MATH 537, Fall 2020
Ordinary Differential Equations

Lecture #4
Chapter 2 Systems of ODEs

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Key ODEs in Chapter 1

$$\frac{dx}{dt} = ax$$

bifurcation at $a = 0$

$$x = x_0 e^{at}$$

$$\boxed{\frac{dx}{dt} = ax\,(1 - x)}$$

bifurcation at $a = 0$

(the Logistic Eq)

$$x = \frac{x_0 e^{at}}{1 - x_0 + x_0 e^{at}}$$

(sigmoid function)

$$\boxed{\frac{dx}{dt} = x(1 - x) - h}$$

bifurcation at $\mathrm{h} = 1/4$

(the Logistic Eq with constant harvesting)

$$\boxed{\frac{dx}{dt} = x(1 - x) - h(1 + \sin(2\pi t))}$$

periodic forcing,
non-autonomous system

(the Logistic Eq with periodic harvesting)

Important Concepts

1. Bifurcation & Bifurcation points
2. Critical points, $f(x_c) = 0$;
3. (equilibrium points = fixed points = critical points)
4. Derivative tests & Perturbation method
5. General solution
6. Initial Value Problem (IVP)
7. Particular solution
8. Phase Line
9. Separable ODEs
10. Sink, Source, an Saddle
11. Stable vs. Unstable Solutions, $f'(x_c)$.
12. Structurally Stable vs. Unstable (i.e., with bifurcation)

Chapter 2: Systems of ODEs

In this chapter we begin the study of *systems of differential equations*. A system of differential equations is a collection of n interrelated differential equations of the form

$$
\begin{aligned}
x_1' &= f_1(t, x_1, x_2, \ldots, x_n) \\
x_2' &= f_2(t, x_1, x_2, \ldots, x_n) \\
&\vdots \\
x_n' &= f_n(t, x_1, x_2, \ldots, x_n).
\end{aligned}
$$

Here the functions f_j are real-valued functions of the $n+1$ variables $x_1, x_2, \ldots,$ x_n, and t. Unless otherwise specified, we will always assume that the f_j are C^∞ functions. This means that the partial derivatives of all orders of the f_j exist and are continuous.

- There are n dependent variables and one independent variable (t).
- The above consists of n ODEs with n functions, $f_j, j = 1,2, \ldots n$.
- f_j are C^∞.

Terminology: C^{∞} and C^{k}

A C^{∞} function is a function that is differentiable for all degrees of differentiation. For instance, $f(x) = e^{2x}$ is C^{∞} because its nth derivative $f^{n}(x) = 2^{n}e^{2x}$ exists and is continuous. All polynomials are C^{∞}. The reason for the notation is that C^{k} have k continuous derivatives.

A function with k continuous derivatives is called a C^{k} function. In order to specify a C^{k} function on a domain X, the notation $C^{k}(\mathrm{X})$ is used. The most common C^{k} space is C^{0}, the space of continuous functions, whereas C^{1} is the space of continuously differentiable functions.

http://mathworld.wolfram.com/C-InfinityFunction.html

Chapter 2. Systems of ODEs in Matrix Form

To simplify notation, we will use vector notation:

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

column vector

We often write the vector X as $(x_1, \ldots, x_n)$ to save space. row vector

Our system may then be written more concisely as

$$X' = F(t, X),$$

where

$$F(t, X) = \begin{pmatrix} f_1(t, x_1, \ldots, x_n) \\ \vdots \\ f_n(t, x_1, \ldots, x_n) \end{pmatrix}.$$

Chapter 2. Systems of ODEs in Matrix Form

Our system may then be written more concisely as

$$X' = F(t, X),$$

- The system of equations is called autonomous if none of the f_j depends on t, so the system becomes $X' = F(X)$.
- In analogy with first-order differential equations, a vector X_c for which $F(X_c)$ is called an equilibrium point for the system. An equilibrium point corresponds to a time-independent solution $X(t) = X_c$ of the system as before.

- ➢ For most of the rest of this book we will be concerned with autonomous systems.
- ➢ We will reserve capital letters for vectors or for vector-valued functions.
- ➢ We will always denote real variables or real-valued functions by lowercase letters.

2.1 Second-Order Differential Equations

Many of the most important differential equations encountered in science and engineering are second-order differential equations. These are differential equations of the form

$$x'' = f(t, x, x').$$

we note that these equations are a special subclass of two-dimensional systems of differential equations that are defined by simply introducing a second variable $y = x'$.

2.1 Second-Order Differential Equations

For example, consider a second-order constant coefficient equation of the form

$$\boxed{x'' + ax' + bx = 0.}$$

If we let $y = x'$, then we may rewrite this equation as a system of first-order equations:

$$\boxed{\begin{aligned} x' &= y \\ y' &= -bx - ay. \end{aligned}}$$

$$y' = x'' = -ax' - bx = -ay - bx = -bx - ay$$

Section 2.2: Planar Systems

In this chapter we will deal with autonomous systems in $\mathbb{R}^2$, which we will write in the form

$$x' = f(x,y)$$
$$y' = g(x,y),$$

thus eliminating the annoying subscripts on the functions and variables. As before, we often use the abbreviated notation $X' = F(X)$, where $X = (x,y)$ and $F(X) = F(x,y) = (f(x,y), g(x,y))$.

$$X' = F(X), \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad F(X) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$

