B: Sink ($\lambda_1 < \lambda_2 < 0$): move toward (0,0)

$$x' = \lambda_1 x \qquad\qquad A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$
$$y' = \lambda_2 y$$

Let $|A - \lambda I| = 0 \Longrightarrow$ $\boxed{\lambda = \lambda_{1,2}}$ $\quad V_1 = \binom{x}{y} = \binom{1}{0} \quad V_2 = \binom{0}{1}$

Thus, a general solution is written as

$$X(t) = \alpha e^{\lambda_1 t}\begin{pmatrix}1\\0\end{pmatrix} + \beta e^{\lambda_2 t}\begin{pmatrix}0\\1\end{pmatrix}$$

$$x = \alpha e^{\lambda_1 t}$$
$$y = \beta e^{\lambda_2 t}$$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\lambda_2 \beta e^{\lambda_2 t}}{\lambda_1 \alpha e^{\lambda_1 t}} = \frac{\lambda_2 \beta}{\lambda_1 \alpha} e^{(\lambda_2 - \lambda_1)t} \qquad \rightarrow \infty \ (or\ -\infty)\ as\ t \rightarrow \infty$$

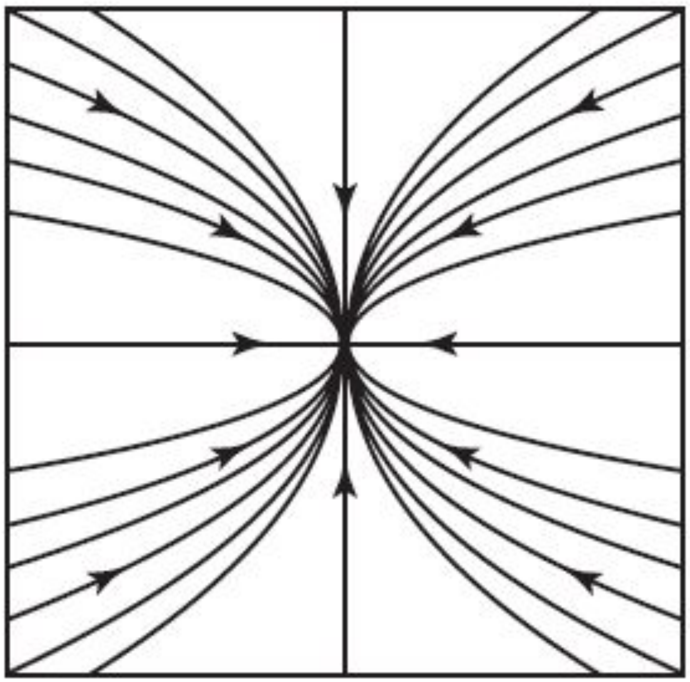
$$(\lambda_2 - \lambda_1) > 0$$

- These solutions tend to the origin (a sink) tangentially to the y axis (i.e., vertical asymptotes)
- x tends to zero much quickly
- λ_1 (λ_2) is referred to as the stronger (weaker) eigenvalue.

B: Sink $(\lambda_1 < \lambda_2 < 0)$

$\lambda_1 < \lambda_2 < 0$

Trajectories (time varying)

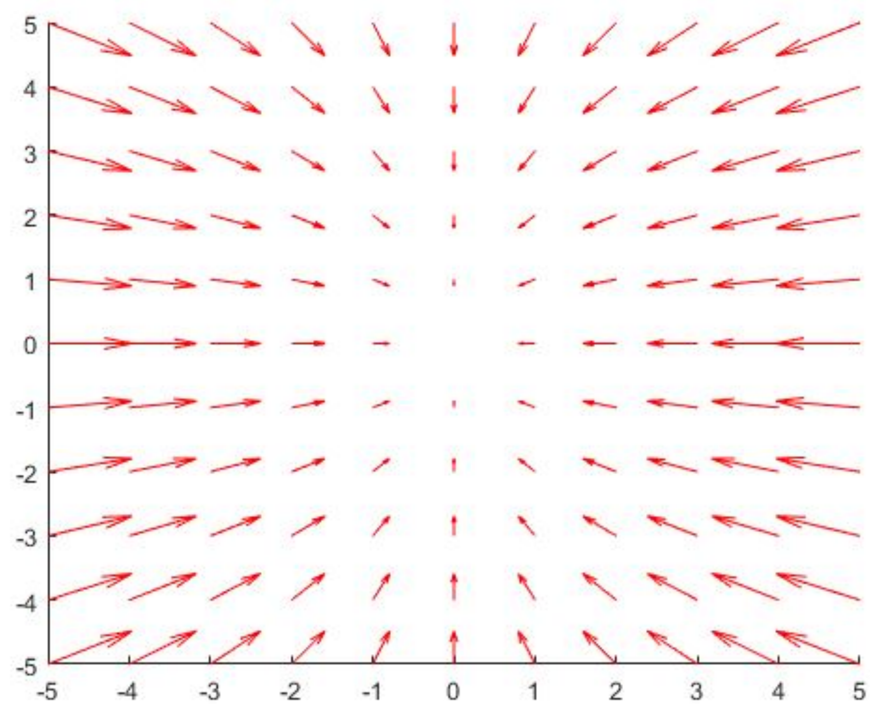


(a)

Figure 3.3 Phase portraits for (a) a sink

X' = -2x   Y' = -y

Vector Fields (at a given time)



MATLAB Plot for Figure 3.3a

- These solutions tend to the origin (a sink) tangentially to the y axis (i.e., vertical asymptotes), associated with the weaker eigenvalue (λ_2).

The Euler Method

From Calc II, we have:

$$\frac{dy}{dx} = F(x,y) \qquad x_n = x_{n-1} + h \qquad h = \Delta x\text{, an increment}$$

$$\boxed{y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})} \qquad n = 1,2,3,\ldots$$

Now, we consider the following:

$$\frac{dy}{dt} = F(y) \qquad y: position \qquad \frac{dy}{dt}: velicity$$

Given a $F(y) = -y$, we have

$$\frac{dy}{dt} = -y \qquad \frac{y_{n+1} - y_n}{\Delta t} = -y_n \qquad y_{n+1} = y_n - \Delta t\, y_n$$

B: Trajectories (derived from Vector Fields)

$$x' = -2x$$
$$y' = -y$$

Apply the Euler Method

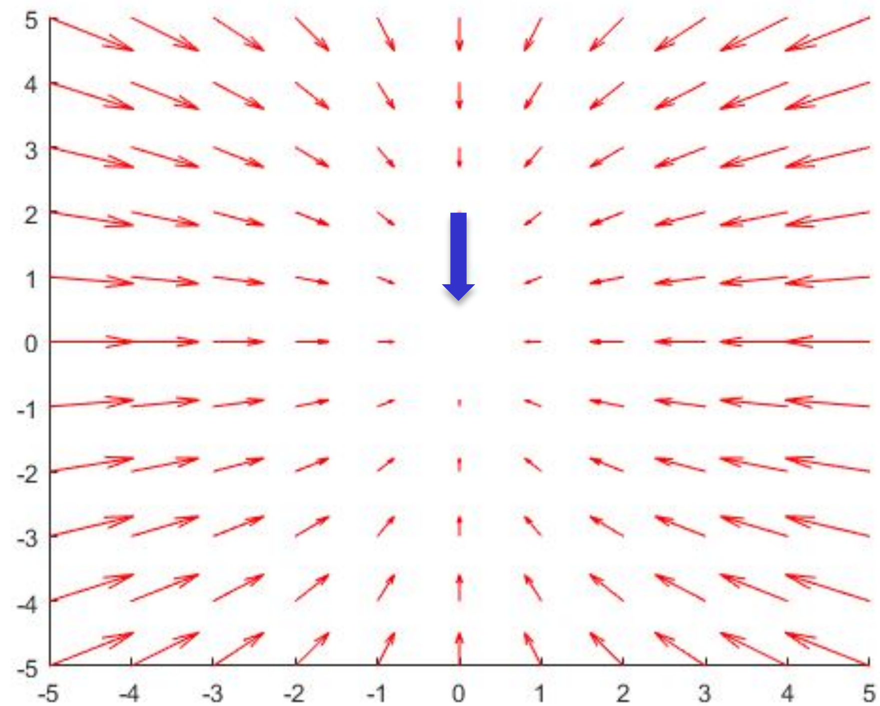
$$y_{n+1} = y_n - \Delta t y_n \qquad \Delta t = 0.5$$

At point B: $(x_n, y_n) = (0, 2)$

$$y_{n+1} = y_n - \Delta t \; y_n$$
$$= 2 - 0.5 * 2$$
$$= 1 < y_n \Rightarrow \text{southward}$$

- In the first quadrant, a particle moves westward ($x' < 0$) and southward ($y' < 0$).
- In the third quadrant, a particle moves eastward westward ($x' > 0$) and northward ($y' > 0$).

