. Solutions may obey continuous dependence on short time intervals and also exhibit sensitive dependence, and diverge from one another on longer time intervals. This is the characteristic of a chaotic differential equation.

Alligood et al.

CDIC vs. SDIC

1. The butterfly effect of the first kind (BE1):

   Indicating *sensitive dependence on initial conditions* (Lorenz, 1963).



- control run (blue): $(X, Y, Z) = (0,1,0)$
- parallel run (red): $(X, Y, Z) = (0,1+\epsilon, 0)$, $\epsilon = 1e-10$.

A Chaotic System

Chaos is defined as aperiodic long-term behavior in a deterministic system that exhibits the sensitive dependence on initial conditions (ICs) (Strogatz 2015)

➢ Sensitive Dependence on Initial Conditions

⇕

1. Positive Lyapunov Exponent (averaged divergence)
2. Boundedness

"Quantitative"

- (LE is defined below)
- Historically, "divergence" of trajectories has been the "main" focus.
- Numerically, it is challenging to apply LE calculations to reveal the degree of solution's periodicity (i.e., LE=0) or quasi-periodicity.

Devaney's Definition on Chaos

➢ Sensitive Dependence on Initial Conditions

Lorenz Chaos
Hamiltonian Chaos

The definitions of chaos of Devaney (1989) include:

(1) sensitivity to initial conditions;

(2) topological transitivity (topological mixing); and

(3) dense periodic points.

By comparison, a system with a positive LE and solution's boundedness is said to be chaotic.

3 +1 Kinds of Chaos

- Steady Chaos (or Full Chaos):
  - i.e., strange chaotic attractor
- Transient Chaos:
  - A local LE is positive initially and it becomes negative later
- Horseshoe Chaos:
  - strange saddle
- Limited Chaos:
  - The property that characterizes a dynamical system in which some special orbits are nonperiodic but most are periodic or almost periodic (Lorenz, 1993)
  - (Li and York, period three implies chaos; Lorenz 1993)

(Relative) Growth Rate

$$y' = \lambda y$$

- The equation is linear in y.
- Its solution, $y = Ce^{\lambda t}$ is exponentially growing or decaying for real λ.
- The growth (or decay) rate of the solution, defined as $\frac{1}{y}\frac{dy}{dt}$, i.e., $\frac{d\ln(y)}{dt}$, is a constant.
- The above computes the growth rate using a single orbit.
- The method in the next slide uses two orbits to compute the growth rate.

Determine Growth Rates using Two near-by Orbits **Supp**

control run

$$y^{(1)} = y_n e^{\lambda(t-t_n)} \qquad y(t=t_n) = y_n$$

parallel run

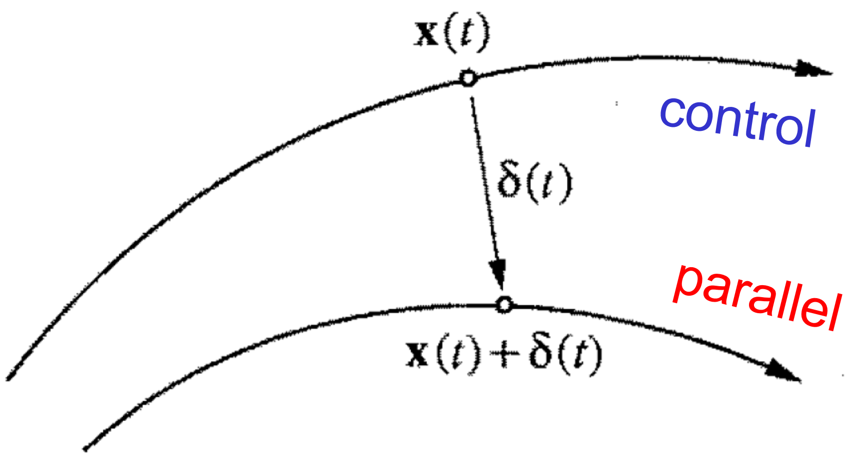
$$y^{(2)} = (y_n + \delta) e^{\lambda(t-t_n)} \qquad y(t=t_n) = y_n + \delta$$

$$y_{n+1}^{(2)} - y_{n+1}^{(1)} = \delta e^{\lambda h}$$

converges if $\lambda < 0$

$$\boxed{\frac{y_{n+1}^{(2)} - y_{n+1}^{(1)}}{y_n^{(2)} - y_n^{(1)}} = e^{\lambda h}} \qquad \lambda = \frac{ln\left(\dfrac{y_{n+1}^{(2)} - y_{n+1}^{(1)}}{y_n^{(2)} - y_n^{(1)}}\right)}{h}$$

Global and Finite-Time Lyapunov Exponent (LE) Supp



$\delta(t)$: a separation vector
$\delta(0)$: an initial separation vector

$$\lambda_{LE} = LE = \lim_{T\to\infty} \frac{1}{T} log \left|\frac{\delta(T)}{\delta(0)}\right|$$

(Global) Lyapunov Exponent
(e.g., climate scale)

- LE is a measure of the average separation speed of nearby trajectories.
- A finite-time LE is defined by a finite value of T.
- Two major features within a chaotic system are positive LE(s) and solution's boundedness.

LE vs. Eigenvalue (1D Linear system)

Supp

$$\boxed{\frac{\delta s_{n+1}}{\delta s_n} = \frac{y_{n+1}^{(2)} - y_{n+1}^{(1)}}{y_n^{(2)} - y_n^{(1)}} = e^{\lambda h}}$$

$$\lambda_{LE} = \lim_{T\to\infty} \frac{1}{T} ln\left(\frac{\delta s_n}{\delta s_0}\right)$$

$$= \lim_{N\to\infty} \frac{1}{Nh} ln\left(\frac{\delta s_n}{\delta s_{n-1}} \frac{\delta s_{n-1}}{\delta s_{n-2}} \cdots \frac{\delta s_1}{\delta s_0}\right)$$

$$= \lim_{N\to\infty} \frac{1}{Nh}\left(ln\frac{\delta s_n}{\delta s_{n-1}} + ln\frac{\delta s_{n-1}}{\delta s_{n-2}} \cdots + ln\frac{\delta s_1}{\delta s_0}\right)$$

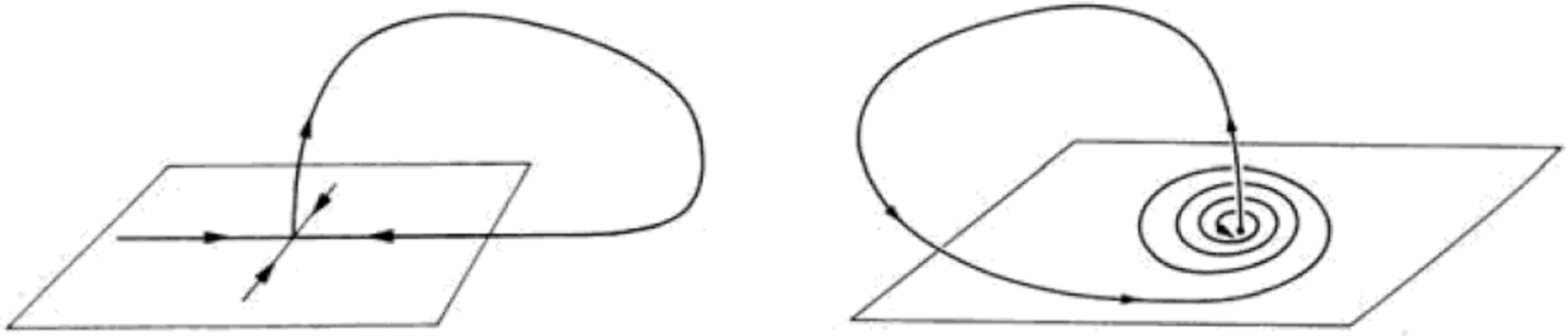
$$= \lim_{N\to\infty} \frac{1}{Nh}(N\lambda h) = \lambda$$

