Section 5.7

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Problem 1

The formula

$$1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$$

is true for all integers $n \geq 1$. Use this fact to solve each of the following problems:

(a) If k is an integer such that $k \geq 2$, find a formula for the expression $1 + 2 + 3 + ... + (k-1)$.

(b) If n is an integer such that $n \geq 1$, find a formula for the expression $3 + 2 + 4 + 6 + 8 + ... + 2n$.

(c) If n is an integer such that $n \geq 1$, find a formula for the expression $3 + 3 \cdot 2 + 3 \cdot 3 + ... + 3 \cdot n + n$.

Solution

(a)
$$\begin{aligned} 1 + 2 + 3 + ... + (k-1) &= \frac{(k-1)((k-1)+1)}{2} \\ &= \frac{(k-1)k}{2} \end{aligned}$$

(b)
$$\begin{aligned} 3 + 2 + 4 + 6 + 8 + ... + 2n &= 3 + 2(1 + 2 + 3 + 4 + ... + n) \\ &= 3 + 2 \cdot \frac{n(n+1)}{2} \\ &= n^2 + n + 3 \end{aligned}$$

(c)
$$\begin{aligned} 3 + 3 \cdot 2 + 3 \cdot 3 + ... + 3 \cdot n + n &= 3(1 + 2 + 3 + ... + n) + n \\ &= 3 \cdot \frac{n(n+1)}{2} + \frac{2n}{2} \\ &= \frac{3n^2 + 5n}{2} \end{aligned}$$

Problem 2

The formula

$$1 + r + r^2 + ... + r^n = \frac{r^{n+1} - 1}{r - 1}$$

is true for all real numbers $r \neq 1$ and for all integers $n \geq 0$. Use this fact to solve each of the following problems:

(a) If i is an integer such that $i \geq 0$, then find a formula for the expression $1 + 2 + 2^2 + ... + 2^{i-1}$.

(b) If n is an integer such that $n \geq 1$, find a formula for the expression $3^{n-1} + 3^{n-2} + ... + 3^2 + 3 + 1$.

(c) If n is an integer such that $n \geq 2$, find a formula for the expression $2^n + 2^{n-2} \cdot 3 + 2^{n-3} \cdot 3 + ... + 2^2 \cdot 3 + 2 \cdot 3 + 3$.

(d) If n is an integer such that $n \geq 1$, find a formula for the expression

$$2^n - 2^{n-1} + 2^{n-2} - 2^{n-3} + ... + (-1)^{n-1} \cdot 2 + (-1)^n$$

Solution

(a) $1 + 2 + 2^2 + ... + 2^{i-1} = \dfrac{2^{(i-1)+1} - 1}{2 - 1} = 2^i - 1$

(b) $3^{n-1} + 3^{n-2} + ... + 3^2 + 3 + 1 = \dfrac{3^{(n-1)+1} - 1}{3 - 1} = \dfrac{3^n - 1}{2}$

(c) $2^n + 2^{n-2} \cdot 3 + 2^{n-3} \cdot 3 + ... + 2^2 \cdot 3 + 2 \cdot 3 + 3$

$$\begin{aligned}
&= 2^n + 3(2^{n-2} + 2^{n-3} + ... + 2^2 + 2 + 1) \\
&= 2^n + 3 \cdot \frac{2^{(n-2)+1} - 1}{2 - 1} \\
&= 2^n + 3 \cdot (2^{n-1} - 1) \\
&= 2 \cdot 2^{n-1} + 3 \cdot 2^{n-1} - 3 \\
&= 2^{n-1}(2 + 3) - 3 \\
&= 5 \cdot 2^{n-1} - 3
\end{aligned}$$

(d) $2^n - 2^{n-1} + 2^{n-2} - 2^{n-3} + ... + (-1)^{n-1} \cdot 2 + (-1)^n$

$$\begin{aligned}
&= (-1)^{2n}2^n + (-1)^{2n-1}2^{n-1} + (-1)^{2n-2}2^{n-2} + (-1)^{2n-3}2^{n-3} + ... + (-1)^{n-1} \cdot 2 + (-1)^n \\
&= (-1)^n[(-1)^n 2^n + (-1)^{n-1}2^{n-1} + (-1)^{n-2}2^{n-2} + (-1)^{2n-2}2^{n-3} + ... + (-1) \cdot 2 + 1] \\
&= (-1)^n[(-2)^n + (-2)^{n-1} + (-2)^{n-2} + (-2)^{n-3} + ... + (-2)^1 + (-2)^0] \\
&= (-1)^n \cdot \frac{(-2)^{n+1} - 1}{-3} \\
&= (-1)^{n+1} \cdot \frac{(-2)^{n+1} - 1}{3} \\
&= \frac{2^{n+1} - (-1)^{n+1}}{3}
\end{aligned}$$

In each of 3-15 a sequence is defined recursively. Use iteration to guess an explicit formula for the sequence. Use the formulas from Section 5.2 to simplify your answers whenever possible.

Problem 3

$a_k = ka_{k-1}, \quad$ for all integers $k \geq 1$

$a_0 = 1$

Solution

$a_0 = 1$

$a_1 = 1 \cdot a_0 = 1 \cdot 1$

$a_2 = 2 \cdot a_1 = 2 \cdot 1$

$a_3 = 3 \cdot a_2 = 3 \cdot (2 \cdot 1)$

$a_4 = 4 \cdot a_3 = 4 \cdot (3 \cdot 2 \cdot 1)$

Guess: $a_n = n(n-1)...3 \cdot 2 \cdot 1 = n! \quad$ for all integers $n \geq 0$.

Problem 4

$b_k = \dfrac{b_{k-1}}{1 + b_{k-1}}, \quad$ for all integers $k \geq 1$

$b_0 = 1$

Solution

$b_0 = 1 = \dfrac{1}{1}$

$b_1 = \dfrac{b_0}{1 + b_0} = \dfrac{1}{1+1} = \dfrac{1}{2}$

$b_2 = \dfrac{b_1}{1 + b_1} = \dfrac{1/2}{1+1/2} = \dfrac{1/2}{3/2} = \dfrac{1}{3}$

$b_3 = \dfrac{b_2}{1 + b_2} = \dfrac{1/3}{1+1/3} = \dfrac{1/3}{4/3} = \dfrac{1}{4}$

Guess: $b_n = \dfrac{1}{n+1} \quad$ for all integers $n \geq 0$.

Problem 5

$c_k = 3c_{k-1} + 1, \quad$ for all integers $k \geq 2$

$c_1 = 1$

Solution

$c_1 = 1$

$c_2 = 3c_1 + 1 = 3 \cdot 1 + 1 = 3 + 1$

$c_3 = 3c_2 + 1 = 3 \cdot (3 + 1) + 1 = 3^2 + 3 + 1$

$c_4 = 3c_3 + 1 = 3 \cdot (3^2 + 3 + 1) + 1 = 3^3 + 3^2 + 3 + 1$

Guess: $c_n = 1 + 3 + 3^2 + ... + 3^{n-1}$

$$= \frac{3^{(n-1)+1} - 1}{3 - 1} \quad \text{by theorem 5.2.3 with } r = 3$$

$$= \frac{3^n - 1}{2} \quad \text{for all integers } n \geq 1$$

Problem 6

$d_k = 2d_{k-1} + 3, \quad \text{for all integers } k \geq 2$

$d_1 = 2$

Solution

$d_1 = 2$

$d_2 = 2d_1 + 3 = 2 \cdot 2 + 3 = 2^2 + 3$

$d_3 = 2d_2 + 3 = 2 \cdot (2^2 + 3) + 3 = 2^3 + 2 \cdot 3 + 3$

$d_4 = 2d_3 + 3 = 2 \cdot (2^3 + 2 \cdot 3 + 3) + 3 = 2^4 + 2^2 \cdot 3 + 2 \cdot 3 + 3$

Guess: $d_n = 2^n + 3(2^{n-2} + 2^{n-3} + ... + 2^2 + 2 + 1)$

$$= 2^n + 3 \cdot \frac{2^{(n-2)+1} - 1}{2 - 1} \quad \text{by theorem 5.2.3 with } r = 2$$

$$= 2^n + 3(2^{n-1} - 1)$$

$$= 2^n + 3 \cdot 2^{n-1} - 3$$

$$= 2^{n-1}(2 + 3) - 3$$

$$= 5 \cdot 2^{n-1} - 3 \quad \text{for all integers } n \geq 1$$

Problem 7

$e_k = 4e_{k-1} + 5, \quad \text{for all integers } k \geq 1$

$e_0 = 2$

Solution

$e_0 = 2$

$e_1 = 4e_0 + 5 = 4 \cdot 2 + 5$

$e_2 = 4e_1 + 5 = 4(4 \cdot 2 + 5) + 5 = 4^2 \cdot 2 + 4 \cdot 5 + 5$

$e_3 = 4e_2 + 5 = 4(4^2 \cdot 2 + 4 \cdot 5 + 5) + 5 = 4^3 \cdot 2 + 4^2 \cdot 5 + 4 \cdot 5 + 5$

Guess: $e_n = 2 \cdot 4^n + 5(4^{n-1} + 4^{n-2} + ... + 4^2 + 4 + 1)$

$$
\begin{aligned}
&= 2 \cdot 4^n + 5 \cdot \frac{4^{(n-1)+1} - 1}{4 - 1} \qquad \text{by theorem 5.2.3 with } r = 4 \\
&= \frac{6 \cdot 4^n}{3} + 5 \cdot \frac{4^n - 1}{3} \\
&= \frac{6 \cdot 4^n + 5 \cdot 4^n - 5}{3} \\
&= \frac{4^n(6 + 5) - 5}{3} \\
&= \frac{11 \cdot 4^n - 5}{3} \quad \text{for all integers } n \geq 0
\end{aligned}
$$

Problem 8

$f_k = f_{k-1} + 2^k, \quad$ for all integers $k \geq 2$

$f_1 = 1$

Solution

$f_1 = 1$

$f_2 = f_1 + 2^2 = 1 + 2^2$

$f_3 = f_2 + 2^3 = 1 + 2^2 + 2^3$

$f_4 = f_3 + 2^4 = 1 + 2^2 + 2^3 + 2^4$

Guess: $f_n = 1 + 2^2(1 + 2 + 2^2 + ... + 2^{n-3} + 2^{n-2})$

$$
\begin{aligned}
&= 1 + 2^2 \cdot \frac{2^{(n-2)+1} - 1}{2 - 1} \qquad \text{by theorem 5.2.3 with } r = 2 \\
&= 1 + 2^2(2^{n-1} - 1) \\
&= 1 + 2^{(n-1)+2} - 2^2 \\
&= 2^{n+1} - 3 \quad \text{for all integers } n \geq 1
\end{aligned}
$$

Problem 9

$g_k = \dfrac{g_{k-1}}{g_{k-1} + 2}, \quad$ for all integers $k \geq 2$

$g_1 = 1$

Solution

$g_1 = 1$

$g_2 = \dfrac{g_1}{g_1+2} = \dfrac{1}{1+2} = \dfrac{1}{3}$

$g_3 = \dfrac{g_2}{g_2+2} = \dfrac{1/3}{1/3+2} = \dfrac{1/3}{7/3} = \dfrac{1}{7}$

$g_4 = \dfrac{g_3}{g_3+2} = \dfrac{1/7}{1/7+2} = \dfrac{1/7}{15/7} = \dfrac{1}{15}$

Guess: $g_n = \dfrac{1}{2^n - 1}$ for all integers $n \geq 1$.

