MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #26

Approximate Solution of Linear Differential Equations

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Topics

7. Asymptotic Series and Local Analysis (Chapter 3 of Bender and Orszag)		2~2.5 weeks ~ 1-2 lectures
	7.1 Classification of Singular Points	
	7.2 Local Behavior Near Ordinary Points	Taylor Series
	7.3 Local Series Expansion About Regular Singular Points	Frobenius Method
	7.4 Local Behavior at Irregular Singular Points	Asymptotic series; asymptotic relations; leading behavior and controlling factor
	7.5 Local Analysis of Inhomogeneous Linear Equations	
	7.6 Asymptotic Relations (for Oscillatory Functions)	
	7.7 Asymptotic Series	

Topics

8. Perturbation Series (Chapter 7of Bender and Orszag)		~ 1 week
	8.1 Perturbation Series	
	8.2 Regular and Singular Perturbation Theory	~ 1 lecture
	8.3 Asymptotic Matching	
9. Boundary Layer Theory (Chapter 9 of Bender and Orszag)		~ 1 week
	9.1 Introduction to Boundary Layer Theory	
	9.2 Mathematical Structure of Boundary Layer	
	9.3 Higher-Order Boundary Layer Theory and Internal Boundary Layers*	
10. WKB Theory (Chapter 10 of Bender and Orszag)		~ 1 week
	10.1 Introduction to WKB Theory (WKB expansion)	~ 1 lectures
	10.2 Conditions for Validity of the WKB Approximation	
	10.3 Patched and Matched Asymptotic Approximations*	

*Optional: these lectures will be presented subject to time availability.

References

- Bender, C. M., and S. A. Orszag, 2010: *Advanced Mathematical Methods for Scientists and Engineers.* Springer-Verlag, 593 pp. ISBN 978-1-4419-3187-0. [BO]

- Boyce W. and R. C. DiPrima, 2012: Elementary Differential Equations. Tenth Edition. John Wiley & Sons, INC. 832 pages ISBN: 978-0-470-45831-0

- Kreyszig, E., 2011: Advanced Engineering Mathematics. 10^{th} edition. John Wiley & Sons, INC. ISBN 978-0-470-45836-5. 1113 pp.

- Haberman, R. 2013: *Applied Partial Differential Equations*, 5^{th} edition, by Publisher: Pearson/Prentice Hall. ISBN-10: 0-321-79705-1. ISBN-13: 978-0321797056.

- Mathews J. and R. L. Walker, 1970: Mathematical Methods of Physics. 2^{nd} edition, 501pp.

Asymptotic Series vs. Convergent Power Series

	convergent power series	divergent asymptotic series
limit to the accuracy	no limit	an upper limit
Improving the accuracy	taking more terms	finding the optimal number of terms

- We can compare this rule with the way we would evaluate the sum of the convergent power series for a fixed value of x. For this series, there is no limit to the accuracy; we can always improve the accuracy by taking more terms in the partial sum.
- However, for a divergent asymptotic series, for each given value of x there is an upper limit to the accuracy and if we take either more or less than the optimal number of terms in the partial sum according to our rule, we usually decrease the accuracy.
- If we are not satisfied with this maximal accuracy, then to improve it we must take x closer to x_o or in the case of the series in (3.5.10) we must take x closer to $+\infty$.)

Power, Frobenius, and Asymptotic Series

Power Series

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Frobenius Series

$$y = x^r \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

leading behavior

$\sim \exp(S(x))$

Asympotic Series

$$w(x) = \sum_{n=0}^{\infty} a_n (x^{\alpha})^n$$

$$y = \boxed{c_1 x^{\frac{3}{4}} e^{\frac{2}{\sqrt{x}}}} \sum_{n=0}^{\infty} a_n (x)^{n/2}$$

α may be non-integers.

leading behavior

asymptotic series

Asymptotic Series vs. Convergent Power Series TBD

$$\boxed{\text{Convergent:} \quad \varepsilon_N(x) = \sum_{n=N+1}^{\infty} a_n(x-x_0)^n \to 0, \qquad N \to \infty;\ x \text{ fixed.}}$$

remainder

error terms

$$\boxed{\text{Asymptotic:} \quad \varepsilon_N(x) \ll (x-x_0)^N, \qquad x \to x_0;\ N \text{ fixed.}}$$

the next term

Convergent vs. Asymptotic vs. Power Series

$$\text{Convergent:} \quad \varepsilon_N(x) = \sum_{n=N+1}^{\infty} a_n(x - x_0)^n \to 0, \qquad N \to \infty;\ x \text{ fixed.}$$

- ϵ_N goes to zero as $N \to \infty$.
- Convergence is an absolute concept.

$$\text{Asymptotic:} \quad \varepsilon_N(x) \ll (x - x_0)^N, \qquad x \to x_0;\ N \text{ fixed.}$$

- ϵ_N goes to zero faster than $(x - x_o)^N$, but needs not to go to zero as $N \to \infty$.
- Asymptoticity is an relative property.

Two Types of 2nd Order ODEs

- Homogeneous Linear ODEs with Constant Coefficients
  - $ay'' + by' + c = 0 \qquad (P, Q, R \rightarrow a, b, c)$
  - $y = e^{rx} \qquad \Rightarrow \qquad ar^2 + br + c = 0$
  - $\left(y = e^{rx} = \sum_{n=0}^{\infty} \frac{1}{n!}(rx)^n\right)$
- Euler-Cauchy Equations
  - $x^2 y'' + axy' + b = 0$
  - $y = x^m \qquad \Rightarrow \qquad m^2 + (a-1)m + b = 0$

Method of Dominant Balance: An Illustration

$$y'' + p(x)y' + q(x)y = 0$$

$\boxed{y = e^{S(x)}}$ $\quad y' = S'e^{S(x)} \quad y'' = S''e^{S(x)} + (S')^2 e^{S(x)}$

$$S''e^{S(x)} + (S')^2 e^{S(x)} + pS'e^{S(x)} + qe^{S(x)} = 0$$

$S'' + (S')^2 + pS' + q = 0$ divide by $e^{s(x)}$

$\boxed{S'' << (S')^2, \quad \text{as } x \to x_0}$

1. drop (all) terms that are small

$\boxed{(S')^2 \sim -pS' - q, \text{ as } x \to x_0}$

2. replace "=" by "~" and solve the system (note that don't have "0" on the RHS)

3. check whether the solution is consistent with the approximation, i.e., whether $S'' << (S')^2$ is valid.

Method of Dominant Balance: How

The method of dominant balance is used to identify those terms in an equation that may be neglected in an asymptotic limit, (i.e., S''). The technique consist of three steps:

1. We drop all terms that appear small and replace the exact equation by an asymptotic relation.
2. We replace the asymptotic relation with an equation by exchanging the ~ sign for an = sign and solve the resulting equation exactly (the solution to this equation automatically satisfies the asymptotic relation although it is certainly not the only function that does so).
3. We check that the solution we have obtained is consistent with the approximation made in (1). If it is consistent, we must still show that the equation for the function obtained by factoring off the dominant balance solution from the exact solution itself has a solution that varies less rapidly than the dominant balance solution. When this happens, we conclude that the controlling factor (and not the leading behavior, i.e., the first term) obtained from the dominant balance relation is the same as that of the exact solution.

``stoppage criteria'' : The leading behavior of y(x) is determined by just those contributions to S(x) that do not vanish as x approaches the irregular singularity.

Perturbation Theory

Perturbation Theory

- Perturbation theory is a large collection of iterative methods for obtaining approximate solutions to problems involving a small parameter ε.

Three typical steps of perturbative analysis are:

1. Convert the original problem into a perturbation problem by introducing the small parameter ε.
2. Assume an expression for the answer in the form of a perturbation series and compute the coefficients of that series.
3. Recover the answer to the original problem by summing the perturbation series for the appropriate value of ε.

Perturbation Theory: An Example

Example 1 *Roots of a cubic polynomial.* Let us find approximations to the roots of

$$x^3 - 4.001x + 0.002 = 0. \tag{7.1.1}$$

(1) By introducing a parameter, ε, we convert the original problem to become:

$$\boxed{x^3 - (4 + \varepsilon)x + 2\varepsilon = 0.} \tag{7.1.2}$$

When $\varepsilon = 0.001$, the original equation (7.1.1) is reproduced.

Perturbation Theory: An Example

$$x^3 - (4+\epsilon)x + 2\epsilon = 0 \qquad \epsilon = 0.001$$

$$x(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n \qquad \boxed{x(\epsilon) = a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots}$$

$$(a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots)^3 - (4+\epsilon)(a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots) + 2\epsilon = 0$$

$O(\epsilon^0)$ or $\epsilon = 0$

$$(a_0)^3 - (4)(a_0) = 0$$

$$a_0 = 0, \pm 2$$

Perturbation Theory: An Example

$$x^3 - (4+\epsilon)x + 2\epsilon = 0 \qquad \epsilon = 0.001$$

$$x(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n \qquad \boxed{x(\epsilon) = a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots}$$

$$(a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots)^3 - (4+\epsilon)(a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots) + 2\epsilon = 0$$

$O(\epsilon^1)$

$$\boxed{3a_0^2 a_1\epsilon - (4a_1\epsilon + a_0\epsilon) + 2\epsilon = 0}$$

$a_0 = -2$

$$12a_1 - (4a_1 - 2) + 2 = 0$$

$$8a_1 + 4 = 0$$

$$a_1 = -\frac{1}{2}$$

Perturbation Theory: An Example

TBD

$$x^3 - (4+\epsilon)x + 2\epsilon = 0 \qquad \epsilon = 0.001$$

$$x(\epsilon) = \sum_{n=0}^{\infty} a_n \epsilon^n \qquad x(\epsilon) = a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots$$

$$(a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots)^3 - (4+\epsilon)(a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots) + 2\epsilon = 0$$

