Section 4.1

Sterling Jeppson

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Problem 1

Assume that k is a particular integer.

(i) Is -17 an odd integer?

(ii) Is 0 an even integer?

(iii) Is $2k - 1$ odd?

Solution

(i) Yes. $-17 = 2 * -9 + 1$. It follows from definition of odd that -17 is odd.

(ii) Yes. $0 = 2 * 0$. It follows from definition of even that 0 is even.

(iii) Yes. $2k - 1 = 2(k - 1) + 1$. It follows from closure under subtraction that $k-1$ is an integer. Let $n = k-1$. Then n is an integer and $2k-1 = 2n+1$. It follows from definition of odd that $2k - 1$ is odd.

Problem 2

Assume that m and n are particular integers

(i) Is $6m + 8n$ even?

(ii) Is $10mn + 7$ odd?

(iii) If $m > n > 0$, is $m^2 - n^2$ composite?

Solution

(i) Yes. $6m + 8n = 2(3m + 4n)$ It follows from closure under addition and multiplication that $(3m + 4n)$ is an integer. Let $p = (3m + 4n)$. Then p is an integer and $6m + 8n = 2p$. It follow from the definition of even that $6m + 8n$ is even.

(ii) Yes. $10mn + 7 = 10mn + 6 + 1 = 2(5mn + 3) + 1$. It follows from closure under addition and multiplication that $5mn + 3$ is an integer. Let $p = 5mn + 3$. Then p is an integer and $10mn + 7 = 2p + 1$. It follows from definition of odd that $10mn + 7$ is odd.

(iii) Not necessarily. Let $m = 2$ and $n = 1$. Then $m > n > 0$ but $m^2 - n^2 = 2^2 - 1^2 = 3$ which is a prime number.

Problem 3

Assume that r and s are particular integers

(i) Is $4rs$ even?

(ii) Is $6r + 4s^2 + 3$ odd?

(iii) If r and s are both positive, is $r^2 + 2rs + s^2$ composite?

Solution

(i) Yes. $4rs = 2(2rs)$ It follows from closure under multiplication that $2rs$ is an integer. Let $t = 2rs$. Then t is an integer and $4rs = 2t$. It follows from the definition of even that $4rs$ is even.

(ii) Yes. $6r + 4s^2 + 3 = 6r + 4s^2 + 2 + 1 = 2(3r + 2s^2 + 1) + 1$. It follows from closure under addition and multiplication that $3r + 2s^2 + 1$ is an integer. Let $t = 3r + 2s^2 + 1$. Then t is an integer and $6r + 4s^2 + 3 = 2t + 1$. It follows from the definition of odd that $6r + 4s^2 + 3$ is odd.

(iii) Yes. $r^2+2rs+s^2 = (r+s)(r+s)$. It follows from closure under addition and multiplication that $r+s$ and $r^2+2rs+s^2$ are integers. Let $t = u = (r+s)$ and let $n = r^2 + 2rs + s^2$. Then t, u, and n are integers and $n = u * t$. As s and r are both positive integers the smallest possible value of both u and t is 2. Thus $1 < u$ and $1 < t$. Furthermore, $n = u * t$, $u * t > t$, and $u * t > u$. Thus $u < n$ and $t < n$. Finally $1 < u < n$, $1 < t < n$, and $n = u * t$. It follows from the definition of composite that $r^2 + 2rs + s^2$ is composite.

Problem 4

Prove that there are integers m and n such that $m > 1$ and $n > 1$ such that $1/m + 1/n$ is an integer.

Theorem: There are integers m and n such that $m > 1$ and $n > 1$ such that $1/m + 1/n$ is an integer.

Proof. Let $m = 2$ and $n = 2$. Then $1/m + 1/n = 1/2 + 1/2 = 1$ which is an integer. □

Problem 5

Prove that there are distinct integers m and n such that $1/m+1/n$ is an integer.

Theorem: There are distinct integers m and n such that $1/m+1/n$ is an integer.

Proof. Let $m = -2$ and $n = 2$. Then $1/m + 1/n = -1/2 + 1/2 = 0$ which is an integer. □

Problem 6

Prove that there are real numbers a and b such that $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$.

Theorem: There are real numbers a and b such that $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$.

