MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #16: Part II

Chapter 5

Higher-Dimensional Linear Algebra

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Section 5.2

A Summary For Section 5.2: Eigenvalues

- Eigenvalue Problems

$$\boxed{AV = \lambda V \quad \Rightarrow \quad (A - \lambda I)V = 0 \quad \Rightarrow \quad |A - \lambda I| = 0}$$

- Linearly Independent eigenvectors associated with real and distinct eigenvalues

$\boxed{det|B| \neq 0}$ $\qquad B$ consists of linearly independent vectors

- Diagonalization

$T = (V_1, V_2, \cdots, V_n),\qquad V_j$ are eigenvectors

$$T^{-1}AT = D$$

5.2 Eigenvalue Problem and LI Eigenvectors

Definition
A vector V is an *eigenvector* of an $n \times n$ matrix A if V is a nonzero solution to the system of linear equations $(A - \lambda I)V = 0$. The quantity λ is called an *eigenvalue* of A, and V is an eigenvector associated to λ.

$$\boxed{AV = \lambda V \quad \Rightarrow \quad (A - \lambda I)V = 0 \quad \Rightarrow \quad |A - \lambda I| = 0}$$

Proposition. *Suppose $\lambda_1, \ldots, \lambda_\ell$ are real and distinct eigenvalues for A with associated eigenvectors $V_1, \ldots, V_\ell$. Then the V_j are linearly independent.* ■

$\boxed{V_j \text{ are Linearly Independent}}$

Sect 5.2 Diagonalization

Corollary. *Suppose A is an $n \times n$ matrix with real, distinct eigenvalues. Then there is a matrix T such that*

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \boxed{= D}$$

where all of the entries off the diagonal are 0.

For example,

$T = (V_1, V_2, \cdots, V_n)$, V_j are eigenvectors

Sect 5.2 Eigenvalues and Eigenvectors

Example. Let

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix}.$$

$$AV = \lambda V \qquad A - \lambda I = \begin{pmatrix} 1-\lambda & 2 & -1 \\ 0 & 3-\lambda & -2 \\ 0 & 2 & -2-\lambda \end{pmatrix}$$

$$|A - \lambda I = 0| \qquad (1-\lambda)(\lambda^2 - \lambda - 2) = 0 \qquad \lambda = 2, 1, -1$$

$$\lambda_1 = 2 \qquad \lambda_2 = 1 \qquad \lambda_3 = -1$$

$$V_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad V_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Construct T and D

Let $D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ λ_j are eigenvalues of the matrix A

Let $T = (V_1, \quad V_2, \quad V_3),$ V_j are the eigenvectors corresponding to λ_j

$\lambda_1 = 2$ $\lambda_2 = 1$ $\lambda_3 = -1$

$V_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ $V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $V_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $T = (V_1, \quad V_2, \quad V_3) = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix}$

Diagnoalization using $T^{-1}AT$

Compute $AT = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

- Pick one and compute
- During the difficult time, the following is added to make our life easier:
  - you will receive additional 10 points for your next homework if your selection matches the one I preselected (to appear later) and your answer is correct.
- Send your results via ”chat” (e.g., a21=???)
- You have 2 minutes

- You are a winner if you select a_{33} and compute it correctly.

$$AT = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 1 & 0 \\ 4 & 0 & -1 \\ 2 & 0 & -2 \end{pmatrix}$$

Diagnoalization using $T^{-1}AT$

$$AT = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 6 & 1 & 0 \\ 4 & 0 & -1 \\ 2 & 0 & -2 \end{pmatrix}$$

What we have done is called "parallel computing":

- Compute each of a_{ij} in parallel
- Load imbalance may appear, e.g., $a_{12} = 1 + 2 * 0 * +(-1) * 0$
- The last element determines the timing for the entire task.