Problem 10

$h_k = 2^k - h_{k-1},$ for all integers $k \geq 1$

$h_0 = 1$

Solution

$h_0 = 1$

$h_1 = 2^1 - h_0 = 2^1 - 1$

$h_2 = 2^2 - h_1 = 2^2 - (2^1 - 1) = 2^2 - 2^1 + 1$

$h_3 = 2^3 - h_2 = 2^3 - (2^2 - 2^1 + 1) = 2^3 - 2^2 + 2^1 - 1$

$h_4 = 2^4 - h_3 = 2^4 - (2^3 - 2^2 + 2^1 - 1) = 2^4 - 2^3 + 2^2 - 2^1 + 1$

$$
\begin{aligned}
\textit{Guess: } h_n &= 2^n - 2^{n-1} + 2^{n-2} - 2^{n-3} + ... + (-1)^{n-1} \cdot 2 + (-1)^n \\
&= (-1)^{2n}2^n + (-1)^{2n-1}2^{n-1} + (-1)^{2n-2}2^{n-2} + (-1)^{2n-3}2^{n-3} + ... + (-1)^{n-1} \cdot 2 + (-1)^n \\
&= (-1)^n[(-1)^n 2^n + (-1)^{n-1}2^{n-1} + (-1)^{n-2}2^{n-2} + (-1)^{2n-2}2^{n-3} + ... + (-1) \cdot 2 + 1] \\
&= (-1)^n[(-2)^n + (-2)^{n-1} + (-2)^{n-2} + (-2)^{n-3} + ... + (-2)^1 + (-2)^0] \\
&= (-1)^n \cdot \frac{(-2)^{n+1} - 1}{-3} \qquad \text{by theorem 5.2.3 with } r = -2 \\
&= (-1)^{n+1} \cdot \frac{(-2)^{n+1} - 1}{3} \\
&= \frac{2^{n+1} - (-1)^{n+1}}{3}
\end{aligned}
$$

Problem 11

$p_k = p_{k-1} + 2 \cdot 3^k$

$p_1 = 2$

Solution

$p_1 = 2$

$p_2 = p_1 + 2 \cdot 3^2 = 2 + 2 \cdot 3^2 = 2(1 + 3^2)$

$p_3 = p_2 + 2 \cdot 3^3 = 2(1 + 3^2) + 2 \cdot 3^3 = 2(1 + 3^2 + 3^3)$

$p_4 = p_3 + 2 \cdot 3^4 = 2(1 + 3^2 + 3^3) + 2 \cdot 3^4 = 2(1 + 3^2 + 3^3 + 3^4)$

$$
\begin{aligned}
\textit{Guess: } p_n &= 2(1 + (3^2(1 + 3 + 3^2 + ... + 3^{n-2}))) \\
&= 2 \cdot \left(1 + 3^2 \cdot \frac{3^{(n-2)+1} - 1}{3 - 1}\right) && \text{by theorem 5.2.3 with } r = -2 \\
&= 2 \cdot \left(1 + 3^2 \cdot \frac{3^{n-1} - 1}{2}\right) \\
&= 2 \cdot \left(1 + \frac{3^{n+1} - 3^2}{2}\right) \\
&= 2 + 3^{n+1} - 9 \\
&= 3^{n+1} - 7 \quad \text{for all integers } n \geq 1
\end{aligned}
$$

Problem 12

$s_k = s_{k-1} + 2k, \quad$ for all integers $k \geq 1$

$s_0 = 3$

Solution

$s_0 = 3$

$s_1 = s_0 + 2 \cdot 1 = 3 + 2$

$s_2 = s_1 + 2 \cdot 2 = (3 + 2) + 2 \cdot 2 = 3 + (2 + 2 \cdot 2)$

$s_3 = s_2 + 2 \cdot 3 = (3 + 2 + 2 \cdot 2) + 2 \cdot 3 = 3 + (2 + 2 \cdot 2 + 2 \cdot 3)$

$$
\begin{aligned}
\textit{Guess: } s_n &= 3 + 2(1 + 2 + 3 + ... + n) \\
&= 3 + 2 \cdot \frac{n(n+1)}{2} && \text{by theorem 5.2.2} \\
&= n^2 + n + 3 \quad \text{for all integers } n \geq 0
\end{aligned}
$$

Problem 13

$t_k = t_{k-1} + 3k + 1, \quad$ for all integers $k \geq 1$

$t_0 = 0$

Solution

$t_0 = 0$

$t_1 = t_0 + 3 \cdot 1 + 1 = 0 + 3 \cdot 1 + 1$

$t_2 = t_1 + 3 \cdot 2 + 1 = 3 \cdot 1 + 1 + 3 \cdot 2 + 1$

$t_3 = t_2 + 3 \cdot 3 + 1 = 3 \cdot 1 + 1 + 3 \cdot 2 + 1 + 3 \cdot 3 + 1$

$$
\begin{aligned}
\textit{Guess: } t_n &= n + 3(1 + 2 + 3 + ... + n) \\
&= n + 3 \cdot \frac{n(n+1)}{2} && \text{by theorem 5.2.2} \\
&= \frac{2n}{2} + \frac{3n^2}{2} + \frac{3n}{2} \\
&= \frac{3}{2}n^2 + \frac{5}{2}n \quad \text{for all integers } n \geq 0
\end{aligned}
$$

Problem 14

$x_k = 3x_{k-1} + k, \quad \text{for all integers } k \geq 2$

$x_1 = 1$

Solution

$x_1 = 1$

$x_2 = 3x_1 + 2 = 3 \cdot 1 + 2 = 3 + 2$

$x_3 = 3x_2 + 3 = 3(3 + 2) + 3 = 3^2 + 2 \cdot 3 + 3$

$x_4 = 3x_3 + 4 = 3(3^2 + 2 \cdot 3 + 3) + 4 = 3^3 + 2 \cdot 3^2 + 3 \cdot 3 + 4$

$x_5 = 3x_4 + 5 = 3(3^3 + 2 \cdot 3^2 + 3 \cdot 3 + 4) + 5 = 3^4 + 2 \cdot 3^3 + 3 \cdot 3^2 + 4 \cdot 3 + 5$

$x_6 = 3x_5 + 6 = 3(3^4 + 2 \cdot 3^3 + 3 \cdot 3^2 + 4 \cdot 3 + 5) = 3^5 + 2 \cdot 3^4 + 3 \cdot 3^3 + 4 \cdot 3^2 + 5 \cdot 3 + 6$

$$
\begin{aligned}
\textit{Guess: } x_n &= 3^{n-1} + 2 \cdot 3^{n-2} + ... + (n-2) \cdot 3^2 + (n-1) \cdot 3 + n \\
&= \underbrace{3^{n-1}}_{\text{1 time}} + \underbrace{3^{n-2} + 3^{n-2}}_{\text{2 times}} + ... + \underbrace{3^2 + 3^2 + ... + 3^2}_{n-2 \text{ times}} + \underbrace{3 + 3 + ... + 3}_{n-1 \text{ times}} + \underbrace{1 + 1 + ... + 1}_{n \text{ times}} \\
&= (3^{n-1} + 3^{n-2} + ... + 3^2 + 3 + 1) \\
&\qquad + (3^{n-2} + ... + 3^2 + 3 + 1) + ... + (3^2 + 3 + 1) + (3 + 1) + 1 \\
&= \frac{3^n - 1}{2} + \frac{3^{n-1} - 1}{2} + ... + \frac{3^3 - 1}{2} + \frac{3^2 - 1}{2} + \frac{3 - 1}{2} && \text{by theorem 5.2.3} \\
&= \frac{1}{2}[3^n + 3^{n-1} + ... + 3^3 + 3^2 + 3 - n] \\
&= \frac{1}{2}[3(3^{n-1} + 3^{n-2} + ... + 3^2 + 3 + 1) - n] \\
&= \frac{1}{2}\left(3\left(\frac{3^n - 1}{2}\right) - n\right) \\
&= \frac{1}{4}(3^{n+1} - 2n - 3) \quad \text{for all integers } n \geq 1
\end{aligned}
$$

Problem 15

$y_k = y_{k-1} + k^2, \quad \text{for all integers } k \geq 2$
$y_1 = 1$

Solution

$y_1 = 1$
$y_2 = y_1 + 2^2 = 1 + 2^2$
$y_3 = y_2 + 3^2 = 1 + 2^2 + 3^2$
$y_4 = y_3 + 4^2 = 1 + 2^2 + 3^2 + 4^2$

$$\begin{aligned} \textit{Guess: } y_n &= 1^2 + 2^2 + 3^2 + ... + n^2 \\ &= \frac{n(n+1)(2n+1)}{6} \qquad \text{by problem 5.2.10} \end{aligned}$$

Problem 16

Solve the recurrence relation obtained from problem 5.6.17c which is

$a_k = 3a_{k-1} + 2, \quad \text{for all integers } k \geq 2$
$a_1 = 2$

Solution

$a_1 = 2$
$a_2 = 3a_1 + 2 = 3 \cdot 2 + 2$
$a_3 = 3a_2 + 2 = 3(3 \cdot 2 + 2) + 2 = 3^2 \cdot 2 + 3 \cdot 2 + 2$
$a_4 = 3a_3 + 2 = 3(3^2 \cdot 2 + 3 \cdot 2 + 2) + 2 = 3^3 \cdot 2 + 3^2 \cdot 2 + 3 \cdot 2 + 2$

$$\begin{aligned} \textit{Guess: } a_n &= 2(3^{n-1} + 3^{n-2} + ... + 3^2 + 3 + 1) \\ &= 2 \cdot \frac{3^n - 1}{2} \qquad \text{by theorem 5.2.3} \\ &= 3^n - 1 \quad \text{for all integers } n \geq 1 \end{aligned}$$

Problem 17

Solve the recurrence relation obtained from problem 5.6.21c which is

$t_k = 2t_{k-1} + 2$
$t_1 = 2$

Solution

$t_1 = 2$

$t_2 = 2t_1 + 2 = 2 \cdot 2 + 2 = 2^2 + 2$

$t_3 = 2t_2 + 2 = 2(2^2 + 2) + 2 = 2^3 + 2^2 + 2$

$t_4 = 2t_3 + 2 = 2(2^3 + 2^2 + 2) + 2 = 2^4 + 2^3 + 2^2 + 2$

$$\begin{aligned} \textit{Guess: } t_n &= 2^n + 2^{n-1} + ... + 2^3 + 2^2 + 2 \\ &= 2(2^{n-1} + 2^{n-2} + ... + 2^2 + 2 + 1) \\ &= 2 \cdot \frac{2^{(n-1)+1} - 1}{2 - 1} && \text{by theorem 5.2.3} \\ &= 2 \cdot 2^n - 2 \\ &= 2^{n+1} - 2 \quad \text{for all integers } n \geq 1 \end{aligned}$$

Problem 18

Suppose d is a fixed constant and $a_0, a_1, a_2, ...$ is a sequence that satisfies the recurrence relation $a_k = a_{k-1} + d$, for all integers $k \geq 1$. Use mathematical induction to prove that $a_n = a_0 + nd$, for all integers $n \geq 0$.

