Section 5.2

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Problem 1

Use mathematical induction (and the proof of Proposition 5.2.1 as a model) to show that any amount of money of at least 14¢ can be made up using 3¢ and 8¢ coins.

Theorem. *For all integers $n \geq 14$, n¢ can be obtained using 3¢ and 8¢ coins.*

Proof. Let the property $P(n)$ be the sentence

$$n\text{¢ can be obtained using 3¢ and 8¢ coins.} \qquad \leftarrow P(n)$$

Show that $P(14)$ is true:
$P(14)$ is true because 14¢ can be obtained using one 8¢ coin and two 3¢ coins.

Show that for all integers $k \geq 14$, $P(k) \implies P(k+1)$: Suppose that k is any integer with $k \geq 14$ such that

$$k\text{¢ can be obtained using 3¢ and 8¢ coins.} \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$(k+1)\text{¢ can be obtained using 3¢ and 8¢ coins.} \qquad \leftarrow P(k+1)$$

Case 1 (There is an 8¢ coin among those used to make up the k¢.): In this case replace the 8¢ coin by three 3¢ coins; the result will be $(k+1)$¢.

Case 2 (There is not an 8¢ coin among those used to make up the k¢.): In this case because $k \geq 14$, at least five 3¢ coins must have been used. So remove five 3¢ coins and replace them by two 8¢ coins; the result will be $(k+1)$¢.

Thus in either case $(k+1)$¢ can be obtained using 3¢ and 8¢ coins. ☐

Problem 2

Use mathematical induction to show that any postage of at least 12¢ can be obtained using 3¢ and 7¢ stamps.

Theorem. *For all integers $n \geq 12$, n¢ can be obtained using 3¢ and 7¢ stamps.*

Proof. Let the property $P(n)$ be the sentence

n¢ can be obtained using 3¢ and 7¢ stamps. $\leftarrow P(n)$

Show that $P(12)$ is true:
$P(12)$ is true because 12¢ of postage can be obtained with four 3¢ stamps.

Show that for any integer $k \geq 12$, $P(k) \implies P(k+1)$: Suppose that k is any integer with $k \geq 12$ such that

k¢ of postage can be obtained using 3¢ and 7¢ stamps $\leftarrow P(k)$ IH

We must show that

$(k+1)$¢ of postage can be obtained using 3¢ and 7¢ stamps $\leftarrow P(k+1)$

Case 1 (There are at least two 7¢ stamps among those used to make up the k¢ of postage): In this case replace two of the 7¢ stamps with five of the 3¢ stamps; the result will be $(k+1)$¢ of postage.

Case 2 (There are less than two 7¢ stamps among those used to make up the k¢ of postage): In this case because $k \geq 12$, at least two 3¢ stamps must have been used. So remove two 3¢ stamps and replace them with one 7¢ stamp; the result will be $(k+1)$¢ of postage.

Thus in either case $(k+1)$¢ of postage can be obtained using 3¢ and 7¢ stamps. □

Problem 3

For each positive integer n, let $P(n)$ be the formula

$$1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(a) Write $P(1)$. Is $P(1)$ true?

(b) Write $P(k)$.

(c) Write $P(k+1)$.

(d) In a proof by mathematical induction that the formula holds for all integers $n \geq 1$, what must be shown in the inductive step?

Solution

(a) $P(1)$ is the sentence that

$$1^2 = \frac{1 \cdot (1+1)(2 \cdot 1 + 1)}{6}$$

$P(1)$ is true because

$$1^2 = 1 \quad \text{and} \quad \frac{1 \cdot (1+1)(2 \cdot 1 + 1)}{6} = \frac{1 \cdot (2)(3)}{6} = \frac{6}{6} = 1$$

(b) $P(k)$ is the sentence that

$$1^2 + 2^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6}$$

(c) $P(k+1)$ is the sentence that

$$1^2 + 2^2 + ... + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

(d) The inductive step must show that for some integer $k \geq 1$, $P(k) \implies P(k+1)$

Problem 4

For each integer n with $n \geq 2$, let $P(n)$ be the formula

$$\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}$$

(a) Write $P(2)$. Is $P(2)$ true?

(b) Write $P(k)$.

(c) Write $P(k+1)$.

(d) In a proof by mathematical induction that the formula holds for all integers $n \geq 2$, what must be shown in the inductive step?

Solution

(a) $P(2)$ is the sentence that

$$\sum_{i=1}^{1} i(i+1) = \frac{2 \cdot (2-1)(2+1)}{3}$$

$P(2)$ is true because

$$\sum_{i=1}^{1} i(i+1) = 2 \quad \text{and} \quad \frac{2 \cdot (2-1)(2+1)}{3} = \frac{2 \cdot (1)(3)}{3} = \frac{6}{3} = 2$$

(b) $P(k)$ is the sentence that

$$\sum_{i=1}^{k-1} i(i+1) = \frac{k(k-1)(k+1)}{3}$$

(c) $P(k+1)$ is the sentence that

$$\sum_{i=1}^{k} i(i+1) = \frac{(k+1)((k+1)-1)((k+1)+1)}{3}$$

(d) The inductive step must show that for some integer $k \geq 2$, $P(k) \implies P(k+1)$

Problem 5 and Solution

Fill in the missing pieces in the following proof that

$$1 + 3 + 5 + ... + (2n-1) = n^2$$

for all integers $n \geq 1$.

Proof. Let the property $P(n)$ be the equation

$$1 + 3 + 5 + ... + (2n-1) = n^2 \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: To establish $P(1)$ we must show that when 1 is substituted in place of n, the left-hand side equals the right hand side. But when $n = 1$, the left hand side is the sum of all the odd integers for 1 to $2 \cdot 1 - 1$, which is the sum of the odd integers from 1 to 1, which is just 1. The right hand side is 1^2 which also equals 1. So $P(1)$ is true.