In the supercomputing world, we may additionally perform the following:

- Decompose data (domain or tasks) into smaller pieces of data;
- Assign sub-tasks (e.g., broadcast data) to different CPUs;
- Gather all of sub-tasks and put them together

Supp

P1

```
for i in range (0, N/2):
    a[i]=c*b[i]
```

P2

```
for i in range (N/2, N):
    a[i]=c*b[i]
```

```
N=1000
N2=N/2
a=np.zeros(N)
b=np.linspace(0, N-1, N)
c=5

for i in range (0, N):
    a[i]=c*b[i]
```

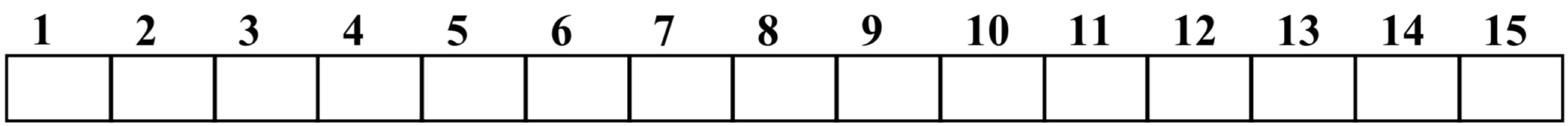
```
x1 = np.linspace(0, 10, N, endpoint=True)
x2 = np.linspace(0, 10, N, endpoint=False)
```

```
xsum = 0.0
 do i=1,15
   xsum =  xsum + x(i)
enddo
```

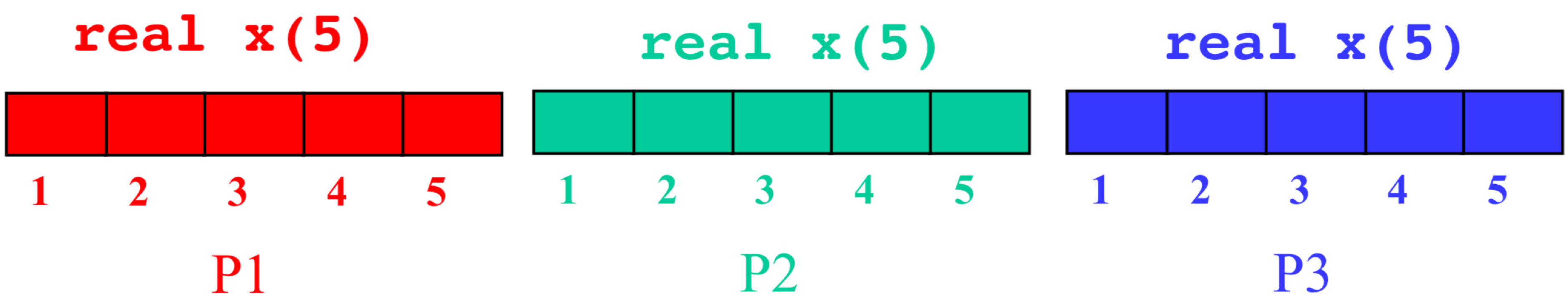
```
real x(15)
```

"Global"	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
"Local"	1	2	3	4	5	1	2	3	4	5	1	2	3	4	5
	P1					P2					P3				

