# Section 4.3

Sterling Jeppson

September 29, 2020

## Problem 1

Is 52 divisible by 13?

## Solution

Yes. $13 \mid 52$ as $52 = 13 \cdot 4$

## Problem 2

Does $7 \mid 56$?

## Solution

Yes. $7 \mid 56$ as $56 = 7 \cdot 8$

## Problem 3

Does $5 \mid 0$?

## Solution

Yes. $5 \mid 0$ as $0 = 5 \cdot 0$

## Problem 4

Does 3 divide $(3k+1)(3k+2)(3k+3)$ for any integer $k$?

## Solution

Yes. $3 \mid (3k+1)(3k+2)(3k+3)$ as

$$(3k+1)(3k+2)(3k+3) = 3 \cdot ((3k+1)(3k+2)(k+1))$$

and $(3k+1)(3k+2)(k+1)$ is an integer as sums and products of integers are integers.

## Problem 5

Is $6m(2m+10)$ divisible by 4 for any integer $m$?

## Solution

Yes.

$$\begin{aligned} 6m(2m+10) &= 12m^2 + 60m \\ &= 4(3m^2 + 15m) \end{aligned}$$

and $3m^2 + 15m$ is an integer as sums and products of integers are integers.

## Problem 6

Is 29 a multiple of 3?

## Solution

No. $29/3 \approx 9.67$, which is not an integer.

## Problem 7

is $-3$ a factor of 66?

## Solution

Yes. $66 = (-3)(-22)$

## Problem 8

Is $6a(a+b)$ a multiple of $3a$ for integers $a$ and $b$?

## Solution

Yes. $6a(a+b) = 3a[2(a+b)]$ and $2(a+b)$ is an integer as sums and products of integers are integers.

## Problem 9

Is 4 a factor of $2a \cdot 34b$?

## Solution

Yes.

$$\begin{aligned} 2a \cdot 34b &= 68 \cdot ab \\ &= 4 \cdot 17ab \end{aligned}$$

and $17ab$ is an integer as products of integers are integers.

## Problem 10

Does $7 \mid 34$?

## Solution

No. $34/7 \approx 4.86$, which is not an integer.

## Problem 11

Does $13 \mid 73$?

## Solution

No. $73/13 \approx 5.62$, which is not an integer.

## Problem 12

If $n = 4k + 1$, does 8 divide $n^2 - 1$ for any integer $k$?

## Solution

Yes.

$$\begin{aligned} n^2 - 1 &= (4k+1)^2 - 1 \\ &= (16k^2 + 8k + 1) - 1 \\ &= 16k^2 + 8k \\ &= 8(2k^2 + k) \end{aligned}$$

and $2k^2 + k$ is an integer as sums and products of integers are integers.

## Problem 13

If $n = 4k + 3$, does 8 divide $n^2 - 1$ for any integer $k$?

## Solution

Yes.

$$\begin{aligned} n^2 - 1 &= (4k+3)^2 - 1 \\ &= (16k^2 + 24K + 9) - 1 \\ &= 16k^2 + 24k + 8 \\ &= 8(2k^2 + 3k + 1) \end{aligned}$$

and $2k^2 + 3k + 1$ is an integer as sums and products of integers are integer.

## Problem 14

Fill in the blanks in the following proof that for all integers $a$ and $b$, if $a \mid b$ then $a \mid -b$.

## Solution

*Proof*: Suppose $a$ and $b$ are any integers such that $a \mid b$. By definition of divisibility, there exists an integer $r$ such that $b = a \cdot r$. By substitution,

$$-b = -ar = a(-r)$$

Let $t = -r$. Then $t$ is an integer because $t = (-1) \cdot r$, and both $-1$ and $r$ are integers. Thus, by substitution, $-b = at$, where $t$ is an integer, and so by definition of divisibility, $a \mid -b$ as was to be shown.

## Problem 15

Prove that for all integers $a, b$, and $c$, if $a \mid b$ and $a \mid c$ then $a \mid (b + c)$.

*Theorem*: For all integers $a, b$, and $c$, if $a \mid b$ and $a \mid c$ then $a \mid (b + c)$.

*Proof.* Suppose that $a, b$, and $c$ are integers such that $a \mid b$ and $a \mid c$. By definition of divisibility $b = ar$ and $c = as$ for some integers $r$ and $s$. By substitution

$$\begin{aligned} b + c &= ar + as \\ &= a(r + s) \end{aligned}$$

If follows from closure under addition that $r+s$ is an integer. Let that integer be $k$. Then $b+c = a \cdot k$ It follows from the definition of divisibility that $a \mid (b+c)$. □

## Problem 16

Prove that for all integers $a, b$, and $c$, if $a \mid b$ and $a \mid c$ then $a \mid (b - c)$.

