# ORDINAL LOGISTIC REGRESSION

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# INTRODUCTION

# Logistic Regression

- What if there are more than two categories?
  - Ordinal Logistic Regression
  - Multinomial / Nominal Logistic Regression

- When the outcomes are **ordered** we can generalize the binary logistic regression model.
- Examples:
  - Disagree, Neutral, Agree
  - Tropical Depression, Tropical Storm, Category 1, 2, 3, 4, 5 Hurricanes

# Ordinal Logistic Regression

- Models are used when the response variable is ordinal.
- Models can also be used when the continuous response variable has a **restricted range** and need to be split into categories.

# Logistic Models

- Binary Logistic Regression (probability that observation *i* has the event):

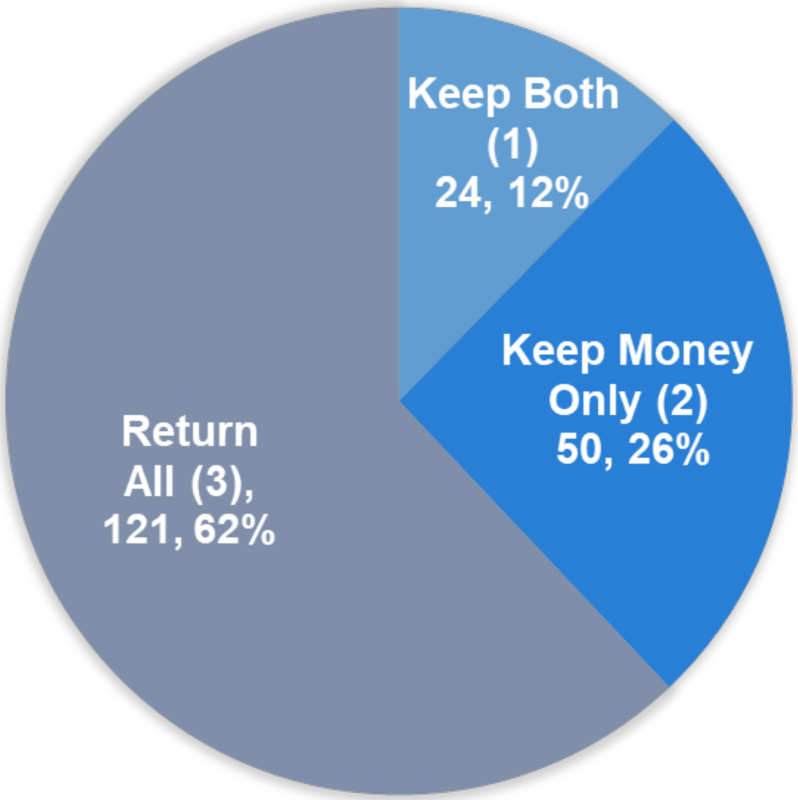
$$= \beta_0 + \beta_1 x_{1,i} + \cdots \beta_k x_{k,i}$$

- Ordinal Logistic Regression (probability that observation *i* has **at most** event *j*, and $j = 1, \ldots, m$ ordered events):

$$= \beta_{0,j} + \beta_1 x_{1,i} + \cdots \beta_k x_{k,i}$$

# "Found a Wallet?" Data Set

- Model the association between various factors and different levels of ethical responses on finding a wallet.
- 195 observations in the data set.



# "Found a Wallet?" Data Set

- Model the association between various factors and different levels of ethical responses on finding a wallet.
- Students at UPenn.
- Predictors:
  - **male:** indicator for a male student
  - **business:** indicator for student enrolled in business school
  - **punish:** how often the student was punished as a child – low (1), moderate (2), high (3)
  - **explain:** indicator of whether explanation for punishment was given

# PROPORTIONAL ODDS MODEL

# Methods for Modeling

- There are three methods for modeling ordinal logistic regression models:
  1. Cumulative Logit Model
  2. Adjacent Categories Model
  3. Continuation Ratio Model

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  2. Adjacent Categories Model
  3. Continuation Ratio Model

Easy to implement and interpret! Also, **most** common…

# Cumulative Logits

- Instead of modeling the typical logit, we will model the cumulative logits.
- If an ordinal variable has *m* levels with probabilities $(p_1, p_2, \ldots, p_m)$, then the cumulative logits are:

$$\log\left(\frac{p_{i,1}}{p_{i,2} + p_{i,3} + \cdots p_{i,m}}\right), \log\left(\frac{p_{i,1} + p_{i,2}}{p_{i,3} + \cdots + p_{i,m}}\right), \ldots, \log\left(\frac{p_{i,1} + \cdots + p_{i,m-1}}{p_m}\right)$$

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***m-1* Binary Logistic Regressions!**

- Event now becomes outcome $\leq j$ for categories $j = 1, \ldots, m$

# Logistic Models

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*m* – 1 Equations!

