Lecture 11

October 1, 2021 3:11 PM

Final Value Theorem

↳ property of the laplace transform

Suppose $y(t)$ is $\mathcal{L}$-transformable and that $\lim_{t\to\infty} y(t)$ exists and is finite. Then

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

e.g. PMDC motor

$$Y(s) = \frac{V_0}{s(s^2+2s+2)}$$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{V_0 \cancel{s}}{\cancel{s}(s^2+2s+2)} = V_0/2$$

Be careful !!

↳ Check conditions for FVT

e.g $\quad Y(s) = \dfrac{1}{s^2+1}$

Apply FVT without checking conditions

$$y(\infty) = \lim_{s\to 0} sY(s) = 0$$

BUT! $\quad y(t) = \mathcal{L}^{-1}\{Y(s)\} = \sin(t)$ ← HAS NO LIMIT $t\to\infty$

↳ FVT applies to signals, not systems (TF)

Assume $Y(s)$ rational and strictly proper

↳ (deg(num) < deg(den))

$$\frac{s+1}{s+2} = 1 + \frac{-1}{s+2} \implies y(t) = \delta(t) - e^{-2t}$$

$$Y(s) = \frac{N(s)}{\text{poly with 4 types of poles}}$$

Types of Poles

Focus

1. Real pole at $-p$
2. Complex conjugate pole at $-\sigma \pm i\omega_d$
3. k repeated poles at $-p$
4. k repeated complex-conjugate poles

$$Y(s) = \frac{N(s)}{\underbrace{(s+p_1)(s+p_2)\dots}_{\text{Type } a_1}\underbrace{[(s+\sigma)^2+\omega_d^2][\quad]}_{\text{Type } b}}$$

only cases a, b

$$\mathcal{L}^{-1}\left\{\frac{d_1 s + d_2}{[(s+\sigma)^2+\omega_d^2]}\right\}$$

$$= e^{-\sigma t}(\lambda_1 \sin(\omega_d t) + \lambda_2 \cos(\omega_d t))$$

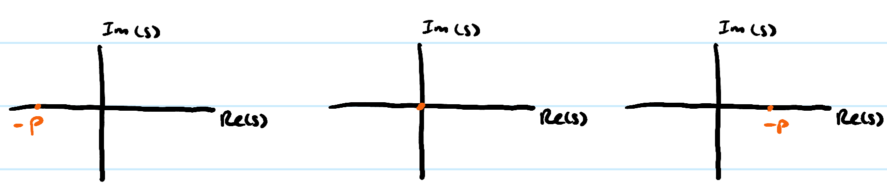
$$= e^{-\sigma t} A \sin(\omega_d t + \phi)$$

PFE

$$= \underbrace{\frac{c_1}{s+p_1} + \frac{c_2}{s+p_2}}_{\text{type } a_1} + \dots + \underbrace{\frac{d_1 s + d_2}{[(s+\sigma)^2+\omega_d^2]}}_{\text{type } b} + \dots$$

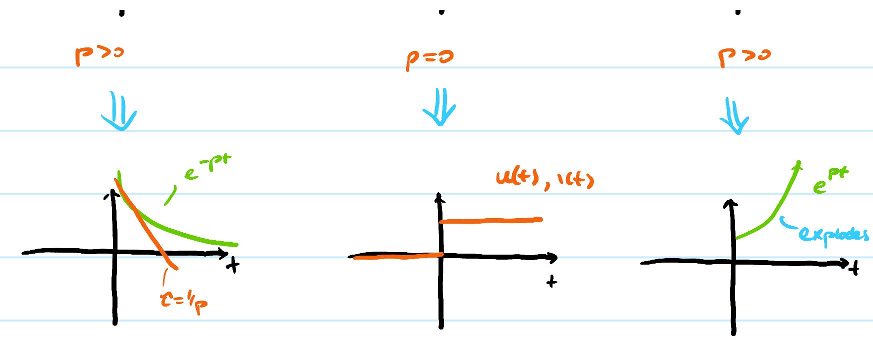
$$y(t) = \underbrace{c_1 e^{-p_1 t} + c_2 e^{-p_2 t}}_{a_1} + \dots + \underbrace{e^{-\sigma t} A \sin(\omega t + \phi)}_{b}$$

