Section 6.3

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For each of 1-4 find a counterexample to show that the statement is false. Assume all sets are subsets of a universal set U.

Problem 1 and Solution

For all sets A, B, and C, $(A \cap B) \cup C = A \cap (B \cup C)$.

Counterexample: Any sets A, B, and C where C contains elements that are not A will serve as a counterexample. For instance, let $A = \{1, 3\}, B = \{2, 3\}$ and $C = \{4\}$. Then $(A \cap B) \cup C = \{3\} \cup \{4\} = \{3, 4\}$ and $A \cap (B \cup C) = \{1, 3\} \cap \{2, 3, 4\} = \{3\}$. Since $\{3, 4\} \neq \{3\}$ it follows that $(A \cap B) \cup C \neq A \cap (B \cup C)$.

Problem 2 and Solution

For all sets A and B, $(A \cup B)^c = A^c \cup B^c$.

Counterexample: Any sets A and B where A or B have any distinct elements will serve as a counterexample. For instance, let $A = \{1\}$ and $B = \{2\}$. Then $(A \cup B)^c = U - \{1, 2\}$ and $A^c \cup B^c = (U - \{1\}) \cup (U - \{2\}) = U$. It follows by definition of U that $1 \in U$ and $2 \in U$. Hence $U \neq U - \{1, 2\}$ and so $(A \cup B)^c \neq A^c \cup B^c$.

Problem 3 and Solution

For all sets A, B, and C, if $A \not\subseteq B$ and $B \not\subseteq C$ then $A \not\subseteq C$.

Counterexample: Any sets A, B, and C where $A \subseteq C$ and B contains at least one element that is not in either A or C will serve as a counterexample. For instance, let $A = \{1\}, B = \{2\}$ and $C = \{1, 2\}$. Then $A \not\subseteq B$ and $B \not\subseteq C$ but $A \subseteq C$ as every element in A is in C.

Problem 4 and Solution

For all sets A, B, and C, if $B \cap C \subseteq A$ then $(A - B) \cap (A - C) = \emptyset$.

Counterexample: Any sets A, B, and C where A has at least one element which is not in B and not in C will serve as a counterexample. For instance, let $A = \{1\}, B = \emptyset$, and $C = \emptyset$. Then $B \cap C = \emptyset \subseteq A$ but $(A - B) \cap (A - C) = (A - \emptyset) \cap (A - \emptyset) = A \cap A = A = \{1\} \neq \emptyset$.

For each of 5 - 21 prove each statement that is true and find a counterexample for each statement that is false. Assume all sets are subsets of a universal set U.

Problem 5 and Solution

For all sets A, B, and C, $A - (B - C) = (A - B) - C$.

Counterexample: Any sets A, B, and C where A and C have at least one element in common that is not in B will serve as a counterexample. For instance, let $A = \{1, 2\}, B = \{3\}$ and $C = \{1\}$. Then $A - (B - C) = \{1, 2\} - \{3\} = \{1, 2\}$ but $(A - B) - C = \{1, 2\} - \{1\} = \{2\}$. It follows that since $\{1, 2\} \neq \{1\}$, $A - (B - C) \neq (A - B) - C$.

Problem 6 and Solution

For all sets A and B, $A \cap (A \cup B) = A$.

Proof. Let A and B be any sets.

(1) Proof that $\boldsymbol{A \cap (A \cup B) \subseteq A}$: Let $x \in A \cap (A \cup B)$. By definition of intersection, $x \in A$ and $x \in A \cup B$. In particular $x \in A$ and so, by definition of subset, $x \subseteq A$.

(2) Proof that $\boldsymbol{A \subseteq A \cap (A \cup B)}$: Let $x \in A$. By definition of union, $x \in A \cup B$. Now by definition of intersection, $x \in A \cap (A \cup B)$.

Conclusion: Since both set containment's have been proved, it follows by definition of set equality that $A \cap (A \cup B) = A$. □

Problem 7 and Solution

For all sets A, B, and C, $(A - B) \cap (C - B) = A - (B \cup C)$.

Counterexample: Any sets A, B, and C where A and C share a common element that is not also in B will serve as a counterexample. For instance, let $A = \{1, 2\}, B = \{3\}$ and $C = \{1\}$. Then $(A - B) \cap (C - B) = \{1, 2\} \cap \{1\} = \{1\}$ but $A - (B \cup C) = \{1, 2\} - \{1, 3\} = \{2\}$. Since $\{1\} \neq \{2\}$ it follows that $(A - B) \cap (C - B) \neq A - (B \cup C)$.

Problem 8 and Solution

For all sets A and B, if $A^c \subseteq B$ then $A \cup B = U$.

Proof. Let A and B be any sets.

(1) Proof that $\boldsymbol{A \cup B \subseteq U}$: Let $x \in A \cup B$. By definition of union, $x \in A$

or $x \in B$. By definition of the universal set, $A \subseteq U$ and $B \subseteq U$. Hence $x \in A \implies x \in U$ and $x \in B \implies x \in U$. In either case, $x \in U$ and so $A \cup B \subseteq U$ by definition of subset.

(2) Proof that $U \subseteq A \cup B$: Let $x \in U$. We must have that either $x \in A$ or $x \notin A$. In the case that $x \in A$ it follows, by definition of union, that $x \in A \cup B$. In the case that $x \notin A$ it follows, by definition of complement that $x \in A^c$. Now since $A^c \subseteq B$ it follows, by definition of subset, that $x \in B$. In either case, $x \in A \cup B$ and so $U \subseteq A \cup B$.

Conclusion: Since both set containment's have been proved, it follows by definition of set equality that $A \cup B = U$. □

Problem 9 and Solution

For all sets A, B, and C, if $A \subseteq C$ and $B \subseteq C$ then $A \cup B \subseteq C$.

Proof. Let A, B, and C be any sets and suppose that $A \subseteq C$ and $B \subseteq C$. Let $x \in A \cup B$. By definition of union $x \in A$ or $x \in B$. In the case that $x \in A$ we have that $x \in C$ since $A \subseteq C$. In the case that $x \in B$ we have that $x \in C$ since $B \subseteq C$. Thus in either case $x \in C$ and so $A \cup B \subseteq C$ by definition of subset. □

Problem 10 and Solution

For all sets A and B, if $A \subseteq B$ then $A \cap B^c = \emptyset$.

Proof. Let A and B be any sets and suppose that $A \cap B^c \neq \emptyset$. Then there exists some $x \in A \cap B^c$. By definition of intersection, $x \in A$ and $x \in B^c$. By definition of complement, $x \notin B$. However, since $A \subseteq B$ it follows that $x \in A \implies x \in B$. Since we have have that $x \in A$ we must therefore have that $x \in B$. But now we have that $x \in B$ and $x \notin B$ which is a contradiction. Hence the supposition that $A \cap B^c \neq \emptyset$ is false and so $A \cap B^c = \emptyset$. □

Problem 11 and Solution

For all sets A, B, and C, if $A \subseteq B$ then $A \cap (B \cap C)^c = \emptyset$.

Counterexample: Any sets A, B, and C where A and B have elements in common that are not also in C will serve as a counterexample. For instance, let $A = \{1, 2\}, B = \{1\}$ and $C = \{4\}$. Then $A \cap (B \cap C)^c = A \cap \emptyset^c = A \cap U = A = \{1, 2\}$. Since $\{1, 2\} \neq \emptyset$ it follows that $A \cap (B \cap C)^c \neq \emptyset$.

Problem 12 and Solution

For all sets A, B, and C, $A \cap (B - C) = (A \cap B) - (A \cap C)$.

Proof. let A, B, and C be any sets.

