ACCELERATED FAILURE TIME MODEL (AFT)

"The bad news is time flies. The good news is you're the pilot." — Michael Altshuler



MODEL STRUCTURE

Parametric Models

- AFT models are parametric we assume failure time (T) has a particular structure and distribution...we will be modeling time until failure (T)
- Kaplan-Meier estimation is nonparametric makes no assumption on failure time.
- Parametric methods allow for more detailed/precise estimation than nonparametric methods IF the distribution is specified correctly.
 - Ex: Easier to estimate medians, survival & hazard functions.

- The accelerated failure time (AFT) model is a regression that relates covariates (independent variables) to the event time T.
- The AFT model is a parametric model depends on knowledge of the underlying distribution of the data.

$$T_i = e^{\beta_0 + \beta_1 x_{i,1} + \dots + \beta_k x_{i,k} + \sigma e_i}$$

 We can transform this model into a linear regression model by taking the natural log of both sides of the equation:

$$T_i = e^{\beta_0 + \beta_1 x_{i,1} + \dots + \beta_k x_{i,k} + \sigma e_i}$$

The equation now becomes:

$$\log T_i = \beta_0 + \beta_1 x_{i,1} + \dots + \beta_k x_{i,k} + \sigma e_i$$

 We can transform this model into a linear regression model by taking the natural log of both sides of the equation:

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Variables used to predict *T*

 We can transform this model into a linear regression model by taking the natural log of both sides of the equation:

$$T_i = e^{\beta_0 + \beta_1 x_{i,1} + \dots + \beta_k x_{i,k} + \sigma e_i}$$

The equation now becomes:

$$\log T_i = \beta_0 + \beta_1 x_{i,1} + \dots + \beta_k x_{i,k} + \sigma e_i$$

Scale parameter of the distribution

Scale vs. Rate

$$\log T_i = \beta_0 + \beta_1 x_{i,1} + \dots + \beta_k x_{i,k} + \sigma e_i$$

Scale parameter for distribution

- Another common form is the multiplicative inverse of the scale, called the **rate**: $(1/\sigma)$.
- If σ is small, then events are not spread out \rightarrow events happening close to one another or rate is large
- If σ is large, then events are spread out \rightarrow smaller rate of events.

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Errors in the model

$$\log T_i = \beta_0 + \beta_1 x_{i,1} + \dots + \beta_k x_{i,k} + \sigma e_i$$

Errors in the model

- The errors in the AFT model can follow many different distributions.
- Assumptions:
 - Specify correct distribution
 - Distribution belongs to the location, scale family
 - Independence across observations

Common distributions for AFT model

Need a distribution that only takes on positive values and is right skewed (survival time distributions):

Weibull distribution

Lognormal distribution

Log-Logistic

Exponential

More to come on distributions later....

AFT Model – R with Lognormal

```
recid.aft.ln <- survreg(Surv(week, arrest) ~ fin + age + mar + prio, data = recid,
dist = 'lognormal')
summary(recid.aft.ln)</pre>
```

```
Call:
survreg(formula = Surv(week, arrest) ~ fin + age + mar + prio,
  data = recid, dist = "lognormal")
          Value
                  Std. Error z
                                р
(Intercept) 4.0146 0.3897 10.30 < 2e-16
fin
          0.3319 0.1657 2.00
                                 0.04524
         0.0333 0.0153 2.18 0.02959
age
    0.5609 0.2928 1.92
                                0.05541
mar
         prio
Log(scale) 0.2723
                 0.0765
                          3.56
                                0.00037
Scale= 1.31
Log Normal distribution
Loglik(model)= -685.5
                        Loglik(intercept only)= -697.9
    Chisq= 24.85 on 4 degrees of freedom, p= 5.4e-05
Number of Newton-Raphson Iterations: 4
n = 432
```

```
Call:
survreg(formula = Surv(week, arrest) ~ fin + age + mar + prio,
  data = recid, dist = "lognormal")
                                                                        Parameter estimates
           Value
                   Std. Error z
                                    р
(Intercept) 4.0146 0.3897
                             10.30
                                    < 2e-16
fin
           0.3319
                    0.1657
                             2.00
                                    0.04524
                             2.18
           0.0333 4 0.0153
                                    0.02959
age
                             1.92
          0.5609
                   0.2928
                                    0.05541
mar
          -0.0743
                    0.0264
                             -2.82
                                    0.00481
prio
Log(scale)
          0.2723
                   0.0765
                             3.56
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                            10.30
                                   < 2e-16
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                  0.1657
                             2.00
                                   0.04524
                                   0.02959
          0.0333 0.0153
                            2.18
age
                            1.92
                                   0.05541
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                            -2.82
                                   0.00481
prio
Log(scale) 0.2723
                   0.0765
                            3.56
                                   0.00037
```

P-values for estimates in the model

Scale= 1.31

```
Log Normal distribution

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                                  р
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          0.3319
                  0.1657
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                  0.2928
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          -0.0743 0.0264
                            -2.82
                                  0.00481
prio
Log(scale) 0.2723
                   0.0765
                            3.56
                                   0.00037
```

Test to see if the "scale" (variance-like parameter) is equal to 1 (or log(1)=0)

Scale= 1.31

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Chisq= 24.85 on 4 degrees of freedom, p= 5.4e-05

Number of Newton-Raphson Iterations: 4
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Scale= 1.31

Log Normal distribution

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Chisq= 24.85 on 4 degrees of freedom, p= 5.4e-05
Number of Newton-Raphson Iterations: 4
n= 432
```

Distribution assumed for the data

