Section 5.4

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Problem 1

Suppose $a_1, a_2, a_3, ...$ is a sequence defined as follows:

$$\begin{aligned} &a_1 = 1,\ a_2 = 3, \\ &a_k = a_{k-2} + 2a_{k-1} \quad \text{for all integers } k \geq 3. \end{aligned}$$

Prove that a_n is odd for all integers $n \geq 1$.

Solution

Proof. Let $a_1, a_2, a_3, ...$ be the sequence defined by specifying that $a_1 = 1$, $a_2 = 3$, and $a_k = a_{k-2} + 2a_{k-1}$ for all integers $k \geq 3$, and let the property $P(n)$ be the statement that

$$a_n \text{ is odd for all integers } n \geq 1. \qquad \leftarrow P(n)$$

Show that $P(1)$ and $P(2)$ are true:

$$a_1 = 1 \text{ and } a_2 = 3 \text{ and 1 and 3 are both odd}$$

Show that for all integers $k \geq 2$, $P(i)$ is true for all integers i from 1 through $k \implies P(k+1)$: Let k be any integer with $k \geq 2$ and suppose that

$$a_i \text{ is odd for all integers } i \text{ with } 1 \leq i \leq k. \qquad \leftarrow \begin{array}{l}\text{inductive}\\ \text{hypothesis}\end{array}$$

We must show that

$$a_{k+1} \text{ is odd.} \qquad \leftarrow P(k+1)$$

But by definition of $a_1, a_2, a_3, ...$ we have that

$$\begin{aligned} a_{k+1} &= a_{k-1} + 2a_k && \text{by definition of } a_1, a_2, a_3, ... \\ &= odd + 2a_k && \text{by inductive hypothesis} \\ &= odd + even && \text{by definition of even} \\ &= odd && \text{by problem 4.1.19} \end{aligned}$$

Thus $P(k+1)$ is true. $\square$

Problem 2

Suppose $b_1, b_2, b_3...$ is a sequence defined as follows:

$$\begin{aligned} &b_1 = 4,\ b_2 = 12 \\ &b_k = b_{k-2} + b_{k-1} \quad \text{for all integers } k \geq 3. \end{aligned}$$

Prove that b_n is divisible by 4 for all integers $n \geq 1$.

Solution

Proof. Let $b_1, b_2, b_3...$ be a sequence defined by specifying that $b_1 = 4$, $b_2 = 12$, and $b_k = b_{k-2} + b_{k-1}$ for all integers $k \geq 3$, and let the property $P(n)$ be the expression

$$4 \mid b_n \quad \text{for all integers } n \geq 1. \qquad \leftarrow P(n)$$

Show that $P(1)$ and $P(2)$ are true:

$$b_1 = 4 \text{ and } b_2 = 12 \text{ and } 4 = 4 \cdot 1 \text{ and } 12 = 4 \cdot 3$$

Show that for all integers $k \geq 2$, $P(i)$ is true for all integers i from 1 through $k \implies P(k+1)$: Let k be any integer with $k \geq 2$ and suppose that

$$4 \mid b_i \text{ for all integers } i \text{ with } 1 \leq i \leq k \qquad \leftarrow \text{inductive hypothesis}$$

The inductive hypothesis implies that $b_i = 4t_i$ for some integer t_i. We must show that

$$4 \mid b_{k+1} \qquad \leftarrow P(k+1)$$

But by the definition of $b_1, b_2, b_3, ...$ we have that

$$\begin{aligned} b_{k+1} &= b_{k-1} + b_k && \text{by definition of } b_1, b_2, b_3, ... \\ &= 4t_{k-1} + 4t_k && \text{by the inductive hypothesis} \\ &= 4(t_{k-1} + t_k) \end{aligned}$$

which implies that $4 \mid b_{k+1}$ and thus $P(k+1)$ is true. ☐

Problem 3

Suppose that $c_0, c_1, c_2, ...$ is a sequence defined as follows:

$$\begin{aligned} &c_0 = 2,\ c_1 = 2,\ c_2 = 6 \\ &c_k = 3c_{k-3} \quad \text{for all integers } k \geq 3. \end{aligned}$$

Prove that c_n is even for all integers $n \geq 0$.

Solution

Proof. Let c_0, c_1, c_2 be the sequence defined by specifying that $c_0 = 2$, $c_1 = 2$, $c_2 = 6$ and $c_k = 3c_{k-3}$ for all integers $k \geq 3$, and let the property $P(n)$ be the statement that

$$c_n \text{ is even for all integers } n \geq 0. \qquad \leftarrow P(n)$$

Show that $P(0)$, $P(1)$, and $P(2)$ are true:

$$c_0 = 2,\ c_1 = 2, \text{ and } c_2 = 6 \text{ and all three are even.}$$

Show that for all integers $k \geq 2$, $P(i)$ is true for all integers i from 0 through $k \implies P(k+1)$: Let k be any integer with $k \geq 2$ and suppose that

$$c_i \text{ is even for all integers } i \text{ with } 0 \leq i \leq k. \qquad \leftarrow \text{inductive hypothesis}$$

The inductive hypothesis implies that $c_i = 2t_i$ for some integer t_i. We must show that

$$c_{k+1} \text{ is even} \qquad \leftarrow P(k+1)$$

But by the definition of $c_0, c_1, c_2, \ldots$ we have that

$$\begin{aligned} c_{k+1} &= 3c_{k-2} && \text{by definition of } c_0, c_1, c_2, \ldots \\ &= 3 \cdot 2t_{k-2} && \text{by inductive hypothesis} \\ &= 2 \cdot 3t_{k-2} \end{aligned}$$

which implies that c_{k+1} is an even integer and so $P(k+1)$ is true. □

Problem 4

Suppose that $d_1, d_2, d_3, \ldots$ is a sequence defined as follows

$$\begin{aligned} &d_1 = \frac{9}{10},\ d_2 = \frac{10}{11}, \\ &d_k = d_{k-1} \cdot d_{k-2} \quad \text{for all integers } k \geq 3. \end{aligned}$$

Prove that $0 < d_n \leq 1$ for all integers $n \geq 1$.

Solution

Proof. Let $d_1, d_2, d_3, \ldots$ be a sequence defined by specifying that $d_1 = \frac{9}{10}$, $d_2 = \frac{10}{11}$, and $d_k = d_{k-1} \cdot d_{k-2}$ for all integers $k \geq 3$ and let the property $P(n)$ be the inequality that

$$0 < d_n \leq 1 \quad \text{for all integers } n \geq 1 \qquad \leftarrow P(n)$$

Show that $P(1)$ and $P(2)$ are true:

$$d_1 = \frac{9}{10} \text{ and } d_2 = \frac{10}{11} \text{ and } 0 < \frac{9}{10} < \frac{10}{11} \leq 1.$$

Show that for all integers $k \geq 2$, $P(i)$ is true for all integers i from 1 through $k \implies P(k+1)$: Let k be any integer with $k \geq 2$ and suppose that

$$0 < d_i \leq 1 \quad \text{for all integers } i \text{ with } 1 \leq i \leq k. \qquad \leftarrow \text{inductive hypothesis}$$

We must show that

$$0 < d_{k+1} \leq 1 \qquad \leftarrow P(k+1)$$

But by the definition of $d_1, d_2, d_3, \ldots$ we have that

$$\begin{aligned}
d_{k+1} &= d_k \cdot d_{k-1} && \text{by the definition of } a_1, a_2, a_3, \ldots \\
&\leq 1 \cdot 1 && \text{by inductive hypothesis} \\
&\leq 1 \\
d_{k+1} &= d_k \cdot d_{k-1} && \text{by the definition of } a_1, a_2, a_3, \ldots \\
&> 0 \cdot 0 && \text{by inductive hypothesis} \\
&> 0
\end{aligned}$$

Thus $0 < d_{k+1} \leq 1$ and so $P(k+1)$ is true. □

Problem 5

Suppose that $e_0, e_1, e_2, \ldots$ is a sequence defined as follows:

$$\begin{aligned}
e_0 &= 12, \; e_1 = 29 \\
e_k &= 5e_{k-1} - 6e_{k-2} \quad \text{for all integers } k \geq 2.
\end{aligned}$$

Prove that $e_n = 5 \cdot 3^n + 7 \cdot 2^n$ for all integers $n \geq 0$.

