Section 5.1

Sterling Jeppson

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Write the first four terms of the sequences defined by the formulas in 1-6.

Problem 1

$a_k = \frac{k}{10+k}$, for all integers $k \geq 1$.

Solution

$\frac{1}{11}, \frac{2}{12}, \frac{3}{13}, \frac{4}{14}$

Problem 2

$b_j = \frac{5-j}{5+j}$, for all integers $j \geq 1$.

Solution

$\frac{4}{6}, \frac{3}{7}, \frac{2}{8}, \frac{1}{9}$

Problem 3

$c_i = \frac{(-1)^i}{3^i}$, for all integers $i \geq 0$.

Solution

$\frac{1}{1}, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}$

Problem 4

$d_m = 1 + \left(\frac{1}{2}\right)^m$, for all integers $m \geq 0$.

Solution

$\frac{2}{1}, \frac{3}{2}, \frac{5}{4}, \frac{9}{8}$

Problem 5

$e_n = \left\lfloor \frac{n}{2} \right\rfloor \cdot 2$, for all integers $n \geq 0$.

Solution

$0, 0, 2, 2$

Problem 6

$f_n = \left\lfloor \frac{n}{4} \right\rfloor \cdot 4$, for all integers $n \geq 1$.

Solution

$0, 0, 0, 4$

Problem 7

Let $a_k = 2k + 1$ and $b_k = (k-1)^3 + k + 2$ for all integers $k \geq 0$. Show that the first three terms of these sequences are identical but that their fourth terms differ.

Solution

$a_0, a_1, a_2, a_3 = 1, 3, 5, 7.$

$b_0, b_1, b_2, b_3 = 1, 3, 5, 13.$

Compute the first fifteen terms of each of the sequences in 8 and 9, and describe the general behavior of these sequences in words.

Problem 8

$g_n = \lfloor \log_2 n \rfloor$, for all integers $n \geq 1$.

Solution

$$\begin{aligned}
g_1 &= 0, & g_2 &= 1, & g_3 &= 1, \\
g_4 &= 2, & g_5 &= 2, & g_6 &= 2, \\
g_7 &= 2, & g_8 &= 3, & g_9 &= 3, \\
g_{10} &= 3, & g_{11} &= 3, & g_{12} &= 3, \\
g_{13} &= 3, & g_{14} &= 3, & g_{15} &= 3
\end{aligned}$$

When n is an integer power of 2, g_n is the exponent of that power. This means that if $n = 2^k$ for some integer k, then $g_n = k$. All terms from g_n up to but not including g_m, where $m = 2^{k+1}$, have the same value as g_n.

Problem 9

$h_n = n \lfloor \log_2 n \rfloor$, for all integers $n \geq 1$.

Solution

$$\begin{array}{lll} h_1 = 0, & h_2 = 2, & h_3 = 3, \\ h_4 = 8, & h_5 = 10, & h_6 = 12, \\ h_7 = 14, & h_8 = 24 & h_9 = 27, \\ h_{10} = 30, & h_{11} = 33, & h_{12} = 36, \\ h_{13} = 39, & h_{14} = 42, & h_{15} = 45 \end{array}$$

When n is an integer power of 2, h_n is the exponent of that power times n. This means that if $n = 2^k$ for some integer k, then $h_n = n \cdot k$. All terms from h_n up to but not including h_m, where $m = 2^{k+1}$, increase from their previous term by that integer k.

Find explicit formulas for sequences of the from $a_1, a_2, a_3, ...$ with the initial terms given in 10-16.

Problem 10

$-1, 1, -1, 1, -1, 1$

Solution

$a_n = (-1)^n$, for all integers $n \geq 1$.

Problem 11

$0, 1, -2, 3, -4, 5$

Solution

$a_n = (n-1)(-1)^n$, for all integers $n \geq 1$.

Problem 12

$\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \frac{5}{36}, \frac{6}{49}$

Solution

$a_n = \frac{n}{(n+1)^2}$, for all integers $n \geq 1$.

