# MATH 537, Fall 2020

# Ordinary Differential Equations

Lecture #13

HW #1 & #2 and the MT Part A

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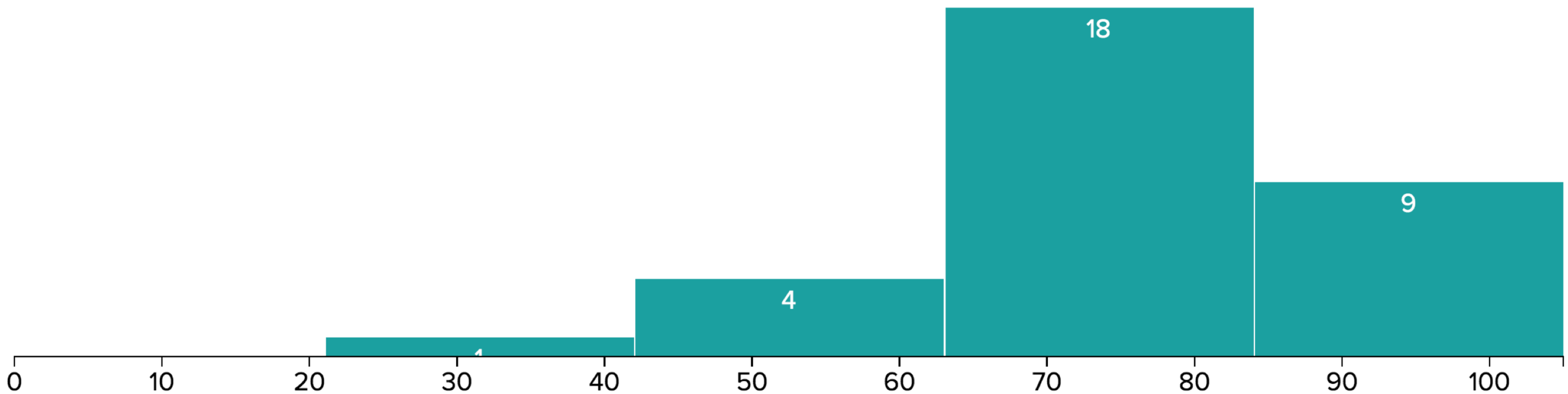
# HW #1

- Derivative Tests vs. Analysis using a Perturbation Method
  - See my whiteboard for additional information

- Source, Sink, Saddle
- The Logistic Eq: $x' = ax(1-x)$, separable
- ➔ $x' = -\beta(x - x_{c+})(x - x_{c-})$

- Association of Uploaded Pages and Specific Problems

# HW #1




| MINIMUM | MEDIAN | MAXIMUM | MEAN | STD DEV |
|---|---|---|---|---|
| 33.0 | 78.0 | 105.0 | 75.38 | 14.73 |

**1**: [20+5 points]

$$\frac{dx}{dt} = f(x),$$

here (i) $f(x) = x$; (ii) $f(x) = x^2$; and (iii) $f(x) = x^3$.

(a) Perform (linear) stability analysis.

(b) Find and analyze the corresponding solutions.

(i-a) one unstable critical point at $x = 0$;

(ii-a) one saddle point at $x = 0$;

(iii-a) one unstable critical point at $x = 0$ (hint: apply a perturbation method)

(i-b) sol: $x(t) = x_o e^t$.

(ii-b) sol: $x(t) = \frac{x_o}{1 - x_o t}$.

(iii-b) sol: $x(t) = \frac{x_o}{\sqrt{1 - 2x_o^2 t}}$

# 1.1 Phase Space and Phase Line

- We may construct a space using dependent variables as coordinates. Such a space is called a phase space (or state space, e.g., Hilborn 2000).
- A 1-D phase space is called a phase line.
- For linear stability analysis of a single first-order ODE (e.g., $x' = f(x;a)$), we analyze the sign of $x'$ near one of the system's critical points.

# 1.1 Linear (Local) Stability Analysis for 1st Order ODEs

consider a general case $\quad \dfrac{dx}{dt} = f(x)$ $\quad \boxed{x' = ax}$

find critical points $\quad f(x_c) = 0$

linearize $f(x)$ wrt a critical pt $\quad \dfrac{dx}{dt} = f(x) = f(x_c) + f'(x_c)(\mathrm{x} - \mathrm{x_c}) + \cdots$

$$\frac{dx}{dt} = f'(x_c)(\mathrm{x} - \mathrm{x_c}) + \cdots \qquad \boxed{x' = ax}$$

find solution $\quad \mathrm{x} - \mathrm{x_c} = \mathrm{c_0}\exp(f'(x_c)t)$

stability

| | |
|---|---|
| the critical point is stable if $f'(x_c) < 0$ | a sink |
| the critical point is unstable if $f'(x_c) > 0$ | a source |

Haberman (2013)

# 1.1 Linear (Local) Stability Analysis for 1st Order ODEs

$$\frac{dx}{dt} = f'(x_c)(\mathrm{x} - \mathrm{x_c}) + \cdots$$

$f'(x_c) < 0$

$x - x_c < 0$

$\frac{dx}{dt} > 0$

positive direction

$x - x_c > 0$

$\frac{dx}{dt} < 0$

negative direction

$\mathrm{x} = x_c$

sink

$f'(x_c) > 0$

$x - x_c < 0$

$\frac{dx}{dt} < 0$

negative direction

$x - x_c > 0$

$\frac{dx}{dt} > 0$

positive direction

$\mathrm{x} = x_c$

source

| | |
|---|---|
| the critical point is stable if $f'(x_c) < 0$ | a sink |
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Haberman (2013)

# 1.1 A Summary: Local Stability Analysis

- Given $x' = f(x)$, equilibrium points or also known as fixed points or critical points are defined when $f(x_c) = 0$.

- A local solution near the critical point is
  - stable for $f'(x_c) < 0$ and
  - unstable for $f'(x_c) > 0$.

- For example, consider $x' = ax$. $x = 0$ is a critical point. $f'(0) = \mathrm{a}$. A local solution is
  - stable $a < 0$ for and
  - unstable $a > 0$.

# $f'(x_c) \to \lambda\ (eigenvalue)$

$$\boxed{x' = ax} \qquad x' = f(x)$$

assume $\quad x = ke^{\lambda t}$

$$\lambda = a = f'(x_c)$$

the solution is stable (unstable) if $\lambda < 0$ ($\lambda > 0$)

consider a general case

$$x' = f(x)$$

linearize $f(x)$ wrt a critical pt

$$\boxed{x' = f(x) \approx \cancel{f(x_c)} + f'(x_c)(\mathrm{x} - \mathrm{x_c}) + \cdots}$$

the critical point is stable if $\quad f'(x_c) < 0$

**(a linearized system)** the critical point is unstable if $f'(x_c) > 0$

assume

$$x - x_c = ke^{\lambda t} \qquad \lambda ke^{\lambda t} \approx f'(x_c)ke^{\lambda t}$$

$$\lambda = f'(x_c) \qquad \lambda : \text{eigenvalue}$$

$$\boxed{\text{the crtical point is stable (unstable) if } \lambda < 0\ (\lambda > 0)}$$

# Stability Analysis

| 1D, linear | 2D, linear |
|---|---|
| $x' = ax$<br><br>$x = ke^{\lambda t}$<br>$\lambda = \mathrm{a}$ | $x' = ax + by$<br>$y' = cx + dy$<br>$X = \begin{pmatrix} x \\ y \end{pmatrix}; A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ |
| $\lambda > 0, source$<br>$\lambda < 0, sink$ | $X' = AX$<br>$AX = \lambda X$ |

| 1D, nonlinear | 2D, nonlinear |
|---|---|
| $x' = f(x, a)$ | $x' = F(x, y)$<br>$y' = G(x, y)$<br>$J = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{X_c}$ |
| $x' \approx f'(x_c)(x - x_c)$<br>$\lambda = f'(x_c)$ | $X' \approx JX$<br>$JX = \lambda X$ |

# When the first derivative test fails:

1. Compute higher derivatives
2. Analyze the equation directly (based on a perturbation method)

# A Saddle Point

$$x' = x^2 = f(x)$$

Is the critical point a sink or source?

critical points $\boxed{f(x) = 0}$

$$\boxed{x = 0}$$

1st derivative $f'(x) = 2x$

$x = 0$ $\quad f'(0) = 0$ ?

