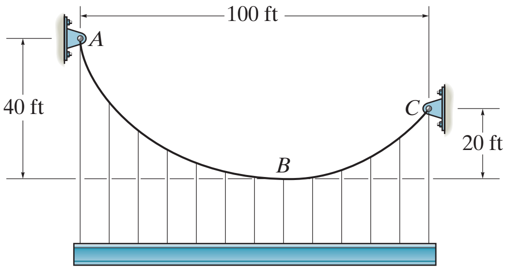
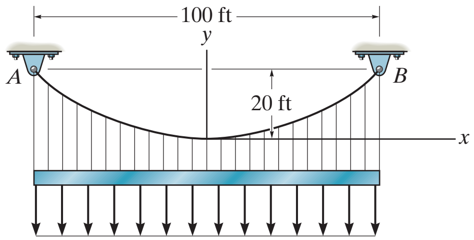
7–107. The cable supports a girder which weighs 850 lb/ft. Determine the tension in the cable at points A, B, and C.



Prob. 7–107

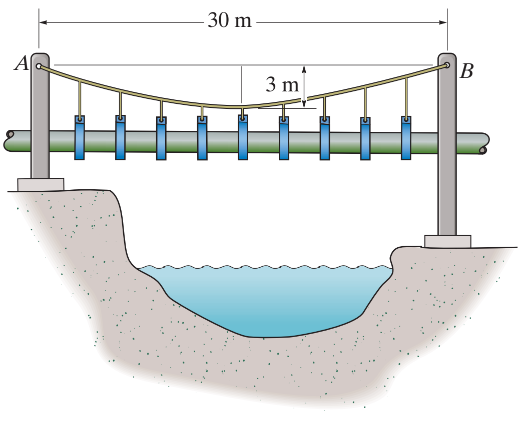
***7–108.** The cable is subjected to a uniform loading of $w = 200$ lb/ft. Determine the maximum and minimum tension in the cable.



Prob. 7–108

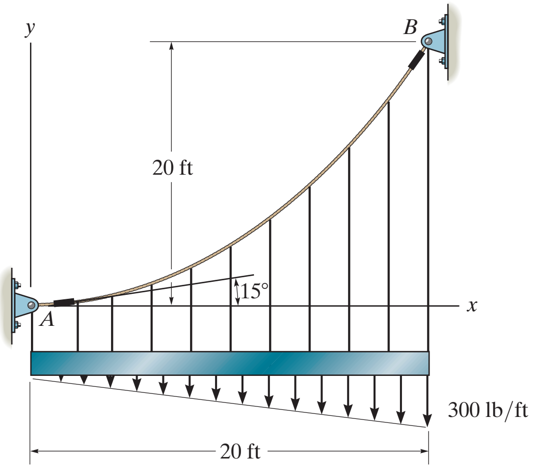
7–109. If the pipe has a mass per unit length of 1500 kg/m, determine the maximum tension developed in the cable.

7–110. If the pipe has a mass per unit length of 1500 kg/m, determine the minimum tension developed in the cable.



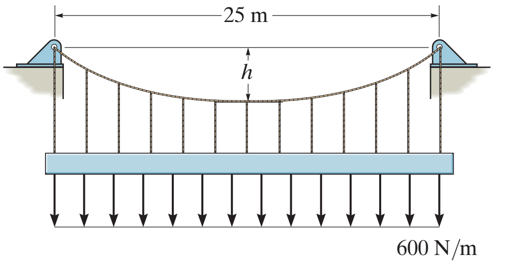
Probs. 7–109/110

7–111. Determine the maximum tension developed in the cable if it is subjected to the triangular distributed load.



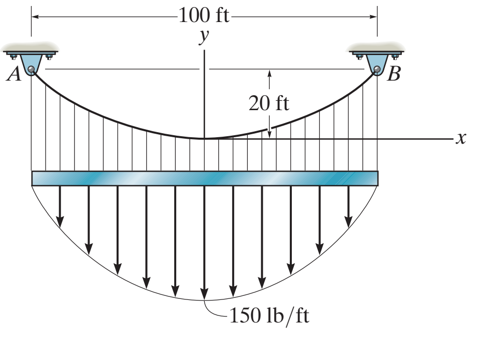
Prob. 7–111

***7–112.** The cable will break when the maximum tension reaches $T_{max} = 10$ kN. Determine the minimum sag h if it supports the uniform distributed load of $w = 600$ N/m.



Prob. 7–112

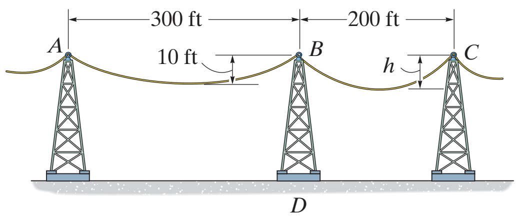
7–113. The cable is subjected to the parabolic loading $w = 150(1 - (x/50)^2)$ lb/ft, where x is in ft. Determine the equation $y = f(x)$ which defines the cable shape AB and the maximum tension in the cable.



Prob. 7–113

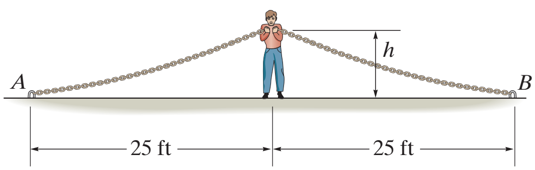
7–114. The power transmission cable weighs 10 lb/ft. If the resultant horizontal force on tower BD is required to be zero, determine the sag h of cable BC.

7–115. The power transmission cable weighs 10 lb/ft. If $h = 10$ ft, determine the resultant horizontal and vertical forces the cables exert on tower BD.



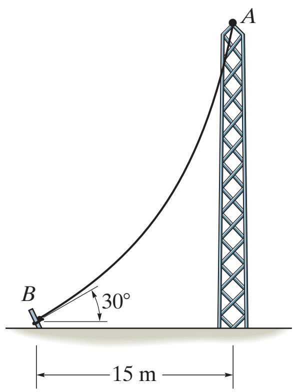
Probs. 7–114/115

***7–116.** The man picks up the 52-ft chain and holds it just high enough so it is completely off the ground. The chain has points of attachment A and B that are 50 ft apart. If the chain has a weight of 3 lb/ft, and the man weighs 150 lb, determine the force he exerts on the ground. Also, how high h must he lift the chain? *Hint*: The slopes at A and B are zero.



Prob. 7–116

7–117. The cable has a mass of 0.5 kg/m and is 25 m long. Determine the vertical and horizontal components of force it exerts on the top of the tower.



Prob. 7–117

7–118. A 50-ft cable is suspended between two points a distance of 15 ft apart and at the same elevation. If the minimum tension in the cable is 200 lb, determine the total weight of the cable and the maximum tension developed in the cable.

7–119. Show that the deflection curve of the cable discussed in Example 7.13 reduces to Eq. 4 in Example 7.12 when the *hyperbolic cosine function* is expanded in terms of a series and only the first two terms are retained. (The answer indicates that the *catenary* may be replaced by a *parabola* in the analysis of problems in which the sag is small. In this case, the cable weight is assumed to be uniformly distributed along the horizontal.)

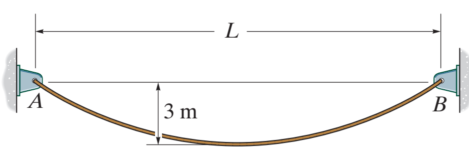
***7–120.** A telephone line (cable) stretches between two points which are 150 ft apart and at the same elevation. The line sags 5 ft and the cable has a weight of 0.3 lb/ft. Determine the length of the cable and the maximum tension in the cable.

7–121. A cable has a weight of 2 lb/ft. If it can span 100 ft and has a sag of 12 ft, determine the length of the cable. The ends of the cable are supported from the same elevation.

7–122. A cable has a weight of 3 lb/ft and is supported at points that are 500 ft apart and at the same elevation. If it has a length of 600 ft, determine the sag.

7–123. A cable has a weight of 5 lb/ft. If it can span 300 ft and has a sag of 15 ft, determine the length of the cable. The ends of the cable are supported at the same elevation.

***7–124.** The 10 kg/m cable is suspended between the supports A and B. If the cable can sustain a maximum tension of 1.5 kN and the maximum sag is 3 m, determine the maximum distance L between the supports.

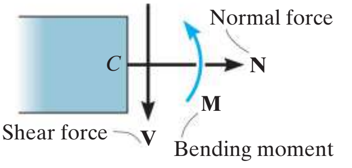


Prob. 7–124

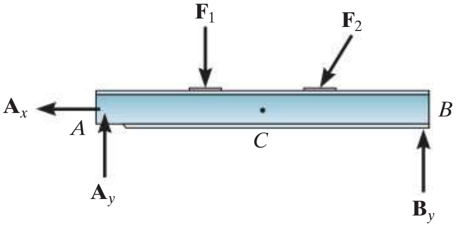
CHAPTER REVIEW

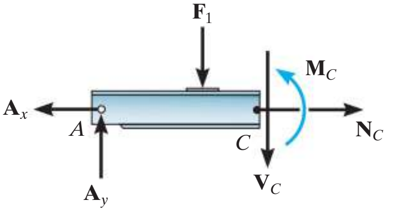
Internal Loadings

If a coplanar force system acts on a member, then in general a resultant internal *normal force* **N**, *shear force* **V**, and *bending moment* **M** will act at any cross section along the member. For two-dimensional problems the positive directions of these loadings are shown in the figure.



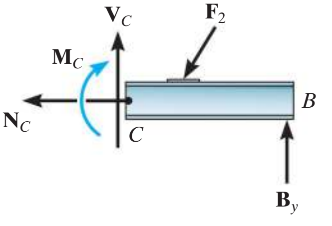
The resultant internal normal force, shear force, and bending moment are determined using the method of sections. To find them, the member is sectioned at the point C where the internal loadings are to be determined. A free-body diagram of one of the sectioned parts is then drawn and the internal loadings are shown in their positive directions.



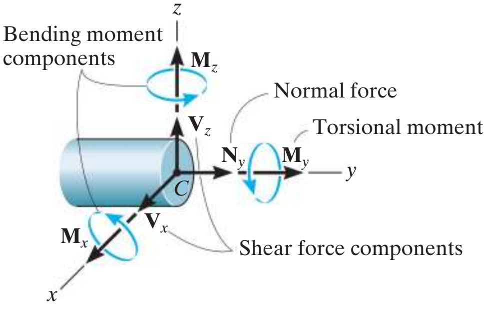


The resultant normal force is determined by summing forces normal to the cross section. The resultant shear force is found by summing forces tangent to the cross section, and the resultant bending moment is found by summing moments about the geometric center or centroid of the cross-sectional area.

$$\Sigma F_x = 0$$
$$\Sigma F_y = 0$$
$$\Sigma M_C = 0$$

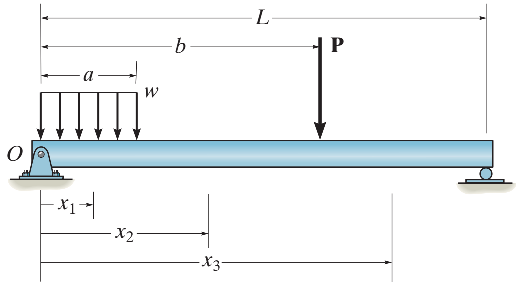


If the member is subjected to a three-dimensional loading, then, in general, a *torsional moment* will also act on the cross section. It can be determined by summing moments about an axis that is perpendicular to the cross section and passes through its centroid.



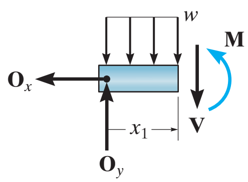
Shear and Moment Diagrams

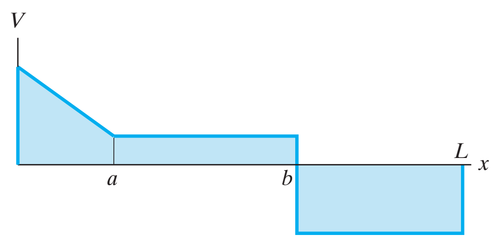
To construct the shear and moment diagrams for a member, it is necessary to section the member at an arbitrary point, located a distance x from the left end.



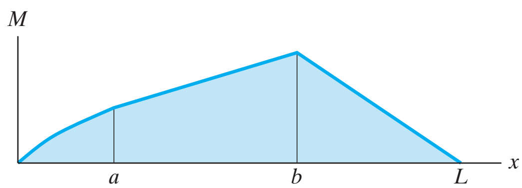
If the external loading consists of changes in the distributed load, or a series of concentrated forces and couple moments act on the member, then different expressions for V and M must be determined within regions between any load discontinuities.

The unknown shear and moment are indicated on the cross section in the positive direction according to the established sign convention, and then the internal shear and moment are determined as functions of x.





Each of the functions of the shear and moment is then plotted to create the shear and moment diagrams.



Relations between Shear and Moment

It is possible to plot the shear and moment diagrams quickly by using differential relationships that exist between the distributed loading *w*, *V* and *M*.

The slope of the shear diagram is equal to the distributed loading at any point. The slope is positive if the distributed load acts upward, and vice-versa.

$$\frac{dV}{dx} = w$$

The slope of the moment diagram is equal to the shear at any point. The slope is positive if the shear is positive, or vice-versa.

$$\frac{dM}{dx} = V$$

The change in shear between any two points is equal to the area under the distributed loading between the points.

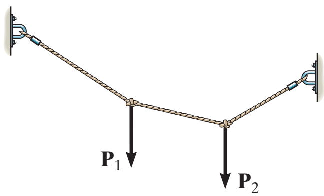
$$\Delta V = \int w\,dx$$

The change in the moment is equal to the area under the shear diagram between the points.

$$\Delta M = \int V\,dx$$

Cables

When a flexible and inextensible cable is subjected to a series of concentrated forces, then the analysis of the cable can be performed by using the equations of equilibrium applied to free-body diagrams of either segments or points of application of the loading.



If external distributed loads or the weight of the cable are to be considered, then the shape of the cable must be determined by first analyzing the forces on a differential segment of the cable and then integrating this result. The two constants, say C_1 and C_2, resulting from the integration are determined by applying the boundary conditions for the cable.

$$y = \frac{1}{F_H}\int\left(\int w(x)\,dx\right)dx$$

Distributed load

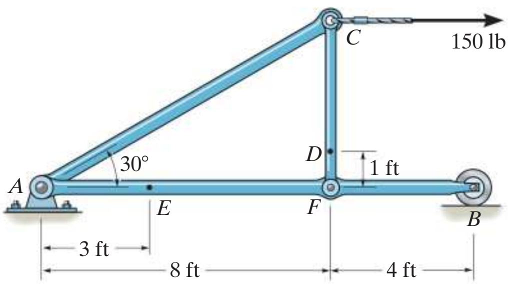
$$x = \int \frac{ds}{\left[1 + \frac{1}{F_H^2}\left(\int w(s)\,ds\right)^2\right]^{1/2}}$$

Cable weight

REVIEW PROBLEMS

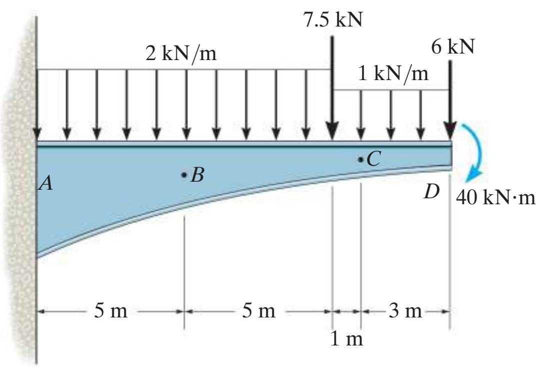
All problem solutions must include FBDs.

R7–1. Determine the internal normal force, shear force, and moment at points *D* and *E* of the frame.



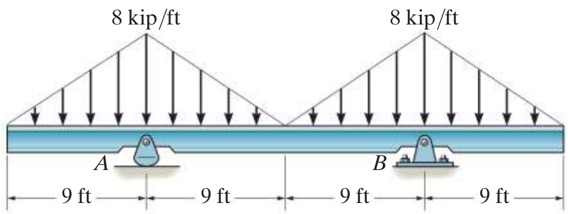
Prob. R7–1

R7–2. Determine the normal force, shear force, and moment at points *B* and *C* of the beam.



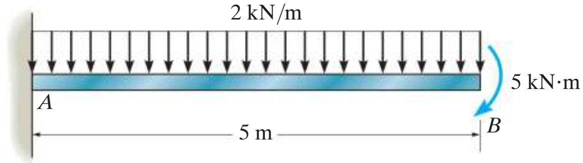
Prob. R7–2

R7–3. Draw the shear and moment diagrams for the beam.



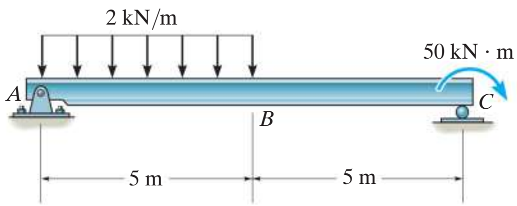
Prob. R7–3

R7–4. Draw the shear and moment diagrams for the beam.



Prob. R7–4

R7–5. Draw the shear and moment diagrams for the beam.



Prob. R7–5

R7–6. A chain is suspended between points at the same elevation and spaced a distance of 60 ft apart. If it has a weight per unit length of 0.5 lb/ft and the sag is 3 ft, determine the maximum tension in the chain.

7

Chapter 8

(© Pavel Polkovnikov/Shutterstock)

The effective design of this brake requires that it resist the frictional forces developed between it and the wheel. In this chapter we will study dry friction, and show how to analyze friction forces for various engineering applications.

Friction

CHAPTER OBJECTIVES

- To introduce the concept of dry friction and show how to analyze the equilibrium of rigid bodies subjected to this force.
- To present specific applications of frictional force analysis on wedges, screws, belts, and bearings.
- To investigate the concept of rolling resistance.

8.1 Characteristics of Dry Friction

Friction is a force that resists the movement of two contacting surfaces that slide relative to one another. This force always acts *tangent* to the surface at the points of contact and is directed so as to oppose the possible or existing motion between the surfaces.

In this chapter, we will study the effects of ***dry friction***, which is sometimes called *Coulomb friction* since its characteristics were studied extensively by the French physicist Charles-Augustin de Coulomb in 1781. Dry friction occurs between the contacting surfaces of bodies when there is no lubricating fluid.*

The heat generated by the abrasive action of friction can be noticed when using this grinder to sharpen a metal blade. (© Russell C. Hibbeler)

*Another type of friction, called fluid friction, is studied in fluid mechanics.