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- Intercept changes, but **slope parameters stays the same** (called **proportional odds** assumption)!

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$$\log\left(\frac{p_{i,1}}{p_{i,2}+p_{i,3}}\right)=\beta_{0,1}+\beta_1\text{male}_i+\beta_2\text{business}_i$$
$$+\beta_3\text{punishM}_i+\beta_4\text{punishH}_i+\beta_5\text{explain}_i$$

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# Proportional Odds Model

```
train <- wallet

train$punish <- factor(train$punish)

library(MASS)

clogit.model <- polr(factor(wallet) ~ male + business + punish + explain,
                     method = "logistic", data = train)

summary(clogit.model)
```

# Proportional Odds Model

```
Call:
polr(formula = factor(wallet) ~ male + business + punish + explain,
    data = train, method = "logistic")

Coefficients:
           Value Std. Error t value
male     -1.0598     0.3274  -3.237
business -0.7389     0.3556  -2.078
punish2  -0.6276     0.4048  -1.551
punish3  -1.4031     0.4823  -2.909
explain   1.0519     0.3408   3.086

Intercepts:
    Value   Std. Error t value
1|2 -2.5679  0.4190    -6.1287
2|3 -0.7890  0.3709    -2.1273

Residual Deviance: 307.3349
AIC: 321.3349
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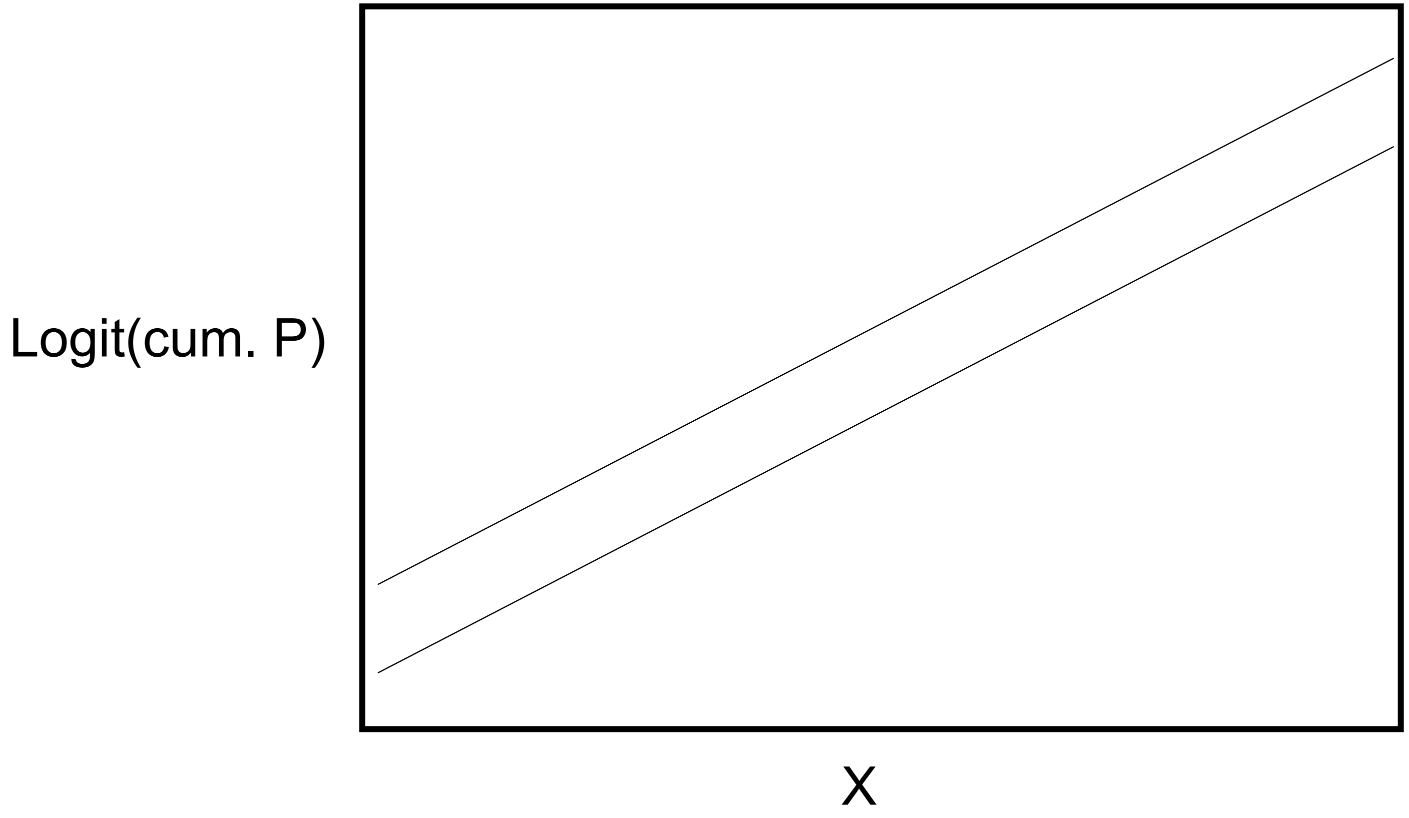
# Testing Assumptions




HOW DO WE TEST IF SLOPES ARE THE SAME?

# Score Test (Brant Test) for Proportional Odds

- Need to test to see if the slopes are statistically different from each other in the proportional odds model.
  - Null: Proportional Odds Correct (Slopes Equal Across Models)
  - Alternative: Proportional Odds Incorrect (Slopes NOT Equal Across Models)

# Brant Test

