Assignment: Random Numbers

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1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat.
Solution: Download the following file and execute the C program or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-
    Assignments/blob/main/Assignment
    \%20-\%20Random\%20Numbers/codes
    /1-1.c
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as:

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

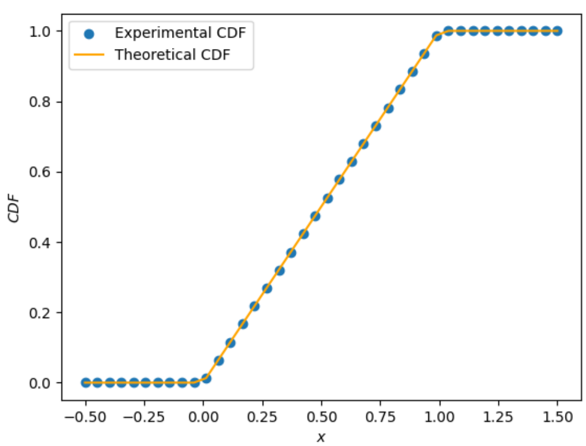

Fig. 1: The CDF of U

Solution: The following python code plots Fig. 1 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-
    Assignments/blob/main/Assignment
    \%20-\%20Random\%20Numbers/codes
    /1-2.py
```

1.3 Find a theoretical expression for $F_U(x)$.
Solution:

$$U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases} \tag{1.2}$$

By (1.2):

$$F_U(x) = \int_0^x U(x)dx \tag{1.3}$$

$$\implies F_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ x, & x \in (0, 1) \\ 1, & x \in (1, \infty) \end{cases} \tag{1.4}$$

1.4 The mean of U is defined as :

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.5}$$

and its variance as:

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.6}$$

Write a C program to find the mean and variance of U.
Solution: Download the following files and execute the C program or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-
    Assignments/blob/main/Assignment
    \%20-\%20Random\%20Numbers/codes
    /1-4.c
```

Values Obtained:

$$\boxed{\text{Mean} = 0.500007} \quad \boxed{\text{Variance} = 0.083301} \tag{1.7}$$

1.5 Verify your result theoretically given that:

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.8}$$

Solution:

$$dF_U(x) = p_U(x)dx \tag{1.9}$$

$$\implies E[U^k] = \int_{-\infty}^{\infty} x^k p_U(x)dx \tag{1.10}$$

Also, by (1.2)

$$p_U(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1, & x \in (0, 1) \\ 0, & x \in (1, \infty) \end{cases} \tag{1.11}$$

Therefore, from Equations 1.2 and 1.10, we have:

$$E[U] = \int_{-\infty}^{\infty} x p_U(x) dx \tag{1.12}$$
$$= \int_0^1 x dx \tag{1.13}$$
$$= \frac{1}{2} \tag{1.14}$$

$$E[U^2] = \int_{-\infty}^{\infty} x^2 p_U(x) dx \tag{1.15}$$
$$= \int_0^1 x^2 dx \tag{1.16}$$
$$= \frac{1}{3} \tag{1.17}$$

$$E[U^2] - E[U]^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 \tag{1.18}$$
$$= \frac{1}{12} \tag{1.19}$$

Therefore, the theoretical mean is $\frac{1}{2}$, and the theoretical variance is $\frac{1}{12}$ which closely matches the experimental values.

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable:

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat.
Solution: Download the following files and execute the C program or type in terminal:

```
wget https://github.com/SterbenVD/AI1110−
    Assignments/blob/main/Assignment
    \%20−\%20Random\%20Numbers/codes
    /2−1.c
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat.

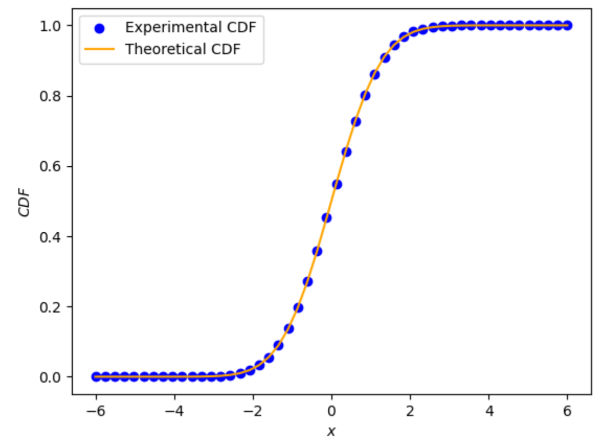

Fig. 2: The CDF of X

Solution: The following python code plots Fig. 2 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110−
    Assignments/blob/main/Assignment
    \%20−\%20Random\%20Numbers/codes
    /2−2.py
```

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as:

