
CHAPTER
SEVEN

PERTURBATION SERIES

You have erred perhaps in attempting to put colour and life
into each of your statements instead of confining yourself
to the task of placing upon record that severe reasoning
from cause to effect which is really the only notable feature
about the thing. You have degraded what should have been
a course of lectures into a series of tales.

—Sherlock Holmes, *The Adventure of the Copper Beeches*
Sir Arthur Conan Doyle

(E) 7.1 PERTURBATION THEORY

Perturbation theory is a large collection of iterative methods for obtaining approximate solutions to problems involving a small parameter ε . These methods are so powerful that sometimes it is actually advisable to introduce a parameter ε temporarily into a difficult problem having no small parameter, and then finally to set $\varepsilon = 1$ to recover the original problem. This apparently artificial conversion to a perturbation problem may be the only way to make progress.

The thematic approach of perturbation theory is to decompose a tough problem into an infinite number of relatively easy ones. Hence, perturbation theory is most useful when the first few steps reveal the important features of the solution and the remaining ones give small corrections.

Here is an elementary example to introduce the ideas of perturbation theory.

Example 1 Roots of a cubic polynomial. Let us find approximations to the roots of

$$x^3 - 4.001x + 0.002 = 0. \quad (7.1.1)$$

As it stands, this problem is not a perturbation problem because there is no small parameter ε . It may not be easy to convert a particular problem into a tractable perturbation problem, but in the present case the necessary trick is almost obvious. Instead of the single equation (7.1.1) we consider the *one-parameter family* of polynomial equations

$$x^3 - (4 + \varepsilon)x + 2\varepsilon = 0. \quad (7.1.2)$$

When $\varepsilon = 0.001$, the original equation (7.1.1) is reproduced.

It may seem a bit surprising at first, but it is easier to compute the approximate roots of the family of polynomials (7.1.2) than it is to solve just the one equation with $\varepsilon = 0.001$. The reason

for this is that if we consider the roots to be functions of ε , then we may further assume a perturbation series in powers of ε :

$$x(\varepsilon) = \sum_{n=0}^{\infty} a_n \varepsilon^n. \quad (7.1.3)$$

To obtain the first term in this series, we set $\varepsilon = 0$ in (7.1.2) and solve

$$x^3 - 4x = 0. \quad (7.1.4)$$

This expression is easy to factor and we obtain in *zeroth-order* perturbation theory $x(0) = a_0 = -2, 0, 2$.

A *second-order* perturbation approximation to the first of these roots consists of writing (7.1.3) as $x_1 = -2 + a_1 \varepsilon + a_2 \varepsilon^2 + O(\varepsilon^3)$ ($\varepsilon \rightarrow 0$), substituting this expression into (7.1.2), and neglecting powers of ε beyond ε^2 . The result is

$$(-8 + 8) + (12a_1 - 4a_1 + 2 + 2)\varepsilon + (12a_2 - a_1 - 6a_1^2 - 4a_2)\varepsilon^2 = O(\varepsilon^3), \quad \varepsilon \rightarrow 0. \quad (7.1.5)$$

It is at this step that we realize the power of generalizing the original problem to a family of problems (7.1.2) with variable ε . It is because ε is *variable* that we can conclude that the coefficient of each power of ε in (7.1.5) is *separately* equal to zero. This gives a sequence of equations for the expansion coefficients a_1, a_2, \dots :

$$\varepsilon^1: \quad 8a_1 + 4 = 0; \quad \varepsilon^2: \quad 8a_2 - a_1 - 6a_1^2 = 0;$$

and so on. The solutions to the equations are $a_1 = -\frac{1}{2}$, $a_2 = \frac{1}{8}$, \dots . Therefore, the perturbation expansion for the root x_1 is

$$x_1 = -2 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + \dots \quad (7.1.6)$$

If we now set $\varepsilon = 0.001$, we obtain x_1 from (7.1.6) accurate to better than one part in 10^9 . The same procedure gives

$$x_2 = 0 + \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + O(\varepsilon^3), \quad x_3 = 2 + 0 \cdot \varepsilon + 0 \cdot \varepsilon^2 + O(\varepsilon^3), \quad \varepsilon \rightarrow 0.$$

(Successive coefficients in the perturbation series for x_3 all vanish because $x_3 = 2$ is the exact solution for all ε .) All three perturbation series for the roots converge for $\varepsilon = 0.001$. Can you prove that they converge for $|\varepsilon| < 1$? (See Prob. 7.6.)

This example illustrates the three steps of perturbative analysis:

1. Convert the original problem into a perturbation problem by introducing the small parameter ε .
2. Assume an expression for the answer in the form of a perturbation series and compute the coefficients of that series.
3. Recover the answer to the original problem by summing the perturbation series for the appropriate value of ε .

Step (1) is sometimes ambiguous because there may be many ways to introduce an ε . However, it is preferable to introduce ε in such a way that the *zeroth-order* solution (the leading term in the perturbation series) is obtainable as a closed-form analytic expression. Perturbation problems generally take the form of a soluble equation [such as (7.1.4)] whose solution is altered slightly by a perturbing term [such as $(2 - x)\varepsilon$]. Of course, step (1) may be omitted when the original problem already has a small parameter if a perturbation series can be developed in powers of that parameter.

Step (2) is frequently a routine iterative procedure for determining successive coefficients in the perturbation series. A *zeroth-order* solution consists of finding the leading term in the perturbation series. In Example 1 this involves solving the *unperturbed problem*, the problem obtained by setting $\varepsilon = 0$ in the perturbation problem. A *first-order* solution consists of finding the first two terms in the perturbation series, and so on. In Example 1 each of the coefficients in the perturbation series is determined in terms of the previous coefficients by a simple linear equation, even though the original problem was a nonlinear (cubic) equation.

Generally it is the existence of a closed-form zeroth-order solution which ensures that the higher-order terms may also be determined as closed-form analytical expressions.

Step (3) may or may not be easy. If the perturbation series converges, its sum is the desired answer. If there are several ways to reduce a problem to a perturbation problem, one chooses the way that is the best compromise between difficulty of calculation of the perturbation series coefficients and rapidity of convergence of the series itself. However, many series converge so slowly that their utility is impaired. Also, we will shortly see that perturbation series are frequently divergent. This is not necessarily bad because many of these divergent perturbation series are asymptotic. In such cases, one obtains a good approximation to the answer when ε is very small by summing the first few terms according to the optimal truncation rule (see Sec. 3.5). When ε is not small, it may still be possible to obtain a good approximation to the answer from a slowly converging or divergent series using the summation methods discussed in Chap. 8.

Let us now apply these three rules of perturbation theory to a slightly more sophisticated example.

Example 2 *Approximate solution of an initial-value problem.* Consider the initial-value problem

$$y'' = f(x)y, \quad y(0) = 1, \quad y'(0) = 1, \quad (7.1.7)$$

where $f(x)$ is continuous. This problem has no closed-form solution except for very special choices for $f(x)$. Nevertheless, it can be solved perturbatively.

First, we introduce an ε in such a way that the unperturbed problem is solvable:

$$y'' = \varepsilon f(x)y, \quad y(0) = 1, \quad y'(0) = 1. \quad (7.1.8)$$

Second, we assume a perturbation expansion for $y(x)$ of the form

$$y(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n(x), \quad (7.1.9)$$

where $y_0(0) = 1$, $y'_0(0) = 1$, and $y_n(0) = 0$, $y'_n(0) = 0$ ($n \geq 1$).

The zeroth-order problem $y'' = 0$ is obtained by setting $\varepsilon = 0$, and the solution which satisfies the initial conditions is $y_0 = 1 + x$. The n th-order problem ($n \geq 1$) is obtained by substituting (7.1.9) into (7.1.8) and setting the coefficient of ε^n ($n \geq 1$) equal to 0. The result is

$$y''_n = y_{n-1} f(x), \quad y_n(0) = y'_n(0) = 0. \quad (7.1.10)$$

Observe that perturbation theory has replaced the intractable differential equation (7.1.7) with a sequence of inhomogeneous equations (7.1.10). In general, any inhomogeneous equation may be solved routinely by the method of variation of parameters whenever the solution of the associated homogeneous equation is known (Sec. 1.5). Here the homogeneous equation is

precisely the unperturbed equation. Thus, it is clear why it is so crucial that the unperturbed equation be soluble.

The solution to (7.1.10) is

$$y_n = \int_0^x dt \int_0^t ds f(s) y_{n-1}(s), \quad n \geq 1. \quad (7.1.11)$$

Equation (7.1.11) gives a simple iterative procedure for calculating successive terms in the perturbation series (7.1.9):

$$\begin{aligned} y(x) &= 1 + x + \varepsilon \int_0^x dt \int_0^t ds (1+s)f(s) \\ &\quad + \varepsilon^2 \int_0^x dt \int_0^t ds f(s) \int_0^s dv \int_0^v du (1+u)f(u) + \dots \end{aligned} \quad (7.1.12)$$

Third, we must sum this series. It is easy to show that when N is large, the N th term in this series is bounded in absolute value by $\varepsilon^N x^{2N} K^N (1 + |x|)/(2N)!$, where K is an upper bound for $|f(t)|$ in the interval $0 \leq |t| \leq |x|$. Thus, the series (7.1.12) is convergent for all x . We also conclude that if $x^2 K$ is small, then the perturbation series is rapidly convergent for $\varepsilon = 1$ and an accurate solution to the original problem may be achieved by taking only a few terms.

How do these perturbation methods for differential equations compare with the series methods that were introduced in Chap. 3? Suppose $f(x)$ in (7.1.7) has a convergent Taylor expansion about $x = 0$ of the form

$$f(x) = \sum_{n=0}^{\infty} f_n x^n. \quad (7.1.13)$$

Then another way to solve for $y(x)$ is to perform a local analysis of the differential equation near $x = 0$ by substituting the series solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n, \quad a_0 = a_1 = 1, \quad (7.1.14)$$

and computing the coefficients a_n . As shown in Chap. 3, the series in (7.1.14) is guaranteed to have a radius of convergence at least as large as that in (7.1.13).

By contrast, the perturbation series (7.1.9) converges for all finite values of x , and not just those inside the radius of convergence of $f(x)$. Moreover, the perturbation series converges even if $f(x)$ has no Taylor series expansion at all.