Saddle, Spiral Saddle, and Homoclinic Orbits

Supp

λ_1, λ_2, and λ_3 are real;
$\lambda_2, \lambda_3 < 0 < \lambda_1$

λ_1 is real
λ_2, and λ_3 are complex;
$Re(\lambda_2), Re(\lambda_3) < 0 < \lambda_1$



(a) (b)

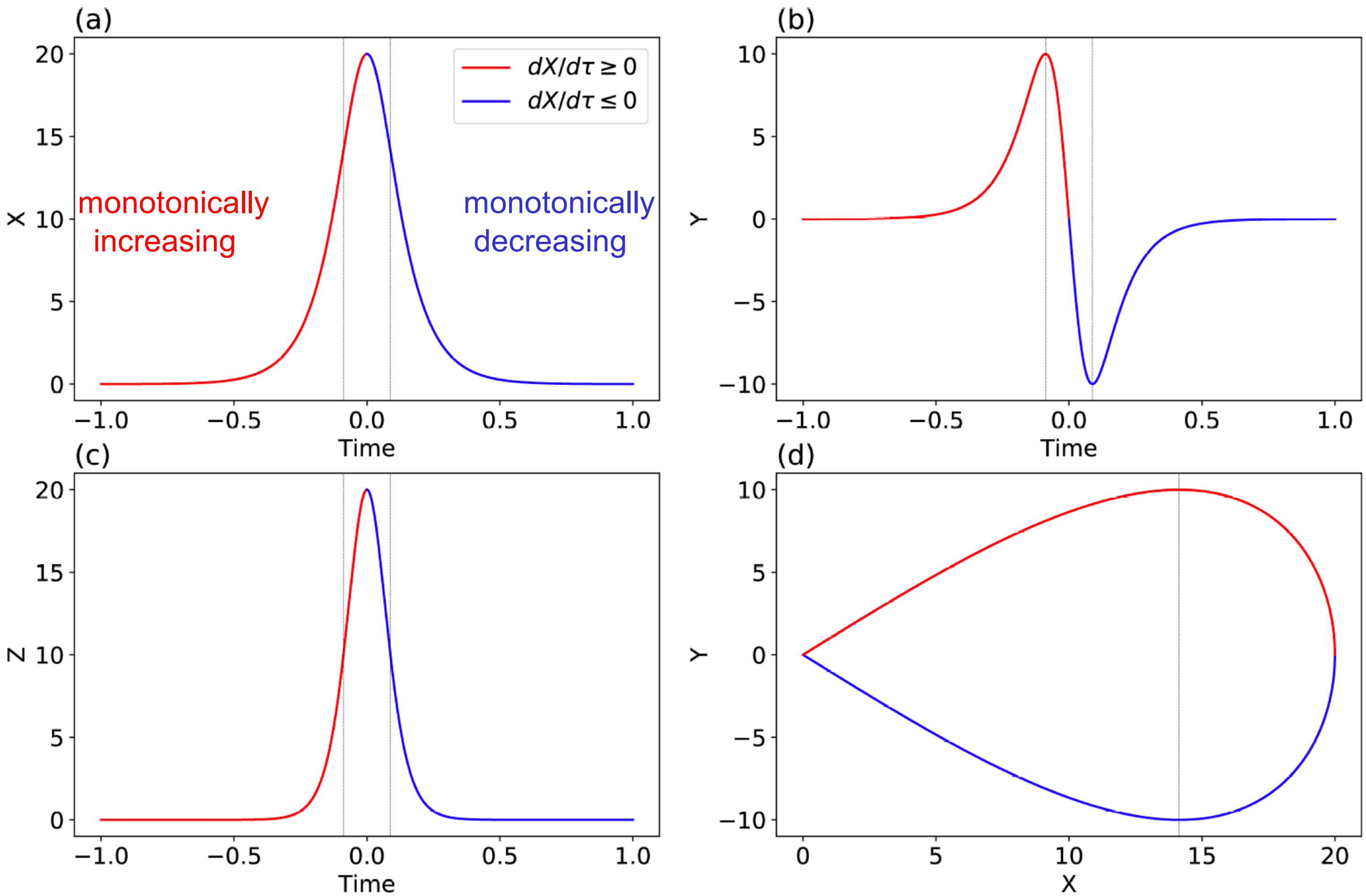
Figure 3.28. Sketch of the two cases of homoclinic bifurcation. In panel (a), there are two stable attracting directions in the plane and one expanding direction perpendicular to it. Both expansion and attraction are exponential, characteristic of the Lorenz case. In panel (b), the attraction is oscillatory whereas the expansion is exponential, characteristic of the Shilnikov case.

Saddle

Spiral Saddle (or Saddle Focus)

Homoclinic Orbit of the Non-dissipative 3DLM

Supp



Pulse width is determined by $[-\tau_t, \tau_t]$

Vertical lines indicate $\tau = \pm\tau_t$ or $X = \pm X_t$ where $\frac{d^2X}{d\tau^2} = 0$

Analytical Solutions of Homoclinic Orbits

Supp

NLS

$$X(\tau) = \frac{4\sqrt{\sigma r}}{e^{\sqrt{\sigma r}\tau} + e^{-\sqrt{\sigma r}\tau}} = 2\sqrt{\sigma r}\,sech(\sqrt{\sigma r}\tau),$$

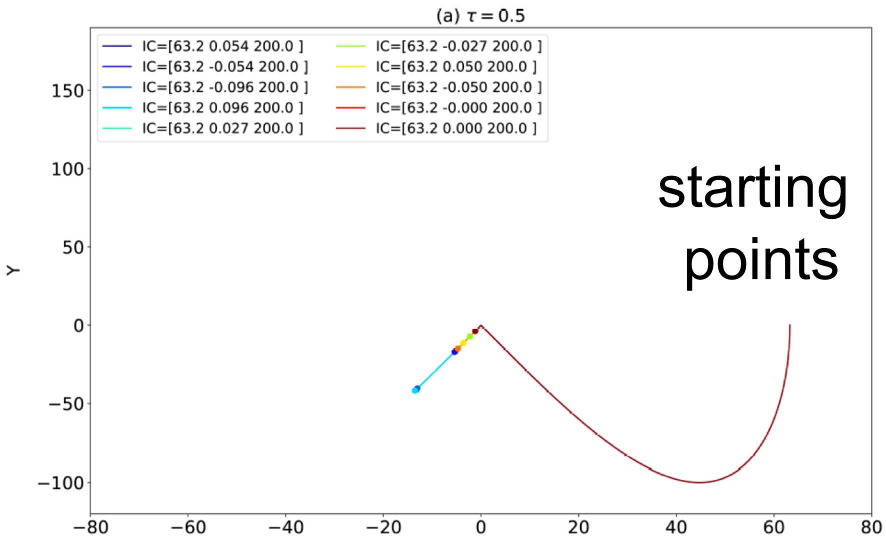
$$Y(\tau) = -4r\frac{e^{\sqrt{\sigma r}\tau} - e^{-\sqrt{\sigma r}\tau}}{\left(e^{\sqrt{\sigma r}\tau} + e^{-\sqrt{\sigma r}\tau}\right)^2} = -(2r)tanh(\sqrt{\sigma r}\tau)sech(\sqrt{\sigma r}\tau),$$

