Homework 8
Partial Differential Equations
Math 531
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Excersise 7.7.1: Solve as simply as possible:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

with $u(a,\theta,t) = 0, u(r,\theta,0) = 0$ and $\frac{\partial u}{\partial t}(r,\theta,0) = \alpha(r)\sin 3\theta$

Let the following be true:

$$u(r,\theta,t) = f(r)g(\theta)h(t)$$

From this, and our original equation, we get the following:

$$\frac{d^2 h}{dt^2} = -\lambda c^2 h \qquad \frac{d^2 g}{d\theta^2} = -\mu g \qquad r\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda r^2 - \mu\right) f = 0$$

Solving the second ODE, we get the following eigenvalues and eigenfunctions:

$$\mu_m = m^2 \qquad g(\theta) = a_m \cos m\theta + b_m \sin m\theta$$

Now we consider the third ODE, we use the product rule, substitute our value of μ_m and then using a simple scaling transformation ($z = \sqrt{\lambda} r$):

$$z^2 \frac{d^2 f}{dz^2} + z\frac{df}{dz} + \left(z^2 - m^2\right) f = 0$$

This is the Bessel's Differential Equation with solution:

$$f(z) = c_1 J_m(z) + c_2 Y_m(z) \quad \Longrightarrow \quad f(r) = c_1 J_m(\sqrt{\lambda} r) + c_2 Y_m(\sqrt{\lambda} r)$$

We know that $|f(0)| < \infty$ and that $\lim_{z\to 0} Y_m(z) = \pm\infty$, we have that $c_2 = 0$. Now we use the fact that $f(a) = 0$:

$$f(a) = c_1 J_m(\sqrt{\lambda} a) = 0 \quad \to \quad \lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2$$

where z_{mn} represents the n^{th} zero of $J_m(z)$.

Now we consider the first ODE:

(a) If we let $\lambda = 0$, we get the following:

$$h(t) = c_1 t + c_2 \quad \to \quad h(0) = c_2 = 0 \quad \to \quad h(t) = c_1 t$$

(b) If we let $\lambda > 0$, we get the following:

$$h(t) = c_1 \cos c\sqrt{\lambda_{mn}} t + c_2 \sin c\sqrt{\lambda_{mn}} t \quad \to \quad h(0) = c_1 = 0 \quad \to \quad h(t) = c_2 \sin c\sqrt{\lambda_{mn}} t$$

Notice our solution for $u(r,\theta,t)$:

$$\begin{aligned}u(r,\theta,t) = c_1 t &+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_m\left(\sqrt{\lambda_{mn}}r\right)\cos m\theta \sin\left(c\sqrt{\lambda_{mn}}t\right)\\ &+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{mn} J_m\left(\sqrt{\lambda_{mn}}r\right)\sin m\theta \sin\left(c\sqrt{\lambda_{mn}}t\right)\end{aligned}$$

Now notice the derivative in respect to t:

$$\begin{aligned}\frac{\partial u}{\partial t}(r,\theta,t) = c_1 &+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c\sqrt{\lambda_{mn}} A_{mn} J_m\left(\sqrt{\lambda_{mn}}r\right)\cos m\theta \cos\left(c\sqrt{\lambda_{mn}}t\right)\\ &+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c\sqrt{\lambda_{mn}} B_{mn} J_m\left(\sqrt{\lambda_{mn}}r\right)\sin m\theta \cos\left(c\sqrt{\lambda_{mn}}t\right)\end{aligned}$$

Now we include our last boundary condition:

$$\begin{aligned}\frac{\partial u}{\partial t}(r,\theta,0) = \alpha(r)\sin 3\theta = c_1 &+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c\sqrt{\lambda_{mn}} A_{mn} J_m\left(\sqrt{\lambda_{mn}}r\right)\cos m\theta\\ &+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c\sqrt{\lambda_{mn}} B_{mn} J_m\left(\sqrt{\lambda_{mn}}r\right)\sin m\theta\end{aligned}$$

From this, we get that $c_1 = 0$ for all m with $A_{mn} = 0$ and $B_{mn} = 0$ for all $m \neq 3$. From here, we solve for $\alpha(r)$:

$$\alpha(r) = \sum_{n=1}^{\infty} c\sqrt{\lambda_{3n}} B_{3n} J_3\left(\sqrt{\lambda_{3n}}r\right)$$

