L10: Smoothing and Mapping (SAM) or Graph SLAM

* Summary of L9 (Data Association)

$$c^i = \underset{j}{\arg\min} \| \overbrace{m_j}^{h(x_t, i)} - z^i \|_2$$ Euclidean Nearest Neighbour

$$c^i = \underset{j}{\arg\min} \| m_j - z^i \|_{\Sigma_j}$$ Mahalanobis N.N.

$$c_t^* = \underset{c_t}{\arg\max} \{ p(z_t | c_t, y_t) \}$$ Maximum Likelihood

$$\max(\Pi p) = \Pi \underbrace{\max(p)}_{\text{equivalent to Mahalanobis N.N.}} \longrightarrow \min(-\overset{\|\cdot\|_{\Sigma}}{\log p})$$

JCBB: Joint Compatibility Branch and Bound

Evaluates hypothesis recursively and eliminates branches

$d^2_{H_t^i} < \chi^2_{d,\alpha}$ \ α confidence level $d = \dim(f_{H_t^i})$

chi squared table for different α, d

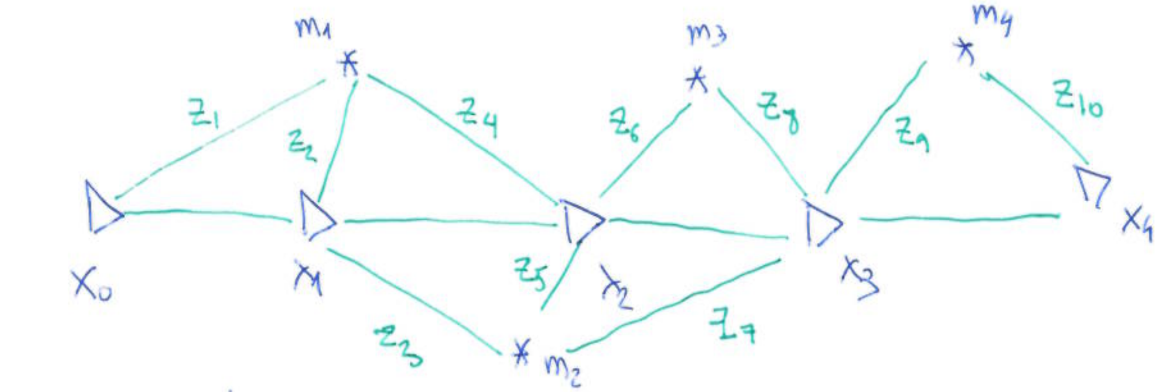
* SAM as a full SLAM problem (almost)

Smoothing $\rightarrow$ The robot trajectory $x_{1:t}$

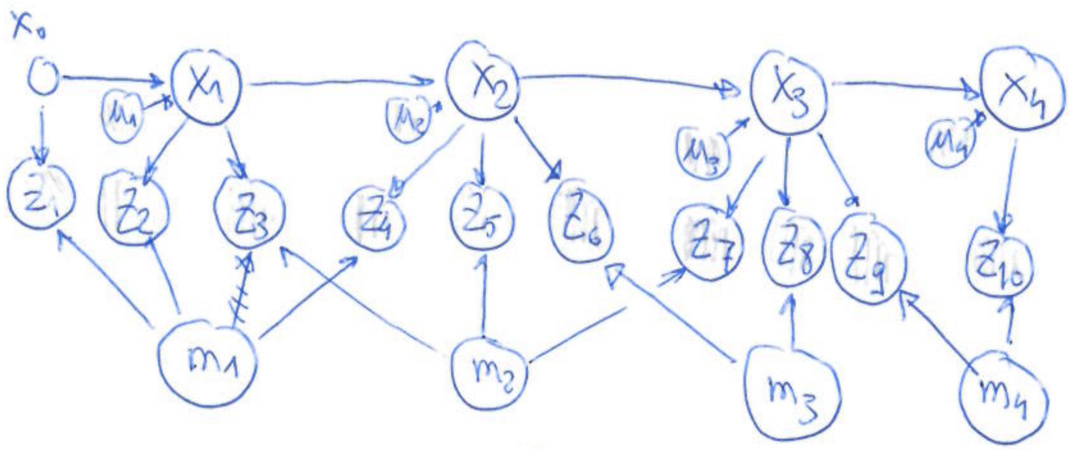
mapping $\rightarrow$ Set of landmarks (today) or any other representation

- Alternative to filter-based (L8) SLAM and still efficient
- Keeping the full trajectory is beneficial rather than just x_t.

