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TECHNICAL NOTE

MIXED-MODE FATIGUE CRACK GROWTH PREDICTION IN BIAXIALLY STRETCHED SHEETS

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Abstract—Mixed-mode fatigue crack growth in biaxially stretched sheets is investigated. The modified fracture criterion proposed by Wang and Du is extended to the case of cyclic loading to predict the mixed-mode fatigue crack growth. The analysis of the mixed-mode fatigue crack growth process is very complex. Software developed recently by the authors can be used to more precisely predict the mixed-mode fatigue crack growth process. The software in which the fatigue crack growth criterion presented here is combined with the displacement discontinuity method, a boundary element method, is described in detail from a number of aspects. The analysis of the fatigue growth process of an inclined crack in biaxially stretched sheets is performed.

1. INTRODUCTION

THE APPLICATION of fracture mechanics concepts to the design and analysis of structural components is still in the developing stage. This is mainly because of the inability to translate laboratory data to the design of complex structures. Sih and Barthlehem [1] pointed out that the commonly used fatigue crack growth equation [2]

$$\frac{\Delta a}{\Delta N} = A(\Delta K_{\mathrm{I}})^m \tag{1}$$

involving two constants A and m may not be adequate for two reasons. First, any crack growth expression, $\Delta a/\Delta N$, should contain, in principle, at least two loading parameters, say the stress amplitude, $\Delta\sigma$, and the means stress level, $\bar{\sigma}$, so that the fatigue loading is properly defined. Equation (1) involves $\Delta\sigma$ only and is restricted to a crack running straight ahead. In practice, the direction of the applied load may change and it would be overly restrictive to assume that the load and crack should always be maintained normal to one another. The majority of the failure in service is of the mixed-mode type where the crack does not propagate in the direction normal to the applied load because of the lack of geometric symmetry. Such an effect has been accounted for in the fatigue crack growth prediction analysis in ref. [1].

In the present paper, as in [1], a modified fracture criterion, presented by Wang and Du [3] for the maximum tangential stress criterion, in which the fracture parameter is accounted for along the Mises elastic–plastic boundary surrounding the crack tip, is extended to the case of cyclic loading to predict the mixed-mode fatigue crack growth. For the purpose of more precisely predicting the mixed-mode fatigue crack growth process, which is very complex, we recently developed software in which the fatigue crack growth criterion is combined with the displacement discontinuity method [4], a boundary element method. The software is described in detail here from a few aspects. The analysis of the fatigue growth process of an inclined crack in biaxially stretched sheets is performed.

2. FATIGUE CRACK GROWTH CRITERION

In this section, as in [1], we will extend the modified fracture criterion presented by Wang and Du [3] for the maximum tangential stress criterion [5] to the case of cyclic loading to predict the mixed-mode fatigue crack growth.

Following the modified fracture criterion [3], fracture parameters should be taken into account along the Mises elastic–plastic boundary surrounding the crack tip, whose equation can be expressed as

$$r = \omega/(6\pi\sigma_S^2), \tag{2}$$

in which

$$\omega = \begin{cases} f_1^2 + f_2^2 - f_1 f_2, & \text{for plane stress conditions} \\ f_1^2 + f_2^2 - f_1 f_2 - \nu(1-\nu) f_3^2, & \text{for plane strain conditions} \end{cases}$$

$$f_{1,2} = f_3 \pm (f_4^2 + 4 f_{xy}^2)^{1/2}$$

$$f_3 = 2\left(K_{\mathrm{I}} \cos\frac{\theta}{2} - K_{\mathrm{II}} \sin\frac{\theta}{2}\right)$$

$$f_4 = -K_{\mathrm{I}} \sin\theta \sin\frac{3\theta}{2} - K_{\mathrm{II}}\left(2\sin\frac{\theta}{2} + \sin\theta\cos\frac{3\theta}{2}\right)$$

$$f_{xy} = \tfrac{1}{2} K_{\mathrm{I}} \sin\theta \cos\frac{3\theta}{2} + K_{\mathrm{II}}\left(\cos\frac{\theta}{2} - \frac{1}{2}\sin\theta\sin\frac{3\theta}{2}\right),$$

where (r, θ) are the polar coordinates around the crack tip, K_{I} and K_{II} the mode I and II stress intensity factors, and ν and σ_S the Poisson ratio and the yield stress of the material, respectively.