Using the orthogonality of the Bessel functions with weight r, we get the following coefficient:

$$B_{3n} = \frac{\int_0^a \alpha(r) J_3\left(\sqrt{\lambda_{3n}}r\right) r\,dr}{c\sqrt{\lambda_{3n}}\int_0^a J_3^2\left(\sqrt{\lambda_{3n}}r\right) r\,dr}$$

Thus, we get the final solution:

$$\boldsymbol{u(r,\theta,t) = \sum_{n=1}^{\infty} B_{3n} J_3\left(\sqrt{\lambda_{3n}}r\right)\sin 3\theta \sin\left(c\sqrt{\lambda_{3n}}t\right)}$$

Excersise 7.7.2a: Solve as simply as possible:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \quad \text{subject to} \quad \frac{\partial u}{\partial r}(a,\theta,t) = 0$$

with initial conditions:

$$u(r,\theta,0) = 0, \qquad \frac{\partial u}{\partial t}(r,\theta,0) = \beta(r)\cos 5\theta$$

Let the following be true:

$$u(r,\theta,t) = f(r)g(\theta)h(t)$$

From this, and our original equation, we get the following:

$$\frac{d^2 h}{dt^2} = -\lambda c^2 h \qquad \frac{d^2 g}{d\theta^2} = -\mu g \qquad r\frac{d}{dr}\left(r\frac{df}{dr}\right) + \left(\lambda r^2 - \mu\right) f = 0$$

Solving the second ODE, we get the following eigenvalues and eigenfunctions:

$$\mu_m = m^2 \qquad g(\theta) = a_m \cos m\theta + b_m \sin m\theta$$

Now we consider the third ODE, we use the product rule, substitute our value of μ_m and then using a simple scaling transformation ($z = \sqrt{\lambda} r$):

$$z^2 \frac{d^2 f}{dz^2} + z\frac{df}{dz} + \left(z^2 - m^2\right) f = 0$$

This is the Bessel's Differential Equation with solution:

$$f(z) = c_1 J_m(z) + c_2 Y_m(z) \quad \Longrightarrow \quad f(r) = c_1 J_m(\sqrt{\lambda} r) + c_2 Y_m(\sqrt{\lambda} r)$$

We know that $|f(0)| < \infty$ and that $\lim_{z\to 0} Y_m(z) = \pm\infty$, we have that $c_2 = 0$. Now we use the fact that $f'(a) = 0$:

$$f'(a) = c_1\sqrt{\lambda_{mn}} J_m'(\sqrt{\lambda_{mn}}a) = 0 \quad \to \quad \lambda_{mn} = \left(\frac{z_{mn}}{a}\right)^2$$

where z_{mn} represents the n^{th} zero of $J_m'(z)$.

(a) If we let $\lambda = 0$, we get the following:

$$h(t) = c_1 t + c_2 \quad \to \quad h(0) = c_2 = 0 \qquad \to \qquad h(t) = c_1 t$$

(b) If we let $\lambda > 0$, we get the following:

$$h(t) = c_1 \cos c\sqrt{\lambda_{mn}}t + c_2 \sin c\sqrt{\lambda_{mn}}t \quad \to \quad h(0) = c_1 = 0 \quad \to \quad h(t) = c_2 \sin c\sqrt{\lambda_{mn}}t$$

Thus we get the following:

$$\begin{aligned}u(r,\theta,t) &= c_1 t + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}r)\cos(m\theta)\left(B_{mn}\sin c\sqrt{\lambda_{mn}}t\right)\\ &+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}r)\sin(m\theta)\left(D_{mn}\sin c\sqrt{\lambda_{mn}}t\right)\\ \frac{\partial u}{\partial t}(r,\theta,t) &= c_1 + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}r)\cos(m\theta)\left(B_{mn}c\sqrt{\lambda_{mn}}\cos c\sqrt{\lambda_{mn}}t\right)\\ &+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}r)\sin(m\theta)\left(D_{mn}c\sqrt{\lambda_{mn}}\cos c\sqrt{\lambda_{mn}}t\right)\end{aligned}$$

Notice the boundary condition:

$$\begin{aligned}\frac{\partial u}{\partial t}(r,\theta,0) = \beta(r)\cos 5\theta &= c_1 + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}r)\cos(m\theta)\left(B_{mn}c\sqrt{\lambda_{mn}}\right)\\ &+ \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}r)\sin(m\theta)\left(D_{mn}c\sqrt{\lambda_{mn}}\right)\end{aligned}$$

