Lecture 3 - Rate of Convergence Examples

In [1]:
```
import matplotlib.pyplot as plt
import numpy as np
```

In [2]:
```
def plot(x_range, f, hass=None, xlim=None, ylim=None):
    x_vals = np.linspace(x_range[0], x_range[1], 10_000)
    y_vals = f(x_vals)

    plt.plot(x_vals, y_vals, '-')
    plt.grid()

    if hass:
        for ass in hass:
            plt.plot(x_vals, ass * np.ones(np.size(x_vals)), 'r--')

    if xlim:
        plt.xlim(xlim[0], xlim[1])
    if ylim:
        plt.ylim(ylim[0], ylim[1])
```

Notice the following rules to calculate the Rate of Convergence, with $\bar{x}$ converges to x^*:

Q-Linear:

$$\frac{||x_{k+1} - x^*||}{||x_k - x^*||} \leq r \in (0,1) \qquad \text{for any large value, } k$$

Q-Superlinear:

$$\lim_{k \to \infty} \frac{||x_{k+1} - x^*||}{||x_k - x^*||} = 0$$

Q-Quadratic:

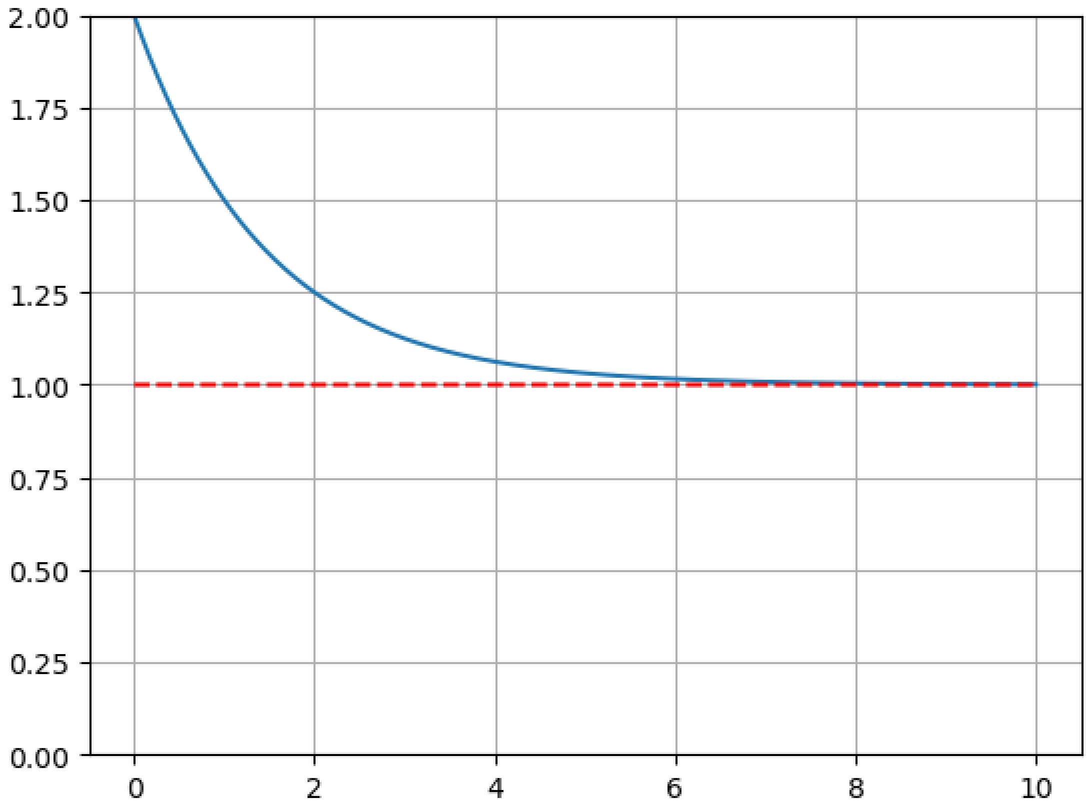
$$\frac{||x_{k+1} - x^*||}{||x_k - x^*||^2} \leq M \in \mathbb{R}^+ \qquad \text{for any large value, } k$$