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$.

$$\text{Suppose } 1 + 3 + 5 + ... + (2k-1) = k^2 \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$1 + 3 + 5 + ... + (2(k+1)+1) = (k+1)^2 \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} 1 + 3 + 5 + ... + (2(k+1)-1) & \\ &= 1 + 3 + 5 + ... + (2k+1) \\ &= (1 + 3 + 5 + ... + (2k-1)) + (2k+1) \\ &\quad \text{the next to last term is } 2k-1 \text{ because it is the last odd integer before } 2k+1 \\ &= k^2 + (2k+1) \quad \text{by the inductive hypothesis} \\ &= (k+1)^2 \end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Prove each statement in 6-9 using mathematical induction. Do not derive them form Theorem 5.2.2 or Theorem 5.2.3.

Problem 6

Theorem. $\forall n \in Z^+, \;\; 2 + 4 + 6 + ... + 2n = n^2 + n.$

Proof. Let the property $P(n)$ be the equation

$$2 + 4 + 6 + ... + 2n = n^2 + n \qquad \leftarrow P(n)$$

Show that $P(1)$ is true:

$$2(1) = 2 \quad \text{and} \quad 1^2 + 1 = 2.$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$.

$$\text{Suppose} \quad 2 + 4 + 6 + ... + 2k = k^2 + k \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$2 + 4 + 6 + ... + 2(k+1) = (k+1)^2 + (k+1) \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
&2 + 4 + 6 + ... + 2(k+1) \\
&\qquad = (2 + 4 + 6 + ... + 2k) + 2(k+1) \\
&\qquad = (k^2 + k) + (2k + 2) \quad \text{by the inductive hypothesis} \\
&\qquad = k(k+1) + 2(k+1) \\
&\qquad = (k+1)(k+2) \\
&\qquad = (k+1)((k+1)+1) \\
&\qquad = (k+1)^2 + (k+1)
\end{aligned}$$

which is the right hand side of $P(k+1)$. $\square$

Problem 7

Theorem. $\forall n \in \mathbb{Z}^+, \;\; 1 + 6 + 11 + 16 + ... + (5n-4) = \frac{n(5n-3)}{2}$

Proof. Let the property $P(n)$ be the equation

$$1 + 6 + 11 + 16 + ... + (5n-4) = \frac{n(5n-3)}{2} \qquad \leftarrow P(n)$$

Show that $P(1)$ is true:

$$5(1) - 4 = 1 \quad \text{and} \quad \frac{1(5 \cdot 1 - 3)}{2} = \frac{2}{2} = 1$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$.

$$\text{Suppose} \quad 1 + 6 + 11 + 16 + ... + (5k-4) = \frac{k(5k-3)}{2} \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$1 + 6 + 11 + 16 + ... + (5(k+1)-4) = \frac{(k+1)(5(k+1)-3)}{2} \quad \leftarrow P(k+1)$$

But the left hand side of $P(k+1)$ is

$$\begin{aligned}
1 + 6 + 11 + 16 + ... + (5(k+1)-4) \\
&= (1 + 6 + 11 + 16 + ... + (5k-4)) + (5(k+1)-4) \\
&= \frac{k(5k-3)}{2} + 5k + 1 \quad \text{by the inductive hypothesis} \\
&= \frac{5k^2-3k}{2} + \frac{10k+2}{2} \\
&= \frac{5k^2+7k+2}{2} \\
&= \frac{(k+1)(5k+2)}{2} \\
&= \frac{(k+1)(5(k+1)-3)}{2}
\end{aligned}$$

which is the right hand side of $P(k+1)$. □

Problem 8

Theorem. $\forall n \in \mathbb{Z}^{nonneg}, \;\; 1 + 2 + 2^2 + ... + 2^n = 2^{n+1} - 1$

Proof. Let the property $P(n)$ be the equation that

$$1 + 2 + 2^2 + ... + 2^n = 2^{n+1} - 1 \qquad \leftarrow P(n)$$

Show that $P(0)$ is true:

$$2^0 = 1 \quad \text{and} \quad 2^{0+1} - 1 = 1$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$. Suppose that

$$1 + 2 + 2^2 + ... + 2^k = 2^{k+1} - 1 \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$1 + 2 + 2^2 + ... + 2^{k+1} = 2^{(k+1)+1} - 1 \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
&1 + 2 + 2^2 + ... + 2^{k+1} \\
&\qquad = (1 + 2 + 2^2 + ... + 2^k) + 2^{k+1} \\
&\qquad = 2^{k+1} - 1 + 2^{k+1} \quad \text{by the inductive hypothesis} \\
&\qquad = 2(2^{k+1}) - 1 \\
&\qquad = 2^{(k+1)+1} - 1
\end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Problem 9

Theorem. *For all integers* $n \geq 3$, $4^3 + 4^4 + 4^5 + ... + 4^n = \frac{4(4^n - 16)}{3}$.

Proof. Let the property $P(n)$ be the equation

$$4^3 + 4^4 + 4^5 + ... + 4^n = \frac{4(4^n - 16)}{3} \qquad \leftarrow P(n)$$

Show that $P(3)$ is true:

$$4^3 = 64 \quad \text{and} \quad \frac{4(4^3 - 16)}{3} = 64$$

Show that for all integers $k \geq 3$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 3$. Suppose that

$$4^3 + 4^4 + 4^5 + ... + 4^k = \frac{4(4^k - 16)}{3} \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$4^3 + 4^4 + 4^5 + ... + 4^{k+1} = \frac{4(4^{k+1} - 16)}{3} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
&4^3 + 4^4 + 4^5 + ... + 4^{k+1} \\
&\qquad = (4^3 + 4^4 + 4^5 + ... + 4^k) + 4^{k+1} \\
&\qquad = \frac{4(4^k - 16)}{3} + 4^{k+1} \quad \text{by the inductive hypothesis} \\
&\qquad = \frac{4(4^k - 16)}{3} + \frac{3 \cdot 4 \cdot 4^k}{3} \\
&\qquad = \frac{4(4^k - 16 + 3 \cdot 4^k)}{3} \\
&\qquad = \frac{4(4 \cdot 4^k - 16)}{3} \\
&\qquad = \frac{4(4^{k+1} - 16)}{3}
\end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Prove each of the statements in 10-17 by mathematical induction.

