PAIA Material and Methods

Selection of Study Samples

Lateral weightbearing X-ray images of 40 feet without IAT, other Achilles insertion deformities, trauma history of the calcaneus or other deformities were included as the control group to study the normal morphology of the calcaneal tuberosity. Lateral weightbearing X-ray images of 40 feet with IAT were used as the diseased group.

Delineating Contour of Contral Calcaneal Tuberosities

The 40 control radiographs were imported into ImageJ (REF). Each calcaneus was circumscribed within a rectangle such that each side of the rectangle corresponded to the anterior, superior, posterior, and inferior border of the calcaneus. Then, 90 positional markers were plotted onto each calcaneus to determine the size of the calcaneus and contour of the calcaneal tuberosity (Figure 1). Among these 90 points, points 1-3 were set up as calibration markers for measurement purposes, some of the rest points represented standardized anatomical markers, while the others did not have specific representation but only being evenly distributed between two anatomical markers along the contour of the bone for mapping purposes (Table 1).

Determining Individualized Standard Circle to Predict the Ideal Calcaneal Tuberosity Contour

These 90 positional markers were collectively mapped to construct and statistically fit an individualized Standard Circle for every subject. To fit the Standard Circle, the positioning and shape of the standard rectangle was first characterized by points 4, 5, 6, and 7, with point 7 denoted as the origin of the x and y coordinates (i.e., x, y = 0 at point 7). The standard rectangle was then normalized to a unit square to mediate differences in foot sizes between the subjects.

The two essential parameters used to define the individualized Standard Circle for each calcaneal tuberosity were relative x and y coordinates of the circle's center, denoted as $O(x_o, y_o)$, and the radius (R) of the Standard Circle determined by bone markers representing dimensions of the calcaneus in terms of height, length, the Calcaneal Pitch angle, etc. The goal of obtaining this individualized Standard Circle was to delineate the contour that best approximates the posterior outline of an “ideally relatively normal” calcaneal tuberosity using dimensions of the calcaneus.

As shown in Figure 2, the center of the Standard Circle was denoted as $O(x_o, y_o)$. Three corners of the rectangle were $A(0,0)$ corresponding to point 7, $B(x_B,\ 0)$ corresponding to point 6, and $C(0, y_C)$ corresponding to point 4 in Figure 1. The width, height and diagonal length of the rectangle were denoted as x_B, y_C, and D respectively. D can be expressed in terms of x_B and y_C as:

$$D = \sqrt{x_B^2 + y_C^2}$$

Each point on the curvature of the control calcaneal tuberosity was standardized along the x and y axes respectively and then approximated using the following equation to determine the Standard Circle:

$$(x - x_0)^2 + (y - y_0)^2 = R^2$$

where the position of the center (O) of the Standard Circle was denoted by its offset from the x axis x_o, and its offset from the y axis y_o.

For all the control calcaneal tuberosities, their respective x_o, y_o, and circle radius R are calculated and used to construct distributions with averages and standard errors to parameterize the Standard Circle. Average x offset, y offset, and radius of the standard circle were then calculated for all control feet. All plotting points as well as the average Standard Circle fitted using the 40 control feet were visualized in Figure 3.

Based on the average circle parameters established using the control feet, for any future given calcaneus, with or without IAT, the individualized Standard Circle can be determined using the same formula with the statistically fitted parameters x_0, y_0, and R.

Developing the Pathologic Achilles Insertion Angle (PAIA)

The calcaneal tuberosities of the 40 feet with IAT were then analyzed by circumscribing the same positional bone markers as defined in the control feet, while intentionally excluding the enlargement of the calcaneal tuberosity (See Figure 4A). For these diseased feet with IAT, only the four corners of the rectangle and those anatomical markers were plotted to collect dimensional information of each calcaneus. Then the individualized Standard Circles were created for all individuals with IAT.

Following that, the enlarged posterior contour of the calcaneal tuberosities with IAT were outlined using positional bone markers 37 – 90 (the purple line in Figure 4A, 4B) to collect the x and y coordinate information of each enlarged calcaneal tuberosity. Then, these enlarged calcaneal tuberosity curves were rotated around the weightbearing point (point 90) to best align with their respective “ideal” curvatures predicted by the Standard Circles using mathematical optimization. The optimization is done by projecting the y coordinates of each plotted dot on the enlarged posterior tuberosity y_i onto the Standard Circle of that calcaneus. In this transformation, the Standard Circle (SC), upon projection, can be written as:

$$SC = \begin{bmatrix} SC_x \\ SC_y \end{bmatrix} = \begin{bmatrix} \sqrt{R^2 - (y_i - y_o)^2} + x_o \\ y_i \end{bmatrix}$$

where SC_x and SC_y are respectively the x and y coordinates of all the points on the Standard Circle. The x coordinates of the projected Standard Curve can be expressed as a function of the y coordinates.

Rotational loss, a mean square error measuring differences between the rotated enlarged calcaneal tuberosities and the Standard Circles of these calcaneal tuberosities in the IAT group was minimized during the optimization. And the most optimum rotation angle was solved during the optimization (Figure 4C). This rotation angle was named Pathologic Achilles Insertion Angle (PAIA) because it represented exactly the enlarged degrees of the calcaneal tuberosity in a foot with IAT, as well as the angle by which the enlarged calcaneal tuberosity curve can be rotated around the weightbearing point to best fit the ideal contour predicted by the Standard Circle of that individual calcaneus.

Mathematically, the counterclockwise θ-angled rotation of the observed curvature can be written as:

$$RV_\theta = \begin{bmatrix} RV_{\theta x} \\ RV_{\theta y} \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} x_i cos\theta - y_i sin\theta \\ x_i sin\theta + y cos\theta \end{bmatrix}$$

where RV_θ is the rotated enlarged contour of the calcaneal tuberosity with x coordinates $RV_{\theta x}$, and y coordinates $RV_{\theta y}$, and x_i and y_i are the coordinates for the original enlarged tuberosities. Then the rotation loss with respect to θ is quantified as a sum of square loss $SSE(\theta)$ which can be used to determine the similarity between the θ-rotated curvature and the standard circle. $SSE(\theta)$ is defined as:

$$SSE(\theta) = \sum_{i=1}^{N} \frac{1}{N} \sqrt{(SC_{xi} - RV_{\theta xi})^2 + \left(SC_{yi} - RV_{\theta yi}\right)^2}$$

where SC_{xi}, SC_{yi} are the x and y coordinates of the Standard Circle, and $RV_{\theta xi}$, $RV_{\theta yi}$ are the x and y coordinates of the rotated enlarged contour. N is typically equal to 54 because from point 37 to point 90, 54 points in total were used to depict the enlarged contour.

And the most optimum rotation angle can be found at the minima of the rotation loss with respect to the rotation angle, or when:

$$\frac{\partial SSE(\theta)}{\partial \theta} = 0$$

Theoretically, since the ideal normal contour and the enlarged contour of the calcaneal tuberosity share the same center of rotation at the weightbearing point of the calcaneus, the size of the PAIA would be the exact size of the Zadek osteotomy if the apex of the osteotomy is chosen at the weightbearing point of the calcaneus.