Section 6.1

Sterling Jeppson

December 25, 2020

Problem 1

In each of (a)-(f), answer the following questions: Is $A \subseteq B$? Is $B \subseteq A$? Is either A or B a proper subset of the other?

(a) $A = \{2, \{2\}, (\sqrt{2})^2\}$, $B = \{2, \{2\}, \{\{2\}\}\}$

(b) $A = \{3, \sqrt{5^2 - 4^2}, 24\, mod\, 7\}$, $B = \{8\, mod\, 5\}$

(c) $A = \{\{1, 2\}, \{2, 3\}\}$, $B = \{1, 2, 3\}$

(d) $A = \{a, b, c\}$, $B = \{\{a\}, \{b\}, \{c\}\}$

(e) $A = \{\sqrt{16}, \{4\}\}$, $B = \{4\}$

(f) $A = \{x \in \mathbb{R} \mid \cos x \in \mathbb{Z}\}$, $B = \{x \in \mathbb{R} \mid \sin x \in \mathbb{Z}\}$

Solution

(a) $A = \{2, \{2\}, (\sqrt{2})^2\} = \{2, \{2\}, 2\} = \{2, \{2\}\}$. It follows that $A \subseteq B$ because every element in A is in B. $B \nsubseteq A$ because $\{\{2\}\} \in B$ but $\{\{2\}\} \notin A$. It follows that $A \subset B$.

(b) $A = \{3, \sqrt{5^2 - 4^2}, 24\, mod\, 7\} = \{3, 3, 3\} = \{3\}$ and $B = \{8\, mod\, 5\} = \{3\}$. It follows that $A \subseteq B$ and $B \subseteq A$ because every element in A is in B and every element in B is in A. It follows that $A \subset B$ and $B \subset A$.

(c) $A \nsubseteq B$ because $\{1, 2\} \in A$ but $\{1, 2\} \notin B$. $B \nsubseteq A$ because $1 \in B$ but $1 \notin A$.

(d) $A \nsubseteq B$ because $a \in A$ but $a \notin B$. $B \nsubseteq A$ because $\{a\} \in B$ but $\{a\} \notin A$.

(e) $A = \{\sqrt{16}, \{4\}\} = \{4, \{4\}\}$. $B \subseteq A$ because every element in B is in A. $A \nsubseteq B$ because $\{4\} \in A$ but $\{4\} \notin B$. It follows that $B \subset B$.

(f) $A = \{x \in \mathbb{R} \mid \cos x \in \mathbb{Z}\} = \{x \in \mathbb{R} \mid x = \frac{\pi}{2}n \text{ for some } n \in \mathbb{Z}\} = B$. It follows that $A \subseteq B$ and $B \subseteq A$.

Problem 2

Complete the proof from example 6.1.3: Prove that $B \subseteq A$ where

$$A = \{m \in \mathbb{Z} \mid m = 2a \text{ for some integer } a\}$$

and

$$B = \{n \in \mathbb{Z} \mid n = 2b - 2 \text{ for some integer } b\}$$

Solution

Part 2, proof that $\boldsymbol{B \subseteq A}$: Suppose x is a particular but arbitrary element of B. By definition of B, there is an integer b such that $x = 2b-2 = 2(b-1)$. It follows from closure under subtraction that $b-1$ is an integer. Let that integer be t. Then $x = 2t$ for some integer t. It follows that $B \subseteq A$.

Problem 3

Let sets R, S, and T be defined as follows:

$$R = \{x \in \mathbb{Z} \mid \ x \text{ is divisible by } 2\}$$
$$S = \{y \in \mathbb{Z} \mid \ y \text{ is divisible by } 3\}$$
$$T = \{z \in \mathbb{Z} \mid \ Z \text{ is divisible by } 6\}$$

a. Is $R \subseteq T$? Explain.

b. Is $T \subseteq R$? Explain.

c. Is $T \subseteq S$? Explain.

Solution

a. $R \nsubseteq T$ because there are elements in R that are not in T. For example, $2 \in R$ but $2 \notin T$ as $6 \nmid 2$.

b. $T \subseteq R$. To see why this is so, suppose that z is a particular but arbitrary element of T. Then there is an integer q such that $z = 6q = 2(3q)$. It follows that $2 \mid z$ and so $z \in R$.

c. $T \subseteq S$. To see why this is so, suppose that z is a particular but arbitrary element of T. Then there is an integer q such that $z = 6q = 3(2q)$. It follows that $3 \mid z$ and so $z \in S$.

Problem 4

Let $A = \{n \in \mathbb{Z} \mid n = 5r \text{ for some integer } r\}$ and
$B = \{m \in \mathbb{Z} \mid m = 20s \text{ for some integer } s\}$.

a. Is $A \subseteq B$? Explain.

b. Is $B \subseteq A$? Explain.

Solution

a. $A \nsubseteq B$ because there are elements in A that are not in $B4$. For example, $5 \in A$ but $5 \notin B$.

b. $B \subseteq A$. To see why this is so suppose that m is any element in B. Then $m = 20s = 5(4s)$. It follows that $m \in A$.

