Section 5.8

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Problem 1

Which of the following are second-order linear homogeneous recurrence relations with constant coefficients?

(a) $a_k = 2a_{k-1} - 5a_{k-2}$

(b) $b_k = kb_{k-1} + b_{k-2}$

(c) $c_k = 3c_{k-1} \cdot c_{k-2}^2$

(d) $d_k = 3d_{k-1} + d_{k-2}$

(e) $r_k = r_{k-1} - r_{k-2} - 2$

(f) $s_k = 10s_{k-2}$

Solution

(a) Yes; $A = 2$ and $B = -5$

(b) No; nonconstant coefficients

(c) No; not linear

(d) Yes; $A = 3$ and $B = 1$

(e) No; not homogeneous

(f) Yes; $A = 0$ and $B = 10$

Problem 2

Which of the following are second-order linear homogeneous recurrence relations with constant coefficients?

(a) $a_k = (k-1)a_{k-1} + 2ka_{k-2}$

(b) $b_k = -b_{k-1} + 7b_{k-2}$

(c) $c_k = 3c_{k-1} + 1$

(d) $d_k = 3d_{k-1}^2 + d_{k-2}$

(e) $r_k = r_{k-1} - 6r_{k-3}$

(f) $s_k = s_{k-1} + 10s_{k-2}$

Solution

(a) No; nonconstant coefficients

(b) Yes; $A = -1$ and $B = 7$

(c) No; not homogeneous

(d) No; not linear

(e) No; not second order

(f) Yes; $A = 1$ and $B = 10$

Problem 3

Let $a_0, a_1, a_2, ...$ be the sequence defined by the explicit formula

$$a_n = C \cdot 2^n + D \quad \text{for all integers } n \geq 0$$

where C and D are real numbers.

(a) Find C and D so that $a_0 = 1$ and $a_1 = 3$. What is a_2 in this case?

(b) Find C and D so that $a_0 = 0$ and $a_1 = 2$. What is a_2 in this case?

Solution

(a) $a_0 = 1 = C \cdot 2^0 + D \implies C + D = 1 \qquad (1)$

$a_1 = 3 = C \cdot 2^1 + D \implies 2C + D = 3 \qquad (2)$

It follows from equation (2) that

$$\begin{aligned} C + (C + D) &= 3 \\ C + 1 &= 3 \qquad \text{by substitution from (1)} \\ C &= 2 \end{aligned}$$

Now by substitution of $C = 2$ into equation (1) we obtain $D = -1$. Hence $a_n = 2 \cdot 2^2 - 1 = 8 - 1 = 7$.

(b) $a_0 = 0 = C \cdot 2^0 + D \implies C + D = 0 \qquad (1)$

$a_1 = 2 = C \cdot 2^1 + D \implies 2C + D = 2 \qquad (2)$

It follows from equation (2) that

$$\begin{aligned} C + (C + D) &= 2 \\ C + 0 &= 2 \qquad \text{by substitution from (1)} \\ C &= 2 \end{aligned}$$

Now by substitution of $C = 2$ into equation (1) we obtain $D = -2$. Hence $a_n = 2 \cdot 2^2 - 2 = 8 - 2 = 6$.

Problem 4

Let $b_0, b_1, b_2, ...$ be the sequence defined by the explicit formula

$$b_n = C \cdot 3^n + D(-2)^n \quad \text{for all integers } n \geq 0.$$

where C and D are real numbers.

(a) Find C and D so that $b_0 = 0$ and $b_1 = 5$. What is b_2 in this case?

(b) Find C and D so that $b_0 = 3$ and $b_1 = 4$. What is b_2 in this case?

Solution

(a) $\left\{\begin{array}{l} b_0 = C \cdot 3^0 + D \cdot (-2)^0 = C + D = 0 \\ b_1 = C \cdot 3^1 + D \cdot (-2)^1 = 3C - 2D = 5 \end{array}\right\}$

$\iff \left\{\begin{array}{l} D = -C \\ 3C - 2(-C) = 5 \end{array}\right\} \iff \left\{\begin{array}{l} C = 1 \\ D = -1 \end{array}\right\}$

$b_2 = 1 \cdot 3^2 - 1(-2)^2 = 9 - 4 = 5$

(b) $\left\{\begin{array}{l} b_0 = C \cdot 3^0 + D \cdot (-2)^0 = C + D = 3 \\ b_1 = C \cdot 3^1 + D \cdot (-2)^1 = 3C - 2D = 4 \end{array}\right\}$

$\iff \left\{\begin{array}{l} D = 3 - C \\ 3C - 2(3 - C) = 4 \end{array}\right\} \iff \left\{\begin{array}{l} C = 2 \\ D = 1 \end{array}\right\}$