Solution

Proof. Let $e_0, e_1, e_2, \ldots$ be the sequence defined by specifying that $e_0 = 12$, $e_1 = 29$, and $e_k = 5e_{k-1} - 6e_{k-2}$ for all integers $k \geq 2$ and let the property $P(n)$ be the equation that

$$e_n = 5 \cdot 3^n + 7 \cdot 2^n \quad \text{for all integers } n \geq 0. \qquad \leftarrow P(n)$$

Show that $P(0)$ and $P(1)$ are true:

$$e_0 = 12 \text{ and } 5 \cdot 3^0 + 7 \cdot 2^0 = 12 \text{ and } e_1 = 29 \text{ and } 5 \cdot 3^1 + 7 \cdot 2^1 = 29$$

Show that for all integers $k \geq 1$, $P(i)$ is true for all integers i from 0 through $k \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$e_i = 5 \cdot 3^i + 7 \cdot 2^i \quad \text{for all integers } i \text{ with } 0 \leq i \leq k \qquad \leftarrow \text{inductive hypothesis}$$

We must show that

$$e_{k+1} = 5 \cdot 3^{k+1} + 7 \cdot 2^{k+1} \qquad \leftarrow P(k+1)$$

But by the definition of $e_0, e_1, e_2, \ldots$ we have that

$$\begin{aligned}
e_{k+1} &= 5e_k - 6e_{k-1} && \text{by definition of } e_0, e_1, e_2 \\
&= 5(5 \cdot 3^k + 7 \cdot 2^k) - 6(5 \cdot 3^{k-1} + 7 \cdot 2^{k-1}) && \text{by inductive hypothesis} \\
&= 25 \cdot 3^k + 35 \cdot 2^k - 30 \cdot 3^{k-1} - 42 \cdot 2^{k-1} \\
&= 25 \cdot 3^k + 35 \cdot 2^k - 10 \cdot 3 \cdot 3^{k-1} - 21 \cdot 2 \cdot 2^{k-1} \\
&= 25 \cdot 3^k + 35 \cdot 2^k - 10 \cdot 3^k - 21 \cdot 2^k \\
&= 25 \cdot 3^k - 10 \cdot 3^k + 35 \cdot 2^k - 21 \cdot 2^k \\
&= 15 \cdot 3^k + 14 \cdot 2^k \\
&= 5 \cdot 3 \cdot 3^k + 7 \cdot 2 \cdot 2^k \\
&= 5 \cdot 3^{k+1} + 7 \cdot 2^{k+1}
\end{aligned}$$

which is the right-hand side of $P(k+1)$. $\square$

Problem 6

Suppose that $f_0, f_1, f_2, \ldots$ is a sequence defined as follows:

$$\begin{aligned}
&f_0 = 5,\ f_1 = 16 \\
&f_k = 7f_{k-1} - 10f_{k-2} \quad \text{for all integers } k \geq 2.
\end{aligned}$$

Prove that $f_n = 3 \cdot 2^n + 2 \cdot 5^n$ for all integers $n \geq 0$.

Solution

Proof. Let the sequence $f_0, f_1, f_2, \ldots$ be defined by specifying that $f_0 = 5$, $f_1 = 16$, $f_k = 7f_{k-1} - 10f_{k-2}$ for all integers $k \geq 2$ and let the property $P(n)$ be the equation

$$f_n = 3 \cdot 2^n + 2 \cdot 5^n \quad \text{for all integers } n \geq 0. \qquad \leftarrow P(n)$$

Show that $P(0)$ and $P(1)$ are true:

$$f_0 = 5 \text{ and } 3 \cdot 2^0 + 2 \cdot 5^0 = 5 \text{ and } f_1 = 16 \text{ and } 3 \cdot 2^1 + 2 \cdot 5^1 = 16$$

Show that for all integers $k \geq 1$, $P(i)$ is true for all integers i from 0 through $k \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$f_i = 3 \cdot 2^i + 2 \cdot 5^i \quad \text{for all integers } i \text{ with } 0 \leq i \leq k \qquad \leftarrow \text{inductive hypothesis}$$

We must show that

$$f_{k+1} = 3 \cdot 2^{k+1} + 2 \cdot 5^{k+1} \qquad \leftarrow P(k+1)$$

But from the definition of $f_1, f_2, f_3, \ldots$ we have that

$$\begin{aligned}
f_{k+1} &= 7f_k - 10f_{k-1} && \text{by the definition of } f_0, f_1, f_2, \ldots \\
&= 7(3\cdot 2^k + 2\cdot 5^k) - 10(3\cdot 2^{k-1} + 2\cdot 5^{k-1}) && \leftarrow \text{inductive hypothesis} \\
&= 21\cdot 2^k + 14\cdot 5^k - 30\cdot 2^{k-1} - 20\cdot 5^{k-1} \\
&= 21\cdot 2^k - 30\cdot 2^{k-1} + 14\cdot 5^k - 20\cdot 5^{k-1} \\
&= 21\cdot 2^k - 15\cdot 2\cdot 2^{k-1} + 14\cdot 5^k - 4\cdot 5\cdot 5^{k-1} \\
&= 21\cdot 2^k - 15\cdot 2^k + 14\cdot 5^k - 4\cdot 5^k \\
&= 6\cdot 2^k + 10\cdot 5^k \\
&= 3\cdot 2\cdot 2^k + 2\cdot 5\cdot 5^k \\
&= 3\cdot 2^{k+1} + 2\cdot 5^{k+1}
\end{aligned}$$

which is the right-hand side of $P(k+1)$. $\square$

Problem 7

Suppose that $g_1, g_2, g_3, \ldots$ is a sequence defined as follows:

$$\begin{aligned}
&g_1 = 3,\ g_2 = 5 \\
&g_k = 3g_{k-1} - 2g_{k-2} \quad \text{for all integers } k \geq 3.
\end{aligned}$$

Prove that $g_n = 2^n + 1$ for all integers $n \geq 1$.

Solution

Proof. Let the sequence $g_1, g_2, g_3, \ldots$ be defined by specifying that $g_1 = 3$, $g_2 = 5$, and $g_k = 3g_{k-1}$ for all integers $k \geq 3 - 2g_{k-2}$, and let $P(n)$ be the equation

$$g_n = 2^n + 1 \quad \text{for all integers } n \geq 1. \qquad \leftarrow P(n)$$

Show that $P(1)$ and $P(2)$ are true:

$$g_1 = 3 \text{ and } 2^1 + 1 = 3 \text{ and } g_2 = 5 \text{ and } 2^2 + 1 = 5$$

Show that for all integers $k \geq 2$, $P(i)$ is true for all integers i from 1 through $k \implies P(k+1)$: Let k be any integer with $k \geq 2$ and suppose that

$$g_i = 2^i + 1 \quad \text{for all integers } i \text{ with } 1 \leq i \leq k \qquad \leftarrow \text{inductive hypothesis}$$

We must show that

$$g_{k+1} = 2^{k+1} + 1 \qquad \leftarrow P(k+1)$$

But from the definition of $g_1, g_2, g_3, \ldots$ we have that

$$
\begin{aligned}
g_{k+1} &= 3g_k - 2g_{k-1} && \text{by the definition of } g_1, g_2, g_3, \ldots \\
&= 3(2^k + 1) - 2(2^{k-1} + 1) && \text{by inductive hypothesis} \\
&= 3 \cdot 2^k + 3 - 2 \cdot 2^{k-1} - 2 \\
&= 3 \cdot 2^k - 2 \cdot 2^{k-1} + 3 - 2 \\
&= 3 \cdot 2^k - 2^k + 1 \\
&= 2^k(3 - 1) + 1 \\
&= 2^k(2) + 1 \\
&= 2^{k+1} + 1
\end{aligned}
$$

which is the right hand side of $P(k+1)$. $\square$

Problem 8

Suppose that $h_0, h_1, h_2, \ldots$ is a sequence defined as follows:

$$
\begin{aligned}
&h_0 = 1,\ h_1 = 2,\ h_2 = 3 \\
&h_k = h_{k-1} + h_{k-2} + h_{k-3} \quad \text{for all integers } k \geq 3.
\end{aligned}
$$

(a) Prove that $h_n \leq 3^n$ for all integers $n \geq 0$.

(b) Suppose that s is any real number such that $s^3 \geq s^2 + s + 1$. (This implies that $3 > s > 1.83$.) Prove that $h_n \leq s^n$ for all $n \geq 2$.

