MA202 (Probability and Statistics)

Assignment 4

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1) p.d.f given = $f_X(x) = \begin{cases} k e^{-b(x-a)} & \text{if } a \le x < \infty \\ 0 & \text{if elsewhere} \end{cases}$

Given: $k, a, b > 0$, constants.

$\boxed{\mu = \text{Mean}}$ $\boxed{\sigma^2 = \text{Variance}}$

So, as given p.d.f, We can say that

$$\int_a^{\infty} k e^{-b(x-a)} \cdot dx = 1.$$

$$\Rightarrow \frac{k}{-b}\left[e^{-b(x-a)}\right]_a^{\infty} = 1$$

$$\Rightarrow -\frac{k}{b}\left[e^{-\infty} - e^{0}\right] = 1$$

$$\Rightarrow -\frac{k}{b}(-1) = 1$$

$$\Rightarrow \boxed{k = b}$$

$f_X(x) = k e^{-b(x-a)}$

$\boxed{f_X(x) = b e^{-b(x-a)}}$ $\therefore k = b.$

$$\text{Mean} = \mu = E(x) = \int x f_X(x) \cdot dx.$$

$$= \int bx\, e^{-b(x-a)} \cdot dx.$$

$$= b \int x e^{-b(x-a)} \cdot dx$$

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(1)

$$\Rightarrow b\int_a^{\infty} x e^{-b(x-a)} \cdot dx = \mu = E(x).$$

$$\Rightarrow E(x) = b\int_a^{\infty} x e^{-b(x-a)} \cdot dx.$$

$$= b\left[x\int e^{-b(x-a)} - \int\left(1\left(\int e^{-b(x-a)} \cdot dx\right)\right) \cdot dx\right]_a^{\infty}$$

$$= b\left[\frac{x e^{-b(x-a)}}{(-b)} - \int \frac{e^{-b(x-a)}}{-b} \cdot dx .\right]_a^{\infty}$$

$$\Rightarrow b\left[-\frac{x e^{-b(x-a)}}{b} + \frac{1}{b}\,\frac{e^{-b(x-a)}}{(-b)}\right]_a^{\infty}$$

$$\Rightarrow b\left[\frac{1}{b^2} e^{-b(x-a)} - \frac{x e^{-b(x-a)}}{b}\right]_a^{\infty}$$

$$\Rightarrow b\left[\frac{a}{b^2} - \frac{a}{b}\, a\right]$$

$$\Rightarrow b\left[-\frac{x e^{-b(x-a)}}{b} - \frac{1}{b^2} e^{-b(x-a)}\right]_a^{\infty}.$$

$$\Rightarrow b\left[-0 - 0 + \frac{a}{b} + \frac{1}{b^2}\right]$$

$$\Rightarrow a + \frac{1}{b}$$

$$\boxed{E(x) = a + 1/b = \mu}$$

$$\boxed{\mu = a + 1/b = \frac{ab+1}{b}}$$

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$$E(x^2) = \int_a^{\infty} x^2 f_X(x) \cdot dx$$

$$= \int_a^{\infty} b\, x^2 e^{-b(x-a)} \cdot dx.$$

$$= b \int_a^{\infty} x^2 e^{-b(x-a)} \cdot dx.$$

$$= b\left[x^2 \int e^{-b(x-a)} \cdot dx - \int \left(2x \int e^{-b(x-a)} \cdot dx\right) \cdot dx \right]_a^{\infty}$$

$$= b\left[\frac{x^2 e^{-b(x-a)}}{-b} - \int 2x \frac{e^{-b(x-a)}}{-b} \cdot dx \right]_a^{\infty}$$

$$= b\left[\frac{x^2 e^{-b(x-a)}}{-b} + 2\int \frac{x e^{-b(x-a)}}{b} \cdot dx \right]_a^{\infty}$$

$$= b\left[\frac{x^2 e^{-b(x-a)}}{-b} + \frac{2}{b}\left(-\frac{x e^{-b(x-a)}}{b} - \frac{1}{b^2} e^{-b(x-a)} \right)\right]_a^{\infty}$$