A saddle point or a half-stable critical point (e.g., Strogatz)

Based on the definition, we can obtain:

| $x < 0$ | $x > 0$ |
| --- | --- |
| $\frac{dx}{dt} > 0$ | $\frac{dx}{dt} > 0$ |
| positive direction | positive direction |

$\mathrm{x} = 0$

A saddle point

Apply a perturbation method

$$x = x_c + \varepsilon(t)$$

$\boxed{x = 0 + \varepsilon}$ $\qquad \varepsilon' = \varepsilon^2$

The above discussions can help determine $x = 1$ is a saddle point within $x' = (x-1)^2$.

# $x' = x^3$

$$x' = x^3 = f(x)$$

Is the critical point a sink or source?

critical points $\boxed{f(x) = 0}$

$$\boxed{x = 0}$$

1st derivative $f'(x) = 3x^2$

$x = 0$ $\quad f'(0) = 0\ ?$

Based on the definition, we can obtain:

| | |
|---|---|
| $x < 0$ | $x > 0$ |
| $\dfrac{dx}{dt} < 0$ | $\dfrac{dx}{dt} > 0$ |
| negative direction | positive direction |

$\mathrm{x} = 0$

a source

# A Perturbation Method $x' = x^3$

$$x' = x^3 = f(x)$$

Is the critical point a sink or source?

critical points $\boxed{f(x) = 0}$

$\boxed{x = 0}$

1st derivative $f'(x) = 3x^2$

$x = 0$ $\quad f'(0) = 0\ ?$

Apply a perturbation method

$$x = x_c + \varepsilon(t)$$

$\boxed{x = 0 + \varepsilon}$ $\qquad \varepsilon' = \varepsilon^3$

The above discussions help determine $x = 1$ is an unstable point within $x' = (x-1)^3$.

$$x = 1 + \varepsilon(t)$$

$$\varepsilon' = \varepsilon^3$$

# Additional Notes: Higher-order Derivative Tests Supp

The Logistic equation includes the following features:

- For a> $0$ the basin of attraction of $x_c = 1$ is $x > 0$, while negative values of $x$ attract to minus infinity.
- In contrast to the logistic map (i.e., difference equation), the logistic equation has no oscillatory nor chaotic solutions.

When both $f$ and $f'$ are zero at the critical point,

- the stability is determined by the sign of the first non-vanishing higher derivatives;
- If that derivative is even (e.g., $f''$), the point is a saddle point, attracting on one side but repelling on the other.
- If that derivative is odd, it follows the same sign rules as $f'$.

Quiz 4 (due Oct 21)

Sprott (2003)

**2**: [20 points]

$$\frac{dx}{dt} = x^2 - 2x.$$

(a) two critical points at $x = 0$ and $x = 2$

(b) a stable critical point at $x = 0$ because of $f'(x = 0) = -2 < 0$.

(c) an unstable critical point at $x = 2$ because of $f'(x = 2) = 2 > 0$.

(d) sol: $x(t) = \frac{2x_o}{x_o - (x_o - 2)e^{2t}}$

**3**: [30 points]

$$\frac{dx}{dt} = -(\alpha x + x^3)$$

for $x \geq 0$ and $x(t=0) = x_o$.

[Hint: set $r = x^2$, solve for $r$ and discuss the results when $\alpha < 0$, $\alpha = 0$ or $0 < \alpha$. ]

(a) $\alpha = 0$, the above Eq. becomes $x' = -x^3$, which has a stable critical point at $x = 0$.

(b) $\alpha > 0$, the above ODE has one stable critical point at $x = 0$ because $f'(x=0) = -\alpha < 0$.

(c) $\alpha < 0$, the system has two critical points (for $x \geq 0$), $x = 0$ and $x = \sqrt{-\alpha}$. The former is a point source because $f'(x=0) = -\alpha > 0$, while the latter is a point sink with $f'(x = \sqrt{-\alpha}) = \alpha < 0$.

(d) $x^2 = \frac{\alpha}{ce^{2\alpha t} - 1}$, $c$ is determined by an initial condition.

**4**: [30 points] Analyze the following ODE with $\beta > 0$:

$$\frac{dx}{dt} = \beta x(1-x) - h$$

for all values of the parameter $h > 0$.

(a) critical points, $x_{c\pm} = \frac{-2\beta \pm \sqrt{\beta^2 - 4\beta h}}{-2\beta} = \frac{1}{2}(1 \pm \sqrt{1 - \frac{4h}{\beta}})$

(b) there are zero, one, and two critical points for $h > \frac{\beta}{4}$, $h = \frac{\beta}{4}$, and $h < \frac{\beta}{4}$, respectively.

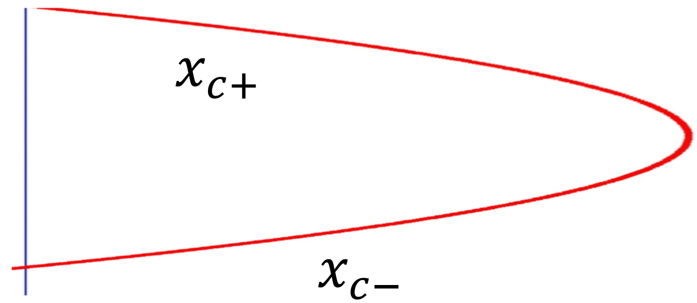
(c) for $h = \frac{\beta}{4}$, $x_c = 1/2$, and $f'(x_c) = 0$. Since $f"(x_c) = -2\beta < 0$, the critical point is half-stable. alternatively, we obtain $x' = -\beta(x - 1/2)^2 < 0$, indicating a saddle at $x = 1/2$.

(d) for $h < \frac{\beta}{4}$, $x_{c+}$ is stable because of $f'(x_{c+}) < 0$; $x_{c-}$ is unstable because of $f'(x_{c-}) > 0$.

(e) $x' = -\beta(x - x_{c+})(x - x_{c-})$, which is separable.

# 1.1 Linear (Local) Stability Analysis for 1st Order ODEs

$$\frac{dx}{dt} = -\beta(\mathrm{x} - \mathrm{x}_{c+})(\mathrm{x} - \mathrm{x}_{c-}) = f(x)$$




Near $\mathrm{x}_{c+}$

$$f(x) = -\beta(\mathrm{x} - \mathrm{x}_{c+})(\mathrm{x} - \mathrm{x}_{c-})$$

+

$x - x_{c+} < 0$

$$\frac{dx}{dt} > 0$$

positive direction

$x - x_{c+} > 0$

$$\frac{dx}{dt} < 0$$

negative direction

$\mathrm{x} = x_{c+}$

sink

Near $\mathrm{x}_{c-}$

$$f(x) = -\beta(\mathrm{x} - \mathrm{x}_{c+})(\mathrm{x} - \mathrm{x}_{c-})$$

−

$x - x_{c-} < 0$

$$\frac{dx}{dt} < 0$$

negative direction

$x - x_{c-} > 0$

$$\frac{dx}{dt} > 0$$

positive direction

$\mathrm{x} = x_{c-}$

source

# HW #2

1. 2nd order ODE vs. a system of first-order ODEs (in a matrix form)
2. Saddle vs. Center for a 2D system
3. Saddle vs. Sink within a Linearized Lorenz model (in preparation for ”understanding” the so-called Lorenz Geometrical model.
4. The SIR model: 3D vs. 1D

**1**: [25 points] Consider the following second-order ordinary differential equations (ODEs) for linear pendulum oscillations:

$$\frac{d^2x}{dt^2} + c\frac{dx}{dt} + Kx = 0, \tag{1}$$

which is a linearized version of the nonlinear system:

$$\frac{d^2x}{dt^2} + c\frac{dx}{dt} + K sin(x) = 0.$$

Assume $c = 5$ and $K = 4$.

(a) Solve Eq. (1) for solutions.

(b) Convert Eq. (1) into a system of first-order ODEs by introducing $y = dx/dt$. Solve the system of the first-order ODEs.

**2**: [25 points] Consider the following system of linear ODEs:

$$\frac{dx}{dt} = \alpha y, \tag{2a}$$

$$\frac{dy}{dt} = -\beta x. \tag{2b}$$

Discuss the region in the $\alpha\beta$-plane where this system has different types of eigenvalues.

**3**: [25 points] Consider the following linearized Lorenz model (Lorenz, 1963):

$$\frac{dX}{dt} = -\sigma X + \sigma Y, \tag{3a}$$

$$\frac{dY}{dt} = rX - Y. \tag{3b}$$

Perform a stability analysis for $\sigma > 0$ (i.e., discuss the cases with $r > 1$ , $r = 1$, and $r < 1$, respectively.)