*Theorem*: For all integers $a, b$, and $c$, if $a \mid b$ and $a \mid c$ then $a \mid (b - c)$.

*Proof.* Suppose that $a, b$, and $c$ are integers such that $a \mid b$ and $a \mid c$. By definition of divisibility $b = ar$ and $c = as$ for some integers $r$ and $s$. By substitution

$$\begin{aligned} b - c &= ar - as \\ &= a(r - s) \end{aligned}$$

It follows from closure under subtraction that $r - s$ is an integer. Let that integer be $k$. Then $b - c = a \cdot k$ It follows from the definition of divisibility that $a \mid (b - c)$. $\square$

## Problem 17

Consider the following statement: The negative of any multiple of 3 is a multiple of 3.

(a) Write the statement formally using a quantifier and a variable.

(b) Determine whether the statement is true or false and justify your answer.

## Solution

(a) $\forall n \in Z$, if $n$ is a multiple of 3 then $-n$ is a multiple of 3.

(b) If $n$ is a multiple of 3 then $n = 3k$ for some integer $k$. By simple algebra $-n = -3k = 3(-k)$. But $-k = (-1) \cdot k$ and so $-k$ is an integer as it is a product of integers. Let this integer be $t$. Then $-n = 3t$. It follows from the definition of divisibility that $-n$ is a multiple of 3.

## Problem 18

Show that the following statement is false: For all integers $a$ and $b$, if $3 \mid (a+b)$, then $3 \mid (a - b)$.

*Counterexample*: Let $a = 4$ and $b = 2$. Then $a + b = 6$ and $3 \mid 6$ as $6 = 3 \cdot 2$. However, $a - b = 2$ and $3 \nmid 2$ as $3/2 = 1.5$ which is not an integer.

## Problem 19

Determine if for all integers $a, b$, and $c$, if $a \mid b$ then $a \mid bc$.

*Theorem*: For all integers $a, b$, and $c$, if $a \mid b$ then $a \mid bc$.

*Proof.* Let $a, b$, and $c$ be integers such that $a \mid b$. Then $b = ar$ for some integer $r$. By substitution

$$\begin{aligned} bc &= (ar)c \\ &= a(rc) \end{aligned}$$

It follows from closure under multiplication that $rc$ is an integer. Let that integer be $t$. Then $bc = at$. It follows from the definition of divisibility that $a \mid bc$. $\square$

## Problem 20

Determine if the sum of any three consecutive integers is divisible by 3.

*Theorem*: The sum of any three consecutive integers is divisible by 3.

*Proof.* Let $n$ be any integer. Then $(n) + (n + 1) + (n + 2)$ is the sum of any three consecutive integers.

$$\begin{aligned}(n) + (n + 1) + (n + 2) &= n + n + 1 + n + 2 \\ &= n + n + n + 1 + 2 \\ &= 3n + 3 \\ &= 3(n + 1)\end{aligned}$$

If follows from closure under addition that $n + 1$ is an integer. Let that integer be $t$. Then $(n)+(n+1)+(n+2) = 3t$. It follows from the definition of divisibility that the sum of any three consecutive integers is divisible by 3. $\square$

## Problem 21

Determine if the product of any two even integers is a multiple of 4.

*Theorem*: The multiple of any two even integers is a multiple of 4.

*Proof.* Let $a$ and $b$ be any two even integers. Then $a = 2c$ and $b = 2d$ for some integers $c$ and $d$.

$$\begin{aligned}a \cdot b &= (2c)(2d) \\ &= 4(cd)\end{aligned}$$

It follows from closure under multiplication that $ab$ is an integer. Let that integer be $t$. Then $ab = 4t$. If follows from the definition of multiple that $ab$ is a multiple of 4. $\square$

## Problem 22

Determine if a necessary condition for an integer to be divisible by 6 is that is be divisible by 2.

*Theorem*: If any integer is divisible by 6 then it is divisible by 2.

*Proof.* Let $n$ be any integer such that $6 \mid n$. Then $n = 6k$ for some integer $k$.

$$\begin{aligned} n &= 6k \\ &= 2(3k) \end{aligned}$$

It follows from closure under multiplication that $3k$ is an integer. Let that integer be $t$. Then $n = 2t$. It follows from the definition of divisibility that $2 \mid n$. $\square$

## Problem 23

Determine if a sufficient condition for an integer to be divisible by 8 is that it is divisible by 16.

