MATH 537, Fall 2020
Ordinary Differential Equations

Lecture #22

Chapter 6
Quasi-Periodic Motions

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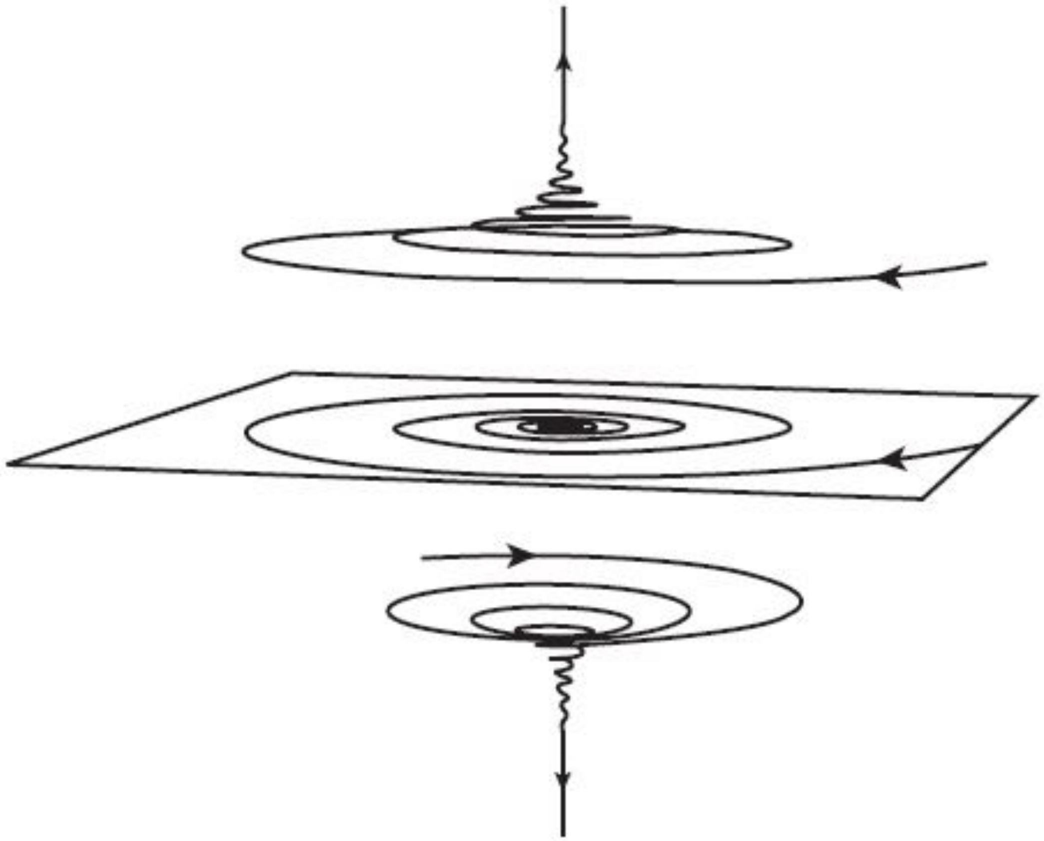
Section 6.1: Saddle Focus (Spiral Saddle)

$$Re(\lambda_{1,2}) < 0$$

$$\lambda_3 > 0$$



Saddle focus
(Ott, p333/334)

Figure 6.5 Typical solutions of the spiral saddle tend to spiral toward the unstable line.

Section 6.1: Spiral Sink and Source

$$\lambda_3 < 0$$
$$Re(\lambda_{1,2}) < 0$$

$$\lambda_3 > 0$$
$$Re(\lambda_{1,2}) > 0$$

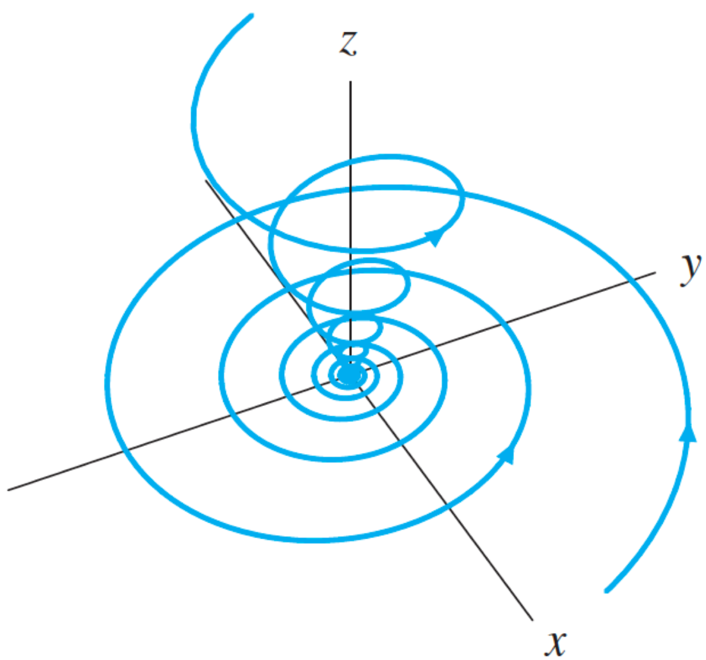


Figure 3.61
Example phase space for spiral sink.

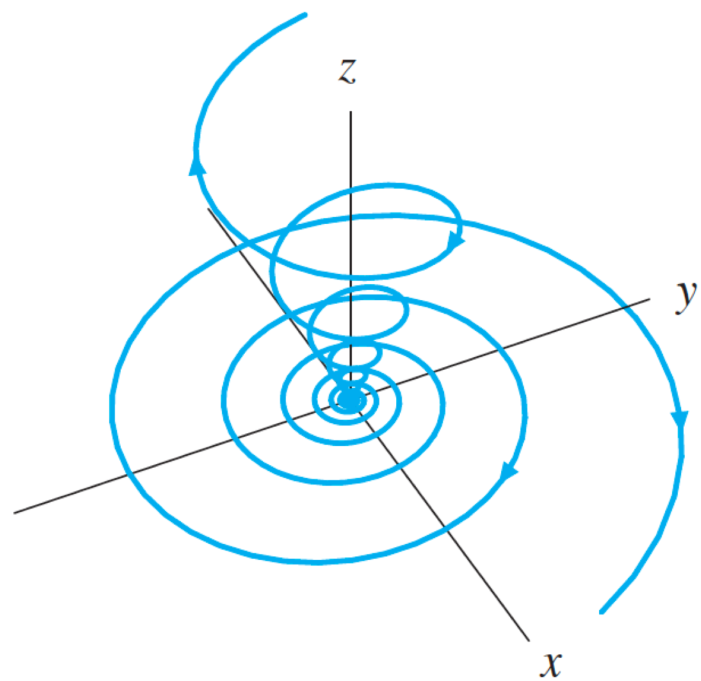


Figure 3.62
Example phase space for spiral source.

Section 6.1: Saddle vs. Spiral Saddle

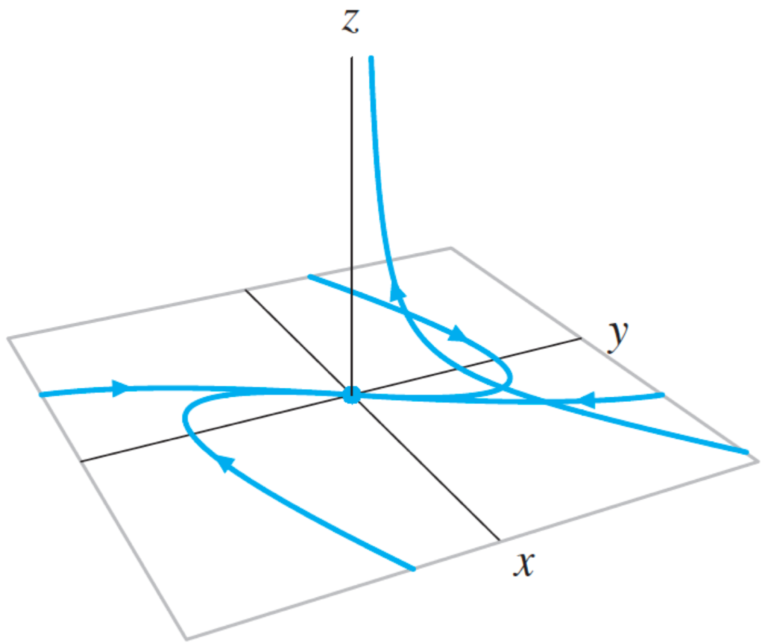
$\lambda_3 > 0$

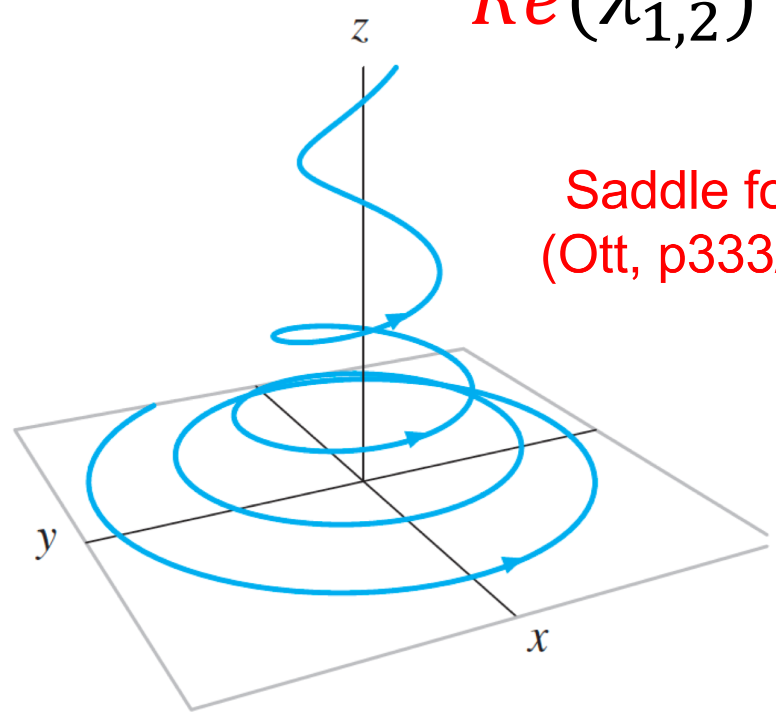
$\lambda_{1,2} < 0$

three real

$\lambda_3 > 0$

$Re(\lambda_{1,2}) < 0$





Saddle focus
(Ott, p333/334)

Figure 3.63
Example of a saddle with one positive and two negative eigenvalues.

Figure 3.64
Example of a saddle with one real eigenvalue and a complex conjugate pair of eigenvalues.

The Phase Portrait for a Spiral Center

(II) $Re(\lambda_{2,3}) = 0$

$Re(\lambda_{2,3}) = 0$

(X, Y)

$\lambda_1 < 0$

(Z)

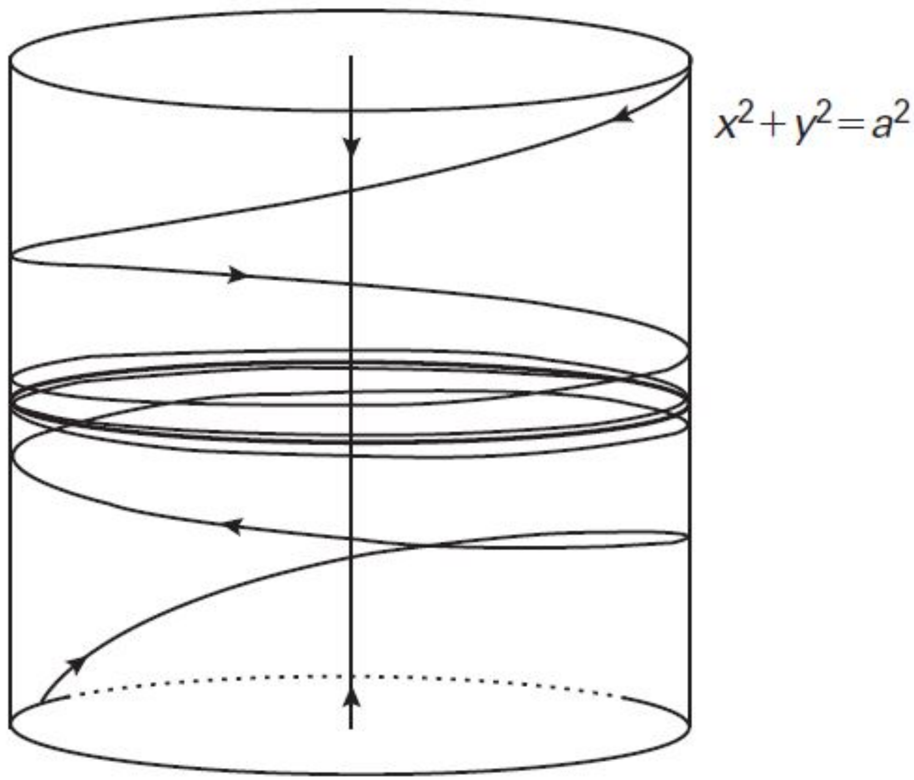


Figure 6.3 The phase portrait for a spiral center.