Example 1

Notice the work to calculate the Rate of Convergence for the following equation:

$$x_k = 1 + (0.5)^k \qquad \to \qquad x^* = 1$$

In [3]:
```
plot((0, 10), lambda k: 1 + (0.5)**k, hass=[1], xlim=None, ylim=(0,2))
```



Q-Linear Test (PASS):

$$\frac{||x_{k+1} - x^*||}{||x_k - x^*||} = \frac{1 + (0.5)^{k+1} - 1}{1 + (0.5)^k - 1} = \frac{(0.5)^{k+1}}{(0.5)^k} = 0.5 \in (0, 1)$$

Q-Superlinear Test (FAIL):

$$\lim_{k \to \infty} \frac{||x_{k+1} - x^*||}{||x_k - x^*||} = \lim_{k \to \infty} \frac{1 + (0.5)^{k+1} - 1}{1 + (0.5)^k - 1} = \lim_{k \to \infty} \frac{(0.5)^{k+1}}{(0.5)^k} = 0.5 \neq 0$$

Q-Quadratic Test (FAIL):

$$\frac{||x_{k+1} - x^*||}{||x_k - x^*||^2} = \frac{1 + (0.5)^{k+1} - 1}{(1 + (0.5)^k - 1)^2} = \frac{(0.5)^{k+1}}{(0.5)^{2k}} = (0.5)^{-k+1} = 2^{k-1} \not\leq M$$
$$\in \mathbb{R}^+ \text{ for any large value k}$$

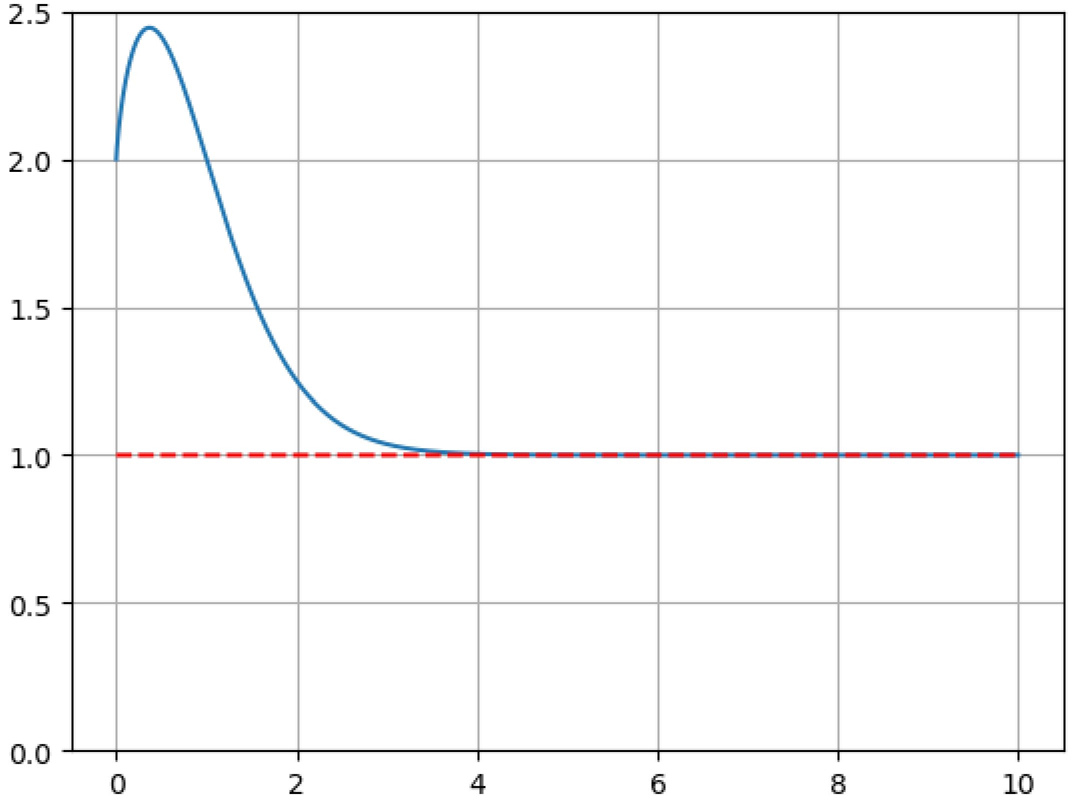
We can come up to a direct conclusion for the all tests, which shows that the Rate of Convergence is only Q-Linear.

