HW 3 Solutions

(1) Scratch:

$$\left| \frac{7n^2}{n^2-n} - 7 \right| < \varepsilon$$

$$\left| \frac{7n^2 - 7n^2 + 7n}{n^2-n} \right| < \varepsilon$$

$$\frac{7}{n-1} < \varepsilon$$

$$\frac{7}{\varepsilon} < n-1$$

$$\frac{7}{\varepsilon} + 1 < n.$$

$$\boxed{\lim_{n\to\infty} \frac{7n^2}{n^2-n} = 7}$$

proof: Let $\varepsilon > 0$.

Let $N \in \mathbb{N}$ where $N > \frac{7}{\varepsilon} + 1$.

Suppose $n > N > \frac{7}{\varepsilon} + 1$.

So $n-1 > \frac{7}{\varepsilon}$

So, $\varepsilon > \frac{7}{n-1}$, since $n-1 > 0$.

Thus $\varepsilon > \left| \frac{7n^2}{n^2-n} - 7 \right|$. ☒

proof:

(2) Let $c_n = a_n - c$ for each index n.

Note $\lim_{n\to\infty} c_n = \lim_{n\to\infty} (a_n - c) = a - c$ by linearity of limits.

Also $a - c \neq 0$ since $a \neq c$.

Thus by set boundedness lemma,

$\exists N \in \mathbb{N}$ and $\beta > 0$ s.t. $\forall n \geq N$

$$\beta < |c_n| = |a_n - c|.$$ ☒

③ proof: Since $\lim_{n\to\infty} a_n = a$, $\exists M \in \mathbb{R}^+$ s.t. $\forall n$

we have $|a_n| < M$ and $|a| < M$ by Boundedness Lemma.

Let $\varepsilon > 0$. $\exists N$ s.t. $\forall n \geq N$, $|a_n - a| < \frac{\varepsilon}{2M}$.

Let $n \geq N$. Then $|a_n - a| < \frac{\varepsilon}{2M}$

$$|a_n - a| \, 2M < \varepsilon.$$

So $|a_n - a|(|a_n| + |a|) < |a_n - a| 2M < \varepsilon$

So $|a_n^2 - a^2| = |(a_n - a)(a_n + a)| \leq |a_n - a|(|a_n| + |a|) < \varepsilon$. ☒

Scratch: $|a_n^2 - a^2| = |(a_n - a)(a_n + a)|$

$$|a_n - a||a_n + a| \leq |a_n - a|(|a_n| + |a|)$$

$$< |a_n - a| \, 2M.$$

$$< \varepsilon$$

(4) (a) Show $S = [20, \infty)$ is closed.

Let $\{a_n\} \subseteq S$ and suppose $\lim_{n \to \infty} a_n = a \in \mathbb{R}$.

By Boundedness Lemma, $\exists M \in \mathbb{R}^+$ s.t. th

$$a_n < M \text{ and } |a| < M.$$

Thus $\forall n$, $20 \le a_n \le M$ so $\{a_n\} \subseteq [20, M]$.

Since $[20, M]$ is closed, $\lim_{n \to \infty} a_n = a \in [20, M]$

Thus $a \in [20, \infty)$. Therefore S is closed.

(b) Let $T = \bigcup_{n=1}^{\infty} [\frac{1}{n}, 1]$. Let $n \geq 1$.

Note $\frac{1}{n} \in [\frac{1}{n}, 1]$ so $\frac{1}{n} \in T$. Thus $\{\frac{1}{n}\} \subseteq T$.

However $0 \notin [\frac{1}{m}, 1]$ for all $m \geq 1$.

Thus $0 \notin T$. Since $\lim_{n \to \infty} \frac{1}{n} = 0 \notin T$,

we have that T is not closed.

(5) Scratch:

$$\left|\sqrt{n+1}-\sqrt{n}\right| = \left|\frac{1}{\sqrt{n+1}+\sqrt{n}}\right| \leq \left|\frac{1}{2\sqrt{n}}\right| = \frac{1}{2\sqrt{n}} < \varepsilon.$$

$$\left(\frac{1}{2\varepsilon}\right)^2 < n.$$

proof: Let $\varepsilon > 0$.

Suppose $N \in \mathbb{N}$ and $N > \left(\frac{1}{2\varepsilon}\right)^2$.

Let $n > N. > \left(\frac{1}{2\varepsilon}\right)^2$.

Then $\sqrt{n} > \frac{1}{2\varepsilon}$.