```
Call:
survreg(formula = Surv(week, arrest) ~ fin + age + mar + prio,
  data = recid, dist = "lognormal")
          Value
                  Std. Error z
                                  р
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          0.3319
                  0.1657
                          2.00
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                                 0.02959
          0.5609 0.2928 1.92
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         -0.0743 0.0264 -2.82 0.00481
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Log Normal distribution
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Loglik(model)= -685.5 Loglik(intercept only)= -697.9 Chisq= 24.85 on 4 degrees of freedom, p= 5.4e-05
```

Number of Newton-Raphson Iterations: 4

n= 432

Similar to an overall F test: H_0 : Intercept model is appropriate (i.e. NO covariates are significant) versus H_A : At least one covariate is significant

INTERPRETATION

AFT Model Parameter Interpretation

- If a parameter estimate is **positive**, increases in that variable **increase** the expected survival time.
- If a parameter estimate is **negative**, increases in that variable **decrease** expected survival times.
- If a parameter estimate is **zero**, increases in that variable have **no impact** on expected survival times.
- $100 \times (e^{\beta} 1)$ is the % increase in the expected survival time for each one-unit increase in the variable.

Variable	β Estimate	$100(e^{eta}-1)$
Financial Aid	0.3319	39.36%
Age at Release	0.0333	3.39%
Marital Status	0.5609	75.22%
Prior Convictions	-0.0743	-7.16%

Variable	β Estimate	$100(e^{eta}-1)$
Financial Aid	0.3319	39.36%
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Marital Status	0.5609	75.22%
Prior Convictions	-0.0743	-7.16%

For individuals who received financial aid, we expect their length of time until recidivism to be 39.4% longer than those who did not receive financial aid.

Variable	β Estimate	$100(e^{eta}-1)$
Financial Aid	0.3319	39.36%
Age at Release	0.0333	3.39%
Marital Status	0.5609	75.22%
Prior Convictions	-0.0743	-7.16%

For every increase in age by one year at time of release, we expect their length of time until recidivism to increase by 3.4%.

Variable	β Estimate	$100(e^{eta}-1)$
Financial Aid	0.3319	39.36%
Age at Release	0.0333	3.39%
Marital Status	0.5609	75.22%
Prior Convictions	-0.0743	-7.16%

For every increase in prior convictions, we expect their length of time until recidivism to decrease by 7.2%.

DISTRIBUTIONS

Common Distributions

Exponential Distribution

- Simplest distribution is the exponential distribution constant hazard that doesn't depend on time.
- Survival function: $S(t) = e^{-\lambda t}$
- Hazard function: $h(t) = \lambda$
- Constant hazard commonly used when failures are completely random:
 - Light bulbs
 - Electronics
 - Etc.

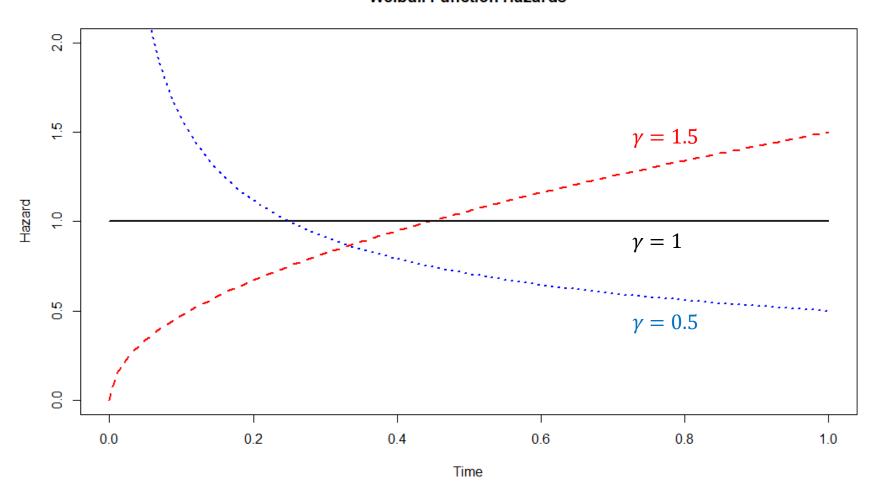
Weibull Distribution

- Most commonly used distribution is the Weibull distribution, which has an additional parameter γ.
- Survival function: $S(t) = e^{-(\lambda t)^{\gamma}}$
- Hazard function: $h(t) = \lambda \gamma (\lambda t)^{\gamma 1}$

- The parameter γ is a rate parameter $(1/\sigma) > 0$ and determines whether the hazard increases or decreases with time:
 - γ > 1: hazard increasing with time (Ex: aging parts "wear out")
 - γ < 1: hazard **decreasing** with time (Ex: post-surgery complications)

Weibull Distribution Hazards

Weibull Function Hazards



Exponential vs. Weibull

- Hazard for Weibull is constant when $\gamma = 1$.
- Weibull distribution **IS** the exponential distribution when $\gamma = 1!$
- R: Log(scale) p-value \rightarrow testing if H_0 : $\log(\gamma) = 0$

Weibull Distribution - R