Classification for 3D Systems

- Saddle (three real eigenvalues), $\lambda_{1,2} < 0$ & $\lambda_3 > 0$
- Sink, $\lambda_{1,2,3} < 0$
- Source, $\lambda_{1,2,3} > 0$
- Spiral center, $Re(\lambda_{1,2}) = 0$ & $\lambda_3 < 0$
- Spiral source, $Re(\lambda_{1,2}) > 0$ & $\lambda_3 > 0$
- Spiral sink, $Re(\lambda_{1,2}) < 0$ & $\lambda_3 < 0$
- Spiral saddle (Saddle focus), $Re(\lambda_{1,2}) < 0$ & $\lambda_3 > 0$

- Stable subspace: $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_k$ are negative
- Unstable subspace: $\lambda_{k+1}, \lambda_{k+2}, \lambda_{k+3} \ldots \lambda_n$ are positive.

Distinct Eigenvalues with 3D Systems

(I)

$$X' = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix} X.$$

Page 109 (and page 84)

$$X(t) = c_1 e^{2t} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Unstable subspace stable subspace

$$\lambda_1 = 2; \lambda_2 = 1; \lambda_3 = -1$$

saddle

(II)

$$X' = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} X$$

Page 111

$$Y(t) = x_0 \begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix} + y_0 \begin{pmatrix} \sin t \\ \cos t \\ 0 \end{pmatrix} + z_0 e^{-t} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda = -1; \lambda = \pm i$$

spiral center

(III)

$$A = \begin{pmatrix} -0.1 & 0 & 1 \\ -1 & 1 & -1.1 \\ -1 & 0 & -0.1 \end{pmatrix}.$$

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$$\lambda_{1,2} = -0.1 \pm i$$

$$\lambda_3 = 1$$

$$Y' = \begin{pmatrix} -0.1 & 1 & 0 \\ -1 & -0.1 & 0 \\ 0 & 0 & 1 \end{pmatrix} Y.$$

spiral saddle (saddle focus)

Review: Saddle, Source and Sink in 2D Systems

$$\lambda^2 - T\lambda + D = 0$$

$$\lambda_{\pm} = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$

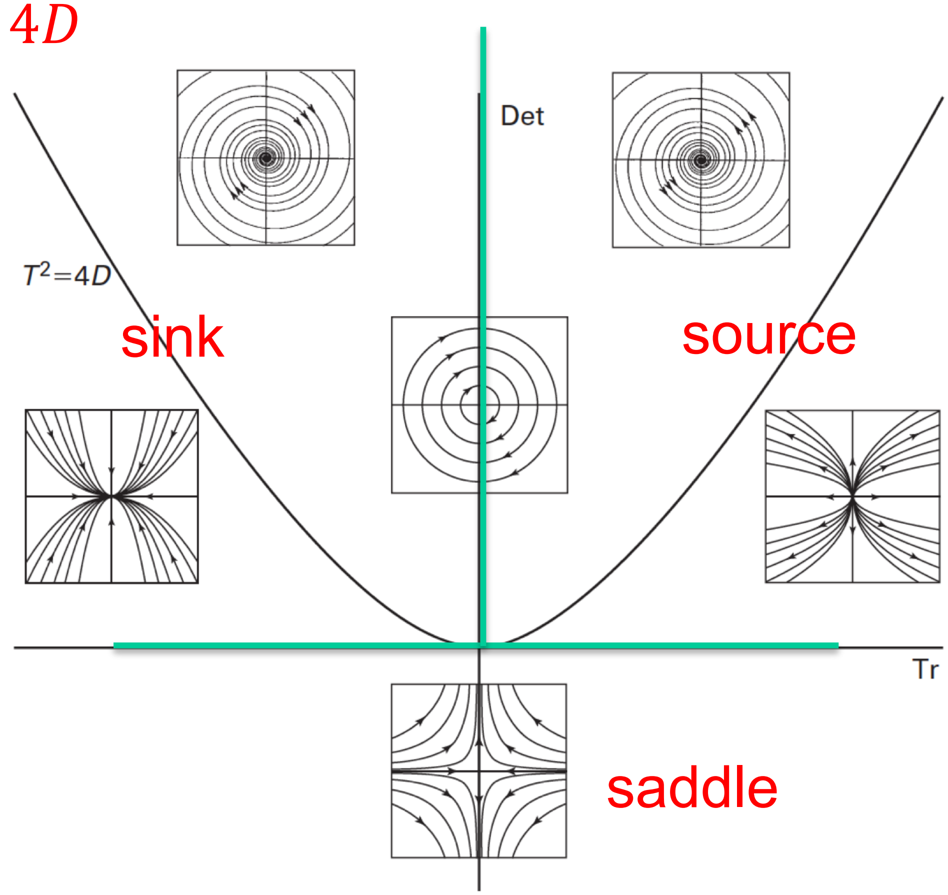
$$\boxed{\lambda_+ + \lambda_- = T = tr}$$

$$(\lambda - \lambda_+)(\lambda - \lambda_-) = 0$$

$$\lambda^2 - (\lambda_+ + \lambda_-)\lambda + \lambda_+\lambda_- = 0$$

$$\boxed{\lambda_+\lambda_- = D = determinat}$$

$T^2 = 4D$



- $D < 0$, λ_+ and λ_- have different signs ➔ saddle
- $D > 0$, λ_+ and λ_- have the same sign ➔ source with $T > 0$
  ➔ sink with $T < 0$

Spiral Point vs. Focus

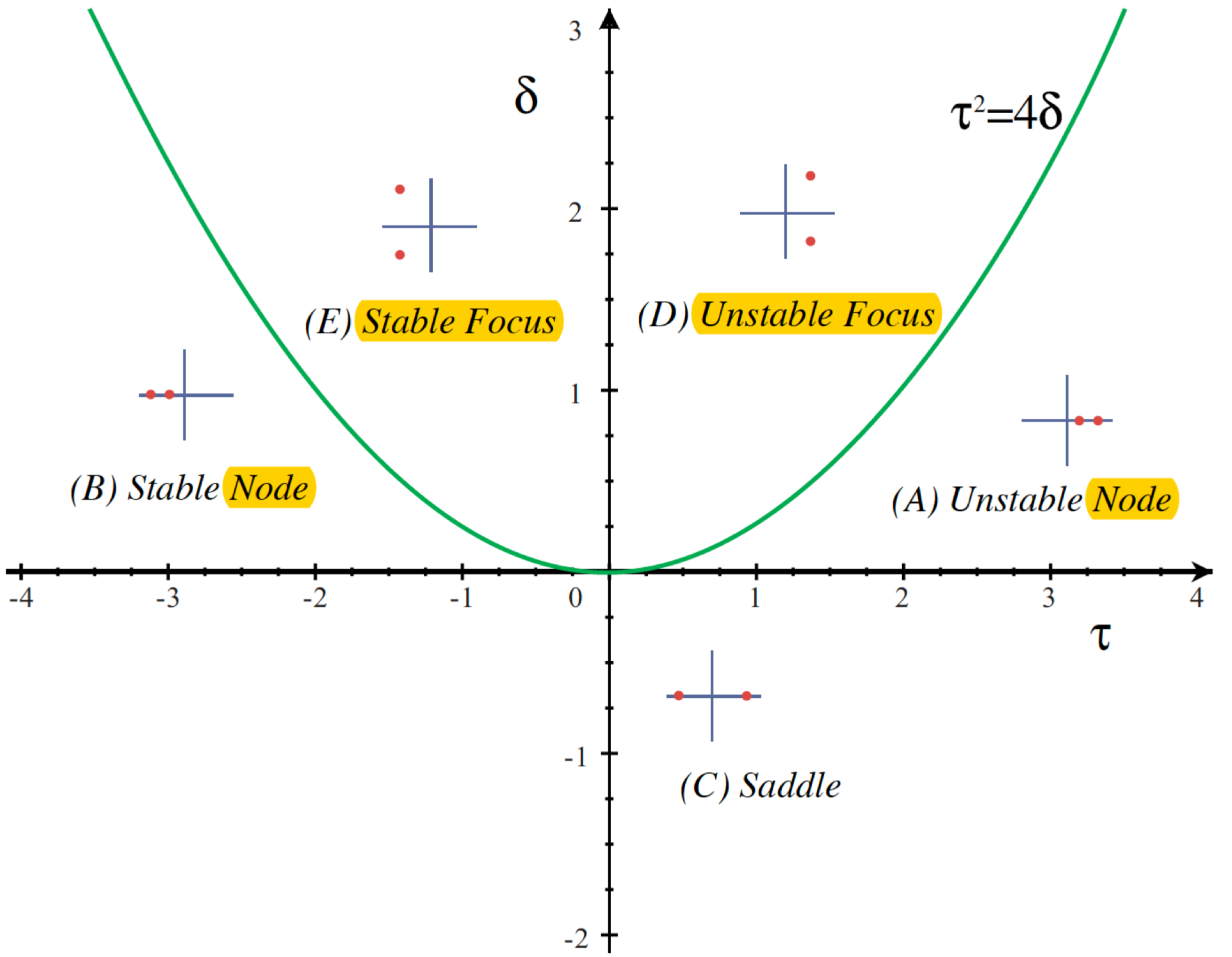


Figure 2.1. *Classification of the eigenvalues for a* 2×2 *linear system in the parameter space of the trace,* τ*, and determinant,* δ.

Meiss, (2007)

Equilibrium (Scholarpedia)

It has two eigenvalues, which are either both real or complex-conjugate. A hyperbolic equilibrium can be a

- **Node** when both eigenvalues are real and of the same sign. The node is stable when the eigenvalues are negative and unstable when they are positive. For the stable node, the eigenvalue(s) with minimal absolute value of the real part is called **principle** or **leading**; when the eigenvalues are different, all orbits but two tend to the node along the leading eigenvector (the picture is reversed for the unstable node);
- **Saddle** when eigenvalues are real and of opposite signs. The saddle is always unstable;
- **Focus** (sometimes called **spiral point**) when eigenvalues are complex-conjugate; The focus is stable when the eigenvalues have negative real part and unstable when they have positive real part.