From here, we can see that:

$$\beta(r) = \sum_{n=1}^{\infty} J_5(\sqrt{\lambda_{5n}}r)\left(B_{5n}c\sqrt{\lambda_{5n}}\right)$$

With this, we can solve for the following coefficient:

$$B_{5n} = \frac{\int_0^a \beta(r) J_5\left(\sqrt{\lambda_{5n}}r\right) r\,dr}{c\int_0^a J_5^2\left(\sqrt{\lambda_{5n}}r\right) r\,dr}$$

Thus we get:

$$\boldsymbol{u(r,\theta,t) = \sum_{n=1}^{\infty} J_5(\sqrt{\lambda_{5n}}r)\cos(5\theta)\left(B_{5n}\sin c\sqrt{\lambda_{5n}}t\right)}$$

Excersise 7.9.1c: Solve Laplace's equation inside a circular cylinder subject to the boundary conditions

$$u(r,\theta,0)=0, \qquad u(r,\theta,H)=\beta(r)\cos 3\theta, \qquad \frac{\partial u}{\partial r}(a,\theta,z)=0$$

Let the following be true:

$$u(r,\theta,z)=f(r)g(\theta)h(z)$$

From this, and our original equation, we get the following:

$$\frac{d^2h}{dt^2}=\lambda h \qquad \frac{d^2g}{d\theta^2}=-\mu g \qquad r\frac{d}{dr}\left(r\frac{df}{dr}\right)+\left(\lambda r^2-\mu\right)f=0$$

Solving the second ODE, we get the following eigenvalues and eigenfunctions:

$$\mu_m=m^2 \qquad g(\theta)=a_m\cos m\theta+b_m\sin m\theta$$

Now we consider the third ODE, we use the product rule, substitute our value of μ_m and then using a simple scaling transformation ($z=\sqrt{\lambda}r$):

$$z^2\frac{d^2f}{dz^2}+z\frac{df}{dz}+\left(z^2-m^2\right)f=0$$

This is the Bessel's Differential Equation with solution:

$$f(z)=c_1J_m(z)+c_2Y_m(z) \quad\Longrightarrow\quad f(r)=c_1J_m(\sqrt{\lambda}r)+c_2Y_m(\sqrt{\lambda}r)$$

We know that $|f(0)|<\infty$ and that $\lim_{z\to 0}Y_m(z)=\pm\infty$, we have that $c_2=0$. Now we use the fact that $f'(a)=0$:

$$f'(a)=c_1\sqrt{\lambda_{0n}}J_m'(\sqrt{\lambda_{0n}}a)=0 \quad\to\quad \lambda_{mn}=\left(\frac{z_{mn}}{a}\right)^2$$

where z_{mn} represents the n^{th} zero of $J_m'(z)$.

(a) If we let $\lambda=0$, we get the following:

$$h(z)=c_1z+c_2 \quad\to\quad h(0)=c_2=0 \quad\to\quad h(z)=c_1z$$

(b) If we let $\lambda>0$, we get the following:

$$h(z)=c_1\cosh(\sqrt{\lambda_{mn}}z)+c_2\sinh(\sqrt{\lambda_{mn}}z) \quad\to\quad h(0)=c_1=0 \quad\to\quad h(z)=c_2\sinh(\sqrt{\lambda_{mn}}z)$$

Thus we get the following:

$$u(r,\theta,z) = c_1 z + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_m(\sqrt{\lambda_{mn}}r)\cos m\theta \sinh(\sqrt{\lambda_{mn}}z) + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{mn} J_m(\sqrt{\lambda_{mn}}r)\sin m\theta \sinh(\sqrt{\lambda_{mn}}z)$$

Notice the boundary condition:

$$u(r,\theta,H) = \beta(r)\cos 3\theta = c_1 H + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_m(\sqrt{\lambda_{mn}}r)\cos m\theta \sinh(\sqrt{\lambda_{mn}}H) + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} B_{mn} J_m(\sqrt{\lambda_{mn}}r)\sin m\theta \sinh(\sqrt{\lambda_{mn}}H)$$

From here we can see that:

$$\beta(r) = \sum_{n=1}^{\infty} A_{3n} J_3(\sqrt{\lambda_{3n}}r)\sinh(\sqrt{\lambda_{3n}}H)$$

With this, we can solve for the following coefficient:

$$A_{3n} = \frac{\int_0^a \beta(r) J_3\left(\sqrt{\lambda_{3n}}r\right) r\,dr}{\int_0^a J_3^2\left(\sqrt{\lambda_{3n}}r\right) r\,dr}$$

