9/19 Finish 3.1 Limit Laws.
Work on 3.2 Boundedness & Closed set properties.

3.1 Product Rule for Limits.

Suppose $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$.

Then $\lim_{n\to\infty} a_n b_n = ab$.

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proof: By the Boundedness Lemma, $\exists M_a, M_b > 0$ s.t.

$\forall n \in \mathbb{N}$, $|a_n| < M_a$, $|a| < M_a$, $|b_n| < M_b$, $|b| < M_b$.

Let $\varepsilon > 0$.

So $\exists N_a, N_b \in \mathbb{N}$ s.t. $\forall n \geq N_a$ and $\forall n \geq N_b$,

$$|a_n - a| < \frac{\varepsilon}{2M_b} \quad \text{and} \quad |b_n - b| < \frac{\varepsilon}{2M_a}.$$

Let $N = \max\{N_a, N_b\}$. Let $n \geq N$.

So $$|a_n b_n - ab| = |a_n b_n - a b_n + a b_n - ab|$$

$$\leq |a_n - a||b_n| + |a||b_n - b|$$

$$\leq |a_n - a| M_b + M_a |b_n - b|$$

$$< \frac{\varepsilon}{2M_b} \cdot M_b + M_a \cdot \frac{\varepsilon}{2M_a} = \varepsilon .$$

SIDE $|a_n b_n - ab| < \varepsilon.$

$|a_n b_n - ab_n + ab_n - ab| \cancel{< \varepsilon}$

$\leq |a_n - a||b_n| + |a||b_n - b| < \varepsilon.$

$M_b \quad M_a.$

Bounded

Prop 2.14 Suppose $\lim_{n\to\infty} b_n = b \neq 0$.

Then $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}$.

Remark: The sequence $\left\{\frac{1}{b_n}\right\}$ makes sense for $n \geq N$ in the Boundedness Lemma, part 2.

proof: Let $\varepsilon > 0$.

By Boundedness Lemma (2), $\exists \beta > 0$, $N_1 \in \mathbb{N}$ s.t. $\forall n \geq N_1$

$$|b_n| > \beta \quad \text{and} \quad |b| > \beta.$$

Since $\lim_{n\to\infty} b_n = b$, $\exists N_2 \in \mathbb{N}$ s.t. $\forall n \geq N_2$,

$$|b_n - b| < \varepsilon \cdot \beta^2.$$

Let $N = \max\{N_1, N_2\}$. Let $n \geq N$.

Consider $\left| \frac{1}{b_n} - \frac{1}{b} \right| = \left| \frac{b - b_n}{b_n \cdot b} \right|$

$$\leq \left| \frac{b - b_n}{\beta^2} \right|$$

$$< \frac{\varepsilon \cdot \cancel{\beta^2}}{\cancel{\beta^2}} = \varepsilon.$$

$\boxed{\varepsilon}$.

Thm 2.15 Quotient Property.

Suppose $\lim_{n \to \infty} a_n = a$ and $\lim_{n \to \infty} b_n = b \neq 0$.

Then $\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{a}{b}$.

proof: Immediate using 2.14 & 2.13.

SIDE: $\left| \frac{1}{b_n} - \frac{1}{b} \right| < \varepsilon.$

$$\left| \frac{b - b_n}{b_n \cdot b} \right| < \frac{|b - b_n|}{\beta^2} < \varepsilon.$$

worthless.

$$\frac{b - b_n}{(M_b)^2} < \left| \frac{b - b_n}{b_n \cdot b} \right|$$

Instead: $|b_n| > \beta$, $|b| > \beta$.

$$|b_n \cdot b| > \beta^2$$

$$\frac{1}{\beta^2} > \frac{1}{|b_n \cdot b|}.$$

3.2

1. Boundedness of Sequences.
2. Sequential Definition of Density.
3. Closed sets ~ sequential perspective.

<u>Definition</u> Let $S \subseteq \mathbb{R}$. We say that S is <u>bounded</u> iff

$$\exists M \in \mathbb{R}^+ \text{ s.t. } \forall x \in S, \quad |x| \leq M.$$

Similar notions exist for "bounded above" and "bounded below".