Spiral Sink/Source and Saddle Focus in the 3D Space

Spiral Sink: $Re(\lambda_{1,2}) < 0$ and $Re(\lambda_3) < 0$

Spiral Source: $Re(\lambda_{1,2}) > 0$ and $Re(\lambda_3) > 0$

Spiral Saddle (Saddle Focus): $Re(\lambda_{1,2}) < 0$ and $Re(\lambda_3) > 0$;

(Ott, p334)

Non-trivial critical points of the Lorenz Model:

$Re(\lambda_{1,2}) > 0$ and $Re(\lambda_3) < 0$, → spiral point

Section 6.2 Harmonic Oscillators (uncoupled)

S1

Consider a pair of undamped harmonic oscillators whose equations are

$$x_1'' = -\omega_1^2 x_1$$
$$x_2'' = -\omega_2^2 x_2.$$

$$Y(t) = \begin{pmatrix} x_1(t) \\ y_1(t) \\ x_2(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} a_1 \cos \omega_1 t + b_1 \sin \omega_1 t \\ -a_1 \sin \omega_1 t + b_1 \cos \omega_1 t \\ a_2 \cos \omega_2 t + b_2 \sin \omega_2 t \\ -a_2 \sin \omega_2 t + b_2 \cos \omega_2 t \end{pmatrix}$$

Polar Coordinates

$$x_j' = \omega_j y_j \qquad y_j' = -\omega_j x_j.$$

$$\Rightarrow$$

$$r_j' = 0 \qquad \theta_j' = -\omega_j.$$

$$\Rightarrow$$

$$\theta_1' = -\omega_1 \qquad \theta_2' = -\omega_2.$$

It is convenient to think of θ_1 and θ_2 as variables in a square of sidelength 2π where we glue together the opposite sides $\theta_j = 0$ and $\theta_j = 2\pi$ to make the torus. In this square our vector field now has constant slope

$$\frac{\theta_2'}{\theta_1'} = \frac{\omega_2}{\omega_1}.$$

Therefore solutions lie along straight lines with slope ω_2/ω_1 in this square. When a solution reaches the edge $\theta_1 = 2\pi$ (say, at $\theta_2 = c$), it instantly reappears on the edge $\theta_1 = 0$ with the θ_2 coordinate given by c, and then continues onward with slope ω_2/ω_1. A similar identification occurs when the solution meets $\theta_2 = 2\pi$.

(Uncoupled) Oscillators

$$x_1{}' = y_1$$

$$y_1{}' = -\omega_1^2 x_1$$

$$x_1'' = -\omega_1^2 x_1 \rightarrow periodic$$

uncoupled

$$x_2'' = -\omega_2^2 x_2 \rightarrow periodic$$

$$x_2{}' = y_2$$

$$y_2{}' = -\omega_2^2 x_2$$

$$\begin{pmatrix} x_1' \\ y_1' \\ x_2' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_2^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} -\lambda & 1 & 0 & 0 \\ -\omega_1^2 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & -\omega_2^2 & -\lambda \end{pmatrix}$$

$$|A - \lambda I| = 0 \qquad -\lambda \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -\omega_2^2 & -\lambda \end{vmatrix} - \begin{vmatrix} -\omega_1^2 & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & -\omega_2^2 & -\lambda \end{vmatrix} = 0$$

$$\lambda_{1,2} = \pm i\ \omega_1$$

$$\lambda_{3,4} = \pm i\ \omega_2$$

Changing Coordinates: Construct the Linear Map

$W_1^T = Re(V_1^T) = (1, 0, 0, 0)$

$\lambda_{1,2} = \pm i\,\omega_1$ $\quad V_1^T = (1, i\omega_1, 0, 0)$ $\quad W_2^T = Im(V_1^T) = (0, \omega_1\,, 0, 0)$

$\lambda_{3,4} = \pm i\,\omega_2$ $\quad V_2^T = (0,0,1, i\omega_2)$ $\quad W_3^T = Re(V_2^T) = (0, 0, 1, 0)$

$W_4^T = Im(V_2^T) = (0,0,0,\omega_2)$

$$T = \begin{array}{c} \begin{matrix} (W_1 & W_2 & W_3 & W_4) \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \omega_2 \end{pmatrix} \end{array} \qquad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_2^2 & 0 \end{pmatrix}$$

$$T^{-1}AT = \begin{pmatrix} 0 & \omega_1 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 \\ 0 & 0 & -\omega_2 & 0 \end{pmatrix}$$

Changing Coordinates: Matlab Code

- syms w1 w2
- T=[1 0 0 0 ; 0 w1 0 0 ; 0 0 1 0; 0 0 0 w2]
- A=[0 1 0 0 ; -w1^2 0 0 0 ; 0 0 0 1 ; 0 0 -w2^2 0]
- inv(T)*A*T

```
ans =


[   0, w1,   0,  0]
[ -w1,  0,   0,  0]
[   0,  0,   0, w2]
[   0,  0, -w2,  0]
```

Changing Coordinates: Construct the Linear Map

$$T^{-1}AT = \begin{pmatrix} \boxed{\begin{matrix} 0 & \omega_1 \\ -\omega_1 & 0 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & \omega_2 \\ -\omega_2 & 0 \end{matrix} \end{pmatrix}$$

$$\begin{pmatrix} x_1' \\ y_1' \end{pmatrix} = \begin{pmatrix} 0 & \omega_1 \\ -\omega_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$x_1{}' = \omega_1 y_1$$

$$y_1{}' = -\omega_1 x_1$$

$$x_1'' = -\omega_1^2 x_1$$

Let $x_1{}' = \omega_1 y_1$

$$y_1' = \frac{x_1''}{\omega_1} = -\omega_1 x_1$$

$$x_1(t) = a_1 \cos(\omega_1 t) + \mathrm{b}_1 \sin(\omega_1 t)$$

$$x_1(t) = e^{\alpha t}(X(\omega t))$$

$$y_1 = \frac{x_1'}{\omega_1} = -a_1 \sin(\omega_1 t) + \mathrm{b}_1 \cos(\omega_1 t)$$

$$y_1(t) = e^{\alpha t}\left(\frac{1}{\omega} X'\right)$$

Solutions & $\frac{dr_j}{dt} = 0$

2nd-order ODE

$$x_1'' = -\omega_1^2 x_1 \rightarrow periodic \qquad x(t) = a_1 \cos(\omega_1 t) + \mathrm{b}_1 \sin(\omega_1 t)$$

2D system

$$x_1' = \omega_1 y_1 \qquad x_1(t) = a_1 \cos(\omega_1 t) + \mathrm{b}_1 \sin(\omega_1 t)$$

$$y_1' = -\omega_1 x_1 \qquad y_1(t) = -a_1 \sin(\omega_1 t) + \mathrm{b}_1 \cos(\omega_1 t)$$

$$x_1^2 = \mathrm{a}_1^2 \cos^2(\omega_1 \mathrm{t}) + 2a_1 \mathrm{b}_1 \cos(\omega_1 \mathrm{t}) \sin(\omega_1 t) + b_1^2 \sin^2(\omega_1 t)$$

$$y_1^2 = \mathrm{a}_1^2 \sin^2(\omega_1 \mathrm{t}) - 2a_1 \mathrm{b}_1 \cos(\omega_1 \mathrm{t}) \sin \omega_1 (t) + b_1^2 \cos^2(\omega_1 t)$$

$$x_1^2 + y_1^2 = a_1^2(\cos^2(\omega_1 \mathrm{t}) + \sin^2(\omega_1 t)) + b_1^2(\cos^2(\omega_1 \mathrm{t}) + \sin^2(\omega_1 t)) = a_1^2 + b_1^2$$

$$r_1^2 = x_1^2 + y_1^2 = a_1^2 + b_1^2 \qquad \frac{dr_1}{dt} = 0$$

Similarly,

$$r_2^2 = x_2^2 + y_2^2 = a_2^2 + b_2^2 \qquad \frac{dr_2}{dt} = 0$$

$$\frac{dr_j}{dt} = 0$$

$$\boxed{\text{Show } x_j y_j' - y_j x_j' = -r_j^2 \omega_j} \qquad r_j^2 = x_j^2 + y_j^2$$

$$x_j = a_j \cos(\omega_j t) + b_j \sin(\omega_j t) \ (a) \qquad x_j' = -a_j\, \omega_j \sin(\omega_j t) + b_j \omega_j \cos(\omega_j t) \ (c)$$

$$y_j = -a_j \sin(\omega_j t) + b_j \cos(\omega_j t) \ (b) \qquad y_j' = -a_j \omega_j \cos(\omega_j t) - b_j\, \omega_j \sin(\omega_j t) \ (d)$$

$$y_j x_j' = (b)(c) = a_j^2 \omega_j \sin^2(\omega_j t) + b_j^2 \omega_j \cos^2(\omega_j t) - 2 a_j b_j \omega_j \sin(\omega_j t) \cos(\omega_j t)$$

$$x_j y_j' = (a)(d) = -a_j^2 \omega_j \cos^2(\omega_j t) - b_j^2 \omega_j \sin^2(\omega_j t) - 2 a_j b_j \omega_j \sin(\omega_j t) \cos(\omega_j t)$$

$$\boxed{x_j y_j' - y_j x_j' = (ad) - (bc) = (-a_j^2 \omega_j - b_j^2 \omega_j) = -r_j^2\, \omega_j}$$

$$\boxed{\text{Show } x_j y_j' - y_j x_j' = -r_j^2 \omega_j}$$

$$r_j^2 = x_j^2 + y_j^2$$

Previously, we learned: $x_1(t) = e^{\alpha t}(X(\omega t))$ $\quad y_1(t) = e^{\alpha t}\left(\frac{1}{\omega} X'\right)$

$$x_j = a_j \cos(\omega_j t) + b_j \sin(\omega_j t) \text{ (a)} \qquad x_j' = -a_j \omega_j \sin(\omega_j t) + b_j \omega_j \cos(\omega_j t) \text{ (c)}$$

$$= \omega_j y_j$$

$$y_j = -a_j \sin(\omega_j t) + b_j \cos(\omega_j t) \ (b) \qquad y_j' = -a_j \omega_j \cos(\omega_j t) - b_j \omega_j \sin(\omega_j t) \ (d)$$

$$= -\omega_j x_j$$

$$y_j x_j' = (b)(c) = \omega_j y_j^2$$

$$x_j y_j' = (a)(d) = -\omega_j x_j^2$$

$$\boxed{x_j y_j' - y_j x_j' = (ad) - (bc) = (-x_j^2 \omega_j - y_j^2 \omega_j) = -\omega_j (a_j^2 + b_j^2) = -r_j^2 \omega_j}$$

Solutions & $\frac{d\theta_j}{dt} = -\omega_j$

$$\tan(\theta_j) = \frac{y_j}{x_j}$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\tan(\theta_j) = \frac{d}{dt}\left(\frac{y_j}{x_j}\right) \qquad \frac{\mathrm{d}\theta_j}{\mathrm{dt}}\sec^2(\theta_j) = \frac{x_j y_j' - y_j x_j'}{x_j^2}$$

(from the previous slide)

$$\boxed{\frac{x_j y_j' - y_j x_j'}{x_j^2} = \frac{-r_j^2\omega_j}{\left(r_j\cos(\theta_j)\right)^2}} = \frac{-\omega_j}{\left(\cos(\theta_j)\right)^2} = -\omega_j\sec^2(\theta_j)$$

$$\frac{\mathrm{d}\theta_j}{\mathrm{dt}}\sec^2(\theta_j) = -\omega_j\sec^2(\theta_j) \qquad \boxed{\frac{\mathrm{d}\theta_j}{\mathrm{dt}} = -\omega_j}$$

$$\theta_j = \theta_{j0} - \omega_j t \qquad \Delta\theta = 2\pi = |-\omega_j|T \qquad \boxed{T = \frac{2\pi}{\omega_j}, \quad period}$$

A linear function of time

Period T and Frequency ω_j

$$x_j = a_j\, cos(\omega_j t) + b_j\, sin(\omega_j t)$$

$$\boxed{\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)}$$

(A) consider $x(t) = a_j\, cos(\omega_j t)$

A period is determined by $T = \frac{2\pi}{\omega_j}$

$$x(t+T) = a_j\, cos(\omega_j (t+T)) = a_j\, cos(\omega_j t + 2\pi) = a_j\, cos(\omega_j t) = x(t)$$

(B) consider $x_j = a_j\, cos(\omega_j t) + b_j\, sin(\omega_j t)$

$$= \sqrt{{a_j}^2 + {b_j}^2}\left(\frac{a_j}{\sqrt{{a_j}^2+{b_j}^2}} cos(\omega_j t) + \frac{b_j}{\sqrt{{a_j}^2+{b_j}^2}} sin(\omega_j t)\right)$$

$$= \sqrt{{a_j}^2 + {b_j}^2}\left(cos(\omega_j t - \theta_0)\right) \qquad \cos(\theta_0) = \frac{a_j}{\sqrt{{a_j}^2 + {b_j}^2}}$$

A period is determined by $T = \frac{2\pi}{\omega_j}$

Periodic Composite Motion

$x_1'' = -\omega_1^2 x_1 \rightarrow periodic$ $\qquad$ $x_1(t) = a_1 \cos(\omega_1 t) + \mathrm{b}_1 \sin(\omega_1 t)$

$x_2'' = -\omega_2^2 x_2 \rightarrow periodic$ $\qquad$ $x_2(t) = a_2 \cos(\omega_2 t) + \mathrm{b}_2 \sin(\omega_2 t)$

Q: under which condition a composite motion with the two frequencies is periodic?

