L05: Motion models (Prob. Rob Ch 5)

* Summary from L04

Bayes Filter
$$\overline{bel}(x_t) = \int p(x_t \mid x_{t-1}, u_t)\, bel(x_{t-1})\, dx_{t-1}$$
$$bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$$

Kalman filter: Bayes filter for linear systems and Gaussians.

prediction (Steps I, II) Marginalization G's

correction (III, IV, V) Conditioning G's.

$$\begin{cases} x_t = g(x_{t-1}, u_t, \varepsilon_t) = A_t x_{t-1} + B_t u_t + \varepsilon_t \\ z_t = h(x_t, \delta_t) = C_t x_t + \delta_t \end{cases}$$

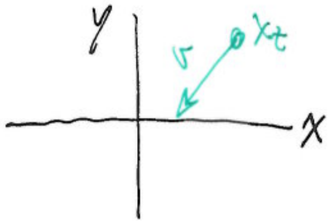
* Introduction to motion models

$x_t = g(x_{t-1}, u_t, \varepsilon_t)$ in general Non-Linear.

As designers we should find the right Transition function $g(\cdot)$ that better describes our system.

Ex: free point 2D, 3D
wheeled robot } Introduction for today.
car,
airplane,
quadrotor,
serial manipulator, ...

Free point 2D: Kinematics



$$x_t = \begin{bmatrix} x \\ y \end{bmatrix} \qquad u_t = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

position velocity.

Free moving point, for now deterministic model (no noise)

→ Discrete-time model $\quad x_t = g(x_{t-1}, u_t)$

Continuous-time model $\quad \dot{x}_t = f(x, u) \quad$ (Transition eq.)

$$\dot{x} = u = \begin{bmatrix} v_x(t) \\ v_y(t) \end{bmatrix} \quad \text{cont-time velocities}$$

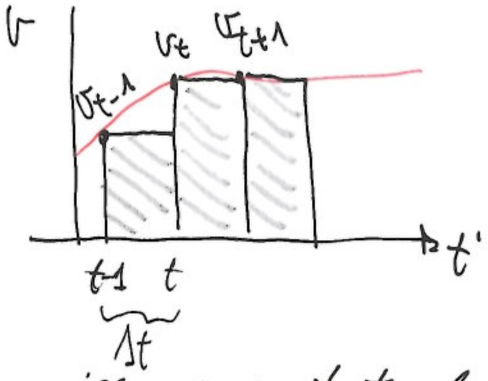
$$x(t) = x(0) + \int_0^t f(x, u)\, dt' \quad \text{continuous-time trajectory}$$

Numerical methods: Euler method, 1st order method

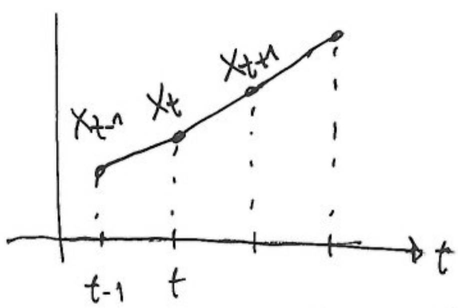
$$\dot{x} = \frac{dx}{dt} \simeq \frac{x(\Delta t) - x(0)}{\Delta t}$$

$$x(\Delta t) = x(0) + \Delta t \cdot \dot{x} = x(0) + \Delta t \cdot f(x(0), u(0))$$

(Runge-Kutta methods: higher order integration methods)



piece-wise constant velocity.



piece-wise linear trajectory.

2D. Kinematic point:

$$x_t = x_{t-1} + \Delta t\, f(x, u) = x_{t-1} + \Delta t \underbrace{\begin{bmatrix} v_x \\ v_y \end{bmatrix}}_{u}$$

Transition function Linear (±)

$$A = \mathbb{I}_{2\times2} \qquad B = \Delta t \cdot \mathbb{I}_{2\times2}$$

* Probabilistic model 2D Kinematic point.

$$x_t = x_{t-1} + \Delta t \cdot \begin{bmatrix} v_x \\ v_y \end{bmatrix} + \eta \quad \backslash \quad \eta \sim N(0, R)$$

