Section 5.5

Sterling Jeppson

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Find the first four terms of the recursively defined sequences in 1-8.

Problem 1

$a_k = 2a_{k-1} + k, \quad$ for all integers $k \geq 2$
$a_1 = 2$

Solution

$a_1 = 1$
$a_2 = 2a_1 + 2 = 2 \cdot 1 + 2 = 4$
$a_3 = 2a_2 + 3 = 2 \cdot 4 + 3 = 11$
$a_4 = 2a_3 + 4 = 2 \cdot 11 + 4 = 26$

Problem 2

$b_k = b_{k-1} + 3k, \quad$ for all integers $k \geq 2$
$b_1 = 1$

Solution

$b_1 = 1$
$b_2 = b_1 + 3 \cdot 2 = 1 + 6 = 7$
$b_3 = b_2 + 3 \cdot 3 = 7 + 9 = 16$
$b_4 = b_3 + 3 \cdot 4 = 16 + 12 = 28$

Problem 3

$c_k = k(c_{k-1})^2, \quad$ for all integers $k \geq 1$
$c_0 = 1$

Solution

$c_0 = 1$

$c_1 = 1(c_0)^2 = 1 \cdot 1^2 = 1 \cdot 1 = 1$

$c_2 = 2(c_1)^2 = 2 \cdot 1^2 = 2 \cdot 1 = 2$

$c_3 = 3(c_2)^2 = 3 \cdot 2^2 = 3 \cdot 4 = 12$

Problem 4

$d_k = k(d_{k-1})^2, \quad \text{for all integers } k \geq 1$

$d_0 = 3$

Solution

$d_0 = 3$

$d_1 = 1(d_0)^2 = 1 \cdot 3^2 = 1 \cdot 9 = 9$

$d_2 = 2(d_1)^2 = 2 \cdot 9^2 = 2 \cdot 81 = 162$

$d_3 = 3(d_2)^2 = 3 \cdot 162^2 = 3 \cdot 26244 = 78732$

Problem 5

$s_k = s_{k-1} + 2s_{k-2}, \quad \text{for all integers } k \geq 2$

$s_0 = 1,\ s_1 = 1$

Solution

$s_0 = 1$

$s_1 = 1$

$s_2 = s_1 + 2s_0 = 1 + 2 \cdot 1 = 1 + 2 = 3$

$s_3 = s_2 + 2s_1 = 3 + 2 \cdot 1 = 3 + 2 = 5$

Problem 6

$t_k = t_{k-1} + 2t_{k-2}, \quad \text{for all integers } k \geq 2$

$t_0 = -1,\ t_1 = 2$

Solution

$t_0 = -1$

$t_1 = 2$

$t_2 = t_1 + 2t_0 = 2 + 2 \cdot -1 = 2 - 2 = 0$

$t_3 = t_2 + 2t_1 = 0 + 2 \cdot 2 = 0 + 4 = 4$

Problem 7

$u_k = ku_{k-1} - u_{k-2}, \quad \text{for all integers } k \geq 3$

$u_1 = 1, \; u_2 = 1$

Solution

$u_1 = 1$

$u_2 = 1$

$u_3 = 3 \cdot u_2 - u_1 = 3 \cdot 1 - 1 = 3 - 1 = 2$

$u_4 = 4 \cdot u_3 - u_2 = 4 \cdot 2 - 1 = 8 - 1 = 7$

Problem 8

$v_k = v_{k-1} + v_{k-2} + 1, \quad \text{for all integers } k \geq 3$

$v_1 = 1, \; v_2 = 3$

Solution

$v_1 = 1$

$v_2 = 3$

$v_3 = v_2 + v_1 + 1 = 3 + 1 + 1 = 5$

$v_4 = v_3 + v_2 + 1 = 5 + 3 + 1 = 9$

Problem 9

Let $a_0, a_1, a_2, ...$ be defined by the formula $a_n = 3n + 1$, for all integers $n \geq 0$. Show that this sequence satisfies the recurrence relation $a_k = a_{k-1} + 3$, for all integers $k \geq 1$.

Solution

By definition of $a_1, a_2, a_3, ...$, for each integer $k \geq 1$,

$$a_k = 3k + 1 \quad \text{and} \quad a_{k-1} = 3(k-1) + 1$$

It follows that

$$\begin{aligned} a_{k-1} + 3 &= 3(k-1) + 1 + 3 \\ &= 3k - 3 + 4 \\ &= 3k + 1 = a_k \end{aligned}$$

Problem 10

Let $b_0, b_1, b_2, ...$ be defined by the formula $b_n = 4^n$, for all integers $n \geq 0$. Show that this sequence satisfies the recurrence relations $b_k = 4b_{k-1}$, for all integers $k \geq 1$.

Solution

By definition of $b_0, b_1.b_2, ...,$ for all integers $k \geq 1$,

$$b_k = 4^k \quad \text{and} \quad b_{k-1} = 4^{k-1}$$

It follows that

$$\begin{aligned} 4b_{k-1} &= 4 \cdot 4^{k-1} \\ &= 4^k = b_k \end{aligned}$$

Problem 11

Let $c_0, c_1, c_2, ...$ be defined by the formula $c_n = 2^n - 1$ for all integers $n \geq 0$. Show that his sequence satisfies the recurrence relation $c_k = 2c_{k-1} + 1$, for all integers $k \geq 1$.

Solution

By definition of $c_0, c_1, c_2, ...,$ for all integers $k \geq 1$,

$$c_k = 2^k - 1 \quad \text{and} \quad c_{k-1} = 2^{k-1} - 1$$

It follows that

$$\begin{aligned} 2c_{k-1} + 1 &= 2(2^{k-1} - 1) + 1 \\ &= 2 \cdot 2^{k-1} - 2 + 1 \\ &= 2^k - 1 = c_k \end{aligned}$$

Problem 12

let $s_0, s_1, s_2, ...$ be defined by the formula $s_n = \frac{(-1)^n}{n!}$ for all integers $n \geq 0$. Show that this sequence satisfies the recurrence relation $s_k = \frac{-s_{k-1}}{k}$, for all integers $k \geq 1$.

Solution

By definition of $s_0, s_1, s_2, ...,$ for all integers $k \geq 1$,

$$s_k = \frac{(-1)^k}{k!} \quad \text{and} \quad s_{k-1} = \frac{(-1)^{k-1}}{(k-1)!}$$

It follows that

$$\begin{aligned} \frac{-s_{k-1}}{k} &= \frac{(-1)^1(-1)^{k-1}}{(k-1)!} \cdot \frac{1}{k} \\ &= \frac{(-1)^k}{k!} = s_k \end{aligned}$$

Problem 13

Let $t_0, t_1, t_2, ...$ be defined by the formula $t_n = 2 + n$ for all integers $n \geq 0$. Show that this sequence satisfies the recurrence relation $t_k = 2t_{k-1} - t_{k-2}$ for all integers $k \geq 2$.

Solution

By definition of $t_0, t_1, t_2, ...$, for all integers $k \geq 2$,

$$t_k = 2 + k \quad \text{and} \quad t_{k-1} = 2 + k - 1 = 1 + k \quad \text{and} \quad t_{k-2} = 2 + k - 2 = k$$

It follows that

$$\begin{aligned} 2t_{k-1} - t_{k-2} &= 2(1 + k) - k \\ &= 2 + 2k - k \\ &= 2 + k = t_k \end{aligned}$$

Problem 14

Let $d_0, d_1, d_2, ...$ be defined by the formula $d_n = 3^n - 2^n$ for all integers $n \geq 0$. Show that this sequence satisfies the recurrence relation $d_k = 5d_{k-1} - 6d_{k-2}$, for all integers $k \geq 2$.