Example 3 *Comparison of Taylor and perturbation series.* The differential equation

$$y'' = -e^{-x} y, \quad y(0) = 1, y'(0) = 1, \quad (7.1.15)$$

may be solved in terms of Bessel functions as

$$y(x) = \frac{[Y_0(2) + Y'_0(2)]J_0(2e^{-x/2}) - [J_0(2) + J'_0(2)]Y_0(2e^{-x/2})}{J_0(2)Y_0(2) - J'_0(2)Y_0(2)}.$$

The local expansion (7.1.14) converges everywhere because e^{-x} has no finite singularities. Nevertheless, a fixed number of terms of the perturbation series (7.1.9) (see Prob. 7.11) gives a much

better approximation than the same number of terms of the Taylor series (7.1.14) if x is large and positive (see Fig. 7.1).

In addition, the perturbation methods of Example 2 are immediately applicable to problems where local analysis cannot be used. For example, an approximate solution of the formidable-looking nonlinear two-point boundary-value problem

$$y'' + y = \frac{\cos x}{3 + y^2}, \quad y(0) = y\left(\frac{\pi}{2}\right) = 2, \quad (7.1.16)$$

may be readily obtained using perturbation theory (see Prob. 7.14).

Thus, the ideas of perturbation theory apply equally well to problems requiring local or global analysis.

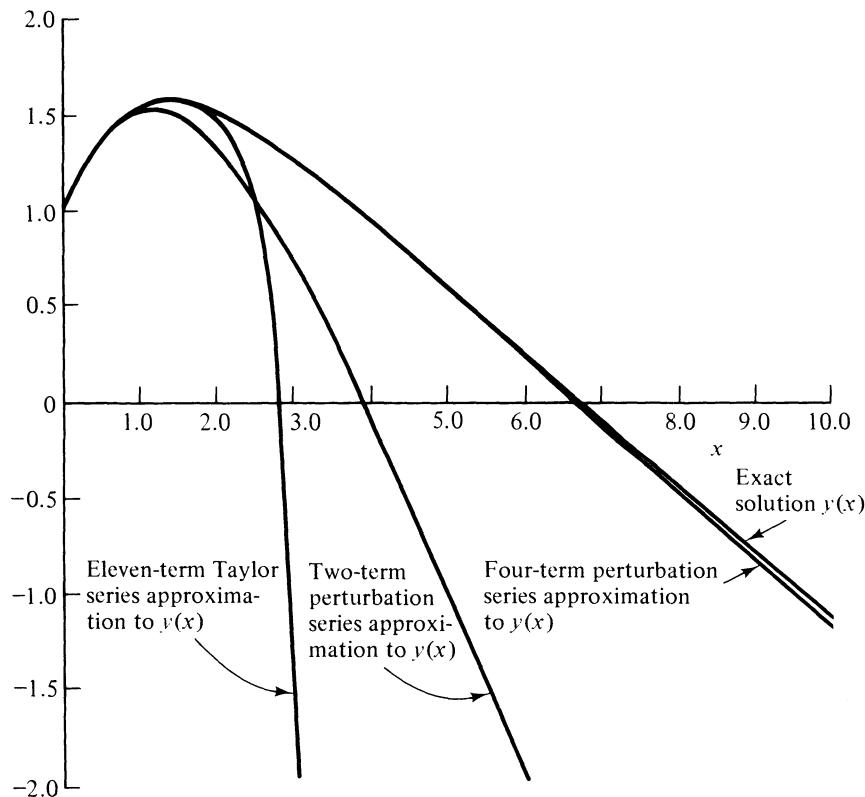


Figure 7.1 A comparison of Taylor series and perturbation series approximations to the solution of the initial-value problem $y'' = -e^{-x}y$ [$y(0) = 1$, $y'(0) = 1$] in (7.1.15). The exact solution to the problem is plotted. Also plotted are an 11-term Taylor series approximation of the form in (7.1.14) and 2- and 4-term perturbation series approximations of the form in (7.1.3) with $\varepsilon = 1$. The global perturbative approximation is clearly far superior to the local Taylor series.

(E) **7.2 REGULAR AND SINGULAR PERTURBATION THEORY**

The formal techniques of perturbation theory are a natural generalization of the ideas of local analysis of differential equations in Chap. 3. Local analysis involves approximating the solution to a differential equation near the point $x = a$ by developing a series solution about a in powers of a small parameter, either $x - a$ for finite a or $1/x$ for $a = \infty$. Once the leading behavior of the solution near $x = a$ (which we would now refer to as the zeroth-order solution!) is known, the remaining coefficients in the series can be computed recursively.

The strong analogy between local analysis of differential equations and formal perturbation theory may be used to classify perturbation problems. Recall that there are two different types of series solutions to differential equations. A series solution about an *ordinary* point of a differential equation is always a Taylor series having a nonvanishing radius of convergence. A series solution about a *singular* point does not have this form (except in rare cases). Instead, it may either be a convergent series not in Taylor series form (such as a Frobenius series) or it may be a divergent series. Series solutions about singular points often have the remarkable property of being meaningful near a singular point yet not existing at the singular point. [The Frobenius series for $K_0(x)$ does not exist at $x = 0$ and the asymptotic series for $\text{Bi}(x)$ does not exist at $x = \infty$.]

Perturbation series also occur in two varieties. We define a *regular* perturbation problem as one whose perturbation series is a power series in ε having a nonvanishing radius of convergence. A basic feature of all regular perturbation problems (which we will use to identify such problems) is that the exact solution for small but nonzero $|\varepsilon|$ smoothly approaches the unperturbed or zeroth-order solution as $\varepsilon \rightarrow 0$.

We define a *singular* perturbation problem as one whose perturbation series either does not take the form of a power series or, if it does, the power series has a vanishing radius of convergence. In singular perturbation theory there is sometimes no solution to the unperturbed problem (the exact solution as a function of ε may cease to exist when $\varepsilon = 0$); when a solution to the unperturbed problem does exist, its qualitative features are distinctly different from those of the exact solution for arbitrarily small but nonzero ε . In either case, the exact solution for $\varepsilon = 0$ is *fundamentally different in character* from the “neighboring” solutions obtained in the limit $\varepsilon \rightarrow 0$. If there is no such abrupt change in character, then we would have to classify the problem as a regular perturbation problem.

When dealing with a singular perturbation problem, one must take care to distinguish between the *zeroth-order* solution (the leading term in the perturbation series) and the solution of the unperturbed problem, since the latter may not even exist. There is no difference between these two in a regular perturbation theory, but in a singular perturbation theory the zeroth-order solution may depend on ε and may exist only for nonzero ε .

The examples of the previous section are all regular perturbation problems. Here are some examples of singular perturbation problems:

Example 1 Roots of a polynomial. How does one determine the approximate roots of

$$\varepsilon^2 x^6 - \varepsilon x^4 - x^3 + 8 = 0? \quad (7.2.1)$$

We may begin by setting $\varepsilon = 0$ to obtain the unperturbed problem $-x^3 + 8 = 0$, which is easily solved:

$$x = 2, 2\omega, 2\omega^2, \quad (7.2.2)$$

where $\omega = e^{2\pi i/3}$ is a complex root of unity. Note that the unperturbed equation has only three roots while the original equation has six roots. This abrupt change in the character of the solution, namely the disappearance of three roots when $\varepsilon = 0$, implies that (7.2.1) is a singular perturbation problem. Part of the exact solution ceases to exist when $\varepsilon = 0$.

The explanation for this behavior is that the three missing roots tend to ∞ as $\varepsilon \rightarrow 0$. Thus, for those roots it is no longer valid to neglect $\varepsilon^2 x^6 - \varepsilon x^4$ compared with $-x^3 + 8$ in the limit $\varepsilon \rightarrow 0$. Of course, for the three roots near 2, 2ω , and $2\omega^2$, the terms $\varepsilon^2 x^6$ and εx^4 are indeed small as $\varepsilon \rightarrow 0$ and we may assume a regular perturbation expansion for these roots of the form

$$x_k(\varepsilon) = 2e^{2\pi i k/3} + \sum_{n=1}^{\infty} a_{n,k} \varepsilon^n, \quad k = 1, 2, 3. \quad (7.2.3)$$

Substituting (7.2.3) into (7.2.1) and comparing powers of ε , as in Example 1 of Sec. 7.1, gives a sequence of equations which determine the coefficients $a_{n,k}$.

To track down the three missing roots we first estimate their orders of magnitude as $\varepsilon \rightarrow 0$. We do this by considering all possible dominant balances between pairs of terms in (7.2.1). There are four terms in (7.2.1) so there are six pairs to consider:

- (a) Suppose $\varepsilon^2 x^6 \sim \varepsilon x^4$ ($\varepsilon \rightarrow 0$) is the dominant balance. Then $x = O(\varepsilon^{-1/2})$ ($\varepsilon \rightarrow 0$). It follows that the terms $\varepsilon^2 x^6$ and εx^4 are both $O(\varepsilon^{-1})$. But $\varepsilon x^4 \ll x^3 = O(\varepsilon^{-3/2})$ as $\varepsilon \rightarrow 0$, so x^3 is the biggest term in the equation and is not balanced by any other term. Thus, the assumption that $\varepsilon^2 x^6$ and εx^4 are the dominant terms as $\varepsilon \rightarrow 0$ is inconsistent.
- (b) Suppose $\varepsilon x^4 \sim x^3$ as $\varepsilon \rightarrow 0$. Then $x = O(\varepsilon^{-1})$. It follows that $\varepsilon x^4 \sim x^3 = O(\varepsilon^{-3})$. But $x^3 \ll \varepsilon^2 x^6 = O(\varepsilon^{-4})$ as $\varepsilon \rightarrow 0$. Thus, $\varepsilon^2 x^6$ is the largest term in the equation. Hence, the original assumption is again inconsistent.
- (c) Suppose $\varepsilon^2 x^6 \sim 8$ so that $x = O(\varepsilon^{-1/3})$ ($\varepsilon \rightarrow 0$). Hence, $x^3 = O(\varepsilon^{-1})$ is the largest term, which is again inconsistent.
- (d) Suppose $\varepsilon x^4 \sim 8$ so that $x = O(\varepsilon^{-1/4})$ ($\varepsilon \rightarrow 0$). Then $x^3 = O(\varepsilon^{-3/4})$ is the biggest term, which is also inconsistent.
- (e) Suppose $x^3 \sim 8$. Then $x = O(1)$. This is a consistent assumption because the other two terms in the equation, $\varepsilon^2 x^6$ and εx^4 , are negligible compared with x^3 and 8, and we recover the three roots of the unperturbed equation $x = 2, 2\omega$, and $2\omega^2$.
- (f) Suppose $\varepsilon^2 x^6 \sim x^3$ ($\varepsilon \rightarrow 0$). Then $x = O(\varepsilon^{-2/3})$. This is consistent because $\varepsilon^2 x^6 \sim x^3 = O(\varepsilon^{-2})$ is bigger than $\varepsilon x^4 = O(\varepsilon^{-5/3})$ and $8 = O(1)$ as $\varepsilon \rightarrow 0$.

