Section 5.9

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Problem 1

Consider the set of Boolean expressions defined in example 5.9.1. Give derivations showing that each of the following is a Boolean expression over the English alphabet $\{a, b, c, ..., x, y, z\}$.

(a) $(\neg p \vee (q \wedge (r \vee \neg s)))$

(b) $((p \vee q) \vee \neg((p \wedge \neg s) \wedge r))$

Solution

(a) (1) By I, p, q, r, and s are Boolean expressions.
  (2) By (1) and II(c), $\neg p$ and $\neg s$ are Boolean expressions.
  (3) By (1) and (2) and II(b) $(r \vee \neg s)$ is a Boolean expression.
  (4) By (1) and (3) and II(a) $q \wedge (r \vee \neg s)$ is a Boolean expression.
  (5) By (2) and (4) and II(b) $(\neg p \vee (q \wedge (r \vee \neg s)))$ is a Boolean expression.

(b) (1) By I, p, q, r, and s are Boolean expressions.
  (2) By (1) and II(b) and (c), $p \vee q$ and $\neg s$ are Boolean expressions.
  (3) By (1) and (2) and II(a), $p \wedge \neg s$ is a Boolean expression.
  (4) By (1) and (3) and II(a), $(p \wedge \neg s) \wedge r$ is a Boolean expression.
  (5) by (4) and II(c) $\neg((p \wedge \neg s) \wedge r)$ is a Boolean expression.
  (6) By (2) and (5) and II(b) $((p \vee q) \vee \neg((p \wedge \neg s) \wedge r))$ is a Boolean expression.

Problem 2

Let S be defined as in example 5.9.2. Give derivations showing that each of the following is in S.

(a) aab

(b) bb

Solution

(a) (1) By I, $\epsilon \in S$.
(2) By (1) and II(a), $\epsilon a \in S$. But $\epsilon a = a$ and so $a \in S$.
(3) By (2) and II(a), $aa \in S$.
(4) By (3) and II(b), $aab \in S$.

(b) (1) By I, $\epsilon \in S$.
(2) By (1) and II(b), $\epsilon b \in S$. But $\epsilon b = b$ and so $b \in S$.
(3) By (2) and II(b), $bb \in S$.

Problem 3

Consider the MIU-system discussed in example 5.9.3. Give derivations showing that each of the following is in the MIU-system.

(a) $MIUI$

(b) $MUIIU$

Solution

(a) (1) By I, MI is in the MIU-system.
(2) By (1) and II(b), MII is in the MIU-system.
(3) By (2) and II(b), $MIIII$ is in the MIU-system.
(4) By (3) and II(b), $MIIIIIIII$ is in the MIU-system.
(5) By (4) and II(c), $MIUIIII$ is in the MIU-system.
(6) By (5) and II(c), $MIUUI$ is in the MIU-system.
(7) By (6) and II(d), $MIUI$ is in the MIU-system.

(b) (1) By I, MI is in the MIU-system.
(2) By (1) and II(b), MII is in the MIU-system.
(3) By (2) and II(b), $MIIII$ is in the MIU-system.
(4) By (3) and II(b), $MIIIIIIII$ is in the MIU-system.
(5) By (4) and II(c), $MUIIIII$ is in the MIU-system.
(6) By (5) and II(c), $MUIIU$ is in the MIU-system.

Problem 4

The set of arithmetic expressions over the real numbers can be defined recursively as follows:

I. BASE: Each real number r is an arithmetic expression.

II. RECURSION: If u and v are arithmetic expressions, then the following are also arithmetic expressions:

a. $(+u)$ d. $(u - v)$

b. $(-u)$ e. $(u \cdot v)$

c. $(u + v)$ f. $\left(\frac{u}{v}\right)$

III. RESTRICTION: There are no arithmetic expressions over the real numbers other than those obtained from I and II.

(Note that the expression $\left(\frac{u}{v}\right)$ is legal even though the value of v may be 0.) Give derivatives showing that each of the following is an arithmetic expression.

a. $((2 \cdot (0.3 - 4.2)) + (-7))$ b. $\left(\dfrac{(9 \cdot (6.1 + 2))}{((4 - 7) \cdot 6)}\right)$

Solution

(a)
(1) By I, $2, 0.3, 4.2$, and 7 are arithmetic expressions.
(2) By (1) and II(d), $(0.3 - 4.2)$ is an arithmetic expression.
(3) By (1) and (2) and II(e), $(2 \cdot (0.3 - 4.2))$ is an arithmetic expression.
(4) By (1) and II(b), (-7) is an arithmetic expression.
(5) By (3) and (4) and II(c), $((2 \cdot (0.3 - 4.2)) + (-7))$ is an arithmetic expression.