$$= b\left[\frac{x^2 e^{-b(x-a)}}{-b} - \frac{2x e^{-b(x-a)}}{b^2} - \frac{2 e^{-b(x-a)}}{b^3} \right]_a^{\infty}$$

$$= b\left[-0 - 0 - 0 + \frac{a^2}{b} + \frac{2a}{b^2} + \frac{2}{b^3} \right]$$

$$= a^2 + \frac{2a}{b} + \frac{2}{b^2}$$

$$\boxed{E(x^2) = a^2 + \frac{2a}{b} + \frac{2}{b^2}}$$

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$$\sigma^2 = E(x^2) - E(x)^2$$

$$= a^2 + \frac{2a}{b} + \frac{2}{b^2} - \left(\frac{ab+1}{b}\right)^2$$

$$= a^2 + \frac{2a}{b} + \frac{2}{b^2} - \frac{1}{b^2}\left[a^2b^2 + 1 + 2ab\right]$$

$$= \cancel{a^2} + \cancel{\frac{2a}{b}} + \frac{2}{b^2} - \cancel{a^2} - \frac{1}{b^2} - \cancel{\frac{2a}{b}}$$

$$= \frac{1}{b^2}$$

$$b^2 = \frac{1}{\sigma^2}$$

$$b = \pm \frac{1}{\sigma}, \quad b > 0,$$

So,

$$\boxed{b = \frac{1}{\sigma}}$$

$$\boxed{k = b = \frac{1}{\sigma}}$$

$$\mu = \frac{ab+1}{b}$$

$$= a + \frac{1}{b}$$

$$a = \mu - \frac{1}{b}$$

$$\boxed{a = \mu - \sigma}$$

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$f_X(x) = b e^{-b(x-a)}$ $\quad x \geq a$.

and it is a similiar exponential distribution ~~othi~~ with origin at $x = a$.

$f_X(x) = \lambda e^{-\lambda(x)}$

$\downarrow$

shift at $x = a$,

We get the required function that is given

Hence, the co-efficient of skewness and Kurtosis of both forms is same as original.

$$\boxed{\text{Co-efficient of skewness} = \frac{\mu_3}{\sigma^3}}$$

$$\mu_3 = \int_a^\infty (x-\mu)^3 f_X(x) \cdot dx.$$

$$= \int_a^\infty k \left[\frac{(x-\mu)^3 e^{-b(x-a)}}{-b} - \frac{3(x-\mu)^2 \cdot e^{-b(x-a)}}{+b^2} + \frac{b(x-\mu) e^{-b(x-a)}}{-b^3} - \frac{6 e^{-b(x-a)}}{b^4} \right]_a^\infty$$

$$= k \left[0 - 0 + 0 - 0 + \frac{(a-\mu)^3}{(+b)} + \frac{3(a-\mu)^2}{b^2} + \frac{6(a-\mu)}{b^3} + \frac{6}{b^4} \right]$$

$$= k \left[\frac{(\mu-\sigma-\mu)^3}{b} + \frac{3(\mu-\sigma-\mu)^2}{b^2} + \frac{6(\mu-\sigma-\mu)}{b^3} + \frac{6}{b^4} \right]$$

$$= k \left[\frac{-\sigma^3}{b} + \frac{3\sigma^2}{b^2} - \frac{6\sigma}{b^3} + \frac{6}{b^4} \right]$$

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$k = b$,

So,

$$\mu_3 = -\sigma^3 + 3\sigma^3 - 6\sigma^3 + 6\sigma^3$$

$$= 2\sigma^3$$

$$\text{Co-efficient of skewness} = \cancel{\mu}\ \frac{\mu_3}{\sigma^3}.$$

$$= \frac{2\sigma^3}{\sigma^3}$$

$$\boxed{\text{Co-efficient of skewness} = 2}$$

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$$\mu_4 = \int_a^\infty (x-\mu)^4 f_x(x) \cdot dx.$$

$$= K \int_a^\infty (x-\mu)^4 k e^{-b(x-a)} \cdot dx.$$

$$= k \left[\frac{(x-\mu)^4 e^{-b(x-a)}}{(-b)} - \frac{4(x-\mu)^3 e^{-b(x-a)}}{b^2} + \frac{12(x-\mu)^2 e^{-b(x-a)}}{-b^3} - \frac{24(x-\mu) e^{-b(x-a)}}{b^4} + \frac{24 \cdot e^{-b(x-a)}}{-b^5} \right]_a^\infty$$

$$= k \left[0 - 0 + 0 - 0 + 0 + \frac{(a-\mu)^4}{b\cancel{4}} + \frac{4(a-\mu)^3}{b^2} + \frac{12(a-\mu)^2}{b^3} + \frac{24(a-\mu)}{b^4} + \frac{24}{b^5} \right]$$