Noise is added to the state space

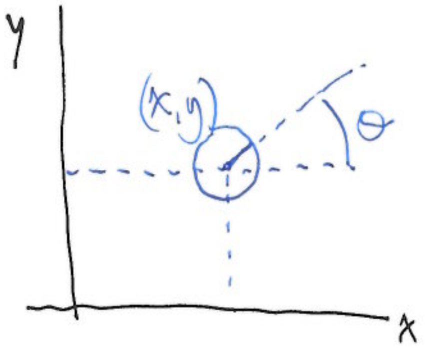
(probRob p. 127) $R = \begin{bmatrix} \alpha_1 v_x^2 + \alpha_2 v_y^2 & 0 \\ 0 & \alpha_3 v_x^2 + \alpha_4 v_y^2 \end{bmatrix}$

$$x_t = \mathbb{I} \cdot x_{t-1} + \Delta t \cdot \mathbb{I} \begin{bmatrix} v_x + \eta_x' \\ v_y + \eta_y' \end{bmatrix} = x_{t-1} + \Delta t \begin{bmatrix} v_x \\ v_y \end{bmatrix} + \underbrace{\Delta t \begin{bmatrix} \eta_x' \\ \eta_y' \end{bmatrix}}_{\eta}$$

if $\eta' \sim N(0, R') \Rightarrow \eta \sim N(0, BR'B^T)$

Noise in the action space.

2D pose: Position and orientation (or heading)



$$x = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \quad \text{State}$$

$$\dot{x} = \underbrace{\begin{bmatrix} v_x \\ v_y \\ w \end{bmatrix}}_{u}$$

$$x_t = x_{t-1} + \Delta t \cdot u = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}_{t-1} + \Delta t \begin{bmatrix} v_x \\ v_y \\ w \end{bmatrix}_t$$

Q: heading and position decoupled.

3 control variables for 3 state variables $\Rightarrow \forall x \in \mathcal{X}$ is reachable.

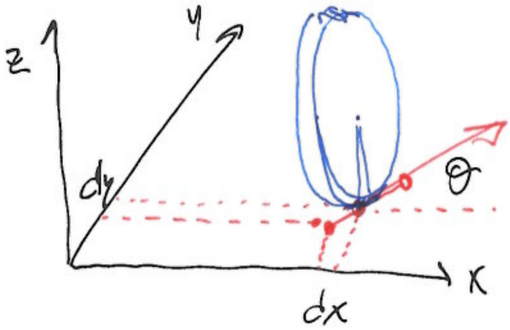
Linear aprox. → build probabilistic model as before.

* Wheeled robots in 2D

$$x_t = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \qquad \text{State is a pose: position and orientation}$$

Why not 3D? Wheeded robots (normally) stay on the ground, which we locally approximate as a plane (2D)

* Kinematic Unicycle



$\dot{x} = f(x, u)$

deterministic c-t transition eq.

Wheels add constraints:

- heading and velocity are related.

$\frac{dy}{dx} = \tan\theta$

any scalar may multiply this.

$\frac{\dot{y}}{\dot{x}} = \frac{\sin\theta}{\cos\theta} \Rightarrow \left(-\dot{x}\sin\theta + \dot{y}\cos\theta\right)^{\cdot v} = 0 \Rightarrow$

$\Rightarrow \dot{x} = v \cdot \cos\theta$

$\dot{y} = v \cdot \sin\theta$

proof: $-\underbrace{v\cos\theta}_{\dot{x}}\sin\theta + \underbrace{v\sin\theta}_{\dot{y}}\cos\theta = 0$ //

$$\dot{x} = f(x, u) = \begin{bmatrix} v\cos\theta \\ v\cdot\sin\theta \\ \omega \end{bmatrix}, \quad u = \begin{bmatrix} v \\ \omega \end{bmatrix}.$$

Still $\forall x \in \mathcal{X}$ is reachable. (proof LaValle book)

$$x_t = \underbrace{\begin{bmatrix} x_{t-1} + \Delta t \cdot v_t \cos\theta_{t-1} \\ y_{t-1} + \Delta t \cdot v_t \sin\theta_{t-1} \\ \theta_{t-1} + \Delta t \cdot w_t \end{bmatrix}}_{g(x_{t-1}, u_t)} = I \cdot x_{t-1} + \begin{bmatrix} \Delta t \cdot \cos\theta_{t-1} & 0 \\ \Delta t \cdot \sin\theta_{t-1} & 0 \\ 0 & \Delta t \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}$$

$B(x)$ Non-Linear!!