(b)
(1) By I, $9, 6.1, 2, 4, 7$, and 6 are arithmetic expressions.
(2) By (1) and II(c), $(6.1 + 2)$ is an arithmetic expression.
(3) By (1) and II(d), $(4 - 7)$ is an arithmetic expression.
(4) By (1) and (2) and II(e), $(9 \cdot (6.1 + 2))$ is an arithmetic expression.
(5) By (1) and (3) and II(e), $((4 - 7) \cdot 6)$ is an arithmetic expression.
(6) By (4) and (5) and II(f), $\left(\dfrac{(9 \cdot (6.1 + 2))}{((4 - 7) \cdot 6)}\right)$ is an arithmetic expression.

Problem 5

Define a set S recursively as follows:

I. BASE: $1 \in S$

II. RECURSION: If $s \in S$, then

a. $0s \in S$ b. $1s \in S$

III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

Use structural induction to prove that every string in S ends in a 1.

Solution

Proof. Define a property ofor S that every string in S ends in a 1.

Show that each object in the BASE for S satisfies the property: The only object in the base for S is 1, which ends in 1.

Show that for each rule in the RECURSION for S, if the rule is applied to an object in S that satisfies the property, then the object defined by the rule also satisfies the property: The recursion for S consists of two rules denoted II(a) and II(b). Suppose that s is a string in S that ends in 1. When rule II(a) is applied to s, the result is the string $0s$, which also ends in s. In the case where rule II(b) is applied to S, the result is the string $1s$ which also ends in a 1. Thus when each rule in the recursion is applied to a string in S that ends in a 1, the result is also a string that ends in a 1. □

Problem 6

Define a set S recursively as follows:

I. BASE: $a \in S$

II. RECURSION: If $s \in S$, then

   a. $sa \in S$ b. $sb \in S$

III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

Use structural induction to prove that every string in S begins with an a.

Solution

Proof. Define a property for S that every string in S begins with an a.

Show that each object in the BASE for S satisfies the property: The only object in the base of S is a, which begins with a.

Show that for each rule in the RECURSION for S, if the rule is applied to an object in S that satisfies the property, then the object defined by the rule also satisfies the property: The recursion for S consists of two rules denoted II(a) and II(b). Suppose that s is a string in S that begins with a. When rule II(a) is applied to s, the result is the string aa, which also begins with a. In the case where rule II(b) is applied to S, the result is the string ab which also begins with a. Thus when each rule in the recursion is applied to a string S that begins with an a, the result is also a string that begins with an a. □

Problem 7

Define a set S recursively as follows:

I. BASE: $\epsilon \in S$

II. RECURSION: If $s \in S$, then

a. $bs \in S$ c. $saa \in S$

b. $sb \in S$ d. $aas \in S$

III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

Use structural induction to prove that every string in S contains an even number of a's

Solution

Proof. Define a property for S that every string in S begins contains an even number of a's.

Show that each object in the BASE for S satisfies the property: The only object in the base of S is ϵ which contains an even number of a's, namely 0 a's.

Show that for each rule in the RECURSION for S, if the rule is applied to an object in S that satisfies the property, then the object defined by the rule also satisfies the property: The recursion for S consists of four rules denoted II(a), II(b), II(c), and II(d). Suppose that s is a string in S that contains an even number of a's. Then the number of a's in s is $2t$ for some integer t. When either rule II(a) or rule II(b) is applied to s, the number of s's in the resulting string is the same as the number of a's in s. Therefore the resulting string will still have an even number of a's. When either rule II(c) or rule II(d) is applied to s, the number of a's in the resulting string is two greater than the number of a's in S. Therefore the number of a's in the resulting string will be $2t + 2 = 2(t + 1)$, which is even. Thus when each rule in the recursion is applied to a string S that contains an even number of a's, the result is also a string that contains an even number of a's. $\square$

Problem 8

Define a set S recursively as follows:

I. BASE: $1 \in S$, $2 \in S$ $3 \in S$, $4 \in S$, $5 \in S$, $6 \in S$, $7 \in S$, $8 \in S$, $9 \in S$

II. RECURSION: If $s \in S$ and $t \in S$ then

a. $s0 \in S$ b. $st \in S$

III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

Use structural induction to prove that no string in S represents an integer with a leading 0.

