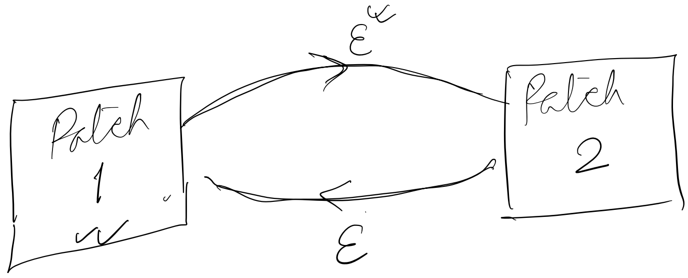
Oct 07, 2024

Goal: To study the impact of spatial dispersal on the dynamics of the whole environment.

Special case:



$0 < \varepsilon << 1$

Model:

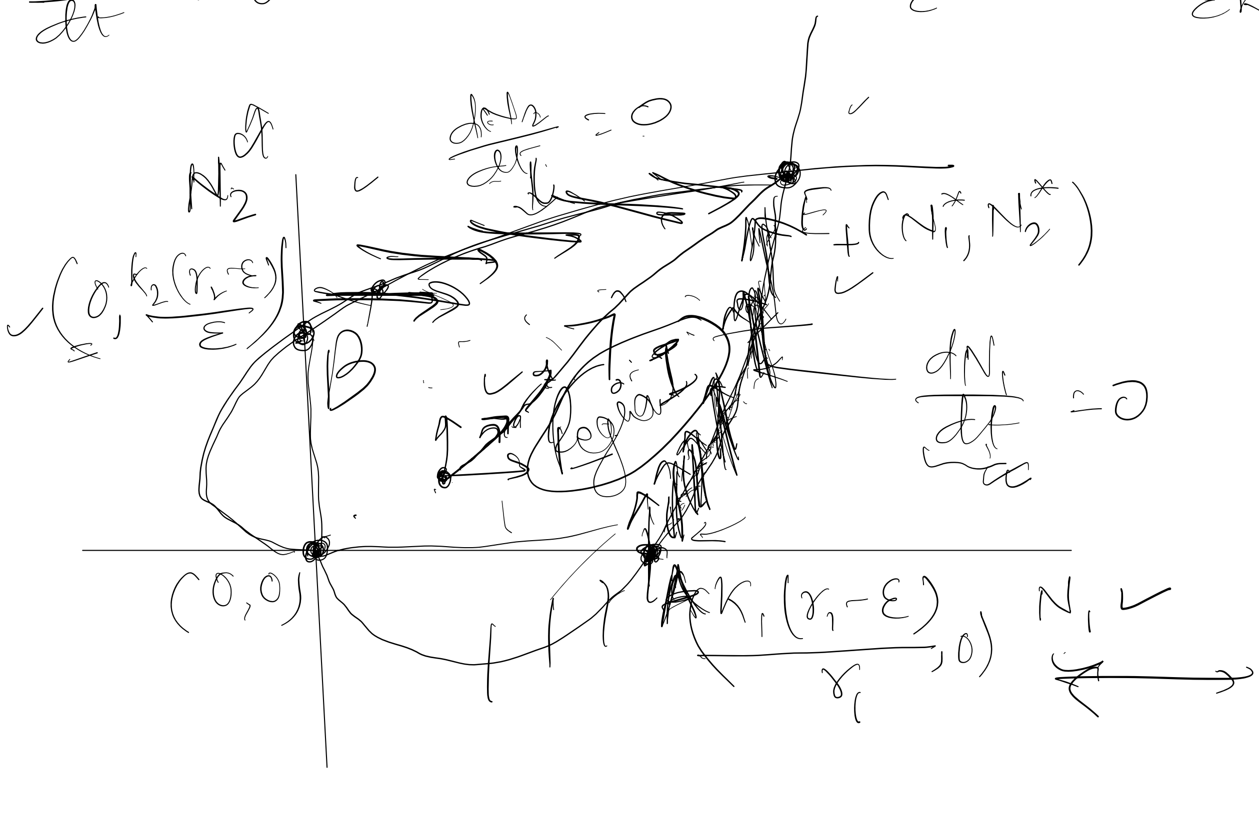
$$\frac{dN_1}{dt} = r_1 N_1\left(1 - \frac{N_1}{K_1}\right) + \varepsilon(N_2 - N_1) \quad \text{--- (I)}$$