Thus, the magnitudes of the three missing roots are $O(\varepsilon^{-2/3})$ as $\varepsilon \rightarrow 0$. This result is a clue to the structure of the perturbation series for the missing roots. In particular, it suggests a *scale transformation* for the variable x :

$$x = \varepsilon^{-2/3} y. \quad (7.2.4)$$

Substituting (7.2.4) into (7.2.1) gives

$$y^6 - y^3 + 8\varepsilon^2 - \varepsilon^{1/3} y^4 = 0. \quad (7.2.5)$$

This is now a *regular* perturbation problem for y in the parameter $\varepsilon^{1/3}$ because the unperturbed problem $y^6 - y^3 = 0$ has six roots $y = 1, \omega, \omega^2, 0, 0, 0$. Now, no roots disappear in the limit $\varepsilon^{1/3} \rightarrow 0$.

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The perturbative corrections to these roots may be found by assuming a regular perturbation expansion in powers of $\varepsilon^{1/3}$ (it would not be possible to match powers in an expansion having only integral powers of ε):

$$y = \sum_{n=0}^{\infty} y_n (\varepsilon^{1/3})^n. \quad (7.2.6)$$

Having established that we are dealing with a singular perturbation problem, it is no surprise that the perturbation series for the roots x is not a series in integral powers of ε .

Nevertheless, when $y_0 = 0$ we find that $y_1 = 0$ and $y_2 = 2, 2\omega$, and $2\omega^2$. Thus, since the first two terms in this series vanish, $x = \varepsilon^{-2/3}y$ is not really $O(\varepsilon^{-2/3})$ but rather $O(1)$ and we have reproduced the three finite roots near $x = 2, 2\omega, 2\omega^2$. Moreover, only every third coefficient in (7.2.6), y_2, y_5, y_8, \dots , is nonvanishing, so we have also reproduced the regular perturbation series in (7.2.3)!

Example 2 *Appearance of a boundary layer.* The boundary-value problem

$$\varepsilon y'' - y' = 0, \quad y(0) = 0, y(1) = 1, \quad (7.2.7)$$

is a singular perturbation problem because the associated unperturbed problem

$$-y' = 0, \quad y(0) = 0, y(1) = 1, \quad (7.2.8)$$

has no solution. (The solution to this first-order differential equation, $y = \text{constant}$, cannot satisfy both boundary conditions.) The solution to (7.2.7) cannot have a regular perturbation expansion of the form $y = \sum_{n=0}^{\infty} y_n(x)\varepsilon^n$ because y_0 does not exist.

There is a close parallel between this example and the previous one. Here, the highest derivative is multiplied by ε and in the limit $\varepsilon \rightarrow 0$ the unperturbed solution loses its ability to satisfy the boundary conditions because a solution is lost. In the previous example the highest power of x is multiplied by ε and in the limit $\varepsilon \rightarrow 0$ some roots are lost.

The exact solution to (7.2.7) is easy to find:

$$y(x) = \frac{e^{x/\varepsilon} - 1}{e^{1/\varepsilon} - 1}. \quad (7.2.9)$$

This function is plotted in Fig. 7.2 for several small positive values of ε . For very small but nonzero ε it is clear from Fig. 7.2 that y is almost constant except in a very narrow interval of thickness $O(\varepsilon)$ at $x = 1$, which is called a *boundary layer*. Thus, outside the boundary layer the exact solution satisfies the left boundary condition $y(0) = 0$ and almost but not quite satisfies the unperturbed equation $y' = 0$.

It is not obvious how to construct a perturbative approximation to a differential equation whose highest derivative is multiplied by ε until it is known how to construct an analytical expression for the zeroth-order approximation. A new technique called asymptotic matching must be introduced (see Sec. 7.4 and Chap. 9) to solve this problem.

Example 3 *Appearance of rapid variation on a global scale.* In the previous example we saw that the exact solution varies rapidly in the neighborhood of $x = 1$ for small ε and develops a discontinuity there in the limit $\varepsilon \rightarrow 0+$. A solution to a boundary-value problem may also develop discontinuities throughout a large region as well as in the neighborhood of a point.

The boundary-value problem $\varepsilon y'' + y = 0$ [$y(0) = 0, y(1) = 1$] is a singular perturbation problem because when $\varepsilon = 0$, the solution to the unperturbed problem, $y = 0$, does not satisfy the boundary condition $y(1) = 1$. The exact solution, when ε is not of the form $(n\pi)^{-2}$ ($n = 0, 1, 2, \dots$), is $y(x) = \sin(x/\sqrt{\varepsilon})/\sin(1/\sqrt{\varepsilon})$. Observe that $y(x)$ becomes discontinuous throughout the inter-

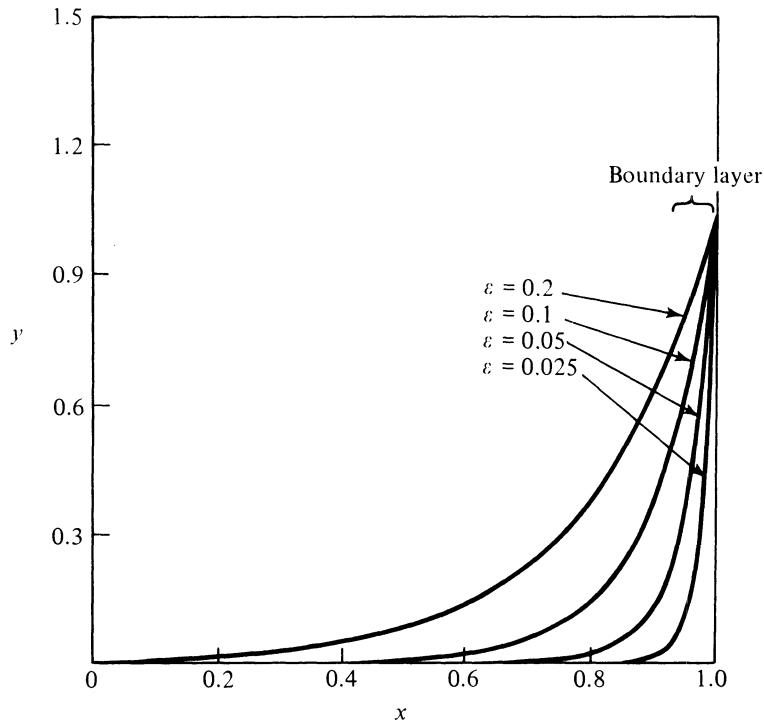


Figure 7.2 A plot of $y(x) = (e^{x/\varepsilon} - 1)/(e^{1/\varepsilon} - 1)$ ($0 \leq x \leq 1$) for $\varepsilon = 0.2, 0.1, 0.05, 0.025$. When ε is small $y(x)$ varies rapidly near $x = 1$; this localized region of rapid variation is called a boundary layer. When ε is negative the boundary layer is at $x = 0$ instead of $x = 1$. This abrupt jump in the location of the boundary layer as ε changes sign reflects the singular nature of the perturbation problem.

val $0 \leq x \leq 1$ in the limit $\varepsilon \rightarrow 0+$ (see Fig. 7.3). When $\varepsilon = (n\pi)^{-2}$, there is no solution to the boundary-value problem.

When the solution to a differential-equation perturbation problem varies rapidly on a global scale for small ε , it is not obvious how to construct a leading-order perturbative approximation to the exact solution. The best procedure that has evolved is called WKB theory (see Chap. 10).

Example 4 *Perturbation theory on an infinite interval.* The initial-value problem

$$y'' + (1 - \varepsilon x)y = 0, \quad y(0) = 1, y'(0) = 0, \quad (7.2.10)$$

is a regular perturbation problem in ε over the finite interval $0 \leq x \leq L$. In fact, the perturbation solution is just

$$\begin{aligned} y(x) &= \cos x + \varepsilon \left(\frac{1}{4}x^2 \sin x + \frac{1}{4}x \cos x - \frac{1}{4} \sin x \right) \\ &\quad + \varepsilon^2 \left(-\frac{1}{32}x^4 \cos x + \frac{5}{48}x^3 \sin x + \frac{7}{16}x^2 \cos x - \frac{7}{16}x \sin x \right) + \dots \end{aligned} \quad (7.2.11)$$

which converges for all x and ε , with increasing rapidity as $\varepsilon \rightarrow 0+$ for fixed x .

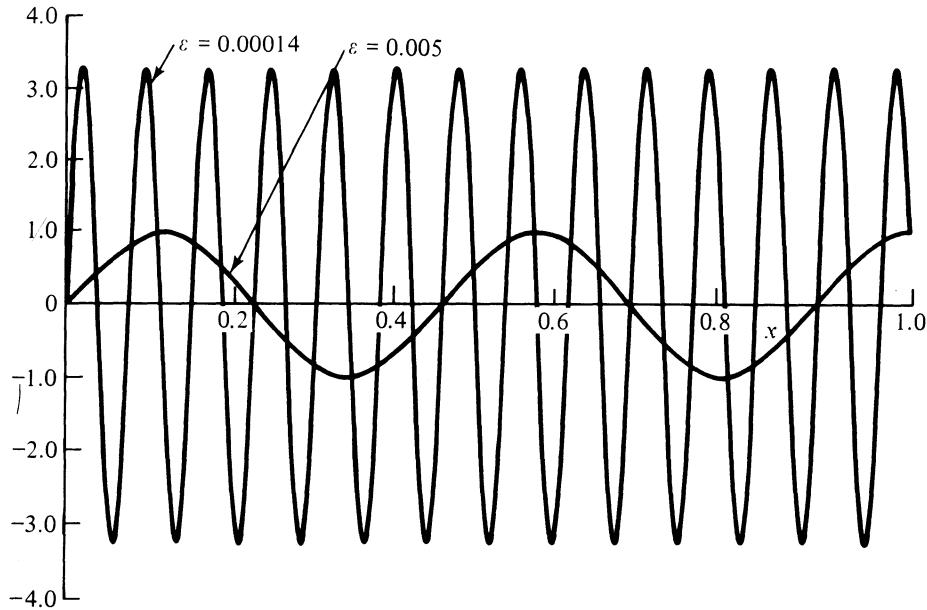


Figure 7.3 A plot of $y(x) = [\sin(x\epsilon^{-1/2})]/[\sin(\epsilon^{-1/2})]$ ($0 \leq x \leq 1$) for $\epsilon = 0.005$ and 0.00014 . As ϵ gets smaller the oscillations become more violent; as $\epsilon \rightarrow 0+$, $y(x)$ becomes discontinuous over the entire interval. The WKB approximation is a perturbative method commonly used to describe functions like $y(x)$ which exhibit rapid variation on a global scale.