Solution

Proof. Define a property for S that no string in S represents an integer with a leading 0.

Show that each object in the BASE for S satisfies the property: By inspection no object in the base of S contains a leading 0.

Show that for each rule in the RECURSION for S, if the rule is applied to an object in S that satisfies the property, then the object defined by the rule also satisfies the property: The recursion for S consists of two rules denoted II(a) and II(b). Suppose that s and t are strings in S that do not contain leading 0's. When rule II(a) is applied to s, a 0 is added to the end of s not to the beginning and so the resulting string will not have a leading 0. When rule II(b) is applied to s and t the resulting string starts with s which does not contain a leading 0. Thus when each rule in the recursion is applied to s or s and t, the result is also a string that does not contain a leading 0. □

Problem 9

Define a set S recursively as follows:

I. BASE: $1 \in S$, $3 \in S$, $5 \in S$, $7 \in S$, $9 \in S$

II. RECURSION: If $s \in S$ and $t \in S$ then

a. $st \in S$ d. $6s \in S$

b. $2s \in S$ e. $8s \in S$

c. $4s \in S$

III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

Use structural induction to prove that every string in S represents an odd integer.

Solution

Proof. Define a property for S that every string in S represents an odd integer.

Show that each object in the BASE for S satisfies the property: By inspection every object in the base of S is an odd integer.

Show that for each rule in the RECURSION for S, if the rule is applied to an object in S that satisfies the property, then the object defined by the rule also satisfies the property: The recursion for S consists of five rules denoted II(a), II(b), II(c), II(d), and II(e). Suppose that s and t are strings in S that are odd. When rule II(a) is applied to s and t, the resulting string will end in the digit that t ends with. Since t ends with an odd digit the resulting string will also be odd. When any other rule is applied to s, the resulting string will end with the digit that s ends with. Since s is odd, the last digit of s must end with an odd integer. Hence the last digit of the resulting string will be an odd integer and so the resulting string will be odd. Thus when each rule in the recursion is applied to s or s and t, the result is also a string that represents an odd integer. □

Problem 10

Define a set S recursively as follows:

I. BASE: $0 \in S$, $5 \in S$

II. RECURSION: If $s \in S$ and $t \in S$ then

   a. $s + t \in S$ b. $s - t \in S$

III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

Use structural induction to prove that every integer in S is divisible by 5.

Solution

Proof. Define a property for S that every integer in S is divisible by 5.

Show that each object in the BASE for S satisfies the property: 0 and 5 are the only objects in the base of S. It follows from the fact that $5 = 5 \cdot 1$ and $0 = 5 \cdot 0$ that every object in the base of S is divisible by 5.

Show that for each rule in the RECURSION for S, if the rule is applied to an object in S that satisfies the property, then the object defined by the rule also satisfies the property: The recursion for S consists of two rules denoted II(a) and II(b). Suppose that s and t are string in S that are divisible by 5. Then $s = 5q$ and $t = 5r$ for some integers q and r. When rule II(a) is applied to s and t the resulting integer is $s+t = 5q+5r = 5(q+r)$. It follows that the resulting integer is divisible by 5. When rule II(b) is applied to s and t, the resulting integer is $s - t = 5q - 5r = 5(q - r)$. It follows that the resulting integer is divisible by 5. Thus when each rule in the recursion is applied to s and t, the resulting integer is also divisible by 5. □

Problem 11

Define a set S recursively as follows:

I. BASE: $0 \in S$

II. RECURSION: If $s \in S$, then

   a. $s + 3 \in S$ b. $s - 3 \in S$

III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

Use structural induction to prove that every integer in S is divisible by 3.