Based on what we discuss, we can simply consider the following:

$$x_1(t) = a_1 \cos(\omega_1 t) \qquad \omega_1(t+T) = \omega_1 t + 2m\pi \qquad T = \frac{2m\pi}{\omega_1}$$

$$x_2(t) = a_2 \cos(\omega_2 t) \qquad \omega_2(t+T) = \omega_2 t + 2n\pi \qquad T = \frac{2n\pi}{\omega_2}$$

$$\frac{2m\pi}{\omega_1} = \frac{2n\pi}{\omega_2}$$

$\boxed{\frac{\omega_2}{\omega_1} = \frac{n}{m}\text{: a rational number}}$

Periodicity vs. Quasi-periodicity

$$\frac{dr_j}{dt} = 0$$

$$\frac{d\theta_j}{dt} = -\omega_j \quad \Rightarrow \quad \begin{aligned} \theta_1' &= -\omega_1 \\ \theta_2' &= -\omega_2 \end{aligned} \qquad \begin{aligned} \theta_1 &= \theta_1(t=0) - \omega_1 t \\ \theta_2 &= \theta_2(t=0) - \omega_2 t \end{aligned}$$

- Periodic: $\frac{\omega_2}{\omega_1}$ is a rational number, e.g., $\frac{\omega_2}{\omega_1} = \frac{5}{2}$

Periodic, $\frac{\omega_2}{\omega_1} = \frac{5}{2}$

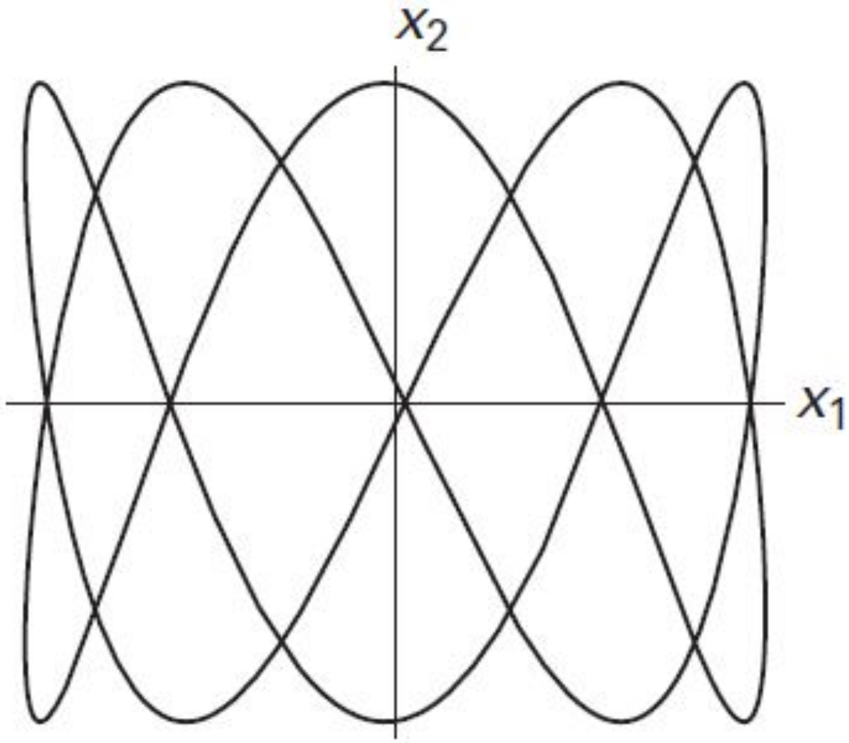
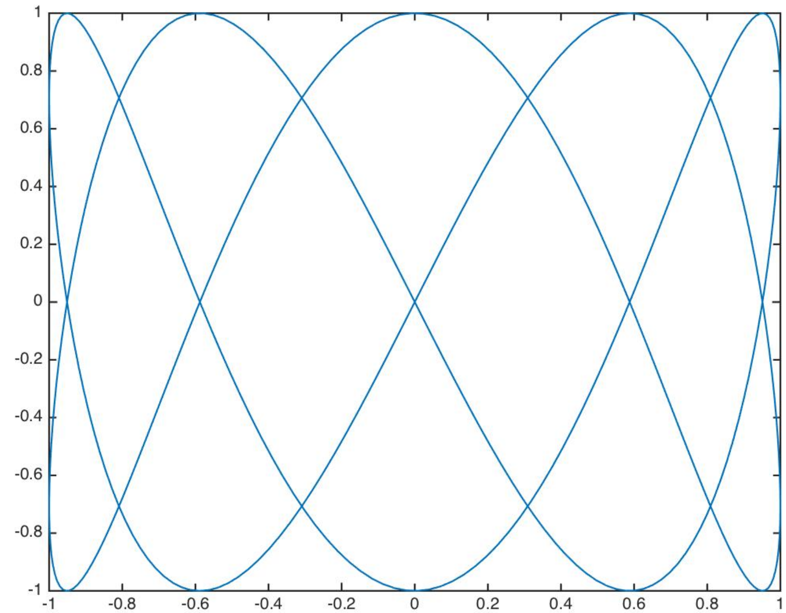


Figure 6.6 A solution with frequency ratio 5/2 projected into the $x_1 x_2$–plane. Note that $x_2(t)$ oscillates five times and $x_1(t)$ only twice before returning to the initial position.

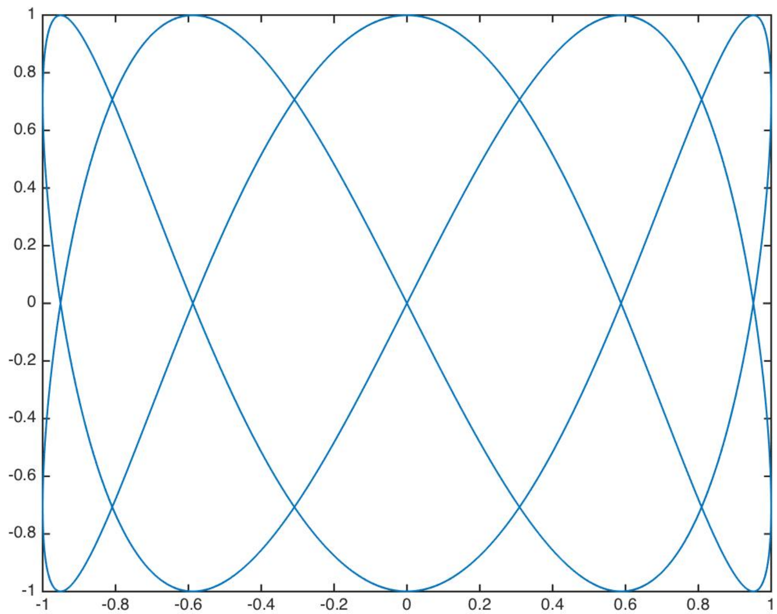



```
time=linspace(0,1, 400) %t=0~1
x1=sin(2*2*pi*time)
x2=sin(5*2*pi*time)
%fig=figure()
plot(x1, x2)
saveas(gcf, 'Figure6_6_numerical', 'jpg' )
```

Section 6.2: Periodic, $\frac{\omega_2}{\omega_1} = \frac{5}{2}$

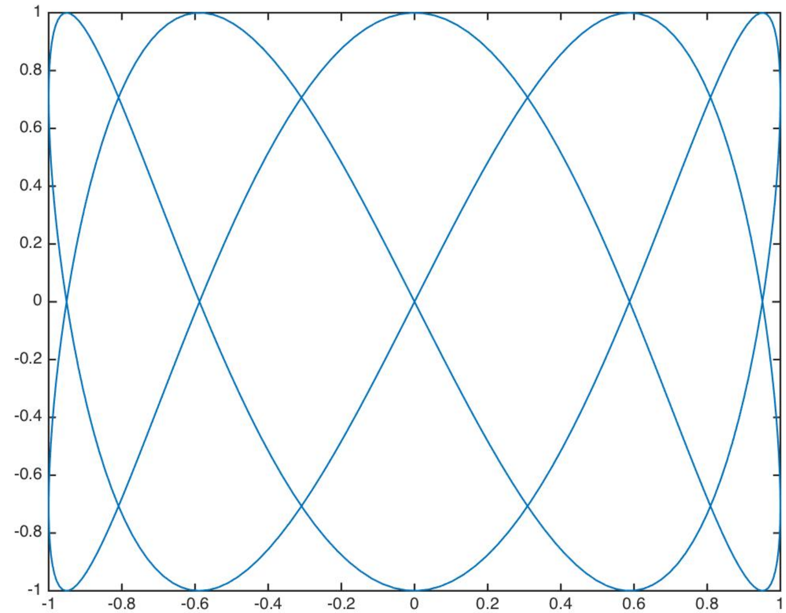
S3

T=0~4



no changes

T=0~1




```
time=linspace(0,4, 1600) %t=0~4
x1=sin(2*2*pi*time)
x2=sin(5*2*pi*time)
%fig=figure()
plot(x1, x2)
saveas(gcf, 'Figure6_6_numerical', 'jpg' )
```

```
time=linspace(0,1, 400) %t=0~1
x1=sin(2*2*pi*time)
x2=sin(5*2*pi*time)
%fig=figure()
plot(x1, x2)
saveas(gcf, 'Figure6_6_numerical', 'jpg' )
```

Dense and Quasi-periodicity

S5

Definition
The set of points $x_0, x_1 = f(x_0), x_2 = f(f(x_0)), \ldots, x_n = f(x_{n-1})$ is called the *orbit* of x_0 under iteration of f.

