AI1110: Probability and Random Variables

Assignment 11

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June 21, 2022

Outline

Question

Every Stochastic matrix corresponds to a Markov chain for which it is the one-step transition matrix.
Show that not every Stochastic matrix can correspond to the two-step transition matrix of a Markov chain.
In particular, a 2 x 2 stochastic matrix is the two-step transition matrix of a Markov chain if and only if the sum of its diagonal elements is greater than or equal to Unity.

Theory

If a stochastic matrix $A = \left(a_{ij}\right)$ where $a_{ij} > 0$, corresponds to the two-step transition matrix of a Markov chain, then there must exist another stochastic matrix $P = \left(p_{ij}\right)$ where $p_{ij} > 0$ such that:

$$A = P^2 \tag{1}$$

Also,

$$\sum_j p_{ij} = 1 \tag{2}$$

Proof

In a two state chain, let $P = \begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix}$ so that,

$$A = P^2 = \begin{pmatrix} \alpha^2 + (1-\alpha)(1-\beta) & (\alpha+\beta)(1-\alpha) \\ (\alpha+\beta)(1-\beta) & \beta^2 + (1-\alpha)(1-\beta) \end{pmatrix} \tag{3}$$

This gives the sum of this its diagonal entries to be:

$$\begin{aligned} a_{11} + a_{22} &= \alpha^2 + 2(1-\alpha)(1-\beta) + \beta^2 && (4) \\ &= (\alpha+\beta)^2 - 2(\alpha+\beta) + 2 && (5) \\ &= 1 + (\alpha+\beta-1)^2 \geq 1 && (6) \end{aligned}$$

Hence proved.