

Assignment: Random Numbers

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1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

- 1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat.

Solution: Download the following file and execute the [C program](#) or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-Assignments/blob/main/Assignment\%20-\%20Random\%20Numbers/codes/1-1.c
```

- 1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as:

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

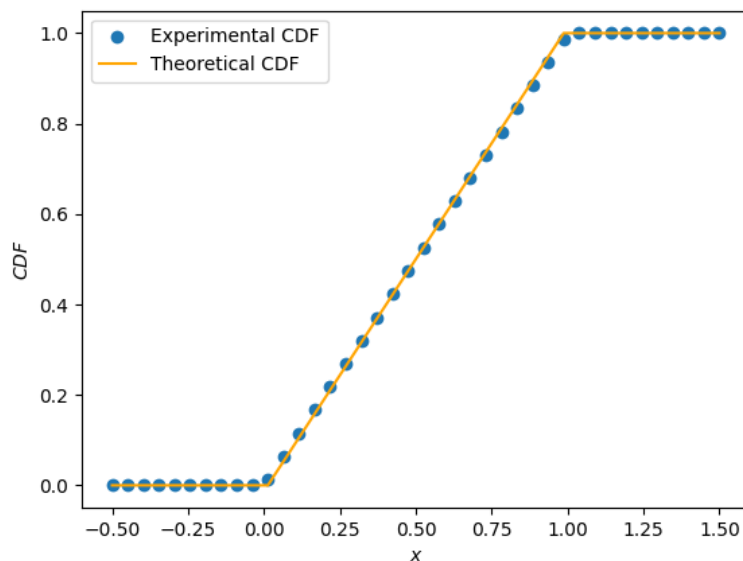


Fig. 1: The CDF of U

Solution: The following [python code](#) plots Fig. 1 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-Assignments/blob/main/Assignment\%20-\%20Random\%20Numbers/codes/1-2.py
```

- 1.3 Find a theoretical expression for $F_U(x)$.

Solution:

$$F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases} \quad (1.2)$$

By (1.2):

$$F_U(x) = \int_0^x U(x)dx \quad (1.3)$$

$$\Rightarrow F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \quad (1.4)$$

1.4 The mean of U is defined as:

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.5)$$

and its variance as:

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.6)$$

Write a C program to find the mean and variance of U .

Solution: Download the following files and execute the [C program](#) or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-Assignments/blob/main/Assignment\%20-\%20
Random\%20Numbers/codes/1-4.c
```

Values Obtained:

$$\boxed{\text{Mean} = 0.500007}$$

$$\boxed{\text{Variance} = 0.083301}$$

(1.7)

1.5 Verify your result theoretically given that:

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.8)$$

Solution:

$$dF_U(x) = p_U(x)dx \quad (1.9)$$

$$\Rightarrow E[U^k] = \int_{-\infty}^{\infty} x^k p_U(x)dx \quad (1.10)$$

Also, by (1.2)

$$p_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases} \quad (1.11)$$

Therefore, from Equations 1.2 and 1.10, we have:

$$E[U] = \int_{-\infty}^{\infty} x p_U(x)dx \quad (1.12)$$

$$= \int_0^1 x dx \quad (1.13)$$

$$= \frac{1}{2} \quad (1.14)$$

$$E[U^2] = \int_{-\infty}^{\infty} x^2 p_U(x) dx \quad (1.15)$$

$$= \int_0^1 x^2 dx \quad (1.16)$$

$$= \frac{1}{3} \quad (1.17)$$

$$E[U^2] - E[U]^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 \quad (1.18)$$

$$= \frac{1}{12} \quad (1.19)$$

Therefore, the theoretical mean is $\frac{1}{2}$, and the theoretical variance is $\frac{1}{12}$ which closely matches the experimental values.

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable:

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat.

Solution: Download the following files and execute the [C program](#) or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-Assignments/blob/main/Assignment\%20-%20Random\%20Numbers/codes/2-1.c
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat.

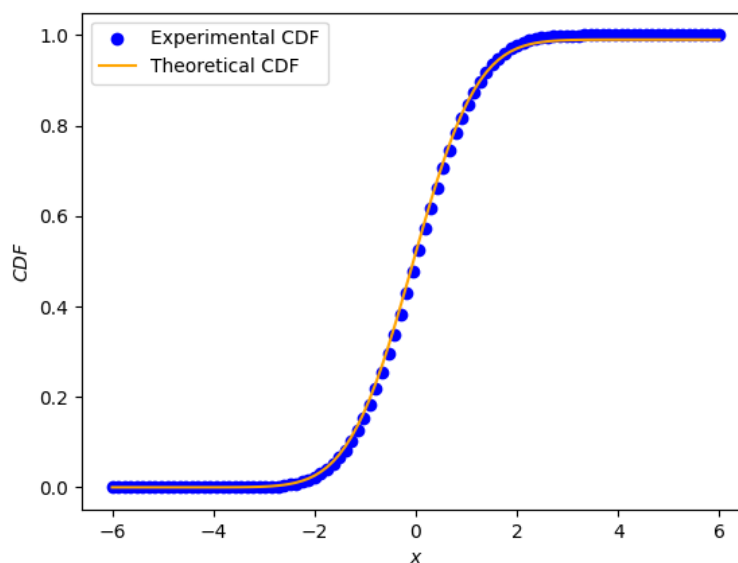


Fig. 2: The CDF of X

Solution: The following [python code](#) plots Fig. 2 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-Assignments/blob/main/Assignment\%20-\%20Random\%20Numbers/codes/2-2.py
```