Solution

(a) *Proof.* Let $h_0, h_1, h_2, \ldots$ be the sequence defined above and let $P(n)$ be the inequality

$$h_n \leq 3^n \quad \text{for all integers } k \geq 3. \qquad \leftarrow P(n)$$

Show that $P(0)$, $P(1)$, and $P(2)$ are true:

$$h_0 = 1 \text{ and } 1 \leq 3^0.\ \ h_1 = 2 \text{ and } 2 \leq 3^1.\ \ h_2 = 3 \text{ and } 3 \leq 3^2$$

Show that for all integers $k \geq 0$, $P(i)$ is true for all integers i from 2 through $k \implies P(k+1)$: Let k be any integer with $k \geq 2$ and suppose that

$$h_i \leq 3^i \quad \text{for all integers } i \text{ with } 0 \leq i \leq k \qquad \leftarrow \text{inductive hypothesis}$$

We must show that

$$h_{k+1} \leq 3^{k+1} \qquad \leftarrow P(k+1)$$

But from the definition of $h_0, h_1, h_2, \ldots$ we have that

$$\begin{aligned}
h_{k+1} &= h_k + h_{k-1} + h_{k-2} && \text{by definition of } h_0, h_1, h_2, \ldots \\
&\leq 3^k + 3^{k-1} + 3^{k-2} && \text{by inductive hypothesis} \\
&= 3^k + 3^{-1} \cdot 3^k + 3^{-2} \cdot 3^k \\
&= 3^k(1 + 3^{-1} + 3^{-2}) \\
&= 3^k \left(\frac{13}{9}\right) \\
&< 3^k(3) \\
&= 3^{k+1}
\end{aligned}$$

We therefore have that $h_{k+1} \leq 3^{k+1}$. $\square$

(b) *Proof.* Let $h_0, h_1, h_2, \ldots$ be the sequence defined above and let s be any real number such that $s^3 \geq s^2 + s + 1$. Now define a property $P(n)$ to be

$$h_n \leq s^n \quad \text{for all } n \geq 2. \qquad \leftarrow P(n)$$

Show that $P(2)$, $P(3)$, and $P(4)$ are true:

$h_2 = 3$ and $s^2 > 3.34$. $h_3 = 6$ and $s^3 > 6.12$. $h_4 = 11$ and $s^4 > 11.21$

Show that for all integers $k \geq 4$, $P(i)$ is true for all integers i from 2 through $k \implies P(k+1)$: Let k be any integer with $k \geq 4$ and suppose that

$$h_i \leq s^i \quad \text{for any integer } i \text{ with } 2 \leq i \leq k \qquad \leftarrow \text{inductive hypothesis}$$

We must show that

$$h_{k+1} \leq s^{k+1} \qquad \leftarrow P(k+1)$$

But by the definition of $h_2, h_3, h_4, \ldots$ we have that

$$\begin{aligned}
h_{k+1} &= h_k + h_{k-1} + h_{k-3} && \text{by definition of } h_0, h_1, h_2, \ldots \\
&\leq s^k + s^{k-1} + s^{k-2} && \text{by inductive hypothesis} \\
&= s^{k-2}(s^2 + s + 1) \\
&\leq s^{k-2}(s^3) \\
&= s^{k+1}
\end{aligned}$$

We therefore have that $h_{k+1} \leq s^{k+1}$. $\square$

Problem 9

Define a sequence $a_1, a_2, a_3, \ldots$ as follows: $a_1 = 1$, $a_2 = 3$, and $a_k = a_{k-1} + a_{k-2}$ for all integers $k \geq 3$. Use strong mathematical induction to prove that $a_n \leq \left(\frac{7}{4}\right)^n$ for all integers $n \geq 1$.

Solution

Proof. Let the property $P(n)$ be the inequality

$$a_n \leq \left(\frac{7}{4}\right)^n \qquad \leftarrow P(n)$$

Show that $P(1)$ and $P(2)$ are true:

$$a_1 = 1 \text{ and } \left(\frac{7}{4}\right)^1 = 1.75. \;\; a_2 = 3 \text{ and } \left(\frac{7}{4}\right)^2 \approx 3.06$$

Show that for all integers $k \geq 2$, $P(i)$ is true for all integers i from 1 through $k \implies P(k+1)$: Let k be any integer with $k \geq 2$ and suppose that

$$a_i \leq \left(\frac{7}{4}\right)^i \quad \text{for all integers } i \text{ with } 1 \leq i \leq k. \qquad \leftarrow \text{inductive hypothesis}$$

We must show that

$$a_{k+1} \leq \left(\frac{7}{4}\right)^{k+1} \qquad \leftarrow P(k+1)$$

But from the definition of $a_1, a_2, a_3, \ldots$ we have that

$$\begin{aligned}
a_{k+1} &= a_k + a_{k-1} && \text{by the definition of } a_1, a_2, a_3, \ldots \\
&\leq \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1} && \text{by inductive hypothesis} \\
&< \left(\frac{33}{28}\right)\left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1} \\
&= \left(\frac{7}{4}\right)^k \left(\left(\frac{33}{28}\right) + \left(\frac{7}{4}\right)^{-1}\right) \\
&= \left(\frac{7}{4}\right)^k \left(\left(\frac{33}{28}\right) + \left(\frac{4}{7}\right)\right) \\
&= \left(\frac{7}{4}\right)^k \left(\left(\frac{33}{28}\right) + \left(\frac{16}{28}\right)\right) \\
&= \left(\frac{7}{4}\right)^k \left(\frac{49}{28}\right) \\
&= \left(\frac{7}{4}\right)^k \left(\frac{7}{4}\right) \\
&= \left(\frac{7}{4}\right)^{k+1}
\end{aligned}$$

We therefore have that $a_{k+1} \leq \left(\frac{7}{4}\right)^{k+1}$. $\square$

Problem 10

The problem that was used to introduce ordinary mathematical induction in section 5.2 can also be solved using strong mathematical induction. Let $P(n)$ be "$n¢$ can be obtained using a combination of $3¢$ and $5¢$ coins." Use strong mathematical induction to prove that $P(n)$ is true for all integers $n \geq 8$.

Solution

Proof. Let the property $P(n)$ be the statement that

"$n¢$ can be obtained using a combination of $3¢$ and $5¢$ coins."

Show that $P(8)$, $P(9)$, and $P(10)$ are true:

$$8¢ = 5¢ + 3¢ \text{ and } 9¢ = 3¢ + 3¢ + 3¢ \text{ and } 10¢ = 5¢ + 5¢.$$

Show that for all integers $k \geq 10$, $P(i)$ is true for all integers i from 8 through $k \implies P(k+1)$: Let k be any integer with $k \geq 10$ and suppose that $i¢$ can be obtained using a combination of $3¢$ and $5¢$ coins for all integers i with $8 \leq i \leq k$. We must show that this implies that we can obtain $(k+1)¢$. Since $k \geq 10$ we have that $k - 2 \geq 8$. Therefore by the inductive hypothesis we can form $(k-2)¢$ and then add a $3¢$ coin and you have $(k-2)+3 = (k+1)¢$. Thus $P(k+1)$ is true. □

Problem 11

You begin a jigsaw puzzle by finding two pieces that match and fitting them together. Each subsequent step of the solution consists of fitting together two blocks made up of one or more pieces that have been previously assembled. Use strong mathematical induction to prove that the number of steps required to put together all n pieces of a jigsaw puzzle is $n - 1$.

Solution

Proof. Let the property $P(n)$ be the statement that

"$n - 1$ steps are required to assemble a jigsaw puzzle of n pieces."

Show that $P(1)$ is true: If there is only 1 piece in the jigsaw puzzle then 0 steps are required to assemble it and $1 - 1 = 0$.

Show that for all integers $k \geq 1$, $P(i)$ is true for all integers i from 1 through $k \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that a jigsaw of i pieces can be assembled in $i - 1$ steps for all integers i with $1 \leq i \leq k$. We must show that this implies that a jigsaw puzzle of $k + 1$ pieces can be assembled in $(k+1)-1 = k$ steps. Imagine that you have a jigsaw puzzle of $k+1$ pieces and you are one step away from completing the puzzle. Then you must have two blocks of jigsaw puzzles one with r pieces and one with s pieces where $r+s=k+1$, $1 \le r \le k$ and $1 \le s \le k$. By the inductive hypothesis the block with r pieces must have taken $r-1$ steps to assemble and the block with s pieces must have taken $s-1$ steps to assemble. Thus the number of steps to complete the puzzle will be $(r-1)+(s-1)+1 = r+s-1 = k$ steps. □

Problem 12

The sides of a circular track contain a sequence of cans of gasoline. The total amount in the cans is sufficient to enable a certain car to make one complete circuit of the track, and it could all fit in the cars gas tank at one time. Use mathematical induction to prove that it is possible to find an initial location for placing the car so that it will be able to traverse the entire track by using the various amounts of gasoline in the cans that it encounters along the way.

Solution

Proof. Let the property $P(n)$ be the statement that with n cans of gasoline distributed around a track it is possible to find a starting location so that a certain car will be able to traverse the entire track using the various amounts of gasoline encountered on the way.

Show that $P(1)$ is true: If there is one can of gasoline then simply start at that can, place all the gas in the car and you will be able to completely traverse the track.

Show that for all integers $k \ge 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \ge 1$ and suppose that with k cans of gas around the track there is a starting place so that you can completely traverse the track with the given amount of gas.

