Oct 02, 2024

Spruce budworm model (contd.)

Model (non-dimensionalized):

$$\boxed{\frac{du}{d\tau} = ru\left(1-\frac{u}{q}\right) - \frac{u^2}{1+u^2}} \qquad r = \frac{Br_B}{A},\ q = \frac{K_B}{A}$$

$$\frac{du}{d\tau} = ru\left(1-\frac{u}{q}\right) - \frac{u^2}{1+u^2} = f(u)$$

- Equilibria structure:

u^* is an equilibrium $\iff f(u^*) = 0$

$$\iff ru\left(1-\frac{u}{q}\right) - \frac{u^2}{1+u^2} = 0$$

$$\iff u\left[r\left(1-\frac{u}{q}\right) - \frac{u}{1+u^2}\right] = 0$$

$\Rightarrow$ • $u^* = 0$ is always an equilibrium

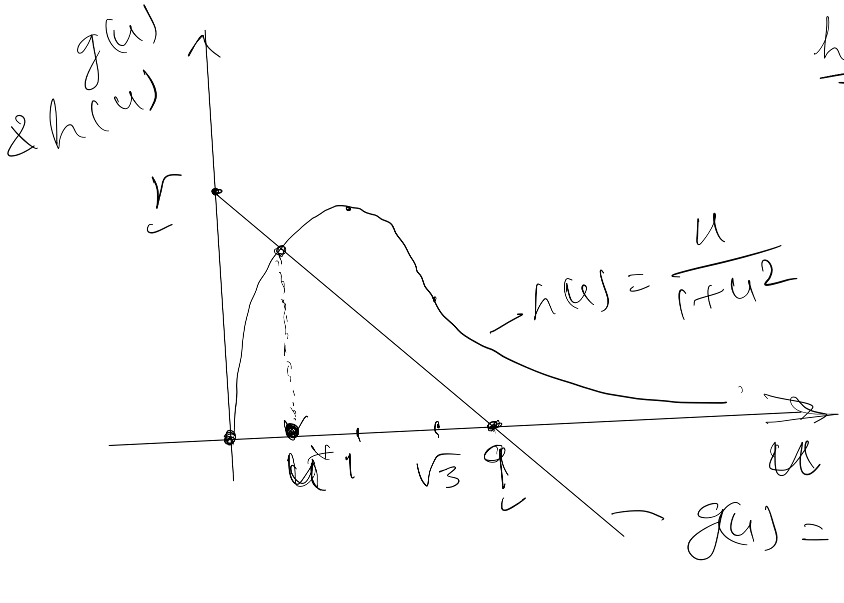
• $u^* \neq 0$ is given by $r\left(1-\frac{u}{q}\right) - \frac{u}{1+u^2} = 0$

