On the Crack Extension in Plates Under Plane Loading and Transverse Shear

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The crack extension in a large plate subjected to general plane loading is examined theoretically and experimentally. It is found that under skew-symmetric plane loading of brittle materials the "sliding" or the crack extension in its own plane does not take place, instead crack grows in the direction approximately 70 deg from the plane of the crack. This is very nearly the direction perpendicular to the maximum tangential stress at the crack tip, which is 70.5 deg. The hypothesis that the crack will grow in the direction perpendicular to the largest tension at the crack tip seems to be verified also by cracked plates under combined tension and shear. In spite of the fact that "sliding" and "tearing" modes of crack extension do not take place in brittle materials it is shown that one can still talk about critical stress intensity factors in plane shear and transverse bending of plates. It is also shown that, in general plane loading, the fracture criterion in terms of stress intensity factors is an ellipse.

Introduction

As pointed out in various papers by Irwin, in fracture mechanics, mostly from a mathematical viewpoint, three basic modes of crack extension are distinguished [1, 2, 3].[1] These are: (*a*) The opening mode encountered in symmetrical extension and bending of cracked materials where displacement discontinuity is perpendicular to the plane of the crack; (*b*) the sliding mode which is presumed to occur in skew-symmetric plane loading of cracked materials where, at the leading edge of the crack, the displacement discontinuity is in the plane and parallel to the direction of the crack; (*c*) the tearing mode which is assumed to occur in skew-symmetric bending (twisting) of cracked plates or skew-symmetric loading of cracked plates by forces perpendicular to their planes where the displacement discontinuity is perpendicular to the plane of the material and in the plane of the crack. In all these cases, it is assumed that the crack is a straight through cut perpendicular to the plane of the material.

In fracture mechanics, each one of the crack extension modes mentioned in the foregoing is associated with a corresponding strain energy release rate, G_I, G_{II}, G_{III}, or crack tip stress intensity factor, k_1, k_2, K_2 (where the G's are proportional to the squares of the respective k's). The values of G may be evaluated by assuming the material to be ideally brittle and, hence, the phenomenon to be reversible and calculating the elastic work for incremental closure of the crack [1, 4].

On the other hand, the stress intensity factors, which are the strengths of stress singularities at the crack tips, are determined from the infinitesimal elasticity solution of the problem [1, 4, 5, 6]. Since the G's and k's are directly related, it suffices to discuss crack instability based on the concept of stress intensity factors.

The generalized Griffith-Irwin fracture theory states that, under the loading conditions as described previously, the slow crack extension will start when the corresponding stress intensity factor reaches to a critical value. These critical stress intensity factors, k_{1c}, k_{2c}, K_{2c}, are mechanical properties of the material in the same sense as, say, its yield stress, that is, they may vary with changing environmental and loading conditions. So far, experimental evidence in support of this theory is restricted mostly to the opening mode of the crack extension ([7] and extensive series of papers and reports published by N.R.L. group under the names Irwin, Kies, Srawley, Smith, Krafft, Boyle, Sullivan, Romauldi). Hence, the questions which may arise in the application of the abovementioned theory in fracture mechanics are: (*a*) Do or can the sliding and tearing modes of crack extension take place in actual structures under proper loading conditions stated previously? (*b*) Are k_{2c} and K_{2c} also mechanical properties of the material, or can one still talk about k_{2c} and K_{2c} if the sliding and tearing modes of crack extension do not take place? (*c*) What is the criterion for fracture if the material is subjected to a combination of various simple loading conditions described in the foregoing? These are the questions which will be discussed in a restrictive sense—the main restrictions being that (*a*) the material is ideally brittle, and (*b*) the structure under consideration is a thin, infinite plate with a straight crack. Both of these restrictions are imposed for purely mathematical reasons.

General Case of Plane Extension

In the case of plane strain or generalized plane stress where material contains a straight crack, the stress state in the neighborhood of the crack tip can be written as follows [1, 8] (see Fig. 1):

[1] Numbers in brackets designate References at end of paper.

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Nomenclature

a = half crack length
b = distance of the shear load from the origin
G = shear modulus
h = plate thickness
H = concentrated twisting moment
k_1, k_2 = stress intensity factors in plane loading
K_1, K_2 = stress intensity factors in bending
k_{ic}, K_{ic} = critical stress intensity factors (i = 1, 2)
M = concentrated edge moment
Q = transverse shear load
r, θ = polar coordinates (with crack tip as the origin)
β = crack angle (as measured from the direction of the load)
κ = material constant, $\kappa = 3 - 4\nu$ for plane strain, $\kappa = (3 - \nu)/(1 + \nu)$ for generalized plane stress
ν = Poisson's ratio
$\sigma_r, \sigma_\theta, \tau_{r\theta}$ = stresses

