Homework 12
Abstract Algebra
Math 320
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Problem 5.3.1: Determine whether the given congruence-class ring is a field. Justify your answer.

(a) $\mathbb{Z}_3[x]/(x^3 + 2x^2 + x + 1)$

Let $p(x) = x^3 + 2x^2 + x + 1$. Notice that the congruence class will be polynomials of degree 2 or less. Also notice that the only factors of the $p(x)$ that meet those requirements are polynomials with degree 2 and its root. Lastly notice the only numbers in $\mathbb{Z}_3$ are $0, 1, 2$

$$\begin{aligned}
p(0) &= 1 \neq 0 \\
p(1) &= 5 \neq 0 \\
p(2) &= 19 \neq 0
\end{aligned}$$

This shows that $p(x)$ is irreducible and has no zero divisors, so by Theorem 5.10, (a) is a field.

(b) $\mathbb{Z}_5[x]/(2x^3 - 4x^2 + 2x + 1)$

Let $p(x) = 2x^3 - 4x^2 + 2x + 1$. Notice that the congruence class will be polynomials of degree 2 or less. Also notice that the only factors of the $p(x)$ that meet those requirements are polynomials with degree 2 and its root. Lastly notice the only numbers in $\mathbb{Z}_5$ are $0, 1, 2, 3, 4$.

$$\begin{aligned}
p(0) &= 1 \\
p(1) &= 1 \\
p(2) &= 5 = [0] \\
p(3) &= 25 = [0] \\
p(4) &= 73
\end{aligned}$$

This shows that $p(x)$ is not irreducible, and has zero divisors, $(x - 2), (x - 3)$, so (b) is NOT a field

(c) $\mathbb{Z}_2[x]/(x^4 + x^2 + 1)$

Let $p(x) = x^4 + x^2 + 1$. Notice the only numbers in $\mathbb{Z}_2$ are 0 and 1. Notice that all factors of $p(x)$ have to be of degree 4 or less. So it can consist of factors of degree 2 with another factor of the same degree or degree 3 with a root.

$$p(0) = 1 \neq 0$$
$$p(1) = 3 \neq 0$$

So this concludes that the only factors of $p(x)$ have to be degree 2. So notice that the only polynomials of degree 2 in $\mathbb{Z}_2[x]$ are:

x^2 $\qquad$ $x^2 + x$ $\qquad$ $x^2 + 1$ $\qquad$ $x^2 + x + 1$

So we can see the multiplication table:

$\times$	x^2	$x^2 + x$	$x^2 + 1$	$x^2 + x + 1$
x^2	x^4	$x^4 + x^3$	$x^4 + x^2$	$x^4 + x^3 + x^2$
$x^2 + x$	$x^4 + x^3$	$x^4 + x^2$	$x^4 + x^3 + x^2 + x$	$x^4 + x$
$x^2 + 1$	$x^4 + x^2$	$x^4 + x$	$x^4 + 1$	$x^4 + x^3 + x + 1$
$x^2 + x + 1$	$x^4 + x^3 + x^2$	$x^4 + x$	$x^4 + x^3 + x + 1$	$x^4 + x^2 + 1$

So because $[x^2 + x + 1]^2 = [x^4 + x^2 + 1] = [0]$, then $p(x)$ is not irreducible, thus meaning (c) is NOT a field

Problem 5.3.5 (b): Show that $\mathbb{Q}(\sqrt{3})$ is isomorphic to $\mathbb{Q}[x]/(x^2-3)$.

Solution. Let $a+b\sqrt{3}, c+d\sqrt{3} \in \mathbb{Q}(\sqrt{3})$, with $a,b,c,d \in \mathbb{Q}$. Let the function $\phi : \mathbb{Q}(\sqrt{3}) \to \mathbb{Q}[x]/(x^2-3)$, such that $\phi(a+b\sqrt{3}) = a+bx$. Also note that in $\mathbb{Q}[x]/(x^2-3)$, $[x^2] = [3]$ Notice the following homomorphic properties:

$$\begin{aligned}
\phi((a+b\sqrt{3})+(c+d\sqrt{3})) &= \phi((a+c)+(b+d)\sqrt{3}) = (a+c)+(b+d)x = a+c+bx+dx \\
&= (a+bx)+(c+dx) = \phi(a+b\sqrt{3})+\phi(c+d\sqrt{3}) \\
\phi(a+b\sqrt{3})\phi(c+d\sqrt{3}) &= (a+bx)(c+dx) = ac+adx+bcx+bdx^2 = ac+adx+bcx+3bd \\
&= (ac+3bd)+(ad+bc)x = \phi((ac+3bd)+(ad+bc)\sqrt{3}) \\
&= \phi((a+b\sqrt{3})(c+d\sqrt{3}))
\end{aligned}$$

Now notice the following bijective properties:

$$\phi(a+b\sqrt{3}) = a+bx = c+dx = \phi(c+d\sqrt{3})$$

The only way for the following to be true is if $a=c$ and $b=d$, thus proving injectivity.

Notice for any function in $\mathbb{Q}[x]/(x^2-3)$, $[e+fx]$, it can always be written as $\phi(e+b\sqrt{3})$. Thus proving surjectivity.