$$\frac{dN_2}{dt} = r_2 N_2\left(1 - \frac{N_2}{K_2}\right) + \varepsilon(N_1 - N_2) \quad \text{--- (II)}$$

Analysis:

$$\frac{dN_1}{dt} = 0 \Rightarrow N_2 = -\frac{(r_1 - \varepsilon)}{\varepsilon} N_1 + \frac{r_1}{\varepsilon K_1} N_1^2$$

$$\frac{dN_2}{dt} = 0 \Rightarrow N_1 = -\frac{(r_2 - \varepsilon)}{\varepsilon} N_2 + \frac{r_2}{\varepsilon K_2} N_2^2$$



Equilibria: intersection of the curve $\frac{dN_1}{dt}=0$ and $\frac{dN_2}{dt}=0$.

$\Rightarrow$ exactly two equilibria:

$$E_0=(0,0), \quad E_+=(N_1^*, N_2^*)$$

$\frac{dN_2}{dt}>0$ on the curve $\frac{dN_1}{dt}=0$ between A and E_+

$\checkmark$

[: put $N_2=0$ in (II), $\frac{dN_2}{dt}=\varepsilon N_1>0$

Similarly,

$\frac{dN_1}{dt}>0$ on the curve $\frac{dN_2}{dt}=0$ between B and E_+.

If $(N_1(0), N_2(0)) \in$ Region (I), where $\frac{dN_1}{dt}>0$, $\frac{dN_2}{dt}>0$, $(N_1(t), N_2(t))$ can not escape from Region (I).

$$\Rightarrow \quad N_1 \uparrow, \quad N_2 \uparrow$$

$$N_1 \leq N_1^*, \quad N_2 \leq N_2^*$$

$$\Rightarrow \lim_{t\to\infty} N_1(t) = N_1^*, \quad \lim_{t\to\infty} N_2(t) = N_2^*.$$

<u>Conclusion</u>:

I. All solutions (except the equilibrium E_0) converge to E_+ as $t \to \infty$.

II. E_0 is unstable and E_+ is asymptotically stable.

Question: what is E_+ and what is the impact of ε on E_+?

<u>One way</u>:

$$\left.\begin{aligned} &\checkmark\ r_1 N_1\left(1 - \frac{N_1}{K}\right) + \varepsilon(N_2 - N_1) = 0 \\ &\checkmark\ r_2 N_2\left(1 - \frac{N_2}{K}\right) + \frac{\varepsilon}{v}(N_1 - N_2) = 0 \end{aligned}\right\} \text{solve}$$

- N_2 = quadratic in N_1 from 1st eqn.
- substitute into the 2nd eqn.
- $\to$ polynomial of N_1 of order $4 = 0$
- $\to$ polynomial of N_1 of order $3 = 0$ and $N_1 = 0$
- $\Rightarrow$ solve for $N_1 \Longrightarrow N_2$.