Solution

Proof. Let the property $P(n)$ be the equation

$$a_n = a_0 + nd \qquad \leftarrow P(n)$$

Show that $P(0)$ is true: Let $n = 0$. Then,

$$a_n = a_0 \quad \text{and} \quad a_0 + 0 \cdot d = a_0 + 0 = a_0$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$a_k = a_0 + kd \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$a_{k+1} = a_0 + (k+1)d \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} a_{k+1} &= a_{(k+1)-1} + d && \text{by definition of } a_0, a_1, a_2, ... \\ &= a_k + d \\ &= a_0 + kd + d && \text{by inductive hypothesis} \\ &= a_0 + d(k+1) \end{aligned}$$

which is the right-hand side of $P(k+1)$. $\square$

Problem 19

A worker is promised a bonus if he can increase his productivity by 2 units a day every day for a period of 30 days. If on day 0 he produces 170 units, how many units must he produce on day 30 to qualify for the bonus.

Solution

Let a_n be the number of units produced on day n. Then,

$$\begin{aligned} a_k &= a_{k-1} + 2, \quad \text{for all integers } k \geq 1 \\ a_0 &= 170 \end{aligned}$$

It follows that $a_0, a_1, a_2, ...$ is an arithmetic sequence and that

$$a_n = 170 + 2n, \quad \text{for all integers } n \geq 0$$

Hence of day 30 the worker must produce $170 + 2 \cdot 30 = 230$ units.

Problem 20

A runner targets herself to improve her time on a certain course by 3 seconds a day. If on day 0 she runs the course in 3 minutes, how fast must she run it on day 14 to stay on target?

Solution

Let a_n be the number of seconds to complete the course on day n. Then,

$$\begin{aligned} a_k &= a_{k-1} - 3, \quad \text{for all integers } k \geq 1 \\ a_0 &= 180 \end{aligned}$$

It follows that $a_0, a_1, a_2, ...$ is an arithmetic sequence and that

$$a_n = 180 - 3n, \quad \text{for all integers } n \geq 0$$

Hence on day 14 the runner must complete the course in $180 - 3 \cdot 14 = 138$ seconds.

Problem 21

Suppose r is a fixed constant and $a_0, a_1, a_2, ...$ is a sequence that satisfies the recurrence relation $a_k = ra_{k-1}$, for all integers $k \geq 1$ and $a_0 = a$. Use mathematical induction to prove that $a_n = ar^n$, for all integers $n \geq 0$.

Solution

Proof. Let the property $P(n)$ be the equation

$$a_n = ar^n \qquad \leftarrow P(n)$$

Show that $P(0)$ is true: Let $n = 0$. Then

$$a_n = a_0 = a \quad \text{and} \quad ar^0 = a \cdot 1 = a$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$a_k = ar^k \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$a_{k+1} = ar^{k+1} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} a_{k+1} &= ra_{(k-1)+1} && \text{by definition of } a_0, a_1, a_2, \ldots \\ &= ra_k \\ &= rar^k && \text{by inductive hypothesis} \\ &= ar^{k+1} \end{aligned}$$

which is the right-hand side of $P(k+1)$. $\square$

Problem 22

As shown in example 5.6.8, if a bank pays interest at a rate of i compounded m times per year, then the amount of money P_k at the end of k time periods (where one time period $= 1/m$th of a year) satisfies the recurrence relation $P_k = [1 + (i/m)]P_{k-1}$ with initial condition $P_0 =$ the initial amount deposited. Find an explicit formula for P_n.

Solution

Let the initial amount be denoted simple as P_0. Then,

$P_0 = P_0$
$P_1 = [1 + (i/m)]P_0$
$P_2 = [1 + (i/m)]P_1 = [1 + (i/m)][1 + (i/m)]P_0$
$P_3 = [1 + (i/m)]P_2 = [1 + (i/m)][1 + (i/m)][1 + (i/m)]P_0$

Guess: $P_n = P_0 \cdot \left(1 + \dfrac{i}{m}\right)^n$ for all integers $n \geq 0$.

Problem 23

Suppose the population of a country increases at a steady rate of 3% per year. If the population is 50 million at a certain time, what will it be 25 years later.

Solution

From problem 22 we have that $P_n = P_0 \cdot \left(1 + \frac{i}{m}\right)^n$ for all integers $n \geq 0$. Let $P_0 = 50$ and let $n = 25$. Then,

$$P_{25} = 50 \cdot (1 + .03)^{25} \approx 105$$

Thus in 25 years the population will be approximately 105 million.

Problem 24

A chain letter works as follows: One person sends a copy of the letter to five friends, each of whom sends a copy to five friends, and so forth. How many people will have received copies of the letter after the twentieth repetition of this process, assuming no person receives more than 1 copy.

Solution

Assuming that the first person that sends out a letter also "receives" the letter, the total number of people that receive the letter will be

$$\sum_{i=0}^{20} 5^i = \frac{5^{21} - 1}{4} \approx 119 \text{ trillion}$$

Problem 25

A certain computer algorithm executes twice as many operations when it is run with an input of size k as when it is run with an input of size $k - 1$ (where k is an integer that is greater than 1). When the algorithm is run with an input of size 1, it executes seven operations. How many operations does it execute when it is run with an input of size 25?

Solution

Let a_n be the number of operations executed when the program is run with an input of size $n + 1$. Then,

$$\begin{aligned} a_k &= 2a_{k-1}, \quad \text{for all integers } k \geq 2 \\ a_1 &= 7 \end{aligned}$$

It follows that $a_1, a_2, a_3, ...$ is a geometric sequence and that

$$a_n = a_1 \cdot 2^n \quad \text{for all integers } n \geq 0.$$

Hence $a_{24} = 7 \cdot 2^{24} \approx 117$ million.

Problem 26

A person saving for a retirement makes an initial deposit of \$1,000 to a bank account earning interest at a rate of 3% per year compounded monthly, and each month she adds an additional \$200 to the account.

(a) For each nonnegative integer n, let A_n be the amount in the account at the end of n months. Find a recurrence relation relating A_k to A_{k-1}.

(b) Use iteration to find an explicit formula for A_n.

(c) Use mathematical induction to prove the correctness of the formula you obtained in part (b).

(d) How much will the account be worth at the end of 20 years? At the end of 40 years?

(e) In how many years will the account be worth \$10,000?

Solution

(a) Since the annual interest rate of 3% is compounded monthly, the monthly interest rate is $3\%/12 = .25\%$. Now the amount in the account at the end of any month is equal to the amount in the account at the end of the previous month + the interest earned on the account during the month + 200. Hence,

$$\begin{aligned}
A_k &= A_{k-1} + .0025A_{k-1} + 200 \\
A_k &= A_{k-1}(1 + .0025) + 200 \\
A_k &= A_{k-1}(1.0025) + 200, \quad \text{for all integers } k \geq 1 \\
A_0 &= 1000
\end{aligned}$$

(b) $A_0 = 1000$

$A_1 = A_0(1.0025) + 200$

$A_2 = A_1(1.0025) + 200 = A_0(1.0025)^2 + 1.0025 \cdot 200 + 200$

$A_3 = A_2(1.0025) + 200 = A_0(1.0025)^3 + 1.0025^2 \cdot 200 + 1.0025 \cdot 200 + 200$

$$\begin{aligned}
\textit{Guess: } A_n &= A_0(1.0025)^n + 200(1.0025^{n-1} + 1.0025^{n-2} + ... + 1.0025^2 + 1.0025 + 1) \\
&= A_0(1.0025)^n + 200 \cdot \frac{1.0025^n - 1}{.0025} \qquad \text{by theorem 5.2.3} \\
&= A_0(1.0025)^n + 80000(1.0025^n - 1) \\
&= A_0(1.0025)^n + 80000 \cdot 1.0025^n - 80000 \\
&= 1.0025^n(A_0 + 80000) - 80000 \\
&= 1.0025^n(81000) - 80000 \quad \text{for all integers } n \geq 0
\end{aligned}$$

(c) *Proof.* Let the property $P(n)$ be the equation

$$A_n = 1.0025^n(81000) - 80000 \qquad \leftarrow P(n)$$

Show that $P(0)$ is true: Let $n = 0$. Then,

$$A_0 = 1000 \quad \text{and} \quad 1.0025^0(81000) - 80000 = 1000$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$A_k = 1.0025^k(81000) - 80000 \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$A_{k+1} = 1.0025^{k+1}(81000) - 80000 \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} A_{k+1} &= A_k(1.0025) + 200 && \text{by definition of } A_0, A_1, A_2, \ldots \\ &= 1.0025^k(81000) - 80000)(1.0025) + 200 && \text{by inductive hypothesis} \\ &= 1.0025^{k+1}(81000) - 80000 \end{aligned}$$

which is the right-hand side of $P(k+1)$. □

(d) At the end of 20 years 240 months will have passed. At the end of 40 years 480 months will have passed. It follows from the results of part a and c that

$$A_{240} = 1.0025^{240}(81000) - 80000 \approx 67481.15$$

and

$$A_{480} = 1.0025^{480}(81000) - 80000 \approx 188527.05$$

(e) Since A_n represents the amount that the account is worth after n months we need to set $A_n = 10000$ and then solve for n.

$$\begin{aligned} 10000 &= 1.0025^n(81000) - 80000 \\ 90000 &= 1.0025^n(81000) \\ \frac{90000}{81000} &= 1.0025^n \\ \ln\left(\frac{10}{9}\right) &= \ln\left(1.0025^n\right) \\ \ln\left(\frac{10}{9}\right) &= n\ln\left(1.0025\right) \\ n &= \frac{\ln\left(\frac{10}{9}\right)}{\ln\left(1.0025\right)} \end{aligned}$$

It follows that $n \approx 42.2$. Hence after $42.2/12 \approx 3.5$ years the account will be worth \$10,000.