Solution

Proof. Define a property for S that every integer in S is divisible by 3.

Show that each object in the BASE for S satisfies the property: 0 is the only object in the base of S. It follows from the fact that $0 = 3 \cdot 0$ that every object in the base of S is divisible by 3.

Show that for each rule in the RECURSION for S, if the rule is applied to an object in S that satisfies the property, then the object defined by the rule also satisfies the property: The recursion for S consists of two rules denoted II(a) and II(b). Suppose that s is a string in S that is divisible by 3. Then $s = 3t$ for some integer t. When rule II(a) is applied to s the resulting integer is $s + 3 = 3t + 3 = 3(t + 1)$. It follows that the resulting integer is divisible by 3. When rule II(b) is applied to s, the resulting integer is $s - 3 = 3t - 3 = 3(t - 1)$. It follows that the resulting integer is divisible by 3. Thus when each rule in the recursion is applied to s, the resulting integer is also divisible by 3. □

Problem 12

Is the string MU in the MIU-system? Use structural induction to prove your answer.

Solution

Conjecture. *The number of I's in any string in the MIU-system cannot be a multiple of 3.*

Proof. Let S be the set of all strings defined by the MIU-system in example 5.9.3. Now define a property for S that the number of I's in every string in S is not a multiple of 3.

Show that each object in the BASE for S satisfies the property: MI is the only object in the base of S. Since MI contains 1 I and since $3 \nmid 1$, the property is true for every object in the base.

Show that for each rule in the RECURSION for S, if the rule is applied to an object in S that satisfies the property, then the object defined by the rule also satisfies the property: The recursion for S consists of four rules denoted II(a), II(b), II(c), and II(d). Suppose that s is a string in S and let t be the number of I's in S for some integer t such that $3 \nmid t$. When rule II(a) is applied to s a U is added to the end of s but this will not affect the number of I's in the result and so it is still true that the number of I's in the result is not a multiple of 3. When rule II(b) is applied to s the number of I's in the result will be twice as many as in s. Hence the number of I's in the result will be $2t$. However since $t \neq 3q$ for some integer q it follows that t does not have a 3 in its prime factorization. It follows that since 2 is prime, $2t$ will also not have a 3 in its prime factorization. Hence by the unique factorization of the integers theorem $2t \neq 3q$ and so the number of I's in the result is not a multiple of 3. When rule II(c) is applied 3 III's are replaced with a U. However since $t \neq 3$ it follows that $t - 3 \neq 3q - 3 = 3(q-1)$. Hence the number of I's in the result will not be a multiple of 3. When rule II(d) is applied to s two U's are replaced by one U. This will not change the number of I's in the result compared to s and so the number of I's in the result will not be a multiple of 3. □

Since the conjecture above showed that the number of I's in any string defined by the MIU-system cannot be a multiple of 3, and since the number of I's in MU is 0 which is a multiple of 3, MU cannot be in the MIU system.

Problem 13

Consider the set P of parenthesis structures defined in example 5.9.4. Give derivations showing that each of the following is in P.

a. ()(())　　　　b. (())(())

Solution

(a) (1) By I, () is in P.

(2) By (1) and II(a), (()) is in P.

(3) By (1) and (2) and II(b), ()(()) is in P.

(b) (1) By I, () is in P.

(2) By (1) and II(a), $(())$ is in P.

(3) By (2) and II(b), $(())(())$ is in P.

Problem 14

Determine whether either of the following parenthesis structures is in the set P defined in example 5.9.4. Use structural induction to prove your answers.

a. $()(()$ b. $((()()))(()$

Solution

(a) **Conjecture.** *The number of left parenthesis is the same as the number of right parenthesis in all grammatical configurations of parenthesis defined in example 5.9.4.*

Proof. Let P be the set of all grammatical configurations of parenthesis defined in example 5.9.4. Now define a property that the number of left and right parenthesis is equal.

Show that each object in the BASE for P satisfies the property: The only object in the base for P is $()$, which has one left and one right parenthesis, so it has an equal number of left and right parenthesis.