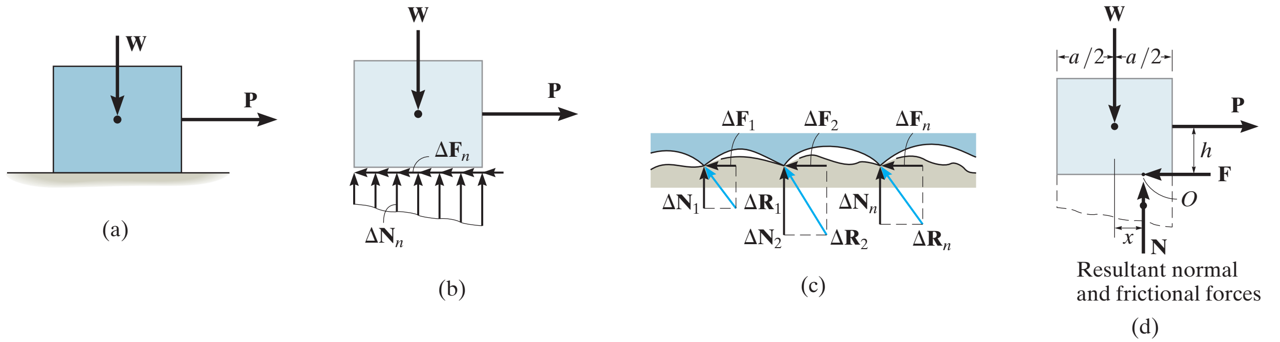
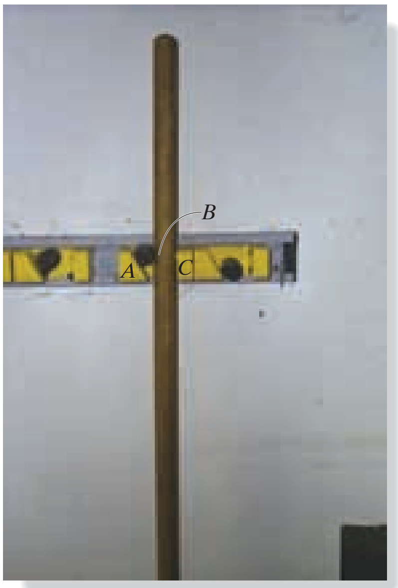


Fig. 8–1



Regardless of the weight of the rake or shovel that is suspended, the device has been designed so that the small roller holds the handle in equilibrium due to frictional forces that develop at the points of contact, A, B, C. (© Russell C. Hibbeler)

Theory of Dry Friction. The theory of dry friction can be explained by considering the effects caused by pulling horizontally on a block of uniform weight $\mathbf{W}$ which is resting on a rough horizontal surface that is *nonrigid or deformable,* Fig. 8–1*a*. The upper portion of the block, however, can be considered rigid. As shown on the free-body diagram of the block, Fig. 8–1*b*, the floor exerts an uneven *distribution* of both *normal force* $\Delta\mathbf{N}_n$ and *frictional force* $\Delta\mathbf{F}_n$ along the contacting surface. For equilibrium, the normal forces must act *upward* to balance the block's weight $\mathbf{W}$, and the frictional forces act to the left to prevent the applied force $\mathbf{P}$ from moving the block to the right. Close examination of the contacting surfaces between the floor and block reveals how these frictional and normal forces develop, Fig. 8–1*c*. It can be seen that many microscopic irregularities exist between the two surfaces and, as a result, reactive forces $\Delta\mathbf{R}_n$ are developed at each point of contact.* As shown, each reactive force contributes both a frictional component $\Delta\mathbf{F}_n$ and a normal component $\Delta\mathbf{N}_n$.

Equilibrium. The effect of the *distributed* normal and frictional loadings is indicated by their *resultants* $\mathbf{N}$ and $\mathbf{F}$ on the free-body diagram, Fig. 8–1*d*. Notice that $\mathbf{N}$ acts a distance x to the right of the line of action of $\mathbf{W}$, Fig. 8–1*d*. This location, which coincides with the centroid or geometric center of the normal force distribution in Fig. 8–1*b*, is necessary in order to balance the "tipping effect" caused by $\mathbf{P}$. For example, if $\mathbf{P}$ is applied at a height h from the surface, Fig. 8–1*d*, then moment equilibrium about point O is satisfied if $Wx = Ph$ or $x = Ph/W$.

*Besides mechanical interactions as explained here, which is referred to as a classical approach, a detailed treatment of the nature of frictional forces must also include the effects of temperature, density, cleanliness, and atomic or molecular attraction between the contacting surfaces. See J. Krim, *Scientific American*, October, 1996.

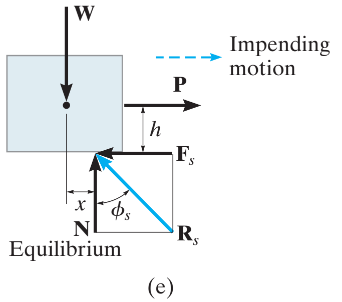
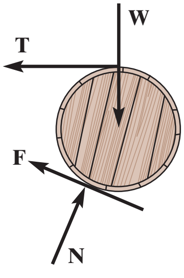


Fig. 8–1 (cont.)



Some objects, such as this barrel, may not be on the verge of slipping, and therefore the friction force **F** must be determined strictly from the equations of equilibrium. (© Russell C. Hibbeler)

Impending Motion. In cases where the surfaces of contact are rather "slippery," the frictional force **F** may *not* be great enough to balance **P**, and consequently the block will tend to slip. In other words, as P is slowly increased, F correspondingly increases until it attains a certain *maximum value* F_s, called the *limiting static frictional force*, Fig. 8–1*e*. When this value is reached, the block is in *unstable equilibrium* since any further increase in P will cause the block to move. Experimentally, it has been determined that this limiting static frictional force F_s is *directly proportional* to the resultant normal force N. Expressed mathematically,

$$\boxed{F_s = \mu_s N} \tag{8–1}$$

where the constant of proportionality, μ_s (mu "sub" s), is called the ***coefficient of static friction***.

Thus, when the block is on the *verge of sliding*, the normal force **N** and frictional force $\mathbf{F}_s$ combine to create a resultant $\mathbf{R}_s$, Fig. 8–1*e*. The angle ϕ_s (phi "sub" s) that $\mathbf{R}_s$ makes with **N** is called the ***angle of static friction***. From the figure,

$$\phi_s = \tan^{-1}\left(\frac{F_s}{N}\right) = \tan^{-1}\left(\frac{\mu_s N}{N}\right) = \tan^{-1}\mu_s$$

Typical values for μ_s are given in Table 8–1. Note that these values can vary since experimental testing was done under variable conditions of roughness and cleanliness of the contacting surfaces. For applications, therefore, it is important that both caution and judgment be exercised when selecting a coefficient of friction for a given set of conditions. When a more accurate calculation of F_s is required, the coefficient of friction should be determined directly by an experiment that involves the two materials to be used.

Table 8–1 Typical Values for μ_s

Contact Materials	Coefficient of Static Friction (μ_s)
Metal on ice	0.03–0.05
Wood on wood	0.30–0.70
Leather on wood	0.20–0.50
Leather on metal	0.30–0.60
Copper on copper	0.74–1.21

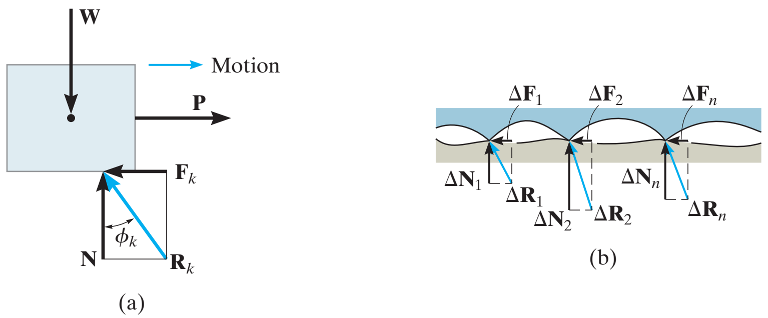


Fig. 8–2

Motion. If the magnitude of **P** acting on the block is increased so that it becomes slightly greater than F_s, the frictional force at the contacting surface will drop to a smaller value F_k, called the *kinetic frictional force*. The block will begin to slide with increasing speed, Fig. 8–2*a*. As this occurs, the block will "ride" on top of these peaks at the points of contact, as shown in Fig. 8–2*b*. The continued breakdown of the surface is the dominant mechanism creating kinetic friction.

Experiments with sliding blocks indicate that the magnitude of the kinetic friction force is directly proportional to the magnitude of the resultant normal force, expressed mathematically as

$$\boxed{F_k = \mu_k N} \qquad (8\text{–}2)$$

Here the constant of proportionality, μ_k, is called the ***coefficient of kinetic friction***. Typical values for μ_k are approximately 25 percent *smaller* than those listed in Table 8–1 for μ_s.

As shown in Fig. 8–2*a*, in this case, the resultant force at the surface of contact, $\mathbf{R}_k$, has a line of action defined by ϕ_k. This angle is referred to as the ***angle of kinetic friction***, where

$$\phi_k = \tan^{-1}\left(\frac{F_k}{N}\right) = \tan^{-1}\left(\frac{\mu_k N}{N}\right) = \tan^{-1}\mu_k$$

By comparison, $\phi_s \geq \phi_k$.

The above effects regarding friction can be summarized by referring to the graph in Fig. 8–3, which shows the variation of the frictional force F versus the applied load P. Here the frictional force is categorized in three different ways:

- F is a *static frictional force* if equilibrium is maintained.
- F is a *limiting static frictional force* F_s when it reaches a maximum value needed to maintain equilibrium.
- F is a *kinetic frictional force* F_k when sliding occurs at the contacting surface.

Notice also from the graph that for very large values of P or for high speeds, aerodynamic effects will cause F_k and likewise μ_k to begin to decrease.

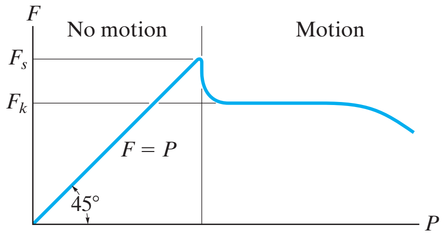


Fig. 8–3

Characteristics of Dry Friction.

As a result of *experiments* that pertain to the foregoing discussion, we can state the following rules which apply to bodies subjected to dry friction.

- The frictional force acts *tangent* to the contacting surfaces in a direction *opposed* to the *motion* or tendency for motion of one surface relative to another.
- The maximum static frictional force F_s that can be developed is independent of the area of contact, provided the normal pressure is not very low nor great enough to severely deform or crush the contacting surfaces of the bodies.
- The maximum static frictional force is generally greater than the kinetic frictional force for any two surfaces of contact. However, if one of the bodies is moving with a *very low velocity* over the surface of another, F_k becomes approximately equal to F_s, i.e., $\mu_s \approx \mu_k$.
- When *slipping* at the surface of contact is *about to occur*, the maximum static frictional force is proportional to the normal force, such that $F_s = \mu_s N$.
- When *slipping* at the surface of contact is *occurring*, the kinetic frictional force is proportional to the normal force, such that $F_k = \mu_k N$.

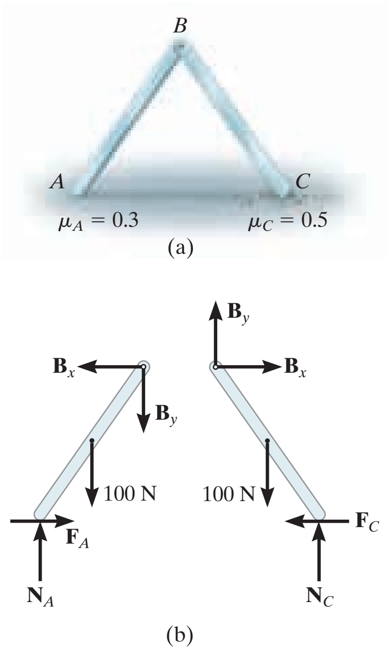


Fig. 8–4

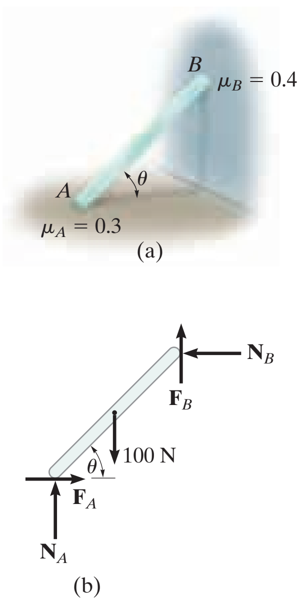


Fig. 8–5

8.2 Problems Involving Dry Friction

If a rigid body is in equilibrium when it is subjected to a system of forces that includes the effect of friction, the force system must satisfy not only the equations of equilibrium but *also* the laws that govern the frictional forces.

Types of Friction Problems. In general, there are three types of static problems involving dry friction. They can easily be classified once free-body diagrams are drawn and the total number of unknowns are identified and compared with the total number of available equilibrium equations.

No Apparent Impending Motion. Problems in this category are strictly equilibrium problems, which require the number of unknowns to be *equal* to the number of available equilibrium equations. Once the frictional forces are determined from the solution, however, their numerical values must be checked to be sure they satisfy the inequality $F \le \mu_s N$; otherwise, slipping will occur and the body will not remain in equilibrium. A problem of this type is shown in Fig. 8–4*a*. Here we must determine the frictional forces at A and C to check if the equilibrium position of the two-member frame can be maintained. If the bars are uniform and have known weights of 100 N each, then the free-body diagrams are as shown in Fig. 8–4*b*. There are six unknown force components which can be determined *strictly* from the six equilibrium equations (three for each member). Once F_A, N_A, F_C, and N_C are determined, then the bars will remain in equilibrium provided $F_A \le 0.3N_A$ and $F_C \le 0.5N_C$ are satisfied.

Impending Motion at All Points of Contact. In this case the total number of unknowns will *equal* the total number of available equilibrium equations *plus* the total number of available frictional equations, $F = \mu N$. When *motion is impending* at the points of contact, then $F_s = \mu_s N$; whereas if the body is *slipping*, then $F_k = \mu_k N$. For example, consider the problem of finding the smallest angle θ at which the 100-N bar in Fig. 8–5*a* can be placed against the wall without slipping. The free-body diagram is shown in Fig. 8–5*b*. Here the *five* unknowns are determined from the *three* equilibrium equations and *two* static frictional equations which apply at *both* points of contact, so that $F_A = 0.3N_A$ and $F_B = 0.4N_B$.

Impending Motion at Some Points of Contact. Here the number of unknowns will be *less* than the number of available equilibrium equations plus the number of available frictional equations or conditional equations for tipping. As a result, several possibilities for motion or impending motion will exist and the problem will involve a determination of the kind of motion which actually occurs. For example, consider the two-member frame in Fig. 8–6*a*. In this problem we wish to determine the horizontal force P needed to cause movement. If each member has a weight of 100 N, then the free-body diagrams are as shown in Fig. 8–6*b*. There are *seven* unknowns. For a unique solution we must satisfy the *six* equilibrium equations (three for each member) and only *one* of two possible static frictional equations. This means that as P increases it will either cause slipping at A and no slipping at C, so that $F_A = 0.3N_A$ and $F_C \leq 0.5N_C$; or slipping occurs at C and no slipping at A, in which case $F_C = 0.5N_C$ and $F_A \leq 0.3N_A$. The actual situation can be determined by calculating P for each case and then choosing the case for which P is *smaller*. If in both cases the *same value* for P is calculated, which would be highly improbable, then slipping at both points occurs simultaneously; i.e., the *seven unknowns* would satisfy *eight equations*.

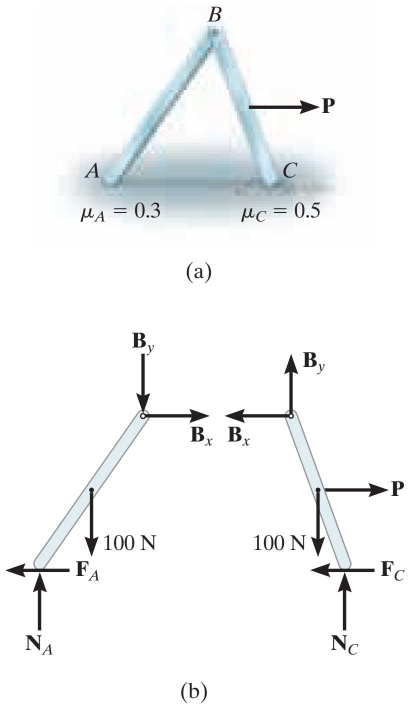


Fig. 8–6

Equilibrium Versus Frictional Equations.

Whenever we solve a problem such as the one in Fig. 8–4, where the friction force F is to be an "equilibrium force" and satisfies the inequality $F < \mu_s N$, then we can assume the sense of direction of F on the free-body diagram. The correct sense is made known *after* solving the equations of equilibrium for F. If F is a negative scalar the sense of **F** is the reverse of that which was assumed. This convenience of *assuming* the sense of **F** is possible because the equilibrium equations equate to zero the *components of vectors* acting in the *same direction*. However, in cases where the frictional equation $F = \mu N$ is used in the solution of a problem, as in the case shown in Fig. 8–5, then the convenience of *assuming* the sense of **F** is *lost*, since the frictional equation relates only the *magnitudes* of two *perpendicular* vectors. Consequently, **F** *must always* be shown acting with its *correct sense* on the free-body diagram, *whenever* the frictional equation is used for the solution of a problem.

Depending upon where the man pushes on the crate, it will either tip or slip. (© Russell C. Hibbeler)

Important Points

- Friction is a tangential force that resists the movement of one surface relative to another.
- If no sliding occurs, the maximum value for the friction force is equal to the product of the coefficient of static friction and the normal force at the surface.
- If sliding occurs at a slow speed, then the friction force is the product of the coefficient of kinetic friction and the normal force at the surface.
- There are three types of static friction problems. Each of these problems is analyzed by first drawing the necessary free-body diagrams, and then applying the equations of equilibrium, while satisfying the conditions of friction or the possibility of tipping.

Procedure for Analysis

Equilibrium problems involving dry friction can be solved using the following procedure.

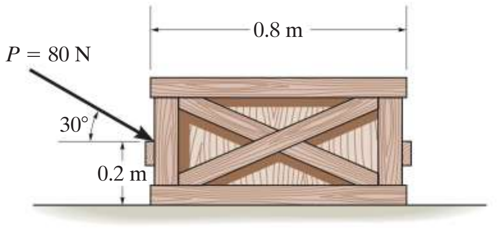
Free-Body Diagrams.

- Draw the necessary free-body diagrams, and unless it is stated in the problem that impending motion or slipping occurs, *always* show the frictional forces as unknowns (i.e., *do not assume* $F = \mu N$).
- Determine the number of unknowns and compare this with the number of available equilibrium equations.
- If there are more unknowns than equations of equilibrium, it will be necessary to apply the frictional equation at some, if not all, points of contact to obtain the extra equations needed for a complete solution.
- If the equation $F = \mu N$ is to be used, it will be necessary to show $\mathbf{F}$ acting in the correct sense of direction on the free-body diagram.

Equations of Equilibrium and Friction.

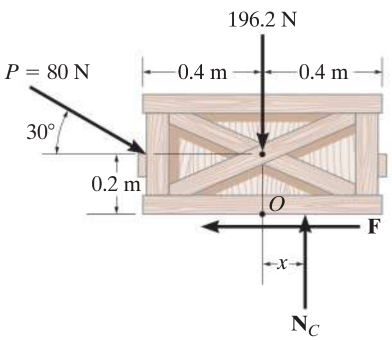
- Apply the equations of equilibrium and the necessary frictional equations (or conditional equations if tipping is possible) and solve for the unknowns.
- If the problem involves a three-dimensional force system such that it becomes difficult to obtain the force components or the necessary moment arms, apply the equations of equilibrium using Cartesian vectors.

The uniform crate shown in Fig. 8–7*a* has a mass of 20 kg. If a force $P = 80$ N is applied to the crate, determine if it remains in equilibrium. The coefficient of static friction is $\mu_s = 0.3$.



(a)

Fig. 8–7



(b)

SOLUTION

Free-Body Diagram. As shown in Fig. 8–7*b*, the *resultant* normal force $\mathbf{N}_C$ must act a distance x from the crate's center line in order to counteract the tipping effect caused by $\mathbf{P}$. There are *three unknowns*, F, N_C, and x, which can be determined strictly from the *three* equations of equilibrium.

Equations of Equilibrium.

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad 80 \cos 30^\circ \text{ N} - F = 0$$

$$+\uparrow \Sigma F_y = 0; \qquad -80 \sin 30^\circ \text{ N} + N_C - 196.2 \text{ N} = 0$$

$$\circlearrowleft + \Sigma M_O = 0; \quad 80 \sin 30^\circ \text{ N}(0.4 \text{ m}) - 80 \cos 30^\circ \text{ N}(0.2 \text{ m}) + N_C(x) = 0$$

Solving,

$$F = 69.3 \text{ N}$$

$$N_C = 236.2 \text{ N}$$

$$x = -0.00908 \text{ m} = -9.08 \text{ mm}$$

Since x is negative it indicates the *resultant* normal force acts (slightly) to the *left* of the crate's center line. No tipping will occur since $x < 0.4$ m. Also, the *maximum* frictional force which can be developed at the surface of contact is $F_{\max} = \mu_s N_C = 0.3(236.2 \text{ N}) = 70.9$ N. Since $F = 69.3 \text{ N} < 70.9$ N, the crate will *not slip*, although it is very close to doing so.

