Homework 4, Math 330
Due on Tuesday, November 5

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ where $f(5) = \frac{1}{2}$ and that f is continuous at 5. Show that $\exists \epsilon > 0$ such that $\forall x \in (5 - \epsilon, 5 + \epsilon)$ we have $f(x) > 0$.

2. Suppose that $f(x) = \begin{cases} 1 - x, & \text{if } x \in \mathbb{Q} \\ 1 + x, & \text{if } x \notin \mathbb{Q} \end{cases}$

   (a) Prove that f is continuous at 0.

   (b) Prove that f is not continuous at 1.

   (c) Is f continuous at any irrational number? Why (brief description is ok here)?

3. Suppose that $S \subseteq \mathbb{R}$ is non-empty and bounded. Prove that if S is not sequentially compact, then there exists a sequence in S that converges to a point outside S.

4. Suppose that $S \subseteq \mathbb{R}$ is non-empty and bounded. Suppose there exists a sequence in S converging to a point x_0 not in S. Show that $f : S \to \mathbb{R}$ by $f(x) = \dfrac{1}{x - x_0}$ is continuous and unbounded.

5. Consider the function $f : (1, \infty) \to \mathbb{R}$ by $f(x) = \dfrac{x+2}{x-1}$.

   (a) Use the sequence definition of continuity and the limit laws of section 2.1 to prove f is continuous at $x = 2$.

   (b) Use the $\epsilon - \delta$ criterion to prove that f is continuous at $x = 2$.

6. Prove that there is a solution to the equation $x^5 + x + 4 = 0$ in $\mathbb{R}$.

7. Prove that the graph of the function $f(x) = \dfrac{2-3x}{x-1}$ intersects the x-axis on the interval [0,2].