Solution

By definition of $d_0, d_1, d_2, ...$ for all integers $k \geq 2$,

$$d_k = 3^k - 2^k \quad \text{and} \quad d_{k-1} = 3^{k-1} - 2^{k-1} \quad \text{and} \quad d_{k-2} = 3^{k-2} - 2^{k-2}$$

It follows that

$$\begin{aligned} 5d_{k-1} - 6d_{k-2} &= 5(3^{k-1} - 2^{k-1}) - 6(3^{k-2} - 2^{k-2}) \\ &= 5 \cdot 3^{k-1} - 5 \cdot 2^{k-1} - 6 \cdot 3^{k-2} + 6 \cdot 2^{k-2} \\ &= 5 \cdot 3^{k-1} - 5 \cdot 2^{k-1} - 2 \cdot 3 \cdot 3^{k-2} + 3 \cdot 2 \cdot 2^{k-2} \\ &= 5 \cdot 3^{k-1} - 5 \cdot 2^{k-1} - 2 \cdot 3^{k-1} + 3 \cdot 2^{k-1} \\ &= 5 \cdot 3^{k-1} - 2 \cdot 3^{k-1} + 3 \cdot 2^{k-1} - 5 \cdot 2^{k-1} \\ &= 3 \cdot 3^{k-1} - 2 \cdot 2^{k-1} \\ &= 3^k - 2^k = d_k \end{aligned}$$

Problem 15

For the sequence of Catalan numbers defined in example 5.6.4, prove that for all integers $n \geq 1$,

$$C_n = \frac{1}{4n + 2} \binom{2n + 2}{n + 1}$$

Solution

Proof. It follows from the definition of the Catalan numbers that

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Now work from the left-hand side of our formula

$$\begin{aligned}
&\frac{1}{4n+2}\binom{2n+2}{n+1} \\
&\quad = \frac{1}{2(2n+1)} \cdot \frac{(2n+2)!}{(n+1)!((2n+2)-(n+1)!)} \\
&\quad = \frac{1}{2(\cancel{2n+1})} \cdot \frac{(2n+2)(\cancel{2n+1})(2n)!}{(n+1)!(n+1)!)} \\
&\quad = \frac{(2n+2)(2n)!}{2 \cdot (n+1)(n+1)(n!)(n!)} \\
&\quad = \frac{(\cancel{2n+2})(2n)!}{(\cancel{2n+2})(n+1)(n!)(n!)} \\
&\quad = \frac{(2n)!}{(n!)(n!)} = \frac{1}{4n+2}\binom{2n+2}{n+1} = C_n
\end{aligned}$$

□

Problem 16

Use the recurrence relation and values for the Tower of Hanoi sequence, $m_1, m_2, m_3, ...$ discussed in example 5.6.5 to compute m_7 and m_8.

Solution

We have from example 5.6.5 that the recurrence relation that describes the Tower of Hanoi sequence $m_1, m_2, m_3, ...$ is

$$\begin{aligned}
m_k &= 2m_{k-1} + 1 \\
m_1 &= 1
\end{aligned}$$

We have from this same example that $m_6 = 63$. Therefore,

$$\begin{aligned}
m_7 &= 2m_6 + 1 = 2 \cdot 63 + 1 = 127 \\
m_8 &= 2m_7 + 1 = 2 \cdot 127 + 1 = 255
\end{aligned}$$

Problem 17

Tower of Hanoi Adjacency Requirement: Suppose that in addition to the requirement that they never move a larger disk on top of a smaller one, the priests who move the disks of the Tower of Hanoi are also allowed only to move disks one by one from one pole to an adjacent pole. Assume poles A and C are at the two ends of the row and pole B is in the middle. Let

$$a_n = \begin{bmatrix} \text{the minimum number of moves} \\ \text{needed to transfer a tower of } n \\ \text{disks from pole } A \text{ to pole } C \end{bmatrix}$$

(a) Find a_1, a_2, and a_3.

(b) Find a_4.

(c) Find a recurrence relation for $a_1, a_2, a_3, ...$

Solution

(a) $a_1 = 2$

$a_2 = 2$ (moves to move the top disk from A to C)
$+ 1$ (move to move the bottom disk from A to B)
$+ 2$ (moves to move top disk from C to A)
$+ 1$ (move to move bottom disk from B to C)
$+ 2$ (moves to move top disk from A to C)
$= 8$

$a_3 = 8 + 1 + 8 + 1 + 8 = 26$

(b) $a_4 = 26 + 1 + 26 + 1 + 26 = 80$

(c) $a_k = 3a_{k-1} + 2$
$a_1 = 2 \quad$ for all integers $k \geq 2$

Problem 18

Tower of Hanoi with Adjacency Requirement: Suppose the same situation as in exercise 17. Let

$$b_n = \begin{bmatrix} \text{the minimum number of moves} \\ \text{needed to transfer a tower of } n \\ \text{disks from pole } A \text{ to pole } B \end{bmatrix}$$

(a) Find b_1, b_2, and b_3.

(b) Find b_4.

(c) Show that $b_k = a_{k-1} + 1 + b_{k-1}$ for all integers $k \geq 2$, where $a_1, a_2, a_3, ...$ is the sequence defined in exercise 17.

(d) Show that $b_k \leq 3b_{k-1} + 1$ for all integers $k \geq 2$.

(e) Show that $b_k = 3b_{k-1} + 1$ for all integers $k \geq 2$.

Solution

(a) $b_1 = 1$

$$\begin{aligned} b_2 &= 2 \text{ (moves to move the top disk from } A \text{ to } C) \\ &\quad + 1 \text{ (move to move the bottom disk from } A \text{ to } B) \\ &\quad\quad + 1 \text{ (move to move top disk from } C \text{ to } B) \\ &= 4 \end{aligned}$$

$$\begin{aligned} b_3 &= 8 \text{ (moves to move the top 2 disks from } A \text{ to } C) \\ &\quad + 1 \text{ (move to move the bottom disk from } A \text{ to } B) \\ &\quad\quad + 4 \text{ (moves to move 2 disks from } C \text{ to } B) \\ &= 13 \end{aligned}$$

(b) $b_4 = 26 + 1 + 13 = 40$

(c) $b_k = \left[\begin{array}{l}\text{minimum number}\\ \text{of moves needed}\\ \text{to move } k-1\\ \text{disks from } A \text{ to } C\end{array}\right] + \left[\begin{array}{l}\text{minimum number}\\ \text{of moves needed}\\ \text{to move top}\\ \text{disk from } A \text{ to } B\end{array}\right] + \left[\begin{array}{l}\text{minimum number}\\ \text{of moves needed}\\ \text{to move } k-1\\ \text{disk from } C \text{ to } B\end{array}\right]$

$b_k = a_{k-1} + 1 + b_{k-1}$

(d) *Proof.* For all integers $k \geq 2$, let $a_1, a_2, a_3, \ldots$ be defined by specifying that $a_1 = 2$ and $a_k = 3a_{k-1} + 2$ and let $b_1, b_2, b_3, \ldots$ be defined by specifying that $b_1 = 1$, $b_2 = 4$, and $b_k = a_{k-1} + 1 + b_{k-1}$. Now let the property $P(k)$ be the inequality

$$b_k \leq 3b_{k-1} + 1 \quad \text{for all integers } k \geq 2$$

Show that $P(2)$ is true: From part (a) $b_2 = 4$ and $b_1 = 1$. Thus $3b_1 + 1 = 3 + 1 = 4$ and $4 \leq 4$.

Show that for all integers $k \geq 2$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 2$ and suppose that

$$b_k \leq 3b_{k-1} + 1 \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$b_{k+1} \leq 3b_k + 1 \qquad \leftarrow P(k+1)$$

But by the definition of $b_1, b_2, b_3, \ldots$ we have that

$$\begin{aligned} b_{k+1} &= a_k + 1 + b_k && \text{by definition of } b_1, b_2, b_3, \ldots \\ &\leq a_k + 1 + 3b_{k-1} + 1 && \text{by inductive hypothesis} \\ &= a_k + 3b_{k-1} + 2 \\ &= 3a_{k-1} + 2 + 3b_{k-1} + 2 && \text{by definition of } a_1, a_2, a_3, \ldots \\ &= 3(b_k - 1 + b_{k-1}) + 2 + 3b_{k-1} + 2 && a_{k-1} = b_k - 1 - b_{k-1} \\ &= 3b_k - 3 + 3b_{k-1} + 2 + 3b_{k-1} + 2 \\ &= 3b_k + 1 \end{aligned}$$

□

(e) **Lemma.** *In the most efficient transfer of n disks from pole A to pole C there is a point at which all disks are on pole B.*

Proof. Let the property $P(n)$ be the lemma to be proved.

Show that $P(1)$ is true: With the same adjacency requirements as in problem 17 a transfer of one disk from pole A to pole C will result in that disk being on pole B after the first step. Thus $P(0)$ is true.

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that in the most efficient transfer of k disks from pole A to pole C, there is a point at which all disks are on pole B.