Problem 5

Let $C = \{n \in \mathbb{Z} \mid n = 6r - 5 \text{ for some integer } r\}$ and
$D = \{m \in \mathbb{Z} \mid m = 3s + 1 \text{ for some integer } s\}$. Prove or disprove each of the following statements

a. $C \subseteq D$ b. $D \subseteq C$

Solution

a. $C \subseteq D$.

Proof. Suppose that n is any element in C. Then

$$n = 6r - 5 = 6r - 6 + 1 = 3(2r - 1) + 1$$

It follows that $n \in D$. □

b. $D \nsubseteq C$ because there are elements in D that are not in C. For example, $4 \in D$ but $4 \notin C$

Problem 6

Let $\{A = x \in \mathbb{Z} \mid x = 5a + 2 \text{ for some integer } a\}$,
$\{B = y \in \mathbb{Z} \mid y = 10b - 3 \text{ for some integer } b\}$, and
$\{C = z \in \mathbb{Z} \mid z = 10c + 7 \text{ for some integer } c\}$. Prove or disprove the following statements.

a. $A \subseteq B$ b. $B \subseteq A$ c. $B = C$

Solution

a. $A \nsubseteq B$ because there are elements in A that are not in B. For example, $2 \in A$ but $2 \notin B$.

b. $B \subseteq A$.

Proof. Suppose that y is any element in B. Then

$$y = 10b - 3 = 10b - 5 + 2 = 5(2b - 1) + 2$$

It follows that $y \in A$. □

c. $B = C$.

Proof. **Part 1, proof that $B \subseteq C$:** Suppose that y is any element in B. Then

$$y = 10b - 3 = 10b - 10 + 7 = 10(b - 1) + 7$$

It follows that $y \in C$.

Part 2, proof that $C \subseteq B$: Suppose that z is any element in C. Then

$$z = 10c + 7 = 10c + 10 - 3 = 10(c + 1) - 3$$

It follows that $z \in B$.

Now since $B \subseteq C$ and $C \subseteq B$ it follows from the definition of set equality that $B = C$. □

Problem 7

Let $\{A = x \in \mathbb{Z} \mid x = 6a + 4$ for some integer $a\}$,
$\{B = y \in \mathbb{Z} \mid y = 18b - 2$ for some integer $b\}$, and
$\{C = z \in \mathbb{Z} \mid z = 18c + 16$ for some integer $c\}$. Prove or disprove the following statements.

a. $A \subseteq B$ b. $B \subseteq A$ c. $B = C$

Solution

a. $A \nsubseteq B$ because there are elements in A that are not in B. For example, $4 \in A$ but $4 \notin B$.

b. $B \subseteq A$.

Proof. Suppose that y is any element in B. Then

$$y = 18b - 2 = 18b - 6 + 4 = 6(3b - 1) + 4$$

It follows that $y \in A$. □

c. $B = C$.

Proof. **Part 1, proof that $B \subseteq C$:** Suppose that y is any element in B. Then

$$y = 18b - 2 = 18b - 18 + 16 = 18(b - 1) + 16$$

It follows that $y \in C$.

Part 2, proof that $C \subseteq B$: Suppose that z is any element in C. Then

$$z = 18c + 16 = 18c + 18 - 2 = 18(c + 1) - 2$$

It follows that $z \in B$.

Now since $B \subseteq C$ and $C \subseteq B$ it follows from the definition of set equality that $B = C$. $\square$

Problem 8

Write in words how to read each of the following out loud. Then write the shorthand notation for each set

a. $\{x \in U \mid x \in A \text{ and } x \in B\}$

b. $\{x \in U \mid x \in A \text{ or } x \in B\}$

c. $\{x \in U \mid x \in A \text{ and } x \notin B\}$

d. $\{x \in U \mid x \notin A\}$

Solution

a. The set of all x in U such that x is in A and x is in B. The shorthand notation is $A \cap B$.

b. The set of all x in U such that x is in A or x is in B. The shorthand notation is $A \cup B$.

c. The set of all x in U such that x is in A and x is not in B. The shorthand notation is $A - B$.

d. The set of all x in U such that x is not in A. The shorthand notation is $\overline{A}$.

Problem 9 and solution

Complete the following sentences without using symbols $\cup, \cap$, or $-$.

a. $x \notin A \cup B$ if, and only if, x is not in A and x is not in B.

b. $x \notin A \cap B$ if, and only if, x is not in A or x is not in B.

c. $x \notin A - B$ if, and only if, x is not in A or x is in B.