Proposition. *Suppose ω_2/ω_1 is irrational. Then the orbit of any initial point x_0 on the circle $\theta_1 = 0$ is dense in the circle.*

Periodicity vs. Quasi-periodicity

$$\frac{dr_j}{dt} = 0$$

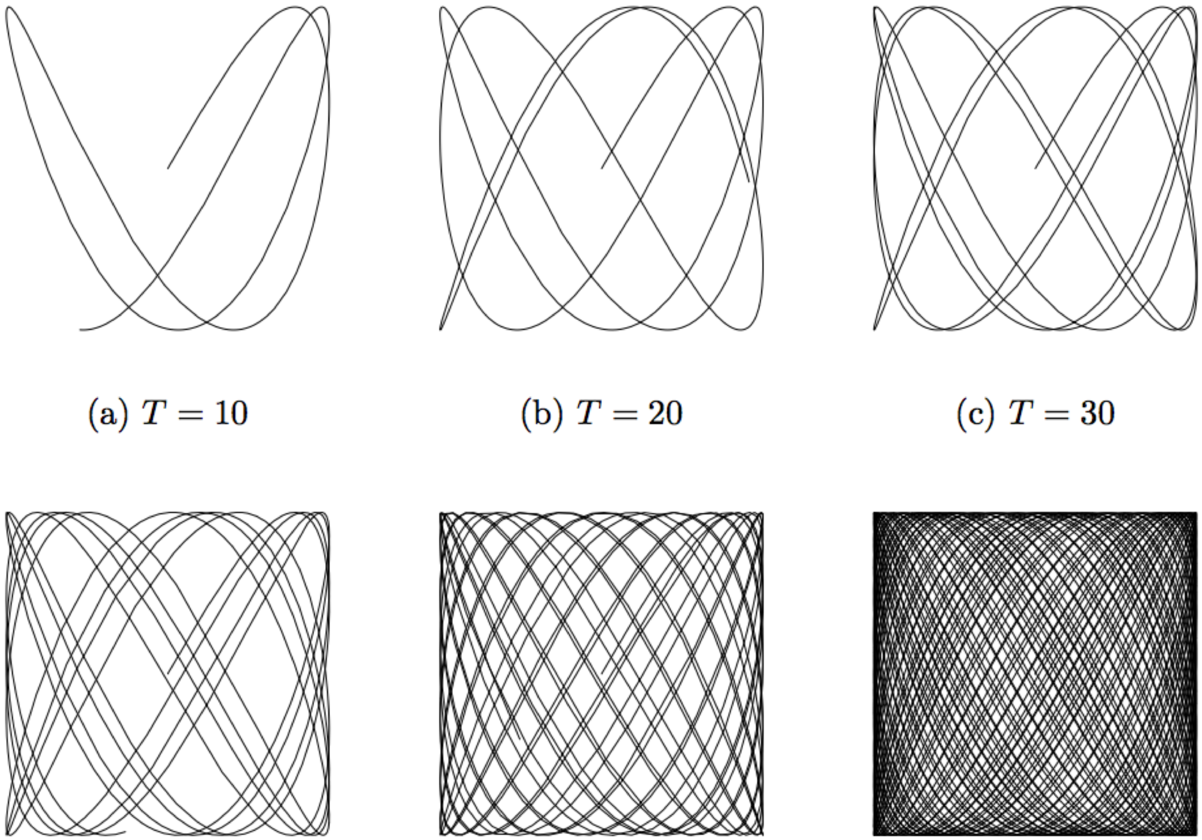
$$\frac{d\theta_j}{dt} = -\omega_j \quad \Rightarrow \quad \begin{aligned} \theta_1' &= -\omega_1 \\ \theta_2' &= -\omega_2 \end{aligned} \qquad \begin{aligned} \theta_1 &= \theta_1(t=0) - \omega_1 t \\ \theta_2 &= \theta_2(t=0) - \omega_2 t \end{aligned}$$

- Periodic: $\frac{\omega_2}{\omega_1}$ is a rational number, e.g., $\frac{\omega_2}{\omega_1} = \frac{5}{2}$
- Quasi-periodic: $\frac{\omega_2}{\omega_1}$ is not a rational number, e.g., $\frac{\omega_2}{\omega_1} = \sqrt{2}$

“Dense” Quasi-periodic Solutions

w1/w2
is irrational

➔
incommensurate
frequencies



(a) $T = 10$ (b) $T = 20$ (c) $T = 30$

(d) $T = 50$ (e) $T = 100$ (f) $T = 300$

Figure 1.2: Trajectory of the mechanical system at different times.

$Quasiperiodic: \dfrac{\omega_2}{\omega_1} = \sqrt{2}$

T=0~50π T=0~100π

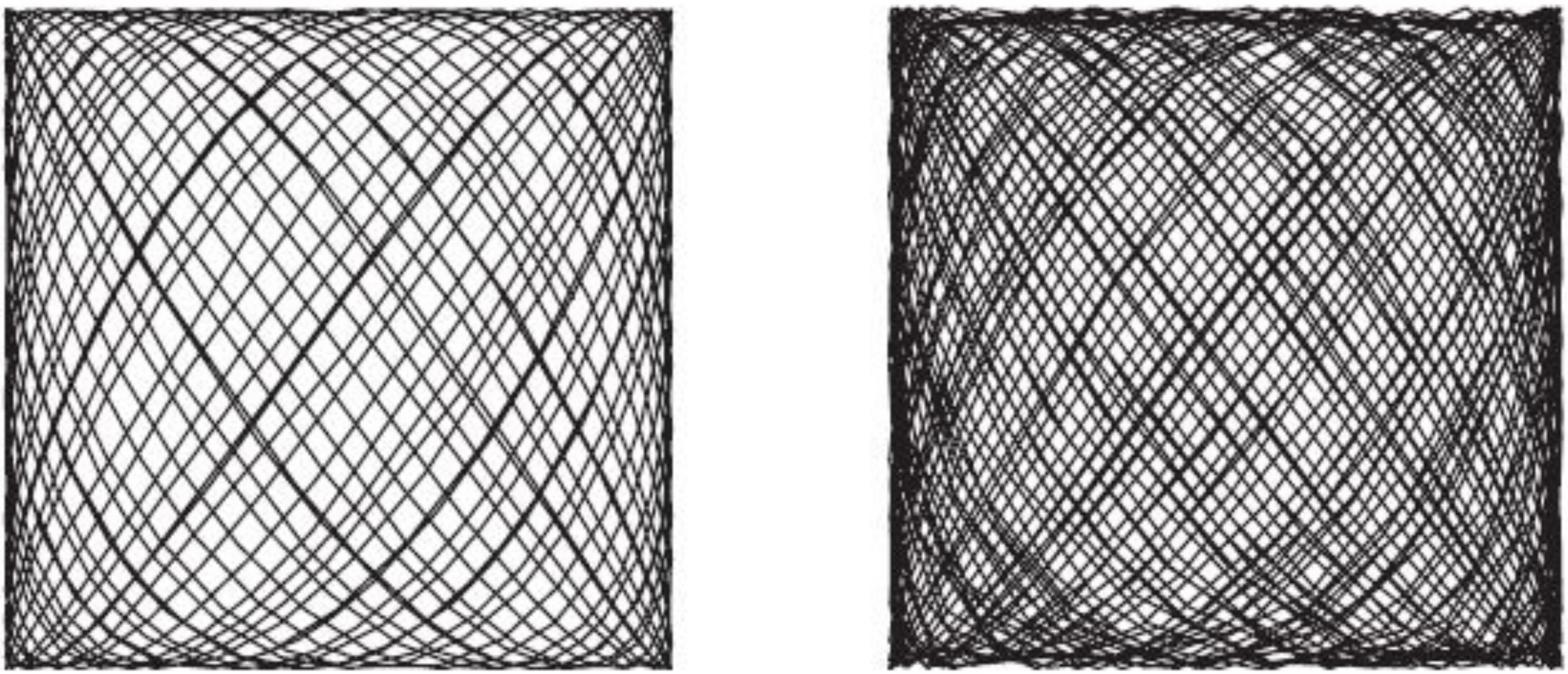
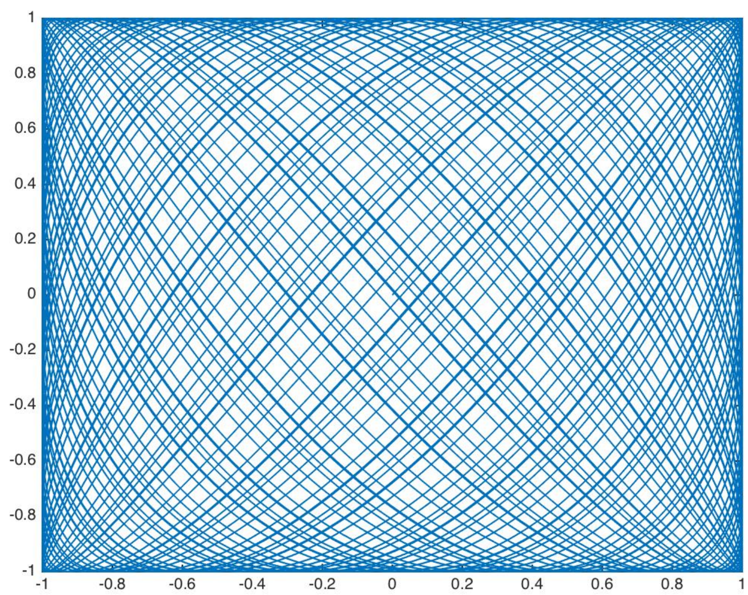


Figure 6.7 A solution with frequency ratio $\sqrt{2}$ projected into the $x_1 x_2$–plane, the left curve computed up to time 50π, the right to time 100π.

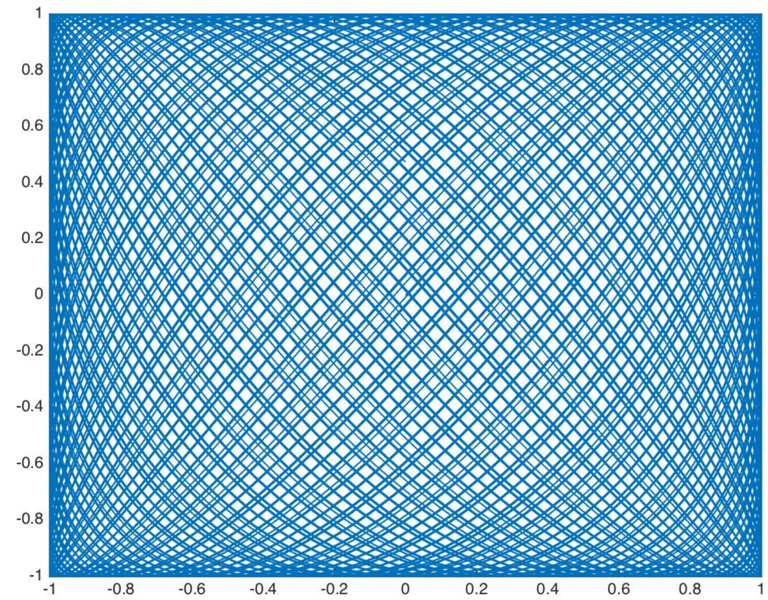
$Quasiperiodic: \dfrac{\omega_2}{\omega_1} = \sqrt{2}$

T=0~50




```
time=linspace(0,50, 50000)
x1=sin(2*pi*time)
x2=sin(sqrt(2)*2*pi*time)
%fig=figure()
plot(x1, x2)
```

T=0~100




```
time=linspace(0,100, 100000)
x1=sin(2*pi*time)
x2=sin(sqrt(2)*2*pi*time)
%fig=figure()
plot(x1, x2)
```