However, this same initial-value problem must be reclassified as a singular perturbation problem over the semi-infinite interval $0 \leq x < \infty$. While the exact solution does approach the solution to the unperturbed problem as $\epsilon \rightarrow 0+$ for fixed x , it does not do so uniformly for all x (see Fig. 7.4). The zeroth-order solution is bounded and oscillatory for all x . But when $\epsilon > 0$, local analysis of the exact solution for large x shows that it is a linear combination of exponentially increasing and decreasing functions (Prob. 7.20). This change in character of the solution occurs because it is certainly wrong to neglect ϵx compared with 1 when x is bigger than $1/\epsilon$. In fact, a more careful argument shows that the term ϵx is not a small perturbation unless $x \ll \epsilon^{-1/2}$ (Prob. 7.20).

Example 4 shows that the interval itself can determine whether a perturbation problem is regular or singular. We examine more examples having this property in the next section on eigenvalue problems. The feature that is common to all such examples is that an n th-order perturbative approximation bears less and less resemblance to the exact solution as x increases.

For these sorts of problems Chap. 11 introduces new perturbative procedures called multiple-scale methods which substantially improve the rather poor predictions of ordinary perturbation theory. The particular problem in Example 4 is reconsidered in Prob. 11.13.

Example 5 Roots of a high-degree polynomial. When a perturbation problem is regular, the perturbation series is convergent and the exact solution is a smooth analytic function of ϵ for

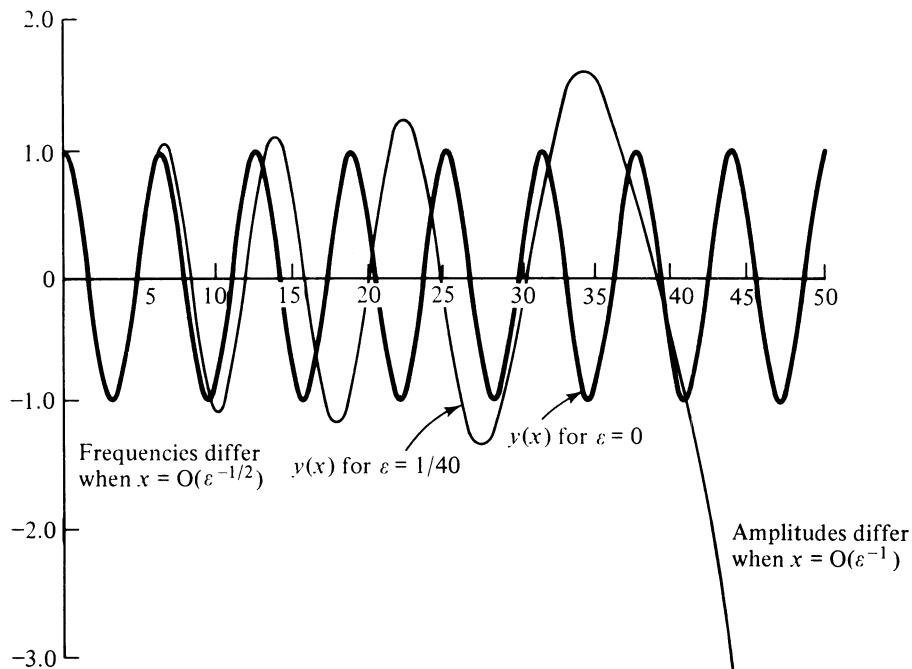


Figure 7.4 Exact solutions to the initial-value problem $y'' + (1 - \varepsilon x)y = 0$ [$y(0) = 1$, $y'(0) = 0$] in (7.2.10) for $\varepsilon = 0$ and $\varepsilon = \frac{1}{40}$. Although this is a regular perturbation problem on the finite interval $0 \leq x \leq L$, it is a singular perturbation problem on the infinite interval $0 \leq x \leq \infty$ because the perturbed solution ($\varepsilon > 0$) is not close to the unperturbed solution ($\varepsilon = 0$), no matter how small ε is. When $x = O(\varepsilon^{-1/2})$ the frequencies begin to differ (the curves become phase shifted) and when $x = O(\varepsilon^{-1})$ the amplitudes differ (one curve remains finite while the other grows exponentially).

sufficiently small ε . However, just what is “sufficiently small” may vary enormously from problem to problem. A striking example by Wilkinson concerns the roots of the polynomial

$$\prod_{k=1}^{20} (x - k) + \varepsilon x^{19} = x^{20} - (210 - \varepsilon)x^{19} + \cdots + 20! \quad (7.2.12)$$

The perturbation εx^{19} is regular, since no roots are lost in the limit $\varepsilon \rightarrow 0$; the roots of the unperturbed polynomial lie at $1, 2, 3, \dots, 20$.

Let us now take $\varepsilon = 10^{-9}$ so that the perturbation in the coefficient of x^{19} is of relative magnitude $10^{-9}/210$, or roughly 10^{-11} . For such a small regular perturbation one might expect the 20 roots to be only very slightly displaced from their $\varepsilon = 0$ values. The actual displaced roots are given in Table 7.1. One is surprised to find that while some roots are relatively unchanged by the perturbation, others have paired into complex conjugates. The qualitative effect on the roots of varying ε is shown in Figs. 7.5 and 7.6. In these plots the paths of the roots are traced as a function of ε . As $|\varepsilon|$ increases, the roots coalesce into pairs of complex conjugate roots. Evidently, a “small” perturbation is one for which $|\varepsilon| < 10^{-11}$, while $|\varepsilon| \gtrsim 10^{-10}$ is a “large” perturbation for at least some of the roots. Low-order regular perturbation theory may be used to understand this behavior (Probs. 7.22 and 7.23).

Table 7.1 Roots of the Wilkinson polynomial (7.2.12) with $\varepsilon = 10^{-9}$

The first column lists the unperturbed ($\varepsilon = 0$) roots 1, 2, ..., 20; the second column gives the results of first-order perturbation theory (see Prob. 7.22); the third column gives the exact roots. The unperturbed roots at 13 and 14, 15 and 16, and 17 and 18 are perturbed into complex-conjugate pairs. Observe that while first-order perturbation theory is moderately accurate for the real perturbed roots near 1, 2, ..., 12, 19, 20, it cannot predict the locations of the complex roots (but see Prob. 7.23)

Unperturbed root	First-order perturbation theory	Exact root
1	1.000 000 000 0	1.000 000 000 0
2	2.000 000 000 0	2.000 000 000 0
3	3.000 000 000 0	3.000 000 000 0
4	4.000 000 000 0	4.000 000 000 0
5	5.000 000 000 0	5.000 000 000 0
6	5.999 999 941 8	5.999 999 941 8
7	7.000 002 542 4	7.000 002 542 4
8	7.999 994 030 4	7.999 994 031 5
9	9.000 839 327 5	9.000 841 033 5
10	9.992 405 941 6	9.992 518 124 0
11	11.046 444 571	11.050 622 592
12	11.801 496 835	11.832 935 987
13	13.605 558 629	13.349 018 036 \pm 0.532 765 750 0 <i>i</i>
14	12.667 031 557	
15	17.119 065 220	15.457 790 724 \pm 0.899 341 526 2 <i>i</i>
16	13.592 486 027	
17	18.904 402 150	17.662 434 477 \pm 0.704 285 236 9 <i>i</i>
18	17.004 413 300	
19	19.309 013 459	19.233 703 334
20	19.956 900 195	19.950 949 654

This example shows that the roots of high-degree polynomials may be extraordinarily sensitive to changes in the coefficients of the polynomial, even though the perturbation problem so obtained is regular. It should serve as ample warning to a “number cruncher” not to trust computer output without sufficient understanding of the nature of the problem being solved.

(I) 7.3 PERTURBATION METHODS FOR LINEAR EIGENVALUE PROBLEMS

In this section we show how perturbation theory can be used to approximate the eigenvalues and eigenfunctions of the Schrödinger equation

$$\left[-\frac{d^2}{dx^2} + V(x) + W(x) - E \right] y(x) = 0, \quad (7.3.1)$$

subject to the boundary condition

$$\lim_{|x| \rightarrow \infty} y(x) = 0. \quad (7.3.2)$$

CHAPTER
NINE

BOUNDARY-LAYER THEORY

His career has been an extraordinary one. He is a man of good birth and excellent education, endowed by nature with a phenomenal mathematical faculty. At the age of twenty-one he wrote a treatise upon the binomial theorem, which has had a European vogue. On the strength of it he won the mathematical chair at one of our smaller universities, and had, to all appearances, a most brilliant career before him. But the man had hereditary tendencies of the most diabolical kind. A criminal strain ran in his blood, which, instead of being modified, was increased and rendered infinitely more dangerous by his extraordinary mental powers.

—Sherlock Holmes, *The Final Problem*
Sir Arthur Conan Doyle

(E) 9.1 INTRODUCTION TO BOUNDARY-LAYER THEORY

In this and the next chapter we discuss perturbative methods for solving a differential equation whose highest derivative is multiplied by the perturbing parameter ε . The most elementary of these methods is called boundary-layer theory.

A *boundary layer* is a narrow region where the solution of a differential equation changes rapidly. By definition, the thickness of a boundary layer must approach 0 as $\varepsilon \rightarrow 0$. In this chapter we will be concerned with differential equations whose solutions exhibit only *isolated* (well-separated) narrow regions of rapid variation. It is possible for a solution to a perturbation problem to undergo rapid variation over a thick region (one whose thickness does *not* vanish with ε). However, such a region is not a boundary layer. We will consider such problems in Chap. 10.

Here is a simple boundary-value problem whose solution exhibits boundary-layer structure.

