Section 6.2

Sterling Jeppson

December 29, 2020

Problem 1 and Solution

a. To say that an element is in $A \cap (B \cup C)$ means that it is in A and in $B \cup C$.

b. To say that an element is in $(A \cap B) \cup C$ means that is is in $A \cap B$ or in C.

c. To say that an element is in $A - (B \cap C)$ means that it is in A and not in $B \cap C$.

Problem 2 and Solution

The following are two proofs that for all sets A and B, $A - B \subseteq A$. The first is less formal, and the second is more formal. Fill in the blanks.

a. *Proof.* Suppose A and B are any two sets. To show that $A - B \subseteq A$, we must show that every element in $A - B$ is in A. But any element in $A - B$ is in A and not in B (by definition of $A - B$). In particular, such an element is in A. □

b. *Proof.* Suppose A and B are any sets and $x \in A - B$. *[We must show that* $x \in A$*]*. By definition of set difference, $x \in A$ and $x \notin B$. In particular, $x \in A$ *[which is what was to be shown]*. □

Problem 3 and Solution

The following is a proof that for all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. Fill in the blanks.

Proof. Suppose A, B, and C are sets and $A \subseteq B$ and $B \subseteq C$. To show that $A \subseteq C$, we must show that every element in A is in C. But given any element in A, that element is in B (because $A \subseteq B$), and so that element is also in C (because $B \subseteq C$). Hence $A \subseteq C$. □

Problem 4 and Solution

The following is a proof that for all sets A and B, if $A \subseteq B$, then $A \cup B \subseteq B$. Fill in the blanks.

Proof. Suppose A and B are any sets and $A \subseteq B$. *[We must show that $A \cup B \subseteq B$].* Let $x \in A \cup B$. *[We must show that $x \in B$.]* By definition of union, $x \in A$ or $x \in B$. In the case $x \in A$, then since $A \subseteq B$, $x \in B$. In the case $x \in B$, then clearly $x \in B$. So in either case, $x \in B$ *[as was to be shown].* $\square$

Problem 5 and Solution

Prove that for all sets A and B, $(B - A) = B \cap A^c$.

Proof. Suppose that A and B are any sets.

(1) Proof that $\boldsymbol{B - A \subseteq B \cap A^c}$: Suppose that $x \in B - A$. By definition of set difference, $x \in B$ and $x \notin A$. It follows from the definition of set complement that $x \in B$ and $x \in A^c$. Now by the definition of set union, $x \in B \cap A^c$. Hence $B - A \subseteq B \cap A^c$ by definition of subset.

(2) Proof that $\boldsymbol{B \cap A^c \subseteq B - A}$: Suppose that $x \in B \cap A^c$. By definition of set intersection, $x \in B$ and $x \in A^c$. But this means, by definition of set complement, that $x \in B$ and $x \notin A$. It now follows by definition of set difference that $x \in B - A$. Hence $B \cap A^c \subseteq B - A$ by definition of subset.

Conclusion: Since both set containment's have been proved, $B - A = B \cap A^c$ by definition of set equality. $\square$

Problem 6 and Solution

The following is a proof that for any sets A, B, and C, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Fill in the blanks.

Proof. Suppose that A, B, and C are any sets.

(1) Proof that $\boldsymbol{A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)}$: Let $x \in A \cap (B \cup C)$. *[We must show that $x \in (A \cap B) \cup (A \cap C)$.]* By definition of intersection, $x \in A$ and $x \in B \cup C$. Thus $x \in A$ and, by definition of union, $x \in B$ or $x \in C$.

Case 1 ($\boldsymbol{x \in A}$ *and* $\boldsymbol{x \in B}$): In this case, by definition of intersection, $x \in A \cap B$, and so, by definition of union, $x \in (A \cap B) \cup (A \cap C)$.

Case 2 ($\boldsymbol{x \in A}$ *and* $\boldsymbol{x \in C}$): In this case, by definition of intersection, $x \in A \cap C$, and so, by definition of union $x \in (A \cap B) \cup (A \cap C)$. Hence in either case, $x \in (A \cap B) \cup (A \cap C)$ *[as was to be shown].*
[So $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ by definition of subset.]

(2) Proof that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$: Let $x \in (A \cap B) \cup (A \cap C)$. *[We must show that $x \in A \cap (B \cup C)$.]* By definition of union, $x \in A \cap B$ or $x \in A \cap C$.

Case 1 ($x \in A \cap B$): In this case, by definition of intersection, $x \in A$ and $x \in B$. Since $x \in B$, then by definition of union, $x \in B \cup C$. Hence $x \in A$ and $x \in B \cup C$, and so, by definition of intersection, $x \in A \cap (B \cup C)$.

Case 2 ($x \in A \cap C$): In this case, by definition of intersection, $x \in A$ and $x \in C$. Since $x \in C$, then by definition of union, $x \in B \cup C$, and so, by definition of intersection, $x \in A \cap (B \cup C)$. In either case, $x \in A \cap (B \cup C)$ *[as was to be shown]. [Thus $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ by definition of subset.]*

Conclusion: *[Since both subset relations have been proved, it follows, be definition of set equality, that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.]* □

Use an element argument to prove each statement in 7-19. Assume that all sets are subsets of a universal set U.

Problem 7 and Solution

For all sets A and B, $(A \cap B)^c = A^c \cup B^c$.

Proof. Suppose that A and B are any sets.

(1) Proof that $(A \cap B)^c \subseteq A^c \cup B^c$: Suppose that $x \in (A \cap B)^c$. It follows from the definition of complement that $x \notin A \cap B$. By definition of intersection this means that it is false to say that ($x \in A$ and $x \in B$). It follows from De Morgan's laws of logic that $x \notin A$ or $x \notin B$. But this means that $x \in A^c \cup B^c$. Hence $(A \cap B)^c \subseteq A^c \cup B^c$ by definition of subset.

