Section 11.2

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Problem 1

The following is a formal definition for Ω-notation, written using quantifiers and variables: $f(x)$ is $\Omega(g(x)) \iff \exists$ positive real numbers a and A such that $\forall x > a$,

$$A|g(x)| \leq |f(x)|.$$

a. Write the formal negation for the definition using the symbols $\forall$ and $\exists$.

b. Restate the negation less formally without using the symbols $\forall$ and $\exists$.

Solution

a. $f(x)$ is not $\Omega(g(x)) \iff \forall$ positive real numbers a and A, $\exists x > a$, such that
$$A|g(x)| > |f(x)|.$$

b. $f(x)$ is not $\Omega(g(x)) \iff$ no matter what positive real numbers a and A are chosen, it is possible to find a number $x > a$ so that
$$A|g(x)| > |f(x)|.$$

Problem 2

The following is a formal definition for O-notation, written using quantifiers and variables: $f(x)$ is $O(g(x)) \iff \exists$ positive real numbers b and B such that $\forall x > b$,

$$|f(x)| \leq B|g(x)|.$$

a. Write the formal negation for the definition using the symbols $\forall$ and $\exists$.

b. Restate the negation less formally without using the symbols $\forall$ and $\exists$.

Solution

a. $f(x)$ is not $O(g(x)) \iff \forall$ positive real numbers b and B, $\exists x > b$, such that
$$|f(x)| > B|g(x)|.$$

b. $f(x)$ is not $\Omega(g(x)) \iff$ no matter what positive real numbers a and A are chosen, it is possible to find a number $x > a$ so that
$$A|g(x)| > |f(x)|.$$

Problem 3

The following is a formal definition for Θ-notation, written using quantifiers and variables: $f(x)$ is $\Theta(g(x)) \iff \exists$ positive real numbers k, A, and B such that $\forall x > k$,

$$A|g(x)| \leq |f(x)| \leq B|g(x)|.$$

a. Write the formal negation for the definition using the symbols $\forall$ and $\exists$.

b. Restate the negation less formally without using the symbols $\forall$ and $\exists$.

Solution

a. $f(x)$ is not $\Theta(g(x)) \iff \forall$ positive real numbers k, A, and B, $\exists x > k$, such that

$$A|g(x)| > |f(x)| \quad \text{or} \quad |f(x)| > B|g(x)|.$$

b. $f(x)$ is not $\Theta(g(x)) \iff$ no matter what positive real numbers k, A, and B are chosen, it is possible to find a number $x > k$ so that either

$$A|g(x)| > |f(x)| \quad \text{or} \quad |f(x)| > B|g(x)|.$$

In 4-9, express each statement using Ω-, O-, or Θ- notation.

Problem 4

$|5x^8 - 9x^7 + 2x^5 + 3x - 1| \leq 6|x^8|$ for all real numbers $x > 3$. (Use O-notation.)

Solution

Let $B = 6$ and $b = 3$. The given statement translates to

$$|5x^8 - 9x^7 + 2x^5 + 3x - 1| \leq B|x^8| \quad \text{for all real numbers } x > b.$$

So by definition of O-notation, $x^8 - 9x^7 + 2x^5 + 3x - 1$ is $O(x^8)$.

Problem 5

$|x| \leq \left|\dfrac{(x^2-1)(12x+25)}{3x^2+4}\right| \leq 6|x|$ for all real numbers $x > 2$. (Use Θ-notation.)

Solution

Let $A = 1$, $B = 6$, and $k = 2$. The given statement translates to

$$A|x| \leq \left|\frac{(x^2-1)(12x+25)}{3x^2+4}\right| \leq B|x| \quad \text{for all real numbers } x > k.$$

So by definition of Θ-notation, $\dfrac{(x^2-1)(12x+25)}{3x^2+4}$ is $\Theta(x)$.

Problem 6

$|x^{7/2}| \le \left|\dfrac{(x^2-7)^2(10x^{1/2}+3)}{x+1}\right|$ for all real numbers $x > 4$. (Use Ω-notation.)

Solution

Let $A = 1$ and $a = 4$. Then the given statement translates to

$$A|x^{7/2}| \le \left|\frac{(x^2-7)^2(10x^{1/2}+3)}{x+1}\right| \quad \text{for all real numbers } x > a$$

So by definition of Ω-notation, $\dfrac{(x^2-7)^2(10x^{1/2}+3)}{x+1}$ is $\Omega(x^{7/2})$.

Problem 7

$|3x^6 + 5x^4 - x^3| \le 9|x^6|$ for all real numbers $x > 1$. (Use O-notation.)

Solution

Let $B = 9$ and $b = 1$. Then the given statement translates to

$$|3x^6 + 5x^4 - x^3| \le B|x^6| \quad \text{for all real numbers } x > b.$$

So by definition of O-notation, $3x^6 + 5x^4 - x^3$ is $O(x^6)$.

Problem 8

$\frac{1}{2}x^4 \le |x^4 - 50x^3 + 1|$ for all real numbers $x > 101$. (Use Ω-notation.)

Solution

First note that $\frac{1}{2}x^4 = \frac{1}{2}|x^4|$. Now let $A = \frac{1}{2}$ and $a = 101$. Then the given statement translates to

$$A|x^4| \le |x^4 - 50x^3 + 1| \quad \text{for all real numbers } x > a.$$

So by definition of Ω-notation, $x^4 - 50x^3 + 1$ is $\Omega(x^4)$.

Problem 9

$\frac{1}{2}x^2 \le |3x^2 - 80x + 7| \le 3|x^2|$ for all real numbers $x > 33$.

Solution

First note that $\frac{1}{2}x^2 = \frac{1}{2}|x^2|$. Now let $A = \frac{1}{2}$, $B = 3$, and $k = 33$. Then the given statement translates to

$$A|x^2| \le |3x^2 - 80x + 7| \le B|x^2| \quad \text{for all real numbers } x > k.$$

So by definition of Θ-notation, $3x^2 - 80x + 7$ is $\Theta(x^2)$.

In each of 10-14 assume f and g are real-valued functions defined on the same set of nonnegative real numbers.

Problem 10 and Solution

Prove that if $g(x)$ is $O(f(x))$, then $f(x)$ is $\Omega(g(x))$.

Proof. Suppose that $g(x)$ is $O(f(x))$. By definition of O-notation, there exists a positive real number B and a nonnegative real number b such that

$$|g(x)| \leq B|f(x)| \quad \text{for all real numbers } x > b.$$

Divide both sides by B to obtain

$$\frac{1}{B}|g(x)| \leq |f(x)| \quad \text{for all real numbers } x > b.$$

Now let $A = 1/B$ and $a = b$. Then A is a positive real number and a is a nonnegative real number a such that

$$A|g(x)| \leq |f(x)| \quad \text{for all real numbers } x > a.$$

It follows by the definition of Ω-notation, that $f(x)$ is $\Omega(g(x))$. $\square$

Problem 11 and Solution

Prove that if $f(x)$ is $O(g(x))$ and c is any nonzero real number, then $cf(x)$ if $O(g(x))$.

Proof. Suppose that $f(x)$ is $O(g(x))$. By definition of O-notation, there exists a positive real number B and a nonnegative real number b such that

$$|f(x)| \leq B|g(x)| \quad \text{for all real numbers } x > b.$$

Multiply both sides by $|c|$ to obtain

$$|c| \cdot |f(x)| \leq |c| \cdot B|g(x)| \quad \text{for all real numbers } x > b.$$

It follows from exercise 4.4.44 that $|c| \cdot |f(x)| = |cf(x)|$. It follows from the associative property of the real numbers that $|c| \cdot B|g(x)| = (|c| \cdot B)|g(x)|$. Also since $c \neq 0$ and $B > 0$, $|c| \cdot B > 0$. Finally we have that

$$|cf(x)| \leq (|c| \cdot B)|g(x)| \quad \text{for all real numbers } x > b.$$

It follows by the definition of O-notation, that $cf(x)$ is $O(g(x))$. $\square$

Problem 12 and Solution

Prove that if $f(x)$ is $O(h(x))$ and $g(x)$ is $O(k(x))$, then $f(x)+g(x)$ is $O(G(x))$, where, for each x in the domain, $G(x) = \max(|h(x)|, |k(x)|)$.

Proof. Suppose that f, g, h, and k are real valued functions defined on the same set D of nonnegative real numbers and suppose that $f(x)$ is $O(h(x))$ and $g(x)$ is $O(k(x))$. By definition of O-notation, there exist positive real numbers B_1 and B_2 and nonnegative real numbers b_1 and b_2 such that

$$|f(x)| \leq B_1|h(x)| \quad \text{for all real numbers } x > b_1$$

and

$$|g(x)| \leq B_2|k(x)| \quad \text{for all real numbers } x > b_2.$$

Now define a function G such that for each $x \in D$, $G(x) = \max(|h(x)|, |k(x)|)$. Also define real numbers $B = B_1 + B_2$ and $b = \max(b_1, b_2)$. Note that by the triangle inequality for absolute value (theorem 4.4.6),

$$|f(x)| + |g(x)| \leq |f(x) + g(x)|$$

for all real numbers $x \in D$. Suppose that $x > b$. Then because $b = \max(b_1, b_2)$,

$$|f(x)| \leq B_1|h(x)| \quad \text{and} \quad |g(x)| \leq B_2|k(x)|$$

Adding the inequalities gives

$$|f(x)| + |g(x)| \leq B_1|h(x)| + B_2|k(x)|$$

Thus, by the transitive law for inequalities,

$$|f(x) + g(x)| \leq B_1|h(x)| + B_2|k(x)|.$$

Now since $G(x) = |G(x)| = \max(|h(x)|, |k(x)|)$,

$$B_1|h(x)| + B_2|k(x)| \leq B_1|G(x)| + B_2|G(x)| = (B_1 + B_2)|G(x)|.$$

Finally by transitive law for inequalities and because $B = B_1 + B_2$,

$$|f(x) + g(x)| \leq B|G(x)| \quad \text{for all real numbers } x > b.$$

Hence, by definition of O-notation, $f(x) + g(x)$ is $O(G(x))$. $\square$

Problem 14 and Solution

Prove that $f(x)$ is $\Theta(f(x))$.

Proof. Let f be a real-valued function defined on a set of nonnegative real numbers D. Also, suppose that $0 < A \leq 1$, $1 \leq B$, and let $k \geq \inf\{D\}$. Then,

$$A|f(x)| \leq |f(x)| \leq B|f(x)| \quad \text{for all real numbers } x > k.$$

Hence, by definition of Θ-notation, $f(x)$ is $\Theta(f(x))$. $\square$

Problem 14 and Solution

Prove that if $f(x)$ is $O(h(x))$ and $g(x)$ is $O(k(x))$, then $f(x)g(x)$ is $O(h(x)k(x))$.

