Sept 25, 2024

Qualitative Analysis (Multi-dimension)

$$\boxed{\text{Model: } \frac{d\vec{x}}{dt} = \vec{f}(\vec{x}, \vec{p})}$$

$\Downarrow$

$$\boxed{\text{Solve: } \vec{f}(\vec{x}, \vec{p}) = \vec{0} \Rightarrow \vec{x}^*}$$

$\Downarrow$

$$J = \left[\frac{\partial f_{ij}}{\partial x_j}\right] \qquad \boxed{\text{Compute Jacobian Matrix: } J\big|_{\vec{x}^*}}$$

$\Downarrow$

$$\boxed{\text{Compute Eigenvalues of } J\big|_{\vec{x}^*}}$$

- If real part of each eigenvalue of $J\big|_{\vec{x}^*}$ is negative, then $\vec{x}^*$ is asymptotically stable.

Local stability of $\vec{x}^*$: 2-dimensional system

$$\vec{x}^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$$

Linearized about $\vec{x}^* \Rightarrow$

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

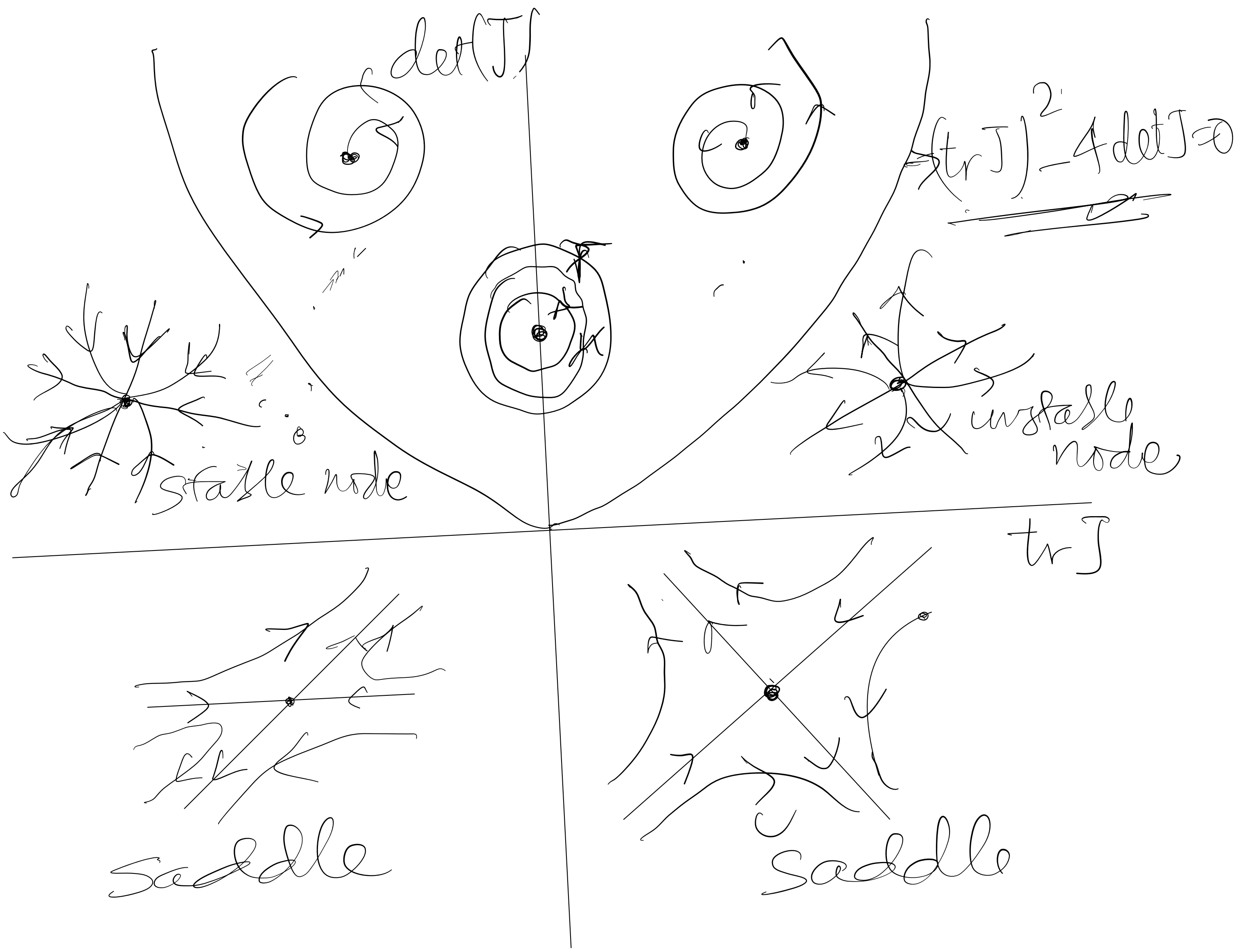
$$J\big|_{x^*} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}\Bigg|_{\vec{x} = \vec{x}^*}$$