i.e. $r\left(1-\frac{u}{q}\right) = \frac{u}{1+u^2}$

i.e., $g(u) = h(u)$

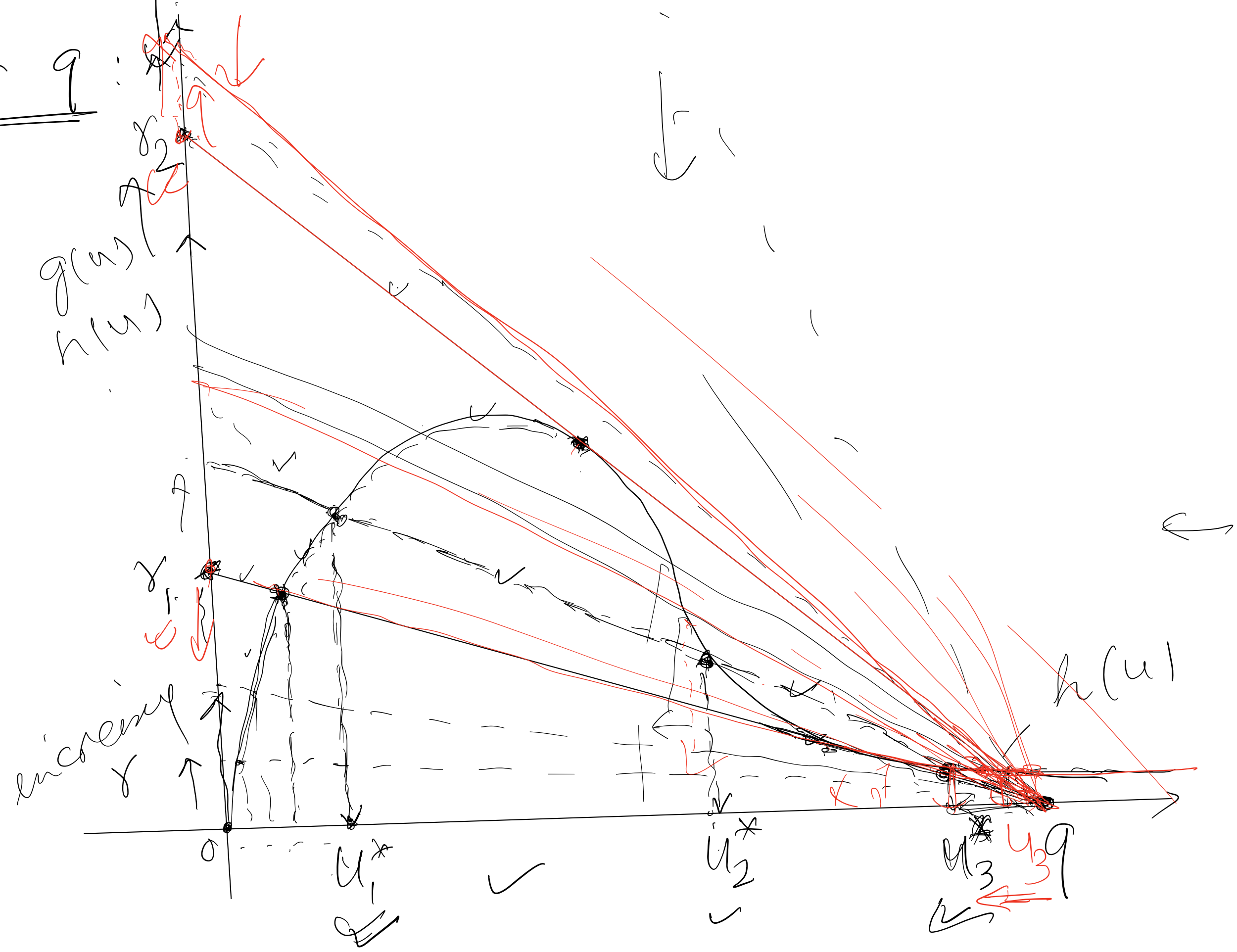
where $g(u) = r\left(1-\frac{u}{q}\right)$ $= 0 \Rightarrow u = q$

$h(u) = \frac{u}{1+u^2}$.



$h(u)$:
- $h(0)=0$, • $h \to 0$ as $u \to \infty$
- $h'(u) = \frac{1-u^2}{(1+u^2)^2}$
- $h''(u) = \frac{-2u}{(1+u^2)^3}(3-u^2)$

• Fix q:



- Increase r from ~~zero~~ until r_1 where $g(u)$ is tangent to $h(u)$.

- to solve for r_1?

$$\left.\begin{aligned} r\left(1-\frac{u}{q}\right) &= \frac{u}{1+u^2} \\ \text{and} \quad -\frac{r}{q} &= \frac{1-u^2}{(1+u^2)^2} \end{aligned}\right\}$$

Conclusion: • for r in $(0, r_1]$, there exists one positive equilibrium u_1^*.