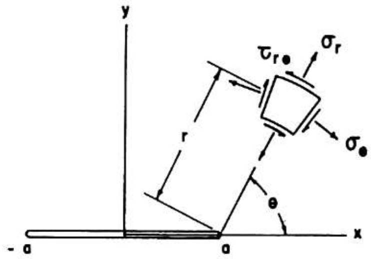


Fig. 1 Stress components near the crack tip in cylindrical coordinates

$$\sigma_r = \frac{1}{(2r)^{1/2}} \cos\frac{\theta}{2}\left[k_1\left(1+\sin^2\frac{\theta}{2}\right)+\frac{3}{2}k_2\sin\theta - 2k_2\tan\frac{\theta}{2}\right]$$

$$\sigma_\theta = \frac{1}{(2r)^{1/2}} \cos\frac{\theta}{2}\left[k_1\cos^2\frac{\theta}{2}-\frac{3}{2}k_2\sin\theta\right] \quad (1)$$

$$\tau_{r\theta} = \frac{1}{2(2r)^{1/2}} \cos\frac{\theta}{2}\ [k_1\sin\theta + k_2(3\cos\theta - 1)]$$

where k_1 and k_2 are symmetric and skew-symmetric components of the stress intensity factors, and are known functions of the external loads. If we are interested in the initiation of the slow crack extension in a given material under plane loading the consideration of equation (1) should be sufficient. Next, we state the following two commonly recognized hypotheses for the extension of cracks in a brittle material under slowly applied plane loads:

(*a*) The crack extension starts at its tip in radial direction.

(*b*) The crack extension starts in the plane perpendicular to the direction of greatest tension.

These hypotheses imply that the crack will start to grow from the tip in the direction along which the tangential stress σ_θ, is maximum and the shear stress $\tau_{r\theta}$, is zero. For two special cases, Figs. 2 and 3 show the stresses around the crack tip as a function of θ in polar plots. In Fig. 2, $k_2 = 0$, i.e., the stress state is symmetric and in Fig. 3, $k_1 = 0$, that is, the stress state is skew-symmetric. To find the angle of crack extension, θ_0, from equations (1), we write the derivative of σ_θ with respect to θ equal to zero. Thus we obtain,

$$k_2 = 0: \quad \theta = \pm\pi,\ \theta_0 = 0$$

$$k_1 = 0: \quad \theta = \pm\pi,\ \theta_0 = -\text{ arc cos }\frac{1}{3} = -70.5 \text{ deg}$$

The first set of values, $\theta = \pm\pi$, correspond to the free surface conditions of the crack and the second to the angles of maximum tangential stress. For the symmetric case, Fig. 2 indicates that $\sigma_\theta(\theta_0)$ is not the largest tensile stress around the crack tip. For example, at $\theta = 60$ deg σ_x, σ_y are the principal stresses with σ_y being a maximum, i.e., $\sigma_y(60 \text{ deg}) = 1.3\ \sigma_\theta(0 \text{ deg})$.

According to the hypotheses (*a*) and (*b*), only the tangential components of these stresses can initiate crack growth, and since $\sigma_\theta(60 \text{ deg})$ is less than $\sigma_\theta(0 \text{ deg})$, crack extends in a direction along the crack plane as verified by experiments.

In the more general case, the angle of maximum tangential stress is calculated from,

$$\cos\frac{\theta}{2}\ [k_1\sin\theta + k_2(3\cos\theta - 1)] = 0 \quad (2)$$

which gives

$$\left.\begin{aligned}\theta &= \pm\ \pi\\ k_1\sin\theta_0 + k_2(3\cos\theta_0 - 1) &= 0\end{aligned}\right\} \quad (3)$$

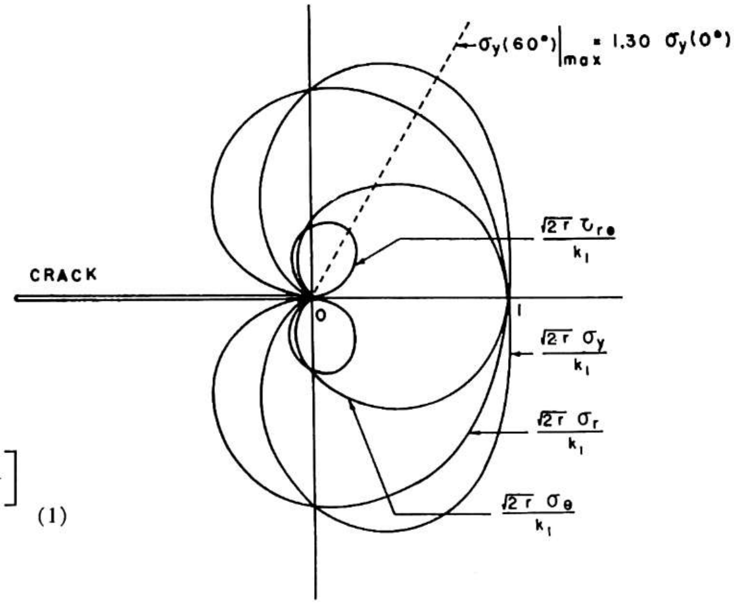


Fig. 2 Polar plot of stresses near the crack tip for symmetric case ($k_2 = 0$)