2.2 Planar Systems: Vector vs. Directional Field

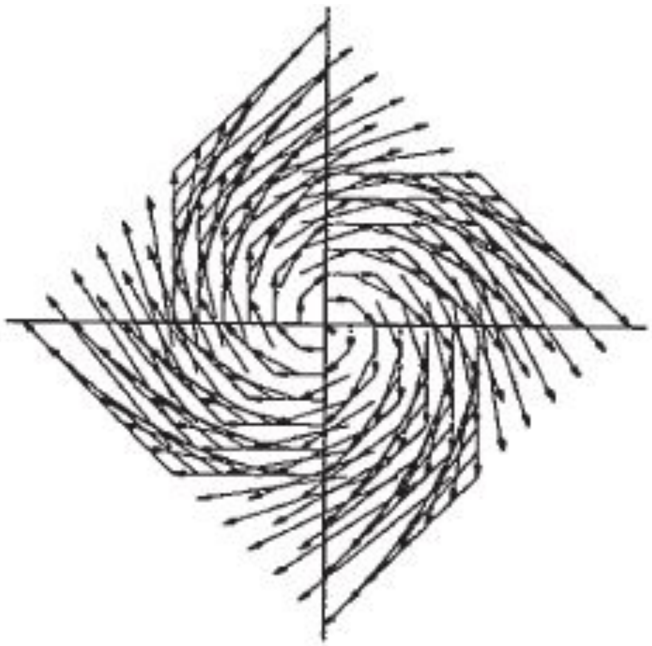
In analogy with the slope fields of Chapter 1, we regard the right side of this equation as defining a *vector field* on $\mathbb{R}^2$. That is, we think of $F(x, y)$ as representing a vector with x- and y-components that are $f(x, y)$ and $g(x, y)$, respectively. We visualize this vector as being based at the point (x, y). For

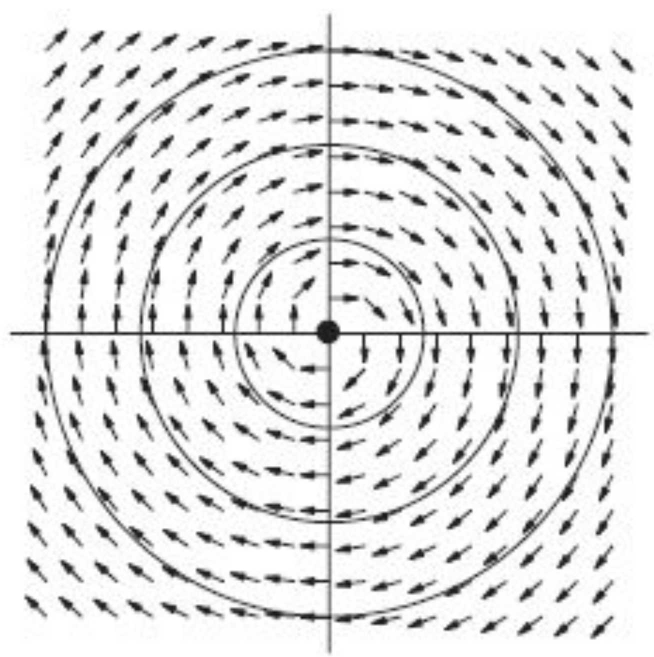
$$x' = y$$
$$y' = -x,$$

$$F(X) = \begin{pmatrix} y \\ -x \end{pmatrix}$$

vector field

direction & magnitude





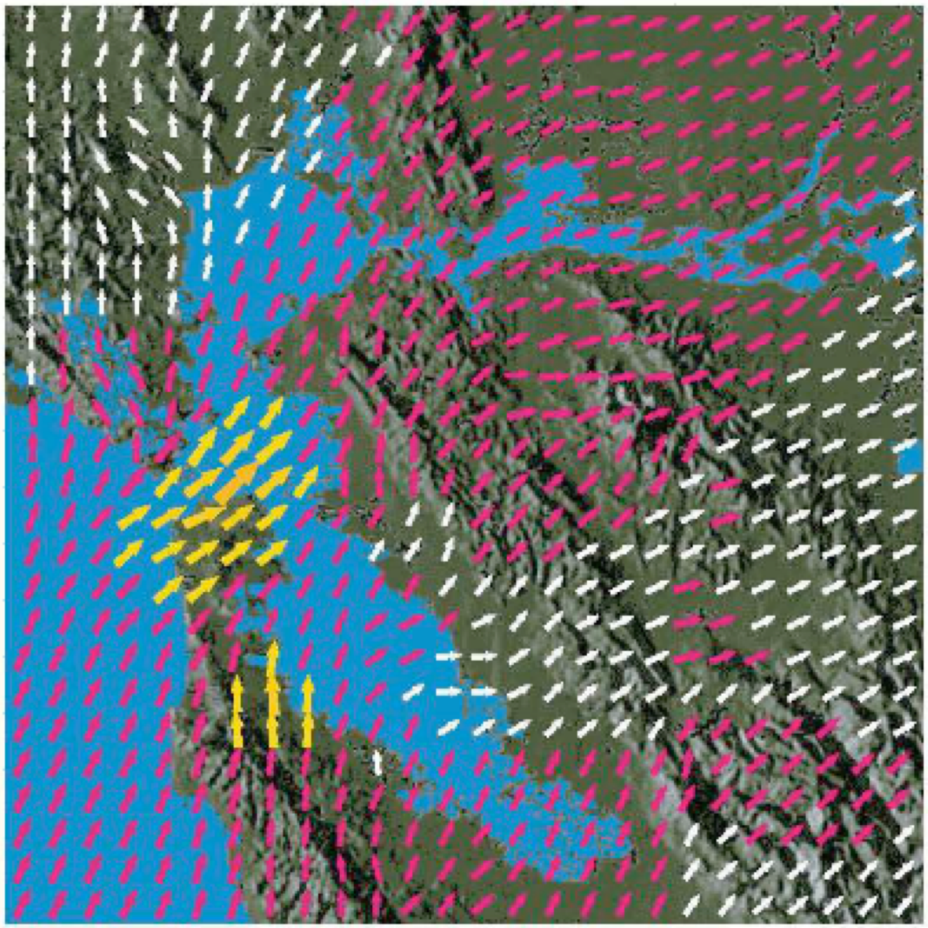
directional field

direction

Review of 16.1 Vector Fields (M252)

Assign a vector to each point. A vector includes a direction and magnitude.