Problem 10 and solution

let $A = \{1, 3, 5, 7, 9\}$, $B = \{3, 6, 9\}$, and $C = \{2, 4, 6, 8\}$. Find each of the following:

a. $A \cup B = \{1, 3, 5, 6, 7, 9\}$

b. $A \cap B = \{3, 0\}$

c. $A \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

d. $A \cap C = \emptyset$

e. $A - B = \{1, 5, 7\}$

f. $B - A = \{6\}$

g. $B \cup C = \{2, 3, 4, 6, 8, 9\}$

h. $B \cap C = \{6\}$

Problem 11 and solution

Let U be the set $\mathbb{R}$ of real numbers and let $A = \{x \in \mathbb{R} \mid 0 < x \leq 2\}$, $B = \{x \in \mathbb{R} \mid 1 \leq x < 4\}$, and $C = \{x \in \mathbb{R} \mid 3 \leq x < 9\}$. Find each of the following:

a. $A \cup B = \{x \in \mathbb{R} \mid 0 < x < 4\}$
b. $A \cap B = \{x \in \mathbb{R} \mid 1 \leq x \leq 2\}$
c. $\overline{A} = \{x \in \mathbb{R} \mid x \leq 0 \text{ or } x > 2\}$
d. $A \cup C$
$= \{x \in \mathbb{R} \mid 0 < x \leq 2 \text{ or } 3 \leq x < 9\}$
e. $A \cap C = \emptyset$
f. $\overline{B} = \{x \in \mathbb{R} \mid x < 1 \text{ or } x \geq 4\}$
g. $\overline{A} \cap \overline{B} = \{x \in \mathbb{R} \mid x \leq 0 \text{ or } x \geq 4\}$
h. $\overline{A} \cup \overline{B} = \{x \in \mathbb{R} \mid x < 1 \text{ or } x > 2\}$
i. $\overline{(A \cap B)} = \{x \in \mathbb{R} \mid x < 1 \text{ or } x > 2\}$
j. $\overline{(A \cup B)} = \{x \in \mathbb{R} \mid x \leq 0 \text{ or } x \geq 4\}$

Problem 12 and solution

Let U be the set $\mathbb{R}$ of real numbers and let $A = \{x \in \mathbb{R} \mid -3 \leq x \leq 0\}$, $B = \{x \in \mathbb{R} \mid -1 < x < 2\}$, and $C = \{x \in \mathbb{R} \mid 6 < x \leq 8\}$. Find each of the following:

a. $A \cup B = \{x \in \mathbb{R} \mid -3 \leq x < 2\}$
b. $A \cap B = \{x \in \mathbb{R} \mid -1 < x \leq 0\}$
c. $\overline{A} = \{x \in \mathbb{R} \mid x < 3 \text{ or } x > 0\}$
d. $A \cup C$
$= \{x \in \mathbb{R} \mid -3 \leq x \leq 0 \text{ or } 6 < x \leq 8\}$
e. $A \cap C = \emptyset$
f. $\overline{B} = \{x \in \mathbb{R} \mid x \leq -1 \text{ or } x \geq 2\}$
g. $\overline{A} \cap \overline{B} = \{x \in \mathbb{R} \mid x < -3 \text{ or } x \geq 2\}$
h. $\overline{A} \cup \overline{B} = \{x \in \mathbb{R} \mid x \leq -1 \text{ or } x > 0\}$
i. $\overline{(A \cap B)} = \{x \in \mathbb{R} \mid x \leq -1 \text{ or } x > 0\}$
j. $\overline{(A \cup B)} = \{x \in \mathbb{R} \mid x < 3 \text{ or } x \geq 2\}$

Problem 13 and solution

Indicate which of the following relationships are true and which are false:

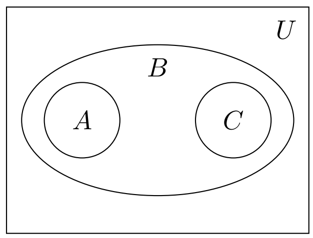
a. $\mathbb{Z}^+ \subseteq \mathbb{Q}$: True
b. $\mathbb{R}^- \subseteq \mathbb{Q}$: False as $-\sqrt{2} \in \mathbb{R}^-$ but $-\sqrt{2} \notin \mathbb{Q}$
c. $\mathbb{Q} \subseteq \mathbb{Z}$: False as $\frac{1}{2} \in \mathbb{Q}$ but $\frac{1}{2} \notin \mathbb{Z}$
d. $\mathbb{Z}^- \cup \mathbb{Z}^+ = Z$: False as $0 \in \mathbb{Z}$ but $0 \notin \mathbb{Z}^+$ and $0 \notin \mathbb{Z}^-$
e. $\mathbb{Z}^- \cap \mathbb{Z}^+ = \emptyset$: True
f. $\mathbb{Q} \cap \mathbb{R} = \mathbb{Q}$: True
g. $\mathbb{Q} \cup \mathbb{Z} = \mathbb{Q}$: True
h. $\mathbb{Z}^+ \cap \mathbb{R} = \mathbb{Q}$: True
i. $\mathbb{Z} \cup \mathbb{Q} = \mathbb{Z}$: False as $\frac{1}{2} \in \mathbb{Z} \cup \mathbb{Q}$ but $\frac{1}{2} \notin \mathbb{Z}$

