Midterm
Intro Math Modeling
Math 336
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Problem 1:

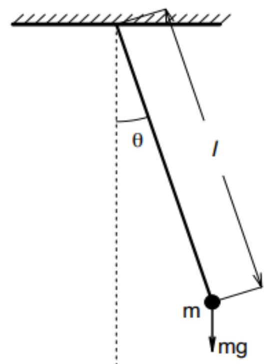


Figure 1: Simple pendulum of mass m and length l under the action of Earth's gravitational force.

(a) Figure 1 shows a simple pendulum with mass m , string length l , and the Earth gravitational acceleration g. Use the dimensional analysis method to determine the period τ of the pendulum as a function of m, l, g, determined up to a constant α, i.e., $\tau = \alpha m^a l^b g^c$

$$\begin{aligned} [\tau] &= \alpha [m]^a [l]^b [g]^c \\ T &= M^a L^b (LT^{-2})^c \\ M^0 L^0 T^1 &= M^a L^{b+c} T^{-2c} \end{aligned}$$

Now we can see $a = 0$, $c = -1/2$ and $b = 1/2$ to get this equality. Thus our solution is:

$$\boldsymbol{\tau = \alpha l^{1/2} g^{-1/2}} = \alpha \sqrt{\frac{l}{g}}$$

(b) Use the conservation law of energy to find an approximate value of α in Part (a) under the condition of $\sin x \approx x$ when x is close to be zero.

Notice we can find our angle θ using a simple harmonic sine function, $A\sin(\omega t + B)$. Let the pendulum reach a max height at A and let the pendulum be released at $t = 0$. This will make $B = 1/2$ because $\sin\frac{\pi}{2} = 1$. Thus we get the following:

$$\theta = A\sin\left(\frac{2\pi}{\tau}t + \frac{\pi}{2}\right)$$

We can calculate the tallest height when $\theta = A$. Also notice we can use the fact that for very small x, $\sin x = x$, so we get:

$$h = l - l\cos A = l(1 - \cos A) = 2l\sin^2\left(\frac{A}{2}\right) = 2l\left(\frac{A}{2}\right)^2 = \frac{lA^2}{2}$$

We can calculate the velocity at the lowest point when $\cos\left(\frac{2\pi}{\tau}t + \frac{\pi}{2}\right) = 1$

$$v = l\,\frac{d\theta}{dt} = \frac{2l\pi A}{\tau}\cos\left(\frac{2\pi}{\tau}t + \frac{\pi}{2}\right) = \frac{2l\pi A}{\tau}$$

Now through the conservation of energy, we know that the potential energy at the highest point is equal to the kinetic energy at the lowest point.

$$\begin{aligned}
E_p = mgh = mg\left[\frac{lA^2}{2}\right] &= \frac{m}{2}\left[\frac{2l\pi A}{\tau}\right]^2 = \frac{1}{2}mv^2 = E_p \\
\frac{1}{2}mglA^2 &= \frac{1}{\tau^2}2ml^2\pi^2A^2 \\
\tau^2 &= \frac{4l\pi^2}{g} \\
\tau &= 2\pi\sqrt{\frac{l}{g}}
\end{aligned}$$

This means we get that:

$$\boldsymbol{\alpha = 2\pi}$$

(c) If the string length is increased to $1.02l$ due to expansion in a higher temperature environment, and its corresponding period is denoted by τ_2. Express τ_2 in terms of τ found in Part (a) of this problem.

$$\boldsymbol{\tau_2 = 2\pi\sqrt{\frac{1.02l}{g}} = \sqrt{1.02}\,2\pi\sqrt{\frac{l}{g}} = \sqrt{1.02}\,\tau}$$

(d) Given that $l = 75$ [cm] and $g = 9.8$ [m/s], calculate τ with unit in second. Write down your steps. You can use a calculator or R to do the calculation. You do not need to submit your R code for this problem even if you use R here.

$$\boldsymbol{\tau = 2\pi\sqrt{\frac{(l = 0.75\,m)}{(g = 9.8\,m/s)}} = 1.73819\,s}$$