$O(\epsilon^2)$

$$3a_0a_1^2\epsilon^2 + 3a_0^2a_2\epsilon^2 - (4a_2\epsilon^2 + a_1\epsilon^2) = 0$$

$a_0 = -2$

$$-6a_1^2 + 12a_2 - (4a_2 + a_1) = 0$$

$a_1 = -\frac{1}{2}$

$$8a_2 = 6a_1^2 + a_1 = 1 \qquad a_2 = \frac{1}{8}$$

$$x(\epsilon) = a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots$$

$$x(\epsilon) = -2 - \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2 + \cdots$$

$$\epsilon = 0.001$$

Perturbation Theory

Perturbation theory is a large collection of iterative methods for obtaining approximate solutions to problems involving a small parameter ε. Three typical steps of perturbative analysis are:

1. Convert the original problem into a perturbation problem by introducing the small parameter ε.

$$x^3 - 4.001x + 0.002 = 0. \quad \Rightarrow \quad x^3 - (4 + \varepsilon)x + 2\varepsilon = 0.$$

1. Assume an expression for the answer in the form of a perturbation series and compute the coefficients of that series.

$$\boxed{x(\varepsilon) = \sum_{n=0}^{\infty} a_n \varepsilon^n.} = a_0 + a_1\varepsilon + a_2\varepsilon^2 + \ldots$$

1. Recover the answer to the original problem by summing the perturbation series for the appropriate value of ε.

$x_1 = -2 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + \cdots.$ If we now set $\varepsilon = 0.001$, we obtain x_1

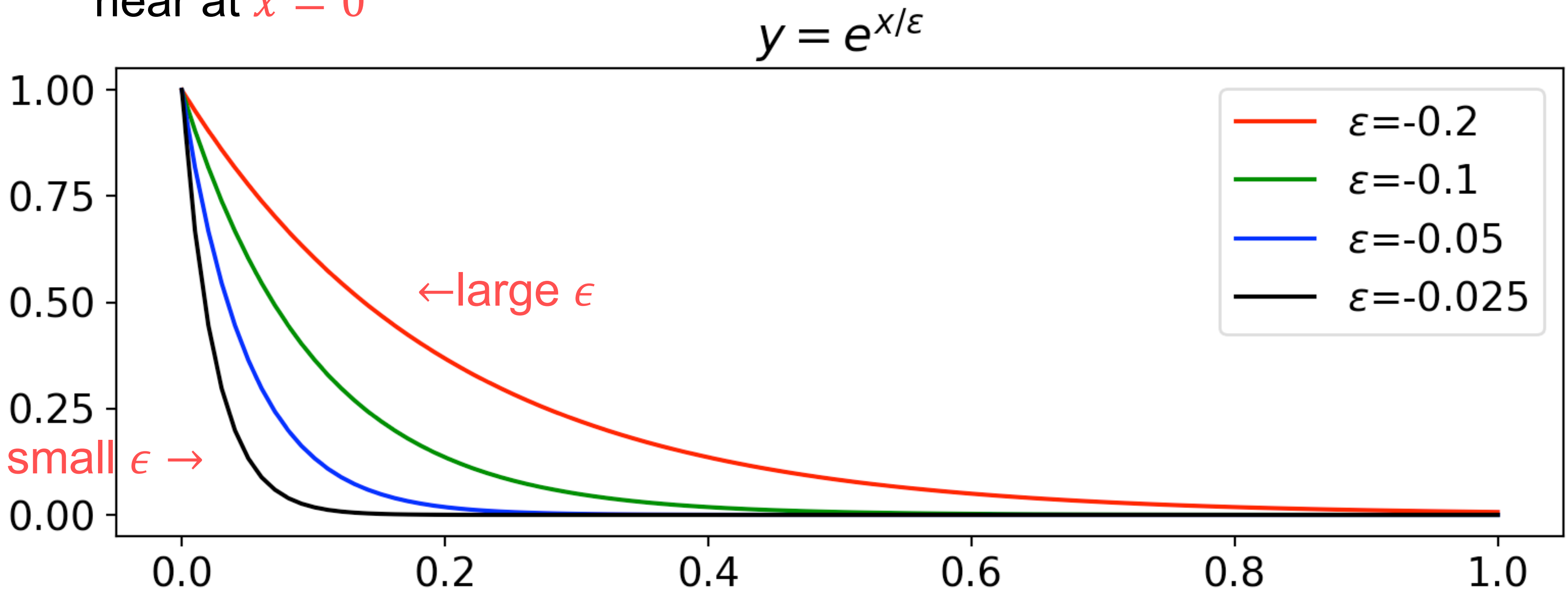
Boundary Layer within a 1st Order ODE

TBD

Stiff ODE $\quad y' = \lambda y \quad \lambda < 0\ and\ |\lambda|$ is large

boundary layer $\quad \varepsilon y' = y \quad \varepsilon = \frac{1}{\lambda} \quad y(x = 0) = 1$