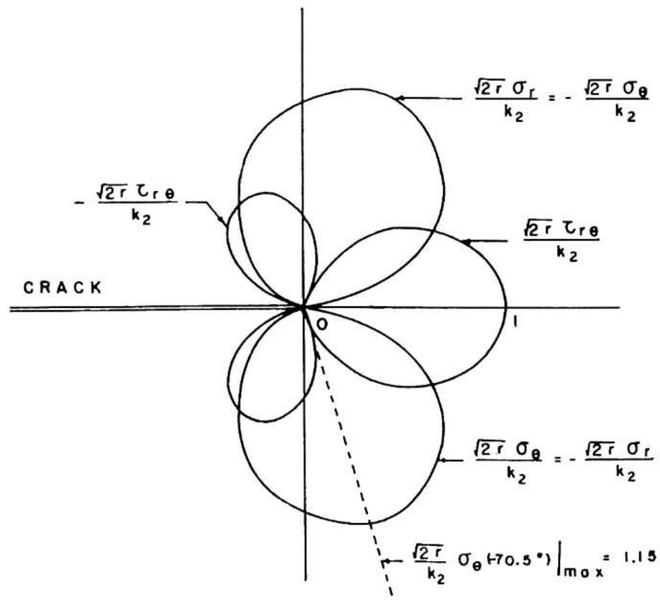


Fig. 3 Polar plot of stresses near the crack tip for skew-symmetric case

The solution, $\theta = \pm\pi$, again corresponds to the free surface conditions and the second equation of (3) gives θ_0 in terms of k_1 and k_2. For a special case shown in Fig. 4, k_1 and k_2 are given by [5]

$$k_1 = \sigma a^{1/2}\sin^2\beta,\ k_2 = \sigma a^{1/2}\sin\beta\cos\beta \quad (4)$$

Hence from (3) and (4) we can write

$$\sin\theta_0 + (3\cos\theta_0 - 1)\cot\beta = 0 \quad (5)$$

provided $\beta \neq 0$, which is a trivial case from the viewpoint of fracture mechanics. From (5), it is seen that for $0 < \beta < \frac{\pi}{2}$, θ_0 is negative; that is, the crack would be expected to start in the directions indicated by the curved lines at the crack tips in Fig. 4. Fig. 5 shows the solution of equation (5) in graphical form.

With reference to the hypotheses (*a*) and (*b*) stated previously, if we add a third well-known hypothesis of brittle fracture,

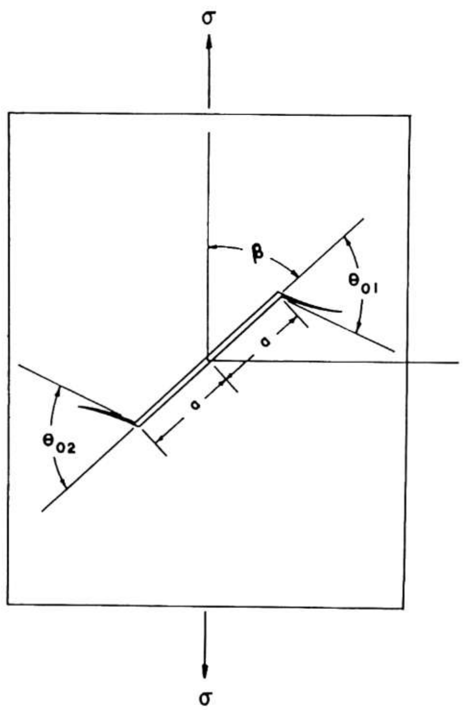


Fig. 4 Cracked plate under uniform tension

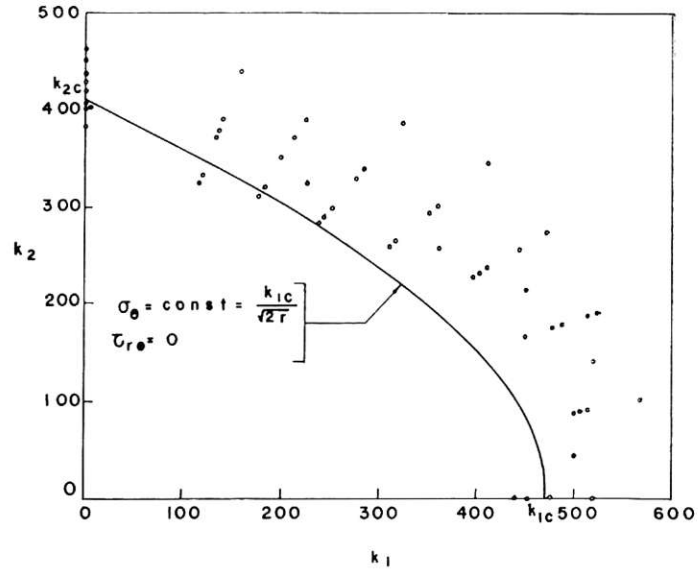


Fig. 6 k_1 versus k_2 at the beginning of crack extension in a cracked plate under plane loading

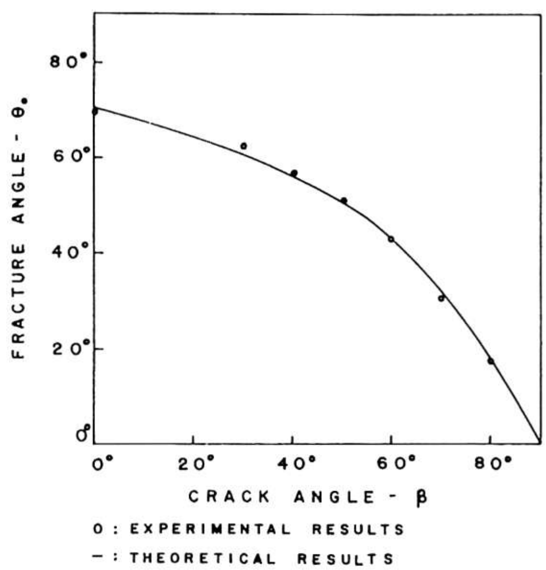


Fig. 5 Fracture angle versus crack angle in a cracked plate under uniform tension