Proof. Let $a = b = 0$. Then

$$\sqrt{a+b} = \sqrt{0+0} = \sqrt{0} = 0$$
$$\sqrt{a} + \sqrt{b} = \sqrt{0} + \sqrt{0} = 0 + 0 = 0$$
$$0 = 0$$

□

Problem 7

Prove that there is an integer $n > 5$ such that $2^n - 1$ is prime.

Theorem: There is an integer $n > 5$ such that $2^n - 1$ is prime.

Proof. Let $n = 7$. Then $2^n - 1 = 127$ which is prime. □

0.1 Problem 8

Prove that there is a real number x such that $x > 1$ and $2^x > x^{10}$.

Theorem: There is a real number x such that $x > 1$ and $2^x > x^{10}$.

Proof. We must show that

$$1 > \frac{x^{10}}{2^x}$$

Let $f(x) = 2^x$. Then

$$f'(x) = 2^x \ln 2$$

As $f'(x)$ is continuous on its domain and $f'(x) > 0$ for all $x \in \mathbb{R}$, $f(x)$ is monotonically increasing. $f(1) = 2^1 = 2$ and $x > 1$. Thus $\forall x > 1$, $f(x) > 2$. Let $x = \frac{101}{100}$. Then $x > 1$ and

$$\frac{x10}{2^x} = \frac{(\frac{101}{100})^{10}}{2^x} < \frac{(\frac{101}{100})^{10}}{2} = \frac{\frac{101^{10}}{100^{10}}}{2}$$

$$101^{10} = 110462212541120451001$$
$$100^{10} = 100000000000000000000$$
$$\frac{110462212541120451001}{100000000000000000000} = 1.10462212541120451001$$
$$\frac{1.10462212541120451001}{2} = .552311062705602255005$$
$$1 > .552311062705602255005 > \frac{x10}{2^x}$$

□

Problem 9

Prove that there is a perfect square that can be written as the sum of two other perfect squares.

Theorem: There is a perfect square that can be written as the sum of two other perfect squares.

Proof. Let $a = 3^2$, and $b = 4^2$. It follows from closure under multiplication that a and b are integers. It follows from the definition of perfect square that a and b are perfect squares. Let $c = a + b$. Then $c = 25 = 5^2$. It follows from closure under addition that c is an integer. It follows from the definition of perfect square that c is a perfect square □

Problem 10

Prove that $\exists n \in \mathbb{Z}$, such that $2n^2 - 5n + 2$ is prime.

Theorem: $\exists n \in \mathbb{Z}$, such that $2n^2 - 5n + 2$ is prime.

Proof. Let $n = 3$. Then

$$2n^2 - 5n + 2 = 2 * 3^2 - 5 * 3 + 2 = 18 - 15 + 2 = 5$$

which is prime. □

Problem 11

Disprove that $\forall a, b \in \mathbb{R}$, $a < b \implies a^2 < b^2$.

Counterexample: Let $a = -2$ and let $b = 2$. Then $a < b$ but $a^2 \nless b^2$.

Problem 12

Disprove that $\forall n \in \mathbb{Z}$, n is odd $\implies \frac{n-1}{2}$ is odd.

Counterexample: Let $n = 1$. Then n is odd and $\frac{n-1}{2} = 0$ which is even.

Problem 13

Disprove that $\forall m$ and n, $2m + 2$ is odd $\implies$ m and n are both odd.

Counterexample: Let $m = 2$ and let $n = 1$. Then $2m + n = 5$ which is odd but $m = 2$ which is even.

Problem 14

Is $(a + b)^2 = a^2 + b^2$ for integers a and b?

Solution

The property is true for some integers and false for other integers. For example, if $a = 1$ and $b = 0$ then

$$(a + b)^2 = (1 + 0)^2 = 1$$
$$a^2 + b^2 = 1^2 + 0^2 = 1$$
$$1 = 1$$

However, if $a = 1$ and $b = 1$ then

$$(a + b)^2 = (1 + 1)^2 = 4$$
$$a^2 + b^2 = 1^2 + 1^2 = 2$$
$$4 \neq 2$$

Problem 15

Is $-(a^n) = (-a)^n$ for $a, n \in \mathbb{Z}$

Solution

The property is true for some integers and false for other integers. If $a = 0$ and $n = 1$ then

$$-(a^n) = -(0^1) = 0$$
$$(-a^n) = (-0^1) = 0$$
$$0 = 0$$

However, if $a = 1$ and $n = 2$ then

$$-(a^n) = -(1^2) = -1$$
$$(-a)^n = (-1)^2 = 1$$
$$-1 \neq 1$$

Problem 16

Is the average of any two odd integers odd?