<u>Alternate</u>: Asymptotic Analysis

Try to find:

$$N_1 = K_1 + \varepsilon x_1 + \cdots,$$

$$N_2 = K_2 + \varepsilon x_2 + \cdots$$

$$\begin{cases} r_1\left[K_1+\varepsilon x_1\right]\left[1-\dfrac{K_1+\varepsilon x_1}{K_1}\right]+\varepsilon\left[K_2-K_1+\varepsilon(x_1-x_2)\right]=0 \\ r_2\left[K_2+\varepsilon x_2\right]\left[1-\dfrac{K_2+\varepsilon x_2}{K_2}\right]+\varepsilon\left[K_1-K_2+\varepsilon(x_1-x_2)\right]=0 \end{cases}$$

$$\overset{I}{\Rightarrow} \quad -\frac{\varepsilon x_1}{K_1} r_1\left[K_1+\varepsilon x_1\right]+\varepsilon\left[K_2-K_1+\varepsilon(x_1-x_2)\right]=0$$

$$-\frac{x_1}{K_1} r_1\left[K_1+\cancel{\varepsilon x_1}\right]+K_2-K_1+\cancel{\varepsilon(x_1-x_2)}=0$$

$$\xrightarrow{\varepsilon<<1} \quad -r_1 x_1+K_2-K_1 \quad [\text{ignoring } \varepsilon]$$

$$\Rightarrow \quad x_1=\frac{K_2-K_1}{r_1}.$$

Similarly, $\quad x_2=\dfrac{K_1-K_2}{r_2}.$

$$\left.\begin{aligned} N_1 &\sim K_1+\varepsilon\frac{K_2-K_1}{r_1} \\ N_2 &\sim K_2+\varepsilon\frac{K_1-K_2}{r_2} \end{aligned}\right\} \xrightarrow{\text{if } K_1>K_2} \begin{aligned} &N_1\sim\ <K_1,\ N_1\downarrow \text{as } \varepsilon\uparrow \\ &N_2\sim\ >K_2,\ N_2\uparrow \text{as } \varepsilon\uparrow \end{aligned}$$

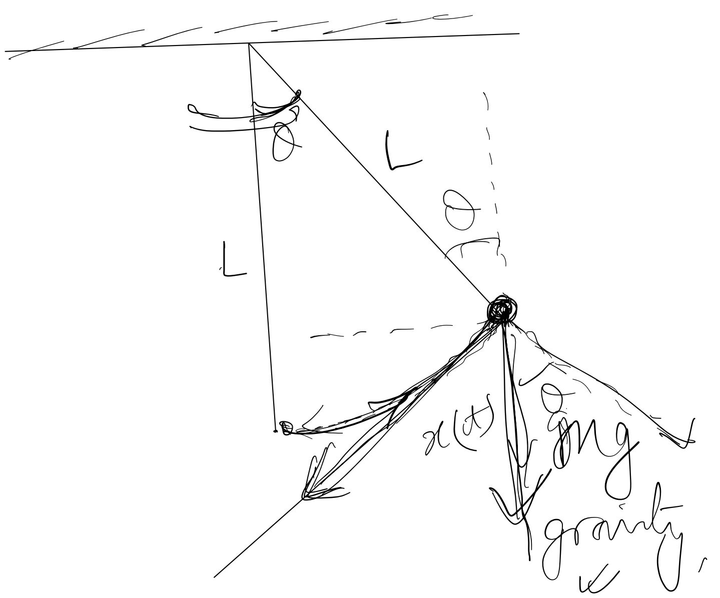
Higher-order ODEs: Simple Pendulum (Harmonic)

objective: - model motion of pendulum (periodic)
- identify period of the motion.

Assumption:
- friction is negligible
- swings in a perfect plane
- arm can not bend or stretch

Compress

- arm is massless
- gravity is constant ($g = 9.8\,m/s^2$)



m = mass

g = acceleration due to gravity

L = length

θ = angle between string position with vertical line (rest)

t = time

T = period of pendulum.

