Chapter 6

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Lemma 6.1.a. $\forall x \in \mathbb{R}, x \neq 3 \implies x^2 - 2x + 3 \neq 0.$

Proof. Let x be any real number such that $x \neq 3$. We must show that $x^2 - 2x + 3 \neq 0$. Assume that it is equal to 0 and derive a contradiction. Completing the square of $x^2 - 2x + 3 = 0$ we obtain

$$\begin{aligned}
x^2 - 2x + 3 &= 0 \\
(x^2 - 2x + 3) + 1 &= 0 + 1 \\
(x^2 - 2x + 1) + 3 &= 1 \\
x^2 - 2x + 1 &= -2 \\
(x-1)^2 &= -2
\end{aligned}$$

which is a contradiction since $\forall y \in \mathbb{R}, y^2 \geq 0$. ☐

Lemma 6.1.b. $\exists x \in \mathbb{C}, x \neq 3 \wedge x^2 - 2x + 3 = 0.$

Proof. When dealing with complex numbers the prior proposition does not hold. To show this we show that the negation is true. Let $x = 1 + i\sqrt{2}$. This is not equal to 3 because the definition of equality on complex numbers is equal real and imaginary parts and $1 \neq 3$ and $\sqrt{2} \neq 0$. Next we verify that $x^2 - 2x + 3 = 0$.

$$\begin{aligned}
x^2 - 2x + 3 &= 0 \\
(1 + i\sqrt{2})^2 - 2(1 + i\sqrt{2}) + 3 &= 0 \\
(1 + i\sqrt{2}) \cdot (1 + i\sqrt{2}) - 2 - i \cdot 2\sqrt{2} + 3 &= 0 \\
1 + i \cdot 2\sqrt{2} + (i\sqrt{2})^2 - i \cdot 2\sqrt{2} + 1 &= 0 \\
(1 + 1) + (i \cdot 2\sqrt{2} - i \cdot 2\sqrt{2}) + (-1 \cdot 2) &= 0 \\
2 + 0 - 2 &= 0 \\
0 &= 0
\end{aligned}$$

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Lemma 6.2. $\forall n \in \mathbb{N}, 2 < n < 3 \implies 7n + 3$ *is odd.*

Proof. The premise is false since there are not natural numbers between 2 and 3 and n is a natural number. Therefore the conclusion follows vacuously. ☐

Lemma 6.3. $\forall x \in \mathbb{Z}$, *if x is odd then x^2 is odd.*

Proof. Let x be some odd integer. Then $x = 2k + 1$ for some integer k. It follows that

$$x^2 = (2k+1)^2 = (4k^2 + 4k + 1) = 2 \cdot (2k^2 + 2k) + 1$$

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Lemma 6.4. $\forall x \in \mathbb{Z}$, *if x is even, then $7x - 5$ is odd.*

Proof. Let x be an even integer, so $x = 2k$ for some integer k. It follows that

$$7x - 5 = 7(2k) - 5 = 14k - 5 = 2 \cdot (7k - 3) + 1$$

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Lemma 6.5. $\forall a, b, c \in \mathbb{Z}$, *if a and c are odd, then $a \cdot b + b \cdot c$ is even.*

Proof. Since all integers are either even or odd b must be either even or odd. If b is even, then both $a \cdot b$ and $b \cdot c$ are even because the product of an even number with any integer is even. The sum of two even numbers is even, so $a \cdot b + b \cdot c$ is even. If b is odd, then both $a \cdot b$ and $b \cdot c$ are odd because the product of two odd numbers is odd. The sum of two odd numbers is even, so $a \cdot b + b \cdot c$ is even. Thus, in both cases, $a \cdot b + b \cdot c$ is even. ☐

Lemma 6.6. *$\forall n \in \mathbb{Z}$, $|n| < 1 \implies 3n - 2$ is even.*

Proof. Given $|n| < 1$, the only integer that satisfies this inequality is $n = 0$. Substituting $n = 0$ into the expression, we have

$$3 \cdot 0 - 2 = -2.$$

Since -2 can be written as $2 \cdot (-1)$, it is an even number. □

Lemma 6.7. *$\forall z \in \mathbb{Z}$, z is odd $\implies \exists\, a, c \in \mathbb{Z}, z = a^2 - c^2$.*

Proof. Since z is odd, we can write $z = 2k + 1$ for some integer k. Let $a = k + 1$ and $c = k$. Then,

$$a^2 - c^2 = (k + 1)^2 - k^2.$$

Expanding this, we get

$$(k + 1)^2 - k^2 = (k^2 + 2k + 1) - k^2 = 2k + 1,$$

which is exactly z. Therefore, there exist integers a and c such that $z = a^2 - c^2$. □