Construct a Torus

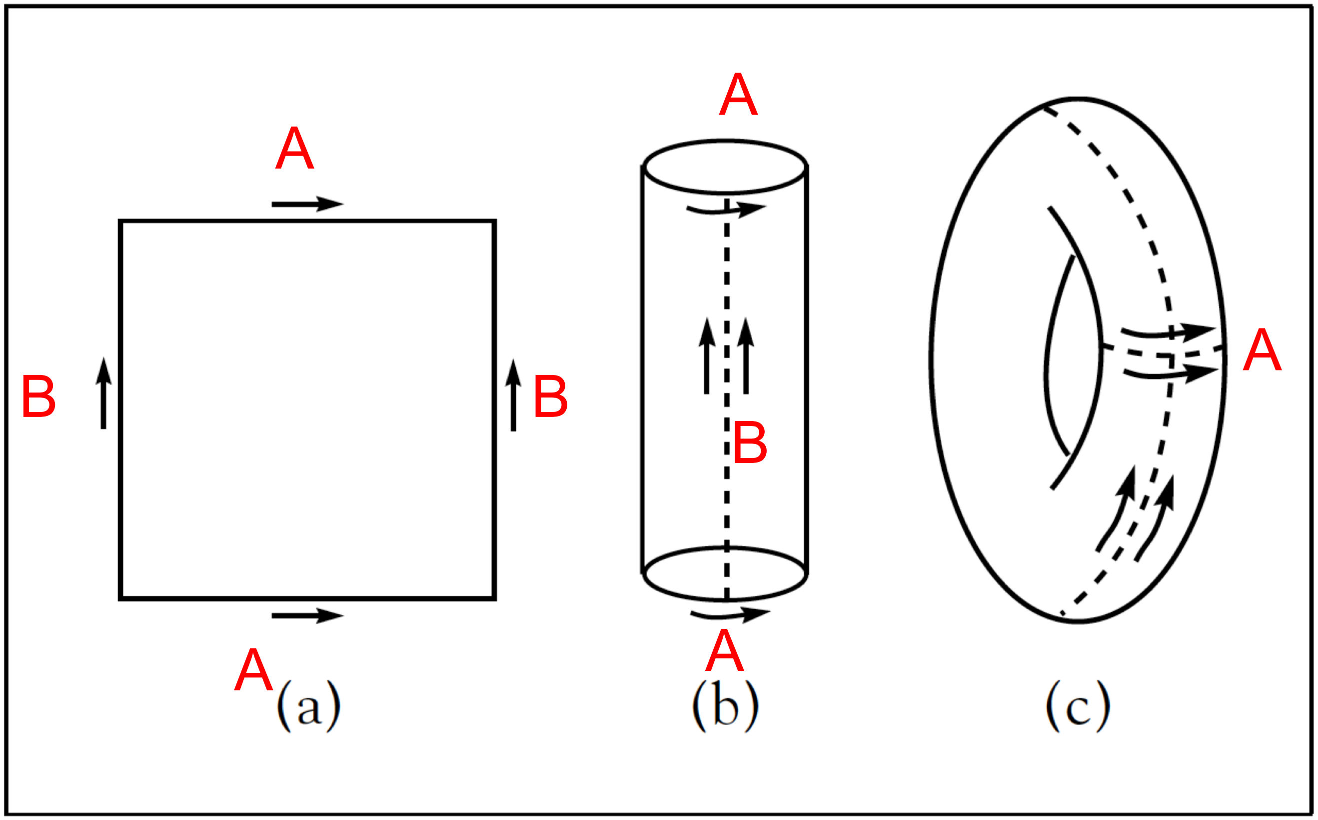


Figure 2.28 Construction of a torus in two easy steps.
(a) Begin with unit square. (b) Identify (glue together) vertical sides. (c) Identify horizontal sides.

Construct a Torus

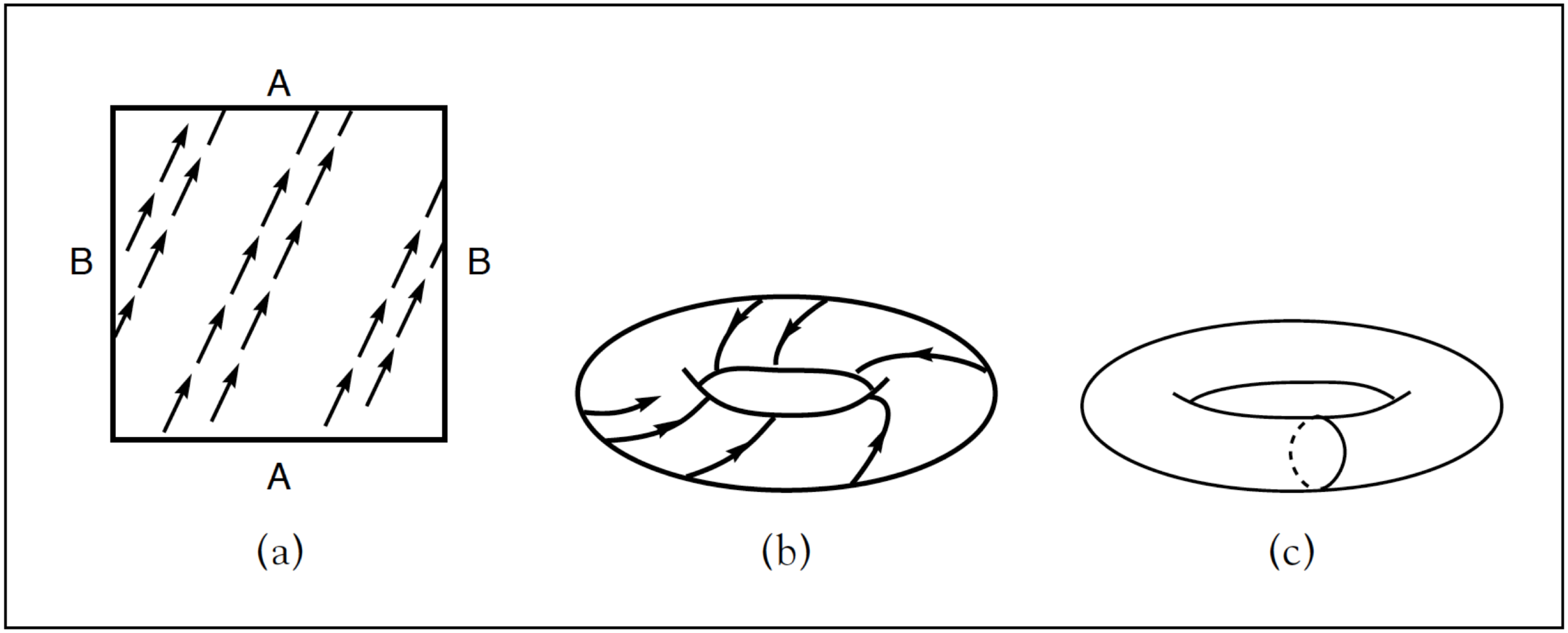


Figure 8.9 A dense orbit on the torus.
(a) The vector field has the same irrational slope at each point in the unit square. (b) The square is made into a torus by gluing together the top and bottom (marked A) to form a cylinder and then gluing together the ends (marked B). Each orbit winds densely around the torus. For each point $\mathbf{u}$ of the torus, $\mathbf{u}$ belongs to $\omega(\mathbf{u})$ and $\alpha(\mathbf{u})$. (c) There is no analogue of the Poincaré-Bendixson Theorem on the torus because there is no Jordan Curve Theorem on the torus. A simple closed curve that does not divide the torus into two parts is shown.

Quasiperiodic: dense

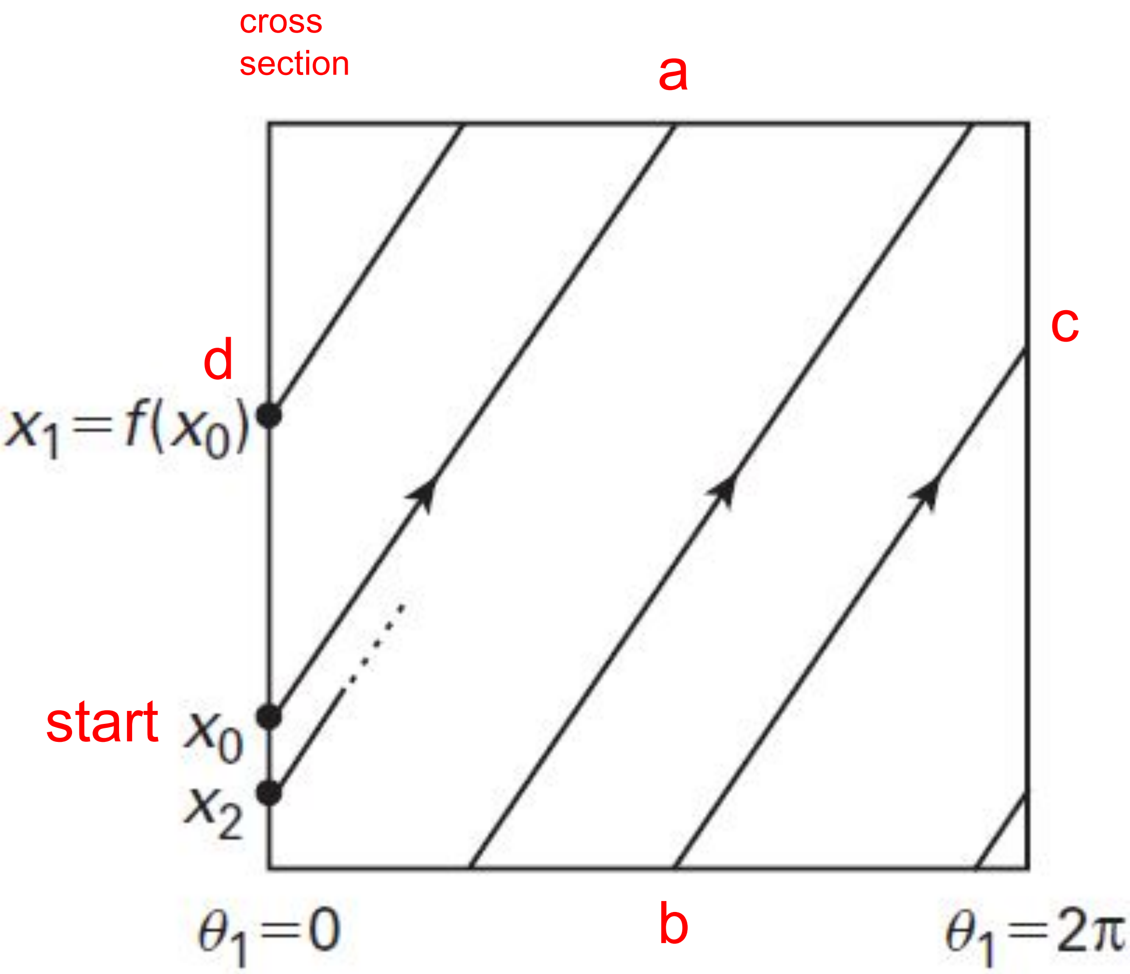
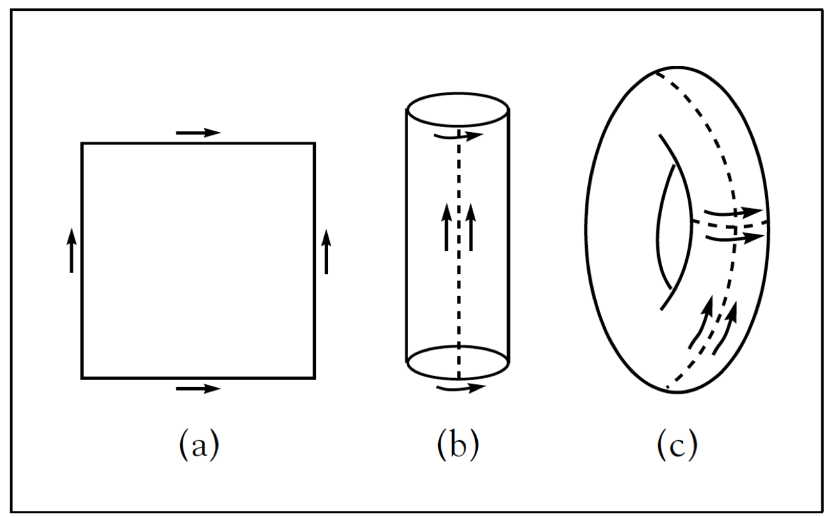


Figure 6.8 The Poincaré map on the circle $\theta_1 = 0$ in the $\theta_1\theta_2$ torus.