We must show that this implies that in the most efficient transfer of $k+1$ disks from pole A to pole C, there is a point at which all disks are on pole B. The first step in such a transfer is to move the top k disks from pole A to pole C. The next step is to move the $(k+1)$st disk from pole A to pole B. Now the k disks on pole C need to be moved back to pole A so that the $(k+1)$st disk can be placed on pole C. But by the inductive hypothesis at some point in this process all k disks will be on pole B and since the $(k+1)$st disk is on the bottom of pole B all disks will be on the middle pole. □

Theorem. $b_k = 3b_{k-1} + 1$ *for all integers* $k \geq 2$.

Proof. Recall that a_k is the least number of steps required to transfer k disks from pole A to pole C and b_k is the least number of steps required to transfer k disks from pole A to pole B. We know from the above lemma that on some step, $step_{\text{mid}}$, in the most efficient process to transfer k disks from pole A to pole C that all disks will be on pole B. This number of steps is the same as b_k. Suppose not. Then either b_k is not the least number of steps to transfer k disks from pole A to pole B or a_k is not the least number of steps to transfer k disks from pole A to pole C. Both of these are contradictions and so $step_{\text{mid}} = b_k$. Now the least number of steps to transfer to transfer the k disks from pole B to C is b_k. From this it follows that $b_k = .5a_k$. Thus,

$$
\begin{aligned}
b_k &= .5(3a_{k-1} + 2) && a_k = 3a_{k-1} + 2 \\
b_k &= 1.5a_{k-1} + 1 \\
b_k &= 1.5(b_k - 1 - b_{k-1}) + 1 && a_{k-1} = b_k - 1 - b_{k-1} \\
b_k &= 1.5b_k - 1.5 - 1.5b_{k-1} + 1 \\
-.5b_k &= -1.5b_{k-1} - .5 \\
b_k &= 3b_{k-1} + 1
\end{aligned}
$$

□

Problem 19

Four-Pole Tower of Hanoi: Suppose that the Tower of Hanoi problem has four poles in a row instead of three. Disks can be transferred one by one from one pole to any other pole, but at no time may a larger disk be placed on top of a smaller disk. Let s_n be the minimum number of moves needed to transfer the entire tower of n disks from the left-most to the right-most pole.

(a) Find s_1, s_2, s_3.

(b) Find s_4.

(c) Show that $s_k \leq 2_{k-2} + 3$ for all integers $k \geq 3$.

Solution

(a) $s_1 = 1$

$$
\begin{aligned}
s_2 &= 1 \text{ (move to move the top disk from } A \text{ to } B\text{)}\\
&\quad + 1 \text{ (move to move the bottom disk from } A \text{ to } D\text{)}\\
&\qquad + 1 \text{ (move to move the top disk from } B \text{ to } D\text{)}\\
&= (1+1+1) = 3
\end{aligned}
$$

$$
\begin{aligned}
s_3 &= s_1 \text{ (moves to move top 3-2 disks from } A \text{ to } B\text{)}\\
&\quad + 1 \text{ (move to move 2nd to last disk from } A \text{ to } D\text{)}\\
&\qquad + 1 \text{ (move to move bottom disk from } A \text{ to } D\text{)}\\
&\qquad\quad + 1 \text{ (move to move 2nd to last disk from } C \text{ to } D\text{)}\\
&\qquad\qquad + s_1 \text{ (moves to move top 3-2 disks from } B \text{ to } D\text{)}\\
&= s_1 + (1+1+1) + s_1 = 5
\end{aligned}
$$

(b)

$$
\begin{aligned}
s_4 &= s_2 \text{ (moves to move top 4-2 disks from } A \text{ to } B\text{)}\\
&\quad + 1 \text{ (move to move 2nd to last disk from } A \text{ to } D\text{)}\\
&\qquad + 1 \text{ (move to move bottom disk from } A \text{ to } D\text{)}\\
&\qquad\quad + 1 \text{ (move to move 2nd to last disk from } C \text{ to } D\text{)}\\
&\qquad\qquad + s_2 \text{ (moves to move top 4-2 disks from } B \text{ to } D\text{)}\\
&= s_2 + (1+1+1) + s_2 = 9
\end{aligned}
$$

(c) For all integers $k \geq 3$, let s_k be defined to be the minimum number of moves required to transfer the entire tower of k disks from pole A to pole D. Also let t_k be some number of moves after which an entire tower of k disks can be transferred from pole A to pole D. More specifically,

$$
t_k = \begin{bmatrix} \text{minimum number} \\ \text{of moves needed} \\ \text{to move top } k-2 \\ \text{disks from } A \text{ to } B \end{bmatrix} + \begin{bmatrix} \text{number of moves} \\ \text{needed to transfer} \\ \text{largest two disks} \\ \text{from pole } A \text{ to } D \end{bmatrix} + \begin{bmatrix} \text{minimum number} \\ \text{of moves needed} \\ \text{to move top } k-2 \\ \text{disks from } B \text{ to } D \end{bmatrix}
$$

$$
t_k = s_{k-2} + 3 + s_{k-2} = 2s_{k-2} + 3
$$

Clearly after t_k steps k disks can be transferred from pole A to pole D. However, it may not be the least number of steps. Thus by definition, $s_k \leq 2s_{k-2} + 3$

Problem 20

Tower of Hanoi Poles in a Circle: Suppose that instead of being lined up in a row, the three poles for the original Tower of Hanoi are placed in a circle. The monks move the disks one by one from one pole to another, but they may only move disks one over in a clockwise direction and they may never move a larger disk on top of a smaller one. Let c_n be the minimum number of moves needed to transfer a pile of n disks from one pole to the next adjacent pole in the clockwise direction.

(a) Justify the inequality $c_k \leq 4c_{k-1} + 1$ for all integers $k \geq 2$.

(b) The expression $4c_{k-1} + 1$ is not the minimum number of moves needed to transfer a pile of k disks from one pole to another. Explain, for example, why $c_3 \neq 4c_2 + 1$.

Solution

(a) Let c_k be defined to be the minimum number of steps needed to transfer k disks from pole A to pole B. Also let t_k be some number of moves after which k disks can be transferred from pole A to pole B. More specifically,

$$t_k = \begin{bmatrix} \text{minimum} \\ \text{number} \\ \text{of moves} \\ \text{needed to} \\ \text{move top} \\ k-1 \text{ disks} \\ \text{from} \\ A \text{ to } B \end{bmatrix} + \begin{bmatrix} \text{minimum} \\ \text{number} \\ \text{of moves} \\ \text{needed to} \\ \text{move top} \\ k-1 \text{ disks} \\ \text{from} \\ B \text{ to } C \end{bmatrix} + \begin{bmatrix} \text{minimum} \\ \text{number} \\ \text{of moves} \\ \text{needed to} \\ \text{move top} \\ \text{disk from} \\ A \text{ to } B \end{bmatrix} + \begin{bmatrix} \text{minimum} \\ \text{number} \\ \text{of moves} \\ \text{needed to} \\ \text{move top} \\ k-1 \text{ disks} \\ \text{from} \\ C \text{ to } A \end{bmatrix} + \begin{bmatrix} \text{minimum} \\ \text{number} \\ \text{of moves} \\ \text{needed to} \\ \text{move top} \\ k-1 \text{ disks} \\ \text{from} \\ A \text{ to } B \end{bmatrix}$$

$t_k = c_{k-1} + c_{k-1} + 1 + c_{k-1} + c_{k-1}$

$t_k = 4c_{k-1} + 1$

Clearly after t_k steps k disks can be transferred from pole A to pole B. However, it may not be the least number of steps. Thus by definition, $c_k \leq 4c_{k-1} + 1$.

(b) First we need to compute c_2.

