MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #19

Chapter 5

Higher-Dimensional Linear Algebra

Instructor: Dr. Bo-Wen Shen*

Department of Mathematics and Statistics

San Diego State University

Section 5.5 Repeated Eigenvalues

5.5: Repeated Eigenvalues: 3×3 & 4×4

In this section we describe the canonical forms that arise when a matrix has repeated eigenvalues. Rather than spending an inordinate amount of time developing the general theory in this case, we will give the details only for 3×3 and 4×4 matrices with repeated eigenvalues. More general cases are relegated to the exercises of this chapter.

We justify this omission in the next section where we show that the "typical" matrix has distinct eigenvalues, and thus can be handled as in the previous section. (If you happen to meet a random matrix while walking down the street, the chances are very good that this matrix will have distinct eigenvalues!)

5.5: Repeated Eigenvalues: General Case

The most general result regarding matrices with repeated eigenvalues is given by the following proposition.

Proposition. *Let A be an $n \times n$ matrix. Then there is a change of coordinates T for which*

Find T such that we have

$$T^{-1}AT = \begin{pmatrix} B_1 & & \\ & \ddots & \\ & & B_k \end{pmatrix},$$

where each of the B_js is a square matrix (and all other entries are zero) of one of the following forms

B_j is a square matrix, as one of the following forms:

$$\text{(i)} \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix} \quad \text{(ii)} \begin{pmatrix} C_2 & I_2 & & & \\ & C_2 & I_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & I_2 \\ & & & & C_2 \end{pmatrix},$$

real eigenvalues complex eigenvalues

The most general result regarding matrices with repeated eigenvalues is given by the following proposition.

$$\text{(ii)} \quad \begin{pmatrix} C_2 & I_2 & & & \\ & C_2 & I_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & I_2 \\ & & & & C_2 \end{pmatrix},$$

coupled

where

$$C_2 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and where $\alpha, \beta, \lambda \in \mathbb{R}$ *with* $\beta \neq 0$. *The special cases where* $B_j = (\lambda)$ *or*

5.5: Special Cases: Why 4D?

TBD

The special cases where $B_j = (\lambda)$ *or*

$$B_j = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

are, of course, allowed. □

real repeated eigenvalue, uncoupled, 2D

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix}$$

complex repeated eigenvalue, uncoupled, 4D

$$B_1 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \quad \text{2D}$$

$$B_2 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \quad \text{2D}$$

Section 5.5: Repeated Eigenvalues in R^3

- We first consider the case of R^3. If A has repeated eigenvalues in R^3, then all eigenvalues must be real.

Proposition. *Suppose A is a 3×3 matrix for which λ is the only eigenvalue. Then we may find a change of coordinates T such that $T^{-1}AT$ assumes one of the following three forms:*

$$\text{(i)} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad \text{(ii)} \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad \text{(iii)} \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

(i) (uncoupled)

$$(A - \lambda I)V = 0$$

Dim K=3
Dim R=0

(ii)

$$(A - \lambda I)V_1 = 0$$

$$(A - \lambda I)V_2 = V_1$$

Dim K=2, V_1 & V_3
Dim R=1

(iii)

$$(A - \lambda I)V_1 = 0$$

$$(A - \lambda I)V_2 = V_1$$

$$(A - \lambda I)^2 V_3 = V_1$$

Dim K=1
Dim R=2

Type (i): Repeated Eigenvalues in R^3

$$\text{(i)} \quad \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

(uncoupled)

- Let K be the kernel of $A - \lambda I$.
- Any vector in K is an eigenvector of A.
- $\mathrm{Dim}\, K = 3$, since $(A - \lambda I) = 0$ and $(A - \lambda I)V = 0$ for any $V \in R^3$.

three "regular" (not generalized) independent eigenvectors.