Problem 10

Theorem. $\forall n \in \mathbb{Z}^+,\ 1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof. Let the property $P(n)$ be the equation

$$1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6} \qquad \leftarrow P(n)$$

Show that $P(1)$ is true:

$$1^2 = 1 \quad \text{and} \quad \frac{1(1+1)(2 \cdot 1 + 1)}{6} = \frac{1(2)(3)}{6} = \frac{6}{6} = 1$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be an integer with $k \geq 1$ and suppose that

$$1^2 + 2^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6} \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$\begin{aligned} 1^2 + 2^2 + ... + (k+1)^2 &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \\ 1^2 + 2^2 + ... + (k+1)^2 &= \frac{(k+1)(k+2)(2k+3)}{6} \qquad \leftarrow P(k+1) \end{aligned}$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned} &1^2 + 2^2 + ... + (k+1)^2 \\ &\qquad = (1^2 + 2^2 + ... + k^2) + (k+1)^2 \\ &\qquad = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{by the inductive hypothesis} \\ &\qquad = \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &\qquad = \frac{(k+1)(k(2k+1) + 6(k+1)}{6} \\ &\qquad = \frac{(k+1)(2k^2 + 7k + 6)}{6} \\ &\qquad = \frac{(k+1)(k+2)(2k+3)}{6} \end{aligned}$$

which is the right-hand side of $P(k+1)$. $\square$

Problem 11

Theorem. $\forall n \in \mathbb{Z}^+,\ 1^3 + 2^3 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)$.

Proof. Let the property $P(n)$ be the equation

$$1^3 + 2^3 + ... + n^3 = \left(\frac{n(n+1)}{2}\right) \qquad \leftarrow P(n)$$

Show that $P(1)$ is true:

$$1^3 = 1 \quad \text{and} \quad \frac{1(1+1)}{2} = \frac{2}{2} = 1$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$1^3 + 2^3 + ... + k^3 = \left(\frac{k(k+1)}{2}\right)^2 \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$1^3 + 2^3 + ... + (k+1)^3 = \left(\frac{(k+1)((k+1)+1)}{2}\right)^2$$

$$1^3 + 2^3 + ... + (k+1)^3 = \left(\frac{(k+1)((k+2)}{2}\right)^2 \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
&1^3 + 2^3 + ... + (k+1)^3 \\
&\qquad = (1^3 + 2^3 + ... + k^3) + (k+1)^3 \\
&\qquad = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \quad \text{by the inductive hypothesis} \\
&\qquad = (k+1)^2\left(\frac{k}{2}\right)^2 + (k+1)^2(k+1) \\
&\qquad = (k+1)^2\left(\left(\frac{k}{2}\right)^2 + (k+1)\right) \\
&\qquad = (k+1)^2\left(\frac{k^2}{4} + \frac{4k+4}{4}\right) \\
&\qquad = (k+1)^2\left(\frac{k^2+4k+4}{4}\right) \\
&\qquad = (k+1)^2\left(\frac{(k+2)^2}{4}\right) \\
&\qquad = (k+1)^2\left(\frac{k+2}{2}\right)^2 = \left(\frac{(k+1)((k+2)}{2}\right)^2
\end{aligned}$$

which is the right hand side of $P(k+1)$. $\square$

Problem 12

Theorem. $\forall n \in \mathbb{Z}^+,\ \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Proof. Let the property $P(n)$ be the equation

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1} \qquad \leftarrow P(n)$$

Show that $P(1)$ is true:

$$\frac{1}{1(1+1)} = \frac{1}{2} \quad \text{and} \quad \frac{1}{1+1} = \frac{1}{2}$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{k(k+1)} = \frac{k}{k+1} \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{(k+1)((k+1)+1)} = \frac{k+1}{(k+1)+1}$$

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{(k+1)((k+1)+1)} = \frac{k+1}{k+2} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
&\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{(k+1)((k+1)+1)} \\
&\qquad = \left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{k(k+1)}\right) + \frac{1}{(k+1)((k+1)+1)} \\
&\qquad = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \qquad \text{by the inductive hypothesis} \\
&\qquad = \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\
&\qquad = \frac{k^2+2k+1}{(k+1)(k+2)} \\
&\qquad = \frac{(k+1)^2}{(k+1)(k+2)} \\
&\qquad = \frac{k+1}{k+2}
\end{aligned}$$

which is the right-hand side of $P(k+1)$. $\square$

Problem 13

Theorem. *For all integers* $n \geq 2$, $\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}$.

Proof. Let the property $P(n)$ be the equation

$$\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3} \qquad \leftarrow P(n)$$

Show that $P(2)$ is true:

$$\sum_{i=1}^{1} i(i+1) = 2 \quad \text{and} \quad \frac{2(2-1)(2+1)}{3} = \frac{6}{3} = 2$$

Show that for all integers $k \geq 2$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 2$ and suppose that

$$\sum_{i=1}^{k-1} i(i+1) = \frac{k(k-1)(k+1)}{3} \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$\begin{aligned}
\sum_{i=1}^{(k+1)-1} i(i+1) &= \frac{(k+1)((k+1)-1)((k+1)+1)}{3} \\
\sum_{i=1}^{k} i(i+1) &= \frac{k(k+1)(k+2)}{3} \qquad \leftarrow P(k+1)
\end{aligned}$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
\sum_{i=1}^{k} i(i+1) &= \sum_{i=1}^{k-1} i(i+1) + k(k+1) \\
&= \frac{k(k-1)(k+1)}{3} + k(k+1) \quad \text{by the inductive hypothesis} \\
&= k(k+1)\left(\frac{k-1}{3} + 1\right) \\
&= k(k+1)\left(\frac{k-1}{3} + \frac{3}{3}\right) \\
&= k(k+1)\left(\frac{k+2}{3}\right) \\
&= \frac{k(k+1)(k+2)}{3}
\end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Problem 14

