Homework 6
Abstract Algebra
Math 320
Stephen Giang

Section 2.3 Problem 11: If $a, b \in \mathbb{Z}_n$ and a is a unit, then the equation ax = b has a unique solution in $\mathbb{Z}_n$ [Note: You must find a solution for the equation and show that this solution is the only one.]

Solution 2.3.11. Let $a, b \in \mathbb{Z}_n$ and a be a unit.

$$\begin{aligned} \text{Consider: } ax &= b \\ a^{-1}ax &= a^{-1}b \\ 1x &= a^{-1}b \\ x &= a^{-1}b \end{aligned}$$

Thus there exists a solution such that ax = b.
Let $ax_1 = b$ and $ax_2 = b$

$$\begin{aligned} ax_1 &= b \\ ax_2 &= b \\ ax_1 &= ax_2 \\ a^{-1}ax_1 &= a^{-1}ax_2 \\ x_1 &= x_2 \end{aligned}$$

Thus there exists a unique solution such that ax = b.

□

Section 2.3 Problem 12: Let a, b, n be integers with $n > 1$ and let $d = (a, n)$. If the equation [a]x = [b] has a solution in $\mathbb{Z}_n$ prove that $d|b$. (Hint: If x = [r] is a solution, then [ar] = [b] so that ar - b = kn for some integer k.]

Solution 2.3.12. Let $a, b, n \in \mathbb{Z}$ with $n > 1$ and Let $d = (a, n)$.Suppose $ax = b$ has a solution,$x = r$, in $\mathbb{Z}_n$.

$$\begin{aligned} [a][r] &= [b] \\ [ar] - [b] &= [0] \\ ar - b &= kn \qquad \text{for some k } \in \mathbb{Z} \\ b &= ar - kn \\ b &= dq_1 + dq_2 \qquad \text{Bc } d = (a, n), \text{ let } ar = dq_1, -kn = dq_2 \\ b &= d(q_1 + q_2) \end{aligned}$$

Thus $d|b$

□

Section 3.1 Problem 15: Write out the addition and multiplication tables for

a) $\mathbb{Z}_2 \times \mathbb{Z}_3$ b) $\mathbb{Z}_2 \times \mathbb{Z}_2$

a)

+	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)
(0,0)	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)
(0,1)	(0,1)	(0,2)	(0,0)	(1,1)	(1,2)	(1,0)
(0,2)	(0,2)	(0,0)	(0,1)	(1,2)	(1,0)	(1,1)
(1,0)	(1,0)	(1,1)	(1,2)	(0,0)	(0,1)	(0,2)
(1,1)	(1,1)	(1,2)	(1,0)	(0,1)	(0,2)	(0,0)
(1,2)	(1,2)	(1,0)	(1,1)	(0,2)	(0,0)	(0,1)

×	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)
(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
(0,1)	(0,0)	(0,1)	(0,2)	(0,0)	(0,1)	(0,2)
(0,2)	(0,0)	(0,2)	(0,1)	(0,0)	(0,2)	(0,2)
(1,0)	(0,0)	(0,0)	(0,0)	(1,0)	(1,0)	(1,0)
(1,1)	(0,0)	(0,1)	(0,2)	(1,0)	(1,1)	(1,2)
(1,2)	(0,0)	(0,2)	(0,1)	(1,0)	(1,2)	(1,1)

b)

+	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,1)	(1,0)	(1,1)
(0,1)	(0,1)	(0,2)	(1,1)	(1,2)
(0,2)	(0,2)	(0,0)	(1,2)	(1,0)
(1,0)	(1,0)	(1,1)	(0,0)	(0,1)
(1,1)	(1,1)	(1,2)	(0,1)	(0,2)
(1,2)	(1,2)	(1,0)	(0,2)	(0,0)

×	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
(0,1)	(0,0)	(0,1)	(0,0)	(0,1)
(0,2)	(0,0)	(0,2)	(0,0)	(0,2)
(1,0)	(0,0)	(0,0)	(1,0)	(1,0)
(1,1)	(0,0)	(0,1)	(1,0)	(1,1)
(1,2)	(0,0)	(0,2)	(1,0)	(1,2)

Section 3.1 Problem 17: Define a new multiplication in $\mathbb{Z}$ by the rule: $ab = 0 \forall a, b \in \mathbb{Z}$. Show that with ordinary addition and this new multiplication, $\mathbb{Z}$ is a commutative ring.

Solution 3.1.17. Define multiplication in $\mathbb{Z}$ by the rule: $ab = 0 \forall a, b \in \mathbb{Z}$. Let $a, b, c \in \mathbb{Z}$

Axiom 6) $ab = 0 \in \mathbb{Z}$
Axiom 7) $a(bc) = a(0) = 0 = (0)c = (ab)c$
Axiom 8) $a(b + c) = 0 = 0 + 0 = ab + bc$
Axiom 9) $ab = 0 = ba$

Thus $\mathbb{Z}$ is a commutative ring.

$\square$

Section 3.1 Problem 19: Let S = a, b, c and let P(S) be the set of all subsets of S; denote the elements of P(S) as follows:

$$\begin{array}{cccc} S = \{a, b, c\} & D = \{a, b\} & E = \{a, c\} & F = \{b, c\} \\ A = \{a\} & B = \{b\} & C = \{c\} & 0 = \emptyset. \end{array}$$

Define addition and multiplication in P(S) by these rules: M + N = (M - N) ∪ (N - M) and MN=M ∩ N.

+	S	D	E	F	A	B	C	0
S	0	C	B	A	F	E	D	S
D	C	0	F	E	B	A	S	D
E	B	F	0	D	C	S	A	E
F	A	E	D	0	S	C	B	F
A	F	B	C	S	0	D	E	A
B	E	A	S	C	D	0	F	B
C	D	S	A	B	E	F	0	C
0	S	D	E	F	A	B	C	0

×	S	D	E	F	A	B	C	0
S	S	D	E	F	A	B	C	0
D	D	D	A	B	A	B	0	0
E	E	A	E	C	A	0	C	0
F	F	B	C	F	0	B	C	0
A	A	A	A	0	A	0	0	0
B	B	B	0	B	0	B	0	0
C	C	0	C	C	0	0	C	0
0	0	0	0	0	0	0	0	0

Section 3.1 Problem 23: Let E be the set of even integers with ordinary addition. Define a new multiplication * on E by the rule "a * b = "ab/2" (where the product on the right is ordinary multiplication). Prove that with these operations E is a commutative ring with identity.

Solution 3.1.17. Define multiplication in E by the rule: $a * b = ab/2$. Let $a, b, c \in E$

Axiom 6) $a * b = ab/2 \in E$

Axiom 7) $a * (b * c) = a * \left(\frac{bc}{2}\right) = \frac{a\left(\frac{1}{2}bc\right)}{2} = \frac{\left(\frac{ab}{2}\right)c}{2} = (a * b) * c$

Axiom 8) $a * (b + c) = \frac{a(b+c)}{2} = \frac{ab}{2} + \frac{ac}{2} = (a * b) + (a * c)$

Axiom 9) $a * b = \frac{ab}{2} = \frac{ba}{2} = b * a$

Thus E is a commutative ring.

□