*SAM as a Bayes Network



Graphical models are a powerful tool to describe SLAM



$$P(\underset{\{x_{0:t}\}}{X}, \underset{\{m_j\}}{M}, Z, U) = p(x_0) \prod_{i=1}^{M} p(x_i \mid x_{i-1}, u_i) \cdot \prod_{k=1}^{K} p(z_k \mid x_{i_k}, m_{j_k})$$

transitions

landmark observations

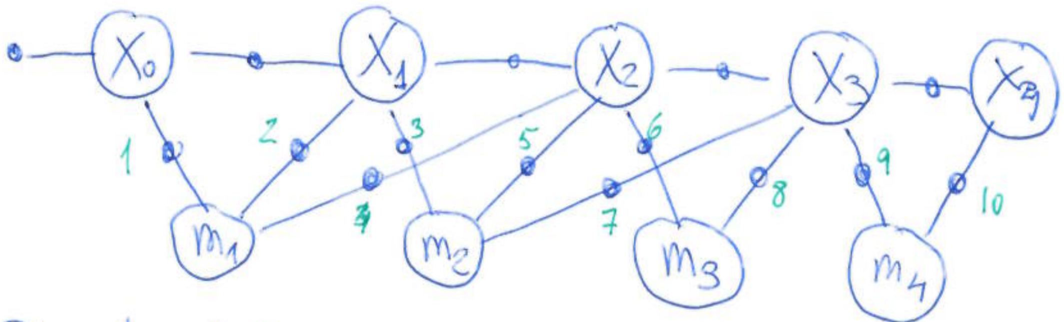
Objective: maximize the joint probability

Optimizing the graphical models drives us into:

- Graph theory
- Linear Algebra.

* SAM as a Factor Graph : (Bipartite graph)

- Bayesian networks are a natural way to express relations
- FG's have a tighter connection to optimization.



Eliminate Z, U as variables, now they are factors expressing the distributions between variables (X, M)

$\Theta = \{X, M, Z, U\}$, $\quad p(\Theta) = \prod \phi_i(\theta_i) \cdot \prod \psi_{ij}(\theta_i, \theta_j)$

$$\phi_0(x_0) \propto p(x_0)$$

$$\psi_{(i-1)i}(x_{i-1}, x_i) \propto p(x_i \mid x_{i-1}, u_i)$$

$$\psi_{i_k, j_k}(x_{i_k}, m_{j_k}) \propto p(z_k \mid x_{i_k}, m_{j_k})$$

Same identical representation as Bayes network if factors are defined this way.

→ Transition model:

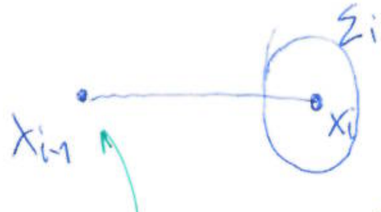
$$(R_i)$$

$$p(x_i \mid x_{i-1}, u_i) = \mathcal{N}(x_i \,;\, g_i(x_{i-1}, u_i),\ \Sigma_{u_i})$$

$$= \eta \cdot \exp\left\{ -\frac{1}{2} \| g_i(x_{i-1}, u_i) - x_i \|^2_{\Sigma_i} \right\}$$

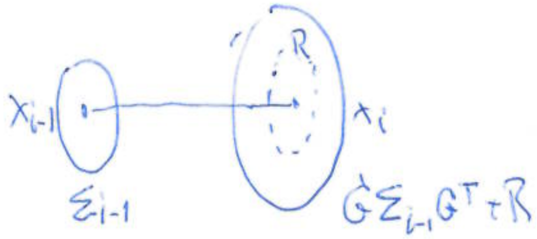
Ex:

$\psi_{(i-1)i}$



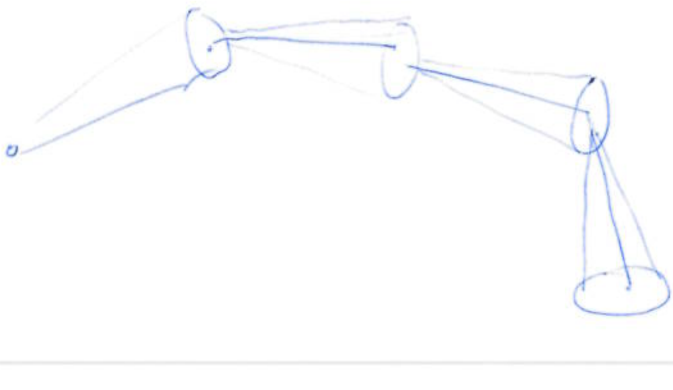
fixed!! (deterministic)

EKF. (Bayesian)



propagation model
- Odometry
- unicycle kinematic
- car kinematic, etc.