(2) Proof that $A^c \cup B^c \subseteq (A \cap B)^c$: Suppose that $x \in A^c \cup B^c$. It follows from the definition of set union and complement that $x \notin A$ or $x \notin B$. But this means by De Morgan's laws of logic that it is false that ($x \in A$ and $x \in B$) and so, by definition of intersection, $x \notin A \cap B$. It follows from the definition of complement that $x \in (A \cap B)^c$. Hence $A^c \cup B^c \subseteq (A \cap B)^c$ by definition of subset.

Conclusion: Since both set containment's have been proved, $(A \cap B)^c = A^c \cup B^c$ by definition of set equality. □

Problem 8 and Solution

For all sets A and B, $(A \cap B) \cup (A \cap B^c) = A$.

Proof. Suppose that A and B be any sets.

(1) Proof that $(A \cap B) \cup (A \cap B^c) \subseteq A$: Suppose that $x \in (A \cap B) \cup (A \cap B^c)$.

It follows from the definition of union that $x \in A \cap B$ or $x \in A \cap B^c$.

Case 1 ($\boldsymbol{x \in A \cap B}$): In this case, by definition of intersection, $x \in A$ and $x \in B$ and so $x \in A$.

Case 2 ($\boldsymbol{x \in A \cap B^c}$): In this case, by definition of intersection, $x \in A$ and $x \in B^c$ and so $x \in A$. In either case, $x \in A$ and so $(A \cap B) \cup (A \cap B^c) \subseteq A$ by definition of subset.

(2) Proof that $\boldsymbol{A \subseteq (A \cap B) \cup (A \cap B^c)}$: Suppose that $x \in A$. Then we either have that $x \in B$ or $x \notin B$.

Case 1 ($\boldsymbol{x \in B}$): In this case $x \in A$ and $x \in B$. It follows from the definition of intersection that $x \in A \cap B$. Hence, by definition of union, $x \in (A \cap B) \cup (A \cap B^c)$.

Case 2 ($\boldsymbol{x \notin B}$): In this case, by definition of complement $x \in B^c$. We now have that $x \in B^c$ and $x \in A$ and so, by definition of intersection, $x \in A \cap B^c$. It follows by definition of union that, $x \in (A \cap B) \cup (A \cap B^c)$. In either case, $x \in (A \cap B) \cup (A \cap B^c)$ and so $A \subseteq (A \cap B) \cup (A \cap B^c)$ by definition of subset.

Conclusion: Since both set containment's have been proved, it follows by definition of set equality that $(A \cap B) \cup (A \cap B^c) = A$. □

Problem 9 and Solution

For all sets A, B, and C, $(A - B) \cup (C - B) = (A \cup C) - B$.

Proof. Suppose that A, B, and C are any sets.

(1) Proof that $\boldsymbol{(A - B) \cup (C - B) \subseteq (A \cup C) - B}$: Let $x \in (A - B) \cup (C - B)$. It follows by definition of union that $x \in A - B$ or $x \in C - B$.

Case 1 ($\boldsymbol{x \in A - B}$): In this case, by definition of difference, $x \in A$ and $x \notin B$. It follows from the definition of union that $x \in A \cup C$ and $x \notin B$. But this means, by definition of difference that $x \in (A \cup C) - B$.

Case 2 ($\boldsymbol{x \in C - B}$): In this case, by definition of difference, $x \in C$ and $x \notin B$. It follows from the definition of union that $x \in A \cup C$ and $x \notin B$. But this means, by definition of difference that $x \in (A \cup C) - B$.

In either case $x \in (A \cup C) - B$ and so $(A - B) \cup (C - B) \subseteq (A \cup C) - B$ by definition of subset.

(2) Proof that $\boldsymbol{(A \cup C) - B \subseteq (A - B) \cup (C - B)}$: Let $x \in (A \cup C) - B$. It follows by definition of difference that $x \in A \cup C$ and $x \notin B$. It follows by definition of set union that $x \in A$ of $x \in C$.

Case 1 ($\boldsymbol{x \in A}$): In this case $x \in A$ and $x \notin B$. It follows by definition of difference that $x \in A - B$. Now, by definition of union, $x \in (A - B) \cup (C - B)$.

Case 2 ($\boldsymbol{x \in C}$): In this case $x \in C$ and $x \notin B$. It follows by definition of difference that $x \in C - B$. Now, by definition of union, $x \in (A - B) \cup (C - B)$.

In either case $x \in (A - B) \cup (C - B)$ and so $(A \cup C) - B \subseteq (A - B) \cup (C - B)$ by definition of subset.

Conclusion: Since both set containment's have been proved, it follows by definition of set equality that $(A - B) \cup (C - B) = (A \cup C) - B$. $\square$

Problem 10 and Solution

For all sets A, B, and C, $(A - B) \cap (C - B) = (A \cap C) - B$.

Proof. Suppose that A, B, and C are any sets.

(1) Proof that $\boldsymbol{(A-B)\cap(C-B) \subseteq (A\cap C)-B}$: Let $x \in (A-B)\cap(C-B)$. It follows by definition of intersection that $x \in A - B$ and $x \in C - B$. By definition of difference this means that $x \in A$ and $x \notin B$ and $x \in C$ and $x \notin B$. But this can be simplified as $x \in A$ and $x \in C$ and $x \notin B$ which means, by definition of union and difference that $x \in (A \cap C) - B$. It now follows from the definition of subset that $(A - B) \cap (C - B) \subseteq (A \cap C) - B$.

(2) Proof that $\boldsymbol{(A \cap C) - B \subseteq (A - B) \cap (C - B)}$: Let $x \in (A \cap C) - B$. It follows by definition of difference that $x \in A \cap C$ and $x \notin B$. It follows by definition of intersection that $x \in A$ and $x \in C$ and $x \notin B$. This can be expanded as $x \in A$ and $x \notin B$ and $x \in C$ and $x \notin B$ which means, by definition of intersection and difference that $x \in (A - B) \cup (C - B)$. It now follows from the definition of subset that $(A \cap C) - B \subseteq (A - B) \cap (C - B)$.

