Section 5.3

Sterling Jeppson

November 13, 2020

Problem 1

Based on the discussion of the product $(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{4})...(1-\frac{1}{n})$ at the beginning of this section, conjecture a formula for general n. Prove your conjecture by mathematical induction.

Solution

Conjecture. *For all integers $n \geq 2$, $\prod_{i=2}^{n}(1-\frac{1}{i}) = \frac{1}{n}$.*

Proof. Let the property $P(n)$ be the equation

$$\prod_{i=2}^{n}\left(1-\frac{1}{i}\right) = \frac{1}{n} \qquad \leftarrow P(n)$$

Show that $P(2)$ is true:

$$\prod_{i=2}^{2}\left(1-\frac{1}{i}\right) = 1-\frac{1}{2} = \frac{1}{2} \quad \text{and with } n=2, \quad \frac{1}{n} = \frac{1}{2}$$

Show that for all integers $k \geq 2$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 2$ and suppose that

$$\prod_{i=2}^{k}\left(1-\frac{1}{i}\right) = \frac{1}{k} \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$\prod_{i=2}^{k+1}\left(1-\frac{1}{i}\right) = \frac{1}{k+1} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
&\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)...\left(1-\frac{1}{k}\right)\left(1-\frac{1}{k+1}\right)\\
&\qquad = \left(\frac{1}{k}\right)\left(1-\frac{1}{k+1}\right) \qquad \text{by inductive hypothesis}\\
&\qquad = \left(\frac{1}{k}\right)\left(\frac{k+1}{k+1}-\frac{1}{k+1}\right)\\
&\qquad = \left(\frac{1}{k}\right)\left(\frac{k}{k+1}\right) = \left(\frac{1}{\cancel{k}}\right)\left(\frac{\cancel{k}}{k+1}\right) = \frac{1}{k+1}
\end{aligned}$$

which is the right-hand side of $P(k+1)$. $\square$

Problem 2

Experiment with computing values of the product $(1+\frac{1}{1})(1+\frac{1}{2})(1+\frac{1}{3})...(1+\frac{1}{n})$ for small values of n to conjecture a formula for this product for general n. Prove your conjecture by mathematical induction.

Solution

$$\begin{aligned}
\left(1+\frac{1}{1}\right) &= 2\\
\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right) &= 3\\
\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right) &= 4
\end{aligned}$$

Conjecture. $\forall n \in \mathbb{Z}^+,\ (1+\frac{1}{1})(1+\frac{1}{2})(1+\frac{1}{3})...(1+\frac{1}{n}) = n+1.$

Proof. Let the property $P(n)$ be the equation

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)...\left(1+\frac{1}{n}\right) = n+1 \qquad \leftarrow P(n)$$

Show that $P(1)$ is true:

$$1+\frac{1}{1} = 2 \quad \text{and} \quad 1+1=2$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)...\left(1+\frac{1}{k}\right) = k+1 \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{k}\right)\left(1+\frac{1}{k+1}\right) = (k+1)+1 \quad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
&\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{k}\right)\left(1+\frac{1}{k+1}\right) \\
&\qquad = (k+1)\left(1+\frac{1}{k+1}\right) && \text{by inductive hypothesis} \\
&\qquad = (k+1)\left(\frac{k+1}{k+1}+\frac{1}{k+1}\right) \\
&\qquad = \cancel{(k+1)}\left(\frac{k+2}{\cancel{k+1}}\right) \\
&\qquad = k+2 = (k+1)+1
\end{aligned}$$

which is the right-hand side of $P(K+1)$. □

Problem 3

Observe that

$$\begin{aligned}
\frac{1}{1\cdot 3} &= \frac{1}{3} \\
\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5} &= \frac{2}{5} \\
\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7} &= \frac{3}{7} \\
\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9} &= \frac{4}{9}
\end{aligned}$$

Guess a general formula and prove it by mathematical induction.

Solution

Conjecture. $\forall n \in \mathbb{Z}^{nonneg},\ \sum_{i=0}^{n} \frac{1}{(2n+1)(2n+3)} = \frac{n+1}{2n+3}$.

Proof. Let the property $P(n)$ be the equation

$$\sum_{i=0}^{n} \frac{1}{(2i+1)(2i+3)} = \frac{n+1}{2n+3} \qquad \leftarrow P(n)$$

Show that $P(0)$ is true:

$$\sum_{i=0}^{0} \frac{1}{(2i+1)(2i+3)} = \frac{1}{(1)(3)} = \frac{1}{3} \quad \text{and} \quad \frac{0+1}{2\cdot 0+3} = \frac{1}{3}$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$\sum_{i=0}^{k} \frac{1}{(2i+1)(2i+3)} = \frac{k+1}{2k+3} \qquad \leftarrow P(k) \text{ IH}$$

we must show that

$$\sum_{i=0}^{k+1} \frac{1}{(2i+1)(2i+3)} = \frac{(k+1)+1}{2(k+1)+3}$$

$$\sum_{i=0}^{k+1} \frac{1}{(2i+1)(2i+3)} = \frac{k+2}{2k+5} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
\sum_{i=0}^{k+1} \frac{1}{(2i+1)(2i+3)} &= \sum_{i=0}^{k} \frac{1}{(2i+1)(2i+3)} + \frac{1}{(2k+3)(2k+5)} \qquad \text{by inductive hypothesis} \\
&= \frac{k+1}{2k+3} + \frac{1}{(2k+3)(2k+5)} \\
&= \frac{(k+1)(2k+5)}{(2k+3)(2k+5)} + \frac{1}{(2k+3)(2k+5)} \\
&= \frac{2k^2+7k+6}{(2k+3)(2k+5)} \\
&= \frac{\cancel{(2k+3)}(k+2)}{\cancel{(2k+3)}(2k+5)} \\
&= \frac{k+2}{2k+5}
\end{aligned}$$

which is the right-hand side of $P(k+1)$. $\square$

Problem 4

Observe that

$$\begin{aligned}
1 &= 1, \\
1-4 &= -(1+2), \\
1-4+9 &= 1+2+3, \\
1-4+9-16 &= -(1+2+3+4), \\
1-4+9-16+25 &= 1+2+3+4+5.
\end{aligned}$$

Guess a general formula and prove it by mathematical induction.

Solution

Conjecture. $\forall n \in \mathbb{Z}^+,\ \sum_{i=1}^{n} (-1)^{n+1} \cdot i^2 = (-1)^{n+1} \cdot \frac{n(n+1)}{2}$.

Proof. Let the property $P(n)$ be the equation

$$\sum_{i=1}^{n} (-1)^{i+1} \cdot i^2 = (-1)^{n+1} \cdot \frac{n(n+1)}{2} \qquad \leftarrow P(n)$$

Show that $P(1)$ is true:

$$\sum_{i=1}^{1} (-1)^{i+1} \cdot i^2 = (-1)^2 \cdot 1 = 1 \quad \text{and} \quad (-1)^{1+1} \cdot \frac{1(1+1)}{2} = 1 \cdot \frac{2}{2} = 1 \cdot 1 = 1$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$\sum_{i=1}^{k} (-1)^{i+1} \cdot i^2 = (-1)^{k+1} \cdot \frac{k(k+1)}{2} \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$\sum_{i=1}^{k+1} (-1)^{i+1} \cdot i^2 = (-1)^{(k+1)+1} \cdot \frac{(k+1)((k+1)+1)}{2}$$
$$\sum_{i=1}^{k+1} (-1)^{i+1} \cdot i^2 = (-1)^{k+2} \cdot \frac{(k+1)(k+2)}{2} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
\sum_{i=1}^{k+1} (-1)^{i+1} \cdot i^2 &= \sum_{i=1}^{k} (-1)^{i+1} \cdot i^2 + (-1)^{k+2} \cdot (k+1)^2 \\
&= (-1)^{k+1} \cdot \frac{k(k+1)}{2} + (-1)^{k+2} \cdot (k+1)^2 && \text{by inductive hypothesis} \\
&= (-1)^1(-1)^k \cdot \frac{k(k+1)}{2} + (-1)^2(-1)^k \cdot (k+1)^2 \\
&= (-1)^k \left(\frac{-k(k+1)}{2} + (k+1)^2 \right) \\
&= \left(1 \cdot (-1)^k\right) \left((k+1) \left(-\frac{k}{2} + k + 1 \right) \right) \\
&= \left((-1)^2 \cdot (-1)^k\right) \left((k+1) \left(-\frac{k}{2} + \frac{2k+2}{2} \right) \right) \\
&= (-1)^{k+2} \cdot \frac{(k+1)(k+2)}{2}
\end{aligned}$$

which is the right-hand side of $P(k+1)$. $\square$

Problem 5

Evaluate the sum $\sum_{k=1}^{n} \frac{k}{(k+1)!}$ for $n = 1, 2, 3, 4,$ and 5. Make a conjecture about a formula for this sum for general n, and prove your conjecture by mathematical induction.

