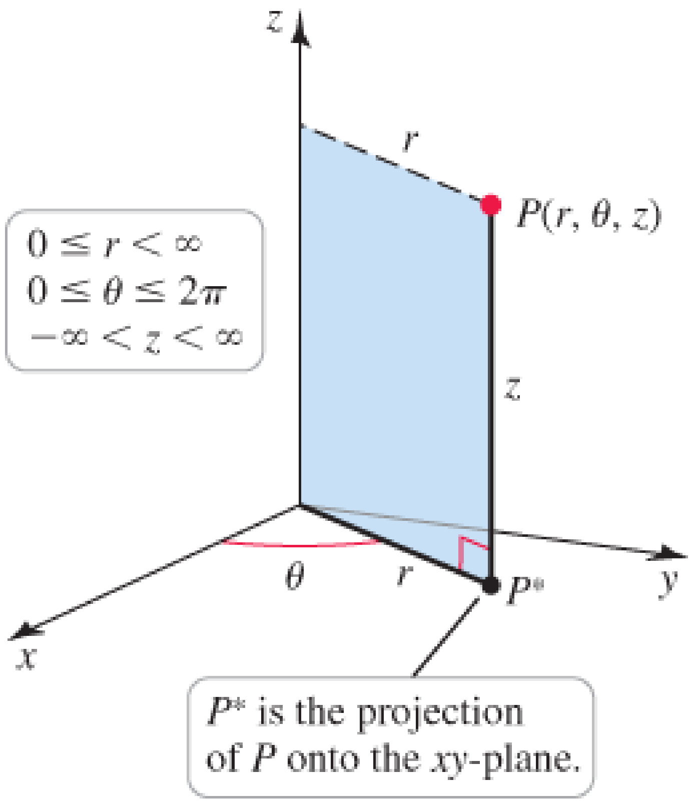
Lecture 15: Triple Integrals in Cylindrical Coordinates

Material Covered: Section 16.5 of text and Reading Assignment 2 Reference material (Section 3.2)
Objectives: Develop Tools for Sketching and Computing Triple Integrals in Cylindrical Coordinates
Concepts/Visualization Skills:

- Integration bounds for radius r, angle θ and z
- Visualization of bounding surfaces in cylindrical coordinates
- Differential volume in cylindrical coordinates

Cylindrical Coordinates



Interactive Demonstration of Constant Surfaces in Cylindrical Coordinates

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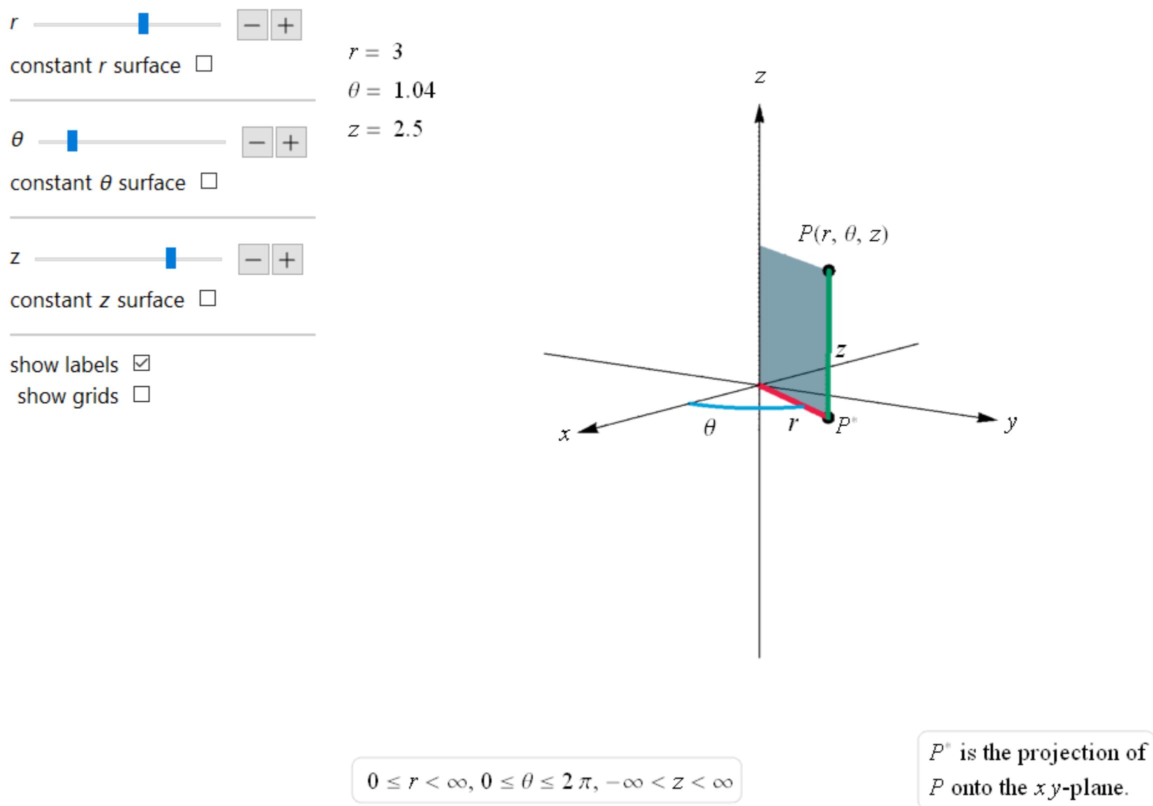
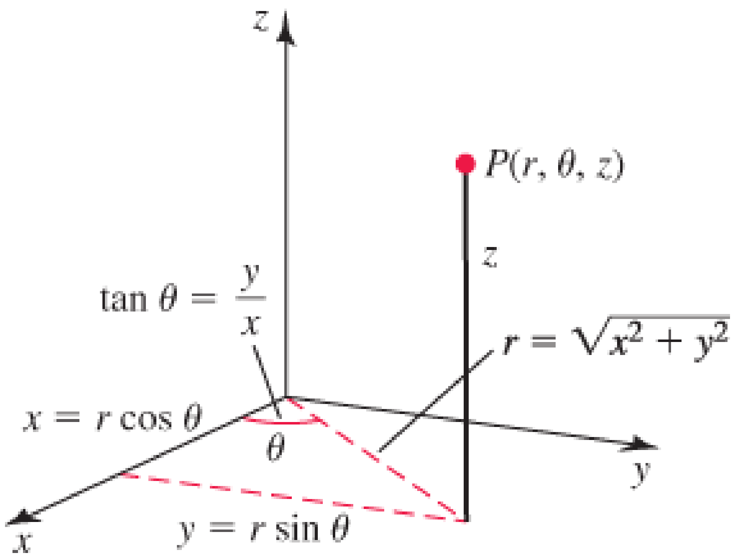