$$\vec{V} = (u, v) = \left(u(x, y), v(x, y)\right)$$



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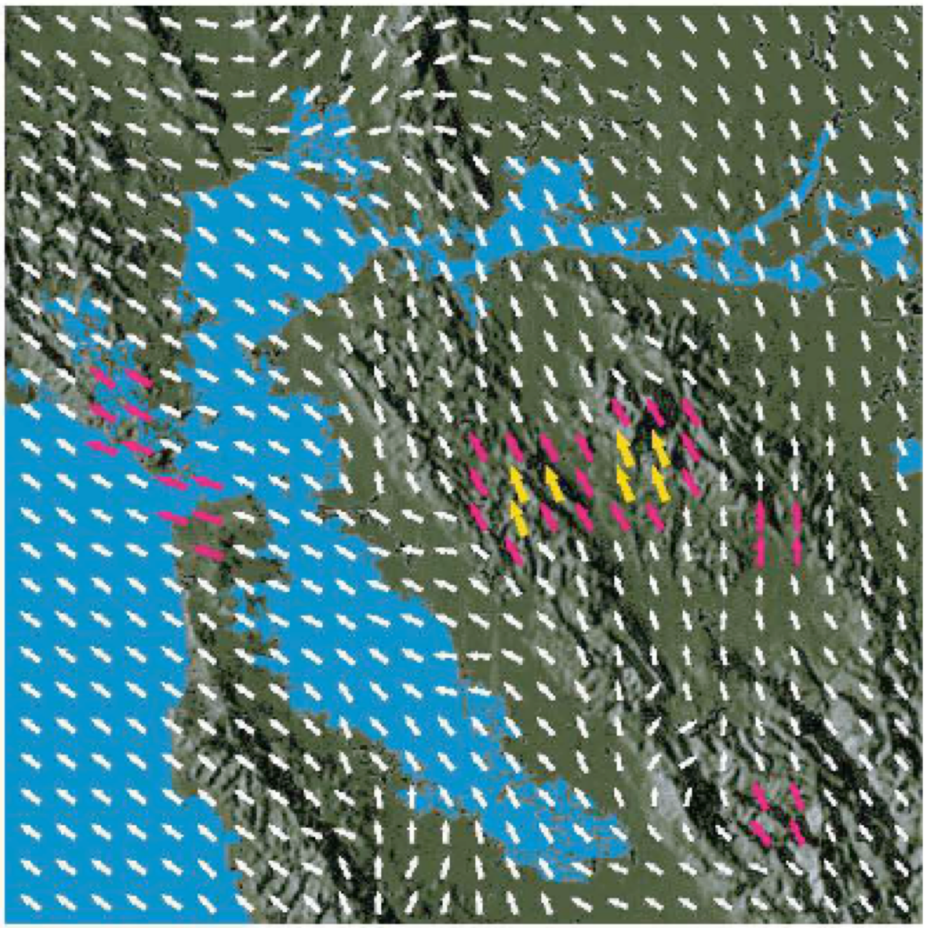

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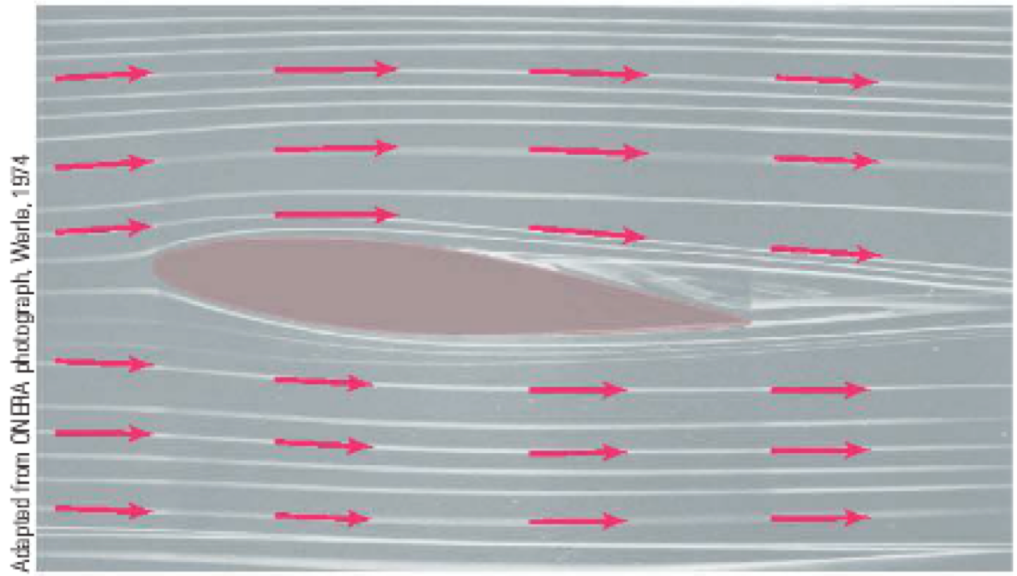
Review of 16.1 Vector Fields (M252)

$$\vec{V} = (u,v) = \big(u(x,y), v(x,y)\big)$$

$$\vec{F} = (P,Q) = \big(P(x,y), Q(x,y)\big)$$

(a) Ocean currents off the coast of Nova Scotia



(b) Airflow past an inclined airfoil

Review of Direction Fields (Math151)

slope only, dy/dx (direction)