Theorem. $\forall n \in \mathbb{Z}^{nonneg}$, $\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$

Proof. Let the property $P(n)$ be the equation

$$\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2 \qquad \leftarrow P(n)$$

Show that $P(0)$ is true:

$$\sum_{i=1}^{1} i \cdot 2^i = 2 \quad \text{and} \quad 0 \cdot 2^{0+2} + 2 = 2$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$\sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{k+2} + 2 \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$\sum_{i=1}^{(k+1)+1} i \cdot 2^i = (k+1) \cdot 2^{(k+1)+2} + 2$$

$$\sum_{i=1}^{k+2} i \cdot 2^i = (k+1) \cdot 2^{k+3} + 2 \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
\sum_{i=1}^{k+2} i \cdot 2^i &= \sum_{i=1}^{k+1} i \cdot 2^i + (k+2) \cdot 2^{k+2} \\
&= k \cdot 2^{k+2} + 2 + (k+2) \cdot 2^{k+2} \quad \text{by the inductive hypothesis} \\
&= k \cdot 2^{k+2} + (k+2) \cdot 2^{k+2} + 2 \\
&= 2^{k+2}(k + (k+2)) + 2 \\
&= 2^{k+2}(2k+2) + 2 \\
&= 2 \cdot 2^{k+2}(k+1) + 2 \\
&= 2^{k+3}(k+1) + 2 \\
&= (k+1) \cdot 2^{k+3} + 2
\end{aligned}$$

which is the right-hand side of $P(k+1)$. $\square$

Problem 15

Theorem. $\forall n \in \mathbb{Z}^+,\ \sum_{i=1}^{n} i(i!) = (n+1)! - 1.$

Proof. Let the property $P(n)$ be the equation

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1 \qquad \leftarrow P(n)$$

Show that $P(1)$ is true:

$$\sum_{i=1}^{1} i(i!) = 1(1!) = 1 \quad \text{and} \quad (1+1)! - 1 = 2! - 1 = 2 - 1 = 1$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$\sum_{i=1}^{k} i(i!) = (k+1)! - 1 \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$\sum_{i=1}^{k+1} i(i!) = ((k+1)+1)! - 1$$
$$\sum_{i=1}^{k+1} i(i!) = (k+2)! - 1 \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
\sum_{i=1}^{k+1} i(i!) &= \sum_{i=1}^{k} i(i!) + (k+1)((k+1)!) \\
&= (k+1)! - 1 + (k+1)((k+1)!) \quad \text{by the inductive hypothesis} \\
&= (k+1)! + (k+1)((k+1)!) - 1 \\
&= (k+1)!(1 + (k+1)) - 1 \\
&= (k+1)!(k+2) - 1 \\
&= (k+2)! - 1
\end{aligned}$$

which is the right-hand side of $P(k+1)$. $\square$

Problem 16

Theorem. *For all integers $n \geq 2$,* $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$.

Proof. Let the property $P(n)$ be the equation

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n} \qquad \leftarrow P(n)$$

Show that $P(2)$ is true:

$$1-\frac{1}{2^2}=\frac{4}{4}-\frac{1}{4}=\frac{3}{4} \quad \text{and} \quad \frac{2+1}{2\cdot 2}=\frac{3}{4}$$

Show that for all integers $k \geq 2$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 2$ and suppose that

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k} \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{(k+1)^2}\right)=\frac{(k+1)+1}{2(k+1)}$$
$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{(k+1)^2}\right)=\frac{k+2}{2k+2} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
&\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{(k+1)^2}\right)\\
&\qquad=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)\\
&\qquad=\left(\frac{k+1}{2k}\right)\left(1-\frac{1}{(k+1)^2}\right) \qquad \text{by the inductive hypothesis}\\
&\qquad=\left(\frac{k+1}{2k}\right)\left(\frac{(k+1)^2}{(k+1)^2}-\frac{1}{(k+1)^2}\right)\\
&\qquad=\left(\frac{k+1}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^2}\right)\\
&\qquad=\left(\frac{\cancel{k+1}}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^{\cancel{2}}}\right)\\
&\qquad=\left(\frac{k^2+2k}{2k(k+1)}\right)\\
&\qquad=\left(\frac{\cancel{k}(k+2)}{\cancel{k}(2k+2)}\right)\\
&\qquad=\frac{k+2}{2k+2}
\end{aligned}$$

which is the right-hand side of $P(k+1)$. $\square$

Problem 17

Theorem. $\forall n \in \mathbb{Z}^{nonneg}$ $\prod_{i=0}^{n}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right) = \frac{1}{(2n+2)!}$.

Proof. Let the property $P(n)$ be the equation

$$\prod_{i=0}^{n}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right) = \frac{1}{(2n+2)!} \qquad \leftarrow P(n)$$

Show that $P(0)$ is true:

$$\frac{1}{2\cdot 0+1}\cdot\frac{1}{2\cdot 0+2} = \frac{1}{1}\cdot\frac{1}{2} = \frac{1}{2} \quad \text{and} \quad \frac{1}{(2\cdot 0+2)!} = \frac{1}{2!} = \frac{1}{2}$$

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$\prod_{i=0}^{k}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right) = \frac{1}{(2k+2)!} \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$\prod_{i=0}^{k+1}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right) = \frac{1}{(2(k+1)+2)!}$$

$$\prod_{i=0}^{k+1}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right) = \frac{1}{(2k+4)!} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
\prod_{i=0}^{k+1}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right) &= \prod_{i=0}^{k}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right)\cdot\left(\frac{1}{2((k+1)+1}\cdot\frac{1}{2(k+1)+2}\right) \\
&= \frac{1}{(2k+2)!}\cdot\left(\frac{1}{2k+3}\cdot\frac{1}{2k+4}\right) \quad \text{by the inductive hypothesis} \\
&= \frac{1}{(2k+4)(2k+3)((2k+2)!)} \\
&= \frac{1}{(2k+4)!}
\end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Problem 18