Smoothing: optimization of the chain trajectory



Relative relations $(i-1, i)$

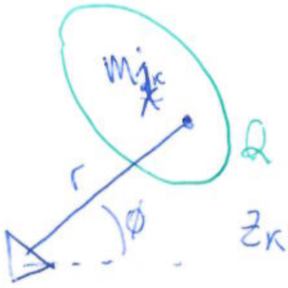
Covariances do not get propagated over the state variables!

Main difference with full SLAM which estimates x_i as distributions

Observation model

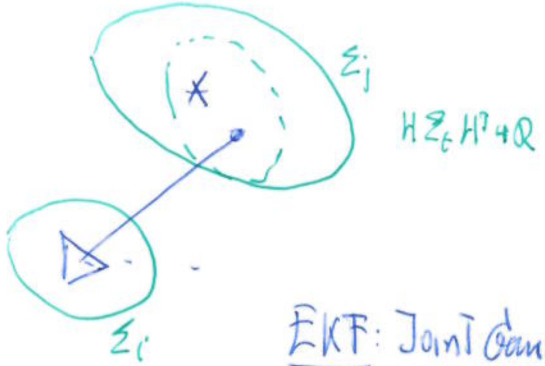
$$p(z_k \mid x_{i_k}, m_{j_k}) = \mathcal{N}\left(z_k \mid h_k(x_{i_k}, m_{j_k}), \underset{(Q)}{\Sigma_k}\right)$$

$$= \eta \exp\left\{ -\frac{1}{2} \| h_k(x_{i_k}, m_{j_k}) - z_k \|^2_{\Sigma_k} \right\}$$



$$z_k = \begin{bmatrix} z_k^r \\ z_k^\phi \end{bmatrix}$$

fixed: x_{i_k}, m_{j_k}



EKF: Joint Gaussian

Solving SAM

$$\theta^* = \underset{\theta}{\operatorname{argmax}}\ p(X, M \mid Z, U) = \underset{\theta}{\operatorname{argmax}}\ p(X, M, Z, U)$$

$$= \underset{\theta}{\operatorname{argmin}} \left\{ -\log p(X, M, Z, U) \right\}$$

$$= \underset{\theta}{\operatorname{argmin}} \left\{ \sum_{i=1}^{M} \| g_i(x_{i-1}, u_i) - x_t \|^2_{\Sigma_i} + \sum_{k=1}^{K} \| h_k(x_{i_k}, m_{j_k}) - z_k \|^2_{\Sigma_k} \right\}$$

Non-linear least squares problem (NLLSQ)

Equivalent to EKF of the augmente state $x_{0:t}$.
for linear systems

Linearize the NLLSQ

$$g_i(x_{i-1}, u_i) - x_i \simeq \Big[\underbrace{g_i(x_{i-1}^0, u_i)}_{\text{residual}} + \underbrace{G_i^{i-1}\, \delta x_{i-1}}_{\text{variables}} \Big] - \Big[\underbrace{x_i^0}_{\text{fixed}} + \underbrace{\delta x_i}_{\text{error perturbation}} \Big]$$

$$= \left(G_i^{i-1}\, \delta x_{i-1} - \delta x_i \right) - a_i$$

Jacobian $G_i^{i-1} = \left. \dfrac{\partial g_i(x_{i-1}, u_i)}{\partial x_{i-1}} \right|_{x_{i-1}^0}$

$$h_k(x_{i_k}, m_{j_k}) - z_k \simeq h_k(x_{i_k}^0, m_{j_k}^0) + H_k^{i_k}\, \delta x_{i_k} + J_k^{j_k}\, \delta m_{j_k}$$

$-z_k$

$$= \left(H_k^{i_k}\, \delta x_{i_k} + J_k^{j_k}\, \delta m_{j_k} \right) - c_k$$

Jacobian: $H_k^{i_k} = \left. \dfrac{\partial h}{\partial x_{i_k}} \right|_{(x_{i_k}^0, m_{j_k}^0)}$

$$J_k^{j_k} = \left. \dfrac{\partial h}{\partial m_{j_k}} \right|_{(x_{i_k}^0, m_{j_k}^0)}$$

* Linearized LSQ

$$\delta^* = \arg\min_{\delta} \left\{ \sum_{i=1}^{M} \| G_i^{i-1} \delta x_{i-1} - I \delta x_i - a_i \|_{\Sigma_i}^2 + \right.$$

$$\left. + \sum_{k=1}^{K} \| H_k^{i_k} \delta x_{i_k} + J_k^{j_k} \delta m_{j_k} - c_k \|_{\Sigma_k}^2 \right\}$$