X' = -2x Y' = -y



MATLAB Plot for Figure 3.3a

B: Sink ($\lambda_1 < \lambda_2 < 0$): a general case

Thus, a general solution is written as

$$X(t) = \binom{x}{y} = \alpha e^{\lambda_1 t}\binom{u_1}{u_2} + \beta e^{\lambda_2 t}\binom{v_1}{v_2}$$

$$x = \alpha e^{\lambda_1 t}u_1 + \beta e^{\lambda_2 t}v_1$$

$$y = \alpha e^{\lambda_1 t}u_2 + \beta e^{\lambda_2 t}v_2$$

$(\lambda_1 - \lambda_2) < 0$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\lambda_1 \alpha e^{\lambda_1 t}u_2 + \lambda_2 \beta e^{\lambda_2 t}v_1}{\lambda_1 \alpha e^{\lambda_1 t}u_1 + \lambda_2 \beta e^{\lambda_2 t}v_1} = \frac{\lambda_1 \alpha e^{(\lambda_1 - \lambda_2)t}u_2 + \lambda_2 \beta v_2}{\lambda_1 \alpha e^{(\lambda_1 - \lambda_2)t}u_1 + \lambda_2 \beta v_1} \rightarrow \frac{v_2}{v_1} \quad as\ t \rightarrow \infty$$

- All solutions (except for those on the straight line corresponding the stronger eigenvalue) tend to the origin (a sink) tangentially to the straight-line solution corresponding to the weaker eigenvalue (λ_2).
- λ_1 (λ_2) is referred to as the stronger (weaker) eigenvalue.

(C): A Source $(0 < \lambda_2 < \lambda_1)$

$$x' = \lambda_1 x$$
$$y' = \lambda_2 y$$

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Let $|A - \lambda I| = 0 \Longrightarrow$ $\boxed{\lambda = \lambda_{1,2}}$

$$V_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus, a general solution is written as

$$X(t) = \alpha X_1 + \beta X_2 = \alpha e^{\lambda_1 t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda_2 t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

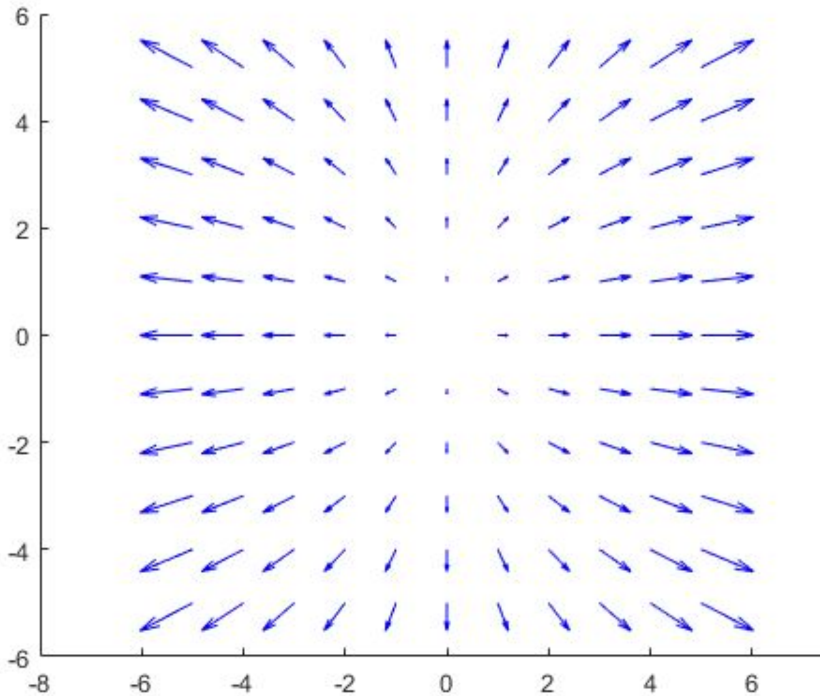
$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\lambda_2 \beta e^{\lambda_2 t}}{\lambda_1 \alpha e^{\lambda_1 t}} = \frac{\lambda_2 \beta}{\lambda_1 \alpha} e^{(\lambda_2 - \lambda_1)t} \qquad \rightarrow \infty \ (or \ -\infty) \ as \ t \rightarrow -\infty$$

tend away from (0,0)

(C): A Source $(0 < \lambda_2 < \lambda_1)$

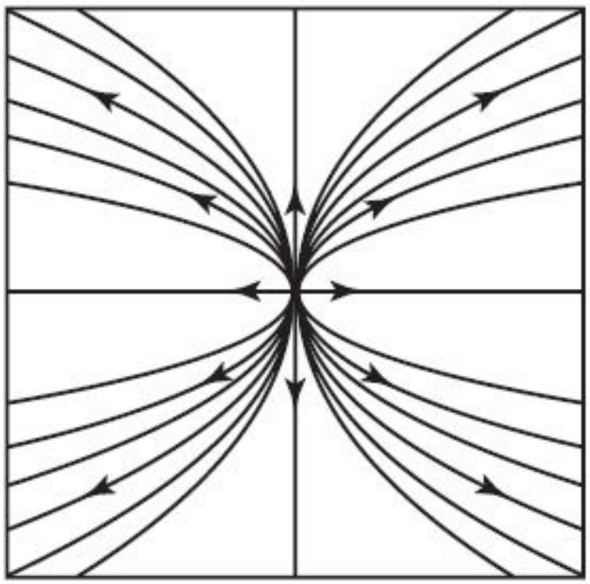
X' = 2x   Y' = y $\lambda_1 > 0$; $\lambda_2 > 0$

Vector Fields (at a given time)



MATLAB Plot for Figure 3.3b

Trajectories (time varying)



(b)

Figure 3.3 Phase portraits for (b) a source.

The general solution and phase portrait remain the same, except that all solutions now tend away from (0,0) along the same paths. See Figure 3.3b.

MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #9

Chapter 3 Phase Portraits for Planar Systems
Complex Eigenvalues
&
Repeated Eigenvalues

Instructor: Dr. Bo-Wen Shen*

Department of Mathematics and Statistics
San Diego State University

Simple 2D Systems with Complex Eigenvalues

$$x' = ax + by \quad (= P(x,y)) \qquad (1)$$

$$y' = cx + dy \quad (= Q(x,y)\ ) \qquad (2)$$

$$X' = AX \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

(I) pure imaginary eigenvalues

$$\boxed{A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}}$$

$$x' = \cancel{\alpha x} + \beta y \quad (= P(x,y))$$

$$y' = \beta x + \cancel{dy} \quad (= Q(x,y))$$

$$\boxed{D. \quad \lambda_{1,2} = \pm i\,\beta: \ \text{center}}$$

(II) complex eigenvalues with $Re \neq 0$

$$\boxed{A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}}$$

$$x' = \alpha x + \beta y \quad (= P(x,y))$$

$$y' = \beta x + \alpha y \quad (= Q(x,y))$$

E. $\lambda_{1,2} = \alpha \pm i\,\beta$: spiral source or sink

Review: Complex Roots

5

$$ay'' + by' + cy = 0$$

where a, b, and c are constants and $a \neq 0$.

