Assignment: Random Numbers

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1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat.
Solution: Download the following file and execute the C program or type in terminal:

```
wget https://github.com/SterbenVD/AI1110−
    Assignments/blob/main/Assignment
    \%20−\%20Random\%20Numbers/codes
    /1−1.c
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as:

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

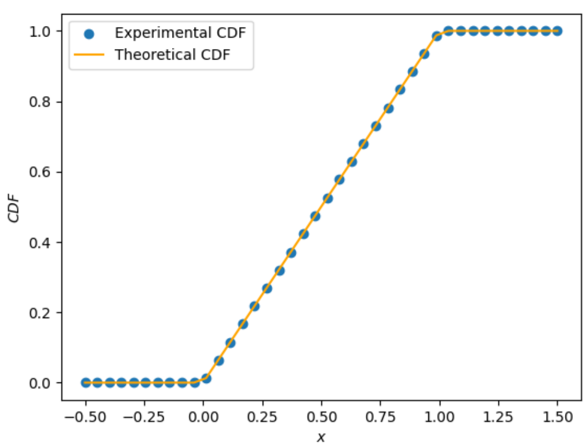

Fig. 1: The CDF of U

Solution: The following python code plots Fig. 1 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110−
    Assignments/blob/main/Assignment
    \%20−\%20Random\%20Numbers/codes
    /1−2.py
```

1.3 Find a theoretical expression for $F_U(x)$.
Solution:

$$U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases} \tag{1.2}$$

By (1.2):

$$F_U(x) = \int_0^x U(x)dx \tag{1.3}$$

$$\implies F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \tag{1.4}$$

1.4 The mean of U is defined as :

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.5}$$

and its variance as:

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.6}$$

Write a C program to find the mean and variance of U.
Solution: Download the following files and execute the C program or type in terminal:

```
wget https://github.com/SterbenVD/AI1110−
    Assignments/blob/main/Assignment
    \%20−\%20Random\%20Numbers/codes
    /1−4.c
```

Values Obtained:

$$\boxed{\text{Mean} = 0.500007} \quad \boxed{\text{Variance} = 0.083301} \tag{1.7}$$

1.5 Verify your result theoretically given that:

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.8}$$

Solution:

$$dF_U(x) = p_U(x)dx \tag{1.9}$$

$$\implies E[U^k] = \int_{-\infty}^{\infty} x^k p_U(x)dx \tag{1.10}$$

Also, by (1.2)

$$p_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1, & x \in (0,1) \\ 0, & x \in (1, \infty) \end{cases} \tag{1.11}$$

Therefore, from Equations 1.2 and 1.10, we have:

$$E[U] = \int_{-\infty}^{\infty} x p_U(x) dx \tag{1.12}$$
$$= \int_0^1 x dx \tag{1.13}$$
$$= \frac{1}{2} \tag{1.14}$$

$$E[U^2] = \int_{-\infty}^{\infty} x^2 p_U(x) dx \tag{1.15}$$
$$= \int_0^1 x^2 dx \tag{1.16}$$
$$= \frac{1}{3} \tag{1.17}$$

$$E[U^2] - E[U]^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 \tag{1.18}$$
$$= \frac{1}{12} \tag{1.19}$$

Therefore, the theoretical mean is $\frac{1}{2}$, and the theoretical variance is $\frac{1}{12}$ which closely matches the experimental values.

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable:

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat.

Solution: Download the following files and execute the C program or type in terminal:

```
wget https://github.com/SterbenVD/AI1110−
    Assignments/blob/main/Assignment
    \%20−\%20Random\%20Numbers/codes
    /2−1.c
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat.

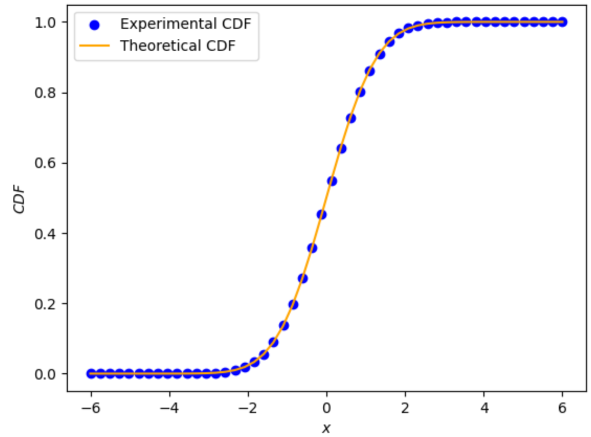

Fig. 2: The CDF of X

Solution: The following python code plots Fig. 2 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110−
    Assignments/blob/main/Assignment
    \%20−\%20Random\%20Numbers/codes
    /2−2.py
```

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as:

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

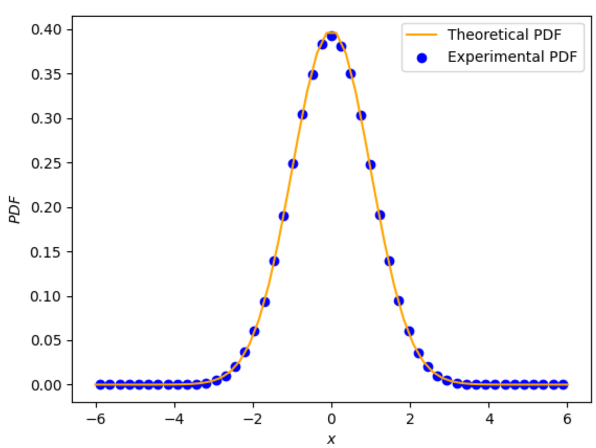

Fig. 3: The PDF of X

Solution: The following python code plots Fig. 3 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-
    Assignments/blob/main/Assignment
    \%20-\%20Random\%20Numbers/codes
    /2-3.py
```

2.4 Find the mean and variance of X by writing a C program.
Solution: Download the following files and execute the C program or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-
    Assignments/blob/main/Assignment
    \%20-\%20Random\%20Numbers/codes
    /2-4.c
```

Values Obtained:

$$\boxed{\text{Mean} = \text{-0.000241}} \quad \boxed{\text{Variance} = 1.000726} \tag{2.3}$$

2.5 Given that:

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.4}$$

repeat the above exercise theoretically
Solution:

$$E[X] = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{2.5}$$

$$= -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\Bigg|_{-\infty}^{\infty} \tag{2.6}$$

$$= 0 \tag{2.7}$$

Also,

$$E[X^2] = \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{2.8}$$

$$= -\frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\Bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{2.9}$$

$$= 0 + \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi} \tag{2.10}$$

$$= 1 \tag{2.11}$$

Thus,

$$\text{var}(X) = E[X^2] - E[X]^2 \tag{2.12}$$

$$= 1 \tag{2.13}$$

Therefore, the mean is 0 and the variance is 1.

$$\Pr(X > x) = Q(Z > x) \tag{2.14}$$

$$= Q(z) \tag{2.15}$$

$$CDF = \Pr(X < x) \tag{2.16}$$

$$= 1 - Q(z) \tag{2.17}$$

3 From Uniform to Other

3.1 Generate samples of:

$$V = -2\ln(1 - U) \tag{3.1}$$

and plot its CDF.
Solution:

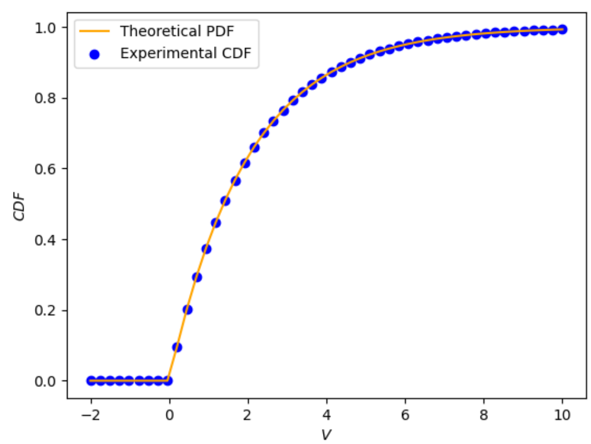

Fig. 4: The CDF of V

The following python code plots Fig. 4 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-
    Assignments/blob/main/Assignment
    \%20-\%20Random\%20Numbers/codes
    /3-1.py
```

3.2 Find a theoretical expression for $F_V(x)$.
Solution:

$$F_V(x) = \Pr(V \le x) \tag{3.2}$$

$$= \Pr(-2\ln(1 - U) \le x) \tag{3.3}$$

$$= \Pr\left(1 - U \ge \exp\left(-\frac{x}{2}\right)\right) \tag{3.4}$$

$$= \Pr\left(U \le 1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.5}$$

$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.6}$$

Therefore,

$$F_V(x) = \begin{cases} 0, & 1 - \exp\left(-\frac{x}{2}\right) \in (-\infty, 0) \\ 1 - \exp\left(-\frac{x}{2}\right), & 1 - \exp\left(-\frac{x}{2}\right) \in (0 \\ 1, & 1 - \exp\left(-\frac{x}{2}\right) \in (1 \end{cases} \tag{3.7}$$

$$\implies F_V(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1 - \exp\left(-\frac{x}{2}\right), & x \in (0, \infty) \end{cases} \tag{3.8}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the following files and execute the C program or type in terminal:

```
wget https://github.com/SterbenVD/AI1110−
    Assignments/blob/main/Assignment
    \%20−\%20Random\%20Numbers/codes
    /4−1.c
```

4.2 Find the CDF of T.
Solution:

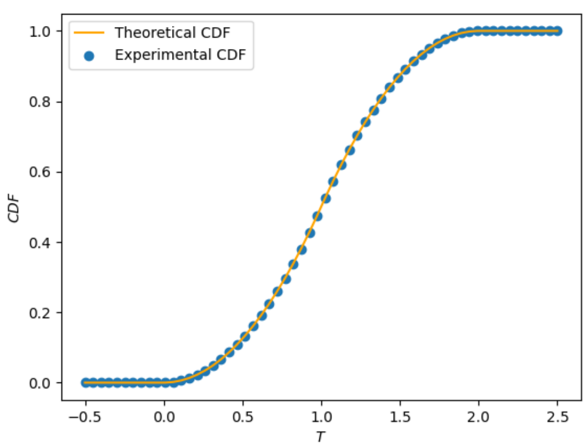

Fig. 5: The CDF of V

The following python code plots Fig. 5 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110−
    Assignments/blob/main/Assignment
    \%20−\%20Random\%20Numbers/codes
    /4−2.py
```