How to simplify this expression?

$$\underset{\substack{\uparrow \\ \text{(Information m.)}}}{\Sigma^{-1} =} \Lambda \underset{\substack{\uparrow \\ \text{(Cholesky)}}}{=} L L^T = \sqrt{\Sigma^{-1}} \cdot \sqrt{\Sigma^{-1}}^T = \Sigma^{-\frac{1}{2}} \cdot \Sigma^{-\frac{T}{2}}$$

$$\|e\|_{\Sigma}^2 = e^T \Sigma^{-1} e = \left(\Sigma^{-\frac{T}{2}} e\right)^T \left(\Sigma^{-\frac{T}{2}} e\right) = \|\Sigma^{-\frac{T}{2}} e\|_2^2$$

1) Invert 2) Cholesky 3) Group. → Transposed squared root inverse.

Mahalanobis dist. becomes Euclidean dist by pre-multiplying.

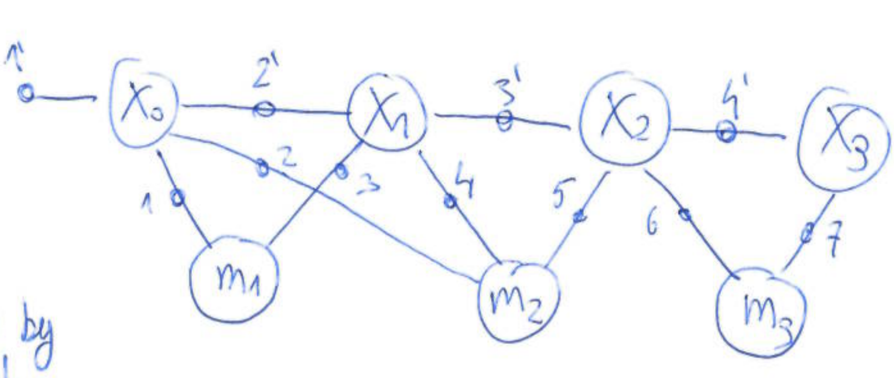
$$\delta^* = \arg\min_{\delta} \left\{ \sum_{i=1}^{M} \| \Sigma_i^{-T/2} \left(G_i^{i-1} \delta x_{i-1} - I \delta x_i \right) - \Sigma_i^{-T/2} a_i \|_2^2 + \right.$$

$$\left. + \sum_{k=1}^{K} \| \Sigma_k^{-T/2} \left(H_k^{i_k} \delta x_{i_k} + J_k^{j_k} \delta m_{j_k} \right) - \Sigma_k^{-T/2} c_k \|_2^2 \right\}$$

$$\boxed{\delta^* = \arg\min_{\delta} \| A\delta - b \|_2^2}$$

LLSQ

Ex:



Premultiply row by information factor W.

$$A = \begin{array}{c|ccccccc|l}
 & (x_0) & (x_1) & (x_2) & (x_3) & (m_1) & (m_2) & (m_3) & \\
(W_1' x) & -I & & & & & & & \\
(W_{2'} x) & G_1^0 & -I & & & & & & \text{Odometry factors} \\
(W_3') & & G_2^1 & -I & & & & & \\
(W_4') & & & G_3^2 & -I & & & & \\
\vdots & H_1^0 & & & & J_1^1 & & & \\
 & H_2^0 & & & & & J_2^2 & & \\
W_3 & & H_3^1 & & & J_3^1 & & & \\
W_4 & & H_4^1 & & & & J_4^2 & & \text{Observation factors} \\
W_5 & & & H_5^2 & & & J_5^2 & & \\
W_6 & & & H_6^2 & & & & J_6^3 & \\
W_7 & & & & H_7^3 & & & J_7^3 &
\end{array}$$

* Solve the LLSQ

$$\min_{\delta} \| A\delta - b \|_2^2$$

$$\frac{\partial}{\partial \delta} \rightarrow \frac{1}{2}(A\delta - b)\cdot A = 0 \iff A\delta = b$$

$$\begin{array}{cc} & N \\ M+n & \boxed{A} \end{array} \boxed{\delta} = \boxed{}\; b$$

A is not square. It is an overconstrained problem.

$$A\delta = b$$

$$A^T A \delta = A^T b$$

$$\delta = (A^T A)^{-1} A^T b$$

Pseudo-inverse (next lecture more on this)

Algorithm call SAM:

- calculate A, b, around $\{X^\circ, M^\circ\} = \theta^\circ$
- $\delta^* = \operatorname{argmin} \| A\delta - b \|_2^2$
- Update X°, M°, $\theta^\circ := \theta^\circ + \delta^*$
- if convergence : return θ