*Theorem*: If any integer is divisible by 16 then it is divisible by 8.

*Proof.* Let $n$ be any integer such that $16 \mid n$. Then $n = 16k$ for some integer $k$.

$$\begin{aligned} n &= 16k \\ &= 8(2k) \end{aligned}$$

It follows from closure under multiplication that $2k$ is an integer. Let that integer be $t$. Then $n = 8t$. It follows from the definition of divisibility that $8 \mid n$. $\square$

## Problem 24

Determine if for all integers $a, b$, and $c$, if $a \mid b$ and $a \mid c$ then $a \mid (2b - 3c)$.

*Theorem*: For all integers $a, b$, and $c$, if $a \mid b$ and $a \mid c$ then $a \mid (2b - 3c)$.

*Proof.* Let $a, b$, and $c$ be integers such that $a \mid b$ and $a \mid c$. Then $b = ak$ and $c = aj$ for some integers $k$ and $j$. By substitution

$$\begin{aligned} 2b - 3c &= 2(ak) - 3(aj) \\ &= 2ak - 3aj \\ &= a(2k - 3j) \end{aligned}$$

It follows from closure under addition and multiplication that $2k - 3j$ is an integer. Let that integer be $t$. Then $2b - 3c = at$. It follows from the definition of divisibility that $a \mid (2b - 3c)$. $\square$

## Problem 25

Determine if for all integers $a, b$, and $c$, if $a$ is a factor of $c$ then $ab$ is a factor of $c$.

*Counterexample*: Let $a = 3$, $b = 3$, and $c = 12$. Then $a \mid c$ but $ab \nmid c$.

## Problem 26

Determine if for all integers $a, b$, and $c$, if $ab \mid c$ then $a \mid c$ and $b \mid c$.

*Theorem*: For all integers $a, b$, and $c$, if $ab \mid c$ then $a \mid c$ and $b \mid c$.

*Proof.* Let $a, b$, and $c$, be integers such that $ab \mid c$. Then $c = abk$ for some integer $k$. It follows from closure under multiplication that $bk$ is an integer and that $ak$ is an integer. Let those integers be $t$ and $s$ respectively. Then $c = at$ and $c = bs$. It follows from the definition of divisibility that $a \mid c$ and $b \mid c$. □

## Problem 27

Determine if for all integers $a, b$, and $c$ if $a \mid (b + c)$ then $a \mid b$ or $a \mid c$.

*Counterexample*: Let $a = 3$, $b = 2$, and $c = 7$. Then $a \mid (b + c)$ but $a \nmid b$ and $a \nmid c$.

## Problem 28

Determine if for all integers $a, b$, and $c$ if $a \mid bc$ then $a \mid b$ or $a \mid c$.

*Counterexample*: Let $a = 4$, $b = 2$, and $c = 2$. Then $a \mid bc$ but $a \nmid b$ and $a \nmid c$.

## Problem 29

Determine if for all integers $a$ and $b$, if $a \mid b$ then $a^2 \mid b^2$.

*Theorem*: For all integers $a$ and $b$, if $a \mid b$ then $a^2 \mid b^2$.

*Proof.* Let $a$ and $b$ be integers such that $a \mid b$. Then $b = ak$ for some integer $k$.

$$\begin{aligned} b^2 &= (ak)^2 \\ &= a^2k^2 \end{aligned}$$

If follows from closure under multiplication that $k^2$ is an integer. Let that integer be $t$. Then $b^2 = a^2t$. It follows from the definition of divisibility that $a^2 \mid b^2$. □

## Problem 30

Determine if for all integers $a$ and $n$, if $a \mid n^2$ and $a \leq n$ then $a \mid n$.

*Counterexample*: Let $a = 9$ and let $n = 21$. Then $a \mid n^2$ and $a \leq n$ but $a \nmid n$.

## Problem 31

Determine if for all integers $a$ and $b$, if $a \mid 10b$ then $a \mid 10$ or $a \mid b$.

*Counterexample*: Let $a = 4$ and let $b = 2$. Then $a \mid 10b$ but $a \nmid 10$ and $a \nmid b$.

## Problem 32

A fast food chain has a contest in which a card with numbers on it is given to each customer who makes a purchase. If some of the numbers on the the card add up to 100, then the customer wins $100. A certain customer receives a card containing the numbers

$$72, 21, 15, 36, 69, 81, 9, 27, 42, \text{ and } 63.$$

Will the customer win $100? Why or Why not?