near at $x = 0$

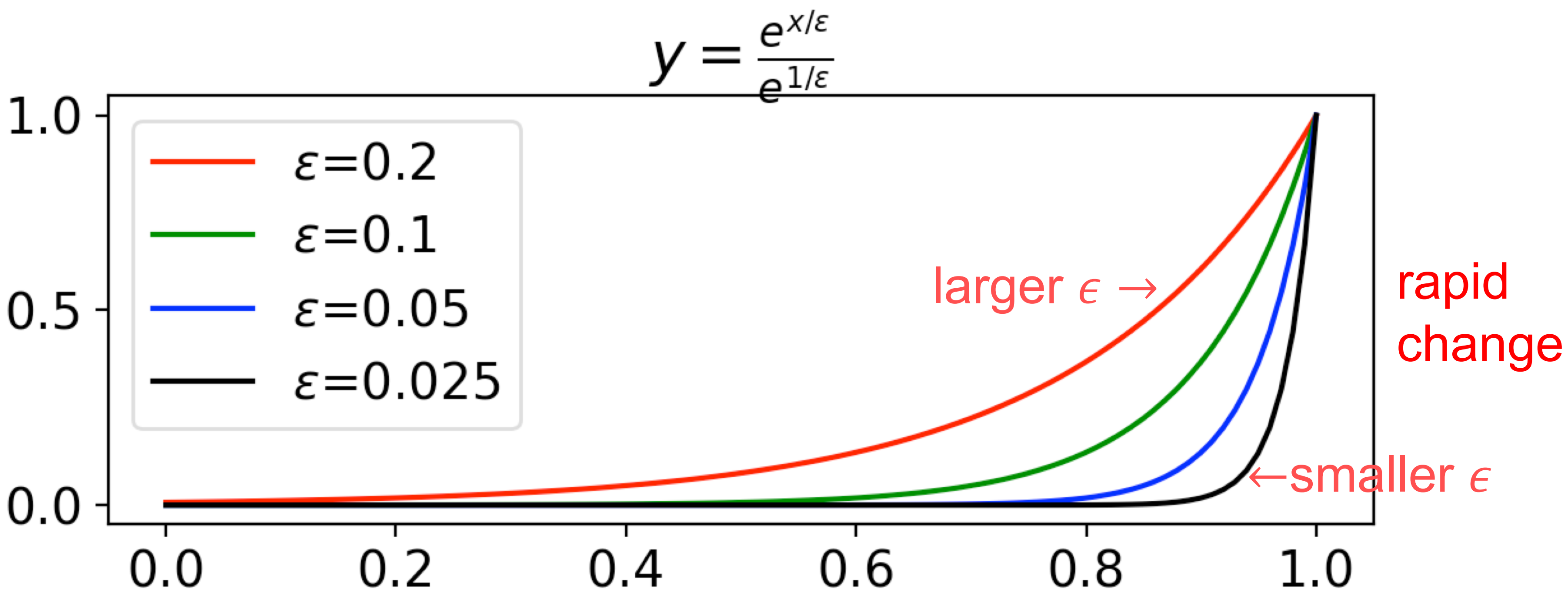


Boundary Layer

TBD

boundary layer $\varepsilon y' = y$ $y(x=0) = \frac{1}{e^{\epsilon}}$

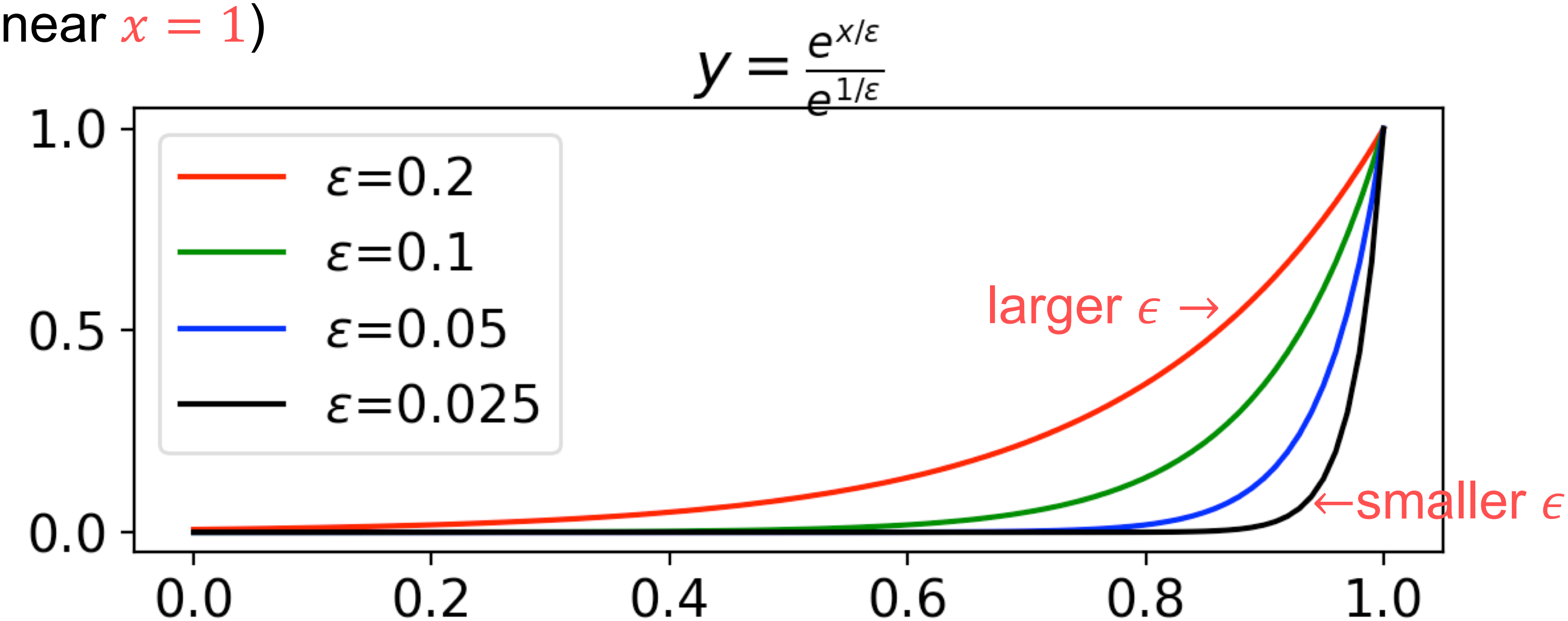
near at $x = 1$ $i.e., y(x=1) = 1$



Boundary Layer

boundary layer $\varepsilon y' = y$ $y(x=0) = e^{-\epsilon}$

(near $x = 1$)



- For a very small ϵ (e.g., a black curve), solution y is almost a constant except for a very narrow interval near $x = 1$ where a boundary layer is defined.
- Stated alternatively, When ϵ is small, y varies rapidly near $x = 1$; this localized region of rapid variation is called a boundary layer.

Singular Perturbation (within a 2nd Order ODE) TBD

Example 2 *Appearance of a boundary layer.* The boundary-value problem

$$\varepsilon y'' - y' = 0, \qquad y(0) = 0,\ y(1) = 1, \tag{7.2.7}$$

is a singular perturbation problem because the associated unperturbed problem

- When $\epsilon = 0$, we have $y' = 0$, yielding no solution that satisfies two boundary conditions (BCs).
- An exact solution is given as follows:

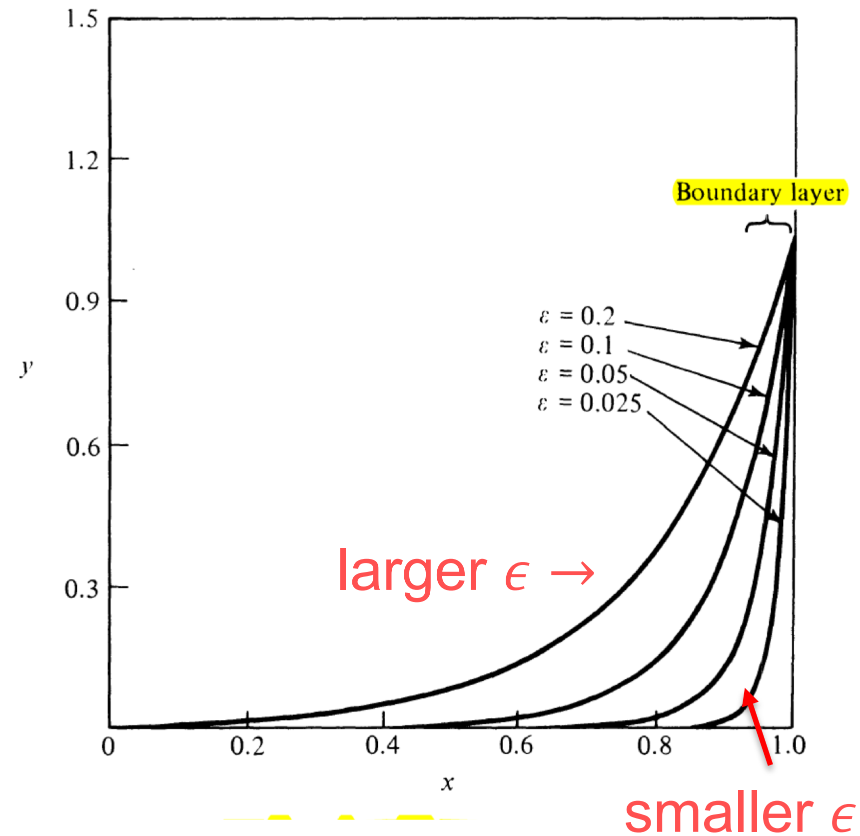
$$y = \frac{e^{x/\epsilon} - 1}{e^{1/\epsilon} - 1}$$

Boundary Layer (*)

TBD

$$\epsilon y'' - y' = 0 \quad y(0) = 0 \;\&\; y(1) = 1$$

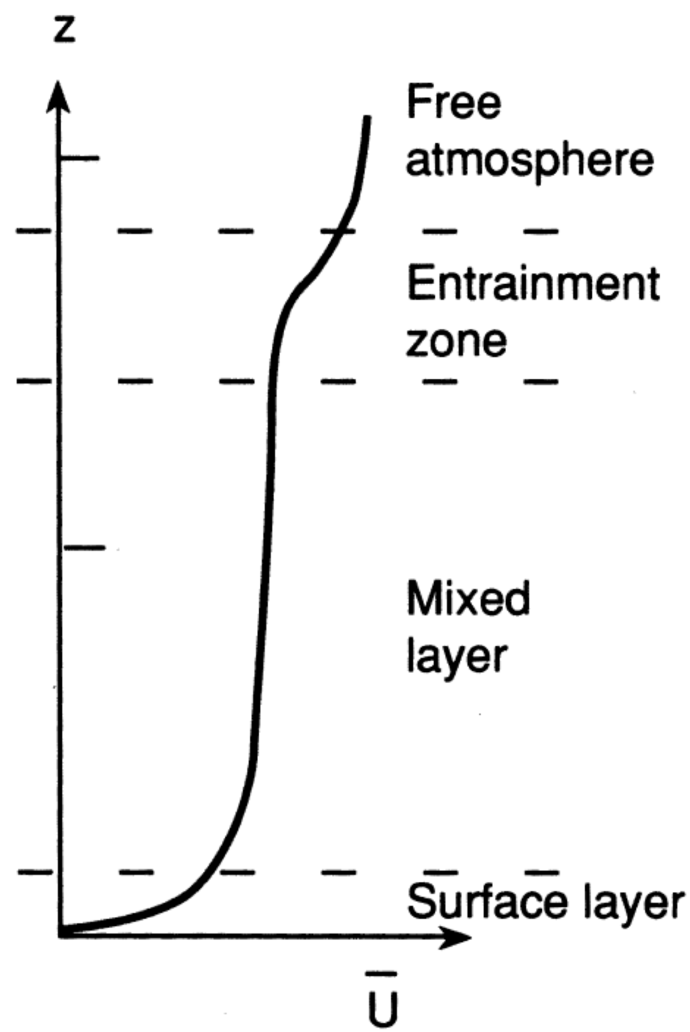
$$y = \frac{e^{x/\epsilon} - 1}{e^{1/\epsilon} - 1}$$



- When ϵ is small, y varies rapidly near $x = 1$; this localized region of rapid variation is called a boundary layer.
- When ϵ is negative, the boundary layer is at $x = 0$ instead of $x = 1$.
- This abrupt jump in the location of the boundary layer as ϵ changes sign reflects the singular nature of the perturbation problem

Atmospheric Surface Layer (Wiki) (*) TBD

- The term boundary layer is used in meteorology and in physical oceanography.
- The atmospheric surface layer is the lowest part of the atmospheric boundary layer (typically the bottom 10% where the log wind profile is valid).
- The surface layer is the layer of a turbulent fluid most affected by interaction with a solid surface or the surface separating a gas and a liquid where the characteristics of the turbulence depend on distance from the interface.
- Surface layers are characterized by large normal gradients of tangential velocity and large concentration gradients of any substances (temperature, moisture, sediments et cetera) transported to or from the interface.



Singular Perturbation Problems: Rapidly Decaying vs. Rapidly Oscillatory

TBD

1. A local breakdown occurs where the approximate solution is exponentially increasing or decreasing. This kind of behavior is called dissipative because the rapidly varying component of the solution decays exponentially (dissipates) away from the point of local breakdown.

$$\boxed{\epsilon y'' - y' = 0, \qquad y(0) = 0, \qquad y(1) = 1,}$$

boundary layer techniques

2. A global breakdown is typically associated with rapidly oscillatory, or dispersive, behavior. A dispersive solution is wavelike with very small and slowly changing wavelengths and slowly varying amplitudes as functions of x.

$$\boxed{\epsilon y'' + y = 0, \qquad y(0) = 0, \qquad y(1) = 1,}$$

WKBJ

Appearance of Rapid Oscillations on a Global Scale TBD

Example 3 *Appearance of rapid variation on a global scale.* In the previous example we saw that the exact solution varies rapidly in the neighborhood of $x = 1$ for small ε and develops a discontinuity there in the limit $\varepsilon \to 0+$. A solution to a boundary-value problem may also develop discontinuities throughout a large region as well as in the neighborhood of a point.