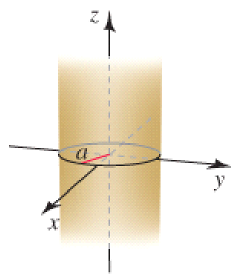
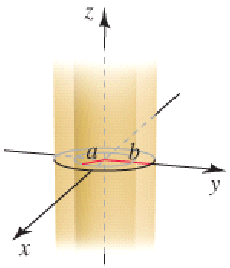
Cylindrical Coordinate Transformations



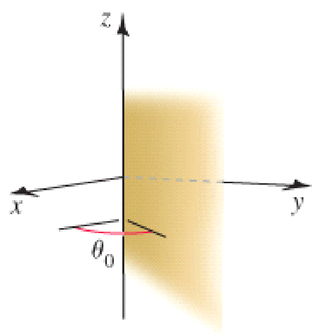
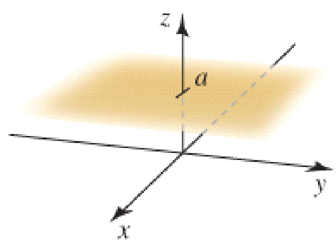
Transformations Between Cylindrical and Rectangular Coordinates

Rectangular $\to$ Cylindrical	Cylindrical $\to$ Rectangular
$r^2 = x^2 + y^2$ $\tan\,\theta = y/x$ $z = z$	$x = r\,\cos\,\theta$ $y = r\,\sin\,\theta$ $z = z$
To Establish Boundary	**To Establish Jacobian**

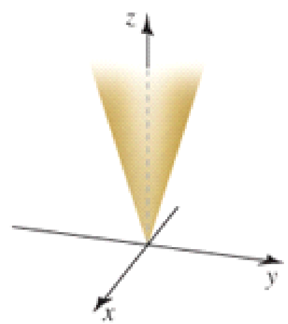
Transforming Surfaces in Cartesian Coordinates to Cylindrical Coordinates

Name	Description	Example
Cylinder	$\{(r, \theta, z) : r = a\}, a > 0$	
Cylindrical shell	$\{(r, \theta, z) : 0 < a \leq r \leq b\}$	

Transforming Surfaces in Cartesian Coordinates to Cylindrical Coordinates

Name	Description	Example
Vertical half-plane	$\{(r, \theta, z) : \theta = \theta_0\}$	
Horizontal plane	$\{(r, \theta, z) : z = a\}$	

