MATH 537, Fall 2020

Ordinary Differential Equations

Lecture #3

Chapter 1 First Order Equations

Constant & Periodic Harvesting and Bifurcations

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Review of Important Concepts

1. Bifurcation;
2. Critical points, $f(x_c) = 0$;
3. (equilibrium points = fixed points = critical points)
4. Derivative tests
5. General solution
6. Initial Value Problem (IVP)
7. Particular solution
8. Phase Line
9. Separable ODEs
10. Sink vs. Source
11. Stable vs. Unstable Solutions, $f'(x_c)$.
12. Structurally Stable vs. Unstable (i.e., with bifurcation)

A Saddle Point

$$x' = x^2 = f(x)$$

Is the critical point a sink or source?

critical points $\boxed{f(x) = 0}$

$\boxed{x = 0}$

1st derivative $f'(x) = 2x$

$x = 0 \qquad f'(0) = 0\ ?$

A saddle point or a half-stable critical point (e.g., Strogatz)

Based on the definition, we can obtain:

$x < 0$	$x > 0$
$\dfrac{dx}{dt} > 0$	$\dfrac{dx}{dt} > 0$
positive direction	positive direction

$\mathrm{x} = 0$

A saddle point

Apply a perturbation method

$$x = x_c + \varepsilon(t)$$

$\boxed{x = 0 + \varepsilon} \qquad \varepsilon' = \varepsilon^2$

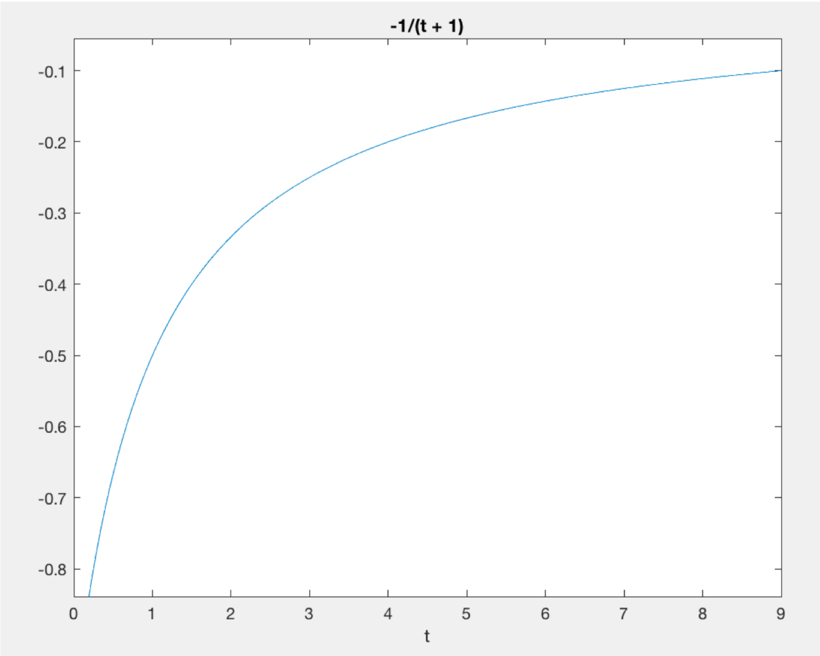
The above discussions can help determine $x = 1$ is a saddle point within $x' = (x-1)^2$.

A Saddle Point

$$x' = x^2$$

$$\boxed{x = \frac{-x_0}{x_0 t - 1}}$$

```
syms t x0
x0=-1
fun=-x0/(x0*t-1)
ezplot (fun, [0, 9])
```