Proof. Suppose that f, g, h, and k are real valued functions defined on the same set D of nonnegative real numbers and suppose that $f(x)$ is $O(h(x))$ and $g(x)$ is $O(k(x))$. By definition of O-notation, there exist positive real numbers B_1 and B_2 and nonnegative real numbers b_1 and b_2 such that

$$|f(x)| \leq B_1|h(x)| \quad \text{for all real numbers } x > b_1$$

and

$$|g(x)| \leq B_2|k(x)| \quad \text{for all real numbers } x > b_2.$$

Now define real numbers $B = B_1B_2$ and $b = \max(b_1, b_2)$. Note that by exercise 4.4.44,

$$|f(x)| \cdot |g(x)| = |f(x)g(x)| \quad \text{and} \quad |h(x)| \cdot |k(x)| = |h(x)k(x)|$$

for all real numbers $x \in D$. Suppose that $x > b$. Then because $b = \max(b_1, b_2)$,

$$|f(x)| \leq B_1|h(x)| \quad \text{and} \quad |g(x)| \leq B_2|k(x)|$$

Multiplying the inequalities gives

$$|f(x)| \cdot |g(x)| \leq B_1|h(x)| \cdot B_2|k(x)| = B_1B_2|h(x)| \cdot |k(x)|$$

Now, by the transitive law for equality, and since $B = B_1B_2$,

$$|f(x)g(x)| \leq B|h(x)k(x)| \quad \text{for all real numbers } x > b.$$

Hence, by definition of O-notation, $f(x)g(x)$ is $O(h(x)k(x))$. □

Problem 15

a. Use mathematical induction to prove that if x is any real number with $x > 1$, then $x^n > 1$ for all integers $n \geq 1$.

b. Prove that if x is any real number with $x > 1$, then $x^m < x^n$ for any integers m and n with $m < n$.

Solution

a. *Proof.* Let x be any real number such that $x > 1$ and suppose that for all integers $n \geq 1$,

$$x^n > 1 \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: Let $n = 1$. Then $x^n = x^1 = x > 1$. Hence $P(0)$ is true.

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$x^k > 1 \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$x^{k+1} > 1 \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} x^{k+1} &= x \cdot x^k \\ &> x \cdot 1 && \text{by inductive hypothesis} \\ &= x \\ &> 1 && \text{by definition of } x \end{aligned}$$

which is the right-hand side of $P(k+1)$. □

b. *Proof.* Let x be any real number such that $x > 1$ and suppose that m and n are any integers such that $m < n$. It follows that $n - m > 0$. However since m and n are integers $n-m$ must also be an integer and so $n-m \geq 1$. Now

$$\begin{aligned} x^{n-m} &> 1 && \text{by part (a)} \\ \frac{x^n}{x^m} &> 1 && \text{by the laws of exponents} \\ x^n &> x^m && \text{multiply both sides by } x^m \end{aligned}$$

□

Problem 16

a. Show that for any real number x, if $x > 1$ then $|x^2| \leq |2x^2 + 15x + 4|$.

b. Show that for any real number x, if $x > 1$ then $|2x^2 + 15x + 4| \leq 21|x^2|$.

c. Use the Ω- and O-notations to express the results of parts (a) and (b).

d. What can you deduce about the order of $2x^2 + 15x + 4$?

Solution

a. For any real number $x > 1$, $0 \leq x^2 + 15x + 4$ because all terms are nonnegative. Adding x^2 to both sides gives $x^2 \leq 2x^2 + 15x + 4$. Because both sides are nonnegative, absolute value signs may be added to both sides of the inequality to give $|x^2| \leq |2x^2 + 15x + 4|$.

b. For all real numbers $x > 1$,

$$\begin{aligned} |2x^2 + 15x + 4| &= 2x^2 + 15x + 4 && \text{all terms are positive} \\ &< 2x^2 + 15x^2 + 4x^2 && \text{because by (11.2.1), } x < x^2 \text{ and } 1 < x^2, \text{ and so } 15x < 15x^2 \text{ and } 4 < 4x^2 \\ &= 21x^2 \\ &= 21|x^2| && x > 1 \text{ and so } x^2 \text{ is positive} \end{aligned}$$

c. Let $A = 1$ and $a = 1$. Then the results of part (a) translates to
$$A|x^2| \leq |2x^2 + 15x + 4| \quad \text{for all real numbers } x > a.$$

Hence, by definition of Ω-notation, $2x^2 + 15x + 4$ is $\Omega(x^2)$.

Let $B = 21$ and $b = 1$. Then the results of part (b) translates to
$$|2x^2 + 15x + 4| \leq 21|x^2| \quad \text{for all integers } x > b.$$

Hence, by definition of O-notation, $2x^2 + 15x + 4$ is $O(x^2)$.

d. It follows by theorem 11.2.1 part 1 that since $2x^2 + 15x + 4$ is $\Omega(x^2)$ and $O(x^2)$, $2x^2 + 15x + 4$ is $\Theta(x^2)$.

Problem 17

a. Show that for any real number x, if $x > 1$ then $|x^4| \leq |23x^4 + 8x^2 + 4x|$.

b. Show that for any real number x, if $x > 1$ then $|23x^4 + 8x^2 + 4x| \leq 35|x^4|$.

c. Use the Ω- and O-notations to express the results of parts (a) and (b).

d. What can you deduce about the order of $23x^4 + 8x^2 + 4x$?

Solution

a. For any real number $x > 1$, $0 \leq 22x^4 + 8x^2 + 4x$ because all terms are nonnegative. Adding x^4 to both sides gives $x^4 \leq 22x^4 + 8x^2 + 4x$. Because both sides are nonnegative, absolute value signs may be added to both sides of the inequality to give $|x^4| \leq |23x^4 + 8x^2 + 4x|$.

b. For all real numbers $x > 1$,

$$\begin{aligned}
|23x^4 + 8x^2 + 4x| &= 23x^4 + 8x^2 + 4x && \text{all terms are positive} \\
&< 23x^4 + 8x^4 + 4x^4 && \text{because by (11.2.1), } x^2 < x^4 \text{ and } x < x^4, \\
& && \text{and so } 8x^2 < 8x^4 \text{ and } 4x < 4x^4 \\
&= 35x^4 \\
&= 35|x^4| && x > 1 \text{ and so } x^4 \text{ is positive}
\end{aligned}$$

c. Let $A = 1$ and $a = 1$. Then the results of part (a) translate to
$$A|x^4| \leq |23x^4 + 8x^2 + 4x| \quad \text{for all real numbers } x > a.$$

Hence, by definition of Ω-notation, $23x^4 + 8x^2 + 4x$ is $\Omega(x^4)$.

Let $B = 35$ and $b = 1$. Then the results of part (b) translate to
$$|23x^4 + 8x^2 + 4x| \leq B|x^4| \quad \text{for all real numbers } x > b.$$

Hence, by definition of O-notation, $23x^4 + 8x^2 + 4x$ is $O(x^4)$.

d. It follows by theorem 11.2.1 part 1 that since $23x^4 + 8x^2 + 4x$ is $\Omega(x^4)$ and $O(x^4)$, $23x^4 + 8x^2 + 4x$ is $\Theta(x^4)$.

Problem 18 and Solution

Use the definition of Θ-notation to show that $5x^3 + 65x + 30$ is $\Theta(x^3)$.

Proof. First note that for any real number $x > 1$, $0 \le 4x^3 + 65x + 30$ because all terms are nonnegative. Adding x^3 to both sides gives $x^3 \le 5x^3 + 65x + 30$. Because both sides are nonnegative, absolute value signs may be added to both sides of the inequality to give $|x^3| \le |5x^3 + 65x + 30|$. Now let $A = 1$ and $a = 1$ and it follows that

$$A|x^3| \le |5x^3 + 65x + 30| \quad \text{for all real numbers } x > a.$$

Hence, by definition of Ω-notation, $5x^3 + 65x + 30$ is $\Omega(x^3)$.

Next note that for any real number $x > 1$,

$$\begin{aligned}
|5x^3 + 65x + 30| &= 5x^3 + 65x + 30 && \text{all terms are positive} \\
&< 5x^3 + 65x^3 + 30x^3 && \text{because by (11.2.1), } x < x^3 \text{ and } 1 < x^3, \\
& && \text{and so } 65x < 65x^3 \text{ and } 30 < 30x^3 \\
&= 100x^3 \\
&= 100|x^3| && x > 1 \text{ and so } x^3 \text{ is positive}
\end{aligned}$$

Now let $B = 100$ and $b = 1$ and it follows that

$$|5x^3 + 65x + 30| \le B|x^3| \quad \text{for all real numbers } x > b.$$

Hence, by definition of O-notation, $5x^3+65x+30$ is $O(x^3)$. It follows by theorem 11.2.1 part 1 that since $5x^3 + 65x + 30$ is $\Omega(x^3)$ and $O(x^3)$, $5x^3 + 65x + 30$ is $\Theta(x^3)$. □

Problem 19 and Solution

Use the definition of Θ-notation to show that $x^2 + 100x + 88$ is $\Theta(x^2)$.

Proof. First note that for any real number $x > 1$, $0 \le 100x + 88$ because all terms are nonnegative. Adding x^2 to both sides gives $x^2 \le x^2 + 100x + 88$. Because both sides are nonnegative, absolute value signs may be added to both sides of the inequality to give $|x^2| \le |x^2 + 100x + 88|$. Now let $A = 1$ and $a = 1$ and it follows that

$$A|x^2| \le |x^2 + 100x + 88| \quad \text{for all real numbers } x > a.$$

Hence, by definition of Ω-notation, $x^2 + 100x + 88$ is $\Omega(x^2)$.