$c_2 = 1$ (moves to move top disk from A to B)
$+ 1$ (move to move top disk from B to C)
$+ 1$ (move to move bottom disk from A to B)
$+ 1$ (move to move top disk from C to A)
$+ 1$ (moves to move top disk from A to B)
$= 1 + 1 + 1 + 1 + 1 = 5$

This is obviously c_2 because the first two moves are the only options, the third move only has two options one of which leads to a repeated position, the fourth move only has one option, and the fifth move has the option to complete the puzzle or not and so the completion move is taken. Thus $c_2 = 5$. Now we need to compute some number of steps that can move 3 disks from A to B which is less than t_3. Start with the following process:

$$\begin{aligned}
&(\text{moves to move top disk from } A \text{ to } B) \\
&\quad + 1\ (\text{move to move top disk from } B \text{ to } C) \\
&\qquad + 1\ (\text{move to move middle disk from } A \text{ to } B) \\
&\qquad\quad + 1\ (\text{move to move top disk from } C \text{ to } A) \\
&\qquad\qquad + 1\ (\text{move to move middle disk from } B \text{ to } C) \\
&\qquad\qquad\quad + 1\ (\text{move to top disk from } A \text{ to } B) \\
&\qquad\qquad\qquad + 1\ (\text{move to top disk from } B \text{ to } C) \\
&= 1+1+1+1+1+1+1 = 7
\end{aligned}$$

After these 7 steps have been completed we can move the bottom disk onto pole B in 1 step. Then with an analogous process as above we can move the top and middle disk from C to B in 7 steps. Now the total number of steps to move all three disks from A to B is $7+7+1 = 15 < 21 = 4c_2+1$.

Problem 21

Double Tower of Hanoi: In this variation of the Tower of Hanoi there are three poles in a row and $2n$ disks, two of each of n different sizes, where n is any positive integer. Initially one of the three poles contains all the disks placed on top of each other in pairs of decreasing size. Disks are transferred one by one from one pole to another, but at no time may a larger disk be placed on top of a smaller disk. However, a disk may be placed on top of one of the same size. Let t_n be the minimum number of moves needed to transfer a tower of $2n$ disks from one pole to another.

(a) Find t_1 and t_2.

(b) Find t_3.

(c) Find a recurrence relation for $t_1, t_2, t_3, ...$

Solution

(a) $t_1 = 2$ (moves to move the top 2 disks from A to B)

$$\begin{aligned}
t_2 &= t_1\ (\text{moves to move top 2 disks from } A \text{ to } C) \\
&\quad + 2\ (\text{moves to move bottom 2 disks from } A \text{ to } B) \\
&\qquad + t_1\ (\text{moves to move top 2 disks from } C \text{ to } B) \\
&= t_1 + 2 + t_1 = 2t_1 + 2 = 6
\end{aligned}$$

(b) $t_3 = t_2$ (moves to move top 2(k-1) disks from A to C)
$\quad + 2$ (moves to move bottom 2 disks from A to B)
$\quad + t_2$ (moves to move 2(k-1) disks from C to B)
$= t_2 + 2 + t_2 = 2t_2 + 2 = 14$

(c) $$t_k = \begin{bmatrix} \text{minimum number} \\ \text{of moves needed} \\ \text{to move } 2(k-1) \\ \text{disks from } A \text{ to } C \end{bmatrix} + \begin{bmatrix} \text{minimum number} \\ \text{of moves needed} \\ \text{to move top 2} \\ \text{disks from } A \text{ to } B \end{bmatrix} + \begin{bmatrix} \text{minimum number} \\ \text{of moves needed} \\ \text{to move } 2(k-1) \\ \text{disks from } C \text{ to } B \end{bmatrix}$$

$$t_k = t_{k-1} + 2 + t_{k-1}$$
$$t_k = 2t_{k-1} + 2 \quad \text{for all integers } k \geq 2$$
$$t_1 = 2$$

Problem 22

Fibonacci Variation: A single pair of rabbits (male and female) is born at the beginning of a year. Assume the following conditions (which are more realistic than Fibonacci's):

1. Rabbit pairs are not fertile during their first month of life but thereafter give birth to four new male/female pairs at the end of every month.
2. No rabbits die.

(a) Let r_n = the number of pairs of rabbits alive at the end of month n, for each integer $n \geq 1$, and let $r_0 = 1$. Find a recurrence relation for $r_0, r_1, r_2, ...$

(b) Compute $r_0, r_1, r_2, r_3, r_4, r_5$, and r_6.

(c) How many rabbits will there be at the end of the year?

Solution

(a) $$\begin{bmatrix} \text{the number} \\ \text{of rabbit} \\ \text{pairs alive} \\ \text{at the end} \\ \text{of month } k \end{bmatrix} = \begin{bmatrix} \text{the number} \\ \text{of rabbit} \\ \text{pairs alive} \\ \text{at the end} \\ \text{of month} \\ k-1 \end{bmatrix} + \begin{bmatrix} \text{the number} \\ \text{of rabbit} \\ \text{pairs born} \\ \text{at the end} \\ \text{of month } k \end{bmatrix}$$

$$r_k = \begin{bmatrix} \text{the number} \\ \text{of rabbit} \\ \text{pairs alive} \\ \text{at the end} \\ \text{of month} \\ k-1 \end{bmatrix} + \begin{bmatrix} \text{4 times the} \\ \text{number} \\ \text{of rabbit} \\ \text{pairs born} \\ \text{at the end} \\ \text{of month } k-2 \end{bmatrix}$$

$$r_k = r_{k-1} + 4r_{k-2} \quad \text{for all integers } k \geq 2$$
$$r_0 = 1, \quad r_1 = 1$$

(b) $r_0 = 1$

$r_1 = 1$

$r_2 = r_1 + 4r_0 = 1 + 4 = 5$

$r_3 = r_2 + 4r_1 = 5 + 4 = 9$

$r_4 = r_3 + 4r_2 = 9 + 20 = 29$

$r_5 = r_4 + 4r_3 = 29 + 27 = 65$

$r_6 = r_5 + 4r_4 = 56 + 116 = 181$

(c) $r_{12} = 49,661$ rabbit pairs which is 99,322 rabbits.

Problem 23

Fibonacci Variation: A single pair of rabbits (male and female) is born at the beginning of the year. Assume the following conditions:

1. Rabbit pairs are not fertile during their first two months of life but thereafter give birth to three new male/female pairs at the end of every month.
2. No rabbits die.

(a) Let s_n = the number of pairs of rabbits alive at the end of month n, for each integer $n \geq 1$, and let $s_0 = 1$. Find a recurrence relation for $s_0, s_1, s_2, \ldots$

(b) Compute $s_0, s_1, s_2, s_3, s_4, s_5$, and s_6.

(c) How many rabbits will there be at the end of the year?

Solution

(a) $$\begin{bmatrix} \text{the number} \\ \text{of rabbit} \\ \text{pairs alive} \\ \text{at the end} \\ \text{of month } k \end{bmatrix} = \begin{bmatrix} \text{the number} \\ \text{of rabbit} \\ \text{pairs alive} \\ \text{at the end} \\ \text{of month} \\ k-1 \end{bmatrix} + \begin{bmatrix} \text{the number} \\ \text{of rabbit} \\ \text{pairs born} \\ \text{at the end} \\ \text{of month } k \end{bmatrix}$$

$$s_k = \begin{bmatrix} \text{the number} \\ \text{of rabbit} \\ \text{pairs alive} \\ \text{at the end} \\ \text{of month} \\ k-1 \end{bmatrix} + \begin{bmatrix} \text{3 times the} \\ \text{number} \\ \text{of rabbit} \\ \text{pairs born} \\ \text{at the end} \\ \text{of month } k-3 \end{bmatrix}$$

$$s_k = s_{k-1} + 4s_{k-2} \quad \text{for all integers } k \geq 3$$

$$s_0 = 1, \quad s_1 = 1, \quad s_2 = 1$$

(b) $s_0 = 1$
$s_1 = 1$
$s_2 = 1$
$s_3 = s_2 + 3s_0 = 1 + 3 = 4$
$s_4 = s_3 + 3s_1 = 4 + 3 = 7$
$s_5 = s_4 + 3s_2 = 7 + 3 = 10$
$s_6 = s_5 + 3s_3 = 10 + 12 = 22$

(c) $s_{12} = 904$ rabbit pairs which is 1,808 rabbits.

In 24-34, $F_0, F_1, F_2, ...$ is the Fibonacci sequence.

Problem 24

Use the recurrence relation and values for $F_0, F_1, F_2, ...$ given in example 5.6.6 to compute F_{13} and F_{14}.