```
recid.aft.w <- survreg(Surv(week, arrest) ~ fin + age + wexp + mar + paro + prio,
data = recid, dist = 'weibull')
summary(recid.aft.w)</pre>
```

R output

	Value	Std. Error	Z	р
(Intercept)	3.8086	0.3762	10.12	< 2e-16
fin	0.2625	0.1376	1.91	0.05650
age	0.0400	0.0159	2.51	0.01210
wexp	0.1115	0.1516	0.74	0.46196
mar	0.3389	0.2725	1.24	0.21366
paro	0.0538	0.1394	0.39	0.69956
prio	-0.0646	0.0210	-3.08	0.00208
Log(scale)	-0.3383	0.0891	-3.80	0.00015

Scale= 0.713

Weibull distribution

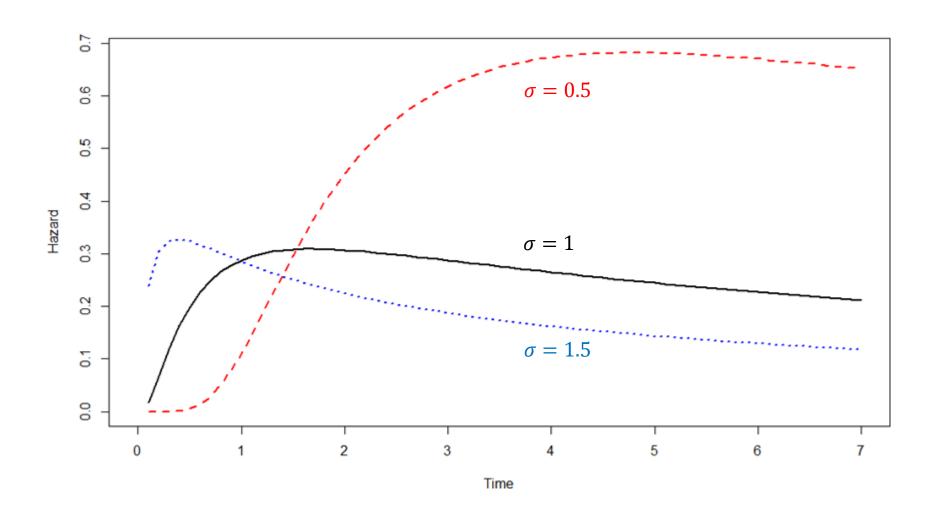
Loglik(model)= -680.5 Loglik(intercept only)= -696.6 Chisq= 32.28 on 6 degrees of freedom, p= 1.4e-05 Number of Newton-Raphson Iterations: 6 n= 432

$$H_0$$
: $\gamma = 1$ (or $Log(\gamma) = 0$)
 H_A : $\gamma \neq 1$ (or $Log(\gamma) \neq 0$)

Other Distributions

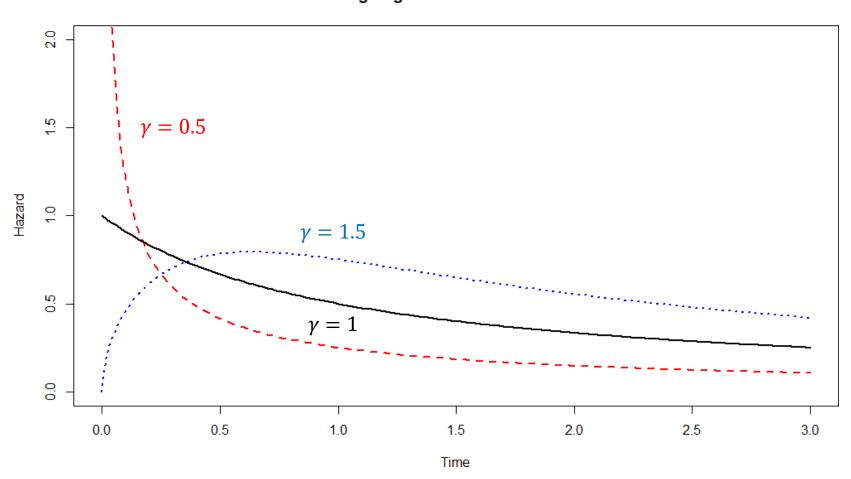
- Log-Normal Distribution: If T has a log-normal distribution, then log(T) follows a normal distribution.
 - IF NO CENSORING, log-normal AFT = linear regression with $y = \log T$ are equivalent.
 - Hazard has different forms depending on the scale parameter
- Log-Logistic Distribution:
 - Hazard takes on different forms depending on the scale parameter
- Generalized Gamma Distribution:
 - Hazard takes on different forms depending on the scale parameter

Log-Normal Hazard



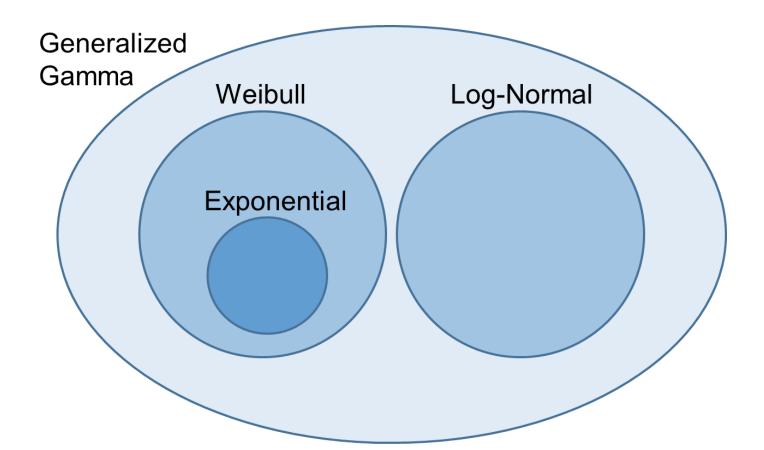
Log-Logistic Hazard

Log-Logistic Function Hazards



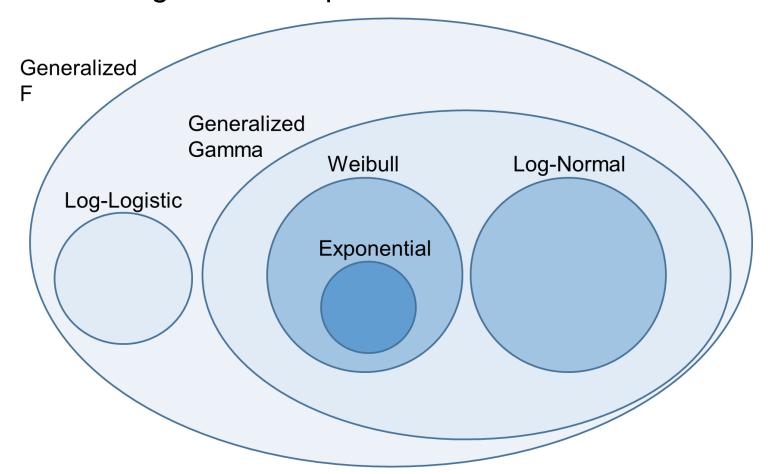
Other Distributions

• Generalized Gamma Distribution: Includes log-normal and Weibull as special cases.



Other Distributions

 Generalized F Distribution: Includes log-logistic and generalized gamma as special cases.

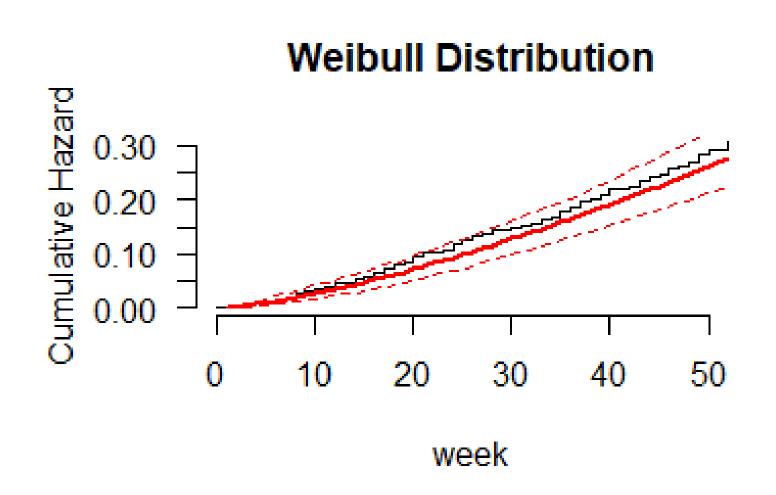


Distributions

- Distributions are commonly checked two ways:
 - Graphically (need to use the flexsurvreg and will compare cumulative hazard functions to actual data)
 - 2. Statistical Tests (if distributions are nested)
- There is no guarantee that your data will adequately match just one
 of the distributions here, or even any of them at all.
- Typically, use full model (all variables) since we don't know which pvalues are correct

Checking distributions - R

Cumulative Hazard - Weibull

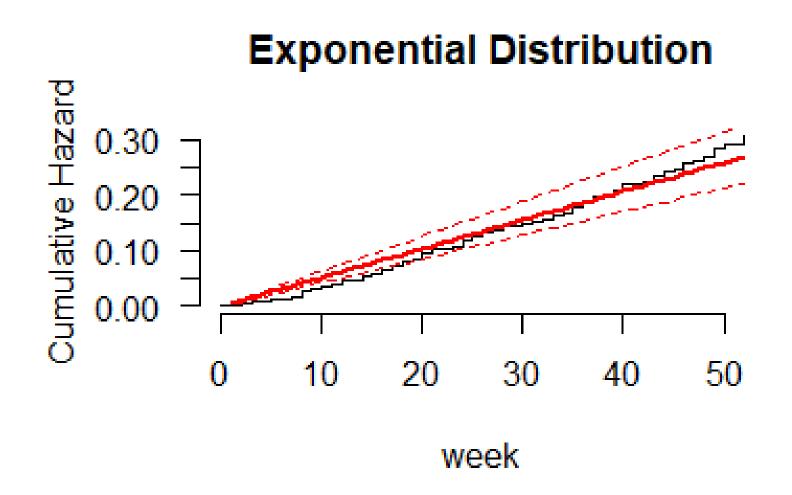


Exponential

