# Lecture 13

October 7, 2021 12:10 PM

## Transient Performance

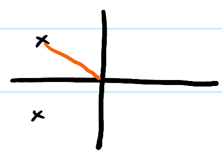
$$\frac{1}{s} \longrightarrow \boxed{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}} \longrightarrow Y(s)$$

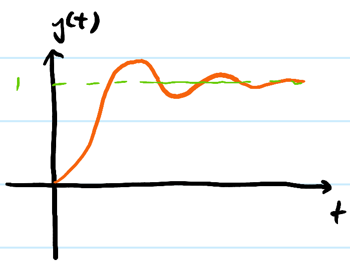
### Conditions for Application of Transient Characteristics

↳ Unit step input

↳ $0 < \zeta < 1$

↳ 2nd Order System





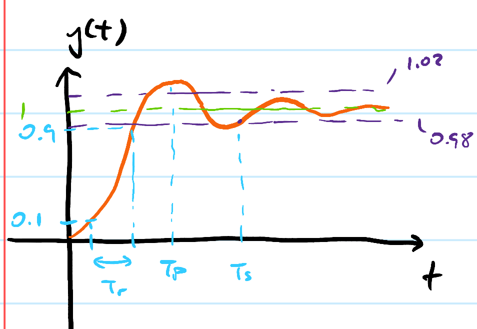

Q:

↳ How long does y(t) take to settle

↳ How much overshoot does y(t)

$T_r$ : Rise Time

↳ duration needed for y(t) to rise from 10% to 90% of steady state value




$T_s$ : Settling Time

↳ time after which $|y(t) - y(\infty)| \leq 0.02\, y(\infty)$

$T_p$ : Peak Time

↳ peak time when y(t) reaches global max

% Overshoot $= \dfrac{y(T_p) - y(\infty)}{y(\infty)}$

Assumptions :

Assumptions:

↳ $u(t)$ is a step input

↳ $G(s)$ has complex conj poles in OLHP

↳ $G(s)$ has no zeroes

Later will discuss if these do not hold

$$Y(s) = \frac{\omega_n^2}{s^2 + s\,\zeta\omega_n + \omega_n^2} \cdot \frac{1}{s}$$

$$\Downarrow$$

$$y(t) = \left[1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \psi)\right] 1(t)$$

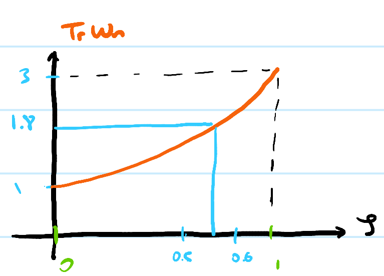
$\psi = \cos^{-1}(\zeta)$

Objective : Characterize $T_r$, $T_s$, % OS in terms of $(\zeta, \omega_n)$ in terms of pole locations

$T_r$ : in principle, could find $t_1$, $t_2$ s.t.

$y(t_1) = 0.1$ $\quad$ $y(t_2) = 0.2$ $\quad\quad$ $T_r = t_2 - t_1$

↳ however, will not get an analytical expression




$T_r \omega_n \propto \zeta$

$T_r \omega_n \doteq \frac{1.8}{\omega_n}$ ← good when $\zeta$ near 0.5

very rough

$T_s$: $(\forall t \geq T_s)$ $\quad$ $|y(t) - 1| \leq 2 \cdot 10^{-2}$

$$\left| \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \psi) \right| \leq 2 \cdot 10^{-2}$$

$$\left| \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \varphi) \right| \leq 2 \cdot 10^{-2}$$


Excellent Approximation : Compute time after which the exp envelope stays below $2 \cdot 10^{-2}$

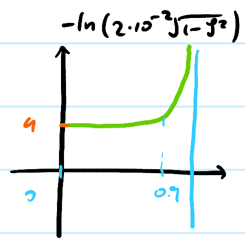
$$\frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}} \leq 2 \cdot 10^{-2}$$

$$T_s \doteq -\frac{\ln\left(2 \cdot 10^{-2} \sqrt{1-\zeta^2}\right)}{\zeta \omega_n}$$

good for all $\zeta$

$$\doteq \frac{4}{\zeta \omega_n} = \frac{4}{\sigma}$$

good for $\zeta < 0.9$




%OS:

$$\frac{y(T_p) - 1}{1}$$

$T_p$ : time of the first peak in $y(t)$

Set $\dot{y}(t) = 0$ and find $T_p$

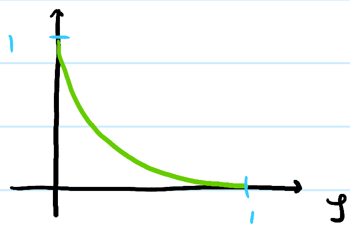
$$\mathcal{L}\{\dot{y}(t)\} = s\,\mathcal{L}\{y(t)\} = s\,Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\dot{y}(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\sigma t} \sin(\omega_d t) \qquad \dot{y}(t) = 0 \iff t = \frac{k\pi}{\omega_d}$$

$$T_p = \frac{\pi}{\omega_d}$$

$$\% OS = y(T_p) - 1 = y\left(\frac{\pi}{\cdots}\right) - 1 = \exp\left(\frac{-\zeta\pi}{\cdots}\right)$$

$$\%OS = y(T_p) - 1 = y\left(\frac{\pi}{\omega_d}\right) - 1 = \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$



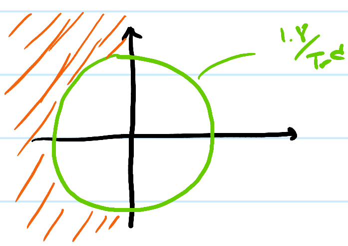

## Transient Performance vs. Pole Locations

$$\frac{1}{s} \longrightarrow \boxed{\frac{a}{s^2 + 2s\zeta\omega_n + \omega_n^2}} \longrightarrow$$

Suppose:

$$\left.\begin{aligned} &\hookrightarrow T_r \le \boxed{T_r^d} \\ &\hookrightarrow T_s \le \boxed{T_s^d} \\ &\hookrightarrow \%OS \le \boxed{\%OS^d} \end{aligned}\right\} \text{numbers}$$

$$T_r \le T_r^d \iff \frac{1.8}{\omega_n} \le T_r^d \qquad \omega_n \ge \frac{1.8}{T_r^d}$$

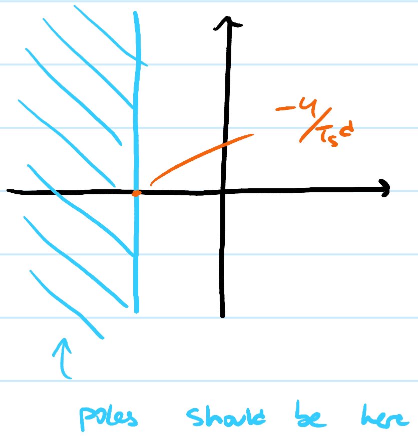



Geometrically, this implies that we want the poles outside a circle with radius 1.8/Trd , and on the OLHP (open left hand plane).

poles should be here

$$T_s \le T_s^d \iff \frac{4}{\zeta\omega_n} = \frac{4}{\sigma} \le T_s^d$$

$$\sigma \ge \frac{4}{T_s^d}$$




$\%OS \le \%OS^d \iff \zeta \ge \zeta^d$