Thus we get:

$$\boldsymbol{u(r,\theta,z) = \sum_{n=1}^{\infty} A_{3n} J_3(\sqrt{\lambda_{3n}}r)\cos 3\theta \sinh(\sqrt{\lambda_{3n}}z)}$$

Excersise 7.9.2b: Solve Laplace's equation inside a semicircular cylinder, subject to the boundary conditions

$$u(r,\theta,0)=0, \qquad \frac{\partial u}{\partial z}(r,\theta,H)=0, \qquad u(r,0,z)=0,$$
$$u(r,\pi,z)=0, \qquad u(a,\theta,z)=\beta(\theta,z)$$

Let the following be true:

$$u(r,\theta,z)=f(r)g(\theta)h(z)$$

From this, and our original equation, we get the following:

$$\frac{d^2h}{dt^2}=-\lambda h \qquad \frac{d^2g}{d\theta^2}=-\mu g \qquad r\frac{d}{dr}\left(r\frac{df}{dr}\right)-\left(\lambda r^2+\mu\right)f=0$$

Solving the second ODE, we get the following eigenvalues and eigenfunctions with the boundary condition of $g(0)=g(\pi)=0$:

$$\mu_m=m^2 \qquad g(\theta)=b_{mn}\sin m\theta$$

Notice the solution for $h(z)$:

(a) If we let $\lambda=0$, we get the following:

$$h(t)=c_1z+c_2 \quad\rightarrow\quad h(0)=c_2=0 \qquad h'(z)=c_1 \quad\rightarrow\quad h'(H)=c_1=0 \quad\rightarrow\quad h(z)=0$$

(b) If we let $\lambda>0$, we get the following:

$$h(z)=c_1\cos\sqrt{\lambda}z+c_2\sin\sqrt{\lambda}z \qquad h'(z)=-c_1\sqrt{\lambda}\sin\sqrt{\lambda}z+c_2\sqrt{\lambda}\cos\sqrt{\lambda}z$$

$$h(0)=c_1=0 \quad\rightarrow\quad h'(H)=c_2\sqrt{\lambda}\cos\sqrt{\lambda}H=0 \quad\rightarrow\quad \lambda_n=\left(\frac{(2n+1)\pi}{2H}\right)^2$$

Notice the Modified Bessel's Differential Equation with solution:

$$f(r)=c_1K_m\left(\frac{(2n+1)\pi\, r}{2H}\right)+c_2I_m\left(\frac{(2n+1)\pi\, r}{2H}\right)$$

We know that K_m is singular at $r=0$ and that I_m is not, we have that $c_1=0$.

$$f(r)=c_2I_m\left(\frac{(2n+1)\pi\, r}{2H}\right)$$

Thus, we get the following:

$$u(r,\theta,z) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} B_{mn} I_m\left(\frac{(2n+1)\pi\, r}{2H}\right)\sin m\theta \sin\left(\frac{(2n+1)\pi\, z}{2H}\right)$$

Now we can use our other boundary condition:

$$u(a,\theta,z) = \beta(\theta,z) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} B_{mn} I_m\left(\frac{(2n+1)\pi\, a}{2H}\right)\sin m\theta \sin\left(\frac{(2n+1)\pi\, z}{2H}\right)$$

Using orthogonality of sines we get the following coefficient:

$$\begin{aligned}
&\int_0^H \int_0^\pi \beta(\theta,z)\sin m\theta \sin\left(\frac{(2n+1)\pi\, z}{2H}\right) d\theta dz \\
&\qquad = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} B_{mn} I_m\left(\frac{(2n+1)\pi\, a}{2H}\right)\int_0^\pi \sin^2 m\theta \, d\theta \int_0^H \sin^2\left(\frac{(2n+1)\pi\, z}{2H}\right) dz \\
&\qquad = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} B_{mn} I_m\left(\frac{(2n+1)\pi\, a}{2H}\right)\left(\frac{\pi}{2}\right)\left(\frac{H}{2}\right) \\
B_{mn} &= \frac{4}{\pi H I_m\left(\frac{(2n+1)\pi\, a}{2H}\right)}\int_0^H \int_0^\pi \beta(\theta,z)\sin m\theta \sin\left(\frac{(2n+1)\pi\, z}{2H}\right) d\theta dz
\end{aligned}$$