Transforming Surfaces in Cartesian Coordinates to Cylindrical Coordinates

Name	Description	Example
Cone	$\{(r, \theta, z) : z = ar\},\ a \neq 0$	

Integration in Cylindrical Coordinates

Riemann Sum: eg. $f(x,y,z)$ is a mass density (units $\left[\frac{kg}{m^3}\right]$)

$$Mass = \lim_{\substack{\Delta V_k \to 0 \\ n\to\infty}} \sum_{k=1}^{n} f(r_k^* cos(\theta_k^*), r_k^* sin(\theta_k^*), z_k^*)\Delta V_k = \iiint_D f(x,y,z)dV = \iiint_D f(x,y,z) r dr d\theta dz$$

$$\Delta V_k = r_k^* \Delta r \Delta\theta \Delta z$$

Theorem 16.6 Change of Variables for Triple Integrals in Cylindrical Coordinates

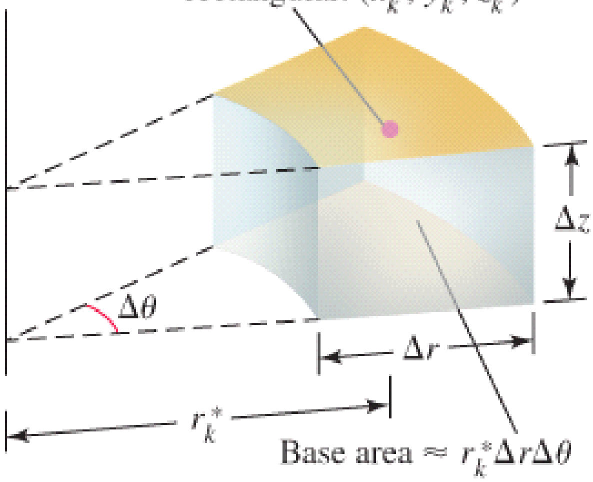
Let f be continuous over the region D, expressed in cylindrical coordinates as

$$D = \{(r,\ \theta,\ z) :\ 0 \le g(\theta) \le r \le h(\theta),\ \alpha \le \theta \le \beta,\ G(x,\ y) \le z \le H(x,\ y)\}.$$

Then f is integrable over D, and the triple integral of f over D is

$$\iiint_D f(x,\ y,\ z)\, dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(r\cos\theta,\, r\sin\theta)}^{H(r\cos\theta,\, r\sin\theta)} f(r\cos\theta,\ r\sin\theta,\ z)\, dz\, r\, dr\, d\theta.$$

Differential Volume in Cylindrical Coordinates



Approximate volume $\Delta V_k = r_k^*\Delta r\Delta\theta\Delta z$

$$dxdydz \rightarrow \left|\left|[J]\right|_{det}\right| dzdrd\theta = \left|\left|\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{bmatrix}\right|_{det}\right| dzdrd\theta$$

$$= \left|\begin{vmatrix} cos\theta & -rsin\theta & 0 \\ sin\theta & rcos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}_{det}\right| dzdrd\theta = rdzdrd\theta$$

Note: equations were set up in a right handed coordinate system so the Jacobian matrix is positive