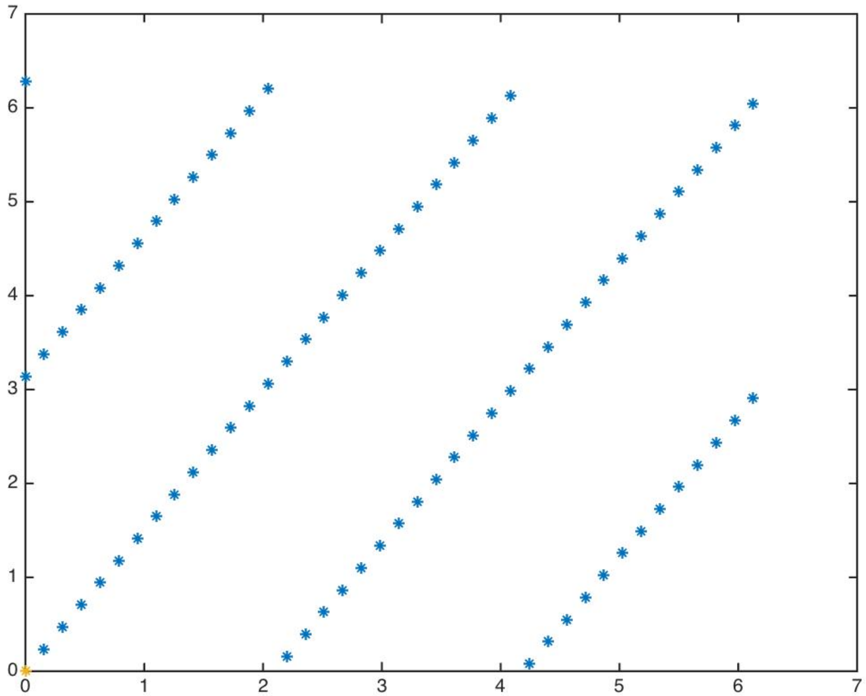
”a” and “b” are the same

”c” and “d” are the same



$Periodic: \frac{\omega_2}{\omega_1} = \frac{3}{2}$ **S6**

T=0~12π

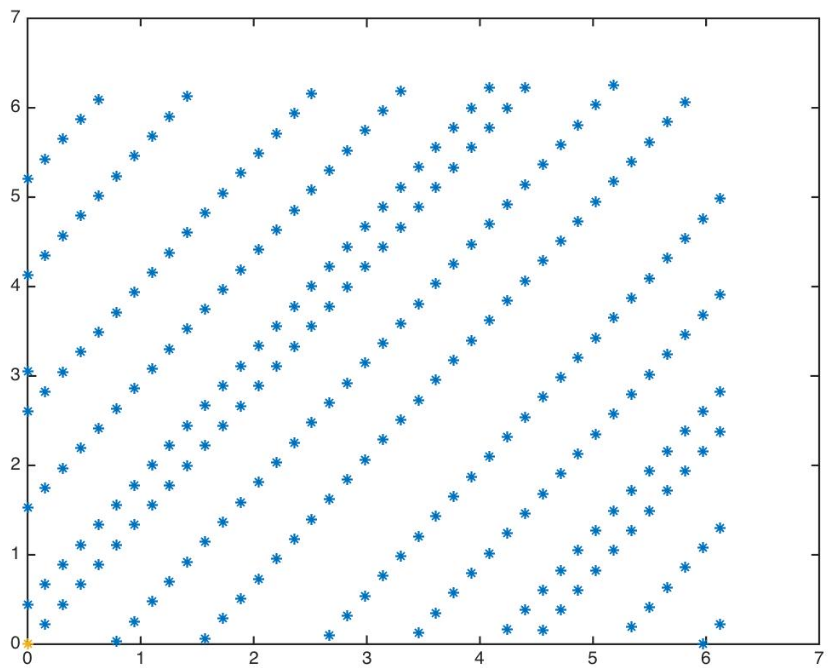



```
NumPTs=241
theta1=zeros(NumPTs)
theta2=zeros(NumPTs)
Ending_time=12*pi
dt=Ending_time/(NumPTs-1)
tau=0.0
omega1=1.
omega2=1.5
i=1
while i <= NumPTs-1
    tau=tau+dt
    theta1(i+1)=theta1(i) + omega1*dt
    theta2(i+1)=theta2(i) + omega2*dt
    if (theta1(i+1) >= 2*pi)
        theta1(i+1)=theta1(i+1)-2*pi
    end
    if (theta2(i+1) >= 2*pi)
        theta2(i+1)=theta2(i+1)-2*pi
    end
    i=i+1
end
plot(theta1, theta2, '*')
```

$Quasiperiodic: \frac{\omega_2}{\omega_1} = \sqrt{2}$

S7

T=0~12π



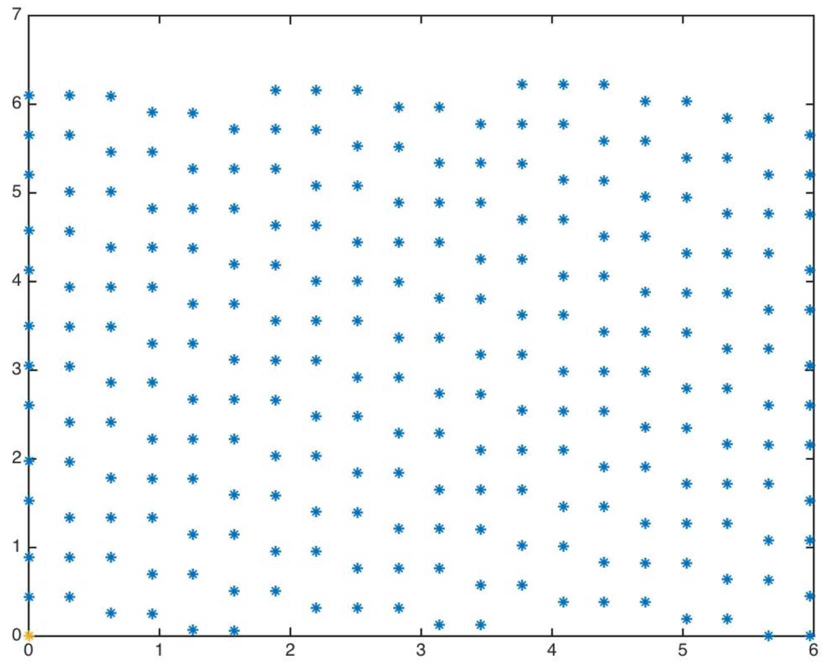
Many points

```
NumPTs=241
theta1=zeros(NumPTs)
theta2=zeros(NumPTs)
Ending_time=12*pi
dt=Ending_time/(NumPTs-1)
tau=0.0
omega1=1
omega2=sqrt(2)
i=1
while i <= NumPTs-1
    tau=tau+dt
    theta1(i+1)=theta1(i) + omega1*dt
    theta2(i+1)=theta2(i) + omega2*dt
    if (theta1(i+1) >= 2*pi)
        theta1(i+1)=theta1(i+1)-2*pi
    end
    if (theta2(i+1) >= 2*pi)
        theta2(i+1)=theta2(i+1)-2*pi
    end
    i=i+1
end
plot(theta1, theta2, '*')
```

$Quasiperiodic: \frac{\omega_2}{\omega_1} = \sqrt{2}$, longer time

S8

T=0~24π




```
NumPTs=241
theta1=zeros(NumPTs)
theta2=zeros(NumPTs)
Ending_time=24*pi
dt=Ending_time/(NumPTs-1)
tau=0.0
omega1=1
omega2=sqrt(2)
i=1
while i <= NumPTs-1
   tau=tau+dt
   theta1(i+1)=theta1(i) + omega1*dt
   theta2(i+1)=theta2(i) + omega2*dt
   if (theta1(i+1) >= 2*pi)
      theta1(i+1)=theta1(i+1)-2*pi
   end
   if (theta2(i+1) >= 2*pi)
      theta2(i+1)=theta2(i+1)-2*pi
   end
   i=i+1
end
plot(theta1, theta2, '*')
```

Review: Harmonic Oscillators (uncoupled)

Consider a pair of undamped harmonic oscillators whose equations are

$$x_1'' = -\omega_1^2 x_1$$
$$x_2'' = -\omega_2^2 x_2.$$

$$Y(t) = \begin{pmatrix} x_1(t) \\ y_1(t) \\ x_2(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} a_1 \cos\omega_1 t + b_1 \sin\omega_1 t \\ -a_1 \sin\omega_1 t + b_1 \cos\omega_1 t \\ a_2 \cos\omega_2 t + b_2 \sin\omega_2 t \\ -a_2 \sin\omega_2 t + b_2 \cos\omega_2 t \end{pmatrix}$$

A Coupled Oscillator: Example 1

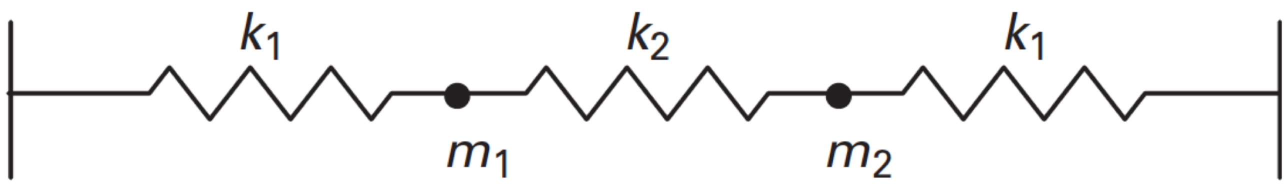


Figure 6.10 A coupled oscillator.

and m_2 attached to springs and walls as shown in Figure 6.10. The springs connecting m_j to the walls both have spring constants k_1, while the spring connecting m_1 and m_2 has spring constant k_2. This coupling means that the motion of either mass affects the behavior of the other.

Let x_j denote the displacement of each mass from its rest position, and assume that both masses are equal to 1. The differential equations for these coupled oscillators are then given by

$$x_1'' = -(k_1 + k_2)x_1 + \boxed{k_2 x_2}$$
$$x_2'' = \boxed{k_2 x_1} - (k_1 + k_2)x_2.$$

coupled

A Coupled Oscillator (.continued)

These equations are derived as follows. If m_1 is moved to the right ($x_1 > 0$), the left spring is stretched and exerts a restorative force on m_1 given by $-k_1 x_1$. Meanwhile, the central spring is compressed, so it exerts a restorative force on m_1 given by $-k_2 x_1$. If the right spring is stretched, then the central spring is compressed and exerts a restorative force on m_1 given by $k_2 x_2$ (since $x_2 < 0$). The forces on m_2 are similar.

(a) Write these equations as a first-order linear system.

(b) Determine the eigenvalues and eigenvectors of the corresponding matrix.

(c) Find the general solution.

(d) Let $\omega_1 = \sqrt{k_1}$ and $\omega_2 = \sqrt{k_1 + 2k_2}$. What can be said about the periodicity of solutions relative to the ω_j? Prove this.

A Coupled Oscillator: Example 2

Kreyszig

$$\frac{d^2y_1}{dt^2} = -k_1y_1 + k_2(y_2 - y_1),$$

$$\frac{d^2y_2}{dt^2} = -k_2(y_2 - y_1).$$

$$\boxed{k_1 = 3;\ k_2 = 2}$$

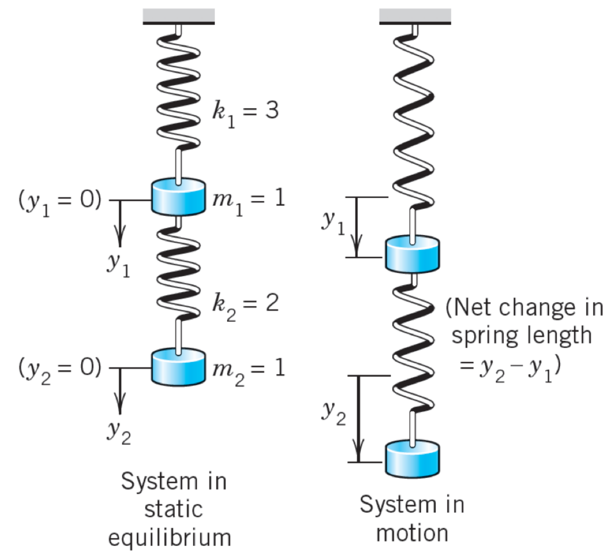


Fig. 161. Masses on springs in Example 4

Vibrating System of Two Masses on Two Springs (Fig. 161)

Mass–spring systems involving several masses and springs can be treated as eigenvalue problems. For instance, the mechanical system in Fig. 161 is governed by the system of ODEs

$$\begin{aligned} y_1'' &= -3y_1 - 2(y_1 - y_2) = -5y_1 + 2y_2 \\ y_2'' &= \qquad\quad -2(y_2 - y_1) = \ \ 2y_1 - 2y_2 \end{aligned} \quad \boxed{= \mathbf{Ay} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.} \tag{6}$$

where y_1 and y_2 are the displacements of the masses from rest, as shown in the figure, and primes denote derivatives with respect to time t.