The tangential stresss is [5]

$$\sigma_\theta = \frac{1}{2\sqrt{(2\pi r)}}\cos\frac{\theta}{2}[K_{\rm I}(1+\cos\theta) - 3K_{\rm II}\sin\theta] = \frac{1}{\sqrt{(2\pi r)}}f(\theta), \tag{3}$$

in which

$$f(\theta) = \frac{1}{2}\cos\frac{\theta}{2}[K_{\rm I}(1+\cos\theta) - 3K_{\rm II}\sin\theta].$$

The fatigue crack growth criterion is assumed to follow the following two conditions.

(1) The fatigue crack growth initiates in the direction of the maximum tangential stress evaluated along the Mises elastic–plastic boundary surrounding the crack tip, i.e.

$$\left(\frac{\partial\sigma_\theta}{\partial\theta}\right)_{\theta=\theta_0} = \frac{1}{\sqrt{(2\pi r)}}\left(\frac{\partial f}{\partial\theta} - \frac{f}{2r}\frac{\partial r}{\partial\theta}\right)_{\theta=\theta_0} = 0 \tag{4a}$$

and

$$\left(\frac{\partial^2\sigma_\theta}{\partial\theta^2}\right)_{\theta=\theta_0} = \frac{1}{\sqrt{(2\pi r)}}\left(\frac{1}{4}\frac{f}{r^2}\left(\frac{\partial r}{\partial\theta}\right)^2 + \frac{\partial^2 f}{\partial\theta^2} - \frac{f}{2r}\cdot\frac{\partial^2 r}{\partial\theta^2}\right)_{\theta=\theta_0} < 0, \tag{4b}$$

where θ_0 is the fatigue crack growth angle; $r = r(\theta)$ is given by eq. (2).

(2) The fatigue crack growth rate equation is

$$\frac{\Delta a}{\Delta N} = C(\Delta K_e)^n, \tag{5}$$

in which

$$\Delta K_e = \frac{1}{2}\cos\frac{\theta_0}{2}[\Delta K_{\rm I}(1+\cos\theta_0) - 3\Delta K_{\rm II}\sin\theta_0],$$

where $\Delta K_{\rm I}$ and $\Delta K_{\rm II}$ are the stress intensity factor ranges, and C and n the material constants which are related to the constants A and m in the Paris equation (1) by the relations

$$C = A \qquad n = m. \tag{6}$$

3. ANALYSIS METHOD

Recently, we developed software in which the displacement discontinuity method [4], a boundary element method, is combined with the improved mixed-mode fatigue crack growth criterion presented in this paper. The software can be used to predict more precisely the mixed-mode fatigue crack growth. Four key techniques involved in the software will now be described.

(1) Based on the equations of the displacement discontinuity method [4], we presented a special displacement discontinuity element (called the special element for short), as shown in Fig. 1, in which the displacement discontinuity functions are taken to be

$$D_S = D_1(\zeta/b)^{1/2}$$
$$D_n = D_2(\zeta/b)^{1/2}, \tag{7}$$

where $2b$ is the element length, as shown in Fig. 1; D_1 and D_2 are, respectively, the tangential and normal displacement discontinuity quantities at the center of the special element. The special element possesses the same unknowns (still two unknowns) as the ordinary (or constant) displacement discontinuity element (called the ordinary element for short). However, it can simulate the displacement field surrounding the crack tip. Based on the displacement field surrounding the crack tip, we can obtain the computed formulas of stress intensity factors $K_{\rm I}$ and $K_{\rm II}$ [6]:

$$K_{\rm I} = -\frac{\sqrt{(2\pi)}\mu D_1}{4\sqrt{b}(1-\nu)}$$
$$K_{\rm II} = -\frac{\sqrt{(2\pi)}\mu D_2}{4\sqrt{b}(1-\nu)} \tag{8}$$

where μ and ν are the shear modulus and the Poisson ratio, respectively.