Conclusion: Since both set containment's have been proved, it follows by definition of set equality that $(A - B) \cap (C - B) = (A \cap C) - B$. $\square$

Problem 11 and Solution

For all sets A and B, $A \cup (A \cap B) = A$.

Proof. Suppose that A and B are any sets.

(1) Proof that $\boldsymbol{A \cup (A \cap B) \subseteq A}$: Let $x \in A \cup (A \cap B)$. It follows by definition of union that $x \in A$ or $x \in A \cap B$.

Case 1 ($\boldsymbol{x \in A}$): In this case there is nothing more to show.

Case 2 ($\boldsymbol{x \in A \cap B}$): In this case, by definition of intersection, $x \in A$ and $x \in B$. In particular $x \in A$.

In either case $x \in A$ and so $A \cup (A \cap B) \subseteq A$ by definition of subset.

(2) Proof that $\boldsymbol{A \subseteq A \cup (A \cap B)}$: Let $x \in A$. Then we have that either $x \in B$ or $x \notin B$.

Case 1 ($\boldsymbol{x \in B}$): In this case $x \in A$ and $x \in B$. It follows by definition of intersection that $x \in A \cap B$. Now by the definition of union $x \in A \cup (A \cap B)$.

Case 2 ($\boldsymbol{x \notin B}$): In this case $x \in A$ and so, by the definition of union, $x \in A \cup (A \cap B)$.

In either case $x \in A \cup (A \cap B)$ and so $A \cup (A \cap B) \subseteq A$ by definition of subset.

Conclusion: Since both set containment's have been proved, it follows by definition of set equality that $A \cup (A \cap B) = A$. □

Problem 12 and Solution

for all sets A, $A \cup \emptyset = A$.

Proof. Let A be any set.

(1) Proof that $\boldsymbol{A \cup \emptyset \subseteq A}$: Suppose that $x \in A \cup \emptyset$. Then, by definition of union, $x \in A$ or $x \in \emptyset$. However, since $x \notin \emptyset$ it must be that $x \in A$. It now follows by definition of subset that $A \cup \emptyset \subseteq A$.

(2) Proof that $\boldsymbol{A \subseteq A \cup \emptyset}$: Suppose that $x \in A$. Then it is true that $x \in A$ or $x \in \emptyset$ and so, by definition of union, $x \in A \cup \emptyset$.

Conclusion: Since both set containment's have been proved, it follows by definition of set equality that $A \cup \emptyset = A$. □

Problem 13 and Solution

For all sets A, B, and C, if $A \subseteq B$ then $A \cap C \subseteq B \cap C$.

Proof. Suppose that A, B, and C are any sets such that $A \subseteq B$. Let $x \in A \cap C$. It follows by definition of intersection that $x \in A$ and $x \in C$. Now since $A \subseteq B$ and $x \in A$ it follows from the definition of subset that $x \in B$. We now have that $x \in C$ and $x \in B$ and so $x \in B \cap C$. Hence $A \cap C \subseteq B \cap C$ by the definition of subset. □

Problem 14 and Solution

For all sets, A, B, and C, if $A \subseteq B$ then $A \cup C \subseteq B \cup C$.

Proof. Let A, B, and C be any sets such that $A \subseteq B$ and let $x \in A \cup C$. It follows by definition of union that $x \in A$ or $x \in C$.

Case 1 ($\boldsymbol{x \in A}$): In this case, since $x \in A$ and $A \subseteq B$, $x \in B$. Hence, by definition of union, $x \in B \cup C$.

Case 2 ($x \in C$): In this case, by definition of union, $x \in B \cup C$.

In either case $x \in B \cup C$ and so $A \cup C \subseteq B \cup C$ by definition of subset. $\square$

Problem 15

For all sets A and B, if $A \subseteq B$ then $B^c \subseteq A^c$.

Proof. Let A and B be any sets such that $A \subseteq B$ and let $x \in B^c$. It follows by definition of complement that $x \notin B$. Now since $A \subseteq B$ it must be the case that $x \notin A$. For if $x \in A$ then $x \in B$ which is false. Hence $x \notin A$ and so, by definition of complement, $x \in A^c$. Now, by definition of subset, $B^c \subseteq A^c$. $\square$

Problem 16 and Solution

For all sets A, B, and C, if $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$.

Proof. Suppose that A, B, and C are any sets such that $A \subseteq B$ and $A \subseteq C$. Now let $x \in A$. It follows by definition of subset that $x \in B$ and $x \in C$. But this means, by definition of intersection, that $x \in B \cap C$. Now, by definition of subset, $A \subseteq B \cap C$. $\square$

Problem 17 and Solution

For all sets A, B, and C, if $A \subseteq C$ and $B \subseteq C$ then $A \cup B \subseteq C$.

Proof. Suppose that A, B, and C are any sets such that $A \subseteq C$ and $B \subseteq C$. Also let $x \in A \cup B$. It follows by definition of union that $x \in A$ or $x \in B$.

Case 1 ($x \in A$): In this case, since $A \subseteq C$, $x \in C$ by definition of subset.

Case 2 ($x \in B$): In this case, since $B \subseteq B$, $x \in C$ by definition of subset.

In either case, $x \in C$ and so $A \cup B \subseteq C$ by definition of subset. $\square$

Problem 18 and Solution

For all sets A, B, and C, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Proof. Suppose that A, B, and C are any sets.

(1) Proof that $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$: Suppose that $(x, y) \in A \times (B \cup C)$. Then $x \in A$ and, by definition of union, $y \in B$ or $y \in C$.

Case 1 ($y \in B$): In this case, since $x \in A$, $(x, y) \in A \times B$ by definition of Cartesian product. It now follows from the definition of union that $(x, y) \in (A \times B) \cup (A \times C)$.