Problem 27

A person borrows \$3,000 on a bank credit card at a nominal rate of 18% per year, which is actually charged at a rate of 1.5% per month

(a) What is the annual percentage rate (APR) for the card?

(b) Assume that the person does not place any additional charges on the card and pays the bank \$150 each month to pay off the loan. Let B_n be the balance owed on the card after n months. Find an explicit formula for B_n.

(c) How long will be required to pay off the debt?

(d) What is the total amount of money the person will have paid for the loan?

Solution

(a) Let A_n denote the amount owed at the end of month n, then $A_k = (1.015)A_{k-1}$, for all integers $k \geq 1$. Let the initial amount borrowed be \$3,000 and assume that no additional deposits or withdrawals are made. It follows that $A_0, A_1, A_2, ...$ is a geometric sequence with initial value 3,000 and constant multiplier 1.015. Hence,

$$A_n = 3000 \cdot (1.015)^n \quad \text{for all integers } n \geq 0$$

After 12 months the amount in the account will be

$$A_{12} = 3000 \cdot (1.015)^{12} \approx 3586.85$$

The APR will be

$$\text{APR} \approx \frac{3586.85 - 3000}{3000} \approx .1956 \approx 19.6\%$$

(b) Let B_k be a recurrence relation for the balance owed on the card after k months. Then,

$$\begin{aligned} A_k &= A_{k-1} + .015A_{k-1} - 150 \\ A_k &= 1.015A_{k-1} - 150, \quad \text{for all integers } k \geq 1 \\ A_0 &= 3000 \end{aligned}$$

We will use iteration to guess an explicit formula for B_n.

$$\begin{aligned} B_0 &= 3000 \\ B_1 &= 1.015B_0 - 150 \\ B_2 &= 1.015B_1 - 150 = 1.015^2 B_0 - 150 \cdot 1.015 - 150 \\ B_3 &= 1.015B_2 - 150 = 1.015^3 B_0 - 150 \cdot 1.015^2 - 150 \cdot 1.015 - 150 \end{aligned}$$

$$\begin{aligned}
\textit{Guess: } B_n &= B_0(1.015)^n - 150(1.015^{n-1} + 1.015^{n-2} + ... + 1.015^2 + 1.015 + 1)\\
&= B_0(1.015)^n - 150 \cdot \frac{1.015^n - 1}{.015} \qquad \text{by theorem 5.2.3}\\
&= B_0(1.015)^n - 10000(1.015^n - 1)\\
&= B_0(1.015)^n - 10000 \cdot 1.015^n + 10000\\
&= 1.015^n(B_0 - 10000) + 10000\\
&= -7000(1.015^n) + 10000 \quad \text{for all integers } n \geq 0
\end{aligned}$$

(c) When the debt is payed of $B_n = 0$.

$$\begin{aligned}
B_n &= -7000(1.015^n) + 10000\\
0 &= -7000(1.015^n) + 10000\\
7000(1.015^n) &= 10000\\
1.015^n &= \frac{10}{7}\\
\ln(1.015^n) &= \ln\left(\frac{10}{7}\right)\\
n\ln(1.015) &= \ln\left(\frac{10}{7}\right)\\
n &= \frac{\ln\left(\frac{10}{7}\right)}{\ln(1.015)}\\
n &\approx 23.9
\end{aligned}$$

It follows that is will take about 24 months or two years to pay off the debt.

(d) Since the person pays \$150 per month simply multiply the number of months to pay off the debt by 150 to obtain \$3593.43.

In 28-42 use mathematical induction to verify the correctness of the formula you obtained in the referenced exercise.

Problem 28

Exercise 3. Suppose that $a_0, a_1, a_2, ...$ is defined as follows:

$$\begin{aligned}
&a_k = ka_{k-1} \quad \text{for all integers } k \geq 1\\
&a_0 = 1
\end{aligned}$$

Prove that $a_n = n(n-1)...3 \cdot 2 \cdot 1 = n!$ for all integers $n \geq 0$.

Solution

Proof. Let the property $P(n)$ be the equation

$$a_n = n! \qquad \leftarrow P(n)$$

Show that $P(0)$ is true: Let $n = 0$. Then,

$$a_n = a_0 = 1 \quad \text{and} \quad 0! = 1$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$a_k = k! \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$a_{k+1} = (k+1)! \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} a_{k+1} &= (k+1)a_k && \text{by definition of } a_0, a_1, a_2, \ldots \\ &= (k+1)k! && \text{by inductive hypothesis} \\ &= (k+1)! \end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Problem 29

Exercise 4. Suppose that $b_0, b_1, b_2, \ldots$ is defined as follows:

$$\begin{aligned} b_k &= \frac{b_{k-1}}{1 + b_{k-1}} \quad \text{for all integers } k \geq 1 \\ b_0 &= 1 \end{aligned}$$

Prove that $b_n = \dfrac{1}{n+1}$ for all integers $n \geq 0$.

Solution

Proof. Let the property $P(n)$ be the equation

$$b_n = \frac{1}{n+1} \qquad \leftarrow P(n)$$

Show that $P(0)$ is true: Let $n = 0$. Then,

$$b_n = b_0 = 1 \quad \text{and} \quad \frac{1}{0+1} = \frac{1}{1} = 1$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$b_k = \frac{1}{k+1} \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$b_{k+1} = \frac{1}{k+2} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} b_{k+1} &= \frac{b_k}{1+b_k} && \text{by definition of } b_0, b_1, b_2, \ldots \\ &= \frac{\frac{1}{k+1}}{1+\frac{1}{k+1}} && \text{by inductive hypothesis} \\ &= \frac{1}{k+1} \cdot \frac{k+1}{k+2} \\ &= \frac{1}{k+2} \end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Problem 30

Exercise 5. Suppose that $c_1, c_2, c_3, \ldots$ is defined as follows:

$$\begin{aligned} c_k &= 3c_{k-1} + 1 \quad \text{for all integers } k \geq 2 \\ c_1 &= 1 \end{aligned}$$

Prove that $c_n = \dfrac{3^n - 1}{2}$ for all integers $n \geq 1$.

Solution

Proof. Let the property $P(n)$ be the equation

$$c_n = \frac{3^n - 1}{2} \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: Let $n = 1$. Then,

$$c_n = c_1 = 1 \quad \text{and} \quad \frac{3^1 - 1}{2} = \frac{3-1}{2} = \frac{2}{2} = 1$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$c_k = \frac{3^k - 1}{2} \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$c_{k+1} = \frac{3^{k+1} - 1}{2} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
c_{k+1} &= 3c_k + 1 && \text{by definition of } c_1, c_2, c_3, \ldots \\
&= 3 \cdot \frac{3^k - 1}{2} + 1 && \text{by inductive hypothesis} \\
&= \frac{3 \cdot 3^k - 3}{2} + \frac{2}{2} \\
&= \frac{3^{k+1} - 1}{2}
\end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Problem 31

Exercise 6. Suppose that $d_1, d_2, d_3, \ldots$ is defined as follows:

$$\begin{aligned}
d_k &= 2d_{k-1} + 3 \quad \text{for all integers } k \geq 2 \\
d_1 &= 2
\end{aligned}$$

Prove that $d_n = 5 \cdot 2^{n-1} - 3$ for all integers $n \geq 1$.

Solution

Proof. Let the property $P(n)$ be the equation

$$d_n = 5 \cdot 2^{n-1} - 3 \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: Let $n = 1$. Then,

$$d_n = d_1 = 2 \quad \text{and} \quad 5 \cdot 2^{1-1} - 3 = 5 \cdot 1 - 3 = 2$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$d_k = 5 \cdot 2^{k-1} - 3 \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$d_{k+1} = 5 \cdot 2^k - 3 \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
d_{k+1} &= 2d_k + 3 && \text{by definition of } d_1, d_2, d_3, \ldots \\
&= 2(5 \cdot 2^{k-1} - 3) + 3 && \text{by inductive hypothesis} \\
&= 5(2 \cdot 2^{k-1}) - 6 + 3 \\
&= 5 \cdot 2^k - 3
\end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Problem 32

Exercise 7. Suppose that $e_0, e_1, e_2, \ldots$ is defined as follows:

$$\begin{aligned} e_k &= 4e_{k-1} + 5 \quad \text{for all integers } k \geq 1 \\ e_0 &= 2 \end{aligned}$$

Prove that $e_n = \dfrac{11 \cdot 4^n - 5}{3}$ for all integers $n \geq 0$.

Solution

Proof. Let the property $P(n)$ be the equation

$$e_n = \frac{11 \cdot 4^n - 5}{3} \qquad \leftarrow P(n)$$

Show that $P(0)$ is true: Let $n = 0$. Then,

$$e_n = e_0 = 2 \quad \text{and} \quad \frac{11 \cdot 4^0 - 5}{3} = \frac{6}{3} = 2$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$e_k = \frac{11 \cdot 4^k - 5}{3} \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$e_{k+1} = \frac{11 \cdot 4^{k+1} - 5}{3} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} e_{k+1} &= 4e_k + 5 && \text{by definition of } e_0, e_1, d_2, \ldots \\ &= 4 \cdot \frac{11 \cdot 4^{k+1} - 5}{3} + 5 && \text{by inductive hypothesis} \\ &= \frac{11 \cdot 4 \cdot 4^{k+1} - 5}{3} + 5 \\ &= \frac{11 \cdot 4^{k+1} - 20}{3} + \frac{15}{3} \\ &= \frac{11 \cdot 4^{k+1} - 5}{3} \end{aligned}$$

which is the right-hand side of $P(k+1)$. ☐

Problem 33

Exercise 8. Suppose that $f_1, f_2, f_3, \ldots$ is defined as follows:

$$\begin{aligned} f_k &= f_{k-1} + 2^k \quad \text{for all integers } k \geq 2 \\ f_1 &= 1 \end{aligned}$$

Prove that $f_n = 2^{n+1} - 3$ for all integers $n \geq 1$.

Solution

Proof. Let the property $P(n)$ be the equation

$$f_n = 2^{n+1} - 3 \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: Let $n = 1$. Then,

$$f_n = f_1 = 1 \quad \text{and} \quad 2^{1+1} - 3 = 4 - 3 = 1$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$f_k = 2^{k+1} - 3 \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$f_{k+1} = 2^{k+2} - 3 \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} f_{k+1} &= f_k + 2^{k+1} && \text{by definition of } f_1, f_2, f_3, \ldots \\ &= 2^{k+1} - 3 + 2^{k+1} && \text{by inductive hypothesis} \\ &= 2 \cdot 2^{k+1} - 3 \\ &= 2^{k+2} - 3 \end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Problem 34

Exercise 9. Suppose that $g_1, g_2, g_3, \ldots$ is defined as follows:

$$\begin{aligned} g_k &= \frac{g_{k-1}}{g_{k-1} + 2} \quad \text{for all integers } k \geq 2 \\ g_1 &= 1 \end{aligned}$$

Prove that $g_n = \dfrac{1}{2^n - 1}$ for all integers $n \geq 1$.