(1) Proof that $A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$: Let $x \in A \cap (B - C)$. By definition of intersection $x \in A$ and $x \in B - C$. By definition of set difference, $x \in B$ and $x \notin C$. By definition of intersection $x \in A \cap B$. By definition of set difference $x \in A - C$. Hence $x \notin A \cap C$. It follows that $x \in A \cap B$ and $x \notin A \cap C$. Finally by definition of set difference, $x \in (A \cap B) - (A \cap C)$. It follows by definition of subset that $A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$.

(2) Proof that $\boldsymbol{(A \cap B) - (A \cap C) \subseteq A \cap (B - C)}$: Let $x \in (A \cap B) - (A \cap C)$. By definition of set difference $x \in A \cap B$ and $x \notin A \cap C$. By definition of intersection, $x \in A$ and $x \in B$. By definition of complement, $x \in (A \cap C)^c$. By De Morgan's law for sets, $x \notin A$ or $x \notin B$. Since we have that $x \in A$ it must be that $x \notin C$. By definition of set difference, $x \in B - C$. By definition of intersection, $x \in A \cap (B - C)$. By definition of subset, $(A \cap B) - (A \cap C) \subseteq A \cap (B - C)$.

Conclusion: Since both set containment's have been proved, it follows by definition of set equality that $A \cap (B - C) = (A \cap B) - (A \cap C)$. □

Problem 13 and Solution

For all sets A, B, and C, $A \cup (B - C) = (A \cup B) - (A \cup C)$.

Counterexample: Any sets A, B, and C where A and B have any elements in common will serve as a counterexample. For instance, let $A = \{1\}, B = \{1, 2\}$ and $C = \{3\}$. Then $A \cup (B - C) = \{1, 2\} \cup \{1\} = \{1, 2\}$ but $(A \cup B) - (A \cup C) = \{1, 2\} - \{1, 3\} = \{2\}$. Since $\{1, 2\} \neq \{2\}$ it follows that $A \cup (B - C) \neq (A \cup B) - (A \cup C)$.

Problem 14 and Solution

For all sets, A, B, and C, if $A \cap C \subseteq B \cap C$ and $A \cup C \subseteq B \cup C$, then $A \subseteq B$.

Proof. Let A, B, and C be any sets and suppose that $x \in A$. Either $x \in C$ or $x \notin C$. In the case that $x \in C$ it follows that $x \in A \cap C$. Now by definition of subset, $x \in B \cap C$. By definition of intersection this means that $x \in B$ and $x \in C$. In particular $x \in B$. In the case that $x \notin C$ it follows that $x \in A \cup C$. Now by definition of subset this means that $x \in B \cup C$. By definition of union, $x \in B$ or $x \in C$. However, since $x \notin C$ it follows that $x \in B$. In either case $x \in B$ and so it follows that $A \subseteq B$. □

Problem 15 and Solution

For all sets, A, B, and C, if $A \cap C = B \cap C$ and $A \cup C = B \cup C$, then $A = B$.

Proof. Let A, B, and C be any sets.

(1) Proof that $\boldsymbol{A \subseteq B}$: Let $x \in A$. Either $x \in C$ or $x \notin C$. In the case that $x \in C$ it follows by definition of intersection that $x \in A \cap C$. Now, by definition of set equality, it follows that $x \in BC$. In particular $x \in B$. In the case that $x \notin C$ it follows by definition of union that $x \in A \cup C$. Now, by definition of set equality, it follows that $x \in B \cup C$. By definition of union it follows that $x \in B$

or $x \in C$. However, $x \notin C$ and so $x \in B$.

(2) Proof that $B \subseteq A$: Let $x \in B$. Either $x \in C$ or $x \notin C$. In the case that $x \in C$ it follows by definition of intersection that $x \in A \cap C$. Now, by definition of set equality, it follows that $x \in AC$. In particular $x \in A$. In the case that $x \notin C$ it follows by definition of union that $x \in B \cup C$. Now, by definition of set equality, it follows that $x \in A \cup C$. By definition of union it follows that $x \in A$ or $x \in C$. However, $x \notin C$ and so $x \in A$.

Conclusion: Since both set containment's have been proved, it follows by definition of set equality that $A = B$. □

Problem 16 and Solution

For all sets A and B, if $A \cap B = \emptyset$ then $A \times B = \emptyset$.

Counterexample: Let $A = \{1\}$ and $B = \{2\}$. Then $A \cap B = \emptyset$ but $A \times B = \{(1, 2)\}$. Since $\{(1, 2)\} \neq \emptyset$ it follows that $A \times B \neq \emptyset$.

Problem 17 and Solution

For all sets A and B, if $A \subseteq B$ then $\mathscr{P}(A) \subseteq \mathscr{P}(B)$.

Proof. Let A and B be any sets and suppose that $X \in \mathscr{P}(A)$. It follows by definition of power set that $X \subseteq A$. Since $A \subseteq B$, it follows from the transitive property of subsets that $X \subseteq B$. Now by definition of power set $X \in \mathscr{P}(B)$. Hence, by definition of subset, $\mathscr{P}(A) \subseteq \mathscr{P}(B)$. □

Problem 18 and Solution

For all sets A and B, $\mathscr{P}(A \cup B) \subseteq \mathscr{P}(A) \cup \mathscr{P}(B)$.

Counterexample: Let $A = \{1\}$ and $B = \{2\}$. Then $\mathscr{P}(A \cup B) = \mathscr{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}$ but $\mathscr{P}(A) \cup \mathscr{P}(B) = \{\emptyset, \{1\}\} \cup \{\emptyset, \{2\}\} = \{\emptyset, \{1\}, \{2\}\}$. It follows from the fact that $\{1, 2\} \in \mathscr{P}(A \cup B)$ but $\{1, 2\} \notin \mathscr{P}(A) \cup \mathscr{P}(B)$ that $\mathscr{P}(A \cup B) \nsubseteq \mathscr{P}(A) \cup \mathscr{P}(B)$.

Problem 19 and Solution

For all sets A and B, $\mathscr{P}(A) \cup \mathscr{P}(B) \subseteq \mathscr{P}(A \cup B)$.

Proof. Let A and B be any sets and suppose that $X \in \mathscr{P}(A) \cup \mathscr{P}(B)$. It follows by definition of union that $X \in \mathscr{P}(A)$ or $X \in \mathscr{P}(B)$. In the case that $X \in \mathscr{P}(A)$ we have, by definition of power set, that $X \subseteq A$. It follows by definition of union that $X \subseteq A \cup B$. Now by definition of power set $X \in \mathscr{P}(A \cup B)$. In the case that $X \in \mathscr{P}(B)$ we have, by definition of power set, that $X \subseteq B$. it follows by definition of union that $X \subseteq A \cup B$. Now by definition of power set $X \in \mathscr{P}(A \cup B)$. Hence, by definition of subset, $\mathscr{P}(A) \cup \mathscr{P}(B) \subseteq \mathscr{P}(A \cup B)$. □

Problem 20 and Solution

For all sets A and B, $\mathscr{P}(A \cap B) = \mathscr{P}(A) \cap \mathscr{P}(B)$.

Proof. Let A and B be any sets.

(1) Proof that $\mathscr{P}(A \cap B) \subseteq \mathscr{P}(A) \cap \mathscr{P}(B)$: Let $X \in \mathscr{P}(A \cap B)$. It follows by definition of power set that $X \subseteq A \cap B$. By definition of intersection this means that $X \subseteq A$ and $X \subseteq B$. It now follows by definition of power set that $X \in \mathscr{P}(A)$ and $X \in \mathscr{P}(B)$. By definition of intersection, $X \in \mathscr{P}(A) \cap \mathscr{P}(B)$.

(2) Proof that $\mathscr{P}(A) \cap \mathscr{P}(B) \subseteq \mathscr{P}(A \cap B)$: Let $X \in \mathscr{P}(A) \cap \mathscr{P}(B)$. By definition of intersection, $X \in \mathscr{P}(A)$ and $X \in \mathscr{P}(B)$. It follows by definition of power set that $X \subseteq A$ and $X \subseteq B$. Now by definition of intersection, $X \subseteq A \cap B$. Finally by definition of power set, $X \in \mathscr{P}(A \cap B)$.