The boundary-value problem $\varepsilon y'' + y = 0$ $[y(0) = 0,\ y(1) = 1]$ is a singular perturbation problem because when $\varepsilon = 0$, the solution to the unperturbed problem, $y = 0$, does not satisfy the boundary condition $y(1) = 1$. The exact solution, when ε is not of the form $(n\pi)^{-2}$ $(n = 0, 1, 2, \ldots)$, is $y(x) = \sin\,(x/\sqrt{\varepsilon})/\sin\,(1/\sqrt{\varepsilon})$. Observe that $y(x)$ becomes discontinuous throughout the inter-

$$\epsilon y'' + y = 0, \qquad y(0) = 0, \qquad y(1) = 1,$$

An exact solution $y = \dfrac{\sin(x/\sqrt{\epsilon})}{\sin(1/\sqrt{\epsilon})}$ when $\epsilon \neq \dfrac{1}{(n\pi)^2}$, $\mathrm{n} = 1,2,3\cdots$

$$\epsilon^2 y'' + y = 0, \qquad y(0) = 0, \qquad y(1) = 1,$$

An exact solution $y = \dfrac{\sin(x/\epsilon)}{\sin(1/\epsilon)}$ when $\epsilon \neq \dfrac{1}{(n\pi)}$, $\mathrm{n} = 1,2,3\cdots$

Appearance of Rapid Oscillations on a Global Scale TBD

An exact solution $y = \dfrac{\sin(x/\sqrt{\epsilon})}{\sin(1/\sqrt{\epsilon})}$ when $\epsilon \neq \dfrac{1}{(n\pi)^2}$, $n = 1,2,3 \cdots$

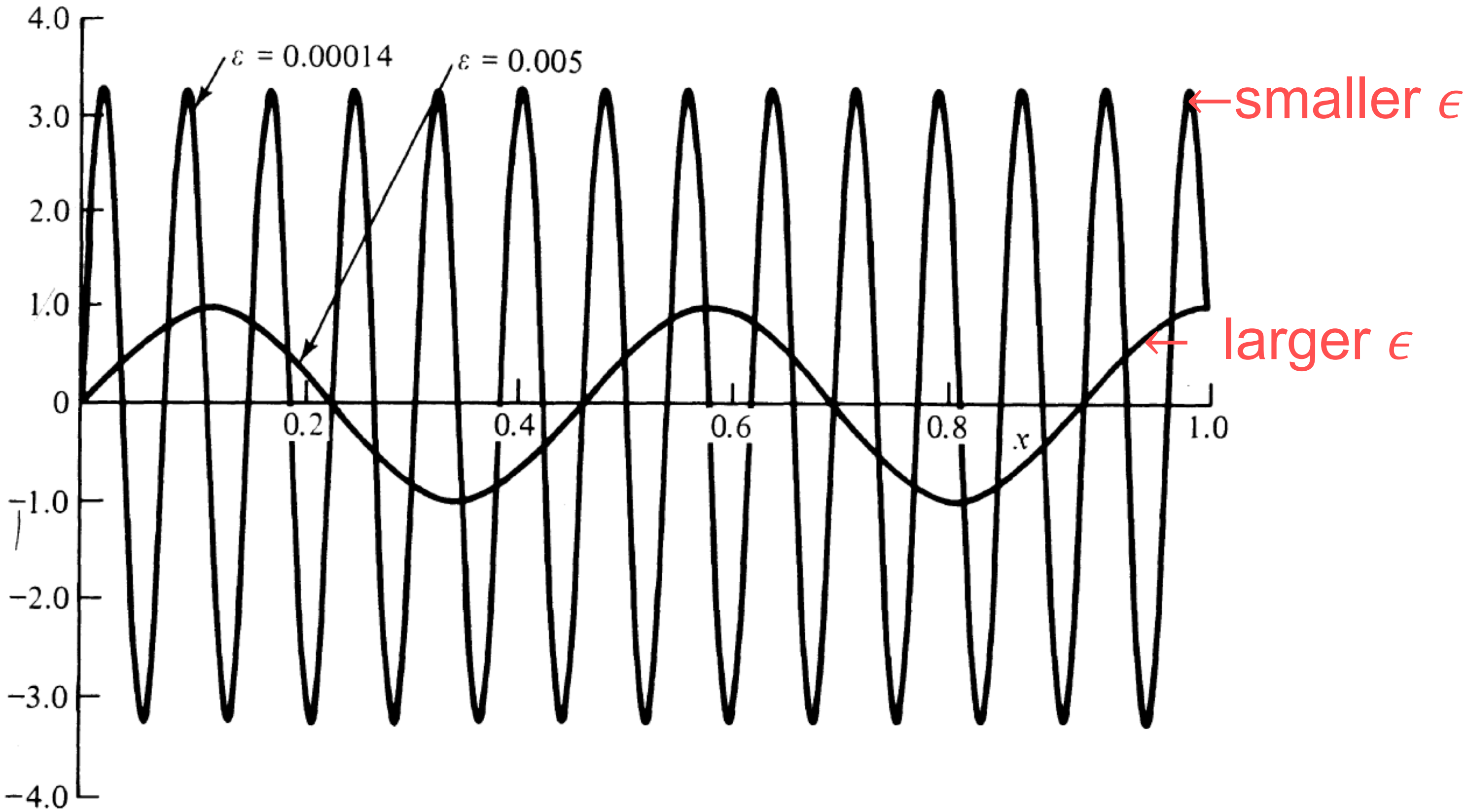


Figure 7.3 A plot of $y(x) = [\sin(x\varepsilon^{-1/2})]/[\sin(\varepsilon^{-1/2})]$ $(0 \le x \le 1)$ for $\varepsilon = 0.005$ and 0.00014. As ε gets smaller the oscillations become more violent; as $\varepsilon \to 0+$, $y(x)$ becomes discontinuous over the entire interval. The WKB approximation is a perturbative method commonly used to describe functions like $y(x)$ which exhibit rapid variation on a global scale.

WKBJ

Example 1 *Approximate solution to a Schrödinger equation.* A second-order homogeneous linear differential equation is in Schrödinger form if the y' term is absent. The approximate solutions to the Schrödinger equation

$$\varepsilon^2 y'' = Q(x)y, \qquad Q(x) \neq 0, \tag{10.1.5}$$

are easy to find using WKB analysis when ε is small. We merely substitute (10.1.4) into (10.1.5).

$$y(x) \sim \frac{c_1}{\sqrt[4]{Q)x)}} exp\left[\frac{1}{\epsilon}\int_a^x \sqrt{Q(t)}dt\right] + \frac{c_2}{\sqrt[4]{Q)x)}} exp\left[\frac{-1}{\epsilon}\int_a^x \sqrt{Q(t)}dt\right]$$

$Q(x) = 1$

$$y(x) \sim c_1 exp\left(\frac{x}{\epsilon}\right) + c_2 exp\left(\frac{-x}{\epsilon}\right)$$

$Q(x) = -1$

$$y(x) \sim c_3 exp\left(\frac{ix}{\epsilon}\right) + c_4 exp\left(\frac{-ix}{\epsilon}\right)$$

Previously, we obtained:

An exact solution $y = \frac{\sin(x/\epsilon)}{\sin(1/\epsilon)}$ when $\epsilon \neq \frac{1}{(n\pi)}$, $\mathrm{n} = 1,2,3\cdots$

WKBJ

TBD

$$\epsilon y'' + y = 0, \qquad y(0) = 0, \qquad y(1) = 0,$$

- The Exponential Approximation (a.k.a. the WKB approximation)

$$y = \exp\left(\frac{S(x)}{\epsilon}\right) \quad \text{or} \quad y = \exp(S(x))$$

- Formal WKB Expansion (using an exponential power series)

$$y = \exp\left[\frac{1}{\epsilon}(S_0 + S_1\epsilon + S_2\epsilon^2 + \cdots)\right]$$

express an exponent as a perturbation series

A Brief Note on WKBJ

$$y'' + Q(x)y = 0$$

$$y = \exp(\lambda x)$$

$\lambda^2 \sim -Q(x)$, $\lambda \sim \pm\sqrt{-Q}$, (to be “proved” later)

λ *is real when* $Q(x) < 0$, ➔ exponential solutions

$\lambda_{1,2}$ *are pure imaginary when* $Q(x) > 0$, ➔ oscillatory solutions

Chapter 3: Approximate Solutions of Linear DEs

1. Classification of Singular Points
   - ✓ $y'' + p(x)y' + q(x)y = 0$
   - ✓ ordinary points: p and q are analytic
   - ✓ regular singular points: xp and x^2q are analytic
   - ✓ irregular singular points: none of the above
2. Local Behavior Near Ordinary Points
   - ✓ Taylor series
3. Local Series Expansions About Regular Singular Points
   - ✓ Frobenius series
4. Local Behavior at Irregular Singular Points
5. Irregular Singular Point at Infinity
6. Asymptotic Relations
7. Asymptotic Series

Chapter 3: Approximate Solutions of Linear DEs

1. Classification of Singular Points
2. Local Behavior Near Ordinary Points
   - ✓ Airy Eq. $y'' = xy$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

3. Local Series Expansions About Regular Singular Points
   - ✓ $4x^2 y'' + y = 0$; $y_1 = \sqrt{x}$ and $y_2 = \sqrt{x}\ln(x)$

$$y = x^r \sum_{n=0}^{\infty} a_n x^n$$

4. Local Behavior at Irregular Singular Points
   - ✓ $x^3 y''' = y$
5. Irregular Singular Point at Infinity
6. Asymptotic Relations
7. Asymptotic Series

Important Concepts

$$y'' + p(x)y' + q(x)y = 0$$

- analytic:
  - ✓ A function is called analytic at a point at $x = x_0$ if it can be represented by a power series in power of $(x - x_0)$ with a positive radius of convergence.
- regular points:
  - $p(x)$ and $q(x)$ are analytic
- regular singular point
  - $xp(x)$ and $\mathrm{x}^2 q(x)$ are analytic
  - $y'' + \frac{y'}{x} + \frac{y}{x^2} = 0$
- irregular singular point
  - $xp(x)$ and $\mathrm{x}^2 q(x)$ are not analytic
  - $x^3 y'' = y$

Analytic

A function f that has a Taylor series expansion about $x = x_0$,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

with a radius of convergence $\rho > 0$, is said to be **analytic** at $x = x_0$. According to statements 6 and 7, if f and g are analytic at x_0, then $f \pm g$, $f \cdot g$, and f/g [provided that $g(x_0) \neq 0$] are analytic at $x = x_0$.