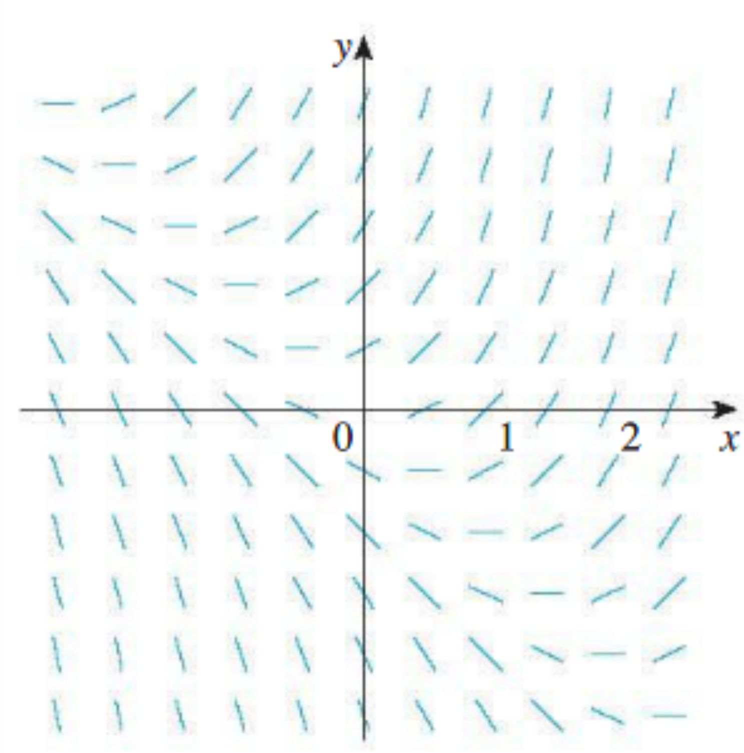


FIGURE 3
Direction field for $y' = x + y$

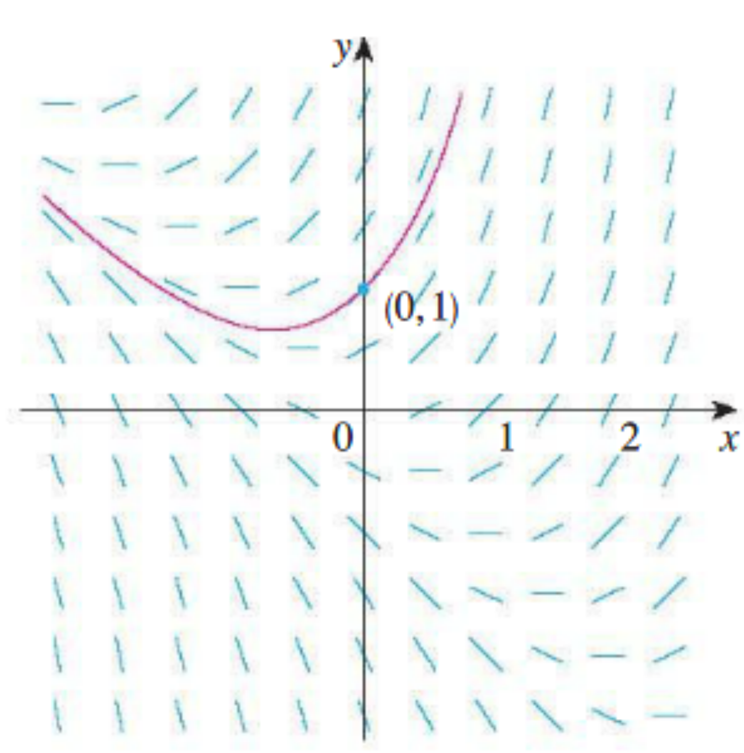


FIGURE 4
The solution curve through (0, 1)

Review of 16.1 2D Vector Fields

1 Definition Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

component functions P and Q as follows:

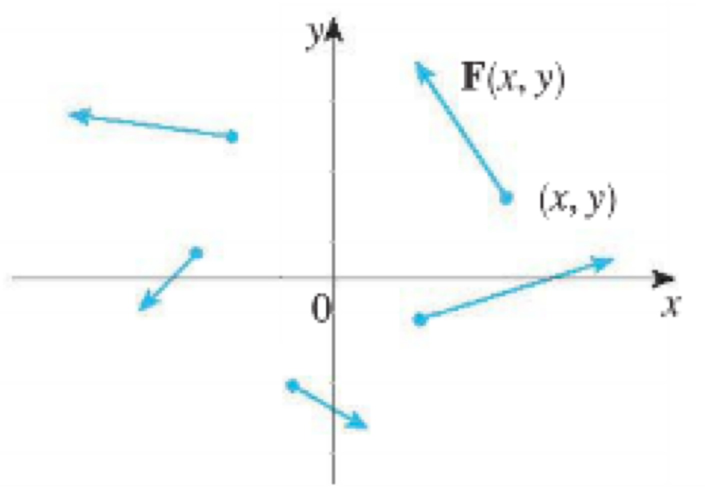
$$\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j} = \langle P(x, y), Q(x, y)\rangle$$

or, for short, $$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$$

Notice that P and Q are scalar functions of two variables and are sometimes called **scalar fields** to distinguish them from vector fields.

Vector
- direction
- length/magnitude

Vector fields
- Vectors + locations



Review: Plots with Vector Fields: How?

Given a vector function, $\vec{F} = (P(x,y), Q(x,y))$, a plot for vector fields can be completed by performing the following:

1) Construct a grid system;
2) Choose sample points, (x_i^*, y_i^*);
3) Use each of the sample points as a starting point;
4) Compute the ending point using:

   ending point = starting point + $\vec{F}(x_i^*, y_i^*)$;

5) Draw a vector from the starting point to the ending point.

Method (I)

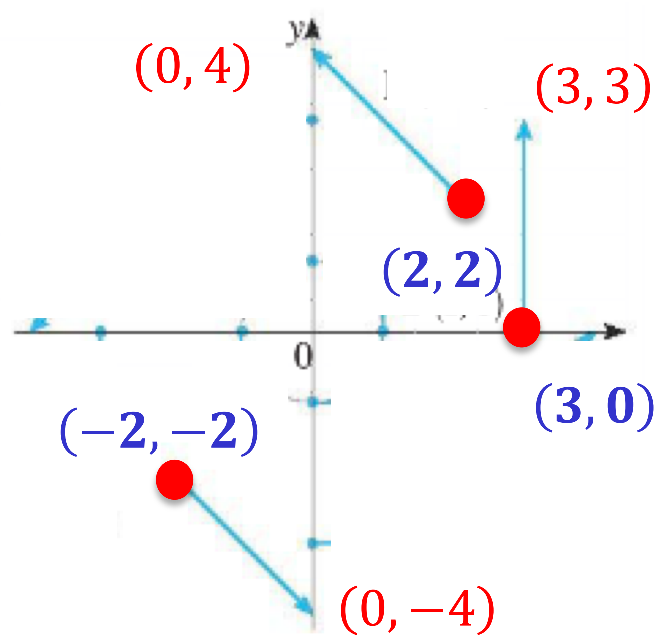
- Alternatively, we view each of the sample points as a new origin (to be discussed below).

Method (II)

Review for the Plot of a Vector Fields: (I)

EXAMPLE 1 A vector field on $\mathbb{R}^2$ is defined by $\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$. Describe $\mathbf{F}$ by sketching some of the vectors $\mathbf{F}(x, y)$ as in Figure 3.