Problem 13

$1-\frac{1}{2}, \frac{1}{2}-\frac{1}{3}, \frac{1}{3}-\frac{1}{4}, \frac{1}{4}-\frac{1}{5}, \frac{1}{5}-\frac{1}{6}, \frac{1}{6}-\frac{1}{7}$

Solution

$a_n = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$, for all integer $n \geq 1$.

Problem 14

$\frac{1}{3}, \frac{4}{9}, \frac{9}{27}, \frac{16}{81}, \frac{25}{243}, \frac{36}{729}$

Solution

$a_n = \frac{n^2}{3^n}$, for all integers $n \geq 1$.

Problem 15

$0, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \frac{6}{7}$

Solution

$a_n = \frac{n \cdot (-1)^n}{n+1}$, for all integer $n \geq 0$.

Problem 16

$3, 6, 12, 24, 48, 96$

Solution

$a_n = 3 \cdot 2^n$, for all integers $n \geq 0$.

Problem 17

Consider the sequence defined by $a_n = \frac{2n+(-1)^n-1}{4}$ for all integers $n \geq 0$. Find an alternative explicit formula for a_n that uses the floor notation.

Solution

$a_n = \left\lfloor \frac{n}{2} \right\rfloor$, for all integers $n \geq 0$.

Problem 18

Let $a_0 = 2, a_1 = 3, a_2 = -2, a_3 = 1, a_4 = 0, a_5 = -1$, and $a_6 = -2$. Compute each of the summations and products below.

(a) $\sum_{i=0}^{6} a_i$ (b) $\sum_{i=0}^{0} a_i$ (c) $\sum_{j=1}^{3} a_{2j}$ (d) $\prod_{k=0}^{6} a_k$ (e) $\prod_{k=2}^{2} a_k$

Solution

(a) $\sum_{i=0}^{6} a_i = 2 + 3 + (-2) + 1 + 0 + (-1) + (-2) = 1$

(b) $\sum_{i=0}^{0} a_i = 2$

(c) $\sum_{j=1}^{3} a_{2j} = (-2) + 0 + (-2) = -4$

(d) $\prod_{k=0}^{6} = 2 \cdot 3 \cdot -2 \cdot 1 \cdot 0 \cdot -1 \cdot -2 = 0$

(e) $\prod_{k=2}^{2} a_k = -2$

Compute the summations and products in 19-28.

Problem 19 and Solution

$\sum_{k=1}^{5} (k+1) = 2 + 3 + 4 + 5 + 6 = 20$

Problem 20 and Solution

$\prod_{k=2}^{4} k^2 = 4 \cdot 9 \cdot 16 = 576$

Problem 21 and Solution

$\sum_{m=0}^{3} \frac{1}{2^m} = \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8}$

Problem 22 and Solution

$\prod_{j=0}^{4} (-1)^j = 1 \cdot -1 \cdot 1 \cdot -1 \cdot 1 = 1$

Problem 23 and Solution

$\sum_{i=1}^{1} i(i+1) = 1(1+1) = 2$

Problem 24 and Solution

$\sum_{j=0}^{0} (j+1) \cdot 2^j = 1$

Problem 25 and Solution

$\prod_{k=2}^{2}(1-\frac{1}{k}) = 1 - \frac{1}{2} = \frac{1}{2}$

Problem 26 and Solution

$\sum_{k=-1}^{1}(k^2+3) = (1+3) + (0+3) + (1+3) = 11$

Problem 27 and Solution

$$\begin{aligned}
&\sum_{i=1}^{10}\left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&\quad = \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + ... + \left(\cancel{\frac{1}{9}} - \cancel{\frac{1}{10}}\right) + \left(\cancel{\frac{1}{10}} - \frac{1}{11}\right) \\
&\quad = 1 - \frac{1}{11} = \frac{10}{11}
\end{aligned}$$

Problem 28 and Solution

$\prod_{i=2}^{5}\frac{i(i+2)}{(i-1)\cdot(i+1)} = \frac{8}{3}\cdot\frac{15}{8}\cdot\frac{24}{15}\cdot\frac{35}{24} = \frac{35}{3}$

Write the summations in 29-32 in expanded form.