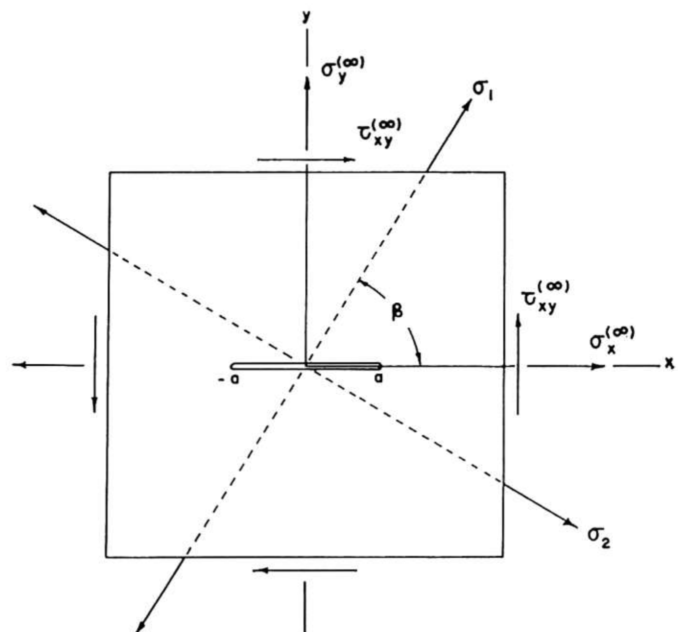


Fig. 7 Cracked plate under biaxial tension

namely, (c) "the maximum stress criterion," from equation (1), we can obtain a criterion for crack extension initiation under combined plane loading.[2] In fact, for $(2r)^{1/2}\sigma_\theta$ = constant and $\tau_{r\theta} = 0$ the second and third equations of (1) give

$$\cos\frac{\theta}{2}\left[k_1\cos^2\frac{\theta}{2} - \frac{3}{2}\,k_2\sin\theta\right] = \text{constant}$$

$$\cos\frac{\theta}{2}\,[k_1\sin\theta + k_2\,(3\cos\theta - 1)] = 0 \tag{6}$$

Equations (6) may be considered as the parametric equations of a curve in k_1, k_2-plane. For $(2r)^{1/2}\sigma_\theta$ = constant = k_{1c} = 472, the curve is shown in Fig. 6 where k_{1c} is the critical stress intensity factor for plexiglass which will be discussed below.

The foregoing analysis is based on very simple notions of brittle fracture. On the other hand, if we accept the Griffith theory as the valid criterion which explains the crack growth, the conditions under which the crack extension will take place should be restated as follows: "The crack will grow in the direction along which the elastic energy release per unit crack extension will be maximum and the crack will start to grow when this energy reaches to a critical value." The application of the Griffith theory in this form requires the calculation of the elastic energy

[2] Even though the elastic theory gives infinite stress at the crack tip and, hence, the maximum stress criterion may not have any meaning; however, if we accept the mechanism as suggested by Barenblatt [9] according to which the actual stresses are finite and the pure elastic solution can still be used, one may be justified in using this criterion.

release per unit crack extension in cases for which the crack extension is not colinear with the crack itself. At the moment the mathematical difficulties for this problem seems to be insurmountable. However, to reconcile the hypothesis (b) made previously with the Griffith theory, one may use the following argument: Consider an infinite plate under uniform biaxial tension with stresses σ_1 and σ_2 (see Fig. 7). Let us assume that the plate, while under stresses, is cut along a straight line of length $2a$ in an arbitrary direction β. In this case, it can be shown that the amount of elastic energy release per unit thickness is

$$\Delta U = \frac{\pi a^2(\kappa + 1)}{16G} [\sigma_1^2 + \sigma_2^2 - (\sigma_1^2 - \sigma_2^2) \cos 2\beta] \quad (7)$$

where G is the shear modulus and $\kappa = (3 - \nu)/(1 + \nu)$ for generalized plane stress and $\kappa = 3 - 4\nu$ for plane strain, ν being the Poisson's ratio. Note that ΔU is a positive definite quantity and its extremums can be obtained by writing $\frac{d}{d\beta} \Delta U = 0$. For $\sigma_1 > \sigma_2$, we have,

$$\begin{aligned} (\Delta U)_{\max} &= (\Delta U)_{\beta=\pi/2} = \frac{\pi a^2(\kappa + 1)}{8G} \sigma_1^2 \\ (\Delta U)_{\min} &= (\Delta U)_{\beta=0} = \frac{\pi a^2(\kappa + 1)}{8G} \sigma_2^2 \end{aligned} \quad (8)$$

The physical interpretation of equation (8) is that, in a plate under biaxial loading, the greatest elastic energy release will result from a straight incision perpendicular to the direction of greatest tension. Heuristically, one may expect that instead of introducing a small incision in a plate under uniform biaxial stresses, if we put a small incision in the plate where stress state is not uniform, such as the immediate neighborhood of a crack tip, the greatest elastic energy release will take place when the incision is made approximately perpendicular to the direction of maximum tension. It should be pointed out that this reasoning should not be taken as a proof of the empirical statements made previously to explain the crack extension, rather it may be thought of just as a supporting argument.