Summary of Cases I–III

Case	Roots of (2)	Basis of (1)	General Solution of (1)
III	Complex conjugate $\lambda_1 = -\frac{1}{2}a + i\omega$, $\lambda_2 = -\frac{1}{2}a - i\omega$	$e^{-ax/2}\cos\omega x$ $e^{-ax/2}\sin\omega x$	$y = e^{-ax/2}(A\cos\omega x + B\sin\omega x)$

Review: 2nd order ODEs with Complex Eigenvalues

$$\lambda_{1,2} = \alpha \pm i\beta$$

$$x = c_1 x_1 + c_2\, x_2 = c_1\, e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$= c_1\, e^{(\alpha+i\beta)t} + c_2 e^{(\alpha-i\beta)t}$$

$$= e^{\alpha x}(c_1\, e^{(i\beta)t} + c_2 e^{(-i\beta)t})$$

$$= e^{\alpha x}(c_1\, (\cos(\beta t) + isin(\beta t)) + c_2(\cos(\beta t) - isin(\beta t))$$

$$= e^{\alpha x}((c_1 + c_2) \cos(\beta t) + i(c_1 - c_2) sin(\beta t))$$

$$= e^{\alpha x}(A\cos(\beta t) + Bsin(\beta t))$$

$A = c_1 + c_2 \qquad B = i(c_1 - c_2)$

(D) A Center with $\lambda = \pm i\beta$

Consider $\quad \begin{aligned} x' &= \beta y \\ y' &= -\beta x \end{aligned}$ $\quad \boxed{A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}}$ $\quad A - \lambda I = \begin{pmatrix} -\lambda & \beta \\ -\beta & -\lambda \end{pmatrix}$

Let $|A - \lambda I| = 0 \Longrightarrow$ $\quad \lambda^2 + \beta^2 = 0$ $\quad \boxed{\lambda = \pm i\beta}$

$$AV_0 = \lambda V_0 \qquad \begin{aligned} \beta y_0 &= \lambda x_0 \\ -\beta x_0 &= \lambda y_0 \end{aligned}$$

$$\boxed{\lambda = i\beta} \qquad \begin{aligned} \beta y_0 &= i\beta x_0 \\ -\beta x_0 &= i\beta y_0 \end{aligned}$$

- Find the eigenvector
- Send your results via ”chat”
- You have 3 minutes

(D) A Center with $\lambda = \pm i\beta$

Consider $\begin{aligned} x' &= \beta y \\ y' &= -\beta x \end{aligned}$ $\boxed{A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}}$ $A - \lambda I = \begin{pmatrix} -\lambda & \beta \\ -\beta & -\lambda \end{pmatrix}$

Let $|A - \lambda I| = 0 \Longrightarrow$ $\lambda^2 + \beta^2 = 0$ $\boxed{\lambda = \pm i\beta}$

$AV_0 = \lambda V_0$ $\begin{aligned} \beta y_0 &= \lambda x_0 \\ -\beta x_0 &= \lambda y_0 \end{aligned}$

$\boxed{\lambda = i\beta}$ $\begin{aligned} \beta y_0 &= i\beta x_0 \\ -\beta x_0 &= i\beta y_0 \end{aligned}$ $y_0 = i x_0$ $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ i x_0 \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ i \end{pmatrix}$

Obtain $\boxed{V_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}}$ as an eigenvector associated with $\lambda = i\beta$

Similarly, $\boxed{V_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}}$ as an eigenvector associated with $\lambda = -i\beta$

(D) A Center with $\lambda = \pm i\beta$

Thus, a general solution is written as

$$\boxed{X(t) = a\boxed{e^{i\beta t}\begin{pmatrix}1\\i\end{pmatrix}} + be^{-i\beta t}\begin{pmatrix}1\\-i\end{pmatrix}}$$

$$= a\big(\cos(\beta t) + i\sin(\beta t)\big)\begin{pmatrix}1\\i\end{pmatrix} + b\big(\cos(\beta t) - i\sin(\beta t)\big)\begin{pmatrix}1\\-i\end{pmatrix}$$

$$= \begin{pmatrix}(a+b)\cos(\beta t)\\(-a-b)\sin(\beta t)\end{pmatrix} + i\begin{pmatrix}(a-b)\sin(\beta t)\\(a-b)\cos(\beta t)\end{pmatrix}$$

$$= (a+b)\begin{pmatrix}\cos(\beta t)\\-\sin(\beta t)\end{pmatrix} + i(a-b)\begin{pmatrix}\sin(\beta t)\\\cos(\beta t)\end{pmatrix}$$

$$\boxed{= c_1 X_{re}(t) + c_2 X_{im}}$$

How to quickly find two LI solutions with trig functions?

$$\boxed{X_{re}(t) = Re\left(e^{i\beta t}\begin{pmatrix}1\\i\end{pmatrix}\right)}$$

$$\boxed{X_{im}(t) = Im\left(e^{i\beta t}\begin{pmatrix}1\\i\end{pmatrix}\right)}$$

(D) A Center with $\lambda = \pm i\beta$

$$e^{i\beta t}\begin{pmatrix}1\\ i\end{pmatrix} = (\cos(\beta t) + isin(\beta t))\begin{pmatrix}1\\ i\end{pmatrix} = \begin{pmatrix}\cos(\beta t) + isin(\beta t)\\ Ans\end{pmatrix}$$

- Find "Ans"
- Send your results via "chat"
- You have 2 minutes

$$e^{i\beta t}\begin{pmatrix}1\\ i\end{pmatrix} = (\cos(\beta t) + isin(\beta t))\begin{pmatrix}1\\ i\end{pmatrix} = \begin{pmatrix}\cos(\beta t) + isin(\beta t)\\ i\cos(\beta t) - \sin(\beta t)\end{pmatrix}$$

$$Re\left(e^{i\beta t}\begin{pmatrix}1\\ i\end{pmatrix}\right) = \begin{pmatrix}\cos(\beta t)\\ -sin(\beta t)\end{pmatrix} = X_{re}$$

$$\mathrm{Im}\left(e^{i\beta t}\begin{pmatrix}1\\ i\end{pmatrix}\right) = \begin{pmatrix}\sin(\beta t)\\ cos(\beta t)\end{pmatrix} = X_{im}$$

(D) A Center with $\lambda = \pm i\beta$

$$Re\left(e^{i\beta t}\begin{pmatrix}1\\ i\end{pmatrix}\right) = Re\left(\left(\cos(\beta t) + i\sin(\beta t)\right)\begin{pmatrix}1\\ i\end{pmatrix}\right) = \begin{pmatrix}\cos(\beta t)\\ -\sin(\beta t)\end{pmatrix} = X_{re}$$

$$\mathrm{Im}\left(e^{i\beta t}\begin{pmatrix}1\\ i\end{pmatrix}\right) = Im\left(\left(\cos(\beta t) + i\sin(\beta t)\right)\begin{pmatrix}1\\ i\end{pmatrix}\right) = \begin{pmatrix}\sin(\beta t)\\ \cos(\beta t)\end{pmatrix} = X_{im}$$

Thus, a general solution is written as

$$X(t) = a\boxed{e^{i\beta t}\begin{pmatrix}1\\ i\end{pmatrix}} + be^{-i\beta t}\begin{pmatrix}1\\ -i\end{pmatrix}$$

$$= \mathrm{c}_1\mathrm{X}_{\mathrm{re}}(\mathrm{t}) + c_2 X_{im}$$

(D) A Center with $\lambda = \pm i\beta$

Thus, a general solution is written as

$$X(t) = a\boxed{e^{i\beta t}\begin{pmatrix}1\\ i\end{pmatrix}} + be^{-i\beta t}\begin{pmatrix}1\\ -i\end{pmatrix}$$

$$X_{re}(t) = \begin{pmatrix}\cos(\beta t)\\ -sin(\beta t)\end{pmatrix}$$

$$= c_1 X_{re}(t) + c_2 X_{im}$$

$$X_{im}(t) = \begin{pmatrix}\sin(\beta t)\\ cos(\beta t)\end{pmatrix}$$

$$= c_1\begin{pmatrix}\cos(\beta t)\\ -sin(\beta t)\end{pmatrix} + c_2\begin{pmatrix}\sin(\beta t)\\ cos(\beta t)\end{pmatrix}$$

$$\boxed{x(t) = c_1 \cos(\beta t) + c_2 \sin(\beta t)}$$

also obtained by solving a 2^{nd}-order ODE for x

$$\boxed{y(t) = -c_1 \sin(\beta t) + c_2 \cos(\beta t)}$$

Note that $\beta y = x'$

Solution for a Center with $\lambda = \pm i\beta$

$\lambda_{1,2} = \pm\, i\, \beta$

$X' = 2y$ $\quad\quad$ $Y' = -2x$

Trajectories (time varying)