Example 1 *Exactly soluble boundary-layer problem.* Consider the differential equation

$$\varepsilon y'' + (1 + \varepsilon)y' + y = 0, \quad y(0) = 0, \quad y(1) = 1. \quad (9.1.1)$$

The exact solution of this equation is

$$y(x) = \frac{e^{-x} - e^{-x/\varepsilon}}{e^{-1} - e^{-1/\varepsilon}}. \quad (9.1.2)$$

In the limit $\varepsilon \rightarrow 0+$, this solution becomes discontinuous at $x = 0$, as is shown in Fig. 9.1. For very small ε the solution $y(x)$ is slowly varying for $\varepsilon \ll x \leq 1$. However, on the small interval $0 \leq x \leq O(\varepsilon)$ ($\varepsilon \rightarrow 0+$) it undergoes an abrupt and rapid change. This small interval of rapid change is called a *boundary layer*. [The notation $0 \leq x \leq O(\varepsilon)$ means that the thickness of the boundary layer is proportional to ε as $\varepsilon \rightarrow 0+$.] The region of slow variation of $y(x)$ is called the *outer region* and the boundary-layer region is called the *inner region*.

Boundary-layer theory is a collection of perturbation methods for solving differential equations whose solutions exhibit boundary-layer structure. When the solution to a differential equation is slowly varying except in isolated boundary layers, then it may be relatively easy to obtain a leading-order approximation to that solution for small ε without directly solving the differential equation.

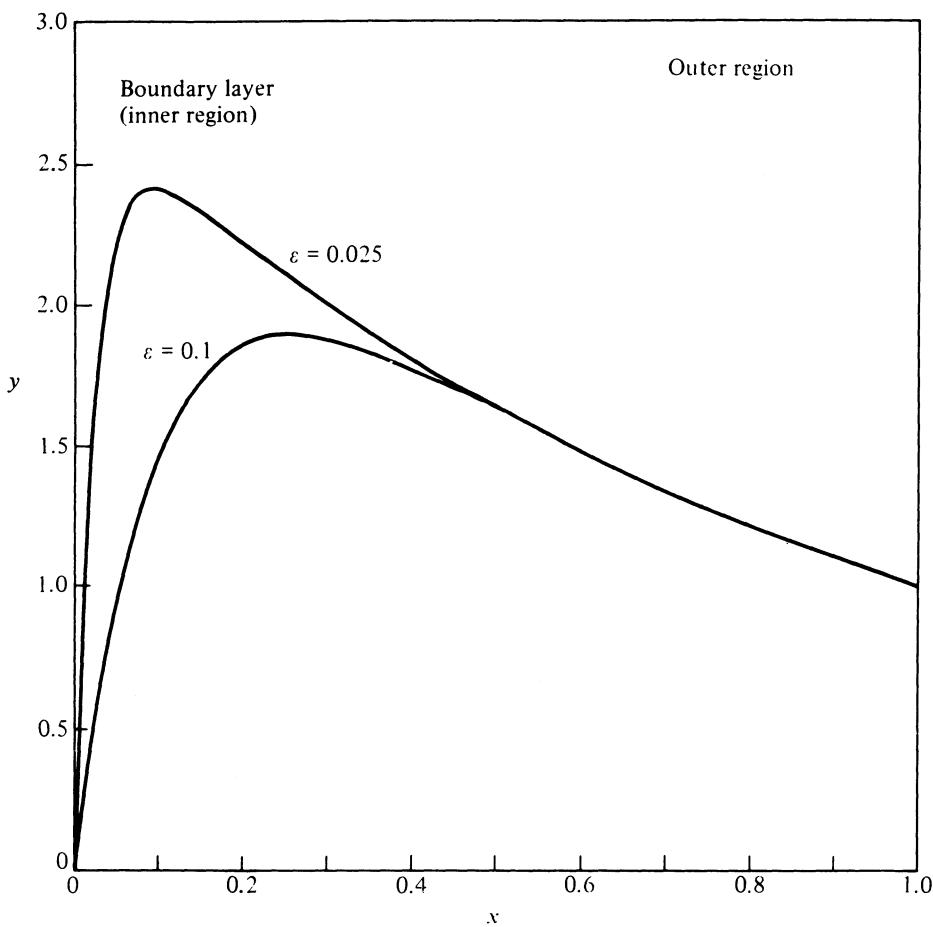


Figure 9.1 A plot of $y(x) = (e^{-x} - e^{-x/\varepsilon})/(e^{-1} - e^{-1/\varepsilon})$ ($0 \leq x \leq 1$) for $\varepsilon = 0.1$ and 0.025 . Note that $y(x)$ is slowly varying for $\varepsilon \ll x \leq 1$ ($\varepsilon \rightarrow 0+$). However, on the interval $0 \leq x \leq O(\varepsilon)$, $y(x)$ rises abruptly from 0 and becomes discontinuous in the limit $\varepsilon \rightarrow 0+$. This narrow and isolated region of rapid change is called a boundary layer.

There are two standard approximations that one makes in boundary-layer theory. In the outer region (away from a boundary layer) $y(x)$ is slowly varying, so it is valid to neglect any derivatives of $y(x)$ which are multiplied by ε . Inside a boundary layer the derivatives of $y(x)$ are large, but the boundary layer is so narrow that we may approximate the coefficient functions of the differential equation by constants. Thus, we can replace a single differential equation by a sequence of much simpler approximate equations in each of several inner and outer regions. In every region the solution of the approximate equation will contain one or more unknown constants of integration. These constants are then determined from the boundary or initial conditions using the technique of asymptotic matching which was introduced in Sec. 7.4.

The following initial-value problem illustrates these ideas.

Example 2 *First-order nonlinear boundary-layer problem.* From the initial-value problem

$$(x - \varepsilon y)y' + xy = e^{-x}, \quad y(1) = 1/e, \quad (9.1.3)$$

we wish to determine a leading-order perturbative approximation to $y(0)$ as $\varepsilon \rightarrow 0+$.

Although this is only a first-order differential equation, it is nonlinear and is much too difficult to solve in closed form. However, in regions where y and y' are not large (such regions are called outer regions), it is valid to neglect $\varepsilon yy'$ compared with e^{-x} . Thus, in outer regions we approximate the solution to (9.1.3) by the solution to the outer equation

$$xy'_\text{out} + xy_\text{out} = e^{-x}.$$

This equation is easy to solve because it is linear. The solution which satisfies $y_\text{out}(1) = 1/e$ is

$$y_\text{out} = (1 + \ln x)e^{-x}. \quad (9.1.4)$$

Note that it is valid to impose the initial condition $y(1) = 1/e$ on $y_\text{out}(x)$ because $x = 1$ lies in an outer region; $x = 1$ is in an outer region because (9.1.3) implies that $y'(1) = 0$, so $y(1)$ and $y'(1)$ are of order 1 as $\varepsilon \rightarrow 0+$.

As $x \rightarrow 0+$, both $y_\text{out}(x)$ and $y'_\text{out}(x)$ become larger. Thus, near $x = 0$ the term $\varepsilon yy'$ is no longer negligible compared with e^{-x} . From the outer solution we can estimate that the thickness δ of the region in which $\varepsilon yy'$ is not small is given by

$$\delta/\ln \delta = O(\varepsilon), \quad \varepsilon \rightarrow 0+.$$

Thus, $\delta \rightarrow 0+$ as $\varepsilon \rightarrow 0+$ [in fact, $\delta = O(\varepsilon \ln \varepsilon)$ as $\varepsilon \rightarrow 0+$] (see Prob. 9.1), and there is a boundary layer of thickness δ at $x = 0$.

In the boundary layer (the inner region), x is small so it is valid to approximate e^{-x} by 1. Furthermore, since y varies rapidly in the narrow boundary layer, we may neglect xy compared with xy' . Hence, in the inner region we approximate the solution to (9.1.3) by the solution to the inner equation

$$(x - \varepsilon y_\text{in})y'_\text{in} = 1.$$

This is a linear equation if we regard x as the dependent variable. Its solution is

$$x = \varepsilon(y_\text{in} + 1) + Ce^{y_\text{in}}, \quad (9.1.5)$$

where C is an unknown constant of integration. Since $x = 0$ is in the inner region, we may use (9.1.5) to find an approximation to $y(0)$.

C is determined by asymptotically matching the outer and inner solutions (9.1.4) and (9.1.5). Take x small but not as small as δ , say $x = O(\varepsilon^{1/2})$. Then (9.1.4) implies that $y_{\text{out}} \sim 1 + \ln x$ as $\varepsilon \rightarrow 0+$ and (9.1.5) implies that $x \sim Ce^{y_{\text{in}}}$ as $\varepsilon \rightarrow 0+$. Thus, $C = 1/e$ and a leading-order implicit equation for $y_{\text{in}}(0)$ is

$$0 = \varepsilon[y_{\text{in}}(0) + 1] + e^{y_{\text{in}}(0) - 1}. \quad (9.1.6)$$

When $\varepsilon = 0.1$ and 0.01 , the numerical solutions of (9.1.6) are $y_{\text{in}}(0) \doteq -1.683$ and $y_{\text{in}}(0) \doteq -2.942$, respectively. These results compare favorably with the numerical solution to (9.1.3) which gives $y(0) \doteq -1.508$ when $\varepsilon = 0.1$ and $y(0) \doteq -2.875$ when $\varepsilon = 0.01$. For both values of ε the relative error between the perturbative and the numerical solutions for $y(0)$ is about $\frac{1}{2}\varepsilon \ln \varepsilon$. Figures 9.2 and 9.3 compare the inner and outer perturbative approximations to $y(x)$ with the numerical solution.

Boundary-layer theory can also be a very powerful tool for determining the behavior of solutions to higher-order equations.