Problem 14

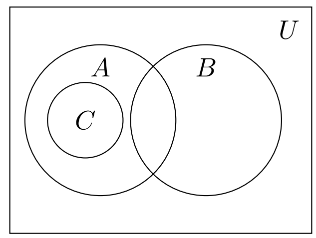
In each of the following, draw a Venn diagram for sets A, B, and C that satisfy the given conditions:

a. $A \subseteq B$; $C \subseteq B$; $A \cap C = \emptyset$
b. $C \subseteq A$; $B \cap C = \emptyset$

Solution



$A \subseteq B$; $C \subseteq B$; $A \cap C = \emptyset$



$C \subseteq A$; $B \cap C = \emptyset$

Problem 15

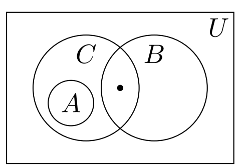
Draw Venn diagrams to describe sets A, B, and C that satisfy the given conditions:

a. $A \cap B = \emptyset$; $A \subseteq C$; $C \cap B \neq \emptyset$

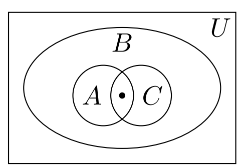
b. $A \subseteq B$; $C \subseteq B$; $A \cap C \neq \emptyset$

c. $A \cap B \neq \emptyset$; $B \cap C \neq \emptyset$; $A \cap C = \emptyset$; $A \nsubseteq B$; $C \nsubseteq B$

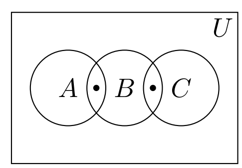
Solution



Venn Diagram A



Venn Diagram B



Venn Diagram C

Problem 16

Let $A = \{a, b, c\}$, $B = \{b, c, d\}$ and $C = \{b, c, e\}$.

a. Find $A \cup (B \cap C)$, $(A \cup B) \cap C$, and $(A \cup B) \cap (A \cup C)$. Which of these sets are equal?

b. Find $A \cap (B \cup C)$, $(A \cap B) \cup C$, and $(A \cap B) \cup (A \cap C)$. Which of these sets are equal?

c. Find $(A - B) - C$ and $A - (B - C)$. Are these sets equal?

Solution

a. $A \cup (B \cap C) = \{a, b, c\}$, $(A \cup B) \cap C = \{b, c\}$, and $(A \cup B) \cap (A \cup C) = \{a, b, c\}$. Hence $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

b. $A \cap (B \cup C) = \{b, c\}$, $(A \cap B) \cup C = \{b, c, e\}$, and $(A \cap B) \cup (A \cap C) = \{b, c\}$. Hence $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

c. $(A - B) - C = \{a\}$ and $A - (B - C) = \{a, b, c\}$. Hence $(A - B) - C \neq A - (B - C)$.

Problem 17

Consider the Venn diagram shown below. For each of (a)-(f), copy the diagram and shade the region corresponding to the indicated set.

a. $A \cap B$

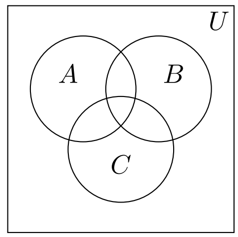
b. $B \cup C$

c. $\overline{A}$

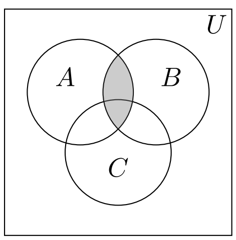
d. $A - (B \cup C)$

e. $\overline{(A \cup B)}$

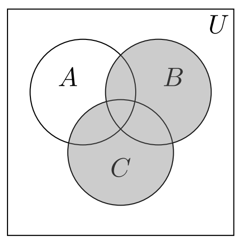
f. $\overline{A} \cap \overline{B}$



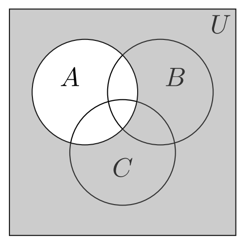
Solution



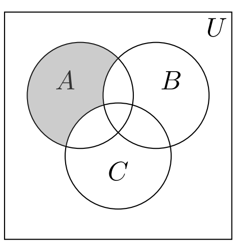
Venn Diagram A



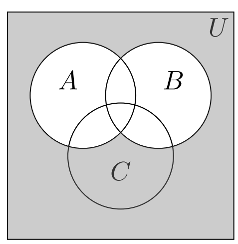
Venn Diagram B



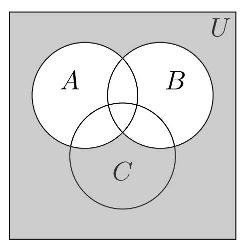
Venn Diagram C



Venn Diagram D



Venn Diagram E



Venn Diagram F

Problem 18

a. Is the number 0 in $\emptyset$? Why?

b. Is $\emptyset = \{\emptyset\}$? Why?

c. Is $\emptyset \in \{\emptyset\}$? Why?

d. Is $\emptyset \in \emptyset$? Why?

Solution

a. No. $0 \notin \emptyset$ because $\emptyset$ has no elements.

b. No. $\emptyset$ is the empty set; it has no elements. $\{\emptyset\}$ is a set with one element: $\emptyset$.

c. Yes. $\{\emptyset\}$ is a set which contains $\emptyset$.

d. No. $\emptyset \notin \emptyset$ because the empty set has no elements.

