Final Exam Part A & Mini Project
Math 537 Ordinary Differential Equations
Due December 11, 2020

Student Name: ______________________________ **ID** ____________

Rules

A. The exam must be taken completely alone. Showing it or discussing it with anybody is forbidden.

B. Make an effort to make your submission clear and readable. Severe readability issues may be penalized by grade.

C. There are four problems. Please complete the first two questions (both Q1 and Q2) and one of the last two questions (i.e., Q3 or Q4).

D. For students who work on mini projects, please submit your reports as responses to the 5th question (Q5).

E. Please submit your work to Gradescope by 11:59 pm on December 11, 2020.

1: [25 points] Consider the following second-order ordinary differential equations (ODEs) for nonlinear pendulum oscillations (e.g., Figure 1):

$$\frac{d^2\theta}{dt^2} + \epsilon\frac{d\theta}{dt} + sin(\theta) = 0. \tag{1.1}$$

By assuming $\theta = z + \pi$, we transform the above equation into the following:

$$\frac{d^2 z}{dt^2} + \epsilon\frac{dz}{dt} - sin(z) = 0. \tag{1.2}$$

Applying Taylor series expansions, Eq. (1.2) with $\epsilon = 0$ or $\epsilon \neq 0$ can be simplified into one of the following systems:

$$\frac{d^2 z}{dt^2} - z = 0. \tag{1.3}$$

$$\frac{d^2 z}{dt^2} + \epsilon\frac{dz}{dt} - z = 0. \tag{1.4}$$

$$\frac{d^2 z}{dt^2} - \left(z - \frac{z^3}{6}\right) = 0. \tag{1.5}$$

(a) [12 points] Perform a linear stability analysis in each of Eqs. (1.3)-(1.5).

(b) [8 points] Compute potential energy functions for Eqs. (1.3) and (1.5). Find extrema of the potential energy functions to reveal the stability of equilibrium points.

(c) [5 points] Discuss the concept of structural stability using results in (1a) and (1b).

2: [50 points] Complete a "mini report" based on the one slide summary (as shown in Figure 2). The following instructions are recommended.

(a) [10 points] Finish a one-paragraph summary.

(b) [10 points] Briefly summarize the major features of the 1st order ODEs (i.e., the column A), including stability of critical points, characteristics of solutions, the relation between the first and second ODEs, etc.

(c) [10 points] Discuss how to analyze a system of linear ODEs (i.e., expanding the information in the columns B and C)

(d) [10 points] Briefly summarize the major features of the 2nd order nonlinear ODEs (i.e., the column D), including stability of critical points, the relation among them, etc.

(e) [10 points] Discuss how to analyze a system of nonlinear high-order ODEs (e.g., stability analysis, a energy method, a perturbation method, the WKBJ or LG method, etc.)

3: [25 points] Consider the following second order ODE:

$$\epsilon^2 \frac{d^2U}{dx^2} = Q(x)U. \tag{3}$$

(a)[8 points] Briefly compare the WKBJ method and the Liouville-Green (LG) approximation.

(b)[12 points] For a small ϵ, find the approximate solution to the above equation using the WKBJ method. [hint: assume $U \sim exp\left(\frac{S(x)}{\epsilon} + C(x)\right)$].

(c)[5 points] Assume $\epsilon = 1$ and $Q(x_t) = 0$ where x_t is a turning point. Linearize $Q(x)$ with respect to the turning point and discuss the characteristic of the simpified system.

4: [25 points] Consider a boundary-layer problem with the following second-order linear differential equation:

$$\epsilon \frac{d^2y}{dx^2} + (1+\epsilon)\frac{dy}{dx} + y = 0,$$

$$y(0) = 0 \text{ and } y(1) = 1.$$

Apply the boundary layer method to solve the above equation for

(a) [5 points] the inner solution in the inner region;

(b) [10 points] the outer solution in the outer region;

(c) [5 points] the solution (y_{match}) in the overlap region;

(d) [5 points] the uniform approximation(y_{unif}) to y (i.e., $y_{unif} \sim y$).

5: [XYZ points.] Please complete the following tasks.

(a) Provide a preferred value of XYZ (here, $0 \leq XYZ \leq 100$).

(b) Submit your report and QuadChart.

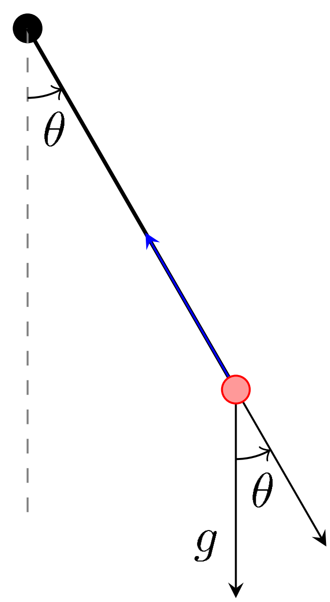

Figure 1: A pendulum consisting of a weightless rod of length L and a bob with a mass of m. The bob and the point of support are marked as a red and black dot, respectively. The parameter "g" denotes the gradational force. The angle θ is measured in the counterclockwise direction. Stable and unstable equilibrium points appear at $\theta = 0$ and $\theta = \pi$, respectively.

One Slide Summary

(A) 1st order	(B) 2nd order	(C) eigenvalue problem
$y' = \alpha y - \beta y^2$ (logistic eq.)	$x'' + \beta x' + \alpha x = 0$	$x' = ax + by$ $y' = cx + dy$
$y' = \alpha y - \beta y^3$	$x' = y$ $y' = -\alpha x - \beta y$	$X' = AX$ $AX = \lambda X$ $X = \begin{pmatrix} x \\ y \end{pmatrix}; A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

(D) nonlinear	(E) a system of ODEs	**(F)**
$x'' - \alpha x + \beta x^3 = 0$ (DE-sech)	$x' = y \equiv F$ $y' = \alpha x - \beta x^3 \equiv G$	$JX = \lambda X$ $J = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{X_c}$
$x'' - \alpha x + \beta x^2 = 0$ (DE-sech^2)	$x' = y \equiv F$ $y' = \alpha x - \beta x^2 \equiv G$	

Figure 2: One slide summary. ($\alpha > 0$ and $\beta > 0$)