$$Z(\tau) = \frac{X^2(\tau)}{2\sigma} = (2r)sech^2(\sqrt{\sigma r}\tau).$$

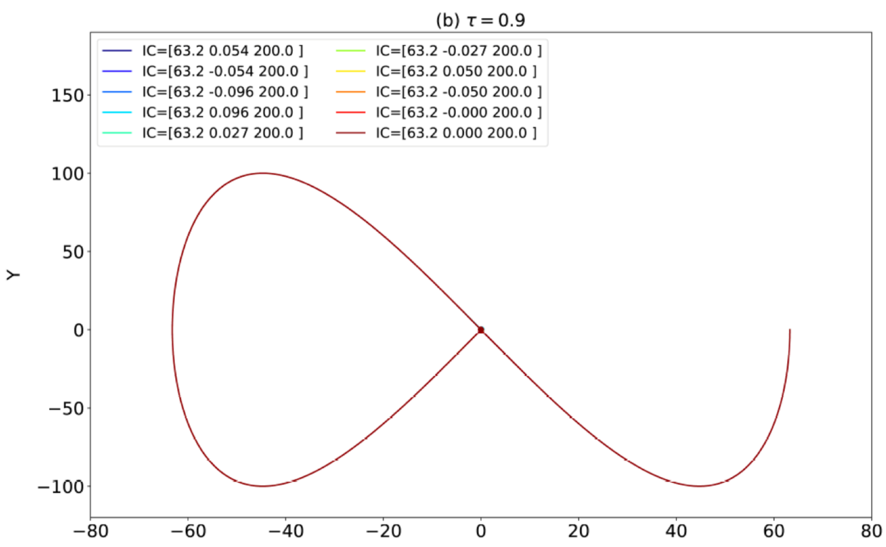
KdV,
simplified SIR

Shen, 2020

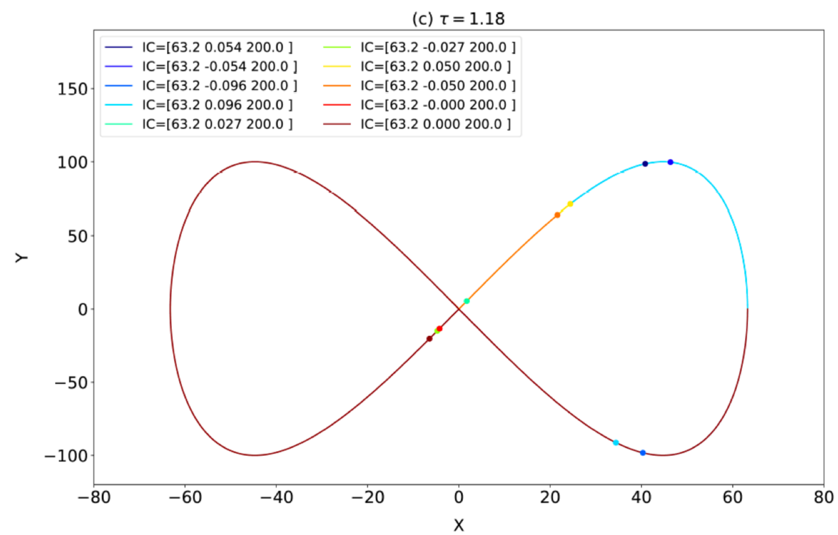
Sensitivity to Initial Conditions (Limited Chaos) **Supp**



https://goo.gl/EUiFX1

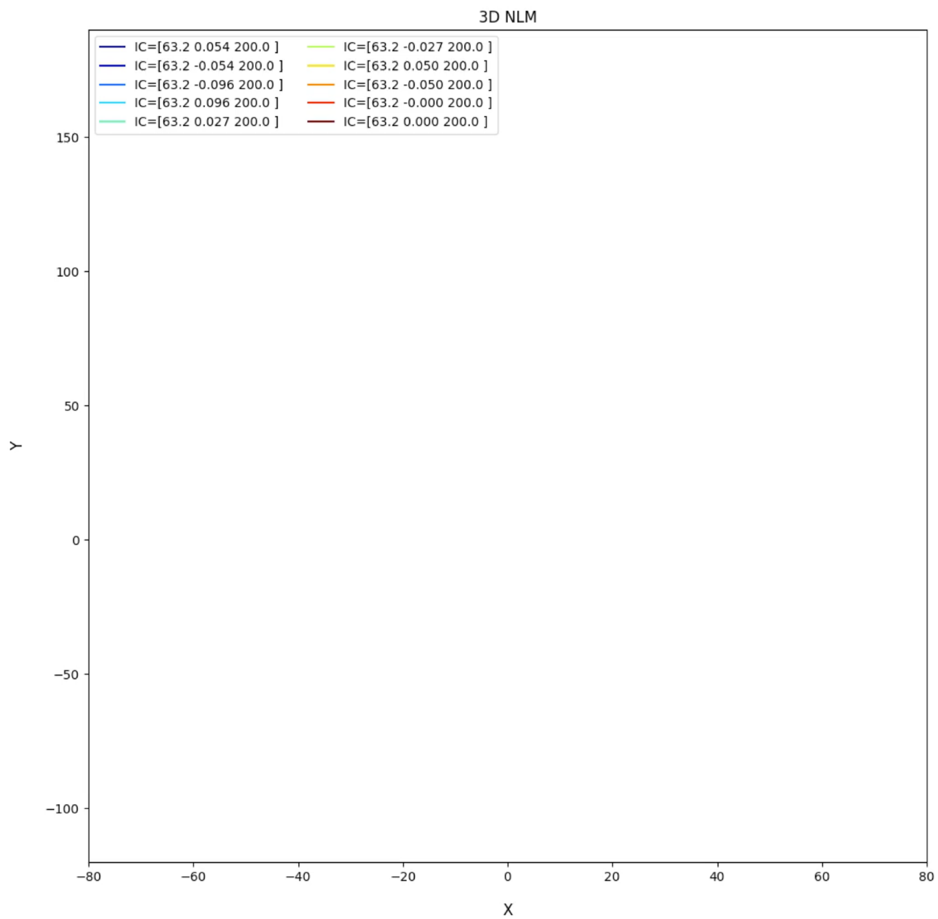


divergence of
initial nearby trajectories



Sensitive Dependence on ICs ($\epsilon = 0$) **Supp**

- Two trajectories with starting points near the homoclinic orbit may be diverged.
- The animation shows 10 orbits initially released near a homoclinic orbit.



CDIC vs. SDIC

Supp

As known, there is no universal definition for chaos. On the other hand, it is accepted that sensitive dependence on ICs is one of the major features for chaos, as shown below from page 8 of Lorenz (1993):

> sequence. Systems in which this is the case are said to be *sensitively dependent on initial conditions*. With a few more qualifications, to be considered presently, sensitive dependence can serve as an acceptable definition of chaos, and it is the one that I shall choose.