Solution

Proof. Let the property $P(n)$ be the equation

$$g_n = \frac{1}{2^n - 1} \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: Let $n = 1$. Then,

$$g_n = g_1 = 1 \quad \text{and} \quad \frac{1}{2^1 - 1} = \frac{1}{2 - 1} = \frac{1}{1} = 1$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$g_k = \frac{1}{2^k - 1} \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$g_{k+1} = \frac{1}{2^{k+1} - 1} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} g_{k+1} &= \frac{g_k}{g_k + 2} && \text{by definition of } g_1, g_2, g_3, \ldots \\ &= \frac{\frac{1}{2^k-1}}{\frac{1}{2^k-1} + 2} \\ &= \frac{1}{2^k - 1} \cdot \frac{2^k - 1}{2(2^k - 1)} \\ &= \frac{1}{2^{k+1} - 1} \end{aligned}$$

which is the right-hand side of $P(k+1)$. $\square$

Problem 35

Exercise 10. Suppose that $h_0, h_1, h_2, \ldots$ is defined as follows:

$$\begin{aligned} h_k &= 2^k - h_{k-1} \quad \text{for all integers } k \geq 1 \\ h_0 &= 1 \end{aligned}$$

Prove that $h_n = \dfrac{2^{n+1} - (-1)^{n+1}}{3}$ for all integers $n \geq 0$.

Solution

Proof. Let the property $P(n)$ be the equation

$$h_n = \frac{2^{n+1} - (-1)^{n+1}}{3} \qquad \leftarrow P(n)$$

Show that $P(0)$ is true: Let $n = 0$. Then,

$$h_n = h_0 = 1 \quad \text{and} \quad \frac{2^{0+1} - (-1)^{0+1}}{3} = \frac{2+1}{3} = \frac{3}{3} = 1$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$h_k = \frac{2^{k+1} - (-1)^{k+1}}{3} \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$h_{k+1} = \frac{2^{k+2} - (-1)^{k+2}}{3} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
h_{k+1} &= 2^{k+1} - h_k && \text{by definition of } h_0, h_1, h_2, \ldots \\
&= 2^{k+1} - \frac{2^{k+1} - (-1)^{k+1}}{3} && \text{by inductive hypothesis} \\
&= \frac{3 \cdot 2^{k+1} - 2^{k+1} + (-1)^{k+1}}{3} \\
&= \frac{2^{k+1}(3-1) + (-1)^{k+1}}{3} \\
&= \frac{2 \cdot 2^{k+1} - (-1)^{k+2}}{3} \\
&= \frac{2^{k+2} - (-1)^{k+2}}{3}
\end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Problem 36

Exercise 11. Suppose that $p_1, p_2, p_3, \ldots$ is defined as follows:

$$\begin{aligned}
&p_k = p_{k-1} + 2 \cdot 3^k \quad \text{for all integers } k \geq 1 \\
&p_1 = 2
\end{aligned}$$

Prove that $p_n = 3^{n+1} - 7$ for all integers $n \geq 1$.

Solution

Proof. Let the property $P(n)$ be the equation

$$p_n = 3^{n+1} - 7 \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: Let $n = 1$. Then,

$$p_n = p_1 = 2 \quad \text{and} \quad 3^{1+1} - 7 = 3^2 - 7 = 9 - 7 = 2$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$p_k = 3^{k+1} - 7 \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$p_{k+1} = 3^{k+2} - 7 \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
p_{k+1} &= p_k + 2 \cdot 3^{k+1} && \text{by definition of } p_1, p_2, p_3, \ldots \\
&= 3^{k+1} - 7 + 2 \cdot 3^{k+1} && \text{by inductive hypothesis} \\
&= 3 \cdot 3^{k+1} - 7 \\
&= 3^{k+2} - 7
\end{aligned}$$

which is the right-hand side of $P(k+1)$. $\square$

Problem 37

Exercise 12. Suppose that $s_0, s_1, s_2, \ldots$ is defined as follows:

$$\begin{aligned}
&s_k = s_{k-1} + 2k \quad \text{for all integers } k \geq 1 \\
&s_0 = 3
\end{aligned}$$

Prove that $s_n = n^2 + n + 3 \quad$ for all integers $n \geq 0$.

Solution

Proof. Let the property $P(n)$ be the equation

$$s_n = n^2 + n + 3 \qquad \leftarrow P(n)$$

Show that $P(0)$ is true: Let $n = 0$. Then,

$$s_n = s_0 = 3 \quad \text{and} \quad 0^2 + 0 + 3 = 3$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$s_k = k^2 + k + 3 \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$s_{k+1} = (k+1)^2 + (k+1) + 3 \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
s_{k+1} &= s_k + 2(k+1) && \text{by definition of } s_1, s_1, s_2, \ldots \\
&= k^2 + k + 3 + 2(k+1) && \text{by inductive hypothesis} \\
&= k^2 + k + 3 + (k+1) + (k+1) \\
&= (k^2 + 2k + 1) + (k+1) + 3 \\
&= (k+1)^2 + (k+1) + 3
\end{aligned}$$

which is the right-hand side of $P(k+1)$. $\square$

Problem 38

Exercise 13. Suppose that $t_0, t_1, t_2, \ldots$ is defined as follows:

$$\begin{aligned} &t_k = t_{k-1} + 3k + 1 \quad \text{for all integers } k \geq 1 \\ &t_0 = 0 \end{aligned}$$

Prove that $t_n = \frac{3}{2}n^2 + \frac{5}{2}n$ for all integers $n \geq 0$.

Solution

Proof. Let the property $P(n)$ be the equation

$$t_n = \frac{3}{2}n^2 + \frac{5}{2}n \qquad \leftarrow P(n)$$

Show that $P(0)$ is true: Let $n = 0$. Then,

$$t_n = t_0 = 0 \quad \text{and} \quad \frac{3}{2} \cdot 0^2 + \frac{5}{2} \cdot 0 = 0$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$t_k = \frac{3}{2}k^2 + \frac{5}{2}k \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$t_{k+1} = \frac{3}{2}(k+1)^2 + \frac{5}{2}(k+1) \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} t_{k+1} &= t_k + 3(k+1) + 1 && \text{by definition of } t_0, t_1, t_2, \ldots \\ &= \frac{3}{2}k^2 + \frac{5}{2}k + 3k + 4 && \text{by inductive hypothesis} \\ &= \frac{3}{2}k^2 + \frac{5}{2}k + 3k + \frac{5}{2} + \frac{3}{2} \\ &= \left(\frac{3}{2}k^2 + 3k + \frac{3}{2}\right) + \left(\frac{5}{2}k + \frac{5}{2}\right) \\ &= \frac{3}{2}(k^2 + 2k + 1) + \frac{5}{2}(k+1) \\ &= \frac{3}{2}(k+1)^2 + \frac{5}{2}(k+1) \end{aligned}$$

which is the right-hand side of $P(k+1)$. ∎

Problem 39

Exercise 14. Suppose that $x_1, x_2, x_3, \dots$ is defined as follows:

$$x_k = 3x_{k-1} + k \quad \text{for all integers } k \geq 2$$
$$x_1 = 0$$

Prove that $x_n = \frac{1}{4}(3^{n+1} - 2n - 3)$ for all integers $n \geq 1$.

Solution

Proof. Let the property $P(n)$ be the equation

$$x_n = \frac{1}{4}(3^{n+1} - 2n - 3) \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: Let $n = 1$. Then,

$$x_n = x_1 = 0 \quad \text{and} \quad \frac{1}{4}(3^{1+1} - 2(1) - 3) = \frac{1}{4}(4) = 1$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$x_k = \frac{1}{4}(3^{k+1} - 2k - 3) \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$x_{k+1} = \frac{1}{4}(3^{k+2} - 2k - 5) \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} x_{k+1} &= 3x_k + k + 1 && \text{by definition of } x_1, x_2, x_3, \dots \\ &= \frac{1}{4}(3 \cdot 3^{k+2} - 6k - 9) + k + 1 && \text{by inductive hypothesis} \\ &= \frac{1}{4}(3 \cdot 3^{k+2} - 6k + 4k - 9 + 4) \\ &= \frac{1}{4}(3^{k+2} - 2k - 5) \end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Problem 40

Exercise 15. Suppose that $y_1, y_2, y_3, \dots$ is defined as follows:

$$y_k = y_{k-1} + k^2 \quad \text{for all integers } k \geq 2$$
$$y_1 = 1$$

Prove that $y_n = \dfrac{n(n+1)(2n+1)}{6}$ for all integers $n \geq 1$.

Solution

Proof. Let the property $P(n)$ be the equation

$$y_n = \frac{n(n+1)(2n+1)}{6} \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: Let $n = 1$. Then,

$$y_n = y_1 = 1 \quad \text{and} \quad \frac{1(1+1)(2(1)+1)}{6} = \frac{(1)(2)(3)}{6} = \frac{6}{6} = 1$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$y_k = \frac{k(k+1)(2k+1)}{6} \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$y_{k+1} = \frac{(k+1)(k+2)(2k+3)}{6} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
y_{k+1} &= y_k + (k+1)^2 && \text{by definition of } y_1, y_2, y_3, \ldots \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{by inductive hypothesis} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 5)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Problem 41

Exercise 41. Suppose that $a_1, a_2, a_3, \ldots$ is defined as follows:

$$\begin{aligned}
a_k &= 3a_{k-1} + 2 \quad \text{for all integers } k \geq 2 \\
a_1 &= 2
\end{aligned}$$

Prove that $a_n = 3^{n-1}$ for all integers $n \geq 1$.

Solution

Proof. Let the property $P(n)$ be the equation

$$a_n = 3^n - 1 \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: Let $n = 1$. Then,

$$a_n = a_1 = 2 \quad \text{and} \quad 3^1 - 1 = 3 - 1 = 2$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$a_k = 3^k - 1 \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$a_{k+1} = 3^{k+1} - 1 \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} a_{k+1} &= 3a_k + 2 && \text{by definition of } a_1, a_2, a_3, \ldots \\ &= 3(3^k - 1) + 2 && \text{by inductive hypothesis} \\ &= 3 \cdot 3^k - 3 + 2 \\ &= 3^{k+1} - 1 \end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Problem 42

Exercise 17. Suppose that $t_1, t_2, t_3, \ldots$ is defined as follows:

$$\begin{aligned} &t_k = 2t_{k-1} + 2 \quad \text{for all integers } k \geq 2 \\ &t_1 = 2 \end{aligned}$$

Prove that $t_n = 2^{n+1} - 2$ for all integers $n \geq 1$.