Alternatively, we can perform the following analysis:

$$A - \lambda I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

zero LI column vectors ➔ Dim (range) = 0

➔ Dim (kernel) = 3

Type (ii): Repeated Eigenvalues in R^3

$$\text{(ii)} \quad \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$(A - \lambda I)V_1 = 0$$

$$(A - \lambda I)V_2 = V_1$$

Dim K=2, V_1 & V_3 two "regular" eigenvector2

Dim R=1

We may first start from the following:

$$A - \lambda I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

One LI column vector ➔ Dim (range) = 1

➔ Dim (kernel) = 2

Type (iii): Repeated Eigenvalues in R^3

$$\text{(iii)} \quad \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

$$(A-\lambda I)V_1 = 0$$

$$(A-\lambda I)V_2 = V_1$$

$$(A-\lambda I)^2 V_3 = V_1$$

Dim K=1
Dim R=2

one "regular" eigenvector

We start from the following:

$$A - \lambda I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Two LI column vector ➔ Dim (range) = 2

➔ Dim (kernel) = 1

5.5: Repeated Eigenvalues in R^3

Example. Suppose

Type III

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & -1 & 2 \end{pmatrix}.$$

$$\text{(iii)} \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Example. Now suppose

Type II

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ -1 & 1 & 2 \end{pmatrix}.$$

$$\text{(ii)} \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

Example 1: Repeated Eigenvalues in R^3

Example. Suppose

Type III

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & -1 & 2 \end{pmatrix}.$$

$$\boxed{A - \lambda I = \begin{pmatrix} 2-\lambda & 0 & -1 \\ 0 & 2-\lambda & 1 \\ -1 & -1 & 2-\lambda \end{pmatrix}}$$

Let $|A - \lambda I| = 0 \Rightarrow$

$$\begin{vmatrix} 2-\lambda & 0 & -1 \\ 0 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} + (-1)\begin{vmatrix} 0 & 2-\lambda \\ -1 & -1 \end{vmatrix} = 0$$

$$= (2-\lambda)\big((2-\lambda)^2 + 1\big) + (-1)(2-\lambda) \;= (2-\lambda)^3 = 0 \qquad \lambda = 2$$

Example 1: Repeated Eigenvalues in R^3

$$\boxed{AV = \lambda V}$$

$$\begin{aligned} 2x - z &= \lambda x \\ 2y + z &= \lambda y \\ -x - y + 2z &= \lambda z \end{aligned}$$

$\lambda = 2$

$$\begin{aligned} 2x - z &= 2x \\ 2y + z &= 2y \\ -x - y + 2z &= 2z \end{aligned} \qquad \begin{aligned} z &= 0 \\ z &= 0 \\ x + y &= 0 \end{aligned}$$

$$\boxed{V_1 = \begin{pmatrix} x_0 \\ -x_0 \\ 0 \end{pmatrix} = x_0 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}$$

Solve the following for V_3

$$\boxed{(A - \lambda I)^2 V_3 = V_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}$$

Compute the LHS:

$$A - \lambda I = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} \qquad (A - \lambda I)^2 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Example 1: Repeated Eigenvalues in R^3

Solve the following for V_3

$$(A-\lambda I)^2 V_3 = V_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$(A-\lambda I)^2 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let $V_3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{array}{c} x+y=1 \\ -x-y=-1 \\ 0=0 \end{array}$$

$$V_3 = \begin{pmatrix} x=1 \\ y=1-x \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Compute the following to obtain V_2

$$\boxed{(A-\lambda I)V_3 = V_2}$$

$$V_2 = (A-\lambda I)V_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

Example 1: Construct a Linear Map

$$\begin{array}{cccc} & V_1 & V_2 & V_3 \\ T = & \multicolumn{3}{c}{\begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}} \end{array}$$

$$\boxed{T^{-1}AT = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}}$$

$$\text{(iii)} \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}. \qquad \lambda = 2$$

An Analysis of Column Vectors

$$\boxed{A - \lambda I = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}}$$