To obtain a criterion for crack extension in general two-dimensional loading, one may also pursue the following reasoning: The case of general plane loading where the crack is assumed to grow in its direction by a small amount δ, on one end, the elastic energy release can be written as[3]

$$\Delta U = \frac{\pi\delta(\kappa + 1)}{8G} (k_1^2 + k_2^2) \quad (9)$$

Even though equation (9) gives the energy release for a special case, since the stress state in the vicinity of the crack tip is entirely controlled by the constants k_1 and k_2 one may also expect ΔU to be entirely dependent on k_1 and k_2 at the instant of growth initiation. Furthermore, since ΔU has to be a homogeneous quadratic form in the stresses for any direction of crack growth, it will also have to be a homogeneous quadratic form in k_1 and k_2; and since ΔU is positive definite, ΔU = constant curve in k_1, k_2-plane will have to be an ellipse and can be written as

$$\frac{\Delta U}{\delta} = a_{11}k_1^2 + 2a_{12}k_1k_2 + a_{22}k_2^2 = \text{constant} \quad (10)$$

where the constants a_{ij} (i, j = 1, 2) are functions of material properties. As will be seen in the discussion of experimental results, two points of this ellipse, namely, its intersections with k_1 and k_2 axes are known. They are ($k_1 = k_{1c}$, $k_2 = 0$) and ($k_1 = 0$, $k_2 = k_{2c}$) where k_{1c} and k_{2c} are mechanical properties of the material.

[3] This result is easily obtained from the elastic solution [6] in the vicinity of the crack tip, $\Phi(z) = \Omega(z) \cong \alpha_0(z - a)^{-1/2}$, which gives $(u + iv)_\theta = 0 = \frac{\kappa + 1}{G} \alpha_0\, i\, r^{1/2}$, $(\sigma_y + i\tau_{xy})_{\theta=0} = 2\bar{\alpha}_0/r^{1/2}$, and $\Delta U = \mathrm{Im}\left[\int_0^\delta \frac{2\bar{\alpha}_0}{\sqrt{r}} \frac{\kappa + 1}{G} \alpha_0 i(\delta - r)^{1/2} dr\right]$; $\alpha_0 = (k_1 - ik_2)/(2\sqrt{2})$.

Experimental Results for the Case of Plane Stress

To test the validity of some of the results found previously for the extension of cracks in brittle materials under plane loading a series of experiments were performed. The material used was 0.12-in-thick plexiglass (commercial version of Plex. II) plates. In addition to being a good approximation to a homogeneous, isotropic, and linearly elastic ideal brittle material, it is very easy to machine and put natural cracks in, to detect the faulty crack tips, and, because of its transparency, to detect accurately the beginning of the crack growth. The preparation of specimens was the same as described in reference [7]. The following were the series of experiments performed:

(a) A large plate with a central straight crack was subjected to equal and opposite concentrated shear loads applied to the crack surface at a distance b from the axis of symmetry (Fig. 8). For this case, the stress intensity factors are [5, 6].

$$k_1 = 0$$

$$k_2 = \frac{Q}{2\pi(a)^{1/2}} \left[\left(\frac{a + b}{a - b}\right)^{1/2} + \left(\frac{a - b}{a + b}\right)^{1/2}\right] \quad (11)$$

where Q is the shear load per unit thickness. The plate was reinforced at the points of load application and was loaded through $^1/_8$-in. pins—lever arms— and dead weights (running water). The tests were stopped at the first detection of crack initiation and the loads were measured. The results of these tests are given in Table 1 and Fig. 8. In Table 1, h is the plate thickness and θ_{0i} are the fracture angles, i.e., the angle between crack direction and crack extension. A photograph of a specimen with the extended crack is shown in Fig. 9(a). Figs. 9(b) and 9(c) show, respectively, the crack extensions in a thin-walled tube with a longitudinal crack under torsion and a large plate with a central crack subjected to shear at infinity. In these cases too, the fracture angles were measured to be around 70 deg with very small scatter.

Table 1

$2a$ (in.)	$2b$ (in.)	h (in.)	Q (lb)	k_2 (lb/in.$^{3/2}$)	θ_{01} (deg)	θ_{02}
1.90	0.5	0.119	167	465		
1.95	0.5	0.116	157	452	71	69
2.18	0.4	0.119	167	436	70.5	71.5
2.80	0.5	0.122	182	408	70	67
3.20	0.41	0.117	185	402	68	71
3.50	0.4	0.112	193	402	69	71
3.60	0.5	0.121	217	430	69.5	69
4.00	0.54	0.118	191	384	70	72.5
4.05	0.45	0.116	216	420	71.5	69

(b) Large plexiglass plates (9 in. × 18 in. in size) with centrally located and arbitrarily oriented 2-in. long cracks were subjected to uniform tension at infinity (Fig. 4). The stress intensity factors for this case are given by equation (4) and theoretical fracture angles are shown in Fig. 5. The results of these experiments are shown in Figs. 5 and 6. At each crack angle β four plates were tested and Fig. 5 contains only the average values of the fracture angles θ_0.