We must show that this implies that we can traverse the track with $k+1$ cans of gas. With the $k+1$ cans of gas around the track there must be at least one can with sufficient fuel to make it to the next can. Suppose not. That is suppose there does not exist any can such that you can make it from that particular can to the next. Let c_i be the ith can, g_i be the distance the gas in c_i can take the car, and d_i be the distance from c_i to the next can for all integers i with $1 \le i \le k+1$. Then the supposition implies that $d_i > g_i$. Thus

$$d_1 + d_2 + ... + d_{k+1} > g_1 + g_2 + ... + g_{k+1} \quad (1)$$

But left side of (1) is the total distance of the track and the right side of (1) is the total distance the fuel can move the car. Thus a contradiction is reached as the problem states the total fuel can take the car the entire duration of the track and so our supposition is false.

Now that we know that it is possible, find a can, $\text{can}_{\text{enough}}$, with enough fuel to move the car to the next can, can_{next}. Take all the fuel out of can_{next} and place it into $\text{can}_{\text{enough}}$ and then remove can_{next} from the track. Now there are only k cans on the track and so by the inductive hypothesis there is a starting point at which the car can completely traverse the track using the gasoline provided in the cans. Now put can_{next} back on the track in its original location and remove from $\text{can}_{\text{enough}}$ the quantity of gas that was removed from can_{next} and put it back into can_{next}. Now the starting location we selected when there were only k cans on the track will be able get us to $\text{can}_{\text{enough}}$. From there we will be able to get to can_{next} as $\text{can}_{\text{enough}}$ was selected for just this purpose. Now once we reach can_{next} the car will be have the same amount of fuel as when it traversed the track with only k cars and so will be able to complete a circuit. Thus $P(k+1)$ is true. □

Problem 13

Use strong mathematical induction to prove the existence part of the unique factorization of integers (Theorem 4.3.5): Every integer greater than 1 is either a prime number or a product of prime numbers.

Solution

Proof. Let the property $P(n)$ be the statement that every integer $n > 1$ is either a prime or a product of prime numbers.

Show that $P(2)$ is true: If $n = 2$ then $n > 1$ and n is prime. Thus $P(2)$ is true.

Show that for all integers $k \geq 2$, $P(i)$ is true for all integers i from 2 through $k \implies P(k+1)$: Let k be any integer with $k \geq 2$ and suppose that i is either a prime number or a product of prime numbers for all integers i with $2 \leq i \leq k$.

We must show that $k+1$ is either a prime number or a product of prime numbers.

Case 1 ($k+1$ is prime): In this case there is nothing more to show.

Case 2 ($k+1$ is not prime): Then $k = rs$ for some integers r and s with $1 < r < k+1$ and $1 < s < k+1$. Now by the inductive hypothesis we have that r and s are both either primes or a product of primes and so $k+1$ is a product of primes. □

Problem 14

Any product of two or more integers is a result of successive multiplications of two integers at a time. For instance here are a few of the ways in which $a_1a_2a_3a_4$ might be computed: (a_1a_2) or $((a_1a_2)a_3)a_4)$ or $a_1((a_2a_3)a_4)$. Use strong mathematical induction to prove that any product of two or more odd integers is odd.

Solution

Proof. Let $P(n)$ be the property that for all integers $n \geq 2$ the product of n odd integers is odd.

Show that $\boldsymbol{P(2)}$ is true: Problem 4.1.43 proves that the product of any two odd integers is odd.

Show that for all integers $\boldsymbol{k \geq 2}$, $\boldsymbol{P(i)}$ is true for all integers $\boldsymbol{i}$ from 2 through $\boldsymbol{k \implies P(k+1)}$: Let k be any integers with $k \geq 2$ and suppose that the product of any i odd integers is odd for all integers i with $1 \leq i \leq k$.

We must show that this implies that the product of any $k+1$ odd integers is odd. Consider a product of $k+1$ odd integers: $x_1, x_2, ..., x_{k+1}$. At some point you will arrive at the final multiplication needed to compute the product. Let L be the product of the left-hand factors and let R be the product of right-hand factors. Also assume that L is made up of l factors and that R is made up of r factors. Then $l + r = k + 1$ and

$$1 \leq l \leq k \quad \text{and} \quad 1 \leq r \leq k$$

By the inductive hypothesis L is odd and R is odd. It now follows from problem 4.1.43 that $L \cdot R$ is odd and so $P(k+1)$ is true. □

Problem 15

Define the "sum" of one integer to be that integer and use strong mathematical induction to prove that for all integers $n \geq 1$, any sum of n even integers is even.

Solution

Proof. Let the property $P(n)$ be the property that for all integers $n \geq 1$ the sum of n even integers is even.

Show that $\boldsymbol{P(1)}$ is true: Let $n = 1$ and then by definition the sum of any integer is itself and all even integers are even.

Show that for all integers $\boldsymbol{k \geq 1}$, $\boldsymbol{P(i)}$ is true for all integers $\boldsymbol{i}$ from 1 through $\boldsymbol{k \implies P(k+1)}$: Let k be any integer with $k \geq 1$ and suppose that for all integers i with $1 \leq i \leq k$ the sum of i even integers is even.

We must show that this implies that the sum of $k+1$ even integers is even. Consider a sum of $k+1$ even integers: $x_1, x_2, ..., x_{k+1}$. At some point you will arrive at the final addition needed to compute the sum. Let L be the sum of the left-handed summands and let R be the sum of the right-handed summands. Also assume that L is made up of l summands and R is made up of r summands. Then $l + r = k + 1$ and

$$1 \leq l \leq k \quad \text{and} \quad 1 \leq r \leq k$$

By the inductive hypothesis L is even and R is even. It now follows from theorem 4.1.1 that $L + R$ is even. □

Problem 16

Use strong mathematical induction to prove that for any integer $n \geq 2$, if n is even, then any sum of n odd integers is even, and if n is odd then any sum of n odd integers is odd.

Solution

Proof. Let $P(n)$ be the property that for all integers $n \geq 2$, the parity of any sum of n odd integers matches the parity of n.

Show that $P(2)$ and $P(3)$ are true: If $n = 2$ then $n \geq 2$ and n is even. Also by problem 4.1.27 the sum of any two odd integers is even. If $n = 3$ then $n \geq 2$ and n is odd. Also for integers a, b, and c

$$(2a + 1) + (2b + 1) + (2c + 1) = 2(a + b + c + 1) + 1 \quad \text{which is odd.}$$

Show that for all integers $k \geq 3$, $P(i)$ is true for all integers i from 2 through $k \implies P(k+1)$: Let k be any integer with $k \geq 3$ and suppose that for all integers i with $2 \leq i \leq k$ the parity of any sum of i odd integers matches the parity of i.

We must show that this implies that the parity of any sum of $k + 1$ odd integers matches the parity of $k + 1$. Consider a sum of $k + 1$ odd integers: $x_1, x_2, ..., x_{k+1}$. At some point you will arrive at the final addition needed to compute the sum. Let L be the sum of the left-handed summands and let R be the sum of the right-handed summands. Also assume that L is made of l summands and R is made up of r summands. Then $l + r = k + 1$ and

$$1 \leq l \leq k \quad \text{and} \quad 1 \leq r \leq k$$

This means that we can not immediately use the inductive hypothesis.

Case 1. $k + 1$ odd.

Case 1a. l odd.

$$\begin{aligned} l + r &= k + 1 \\ \text{odd} + r &= \text{odd} \implies r \text{ is even} \implies r \geq 2 \end{aligned}$$

Now since $r \geq 2$ we can use the inductive hypothesis to show that R is even. We do not know if $l \geq 2$ so we may not be able to use the inductive hypothesis but if $l < 2$ then $l = 1$ and so L is just an odd integer. If $l \geq 2$ then l by the inductive hypothesis L is an odd integer. Now we have that $L + R =$ odd + even = odd.

Case 1b. l even.

$$\begin{aligned} l + r &= k + 1 \\ \text{even} + r &= \text{odd} \implies r \text{ is odd} \implies r \geq 2 \end{aligned}$$

Now by the inductive hypothesis since $r \geq 2$, R is odd. If $l < 2$ then L is just an even integer. If $l > 2$ then by the inductive hypothesis L is even. Thus $L + R =$ even + odd = odd.

Case 2. $k + 1$ even.

Case 2a. l odd.

$$\begin{aligned} l + r &= k + 1 \\ \text{odd} + r &= \text{even} \implies r \text{ is odd} \end{aligned}$$

If $l < 2$ then L is just an odd integer. If $l > 2$ then by the inductive hypothesis L is odd. If $r < 2$ then L is just an odd integer. If $r > 2$ then by the inductive hypothesis R is odd. Thus $L+R =$ odd + odd = even.

Case 2b. l even $\implies l \geq 2$.

$$\begin{aligned} l + r &= k + 1 \\ \text{even} + r &= \text{even} \implies r \text{ is even} \implies r \geq 2 \end{aligned}$$

By the inductive hypothesis L and R are even. Thus
$L + R =$ even + even = even.

