Discrete Mathematics
Lecture 5 - Induction and Recursion

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Mathematical Induction

One of the most basic methods of proof is *mathematical induction,* which is a way to establish the truth of a statement about all the natural numbers or, sometimes, all sufficiently large integers.

We will first consider one example where will be used the idea of mathematical induction in its solution and then define *The Principle of Mathematical Induction*.

Examples

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Solution. If you want to purchase 44 envelopes, you can buy 2 packages of twelve and 4 packages of five ($2 \cdot 12 + 4 \cdot 5 = 44$). If you want to purchase 45 envelopes, you can buy 9 packages of five ($9 \cdot 5 = 45$). If you want to purchase 46 envelopes, pick up 3 packages of twelve and 2 packages of five ($3 \cdot 12 + 2 \cdot 5 = 46$). If you want to buy 47 envelopes, get 1 package of twelve and 7 packages of five ($1 \cdot 12 + 7 \cdot 5 = 47$) and, if you want 48 envelopes, purchase 4 packages of twelve ($4 \cdot 12 = 48$).

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The obvious difficulty with this way of attacking the problem is that it never ends. Even supposing that we continued laboriously to answer the question for n as big as 153, say, could we be sure of a solution for $n = 154$? What is needed is a general, not ad hoc, way to continue; that is, if it is possible to fill an order for exactly k envelopes at this store, we would like to be able to deduce that the store can also fill order for $k + 1$ envelopes.

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The Principle of Mathematical Induction (PMI)

Given a statement $\mathcal{P}$ concerning the integer n, suppose

1. $\mathcal{P}$ is true for some particular integer n_0;

2. if $k \geq n_0$ is an integer and $\mathcal{P}$ is true for k, then $\mathcal{P}$ is true for the next integer $k+1$.

Then $\mathcal{P}$ is true for all integers $n \geq n_0$.

Part 1 of PMI is called *a basis of induction* and Part 2 is known as the *induction hypothesis*.

In our previous example, we had to prove that any order of n envelopes, $n \geq 44$, could be filled with packages of five and twelve; n_0 was 44 and the induction hypothesis was the assumption that there was a way to purchase k envelopes with packages of five and twelve.

Examples

2. Prove that for any integer $n \geq 1$ the sum of the odd integers from 1 to $2n-1$ is n^2.

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Solution. It can be also rephrased that for any integer $n \geq 1$

$$1 + 3 + \ldots + (2n-1) = n^2.$$

In this question, $n_0 = 1$. When $n = 1$, then

$$2 \cdot 1 - 1 = 1^2 \quad \textbf{basis of induction.}$$

Now suppose that k is an integer, $n \geq 1$ and the statement is true for $n = k$; in other words, suppose

$$1 + 3 + 5 + \ldots + (2k-1) = k^2 \quad \textbf{induction hypothesis.}$$

We must show that the statement is true for the next integer, $n = k+1$; that is, we must show that

$$1 + 3 + 5 + \ldots + (2(k+1)-1) = (k+1)^2.$$

Examples

We write the the left-hand side as

$$1 + 3 + 5 + \ldots + (2(k+1) - 1) = [1 + 3 + 5 + \ldots + (2k-1)] + (2k+1).$$

By the induction hypothesis, we know that

$$= k^2 + (2k+1) = k^2 + 2k + 1 = (k+1)^2,$$

this is the result we wanted. By PMI, the statement is true for all integers $n \geq 1$
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Examples

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Solution. When $n = 1$, the sum of the integers from 1^2 to 1^2 is 1. Also

$$\frac{1(1+1)(2\cdot 1+1)}{6} = 1,$$

so the statement is true for $n = 1$. Now suppose that $k \geq 1$ and the statement is true for $n = k$; that is, suppose that

$$1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

We have to show that the statement is true for $n = k + 1$; that is, we have to show that

$$1^2 + 2^2 + 3^2 + \ldots + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$

Examples

Observe that

$$1^2 + 2^2 + 3^2 + \ldots + (k+1)^2 = (1^2 + 2^2 + 3^2 + \ldots + k^2) + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} = \frac{(k+1)(k(2k+1) + 6(k+1))}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

which is just what we wanted. By PMI, the statement is true for all integers $n \geq 1$ $\square$

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Solution.
Basis of induction. When $n = 1$, $2^{2\cdot 1} - 1 = 4 - 1 = 3$ is divisible by 3.

