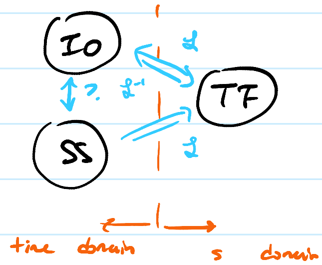
Lecture 9

September 28, 2021 3:08 PM

The Transfer Function of $\dot{x} = Ax + Bu$

$y = Cx + Du$



$G(s) = C(sI - A)^{-1}B + D$

↳ Proper, rational function

↳ degree (num) ≤ degree (denom)

Converting from TF to SS representation

Problem:

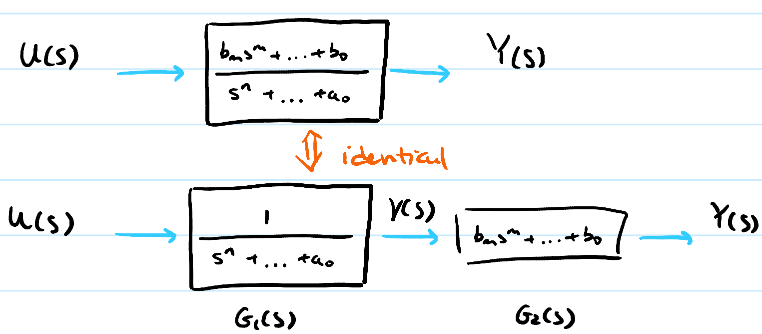
↳ Given a rational, proper TF $G(s) = \frac{b_m s^m + \dots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0}$

$m \le n$ ← proper rational

find A, B, C, D s.t. G(s) is the TF

$\dot{x} = Ax + Bu$ <=> $G(s) = C(sI - A)^{-1}B + D$

$y = Cx + Du$



$G(s) = G_1(s)G_2(s)$

$G_1(s)$: $Y(s) = \frac{1}{s^n + a_{n-1}s^{n-1} + \ldots + a_0} U(s)$

$$\Downarrow$$

$$(s^n + a_{n-1}s^{n-1} + \ldots + a_0) Y(s) = U(s)$$

$$\Downarrow \mathcal{L}^{-1}$$

$$\frac{d^n v}{dt^n} + a_{n-1}\frac{d^{n-1}v}{dt^{n-1}} + \ldots + a_1\frac{dv}{dt} + a_0 v = u$$

Define n states as :

$x_1 = v$ $\qquad$ $\dot{x}_1 = \frac{dv}{dt} = x_2$

$x_2 = \frac{dv}{dt}$ $\qquad$ $\dot{x}_2 = \frac{d^2v}{dt^2} = x_3$

generally , $\dot{x}_n = x_{n+1}$, $i \in [1, n-1]$

$\vdots$

$x_n = \frac{d^{n-1}v}{dt^{n-1}}$ $\qquad$ $\dot{x}_n = \frac{d^n v}{dt^n} = -a_0 v - a_1\dot{v} - \ldots - a_{n-1}\frac{d^{n-1}v}{dt^{n-1}} + u$

$$= -a_0 x_1 - a_1 x_2 - \ldots - a_{n-1} x_n + u$$

Output:

$v = x_1$

Matrix Form:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ & & & \ddots & & \\ & & & & & 1 \\ -a_0 & -a_1 & \ldots & & & -a_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$v = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \end{bmatrix} x$$

$G_2(s)$: $Y(s) = (b_m s^m + \ldots + b_1 s + b_0) V(s)$

Recall : $m \leq n$

Actually, assume $m < n$ (case $m = n$ left as an exercise)

$$y(t) = b_m \underbrace{\frac{d^m v}{dt^m}}_{x_{m+1}} + b_{m-1}\underbrace{\frac{d^{m-1}v}{dt^{m-1}}}_{x_m} + \ldots + b_1 \underbrace{\frac{dv}{dt}}_{x_2} + b_0 \underbrace{v}_{x_1}$$

Case $m = n$:

Grewrite $G(s) = \underbrace{b_m}_{\text{yields non-zero } D} + G_1(s)$

$$y(t) = [\underbrace{b_0 \quad b_1 \quad \ldots \quad b_m}_{\text{"states"}} \quad \underbrace{0 \ldots 0}_{n-m-1}] \, x(t)$$

Conclusion:

↳ A state space realization of $G = \frac{b_m s^m + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0}$ is

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ & & & \ddots & & \\ & & & & & 1 \\ -a_0 & -a_1 & \dots & \dots & \dots & -a_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_0 & b_1 & \dots & b_m & 0 & \dots & 0 \end{bmatrix} x \qquad D = 0$$

Note: This state space model is not the realization of $G(s)$

↳ in other words, S.S. model of $G(s)$ is not unique!

e.g.

$$\underbrace{LC}_{a_2}\frac{d^2y}{dt^2} + \underbrace{RC}_{a_1}\frac{dy}{dt} + \underbrace{1}_{a_0}y = \underbrace{1}_{b_0}u$$

can convert easily

$$G(s) = \frac{1}{LCs^2 + RCs + 1}$$

$$= \frac{1/LC}{s^2 + R/L\, s + \frac{1}{LC}}$$

Previously, we had

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{-1}{LC} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \dot{x} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} u$$

$$y = \begin{bmatrix} \frac{1}{LC} & 0 \end{bmatrix} x \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

difference?

$x_1 = LC\, v_c$ $\qquad$ $x_1 = v_c$

$x_2 = LC \frac{dv_c}{dt} = L i_c$ $\qquad$ $x_2 = i = C \frac{dv_c}{dt}$

$x_2 = LC\frac{dv_C}{dt} = Li_C$ $\qquad\qquad$ $x_2 = i = C\frac{dv_C}{dt}$

Choice of states is purely mathematical

Mega-Summary

Physical System $\Longrightarrow \left(\frac{d^n y}{dt^n} + \dots + a_0 y = b_m \frac{d^m y}{dt^m} + \dots\right)$ IO

(Non-unique!!)

$\dot{x} = f(x,u)$ $\qquad\Longrightarrow\qquad$ $\dot{x} = Ax + Bu$

$y = h(x,u)$ $\quad$ Linearization $\quad$ $y = Cx + Du$ $\quad$ Nonliner S.S.

Nonliner Model

non-unique!! $\qquad$ TF: $G(s)$