- A Taylor series expansion
- A non-zero radius of convergence

(I) Regular Points: Example

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)} + \cdots p_1 y^{(1)} + p_0 y = 0, \qquad (3.1.1)$$

$$y^{(k)}(x) = \frac{d^k y}{dx^k}$$

Ordinary Points

The point x_0 ($x_0 \neq \infty$) is called an *ordinary point* of (3.1.1) if the coefficient functions $p_0(x), \ldots, p_{n-1}(x)$ are all analytic in a neighborhood of x_0 in the complex plane.

Example 1 *Ordinary points.*

(*a*) $y'' = e^x y$. Every point $x_0 \neq \infty$ is an ordinary point because e^x is entire.
(*b*) $x^5 y''' = y$. Every point x_0 except for $x_0 = 0$ and ∞ is an ordinary point.

Airy Equation ($y'' = xy$) and Airy Functions

$$\mathrm{Ai}(x) \equiv 3^{-2/3} \sum_{n=0}^{\infty} \frac{x^{3n}}{9^n n!\, \Gamma(n+\frac{2}{3})} - 3^{-4/3} \sum_{n=0}^{\infty} \frac{x^{3n+1}}{9^n n!\, \Gamma(n+\frac{4}{3})}, \tag{3.2.1}$$

$$\mathrm{Bi}(x) \equiv 3^{-1/6} \sum_{n=0}^{\infty} \frac{x^{3n}}{9^n n!\, \Gamma(n+\frac{2}{3})} + 3^{-5/6} \sum_{n=0}^{\infty} \frac{x^{3n+1}}{9^n n!\, \Gamma(n+\frac{4}{3})}. \tag{3.2.2}$$

APPROXIMATE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS **69**

$$y = \sum_{n=0}^{\infty} a_n x^n$$

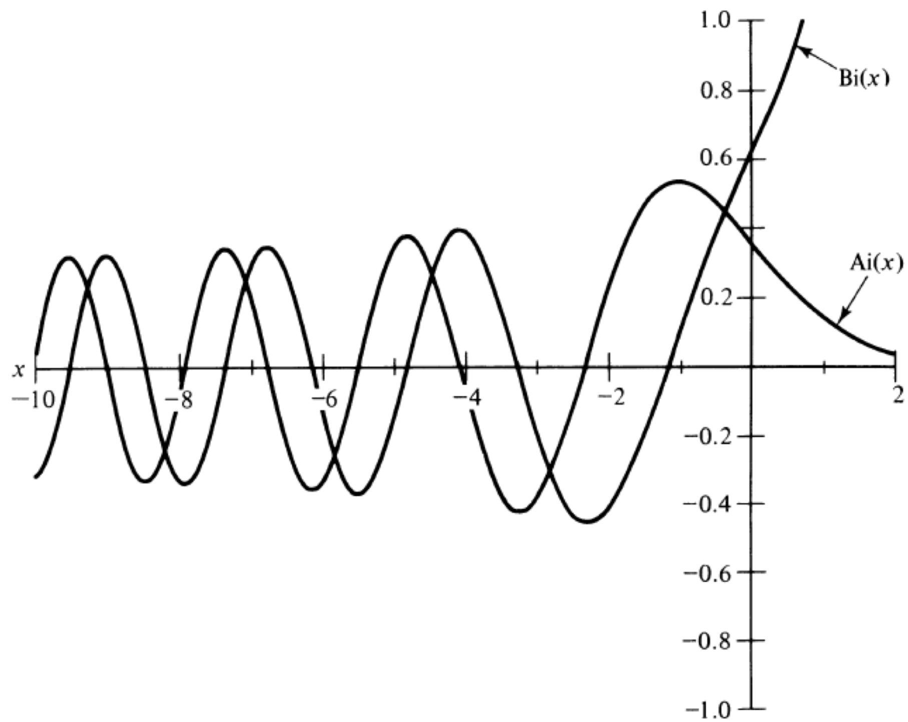


Figure 3.1 A plot of the Airy functions Ai(x) and Bi(x) for $-10 \le x \le 2$. Both functions are oscillatory for negative x; Bi(x) grows exponentially and Ai(x) decays exponentially as $x \to +\infty$.

Review: A Brief Note on WKBJ

$$y'' + Q(x)y = 0$$

$$y = \exp(\lambda x)$$

$\lambda^2 \sim -Q(x)$, $\lambda \sim \pm\sqrt{-Q}$, (to be "proved" later)

$\lambda\ is\ real\ when\ Q(x) < 0,$ ➔ exponential solutions

$\lambda_{1,2}\ are\ pure\ imaginary\ when\ Q(x) > 0,$ ➔ oscillatory solutions

Airy Equation $(y'' = xy)$

Example 2 *Local analysis of the Airy equation at $x = 0$.* To find the local behavior of the solutions to the Airy equation (1.4.8) $y'' = xy$ near $x = 0$ we substitute the series $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and differentiate term by term. The result is

$$\sum_{n=0}^{\infty} a_n n(n-1)x^{n-2} = \sum_{n=0}^{\infty} a_n x^{n+1}.$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

After a shift of indices, the right side becomes $\sum_{n=3}^{\infty} a_{n-3} x^{n-2}$.

Equating the coefficients of x^{n-2} in y'' and xy gives

$$a_n n(n-1) = 0, \qquad n = 0, 1, 2,$$
$$a_n n(n-1) = a_{n-3}, \qquad n = 3, 4, \ldots,$$

which can be solved in closed form. The first equation is already satisfied for $n = 0$ and $n = 1$. Hence, a_0 and a_1 are *arbitrary* constants. Also, $a_2 = 0$. The solutions of the second equation are

$$a_{3n} = \frac{a_0}{3n(3n-1)(3n-3)(3n-4)\cdots 9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2}$$
$$= \frac{a_0}{3^n n!\, 3^n(n-\frac{1}{3})(n-1-\frac{1}{3})(n-2-\frac{1}{3})\cdots(\frac{5}{3})(\frac{2}{3})}$$
$$= \frac{a_0\Gamma(\frac{2}{3})}{3^n n!\, 3^n\Gamma(n+\frac{2}{3})},$$

$$a_{3n+1} = \frac{a_1}{(3n+1)(3n)(3n-2)(3n-3)\cdots 10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3}$$
$$= \frac{a_1}{3^n n!\, 3^n(n+\frac{1}{3})(n-1+\frac{1}{3})(n-2+\frac{1}{3})\cdots(\frac{7}{3})(\frac{4}{3})}$$
$$= \frac{a_1\Gamma(\frac{4}{3})}{3^n n!\, 3^n\Gamma(n+\frac{4}{3})},$$

$$a_{3n+2} = 0.$$

Gamma function

Main definition [edit]

The notation $\Gamma(z)$ is due to Legendre. If the real part of the complex number z is positive $(\mathrm{Re}(z) > 0)$, then the integral

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\,dx.$$

converges absolutely, and is known as the **Euler integral of the second kind** (the Euler integral of the first kind defines the beta function). Using integration by parts, one sees $\Gamma(z)$ satisfies the functional equation:

$$\Gamma(z+1) = z\Gamma(z).$$

Combining this with $\Gamma(1) = 1$, one gets:

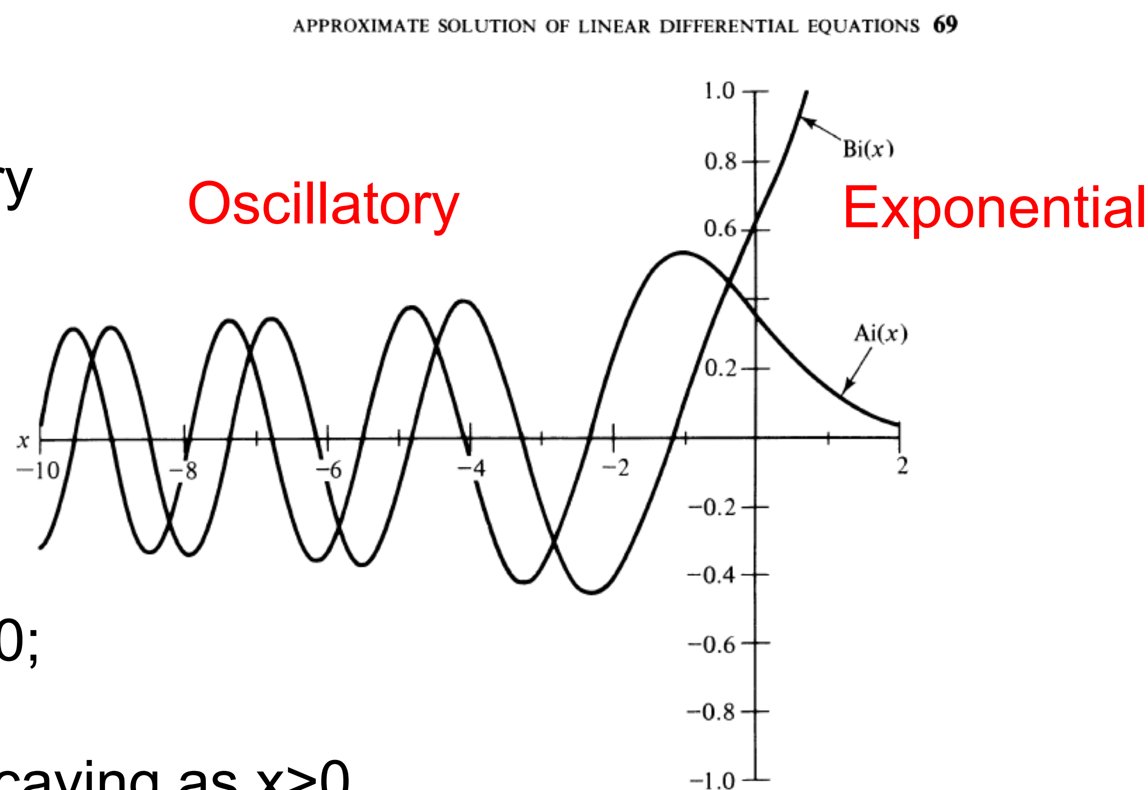
$$\Gamma(n) = 1\cdot 2\cdot 3\cdots(n-1) = (n-1)!$$

for all positive integers n.