An infinite plate with a through crack of length $2a$ which is subjected to uniform stress normal to the crack plane at distances sufficiently far away from the crack is taken for example to compute the stress intensity factor $K_{\rm I}$. Owing to its symmetry, only half is taken for analysis. Figures 2 and 3, respectively, show the error to vary with the number of the elements which all, including the special element, possess the same size, and with the ratio of the size of the special element to that of the ordinary elements. In the latter case, the total number of elements is 11. It can be seen from Fig. 2 that a good result for the stress intensity factor $K_{\rm I}$ can be obtained using the special element placed at the crack tip. It can be seen from Fig. 3 that the ratio of the size of the special element to that of the ordinary elements is necessarily taken to be about unity in order to obtain a good result. This can be regarded as the limitation to placing the special element at the crack tip.

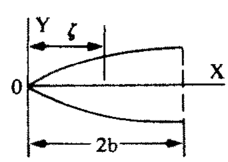

Fig. 1. The special element.

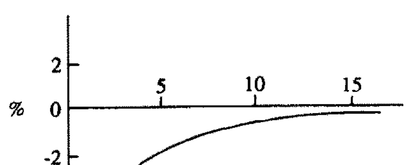

Fig. 2. Variation of error with the number of elements.

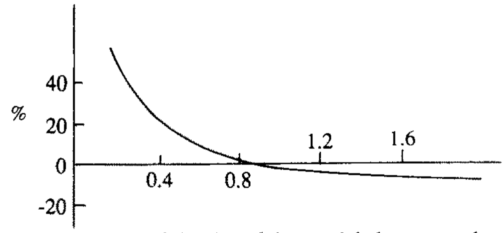

Fig. 3. Variation of error with the ratio of the size of the special element to that of the ordinary elements.

(2) Because the mixed-mode fatigue crack grows in the form of a zig-zag, its growth angle is varied with the mixed-mode fatigue crack growth. Thus when it is assumed that it grows step by step, its growth angle must be determined for each growth increment following the condition (1) of the mixed-mode fatigue crack growth criterion, which is very important in predicting the mixed-mode fatigue crack growth. Here, the method for determining the growth angle of the mixed-mode fatigue crack is as follows.

First, we assume the mixed-mode fatigue crack to grow step by step, its growth increments denoted by Δa_1, $\Delta a_2 \ldots \Delta a_i \ldots$, its growth angles by $\theta_{01}, \theta_{02} \ldots \theta_{0i} \ldots$, and its stress intensity factors by (K_{I_1}, K_{II_1}), $(K_{I_2}, K_{II_2}) \ldots$ $(K_{I_i}, K_{II_i}) \ldots$, in which K_{I_1}, K_{II_1} and θ_{01} are the stress intensity factors and the growth angle of the original crack. The following equation is used to give the approximate value (absolute) of the growth angle θ_{0i}, denoted by $\bar{\theta}_i$:

$$\bar{\theta}_i = \varphi(\bar{\beta}_i), \tag{9}$$

in which

$$\bar{\beta}_i = \mathrm{arctg}(K_{I_i}/|K_{II_i}|) \times 180/\pi \quad (i = 1, 2 \ldots), \tag{10}$$

while the function φ, which is a polynomial which gives the growth angle on the inclined crack plate subjected to static uniform stress at distances sufficiently far away from the crack, is obtained by means of the Lagrangian interpolation. For example, the function $\varphi(\beta)$ for the improved tangential stress criterion is

$$\begin{aligned}\varphi(\beta) = &-85.52922 + 1.101227\beta - 0.6458092 \times 10^{-1}\beta^2 \\ &+ 0.360623 \times 10^{-2}\beta^3 - 0.1166826 \times 10^{-3}\beta^4 + 0.2317131 \times 10^{-5}\beta^5 \\ &- 0.2744923 \times 10^{-7}\beta^6 + 0.1786828 \times 10^{-9}\beta^7 - 0.4886161 + 10^{-12}\beta^8.\end{aligned} \tag{11}$$

