Lab Nr. 11, Numerical Calculus

Numerical Integration II

Romberg's Method; Gaussian Quadratures

1. Implement Romberg's method for the trapezoidal rule. Keep also track of the number of function evaluations.

2. For the five weight functions and intervals given in the table, implement a Matlab routine that uses Gaussian quadratures to approximate the value of an integral, with a given number of nodes. Compute the nodes and the coefficients of the Gaussian quadrature formula using the Jacobi matrix (see Theorem 3.18, Lecture 10).

Applications

1. Approximate the integral

$$I = \int_0^{\pi/2} \frac{dx}{2+\sin x}$$

(whose exact value is $\dfrac{\pi\sqrt{3}}{9}$), with 5 correct decimals using Romberg's method for the trapezium rule. Compute the error of the approximation and the number of function evaluations.

2. Approximate π using a suitable Gaussian quadrature with 2 nodes. Find the error of the approximation.

3. Approximate the integral

$$I = \int_0^{\pi/4} e^{\cos x}\, dx,$$

using a Gaussian formula with $n = 1, 2, \ldots, 5$ nodes. Display the approximate values of I and the errors of each approximation.

4. Use appropriate Gaussian quadratures with $n = 2, 4, 6, 8$ and 10 nodes to approximate the integrals

a) $\displaystyle\int_0^{\infty} e^{-x} \sin x\, dx \left(= \frac{1}{2}\right);$

b) $\displaystyle\int_{\mathbb{R}} e^{-x^2} \cos x\, dx;$

c) $\displaystyle\int_{-1}^{1} \frac{\sin\left(x^2\right)}{\sqrt{1-x^2}}\, dx.$

Optional

5. Consider the integral

$$I = \int_0^1 \frac{\sin x}{x}\, dx.$$

a) Compute I with the Matlab function *integral*. So the integral exists.

b) Use Romberg's method for the trapezoidal rule to approximate I. What is happening and why? Find a solution and fix the problem.

Name	Notation	Polynomial	Weight Fn.	Interval	$\boldsymbol{\alpha_k}$	$\boldsymbol{\beta_k}$
Legendre	l_m	$\left[(x^2-1)^m\right]^{(m)}$	1	$[-1,1]$	0	$\beta_0 = 2$, $\beta_k = (4-k^{-2})^{-1}, k \geq 1$
Chebyshev 1^{st}	T_m	$\cos\left(m \arccos x\right)$	$(1-x^2)^{-\frac{1}{2}}$	$[-1,1]$	0	$\beta_0 = \pi$, $\beta_1 = \frac{1}{2}$, $\beta_k = \frac{1}{4}, k \geq 2$
Chebyshev 2^{nd}	Q_m	$\dfrac{\sin\left[(m+1)\arccos x\right]}{\sqrt{1-x^2}}$	$(1-x^2)^{\frac{1}{2}}$	$[-1,1]$	0	$\beta_0 = \frac{\pi}{2}$, $\beta_k = \frac{1}{4}, k \geq 1$
Laguerre	L_m^a	$x^{-a}e^x\left(x^{m+a}e^{-x}\right)^{(m)}$	$x^a e^{-x},\ a > -1$	$[0,\infty)$	$2k+a+1$	$\beta_0 = \Gamma(1+a)$, $\beta_k = k(k+a), k \geq 1$
Hermite	H_m	$(-1)^m e^{x^2}\left(e^{-x^2}\right)^{(m)}$	e^{-x^2}	$(-\infty,\infty)$	0	$\beta_0 = \sqrt{\pi}$, $\beta_k = \frac{k}{2}, k \geq 1$