Airy Equation ($y'' = xy$): WKB Analysis

$$y'' = xy$$

- $x > 0, \sqrt{-Q} = \sqrt{x}$, real
- $x < 0, \sqrt{-Q} = \sqrt{x}$, imaginary



- x=0 is a turning point;
- Oscillatory solutions as x<0;
- Exponential growing or decaying as x>0

Radius of Convergence vs. Singularity

Supp

- Fuchs proved in 1866 that all n linearly independent solutions of (3.1.1) are analytic in a neighborhood of an ordinary point.
- Moreover, he proved that if any solution is expanded in a Taylor series about the ordinary point x_o, $y(x)= \sum_{n=0}^{\infty} a_n(x-x_0)^n$, then the radius of convergence of this series is at least as large as the distance to the nearest singularity of the coefficient functions in the complex plane (see Prob. 3.9).
- The location of a singularity of a solution must coincide with the location of a singularity of a coefficient function

Regular Point and Power Series Solutions

- regular points are defined when $p(x)$ and $q(x)$ are analytic.

  (a Taylor series expansion with A non-zero radius of convergence)
- every solution is analytic at the regular points.

When do power series solutions exist? *Answer:* if p, q, r in the ODEs

$$y'' + p(x)y' + q(x)y = r(x) \tag{12}$$

have power series representations (Taylor series). More precisely, a function $f(x)$ is called **analytic** at a point $x = x_0$ if it can be represented by a power series in powers of $x - x_0$ with positive radius of convergence. Using this concept, we can state the following basic theorem, in which the ODE (12) is **in standard form**, that is, it begins with the y''. If your ODE begins with, say, $h(x)y''$, divide it first by $h(x)$ and then apply the theorem to the resulting new ODE.

Existence of Power Series Solutions

If p, q, and r in (12) are analytic at $x = x_0$, then every solution of (12) is analytic at $x = x_0$ and can thus be represented by a power series in powers of $x - x_0$ with radius of convergence $R > 0$.

Analytic (Complex) Functions

The Cauchy–Riemann equations are the most important equations in this chapter and one of the pillars on which complex analysis rests. They provide a criterion (a test) for the analyticity of a complex function

$$w = f(z) = u(x, y) + iv(x, y).$$

Roughly, f is analytic in a domain D if and only if the first partial derivatives of u and v satisfy the two **Cauchy–Riemann equations**[4]

$$\textbf{(1)} \qquad u_x = v_y, \qquad u_y = -v_x$$

everywhere in D; here $u_x = \partial u/\partial x$ and $u_y = \partial u/\partial y$ (and similarly for v) are the usual notations for partial derivatives. The precise formulation of this statement is given in Theorems 1 and 2.

Example: $f(z) = z^2 = x^2 - y^2 + 2ixy$ is analytic for all z (see Example 3 in Sec. 13.3), and $u = x^2 - y^2$ and $v = 2xy$ satisfy (1), namely, $u_x = 2x = v_y$ as well as $u_y = -2y = -v_x$. More examples will follow.

Cauchy–Riemann Equations

Let $f(z) = u(x, y) + iv(x, y)$ be defined and continuous in some neighborhood of a point $z = x + iy$ and differentiable at z itself. Then, at that point, the first-order partial derivatives of u and v exist and satisfy the Cauchy–Riemann equations (1).

Hence, if $f(z)$ is analytic in a domain D, those partial derivatives exist and satisfy (1) *at all points of D.*

Entire function (Wiki)

Supp

- In complex analysis, an entire function, also called an integral function, is a complex-valued function that is holomorphic over the whole complex plane.
- Typical examples of entire functions are polynomials and the exponential function, and any finite combinations of sums, products and compositions of these, such as the trigonometric functions sine and cosine and their hyperbolic counterparts sinh and cosh, as well as derivatives and integrals of entire functions such as the error function.
- If an entire function $f(z)$ has a root at w, then $f(z)/(z-w)$ is an entire function.
- On the other hand, neither the natural logarithm nor the square root is an entire function, nor can they be continued analytically to an entire function.

Holomorphic Functions

Supp

In mathematics, a holomorphic function is a complex-valued function of one or more complex variables that is complex differentiable in a neighborhood of every point in its domain. The existence of a complex derivative in a neighborhood is a very strong condition, for it implies that any holomorphic function is actually infinitely differentiable and equal to its own Taylor series (analytic). Holomorphic functions are the central objects of study in complex analysis.

Though the term analytic function is often used interchangeably with "holomorphic function", the word "analytic" is defined in a broader sense to denote any function (real, complex, or of more general type) that can be written as a convergent power series in a neighborhood of each point in its domain. The fact that all holomorphic functions are complex analytic functions, and vice versa, is a major theorem in complex analysis.[1]

Holomorphic functions are also sometimes referred to as regular functions[2] or as conformal maps. A holomorphic function whose domain is the whole complex plane is called an entire function. The phrase "holomorphic at a point z0" means not just differentiable at z0, but differentiable everywhere within some neighborhood of z0 in the complex plane.

Isolated Singularity, Poles, and Essential Singularity Supp

We say that a function $f(z)$ **is singular** or **has a singularity** at a point $z = z_0$ if $f(z)$ is not analytic (perhaps not even defined) at $z = z_0$, but every neighborhood of $z = z_0$ contains points at which $f(z)$ is analytic. We also say that $z = z_0$ is a **singular point** of $f(z)$.

We call $z = z_0$ an **isolated singularity** of $f(z)$ if $z = z_0$ has a neighborhood without further singularities of $f(z)$. *Example:* $\tan z$ has isolated singularities at $\pm\pi/2, \pm 3\pi/2$, etc.; $\tan(1/z)$ has a nonisolated singularity at 0. (Explain!)

Isolated singularities of $f(z)$ at $z = z_0$ can be classified by the Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \qquad (1) \quad \text{(Sec. 16.1)}$$

valid ***in the immediate neighborhood*** of the singular point $z = z_0$, except at z_0 itself, that is, in a region of the form

$$0 < |z - z_0| < R.$$

The sum of the first series is analytic at $z = z_0$, as we know from the last section. The second series, containing the negative powers, is called the **principal part** of (1), as we remember from the last section. If it has only finitely many terms, it is of the form

$$\frac{b_1}{z - z_0} + \cdots + \frac{b_m}{(z - z_0)^m} \qquad (2) \quad (b_m \neq 0).$$

Then the singularity of $f(z)$ at $z = z_0$ is called a **pole**, and m is called its **order**. Poles of the first order are also known as **simple poles**.

If the principal part of (1) has infinitely many terms, we say that $f(z)$ has at $z = z_0$ an **isolated essential singularity**.

We leave aside nonisolated singularities.

Poles and Essential Singularity

Poles. Essential Singularities

The function

poles

$$f(z) = \frac{1}{z(z-2)^5} + \frac{3}{(z-2)^2}$$

has a simple pole at $z = 0$ and a pole of fifth order at $z = 2$. Examples of functions having an isolated essential singularity at $z = 0$ are

essential singularity

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots$$

(II) Regular Singular Points

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)} + \cdots p_1 y^{(1)} + p_0 y = 0, \qquad (3.1.1)$$

Regular Singular Points

The point x_0 ($x_0 \neq \infty$) is called a *regular singular point* of (3.1.1) if not all of $p_0(x), \ldots, p_{n-1}(x)$ are analytic but if all of $(x - x_0)^n p_0(x)$, $(x - x_0)^{n-1} p_1(x), \ldots,$ $(x - x_0)p_{n-1}(x)$ *are* analytic in a neighborhood of x_0.

Example 3 *Regular singular points.*

(*a*) $(x - 1)y''' = y$ has a regular singular point at 1.
(*b*) $x^2 y'' + xy' = y$ has a regular singular point at 0.
(*c*) $x^3 y' = (x + 1)y$ does *not* have a regular singular point at 0.

a) $p_0 = -\frac{1}{x-1}$; $n = 3$; $(x-1)^3 p_0$ is analytic;

b) $p_0 = -\frac{1}{x^2}$; $p_1 = \frac{1}{x}$; $n = 2$; $x^2 p_0$ and $x p_1$ are analytic

c) Irregular singular point

Classification of the Point $X_0 = \infty$

Classification of the Point $x_0 = \infty$

We have completed the classification of points x_0 in the finite complex plane, but it is also useful to classify the point $x_0 = \infty$. We do this by analytically mapping the point at infinity into the origin using the inversion transformation

$$x = \frac{1}{t},$$

$$\frac{d}{dx} = -t^2 \frac{d}{dt}, \tag{3.1.6}$$

$$\frac{d^2}{dx^2} = t^4 \frac{d^2}{dt^2} + 2t^3 \frac{d}{dt},$$

and so on, and then classifying the point $t = 0$. *The point $x_0 = \infty$ is called an ordinary, a regular singular, or an irregular singular point if the point at $t = 0$ is correspondingly classified.*

Classification of the Point $X_0 = \infty$

Example 5 *Comparison of ordinary, regular singular, and irregular singular points.* Consider the three equations

$$\frac{dy}{dx} - \frac{1}{2}y = 0, \qquad x = \frac{1}{t}, \tag{3.1.7}$$

$$\frac{dy}{dx} - \frac{1}{2x}y = 0, \qquad \frac{d}{dx} = -t^2\frac{d}{dt}, \tag{3.1.8}$$

$$\frac{dy}{dx} - \frac{1}{2x^2}y = 0. \qquad \frac{d^2}{dx^2} = t^4\frac{d^2}{dt^2} + 2t^3\frac{d}{dt}, \tag{3.1.9}$$

$$\frac{dy}{dx} - \frac{1}{2}y = 0 \qquad (-t^2)\frac{dy}{dt} - \frac{1}{2}y = 0$$

irregular singularity at $t = 0$
(regular at $x = 0$)

$$\frac{dy}{dx} - \frac{1}{2x}y = 0 \qquad (-t^2)\frac{dy}{dx} - \frac{t}{2}y = 0$$

regular singularity at $t = 0$
(regular singularity at $x = 0$)

$$\frac{dy}{dx} - \frac{1}{2x^2}y = 0 \qquad (-t^2)\frac{dy}{dx} - \frac{t^2}{2}y = 0$$

regular at $t = 0$
(irregular singularity at $x = 0$)

Regular Singular Point and Frobenius Series Solutions

Frobenius Method

Let $b(x)$ and $c(x)$ be any functions that are analytic at $x = 0$. Then the ODE

(1) $$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

has at least one solution that can be represented in the form

(2) $$\boxed{y(x) = x^r \sum_{m=0}^{\infty} a_m x^m} = x^r(a_0 + a_1 x + a_2 x^2 + \cdots) \qquad (a_0 \neq 0)$$

where the exponent r may be any (real or complex) number (and r is chosen so that $a_0 \neq 0$).

