Lecture 7

September 23, 2021 8:49 PM

UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE & ENGINEERING

DATE: Sept. 23/21 | 1 | PAGE

NAME: Stephen Yang

COURSE NO. | COURSE NAME: ECE311 | 2

Residue Theorem

$\mathcal{L}^{-1}\{F(s)\} = \sum_{i=0}^{N} \mathrm{Res}(F(s)e^{st}, s=p_i)$ where $p_i, \dots, p_N$ are poles of F

$\mathrm{Res}(G(s), s=p) := \frac{1}{(r-1)!} \frac{d^{r-1}}{ds^{r-1}} \left[G(s)(s-p)^r \right] \Big|_{s=p}$ where r = multiplicity of p

e.g. Find $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+2s+2)}\right\}$

$F(s) = \frac{1}{(s+1)(s^2+2s+2)}$

$= \frac{1}{(s+1)(s+1-i)(s+1+i)}$

$\mathcal{L}^{-1}\{F(s)\} = \mathrm{Res}(F(s)e^{st}, s=-1) + \mathrm{Res}(F(s)e^{st}, s=-1+i) + \mathrm{Res}(F(s)e^{st}, s=-1-i)$

$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{e^{-t}}{1-2+2} + \frac{e^{(-1+i)t}}{(-1+i+1)(-1+i+1+i)} + \frac{e^{(-1-i)t}}{(-1-i+1)(-1-i+1-i)}$

$= e^{-t} + e^{-t}\left(\frac{e^{it}}{(2i)(i)} + \frac{e^{-it}}{(-i)(-2i)} \right) = e^{-t}\left[1 - \cos(t)\right] 1(t)$

Alternatively; using PF

$F(s) = \frac{1}{(s+1)(s^2+2s+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+2}$: Find $A = 1$, $B = -1$, $C = -1$

$= \frac{1}{s+1} - \frac{s+1}{s^2+2s+2}$

$= \frac{1}{s+1} - \frac{s+1}{(s+1)^2+1} \implies f(t) = \left(e^{-t} - e^{-t}\cos(t)\right) 1(t)$

e.g. RLC circuit $\quad LC\frac{d^2y}{dt^2} + RC\frac{dy}{dt} + y = u(t) \quad y = V_c$

Note: Use Laplace Transforms to solve constant coeff. ODEs (I/O Model)

$u(t) = 1(t) \quad y(0) = \dot{y}(0) = 0$

$\mathcal{L}\{\dot{y}(t)\} = sY(s) - y(0^-) = sY(s)$

$\mathcal{L}\{\ddot{y}(t)\} = s^2Y(s) - sy(0^-) - \dot{y}(0^-) = s^2Y(s)$

$\mathcal{L}\{1(t)\} = \frac{1}{s}$

$LCs^2Y(s) + RCsY(s) + Y(s) = \frac{1}{s}$

$Y(s) = \frac{1/s}{LCs^2 + RCs + 1}$

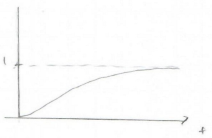
$= \frac{1/LC}{s(s^2 + \frac{R}{L}s + \frac{1}{LC})}$

pick $LC = 1/3$, $R/L = 4$

$= \frac{3}{s(s^2+4s+3)} = \frac{3}{s(s+1)(s+3)}$

$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \left[1 - \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}\right] 1(t)$

Check: $y(0)=0 \quad \dot{y}(0)=0$



What if $y(0)=2 \quad \dot{y}(0)=1$

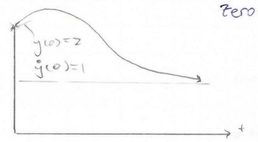
$LC\ddot{y} + RC\dot{y} + y = u$

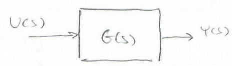
$$LC[\,s^2Y(s) \underbrace{-s(2)-1}_{\text{Initial}}\,] + RC[\,sY(s) \underbrace{-2}_{\text{Nonzero!}}\,] + Y(s) = 1/s$$

Initial conditions Nonzero!

$$Y(s) = \frac{1/s}{LCs^2+RCs+1} + \frac{LC(2s+1)+2RC}{LCs^2+RCs+1}$$

$y(t) = y_1 + y_2$ $\quad \underbrace{y_1 = \mathcal{L}^{-1}\left\{\frac{1/s}{LCs^2+RCs+1}\right\}}_{\text{zero-state response}}$ $\quad \underbrace{y_2 = \mathcal{L}^{-1}\left\{\frac{LC(2s+1)+2RC}{LCs^2+RCs+1}\right\}}_{\text{zero-input response}}$





Transfer Function (TF)

$$\frac{d^ny}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_0y = b_m\frac{d^mu}{dt^m} + \ldots + b_0u \quad , \; m \leq n \quad (I/O \text{ Model})$$

Assume zero initial conditions: $y(0) = \dot{y}(0) = \ddot{y}(0) = \ldots = \frac{d^{n-1}y}{dt^{n-1}}(0) = 0$

$u(0) = \dot{u}(0) = \ddot{u}(0) = \ldots = \frac{d^{n-1}u}{dt^{n-1}}(0) = 0$

TF does NOT take into account initial conditions

$\mathcal{L}\left\{\frac{d^ky}{dt^k}\right\} = s^kY(s)$; $\quad \mathcal{L}\left\{\frac{d^ku}{dt^k}\right\} = s^kU(s)$

$$Y(s)\left[s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_0\right] = U(s)\left[b_ms^m + \ldots + b_0\right]$$

$$Y(s) = \underbrace{\frac{b_ms^m + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + a_0}}_{G(s)} U(s)$$

$(IO) \Leftrightarrow (TF)$

Time Domain $\qquad$ s Domain

$\leftarrow G(s)$ is the Transfer Function!!

$$\boxed{Y(s) = G(s)U(s)}$$