Solution

Proof. Let the property $P(n)$ be the equation

$$t_n = 2^{n+1} - 2 \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: Let $n = 1$. Then,

$$t_n = t_1 = 2 \quad \text{and} \quad 2^{1+1} - 2 = 2^2 - 2 = 4 - 2 = 2$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$t_k = 2^{k+1} - 2 \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$t_{k+1} = 2^{k+2} - 2 \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} t_{k+1} &= 2t_k + 2 && \text{by definition of } t_1, t_2, t_3, \ldots \\ &= 2(2^{k+1} - 2) + 2 && \text{by inductive hypothesis} \\ &= 2^{k+2} - 4 + 2 \\ &= 2^{k+2} - 2 \end{aligned}$$

which is the right-hand side of $P(k+1)$. □

In each of 43-49 a sequence is defined recursively. (a) Use iteration to guess an explicit formula for the sequence. (b) Use strong mathematical induction to verify that the formula of part (a) is correct.

Problem 43

$a_k = \dfrac{a_{k-1}}{2a_{k-1} - 1}$ for all integers $k \geq 1$

$a_0 = 2$

Solution

(a) $a_0 = 2$

$$a_1 = \frac{a_0}{2a_0 - 1} = \frac{2}{2 \cdot 2 - 1} = \frac{2}{3}$$

$$a_2 = \frac{2/3}{2(2/3) - 1} = \frac{2/3}{4/3 - 3/3} = \frac{2}{3} \cdot \frac{3}{1} = 2$$

$$a_3 = \frac{a_2}{2a_2 - 1} = \frac{2}{2 \cdot 2 - 1} = \frac{2}{3}$$

Guess: $a_n = \begin{cases} 2 & \text{if } n \text{ is even} \\ \frac{2}{3} & \text{if } n \text{ is odd} \end{cases}$

(b) *Proof.* Let the property $P(n)$ be the equation

$$a_n = \begin{cases} 2 & \text{if } n \text{ is even} \\ \frac{2}{3} & \text{if } n \text{ is odd} \end{cases} \qquad \leftarrow P(n)$$

Show that $P(0)$ and $P(1)$ are true: From part (a) we have that $a_0 = 2$ and $a_1 = \frac{2}{3}$. Hence $P(0)$ and $P(1)$ are true.

Show that for all integers $k \geq 1$, $P(i)$ is true for all integers i from 0 through $k \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$a_i = \begin{cases} 2 & \text{if } i \text{ is even} \\ 2/3 & \text{if } i \text{ is odd} \end{cases} \qquad \leftarrow \text{inductive hypothesis}$$

We must show that this implies that

$$a_{k+1} = \begin{cases} 2 & \text{if } k+1 \text{ is even} \\ 2/3 & \text{if } k+1 \text{ is odd} \end{cases} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} a_{k+1} &= \frac{a_k}{2a_k - 1} && \text{by definition of } a_0, a_1, a_2, \ldots \\ &= \begin{cases} \dfrac{2}{2 \cdot 2 - 1} & \text{if } k \text{ is even} \\ \dfrac{2/3}{2(2/3) - 1} & \text{if } k \text{ is odd} \end{cases} && \text{by inductive hypothesis} \\ &= \begin{cases} 2/3 & \text{if } k \text{ is even} \\ \dfrac{2}{3} \cdot \dfrac{3}{1} & \text{if } k \text{ is odd} \end{cases} \\ &= \begin{cases} 2/3 & \text{if } k+1 \text{ is odd} \\ 2 & \text{if } k+1 \text{ is even} \end{cases} \end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Problem 44

$b_k = \dfrac{2}{b_{k-1}}$ for all integers $k \geq 2$

$b_1 = 1$

Solution

(a) $b_1 = 1$

$$b_2 = \frac{2}{b_1} = \frac{2}{1} = 2$$

$$b_3 = \frac{2}{b_2} = \frac{2}{2} = 1$$

$$b_4 = \frac{2}{b_3} = \frac{2}{1} = 2$$

Guess: $b_n = \begin{cases} 2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$

(b) *Proof.* Let the property $P(n)$ be the equation

$$b_n = \begin{cases} 2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

Show that $P(1)$ and $P(2)$ are true: From part (a) we have that $b_1 = 1$ and $b_2 = 2$. Hence $P(1)$ and $P(2)$ are true.

Show that for all integers $k \geq 2$, $P(i)$ is true for all integers i from 1 through $k \implies P(k+1)$: Let k be any integer with $k \geq 2$ and suppose that

$$b_i = \begin{cases} 2 & \text{if } i \text{ is even} \\ 1 & \text{if } i \text{ is odd} \end{cases} \qquad \leftarrow \text{inductive hypothesis}$$

We must show that this implies that

$$b_{k+1} = \begin{cases} 2 & \text{if } k+1 \text{ is even} \\ 1 & \text{if } k+1 \text{ is odd} \end{cases} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} b_{k+1} &= \frac{2}{b_k} && \text{by definition of } b_1, b_2, b_3, \ldots \\ &= \begin{cases} 1 & \text{if } k \text{ is even} \\ 2 & \text{if } k \text{ is odd} \end{cases} && \text{by inductive hypothesis} \\ &= \begin{cases} 1 & \text{if } k+1 \text{ is odd} \\ 2 & \text{if } k+1 \text{ is even} \end{cases} \end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Problem 45

$v_k = v_{\lfloor k/2 \rfloor} + v_{\lfloor (k+1)/2 \rfloor} + 2 \quad \text{for all integers } k \geq 2$
$v_1 = 1$

Solution

(a) $v_1 = 1$

$v_2 = v_{\lfloor 2/2 \rfloor} + v_{\lfloor (2+1)/2 \rfloor} + 2 = v_1 + v_1 + 2 = 1 + 1 + 2 = 2 + 2$

$v_3 = v_{\lfloor 3/2 \rfloor} + v_{\lfloor (3+1)/2 \rfloor} + 2 = v_1 + v_2 + 2 = 1 + 2 + 2 + 2 = 3 + 4$

$v_4 = v_{\lfloor 4/2 \rfloor} + v_{\lfloor (4+1)/2 \rfloor} + 2 = v_2 + v_2 + 2 = 2 + 2 + 2 + 2 + 2 = 4 + 6$

Guess: $v_n = n + 2(n-1)$
$\quad = 3n - 2 \quad \text{for all integers } n \geq 1$

(b) *Proof.* Let the property $P(n)$ be the equation

$$v_n = 3n - 2 \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: Let $n = 1$. Then

$$v_n = v_1 = 1 \quad \text{and} \quad 3(1) - 2 = 3 - 2 = 1$$

Show that for all integers $k \geq 1$, $P(i)$ is true for all integers i from 1 through $k \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$v_i = 3i - 2 \qquad \leftarrow \text{inductive hypothesis}$$

We must show that this implies that

$$\begin{aligned} v_{k+1} &= 3(k+1) - 2 \\ &= 3k + 1 \qquad \leftarrow P(k+1) \end{aligned}$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} v_{k+1} &= v_{\lfloor (k+1)/2 \rfloor} + v_{\lfloor (k+2)/2 \rfloor} + 2 && \text{by definition of } v_1, v_2, v_3, \ldots \\ &= \left(3\left\lfloor \frac{k+1}{2} \right\rfloor - 2\right) + \left(3\left\lfloor \frac{k+2}{2} \right\rfloor - 2\right) + 2 && \text{by inductive hypothesis} \\ &= \begin{cases} 3\left(\dfrac{k}{2} + \dfrac{k+2}{2}\right) - 2 & \text{if } k \text{ is even} \\ 3\left(\dfrac{k+1}{2} + \dfrac{k+1}{2}\right) - 2 & \text{if } k \text{ is odd} \end{cases} \\ &= 3\left(\frac{2k+2}{2}\right) - 2 \\ &= 3(k+1) - 2 = 3k + 1 \end{aligned}$$

which is the right-hand side of $P(k+1)$. $\square$

Problem 46

$s_k = 2s_{k-2}$ for all integers $k \geq 2$

$s_0 = 1, \ s_1 = 2$

Solution

(a) $s_0 = 1$

$s_1 = 2$

$s_2 = 2s_0 = 2(1)$

$s_3 = 2s_1 = 2(2)$

$s_4 = 2s_2 = 2(2)$

$s_5 = 2s_3 = 2(2)(2)$

Guess: $s_n = 2^{\lceil n/2 \rceil}$ for all integers $n \geq 0$.

(b) *Proof.* Let the property $P(n)$ be the equation

$$s_n = 2^{\lceil n/2 \rceil} \qquad \leftarrow P(n)$$

Show that $P(0)$ and $P(1)$ are true: It follows from the definition of $s_0, s_1, s_2, \ldots$ that $s_0 = 1$ and $s_1 = 2$. Also $2^{\lceil 0/2 \rceil} = 1$ and $2^{\lceil 1/2 \rceil} = 2$. Hence $P(0)$ and $P(1)$ are true.

Show that for all integers $k \geq 1$, $P(i)$ is true for all integers i from 0 through $k \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$s_i = 2^{\lceil i/2 \rceil} \qquad \leftarrow \text{inductive hypothesis}$$

We must show that this implies that

$$s_{k+1} = 2^{\lceil (k+1)/2 \rceil} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
s_{k+1} &= 2s_{k-1} && \text{by definition of } s_0, s_1, s_2, \ldots \\
&= 2 \cdot 2^{\lceil (k-1)/2 \rceil} && \text{by inductive hypothesis} \\
&= \begin{cases} 2 \cdot 2^{k/2} & \text{if } k \text{ is even} \\ 2 \cdot 2^{(k-1)/2} & \text{if } k \text{ is odd} \end{cases} \\
&= \begin{cases} 2^{(k+2)/2} & \text{if } k \text{ is even} \\ 2^{(k+1)/2} & \text{if } k \text{ is odd} \end{cases} \\
&= \begin{cases} 2^{\lceil (k+1)/2 \rceil} & \text{if } k \text{ is even} \\ 2^{\lceil (k+1)/2 \rceil} & \text{if } k \text{ is odd} \end{cases} \\
&= 2^{\lceil (k+1)/2 \rceil}
\end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Problem 47

$t_k = k - t_{k-1}$ for all integers $k \geq 1$

$t_0 = 0$

Solution

(a) $t_0 = 0$

$t_1 = 1 - t_0 = 1 - 0 = 1$

$t_2 = 2 - t_1 = 2 - 1 = 1$

$t_3 = 3 - t_2 = 3 - 1 = 2$

$t_4 = 4 - t_3 = 4 - 2 = 2$

Guess: $t_n = \lceil n/2 \rceil$ for all integers $n \geq 0$.

