Bra-Ket Notation

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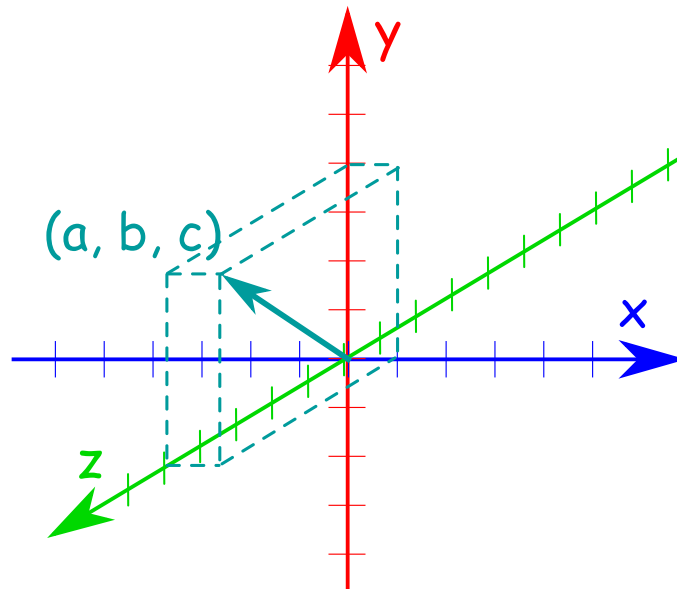
Also called Dirac Notation.

Bra-Ket is a way of writing special vectors used in Quantum Physics that looks like this:

$$\langle bra | ket \rangle$$

Here is a vector in 3 dimensions:

We can write this as a column vector like this:

$$r = \begin{vmatrix} a \\ b \\ c \end{vmatrix}$$

Or we can write it as a "ket":

$$|r\rangle = \begin{vmatrix} a \\ b \\ c \end{vmatrix}$$

But kets are special:

- The values (a, b and c above) are complex numbers (they may be real numbers, imaginary numbers or a combination of both)
- A ket is a quantum state
- Kets can have any number of dimensions, including infinite dimensions!

The "bra" is similar, but the values are in a **row**, and each element is the complex conjugate of the ket's elements.

> Example: This ket:
>
> $$|a\rangle = \begin{vmatrix} 2-3i \\ 6+4i \\ 3-i \end{vmatrix}$$
>
> Has this bra:
>
> $$\langle a| = \begin{vmatrix} 2+3i & 6-4i & 3+i \end{vmatrix}$$
>
> The values are now in a row, and we also **changed the sign** (+ to −, and − to +) in the middle of each element.

In "matrix language", changing a ket into a bra (or bra into a ket) is a "conjugate transpose":

- conjugate: 2−3i becomes 2+3i, etc...
- transpose: rows swap with columns

Read more at Matrix Types .

Multiplying

Multiplying a bra $\langle a|$ and ket $|b\rangle$ looks like this:

$$\langle a|b\rangle$$

We use matrix multiplication , in particluar the dot product :

> The "Dot Product" is where we **multiply matching members**, then sum up:
>
> $$\begin{vmatrix} 1 & 2 & 3 \end{vmatrix} \begin{vmatrix} 7 \\ 9 \\ 11 \end{vmatrix} = 1 \times 7 + 2 \times 9 + 3 \times 11 = 58$$
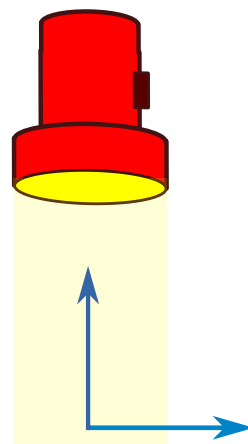
We match the 1st members (1 and 7), multiply them, likewise for the 2nd members (2 and 9) and the 3rd members (3 and 11), and finally sum them up.

In effect, the dot product "projects" one vector on to the other before multiplying the lengths:
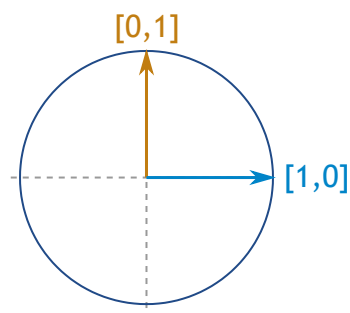
a projected on to **b**
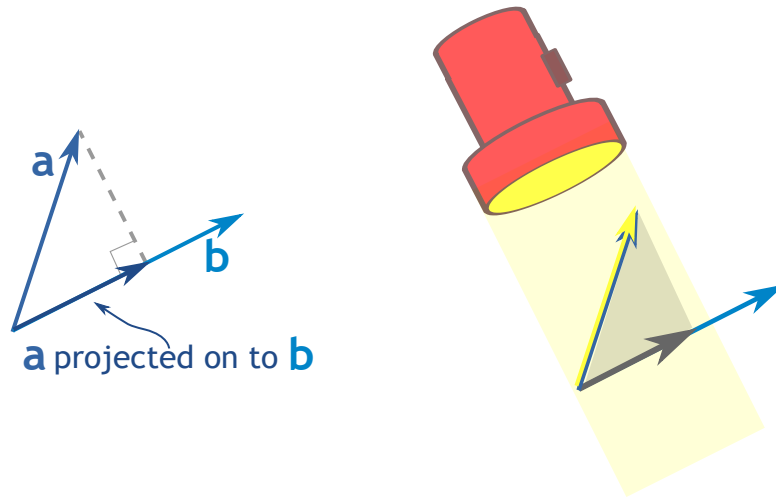
Like shining a light to see
where the shadow lies

