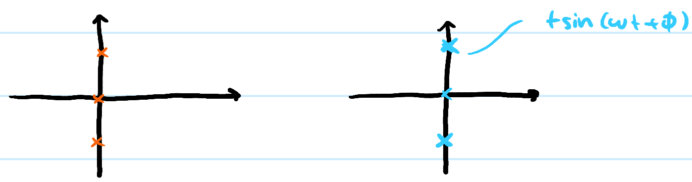
Lecture 17

October 15, 2021 3:09 PM

Observation: A signal $f(t)$ is bounded iff $F(s)$ has poles with real part ≤ 0 and non-repeated poles with real part $= 0$.
↳ Property B

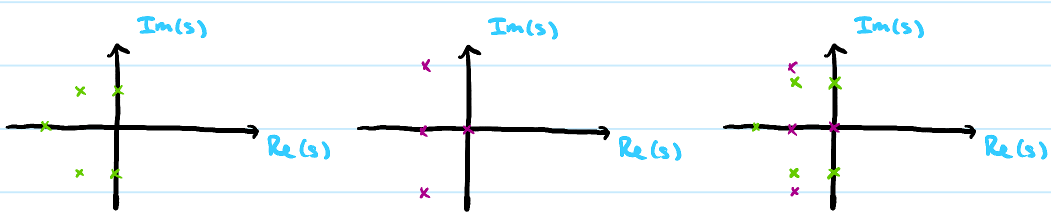


Goal: Find conditions for G under which the following implication holds:

$\forall u(t)$ ($u(t)$ bounded $\Rightarrow$ $y(t)$ bounded)

$\Updownarrow$

$\forall U(s)$ with property B, $Y(s) = G(s)U(s)$ also has property B



poles of U poles of G poles of $Y = GU$

If G has a pole anywhere on the imag. axis, $\exists U(s)$ s.t. $Y = GU$ is unbound

↳ e.g. $\exists U(s)$ s.t. U, G share a pole on the imag. axis $\Rightarrow$ yielding a repeated pole on imag. axis

WLOG, pick $u(t)$ s.t. $U(s)$ has no zeroes

↳ Then $\{\text{poles of } Y(s)\} = \{\text{poles of } G(s) \cup \text{poles of } U(s)\}$

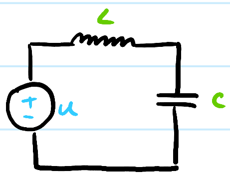
Want poles of Y to satisfy B for any $U(s)$ satisfying B

↳ $G(s)$ cannot have poles with real part ≥ 0

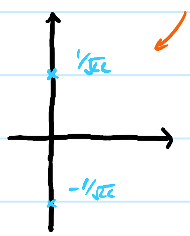
Theorem 2:

↳ $Y(s) = G(s)U(s)$ is BIBO Stable iff all poles G are in OLHP

e.g. Ideal LC circuit



$$Y(s) = \frac{1/LC}{s^2 + 1/LC} U(s)$$



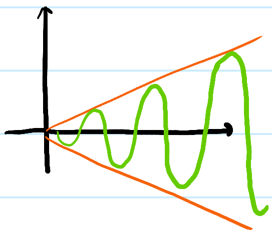
Choose $U(s) = \dfrac{s}{s^2 + 1/LC}$

duplicates poles of G on imag. axis

$u(t) = \cos\left(\frac{1}{\sqrt{LC}} t\right)$

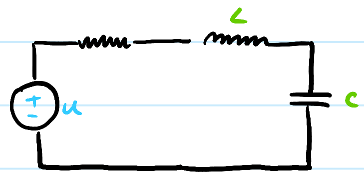
$$Y(s) = \frac{s/LC}{(s^2 + 1/LC)^2}$$

$$y(t) = \frac{1}{2\sqrt{LC}} t \sin\left(\frac{t}{\sqrt{LC}}\right)$$



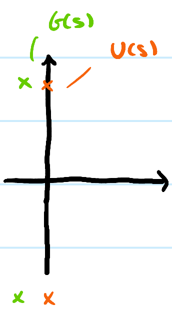
Resonance

Now consider - RLC circuit



$$\frac{Y(s)}{U(s)} = \frac{1/LC}{s^2 + R/L\, s + 1/LC}$$

poles in OLHP $\Rightarrow$ BIBO stable, no resonance



Try again: $U(s) = \frac{s}{s^2 + 1/LC}$

$$y(t) = Ce^{-\sigma t}\cos(\omega_d t + \phi) + D\sin\left(\frac{1}{\sqrt{LC}}t + \varphi\right)$$

Theorem 1 - Asymptotic Stability

↳ eigenvalues of A in OLHP

Theorem 2 - BIBO Stability

↳ poles of G(s) in OLHP

Q: Is there a relationship between the eigenvalues of A and the poles of G?

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \Longrightarrow G(s) = C(sI - A)^{-1}B + D$$

$\underset{n}{\left\{ \text{eigenvalues of } A \right\}} \longleftrightarrow \underset{\leq n}{\left\{ \text{Poles of } G \right\}}$

$$G(s) = \frac{1}{\det(sI-A)}\left[\underbrace{C\,\mathrm{Adj}(sI-A)B + D}_{\text{poly of degree } \leq n}\right]$$

The poles of G are necessarily the roots of $\det(sI-A)$

↳ these are the eigenvalues of A

↳ if p is a pole in G, then p is an eigenvalue of A

↳ $\{\text{Poles of } G\} \subset \{\text{eigenvalues of } A\}$ = spectrum of A

↳ some singularities may be cancelled by zeros

↳ singularities — eigenvalues of A

e.g.

$\dot{x}_1 = -x_1 + u$

$\dot{x}_2 = x_2$

$y = x_1$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

eigenvalues $= \{-1, 1\}$

sys unstable

↳ singularity at 1

$x_2(t) = e^t x_2(0)$

$$G(s) = [1 \quad 0]\begin{bmatrix} & \\ & \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{s+1}$$

∴ System is actually BIBO stable