Next note that for any real number $x > 1$,

$$\begin{aligned}
|x^2 + 100x + 88| &= x^2 + 100x + 88 && \text{all terms are positive} \\
&< x^2 + 100x^2 + 88x^2 && \text{because by (11.2.1), } x < x^2 \text{ and } 1 < x^2, \\
& && \text{and so } 100x < 100x^2 \text{ and } 88 < 88x^2 \\
&= 189x^2 \\
&= 189|x^2| && x > 1 \text{ and so } x^2 \text{ is positive}
\end{aligned}$$

Now let $B = 189$ and $b = 1$ and it follows that
$$|x^2 + 100x + 88| \le B|x^2| \quad \text{for all real numbers } x > b.$$

Hence, by definition of O-notation, $x^2+100x+88$ is $O(x^2)$. It follows by theorem 11.2.1 part 1 that since $x^2 + 100x + 88$ is $\Omega(x^2)$ and $O(x^2)$, $x^2 + 100x + 88$ is $\Theta(x^2)$. □

Problem 20

a. Show that for any real number x, if $x > 1$ then $|x^2| \le |\lceil x^2 \rceil|$.

b. Show that for any real number x, if $x > 1$ then $|\lceil x^2 \rceil| \le 2|x^2|$.

c. Use the Ω- and O-notations to express the results of parts (a) and (b).

d. What can you deduce about the order of $\lceil x^2 \rceil$?

Solution

a. By definition of ceiling, for any real number x, $\lceil x^2 \rceil$ is that integer n such that $n - 1 < x^2 \le n$. Hence, by substitution, $x^2 \le \lceil x^2 \rceil$. Since $x > 1$, both sides of the inequality are positive, and so $|x^2| \le |\lceil x^2 \rceil|$.

b. By definition of ceiling, for any real number x, $\lceil x^2 \rceil$ is that integer n such that $n - 1 < x^2 \le n$. Adding 1 to all parts of this inequality gives $n < x^2 + 1 \le n + 1$. It follows that $\lceil x^2 \rceil < x^2 + 1$. Thus if x is any real number with $x > 1$, then

$$\begin{aligned}
|\lceil x^2 \rceil| &= \lceil x^2 \rceil && \text{$\lceil x^2 \rceil$ is positive} \\
&< x^2 + 1 && \text{by the argument above} \\
&< x^2 + x^2 && \text{by (11.2.1), $1 < x^2$} \\
&= 2x^2 \\
&= 2|x^2| && \text{because x^2 is positive}
\end{aligned}$$

c. Let $A = 1$ and $a = 1$. Then the results of part (a) translate to
$$A|x^2| \le |\lceil x^2 \rceil| \quad \text{for all real numbers } x > a.$$

Hence, by definition of Ω-notation, $\lceil x^2 \rceil$ is $\Omega(x^2)$.

Let $B = 2$ and $b = 1$. Then the results of part (b) translate to
$$\lceil x^2 \rceil \le B|x^2| \quad \text{for all real numbers } x > b.$$

Hence, by definition of O-notation, $\lceil x^2 \rceil$ is $O(x^2)$.

d. It follows by theorem 11.2.1 part 1 that since $\lceil x^2 \rceil$ is $\Omega(x^2)$ and $O(x^2)$, $\lceil x^2 \rceil$ is $\Theta(x^2)$.

Problem 21

a. Show that for any real number x, if $x > 1$ then $|\lfloor\sqrt{x}\rfloor| \leq |\sqrt{x}|$.

b. Show that for any real number x, if $x > 1$ then $|\sqrt{x}| \leq 2|\lfloor\sqrt{x}\rfloor|$.

c. Use the Ω- and O-notations to express the results of parts (a) and (b).

d. What can you deduce about the order of $\lfloor\sqrt{x}\rfloor$?

Solution

a. By definition of floor, for any real number x, $\lfloor\sqrt{x}\rfloor$ is that integer n such that $n \leq \sqrt{x} < n + 1$. Hence, by substitution, $\lfloor\sqrt{x}\rfloor \leq \sqrt{x}$. Since $x > 1$, both sides of the inequality are positive, and so $|\lfloor\sqrt{x}\rfloor| \leq |\sqrt{x}|$.

b. By definition of floor, for any real number x, $\lfloor\sqrt{x}\rfloor$ is that integer n such that $n \leq \sqrt{x} < n + 1$. It follows that $\sqrt{x} \leq \lfloor\sqrt{x}\rfloor + 1$. Thus if x is any real number with $x > 1$, then

$$
\begin{aligned}
|\sqrt{x}| &= \sqrt{x} && \lfloor\sqrt{x}\rfloor \text{ is positive} \\
&\leq \lfloor\sqrt{x}\rfloor + 1 && \text{by the argument above} \\
&\leq \lfloor\sqrt{x}\rfloor + \lfloor\sqrt{x}\rfloor && \text{by (11.2.1), } 1 < \sqrt{x} \text{ and so } 1 \leq \lfloor\sqrt{x}\rfloor \\
&= 2\lfloor\sqrt{x}\rfloor && \\
&= 2|\lfloor\sqrt{x}\rfloor| && \lfloor\sqrt{x}\rfloor \text{ is positive}
\end{aligned}
$$

c. Let $B = 1$ and $b = 1$. Then the results are part (a) translate to

$$|\lfloor\sqrt{x}\rfloor| \leq B|\sqrt{x}| \quad \text{for all integers } x > b.$$

Hence, by definition of O-notation, $\lfloor\sqrt{x}\rfloor$ is $O(\sqrt{x})$.

Let $A = \frac{1}{2}$ and $a = 1$. Then the results of part (b) translate to

$$A|\sqrt{x}| \leq |\lfloor\sqrt{x}\rfloor| \quad \text{for all integers } x > b.$$

Hence, by definition of Ω-notation, $\lfloor\sqrt{x}\rfloor$ is $\Omega(\sqrt{x})$.

d. It follows by theorem 11.2.1 part 1 that since $\lfloor\sqrt{x}\rfloor$ is $\Omega(\sqrt{x})$ and $O(\sqrt{x})$, $\lfloor\sqrt{x}\rfloor$ is $\Theta(\sqrt{x})$.

Problem 22

a. Show that for any real number x, if $x > 1$ then $|7x^4 - 95x^3 + 3| \leq 105|x^4|$.

b. Use O-notation to express the result of part (a).

Solution

a. *Proof.* For all real numbers $x > 1$,

$$\begin{aligned}
|7x^4 - 95x^3 + 3| &\le |7x^4| + |95x^3| + |3| && \text{by the triangle inequality} \\
&= 7x^4 + 95x^3 + 3 && \text{all terms are positive} \\
&< 7x^4 + 95x^4 + 3x^4 && \text{because by (11.2.1), } x^3 < x^4 \text{ and } 1 < x^4, \\
& && \text{and so } 95x^3 < 95x^4 \text{ and } 3 < 3x^4 \\
&= 105x^4 \\
&= 105|x^4| && x > 1 \text{ and so } x^4 \text{ is positive} \qquad \square
\end{aligned}$$

b. Now let $B = 105$ and $b = 1$ and it follows that

$$|7x^4 - 95x^3 + 3| \le 105|x^4| \quad \text{for all real numbers } x > b.$$

Hence, by definition of O-notation, $7x^4 - 95x^3 + 3$ is $O(x^4)$.

Problem 23

a. Show that for any real number x, if $x > 1$ then $|\frac{1}{5}x^2 - 42x - 8| \le 51|x^2|$.

b. Use O-notation to express the result of part (a).

Solution

a. *Proof.* For all real numbers $x > 1$,

$$\begin{aligned}
|\frac{1}{5}x^2 - 42x - 8| &\le |\frac{1}{5}x^2| + |42x| + |8| && \text{by the triangle inequality} \\
&= \frac{1}{5}x^2 + 42x + 8 && \text{all terms are positive} \\
&< \frac{1}{5}x^2 + 42x^2 + 8x^2 && \text{because by (11.2.1), } x < x^2 \text{ and } 1 < x^2, \\
& && \text{and so } 42x < 42x^2 \text{ and } 8 < 8x^2 \\
&< x^2 + 42x^2 + 8x^2 && \frac{1}{5} < 1. \\
&= 51x^2 \\
&= 51|x^2| && x > 1 \text{ and so } x^2 \text{ is positive} \qquad \square
\end{aligned}$$

b. Now let $B = 51$ and $b = 1$ and it follows that

$$|\frac{1}{5}x^2 - 42x - 8| \le 51|x^2| \quad \text{for all real numbers } x > b.$$

Hence, by definition of O-notation, $\frac{1}{5}x^2 - 42x - 8$ is $O(x^2)$.

Problem 24

a. Show that for any real number x, if $x > 1$ then $|\frac{1}{4}x^5 - 50x^3 + 3x + 12| \le 66|x^5|$.

b. Use O-notation to express the result of part (a).

Solution

a. *Proof.* For all real numbers $x > 1$,

$$\begin{aligned}
&|\frac{1}{4}x^5 - 50x^3 + 3x + 12| \\
&\quad \le |\frac{1}{4}x^5| + |50x^3| + |3x| + |12| && \text{by the triangle inequality} \\
&\quad = \frac{1}{4}x^5 + 50x^3 + 3x + 12 && \text{all terms are positive} \\
&\quad < \frac{1}{4}x^5 + 50x^5 + 3x^5 + 12x^5 && \text{by (11.2.1), } x^3 < x^5,\ x < x^5, \text{ and } 1 < x^5 \text{ and so } 50x^3 < 50x^5,\ 3x < 3x^5, \text{ and } 12 < 12x^5 \\
&\quad < x^5 + 50x^5 + 3x^5 + 12x^5 && \frac{1}{4} < 1. \\
&\quad = 66x^2 \\
&\quad = 66|x^5| && x > 1 \text{ and so } x^5 \text{ is positive}
\end{aligned}$$

□

b. Now let $B = 66$ and $b = 1$ and it follows that

$$|\frac{1}{4}x^5 - 50x^3 + 3x + 12| \le 66|x^5| \quad \text{for all real numbers } x > b.$$

Hence, by definition of O-notation, $\frac{1}{4}x^5 - 50x^3 + 3x + 12$ is $O(x^5)$.

Problem 25

Show that x^5 is not $O(x^2)$.