P (6)

$$\mu_4 = b\left[\frac{\sigma^4}{b} - \frac{4\sigma^3}{b^2} + \frac{12\sigma^2}{b^3} - \frac{24\sigma}{b^4} + \frac{24}{b^5}\right]$$

$$= \sigma^4 - \frac{4\sigma^3}{b} + \frac{12\sigma^2}{b^2} - \frac{24\sigma}{b^3} + \frac{24}{b^4}$$

$$\mu_4 = \sigma^4 - 4\sigma^4 + 12\sigma^4 - \cancel{24}\sigma^4 + \cancel{24}\sigma^4$$

$$= 9\sigma^4$$

$$\boxed{\mu_4 = 9\sigma^4}$$

Co-efficient of Kurtosis $= \dfrac{\mu_4}{\sigma^4}$

$$\boxed{\text{Co-efficient of Kurtosis} = \frac{9\sigma^4}{\sigma^4} = 9}$$

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(2) p.d.f given :

$$f_X(x) = \begin{cases} \sin x & \text{if } a < x < \pi/2 \\ 0 & \text{elsewhere} \end{cases}$$

Given p.d.f, we can say that,

$$\int_a^{\pi/2} \sin x = 1 :$$

$$[-\cos x]_a^{\pi/2} = 1.$$

$$-\cos\frac{\pi}{2} + \cos a = 1$$

$$\boxed{\cos a = 1}$$

$$\boxed{\text{So, } a = 0}$$

Let M be median,

$$\int_0^M f_X(x) \cdot dx = 1/2$$

$$\int_0^M \sin x \cdot dx = 1/2$$

$$[-\cos x]_0^M = 1/2$$

$$-\cos M + 1 = 1/2$$

$$\cos M = 1/2$$

$$\boxed{M = \pi/3}$$

$$\boxed{\text{Median} = \pi/3}$$

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for Mode,

$$f'(x) = 0.$$

$$\cos x = 0$$

$$\boxed{x = \pi/2}$$

~~Hence median = $\pi/3$ & mode = $\pi/2$~~

So, $0 < x < \pi/2$ as given in p.d.f.

There mode here does not exist, as $x = \pi/2 \notin (0, \pi/2)$

(3) Given:

$$X \sim N(\mu_X, \sigma_X^2)$$

$$Y \sim N(\mu_Y, \sigma_Y^2)$$

~~Find~~

Using characterstic fn,

$$\phi_u(t) = \phi_{X+Y}(t) = E(e^{it(x+y)})$$

Here, X and Y are independent variables.

The characterstic fn. of normal distribution with mean μ and variance σ^2 is.

$$\phi(t) = \exp\left(it\mu - \frac{\sigma^2 t^2}{2}\right)$$

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So, $\phi_{(X+Y)}(t) = \phi_X(t) \cdot \phi_{(Y)}(t)$

$$= \exp\left(it\mu_x - \frac{\sigma_x^2 t^2}{2}\right) \cdot \exp\left(it\mu_y - \frac{\sigma_y^2 t^2}{2}\right)$$

$$= \exp\left(it(\mu_x + \mu_y) - \frac{(\sigma_x^2 + \sigma_y^2)t^2}{2}\right)$$

$U = X + Y$ also gives a characteristic fn of normal distribution with mean as $\mu_x + \mu_y$ and

Variance as $\sigma_x^2 + \sigma_y^2$.