$x_0 < x_c = 0,$ move toward the critical point

```
syms t x0
x0=0.1
fun=-x0/(x0*t-1)
ezplot (fun, [0, 10])
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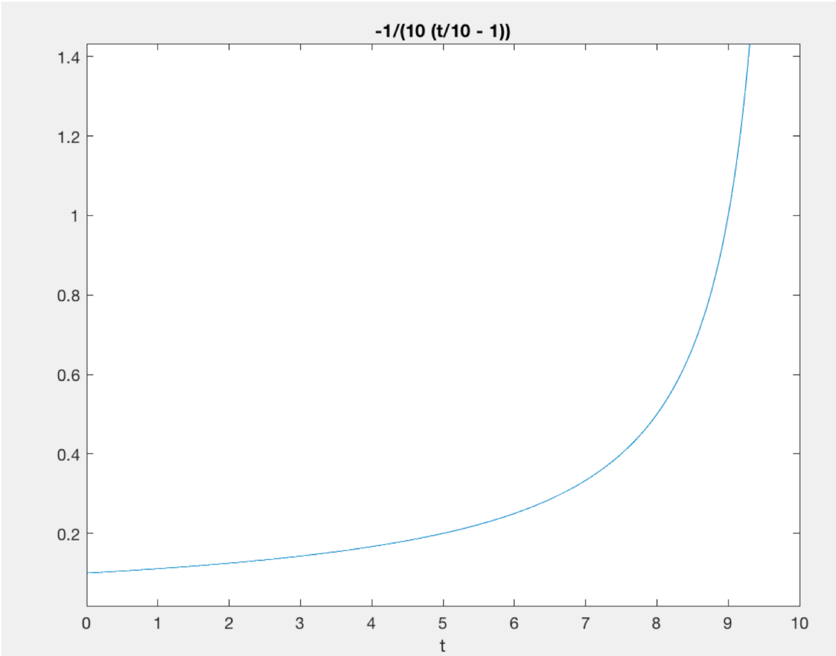

$x_0 > x_c = 0,$ move away from the critical point (x_0 is close to x_c)

Additional Notes

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The Logistic equation includes the following features:

- For a> 0 the basin of attraction of $x_c = 1$ is $x > 0$, while negative values of x attract to minus infinity.
- In contrast to the logistic map (i.e., difference equation), the logistic equation has no oscillatory nor chaotic solutions.

When both f and f' are zero at the critical point,

- the stability is determined by the sign of the first non-vanishing higher derivatives;
- If that derivative is even (e.g., f''), the point is a saddle point, attracting on one side but repelling on the other.
- If that derivative is odd, it follows the same sign rules as f'.

Sprott (2003)

1.3: Constant Harvesting and Bifurcations

$$\boxed{x' = x(1-x) - h = f(x,h)}$$

$$\boxed{\begin{aligned} &f(x,h) = 0 \ \& \\ &f_x(x,h) = 0 \end{aligned}}$$

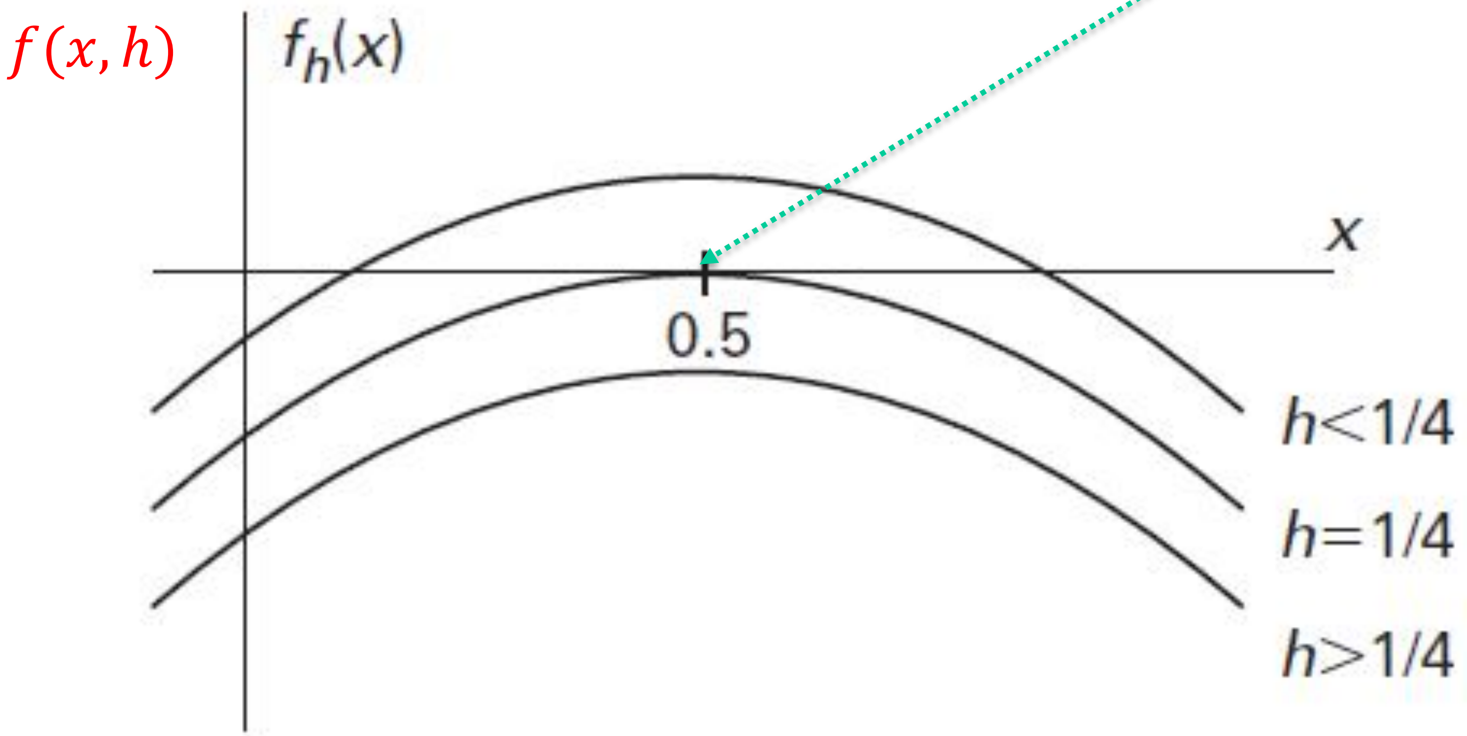


Figure 1.6 The graphs of the function $f_h(x) = x(1-x) - h$.

Bifurcations

- A Bifurcation occurs when there is a “significant” change in the structure of the solutions of the system as system parameter “a” varies.
- The simplest types of bifurcations occur when the number of equilibrium solutions changes as “a” varies.