Solution

Proof. Suppose that x^5 is $O(x^2)$. Then, by definition of O-notation, there exists a positive real number B and a nonnegative real number b such that

$$|x^5| \le B|x^2| \quad \text{for all real numbers } x > b.$$

Now let x be a real number such that $x > b$, and $x > B^{1/3}$. Then

$$\begin{aligned}
|x^5| &= x^5 && x > 0 \text{ and so } x^5 > 0. \\
&= x^3 \cdot x^2 \\
&> B \cdot x^2 && x > B^{1/3} \text{ and so } x^3 > B \\
&= B|x^2| && x > 0 \text{ and so } x^2 > 0
\end{aligned}$$

But now we have that $|x^5| \le B|x^2|$ and $|x^5| > B|x^2|$ which is a contradiction. Hence the supposition that x^5 is $O(x^2)$ is false. □

Problem 26

Suppose that $a_0, a_1, a_2, ..., a_n$ are real numbers and $a_n \neq 0$. Use the generalization of the triangle inequality to n integers (exercise 43, section 5.5) to show that

$$a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0 \text{ is } O(x^n).$$

Solution

Proof. Suppose that $a_0, a_1, a_2, ..., a_n$ are real numbers and $a_n \neq 0$. Then,

$$
\begin{aligned}
|a_n x^n + & a_{n-1}x^{n-1} + ... + a_1 x + a_0| \\
&\leq |a_n x^n| + |a_{n-1}x^{n-1}| + ... + |a_1 x| + |a_0| && \text{by the generalized triangle inequality} \\
&= |a_n| \cdot |x^n| + |a_{n-1}| \cdot |x^{n-1}| + ... + |a_1| \cdot |x| + |a_0| && \text{by exercise 4.4.44} \\
&< |a_n| \cdot |x^n| + |a_{n-1}| \cdot |x^n| + ... + |a_1| \cdot |x^n| + |a_0| \cdot |x^n| && \text{by theorem 11.2.1} \\
&= |x^n| \cdot (|a_n| + |a_{n-1}| + ... + |a_1| + |a_0|)
\end{aligned}
$$

Now let $B = (|a_n| + |a_{n-1}| + ... + |a_1| + |a_0|)$ and $b = 1$ and it follows that for all real numbers $x > b$,

$$|a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0| \leq B \cdot |x^n|$$

Hence, by definition of O-notation,

$$a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 \text{ is } O(x^n).$$

□

Problem 27

Suppose $a_0, a_1, a_2, ..., a_n$ are any real numbers and that $a_n > 0$. Show that $a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ is $\Omega(x^n)$ by letting

$$d = 2\left(\frac{|a_0| + |a_1| + |a_2| + ... + |a_{n-1}|}{|a_n|}\right)$$

and letting $a = \max(d, 1)$.

Solution

Proof. Since $x > a$, it follows that

$$
\begin{aligned}
x &> 2\left(\frac{|a_0| + |a_1| + |a_2| + ... + |a_{n-1}|}{|a_n|}\right) \\
x &> 2\frac{|a_0|}{|a_n|} + 2\frac{|a_1|}{|a_n|} + 2\frac{|a_2|}{|a_n|} + ... + 2\frac{|a_{n-1}|}{|a_n|} \\
x &> 2\frac{|a_0|}{|a_n|}\frac{1}{x^{n-1}} + 2\frac{|a_1|}{|a_n|}\frac{1}{x^{n-2}} + 2\frac{|a_2|}{|a_n|}\frac{1}{x^{n-3}} + ... + 2\frac{|a_{n-1}|}{|a_n|} && x > 1 \implies 1/x^i < 1 \text{ for all } i = 1, 2, ... n-1 \\
\frac{|a_n|x^n}{2} &> |a_0| + |a_1|x + |a_2|x^2 + ... + |a_{n-1}|x^{n-1} && \text{multiply both sides by } \frac{|a_n|x^n}{2} \\
\frac{a_n x^n}{2} &> |a_0| + |a_1|x + |a_2|x^2 + ... + |a_{n-1}|x^{n-1} && a_n > 0 \text{ and so } |a_n| = a_n \\
\frac{a_n x^n}{2} &> -a_0 - a_1 x - a_2 x^2 - ... - a_{n-1}x^{n-1} && |a_i| \geq -a_i \text{ for all } i = 1, 2, ..., n-1 \\
a_n x^n - \frac{a_n x^n}{2} &> -a_0 - a_1 x - a_2 x^2 - ... - a_{n-1}x^{n-1}
\end{aligned}
$$

Now by adding $\dfrac{a_n x^n}{2}$ and $a_0 + a_1 x + a_2 x + ... + a_{n-1}x^{n-1}$ to both sides we obtain

$$a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0 > \frac{a_n x^n}{2}$$

Now since $a_n > 0$ and $x^n > 0$ it follows that $\dfrac{a_n x^n}{2} > 0$. Since this is the case it must also be true that

$$a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0 > 0$$

Finally let $A = \dfrac{a_n}{2}$ and it follows that

$$|a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0| \geq A|x^n| \quad \text{for all real } x > a$$

Hence, by definition of Ω-notation,

$$a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 \text{ is } \Omega(x^n). \qquad \square$$

In 28-30: (a) Let d be the number obtained by adding up the absolute values of the coefficients of the lower-order terms of the given polynomial, dividing by the absolute value of the highest-order term, and multiplying the result by 2. Let a be the maximum number of d and 1, and let A be half the coefficient of the absolute value of the highest-order term of the polynomial. (b) Show that if $x > a$, then the absolute value of the polynomial will be greater than the product of A and the absolute value of x^4, where n is the degree of the polynomial. (c) Deduce the result given in the exercise.

Problem 28 and Solution

Use the definition of Ω-notation to show that $7x^4 - 95x^3 + 3$ is $\Omega(x^4)$.

Proof. Let $a = 2\left(\dfrac{95+3}{7}\right) = 28$ and let $A = \dfrac{7}{2}$. If $x > a$ then

$$\begin{aligned}
x &> 2\left(\frac{95+3}{7}\right) \\
x &> 2\cdot\frac{95}{7} + 2\cdot\frac{3}{7} \\
x &> 2\cdot\frac{95}{7} + 2\cdot\frac{3}{7}\cdot\frac{1}{x^3} && x > 28 \implies 1/x^3 < 1 \\
\frac{7}{2}\cdot x^4 &> 95x^3 + 3 && \text{multiply both sides by } \frac{7x^3}{2} \\
\left(7 - \frac{7}{2}\right)x^4 &> 95x^3 - 3 && \frac{7}{2} = 7 - \frac{7}{2} \text{ and } -3 < 3 \\
7x^4 - \frac{7}{2}x^4 &> 95x^3 - 3 \\
7x^4 - 95x^3 + 3 &> \frac{7}{2}x^4 && \text{add } \tfrac{7}{2}x^4 \text{ to both sides and subtract } (95x^3 - 3) \text{ from both sides}
\end{aligned}$$

$$
\begin{aligned}
7x^4 - 95x^3 + 3 &> Ax^4 && A = \frac{7}{2} \\
|7x^4 - 95x^3 + 3| &> A|x^4| && 7x^4 - 95x^3 + 3 > Ax^4 > 0 \text{ and so } |7x^4 - 95x^3 + 3| = 7x^4 - 95x^3 + 3
\end{aligned}
$$

Hence, by definition of Ω-notation, $7x^4 - 95x^3 + 3$ is $\Omega(x^4)$. $\square$

Problem 29 and Solution

Use the definition of Ω-notation to show that $\frac{1}{5}x^2 - 42x - 8$ is $\Omega(x^2)$.

Proof. Let $a = 2\left(\dfrac{42+8}{1/5}\right) = 500$ and let $A = \frac{1/5}{2}$. If $x > a$ then

$$
\begin{aligned}
x &> 2\left(\frac{42+8}{1/5}\right) \\
x &> 2 \cdot \frac{42}{1/5} + 2 \cdot \frac{8}{1/5} \\
x &> 2 \cdot \frac{42}{1/5} + 2 \cdot \frac{8}{1/5} \cdot \frac{1}{x} && x > 500 \implies \frac{1}{x} < 1 \\
\frac{1/5}{2} \cdot x^2 &> 42x + 8 && \text{multiply both sides by } \frac{1/5x}{2} \\
\left(\frac{1}{5} - \frac{1/5}{2}\right) x^2 &> 42x + 8 && \frac{1}{5} - \frac{1/5}{2} = \frac{1/5}{2} \\
\frac{1}{5}x^2 - \frac{1/5}{2}x^2 &> 42x + 8 \\
\frac{1}{5}x^2 - 42x - 8 &> \frac{1/5}{2}x^2 \\
\frac{1}{5}x^2 - 42x - 8 &> Ax^2 && A = \frac{1/5}{2} \\
|\frac{1}{5}x^2 - 42x - 8| &> A|x^2| && \tfrac{1}{5}x^2 - 42x - 8 > Ax^2 > 0 \text{ and so } |\tfrac{1}{5}x^2 - 42x - 8| = \tfrac{1}{5}x^2 - 42x - 8
\end{aligned}
$$

Hence, by definition of Ω-notation, $\frac{1}{5}x^2 - 42x - 8$ is $\Omega(x^2)$. $\square$

Problem 30

Use the definition of Ω-notation to show that $\frac{1}{4}x^5 - 50x^3 + 3x + 12$ is $\Omega(x^5)$.

Proof. Let $a = 2\left(\dfrac{50+3+12}{1/4}\right) = 520$ and let $A = \frac{1/4}{2}$. If $x > a$ then

$$
\begin{aligned}
x &> 2\left(\frac{50+3+12}{1/4}\right) \\
x &> 2 \cdot \frac{50}{1/4} + 2 \cdot \frac{3}{1/4} + 2 \cdot \frac{12}{1/4} \\
x &> 2 \cdot \frac{50}{1/4} \cdot \frac{1}{x} + 2 \cdot \frac{3}{1/4} \cdot \frac{1}{x^3} + 2 \cdot \frac{12}{1/4} \cdot \frac{1}{x^4} && x > 520 \implies \frac{1}{x^4} < \frac{1}{x^3} < \frac{1}{x} < 1
\end{aligned}
$$

$$\frac{1/4}{2} \cdot x^5 > 50x^3 + 3x + 12 \quad \text{multiply both sides by } \frac{1/4x^4}{2}$$

$$\left(\frac{1}{4} - \frac{1/4}{2}\right) x^5 > 50x^3 + 3x + 12 \quad \frac{1}{4} - \frac{1/4}{2} = \frac{1/4}{2}$$

$$\frac{1}{4}x^5 - \frac{1/4}{2}x^5 > 50x^3 - 3x - 12 \quad -3x < 3x \text{ and } -12 < 12$$

$$\frac{1}{4}x^5 - 50x^3 + 3x + 12 > \frac{1/4}{2}x^5$$

$$\frac{1}{4}x^5 - 50x^3 + 3x + 12 > Ax^5 \quad A = \frac{1/4}{2}$$

$$|\frac{1}{4}x^5 - 50x^3 + 3x + 12| > A|x^5| \quad \tfrac{1}{4}x^5 - 50x^3 + 3x + 12 > Ax^5 > 0 \text{ and so } |\tfrac{1}{4}x^5 - 50x^3 + 3x + 12| = \tfrac{1}{4}x^5 - 50x^3 + 3x + 12$$

Hence, by definition of Ω-notation, $\frac{1}{4}x^5 - 50x^3 + 3x + 12$ is $\Omega(x^5)$. $\square$

Problem 31

Refer to the results of exercises 22 and 28 to find an order for $7x^4 - 95x^3 + 3$ from among the set of power function.