$b_2 = 2 \cdot 3^2 + 1(-2)^2 = 18 + 4 = 22$

Problem 5

Let $a_0, a_1, a_2, ...$ be the sequence defined by the explicit formula

$$a_n = C \cdot 2^n + D \quad \text{for all integers } n \geq 0,$$

where C and D are real numbers. Show that for any choice of C and D,

$$a_k = 3a_{k-1} - 2a_{k-2} \quad \text{for all integers } k \geq 2$$

Solution

Let $a_n = C \cdot 2^n + D$ for any real numbers C and D and for all integers $n \geq 0$. Then

$$a_k = C \cdot 2^k + D \qquad a_{k-1} = C \cdot 2^{k-1} + D \qquad a_{k-2} = C \cdot 2^{k-2} + D$$

Hence

$$\begin{aligned} 3a_{k-1} - 2a_{k-2} &= 3(C \cdot 2^{k-1} + D) - 2(C \cdot 2^{k-2} + D) \\ &= 3C \cdot 2^{k-1} + 3D - 2C \cdot 2^{k-2} - 2D \\ &= 3C \cdot 2^{k-1} - 2C \cdot 2^{k-2} + 3D - 2D \\ &= C(3 \cdot 2^{k-1} - 2 \cdot 2^{k-2}) + D \\ &= C(3 \cdot 2^{k-1} - 2^{k-1}) + D \\ &= C \cdot 2^{k-1} + D \\ &= a_k \end{aligned}$$

Problem 6

Let $b_0, b_1, b_2, ...$ be the sequence defined by the explicit formula

$$b_n = C \cdot 3^n + D(-2)^n \quad \text{for all integers } n \geq 0,$$

where C and D are real numbers. Show that for any choice of C and D,

$$b_k = b_{k-1} + 6b_{k-2} \quad \text{for all integers } k \geq 2.$$

Solution

Let $b_n = C \cdot 3^n + D(-2)^n$ for any real numbers C and D and for all integers $n \geq 0$. Then

$$b_k = C{\cdot}3^k{+}D(-2)^k \qquad b_{k-1} = C{\cdot}3^{k-1}{+}D(-2)^{k-1} \qquad b_{k-2} = C{\cdot}3^{k-2}{+}D(-2)^{k-2}$$

Hence

$$\begin{aligned}
b_{k-1} + 6b_{k-2} &= C \cdot 3^{k-1} + D(-2)^{k-1} + 6(C \cdot 3^{k-2} + D(-2)^{k-2}) \\
&= C \cdot 3^{k-1} + D(-2)^{k-1} + 6C \cdot 3^{k-2} + 6D(-2)^{k-2} \\
&= C \cdot 3^{k-1} + 6C \cdot 3^{k-2} + D(-2)^{k-1} + 6D(-2)^{k-2} \\
&= C(3^{k-1} + 2 \cdot 3 \cdot 3^{k-2}) + D((-2)^{k-1} + (-3)(-2)(-2)^{k-2}) \\
&= C(3^{k-1} + 2 \cdot 3^{k-1}) + D((-2)^{k+1} + (-3)(-2)^{k-1}) \\
&= C(3 \cdot 3^{k-1}) + D((-2)(-2)^{k-1}) \\
&= C \cdot 3^k + D(-2)^k = b_k
\end{aligned}$$

Problem 7

Solve the system of equations in Example 5.8.4 to obtain

$$C = \frac{1+\sqrt{5}}{2\sqrt{5}} \quad \text{and} \quad D = \frac{-(1-\sqrt{5})}{2\sqrt{5}}$$

Solution

$$\left\{\begin{aligned} &C + D = 1 \\ &C\left(\frac{1+\sqrt{5}}{2}\right) + D\left(\frac{1-\sqrt{5}}{2}\right) = 1 \end{aligned}\right\} \iff \left\{\begin{aligned} &C = 1 - D \\ &(1-D)\left(\frac{1+\sqrt{5}}{2}\right) + D\left(\frac{1-\sqrt{5}}{2}\right) = 1 \end{aligned}\right\}$$

$$\iff \left\{\begin{aligned} &C = 1 - D \\ &\frac{1+\sqrt{5}-D-D\sqrt{5}+D-D\sqrt{5}}{2} = 1 \end{aligned}\right\} \iff \left\{\begin{aligned} &C = 1 - D \\ &\frac{1+\sqrt{5}-2D\sqrt{5}}{2} = 1 \end{aligned}\right\}$$

$$\iff \left\{\begin{aligned} &C = 1 - D \\ &D = \frac{(1-\sqrt{5})}{2\sqrt{5}} \end{aligned}\right\} \iff \left\{\begin{aligned} &C = \frac{2\sqrt{5}}{2\sqrt{5}} - \frac{-(1-\sqrt{5})}{2\sqrt{5}} \\ &D = \frac{-(1-\sqrt{5})}{2\sqrt{5}} \end{aligned}\right\} \iff \left\{\begin{aligned} &C = \frac{1+\sqrt{5}}{2\sqrt{5}} \\ &D = \frac{-(1-\sqrt{5})}{2\sqrt{5}} \end{aligned}\right\}$$