Example 3 Second-order linear boundary-value problem. Let us find an approximate solution to the boundary-value problem

$$\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = 0, \quad 0 \leq x \leq 1, \quad y(0) = A, \quad y(1) = B. \quad (9.1.7)$$

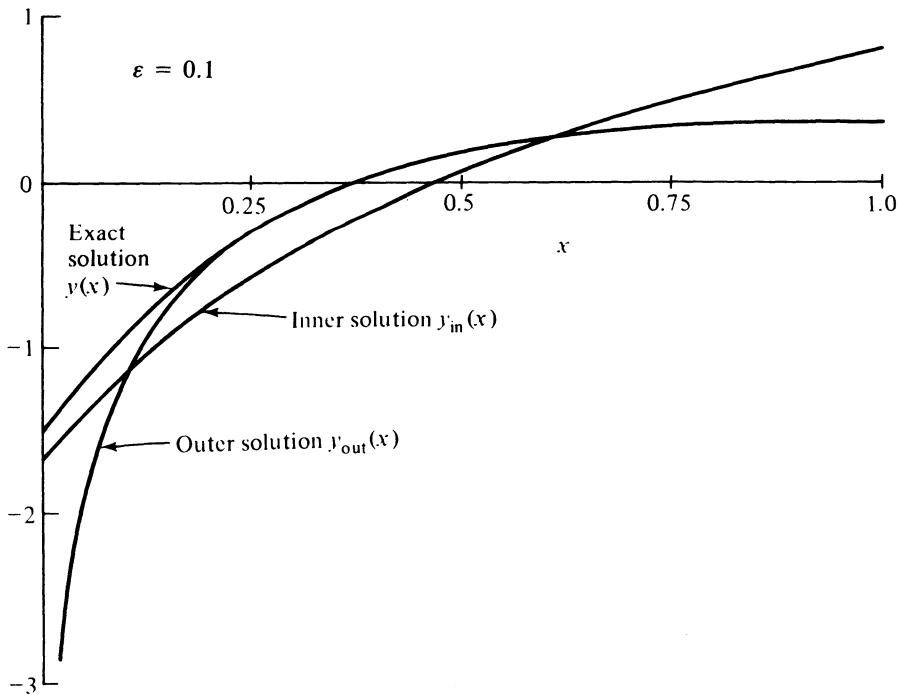


Figure 9.2 A comparison for $\varepsilon = 0.1$ of the exact solution $y(x)$ to the nonlinear differential equation (9.1.3) and the inner and outer approximations to $y(x)$ using boundary-layer theory. The integration constant in y_{out} is determined from the initial condition $y(1) = 1/e$. The integration constant in y_{in} is determined from asymptotic matching. A measure of the accuracy of the boundary-layer approximation is the magnitude of the error in the predicted value of $y(0)$. When $\varepsilon = 0.1$, $y(0) \doteq -1.508$ and $y_{\text{in}}(0) \doteq -1.683$, an error of about 10 percent.

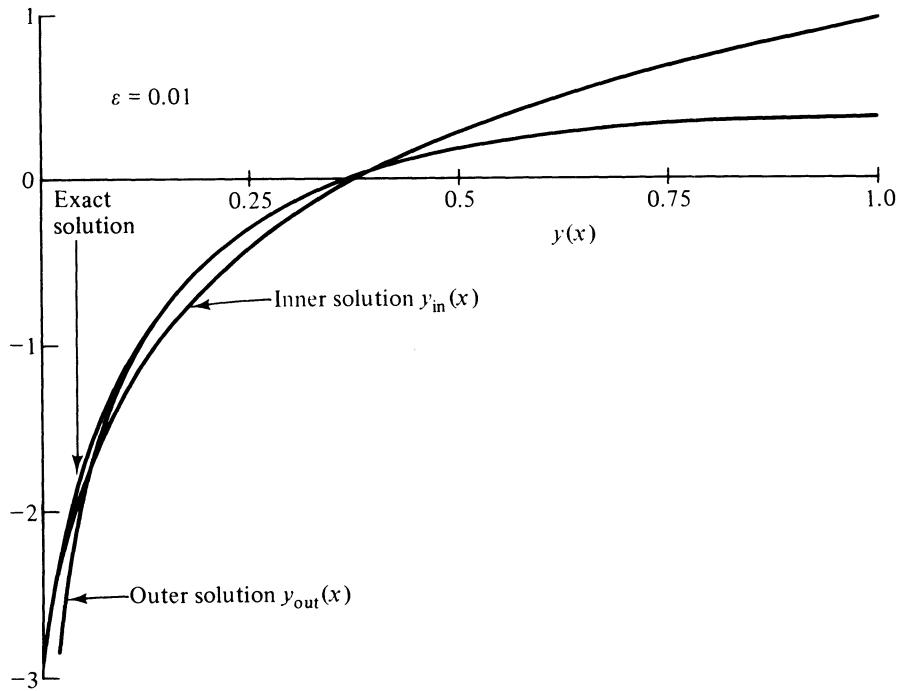


Figure 9.3 Same as Fig. 9.2 with $\varepsilon = 0.1$ replaced by $\varepsilon = 0.01$. Here, $y(0) \doteq -2.875$ and $y_{in}(0) \doteq -2.942$, an error of about 2 percent. Observe that as $\varepsilon \rightarrow 0+$, the inner and outer approximations $y_{in}(x)$ and $y_{out}(x)$ hug the exact solution $y(x)$ more closely. The error appears to be of order $\frac{1}{2}\varepsilon \ln \varepsilon$ (see Prob. 9.5).

as $\varepsilon \rightarrow 0+$. We assume for reasons to be made clear later that $a(x) \neq 0$ for $0 \leq x \leq 1$, and for definiteness we choose $a(x) > 0$; otherwise $a(x)$ and $b(x)$ are arbitrary continuous functions.

We shall analyze the behavior of $y(x)$ as $\varepsilon \rightarrow 0+$ by assuming that in this limit the solution $y(x)$ develops an isolated boundary layer in the neighborhood of $x = 0$ and that there are no other regions of rapid change of $y(x)$ ($\varepsilon \rightarrow 0+$). We will then justify these assumptions by showing that no other possibility is mathematically consistent.

The outer region is characterized by the absence of rapid variation of $y(x)$: $y(x)$, $y'(x)$, and $y''(x)$ are all of order 1 (assuming that A and B are finite) as $\varepsilon \rightarrow 0+$. Thus, in the outer region a good approximation to (9.1.7) is the first-order linear equation

$$a(x)y'_{out}(x) + b(x)y_{out}(x) = 0. \quad (9.1.8)$$

Observe that the outer approximation has reduced the order of the differential equation, thereby making it soluble. The solution to (9.1.8) is $y_{out}(x) = K \exp\left[\int_x^1 b(t)/a(t) dt\right]$, where K is an integration constant. In general, it is not possible for $y_{out}(x)$ to satisfy both boundary conditions $y(0) = A$ and $y(1) = B$. However, we have assumed that $x = 1$ lies within the outer region and that $x = 0$ does not. Thus, we should require that $y_{out}(1) = B$, but not $y_{out}(0) = A$. It follows that $K = B$:

$$y_{out}(x) = B \exp\left[\int_x^1 b(t)/a(t) dt\right]. \quad (9.1.9)$$

The outer solution (9.1.9) is a uniform approximation to the solution $y(x)$ as $\varepsilon \rightarrow 0+$ on the subinterval $\delta \ll x \leq 1$ of $[0, 1]$, where $\delta(\varepsilon)$ is the thickness of the boundary layer. It is now becoming clear why we have assumed that $a(x_0) \neq 0$ for $0 \leq x_0 \leq 1$. If $a(x_0) = 0$ for some x_0 on this interval, then $y_{\text{out}}(x)$ would be singular at x_0 , assuming that $b(x_0) \neq 0$. This would violate the assumption that y , y' , and y'' are all of order 1.

The outer solution $y_{\text{out}}(x)$ is not valid in the neighborhood of $x = 0$ unless $y_{\text{out}}(0) = A$, in which case $y_{\text{out}}(x)$ is a uniformly valid leading-order approximation to $y(x)$ for $0 \leq x \leq 1$. However, since A is arbitrary, in general $y_{\text{out}}(0) \neq A$. Thus, the boundary condition $y(0) = A$ must be achieved through a boundary layer at $x = 0$. In other words, the outer solution $y_{\text{out}}(x)$ is approximately equal to $y(x)$ as x approaches 0 from above until $x = O(\delta)$. At this point $y_{\text{out}}(x)$ is approaching and already very close to $y_{\text{out}}(0)$, while the actual solution $y(x)$ rapidly veers off and approaches $y(0) = A$ (see Fig. 9.4).

To determine the behavior of $y(x)$ when $x = O(\delta)$, we may approximate the functions of $a(x)$ and $b(x)$ in the original differential equation (9.1.7) by $a(0) = \alpha \neq 0$ and $b(0) = \beta$ because δ vanishes as $\varepsilon \rightarrow 0$. Also, in the inner region, y is much smaller than y' because y is rapidly varying. Therefore, we may neglect y compared with y' . Thus, the inner approximation to (9.1.7) is the constant coefficient differential equation

$$\varepsilon y''_{\text{in}} + \alpha y'_{\text{in}} = 0, \quad (9.1.10)$$

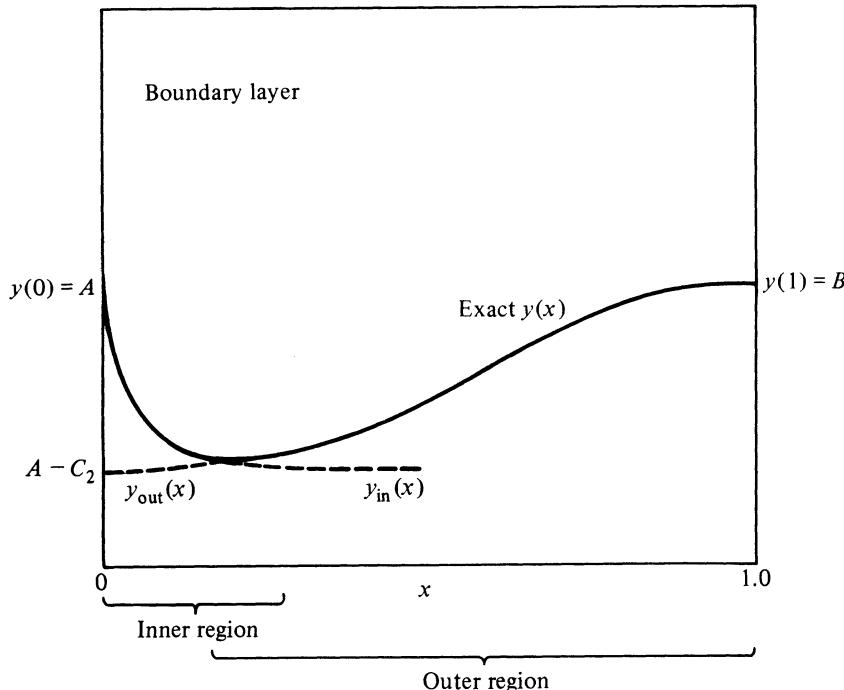


Figure 9.4 A schematic plot of the solution to the boundary-value problem $\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = 0$ [$0 \leq x \leq 1$; $a(x) > 0$] with $y(0) = A$, $y(1) = B$, in (9.1.7). The exact solution satisfies the boundary conditions $y(0) = A$ and $y(1) = B$ and has a boundary layer (region of rapid variation) of thickness $O(\varepsilon)$ at $x = 0$. The outer solution $y_{\text{out}}(x)$ is a good approximation to $y(x)$ in the outer region, but $y_{\text{out}}(0) = A - C_2$. The inner solution is a good approximation to $y(x)$ in the inner region. The asymptotic match of $y_{\text{in}}(x)$ and $y_{\text{out}}(x)$ occurs in the overlap of the inner and outer regions; in the overlap region $y_{\text{in}}(x)$ and $y_{\text{out}}(x)$ both approach the constant $A - C_2$.

which is soluble. The most general solution to (9.1.10) is

$$y_{in}(x) = C_1 + C_2 e^{-\alpha x/\varepsilon}.$$

Finally, we must require that $y_{in}(0) = y(0) = A$. Thus,

$$y_{in}(x) = A + C_2(e^{-\alpha x/\varepsilon} - 1). \quad (9.1.11)$$

The remaining constant of integration C_2 will be determined by asymptotic matching.