Theorem. *If x is a real number not divisible by π, then for all integers $n \geq 1$,*

$$\sin x + \sin 3x + \sin 5x + \ldots + \sin(2n-1)x = \frac{1 - \cos 2nx}{2\sin x}$$

Proof. Let the property $P(n)$ be the equation

$$\sin x + \sin 3x + \sin 5x + ... + \sin((2n-1)x) = \frac{1-\cos 2nx}{2\sin x} \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: On the left side of $P(1)$ we have

$$\sin((2\cdot 1 - 1)x) = \sin x$$

On the right-hand side of $P(1)$ we have

$$\frac{1-\cos(2\cdot 1\cdot x)}{2\sin x} = \frac{1-\cos 2x}{2\sin x} = \frac{1-(1-2\sin^2 x)}{2\sin x} = \frac{\cancel{2}\sin^{\cancel{2}} x}{\cancel{2}\sin x} = \sin x$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$\sin x + \sin 3x + \sin 5x + ... + \sin((2k-1)x) = \frac{1-\cos 2kx}{2\sin x} \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$\sin x + \sin 3x + \sin 5x + ... + \sin((2(k+1)-1)x) = \frac{1-\cos 2(k+1)x}{2\sin x}$$
$$\sin x + \sin 3x + \sin 5x + ... + \sin((2k+1)x) = \frac{1-\cos((2k+2)x)}{2\sin x} \qquad \leftarrow P(k+1)$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
&\sin x + \sin 3x + \sin 5x + ... + \sin((2k+1)x) \\
&= (\sin x + \sin 3x + \sin 5x + ... + \sin((2k-1)x)) + \sin((2k+1)x) \\
&= \frac{1-\cos 2kx}{2\sin x} + \sin((2k+1)x) \qquad \text{by the inductive hypothesis} \\
&= \frac{1-\cos 2kx}{2\sin x} + \sin(2kx + x) \\
&= \frac{1-\cos 2kx}{2\sin x} + \sin 2kx\cdot\cos x + \cos 2kx\cdot\sin x \qquad \begin{array}{l}\sin(a+b) = \\ \sin a\cos b + \cos a \sin b\end{array} \\
&= \frac{1-\cos 2kx}{2\sin x} + \frac{2\sin 2kx\cdot\sin x\cdot\cos x + 2\cos 2kx\cdot\sin^2 x}{2\sin x} \\
&= \frac{1-\cos 2kx}{2\sin x} + \frac{\sin 2kx\cdot\sin 2x + 2\cos 2kx\cdot\sin^2 x}{2\sin x} \qquad \begin{array}{l}\sin 2a = \\ 2\sin a\cdot\cos a\end{array} \\
&= \frac{1-\cos 2kx}{2\sin x} + \frac{\sin 2kx\cdot\sin 2x + 2\cos 2kx\cdot(\cos^2 x - \cos 2x)}{2\sin x} \qquad \begin{array}{l}\sin^2 a = \\ \cos^2 a - \cos 2a\end{array} \\
&= \frac{1-\cos 2kx}{2\sin x} + \frac{\sin 2kx\cdot\sin 2x + 2\cos 2kx\cdot\cos^2 x - 2\cos 2kx\cdot\cos 2x}{2\sin x} \\
&= \frac{1-\cos 2kx}{2\sin x} + \frac{-\cos(2kx+2x) + 2\cos 2kx\cdot\cos^2 x - \cos 2kx\cdot\cos 2x}{2\sin x} \qquad \begin{array}{l}-\cos(a+b) = \\ \sin a\cdot\sin b - \cos a\cdot\cos b\end{array} \\
&= \frac{1-\cos(2kx+2x) - \cos 2kx + 2\cos 2kx\cdot\cos^2 x - \cos 2kx\cdot\cos 2x}{2\sin x}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1 - \cos(2kx + 2x) + \cos 2kx(-1 + 2\cos^2 x - \cos 2x)}{2\sin x} \\
&= \frac{1 - \cos(2kx + 2x) + \cos 2kx(-1 + 2\cos^2 x - (1 - 2\sin^2 x)}{2\sin x} && \cos 2a = 1 - 2\sin^2 a \\
&= \frac{1 - \cos(2kx + 2x) + \cos 2kx(-2 + 2\cos^2 x + 2\sin^2 x)}{2\sin x} \\
&= \frac{1 - \cos(2kx + 2x) + \cos 2kx(-2 + 2(\cos^2 x + \sin^2 x))}{2\sin x} \\
&= \frac{1 - \cos(2kx + 2x) + \cos 2kx(-2 + 2(1))}{2\sin x} && \cos^2 a + \sin^2 a = 1 \\
&= \frac{1 - \cos(2kx + 2x) + \cos 2kx(0)}{2\sin x} \\
&= \frac{1 - \cos((2k + 2)x)}{2\sin x}
\end{aligned}
$$

which is the right-hand side of $P(k+1)$. □

Problem 19

Use mathematical induction, the product rule from calculus, and the facts that $\frac{d(x)}{dx} = 1$ and that $x^{k+1} = x \cdot x^k$ to prove that for all integers $n \geq 1$, $\frac{d(x^n)}{dx} = nx^{n-1}$.