Problem 19 and solution

let $A_i = \{i, i^2\}$ for all integers $i = 1, 2, 3, 4$.

a. $A_1 \cup A_2 \cup A_3 \cup A_4 = \{1, 2, 3, 4, 9, 16\}$

b. $A_1 \cap A_2 \cap A_3 \cap A_4 = \emptyset$

c. A_1, A_2, A_3, and A_4 are not mutually disjoint because $A_2 \cap A_4 = \{4\} \neq \emptyset$.

Problem 20 and solution

Let $B_i = \{x \in \mathbb{R} \mid 0 \leq x \leq i\}$ for all integers $i = 1, 2, 3, 4$.

a. $B_1 \cup B_2 \cup B_3 \cup B_4 = \{x \in \mathbb{R} \mid 0 \leq x \leq 4\}$

b. $B_1 \cap B_2 \cap B_3 \cap B_4 = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$

c. B_1, B_2, B_3, and B_4 are not mutually disjoint because $B_1 \cap B_2 \cap B_3 \cap B_4 \neq \emptyset$.

Problem 21 and solution

Let $C_i = \{i, -i\}$ for all nonnegative integers i.

a. $\bigcup_{i=0}^{4} C_i = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$

b. $\bigcap_{i=0}^{4} C_i = \emptyset$

c. $C_0, C_1, C_2, ...$ are mutually disjoint because $C_i \cap C_j = \emptyset$ for all $i, j = 1, 2, 3, ...$ whenever $i \neq j$.

d. $\bigcup_{i=0}^{n} C_i = \{-n, -(n-1), ..., -2, -1, 0, 1, 2, ..., (n-1), n\}$

e. $\bigcap_{i=0}^{n} C_i = \emptyset$

f. $\bigcup_{i=0}^{\infty} C_i = \mathbb{Z}$, the set of all integers.

g. $\bigcap_{i=0}^{\infty} C_i = \emptyset$

Problem 22 and solution

Let $D_i = \{x \in \mathbb{R} \mid -i \leq x \leq i\} = [-i, i]$ for all nonnegative integers i.

a. $\bigcup_{i=0}^{4} D_i = \{x \in \mathbb{R} \mid 4 \leq x \leq 4\} = [4, 4]$

b. $\bigcap_{i=0}^{4} D_i = \{0\}$

c. $D_0, D_1, D_2, ...$ are not mutually disjoint because $0 \in D_0$ and $0 \in D_1$.

d. $\displaystyle\bigcup_{i=0}^{n} D_i = \{x \in \mathbb{R} \mid -n \le x \le n\} = [-n, n]$

e. $\displaystyle\bigcap_{i=0}^{n} D_i = \{0\}$

f. $\displaystyle\bigcup_{i=0}^{\infty} C_i = \mathbb{R}$, the set of all real numbers.

g. $\displaystyle\bigcap_{i=0}^{\infty} C_i = \{0\}$

Problem 23 and solution

Let $V_i = \left\{x \in \mathbb{R} \mid -\frac{1}{i} \le x \le \frac{1}{i}\right\} = \left[-\frac{1}{i}, \frac{1}{i}\right]$ for all positive integers i.

a. $\displaystyle\bigcup_{i=1}^{4} V_i = \{x \in \mathbb{R} \mid -1 \le x \le 1\} = [-1, 1]$

b. $\displaystyle\bigcap_{i=1}^{4} V_i = \left\{x \in \mathbb{R} \mid -\frac{1}{4} \le x \le \frac{1}{4}\right\} = \left[-\frac{1}{4}, \frac{1}{4}\right]$

c. $V_0, V_1, V_2, \ldots$ are not mutually disjoint because $1/4 \in V_1$ and $1/4 \in V_4$.

d. $\displaystyle\bigcup_{i=1}^{n} V_i = \{x \in \mathbb{R} \mid -1 \le x \le 1\} = [-1, 1]$

e. $\displaystyle\bigcap_{i=1}^{n} V_i = \left\{x \in \mathbb{R} \mid -\frac{1}{n} \le x \le \frac{1}{n}\right\} = \left[-\frac{1}{n}, \frac{1}{n}\right]$

f. $\displaystyle\bigcup_{i=1}^{\infty} V_i = \{x \in \mathbb{R} \mid -1 \le x \le 1\} = [-1, 1]$

g. $\displaystyle\bigcap_{i=1}^{\infty} V_i = \{0\}$

Problem 24 and solution

Let $W_i = \{x \in \mathbb{R} \mid x > i\} = (i, \infty)$ for all nonnegative integers i.

a. $\displaystyle\bigcup_{i=0}^{4} W_i = \{x \in \mathbb{R} \mid x > 0\} = (0, \infty) = \mathbb{R}^+$

b. $\displaystyle\bigcap_{i=0}^{4} W_i = \{x \in \mathbb{R} \mid x > 4\} = (4, \infty)$

c. $W_0, W_1, W_2, \ldots$ are not mutually disjoint because $2 \in W_0$ and $2 \in W_1$.