Solution

By exercise 22, $7x^4 - 95x^3 + 3$ is $O(x^4)$, and by exercise 28, $7x^4 - 95x^3 + 3$ is $\Omega(x^4)$. Hence, by theorem 11.2.1 (1) $7x^4 - 95x^3 + 3$ is $\Theta(x^4)$.

Problem 32

Refer to the results of exercises 23 and 29 to find an order for $\frac{1}{5}x^2 - 42x - 8$ from among the set of power function.

Solution

By exercise 23, $\frac{1}{5}x^2 - 42x - 8$ is $O(x^2)$, and by exercise 29, $\frac{1}{5}x^2 - 42x - 8$ is $\Omega(x^2)$. Hence, by theorem 11.2.1 (1) $\frac{1}{5}x^2 - 42x - 8$ is $\Theta(x^2)$.

Problem 33

Refer to the results of exercises 24 and 30 to find an order for $\frac{1}{4}x^5 - 50x^3 + 3x + 12$ from among the set of power function.

Solution

By exercise 24, $\frac{1}{4}x^5 - 50x^3 + 3x + 12$ is $O(x^5)$, and by exercise 30, $\frac{1}{4}x^5 - 50x^3 + 3x + 12$ is $\Omega(x^5)$. Hence, by theorem 11.2.1 (1) $\frac{1}{4}x^5 - 50x^3 + 3x + 12$ is $\Theta(x^5)$.

Use the theorem on polynomial orders to prove each of the statements in 34-39.

Problem 34 and Solution

Prove that $\dfrac{(x+1)(x-2)}{4}$ is $\Theta(x^2)$.

Proof.

$$\frac{(x+1)(x-2)}{4} = \frac{x^2-x-2}{4} = \frac{1}{4}x^2 - \frac{1}{4}x - \frac{1}{2}$$

Hence $\dfrac{(x+1)(x-2)}{4}$ is $\Theta(x^2)$ be the theorem on polynomial orders. □

Problem 35 and Solution

Prove that $\dfrac{x}{3}(4x^2-1)$ is $\Theta(x^3)$.

Proof.

$$\frac{x}{3}(4x^2-1) = \frac{4}{3}x^3 - \frac{1}{3}x$$

Hence $\dfrac{x}{3}(4x^2-1)$ is $\Theta(x^3)$ by the theorem on polynomial orders. □

Problem 36 and Solution

Prove that $\dfrac{x(x-1)}{2} + 3x$ is $\Theta(x^2)$.

Proof.

$$\frac{x(x-1)}{2} + 3x = \frac{x(x-1)}{2} + \frac{6x}{2} = \frac{x^2+5x}{2} = \frac{1}{2}x^2 + \frac{5}{2}x$$

Hence $\dfrac{x(x-1)}{2} + 3x$ is $\Theta(x^2)$ by the theorem on polynomial orders. □

Problem 37 and Solution

Prove that $\dfrac{n(n+1)(2n+1)}{6}$ is $\Theta(n^3)$.

Proof.

$$\frac{n(n+1)(2n+1)}{6} = \frac{2n^3+3n^2+n}{6} = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

Hence $\dfrac{n(n+1)(2n+1)}{6}$ is $\Theta(n^3)$ by the theorem on polynomial orders. □

Problem 38 and Solution

Prove that $\left[\dfrac{n(n+1)}{2}\right]^2$ is $\Theta(n^4)$.

Proof.

$$\left[\frac{n(n+1)}{2}\right]^2 = \left[\frac{n^2+n}{2}\right]^2 = \frac{n^4+2n^3+n^2}{4} = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n$$

Hence $\left[\dfrac{n(n+1)}{2}\right]^2$ is $\Theta(n^4)$ by the theorem on polynomial orders. □

Problem 39 and Solution

Prove that $2(n-1) + \dfrac{n(n+1)}{2} + 4\left(\dfrac{n(n-1)}{2}\right)$ is $\Theta(n^2)$.

Proof.

$$\begin{aligned}
2(n-1)+&\frac{n(n+1)}{2} + 4\left(\frac{n(n-1)}{2}\right) \\
&= (n-1)(2n+2) + \frac{n^2+n}{2} \\
&= 2n^2 - 2 + \frac{n^2+n}{2} \\
&= \frac{4n^2-4}{2} + \frac{n^2+n}{2} \\
&= \frac{5n^2+n-4}{2} = \frac{5}{2}n^2 + \frac{1}{2}n - 2
\end{aligned}$$

Hence $2(n-1) + \dfrac{n(n+1)}{2} + 4\left(\dfrac{n(n-1)}{2}\right)$ is $\Theta(n^2)$ by the theorem on polynomial orders. □

Prove each of the statements in 40-47, assuming n is a variable that takes positive integer values. (Use formulas from the exercise set of section 5.2 and the theorem on polynomial orders as appropriate.)

Problem 40 and Solution

Prove that $1^2 + 2^2 + 3^2 + ... + n^2$ is $\Theta(n^3)$.

Proof. It follows from exercise 5.2.10 that

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Hence by exercise 37, $1^2 + 2^2 + 3^2 + ... + n^2$ is $\Theta(n^3)$. □

Problem 41 and Solution

Prove that $1^3 + 2^3 + 3^3 + ... + n^3$ is $\Theta(n^4)$.

Proof. It follows from exercise 5.2.11 that

$$1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Hence by exercise 38, $1^3 + 2^3 + 3^3 + ... + n^3$ is $\Theta(n^4)$. □

Problem 42 and Solution

Prove that $2 + 4 + 6 + ... + 2n$ is $\Theta(n^2)$.

Proof.

$$\begin{aligned} 2 + 4 + 6 + ... + 2n &= 2(1 + 2 + 3 + ... + n) \\ &= 2 \cdot \frac{n(n+1)}{2} && \text{by theorem 5.2.2} \\ &= n(n+1) \\ &= n^2 + n \end{aligned}$$

Hence $2 + 4 + 6 + ... + 2n$ is $\Theta(n^2)$ by the theorem on polynomial orders. □

problem 43 and Solution

Prove that $5 + 10 + 15 + 20 + 25 + ... + 5n$ is $\Theta(n^2)$.

Proof.

$$\begin{aligned} 5 + 10 + 15 + 20 + 25 + ... + 5n &= 5(1 + 2 + 3 + 4 + 5 + ... + n) \\ &= 5 \cdot \frac{n(n+1)}{2} && \text{by theorem 5.2.2} \\ &= \frac{5}{2}n^2 + \frac{5}{2}n \end{aligned}$$

Hence $5 + 10 + 15 + 20 + 25 + ... + 5n$ is $\Theta(n^2)$ by the theorem on polynomial orders. □

Problem 44 and Solution

Prove that $\sum_{i=1}^{n}(4i - 9)$ is $\Theta(n^2)$.

Proof.

$$\begin{aligned}
\sum_{i=1}^{n}(4i-9) &= 4\sum_{i=1}^{n} i - \sum_{i=1}^{n} 9 && \text{by theorem 5.1.1}\\
&= 4 \cdot \frac{n(n+1)}{2} - 9n && \text{by theorem 5.2.2}\\
&= 2(n^2+n) - 9n\\
&= 4n^2 + 2n - 9n\\
&= 4n^2 - 7n
\end{aligned}$$

Hence $\sum_{i=1}^{n}(4i-9)$ is $\Theta(n^2)$ by the theorem on polynomial orders. □

Problem 45 and Solution

Prove that $\sum_{k=1}^{n}(k+3)$ is $\Theta(n^2)$.

Proof.

$$\begin{aligned}
\sum_{k=1}^{n}(k+3) &= \sum_{k=1}^{n} k + \sum_{k=1}^{n} 3 && \text{by theorem 5.1.1}\\
&= \frac{n(n+1)}{2} + 3n && \text{by theorem 5.2.2}\\
&= \frac{n^2+n}{2} + \frac{6n}{2}\\
&= \frac{n^2+7n}{2}\\
&= \frac{1}{2}n^2 + \frac{7}{2}n
\end{aligned}$$

Hence $\sum_{k=1}^{n}(k+3)$ is $\Theta(n^2)$ by the theorem on polynomial orders. □

Problem 46 and Solution

Prove that $\sum_{i=1}^{n} i(i+1)$ is $\Theta(n^3)$.

Proof.

$$\begin{aligned}
\sum_{i=1}^{n} i(i+1) &= \sum_{i=1}^{n} i^2 + i\\
&= \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i && \text{by theorem 5.1.1}
\end{aligned}$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \qquad \text{by exercise 5.2.10 and theorem 5.2.2}$$
$$= \frac{2n^3+3n^2+n}{6} + \frac{n^2+n}{2}$$
$$= \frac{2n^3+3n^2+n}{6} + \frac{3n^2+3n}{6}$$
$$= \frac{2n^3+6n^2+4n}{6}$$
$$= \frac{1}{3}n^3 + n^2 + \frac{2}{3}n$$

Hence $\sum_{i=1}^{n} i(i+1)$ is $\Theta(n^3)$ by the theorem on polynomial orders. □

Problem 47 and Solution

Prove that $\sum_{k=3}^{n}(k^2-2k)$ is $\Theta(n^3)$.