Solution

From example 5.6.6 we have that the recurrence relation for $F_0, F_1, F_2, ...$ is

$$F_k = F_{k-1} + F_{k-2} \quad \text{for all integers } k \geq 2$$
$$F_0 = 1, \quad F_1 = 1$$

We also know from example 5.6.6 that $F_11 = 144$ and $F_12 = 233$. Thus,

$$F_{13} = F_{12} + F_{11} = 233 + 144 = 377$$
$$F_{14} = F_{13} + F_{12} = 377 + 233 = 610$$

Problem 25

The Fibonacci sequence satisfies the recurrence relation $F_k = F_{k-1} + F_{k-2}$, for all integers $k \geq 2$.

(a) Explain why the following is true:

$$F_{k+1} = F_k + F_{k-1} \quad \text{for all integers } k \geq 1.$$

(b) Write an equation expressing F_{k+2} in terms of F_{k+1} and F_k.

(c) Write an equation expressing F_{k+3} in terms of F_{k+2} and F_{k+1}.

Solution

(a) Each term in the Fibonacci sequence beyond the second is equal to the sum of the previous terms. For any integer $k \geq 1$, the previous terms are F_k and F_{k-1}. Hence for all integers $k \geq 1$, $F_{k+1} = F_k + F_{k-1}$.

(b) $F_{k+2} = F_{k+1} + F_k, \quad$ for all integers $k \geq 2$.

(c) $F_{k+3} = F_{k+2} + F_{k+1}, \quad$ for all integers $k \geq 3$.

Problem 26

Prove that $F_k = 3F_{k-3} + 2F_{k-4}$ for all integers $k \geq 4$.

Solution

Proof. We show that for all integers $k \geq 4$, $F_k = 3F_{k-3} + 2F_{k-4}$ with repeated use of the definition of the Fibonacci sequence.

$$\begin{aligned}
F_k &= F_{k-1} + F_{k-2} \\
&= F_{k-2} + F_{k-3} + F_{k-3} + F_{k-4} \\
&= F_{k-3} + F_{k-4} + F_{k-3} + F_{k-3} + F_{k-4} \\
&= 3F_{k-3} + 2F_{k-4}
\end{aligned}$$

□

Problem 27

Prove that $F_k^2 - F_{k-1}^2 = F_k F_{k+1} - F_{k-1} F_{k+1}$, for all integers $k \geq 1$.

Solution

Proof.

$$\begin{aligned}
F_k^2 - F_{k-1}^2 &= (F_k + F_{k-1})(F_k - F_{k-1}) && (a^2 - b^2) = (a+b)(a-b) \\
&= F_{k+1}(F_k - F_{k-1}) && \text{by definition of } F_0, F_1, F_2, \ldots \\
&= F_k F_{k+1} - F_{k-1} F_{k+1}
\end{aligned}$$

□

Problem 28

Prove that $F_{k+1}^2 - F_k^2 - F_{k-1}^2 = 2F_k F_{k-1}$, for all integers $k \geq 1$.

Solution

Proof.

$$\begin{aligned}
F_{k+1}^2 - F_k^2 - F_{k-1}^2 &= (F_k + F_{k-1})^2 - F_k^2 - F_{k-1}^2 && F_{k+1} = F_k + F_{k-1} \\
&= F_k^2 + 2F_k F_{k-1} + F_{k-1}^2 - F_k^2 - F_{k-1}^2 \\
&= 2F_k F_{k-1} + (F_k^2 - F_k^2) + (F_{k-1}^2 - F_{k-1}^2) \\
&= 2F_k F_{k-1}
\end{aligned}$$

□

Problem 29

Prove that $F_{k+1}^2 - F_k^2 = F_{k-1}F_{k+2}$, for all integers $k \geq 1$.

Solution

Proof.

$$\begin{aligned} F_{k+1}^2 - F_k^2 &= (F_{k+1} + F_k)(F_{k+1} - F_k) && a^2 - b^2 = (a+b)(a-b) \\ &= F_{k+2}(F_{k+1} - F_k) && \text{by definition of } F_0, F_1, F_2, \ldots \\ &= F_{k+2}F_{k-1} && F_{k+1} - F_k = F_{k-1} \end{aligned}$$

□

Problem 30

Use mathematical induction to prove that for all integers $n \geq 0$,

$$F_{n+2}F_n - F_{n+1}^2 = (-1)^n$$

Solution

Proof. Let n be any integer with $n \geq 0$ and let the property $P(n)$ be the equation

$$F_{n+2}F_n - F_{n+1}^2 = (-1)^n \qquad \leftarrow P(n)$$

Show that $P(0)$ is true: It follows from the definition of $F_0, F_1, F_2, \ldots$ that

$$F_2F_0 - F_1^2 = 2 \cdot 1 - 1^2 = 2 - 1 = 1 \quad \text{and} \quad (-1)^0 = 1$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$F_{k+2}F_k - F_{k+1}^2 = (-1)^k \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$F_{k+3}F_{k+1} - F_{k+2}^2 = (-1)^{k+1} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} &F_{k+3}F_{k+1} - F_{k+2}^2 \\ &\quad = (F_{k+2} + F_{k+1})F_{k+1} - F_{k+2}^2 && \text{by definition of } F_0, F_1, F_2, \ldots \\ &\quad = F_{k+1}^2 + F_{k+1}F_{k+2} - F_{k+2}^2 \\ &\quad = F_{k+2}F_k - (-1)^k + F_{k+1}F_{k+2} - F_{k+2}^2 && \text{by inductive hypothesis } F_{k+1}^2 = F_{k+2}F_k - (-1)^k \\ &\quad = F_{k+2}(F_k + F_{k+1} - F_{k+2}) + (-1)^1(-1)^k \\ &\quad = F_{k+2}(F_{k+2} - F_{k+2}) + (-1)^1(-1)^k && \text{by definition of } F_0, F_1, F_2, \ldots \\ &\quad = F_{k+2}(0) + (-1)^{k+1} && (-1)^1(-1)^k = (-1)^{k+1} \\ &\quad = (-1)^{k+1} \end{aligned}$$

Hence $P(k+1)$ is true. □

Problem 31

Use Strong mathematical induction to prove that $F_n < 2^n$, for all integers $n \geq 1$.

Solution

Proof. Let n be any integer with $n \geq 1$ and let the property $P(n)$ be the inequality

$$F_n < 2^n \qquad \leftarrow P(n)$$

Show that $P(1)$ and $P(2)$ are true: It follows from the definition of $F_0, F_1, F_2, \ldots$ that $F_1 = 1$ and $F_2 = 2$. Since $2^1 = 2$ and $2^2 = 4$ it follows that $F_1 < 2^1$ and $F_2 < 2^2$ and so $P(1)$ and $P(2)$ are true.

Show that for all integers $k \geq 2$, $P(i)$ is true for all integers i from 1 through $k \implies P(k+1)$: Let k be any integer with $k \geq 2$ and suppose that for all integers i with $1 \leq i \leq k$

$$F_i < 2^i \qquad \leftarrow \text{inductive hypothesis}$$

We must show that this implies that

$$F_{k+1} < 2^{k+1} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
F_{k+1} &= F_k + F_{k-1} && \text{by definition of } F_0, F_1, F_2, \ldots \\
&< 2^k + 2^{k-1} && \text{by inductive hypothesis} \\
&= 2^k + 2^{-1} \cdot 2^k \\
&= 2^k(1 + 2^{-1}) \\
&= 2^k \left(\frac{3}{2}\right) \\
&< 2^k(2) && \tfrac{3}{2} < 2 \\
&= 2^{k+1}
\end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Problem 32

Let $F_0, F_1, F_2, \ldots$ be the Fibonacci sequence defined in section 5.6. Prove that for all integers $n \geq 0$, $\gcd(F_{n+1}, F_n) = 1$.

Solution

Lemma 4.8.3. If a and b are any integers not both zero, and if q and r are any integers such that

$$a = bq + r,$$

then

$$\gcd(a, b) = \gcd(b, r).$$

Theorem. $\forall n \in \mathbb{Z}^{nonneg}$, *gcd*$(F_{n+1},\ F_n) = 1$.

Proof. Let n be any integer with $n \geq 0$ and let the property $P(n)$ be the equation

$$\gcd(F_{n+1},\ F_n) = 1 \qquad \leftarrow P(n)$$

Show that $P(0)$ is true: It follows from the definition of $F_0, F_1, F_2, \ldots$ that $F_0 = 1$ and $F_1 = 1$. It follows from the definition of gcd that $\gcd(1,\ 1) = 1$. Hence $\gcd(F_1,\ F_0) = 1$ and so $P(0)$ is true.