Solution

The property is true for some integers and false for others. Let $a = 5$ and let $b = 5$. Then $\frac{a+b}{2} = \frac{5+5}{2} = 5$ which is odd. However, if $a = 5$ and $b = 7$ then $\frac{a+b}{2} = \frac{5+7}{2} = 6$ which is even.

Problem 17

Prove that every Positive even integer less than 26 can be expressed as a sum of three or fewer perfect squares.

Theorem: Every positive even integer less than 26 can be expressed as a sum of three or fewer perfect squares.

Proof.

$$\begin{aligned}
2 &= 1^2 + 1^2 \\
4 &= 2^2 \\
6 &= 2^2 + 1^2 + 1^2 \\
8 &= 2^2 + 2^2 \\
10 &= 3^2 + 1^2 \\
12 &= 2^2 + 2^2 + 2^2 \\
14 &= 3^2 + 2^2 + 1^2 \\
16 &= 4^2 \\
18 &= 4^2 + 1^2 + 1^2 \\
20 &= 4^2 + 2^2 \\
24 &= 4^2 + 2^2 + 2^2 \\
26 &= 4^2 + 3^2 + 1^2
\end{aligned}$$

□

Problem 18

Prove that for each integer n with $1 \leq n \leq 10$, $n^2 - n + 11$ is a prime number.

Theorem: For each integer n with $1 \leq n \leq 10$, $n^2 - n + 11$ is a prime number.

Proof.

$$\begin{aligned}
1 &: 1^2 - 1 + 11 = 11\\
2 &: 2^2 - 2 + 11 = 13\\
3 &: 3^2 - 3 + 11 = 17\\
4 &: 4^2 - 4 + 11 = 23\\
5 &: 5^2 - 5 + 11 = 31\\
6 &: 6^2 - 6 + 11 = 41\\
7 &: 7^2 - 7 + 11 = 53\\
8 &: 8^2 - 8 + 11 = 67\\
9 &: 9^2 - 9 + 11 = 83\\
10 &: 10^2 - 10 + 11 = 101
\end{aligned}$$

□

Problem 19

Write the following proof in 3 different ways and then prove.

(i) $\forall$ integers m and n, if m is even and n is odd then $m + n$ is odd.

(ii) $\forall m, n \in \mathbb{Z}$, m is even and n is odd $\implies$ $m + n$ is odd.

(iii) If m and n are integers and m is even and n is odd then $m + n$ is odd.

Theorem: The sum of any even integer and any odd integer is odd.

Proof. Suppose m is any even integer and n is any odd integer. By definition of even, $m = 2r$ for some integer r. By definition of odd, $n = 2s + 1$ for some integer s. By substitution and algebra,

$$m + n = 2r + 2s + 1 = 2(r + s) + 1.$$

It follows from closure under addition that $r + s$ is an integer. Hence $m + n$ has the form twice some integer plus one, and so $m + n$ is odd by the definition of odd. □

Problem 20

For all integers m, if $m > 1$ then $0 < \frac{1}{m} < 1$.

(i) If an integer is greater than 1, then its reciprocal is between 0 and 1.

(ii) *Start of proof*: Suppose m is an integer such that $m > 1$.
Conclusion to be shown: $0 < \frac{1}{m} < 1$.

Problem 21

For all real numbers x, if $x > 1$ then $x^2 > x$.

(i) If a real number is greater than 1, then its square is greater than itself.

(ii) *Start of proof*: Suppose x is a real number such that $x > 1$.
Conclusion to be shown: $x^2 > x$.

Problem 22

For all integers m and n, if $mn = 1$ then $m = n = 1$ or $m = n = -1$.

(i) If the product of two integers is 1 then both integers are either 1 or -1.

(ii) *Start of Proof*: Suppse that m and n are integers such that $mn = 1$.
Conclusion to be shown: $m = n = 1$ or $m = n = -1$.

Problem 23

For all real numbers x, if $0 < x < 1$ then $x^2 < x$.

(i) If a real number is greater than 0 and less than 1 then its square is less than itself

(ii) *Start of proof*: Suppose that x is a real number such that $0 < x < 1$.
Conclusion to be shown: $x^2 < x$.

Problem 24

Prove that the negative or any even integer is even.