- 2.3 Load `gau.dat` in python and plot the empirical PDF of X using the samples in `gau.dat`. The PDF of X is defined as:

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

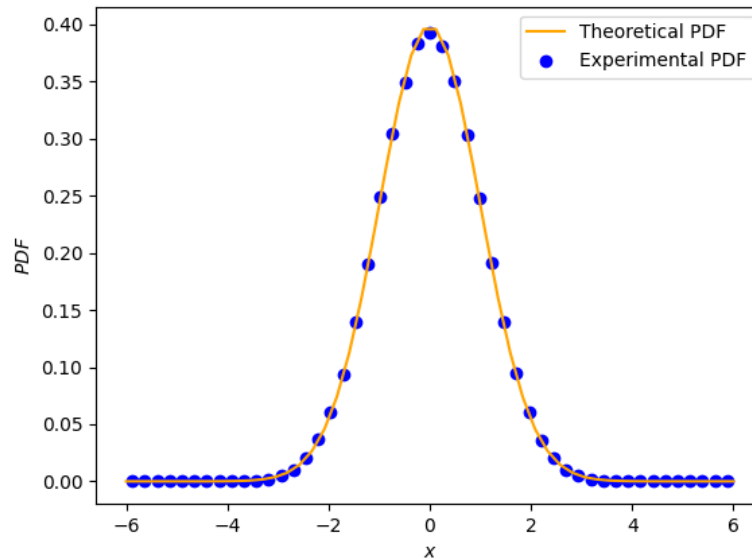


Fig. 3: The PDF of X

Solution: The following [python code](#) plots Fig. 3 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-Assignments/blob/main/Assignment\%20-\%20Random\%20Numbers/codes/2-3.py
```

- 2.4 Find the mean and variance of X by writing a C program.

Solution: Download the following files and execute the [C program](#) or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-Assignments/blob/main/Assignment\%20-\%20Random\%20Numbers/codes/2-4.c
```

Values Obtained:

Mean = -0.000241

Variance = 1.000726

(2.3)

- 2.5 Given that:

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.4)$$

repeat the above exercise theoretically

Solution:

$$E[X] = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.5)$$

$$= -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \Big|_{-\infty}^{\infty} \quad (2.6)$$

$$= 0 \quad (2.7)$$

Also,

$$E[X^2] = \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.8)$$

$$= -\frac{x}{\sqrt{2\pi}} e^{\left(-\frac{x^2}{2}\right)} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\left(-\frac{x^2}{2}\right)} dx \quad (2.9)$$

$$= 0 + \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi} \quad (2.10)$$

$$= 1 \quad (2.11)$$

Thus,

$$\text{var}(X) = E[X^2] - E[X]^2 \quad (2.12)$$

$$= 1 \quad (2.13)$$

Therefore, the mean is 0 and the variance is 1.

3 FROM UNIFORM TO OTHER

3.1 Generate samples of:

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

Solution:

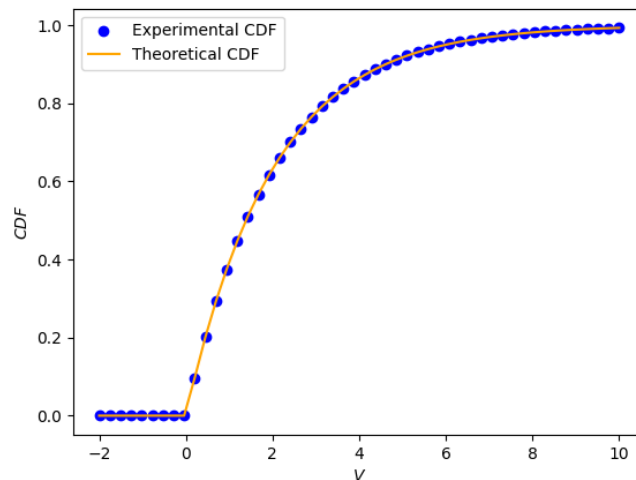


Fig. 4: The CDF of V

The following [python code](#) plots Fig. 4 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-Assignments/blob/main/Assignment\%20-\%20Random\%20Numbers/codes/3-1.py
```

3.2 Find a theoretical expression for $F_V(x)$.

Solution:

$$F_V(x) = \Pr(V \leq x) \quad (3.2)$$

$$= \Pr(-2 \ln(1 - U) \leq x) \quad (3.3)$$

$$= \Pr\left(1 - U \geq \exp\left(-\frac{x}{2}\right)\right) \quad (3.4)$$

$$= \Pr\left(U \leq 1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.5)$$

$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.6)$$

Therefore,

$$F_V(x) = \begin{cases} 0, & 1 - \exp\left(-\frac{x}{2}\right) \in (-\infty, 0) \\ 1 - \exp\left(-\frac{x}{2}\right), & 1 - \exp\left(-\frac{x}{2}\right) \in (0, 1) \\ 1, & 1 - \exp\left(-\frac{x}{2}\right) \in (1, \infty) \end{cases} \quad (3.7)$$

$$\Rightarrow F_V(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1 - \exp\left(-\frac{x}{2}\right), & x \in (0, \infty) \end{cases} \quad (3.8)$$

4 TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution: Download the following files and execute the [C program](#) or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-Assignments/blob/main/Assignment\%20-\%20Random\%20Numbers/codes/4-1.c
```

4.2 Find the CDF of T .

Solution:

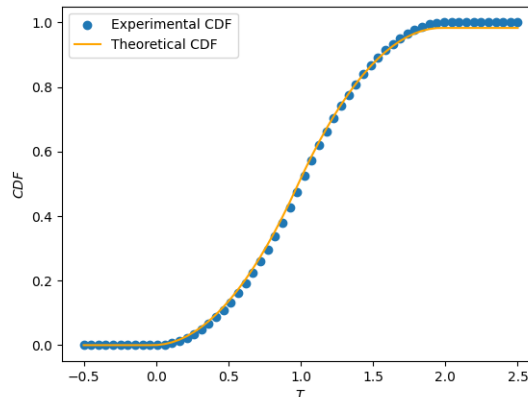


Fig. 5: The CDF of V

The following [python code](#) plots Fig. 5 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-Assignments/blob/main/Assignment\%20-\%20Random\%20Numbers/codes/4-2.py
```

4.3 Find the PDF of T .

Solution:

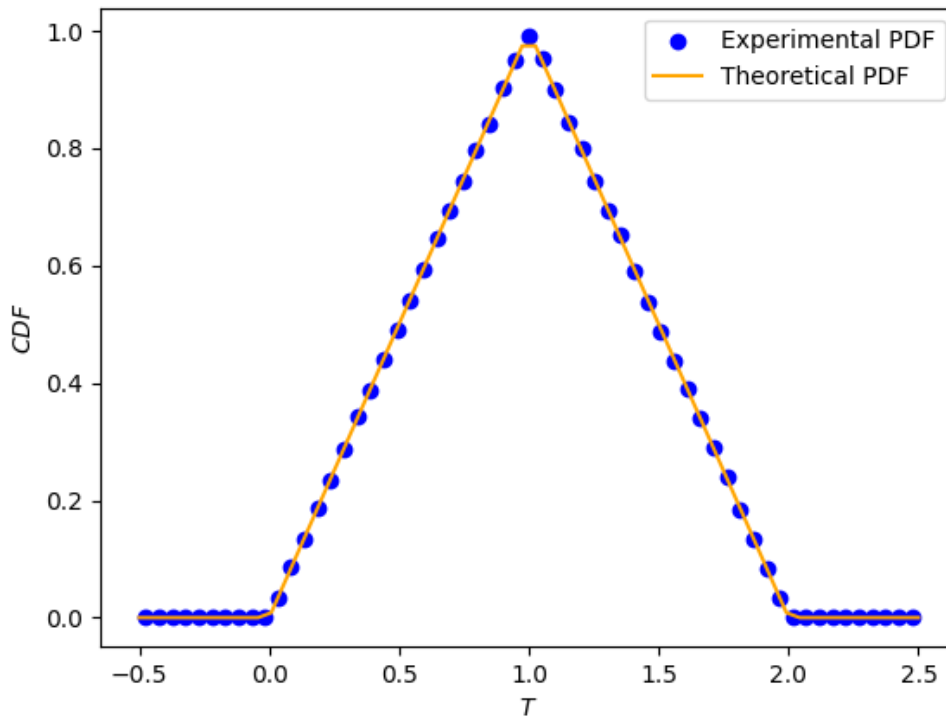


Fig. 6: The PDF of T

Solution: The following [python code](#) plots Fig. 6 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-Assignments/blob/main/Assignment\%20-\%20Random\%20Numbers/codes/4-3.py
```