In each of 8-10: (a) suppose a sequence of the form $1, t, t^2, t^3, ..., t^n, ...$ where $t \neq 0$ satisfies the given recurrence relation (but not necessarily the initial conditions), and find all possible values of t: (b) suppose a sequence satisfies the given initial conditions as well as the recurrence relation, and find an explicit formula for the sequence.

Problem 8

$a_k = 2a_{k-1} + 3a_{k-2}, \quad$ for all integers $k \geq 2$

$a_0 = 1,\ a_1 = 2$

Solution

(a) It follows from definition that a_k is a second-order linear homogeneous recurrence relation with constant coefficients $A = 2$ and $B = 3$. Hence by lemma 5.8.1 $1, t, t^2, t^3, ..., t^n, ...$ satisfies $a_k \iff t$ satisfies $t^2 - At - B = 0$.

$$t^2 - 2t - 3 = 0 \implies (t-3)(t+1) = 0 \implies t = 3 \quad \text{or} \quad t = -1$$

(b) It follows from (a) and the distinct roots theorem that for some numbers C and D, $a_0, a_1, a_2, ...$ is given by the explicit formula

$$a_n = C \cdot 3^n + D \cdot (-1)^n \quad \text{for all integers } n \geq 0.$$

Since $a_0 = 1$ and $a_1 = 2$ we have that

$$\begin{Bmatrix} a_0 = C \cdot 3^0 + D \cdot (-1)^0 = C + D = 1 \\ a_1 = C \cdot 3^1 + D \cdot (-1)^1 = 3C - D = 2 \end{Bmatrix}$$
$$\iff \begin{Bmatrix} D = 1 - C \\ 3C - (1 - C) = 2 \end{Bmatrix} \iff \begin{Bmatrix} D = 1 - C \\ 4C - 1 = 2 \end{Bmatrix} \iff \begin{Bmatrix} D = 1/4 \\ C = 3/4 \end{Bmatrix}$$

Thus $a_n = \dfrac{3}{4}(3^n) + \dfrac{1}{4}(-1)^n \quad$ for all integers $n \geq 0$.

Problem 9

$b_k = 7b_{k-1} - 10b_{k-2}, \quad$ for all integers $k \geq 2$

$b_0 = 2,\ b_1 = 2$

Solution

(a) It follows from definition that b_k is a second-order linear homogeneous recurrence relation with constant coefficients $A = 7$ and $B = -10$. Hence by lemma 5.8.1 $1, t, t^2, t^3, ..., t^n, ...$ satisfies $b_k \iff t$ satisfies $t^2 - At - B = 0$.

$$t^2 - 7t + 10 = 0 \implies (t-5)(t-2) = 0 \implies t = 5 \quad \text{or} \quad t = 2$$

(b) It follows from (a) and the distinct roots theorem that for some numbers C and D, $b_0, b_1, b_2, ...$ is given by the explicit formula

$$b_n = C \cdot 5^n + D \cdot 2^n \quad \text{for all integers } n \geq 0.$$

Since $b_0 = 2$ and $b_1 = 2$ we have that

$$\left\{\begin{aligned} b_0 &= C \cdot 5^0 + D \cdot 2^0 = C + D = 2 \\ b_1 &= C \cdot 5^1 + D \cdot 2^1 = 5C + 2D = 2 \end{aligned}\right\}$$
$$\iff \left\{\begin{aligned} &D = 2 - C \\ &5C + 2(2 - C) = 2 \end{aligned}\right\} \iff \left\{\begin{aligned} &D = 2 - C \\ &3C + 4 = 2 \end{aligned}\right\} \iff \left\{\begin{aligned} &D = 8/3 \\ &C = -2/3 \end{aligned}\right\}$$

Thus $b_n = -\dfrac{2}{3}(5^n) + \dfrac{8}{3}(2)^n \quad$ for all integers $n \geq 0$.