λ: eigenvalue of J

$$|J-\lambda I| = 0 \Rightarrow \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

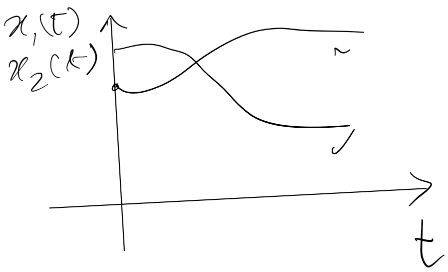
$$\Rightarrow \lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

$$\Rightarrow \lambda^2 - tr(J)\lambda + det(J) = 0$$

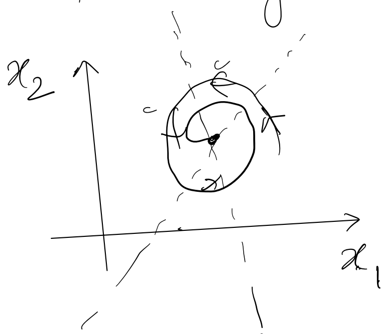


Graphical Analyses (2D):

- Solution curve

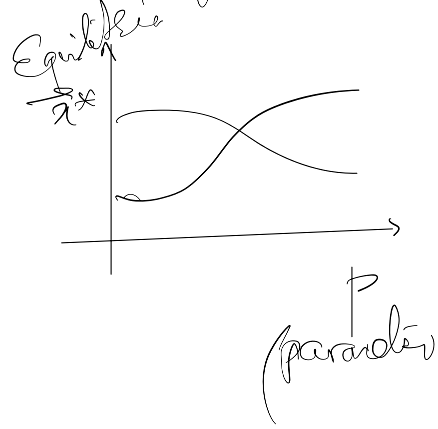


- Phase diagram



null clines
$\vec{F}(x,p)=0$

- Bifurcation



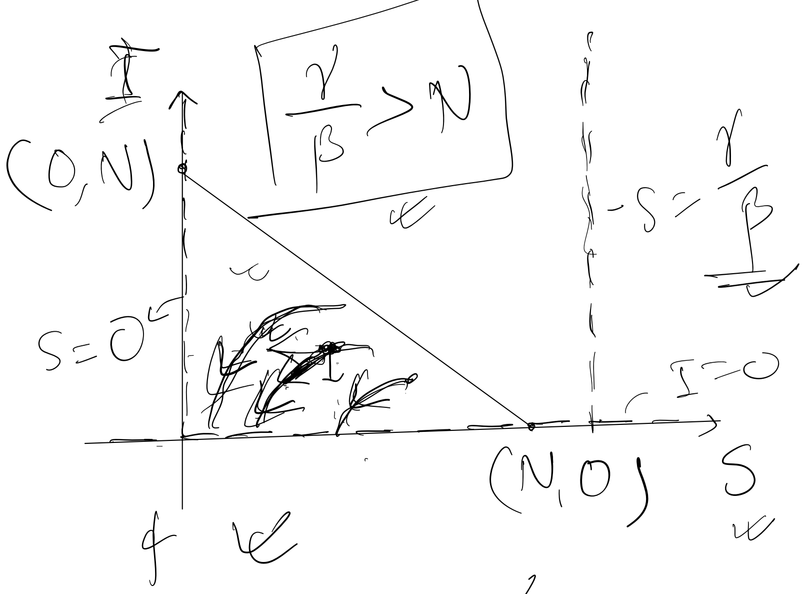
- Example: SIR model $\left(\underline{N} = \underline{S+I+R} = \text{constant}\right)$