Bifurcation Point(s)

$$x' = x(1-x) - h = f(x,h)$$

bifurcation points

$$f(x,h) = 0 \ \& \ f_x(x,h) = 0$$

$$f(x,h) = 0 \qquad x(1-x) - h = 0 \qquad h = 1/4$$

$$x_c = \frac{1 \pm \sqrt{1-4h}}{2}$$

$$f_x(x,h) = 0 \qquad 1 - 2x = 0 \qquad x = \frac{1}{2}$$

1.3: Stability Analysis

$$\boxed{x' = x(1-x) - h = f(x,h)}\qquad \boxed{x_c = \frac{1 \pm \sqrt{1-4h}}{2}}\qquad \frac{dx}{dt} = f'(x_c)(\mathrm{x} - \mathrm{x_c}) + \cdots$$

$h > \frac{1}{4}, \rightarrow$ *no critical points* because of $f(x,h) \neq 0$

$$x' = -x^2 + x - h = -\left(x - \frac{1}{2}\right)^2 + \frac{1}{4} - h < 0$$

$h = \frac{1}{4}, \rightarrow$ *critical point* $x_c = 1/2$

$$x' = -\left(x - \frac{1}{2}\right)^2 < 0$$

a saddle at $x_c = 1/2$

$h < \frac{1}{4}, \rightarrow$ *two critical points*, $x_{c1,2} = \dfrac{1 \pm \sqrt{1-4h}}{2}$

$$\boxed{f_x(x) = -2\mathrm{x} + 1}$$

$f_x(x_{c1}) < 0$	stable	a sink
$f_x(x_{c2}) > 0$	unstable	a source

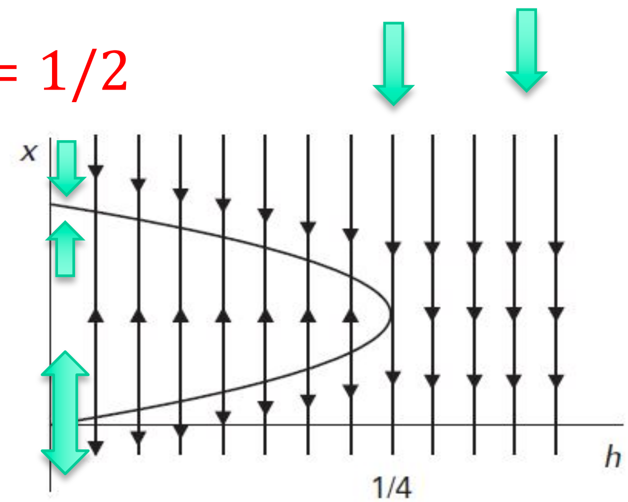


Figure 1.7 The bifurcation diagram for $f_h(x) = x(1-x) - h$.

Section 1.3: Constant Harvesting and Bifurcations

$$\frac{dx}{dt} = x(1-x) - h$$

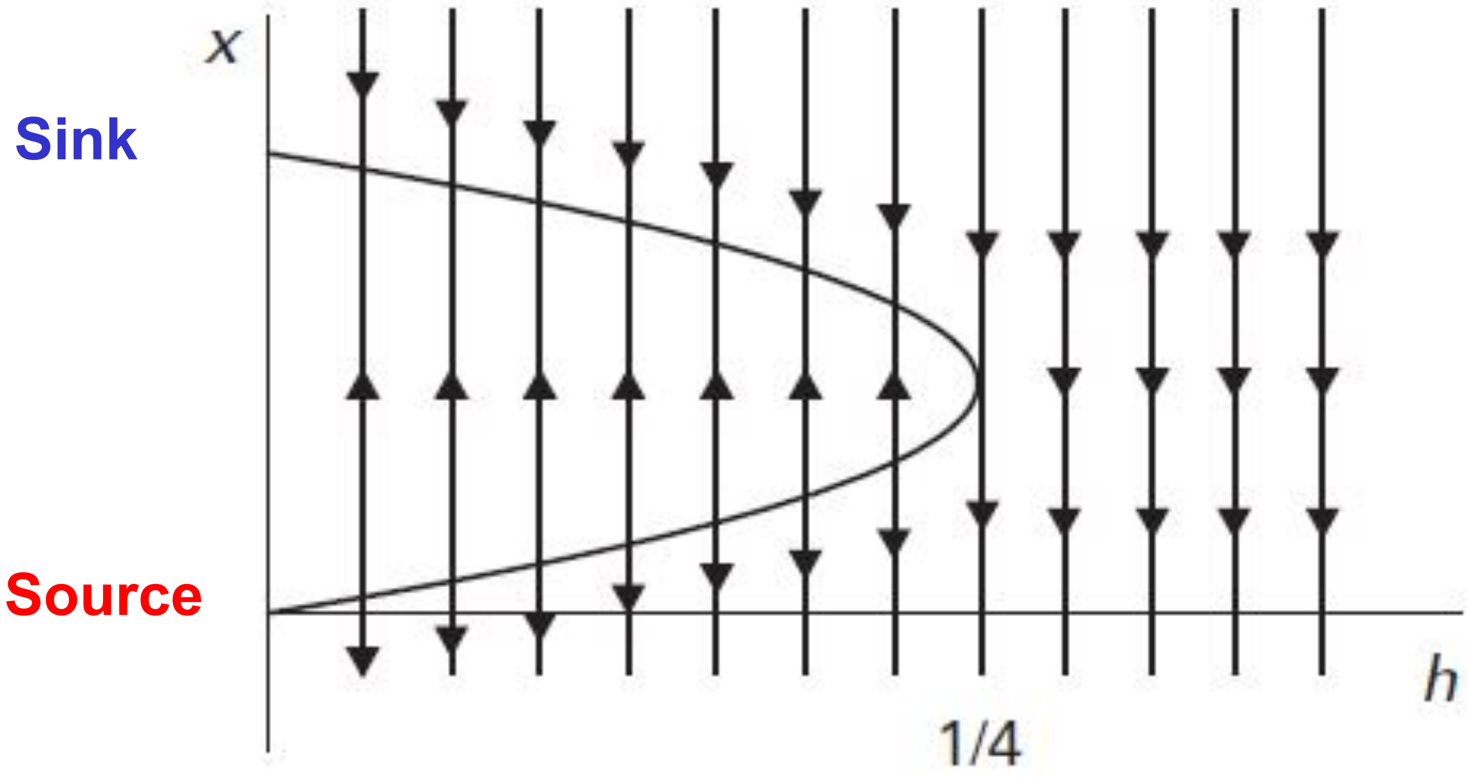


Figure 1.7 The bifurcation diagram for $f_h(x) = x(1-x) - h$.

1.3 Another Example of a Bifurcation

$$\boxed{x' = x^2 - ax}$$

bifurcation points

$$\boxed{f(x,a) = 0 \ \& \ f_x(x,a) = 0}$$

$$f(x,a) = 0 \qquad \boxed{x^2 - ax = 0} \qquad \boxed{a = 0}$$

$$f_x(x,a) = 0 \qquad \boxed{2x - a = 0} \qquad \boxed{x = \frac{a}{2}}$$

1.3 Another Example of a Bifurcation