Problem 10

$c_k = c_{k-1} + 6c_{k-2}, \quad$ for all integers $k \geq 2$

$c_0 = 0,\ c_1 = 3$

Solution

(a) It follows from definition that c_k is a second-order linear homogeneous recurrence relation with constant coefficients $A = 1$ and $B = 6$. Hence by lemma 5.8.1 $1, t, t^2, t^3, ..., t^n, ...$ satisfies $c_k \iff t$ satisfies $t^2 - At - B = 0$.

$$t^2 - t - 6 = 0 \implies (t - 3)(t + 2) = 0 \implies t = 3 \quad \text{or} \quad t = -2$$

(b) It follows from (a) and the distinct roots theorem that for some numbers C and D, $c_0, c_1, c_2, ...$ is given by the explicit formula

$$c_n = C \cdot 3^n + D \cdot (-2)^n \quad \text{for all integers } n \geq 0.$$

Since $c_0 = 0$ and $c_1 = 3$ we have that

$$\left\{\begin{aligned} c_0 &= C \cdot 3^0 + D \cdot (-2)^0 = C + D = 0 \\ c_1 &= C \cdot 3^1 + D \cdot (-2)^1 = 3C - 2D = 3 \end{aligned}\right\}$$
$$\iff \left\{\begin{aligned} &D = -C \\ &3C - 2(-C) = 3 \end{aligned}\right\} \iff \left\{\begin{aligned} &D = -C \\ &5C = 3 \end{aligned}\right\} \iff \left\{\begin{aligned} &D = -3/5 \\ &C = 3/5 \end{aligned}\right\}$$

Thus $c_n = \dfrac{3}{5}(3^n) - \dfrac{3}{5}(-2)^n \quad$ for all integers $n \geq 0$.

In each of 11 - 16 suppose a sequence satisfies the given recurrence relation and initial conditions. Find an explicit formula for the sequence.

Problem 11

$d_k = 4d_{k-2}, \quad$ for all integers $k \geq 2$

$d_0 = 1, \; d_1 = -1$

Solution

Characteristic equation: $t^2-4=0 \implies (t-2)(t+2)=0 \implies t=2$ or $t=-2$. Hence by the distinct roots theorem for some numbers C and D, $d_0, d_1, d_2, \ldots$ is given by the explicit formula

$$d_n = C \cdot 2^n + D \cdot (-2)^n \quad \text{for all integers } n \geq 0$$

Since $d_0 = 1$ and $d_1 = -1$ we have that

$$\begin{Bmatrix} d_0 = C \cdot 2^0 + D \cdot (-2)^0 = C + D = 1 \\ d_1 = C \cdot 2^1 + D \cdot (-2)^1 = 2C - 2D = -1 \end{Bmatrix}$$
$$\iff \begin{Bmatrix} D = 1 - C \\ 2C - 2(1-C) = -1 \end{Bmatrix} \iff \begin{Bmatrix} D = 1 - C \\ 4C - 2 = -1 \end{Bmatrix} \iff \begin{Bmatrix} D = 3/4 \\ C = 1/4 \end{Bmatrix}$$

Thus $d_n = \dfrac{1}{4}(2^n) + \dfrac{3}{4}(-2)^n \quad$ for all integers $n \geq 0$.

Problem 12

$e_k = 9e_{k-2}, \quad$ for all integers $k \geq 2$

$e_0 = 0, \; e_1 = 2$

Solution

Characteristic equation: $t^2-9=0 \implies (t-3)(t+3)=0 \implies t=3$ or $t=-3$. Hence by the distinct roots theorem for some numbers C and D, $e_0, e_1, e_2, \ldots$ is given by the explicit formula

$$e_n = C \cdot 3^n + D \cdot (-3)^n \quad \text{for all integers } n \geq 0$$

Since $e_0 = 0$ and $e_1 = 2$ we have that

$$\begin{Bmatrix} e_0 = C \cdot 3^0 + D \cdot (-3)^0 = C + D = 0 \\ e_1 = C \cdot 3^1 + D \cdot (-3)^1 = 3C - 3D = 2 \end{Bmatrix}$$
$$\iff \begin{Bmatrix} D = -C \\ 3C - 3(-C) = 2 \end{Bmatrix} \iff \begin{Bmatrix} D = -C \\ 6C = 2 \end{Bmatrix} \iff \begin{Bmatrix} D = -1/3 \\ C = 1/3 \end{Bmatrix}$$

Thus $d_n = \dfrac{1}{3}(3^n) - \dfrac{1}{3}(-3)^n \quad$ for all integers $n \geq 0$.

