HW 4 SOLUTIONS

① Since f is continuous at $x=5$ and $f(5)=\frac{1}{2}$, we know

$$\boxed{\forall \varepsilon>0,\ \exists \delta>0 \text{ s.t. } \forall x\in\mathbb{R},\ \text{if } |x-5|<\delta, \text{ then } |f(x)-\tfrac{1}{2}|<\varepsilon.}$$

Let $\varepsilon=\frac{1}{4}$. Then $\exists\, \delta>0$ s.t.

$\forall x\in\mathbb{R}$ if $|x-5|<\delta$, then $|f(x)-\frac{1}{2}|<\frac{1}{4}$

Let $x\in(5-\delta, 5+\delta)$.

Then $|f(x)-\frac{1}{2}|<\frac{1}{4}$

So $-\frac{1}{4}<f(x)-\frac{1}{2}<\frac{1}{4}$

So $\frac{1}{4}<f(x)$. ☐

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② (a) Suppose $\{x_n\}\subseteq\mathbb{R}$ and $\lim_{n\to\infty} x_n=0$.

Consider $|f(x_n)-f(0)|$.

Case 1: Suppose $x_n\in\mathbb{Q}$.

Then $|f(x_n)-f(0)|=|1-x_n-1|=|x_n|$.

Case 2: Suppose $x_n\notin\mathbb{Q}$.

Then $|f(x_n)-f(0)|=|1+x_n-1|=|x_n|$.

Thus $\lim_{n\to\infty}|f(x_n)-f(0)|=\lim_{n\to\infty}|x_n|=0$.

I.e. $\lim_{n\to\infty}f(x_n)=f(0)$. ☐

(b) Recall irrational numbers are dense in $\mathbb{R}$

Choose $\{x_n\} \subseteq \mathbb{R} \setminus \mathbb{Q}$ s.t. $\lim_{n \to \infty} x_n = 1$.

Then $\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} (1 + x_n) = 2 \neq 0 = f(1)$.

So $\exists \{x_n\} \subseteq \mathbb{R}$ s.t. $\lim_{n \to \infty} x_n = 1$ and $\lim_{n \to \infty} f(x_n) \neq f(1)$.

(c) No. Let $x_0 \in \mathbb{R} \setminus \mathbb{Q}$. By density of $\mathbb{Q}$ in $\mathbb{R}$, choose $\{x_n\} \subseteq \mathbb{Q}$ s.t. $\lim_{n \to \infty} x_n = x_0$.

Then $\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} (1 - x_n) = 1 - x_0 \neq 1 + x_0$

when $x_0 \neq 0$.

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③ Suppose S is not sequentially compact.

Then $\exists \{x_n\} \subseteq S$ s.t. $\forall \{x_{n_k}\}$,

$$\lim_{k \to \infty} x_{n_k} \notin S.$$

Since S is bounded, $\{x_n\}$ is bounded.

Thus $\exists \{x_{n_k}\}$ a convergent subsequence.

So $\exists x_0 \in \mathbb{R}$ s.t. $\lim_{k \to \infty} x_{n_k} = x_0 \notin S$.

④ We know from Cor. 3.5, f is continuous on S since $x_0 \notin S$. Since $\exists \{x_n\} \subseteq S$ s.t.

$\lim_{n\to\infty} x_n = x_0$, consider $f(x_n)$.

Let $N \in \mathbb{N}^+$. Then $\exists M \in \mathbb{N}$ s.t. $\forall m \geq M$,

$|x_m - x_0| < \frac{1}{N}$.

Thus letting $m \geq M$, $|f(x_m)| = \left|\frac{1}{x_m - x_0}\right| > N$.

Thus $\{f(x_n)\}$ is an unbounded sequence.

⑤ (a) Let $\{x_n\} \subseteq (1,\infty)$ and suppose

$\lim_{n\to\infty} x_n = 2$.

Then $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{x_n + 2}{x_n - 1} = \frac{4}{1} = f(2)$. $\square$

(b) Scratch: $\left|\frac{x+2}{x-1} - 4\right| < \varepsilon$

$\left|\frac{x+2-4x+4}{x-1}\right| < \varepsilon$

$1.9 < x < 2.1$ $\left|\frac{6-3x}{x-1}\right| < \varepsilon$

$0.9 < x-1 < 1.1$

$\left|\frac{2-x}{x-1}\right| < \frac{|2-x|}{0.9} < \varepsilon/3$

⑤ (b) Let $\varepsilon > 0$.

Let $\delta = \min\left\{0.1, \frac{0.9\varepsilon}{3}\right\}$.

Let $x \in (1, \infty)$ and suppose $|2 - x| < \delta$.

Then $|x - 2| < 0.1$ and $1.9 < x < 2.1$.

So $0.9 < x - 1 < 1.1$. ☆

Also $|2 - x| < \frac{0.9\varepsilon}{3}$

So $\left|\frac{6 - 3x}{0.9}\right| < \varepsilon$. From ☆ we have

$$\left|\frac{x+2}{x-1} - 4\right| = \left|\frac{6-3x}{x-1}\right| < \left|\frac{6-3x}{0.9}\right| < \varepsilon. \quad \boxed{\varepsilon}.$$

I.e. $|f(x) - f(2)| < \varepsilon$.

(6) Note that $f(x) = x^5 + x + 4$ for

~~$f: [0, 2]$~~

$f : [-2, 0] \to \mathbb{R}$ is

continuous. Also $f(-2) = -30$

and $f(0) = 4$.

Since $-30 < 0 < 4$, the I.V.T.

says $\exists\, c \in (-2, 0)$ s.t. $f(c) = 0$. ted

(7) Note $f\left(\frac{2}{3}\right) = \dfrac{2 - 3\left(\frac{2}{3}\right)}{\frac{2}{3} - 1} = \dfrac{0}{-1/3} = 0.$

Since $\frac{2}{3} \in [0, 2]$, the graph of f

intersects the x-axis on the interval $[0, 2]$.