Solution

Theorem. $\forall n \in \mathbb{Z}^+,\ \frac{d(x^n)}{dx} = nx^{n-1}$

Proof. Let the property $P(n)$ be the equation

$$\frac{d(x^n)}{dx} = nx^{n-1} \qquad \leftarrow P(n)$$

Show that $P(1)$ is true:

$$\frac{d(x^1)}{dx} = 1 \quad \text{and} \quad 1 \cdot x^{1-1} = 1 \cdot x^0 = 1 \cdot 1 = 1$$

Show that for all integers $k \geq 1$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 1$ and suppose that

$$\frac{d(x^k)}{dx} = kx^{k-1} \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$
\begin{aligned}
\frac{d(x^{k+1})}{dx} &= (k+1)x^{(k+1)-1} \\
\frac{d(x^{k+1})}{dx} &= (k+1)x^k && \leftarrow P(k+1)
\end{aligned}
$$

But the left-hand side of $P(k+1)$ is

$$\begin{aligned}
\frac{d(x^{k+1})}{dx} &= \frac{d(x \cdot x^k)}{dx} \\
&= \frac{d(x)}{dx} \cdot x^k + \frac{d(x^k)}{dx} \cdot x \quad \text{by the product rule} \\
&= \frac{d(x)}{dx} \cdot x^k + kx^{k-1} \cdot x \quad \text{by the inductive hypothesis} \\
&= 1 \cdot x^k + kx^{k-1} \cdot x \quad \tfrac{d(x)}{dx} = 1 \\
&= x^k + kx^k \\
&= x^k(1+k) \\
&= (k+1)x^k
\end{aligned}$$

which is the right-hand side of $P(k+1)$. □

Use the formula for the sum of the first n integers and/or the formula for the sum of a geometric sequence to evaluate the sums in 20-29 or to write them in closed form.

Problem 20 and Solution

$$\begin{aligned}
4+8+12+16+...+200 &= 4(1+2+3+...+50) \\
&= 4\left(\frac{50(50+1)}{2}\right) \\
&= 5,100
\end{aligned}$$

Problem 21 and Solution

$$\begin{aligned}
5+10+15+20+...+300 &= 5(1+2+3+...+60) \\
&= 5\left(\frac{60(60+1)}{2}\right) \\
&= 9,150
\end{aligned}$$

Problem 22 and Solution

$$\begin{aligned}
3+4+5+6+...+1000 &= (1+2+3+...+1000)-(1+2) \\
&= \frac{1000 \cdot 10001}{2} - 3 \\
&= 500,497
\end{aligned}$$

Problem 23 and Solution

$$\begin{aligned} 7+8+9+10+...+600 &= (1+2+3+...+600)-(1+2+3+4+5+6) \\ &= \frac{600\cdot 601}{2}-21 \\ &= 180,279 \end{aligned}$$

Problem 24 and Solution

$1+2+3+...+(k-1),$ where k is an integer and $k \geq 2$

$$= \frac{(k-1)((k-1)+1)}{2} = \frac{(k)(k-1)}{2}$$

Problem 25 and Solution

(a) $$\begin{aligned} 1+2+2^2+...+2^{25} &= \frac{2^{25+1}-1}{2-1} \\ &= 67,108,863 \end{aligned}$$

(b) $$\begin{aligned} 2+2^2+2^3+...+2^{26} &= 2(1+2+2^2+...+2^{25}) \\ &= 2(67,108,863) \\ &= 134,217,726 \end{aligned}$$

Problem 26 and Solution

$3+3^2+3^3+...+3^n,$ where n is an integer with $n \geq 1$

$$\begin{aligned} &= 3\left(\frac{3^n-1}{3-1}\right) \\ &= \frac{3^{n+1}-3}{2} \end{aligned}$$

Problem 27 and Solution

$5^3+5^4+5^5+...+5^k,$ where k is any integer with $k \geq 3$

$$\begin{aligned} &= 5^3\left(\frac{5^{k-2}-1}{5-1}\right) \\ &= \frac{5^{k+1}-5^3}{4} \end{aligned}$$

Problem 28 and Solution

$1+\frac{1}{2}+\frac{1}{2^2}+...+\frac{1}{2^n}$, where n is a positive integer

$$= \frac{\frac{1}{2}^{n+1}-1}{\frac{1}{2}-1} = \frac{\frac{1}{2}^{n+1}-1}{-\frac{1}{2}} = \frac{\frac{1}{2}^{n+1}}{-\frac{1}{2}^{1}} + \frac{-1}{-\frac{1}{2}}$$
$$= -\frac{1}{2^n}+2$$

Problem 29 and Solution

$1-2+2^2-2^3+...+(-1)^n 2^n$, where n is a positive integer

$$= 1-2+2^2-2^3+...+(-1\cdot 2)^n$$
$$= \frac{(-2)^{n+1}-1}{-3}$$

Problem 30

Find a formula in n, a, m, and d for the sum

$$(a+md)+(a+(m+1)d)+(a+(m+2)d)+...+(a+(m+n)d)$$

where m and n are integers, $n \geq 0$, and a and d are real numbers. Justify your answer.

Solution

Think of this sum as 2 separate sums. We have

$$(a+md)+(a+(m+1)d)+(a+(m+2)d)+...+(a+(m+n)d) =$$
$$(a+md)+(a+(md+d))+(a+(md+2d))+...+(a+(md+nd))$$
$$= \sum_{i=1}^{n+1}(a+md)+\sum_{i=1}^{n} id$$
$$= \sum_{i=1}^{n+1} a+\sum_{i=1}^{n+1} md + d\sum_{i=1}^{n} i$$
$$= a\sum_{i=1}^{n+1} 1 + md\sum_{i=1}^{n+1} 1 + d\sum_{i=1}^{n} i$$
$$= a(n+1)+md(n+1)+d\cdot\frac{n(n+1)}{2} \quad \text{by theorem 5.2.2}$$
$$= (n+1)\left(a+md+\frac{dn}{2}\right)$$

Problem 31

Find a formula in a, r, m and n for the sum

$$ar^m + ar^{m+1} + ar^{m+2} + ... + ar^{m+n}$$

where m and n are integers, $n \geq 0$ and a and r are real numbers. Justify your answer.