Second, based on the fact that the mixed-mode fatigue crack grows in the form of a zig-zag and the postive and negative definitions of its growth angle (when the crack turns counterclockwise the growth angle is negative, otherwise, positive), we can decide that when $K_{II_i} > 0$ the approximate value of the growth angle θ_i' is $\theta_i' = \bar{\theta}_i$, otherwise $\theta_i' = -\bar{\theta}_i$.

Third, based on the approximate value of the growth angle, the growth angle equation is solved by means of the bipartition method to find the accurate value of the growth angle, θ_{0i}.

(3) When an analysis for fatigue growth of an inclined crack in uniaxial cyclic stress or biaxial cyclic stresses is performed, the cracked plate can be regarded as an infinite plate. Therefore, the boundary requiring to be treated is only the crack plane. Then, computed quantities are largely decreased. Taking into account the fact that the cracked plate is finite, the stress intensity factors K_I and K_{II} which are calculated by considering the cracked plate to be infinite are modified by means of the following modification coefficients [6]:

$$f_I = 1 + 0.8692(2a_x/W)^{1.9}$$

$$f_{II} = 1 + 0.32096(2a_x/W)^{1.9},$$

where a_x is the projection length of the half-crack on the x-axis, and W is the width of the cracked plate.

(4) In respect of the program design, an element at each crack tip is increased for each crack growth increment and the element is taken to be the special element and the original element is modified to be the ordinary element.

The flow chart of the analysis of the mixed-mode fatigue crack growth is referred to ref. [6].

4. RESULTS

In this section, we will use the software developed recently by the authors, in which the fatigue crack growth criterion presented in Section 2 is combined with the displacement discontinuity method, to predict the fatigue growth process of an inclined crack in biaxially stretched sheets, as shown in Fig. 5. In analysis, the cracked plate is assumed to be under plane strain conditions. Moreover, the material parameters, the shear modulus, μ, the Poisson ratio, v, the fracture toughness, K_{Ic}, the constants A and m in the Paris equation, the cyclic loading parameters, the mean values of σ_y, σ_{ym}, the cyclic characteristic, R, and the geometric parameters, the original half-crack length, a_0, and the cracked plate width, W, are as follows:

$$\begin{aligned}&\mu = 2744\ \mathrm{kg/(mm)^2}, \quad v = 0.321, \quad K_{Ic} = 116\ \mathrm{kg/(mm)^{3/2}} \\ &A = 1.039 \times 10^{-10}, \quad m = 2.7438, \quad \sigma_{ym} = 15.33\ \mathrm{kg/(mm)^2} \\ &R = 0.048, \quad a_0 = 7\ \mathrm{mm}, \quad W = 150\ \mathrm{mm}.\end{aligned} \tag{12}$$

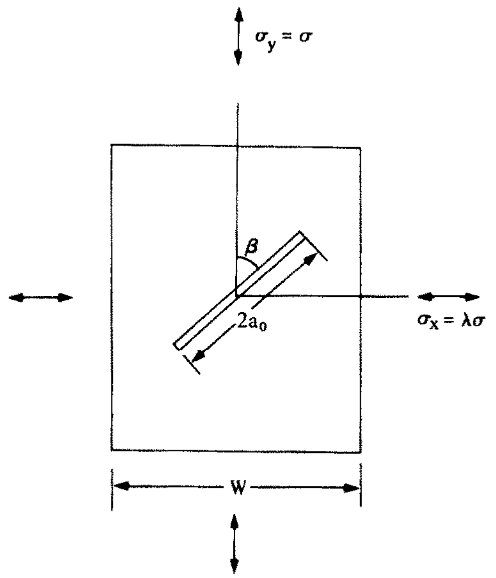

Fig. 4. An inclined crack in a biaxially stretched sheet.