d. $\displaystyle\bigcup_{i=0}^{n} W_i = \{x \in \mathbb{R} \mid x > 0\} = (0, \infty) = \mathbb{R}^+$

e. $\displaystyle\bigcap_{i=0}^{n} W_i = \{x \in \mathbb{R} \mid x > n\} = (n, \infty)$

f. $\displaystyle\bigcup_{i=0}^{\infty} W_i = \{x \in \mathbb{R} \mid x > 0\} = (0, \infty) = \mathbb{R}^+$

g. $\displaystyle\bigcap_{i=0}^{\infty} W_i = \emptyset$

Problem 25 and solution

Let $R_i = \left\{x \in \mathbb{R} \,\middle|\, 1 \leq x \leq 1 + \frac{1}{i}\right\} = \left[1, 1 + \frac{1}{i}\right]$ for all positive integers i.

a. $\displaystyle\bigcup_{i=1}^{4} R_i = \{x \in \mathbb{R} \mid 1 \leq x \leq 2\} = [1, 2]$

b. $\displaystyle\bigcap_{i=1}^{4} R_i = \left\{x \in \mathbb{R} \,\middle|\, 1 \leq x \leq \frac{5}{4}\right\} = \left[1, \frac{5}{4}\right]$

c. $R_0, R_1, R_2, \ldots$ are not mutually disjoint because $1 \in R_1$ and $1 \in R_2$.

d. $\displaystyle\bigcup_{i=1}^{n} R_i = \{x \in \mathbb{R} \mid 1 \leq x \leq 2\} = [1, 2]$

e. $\displaystyle\bigcap_{i=1}^{n} R_i = \left\{x \in \mathbb{R} \,\middle|\, 1 \leq x \leq 1 + \frac{1}{n}\right\} = \left[1, 1 + \frac{1}{n}\right]$

f. $\displaystyle\bigcup_{i=1}^{\infty} R_i = \{x \in \mathbb{R} \mid 1 \leq x \leq 2\} = [1, 2]$

g. $\displaystyle\bigcap_{i=1}^{\infty} R_i = \{1\}$

Problem 26 and solution

Let $S_i = \left\{x \in \mathbb{R} \,\middle|\, 1 < x < 1 + \frac{1}{i}\right\} = \left(1, 1 + \frac{1}{i}\right)$ for all positive integers i.

a. $\displaystyle\bigcup_{i=1}^{4} S_i = \{x \in \mathbb{R} \mid 1 < x < 2\} = (1, 2)$

b. $\displaystyle\bigcap_{i=1}^{4} S_i = \left\{x \in \mathbb{R} \,\middle|\, 1 < x < \frac{5}{4}\right\} = \left(1, \frac{5}{4}\right)$

c. $S_0, S_1, S_2, \ldots$ are not mutually disjoint because $1.2 \in S_1$ and $1.2 \in S_2$.

d. $\displaystyle\bigcup_{i=1}^{n} S_i = \{x \in \mathbb{R} \mid 1 < x < 2\} = (1, 2)$

e. $\displaystyle\bigcap_{i=1}^{n} S_i = \left\{x \in \mathbb{R} \,\middle|\, 1 < x < 1 + \frac{1}{n}\right\} = \left(1, 1 + \frac{1}{n}\right)$

f. $\displaystyle\bigcup_{i=1}^{\infty} S_i = \{x \in \mathbb{R} \mid 1 < x < 2\} = (1, 2)$

g. $\displaystyle\bigcap_{i=1}^{\infty} S_i = \emptyset$

Problem 27 and Solution

a. $\{\{a,d,e\},\{b,c\},\{d,f\}\}$ is not a partition of $\{a,b,c,d,e,f\}$ because the sets $\{\{a,d,e\},\{b,c\},\{d,f\}\}$ are not mutually disjoint as $\{b,c\}$ and $\{d,f\}$ both contain the element d.

b. $\{\{w,x,v\},\{u,y,q\},\{p,z\}\}$ is a partition of $\{p,q,u,v,w,x,y,z\}$ because $\{w,x,v\}$, $\{u,y,q\}$, and $\{p,z\}$ are mutually disjoint and $\{w,x,v\}\cup\{u,y,q\}\cup\{p,z\} = \{p,q,u,v,w,x,y,z\}$.

c. $\{\{5,4\},\{7,2\},\{1,3,4\},\{6,8\}\}$ is not a partition of $\{1,2,3,4,5,6,7,8\}$ because the sets $\{\{5,4\},\{7,2\},\{1,3,4\},\{6,8\}\}$ are not mutually disjoint as $\{5,4\}$ and $\{1,3,4\}$ both contain the element 4.

d. $\{\{3,7,8\},\{2,9\},\{1,4,5\}\}$ is not a partition of $\{1,2,3,4,5,6,7,8,9\}$ as $\{3,7,8\}\cup\{2,9\}\cup\{1,4,5\} = \{1,2,3,4,5,7,8,9\} \neq \{1,2,3,4,5,6,7,8,9\}$.

e. $\{\{1,5\},\{4,7\},\{2,8,6,3\}\}$ is a partition of $\{1,2,3,4,5,6,7,8\}$ as $\{1,5\}$, $\{4,6\}$ and $\{2,8,6,3\}$ are mutually disjoint and $\{1,5\}\cup\{4,7\}\cup\{2,8,6,3\} = \{1,2,3,4,5,6,7,8\}$.