Show that for each rule in the RECURSION for P, if the rule is applied to an object in P that satisfies the property, then the object defined by the rule also satisfies the property: The recursion for P consists of two rules denoted II(a) and II(b). Suppose E and F are parenthesis configurations in P that have an equal number of left and right parenthesis. Say E has m left and right parenthesis and F has n left and right parenthesis. When rule II(a) is applied to E, the result is (E), so both the number of left parenthesis and the number of right parenthesis are increased by one. Since these numbers were equal to begin with they remain equal when each is increased by one. When rule II(b) is applied to E and F the result is EF, which has the same number of left and right parenthesis, namely $m + n$ left and right parenthesis. Thus when each rule of the recursion is applied to a configuration of parenthesis in P with an equal number of left and right parenthesis, the result is a configuration with an equal number of left and right parenthesis. □

Since the parenthesis configuration $()(()$ has 3 left and 2 right parenthesis it does not belong in P.

(b) **Conjecture.** *In all grammatical configurations of parenthesis defined in example 5.9.4, if a configuration can be written as AB where A and B are possibly-empty sub-configurations of parenthesis, then A always at least as many left parenthesis as right parenthesis.*

Proof. Let P be the set of all grammatical configurations of parenthesis defined in example 5.9.4. Now define a property of P that if AB is in P where A and B are possibly-empty sub-configurations of parenthesis, then A always has at least as many left parenthesis as right parenthesis.

Show that each object in the BASE for P satisfies the property: The only object in the base for P is $()$. This means that A can only be ϵ, $($, or $()$ each of which clearly has at least as many left parenthesis as right parenthesis.

Show that for each rule in the RECURSION for P, if the rule is applied to an object in P that satisfies the property, then the object defined by the rule also satisfies the property: The recursion for P consists of two rules denoted II(a) and II(b). Suppose E and F are parenthesis configurations in P such that if E is written as AB and F is written as CD then A and C have at least as many left parenthesis as right parenthesis. When rule II(a) is applied to E the result will be $(E) = (AB)$. Now the resulting leftmost sub-configuration can only be $(A$ or (AB). Since A has at least as many left as right parenthesis (A will also since we simply added a left parenthesis. Since AB is in P we know from part(a) that is must contain an equal number of left and right parenthesis and so (AB) will contain an equal number of left and right parenthesis. When rule II(b) is applied to E and F the result will be $EF = ABCD$. Now the resulting leftmost sub-configuration can only be A or ABC. If it is A then we already know that is has at least as many left parenthesis as right parenthesis. It it is ABC then we apply the proposition from part (a) to AB to conclude that AB has the same number of left and right parenthesis. We know that C has at least as many left as right parenthesis and so when we add C to AB there will be at least as many left as right parenthesis. Thus when each rule in the recursion is applied to a configuration of parenthesis with every possible leftmost sub-configuration having at least as many left parenthesis as right parenthesis, the result is a parenthesis configuration that follows the same property. $\square$

Since the parenthesis configuration $(()()))(()$ can be divided into AB where $A = (()()))$ and $B = (()$ and since A has 3 left parenthesis and 4 right parenthesis, $(()()))(()$ does not belong in P.

Problem 15

Give a recursive definition for the set of all strings of 0's and 1's that have the same number of 0's as 1's.

Solution

I. BASE: $\epsilon \in S$

II. RECURSION: If $s \in S$, then

a. $01s \in S$ c. $s01 \in S$ e. $0s1 \in S$

b. $10s \in S$ d. $s10 \in S$ f. $1s0 \in S$

III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

Problem 16

Give a recursive definition for the set of all strings of 0's and 1's for which all the 0's precede all the 1's.

Solution

I. BASE: $\epsilon \in S$

II. RECURSION: If $s \in S$, then

a. $0s \in S$ b. $s1 \in S$

III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

Problem 17

Give a recursive definition for the set of all strings of a's and b's that contain an odd number of a's.

Solution

I. BASE: $a \in S$

II. RECURSION: If $s \in S$, then

a. $aas \in S$ d. $bs \in S$

b. $asa \in S$ e. $sb \in S$

c. $saa \in S$

III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

Problem 18

Give a recursive definition for the set of all strings of a's and b's that contain exactly one a.