Now we have that the parity of the sum of $k+1$ odd integers matches the parity of $k + 1$. $\square$

Problem 17

Compute 4^1, 4^2, 4^3, 4^4, 4^5, 4^6, 4^7, and 4^8. Make a conjecture about the units digit of 4^n where n is a positive integer. Use strong mathematical induction to prove your conjecture.

Solution

$4^1 = 4$, $4^2 = 16$, $4^3 = 64$, $4^4 = 256$, $4^5 = 1024$, $4^6 = 4096$, $4^7 = 16384$, $4^8 = 65536$.

Conjecture. *For all integers $n > 0$ the units digit of 4^n is 4 if n is odd and 6 if n is even.*

Proof. Let the property $P(n)$ be the conjecture to be proved.

Show that $P(1)$ and $P(2)$ are true:

$$4^1 = 4 \text{ and } 4^2 = 16$$

Show that for all integers $k \geq 2$, $P(i)$ is true for all integers i from 1 through $k \implies P(k+1)$: Let k be any integer with $k \geq 2$ and suppose that the units digit of 4^i is 4 if i is odd and 6 if i is even for all integers i with $1 \leq i \leq k$.

We must show that this implies that the units digit of 4^{k+1} is 4 if $k+1$ is odd and 6 if $k+1$ is even.

Case 1 ($k+1$ is odd): In this case k must be even and so by the inductive hypothesis the units digit of 4^k must be 6. Thus $4^k = 10q + 6$ for some integer q.

$$4^{k+1} = 4 \cdot 4^k = 4(10q + 6) = 40q + 24 = 10(4q + 2) + 4$$

And so if $k+1$ is odd the units digit of 4^{k+1} is 4.

Case 2 ($k+1$ is even): In this case k must be odd and so by the inductive hypothesis the units digit of 4^k must be 4. Thus $4^k = 10r + 4$ for some integer r.

$$4^{k+1} = 4 \cdot 4^k = 4(10r + 4) = 40r + 16 = 10(4r + 1) + 6$$

And so if $k+1$ is even the units digit of 4^{k+1} is 6.

Thus $P(k+1)$ is true. □

Problem 18

Compute 9^0, 9^1, 9^2, 9^3, 9^4, and 9^5. Make a conjecture about the units digit of 9^n where n is a positive integer. Use strong mathematical induction to prove your conjecture.

Solution

$9^0 = 1$, $9^1 = 9$, $9^2 = 81$, $9^3 = 729$, $9^4 = 6561$, and $9^5 = 59049$.

Conjecture. *For all integers $n \geq 0$ the units digit of 9^n is 1 if n is even and 9 if n is odd.*

Proof. Let the property $P(n)$ be the conjecture to be proved.

Show that $P(0)$ and $P(1)$ are true:

$$9^0 = 1 \text{ and } 9^1 = 9$$

Show that for all integers $k \geq 1$, $P(i)$ is true for all integers i from 0 through $k \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that for all integers i with $1 \leq i \leq k$ the units digit of 9^i is 1 if i is even and 9 if i is odd.

We must show that this implies that the units digit of 9^{k+1} is 1 if i is even and 9 if i is odd.

Case 1 ($k + 1$ is odd): In this case k must be even and so by the inductive hypothesis the units digit of 9^k must be 1. Thus $9^k = 10q + 1$ for some integer q.

$$9^{k+1} = 9 \cdot 9^k = 9(10q + 1) = 90q + 9 = 10(9q) + 9$$

And so if $k + 1$ is odd the units digit of 9^{k+1} is 9.

Case 1 ($k + 1$ is even): In this case k must be odd and so by the inductive hypothesis the units digit of 9^k must be 9. Thus $9^k = 10r + 9$ for some integer r.

$$9^{k+1} = 9 \cdot 9^k = 9(10r + 9) = 90r + 81 = 10(9r + 8) + 1$$

And so if $k + 1$ is even the units digit of 9^{k+1} is 1.

Thus $P(k + 1)$ is true. □

Problem 19

Find the mistake in the following "proof" that purports to show that every nonnegative integer power of every nonzero real number is 1.

"proof:" Let r be any nonzero real number and let the property $P(n)$ be the equation $r^n = 1$.

Show that $P(0)$ is true: $P(0)$ is true because $r^0 = 1$ by definition of zeroth power.
Show that for all integers $k \geq 0$, $P(i)$ is true for all integers i from 0 through $k \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that $r^i = 1$ for all integers i with $0 \leq i \leq k$. This is the inductive hypothesis. We

must show that $r^{k+1} = 1$. Now

$$\begin{aligned}
r^{k+1} &= r^{k+k-(k-1)} && \text{because } k+k-(k-1) \\
& && = k+k-k+1 = k+1 \\
&= \frac{r^k \cdot r^k}{r^{k-1}} && \text{by the laws of exponents} \\
&= \frac{1 \cdot 1}{1} && \text{by the inductive hypothesis} \\
&= 1
\end{aligned}$$

Thus $r^{k+1} = 1$.

Solution

The mistake in this false proof is that the inductive hypothesis was used to show that $r^{k-1} = 1$. This is not justified because the inductive step supposed that $k \geq 0$ and that $r^i = 1$ is true for all integers i with $0 \leq i \leq k$. However since $k \geq 0$, k could equal 0 and then r^{k-1} would be r^{-1} which is outside the bounds of i in the inductive step. An attempt to remedy this could be to let $-1 \leq i \leq k$ but then $P(-1)$ would need to be shown to be true but $r^{-1} = \frac{1}{r} \neq 1$.

Problem 20

Use the well-ordering principle for the integers to prove Theorem 4.3.4: Every integer greater than 1 is divisible by a prime number.

Solution

Proof. Let n be any integer with $n > 1$ and let S be the set of all integers which divide n and are greater than 1. It follows from the fact the $n \mid n$ and $n > 1$ that S is not empty. We now have from the well-ordering principle for the integers that S contains a least element. Let this element be p. It follows that p is prime. Suppose not. That is, suppose that p is not prime. Then there exist integers a and b such that $p = ab$ and $1 < a < p$ and $1 < b < p$. It follows from the definition of divisibility that $a \mid p$. Now since $a \mid p$ and $p \mid n$ it follows from the transitivity of divisibility that $a \mid n$. We now have that $a > 1$ and $a \mid n$ and so $a \in S$. However this is a contradiction as $a < p$ and $a \in S$ but p was selected to be the least element of S. $\square$

Problem 21

Use the well-ordering principle for the integers to prove the existence part of the unique factorization of integers theorem: Every integer greater than 1 is either a prime or a product of prime numbers.

Solution

Let S be the set of all integers greater than 1 which are not prime and cannot be factored into a product of primes. Suppose that S is not empty. Then by the well-ordering principle for the integers we have that S has a least element n. Since n is not a prime n can be written as a product of two integers $n = ab$ such that $1 < a < n$ and $1 < b < n$. From the definition of S we know that a and b are not prime. Since $a < n$ and $b < n$ they cannot be in S as n is the least element of S. However since $1 < a$ and $1 < b$ we know the reason that a and b were excluded from S must be that a and b are both either prime or can be factored into primes. But both cases are contradictions since both cases imply that n can be factored into a product of primes. Thus our supposition that S is not empty is false and so there do not exist integers greater than 1 which are not prime and cannot be factored into a product of primes. Hence all integers are either prime or can be factored into a product of primes.

Problem 22

(a) The Archimedean property for the rational numbers states that for all rational numbers r, there is an integer n such that $n > r$. Prove this property.

(b) Prove that given any rational number r, the number $-r$ is also rational.

(c) Use the results of parts (a) and (b) above to prove that given any rational number r, there is an integer m such that $m < r$.

Solution

(a) *Proof.* Let r be any rational number and suppose that n is an integer.

Case 1 ($r \leq 0$): In this case let $n \geq 1$ and then $n > r$.

Case 2 ($r > 0$): In this case $r = \frac{a}{b}$ for $a, b \in \mathbb{Z}^+$. Let $n = 2a$.

$$\begin{aligned} 2 &> 1 \\ 2a &> a && a \geq 0 \\ 2a &> \frac{a}{b} && b \geq 1 \\ n &> r \end{aligned}$$

Thus for every rational number there is an integer that is greater. □

(b) *Proof.* Let r be any rational number. From the definition of rational we have that $r = \frac{a}{b}$ for some integers a and b with $b \neq 0$. It follows that $-r = -\frac{a}{b} = \frac{-1 \cdot a}{b}$. Since integers are closed under multiplication we have that $-r$ is a ratio of two integers with the denominator not 0 and so $-r$ is rational. □

(c) *Proof.* Let r be any rational number and suppose that m is an integer. By part (a) we have that there exists an integer n and a rational number s such that $n > s$. It follows that $-n < -s$. Now from closure under multiplication and from part (b) we have that $-n$ is an integer and $-r$ is a rational number. Let $m = -n$ and let $r = -s$. Now we have that $m < r$ as was to be shown. $\square$

Problem 23

Use the results of exercise 22 and the well-ordering principle of the integers to show that given any rational number r, there exists an integer m such that $m \leq r < m + 1$.