Interactive Demonstration of Differential Volume in Cylindrical Coordinates

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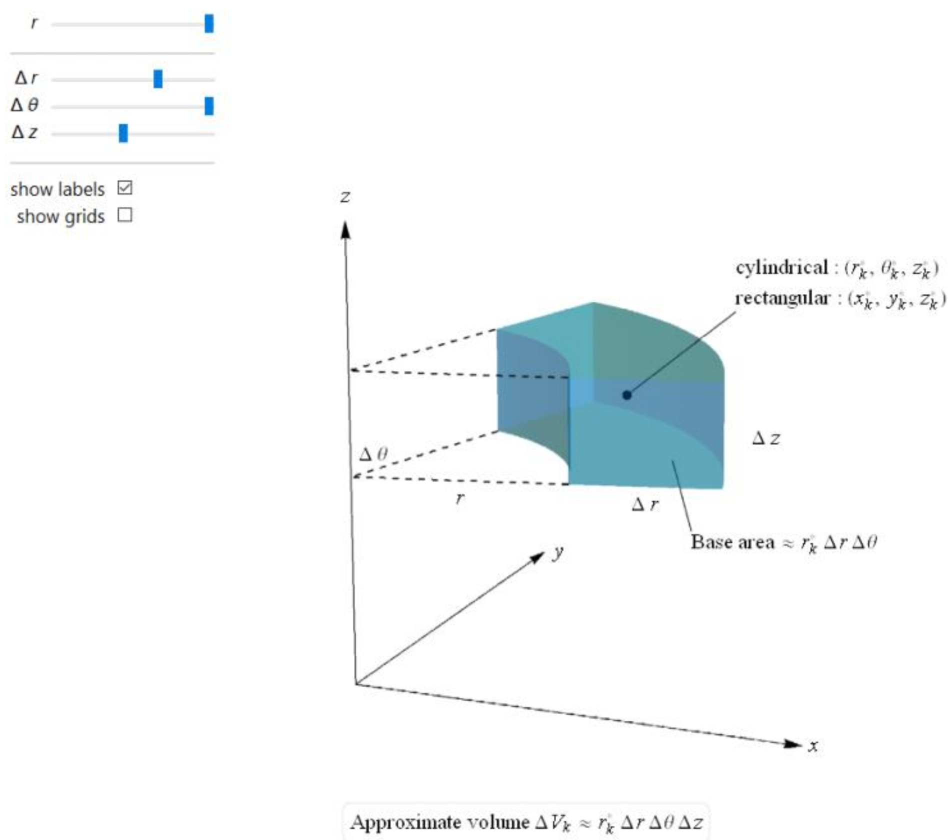
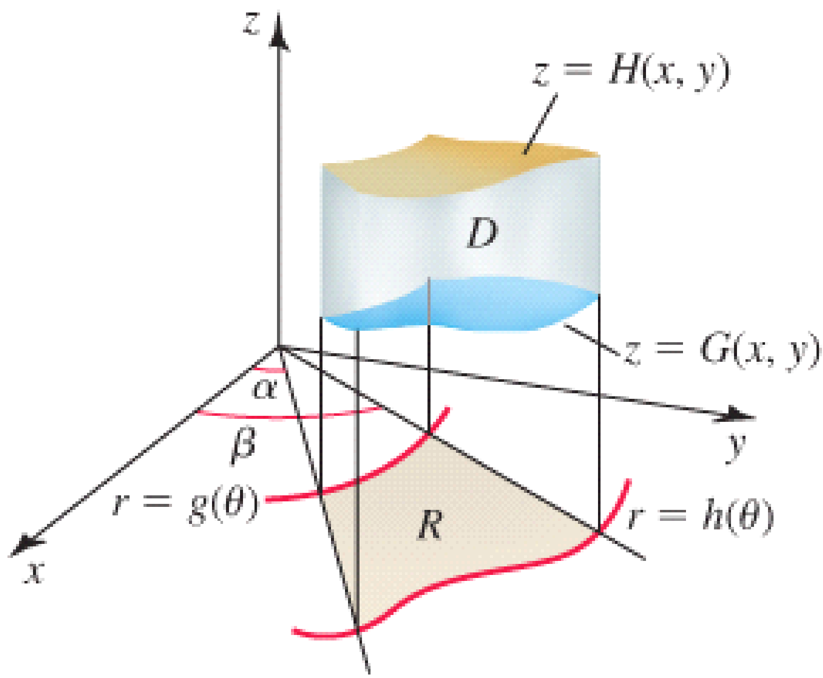
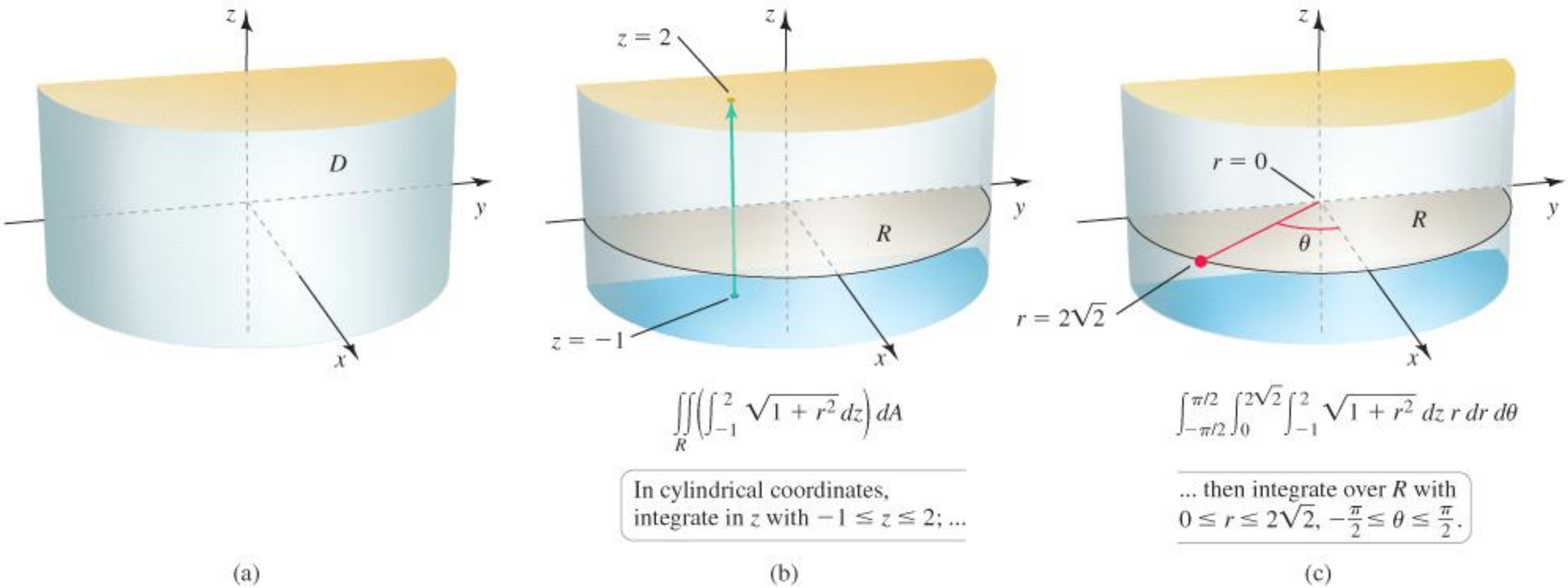


