SecondDE
Differential Equations
Math 337
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Problem 9: Find the Solution for

$$y'' + 7y' + 10y = 36e^t \qquad\qquad y(0) = 4, y'(0) = 1$$

To find the eigenvalues, we can write the equation like below and solve for λ.

$$\lambda^2 + 7\lambda + 10 = (\lambda + 5)(\lambda + 2) = 0$$
$$\lambda = -5, -2$$

Now we can write the homogeneous solution as:

$$y_h = c_1 e^{-5t} + c_2 e^{-2t}$$

To solve for the particular solution, we can write y_p as below

$$y_p' = Ae^t \qquad y_p' = Ae^t \qquad y_p'' = Ae^t$$

By plugging in the particular solution:

$$y_p'' + 7y_p' + 10y_p = Ae^t + 7Ae^t + 10Ae^t$$
$$18Ae^t = 36e^t$$
$$A = 2$$

So thus the particular solution is:

$$y_p = 2e^t$$

We can now have the complete solution, $y(t)$ to this differential equation:

$$y(t) = c_1 e^{-5t} + c_2 e^{-2t} + 2e^t$$

We now use $y'(t)$ and the initial conditions to solve for c_1 and c_2

$$y'(t) = -5c_1 e^{-5t} + -2c_2 e^{-2t} + 2e^t$$

$$y(0) = c_1 + c_2 + 2 = 4$$
$$y'(0) = -5c_1 + -2c_2 + 2 = 1$$

$$\text{rref}\begin{pmatrix} 1 & 1 & 2 \\ -5 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix}$$

Thus the solution is:

$$y(t) = -e^{-5t} + 3e^{-2t} + 2e^t$$

Problem 14: Write a complete solution with details on how you found the general solution

$$y'' + 16y = -16\sin(4t)$$

To solve for the particular solution, we can write y_p as below

$$\begin{aligned}
y_p &= At\cos(4\,t) + Bt\sin(4\,t)\\
y_p' &= A\cos(4\,t) - 4\,At\sin(4\,t) + B\sin(4\,t) + 4\,Bt\cos(4\,t)\\
y_p'' &= -8\,A\sin(4\,t) - 16\,At\cos(4\,t) + 8\,B\cos(4\,t) - 16\,Bt\sin(4\,t)
\end{aligned}$$

By plugging in the particular solution:

$$y_p'' + 16yp = -8\,A\sin(4\,t) + 8\,B\cos(4\,t) = -16\sin(4t)$$

$$A = 2, B = 0$$

Thus the particular solution is:

$$y_p = 2t\cos(4\,t)$$

To find the eigenvalues, we can write the equation like below and solve for λ.

$$\begin{aligned}
\lambda^2 + 16 = 0\lambda^2 &\qquad = -16\\
\lambda &= \pm 4i
\end{aligned}$$

So we can now right this in its homogeneous solution form using $\lambda = 0 \pm 4i$

$$y_h = c_1\cos(4t) + c_2\sin(4t)$$

Thus the complete solution is

$$y(t) = c_1\cos(4t) + c_2\sin(4t) + 2t\cos(4\,t)$$

Problem 15: Write a complete solution with details on how you found the general solution

$$y'' - 10y' + 25y = -10.5e^{5t}$$

To solve for the particular solution, we can write y_p as below

$$\begin{aligned} y_p &= At^2e^{5\,t} \\ y_p' &= 2\,Ate^{5\,t} + 5\,At^2e^{5\,t} \\ y_p'' &= 2\,Ae^{5\,t} + 20\,Ate^{5\,t} + 25\,At^2e^{5\,t} \end{aligned}$$

By plugging in the particular solution:

$$y_p'' - 10y_p' + 25yp = 2\,Ae^{5\,t} = -10.5e^{5t}$$

$$A = \frac{-2}{10.5}$$

Thus the particular solution is:

$$y_p = \frac{-2}{10.5}t^2e^{5\,t}$$

To find the eigenvalues, we can write the equation like below and solve for λ.

