Lab Nr. 12, Numerical Calculus

Numerical Methods for Nonlinear Equations

Matlab functions

- *solve*: solves nonlinear equations and systems symbolically
- *fsolve*: solves nonlinear equations and systems numerically
- *roots*: finds roots of polynomials
- *factor*: finds factors of numbers or symbolic expressions

Implement the bisection, secant and Newton's methods for approximating numerical solutions of a nonlinear equation $f(x) = 0$, with a given precision. Keep also track of the number of iterations needed for a desired precision.

Applications

1. Let $f(x) = x - e^{-x}$.

a) Locate the real roots of the equation $f(x) = 0$ analytically and graphically.

b) Approximate the real roots of the equation using the three methods above, with appropriate starting values. Compare the speed of convergence of the three methods.

2. Consider the polynomial equation

$$p(x) \equiv x^4 + 2x^3 - 4x^2 - 14x - 5 = 0.$$

a) Use Matlab functions *roots, solve* and *fsolve* to find all (real and complex) roots of p.

b) Factor p and plot its graph for $x \in [-1, 3]$.

c) Approximate all (real and complex) roots of p, using the three methods above.

d) Use the secant method with values $x_0 = -3 - 2i,\ x_1 = -1 - i$, then Newton's method with $x_0 = 3 + i$. What do you notice?

3. Let

$$f(x) = x + e^{-Bx^2} \cos x, \text{ for some parameter } B > 0.$$

a) Show that the equation $f(x) = 0$ has a real root in the interval $(-1, 0)$, for any $B > 0$ (it can be shown that that is the *only* root). For $B = 1, 5, 50$ and 100, plot the function f for $x \in [-1.2, 0.5]$.

b) For $B = 1$ and 5, approximate the root using Newton's method with starting value $x_0 = 0$.

b) Can you do the same for $B = 50$ and 100? Explain what is happening and find a way to approximate the root using Newton's method.

Optional

4. An interesting rootfinding problem occurs in the computation of *annuities* (which are savings accounts or trust funds). Suppose you are paying into an account an amount of P_{in} per period of time, for N_{in} periods of time. The amount you deposit is compounded at an interest rate of r per period of time. Then at the beginning of period $N_{in}+1$, you will withdraw an amount of P_{out} per time period, for N_{out} periods. It is assumed that the interest rate r holds over all $N_{in}+N_{out}$ periods. In order that the amount you withdraw balance that which has been deposited, including interest, what is the needed interest rate? The mathematical model of this problem leads to the equation

$$P_{in}\Big[(1+r)^{N_{in}}-1\Big] \;=\; P_{out}\Big[1-(1+r)^{-N_{out}}\Big].$$

As a particular case, suppose you are paying in $P_{in}=1000$ currency units (CU = RON, EUR, USD, etc.) each month for 40 years. Then you wish to withdraw $P_{out}=5000$ CU's per month for 20 years. If the interest rate is R per year, compounded monthly, then $r=R/12$. Also, $N_{in}=40\cdot 12=480$ and $N_{out}=20\cdot 12=240$. Thus, we want to solve

$$f(r) \;\equiv\; 1000\left[\left(1+\frac{R}{12}\right)^{480}-1\right]-5000\left[1-\left(1+\frac{R}{12}\right)^{-240}\right] \;=\; 0.$$

What is the needed yearly interest rate R?

a) Plot the graph of f on an appropriate interval and locate the real roots of f.

b) Approximate the positive root, using the bisection and secant methods with various starting values. Compare the results. Would Newton's method be appropriate in this case?

Note: This example shows the power of *compound* interest (compound interest is when you earn interest on the money you've saved *and* on the interest you earn along the way).