CS 413 Homework 6

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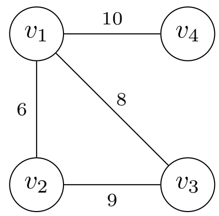
Problem 9

a. **Counterexample.** Consider the graph G below:



Since the weight of the only edge adjacent to v_4 in G is 10, it must be that any spanning tree T of G has a bottleneck edge of weight no less than 10. It follows then that every spanning tree of G which has a maximum edge weight of 10 must be a minimum-bottleneck spanning tree. The tree with edge set $E = \{6, 9, 10\}$ is one such tree. However, by Kruskal's algorithm, the edge set of the minimum spanning tree of G is $E = \{6, 8, 10\}$. Hence, not every minimum-bottleneck tree T of G is a minimum spanning tree of G.

b. **Proposition.** *Every minimum spanning tree of G is a minimum-bottleneck tree of G.*

Proof. Define $b(T)$ to be the weight of the bottleneck edge of a spanning tree of G. Now let T_1 be a minimum spanning tree of G and suppose that T_1 is not a minimum-bottleneck tree of G. Then there exists a spanning tree T_2 of G, such that $b(T_1) > b(T_2)$. Let e be the bottleneck edge of T_1. Since T_1 is a tree, removing e will disconnect T_1. Now using the remaining edges of T_1, let V_1 be the subset of $V(G)$ which are reachable from one end of e and let V_2 be the subset of $V(G)$ which are reachable from the other end of e. It follows that e is the only edge in T_1 that connects V_1 and V_2. Hence by the Cut Property, e has the minimum weight of all the edges in G which connect V_1 and V_2. Since $e \notin T_2$ it must be that there is some other edge in T_2 that connects V_1 and V_2. But we have just established that e is the edge of minimum weight in G that can do this. And since e is the bottleneck edge of T_1 it must be that $b(T_1) \leq b(T_2)$. We have reached a contradiction. $\square$