Assignment 3 (Part II): November 20, 2023

Due December 5, 2023.

(1) Given the system of differential equations

$$\dot{x} = a - bx + x^2y - x;$$
$$\dot{y} = bx - x^2y,$$

and constants $a, b > 0$. Consider the equations on the space $X = [0, \infty) \times [0, \infty)$.

(a) Consider a compact quadrilateral U bounded by the edges $x = 0$, $y = 0$, $y = A - x$ and $y = B + x$. Determine constants $A > 0$ and $B > 0$ such that U is a trapping region for the above system of equations;

(b) Show that $(x, y) = (a, b/a)$ is the only equilibrium point in X and give conditions on a and b such that $(a, b/a)$ is a repeller;

(c) Restrict the differential equations to the quadrilateral U. Explain that the differential equations define a semi-flow $\phi \colon \mathbb{R}^+ \times U \to U$;

(d) Use (a), (b) and (c) to find an attractor $A \subset U$ that does not contain equilibrium points and indicate a non-trivial attractor-repeller pair.

(2) Consider the piecewise linear map $f \colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 3x & \text{for} \quad x \leq 1/2; \\ 3(1-x) & \text{for} \quad x \geq 1/2. \end{cases}$$

Consider the dynamical system $\phi \colon \mathbb{Z}^+ \times \mathbb{R} \to \mathbb{R}$ determined by the iteration of f, i.e. $x_{n+1} = f(x_n)$. In particular we can write $\phi(1, x) = f(x)$ and $\phi(n, x) = f^n(x)$, $n \geq 0$, the nth iterate of f.

(a) A repelling region for f with $\tau = -1$ is called a *repelling block*. Show that a subset $U \subset \mathbb{R}$ is a repelling block if and only if $f^{-1}(\mathrm{cl}\, U) \subset \mathrm{int}\, U$;

(b) Find constants $\alpha, \beta > 0$ such that $U_{\alpha,\beta} = [-\alpha, 1 + \beta]$ is a repelling region for ϕ;

(c) By definition $R = \alpha(U_{\alpha,\beta})$ is a (compact, forward-backward invariant) repeller for ϕ. Show that

$$R = \bigcap_{n \geq 0} f^{-n}\big([0, 1]\big),$$

(hint: use Proposition 3.2.14 in the course notes);

(d) Prove, using the expression for R in (c), that the repeller R is the Cantor middle-thirds set;

(e) Show that R is strongly invariant (hint: construct an appropriate surjective map).

(3) Consider the Lorenz equations

$$\begin{aligned}\dot{x} &= -\sigma x + \sigma y;\\ \dot{y} &= rx - y - xz;\\ \dot{z} &= xy - bz,\end{aligned}$$

with $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ and $\sigma, r, b > 0$.

(a) Consider the function $V(\mathbf{x}) = V(x, y, z) = x^2 + y^2 + (z - r - \sigma)^2$. Prove, using Gronwall's inequality, that for every $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$, with $V(\mathbf{x}) \leq C_0$, $C_0 > 0$, there exists a $\tau_{C_0} > 0$ such that

$$V\big(\phi(t, \mathbf{x})\big) \leq \frac{2b(r+\sigma)^2}{\alpha}, \quad \forall t \geq \tau_{C_0},$$

for some $\alpha > 0$.

Let $\phi\colon \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ be a semi-flow on $\mathbb{R}^n$. An *absorbing set* for ϕ is a compact set $N \subset \mathbb{R}^n$ such that for every bounded set $U \subset \mathbb{R}^n$ there exists a $\tau_U > 0$ so that

$$\phi(t, U) \subset N, \quad \forall t \geq \tau_U.$$

An open, bounded neighborhood V of N has the property that $\omega(V) = \omega(N) \subset \operatorname{int} V$, and thus $\omega(N)$ is an attractor.

(b) Use (a) to find an absorbing set in order to show that the Lorenz equations have a non-trivial, compact attractor, i.e. a compact set A such that $A = \omega(U)$ for some neighborhood $U \supset A$.

Show all calculations and details!

Good luck