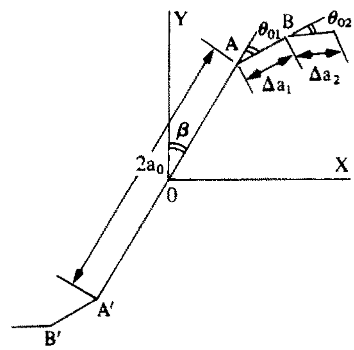

Fig. 5. Schematic illustration of mixed-mode fatigue crack growth.

In order to investigate the fatigue growth laws of the cracked plate shown in Fig. 5 with the variations of β and λ, the values of β and λ (see Fig. 5) are taken to be

$$\beta = 10^\circ, 20^\circ, 30^\circ, 40^\circ, 50^\circ, 60^\circ, 70^\circ, 80^\circ, 90^\circ$$
$$\lambda = 0.00, 0.25, 0.50, 0.75, 1.00. \tag{13}$$

Some results are given in Figs 6 and 7, from which it can be seen that the fatigue growth laws of the cracked plates are as follows.

(1) Under the uniaxial cyclic loading case ($\lambda = 0.00$), the fatigue crack growth path always tends rapidly to be normal to σ_y, whether β is large or small, once the crack grows.

(2) Under the equal-biaxial cyclic loading case ($\lambda = 1.00$), the fatigue crack growth path is always along the original crack plane, whether β is large or small, after the crack grows.

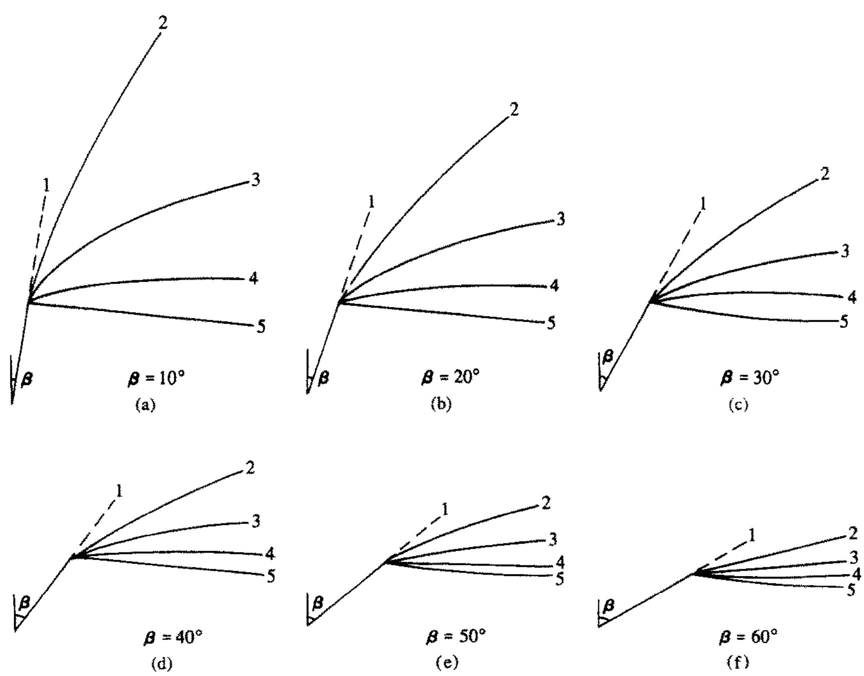

Fig. 6. Fatigue crack growth paths (1: $\lambda = 1.00$; 2: $\lambda = 0.75$; 3: $\lambda = 0.50$; 4: $\lambda = 0.25$; 5: $\lambda = 0.00$).