Problem 2: The SVD result of a matrix A is below

```
svdA=svd(A)

svdA$d
[1] 2.0 1.0

svdA$u
     [,1] [,2]
[1,]    0    1
[2,]    1    0

svdA$v
     [,1] [,2]
[1,]  0.0    1
[2,]  0.4    0
[3,]  0.9    0
```

(a) Write down three matrices U, D, V in the SVD formula $A = UDV'$ where V' denotes the transpose matrix of V.

$$\boldsymbol{A = UDV' =} \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0.4} & \mathbf{0.9} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

(b) Use the SVD formula $A = UDV'$ to approximately recover the original matrix A by hand calculation for the multiplication of the three matrices. Show your work and steps. You can use a calculator to do your number multiplication, but you still need to show your work. You may use R to verify your solution, but that is not required.

$$\begin{aligned} A = UDV' &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0.4 & 0.9 \\ 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0(2)+1(0) & 0(0)+1(1) \\ 1(2)+0(0) & 1(0)+0(1) \end{bmatrix} \begin{bmatrix} 0 & 0.4 & 0.9 \\ 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0.4 & 0.9 \\ 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0(0)+1(1) & 0(0.4)+1(0) & 0(0.9)+1(0) \\ 2(0)+0(1) & 2(0.4)+0(0) & 2(0.9)+0(0) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{.8} & \mathbf{1.8} \end{bmatrix} \end{aligned}$$

Problem 3:

(a) Derive a formula for the monthly mortgage payment x, expressed in terms of the principal amount P, monthly interest rate r, and total number of months of the loan n. Show your work and the detailed steps. The answer for this step is a formula.

$$\begin{aligned}
P_1 &= P(1+r) - x \\
P_2 &= P(1+r)^2 - x(1+r) - x \\
\vdots \quad & \vdots \quad \vdots \quad \vdots \\
P_k &= P(1+r)^k - x(1+r)^{k-1} - \cdots - x(1+r) - x \\
&= P(1+r)^k - x\left(\frac{1-(1+r)^k}{1-(1+r)}\right)
\end{aligned}$$

Because n is the total number of months on the loan, then at $k = n$, $P_{k=n} = 0$. Thus we get the equation below:

$$\boldsymbol{P_n = 0 = P(1+r)^n - x\left(\frac{1-(1+r)^n}{1-(1+r)}\right)}$$

(b) Given the data: The principal amount (i.e., the total loan) is $P = \$250,000$, the annual interest rate is 3.6% (converted into the monthly rate 0.3%), and the loan is to be paid off in 30 years (equivalent to 360 months). Use the above derived formula and the data to compute the monthly mortgage payment x by a calculator or R. The answer should be an amount of money per month. You do not need to submit the R code for this problem.

$$\begin{aligned}
P_{360} = 0 &= 250,000(1+0.003)^{360} - x\left(\frac{1-(1+0.003)^{360}}{1-(1+0.003)}\right) \\
x &= 250,000(1+0.003)^{360} \times \frac{1-(1+0.003)}{1-(1+0.003)^{360}} \\
&= \mathbf{\$1,136.61/month}
\end{aligned}$$

(c) If the annual rate is reduced to 2.9% in the above data, what is the monthly mortgage payment?

Our monthly interest rate will be $r = 0.029/12$, so our monthly payment will be:

$$\begin{aligned}
x &= 250,000(1+(0.029/12))^{360} \times \frac{1-(1+(0.029/12))}{1-(1+(0.029/12))^{360}} \\
&= \mathbf{\$1,040.57/month}
\end{aligned}$$

(d) If the principal is increased to $P = \$270,000$, the annual rate is 2.9%, and the loan period is still 30 years, what is the monthly mortgage payment now?

$$\begin{aligned}
x &= 270,000(1+(0.029/12))^{360} \times \frac{1-(1+(0.029/12))}{1-(1+(0.029/12))^{360}} \\
&= \mathbf{\$1,123.82/month}
\end{aligned}$$

Problem 4: The R programming part

(i) Use R to solve the following linear equations for x, y, z:

$$\begin{cases} -x + 2.9y + z = & 1 \\ -1.5x - y + z = & 2.1 \\ 2.2x + y - 4z = & 0 \end{cases}$$

Copy your R solution result to your R code as comment lines after #.