Proof.

$$\sum_{k=3}^{n}(k^2-2k) = \sum_{k=3}^{n}k^2 - 2\sum_{k=3}^{n}k \qquad \text{by theorem 5.1.1}$$
$$= \sum_{k=1}^{n}k^2 - 1^2 - 2^2 - 2\left(\sum_{k=1}^{n}k - 1 - 2\right)$$
$$= \sum_{k=1}^{n}k^2 - 5 - 2\left(\sum_{k=1}^{n}k - 3\right)$$
$$= \sum_{k=1}^{n}k^2 - 5 - 2\sum_{k=1}^{n}k + 6$$
$$= \sum_{k=1}^{n}k^2 - 2\sum_{k=1}^{n}k + 1$$
$$= \frac{2n^3+3n^2+n}{6} - 2\cdot\frac{n^2+n}{2} + 1 \qquad \text{by exercise 5.2.10 and theorem 5.2.2}$$
$$= \frac{2n^3+3n^2+n}{6} - \frac{6n^2+6n}{6} + \frac{6}{6}$$
$$= \frac{2n^3-3n^2-5n+6}{6} = \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{7}{6}n + 1$$

Hence $\sum_{k=3}^{n}(k^2-2k)$ is $\Theta(n^3)$ by the theorem on polynomial orders. □

Problem 48

a. Let $a_0, a_1, a_2, ..., a_n$ be real numbers with $a_n \neq 0$. Prove that

$$\lim_{x\to\infty} \left| \frac{a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0}{a_n x^n} \right| = 1$$

b. Use the results of part (a) and the definition of limit to prove that

$$a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x_1 + a_0 \quad \text{is } \Theta(x^n)$$

Solution

a. *Proof.* Let $a_0, a_1, a_2, ..., a_n$ be real numbers with $a_n \neq 0$. Then,

$$\begin{aligned}
&\lim_{x\to\infty} \left| \frac{a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0}{a_n x^n} \right| \\
&= \lim_{x\to\infty} \left| \frac{a_n x^n}{a_n x^n} + \frac{a_{n-1}x^{n-1}}{a_n x^n} + ... + \frac{a_1 x}{a_n x^n} + \frac{a_0}{a_n x^n} \right| \\
&= \lim_{x\to\infty} \left| 1 + \frac{a_{n-1}}{a_n} \cdot \frac{1}{x} + ... + \frac{a_1}{a_n} \cdot \frac{1}{x^{n-1}} + \frac{a_0}{a_n} \cdot \frac{1}{x^n} \right| \\
&= \left| 1 + \frac{a_{n-1}}{a_n} \cdot \lim_{x\to\infty} \frac{1}{x} + ... + \frac{a_1}{a_n} \cdot \lim_{x\to\infty} \frac{1}{x^{n-1}} + \frac{a_0}{a_n} \cdot \lim_{x\to\infty} \frac{1}{x^n} \right| \\
&= \left| 1 + \frac{a_{n-1}}{a_n} \cdot 0 + ... + \frac{a_1}{a_n} \cdot 0 + \frac{a_0}{a_n} \cdot 0 \right| \\
&= |1 + 0 + ... + 0 + 0| = |1| = 1
\end{aligned}$$

□

b. *Proof.* To say that $\lim_{x\to\infty} f(x) = L$ means that given any real number $\epsilon > 0$, there is a real number $M > 0$ such that $L - \epsilon < f(x) < L + \epsilon$ for all real numbers $x > M$. Now define a function f as

$$f(x) = \left| \frac{a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0}{a_n x^n} \right|$$

Since $\lim_{x\to\infty} f(x) = 1$, we can select $\epsilon = \frac{1}{2}$ and it follows that there exists a real number $M > 0$ such that for all real numbers $x > M$,

$$\begin{aligned}
L - \epsilon &< f(x) < L + \epsilon \\
1 - \frac{1}{2} &< f(x) < 1 + \frac{1}{2} \\
\frac{1}{2} &< f(x) < \frac{3}{2} \\
\frac{1}{2} &< \left| \frac{a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0}{a_n x^n} \right| < \frac{3}{2} \qquad (1)
\end{aligned}$$

Proof that $a_nx^n + a_{n-1}x^{n-1} + ... + a_1x_1 + a_0$ is $O(x^n)$:

$$\left|\frac{a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0}{a_nx^n}\right| < \frac{3}{2} \qquad \text{right side of inequality (1)}$$

$$\frac{|a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0|}{|a_nx^n|} < \frac{3}{2} \qquad \left|\frac{a}{b}\right| = \frac{|a|}{|b|}$$

$$|a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0| < \frac{3}{2} \cdot |a_nx^n|$$

$$|a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0| < \frac{3}{2} \cdot |a_n| \cdot |x^n| \qquad \text{by exercise 4.4.44}$$

Finally Let $B = \frac{3}{2} \cdot |a_n|$ and it follows that for all real numbers $x > M$,

$$|a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0| \leq B|x^n|$$

Hence, by definition of O-notation,

$$a_nx^n + a_{n-1}x^{n-1} + ... + a_1x_1 + a_0 \text{ is } O(x^n).$$

Proof that $a_nx^n + a_{n-1}x^{n-1} + ... + a_1x_1 + a_0$ is $\Omega(x^n)$:

$$\frac{1}{2} < \left|\frac{a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0}{a_nx^n}\right| \qquad \text{right side of inequality (1)}$$

$$\frac{1}{2} < \frac{|a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0|}{|a_nx^n|} \qquad \left|\frac{a}{b}\right| = \frac{|a|}{|b|}$$

$$\frac{1}{2} \cdot |a_nx^n| < |a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0|$$

$$\frac{1}{2} \cdot |a_n| \cdot x^n < |a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0| \qquad \text{by exercise 4.4.44}$$

Finally let $A = \frac{1}{2} \cdot |a_n|$ and it follows that for all real numbers $x > M$,

$$A|x^n| \leq |a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0|$$

Hence, by definition of Ω-notation,

$$a_nx^n + a_{n-1}x^{n-1} + ... + a_1x_1 + a_0 \text{ is } \Omega(x^n).$$

Conclusion: Since $a_nx^n + a_{n-1}x^{n-1} + ... + a_1x_1 + a_0$ is $O(x^n)$ and $\Omega(x^n)$ it follows by theorem 11.2.1 (1) that $a_nx^n + a_{n-1}x^{n-1} + ... + a_1x_1 + a_0$ is $\Theta(x^n)$. □

Problem 49

Another approach to proving part of the theorem of polynomial orders uses properties of O-notation.

a. Show that if f, g, and h are functions from $\mathbb{R}$ to $\mathbb{R}$ and $f(x)$ if $O(h(x))$ and $g(x)$ is $O(h(x))$, then $f(x) + g(x)$ is $O(h(x))$.

b. How does it follow from part (a) and property 11.2.1 that x^4+x^2 is $O(x^4)$?

c. The result of exercise 11 states that if f is a function from $\mathbb{R}$ to $\mathbb{R}$, $f(x)$ is $O(g(x))$, and c is any nonzero real number, then $cf(x)$ is $O(g(x))$. How does it follows from this result and part (a) that $12x^5 - 34x^2 + 7$ is $O(x^5)$?

d. Use the results of part (a) and exercise 11 to show that if n is any positive integer and $a_0, a_1, ..., a_n$ are real numbers with $a_n \neq 0$, then

$$a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 \text{ is } O(x^n)$$

Solution

a. *Proof.* Let f, g, and h be functions from $\mathbb{R}$ to $\mathbb{R}$ and suppose that $f(x)$ is $O(h(x))$ and $g(x)$ is $O(h(x))$. Then there exist nonnegative real numbers b_1 and b_2 and positive integers B_1 and B_2 such that

$$|f(x)| \leq B_1|h(x)| \quad \text{for all real numbers } x > b_1,$$

and

$$|g(x)| \leq B_2|h(x)| \quad \text{for all real numbers } x > b_2.$$

Let $B = B_1 + B_2$ and let $b = \max(b_1, b_2)$. Now, for all $x > b$,

$$\begin{aligned} |f(x)| + |g(x)| &\leq B_1|h(x)| + B_2|h(x)| && \text{by hypothesis} \\ |f(x) + g(x)| &\leq B_1|h(x)| + B_2|h(x)| && \text{by the triangle inequality} \\ &= (B_1 + B_2)|h(x)| \\ &= B|h(x)| && B = B_1 + B_2 \end{aligned}$$

Hence, by definition of O-notation, $f(x) + g(x)$ is $O(h(x))$. □

b. By property 11.2.1, for all $x > 1$, $x^2 < x^4$. Hence $|x^2| \leq 1 \cdot |x^4|$ for all $x > 1$. Thus, by definition of O-notation, x^2 is $O(x^4)$. Also $|x^4| \leq 1 \cdot |x^4|$ for all x and so x^4 is $O(x^4)$. It follows by part (a) that $x^4 + x^2$ is $O(x^4)$.

c. By property 11.2.1, for all $x > 1$, $1 < x^5$ and $x^2 < x^5$. Hence $|1| \leq 1 \cdot |x^5|$ and $|x^2| \leq 1 \cdot |x^5|$ for all $x > 1$. Thus, by definition of O-notation, 1 is $O(x^5)$ and x^2 is $O(x^5)$. Also $|x^5| \leq 1 \cdot |x^5|$ for all x and so x^5 is $O(x^5)$. It now follows by exercise 11 that 7 is $O(x^5)$, $-34x^2$ is $O(x^5)$, and $12x^5$ is $O(x^5)$. Finally it follows by part (a) that $12x^5 - 34x^2 + 7$ is $O(x^5)$.

d. *Proof.* Let n be any positive integer and let $a_0, a_1, ..., a_n$ be any real numbers with $a_n \neq 0$. By property 11.2.1, for all $x > 1$,

$$x^{n-1} < x^n, \quad x^{n-2} < x^n, \ \ ..., \ \ x < x^n, \ \text{and} \ \ 1 < x^n$$

Hence,

$$|x^{n-1}| \leq 1 \cdot |x^n|, \quad |x^{n-2}| \leq 1 \cdot |x^n|, \ \ ..., \ \ |x| \leq 1 \cdot |x^n|, \ \text{and} \ \ |1| \leq 1 \cdot |x^n|$$

Thus, by definition of O-notation,

$$x^{n-1} \text{ is } O(x^n), \quad x^{n-2} \text{ is } O(x^n), \quad ..., \quad x \text{ is } O(x^n), \quad \text{and} \quad 1 \text{ is } O(x^n)$$

Also, $|x^n| \leq 1 \cdot |x^n|$ for all x and so x^n is $O(x^n)$. It now follows by exercise 11 that

$$a_n x^n \text{ is } O(x^n), \quad a_{n-1}x^{n-1} \text{ is } O(x^n), \quad ..., \quad a_1 x \text{ is } O(x^n), \quad \text{and} \quad a_0 \text{ is } O(x^n)$$

Finally it follows by part (a) that

$$a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 \text{ is } O(x^n).$$

□

Problem 50

a. Let x be any positive real number. Use mathematical induction to prove that for all integers $n \geq 1$, if $x \leq 1$ then $x^n \leq 1$.

b. Explain how it follows from part (a) that if x is any positive real number, then for all integers $n \geq 1$, if $x^n > 1$ then $x > 1$.

c. Explain how it follows from part (b) that if x is any positive real number, then for all integers $n \geq 1$, if $x > 1$ then $x^{1/n} > 1$.

d. Let p, q, and s be positive integers, let r be a nonnegative integer, and suppose $p/q > r/s$. Use part (c) and the result of exercise 15 to prove property 11.2.1. In other words show that for any real number x, if $x > 1$ then $x^{p/q} > x^{r/s}$.