Similarly a sequence $\{a_n\}$ is <u>bounded</u> iff

$$\exists M \in \mathbb{R}^+ \text{ s.t. } \forall n \in \mathbb{N}, \quad |a_n| \leq M.$$

<u>Remark</u>: We proved "All convergent sequences are bounded."

(If $\{a_n\}$ is unbounded, then $\{a_n\}$ does not converge.)

2. Sequences & ~~Density~~ Density.

Recall: $S \subseteq \mathbb{R}$ is dense in $\mathbb{R}$

iff

$\forall a<b,\quad (a,b) \cap S \neq \emptyset.\quad$ (I.e. $\exists x \in S$ s.t. $x \in (a,b)$.)

Prop 2.19 Let $S \subseteq \mathbb{R}$.

Then S is dense in $\mathbb{R}$ iff $\forall x \in \mathbb{R}, \exists \{a_n\} \subseteq S$ s.t. $\lim_{n\to\infty} a_n = x$.

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proof: $(\rightarrow)$ Suppose S is dense in $\mathbb{R}$.

Let $x \in \mathbb{R}$.

Let $n \geq 1$. Since S is dense in $\mathbb{R}$,

$\exists a_n \in S$ s.t. $x - \frac{1}{n} < a_n < x + \frac{1}{n}$.

So $-\frac{1}{n} < a_n - x < \frac{1}{n}$.

So $\forall n \in \mathbb{N}^+,\quad |a_n - x| < \frac{1}{n}$.

Since $\lim_{n\to\infty} \frac{1}{n} = 0$, we have $\lim_{n\to\infty} a_n = x$

by Lemma 2.9.

($\Longleftarrow$) Suppose $\forall x \in \mathbb{R}$, $\exists \{a_n\} \subseteq S$ s.t. $\lim_{n\to\infty} a_n = x$.

Suppose $a < b$.

$x = \frac{a+b}{2}$.

$\exists \{a_n\}$ s.t. $\lim_{n\to\infty} a_n = x$.

Let $x = \frac{a+b}{2}$. Then $\exists \{a_n\} \subseteq S$ s.t. $\lim_{n\to\infty} a_n = x$.

Thus $\exists N \in \mathbb{N}$ s.t. $n \geq N$, $|a_n - x| < \varepsilon := \frac{b-a}{2}$.

Thus $$-\frac{b-a}{2} < a_N - \frac{a+b}{2} < \frac{b-a}{2}$$

$$a < a_N < b$$ $\underline{\text{Thus}}$ $\exists a_N \in S$ s.t. $a_N \in (a,b)$.

So S is dense in $\mathbb{R}$.

Thm 2.20 (Immediate Consequence.)

$\forall x \in \mathbb{R}$, $\exists \{a_n\} \subseteq \mathbb{Q}$ s.t. $\lim_{n\to\infty} a_n = x$.

(We already proved $\mathbb{Q}$ is dense in $\mathbb{R}$).

3. Closed sets, sequential perspective.

Definition: Suppose $S \subseteq \mathbb{R}$. We say S is closed iff

$\forall \{c_n\} \subseteq S$, if $\{c_n\}$ converges, then $\lim_{n\to\infty} c_n \in S$.

Examples: $S = (1, 3]$ is not closed

$\exists \{c_n\} \subseteq S$ but $\lim_{n \to \infty} c_n = 1 \notin S$.

v.s.

$T = [1, 4]$ is closed (proof next time)

Hw: to be added:

Show: ① $[20, \infty)$ is closed.

② $\bigcup_{n=2}^{\infty} [\frac{1}{n}, 1]$ is not closed.