Solution

$$\frac{1}{(1+1)!} = \frac{1}{2}$$

$$\frac{1}{(1+1)!} + \frac{2}{(2+1)!} = \frac{5}{6}$$

$$\frac{1}{(1+1)!} + \frac{2}{(2+1)!} + \frac{3}{(3+1)!} = \frac{23}{24}$$

$$\frac{1}{(1+1)!} + \frac{2}{(2+1)!} + \frac{3}{(3+1)!} + \frac{4}{(4+1)!} = \frac{119}{120}$$

$$\frac{1}{(1+1)!} + \frac{2}{(2+1)!} + \frac{3}{(3+1)!} + \frac{4}{(4+1)!} + \frac{5}{(5+1)!} = \frac{719}{720}$$

Conjecture. $\forall n \in \mathbb{Z}^+,\ \sum\limits_{i=1}^{n} \frac{i}{(i+1)!} = \frac{(n+1)!-1}{(n+1)!}$.

Proof. Let the property $P(n)$ be the equation

$$\sum_{i=1}^{n} \frac{i}{(i+1)!} = \frac{(n+1)!-1}{(n+1)!} \qquad \leftarrow P(n)$$

Show that $P(1)$ is true:

$$\sum_{i=1}^{1} \frac{i}{(i+1)!} = \frac{1}{2!} = \frac{1}{2} \quad \text{and} \quad \frac{(1+1)!-1}{(1+1)!} = \frac{2!-1}{2!} = \frac{1}{2}$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$\sum_{i=1}^{k} \frac{i}{(i+1)!} = \frac{(k+1)!-1}{(k+1)!} \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$\sum_{i=1}^{k+1} \frac{i}{(i+1)!} = \frac{((k+1)+1)!-1}{((k+1)+1)!}$$

$$\sum_{i=1}^{k+1} \frac{i}{(i+1)!} = \frac{(k+2)!-1}{(k+2)!} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{i}{(i+1)!} &= \sum_{i=1}^{k} \frac{i}{(i+1)!} + \frac{k+1}{(k+2)!} \\
&= \frac{(k+1)!-1}{(k+1)!} + \frac{k+1}{(k+2)!} && \text{by inductive hypothesis} \\
&= \frac{k+2}{k+2} \cdot \frac{(k+1)!-1}{(k+1)!} + \frac{k+1}{(k+2)!} \\
&= \frac{(k+2)!-(k+2)}{(k+2)!} + \frac{k+1}{(k+2)!} \\
&= \frac{(k+2)!-1}{(k+2)!}
\end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Problem 6

For each nonnegative integer n, let $P(n)$ be the property

$$5^n - 1 \text{ is divisible by } 4.$$

(a) Write $P(0)$. Is $P(0)$ true.

(b) Write $P(k)$.

(c) Write $P(k+1)$.

(d) In a proof by mathematical induction that this divisibility property holds for all integers $n \geq 0$, what must be shown in the inductive step?

Solution

(a) $P(0)$ is the expression $4 \mid (5^0 - 1)$. $P(0)$ is true because $5^0 - 1 = 0$ and $0 = 4 \cdot 0$.

(b) $P(k)$ is the expression $4 \mid (5^k - 1)$.

(c) $P(k+1)$ is the expression $4 \mid (5^{k+1} - 1)$.

(d) What must be shown in the inductive step is that if k is any integer with $k \geq 0$ then $4 \mid (5^k - 1) \implies 4 \mid (5^{k+1} - 1)$.

Problem 7

For each positive integer n, let $P(n)$ be the property

$$2^n < (n+1)!$$

(a) Write $P(2)$. Is $P(2)$ true.

(b) Write $P(k)$.

(c) Write $P(k+1)$.

(d) In a proof by mathematical induction that this inequality holds for all integers $n \geq 2$, what must be shown in the inductive step?

Solution

(a) $P(2)$ is the inequality $2^2 < (2+1)!$. $P(2)$ is true because $2^2 = 4$ and $(2+1)! = 6$ and $4 < 6$.

(b) $P(k)$ is the expression $2^k < (k+1)!$.

(c) $P(k+1)$ is the expression $2^{k+1} < ((k+1)+1)!$.

(d) What must be shown in the inductive step is that if k is any integer with $k \geq 2$ then $2^k < (k+1)! \implies 2^{k+1} < ((k+1)+1)!$.

Prove each statement in 8-23 by mathematical induction.

Problem 8

Theorem. $\forall n \in \mathbb{Z}^{nonneg},\ 4 \mid (5^n - 1)$.

Proof. Let the property $P(n)$ be the expression

$$4 \mid (5^n - 1) \qquad \leftarrow P(n)$$

Show that $P(0)$ is true:

$$5^0 - 1 = 1 - 1 = 0 \quad \text{and} \quad 0 = 4 \cdot 0$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$4 \mid (5^k - 1) \qquad \leftarrow P(k) \text{ IH}$$

By definition of divisibility this means that $5^k - 1 = 4r$ for some integer r. We must show that

$$4 \mid (5^{k+1} - 1) \qquad \leftarrow P(k+1)$$

But the right-hand side of $P(k+1)$ is

$$\begin{aligned} 5^{k+1} - 1 &= 5 \cdot 5^k - 1 \\ &= (4+1) \cdot 5^k - 1 \\ &= 4 \cdot 5^k + (5^k - 1) \\ &= 4 \cdot 5^k + 4r && \text{by inductive hypothesis} \\ &= 4(5^k + r) \end{aligned}$$

which implies that $4 \mid (5^{k+1} - 1)$. $\square$

Problem 9

Theorem. $\forall n \in \mathbb{Z}^{nonneg},\ 6 \mid (7^n - 1)$.

Proof. Let the property $P(n)$ be the expression

$$6 \mid (7^n - 1) \qquad\qquad \leftarrow P(n)$$

Show that $P(0)$ is true:

$$7^0 - 1 = 1 - 1 = 0 \quad \text{and} \quad 0 = 6 \cdot 0$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$6 \mid (7^k - 1) \qquad\qquad \leftarrow P(k) \text{ IH}$$

By definition of divisibility this means that $7^k - 1 = 6r$ for some integer r. We must show that

$$6 \mid (7^{k+1} - 1) \qquad\qquad \leftarrow P(k+1)$$

But the right-hand side of $P(k+1)$ is

$$\begin{aligned} 7^{k+1} - 1 &= 7 \cdot 7^k - 1 \\ &= (6+1) \cdot 7^k - 1 \\ &= 6 \cdot 7^k + (7^k - 1) \\ &= 6 \cdot 7^k + 6r && \text{by inductive hypothesis} \\ &= 6(7^k + r) \end{aligned}$$

which implies that $6 \mid (7^{k+1} - 1)$. $\square$

Problem 10

Theorem. $\forall n \in \mathbb{Z}^{nonneg},\ 3 \mid (n^3 - 7n + 3)$.

Proof. Let the property $P(n)$ be the expression

$$3 \mid (n^3 - 7n + 3) \qquad\qquad \leftarrow P(n)$$

Show that $P(0)$ is true:

$$0^3 - 7 \cdot 0 + 3 = 3 \quad \text{and} \quad 3 = 3 \cdot 1$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$3 \mid (k^3 - 7k + 3) \qquad\qquad \leftarrow P(k) \text{ IH}$$

It follows from the definition of divisibility that $k^3-7k+3 = 3r$ for some integer r. We must show that

$$3 \mid ((k+1)^3 - 7(k+1) + 3) \qquad \leftarrow P(k+1)$$

But the right-hand side of $P(k+1)$ is

$$\begin{aligned}
(k+1)^3 - 7(k+1) + 3 &= (k^3 + 3k^2 + 3k + 1) + (-7k - 7) + 3 \\
&= (k^3 - 7k + 3) + (3k + 1 - 7) \\
&= 3r + (3k - 6) && \text{by inductive hypothesis} \\
&= 3(r + k - 2)
\end{aligned}$$

which implies that $3 \mid ((k+1)^3 - 7(k+1) + 3)$. □

Problem 11

Theorem. $\forall n \in \mathbb{Z}^{nonneg}$, $8 \mid (3^{2n} - 1)$.