Since $y_{in}(x)$ in (9.1.11) varies rapidly when $x = O(\varepsilon)$, we conclude that the boundary-layer thickness δ is of order ε . The asymptotic match of the inner and outer solutions takes place between the rightmost edge of the inner region and the leftmost edge of the outer region, say for values of $x = O(\varepsilon^{1/2})$. For such values of x ,

$$y_{in}(x) \sim A - C_2, \quad \varepsilon \rightarrow 0+,$$

and

$$y_{out}(x) \sim y_{out}(0) = B \exp \left[\int_0^1 b(t)/a(t) dt \right], \quad \varepsilon \rightarrow 0+.$$

Thus, if $y_{in}(x)$ and $y_{out}(x)$ are to be good approximations to $y(x)$ in the overlap of the inner and outer regions, then we must require that $C_2 = A - y_{out}(0)$.

To summarize, the boundary-layer approximation is

$$\begin{aligned} y(x) &\sim B \exp \left[\int_x^1 b(t)/a(t) dt \right], & 0 < x \leq 1, \varepsilon \rightarrow 0+; \\ y(x) &\sim Ae^{-a(0)x/\varepsilon} + B(1 - e^{-a(0)x/\varepsilon}) \exp \left[\int_0^1 b(t)/a(t) dt \right], & x = O(\varepsilon), \varepsilon \rightarrow 0+. \end{aligned} \quad (9.1.12)$$

We may proceed further by combining the above two expressions into a single, *uniform* approximation y_{unif} , valid for all $0 \leq x \leq 1$. A suitable expression is

$$y_{unif}(x) = y_{out}(x) + y_{in}(x) - y_{match}(x),$$

where $y_{match} = A - C_2$. Hence,

$$y_{unif}(x) = B \exp \left[\int_x^1 b(t)/a(t) dt \right] + \left\{ A - B \exp \left[\int_0^1 b(t)/a(t) dt \right] \right\} e^{-a(0)x/\varepsilon}. \quad (9.1.13)$$

To verify that $y_{unif}(x) \sim y(x)$ ($\varepsilon \rightarrow 0+$), one must examine it for values of x in the inner and outer regions and check that it reduces to the two expressions in (9.1.12). Equation (9.1.13) is a uniform approximation in the sense that the difference between $y(x)$ and $y_{unif}(x)$ is uniformly $O(\varepsilon)$ ($0 \leq x \leq 1, \varepsilon \rightarrow 0+$) (see Prob. 9.2).

We conclude this example with several observations. First, if $a(x) < 0$ throughout $[0, 1]$, then no match is possible with the boundary layer solution (9.1.11) at $x = 0$ because $y_{in}(x)$ grows exponentially with x/ε unless $C_2 = 0$. On the other hand, if the boundary layer occurs at $x = 1$, then matching is possible if $a(x) < 0$ (see Prob. 9.3). If $a(x) > 0$, it is impossible to match to a boundary layer at $x = 1$ for the same reason that a match cannot be made at $x = 0$ when $a(x) < 0$.

Second, there can be no boundary layer at an internal point x_0 ($0 < x_0 < 1$) if $a(x_0) \neq 0$. If a boundary layer did exist at x_0 , then within this narrow layer we could approximate the original differential equation (9.1.7) by $\varepsilon y''_{in} + a(x_0)y'_{in} = 0$. The general solution to this equation is

$$y_{in} = C_1 + C_2 e^{-a(x_0)(x-x_0)/\varepsilon}.$$

If $a(x_0) > 0$ (< 0), then no asymptotic match is possible at the left (right) edge of the boundary layer unless $C_2 = 0$ because the approximation must remain finite. Thus, the matching conditions

require that $C_2 = 0$. Hence, the outer solutions to the left and right of the boundary layer both approach the same constant C_1 as $x \rightarrow x_0$ from below and above. Thus, the outer solutions approach each other and there is no internal region of rapid change.

In summary, then, when $a(x)$ in (9.1.7) satisfies $a(x) > 0$ for $0 \leq x \leq 1$ the boundary layer always lies at $x = 0$ and when $a(x) < 0$ for $0 \leq x \leq 1$ the boundary layer always lies at $x = 1$.

This completes our heuristic introduction to boundary-layer theory. Our purpose in this section was to show how to convert difficult differential equations into easy ones by seeking approximate rather than exact solutions. However, several questions must be answered before the ideas of boundary-layer theory can really be applied with confidence. For example, how can one know *a priori* whether the solution to a differential equation has boundary-layer structure? How can one predict the locations of the boundary layers? How does one estimate δ , the thickness of the boundary layer? How can we be sure that there is an overlap region between the inner and outer regions on which to perform asymptotic matching? Is it useful to decompose a solution into its inner and outer parts if one is seeking a high-order approximation to the exact answer? These questions will be answered in the next two sections.

(E) 9.2 MATHEMATICAL STRUCTURE OF BOUNDARY LAYERS: INNER, OUTER, AND INTERMEDIATE LIMITS

Having demonstrated the power and broad applicability of boundary-layer analysis in Sec. 9.1, it is now appropriate to formalize and restate more carefully some of the rather loosely defined concepts. This section deals with the questions about boundary-layer theory that were raised at the end of the previous section.

To keep our presentation as concrete as possible we will use Example 1 of Sec. 9.1 as a model boundary-layer problem and will analyze its mathematical structure in detail. You will recall that the function

$$y(x) = \frac{e^{-x} - e^{-x/\varepsilon}}{e^{-1} - e^{-1/\varepsilon}}, \quad (9.2.1)$$

which is the exact solution of the boundary-value problem

$$\varepsilon y'' + (1 + \varepsilon)y' + y = 0, \quad y(0) = 0, \quad y(1) = 1, \quad (9.2.2)$$

has a boundary layer at $x = 0$ when $\varepsilon \rightarrow 0+$. The function $y(x)$ has two components: e^{-x} , a slowly varying function on the entire interval $[0, 1]$, and $e^{-x/\varepsilon}$, a rapidly varying function in the boundary layer $x \leq O(\delta)$, where $\delta = O(\varepsilon)$ is the thickness of the boundary layer.

In boundary-layer theory we treat the solution y of the differential equation as a function of two independent variables, x and ε . The goal of the analysis is to find a global approximation to y as a function of x ; this is achieved by performing a local analysis of y as $\varepsilon \rightarrow 0+$.

To explain the appearance of the boundary layer we introduce the notion of

an inner and outer limit of the solution. The *outer limit* of the solution (9.2.1) is obtained by choosing a fixed x *outside* the boundary layer, that is, $\delta \ll x \leq 1$, and allowing $\varepsilon \rightarrow 0+$. Thus, the outer limit is

$$y_{\text{out}}(x) \equiv \lim_{\varepsilon \rightarrow 0+} y(x) = e^{1-x}. \quad (9.2.3)$$

The difference between the outer limit of the exact solution and the exact solution itself, $|y(x) - y_{\text{out}}(x)|$, is exponentially small in the limit $\varepsilon \rightarrow 0$ when $\delta \ll x$.

Similarly, we can formally take the outer limit of the differential equation (9.2.2); the result of keeping x fixed and letting $\varepsilon \rightarrow 0+$ is simply

$$y'_{\text{out}} + y_{\text{out}} = 0, \quad (9.2.4)$$

which is satisfied by (9.2.3). Because the outer limit of (9.1.2) is a *first-order* differential equation, its solution cannot in general be required to satisfy both boundary conditions $y(0) = 0$ and $y(1) = 1$; the outer limit of (9.2.1) satisfies $y(1) = 1$ but not $y(0) = 0$.

In other words, the small- ε limit of the solution is *not* everywhere close to the solution of the unperturbed differential equation (9.2.4) [the differential equation (9.2.2) with $\varepsilon = 0$]. Thus, the problem (9.2.2) is a singular perturbation problem. The singular behavior [the appearance of a discontinuity in $y(x)$ as $\varepsilon \rightarrow 0+$] occurs because the highest-order derivative in (9.2.2) disappears when $\varepsilon = 0$.

The exact solution satisfies the boundary condition $y(0) = 0$ by developing a boundary layer in the neighborhood of $x = 0$. To examine the nature of this boundary layer, we consider the *inner limit* in which $\varepsilon \rightarrow 0+$ with $x \leq O(\varepsilon)$. In this case x lies inside the boundary layer at $x = 0$. For this limit it is convenient to let $x = \varepsilon X$ with X fixed and finite. X is called an *inner variable*. X is a better variable than x to describe y in the boundary layer because, as $\varepsilon \rightarrow 0+$, y varies rapidly as a function of x but slowly as a function of X . It follows from (9.2.1) that

$$y_{\text{in}}(x) = Y_{\text{in}}(X) = \lim_{\varepsilon \rightarrow 0+} y(\varepsilon X) = e - e^{1-X}. \quad (9.2.5)$$

Similarly, the inner limit of the differential equation is obtained by rewriting (9.2.2) as

$$\frac{1}{\varepsilon} \frac{d^2 Y}{dX^2} + \left(\frac{1}{\varepsilon} + 1 \right) \frac{dY}{dX} + Y = 0, \quad (9.2.6)$$

where we define $Y(X) \equiv y(x)$ and use

$$\varepsilon \frac{dy}{dx} = \frac{dY}{dX}, \quad \varepsilon^2 \frac{d^2 y}{dx^2} = \frac{d^2 Y}{dX^2}.$$

Thus, taking the inner limit, $\varepsilon \rightarrow 0+$, X fixed, gives

$$\frac{d^2 Y_{\text{in}}(X)}{dX^2} + \frac{dY_{\text{in}}(X)}{dX} = 0. \quad (9.2.7)$$

Observe that the inner limit function (9.2.5) does satisfy (9.2.7) together with the boundary condition $Y_{\text{in}}(0) = 0$.