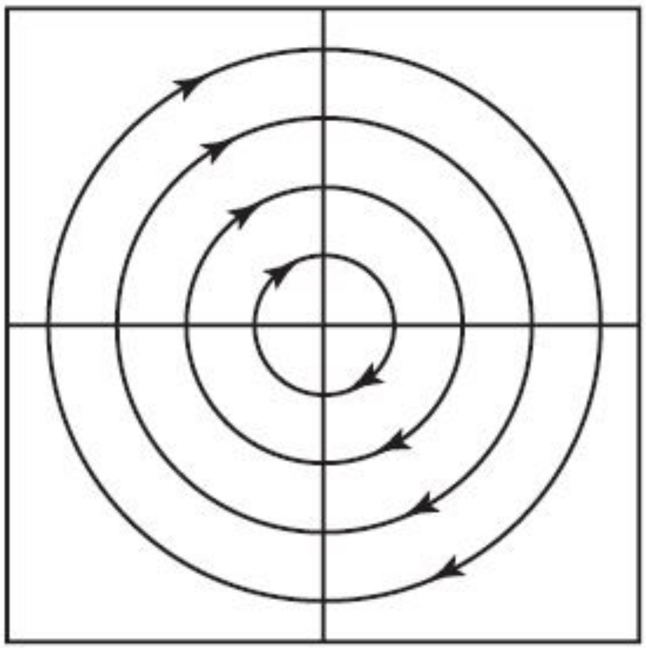
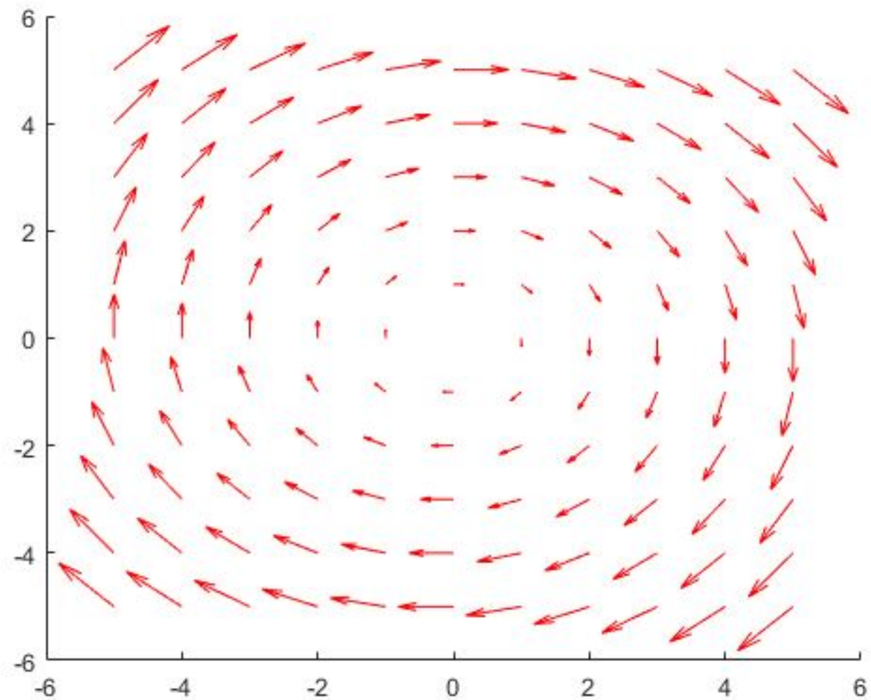


Figure 3.4 Phase portrait for a center.

Vector Fields (at a given time)



MATLAB Plot for Figure 3.4

$$\nabla \cdot \vec{F} = P_x + Q_y = 0$$

$$\nabla \times \vec{F} = Q_x - P_y = -2 < 0$$

A Planar System on a Complex Plane

$$\frac{d}{dt}$$

$$\boxed{\begin{aligned} x' &= y \\ y' &= -x, \end{aligned}}$$

$$t = \beta\tau$$

$$\frac{d}{d\tau}$$

$$x' = \beta y$$
$$y' = -\beta x,$$

$$u = x + iy$$

$$w = x - iy$$

$$u' = x' + iy'$$
$$= y - ix = -i(x + iy)$$
$$= -iu$$

$$\boxed{u' = -iu}$$

$$u = c_1 e^{-it} = c_1\cos(\mathrm{t}) - ic_1 \sin(\mathrm{t})$$

$$x = c_1\cos(\mathrm{t})$$

$$y = -c_1\sin(\mathrm{t})$$

- Find the ODE for *w*
- Send your results via "chat"
- You have 3 (up to 5) minutes

A Planar System on a Complex Plane

$$\frac{d}{dt} \qquad\qquad \frac{d}{d\tau}$$

$$\boxed{\begin{aligned} x' &= y \\ y' &= -x, \end{aligned}} \qquad t = \beta\tau \qquad \Leftarrow \qquad \begin{aligned} x' &= \beta y \\ y' &= -\beta x, \end{aligned}$$

$$u = x + iy$$

$$u' = x' + iy'$$

$$= y - ix = -i(x + iy)$$

$$= -iu$$

$$\boxed{u' = -iu} \qquad c = c_1 + ic_2$$

$$u = ce^{-it} = c\cos(\mathrm{t}) - ic\sin(\mathrm{t})$$

$$x = \mathrm{c}_1\cos(\mathrm{t}) + \mathrm{c}_2\sin(t)$$

$$y = -\mathrm{c}_1\sin(\mathrm{t}) + \mathrm{c}_2\cos(t)$$

$$w = x - iy$$

$$w' = x' - iy'$$

$$= y + ix = i(x - iy)$$

$$= iw$$

$$\boxed{w' = iw} \qquad \mathrm{d} = c_1 - ic_2$$

$$\mathrm{w} = d\, e^{it} = d\cos(\mathrm{t}) + id\ \sin(\mathrm{t})$$

$$x = \mathrm{c}_1\cos(\mathrm{t}) + \mathrm{c}_2\sin(t)$$

$$y = -\mathrm{c}_1\sin(\mathrm{t}) + \mathrm{c}_2\cos(t)$$

Solution for a Center with $\lambda = \pm i\beta$

$$\boxed{\begin{aligned} x' &= y = P \\ y' &= -x = Q \end{aligned}}$$

$$x = c_1\cos(t) + c_2\sin(t)$$
$$y = -c_1\sin(t) + c_2\cos(t)$$

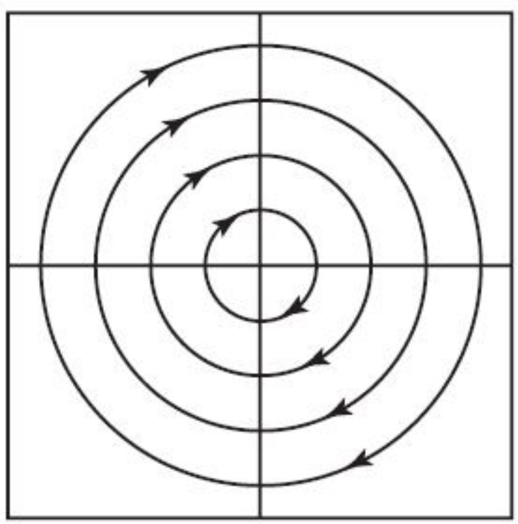
$$\nabla\cdot\vec{F} = P_x + Q_y = 0$$
$$\nabla\times\vec{F} = Q_y - P_x = -2$$

$$x^2 = (c_1\cos(t) + c_2\sin(t))^2 = c_1^2\cos^2(t) + 2c_1c_2\cos(t)\sin(t) + c_2^2\sin^2(t)$$