Proof. Let the property $P(n)$ be the expression

$$8 \mid (3^{2n} - 1) \qquad \leftarrow P(n)$$

Show that $P(0)$ is true:

$$3^{2\cdot 0} - 1 = 3^0 - 1 = 1 - 1 = 0 \quad \text{and} \quad 0 = 8 \cdot 0$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$8 \mid (3^{2k} - 1) \qquad \leftarrow P(k) \text{ IH}$$

By definition of divisibility this means that $3^{2k} - 1 = 8r$ for some integer r. We must show that

$$\begin{aligned}
&8 \mid (3^{2(k+1)} - 1) \\
&8 \mid (3^2 \cdot 3^{2k} - 1) && \leftarrow P(k+1)
\end{aligned}$$

But the right-hand side of $P(k+1)$ is

$$\begin{aligned}
3^2 \cdot 3^{2k} - 1 &= 9 \cdot 3^{2k} - 1 \\
&= (8+1) \cdot 3^{2k} - 1 \\
&= 8 \cdot 3^{2k} + 1 \cdot 3^{2k} - 1 \\
&= 8 \cdot 3^{2k} + (3^{2k} - 1) \\
&= 8 \cdot 3^{2k} + 8r && \text{by inductive hypothesis} \\
&= 8(3^{2k} + r)
\end{aligned}$$

which implies that $8 \mid (3^2 \cdot 3^{2k} - 1)$. □

Problem 12

Theorem. $\forall n \in \mathbb{Z}^{nonneg},\ 5 \mid (7^n - 2^n)$.

Proof. Let the property $P(n)$ be the expression

$$5 \mid (7^n - 2^n) \qquad \leftarrow P(n)$$

Show that $P(0)$ is true:

$$7^0 - 2^0 = 1 - 1 = 0 \quad \text{and} \quad 0 = 5 \cdot 0$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$5 \mid (7^k - 2^k) \qquad \leftarrow P(k) \text{ IH}$$

By definition of divisibility this means that $7^k - 2^k = 5r$ for some integer r. We must show that

$$5 \mid (7^{k+1} - 2^{k+1}) \qquad \leftarrow P(k+1)$$

But the right-hand side of $P(k+1)$ is

$$\begin{aligned}
7^{k+1} - 2^{k+1} &= 7 \cdot 7^k - 2 \cdot 2^k \\
&= (5 + 1 + 1) \cdot 7^k - (1 + 1) \cdot 2^k \\
&= 5 \cdot 7^k + 7^k + 7^k - 2^k - 2^k \\
&= 5 \cdot 7^k + (7^k - 2^k) + (7^k - 2^k) \\
&= 5 \cdot 7^k + 5r + 5r && \text{by inductive hypothesis} \\
&= 5(7^k + r + r)
\end{aligned}$$

which implies that $5 \mid (7^{k+1} - 2^{k+1})$. $\square$

Problem 13

Theorem. *For any integer $n \geq 0$, $x^n - y^n$ is divisible by $x - y$, where x and y are any integers with $x \neq y$.*

Proof. Let the property $P(n)$ be the expression

$$(x - y) \mid (x^n - y^n) \qquad \leftarrow P(n)$$

Show that $P(0)$ is true:

$$x^0 - y^0 = 1 - 1 = 0 \quad \text{and} \quad 0 = (x - y) \cdot 0$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$(x - y) \mid (x^k - y^k) \qquad \leftarrow P(k) \text{ IH}$$

By definition of divisibility this means that $x^k - y^k = (x - y) \cdot r$ for some integer r. We must show that

$$(x - y) \mid (x^{k+1} - y^{k+1}) \qquad \leftarrow P(k+1)$$

But the right-hand side of $P(k + 1)$ is

$$\begin{aligned} x^{k+1} - y^{k+1} &= x \cdot x^k - y \cdot y^k \\ &= x \cdot x^k - x \cdot y^k + x \cdot y^k - y \cdot y^k \\ &= x(x^k - y^k) + y^k(x - y) \\ &= x((x - y) \cdot r) + y^k(x - y) && \text{by inductive hypothesis} \\ &= (x - y)(xr + yk) \end{aligned}$$

which implies that $(x - y) \mid (x^{k+1} - y^{k+1})$ □

Problem 14

Theorem. $\forall n \in \mathbb{Z}^{nonneg},\ 6 \mid (n^3 - n)$.

Proof. Let the Property $P(n)$ be the expression

$$6 \mid (n^3 - n) \qquad \leftarrow P(n)$$

Show that $P(0)$ is true: Let k be any integer with $k \geq 0$ and suppose that

$$6 \mid (k^3 - k) \qquad \leftarrow P(k) \text{ IH}$$

By definition of divisibility this means that $k^3 - k = 6r$ for some integer r. We must show that

$$6 \mid ((k + 1)^3 - (k + 1)) \qquad \leftarrow P(k+1)$$

But the right-hand side of $P(k + 1)$ is

$$\begin{aligned} (k + 1)^3 - (k + 1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= (k^3 - k) + 3k(k + 1) \\ &= 6r + 3k(k + 1) && \text{by inductive hypothesis} \end{aligned}$$

Since k and $k + 1$ are two consecutive integers one of them must be even and the other must be odd by the parity of consecutive integers theorem (4.4.2). It now follows from problem 4.2.46 that $k(k + 1)$ must be an even integer. Thus $k(k + 1) = 2n$ for some integer n and so by substitution

$$\begin{aligned} (k + 1)^3 - (k + 1) &= 6r + 3(2n) \\ &= 6r + 6n \\ &= 6(r + n) \end{aligned}$$

which implies that $6 \mid ((k + 1)^3 - (k + 1))$ □

Problem 15

Theorem. $\forall n \in \mathbb{Z}^{nonneg},\ 6 \mid n(n^2+5)$.

Proof. Let the property $P(n)$ be the equation

$$6 \mid n(n^2+5) \qquad \leftarrow P(n)$$

Show that $P(0)$ is true:

$$0(0^2+5) = 0 \quad \text{and} \quad 0 = 6 \cdot 0$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$6 \mid k(k^2+5) \qquad \leftarrow P(k) \text{ IH}$$

By definition of divisibility this means that $k(k^2+5) = 6r$ for some integer r. We must show that

$$6 \mid (k+1)((k+1)^2+5) \qquad \leftarrow P(k+1)$$

But the right-hand side of $P(k+1)$ is

$$\begin{aligned}
(k+1)((k+1)^2+5) &= k(k^2+5) + 3k^2 + 3k + 6 \\
&= 6r + 3k^2 + 3k + 6 && \text{by inductive hypothesis} \\
&= 6r + 6 + 3k^2 + 3k \\
&= 6r + 6 + 3k(k+1) \\
&= 6r + 6 + 3(2n) && \text{the product of any 2 consecutive integers is even} \\
&= 6(r + 1 + n)
\end{aligned}$$

which implies that $6 \mid (k+1)((k+1)^2+5)$. □

Problem 16

Theorem. *For all integer $n \geq 2$, $2^n < (n+1)!$.*

Proof. Let the property $P(n)$ be the inequality

$$2^n < (n+1)! \qquad \leftarrow P(n)$$

Show that $P(2)$ is true:

$$2^2 = 4 \quad \text{and} \quad (2+1)! = 3! = 6$$

Show that for all integers $k \geq 2$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 2$ and suppose that

$$2^k < (k+1)! \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$2^{k+1} < ((k+1)+1)!$$
$$2^{k+1} < (k+2)! \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
2^{k+1} &= 2 \cdot 2^k \\
&< 2 \cdot (k+1)! && \text{by inductive hypothesis} \\
&< (k+2)(k+1)! && k \geq 2,\ k+2 > 2 \\
&= (k+2)!
\end{aligned}$$

Thus $2^{k+1} < (k+2)!$. ☐

Problem 17

Theorem. $\forall n \in \mathbb{Z}^{nonneg},\ 1 + 3n \leq 4^n$.

Proof. Let the property $P(n)$ be the inequality

$$1 + 3n \leq 4^n \qquad \leftarrow P(n)$$

Show that $P(0)$ is true:

$$1 + 3 \cdot 0 = 1 \quad \text{and} \quad 4^0 = 1$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$1 + 3k \leq 4^k \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$1 + 3(k+1) \leq 4^{k+1} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
1 + 3(k+1) &= 1 + 3k + 3 \\
&= 3 + (1 + 3k) \\
&\leq 3 + 4^k && \text{by inductive hypothesis} \\
&\leq 3 \cdot 4^k + 4^k && 3 \leq 3 \cdot 4^k \text{ if } k \geq 0 \\
&= 4 \cdot 4^k \\
&= 4^{k+1}
\end{aligned}$$

Thus $1 + 3(k+1) \leq 4^{k+1}$. ☐

Problem 18

Theorem. $5^n + 9 < 6^n$, *for all integers* $n \geq 2$.