Case 2 ($y \in C$): In this case, since $x \in A$, $(x, y) \in A \times C$ by definition of Cartesian product. It now follows from the definition of union that $(x, y) \in (A \times B) \cup (A \times C)$.

In either case $(x, y) \in (x, y) \in (A \times B) \cup (A \times C)$ and so $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$.

(2) Proof that $\boldsymbol{(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)}$: Let $(x, y) \in (A \times B) \cup (A \times C)$. Then, by definition of union, $(x, y) \in A \times B$ or $(x, y) \in A \times C$.

Case 1 ($\boldsymbol{(x, y) \in A \times B}$): In this case it follows from the definition of Cartesian product that $x \in A$ and $y \in B$. Now, by definition of union, $y \in B \cup C$ and so, by definition of Cartesian product, $(x, y) \in A \times (B \cup C)$.

Case 2 ($\boldsymbol{(x, y) \in A \times C}$): In this case it follows from the definition of Cartesian product that $x \in A$ and $y \in C$. Now, by definition of union, $y \in B \cup C$ and so, by definition of Cartesian product, $(x, y) \in A \times (B \cup C)$.

In either case $(x, y) \in A \times (B \cup C)$ and so $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$.

Conclusion: Since both set containment's have been proved, it follows by definition of set equality that $A \times (B \cup C) = (A \times B) \cup (A \times C)$. □

Problem 19 and Solution

For all sets A, B, and C, $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Proof. Suppose that A, B, and C are any sets.

(1) Proof that $\boldsymbol{A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)}$: Let $(x, y) \in A \times (B \cap C)$. Then $x \in A$ and, by definition of intersection, $y \in B$ and $y \in C$. This can be expanded as $x \in A$ and $y \in B$ and $x \in A$ and $y \in C$. But this means, by definition of Cartesian product, that $(x, y) \in A \times B$ and $(x, y) \in A \times C$. It follows from definition of intersection that $(x, y) \in (A \times B) \cap (A \times C)$. Hence $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$.

(2) Proof that $\boldsymbol{(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)}$: Let $(x, y) \in (A \times B) \cap (A \times C)$. Then, by definition of intersection, $(x, y) \in A \times B$ and $(x, y) \in A \times C$. It follows from the definition of Cartesian product that $x \in A$ and $y \in B$ and $x \in A$ and $y \in C$. This can be simplified as $x \in A$ and $y \in B$ and $y \in C$. It now follows by definition of union that $x \in A$ and $y \in B \cap C$. Finally, by definition of Cartesian product, $(x, y) \in A \times (B \cap C)$. Hence $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$.

Conclusion: Since both set containment's have been proved, it follows by definition of set equality that $A \times (B \cap C) = (A \times B) \cap (A \times C)$. □

Problem 20

Find the mistake in the following "proof" that for all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

"Proof:" Suppose A, B, and C are sets such that $A \subseteq B$ and $B \subseteq C$. Since

$A \subseteq B$, there is an element x such that $x \in A$ and $x \in B$. Since $B \subseteq C$, there is an element x such that $x \in B$ and $x \in C$. Hence there is an element x such that $x \in A$ and $x \in C$ and so $A \subseteq C$.

Solution

The first problem with this false proof is that it has the wrong definition of subset. For any sets A and B, $A \subseteq B$ does not means that there exists an element $x \in A$ such that $x \in B$. This is not only an insufficient condition but also an unnecessary condition. For example, for any set C, $\emptyset \subseteq C$ but $\emptyset$ has no elements. Instead the correct way to define $A \subseteq B$ is to say that for all x, $x \in A \implies x \in B$. Even if the definition of subset provided by the proof was correct the proof would still not have shown that $A \subseteq B$. This is because the proof assumes that the x which is in A and B is the same x which is in both B and C. However this is not necessarily the case. For example, let $A = \{1, 2\}$, $B = \{1, 5\}$, and $C = \{5, 6\}$. Then there is an element in A which is also in B, namely 1, and there is an element in B which is in C, namely 5, but there is no element in A which is also in C.

Problem 21

Find the mistake in the following "proof" that for all sets A and B, $A^c \cup B^c \subseteq (A \cup B)^c$.

"Proof:" Suppose A and B are sets, and $x \in A^c \cup B^c$. Then $x \in A^c$ or $x \in B^c$ by definition of union. It follows that $x \notin A$ or $x \notin B$ by definition of complement, and so $x \notin A \cup B$ by definition of union. Thus $x \in (A \cup B)^c$ by definition of complement, and hence $A^c \cup B^c \subseteq (A \cup B)^c$.

Solution

The problem with this proof is that $(x \notin A \text{ or } x \notin B) \not\Longrightarrow x \notin A \cup B$. To say that $x \notin A$ or $x \notin B$ means, by De Morgan's laws of logic, that it is false that $x \in A$ and $x \in B$. Hence, by definition of intersection, $x \notin A \cap B$. As a concrete example that their reasoning is flawed, consider sets $A = \{1, 2\}$ and $B = \{3, 4\}$. Now let $x = 2$ and the statement $x \notin A$ or $x \notin B$ is true and yet it is not true that $x \notin A \cup B$.

Problem 22

Find the mistake in the following "proof" that for all sets A and B, $(A - B) \cup (A \cap B) \subseteq A$.

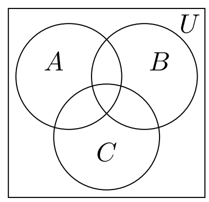
"Proof:" Suppose A and B are sets, and suppose $x \in (A - B) \cup (A \cap B)$. If $x \in A$ then $x \in A - B$. Then, by definition of difference, $x \in A$ and $x \notin B$. Hence $x \in A$, and so $(A - B) \cup (A \cap B) \subseteq A$ by definition of subset.