Theorem: The negative of any even integer is even.

Proof. Let n be any even integer. Then by definition of even $n = 2r$ for some integer r. Multiplying both sides by -1 gives

$$-(n) = -(2r)$$

$$-n = 2(-r)$$

$$-r = -1(r)$$

If follows from closure under multiplication that $-1(r)$ is an integer. Let that integer be s. Then

$$-n = 2s$$

It follows from the definition of even than $-n$ is even. □

Problem 25

Prove that the difference of any even integer minus any odd integer is odd.

Theorem: The difference of any even integer minus any odd integer is odd.

Proof. Let m be any even integer and let n be any odd integer. Then $m = 2s$ for some integer s and $n = 2r + 1$ for some integer r.

$$m - n = 2s - (2r + 1) = (2s - 2r) - 1 = 2(s - r - 1) + 1$$

It follows from closure under subtraction that $s - r - 1$ is an integer. Let that integer be t. Then

$$m - n = 2t + 1$$

It follows from the definition of odd that $m - n$ is odd. □

Problem 26

Prove that the difference between any odd integer and any even integer is odd.

Theorem: The difference between any odd integer and any even integer is odd.

Proof. Let m be any even integer and let n be any odd integer. Then $m = 2s$ for some integer s and $n = 2r + 1$ for some integer r.

$$n - m = (2r + 1) - 2s = (2r - 2s) + 1 = 2(r - s) + 1$$

It follows from closure under subtraction that $r - s$ is an integer. Let that integer be t. Then

$$n - m = 2t + 1$$

It follows from the definition of odd that $n - m$ is odd. □

Problem 27

Prove that the sum of any two odd integers is even.

Theorem: The sum of any two odd integers is even.

Proof. Let m and n be any two odd integers. Then $m = 2r + 1$ and $n = 2s + 1$ for some integers r and s.

$$m + n = (2r + 1) + (2s + 1) = (2r + 2s + 2) = 2(r + s + 1)$$

It follows from closure under addition that $r + s + 1$ is an integer. Let that integer be t. Then $m + n = 2t$. It follows from the definition of even that $m + n$ is even. □

Problem 28

Prove that for all integers n, if n is odd, then n^2 is odd.

Theorem: For all integers n, if n is odd, then n^2 is odd.

Proof. Let n be an odd integer. Then $n = 2s + 1$ for some integer s.

$$n^2 = (2s + 1)^2 = 4s^2 + 4s + 1$$

It follows from closure under multiplication that s^2 is an integer. Let that integer be t. Then

$$n^2 = 4t + 4s + 1 = 2(2t + 2s) + 1$$

It follows from closure under addition and multiplication that $2t + 2s$ is an integer. Let that integer be q. Then

$$n^2 = 2q + 1$$

It follows from the definition of odd the n^2 is odd. □

Problem 29

Prove that for all integers n, if n is odd then $3n + 5$ is even.

Theorem: For all integers n, if n is odd then $3n + 5$ is even.

Proof. Let n be an odd integer. Then $n = 2r + 1$ for some integer r.

$$3n + 5 = 3(2r + 1) + 5 = (6r + 3) + 5 = 6r + 8 = 2(3r + 4)$$

It follows from closure under addition and multiplication that $3r+4$ is an integer. Let this integer be s. Then $3n + 5 = 2s$. It follow from the definition of even that $3n + 5$ is even. □

Problem 30

Prove that for all integers m, if m is even then $3m + 5$ is odd.

Theorem: For all integers m, if m is even then $3m + 5$ is odd.

Proof. Let m be an even integer. Then $m = 2r$ for some integer r.

$$3m + 5 = 3(2r) + 5 = 6r + 5 = 2(3r + 2) + 1$$

It follows from closure under addition and multiplication that $3r+2$ is an integer. Let that integer be s. Then $3m + 5 = 2s + 1$. It follows from the definition of odd that $3m + 5$ is odd. □

Problem 31

Prove that if k is any odd integer and m is any even integer, then k^2+m^2 is odd.

Theorem: If k is any odd integer and m is any even integer, then $k^2 + m^2$ is odd.

