Fibonacci Sequence Convergence Theorem

Sterling Jeppson

October 15, 2023

Lemma 1. *For all integers $n \geq 0$, $F_{n+2}F_n - F_{n+1}^2 = (-1)^n$.*

Proof. Let n be any integer with $n \geq 0$ and let the property $P(n)$ be the equation to be proved.

Show that $P(0)$ is true: It follows from the definition of $F_0, F_1, F_2, ...$ that

$$F_2F_0 - F_1^2 = 2 \cdot 1 - 1^2 = 2 - 1 = 1 \quad \text{and} \quad (-1)^0 = 1$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$F_{k+2}F_k - F_{k+1}^2 = (-1)^k \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$F_{k+3}F_{k+1} - F_{k+2}^2 = (-1)^{k+1} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
&F_{k+3}F_{k+1} - F_{k+2}^2 \\
&= (F_{k+2} + F_{k+1})F_{k+1} - F_{k+2}^2 && \text{by definition of } F_0, F_1, F_2, ... \\
&= F_{k+1}^2 + F_{k+1}F_{k+2} - F_{k+2}^2 \\
&= F_{k+2}F_k - (-1)^k + F_{k+1}F_{k+2} - F_{k+2}^2 && \text{by inductive hypothesis } F_{k+1}^2 = F_{k+2}F_k - (-1)^k \\
&= F_{k+2}(F_k + F_{k+1} - F_{k+2}) + (-1)^1(-1)^k \\
&= F_{k+2}(F_{k+2} - F_{k+2}) + (-1)^1(-1)^k && \text{by definition of } F_0, F_1, F_2, ... \\
&= F_{k+2}(0) + (-1)^{k+1} && (-1)^1(-1)^k = (-1)^{k+1} \\
&= (-1)^{k+1}
\end{aligned}$$

Hence $P(k+1)$ is true. □

Lemma 2. *The sequence of Fibonacci ratios defined as*

$$a_n = \frac{F_{2n}}{F_{2n-1}}, \quad n \geq 1$$

is decreasing and bounded below.

Proof. It follows from lemma 1 that $\forall n \in \mathbb{Z}^{nonneg}$, $F_{n+2}F_n - F_{n+1}^2 = (-1)^n$. Hence,

$$\begin{aligned}
F_{2n+1}F_{2n-1} - F_{2n}^2 &= (-1)^{2n-1} && \text{substitute } 2n-1 \text{ for } n \\
F_{2n+1}F_{2n-1} - F_{2n}^2 &= -1 && (-1)^{2n-1} = (-1)^{2n}(-1)^{-1} = 1(-1) = -1 \\
F_{2n+1}F_{2n-1} - F_{2n}^2 &< 0 \\
F_{2n+1}F_{2n-1} &< F_{2n}^2 \\
F_{2n}F_{2n-1} + F_{2n+1}F_{2n-1} &< F_{2n}^2 + F_{2n}F_{2n-1} && \text{add } F_{2n}F_{2n-1} \text{ to both sides} \\
F_{2n-1}(F_{2n} + F_{2n+1}) &< F_{2n}(F_{2n} + F_{2n-1}) \\
F_{2n-1}F_{2n+2} &< F_{2n}F_{2n+1} && \text{by definition of } F_0, F_1, F_2, ... \\
\frac{F_{2n+2}}{F_{2n+1}} &< \frac{F_{2n}}{F_{2n-1}}
\end{aligned}$$

We have shown that a_n is decreasing. Now we need to show that it is bounded below. But this is so because all Fibonacci numbers are positive and the quotient of any two positive numbers is positive. Hence $a_n > 0$. □

Lemma 3. *The sequence of Fibonacci ratios defined as*

$$b_n = \frac{F_{2n+1}}{F_{2n}}, \quad n \geq 0$$

is increasing and bounded above.

Proof. It follows from lemma 1 that $\forall n \in \mathbb{Z}^{nonneg}$, $F_{n+2}F_n - F_{n+1}^2 = (-1)^n$. Hence,

$$\begin{aligned}
F_{2n+2}F_{2n} - F_{2n+1}^2 &= (-1)^{2n} && \text{substitute } 2n \text{ for } n \\
F_{2n+2}F_{2n} - F_{2n+1}^2 &= 1 && (-1)^2 = 1,\ \ 1^n = 1 \\
F_{2n+2}F_{2n} - F_{2n+1}^2 &> 0 \\
F_{2n+2}F_{2n} &> F_{2n+1}^2 \\
F_{2n}F_{2n+1} + F_{2n+2}F_{2n} &> F_{2n+1}^2 + F_{2n}F_{2n+1} && \text{add } F_{2n}F_{2n+1} \text{ to both sides} \\
F_{2n}(F_{2n+1} + F_{2n+2}) &> F_{2n+1}(F_{2n+1} + F_{2n}) \\
F_{2n}F_{2n+3} &> F_{2n+1}F_{2n+2} && \text{by definition of } F_0, F_1, F_2 \ldots \\
\frac{F_{2n+3}}{F_{2n+2}} &> \frac{F_{2n+1}}{F_{2n}}
\end{aligned}$$