8

EXAMPLE 8.2

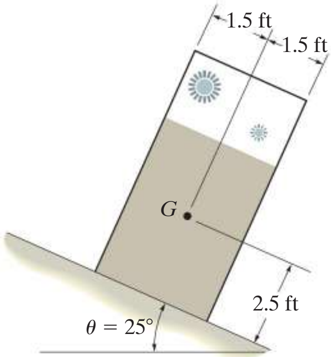
It is observed that when the bed of the dump truck is raised to an angle of $\theta = 25^\circ$ the vending machines will begin to slide off the bed, Fig. 8–8*a*. Determine the static coefficient of friction between a vending machine and the surface of the truckbed.

(a)

(© Russell C. Hibbeler)

SOLUTION

An idealized model of a vending machine resting on the truckbed is shown in Fig. 8–8*b*. The dimensions have been measured and the center of gravity has been located. We will assume that the vending machine weighs W.



(b)

Free-Body Diagram. As shown in Fig. 8–8*c*, the dimension x is used to locate the position of the resultant normal force $\mathbf{N}$. There are four unknowns, N, F, μ_s, and x.

Equations of Equilibrium.

$$+\searrow \Sigma F_x = 0; \qquad W \sin 25^\circ - F = 0 \qquad (1)$$

$$+\nearrow \Sigma F_y = 0; \qquad N - W \cos 25^\circ = 0 \qquad (2)$$

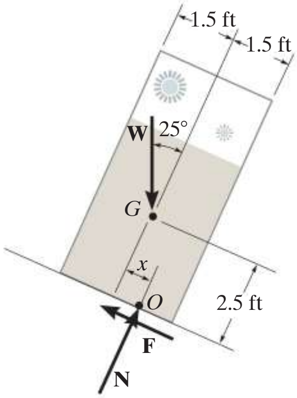
$$\circlearrowleft + \Sigma M_O = 0; \qquad -W \sin 25^\circ(2.5 \text{ ft}) + W \cos 25^\circ(x) = 0 \qquad (3)$$

Since slipping impends at $\theta = 25^\circ$, using Eqs. 1 and 2, we have

$$F_s = \mu_s N; \qquad W \sin 25^\circ = \mu_s(W \cos 25^\circ)$$

$$\mu_s = \tan 25^\circ = 0.466 \qquad \textit{Ans.}$$

The angle of $\theta = 25^\circ$ is referred to as the ***angle of repose***, and by comparison, it is equal to the angle of static friction, $\theta = \phi_s$. Notice from the calculation that θ is independent of the weight of the vending machine, and so knowing θ provides a convenient method for determining the coefficient of static friction.

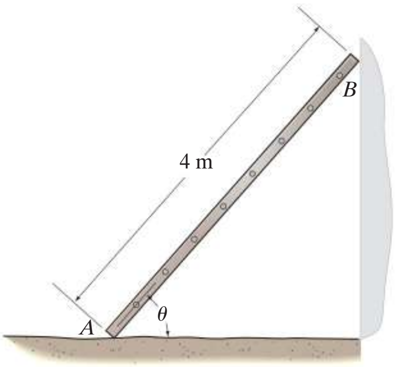


(c)

Fig. 8–8

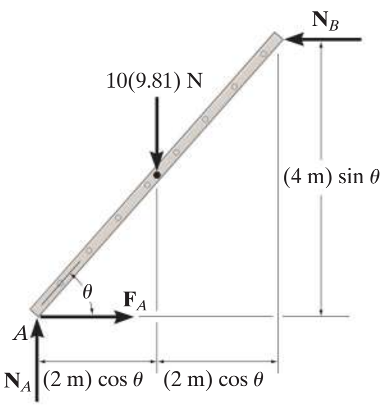
NOTE: From Eq. 3, we find $x = 1.17$ ft. Since $1.17 \text{ ft} < 1.5 \text{ ft}$, indeed the vending machine will slip before it can tip as observed in Fig. 8–8*a*.

The uniform 10-kg ladder in Fig. 8–9*a* rests against the smooth wall at B, and the end A rests on the rough horizontal plane for which the coefficient of static friction is $\mu_s = 0.3$. Determine the angle of inclination θ of the ladder and the normal reaction at B if the ladder is on the verge of slipping.



(a)

Fig. 8–9



(b)

SOLUTION

Free-Body Diagram. As shown on the free-body diagram, Fig. 8–9*b*, the frictional force $\mathbf{F}_A$ must act to the right since impending motion at A is to the left.

Equations of Equilibrium and Friction. Since the ladder is on the verge of slipping, then $F_A = \mu_s N_A = 0.3 N_A$. By inspection, N_A can be obtained directly.

$$+\uparrow \Sigma F_y = 0; \qquad N_A - 10(9.81)\text{ N} = 0 \qquad N_A = 98.1\text{ N}$$

Using this result, $F_A = 0.3(98.1\text{ N}) = 29.43\text{ N}$. Now N_B can be found.

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad 29.43\text{ N} - N_B = 0$$

$$N_B = 29.43\text{ N} = 29.4\text{ N} \qquad \textit{Ans.}$$

Finally, the angle θ can be determined by summing moments about point A.

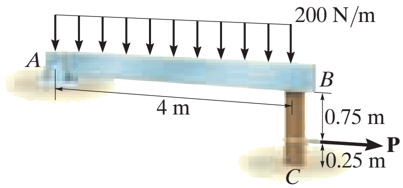
$$\circlearrowleft + \Sigma M_A = 0; \qquad (29.43\text{ N})(4\text{ m})\sin\theta - [10(9.81)\text{ N}](2\text{ m})\cos\theta = 0$$

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = 1.6667$$

$$\theta = 59.04^\circ = 59.0^\circ \qquad \textit{Ans.}$$

EXAMPLE 8.4

Beam AB is subjected to a uniform load of 200 N/m and is supported at B by post BC, Fig. 8–10a. If the coefficients of static friction at B and C are $\mu_B = 0.2$ and $\mu_C = 0.5$, determine the force $\mathbf{P}$ needed to pull the post out from under the beam. Neglect the weight of the members and the thickness of the beam.



(a)

SOLUTION

Free-Body Diagrams. The free-body diagram of the beam is shown in Fig. 8–10b. Applying $\Sigma M_A = 0$, we obtain $N_B = 400$ N. This result is shown on the free-body diagram of the post, Fig. 8–10c. Referring to this member, the *four* unknowns F_B, P, F_C, and N_C are determined from the *three* equations of equilibrium and *one* frictional equation applied either at B or C.

Equations of Equilibrium and Friction.

$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad P - F_B - F_C = 0 \qquad (1)$$

$$+\uparrow\Sigma F_y = 0; \qquad N_C - 400\text{ N} = 0 \qquad (2)$$

$$\circlearrowleft + \Sigma M_C = 0; \qquad -P(0.25\text{ m}) + F_B(1\text{ m}) = 0 \qquad (3)$$

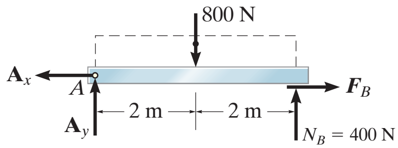
(Post Slips at *B* and Rotates about *C*.) This requires $F_C \le \mu_C N_C$ and

$$F_B = \mu_B N_B; \qquad F_B = 0.2(400\text{ N}) = 80\text{ N}$$

Using this result and solving Eqs. 1 through 3, we obtain

$$P = 320\text{ N}$$
$$F_C = 240\text{ N}$$
$$N_C = 400\text{ N}$$

Since $F_C = 240\text{ N} > \mu_C N_C = 0.5(400\text{ N}) = 200\text{ N}$, slipping at C occurs. Thus the other case of movement must be investigated.



(b)

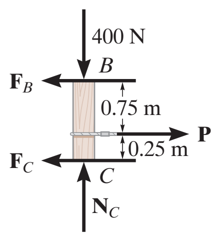
(Post Slips at *C* and Rotates about *B*.) Here $F_B \le \mu_B N_B$ and

$$F_C = \mu_C N_C; \qquad F_C = 0.5 N_C \qquad (4)$$

Solving Eqs. 1 through 4 yields

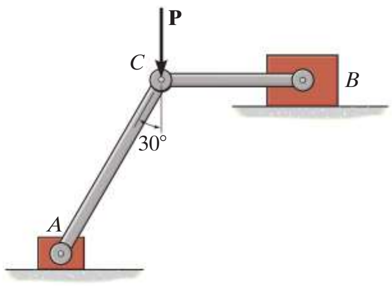
$$P = 267\text{ N} \qquad \textit{Ans.}$$
$$N_C = 400\text{ N}$$
$$F_C = 200\text{ N}$$
$$F_B = 66.7\text{ N}$$

Obviously, this case occurs first since it requires a *smaller* value for P.



(c)

Fig. 8–10



(a)

Blocks A and B have a mass of 3 kg and 9 kg, respectively, and are connected to the weightless links shown in Fig. 8–11*a*. Determine the largest vertical force **P** that can be applied at the pin C without causing any movement. The coefficient of static friction between the blocks and the contacting surfaces is $\mu_s = 0.3$.

SOLUTION

Free-Body Diagram. The links are two-force members and so the free-body diagrams of pin C and blocks A and B are shown in Fig. 8–11*b*. Since the horizontal component of $\mathbf{F}_{AC}$ tends to move block A to the left, $\mathbf{F}_A$ must act to the right. Similarly, $\mathbf{F}_B$ must act to the left to oppose the tendency of motion of block B to the right, caused by $\mathbf{F}_{BC}$. There are seven unknowns and six available force equilibrium equations, two for the pin and two for each block, so that *only one* frictional equation is needed.

Equations of Equilibrium and Friction. The force in links AC and BC can be related to P by considering the equilibrium of pin C.

$$+\uparrow \Sigma F_y = 0; \qquad F_{AC}\cos 30^\circ - P = 0; \qquad F_{AC} = 1.155P$$

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad 1.155P \sin 30^\circ - F_{BC} = 0; \qquad F_{BC} = 0.5774P$$

Using the result for F_{AC}, for block A,

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad F_A - 1.155P\sin 30^\circ = 0; \qquad F_A = 0.5774P \tag{1}$$

$$+\uparrow \Sigma F_y = 0; \qquad N_A - 1.155P\cos 30^\circ - 3(9.81\text{ N}) = 0;$$

$$N_A = P + 29.43\text{ N} \tag{2}$$

Using the result for F_{BC}, for block B,

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad (0.5774P) - F_B = 0; \qquad F_B = 0.5774P \tag{3}$$

$$+\uparrow \Sigma F_y = 0; \qquad N_B - 9(9.81)\text{ N} = 0; \qquad N_B = 88.29\text{ N}$$

Movement of the system may be caused by the initial slipping of *either* block A or block B. If we assume that block A slips first, then

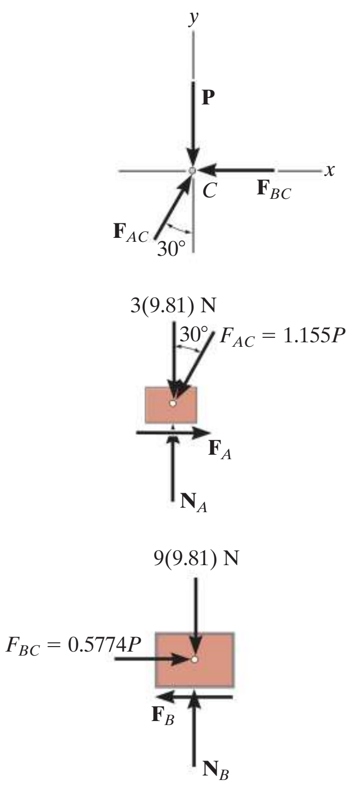
$$F_A = \mu_s N_A = 0.3N_A \tag{4}$$

Substituting Eqs. 1 and 2 into Eq. 4,

$$0.5774P = 0.3(P + 29.43)$$

$$P = 31.8\text{ N} \qquad \textit{Ans.}$$

Substituting this result into Eq. 3, we obtain $F_B = 18.4$ N. Since the maximum static frictional force at B is $(F_B)_{max} = \mu_s N_B = 0.3(88.29\text{ N}) = 26.5\text{ N} > F_B$, block B will not slip. Thus, the above assumption is correct. Notice that if the inequality were not satisfied, we would have to assume slipping of block B and then solve for P.

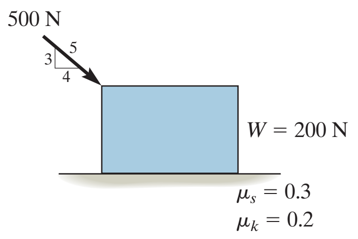


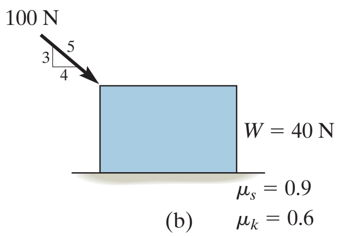
(b)

Fig. 8–11

PRELIMINARY PROBLEMS

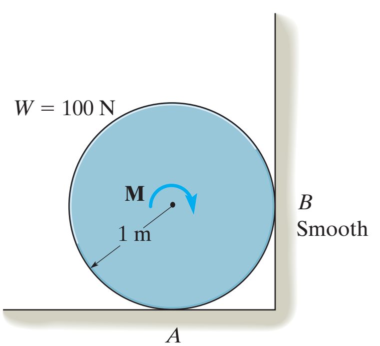
P8–1. Determine the friction force at the surface of contact.





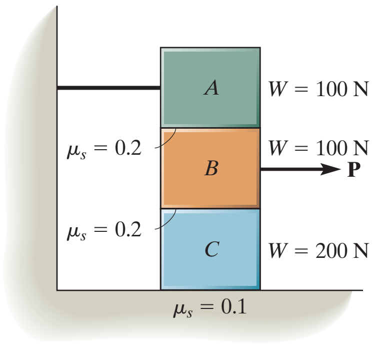
Prob. P8–1

P8–2. Determine **M** to cause impending motion of the cylinder.



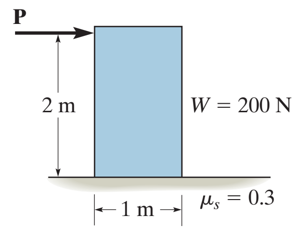
Prob. P8–2

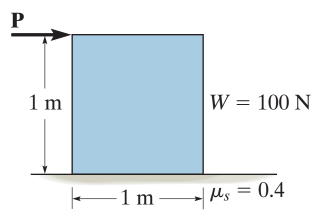
P8–3. Determine the force P to move block B.



Prob. P8–3

P8–4. Determine the force P needed to cause impending motion of the block.

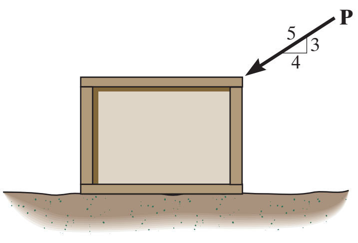




Prob. P8–4

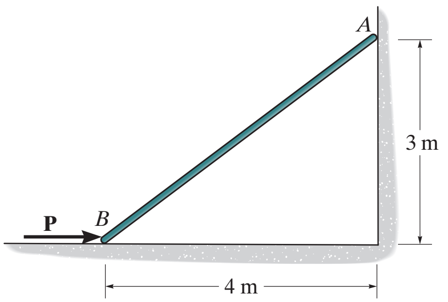
All problem solutions must include FBDs.

F8–1. Determine the friction developed between the 50-kg crate and the ground if a) $P = 200$ N, and b) $P = 400$ N. The coefficients of static and kinetic friction between the crate and the ground are $\mu_s = 0.3$ and $\mu_k = 0.2$.



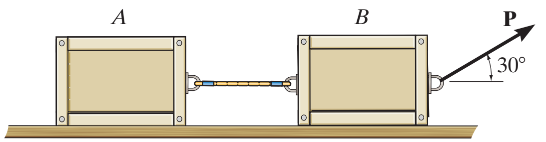
Prob. F8–1

F8–2. Determine the minimum force P to prevent the 30-kg rod AB from sliding. The contact surface at B is smooth, whereas the coefficient of static friction between the rod and the wall at A is $\mu_s = 0.2$.



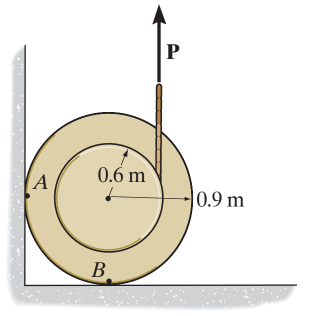
Prob. F8–2

F8–3. Determine the maximum force P that can be applied without causing the two 50-kg crates to move. The coefficient of static friction between each crate and the ground is $\mu_s = 0.25$.



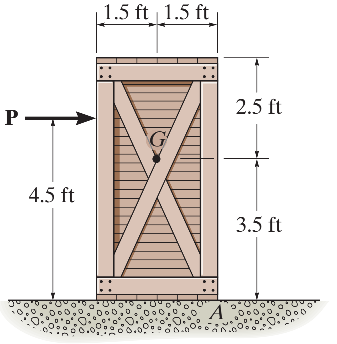
Prob. F8–3

F8–4. If the coefficient of static friction at contact points A and B is $\mu_s = 0.3$, determine the maximum force P that can be applied without causing the 100-kg spool to move.



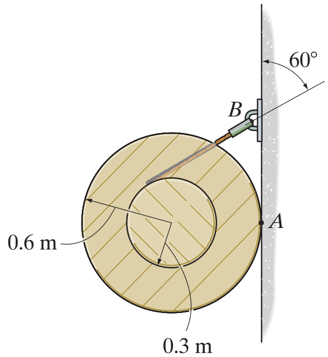
Prob. F8–4

F8–5. Determine the maximum force P that can be applied without causing movement of the 250-lb crate that has a center of gravity at G. The coefficient of static friction at the floor is $\mu_s = 0.4$.



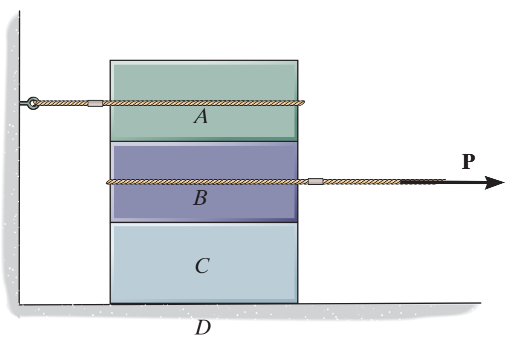
Prob. F8–5

F8–6. Determine the minimum coefficient of static friction between the uniform 50-kg spool and the wall so that the spool does not slip.



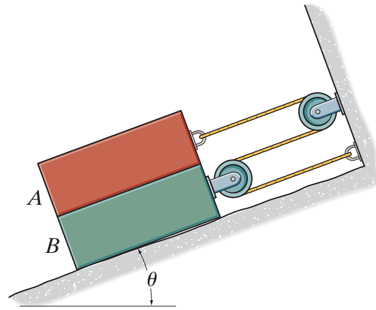
Prob. F8–6

F8–7. Blocks A, B, and C have weights of 50 N, 25 N, and 15 N, respectively. Determine the smallest horizontal force P that will cause impending motion. The coefficient of static friction between A and B is $\mu_s = 0.3$, between B and C, $\mu_s' = 0.4$, and between block C and the ground, $\mu_s'' = 0.35$.



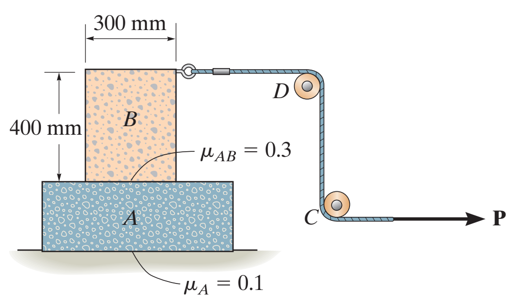
Prob. F8–7

F8–8. If the coefficient of static friction at all contacting surfaces is μ_s, determine the inclination θ at which the identical blocks, each of weight W, begin to slide.



Prob. F8–8

F8–9. Blocks A and B have a mass of 7 kg and 10 kg, respectively. Using the coefficients of static friction indicated, determine the largest force P which can be applied to the cord without causing motion. There are pulleys at C and D.