Solution

$$\begin{aligned} ar^m + & ar^{m+1} + ar^{m+2} + ... + ar^{m+n} \\ &= ar^m(1 + r + r^2 + ... + r^n) \\ &= ar^m \cdot \left(\frac{r^{n+1} - 1}{r - 1}\right) \quad \text{by theorem 5.2.3} \end{aligned}$$

Problem 32

You have two parents, four grandparents, eight great-grandparents, and so forth.

(a) If all of your ancestors were distinct, what would be the total number of your ancestors for the past 40 generations (counting your parents generation as number one)?

(b) Assuming that each generation represents 25 years, how long is 40 generations?

(c) The total number of people who have ever lived is approximately 10 billion, which equals 10^{10} people. Compare this fact with the answer to part (a). What do you deduce?

Solution

(a) $$\begin{aligned} 2^1 + 2^2 + 2^3 + ... + 2^40 &= 2(2^0 + 2^1 + 2^2 + ... + 2^{39}) \\ &= 2 \cdot \left(\frac{2^{39+1} - 1}{2 - 1}\right) = 2 \cdot 2^{40} - 2 = 2^{41} - 2 \\ &= 2,199,023,255,550 \end{aligned}$$

(b) $\frac{25 \text{ years}}{1 \text{ generation}} \cdot 40$ generations $= 1000$ years

(c) $10^{10} < 2^{41} - 2$ so the ancestors cannot all be distinct

Find the mistakes in the proof fragments in 33-35

Problem 33

Theorem. $\forall n \in \mathbb{Z}^+,\ 1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$.

"Proof": Certainly the theorem is true for $n = 1$ because

$$1^2 = 1 \quad \text{and} \quad \frac{1(1+1)(2 \cdot 1 + 1)}{6} = 1.$$

So the basis step is true. For the inductive step, suppose that for some integer $k \geq 1$,

$$k^2 = \frac{k(k+1)(2k+1)}{6}$$

We must show that

$$(k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

Solution

In this incorrect proof fragment both the inductive hypothesis and what is to be shown are wrong. The correct inductive hypothesis is that

$$1^2 + 2^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6} \qquad \leftarrow P(k) \text{ IH}$$

and what is to be shown is that

$$1^2 + 2^2 + ... + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \qquad \leftarrow P(k+1)$$

Problem 34

Theorem. $\forall n \in \mathbb{Z}^{nonneg},\ 1 + 2 + 2^2 + ... + 2^n = 2^{n+1} - 1$.

"Proof": Let the property $P(n)$ be

$$1 + 2 + 2^2 + ... + 2^n = 2^{n+1} - 1$$

Show that $\boldsymbol{P(0)}$ is true: The left-hand side of $P(0)$ is $1+2+2^2+...+2^0 = 1$. and the right-hand side is $2^{0+1} - 1 = 2 - 1 = 1$ also. So $P(0)$ is true.

Solution

The problem with this incorrect proof fragment is that the left hand side of $P(0)$ is not expressed correctly. When n is 0 then the left-hand side of $P(n)$ is simply

$$\sum_{i=0}^{0} 2^i = 1$$

It should be expressed simply as 1 even in expanded form.

Problem 35

Theorem. *For any integer* $n \geq 1$,

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1$$

"Proof": Let the property $P(n)$ be the equation

$$\sum_{i=1}^{n} i(i!) = (n+1)! - 1$$

Show that $P(1)$ is true: When $n = 1$

$$\begin{aligned} &\sum_{i=1}^{1} i(i!) = (1+1)! - 1 \\ \text{So} \qquad & 1(1!) = 2! - 1 \\ \text{and} \qquad & 1 = 1 \end{aligned}$$

Thus $P(1)$ is true.

Solution

The problem with this incorrect proof fragment is that the left-hand and the right-hand side of $P(1)$ were set equal to each other. If you are trying to prove that two expressions are equal to each other you should never assert that they are until that has been independently shown on one or both parts of the expression.

Problem 36

Use theorem 5.2.2 to prove that if m and n are any positive integers and m is odd, then $\sum_{k=0}^{m-1}(n+k)$ is divisible by m. Does the conclusion hold if m is even? Justify your answer.

Solution

Theorem. $\forall m, n \in \mathbb{Z}^+,\ m$ *is odd* $\implies m \mid \left(\sum_{k=0}^{m-1}(n+k)\right)$.

Proof. Let the property $P(n)$ be the expression

$$m \text{ is odd } \implies m \left| \left(\sum_{k=0}^{m-1} (n+k) \right) \right. \qquad \leftarrow P(n)$$

Show that $P(1)$ is true: Let m be any odd integer with $m \geq 1$. It follows from the definition of odd that $m = 2s + 1$ for some integer s.

$$\sum_{k=0}^{m-1}(1+k) = \sum_{i=1}^{m} i = \frac{m(m+1)}{2} = \frac{m((2s+1)+1)}{2} = \frac{m(\cancel{2}(s+1))}{\cancel{2}} = m(s+1)$$

Show that for all integers $q \geq 1$, $P(q) \implies P(q+1)$: Let q be any integer with $q \geq 1$ and suppose that

$$m \text{ is odd } \implies m \,\Big|\, \left(\sum_{k=0}^{m-1}(q+k)\right) \qquad \leftarrow P(q) \text{ IH}$$

We must show that

$$m \text{ is odd } \implies m \,\Big|\, \left(\sum_{k=0}^{m-1}((q+1)+k)\right) \qquad \leftarrow P(q+1)$$

Let m be any odd integer with $m \geq 1$.

$$\begin{aligned}
\sum_{k=0}^{m-1}((q+1)+k) &= \sum_{k=0}^{m-1}(q+k) + \sum_{k=0}^{m-1} 1 \\
&= mr + \sum_{i=1}^{m} 1 && \text{by the inductive hypothesis } \textstyle\sum_{k=0}^{m-1}(q+k) = mr, \text{ for some integer } r \\
&= mr + m \\
&= m(r+1)
\end{aligned}$$

which implies that $m \mid \left(\sum_{k=0}^{m-1}((q+1)+k)\right)$. □

Problem 37

Use theorem 5.2.2 and the result of exercise 10 to prove that if p is any prime number with $p \geq 5$, then the sum of squares of any p consecutive integers is divisible by p.