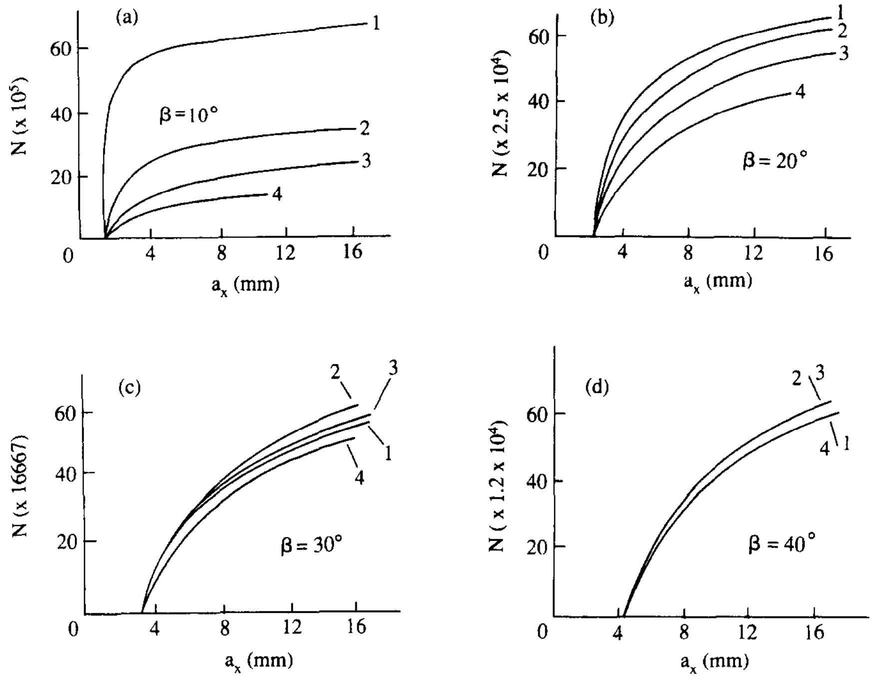

Fig. 7. N–a_x curves (1: $\lambda = 1.00$; 2: $\lambda = 0.25$; 3: $\lambda = 0.50$; 4: $\lambda = 0.75$).

(3) For the cases, $1 > \lambda > 0$, the fatigue crack growth paths transmit from the ones roughly normal to σ_1 to the one along the original crack plane with the increase of λ from zero to unity. The fatigue crack growth paths are affected by λ: the effect is significant when β is smaller and is decreased with the increase of β.

(4) The N–a_x curves (N is the cyclic times of fatigue loading; a_x is the projection length of the half-crack on the x-axis) are affected by λ. The effect is obvious when β is smaller and can be neglected when $\beta > 50°$.

5. CONCLUSION

In the present paper, a fatigue crack growth criterion was presented to predict the mixed-mode fatigue crack growth process, which is very complex. Software, developed recently by the authors, which can be used to more precisely predict the mixed-mode fatigue crack growth process, was described in detail from a number of aspects. The displacement discontinuity method which is contained in the software is very effective for computing the stress intensity factors of the growing crack tip. The analysis of the fatigue growth process of an inclined crack in biaxially stretched sheets was performed. The effects of the biaxial load ratio, λ (see Fig. 4), on the fatigue crack growth process were given.

REFERENCES

[1] G. C. Sih and B. M. Barthlehem, Mixed mode fatigue crack growth predictions. *Engng Fracture Mech.* **13,** 439–451 (1980).
[2] P. C. Paris and F. Erdogan, A critical analysis of crack propagation laws. *J. bas. Engng* **88,** 528–353 (1963).
[3] D. Wang and S. Y. Du, On the modified fracture criterion of the maximum tangential stress criterion. *J. Harbin Inst. Technol.* **3,** 58–64 (1976).
[4] S. L. Srouch and A. M. Starfield, *Boundary Element Methods in Solid Mechanics, with Application in Rock Mechanics and Geological Engineering.* George Allen & Unwin, Boston, MA (1983).
[5] F. Erdogan and G. C. Sih, On the crack extension in plates under plane loading and transverse shear. *J. bas. Engng* **88,** 519–227 (1963).
[6] X. Q. Yan, Mixed mode fracture and fatigue investigation for the material with different yield strengths in tension and compression. M.S. thesis, Harbin Institute of Technology, Harbin, P.R.C. (1986).

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