• If $r > r_1$ and $r < r_2$,

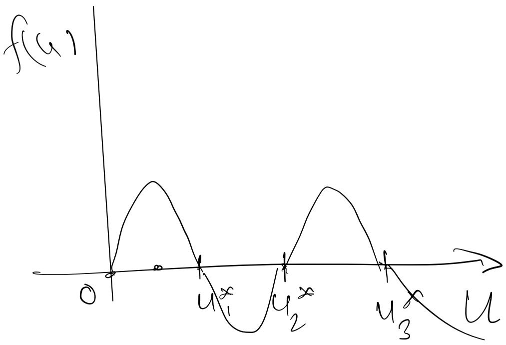
there are exactly three positive equilibria

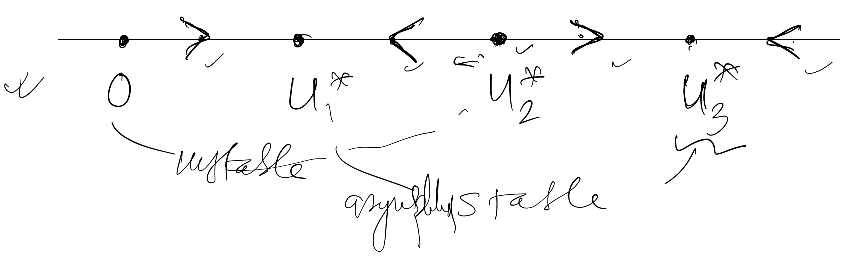
$$u_1^* < u_2^* < u_3^*.$$

• If $r > r_2$, there is exactly one positive equilibrium u_3^*.

• Stability of Equilibria (Global dynamics)

– Phase diagram





Then $f(u) = u\left[g(u) - h(u)\right]$

$= u\left[\text{straight line} - \text{curve}\right]$

$$f(u) = \begin{cases} >0 & u \in (0, u_1^*) \\ <0 & u \in (u_1^*, u_2^*) \\ >0 & u \in (u_2^*, u_3^*) \\ <0 & u \in (u_3^*, \infty) \end{cases}$$

Conclusion: • u_1^* and u_3^* are asymptotically stable (biologically important)

- $u_0 = 0$, u_2 are unstable.

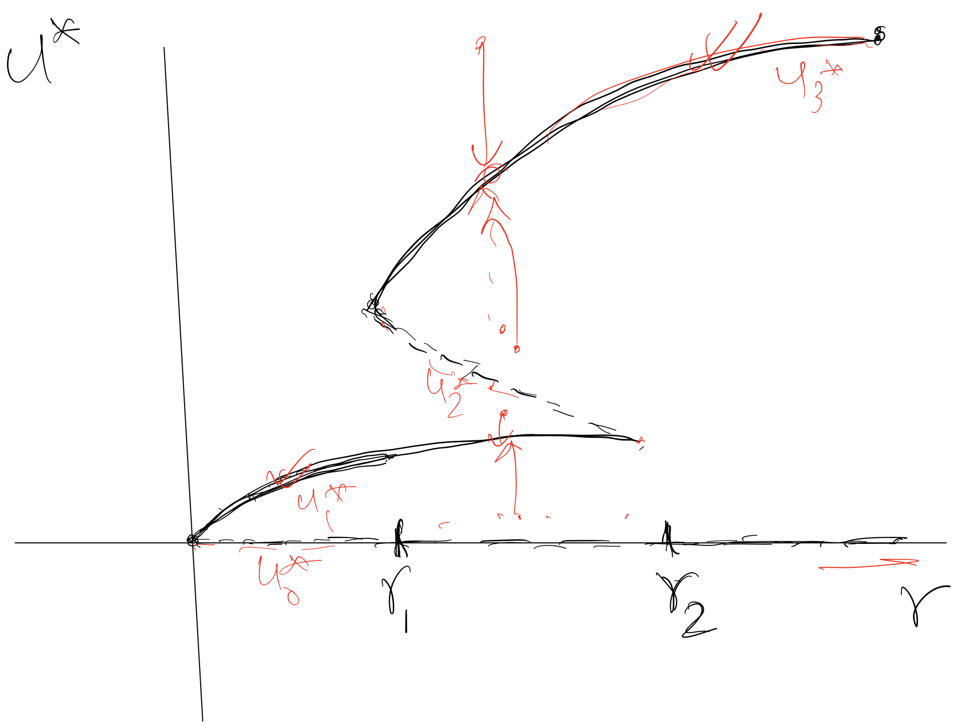
u_1^*: refugee stable (not so important for control)

u_3^*: outbreak

u_0^*, u_3^*: critical values (u_0^* is not so important but u_2^* is very important)