The ODE (1) *also has a second solution (such that these two solutions are linearly independent) that may be similar to* (2) *(with a different r and different coefficients) or may contain a logarithmic term.* (Details in Theorem 2 below.)

---

[3]OSSIAN BONNET (1819–1892), French mathematician, whose main work was in differential geometry.

[4]GEORG FROBENIUS (1849–1917), German mathematician, professor at ETH Zurich and University of Berlin, student of Karl Weierstrass (see footnote, Sect. 15.5). He is also known for his work on matrices and in group theory.

In this theorem we may replace x by $x - x_0$ with any number x_0. The condition $a_0 \neq 0$ is no restriction; it simply means that we factor out the highest possible power of x.

The singular point of (1) at $x = 0$ is often called a **regular singular point**, a term confusing to the student, which we shall not use.

Kreyszig

Extended Power Series Method: Frobenius Method

$$x^2y'' + xb(x)y' + c(x)y = 0.$$

$$y = x^r \sum_{m=0}^{\infty} a_m (x)^m$$

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r(a_0 + a_1 x + a_2 x^2 + \cdots)$$

indicial equation

$$r(r-1) + b_0 r + c_0 = 0.$$

Three cases that depend on the roots r of the indicial equation include:

Case 1. Distinct roots not differing by an integer $1, 2, 3, \cdots$.

Case 2. A double root.

Case 3. Roots differing by an integer $1, 2, 3, \cdots$.

Cases 1 and 2 are not unexpected because of the Euler–Cauchy equation (Sec. 2.5),

Frobenius Method. Case 1: Distinct Roots

Frobenius Method. Basis of Solutions. Three Cases

Suppose that the ODE (1) *satisfies the assumptions in Theorem* 1. *Let* r_1 *and* r_2 *be the roots of the indicial equation* (4). *Then we have the following three cases.*

***Case* 1. *Distinct Roots Not Differing by an Integer*.** *A basis is*

$$y_1(x) = x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots) \tag{5}$$

and

$$y_2(x) = x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots) \tag{6}$$

with coefficients obtained successively from (3) *with* $r = r_1$ *and* $r = r_2$, *respectively.*

Cases 2 & 3: Double Root & Roots Differing by an Integer

Case 2. Double Root $\boldsymbol{r_1 = r_2 = r}$. *A basis is*

$$y_1(x) = x^r(a_0 + a_1x + a_2x^2 + \cdots) \qquad [r = \tfrac{1}{2}(1 - b_0)] \tag{7}$$

(of the same general form as before) and

$$y_2(x) = y_1(x) \ln x + x^r(A_1x + A_2x^2 + \cdots) \qquad (x > 0). \tag{8}$$

Case 3. Roots Differing by an Integer. *A basis is*

$$y_1(x) = x^{r_1}(a_0 + a_1x + a_2x^2 + \cdots) \tag{9}$$

(of the same general form as before) and

$$y_2(x) = ky_1(x) \ln x + x^{r_2}(A_0 + A_1x + A_2x^2 + \cdots), \tag{10}$$

where the roots are so denoted that $r_1 - r_2 > 0$ *and* k *may turn out to be zero.*

"Two Parts" of the Frobenius Series Solution

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)} + \cdots p_1 y^{(1)} + p_0 y = 0, \qquad (3.1.1)$$

Regular Singular Points

The point x_0 ($x_0 \neq \infty$) is called a *regular singular point* of (3.1.1) if not all of $p_0(x), \ldots, p_{n-1}(x)$ are analytic but if all of $(x - x_0)^n p_0(x)$, $(x - x_0)^{n-1} p_1(x), \ldots,$ $(x - x_0)p_{n-1}(x)$ *are* analytic in a neighborhood of x_0.

Fuchs showed that there is always at least one solution of the form

$$y = (x - x_0)^{\alpha} A(x)$$

where

- α is a number called the indicial exponent and
- $A(x)$ is a function which is analytic at x_0 and which has a Taylor series with a positive radius of convergence.

Review: Power, Frobenius, and Asymptotic Series

Power Series

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Frobenius Series

$$y = x^r \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

Asympotic Series

$$w(x) = \sum_{n=0}^{\infty} a_n (x^{\alpha})^n$$

α may be non-integers.

leading behavior

$\sim \exp(S(x))$

$$y = \boxed{c_1 x^{\frac{3}{4}} e^{\frac{2}{\sqrt{x}}}} \sum_{n=0}^{\infty} a_n (x)^{n/2}$$

leading behavior

asymptotic series

Asympotic Series

- The formulas which express the local behavior of a solution near irregular singular points are generalizations of Frobenius series in much the same way that Frobenius series are generalizations of Taylor series.

(III) Irregular Singular Points

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)} + \cdots p_1 y^{(1)} + p_0 y = 0, \qquad (3.1.1)$$

Classification of Singular Points

- ✓ $y'' + p(x)y' + q(x)y = 0$
- ✓ ordinary points: p and q are analytic
- ✓ regular singular points: xp and x^2q are analytic
- ✓ irregular singular points: none of the above

Example 3 *Regular singular points.*

(*a*) $(x-1)y''' = y$ has a regular singular point at 1.
(*b*) $x^2y'' + xy' = y$ has a regular singular point at 0.
(*c*) $x^3y' = (x+1)y$ does *not* have a regular singular point at 0.

c) Irregular singular point

Failure of Frobenius Method at a Irregular Singular Point

Example 2 *Irregular singular point at which there are no solutions of Frobenius form.* What happens if we try to expand a solution of the differential equation

$$x^3 y'' = y \tag{3.4.1}$$

in a Frobenius series about the irregular singular point at 0? If a solution of Frobenius form $y = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$ with $a_0 \neq 0$ exists, then $\sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1)a_n x^{n+\alpha+1} = \sum_{n=0}^{\infty} a_n x^{n+\alpha}$. Equating coefficients of x^α gives $a_0 = 0$, which is a contradiction. Therefore, no solution of Frobenius form exists about $x = 0$.

- No solution of Frobenius form exists at about $x = 0$.

We apply the method of dominant balance to obtaining an approximate solution, including the leading behavior of the series, which is the first term in such a series, and the controlling factor, which represents the most rapidly changing component of the leading behavior in the limit $x \to x_o$ as

An Introduction to Asymptotics

(I) We must introduce two new symbols which express the relative behavior of two functions. The notation

$$f(x) \ll g(x), \qquad x \to x_0,$$

which is read "$f(x)$ is much smaller than $g(x)$ as x tends to x_0," means

much smaller than $\boxed{\lim_{x\to x_0} f(x)/g(x) = 0.}$

(II) Second, the notation

$$f(x) \sim g(x), \qquad x \to x_0,$$

which is read "$f(x)$ is asymptotic to $g(x)$ as x tends to x_0," means that the relative error between f and g goes to zero as $x \to x_0$:

$$f(x) - g(x) \ll g(x), \qquad x \to x_0,$$

or, equivalently,

asymptotic to $\boxed{\lim_{x\to x_0} f(x)/g(x) = 1.}$

Note that if $f(x) \sim g(x)$ $(x \to x_0)$ then $g(x) \sim f(x)$ $(x \to x_0)$.

Method of Dominant Balance

The method of dominant balance is used to identify those terms in an equation that may be neglected in an asymptotic limit, (i.e., S''). The technique consist of three steps:

1. We drop all terms that appear small and replace the exact equation by an asymptotic relation.
2. We replace the asymptotic relation with an equation by exchanging the ~ sign for an = sign and solve the resulting equation exactly (the solution to this equation automatically satisfies the asymptotic relation although it is certainly not the only function that does so).
3. We check that the solution we have obtained is consistent with the approximation made in (1). If it is consistent, we must still show that the equation for the function obtained by factoring off the dominant balance solution from the exact solution itself has a solution that varies less rapidly than the dominant balance solution. When this happens, we conclude that the controlling factor (and not the leading behavior, i.e., the first term) obtained from the dominant balance relation is the same as that of the exact solution.

``stoppage criteria'' : The leading behavior of y(x) is determined by just those contributions to S(x) that do not vanish as x approaches the irregular singularity.

Method of Dominant Balance: An Illustration

$$y'' + p(x)y' + q(x)y = 0$$

$\boxed{y = e^{S(x)}}$ $\quad y' = S'e^{S(x)}$ $\quad y'' = S''e^{S(x)} + (S')^2 e^{S(x)}$

$$S''e^{S(x)} + (S')^2 e^{S(x)} + pS'e^{S(x)} + qe^{S(x)} = 0$$

$S'' + (S')^2 + pS' + q = 0$ divide by $e^{s(x)}$

$\boxed{S'' << (S')^2, \quad \text{as } x \to x_0}$

1. drop (all) terms that are small

$\boxed{(S')^2 \sim -pS' - q, \text{ as } x \to x_0}$

2. replace "=" by "~" and solve the system (note that don't have "0" on the RHS)

Asymptotic differential equations

3. check whether the solution is consistent with the approximation, i.e., whether $S'' << (S')^2$ is valid.