```
recid.aft.e <- flexsurvreg(Surv(week, arrest) ~ fin + age + wexp + mar + paro + prio, data = recid, dist = "exp")
```

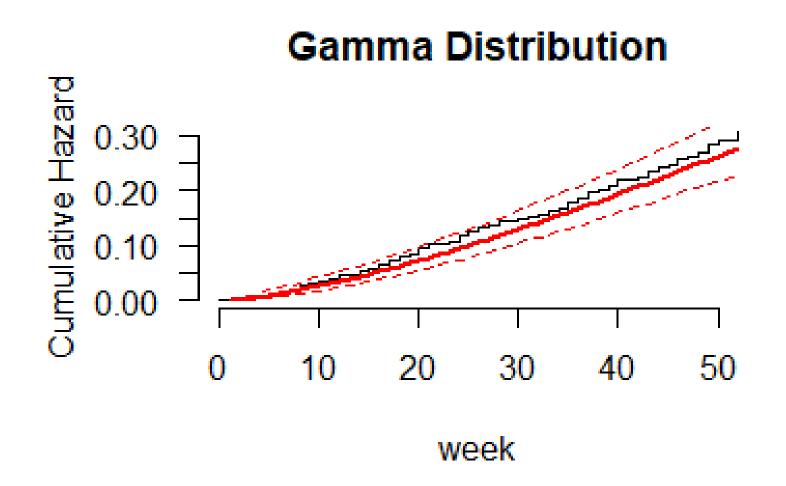
```
plot(recid.aft.e, type = "cumhaz", ci = TRUE, conf.int = FALSE, las = 1, bty = "n", xlab = "week", ylab = "Cumulative Hazard", main = "Exponential Distribution")
```

Cumulative Hazard - Exponential



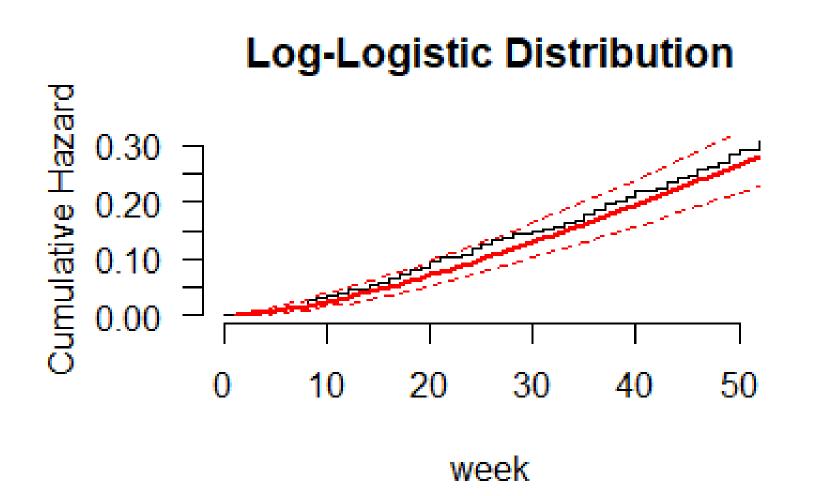
Gamma

Cumulative Hazard - Gamma



Log-Logistic

Cumulative Hazard – Log Logistic

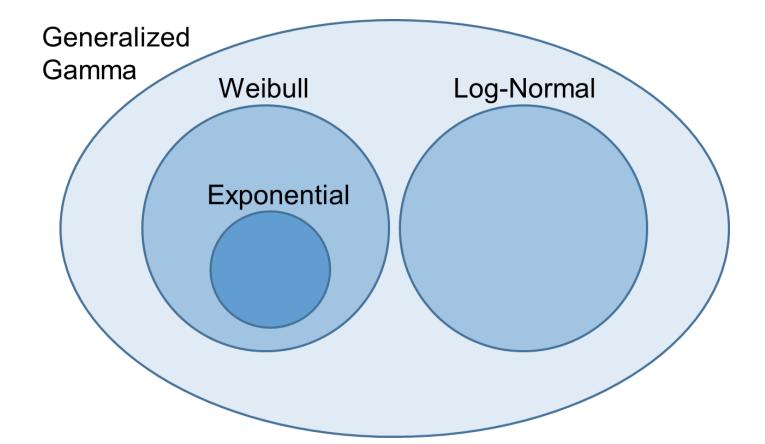


Distributions

- Distributions are commonly checked two ways:
 - Graphically
 - 2. Statistical Tests.... For nested distributions
- There is no guarantee that your data will adequately match just one of the distributions here, or even any of them at all.

Nested Distributions!

• Generalized Gamma Distribution: Includes log-normal and Weibull as special cases.



Goodness-of-Fit Tests

- Since these models are nested within the generalized gamma, we can use the likelihood ratio test.
- Likelihood Ratio Test:

$$LRT = -2(\log L_{Nested} - \log L_{Full})$$

 Typically, use full model (all variables) since we don't know which p-values are correct.

Code to get likelihoods...