Example 2

Notice the work to calculate the Rate of Convergence for the following equation:

$$x_k = 1 + k^{-k} \qquad \rightarrow \qquad x^* = 1$$

In [4]:
```
plot((0, 10), lambda k: 1 + (k)**(-k), hass=[1], xlim=None, ylim=(0,2.5))
```



Q-Linear Test (PASS):

$$\frac{||x_{k+1} - x^*||}{||x_k - x^*||} = \frac{1 + (k+1)^{-(k+1)} - 1}{1 + k^{-k} - 1} = \frac{(k+1)^{-(k+1)}}{k^{-k}} = \frac{k^k}{(k+1)^{k+1}} \leq r \in (0, 1)$$

Q-Superlinear Test (PASS):

$$\lim_{k\to\infty} \frac{||x_{k+1} - x^*||}{||x_k - x^*||} = \lim_{k\to\infty} \frac{1 + (k+1)^{-(k+1)} - 1}{1 + k^{-k} - 1} = \lim_{k\to\infty} \frac{(k+1)^{-(k+1)}}{k^{-k}} = \lim_{k\to\infty} \frac{k^k}{(k+1)^{k+1}} = 0$$

Q-Quadratic Test (FAIL):

$$\frac{||x_{k+1} - x^*||}{||x_k - x^*||^2} = \frac{1 + (k+1)^{-(k+1)} - 1}{(1 + k^{-k} - 1)^2} = \frac{(k+1)^{-(k+1)}}{k^{-2k}} = \frac{k^{2k}}{(k+1)^{k+1}} \not\leq M$$
$$\in \mathbb{R}^+ \text{ for any large value k}$$

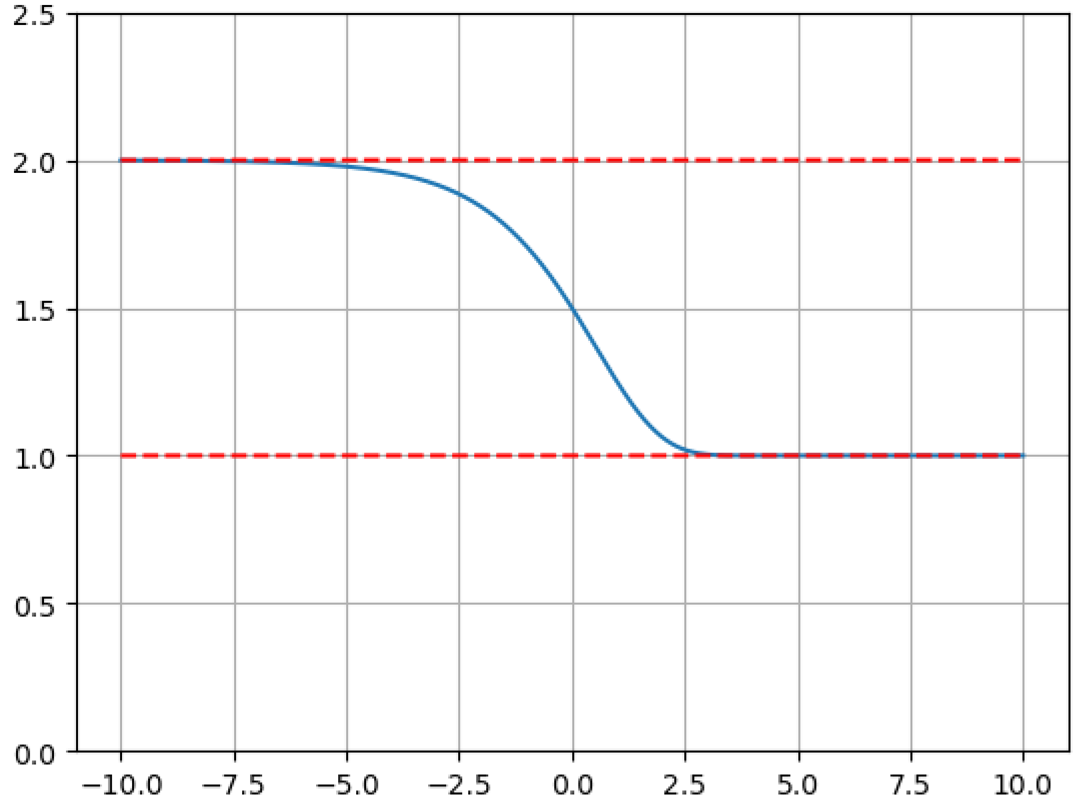
We can come up to a direct conclusion for the Q-Superlinear Test which shows that the Rate of Convergence is Q-Superlinear AND Q-Linear.