Proof. Let the property $P(n)$ be the inequality

$$5^n + 9 < 6^n \qquad \leftarrow P(n)$$

Show that $P(2)$ is true:

$$5^2 + 9 = 25 + 9 = 34 \quad \text{and} \quad 6^2 = 36$$

Show that for all integers $k \geq 2$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 2$ and suppose that

$$5^k + 9 < 6^k \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$5^{k+1} + 9 < 6^{k+1} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} 5^{k+1} + 9 &= 5 \cdot 5^k + 9 \\ &= (4+1) \cdot 5^k + 9 \\ &= 4 \cdot 5^k + 5^k + 9 \\ 5^{k+1} + 9 &< 4 \cdot 5^k + 6^k \quad (1) \qquad \text{by inductive hypothesis} \end{aligned}$$

Also from the inductive hypothesis we have that

$$\begin{aligned} 5^k + 9 &< 6^k \\ 5^k &< 6^k - 9 \\ 4 \cdot 5^k &< 4 \cdot 6^k - 36 \\ &< 4 \cdot 6^k \\ 4 \cdot 5^k &< 5 \cdot 6^k \quad (2) \end{aligned}$$

Now from inequality (1) and (2) we have that

$$\begin{aligned} 5^{k+1} + 9 &< 4 \cdot 5^k + 6^k \\ &< 5 \cdot 6^k + 6^k \\ &= 6 \cdot 6^k \\ &= 6^{k+1} \end{aligned}$$

Thus $5^{k+1} + 9 < 6^{k+1}$. $\square$

Problem 19

Theorem. *$n^2 < 2^n$, for all integers $n \geq 5$.*

Proof. Let the property $P(n)$ be the equation

$$n^2 < 2^n \qquad \leftarrow P(n)$$

Show that $P(5)$ is true:

$$5^2 = 25 \quad \text{and} \quad 2^5 = 32$$

Show that for all integers $k \geq 5$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 5$ and suppose that

$$k^2 < 2^k \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$(k+1)^2 < 2^{k+1} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
(k+1)^2 &= k^2 + 2k + 1 \\
&< 2^k + 2k + 1 && \text{by inductive hypothesis} \\
&< 2^k + 2^k && \text{by proposition 5.3.2} \\
&= 2 \cdot 2^k \\
&= 2^{k+1}
\end{aligned}$$

Thus $(k+1)^2 < 2^{k+1}$. □

Problem 20

Theorem. $\forall n \in \mathbb{Z}^{nonneg}$, $2^n < (n+2)!$.

Proof. Let the property $P(n)$ be the equation

$$2^n < (n+2)! \qquad \leftarrow P(n)$$

Show that $P(0)$ is true:

$$2^0 = 1 \quad \text{and} \quad (0+2)! = 2! = 2$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$2^k < (k+2)! \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$\begin{aligned} 2^{k+1} &< ((k+1)+2)! \\ 2^{k+1} &< (k+3)! && \leftarrow P(k+1) \end{aligned}$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &< 2 \cdot (k+2)! && \text{by inductive hypothesis} \\ &< (k+3)(k+2)! && k \geq 0 \\ &= (k+3)! \end{aligned}$$

Thus $2^{k+1} < (k+3)!$. □

Problem 21

Theorem. *For all integers $n \geq 2$, $\sqrt{n} < \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$.*

Proof. Let the property $P(n)$ be the inequality

$$\sqrt{n} < \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \qquad \leftarrow P(n)$$

Show that $P(2)$ is true:

$$\sqrt{2} \approx 1.4 \quad \text{and} \quad 1 + \frac{1}{\sqrt{2}} \approx 1.7$$

Show that for all integers $k \geq 2$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 2$ and suppose that

$$\sqrt{k} < \sum_{i=1}^{k} \frac{1}{\sqrt{i}} \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$\sqrt{k+1} < \sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} \qquad \leftarrow P(k+1)$$

But the right-hand side of $P(k+1)$ is

$$\begin{aligned} \sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} &= \sum_{i=1}^{k} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}} \\ &> \sqrt{k} + \frac{1}{\sqrt{k+1}} && \text{by inductive hypothesis} \\ &> \sqrt{k} \cdot \frac{\sqrt{k}}{\sqrt{k+1}} + \frac{1}{\sqrt{k+1}} && \sqrt{k} < \sqrt{k+1} \\ &= \frac{k+1}{\sqrt{k+1}} = \sqrt{k+1} \end{aligned}$$

□

Problem 22

Theorem. $1 + nx \leq (1+x)^n$, *for all real numbers* $x > -1$ *and integers* $n \geq 2$.

Proof. Let the property $P(n)$ be the inequality

$$1 + nx \leq (1+x)^n \qquad \leftarrow P(n)$$

Show that $P(2)$ is true:

$$\text{For } n = 2,\ 1 + nx = 1 + 2x \quad \text{and} \quad (1+x)^2 = x^2 + 2x + 1$$

Show that for all integers $k \geq 2$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 2$ and suppose that

$$1 + kx \leq (1+x)^k \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$1 + (k+1)x \leq (1+x)^{k+1} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
1 + (k+1)x &= x + (1 + kx) \\
&\leq x + (1+x)^k && \text{by inductive hypothesis} \\
&\leq x(1+x)^k + (1+x)^k && (1+x)^k \geq 1 \text{ when } x \in \mathbb{R}^{nonneg} \\
& && (1+x)^k < 1 \text{ when } x \in \mathbb{R}^{-} \\
&= (1+x)^k(x+1) \\
&= (1+x)^{k+1}
\end{aligned}$$

□

Problem 23

Theorem. *For all integers* $n \geq 2$, $n^3 > 2n + 1$.

Proof. Let the property $P(n)$ be the inequality

$$n^3 > 2n + 1 \qquad \leftarrow P(n)$$

Show that $P(2)$ is true:

$$2^3 = 8 \quad \text{and} \quad 2 \cdot 2 + 1 = 5$$

Show that for all integers $k \geq 2$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 2$ and suppose that

$$k^3 > 2k + 1 \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$(k+1)^3 > 2(k+1) + 1 \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
(k+1)^3 &= k^3 + 3k^2 + 3k + 1 \\
&> 2k + 1 + 3k^2 + 3k + 1 && \text{by inductive hypothesis} \\
&= 3k^2 + 3k + 2k + 2 \\
&> 2k + 3 && 3k^2 + 3k > 1 \text{ if } k \geq 2 \\
&= 2(k+1) + 1
\end{aligned}$$

□

Theorem. *For all integers $n \geq 4$, $n! > n^2$.*

Proof. Let the property $P(n)$ be the inequality

$$n! > n^2 \qquad \leftarrow P(n)$$

Show that $P(4)$ is true:

$$4! = 24 \quad \text{and} \quad 4^2 = 16$$

Show that for all integers $k \geq 4$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 4$ and suppose that

$$k! > k^2 \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$(k+1)! > (k+1)^2 \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
(k+1)! &= (k+1)(k!) \\
&> (k+1)(k^2) \quad (1) && \text{by inductive hypothesis}
\end{aligned}$$

We need to show that $(k^2) > (k+1)$.

$$\begin{aligned}
k + 1 &> k \\
(k-1)(k+1) &> k && k - 1 > 1 \text{ if } k \geq 4 \\
k^2 - 1 &> k \\
k^2 &> k + 1
\end{aligned}$$

Now from inequality (1) we have that $(k+1)! > (k+1)^2$. □

Problem 24

A sequence $a_1, a_2, a_3, \ldots$ is defined by letting $a_1 = 3$ and $a_k = 7a_{k-1}$ for all integers $k \geq 2$. Show that $a_n = 3 \cdot 7^{n-1}$ for all integers $n \geq 1$.

Theorem. $\forall n \in \mathbb{Z}^+$, $a_n = 3 \cdot 7^{n-1}$.

Proof. Let the property $P(n)$ be the equation

$$a_n = 3 \cdot 7^{n-1} \qquad \leftarrow P(n)$$

Show that $P(1)$ is true:

$$a_1 = 3 \quad \text{and} \quad 3 \cdot 7^{1-1} = 3 \cdot 7^0 = 3 \cdot 1 = 3$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$a_k = 3 \cdot 7^{k-1} \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$a_{k+1} = 3 \cdot 7^{(k+1)-1}$$
$$a_{k+1} = 3 \cdot 7^k \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} a_{k+1} &= 7a_{(k+1)-1} && \text{by definition of } a_1, a_2, a_3, \ldots \\ &= 7a_k \\ &= 7 \cdot (3 \cdot 7^{k-1}) && \text{by inductive hypothesis} \\ &= 3 \cdot (7 \cdot 7^{k-1}) \\ &= 3 \cdot 7^k \end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Problem 25

A sequence $b_0, b_1, b_2, \ldots$ is defined by letting $b_0 = 5$ and $b_k = 4 + b_{k-1}$ for all integers $k \geq 1$. Show that $b_n > 4n$ for all integers $n \geq 0$.