```
like.e <- flexsurvreg(Surv(week, arrest) ~ fin + age + wexp + mar + paro + prio, data = recid, dist =
"exp")$loglik
like.w <- flexsurvreg(Surv(week, arrest) ~ fin + age + wexp + mar + paro + prio, data = recid, dist =
"weibull")$loglik
like.ln <- flexsurvreg(Surv(week, arrest) ~ fin + age + wexp + mar + paro + prio, data = recid, dist =
"Inorm")$loglik
like.g <- flexsurvreg(Surv(week, arrest) ~ fin + age + wexp + mar + paro + prio, data = recid, dist =
"gamma")$loglik
like.ll <- flexsurvreg(Surv(week, arrest) ~ fin + age + wexp + mar + paro + prio, data = recid, dist =
"llogis")$loglik
######DO NOT RUN the F –distribution....does NOT converge!!
like.f <- flexsurvreg(Surv(week, arrest) ~ fin + age + wexp + mar + paro + prio, data = recid, dist =
"genf")$loglik
```

Code to get likelihoods...

```
like.e <- flexsurvreg(Surv(week, arrest) ~ fin + age + wexp + mar + paro + prio, data = recid, dist =
"exp")$loglik
like.w <- flexsurvreg(Surv(week, arrest) ~ fin + age + wexp + mar + paro + prio, data = recid, dist =
"weibull")$loglik
like.ln <- flexsurvreg(Surv(week, arrest) ~ fin + age + wexp + mar + paro + prio, data = recid, dist =
"Inorm")$loglik
like.g <- flexsurvreg(Surv(week, arrest) ~ fin + age + wexp + mar + paro + prio, data = recid, dist =
"gamma")$loglik
like.ll <- flexsurvreg(Surv(week, arrest) ~ fin + age + wexp + mar + paro + prio, data = recid, dist =
"llogis")$loglik
######DO NOT RUN the F -distribution....does NOT converge!!
like.f <- flexsurvreg(Surv(week, arrest) ~ fin + age + wexp + mar + paro + prio, data = recid, dist =
"genf")$loglik
```

Calculate p-values

```
Tests = c('Exp vs. Gam', 'Wei vs. Gam', 'LogN vs. Gam')
P_values = c(pval.e.g, pval.w.g, pval.ln.g)
cbind(Tests, P_values)
```

Tests P_values
"Exp vs. Gam" 0.0019
"Wei vs. Gam" 1
"LogN vs. Gam" 0.0077

H₀: No "significant difference" in fit between the two distributions (Simpler distribution is fine)

H_A: Significant difference in fit between the two distributions (More complex distribution is needed)

After finding best distribution

After finding best distribution, we can select the best model (by hand...forward selection or backward elimination)

Using the Weibull distribution, we will use the following model moving forward:

	Value	Std. Error	Z	p
(Intercept)	3.7738	0.3581	10.54	< 2e-16
fin	0.2495	0.1372	1.82	0.06901
age	0.0478	0.0154	3.11	0.00189
prio	-0.0698	0.0201	-3.47	0.00051
Log(scale)	-0.3367	0.0892	-3.77	0.00016

Scale= 0.714

PREDICTING SURVIVAL & EVENT TIMES

Making Predictions

- AFT models assume a distribution for T, meaning that we expect event times to behave in a certain way.
- IF WE ASSUME CORRECT DISTRIBUTION we can predict quantiles, survival probabilities, event times, survival curves, and changes in expected values as predictor variable values change.

Example Predictions

- Median survival time:
 - Find t such that $\hat{S}_i(t) = 0.5$
- The time by which q% of people with the same values for predictor variables have the event:
 - Find t such that $\hat{S}_i(t) = 1 q$
- 20 week predicted survival probability:
 - $\hat{S}_i(20)$
- **CAREFUL:** $\hat{S}_i(t)$ is entirely determined by the distribution used so estimates WON'T be the same across different distributions.

Predicting Survival Quantiles