Conclusion: Since both set containment's have been proved, it follows by definition of set equality that $\mathscr{P}(A \cap B) = \mathscr{P}(A) \cap \mathscr{P}(B)$. □

Problem 21 and Solution

For all sets A and B, $\mathscr{P}(A \times B) = \mathscr{P}(A) \times \mathscr{P}(B)$.

Counterexample: Let $A = \{1\}$ and let $B = \{2\}$. Then $\mathscr{P}(A \times B) = \mathscr{P}(\{(1,2)\}) = \{\emptyset, \{(1,2)\}\}$ but $\mathscr{P}(A) \times \mathscr{P}(B) = \{\emptyset, \{1\}\} \times \{\emptyset, \{2\}\} = \{(\emptyset, \emptyset), (\emptyset, \{2\}), (\{1\}, \emptyset), (\{1\}, \{2\})\}$.

Problem 22

Write a negation for each of the following statements. Indicate which is true, the statement or its negation. Justify your answers.

a. $\forall$ sets S, $\exists$ a set T such that $S \cap T = \emptyset$.

b. $\exists$ a set S such that $\forall$ sets T, $S \cup T = \emptyset$.

Solution

a. $\exists$ a set S such that $\forall$ sets T, $S \cap T \neq \emptyset$. The original statement is true and the negation is false. Let $S = \emptyset$ and then $S \cap T = \emptyset$.

b. $\forall$ sets S, $\exists$ a set T such that $S \cup T \neq \emptyset$. The negation is true and the original statement is false. Let T be any set such that $T \neq \emptyset$ and then $S \cup T \neq \emptyset$.

Problem 23

Let $S = \{a, b, c\}$ and for each integer $i = 0, 1, 2, 3$, let S_i be the set of all subsets of S that have i elements. List the elements in S_0, S_1, S_2, and S_3. Is $\{S_0, S_1, S_2, S_3\}$ a partition of $\mathscr{P}(S)$?

Solution

$S_0 = \{\emptyset\}$
$S_1 = \{\{a\}, \{b\}, \{c\}\}$
$S_1 = \{\{a, b\}, \{a, c\}, \{b, c\}\}$
$S_2 = \{\{a, b, c\}\}$

$\{S_0, S_1, S_2, S_3\}$ is a partition of $\mathscr{P}(S)$ because $\mathscr{P}(S) = S_0 \cup S_1 \cup S_2 \cup S_3$ and $S_i \cap S_j = \emptyset$ for all $i, j = 0, 1, 2, 3$ and $i \neq j$.

Problem 24

Let $S = \{a, b, c\}$ and let S_a be the set of all subsets of S that contain a, let S_b be the set of all subsets of S that contain b, let S_c be the set of all subsets of S that contain c, and let $S_\emptyset$ be the set whose only element is $\emptyset$. Is $\{S_a, S_b, S_c, S_\emptyset\}$ a partition of $\mathscr{P}(S)$?

Solution

$S_\emptyset = \{\emptyset\}$
$S_a = \{\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$
$S_b = \{\{b\}, \{a, b\}, \{b, c\}.\{a, b, c\}\}$
$S_b = \{\{c\}, \{a, c\}, \{b, c\}.\{a, b, c\}\}$

$\{S_a, S_b, S_c, S_\emptyset\}$ is not a partition of $\mathscr{P}(S)$ because $\{S_a, S_b, S_c, S_\emptyset\}$ is not mutually disjoint. For example, $S_a \cap S_b = \{\{a, b\}, \{a, b, c\}\}$.

Problem 25

Let $A = \{t, u, v, w\}$ and let S_1 be the set of all subsets of A that do not contain w and S_2 the set of all subsets of A that contain w.

a. Find S_1
b. Find S_2
c. Are S_1 and S_2 disjoint?
d. Compare the sizes of S_1 and S_2.
e. How many elements are in S_1 and S_2?
f. What is the relation between $S_1 \cup S_2$ and $\mathscr{P}(A)$?

Solution

a. $S_1 = \{\emptyset, \{t\}, \{u\}, \{v\}, \{t, u\}, \{t, v\}, \{u, v\}, \{t, u, v\}$
b. $S_2 = \{\{w\}, \{t, w\}, \{u, w\}, \{v, w\}, \{t, u, w\}, \{t, v, w\}, \{u, v, w\}, \{t, u, v, w\}$
c. S_1 and S_2 are disjoint because $S_1 \cap S_2 = \emptyset$.
d. S_1 and S_2 have the same size.
e. There are 16 elements in $S_1 \cup S_2$.
f. $S_1 \cup S_2 = \mathscr{P}(A)$.

Problem 26

The following problem, which was devised by Ginger Bolton, appeared in the January 1989 issue of the *College Mathematics Journal* (Vol. 20, No. 1, p.68): Given a positive integer $n \geq 2$, let S be the set of all nonempty subsets of $\{2, 3, ..., n\}$. For each $S_i \in S$, let P_i be the product of the elements of S_i. Prove or disprove that

$$\sum_{i=1}^{2^{n-1}-1} P_i = \frac{(n+1)!}{2} - 1$$

Solution

Proof. Let the property $P(n)$ be the equation

$$\sum_{i=1}^{2^{n-1}-1} P_i = \frac{(n+1)!}{2} - 1 \qquad \leftarrow P(n)$$

Show that $P(2)$ is true: Let $n = 2$. Then $S = \{\{2\}\}$ and so it follows that

$$\sum_{i=1}^{1} P_i = 2 \quad \text{and} \quad \frac{(2+1)!}{2} - 1 = \frac{6}{2} - 1 = 3 - 1 = 2$$

Show that for all integers $k \geq 2$, $P(k) \implies P(k+1)$: Let k be any integer such that $k \geq 2$ and suppose that

$$\sum_{i=1}^{2^{k-1}-1} P_i = \frac{(k+1)!}{2} - 1 \qquad \leftarrow P(k) \text{ IH}$$

We must show that this implies that

$$\sum_{i=1}^{2^{k}-1} P_i = \frac{(k+2)!}{2} - 1 \qquad \leftarrow P(k+1)$$

Consider the set of all nonempty subsets of the set $\{2, ..., k+1\}$. Any subset of $\{2, ..., k+1\}$ will either contain $k+1$ or will not contain $k+1$. It follows that

$$\begin{bmatrix} \text{the sum of all products} \\ \text{of elements of nonempty} \\ \text{subsets of } \{2,...,k+1\} \end{bmatrix} = \begin{bmatrix} \text{the sum of all products} \\ \text{of elements of nonempty} \\ \text{subsets of } \{2,...,k+1\} \\ \text{that do not contain } k+1 \end{bmatrix} + \begin{bmatrix} \text{the sum of all products} \\ \text{of elements of nonempty} \\ \text{subsets of } \{2,...,k+1\} \\ \text{that contain } k+1 \end{bmatrix}$$

But any subset of $\{2, ..., k+1\}$ that does not contain $k+1$ is a subset of $\{2, ..., k\}$. And any subset of $\{2, ..., k+1\}$ that does contain $k+1$ is the union of some subset of $\{2, ..., k+1\}$ and $\{k+1\}$. We must be careful to add $k+1$ to our result as the inductive hypothesis result does not include the empty set and so a valid subset that contains $k+1$ namely $\{k+1\}$ will not be included if we simply multiply the sum of products of elements in nonempty subsets by $k+1$. Hence we have that

$$\begin{aligned}
\sum_{i=1}^{2^k-1} P_i &= \frac{(k+1)!}{2} - 1 + (k+1)\left(\frac{(k+1)!}{2} - 1\right) + k + 1 && \text{by inductive hypothesis} \\
&= \frac{(k+1)!}{2} - 1 + \frac{(k+1)(k+1)!}{2} - (k+1) + k + 1 \\
&= \frac{(k+1)!}{2} + \frac{(k+1)(k+1)!}{2} - 1 \\
&= \frac{(k+1)! + (k+1)(k+1)!}{2} - 1 \\
&= \frac{(k+1)!(1+(k+1))}{2} - 1 \\
&= \frac{(k+1)!(k+2)}{2} - 1 \\
&= \frac{(k+2)!}{2} - 1
\end{aligned}$$

which is the right-hand side of $P(k+1)$. □

In 27 and 28 supply a reason for each step in the derivation.