Problem 13

$r_k = 2r_{k-1} - r_{k-2}, \quad$ for all integers $k \geq 2$
$r_0 = 1, \; r_1 = 4$

Solution

Characteristic equation: $t^2 - 2t + 1 = 0 \implies (t-1)^2 = 0 \implies t = 1$. Hence by the single root theorem for some numbers C and D, $r_0, r_1, r_2, ...$ is given by the explicit formula

$$r_n = C + nD \quad \text{for all integers } n \geq 0$$

Since $r_0 = 1$ and $r_1 = 4$ we have that

$$\begin{Bmatrix} e_0 = C = 1 \\ e_1 = C + D = 4 \end{Bmatrix} \iff \begin{Bmatrix} C = 1 \\ D = 3 \end{Bmatrix}$$

Thus $r_n = 3n + 1 \quad$ for all integers $n \geq 0$.

Problem 14

$s_k = -4s_{k-1} - 4s_{k-2}, \quad$ for all integers $k \geq 2$
$s_0 = 0, \; s_1 = -1$

Solution

Characteristic equation: $t^2 + 4t + 4 = 0 \implies (t+2)^2 = 0 \implies t = -2$. Hence by the single root theorem for some numbers C and D, $s_0, s_1, s_2, ...$ is given by the explicit formula

$$s_n = C \cdot (-2)^n + nD \cdot (-2)^n \quad \text{for all integers } n \geq 0$$

Since $s_0 = 0$ and $s_1 = -1$ we have that

$$\begin{Bmatrix} s_0 = C \cdot (-2)^0 + 0 \cdot D \cdot (-2)^0 = C = 0 \\ s_1 = C \cdot (-2)^1 + D \cdot (-2)^1 = -2C - 2D = -1 \end{Bmatrix} \iff \begin{Bmatrix} C = 0 \\ D = 1/2 \end{Bmatrix}$$

Thus $s_n = \dfrac{1}{2} n \cdot (-2)^n \quad$ for all integers $n \geq 0$.

Problem 15

$t_k = 6t_{k-1} - 9t_{k-2}, \quad$ for all integers $k \geq 2$
$t_0 = 1, \; t_1 = 3$

Solution

Characteristic equation: $t^2 - 6t + 9 = 0 \implies (t-3)^2 = 0 \implies t = 3$. Hence by the single root theorem for some numbers C and D, $t_0, t_1, t_2, ...$ is given by the explicit formula

$$t_n = C \cdot 3^n + nD \cdot 3^n \quad \text{for all integers } n \geq 0$$

Since $t_0 = 1$ and $t_1 = 3$ we have that

$$\left\{\begin{matrix} t_0 = C \cdot 3^0 + 0 \cdot D \cdot 3^0 = C = 1 \\ t_1 = C \cdot 3^1 + D \cdot 3^1 = 3C + 3D = 3 \end{matrix}\right\} \iff \left\{\begin{matrix} C = 1 \\ D = 0 \end{matrix}\right\}$$

Thus $t_n = 3^n \quad$ for all integers $n \geq 0$.

Problem 16

$s_k = 2s_{k-1} + 2s_{k-2}, \quad$ for all integers $k \geq 2$
$s_0 = 1,\ s_1 = 3$

Solution

Characteristic equation: $t^2 - 2t - 2 = 0$. By the quadratic equation

$$t = \frac{2 \pm \sqrt{4 - 4(1)(-2)}}{2} = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$$

Hence by the distinct roots theorem for some numbers C and D, $s_0, s_1, s_2, ...$ is given by the explicit formula

$$s_n = C \cdot (1 + \sqrt{3})^n + D \cdot (1 - \sqrt{3})^n \quad \text{for all integers } n \geq 0$$

Since $s_0 = 1$ and $s_1 = 3$ we have that

$$\left\{\begin{matrix} s_0 = C \cdot (1+\sqrt{3})^0 + D \cdot (1-\sqrt{3})^0 = C + D = 1 \\ s_1 = C \cdot (1+\sqrt{3})^1 + D \cdot (1-\sqrt{3})^1 = C \cdot (1+\sqrt{3}) + D \cdot (1-\sqrt{3}) = 3 \end{matrix}\right\}$$

$$\iff \left\{\begin{matrix} D = 1 - C \\ C \cdot (1+\sqrt{3}) + (1-C) \cdot (1-\sqrt{3}) = 3 \end{matrix}\right\} \iff \left\{\begin{matrix} D = 1 - C \\ 2C\sqrt{3} + 1 - \sqrt{3} = 3 \end{matrix}\right\}$$

$$\iff \left\{\begin{matrix} D = \dfrac{\sqrt{3} - 2}{2\sqrt{3}} \\ C = \dfrac{\sqrt{3} + 2}{2\sqrt{3}} \end{matrix}\right\}$$

Thus $s_n = \dfrac{\sqrt{3}+2}{2\sqrt{3}}(1+\sqrt{3})^n + \dfrac{\sqrt{3}-2}{2\sqrt{3}}(1-\sqrt{3})^n \quad$ for all integers $n \geq 0$.