When the two vectors are at right angles the dot product is zero:

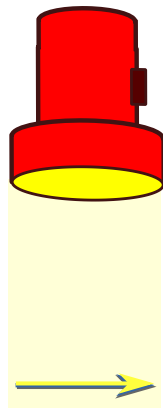No shadow is cast!

Example:

[0,1]

[1,0]

$$|a\rangle = \begin{vmatrix} 1 \\ 0 \end{vmatrix}, \text{ and } |b\rangle = \begin{vmatrix} 0 \\ 1 \end{vmatrix}$$

So:

$$\langle a|b\rangle = \begin{vmatrix} 1 & 0 \end{vmatrix}\begin{vmatrix} 0 \\ 1 \end{vmatrix} = 1\times 0 + 0\times 1 = 0$$

This can be a simple test to see if vectors are **orthogonal** (the more general concept of "at right angles")

The dot product of a vector **with itself** is the length of the vector times the length of the vector. In other words it is **length^2**:

Full shadow is cast!

Example:

$$c = \begin{vmatrix} 2 \\ 1 \end{vmatrix}$$

So:

$$\langle c|c\rangle = \begin{vmatrix} 2 & 1 \end{vmatrix}\begin{vmatrix} 2 \\ 1 \end{vmatrix} = 2\times 2 + 1\times 1 = 5$$

The dot product is **5**

And we can also work out **c**'s length to be $\sqrt{5}$

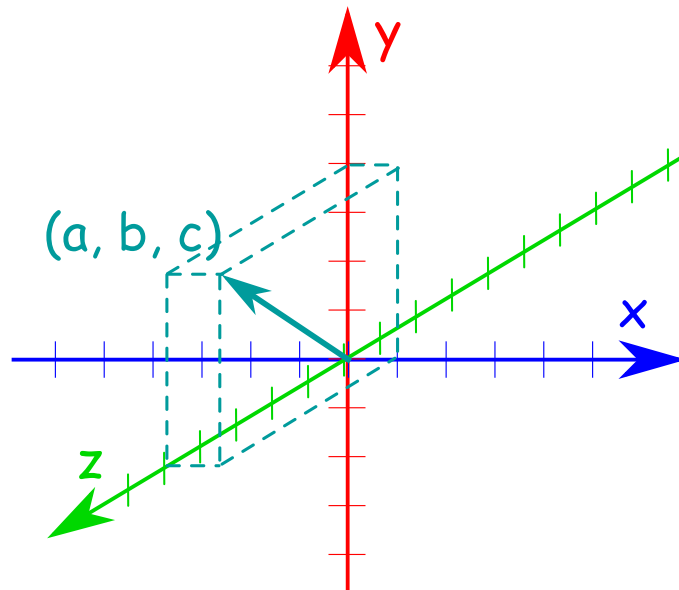Example: What is the length of the vector [1, 2, −2, 5] ?

$$\begin{vmatrix} 1 & 2 & -2 & 5 \end{vmatrix} \begin{vmatrix} 1 \\ 2 \\ -2 \\ 5 \end{vmatrix} = 1{\times}1 + 2{\times}2 + (-2){\times}(-2) + 5{\times}5$$

$$= 1 + 4 + 4 + 25 = 34$$

The dot product is 34, so the vector's length is **√34**

Note: we can also use [Pythagoras' Theorem] to calculate its length:

$$\sqrt{(1^2 + 2^2 + (-2)^2 + 5^2)} = \sqrt{34}$$

Basis

We can separate the parts of a vector like this:

$$|r\rangle = \begin{vmatrix} a \\ b \\ c \end{vmatrix} = a \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} + b \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix} + c \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}$$

The vectors "1, 0, 0", "0, 1, 0" and "0, 0, 1" form the **basis**: the vectors that we measure things against.

In this case they are simple unit vectors, but any set of vectors can be used when they are **independent** of each other (being at right angles achieves this) and can together **span** every part of the space.
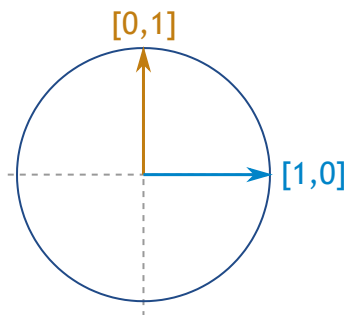
Matrix Rank has more details about linear dependence, span and more.