Example 3

Notice the work to calculate the Rate of Convergence for the following equation:

$$x_k = 1 + (0.5)^{2^k} \qquad \rightarrow \qquad x^* = 1, 2$$

In [5]:
```
plot((-10, 10), lambda k: 1 + (0.5)**(2**k), hass=[2, 1], xlim=None, ylim=(0,2.5))
```



Notice the following tests for $x^* = 1$:

Q-Linear Test:

$$\frac{||x_{k+1} - x^*||}{||x_k - x^*||} = \frac{1 + (0.5)^{2^{k+1}} - 1}{1 + (0.5)^{2^k} - 1} = \frac{(0.5)^{2^{k+1}}}{(0.5)^{2^k}} = 0.5^{2^{k+1} - 2^k}$$

Q-Superlinear Test:

$$\lim_{k\to\infty} \frac{||x_{k+1} - x^*||}{||x_k - x^*||} = \lim_{k\to\infty} \frac{1 + (0.5)^{2^{k+1}} - 1}{1 + (0.5)^{2^k} - 1} = \lim_{k\to\infty} \frac{(0.5)^{2^{k+1}}}{(0.5)^{2^k}} = \lim_{k\to\infty} 0.5^{2^{k+1} - 2^k}$$

Q-Quadratic Test (PASS):

$$\frac{||x_{k+1} - x^*||}{||x_k - x^*||^2} = \frac{1 + (0.5)^{2^{k+1}} - 1}{(1 + (0.5)^{2^k} - 1)^2} = \frac{(0.5)^{2^{k+1}}}{(0.5)^{2^{k+1}}} = 0.5^{2^{k+1} - 2^{k+1}} = 1 \leq M$$
$$\in \mathbb{R}^+ \text{ for any large value k}$$

We can come up to a direct conclusion for the Q-Quadratic Test which shows that the Rate of Convergence with $x^* = 1$ is Q-Quadratic, Q-Superlinear, AND Q-Linear.

Notice the Q-Quadratic Test for $x^* = 2$:

$$\frac{||x_{k+1} - x^*||}{||x_k - x^*||^2} = \frac{1 + (0.5)^{2^{k+1}} - 2}{(1 + (0.5)^{2^k} - 2)^2} = \frac{(0.5)^{2^{k+1}} - 1}{(0.5)^{2^{k+1}} + 1 - 2(0.5)^{2^k}}$$
$$= \frac{(0.5)^{2^{k+1}} - 1}{(0.5)^{2^{k+1}} + 1 - (0.5)^{-1}(0.5)^{2^k}} = \frac{(0.5)^{2^{k+1}} - 1}{(0.5)^{2^{k+1}} + 1 - (0.5)^{2^k - 1}} \leq M$$
$$\in \mathbb{R}^+ \text{ for any large value k}$$

We can see that for large values, k, we know that the numerator is smaller than the denominator:

$$(0.5)^{2^{k+1}} - 1 < (0.5)^{2^{k+1}} + 1 - (0.5)^{2^k - 1}$$

such that the quotient converges which implies there exists a real positive value, M, that the quotient will always be less than.

We can come up to a direct conclusion for the Q-Quadratic Test which shows that the Rate of Convergence with $x^* = 1$ is Q-Quadratic, Q-Superlinear, AND Q-Linear.