To improve our understanding of the dependence of the predictability on ICs, as well as initial errors, it is important to understand the differences of the following two terms: (1) continuous dependence on ICs (CDIC) and (2) sensitive dependence on ICs (SDIC). As shown in Table R4, we briefly discuss both terms and present major features of the Lorenz 1963 model. Two terms are defined as follows (see details in Shen 2017, lecture notes):

- Continuous dependence on ICs: solutions through nearby ICs remain close over short time intervals. Mathematically, $|X(t) - Y(t)| < |X(t = t_o) - Y(t = t_o)|e^{K(t - t_o)}$.
- Sensitive dependence on ICs: "*The property characterizing an orbit if most other orbits that pass close to it at some point do not remain close to it as time advances*". *(Lorenz, 1993). Mathematically, given* δ*, we may find n such that* $|X(t = t_n) - Y(t = t_n)| > \delta$ *(e.g., Drazin, 1992).*

A Need of 3D DE systems for Chaos

Supp

- Here, it should be noted that “the most common examples of chaos in continuous-time system come from three first order ODEs with no explicit time dependence (Sprott 2010)”.
- Second order forced ODEs (e.g., for a forced Pendulum), that can be represented using three-dimensional or higher autonomous systems, can possess chaos.
- Similarly, on page 441, Strogatz (2015) pointed out that “Neighboring trajectories separate by spiraling out (“stretching”), then cross without intersecting by going into the third dimension (“folding”) and then circulate back near their starting places (“re-injection”). We can now see why three dimensions are needed for a flow to be chaotic.”

Fundamental Concepts

1. **Existence:** Each point in the (t, x)-plane has a solution passing through it. The solution has slope given by the differential equation at that point.
2. **Uniqueness:** Only one solution passes through any particular (t, x).
3. **Continuous dependence:** Solutions through nearby initial conditions remain close over short time intervals. In other words, the flow $F(t, x_0)$ is a continuous function of x_0 as well as t. $|X(t) - Y(t)| < |X_0 - Y_0| e^{K(t-t_0)}$

f & f_y bounded, a convex set ➔ (1) & (2)

- *"SDIC" means that nearby trajectories separate exponentially fast, i.e., the system has a positive Liapunov exponent. (Strogatz, p331)*
- *We may define an infinite invariant set of a map $F: R \rightarrow R$ to have sensitive dependence on initial conditions if there exists $\delta > 0$ such that for all $x \in S$ and all neighbourhoods N (however small) of x there exists $y \in N$ and $n > 0$ such that $|F^n(x) - F^n(y)| > \delta$. So neighbouring orbits, however close initially, separate from one another, although each keeps close to the invariant set. (Drazin, p140; Devaney, p49)*

Alligood et al.

CDIC vs. SDIC

	CDIC	SDIC
Definition	solutions through nearby ICs remain close over short time intervals	*The property for an orbit when most other orbits that pass close to it at some point do not remain close to it as time advances.*

- nearby trajectories diverge no faster than an exponential rate
- nearby trajectories separate exponentially fast

- *Continuous dependence on initial conditions (CDIC)*
- *"Sensitive dependence on initial conditions" (SDIC) means that nearby trajectories separate exponentially fast, i.e., the system has a positive Liapunov exponent. (Strogatz, p331)*
- *We may define an infinite invariant set of a map $F: R \to R$ to have sensitive dependence on initial conditions if there exists $\delta > 0$ such that for all $x \in S$ and all neighbourhoods N (however small) of x there exists $y \in N$ and $n > 0$ such that $|F^n(x) - F^n(y)| > \delta$. So neighbouring orbits, however close initially, separate from one another, although each keeps close to the invariant set. (Drazin, p140; Devaney, p49)*

MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #12

Linearization Theorem

Last updated: 2020/09/24

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Outline

1. Existence and Uniqueness Theorems
2. Lipschitz Condition
3. Continuous Dependence of Solutions on Initial Conditions (CDIC)
4. Sensitive Dependence of Solutions on Initial Conditions (SDIC)
5. Linearization Theorems & Linearized Systems

2D Linear vs. Nonlinear Systems

In Chapters 2-4, we consider the following linear 2D system:

$$x' = ax + by \quad (= P(x, y)) \qquad (1)$$

$$y' = cx + dy \quad (= Q(x, y)\ ) \qquad (2)$$

$$X' = AX \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

1. Eigenvalue problem: $|A - \lambda I| = 0$
2. Two linearly independent solutions, $e^{\lambda_1 t} V_1$ and $e^{\lambda_2 t} V_2$
3. Real eigenvalues for a source, sink, or saddle
4. Complex eigenvalues for a center, spiral sink or spiral source

In the following, we consider the following nonlinear 2D system:

$$x' = F(x, y) \qquad (3)$$

$$y' = G(x, y) \qquad (4)$$

Hyperbolic

Definition
A matrix A is *hyperbolic* if none of its eigenvalues has real part 0. We also say that the system $X' = AX$ is *hyperbolic.*

$$\boxed{Re(\lambda_j) \neq 0}$$

Hyperbolicity

If $\text{Re}(\lambda) \neq 0$ for both eigenvalues, the fixed point is often called ***hyperbolic***. (This is an unfortunate name—it sounds like it should mean "saddle point"—but it has become standard.) Hyperbolic fixed points are sturdy; their stability type is unaffected by small nonlinear terms. Nonhyperbolic fixed points are the fragile ones.

We've already seen a simple instance of hyperbolicity in the context of vector fields on the line. In Section 2.4 we saw that the stability of a fixed point was accurately predicted by the linearization, *as long as* $f'(x^*) \neq 0$. This condition is the exact analog of $\text{Re}(\lambda) \neq 0$.

Strogatz (2015), p156

Linearization Theorem

THEOREM 6.2.3
Let $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ be a nonlinear system of n first-order equations with $\mathbf{x}^*$ as an equilibrium solution and $\mathbf{f}$ a sufficiently smooth vector function. If $\mathrm{Re}(\lambda_i) \neq 0$ for all i, then the predictions given by the linear stability results of Theorem 6.2.2 hold for the equilibrium solution in the nonlinear system.

Hyperbolic

Wirkus and Swift

Conjugacy and Linearization Theorem

We can now conjugate the flow of a nonlinear system near a hyperbolic equilibrium point that is a sink to the flow of its linearized system. Indeed, the argument used in the second example of the previous section goes over essentially unchanged. In similar fashion, nonlinear systems near a hyperbolic source are also conjugate to the corresponding linearized system.

This result is a special case of the following more general theorem.

The Linearization Theorem. *Suppose the n-dimensional system $X' = F(X)$ has an equilibrium point at X_0 that is hyperbolic. Then the nonlinear flow is conjugate to the flow of the linearized system in a neighborhood of X_0.* ■

a.k.a. Hartman–Grobman Theorem

Dynamical equivalence

Hartman–Grobman Theorem

An underlying theme throughout the first chapter of this book was that the orbit structure near a hyperbolic fixed point was qualitatively the same as the orbit structure given by the associated linearized dynamical system. A theorem proved independently by Hartman [1960] and Grobman [1959] makes this precise. We will describe the situation for vector fields.

Consider a $\mathbf{C}^r$ $(r \geq 1)$ vector field

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n, \tag{19.12.18}$$

where f is defined on a sufficiently large open set of $\mathbb{R}^n$. Suppose that (19.12.18) has a *hyperbolic* fixed point at $x = x_0$, i.e.,

$$f(x_0) = 0,$$

and $Df(x_0)$ has no eigenvalues on the imaginary axis. Consider the associated linear vector field

$$\dot{\xi} = Df(x_0)\xi, \qquad \xi \in \mathbb{R}^n. \tag{19.12.19}$$

Then we have the following theorem.