Orthonormal Basis

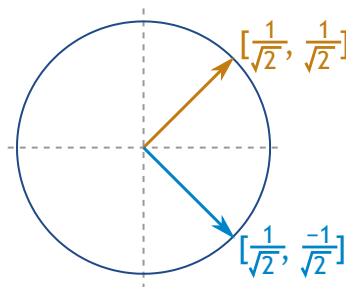In most cases we want an **orthonormal** basis which is:

- **Orthogonal**: each basis vector is at **right angles** to all others.
  We can test it by making sure any pairing of basis vectors has a dot product **a·b = 0**

- **Normalized**: each basis vector has length **1**

Our simple example from above works nicely:

The vectors are at right angles,
and each vector has a length of 1

And this one also works:

Let's check it!

Is the dot product zero?

$$a \cdot b = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \times \frac{-1}{\sqrt{2}}$$

$$= \frac{1}{2} - \frac{1}{2} = 0$$

Is each length 1?

$$|a| = (\frac{1}{\sqrt{2}})^2 + (\frac{1}{\sqrt{2}})^2 = \frac{1}{2} + \frac{1}{2} = 1$$

$$|b| = (\frac{1}{\sqrt{2}})^2 + (\frac{-1}{\sqrt{2}})^2 = \frac{1}{2} + \frac{1}{2} = 1$$

So yes it **is** an orthonormal basis!

Schrödinger's Cat

A famous example is "Schrödinger's Cat": a thought experiment where a cat is in a box with a quantum-triggered container of gas. There is an equal chance of it being alive or dead (until we open the box).
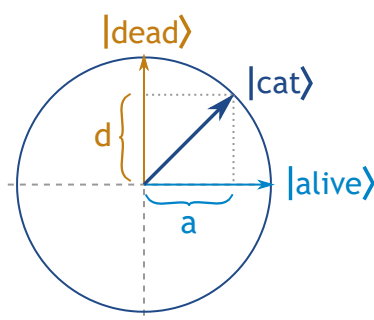
It can be written like this:

$$|cat\rangle = \frac{1}{\sqrt{2}}|alive\rangle + \frac{1}{\sqrt{2}}|dead\rangle$$

It says the state of the cat is in a **superposition** of the two states "alive" and "dead".

But why the $\frac{1}{\sqrt{2}}$?

First let us illustrate it like this:

The **basis** is the two vectors **alive** and **dead**. The cat is shown in that probability space as a vector with equal components a and d.

Now let us normalize it!

Normalized

A **normalized** vector has a length of 1.

We know the dot product of a vector with itself is length^2, so a normalized vector

has:

$$\langle a|a \rangle = 1^2 = 1$$

Example: Normalizing the cat vector

If we assume a = d = 1 we get this:

$$\langle cat|cat \rangle = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \times 1 + 1 \times 1 = 2$$

But it **should be 1**, right?

Let us try $\frac{1}{\sqrt{2}}$:

$$\langle cat|cat \rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{2} + \frac{1}{2} = 1$$

So a = d = $\frac{1}{\sqrt{2}}$, and we get:

$$|cat\rangle = \frac{1}{\sqrt{2}}|alive\rangle + \frac{1}{\sqrt{2}}|dead\rangle$$

And it now has a length of 1

Probability

Let us try to find the probability by adding the component lengths a and d:

$$Probability = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}} = \sqrt{2} \; ???$$

But that can't be right, probability **can't be greater than 1**

In fact we need to take the **magnitude** of each vector (shown using ||) and **square** it:

$$\text{Probability} = |\frac{1}{\sqrt{2}}|^2 + |\frac{1}{\sqrt{2}}|^2$$
$$= \frac{1}{2} + \frac{1}{2} = \mathbf{1} \text{ (yay!)}$$

This is a general rule in Quantum Physics:

> The probability equals the amplitude magnitude squared, in other words:
>
> $$\text{Probability} = |\text{Amplitude}|^2$$
>
> The || means <u>magnitude of a vector</u>, not absolute value.

Naming Kets

Notice how we are free to use any word or symbol inside the ket. In some cases numbers are also used, but they are used as labels so don't try to do arithmetic with them.

Many Dimensions

We can **easily** have many dimensions.

Imagine "Quantum Dice" that are in a superposition of 1, 2, 3, 4, 5 and 6.

The ket looks like this:

$$|\text{die}\rangle = \begin{vmatrix} a \\ b \\ c \\ d \\ e \\ f \end{vmatrix}$$

For a [fair die] all elements (a, b, c, d, e, f) are equal, but **your** dice may be loaded!

Why?

Why do we do all this?

So we can "map" some real world case (usually one with probabilities) onto a well-defined mathematical basis. This then gives us the power to use all the math tools to study it.

Conclusion

The bra-ket notation is a simple way to refer to a vector with complex elements, any number of dimensions, that represents one state in a state space. The probability of any state equals the magnitude of its vector squared.