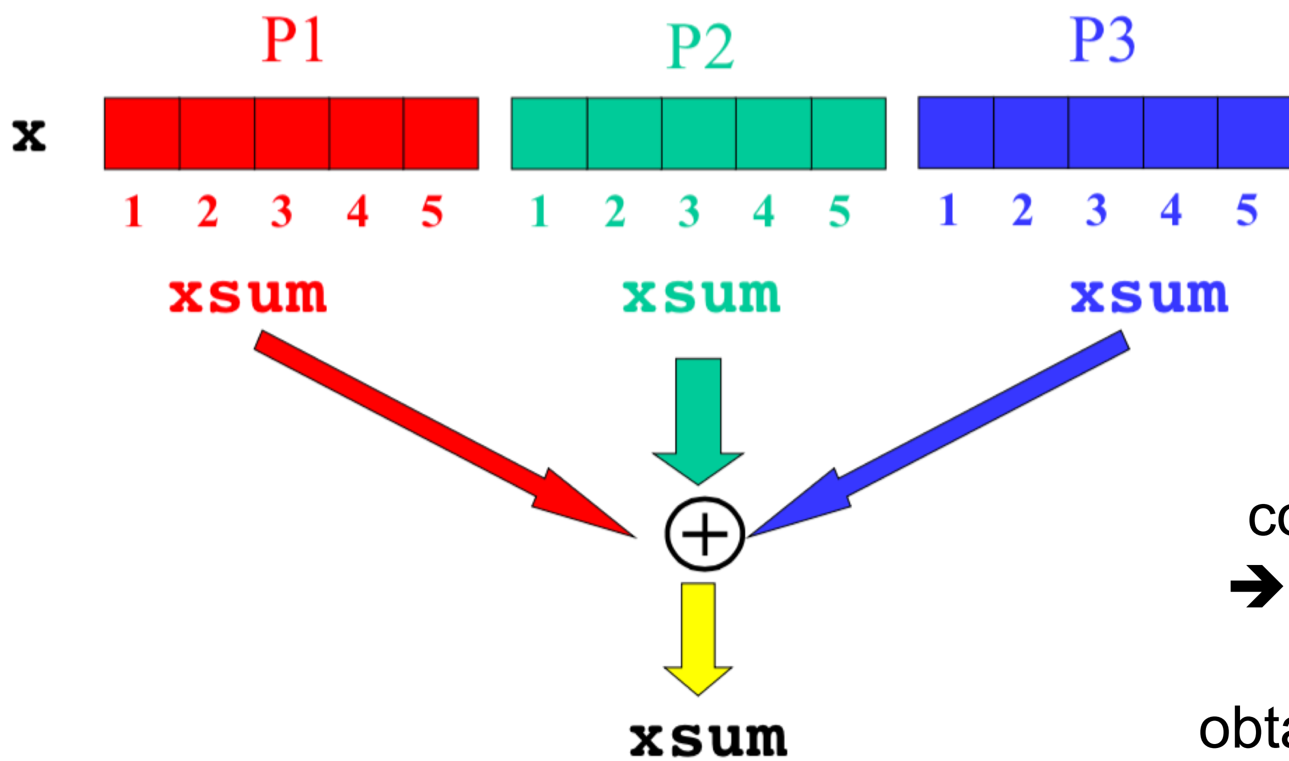
`real x(15)`



Parallel Code



```
xsum = 0.0
 do i=1,15
   xsum = xsum + x(i)
enddo
```

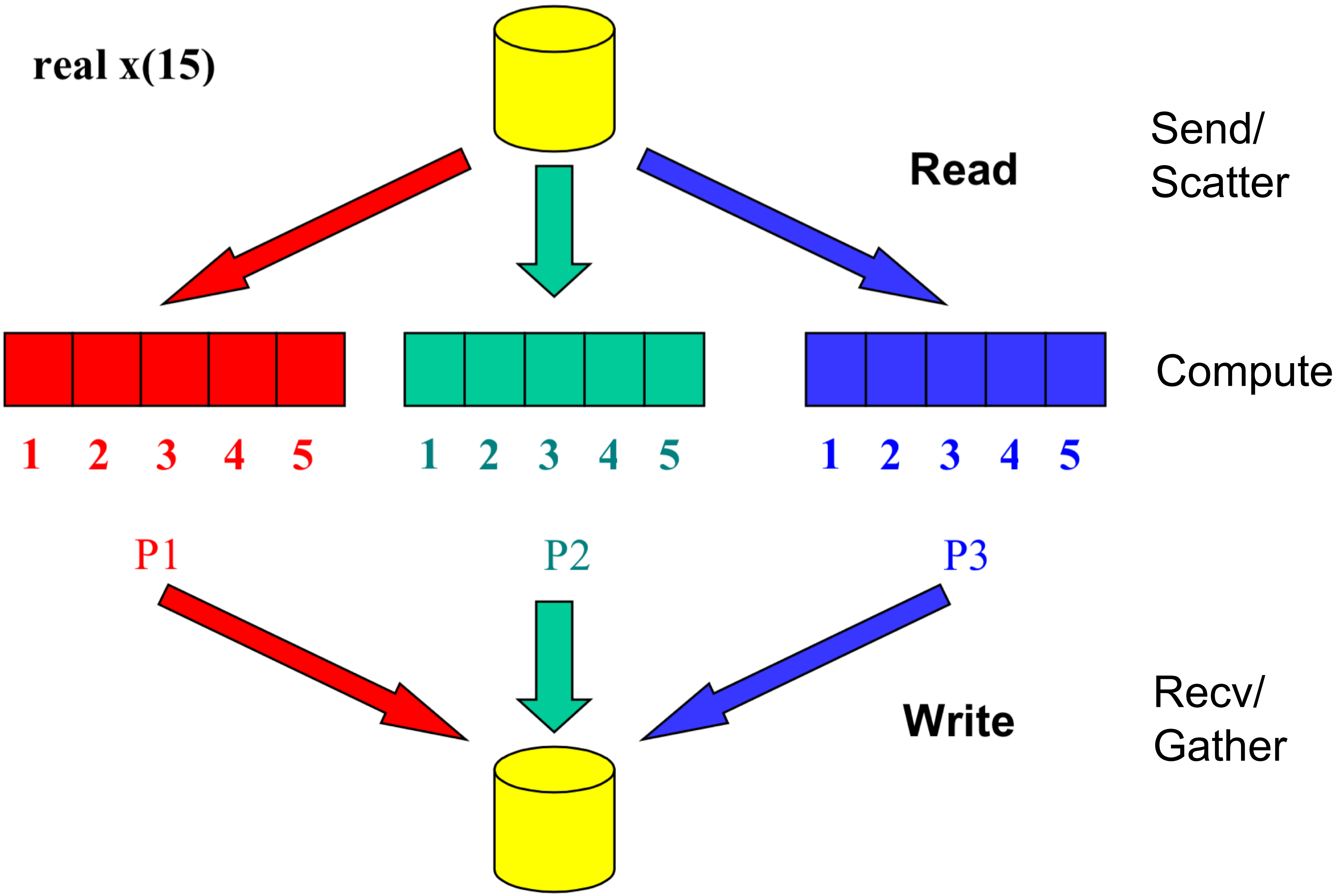


domain decomposition

compute local sums

collect local sums
➔ synchronization

obtain the global sum


real x(15)
Read
Send/
Scatter
1 2 3 4 5
1 2 3 4 5
1 2 3 4 5
Compute
P1
P2
P3
Write
Recv/
Gather