Solve for the Solutions

$$\boxed{\begin{aligned} y_1'' &= \\ y_2'' &= \end{aligned}} \begin{aligned} -3y_1 - 2(y_1 - y_2) &= -5y_1 + 2y_2 \\ -2(y_2 - y_1) &= 2y_1 - 2y_2 \end{aligned} \quad \boxed{= \mathbf{Ay} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.}$$

$$Y'' = AY\ ; \qquad Y = \begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix} e^{\omega t}\ ;$$

$$\omega^2 \begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix} e^{\omega t} = A \begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix} e^{\omega t}; \qquad (A - \omega^2 I) \begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix} = 0;$$

$\begin{pmatrix} y_{01} \\ y_{02} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a solution. To obtain non-trivial solutions, we require

$|A - \omega^2 I| = 0$, which is the same as the following:

$\boxed{|A - \lambda I| = 0 \text{ if we introduce } \lambda = \omega^2 \text{ for convenience.}}$

A Brief Note for the Special System of 2nd ODEs

$$\boxed{\begin{aligned} y_1'' &= \\ y_2'' &= \end{aligned}} \begin{aligned} -3y_1 - 2(y_1 - y_2) &= -5y_1 + 2y_2 \\ -2(y_2 - y_1) &= \quad 2y_1 - 2y_2 \end{aligned} \boxed{= \mathbf{A}\mathbf{y} = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.}$$

$Y' = AY$; $\qquad Y = \binom{y_{01}}{y_{02}} e^{\omega t}$;

$|A - \omega^2 I| = 0$, which is the same as the following:

$\boxed{|A - \lambda I| = 0 \text{ if we introduce } \lambda = \omega^2 \text{ for convenience.}}$

The above indicates that

1. Let A represent a 2x2 matrix for a system of "two" second-order ODEs (for two oscillators in this case; no first derivatives in the above system);
2. The eigenvalues of the matrix A represent ω^2. Here, ω may represent the frequency of the oscillators.

Section 6.2: Quasi-Periodic

Supp

Consider two frequencies, ω and Ω.

- T1, 2π/ω, is the the time required to transit the torus the short way;
- T2, 2π/Ω, is the time required to transit the torus the long way.

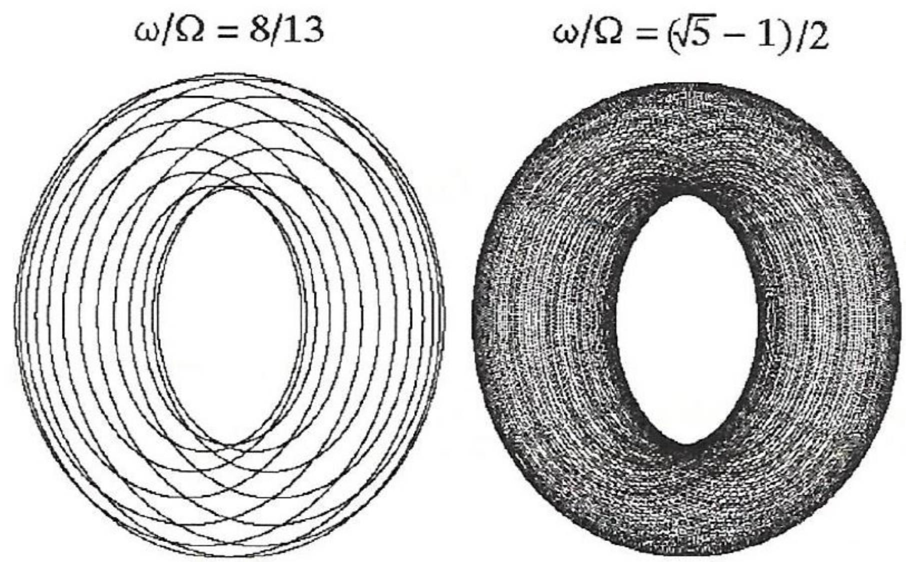


Fig. 3.5 Invariant torus for commensurate and incommensurate frequencies.

- The ratio ω/Ω, called winding number (or rotation number), is the number of times the trajectory transits the short way around for each time the long way.
- If the ratio is rational, (the frequencies are commensurate), the motion is periodic with a period equal to the smallest common multiple of the two periods. This condition is variously called frequency locking, phase locking, mode locking, frequency pulling etc.
- If the ratio is irrational, (the frequencies are incommensurate), the trajectory fills the whole toroidal surface without ever intersecting itself and the motion is quasiperiodic. **There is no common multiple of the frequencies**, and the period is infinite.

Section 6.2: Harmonic Oscillators

Supp

The second possibility is that the slope is ***irrational*** (Figure 8.6.6).Then the flow is said to be ***quasiperiodic***. Every trajectory winds around endlessly on the torus, never intersecting itself and yet never quite closing.

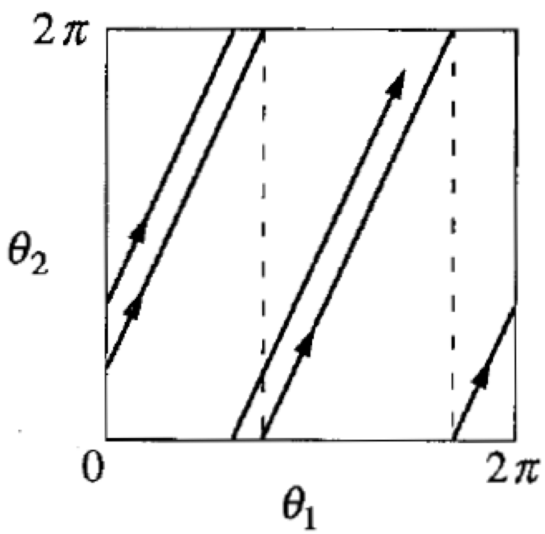


Figure 8.6.6

How can we be sure the trajectories never close? Any closed trajectory necessarily makes an integer number of revolutions in both θ_1 and θ_2; hence the slope would have to be rational, contrary to assumption.

Furthermore, when the slope is irrational, each trajectory is ***dense*** on the torus: in other words, each trajectory comes arbitrarily close to any given point on the torus. This is *not* to say that the trajectory passes *through* each point; it just comes arbitrarily close (Exercise 8.6.3).

Quasiperiodicity is significant because it is a new type of long-term behavior. Unlike the earlier entries (fixed point, closed orbit, homoclinic and heteroclinic orbits and cycles), quasiperiodicity occurs only on the torus.

Quasi-Periodicity

- In the quasi-periodic case the motion, strictly speaking, never exactly repeats itself (hence, the modifier quasi), but the motion is not chaotic;
- it is composed of two (or more) periodic components, whose presence could be made known by measuring the frequency spectrum (Fourier power spectrum) of the motion.
- We should point out that detecting the difference between quasi-periodic motion and motion with a rational ratio of frequencies, when the integers are large, is a delicate question.
- Whether a given experiment can distinguish the two cases depends on the resolution of the experimental equipment. As we shall see later, the behavior of the system can switch abruptly back and forth between the two cases as a parameter of the system is varied.
- The important point is that the attractor for the system is a two-dimensional surface of the torus for quasi-periodic behavior.

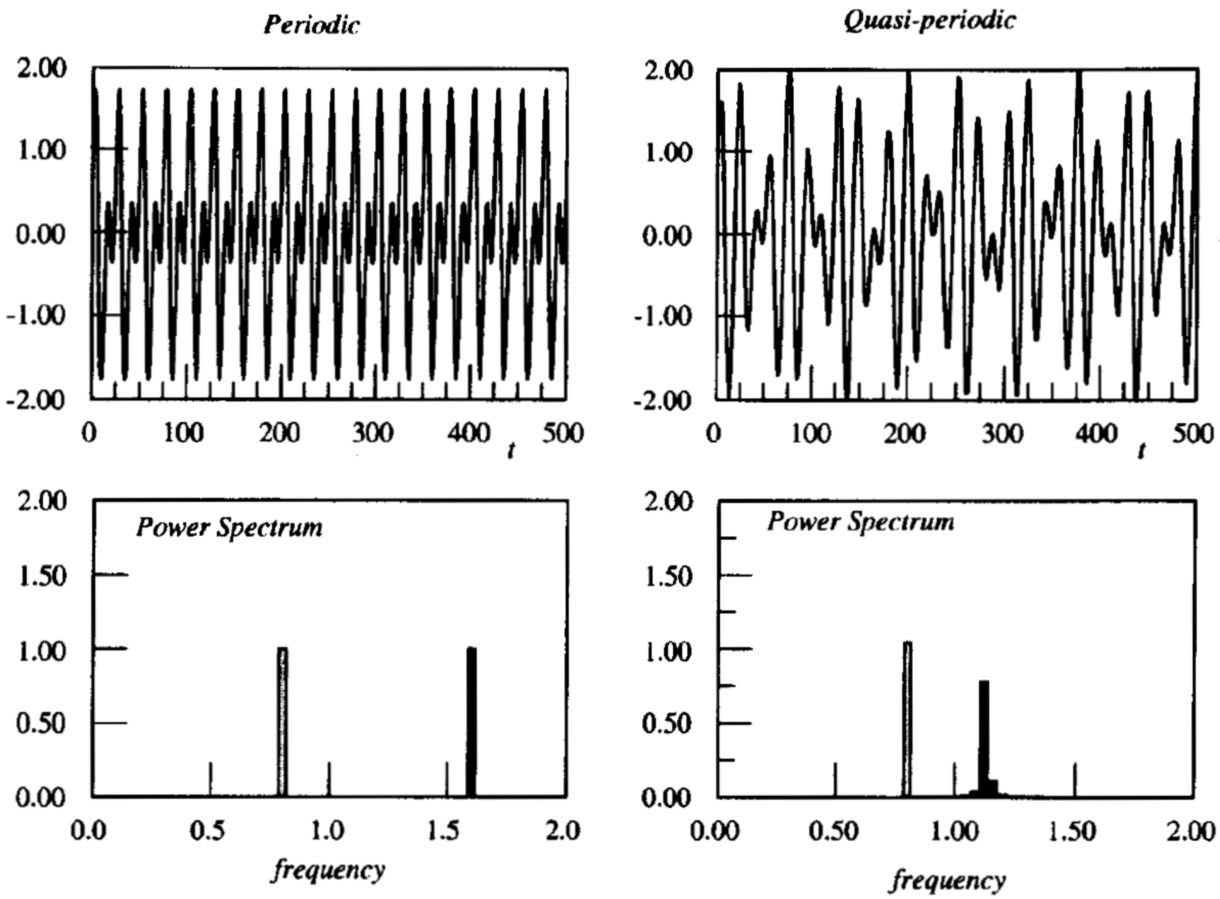
Hilborn (2000, p135)

Quasi-Periodicity

- If the ratio is irrational, then we say that the system's behavior is quasi-periodic (The terms conditionally periodic, almost periodic are sometimes used in place of quasi-periodic.)
- We must ask, therefore, how do we know in practice whether two frequencies f_1 and f_2 are commensurate or incommensurate? The problem is that any actual measurement of the frequencies has some finite precision.
- Similarly, any numerical calculation, say on a computer, has only finite arithmetical precision: any number used by the computer is effectively a rational number.
- All we can say is that to within the precision of our measurements or within the precision of our numerical calculations, a given frequency ratio is equal to a particular irrational number or a particular rational number, which is close to that irrational number.
- Beyond that we cannot say whether the ratio is "really" rational or irrational.

Hilborn (2000, p211)

Spectra of Periodic and Quasi-Periodic Motions



The second frequency is incommensurate with the first

Fig. 6.1. On the left is the time evolution of a system with two frequencies. Here $f_1 = 2 f_2$. On the right is the time evolution when the two frequencies are incommensurate $f_2 = \sqrt{2} f_1$. The behavior on the right looks quite irregular, but the power spectrum shown in the lower part of the figure indicates that only two frequencies (with different amplitudes) are contributing to the behavior. The crucial point is that in the case on the right has two incommensurate frequencies. The widths of the power spectrum "peaks" are due to the relatively short time interval of data used in the analysis.

Hilborn (2000, p212)

Backup