Lecture 18

October 19, 2021 3:14 PM

Recap:

THM 1: Sys(2) is asymptotically stable iff all eigenvalues of A are in OLHP

THM 2: Sys(1) or (3) is BIBO stable iff all poles of $G(s) = C(sI-A)^{-1}B+D$ are in OLHP

Pole / Eigenvalue Relationship

Sys(1): $\dot{x} = Ax + Bu$

$y = Cx + Du$ $\quad x(0) = 0$

Sys(2): $\dot{x} = Ax$

Sys(3): $Y(s) = G(s)U(s)$

$\{\text{Poles of } \underbrace{G(s)}_{C(sI-A)^{-1}B+D}\} \subset \{\text{eigenvalues of } A\}$

Direct Consequence:

↳ if sys(2) is asymptotically stable, then automatically the poles of G(s) are in OLHP

eigenvalues of A in OLHP

↓

poles of G(s) in OLHP

THM 3: If sys(2) is asymptotically stable, then for any B, C, D, sys(1) is BIBO stable

$\dot{x} = Ax + Bu$

$y = Cx + Du$

Asymptotic Stability ⇒ BIBO Stability

BIBO Stability $\not\Rightarrow$ Asymptotic Stability

Verifying Theorem 1, 2

Q: Given a real poly $a(s) = s^n + a_{n-1}s^{n-1} + \dots + a_1 s_1 + a_0$, when are the roots of a(s) in the OLHP?

↳ Want a Yes/No answer, without computing roots when are the roots of a(s) in the OLHP?

↳ Want a Yes/No answer, without computing roots

Routh Criterion

↳ generalization of criterion saying that the roots of $s^2+a_1s+a_0$ are in OLHP iff $a_1, a_0 > 0$

Routh Array

s^n	1	a_{n-2}	a_{n-4}	$\cdots$	0
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$	0
s^{n-2}	b_1	b_2	b_3		0
s^{n-3}	c_1	c_2	c_3		0
$\vdots$					0
s^2					0
s^1					0
1					0

$$b_1 = \frac{-1}{a_{n-1}} \det \begin{bmatrix} 1 & a_{n-2} \\ a_{n-1} & a_{n-3} \end{bmatrix} \qquad b_2 = \frac{-1}{a_{n-1}} \det \begin{bmatrix} 1 & a_{n-4} \\ a_{n-1} & a_{n-5} \end{bmatrix}$$

$$c_1 = -\frac{1}{b_1} \det \begin{bmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{bmatrix} \qquad c_2 = \frac{-1}{b_1} \det \begin{bmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{bmatrix}$$

---

s^n	1	a_{n-2}	a_{n-4}	$\cdots$	0
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$	0
s^{n-2}	b_1	b_2	b_3		0
s^{n-3}	c_1	c_2	c_3		0
$\vdots$		$\vdots$			0
s^2	$\times$	$\times$	0	$\cdots$	0
s^1	$\times$	0	0	$\cdots$	0
1	$\times$	0	0	$\cdots$	0

Routh Criterion

The # of sign variations in the first col of the Routh array is equal to the # of roots of a(s) with real part ≥ 0. So the roots of a(s) are in OLHP iff there are no sign variations

e.g.

$a(s) = s^5 + s^4 - s^3 - 2s^2 + s - 2$

s^5	1	-1	1	0
s^4	1	-2	-2	0
s^3	1	3	0	0
s^2	-5	-2	0	0
s^1	13/5	0	0	0
1	-2	0	0	0

1 sign variation

1 sign variation

1 sign variation

= 3 total sign variations

3 sign variations $\Rightarrow$ 3 roots with real part ≥ 0

The Routh Algorithm fails when 1st coeff of a row is 0, OR an entire row is 0

Fact: If such an exception occurs, then a(s) has roots with real part ≥ 0 so we do not have BIBO stability or Asymptotic Stability

e.g.

$a(s) = s(s+1) = s^2 + s$

s^2	1	0
s	1	0
1	0	0

zeroes:

↳ some roots have real part ≥ 0

↳ root at $s=0$

e.g.

$a(s) = s^2 + 1$

s^2	1	1
s	0	0
1	STOP	

2 roots on Imag Axis

e.g.

s^3	1	1	0
s^2	3	3	0
s^1	0	0	0
1		STOP	

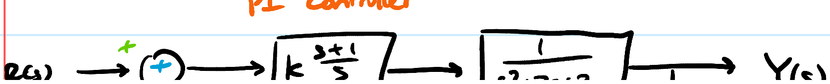
indeed, 2 roots on imag axis

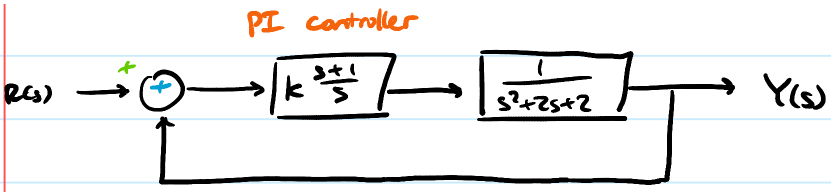
Utility of Routh Array

↳ stability determination in the presence of design parameters

e.g.

PI controller





Find all values of k for which the closed loop TF is BIBO stable

$$\frac{G}{1+CG}$$