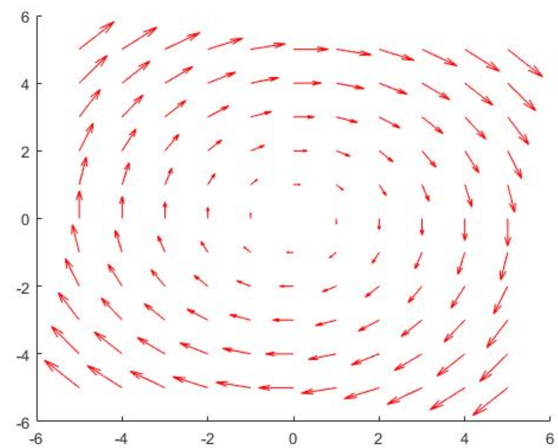
$$y^2 = (-c_1\cos(t) + c_2\sin(t))^2 = c_1^2\sin^2(t) - 2c_1c_2\cos(t)\sin(t) + c_2^2\cos^2(t)$$

$$x^2 + y^2 = c_1^2(\cos^2(t) + \sin^2(t)) + c_2^2(\cos^2(t) + \sin^2(t)) = c_1^2 + c_2^2$$

concentric circles



$$\vec{F} = (P, Q) = (y, -x)$$



Simple 2D Systems with Complex Eigenvalues

$$x' = ax + by \quad (= P(x,y)) \qquad (1)$$

$$y' = cx + dy \quad (= Q(x,y)\ ) \qquad (2)$$

$$X' = AX \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

(I) pure imaginary eigenvalues

$$A = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}$$

$$x' = \cancel{ax} + \beta y \quad (= P(x,y))$$

$$y' = \beta x + \cancel{dy} \quad (= Q(x,y))$$

$D.$ $\lambda_{1,2} = \pm i\,\beta$: center

(II) complex eigenvalues with $Re \neq 0$

$$\boxed{A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}}$$

$$x' = \alpha x + \beta y \quad (= P(x,y))$$

$$y' = \beta x + \alpha y \quad (= Q(x,y))$$

$E.$ $\lambda_{1,2} = \alpha \pm i\,\beta$: spiral source or sink

(E): Spiral Sink or Spiral Source with $\lambda = \alpha \pm i\beta$

Consider $\begin{aligned} x' &= \alpha x + \beta y \\ y' &= -\beta x + \alpha y \end{aligned}$ $\qquad A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$

Let $|A - \lambda I| = 0 \Longrightarrow$ $\boxed{\lambda = \alpha \pm i\beta}$

$$AV_0 = \lambda V_0 \qquad \begin{aligned} \alpha x_0 + \beta y_0 &= \lambda x_0 \\ -\beta x_0 + \alpha y_0 &= \lambda y_0 \end{aligned}$$

$\boxed{\text{Consider } \lambda = \alpha + i\beta}$

$$\begin{aligned} \alpha x_0 + \beta y_0 &= (\alpha + i\beta) x_0 \\ -\beta x_0 + \alpha y_0 &= (\alpha + i\beta) y_0 \end{aligned} \qquad y_0 = i x_0 \qquad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \\ i x_0 \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Obtain

$\boxed{V_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}}$ as an eigenvector associated with $\lambda = \alpha + i\beta$

(E): Spiral Sink or Spiral Source with $\lambda = \alpha \pm i\beta$

Thus, a general solution is written as

$$X(t) = ae^{\alpha \boxed{+i\beta t} \begin{pmatrix} 1 \\ i \end{pmatrix}} + be^{\alpha - i\beta t} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$= e^{\alpha t}(\mathrm{c}_1 \mathrm{X}_{\mathrm{re}}(\mathrm{t}) + c_2 X_{im})$$

$$\mathrm{X}_{\mathrm{re}}(\mathrm{t}) = \begin{pmatrix} \cos(\beta t) \\ -sin(\beta t) \end{pmatrix} \quad \mathrm{X}_{im}(\mathrm{t}) = \begin{pmatrix} \sin(\beta t) \\ cos(\beta t) \end{pmatrix}$$

$$Re\left(e^{i\beta t}\begin{pmatrix} 1 \\ i \end{pmatrix}\right) = Re\left((\cos(\beta t) + isin(\beta t))\begin{pmatrix} 1 \\ i \end{pmatrix}\right) = \begin{pmatrix} \cos(\beta t) \\ -sin(\beta t) \end{pmatrix} = X_{re}$$

$$\mathrm{Im}\left(e^{i\beta t}\begin{pmatrix} 1 \\ i \end{pmatrix}\right) = Im\left((\cos(\beta t) + isin(\beta t))\begin{pmatrix} 1 \\ i \end{pmatrix}\right) = \begin{pmatrix} \sin(\beta t) \\ cos(\beta t) \end{pmatrix} = X_{im}$$

(E): Spiral Sink or Spiral Source with $\lambda = \alpha \pm i\beta$

Thus, a general solution is written as

$$X(t) = ae^{\alpha+i\beta t}\begin{pmatrix}1\\ i\end{pmatrix} + be^{\alpha-i\beta t}\begin{pmatrix}1\\ -i\end{pmatrix}$$

$$X_{re}(t) = \begin{pmatrix}\cos(\beta t)\\ -sin(\beta t)\end{pmatrix}$$

$$= e^{\alpha t}(c_1 X_{re}(t) + c_2 X_{im})$$

$$X_{im}(t) = \begin{pmatrix}\sin(\beta t)\\ cos(\beta t)\end{pmatrix}$$

decaying or growing

oscillatory

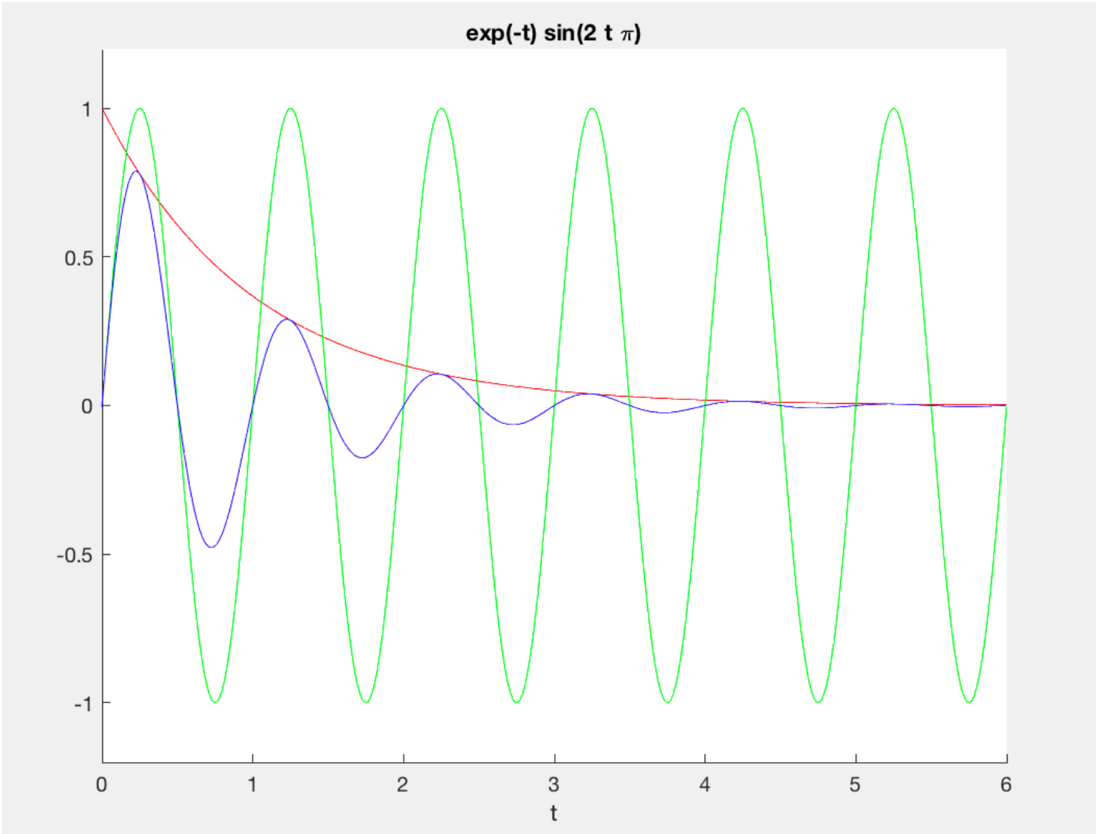
- $\alpha < 0$ spirals into the origin, a spiral sink

oscillatory with time varying radii

- $\alpha > 0$ spirals away the origin, a spiral source

A Spiral Sink with $\lambda = \alpha \pm i\beta$: Oscillatory Decay

- red: $e^{\alpha t}, \alpha < 0$
- green: $\sin(\beta t)$
- blue: $e^{\alpha t}\sin(\beta t)$



Time

```
syms t a b
a=-1
b=2*pi
fexp=exp(a*t)
fosc=sin(b*t)
hold on
ezplot (fexp, [0, 6, -1.2, 1.2])
ezplot (fosc, [0, 6, -1.2, 1.2])
ezplot (fexp*fosc, [0, 6, -1.2, 1.2])
```

Codes

```
clear
syms t a b
a=-1
b=2*pi
fexp=exp(a*t)
fosc=sin(b*t)
hold on
ez1=ezplot (fexp, [0, 6, -1.2, 1.2])
ez2=ezplot (fosc, [0, 6, -1.2, 1.2])
ez3=ezplot (fexp*fosc, [0, 6, -1.2, 1.2])
set(ez1,'color',[1 0 0])
set(ez2,'color’,[0 1 0])
set(ez3,'color’,[0 0 1])
```