$\boxed{x' = x^2 - ax = f(x,a)}$ $\boxed{f_x = 2x - a}$

bifurcation points $f(x,a) = 0$ & $f_x(x,a) = 0$

$\boxed{a = 0}$ $\boxed{x = 0}$

$critical\ points\ x_{c1,2} = a, or\ 0$

$a > 0$:

$x_{c1} = a$: $f_x(a) > 0, source$

$x_{c2} = 0$: $f_x(0) < 0, sink$

$a < 0$:

$x_{c1} = a$: $f_x(a) < 0, sink$

$x_{c2} = 0$: $f_x(0) > 0, source$

$a = 0$: $f_x(0) = 0 \rightarrow x' = x^2 \geq 0$

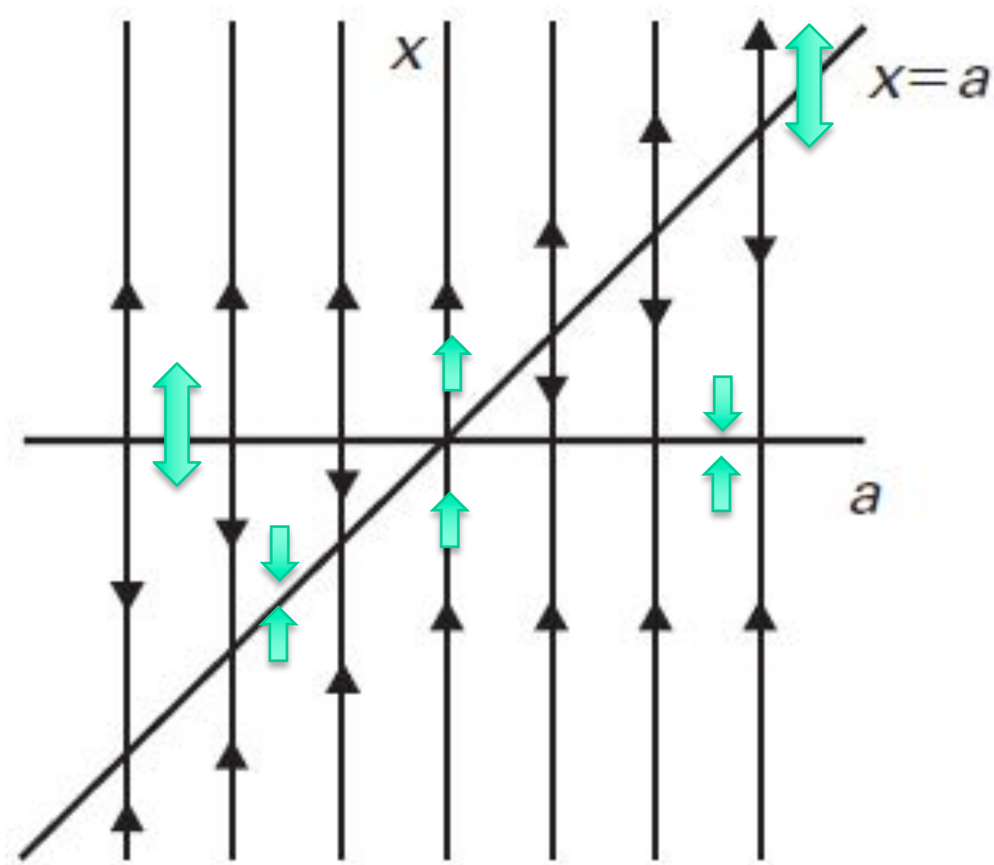
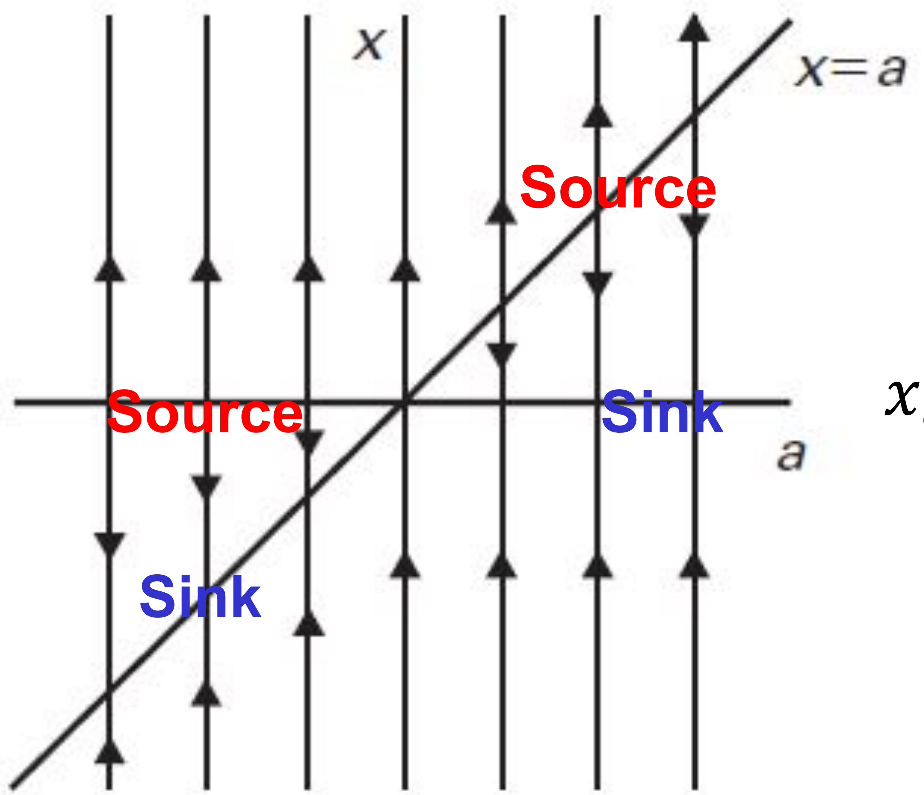


Figure 1.8 The bifurcation diagram for $x' = x^2 - ax$.

Section 1.3: Constant Harvesting and Bifurcations

$$\frac{dx}{dt} = x(x-a)$$

$x_c = a$



$x_c = 0$

Figure 1.8 The bifurcation diagram for $x' = x^2 - ax$.

1.4: Periodic Harvesting and Periodic Solutions

$$x' = x(1-x) - h(1+\sin(2\pi t)))$$

$$g(t) = h(1+\sin(2\pi t)))$$

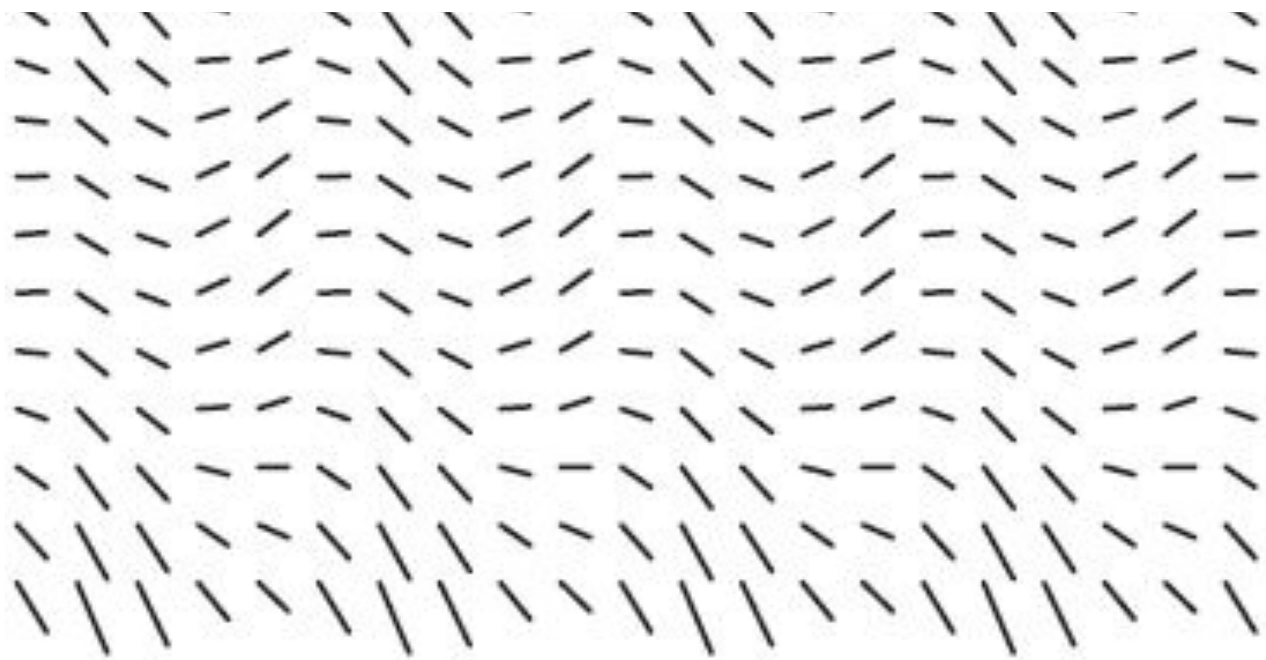


Figure 1.9 The slope field for $f(x) = x(1-x) - h(1+\sin(2\pi t))$.

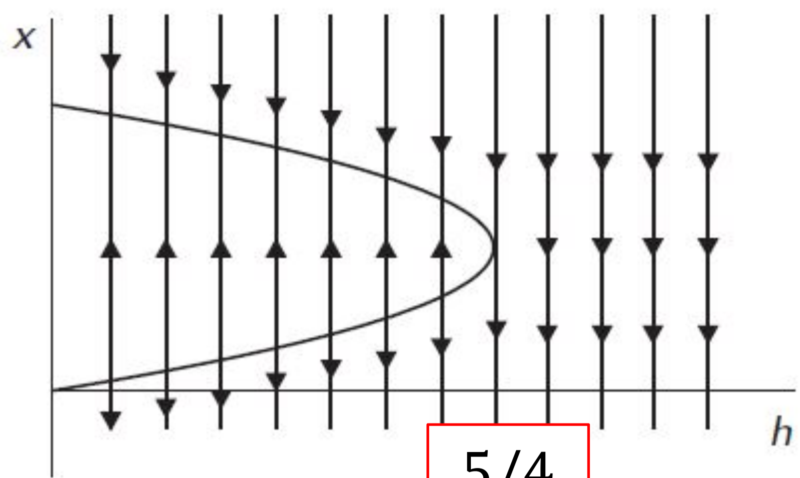
Bifurcation Point(s)

consider $\boxed{x' = 5x(1-x) - h = f(x,h)}$

bifurcation points $\boxed{f(x,h) = 0 \ \& \ f_x(x,h) = 0}$

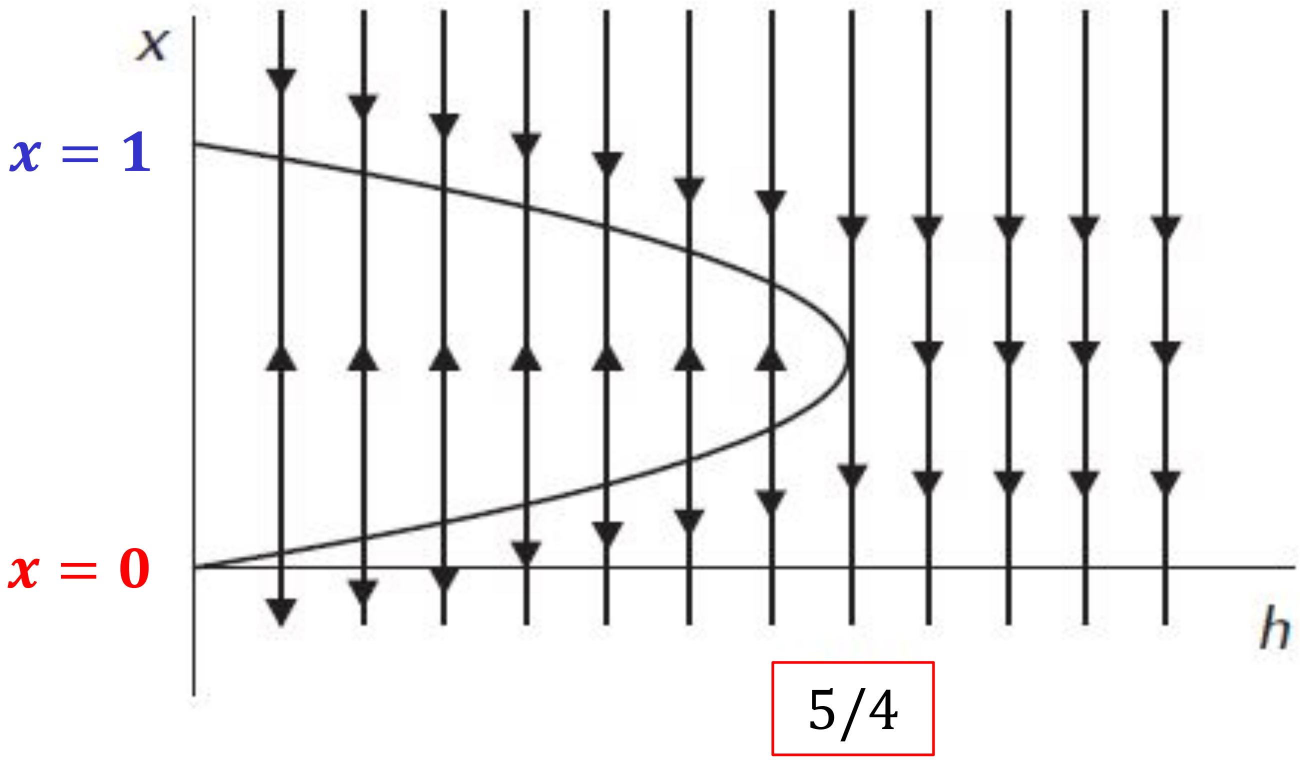
$f(x,h) = 0$ $\boxed{5x(1-x) - h = 0}$ $\boxed{h = 5/4}$

$f_x(x,h) = 0$ $\boxed{5 - 10x = 0}$ $\boxed{x = \frac{1}{2}}$



$\boxed{5/4}$

Potential Appearance of Stable and Unstable Points



For a time varying g(t) that is less than $\frac{5}{4}$,

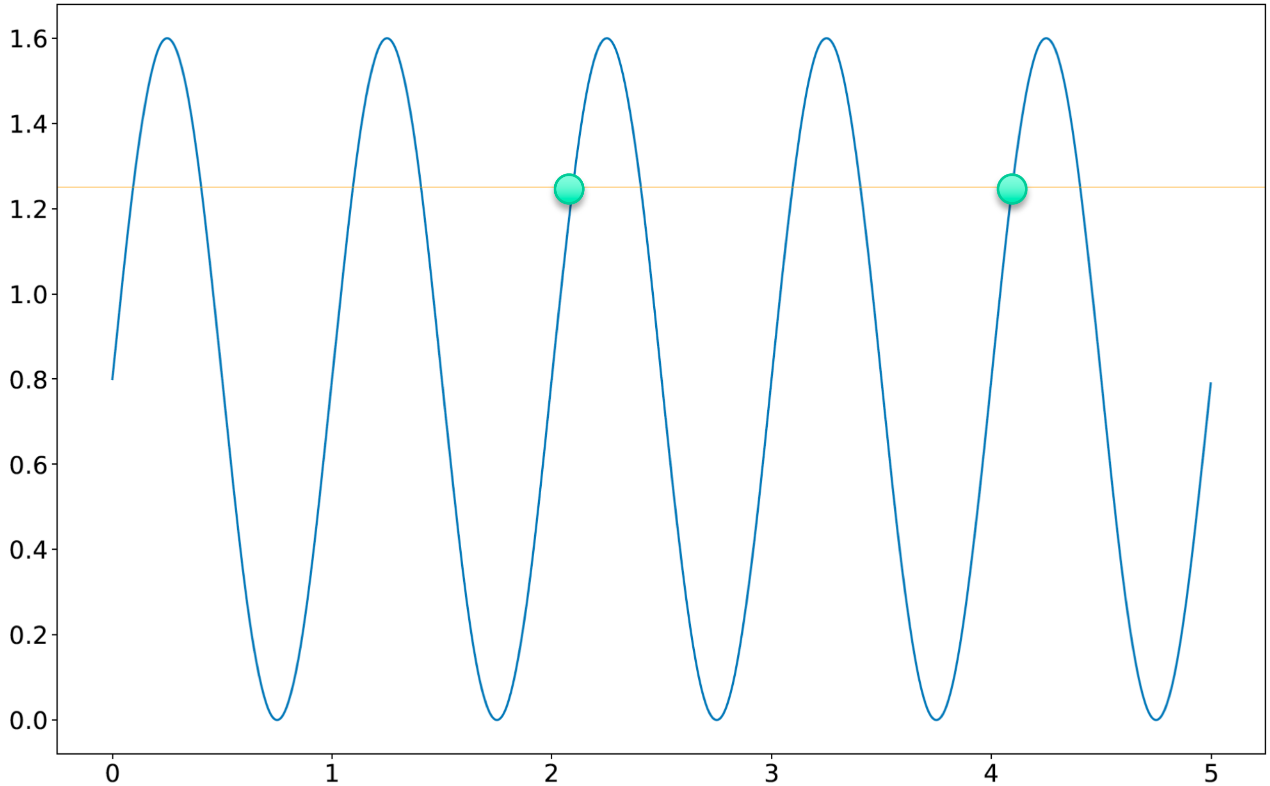
- stable points may appear between $\frac{1}{2}$ and 1, and