Induction hypothesis. We suppose that $k \geq 1$, and the statement is true for $n = k$; We must prove that the statement is true for $n = k + 1$; that is, we must prove that $2^{2(k+1)} - 1$ is divisible by 3. Thanks to the Induction hypothesis, there exists an integer t such that $2^{2k} - 1 = 3t$.
We have

$$2^{2(k+1)} - 1 = 4(2^{2k}) - 1 = 4(3t + 1) - 1 = 12t + 4 - 1 = 12t + 3 = 3(4t + 1).$$

Thus, $2^{2(k+1)} - 1$ is divisible by 3, as required. By the PMI, $2^{2n} - 1$ is divisible by 3 for all integers $n \geq 1$ $\square$

An Information on Factorials

Definition (Factorial)

Define $0! = 1$ and, for any integer $n \geq 1$, define

$$n! = n(n-1)(n-2)\cdots(3)(2)(1).$$

Factorials grow very quickly.

Theorem (James Stirling (1730))

$lim_{n\to\infty}\frac{n!}{\sqrt{2\pi n}(n/e)^n} = 1;$ equivalently, $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$.

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Solution. In this question, $n_0 = 4$ and certainly $4! = 24 > 16 = 2^4$. Thus, the statement is true for n_0. Now suppose that $k \geq 4$ and the statement is true for $n = k$. Thus, we suppose that $k! > 2^k$. We must prove that the statement is true for $n = k + 1$; that is, we must prove that $(k + 1)! > 2^{k+1}$. Now

$$(k + 1)! = (k + 1)k! > (k + 1)2^k$$

using the induction hypothesis. Since $k \geq 4$, certainly

$$k + 1 > 2, \text{ so } (k + 1)2^k > 2 \cdot 2^k = 2^{k+1}.$$

We conclude that $(k + 1)! > 2^{k+1}$ as desired. By the PMI, we conclude that $n! > 2^n$ for all integers $n \geq 4$ $\square$

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The PMI is one of the most powerful tools of mathematics. With it, we can prove many interesting things, but it is not applied correctly, we can also prove some interesting things that are not true!

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“Assume that $2 + 4 + 6 + \ldots + 2k = (k - 1)(k + 2)$ for some integer k. Then

$$2 + 4 + 6 + \ldots + 2(k + 1) = (2 + 4 + 6 + \ldots + 2k) + 2(k + 1)$$

$$= (k - 1)(k + 2) + 2(k + 1) \text{ (by the induction hypothesis)}$$

$$= k^2 + 3k = [(k + 1) - 1][(k + 1) + 2].$$

which is the given statement for $n = k + 1$. It follows by the PMI that the statement is true for all positive integers n”.

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Solution. The inductive step, as given, is correct, but we neglected to check the case $n = 1$, for which the statement is most definitely false $\square$

Is your name Smith? Let's "prove"

Example 2. In a group of n people, everyone has the same name.

"**Proof.** Let's show that the basis of induction is true. In a group of $n = 1$ people, all of them have the same name.
Let's show that induction hypothesis holds. Since the positive integer k is in the set, this means that in a group of any k people, all of them have the same name. So let's consider a group of $k + 1$ people. Since the first k people have the same name, and the last k people also have the same name, it means that all $k + 1$ of them must have the same name by the induction hypothesis.
Thus, by PMI, in a group of n people, everyone has the same name.

In particular, say my name is Smith, and you are in the set of all N people on Earth, it follows that your name is also Smith."

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Question. What is the flaw in this "proof"?

Questions for Self-Study

- Establish the *arithmetic mean-geometric mean inequality:* For any $n \geq 1$ and any n nonnegative real numbers $a_1, a_2, \ldots a_n$

$$\frac{a_1+a_2+\cdots+a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n}$$

- Prove that two Principles of Mathematical Induction and Well-Ordering are equivalent.

Sequences

Suppose that n is a natural number, How to define 2^n? We could write

$$2^n = \underbrace{2 \cdot 2 \cdots 2}_{n\,2's}$$

or

$$2^1 = 2 \text{ and, for } k \geq 1,\ 2^{k+1} = 2 \cdot 2^k.$$

It explicitly defines 2^n when $n = 1$ and then, assuming 2^n has been defined for $n = k$, defines it for $n = k + 1$. By PMI, we know that 2^n has been defined for all integers $n \geq 1$.

The last definition of 2^n is an example of *recursive definition*.