Problem 17

Find an explicit formula for the sequence of exercise 39 in section 5.6.

$c_k = c_{k-1} + c_{k-2}$ for all integers $k \geq 3$
$c_1 = 1, \ c_2 = 2$

Solution

We adjust the formula that we obtained in 5.6.39 to be

$c_k = c_{k-1} + c_{k-2}$ for all integers $k \geq 2$
$c_0 = 1, \ c_1 = 1$

This reflects the fact that there is 1 way to climb a staircase of 0 steps. Further this allows the recurrence relation to hold for $k \geq 2$. While the distinct roots theorem will work for the exact relation derived in 5.6.39 we would need to justify why $k \geq 3$ instead of $k \geq 2$ is valid which is not the purpose of the problem.

Characteristic equation: $t^2 - t - 1 = 0$. By the quadratic equation

$$t = \frac{1 \pm \sqrt{1^2 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Hence by the distinct roots theorem for some numbers C and D, $c_0, c_1, c_2, \ldots$ is given by the explicit formula

$$c_n = C\left(\frac{1+\sqrt{5}}{2}\right)^n + D\left(\frac{1+\sqrt{5}}{2}\right)^n \quad \text{for all integers } n \geq 0.$$

Since $c_0 = 1$ and $c_1 = 1$ we have that

$$C + D = 1 \quad \text{and} \quad C\left(\frac{1+\sqrt{5}}{2}\right) + D\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

However this system of two equations was solved for C and D in problem 5.8.7 with the result that

$$C = \frac{1+\sqrt{5}}{2\sqrt{5}} \quad \text{and} \quad D = \frac{-(1-\sqrt{5})}{2\sqrt{5}}$$

Thus $c_n = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{-(1-\sqrt{5})}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n$

This is recognized as the explicit formula for the Fibonacci sequence derived in example 5.8.4. However the problem as stated in 5.6.39 required that $n \geq 1$ and so our formula will only apply for all integers $n \geq 1$.

Problem 18

Suppose that the sequence $s_0, s_1, s_2, \ldots$ and $t_0, t_1, t_2, \ldots$ both satisfy the same second-order linear homogeneous recurrence relation with constant coefficients:

$$\begin{aligned} s_k &= 5s_{k-1} - 4s_{k-2} \quad \text{for all integers } k \geq 2, \\ t_k &= 5t_{k-1} - 4t_{k-2} \quad \text{for all integers } k \geq 2. \end{aligned}$$

Show that the sequence $2s_0 + 3t_0, 2s_1 + 3t_1, 2s_2 + 3t_2, \ldots$ also satisfies the same relation. In other words, show that

$$2s_k + 3t_k = 5(2s_{k-1} + 3t_{k-1}) - 4(2s_{k-2} + 3t_{k-2})$$

for all integers $k \geq 2$. Do *not* use lemma 5.8.2.

Solution

$$\begin{aligned} 2s_k + 3t_k &= 2(5s_{k-1} - 4s_{k-2}) + 3(5t_{k-1} - 4t_{k-2}) && \text{by definition of } s_0, s_1, s_2, \ldots \text{ and } t_0, t_1, t_2, \ldots \\ &= 2 \cdot 5s_{k-1} - 2 \cdot 4s_{k-2} + 3 \cdot 5t_{k-1} - 3 \cdot 4t_{k-2} \\ &= 2 \cdot 5s_{k-1} + 3 \cdot 5t_{k-1} - 2 \cdot 4s_{k-2} - 3 \cdot 4t_{k-2} \\ &= 5(2s_{k-1} + 3t_{k-1}) - 4(2s_{k-2} + 3t_{k-2}) \end{aligned}$$

Problem 19

Show that if r, s, a_0, and a_1 are numbers with $r \neq s$, then there exist unique numbers C and D so that

$$\begin{aligned} C + D &= a_0 \\ Cr + Ds &= a_1 \end{aligned}$$

Solution

From linear algebra it is known that a $n \times n$ system of linear equations has a unique solution if and only if the determinant of its coefficient matrix in non-zero. Let A be the coefficient matrix of the system of linear equations above. Then,

$$A = \begin{bmatrix} 1 & 1 \\ r & s \end{bmatrix} \quad \text{and} \quad \det(A) = s \cdot 1 - r \cdot 1 = s - r$$

However, $s \neq r$ and so $s - r \neq 0$. Hence, $\det(A) \neq 0$ and so the system of linear equations above must have a unique solution.