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

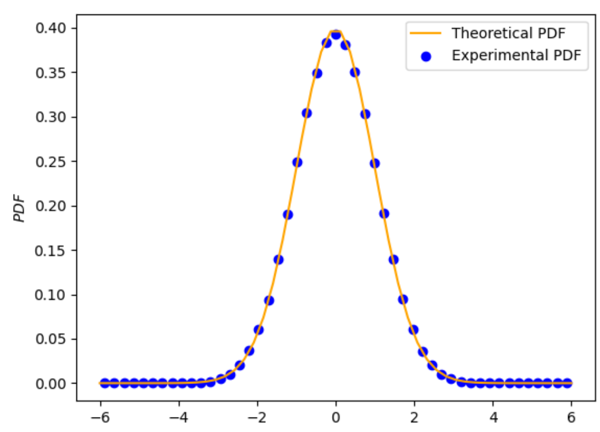

Fig. 3: The PDF of X

Solution: The following python code plots Fig. 3 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-
    Assignments/blob/main/Assignment
    \%20-\%20Random\%20Numbers/codes
    /2-3.py
```

2.4 Find the mean and variance of X by writing a C program.
Solution: Download the following files and execute the C program or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-
    Assignments/blob/main/Assignment
    \%20-\%20Random\%20Numbers/codes
    /2-4.c
```

Values Obtained:

$$\boxed{\text{Mean} = -0.000241} \quad \boxed{\text{Variance} = 1.000726} \tag{2.3}$$

2.5 Given that:

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.4}$$

repeat the above exercise theoretically
Solution:

$$E[X] = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{2.5}$$

$$= -\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\Bigg|_{-\infty}^{\infty} \tag{2.6}$$

$$= 0 \tag{2.7}$$

Also,

$$E[X^2] = \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{2.8}$$

$$= -\frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\Bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{2.9}$$

$$= 0 + \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi} \tag{2.10}$$

$$= 1 \tag{2.11}$$

Thus,

$$\text{var}(X) = E[X^2] - E[X]^2 \tag{2.12}$$

$$= 1 \tag{2.13}$$

Therefore, the mean is 0 and the variance is 1.

$$\Pr(X > x) = Q(Z > x) \tag{2.14}$$

$$= Q(z) \tag{2.15}$$

$$CDF = \Pr(X < x) \tag{2.16}$$

$$= 1 - Q(z) \tag{2.17}$$

3 From Uniform to Other

3.1 Generate samples of:

$$V = -2\ln(1 - U) \tag{3.1}$$

and plot its CDF.
Solution:

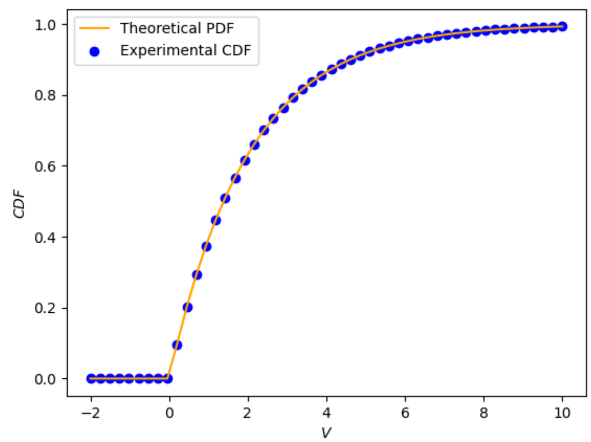

Fig. 4: The CDF of V

The following python code plots Fig. 4 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-
    Assignments/blob/main/Assignment
    \%20-\%20Random\%20Numbers/codes
    /3-1.py
```

3.2 Find a theoretical expression for $F_V(x)$.
Solution:

$$F_V(x) = \Pr(V \le x) \tag{3.2}$$

$$= \Pr(-2\ln(1-U) \le x) \tag{3.3}$$

$$= \Pr\left(1 - U \ge \exp\left(-\frac{x}{2}\right)\right) \tag{3.4}$$

$$= \Pr\left(U \le 1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.5}$$

$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.6}$$

Therefore,

$$F_V(x) = \begin{cases} 0, & 1 - \exp\left(-\frac{x}{2}\right) \in (-\infty, 0) \\ 1 - \exp\left(-\frac{x}{2}\right), & 1 - \exp\left(-\frac{x}{2}\right) \in (0 \\ 1, & 1 - \exp\left(-\frac{x}{2}\right) \in (1 \end{cases} \tag{3.7}$$

$$\implies F_V(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1 - \exp\left(-\frac{x}{2}\right), & x \in (0, \infty) \end{cases} \tag{3.8}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: Download the following files and execute the C program or type in terminal:

```
wget https://github.com/SterbenVD/AI1110−
    Assignments/blob/main/Assignment
    \%20−\%20Random\%20Numbers/codes
    /4−1.c
```

4.2 Find the CDF of T.

Solution:

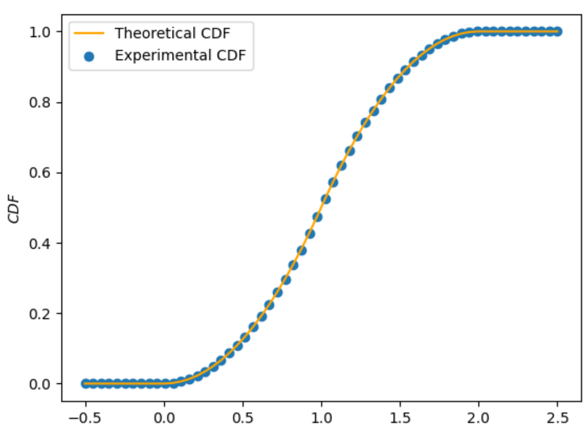

Fig. 5: The CDF of T

The following python code plots Fig. 5 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110−
    Assignments/blob/main/Assignment
    \%20−\%20Random\%20Numbers/codes
    /4−2.py
```