The largest deviation was observed to be 3 deg. The values of k_1 and k_2 shown in Fig. 6 were calculated from equation (4). All the crack lengths were approximately 2 in. Fig. 10 shows a typical specimen with a rotated crack. In Fig. 6, the experimental results corresponding to $k_1 = 0$ are those given in Table 1, which shows the results of the tests with concentrated shear loads.

Results for Transverse Bending of Plates

The transverse bending problem of an infinite plate with a

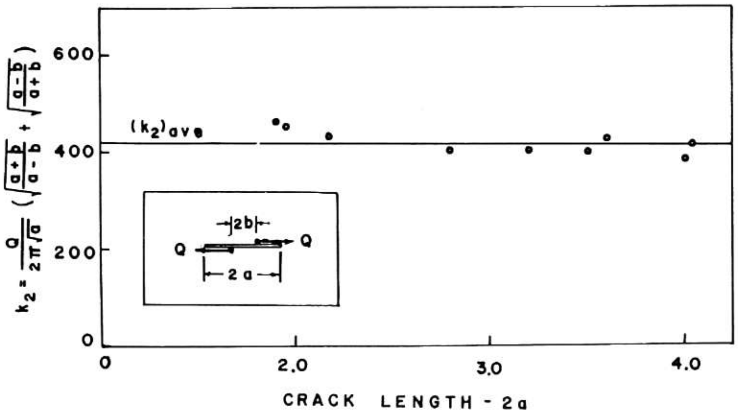


Fig. 8 Results of the tests with plane shear loads

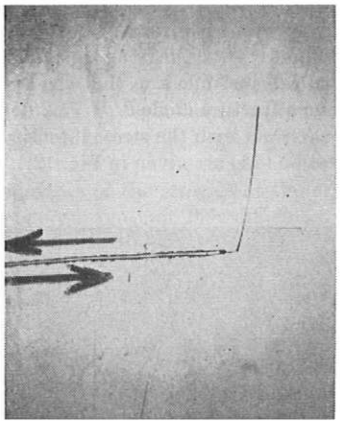


Fig. 9 (a) Crack tip in a plate under concentrated plane shear load

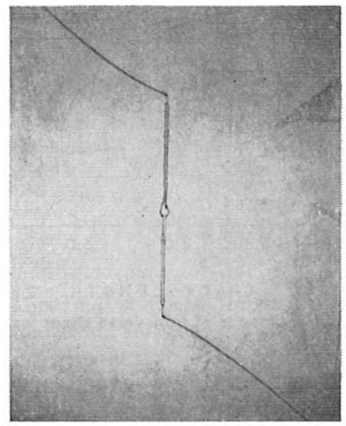


Fig. 9 (c) Cracked plate subjected to distributed shear away from the crack

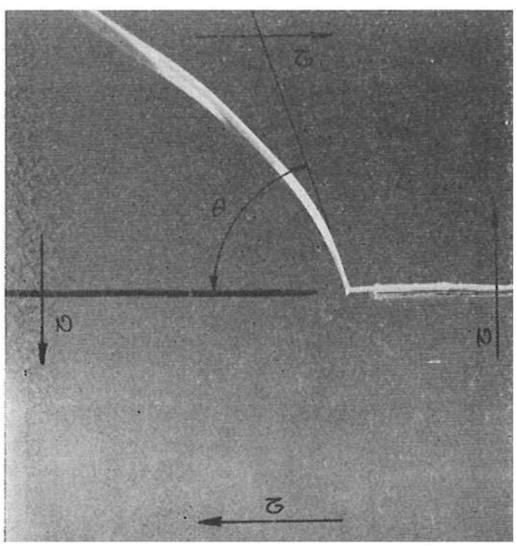


Fig. 9 (b) Crack tip in a thin-walled tube under torsion

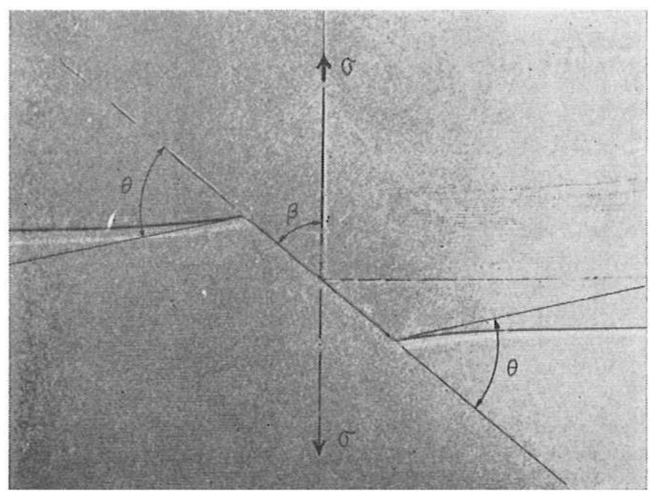


Fig. 10 Plate with a rotated crack under uniform tension

concentrated couple, $C = M + iH$, applied to one of the crack surfaces at $x = b$ of a central crack of length $2a$, can be solved in a manner similar to that of the extensional problem [5]. The method of analysis is described in detail in reference [5] and, therefore, only the final results, which are based on the Kirchhoff's theory, will be given here.