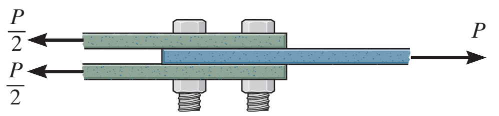


Prob. F8–9

PROBLEMS

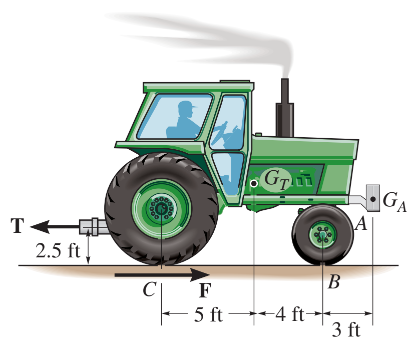
All problem solutions must include FBDs.

8–1. Determine the maximum force P the connection can support so that no slipping occurs between the plates. There are four bolts used for the connection and each is tightened so that it is subjected to a tension of 4 kN. The coefficient of static friction between the plates is $\mu_s = 0.4$.



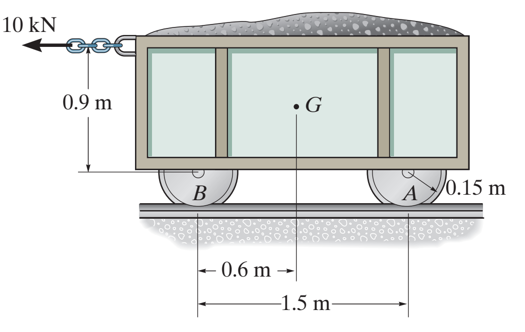
Prob. 8–1

8–2. The tractor exerts a towing force $T = 400$ lb. Determine the normal reactions at each of the two front and two rear tires and the tractive frictional force **F** on each rear tire needed to pull the load forward at constant velocity. The tractor has a weight of 7500 lb and a center of gravity located at G_T. An additinal weight of 600 lb is added to its front having a center of gravity at G_A. Take $\mu_s = 0.4$. The front wheels are free to roll.



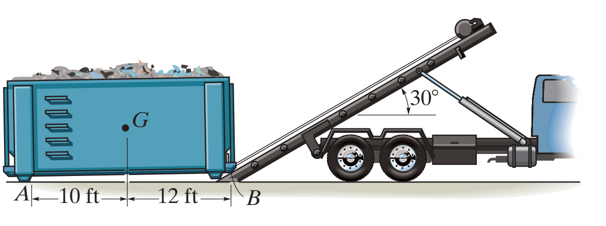
Prob. 8–2

8–3. The mine car and its contents have a total mass of 6 Mg and a center of gravity at G. If the coefficient of static friction between the wheels and the tracks is $\mu_s = 0.4$ when the wheels are locked, find the normal force acting on the front wheels at B and the rear wheels at A when the brakes at both A and B are locked. Does the car move?



Prob. 8–3

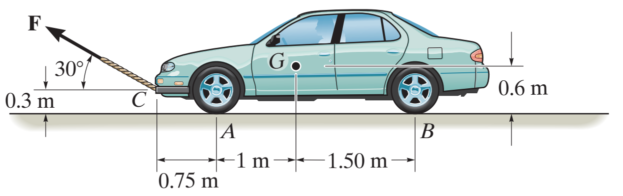
***8–4.** The winch on the truck is used to hoist the garbage bin onto the bed of the truck. If the loaded bin has a weight of 8500 lb and center of gravity at G, determine the force in the cable needed to begin the lift. The coefficients of static friction at A and B are $\mu_A = 0.3$ and $\mu_B = 0.2$, respectively. Neglect the height of the support at A.



Prob. 8–4

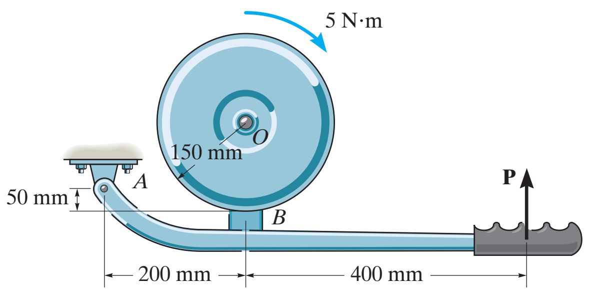
8–5. The automobile has a mass of 2 Mg and center of mass at G. Determine the towing force **F** required to move the car if the back brakes are locked, and the front wheels are free to roll. Take $\mu_s = 0.3$.

8–6. The automobile has a mass of 2 Mg and center of mass at G. Determine the towing force **F** required to move the car. Both the front and rear brakes are locked. Take $\mu_s = 0.3$.



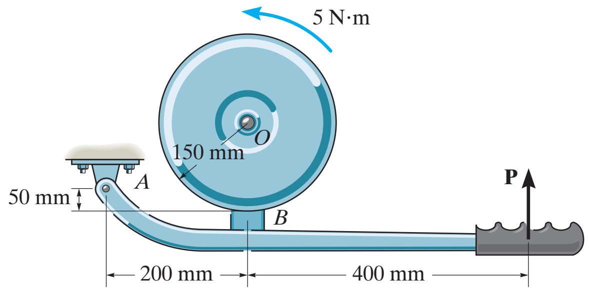
Probs. 8–5/6

8–7. The block brake consists of a pin-connected lever and friction block at B. The coefficient of static friction between the wheel and the lever is $\mu_s = 0.3$, and a torque of 5 N · m is applied to the wheel. Determine if the brake can hold the wheel stationary when the force applied to the lever is (a) $P = 30$ N, (b) $P = 70$ N.



Prob. 8–7

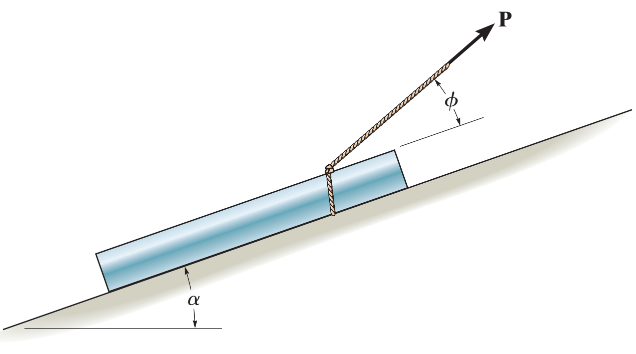
***8–8.** The block brake consists of a pin-connected lever and friction block at B. The coefficient of static friction between the wheel and the lever is $\mu_s = 0.3$, and a torque of 5 N · m is applied to the wheel. Determine if the brake can hold the wheel stationary when the force applied to the lever is (a) $P = 30$ N, (b) $P = 70$ N.



Prob. 8–8

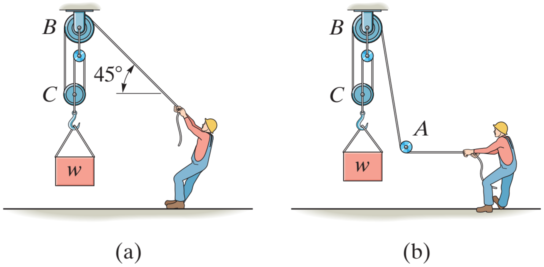
8–9. The pipe of weight W is to be pulled up the inclined plane of slope α using a force **P**. If **P** acts at an angle ϕ, show that for slipping $P = W \sin(\alpha + \theta)/\cos(\phi - \theta)$, where θ is the angle of static friction; $\theta = \tan^{-1} \mu_s$.

8–10. Determine the angle ϕ at which the applied force **P** should act on the pipe so that the magnitude of **P** is as small as possible for pulling the pipe up the incline. What is the corresponding value of P? The pipe weighs W and the slope α is known. Express the answer in terms of the angle of kinetic friction, $\theta = \tan^{-1} \mu_k$.



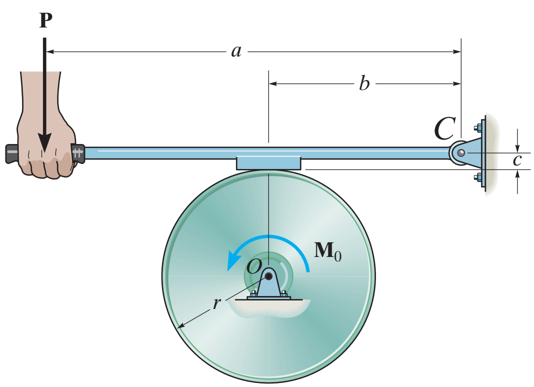
Probs. 8–9/10

8–11. Determine the maximum weight W the man can lift with constant velocity using the pulley system, without and then with the “leading block” or pulley at A. The man has a weight of 200 lb and the coefficient of static friction between his feet and the ground is $\mu_s = 0.6$.



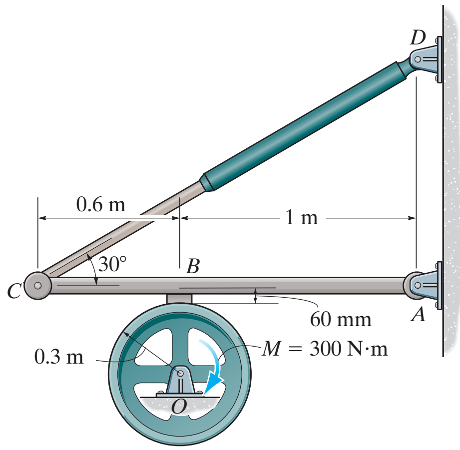
Prob. 8–11

***8–12.** The block brake is used to stop the wheel from rotating when the wheel is subjected to a couple moment $\mathbf{M}_0$. If the coefficient of static friction between the wheel and the block is μ_s, determine the smallest force P that should be applied.



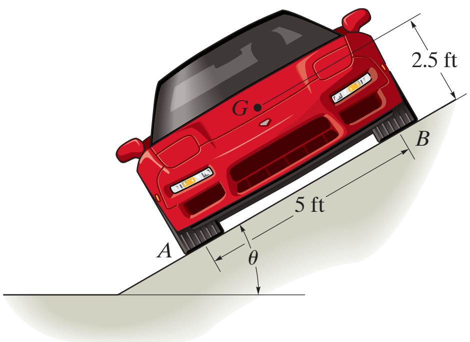
Prob. 8–12

8–13. If a torque of $M = 300\ \text{N}\cdot\text{m}$ is applied to the flywheel, determine the force that must be developed in the hydraulic cylinder CD to prevent the flywheel from rotating. The coefficient of static friction between the friction pad at B and the flywheel is $\mu_s = 0.4$.



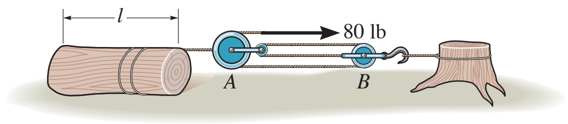
Prob. 8–13

8–14. The car has a mass of 1.6 Mg and center of mass at G. If the coefficient of static friction between the shoulder of the road and the tires is $\mu_s = 0.4$, determine the greatest slope θ the shoulder can have without causing the car to slip or tip over if the car travels along the shoulder at constant velocity.



Prob. 8–14

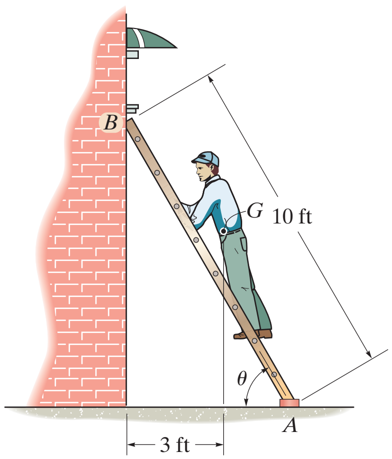
8–15. The log has a coefficient of state friction of $\mu_s = 0.3$ with the ground and a weight of 40 lb/ft. If a man can pull on the rope with a maximum force of 80 lb, determine the greatest length l of log he can drag.



Prob. 8–15

***8–16.** The 180-lb man climbs up the ladder and stops at the position shown after he senses that the ladder is on the verge of slipping. Determine the inclination θ of the ladder if the coefficient of static friction between the friction pad A and the ground is $\mu_s = 0.4$. Assume the wall at B is smooth. The center of gravity for the man is at G. Neglect the weight of the ladder.

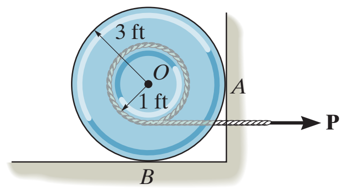
8–17. The 180-lb man climbs up the ladder and stops at the position shown after he senses that the ladder is on the verge of slipping. Determine the coefficient of static friction between the friction pad at A and ground if the inclination of the ladder is $\theta = 60°$ and the wall at B is smooth. The center of gravity for the man is at G. Neglect the weight of the ladder.



Probs. 8–16/17

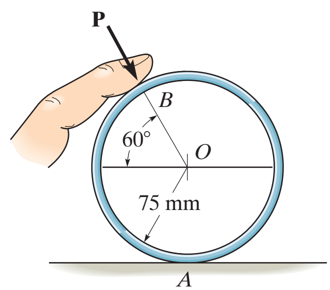
8–18. The spool of wire having a weight of 300 lb rests on the ground at B and against the wall at A. Determine the force P required to begin pulling the wire horizontally off the spool. The coefficient of static friction between the spool and its points of contact is $\mu_s = 0.25$.

8–19. The spool of wire having a weight of 300 lb rests on the ground at B and against the wall at A. Determine the normal force acting on the spool at A if $P = 300$ lb. The coefficient of static friction between the spool and the ground at B is $\mu_s = 0.35$. The wall at A is smooth.



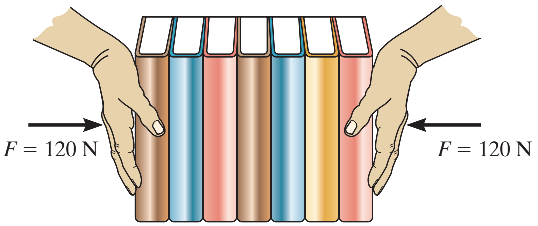
Probs. 8–18/19

***8–20.** The ring has a mass of 0.5 kg and is resting on the surface of the table. In an effort to move the ring a normal force **P** from the finger is exerted on it. If this force is directed towards the ring's center O as shown, determine its magnitude when the ring is on the verge of slipping at A. The coefficient of static friction at A is $\mu_A = 0.2$ and at B, $\mu_B = 0.3$.



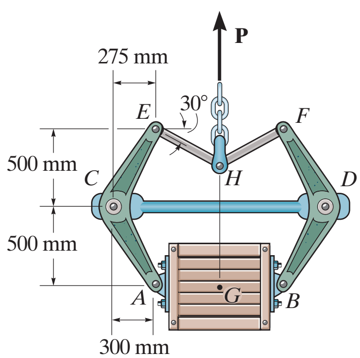
Prob. 8–20

8–21. A man attempts to support a stack of books horizontally by applying a compressive force of $F = 120$ N to the ends of the stack with his hands. If each book has a mass of 0.95 kg, determine the greatest number of books that can be supported in the stack. The coefficient of static friction between his hands and a book is $(\mu_s)_h = 0.6$ and between any two books $(\mu_s)_b = 0.4$.



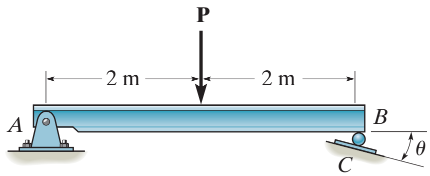
Prob. 8–21

8–22. The tongs are used to lift the 150-kg crate, whose center of mass is at G. Determine the least coefficient of static friction at the pivot blocks so that the crate can be lifted.



Prob. 8–22

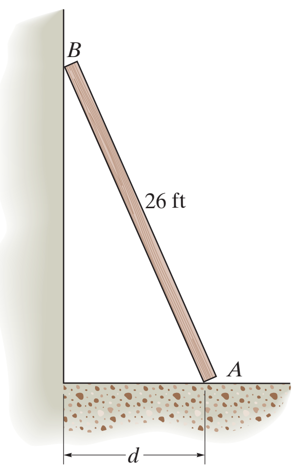
8–23. The beam is supported by a pin at A and a roller at B which has negligible weight and a radius of 15 mm. If the coefficient of static friction is $\mu_B = \mu_C = 0.3$, determine the largest angle θ of the incline so that the roller does not slip for any force **P** applied to the beam.



Prob. 8–23

***8–24.** The uniform thin pole has a weight of 30 lb and a length of 26 ft. If it is placed against the smooth wall and on the rough floor in the position $d = 10$ ft, will it remain in this position when it is released? The coefficient of static friction is $\mu_s = 0.3$.

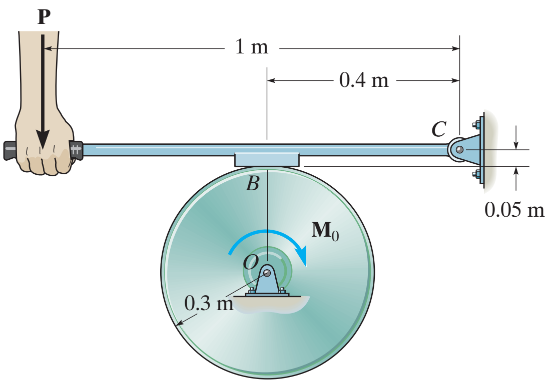
8–25. The uniform pole has a weight of 30 lb and a length of 26 ft. Determine the maximum distance d it can be placed from the smooth wall and not slip. The coefficient of static friction between the floor and the pole is $\mu_s = 0.3$.



Probs. 8–24/25

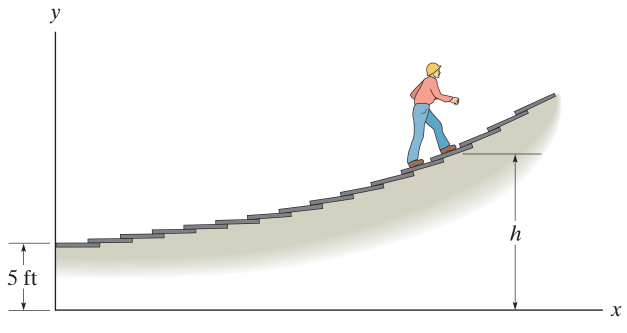
8–26. The block brake is used to stop the wheel from rotating when the wheel is subjected to a couple moment $M_0 = 360\ \text{N}\cdot\text{m}$. If the coefficient of static friction between the wheel and the block is $\mu_s = 0.6$, determine the smallest force P that should be applied.

8–27. Solve Prob. 8–26 if the couple moment $\mathbf{M}_0$ is applied counterclockwise.



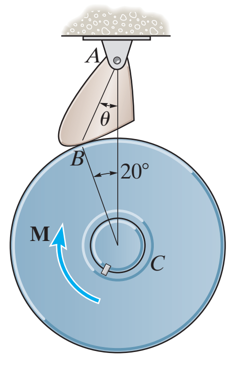
Probs. 8–26/27

***8–28.** A worker walks up the sloped roof that is defined by the curve $y = (5e^{0.01x})$ ft, where x is in feet. Determine how high h he can go without slipping. The coefficient of static friction is $\mu_s = 0.6$.



Prob. 8–28

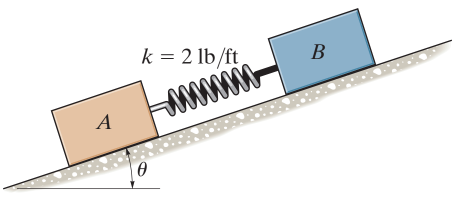
8–29. The friction pawl is pinned at A and rests against the wheel at B. It allows freedom of movement when the wheel is rotating counterclockwise about C. Clockwise rotation is prevented due to friction of the pawl which tends to bind the wheel. If $(\mu_s)_B = 0.6$, determine the design angle θ which will prevent clockwise motion for any value of applied moment M. *Hint*: Neglect the weight of the pawl so that it becomes a two-force member.



Prob. 8–29

8–30. Two blocks A and B have a weight of 10 lb and 6 lb, respectively. They are resting on the incline for which the coefficients of static friction are $\mu_A = 0.15$ and $\mu_B = 0.25$. Determine the incline angle θ for which both blocks begin to slide. Also find the required stretch or compression in the connecting spring for this to occur. The spring has a stiffness of $k = 2\ \text{lb/ft}$.