Solution

Proof. Let r be any rational number and let S be the set of all integers which are greater than r. It follows from problem 5.4.22a that there is an integer n such that $n > r$ and so S is not empty. Also by 5.4.22c we have that there exists an integer p such that $p < r$. Now since every element in S is greater than r and $r > p$ we have that every element in S is greater than a fixed integer p. Thus by the well-ordering principle of the integers S has a least element v. It follows from the definition of S that $r < v$. We also have that $v - 1 \leq r$. Suppose not. That is suppose that $v - 1 > r$. If this were true then $v - 1 \in S$. But this is a contradiction as $v - 1 < v$ but v is the least element of S and so our supposition is false and $v - 1 \leq r$. Finally we have that $v - 1 \leq r < v$. Let $m = v - 1$ and then $m \leq r < m + 1$ as was to be shown. $\square$

Problem 24

Use the well-ordering principle to prove that given any integer $n \geq 1$, there exists an odd integer m and a nonnegative integer k such that $n = 2^k \cdot m$.

Solution

Proof. Let S be the set of all nonnegative integers of the form

$$\frac{n}{2^i} \quad \text{where } i \in \mathbb{Z}^{nonneg}$$

This set has at least one element because if $i = 0$ then

$$\frac{n}{2^i} = \frac{n}{2^0} = \frac{n}{1} = n \geq 1$$

It now follows from the well-ordering principle of the integers that S has a least element m. Then for some specific $i = k$,

$$m = \frac{n}{2^k}$$

Multiplying both sides by 2^k gives

$$n = 2^k \cdot m$$

We claim that m is odd. Suppose not. That is, suppose that m is not odd. Then m must be even and so $m = 2p$ for some integer p.

$$n = 2^k \cdot 2p = (2 \cdot 2^k)p = (2^{k+1})p$$

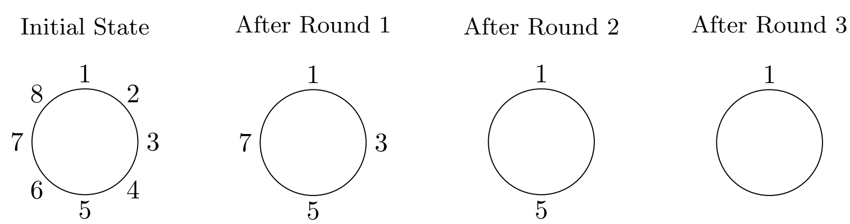
But now we can write that

$$p = \frac{n}{2^{k+1}}$$

which is a form of S. But this is a contradiction as $p < m$ as $m = 2p$ and m is the least element of S. Therefore our supposition that m is not odd is false and so m is odd. Thus $n = 2^k \cdot m$ for some odd integer m and some nonnegative integer k. $\square$

Problem 25

Imagine a situation in which 8 people, numbered consecutively 1-8, are arranged in a circle. Starting from person number 1, every second person in the circle is eliminated. The elimination process continues until only one person remains. In the first round the people numbered 2, 4, 6, and 8 are eliminated, in the second round the people numbered 3 and 7 are eliminated, and in the third round person number 5 is eliminated. So after the third round only person number 1 remains, as shown below.



a) Given a set of sixteen people arranged in a circle and numbered, consecutively 1-16, list the numbers of the people who are eliminated in each round if every second person is eliminated and the elimination process continues until only one person remains. Assume that the starting point is person number 1.

b) Use mathematical induction to prove that for all integers $n \geq 1$, given any set of 2^n people arranged in a circle and numbered consecutively 1 through 2^n, if one starts from person number 1 and goes repeatedly around the circle successively eliminating every second person, eventually only person number 1 will remain.

c) Use the result of part (b) to prove that for any nonnegative integers n and m with $2^n \leq 2^n + m < 2^{n+1}$, if $r = 2^n + m$, then given any set of r people arranged in a circle and numbered consecutively 1 through r, if one starts from person number 1 and goes repeatedly around the circle successively eliminating every second person, eventually only person number $(2m+1)$ will remain.

Solution

(a) **Round 1:** Person 2, 4, 6, 8, 10, 12, 14, and 16 will be eliminated.
Round 2: Person 3, 7, 11, and 15 will be eliminated.
Round 3: Person 5 and 13 will be eliminated.
Round 4: Person 9 will be eliminated.

(b) *Proof.* Let the property $P(n)$ be the statement that for all integers $n \geq 1$, given any set of 2^n people arranged in a circle and numbered consecutively 1 through 2^n, if one starts from person number 1 and goes repeatedly around the circle successively eliminating every second person, eventually only person number 1 will remain.

Show that $P(1)$ is true: If there are $2^1 = 2$ people arranged in a circle and numbered consecutively from 1 through 2 then eliminating every second person will result in eliminating person 2 and thereby ending the game with only person 1 remaining. Thus $P(1)$ is true.

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that if 2^k people are arranged and numbered consecutively around a circle then the elimination of every second person starting from person one will eventually result in only person 1 remaining.

We must show that this implies that if 2^{k+1} people are arranged and numbered consecutively around a circle then the elimination of every second person starting from person 1 will eventually result in only person one remaining. However since $2^{k+1} = 2 \cdot 2^k$ there will be twice as many people in $P(k+1)$ as there were in $P(k)$. So on our first pass of $P(k+1)$ all of the even labeled people will be eliminated. This will not eliminate the first person as 1 is odd and it will eliminate half of the people from the circle. Now there are only 2^k people on the circle and we will be starting at person 1. Then by the inductive hypothesis since there are only 2^k people on the circle and we are starting at person 1, person one will be the last person remaining on the circle when the elimination is over. $\square$

(c) *Proof.* Let the property $P(m)$ be statement that for any nonnegative integers n and m with $2^n \leq 2^n + m < 2^{n+1}$, if $r = 2^n + m$, then given any set of r people arranged in a circle and numbered consecutively 1 through r, if one starts from person number 1 and goes repeatedly around the circle successively eliminating every second person, eventually only person number $(2m+1)$ will remain.

Show that $P(0)$ is true: If $n = 0$ and $m = 0$ then there are $2^0 + 0 = 1$ people on the circle and so the game will finish before it starts with only person number 1 on the circle and $2m + 1 = 2(0) + 1 = 1$ and so $P(0)$ is true for this case. For all other values of $n \geq 1$ and $m = 0$ we have $r = 2^n + m = 2^n + 0 = 2^n$ and from part (b) we know that in this configuration person 1 will remain. This matches the prediction person $2m + 1$ will remain as $2m + 1 = 2(0) + 1 = 1$.

Show that for all integers $k \geq 0$, $P(i)$ is true for all integers i from 0 through $k \implies P(k + 1)$: Let k be any integer with $k \geq 0$ and suppose that for all integers i with $0 \leq i \leq k$ and $2^n \leq 2^n + i < 2^{n+1}$, if $r = 2^n + i$, then given any set of r people arranged in a circle and numbered consecutively 1 through r, starting at person 1 and successively eliminating every second person will eventually result in only person $2i+1$ remaining.

We must show that this implies that $P(k + 1)$ is true. That is, we must show that if there are a certain number of people $r = 2^n + (k + 1)$ arranged and numbered consecutively around a circle from 1 through r, then starting at person 1 and successively eliminating every second person will eventually result in only person $2(k + 1) + 1$ remaining.

Case 1 ($r = 2^n + (k + 1) = 2^{n+1}$): This occurs when $i = k = 2^n - 1$.

$$r = 2^n + (k + 1) = 2^n + ((2^n - 1) + 1) = 2^n + 2^n = 2 \cdot 2^n = 2^{n+1}$$

This is the maximum value of i that is allowed in the inductive step so that in the next step, $P(k + 1)$, r will be the next power of n. However since r would then be a positive integer power of 2 we can conclude from the results of part (b) that the last person remaining in this case would be person 1. The formula for the person remaining, $2(k + 1) + 1$ would give

$$2(k + 1) + 1 = 2((2^n - 1) + 1) + 1 = 2^{n+1} + 1$$

While this answer seems incorrect because $2^{n+1} + 1$ will always be greater than 1 the formula is actually working fine. What the result means is that to find the last remaining person we would traverse the first 2^n people and then have $k + 1$ people left to traverse. But since $k + 1 = 2^n$ we would traverse the rest of the people on the circle and then stop at the next person which is where the +1 comes from and matches the fact that the remaining person should be person 1. In reality if such a circumstance occurred that $k + 1 = 2^n$ then you would relabel 2^n as 2^{n+1} and relabel m from 2^n to 0. Then $2m+1$ would be $2(0)+1 = 1$ which is correct.