A Spiral Sink with $\lambda = \alpha \pm i\beta$

$$X(t) = e^{\alpha t}(c_1 X_{re}(t) + c_2 X_{im})$$

$$X_{im}(t) = \begin{pmatrix} \sin(\beta t) \\ cos(\beta t) \end{pmatrix}$$

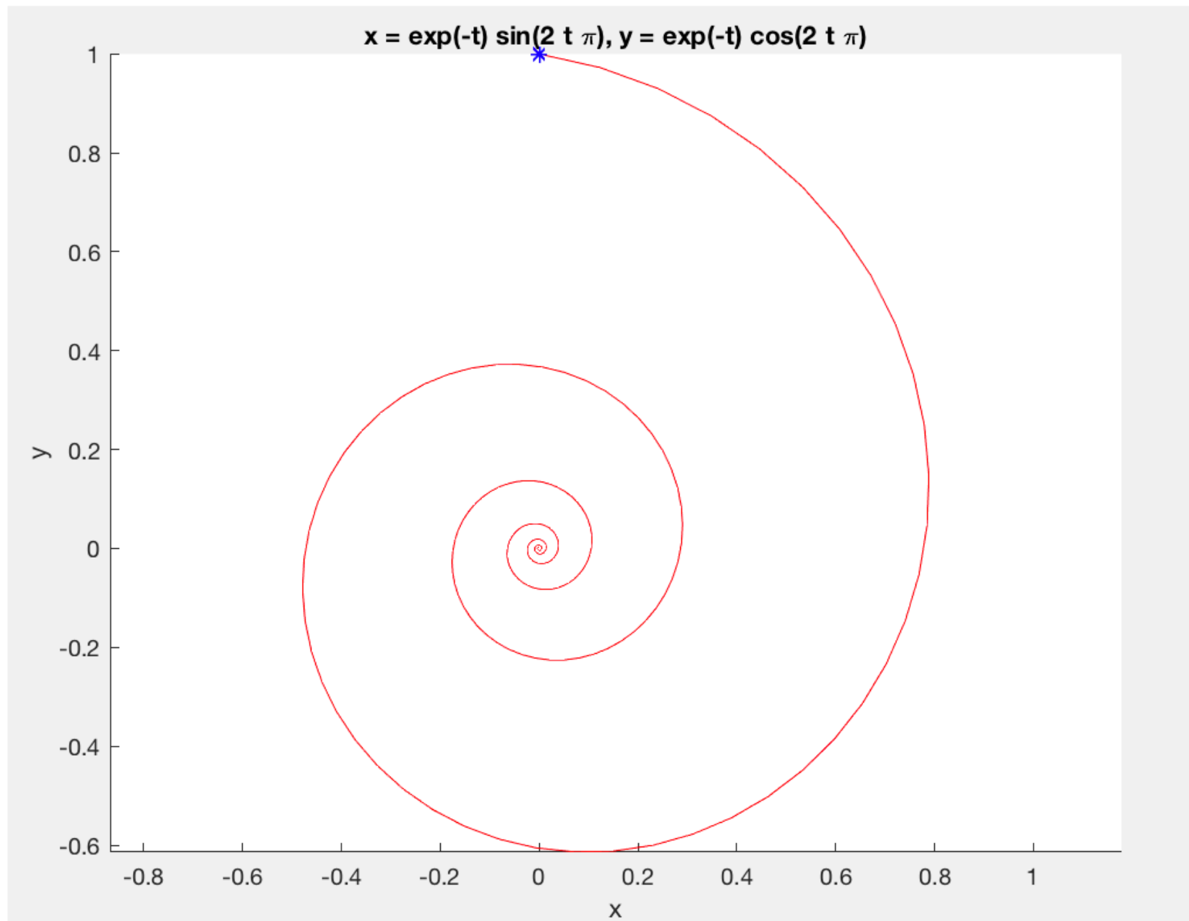
$$c_1 = 0\,;\, c_2 = 1$$

$$x(t) = e^{\alpha t}\sin(\beta t)$$
$$y(t) = e^{\alpha t}\cos(\beta t)$$

```
clear
syms t a b x y
a=-1
b=2*pi
x=exp(a*t)*sin(b*t)
y=exp(a*t)*cos(b*t)

x0=0
y0=1
hold on
ez1=ezplot (x,y, [0, 6])
plot(x0, y0, 'b*')
set(ez1,'color',[1 0 0])
```

y



x

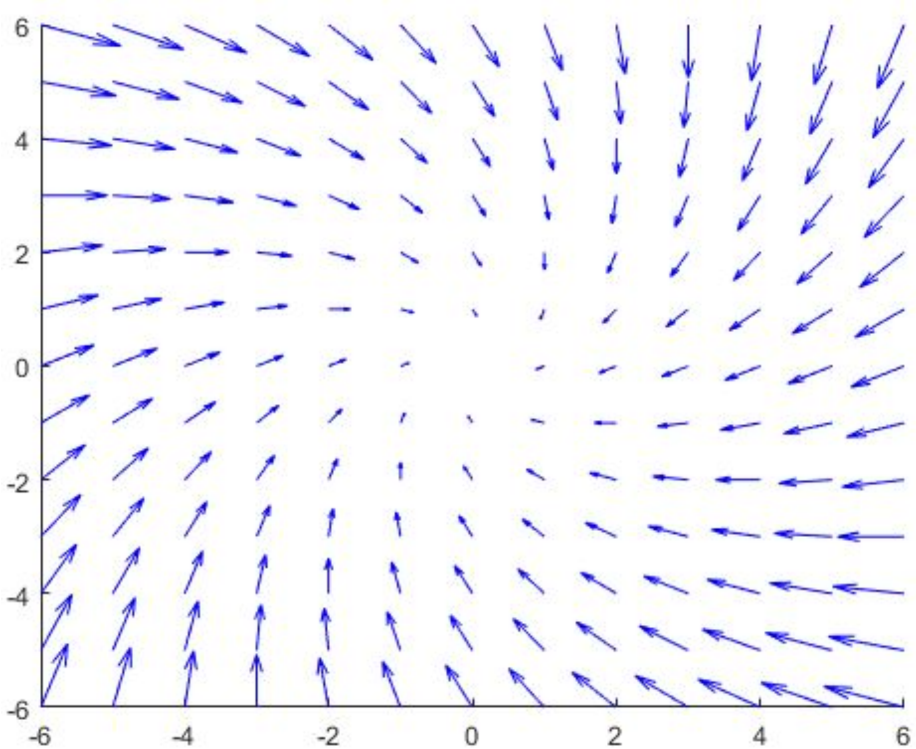
A Spiral **Sink** with $\lambda = \alpha \pm i\beta$ and $\alpha < \mathbf{0}$

X' = -2x + y Y' = -x – 2y

$\lambda_{1,2} = \alpha \pm i\,\beta$ $\alpha = -2 < 0$

Vector Fields (at a given time)

Trajectories (time varying)



MATLAB Plot for Figure 3.5a

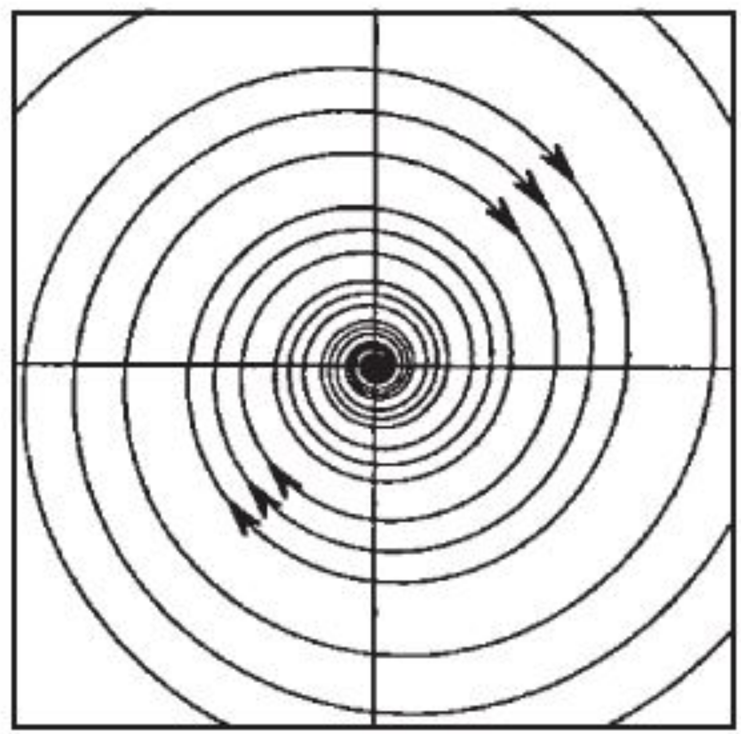


Figure 3.5a Phase portrait for spiral sink

A Spiral Source with $\lambda = \alpha \pm i\beta$ and $\alpha > 0$

$\lambda_{1,2} = \alpha \pm i\,\beta$ $\alpha = 2 > 0$ $X' = 2x + y$ $Y' = -x + 2y$