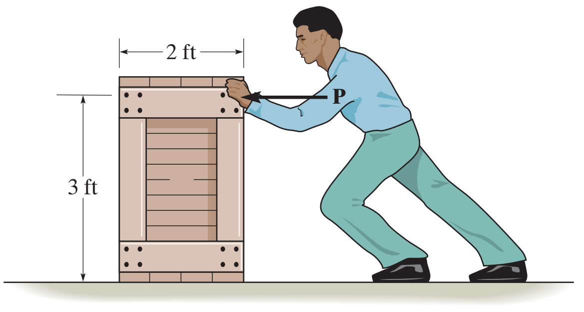
8–31. Two blocks A and B have a weight of 10 lb and 6 lb, respectively. They are resting on the incline for which the coefficients of static friction are $\mu_A = 0.15$ and $\mu_B = 0.25$. Determine the angle θ which will cause motion of one of the blocks. What is the friction force under each of the blocks when this occurs? The spring has a stiffness of $k = 2\ \text{lb/ft}$ and is originally unstretched.



Probs. 8–30/31

***8–32.** Determine the smallest force P that must be applied in order to cause the 150-lb uniform crate to move. The coefficent of static friction between the crate and the floor is $\mu_s = 0.5$.

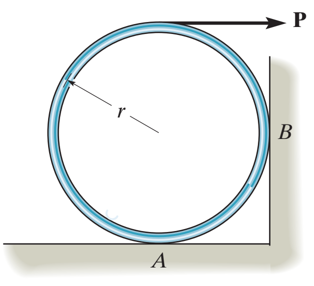
8–33. The man having a weight of 200 lb pushes horizontally on the crate. If the coefficient of static friction between the 450-lb crate and the floor is $\mu_s = 0.3$ and between his shoes and the floor is $\mu'_s = 0.6$, determine if he can move the crate.



Probs. 8–32/33

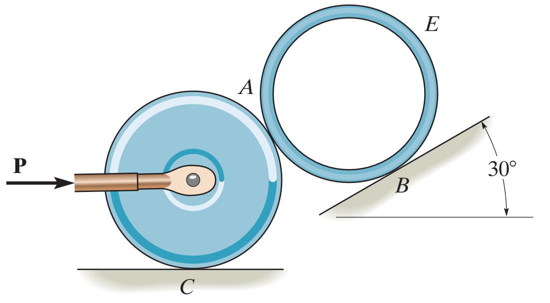
8–34. The uniform hoop of weight W is subjected to the horizontal force P. Determine the coefficient of static friction between the hoop and the surface of A and B if the hoop is on the verge of rotating.

8–35. Determine the maximum horizontal force **P** that can be applied to the 30-lb hoop without causing it to rotate. The coefficient of static friction between the hoop and the surfaces A and B is $\mu_s = 0.2$. Take $r = 300$ mm.



Probs. 8–34/35

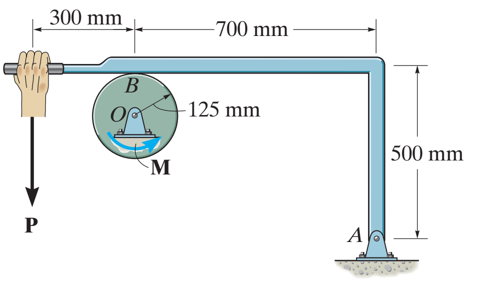
***8–36.** Determine the minimum force P needed to push the tube E up the incline. The force acts parallel to the plane, and the coefficients of static friction at the contacting surfaces are $\mu_A = 0.2$, $\mu_B = 0.3$, and $\mu_C = 0.4$. The 100-kg roller and 40-kg tube each have a radius of 150 mm.



Prob. 8–36

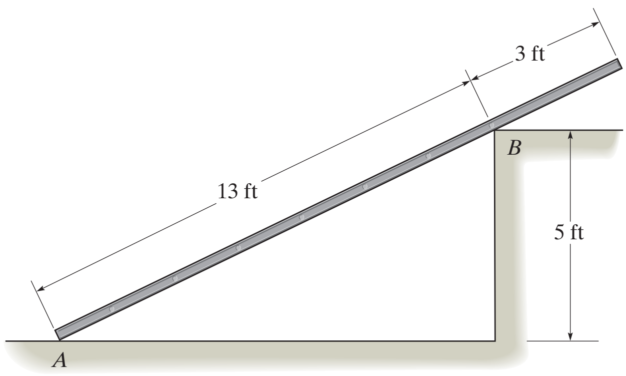
8–37. The coefficients of static and kinetic friction between the drum and brake bar are $\mu_s = 0.4$ and $\mu_k = 0.3$, respectively. If $M = 50\ \text{N}\cdot\text{m}$ and $P = 85$ N, determine the horizontal and vertical components of reaction at the pin O. Neglect the weight and thickness of the brake. The drum has a mass of 25 kg.

8–38. The coefficient of static friction between the drum and brake bar is $\mu_s = 0.4$. If the moment $M = 35\ \text{N}\cdot\text{m}$, determine the smallest force P that needs to be applied to the brake bar in order to prevent the drum from rotating. Also determine the corresponding horizontal and vertical components of reaction at pin O. Neglect the weight and thickness of the brake bar. The drum has a mass of 25 kg.



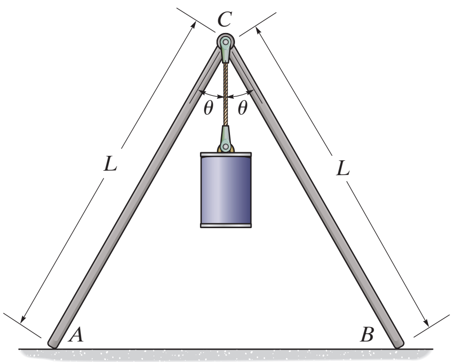
Probs. 8–37/38

8–39. Determine the smallest coefficient of static friction at both A and B needed to hold the uniform 100-lb bar in equilibrium. Neglect the thickness of the bar. Take $\mu_A = \mu_B = \mu$.



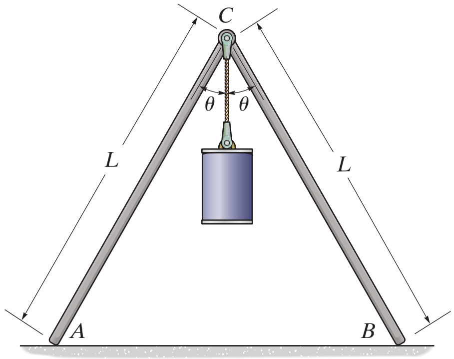
Prob. 8–39

***8–40.** If $\theta = 30°$, determine the minimum coefficient of static friction at A and B so that equilibrium of the supporting frame is maintained regardless of the mass of the cylinder. Neglect the mass of the rods.



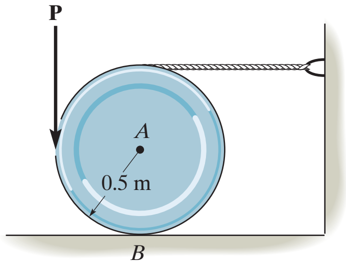
Prob. 8–40

8–41. If the coefficient of static friction at A and B is $\mu_s = 0.6$, determine the maximum angle θ so that the frame remains in equilbrium, regardless of the mass of the cylinder. Neglect the mass of the rods.



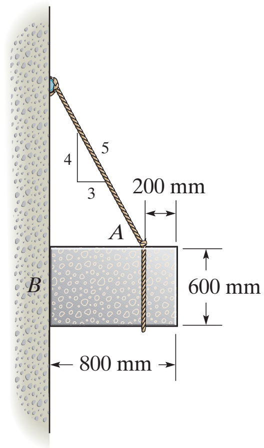
Prob. 8–41

8–42. The 100-kg disk rests on a surface for which $\mu_B = 0.2$. Determine the smallest vertical force **P** that can be applied tangentially to the disk which will cause motion to impend.



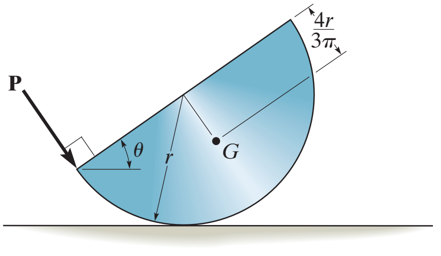
Prob. 8–42

8–43. Investigate whether the equilibrium can be maintained. The uniform block has a mass of 500 kg, and the coefficient of static friction is $\mu_s = 0.3$.



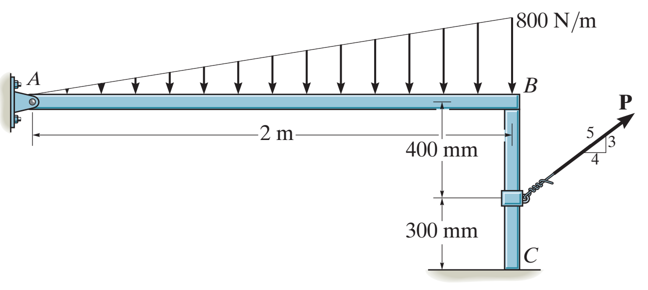
Prob. 8–43

***8–44.** The homogenous semicylinder has a mass of 20 kg and mass center at G. If force $\mathbf{P}$ is applied at the edge, and $r = 300$ mm, determine the angle θ at which the semicylinder is on the verge of slipping. The coefficient of static friction between the plane and the cylinder is $\mu_s = 0.3$. Also, what is the corresponding force P for this case?



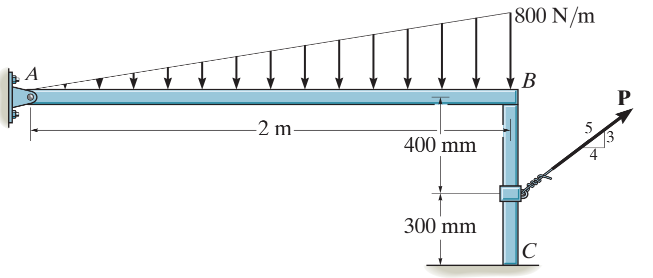
Prob. 8–44

8–45. The beam AB has a negligible mass and thickness and is subjected to a triangular distributed loading. It is supported at one end by a pin and at the other end by a post having a mass of 50 kg and negligible thickness. Determine the minimum force P needed to move the post. The coefficients of static friction at B and C are $\mu_B = 0.4$ and $\mu_C = 0.2$, respectively.



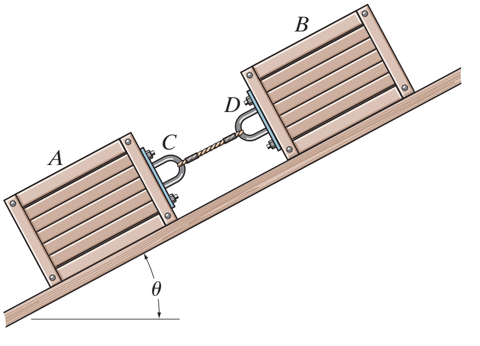
Prob. 8–45

8–46. The beam AB has a negligible mass and thickness and is subjected to a triangular distributed loading. It is supported at one end by a pin and at the other end by a post having a mass of 50 kg and negligible thickness. Determine the two coefficients of static friction at B and at C so that when the magnitude of the applied force is increased to $P = 150$ N, the post slips at both B and C simultaneously.



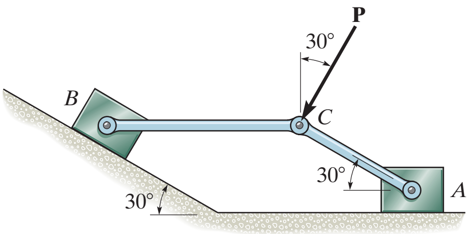
Prob. 8–46

8–47. Crates A and B weigh 200 lb and 150 lb, respectively. They are connected together with a cable and placed on the inclined plane. If the angle θ is gradually increased, determine θ when the crates begin to slide. The coefficients of static friction between the crates and the plane are $\mu_A = 0.25$ and $\mu_B = 0.35$.



Prob. 8–47

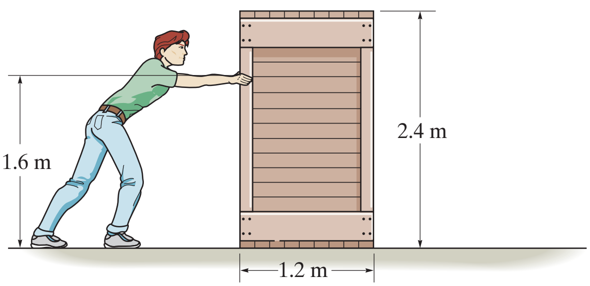
***8–48.** Two blocks A and B, each having a mass of 5 kg, are connected by the linkage shown. If the coefficient of static friction at the contacting surfaces is $\mu_s = 0.5$, determine the largest force P that can be applied to pin C of the linkage without causing the blocks to move. Neglect the weight of the links.



Prob. 8–48

8–49. The uniform crate has a mass of 150 kg. If the coefficient of static friction between the crate and the floor is $\mu_s = 0.2$, determine whether the 85-kg man can move the crate. The coefficient of static friction between his shoes and the floor is $\mu'_s = 0.4$. Assume the man only exerts a horizontal force on the crate.

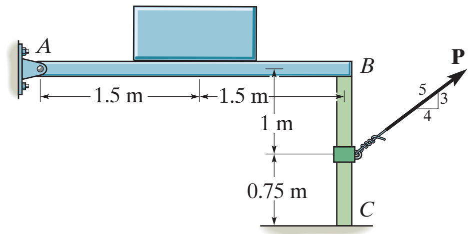
8–50. The uniform crate has a mass of 150 kg. If the coefficient of static friction between the crate and the floor is $\mu_s = 0.2$, determine the smallest mass of the man so he can move the crate. The coefficient of static friction between his shoes and the floor is $\mu'_s = 0.45$. Assume the man exerts only a horizontal force on the crate.



Probs. 8–49/50

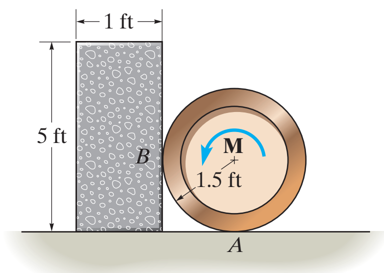
8–51. Beam AB has a negligible mass and thickness, and supports the 200-kg uniform block. It is pinned at A and rests on the top of a post, having a mass of 20 kg and negligible thickness. Determine the minimum force P needed to move the post. The coefficients of static friction at B and C are $\mu_B = 0.4$ and $\mu_C = 0.2$, respectively.

***8–52.** Beam AB has a negligible mass and thickness, and supports the 200-kg uniform block. It is pinned at A and rests on the top of a post, having a mass of 20 kg and negligible thickness. Determine the two coefficients of static friction at B and at C so that when the magnitude of the applied force is increased to $P = 300$ N, the post slips at both B and C simultaneously.



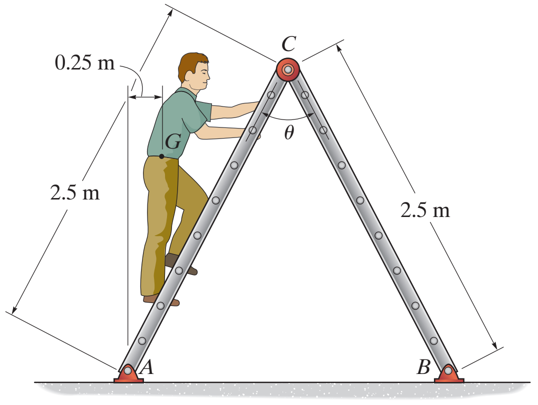
Probs. 8–51/52

8–53. Determine the smallest couple moment that can be applied to the 150-lb wheel that will cause impending motion. The uniform concrete block has a weight of 300 lb. The coefficients of static friction are $\mu_A = 0.2$, $\mu_B = 0.3$, and between the concrete block and the floor, $\mu = 0.4$.



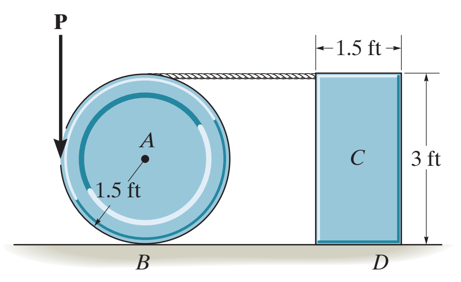
Prob. 8–53

8–54. Determine the greatest angle θ so that the ladder does not slip when it supports the 75-kg man in the position shown. The surface is rather slippery, where the coefficient of static friction at A and B is $\mu_s = 0.3$.



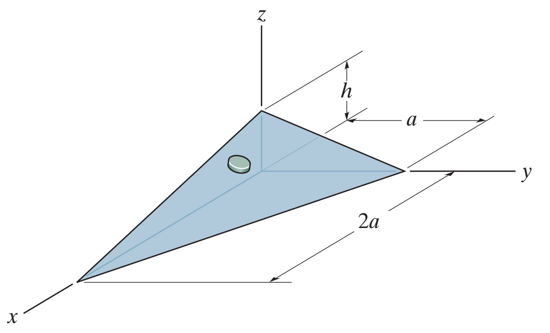
Prob. 8–54

8–55. The wheel weighs 20 lb and rests on a surface for which $\mu_B = 0.2$. A cord wrapped around it is attached to the top of the 30-lb homogeneous block. If the coefficient of static friction at D is $\mu_D = 0.3$, determine the smallest vertical force that can be applied tangentially to the wheel which will cause motion to impend.



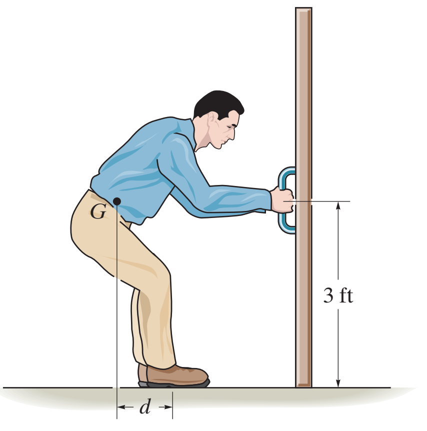
Prob. 8–55

***8–56.** The disk has a weight W and lies on a plane that has a coefficient of static friction μ. Determine the maximum height h to which the plane can be lifted without causing the disk to slip.



Prob. 8–56

8–57. The man has a weight of 200 lb, and the coefficient of static friction between his shoes and the floor is $\mu_s = 0.5$. Determine where he should position his center of gravity G at d in order to exert the maximum horizontal force on the door. What is this force?



Prob. 8–57

CONCEPTUAL PROBLEMS

C8–1. Draw the free-body diagrams of each of the two members of this friction tong used to lift the 100-kg block.

C8–1 (© Russell C. Hibbeler)

C8–2. Show how to find the force needed to move the top block. Use reasonable data and use an equilibrium analysis to explain your answer.

C8–2 (© Russell C. Hibbeler)

C8–3. The rope is used to tow the refrigerator. Is it best to pull slightly up on the rope as shown, pull horizontally, or pull somewhat downwards? Also, is it best to attach the rope at a high position as shown, or at a lower position? Do an equilibrium analysis to explain your answer.

C8–4. The rope is used to tow the refrigerator. In order to prevent yourself from slipping while towing, is it best to pull up as shown, pull horizontally, or pull downwards on the rope? Do an equilibrium analysis to explain your answer.

C8–3/4 (© Russell C. Hibbeler)

C8–5. Explain how to find the maximum force this man can exert on the vehicle. Use reasonable data and use an equilibrium analysis to explain your answer.

C8–5 (© Russell C. Hibbeler)

8.3 Wedges

Wedges are often used to adjust the elevation of structural or mechanical parts. Also, they provide stability for objects such as this pipe. (© Russell C. Hibbeler)

A ***wedge*** is a simple machine that is often used to transform an applied force into much larger forces, directed at approximately right angles to the applied force. Wedges also can be used to slightly move or adjust heavy loads.

Consider, for example, the wedge shown in Fig. 8–12*a*, which is used to *lift* the block by applying a force to the wedge. Free-body diagrams of the block and wedge are shown in Fig. 8–12*b*. Here we have excluded the weight of the wedge since it is usually *small* compared to the weight **W** of the block. Also, note that the frictional forces $\mathbf{F}_1$ and $\mathbf{F}_2$ must oppose the motion of the wedge. Likewise, the frictional force $\mathbf{F}_3$ of the wall on the block must act downward so as to oppose the block's upward motion. The locations of the resultant normal forces are not important in the force analysis since neither the block nor wedge will "tip." Hence the moment equilibrium equations will not be considered. There are seven unknowns, consisting of the applied force **P**, needed to cause motion of the wedge, and six normal and frictional forces. The seven available equations consist of four force equilibrium equations, $\Sigma F_x = 0$, $\Sigma F_y = 0$ applied to the wedge and block, and three frictional equations, $F = \mu N$, applied at each surface of contact.