Problem 27 and Solution

For all sets A, B, and C, $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

Proof. Suppose A, B, and C are any sets. Then

$$\begin{aligned}
(A \cup B) \cap C &= C \cap (A \cup B) && \text{by commutative law for } \cap \\
&= (C \cap A) \cup (C \cap B) && \text{by distributive law} \\
&= (A \cap C) \cup (B \cap C) && \text{by commutative law for } \cap
\end{aligned}$$

□

Problem 28 and Solution

For all sets A, B, and C, $(A \cup B) - (C - A) = A \cup (B - C)$.

Proof. Suppose A, B, and C are any sets. Then

$$\begin{aligned}
(A \cup B) - (C - A) &= (A \cup B) \cap (C - A)^c && \text{by set difference law} \\
&= (A \cup B) \cap (C \cap A^c)^c && \text{by set difference law} \\
&= (A \cup B) \cap (A^c \cap C)^c && \text{by commutative law for } \cap \\
&= (A \cup B) \cap ((A^c)^c \cup C^c) && \text{by De Morgan's law} \\
&= (A \cup B) \cap (A \cup C^c) && \text{by double complement law} \\
&= A \cup (B \cap C^c) && \text{by distributive law} \\
&= A \cup (B - C) && \text{by set difference law}
\end{aligned}$$

□

Problem 29

Some steps are missing from the following proof that for all sets $(A \cup B) - C = (A - C) \cup (B - C)$. Indicate what they are, and then write the proof correctly.

Proof. Let A, B, and C be any sets. Then

$$\begin{aligned}
(A \cup B) - C &= (A \cup B) \cap C^c && \text{by the set difference law} \\
&= (A \cap C^c) \cup (B \cap C^c) && \text{by the distributive law} \\
&= (A - C) \cup (B - C) && \text{by the set difference law}
\end{aligned}$$
$\square$

The proof did not use the identities listed in theorem 6.2.2 exactly. The following proof corrects the errors.

Proof. Let A, B, and C be any sets. Then

$$\begin{aligned}
(A \cup B) - C &= (A \cup B) \cap C^c && \text{by the set difference law} \\
&= C^c \cap (A \cup B) && \text{by commutative law for } \cap \\
&= (C^c \cap A) \cup (C^c \cap B) && \text{by the distributive law} \\
&= (A \cap C^c) \cup (B \cap C^c) && \text{by commutative law for } \cap \\
&= (A - C) \cup (B - C) && \text{by the set difference law}
\end{aligned}$$
$\square$

In 30-40, construct an algebraic proof for the given statement. Cite a property from Theorem 6.2.2 for every step.

Problem 30 and Solution

For all sets A, B, and C, $(A \cap B) \cup C = (A \cup C) \cap (B \cap C)$.

Proof. Let A, B, and C be any sets. Then

$$\begin{aligned}
(A \cap B) \cup C &= C \cup (A \cap B) && \text{by commutative law for } \cup \\
&= (C \cup A) \cap (C \cup B) && \text{by distributive law} \\
&= (A \cup C) \cap (B \cap C) && \text{by commutative law for } \cup
\end{aligned}$$
$\square$

Problem 31 and Solution

For all sets A and B, $A \cup (B - A) = A \cup B$.

Proof. Let A and B be any sets. Then

$$\begin{aligned}
A \cup (B - A) &= A \cup (B \cap A^c) && \text{by set difference law} \\
&= (A \cup B) \cap (A \cup A^c) && \text{by distributive law} \\
&= (A \cup B) \cap U && \text{by complement law} \\
&= A \cup B && \text{by identity law}
\end{aligned}$$
$\square$

Problem 32 and Solution

For all sets A and B, $(A - B) \cup (A \cap B) = A$.

Proof. Let A and B be any sets. Then

$$\begin{aligned}
(A - B) \cup (A \cap B) &= (A \cap B^c) \cup (A \cap B) && \text{by set difference law} \\
&= A \cap (B^c \cup B) && \text{by distributive law} \\
&= A \cap (B \cup B^c) && \text{by commutative law for } \cup \\
&= A \cap U && \text{by complement law} \\
&= A && \text{by identity law} \qquad \square
\end{aligned}$$

Problem 33 and Solution

For all sets A, and B, $(A - B) \cap (A \cap B) = \emptyset$,

Proof. Let A and B be any sets. Then

$$\begin{aligned}
(A - B) \cap (A \cap B) &= (A \cap B^c) \cap (A \cap B) && \text{by set difference law} \\
&= (A \cap B^c) \cap (B \cap A) && \text{by commutative law for } \cap \\
&= (A \cap (B^c \cap B)) \cap A && \text{by associative law} \\
&= (A \cap \emptyset) \cap A && \text{by complement law} \\
&= \emptyset \cap A && \text{by complement law} \\
&= \emptyset && \text{by complement law} \qquad \square
\end{aligned}$$

Problem 34 and Solution

For all sets A, B, and C, $(A - B) - C = A - (B \cup C)$.

Proof. Let A, B, and C, be any sets. Then

$$\begin{aligned}
(A - B) - C &= (A \cap B^c) - C && \text{by set difference law} \\
&= (A \cap B^c) \cap C^c && \text{by set difference law} \\
&= A \cap (B^c \cap C^c) && \text{by associative law} \\
&= A \cap ((B^c)^c \cup (C^c)^c)^c && \text{by De Morgan's law} \\
&= A \cap (B \cup C)^c && \text{by double complement law} \\
&= A - (B \cup C) && \text{by set difference law} \qquad \square
\end{aligned}$$

Problem 35

For all sets A and B, $A - (A - B) = A \cap B$.

Proof. Let A and B be any sets. Then

$$\begin{aligned}
A - (A - B) &= A \cap (A - B)^c && \text{by set difference law} \\
&= A \cap (A \cap B^c)^c && \text{by set difference law} \\
&= A \cap (A^c \cup (B^c)^c) && \text{by De Morgan's law} \\
&= A \cap (A^c \cup B) && \text{by double complement law} \\
&= (A \cap A^c) \cup (A \cap B) && \text{by distributive law} \\
&= \emptyset \cup (A \cap B) && \text{by complement law} \\
&= (A \cap B) \cup \emptyset && \text{by commutative law for } \cup \\
&= A \cap B && \text{by identity law}
\end{aligned}$$

□

Problem 36 and Solution

For all sets A and B, $((A^c \cup B^c) - A)^c = A$.

Proof. Let A and B be any sets. Then

$$\begin{aligned}
((A^c \cup B^c) - A)^c &= ((A^c \cup B^c) \cap A^c)^c && \text{by set difference law} \\
&= (A^c \cup B^c)^c \cup (A^c)^c && \text{by De Morgan's law} \\
&= ((A^c)^c \cap (B^c)^c) \cup (A^c)^c && \text{by De Morgan's law} \\
&= (A \cap B) \cup A && \text{by double complement law} \\
&= A \cup (A \cap B) && \text{by commutative law for } \cup \\
&= A && \text{by absorption law}
\end{aligned}$$

□

Problem 37 and Solution

For all sets A and B, $(B^c \cup (B^c - A))^c = B$.