SPMD: Single Program Multiple Data

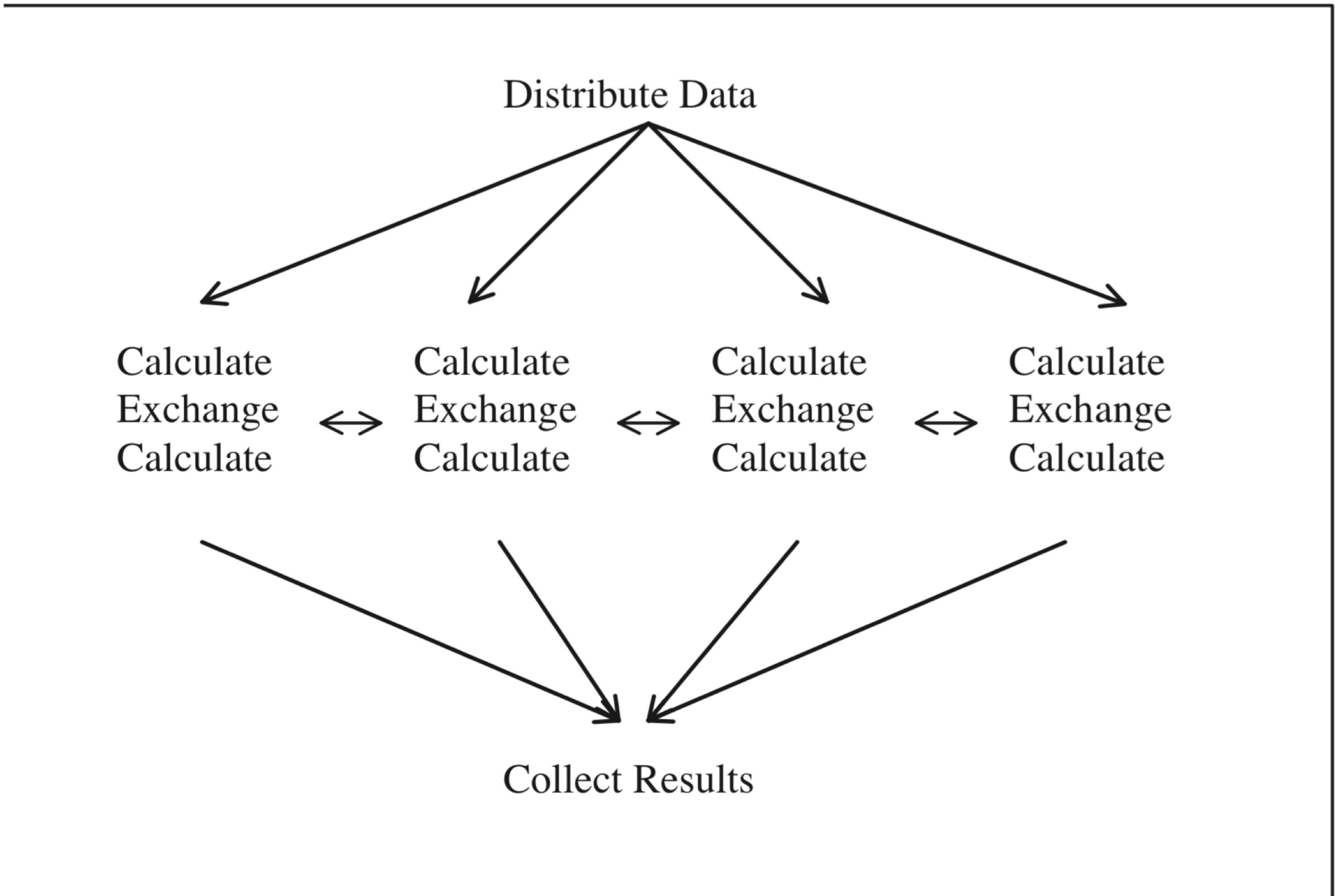


Figure 1.5 Basic structure of a SPMD program.

A Minimal Set for MPI

A minimal set of routines that most parallel codes run with are:

- MPI_INIT:

  Initialization. MPI spawns an identical copy of my_proc when mpirun -np N my_proc is issued.

- MPI_COMM_SIZE:

  returns the size or number (i.e., N) of processes in the application.

- MPI_COMM_RANK:

  returns the rank ("id", 0~N-1) of the calling process.

- MPI_SEND
- MPI_RECV
- MPI_WAIT
- MPI_FINALIZE:

  terminates all MPI processing

A Minimal Set for MPI in Python

A minimal set of routines that most parallel codes run with are:

	Fortran	MPI4PY	
	MPI_INIT	----------	comm = MPI.COMM_WORLD
	MPI_COMM_SIZE	comm.Get_size()	
	MPI_COMM_RANK	comm.Get_rank()	
	MPI_SEND	comm.send(…)	comm.Send(…)
	MPI_RECV	comm.recv(…)	comm.Recv(…)
	MPI_WAIT	obj.wait()	
	MPI_FINALIZE	-----------	

Diagnoalization using $T^{-1}AT$

$$AT = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{pmatrix}\begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 6 & 1 & 0 \\ 4 & 0 & -1 \\ 2 & 0 & -2 \end{pmatrix}$$

$$TD = \begin{pmatrix} 3 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 1 & 0 \\ 4 & 0 & -1 \\ 2 & 0 & -2 \end{pmatrix}$$

$$TD = AT$$

$$T^{-1}TD = T^{-1}AT$$

$$ID = T^{-1}AT$$

$$D = T^{-1}AT$$

Section 5.3 Complex Eigenvalues

if $AV = (\alpha + i\beta)V$, we have $A\bar{V} = (\alpha - i\beta)\bar{V}$

- Let V be the eigenvector associated with the complex eigenvalue $\alpha + i\beta$
- Show that $\bar{V}$ is an eigenvector associated with the complex eigenvalue $\alpha - i\beta$
- V and $\bar{V}$ are linearly independent, i.e., $\mathrm{c}V + \mathrm{d}\bar{V} = 0 \iff c = d = 0$
    - c & d are complex numbers.

➢ Note that V and $\bar{V}$ yield the same indepdent real functions, because $\mathrm{Re}(V) = Re(\bar{V})$ and $\mathrm{Im}(V) = -Im(\bar{V})$.

➢ From the first bullet, we have $\mathrm{A}V = (\alpha + i\beta\,)V$.