1	2	3	4	5
	(x_i^*, y_i^*)	$\vec{F} = (-y, x)$	starting	ending (c3+c4)
A	$(3, 0)$	$(0, 3)$	$(3\ 0)$	$(3\ 3)$
B	$(2, 2)$	$(-2, 2)$	$(2, 2)$	$(0, 4)$
C	$(-2, -2)$	$(2, -2)$	$(-2, -2)$	$(0, -4)$

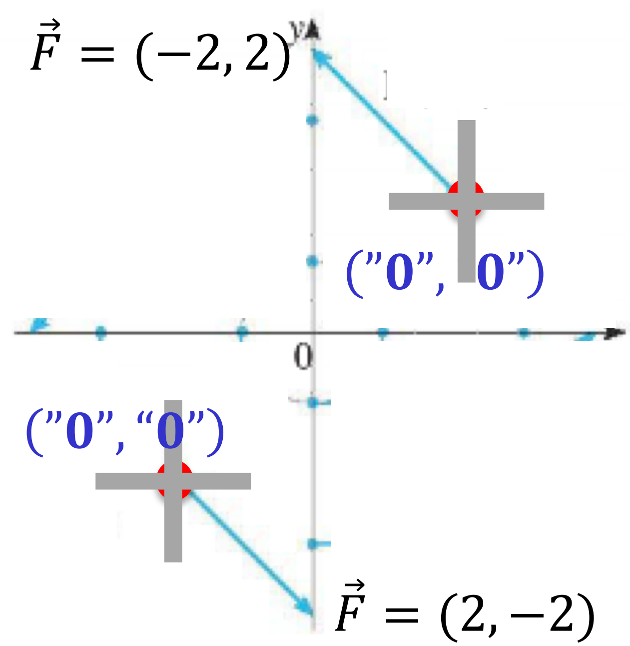


Review for the Plot of a Vector Field: (II)

V EXAMPLE 1 A vector field on $\mathbb{R}^2$ is defined by $\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$. Describe $\mathbf{F}$ by sketching some of the vectors $\mathbf{F}(x, y)$ as in Figure 3.

1	2	3	4	5
	(x_i^*, y_i^*)	$\vec{F} = (-y, x)$	starting	ending (c3+c4)
B	$(2, 2)$	$(-2, 2)$	$(2, 2)$	$(0, 4)$

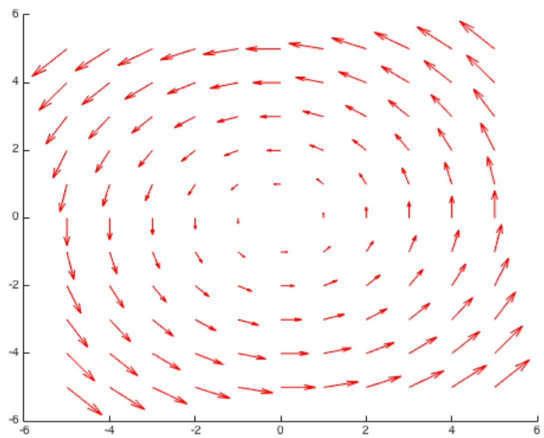
- Alternatively, we view each of the sample points as a new origin.



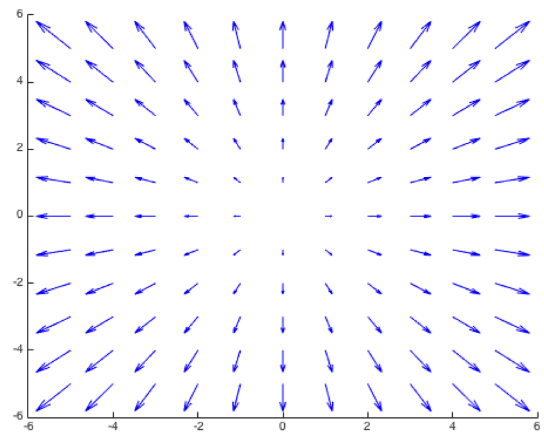
$$magnitude = \sqrt{x^2 + y^2} = r$$

Review of Four Vector Fields

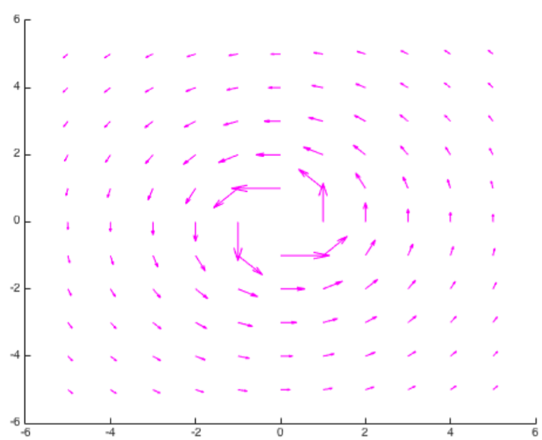
1. Uniform Rotation Field



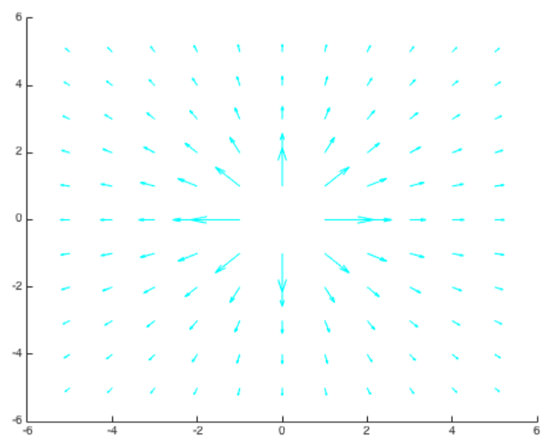
2. Uniform Expansion Field



3. Whirlpool Field



4. 2D Electrical Field



$$1:\ \vec{F}=\left(\frac{-y}{2},\frac{x}{2}\right)$$

(example 1 of section 16.1)

$$2:\ \vec{F}=\left(\frac{x}{2},\frac{y}{2}\right)$$

$$3:\ \vec{F}=\left(\frac{-y}{x^2+y^2},\frac{x}{x^2+y^2}\right)$$

(example 3 of section 16.4)

$$4:\ \vec{F}=\left(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2}\right)$$