Proof. Let A and B be any sets. Then

$$\begin{aligned}
(B^c \cup (B^c - A))^c &= (B^c \cup (B^c \cap A^c))^c && \text{by set difference law} \\
&= (B^c)^c \cap (B^c \cap A^c)^c && \text{by De Morgan's law} \\
&= (B^c)^c \cap ((B^c)^c \cup (A^c)^c) && \text{by De Morgan's law} \\
&= B \cap (B \cup A) && \text{by double complement law} \\
&= B && \text{by absorptive law}
\end{aligned}$$

□

Problem 38 and Solution

For all sets A and B, $A - (A \cap B) = A - B$.

Proof. Let A and B be any sets. Then

$$\begin{aligned}
A-(A\cap B) &= A\cap(A\cap B)^c && \text{by set difference law}\\
&= A\cap(A^c\cup B^c) && \text{by De Morgan's law}\\
&= (A\cap A^c)\cup(A\cap B^c) && \text{by distributive law}\\
&= \emptyset\cup(A\cap B^c) && \text{by complement law}\\
&= (A\cap B^c)\cup\emptyset && \text{by commutative law for } \cup\\
&= (A\cap B^c) && \text{by identity law}\\
&= A-B && \text{by set difference law} \qquad \square
\end{aligned}$$

Problem 39 and Solution

For all sets A and B, $(A-B)\cup(B-A)=(A\cup B)-(A\cap B)$.

Proof. Let A and B be any sets. Then

$$\begin{aligned}
&(A-B)\cup(B-A)\\
&\quad= (A\cap B^c)\cup(B\cap A^c) && \text{by set difference law}\\
&\quad= ((A\cap B^c)\cup B)\cap((A\cap B^c)\cup A^c) && \text{by distributive law}\\
&\quad= ((B\cup(A\cap B^c))\cap((A^c\cup(A\cap B^c)) && \text{by commutative law for } \cup\\
&\quad= ((B\cup A)\cap(B\cup B^c))\cap((A^c\cup A)\cap(A^c\cup B^c)) && \text{by distributive law}\\
&\quad= ((A\cup B)\cap(B\cup B^c))\cap((A\cup A^c)\cap(A^c\cup B^c)) && \text{by commutative law}\\
&\quad= ((A\cup B)\cap U)\cap(U\cap(A^c\cup B^c)) && \text{by complement law}\\
&\quad= ((A\cup B)\cap U)\cap((A^c\cup B^c)\cap U) && \text{by commutative law}\\
&\quad= (A\cup B)\cap(A^c\cup B^c) && \text{by identity law}\\
&\quad= (A\cup B)\cap((A\cap B)^c && \text{by De Morgan's law}\\
&\quad= (A\cup B)-(A\cap B) && \text{by set difference law} \qquad \square
\end{aligned}$$

Problem 40 and Solution

For all sets A, B, and C, $(A-B)-(B-C)=A-B$.

Proof. Let A, B, and C be any sets. Then

$$\begin{aligned}
(A-B)-(B-C) &= (A\cap B^c)\cap(B\cap C^c)^c && \text{by set difference law}\\
&= (A\cap B^c)\cap(B^c\cup(C^c)^c) && \text{by De Morgan's law}\\
&= (A\cap B^c)\cap(B^c\cup C) && \text{by double complement law}\\
&= ((A\cap B^c)\cap B^c)\cup((A\cap B^c)\cap C) && \text{by distributive law}\\
&= (A\cap(B^c\cap B^c))\cup((A\cap B^c)\cap C) && \text{by associative law}\\
&= (A\cap B^c)\cup((A\cap B^c)\cap C) && \text{by idempotent law}\\
&= (A\cap B^c) && \text{by absorptive law}\\
&= A-B) && \text{by set difference law} \qquad \square
\end{aligned}$$

In 41-43 simplify the given expression. Cite a property from theorem 6.2.2 for every step.

Problem 41 and Solution

$$
\begin{aligned}
A \cap ((B \cup A^c) \cap B^c) &= A \cap (B^c \cap (B \cup A^c)) && \text{by commutative law for } \cap \\
&= A \cap ((B^c \cap B) \cup (B^c \cap A^c)) && \text{by distributive law} \\
&= A \cap (\emptyset \cup (B^c \cap A^c)) && \text{by complement law} \\
&= A \cap (B^c \cap A^c) && \text{by identity law} \\
&= A \cap (A^c \cap B^c) && \text{by commutative law} \\
&= (A \cap A^c) \cap B^c && \text{by associative law} \\
&= \emptyset \cap B^c && \text{by complement law} \\
&= \emptyset && \text{by universal bound law}
\end{aligned}
$$

Problem 42 and Solution

$$
\begin{aligned}
&(A - (A \cap B)) \cap (B - (A \cap B)) \\
&\quad = (A \cap (A \cap B)^c) \cap (B \cap (A \cap B)^c) && \text{by set difference law} \\
&\quad = (A \cap (A^c \cup B^c)) \cap (B \cap (A^c \cup B^c)) && \text{by De Morgan's law} \\
&\quad = ((A \cap A^c) \cup (A \cap B^c)) \cap ((B \cap A^c) \cup (B \cap B^c)) && \text{by distributive law} \\
&\quad = (\emptyset \cup (A \cap B^c)) \cap ((B \cap A^c) \cup \emptyset) && \text{by complement law} \\
&\quad = (A \cap B^c) \cap (B \cap A^c) && \text{by identity law} \\
&\quad = (A \cap (B^c \cap B)) \cap A^c && \text{by associative law} \\
&\quad = (A \cap (B \cap B^c)) \cap A^c && \text{by commutative law for } \cap \\
&\quad = (A \cap \emptyset) \cap A^c && \text{by complement law} \\
&\quad = \emptyset \cap A^c && \text{by universal bound law} \\
&\quad = \emptyset && \text{by universal bound law}
\end{aligned}
$$

Problem 43 and Solution

$$
\begin{aligned}
&((A \cap (B \cup C)) \cap (A - B)) \cap (B \cap C^c) \\
&\quad = (((A \cap B) \cup (A \cap C)) \cap (A - B)) \cap (B \cap C^c) && \text{by distributive law} \\
&\quad = ((A - B) \cap ((A \cap B) \cup (A \cap C))) \cap (B \cap C^c) && \text{by commutative law for } \cap \\
&\quad = (((A - B) \cap (A \cap B)) \cup ((A - B) \cap (A \cap C))) \cap (B \cap C^c) && \text{by distributive law} \\
&\quad = (((A \cap B^c) \cap (A \cap B)) \cup ((A \cap B^c) \cap (A \cap C))) \cap (B \cap C^c) && \text{by set difference law} \\
&\quad = (((A \cap B^c) \cap (B \cap A)) \cup ((A \cap B^c) \cap (A \cap C))) \cap (B \cap C^c) && \text{by commutative law for } \cap \\
&\quad = (((A \cap (B^c \cap B)) \cap A)) \cup ((A \cap B^c) \cap (A \cap C))) \cap (B \cap C^c) && \text{by associative law} \\
&\quad = (((A \cap (B \cap B^c)) \cap A)) \cup ((A \cap B^c) \cap (A \cap C))) \cap (B \cap C^c) && \text{by commutative law for } \cap \\
&\quad = (((A \cap \emptyset) \cap A)) \cup ((A \cap B^c) \cap (A \cap C))) \cap (B \cap C^c) && \text{by complement law}
\end{aligned}
$$