Case 2 ($r = 2^n + (k + 1) \neq 2^{n+1}$): In this case $2^n + (k + 1)$ will be in between the two consecutive integer powers of 2 so that

$$2^n < 2^n + (k+1) < 2^{n+1}$$

Now consider the result of the first round of eliminations. In the first round of eliminations think of starting just behind person 1 and jumping over every odd labeled person once. Thus every odd numbered person corresponds with a jump and every jump corresponds with an elimination. However in a sequence from 1 through $a \geq 1$ where a is an odd integer there are $\frac{a+1}{2}$ odds and $\frac{a}{2}$ evens. We only consider the case that 2^n is even since if $2^n = 1$ then $n = 0$ but then $m = 0$ and the basis step showed that this works. Therefore since 2^n is even the parity of $r = 2^n + (k+1)$ depends only on the parity of $k+1$ as even + even = even and even + odd = odd.

Case 2a ($k+1$ odd): This implies that r is odd and that k is even. Now after the first round of eliminations the number of people remaining, r', will be

$$r' = r - \frac{r+1}{2} = \frac{2r}{2} - \frac{r+1}{2} = \frac{r-1}{2} = \frac{2^n + (k+1) - 1}{2} = 2^{n-1} + \frac{k}{2}$$

Now imagine that we have a new circle with $2^{n-1} + \frac{k}{2}$ people on it. By the inductive hypothesis the last person remaining will be person number

$$2 \cdot \frac{k}{2} + 1 = k + 1$$

However we need to be careful as the inductive hypothesis says that the people must be numbered according to their order on the circle currently but that is not the same as their order when there were r people still on the circle. For example if 5 people are on a circle and the first round of elimination occurs there will remain person 3 and 5. But when we applied the inductive hypothesis we needed to say that person 3 was the first person on the current circle and person 5 was the second person. Therefore now that we know that person $k+1$ will remain among all the people that survived the first elimination we need to know what that position was among all of the people. Since r is odd we will have eliminated 1 in the first round. We guarantee the existence of 3 since $n \geq 1$ and $k+1 \geq 1$ and so $r \geq 2^1 + 1 = 3$. Therefore the correspondence between the numbering of the first circle and the second circle is as follows

Circle 2: $1, 2, 3, 4, ..., r'$

Circle 1: $3, 5, 7, 9, ..., r$

It follows then that for all integers $1 \leq j \leq r'$ the formula that relates the two positions is

$$2 \cdot r'_j + 1 = r_j$$

Therefore if the position in the circle after the first elimination that will remain is position $k+1$ then the position in the original circle that will remain is

$$2(k+1)+1$$

Case 2b ($\boldsymbol{k+1}$ even): This implies that r is even and that k is odd. Thus after the first round of eliminations the number of people remaining, r', will be

$$r' = r - \frac{r}{2} = \frac{2r}{2} - \frac{r}{2} = \frac{r}{2} = \frac{2^n + (k+1)}{2} = 2^{n-1} + \frac{k+1}{2}$$

Since $k+1$ is even $\frac{k+1}{2}$ is an integer. Also

$$\begin{aligned} 1 &\leq k && \text{$k \geq 0$ and k is odd} \\ k+1 &\leq k+k \\ k+1 &\leq 2k \\ \frac{k+1}{2} &\leq k \end{aligned}$$

Now the inductive hypothesis gives that the person that will remain among those that survived the first elimination is

$$2 \cdot \frac{k+1}{2} + 1 = k+2$$

Since r is even person number 1 will not be eliminated and so the starting position among the people that survived the first round of eliminations will be 1. The correspondence between the two circles is as follows

Circle 2: $1, 2, 3, 4, ..., r'$

Circle 1: $1, 3, 5, 7, ..., r$

It follows then that for all integers $1 \leq j \leq r'$ the formula that relates the two positions is

$$2 \cdot r'_j - 1 = r_j$$

Therefore if the position in the circle after the first elimination that will remain is position $k+2$ then the position in the original circle that will remain is

$$2(k+2) - 1 = 2k + 4 - 1 = 2k + 3 = 2k + 2 = 1 = 2(k+1) + 1$$

Hence in all possible cases $P(k) \implies P(k+1)$. $\square$

Problem 26

Suppose $P(n)$ is a property such that

(a) $P(0)$, $P(1)$, $P(2)$ are all true,

(b) for all integers $k \geq 0$, if $P(k)$ is true, then $P(3k)$ is true. Must it follows that $P(n)$ is true for all integers $n \geq 0$? If yes, explain why, if no, give a counterexample.

Solution

It does not follow that $P(n)$ is true for all integers $n \geq 0$. For example suppose that the property $P(n)$ is the statement that n is either 0, 1, or 2, or n is a multiple of 3. Then $P(0)$, $P(1)$, $P(2)$ are all true and $P(k) \implies P(3k)$ as by definition $3k$ is a multiple of 3 for any integer k. But $P(4)$ is not true as $4 \neq 3k$ for any integer k.

Problem 27

Prove that if a statement can be proved by strong mathematical induction, then it can be proved by ordinary mathematical induction. To do this, let $P(n)$ be a property that is defined for integers n, and suppose the following two statements are true:

(a) $P(a), P(a+1), P(a+2), ..., P(b)$.

(b) For any integer $k \geq b$, if $P(i)$ is true for all integers i from a through k, then $P(k+1)$ is true.

The principle of strong mathematical induction would allow us to conclude immediately that $P(n)$ is true for all integers $n \geq a$. Can we reach the same conclusion using the principle of ordinary mathematical induction? Yes! To see this, let $Q(n)$ be the property

$$P(j) \text{ is true for all integers } j \text{ with } a \leq j \leq n.$$

Then use ordinary mathematical induction to show that $Q(n)$ is true for all integers $n \geq b$. That is, prove

(a) $Q(b)$ is true.

(b) For any integer $k \geq b$, if $Q(k)$ is true then $Q(k+1)$ is true.

Solution

Theorem. *Any statement that can be proved by strong mathematical induction can be proved by weak mathematical induction.*

Proof. Let $P(n)$ be a property that is defined for integers n and suppose that the following two statements are true:

(a) $P(a), P(a+1), P(a+2), ..., P(b)$ are all true.

(b) For any integer $k \geq b$, if $P(i)$ is true for all integers i from a through k, then $P(k+1)$ is true.

Now let the property $Q(n)$ be the statement that

$$P(j) \text{ is true for all integers } j \text{ with } a \leq j \leq n. \qquad \leftarrow Q(n)$$

Show that $Q(b)$ is true: This immediately follows from statement (a).

Show that for all integers $k \geq b$, $Q(k) \implies Q(k+1)$: Let k be any integer with $k \geq b$ and suppose that

$$P(j) \text{ is true for all integers } j \text{ with } a \leq j \leq k. \qquad \leftarrow Q(k) \text{ IH}$$

We must show that this implies that $Q(k+1)$ is true. That is, we must show that

$$P(j) \text{ is true for all integers } j \text{ with } a \leq j \leq k+1. \qquad \leftarrow Q(k+1)$$

It follows from the inductive hypothesis and from the truth of statement (b) that $P(k+1)$ is true. But now from the inductive hypothesis and the fact that $P(k+1)$ is true we have that $P(j)$ is true for all integers j with $a \leq j \leq k+1$. But this is $Q(k+1)$ and so we have shown that $Q(k) \implies Q(k+1)$. Now that we have proven the basis step and the inductive step we can conclude that the theorem is true. That is, we can conclude that $Q(n)$ is true for all integers $n \geq b$. But this means that for all $n \geq b$, $P(j)$ is true for all integers j with $a \leq j \leq n$. And since $a \leq b \leq n$ we have that $P(n)$ is true for all integers $n \geq a$ which is the same result obtained by strong mathematical induction. $\square$

Problem 28

Give examples to illustrate the proof of theorem 5.4.1.

Solution

For the existence portion of the proof two example are given $k+1=7$ which is odd and $k+1=4$ which is even.

Case 1 $(k+1)$ odd: Let $k+1=7$. Then $k=6$ and $\frac{k}{2}=\frac{6}{2}=3$. To represent 3 in binary (which we can do by the inductive hypothesis) we have

$$3 = 1 \cdot 2^1 + 1 \cdot 2^0$$
$$2 \cdot 3 + 1 = 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0$$

which is a sum of powers of 2 of the required form.

Case 2 $(k+1)$ even: Let $k+1=4$. Then $\frac{k+1}{2}=\frac{4}{2}=2$. To represent 3 in binary (which we can do by the inductive hypothesis) we have

$$2 = 1 \cdot 2^1 + 0 \cdot 2^0$$
$$2 \cdot 2 = 1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$$

which is a sum of powers of 2 of the required form.

To prove the uniqueness of the binary representation of any given integer n, suppose that $n = 15$. The binary representation of $(15)_{10} = (1111)_2$. Observe that if another binary integer has a most significant digit in just 1 place higher than our binary digit we will not be able to make up the difference even if all of our other digits are 1 and the other numbers digits are 0. For example $(1111)_2 < (10000)_2$.