Solution

Theorem. *If p is any prime number with $p \geq 5$, then the sum of squares of any p consecutive integers is divisible by p.*

Proof. Let n be any integer and let p be any prime number with $p \geq 5$. Then

$$\sum_{i=0}^{p-1}(n+i)^2$$

defines a sum of squares of any p consecutive integers of at least 5 terms. Let the property $P(n)$ be the expression

$$p \,\Big|\, \sum_{i=0}^{p-1}(n+i)^2 \qquad \leftarrow P(n)$$

Show that $P(0)$ is true: The right-hand side of $P(0)$ is

$$\begin{aligned}
\sum_{i=0}^{p-1}(0+i)^2 &= \sum_{j=1}^{p}(j-1)^2 \\
&= \sum_{j=1}^{p}(j^2-2j+1) \\
&= \sum_{j=1}^{p}j^2 - 2\sum_{j=1}^{p}j + \sum_{j=1}^{p}1 \\
&= \frac{p(p+1)(2p+1)}{6} - p(p+1) + p \\
\sum_{i=0}^{p-1} i^2 &= \frac{(p)(2p-1)(p-1)}{6}
\end{aligned}$$

Multiply both sides by 6 to obtain

$$2 \cdot 3 \cdot \sum_{i=0}^{p-1} i^2 = (p)(2p-1)(p-1) \qquad (1)$$

It follows from closure under multiplication and addition that $2 \cdot 3 \cdot \sum_{i=0}^{p-1} i^2$ and $(p)(2p-1)(p-1)$ are integers. It now follows from the unique factorization of the integers theorem that the prime factor p on the right-hand side must be included in left-hand side of equation (1). However since $2 < 3 < p \geq 5$, p must be a factor of $\sum_{i=0}^{p-1} i^2$ and hence $p \,\Big|\, \sum_{i=0}^{p-1} i^2$.

Show that for all integers $k \geq 0$, $P(k) \implies P(k+1)$: Let k be any integer with $k \geq 0$ and suppose that

$$p \,\Bigg|\, \sum_{i=0}^{p-1}(k+i)^2 \qquad \leftarrow P(k) \text{ IH}$$

We must show that

$$p \,\Bigg|\, \sum_{i=0}^{p-1}((k+1)+i)^2 \qquad \leftarrow P(k+1)$$

But the right-hand side of $P(k+1)$ is

$$\begin{aligned}
\sum_{i=0}^{p-1}((k+1)+i)^2 &= \sum_{i=0}^{p-1}((k+i)+1)^2 \\
&= \sum_{i=0}^{p-1}(k+i)^2 + 2\sum_{i=0}^{p-1}(k+i) + \sum_{i=0}^{p-1}1
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{p-1}(k+i)^2 + 2\sum_{i=0}^{p-1} k + 2\sum_{i=0}^{p-1} i + p \\
&= \sum_{i=0}^{p-1}(k+i)^2 + 2kp + 2\sum_{i=1}^{p} i - 2p + p \\
&= \sum_{i=0}^{p-1}(k+i)^2 + 2kp + \cancel{2}\cdot\frac{p(p+1)}{\cancel{2}} - 2p + p \\
&= \sum_{i=0}^{p-1}(k+i)^2 + 2kp + p^2 + p - 2p + p \\
&= \sum_{i=0}^{p-1}(k+i)^2 + 2kp + p^2 \\
&= sp + 2kp + p^2 && \text{by the inductive hypothesis } \textstyle\sum_{i=0}^{p-1}(k+i)^2 = sp, \text{ for some integer } s \\
&= p(s+2k+p)
\end{aligned}
$$

which implies that $p \mid \sum_{i=0}^{p-1}((k+1)+i)^2$.

Show that for all integers $k \le 0$, $P(k) \implies P(k-1)$: Let k be any integer such that $k \le 0$ and suppose that the inductive hypothesis is true. We must show that

$$p \,\Bigg|\, \sum_{i=0}^{p-1}((k-1)+i)^2 \qquad \leftarrow P(k-1)$$

But the right-hand side of $P(k-1)$ is

$$
\begin{aligned}
\sum_{i=0}^{p-1}((k-1)+i)^2 &= \sum_{i=0}^{p-1}((k+i)-1)^2 \\
&= \sum_{i=0}^{p-1}(k+i)^2 - 2\sum_{i=0}^{p-1}(k+i) + \sum_{i=0}^{p-1} 1 \\
&= \sum_{i=0}^{p-1}(k+i)^2 - 2\sum_{i=0}^{p-1} k - 2\sum_{i=0}^{p-1} i + p \\
&= \sum_{i=0}^{p-1}(k+i)^2 - 2kp - 2\sum_{i=1}^{p} i + 2p + p \\
&= \sum_{i=0}^{p-1}(k+i)^2 + 2kp - \cancel{2}\cdot\frac{p(p+1)}{\cancel{2}} + 2p + p \\
&= \sum_{i=0}^{p-1}(k+i)^2 + 2kp - p^2 - p + 2p + p
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=0}^{p-1} (k+i)^2 + 2kp - p^2 + 2p \\
&= sp + 2kp - p^2 + 2p && \text{by the inductive hypothesis } \textstyle\sum_{i=0}^{p-1}(k+i)^2 = sp, \text{ for some integer } s \\
&= p(s + 2k - p + 2)
\end{aligned}
$$

which implies that $p \mid \sum_{i=0}^{p-1}((k-1)+i)^2$.

Since we have proved that $p \mid \sum_{i=0}^{p-1}(n+i)^2$ for all nonnegative integers and for all nonpositive integers we can conclude that if p is any prime number with $p \geq 5$, then the sum of squares of any p consecutive integers is divisible by p. $\square$