Problem 29 and Solution

$\sum_{i=1}^{n}(-2)^i = (-2)^1 + (-2)^2 + (-2)^3 + ... + (-2)^n$

Problem 30 and Solution

$\sum_{j=1}^{n} j(j+1) = 1(1+1) + 2(2+1) + 3(3+1) + ... + n(n+1)$

Problem 31 and Solution

$\sum_{k=0}^{n+1}\frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{(n+1)!}$

Problem 32 and Solution

$\sum_{i=1}^{k+1} i(i!) = 1(1!) + 2(2!) + 3(3!) + ... + (k+1)((k+1)!)$

Evaluate the summations and products in 33-36 for the indicated values of the variable.

Problem 33 and Solution

$n = 1$

$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} = \frac{1}{1^2} = 1$

Problem 34 and Solution

$m = 2$

$1(1!) + 2(2!) + 3(3!) + ... + m(m!) = 1(1!) + 2(2!) = 5$

Problem 35 and Solution

$k = 3$

$\left(\frac{1}{1+1}\right)\left(\frac{2}{2+1}\right)\left(\frac{3}{3+1}\right) ... \left(\frac{k}{k+1}\right) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{4}$

Problem 36 and Solution

$m = 1$

$\left(\frac{1\cdot2}{3\cdot4}\right)\left(\frac{4\cdot5}{6\cdot7}\right)\left(\frac{6\cdot7}{8\cdot9}\right) ... \left(\frac{m\cdot(m+1)}{(m+2)\cdot(m+3)}\right) = \left(\frac{1\cdot2}{3\cdot4}\right) = \frac{1}{6}$

Rewrite 37-39 by separating off the final term.

Problem 37 and Solution

$\sum_{i=1}^{k+1} i(i!) = \sum_{i=1}^{k} i(i!) + (k+1)((k+1)!)$

Problem 38 and Solution

$\sum_{k=1}^{m+1} k^2 = \sum_{k=1}^{m} k^2 + (m+1)^2$

Problem 39 and Solution

$\sum_{m=1}^{n+1} m(m+1) = \sum_{m=1}^{n} m(m+1) + n^2 + 3n + 2$

Write each of 40-42 as a single summation.

Problem 40 and Solution

$\sum_{i=1}^{k} i^3 + (k+1)^3 = \sum_{i=1}^{k+1} i^3$

Problem 41 and Solution

$\sum_{k=1}^{m} \frac{k}{k+1} + \frac{m+1}{m+2} = \sum_{k=1}^{m+1} \frac{k}{k+1}$

Problem 42 and Solution

$\sum_{m=0}^{n} (m+1) \cdot 2^m + (n+2) \cdot 2^{n+1} = \sum_{m=0}^{n+1} (m+1) \cdot 2^m$

Write each of 43-52 using summations or product notation.

Problem 43 and Solution

$1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + 7^2 = \sum_{i=1}^{7} (-1)^{i+1} \cdot i^2$

Problem 44 and Solution

$(1^3 - 1) - (2^3 - 1) + (3^3 - 1) - (4^3 - 1) + (5^3 - 1) = \sum_{i=1}^{5} (-1)^{i+1} \cdot (i^3 - 1)$

Problem 45 and Solution

$(2^2 - 1) \cdot (3^2 - 1) \cdot (4^2 - 1) = \prod_{i=2}^{4} (i^2 - 1)$

Problem 46 and Solution

$\frac{2}{3 \cdot 4} - \frac{3}{4 \cdot 5} + \frac{4}{5 \cdot 6} - \frac{5}{6 \cdot 7} + \frac{6}{7 \cdot 8} = \sum_{i=2}^{6} (-1)^i \cdot \frac{i}{i^2+3i+2}$

Problem 47 and Solution

$1 - r + r^2 - r^3 + r^4 - r^5 = \sum_{i=0}^{5} (-r)^i$

Problem 48 and Solution

$(1-t) \cdot (1-t^2) \cdot (1-t^3) \cdot (1-t^4) = \prod_{i=1}^{4} (1 - t^i)$

Problem 49 and Solution

$1^3 + 2^3 + 3^3 + ... + n^3 = \sum_{i=1}^{n} i^3$

Problem 50 and Solution

$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + ... + \frac{n}{(n+1)!} = \sum_{i=1}^{n} \frac{i}{(i+1)!}$