## Solution

The numbers on the customers card are all divisible by three so an equivalent list would be

$$(3 \cdot 24), (3 \cdot 7), (3 \cdot 5), (3 \cdot 12), (3 \cdot 23), (3 \cdot 27), (3 \cdot 3), (3 \cdot 9), (3 \cdot 14), (3 \cdot 21)$$

Any sum of $k$ numbers from the list can be written as

$$Sum = (n_1 + n_2 + ... + n_k) \text{ where } n_1, n_2, ..., n_k \in \text{the list}$$

Since each $n_i$ in the list is a multiple of 3 for $1 \leq i \leq k$ we can write an equivalent sum as

$$Sum = 3 \cdot (m_1 + m_2 + ... + m_k) \text{ where } m_i = \tfrac{n_i}{3}$$

It follows from the fact that each element in the list is divisible by 3 that $m_i$ is an integer. It follows from this fact and from closure under addition that $m_1 + m_2 + ... + m_k$ is an integer. Let this integer be $t$. Then any sum of $k$ elements from the list can be written as $3t$. If the customer wins the prize then there exists a sum of elements from their list such that $100 = 3t$. But this cannot be so as $t$ is an integer and $100/3 = 33.\overline{33}$ is not an integer. Thus the customer will not win $100.

## Problem 33

Is it possible to have a combination of nickels, dimes, and quarters, that add up to $4.72? Explain.

## Solution

\$4.72 is equivalent to 472 cents. A nickel is 5 cents, a dime is 10 cents and a quarter is 25 cents. We need to determine if a sum composed of nickels, dimes, and quarters divides 472 such that 472 divided by the sum is 1. But because 5, 10, and 25 are all divisible by 5 any sum of these coins will also be divisible by 5. It follows from the transitivity of divisibility that we only need to determine whether $5 \mid 472$. But $5 \nmid 472$ as $472/5 = 94.4$ which is not an integer. Therefore \$4.72 can never be composed of some collection of nickels, dimes, and quarters.

## Problem 34

Is it possible to have 50 coins, made up of pennies, dimes, and quarters, that add up to \$3? Explain.

## Solution

Let $x, y$, and $z$ represent the number of pennies, dimes, and quarters respectively. If it is possible to have 50 coins, made up of pennies, dimes, and quarters, that add up to \$3, then $x, y$, and $z$ must all be integers and the following equations must be true.

$$\begin{aligned} x + y + z &= 50 \\ x + 10y + 25z &= 300 \end{aligned}$$

Solve for $x$ to obtain

$$x = 50 - y - z$$

By substitution,

$$\begin{aligned} x + 10y + 25z &= 300 \\ 50 - y - z + 10y + 25z &= 300 \\ 9y + 24z &= 250 \\ 3(3y + 8z) &= 250 \\ 3y + 8z &= \frac{250}{3} \\ 3y + 8z &= 83.\overline{33} \end{aligned}$$

But we said that $y$ and $z$ must be integers and it is impossible for a sum of two products of integers to be a non integer as integers are closed under addition and multiplication. Therefore either $y$ or $z$ or both are not integers. It follows then that only one of the following can be true. Either

$$\begin{aligned} x + y + z &= 50 \\ x + 10y + 25z &= 300 \end{aligned}$$

or $x, y$, and $z$ are all integers. Thus it is impossible to have 50 coins, made up of pennies, dimes, and quarters, that add up to \$3.

## Problem 35

Two athletes run a circular track at a steady pace so that the first completes one round in 8 minutes and the second in 10 minutes. If they both start from the same spot at 4 P.M., when will be the first time they return to the start together?

## Solution

We must find the smallest positive number of minutes $x$ such that $8 \mid x$ and $10 \mid x$. The first 5 positive numbers that 8 divides are 8, 16, 24, 32 and 40. The first 4 positive numbers that 10 divides are 10, 20, 30, and 40. Thus the smallest positive number of minutes that both 8 and 10 divide is 40. And so the two athletes will both return to the start together at 4:40 P.M.

## Problem 36

It can be shown that an integer is divisible by 3 if, and only if, the sum of its digits is divisible by 3. An integer is divisible by 9 if, and only if, the sum of its digits is divisible by 9. An integer is divisible by 5 if, and only if, its right most digit is 5 or 0. And an integer is divisible by 4 if, and only if, the number formed by its rightmost two digits is divisible by 4. Check the following integers for divisibility by 3, 4, 5, and 9.