Example

Example 2 *Irregular singular point at which there are no solutions of Frobenius form.* What happens if we try to expand a solution of the differential equation

$$x^3 y'' = y \tag{3.4.1}$$

$$y'' - \frac{y}{x^3} = 0$$

$$y'' + p(x)y' + q(x)y = 0$$

$$\boxed{(S')^2 \sim -pS' - q, \text{ as } x \to x_0}$$

$$\boxed{y = e^{S(x)}}$$

Asymptotic differential EQ

$$(S')^2 \sim$$

Example: Controlling Factor

Example 2 *Irregular singular point at which there are no solutions of Frobenius form.* What happens if we try to expand a solution of the differential equation

$$x^3 y'' = y \tag{3.4.1}$$

$$y'' - \frac{y}{x^3} = 0$$

$$y'' + p(x)y' + q(x)y = 0$$

$$\boxed{(S')^2 \sim -pS' - q, \text{ as } x \to x_0}$$

$$\boxed{y = e^{S(x)}}$$

Asymptotic differential EQ

$$(S')^2 \sim \frac{1}{x^3} \qquad S' \sim \pm x^{-3/2}$$

$$S \sim \mp 2x^{-1/2} + C(x) \qquad C(x) << 2x^{-1/2}$$ (to be verified)

$$\boxed{y = e^{2x^{-1/2}}} \quad \text{or} \quad \boxed{y = e^{-2x^{-1/2}}}$$

Leading Behavior

$$y'' - \frac{y}{x^3} = 0$$

$$y'' + p(x)y' + q(x)y = 0$$

$$\boxed{y = e^{S(x)}}$$

$$S = 2x^{-1/2} + C(x) \qquad C(x) << 2x^{-1/2}$$

$$S'' + (S')^2 + pS' + q = 0$$
3.4.8

$$S' = -x^{-\frac{3}{2}} + C'(x)$$

$$S'' = \frac{3}{2}x^{-\frac{5}{2}} + C''(x)$$

$$\left(\frac{3}{2}x^{-\frac{5}{2}} + C''(x)\right) + \left(-x^{-\frac{3}{2}} + C'(x)\right)^2 - \frac{1}{x^3} = 0$$

$$\left(\frac{3}{2}x^{-\frac{5}{2}} + C''(x)\right) + (C'(x))^2 - 2x^{-\frac{3}{2}}C' + \frac{1}{x^3} - \frac{1}{x^3} = 0$$

Leading Behavior

$$\boxed{\frac{3}{2}x^{-\frac{5}{2}}} + C''(x)\Big) + (C'(x))^2 \boxed{- 2x^{-\frac{3}{2}}C'} + \frac{1}{x^3} - \frac{1}{x^3} = 0$$

$C(x) \ll 2x^{-1/2}$ (magnitude)

$C' \ll -x^{-3/2}$ $\quad C'C' \ll -x^{-3/2}C'$

$$C'' \ll \frac{3}{2}x^{-5/2}$$

$$\frac{3}{2}x^{-\frac{5}{2}} \sim 2x^{-\frac{3}{2}}C'$$

$$C' \sim \frac{3}{4}x^{-1}$$

$$C \sim \frac{3}{4}\ln(x)$$

Double check: $C'' = -\frac{3}{4}x^{-3/2} \ll \frac{3}{2}x^{-5/2}$

Leading Behavior

$$y'' - \frac{y}{x^3} = 0$$

$$S'' + (S')^2 + pS' + q = 0$$
3.4.8

$$S = 2x^{-1/2} + \frac{3}{4}\ln(x) + D(x) \quad D(x) \ll \frac{3}{4}\ln(x)$$ (magnitude)

1. compute S' and S''
2. plug into Eq. 3.4.8
3. solve for $D(x)$

$$D(x) \sim d + \frac{-3}{16}x^{1/2} = d + \delta(x)$$

``stoppage criteria'' : The leading behavior of y(x) is determined by just those contributions to S(x) that do not vanish as x approaches the irregular singularity.

$$\lim_{x\to 0}\delta(x) = \lim_{x\to 0}\frac{-3}{16}x^{1/2} = 0$$ (stoppage criteria, "zero contribution")

$$S = 2x^{-1/2} + \frac{3}{4}\ln(x) + d \quad = 2x^{-1/2} + \ln(x)^{3/4} + d$$

$$y \sim \exp(S(x)) = \exp\left(2x^{-\frac{1}{2}} + \ln(x)^{3/4} + d\right) = c_1 x^{3/4} e^{2/\sqrt{x}}$$

Leading Behavior

$$y'' - \frac{y}{x^3} = 0$$

$$S = 2x^{-1/2} + \frac{3}{4}\ln(x) + d$$

$$\boxed{y_1 \sim c_1 x^{3/4} e^{2/\sqrt{x}}} \qquad c_1 = \exp(d)$$

controlling factor

leading behavior (three terms, non-zero contributions)

$$\boxed{y_2 \sim c_2 x^{3/4} e^{-2/\sqrt{x}}}$$

Leading Behavior vs. Controlling Factor

- We will refer to the first term in a series as the leading behavior of the series.
- The leading behavior is determined by those contributions to $S(x)$ that do not vanish as x approaches the irregular singularity.
- We will also refer to the most rapidly changing component of the leading behavior in the limit $x \to x_o$ as the controlling factor.

The following represent the first terms of infinite series…

y_1 $$y(x) \sim c_1 x^{3/4} \boxed{e^{2x^{-1/2}}}, \qquad x \to 0+; \qquad \boxed{(3.4.4a)}$$

(discussed below)

the other solution has the behavior

y_2 $$y(x) \sim \boxed{c_2 x^{3/4} \boxed{e^{-2x^{-1/2}}}}, \qquad x \to 0+. \qquad (3.4.4b)$$

leading behavior (three terms)

controlling factor (exponential)

- Observe that these behaviors all involve exponentials of functions which become singular at the irregular singularity of the differential equation.
- Thus, these two functions have essential singularities at x = 0.

Review: Poles and Essential Singularity

Poles. Essential Singularities

The function

poles

$$f(z) = \frac{1}{z(z-2)^5} + \frac{3}{(z-2)^2}$$

has a simple pole at $z = 0$ and a pole of fifth order at $z = 2$. Examples of functions having an isolated essential singularity at $z = 0$ are

essential singularity

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots$$

Correction to the Leading Behavior

$$\boxed{y'' - \frac{y}{x^3} = 0 \quad (3.4.1)}$$

$$y_1 \sim c_1 (x)^{3/4} e^{2/\sqrt{x}}$$

$$\boxed{y = c_1 x^{\frac{3}{4}} e^{\frac{2}{\sqrt{x}}} (w(x))}$$

$$\boxed{y = c_1 x^{\frac{3}{4}} e^{\frac{2}{\sqrt{x}}} \sum_{n=0}^{\infty} a_n (x)^{n/2}}$$

Plugging into (3.4.1), we have

$$w'' + \left(\frac{3}{2x} - \frac{2}{x^{3/2}}\right) w' - \frac{3}{16x^2} w = 0$$

(very tedious)

Let $w(x) = 1 + \epsilon(x)$

$$w(x) = 1 - \frac{3}{16} \mathrm{x}^{\frac{1}{2}} + \cdots$$

$$\boxed{w(x) = \sum_{n=0}^{\infty} a_n (x^{\alpha})^n} \qquad \alpha = \frac{1}{2}$$

(asymptotic series)

Correction to the Leading Behavior

$$y'' - \frac{y}{x^3} = 0 \quad (3.4.1)$$

$$y = c_1 x^{\frac{3}{4}} e^{\frac{2}{\sqrt{x}}} \sum_{n=0}^{\infty} a_n (x)^{n/2} \qquad a_0 = 1$$

Equating coefficients of powers of $x^{1/2}$

$$a_{n+1} = \frac{(2n-1)(2n+3)}{16(n+1)} a_n \qquad a_n = -\frac{\Gamma\left(n-\frac{1}{2}\right)\Gamma(\mathrm{n}+\frac{3}{2})}{\pi 4^n n!}$$

$$y = -c_1 x^{\frac{3}{4}} e^{\frac{2}{\sqrt{x}}} \sum_{n=0}^{\infty} \frac{\Gamma\left(n-\frac{1}{2}\right)\Gamma(\mathrm{n}+\frac{3}{2})}{\pi 4^n n!} (x)^{n/2}$$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{16(n+1)}{(2n-1)(2n+3)}\right| = 0$$

Procedures for Obtaining Asymptotic Solutions

$$y = \boxed{c_1 x^{\frac{3}{4}} e^{\frac{2}{\sqrt{x}}}} \boxed{\sum_{n=0}^{\infty} a_n (x)^{n/2}}$$

leading behavior   asymptotic series

- First, by means of the exponential substitution $y = \exp(S(x))$ we determined the leading behavior of y(x) as $x \to x_o$.

$$S = 2x^{-1/2} + \frac{3}{4}\ln(x) + d \qquad y_{leading} \sim c_1 x^{3/4} e^{2/\sqrt{x}}$$

leading behavior

- Next, we refined this approximation to y(x) by
  (1) peeling or factoring off the leading behavior and

$$y = y_{leading} * w(x), \qquad w(x) = 1 + \epsilon(x)$$

  (2) expanding what remains as a series of fractional powers of $(x - x_0)$

$$\boxed{w(x) = \sum_{n=0}^{\infty} a_n (x^{\alpha})^n}$$

asymptotic series