Problem 51 and Solution

$n + (n-1) + (n-2) + ... + 1 = \sum_{i=0}^{n-1} (n-i)$

Problem 52 and Solution

$n + \frac{n-1}{2!} + \frac{n-2}{3!} + \frac{n-3}{4!} + ... + \frac{1}{n!} = \sum_{i=0}^{n-1} \frac{n-i}{(i+1)!}$

Transform each of 53 and 54 by making the change of variable $i = k + 1$.

Problem 53 and Solution

$\sum_{k=0}^{5} k(k-1) = \sum_{i=1}^{6} (i^2 - 3i + 2)$

Problem 54 and Solution

$\prod_{k=1}^{n} \frac{k}{k^2+4} = \prod_{i=2}^{n+1} \frac{i^2}{i^2-2i+5}$

Transform each of 55-58 by making the change of variable $j = i - 1$.

Problem 55 and Solution

$\sum_{i=1}^{n+1} \frac{(i-1)^2}{i \cdot n} = \sum_{j=0}^{n} \frac{j^2}{n(j+1)}$

Problem 56 and Solution

$\sum_{i=3}^{n} \frac{i}{i+n-1} = \sum_{j=2}^{n-1} \frac{j+1}{j+n}$

Problem 57 and Solution

$\sum\limits_{i=1}^{n-1} \frac{i}{(n-i)^2} = \sum\limits_{j=0}^{n-2} \frac{j+1}{(n-j-1)^2}$

Problem 58 and Solution

$\prod\limits_{i=n}^{2n} \frac{n-i+1}{n+i} = \prod\limits_{j=n-1}^{2n-1} \frac{n-j}{n+j+1}$

Write each of 59-61 as a single summation or product.

Problem 59

$3 \cdot \sum\limits_{k=1}^{n} (2k-3) + \sum\limits_{k=1}^{n} (4-5k)$

Solution

$$\begin{aligned} 3 \cdot \sum_{k=1}^{n}(2k-3) + \sum_{k=1}^{n}(4-5k) &= \sum_{k=1}^{n}(6k-9) + \sum_{k=1}^{n}(4-5k) \\ &= \sum_{k=1}^{n}((6k-9)+(4-5k)) \\ &= \sum_{k=1}^{n}(k-5) \end{aligned}$$

Problem 60

$2 \cdot \sum\limits_{k=1}^{n} (3k^2+4) + 5 \cdot \sum\limits_{k=1}^{n} (2k^2-1)$

Solution

$$\begin{aligned} 2 \cdot \sum_{k=1}^{n}(3k^2+4) + 5 \cdot \sum_{k=1}^{n}(2k^2-1) &= \sum_{k=1}^{n}(6k^2+8) + \sum_{k=1}^{n}(10k^2-5) \\ &= \sum_{k=1}^{n}((6k^2+8)+(10k^2-5)) \\ &= \sum_{k=1}^{n}(16k^2+3) \end{aligned}$$

Problem 61 and Solution

$\left(\prod_{k=1}^{n} \frac{k}{k+1}\right) \cdot \left(\prod_{k=1}^{n} \frac{k+1}{k+2}\right) = \prod_{k=1}^{n} \frac{k}{k+1} \cdot \frac{k+1}{k+2} = \prod_{k=1}^{n} \frac{k}{k+2}$

Compute each of 62-67. Assume the values of the variables are restricted so that the expressions are defined.

