1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Supposing that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = f(y),$$

find an equation for $y(x)$ in terms of f.

Multiplying by $\mathrm{d}y/\mathrm{d}x$, we obtain

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= \frac{\mathrm{d}y}{\mathrm{d}x} f(y) \\ \implies \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right] &= \frac{\mathrm{d}}{\mathrm{d}x}[F(y)] \end{aligned} \tag{1}$$

for F an antiderivative of f - eg

$$F(t) := \int_0^t f(\nu)\,\mathrm{d}\nu.$$

Then we can integrate (1) to get

$$\begin{aligned} \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 &= 2F(y) + C_1 \\ \implies \qquad \frac{\mathrm{d}y}{\mathrm{d}x} &= \pm\sqrt{2F(y) + C_1} \\ \implies \int (2F(y) + C_1)^{-1/2}\,\mathrm{d}y &= \pm x + C_2. \end{aligned}$$

2. Solve the differential equation for y in terms of x:

$$x^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0$$

Use substitution $u = y + x\frac{\mathrm{d}y}{\mathrm{d}x}$. This transforms the equation into: $x\frac{\mathrm{d}u}{\mathrm{d}x} - u = 0$ which is separable with solution $u = 2Ax$ for some constant A. Observe that we also have a first integral for y: $u = 2Ax = \frac{\mathrm{d}}{\mathrm{d}x}(xy)$, so integrating, we get $xy = Ax^2 + B$ for some constant B. So we are left with the solution $y = Ax + Bx^{-1}$.

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$, which satisfy the equation $f(x)f(y) = f(x+y) + xy$ for all real numbers x and y.

Letting $x = y = 0$, we see that $f(0)^2 = f(0)$. Then either

- $f(0) = 0$. In this case, letting $y = 0$ shows that $f(x) = f(x+0)+x\cdot 0 = f(x)f(0) = 0$ for all x. So f is identically zero. However, letting $x = y = 1$, we see that $0 \neq 0+1$, so this does not provide a solution.
- or, $f(0) = 1$. In this case, letting $y = -x$ shows that $f(x)f(-x) = f(0)-x^2 = 1-x^2$, for all x.

  Particularly it follows that $f(1)f(-1) = 0$. So either

- $f(1) = 0$. In this case, letting $y = 1$, we have $f(x+1) + x = f(x)f(1) = 0$ for all x, so $f(x+1) = -x$ for all x, ie $f(x) = 1 - x$ for all x. And indeed, $(1-x)(1-y) = (1-(x+y)) + xy$.
- or, $f(-1) = 0$. In this case, letting $y = -1$, we have $f(x-1) - x = f(x)f(-1) = 0$ for all x, so $f(x-1) = x$ for all x, ie $f(x) = 1 + x$ for all x. And indeed, $(1+x)(1+y) = (1+(x+y)) + xy$.

So either $f(x) = 1 + x$ or $f(x) = 1 - x$.

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$, which satisfy the equation

$$f(x^3) + f(y^3) = (x+y)(f(x^2) + f(y^2) - f(xy))$$

for all real numbers x and y.

Letting $x = y = 0$ we discover the value of $f(0)$ has to be 0. Letting $y = -x$ gives $f(x^3) = -f(-x^3)$, and since x^3 surjective we therefore have f is odd, so for all x, $f(-x) = -f(x)$.

Letting $y = x$ gives $2f(x^3) = 2x(2f(x^2) - f(x^2))$, so we have $f(x^3) = xf(x^2)$ for all x. Substituting this expression in for $f(x^3)$ and $f(y^3)$ in the original equation gives $xf(x^2) + yf(y^2) = (x+y)(f(x^2) + f(y^2) - f(xy))$, which after expanding and cancelling gives

$$yf(x^2) + xf(y^2) = (x+y)f(xy) \tag{2}$$

for all x and y.

Now let $y = 1$, in this equation and we have $f(x^2) + xf(1) = (x+1)f(x)$ for all x, which gives us an expression for $f(x^2)$: $f(x^2) = (x+1)f(x) - xf(1)$. We can substitute these into (2), giving us:

$$y(x+1)f(x) + x(y+1)f(y) - 2xyf(1) = (x+y)f(xy)$$

Now substituting $y = -x$ and using the fact that f is odd, we get:

$$-2xf(x) + 2x^2f(1) = 0$$

So we have for $x \neq 0$, $f(x) = f(1)x$, and at 0 we know f takes the value 0 anyway. And we can check to see that this works for any chosen value of $f(1)$. So $f(x) = kx$ for some $k \in \mathbb{R}$.