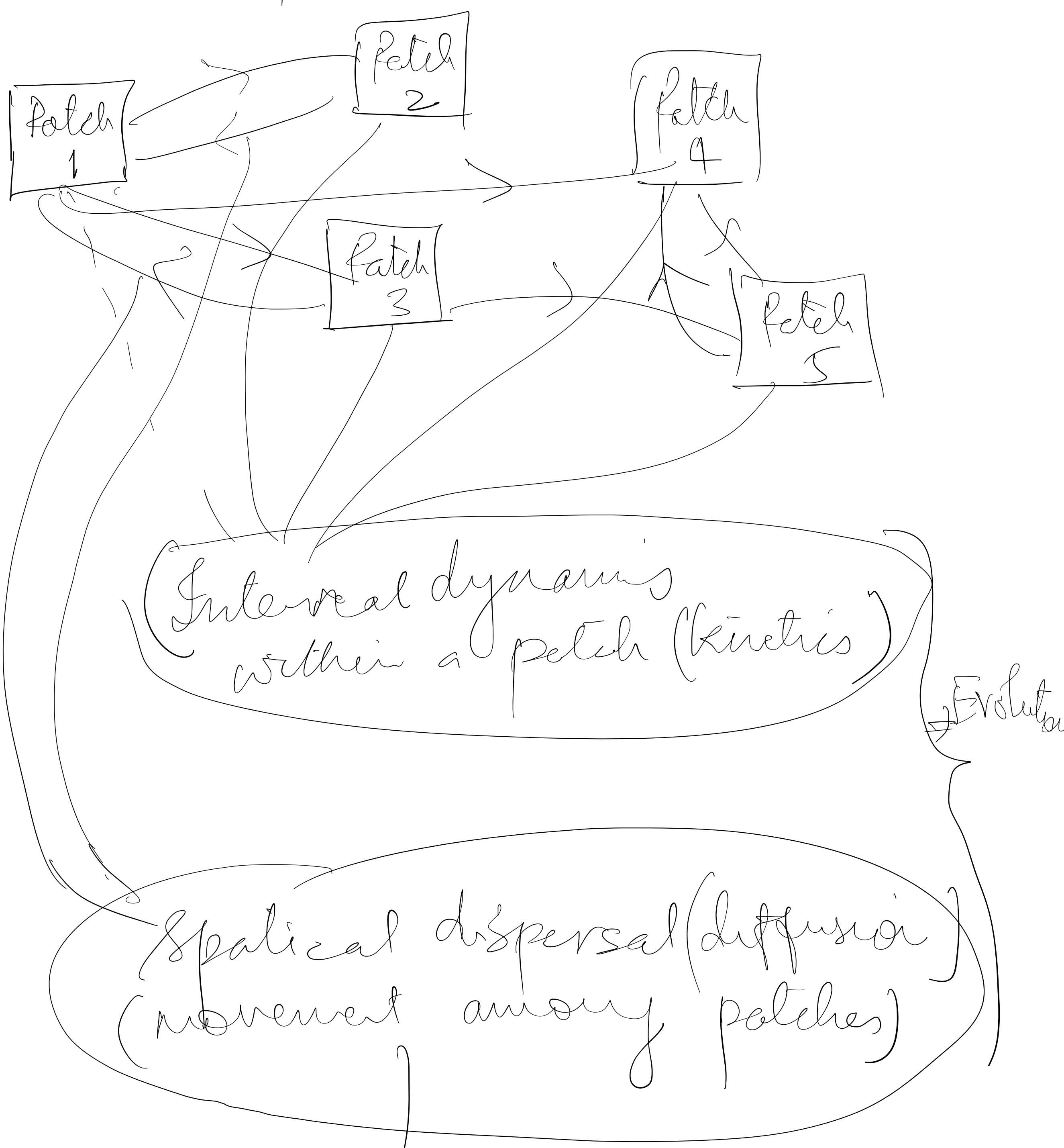
– $\exists$ two ways for outbreak control
  - control the initial value u (below u_2^*)
  - decrease r (below r_1).