Problem 20

Show that if r is a nonzero real number, k and m are distinct integers, and a_k, and a_m are any real numbers, then there exist unique real numbers C and D so that

$$\begin{aligned} Cr^k + kDr^k &= a_k \\ Cr^m + mDr^m &= a_m \end{aligned}$$

Solution

Let A be the coefficient matrix of the system of linear equations above. Then,

$$A = \begin{bmatrix} r^k & kr^k \\ r^m & mr^m \end{bmatrix} \quad \text{and} \quad \det(A) = mr^{k+m} - kr^{k+m} = r^{k+m}(m-k)$$

However, $k \neq m$ and $r \neq 0$ and so $\det(A) = r^{k+m}(m-k) \neq 0$. Hence the system of linear equations above must have a unique solution.

Problem 21

Prove theorem 5.8.5 for the case where the values of C and D are determined by a_0 and a_1.

Solution

Theorem. *Suppose a sequence $a_0, a_1, a_2, \ldots$ satisfies a recurrence relation*

$$a_k = Aa_{k-1} + Ba_{k-2}$$

for some real numbers A and B with $B \neq 0$ and for all integers $k \geq 2$. If the characteristic equation $t^2 - At - B = 0$ has a single (real) root r, then $a_0, a_1, a_2, \ldots$ is given by the explicit formula

$$a_n = Cr^n + Dnr^n,$$

where C and D are the real numbers whose values are determined by the values of a_0 and a_1.

Proof. Let the property $P(n)$ be the equation that for all integers $n \geq 0$,

$$a_n = Cr^n + Dnr^n \qquad \leftarrow P(n)$$

Show that $P(0)$ and $P(1)$ are true: C and D are selected to be those numbers that make the following equations true:

$$a_0 = Cr^0 + D(0)r^0 \quad \text{and} \quad a_1 = Cr^1 + D(1)r^1$$

Now note that $r \neq 0$ as r is the root of the characteristic equation $t^2 - At - B = 0$, and so

$$r = 0 \implies 0^2 - A(0) - B = 0 \implies B = 0 \text{ but } B \neq 0.$$

Finally it follows from problem 5.8.20 that there must be a unique solution to the above linear equations and so $P(1)$ and $P(2)$ are true.

Show that for all integers $k \geq 1$, $P(i)$ is true for all integers i from 0 through $k \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$a_i = Cr^i + Dir^i \qquad \leftarrow \text{inductive hypothesis}$$

We must show that this implies that

$$a_{k+1} = Cr^{k+1} + D(k+1)r^{k+1} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
a_{k+1} &= Aa_k + Ba_{k-1} && \text{by definition of } a_0, a_1, a_2, \ldots \\
&= A(Cr^k + Dkr^k) + B(Cr^{k-1} + D(k-1)r^{k-1}) && \text{by inductive hypothesis} \\
&= ACr^k + ADkr^k + BCr^{k-1} + BD(k-1)r^{k-1} \\
&= (ACr^k + BCr^{k-1}) + (ADkr^k + BD(k-1)r^{k-1}) \\
&= C(Ar^k + Br^{k-1}) + D(Akr^k + B(k-1)r^{k-1}) \\
&= Cr^{k+1} + D(k+1)r^{k+1} && \text{by lemma 5.8.4}
\end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Exercises 22 and 23 are intended for students who are familiar with complex numbers.

Problem 22

Find an explicit formula for a sequence $a_0, a_1, a_2, \ldots$ that satisfies

$$a_k = 2a_{k-1} - 2a_{k-2} \quad \text{for all integers } k \geq 2$$

with initial conditions $a_0 = 1$ and $a_1 = 2$.

Solution

Characteristic equation: $t^2 - 2t + 2 = 0$. By the quadratic equation

$$t = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

Hence by the distinct roots theorem for some numbers C and D, $a_0, a_1, a_2, \ldots$ is given by the explicit formula

$$a_n = C(1+i)^n + D(1-i)^n \quad \text{for all integers } n \geq 0$$

Since $a_0 = 1$ and $a_1 = 2$ we have that

$$\begin{Bmatrix} a_0 = C \cdot (1+i)^0 + D \cdot (1-i)^0 = C + D = 1 \\ a_1 = C \cdot (1+i)^1 + D \cdot (1-i)^1 = C \cdot (1+i) + D \cdot (1-i) = 2 \end{Bmatrix}$$

$$\iff \begin{Bmatrix} D = 1 - C \\ C + Ci + (1-C)(1-i) = 2 \end{Bmatrix} \iff \begin{Bmatrix} D = 1 - C \\ C + Ci + 1 - i - C + Ci = 2 \end{Bmatrix}$$

$$\iff \begin{Bmatrix} D = 1 - C \\ C(1 + i - 1 + i) + 1 - i = 2 \end{Bmatrix} \iff \begin{Bmatrix} D = 1 - C \\ C(2i) = 1 + i \end{Bmatrix}$$

$$\iff \left\{\begin{array}{l} D = 1 - C \\ C = \dfrac{1+i}{2i} = \dfrac{1+i}{2i} \cdot \dfrac{i}{i} = \dfrac{i-1}{-2} = \dfrac{1-i}{2} \end{array}\right\}$$

$$\iff \left\{\begin{array}{l} D = 1 - \dfrac{1-i}{2} = \dfrac{2-1+i}{2} = \dfrac{1+i}{2} \\ C = \dfrac{1-i}{2} \end{array}\right\}$$

