Lecture 4

September 16, 2021 12:05 PM

Classes of Control System Models

Nonlinear State Space

$$\left.\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned}\right\} \begin{aligned} &x \in \mathbb{R}^n \\ &u \in \mathbb{R}^m \end{aligned} \qquad y \in \mathbb{R}^p$$

MIMO

↳ Multiple Input Multiple Output

LTI State Space

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}\right\} \begin{aligned} &x \in \mathbb{R}^n \\ &u \in \mathbb{R}^m \end{aligned} \qquad y \in \mathbb{R}^p$$

LTI Input/Output

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_0 y = b_k \frac{d^k u}{dt^k} + b_0 u \qquad (IO)$$

$k \leq n$

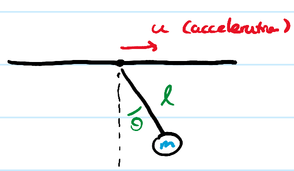
linear constant coefficient diffeq

SISO

↳ single input single output

Linearization of NL at Equilibrium

e.g. Crane Model



$$\ddot{\theta} = -\frac{g}{\ell}\sin(\theta) - \frac{1}{\ell}\cos(\theta)u$$

States:

$$\left.\begin{aligned} x_1 &= \theta \\ x_2 &= \dot{\theta} \end{aligned}\right\}$$ choose all differentiated variables in model eqn and $n-1$ of its time derivatives

$$\dot{x}_1 = x_2 \qquad f(x,u) = \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin(x_1) - \frac{1}{\ell}\cos(x_1)u \end{bmatrix} \quad (*)$$

$$\dot{x}_2 = -\frac{g}{\ell}\sin(\theta) - \frac{1}{\ell}\cos(\theta)u$$

$$h(x,u) = x_1$$

Equilibrium

↳ Handout on quercus covers this topic

$\dot{x} = f(x,u)$ set $u(t) \equiv \bar{u}$ constant $\qquad \equiv$: equal for all time t (identically equal)

$y = h(x,u)$

what are the equilibrium of NL for this control "input"

Defn: Consider (NL) and set $u(t) \equiv \bar{u}$ constant

A state $\bar{x} \in \mathbb{R}^n$ is an equilibrium if $f(\bar{x}, \bar{u}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Explanation: if we set $u(t) \equiv \bar{u}$ and we initialize (NL) at $x(0) = \bar{x}$ then a solution of (NL) for this initial condition is $x(t) \equiv \bar{x}$ (constant solution)

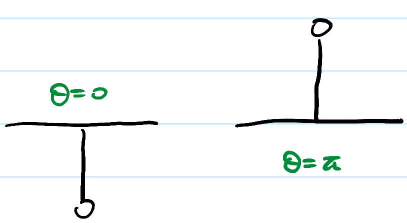
(i) $x(t)$ satisfies the initial condition : $x(0) = \bar{x}$

(ii) $x(t)$ satisfies the ODE : $\frac{dx(t)}{dt} = \frac{d\bar{x}}{dt} = 0 = f(\bar{x}, \bar{u}) = f(x(t), u(t))$

Moral: Equilibrium are constant solutions of (NL) when the control is kept constant

e.g. the crane model set $u(t) \equiv 0 \quad (= \bar{u})$

Equilibriums of crane model:



set $f(x,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$f(x,0) = \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin(x_1) \end{bmatrix} \quad \leftarrow \text{from } (*)$$

$$\begin{bmatrix} 0 \\ \end{bmatrix} = \begin{bmatrix} x_2 \\ \end{bmatrix}$$

$x_2 = 0$ makes sense, $x_2 = \dot{\theta} = 0$ (no accel.)

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin(x_1) \end{bmatrix}$$

$x_2 = 0$ makes sense, $x_2 = \dot{\theta} = 0$ (no accel.)

$\sin(x_1) = 0 \iff x_1 = k\pi$

e.g. find all equilibria when $u(t) \equiv g$

$$f(x,g) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin(x_1) - \frac{1}{\ell}\cos(x_1)\cdot g \end{bmatrix}$$

