Lecture 6

September 21, 2021 7:00 PM

Laplace Transform

$$\mathcal{L}\{f(t)\} = F(s) := \int_0^{+\infty} f(t) e^{-st} dt$$

Existence Theorem

F(s) exists if
- ↳ f(t) is PWC (Piecewise Linear)
- ↳ $\exists M$, $f(t) \leq Me^{at}$, $t \geq 0$

Region of Convergence
- ↳ We will not discuss RoC in ECE311
- ↳ Review in TB

Laplace Transform Pairs

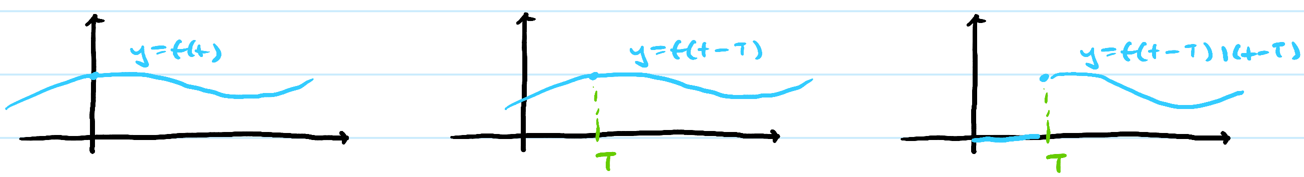
$f(t)$	$F(s)$
$1(t)$ or $u(t)$	$1/s$
$t\,1(t)$	$1/s^2$
$\frac{t^k}{k!} 1(t)$	$1/s^{k+1}$
$e^{at} 1(t)$	$\frac{1}{s-a}$
$\frac{t^k}{k!} e^{at} 1(t)$	$\frac{1}{(s-a)^{k+1}}$
$\sin(\omega t) 1(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t) 1(t)$	$\frac{s}{s^2+\omega^2}$

Rmk: we multiply each signal $f(t)$ by the unit step ($1(t)$) to ensure $f(t) = 0$ $\forall$ $t < 0$

↳ This is useful if we shift signals in time

Fundamental Properties of L.T

- Linearity
  - $\mathcal{L}\{c_1 f(t) + c_2 g(t)\} = c_1 F(s) + c_2 G(s) \qquad c_1, c_2 \in \mathbb{R}$
- Differentiation
  - $\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - \underbrace{\lim_{t\to 0^-} f(t)}_{f(0^-)}$
  - $\mathcal{L}\left\{\frac{d^k f(t)}{dt}\right\} = s^k F(s) - s^{k-1} f(0^-) - s^{k-2} f'(0^-) - \dots - \frac{d^{k-1} f}{dt^{k-1}}(0^-)$
- Integration
  - $\mathcal{L}\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s} F(s)$
- Convolution
  - $\mathcal{L}\{f(t) * g(t)\} := \mathcal{L}\left\{\int_0^t f(t-\tau) g(\tau)\, d\tau\right\} = F(s) G(s)$
- Time Shift
  - $\mathcal{L}\{f(t-\tau)\, 1(t-\tau)\} = e^{-Ts} F(s) \qquad , T \geq 0$



- Multiplication by T
  - $\mathcal{L}\{t f(t)\} = -\frac{dF(s)}{ds}$
- Shift in S
  - $\mathcal{L}\{e^{at} f(t)\} = F(s-a) \qquad a \in \mathbb{R}$

e.g.

$f(t) = [3e^{-t} + \frac{1}{2} e^{2t}]\, 1(t)$

$\Downarrow$ Linearity!

$F(s) = 3\,\frac{1}{s+1} + \frac{1}{2}\,\frac{1}{s-2}$

e.g.

$$f(t) = 1(t) - 1(t-1)$$

$\Downarrow$ Linearity , Time-Shift

$$F(s) = 1/s - e^{-s}\, 1/s$$

e.g.

$$f(t) = e^{3t}\sin(5t)\,1(t)$$

$$F(s) = \mathcal{L}\{e^{3t}\sin(5t)\,1(t)\}(s) = \mathcal{L}\{\sin(5t)\,1(t)\}(s-3)$$

$\uparrow$ Shift in S

$$= \frac{5}{s^2+25}\bigg|_{s=s-3}$$

$$= \frac{5}{(s-3)^2+25}$$

More Generally:

$$\mathcal{L}\{e^{at}\sin(\omega t)\,1(t)\} = \frac{\omega}{(s-a)^2+\omega^2}$$

$$\mathcal{L}\{e^{at}\cos(\omega t)\,1(t)\} = \frac{s-a}{(s-a)^2+\omega^2}$$

e.g.