(example 1 of section 16.3)
(example 3 of section 16.9)

2.2 Planar Systems: Vector Field

$F(X) = (y, -x) = (P, Q)$

	(x_i^*, y_i^*)	$\vec{F}$
A	$(0.5, 0)$	$(0, -0.5)$
B	$(0, 0.5)$	$(0.5, 0)$
C	$(-0.5, 0)$	$(0, 0.5)$
D	$(0, -0.5)$	$(-0.5, 0)$

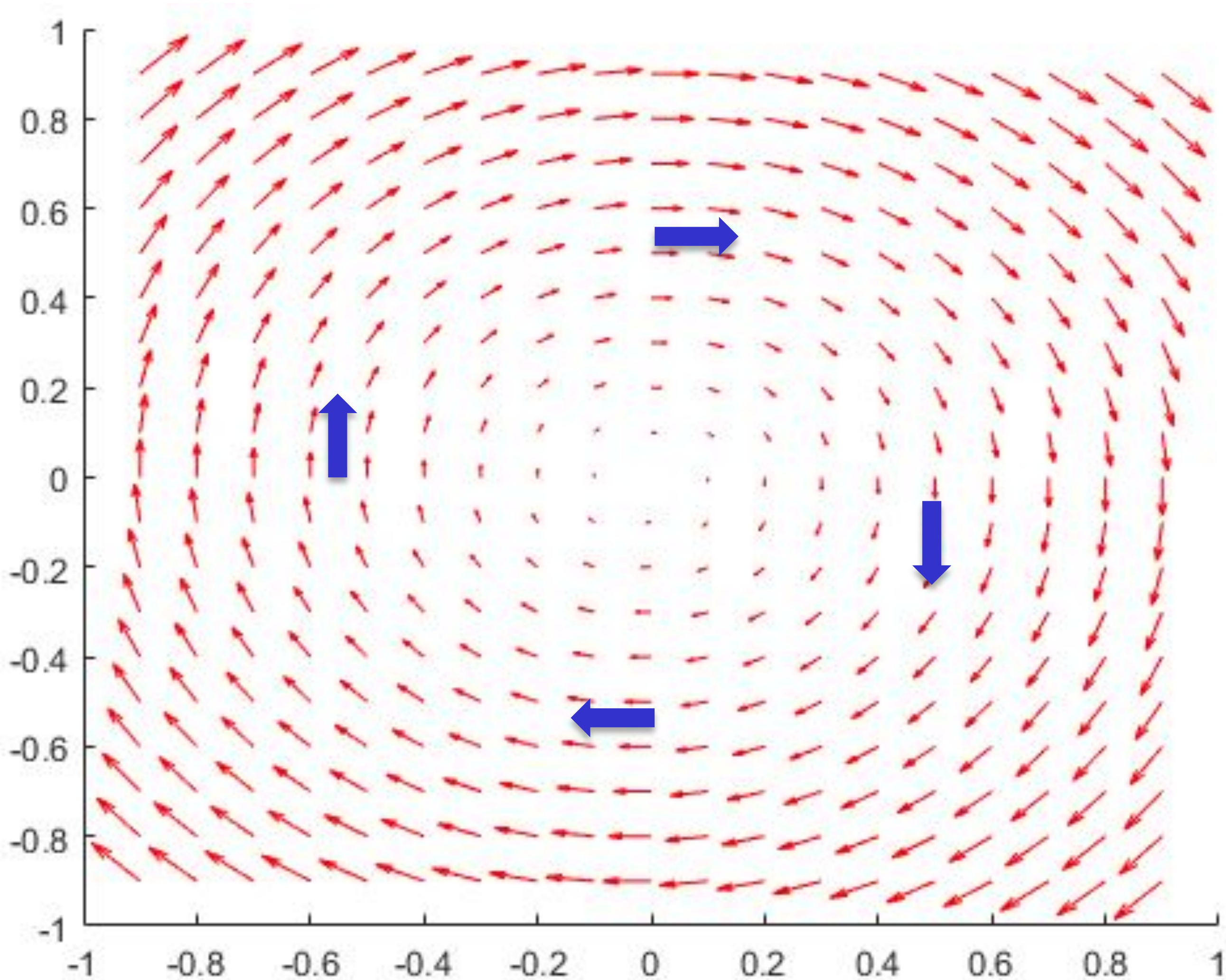
$\nabla \times F = -2$, *clockwise*

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$\nabla \cdot F = 0$

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

MATLAB Plot (Direction Field) for Figure 2.1



Review: A "Meta" Vector: $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$

- Consider a "meta" vector $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$, a function $f = f(x, y)$ and a vector $\vec{F} = (P(x,y), Q(x,y))$

We can define the following:

$\nabla: nabla$

- Gradient:

$$\nabla f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (f_x, f_y)$$

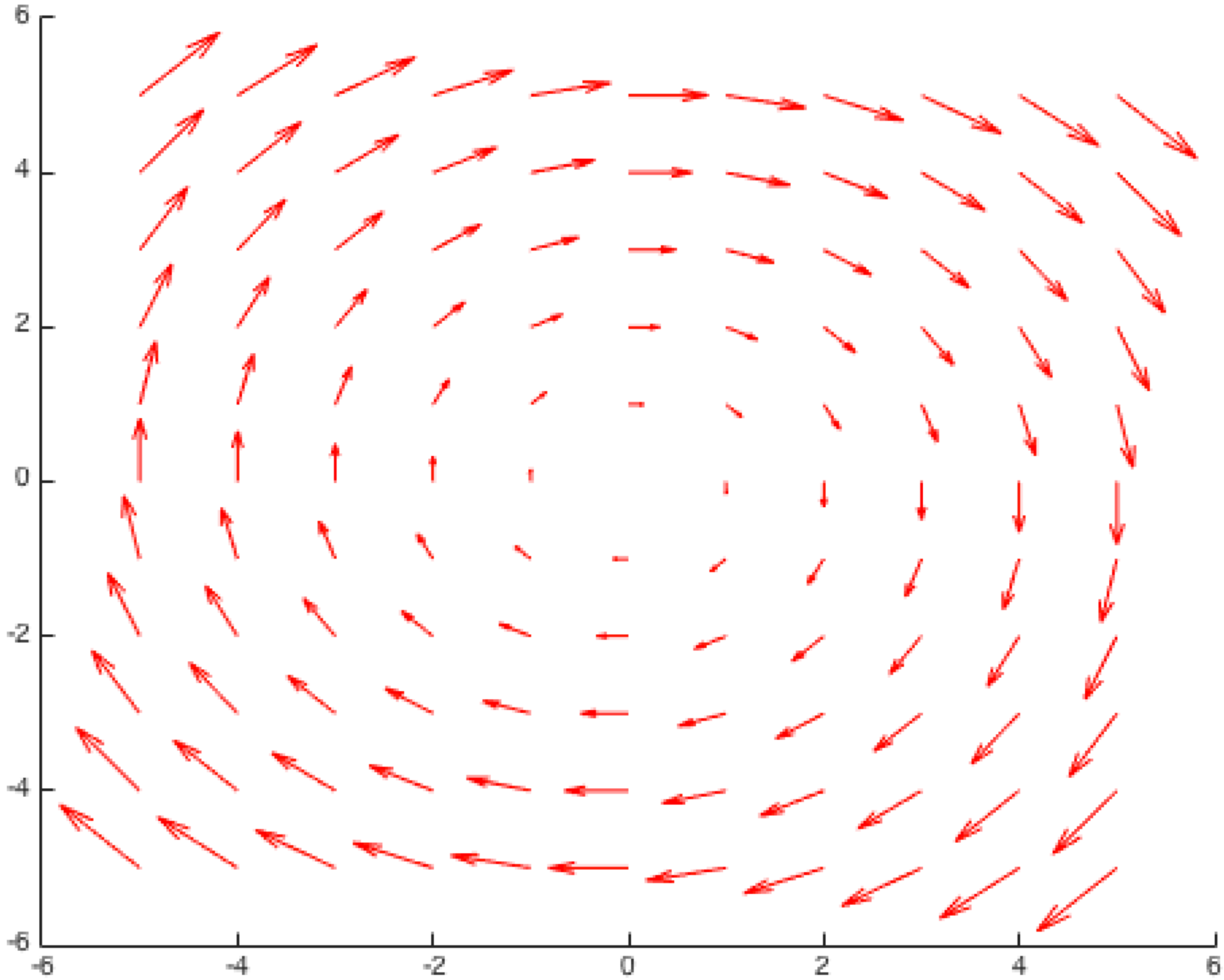
- Curl (a Cross product of ∇ and $\vec{F}$):

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = k\left(Q_x - P_y\right)$$

- Divergence (a Dot product of ∇ and $\vec{F}$):

$$\nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot (P, Q) = P_x + Q_y$$

A Vector Plot



Vector Plots

```
clear all; clc; close all;

% Set scale and start/final points of axes
dx = 1.0;
dy = 1.0;
xStart = -5.0;
xFinal = 5.0;
yStart = -5.0;
yFinal = 5.0;

x = xStart : dx : xFinal;
y = yStart : dx : yFinal;

% Create 2-D array (graph)
[X, Y] = meshgrid(x,y);

% Set to the length of axes
xLen = length(x);
yLen = length(y);

% Fill with 0's
z = zeros(xLen,yLen);
P = zeros(xLen,yLen);
Q = zeros(xLen,yLen);
```

```
% Nested for loop to display vectors
for i = 1 : xLen
    xi = xStart + (i-1) * dx;
    for j = 1 : yLen
        yj = yStart + (j-1) * dy;
        display ('xi yj ');
        display ([xi, yj]);
        P(j,i) = yj;
        Q(j,i) = -xi;
    end
end

% White background with phase diagram in red
figure('Color','w');
hold on;
quiver(X, Y, P, Q,'r');
hold off
```

2.2 A System of 1st Order ODEs

$$\boxed{\begin{aligned} x' &= y \\ y' &= -x \end{aligned}}$$

How to solve?

For now, let's transform the above system into a single ODE

$$\boxed{x'' = y' = -x}$$

Assume

$$x = ke^{\lambda t}$$

$$\lambda = \pm i$$

$$x = c_1 \cos(t) \qquad\qquad x = c_2\, sin(t)$$

How to obtain y?

$$\boxed{x' = y}$$

$$y = -c_1 \sin(t) \qquad\qquad y = c_2 \cos(t)$$

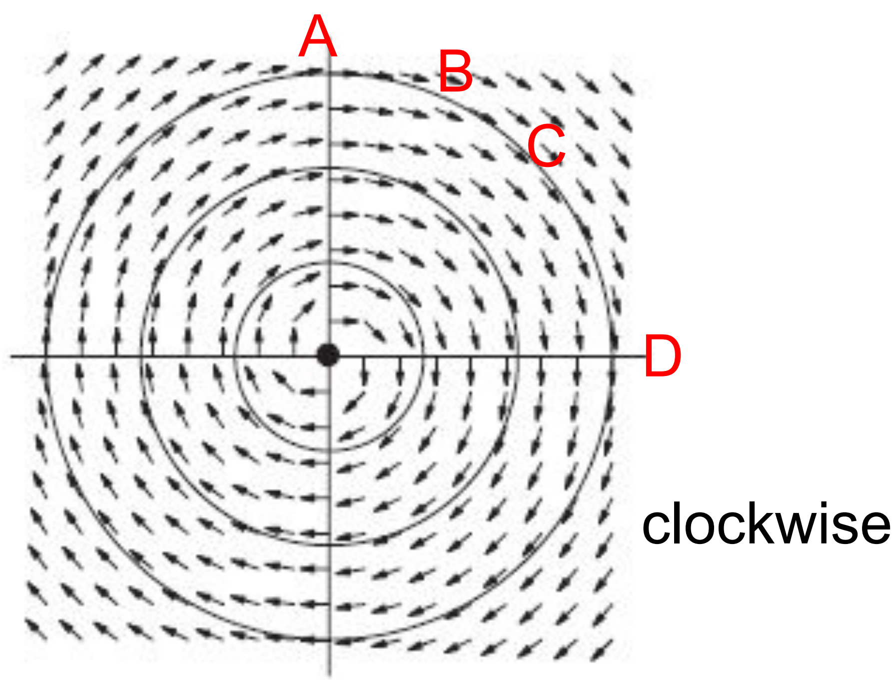
2.2 A System of 1st Order ODEs:

$$x' = y$$
$$y' = -x$$

Verify whether the following are the solutions to the above system

$$x = a \sin(t) \qquad y = a\, cos(t)$$

	t	$(\sin(t),\, cos(t))$
A	0	$(0, 1)$
B	$\pi/6$	$(1/2, \sqrt{3}/2)$
C	$\pi/4$	$(\sqrt{2}/2, \sqrt{2}/2)$
D	$\pi/2$	$(1, 0)$