# The Trace-Determinant Plane

$T^2 = 4D$

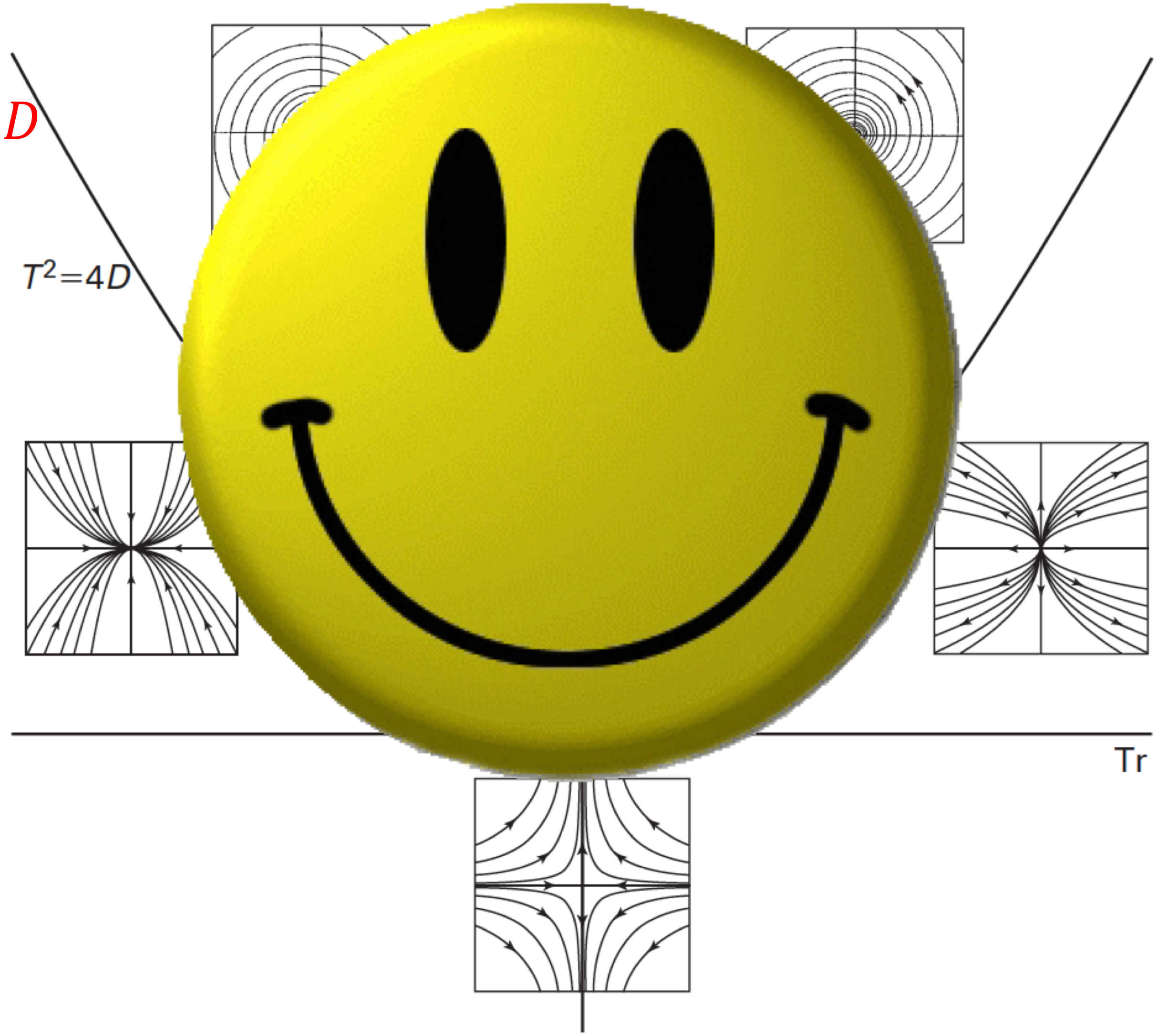



Figure 4.1 The trace-determinant plane. Any resemblance to any of the authors' faces is purely coincidental.

# Classification: Smiling Curve $T^2 - 4D = 0$

$$\lambda^2 - T\lambda + \mathrm{D} = 0$$

$$\lambda_\pm = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

$$f(T, D) = T^2 - 4D$$

Define a smiling curve:

$$T^2 = 4D$$

Sample points

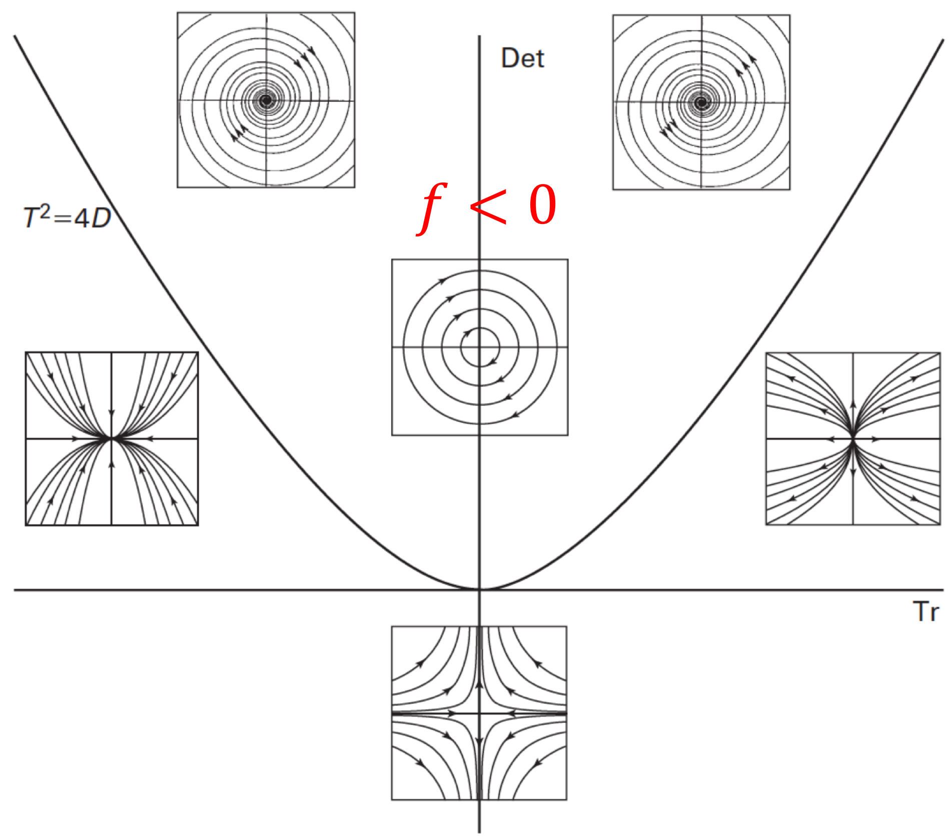
$$f(0,1) = -4 < 0$$

complex eigenvalues

$$f(0,-1) = 4 > 0$$

real eigenvalues

$T^2 = 4D$




$f < 0$

$f > 0$

# Classification: Saddle, Source and Sink

$$\lambda^2 - T\lambda + \mathrm{D} = 0$$

$$\lambda_\pm = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

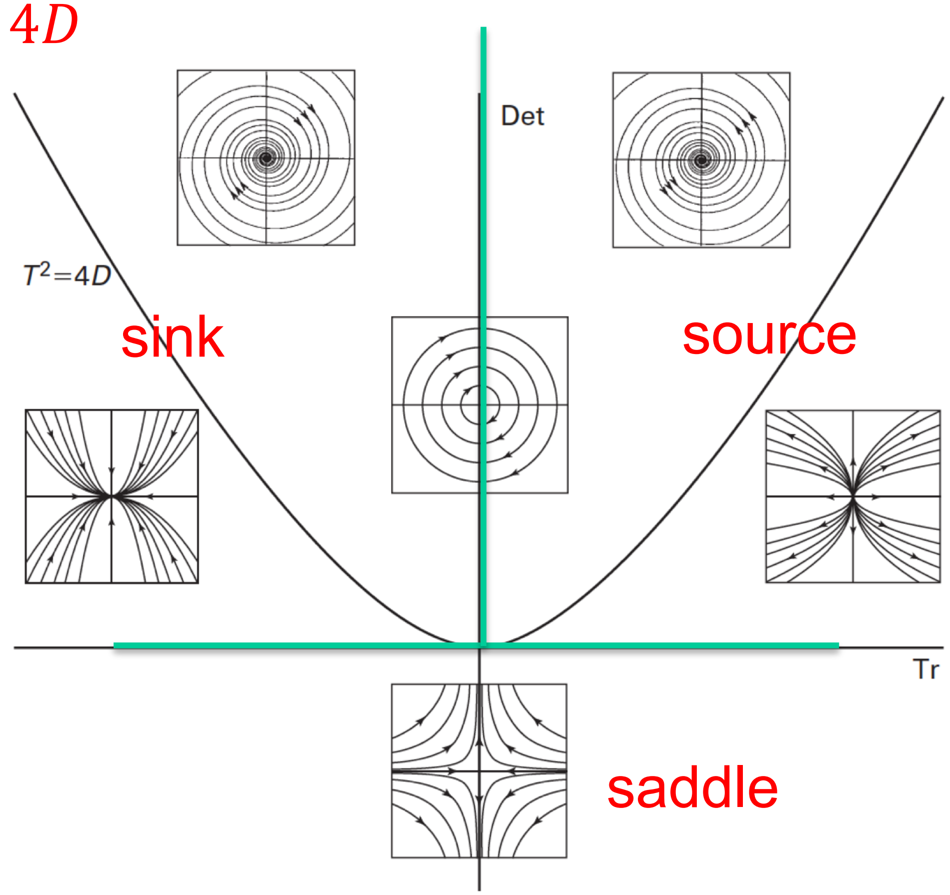
$$\boxed{\lambda_+ + \lambda_- = T = tr}$$

