Lecture 8

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Input/output Model

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} + \dots + a_0 y = b_m \frac{d^m u}{dt^m} + \dots + b_0 u$$

Assume $y(0) = \dot{y}(0) = \ddot{y}(0) = \dots = \frac{d^{n-1}}{dt^{n-1}}(0) = 0$

Apply $\mathcal{L}$ to I/O we get

$$Y(s) = \underbrace{\frac{b_m s^m + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0}}_{G(s)} U(s)$$

$$U(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$

In time domain

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{G(s)U(s)\}$$

$$= \int_0^t g(t-\tau)u(\tau)\,d\tau$$

$g(t) := \mathcal{L}^{-1}\{G(s)\}$ is the impulse response

Q: What is the Transfer Function of a state-space model?

$\dot{x} = Ax + Bu \qquad y = Cx + Du$

A:

Assume zero initial conditions:

$\vec{x}(0) = 0$

state ($\vec{x}$) at time $t=0$ is 0

Consider SISO (single input single output)

A: $n \times n$ B: $n \times 1$ C: $1 \times n$ D: 1×1

Take $\mathcal{L}$ of $\dot{x} = Ax + Bu$

$$X(s) := \mathcal{L}\{x(t)\} := \begin{bmatrix} \mathcal{L}\{x_1(t)\} \\ \vdots \\ \mathcal{L}\{x_n(t)\} \end{bmatrix}$$

$$\mathcal{L}\{\dot{x}(t)\} = sX(s) - \cancel{x(0)}^{\,0}$$ (zero initial conditions)

$$sX(s) = AX(s) + BU(s)$$

$$(sI - A)X(s) = BU(s)$$

$$X(s) = \underbrace{(sI-A)^{-1}}_{n \times n} BU(s)$$

$$Y(s) = CX(s) + DU(s)$$
$$= C(sI - A^{-1})BU(s) + DU(s)$$
$$= \underbrace{[C(sI-A^{-1})B + D]}_{G(s)} U(s)$$

$G(s)$ the TF of SS

$$Y(s) = [\underset{1\times n}{C}\,\underset{n\times n}{(sI-A^{-1})}\,\underset{n\times 1}{B} + \underset{1\times 1}{D}]\,U(s)$$

rmk: $G(s)$ is rational (just as in the case of I/O models)

rmk: the values of $s \in \mathbb{C}$ for which $sI - A$ is not invertible are poles of $G(s)$

e.g. RLC Circuit

Choosing $x_1 = v$, $x_2 = i$

$$\dot{x} = \begin{bmatrix} 0 & 1/C \\ -\frac{1}{L} & -R/L \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Find the TF:

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} s & -1/C \\ 1/L & s+\frac{R}{L} \end{bmatrix}^{-1}}_{\frac{1}{(s^2+R/L\, s+\frac{1}{LC})} \cdot \begin{bmatrix} s+\frac{R}{L} & 1/C \\ -1/L & s \end{bmatrix}} \begin{bmatrix} 0 \\ 1/L \end{bmatrix} = \frac{1/LC}{s^2+R/L\, s+\frac{1}{LC}}$$

e.g. Crane Model linearized at $(\pi, 0)$

$$\dot{\tilde{x}} = \begin{bmatrix} 0 & 1 \\ g/l & 0 \end{bmatrix} \tilde{x} + \begin{bmatrix} 0 \\ 1/l \end{bmatrix} \tilde{u} \qquad \tilde{Y}(s) = G(s)\tilde{U}(s)$$

$$\tilde{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \tilde{x}$$

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} s & -1 \\ -\frac{g}{l} & s \end{bmatrix}^{-1}}_{\frac{1}{s^2-g/l} \begin{bmatrix} s & 1 \\ g/l & s \end{bmatrix}} \begin{bmatrix} 0 \\ 1/l \end{bmatrix} = \frac{1/l}{s^2-g/l}$$

e.g. PMDC speed control mode

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \\ i_a \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -b/I & K_t/I \\ 0 & -K_e/L_a & -R_a/L_a \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1/L_a \end{bmatrix} u \qquad y = [0 \quad 1 \quad 0]\, x$$

$$G(s) = [0 \quad 1 \quad 0] \underbrace{\begin{bmatrix} s & 1 & 0 \\ 0 & s+b/I & -K_t/I \\ 0 & K_e/L_a & s+R_a/L_a \end{bmatrix}^{-1}}_{A^{-1}} \begin{bmatrix} 0 \\ 0 \\ 1/L_a \end{bmatrix}$$

☆

chrisp rett ?

$$A^{-1} = \frac{1}{\det(A)} \mathrm{Adj}(A) \qquad \mathrm{Adj}(A) = C(A)^T$$

↑ cofactor matrix

$$A^{-1} = \frac{1}{s\det(☆)} \cdot \begin{bmatrix} \det(☆) & \det(\otimes) & \cdots \\ \vdots & \ddots & \vdots \end{bmatrix}$$

$$G(s) = \frac{1}{s\left(s^2 + \left(\frac{b}{I} + \frac{R_a}{L_a}\right)s + \frac{K_e K_t}{I L_a}\right)} \cdot \frac{1}{L_a} \underbrace{\left(\mathrm{Adj}(sI - A)\right)_{2,3}}_{C_{3,2}}$$

$$-\left(-\frac{K_t}{I} s\right)$$

$$\det \otimes = \begin{vmatrix} 0 & -K_t/I \\ 0 & s + R_a/L_a \end{vmatrix}$$

$$G(s) = \frac{K_t/L_a I}{s^2 + (b/I + R_a/L_a)s + \frac{K_e K_t}{L_a I}}$$

Summary

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