```
# Predicted Survival Quantiles recid.aft.w = survreg(Surv(week, arrest) ~ fin + age +prio, data = recid, dist = 'weibull') summary(recid.aft.w)
```

#####Find the 25th, 50th and 75th percentile of survival curve for each individual

```
survprob.75.50.25 = predict(recid.aft.w, type = "quantile", se.fit = TRUE,p = c(0.25, 0.5, 0.75))
```

head(survprob.75.50.25\$fit)

```
[,1] [,2] [,3]
```

- [1,] 52.68849 98.72758 161.95827
- [2,] 24.17956 45.30760 74.32514
- [3,] 17.89085 33.52383 54.99438
- [4,] 64.22717 120.34873 197.42682
- [5,] 35.95471 67.37185 110.52057
- [6,] 48.95457 91.73097 150.48064

Survival Curve for First Individual

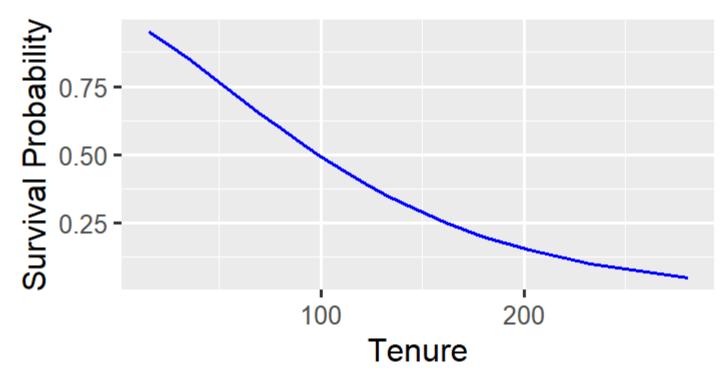
```
quant.prob=seq(0.05,0.95,by=0.05)
survprob = predict(recid.aft.w, type = "quantile",
se.fit = TRUE,p = quant.prob)
surv.prob=rev(quant.prob)
```

graph.dat=data.frame(cbind(survprob\$fit[1,],surv.
prob))

colnames(graph.dat)=c("Tenure","SurvivalProb")

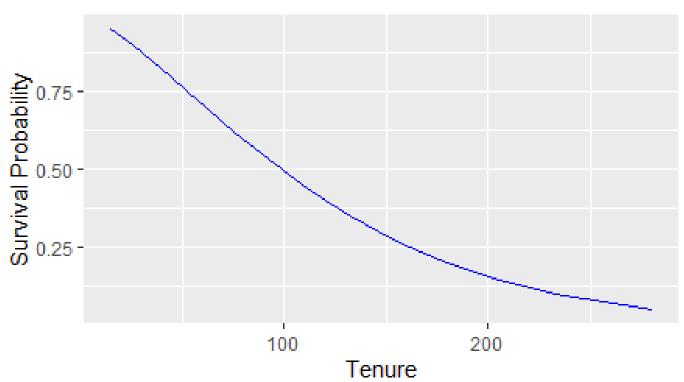
ggplot(graph.dat,aes(x=Tenure,y=SurvivalProb))+g
eom_line(color="blue")+labs(title="Survival Curve
for Person 1",x="Tenure",y="Survival Probability")

Survival Curve for Person 1

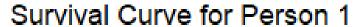


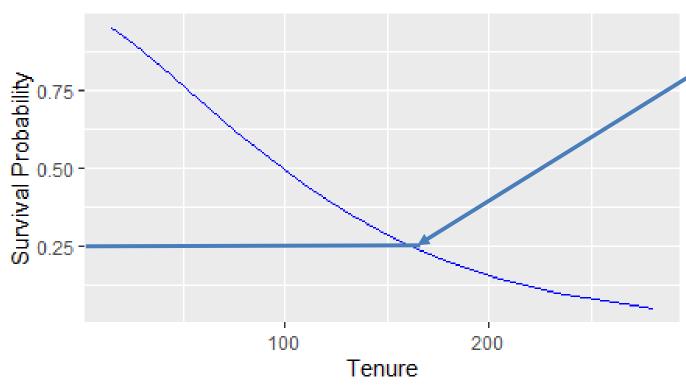
0.25 0.5 0.75 [1,] 52.68849 98.72758 161.95827

Survival Curve for Person 1



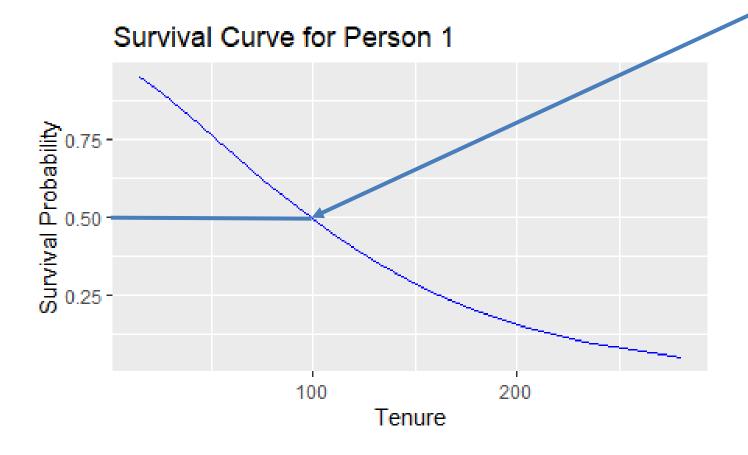
0.25 0.5 0.75 [1,] 52.68849 98.72758 16 1.95827





Probability of event occurring on or before 161.95 is 0.75 (probability of "surviving" beyond 161.95 is 0.25!)

0.25 0.5 0.75 [1,] 52.68849 98.72758 161.95827



Probability of event occurring on or before 98.73 is 0.5 (probability of "surviving" beyond 98.73 is 0.5!)

Predict mean survival time

p.time.mean <- predict(recid.aft.w, type = "response", se.fit = TRUE)
head(p.time.mean\$fit, n = 10)</pre>

[1] 128.26394 58.86229 43.55317 156.35349 87.52751 [6] 119.17415 143.73152 115.26040 81.92984 113.19494

Predict survival probability at 10 weeks:

```
survprob.10wk = 1 - psurvreg(10,
   mean = predict(recid.aft.w, type = "lp"),
   scale = recid.aft.w$scale,
   distribution = recid.aft.w$dist)
head(survprob.10wk)
```

[1] 0.9723202 0.9198457 0.8803901 0.9789527 0.9531961 [6] 0.9693657

Predict Change in Event time (for a given variable)

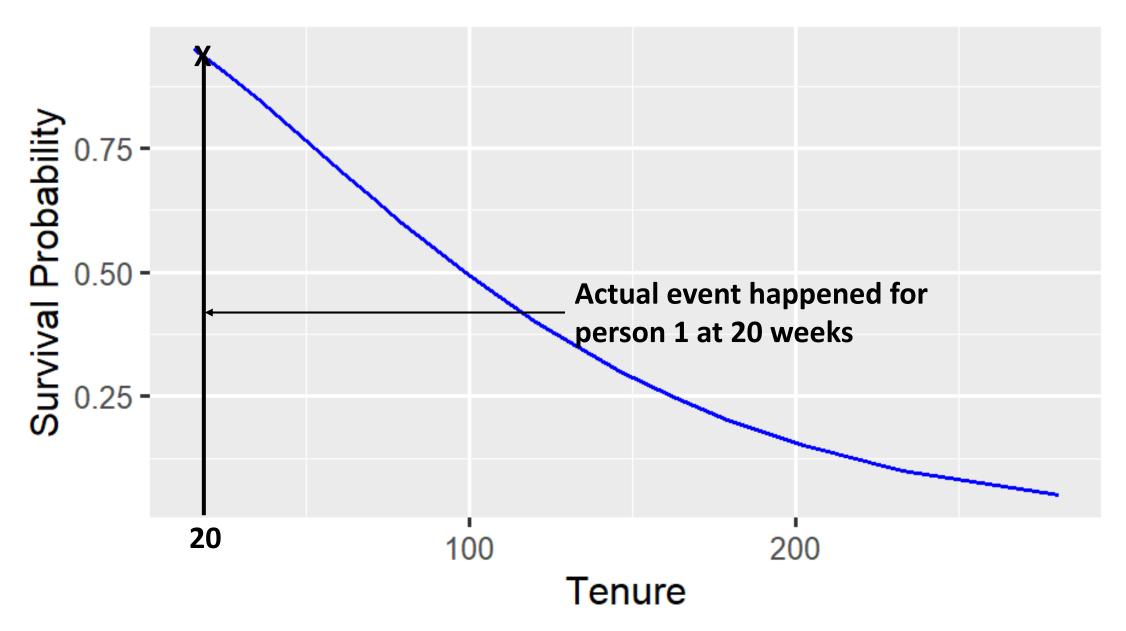
How much of an *impact* do we think there would be if we gave financial aid to those individuals who did NOT it? We can use the tools that we just learned to explore this idea...

We will assume the "quantile" they fell on is still the same quantile, but we will use a new "linear predictor" (with financial aid =1)

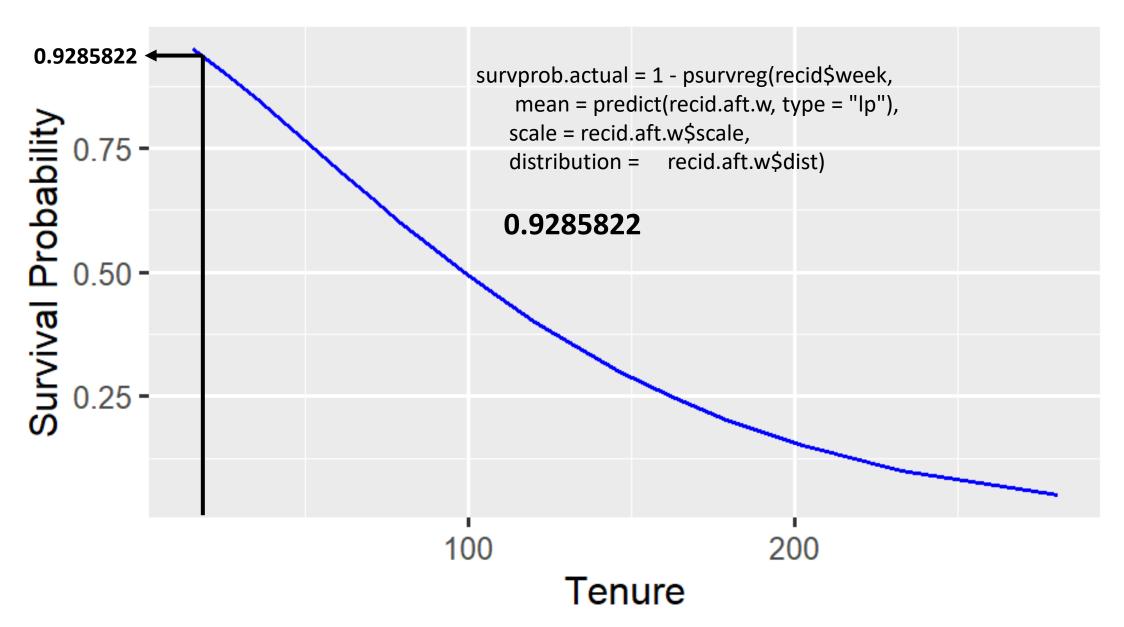
```
survprob.actual = 1 - psurvreg(recid$week,
mean = predict(recid.aft.w, type = "lp"),
scale = recid.aft.w$scale,
distribution = recid.aft.w$dist)
```

First find where the "event" occurred on their survival curve.

Survival Curve for Person 1



Survival Curve for Person 1



Predict Change in Event time (for a given variable)

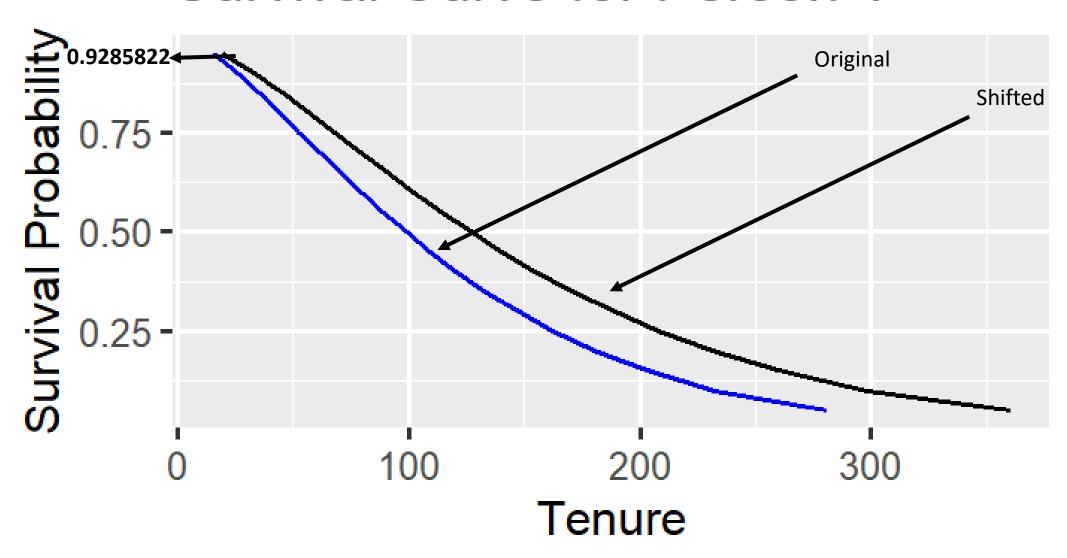
How much of an *impact* do we think there would be if we gave financial aid to those individuals who did NOT it? We can use the tools that we just learned to explore this idea...