$$
\begin{aligned}
&= ((\emptyset \cap A) \cup ((A \cap B^c) \cap (A \cap C))) \cap (B \cap C^c) && \text{by universal bound law} \\
&= ((A \cap \emptyset) \cup ((A \cap B^c) \cap (A \cap C))) \cap (B \cap C^c) && \text{by commutative law for } \cap \\
&= (\emptyset \cup ((A \cap B^c) \cap (A \cap C))) \cap (B \cap C^c) && \text{by universal bound law} \\
&= (((A \cap B^c) \cup \emptyset) \cap (A \cap C))) \cap (B \cap C^c) && \text{by commutative law for } \cup \\
&= ((A \cap B^c) \cap (A \cap C)) \cap (B \cap C^c) && \text{by identity law} \\
&= ((A \cap B^c) \cap (A \cap C)) \cap (C^c \cap B) && \text{by commutative law for } \cap \\
&= (A \cap B^c) \cap ((A \cap (C \cap C^c)) \cap B) && \text{by associative law} \\
&= (A \cap B^c) \cap ((A \cap \emptyset) \cap B) && \text{by complement law} \\
&= (A \cap B^c) \cap (\emptyset \cap B) && \text{by universal bound law} \\
&= (A \cap B^c) \cap (B \cap \emptyset) && \text{by commutative law for } \cap \\
&= (A \cap B) \cap \emptyset && \text{by identity law} \\
&= \emptyset && \text{by identity law}
\end{aligned}
$$

Problem 44

Consider the following set property: For all sets A and B, $A - B$ and B are disjoint.

a. Use an element argument to derive the property.

b. Use an algebraic argument to derive the property.

c. Comment on which method you found easier.

Solution

a. *Proof.* Let A and B be any sets and suppose that $x \in (A - B) \cap B$. It follows by definition of union that $x \in A - B$ and $x \in B$. By definition of set difference $x \in A$ and $x \notin B$. In particular $x \notin B$. But now we have that $x \in B$ and $x \notin B$ which is a contradiction. Hence the supposition that $x \in (A - B) \cap B$ is false and so $(A - B) \cap B = \emptyset$. □

b. *Proof.* Let A and B be any sets. Then

$$
\begin{aligned}
(A - B) \cap B &= (A \cap B^c) \cap B && \text{by set difference law} \\
&= A \cap (B^c \cap B) && \text{by associative law} \\
&= A \cap (B \cap B^c) && \text{by commutative law for } \mathit{cap} \\
&= A \cap \emptyset && \text{by complement law} \\
&= \emptyset && \text{by universal bound law}
\end{aligned}
$$

□

c. I found that the element argument was easier to derive because it required none of the tedious associative and commutative manipulations required by the algebraic argument.

Problem 45

Consider the following set property: For all sets A, B, and C, $(A-B)\cup(B-C) = (A \cup B) - (B \cap C)$.

a. Use an element argument to derive the property.

b. Use an algebraic argument to derive the property.

c. Comment on which method you found easier.

Solution

a. *Proof.* Let A, B, and C be any sets.

(1) Proof that $(A - B) \cup (B - C) \subseteq (A \cup B) - (B \cap C)$: Let $x \in (A - B) \cup (B - C)$. By definition of union, $x \in A - B$ or $x \in B - C$.

Case 1 ($x \in B - C$): By definition of set difference $x \in B$ and $x \notin C$. Since $x \in B$, by definition of union, $x \in A \cup B$. Since $x \notin C$, it follows by definition of intersection that $x \notin B \cap C$. Now by definition of set difference, $x \in (A \cup B) - (B \cap C)$.

Case 2 ($x \in A - B$): By definition of set difference $x \in A$ and $x \notin B$. Since $x \in A$, by definition of union, $x \in A \cup B$. Since $x \notin B$, it follows by definition of intersection that $x \notin B \cap C$. Now by definition of set difference, $x \in (A \cup B) - (B \cap C)$.

(2) Proof that $(A \cup B) - (B \cap C) \subseteq (A - B) \cup (B - C)$: Let $x \in (A \cup B) - (B \cap C)$. By definition of set difference $x \in A \cup B$ and $x \notin B \cap C$. By definition of union, $x \in A$ or $x \in B$. Since $x \notin B \cap C$ it is false to say that $x \in B$ and $x \in C$. By De Morgan's laws of logic, $x \notin B$ or $x \notin C$.

Case 1 ($x \in A$ and $x \notin B$): By definition of set difference, $x \in A - B$. Hence, by definition of union, $x \in (A - B) \cup (B - C)$.

Case 2 ($x \in A$ and $x \in B$): Since $x \in B$ it follows that $x \notin C$ as $x \notin B$ or $x \notin C$. Hence $x \in B$ and $x \notin C$. It follows by definition of set difference that $x \in B - C$. Hence, by definition of union, $x \in (A - B) \cup (B - C)$.

Case 3 ($x \notin A$ and $x \in B$): Since $x \in B$ it follows that $x \notin C$ as $x \notin B$ or $x \notin C$. Hence $x \in B$ and $x \notin C$. It follows by definition of set difference that $x \in B - C$. Hence, by definition of union, $x \in (A - B) \cup (B - C)$.

Conclusion: Since both set containment's have been proved, it follows by definition of set equality that $(A-B)\cup(B-C) = (A\cup B)-(B\cap C)$. $\square$

b. *Proof.*

$$
\begin{aligned}
&(A - B) \cup (B - C) \\
&\quad = (A \cap B^c) \cup (B \cap C^c) && \text{by set difference law} \\
&\quad = ((A \cap B^c) \cup B) \cap ((A \cap B^c) \cup C^c) && \text{by distributive law} \\
&\quad = (B \cup (A \cap B^c)) \cap (C^c \cup (A \cap B^c)) && \text{by commutative law for } \cup \\
&\quad = ((B \cup A) \cap (B \cup B^c)) \cap ((C^c \cup A) \cap (C^c \cup B^c)) && \text{by distributive law} \\
&\quad = ((B \cup A) \cap U) \cap ((C^c \cup A) \cap (C^c \cup B^c)) && \text{by complement law} \\
&\quad = (B \cup A) \cap ((C^c \cup A) \cap (C^c \cup B^c)) && \text{by identity law} \\
&\quad = ((B \cup A) \cap (C^c \cup A)) \cap (C^c \cup B^c) && \text{by associative law} \\
&\quad = ((A \cup B) \cap (A \cup C^c)) \cap (B^c \cup C^c) && \text{by commutative law} \\
&\quad = ((A \cup B) \cap ((A \cup \emptyset) \cup C^c)) \cap (B^c \cup C^c) && \text{by identity law} \\
&\quad = ((A \cup B) \cap ((A \cup (B \cap B^c)) \cup C^c)) \cap (B^c \cup C^c) && \text{by complement law} \\
&\quad = ((A \cup B) \cap (((A \cup B) \cap (A \cup B^c)) \cup C^c)) \cap (B^c \cup C^c) && \text{by distributive law} \\
&\quad = ((A \cup B) \cap (C^c \cup ((A \cup B) \cap (A \cup B^c)))) \cap (B^c \cup C^c) && \text{by commutative law} \\
&\quad = ((A \cup B) \cap ((C^c \cup A \cup B) \cap (C^c \cup A \cup B^c))) \cap (B^c \cup C^c) && \text{by distributive law} \\
&\quad = ((A \cup B) \cap ((A \cup B \cup C^c) \cap (C^c \cup A \cup B^c))) \cap (B^c \cup C^c) && \text{by commutative law} \\
&\quad = ((A \cup B) \cap (A \cup B \cup C^c) \cap (C^c \cup A \cup B^c)) \cap (B^c \cup C^c) && \text{by associative law} \\
&\quad = ((A \cup B) \cap (C^c \cup A \cup B^c)) \cap (B^c \cup C^c) && \text{by absorptive law} \\
&\quad = ((A \cup B) \cap (B^c \cup C^c \cup A)) \cap (B^c \cup C^c) && \text{by commutative law} \\
&\quad = (A \cup B) \cap (B^c \cup C^c \cup A) \cap (B^c \cup C^c) && \text{by associative law} \\
&\quad = (A \cup B) \cap (B^c \cup C^c) \cap (B^c \cup C^c \cup A) && \text{by commutative law} \\
&\quad = (A \cup B) \cap (B^c \cup C^c) && \text{by absorptive law} \\
&\quad = (A \cup B) \cap ((B^c \cup C^c)^c)^c && \text{by double complement law} \\
&\quad = (A \cup B) \cap ((B^c)^c \cap (C^c))^c && \text{by De Morgan's law} \\
&\quad = (A \cup B) \cap (B \cap C)^c && \text{by double complement law} \\
&\quad = (A \cup B) - (B \cap C) && \text{by set difference law}
\end{aligned}
$$

□

c. The element method was much easier as the algebraic method of proof required a very tedious number of associative an commutative operations which were extremely obvious.