- unstable points may appear between 0 and $\frac{1}{2}$.

1.4: Periodic Harvesting (Forcing)

$$x' = x(1-x) - 0.8(1+\sin(2\pi t))$$

$$g(t) = 0.8(1+\sin(2\pi t))$$

$g(t)$



t

“”no critical point”

$\frac{5}{4}$

”two critical points”

1.4: Periodic Harvesting and Periodic Solutions

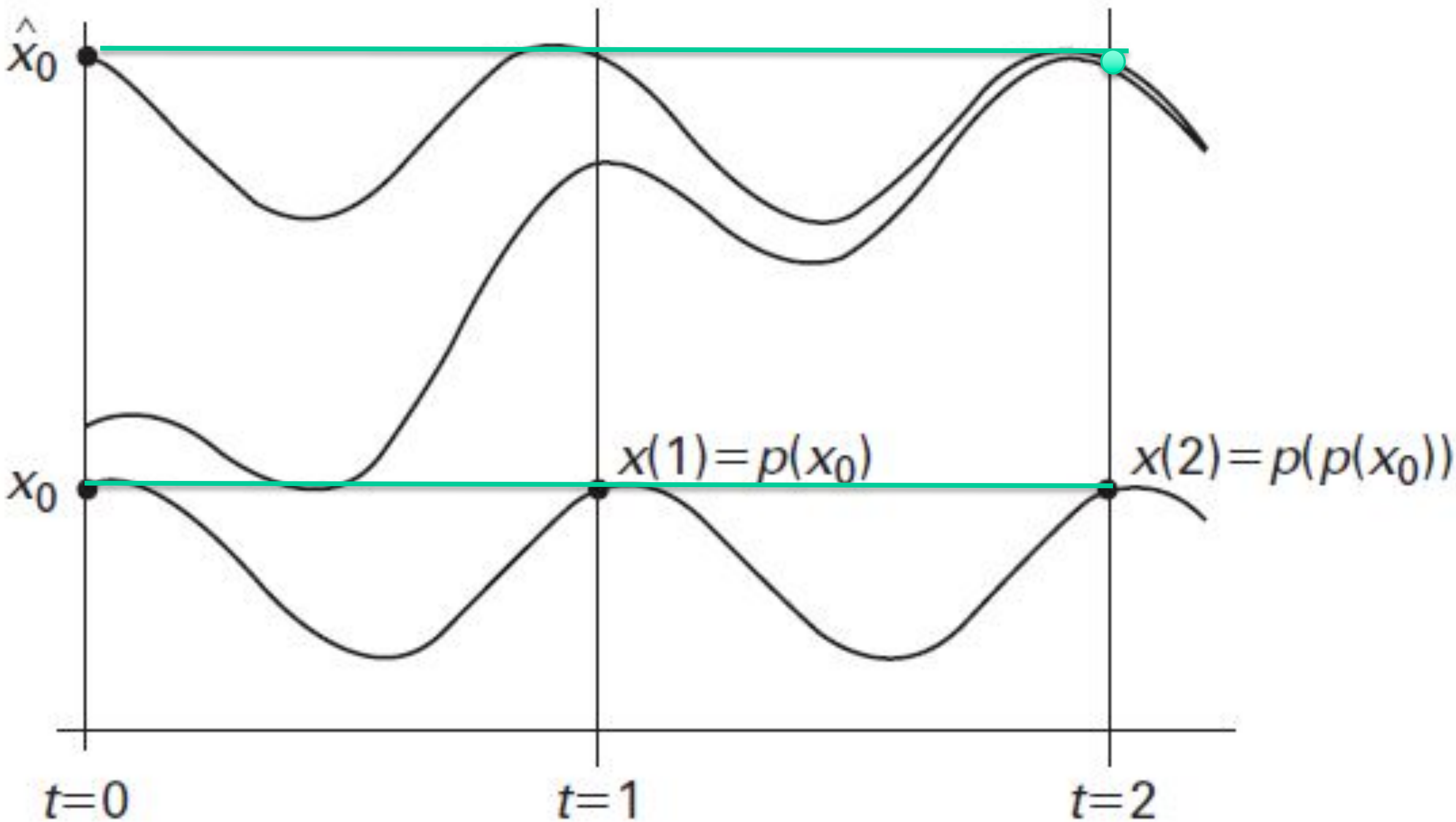


Figure 1.10 The Poincaré map for $x' = 5x(1-x) - 0.8(1 + \sin(2\pi t))$.

Section 1.5: Computing the Poincare Map

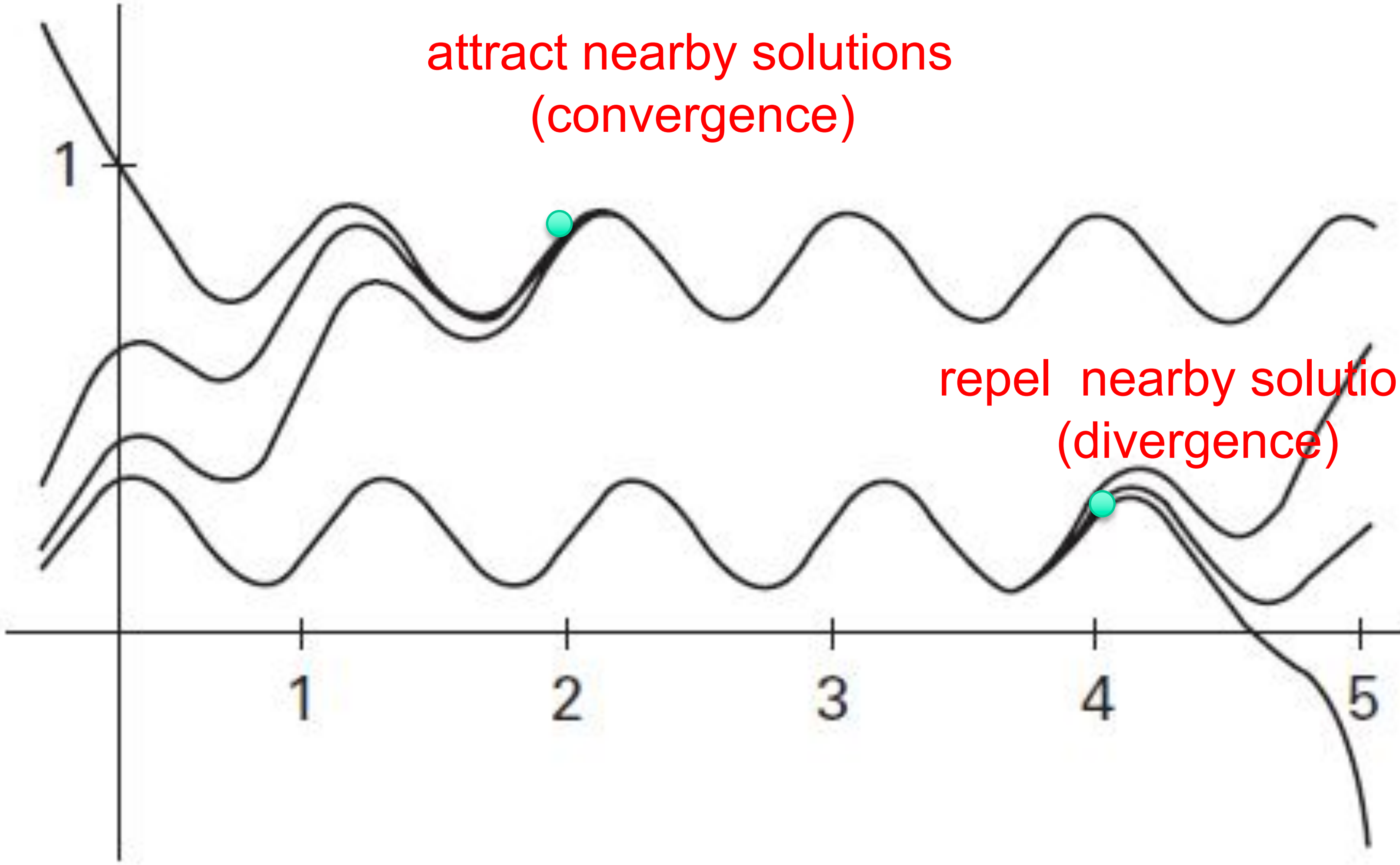
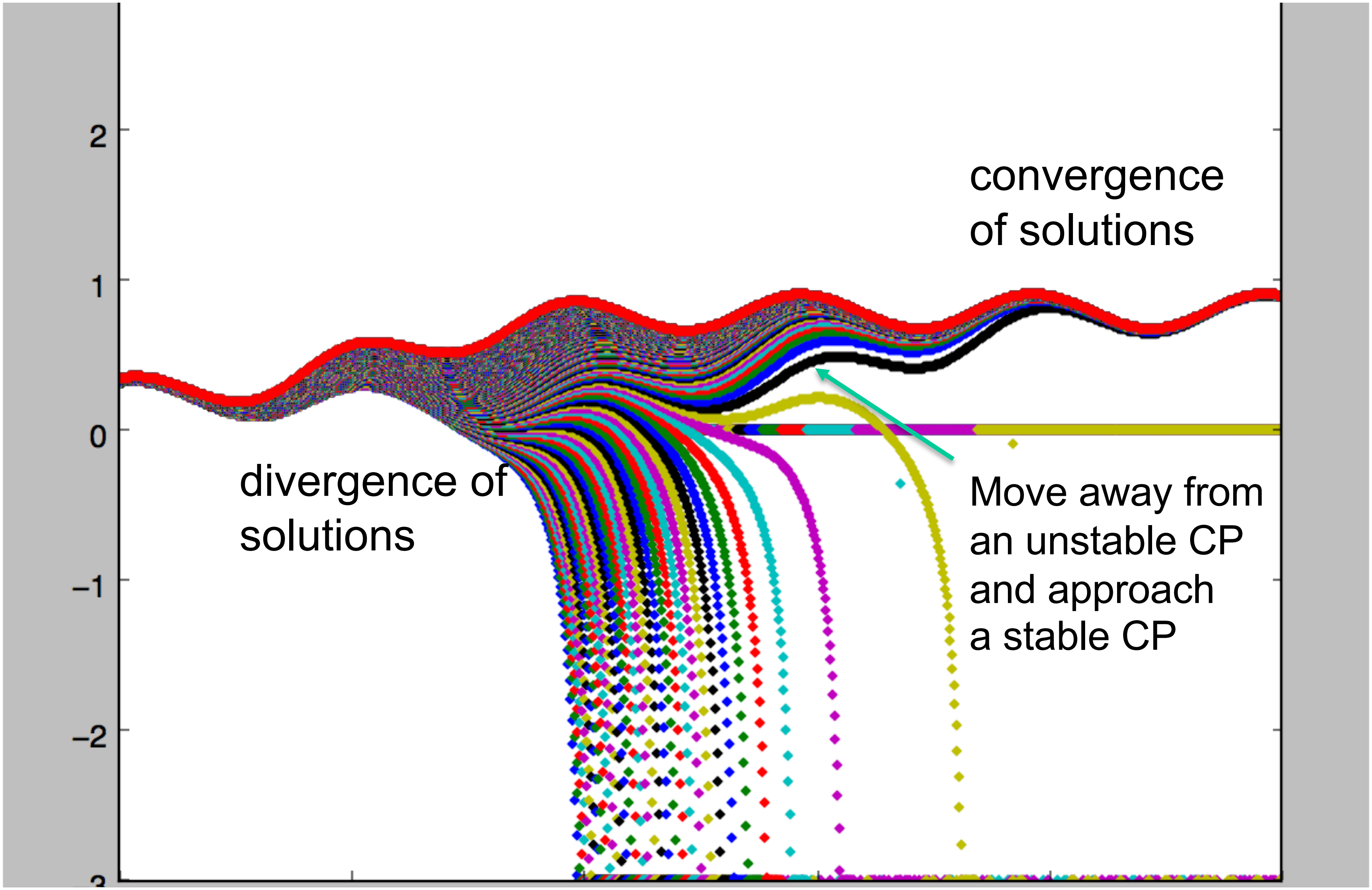
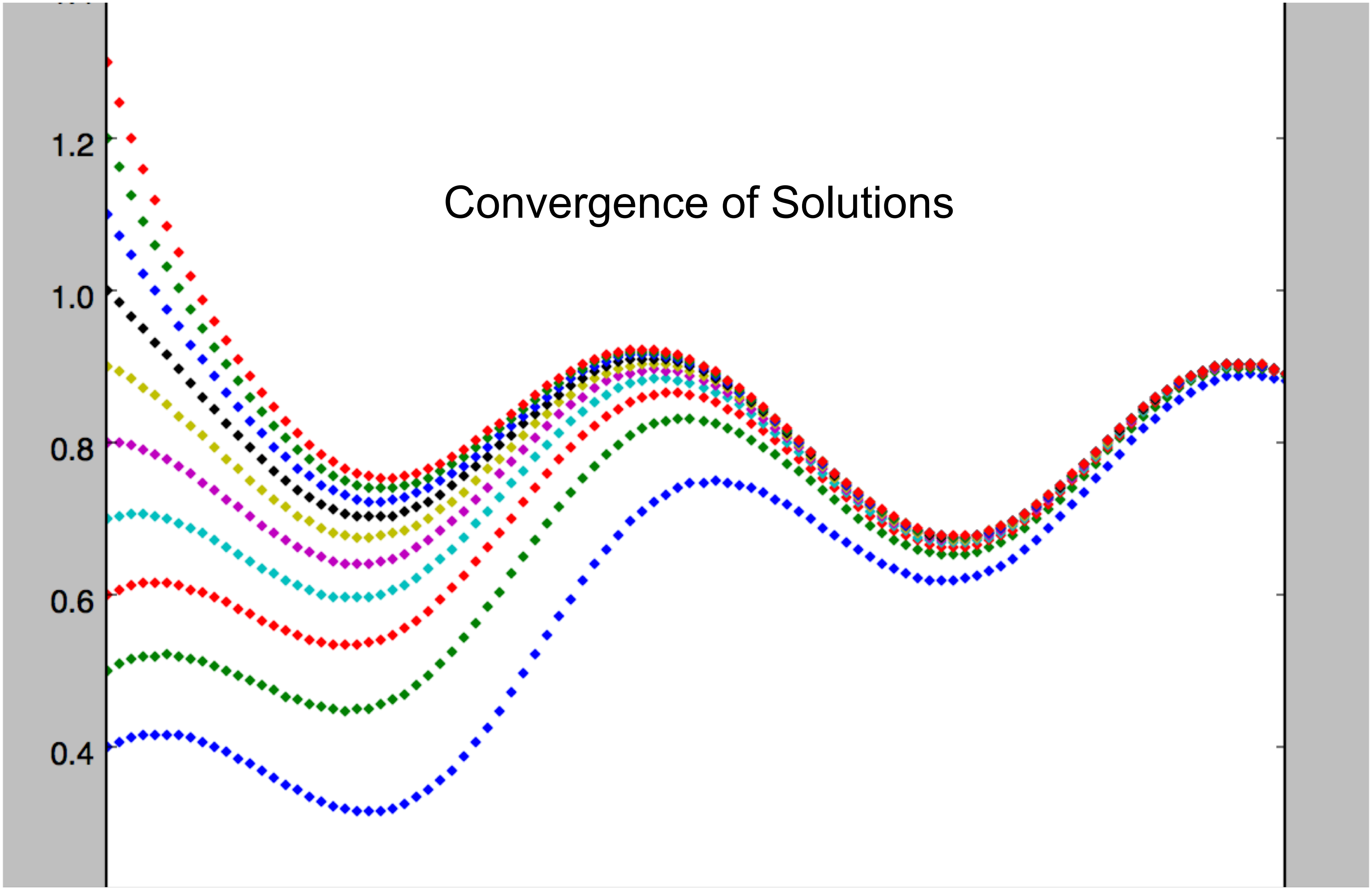


Figure 1.11 Several solutions of $x' = 5x(1-x) - 0.8(1+\sin(2\pi t))$.

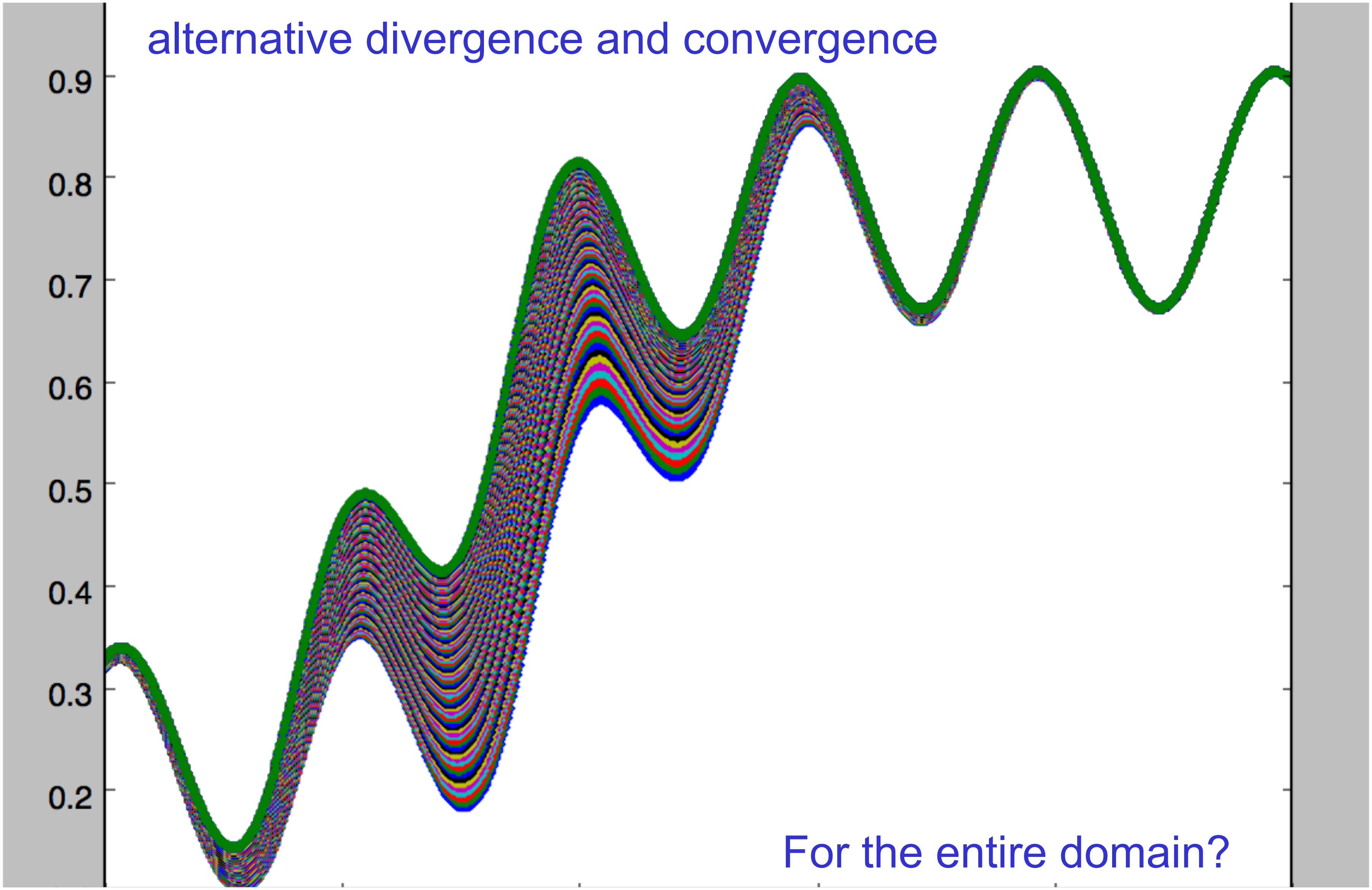
$$x' = 5x(1-x) - 0.8(1+\sin(2\pi t))$$



$$x' = 5x(1-x) - 0.8(1+\sin(2\pi t))$$



$$x' = 5x(1-x) - 0.8(1+\sin(2\pi t))$$



Logistic Equation with periodic Harvesting (Strogatz, 2015)

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8.5.4 (Logistic equation with sinusoidal harvesting) In Exercise 3.7.3 you were asked to consider a simple model of a fishery with constant harvesting. Now consider a generalization in which the harvesting varies periodically in time, perhaps due to daily or seasonal variations. To keep things simple, assume the periodic harvesting is purely sinusoidal (Benardete et al. 2008). Then, if the fish population grows logistically in the absence of harvesting, the model is given in dimensionless form by $\dot{x} = rx(1-x) - h(1+\alpha \sin t)$. Assume that $r, h > 0$ and $0 < \alpha < 1$.