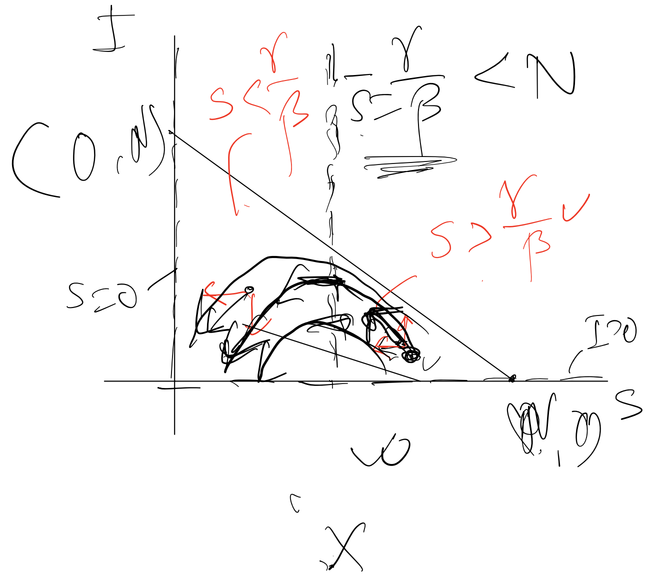
$$
\begin{cases}
\dfrac{dS}{dt} = -\beta I S \\[2ex]
\dfrac{dI}{dt} = \underbrace{\beta I S - \gamma I} = I(\beta S - \gamma)
\end{cases}
$$

Nullclines:

$$\beta I S = 0 \Rightarrow I = 0, \quad S = 0$$

$$\beta I S - \gamma I = 0 \Rightarrow I = 0, \quad \underbrace{S = \frac{\gamma}{\beta}}_{\mathcal{R}}$$