Theorem 19.12.6 (Hartman and Grobman) *The flow generated by (19.12.18) is* $\mathbf{C}^0$ conjugate *to the flow generated by (19.12.19) in a neighborhood of the fixed point* $x = x_0$.

nonlinear system

linearized system

C^0 conjugate

Wiggins (2003)

Hartmann-Grobmann Theorem

By the Hartmann-Grobmann Theorem, see Hartman [12], a

nonlinear system

$$\dot{x} = f(x)$$

is topologically conjugate to its linearization

$$\dot{y} = Df(x)y, \quad y = x - \bar{x},$$

at a hyperbolic stationary solution $\bar{x}$. Then the Rectification Theorem 3.2, see Arnold [2], says that the flow on the linear unstable manifold E^u and the linear stable manifold E^s are topologically equivalent to the flows of

$$\dot{x}_u = x_u, \text{ and } \dot{x}_s = -x_s$$

respectively, where $x_u = p_u \circ x$ and $x_s = p_s \circ x$ denote the projections onto E^u and E^s. This proves the following theorem.

Birnir, UCSB

Hartman–Grobman Theorem

Theorem 1.3.1 (Hartman–Grobman). *If $Df(\bar{x})$ has no zero or purely imaginary eigenvalues then there is a homeomorphism h defined on some neighborhood U of $\bar{x}$ in $\mathbb{R}^n$ locally taking orbits of the nonlinear flow ϕ_t of* (1.3.1), *to those of the linear flow $e^{tDf(\bar{x})}$ of* (1.3.2). *The homeomorphism preserves the sense of orbits and can also be chosen to preserve parametrization by time.*

A more delicate situation in which the nonlinear and linear flows are related via *diffeomorphisms* (Sternberg's theorem) requires certain nonresonance conditions among the eigenvalues of $Df(\bar{x})$. We shall not consider this here, but see the discussion of normal forms in Chapter 3.

When $Df(\bar{x})$ has no eigenvalues with zero real part, $\bar{x}$ is called a *hyperbolic* or *nondegenerate* fixed point and the asymptotic behavior of solutions near it (and hence its stability type) is determined by the linearization. If any one of the eigenvalues has zero real part, then stability cannot be determined by linearization.

Guckenheimer and Holmes (1983)

Hartman–Grobman Theorem

Theorem 2.2.1 (Hartman–Grobman) *Let* $\mathbf{x}^*$ *be a hyperbolic fixed point of the diffeomorphism* $\mathbf{f}: U \to \mathbb{R}^n$. *Then there is a neighbourhood* $N \subseteq U$ *of* $\mathbf{x}^*$ *and a neighbourhood* $N' \subseteq \mathbb{R}^n$ *containing the origin such that* $\mathbf{f}|N$ *is topologically conjugate to* $\mathbf{Df}(\mathbf{x}^*)|N'$.

Arrowsmitch and Place

Hartman–Grobman Theorem

In mathematics, in the study of dynamical systems, the **Hartman–Grobman theorem** or **linearization theorem** is a theorem about the local behavior of dynamical systems in the neighbourhood of a hyperbolic equilibrium point. It asserts that linearization—a natural simplification of the system—is effective in predicting qualitative patterns of behavior.

The theorem states that the behavior of a dynamical system in a domain near a hyperbolic equilibrium point is qualitatively the same as the behavior of its linearization near this equilibrium point, where hyperbolicity means that no eigenvalue of the linearization has real part equal to zero. Therefore, when dealing with such dynamical systems one can use the simpler linearization of the system to analyze its behavior around equilibria.[1]

(wikipedia)

Hartman–Grobman Theorem

Main theorem [edit]

Consider a system evolving in time with state $u(t) \in \mathbb{R}^n$ that satisfies the differential equation $du/dt = f(u)$ for some smooth map $f : \mathbb{R}^n \to \mathbb{R}^n$. Suppose the map has a hyperbolic equilibrium state $u^* \in \mathbb{R}^n$: that is, $f(u^*) = 0$ and the Jacobian matrix $A = [\partial f_i / \partial x_j]$ of f at state u^* has no eigenvalue with real part equal to zero. Then there exists a neighborhood N of the equilibrium u^* and a homeomorphism $h : N \to \mathbb{R}^n$, such that $h(u^*) = 0$ and such that in the neighbourhood N the flow of $du/dt = f(u)$ is topologically conjugate by the continuous map $U = h(u)$ to the flow of its linearization $dU/dt = AU$.[2][3][4][5]

Even for infinitely differentiable maps f, the homeomorphism h need not to be smooth, nor even locally Lipschitz. However, it turns out to be Hölder continuous, with an exponent depending on the constant of hyperbolicity of A.[6]

The Hartman–Grobman theorem has been extended to infinite dimensional Banach spaces, non-autonomous systems $du/dt = f(u, t)$ (potentially stochastic), and to cater for the topological differences that occur when there are eigenvalues with zero or near-zero real-part.[7][8][9][10]

(wikipedia)

Hyperbolicity ($Re(\lambda) \neq 0$) and Linearization Theorem

- Thus the dynamics near a hyperbolic fixed point are structurally stable, while the non-hyperbolic fixed is not structurally stable.
- This assures that the local linearization is a valid approximation for hyperbolic fixed points in any number of dimensions.
- (in other words) the validity of local linearization of nonlinear dynamics near an equilibrium is guaranteed only in the generic case of hyperbolic fixed point. (p 200)
- (p 206) For example, a generalized hyperbolic structure, the horseshoe map, enabled Smale to prove that what is called chaos may be structurally stable.
- (p 206) In higher-dimensional phase space, non-hyperbolic structure can be a more serious problem.

Thompson and Stewart, p 198

Hyperbolicity, Linearization, and Homemorphism

These ideas also generalize neatly to higher-order systems. A fixed point of an nth-order system is *hyperbolic* if all the eigenvalues of the linearization lie off the imaginary axis, i.e., $\text{Re}(\lambda_i) \neq 0$ for $i = 1, \ldots, n$. The important *Hartman–Grobman theorem* states that the local phase portrait near a hyperbolic fixed point is "topologically equivalent" to the phase portrait of the linearization; in particular, the stability type of the fixed point is faithfully captured by the linearization. Here ***topologically equivalent*** means that there is a ***homeomorphism*** (a continuous deformation with a continuous inverse) that maps one local phase portrait onto the other, such that trajectories map onto trajectories and the sense of time (the direction of the arrows) is preserved.

Homeomorphism: a continuous deformation with a continuous inverse.

Strogatz (2015), p156

Hyperbolicity, Linearization, Homemorphism

Intuitively, two phase portraits are topologically equivalent if one is a distorted version of the other. Bending and warping are allowed, but not ripping, so closed orbits must remain closed, trajectories connecting saddle points must not be broken, etc.

Hyperbolic fixed points also illustrate the important general notion of structural stability. A phase portrait is ***structurally stable*** if its topology cannot be changed by an arbitrarily small perturbation to the vector field. For instance, the phase portrait of a saddle point is structurally stable, but that of a center is not: an arbitrarily small amount of damping converts the center to a spiral.

non-hyperbolic

- A phase portrait is structurally stable if its topology cannot be changed by an arbitrarily small perturbation to the vector field.
- The phase portrait of a saddle point is structurally stable, but
- the phase of portrait of a center is not.