$$\begin{aligned} \lambda^2 - 10\lambda + 25 &= 0 \\ (\lambda - 5)^2 &= 0 \\ \lambda &= 5 \end{aligned}$$

So we can now right this in its homogeneous solution form using $\lambda = 5$

$$y_h = c_1e^{5t} + c_2te^{5t}$$

Thus the complete solution is

$$y(t) = c_1e^{5t} + c_2te^{5t} + \frac{-2}{10.5}t^2e^{5\,t}$$

Problem 16: Write a complete solution with details on how you found the general solution

$$y'' - 7y' + 12y = -288t^3$$

To solve for the particular solution, we can write y_p as below

$$\begin{aligned} y_p &= At^3 + Bt^2 + Ct + D \\ y_p' &= 3\,At^2 + 2\,Bt + C \\ y_p'' &= 6\,At + 2\,B \end{aligned}$$

By plugging in the particular solution:

$$\begin{aligned} y_p'' - 7y_p' + 12y_p &= (12A)t^3 + (-21A + 12B)t^2 + (6A - 14B + 12C)t + (2B - 7C + 12D) \\ &= -288t^3 \end{aligned}$$

$$\text{rref}\begin{pmatrix} 12 & 0 & 0 & 0 & -288 \\ -21 & 12 & 0 & 0 & 0 \\ 6 & -14 & 12 & 0 & 0 \\ 0 & 2 & -7 & 12 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & -24 \\ 0 & 1 & 0 & 0 & -42 \\ 0 & 0 & 1 & 0 & -37 \\ 0 & 0 & 0 & 1 & -14.5833 \end{pmatrix}$$

Thus the particular solution is:

$$y_p = -24t^3 + -42t^2 + -37t + -14.5833$$

To find the eigenvalues, we can write the equation like below and solve for λ.

$$\begin{aligned} \lambda^2 - 7\lambda + 12 &= 0 \\ (\lambda - 3)(\lambda - 4) &= 0 \\ \lambda &= 3, 4 \end{aligned}$$

So we can now right this in its homogeneous solution form using $\lambda = 3, 4$

$$y_h = c_1 e^{3t} + c_2 e^{4t}$$

Thus the complete solution is

$$y(t) = c_1 e^{3t} + c_2 e^{4t} + -24t^3 + -42t^2 + -37t + -14.5833$$

Problem 21: Write a complete solution with details on how you found the general solution

$$x^2y'' + 13xy' + 36y = x^6 \qquad y(1) = -8, y'(1) = -2$$

Let the following be true:

$$y = x^r \qquad y' = rx^{r-1} \qquad y'' = (r^2 - r)x^{r-2}$$

So then by plugging in the following into the homogeneous equation:

$$\begin{aligned} x^2y'' + 13xy' + 36y &= x^r(r^2 - r + 13r + 36) = 0 \\ &= x^r(r^2 + 12r + 36) = 0 \\ &= x^r(r+6)^2 = 0 \\ r &= -6 \end{aligned}$$

So now we have the homogeneous solution:

$$y_h = (c_1 + c_2 \ln|x|)x^{-6}$$

To solve for the particular solution, we can write y_p as below

$$\begin{aligned} y_p &= Ax^6 \\ y_p' &= 6\,Ax^5 \\ y_p'' &= 30\,Ax^4 \end{aligned}$$

By plugging in the particular solution:

$$x^2y_p'' + 13xy_p' + 36y_p = 144\,Ax^6 = x^6$$

$$A = \frac{1}{144}$$

Thus the particular solution is:

$$y_p = \frac{1}{144}x^6$$

We can now have the complete solution, $y(t)$ to this differential equation:

$$y(t) = (c_1 + c_2 \ln|x|)x^{-6} + \frac{1}{144}x^6$$

We now use $y'(t)$ and the initial conditions to solve for c_1 and c_2

$$y'(t) = -6c_1x^{-7} + -6c_2 \ln|x|x^{-7} + c_2x^{-7} + \frac{6}{144}x^5$$

$$y(1) = c_1 + \frac{1}{144} = -8 \qquad c_1 = -8 + -\frac{1}{144} = \frac{-1153}{144}$$

$$y'(1) = -6c_1 + c_2 + \frac{6}{144} = -2 \qquad c_2 = -2 + -\frac{6}{144} + \frac{6(-1153)}{144} = \frac{-601}{12}$$

