Midterm Part B
Ordinary Differential Equations
Math 537
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Problem 1 (9:00am - 9:05am), (9:37am - 9:50am): Consider the following two systems, including a single first order ODE:

$$\frac{dz}{dt} = f(z) = \alpha z^2 + \beta z \tag{1.1}$$

and a system of two first-order ODEs:

$$\begin{aligned}\frac{dx}{dt} &= ax, \\ \frac{dy}{dt} &= by.\end{aligned} \tag{1.2}$$

(a) The trivial critical point $z = 0$ in Eq. (1.1) may be a source, a sink, or a saddle point. Select a pair of (α, β) that produces each of the three types of critical points. [Hint: Either α or β can be zero for simplicity.]

Let $(\alpha, \beta) = (0, 1)$. Notice that we get:

$$\frac{dz}{dt} = z$$

From $z < 0$, we get $\frac{dz}{dt} < 0$, and from $z > 0$, we get $\frac{dz}{dt} > 0$. This means we get a **sink**.

Let $(\alpha, \beta) = (0, -1)$. Notice that we get:

$$\frac{dz}{dt} = -z$$

From $z < 0$, we get $\frac{dz}{dt} > 0$, and from $z > 0$, we get $\frac{dz}{dt} < 0$. This means we get a **source**.

Let $(\alpha, \beta) = (1, 0)$. Notice that we get:

$$\frac{dz}{dt} = z^2$$

From $z < 0$, we get $\frac{dz}{dt} > 0$, and from $z > 0$, we get $\frac{dz}{dt} > 0$. This means we get a **saddle**.

(b) Within the planar system in Eq. (1.2), discuss the characteristics of the trivial critical point $(x, y) = (0, 0)$ in $a - b$ space.

Notice we get an upper triangular matrix such that $\lambda_1 = a$, $\lambda_2 = b$.

When ($a > 0$, $b > 0$), we get two positive eigenvalues leading to a **source**

When ($a < 0$, $b > 0$), we get two positive eigenvalues leading to a **saddle**

When ($a < 0$, $b < 0$), we get two positive eigenvalues leading to a **sink**

When ($a > 0$, $b < 0$), we get two positive eigenvalues leading to a **saddle**

Problem 3 (9:05am - 9:37am): Consider the following system:

$$X' = AX, \tag{3.1}$$

where

$$A = \begin{pmatrix} -0.1 & 1.1 \\ 1.1 & -0.1 \end{pmatrix} \text{ and } X = \begin{pmatrix} x \\ y \end{pmatrix}$$

(a) Solve for eigenvalue(s) and eigenvector(s).

Notice we can get the characteristic equation from $A - \lambda I$:

$$\begin{aligned} (\lambda + 0.1)(\lambda + 0.1) - 1.1(1.1) &= 0 \\ \lambda^2 + 0.2\lambda + .01 - 1.21 &= 0 \\ \lambda^2 + 0.2\lambda - 1.2 &= 0 \end{aligned}$$

Notice the following eigenvalues:

$$\begin{aligned} \lambda_1 &= \frac{-0.2 + \sqrt{.04 - 4(-1.2)}}{2} = \frac{-0.2 + 2.2}{2} = 1 \\ \lambda_2 &= \frac{-0.2 - \sqrt{.04 - 4(-1.2)}}{2} = \frac{-0.2 - 2.2}{2} = -1.2 \end{aligned}$$

Notice the eigenvectors found from $A - \lambda I$

$$\begin{pmatrix} -0.1 - \lambda_1 & 1.1 \\ 1.1 & -0.1 - \lambda_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1.1 & 1.1 \\ 1.1 & -1.1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad v_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -0.1 - \lambda_2 & 1.1 \\ 1.1 & -0.1 - \lambda_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1.1 & 1.1 \\ 1.1 & 1.1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(b) Construct T using the results from problem (3a) and calculate $T^{-1}AT$

Notice that the eigenvalues were real and different. So we can construct T from the eigenvectors, such that:

$$T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Notice $T^{-1}AT$:

$$\begin{aligned} T^{-1}AT &= \begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -0.1 & 1.1 \\ 1.1 & -0.1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1.2 \end{pmatrix} \end{aligned}$$

(c) Let $X = TY$. Show

$$Y' = (T^{-1}AT)Y, \tag{3.2}$$

Here Y is a column vector and its transpose is defined as $Y^T = (u, w)$.

Notice the following:

$$\begin{pmatrix} u' \\ w' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1.2 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} u \\ -1.2w \end{pmatrix}$$

Because we have that $u' = \lambda_1 u$ and $w' = \lambda_2 w$, we have shown the above statement to be true.

(d) Solve Eq. (3.2) for Y.

We can see the eigenvalues because $T^{-1}AT$ is an upper triangular matrix. So we get that $\lambda_1 = 1$ and $\lambda_2 = -1.2$. We can also easily see the eigenvectors being $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

So we get

$$Y = Ae^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + Be^{-1.2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(e) Find the solution X to Eq. (3.1).

$$X = TY = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} Ae^t & 0 \\ 0 & Be^{-1.2t} \end{pmatrix} = \begin{pmatrix} Ae^t & Be^{-1.2t} \\ Ae^t & -Be^{-1.2t} \end{pmatrix}$$

Problem 6 (9:50am - 10am):

(a) Notice the characteristic equation:

$$\begin{aligned}(\lambda-2)(\lambda-1)+1/4&=0\\ \lambda^2-3\lambda+9/4&=0\end{aligned}$$

So we get the eigenvalues:

$$\lambda=\frac{3\pm\sqrt{9-4(9/4)}}{2}=\frac{3}{2}$$

So we can get the eigenvalues from $A-\lambda I$

$$v_1=\begin{pmatrix}1\\-1/2\end{pmatrix},v_2=\begin{pmatrix}0\\1\end{pmatrix}$$

.

(b)

$$T=\begin{pmatrix}1&0\\-1/2&1\end{pmatrix}, \qquad T^{-1}AT=\begin{pmatrix}3/2&1\\0&3/2\end{pmatrix}$$