Problem 46 and Solution

Let $A = \{1, 2, 3, 4\}, B = \{3, 4, 5, 6\}$, and $C = \{5, 6, 7, 8\}$. Find each of the following sets:

a. $A \triangle B = (A - B) \cup (B - A) = \{1, 2\} \cup \{5, 6\} = \{1, 2, 5, 6\}$

b. $B \triangle C = (B - C) \cup (C - B) = \{3, 4\} \cup \{7, 8\} = \{3, 4, 7, 8\}$

c. $A \triangle C = (A - C) \cup (C - A) = \{1,2,3,4\} \cup \{5,6,7,8\} = \{1,2,3,4,5,6,7,8\}$

d. $$\begin{aligned}(A \triangle B) \triangle C &= ((A - B) \cup (B - A)) \triangle C \\ &= (((A - B) \cup (B - A)) - C) \cup (C - ((A - B) \cup (B - A))) \\ &= (\{1,2,5,6\} - \{5,6,7,8\}) \cup (\{5,6,7,8\} - \{1,2,5,6\}) \\ &= \{1,2\} \cup \{7,8\} \\ &= \{1,2,7,8\}\end{aligned}$$

Refer to the definition of symmetric difference given above. Prove each of 47-52, assuming that A, B, and C are all subsets of a universal set U.

Problem 47 and Solution

$A \triangle B = B \triangle A$.

Proof. Let A and B be any subsets of a universal set. Then

$$\begin{aligned}A \triangle B &= (A - B) \cup (B - A) && \text{by definition of } \triangle \\ &= (B - A) \cup (A - B) && \text{by commutative law} \\ &= B \triangle A && \text{by definition of } \triangle\end{aligned}$$ □

Problem 48 and Solution

$A \triangle \emptyset = A$.

Proof. Let A be any subset of a universal set. Then

$$\begin{aligned}A \triangle \emptyset &= (A - \emptyset) \cup (\emptyset - A) && \text{by definition of } \triangle \\ &= (A \cap \emptyset^c) \cup (\emptyset \cap A^c) && \text{by set difference law} \\ &= (A \cap U) \cup (\emptyset \cap A^c) && \text{by completeness of } \emptyset \\ &= A \cup (\emptyset \cap A^c) && \text{by identity law} \\ &= A \cup \emptyset && \text{by universal bound law} \\ &= A && \text{by identity law}\end{aligned}$$ □

Problem 49 and Solution

$A \triangle A^c = U$.

Proof. Let A be any subset of a universal set. Then

$$\begin{aligned}A \triangle A^c &= (A - A^c) \cup (A^c - A) && \text{by definition of } \triangle \\ &= (A \cap (A^c)^c) \cup (A^c \cap A^c) && \text{by set difference law} \\ &= (A \cap A) \cup (A^c \cap A^c) && \text{by double complement law} \\ &= A \cup A^c && \text{by idempotent law} \\ &= U && \text{by complement law}\end{aligned}$$ □

Problem 50 and Solution

$A \triangle A = \emptyset$.

Proof. Let A be any subset of a universal set. Then

$$\begin{aligned} A \triangle A &= (A - A) \cup (A - A) && \text{by definition of } \triangle \\ &= (A \cap A^c) \cup (A \cap A^c) && \text{by set difference law} \\ &= \emptyset \cup \emptyset && \text{by complement law} \\ &= \emptyset && \text{by idempotent law} \end{aligned}$$

□

Problem 51 and Solution

If $A \triangle C = B \triangle C$, then $A = B$.

Lemma 1. *For any sets A and B and for any element x,*

$$x \in A \triangle B \iff (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)$$

Proof. Let A and B be any sets.

Proof that $x \in A \triangle B \implies (x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A)$: Let $x \in A \triangle B$. By definition of $\triangle$, $x \in (A - B) \cup (B - A)$. By definition of union, $x \in (A - B)$ or $X \in (B - A)$ by definition of set difference, $x \in A$ and $x \notin B$ or $x \in B$ and $x \notin A$.

Proof that $(x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A) \implies x \in A \triangle B$: Let $x \in A$ and $x \notin B$. By definition of complement, $x \in B^c$. By definition of intersection, $x \in A \cap B^c$. By definition of set difference, $x \in A - B$. By definition of union, $x \in (A - B) \cup (B - A)$. By definition of $\triangle$, $x \in A \triangle B$. Now let $x \in B$ and $x \notin A$. By definition of set complement $x \in A^c$. By definition of intersection, $x \in B \cap B^c$. By definition of set difference, $x \in B - A$. By definition of union, $x \in (A - B) \cup (B - A)$. By definition of $\triangle$, $x \in A \triangle B$. □

Lemma 2. *For any sets A and B and for any element x,*

$$x \notin A \triangle B \iff (x \in A \text{ and } x \in B) \text{ or } (x \notin A \text{ and } x \notin B)$$

Proof. Let A and B be any sets.

Proof that $x \notin A \triangle B \implies (x \in A$ and $x \in B)$ or $(x \notin A$ and $x \notin B)$: Let $x \notin A \triangle B$. By definition of $\triangle$, $x \notin (A - B) \cup (B - A)$. By definition of complement, $x \in ((A - B) \cup (B - A))$. By De Morgan's law for sets, $x \in (A - B)^c \cap (B - A)^c$. By set difference law, $x \in (A \cap B^c)^c \cap (B \cap A^c)^c$. By De Morgan's law for sets, $x \in (A^c \cup B) \cap (B^c \cup A)$. By definition of intersection, $x \in A^c \cup B$ and $x \in B^c \cup A$. By definition of union, $x \in A^c$ or $x \in B$ and $x \in B^c$ or $x \in A$. By definition of complement, $x \notin A$ or $x \in B$ and $x \notin B$ or $x \in A$. But the only combination that does not lead to a contradiction is $x \in A$ and $x \in B$ or $x \notin A$ and $x \notin B$.

Proof that ($x \in A$ and $x \in B$) or ($x \notin A$ and $x \notin B$) $\implies$ $x \notin A \triangle B$: Let $x \in A$ and $x \in B$. By definition of set difference, $x \notin A - B$ and $x \notin B - A$. Hence, by definition of $\triangle$, $x \notin A \triangle B$. Let $x \notin A$ and $x \notin B$. By definition of set difference, $x \notin A - B$ and $x \notin B - A$. Hence, by definition of $\triangle$, $x \notin A \triangle B$. □

Proposition. *For any sets A, B, and C, if $A \triangle C = B \triangle C$, then $A = B$.*

Proof. Let A, B, and C be any subsets of a universal set.

(1) Proof that $A \subseteq B$: Let $x \in A$. Then we have that either $x \in A \triangle C$ or $x \notin A \triangle C$.

