Lecture 27

November 16, 2021 3:10 PM

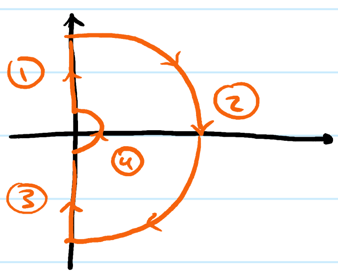
Review:

↳ Nyquist Stability

↳ Case when $L(s)$ has poles on imag axis

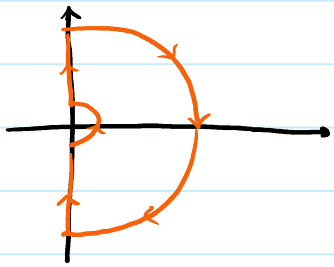
Idea:

↳ Use contours to exclude the poles on the imag axis. The Nyquist criterion then remains the same.



1, 2, 3 as before

4: $s = re^{i\theta}$ $\theta \in [-\pi/2, \pi/2]$



Suppose $L(s)$ has k poles at 0

↳ What is the effect of part 4 of the contour on the Nyquist plot

$$L(s) = \frac{N_L(s)}{s^k D_L(s)}$$

On part 4:

$$L(re^{i\theta}) = \frac{N_L(re^{i\theta})}{(re^{i\theta})^k D_L(re^{i\theta})}$$

circle of radius $\frac{1}{r^k}$

$\to 0$

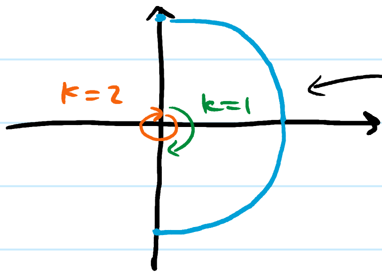
$$\ldots \quad (re^{i\theta})^k D_c(re^{i\theta})$$

As $r \to 0$, $\dfrac{N_c(0)}{(re^{i\theta})^k D_c(0)} = \dfrac{1}{r} e^{-ik\theta}$ $\to 0$

$\dfrac{N_c(0)}{D_c(0)}$ an integer

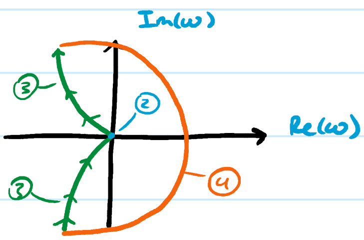
describing angle of $k\pi$ oriented CW

angle of circular contour given by $k\pi$



e.g. $L(s) = \dfrac{10}{s(s+1)}$

(Last time we drew parts of ①, ②, ③ of γ)



L has $k=1$ pole at 0

↳ part ④ will give a semicircle of ∞ radius, oriented CW

Nyquist Criterion

↳ L has 0 poles in ORHP

↳ roots of $1+L$ are in OLHP iff γ encircles -1 0 times CCW.

↳ This is the case in above example

Frequency Response

↳ In order to plot the Nyquist contour of a TF $L(s)$ we need to plot the complex # $L(i\omega)$ as ω varies in $(-\infty, +\infty)$

Suppose $L(s)$ is BIBO stable (i.e. all poles in OLHP).

↳ any bounded input yields bounded output

$L(i\omega)$ is the Frequency Response

$u(t)=\sin(\omega t) \longrightarrow \boxed{L(s)} \longrightarrow y(t)$ $\qquad U(s)=\frac{\omega}{s^2+\omega^2}$

$$Y(s)=\underbrace{L(s)}_{\text{poles in OLHP}} \cdot \underbrace{\frac{\omega}{s^2+\omega^2}}_{\text{poles at } \pm i\omega}$$

$$y(t)=\sum_i \text{Res}(Y(s)e^{st}, p_i)$$

$$= \sum \text{Res}(Ye^{st}, \text{poles of } L \text{ in OLHP}) \leftarrow \tilde{y}(t)$$

$$+ \underbrace{\sum \text{Res}(Ye^{st}, s=i\omega)}_{*} + (\bar{*}) \leftarrow \bar{y}(t)$$

Important Property

↳ each residue in $\tilde{y}(t)$ gives a function that tends to 0 exponentially. Say that a general pole of $L(s)$ is at $-a \pm ib$ $(a>0)$.

$$\text{Res}(Y(s)e^{st}, -a \pm ib) = \ldots e^{(-a \pm ib)t} = e^{-at}e^{\pm ibt} \rightarrow 0$$

$\tilde{y}(t)$ is the transient component of $y(t)$

$\bar{y}(t)$ is the steady-state behaviour

$$y(t) \rightarrow \bar{y}(t) = \text{Res}(Y(s)e^{st}, s=+\omega i) + \text{Res}(Y(s)e^{st}, s=-\omega i)$$

$$\bar{y}(t) = \text{Res}\left(\frac{L(s)\omega}{s^2+\omega^2}e^{st}, s=i\omega\right) + (\overline{\ldots})$$

$$= \frac{L(i\omega)\cancel{\omega}e^{i\omega t}}{2i\cancel{\omega}} + \frac{L(-i\omega)\omega e^{-i\omega t}}{-2i\omega}$$

$$L(i\omega) = |L(i\omega)|\, e^{i \angle L(i\omega)}$$

$$\bar{y}(t) = |L(i\omega)| \frac{e^{i(\omega t + \angle L(i\omega))} - e^{i(\omega t + \angle L(i\omega))}}{2i}$$

$$= |L(i\omega)| \sin(\omega t + \angle L(i\omega))$$

More generally

$u(t) = A \sin(\omega t + \phi)$

$y(t)$ will converge exponentially to

$\bar{y}(t) = A |L(i\omega)| \sin(\omega t + \phi + \angle L(i\omega))$

RMK:

1. If $u(t)$ sinusoidal, then $y(t)$ is too (in S.S.)
   ↳ same frequency
2. $|L(i\omega)|$ scales the input magnitude
3. $\angle L(i\omega)$ shifts the phase of the input signal

$\therefore$ $L(i\omega)$ is the frequency response

Plotting N is hard since we don't know how $L(i\omega)$ moves in w-plane

Idea:

↳ produce reasonably accurate plots of $|L(i\omega)|$ and $\angle L(i\omega)$ for $\omega \geq 0$ in logarithmic scale.

↳ Bode Plots

Bode Mag Plot : $20\log|L(i\omega)|$ vs $\log(\omega)$

Bode Phase Plot : $\angle L(i\omega)$ vs $\log(\omega)$