$$f(t) = [te^{-2t} + t\sin(t)] = t\underbrace{[e^{-2t} + \sin(t)]\,1(t)}_{g(t)}$$

$$F(s) = -\frac{dG(s)}{ds}$$

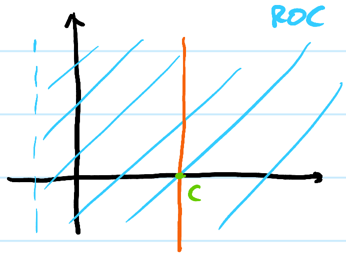
$$G(s) = \frac{1}{s+2} + \frac{1}{s^2+1}$$

$\uparrow$ Linearity

$$F(s) = \frac{1}{(s+2)^2} + \frac{2s}{(s^2+1)^2} = -\frac{dG(s)}{ds}$$

Inverse Laplace Transform

Inversion Formula



$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \int_{c-i\infty}^{c+i\infty} F(s)e^{st}\,ds$$

If $F(s)$ analytic everywhere except for a finite # of poles $\{p_1, \dots, p_n\}$, then $\mathcal{L}^{-1}$ can be computed using the Residue Theorem

$$f(t) = \sum_{i=1}^{N} \mathrm{Res}(\underbrace{F(s)e^{st}}_{G(s)}, s=p_i)\, 1(t)$$

Residue Theorem

Let $G(s)$ be an analytic function w/ a pole at $s=p$ of multiplicity $r \geq 1$

Then the $\mathrm{Res}(G(s), s=p) = \left[\frac{d^{r-1}}{ds^{r-1}}\left(G(s)(s-p)^r\right)\right]\Big|_{s=p} \frac{1}{(r-1)!}$

$r=1$ $\quad \mathrm{Res}(G(s), s=p) = \left[G(s)(s-p)\right]\Big|_{s=p}$

$r=2$ $\quad \mathrm{Res}(G(s), s=p) = \frac{d}{ds}\left[G(s)(s-p)^2\right]\Big|_{s=p}$

e.g.

$$F(s) = \frac{3}{(s+3)(s+9)}$$

2 Methods to solve:

↳ Partial Fraction Decomposition

↳ Residue Formula

Method 1 : $\frac{3}{(s+3)(s+9)} = \frac{A}{s+3} + \frac{B}{s+9}$

$\Downarrow$ ...

$$f(t) = \left[Ae^{-3t} + Be^{-9t}\right] 1(t)$$

Method 2 :

$$f(t) = \Big[\underbrace{\mathrm{Res}\Big(\frac{3e^{st}}{(s+3)(s+9)}, s=-3\Big)}_{} + \underbrace{\mathrm{Res}\Big(\frac{3e^{st}}{(s+3)(s+9)}, s=-9\Big)}_{} \Big] 1(t)$$

$$\frac{3e^{st}}{s+9}\Big|_{s=-3} \qquad\qquad \frac{3e^{st}}{s+3}\Big|_{s=-9}$$

$$\frac{3e^{-3t}}{6} = \tfrac{1}{2}e^{-3t} \qquad\qquad \frac{3e^{-9t}}{-6}$$

$$f(t) = \tfrac{1}{2}\left(e^{-3t} - e^{-9t}\right) 1(t)$$

e.g.

$$F(s) = \frac{3}{(s+3)(s+9)^2}$$

you could solve with P.F.
↳ but a lot more work

$$f(t) = \Big[\underbrace{\mathrm{Res}\Big(\frac{3e^{st}}{(s+3)(s+9)^2}, s=-3\Big)}_{} + \underbrace{\mathrm{Res}\Big(\frac{3e^{st}}{(s+3)(s+9)^2}, s=-9\Big)}_{} \Big] 1(t)$$

$$\frac{3e^{st}}{(s+9)^2}\Big|_{s=-3}$$

$$\frac{d}{ds}\Big(\frac{3e^{st}}{s+3}\Big)\Big|_{s=-9}$$

$(s+4)^2 \,|_{s=-3}$

$$\frac{d}{ds}\left(\frac{3e^{st}}{s+3}\right)\Big|_{s=-4}$$

$$= \frac{1}{(s+3)^2}\left((s+3)\cdot 3te^{st} - 3e^{st}(1)\right)\Big|_{s=-4}$$

$$f(t) = \left[\frac{1}{12}e^{-3t} - \frac{1}{2}te^{-4t} - \frac{1}{12}e^{-4t}\right]1(t)$$