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$\gcd(F_{k+1},\ F_k) = 1 \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$\gcd(F_{k+2},\ F_{k+1}) = 1 \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
\gcd(F_{k+2},\ F_{k+1}) &= \gcd(F_{k+1} + F_k,\ F_{k+1}) && \text{by definition of } F_0, F_1, F_2, \ldots \\
&= \gcd(F_{k+1},\ F_k) && \text{by lemma 4.8.3 with } q = 1 \text{ and } r = F_k \\
&= 1 && \text{by inductive hypothesis}
\end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Problem 33

It turns out that the Fibonacci sequence satisfies the following explicit formula: For all integers $F_n \geq 0$,

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

Verify that the sequence defined by this formula satisfies the recurrence relation $F_k = F_{k-1} + F_{k-2}$ for all integers $k \geq 2$.

Solution

By definition of the explicit formula for the Fibonacci sequence, for all integers $k \geq 2$,

$$F_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right] \quad F_{k-1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\right]$$

$$F_{k-2} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right]$$

It follows that

$$
\begin{aligned}
F_{k-1}+&F_{k-2}\\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right]\\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k} + \left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right]\\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k} + \left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right]\\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k}\left(1+\frac{2}{1+\sqrt{5}}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\left(1+\frac{2}{1-\sqrt{5}}\right)\right]\\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k}\cdot\frac{3+\sqrt{5}}{1+\sqrt{5}} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\cdot\frac{3-\sqrt{5}}{1-\sqrt{5}}\right]\\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k}\cdot\frac{3+\sqrt{5}}{1+\sqrt{5}}\cdot\frac{1-\sqrt{5}}{1-\sqrt{5}} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\cdot\frac{3-\sqrt{5}}{1-\sqrt{5}}\cdot\frac{1+\sqrt{5}}{1+\sqrt{5}}\right]\\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k}\cdot\frac{1+\sqrt{5}}{2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\cdot\frac{1-\sqrt{5}}{2}\right]\\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]\\
&= F_k
\end{aligned}
$$

Problem 34

Find $\lim\limits_{n\to\infty}\left(\frac{F_{n+1}}{F_n}\right)$, assuming that the limit exists.

Solution

Assuming that the limit exists let $L = \lim\limits_{n\to\infty}\left(\frac{F_{n+1}}{F_n}\right)$. Now,

$$
\begin{aligned}
L &= \lim_{n\to\infty}\left(\frac{F_{n+1}}{F_n}\right)\\
&= \lim_{n\to\infty}\left(\frac{F_n + F_{n-1}}{F_n}\right) && \text{by definition of } F_0, F_1, F_2, \ldots\\
&= \lim_{n\to\infty}\left(1+\frac{F_{n-1}}{F_n}\right)\\
&= 1+\lim_{n\to\infty}\left(\frac{1}{\frac{F_n}{F_{n-1}}}\right)\\
&= 1+\frac{1}{\lim\limits_{x\to\infty}\frac{F_x+1}{F_x}} = 1+\frac{1}{L} && \text{let } x = n-1
\end{aligned}
$$

Place the equation for L above in a form that can be solved by the quadratic equation.

$$1 + \frac{1}{L} = L \iff L - 1 - \frac{1}{L} = 0 \iff L^2 - L - 1 = 0$$

Now solve by the quadratic equation

$$L = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

However since

$$L = \lim_{n\to\infty} \left(\frac{F_{n+1}}{F_n}\right)$$

and since every term in the Fibonacci sequence is at least as large as the previous term we cannot have $L < 1$. It Follows that

$$L = \frac{1+\sqrt{5}}{2}$$

Problem 35

Prove that $\lim_{n\to\infty} \left(\frac{F_{n+1}}{F_n}\right)$ exists.

Solution

Lemma 0.1. *The sequence of Fibonacci ratios defined as*

$$a_n = \frac{F_{2n}}{F_{2n-1}}, \quad n \geq 1$$

is decreasing and bounded below.

Proof. Recall from exercise 5.6.30 that $\forall n \in \mathbb{Z}^{nonneg}$, $F_{n+2}F_n - F_{n+1}^2 = (-1)^n$. It follows that,

$$\begin{aligned}
F_{2n+1}F_{2n-1} - F_{2n}^2 &= (-1)^{2n-1} && \text{substitute } 2n-1 \text{ for } n \\
F_{2n+1}F_{2n-1} - F_{2n}^2 &= -1 && (-1)^{2n-1} = (-1)^{2n}(-1)^{-1} = 1(-1) = -1 \\
F_{2n+1}F_{2n-1} - F_{2n}^2 &< 0 \\
F_{2n+1}F_{2n-1} &< F_{2n}^2 \\
F_{2n}F_{2n-1} + F_{2n+1}F_{2n-1} &< F_{2n}^2 + F_{2n}F_{2n-1} && \text{add } F_{2n}F_{2n-1} \text{ to both sides} \\
F_{2n-1}(F_{2n} + F_{2n+1}) &< F_{2n}(F_{2n} + F_{2n-1}) \\
F_{2n-1}F_{2n+2} &< F_{2n}F_{2n+1} && \text{by definition of } F_0, F_1, F_2, \ldots \\
\frac{F_{2n+2}}{F_{2n+1}} &< \frac{F_{2n}}{F_{2n-1}}
\end{aligned}$$

We have shown that a_n is decreasing. Now we need to show that it is bounded below. But this is so because all Fibonacci numbers are positive and the quotient of any two positive numbers is positive. Hence $a_n > 0$. $\square$

Lemma 0.2. *The sequence of Fibonacci ratios defined as*

$$b_n = \frac{F_{2n+1}}{F_{2n}}, \quad n \geq 0$$

is increasing and bounded above.

Proof. Recall from exercise 5.6.30 that $\forall n \in \mathbb{Z}^{nonneg}$, $F_{n+2}F_n - F_{n+1}^2 = (-1)^n$. It follows that,

$$\begin{aligned}
F_{2n+2}F_{2n} - F_{2n+1}^2 &= (-1)^{2n} && \text{substitute } 2n \text{ for } n \\
F_{2n+2}F_{2n} - F_{2n+1}^2 &= 1 && (-1)^2 = 1,\ 1^n = 1 \\
F_{2n+2}F_{2n} - F_{2n+1}^2 &> 0 \\
F_{2n+2}F_{2n} &> F_{2n+1}^2 \\
F_{2n}F_{2n+1} + F_{2n+2}F_{2n} &> F_{2n+1}^2 + F_{2n}F_{2n+1} && \text{add } F_{2n}F_{2n+1} \text{ to both sides} \\
F_{2n}(F_{2n+1} + F_{2n+2}) &> F_{2n+1}(F_{2n+1} + F_{2n}) \\
F_{2n}F_{2n+3} &> F_{2n+1}F_{2n+2} && \text{by definition of } F_0, F_1, F_2 \ldots \\
\frac{F_{2n+3}}{F_{2n+2}} &> \frac{F_{2n+1}}{F_{2n}}
\end{aligned}$$

We have shown that b_n is increasing. Now we need to show that it is bounded above. But for all integers $n > 1$,

$$b_n = \frac{F_{2n+1}}{F_{2n}} = \frac{F_{2n} + F_{2n-1}}{F_{2n}} = 1 + \frac{F_{2n-1}}{F_{2n}} \leq 2$$

The previous inequality follows from the fact that the $(n+1)$st term in the Fibonacci sequence is greater than the nth term for all $n \geq 1$. ☐

Theorem. $\lim_{n\to\infty} \left(\frac{F_{n+1}}{F_n}\right)$ *exists.*

Proof. Define a sequence of Fibonacci ratios as

$$c_n = \frac{F_{n+1}}{F_n}, \quad n \geq 0$$

Now by lemma 1 and 2 above we know that a_n and b_n are bounded monotonic sequences. It follows from the monotonic sequence theorem that a_n and b_n are convergent. This implies that there exist limits L_a and L_b such that $\lim_{n\to\infty} a_n = L_a$ and $\lim_{n\to\infty} b_n = L_b$. We must show that $L_a = L_b$. In order to do this we first show that for all $n \in \mathbb{Z}^+$,

$$b_n - a_n = \frac{F_{2n+1}}{F_{2n}} - \frac{F_{2n}}{F_{2n-1}} = \frac{F_{2n+1}F_{2n-1} - F_{2n}^2}{F_{2n}F_{2n-1}} = \frac{1}{F_{2n}F_{2n-1}}$$