(b) *Proof.* Let the property $P(n)$ be the equation

$$t_n = \lceil n/2 \rceil \qquad \leftarrow P(n)$$

Show that $P(0)$ is true: Let $n = 0$. Then,

$$t_n = t_0 = 0 \quad \text{and} \quad \lceil 0/2 \rceil = 0$$

Show that for all integers $k \geq 0$, $P(i)$ is true for all integers i from 0 through $k \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$t_i = \lceil i/2 \rceil \qquad \leftarrow P(n)$$

We must show that this implies that

$$t_{k+1} = \lceil (k+1)/2 \rceil \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} t_{k+1} &= k + 1 - t_k && \text{by definition of } t_0, t_1, t_2, \ldots \\ &= k + 1 - \lceil k/2 \rceil && \text{by inductive hypothesis} \\ &= \begin{cases} k/2 + 1 & \text{if } k \text{ is even} \\ (k+1)/2 & \text{if } k \text{ is odd} \end{cases} \\ &= \begin{cases} \lceil (k+1)/2 \rceil & \text{if } k \text{ is even} \\ \lceil (k+1)/2 \rceil & \text{if } k \text{ is odd} \end{cases} \\ &= \lceil (k+1)/2 \rceil \end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Problem 48

$w_k = w_{k-2} + k \quad$ for all integers $k \geq 3$

$w_1 = 1, \ w_2 = 2$

Solution

(a) $w_1 = 1 = 1^2$

$w_2 = 2 = 1(2)$

$w_3 = w_1 + 3 = 4 = 2^2$

$w_4 = w_2 + 4 = 6 = 2(3)$

$w_5 = w_3 + 5 = 9 = 3^2$

$w_6 = w_4 + 6 = 12 = 3(4)$

$w_7 = w_5 + 7 = 16 = 4^2$

$w_8 = w_6 + 8 = 20 = 4(5)$

Guess: $w_n = \begin{cases} \left(\frac{n+1}{2}\right)^2 & \text{if } n \text{ is odd} \\ \frac{n}{2}\left(\frac{n}{2} + 1\right) & \text{if } n \text{ is even} \end{cases}$ for all integers $n \geq 1$.

(b) *Proof.* Let the property $P(n)$ be the equation

$$w_n = \begin{cases} \left(\frac{n+1}{2}\right)^2 & \text{if } n \text{ is odd} \\ \frac{n}{2}\left(\frac{n}{2}+1\right) & \text{if } n \text{ is even} \end{cases} \qquad \leftarrow P(n)$$

Show that $P(1)$ and $P(2)$ are true: From part (a) we have that $w_1 = 1$ and $w_2 = 2$. Hence $P(1)$ and $P(2)$ are true.

Show that for all integers $k \geq 2$, $P(i)$ is true for all integers i from 1 through $k \implies P(k+1)$: Let k be any integer with $k \geq 2$ and suppose that

$$w_i = \begin{cases} \left(\frac{i+1}{2}\right)^2 & \text{if } i \text{ is odd} \\ \frac{i}{2}\left(\frac{i}{2}+1\right) & \text{if } i \text{ is even} \end{cases} \qquad \leftarrow \text{inductive hypothesis}$$

We must show that this implies that

$$w_{k+1} = \begin{cases} \left(\frac{k+2}{2}\right)^2 & \text{if } k+1 \text{ is odd} \\ \frac{k+1}{2}\left(\frac{k+1}{2}+1\right) & \text{if } k+1 \text{ is even} \end{cases} \qquad \leftarrow P(k+1)$$

Case 1 ($k+1$ is odd): $k+1$ is odd $\implies k-1$ is odd. Then,

$$\begin{aligned} w_{k+1} &= w_{k-1} + k + 1 && \text{by definition of } w_1, w_2, w_3, \ldots \\ &= \left(\frac{k}{2}\right)^2 + k + 1 && \text{by inductive hypothesis} \\ &= \frac{k^2+4k+4}{4} = \left(\frac{k+2}{2}\right)^2 \end{aligned}$$

Case 2 ($k+1$ is even): $k+1$ is even $\implies k-1$ is even. Then,

$$\begin{aligned} w_{k+1} &= w_{k-1} + k + 1 && \text{by definition of } w_1, w_2, w_3, \ldots \\ &= \frac{k-1}{2}\left(\frac{k-1}{2}+1\right) + k + 1 && \text{by inductive hypothesis} \\ &= \left(\frac{k-1}{2}\right)^2 + \frac{k-1}{2} + k + 1 \\ &= \frac{k^2-2k+1}{4} + \frac{2k-2}{4} + \frac{4k}{4} + \frac{4}{4} \\ &= \left(\frac{k^2-2k+1}{4} + \frac{4k}{4}\right) + \left(\frac{2k-2}{4} + \frac{4}{4}\right) \\ &= \left(\frac{k^2+2k+1}{4}\right) + \left(\frac{2k+2}{4}\right) \\ &= \left(\frac{k+1}{2}\right)^2 + \frac{k+1}{2} = \frac{k+1}{2}\left(\frac{k+1}{2}+1\right) \end{aligned}$$

We arrive at the right-hand side of $P(k+1)$ in both cases and so $P(k+1)$ is true. $\square$

Problem 49

$u_k = u_{k-2} \cdot u_{k-1}$ for all integers $k \geq 2$

$u_0 = u_1 = 2$

Solution

(a) $u_0 = u_1 = 2$

$u_2 = u_0 \cdot u_1 = 2^1 \cdot 2^1$

$u_3 = u_1 \cdot u_2 = 2^1 \cdot 2^2$

$u_4 = u_2 \cdot u_3 = 2^2 \cdot 2^3 = 32$

Guess:
$$\begin{aligned} u_n &= 2^{F_{n-2}} \cdot 2^{F_{n-1}} \\ &= 2^{F_{n-2}+F_{n-1}} \\ &= 2^{F_n} \quad \text{for all integers } n \geq 0. \end{aligned}$$

(b) *Proof.* Let F_n be the explicit formula for the sequence of Fibonacci numbers which was verified in problem 5.6.33 and let the property $P(n)$ be the equation

$$u_n = 2^{F_n} \qquad \leftarrow P(n)$$

Show that $P(0)$ and $P(1)$ are true: From part (a) we have that $u_0 = u_1 = 2$. Also $2^{F_0} = 2^{F_1} = 2^1 = 2$. Hence $P(0)$ and $P(1)$ are true.

Show that for all integers $k \geq 1$, $P(i)$ is true for all integers i from 0 through $k \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$u_i = 2^{F_i} \qquad \leftarrow \text{inductive hypothesis}$$

We must show that this implies that

$$u_{k+1} = 2^{F_{k+1}} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} u_{k+1} &= u_{k-1} \cdot u_k && \text{by definition of } u_0, u_1, u_2, \ldots \\ &= 2^{F_{k-1}} \cdot 2^{F_k} && \text{by inductive hypothesis} \\ &= 2^{F_{k-1}+F_k} \\ &= 2^{F_{k+1}} && \text{by definition of } F_0, F_1, F_2, \ldots \end{aligned}$$

which is the right-hand side of $P(k+1)$. □

In 50 and 51 determine whether the given recursively defined sequence satisfies the explicit formula $a_n = (n-1)^2$, for all integers $n \geq 1$.

Problem 50

$a_k = 2a_{k-1} + k - 1 \quad$ for all integers $k \geq 2$

$a_1 = 0$

Solution

$a_2 = 2a_1 + 2 - 1 = 2(0) + 2 - 1 = 1$

$a_3 = 2a_2 + 3 - 1 = 2(1) + 3 - 1 = 4$

$a_4 = 2a_3 + 4 - 1 = 2(4) + 4 - 1 = 11$

The recurrence relation given for a_k does not satisfy the explicit formula for a_n. As a counterexample $a_4 = 11$ as given by the recurrence relation. However the explicit formula for the sequence gives $a_4 = (4-1)^2 = 3^2 = 9 \neq 11$.

Problem 51

$a_k = (a_{k-1} + 1)^2 \quad$ for all integers $k \geq 2$

$a_1 = 0$

Solution

$a_2 = (a_1 + 1)^2 = (0+1)^2 = 1^2 = 1$

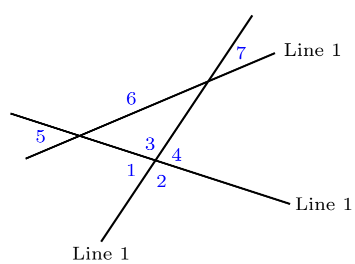
$a_3 = (a_2 + 1)^2 = (1+1)^2 = 2^2 = 4$

$a_4 = (a_3 + 1)^2 = (4+1)^2 = 5^2 = 25$

The recurrence relation given for a_k does not satisfy the explicit formula for a_n. As a counterexample $a_4 = 25$ as given by the recurrence relation. However the explicit formula for the sequence gives $a_4 = (4-1)^2 = 3^2 = 9 \neq 25$.

Problem 52

A single line divides a plane into two regions. Two lines (by crossing) can divide a plane into four regions; three lines can divide it into seven regions(see the figure below). Let P_n be the maximum number of regions into which n lines divide a plane, where n is a positive integer.



(a) Derive a recurrence relation for P_k in terms of P_{k-1}, for all integers $k \geq 2$.

(b) Use iteration to guess an explicit formula for P_n.

Solution

(a) In order to obtain the maximum number of regions in the plane, each new line added must intersect all existing lines but not at an already existing intersection. The only requirement for each line to intersect each other is that none of them are parallel. Since there are an infinite number of angles it is possible to place k lines in such a way that none have common angles (i.e. are parallel). Furthermore every additional line can be placed in such a way that it does not intersect any other already existing intersection. The procedure is as follows: first select a unique angle to hold the line at and then slide the line back and forth until there are no common intersections. This can be done because there are more real numbers than integers and the position of the line can be shifted by any real number in a direction perpendicular to the line but there can only be an integer number of intersections. Now that we know that it is always possible to place a line that will produce the maximum number of regions consider the result of such an action. Every region the new line passes through will be divided into two regions and the number of regions that the line passes through will be one more than the number of lines that the new line crosses. Since the new line, the kth line, crosses every other line it will cross $k-1$ lines and so pass through k regions. This will then divide k existing regions into two new regions each and increase the total number of regions by k. It follows that,

$$\begin{aligned} &P_k = P_{k-1} + k \quad \text{for all integers } k \geq 2 \\ &P_1 = 2 \end{aligned}$$

(b) $P_1 = 2$
$P_2 = P_1 + 2 = 2 + 2$
$P_3 = P_2 + 3 = 2 + 2 + 3$
$P_4 = P_3 + 4 = 2 + 2 + 3 + 4$
$P_5 = P_4 + 5 = 2 + 2 + 3 + 4 + 5$

$$\begin{aligned} \textit{Guess: } P_n &= 1 + (1 + 2 + 3 + \ldots + n) \\ &= 1 + \frac{n(n+1)}{2} \qquad \text{by theorem 5.2.2} \\ &= \frac{2}{2} + \frac{n^2+n}{2} \\ &= \frac{n^2+n+2}{2} \end{aligned}$$

Problem 53

Compute $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$ for small values of n. Conjecture explicit formulas for the entries in this matrix, and prove your conjecture using mathematical induction.