We will assume the "quantile" they fell on is still the same quantile, but we will use a new "linear predictor" (with financial aid =1)

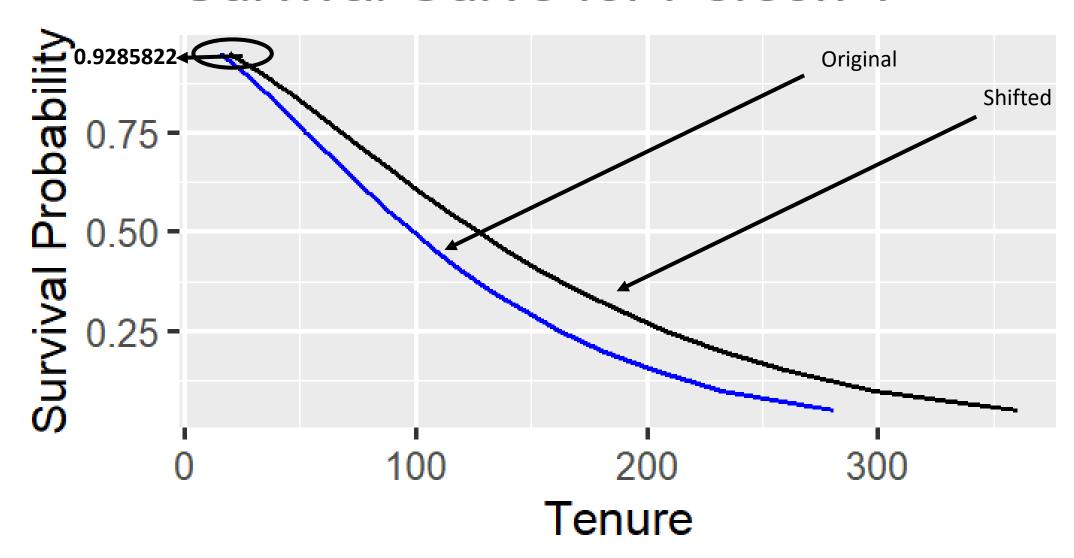
```
new_time = qsurvreg(1 - survprob.actual,
mean = predict(recid.aft.w, type = "lp") +
    coef(recid.aft.w)['fin'],
    scale = recid.aft.w$scale,
    distribution = recid.aft.w$dist)
```

Assume the same survival curve and want to find the quantile of when the event occurred.

Survival Curve for Person 1



Survival Curve for Person 1



Predict Change in Event time

Now save new predict time to data set and find difference

Subset to only look at those that did NOT have financial to start!

```
recid$new_time = new_time
recid$diff = recid$new_time - recid$week

impact.fin=data.frame(recid$week, recid$new_time,
recid$diff,recid$arrest,recid$fin)
colnames(impact.fin)=c("O.Week","N.Week","Diff","Arrest","F
in")
impact.fin2=subset(impact.fin,Arrest==1 & Fin==0)
head(impact.fin2)
```

Output

0.\	Neek	N.Week	Diff A	Arrest	Fin
1	20	25.66776	5.667764	1	0
2	17	21.81760	4.817600	1	0
3	25	32.08471	7.084706	1	0
7	23	29.51793	6.517929	1	0
13	37	47.48536	10.485364	1 1	0
15	25	32.08471	7.084706	1	0