a) Show that if $h > r/4$ the system has no periodic solutions, even though the fish are being harvested periodically with period $T = 2\pi$. What happens to the fish population in this case?

b) By using a Poincaré map argument like that in the text, show that if $h < \dfrac{r}{4(1+\alpha)}$, there exists a 2π-periodic solution—in fact, a stable limit cycle—in the strip $1/2 < x < 1$. Similarly, show there exists an unstable limit cycle in the strip $0 < x < 1/2$. Interpret your results biologically.

c) What happens in between cases (a) and (b), i.e., for $\dfrac{r}{4(1+\alpha)} < h < \dfrac{r}{4}$?

$$\frac{dr}{dt} = r(1 - r^3)$$

Limit Cycle

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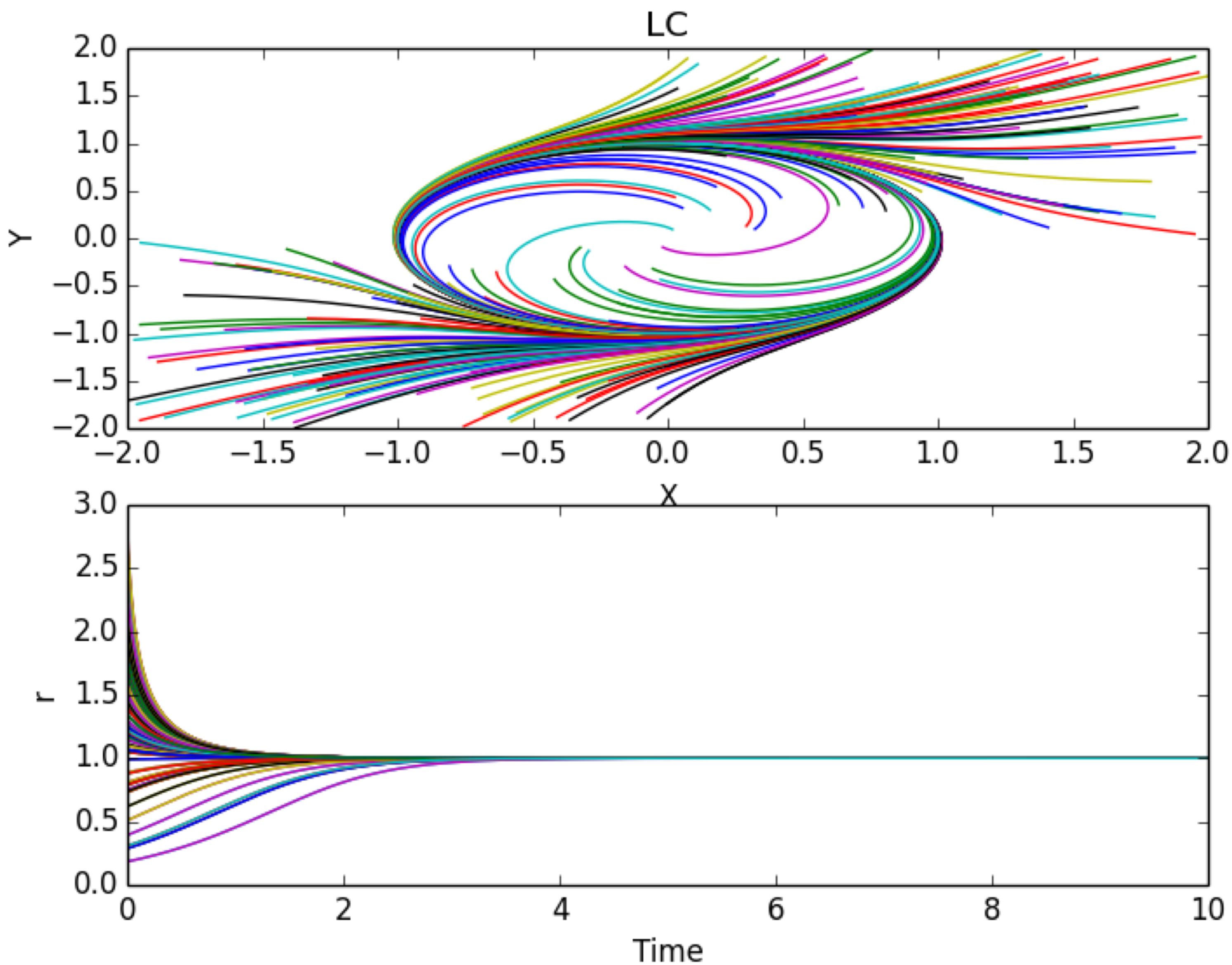

Key ODEs in Chapter 1

$$\frac{dx}{dt} = ax$$

bifurcation at $a = 0$

$$x = x_0 e^{at}$$

$$\boxed{\frac{dx}{dt} = ax\,(1 - x)}$$

bifurcation at $a = 0$

(the Logistic Eq)

$$x = \frac{x_0 e^{at}}{1 - x_0 + x_0 e^{at}}$$

(sigmoid function)

$$\boxed{\frac{dx}{dt} = x(1 - x) - h}$$

bifurcation at $\mathrm{h} = 1/4$

(the Logistic Eq with constant harvesting)

$$\boxed{\frac{dx}{dt} = x(1 - x) - h(1 + \sin(2\pi t))}$$

periodic forcing,
non-autonomous system

(the Logistic Eq with periodic harvesting)

Important Concepts

1. Bifurcation & Bifurcation points
2. Critical points, $f(x_c) = 0$;
3. (equilibrium points = fixed points = critical points)
4. Derivative tests & Perturbation method
5. General solution
6. Initial Value Problem (IVP)
7. Particular solution
8. Phase Line
9. Separable ODEs