Solution

In contrast to the way the Fibonacci sequence is defined in 5.6 let $F_0 = 0$ and let $F_1 = 1$. Now,

$$\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}^1 = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}^2 = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}^3 = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}^2 = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}2 & 1\\1 & 1\end{bmatrix} = \begin{bmatrix}3 & 2\\2 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}^4 = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}^3 = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}3 & 2\\2 & 1\end{bmatrix} = \begin{bmatrix}5 & 3\\3 & 2\end{bmatrix}$$

Conjecture. $\forall n \in \mathbb{Z}^{nonneg}$, $\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}^n = \begin{bmatrix}F_{n+1} & F_1\\F_1 & F_{n-1}\end{bmatrix}$.

Proof. Let the property $P(n)$ be the equation

$$\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}^n = \begin{bmatrix}F_{n+1} & F_n\\F_n & F_{n-1}\end{bmatrix} \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: It follows from the definition of the Fibonacci sequence that $F_0 = 0$, $F_1 = 1$, and $F_2 = 1$. Hence if $n = 1$ then,

$$\begin{bmatrix}F_{n+1} & F_n\\F_n & F_{n-1}\end{bmatrix} = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}^n = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}^k = \begin{bmatrix}F_{k+1} & F_k\\F_k & F_{k-1}\end{bmatrix} \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}^{k+1} = \begin{bmatrix}F_{k+2} & F_{k+1}\\F_{k+1} & F_k\end{bmatrix} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}^{k+1} &= \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}^1\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}^k\\ &= \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}F_{k+1} & F_k\\F_k & F_{k-1}\end{bmatrix} && \text{by inductive hypothesis}\\ &= \begin{bmatrix}F_{k+1}+F_k & F_k+F_{k-1}\\F_{k+1} & F_k\end{bmatrix} = \begin{bmatrix}F_{k+2} & F_{k+1}\\F_{k+1} & F_k\end{bmatrix}\end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Problem 54

In economics the behavior of an economy from one period to another is often modeled by recurrence relations. Let Y_k be the income in period k and C_k be the consumption in period k. In one economic model, income in any period is assumed to be the sum of consumption in that period plus investment and government expenditures(which are assumed to constant from period to period), and consumption in each period is assumed to be a linear function of the income of the preceding period. That is,

$Y_k = C_k + E$	where E is the sum of investment plus government expenditures
$C_k = c + mY_{k-1}$	where c and m are constants

Substituting the second equation into the first gives $Y_k = E + c + mY_{k-1}$.

(a) Use iteration on the above recurrence relation to obtain

$$Y_n = (E + c)\left(\frac{m^n - 1}{m - 1}\right) + m^n Y_0$$

for all integers $n \geq 0$.

(b) Show that if $0 < m < 1$, then $\lim_{n\to\infty} Y_n = \dfrac{E + c}{m - 1}$.

Solution

(a) $$\begin{aligned}
Y_0 &= Y_0\\
Y_1 &= E + c + mY_0\\
Y_2 &= E + c + mY_1 = E + c + (E + c)(m) + m^2Y_0\\
&= (E + c)(1 + m) + m^2Y_0\\
Y_3 &= E + c + mY_2 = E + c + (E + c)(m + m^2) + m^3Y_0\\
&= (E + c)(1 + m + m^2) + m^3Y_0\\
Y_4 &= E + c + mY_3 = E + c + (E + c)(m + m^2 + m^3) + m^4Y_0\\
&= (E + c)(1 + m + m^2 + m^3) + m^4Y_0
\end{aligned}$$

Guess: $$\begin{aligned}
Y_n &= (E + c)(1 + m + m^2 + ... + m^{n-1}) + m^nY_0\\
&= (E + c)\left(\frac{m^n - 1}{m - 1}\right) + m^nY_0 \quad \text{by theorem 5.2.3}
\end{aligned}$$

(b)

Lemma 1. *If $0 < m < 1$ and $E + c < Y_0 - mY_0$ then the sequence Y_n defined recursively as*

$$\begin{aligned}
Y_k &= E + c + mY_{k-1}, \quad k \geq 1\\
Y_0 &= Y_0
\end{aligned}$$

is monotonically decreasing and bounded below.

Proof. Let the property $P(n)$ be the inequality

$$Y_n > Y_{n+1} \qquad \leftarrow P(n)$$

Show that $P(0)$ is true: Let $n = 0$. Then $Y_n = Y_0$ and

$$\begin{aligned} Y_{n+1} = Y_1 &= E + c + mY_0 \\ &< Y_0 - mY_0 + mY_0 && E + c < Y_0 - mY_0 \\ &= Y_0 \end{aligned}$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$Y_k > Y_{k+1} \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that $P(k+1)$ is true. But the left-hand side of $P(k+1)$ is

$$\begin{aligned} Y_{k+1} &= E + c + mY_k && \text{by definition of } Y_0, Y_1, Y_2, \ldots \\ &> E + c + mY_{k+1} && \text{by inductive hypothesis} \\ &= Y_{k+2} \end{aligned}$$

which is the right-hand side of $P(k+1)$.

We have shown that Y_n is monotonically decreasing. Next we verify that Y_n is bounded below for all $n \geq 0$. Let the property $P(n)$ be the inequality

$$Y_n > \frac{E+c}{1-m} \qquad \leftarrow P(n)$$

Show that $P(0)$ is true: Let $n = 0$. Then $Y_n = Y_0$. But

$$\begin{aligned} E + c &< Y_0 - mY_0 \\ E + c &< Y_0(1-m) \\ Y_0 &> \frac{E+c}{1-m} && m < 1 \end{aligned}$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$Y_k > \frac{E+c}{1-m} \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that $P(k+1)$ is true. But the left-hand side of $P(k+1)$ is

$$\begin{aligned} Y_{k+1} &= E + c + mY_k \\ &> E + c + m\left(\frac{E+c}{1-m}\right) && \text{by inductive hypothesis} \\ &= (E+c)\left(1 + \frac{m}{1-m}\right) = \frac{E+c}{1-m} \end{aligned}$$

which is the right-hand side of $P(k+1)$. $\square$

Lemma 2. *If $0 < m < 1$ and $E + c > Y_0 - mY_0$ then the sequence Y_n defined recursively as*

$$\begin{aligned} Y_k &= E + c + mY_{k-1}, \quad k \geq 1 \\ Y_0 &= Y_0 \end{aligned}$$

is monotonically increasing and bounded above.

Proof. Let the property $P(n)$ be the inequality

$$Y_n < Y_{n+1} \qquad \leftarrow P(n)$$

Show that $P(0)$ is true: Let $n = 0$. Then $Y_n = Y_0$ and

$$\begin{aligned} Y_{n+1} = Y_1 &= E + c + mY_0 \\ &> Y_0 - mY_0 + mY_0 && E + c > Y_0 + mY_0 \\ &= Y_0 \end{aligned}$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$Y_k < Y_{k+1} \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that $P(k+1)$ is true. But the left-hand side of $P(k+1)$ is

$$\begin{aligned} Y_{k+1} &= E + c + mY_k && \text{by definition of } Y_0, Y_1, Y_2, \ldots \\ &< E + c + mY_{k+1} && \text{by inductive hypothesis} \\ &= Y_{k+2} \end{aligned}$$

which is the right-hand side of $P(k+1)$.

We have shown that Y_n is monotonically increasing. Next we verify that Y_n is bounded above for all $n \geq 0$. Let the property $P(n)$ be the inequality

$$Y_n < \frac{E + c}{1 - m} \qquad \leftarrow P(n)$$

Show that $P(0)$ is true: Let $n = 0$. Then $Y_n = Y_0$. But

$$\begin{aligned} E + c &> Y_0 - mY_0 \\ E + c &> Y_0(1 - m) \\ Y_0 &< \frac{E + c}{1 - m} && m < 1 \end{aligned}$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$Y_k < \frac{E + c}{1 - m} \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that $P(k+1)$ is true. But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
Y_{k+1} &= E + c + mY_k \\
&< E + c + m\left(\frac{E+c}{1-m}\right) && \text{by inductive hypothesis} \\
&= (E+c)\left(1+\frac{m}{1-m}\right) = \frac{E+c}{1-m}
\end{aligned}$$

which is the right-hand side of $P(k+1)$. $\square$

Lemma 3. *If $0 < m < 1$ and $E + c = Y_0 - mY_0$ then the sequence Y_n defined recursively as*

$$\begin{aligned}
&Y_k = E + c + mY_{k-1}, \quad k \geq 1 \\
&Y_0 = Y_0
\end{aligned}$$

is constant.

Proof. Let the property $P(n)$ be the inequality

$$Y_n = Y_{n+1} \qquad \leftarrow P(n)$$

Show that $P(0)$ is true: Let $n = 0$. Then $Y_n = Y_0$ and

$$\begin{aligned}
Y_{n+1} = Y_1 &= E + c + mY_0 \\
&= Y_0 - mY_0 + mY_0 && E + c > Y_0 + mY_0 \\
&= Y_0
\end{aligned}$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$Y_k = Y_{k+1} \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that $P(k+1)$ is true. But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
Y_{k+1} &= E + c + mY_k && \text{by definition of } Y_0, Y_1, Y_2, \ldots \\
&= E + c + mY_{k+1} && \text{by inductive hypothesis} \\
&= Y_{k+2}
\end{aligned}$$

which is the right-hand side of $P(k+1)$. $\square$

It follows from lemma 1 and 2 that if $E + c \neq Y_0 - mY_0$ and $0 < m < 1$ then Y_n is a bounded and monotonic sequence. Hence by the monotonic sequence theorem, if $E + c \neq Y_0 - mY_0$ and $0 < m < 1$ then Y_n is convergent. From lemma 3 we have that if $E+c = Y_0 - mY_0$ and $0 < m < 1$

then Y_n is a constant sequence. It follows that for some constant C and for every $\epsilon > 0$,

$$n > -1 \implies |Y_n - C| = 0 < \epsilon$$

Hence if $E + c = Y_0 - mY_0$ and $0 < m < 1$, then Y_n is convergent. Thus in all cases Y_n is convergent and so $L = \lim_{n\to\infty} Y_n$ exists. Finally,

$$\begin{aligned}
L &= \lim_{n\to\infty} Y_n \\
L &= \lim_{n\to\infty} E + c + mY_{n-1} \\
L &= E + c + m \lim_{n\to\infty} Y_{n-1} \\
L &= E + c + m \lim_{x\to\infty} Y_x && \text{let } x = n - 1 \\
L &= E + c + mL
\end{aligned}$$

Now we solve the equation $L = E + c + mL$ to obtain a value for L.

$$L = E + c + ml \implies L - mL = E + c \implies L = \frac{E + c}{1 - m}.$$