Quiz 7
Differential Equations
Math 337
Stephen Giang

Problem 1: Consider the linear nonhomogeneous ODE given by:

$$y'' + y' - 2y = 8t^2$$

Show how to solve this problem using both the *Method of Undetermined Coefficients* and the *Variation of Parameters*. Which technique do you consider easier to perform and why?

We can get the 2 solutions of this equation by finding the eigenvalues and solving the homogeneous equation.

$$\lambda^2 + \lambda - 2 = 0$$
$$(\lambda + 2)(\lambda - 1) = 0$$

So we get $\lambda = -2, 1$ with the solutions being $y_1(t) = e^{-2t}$ and $y_2(t) = e^t$

(a) *Method of Undetermined Coefficients*

Let the following be true:

$$y_p = At^2 + Bt + C \qquad\qquad y_p' = 2At + B \qquad\qquad y_p'' = 2A$$

Now we can evaluate the following and solve for A, B, C:

$$y_p'' + y_p' - 2y_p = -2At^2 + (2A - 2B)t + (2A + B - 2C) = 8t^2$$

Thus we get $A = -4$, $B = -4$, $C = -6$, so our particular solution is $y_p = -4t^2 - 4t - 6$

(b) *Variation of Parameters*

Notice:

$$W_{[e^{-2t},e^t]}(t) = \begin{vmatrix} e^{-2t} & e^t \\ -2e^{-2t} & e^t \end{vmatrix} = 3e^{-t}$$

To get the particular solution:

$$\begin{aligned} y_p &= -e^{-2t}\int^t \frac{8e^s s^2}{3e^{-s}}ds + e^t\int^t \frac{8e^{-2s}s^2}{3e^{-s}}ds \\ &= -\frac{8}{3}e^{-2t}\int^t e^{2s}s^2 ds + \frac{8}{3}e^t\int^t e^{-s}s^2 ds \end{aligned}$$

Using Integration by Parts, we get:

$$\int^t e^{2s}s^2 ds = \frac{t^2e^{2t}}{2} - \int se^{2s}ds = \frac{t^2e^{2t}}{2} - \frac{te^{2t}}{2} + \frac{e^{2t}}{4} = \frac{e^{2t}}{4}(2t^2 - 2t + 1)$$

$$\int^t e^{-s}s^2 ds = -t^2e^{-t} + 2\int se^{-s}ds = -t^2e^{-t} - 2te^{-t} - 2e^{-t} = -e^{-t}(t^2 + 2t + 2)$$

Now we get:

$$\begin{aligned} y_p &= -\frac{4}{3}t^2 + \frac{4}{3}t - \frac{2}{3} - \frac{8}{3}t^2 - \frac{16}{3}t - \frac{16}{3} \\ &= -4t^2 - 4t - 6 \end{aligned}$$

Thus our complete solution to this problem is: $y(t) = c_1e^{-2t} + c_2e^t - 4t^2 - 4t - 6$

I consider the Undetermined Coefficients method easier in this case. With the solutions and the Wronskian being exponential made us do integration by parts four times when using Variation of Parameters. The Undetermined Coefficients method was simple to find our A, B, C as our $g(t)$ was simply a quadratic.

Problem 2 (a): Consider the linear homogeneous ODE given by:

$$ty'' - y' + 4t^3 y = 0$$

Show that $y_1(t) = \cos(t^2)$ and $y_2(t) = \sin(t^2)$ are solutions to this ODE. Find the Wronskian of these solutions, $W_{[y_1,y_2]}(t)$ and use this to prove that these solutions form a *fundamental set of solutions* to this ODE.