4.3 Find the PDF of T.

Solution:

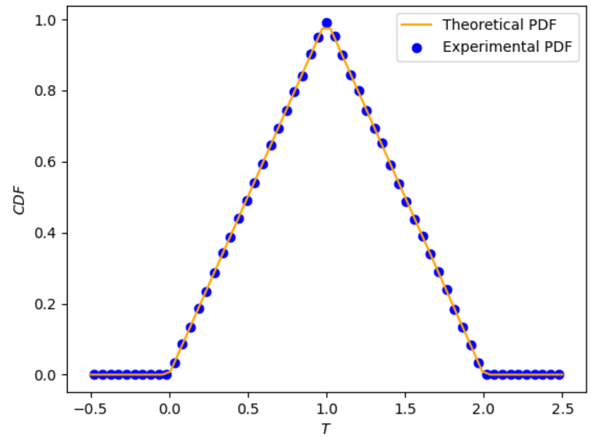

Fig. 6: The PDF of T

Solution: The following python code plots Fig. 6 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110−
    Assignments/blob/main/Assignment
    \%20−\%20Random\%20Numbers/codes
    /4−3.py
```

4.4 Find the theoretical expressions for the PDF and CDF of T.

Solution:

$$T = U_1 + U_2 \tag{4.2}$$

Thus we have:

$$p_T(t) = (p_{U_1} * p_{U_2})(t) \tag{4.3}$$
$$= \int_{-\infty}^{\infty} p_U(u) p_U(t-u) du \tag{4.4}$$
$$= \int_0^1 p_U(t-u) du \tag{4.5}$$

When $0 < t < 1$:

$$p_T(t) = \int_0^1 p_U(t-u) du \tag{4.6}$$
$$= \int_0^t p_U(t-u) du \tag{4.7}$$
$$= \int_0^t du \tag{4.8}$$
$$= t \tag{4.9}$$

when $1 < t < 2$:

$$p_T(t) = \int_0^1 p_U(t-u)du \tag{4.10}$$
$$= \int_{t-1}^1 p_U(t-u)du \tag{4.11}$$
$$= \int_{t-1}^1 du \tag{4.12}$$
$$= 2-t \tag{4.13}$$

When $t < 0$ and $t > 2$, the integral evaluates to 0. Thus,

$$p_T(t) = \begin{cases} 0, & t \in (-\infty, 0) \\ t, & t \in (0,1) \\ 2-t, & t \in (1,2) \\ 0, & t \in (2,\infty) \end{cases} \tag{4.14}$$

For CDF of T:

$$F_T(t) = \int_{-\infty}^t p_T(x)dx \tag{4.15}$$

$$\implies F_T(t) = \begin{cases} 0, & t \in (-\infty, 0) \\ \frac{t^2}{2}, & t \in (0,1) \\ -\frac{t^2}{2} + 2t - 1, & t \in (1,2) \\ 1, & t \in (2,\infty) \end{cases} \tag{4.16}$$

4.5 Verify your results through a plot.
Solution: Fig.5 and Fig.6 plots the theoretical cdf and pdf respectively, which closely matches the experimental values.

5 Maximul Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$.
Solution: Download the following files and execute the C program or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-
    Assignments/blob/main/Assignment
    \%20-\%20Random\%20Numbers/codes
    /5-1.c
```

5.2 Generate

$$Y = AX + N, \tag{5.1}$$

where $A = 5$ dB, $X \in \{1, -1\}$, is Bernoulli and $N \sim \mathcal{N}(0, 1)$.
Solution: Download the following files and execute the C program or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-
    Assignments/blob/main/Assignment
    \%20-\%20Random\%20Numbers/codes
    /5-2.c
```

5.3 Plot Y.
Solution:

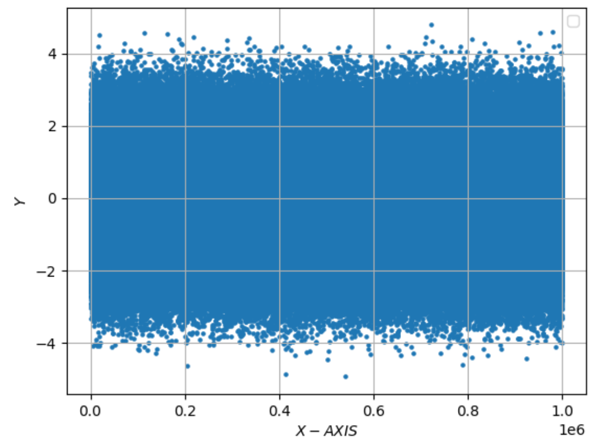

Fig. 7: Plot of Y

The following python code plots Fig. 7 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-
    Assignments/blob/main/Assignment
    \%20-\%20Random\%20Numbers/codes
    /5-3.py
```

5.4 Guess how to estimate X from Y.
Solution: Since X is Bernoulli with values $\{-1, 1\}$, the function $sgn(y)$ can be defined to estimate X from Y.

$$sgn(y) = \begin{cases} -1, & y \in (-\infty, 0) \\ 1, & y \in [0, \infty) \end{cases} \tag{5.2}$$

Using $sgn(y)$, we can estimate the corresponding values of X.