Unicycle probabilistic model

$$x_t = g(x_{t-1}, u_t) \underset{\text{Taylor}}{\simeq} g(\mu_{t-1}, u_t) + G_t(x_{t-1} - \mu_{t-1}) + V_t(u_t - \bar{u}_t)$$

mean value $u_t = \bar{u}_t + \delta u_t$

$$G_t = \left.\frac{\partial g(x_{t-1}, u_t)}{\partial x_{t-1}}\right|_{\mu_{t-1}} = \begin{array}{c} \begin{bmatrix} 1 & 0 & -\sin\theta_{t-1}\cdot\Delta t\, v_t \\ 0 & 1 & \cos\theta_{t-1}\,\Delta t\, v_t \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{array}{ccc} \partial/\partial x & \partial/\partial y & \partial/\partial\theta \end{array} \end{array}$$

$$V_t = \left.\frac{\partial g(x_{t-1}, u_t)}{\partial u_t}\right|_{\bar{u}_t} = \begin{bmatrix} \cos\theta_{t-1}\cdot\Delta t & 0 \\ \sin\theta_{t-1}\,\Delta t & 0 \\ 0 & \Delta t \end{bmatrix}$$

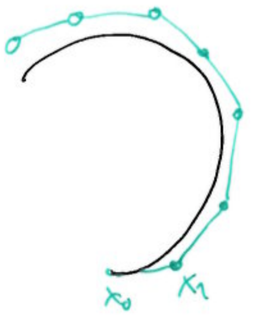
Add noise: $x_t = g(x_{t-1}, u_t, \varepsilon_t) = g(x_{t-1}, u_t) + \varepsilon_t$

$x_{t-1} \sim N(\mu_{t-1}, \Sigma_{t-1})$, $\varepsilon_t \sim N(0, R)$ state space noise.

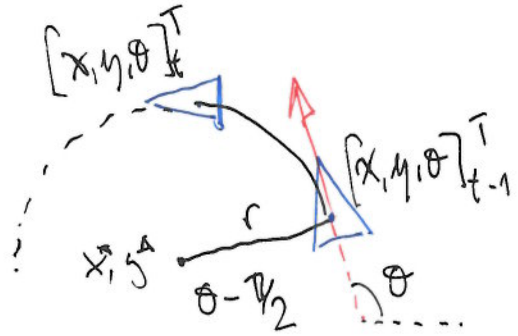
$$\Rightarrow x_t \sim N\left(g(\mu_{t-1}, u_t),\ \boxed{G_t \Sigma G_t^T + R}\right)$$

Q: home: show this!

Ex: Circular path. $v = 1$, $w = 1$



Can we do better than Euler Integration?
(Ch 5.3)
in ProbRob they use a sequence of arcs to approximate the unicycle.



$$\begin{bmatrix} x \\ y \\ \theta \end{bmatrix}_{t-1} = x_{t-1}$$

$$\begin{bmatrix} x \\ y \\ \theta \end{bmatrix}_t = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}_{t-1} + \begin{bmatrix} -\frac{v}{w}\sin\theta + \frac{v}{w}\sin(\theta + w\Delta t) \\ \frac{v}{w}\cos\theta - \frac{v}{w}\cos(\theta + w\Delta t) \\ w\Delta t + \underbrace{\gamma \Delta t}_{} \end{bmatrix} = g(x_{t-1}, u_t)$$

extra term: final orientation

* Probabilistic arc Kinematic motion model

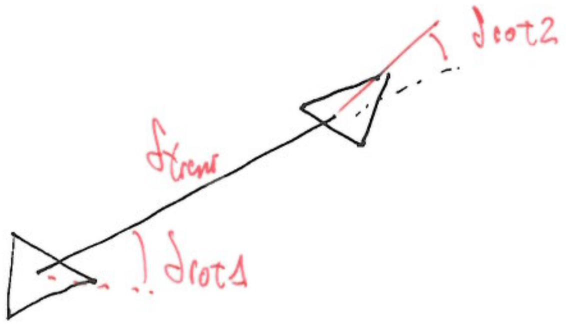
$$\begin{bmatrix} \hat{v} \\ \hat{w} \end{bmatrix} = \begin{bmatrix} v + \varepsilon_v \\ w + \varepsilon_w \end{bmatrix}, \quad \varepsilon \sim N(0, R)$$ not diagonal. (ProbRob 127.)

$$x_t \sim N\left(g(\mu_{t-1}, u_t),\ G_t \Sigma_{t-1} G_t^T + V_t R V_t^T\right)$$

* Odometry model

counts of increments on spinning wheels (observation)

more accurate than velocity models



reminds the 1st order unicycle.