Solution

The problem with this proof is that $x \in A \not\Rightarrow x \in A-B$. As a counterexample consider sets $A = \{1, 2, 3\}$ and $B = \{1, 5, 6\}$. Then $A - B = \{2, 3\}$. However is $x = 1$ them $x \in A$ but $x \notin A - B$.

Problem 23

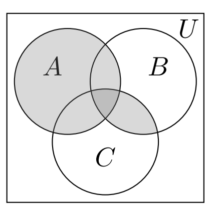
Consider the Venn diagram below



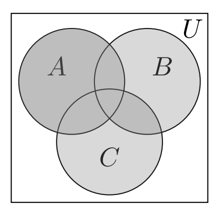
a. Illustrate one of the distributive laws by shading in the region corresponding to $A \cup (B \cap C)$ on one copy of the diagram and $(A \cup B) \cap (A \cup C)$ on another.
b. Illustrate the other distributive law by shading in the region corresponding to $A \cap (B \cup C)$ on one copy of the diagram and $(A \cap B) \cup (A \cap C)$ on another.
c. Illustrate one of De Morgan's laws by shading in the region corresponding to $(A \cup B)^c$ on one copy of the diagram and $A^c \cap B^c$ on the other. (Leave the set C out of your diagrams.)
d. Illustrate the other De Morgan's law by shading in the region corresponding to $(A \cap B)^c$ on one copy of the diagram and $A^c \cup B^c$ on the other. (Leave the set C out of your diagrams.)

Solution

a.

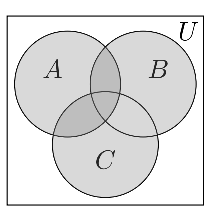


Entire shaded region is $A \cup (B \cap C)$

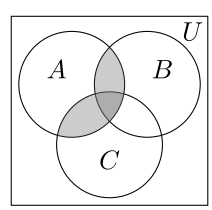


Dark shaded region is $(A \cup B) \cap (A \cup C)$

b.

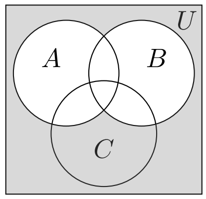


Dark shaded region is $A \cap (B \cup C)$

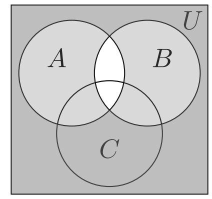


Entire shaded region is $(A \cap B) \cup (A \cap C)$

c.

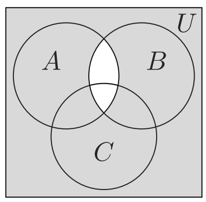


Entire shaded region is $(A \cup B)^c$

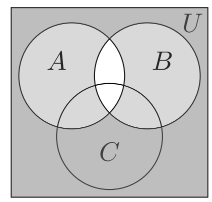


Dark shaded region is $A^c \cap B^c$

d.



Entire shaded region is $(A \cap B)^c$



Entire shaded region is $A^c \cup B^c$

Problem 24

Fill in the blanks in the following proof that for all sets A and B, $(A - B) \cap (B - A) = \emptyset$.

Proof. Let A and B be any sets and suppose $(A - B) \cap (B - A) \neq \emptyset$. That is, suppose there were an element $x \in (A - B) \cap (B - A)$. By definition of union, $x \in A - B$ and $x \in B - A$. Then by definition of set difference, $x \in A$ and $x \notin B$ and $x \in B$ and $x \notin A$. In particular, $x \in A$ and $x \notin A$ which is a contradiction. Hence the supposition that $(A - B) \cap (B - A) \neq \emptyset$ is false and so $(A - B) \cap (B - A) = \emptyset$. □

Use the element method for proving a set equals the empty set to prove each statement in 25-35. Assume that all sets are subsets of a universal set U.

Problem 25 and Solution

For all sets A and B, $(A \cap B) \cap (A \cap B^c) = \emptyset$.

Proof. Let A and B be any sets and suppose that $(A \cap B) \cap (A \cap B^c) \neq \emptyset$. That is suppose that there exists an element $x \in (A \cap B) \cap (A \cap B^c)$. By definition of intersection, $x \in A \cap B$ and $x \in A \cap B^c$. Then by definition of intersection, $x \in A$ and $x \in B$ and $x \in A$ and $x \in B^c$. Finally by definition of complement, $x \notin B$. Now $x \in B$ and $x \notin B$ which is a contradiction. Hence the supposition that $x(A \cap B) \cap (A \cap B^c) \neq \emptyset$ is false and so $(A \cap B) \cap (A \cap B^c) = \emptyset$. □

Problem 26 and Solution

For all sets, A, B, and C, $(A-B)\cap(B-C)\cap(A-B)=\emptyset$.

Proof. Let A, B, and C, be any sets and suppose that $(A-B)\cap(B-C)\cap(A-B)\neq\emptyset$. That is suppose that there exists an element $x\in(A-B)\cap(B-C)\cap(A-B)$. By definition of intersection, $x\in A-B$ and $x\in B-C$ and $x\in A-B$. Then by definition of set difference, $x\in A$ and $x\notin B$ and $x\in B$ and $x\notin C$ and $x\in A$ and $X\notin B$. In particular, $x\in B$ and $x\notin B$ which is a contradiction. Hence the supposition that $(A-B)\cap(B-C)\cap(A-B)\neq\emptyset$ is false and so $(A-B)\cap(B-C)\cap(A-B)=\emptyset$ □

Problem 27 and Solution

For all subsets A of a universal set U, $A\cap A^c=\emptyset$.

Proof. Let A be any set and let U be a universal set so that $A\subseteq U$. Suppose that $A\cap A^c\neq\emptyset$. Then there exists an element $x\in A\cap A^c$. It follows by definition of intersection that $x\in A$ and $x\in A^c$. It follows by definition of complement that $x\notin U$. But now we have that $x\in A$ and $x\notin A$ which is a contradiction. Hence the supposition that $A\cap U^c\neq\emptyset$ is false and so $A\cap U^c=\emptyset$. □

Problem 28 and Solution

If U denotes a universal set, then $U^c=\emptyset$.