Theorem. *$\forall n \in \mathbb{Z}^{nonneg}$, $b_n > 4n$.*

Proof. Let the property $P(n)$ be the inequality

$$b_n > 4n \qquad \leftarrow P(n)$$

Show that $P(0)$ is true:

$$b_0 = 5 \quad \text{and} \quad 4 \cdot 0 = 0$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$b_k > 4k \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$b_{k+1} > 4(k+1) \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
b_{k+1} &= 4 + b_{(k+1)-1} && \text{by definition of } b_0, b_1, b_2, \ldots \\
&= 4 + b_k \\
&> 4 + 4k && \text{by inductive hypothesis} \\
&= 4(k+1)
\end{aligned}$$

which is the right-hand side of $P(k+1)$. $\square$

Problem 26

A sequence $c_0, c_1, c_2, \ldots$ is defined by letting $c_0 = 3$ and $c_k = (c_{k-1})^2$ for all integers $k \geq 1$. Show that $c_n = 3^{2^n}$ for all integers $n \geq 0$.

Theorem. $\forall n \in \mathbb{Z}^{nonneg}$, $c_n = 3^{2^n}$.

Proof. Let the property $P(n)$ be the equation

$$c_n = 3^{2^n} \qquad \leftarrow P(n)$$

Show that $P(0)$ is true:

$$c_0 = 3 \quad \text{and} \quad 3^{2^0} = 3^1 = 3$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$c_k = 3^{2^k} \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$c_{k+1} = 3^{2^{k+1}} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
c_{k+1} &= (c_{(k+1)-1})^2 && \text{by definition of } c_0, c_1, c_2, \ldots \\
&= (c_k)^2 \\
&= (3^{2^k})^2 && \text{by inductive hypothesis} \\
&= 3^{2 \cdot 2^k} \\
&= 3^{2^{k+1}}
\end{aligned}$$

which is the right-hand side of $P(k+1)$. $\square$

Problem 27

A sequence $d_1, d_2, d_3, ...$ is defined by letting $d_1 = 2$ and $d_k = \frac{d_{k-1}}{k}$ for all integers $k \geq 2$. Show that for all integers $n \geq 1$, $d_n = \frac{2}{n!}$.

Theorem. $\forall n \in \mathbb{Z}^+$, $d_n = \frac{2}{n!}$.

Proof. Let the property $P(n)$ be the equation

$$d_n = \frac{2}{n!} \qquad \leftarrow P(n)$$

Show that $P(1)$ is true:

$$d_1 = 2 \quad \text{and} \quad \frac{2}{1!} = \frac{2}{1} = 2$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$d_k = \frac{2}{k!} \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$d_{k+1} = \frac{2}{(k+1)!} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} d_{k+1} &= \frac{d_{(k+1)-1}}{k+1} && \text{by definition of } d_1, d_2, d_3, ... \\ &= \frac{d_k}{k+1} \\ &= \frac{2}{(k+1)(k!)} && \text{by inductive hypothesis} \\ &= \frac{2}{(k+1)!} \end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Problem 28

Prove that for all integers $n \geq 1$,

$$\begin{aligned} \frac{1}{3} &= \frac{1+3}{5+7} + \frac{1+3+5}{7+9+11} = ... \\ &= \frac{1+3+...+(2n-1)}{(2n+1)+...+(4n-1)} \end{aligned}$$

Proof. Let the property $P(n)$ be the equation

$$\frac{\sum_{i=1}^{n}(2i-1)}{\sum_{i=n}^{2n-1}(2i+1)} = \frac{1}{3} \qquad \leftarrow P(n)$$

Show that $P(1)$ is true:

$$\sum_{i=1}^{1}(2i-1) = 1 \quad \text{and} \quad \sum_{i=1}^{1}(2i+1) = 3$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$\frac{\sum_{i=1}^{k}(2i-1)}{\sum_{i=k}^{2k-1}(2i+1)} = \frac{1}{3} \qquad \leftarrow P(k) \text{ IH}$$

It follows from closure under multiplication and addition that $\sum_{i=1}^{k}(2i-1)$ is an integer. Let this integer be t. It now follows from closure under multiplication and addition and the inductive hypothesis that $\sum_{i=k}^{2k-1}(2i+1) = 3t$. We must show that

$$\frac{\sum_{i=1}^{k+1}(2i-1)}{\sum_{i=k+1}^{2(k+1)-1}(2i+1)} = \frac{1}{3} \qquad \leftarrow P(k+1)$$

But the numerator of the left-hand side of $P(k+1)$ is

$$\begin{aligned}\sum_{i=1}^{k+1}(2i-1) &= \sum_{i=1}^{k}(2i-1) + 2k + 1 \\ &= t + 2k + 1 && \text{by inductive hypothesis}\end{aligned}$$

And the denominator of the left-hand side of $P(k+1)$ is

$$\begin{aligned}\sum_{i=k+1}^{2(k+1)-1}(2i+1) &= \sum_{i=k}^{2k-1}(2i-1) - (2k+1) + (4k+3) + (4k+1) \\ &= 3t + 6k + 3 && \text{by inductive hypothesis}\end{aligned}$$

Dividing the numerator by the denominator gives

$$\frac{t+2k+1}{3t+6k+3} = \frac{t+2k+1}{3(t+2k+1)} = \frac{1}{3}$$

which is the right-hand side of $P(k+1)$. $\square$

Problem 29

As each of a group of businesspeople arrives at a meeting, each shakes hands with all the other people present. Use mathematical induction to show that if n people come to the meeting then $[n(n-1)]/2$ handshakes occur.

Proof. Let the property $P(n)$ be the equation

$$\text{number of handshakes with } n \text{ people} = \frac{n(n-1)}{2} \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: If one businessman shows up then 0 handshakes occur and $[1(1-1)]/2 = [1(0)]/2 = 0/2 = 0$. **Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$:** Let k be any integer with $k \geq 1$ and suppose that

$$\text{number of handshakes with } k \text{ people} = \frac{k(k-1)}{2} \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$\text{number of handshakes with } k+1 \text{ people} = \frac{(k+1)((k+1)-1)}{2} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
&\text{number of handshakes with } k+1 \text{ people} = \\
&\qquad \text{number of handshakes with } k \text{ people} + k \text{ handshakes} \\
&\qquad = \frac{k(k-1)}{2} + k \qquad \text{by inductive hypothesis} \\
&\qquad = \frac{k(k-1)}{2} + \frac{2k}{2} \\
&\qquad = \frac{k^2 - k + 2k}{2} \\
&\qquad = \frac{k^2 + k}{2} \\
&\qquad = \frac{(k+1)((k+1)-1)}{2}
\end{aligned}$$

which is the right-hand side of $P(k+1)$. $\square$

In order for a proof by mathematical induction to be valid, the basis statement must be true for $n = a$ and the argument of the inductive step must be correct for every integer $k \geq a$. In 30 and 31 find the mistakes in the "proofs" by mathematical induction.

Problem 30

"Theorem:" For any integer $n \geq 1$, all the numbers in a set of n numbers are equal to each other.
"Proof (by mathematical induction:" It is obviously true that all numbers in a set consisting of just one number are equal to each other, so the basis step is true. For the inductive step, let $A = \{a_1, a_2, ..., a_k, a_{k+1}\}$ be any set of $k+1$ numbers. Form two subsets each of size k :

$$B = \{a_1, a_2, a_3, ..., a_k\} \quad \text{and}$$
$$C = \{a_1, a_3, a_4, ..., a_{k+1}\}.$$

(B consists of all the numbers in A except a_{k+1}, and C consists of all the numbers in A except a_2.) By inductive hypothesis, all the numbers in B equal a_1 and all the numbers in C equal a_1 (since both sets have only k numbers). But every number in A is in B or C, so all the numbers in A equal a_1; hence all are equal to each other.

Solution

Consider that case that $k = 1$. Then $A = \{a_1, a_2\}$ and $B = \{a_1\}$ and $C = \{a_1\}$. Thus there is an element in A namely a_2 which is not in B or C. This means that $P(1) \not\Longrightarrow P(2)$. The inductive part of the proof would work for all other $k > 1$ but then the basis step could not be proved.

Problem 31

"Theorem:" For all integers $n \geq 1$, $3^n - 2$ is even.

"Proof (by mathematical induction:" Suppose the theorem is true for an integer k, where $k \geq 1$. That is, suppose that $3^k - 2$ is even. We must show that $3^{k+1} - 2$ is even. But

$$\begin{aligned} 3^{k+1} - 2 &= 3^k \cdot 3 - 2 = 3^k(1+2) - 2 \\ &= (3^k - 2) + 3^k \cdot 2 \end{aligned}$$

Now $3^k - 2$ is even by the inductive hypothesis and $3^k \cdot 2$ is even by inspection. Hence the sum of the two quantities is even (by Theorem 4.1.1). It follows that $3^{k+1} - 2$ is even, which is what we needed to show.

Problem 32

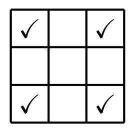
The basis step was never shown to be true for $n = 1$. And when 1 is plugged in for n we obtain $3^1 - 2 = 1$ which is not even. Thus the basis step is false and the inductive hypothesis can not be used in the proof.