$$K_1 = \frac{3M}{\pi h^2\sqrt{a}}\left(\frac{a+b}{a-b}\right)^{1/2} - \frac{3H(1+\nu)}{2\pi h^2\sqrt{a}}$$
$$K_2 = \frac{3M(1+\nu)}{2\pi h^2\sqrt{a}} + \frac{3H}{\pi h^2\sqrt{a}}\left(\frac{a+b}{a-b}\right)^{1/2} \tag{12}$$

where K_1 and K_2 are the crack tip stress intensity factors for symmetrical and skew-symmetrical stress distributions, respectively. In the case of equal and opposite twisting couples H applied to the crack surfaces at $x = b$ (Fig. 11) equation (12) reduces to,

$$K_1 = 0$$
$$K_2 = \frac{6H}{\pi h^2\sqrt{a}}\left(\frac{a+b}{a-b}\right)^{1/2} \tag{13}$$

Equation (13) may be taken as a Green's function to solve the problem illustrated in Fig. 12. This is accomplished by adding equal and opposite uniform twisting couples, $-H$ from $-a$ to 0, and H from 0 to a. This loading in effect is equivalent to equal and opposite concentrated edge forces $Q = 2H$ applied normal to the plate at the origin (Fig. 12). The result is,

$$K_2 = \frac{3Q}{\pi h^2 \sqrt{a}} \left(\int_0^a \left(\frac{a+x}{a-x}\right)^{1/2} dx + \int_0^{-a} \left(\frac{a+x}{a-x}\right)^{1/2} dx \right)$$

$$= \frac{6Q\sqrt{a}}{\pi h^2} \quad (14)$$

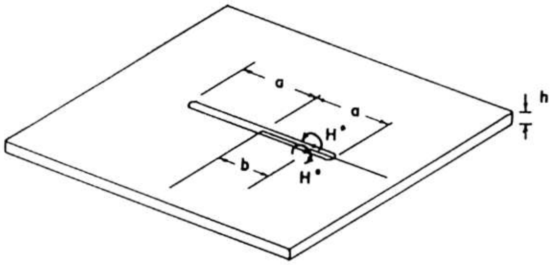


Fig. 11 Cracked plate under concentrated twist

Experiments were performed with 9 × 18-in. plexiglass plates containing a central crack parallel to the longer side subjecting them to concentrated skew-symmetric twist and transverse shear loads (Figs. 11 and 12). Fig. 13 shows the typical specimens which are loaded by concentrated couples (a), and by concentrated transverse shear loads (b), (c), and (d). Since Kirchhoff's theory does not represent the conditions in the immediate vicinity of the crack tips and since a more refined solution is not available at present for the general case, we will not attempt to analyze the crack extension. However, leaving the analysis for this case to a later time, we will just mention that the average of sixteen measurements gave a fracture angle $\theta_0 = 71.2$ deg.

The results of the tests with the stress intensity factors as calculated from equation (14) are given in Fig. 12.

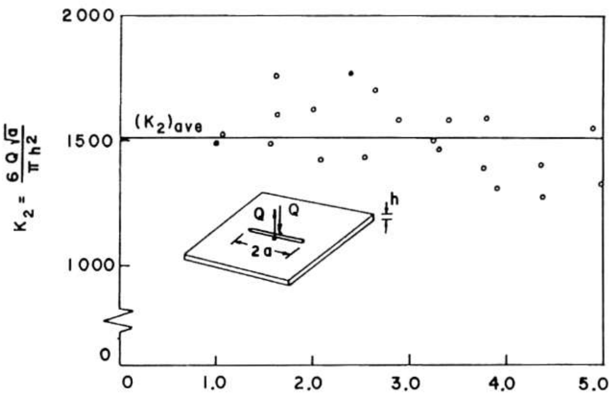
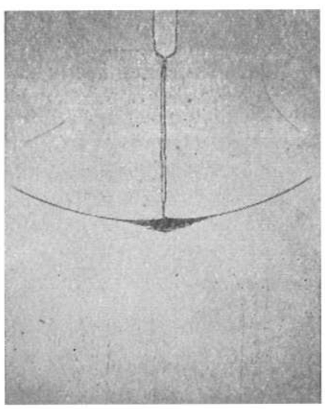
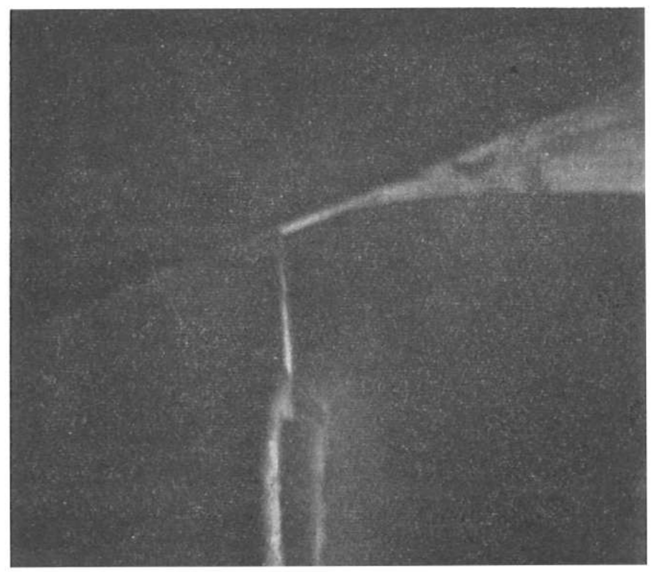