➢ To prove the statement in the 2nd bullet, we consider $A\bar{V}$:

$$A\bar{V} = \overline{AV} = \overline{(\alpha + i\beta\,)V} = (\alpha - i\beta)\bar{V}$$

$\boldsymbol{Re(V)}$ and $\boldsymbol{Im(V)}$ Are Linearly Independent

$V = u + i\,w$ Show that u and w are LI

$\boxed{\text{Assume } w = cu,\ c \in R}$ $\lambda = \alpha + i\beta$

$$AV = A(u + iw) = A(u + icu) \quad (1)$$

$$\lambda V = (\alpha + i\beta)(u + icu) = (\alpha u - c\beta u) + i(\beta u + c\alpha u) \quad (2)$$

Eq. (1) = (2):

real part: $Au = \alpha u - c\beta u$ (3)

imaginary part: $Acu = \beta u + c\alpha u$ (4)

(3)*c = (4),

$-c^2\beta u = \beta u$ $\boxed{c^2 = -1, \quad c = \pm i}$ contradiction

Construct a Linear Map, T, using Real and Imaginary Parts

Assume A to be a $(2n\times 2n)$ matrix that has the following eigenvalues and eigenvectors:

- $\alpha_j \pm \beta_j$ and $V_j, \bar{V}_j$, $j = 1,2$;
- $AV_j = (\alpha_j + i\beta)V_j$ and $A\bar{V}_j = (\alpha_j - i\beta)\bar{V}_j$

Define the following

$$\boxed{W_{2j-1} = \frac{1}{2}(V_j + \bar{V}_j) = Re(V_j)} \qquad \boxed{W_{2j} = \frac{-i}{2}(V_j - \bar{V}_j) = Im(V_j)}$$

Show that (TBD in the next slide)

$$AW_{2j-1} = \alpha_j W_{2j-1} - \beta_j W_{2j} \qquad AW_{2j} = \beta_j W_{2j-1} + \alpha_j W_{2j}$$

Construct T as follows:

$$T = [W_1, W_2, \cdots W_{2j-1}, W_{2j} \cdots W_{2n-1}, W_{2n}]$$

Obtain $Y' = BY, X = TY$, and B is defined as follows:

$$B = T^{-1}AT$$

Find AW_{2j-1} and AW_{2j}

Show that

$$AW_{2j-1} = \alpha_j W_{2j-1} - \beta_j W_{2j} \qquad AW_{2j} = \beta_j W_{2j-1} + \alpha_j W_{2j}$$

$$AW_{2j-1} = \frac{1}{2}\left(AV_j + A\bar{V}_j\right) = \frac{1}{2}\left(\left(\alpha_j + i\beta\right)V_j + \left(\alpha_j - i\beta\right)\bar{V}_j\right)$$

$$= \frac{1}{2}\left(\alpha_j\left(V_j + \bar{V}_j\right) + i\beta_j\left(V_j - \bar{V}_j\right)\right) = \alpha_j W_{2j-1} - \beta_j W_{2j}$$

$$AW_{2j} = \frac{-i}{2}\left(AV_j - A\bar{V}_j\right) = \frac{-i}{2}\left(\left(\alpha_j + i\beta\right)V_j - \left(\alpha_j - i\beta\right)\bar{V}_j\right)$$

$$= \frac{-i}{2}\left(\alpha_j\left(V_j - \bar{V}_j\right) + i\beta_j\left(V_j + \bar{V}_j\right)\right) = \beta_j W_{2j-1} + \alpha_j W_{2j}$$

$W_1, W_2, \cdots W_{2j-1}, W_{2j} \cdots W_{2n-1}, W_{2n}$ are LI

$$W_{2j-1} = \frac{1}{2}(V_j + \bar{V}_j) = Re(V_j)$$

$$W_{2j} = \frac{-i}{2}(V_j - \bar{V}_j) = Im(V_j)$$

Proposition. *The vectors $W_1, \ldots, W_{2n}$ are linearly independent.*

Form a linear combination:

$$\sum_{j=1}^{n}(c_j W_{2j-1} + d_j W_{2j}) = 0$$

They are LD for some non-zero $c_j, d_j \in R$

Plug the Eqs. on the top into the above Eq.:

$$\frac{1}{2}\sum_{j=1}^{n}(c_j - id_j)V_j + (c_j + id_j)\bar{V}_j = 0$$

Since the V_j and the $\bar{V}_j$ are LI, we have

$$(c_j - id_j) = 0 = (c_j + id_j) \quad \Rightarrow c_j = d_j = 0 \quad \text{contradiction}$$