(a) 637,425,403,705,125 = x

(b) 12,858,306,120,312 = y

(c) 517,924,440,926,512 = z

(d) 14,328,083,360,232 = w

## Solution

(a) The sum of the digits of x is 54 and $9 \mid 54$ so $9 \mid x$. By transitivity of divisibility $3 \mid x$. The two rightmost digits of x are 25 but $4 \nmid 24$ and so $4 \nmid x$. The rightmost digit of x is 5 and so $5 \mid x$.

(b) The sum of the digits of y is 42 and $9 \nmid 42$ so $9 \nmid y$. $3 \mid 42$ so $3 \mid y$. The two rightmost digits of y are 12 and $4 \mid 12$ so $4 \mid y$. The rightmost digit of y is 2 and so $5 \nmid y$.

(c) The sum of the digits of z is 61 but $3 \nmid 61$ and $9 \nmid 61$. Thus $3 \nmid z$ and $9 \nmid z$. The two rightmost digits of z are 12 and $4 \mid 12$ so $4 \mid z$. The rightmost digit of z is 2 and so $5 \nmid z$.

(d) The sum of the digits of w is 45 and $9 \mid 45$ so $9 \mid w$. By transitivity of divisibility $3 \mid w$. The two rightmost digits of w are 32 and $4 \mid 32$ so $4 \mid w$. The rightmost digit of w is 2 and so $5 \nmid w$.

## Problem 37

Use the unique factorization theorem to write the following integers in standard factored form.

(a) 1,176

(b) 5,733

(c) 3,675

## Solution

(a) $1,176 = 2^3 \cdot 3^1 \cdot 7^2$

(b) $5,733 = 3^2 \cdot 7^2 \cdot 13^1$

(c) $3,675 = 3^1 \cdot 5^2 \cdot 7^2$

## Problem 38

Suppose that in standard factored form $a = p_1^{e_1} p_2^{e_2} ... p_k^{e_k}$, where $k$ is a positive integer; $p_1$, $p_2$,...,$p_k$ are prime numbers; and $e_1$, $e_2$,...,$e_k$ are positive integers.

(a) What is the standard factored form for $a^2$?

(b) Find the least positive integer $n$ such that $2^5 \cdot 3 \cdot 5^2 \cdot 7^3 \cdot n$ is a perfect square. Write the resulting product as a perfect square.

(c) Find the least positive integer $m$ such that $2^2 \cdot 3^5 \cdot 7 \cdot 11 \cdot m$ is a perfect square. Write the resulting product as a perfect square.

## Solution

(a) $p_1^{2e_1} p_2^{2e_2} ... p_k^{2e_k}$

(b) $n = 2 \cdot 3 \cdot 7$
$2^5 \cdot 3 \cdot 5^2 \cdot 7^3 \cdot n = (2^3 \cdot 3 \cdot 5 \cdot 7^2)^2$

(c) $m = 3 \cdot 7 \cdot 11$
$2^2 \cdot 3^5 \cdot 7 \cdot 11 \cdot m = (2 \cdot 3^3 \cdot 7 \cdot 11)^2$

## Problem 39

Suppose that in standard factored form $a = p_1^{e_1} p_2^{e_2} ... p_k^{e_k}$, where $k$ is a positive integer; $p_1$, $p_2$,...,$p_k$ are prime numbers; and $e_1$, $e_2$,...,$e_k$ are positive integers.

(a) What is the standard factored form for $a^3$?

(b) Find the least positive integer $k$ such that $2^4 \cdot 3^5 \cdot 7 \cdot 11^2 \cdot k$ is a perfect cube. Write the resulting product as a perfect cube.

## Solution

(a) $p_1^{3e_1} p_2^{3e_2} ... p_k^{3e_k}$

(b) $k = 2^2 \cdot 3 \cdot 7^2 \cdot 11$
$2^4 \cdot 3^5 \cdot 7 \cdot 11^2 \cdot k = (2^2 \cdot 3^2 \cdot 7 \cdot 11)^3$

## problem 40

(a) If $a$ and $b$ are integers and $12a = 25b$, does $12 \mid b$? Does $25 \mid a$? Explain.

(b) If x and y are integers and 10x = 9y, does $10 \mid y$? Does $9 \mid x$? Explain.

## Solution

(a) Because $12a = 25b$ the unique factorization theorem guarantees that the standard factored form of $12a$ and $25b$ must be the same. Thus $25b$ must contain the factors $2^2 \cdot 3 = 12$. But the prime factorization of $25 = 5^2$. Thus $b$ must be the part of the number $25b$ that contains the factors $3 \cdot 2^2$. And so $12 \mid b$. Similarly, $12a$ must contain the factors $5^2$ but the prime factorization of $12 = 2^2 \cdot 3$ and so $a$ must provide these factors. Thus $25 \mid a$.