Two LI column vector ➔ Dim (range) = 2

➔ Dim (kernel) = 1

➔ One “regular” eigenvector

Example 2: Repeated Eigenvalues in R^3

Example. Now suppose

Type II

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ -1 & 1 & 2 \end{pmatrix}.$$

$$\boxed{A - \lambda I = \begin{pmatrix} 1-\lambda & 1 & 0 \\ -1 & 3-\lambda & 0 \\ -1 & 1 & 2-\lambda \end{pmatrix}}$$

Let $|A - \lambda I| = 0 \Longrightarrow$
$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ -1 & 3-\lambda & 0 \\ -1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} = (\lambda^2 - 4\lambda + 4)\,(2-\lambda) = (2-\lambda)^3 = 0 \qquad \lambda = 2$$

An Analysis of Column Vectors

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 1 & 0 \\ -1 & 3-\lambda & 0 \\ -1 & 1 & 2-\lambda \end{pmatrix}$$

$\lambda = 2$

$$\boxed{A - \lambda I = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}}$$

One LI column vector ➔ Dim (range) = 1

➔ Dim (kernel) = 2

➔ Two "regular" eigenvectors

Example 2: Repeated Eigenvalues in R^3

Example. Now suppose

Type II

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ -1 & 1 & 2 \end{pmatrix}.$$

$$\boxed{A - \lambda I = \begin{pmatrix} 1-\lambda & 1 & 0 \\ -1 & 3-\lambda & 0 \\ -1 & 1 & 2-\lambda \end{pmatrix}}$$

$\lambda = 2$

$$\boxed{A - \lambda I = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}}$$

$$A - \lambda I = 0$$

$$\begin{aligned} -x + y &= 0 \\ -x + y &= 0 \\ -x + y &= 0 \end{aligned}$$

$$V = \begin{pmatrix} x \\ y = x \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

An eigenvector is a linear combination of two LI vectors.

➔ Dim (kernel) = 2 ➔ 2 "regular" eigenvectors

Fine a Generalized Eigenvector for dim (kernel)=2

- We have two regular eigenvectors, and, thus, one generalized eigenvector, V_2
- To compute V_2, we solve the following:

$\boxed{(A-\lambda I)V_2 = V_j}$ here, V is from $AV_j = \lambda V_j$

Thus, we have

$\boxed{(A-\lambda I)^2 V_2 = 0}$ ← Solve this with reasons provided below

$$Since\ (A-\lambda I)^2 = 0, we\ can\ have\ any\ V_2$$

$$we\ choose\ V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Verify:

$$V_1 = (A-\lambda I)V_2 = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$\rightarrow V = \begin{pmatrix} x \\ x \\ z \end{pmatrix}$,

$\mathrm{x} = -1 \ \& \ \mathrm{z} = -1$

Find a Linear Map

$$V_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad V_1 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \qquad V = x\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{array}{c} \quad\; V_1 \quad V_2 \quad V_2 \\ T = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}. \end{array}$$

Matlab Code for $T^{-1}AT$

```
>> T=[-1 1 0; -1 0 0; -1 0 1]

T =

    -1     1     0
    -1     0     0
    -1     0     1

>> Tinv=inv(T)

Tinv =

     0    -1     0
     1    -1     0
     0    -1     1

>> Tinv*A*T

ans =

     2     1     0
     0     2     0
     0     0     2
```

$$T = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

$$\boxed{T^{-1}AT = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}}$$

Properties for dim (kernel)=2

For the case repeated eigenvalue, we have

$$\lambda = 2 \qquad \boxed{A - \lambda I = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}} \qquad V = \begin{pmatrix} x \\ y = x \\ z \end{pmatrix} = x \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{U_1} + z \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{U_2}$$