$$(\lambda - \lambda_+)(\lambda - \lambda_-) = 0$$

$$\lambda^2 - (\lambda_+ + \lambda_-)\lambda + \lambda_+\lambda_- = 0$$

$$\boxed{\lambda_+\lambda_- = D = determinat}$$

$T^2 = 4D$




- $D < 0$, $\lambda_+$ and $\lambda_-$ have different signs ➔ saddle
- $D > 0$, $\lambda_+$ and $\lambda_-$ have the same sign ➔ source with $T > 0$

  ➔ sink with $T < 0$

# Mathematical Problems for the Next Century

- Smale, S., 1998: Mathematical Problems for the Next Century. The Mathematical Intelligencer 20, no. 2, pages 7–15. Smale's List of 1998.

*V. I. Arnold, on behalf of the International Mathematical Union has written to a number of mathematicians with a suggestion that they describe some great problems for the next century.*

*Arnold's invitation is inspired in part by Hilbert's list of 1900 (see e.g. (Browder, 1976)) and I have listed 18 problems*

**Problem 14: Lorenz attractor.**

*Is the dynamics of the ordinary differential equations of Lorenz (1963), that of the geometric Lorenz attractor of Williams, Guckenheimer and Yorke?*

- Problem 14 asks if the dynamics of the original equations is the same as that of the geometric model.
- The most complete positive answer would be to describe a homeomorphism of $R^3$ to $R^3$ which would take solutions of the Lorenz equations to solutions of the geometric attractor.

# The Lorenz Attractor Exists

## The Lorenz attractor exists

**Warwick TUCKER**

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(Reçu le 15 janvier 1999, accepté après révision le 12 avril 1999)

**Abstract.** We prove that the Lorenz equations support a strange attractor, as conjectured by Edward Lorenz in 1963. We also prove that the attractor is robust, i.e., it persists under small perturbations of the coefficients in the underlying differential equations. The proof is based on a combination of normal form theory and rigorous numerical computations. 

of unpredictability in many systems. Numerical simulations for an open neighbourhood of the classical parameter values $\sigma = 10$, $\beta = 8/3$ and $\varrho = 28$ suggest that almost all points in phase space tend to a strange attractor $\mathcal{A}$—*the Lorenz attractor*. Based on numerical data, a geometric model describing the dynamics of the flow was introduced by Guckenheimer and Williams (*see* [2], [7]). We prove that this model does indeed give an accurate description of the dynamics of (1).

# A Geometric Model for the Lorenz Attractor

1. The following geometric model for the Lorenz attractor was originally proposed by Guckenheimer and Williams (1979).

$$x' = -3x$$
$$y' = 2y$$
$$z' = -z.$$

- Document the major features of the model
- Send your results via ”chat”
- You have 3 minutes

2. Tucker (1999) showed that this model does indeed correspond to the Lorenz system for certain parameters.

3. Stewart (2002) stated *“thanks to Tucker, dynamical systems theorists can at last stop worrying about whether their most potent icon might suddenly fall apart. And Lorenz’s original insight, that the strange behavior of his equations was not a numerical artefact, can no longer be disputed.”*


- Guckenheimer, J., and Williams, R. F., 1979: Structural stability of Lorenz attractors. Publ. Math. IHES . 50 (1979), 59.
- Tucker, W., 1999: The Lorenz attractor exists. C. R. Acad. Sci. Paris Sér. I Math. 32, 1197.
- Stewart, I., 2000: The Lorenz attractor exists. Nature, vol 406, No 6799, 948-949.

# A Geometric Model for the Lorenz Attractor

1. The following geometric model for the Lorenz attractor was originally proposed by Guckenheimer and Williams (1979).

$$x' = -3x$$
$$y' = 2y$$
$$z' = -z.$$

- Linear,
- Uncoupled,
- Two negative eigenvalues
- **One critical point (a saddle)**
- **No recurrence**

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# The Lorenz Model and the Geometric Model

The Lorenz Model

$$\frac{dX}{d\tau} = \boxed{-\sigma X} + \boxed{\sigma Y},$$

$$\frac{dY}{d\tau} = -XZ + \boxed{rX} - Y,$$

$$\frac{dZ}{d\tau} = XY \boxed{- bZ.}$$

A Geometric Model by

Guckenheimer and Williams (1979)

$$x' = \boxed{-3x}$$

$$y' = \boxed{2y}$$

$$z' = \boxed{-z.}$$

- Missing nonlinear terms (no -XZ and XY)
- Missing recurrence (no complex eigenvalues)

➔ Understanding Butterfly Effects, Searching for Recurrence

**4**: [25 points] Consider the following epidemic model (Kermack and McKendrick, 1927), which is called the "SIR" model:

$$\frac{dS}{dt} = -\frac{\beta}{N} SI, \tag{4.1}$$

$$\frac{dI}{dt} = \frac{\beta}{N} SI - \nu I, \tag{4.2}$$

$$\frac{dR}{dt} = \nu I. \tag{4.3}$$

Here, $S$, $I$, and $R$ denote susceptible, infected, and recovered individuals, respectively. Three parameters, $\beta > 0$, $\nu > 0$, and $N > 0$, represent a transmission rate, a recovery rate, and a fixed population ($N = S + I + R$), respectively. Complete the following derivations to convert Eqs. (4.1)-(4.3) into the following equations:

# Periodic Solution?

$$S = S(0)e^{-\frac{\beta}{N\nu}(R(t)-R(0))}, \tag{4.4}$$

$$I = N - S(0)e^{-\frac{\beta}{N\nu}(R(t)-R(0))} - R, \tag{4.5}$$

$$\frac{dR}{dt} = \nu\left(N - R - S(0)e^{-\frac{\beta}{N\nu}(R(t)-R(0))}\right), \tag{4.6}$$

- Does the above system posses a periodic solution?
- Send your results via ”chat”
- You have 1 minute

# MT Part A

1. The SIR model: a simplified version vs. Logistic Eq.
2. A nonlinear, non-dissipative Lorenz model: linearization and linear stability analysis
3. A general linear 2D system:
   - eigenvalue problems
   - changing coordinates
   - canonical form
4. Show off your skills and knowledge