Proof. Let k be any odd integer. Then $k = 2r + 1$ for some integer r. Let m be any even integer. Then $m = 2s$ for some integer s.

$$k^2 + m^2 = (2r + 1)^2 + (2s)^2 = 4r^2 + 4r + 1 + 4s^2 = 2(2r^2 + 2r + 2s^2) + 1$$

It follows from closure under multiplication and addition that $2r^2 + 2r + 2s^2$ is an integer. Let that integer be t. Then $k^2 + m^2 = 2t + 1$. It follows from the definition of odd that $k^2 + m^2$ is odd. □

Problem 32

Prove that if a is any odd integer and b is any even integer, then $2a+3b$ is even.

Theorem: If a is any odd integer and b is any even integer, then $2a + 3b$ is even.

Proof. Let a be any odd integer and let b be any even integer. Then $a = 2r + 1$ for some integer r and $b = 2s$ for some integer s.

$$2a + 3b = 2(2r + 1) + 3(2s) = 2(2r + 1 + 3s)$$

It follows from closure under multiplication and addition that $(2r+1+3s)$ is an integer. Let that integer be t. Then $2a + 3b = 2t$. It follows from the definition of even that $2a + 3b$ is even. □

Problem 33

Prove that if n is any even integer, then $(-1)^n = 1$.

Theorem: If n is any even integer, then $(-1)^n = 1$.

Proof. Let n be any even integer. Then $n = 2r$ for some integer r.

$$(-1)^n = (-1)^{2r} = ((-1)^2)^r = (1)^r = 1$$

Thus $(-1)^n = 1$. □

Problem 34

Prove that if n is any odd integer then $(-1)^n = -1$.

Theorem: If n is any odd integer then $(-1)^n = -1$.

Proof. Let n be any odd integer. Then $n = 2s + 1$ for some integer s.

$$(-1)^{2s+1} = (-1)^{2s} * (-1)^1 = 1 * -1 = -1$$

Thus $(-1)^n = -1$. □

Problem 35

Prove that there does not exist an integer $m \geq 3$ such that $m^2 - 1$ is prime.

Theorem: There does not exist an integer $m \geq 3$ such that $m^2 - 1$ is prime.

Proof. Let m be any integer such that $m \geq 3$.

$$m^2 - 1 = (m - 1)(m + 1)$$

It follows from closure under addition and subtraction that $m-1$ and $m+1$ are both integers. Because $m \geq 3$ both integers are greater than 1 and less than m^2-1. It follows from the definition of composite that m^2-1 is composite. □

Problem 36

Prove that there does not exist an integer n, such that $6n^2 + 27$ is prime.

Theorem There does not exist an integer n, such that $6n^2 + 27$ is prime.

Proof. $6n^2 + 27 = 3(2n^2 + 9)$. It follows from closure under addition and multiplication that $2n^2 + 9$ is an integer. Let that integer be s and let $t = 3$. Then $6n^2 + 27 = st$. It follows from the fact that the square of any integer is non negative that $s = 2n^2 + 9 \geq 9 > 1$. Thus $1 < s$ and $1 < t$. It follows from the previous sentence and the fact that s and t are both integer factors of $6n^2 + 27$ that $s < 6n^2 + 27$ and $t < 6n^2 + 27$. Finally $1 < s < 6n^2 + 27$ and $1 < t < 6n^2 + 27$. It follows from the definition of composite that $6n^2 + 27$ is composite. □

Problem 37

Prove that there does not exist an integer $k \geq 4$ such that $2k^2 - 5k + 2$ is prime.

Theorem: There does not exist an integer $k \geq 4$ such that $2k^2 - 5k + 2$ is prime.

Proof. $2k^2 - 5k + 2 = (2k - 1)(k - 2)$. It follows from closure under subtraction and multiplication that $2k-1$ and $k-2$ are integers. Let those integers be s and t respectively. Then $2k^2 - 5k + 2 = st$. If $k \geq 4$ then both s and t are greater than 1. It follows from the previous sentence and the fact that s and t are integer factors of $2k^2 - 5k + 2$ that $1 < s < 2k^2 - 5k + 2$ and $1 < t < 2k^2 - 5k + 2$. Thus by the definition of composite $2k^2 - 5k + 2$ is composite. □

Problem 38

Theorem: For all integers k, if $k > 0$ then $k^2 + 2k + 1$ is composite.

"Proof": For $k = 2$, $k^2 + 2k + 1 = 2^2 + 2 * 2 + 1 = 9$. But $9 = 3 * 3$, and so 9 is composite. Hence the theorem is true.

