Homework 5
Numerical Matrix Analysis
Math 543
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Section 12 Problem 3: The goal of this problem is to explore some properties of random matrices. Your job is to be a laboratory scientist, performing experiments that lead to conjectures and more refined experiments. Do not try to prove anything. Do produce well-designed plots, which are worth a thousand numbers. Define a random matrix to be an mxm matrix whose entries are independent samples from the real normal distribution with mean zero and standard deviation $m^{-1/2}$. (In MATLAB, A = randn(m,m)/sqrt (m).) The factor $\sqrt{m}$ is introduced to make the limiting behavior clean as m $\to \infty$.

(a) What do the eigenvalues of a random matrix look like? What happens, say, if you take 100 random matrices and superimpose all their eigenvalues in a single plot? If you do this for m = 8, 16, 32, 64, ... , what pattern is suggested? How does the spectral radius $\rho(A)$ (Exercise 3.2) behave as m $\to \infty$?

(b) What about norms? How does the 2-norm of a random matrix behave as m $\to \infty$? Of course, we must have $\rho(A) < ||A||$ (Exercise 3.2). Does this inequality appear to approach an equality as m $\to \infty$?

(c) What about condition numbers—or more simply, the smallest singular value σ_{min} ? Even for fixed m this question is interesting. What proportions of random matrices in $\mathbb{R}^{m\times m}$ seem to have $\sigma_{min} < 2^{-1}, 4^{-1}, 8^{-1}$, ...? In other words, what does the tail of the probability distribution of smallest singular values look like? How does the scale of all this change with m ?

(a) The eigenvalues of random matrices take the shape of circles. As m $\to \infty$, the spectral radius, $\rho(A)$ approaches 1.

(b) As m $\to \infty$, the 2-norm of the random matrices approach 2. The inequality $\rho(A) < ||A||$, remains true as m $\to \infty$. The inequality does not approach an equality as m approaches infinity.

(c) As m $\to \infty$, the proportion between $\delta_{min} < 2^{-1}, 4^{-1}, 8^{-1}$ and the number of random matrices approach 100 %. The proportion between $\delta_{min} < 2^{-1}$ and the number of iterations approaches 100% the fastest as m increases.