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.3}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{5.4}$$

Solution: Download the following files and execute the C program or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-
    Assignments/blob/main/Assignment
    \%20-\%20Random\%20Numbers/codes
    /5-4.c
```

Values Obtained:

$$\boxed{P_{e|0} = 0.310412} \quad \boxed{P_{e|1} = 0.310724} \tag{5.5}$$

5.6 Find P_e, assuming that X has equiprobable symbols.
Solution:

$$P_e = P_{e|0} \times \Pr(X = 1) + P_{e|1} \times \Pr(X = -1) \tag{5.6}$$

$$= \frac{P_{e|0} + P_{e|1}}{2} \tag{5.7}$$

Also,

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.8}$$
$$= \Pr(AX + N < 0 | X = 1) \tag{5.9}$$
$$= \Pr(A + N < 0) \tag{5.10}$$
$$= \Pr(N < -A) \tag{5.11}$$

Since, $N \sim \mathcal{N}(0, 1)$

$$\implies P_{e|0} = \int_{-\infty}^{-A} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \tag{5.12}$$
$$= \int_{A}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \tag{5.13}$$
$$= Q_N(A) \tag{5.14}$$

Similarly, $P_{e|1} = Q_N(A)$

$$\implies P_e = Q_N(A) \tag{5.15}$$

5.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.
Solution:

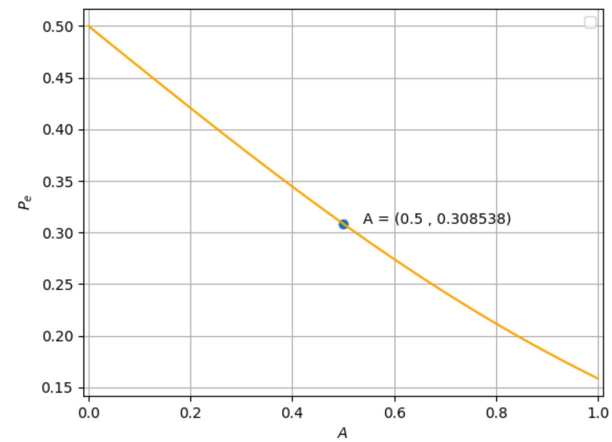

Fig. 8: Plot of P_e

The following python code plots Fig. 8 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-
    Assignments/blob/main/Assignment
    \%20-\%20Random\%20Numbers/codes
    /5-7.py
```