No, two distinct distributions can both have a same characteristic fn, so the distribution of $X + Y$ must be this normal distribution.

$$\boxed{U \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)}$$

Probability density fn for U is:

$$\boxed{f_x(x; \mu_x + \mu_y, \sigma_x^2 + \sigma_y^2) = \frac{1}{\sqrt{2\pi}\sqrt{(\sigma_x^2 + \sigma_y^2)}} e^{-\frac{(x - \mu_x - \mu_y)^2}{2(\sigma_x^2 + \sigma_y^2)}}}$$

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4. Given: p.d.f

$$f_{XY}(x,y) = \begin{cases} e^{-x-y} & , x \geq 0, y \geq 0 \\ 0 & , \text{elsewhere} \end{cases}$$

(a) $F_{XY}(x,y)$ = Joint distribution fn.

$$F_{XY}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{XY}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{0}\int_{-\infty}^{x} f_{XY}(x,y)\, dx\, dy + \int_{0}^{y}\int_{-\infty}^{x} f_{XY}(x,y)\, dx\, dy$$

$$= 0 + \int_{0}^{y}\left(\int_{-\infty}^{0} f_{XY}(x,y)\, dx + \int_{0}^{x} f_{XY}(x,y)\, dx\right) dy$$

$$= \int_{0}^{y}\int_{0}^{x} f_{XY}(x,y)\, dx\, dy$$

$$= \int_{0}^{y}\int_{0}^{x} e^{-x-y}\, dx\, dy$$

$$= \int_{0}^{y}\left[-e^{-x-y}\right]_{0}^{x} dy$$

$$= \int_{0}^{y}\left(-e^{-x-y} + e^{-y}\right) dy$$

$$= \left[e^{-x-y} + (-e^{-y})\right]_{0}^{y}$$

$$= \left[e^{-x-y} - e^{-y}\right]_{0}^{y}$$

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(11)

$$F_{XY}(x,y) = (e^{-y-x} - e^{-y} - e^{-x} + 1)$$

$$\boxed{F_{XY}(x,y) = 1 + e^{-x-y} - e^{-x} - e^{-y}}$$

(b) The marginal distribution fn. of X is $F_X(x)$

$$F_X(x) = F_{XY}(x,\infty)$$

$$= 1 + e^{-x-\infty} - e^{-x} - e^{-\infty}$$

$$\boxed{F_X(x) = 1 - e^{-x}}$$

$$F_Y(y) = F_{XY}(\infty, y)$$

$$= 1 + e^{-\infty-y} - e^{-\infty} - e^{-y}$$

$$\boxed{F_Y(y) = 1 - e^{-y}}$$

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(c) $P(X=Y) = \int_0^\infty F_{XY}(x,x)\cdot dx$

$$= \int_0^\infty e^{-2x}\, dx$$

$$= \int_0 \left[\frac{e^{-2x}}{-2}\right]_0^\infty$$

$$\boxed{P(X=Y) = 0 + \frac{1}{2} = 1/2}$$

P(12)

d) $P(X+Y \leq 4) = \int_0^4 \int_0^{4-y} f_{XY}(x,y)\,dx\,dy$

$= \int_0^4 \left[-e^{-x-y}\right]_0^{4-y} \cdot dy$

$= \int_0^4 (-e^{-4} + e^{-y})\,dy$

$= \left[-e^{-4}y - e^{-y}\right]_0^4$

$= (-4e^{-4} - e^{-4}) - (-1)$

$= -5e^{-4} + 1$

$$\boxed{P(X+Y \leq 4) = 1 - 5e^{-4} = 1 - \frac{5}{e^4}}$$

e) $P(X \geq 1) = F_X(\infty) - F_X(1)$

$= (1 - e^{-\infty}) - (1 - e^{-1})$

$= e^{-1} = 1/e$

$$\boxed{P(X \geq 1) = 1/e}$$

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(f) $P(X \le Y) = \int_0^{\infty} \int_0^{y} f_{XY}(x, y)\, dx\, dy$

$$= \int_0^{\infty} \int_0^{y} e^{-x-y}\, dx\, dy$$

$$= \int_0^{\infty} \left[-e^{-x-y}\right]_0^{y} dy$$

$$= \int_0^{\infty} \left(-e^{-2y} + e^{-y}\right) dy$$

$$= \left[\frac{-e^{-2y}}{(-2)} - e^{-y}\right]_0^{\infty}$$

$$= \left[\frac{e^{-2y}}{2} - e^{-y}\right]_0^{\infty}$$

$$= -\left[\tfrac{1}{2} - 1\right]$$

$$\boxed{P(X \le Y) = \tfrac{1}{2}}$$

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(g) $P(a < X+Y < b) \quad 0 < x < b$