Solution

This proof is incorrect because it only shows $k^2 + 2k + 1$ to be composite in the case that $k = 2$ and not in the case that k is any integer greater than 0 as the theorem states.

Problem 39

Theorem: The difference between any odd integer and any even integer is odd.

"Proof": Suppose n is any odd integer, and m is any even integer. By definition of odd, $n = 2k + 1$ where k is an integer, and be definition of even, $m = 2k$ where k is an integer. Then

$$n - m = (2k + 1) - 2k = 1$$

But 1 is odd. Therefore, the difference between any odd integer and any even integer is odd.

Solution

This proof is incorrect because it defines the integers n and m in terms of the same integer k. This proof only shows that the difference between any odd integer and the previous even integer is 1 not that that difference between any odd integer and any even integer is odd as the theorem claims.

Problem 40

Theorem: For all integers k, if $k > 0$ then $k^2 + 2k + 1$ is composite.

"Proof": Suppose l is an integer such that $k > 0$. If $k^2 + 2k + 1$ is composite, then $k^2 + 2k + 1 = rs$ for some integers r and s such that

$$1 < r < (k^2 + 2k + 1)$$

and

$$1 < s < k^2 + 2k + 1$$

since

$$k^2 + 2k + 1 = rs$$

and both r and s are strictly between 1 and k^2+2k+1, k^2+2k+1 is not prime. Hence $k^2 + 2k + 1$ is composite as was to be shown.

Solution

This proof is incorrect because it simply states what would be true if the theorem were true. In other words an expression is given $k^2 + 2k + 1$ and the writer of the proof must show that the conditions of a composite number hold for this expression. Instead the writer supposes that the expression is composite then states what would be true if the expression is composite and then claims that these conditions hold without justification.

Problem 41

Theorem: The Product of an even integer and an odd integer is even.

"Proof": Suppose m is an even integer and n is an odd integer. If mn is even, then by definition of even there exists an integer r such that $mn = 2r$. Also, since m is even, there exists an integer p such that $m = 2p$, and since n is odd there exists an integer q such that $n = 2q + 1$. Thus

$$mn = (2p)(2q + 1) = 2r$$

where r is an integer. By definition of even, then, mn is even as was to be shown.

Solution

This proof is incorrect because it simply states what would be true if the theorem were true. It then claimed that these conditions were true without providing any justification.

Problem 43

Determine if the product of any two odd integers is odd

Theorem: The product of any two odd integers is odd

Proof. Suppose that m and n are any odd integers. Then $m = 2p + 1$ and $n = 2q + 1$ for some integers p and q.

$$mn = (2p + 1)(2q + 1) = 4pq + 2p + 2q + 1 = 2(2pq + p + q) + 1$$

It follows from closure under addition and multiplication that $2pq + p + q$ is an integer. Let that integer r. Then $mn = 2t + 1$. It follows from the definition of odd that mn is odd. □

Problem 44

Determine if the negative of any odd integer is odd

Theorem: The negative of any odd integer is odd

Proof. Let n be any odd integer. Then $n = 2p + 1$

$$-n = -(2p + 1) = (-2p - 2) + 1 = 2(-p - 1) + 1$$

It follows from closure under multiplication and subtraction that $-p - 1$ is an integer. Let that integer be t. Then $-n = 2t + 1$. It Follows from the definition of odd that $-n$ is odd. □

Problem 45

Determine if the difference of any two odd integers is odd.

Theorem: The difference of any two odd integers is even.

Proof. Let m and n be any two odd integers. Then $m = 2p + 1$ and $n = 2q + 1$ for some integers p and q.

$$m - n = (2p + 1) - (2q + 1) = 2p - 2q + 1 - 1 = 2(p - q)$$

It follows from closure under subtraction that $p - q$ is an integer. Let this integer be t. Then $m - n = 2t$. It follows from the definition of even that $m - n$ is even. □

Problem 46

Determine if the product of any even integer and any integer is even.

Theorem: The product of any even integer and any integer is even.

Proof. Let m be any even integer. Then $m = 2p$ for some integer p. Let the other integer be n.

(i) Case 1: Suppose that n is even. Then $n = 2q$ for some integer q.

$$nm = (2p)(2q) = 2(2pq)$$

It follows from closure under multiplication that $2pq$ is an integer. Let this integer be t. Then $nm = 2t$. It follows from the definition of even that nm is even.