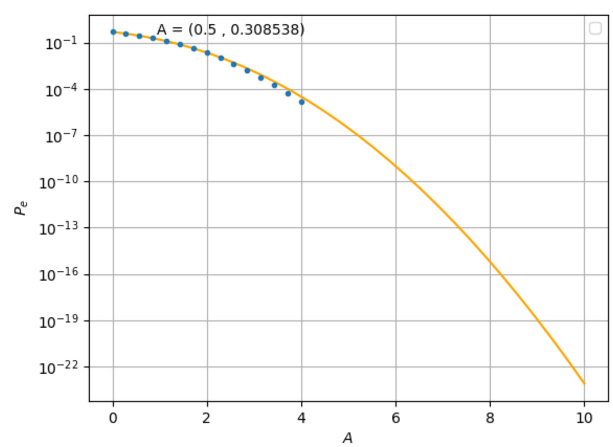

Fig. 9: Plot of P_e with semilog axis

The following python code plots Fig. 9 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-
    Assignments/blob/main/Assignment
    \%20-\%20Random\%20Numbers/codes
    /5-7_semilog.py
```

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes

the theoretical P_e.
Solution: To estimate X from Y:

$$X = \begin{cases} 1, & Y > \delta \\ -1, & Y < \delta \end{cases} \tag{5.16}$$

Thus,

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{5.17}$$
$$= \Pr(AX + N < \delta | X = 1) \tag{5.18}$$
$$= \Pr(N + A < \delta) \tag{5.19}$$
$$= \Pr(N < \delta - A) \tag{5.20}$$
$$= \int_{-\infty}^{\delta - A} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx \tag{5.21}$$
$$= \int_{A-\delta}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx \tag{5.22}$$
$$= Q_N(A - \delta) \tag{5.23}$$
$$\tag{5.24}$$

Similarly,

$$P_{e|1} = Q_N(A + \delta) \tag{5.25}$$

Thus,

$$P_e = P_{e|0} \Pr(X = 1) + P_{e|1} \Pr(X = -1) \tag{5.26}$$
$$= \frac{Q_N(A - \delta) + Q_N(A + \delta)}{2} \tag{5.27}$$

To maximize P_e, differentiate the above equation w.r.t δ and equate it to 0:

$$0 = \frac{d}{d\delta}\left(\frac{Q_N(A - \delta) + Q_N(A + \delta)}{2}\right) \tag{5.28}$$
$$= \frac{1}{2}\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{(\delta - A)^2}{2}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(A+\delta)^2}{2}}\right) \tag{5.29}$$
$$\tag{5.30}$$

Thus,

$$(\delta - A)^2 = (\delta + A)^2 \tag{5.31}$$
$$\implies \delta = 0 \tag{5.32}$$

5.9 Repeat the above exercise when

$$p_X(0) = p \tag{5.33}$$

Solution: Using Eq. (5.26), we have:

$$P_e = P_{e|0} p + P_{e|1}(1 - p) \tag{5.34}$$
$$= pQ_N(A - \delta) + (1 - p)Q_N(A + \delta) \tag{5.35}$$

To maximize P_e, differentiate the above equation w.r.t δ and equate it to 0:

$$0 = \frac{p}{\sqrt{2\pi}} e^{-\frac{(\delta - A)^2}{2}} - \frac{(1 - p)}{\sqrt{2\pi}} e^{-\frac{(A+\delta)^2}{2}} \tag{5.36}$$
$$\implies pe^{-\frac{(\delta - A)^2}{2}} = (1 - p)e^{-\frac{(A+\delta)^2}{2}} \tag{5.37}$$

Taking ln on both sides:

$$\ln p - \frac{(\delta - A)^2}{2} = \ln(1 - p) - \frac{(\delta + A)^2}{2} \tag{5.38}$$
$$\implies 2A\delta = \ln \frac{1 - p}{p} \tag{5.39}$$
$$\implies \delta = \frac{1}{2A} \ln \frac{1 - p}{p} \tag{5.40}$$

5.10 Repeat the above exercise using the MAP criterion.
Solution:

$$\Pr(X = -1) = p \tag{5.41}$$
$$\Pr(X = 1) = (1 - p) \tag{5.42}$$

$$\implies p_Y(y) = p_{Y|X=-1}(y| - 1) \Pr(X = -1)$$
$$+ p_{Y|X=1}(y|1) \Pr(X = 1) \tag{5.43}$$
$$= p \times p_{(-A+N)}(y)$$
$$+ (1 - p) \times p_{(A+N)}(y) \tag{5.44}$$

Also:

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x) \times p_X(x)}{p_Y(y)} \tag{5.45}$$

When $X = 1$:

$$p_{X|Y}(1|y) = \frac{p_{Y|X}(y|1) \times p_X(1)}{p_Y(y)} \tag{5.46}$$
$$= \frac{(1 - p)\,\frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}}}{p\frac{e^{-\frac{(y+A)^2}{2}}}{\sqrt{2\pi}} + (1 - p)\,\frac{e^{-\frac{(y-A)^2}{2}}}{\sqrt{2\pi}}} \tag{5.47}$$
$$= \frac{1 - p}{pe^{-2yA} + (1 - p)} \tag{5.48}$$

Similarly, when $X = -1$:

$$p_{X|Y}(-1|y) = \frac{p}{p + (1 - p)\,e^{2yA}} \tag{5.49}$$

Thus,

$$\frac{p}{p + (1 - p)\,e^{-2yA}} \gtrless \frac{1 - p}{(1 - p) + pe^{2yA}} \tag{5.50}$$
$$\implies p^2 e^{2yA} \gtrless (1 - p)^2\, e^{-2yA} \tag{5.51}$$
$$\implies y \gtrless \frac{1}{2A} \ln\left(\frac{1 - p}{p}\right) \tag{5.52}$$

By (5.52), if $y > 0$, then we can estimate $X = 1$. If $y < 0$, then we can estimate $X = -1$.

6 Gaussian to Other

6.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

Solution: Download the following files and execute the C program or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-Assignments/blob/main/Assignment\%20-\%20Random\%20Numbers/codes/6-1.c
```