Definition

A *sequence* is a function whose domain is some infinite set of integers (often $\mathbb{N}$) and whose range is a set of real numbers. $f(n)$ is called a *term* of the sequence.

Example

$f : \mathbb{N} \to \mathbb{R}$ defined by $f(n) = n^2$. It defines the following list of numbers

$$1, 4, 9, 16, \ldots$$

The sequence $2, 4, 8, 16, \ldots$ can be defined recursively like this:

$$a_1 = 2 \text{ and for } k \geq 1,\ a_{k+1} = 2a_k.$$

By this, we understand that $a_1 = 2$ and then, setting $k = 1$ in the second part of the definition that $a_2 = 2a_1 = 2 \cdot 2 = 4$.
With $k = 2$, $a_3 = 2a_2 = 2 \cdot 4 = 8$; with $k = 3$, $a_4 = 2 \cdot a_3 = 2 \cdot 8 = 16$ and so on.
The equation $a_{k+1} = 2a_k$ is said to be *recurrence relation* and $a_1 = 2$ is called *an initial condition*.

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One sequence can be defined by two recursive definitions. For example, we could write

$$a_0 = 2 \text{ and, for } k \geq 0,\ a_{k+1} = 2a_k,$$

or we could say

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Sometimes, after computing a few terms of a sequence that has been defined recursively, we can guess an explicit formula for a_n. In the preceding example, $a_n = 2^n$. We say that $a_n = 2^n$ is *the solution* to the recurrence relation.

Example

$a_1 = 1$, $a_{k+1} = 3a_k + 1$ for $k \geq 1$. Let's try to guess a formula for a_n and prove it.

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Now assume that $k \geq 1$ and that $a_k = \frac{1}{2}(3^k - 1)$. We wish to prove that $a_{k+1} = \frac{1}{2}(3^{k+1} - 1)$. We have

$$a_{k+1} = 3a_k + 1 = 3[\tfrac{1}{2}(3^k - 1)] + 1$$

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using the induction hypothesis. Hence,

$$a_{k+1} = \tfrac{1}{2}3^{k+1} - \tfrac{3}{2} + 1 = \tfrac{1}{2}(3^{k+1} - 1)$$

as required. By the PMI, our guess is correct.

Special sequence: Arithmetic sequence

The *arithmetic sequence* with first term a and common *difference* d is the sequence defined by

$$a_1 = a \text{ and, for } k \geq 1,\ a_{k+1} = a_k + d.$$

The general arithmetic sequence thus takes the form

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots$$

and it is easy to see that, for $n \geq 1$, the nth term of the sequence is

$$a_n = a + (n-1)d.$$

The sum of first n terms S_n of the arithmetic sequence with first term a and common difference d is

$$S_n = \frac{n}{2}[2a + (n-1)d].$$

Special sequence: Geometric sequence

The *geometric sequence* with first term a and common *ratio* r is the sequence defined by

$$a_1 = a \text{ and, for } k \geq 1,\ a_{k+1} = ra_k.$$

The general geometric sequence thus has the form

$$a,\ ar,\ ar^2,\ ar^3, \ldots$$

and it is easy to see that, for $n \geq 1$, the nth term of the sequence is

$$a_n = ar^{n-1}.$$

The sum of first n terms S_n of the geometric sequence with first term a and common ratio $r \neq 1$ is

$$S_n = \frac{a(1 - r^n)}{1 - r}.$$

Without guesswork

Given $a_1 = 1$ and $a_{k+1} = 3a_k + 1$ for $k \geq 1$. Let us try to find formula for a_n (depending on only n) without guesswork.

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So eventually we will obtain

$$a_n = 3^{n-1} a_1 + (3^{n-2} + \ldots + 3^2 + 3 + 1).$$

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Since $a_1 = 1$ and $3^{n-2} + \ldots + 3^2 + 3 + 1 = \frac{1(1-3^{n-1})}{1-3} = \frac{1}{2}(3^{n-1} - 1)$, we obtain

$$a_n = 3^{n-1} + \frac{1}{2}(3^{n-1} - 1) = \frac{1}{2}(2 \cdot 3^{n-1} + 3^{n-1} - 1)$$

$$= \frac{1}{2}(3 \cdot 3^{n-1} - 1) = \frac{1}{2}(3^n - 1).$$

Solving Recurrence Relations

Recall that solving a recurrence relation means to find a formula for a_n which depends on only n, sometimes we call it *explicit* formula.
In our lecture we consider solutions to recurrence relations of the form