$x_2 = 0$

$$0 = -\cancel{\frac{g}{\ell}}\sin(x_1) - \cos(x_1)\cdot\cancel{\frac{g}{\ell}}$$

$$0 = \sin(x_1) + \cos(x_1)$$

$x_1 = \pi n - \pi/4$

$\sin(x_1) = -\cos(x_1)$

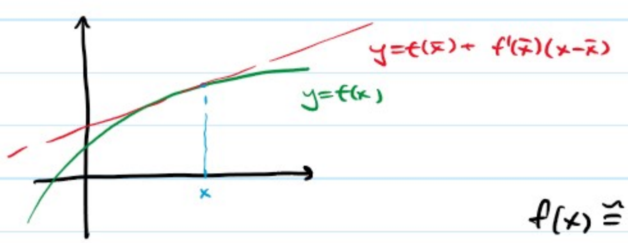
$\boxed{x_1 = -\pi/4,\ 3\pi/4}$

$\tan(x_1) = -1$

Linearization

Approximate $f(x,u)$ and $h(x,u)$ using 1st order Taylor Series

Consider a function $f: \mathbb{R} \to \mathbb{R}$



$$f(x) \cong f(\bar{x}) + f^{-1}(\bar{x})(x-\bar{x})$$

generalise with vector calc

$$f(x) \doteq f(\bar{x}) + \left[\frac{\partial f}{\partial x}(\bar{x})\right](x-\bar{x})$$

$k \times n$

$k \times n$

Jacobian Matrix

Not the gradient (∇f)!
↳ gradient applies iff $f \in \mathbb{R}$

$$\frac{\partial f}{\partial x}(\bar{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\bar{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\bar{x}) \\ \frac{\partial f_2}{\partial x_1}(\bar{x}) & \cdots & \frac{\partial f_2}{\partial x_n}(\bar{x}) \\ \frac{\partial f_k}{\partial x_1}(\bar{x}) & \cdots & \frac{\partial f_k}{\partial x_n}(\bar{x}) \end{bmatrix}$$

Jacobian: $\left[\frac{\partial f}{\partial x}\right]_{(i,j)} = \frac{\partial f_i}{\partial x_j}$

Now: Apply to $f(x,u)$

$$f(x,u) - f\left(\begin{bmatrix} x \\ u \end{bmatrix}\right) \doteq \underbrace{f\left(\begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix}\right)}_{=0 \text{ since } f\left(\begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix}\right) \text{ is an equilibrium}} + \frac{\partial f}{\partial \begin{bmatrix} x \\ u \end{bmatrix}}\left(\begin{bmatrix} x \\ u \end{bmatrix} - \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix}\right)$$

$$\frac{\partial f}{\partial \begin{bmatrix} x \\ u \end{bmatrix}} = \left[\frac{\partial f}{\partial x} \;\vdots\; \frac{\partial f}{\partial u}\right]$$

$$f(x,u) \doteq 0 + \left[\frac{\partial f}{\partial x}(\bar{x},\bar{u}) \;\vdots\; \frac{\partial f}{\partial u}(\bar{x},\bar{u})\right] \cdot \begin{bmatrix} x - \bar{x} \\ u - \bar{u} \end{bmatrix}$$

$$\doteq \underbrace{\left[\frac{\partial f}{\partial x}(\bar{x},\bar{u})\right]}_{A}(x-\bar{x}) + \underbrace{\left[\frac{\partial f}{\partial u}(\bar{x},\bar{u})\right]}_{B}(u-\bar{u})$$

$$f(x,u) \doteq A(x-\bar{x}) + B(u-\bar{u})$$

Define $\tilde{x} := x - \bar{x}$ then $\dot{\tilde{x}} = \dot{x} - \cancelto{0}{\frac{d\bar{x}}{dt}}$ $\bar{x}, \bar{u}$ constant

$\hat{u} := u - \bar{u}$ $\dot{\hat{u}} = \dot{u} - \cancelto{0}{\frac{d\bar{u}}{dt}}$

$$f(x,u) \doteq A\tilde{x} + B\tilde{u}$$

Conclusion

if $\bar{x}$ is an equilibrium of (NL) where $u(t) \equiv \bar{u}$, then the linearization of NL at this equilibrium is

$\dot{\tilde{x}} = A\tilde{x} + B\tilde{u}$ where $\tilde{x} = x - \bar{x}$

$\tilde{y} = C\tilde{x} + D\tilde{u}$ $\hat{u} = u - \bar{u}$

$\tilde{y} = y - h(\bar{x},\bar{u})$

$A = \frac{\partial f}{\partial x}(\bar{x},\bar{u})$ $B = \frac{\partial f}{\partial u}(\bar{x},\bar{u})$ $C = \frac{\partial h}{\partial x}(\bar{x},\bar{u})$ $D = \frac{\partial h}{\partial u}(\bar{x},\bar{u})$

$n \times n$ $n \times m$ $p \times n$ $p \times m$