Problem 28

Let E be the set of all even integers and O be the set of all odd integers. Is $\{E,O\}$ a partition of $\mathbb{Z}$ the set of all integers? Explain your answer.

Solution

We can rewrite the set E as $E = \{a \in \mathbb{Z} \mid a = 2c \text{ for some integer c}\}$ and the set O as $O = \{b \in \mathbb{Z} \mid b = 2d+1 \text{ for some integer d}\}$. It follows from theorem 4.6.1 which states that there is no integer that is both even an odd that $E \cap O = \emptyset$. Hence E and O are mutually disjoint. Finally, $E \cup O = \{e \in \mathbb{Z} \mid e = 2g \text{ or } e = 2g+1 \text{ for some integer } g\}$. It follows from the quotient-remainder theorem that every integer can be expressed in the form $2g$ or $2g+1$ for some integer g and so $\{E,O\}$ is a partition of Z.

Problem 29

Let R be the set of all real numbers. Is $\{\mathbb{R}^+,\mathbb{R}^-,\{0\}\}$ a partition of $\mathbb{R}$? Explain your answer.

Solution

It follows from definition that $\mathbb{R}^+ \cap \mathbb{R}^- \cap \{0\} = \emptyset$. Hence $\{\mathbb{R}^+,\mathbb{R}^-,\{0\}\}$ are mutually disjoint. Finally $\{\mathbb{R}^+ \cup \mathbb{R}^- \cup \{0\}\} = \mathbb{R}$ and so $\{\mathbb{R}^+,\mathbb{R}^-,\{0\}\}$ form a partition of $\mathbb{R}$.

Problem 30

Let $\mathbb{Z}$ be the set of all integers and let

$$\begin{aligned}
A_0 &= \{n \in \mathbb{Z} \mid n = 4k, \quad \text{for some integer } k\} \\
A_1 &= \{n \in \mathbb{Z} \mid n = 4k+1, \quad \text{for some integer } k\} \\
A_2 &= \{n \in \mathbb{Z} \mid n = 4k+2, \quad \text{for some integer } k\} \\
A_3 &= \{n \in \mathbb{Z} \mid n = 4k+3, \quad \text{for some integer } k\}
\end{aligned}$$

Is $\{A_0, A_1, A_2, A_3\}$ a partition of $\mathbb{Z}$? Explain your answer.

Solution

It follows from the quotient remainder theorem that any integer n can be represented in exactly one of the following ways

$$n = 4k \quad \text{or} \quad n = 4k+1 \quad \text{or} \quad n = 4k+2 \quad \text{or} \quad 4k+3$$

for some integer k. Hence $A_0 \cap A_1 \cap A_2 \cap A_3 = \emptyset$ and $A_0 \cup A_1 \cup A_2 \cup A_3 = \mathbb{Z}$. It follows that $\{A_0, A_1, A_2, A_3\}$ is a partition of $\mathbb{Z}$.

Problem 31 and solution

Suppose $A = \{1, 2\}$ and $B = \{2, 3\}$. Find each of the following:

a. $\mathscr{P}(A \cap B) = \{\emptyset, \{2\}\}$

b. $\mathscr{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

c. $\mathscr{P}(A \cup B) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$

d. $\mathscr{P}(A \times B) = \{\emptyset, \{(1, 2)\}, \{(1, 3)\}, \{(2, 2)\}, \{(2, 3)\}, \{(1, 2), (1, 3)\}, \{(1, 2), (2, 2)\}$
$\{(1, 2), (2, 3)\}, \{(1, 3), (2, 2)\}, \{(1, 3), (2, 3)\}, \{(2, 2), (2, 3)\}$
$\{(1, 2), (1, 3), (2, 2)\}, \{(1, 2), (1, 3), (2, 3)\}, \{(1, 2), (2, 2), (2, 3)\}$
$\{(1, 3), (2, 2), (2, 3)\}, \{(1, 2), (1, 3), (2, 2), (2, 3)\}\}$

Problem 32

a. Suppose $A = \{1\}$ and $B = \{u, v\}$. Find $\mathscr{P}(A \times B)$.

b. Suppose $X = \{a, b\}$ and $Y = \{x, y\}$. Find $\mathscr{P}(X \times Y)$.