Boundary of Domain in Cylindrical Coordinates



Example Problem 1



Problem Statement:

Evaluate the integral $I = \int_0^{2\sqrt{2}} \int_{-\sqrt{8-x^2}}^{\sqrt{8-x^2}} \int_{-1}^{2} \sqrt{1 + x^2 + y^2}\, dzdydx$

Interactive Demonstration of Problem 1

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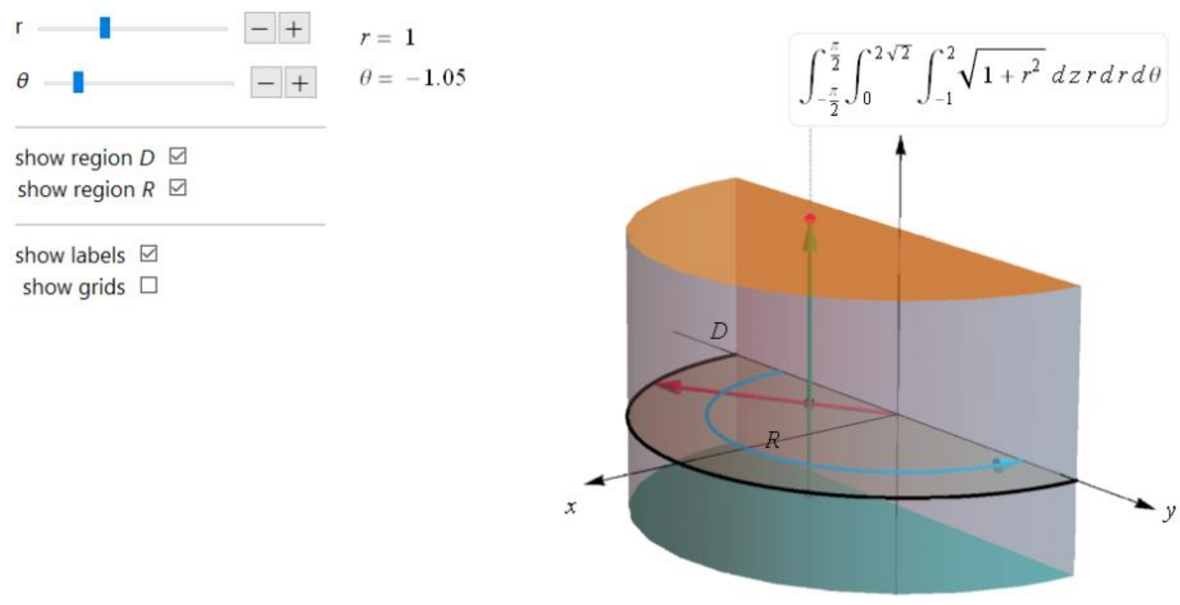


In cylindrical coordinates, integrate in z with $-1 \le z \le 2$; then integrate over R with $0 \le r \le 2\sqrt{2}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