Case 1 ($x \in A \triangle C$): By lemma 1, $x \in A$ and $x \notin C$ or $x \in C$ and $x \notin A$. But $x \in A$ and so it must be that $x \notin C$. Since $A \triangle C = B \triangle C$, by definition of set equality, $x \in B \triangle C$. Now by lemma 1, $x \in B$ and $x \notin C$ or $x \in C$ and $x \notin B$. But $x \notin C$ and so it must be that $x \in B$.

Case 2 ($x \notin A \triangle C$): By lemma 2, $x \in A$ and $x \in C$ or $x \notin A$ and $xinC$. But $x \in A$ and so it must be that $x \in C$. Since $A \triangle C = B \triangle C$, by definition of set equality, $x \notin B \triangle C$. Now by lemma 2, $x \in B$ and $x \in C$ or $x \notin B$ and $x \notin C$. But $x \in C$ and so it must be that $x \in B$.

(2) Proof that $B \subseteq A$: Let $x \in B$. Then we have that either $x \in B \triangle C$ or $x \notin B \triangle C$.

Case 1 ($x \in B \triangle C$): By lemma 1, $x \in B$ and $x \notin C$ or $x \in C$ and $x \notin B$. But $x \in B$ and so it must be the case that $x \notin C$. Since $A \triangle C = B \triangle C$, by definition of set equality, $x \in A \triangle C$. Now by lemma 1, $x \in A$ and $x \notin C$ or $x \in C$ and $x \notin A$. But $x \notin C$ and so it must be the case that $x \in A$.

Case 2 ($x \notin B \triangle C$): By lemma 2, $x \in B$ and $x \in C$ or $x \notin B$ and $x \notin C$. But $x \in B$ and so it must be the case that $x \in C$. Since $A \triangle C = B \triangle C$, by definition of set equality, $x \notin A \triangle C$. Now by lemma 2, $x \in A$ and $x \in C$ or $x \notin A$ and $x \notin C$. But $x \in C$ and so it must be the case that $x \in A$.

Conclusion: Since both set containment's have been proved, it follows by definition of set equality that $A = B$. □

Problem 52 and Solution

$(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

Proof. Let A, B, and C be any sets.

(1) Proof that $(A \triangle B) \triangle C \subseteq A \triangle (B \triangle C)$: Let $x \in (A \triangle B) \triangle C$. By definition of $\triangle$, $x \in A \triangle B$ and $x \notin C$ or $x \in C$ and $x \notin A \triangle B$.

Case 1 ($x \in A \triangle B$ and $x \notin C$): By definition of $\triangle$, $x \in A$ and $x \notin B$ or $x \in B$ and $x \notin A$.

Case 1a ($\boldsymbol{x \in A}$ and $\boldsymbol{x \notin B}$): In this case $x \notin C$, $x \in A$ and $x \notin B$ and so by lemma 2 of problem 51 $x \in A$ and $x \notin B \triangle C$. Now by lemma 1 of problem 51, $x \in A \triangle (B \triangle C)$.

Case 1b ($\boldsymbol{x \in B}$ and $\boldsymbol{x \notin A}$): In this case $x \notin C$, $x \in B$ and $x \notin A$ and so by lemma 1 of problem 51 $x \notin A$ and $x \in B \triangle C$. Now by lemma 1 of problem 51, $x \in A \triangle (B \triangle C)$.

Case 2 ($\boldsymbol{x \in C}$ and $\boldsymbol{x \notin A \triangle B}$): By definition of $\triangle$, $x \in A$ and $x \in B$ or $x \notin A$ and $x \notin B$.

Case 2a ($\boldsymbol{x \in A}$ and $\boldsymbol{x \in B}$): In this case $x \in C$, $x \in A$, and $x \in B$ and so by lemma 2 of problem 51, $x \in A$ and $x \notin B \triangle C$. Now by lemma 1 of problem 51, $x \in A \triangle (B \triangle C)$.

Case 2b ($\boldsymbol{x \notin A}$ and $\boldsymbol{x \notin B}$): In this case $x \in C$, $x \notin A$, and $x \notin B$ and so by lemma 1 of problem 51, $x \in B \triangle C$ and $x \notin A$. Now by lemma 1 of problem 51, $x \in A \triangle (B \triangle C)$.

(2) Proof that $\boldsymbol{A \triangle (B \triangle C) \subseteq (A \triangle B) \triangle C}$: Let $x \in A \triangle (B \triangle C)$. By definition of $\triangle$, $x \in A$ and $x \notin B \triangle C$ or $x \notin A$ and $x \in B \triangle C$.

Case 1 ($\boldsymbol{x \in A}$ and $\boldsymbol{x \notin B \triangle C}$): By definition of $\triangle$, $x \in B$ and $x \in C$ or $x \notin B$ and $x \notin C$.

Case 1a ($\boldsymbol{x \in B}$ and $\boldsymbol{x \in C}$): In this case $x \in A$, $x \in B$ and $x \in C$ and so by lemma 2 of problem 51, $x \notin A \triangle B$ and $x \in C$. Now by lemma 1 of problem 51, $x \in (A \triangle B) \triangle C$.

Case 2b ($\boldsymbol{x \notin B}$ and $\boldsymbol{x \notin C}$): In this case $x \in A$, $x \notin B$, and $x \notin C$ and so by lemma 1 or problem 51, $x \in A \triangle B$ and $x \notin C$. Now by lemma 1 or problem 51, $x \in (A \triangle B) \triangle C$.

Case 2 ($\boldsymbol{x \in B \triangle C}$ and $\boldsymbol{x \notin A}$): By definition of $\triangle$, $x \in B$ and $x \notin C$ or $x \notin B$ and $x \in C$.

Case 2a ($\boldsymbol{x \in B}$ and $\boldsymbol{x \notin C}$): In this case $x \notin A$, $x \in B$, and $x \notin C$ and so by lemma 1 of problem 51, $x \in A \triangle B$ and $x \notin C$. Now by lemma 1 of problem 51, $x \in (A \triangle B) \triangle C$.

Case 2b ($\boldsymbol{x \notin B}$ and $\boldsymbol{x \in C}$): In this case $x \notin A$, $x \notin B$, and $x \in C$ and so by lemma 2 of problem 51, $x \notin A \triangle B$ and $x \in C$. It follows by lemma 1 of problem 51 that $x \in (A \triangle B) \triangle C$.

Conclusion: Since both set containment's have been proved, it follows by definition of set equality that $(A \triangle B) \triangle C = A \triangle (B \triangle C)$. $\square$

Problem 53

Derive the set identity $A \cup (A \cap B) = A$ from the properties listed in theorem 6.2.2(1)-(9). Start by showing that for all subsets B of a universal set U, $U \cup B = U$. Then intersect both sides with A and deduce the identity.

Solution

Let A and B be any subsets of a universal set U. Then

$$
\begin{aligned}
B \cup U &= U && \text{by universal bound law} \\
U \cup B &= U && \text{by commutative law} \\
A \cap (U \cup B) &= A \cap U && \\
(A \cap U) \cup (A \cap B) &= A \cap U && \text{by distributive law} \\
A \cup (A \cap B) &= A && \text{by identity law}
\end{aligned}
$$

Problem 54

Derive the set identity $A \cap (A \cup B) = A$ from the properties listed in theorem 6.2.2(1)-(9). Start by showing that for all subsets B of a universal set U, $\emptyset = \emptyset \cap B$. Then take the union of both sides with A and deduce the identity.

Solution

Let B be any subset of a universal set U. Then

$$
\begin{aligned}
B \cap \emptyset &= \emptyset && \text{by universal bound law} \\
\emptyset \cap B &= \emptyset && \text{by commutative law} \\
A \cup (\emptyset \cap B) &= A \cup \emptyset && \\
A \cup (\emptyset \cap B) &= A && \text{by identity law} \\
(A \cup \emptyset) \cap (A \cup B) &= A && \text{by distributive law} \\
A \cap (A \cup B) &= A && \text{by identity law}
\end{aligned}
$$