If the block is to be *lowered*, then the frictional forces will all act in a sense opposite to that shown in Fig. 8–12*b*. Provided the coefficient of friction is very *small* or the wedge angle θ is *large,* then the applied force **P** must act to the right to hold the block. Otherwise, **P** may have a reverse sense of direction in order to *pull* on the wedge to remove it. If **P** is *not applied* and friction forces hold the block in place, then the wedge is referred to as *self-locking*.

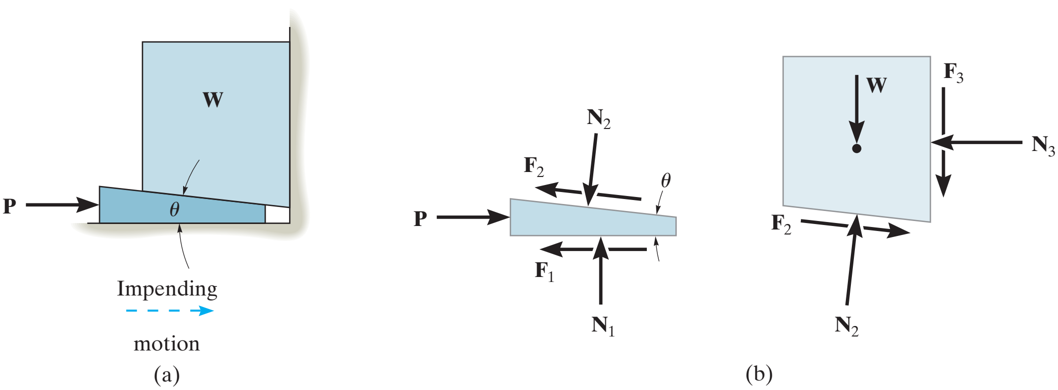


Fig. 8–12

EXAMPLE 8.6

The uniform stone in Fig. 8–13a has a mass of 500 kg and is held in the horizontal position using a wedge at B. If the coefficient of static friction is $\mu_s = 0.3$ at the surfaces of contact, determine the minimum force P needed to remove the wedge. Assume that the stone does not slip at A.

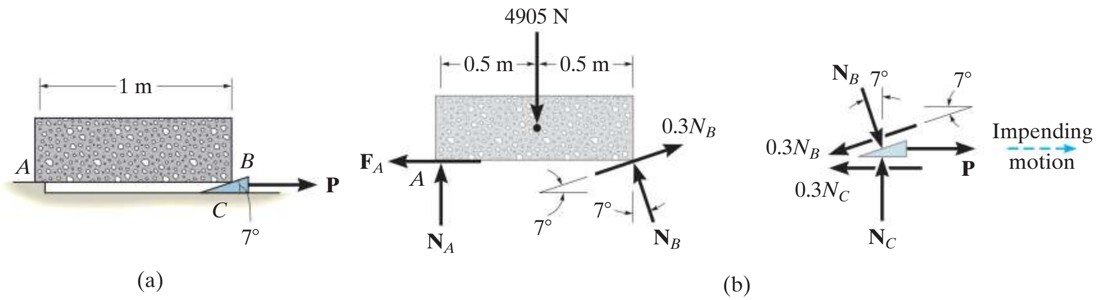


Fig. 8–13

SOLUTION

The minimum force P requires $F = \mu_s N$ at the surfaces of contact with the wedge. The free-body diagrams of the stone and wedge are shown in Fig. 8–13b. On the wedge the friction force opposes the impending motion, and on the stone at A, $F_A \leq \mu_s N_A$, since slipping does not occur there. There are five unknowns. Three equilibrium equations for the stone and two for the wedge are available for solution. From the free-body diagram of the stone,

$$\circlearrowleft + \Sigma M_A = 0; \quad -4905\text{ N}(0.5\text{ m}) + (N_B \cos 7^\circ\text{ N})(1\text{ m}) + (0.3N_B \sin 7^\circ\text{ N})(1\text{ m}) = 0$$

$$N_B = 2383.1\text{ N}$$

Using this result for the wedge, we have

$$+\uparrow \Sigma F_y = 0; \quad N_C - 2383.1 \cos 7^\circ\text{ N} - 0.3(2383.1 \sin 7^\circ\text{ N}) = 0$$

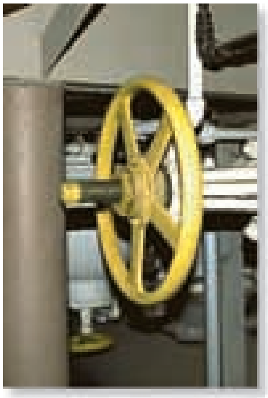
$$N_C = 2452.5\text{ N}$$

$$\xrightarrow{+} \Sigma F_x = 0; \quad 2383.1 \sin 7^\circ\text{ N} - 0.3(2383.1 \cos 7^\circ\text{ N}) + P - 0.3(2452.5\text{ N}) = 0$$

$$P = 1154.9\text{ N} = 1.15\text{ kN} \qquad \textit{Ans.}$$

NOTE: Since P is positive, indeed the wedge must be pulled out. If P were zero, the wedge would remain in place (self-locking) and the frictional forces developed at B and C would satisfy $F_B < \mu_s N_B$ and $F_C < \mu_s N_C$.

8.4 Frictional Forces on Screws



Square-threaded screws find applications on valves, jacks, and vises, where particularly large forces must be developed along the axis of the screw. (© Russell C. Hibbeler)

In most cases, screws are used as fasteners; however, in many types of machines they are incorporated to transmit power or motion from one part of the machine to another. A ***square-threaded screw*** is commonly used for the latter purpose, especially when large forces are applied along its axis. In this section, we will analyze the forces acting on square-threaded screws. The analysis of other types of screws, such as the V-thread, is based on these same principles.

For analysis, a square-threaded screw, as in Fig. 8–14, can be considered a cylinder having an inclined square ridge or ***thread*** wrapped around it. If we unwind the thread by one revolution, as shown in Fig. 8–14b, the slope or the ***lead angle*** θ is determined from $\theta = \tan^{-1}(l/2\pi r)$. Here l and $2\pi r$ are the vertical and horizontal distances between A and B, where r is the mean radius of the thread. The distance l is called the ***lead*** of the screw and it is equivalent to the distance the screw advances when it turns one revolution.

Upward Impending Motion.

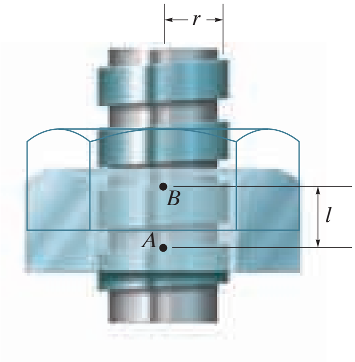
Let us now consider the case of the square-threaded screw jack in Fig. 8–15 that is subjected to upward impending motion caused by the applied torsional moment ***M**. A free-body diagram of the *entire unraveled thread h* in contact with the jack can be represented as a block, as shown in Fig. 8–16a. The force **W** is the vertical force acting on the thread or the axial force applied to the shaft, Fig. 8–15, and M/r is the resultant horizontal force produced by the couple moment M about the axis of the shaft. The reaction **R** of the groove on the thread has both frictional and normal components, where $F = \mu_s N$. The angle of static friction is $\phi_s = \tan^{-1}(F/N) = \tan^{-1}\mu_s$. Applying the force equations of equilibrium along the horizontal and vertical axes, we have

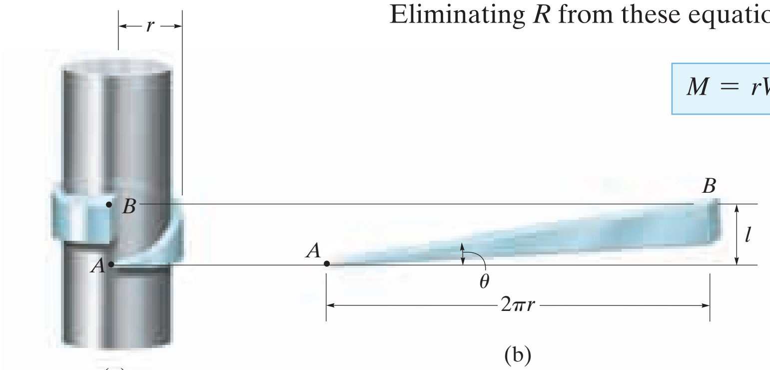
$$\overset{+}{\rightarrow}\Sigma F_x = 0; \qquad M/r - R\sin(\theta + \phi_s) = 0$$

$$+\uparrow\Sigma F_y = 0; \qquad R\cos(\theta + \phi_s) - W = 0$$

Eliminating R from these equations, we obtain

$$\boxed{M = rW\tan(\theta + \phi_s)} \tag{8–3}$$





(a)

(b)

Fig. 8–14

*For applications, **M** is developed by applying a horizontal force **P** at a right angle to the end of a lever that would be fixed to the screw.

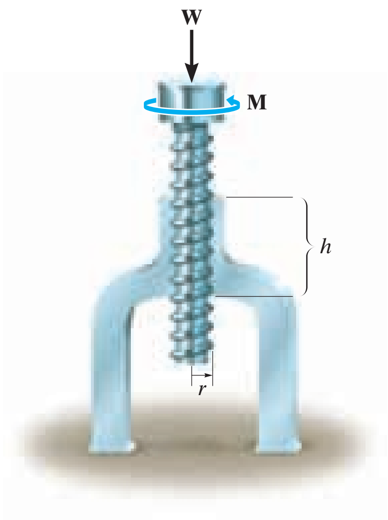


Fig. 8–15

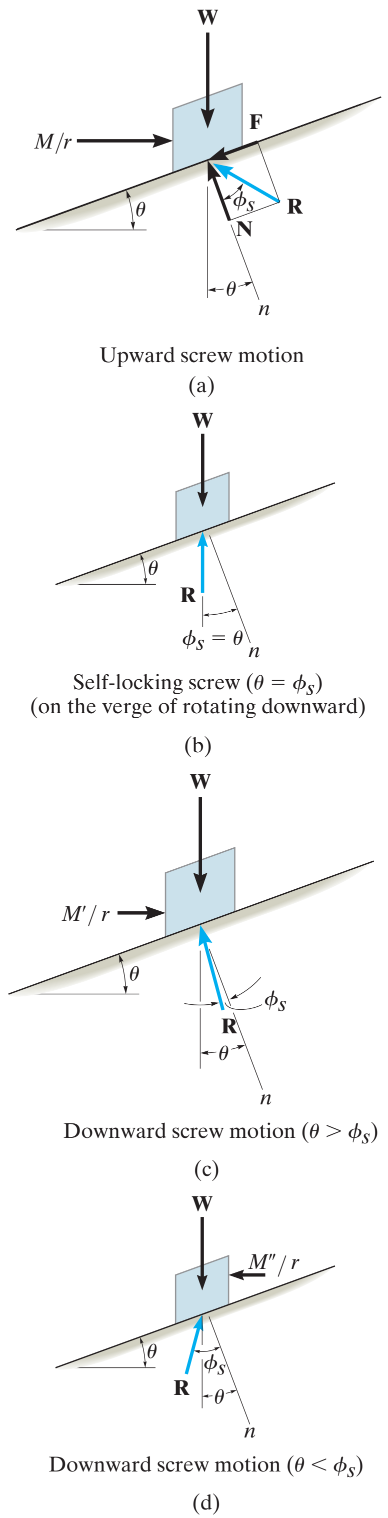


Fig. 8–16

Self-Locking Screw. A screw is said to be *self-locking* if it remains in place under any axial load **W** when the moment **M** is removed. For this to occur, the direction of the frictional force must be reversed so that **R** acts on the other side of **N**. Here the angle of static friction ϕ_s becomes greater than or equal to θ, Fig. 8–16*d*. If $\phi_s = \theta$, Fig. 8–16*b*, then **R** will act vertically to balance **W**, and the screw will be on the verge of winding downward.

Downward Impending Motion, ($\theta > \phi_s$). If the screw is not self-locking, it is necessary to apply a moment **M′** to prevent the screw from winding downward. Here, a horizontal force M'/r is required to push against the thread to prevent it from sliding down the plane, Fig. 8–16*c*. Using the same procedure as before, the magnitude of the moment **M′** required to prevent this unwinding is

$$\boxed{M' = rW \tan(\theta - \phi_s)} \qquad (8\text{–}4)$$

Downward Impending Motion, ($\phi_s > \theta$). If a screw is self-locking, a couple moment **M″** must be applied to the screw in the opposite direction to wind the screw downward ($\phi_s > \theta$). This causes a reverse horizontal force M''/r that pushes the thread down as indicated in Fig. 8–16*d*. In this case, we obtain

$$\boxed{M'' = rW \tan(\phi_s - \theta)} \qquad (8\text{–}5)$$

If *motion of the screw* occurs, Eqs. 8–3, 8–4, and 8–5 can be applied by simply replacing ϕ_s with ϕ_k.

EXAMPLE 8.7

The turnbuckle shown in Fig. 8–17 has a square thread with a mean radius of 5 mm and a lead of 2 mm. If the coefficient of static friction between the screw and the turnbuckle is $\mu_s = 0.25$, determine the moment **M** that must be applied to draw the end screws closer together.

(© Russell C. Hibbeler)

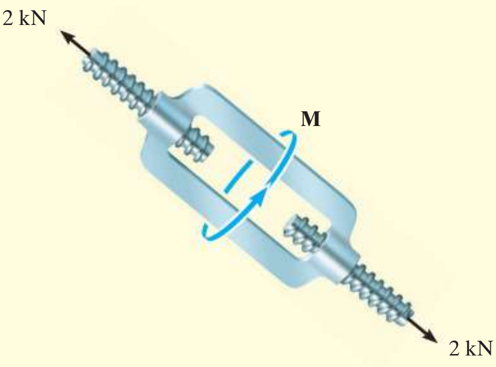


Fig. 8–17

SOLUTION

The moment can be obtained by applying Eq. 8–3. Since friction at *two screws* must be overcome, this requires

$$M = 2[rW\tan(\theta + \phi_s)] \tag{1}$$

Here $W = 2000$ N, $\phi_s = \tan^{-1}\mu_s = \tan^{-1}(0.25) = 14.04°$, $r = 5$ mm, and $\theta = \tan^{-1}(l/2\pi r) = \tan^{-1}(2\text{ mm}/[2\pi(5\text{ mm})]) = 3.64°$. Substituting these values into Eq. 1 and solving gives

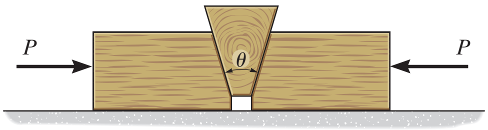
$$M = 2[(2000\text{ N})(5\text{ mm})\tan(14.04° + 3.64°)]$$

$$= 6374.7\text{ N}\cdot\text{mm} = 6.37\text{ N}\cdot\text{m} \qquad \textit{Ans.}$$

NOTE: When the moment is *removed*, the turnbuckle will be self-locking; i.e., it will not unscrew since $\phi_s > \theta$.

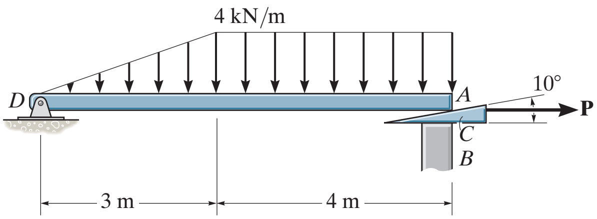
PROBLEMS

8–58. Determine the largest angle θ that will cause the wedge to be self-locking regardless of the magnitude of horizontal force P applied to the blocks. The coefficient of static friction between the wedge and the blocks is $\mu_s = 0.3$. Neglect the weight of the wedge.



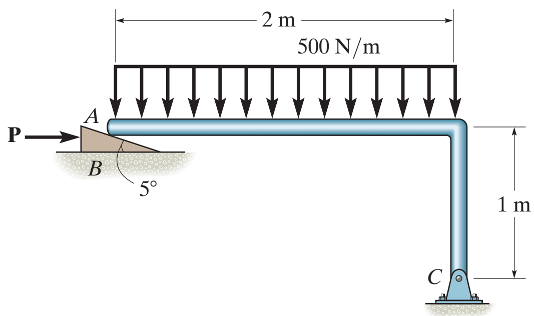
Prob. 8–58

8–59. If the beam AD is loaded as shown, determine the horizontal force P which must be applied to the wedge in order to remove it from under the beam. The coefficients of static friction at the wedge's top and bottom surfaces are $\mu_{CA} = 0.25$ and $\mu_{CB} = 0.35$, respectively. If $P = 0$, is the wedge self-locking? Neglect the weight and size of the wedge and the thickness of the beam.



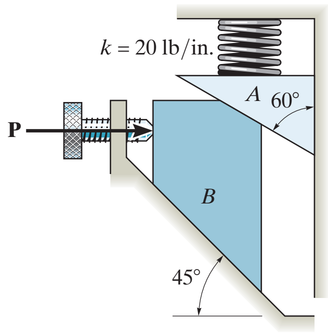
Prob. 8–59

***8–60.** The wedge is used to level the member. Determine the horizontal force **P** that must be applied to begin to push the wedge forward. The coefficient of static friction between the wedge and the two surfaces of contact is $\mu_s = 0.2$. Neglect the weight of the wedge.



Prob. 8–60

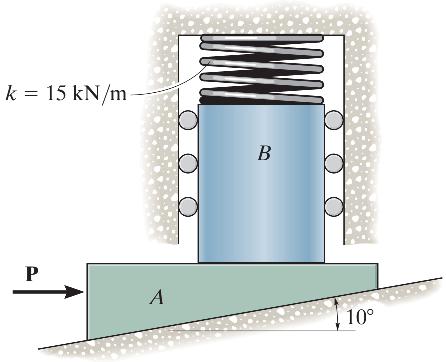
8–61. The two blocks used in a measuring device have negligible weight. If the spring is compressed 5 in. when in the position shown, determine the smallest axial force P which the adjustment screw must exert on B in order to start the movement of B downward. The end of the screw is *smooth* and the coefficient of static friction at all other points of contact is $\mu_s = 0.3$.



Prob. 8–61

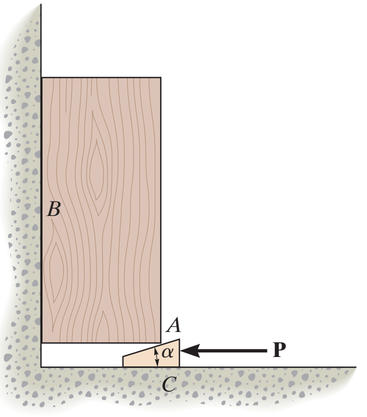
8–62. If $P = 250$ N, determine the required minimum compression in the spring so that the wedge will not move to the right. Neglect the weight of A and B. The coefficient of static friction for all contacting surfaces is $\mu_s = 0.35$. Neglect friction at the rollers.

8–63. Determine the minimum applied force **P** required to move wedge A to the right. The spring is compressed a distance of 175 mm. Neglect the weight of A and B. The coefficient of static friction for all contacting surfaces is $\mu_s = 0.35$. Neglect friction at the rollers.



Probs. 8–62/63

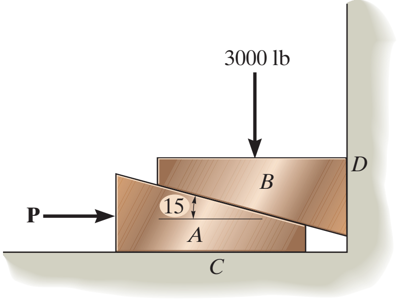
***8–64.** If the coefficient of static friction between all the surfaces of contact is μ_s, determine the force **P** that must be applied to the wedge in order to lift the block having a weight W.



Prob. 8–64

8–65. Determine the smallest force P needed to lift the 3000-lb load. The coefficient of static friction between A and C and between B and D is $\mu_s = 0.3$, and between A and B $\mu'_s = 0.4$. Neglect the weight of each wedge.