- Bifurcation diagram:



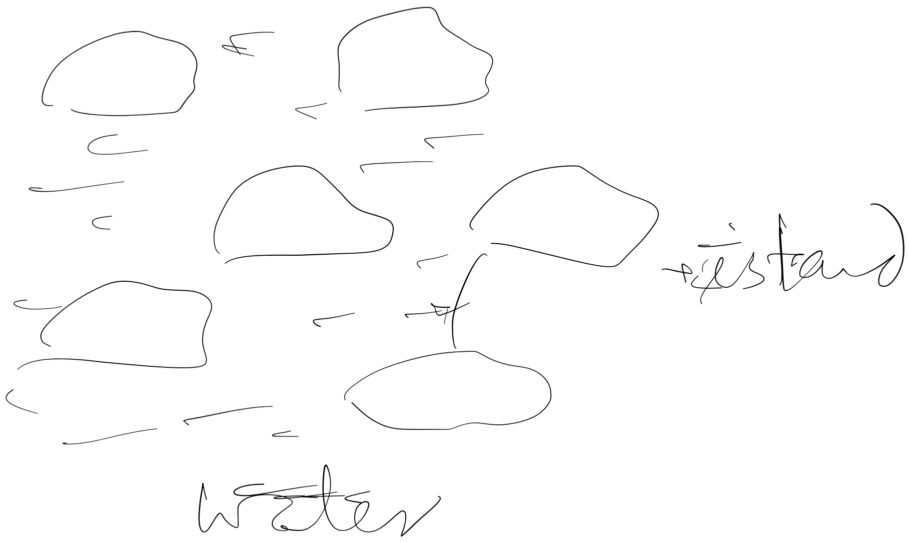
☒ ODE model for spatial variation (second independent variable)

- discreate approach: Patchy model.



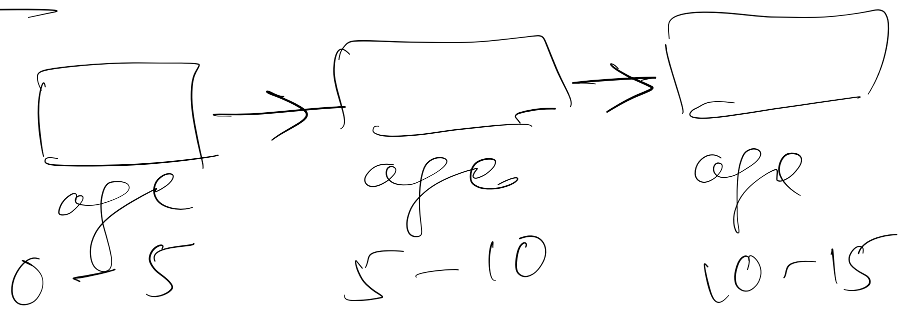
- radom movement
- directed movement

Example:



random.

directed:



traffic:

$N_i(t)$: the density/size at time t and patch i

$\underbrace{}_{\Downarrow}$

i : second variable (independent)

$$\frac{dN_i(t)}{dt} = \text{kinetics (internal/local dynamics)} + \text{dispersal/diffusion}$$

$$\Rightarrow \frac{dN_i(t)}{dt} = f_i(N_i) + \sum_{j \neq i} D_{ij} N_j - D_{ii} N_i$$

$i = 1, 2, \ldots, n$, $n = \#$ of patches

Note: $D_{ii} = \sum_{j \neq i} D_{ji}$