- If we define U_1 and U_2 as the above, both are eigenvectors with the same eigenvalues, e.g.,

$$(A - \lambda I)U_1 = 0 \qquad (A - \lambda I)U_2 = 0$$

- Note that the "original" eigenvector" V is a linear combination of U_1 and U_2 as the above. Is this property special? Let $W = xU_1 + yU_2$. Consider the following:

$$AW = A(xU_1 + yU_2) = (xAU_1 + yAU_2) = (x\lambda_1 U_1 + y\lambda_2 U_2)$$

$$= \lambda(xU_1 + yU_2) = \lambda W \quad \text{only when } \lambda_1 = \lambda_2 = \lambda$$

- When W is a linear combination of two eigenvectors U_1 and U_2 with the same eigenvalue, W is also an eigenvector.

Important Notes for the Cases with dim (kernel)=2

- We have two regular eigenvectors, and, thus, one generalized eigenvector, V_2

1. To compute V_2, we solve the following:

   $\boxed{(A-\lambda I)V_2 = V_j}$ here, V is from $AV_j = \lambda V_j$

2. Thus, we have

   $\boxed{(A-\lambda I)^2 V_2 = 0}$

   $Since\ (A-\lambda I)^2 = 0\ for\ this\ case, we\ can\ have\ any\ V_2$

Why we perform step (2) instead of step (1)?

- Since dim (kernel)=2, we have two regular eigenvectors whose linear combinations are also eigenvectors (as shown earlier).
- The above leads to dim (range)=1. It means when V_j is an eigenvector and $V_j = (A-\lambda I)V_2$, $V_j \in R^1$. Therefore, only a special combination of the two regular eigenvectors can have V_j that satisfies $(A-\lambda I)V_2 = V_j$. (we need to find a good V_j first, as illustrated below).

Pre-Select a V_1 (not work)

$$V = \begin{pmatrix} x \\ y = x \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$we\ choose\ V_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\ \&\ V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{(A - \lambda I)V_2 = V_1}$$

$$(A - \lambda I)V_2 = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = V_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -x + y &= 1 \\ -x + y &= 1 \\ -x + y &= 0 \end{aligned}$$

No Solution for V_2

➔ The above preselected V_1 is not in the Range

Another Section of V_1

$$V = \begin{pmatrix} x \\ y = x \\ z \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$we\ choose\ V_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\ \&\ V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\boxed{(A - \lambda I)V_2 = V_1}$$

$$(A - \lambda I)V_2 = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = V_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{aligned} -x + y &= 1 \\ -x + y &= 1 \\ -x + y &= 1 \end{aligned}$$

$$V_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x = -1 \\ y = 1 + x \\ z \end{pmatrix}$$

$$V_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

Construct a Linear Map T

$$V_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \ \& \ V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad V_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{T = \begin{array}{c} \begin{array}{ccc} V_1 & V_2 & V_3 \end{array} \\ \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array}}$$

$$\boxed{T^{-1}AT = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}}$$

$$\text{(ii)} \quad \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \qquad \lambda = 2$$

Matlab for $T^{-1}AT$ with the Previously Constructed T

```
>> A=[1 1 0; -1 3 0; -1 1 2]

A =

     1     1     0
    -1     3     0
    -1     1     2

>> T=[1 -1 0; 1 0 0; 1 0 1]

T =

     1    -1     0
     1     0     0
     1     0     1

>> Tinv=inv(T)

Tinv =

     0     1     0
    -1     1     0
     0    -1     1

>> Tinv*A*T

ans =

     2     1     0
     0     2     0
     0     0     2
```

A=[1 1 0; -1 3 0; -1 1 2]

T=[1 -1 0; 1 0 0; 1 0 1]

Tinv=inv(T)

Tinv*A*T

$$\text{(ii)} \quad \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$\boxed{T^{-1}AT = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}}$$

Same as the textbook illustration

Generalized Eigenvector

Definition 5.6.1
We say that $\mathbf{u} \neq \mathbf{0}$ is a **generalized eigenvector** of $\mathbf{A}$ associated to the eigenvalue λ if

$$(\mathbf{A} - \lambda \mathbf{I})^k \mathbf{u} = \mathbf{0} \tag{5.78}$$

for some integer $k > 0$. The index of the generalized eigenvector is the smallest k satisfying (5.78).

