Homework 2
Abstract Algebra
Math 320
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Section 1.2 Problem 11a: If $n \in \mathbb{Z}$, what are the possible values of $(n, n+2)$.

Solution:
Let $n \in \mathbb{Z}$

$$\begin{aligned}
&n + 2 = n(1) + 2 && (1)\\
&\text{By the Euclidean Algorithm:}(n+2, n) = (n, 2) && (2)\\
&\text{By divisibility rules, the only divisors of 2 is } \pm 1, \pm 2 && (3)\\
&\text{Also by divisibility rules, the only divisors of n have to be } \leq |n| && (4)\\
&\text{Thus the common divisors would be } \pm 1, \pm 2 && (5)\\
&\text{The greatest would be their positive counterparts: } \mathbf{1, 2} && (6)\\
& && (7)
\end{aligned}$$

Section 1.2 Problem 15c: Use the Euclidean Algorithm to find (1003,456)

Solution:

$$\begin{aligned}
1003 &= 456(2) + 91 && (8)\\
456 &= 91(5) + 1 && (9)\\
91 &= 1(91) && (10)
\end{aligned}$$

GCD $= (1003, 456) = \mathbf{1}$

Section 1.2 Problem 15j: Use the method described in parts (f)-(i) to express the GCD in part (c) as a linear combination of 1003 and 456.

Solution:
Let $u, v \in \mathbb{Z}$
Show : $1 = 1003u + 456v$

$$\begin{aligned}
1 &= 456 - 91(5) && (11)\\
&= 456 - (1003 - 456(2))(5) && (12)\\
&= 1003(-5) + (456)(11) && (13)
\end{aligned}$$

(1003,456) can be written as a **Linear Combination when $\boldsymbol{u = -5, v = 11}$**

Section 1.2 Problem 17: Suppose $(a, b) = 1$. If $a|c$ and $b|c$, prove that $ab|c$. [Hint: $c = bt$ (why?), so $a|bt$. Use Theorem 1.4.]
Theorem 1.4 : If $a|bc$ and $(a, b) = 1$, then $a|c$.

Solution:
Suppose $(a, b) = 1$. Let $a|c$ and $b|c$ for some $a, b, c, r, t \in \mathbb{Z}$

$$\begin{gathered} c = br \\ \text{so } a|br \\ \text{by Theorem 1.4, } a|r \\ \text{so } r = at \\ \text{so } c = b(at) \\ c = ab(t) \\ \text{Thus } \boldsymbol{ab|c} \end{gathered}$$

Section 1.2 Problem 19: If $a|(b + c)$ and $(b, c) = 1$, prove that $(a, b) = 1 = (a, c)$.

Solution:
Let $a|(b + c)$ and $(b, c) = 1$ for some $a, b, c, r \in \mathbb{Z}$

$$b + c = ar \tag{14}$$
$$c = ar - b \tag{15}$$
$$bu + cv = 1 \tag{16}$$
$$bu + (ar - b)v = 1 \tag{17}$$
$$bu + arv - bv = 1 \tag{18}$$
$$b(u - v) + a(rv) = 1 \tag{19}$$

$$b + c = ar \tag{20}$$
$$b = ar - c \tag{21}$$
$$bu + cv = 1 \tag{22}$$
$$(ar - c)u + cv = 1 \tag{23}$$
$$aru - cu + cv = 1 \tag{24}$$
$$a(ru) + c(v - u) = 1 \tag{25}$$

Thus $\boldsymbol{(a, b) = 1 = (a, c)}$