```
library(brant)
brant(clogit.model)
```

```
--------------------------------------------
Test for     X2   df   probability
--------------------------------------------
Omnibus      5.46    5    0.36
male         0.51    1    0.47
business     0.58    1    0.45
punish2      0.99    1    0.32
punish3      2.81    1    0.09
explain      0.25    1    0.62
--------------------------------------------

H0: Parallel Regression Assumption holds
```

# What if Assumption Fails?

- The proportional odds assumption may not be met for all variables.
- 2 Approaches:
  1. Partial Proportional Odds Model
  2. Multinomial Logistic Regression

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**Some** variables
fail assumption

**All** variables
fail assumption

# Partial Proportional Odds

```
library(VGAM)

plogit.model <- vglm(factor(wallet) ~ male + business + punish + explain,
                     data = train,
                     family = cumulative(parallel = F ~ business))

summary(plogit.model)
```

# Partial Proportional Odds

```
Call:
vglm(formula = factor(wallet) ~ male + business + punish + explain,
    family = cumulative(parallel = F ~ business), data = train)

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept):1  -2.6695     0.4466  -5.978 2.26e-09 ***
(Intercept):2  -0.7730     0.3678  -2.102  0.03557 *
male            1.0707     0.3258   3.287  0.00101 **
business:1      0.9722     0.4789   2.030  0.04236 *
business:2      0.6376     0.3810   1.674  0.09423 .
punish2         0.6300     0.4008   1.572  0.11594
punish3         1.3956     0.4727   2.952  0.00316 **
explain        -1.0532     0.3413  -3.086  0.00203 **
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# INTERPRETATION

# Model Notation

- With cumulative logits, increasing the right-hand side of the equation leads to an increased log(odds) of **higher** outcome category:

$$\log\left(\frac{p_{i,3}}{p_{i,1}+p_{i,2}}\right) = \beta_{0,1} + \beta_1 \mathrm{male}_i + \beta_2 \mathrm{business}_i$$
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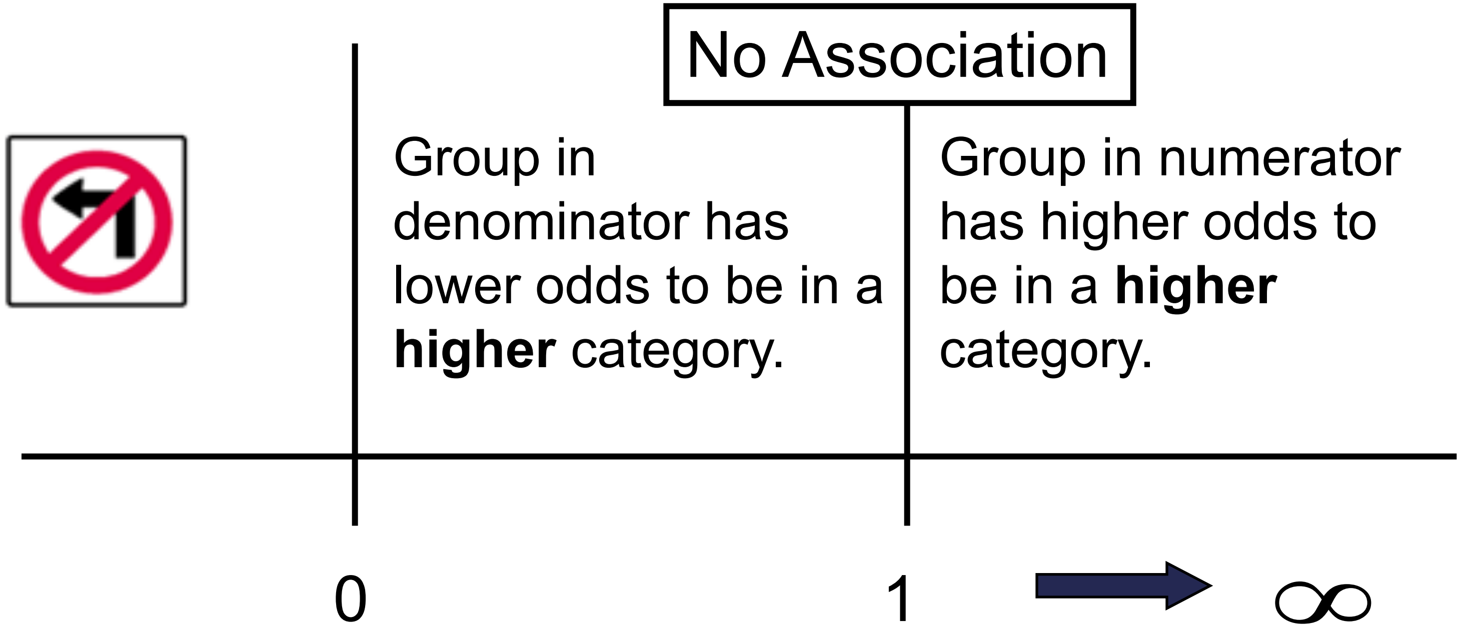
# Odds Ratio Interpretation – Descending