- $a_{n+1} = ra_n + s$, $n \geq 0$ and $r \in \mathbb{R}$
- $a_{n+2} = ra_{n+1} + sa_n$, $n \geq 0$ and $r, s \in \mathbb{R}$

Theorem

Let $a_{n+1} = ra_n + s$, $n \geq 0$ where $r \neq 1$ and $a_0 = t$. Then

$$a_n = r^n t + \frac{r^n - 1}{r - 1} s.$$

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Proof.

$$a_{n+1} = ra_n + s = r(ra_{n-1} + s) + s = r^2 a_{n-1} + rs + s$$
$$= r^2(ra_{n-2} + s) + rs + s = r^3 a_{n-2} + r^2 s + rs + s.$$

So eventually we will obtain

$$a_{n+1} = r^{n+1} a_0 + (r^n + \ldots + r + 1)s = r^{n+1} a_0 + \frac{r^{n+1} - 1}{r - 1} s.$$

Since $a_0 = t$,

$$a_{n+1} = r^{n+1} t + \frac{r^{n+1} - 1}{r - 1} s, \quad n \geq 0.$$

It is equivalent to

$$a_n = r^n t + \frac{r^n - 1}{r - 1} s, \quad n \geq 0 \, \square$$

Theorem

Let x_1 and x_2 be the roots of the polynomial $x^2 - rx - s$. Then the solution of the recurrence relation $a_{n+2} = ra_{n+1} + sa_n$, $n \geq 0$, is

$$a_n = \begin{cases} c_1 x_1^n + c_2 x_2^n & \text{if} \quad x_1 \neq x_2 \\ c_1 x^n + c_2 n x^n & \text{if} \quad x_1 = x_2 = x. \end{cases}$$

In each case, c_1 and c_2 are constants determined by initial conditions. $x^2 - rx - s$ is called *characteristic polynomial* and its roots are called *characteristic roots* of the recurrence relation.

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We leave its proof for self-study.

Examples

1. $a_n = 5a_{n-1} - 6a_{n-2}$, $n \geq 2$, given $a_0 = -3$, $a_1 = -2$ or we could write it $a_{n+2} = 5a_{n+1} - 6a_n$, $n \geq 0$.

Solution. The characteristic polynomial $x^2 - 5x + 6$ has distinct roots $x_1 = 2$, $x_2 = 3$. Then last Theorem tells us that the solution is

$$a_n = c_1(2^n) + c_2(3^n).$$

Since $a_0 = -3$, we must have $c_1(2^0) + c_2(3^0) = -3$ and since $a_1 = -2$, we have $c_1(2^1) + c_2(3^1) = -2$.
Therefore,

$$\begin{cases} c_1(2^0) + c_2(3^0) = -3 \\ c_1(2^1) + c_2(3^1) = -2. \end{cases}$$

Solving, we have $c_1 = -7$, $c_2 = 4$, so the solution is $a_n = -7(2^n) + 4(3^n)$.

2. $a_{n+2} = 4a_{n+1} - 4a_n$, $n \geq 0$, given $a_0 = 1$, $a_1 = 4$.

Solution. The characteristic polynomial $x^2 - 4x + 4$ has the repeated root $x = 2$. Hence, the solution is

$$a_n = c_1(2^n) + c_2 n(2^n).$$

Initial conditions yield $c_1 = 1$, $2c_1 + 2c_2 = 4$, so $c_2 = 1$. Thus,

$$a_n = 2^n + n2^n = (n+1)2^n.$$

The Fibonacci sequence: $1, 1, 2, 3, 5, 8, 13, \ldots$

3. The sequence defined by the recurrence relation $f_{n+2} = f_{n+1} + f_n$ with $f_0 = f_1 = 1$ for $k \geq 2$ is called the *Fibonacci sequence*. Let us try to find explicit formula for f_n.

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$$f_n = c_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + c_2 \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

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$$f_n = c_1 \left(\frac{1 + \sqrt{5}}{2} \right)^n + c_2 \left(\frac{1 - \sqrt{5}}{2} \right)^n .$$

The initial conditions give

$$\begin{cases} c_1 + c_2 = 1 \\ c_1 \left(\frac{1+\sqrt{5}}{2} \right) + c_2 \left(\frac{1-\sqrt{5}}{2} \right) = 1 \end{cases}$$

yielding $c_1 = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)$ and $c_2 = -\frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)$.