$$u = \begin{bmatrix} \delta_{rot1} \\ \delta_{trans} \\ \delta_{rot2} \end{bmatrix} = \begin{bmatrix} \operatorname{atan2}(y_t - y_{t-1},\ x_t - x_{t-1}) \\ \sqrt{(x_t - x_{t-1})^2 + (y_t - y_{t-1})^2} \\ \theta_t - \theta_{t-1} - \delta_{rot1} \end{bmatrix}$$

D-time transition function

$$x_t = g(x_{t-1}, u_t) = \begin{bmatrix} x_{t-1} + \delta_{trans} \cos(\theta + \delta_{rot1}) \\ y_{t-1} + \delta_{trans} \sin(\theta + \delta_{rot1}) \\ \theta_{t-1} + \delta_{rot1} + \delta_{rot2} \end{bmatrix}$$

* Probabilistic odometry model

(noise in action space)

$$x_t = g(x_{t-1}, u_t, \varepsilon_t) \qquad \varepsilon_t \sim N(0, R) \quad \text{(ProbRob 139)}$$

$$\varepsilon_t = \begin{bmatrix} \varepsilon_{\delta rot1} \\ \varepsilon_{\delta trans} \\ \varepsilon_{\delta rot2} \end{bmatrix} = N\left(0, \begin{bmatrix} \alpha_1 \delta_{rot1}^2 + \alpha_2 \delta_{trans}^2 & 0 & 0 \\ 0 & \alpha_3 \delta_{trans}^2 + \alpha_4(\delta_{rot1}^2 + \delta_{rot2}^2) & 0 \\ 0 & 0 & \alpha_1 \delta_{rot2}^2 + \alpha_2 \delta_{trans}^2 \end{bmatrix}\right)$$

Jacobian for odometry model

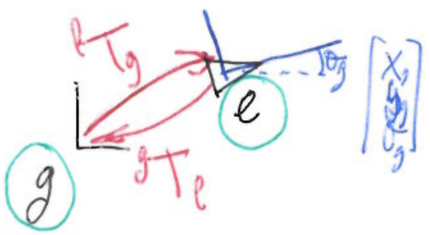
$$G_t = \left.\frac{\partial g(x_{t-1}, u_t)}{\partial x_{t-1}}\right|_{\mu_{t-1}} = \begin{array}{c} \begin{bmatrix} 1 & 0 & -\delta_{trans} \sin(\theta + \delta_{rot1}) \\ 0 & 1 & \delta_{trans} \cos(\theta + \delta_{rot1}) \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{array}{ccc} \partial/\partial x & \partial/\partial y & \partial/\partial\theta \end{array} \end{array}$$

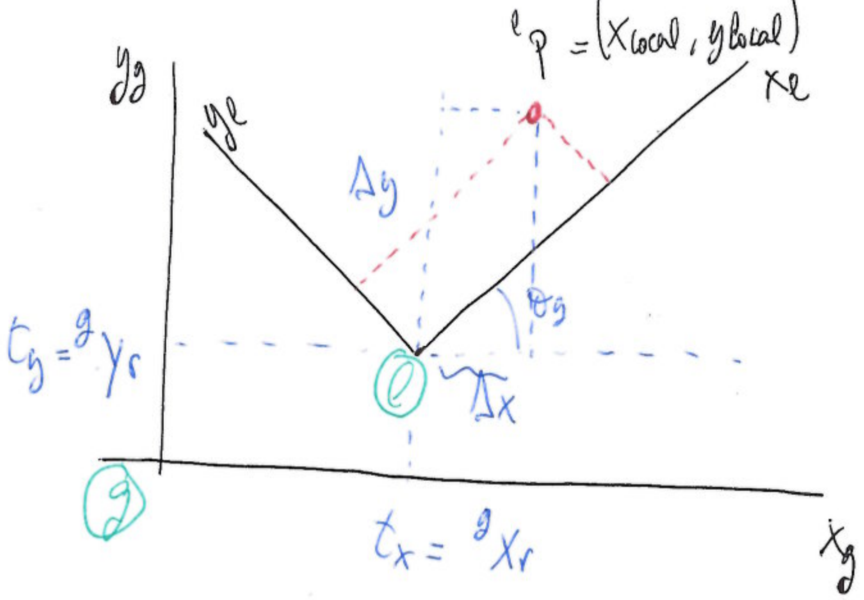
$$V_t = \left.\frac{\partial g(x_{t-1}, u_t)}{\partial u_t}\right|_{\bar{\mu}_t} = \begin{array}{c} \begin{bmatrix} -\delta_{trans} \cdot \sin(\theta + \delta_{rot1}) & \cos(\theta + \delta_{rot1}) & 0 \\ \delta_{trans} \cos(\theta + \delta_{rot1}) & \sin(\theta + \delta_{rot1}) & 0 \\ 1 & 0 & 1 \end{bmatrix} \\ \begin{array}{ccc} \partial/\partial\delta_{rot1} & \partial/\partial\delta_{trans} & \partial/\partial\delta_{rot2} \end{array} \end{array}$$