Example Problem 1 Solution

Integrand: $f(x, y, z) = \sqrt{1 + x^2 + y^2} = \sqrt{1 + r^2}$

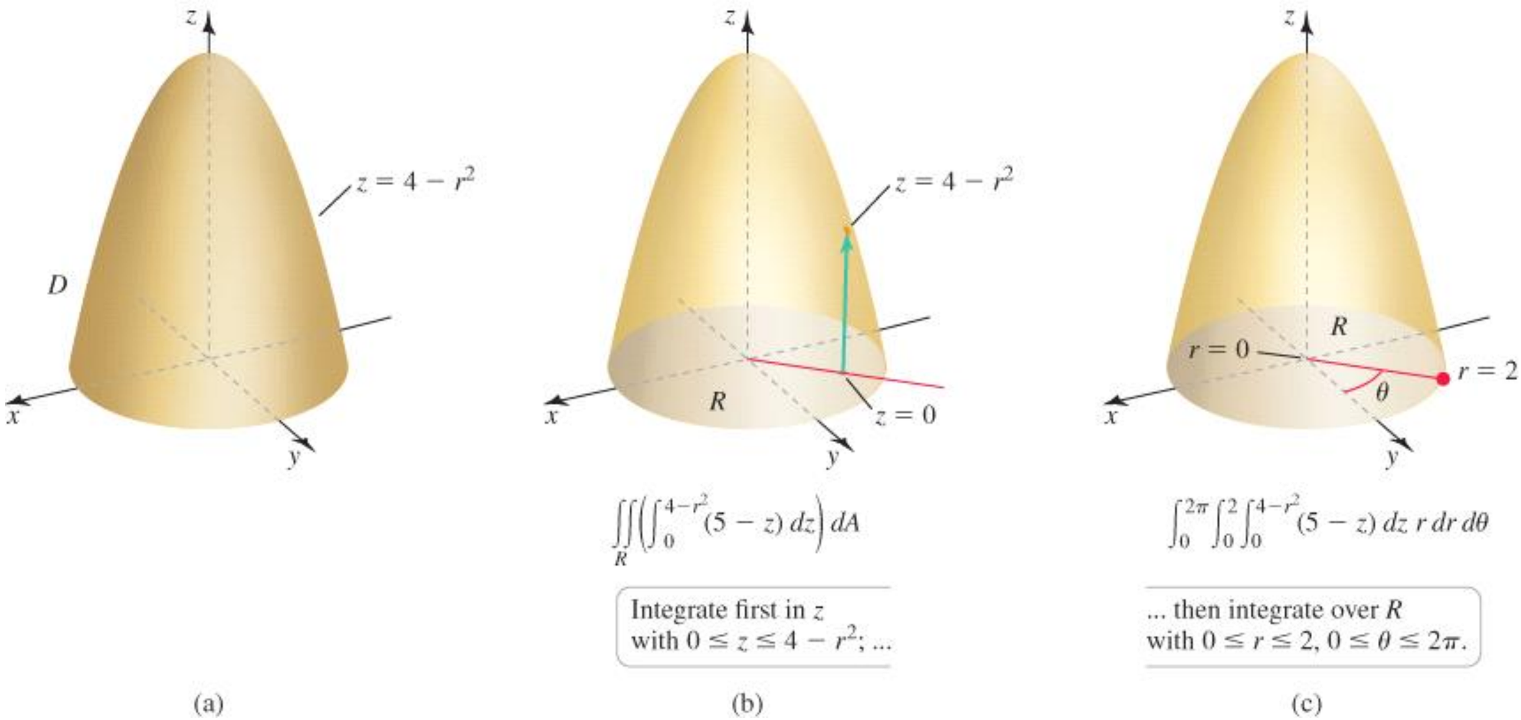
Region of Integration: $D = \left\{(r, \theta, z): 0 \leq r \leq 2\sqrt{2},\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, -1 \leq z \leq 2\right\}$

Solution to Integral:

$$\iiint_D f(x, y, z)dV = \int_{-\pi/2}^{\pi/2} \int_0^{2\sqrt{2}} \int_{-1}^{2} \sqrt{1 + r^2}dzrdrd\theta = 3\int_{-\pi/2}^{\pi/2} \int_0^{2\sqrt{2}} \sqrt{1 + r^2}rdrd\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(1 + r^2\right)^{3/2}\Big|_0^{2\sqrt{2}} d\theta = \int_{-\pi/2}^{\pi/2} 26d\theta = 26\pi$$

Example Problem 2



Problem Statement:

Find the mass of the cylinder D bounded by the paraboloid $z = 4 - r^2$ and the plane $z = 0$, where the mass density of the solid in units of $\left[\frac{kg}{m^3}\right]$, given in cylindrical coordinates, is $f(x, y, z) = 5 - z$ (heavy near the base and light near the vertex).