(ii) Case 2: Suppose that n is odd. Then $n = 2q + 1$ for some integer q.

$$nm = (2p)(2q + 1) = 2(2pq + p)$$

It follows from closure under multiplication and addition that $2pq+p$ is an integer. Let this integer be t. Then $nm = 2t$. It follows from the definition of even that nm is even.

Because nm is even in the case that n is even and in the case that n is odd it follows that the product of any even integer and any integer is even. □

Problem 47

Determine whether the sum of two integers being even implies that one of the summands is even.

Counterexample: Let $a = 1$ and let $b = 1$. Then $a + b = 1 + 1 = 2$ which is even yet both a and b are odd.

Problem 48

Determine if the difference of any two even integers is even.

Theorem: The difference of any two even integers is even.

Proof. Let a and b be any even integers. Then $a = 2p$ and $b = 2q$ for some integers p and q.

$$a - b = 2p - 2q = 2(p - q)$$

It follows from closure under subtraction that $p - q$ is an integer. Let that integer be t. Then $a - b = 2t$. It follows from the definition of even that $a - b$ is even. □

Problem 49

Determine if the difference of any two odd integers is even.

Theorem: The difference of any two odd integers is even.

Proof. Let a and b be odd integers. Then $a = 2p + 1$ and $b = 2q + 1$ for some integers p and q.

$$a - b = (2p + 1) - (2q + 1) = 2(p - q)$$

It follows from closure under subtraction that $p - q$ is an integer. Let that integer be t. Then $a - b = 2t$. It follows from the definition of even that $a - b$ is even. □

Problem 50

Determine if for all integers n and m, if $n - m$ is even then $n^3 - m^3$ is even.

Theorem: For all integers n and m, if $n - m$ is even then $n^3 - m^3$ is even.

Proof. If follows from Problem 50 and Problem 49 this if $n - m$ is even then either n and m are both even or n and m are both odd.

(i) Case 1: Suppose that n and m are both even. Then $n = 2p$ and $m = 2q$ for some integers p and q.

$$n^3 - m^3 = (2p)^3 - (2q)^3 = 8p^3 - 8q^3 = 2(4p^3 - 4q^3)$$

It follows from closure under multiplication that $4p^3 - 4q^3$ is an integer. Let this integer be t. Then $n^3 - m^3 = 2t$. It follows from the definition of even that $n^3 - m^3$ is even.

(ii) Case 2: Suppose that n and m are both odd. Then $n = 2p + 1$ and $m = 2q + 1$ for some integers p and q.

$$n^3 - m^3 = (2p + 1)^3 - (2q + 1)^3 = 2(4p^3 + 6p^2 + 3p - 4q^3 - 6q^2 - 3q)$$

It follows from closure under multiplication, addition, and subtraction that $4p^3 + 6p^2 + 3p - 4q^3 - 6q^2 - 3q$ is an integer. Let this integer be t. Then $n^3 - m^3 = 2t$. It follows from the definition of even that $n^3 - m^3$ is even.

Because $n^3 - m^3$ is even in the case that n and m are both even and in the case that n and m are both odd, $n^3 - m^3$ is even in the case that $n - m$ is even. □

Problem 51

Determine if for all integers n, if n is prime then $(-1)^n = -1$.

Counterexample: Let $n = 2$. Then n is prime but $(-1)^2 = 1$.

Problem 52

Determine if for all integers m, if $m > 2$, then $m^2 - 4$ is composite.

Counterexample: Let $m = 3$. Then $m > 2$ but

$$m^2 - 4 = 3^2 - 4 = 9 - 4 = 5$$

which is prime.

Problem 53

Determine if for all integers n, $n^2 - n + 11$ is a prime number.

Counterexample: Let $n = 11$. Then

$$n^2 - n + 11 = 11^2 - 11 + 11 = 121$$

which is composite.

Problem 54

Determine if for all integers n, $4(n^2 + n + 1) - 3n^2$ is a perfect square.

Theorem: For all integers n, $4(n^2 + n + 1) - 3n^2$ is a perfect square.