Problem 62 and Solution

$\frac{4!}{3!} = \frac{4 \cdot 3!}{3!} = 4$

Problem 63 and Solution

$\frac{6!}{8!} = \frac{6!}{8 \cdot 7 \cdot 6!} = \frac{1}{56}$

Problem 64 and Solution

$\frac{4!}{0!} = \frac{4!}{1} = 24$

Problem 65 and Solution

$\frac{n!}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n$

Problem 66 and Solution

$\frac{(n-1)!}{(n+1)!} = \frac{(n-1)!}{(n+1)(n)(n-1)!} = \frac{1}{n^2+n}$

Problem 67 and Solution

$\frac{n!}{(n-2)!} = \frac{n(n-1)(n-2)!}{(n-2)!} = n^2 - n$

Problem 68 and Solution

$\frac{((n+1)!)^2}{(n!)^2} = \frac{(n+1)!(n+1)!}{n!n!} = \frac{((n+1)n!)((n+1)n!)}{n!n!} = (n+1)^2 = n^2 + 2n + 1$

Problem 69 and Solution

$\frac{n!}{(n-k)!} = \frac{n(n-1)(n-2)...(n-k+1)(n-k)(n-k-1)...3 \cdot 2 \cdot 1}{(n-k)(n-k-1)...3 \cdot 2 \cdot 1} = n(n-1)(n-2)...(n-k+1)$

Problem 70 and Solution

$\frac{n!}{(n-k+1)!} = \frac{n(n-1)(n-2)...(n-k+2)(n-k+1)(n-k)...3 \cdot 2 \cdot 1}{(n-k+1)(n-k)...3 \cdot 2 \cdot 1} = n(n-1)(n-2)...(n-k+2)$

Problem 71 and Solution

$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5\cdot4\cdot3!}{3!\cdot2!} = \frac{5\cdot4}{2!} = \frac{20}{2} = 10$

Problem 72 and Solution

$\binom{7}{4} = \frac{7!}{4!(7-4)!} = \frac{7\cdot6\cdot5\cdot4!}{4!\cdot3!} = \frac{7\cdot6\cdot5}{3\cdot2\cdot1} = 35$

Problem 73 and Solution

$\binom{3}{0} = \frac{3!}{0!(3-0)!} = \frac{3!}{1\cdot3!} = 1$

Problem 74 and Solution

$\binom{5}{5} = \frac{5!}{5!(5-5)!} = \frac{5!}{5!(0!)} = \frac{5!}{5!(1)} = 1$

Problem 75 and Solution

$\binom{n}{n-1} = \frac{n!}{(n-1)!(n-(n-1))!} = \frac{n!}{(n-1)!(1)!} = \frac{n(n-1)!}{(n-1)!} = n$

Problem 76 and Solution

$\binom{n+1}{n-1} = \frac{(n+1)!}{(n-1)!((n+1)-(n-1))!} = \frac{(n+1)!}{(n-1)!2!} = \frac{(n+1)(n)(n-1)!}{(n-1)!2!} = \frac{n^2+n}{2}$

Problem 77

(a) Prove that $n! + 2$ is divisible by 2, for all integers $n \geq 2$.

(b) Prove that $n! + k$ is divisible by k, for all integers $n \geq 2$ and $k = 2, 3, ..., n$.

(c) Given any integer $m \geq 2$, is it possible to find a sequence of $m-1$ consecutive positive integers none of which is prime? Explain your answer.

Solution

(a) **Theorem.** $\forall n \in \mathbb{Z}, n \geq 2 \implies 2 \mid (n! + 2)$.

Proof. Let n be any integer such that $n \geq 2$. It follows from the definition of factorial that

$$n! = \begin{cases} 2 \cdot 1 & \text{if } n = 2 \\ 3 \cdot 2 \cdot 1 & \text{if } n = 3 \\ n(n-1)...2 \cdot 1 & \text{if } n > 3 \end{cases}$$

In each case $n!$ has a factor of 2 and so $n! = 2k$ for some integer k. Then

$$\begin{aligned} n! + 2 &= 2k + 2 \\ &= 2(k+1) \end{aligned}$$

Since $k + 1$ is an integer it follows that $2 \mid (n! + 2)$. □

(b) **Theorem.** $\forall n \in \mathbb{Z}, n \geq 2$ and $k = 2, 3, ..., n \implies k \mid (n! + k)$.