Strogatz (2015), p156

Linearization: An Illustration

Consider the following system of first-order ODEs

$$x' = F(x,y) \qquad\qquad y' = G(x,y)$$

Find critical points

$$F(x_c,y_c) = 0 \qquad\qquad G(x_c,y_c) = 0$$

Linearize F and G with respect to one of the critical points

$$F(x,y) = F(x_c,y_c) + \boxed{F_x(x_c,y_c)(x-x_c) + F_y(x_c,y_c)(y-y_c)}$$

$$G(x,y) = G(x_c,y_c) + \boxed{G_x(x_c,y_c)(x-x_c) + G_y(x_c,y_c)(y-y_c)}$$

Express the above in a matrix form

$$\begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix} = \boxed{\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{x_c,y_c} \begin{pmatrix} x-x_c \\ y-y_c \end{pmatrix}} = J(F,G)\begin{pmatrix} x-x_c \\ y-y_c \end{pmatrix}$$

Example

Consider the following system of first-order ODEs

$$x' = ax + by \quad (= F(x, y)) \qquad (1)$$

$$y' = cx + dy \quad (= G(x, y)\ ) \qquad (2)$$

$$X = \binom{x}{y}$$

Find critical points

$$x_c = 0 \qquad\qquad y_c = 0$$

Compute the Jacobian matrix at a critical point

$$J(F, G) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{x_c, y_c} =$$

- Find F_x, F_y, G_x, and G_y
- Send your results via ”chat”
- You have 3 minutes

Example

Consider the following system of first-order ODEs

$$x' = ax + by \quad (= F(x, y)) \qquad (1)$$

$$y' = cx + dy \quad (= G(x, y)\ ) \qquad (2)$$

$$X = \binom{x}{y}$$

Find critical points

$$x_c = 0 \qquad\qquad y_c = 0$$

Compute the Jacobian matrix at a critical point

$$J(F, G) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{x_c, y_c} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \text{“}A\text{”}$$

Linearization: An Illustration

Consider the following system of first-order ODEs

$$\boxed{x' = F(x,y) \qquad\qquad y' = G(x,y)} \qquad (1)$$

Express the above in a matrix form with a Jacobian matrix

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{x_c, y_c} \begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix} = J(F,G) \begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix} \qquad (2)$$

Define perturbations, u and v, with respect to their critical point

$$u = x - x_c \qquad\qquad v = y - y_c \qquad (3)$$

Obtain

$$u' = x' \qquad\qquad v' = y' \qquad (4)$$

Plug Eqs. (3) and (4) into Eq. (2), yielding:

$$\boxed{\begin{pmatrix} u' \\ v' \end{pmatrix} = J(F,G) \begin{pmatrix} u \\ v \end{pmatrix}} \qquad\qquad J(F,G) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{x_c, y_c} \qquad (5)$$

Linear Stability Analysis

Consider the following system of first-order ODEs

$$\boxed{x' = F(x,y) \qquad\qquad y' = G(x,y)}$$

Find critical points

$$F(x_c, y_c) = 0 \qquad\qquad G(x_c, y_c) = 0$$

Compute the Jacobian matrix and evaluate it at a critical point

$$J(F,G) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{x_c, y_c}$$

Solve an eigenvalue problem:

$$JV = \lambda V \qquad V = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\boxed{|J - \lambda I| = 0}$$

Linear Stability and Jacobian

THEOREM 6.2.2

Let $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ be a nonlinear system of n first-order equations with $\mathbf{x}^*$ as an equilibrium solution and $\mathbf{f}$ a sufficiently smooth vector function. Let $\mathbf{J}$ be the Jacobian (the matrix of partial derivatives) evaluated at this equilibrium solution:

$$\mathbf{J}(\mathbf{x}^*) = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 & \cdots & \partial f_1/\partial x_n \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 & \cdots & \partial f_2/\partial x_n \\ \vdots & & \ddots & \\ \partial f_n/\partial x_1 & \partial f_n/\partial x_2 & \cdots & \partial f_n/\partial x_n \end{bmatrix}_{\mathbf{x}=\mathbf{x}^*}. \tag{6.13}$$

Let $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ be the n (real or complex, possibly repeated) eigenvalues of the Jacobian matrix.

a. If the real part of the eigenvalue $\text{Re}(\lambda_i) < 0$ for all i, then the equilibrium is stable.

b. If the real part of the eigenvalue $\text{Re}(\lambda_i) < 0$ for at least one i and $\text{Re}(\lambda_j) > 0$ for at least one j, then the equilibrium is a saddle.

c. If the real part of the eigenvalue $\text{Re}(\lambda_i) > 0$ for all i, then the equilibrium is unstable.

d. If any of the eigenvalues are complex, then the stable or unstable equilibria is a spiral; if all of the eigenvalues are real, it is a node.

e. If a pair of complex conjugate eigenvalues $\lambda_i, \overline{\lambda_i}$ satisfy $\text{Re}(\lambda_i) = 0$, then the equilibrium is a linear center in the plane containing the corresponding eigenvectors.

Wirkus and Swift

Derivatives of $f(x)$, $f(x, y)$ and $(F(x, y), G(x, y))$

$f(x)$: $\quad \dfrac{df}{dx}$

$f(x,y)$: $\quad \dfrac{\partial f}{\partial x}$ & $\dfrac{\partial f}{\partial y}$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

Gradient

$F(x,y)$ $\quad \dfrac{\partial F(x,y)}{\partial x}$ & $\dfrac{\partial F(x,y)}{\partial y}$

$G(x,y)$ $\quad \dfrac{\partial G(x,y)}{\partial x}$ & $\dfrac{\partial G(x,y)}{\partial y}$

$$J(F,G) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}$$

Jacobian

Example: A Linearized Lorenz Model

$$\frac{dX}{dt} = -\sigma X + \sigma Y,$$
$$\frac{dY}{dt} = rX - Y.$$

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \qquad A = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix}$$

Complete the following to obtain the Jacobian matrix:

$$F(X_c, Y_c) = 0 \implies X_c = Y_c = 0$$

$$J(F,G) = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{x_c, y_c} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix}$$

Example: 3-Dimenaional Lorenz Model (3DLM)

1) **r** – Rayleigh number: (Ra/Rc)

a dimensionless measure of temperature difference between top and bottom surfaces of liquid; proportional to effective force on fluid

2) **σ** – Prandtl number: (v/κ)

the ratio of the kinetic viscosity (κ, momentum diffusivity) to thermal diffusivity (v)

3) **b** – Physical proportion: $(4/(1+a^2))$, b=8/3.

4) **a** – a=l/m, the ratio of the vertical height h of the fluid layer to the horizontal size of the convection rolls. b =8/3.

- l=aπ/H and m=π/H,

$$\frac{dX}{d\tau} = -\sigma X + \sigma Y, \qquad M_1$$

$$\frac{dY}{d\tau} = -XZ + rX - Y, \qquad M_2$$

$$\frac{dZ}{d\tau} = XY - bZ. \qquad M_3$$

-XZ is associated with the J(M1, M3), indicating the impact of the M3 mode. With no –XZ, the above system is reduced to become a system with linear terms only, leading to an unstable solution as r>1.

