HW 6 Solutions

(1) Let $\{x_n\} \subseteq \mathbb{R} \setminus \{0\}$.

Suppose $\lim_{n\to\infty} x_n = 0$.

Then $\lim_{n\to\infty} a x_n = 0$.

Since $f'(0)$ exists,

$$\lim_{n\to\infty} \frac{f(ax_n) - f(0)}{ax_n} = f'(0).$$

Thus
$$\lim_{n\to\infty} \frac{f(ax_n) - f(0)}{cx_n} = \frac{a}{c} \lim_{n\to\infty} \frac{f(ax_n) - f(0)}{ax_n}$$
$$= \frac{a}{c} f'(0). \quad \blacksquare$$

② Let $\{x_n\} \subseteq \mathbb{R} \setminus \{0\}$ and suppose $\lim_{n\to\infty} x_n = 0$.

Then $\lim_{n\to\infty} \dfrac{x_n^2 h(x_n) - 0}{x_n} = \lim_{n\to\infty} x_n h(x_n)$.

Since h is bounded, let $M \in \mathbb{R}$ be such that

$$-M \le h(x) \le M \quad , \quad \forall x \in \mathbb{R}.$$

Then $-M|x_n| \le x_n h(x_n) \le M|x_n|$.

Since $\lim_{n\to\infty} -M|x_n| = \lim_{n\to\infty} M|x_n| = 0$,

We have $\lim_{n\to\infty} x_n h(x_n) = 0$.

So $f'(0) = \lim_{n\to\infty} \dfrac{f(x_n) - f(0)}{x_n} = \lim_{n\to\infty} x_n h(x_n) = 0$.

$\boxed{qed}$

③ Let $f : [1, 2] \to \mathbb{R}$ by

$$f(x) = x^3 + 2x^2 - 10.$$

Note f is continuous and differentiable.

Since $f(1) = -7$, $f(2) = 6$, and $-7 < 0 < 6$, the IVT says $\exists x_0 \in (1,2)$ s.t.

$$f(x_0) = 0.$$

Suppose $\exists x_1, x_2 \in (1,2)$ s.t.

$$f(x_1) = f(x_2) = 0$$

Then by Rolle's Thm, $\exists x_3 \in (x_1, x_2)$ s.t.

$$f'(x_3) = 0.$$

But $f'(x) = 3x^2 + 4x$ and $\forall x \in [1,2]$,

$$f'(x) > 0. \text{ Thus } f'(x_3) > 0. \Rightarrow\Leftarrow.$$

Thus $x_0 \in (1,2)$ is the unique solution to

$$f(x) = 0 \text{ on } (1,2).$$

(4) Suppose $\forall x \in D$, $f(x) = 10$.

Let $x_0 \in \mathbb{R}$. Since D is dense in $\mathbb{R}$,

$\exists \{x_n\} \subseteq D$ s.t. $\lim_{n\to\infty} x_n = x_0$.

Since f is continuous at x_0,

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 10 = 10 = f(x_0).$$

Thus $\forall x \in \mathbb{R}$, $f(x) = 10$.

(5) Note $f(0) = 6$.

Let $x \in \mathbb{R}$. Then $x^2+2 \geq 2$ and so

$$f(x) = \frac{12}{x^2+2} \leq \frac{12}{2} = 6.$$

Thus a max value of 6 is attained.

Notice $\forall x \in \mathbb{R}$, $f(x) = \frac{12}{x^2+2} > 0$.

Let $0 < a < 6$.

Suppose $x > \sqrt{\frac{12}{a} - 2} \geq 0$

Then $x^2 > \frac{12}{a} - 2$ and $a > \frac{12}{x^2+2} = f(x)$.

Thus a is not a minimum of f.

Thus f does not attain a minimum value.

(6) Let $\varepsilon > 0$.

Let $\delta = \min\{0.1, \frac{0.9\varepsilon}{2}\} > 0$.

Suppose $x \in \mathbb{R}$ and $0 < |x-1| < \delta$.

Then $0.9 < x < 1.1$

and $-1.1 < x-2 < -0.9$

Also $|x-1| < \frac{0.9\varepsilon}{2}$

So $$\frac{2|x-1|}{0.9} < \varepsilon$$

Thus $$\frac{2|x-1|}{|x-2|} < \frac{2|x-1|}{0.9} < \varepsilon.$$

I.e. $\left|\frac{3x-4}{x-2} - 1\right| < \varepsilon$. ∎

Scratch:

$$\left|\frac{3x-4}{x-2} - \frac{x-2}{x-2}\right| < \varepsilon$$

$$\frac{2|x-1|}{|x-2|} = \left|\frac{2x-2}{x-2}\right| < \varepsilon$$

$.9 < x < 1.1$

$-1.1 < x-2 < -0.9$.

$$\frac{2|x-1|}{|x-2|} < |x-1| \cdot \frac{2}{0.9} < \varepsilon.$$