Here u may be V_2, V_3, … V_k etc.

Let V_1 be the eigenvector of A associated to the eigenvalue λ.
Namely, $(A - \lambda I)V_1 = 0$. (a notation used in the HSD)

Consider u to be V_2 and $(A - \lambda I)V_2 = V_1$. Therefore, we have $(A - \lambda I)^2 V_2 = (A - \lambda I)V_1 = 0$. V_2 is a generalized eigenvector.

Wirkus and Swift

Generalized Eigenvspace

TBD

For k > 1, we see that the original eigenvector (v) gives rise to a set of generalized eigenvectors (u_1, $u_2 \ldots u_k$):

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{u}_{k-1} = \mathbf{u}_{k-2},$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{u}_{k-2} = \mathbf{u}_{k-3}, \quad \cdots,$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{u}_3 = \mathbf{u}_2,$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{u}_2 = \mathbf{u}_1,$$

$$\boxed{(\mathbf{A} - \lambda\mathbf{I})\mathbf{u}_1 = \mathbf{v}.}$$

Wirkus and Swift

The set $\{\mathbf{v}, \mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{k-1}\}$ satisfying (5.79) is called a **chain** of generalized eigenvectors. The chain is determined entirely by the choice of $\mathbf{v}$, which is referred to as the **bottom of the chain**. For those that have read Section 5.3, we define

$$\tilde{E}_\lambda = \{v | (\mathbf{A} - \lambda\mathbf{I})^k \mathbf{v} = \mathbf{0} \text{ for some } k\} \qquad (5.80)$$

as the generalized eigenspace of λ. From (5.50), we note that $E_\lambda \subseteq \tilde{E}_\lambda$. If λ is an eigenvalue of $\mathbf{A}$ of multiplicity m, then $\tilde{E}_\lambda$ is a subspace of dimension m.

THEOREM 5.6.3

Consider the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ for the $k \times k$ matrix $\mathbf{A}$ in which (i) λ is an eigenvalue of multiplicity k with a single eigenvector $\mathbf{v}$ and (ii) $\{\mathbf{v}, \mathbf{u}_1, \cdots, \mathbf{u}_{k-1}\}$ is the corresponding chain of generalized eigenvectors. Set

$$
\begin{aligned}
\mathbf{x}_1 &= te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{u}_1 \\
\mathbf{x}_2 &= \frac{t^2}{2!}e^{\lambda t}\mathbf{v} + te^{\lambda t}\mathbf{u}_1 + e^{\lambda t}\mathbf{u}_2 \\
&\ \vdots \\
\mathbf{x}_{k-1} &= \frac{t^{k-1}}{(k-1)!}e^{\lambda t}\mathbf{v} + \frac{t^{k-2}}{(k-2)!}e^{\lambda t}\mathbf{u}_1 \cdots + te^{\lambda t}\mathbf{u}_{k-2} + e^{\lambda t}\mathbf{u}_{k-1}^{(k)}.
\end{aligned}
$$

Then $e^{\lambda t}\mathbf{v}, \mathbf{x}_1, \cdots, \mathbf{x}_{k-1}$ are linearly independent and the general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ can be written as

$$
\mathbf{x} = c_1 e^{\lambda t}\mathbf{v} + c_2\mathbf{x}_1 + c_3\mathbf{x}_2 + \cdots + c_k\mathbf{x}_{k-1}.
$$