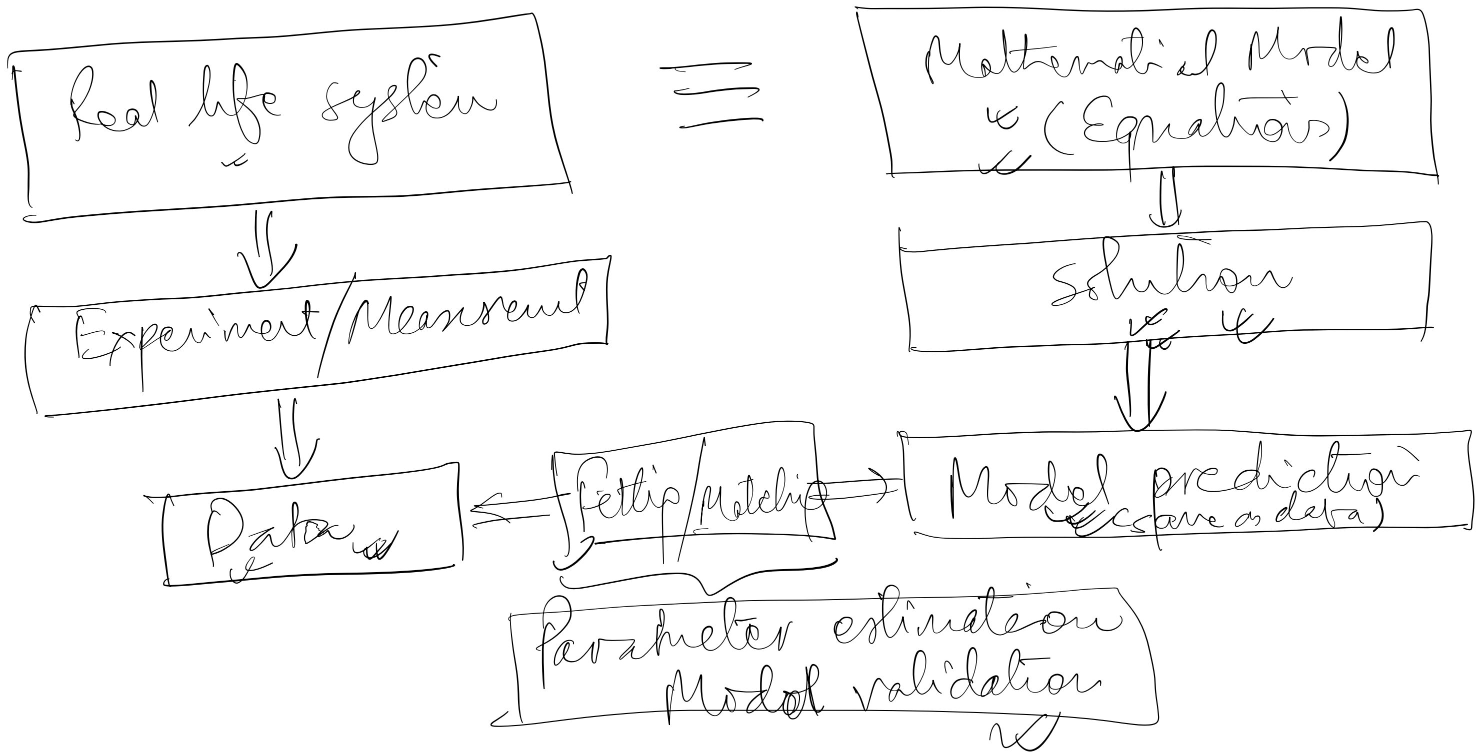
$S < \frac{\gamma}{\beta}$

Control & prevention : $\frac{\gamma}{\beta} > N$

strategy:

- increase γ $\equiv$ treatment
- decrease β $\equiv$ lockdown (isolation)
- decrease N $\equiv$ vaccination

Mathematical Model, Data Fitting, and Parameter Estimation



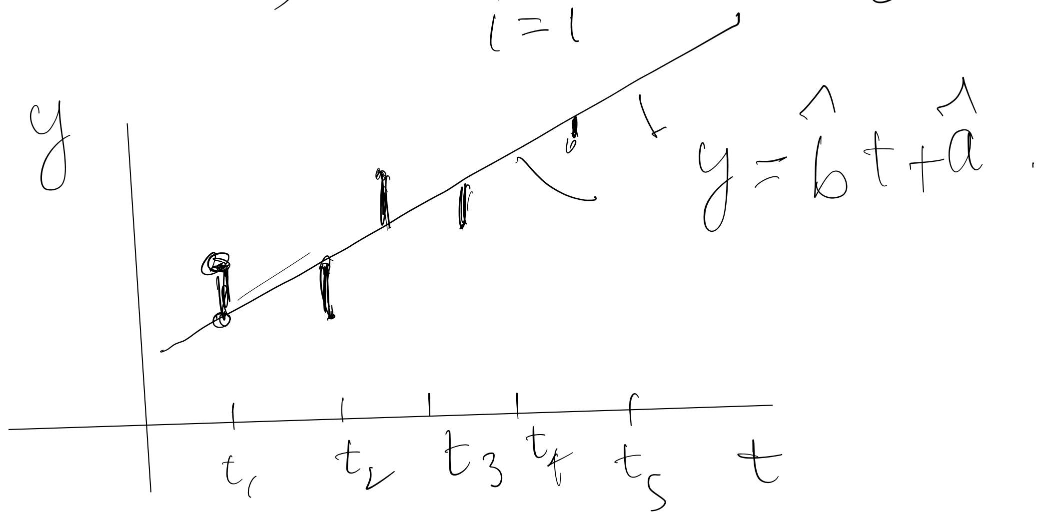
Idea (Linear model):

$$\begin{cases} \dfrac{dy}{dt} = b \\ y(0) = a \end{cases} \Rightarrow \boxed{y(t) = bt + a}$$

a, b ?

- Data: (t_i, y_i), $i = 1, 2, \dots, n$
- Minimizing the distance between the data and the model (SSR: Sum of squared residuals)

$$(\hat{b}, \hat{a}) = \underset{(a,b)\in\mathbb{R}^2}{\operatorname{argmin}} \sum_{i=1}^{n} (y_i - \underline{b t_i} - a)^2$$



· Analytical expression of $\hat{b}$ and $\hat{a}$ for linear model

$$S = \sum_{i=1}^{n} (y_i - b t_i - a)^2$$

$$\frac{\partial S}{\partial b} = 0, \quad \frac{\partial S}{\partial a} = 0$$

$$\Rightarrow \quad \hat{b} = \frac{\sum t_i y_i - n \bar{t}\, \bar{y}_i}{\sum t_i^2 - n \bar{t}^2}$$

$$\hat{a} = \bar{y} - b\bar{t}$$