Proof. Let n be an integer such that $n \geq 2$ and let k be any integer from $[2, n]$. It follows from the definition of factorial that

$$n! = \begin{cases} 2 \cdot 1 & \text{if } n = 2 \\ 3 \cdot 2 \cdot 1 & \text{if } n = 3 \\ n(n-1)...2 \cdot 1 & \text{if } n > 3 \end{cases}$$

In each case $n!$ contains every possible value of k and so $n! = kj$ for some integer j. Then

$$\begin{aligned} n! + k &= kj + k \\ &= k(j+1) \end{aligned}$$

Since $j + 1$ is an integer it follows that $k \mid (n! + k)$. $\square$

(c) Yes. Let m be any integer such that $m \geq 2$. It follows from part (b) that if k is any integer from $[2, m]$ then $k \mid (m! + k)$. Since the interval in which k exists contains $(m - 2) + 1 = m - 1$ integers it follows that $m! + k$ can be any of $m - 1$ consecutive integers all of which are divisible by k and are therefore not prime.

Problem 78

Prove that for all nonnegative integers n and r with $r + 1 \leq n$, $\binom{n}{r+1} = \frac{n-r}{r+1}\binom{n}{r}$.

Theorem. $\forall n, r \in \mathbb{Z}^{nonneg}$, $r + 1 \leq n \implies \binom{n}{r+1} = \frac{n-r}{r+1}\binom{n}{r}$.

Proof. Suppose that n and r are any nonnegative integers such that $r + 1 \leq n$.

$$\begin{aligned} \frac{n-r}{r+1}\binom{n}{r} &= \frac{n-r}{r+1} \cdot \frac{n!}{r!(n-r)!} \\ &= \frac{\cancel{n-r}}{r+1} \cdot \frac{n!}{r!(\cancel{n-r})(n-r-1)!} \\ &= \frac{n!}{r!(r+1)(n-r-1)!} \\ &= \frac{n!}{(r+1)!(n-(r+1))!} \\ &= \binom{n}{r+1} \end{aligned}$$

$\square$

Problem 79

Prove that if p is a prime number and r is an integer with $0 < r < p$, then $\binom{p}{r}$ is divisible by p.

Theorem. *If p is a prime number and r is an integer with $0 < r < p$, then $p \mid \binom{p}{r}$*

Proof. Let p be any prime number and let r be any integer such that $0 < r < p$.

$$\begin{aligned}
\binom{p}{r} &= \frac{p!}{r!(p-r)!} \\
&= \frac{p(p-1)!}{r(r-1)!(p-r)!} \\
&= \frac{p}{r} \cdot \frac{(p-1)!}{(r-1)!(p-r)!} \\
&= \frac{p}{r} \cdot \frac{(p-1)!}{(r-1)!((p-1)-(r-1))!} \\
&= \frac{p}{r} \cdot \binom{p-1}{r-1} \\
r \cdot \binom{p}{r} &= p \cdot \binom{p-1}{r-1}
\end{aligned}$$

It follows that since $0 < r < p$, $\binom{p}{r}$ and $\binom{p-1}{r-1}$ are both integers. It now follows form the unique factorization of the integers that p is a factor in at least one of r and $\binom{p}{r}$. However by theorem 4.3.1 p cannot be a factor of r as $r < p$. Thus p must be a factor of $\binom{p}{r}$ and hence $p \mid \binom{p}{r}$. □

Problem 80 and Solution

Suppose that $a[1], a[2], a[3], ..., a[m]$ is a one dimensional array and consider the following algorithm segment:

```
sum := 0
for k := 1 to m do
    sum := sum + a[k]
end for
```

Fill in the blanks below so that each algorithm segment performs the same job as the one given previously.

```
sum := 0
for i := 0 to m − 1 do
    sum := sum + a[i + 1]
end for
```

```
sum := 0
for j := 2 to m + 1 do
    sum := sum + a[j − 1]
end for
```

Use repeated division by 2 to convert (by hand) the integers in 81-83 from base 10 to base 2.