2.2: Alternative Method

$$\boxed{\begin{aligned} x' &= y \\ y' &= -x \end{aligned}}$$ ODEs

How to solve?

Previously, we assume $x = ke^{\lambda t}$

Now, we asume $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} e^{\lambda t}$ $\quad$ $\begin{aligned} x &= x_0 e^{\lambda t} \\ y &= y_0 e^{\lambda t} \end{aligned}$

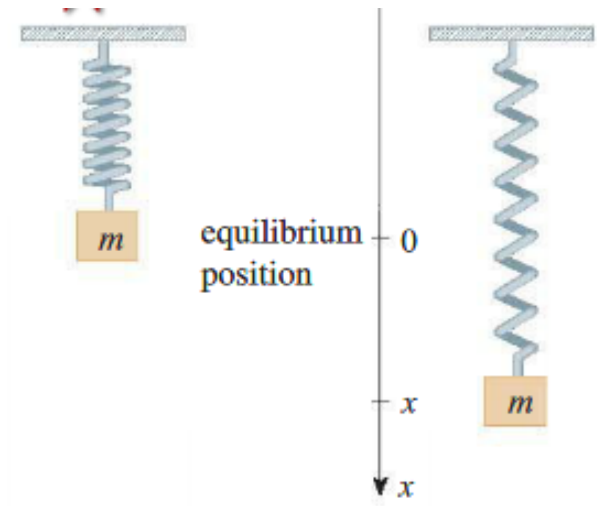
Plug into the above Eq. $\boxed{\begin{aligned} \lambda x_0 &= y_0 \quad (1) \\ \lambda y_0 &= -x_0 \quad (2) \end{aligned}}$ Algebraic Eq.

$\lambda \times (1) + (2)$ $\quad$ $\lambda^2 x_0 = -x_0$

$$\lambda = \pm i$$

A Model for an Oscillatory Motion

- Hooke's Law: if the spring is stretched (or compressed) x units from its natural length, then it exerts a forcing that is proportional to x:



- A second-order differential equation

$$\boxed{m\frac{d^2x}{dt^2} = -kx}$$

$$F = ma = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} = \frac{-k}{m}x = -l^2x$$

- The second derivative of x is proportional to x but has the opposite sign.
- Its solutions are trigonometric functions.

With a Damping Term

There are three parameters for this system: m denotes the mass of the oscillator, $b \geq 0$ is the *damping constant*, and $k > 0$ is the *spring constant*. Newton's law states that the force acting on the oscillator is equal to mass times acceleration. Therefore the differential equation for the damped harmonic oscillator is

$$mx'' + bx' + kx = 0.$$

If $b = 0$, the oscillator is said to be *undamped*; otherwise, we have a *damped* harmonic oscillator. This is an example of a second-order, linear, constant coefficient, homogeneous differential equation. As a system, the harmonic oscillator equation becomes

$$\begin{aligned} x' &= y \\ y' &= -\frac{k}{m}x - \frac{b}{m}y. \end{aligned}$$

With a Damping Term

$$\boxed{mx'' + bx' + kx = 0} \qquad m, b, k > 0$$

Divide by m

$$x'' + \frac{b}{m} x' + \frac{k}{m} x = 0$$

Assume

$$\boxed{x = x_0 e^{\lambda t}}$$

Obtain the characteristic equation

$$\lambda^2 + \frac{b}{m}\lambda + \frac{k}{m} = 0$$

$$\boxed{\lambda = \frac{-\frac{b}{m} \pm \sqrt{(\frac{b}{m})^2 - \frac{4k}{m}}}{2}}$$

Represent the solution: $x = x_0 e^{\lambda t} \propto e^{\frac{-b}{2m}t}$

$x \downarrow \ \ as\ t \uparrow$

b is a damping constant!

$b > 0$ Positive damping

$b < 0$ Negative damping

With an External Force

More generally, the motion of the mass-spring system can be subjected to an external force (such as moving the vertical wall back and forth periodically). Such an external force usually depends only on time, not position, so we have a more general forced harmonic oscillator system,

$$mx'' + bx' + kx = f(t),$$

where $f(t)$ represents the external force. This is now a nonautonomous, second-order, linear equation. ■

$f'(x_c) \rightarrow \lambda\ (eigenvalue)$

$$\boxed{x' = ax} \qquad x' = f(x)$$

assume $\quad x = ke^{\lambda t}$

$$\lambda = a = f'(x_c)$$

$$\boxed{\text{the solution is stable (unstable) if } \lambda < 0\ (\lambda > 0)}$$

consider a general case

$$x' = f(x)$$

linearize $f(x)$ wrt a critical pt

$$x' = f(x) \approx \cancel{f(x_c)} + f'(x_c)(\mathrm{x} - \mathrm{x_c}) + \cdots$$

$$\boxed{\begin{array}{l}\text{the critical point is stable if } \quad f'(x_c) < 0 \\ \text{the critical point is unstable if } f'(x_c) > 0\end{array}}$$

assume

$$x - x_c = ke^{\lambda t} \qquad \lambda ke^{\lambda t} \approx f'(x_c)ke^{\lambda t}$$

$$\lambda = f'(x_c) \qquad \lambda : \text{eigenvalue}$$

$$\boxed{\text{the crtical point is stable (unstable) if } \lambda < 0\ (\lambda > 0)}$$