It follows that since the Fibonacci sequence is unbounded and increasing we can make $b_n - a_n$ arbitrarily small. Suppose that $L_a \neq L_b$. Then there exists a real number $\epsilon > 0$ such that $\epsilon = |L_a - L_b|$. Now since $\lim_{n\to\infty} a_n = L_a$ there exists an integer N_a such $n > N_a \implies |a_n - L_a| < \epsilon/4$. Also since $\lim_{n\to\infty} b_n = L_b$ there exists an integer N_b such $n > N_b \implies |b_n - L_b| < \epsilon/4$. Finally since $b_n - a_n$ can be made arbitrarily small there exists an integer N_d such such that $n > N_d \implies |b_n - a_n| < \epsilon/4$. Now let $N = \max\{N_a, N_b, N_d\}$ and it follows that for any $n > N$,

$$\begin{aligned}
\epsilon &= |L_a - L_b| \\
&= |L_a - L_b + (a_n - a_n) + (b_n - b_n)| \\
&= |(a_n - L_a) + (L_b - b_n) + (b_n - a_n)| \\
&\leq |a_n - L_a| + |L_b - b_n| + |b_n - a_n| && \text{by the triangle inequality} \\
&= |a_n - L_a| + |b_n - L_b| + |b_n - a_n| && \text{lemma 4.4.5} \\
&< \epsilon/4 + \epsilon/4 + \epsilon/4 \\
&= 3\epsilon/4
\end{aligned}$$

But this is a contradiction as $\epsilon \not< 3\epsilon/4$ and so our supposition that $L_a \neq L_b$ is false and hence $L_a = L_b$.

Finally let $L = L_a = L_b$. It follows that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$. Let a real number $\epsilon > 0$ be given. Then there exist integers N_a and N_b such that $n > N_a \implies |a_n - L| < \epsilon$ and $n > N_b \implies |b_n - L| < \epsilon$. Let $N = \max\{2N_a - 1, 2N_b\}$ and let $n > N$. Then $c_n = a_k$ for some $k > N_a$ or $c_n = b_k$ for some $k > N_b$. Thus no matter if n is even or odd $|c_n - L| < \epsilon$ and so c_n is convergent and $\lim_{n\to\infty} c_n = L$. □

Problem 36

Define $x_0, x_1, x_2, \ldots$ as follows:

$$\begin{aligned}
x_k &= \sqrt{2 + x_{k-1}} \quad \text{for all integers } k \geq 1 \\
x_0 &= 0
\end{aligned}$$

Find $\lim_{n\to\infty} x_n$. (Assume that the limit exists.)

Solution

Assuming that the limit exists let $L = \lim\limits_{n\to\infty} \sqrt{2 + x_{n-1}}$. Now,

$$\begin{aligned}
L &= \lim_{n\to\infty} \sqrt{2 + x_{n-1}} \\
L^2 &= 2 + \lim_{n\to\infty} x_{n-1} \\
L^2 &= 2 + \lim_{j\to\infty} x_j && \text{let } j = n - 1 \\
L^2 &= 2 + L
\end{aligned}$$

It follows that $L^2 - L - 2 = 0$. By factoring we obtain $(L-2)(L+1) = 0$. Hence $L = 2$ or $L = -1$. However, the sequence x_n is an increasing sequence and so $L \not< 0$. It follows that $\lim\limits_{n\to\infty} x_n = 2$.

Problem 37

Compound Interest: Suppose a certain amount of money is deposited in an account paying 4% annual interest compounded quarterly. For each positive integer n, let R_n = the amount on deposit at the end of the nth quarter, assuming no additional deposits or withdrawals, and let R_0 be the initial amount deposited.

(a) Find a recurrence relation for $R_0, R_1, R_2, ...$

(b) If $R_0 = \$5000$, find the amount of money on deposit at the end of one year.

(c) Find the APR for the account.

Solution

(a) Since the annual interest of 4% is compounded quarterly, the quarterly interest rate is $4\%/4 = 1\%$. Now the amount in the account at the end of any quarter is equal to the amount in the account at the end of the previous quarter + the interest earned on the account during the quarter. Hence,

$$\begin{aligned}
R_k &= R_{k-1} + .01R_{k-1} \\
R_k &= R_{k-1}(1 + .01) \\
R_k &= 1.01R_{k-1} \quad \text{for all integers } k \geq 1
\end{aligned}$$

(b) We need to find the amount of money on deposit at the end of four quarters. Hence we need to find R_4 with $R_0 = \$5000$.

$$\begin{aligned}
R_0 &= 5000 \\
R_1 &= 1.01R_0 = 1.01(5000) = 5050 \\
R_2 &= 1.01R_1 = 1.01(5050) = 5100.5 \\
R_3 &= 1.01R_2 = 1.01(5100.5) \approx 5151.51 \\
R_4 &= 1.01R_3 \approx 1.01(5151.51) \approx 5203.02
\end{aligned}$$

Thus after one year there will be \$5203.02 on deposit(rounded to the nearest cent).

(c) The annual percentage rate (APR) is the percentage increase in the value of the account over a one-year period. It follows that the APR is

$$\frac{5203.02 - 5000}{5000} = .040604 = 4.0604\%$$

Problem 38

Compound Interest: Suppose a certain amount of money is deposited in an account paying 3% annual interest compounded monthly. For each positive integer n, let $S_n =$ the amount on deposit at the end of the nth month, and let S_0 be the initial amount deposited.

(a) Find a recurrence relation for $S_0, S_1, S_2, ...$ assuming no additional deposits or withdrawals during the year.

(b) If $S_0 = \$10,000$, find the amount of money on deposit at the end of one year.

(c) Find the APR for the account.

Solution

(a) Since the annual interest of 3% is compounded monthly, the monthly interest rate is $3\%/12 = .25\%$. Now the amount in the account at the end of any month is equal to the amount in the account at the end of the previous month + the interest earned on the account during the month. Hence,

$$\begin{aligned}
S_k &= S_{k-1} + .0025S_{k-1} \\
S_k &= S_{k-1}(1 + .0025) \\
S_k &= 1.0025S_{k-1} \quad \text{for all integers } k \geq 1
\end{aligned}$$

(b) We need to find the amount of money on deposit at the end of twelve months. Hence we need to find S_{12} with $S_0 = \$10,000$

$$\begin{aligned}
S_0 &= 10000 \\
S_1 &= 1.0025S_0 = 1.0025(10000) = 10025 \\
S_2 &= 1.0025S_1 = 1.0025(10025) \approx 10050.06 \\
S_3 &= 1.0025S_2 \approx 1.0025(10050.06) \approx 10075.19 \\
&\quad\ldots\ldots \\
S_{11} &= 1.0025S_{10} \approx 1.0025(10252.83) \approx 10278.46 \\
S_{12} &= 1.0025S_{11} \approx 1.0025(10278.46) \approx 10304.16
\end{aligned}$$

Thus after one year there will be \$10304.16 on deposit(rounded to the nearest cent).

(c) The annual percentage rate (APR) is the percentage increase in the value of the account over a one-year period. It follows that the APR is

$$\frac{10304.16 - 10000}{10000} = .030416 = 3.0416\%$$

Problem 39

With each step you take when climbing a staircase, you can move up either one stair of two stairs. As a result, you can climb the entire staircase taking one stair at a time, taking two at a time, or taking a combination of one and two-stair increments. For each integer $n \geq 1$, if the staircase consists of n stairs, let c_n be the number of different ways to climb the staircase. Find a recurrence relation for c_1, c_2, c_3.