Wirkus and Swift

Review: Repeated Eigenvalues in R^2 TBD

$$T^{-1}AT = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

$$(A - \lambda I)V_1 = 0$$

$$\boxed{(A - \lambda I)V_2 = V_1} \qquad \boxed{AV_2 = \lambda V_2 + V_1}$$

$$T = [V_1, V_2]$$

$$T^{-1}AT = T^{-1}A[V_1, V_2] = T^{-1}[AV_1, AV_2]$$

$$= T^{-1}[\lambda V_1, \lambda V_2 + V_1] = [\lambda T^{-1}V_1, \lambda T^{-1}V_2 + T^{-1}V_1]$$

$$= [\lambda E_1, \lambda E_2 + E_1] = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Review: Repeated Eigenvalues in R^2 TBD

$$x' = \lambda x + y$$
$$y' = \lambda y.$$

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

$$\alpha e^{\lambda t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta e^{\lambda t} \begin{pmatrix} t \\ 1 \end{pmatrix}.$$

Chap 6: A 3D System with Repeated Eigenvalues TBD

Example. Let

$$X' = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} X.$$

The only eigenvalue for this system is λ, and its only eigenvector is $(1,0,0)$. We

Altogether, we find

$$X(t) = c_1 e^{\lambda t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{\lambda t} \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} + c_3 e^{\lambda t} \begin{pmatrix} t^2/2 \\ t \\ 1 \end{pmatrix},$$

which is the general solution. Despite the presence of the polynomial terms

Example first: Repeated Eigenvalues (4x4)

Example. Let

$$A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

The characteristic equation, after a little computation, is

$$(\lambda^2 + 1)^2 = 0.$$

Hence A has eigenvalues $\pm i$, each repeated twice.

5.5: Repeated Eigenvalues (4x4)

$$A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & 1 \end{pmatrix} \qquad A - \lambda I = \begin{pmatrix} 1-\lambda & -1 & 0 & 1 \\ 2 & -1-\lambda & 1 & 0 \\ 0 & 0 & -1-\lambda & 2 \\ 0 & 0 & -1 & 1-\lambda \end{pmatrix}$$

$$|A - \lambda I| = 0 \qquad (1-\lambda)\begin{vmatrix} -1-\lambda & 1 & 0 \\ 0 & -1-\lambda & 2 \\ 0 & -1 & 1-\lambda \end{vmatrix} - 2\begin{vmatrix} -1 & 0 & 1 \\ 0 & -1-\lambda & 2 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$-(1-\lambda)(1+\lambda)\begin{vmatrix} -1-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix} + 2\begin{vmatrix} -1-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix} = 0$$

$$(2-(1-\lambda)(1+\lambda))\begin{vmatrix} -1-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix} = 0$$

$$(1+\lambda^2)(\lambda^2 - 1 + 2) = 0$$

$$(1+\lambda^2)^2 = 0$$

$\boxed{\lambda = \pm i\text{, each repeats twice}}$

Eigenvectors (4x4)

$$A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\lambda = \pm i$$

$$V = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$$

$$AV = \lambda V$$

$$\lambda = i$$

$$\begin{aligned} x - y + w &= ix \\ 2x - y + z &= iy \\ -z + 2w &= iz \\ -z + w &= iw \end{aligned}$$

$$\begin{aligned} -(1+i)z + 2w &= 0 \\ -z + (1-i)w &= 0 \end{aligned}$$

$$Det = 0, w \,\&\, z \text{ are } LD$$

$$w = \frac{1+i}{2} z$$

An Eigenvector

- We first analyze Eqs. (3) and (4) as follows:

$x - y + w = ix \quad (1)$

$2x - y + z = iy \quad (2)$

$-z + 2w = iz \quad (3) \qquad\qquad -(1+i)z + 2w = 0 \quad (3a)$

$-z + w = iw \quad (4) \qquad\qquad -z + (1-i)w = 0 \quad (4a)$

- From Eqs. (3a) and (4a), we have a 2nd order determinant:

$$\begin{vmatrix} -(1+i) & 2 \\ -1 & 1-i \end{vmatrix} = i^2 - 1 + 2 = 0 \qquad Det = 0, w \,\&\, z \text{ are } LD$$

- Thus, we can use either Eq. (3) or Eq. (4) to obtain a relation between z and w. From Eq. (3a), we have $w = \frac{1+i}{2} z$.