Three Dimensional Systems

$$\frac{dx}{dt} = F(x,y,z)$$

$$\frac{dy}{dt} = G(x,y,z)$$

$$\frac{dz}{dt} = H(x,y,z)$$

$$J = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} & \frac{\partial H}{\partial z} \end{pmatrix}_{(x_c, y_c, z_c)} = A$$

$$x = x_c + \boxed{x_1}$$

$$y = y_c + \boxed{y_1}$$

$$z = z_c + \boxed{z_1}$$

$$F(x_c, y_c, z_c) = 0$$

$$G(x_c, y_c, z_c) = 0$$

$$H(x_c, y_c, z_c) = 0$$

Example: The Lorenz Model

$$F(X,Y,Z) = \sigma Y - \sigma X$$

$$G(X,Y,Z) = -XZ + rX - Y$$

$$H(X,Y,Z) = XY - bZ$$

$$\begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} & \frac{\partial H}{\partial z} \end{pmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ -Z + r & -1 & -X \\ Y & X & -b \end{bmatrix}$$

A Locally Linear System

$$\begin{pmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dy_1}{dt} \\ \dfrac{dz_1}{dt} \end{pmatrix} = \boxed{\begin{pmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} & \dfrac{\partial F}{\partial z} \\ \dfrac{\partial G}{\partial x} & \dfrac{\partial G}{\partial y} & \dfrac{\partial G}{\partial z} \\ \dfrac{\partial H}{\partial x} & \dfrac{\partial H}{\partial y} & \dfrac{\partial H}{\partial z} \end{pmatrix}_{(x_c, y_c, z_c)}} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$$\frac{d\vec{U}}{dt} = \boxed{A}\vec{U}; \quad \vec{U} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \qquad assume \quad \vec{U} = e^{\lambda t} \begin{pmatrix} x_{eigen} \\ y_{eigen} \\ z_{eigen} \end{pmatrix} = e^{\lambda t} \vec{V}$$

A Locally Linear System

$$\frac{dx}{dt} = F(x,y,z)$$

$$\frac{dy}{dt} = G(x,y,z)$$

$$\frac{dz}{dt} = H(x,y,z)$$

$$J = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} & \frac{\partial H}{\partial z} \end{pmatrix}_{(x_c, y_c, z_c)} = \boxed{A}$$

$$A\vec{V} = \lambda\vec{V}; \quad \vec{V} = \begin{pmatrix} x_{eigen} \\ y_{eigen} \\ z_{eigen} \end{pmatrix}$$

$$F(x_c, y_c, z_c) = 0$$

$$G(x_c, y_c, z_c) = 0$$

$$H(x_c, y_c, z_c) = 0$$

Example: Jacobian Matrix for the 3DLM

$$\frac{dX}{d\tau} = -\sigma X + \sigma Y, \qquad F(X_c, Y_c, Z_c) = 0 \;\Rightarrow\; X_c = Y_c$$

$$\frac{dY}{d\tau} = -XZ + rX - Y, \qquad G(X_c, Y_c, Z_c) = 0 \;\Rightarrow\; Z_c = r - 1$$

$$\frac{dZ}{d\tau} = XY - bZ. \qquad H(X_c, Y_c, Z_c) = 0 \;\Rightarrow\; X_c = \pm\sqrt{b(r-1)}$$

$$J(F,G,H) = \left.\begin{pmatrix} \frac{\partial(1)}{\partial x} & \frac{\partial(1)}{\partial y} & \frac{\partial(1)}{\partial z} \\ \frac{\partial(2)}{\partial x} & \frac{\partial(2)}{\partial y} & \frac{\partial(2)}{\partial z} \\ \frac{\partial(3)}{\partial x} & \frac{\partial(3)}{\partial y} & \frac{\partial(3)}{\partial z} \end{pmatrix}\right|_{(x_c,\, y_c,\, z_c)} = \begin{pmatrix} -\sigma, & \sigma, & 0 \\ r - Z_c & -1 & -X_c \\ Y_c & X_c & -b \end{pmatrix}$$

Determine the Jacobian using Matlab

```
syms x y z r  theta real

x=r * cos(theta)
y=r * sin(theta)
vec1=[x, y]
vec2=[r, theta]
ja=jacobian(vec1, vec2)
jadet=abs(det(ja))
simplify(jadet)
```

➔

```
ans =
abs(r)
```

$$|J| = \mathrm{r}$$

```
syms x y z rho phi theta real

assume (phi > 0)

x=rho * sin(phi) * cos(theta)
y=rho * sin(phi) * sin(theta)
z=rho * cos(phi)
vec1=[x, y, z]
vec2=[rho, theta, phi]
ja=jacobian(vec1, vec2)
jadet=abs(det(ja))
simplify(jadet)
```

➔

```
ans =
rho^2*abs(sin(phi))
```

$$|J| = \rho^2 \sin(\phi)$$

Perturbation Theory

Perturbation theory is a large collection of iterative methods for obtaining approximate solutions to problems involving a small parameter ε.

Three typical steps of perturbative analysis are:

1. Convert the original problem into a perturbation problem by introducing the small parameter ε.
2. Assume an expression for the answer in the form of a perturbation series and compute the coefficients of that series.
3. Recover the answer to the original problem by summing the perturbation series for the appropriate value of ε.

3D Lorenz Model (3DLM)

$$\frac{dX}{d\tau} = \sigma Y - \sigma X = F,$$

$$\frac{dY}{d\tau} = -XZ + rX - Y = G,$$

$$\frac{dZ}{d\tau} = XY - bZ = H.$$

$$X = X_c + \epsilon X'$$
$$Y = Y_c + \epsilon Y'$$
$$Z = Z_c + \epsilon Z'$$

reference (or basic) state

perturbations

From Eqs. (1-3), the Jacobian matrix is written as follows:

$$A_1 = \begin{pmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \\ H_x & H_y & H_z \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - Z_c & -1 & -X_c \\ Y_c & X_c & -b \end{pmatrix}_{(X_c, Y_c, Z_c)} .$$

3DLM: Eqs for Perturbations (ε^1)

Supp

ε^1:

$$\frac{dX'}{d\tau} = \sigma Y' - \sigma X'$$

$$\frac{dY'}{d\tau} = -Z_c X' + r X' - Y' - X_c Z'$$

$$\frac{dZ'}{d\tau} = X' Y_c + X_c Y' - b Z'$$

$$A_2 = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - Z_c & -1 & -X_c \\ Y_c & X_c & -b) \end{pmatrix}$$

- The above derivations show that the Jacobian matrix A1 in the (nonlinear) 3DLM is the same as the matrix A2.
- In other words, the system with the equations associated with ε^1 represents the locally linearized equations of the 3DLM.