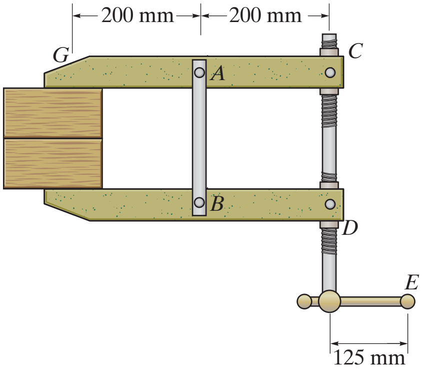
8–66. Determine the reversed horizontal force $-\mathbf{P}$ needed to pull out wedge A. The coefficient of static friction between A and C and between B and D is $\mu_s = 0.2$, and between A and B $\mu'_s = 0.1$. Neglect the weight of each wedge.



Probs. 8–65/66

8–67. If the clamping force at G is 900 N, determine the horizontal force **F** that must be applied perpendicular to the handle of the lever at E. The mean diameter and lead of both single square-threaded screws at C and D are 25 mm and 5 mm, respectively. The coefficient of static friction is $\mu_s = 0.3$.

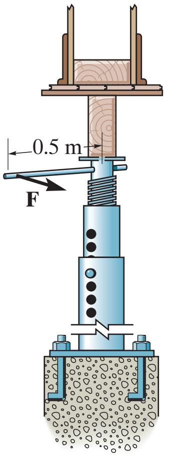
***8–68.** If a horizontal force of $F = 50$ N is applied perpendicular to the handle of the lever at E, determine the clamping force developed at G. The mean diameter and lead of the single square-threaded screw at C and D are 25 mm and 5 mm, respectively. The coefficient of static friction is $\mu_s = 0.3$.



Probs. 8–67/68

8–69. The column is used to support the upper floor. If a force $F = 80$ N is applied perpendicular to the handle to tighten the screw, determine the compressive force in the column. The square-threaded screw on the jack has a coefficient of static friction of $\mu_s = 0.4$, mean diameter of 25 mm, and a lead of 3 mm.

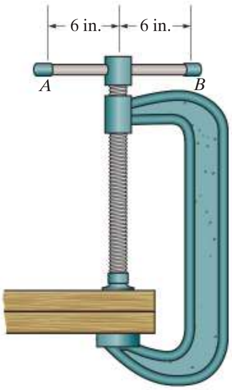
8–70. If the force **F** is removed from the handle of the jack in Prob. 8–69, determine if the screw is self-locking.



Probs. 8–69/70

8–71. If couple forces of $F = 10$ lb are applied perpendicular to the lever of the clamp at A and B, determine the clamping force on the boards. The single square-threaded screw of the clamp has a mean diameter of 1 in. and a lead of 0.25 in. The coefficient of static friction is $\mu_s = 0.3$.

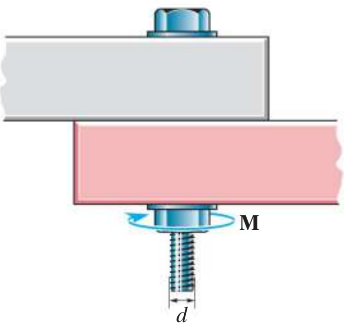
***8–72.** If the clamping force on the boards is 600 lb, determine the required magnitude of the couple forces that must be applied perpendicular to the lever AB of the clamp at A and B in order to loosen the screw. The single square-threaded screw has a mean diameter of 1 in. and a lead of 0.25 in. The coefficient of static friction is $\mu_s = 0.3$.



Probs. 8–71/72

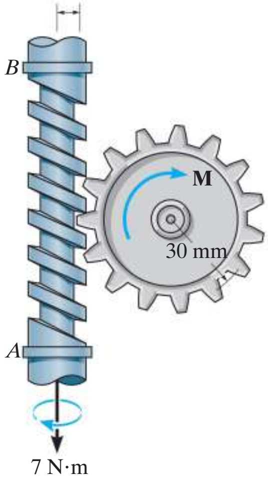
8–73. Prove that the lead l must be less than $2\pi r\mu_s$ for the jack screw shown in Fig. 8–15 to be "self-locking."

8–74. The square-threaded bolt is used to join two plates together. If the bolt has a mean diameter of $d = 20$ mm and a lead of $l = 3$ mm, determine the smallest torque M required to loosen the bolt if the tension in the bolt is $T = 40$ kN. The coefficient of static friction between the threads and the bolt is $\mu_s = 0.15$.



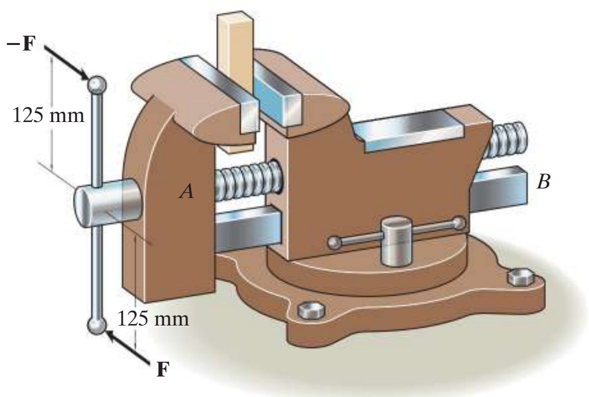
Prob. 8–74

8–75. The shaft has a square-threaded screw with a lead of 8 mm and a mean radius of 15 mm. If it is in contact with a plate gear having a mean radius of 30 mm, determine the resisting torque **M** on the plate gear which can be overcome if a torque of 7 N · m is applied to the shaft. The coefficient of static friction at the screw is $\mu_B = 0.2$. Neglect friction of the bearings located at A and B.



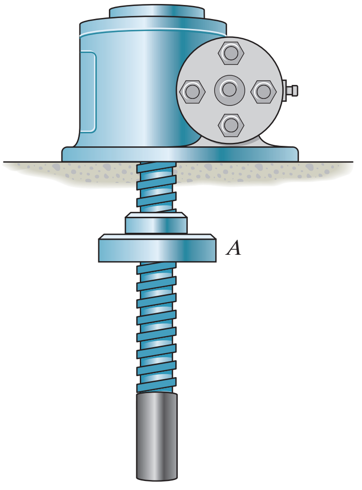
Prob. 8–75

***8–76.** If couple forces of $F = 35$ N are applied to the handle of the machinist's vise, determine the compressive force developed in the block. Neglect friction at the bearing A. The guide at B is smooth. The single square-threaded screw has a mean radius of 6 mm and a lead of 8 mm, and the coefficient of static friction is $\mu_s = 0.27$.



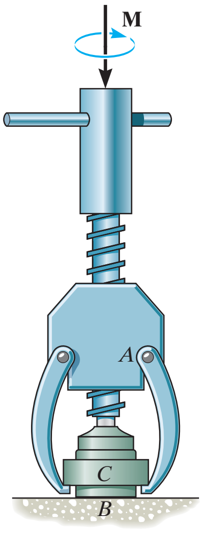
Prob. 8–76

8–77. The square-threaded screw has a mean diameter of 20 mm and a lead of 4 mm. If the weight of the plate A is 5 lb, determine the smallest coefficient of static friction between the screw and the plate so that the plate does not travel down the screw when the plate is suspended as shown.



Prob. 8–77

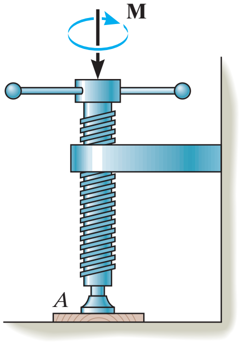
8–78. The device is used to pull the battery cable terminal C from the post of a battery. If the required pulling force is 85 lb, determine the torque **M** that must be applied to the handle on the screw to tighten it. The screw has square threads, a mean diameter of 0.2 in., a lead of 0.08 in., and the coefficient of static friction is $\mu_s = 0.5$.



Prob. 8–78

8–79. Determine the clamping force on the board A if the screw is tightened with a torque of $M = 8\ \text{N}\cdot\text{m}$. The square-threaded screw has a mean radius of 10 mm and a lead of 3 mm, and the coefficient of static friction is $\mu_s = 0.35$.

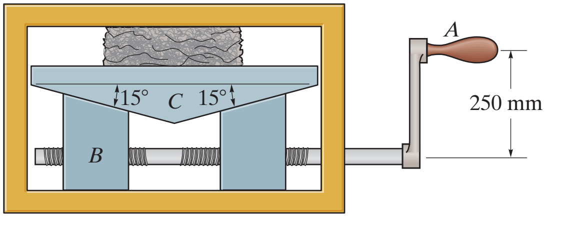
***8–80.** If the required clamping force at the board A is to be 2 kN, determine the torque M that must be applied to the screw to tighten it down. The square-threaded screw has a mean radius of 10 mm and a lead of 3 mm, and the coefficient of static friction is $\mu_s = 0.35$.



Probs. 8–79/80

8–81. If a horizontal force of $P = 100$ N is applied perpendicular to the handle of the lever at A, determine the compressive force **F** exerted on the material. Each single square-threaded screw has a mean diameter of 25 mm and a lead of 7.5 mm. The coefficient of static friction at all contacting surfaces of the wedges is $\mu_s = 0.2$, and the coefficient of static friction at the screw is $\mu_s' = 0.15$.

8–82. Determine the horizontal force **P** that must be applied perpendicular to the handle of the lever at A in order to develop a compressive force of 12 kN on the material. Each single square-threaded screw has a mean diameter of 25 mm and a lead of 7.5 mm. The coefficient of static friction at all contacting surfaces of the wedges is $\mu_s = 0.2$, and the coefficient of static friction at the screw is $\mu_s' = 0.15$.



Probs. 8–81/82

8.5 Frictional Forces on Flat Belts

Whenever belt drives or band brakes are designed, it is necessary to determine the frictional forces developed between the belt and its contacting surface. In this section we will analyze the frictional forces acting on a flat belt, although the analysis of other types of belts, such as the V-belt, is based on similar principles.

Consider the flat belt shown in Fig. 8–18*a*, which passes over a fixed curved surface. The total angle of belt-to-surface contact in radians is β, and the coefficient of friction between the two surfaces is μ. We wish to determine the tension T_2 in the belt, which is needed to pull the belt counterclockwise over the surface, and thereby overcome both the frictional forces at the surface of contact and the tension T_1 in the other end of the belt. Obviously, $T_2 > T_1$.

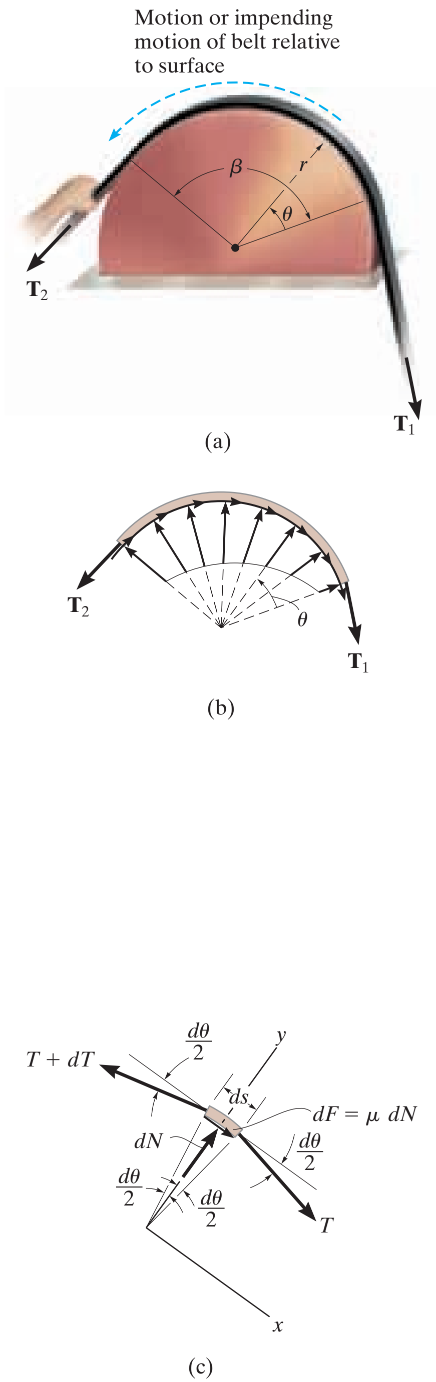


Fig. 8–18

Frictional Analysis. A free-body diagram of the belt segment in contact with the surface is shown in Fig. 8–18*b*. As shown, the normal and frictional forces, acting at different points along the belt, will vary both in magnitude and direction. Due to this *unknown* distribution, the analysis of the problem will first require a study of the forces acting on a differential element of the belt.

A free-body diagram of an element having a length ds is shown in Fig. 8–18*c*. Assuming either impending motion or motion of the belt, the magnitude of the frictional force $dF = \mu\, dN$. This force opposes the sliding motion of the belt, and so it will increase the magnitude of the tensile force acting in the belt by dT. Applying the two force equations of equilibrium, we have

$$\searrow + \Sigma F_x = 0; \qquad T\cos\left(\frac{d\theta}{2}\right) + \mu\, dN - (T + dT)\cos\left(\frac{d\theta}{2}\right) = 0$$

$$+\nearrow \Sigma F_y = 0; \qquad dN - (T + dT)\sin\left(\frac{d\theta}{2}\right) - T\sin\left(\frac{d\theta}{2}\right) = 0$$

Since $d\theta$ is of *infinitesimal size*, $\sin(d\theta/2) = d\theta/2$ and $\cos(d\theta/2) = 1$. Also, the *product* of the two infinitesimals dT and $d\theta/2$ may be neglected when compared to infinitesimals of the first order. As a result, these two equations become

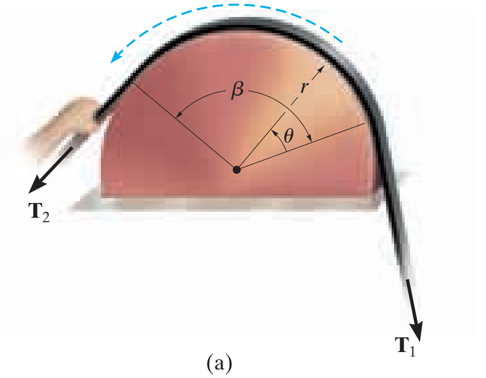
$$\mu\, dN = dT$$

and

$$dN = T\, d\theta$$

Eliminating dN yields

$$\frac{dT}{T} = \mu\, d\theta$$



(a)

Fig. 8–18 (Repeated)

Flat or V-belts are often used to transmit the torque developed by a motor to a wheel attached to a pump, fan, or blower. (© Russell C. Hibbeler)

Integrating this equation between all the points of contact that the belt makes with the drum, and noting that $T = T_1$ at $\theta = 0$ and $T = T_2$ at $\theta = \beta$, yields

$$\int_{T_1}^{T_2} \frac{dT}{T} = \mu \int_0^{\beta} d\theta$$

$$\ln \frac{T_2}{T_1} = \mu\beta$$

Solving for T_2, we obtain

$$\boxed{T_2 = T_1 e^{\mu\beta}} \tag{8–6}$$

where

T_2, T_1 = belt tensions; T_1 opposes the direction of motion (or impending motion) of the belt measured relative to the surface, while T_2 acts in the direction of the relative belt motion (or impending motion); because of friction, $T_2 > T_1$

μ = coefficient of static or kinetic friction between the belt and the surface of contact

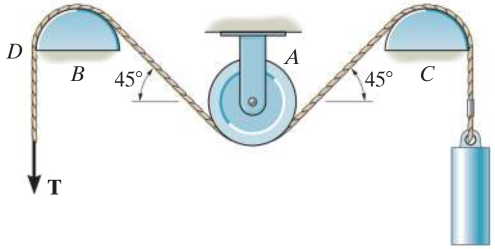
β = angle of belt-to-surface contact, measured in radians

e = 2.718 . . . , base of the natural logarithm

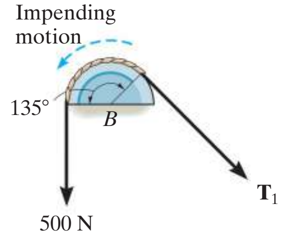
Note that T_2 is *independent* of the *radius* of the drum, and instead it is a function of the angle of belt to surface contact, β. As a result, this equation is valid for flat belts passing over any curved contacting surface.

EXAMPLE 8.8

The maximum tension that can be developed in the cord shown in Fig. 8–19*a* is 500 N. If the pulley at A is free to rotate and the coefficient of static friction at the fixed drums B and C is $\mu_s = 0.25$, determine the largest mass of the cylinder that can be lifted by the cord.



(a)



(b)

SOLUTION

Lifting the cylinder, which has a weight $W = mg$, causes the cord to move counterclockwise over the drums at B and C; hence, the maximum tension T_2 in the cord occurs at D. Thus, $F = T_2 = 500$ N. A section of the cord passing over the drum at B is shown in Fig. 8–19*b*. Since $180° = \pi$ rad the angle of contact between the drum and the cord is $\beta = (135°/180°)\pi = 3\pi/4$ rad. Using Eq. 8–6, we have

$$T_2 = T_1 e^{\mu_s \beta}; \qquad 500 \text{ N} = T_1 e^{0.25[(3/4)\pi]}$$

Hence,

$$T_1 = \frac{500 \text{ N}}{e^{0.25[(3/4)\pi]}} = \frac{500 \text{ N}}{1.80} = 277.4 \text{ N}$$

Since the pulley at A is free to rotate, equilibrium requires that the tension in the cord remains the *same* on both sides of the pulley.

The section of the cord passing over the drum at C is shown in Fig. 8–19*c*. The weight $W < 277.4$ N. Why? Applying Eq. 8–6, we obtain

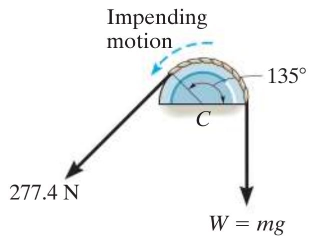
$$T_2 = T_1 e^{\mu_s \beta}; \qquad 277.4 \text{ N} = W e^{0.25[(3/4)\pi]}$$

$$W = 153.9 \text{ N}$$

so that

$$m = \frac{W}{g} = \frac{153.9 \text{ N}}{9.81 \text{ m/s}^2}$$

$$= 15.7 \text{ kg} \qquad \textit{Ans.}$$



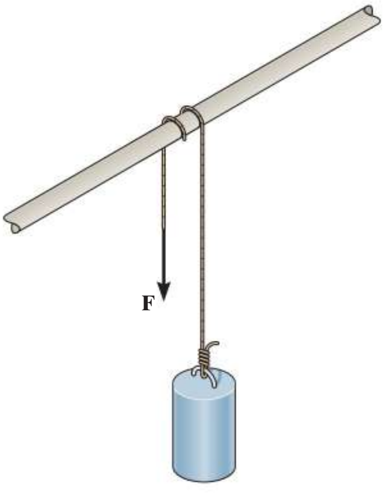
(c)

Fig. 8–19

PROBLEMS

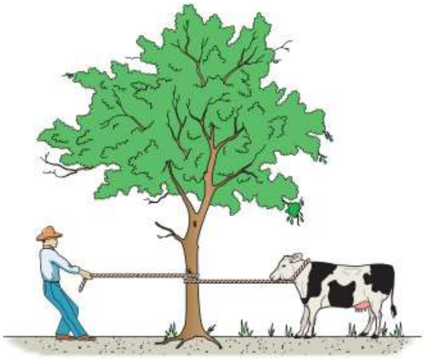
8–83. A cylinder having a mass of 250 kg is to be supported by the cord that wraps over the pipe. Determine the smallest vertical force **F** needed to support the load if the cord passes (a) once over the pipe, $\beta = 180°$, and (b) two times over the pipe, $\beta = 540°$. Take $\mu_s = 0.2$.

***8–84.** A cylinder having a mass of 250 kg is to be supported by the cord that wraps over the pipe. Determine the largest vertical force **F** that can be applied to the cord without moving the cylinder. The cord passes (a) once over the pipe, $\beta = 180°$, and (b) two times over the pipe, $\beta = 540°$. Take $\mu_s = 0.2$.