10. Sink, Source, an Saddle
11. Stable vs. Unstable Solutions, $f'(x_c)$.
12. Structurally Stable vs. Unstable (i.e., with bifurcation)

Terminology

- We will study equations of the following form:

$$x' = f(x, t; a) \qquad \text{(ordinary differential equation)}$$

and

$$x \to g(x; a), \qquad \text{(difference equation)}$$

with $x \in U \subset R^n$, $t \in R^1$, and $\mathrm{a} \in R^p$. We refer to $x, t,$ and a ad dependent variables, independent variables and parameter.

- By a solution of the above differential equation, we mean a map, x, from some interval, $I \in R^1$ into R^n, written as follows:

$$x: I \to R^n$$

$$t \to x(t).$$

- System with $f = f(x; a)$ that is not a function of time is referred to as autonomous systems.

Notation

- To emphasize the dependence of solutions on initial values, x_0, we denote the corresponding solution by $\phi(t, x_0)$.
- This function, $\phi: R \times R \rightarrow R$, is called the flow associated with the differential equations.
- If we hold the variable x_0 fixed, the function

$$\mathrm{t} \rightarrow \phi(t, x_0)$$

is just an alternative expression of the solution of the differential equations satisfying the ICs.

- Alternatively, the solution can be expressed as $\phi(t, x_0)$ or $\phi_t(x_0)$.

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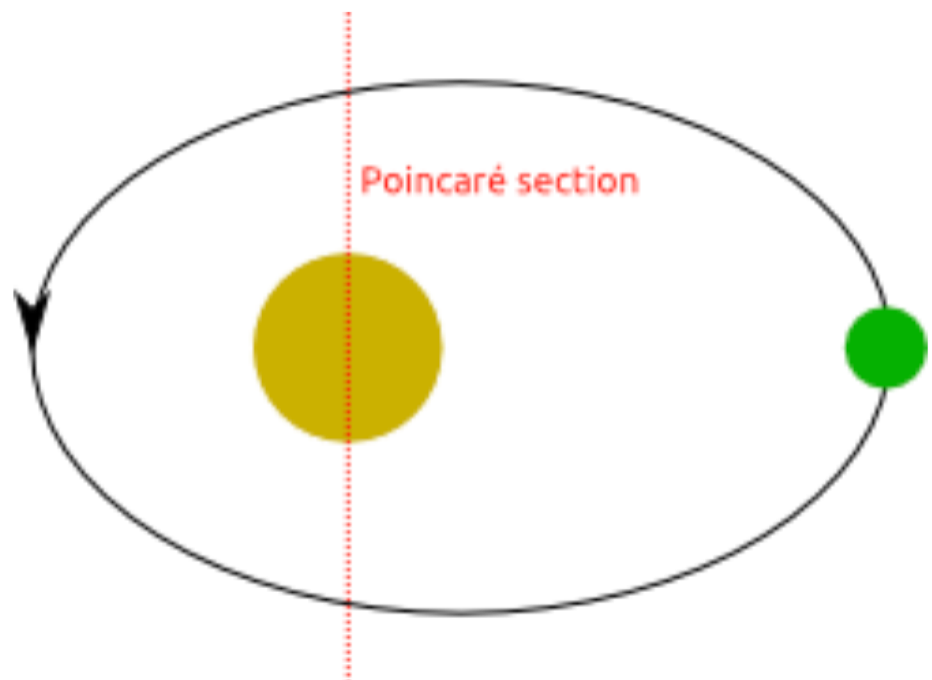
Poincare Section: Cross Section

- Instead of analyzing the entire trajectory of a planet, you would only look at its position once a year, whenever it intersects (with a given direction) a plane.