Solution

With the given rules, the last step in climbing a staircase consisting of n stairs involves either taking a one stair step or a two stair step. There are c_{n-1} ways to climb the n stairs with a final one stair step and c_{n-2} ways to climb the n stairs with a final two stair step. Thus $c_n = c_{n-1} + c_{n-2}$. Now $c_1 = 1$ because if there is only one step then there is only one way to climb it, namely taking a single stair step. Also $c_2 = 2$ because if there are only 2 steps then there are two ways to climb it, namely taking two single stair steps or a single double stair step. Finally we have that

$$
\begin{aligned}
&c_k = c_{k-1} + c_{k-2} \quad \text{for all integers } k \geq 3 \\
&c_1 = 1,\ c_2 = 2
\end{aligned}
$$

Problem 40

A set of blocks contains blocks of heights 1, 2, and 4 centimeters. Imagine constructing towers by piling blocks of different heights directly on top of one another. (A tower of height 6 cm could be obtained using six 1-cm blocks, three 2-cm blocks, one 2-cm block with one 4-cm block on top, one 4-cm block with one 2-cm block on top, and so forth.) Let t_n be the number of ways to construct a tower of height n cm using blocks from the set. (Assume an unlimited supply of blocks of each size.) Find a recurrence relation for $t_1, t_2, t_3, ...$

Solution

With the given rules, the last step in constructing a block of n-cm involves either adding adding a 1-cm, 2-cm, or 4-cm block. There are t_{n-1} ways to complete the construction with a 1-cm block, t_{n-2} ways to complete the construction with a 2-cm block, and t_{n-4} ways to complete the construction with a 4-cm block. Thus $t_n = t_{n-1} + t_{n-2} + t_{n-4}$. Also $t_1 = 1$, $t_2 = 2$, $t_3 = 3$, $t_4 = 6$. The initial conditions are derived using the recurrence relation from problem 39 and then adding 1 to t_4 to account for the fact that the construction can be completed using a single 4-cm block. Finally we have that

$$
\begin{aligned}
&t_k = t_{k-1} + t_{k-2} + t_{k-4} \quad \text{for all integers } k \geq 5 \\
&t_1 = 1,\ t_2 = 2,\ t_3 = 3,\ t_4 = 6
\end{aligned}
$$

Problem 41

Use the recursive definition of summation, together with mathematical induction, to prove the generalized distributive law that for all positive integers n, if $a_1, a_2, ..., a_n$ and c are real numbers, then

$$\sum_{i=1}^{n} ca_i = c\left(\sum_{i=1}^{n} a_i\right)$$

Solution

Proof. Let the property $P(n)$ be the equation

$$\sum_{i=1}^{n} ca_i = c\left(\sum_{i=1}^{n} a_i\right) \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: Let $n = 1$ and then

$$\sum_{i=1}^{1} ca_i = ca_1 \quad \text{and} \quad c\left(\sum_{i=1}^{1} a_i\right) = c(a_1) = ca_1$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$\sum_{i=1}^{k} ca_i = c\left(\sum_{i=1}^{k} a_i\right) \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$\sum_{i=1}^{k+1} ca_i = c\left(\sum_{i=1}^{k+1} a_i\right) \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
\sum_{i=1}^{k+1} ca_i &= ca_{k+1} + \sum_{i=1}^{k} ca_i && \text{by definition of } \sum \\
&= ca_{k+1} + c\left(\sum_{i=1}^{k} a_i\right) && \text{by inductive hypothesis} \\
&= c\left(a_{k+1} + \sum_{i=1}^{k} a_i\right) && \text{by distributive property} \\
&= c\left(\sum_{i=1}^{k+1} a_i\right) && \text{by definition of } \sum
\end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Problem 42

use the recursive definition of product, together with mathematical induction, to prove that for all positive integers n, if $a_1, a_2, ..., a_n$ and $b_a, b_2, ...b_n$ are real numbers, then

$$\prod_{i=1}^{n}(a_i b_i) = \left(\prod_{i=1}^{n} a_i\right)\left(\prod_{i=1}^{n} b_i\right)$$

Solution

Proof. Let the property $P(n)$ be the equation

$$\prod_{i=1}^{n}(a_i b_i) = \left(\prod_{i=1}^{n} a_i\right)\left(\prod_{i=1}^{n} b_i\right) \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: Let $n = 1$, then

$$\prod_{i=1}^{1}(a_i b_i) = a_1 b_1 \quad \text{and} \quad \left(\prod_{i=1}^{1} a_i\right)\left(\prod_{i=1}^{1} b_i\right) = (a_1)(b_1) = a_1 b_1$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$\prod_{i=1}^{k}(a_i b_i) = \left(\prod_{i=1}^{k} a_i\right)\left(\prod_{i=1}^{k} b_i\right) \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$\prod_{i=1}^{k+1}(a_i b_i) = \left(\prod_{i=1}^{k+1} a_i\right)\left(\prod_{i=1}^{k+1} b_i\right) \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
\prod_{i=1}^{k+1}(a_i b_i) &= a_{k+1}b_{k+1} \cdot \prod_{i=1}^{k}(a_i b_i) && \text{by definition of } \prod \\
&= a_{k+1}b_{k+1} \cdot \left(\prod_{i=1}^{k} a_i\right)\left(\prod_{i=1}^{k} b_i\right) && \text{by inductive hypothesis} \\
&= \left(a_{k+1}\prod_{i=1}^{k} a_i\right)\left(b_{k+1}\prod_{i=1}^{k} b_i\right) && \text{by associative and commutative laws of algebra} \\
&= \left(\prod_{i=1}^{k+1} a_i\right)\left(\prod_{i=1}^{k+1} b_i\right) && \text{by definition of } \prod
\end{aligned}$$

which is the right hand side of $P(k+1)$. ☐

Problem 43

Proof. Use the recursive definition of product, together with mathematical induction, to prove that for all positive integers n, if $a_1, a_2, ..., a_n$ and c are real numbers then

$$\prod_{i=1}^{n}(ca_i) = c^n \left(\prod_{i=1}^{n} a_i\right)$$

Solution

Let the property $P(n)$ be the equation

$$\prod_{i=1}^{n}(ca_i) = c^n \left(\prod_{i=1}^{n} a_i\right) \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: Let $n = 1$, then

$$\prod_{i=1}^{1}(ca_i) = ca_1 \quad \text{and} \quad c^1 \left(\prod_{i=1}^{1} a_i\right) = c^1(a_1) = ca_1$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$\prod_{i=1}^{k}(ca_i) = c^k \left(\prod_{i=1}^{k} a_i\right) \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$\prod_{i=1}^{k+1}(ca_i) = c^{k+1} \left(\prod_{i=1}^{k+1} a_i\right) \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
\prod_{i=1}^{k+1}(ca_i) &= ca_{k+1} \cdot \prod_{i=1}^{k}(ca_i) && \text{by definition of } \prod \\
&= ca_{k+1} \cdot c^k \left(\prod_{i=1}^{k} a_i\right) && \text{by inductive hypothesis} \\
&= (c \cdot c^k)\left(a_{k+1} \prod_{i=1}^{k} a_i\right) && \text{by associative and commutative laws of algebra} \\
&= c^{k+1} \left(\prod_{i=1}^{k+1} a_i\right) && \text{by definition of } \prod
\end{aligned}$$

which is the right-hand side of $P(k+1)$. $\square$

Problem 44

The triangle inequality for absolute value states that for all real numbers a and b, $|a+b| \leq |a| + |b|$. Use the recursive definition of summation, the triangle inequality, the definition of absolute value, and mathematical induction to prove that for all positive integers n, if $a_1, a_2, \ldots, a_n$ are real numbers, then

$$\left|\sum_{i=1}^{n} a_i\right| \leq \sum_{i=1}^{n} |a_i|$$

Solution

Proof. Let the property $P(n)$ be the inequality

$$\left|\sum_{i=1}^{n} a_i\right| \leq \sum_{i=1}^{n} |a_i| \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: Let $n = 1$, then

$$\left|\sum_{i=1}^{1} a_i\right| = |a_1| \quad \text{and} \quad \sum_{i=1}^{1} |a_i| = |a_1|$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$\left|\sum_{i=1}^{k} a_i\right| \leq \sum_{i=1}^{k} |a_i| \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$\left|\sum_{i=1}^{k+1} a_i\right| \leq \sum_{i=1}^{k+1} |a_i| \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
\left|\sum_{i=1}^{k+1} a_i\right| &= \left|a_{k+1} + \sum_{i=1}^{k} a_i\right| && \text{by definition of } \textstyle\sum \\
&\leq |a_{k+1}| + \left|\sum_{i=1}^{k} a_i\right| && \text{by the triangle inequality} \\
&= |a_{k+1}| + \sum_{i=1}^{k} |a_i| && \text{by inductive hypothesis} \\
&= \sum_{i=1}^{k+1} |a_i| && \text{by definition of } \textstyle\sum
\end{aligned}$$

which is the right-hand side of $P(k+1)$. $\square$