Boundary-layer analysis is extremely useful because it allows one to construct an approximate solution to a differential equation, even when an exact answer is not attainable. This is because the inner and outer limits of an insoluble differential equation are often soluble. Once y_{in} and y_{out} have been determined, they must be asymptotically matched. This asymptotic match occurs on the overlap region which is defined by the *intermediate* limit $x \rightarrow 0$, $X = x/\varepsilon \rightarrow \infty$, $\varepsilon \rightarrow 0+$. For example, if $x = \varepsilon^{1/2}z$ with z fixed as $\varepsilon \rightarrow 0$, then an intermediate limit is obtained. A glance at (9.2.3) and (9.2.5) shows that the intermediate limits of $y_{\text{out}}(x)$ and $y_{\text{in}}(x) = Y_{\text{in}}(X)$ agree: $\lim_{x \rightarrow 0} y_{\text{out}}(x) = \lim_{X \rightarrow \infty} Y_{\text{in}}(X) = e$. This common limit verifies the asymptotic match between the inner and outer solutions. (It is *not* generally the case in boundary-layer theory that the intermediate limit is a *number* independent of x and X , as we will shortly see.) The above match condition provides the second boundary condition for the solution of (9.2.7): $Y_{\text{in}}(\infty) = e$. Observe that although the x region is finite, $0 \leq x \leq 1$, the size of the matching region in terms of the inner variable is infinite. As we emphasized in Chap. 7, the extent of the matching region must always be infinite.

A very subtle aspect of boundary-layer theory is the question of whether or not an overlap region for any given problem actually exists. Since one's ability to construct a matched asymptotic expansion depends on the presence of this overlap region, its existence is crucial to the solution of the problem. How did we know, for example, that the intermediate limits of y_{out} and Y_{in} would agree? That is, how did we know that the inner and outer limits of the differential equation (9.2.2) have a common region of validity?

To answer these questions we will perform a complete perturbative solution of (9.2.2) to all orders in powers of ε , and not just to leading order. First, we examine the outer solution. We seek a perturbation expansion of the outer solution of the form

$$y_{\text{out}}(x) \sim \sum_{n=0}^{\infty} y_n(x)\varepsilon^n, \quad \varepsilon \rightarrow 0+, \quad (9.2.8)$$

and restate the boundary condition $y(1) = 1$ as

$$y_0(1) = 1, \quad y_1(1) = 0, \quad y_2(1) = 0, \quad y_3(1) = 0, \quad \dots \quad (9.2.9)$$

Note that $y_{\text{out}}(x)$ in (9.2.8) is *not* the same as $y_{\text{out}}(x)$ in (9.2.3); $y_{\text{out}}(x)$ in (9.2.3) is the first term $y_0(x)$ in (9.2.8).

Substituting (9.2.8) into (9.2.2) and collecting powers of ε gives a sequence of differential equations:

$$\begin{aligned} y'_0 + y_0 &= 0, & y_0(1) &= 1, \\ y'_n + y_n &= -y''_{n-1} - y'_{n-1}, & y_n(1) &= 0, n \geq 1. \end{aligned}$$

The solution to these equations is

$$\begin{aligned} y_0 &= e^{1-x}, \\ y_n &= 0, \quad n \geq 1. \end{aligned} \quad (9.2.10)$$

Thus, the leading-order outer solution, $y_{\text{out}} = e^{1-x}$, is correct to *all* orders in perturbation theory. This is the reason why in the outer region, $x \gg \varepsilon$, the difference between $y(x)$ and $y_{\text{out}}(x)$ is at most exponentially small (subdominant): $|y - y_{\text{out}}| = O(\varepsilon^n)$ for *all* n as $\varepsilon \rightarrow 0+$.

Next, we perform a similar expansion of the inner solution. We assume a perturbation series of the form

$$Y_{\text{in}}(X) \sim \sum_{n=0}^{\infty} \varepsilon^n Y_n(X), \quad \varepsilon \rightarrow 0+, \quad (9.2.11)$$

and restate the boundary condition $Y_{\text{in}}(0) = y(0) = 0$ as

$$Y_n(0) = 0, \quad n \geq 0. \quad (9.2.12)$$

Substituting (9.2.11) into (9.2.6) gives the sequence of differential equations:

$$\begin{aligned} Y_0'' + Y_0' &= 0, & Y_0(0) &= 0, \\ Y_n'' + Y_n' &= -Y_{n-1}' - Y_{n-1}, & Y_n(0) &= 0, \quad n \geq 1. \end{aligned}$$

These equations may be solved by means of the integrating factor e^X . The results are

$$\begin{aligned} Y_0(X) &= A_0(1 - e^{-X}), \\ Y_n(X) &= \int_0^X [A_n e^{-z} - Y_{n-1}(z)] dz, \quad n \geq 1, \end{aligned} \quad (9.2.13)$$

where the A_n are arbitrary integration constants.

Does this inner solution match asymptotically, order by order in powers of ε , to $y_{\text{out}}(x)$? To see if this is so, we substitute $x = \varepsilon X$ into y_{out} in (9.2.10) and expand in powers of ε :

$$y_{\text{out}}(x) = e^{1-x} = e \left(1 - \varepsilon X + \frac{\varepsilon^2 X^2}{2!} - \frac{\varepsilon^3 X^3}{3!} + \dots \right). \quad (9.2.14)$$

Returning to equation (9.2.13), we take X large ($X \rightarrow \infty$) and obtain $Y_0(X) \sim A_0$ ($X \rightarrow \infty$). Thus, comparing with the first term of (9.2.14), we have $A_0 = e$, as we already know. Now that Y_0 is known, we may compute Y_1 from (9.2.13):

$$Y_1(X) = (A_1 + A_0)(1 - e^{-X}) - eX.$$

Asymptotic matching with y_{out} [comparing $Y_1(X)$, when $X \rightarrow \infty$, with the second term of (9.2.14)] gives $A_1 = -e$, so $Y_1(X) = -eX$. Similarly, $Y_n(X) = e[(-1)^n/n!]X^n$. Hence the full inner expansion is

$$Y_{\text{in}}(X) = e \sum_{n=0}^{\infty} \varepsilon^n \frac{(-1)^n X^n}{n!} - e^{1-X} = e^{1-\varepsilon X} - e^{1-X}. \quad (9.2.15)$$

Evidently, the inner expansion is a valid asymptotic expansion not only for values of X inside the boundary layer [$X = O(1)$] but also for large values of X

$[X = O(\varepsilon^{-\alpha}), 0 < \alpha < 1]$. At the same time the expansion for $y_{\text{out}}(x)$ is valid for $\varepsilon \ll x \leq 1$ ($\varepsilon \rightarrow 0+$). [$y_{\text{out}}(x)$ is not valid for $x = O(\varepsilon)$ because it does not satisfy the boundary condition $y(0) = 0$; nor does it have the boundary-layer term e^{1-x} which is present in $Y_{\text{in}}(X)$.] We conclude that to all orders in powers of ε it is possible to match asymptotically the inner and outer expansions because they have a common region of validity: $\varepsilon \ll x \ll 1$ ($\varepsilon \rightarrow 0+$).

We have been able to demonstrate explicitly the existence of the overlap region for this problem because it is soluble to all orders in perturbation theory. In general, however, such a calculation is too difficult. Instead, our approach will always be to assume that an overlap region exists and then to verify the consistency of this assumption by performing an asymptotic match. In the above simple boundary-value problem, we found that the size of the overlap region was independent of the order of perturbation theory. In general, however, the extent of the matching region may vary with the order of perturbation theory (see Sec. 7.4 and Example 1 of Sec. 9.3).

One final point concerns the construction of the uniform approximation to $y(x)$. The formula used in the previous section to construct a uniform approximation is $y_{\text{unif}}(x) = y_{\text{in}}(x) + y_{\text{out}}(x) - y_{\text{match}}(x)$, where $y_{\text{match}}(x)$ is the approximation to $y(x)$ in the matching region and $y_{\text{unif}}(x)$ is a uniform approximation to $y(x)$. This formula is applicable here too. For the boundary-layer solution to (9.2.2), it is easy to verify that if $y_{\text{in}}(x)$, $y_{\text{out}}(x)$, and $y_{\text{match}}(x)$ are calculated to n th order in perturbation theory, then $|y(x) - y_{\text{unif}}(x)| = O(\varepsilon^{n+1})$ ($\varepsilon \rightarrow 0+$; $0 \leq x \leq 1$).

The differential equation (9.2.2) is sufficiently simple that $y_{\text{unif}}(x)$ can be calculated to all orders in perturbation theory. It follows from (9.2.10) for $y_{\text{out}}(x)$, (9.2.15) for $y_{\text{in}}(x)$, and the result $y_{\text{match}}(x) = e^{1-x}$ that

$$y_{\text{unif}} = e^{1-x} - e^{1-x}$$

is the infinite-order uniform approximation to $y(x)$.

It is remarkable, however, that this expression, which is the result of summing up perturbation theory to infinite order, is actually *not* equal to $y(x)$ in (9.2.1). Thus, although the perturbation series for y_{unif} is *asymptotic* to $y(x)$ as $\varepsilon \rightarrow 0+$, the asymptotic series does not *converge* to $y(x)$ as n , the order of perturbation theory, tends to ∞ ; there is an exponentially small error, of order $e^{-1/\varepsilon}$, which remains undetermined. Boundary-layer theory is indeed a singular, and not a regular, perturbation theory.

Why is boundary-layer theory a singular perturbation theory? The singular nature of boundary-layer theory is intrinsic to both the inner and outer expansions. The outer expansion is singular because there is an abrupt change in the order of the differential equation when $\varepsilon = 0$. By contrast, the inner expansion is a regular perturbation expansion for finite X (see Example 2 of Sec. 7.1). However, since asymptotic matching takes place in the limit $X \rightarrow \infty$, the inner expansion is also singular (see Example 4 of Sec. 7.2). Another manifestation of the singular limit $\varepsilon \rightarrow 0$ is the location of the boundary layer in (9.2.1); when the limit $\varepsilon \rightarrow 0+$ is replaced by $\varepsilon \rightarrow 0-$, the boundary layer abruptly jumps from $x = 0$ to $x = 1$.