Proof. Let U be a universal set and suppose that $U^c\neq\emptyset$. Then there exists an element $x\in U^c$. It now follows from the definition of complement that $x\notin U$. However this is a contradiction as x can only be chosen from the universal set U. Hence the supposition that $U^c\neq\emptyset$ is false and so $U^c=\emptyset$. □

Problem 29 and Solution

For all sets A, $A\times\emptyset=\emptyset$.

Proof. Let A be any set and suppose that $A\times\emptyset\neq\emptyset$. Then there exists an ordered pair $(x,y)\in A\times\emptyset$. It follows by definition of Cartesian product that $x\in A$ and $y\in\emptyset$. In particular $y\in\emptyset$ which is a contradiction as $\emptyset$ is defined to be that set which has no elements. Hence the supposition that $A\times\emptyset\neq\emptyset$ is false and so $A\times\emptyset=\emptyset$. □

Problem 30 and Solution

For all sets A and B, if $A\subseteq B$ then $A\cap B^c=\emptyset$.

Proof. Let A and B be any sets such that $A\subseteq B$ and suppose that $A\cap B^c\neq\emptyset$. Then there exists an element $x\in A\cap B^c$. It follows by definition of intersection that $x\in A$ and $x\in B^c$. By the definition of complement $x\notin B$. But now we have that $x\in A$ and $x\notin B$ which is a contradiction as $A\subseteq B$ which means that $x\in A\implies x\in B$. Hence the supposition that $A\cap B^c\neq\emptyset$ is false and so $A\times\emptyset=\emptyset$. □

Problem 31 and Solution

For all sets A and B, if $B \subseteq A^c$ then $A \cap B = \emptyset$.

Proof. Let A and B be any sets and suppose that $A \cap B \neq \emptyset$. Then there exists an element $x \in A \cap B$. It follows by definition of intersection that $x \in A$ and $x \in B$. But this is a contradiction as $B \subseteq A^c$ which means that $x \in B \implies x \notin A$. Hence the supposition that $A \cap B \neq \emptyset$ is false and so $A \cap B = \emptyset$. $\square$

Problem 32 and Solution

For all sets A, B, and C, if $A \subseteq B$ and $B \cap C = \emptyset$ then $A \cap C = \emptyset$.

Proof. Let A, B, and C be any sets such that $A \subseteq B$ and $B \cap C = \emptyset$ and suppose that $A \cap C \neq \emptyset$. Then there exists an element $x \in A \cap C$. By definition of intersection that means that $x \in A$ and $x \in C$. But this is a contradiction as $A \subseteq B$ and $B \cap C = \emptyset$ which means that $x \in A \implies x \in B \implies x \notin C$. Hence the supposition that $A \cap C \neq \emptyset$ is false and so $A \cap C = \emptyset$. $\square$

Problem 33 and Solution

For all sets A, B, and C, if $C \subseteq B - A$, then $A \cap C = \emptyset$.

Proof. Let A, B, and C be any sets such that $C \subseteq B - A$ and suppose that $A \cap C \neq \emptyset$. Then there exists some element $x \in A \cap C$. By definition of intersection, $x \in A$ and $x \in C$. But this is a contradiction as $C \subseteq B - A$ which means that $x \in C \implies x \in B$ and $x \notin A$. Hence the supposition that $A \cap C \neq \emptyset$ is false and so $A \cap C = \emptyset$. $\square$

Problem 34 and Solution

For all sets A, B, and C, if $B \cap C \subseteq A$, then $(C - A) \cap (B - A) = \emptyset$.

Proof. Let A, B, and C be any sets such that $B \cap C \subseteq A$ and suppose that $(C - A) \cap (B - A) \neq \emptyset$. Then there exists an element $x \in (C - A) \cap (B - A)$. By definition of intersection this means that $x \in C - A$ and $x \in B - A$. By definition of difference this means that $x \in C$ and $x \notin A$ and $x \in B$ and $x \notin A$. In particular, $x \in B$ and $x \in C$ and $x \notin A$. But this is a contradiction as $B \cap C \subseteq A$ which means that $x \in B$ and $x \in C \implies x \in A$. Hence the supposition that $(C - A) \cap (B - A) \neq \emptyset$ is false and so $(C - A) \cap (B - A) = \emptyset$. $\square$

Problem 35 and Solution

For all sets A, B, C, and D, if $A \cap C = \emptyset$ then $(A \times B) \cap (C \times D) = \emptyset$.

Proof. let A, B, C, and D be any sets such that $A \cap C = \emptyset$ and suppose that $(A \times B) \cap (C \times D) \neq \emptyset$. Then there exists an ordered pair $(x, y) \in (A \times B) \cap (C \times D)$. By definition of intersection this means that $(x, y) \in (A \times B)$ and $(x, y) \in (C \times D)$. By definition of Cartesian product this means that $x \in A$ and $y \in B$ and $x \in C$ and $y \in D$. In particular, $x \in A$ and $x \in C$. But this is a contradiction as $A \cap C = \emptyset$ which means that $x \in A \implies x \notin C$. Hence the supposition that $(A \times B) \cap (C \times D) \neq \emptyset$ is false and so $(A \times B) \cap (C \times D) = \emptyset$. $\square$

Prove each statement in 36 - 41

Problem 36

For all sets A and B,

a. $(A - B) \cup (B - A) \cup (A \cap B) = A \cup B$.

b. The sets $(A - B)$, $(B - A)$, and $(A \cap B)$ are mutually disjoint.

Solution

a. *Proof.* Let A and B be any sets.

(1) Proof that $\boldsymbol{(A - B) \cup (B - A) \cup (A \cap B) \subseteq A \cup B}$: Let $x \in (A - B) \cup (B - A) \cup (A \cap B)$. It follows by definition of union that $x \in A - B$ or $x \in B - A$ or $x \in A \cap B$.