Solution

a. *Proof.* Let x be any positive real number and let $P(n)$ be the property that for all integers $n \geq 1$, if $x \leq 1$ then

$$x^n \leq 1 \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: Let $n = 1$. Then $x^n = x^1 = x \leq 1$. Hence $P(1)$ is true.

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$x^k \leq 1 \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$x^{k+1} \leq 1 \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} x^{k+1} &= x \cdot x^k \\ &\leq x \cdot 1 && \text{by inductive hypothesis} \\ &\leq x \\ &\leq 1 && \text{by definition of } x \end{aligned}$$

which is the right-hand side of $P(k+1)$. □

b. The statement in part(b) is the contrapositive of the statement that was proved in part (a). Since a statement and its contrapositive have the same truth values, proving a statement proves its contrapositive and vice versa.

c. In part (b) select $x^{1/n}$ to be the positive real number in place of x. Then the statement in (b) becomes if $(x^{1/n})^n = x^{n/n} = x > 1$ then $x^{1/n} > 1$.

d. *Proof.* Let p, q, and s be positive integers, let r be a nonnegative integer, let x be a positive real number, and suppose $p/q > r/s$ and $x > 1$. Then $ps > rq$ and so $ps - rq > 0$. Also $\frac{x^{p/q}}{x^{r/s}} = x^{(p/q-r/s)} = x^{(pq-rs)/qs}$. Since p, q, r, and s are integers it follows by closure under multiplication and subtraction that $ps - rq$ is also an integer. Since $ps - rq > 0$ and $ps - rq$ is an integer it must be that $ps - rq \geq 1$. Also since q and s are positive integers qs is also a positive integer. Now since $x > 1$, it follows from exercise 15 that $x^{ps-rq} > 1$. From part (c), substitute x^{ps-rq} in place of x and $1/qs$ in place of n, and it follows that

$$\begin{aligned}
(x^{ps-rq})^{1/qs} &> 1 \\
\left(\frac{x^{ps}}{x^{rq}}\right)^{1/qs} &> 1 \\
\frac{x^{ps/qs}}{x^{rq/qs}} &> 1 \\
\frac{x^{p/q}}{x^{r/s}} &> 1 \\
x^{p/q} &> x^{r/s}
\end{aligned}$$

□

Explain how each statement in 51 and 52 follows from exercise 50, exercise 13, and parts (a) and (c) of exercise 49.

Problem 51 and Solution

Prove that $4x^{4/3} - 15x + 7$ is $O(x^{4/3})$.

Proof. By part (d) of exercise 50, for all $x > 1$, $x \leq x^{4/3}$ and $x^0 = 1 \leq x^{4/3}$. Hence, by definition of O-notation (since all expressions are positive), x is $O(x^{4/3})$ and 1 is $O(x^{4/3})$. Also, by exercise 13, $x^{4/3}$ is $\Theta(x^{4/3})$ and hence, by theorem 11.2.1 (1), $x^{4/3}$ is $O(x^{4/3})$. Now by part (c) of exercise 49, $4x^{4/3}$ is $O(x^{4/3})$, $-15x$ is $O(x^{4/3})$ and $7 = O(x^{4/3})$. Finally by part (a) of exercise 49, $4x^{4/3} - 15x + 7$ is $O(x^{4/3})$. □

Problem 52 and Solution

Prove that $\sqrt{x}(38x^5 + 9)$ is $O(x^{11/2})$.

Proof. First note that $\sqrt{x}(38x^5 + 9) = 38x^{11/2} + 9x^{1/2}$. By part (d) of exercise 50, for all $x > 1$, $x^{1/2} < x^{11/2}$. Also, by exercise 13, $x^{11/2}$ is $\Theta(x^{11/2})$ and hence, by theorem 11.2.1 (3), $x^{11/2}$ is $O(x^{11/2})$. Now by part (c) of exercise 49, $38x^{11/2}$ is $O(x^{11/2})$ and $9x^{11/2}$ is $O(x^{11/2})$. Finally by part (a) of exercise 49, $38x^{11/2} + 9x^{1/2}$ is $O(x^{11/2})$ and so $\sqrt{x}(38x^5 + 9)$ is $O(x^{11/2})$. □

Problem 53 and Solution

Prove that if r and s are rational numbers with $r > s$, then x^r is not $O(x^s)$.

Proof. Suppose that x^r is $O(x^s)$. Then, by definition of O-notation, there exists a positive real number B and a nonnegative real number b such that

$$|x^r| \le B|x^s| \quad \text{for all real numbers } x > b$$

Now let x be a real number such that $x > b$, $x > 1$, and $x > B^{1/(r-s)}$. Then

$$\begin{aligned}
|x^r| &= x^r && x > 1 \implies x^r > 0 \\
&= x^{r-s} \cdot x^s \\
&> B \cdot x^s && x > B^{1/(r-s)} \implies x^{r-s} > B \\
&= B \cdot |x^s| && x > 0 \implies x^s > 0
\end{aligned}$$

Thus there is a real number $x > b$ such that $|x^r| > B|x^x|$ which is a contradiction. □

In 54-56, use theorem 11.2.4 to find an order for each of the given functions from among the set of rational power functions.

Problem 54 and Solution

$f(x) = \dfrac{\sqrt{x}(3x+5)}{2x+1} = \dfrac{3x^{3/2} + 5x^{1/2}}{2x+1}$. The numerator of $f(x)$ is a sum of rational power functions with highest power $3/2$, and the denominator is a sum of rational power functions with highest power 1. Because $3/2 - 1 = 1/2$, theorem 11.2.4 implies that $f(x)$ is $\Theta(x^{1/2})$.

Problem 55 and Solution

$f(x) = \dfrac{(2x^{7/2}+1)(x-1)}{(x^{1/2}+1)(x+1)} = \dfrac{2x^{9/2} - 2x^{7/2} + x - 1}{x^{3/2} + x + x^{1/2} + 1}$. The numerator of $f(x)$ is a sum of rational power functions with highest power $9/2$, and the denominator is a sum of rational power functions with highest power $3/2$. Because $9/2 - 3/2 = 6/2 = 3$, theorem 11.2.4 implies that $f(x)$ is $\Theta(x^3)$.

Problem 56 and Solution

$f(x) = \dfrac{(5x^2+1)(\sqrt{x}-1)}{4x^{3/2} - 2x} = \dfrac{5x^{5/2} - 5x^2 + x^{1/2} - 1}{4x^{3/2} - 2x}$. The numerator of $f(x)$ is a sum of rational power functions with highest power $5/2$, and the denominator is a sum of rational power functions with highest power $3/2$. Because $5/2 - 3/2 = 2/2 = 1$, theorem 11.2.4 implies that $f(x)$ is $\Theta(x)$.

Problem 57

a. Use mathematical induction to prove that for all integers $n \geq 1$,

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + ... + \sqrt{n} \leq n^{3/2}$$

b. Use mathematical induction to prove that for all integers $n \geq 1$,

$$\frac{1}{2}n^{3/2} \leq \sqrt{1} + \sqrt{2} + \sqrt{3} + ... + \sqrt{n}$$

c. What can you conclude from parts (a) and (b) about an order for $\sqrt{1} + \sqrt{2} + \sqrt{3} + ... + \sqrt{n}$?

solution

a. *Proof.* Let the property $P(n)$ be the inequality

$$\sum_{i=1}^{n} \sqrt{i} \leq n^{3/2} \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: Let $n = 1$. Then $\sum_{i=1}^{n} \sqrt{i} = \sum_{i=1}^{1} \sqrt{i} = \sqrt{1} = 1$ and $n^{3/2} = 1^{3/2} = 1$. Since $1 \leq 1$ it follows that $P(1)$ is true.

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$\sum_{i=1}^{k} \sqrt{i} \leq k^{3/2} \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$\sum_{i=1}^{k+1} \sqrt{i} \leq (k+1)^{3/2} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
\sum_{i=1}^{k+1} \sqrt{i} &= \sum_{i=1}^{k} \sqrt{i} + \sqrt{k+1} && \text{by definition of } \sum \\
&\leq k^{3/2} + (k+1)^{1/2} && \text{by inductive hypothesis} \\
&= k \cdot k^{1/2} + (k+1)^{1/2} && k^{3/2} = k \cdot k^{1/2} \\
&\leq k \cdot (k+1)^{1/2} + (k+1)^{1/2} && k^{1/2} < (k+1)^{1/2} \\
&= (k+1)^{1/2}(k+1) \\
&= (k+1)^{3/2}
\end{aligned}$$

which is the right-hand side of $P(k+1)$. □

b. *Proof.* Let the property $P(n)$ be the inequality

$$\sum_{i=1}^{n} \sqrt{i} \geq \frac{1}{2} n^{3/2} \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: Let $n = 1$. Then $\sum_{i=1}^{n} \sqrt{i} = \sum_{i=1}^{1} \sqrt{i} = \sqrt{1} = 1$ and $\frac{1}{2} n^{3/2} = \frac{1}{2} \cdot 1^{3/2} = \frac{1}{2} \cdot 1 = 1/2$. Since $1 \geq 1/2$ it follows that $P(1)$ is true.

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$\sum_{i=1}^{k} \sqrt{i} \geq \frac{1}{2} k^{3/2} \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$\sum_{i=1}^{k+1} \sqrt{i} \geq \frac{1}{2} (k+1)^{3/2} \qquad \leftarrow P(k+1)$$

Since $k \geq 1$ it follows that,

$$\begin{aligned} k^2 &\geq k^2 - 1 \\ k^2 &\geq (k-1)(k+1) \\ \frac{k}{k-1} &\geq \frac{k+1}{k} && \text{divide both sides by } k(k+1) \end{aligned}$$

But $\dfrac{k+1}{k} \geq 1$ and any number which is greater than or equal to 1 is greater than or equal to its square root. Hence,

$$\frac{k}{k-1} \geq \frac{k+1}{k} \geq \sqrt{\frac{k+1}{k}} = \frac{\sqrt{k+1}}{\sqrt{k}}$$

It follows that

$$k\sqrt{k} \geq (k-1)\sqrt{k+1} = (k+1-2)\sqrt{k+1} = (k+1)\sqrt{k+1} - 2\sqrt{k+1}$$

Thus,

$$k\sqrt{k} + 2\sqrt{k+1} \geq (k+1)\sqrt{k+1}$$

Divide both sides by 2 to obtain

$$\frac{1}{2} k^{3/2} + (k+1)^{1/2} \geq \frac{1}{2} (k+1)^{3/2} \qquad (1)$$

Now the left-hand side of $P(k+1)$ is

$$\begin{aligned}
\sum_{i=1}^{k+1} \sqrt{i} &= \sum_{i=1}^{k} \sqrt{i} + \sqrt{k+1} && \text{by definition of } \sum \\
&\geq \frac{1}{2}k^{3/2} + (k+1)^{1/2} && \text{by inductive hypothesis} \\
&\geq \frac{1}{2}(k+1)^{3/2} && \text{by inequality (1)}
\end{aligned}$$

which is the right-hand side of $P(k+1)$. ☐

c. From (a) and (b) we conclude that $\sqrt{1} + \sqrt{2} + \sqrt{3} + ... + \sqrt{n}$ is $\Theta(n^{3/2})$.