# An Epidemic Model: SIR

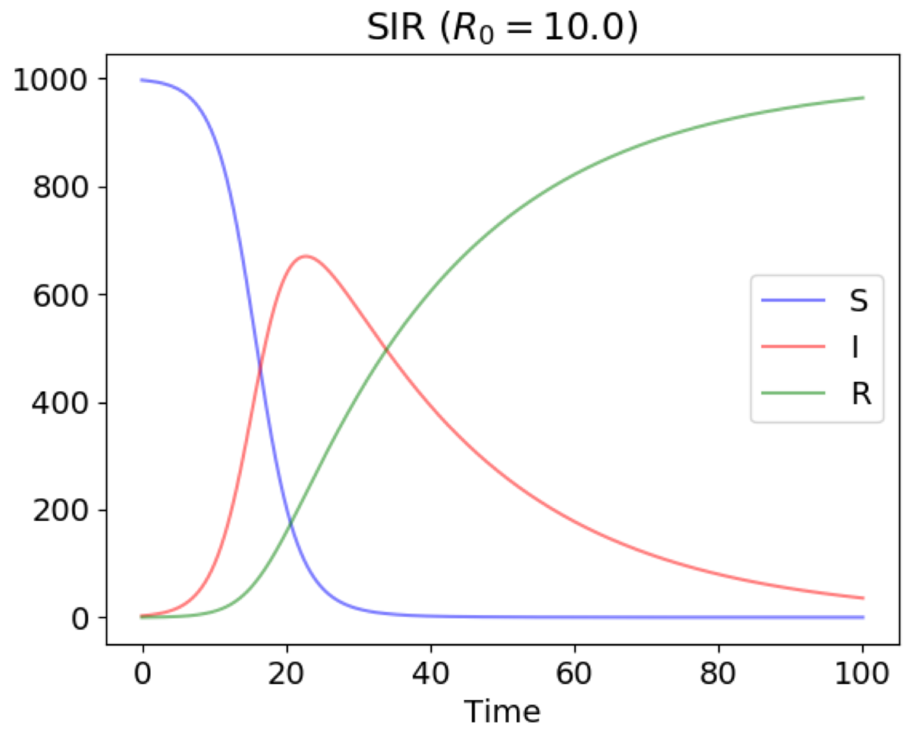
Susceptible | Infectious | Recovered

$$\frac{dS}{dt} = -\frac{\beta}{N} SI,$$

$$\frac{dI}{dt} = \frac{\beta}{N} SI - \nu I,$$

$$\frac{dR}{dt} = \nu I.$$

$$N = S + I + R = constant$$




$\beta > 0$: infection rate (transmission rate, or transmission coefficients);

$\nu > 0$: recovery rate;

$R_0 = \frac{\beta}{\nu}$: basic reproduction #, or contact #.

Reproduction of Wikipedia

# Problem 4: An Epidemic Model

**4**: [25 points] Consider the following epidemic model (Kermack and McKendrick, 1927), which is called the "SIR" model:

2D

$$\boxed{\begin{aligned}\frac{dS}{dt} &= -\frac{\beta}{N}SI, \\ \frac{dI}{dt} &= \frac{\beta}{N}SI - \nu I,\end{aligned}} \quad \begin{aligned} &= F(S,I) \quad (4.1) \\ &= G(S,I) \quad (4.2)\end{aligned}$$

$$\frac{dR}{dt} = \nu I. \qquad = H(I) \tag{4.3}$$

Here, $S$, $I$, and $R$ denote susceptible, infected, and recovered individuals, respectively. Three parameters, $\beta > 0$, $\nu > 0$, and $N > 0$, represent a transmission rate, a recovery rate, and a fixed population ($N = S + I + R$), respectively. Complete the following derivations to convert Eqs. (4.1)-(4.3) into the following equations:

1$D$ with one ODE

$$S = S(0)e^{-\frac{\beta}{N\nu}(R(t)-R(0))}, \tag{4.4}$$

$$I = N - S(0)e^{-\frac{\beta}{N\nu}(R(t)-R(0))} - R, \tag{4.5}$$

$$\frac{dR}{dt} = \nu\left(N - R - S(0)e^{-\frac{\beta}{N\nu}(R(t)-R(0))}\right), \tag{4.6}$$

where $S(0)$ and $R(0)$ represent the initial values of $S$ and $R$, respectively.

# Problem 4: An Epidemic Model

(a) Show

$$S + I + R = constant = N \tag{4.7}$$

(i.e., $\frac{d(S+I+R)}{dt} = 0$).

(b) Apply Eqs (4.1) and (4.2) to obtain the following:

$$\frac{S'}{S} = -\frac{\beta}{N\nu} R'.$$

Integrate the above Eq. to obtain Eq. (4.4), yielding $S = S(R)$.

(c) Apply Eqs. (4.4) and (4.7) to find Eq. (4.5) for $I$, which is a function of $R$.

(d) Apply the above to obtain Eq. (4.6).

(e) Briefly discuss how to analyze Eq. (4.6) to reveal the characteristics of the solution.

Note that based on Eqs. (4.4)-(4.6), we can obtain the solutions by solving a single first order ODE in Eq. (4.6) for $R(t)$, and then compute $S(t)$ and $R(t)$ using Eqs. (4.4) and (4.5), respectively.

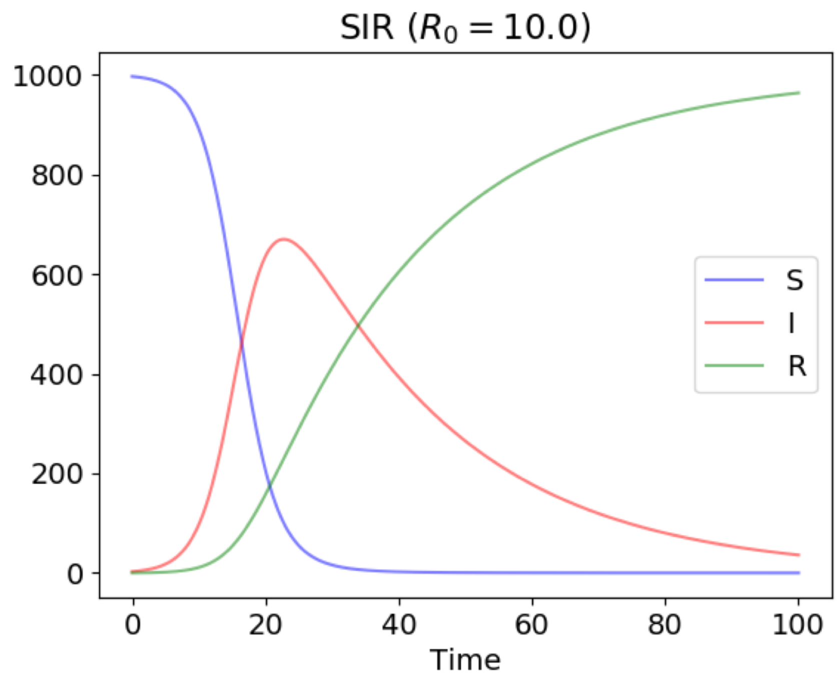
# Flattening the Curve: Impact of $R_0 = \frac{\beta}{\nu}$

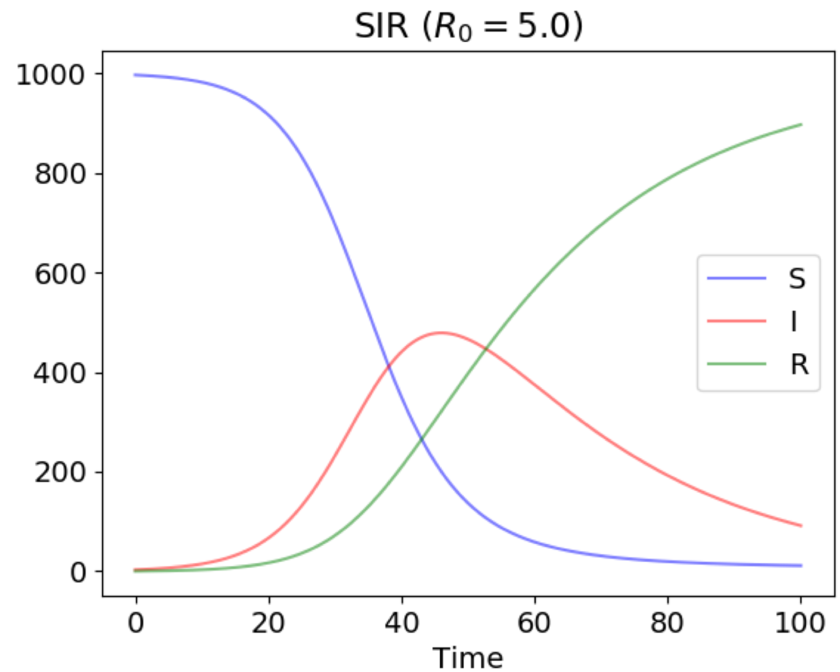
Supp

$R_0 = \frac{\beta}{\nu}$: basic reproduction #, or contact #.