Thus $a_n = \left(\dfrac{1-i}{2}\right)(1+i)^n + \left(\dfrac{1+i}{2}\right)(1-i)^n$ for all integers $n \geq 0$.

Problem 23

Find an expression for a sequence $b_0, b_1, b_2, ...$ that satisfies

$$b_k = 2b_{k-1} - 5b_{k-2} \quad \text{for all integers } k \geq 2$$

with initial conditions $b_0 = 1$ and $b_1 = 1$.

Solution

Characteristic equation: $t^2 - 2t + 5 = 0$. By the quadratic equation

$$t = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(5)}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

Hence by the distinct roots theorem for some numbers C and D, $b_0, b_1, b_2, ...$ is given by the explicit formula

$$b_n = C(1+2i)^n + D(1-2i)^n \quad \text{for all integers } n \geq 0$$

Since $b_0 = 1$ and $b_1 = 1$ we have that

$$\left\{\begin{array}{l} b_0 = C \cdot (1+2i)^0 + D \cdot (1-2i)^0 = C + D = 1 \\ b_1 = C \cdot (1+2i)^1 + D \cdot (1-2i)^1 = C \cdot (1+2i) + D \cdot (1-2i) = 1 \end{array}\right\}$$

$$\iff \left\{\begin{array}{l} D = 1 - C \\ C + 2Ci + (1-C)(1-2i) = 1 \end{array}\right\}$$

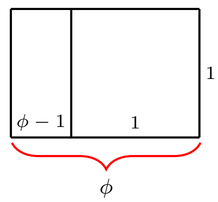
$$\iff \left\{\begin{array}{l} D = 1 - C \\ C + 2Ci + 1 - 2i - C + 2Ci = 1 \end{array}\right\}$$

$$\iff \left\{\begin{array}{l} D = 1 - C \\ C(1 + 2i - 1 + 2i) + 1 - 2i = 1 \end{array}\right\} \iff \left\{\begin{array}{l} D = 1 - C \\ C(4i) = 2i \end{array}\right\} \iff \left\{\begin{array}{l} D = 1/2 \\ C = 1/2 \end{array}\right\}$$

Thus $b_n = \dfrac{1}{2}(1+2i)^n + \dfrac{1}{2}(1-2i)^n$ for all integers $n \geq 0$.

Problem 24

The numbers $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$ that appear in the explicit formula for the Fibonacci sequence are related to a quantity called the golden ratio in Greek mathematics. Consider a rectangle of length ϕ units and height 1, where $\phi > 1$.



Divide the rectangle into a rectangle and a square as shown in the preceding diagram. The square is 1 unit on each side, and the rectangle has sides of length 1 and $\phi - 1$.

The ancient Greeks considered the outer rectangle to be perfectly proportioned (saying that the lengths of its sides were in a golden ratio to each other) if the ratio of the length to the width of the outer rectangle equaled the ratio of the length to the width of the inner rectangle. That is,

$$\frac{\phi}{1} = \frac{1}{\phi - 1}$$

(a) Show that ϕ satisfies the following quadratic equation:
$t^2 - t - 1 = 0$.

(b) Find the two solutions of $t^2 - t - 1 = 0$ and call them ϕ_1 and ϕ_2.

(c) Express the explicit formula for the Fibonacci sequence in terms of ϕ_1 and ϕ_2.

Solution

(a) $\frac{\phi}{1} = \frac{1}{\phi - 1} \implies \phi = \frac{1}{\phi - 1} \implies \phi(\phi - 1) = 1 \implies \phi^2 - \phi - 1 = 0$

(b) By the quadratic equation

$$t = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Hence $\phi_1 = \frac{1+\sqrt{5}}{2}$ and $\phi_2 = \frac{1-\sqrt{5}}{2}$

(c) From the explicit formula for the Fibonacci sequence derived in Example 5.8.4 we have that

$$F_n = \frac{\phi_1^{n+1}}{\sqrt{5}} - \frac{\phi_2^{n+1}}{\sqrt{5}} = \frac{1}{\sqrt{5}}(\phi_1^{n+1} - \phi_2^{n+1})$$