Problem 58

a. Use mathematical induction to prove that for all integers $n \geq 1$,

$$1^{1/3} + 2^{1/3} + ... + n^{1/3} \leq n^{4/3}$$

b. Use mathematical induction to prove that for all integers $n \geq 1$,

$$\frac{1}{2}n^{4/3} \leq 1^{1/3} + 2^{1/3} + ... + n^{1/3}$$

c. What can you conclude from parts (a) and (b) about an order for $1^{1/3} + 2^{1/3} + ... + n^{1/3}$?

Solution

a. *Proof.* Let the property $P(n)$ be the inequality

$$\sum_{i=1}^{n} i^{1/3} \leq n^{4/3} \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: Let $n = 1$. Then $\sum_{i=1}^{n} i^{1/3} = \sum_{i=1}^{1} i^{1/3} = 1^{1/3} = 1$ and $1^{4/3} = 1$. Since $1 \leq 1$ it follows that $P(1)$ is true.

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$\sum_{i=1}^{k} i^{1/3} \leq k^{4/3} \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$\sum_{i=1}^{k+1} i^{1/3} \leq (k+1)^{4/3} \qquad \leftarrow P(k) \text{ IH}$$

But the left-hand side of $P(k+1)$ is

$$
\begin{aligned}
\sum_{i=1}^{k+1} i^{1/3} &= \sum_{i=1}^{k} i^{1/3} + (k+1)^{1/3} && \text{by definition of } \textstyle\sum \\
&\le k^{4/3} + (k+1)^{1/3} && \text{by inductive hypothesis} \\
&= k \cdot k^{1/3} + (k+1)^{1/3} \\
&\le k \cdot (k+1)^{1/3} + (k+1)^{1/3} && k^{1/3} \le (k+1)^{1/3} \\
&= (k+1)^{1/3}(k+1) \\
&= (k+1)^{4/3}
\end{aligned}
$$

which is the right-hand side of $P(k+1)$. $\square$

b. *Proof.* Let the property $P(n)$ be the inequality

$$\sum_{i=1}^{n} i^{1/3} \ge \frac{1}{2} n^{4/3} \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: Let $n = 1$. Then $\sum_{i=1}^{n} i^{1/3} = \sum_{i=1}^{1} i^{1/3} = 1$ and $\frac{1}{2} \cdot 1^{4/3} = 1/2$. Since $1 \ge \frac{1}{2}$ it follows that $P(1)$ is true.

Show that for all integers $k \ge 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \ge 1$ and suppose that

$$\sum_{i=1}^{k} i^{1/3} \ge \frac{1}{2} k^{4/3} \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$\sum_{i=1}^{k+1} i^{1/3} \ge \frac{1}{2} (k+1)^{4/3} \qquad \leftarrow P(k+1)$$

Since $k \ge 1$ it follows that,

$$
\begin{aligned}
k^2 &\ge k^2 - 1 \\
k^2 &\ge (k-1)(k+1) \\
\frac{k}{k-1} &\ge \frac{k+1}{k} && \text{divide both sides by } k(k+1)
\end{aligned}
$$

But $\dfrac{k+1}{k} \ge 1$ and any number which is greater than or equal to 1 is greater than or equal to its cube root. Hence,

$$\frac{k}{k-1} \ge \frac{k+1}{k} \ge \sqrt[3]{\frac{k+1}{k}} = \frac{\sqrt[3]{k+1}}{\sqrt[3]{k}}$$

It follows that

$$k \cdot k^{1/3} \ge (k-1)(k+1)^{1/3} = (k+1-2)(k+1)^{1/3} = (k+1)(k+1)^{1/3} - 2(k+1)^{1/3}$$

Thus,

$$k \cdot k^{1/3} + 2(k+1)^{1/3} \geq (k+1)(k+1)^{1/3}$$

Divide both sides by 2 to obtain

$$\frac{1}{2}k^{4/3} + (k+1)^{1/3} \geq \frac{1}{2}(k+1)^{4/3} \qquad (1)$$

Now the left-hand side of $P(k+1)$ is

$$\begin{aligned}
\sum_{i=1}^{k+1} i^{1/3} &= \sum_{i=1}^{k} i^{1/3} + (k+1)^{1/3} && \text{by definition of } \sum \\
&\geq \frac{1}{2}k^{4/3} + (k+1)^{1/3} && \text{by inductive hypothesis} \\
&\geq \frac{1}{2}(k+1)^{4/3} && \text{by inequality (1)}
\end{aligned}$$

which is the right-hand side of $P(k+1)$. □

c. From (a) and (b) we conclude that $1^{1/3} + 2^{1/3} + ... + n^{1/3}$ is $\Theta(n^{4/3})$.

Exercises 59-61 use the following definition, which requires the concept of limit from calculus.

Definition: If f and g are real-valued functions of a real variable and $\lim_{x\to\infty} g(x) \neq 0$, then

$$f(x) \text{ is } o(g(x)) \iff \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

The notation $f(x)$ is $o(g(x))$ is read "$f(x)$ is little-oh of $g(x)$."

Problem 59 and Solution

Prove that if $f(x)$ is $o(g(x))$, then $f(x)$ is $O(g(x))$.

Proof. Suppose that $f(x)$ is $o(g(x))$. By definition of o-notation, $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. By definition of limit this implies that given any real number $\epsilon > 0$ there exists a real number x_0 such that

$$\left|\frac{f(x)}{g(x)} - 0\right| = \left|\frac{f(x)}{g(x)}\right| = \frac{|f(x)|}{|g(x)|} < \epsilon \quad \text{for all } x > x_0.$$

Define a real number $b = \max(0, x_0)$ and let $B = \epsilon$ and it follows that there exists a positive real number B and a nonnegative real number b such that

$$|f(x)| \leq B|g(x)| \quad \text{for all } x > b.$$

Hence, by definition of O-notation, $f(x)$ is $O(g(x))$. □

Problem 60 and Solution

Prove that if $f(x)$ and $g(x)$ are both $o(h(x))$, then for all real numbers a and b, $af(x) + bg(x)$ is $o(h(x))$.

Proof. Suppose that $f(x)$ and $g(x)$ are both $o(h(x))$. By definition of o-notation, $\lim_{x\to\infty} \frac{f(x)}{h(x)} = 0$ and $\lim_{x\to\infty} \frac{g(x)}{h(x)} = 0$. By definition of limit this implies that given any real number $\epsilon > 0$ there exist real numbers x_1 and x_2 such that

$$\left|\frac{f(x)}{h(x)}\right| < \epsilon \quad \text{for all } x > x_1 \quad \text{and} \quad \left|\frac{g(x)}{h(x)}\right| < \epsilon \quad \text{for all } x > x_2$$

Now define a real number $x_0 = \max(x_1, x_2)$ and it follows that for all $x > x_0$,

$$\left|\frac{f(x)}{h(x)}\right| < \epsilon \quad \text{and} \quad \left|\frac{g(x)}{h(x)}\right| < \epsilon$$

Let a and b be any real numbers such that $a \neq 0$ and $b \neq 0$ and it follows that

$$|a| \cdot \left|\frac{f(x)}{h(x)}\right| < \epsilon|a| \quad \text{and} \quad |b| \cdot \left|\frac{g(x)}{h(x)}\right| < \epsilon|b|$$

Adding the two inequalities give

$$\begin{aligned}
|a| \cdot \left|\frac{f(x)}{h(x)}\right| + |b| \cdot \left|\frac{g(x)}{h(x)}\right| &< \epsilon|a| + \epsilon|b| \\
\left|\frac{af(x)}{h(x)}\right| + \left|\frac{bg(x)}{h(x)}\right| &< \epsilon|a| + \epsilon|b| && \text{by exercise 4.4.44} \\
\frac{|af(x)|}{|h(x)|} + \frac{|bg(x)|}{|h(x)|} &< \epsilon|a| + \epsilon|b| && \text{by definition of absolute value} \\
\frac{|af(x)| + |bg(x)|}{|h(x)|} &< \epsilon|a| + \epsilon|b| \\
\frac{|af(x) + bg(x)|}{|h(x)|} &< \epsilon|a| + \epsilon|b| && \text{by the triangle inequality} \\
\left|\frac{af(x) + bg(x)}{h(x)}\right| &< \epsilon|a| + \epsilon|b| && \text{by definition of absolute value} \\
\left|\frac{af(x) + bg(x)}{h(x)}\right| &< \epsilon(|a| + |b|)
\end{aligned}$$

This implies that $\lim_{x\to\infty} \frac{af(x) + bg(x)}{h(x)} = 0$. In the case that $a = 0$ and $b = 0$, $af(x) = 0$ and $bg(x) = 0$ and so $\lim_{x\to\infty} \frac{af(x) + bg(x)}{h(x)} = 0$. In this case that $b = 0$ and $a \neq 0$ we only need to show that $\lim_{x\to\infty} \frac{af(x)}{h(x)} = 0$. But by limit laws $\lim_{x\to\infty} \frac{af(x)}{h(x)} = a \cdot \lim_{x\to\infty} \frac{f(x)}{h(x)} = a \cdot 0 = 0$. The case in which $a = 0$ and $b \neq 0$ is analogous. Hence in every case, $af(x) + bg(x)$ is $o(h(x))$. □

Problem 61 and Solution

Prove that for any positive real numbers a and b, if $a < b$ then x^a is $o(x^b)$.

Proof. Let $n = b - a > 0$ and let a real number $\epsilon > 0$ be given. Now define a positive real number $N = \dfrac{1}{\sqrt[n]{\epsilon}}$ and suppose that $x > N$. Then,

$$x > \frac{1}{\sqrt[n]{\epsilon}} \implies x^n > \frac{1}{\epsilon} \implies \frac{1}{x^n} < \epsilon \implies \left| \frac{x^a}{x^b} - 0 \right| < \epsilon$$

This means that $\lim_{x \to \infty} \dfrac{x^a}{x^b} = 0$ and so x^a is x^b. $\square$