Notice we can solve this using a coefficient matrix, A, constant matrix, b, and our solution matrix, X:

$$A = \begin{bmatrix} -1 & 2.9 & 1 \\ -1.5 & -1 & 1 \\ 2.2 & 1 & -4 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2.1 \\ 0 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Now we use R to solve $AX = b$ and get the following solutions:

```r
# Problem 4i

A = matrix( c(-1, -1.5, 2.2, 2.9, -1, 1, 1, 1, -4), nrow = 3)
b = matrix( c(1, 2.1, 0), ncol = 1)
X = solve(A,b)
sprintf("x = %.4f", X[1]) # x = -2.2117
```

```
## [1] "x = -2.2117"
```

```r
sprintf("y = %.4f", X[2]) # y = 0.0015
```

```
## [1] "y = 0.0015"
```

```r
sprintf("z = %.4f", X[3]) # z = -1.2161
```

```
## [1] "z = -1.2161"
```

(ii) Figure 2 shows the history of the global average December mean temperature anomalies. Use R and the dataset `EarthTemperatureData.txt` or `EarthTemperatureData.csv` downloadable from BB's Assignment/Midterm block (or from the instructor's email) to plot a similar figure but for June and with the following requirements.

(a) Replace "Samuel Shen" and "December" in the main title by your name and June.

(b) Change the curve's color from black to purple and use `lwd=5`.

(c) Compute the linear trend of the June temperature anomalies for the entire time span from 1850 to 2015.

(d) Change the linear trend line's color from black to blue, and use `lwd=3` for the trend line.

(e) Change the text "December trend = 0.52 deg C/century" to "June trend = ?? deg /Century", and use the trend calculated from Step c) in the position "??".

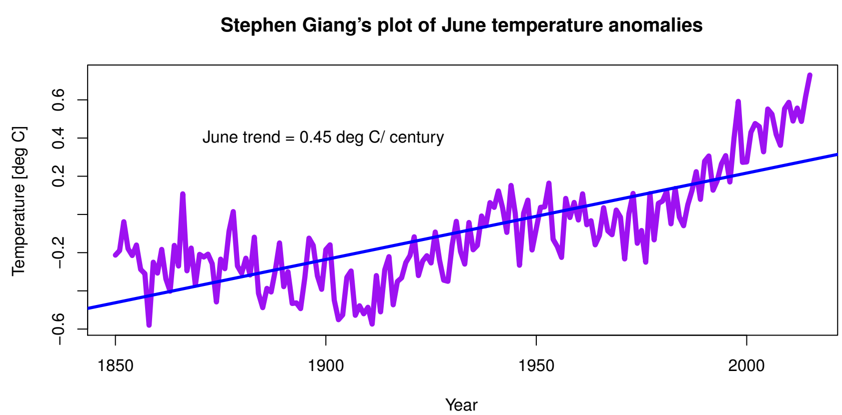
(f) Save your plot as a png file with the filename as "first2letters-of-your-last-name-temp.png". If your `Compile Report ...` is successful, you do not need to do this figure saving step.

(g) Plot the histogram of the June temperature anomalies from 1850 to 2015. The title of the figure is "Histogram of the June Temperature Anomalies." The x-label is "Temperature anomalies [deg C]." Save this figure as the above step (f) but with a file name "first2letters-of-your-last-name-histogram.png"

(h) Save all your work as a pdf file and combine all your work for this exam into a single pdf file

(i) Submit both pdf and R files into BB.

```
# Problem 4ii

setwd('C:/Users/Stephen Giang/Documents/Math336Files')
readData = read.csv('EarthTemperatureData.csv')

xvals = as.numeric(readData$YEAR)
yvals = as.numeric(readData$JUN)

plot(xvals, yvals, 'l', col = 'purple', lwd = 5,
     xlab = 'Year', ylab = 'Temperature [deg C]',
     main = 'Stephen Giang\'s plot of June temperature anomalies')

linmod = lm(yvals ~ xvals)
abline(linmod, col = 'blue', lwd = 3)

slope = linmod$coefficients[2] * 100 # .4525051
text(1900, 0.4, 'June trend = 0.45 deg C/ century ')
```




```
hist(yvals, main = 'Histogram of the June Temperature Anomalies.',
     xlab = 'Temperature anomalies [deg C]')
```