Construct a Linear Map, T

$$\boxed{AW_{2j-1} = \alpha_j W_{2j-1} - \beta_j W_{2j}} \qquad AW_{2j} = \beta_j W_{2j-1} + \alpha_j W_{2j}$$

Construct $\quad T = [W_1, W_2, \cdots W_{2j-1}, W_{2j} \cdots W_{2n-1}, W_{2n}]$

Obtain $\quad TE_j = W_j, \quad j = 1 \sim 2n, \qquad T^{-1}W_j = E_j, \quad j = 1 \sim 2n,$

$$for\ 2j-1, \quad (T^{-1}AT)E_{2j-1} = T^{-1}ATE_{2j-1} = T^{-1}\boxed{AW_{2j-1}}$$

$$= T^{-1}(\alpha_j W_{2j-1} - \beta_j W_{2j})$$

$$= (\alpha_j T^{-1} W_{2j-1} - \beta_j T^{-1} W_{2j})$$

$$= (\alpha_j E_{2j-1} - \beta_j E_{2j})$$

$$for\ 2j, \quad (T^{-1}AT)E_{2j} = (\beta_j E_{2j-1} + \alpha_j E_{2j})$$

Construct a Linear Map, T

$$(T^{-1}AT)E_{2j-1} = \left(\alpha_j E_{2j-1} - \beta_j E_{2j}\right)$$

$$(T^{-1}AT)E_{2j} = \left(\beta_j E_{2j-1} + \alpha_j E_{2j}\right)$$

j=1, we have

$$(T^{-1}AT)E_1 = (\alpha_1 E_1 - \beta_1 E_2) = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} - \beta_1 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ -\beta_1 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

$$(T^{-1}AT)E_2 = (\beta_1 E_1 + \alpha_1 E_2) = \beta_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

Construct a Linear Map, T

$$(T^{-1}AT)E_1 = (\alpha_1 E_1 - \beta_1 E_2) = \alpha_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} - \beta_1 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ -\beta_1 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

$$(T^{-1}AT)E_2 = (\beta_1 E_1 + \alpha_1 E_2) = \beta_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \alpha_1 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

$$(T^{-1}AT)[E_1, E_2, \cdots E_{2n-1}, E_{2n}] = \begin{pmatrix} \alpha_1 & \beta_1 & & \\ -\beta_1 & \alpha_1 & & \\ & & & \\ & & & \end{pmatrix}$$

Construct a Linear Map, T

$$(T^{-1}AT)\boxed{[E_1, E_2, \cdots E_{2n-1}, E_{2n}]} = \begin{pmatrix} \alpha_1 & \beta_1 & & \\ -\beta_1 & \alpha_1 & & \\ & & & \\ & & & \end{pmatrix}$$

Identity matrix I

$$T^{-1}AT = \begin{array}{c} \begin{pmatrix} \alpha_1 & \beta_1 & & & & \\ -\beta_1 & \alpha_1 & & & & \\ & & & & & \\ & & & \alpha_j & \beta_j & \\ & & & -\beta_j & \alpha_j & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array} \begin{array}{l} \\ \\ \\ \leftarrow 2j-1 \\ \leftarrow 2j \\ \\ \\ \end{array}$$

$$D_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{pmatrix} \qquad D_j = \begin{pmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{pmatrix}$$

A Summary with Complex Eigenvalues

Assume A to be a $(2n\times 2n)$ matrix that has the following eigenvalues and eigenvectors:

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Define the following

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Construct T as follows:

$$T = [W_1, W_2, \cdots W_{2j-1}, W_{2j} \cdots W_{2n-1}, W_{2n}]$$

Obtain $Y' = BY, X = TY$, and B is defined as follows:

$$B = T^{-1}AT = \begin{pmatrix} D_1 & & \\ & \ddots & \\ & & D_n \end{pmatrix} \qquad D_n = \begin{pmatrix} \alpha_n & \beta_n \\ -\beta_n & \alpha_n \end{pmatrix}$$