$\beta > 0$: infection rate
(transmission rate, or transmission coefficients);

$\nu > 0$: recovery rate;



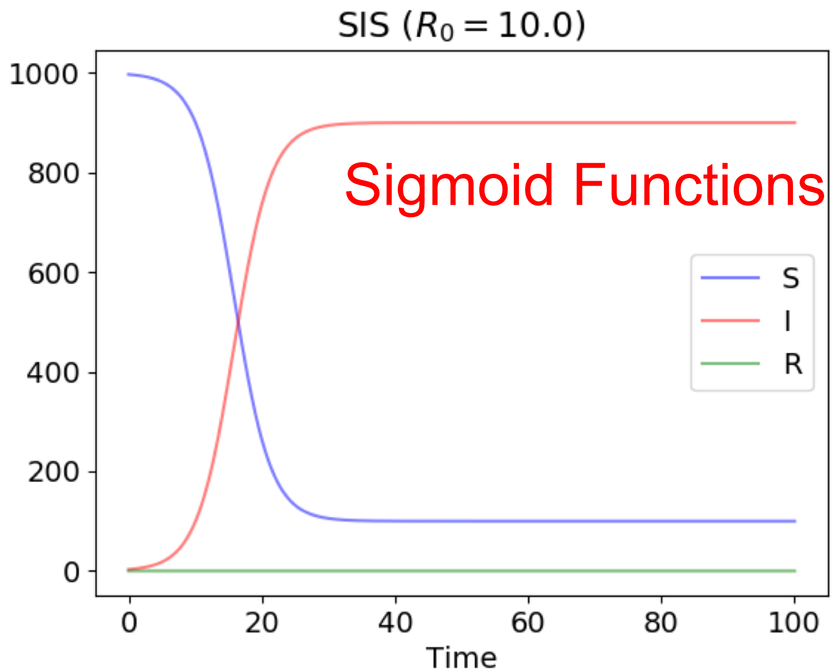

/Users/bwshen/Public/Math537_Fall2020/Slides/Week_SIR/Codes/SIR/C2_wiki

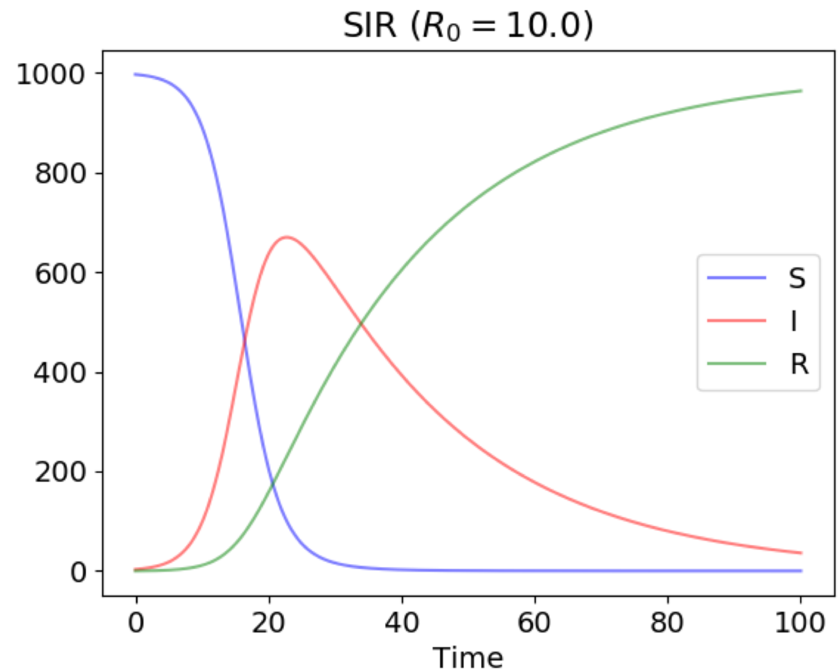
# SIS vs. SIR: Sigmoid Functions

Supp

$$\frac{dS}{dt} = -\frac{\beta}{N}SI + \nu I,$$
$$\frac{dI}{dt} = \frac{\beta}{N}SI - \nu I,$$

$$\frac{dS}{dt} = -\frac{\beta}{N}SI,$$
$$\frac{dI}{dt} = \frac{\beta}{N}SI - \nu I,$$
$$\frac{dR}{dt} = \nu I.$$



# A Simplified SIR for "Weak Outbreak"

| | |
|---|---|
| The Logistic Equation<br>$\frac{df}{dt} = f(1-f)$ | $sigmoid$ |
| $\frac{dg}{dt} = \frac{1}{4} - g^2$ | $tanh$ |

# A Simplified SIR for "Weak Outbreak"

| The Equation for $R'$ |
|---|
| $\frac{dR}{dt} = \nu\left(N - R - S(0)e^{-\frac{\beta}{N\nu}(R(t)-R(0))}\right)$ |
| A simplified Eq. for "weak outbreak" |
| $\frac{dR}{dt} = \nu\,(N - R - S(0)(1 - x + x^2/2))$ |
| $x = \frac{\beta}{N\nu}(R(t) - R(0))$ |
| $e^{-x} \approx 1 - x + x^2/2$ |

