Quiz 6
Differential Equations
Math 337
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Problem 1: Consider the 3^{rd} order linear homogeneous ODE given by:

$$t^2y''' - ty'' + 2y' = 0$$

Use similar techniques for solving the *Cauchy-Euler* problem to solve this problem. Find 3 linearly independent solutions to this problem. How would one establish that these are 3 linearly independent solutions.

Let the following be true:

$$y = t^{r+1} \qquad y' = (r+1)t^r \qquad y'' = (r^2+r)t^{r-1} \qquad y''' = (r^3-r)t^{r-2}$$

When we now evaluate the original problem with our $y = t^{r+1}$, we get

$$\begin{aligned} t^2t^{r-2}(r^3-r) - tt^{r-1}(r^2+r) + t^r(2r+2) &= 0 \\ t^r(r^3-r^2+2) &= 0 \\ t^r(r+1)(r^2-2r+2) &= 0 \\ r = -1 \qquad r &= 1 \pm i \end{aligned}$$

So now we get the 3 solutions:

$$y_1 = t^{-1+1} = 1 \qquad y_2 = t^2\cos(\ln t) \qquad y_3 = t^2\sin(\ln t)$$

We can see that these solutions are linearly independent by seeing that the Wronskian is nonzero:

$$\begin{aligned} W_{[y_1,y_2,y_3]}(t) &= \begin{vmatrix} 1 & t^2\cos(\ln(t)) & t^2\sin(\ln(t)) \\ 0 & 2\,t\cos(\ln(t)) - t\sin(\ln(t)) & 2\,t\sin(\ln(t)) + t\cos(\ln(t)) \\ 0 & -3\,\sin(\ln(t)) + \cos(\ln(t)) & \sin(\ln(t)) + 3\,\cos(\ln(t)) \end{vmatrix} \\ &= 5\,(\sin(\ln(t)))^2\,t + 5\,(\cos(\ln(t)))^2\,t \\ &= 5t \qquad t > 0 \end{aligned}$$

So we can see that the Wronskian is nonzero for $t > 0$ thus the solutions are linearly independent.

Problem 2: If $y_1(x)$ is known for the linear ODE:

$$y'' + p(x)y' + q(x)y = 0$$

Then one attempts a solution of the form $y(x) = v(x)y_1(x)$. Provided $y_1(x) \neq 0$, show that

$$\frac{dv}{dx} = \frac{1}{[y_1(x)]^2} e^{-\int^x p(s)ds}$$

Solve for $v(x)$ to obtain the 2^{nd} linearly independent solution, $y_2(x)$.

Let $y_1(x)$ be a known solution to the original equation such that $y_1'' + p(x)y_1' + q(x)y_1 = 0$. Notice the following:

$$\begin{aligned}
y(x) &= v(x)y_1(x) \\
y'(x) &= v'(x)y_1(x) + v(x)y_1'(x) \\
y''(x) &= 2v'(x)y_1'(x) + v''(x)y_1(x) + v(x)y_1''(x)
\end{aligned}$$

We can also see that the second solution $y(x)$ will also satisfy the original equation.

$$\begin{aligned}
2v'(x)y_1'(x) + v''(x)y_1(x) + v(x)y_1''(x) + p(x)v'(x)y_1(x) + p(x)v(x)y_1'(x) + q(x)v(x)y_1(x) &= 0 \\
y_1(x)v''(x) + [p(x)y_1(x) + 2y_1'(x)]\, v'(x) + [y_1''(x) + p(x)y_1'(x) + q(x)y_1(x)]\, v(x) &= 0
\end{aligned}$$

Notice the last term equals zero from earlier observations. Now if we let $w(x) = v'(x)$, we get:

$$y_1(x)w'(x) + (p(x)y_1(x) + 2y_1'(x))w(x) = 0$$

Using the method of linear separation, we get:

$$\begin{aligned}
\frac{dw}{w(x)} &= \frac{-p(x)y_1(x) - 2y_1'(x)}{y_1(x)} dx \\
\ln(w(x)) &= \int -p(x)dx - 2\int \frac{y_1'(x)}{y_1(x)} dx \\
w(x) &= e^{-\int p(x)dx} e^{-2\ln(y_1(x))} \\
w(x) &= \frac{1}{[y_1(x)]^2} e^{-\int p(x)dx}
\end{aligned}$$

Thus we get the results:

$$\frac{dv}{dx} = \frac{1}{[y_1(x)]^2} e^{-\int^x p(s)ds} \qquad\qquad v(x) = \int \frac{1}{[y_1(x)]^2} e^{-\int^x p(s)ds}$$

With the second solution being:

$$y_2(x) = y_1(x) \int \frac{1}{[y_1(x)]^2} e^{-\int^x p(s)ds}$$

Problem 3: Consider the following ODE:

$$xy'' + (1-2x)y' + (x-1)y = 0 \tag{1}$$

(a) Show that $y_1(x) = e^x$ is a solution to this differential equation.

(b) In Part a, $y_1(x) = e^x$ was found as one solution to (1). Use the **Reduction of Order** method to find $y_2(x)$ for (1). Use the Wronskian to show this is a fundamental set of solutions.

(a) Notice that when evaluating $y_1 = e^x$ into the original equation, we get:

$$xe^x + e^x - 2xe^x + xe^x - e^x = 0$$

Thus $y_1(x) = e^x$ is a solution.

(b) Using the Reduction of Order, we get

$$y_2(x) = e^x \int \frac{e^{\int \left(\frac{-1}{x}+2\right)dx}}{e^{2x}} dx = e^x \int \frac{x^{-1}e^{2x}}{e^{2x}} dx = e^x \ln(x)$$

We can show that these solutions make a fundamental set of solutions by showing that the Wronskian of the two are nonzero.

$$W_{[y_1,y_2]} = \begin{vmatrix} e^x & e^x \ln x \\ e^x & e^x \ln x + \frac{e^x}{x} \end{vmatrix} = \frac{e^{2x}}{x}$$

We can see that $W_{[y_1,y_2]} \neq 0$ for all x, thus making y_1, y_2 a fundamental set of solutions.