An Eigenvector

- We then analyze Eqs. (1) and (2), assuming known z and w:

$$\begin{aligned} x - y + w &= ix \quad (1) \\ 2x - y + z &= iy \quad (2) \\ -z + 2w &= iz \quad (3) \\ -z + w &= iw \quad (4) \end{aligned} \qquad \begin{aligned} (1-i)x - y &= -w \quad (1\text{a}) \\ 2x - (1+i)y &= -z \quad (2\text{a}) \end{aligned}$$

- From Eqs. (1a) and (2a), we have a 2nd order determinant:

$$\begin{vmatrix} (1-i) & -1 \\ 2 & -(1+i) \end{vmatrix} = i^2 - 1 + 2 = 0 \qquad Det = 0, x \& y \ are \ LD$$

- Thus, we can use either Eq. (1a) or Eq. (2a) to obtain a relation among x, y, z, and w.

An Eigenvector

Previously, we have obtained the following:

$$(1-i)x - y = -w \quad (1a)$$
$$2x - (1+i)y = -z \quad (2a)$$

$$w = \frac{1+i}{2}z.$$

- If $(z, w) = (0,0)$, Eqs. (1a) and (2a) contain infinitely many solutions because its determinant is zero.
- If $w = \frac{1+i}{2}z \neq 0$, Eqs. (1a) and (2a) become

$$(1-i)x - y = -(\frac{1+i}{2}z) \quad (1a) \qquad * (1+i)$$
$$2x - (1+i)y = -z \quad (2a)$$

$$(1-i^2)x - (1+i)y = -\frac{1}{2}(1+i)^2 \mathrm{z} \quad (1aa)$$
$$2x - (1+i)y = -z \quad (2a)$$

No solutions if $\mathrm{z} \neq 0$.
Thus, $\mathrm{z} = 0 = w$.

An Eigenvector

- Thus, we have $(z, w) = (0,0)$ and

$$(1-i)x - y = 0 \quad (1a)$$

$$2x - (1+i)y = 0 \quad (2a)$$

$$w = \frac{1+i}{2} z.$$

- From Eq. (1a), we have $y = (1-i)x$.

$$V = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = x \begin{pmatrix} 1 \\ 1-i \\ 0 \\ 0 \end{pmatrix}$$

A Generalized Eigenvector

$$V_1 = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 1-i \\ 0 \\ 0 \end{pmatrix} \quad \lambda = i \quad \boxed{A - \lambda I = \begin{pmatrix} 1-i & -1 & 0 & 1 \\ 2 & -1-i & 1 & 0 \\ 0 & 0 & -1-i & 2 \\ 0 & 0 & -1 & 1-i \end{pmatrix}}$$

A generalized eigenvector, V_2:

$$\boxed{(A-\lambda I)V_2 = V_1} \qquad V_2 = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$V_2 = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1-i \\ 1 \end{pmatrix}$$

Construct a Linear Map T

$$V_1 = \begin{pmatrix} 1 \\ 1-i \\ 0 \\ 0 \end{pmatrix} \qquad V_2 = \begin{pmatrix} 0 \\ 0 \\ 1-i \\ 1 \end{pmatrix}$$

$$W_1 = Re(V_1)\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad W_2 = Im(V_2)\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$W_3 = Re(V_2)\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad W_4 = Im(V_2)\begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

$$\boxed{T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}}$$

Compute $T^{-1}AT$

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

```
>> T=[1 0 0 0; 1 -1 0 0; 0 0 1 -1; 0 0 1 0]

T =

     1     0     0     0
     1    -1     0     0
     0     0     1    -1
     0     0     1     0

>> A=[1 -1 0 1; 2 -1 1 0; 0 0 -1 2; 0 0 -1 1]

A =

     1    -1     0     1
     2    -1     1     0
     0     0    -1     2
     0     0    -1     1

>> inv(T)*A*T

ans =

     0     1     1     0
    -1     0     0     1
     0     0     0     1
     0     0    -1     0
```

$$T^{-1}AT = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

5.5: Repeated Eigenvalues (4x4)