(b) Because $10x = 9y$ the unique factorization theorem guarantees that the standard factored form of $10x$ and $9y$ must be the same. Thus $9y$ must contain the factors $2 \cdot 5 = 10$. But the prime factorization of $9 = 3^2$. Thus $y$ must be the part of the number $9y$ that contains the factors $2 \cdot 5$. And so $10 \mid y$. Similarly, $10x$ must contain the factors $3^2$ but the prime factorization of $10 = 2 \cdot 5$ and so $x$ must provide these factors. Thus $9 \mid x$.

## Problem 41

How many zeroes are at the end of $45^8 \cdot 88^5$? Explain how you can answer this question without actually computing the number.

## Solution

First note that the number of zeroes at the end of any integer is equal to the number of times that number can be evenly divided by 10. Also note that $10 = 2 \cdot 5$

$$\begin{aligned}
45^8 \cdot 88^5 &= (3^2 \cdot 5)^8 \cdot (2^3 \cdot 11)^5 \\
&= 3^{16} \cdot 5^8 \cdot 2^{15} \cdot 11^5 \\
&= 3^{16} \cdot 11^5 \cdot (5^8 \cdot 2^8) \cdot 2^7 \\
&= 3^{16} \cdot 11^5 \cdot (5 \cdot 2)^8 \cdot 2^7 \\
&= 3^{16} \cdot 11^5 \cdot 10^8 \cdot 2^7
\end{aligned}$$

The $10^8$ contributes 8 zeroes to the end of the number. Now we need to determine if the other part of the number $(3^{16} \cdot 11^5 \cdot 2^7)$ can add any more zeroes to the end of the number. If it could then it would be divisible by 10. Does $10 \mid (3^{16} \cdot 11^5 \cdot 2^7)$? No because if it was divisible 10 it would also be divisible by 5 by transitivity of divisibility. And $(3^{16} \cdot 11^5 \cdot 2^7)$ is not divisible by 5 as 5 is not part of its prime factorization. Therefore $45^8 \cdot 88^5$ has 8 zeroes at the end of it.

## Problem 42

If $n$ is an integer and $n > 1$, then $n!$ is the product of $n$ and every other positive integer that is less than $n$. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

(a) Write 6! in standard factored form.

(b) write 20! in standard factored form.

(c) Without computing the value of $(20!)^2$ determine how many zeroes are at the end of this number when it is written in decimal form. Justify your answer.

## Solution

(a) $6! = 2^4 \cdot 3^2 \cdot 5$

(b) $20! = 2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$

(c)

$$\begin{aligned}
(20!)^2 &= (2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19)^2 \\
&= 2^{36} \cdot 3^{16} \cdot 5^8 \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \\
&= (2^8 \cdot 5^8) \cdot (2^{28} \cdot 3^{16} \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2) \\
&= (2 \cdot 5)^8 \cdot (2^{28} \cdot 3^{16} \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2) \\
&= 10^8 \cdot (2^{28} \cdot 3^{16} \cdot 7^4 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2)
\end{aligned}$$

Because the number has 8 10's in its factorization it has 8 zeroes at the end of it.

## Problem 43

In a certain town 2/3 of the adult men are married to 3/5 of the adult women. Assume that all marriages are monogamous. Also assume that there are at least 100 adult men in the town. What is the least possible number of adult men in the town? of adult women in the town?

## Solution

Let $m$ and $w$ be integers that represent the number of men and women in the town respectively. Since the marriages are monogamous

$$\frac{2}{3}m = \frac{3}{5}w$$

$$10m = 9w$$

$$2 \cdot 5 \cdot m = 3^2 \cdot w$$

By the unique factorization of integers theorem these numbers must have the same prime factors. Thus $10 \mid w$ and $9 \mid m$. But there must be at least 100 adult men in the town. Thus the least number of men $m$ such that $m \geq 100$ and $9 \mid m$ is 108. By substitution

$$10 \cdot 108 = 9 \cdot w$$

$$w = 10 \cdot \frac{108}{9} = 10 \cdot 12 = 120$$

Thus the least possible number of men in the town is 108 and the least possible number of women in the town in 120.

## Problem 44

Prove that if $n$ is any nonnegative integer whose decimal representation ends in 0, then $5 \mid n$.

*Theorem*: If $n$ is any nonnegative integer whose decimal representation ends in 0, then $5 \mid n$.