4.3 Find the PDF of T.
Solution:

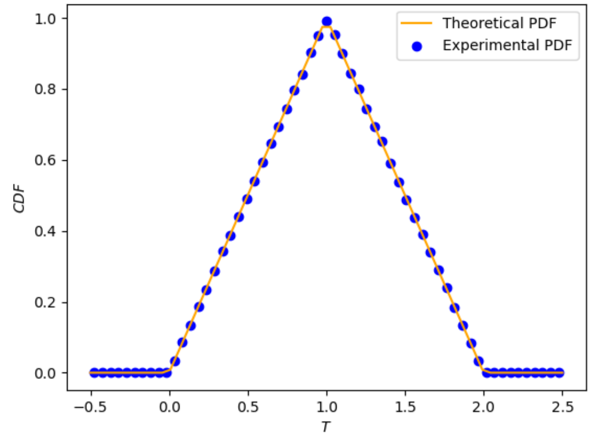

Fig. 6: The PDF of T

Solution: The following python code plots Fig. 6 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110−
    Assignments/blob/main/Assignment
    \%20−\%20Random\%20Numbers/codes
    /4−3.py
```

4.4 Find the theoretical expressions for the PDF and CDF of T.
Solution:

$$T = U_1 + U_2 \tag{4.2}$$

Thus we have:

$$p_T(t) = (p_{U_1} * p_{U_2})(t) \tag{4.3}$$

$$= \int_{-\infty}^{\infty} p_U(u) p_U(t-u) du \tag{4.4}$$

$$= \int_0^1 p_U(t-u) du \tag{4.5}$$

When $0 < t < 1$:

$$p_T(t) = \int_0^1 p_U(t-u) du \tag{4.6}$$

$$= \int_0^t p_U(t-u) du \tag{4.7}$$

$$= \int_0^t du \tag{4.8}$$

$$= t \tag{4.9}$$

when $1 < t < 2$:

$$p_T(t) = \int_0^1 p_U(t-u)du \tag{4.10}$$
$$= \int_{t-1}^1 p_U(t-u)du \tag{4.11}$$
$$= \int_{t-1}^1 du \tag{4.12}$$
$$= 2-t \tag{4.13}$$

When $t < 0$ and $t > 2$, the integral evaluates to 0. Thus,

$$p_T(t) = \begin{cases} 0, & t \in (-\infty, 0) \\ t, & t \in (0,1) \\ 2-t, & t \in (1,2) \\ 0, & t \in (2,\infty) \end{cases} \tag{4.14}$$

For CDF of T:

$$F_T(t) = \int_{-\infty}^t p_T(x)dx \tag{4.15}$$

$$\implies F_T(t) = \begin{cases} 0, & t \in (-\infty, 0) \\ \frac{t^2}{2}, & t \in (0,1) \\ -\frac{t^2}{2} + 2t - 1, & t \in (1,2) \\ 1, & t \in (2,\infty) \end{cases} \tag{4.16}$$

4.5 Verify your results through a plot.
Solution: Fig.5 and Fig.6 plots the theoretical cdf and pdf respectively, which closely matches the experimental values.

5 Maximul Likelihood

5.1 Generate

$$Y = AX + N, \tag{5.1}$$

where $A = 5$ dB, X ı$\{1, -1\}$, is Bernoulli and $N \sim \mathcal{N}(0, 1)$.

5.2 Plot Y.

5.3 Guess how to estimate X from Y.

5.4 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{5.2}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{5.3}$$

5.5 Find P_e.

5.6 Verify by plotting the theoretical P_e.

6 Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 \sim \mathcal{N}(0, 1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.2}$$

find α.

6.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{6.3}$$

7 Conditional Probability

7.1

7.2 Plot

$$P_e = \Pr\left(\hat{X} = -1|X = 1\right) \tag{7.1}$$

for

$$Y = AX + N, \tag{7.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0, 1), X \in (-1, 1)$ for $0 \leq \gamma \leq 10$ dB.

7.3 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

7.4 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x)p_X(x)\,dx \tag{7.3}$$

Find $P_e = E\left[P_e(N)\right]$.

7.5 Plot P_e in problems 7.2 and 7.4 on the same graph w.r.t γ. Comment.

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{8.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \tag{8.3}$$

8.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.4}$$

on the same graph using a scatter plot.

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

8.3 Plot
$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{8.5}$$
with respect to the SNR from 0 to 10 dB.

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.