Interactive Demonstration of Problem 2

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r

θ

show region D ☑

show region R ☑

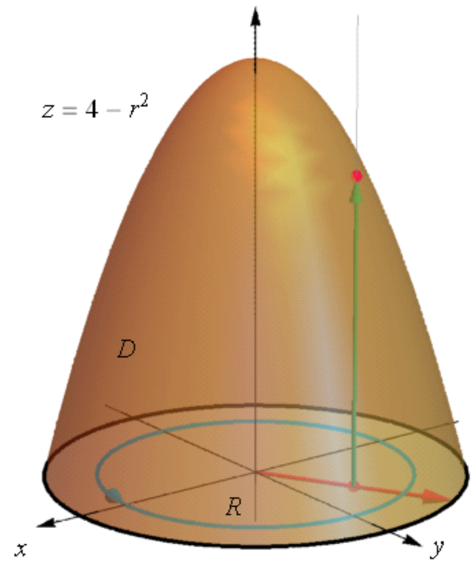
show labels ☑

show grids ☐

$r = 1$

$\theta = 2.09$

$$\int_0^{2\pi}\int_0^{2}\int_0^{4-r^2}(5-z)\,dz\,r\,dr\,d\theta$$



Integrate first in z with $0 \le z \le 4 - r^2$;
then integrate over R with $0 \le r \le 2$, $0 \le \theta \le 2\pi$.

Example Problem 2 Solution

Integrand: $f(x, y, z) = 5 - z$

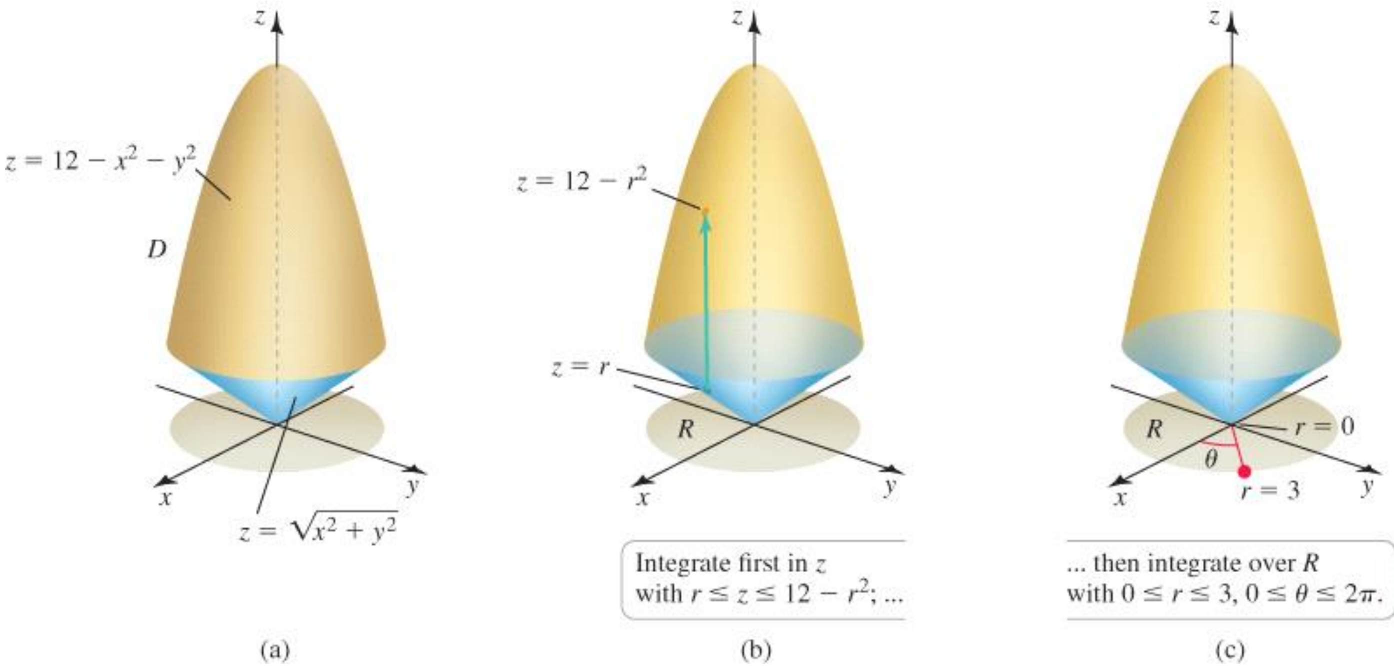
Region of Integration: $D = \{(r, \theta, z): 0 \le r \le 2,\ 0 \le \theta \le 2\pi, 0 \le z \le 4 - r^2\}$

Solution to Integral:

$$\iiint_D f(x, y, z)dV = \int_0^{2\pi}\int_0^{2}\int_0^{4-r^2}(5 - z)dzrdrd\theta = \int_0^{2\pi}\int_0^{2}\left(5z - \frac{z^2}{2}\right)\Bigg|_0^{4-r^2} rdrd\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{2}(24r - 2r^3 - r^5)drd\theta = \int_0^{2\pi}\frac{44}{3}d\theta = \frac{88\pi}{3}$$

Example Problem 3



Problem Statement:

Find the volume of the solid D between the cone $z = \sqrt{x^2 + y^2}$ and the inverted paraboloid $z = 12 - x^2 - y^2$.