- Interpretation is still an odds ratio: $\mathbf{100 * \left(e^{\hat{\beta}_j} - 1\right)\%}$ **higher expected odds** of being in a higher category.



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  - Same increase in odds across all singular jumps in category.
  - Wallet example: OR same comparing 3 to 1,2 and from 3,2 to 1.

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- Male variable (coefficient = $-1.0598$, $\mathbf{100 * \left(e^{-1.0598} - 1\right) = -65.35\%}$ )
  - Males have $\mathbf{65.35\%}$ **lower expected odds** of being in a **higher** ethical category as compared to non-males.

# Odds Ratio Interpretation – Descending

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  - Wallet example: OR same comparing 3 to 1,2 and from 3,2 to 1.

- Business variable (coefficient = $-0.7389$, $\mathbf{100} * \left(e^{-\mathbf{0.7389}} - \mathbf{1}\right) = -\mathbf{52.24}\ \%$ )
  - Business school students have $\mathbf{52.24\%}$ **lower expected odds** of being in a **higher** ethical category as compared to students not in the business school.

# PREDICTIONS AND DIAGNOSTICS

# Similarities

- Ordinal logistic regression has a lot of the same aspects/issues as a binary logistic regression:
  - Multicollinearity still exists.
  - Non-convergence problems still exist.
  - Concordance, Discordance, Tied pairs still exist – so the c statistic still exists.
  - Generalized $R^2$ remains the same.

# Differences

- Ordinal logistic regression has a few aspects/issues that differ from a binary logistic regression:
  - A lot of the diagnostics for binary regression cannot be calculated easily since there are actually **multiple** models – ROC curves for each model?
  - Diagnostics / Influence plots are not available – residuals for each model?
  - Predicted probabilities are for **each** category.

# Predicted Probabilities

```
pred_probs <- predict(clogit.model, newdata = train, type = "probs")

head(pred_probs)
```

```
           1         2         3
1 0.12562481 0.3341195 0.5402557
2 0.04778463 0.1813420 0.7708734
3 0.02609095 0.1108549 0.8630542
4 0.12562481 0.3341195 0.5402557
5 0.07176375 0.2423258 0.6859105
6 0.02609095 0.1108549 0.8630542
```

# Confusion Matrix

- A confusion matrix is a matrix of all predicted responses compared to actual responses in terms of correct percentage.

| | **Predicted** | | |
|---|---|---|---|
| **Actual** | 4 | 11 | 9 |
| | 3 | 9 | 38 |
| | 0 | 12 | 109 |

# Confusion Matrix

- A confusion matrix is a matrix of all predicted responses compared to actual responses in terms of correct percentage.

| | **Predicted** | | |
|---|---|---|---|
| **Actual** | 16.7% | 45.8% | 37.5% |
| | 6.0% | 18.0% | 76.0% |
| | 0.0% | 9.9% | 90.1% |

# Good Confusion Matrix

| | Predicted | | |
|---|---|---|---|
| **Actual** | 100% | 0.0% | 0.0% |
| | 0.0% | 100% | 0.0% |
| | 0.0% | 0.0% | 100% |

# Confusion Matrix

- Weighted accuracy scores are common.

| | **Predicted** | | |
|---|---|---|---|
| **Actual** | 1 | 0.5 | 0 |
| | 0.5 | 1 | 0.5 |
| | 0 | 0.5 | 1 |

# Notch Graph

- Some people use notch graphs to show the “accuracy” gains the further out the prediction is from the truth.