We have shown that b_n is increasing. Now we need to show that it is bounded above. But $b_0 = 1 < 2$ and for all integers $n \geq 1$,

$$b_n = \frac{F_{2n+1}}{F_{2n}} = \frac{F_{2n} + F_{2n-1}}{F_{2n}} = 1 + \frac{F_{2n-1}}{F_{2n}} < 2$$

The previous inequality follows from the fact that the $(n+1)$st term in the Fibonacci sequence is greater than the nth term for all $n \geq 1$. ☐

Theorem. $\lim_{n\to\infty} \left(\frac{F_{n+1}}{F_n}\right)$ *exists.*

Proof. Define a sequence of Fibonacci ratios as

$$c_n = \frac{F_{n+1}}{F_n}, \quad n \geq 0$$

Now by lemma 2 and 3 above we know that a_n and b_n are bounded monotonic sequences. It follows from the monotonic sequence theorem that a_n and b_n are convergent. This implies that there exist limits L_a and L_b such that $\lim_{n\to\infty} a_n = L_a$ and $\lim_{n\to\infty} b_n = L_b$. We must show that $L_a = L_b$. In order to do this we first show that for all $n \in \mathbb{Z}^+$,

$$a_n - b_n = -(b_n - a_n) = -\left(\frac{F_{2n+1}}{F_{2n}} - \frac{F_{2n}}{F_{2n-1}}\right) = -\frac{F_{2n+1}F_{2n-1} - F_{2n}^2}{F_{2n}F_{2n-1}} = -\frac{-1}{F_{2n}F_{2n-1}} = \frac{1}{F_{2n}F_{2n-1}}$$

It follows that since the Fibonacci sequence is unbounded and increasing we can make $a_n - b_n$ arbitrarily small. Suppose that $L_a \neq L_b$. Then there exists a real number $\epsilon > 0$ such that $\epsilon = |L_a - L_b|$. Now since $\lim_{n\to\infty} a_n = L_a$ there exists an integer N_a such that $n > N_a \implies |a_n - L_a| < \epsilon/4$. Also since $\lim_{n\to\infty} b_n = L_b$ there exists an integer N_b such that $n > N_b \implies |b_n - L_b| < \epsilon/4$. Finally since $a_n - b_n$ can be made arbitrarily small there exists an integer N_c such such that $n > N_c \implies |a_n - b_n| < \epsilon/4$. Now let $N = \max\{N_a, N_b, N_c\}$ and it follows that for any $n > N$,

$$\begin{aligned}
\epsilon &= |L_a - L_b| \\
&= |(L_a - L_b) + (a_n - a_n) + (b_n - b_n)| \\
&= |(L_a - a_n) + (b_n - L_b) + (a_n - b_n)| \\
&\leq |L_a - a_n| + |b_n - L_b| + |a_n - b_n| && \text{by the triangle inequality} \\
&= |a_n - L_a| + |b_n - L_b| + |a_n - b_n| && \text{by lemma 4.4.5(in my book)} \\
&< \epsilon/4 + \epsilon/4 + \epsilon/4 \\
&= 3\epsilon/4
\end{aligned}$$

But this is a contradiction as $\epsilon \not< 3\epsilon/4$ and so our supposition that $L_a \neq L_b$ is false and hence $L_a = L_b$.

Finally let $L = L_a = L_b$. It follows that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$. Let a real number $\epsilon > 0$ be given. Then there exist integers N_a and N_b such that $n > N_a \implies |a_n - L| < \epsilon$ and $n > N_b \implies |b_n - L| < \epsilon$. Now let $N = \max\{2N_a - 1, 2N_b\}$ and let $n > N$. Then $c_n = a_k$ for some $k > N_a$ or $c_n = b_k$ for some $k > N_b$. Thus no matter if n is even or odd $|c_n - L| < \epsilon$ and so c_n is convergent and $\lim_{n\to\infty} c_n = L$. $\square$

Since it is proven that the limit exists we can say $L = \lim\limits_{n\to\infty} \left(\frac{F_{n+1}}{F_n}\right)$. Now,

$$\begin{aligned}
L &= \lim_{n\to\infty} \left(\frac{F_{n+1}}{F_n}\right) \\
&= \lim_{n\to\infty} \left(\frac{F_n + F_{n-1}}{F_n}\right) && \text{by definition of } F_0, F_1, F_2, \ldots \\
&= \lim_{n\to\infty} \left(1 + \frac{F_{n-1}}{F_n}\right) \\
&= 1 + \lim_{n\to\infty} \left(\frac{1}{\frac{F_n}{F_{n-1}}}\right) \\
&= 1 + \frac{1}{\lim\limits_{x\to\infty} \frac{F_{x+1}}{F_x}} = 1 + \frac{1}{L} && \text{let } x = n - 1
\end{aligned}$$

Place the equation for L above in a form that can be solved by the quadratic equation.

$$1 + \frac{1}{L} = L \iff L - 1 - \frac{1}{L} = 0 \iff L^2 - L - 1 = 0$$

Now solve by the quadratic equation

$$L = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

However since

$$L = \lim_{n\to\infty} \left(\frac{F_{n+1}}{F_n}\right)$$

and since every term in the Fibonacci sequence is at least as large as the previous term we cannot have $L < 1$. It Follows that

$$L = \frac{1+\sqrt{5}}{2}$$