Solution

I. BASE: $a \in S$

II. RECURSION: If $s \in S$, then

   a. $bs \in S$ b. $sb \in S$

III. RESTRICTION: Nothing is in S other than objects defined in I and II above.

Problem 19

Use the definition of McCarthy's 91 function in example 5.9.6 to show the following:

a. $M(86) = M(91)$ b. $M(91) = 91$

Solution

(a) By repeated use of the definition of M,

$$\begin{aligned} M(86) &= M(M(97)) \\ &= M(M(M(108))) \\ &= M(M(98)) \\ &= M(M(M(109))) \\ &= M(M(99)) \\ &= M(91) && \text{by example 5.9.6} \end{aligned}$$

(b) By repeated use of the definition of M,

$$\begin{aligned} M(91) &= M(M(102)) \\ &= M(92) \\ &= M(M(103)) \\ &= M(93) \\ &= M(M(104)) \\ &= M(94) \\ &= M(M(105)) \\ &= M(95) \\ &= M(M(106)) \\ &= M(96) \\ &= M(M(107)) \\ &= M(97) \\ &= 91 && \text{by problem 19(a)} \end{aligned}$$

Problem 20

Prove that McCarthy's 91 function equals 91 for all integers less than or equal to 101.

Solution

Proof. Let the property $P(n)$ be that for all integers $n \leq 101$,

$$M(n) = 91 \qquad \leftarrow P(n)$$

Show that $P(n)$ is true for $91 \leq n \leq 101$: Problem 19 together with example 5.9.6 referenced therein show that $P(91), P(92), P(93), ..., P(101)$ are all true.

Show that for all integers $k \leq 91$, $P(i)$ is true for all integers i from k through 101 $\implies P(k-1)$: Let k be any integer with $k \leq 91$ and suppose that

$$M(i) = 91 \qquad \leftarrow \text{inductive hypothesis}$$

We must show that this implies that

$$M(k-1) = 91 \qquad \leftarrow P(k-1)$$

But the left-hand side of $P(k-1)$ is

$$\begin{aligned} M(k-1) &= M(M(k-1+11)) && \text{by definition of McCarthy's 91 function} \\ &= M(M(k+10)) \\ &= M(91) && \text{by inductive hypothesis since } k+10 \leq 101 \\ &= 91 \end{aligned}$$

which is the right-hand side of $P(k-1)$. □

Problem 21

Use the definition of the Ackermann function in example 5.9.7 to compute the following:

a. $A(1,1)$ b. $A(2,1)$

Solution

(a) By repeated use of the definition of the Ackermann function,

$$\begin{aligned} A(1,1) &= A(0, A(1,0)) \\ &= A(0, A(0,1)) \\ &= A(0,2) \\ &= 3 \end{aligned}$$

(b) By repeated use of the definition of the Ackermann function,

$$\begin{aligned}
A(2,1) &= A(1, A(2,0)) \\
&= A(1, A(1,1)) \\
&= A(1,3) && \text{by part(a)} \\
&= A(0, A(1,2)) \\
&= A(0,4) && \text{by example 5.9.7} \\
&= 5
\end{aligned}$$

Problem 22

Use the definition of the Ackermann function to show the following:

a. $A(1,n) = n+2$, for all nonnegative integers n.

b. $A(2,n) = 3+2n$, for all nonnegative integers n.

c. $A(3,n) = 8 \cdot 2^n - 3$, for all nonnegative integers n.

Solution

(a) *Proof.* Let the property $P(n)$ be the equation

$$A(1,n) = n+2 \qquad \leftarrow P(n)$$

Show that $P(0)$ is true:

$$A(1,0) = A(0,1) = 2 \quad \text{and} \quad 0+2=2$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$A(1,k) = k+2 \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$A(1,k+1) = k+3 \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
A(1,k+1) &= A(0, A(1,k)) \\
&= A(1,k)+1 \\
&= k+2+1 && \text{by inductive hypothesis} \\
&= k+3
\end{aligned}$$

which is the right-hand side of $P(k+1)$. □

(b) *Proof.* Let the property $P(n)$ be the equation

$$A(2, n) = 3 + 2n \qquad \leftarrow P(n)$$

Show that $P(0)$ is true:

$$A(2, 0) = A(1, 1) = 3 \quad \text{and} \quad 3 + 2 \cdot 0 = 3$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$A(2, k) = 3 + 2k \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$A(2, k + 1) = 2k + 5 \qquad \leftarrow P(k + 1)$$