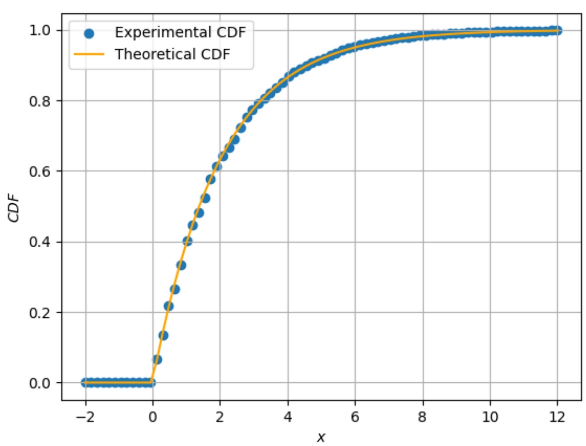

Fig. 10: The CDF of V

The following python code plots Fig. 10 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-Assignments/blob/main/Assignment\%20-\%20Random\%20Numbers/codes/6-1_cdf.py
```

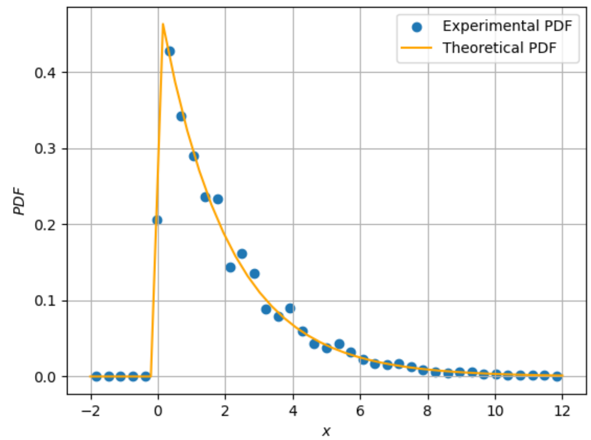

Fig. 11: The PDF of V

The following python code plots Fig. 11 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-Assignments/blob/main/Assignment\%20-\%20Random\%20Numbers/codes/6-1_pdf.py
```

6.2 If:

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.2}$$

find α.

Solution: X_1 and X_2 are i.i.d that can be transformed as:

$$X_1 = R\cos\Theta \tag{6.3}$$
$$X_2 = R\sin\Theta \tag{6.4}$$

where $R \in [0,\infty), \Theta \in [0, 2\pi)$. Thus,

$$\mathbf{J} = \begin{pmatrix} \frac{\partial X_1}{\partial R} & \frac{\partial X_2}{\partial R} \\ \frac{\partial X_1}{\partial \Theta} & \frac{\partial X_2}{\partial \Theta} \end{pmatrix} \tag{6.5}$$
$$= \begin{pmatrix} \cos\Theta & \sin\Theta \\ -R\sin\Theta & R\cos\Theta \end{pmatrix} \tag{6.6}$$
$$\implies |\mathbf{J}| = R \tag{6.7}$$

Also,

$$p_{R,\Theta}(r,\theta) = |\mathbf{J}| p_{X_1,X_2}(x_1,x_2) \tag{6.8}$$
$$= R p_{X_1}(x_1) p_{X_2}(x_2) \tag{6.9}$$
$$= \frac{R}{2\pi}\exp\left(-\frac{X_1^2+X_2^2}{2}\right) \tag{6.10}$$
$$= \frac{R}{2\pi}\exp\left(-\frac{R^2}{2}\right) \tag{6.11}$$

Thus,

$$p_R(r) = \int_0^{2\pi} p_{R,\Theta}(r,\theta)d\theta \quad (6.12)$$

$$= R\exp\left(-\frac{R^2}{2}\right) \quad (6.13)$$

However, $V = X_1^2+X_2^2 = R^2 \geq 0$, thus $F_V(x) = 0$ for $x \leq 0$.

$$F_V(x) = F_R(\sqrt{x}) \quad (6.14)$$

$$= \int_0^{\sqrt{x}} r\exp\left(-\frac{r^2}{2}\right)dr \quad (6.15)$$

$$= \int_0^{\frac{x}{2}} e^{-t}dt = 1 - e^{-\frac{x}{2}} \quad (6.16)$$

For $x \geq 0$,

$$p_V(x) = \frac{1}{2}e^{-\frac{x}{2}} \quad (6.17)$$

Hence,

$$F_V(x) = \begin{cases} 1 - e^{-\frac{x}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (6.18)$$

$$p_V(x) = \begin{cases} \frac{1}{2}e^{-\frac{x}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (6.19)$$

6.3 Plot the CDF and PDf of

$$A = \sqrt{V} \quad (6.20)$$

Solution:

$$F_A(x) = \Pr(A \leq x) \quad (6.21)$$

$$= \Pr\left(\sqrt{V} \leq x\right) \quad (6.22)$$

$$(6.23)$$

For $x \geq 0$:

$$F_A(x) = \Pr\left(V \leq x^2\right) \quad (6.24)$$

$$= F_V(x^2) = 1 - e^{-\frac{x^2}{2}} \quad (6.25)$$

and so,

$$p_A(x) = xe^{-\frac{x^2}{2}} \quad (6.26)$$

Thus, the CDF and PDF of A is given by

$$F_V(x) = \begin{cases} 1 - e^{-\frac{x^2}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (6.27)$$

$$p_V(x) = \begin{cases} xe^{-\frac{x}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (6.28)$$