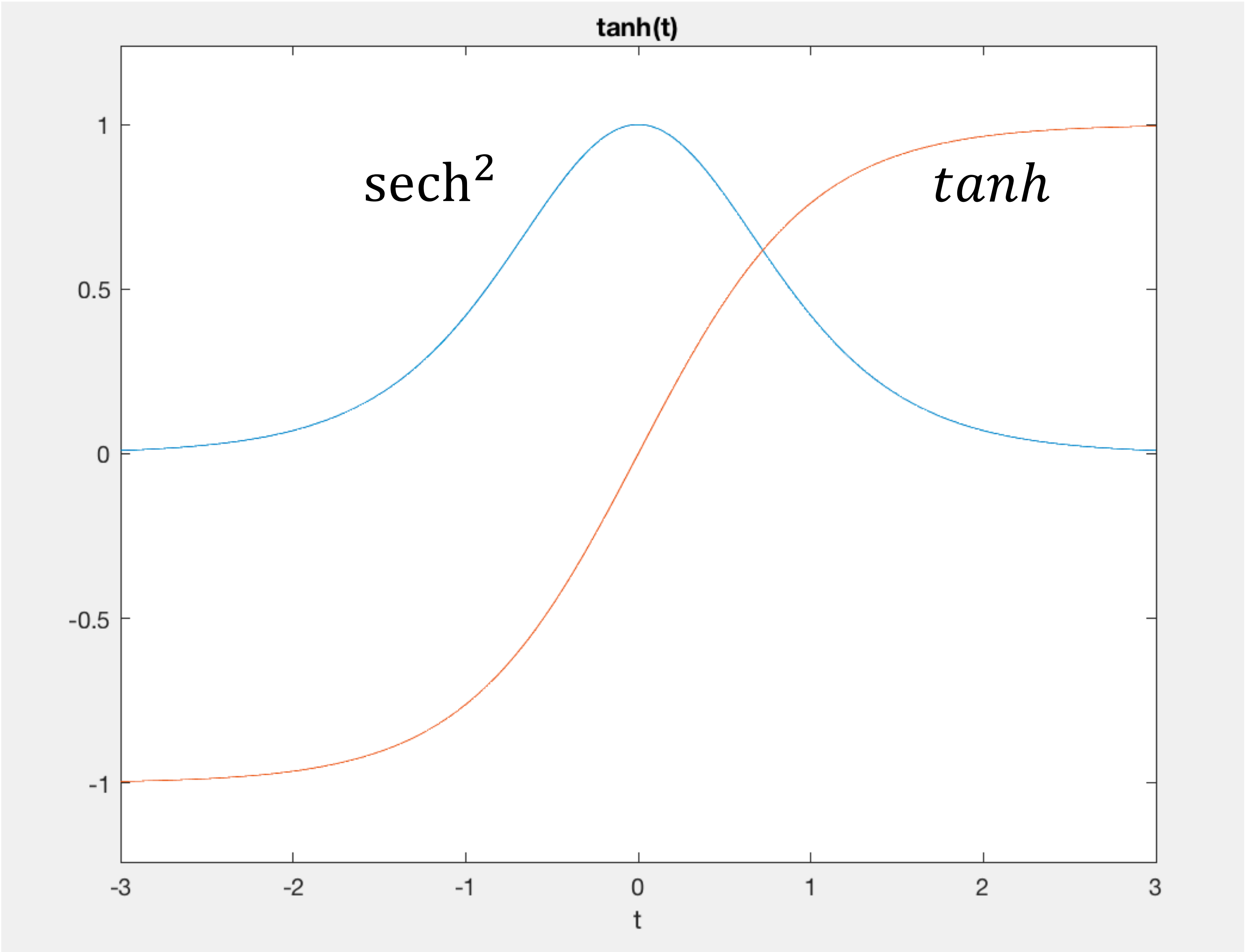
*small* $\beta R/(N\nu)$

$R \approx tanh$

$$\frac{dR}{dt} = \nu I.$$

$I \approx \mathrm{sech}^2$

# tanh vs. sech



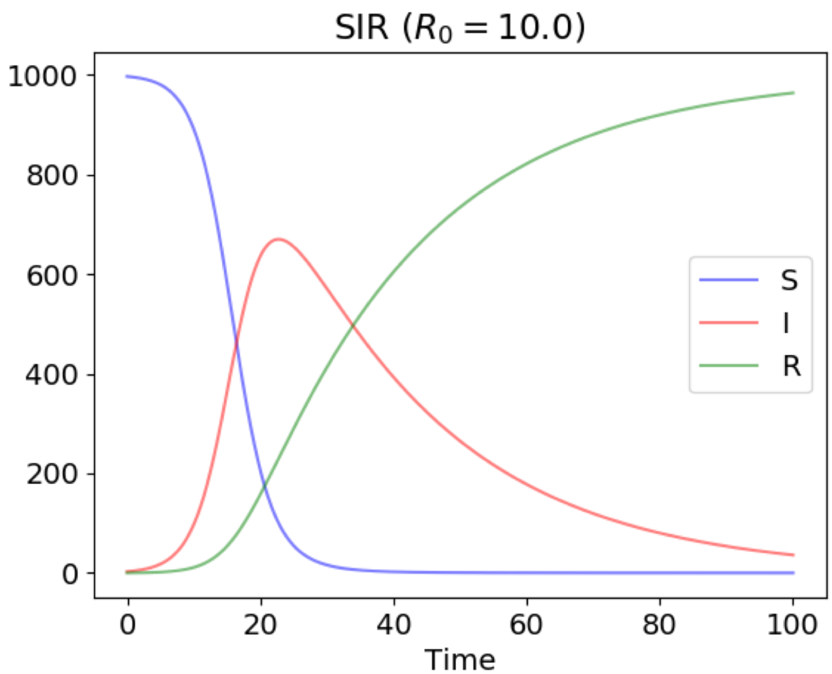
# “tanh” and “sech”

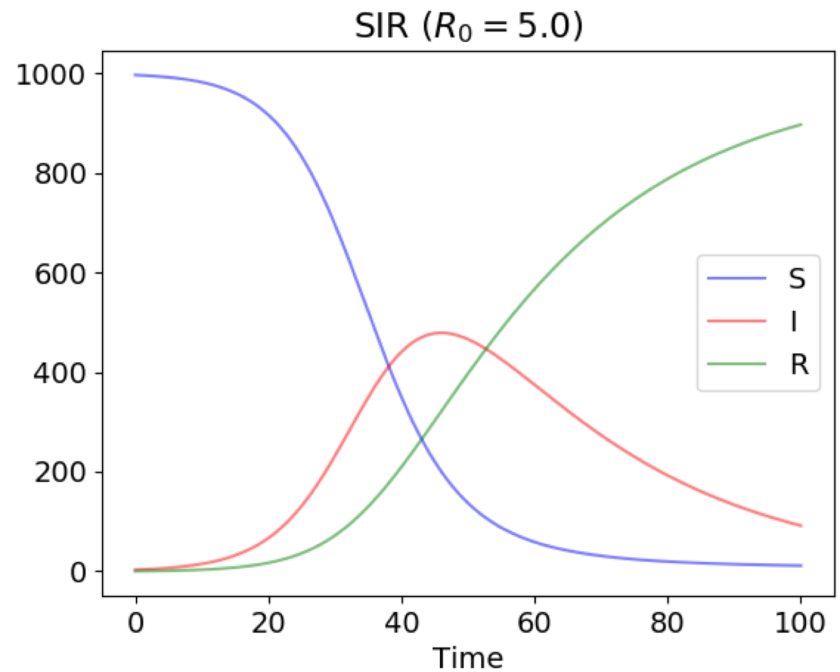
Supp

$\mathrm{R}_0 = \frac{\beta}{\nu}$: basic reproduction #, or contact #.

$\beta > 0$: infection rate
(transmission rate, or transmission coefficients);

$\nu > 0$: recovery rate;



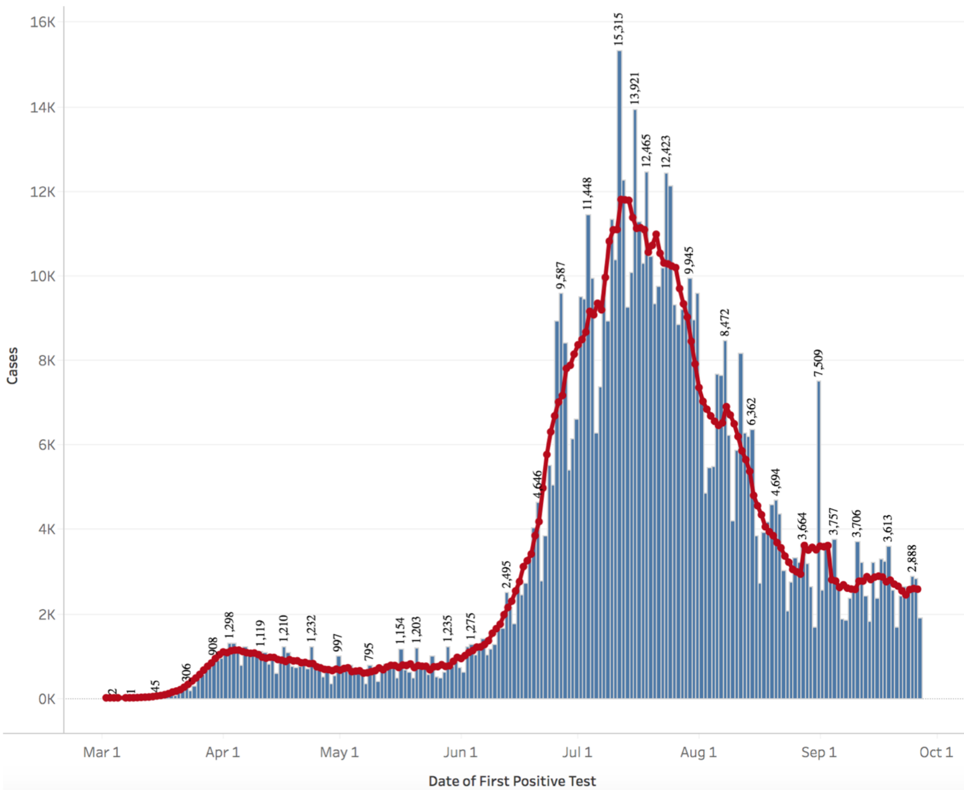

$$I \approx \mathrm{sech}^2$$

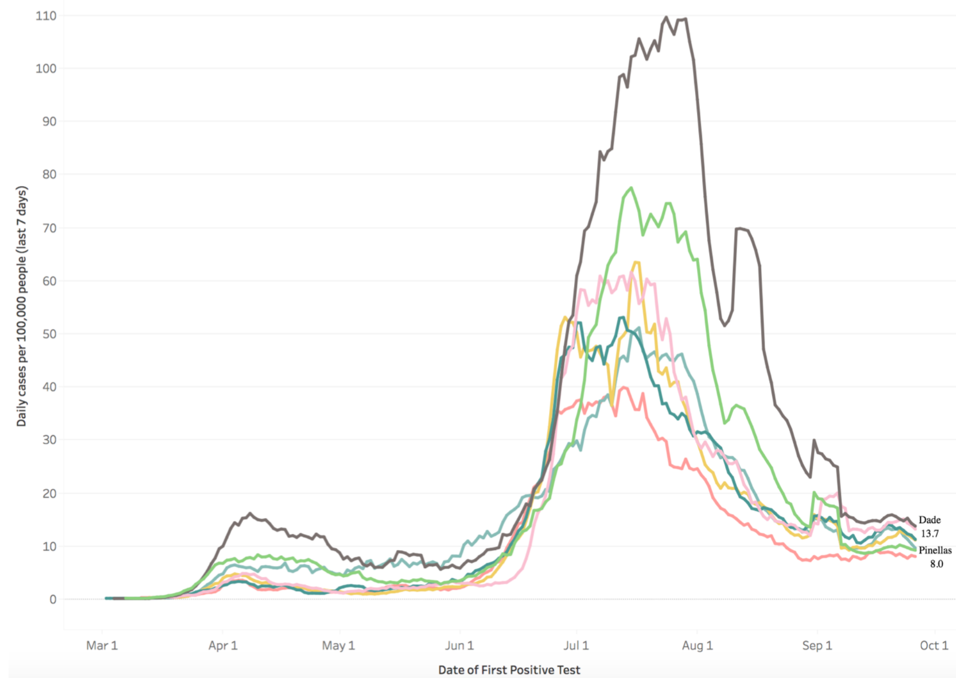
$$R \approx tanh$$

# A Simplified SIR, Logistic Eq., and KdV Eq.

| The SIR Model or KdV Eq. | Fundamental Models | Solutions |
|---|---|---|
| The Equation for $R'$ <br> $\frac{dR}{dt} = \nu\left(N - R - S(0)e^{-\frac{\beta}{N\nu}(R(t)-R(0))}\right)$ | | |
| A simplified Eq. for "weak outbreak" <br> $\frac{dR}{dt} = \nu\left(N - R - S(0)(1 - x + x^2/2)\right)$ <br> $x = \frac{\beta}{N\nu}(R(t) - R(0))$ <br> $e^{-x} \approx 1 - x + x^2/2$ MT Part A | The Logistic Equation <br> $\frac{df}{dt} = f(1-f)$ | $sigmoid$ |
| | $\frac{dg}{dt} = \frac{1}{4} - g^2$ | $tanh$ |
| The Korteweg-de Vries Equation <br> $\frac{d^2J}{d\xi^2} + 3J^2 - cJ = 0$ <br> $c = 1/2$ Quiz 3 | $\frac{d^2Z}{d\tau^2} + 3Z^2 - Z/2 = 0$ <br> $Z = dg/dt$, $\tau = \sqrt{2}t$ | $\boxed{sech^2}$ |

