Lecture 5

September 16, 2021 1:40 PM

Linearization of NL Control Systems

$\dot{x} = f(x,u)$

$y = h(x,u)$

Equilibrium for a constant control $\bar{u}$:

↳ a vector $\bar{x} \in \mathbb{R}^n$ s.t. $f(\bar{x},\bar{u}) = \vec{0}$

Linear Approximation at $(\bar{x},\bar{u})$

$$\dot{x} = f(x,u) \doteq \cancelto{0}{f(\bar{x},\bar{u})} + \Big[\underbrace{\frac{\partial f}{\partial x}(\bar{x},\bar{u})}_{A}\Big](x-\bar{x}) + \Big[\underbrace{\frac{\partial f}{\partial u}(\bar{x},\bar{u})}_{B}\Big](u-\bar{u})$$

$$y = h(x,u) \doteq h(\bar{x},\bar{u}) + \Big[\underbrace{\frac{\partial h}{\partial x}(\bar{x},\bar{u})}_{C}\Big](x-\bar{x}) + \Big[\underbrace{\frac{\partial h}{\partial u}(\bar{x},\bar{u})}_{D}\Big](u-\bar{u})$$

$\hat{x} := x - \bar{x} \qquad \hat{u} := u - \bar{u} \qquad \hat{y} := y - \bar{y}$

$$\dot{\hat{x}} = \dot{x} \doteq A\hat{x} + B\hat{u}$$
$$\hat{y} \doteq C\hat{x} + D\hat{u}$$

$A = \frac{\partial f}{\partial x}(\bar{x},\bar{u}) \qquad B = \frac{\partial f}{\partial u}(\bar{x},\bar{u}) \qquad C = \frac{\partial h}{\partial x}(\bar{x},\bar{u}) \qquad D = \frac{\partial h}{\partial u}(\bar{x},\bar{u})$

Rmk: If $\bar{x}$ were not an equilibrium,

$f(\bar{x},\bar{u}) \neq 0$, and linear approx. of $f(x,u)$

would be:

$$f(x,u) \doteq \overbrace{f(\bar{x},\bar{u})}^{r \text{ (nonzero!!)}} + A\hat{x} + B\hat{u}$$

$\dot{\hat{x}} = A\hat{x} + B\hat{u} + r$ ← This ODE does not satisfy the axioms of linearity

∴ we can only linearize at equilibrium

e.g. Crane Example

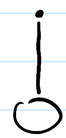
e.g. Crane Example

$\dot{x}_1 = x_2$

$\dot{x}_2 = -\frac{g}{\ell}\sin(x_1) - \frac{1}{\ell}\cos(x_1)\cdot u$

$y = x_1$

when $u(t) \equiv 0$, we have equilibrium at $\bar{x} = \begin{bmatrix} 0 \mod 2\pi \\ 0 \end{bmatrix}$ and $\bar{x} = \begin{bmatrix} \pi \mod 2\pi \\ 0 \end{bmatrix}$



Linearize at upward equilibrium

$\bar{x} = \begin{bmatrix} \pi \\ 0 \end{bmatrix} \qquad \bar{u} = 0$

Important : Define $\hat{x}, \hat{u}, \hat{y}$

$\hat{x} = x - \bar{x} = \begin{bmatrix} x_1 - \pi \\ x_2 \end{bmatrix}$

$$f(x,u) = \begin{bmatrix} x_2 \\ -\frac{g}{\ell}\sin(x_1) - \frac{1}{\ell}\cos(x_1)u \end{bmatrix} \begin{matrix} f_1() \\ f_2() \end{matrix}$$

$\hat{u} = u - \bar{u} = u$

$\hat{y} = y - \bar{y} = y - h(\bar{x}, \bar{u}) = x_1 - \pi$

$$A = \frac{\partial f}{\partial x}(\bar{x}, \bar{u}) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos(\bar{x}_1) + \frac{1}{\ell}\sin(\bar{x}_1)\cdot\bar{u} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ g/\ell & 0 \end{bmatrix}$$

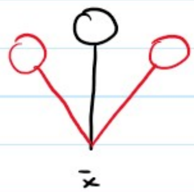
$$B = \frac{\partial f}{\partial u}(\bar{x}, \bar{u}) = \begin{bmatrix} 0 \\ -1/\ell \cos(\bar{x}_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\ell \end{bmatrix}$$

$C = [1 \quad 0]$

$D = 0$

$$\dot{\hat{x}} = \begin{bmatrix} 0 & 1 \\ g/\ell & 0 \end{bmatrix}\hat{x} + \begin{bmatrix} 0 \\ 1/\ell \end{bmatrix}\hat{u} \qquad \hat{y} = [1 \quad 0]\hat{x}$$

Rmk: Just as in calculus, the first order Taylor Series approx. of a function is good only near the point at which we approx. the function, the linearization of (NL) at $\bar{x}$ is only valid when $x(t)$ is close to equilibrium $\bar{x}$



In the e.g., model is valid if the pendulum is close to being upright and has low speed

Rmk: The linearization procedure gives an LTI state-space model. We will convert it to a transfer function and then apply the tools we are about to learn

Laplace Transform

We need the Laplace Transform because with it we can represent any LTI system

↳ I/O or

↳ State space

Moreover, with the transfer function, we avoid having to deal with ODE's

In ECE216 the focus was the Fourier Transform

Laplace Transforms are generalizations of FT's, applicable to a larger class of signals

Defn:

Let $f(t)$ be a function $f: \mathbb{R} \to \mathbb{R}$. Then

$$\mathcal{L}\{f(t)\} = F(s) := \int_{0}^{+\infty} f(t) e^{-st} \, dt \quad , \quad s \in \mathbb{C}$$

$F: \mathbb{C} \to \mathbb{C}$

right-sided laplace transform

↳ time starts at $t=0$

Existence of the Integral:

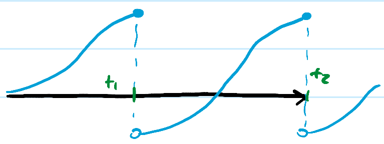
↳ $F(s)$ exists if

a) $f(t)$ is piecewise continuous (PWC)

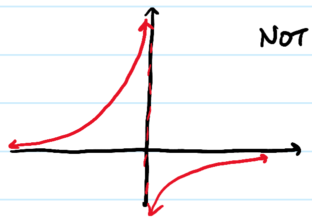
↳ it on every finite interval of time, $f(t)$ has at most a finite # of discontinuous

↳ $f(t)$ has finite left and right limits at such discontinuities

PWC



NOT PWC

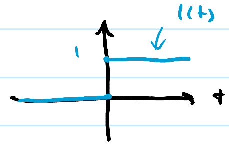


b) $f(t)$ is "exponentially bounded": $\exists M \geq 0, a \in \mathbb{R}$ s.t. $|f(t)| \leq Me^{at} \; \forall t \geq 0$

↳ e.g.

Unit Step $1(t) := \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$ PWC

use 1 since "u" reserved for input



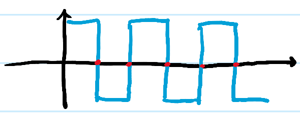
$|f(t)| \leq 1 = 1 \cdot e^{0t}$ ← a, ↑ M

Exponentially Bounded

↳ e.g.

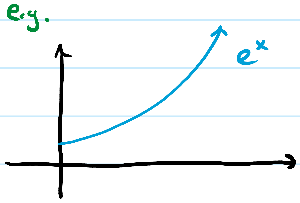
square wave

PWC



Exponentially Bounded

↳ e.g.



cts. → PWC

Exponentially Bounded

↳ $M=1$ $a=1$