Problem 81

$(90)_{10}$

Solution

$$\begin{aligned}
90 &= 45(2)+0\\
45 &= 22(2)+1\\
22 &= 11(2)+0\\
11 &= 5(2)+1\\
5 &= 2(2)+1\\
2 &= 1(2)+0\\
1 &= 0(2)+1
\end{aligned}$$

Therefore $(90)_{10} = (1011010)_2$

Problem 82

$(98)_{10}$

Solution

$$\begin{aligned}
98 &= 49(2)+0\\
49 &= 24(2)+1\\
24 &= 12(2)+0\\
12 &= 6(2)+0\\
6 &= 3(2)+0\\
3 &= 1(2)+1\\
1 &= 0(2)+1
\end{aligned}$$

Therefore $(98)_{10} = (1100010)_2$

Problem 83

$(205)_{10}$

Solution

$205 = 102(2)+1$
$102 = 51(2)+0$
$51 = 25(2)+1$
$25 = 12(2)+1$
$12 = 6(2)+0$
$6 = 3(2)+0$
$3 = 1(2)+1$
$1 = 0(2)+1$

Therefore $(205)_{10} = (11001101)_2$

Make a trace table to trace the action of Algorithm 5.1.1 on the input in 84-86.

Problem 84 and Solution

Trace table for algorithm 5.1.1 with $a = 23$

	0	**1**	**2**	**3**	**4**	**5**
a	23					
i	0	1	2	3	4	5
q	23	11	5	2	1	0
$r[0]$		1				
$r[1]$			1			
$r[2]$				1		
$r[3]$					0	
$r[4]$						1

Problem 85 and Solution

Trace table for algorithm 5.1.1 with $a = 28$

	0	**1**	**2**	**3**	**4**	**5**
a	28					
i	0	1	2	3	4	5
q	28	14	7	3	1	0
$r[0]$		0				
$r[1]$			0			
$r[2]$				1		
$r[3]$					1	
$r[4]$						1

Problem 86 and Solution

Trace table for algorithm 5.1.1 with $a = 44$

	0	**1**	**2**	**3**	**4**	**5**	**6**
a	44						
i	0	1	2	3	4	5	6
q	44	22	11	5	2	1	0
$r[0]$		0					
$r[1]$			0				
$r[2]$				1			
$r[3]$					1		
$r[4]$						0	
$r[5]$							1

Problem 87

Write an informal description of an algorithm (using repeated division by 16) to convert a nonnegative integer from decimal notation to a hexadecimal notation (base 16).

Solution

The algorithm would accept a nonnegative integer a as input. Another integer q which stands for quotient would be set to a. A third integer i would monitor the current position in an array and would be set to 0 initially. In a loop you would modulus q with 16 and place the results in position i of a one dimensional array. Then q would be divided by 16 and the result truncated and placed into q. Finally i would be incremented. These operations would repeat until $q = 0$. In the case that $a = 0$ and therefore $q = 0$ from the start, the loop would run 1 time.

Use the algorithm you developed for exercise 87 to convert the integers in 88-90 to hexadecimal notation.

Problem 88

$(287)_{10}$

Solution

$$\begin{aligned} 287 &= 17(16) + 15 \\ 17 &= 1(16) + 1 \\ 1 &= 0(16) + 1 \end{aligned}$$

Therefore $(287)_{10} = (11\text{F})_{16}$

Problem 89

$(693)_{10}$

Solution

$$\begin{aligned} 693 &= 43(16) + 5 \\ 43 &= 2(16) + 11 \\ 2 &= 0(16) + 2 \end{aligned}$$

Therefore $(693)_{10} = (2\text{B}5)_{16}$

Problem 90

$(2,301)_{10}$

Solution

$$\begin{aligned} 2,301 &= 143(16) + 13 \\ 143 &= 8(16) + 15 \\ 8 &= 0(16) + 8 \end{aligned}$$

Therefore $(2,301)_{10} = (8\text{FD})_{16}$

Problem 91

Write a formal version of the algorithm you developed for exercise 87.

Solution

Algorithm to convert integers from base 10 to base 16

```
procedure HEXADECIMAL(a)
    q := a, i := 0
    while i = 0 || q ≠ 0 do
        r[i] := q mod 16
        q := q div 16
        i := i + 1
    end while
    return r[0], r[1], r[2], ..., r[i − 1]
end procedure
```