$$p(a < X+Y < b) = \int_a^b \int_0^{b-y} (e^{-x-y})\, dx\, dy$$

$$= \int_a^b \left[-e^{-x-y}\right]_0^{b-y} dy$$

$$= \int_a^b (-e^{-b} + e^{-y})\, dy$$

$$= \left[-e^{-b} y - e^{-y}\right]_a^b$$

$$= -be^{-b} - e^{-b} - (-e^{-b} \cdot a - e^{-a})$$

$$= -be^{-b} - e^{-b} + e^{-b} \cdot a + e^{-a}$$

$$= e^{-b}(a-b) + e^{-a} - e^{-b}$$

$$\boxed{P(a < X+Y < b) = e^{-b}(a-b) + e^{-a} - e^{-b}}$$

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(h) $F_{XY}(x,y) = 1 - e^{-x} - e^{-y} + e^{-x-y}$

$= (1-e^{-x}) \cdot (1-e^{-y})$

$= F_X(x) \cdot F_Y(y)$

Hence, X and Y are independent variables.

(5) p.d.f given: $f_X(x) = \frac{1}{2} e^{-|x|}, -\infty < x < \infty$

$F_Y(y) = \cancel{P(y \le Y)}\ P(Y \le y)$

$= P(X^2 \le y)$

$= P(-\sqrt{y} \le X \le \sqrt{y})$

$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2} e^{-|x|} dx$

$= 2 \times \frac{1}{2} \int_0^{\sqrt{y}} e^{-|x|} dx$

$= \int_0^{\sqrt{y}} e^{-|x|} dx$

$= \int_0^{\sqrt{y}} e^{-x} \cdot dx$

$= \left[-e^{-x}\right]_0^{\sqrt{y}}$

$= -e^{-\sqrt{y}} + 1$

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Now, we know.

$$f_Y(Y) = F_y'(Y)$$
$$= \frac{d}{dy}\left(1 - e^{-\sqrt{y}}\right)$$
$$= e^{-\sqrt{y}} \cdot \frac{1}{2\sqrt{y}}$$
$$= \frac{1}{2\sqrt{y}} e^{-\sqrt{y}}$$

Hence, Probability density fn, ~~of~~ $Y = X^2$ is

$$f_Y(Y) = \begin{cases} \frac{1}{2\sqrt{y}} e^{-\sqrt{y}}, & y > 0; \\ 0 & \text{otherwise} \end{cases}$$

16) Given: $f_X(x) = \begin{cases} 6x(1-x) & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$

~~f_X~~ p.d.f.

With this data we can say that,

$$\Rightarrow \int_0^1 cx(1-x)\,dx = 1.$$

$$\Rightarrow \int_0^1 (cx - cx^2)\,dx = 1$$

$$\Rightarrow \left[\frac{cx^2}{2} - \frac{cx^3}{3}\right]_0^1 = 1$$

$$\Rightarrow \frac{c}{2} - \frac{c}{3} = 1 \Rightarrow \frac{c}{6} = 1$$

$$\Rightarrow \boxed{c = 6}$$

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P(17)

$$F_X(x) = \int_0^{x} f_X(x)\cdot dx$$

$$= \int_0^{x} 6x(1-x)\,dx$$

$$= \left[\frac{6x^2}{2} - \frac{6x^3}{3}\right]_0^{x}$$

$$= 3x^2 - 2x^3$$

$$\boxed{F_X(x) = 3x^2 - 2x^3}$$

$$P(X > 1/2) = F_X(1) - F_X(1/2)$$

$$= (3-2) - \left(3\cdot\frac{1}{2^2} - \frac{2}{2^3}\right)$$

$$= 1 - \left(\frac{3}{4} - \cancel{\frac{2}{8}}\frac{1}{4}\right)$$

$$= 1 - 2/4$$

$$= 1/2$$

$$\boxed{P(X > 1/2) = 1/2}$$

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