*Proof.* If the decimal representation of a nonnegative integer $n$ ends in $d_0$, then $n = 10m + d_0$ for some integer $m$. Thus if the decimal representation of $n$ ends in 0 then $n = 10m + 0 = 10m = 5(2m)$. It follows from closure under multiplication that $2m$ is an integer. Let that integer be $t$. Then $n = 5t$. It follows from the definition of divisibility that $5 \mid n$. □

## Problem 45

Prove that if $n$ is any nonnegative integer whose decimal representation ends in 5, then $5 \mid n$.

*Theorem*: If $n$ is any nonnegative integer whose decimal representation ends in 5, then $5 \mid n$.

*Proof.* It follows from Problem 43 that if $m$ is any nonnegative integer whose decimal representation ends in 0, then $5 \mid m$. Let $n = m + 5$. Then $n$ is any nonnegative integer whose decimal representation ends in 5. It follows from the fact that $5 \mid m$ that $m = 5t$ for some integer $t$. By substitution

$$n - 5 = 5t$$

$$n = 5t + 5 = 5(t + 1)$$

It follows from closure under addition that $t + 1$ is an integer. Let that integer be $s$. Then $n = 5s$. It follows from the definition of divisibility that $5 \mid n$. $\square$

## Problem 46

Prove that if the decimal representation of a nonnegative integer $n$ ends in $d_0 d_1$ and if $4 \mid (10d_1 + d_0)$, then $4 \mid n$.

*Theorem*: If the decimal representation of a nonnegative integer $n$ ends in $d_1 d_0$ and if $4 \mid (10d_1 + d_0)$, then $4 \mid n$.

*Proof.* If the decimal representation of a nonnegative integer $n$ ends in $d_1 d_0$, then there is an integer $s$ such that $n = 100s + 10d_1 + d_0$. Let $n$ be the decimal representation of any nonnegative integer, such that $n$ end in $d_1 d_0$, and $4 \mid (10d_1 + d_0)$. Then

$$10d_1 + d_0 = 4k$$

By substitution

$$n - 100s = 4k$$

$$n = 4k + 100s = 4(k + 25s)$$

It follows from closure under addition and multiplication that $k + 25s$ is an integer. Let that integer be $t$. Then $n = 4t$. It follows from the definition of divisibility that $4 \mid n$. $\square$

## Problem 47

Observe that

$$\begin{aligned}
7524 &= 7 \cdot 1000 + 5 \cdot 100 + 2 \cdot 10 + 4 \\
&= 7(999 + 1) + 5(99 + 1) + 2(9 + 1) + 4 \\
&= (7 \cdot 999 + 7) + (5 \cdot 99 + 5) + (2 \cdot 9 + 2) + 4 \\
&= (7 \cdot 999 + 5 \cdot 99 + 2 \cdot 9) + (7 + 5 + 2 + 4) \\
&= (7 \cdot 111 \cdot 9 + 5 \cdot 11 \cdot 9 + 2 \cdot 9) + (7 + 5 + 2 + 4) \\
&= (7 \cdot 111 + 5 \cdot 11 + 2) \cdot 9 + (7 + 5 + 2 + 4) \\
&= (\text{an integer divisible by } 9) + (\text{the sum of the digits of } 7524).
\end{aligned}$$

Since the sum of the digits of 7524 is divisible by 9, 7524 can be written as a sum of two integers each of which is divisible by 9. It follows from exercise 15 that 7524 is divisible by 9.

Generalize the argument given in this example to any nonnegative integer $n$. In other words, prove that for any nonnegative integer $n$, if the sum of the digits of $n$ is divisible by 9, then $n$ is divisible by 9.

*Theorem*: For any nonnegative integer $n$, if the sum of the digits of $n$ is divisible by 9, then $n$ is divisible by 9.

*Proof.* Assume that $n$ is a nonnegative integer whose whole number representation is $d_k...d_2d_1d_0$ and the sum of the digits of $n$ is divisible by 9. Then

$$9 \mid (d_k + ... + d_2 + d_1 + d_0)$$

$$\begin{aligned} n &= d_k 10^k + ... + d_2 10^2 + d_1 10 + d_0 \\ &= d_k(10^k - 1 + 1) + ... + d_2(10^2 - 1 + 1) + d_1(10 - 1 + 1) + d_0 \\ &= d_k(10^k - 1) + ... + d_2(10^2 - 1) + d_1(10 - 1) + (d_k + ... + d_2 + d_1 + d_0) \end{aligned}$$