S7

- 2a: a pair of L.I. eigenvectors corresponding to $(\alpha + i\beta\ )$
  - V_1 (V_2) and $\overline{V_1}$ $(\ \overline{V_2})$ are eigenvectors
  - $W_1 = \mathrm{Re}(V_1) \ \Rightarrow AW_1 = \alpha W_1 - \beta W_2$
  - $W_2 = \mathrm{Im}(V_1) \Rightarrow AW_2 = \beta W_1 + \alpha W_2$
  - $W_3 = \mathrm{Re}(V_2) \ \Rightarrow AW_3 = \alpha W_3 - \beta W_4$
  - $W_4 = \mathrm{Im}(V_2) \Rightarrow AW_4 = \beta W_3 + \alpha W_4$

$$\begin{bmatrix} \alpha & \beta & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{bmatrix}$$

- 2b: only one eigenvector corresponding to $(\alpha + i\beta\ )$
  - V_1 and $\overline{V_1}$ are eigenvectors
  - $(A - \lambda I)V_2 = V_1$
  - $W_3 = \mathrm{Re}(V_2) \Rightarrow AW_3 = \alpha W_3 - \beta W_4 + W_1$
  - $W_4 = \mathrm{Im}(V_2) \Rightarrow AW_4 = \beta W_3 + \alpha W_4 + W_2$

$$\begin{bmatrix} \alpha & \beta & 1 & 0 \\ -\beta & \alpha & 0 & 1 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{bmatrix}$$

5.5: Repeated Eigenvalues (4x4)

Consider the 4x4 case:

1. Real repeated eigenvalues ➔ similar to the 3x3 case

2. Repeated complex eigenvalues*, i.e., only $(\alpha + i\beta\ )$ (and $\alpha - i\beta$)

- 2a: a pair of L.I. eigenvectors corresponding to $(\alpha + i\beta\ )$

“uncoupled”

  - V_1 and V_2 are eigenvectors corresponding to $(\alpha + i\beta\ )$
  - $\overline{V_1}$ and $\overline{V_2}$ are eigenvectors corresponding to $(\alpha - i\beta\ )$

- 2b: only one eigenvector corresponding to $(\alpha + i\beta\ )$

coupled

  - V_1 is the eigenvector corresponding to $(\alpha + i\beta\ )$
  - $\overline{V_1}$ is the eigenvector corresponding to $(\alpha - i\beta\ )$
  - $(A - \lambda I)V_2 = V_1$

- For the 4x4 case, we have two pairs of complex eigenvalues.
- For the case with repeated complex eigenvalues, it contains only one pair.

A Summary for a 4D System with Repeated Eigenvalue

Example. Let

$$A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

The characteristic equation, after a little computation, is

$$(\lambda^2 + 1)^2 = 0.$$

Hence A has eigenvalues $\pm i$, each repeated twice.

Next we solve the system $(A - iI)X = V_1$ to find $V_2 = (0, 0, 1 - i, 1)$. Then $\overline{V_2}$ solves the system $(A - iI)X = \overline{V_1}$. Finally, choose

$$W_1 = \left(V_1 + \overline{V_1}\right)/2 = \text{Re } V_1$$

$$W_2 = -i\left(V_1 - \overline{V_1}\right)/2 = \text{Im } V_1$$

$$W_3 = \left(V_2 + \overline{V_2}\right)/2 = \text{Re } V_2$$

$$W_4 = -i\left(V_2 - \overline{V_2}\right)/2 = \text{Im } V_2$$