4.4 Find the theoretical expressions for the PDF and CDF of T .

Solution:

$$T = U_1 + U_2 \quad (4.2)$$

Thus we have:

$$p_T(t) = (p_{U_1} * p_{U_2})(t) \quad (4.3)$$

$$= \int_{-\infty}^{\infty} p_U(u) p_U(t-u) du \quad (4.4)$$

$$= \int_0^1 p_U(t-u) du \quad (4.5)$$

When $0 < t < 1$:

$$p_T(t) = \int_0^1 p_U(t-u)du \quad (4.6)$$

$$= \int_0^t p_U(t-u)du \quad (4.7)$$

$$= \int_0^t du \quad (4.8)$$

$$= t \quad (4.9)$$

when $1 < t < 2$:

$$p_T(t) = \int_0^1 p_U(t-u)du \quad (4.10)$$

$$= \int_{t-1}^1 p_U(t-u)du \quad (4.11)$$

$$= \int_{t-1}^1 du \quad (4.12)$$

$$= 2 - t \quad (4.13)$$

When $t < 0$ and $t > 2$, the integral evaluates to 0. Thus,

$$p_T(t) = \begin{cases} 0, & t \in (-\infty, 0) \\ t, & t \in (0, 1) \\ 2 - t, & t \in (1, 2) \\ 0, & t \in (2, \infty) \end{cases} \quad (4.14)$$

For CDF of T:

$$F_T(t) = \int_{-\infty}^t p_T(x)dx \quad (4.15)$$

$$\Rightarrow F_T(t) = \begin{cases} 0, & t \in (-\infty, 0) \\ \frac{t^2}{2}, & t \in (0, 1) \\ -\frac{t^2}{2} + 2t - 1, & t \in (1, 2) \\ 1, & t \in (2, \infty) \end{cases} \quad (4.16)$$

4.5 Verify your results through a plot.

Solution: Fig.5 and Fig.6 plots the theoretical cdf and pdf respectively, which closely matches the experimental values.

5 MAXIMUM LIKELIHOOD

5.1 Generate

$$Y = AX + N, \quad (5.1)$$

where $A = 5$ dB, $X \in \{1, -1\}$, is Bernoulli and $N \sim \mathcal{N}(0, 1)$.

5.2 Plot Y .

5.3 Guess how to estimate X from Y .

5.4 Find

$$P_{e|0} = \Pr(\hat{X} = -1 | X = 1) \quad (5.2)$$

and

$$P_{e|1} = \Pr(\hat{X} = 1 | X = -1) \quad (5.3)$$

5.5 Find P_e .

5.6 Verify by plotting the theoretical P_e .

6 GAUSSIAN TO OTHER

6.1 Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \quad (6.1)$$

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad (6.2)$$

find α .

6.3 Plot the CDF and PDF of

$$A = \sqrt{V} \quad (6.3)$$

7 CONDITIONAL PROBABILITY

7.1

7.2 Plot

$$P_e = \Pr(\hat{X} = -1 | X = 1) \quad (7.1)$$

for

$$Y = AX + N, \quad (7.2)$$

where A is Rayleigh with $E[A^2] = \gamma$, $N \sim \mathcal{N}(0, 1)$, $X \in (-1, 1)$ for $0 \leq \gamma \leq 10$ dB.

7.3 Assuming that N is a constant, find an expression for P_e . Call this $P_e(N)$

7.4 For a function g ,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)p_X(x)dx \quad (7.3)$$

Find $P_e = E[P_e(N)]$.

7.5 Plot P_e in problems 7.2 and 7.4 on the same graph w.r.t γ . Comment.

8 TWO DIMENSIONS

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \quad (8.1)$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (8.2)$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \quad (8.3)$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \quad (8.4)$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols \mathbf{s}_0 and \mathbf{s}_1 .

8.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \quad (8.5)$$

with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e . Verify this by comparing the theory and simulation plots on the same graph.