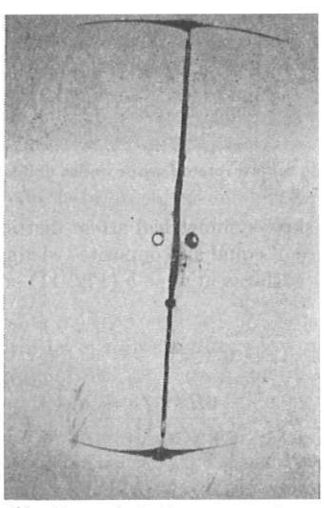
Fig. 12 Results of transverse shear tests



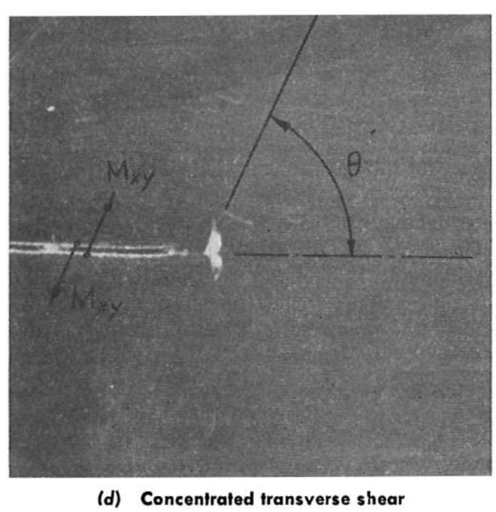
(a) Concentrated twist



(c) Concentrated transverse shear



(b) Concentrated transverse shear



(d) Concentrated transverse shear

Fig. 13 Plate under transverse bending

The Discussion of the Results and Conclusions

In light of the analysis and the experimental results described previously we may attempt to give some cautious answers to the questions set forth in the Introduction to this paper:

Figs. 9, 10, and 13 and the related experiments clearly indicate that, in ideal brittle materials, the so-called "sliding" and "tearing" modes of crack extension do not take place. The mode of fracture seems to be always a crack opening.

The simple explanation of the crack extension in two-dimensional problems given in paragraph 2 is based on the three hypotheses: (*a*) crack grows radially; (*b*) growth direction is perpendicular to maximum tension; (*c*) maximum stress theory is applicable, and may be considered as a satisfactory model for brittle materials. The first one is quite plausible and does not require any supporting argument. In support of (*b*), we mention the results given in Table 1 and Fig. 5. Table 1 gives an average fracture angle of 70 deg as compared to the calculated value of 70.5 deg. In the case of biaxial loading, even though there is more scatter, the agreement between observed and calculated fracture angles given in Fig. 5 seems to be satisfactory.

The results shown in Fig. 6 indicate that the maximum stress hypothesis (*c*) should be regarded as a practical design criterion only. At least for the material under consideration, it seems to be a conservative theory. Unless Barenblatt's suggestions concerning the finiteness of the stresses at the crack tip is blended into Griffith's energy concepts, it is difficult to explain the presence of the maximum stress theory in fracture mechanics. Obviously, more work is needed to be done in this area. The test results of Fig. 6 suggest somewhat more strongly the validity of a more plausible criterion as given by equation (10). Here the difficulty is mathematical in nature and lies in the evaluation of the coefficients a_{ij} corresponding to maximum elastic energy release per unit crack extension $\Delta U/\delta$.

As pointed out in the reasoning which led to equation (10), $\Delta U/\delta$ is a quadratic form in the stress intensity factors irrespective of the direction of the crack extension. Hence, in simple cases like pure shear or pure twist $\Delta U/\delta$ will be proportional to the square of the respective stress intensity factors, k_2 or K_2, and since $\Delta U/\delta$ = constant initiates the crack growth, one may talk about critical stress intensity factors. For example, in plane shear $k_1 = 0$ and $\Delta U/\delta = a_{22}k_{2c}^2$ = constant would initiate the crack extension. It has to be noted that the constant a_{22} is not equal to $\pi(\kappa + 1)/(4G)$ as given in equation (9) and for plexiglass $a_{22} > \pi(\kappa + 1)/(4G)$. The results given in Figs. 8 and 12 seem to support this conclusion. The wide scatter in Fig. 12 is mostly due to the difficulty in the detection of the initiation of crack extension. The cracks shown in Fig. 13 are very stable and at a reasonable loading rate (say $^1/_2$ hour for k_{2c}), they grow very slowly.

The bending problem needs to be studied theoretically first. Its physical picture also seems to be very complicated. We may mention an easily observable peculiarity which is that the plane tangent to the crack extension is not perpendicular to the plane of the plate, it is inclined toward the crack itself on the tension side. In plane loading, all the fracture surfaces were perpendicular to the plane of the plate.

In conclusion, it should be pointed out that most of the foregoing conclusions cannot be expected to hold if any plastic zone develops around the crack tip. Preliminary tests with aluminum plates indicate that the only conclusion which may be applicable to ductile materials is the existence of a possible fracture criterion as given by equation (10); in fact, the scatter for aluminum is considerably smaller than that for plexiglass shown in Fig. 6.

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