Problem 32

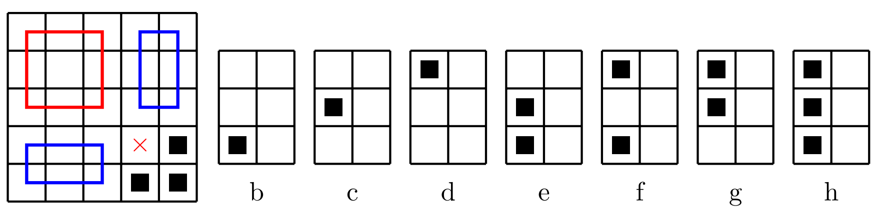
Some 5×5 checkerboards with one square removed can be completely covered by L-shaped trominoes, whereas other 5×5 checkerboards cannot. Find examples of both kinds of checkerboards. Justify your answer.

Solution

Consider the problem of trying to cover a 3×3 checkerboard with L-shaped trominoes. Observe that each of the squares below with a check in them cannot be covered by the same tromino.



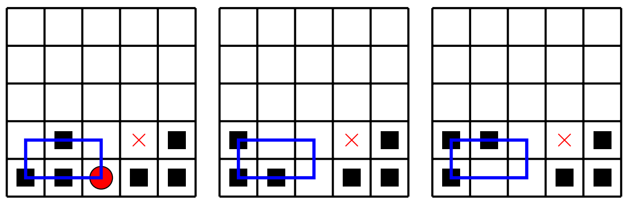
Therefore we need at least 4 trominoes to cover the entire board. But each tromino has 3 squares and $3 \cdot 4 = 12$ and the board only has 9 squares. This means that we cannot perfectly cover a 3×3 grid with L-shaped trominoes (some squares will hang off the edge). Now consider the following diagrams below. If we remove the square marked with a red $\times$ in figure a then we will not



b c d e f g h

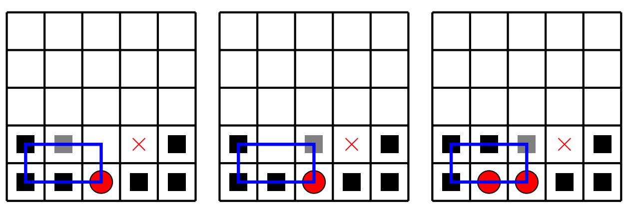
a

be able to completely cover the 5×5 board with L-shaped trominoes. Clearly the bottom right square of figure a can only be covered if a trominoe is placed exactly as shown in black in figure a. Now the red portion of figure a is a 3×3 checkerboard. We know from above that if this region is totally covered then there will be 3 squared of trominoes hanging off of the red colored region. Assuming that they all fell inside the bounds of the 5×5 checkerboard, one of the blue regions must have either 1, 2, or 3 squares already filled by the time the red $\times$ square is removed, and the lower-right and red circled region are filled. The squares that fall into the blue regions will all be on one side of the blue regions since the trominoes are 2 squares long and at least one square must be in the red region. Since the red circled region is already filled no trominoe can exist in both of the blue regions as it would either overlap with the red $\times$ or the red circled region. Therefore the blue regions must be covered independently of each other. Now if it happens to be the case that one of the blue regions has 1 square already filled then the region cannot be filled as there are 5 squares to be filled but trominoes can only fill multiples of 3. The same reasoning explains why e, f, and g can not be filled. Figure h has a multiple of 3 squares remaining to be covered but these clearly cannot be covered by an L-shaped trominoe. Thus if the red region is covered then not both of the blue regions can be covered. Now we must show that failing to cover the two blue regions independently of any other square will lead to a failure to cover the entire board. Without loss of generality consider the lower-left blue region. The



i j k

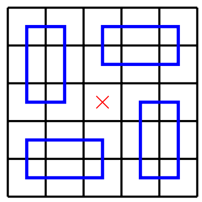
only way that the lower-left square in the 5×5 checkerboard can be covered is if a trominoe is placed in one of the configurations in figures i, j, or k. However if it it is covered as seen in figure i then a region will be created which cannot be covered as marked by the red circle in figure i. Thus if you want to cover the lower left square and the entire blue region you must place a tromino as seen in figure j or k. Now consider the diagrams below. Figure l and m show that if



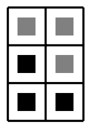
l m n

you choose to cover the lower-left square of the 5×5 checkerboard as is done in figure j and you then cover either of the squares which are colored gray and you do not cover the remainder of the blue circled region then you will create a region which cannot be covered. Figure n shows that if you choose to cover the lower-left square of the 5×5 checkerboard as is done in figure k and you then cover the square which is colored gray and you do not cover the remainder of the blue circled region then you will create a region which cannot be covered. This argument applies to both blue circled regions and shows that failure to cover both of the blue circled regions independently of all other regions on the 5×5 checkerboard will result in a region that cannot be covered. We have now shown that we will be unable to fully cover the board no matter which action we take. If we choose to fully cover the red circled region as shown in figure a then at least one of the blue regions cannot be covered. If we instead choose to cover the blue regions first it must be done in a way such that that they are independently covered from all other regions in the board. This now leaves only the 3×3 red-colored region to be covered which we showed to be impossible.

Consider the diagrams below which give an example of a square that can be removed such that the 5×5 checkerboard can still be fully covered. Figure o



o



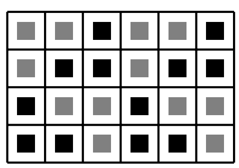
p

shows that if the center square in a 5×5 checkerboard is removed then the remaining squares can be partitioned into 4 2×3 checkerboards. Figure p shows that a 2×3 checkerboard can be independently covered by 2 L-shaped trominoes. It follows that then that the entire board can be covered.

Problem 33

Consider a 4×6 checkerboard. Draw a covering of the board by L-shaped trominoes.

Solution



Problem 34

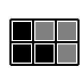
(a) Use mathematical induction to prove that any checkerboard with dimensions $2 \times 3n$ can be completely covered with L-shaped trominoes for any integer $n \geq 1$.

(b) Use the results of part (a) to prove by mathematical induction that for all integers $n \geq 1$ and $m \geq 1$, any checkerboard with dimensions $2m \times 3n$ can be completely covered with L-shaped trominoes.

Solution

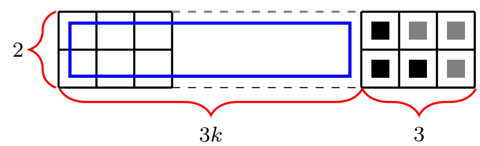
(a) **Theorem.** Any checkerboard with dimensions $2 \times 3n$ can be completely covered with L-shaped trominoes for any integer $n \geq 1$.

Proof. Let the property $P(n)$ be the theorem to be proved.

Show that $P(1)$ is true: 

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that any checkerboard with dimensions $2 \times 3k$ can be covered with L-shaped trominoes. We must show that any checkerboard with dimensions $2 \times 3(k+1)$ can be covered with L-shaped trominoes. But a $2 \times 3(k+1)$ checkerboard is equivalent to a $2 \times 3k$ checkerboard with a 2×3 checkerboard tagged on.

$2 \times 3(k+1)$ checkerboard



It follows from the inductive hypothesis that we can cover the blue circled region in the $2 \times 3(k+1)$ checkerboard. It follows from inspection that the remaining 2×3 region can be covered. □

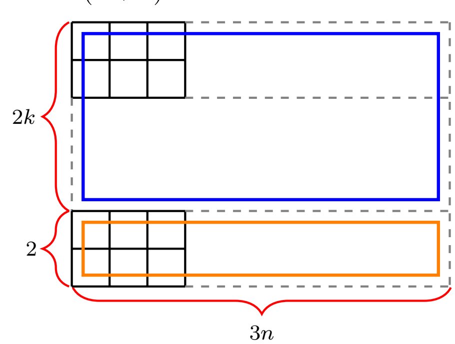
(b) **Theorem.** Any checkerboard with dimensions $2m \times 3n$ can be completely covered with L-shaped trominoes for any integers $n \geq 1$ and $m \geq 1$.

Proof. Let the property $P(m)$ be the theorem to be proved.

Show that $P(1)$ is true: If $m = 1$ then $P(m)$ is equivalent to part (a) above and so $P(1)$ is true.

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that any checkerboard with dimensions $2k \times 3n$ can be covered with L-shaped trominoes. We must show that any checkerboard with dimensions $2(k+1) \times 3n$ can be covered with L-shaped trominoes. But a $2(k+1) \times 3n$ checkerboard is equivalent to a $2k \times 3n$ checkerboard with a $2 \times 3n$ checkerboard tagged on.