Solution

a. $\mathscr{P}(A \times B) = \{\emptyset, \{(1, u)\}, \{(1, v)\}, \{(1, u), (1, v)\}$

b. $\mathscr{P}(X \times Y) = \{\emptyset, \{(a, x)\}, \{(a, y)\}, \{(b, x)\}, \{(b, y)\}, \{(a, x), (a, y)\}, \{(b, x), (b, y)\}$
$\{(a, x), (b, x)\}, \{(a, x), (b, y)\}, \{(a, y), (b, y)\}, \{(a, y), (b, x)\}$
$\{(a, x), (a, y), (b, x)\}, \{(a, x), (b, x), (b, y)\}, \{(a, x), (a, y), (b, y)\}$
$\{(a, y), (b, x), (b, y)\}, \{(a, x), (a, y), (b, x), (b, y)\}\}$

Problem 33 and solution

a. $\mathscr{P}(\emptyset) = \{\emptyset\}$

b. $\mathscr{P}(\mathscr{P}(\emptyset)) = \mathscr{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$

c. $\mathscr{P}(\mathscr{P}(\mathscr{P}(\emptyset))) = \mathscr{P}(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

Problem 34

Let $A_1 = \{1, 2, 3\}, A_2 = \{u, v\}$, and $A_3 = \{m, n\}$. Find each of the following sets.

a. $A_1 \times (A_2 \times A_3)$

b. $(A_1 \times A_2) \times A_3$

c. $A_1 \times A_2 \times A_3$

Solution

a. $$\begin{aligned} A_1 \times (A_2 \times A_3) &= \{1,2,3\} \times \{(u,m),(u,n),(v,m),(v,n)\} \\ &= \{(1,(u,m)),(1,(u,n)),(1,(v,n)),(1,(v,n)) \\ &\quad (2,(u,m)),(2,(u,n)),(2,(v,n)),(2,(v,n)) \\ &\quad (3,(u,m)),(3,(u,n)),(3,(v,n)),(3,(v,n))\} \end{aligned}$$

b. $$\begin{aligned} (A_1 \times A_2) \times A_3 &= \{(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)\} \times \{m,n\} \\ &\quad ((1,u),m),((1,v),m),((2,u),m),((2,v),m),((3,u),m),((3,v),m) \\ &\quad ((1,u),n),((1,v),n),((2,u),n),((2,v),n),((3,u),n),((3,v),n)\} \end{aligned}$$

c. $$\begin{aligned} A_1 \times A_2 \times A_3 &= \{1,2,3\} \times \{u,v\} \times \{m,n\} \\ &= \{(1,u,m),(1,u,n),(1,v,m),(1,v,n) \\ &\quad (2,u,m),(2,u,n),(2,v,m),(2,v,n) \\ &\quad (3,u,m),(3,u,n),(3,v,m),(3,v,n)\} \end{aligned}$$

Problem 35

Let $A = \{a,b\}, B = \{1,2\}$ and $C = \{2,3\}$. Find each of the following sets.

a. $A \times (B \cup C)$

b. $(A \times B) \cup (A \times C)$

c. $(A \times (B \cap C)$

d. $(A \times B) \cap (A \times C)$

Solution

a. $A \times (B \cup C) = \{a,b\} \times \{1,2,3\} = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\}$

b. $$\begin{aligned} (A \times B) \cup (A \times C) &= \{(a,1),(a,2),(b,1),(b,2)\} \cup \{(a,2),(a,3),(b,2),(b,3)\} \\ &= \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3)\} \end{aligned}$$

c. $(A \times (B \cap C) = \{a,b\} \times \{2\} = \{(a,2),(b,2)\}$

d. $$\begin{aligned} (A \times B) \cap (A \times C) &= \{(a,1),(a,2),(b,1),(b,2)\} \cap \{(a,2),(a,3),(b,2),(b,3)\} \\ &= \{(a,2),(b,2)\} \end{aligned}$$

Problem 36

Trace the action of algorithm 6.1.1 on the variables i, j, $found$, and $answer$ for $m = 3$, $n = 3$, and sets A and B represented as the arrays $a[1] = u, a[2] = v, a[3] = w, b[1] = w, b[2] = u, b[3] = v$.

Solution

i	1			2				3	4
j	1	2	3	1	2	3	4	1	
$found$	no	yes		no		yes		yes	
$answer$	$A \subseteq B$								

Problem 36

Trace the action of algorithm 6.1.1 on the variables i, j, $found$, and $answer$ for $m = 4$, $n = 4$, and sets A and B represented as the arrays $a[1] = u, a[2] = v, a[3] = w, a[4] = x, b[1] = r, b[2] = u, b[3] = y, b[4] = z$.

Solution

i	1			2				
j	1	2	3	1	2	3	4	5
$found$	no	yes		no				
$answer$	$A \subseteq B$							$A \nsubseteq B$

Problem 37

Write an algorithm to determine whether a given element x, belongs to a given set, which is represented as an array $a[1], a[2], ..., a[n]$.

Solution

Algorithm 11

Input: x [an element to be found], $a[1], a[2], a[3], ...a[n]$ [an array of n elements which represents the set], n [a positive integer]

$i := 1$
$answer := False$
while ($i \leq n$ and $answer = False$) **do**
    if $a[i] = x$ **then**
        $answer := True$
    end if
    $i := i + 1$
end while

Output: $answer$