Let the following be true:

$$\begin{aligned} y_1(t) &= \cos(t^2) & y_1'(t) &= -2t\sin(t^2) & y_1''(t) &= -2\sin(t^2) - 4t^2\cos(t^2) \\ y_2(t) &= \sin(t^2) & y_2'(t) &= 2t\cos(t^2) & y_2''(t) &= 2\cos(t^2) - 4t^2\sin(t^2) \end{aligned}$$

Now if we evaluate our original equation with each, we will get zero:

$$\begin{aligned} -2t\sin(t^2) - 4t^3\cos(t^2) + 2t\sin(t^2) + 4t^3\cos(t^2) &= 0 \\ 2t\cos(t^2) - 4t^3\sin(t^2) - 2t\cos(t^2) + 4t^3\sin(t^2) &= 0 \end{aligned}$$

Thus $y_1(t)$ and $y_2(t)$ are solutions

Now we can prove that these form a *fundamental set of solutions* to this ODE by proving their Wronskian is nonzero.

$$W_{[y_1,y_2]} = \begin{vmatrix} \cos(t^2) & \sin(t^2) \\ -2t\sin(t^2) & 2t\cos(t^2) \end{vmatrix} = 2t(\cos^2(t^2) + \sin^2(t^2)) = 2t$$

We can rewrite the original equation to find the interval I

$$y'' - t^{-1}y' + 4t^2 y = 0$$

Thus we can see that $-t^{-1}$ and $4t^2$ are both continuous on the interval where $t \neq 0$

Because the Wronskian is nonzero for $t \neq 0$, the solutions form a *fundamental set of solutions*

Problem 2 (b): Consider the linear nonhomogeneous ODE given by:

$$ty'' - y' + 4t^3 y = 8t^3$$

Use the *Variation of Parameters* method to solve this problem.

From part (a). we can see that the 2 solutions are: $y_1(t) = \cos(t^2)$ and $y_2(t) = \sin(t^2)$

We can also see from part (a), that the Wronskian $W_{[y_1,y_2]} = 2t$.

We can also rewrite the original equation as:

$$y'' - \frac{1}{t}y' + 4t^2 y = 8t^2$$

Now using the *Variation of Parameters*, we can find the particular solution:

$$\begin{aligned}
y_p &= -\cos(t^2)\int^t \frac{\sin(s^2)8s^2}{2s}\,ds + \sin(t^2)\int^t \frac{\cos(s^2)8s^2}{2s}\,ds \\
&= -4\cos(t^2)\int^t \sin(s^2)s\,ds + 4\sin(t^2)\int^t \cos(s^2)s\,ds \\
&= -4\cos(t^2)\frac{-\cos(t^2)}{2} + 4\sin(t^2)\frac{\sin(t^2)}{2} \\
&= 2\left(\cos^2(t^2) + \sin^2(t^2)\right) \\
&= 2
\end{aligned}$$

Thus we get the complete solution:

$$y(t) = c_1\cos(t^2) + c_2\sin(t^2) + 2$$

Problem 3: Consider the following ODE:

$$y'' + 16y = 32\csc^2(4t)$$

Find the solution to this problem.

First, we find the homogeneous solution:

$$\begin{aligned} \lambda^2 + 16 &= 0 \\ \lambda &= \pm 4i \end{aligned}$$

So our solutions are $y_1(t) = \cos(4t)$ and $y_2(t) = \sin(4t)$

Now we can use the *Variation of Parameters* to find our particular solution.

Notice:

$$W_{[y_1,y_2]}(t) = \begin{vmatrix} \cos(4t) & \sin(4t) \\ -4\sin(4t) & 4\cos(4t) \end{vmatrix} = 4$$

Now *Variation of Parameters* tells us our particular solution is as follows:

$$\begin{aligned} y_p &= -\cos(4t)\int^t \frac{\sin(4s)32\csc^2(4s)}{4}\,ds + \sin(4t)\int^t \frac{\cos(4s)32\csc^2(4s)}{4}\,ds \\ &= -8\cos(4t)\int^t \csc(4s)\,ds + 8\sin(4t)\int^t \cot(4s)\csc(4s)\,ds \\ &= -8\cos(4t)\left(-\frac{1}{4}\ln|\csc(4t)+\cot(4t)|\right) + 8\sin(4t)\left(-\frac{1}{4}\csc(4t)\right) \\ &= 2\cos(4t)\ln|\csc(4t)+\cot(4t)| - 2 \end{aligned}$$

Thus we get the complete solution:

$$y(t) = c_1\cos(4t) + c_2\sin(4t) + 2\cos(4t)\ln|\csc(4t)+\cot(4t)| - 2$$