3DLM: Eqs for the Basic/Reference State (ε^0) Supp

Plugging the above into Eqs. (1-3), we have

$$\frac{dX_c}{d\tau} + \epsilon\frac{dX'}{d\tau} = \sigma(Y_c + \epsilon Y') - \sigma(X_c + \epsilon X') \tag{4}$$

$$\frac{dY_c}{d\tau} + \epsilon\frac{dY'}{d\tau} = -(X_cZ_c + \epsilon X_cZ' + \epsilon Z_cX' + \epsilon^2X'Z') + r(X_c + \epsilon X') - (Y_c + \epsilon Y') \tag{5}$$

$$\frac{dZ_c}{d\tau} + \epsilon\frac{dZ'}{d\tau} = (X_cY_c + \epsilon X_cY' + \epsilon Y_cX' + \epsilon^2X'Y') - b(Z_c + \epsilon Z') \tag{6}$$

ε^0:

$$\cancel{\frac{dX_c}{d\tau}} = \sigma Y_c - \sigma X_c$$

$$\cancel{\frac{dY_c}{d\tau}} = -X_cZ_c + rX_c - Y_c$$

$$\cancel{\frac{dZ_c}{d\tau}} = X_cY_c - bZ_c$$

A Perturbation Method

For the "linear" case (FN=0), we have

$$\frac{dX'}{d\tau} = -\sigma X' + \sigma Y',$$

$$\frac{dY'}{d\tau} = (r - Z_c)X' - Y' - X_c Z'$$

$$\frac{dZ'}{d\tau} = Y_c X' + X_c Y' - bZ'$$

$$X = X_c + X'$$

$$Y = Y_c + Y'$$

$$Z = Z_c + Z'$$

ODE Solver

and

$$\begin{pmatrix} \dfrac{dX'}{d\tau} \\ \dfrac{dY'}{d\tau} \\ \dfrac{dZ'}{d\tau} \end{pmatrix} = \begin{pmatrix} -\sigma, & \sigma, & 0 \\ r - Z_c & -1 & -X_c \\ Y_c & X_c & -b \end{pmatrix} \begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix}$$

Eigenvalue Analysis

A Perturbative Analysis of Van del Pol Eq.

Consider the following Van der Pol equation

$$\frac{d^2X}{dt^2} + b(X^2 - 1)\frac{dX}{dt} + X = 0, \tag{1}$$

which can be written as follows:

$$\frac{d^2X}{dt^2} + X = \epsilon\frac{dX}{dt} - \epsilon X^2\frac{dX}{dt}, \tag{2}$$

where ϵ is introduced to replace b in order to perform a perturbative analysis.
We seek a first-order expansion for the solution in the form

$$X = X_o + \epsilon X_1 + \dots \tag{3}$$

Plugging the above into Eq. (2), we have

$$\left(X_o'' + \epsilon X_1''\right) + (X_o + \epsilon X_1) = \epsilon\left(\frac{dX_o}{dt} + \epsilon\frac{dX_1}{dt}\right) - \epsilon\,(X_o + \epsilon X_1)^2\left(\frac{dX_o}{dt} + \epsilon\frac{dX_1}{dt}\right). \tag{4}$$

A Perturbative Analysis of the Solution

$$\boxed{\left(X_o''\right.} + \epsilon X_1''\Big) + \boxed{(X_o} + \epsilon X_1) = \epsilon\left(\frac{dX_o}{dt} + \epsilon\frac{dX_1}{dt}\right) - \epsilon\left(X_o + \epsilon X_1\right)^2\left(\frac{dX_o}{dt} + \epsilon\frac{dX_1}{dt}\right). \tag{4}$$

Considering terms with ϵ^0, we have

$$\boxed{X_o'' + X_o = 0.} \tag{5}$$

A Perturbative Analysis of the Solution

$$\left(X_o'' + \boxed{\epsilon X_1''}\right) + (X_o + \boxed{\epsilon X_1}) = \epsilon \left(\boxed{\frac{dX_o}{dt}} + \epsilon \frac{dX_1}{dt}\right) - \epsilon \left(\boxed{X_o} + \epsilon X_1\right)^2 \left(\boxed{\frac{dX_o}{dt}} + \epsilon \frac{dX_1}{dt}\right). \tag{4}$$

Considering terms with ϵ^0, we have

$$X_o'' + X_o = 0. \tag{5}$$

Considering terms with ϵ^1, we have

$$\boxed{X_1'' + X_1} = \boxed{X_o'} - \boxed{X_o^2 X_o'}. \tag{6}$$

X_0 acts as a forcing term

From Eq. (5), we have the solution of X_o as follows:

$$X_o = acos(t + \beta), \tag{7}$$

A Perturbative Analysis of the Solution

$$X_1'' + X_1 = \left(-a + \frac{1}{4}a^3\right) sin(t+\beta) + \frac{1}{4}a^3 sin(3(t+\beta)). \qquad (12)$$

Consider $X_1 = u + v$, which satisfy the following:

$$u'' + u = \left(-a + \frac{1}{4}a^3\right) sin(t+\beta), \qquad (13)$$

$$v'' + v = \frac{a^3}{4} sin(3(t+\beta)). \qquad (14)$$

A particular solution of Eq. (13) is:

$$u_p = \frac{at}{2}\left(1 - \frac{1}{4}a^2\right) cos(t+\beta), \qquad (15)$$

A particular solution of Eq. (14) is:

$$v_p = \frac{1}{32}a^3 sin(3(t+\beta)). \qquad (16)$$

A red circle indicates a nonlinear term.

A Perturbative Analysis of the Solution

From Eqs. (3), (7), (15-16), we have

$$X = acos(t+\beta) + \epsilon \left[\frac{at}{2} \left(1 - \frac{1}{4} a^2 \right) cos(t+\beta) + \frac{1}{32} a^3 sin(3(t+\beta)) \right] + \cdot \quad (17)$$

As a result of the presence of the mixed-secular term, the above expansion is non-uniform for $t \geq O(\epsilon^{-1})$ because the correction term is the order or larger than the first term. The mixed-secular term in Eq. (17) disappears if

$$a(1 - \frac{1}{4}a^2) = 0, \qquad \boxed{\frac{1}{4}a^2 - 1 = 0} \quad (18)$$

leading to $a = 0$ $a = \pm 2$, the latter of which provides an estimate on the amplitude (a). Therefore, the solution becomes

$$X = 2cos(t+\beta) + \frac{1}{4}\epsilon sin(3(t+\beta)) + \cdots \quad (19)$$

Eigenvectors vs. Manifolds

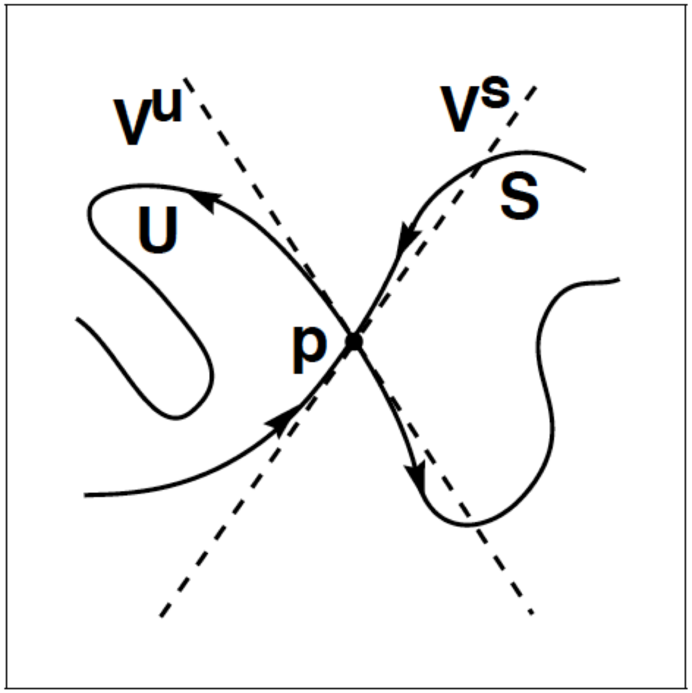

Figure 10.3 Illustration of the Stable Manifold Theorem.
The eigenvector V^s is tangent to the stable manifold S at **p**, and the eigenvector V^u is tangent to the unstable manifold $\mathcal{U}$. The manifolds are curves that can wind through a region infinitely many times. Here we show a finite segment of these manifolds.

Alligood, Sauer and Yorke (p402)