Problem 29

It is a fact that every integer $n \geq 1$ can be written in the form

$$c_r \cdot 3^r + c_{r-1} \cdot 3^{r-1} + ... + c_2 \cdot 3^2 + c_1 \cdot 3 + c_0$$

where $c_r = 1$ or 2 and $c_i = 0, 1,$ or 2 for all integer $i = 0, 1, 2, ..., r-1$. Sketch a proof of this fact.

Solution

Proof. Let the property $P(n)$ be the equation

$$n = c_r \cdot 3^r + c_{r-1} \cdot 3^{r-1} + ... + c_2 \cdot 3^2 + c_1 \cdot 3 + c_0 \qquad \leftarrow P(n)$$

where $c_r = 1$ or 2 and $c_i = 0, 1,$ or 2 for all integer $i = 0, 1, 2, ..., r-1$.

Show that $P(1)$ is true: Let $r = 0$ and $c_0 = 1$. Then $1 \cdot 3^0 = 1 \cdot 1 = 1$ and $n = 1$.

Show that for all integers $k \geq 1$, $P(i)$ is true for all integers i from 1 through $k \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that for all integers i with $1 \leq i \leq k$

$$i = c_r \cdot 3^r + c_{r-1} \cdot 3^{r-1} + ... + c_2 \cdot 3^2 + c_1 \cdot 3 + c_0 \qquad \leftarrow \text{inductive hypothesis}$$

We must show that this implies that $P(k+1)$ is true. That is, we must show that

$$k + 1 = c_r \cdot 3^r + c_{r-1} \cdot 3^{r-1} + ... + c_2 \cdot 3^2 + c_1 \cdot 3 + c_0 \qquad \leftarrow P(k+1)$$

Case 1 ($\boldsymbol{k+1 = 3x}$ for some integer x): In this case $\frac{k+1}{3}$ is an integer. We must show that $\frac{k+1}{3} \leq k$.

$$\begin{aligned} 1 &\leq k && \color{blue}{k \geq 1} \\ k+1 &\leq k+k+k \\ k+1 &\leq 3k \\ \frac{k+1}{3} &\leq k \end{aligned}$$

It follows from the inductive hypothesis that

$$\begin{aligned} \frac{k+1}{3} &= c_r \cdot 3^r + c_{r-1} \cdot 3^{r-1} + ... + c_2 \cdot 3^2 + c_1 \cdot 3 + c_0 \\ k+1 &= c_r \cdot 3^{r+1} + c_r \cdot 3^{r-1} + ... + c_2 \cdot 3^3 + c_1 \cdot 3^2 + c_0 \cdot 3 \end{aligned}$$

which is a sum of powers of 3 of the form required.

Case 2 ($\boldsymbol{k+1 = 3x+1}$ for some integer x): In this case $\frac{k}{3}$ is an integer. It immediately follows from the inductive hypothesis that

$$\begin{aligned} \frac{k}{3} &= c_r \cdot 3^r + c_{r-1} \cdot 3^{r-1} + ... + c_2 \cdot 3^2 + c_1 \cdot 3 + c_0 \\ k+1 &= c_r \cdot 3^{r+1} + c_r \cdot 3^{r-1} + ... + c_2 \cdot 3^3 + c_1 \cdot 3^2 + c_0 \cdot 3 + 1 \end{aligned}$$

which is a sum of powers of 3 of the form required.

Case 3 ($\boldsymbol{k+1 = 3x+2}$ for some integer x): In this case $\frac{k-1}{3}$ is an integer. It follows that since $k \geq 1$, $\frac{k-1}{3} \geq 0$. Now by the inductive hypothesis we have

$$\begin{aligned} \frac{k-1}{3} &= c_r \cdot 3^r + c_{r-1} \cdot 3^{r-1} + ... + c_2 \cdot 3^2 + c_1 \cdot 3 + c_0 \\ k+1 &= c_r \cdot 3^{r+1} + c_r \cdot 3^{r-1} + ... + c_2 \cdot 3^3 + c_1 \cdot 3^2 + c_0 \cdot 3 + 2 \end{aligned}$$

which is a sum of powers of 3 of the form required.

The quotient remainder theorem guarantees that every integer can be expressed as either $3x$, $3x+1$, or $3x+2$ for some integer x. Therefore our three cases include every possible integer. It now follows that since $P(k+1)$ is true in every possible case $P(k) \implies P(k+1)$ and so the theorem is true. $\square$

Problem 30

Use mathematical induction to prove the existence part of the quotient-remainder theorem for integers $n \geq 0$.

Solution

Theorem. *Given any integer $n \geq 0$ and any positive integer d, there exist integers q and r such that*

$$n = dq + r \quad \textit{and} \quad 0 \leq r < d$$

Proof. Let the property $P(n)$ be the statement that given any integer $n \geq 0$ and any positive integer d, there exist integers q and r such that

$$n = dq + r \quad \text{and} \quad 0 \leq r < d \qquad \leftarrow P(n)$$

Show that $P(0)$ is true: Let $r = 0$ and let $q = 0$ then $d(0) + 0 = 0$ and $n = 0$.

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that there exist integers q' and r' such that

$$k = dq' + r' \quad \text{and} \quad 0 \leq r' < d \qquad \leftarrow P(k+1)$$

We must show that this implies that $P(k+1)$ is true. That is, we must show that

$$k + 1 = dq + r \quad \text{and} \quad 0 \leq r < d \qquad \leftarrow P(k+1)$$

Case 1 ($r' = d - 1$): By the inductive hypothesis we have that

$$\begin{aligned} k &= dq' + r' && \text{by inductive hypothesis} \\ k + 1 &= dq' + r' + 1 \\ k + 1 &= dq' + (d - 1) + 1 && \text{by substitution} \\ k + 1 &= dq' + d \\ k + 1 &= d(q' + 1) + 0 \end{aligned}$$

Therefore let $q = q' + 1$ and let $r = 0$ and then there exist integers q and r such that $k + 1 = dq + r$ and $0 \leq r < d$.

Case 2 ($r' < d - 1$): By the inductive hypothesis we have that

$$\begin{aligned} k &= dq' + r' && \text{by inductive hypothesis} \\ k + 1 &= dq' + r' + 1 \\ k + 1 &= dq + (r' + 1) \end{aligned}$$

It follows that since $r' < d - 1$, $r' + 1 < d$. Therefore let $q = q'$ and let $r = r' + 1$ and then there exist integers q and r such that $k + 1 = dq + r$ and $0 \leq r < d$.

Since we have proved the basis step and since $P(k+1)$ is shown in every possible case we conclude that the theorem is true. $\square$

Problem 31

Prove that if a statement can be proved by ordinary mathematical induction, then it can be proved by the well-ordering principle.

Solution

Theorem. *Any statement that can be proved by weak induction can be proved by the well-ordering principle.*

Proof. Let $P(n)$ be a property that is defined for integers n and let a be a fixed integer. Suppose that the following two statements are true:

(a) $P(a)$ is true.

(b) For any integer $k \geq a$, $P(k) \implies P(k+1)$.

Now let S be the set of all integers greater than or equal to a for which $P(n)$ is false. Suppose that S is not empty. Then by the well ordering principle S contains a least element r. It follows that $r \geq a$ and $P(r)$ is false. More specifically, $r > a$ because $P(a)$ is true. Now from statement (b) and the definition of S we know that $P(r-1)$ must be false because since $r - 1 \geq a$ and $P(r-1) \implies P(r)$. But this is a contradiction as we supposed that r was the least element in S. Thus our supposition that S is non-empty is false and S is empty. It follows that $P(n)$ is true for all integers $n \geq a$ which is the same result obtained by weak induction. □

Problem 32

Use the principle of ordinary induction to prove the well-ordering principle for the integers.

Solution

Theorem. *Weak mathematical induction $\implies$ the well-ordering principle.*

Proof. The question to ask is whether there exists any set that contradicts the well-ordering principle. If this is true then there exists a non-empty set $S = \{x \in \mathbb{Z} \mid x \geq a\}$ and S has no least element. Now define a property $P(n)$ that $i \notin S$ for all integers i with $a \leq i \leq n$. From the definition of S we verify $P(a)$ is true because if $a \in S$ then a would be the least element of S. Now suppose that $P(n)$ is true and show that $P(n) \implies P(n+1)$. It follows from the inductive hypothesis that $i \notin S$ for all $i = a, a+1, a+2, ..., n$. Therefore $n + 1 \notin S$ as this would then be the least element of S. It follows that $P(n)$ is true which is the property that there are no elements in S which are greater than or equal to a. But this means that S has no elements which is a contradiction of the definition of S. Hence our supposition that set S exists is false. It follows that weak mathematical induction $\implies$ the well-ordering principle. □