Thus the solution is:

$$y(t) = \left(\frac{-1153}{144} + \frac{-601}{12}\ln|x|\right)x^{-6} + \frac{1}{144}x^6$$

Problem 22: Write a complete solution with details on how you found the general solution

$$y'' - 4y' + 4y = \frac{16.5e^{2t}}{t^2+1}$$

To find the eigenvalues, we can write the equation like below and solve for λ.

$$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$$
$$\lambda = 2$$

Now we can write the homogeneous solution as:

$$y_h = c_1e^{2t} + c_2te^{2t}$$

We can now generalize it to:

$$y(t) = u_1(t)e^{2t} + u_2(t)te^{2t}$$

We now set the following to be true"

$$u_1'(t)e^{2t} + u_2'(t)te^{2t} = 0$$

Now we take the first and second derivative of $y(t)$:

$$\begin{aligned}
y'(t) &= u_1'(t)e^{2t} + 2u_1(t)e^{2t} + u_2'(t)te^{2t} + u_2(t)e^{2t} + 2u_2(t)te^{2t} \\
&= 2u_1(t)e^{2t} + u_2(t)e^{2t} + 2u_2(t)te^{2t} \\
y''(t) &= 2u_1'(t)e^{2t} + 4u_1(t)e^{2t} + u_2'(t)e^{2t} + 4u_2(t)e^{2t} + 2u_2'(t)te^{2t} + 4u_2(t)te^{2t} \\
&= 4u_1(t)e^{2t} + u_2'(t)e^{2t} + 4u_2(t)e^{2t} + 4u_2(t)te^{2t}
\end{aligned}$$

By plugging into the differential equation now:

$$\begin{aligned}
y'' - 4y' + 4y = {} & 4u_1(t)e^{2t} + u_2'(t)e^{2t} + 4u_2(t)e^{2t} + 4u_2(t)te^{2t} \\
& - 8u_1(t)e^{2t} + -4u_2(t)e^{2t} + -8u_2(t)te^{2t} \\
& + 4u_1(t)e^{2t} + 4u_2(t)te^{2t} \\
= {} & u_2'(t)e^{2t}
\end{aligned}$$

So now we solve for the $u_1(t)$ and $u_2(t)$

$$\begin{aligned}
u_2'(t)e^{2t} &= \frac{16.5e^{2t}}{t^2+1} \\
u_2'(t) &= \frac{16.5}{t^2+1} \\
u_2(t) &= 16.5\arctan(t) + c_2
\end{aligned}$$

$$\begin{aligned}
u_1'(t)e^{2t} + u_2'(t)te^{2t} &= 0 \\
u_1'(t)e^{2t} &= -u_2'(t)te^{2t} \\
&= \frac{-16.5te^{2t}}{t^2+1} \\
u_1'(t) &= \frac{-16.5t}{t^2+1} \\
u_1(t) &= \frac{-16.5}{2}\ln(t^2+1) + c_1
\end{aligned}$$

Now we take our generalized solution and plug everything in:

$$\begin{aligned}
y(t) &= \frac{-16.5}{2}\ln(t^2+1)e^{2t} + c_1e^{2t} + 16.5\arctan(t)te^{2t} + c_2te^{2t} \\
&= c_1e^{2t} + c_2te^{2t} + \frac{-16.5}{2}\ln(t^2+1)e^{2t} + 16.5\arctan(t)te^{2t}
\end{aligned}$$