Trajectories (time varying)

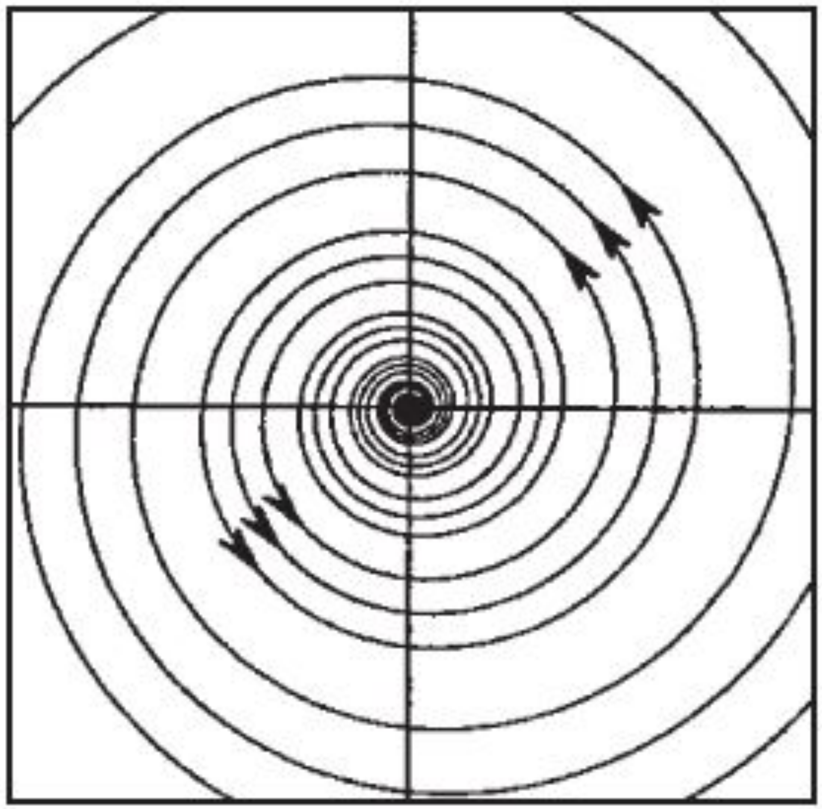


Figure 3.5b Phase portrait for spiral source

Is Weather Chaotic?

Coexistence of Chaos and Order within a Generalized Lorenz Model

by

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"A Paradigm Shift" in Predictability Study

- *``As with Poincare and Birkhoff, everything centers around periodic solutions" (Lorenz, 1993).*

- After Lorenz (1963, 1972), Prof. Lorenz and chaos advocates focused on the existence of non-periodic solutions and their complexities.

- Based on the concept of attractor coexistence within the original and generalized Lorenz models (Shen, 2019a), we (Shen et al., 2019) propose a revised view that focus on the duality of chaos and order.

Review: Double Root

5

$$ay'' + by' + cy = 0$$

where a, b, and c are constants and $a \neq 0$.

Summary of Cases I–III

Case	Roots of (2)	Basis of (1)	General Solution of (1)
II	Real double root $\lambda = -\frac{1}{2}a$	$e^{-ax/2}$, $xe^{-ax/2}$	$y = (c_1 + c_2 x)e^{-ax/2}$

Sec 3.3 Repeated Eigenvalues

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$x' = \lambda_1 x$$
$$y' = \lambda_1 y$$

Uncoupled System

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$x' = \lambda_1 x + y$$
$$y' = \lambda_1 y$$

Coupled System

x: responder

y: driver

Sec 3.3 Repeated Eigenvalues: (I)

Consider

$$x' = \lambda_1 x$$
$$y' = \lambda_1 y$$

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$$

Uncoupled

Let $|A - \lambda I| = 0 \Longrightarrow$ $(\lambda - \lambda_1)^2 = 0$ $\boxed{\lambda = \lambda_1}$

$$AV_0 = \lambda V_0$$

$$\lambda_1 x_0 = \lambda x_0$$
$$\lambda_1 y_0 = \lambda y_0$$

$\boxed{\lambda = \lambda_1}$

$\lambda_1 x_0 = \lambda_1 x_0$ Any number

$\lambda_1 y_0 = \lambda_1 y_0$ Any number

$V_0 = \begin{pmatrix} x \\ y \end{pmatrix}$, any x and y any straight line through (0,0)

Alternatively, $x = c_1 e^{\lambda_1 t}$ $y = c_2 e^{\lambda_1 t}$

Sec 3.3 Repeated Eigenvalues: (II)

Consider

$$x' = \lambda_1 x + y$$
$$y' = \lambda_1 y$$

$$A = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}$$

coupled

Let $|A - \lambda I| = 0 \Longrightarrow \quad (\lambda - \lambda_1)^2 = 0 \quad \boxed{\lambda = \lambda_1}$

$$AV_0 = \lambda V_0 \qquad \begin{aligned} \lambda_1 x_0 + y_0 &= \lambda x_0 \\ \lambda_1 y_0 &= \lambda y_0 \end{aligned}$$

$$\boxed{\lambda = \lambda_1} \qquad \begin{aligned} y_0 &= 0 \\ \lambda_1 y_0 &= \lambda_1 y_0 \end{aligned} \qquad V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

How to find V_2?

(1) Apply the concept of generalized eigenvalue problem
(2) Solve the ODE for the driver that is uncoupled with the ODE for the responder; then solve the ODE for the responder
(3) Transform the above system to a 2nd order ODE

Generalized Eigenvector

TBD

Definition 5.6.1

We say that $\mathbf{u} \neq \mathbf{0}$ is a **generalized eigenvector** of $\mathbf{A}$ associated to the eigenvalue λ if

$$(\mathbf{A} - \lambda\mathbf{I})^k \mathbf{u} = \mathbf{0} \tag{5.78}$$

for some integer $k > 0$. The index of the generalized eigenvector is the smallest k satisfying (5.78).

Here u may be V_2, V_3, … V_k etc.

Let V_1 be the eigenvector of A associated to the eigenvalue λ.
Namely, $(A - \lambda I)V_1 = 0$. (a notation used in the HSD)

Consider u to be V_2 and $(A - \lambda I)V_2 = V_1$. Therefore, we have
$(A - \lambda I)^2 V_2 = (A - \lambda I)V_1 = 0$. V_2 is a generalized eigenvector.

Wirkus and Swift

Generalized Eigenvspace

TBD

For k > 1, we see that the original eigenvector (v) gives rise to a set of generalized eigenvectors (u_1, $u_2 \ldots u_k$):

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{u}_{k-1} = \mathbf{u}_{k-2},$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{u}_{k-2} = \mathbf{u}_{k-3}, \quad \cdots,$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{u}_3 = \mathbf{u}_2,$$

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{u}_2 = \mathbf{u}_1,$$

$$\boxed{(\mathbf{A} - \lambda \mathbf{I})\mathbf{u}_1 = \mathbf{v}.}$$

The set $\{\mathbf{v}, \mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{k-1}\}$ satisfying (5.79) is called a **chain** of generalized eigenvectors. The chain is determined entirely by the choice of $\mathbf{v}$, which is referred to as the **bottom of the chain**. For those that have read Section 5.3, we define

$$\tilde{E}_\lambda = \{v | (\mathbf{A} - \lambda \mathbf{I})^k \mathbf{v} = \mathbf{0} \text{ for some } k\} \tag{5.80}$$

as the generalized eigenspace of λ. From (5.50), we note that $E_\lambda \subseteq \tilde{E}_\lambda$. If λ is an eigenvalue of $\mathbf{A}$ of multiplicity m, then $\tilde{E}_\lambda$ is a subspace of dimension m.