Case 1 ($\boldsymbol{x \in A - B}$): It follows from definition of set difference that $x \in A$ and $x \notin B$. In particular $x \in A$ and so, by definition of union, $x \in A \cup B$.

Case 2 ($\boldsymbol{x \in B - A}$): It follows from definition of set difference that $x \in B$ and $x \notin A$. In particular $x \in B$ and so, by definition of union, $x \in A \cup B$.

Case 3 ($\boldsymbol{x \in A \cap B}$): It follows from definition of intersection that $x \in A$ and $x \in B$. It follows by definition of union that $x \in A \cup B$.

In all cases $x \in A \cup B$ and so, by definition of subset, $(A - B) \cup (B - A) \cup (A \cap B) \subseteq A \cup B$.

(2) Proof that $\boldsymbol{A \cup B \subseteq (A - B) \cup (B - A) \cup (A \cap B)}$: Let $x \in A \cup B$. It follows by definition of union that $x \in A$ or $x \in B$.

Case 1 ($\boldsymbol{x \in A}$): In this case we can either have that $x \in B$ or $x \notin B$. If $x \in B$ then it follows by definition of intersection that $x \in A \cap B$. Hence $x \in (A - B) \cup (B - A) \cup (A \cap B)$ by definition of union. In the case that $x \notin B$, then it follows by definition of set difference that $x \in A - B$. Hence $x \in (A - B) \cup (B - A) \cup (A \cap B)$ by definition of union.

Case 2 ($\boldsymbol{x \in B}$): In this case we can either have that $x \in A$ or $x \notin A$. If $x \in A$ then it follows by definition of intersection that $x \in A \cap B$. Hence $x \in (A - B) \cup (B - A) \cup (A \cap B)$ by definition of union. In the case that $x \notin A$, then it follows by definition of set difference that $x \in B - A$. Hence $x \in (A - B) \cup (B - A) \cup (A \cap B)$ by definition of union.

In either case $x \in (A - B) \cup (B - A) \cup (A \cap B)$ and so $A \cup B \subseteq (A - B) \cup (B - A) \cup (A \cap B)$.

Conclusion: Since both set containment's are proved, it follows by definition of set equality that $(A - B) \cup (B - A) \cup (A \cap B) = A \cup B$. $\square$

b. *Proof.* If $(A-B)$, $(B-A)$, and $(A\cap B)$ are mutually disjoint then the intersection of any two of these sets must be the empty set. But by definition of set difference and set intersection, $x \in A - B \implies x \in A$ and $x \notin B$, $x \in B - A \implies x \in B$ and $x \notin A$, and $x \in A \cap B \implies x \in A$ and $x \in B$. It follows that no two of these conditions can be satisfied at the same time and so no element can be in the intersection of any two of these sets. Hence the intersection of any two of these sets is the empty set and so the sets $(A - B)$, $(B - A)$, and $(A \cap B)$ are mutually disjoint. □

Problem 37 and Solution

For all integers $n \geq 1$, if A and $B_1, B_2, B_3, ...$ are any sets, then

$$A \cap \left(\bigcup_{i=1}^{n} B_i\right) = \bigcup_{i=1}^{n}(A \cap B_i)$$

Proof. Let A and $B_1, B_2, B_3, ...$ be any sets.

(1) Proof that $\boldsymbol{A \cap (\bigcup_{i=1}^{n} B_i) \subseteq \bigcup_{i=1}^{n}(A \cap B_i)}$: Let $x \in A \cap (\bigcup_{i=1}^{n} B_i)$. By definition of intersection, $x \in A$ and $x \in \bigcup_{i=1}^{n} B_i$. By definition of general union, $x \in B_i$ for some $i = 1, 2, ..., n$, and so, since $x \in A$, the definition of intersection implies that $x \in A \cap B_i$ for some $i = 1, 2, ..., n$. Thus, by definition of general union, $x \in \bigcup_{i=1}^{n}(A \cap B_i)$.

(2) Proof that $\boldsymbol{\bigcup_{i=1}^{n}(A \cap B_i) \subseteq A \cap (\bigcup_{i=1}^{n} B_i)}$: Let $x \in \bigcup_{i=1}^{n}(A \cap B_i)$. By definition of general union, $x \in (A \cap B_i)$ for some $i = 1, 2, ..., n$. It follows by definition of intersection that $x \in A$ and $x \in B_i$ for some $i = 1, 2, ..., n$. Since $x \in B_i$ for some $i = 1, 2, ..., n$ it follows by definition of general union that $x \in \bigcup_{i=1}^{n} B_i$. Now we have that $x \in A$ and $x \in \bigcup_{i=1}^{n} B_i$ and so, by definition of intersection, $x \in A \cap \bigcup_{i=1}^{n} B_i$.

Conclusion: Since both set containment's have been proved, it follows by definition of set equality that $A \cap \left(\bigcup_{i=1}^{n} B_i\right) = \bigcup_{i=1}^{n}(A \cap B_i)$. □

Problem 38 and Solution

For all integers $n \geq 1$, if $A_1, A_2, A_3, ...$ and B are any sets, then

$$\bigcup_{i=1}^{n}(A_i - B) = \left(\bigcup_{i=1}^{n} A_i\right) - B$$

Proof. Let $A_1, A_2, A_3, ...$ and B be any sets.

(1) Proof that $\boldsymbol{\bigcup_{i=1}^{n}(A_i - B) \subseteq (\bigcup_{i=1}^{n} A_i) - B}$: Let $x \in \bigcup_{i=1}^{n}(A_i - B)$. By definition of general union, $x \in A_i - B$ for some $i = 1, 2, ...n$. By definition of set difference, $x \in A_i$ for some $i = 1, 2, ..., n$ and $x \notin B$. It now follows from the definition of general union and set difference that $x \in (\bigcup_{i=1}^{n} A_i) - B$.