Interactive Demonstration of Problem 3

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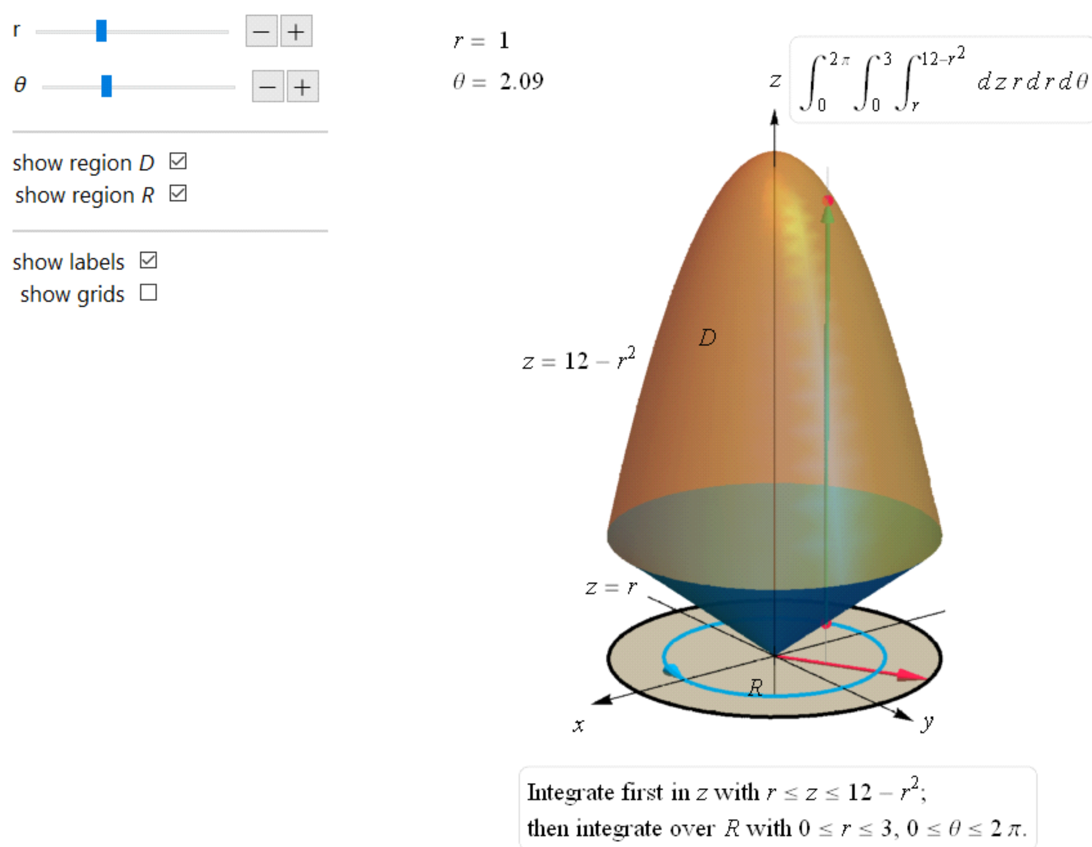

Example Problem 3 Solution

Integrand: $f(x,y,z) = 1$ (no units)

Region of Integration: $D = \{(r,\theta,z): 0 \le r \le 3,\ 0 \le \theta \le 2\pi,\ r \le z \le 12 - r^2\}$

Note: region R on the xy plane is determined by the intersection line obtained by equating the upper and lower bounding surfaces

$z = \sqrt{x^2 + y^2} = 12 - x^2 - y^2.$

The only valid solution is $x^2 + y^2 = 9$ therefore the region R is $x^2 + y^2 \le 9$

Solution to Integral:

$$\iiint_D f(x,y,z)dV = \int_0^{2\pi}\int_0^3\int_r^{12-r^2} dz r dr d\theta = \int_0^{2\pi}\int_0^3 (12 - r^2 - r) r dr d\theta = \int_0^{2\pi} \frac{99}{4} d\theta = \frac{99\pi}{2}$$

Lect 15 Q&A Scratch Pad