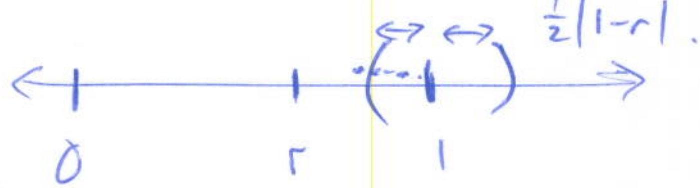
So $\varepsilon > \frac{1}{2\sqrt{n}} > \frac{1}{\sqrt{n}+\sqrt{n+1}} = \left|\sqrt{n+1}-\sqrt{n}\right|$. ∎

(6) (a) Scratch:

$$\frac{(n+1)r^{n+1}}{n\ r^n} < 1.$$

$$\frac{(n+1)r}{n} < 1.$$

$$r < \frac{n}{n+1} \to 1.$$



proof: Notice that $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1$.

Thus $\exists N$ s.t. $\forall n \geq N$,

$$\left|\frac{n}{n+1} - 1\right| < \frac{1}{2}(1-r).$$ Let $n \geq N$.

So $-\frac{1}{2}(1-r) < \frac{n}{n+1} - 1 < \frac{1}{2}(1-r)$.

So $1 - \frac{1}{2}(1-r) < \frac{n}{n+1}$.

So $\frac{1}{2} + \frac{1}{2}r < \frac{n}{n+1}$

And $\frac{1}{2}r + \frac{1}{2}r < \frac{1}{2} + \frac{1}{2}r < \frac{n}{n+1}$

Thus $r < \frac{n}{n+1}$

So $\frac{n+1}{n} \cdot r < 1$. Therefore $\frac{(n+1)r^{n+1}}{n \cdot r^n} < 1$. $\blacksquare$

Scratch

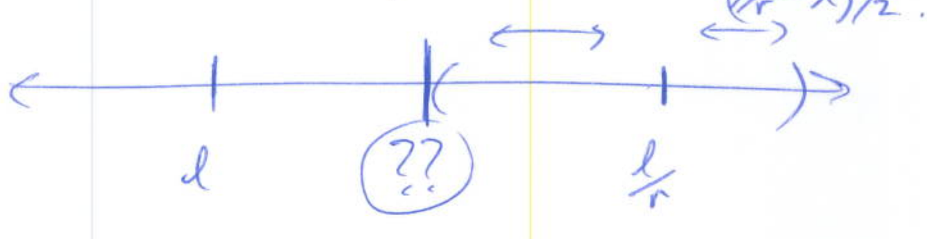
(6)

$l = \inf\{n r^n\}$.

$0 < l < (n+1) r^{n+1}$.

$\frac{l}{r} < n r^n + r^n$.

$?? < \frac{l}{r} - r^n < n r^n$



$\exists N_1$ s.t. $\forall n \geq N_1$, $r^n < \left(\frac{l}{r} - l\right)/2$.

So

$$\frac{l}{r} - r^n > \frac{l}{r} - \left(\frac{l}{r} - l\right)/2.$$

$$\frac{l}{r} - r^n > \frac{l}{2r} + \frac{l}{2} > \frac{l}{2} + \frac{l}{2} = l.$$

??

(6) ($\star\star$) proof: By ($\star$) $\exists N_2$ st $\forall n \geq N_2$ we have $\frac{(n+1)r^{n+1}}{n r^n} < 1$ and thus $(n+1)r^{n+1} < nr^n$.

Thus $\{nr^n\}_{n=N_2}^{\infty}$ is bounded below by 0 and decreasing.

By MCT, $\lim_{n\to\infty} nr^n = \inf\{nr^n\}_{n=N_2}^{\infty} = l$.

Since $\forall n \geq N_2$, $nr^n > 0$, $l \geq 0$.

Suppose $l > 0$. Since $\lim_{n\to\infty} r^n = 0$, $\exists N_1$ st. $\forall n > N_1$, $r^n < \left(\frac{l}{r} - l\right)/2$.

Thus $\forall n > N_1$, $\frac{l}{r} - r^n > \frac{l}{r} - \left(\frac{l}{r} - l\right)/2$.

And so $\forall n > N_1$, $\frac{l}{r} - r^n > \frac{l}{2r} + \frac{l}{2} > \frac{l}{2} + \frac{l}{2} = l$. ($\dagger$).

Since $l = \inf\{nr^n\}_{n=N_2}^{\infty}$, letting $n > \max\{N_1, N_2\}$,

we have $l < (n+1)r^{n+1}$

So $\frac{l}{r} < (n+1)r^n$

So $\frac{l}{r} - r^n < nr^n$.

But also by ($\dagger$),

$l < \frac{l}{2r} + \frac{l}{2} < nr^n$ when $n > \max\{N_1, N_2\}$.

Since $\{nr^n\}_{n=N_2}^{\infty}$ decreases, $\frac{l}{2r} + \frac{l}{2}$ is a lower bound and $\frac{l}{2r} + \frac{l}{2} > l$. $(\Rightarrow\Leftarrow)$. Thus $l = 0$.