# Epidemic Curve for Covid-19




http://covid19florida.mystrikingly.com/ (link provided by Prof. Joey Lin)

# References


- Wikipdeia, https://en.wikipedia.org/wiki/Compartmental_models_in_epidemiology
- Kermack, W. O.; McKendrick, A. G. (1927). "A Contribution to the Mathematical Theory of Epidemics". Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences. 115 (772): 700. Bibcode:1927RSPSA.115..700K. doi:10.1098/rspa.1927.0118.
- Shen, B.-W.*, 2020: Homoclinic Orbits and Solitary Waves within the non-dissipative Lorenz Model and KdV Equation. http://doi.org/10.13140/RG.2.2.13395.53287 (Accepted, Sep. 4, 2020)
- Shen, B.-W.*, R. A. Pielke Sr., X. Zeng, J.-J. Baik, S. Faghih-Naini#, J. Cui#, and R. Atlas, 2020: Is Weather Chaotic? Coexistence of Chaos and Order within a Generalized Lorenz Model. Bulletin of American Meteorological Society. Available from ResearchGate: http://doi.org/10.13140/RG.2.2.21811.07204 (Accepted, Sep. 18, 2020)

# One Slide Summary

| 1st order | 2nd order | eigenvalue problem |
|---|---|---|
| $y' = \alpha y - \beta y^{\mathbf{2}}$<br>(logistic eq.) | $x'' + \beta x' + \alpha x = 0$ | $x' = ax + by$<br>$y' = cx + dy$ |
| $y' = \alpha y - \beta y^3$ | $x' = y$<br>$y' = -\alpha x - \beta y$ | $X' = AX$<br>$AX = \lambda X$<br>$X = \begin{pmatrix} x \\ y \end{pmatrix}; A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ |

| nonlinear | a system of ODEs | |
|---|---|---|
| $x'' - \alpha x + \beta x^3 = 0$<br>(DE-sech) | $x' = y \equiv F$<br>$y' = \alpha x - \beta x^3 \equiv G$ | $JX = \lambda X$<br>$J = \begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}_{X_c}$ |
| $x'' - \alpha x + \beta x^{\mathbf{2}} = 0$<br>(DE-sech$^2$) | $x' = y \equiv F$<br>$y' = \alpha x - \beta x^2 \equiv G$ | |

# Mid Term Part A
## Math 537 Ordinary Differential Equations
## Due Sep 30, 2020

**Student Name:** ______________________ **ID** ____________

### Rules

**A.** The exam must be taken completely alone. Showing it or discussing it with anybody is forbidden.

**B.** Make an effort to make your submission clear and readable. Severe readability issues may be penalized by grade.

**C.** Please submit your work to Gradescope by 11:59 pm on Sep. 30, 2020.

# MT Part A: The SIR Model

**1**: [30 points] The well-known "SIR" epidemic model (Kermack and McKendrick, 1927) consists of three first-order ordinary differential equations (ODEs) for three time dependent variables, $S$, $I$, and $R$, that represent susceptible, infected, and recovered individuals, respectively. In HW2, we have reduced the system of three ODEs into a single ODE with one time dependent variable $R$, as follows:

$$\frac{dR}{dt} = \nu\left(N - R - S(0)e^{-\frac{\beta}{N\nu}(R(t)-R(0))}\right). \tag{1.1}$$

Here, three parameters, $\beta > 0$, $\nu > 0$, and $N > 0$, represent a transmission rate, a recovery rate, and a fixed population ($N = S + I + R$), respectively. $S(0)$ and $R(0)$ denote the initial values of $S$ and $R$, respectively. Complete the following problems.

# MT Part A: The SIR Model

(a) [6 points] Consider the following logistic equation:

$$\frac{df}{dt} = f(1-f). \tag{1.2}$$

Introduce a new dependent variable ($g$) to transform Eq. (1.2) into the following ODE:

$$\frac{dg}{dt} = \frac{1}{4} - g^2. \tag{1.3}$$

(b) [8 points] Express the solutions of Eqs. (1.2) and (1.3) in terms of the sigmoid and hyperbolic tangent functions, respectively.

(c) [6 points] Apply a Taylor series expansion with $e^{-x} \approx 1 - x + x^2/2$ to simplify the term $e^{-\frac{\beta}{N\nu}(R(t)-R(0))}$ in Eq. (1.1). Then, perform a (linear) stability analysis.

(d) [10 points] Solve the ODE derived in problem (1c) using a small non-negative $R(0)$.

- ➔ $x' = -\beta(x - x_{c+})(x - x_{c-})$

# MT Part A: The Non-dissipative Lorenz Model

**2**: [25 points] A nonlinear, non-dissipative Lorenz model is written as follows:

$$\frac{d^2X}{dt^2} - (\sigma r + C)\, X + \frac{X^3}{2} = 0. \tag{2}$$

Here, we assume that both $\sigma$ and $r$ are positive, and choose $C = 0$ for convenience. Complete the following problems.

(a) [3 points] Transform the 2nd order ODE in Eq. (2) into a system of the first order ODEs, (i.e., $Y = X'$).

(b) [3 points] Find critical points in the above 2D system in problem (2a).

(c) [6 points] Compute the Jacobian matrix of the above 2D system.

(d) [13 points] Perform a linear stability analysis for all of the critical points.

# MT Part A: Eigenvalue Problem and Changing Coordinates

**3**: [25 points] Consider the general, linear, 2D system as follows:

$$X' = AX, \tag{3.1}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } X = \begin{pmatrix} x \\ y \end{pmatrix}.$$

By properly choosing a linear map (or linear transformation) $T$, the above system can be transformed into the system with its matrix in one of the following three forms:

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}. \tag{3.2}$$

(a) [5 points] Discuss the conditions under which the general system in Eq. (3.1) can be transformed into the system with one of of the matrices in Eq. (3.2).

(b) [10 points] Discuss how to construct a linear map to achieve the goals in problem (3a) for all of the three cases. [Hints: construct a 2x2 matrix $T$ that can convert the given linear system into one with a different coefficient matrix that is in canonical form.]

(c) [10 points] Apply $(a, b, c, d) = (-2, 1, -9/4, 1)$ to illustrate the above procedures in problem (3b). [Hint: construct $T$ and compute $T^{-1}AT$.]

# MT Part A: Show off Your Skills and Knowledge

**4**: [20 points] Show off Your Skills and and Knowledge.

(a) [7 points] Design your problem using the skills and knowledge that have been discussed in the textbook or lectures.

(b) [7 points] Discuss why your problem is unique, as compared to the above problems and/or problems in homework (1-2).

(c) [6 points] Solve the problem.

# A Summary for Chapter 2

| $AX = \gamma$ | $X' = AX$ |
|---|---|
| | $(A - \lambda I)V_0 = 0$ |
| $\lvert A\rvert \neq 0,$ unique sol<br>$\lvert A\rvert \neq 0$ & $\gamma=0,$ trivial sol | $\lvert A - \lambda I\rvert \neq 0,$ trivial sol |
| | |
| $\lvert A\rvert = 0$<br>• no solution<br>• Infinitely many solutions | $\lvert A - \lambda I\rvert = 0$<br>• Infinitely many solutions |
| | • The above is called an eigenvalue problem<br>• $Let\ AV_1 = \lambda_1 V_1;\ AV_2 = \lambda_2 V_2,$ we have a general solution as follows: $X = c_1 V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t}$ |
| | • 1D $x' = f(x)$<br>• $x' = f(x) \approx f'(\mathrm{x_c})(\mathrm{x} - \mathrm{x_c})$<br>• $\lambda = f'(\mathrm{x_c})$ |