# Lecture 16

October 14, 2021 12:14 PM

## Internal stability

↳ sys (1) stable if all $x(t)$ bounded

↳ asymptotically stable if all solutions $\to \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

## Input / Output stability

↳ Sys (2),(3) are BIBO stable if every bounded input gives a bounded output

↳ BIBO unstable if any bounded input gives unbounded output

## Characterize Asymptotic Stability

$\dot{x} = Ax$ Goal: find conditions s.t. $\forall x \in \mathbb{R}^n$,

$x(0) = x_0$ $x(t) \to 0$ as $t \to \infty$

$\mathcal{L}\{\dot{x}(t)\} = sX(s) - x_0 \longrightarrow sX(s) - x_0 = AX(s)$

$(sI - A)X(s) = x_0 \longrightarrow \underset{n\times 1}{X(s)} = \underset{n\times n}{(sI-A)^{-1}} \underset{n\times 1}{x_0}$

$$X(s) = \underbrace{\frac{1}{\det(sI-A)}}_{\text{A polynomial of degree } n} \operatorname{Adj}(sI-A)\, x_0$$

( each entree is a poly in $s$ of order $\leq n$

$X(s) = \begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix}$ $X_i(s)$ has the form $X_i(s) = \dfrac{N_i(s)}{\det(sI-A)}$

so $X_i(s)$ is rational, proper

The poles of $X_i(s)$ are roots of $\det(sI-A)$
↳ the eigenvalues of A !!

In order for $x(t) \to 0$
↳ we need poles of $X_i(s)$
to have strictly -ve real poles
↳ poles of $X_i(s)$ in OLHP

Theorem 1

1. sys(1) is Asymptotically stable iff all values of A have real part $< 0$ (poles in OLHP)
2. If A has one or more eigenvalues with real part $> 0$, then (1) is unstable

Note: The instability criterion is not a necessary condition
↳ sys (1) could be unstable even if A has no values with real part $> 0$
↳ poles at real $= 0$ (Imaginary Axis)

e.g. Crane Model

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \underbrace{k x_2}_{\text{viscous friction}}$$

$$A = \begin{bmatrix} 0 & 1 \\ -g/l & -k \end{bmatrix}$$

**Eigenvalues** : $\det(\lambda I - A) = \det\begin{bmatrix} \lambda & -1 \\ g/l & \lambda + k \end{bmatrix} = \lambda^2 + k\lambda + g/l$

Useful fact: The roots of $\lambda^2 + a\lambda + b$ have -ve real part iff $a, b > 0$

The linearization is asy. stable, as expected

## Equilibrium #2

$\begin{bmatrix} \pi \\ 0 \end{bmatrix}$, $\bar{u} = 0$

$$A = \begin{bmatrix} 0 & 1 \\ g/l & -k \end{bmatrix}$$

$$\det(\lambda I - A) = \lambda(\lambda + k) - g/l$$
$$= \lambda^2 + \lambda k - g/l$$

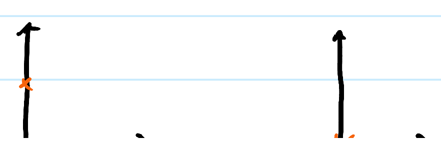
$$\lambda_{roots} = -k/2 \pm \frac{\sqrt{k^2 + 4g/l}}{2}$$

↳ Linearization is Unstable

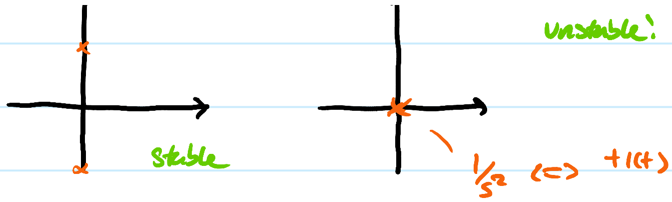
rmk: when $A$ has all eigenvalues with real part $\leq 0$
and some eigenvalues with real part $= 0$

Thm 1 asserts that the sys is not asympt stable
↳ is it stable or unstable?
↳ Ans: depends

## Arithmetic vs Geometric Multiplicity



unstable!

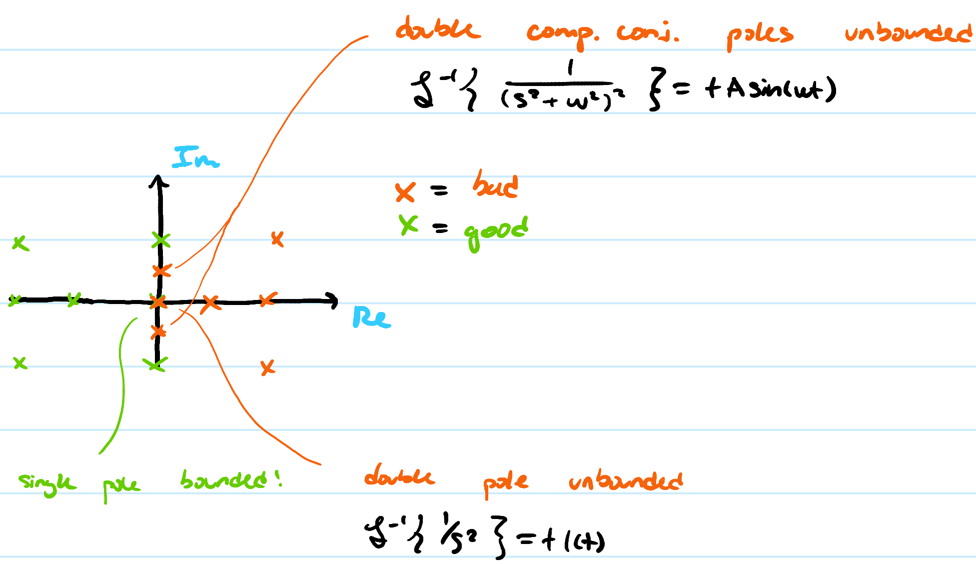



## BIBO Stability

$Y(s) = G(s)U(s)$

goal: find condition s.t.
$u(t)$ bounded $\Rightarrow$ $y(t)$ bounded

Q: what conditions of $F(s)$ guarantees $f(t)$ bounded?




Q: Which conditions on $F(s)$ is $f(t)$ bounded?

A: $f(t)$ is bounded iff all poles of $F(s)$ have real part $\le 0$ AND poles on imag. axis are

NOT repeated