Type A Term



$p>0$ $\qquad$ $p=0$ $\qquad$ $p>0$

I. $\qquad$ II. $\qquad$ I.



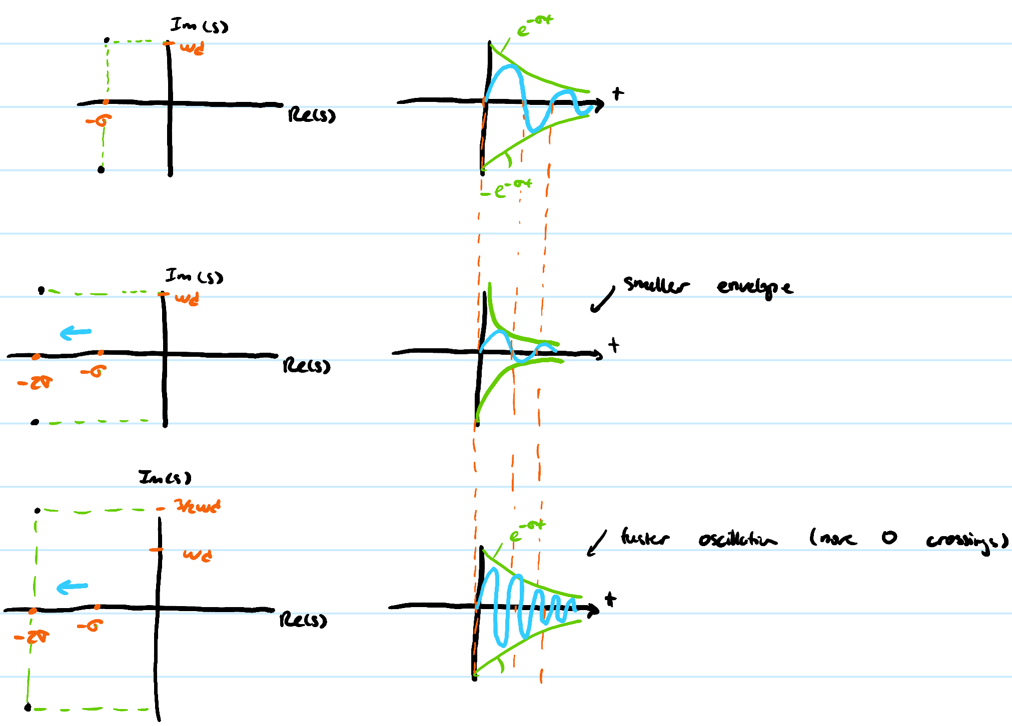
τ: Time Constant

Type B term

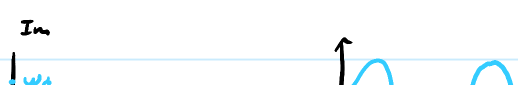
$$Y(s) = \frac{\omega_d}{(s+\sigma)^2 + \omega_d^2} \overset{\mathcal{L}^{-1}}{\Longleftrightarrow} y(t) = e^{-\sigma t} \sin(\omega_d t) 1(t)$$

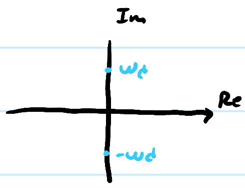
poles at $-\sigma \pm i\omega_d$

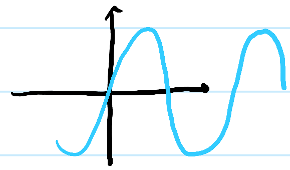
$\sigma > 0$ (underdamped):



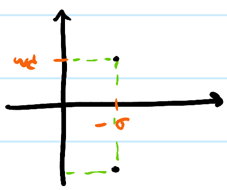
$\sigma = 0$ (undamped)

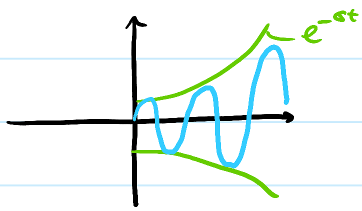






$\sigma > 0$





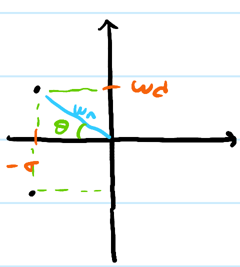
Rmk: for terms of type B, if we have a generic first order poly in the numerator,

$$e^{-\sigma t}\sin(\omega_d t) \longrightarrow Ae^{-\sigma t}\sin(\omega_d t+\phi)$$

numerator is ω_d numerator is $c_1 s + c_2$

Poles are qualitatively unchanged

Alternative Representation of C.C poles



$$(\sigma, \omega_d) \Rightarrow (\zeta, \omega_n)$$

$$\zeta := \cos(\theta)$$

$$\sigma = \omega_n \cos(\theta) = \zeta\omega_n$$

$$\omega_d = \omega_n \sin(\theta) = \omega_n\sqrt{1-\zeta^2}$$