$2(k+1) \times 3n$ checkerboard



It follows from the inductive hypothesis that the blue region in the $2(k+1) \times 3n$ checkerboard can be covered. It follows from part (a) that the orange region in the $2(k+1) \times 3n$ checkerboard can be covered. □

Problem 35

Let m and n be any integers that are greater than or equal to 1.

(a) Prove that a necessary condition for an $m \times n$ checkerboard to be completely coverable by L-shaped trominoes is that mn be divisible by 3.

(b) Prove that having mn be divisible by 3 is not a sufficient condition for an $m \times n$ checkerboard to be completely coverable by L-shaped trominoes.

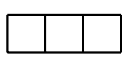
Solution

(a) **Theorem.** $\forall m, n \in \mathbb{Z}^+,\ 3 \nmid mn \implies m \times n$ checkerboard cannot be completely covered by L-shaped trominoes.

Proof. Suppose not. That is suppose that m and n are any positive integers such that $3 \nmid mn$ and an $m \times n$ checkerboard can be completely covered by L-shaped trominoes. If the checkerboard is completely covered by L-shaped trominoes then it must have an integer number of L-shaped trominoes on it. Let this integer be k. Since L-shaped trominoes have 3 squares each the board must contain $3k$ squares. However this means that $3 \mid mn$ which is a contradiction. □

(b) **Theorem.** $\forall m, n \in \mathbb{Z}^+,\ 3 \mid mn \not\Longrightarrow m \times n$ checkerboard can be completely covered by L-shaped trominoes.

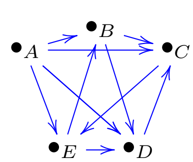
Proof. Let $m = 1$ and let $n = 3$. Then $m \geq 1$ and $n \geq 1$ and $3 \mid mn$. But the following 1×3 checkerboard cannot be fully covered by L-shaped trominoes.



□

Problem 36

In a round-robin tournament each team plays every other team exactly once. If the teams are labeled $T_1, T_2, ..., T_n$, then the outcome of such a tournament can be represented by a drawing, called a directed graph, in which the teams are represented as dots and an arrow is drawn from one dot to another if, and only if, the team represented by the first dot beats the team represented by the second dot. For example, the directed graph below shows one outcome of a round-robin tournament involving five teams $A, B, C, D,$ and E.



Use mathematical induction to show that in any round-robin tournament involving n teams, where $n \geq 2$, it is possible to label the teams $T_1, T_2, ..., T_n$ so that T_i beats T_{i+1} for all $i = 1, 2, ..., n-1$. (For instance, one such labeling in the example above is $T_1 = A$, $T_2 = B$, $T_3 = C$, $T_4 = E$, and $T_5 = D$.)

Solution

Proof. Let the property $P(n)$ be the theorem to be proved.

Show that $P(2)$ is true: If $n = 2$ then two teams play each other exactly once. Let the winner be called T_1 and the looser T_2. Then T_1, T_2 is an ordering which satisfies the conditions of the theorem and so $P(2)$ is true.

Show that for all integers $k \geq 2$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 2$ and suppose that in any round robin tournament involving k teams it is possible to label the teams $T_2, T_2, ..., T_k$ so that T_i beats T_{i+1} for all $i = 1, 2, ..., k-1$. We must show that $P(k+1)$ is true. From the $k+1$ teams that played in tournament select any one team and call it T'. From the inductive hypothesis we know that we can order the remaining k teams $T_1, T_2, ..., T_k$ such that T_i beats T_{i+1} for all $i = 1, 2, ..., k-1$. We now need to see if we can insert T' somewhere into our ordering and still maintain the conditions of the theorem.

Case 1 (T' beats T_1): In this case simply rename T' as T_1 and rename each of $T_1, T_2, ..., T_k$ as $T_2, T_3, ..., T_{k+1}$. Then we will have an ordering $T_1, T_2, ..., T_k, T_{k+1}$ that satisfies the conditions of the theorem.

Case 2 (T' looses to T_k): In this case simply rename T' as T_{k+1}. Then we will have an ordering $T_1, T_2, ..., T_k, T_{k+1}$ that satisfies the conditions of the theorem.

Case 2 (T' looses to the first m teams and beats team $m+1$ where $1 \leq m \leq k-1$): In this case Simply rename T' as T_{m+1} and rename teams $T_{m+1}, T_{m+2}, ..., T_k$ as $T_{m+2}, T_{m+3}, ..., T_{k+1}$. Then we will have an ordering $T_1, T_2, ..., T_k, T_{k+1}$ that satisfies the conditions of the theorem.

Thus $P(k+1)$ is true. □

Problem 37

On the outside rim of a circular disk the integers from 1 through 30 are painted in random order. Show that no matter what this order is, there must be three successive integers whose sum is at least 45.

Solution

Proof. Let $x_1, x_2, x_3, ..., x_{30}$ be an arbitrary ordering of the integers $1, 2, 3, ..., 30$. Now we must define every possible sum of three consecutive integers on the circular disk.

$$
\begin{gathered}
x_1 + x_2 + x_3 \\
x_2 + x_3 + x_4 \\
x_3 + x_4 + x_5 \\
\ldots.. \\
x_{28} + x_{29} + x_{30} \\
x_{29} + x_{30} + x_1 \\
x_{30} + x_1 \ \ + x_2
\end{gathered}
$$

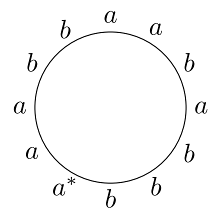
Notice that every number occurs in 3 separate sums so that if we were to sum together all of the sums we would obtain

$$
\begin{aligned}
(x_1 + x_2 + ... + x_{30}) &+ (x_1 + x_2 + ... + x_{30}) + (x_1 + x_2 + ... + x_{30}) \\
&= 3(x_1 + x_2 + ... + x_{30}) \\
&= 3\sum_{i=1}^{30} i = 3 \cdot \frac{30(31)}{2} = 1395 \quad (1)
\end{aligned}
$$

Now suppose that the theorem is false. That is suppose that for all arbitrary orderings of the integers from 1 to 30 on a disk there are not 3 consecutive integers whose sum is at least 45. Since all of the numbers are integers all of the sums of the three consecutive numbers must be integers. Therefore if none of the sums are at least 45 they must be a maximum of 44. Since there are 30 such sums the maximum value of all the sums together would be $30 \cdot 44 = 1320$. However this is a contradiction as we know from equation (1) above that the true sum of all the integers is 1395 and $1320 < 1395$. Thus our supposition that for all arbitrary orderings of the integers from 1 to 30 on a disk there are not 3 consecutive integers whose sum is at least 45 is false and the theorem is true. $\square$

Problem 38

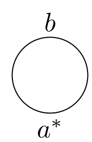
Suppose that n a's and n, b's are distributed around the outside of a circle. Use mathematical induction to prove that for all integers $n \geq 1$, given any such arrangement, it is possible to find a starting point so that if one travels around the circle in a clockwise direction, the number of a's one has passed is never less than the number of b's one has passed. For example, in the diagram shown below, one could start at the a with an asterisk.



Solution

Proof. Let the Property $P(n)$ be the theorem to be proved.

Show that $P(1)$ is true: From the diagram below we can see that if there is one a and one b around the circle then by starting just behind the a and traversing clockwise the number of b's crossed will never be more than the number of a's crossed. Thus $P(1)$ is true.



Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that if k a's and k b's are distributed around the outside of a circle there will always be a starting point such that the number of a's passed traversing in a clockwise direction is never less than the number of b's passed. We must show that $P(k+1)$ is true.

Imagine that $k+1$ a's and $k+1$ b's have placed somewhere around a circle. Traversing in a clockwise direction there must be at least one instance in which an a is immediately followed by a b. Now there are only k a's and k b's on the circle and so by the inductive hypothesis there is a location at which a clockwise traversal will never pass more b's than a's. Once this locations is found add back in the a and b. They will not violate the theorem as the a will be crossed before the b. Thus $P(k+1)$ is true. $\square$

Problem 39

For a polygon to be convex means that given any two points on or inside the polygon, the line joining the points lies entirely inside the polygon. Use mathematical induction to prove that for all integers $n \geq 3$, the angles of any $n-$sided convex polygon add up to $180(n-2)$ degrees.

Solution

Proof. Let the property $P(n)$ be the theorem to be proved.

Show that $P(3)$ is true: A 3 sided convex polygon is a triangle and all triangles are known to have interior angles that sum to $180°$. Thus $P(3)$ is true.

Show that for all integers $k \geq 3$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 3$ and suppose that the angles of any $k-$sided convex polygon add up to $180(k-2)$ degrees. We must show that $P(k+1)$ is true. That is we must show that any convex polygon with $k+1$ sides has a sum of interior angles of $180((k+1)-2)$ degrees.