* 2D-Rigid Body Transformations

$$\underset{(\text{global frame})}{{}^{g}}X_{\underset{(\text{robot frame})}{r}} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

2D poses can be interpreted as a transformation between coordinate frames → XYT parametrization





We need to project point ${}^{e}p$ in the local frame to the global frame.

$${}^{g}P = \begin{bmatrix} x_{local} \cdot \cos\theta_g - y_{local} \sin\theta_g + t_x \\ x_{local} \cdot \sin\theta_g + y_{local} \cos\theta_g + t_y \end{bmatrix} =$$

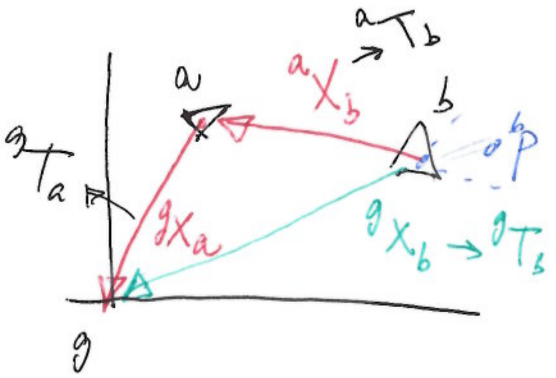
$$= \begin{bmatrix} \cos\theta_g & -\sin\theta_g & t_x \\ \sin\theta_g & \cos\theta_g & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{local} \\ y_{local} \\ 1 \end{bmatrix}$$

(homogeneous coordinates)

$$ {}^g p = \begin{bmatrix} {}^gR_l(\theta_r) & t \\ 0_{1\times 2} & 1 \end{bmatrix} \cdot {}^l p = {}^gT_l \cdot {}^l p = {}^g p $$

from local (subscript) to global (superscript).

RBT composition



$$ {}^g p = {}^gT_b \cdot {}^b p $$

Non commutative.

$$ {}^a p = {}^aT_b \cdot {}^b p $$

$$ {}^g p = {}^gT_a \cdot {}^a p = {}^gT_a \, {}^aT_b \, {}^b p $$

Sensor Models

$p(z_t \mid x_t)$ Observation distribution (LO9)

$z_t = h(x_t, \delta_t)$ Observation function

Ex: Accelerometer, sonar, radar, Lidar, camera, ...

we will propose a sensor model, but any function relating state and observation is a valid sensor model.

* Feature-based measurements model

extract features f from observations $f(z_t) = \{f_1, f_2, \dots f_n\}$

Ex: lines, corners, point descriptors, objects, etc...

$\underbrace{\text{lines, corners}}_{\text{laser}}$ — C. Vision.

∘ Landmark: feature which corresponds to physical objects

(2D) $f_i = \begin{bmatrix} \text{'range'} \\ \text{'bearing'} \\ \text{'signature'} \end{bmatrix} = \begin{bmatrix} r \\ \phi \\ s \end{bmatrix}$ optional to alleviate data association problem. (color, size, $\mathbb{R}^{60}$ embedding, etc...)

→ each feature corresponds to a location $[m_{i,x}, m_{i,y}]^T$

$$\begin{bmatrix} r_t^j \\ \phi_t^j \\ s_t^j \end{bmatrix} = \begin{bmatrix} \sqrt{(m_{i,x}-x)^2 + (m_{i,y}-y)^2} \\ \text{atan2}(m_{i,y}-y,\ m_{i,x}-x) - \theta \\ s_i \end{bmatrix} + \begin{bmatrix} \delta_{\sigma_r^2} \\ \delta_{\sigma_\phi^2} \\ \delta_{\sigma_s^2} \end{bmatrix}, \quad \delta \sim N(0, \Sigma_\delta)$$

data association i^{th} map location m_i corresponds to the j^{th} feature.

$$f^j = h(x_t, m_i) + \delta_t$$

$$f \sim N(f(z_t);\ h(x_t, m_i),\ \Sigma_\delta) \quad \text{Probabilistic model}$$

Sampling x_t from m_i: ProbRob 180

(Next class. ProbRob, Ch 3)