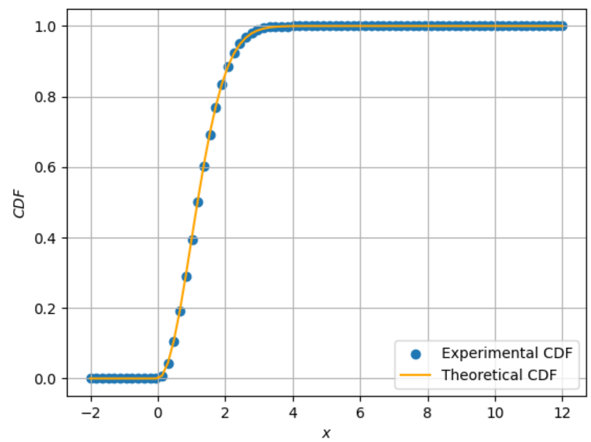

Fig. 12: The CDF of A

The following python code plots Fig. 12 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-Assignments/blob/main/Assignment\%20-\%20Random\%20Numbers/codes/6-3_cdf.py
```

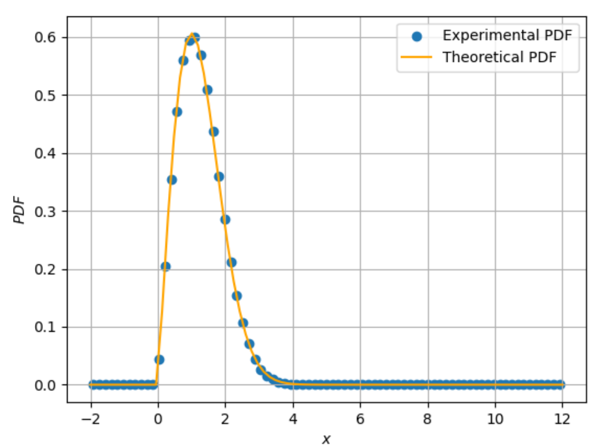

Fig. 13: The PDF of A

The following python code plots Fig. 13 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-Assignments/blob/main/Assignment\%20-\%20Random\%20Numbers/codes/6-3_pdf.py
```

7 Conditional Probability

7.1 Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{7.1}$$

for

$$Y = AX + N \tag{7.2}$$

where A is Rayleigh with $E\left[A^2\right] = \gamma$, $N \sim \mathcal{N}(0, 1)$, $X \in \{1, -1\}$ for $0 \leq \gamma \leq 10$ dB.
Solution: Download the following files and execute the C program or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-
     Assignments/blob/main/Assignment
     \%20-\%20Random\%20Numbers/codes
     /7-1.c
```

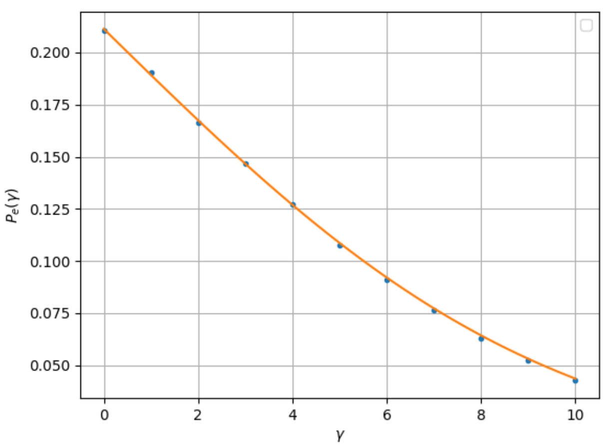

Fig. 14: Plot of P_e wrt γ

The following python code plots Fig. 14 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-
     Assignments/blob/main/Assignment
     \%20-\%20Random\%20Numbers/codes
     /7-1.py
```

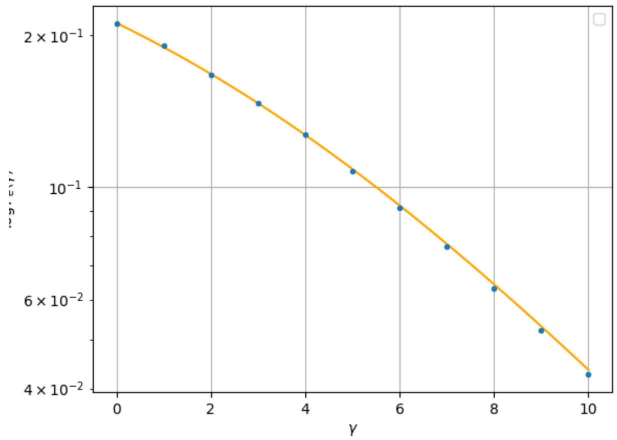

Fig. 15: Semilog Plot of P_e wrt γ

The following python code plots Fig. 15 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110-
     Assignments/blob/main/Assignment
     \%20-\%20Random\%20Numbers/codes
     /7-1_semilog.py
```

7.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$.
Solution: We rewrite the previous expression for P_e as

$$P_e(N) = \Pr(A + N < 0) = F_A(-N) \tag{7.3}$$
$$= \begin{cases} 1 - e^{-\frac{N^2}{\gamma}} & N \leq 0 \\ 0 & N > 0 \end{cases} \tag{7.4}$$

7.3 For a function g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) p_X(x) dx \tag{7.5}$$

Find $P_e = E[P_e(N)]$.
Solution: We write,

$$P_e = \int_0^{\infty} F_A(x) f_N(x) dx \tag{7.6}$$
$$= \int_0^{\infty} (1 - e^{-\frac{x^2}{\gamma}}) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \tag{7.7}$$
$$= \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \exp\left(-\frac{x^2}{\frac{2\gamma}{\gamma+2}}\right) dx \tag{7.8}$$
$$= \frac{1}{2}\left(1 - \sqrt{\frac{\gamma}{\gamma + 2}}\right) \tag{7.9}$$

where f_N denotes the standard normal distribution.

7.4 Plot P_e in problems 7.1 and 7.3 on the same graph w.r.t γ. Comment.
Solution: Since $P_{e|0} = E\left[P_e(N)\right]$, the error rate is independent of the noise.

8 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n} \tag{8.1}$$

where

$$\mathbf{x} \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{8.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1) \tag{8.3}$$

8.1 Plot $\mathbf{y}|\mathbf{s_0}$ and $\mathbf{y}|\mathbf{s_1}$ on the same graph using a scatter plot.
Solution: Download the following files and execute the C program or type in terminal:

```
wget https://github.com/SterbenVD/AI1110−
    Assignments/blob/main/Assignment
    \%20−\%20Random\%20Numbers/codes
    /8−1.c
```

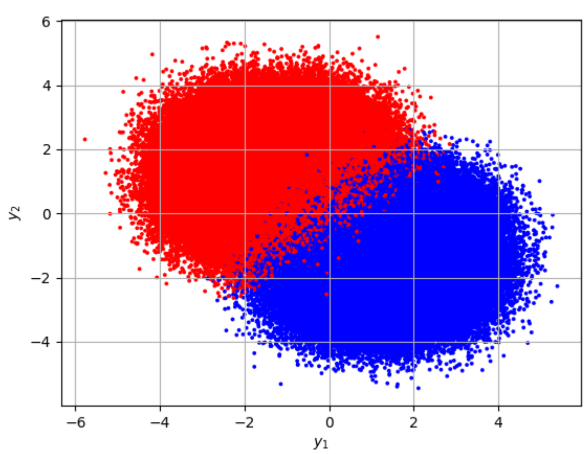

Fig. 16: Plot of y_2 vs y_1 at $A = 5dB$

The following python code plots Fig. 16 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110−
    Assignments/blob/main/Assignment
    \%20−\%20Random\%20Numbers/codes
    /8−1.py
```

8.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.
Solution: Let $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. By figure , $y_1 - y_2 = 0$ seperates the two plots.
Thus,

$$\hat{\mathbf{x}} = \begin{cases} \mathbf{s}_0 & y_1 > y_2 \\ \mathbf{s}_1 & y_1 < y_2 \end{cases} \tag{8.4}$$

8.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \tag{8.5}$$

with respect to the SNR from 0 to 10 dB.
Solution:

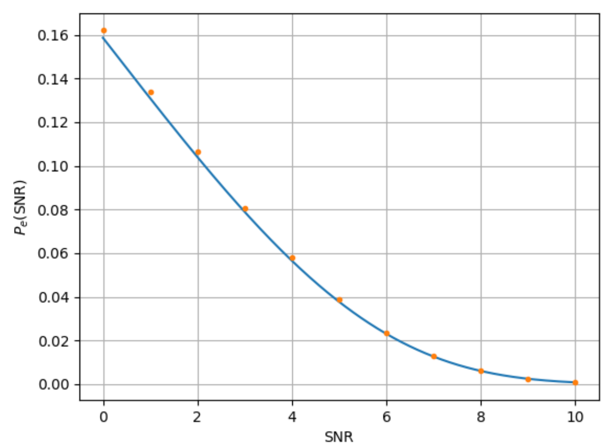

Fig. 17: Plot of P_e w.r.t to A

The following python code plots Fig. 17 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110−
    Assignments/blob/main/Assignment
    \%20−\%20Random\%20Numbers/codes
    /8−3.py
```

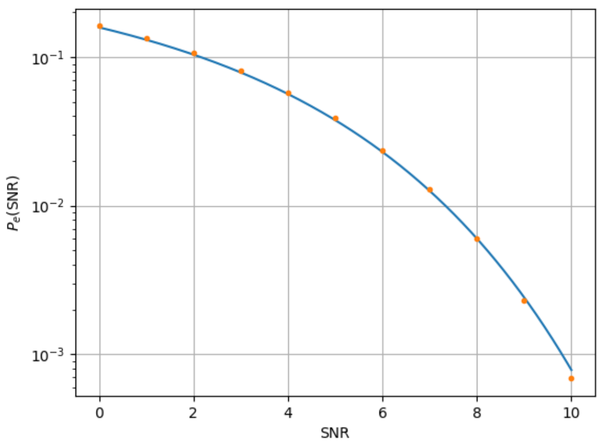

Fig. 18: Semilog Plot of P_e w.r.t to A

The following python code plots Fig. 18 or type in terminal:

```
wget https://github.com/SterbenVD/AI1110−
    Assignments/blob/main/Assignment
    \%20−\%20Random\%20Numbers/codes
    /8−3_semilog.py
```

8.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.
Solution: We have,

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{8.6}$$
$$= \Pr\left(y_1 < y_2 | \mathbf{x} = \mathbf{s}_0\right) \tag{8.7}$$
$$= \Pr\left(A + n_1 < -A + n_2\right) \tag{8.8}$$
$$= \Pr\left(n_2 - n_1 > 2A\right) \tag{8.9}$$
$$= \Pr\left(N > 2A\right) = Q\left(2A\right) \tag{8.10}$$

where $N = n_2 - n_1 \sim \mathcal{N}(0, 2)$ and SNR $= \frac{E\left[A^2\right]}{\sigma_N^2}$.