Proof.

$$4(n^2 + n + 1) - 3n^2 = 4n^2 + 4n + 4 - 3n^2 = n^2 + 4n + 4 = (n + 2)^2$$

It follows from the closure under addition that $n + 2$ is an integer. Let this integer be t. Then $4(n^2 + n + 1) - 3n^2 = t^2$. It follows from the definition of perfect square that $4(n^2 + n + 1) - 3n^2$ is a perfect square. $\square$

Problem 55

Determine if every positive integer can be expressed as a sum of three or fewer perfect squares

Counterexample: let the integer be 7. The first 3 positive perfect squares are 1, 4, and 9. Clearly 9 cannot be part of the sum as $9 > 7$. If the sum 7 is to be reached then four must be included as $1 + 1 + 1 = 3 < 7$. If 4 is used once then you have $4 + 1 + 1 = 6 < 7$. If 4 is used twice then you have $4 + 4 > 8$. Thus 7 cannot be expressed as a sum of three or fewer perfect squares.

0.2 Problem 56

Determine if all products of four consecutive integers is one less than a perfect square.

Theorem: All product of four consecutive integers is one less than a perfect square.

Proof. Let n be an integer. We will define a product of four consecutive integers.

$$\begin{aligned}(n-1)(n)(n+1)(n+2) &= n^4 + 2n^3 - n^2 - 2n \\ &= (n^4 + 2n^3 - n^2 - 2n + 1) - 1 \\ &= (n^2 + n - 1)^2 - 1\end{aligned}$$

It follows from closure under addition subtraction and multiplication that $n^2 + n - 1$ is an integer. Let this integer be t. Then $(n-1)(n)(n+1)(n+2) = t^2 - 1$. Thus the product of any four consecutive integers is one less than a perfect square. $\square$

Problem 57

If m and n are positive integers and mn is a perfect square, does it follows that m and n are both perfect squares?

Counterexample: Let $m = 2$ and let $n = 8$. Then m and n are both positive integers and mn is a perfect square as $mn = 2 * 8 = 16$ but neither 2 nor 8 are perfect squares.

Problem 58

Determine if the difference of the squares of any two consecutive integers is odd.

Theorem: The difference of the squares of any two consecutive integers is odd.

Proof. Let the first integer be n and let the second integer be $n + 1$. Then the two integers are consecutive.

$$(n+1)^2 - n^2 = n^2 + 2n + 1 - n^2 = 2n + 1$$

If follows from the definition of odd that $(n+1)^2 - n^2$ is odd. $\square$

Problem 59

Determine if for all nonnegative real numbers a and b, $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

Theorem: For all nonnegative real numbers a and b, $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

Proof. Let a and b be nonnegative real numbers. Then there exists unique nonnegative real numbers p and q denoted $\sqrt{a}$ and $\sqrt{b}$ respectively such that $p^2 = a$ and $q^2 = b$. Because a and b are real numbers there exists a real number r denoted $\sqrt{ab}$ such that $r^2 = ab$. It follows that $r^2 = p^2 * q^2 = (pq)^2$. It follows by taking the square root of both sides that $r = pq$. By substituting r, p, and q with what they are denoted be(defined above) we obtain $\sqrt{ab} = \sqrt{a}\sqrt{b}$. $\square$

Problem 60

Determine if for all nonnegative real numbers a and b, $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$.

Counterexample: Let $a = 1$ and let $b = 1$ then

$$\sqrt{a+b} = \sqrt{1+1} = \sqrt{2}$$

$$\sqrt{1} + \sqrt{1} = 1 + 1 = 2$$
$$\sqrt{2} \neq 2$$

Problem 61

Suppose that integers m and n are perfect squares, Then $m + n + 2\sqrt{mn}$ is also a perfect square, Why?

Solution

If m and n are perfect squares then $m = p^2$ for some integer p and $n = q^2$ for some integer q.

$$m + n + 2\sqrt{mn}$$
$$p^2 + q^2 + 2\sqrt{(p^2)(q^2)}$$

By the results of problem 59 $\sqrt{(p^2)(q^2)} = \sqrt{p^2}\sqrt{r^2}$

$$p^2 + q^2 + 2pq$$

$$(p + q)^2$$

Problem 62

If p is a prime number, must $2^p - 1$ also be prime?

Counterexample: Let $p = 11$. Then p is a prime but $2^{11} - 1 = 2047$ which is composite as $2047 = 23 * 89$.

Problem 62

If n is a nonnegative integer, must $2^{2^n} + 1$ be prime?

Counterexample: Let $n = 5$. Then $2^{2^n} + 1 = 2^{2^5} + 1 = 4294967297$ but this number is not prime as $4294967297 = 641 * 6700417$.