But the left-hand side of $P(k + 1)$ is

$$\begin{aligned} A(2, k + 1) &= A(1, A(2, k)) \\ &= A(1, 3 + 2k) && \text{by inductive hypothesis} \\ &= 3 + 2k + 2 && \text{by part(a)} \\ &= 2k + 5 \end{aligned}$$

which is the right-hand side of $P(k + 1)$. □

(c) *Proof.* Let the property $P(n)$ be the equation

$$A(3, n) = 8 \cdot 2^n - 3 \qquad \leftarrow P(n)$$

Show that $P(0)$ is true:

$$A(3, 0) = A(2, 1) = 5 \quad \text{and} \quad 8 \cdot 2^0 - 3 = 8 - 3 = 5$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$A(3, k) = 8 \cdot 2^k - 3 \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$A(3, k + 1) = 8 \cdot 2^{k+1} - 3 \qquad \leftarrow P(k + 1)$$

But the left-hand side of $P(k + 1)$ is

$$\begin{aligned} A(3, k + 1) &= A(2, A(3, k)) \\ &= A(2, 8 \cdot 2^k - 3) && \text{by inductive hypothesis} \\ &= 3 + 2(8 \cdot 2^k - 3) && \text{by part(b)} \\ &= 3 + 8 \cdot 2^{k+1} - 6 \\ &= 8 \cdot 2^{k+1} - 3 \end{aligned}$$

which is the right-hand side of $P(k + 1)$. □

Problem 23

Compute $T(2), T(3), T(4), T(5), T(6)$, and $T(7)$ for the "function" T defined after example 5.9.8

Solution

$$
\begin{aligned}
&T(2) = T(1) = 1\\
&T(3) = T(10) = T(5) = T(16) = T(8) = T(4) = T(2) = 1\\
&T(4) = T(2) = 1\\
&T(5) = T(16) = T(8) = T(4) = 1\\
&T(6) = T(3) = 1\\
&T(7) = T(22) = T(11) = T(34) = T(17) = T(52) = T(26)\\
&\qquad = T(13) = T(40) = T(20) = T(10) = T(5) = 1
\end{aligned}
$$

Problem 24

Student A tries to define a function $F : \mathbb{Z}^+ \to \mathbb{Z}$ by the rule

$$
F(n) = \begin{cases} 1 & \text{if } n \text{ is } 1 \\ F\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \\ 1 + F(5n-9) & \text{if } n \text{ is odd and } n > 1 \end{cases}
$$

for all integers $n \geq 1$. Student B claims that F is not well defined. Justify student B's claim.

Solution

Suppose that F is a function. Then by the definition of F,

$$
\begin{aligned}
F(1) &= 1\\
F(2) &= F(1) = 1\\
F(3) &= 1 + F(6) = 1 + F(3)
\end{aligned}
$$

But Subtracting $F(3)$ from both sides of $F(3)$ gives $0 = 1$, which is false. Since the supposition that F leads to a logically false statement, it follows that F is not a function.

Problem 25

Student A tries to define a function $F : \mathbb{Z}^+ \to \mathbb{Z}$ by the rule

$$
G(n) = \begin{cases} 1 & \text{if } n \text{ is } 1 \\ G\left(\frac{n}{2}\right) & \text{if } n \text{ is even} \\ 2 + G(3n-5) & \text{if } n \text{ is odd and } n > 1 \end{cases}
$$

for all integers $n \geq 1$. Student D claims that G is not well defined. Justify student D's claim.

Solution

Suppose that G is a function. Then by the definition of G,

$$
\begin{aligned}
&G(1) = 1 \\
&G(2) = G(1) = 1 \\
&G(3) = 2 + G(4) = 2 + G(2) = 2 + 1 = 3 \\
&G(4) = G(2) = 1 \\
&G(5) = 2 + G(10) = 2 + G(5)
\end{aligned}
$$

But subtracting $G(5)$ from both sides of $G(5)$ gives $0 = 2$, which is false. Since the supposition that G leads to a logically false statement, it follows that G is not a function.