THEOREM 5.6.3

Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for the $k \times k$ matrix $\mathbf{A}$ in which (i) λ is an eigenvalue of multiplicity k with a single eigenvector $\mathbf{v}$ and (ii) $\{\mathbf{v}, \mathbf{u}_1, \cdots, \mathbf{u}_{k-1}\}$ is the corresponding chain of generalized eigenvectors. Set

$$\begin{aligned}
\mathbf{x}_1 &= te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{u}_1 \\
\mathbf{x}_2 &= \frac{t^2}{2!}e^{\lambda t}\mathbf{v} + te^{\lambda t}\mathbf{u}_1 + e^{\lambda t}\mathbf{u}_2 \\
&\vdots \\
\mathbf{x}_{k-1} &= \frac{t^{k-1}}{(k-1)!}e^{\lambda t}\mathbf{v} + \frac{t^{k-2}}{(k-2)!}e^{\lambda t}\mathbf{u}_1 \cdots + te^{\lambda t}\mathbf{u}_{k-2} + e^{\lambda t}\mathbf{u}_{k-1}^{(k)}.
\end{aligned}$$

Then $e^{\lambda t}\mathbf{v}, \mathbf{x}_1, \cdots, \mathbf{x}_{k-1}$ are linearly independent and the general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ can be written as

$$\mathbf{x} = c_1 e^{\lambda t}\mathbf{v} + c_2\mathbf{x}_1 + c_3\mathbf{x}_2 + \cdots + c_k\mathbf{x}_{k-1}.$$

(1) Solutions using the Concept of Generalized Eigenvector

Solve for V_1 $\quad (A-\lambda I)V_1 = 0 \quad A = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \quad \boxed{\lambda = \lambda_1} \quad V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Solve for V_2 $\quad \boxed{(A-\lambda I)V_2 = V_1}$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \boxed{y = 1} \qquad V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Obtain X_1 $\quad X_1 = e^{\lambda t} V_1 = e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Construct X_2 $\quad \boxed{X_2 = t e^{\lambda t} V_1 + e^{\lambda t} V_2} = e^{\lambda t} \begin{pmatrix} t \\ 0 \end{pmatrix} + e^{\lambda t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = e^{\lambda t} \begin{pmatrix} t \\ 1 \end{pmatrix}$

Obtain X $\quad \boxed{X = \alpha X_1 + \beta X_2 = \alpha e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda t} \begin{pmatrix} t \\ 1 \end{pmatrix}}$

(2) Solve the ODEs for the Driver and Responders

Consider

$$x' = \lambda_1 x + y$$
$$y' = \lambda_1 y$$

uncoupled

Start with the uncoupled equation

$$y' = \lambda_1 y$$

$$\boxed{y = \beta e^{\lambda_1 t}}$$

The 1st eq becomes:

$$x' - \lambda_1 x = \beta e^{\lambda_1 t}$$

$$x' + P(t)x = Q(t)$$

$$I = e^{-\lambda_1 t}$$

$$\boxed{x = \frac{1}{I}\left[\int IQdt + C\right] = \frac{1}{I}(\beta \mathrm{t} + \mathrm{C}) = Ce^{\lambda_1 t} + \beta \mathrm{t}e^{\lambda_1 t}}$$

- Forced problem
- Non-autonomous system
- The system's "frequency" is the same as the "frequency" of the forcing.

(2) Solve the ODEs for the Driver and Responders

$$\boxed{x = \frac{1}{I}\left[\int IQdt + C\right] = \frac{1}{I}(\beta \mathrm{t} + \mathrm{C}) = Ce^{\lambda_1 t} + \beta \mathrm{t}e^{\lambda_1 t}}$$

$$\boxed{y = \beta e^{\lambda_1 t}}$$

$$X = \binom{x}{y} = \begin{pmatrix} Ce^{\lambda_1 t} + \beta \mathrm{t}e^{\lambda_1 t} \\ \beta e^{\lambda_1 t} \end{pmatrix} = Ce^{\lambda_1 t}\binom{1}{0} + \beta e^{\lambda_1 t}\binom{t}{1}$$

$\lambda_1 < 0$

$$\lim_{t\to\infty} te^{\lambda_1 t} = \lim_{t\to\infty} \frac{t}{e^{-\lambda_1 t}} = \lim_{t\to\infty} \frac{1}{(-\lambda_1)e^{-\lambda_1 t}} = 0$$

Low-D Non-autonomous → High-D autonomous

Consider

$$\boxed{x' - \lambda_1 x = \beta e^{\lambda_1 t}}$$

Low dimensional, Non-autonomous

Introduce a new variable to represent the (time varying) forcing term

$$y = \beta e^{\lambda_1 t}$$

$$x' = \lambda_1 x + y$$
$$y' = \lambda_1 y$$

$$\boxed{\begin{aligned} x' &= \lambda_1 x + y \\ y' &= \lambda_1 y \end{aligned}}$$

High dimensional, autonomous

(3) Convert and Solve a 2nd Order ODE

Consider

$$x' = \lambda_1 x + y$$
$$y' = \lambda_1 y$$

uncoupled

Transform into a 2nd order ODE

$$x'' = \lambda_1 x' + y'$$

$$x'' = \lambda_1 x' + \lambda_1 y \qquad x'' = \lambda_1 x' + \lambda_1 (x' - \lambda_1 x)$$

$$\boxed{x'' - 2\lambda_1 x' + \lambda_1^2 x = 0}$$

Assume

$$\boxed{x = ke^{\lambda t}}$$

Obtain

$$(\lambda - \lambda_1)^2 = 0 \qquad \boxed{\lambda = \lambda_1}$$

$$\boxed{x_1 = e^{\lambda_1 t}} \qquad \boxed{x_2 = te^{\lambda_1 t} = \lim_{\lambda \to \lambda_1} \frac{e^{\lambda t} - e^{\lambda_1 t}}{\lambda - \lambda_1}}$$

Repeated Negative Eigenvalue

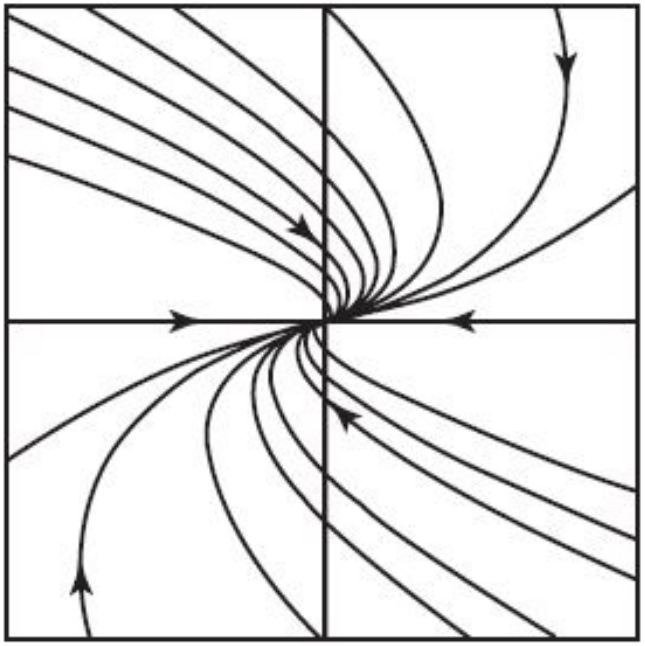

Figure 3.6 Phase portrait for a system with repeated negative eigenvalues.

A Summary: Repeated Eigenvalues

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$x' = \lambda_1 x + y$$
$$y' = \lambda_1 y$$

Coupled System

y: driver

x: responder

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$x' = \lambda_1 x$$
$$y' = \lambda_1 y$$

Uncoupled System

$$x = c_1 e^{\lambda_1 t}$$

$$y = c_2 e^{\lambda_1 t}$$

$$\boxed{X = \alpha X_1 + \beta X_2 = \alpha e^{\lambda t} \overset{V_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} + \beta e^{\lambda t} \overset{V_2}{\begin{pmatrix} t \\ 1 \end{pmatrix}}}$$

How to find V_2?

(1) Apply the concept of generalized eigenvalue problem
(2) Solve the ODE for the driver that is uncoupled with the ODE for the responder; then solve the ODE for the responder
(3) Transform the above system to a 2nd order ODE