So $\mathbb{Q}(\sqrt{3})$ is isomorphic to $\mathbb{Q}[x]/(x^2-3)$

□

Problem 6.1.2: Show that the set I of all polynomials with even constant terms is an ideal in $Z[x]$.

$$I = \{ax^n + ... + 2k | a \in \mathbb{F}, k \in \mathbb{Z}\}$$

Notice that zero is in this set:

$$0_{\mathbb{Z}} = 0x^n + ... + 2(0) \in I$$

Notice that the set is closed under subtraction, and let $r = a_1x^n + ... + 2k$, $s = a_2x^m + ... + 2j \in I$. Note: (It is implied that $a_1, a_2 \in \mathbb{F}$ and $k, j \in \mathbb{Z}$ because $r, s \in I$. This is implied with other sets in other problems as well :))

$$r - s = a_1x^n + ... + 2k - (a_2x^m + ... + 2j) = a_1x^n - a_2x^m + ... + 2(k - j) \in I$$

Notice that the set satisfies the absorption property, and let $r = a_1x^n + ... + 2k \in I$, $s \in \mathbb{Z}$.

$$rs = a_1sx^n + ... + 2sk = sr \in I$$

Thus I is an ideal in $Z[x]$.

Problem 6.1.3:

(a) Show that the set $I = \{(k, 0), k \in \mathbb{Z}\}$ is an ideal in the ring $\mathbb{Z} \times \mathbb{Z}$

Notice that zero is in this set:

$$0_{\mathbb{Z}\times\mathbb{Z}} = (0, 0) \in I$$

Notice that the set is closed under subtraction, and let $r = (a, 0), s = (b, 0) \in I$.

$$r - s = (a, 0) - (b, 0) = (a - b, 0) \in I$$

Notice that the set satisfies the absorption property, and let $r = (a, 0) \in I$ and $s = (b, c) \in \mathbb{Z} \times \mathbb{Z}$

$$rs = (a, 0)(b, c) = (ab, 0) = (b, c)(a, 0) = (ba, 0) = sr \in I$$

Thus I is an ideal in the ring $\mathbb{Z} \times \mathbb{Z}$

(b) Show that the set $T = \{(k, k), k \in \mathbb{Z}\}$ is not ideal in the ring $\mathbb{Z} \times \mathbb{Z}$

Notice that T does not satisfies the absorption property, and let $r = (1, 1) \in I$ and $s = (2, 3) \in \mathbb{Z} \times \mathbb{Z}$:

$$rs = (1, 1)(2, 3) = (2, 3) \notin T$$

Thus I is not an ideal in the ring $\mathbb{Z} \times \mathbb{Z}$

Problem 6.1.8: If I is an ideal in R and J is an ideal in the ring S, prove that $I \times J$ is an ideal in the ring $R \times S$.

Let the following be true:

$$T = \{(i, j) | i \in I, j \in J\} = I \times J$$

Because I is an ideal in R and J is an ideal in the ring S, $0_R \in I$ and $0_S \in J$, such that

$$0_{R \times S} = (0_R, 0_S) \in T$$

Notice that the set is closed under subtraction, and let $a = (i_1, j_1), b = (i_2, j_2) \in T$.

$$a - b = (i_1, j_1) - (i_2, j_2) = (i_1 - i_2, j_1 - j_2) \in T$$

Because I and R are ideals, notice that they also closed under subtraction with $i_1 - i_2 \in I$ and $j_1 - j_2 \in J$.

Notice that the set satisfies the absorption property, and let $a = (i, j) \in T$ and $b = (r, s) \in R \times S$, with $r \in R, s \in S$.

$$rs = (i, j)(r, s) = (ir, js) = (ri, sj) = sr \in T$$

Because I is an ideal of R, $ir \in I$ and because J is an ideal of S, $js \in R$

Thus $I \times J$ is an ideal in the ring $R \times S$.

Problem 6.1.41:

(a) Prove that the set S of rational numbers (in lowest terms) with odd denominators is a subring of $\mathbb{Q}$.

Let the following be true:

$$S = \left\{ \frac{a}{2k+1} \,\middle|\, a \nmid (2k+1), k \in \mathbb{Z} \right\}$$

Notice that zero is in this set:

$$0_{\mathbb{Q}} = \frac{0}{2k+1} \in S$$

Notice that the set is closed under subtraction, and let $r = \frac{a}{2k+1}, s = \frac{b}{2j+1} \in S$

$$r - s = \frac{a}{2k+1} - \frac{b}{2j+1} = \frac{a(2j+1) - b(2k+1)}{2(2kj+k+j)+1} \in S$$

Notice that the set is closed under multiplication, and let $r = \frac{a}{2k+1}, s = \frac{b}{2j+1} \in S$

$$rs = \frac{a}{2k+1} * \frac{b}{2j+1} = \frac{ab}{2(2kj+k+j)+1} \in S$$

Thus S is a subring of $\mathbb{Q}$

(b) Let I be the set of elements of S with even numerators. Prove that I is an ideal in S.

Let the following be true:

$$I = \left\{ \frac{2a}{2k+1} \,\middle|\, a, k \in \mathbb{Z} \right\}$$

Notice that zero is in this set:

$$0_S = \frac{2(0)}{2k+1} \in I$$

Notice that the set is closed under subtraction, and let $r = \frac{2a}{2k+1}, s = \frac{2b}{2j+1} \in I$

$$r - s = \frac{2a}{2k+1} - \frac{2b}{2j+1} = \frac{2(a(2j+1) - b(2k+1))}{2(2kj+k+j)+1} \in I$$

Notice that the set satisfies the absorption property and let $r = \frac{2a}{2k+1} \in I, s = \frac{b}{2j+1} \in S$

$$rs = \frac{2a}{2k+1} * \frac{b}{2j+1} = \frac{2ab}{2(2kj+k+j)+1} = \frac{2ba}{2(2kj+k+j)+1} = sr \in I$$

Thus I is an ideal in S