- This plane is a Poincaré section for the orbit of this planet.
- if the planet’s orbit is exactly periodic with a period length corresponding to one year, our yearly recording would always yield the same result. Namely, our planet would intersect the Poincaré section at the same point every year.

flow vs. map

continue vs. discrete in time



https://physics.stackexchange.com/questions/141493/poincar%C3%A9-maps-and-interpretation

Poincare Section: Cross Section

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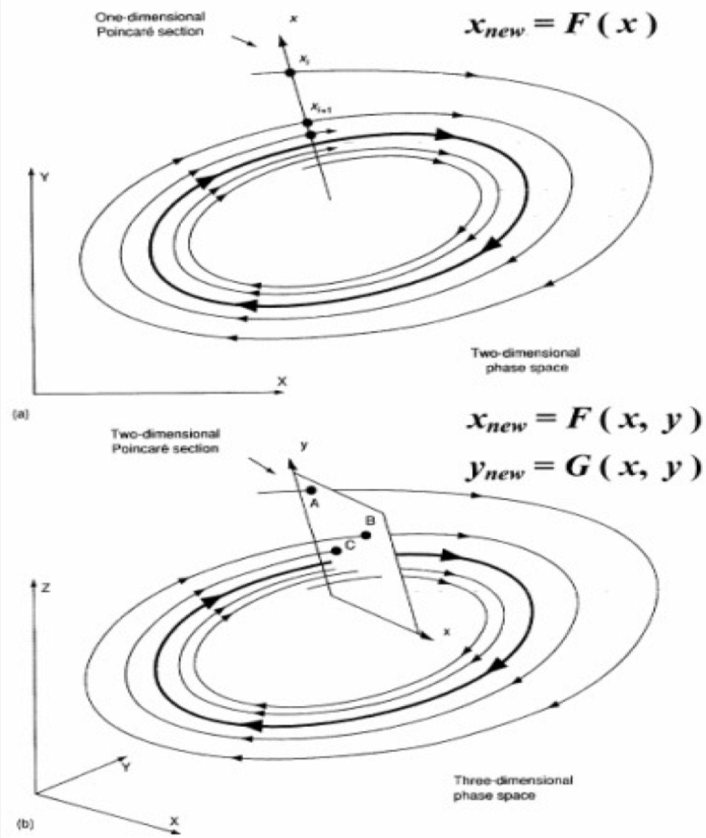
Poincaré Section

- To examine chaos, Poincare used the idea of a section
- This cuts across the phase-space orbits
- The original system flows in continuous time
- On the section, we observe steps in discrete time
- The flow is replaced by what is called an iterated map
- The dimension of the phase-space is reduced by one



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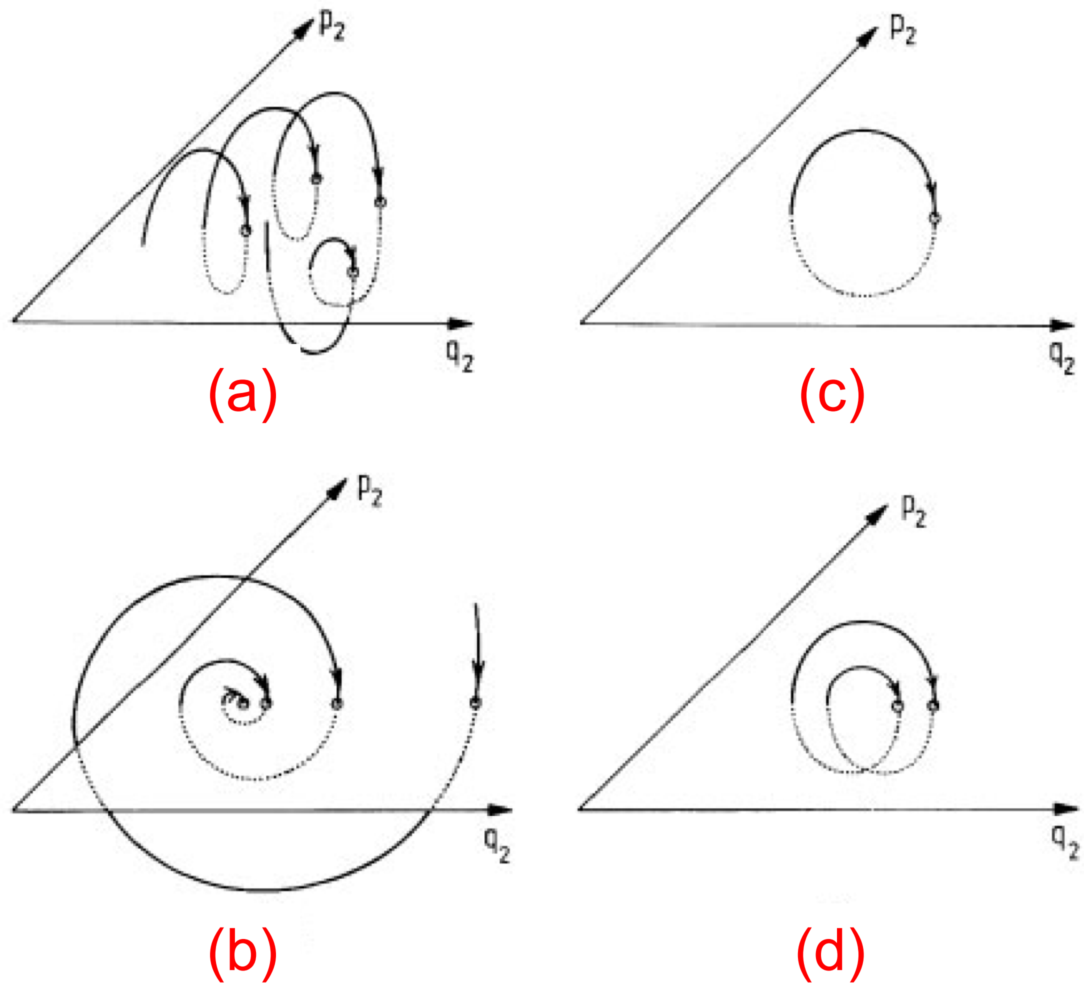
PHYS 460/660: Introduction to deterministic chaos

Poincare Section vs. Cross Section Supp

Chaotic

Steady state



Periodic

Half vs. full cycle

Cycle of period two

$$P_{n+1} = f(P_n)$$

Figure 6: Qualitatively different trajectories can be distinguished by their Poincaré sections: a) chaotic motion; b) approach of a fixed point; c) cycle; d) cycle of period two.

(Schuster)

Poincare Section

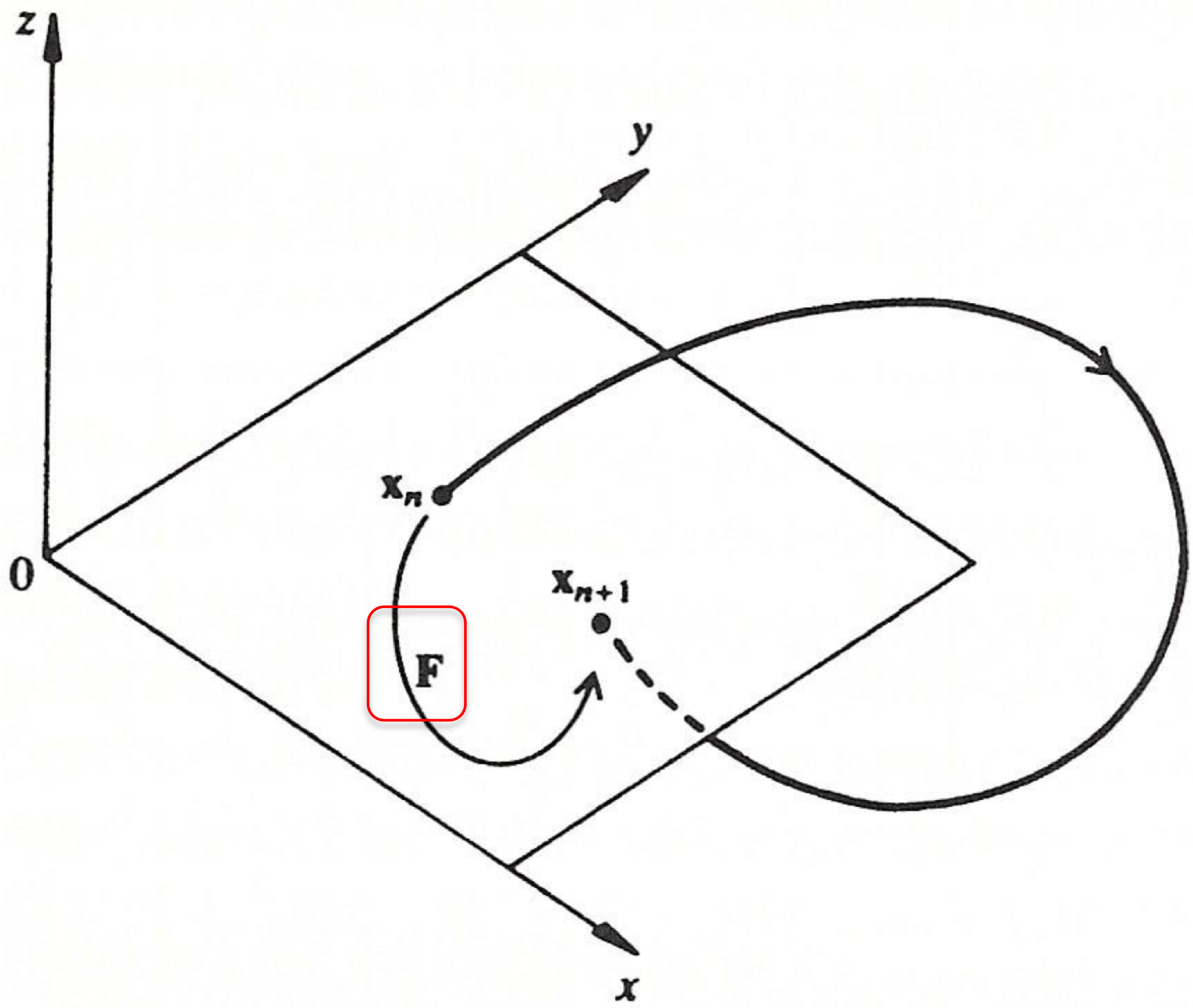


Fig. 1.17 A sketch of the return map.

Backup