There exists integers $c_1$, $c_2$,...,$c_k$ such that $10^i - 1 = 9c_i$.

$$n = 9c_k d_k + ... + 9c_2 d_2 + 9c_1 d_1 + (d_k + ... + d_2 + d_1 + d_0)$$

It follows from the definition of divisibility that $9 \mid (9c_k d_k + ... + 9c_2 d_k + 9c_1 d_1)$. It follows from the conditions of the problem that $9 \mid (d_k + ... + d_2 + d_1 + d_0)$. It follows from problem 15 that if two integers are divisible by 9 then their sum is also divisible by 9. Thus $9 \mid n$. □

## Problem 48

Prove that for any nonnegative integer $n$m if the sum of the digits of $n$ is divisible by 3, then $n$ is divisible by 3.

*Theorem*: If the sum of the digits of any nonnegative integer is divisible by 3 then the integer is divisible by 3.

*Proof.* Assume that $n$ is a nonnegative integer whose whole number representation is $d_k...d_2d_1d_0$ and the sum of the digits of $n$ is divisible by 3. Then

$$3 \mid (d_k + ... + d_2 + d_1 + d_0)$$

$$\begin{aligned} n &= d_k 10^k + ... + d_2 10^2 + d_1 10 + d_0 \\ &= d_k(10^k - 1 + 1) + ... + d_2(10^2 - 1 + 1) + d_1(10 - 1 + 1) + d_0 \\ &= d_k(10^k - 1) + ... + d_2(10^2 - 1) + d_1(10 - 1) + (d_k + ... + d_2 + d_1 + d_0) \end{aligned}$$

There exists integers $c_1$, $c_2$,...,$c_k$ such that $10^i - 1 = 9c_i$.

$$n = 9c_k d_k + ... + 9c_2 d_2 + 9c_1 d_1 + (d_k + ... + d_2 + d_1 + d_0)$$

It follows from the definition of divisibility that $9 \mid (9(c_k d_k + ... + c_2 d_k + c_1 d_1))$. It follows from the fact that $3 \mid 9$ and the transitivity of divisibility that $3 \mid (9c_k d_k + ... + 9c_2 d_k + 9c_1 d_1)$. It follows from the conditions of the problem that $3 \mid (d_k + ... + d_2 + d_1 + d_0)$. It follows from problem 15 that if two integers are divisible by 3 then their sum is also divisible by 3. Thus $3 \mid n$ □

## Problem 49

Given a positive integer $n$ written in decimal form, the alternating sum of the digits of n is obtained by starting with the right mist digits, subtracting the digits immediately to its left, adding the next digit to the left, subtracting the next digit, and so forth. For example, the alternating sum of the digits of 180,928 is 8 - 2 + 9 - 0 + 8 - 1 = 22. Justify the fact that for any nonnegative integer $n$, if the alternating sum of the digits of $n$ is divisible by 11, then $n$ is divisible by 11.

*Theorem*: For any nonnegative integer $n$, if the alternating sum of the digits of $n$ is divisible by 11, then $n$ is divisible by 11.

*Proof.* Assume that $n$ is a nonnegative integer whose whole number representation is $d_k ... d_2 d_1 d_0$ and the alternating sum of the digits of $n$ is divisible by 11.

$$11 \mid (d_0 - d_1 + d_2 - ... + d_k(-1)^k)$$

$$\begin{aligned} n &= d_k 10^k + ... + d_2 10^2 + d_1 10 + d_0 \\ &= d_k(10^k - (-1)^k + (-1)^k) + ... + d_2(10^2 - 1 + 1) + d_1(10 - 1 + 1) + d_0 \\ &= d_k(10^k - (-1)^k) + ... + d_2(10^2 - 1) + d_1(10 + 1) + (d_0 - d_1 + d_2 - ... + d_k(-1)^k) \end{aligned}$$

There exists integers $c_1, c_2, ..., c_k$ such that $10^i - 1 = 11c_i$ when $i$ is even and $10^i + 1 = 11c_i$ when $i$ is odd. Thus

$$n = 11c_k d_k + ... + 11c_2 d_2 + 11c_1 d_1 + (d_0 - d_1 + d_2 - ... + d_k(-1)^k)$$

It follows from the definition of divisibility that $11 \mid (11(c_k d_k + ... + c_2 d_2 + c_1 d_1))$. It follows from the definition of the problem that $11 \mid (d_0 - d_1 + d_2 - ... + d_k(-1)^k)$. It follows from problem 15 that if two integers are divisible by 11 then their sum is also divisible by 11. Thus $11 \mid n$. □