$x(t) = L\theta(t)$

$\Rightarrow$ acceleration, $\underline{a} = \dfrac{d^2x}{dt^2} = \underbrace{L\dfrac{d^2\theta}{dt^2}}$

Force, $F = -mg\sin\theta$

Newton's 2nd Law: $F = ma$

$\Rightarrow -mg\sin\theta = m \cdot L\dfrac{d^2\theta}{dt^2}$

$\Rightarrow \boxed{\dfrac{d^2\theta}{dt^2} = -\dfrac{g}{L}\sin\theta}$

$$\begin{cases} \dfrac{d^2\theta}{dt^2} + \dfrac{g}{L}\sin\theta = 0 \longrightarrow \left(2^{nd} \text{ order ODE}\right) \\ \theta(0) = \theta_0 \\ \theta'(0) = 0 \end{cases}$$

$$\equiv \begin{cases} x_1(t) = \theta(t) \\ x_2(t) = \dfrac{d\theta}{dt} \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{dx_1(t)}{dt} = x_2(t), \quad x_1(0) = \theta_0 \\ \dfrac{dx_2(t)}{dt} = -\dfrac{g}{L}\sin x_1(t), \quad x_2(0) = 0 \end{cases}$$

$$\xrightarrow{\text{Solve}} x_1(t) = \theta(t)$$

- small angle approximation:

$$\theta \simeq \sin(\theta), \text{ i.e., } x_1 \simeq \sin x_1$$

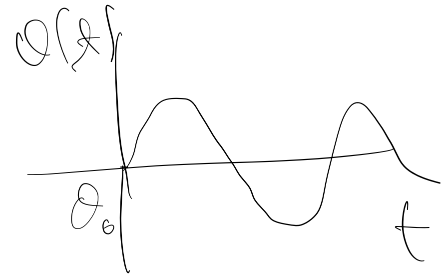
$$\Rightarrow \quad \frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

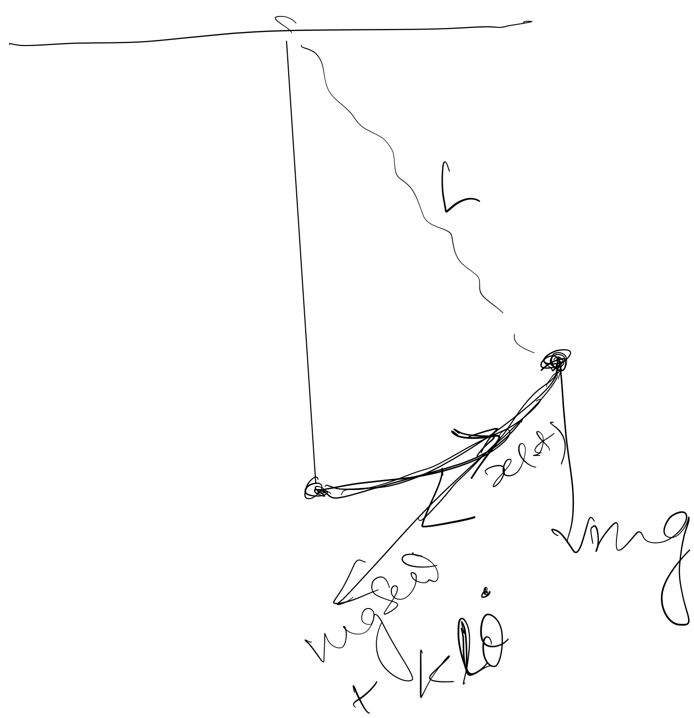
Solve (Lin Diff. Eq)

$$\theta(t) \simeq \theta_0 \cos\Big(\underbrace{\sqrt{\tfrac{g}{L}}}\, t\Big) \rightarrow \omega = \frac{2\pi}{T}$$

$$\Rightarrow T \simeq \underbrace{2\pi\sqrt{\frac{L}{g}}}$$



☐ Damped pendulum (oscillation)



$$\therefore \quad \frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta + \frac{k}{m}\frac{d\theta}{dt} = 0$$

$$\Rightarrow \begin{cases} \dfrac{dx_1}{dt} = x_2, \\[2ex] \dfrac{dx_2}{dt} = -\dfrac{g}{L}\sin x_1 - \dfrac{k}{m}x_2 \end{cases}$$