Continued. . .

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Thus, the solution is

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}.$$

Continued. . .

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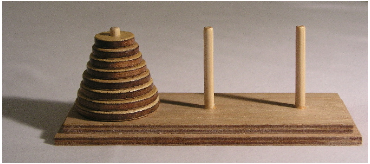
$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}.$$

The nth term of the Fibonacci sequence is

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}.$$

Question 1: The Towers of Hanoi

The Towers of Hanoi is a popular puzzle. It consists of three pegs and a n discs of different diameters each with a hole in the center. The discs initially sit on one of the pegs in order of decreasing diameter(smallest



at top, largest at bottom as in the picture from Wikipedia). The object is to move the tower to one of the other pegs by transforming the discs to any peg one at a time in a such way that no disc is ever placed upon a smaller one. Let T_n be the minimum number of moves that will transfer n disks from one peg to another.

Find a recurrence relation of T_n for $n \geq 1$ and its explicit formula.

For example, $T_1 = 1$, $T_2 = 3$ and $T_3 = 7$.

Solution of Question 1

We first transfer the $n-1$ smallest to a different peg that requires T_{n-1} moves, then move the largest that requires one move, and finally transfer the $n-1$ smallest back onto the largest that requires another T_{n-1} moves. Thus we can transfer n discs for $n > 0$ in

$$T_n = T_{n-1} + 1 + T_{n-1} = 2T_{n-1} + 1, \quad \text{for} \quad n > 0 \quad \text{with} \quad T_1 = 1.$$

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Consequently,

$$T_{n+1} = 2^n - 1. \quad \text{(Why?)}$$

Question 2

For $n \geq 0$ let $S = \{1, 2, 3, \dots n\}$ (if $n = 0$, then $S = \emptyset$). Let a_n denote the number of subsets of S that contain no consecutive integers.

Find a recurrence relation of a_n.

For small n, we have the following answers.

If $n = 0$, then there is just $\emptyset$; so $a_0 = 1$.

If $n = 1$, then there are $\emptyset$, $\{1\}$; so $a_1 = 2$.

If $n = 2$, then there are $\emptyset$, $\{1\}$, $\{2\}$; so $a_2 = 3$.

If $n = 3$, then there are $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 3\}$; so $a_3 = 5$.

If $n = 4$, then there are $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 4\}$; so $a_4 = 8$.

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Case 1: Suppose $n \in A$. Then $n-1 \notin A$ and A would be counted in a_{n-2}.

Case 2: Suppose $n \notin A$. Then A would be counted in a_{n-1}.

Two cases are mutually disjoint and exhaustive; so

$$a_n = a_{n-2} + a_{n-1} \quad \text{for} \quad n \geq 2, \quad a_0 = 1, \quad a_1 = 2.$$

Question 3

Find a recurrence relation for the number of the binary sequences(consisting of 1's and 0's) of length n that have no consecutive 0's.

For example,

if $n = 1$, there are 0, 1; so there is 2 binary sequence.

If $n = 2$, there are 01, 10, 11; so there are 3 binary sequences.

If $n = 3$, there are 010, 101, 011, 110, 111; so there are 5 binary sequences

Solution of Question 3

Let $a_n^{(0)}$ be the numbers of binary sequences that end in 0 and $a_n^{(1)}$ be the numbers of binary sequences that end in 1.

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Hence

$$a_n = a_{n-2} + a_{n-1} \quad \text{for} \quad n \geq 3, \quad a_1 = 2, \quad a_2 = 3.$$

Interesting open problem: The Collatz conjecture

Consider the following process for any arbitrary positive integer number n:

- If the number n is even, divide it by two (that is, $\frac{n}{2}$)
- If the number is odd, triple it and add one (that is, $3n+1$)

Conjecture (Collatz (1937))

This process will eventually reach the number 1, regardless of which positive integer is chosen initially.

Examples

If $n = 12$, then one gets the sequence

$$12, 6, 3, 10, 5, 16, 8, 4, 2, \mathbf{1}.$$

If $n = 19$, then

$$19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, \mathbf{1}.$$

Online database on sequence of integers

A link to an online database of integer sequences:

$$https://oeis.org/$$

There you can find all known information about a sequence of integers entering them or name of sequence of integers.

The End of Lecture 5