(2) Proof that $(\bigcup_{i=1}^n A_i) - B \subseteq \bigcup_{i=1}^n (A_i - B)$: Let $x \in (\bigcup_{i=1}^n A_i) - B$. By definition of set difference, $x \in \bigcup_{i=1}^n A_i$ and $x \notin B$. By definition of general union, $x \in A_i$ for some $i = 1, 2, ..., n$. It now follows from the definition of general union that $x \in \bigcup_{i=1}^n (A_i - B)$.

Conclusion: Since both set containment's have been proved, it follows by definition of set equality that $\displaystyle\bigcup_{i=1}^n (A_i - B) = \left(\bigcup_{i=1}^n A_i\right) - B$. □

Problem 39 and Solution

For all integers $n \geq 1$, if $A_1, A_2, A_3, ...$ and B are any sets, then

$$\bigcap_{i=1}^n (A_i - B) = \left(\bigcap_{i=1}^n A_i\right) - B$$

Proof. Let $A_1, A_2, A_3, ...$ and B be any sets.

(1) Proof that $\bigcap_{i=1}^n (A_i - B) \subseteq (\bigcap_{i=1}^n A_i) - B$: Let $x \in \bigcap_{i=1}^n (A_i - B)$. By definition of general intersection, $x \in A_i - B$ for all $i = 1, 2, ...n$. Now by definition of set difference $x \in A_i$ and $x \notin B$. Hence, by definition of general intersection, $x \in (\bigcap_{i=1}^n A_i) - B$.

(2) Proof that $(\bigcap_{i=1}^n A_i) - B \subseteq \bigcap_{i=1}^n (A_i - B)$: Let $x \in (\bigcap_{i=1}^n A_i) - B$. By definition of set difference, $x \in \bigcap_{i=1}^n A_i$ and $x \notin B$. Now by definition of general intersection, $x \in A_i$ for all $i = 1, 2, ..., n$. Hence, by definition of general intersection, $x \in \bigcap_{i=1}^n (A_i - B)$.

Conclusion: Since both set containment's have been proved, it follows by definition of set equality that $\displaystyle\bigcap_{i=1}^n (A_i - B) = \left(\bigcap_{i=1}^n A_i\right) - B$. □

Problem 40 and Solution

For all integers, $n \geq 1$, if A and $B_1, B_2, B_3...$ are any sets, then

$$\bigcup_{i=1}^n (A \times B_i) = A \times \left(\bigcup_{i=1}^n B_i\right)$$

Proof. Let A and $B_1, B_2, B_3, ...$ be any sets.

(1) Proof that $\bigcup_{i=1}^n (A \times B_i) \subseteq A \times (\bigcup_{i=1}^n B_i)$: Let $(x, y) \in \bigcup_{i=1}^n (A \times B_i)$. By definition of general union, $(x, y) \in A \times B_i$ for some $i = 1, 2, ..., n$. By definition of Cartesian product, $x \in A$ and $y \in B_i$ for some $i = 1, 2, ..., n$. It now follows by definition of general union that $y \in \bigcup_{i=1}^n B_i$. Hence, by definition of Cartesian product $(x, y) \in A \times (\bigcup_{i=1}^n B_i)$.

(2) Proof that $A \times (\bigcup_{i=1}^n B_i) \subseteq \bigcup_{i=1}^n (A \times B_i)$: Let $(x, y) \in A \times (\bigcup_{i=1}^n B_i)$. By definition of Cartesian product, $x \in A$ and $y \in \bigcup_{i=1}^n B_i$. By definition of general union, $y \in B_i$ for some $i = 1, 2, ..., n$. It follows by definition of Cartesian product that $(x, y) \in A \times B_i$ for some $i = 1, 2, ..., n$. Hence by definition of general union $(x, y) \in \bigcup_{i=1}^n (A \times B_i)$.

Conclusion: Since both set containment's have been proved, it follows by definition of set equality that $\displaystyle\bigcup_{i=1}^{n}(A \times B_i) = A \times \left(\bigcup_{i=1}^{n} B_i\right)$. $\square$

Problem 41 and Solution

For all integers $n \geq 1$, if A and $B_1, B_2, B_3, ...$ are any sets, then

$$\bigcap_{i=1}^{n}(A \times B_i) = A \times \left(\bigcap_{i=1}^{n} B_i\right)$$

Proof. Let A and $B_1, B_2, B_3, ...$ be any sets.

(1) Proof that $\bigcap_{i=1}^n (A \times B_i) \subseteq A \times (\bigcap_{i=1}^n B_i)$: Let $(x, y) \in \bigcap_{i=1}^n (A \times B_i)$. By definition of general intersection, $(x, y) \in A \times B_i$ for all $i = 1, 2, ...n$. By definition of Cartesian product, $x \in A$ and $y \in B_i$. It follows by definition of general union that $y \in \bigcap_{i=1}^n B_i$. Now by definition of Cartesian product $(x, y) \in A \times (\bigcap_{i=1}^n B_i)$.

(2) Proof that $A \times (\bigcap_{i=1}^n B_i) \subseteq \bigcap_{i=1}^n (A \times B_i)$: Let $(x, y) \in A \times (\bigcap_{i=1}^n B_i)$. By definition of Cartesian product, $x \in A$ and $y \in \bigcap_{i=1}^n B_i$. By definition of general intersection, $y \in B_i$ for all $i = 1, 2, ..., n$. It follows by definition of Cartesian product that $(x, y) \in A \times B_i$ for all $i = 1, 2, ..., n$. Hence by definition of general intersection $(x, y) \in \bigcap_{i=1}^n (A \times B_i)$.

Conclusion: Since both set containment's have been proved, it follows by definition of set equality that $\displaystyle\bigcap_{i=1}^{n}(A \times B_i) = A \times \left(\bigcap_{i=1}^{n} B_i\right)$. $\square$