Simply select any angle on a $k+1$ sided polygon and then proceed to trace the polygon in a clockwise direction and stop when you run into the second angle not including the one you started on. Draw a line from the second angle to the original angle you were on and you have a triangle. Remove the triangle from polygon and you are left with a convex polygon with k sides. By the inductive hypothesis this has interior angle sum of $180(k-2)$ degrees. The sum of the triangle's interior angles and new shapes interior angles is the sum of the original shapes interior angles. Since a triangle has interior angles of $180°$ our original shape has interior angles of

$$\begin{aligned}180(k-2)+180 &= 180k - 2\cdot 180 + 180\\ &= 180k - 180\\ &= 180(k-1)\\ &= 180((k+1)-2)\end{aligned}$$

□

Problem 40

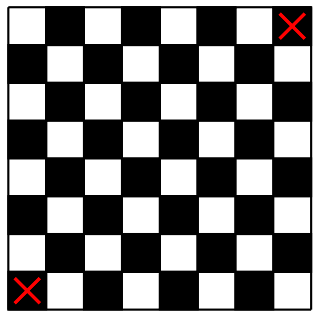
(a) Prove that in an 8×8 checkerboard with alternating black and white squares, if the squares in the top right and bottom left corners are removed the remaining board cannot be covered with dominoes.

(b) Use mathematical induction to prove that for all integers $n \geq 1$, if a $2n \times 2n$ checkerboard with alternating black and white squares has one white square and one black square removed anywhere on the board, the remaining squares can be covered with dominoes.

Solution

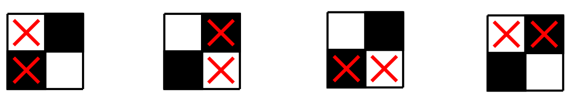
(a) *Proof.* Observe that in such a board as in the figure below no domino can cover two squares of the same color. Further observe that all diagonals on this board are the same color. Thus if we eliminate the top right and bottom left corners then we will be removing two squares of the same color. This will leave 32 white squares and 30 black squares. However since placing a domino(assuming that it is possible) always removes one black square and one white square we will eventually be left with 2 white squares which cannot be covered by a single domino. But we only have two spaces left and a single domino covers two spaces and so we cannot cover the board. □

8×8 black and white checkerboard



(b) *Proof.* Let $P(n)$ be the property that if a $2n \times 2n$ checkerboard with alternating black and white squares has one white square and one black square removed anywhere on the board, the remaining squares can be covered with dominoes.

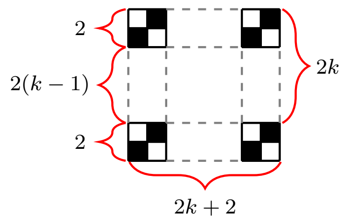
Show that $P(1)$ is true: If $n = 1$ then we have a 2×2 checkerboard. Since every black square is adjacent to every white square on this board the removal of any white square and any black square will result in a white square and black square that are adjacent and can therefore be covered by a domino. Thus $P(1)$ is true.



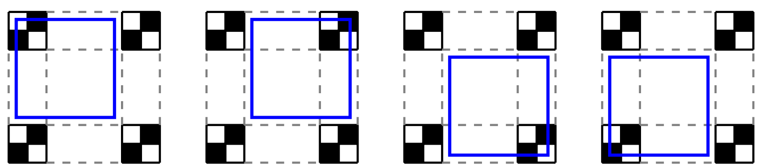
Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that if a $2k \times 2k$ checkerboard with alternating black and white squares has one white square and one black square removed anywhere on the board, the remaining squares can be covered with dominoes.

We must show that $P(k+1)$ is true. That is we must show that if one black and one white square of a $2(k+1) \times 2(k+1)$ checkerboard is removed the board can be covered with dominoes. But a $2(k+1) \times 2(k+1)$ checkerboard is the same as a $2k+2 \times 2k+2$ checkerboard. The diagram below shows such a board.

$2k+2 \times 2k+2$ black and white checkerboard

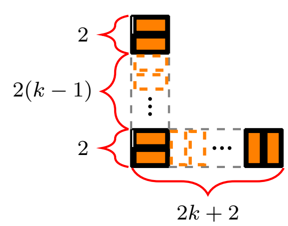


Now with the $2k+2 \times 2k+2$ board in a constant orientation we can draw four $2k \times 2k$ regions such that one of the corners of each $2k \times 2k$ region is pressed against a separate corner of the $2k+2 \times 2k+2$ board. The diagrams below illustrate these four regions If it so happens that the

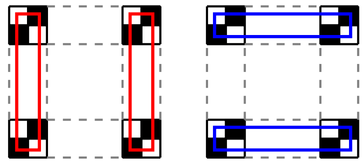


a b c d

black square and white square that are removed from the $2k+2 \times 2k+2$ checkerboard both come from one of the blue regions shown above then by the inductive hypothesis the blue region can be covered by dominoes. We now need to show that the regions that remain(the L-shaped regions) can be covered by dominoes. The L-shaped regions have different orientations but they all have the same number of squares and the same shape and so if one of them can be shown to be covered they can all be covered.



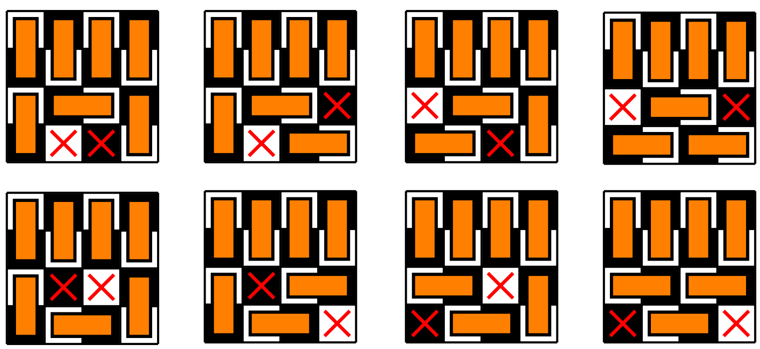
The diagram above shows that such an L can be covered with dominoes. For the remainder of this proof we will be discussing the case that the white and black square are removed such that none of the blue regions shown in a, b, c, and d contain them both.



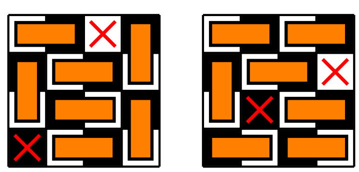
e f

In this case the white and black squares must have been removed from opposite sides of the board as in figure e and f above. Suppose that this was not the case. That is suppose that there exist two squares that are not on opposite edges (top and bottom or left and right) such that no blue circled region in figures a, b, c, and d contains them both. By inspection this is clearly a contradiction and so figures e and f are correct.

Consider the case that $k = 1$. In this case the dimensions of our checkerboard are $2(1) + 2 \times 2(1) + 2 = 4 \times 4$. Note that the interior region of the checkerboard which is dashed and has dimensions $2(k-1) \times 2(k-1)$ does not exist as $2(1-1) = 2(0) = 0$. Therefore if $k = 1$ the black and white squares must have been removed from separate 4×4 corners on the checkerboard. They could have been removed from diagonal corners or from adjacent corners. The coverings for the adjacent cases are given first.

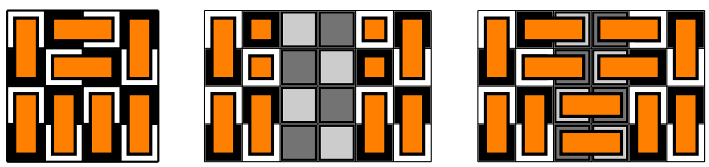


Any other removal of a red and black square from adjacent 2×2 corners on a 4×4 checkerboard can be achieved by rotating the eight boards given here. The coverings for the diagonal cases are given next.



Any other removal of a red and black square from diagonal 2×2 corners on a 4×4 checkerboard can be achieved by rotating or mirroring the two boards given here.

If the theorem is true (the inductive hypothesis works) then we have just shown that any white and any black square can be removed from a 4×4 checkerboard and the board can still be covered with dominoes. The basis step showed that this was true for a 2×2 board. We now need to show that this is true for $k \geq 2$. When $k \geq 2$ our checkerboard is at least $2(2)+2 \times 2(2)+2 = 6 \times 6$. This means that we can find 2 adjacent rows and two adjacent columns which have no squares removed from them. Remove two such columns and rows from the $2k+2 \times 2k+2$ checkerboard and the board will become a $2k \times 2k$ board. By the inductive hypothesis there is a covering for the $2k \times 2k$ board. Now add back in the two columns and two rows which were removed. If they were removed from the edge then fill them independently and the $2k+2 \times 2k+2$ board will be filled. If they are inserted somewhere in the interior of the board then they may have end up cutting a domino in half. Remove the two halves of the domino and place two new ones wherever this occurs and then fill the remainder of the board. An example is shown below.



Thus $P(k+1)$ is true. $\square$