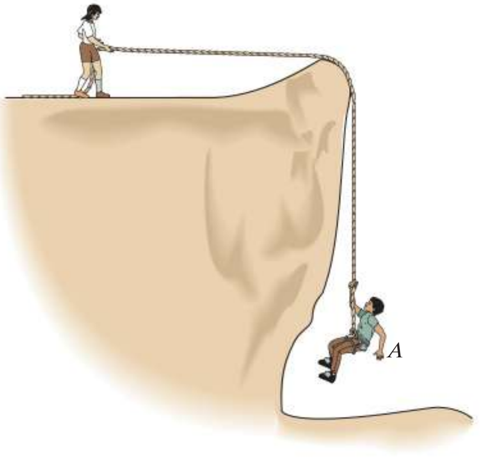
Probs. 8–83/84

8–85. A 180-lb farmer tries to restrain the cow from escaping by wrapping the rope two turns around the tree trunk as shown. If the cow exerts a force of 250 lb on the rope, determine if the farmer can successfully restrain the cow. The coefficient of static friction between the rope and the tree trunk is $\mu_s = 0.15$, and between the farmer's shoes and the ground $\mu'_s = 0.3$.



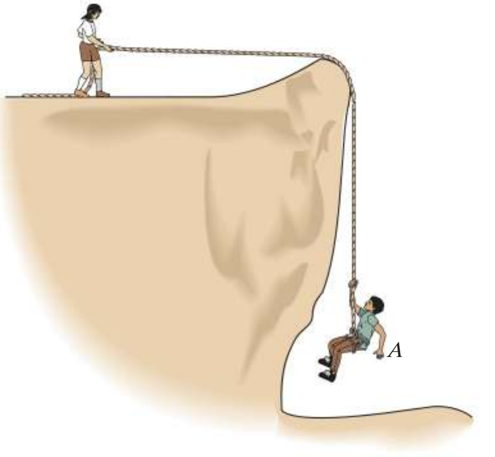
Prob. 8–85

8–86. The 100-lb boy at A is suspended from the cable that passes over the quarter circular cliff rock. Determine if it is possible for the 185-lb woman to hoist him up; and if this is possible, what smallest force must she exert on the horizontal cable? The coefficient of static friction between the cable and the rock is $\mu_s = 0.2$, and between the shoes of the woman and the ground $\mu'_s = 0.8$.



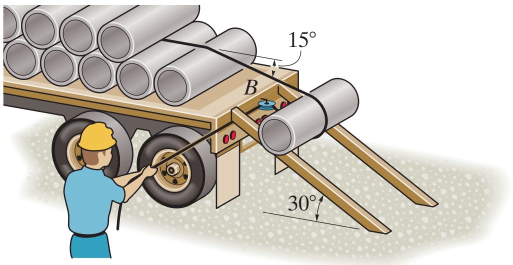
Prob. 8–86

8–87. The 100-lb boy at A is suspended from the cable that passes over the quarter circular cliff rock. What horizontal force must the woman at A exert on the cable in order to let the boy descend at constant velocity? The coefficients of static and kinetic friction between the cable and the rock are $\mu_s = 0.4$ and $\mu_k = 0.35$, respectively.



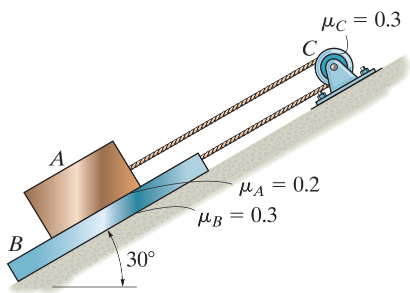
Prob. 8–87

***8–88.** The uniform concrete pipe has a weight of 800 lb and is unloaded slowly from the truck bed using the rope and skids shown. If the coefficient of kinetic friction between the rope and pipe is $\mu_k = 0.3$, determine the force the worker must exert on the rope to lower the pipe at constant speed. There is a pulley at B, and the pipe does not slip on the skids. The lower portion of the rope is parallel to the skids.



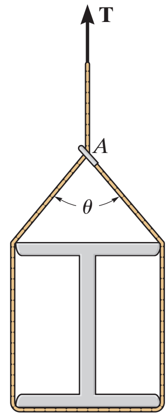
Prob. 8–88

8–89. A cable is attached to the 20-kg plate B, passes over a fixed peg at C, and is attached to the block at A. Using the coefficients of static friction shown, determine the smallest mass of block A so that it will prevent sliding motion of B down the plane.



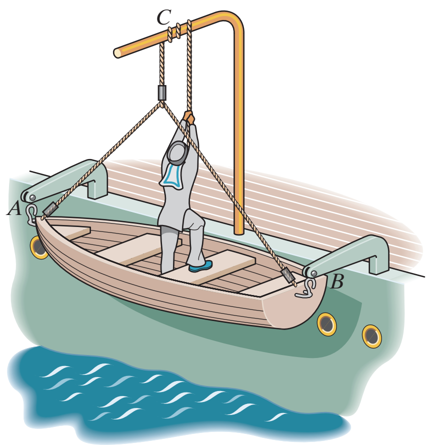
Prob. 8–89

8–90. The smooth beam is being hoisted using a rope that is wrapped around the beam and passes through a ring at A as shown. If the end of the rope is subjected to a tension $\mathbf{T}$ and the coefficient of static friction between the rope and ring is $\mu_s = 0.3$, determine the smallest angle of θ for equilibrium.



Prob. 8–90

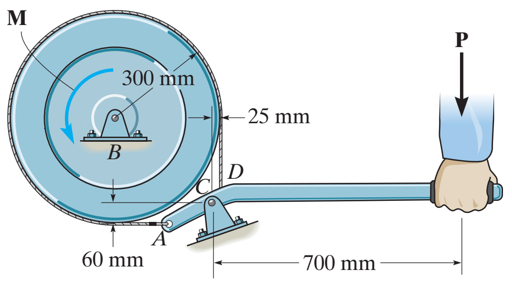
8–91. The boat has a weight of 500 lb and is held in position off the side of a ship by the spars at A and B. A man having a weight of 130 lb gets in the boat, wraps a rope around an overhead boom at C, and ties it to the end of the boat as shown. If the boat is disconnected from the spars, determine the *minimum number of half turns* the rope must make around the boom so that the boat can be safely lowered into the water at constant velocity. Also, what is the normal force between the boat and the man? The coefficient of kinetic friction between the rope and the boom is $\mu_s = 0.15$. *Hint*: The problem requires that the normal force between the man's feet and the boat be as small as possible.



Prob. 8–91

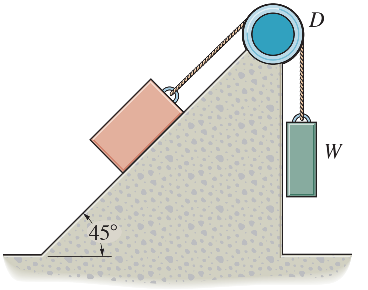
*8–92. Determine the force P that must be applied to the handle of the lever so that the wheel is on the verge of turning if $M = 300\ \text{N} \cdot \text{m}$. The coefficient of static friction between the belt and the wheel is $\mu_s = 0.3$.

8–93. If a force of $P = 30\ \text{N}$ is applied to the handle of the lever, determine the largest couple moment **M** that can be resisted so that the wheel does not turn. The coefficient of static friction between the belt and the wheel is $\mu_s = 0.3$.



Probs. 8–92/93

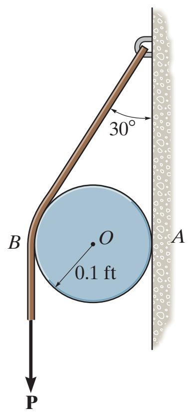
*8–96. Determine the maximum and the minimum values of weight W which may be applied without causing the 50-lb block to slip. The coefficient of static friction between the block and the plane is $\mu_s = 0.2$, and between the rope and the drum D is $\mu_s' = 0.3$.



Prob. 8–96

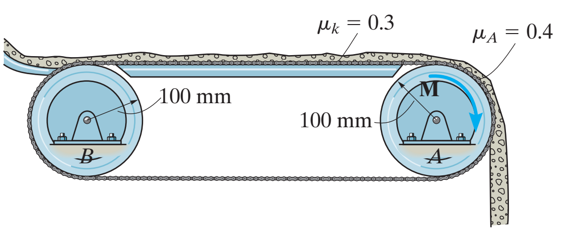
8–94. A minimum force of $P = 50$ lb is required to hold the cylinder from slipping against the belt and the wall. Determine the weight of the cylinder if the coefficient of friction between the belt and cylinder is $\mu_s = 0.3$ and slipping does not occur at the wall.

8–95. The cylinder weighs 10 lb and is held in equilibrium by the belt and wall. If slipping does not occur at the wall, determine the minimum vertical force P which must be applied to the belt for equilibrium. The coefficient of static friction between the belt and the cylinder is $\mu_s = 0.25$.



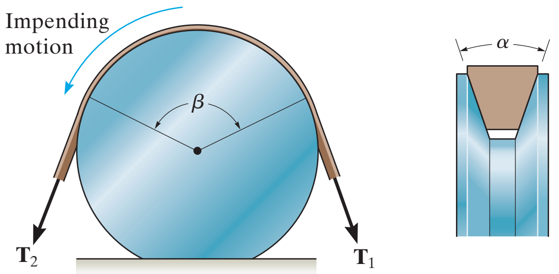
Probs. 8–94/95

8–97. Granular material, having a density of $1.5\ \text{Mg/m}^3$, is transported on a conveyor belt that slides over the fixed surface, having a coefficient of kinetic friction of $\mu_k = 0.3$. Operation of the belt is provided by a motor that supplies a torque **M** to wheel A. The wheel at B is free to turn, and the coefficient of static friction between the wheel at A and the belt is $\mu_A = 0.4$. If the belt is subjected to a pretension of 300 N when no load is on the belt, determine the greatest volume V of material that is permitted on the belt at any time without allowing the belt to stop. What is the torque **M** required to drive the belt when it is subjected to this maximum load?



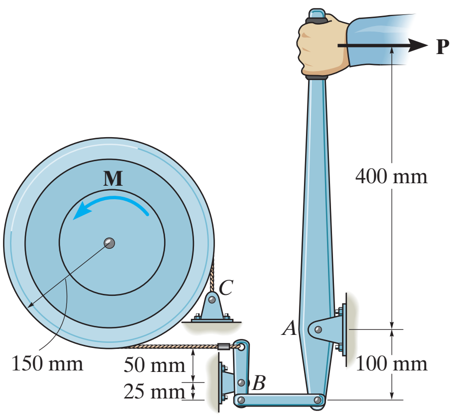
Prob. 8–97

8–98. Show that the frictional relationship between the belt tensions, the coefficient of friction μ, and the angular contacts α and β for the V-belt is $T_2 = T_1 e^{\mu\beta/\sin(\alpha/2)}$.



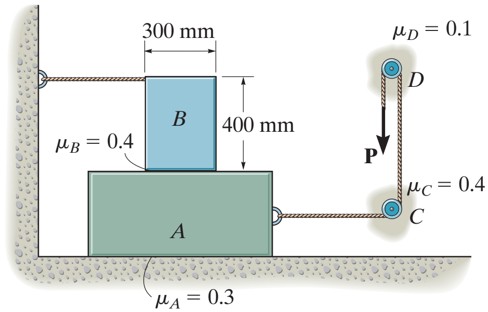
Prob. 8–98

8–99. The wheel is subjected to a torque of $M = 50\ \mathrm{N\cdot m}$. If the coefficient of kinetic friction between the band brake and the rim of the wheel is $\mu_k = 0.3$, determine the smallest horizontal force P that must be applied to the lever to stop the wheel.



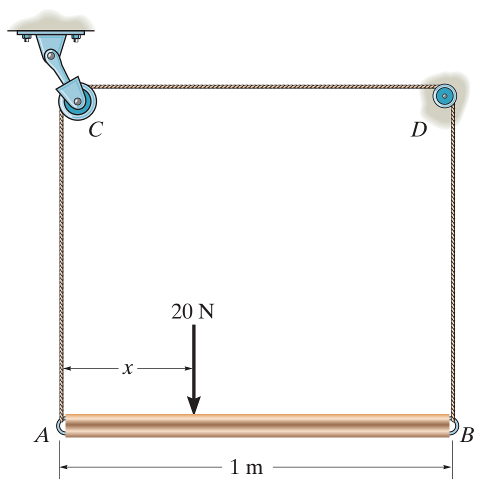
Prob. 8–99

***8–100.** Blocks A and B have a mass of 7 kg and 10 kg, respectively. Using the coefficients of static friction indicated, determine the largest vertical force P which can be applied to the cord without causing motion.



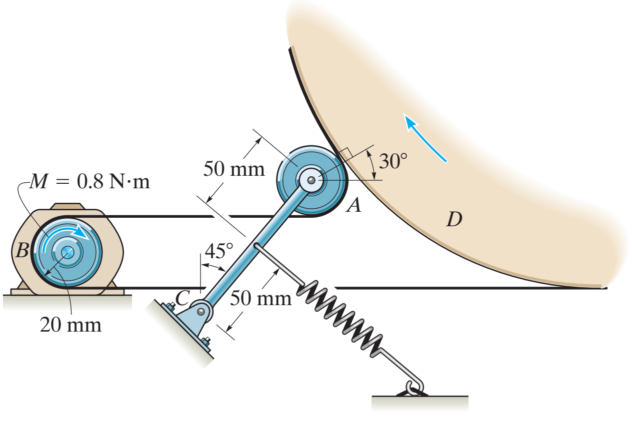
Prob. 8–100

8–101. The uniform bar AB is supported by a rope that passes over a frictionless pulley at C and a fixed peg at D. If the coefficient of static friction between the rope and the peg is $\mu_D = 0.3$, determine the smallest distance x from the end of the bar at which a 20-N force may be placed and not cause the bar to move.



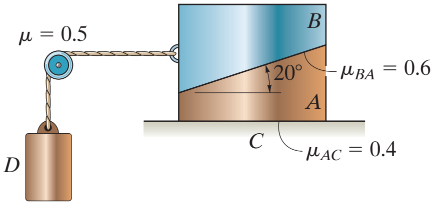
Prob. 8–101

8–102. The belt on the portable dryer wraps around the drum D, idler pulley A, and motor pulley B. If the motor can develop a maximum torque of $M = 0.80\ \text{N}\cdot\text{m}$, determine the smallest spring tension required to hold the belt from slipping. The coefficient of static friction between the belt and the drum and motor pulley is $\mu_s = 0.3$.



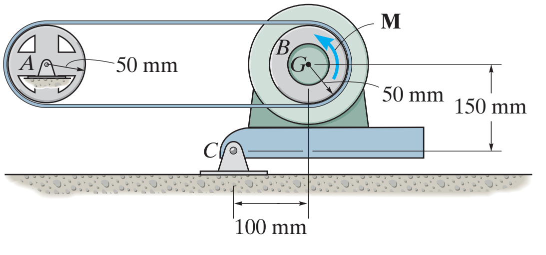
Prob. 8–102

8–103. Blocks A and B weigh 50 lb and 30 lb, respectively. Using the coefficients of static friction indicated, determine the greatest weight of block D without causing motion.



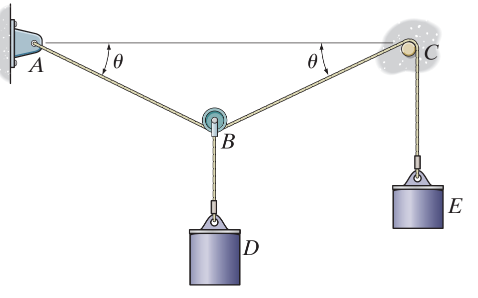
Prob. 8–103

***8–104.** The 20-kg motor has a center of gravity at G and is pin connected at C to maintain a tension in the drive belt. Determine the smallest counterclockwise twist or torque **M** that must be supplied by the motor to turn the disk B if wheel A locks and causes the belt to slip over the disk. No slipping occurs at A. The coefficient of static friction between the belt and the disk is $\mu_s = 0.3$.



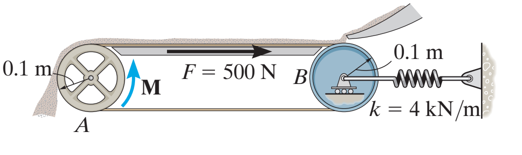
Prob. 8–104

8–105. A 10-kg cylinder D, which is attached to a small pulley B, is placed on the cord as shown. Determine the largest angles θ so that the cord does not slip over the peg at C. The cylinder at E also has a mass of 10 kg, and the coefficient of static friction between the cord and the peg is $\mu_s = 0.1$.



Prob. 8–105

8–106. A conveyer belt is used to transfer granular material and the frictional resistance on the top of the belt is $F = 500$ N. Determine the smallest stretch of the spring attached to the moveable axle of the idle pulley B so that the belt does not slip at the drive pulley A when the torque **M** is applied. What minimum torque **M** is required to keep the belt moving? The coefficient of static friction between the belt and the wheel at A is $\mu_s = 0.2$.



Prob. 8–106

*8.6 Frictional Forces on Collar Bearings, Pivot Bearings, and Disks

Pivot and ***collar bearings*** are commonly used in machines to support an *axial load* on a rotating shaft. Typical examples are shown in Fig. 8–20. Provided these bearings are not lubricated, or are only partially lubricated, the laws of dry friction may be applied to determine the moment needed to turn the shaft when it supports an axial force.

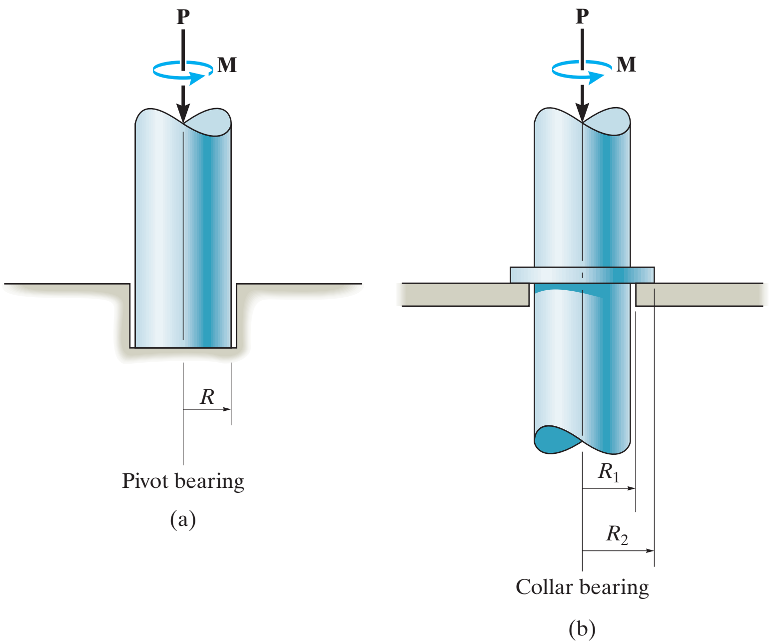


Fig. 8–20

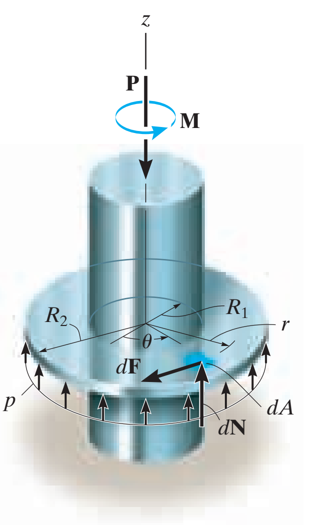


Fig. 8–21

Frictional Analysis. The collar bearing on the shaft shown in Fig. 8–21 is subjected to an axial force **P** and has a total bearing or contact area $\pi(R_2^2 - R_1^2)$. Provided the bearing is new and evenly supported, then the normal pressure p on the bearing will be *uniformly distributed* over this area. Since $\Sigma F_z = 0$, then p, measured as a force per unit area, is $p = P/\pi(R_2^2 - R_1^2)$.

The moment needed to cause impending rotation of the shaft can be determined from moment equilibrium about the z axis. A differential area element $dA = (r\,d\theta)(dr)$, shown in Fig. 8–21, is subjected to both a normal force $dN = p\,dA$ and an associated frictional force,

$$dF = \mu_s\,dN = \mu_s p\,dA = \frac{\mu_s P}{\pi(R_2^2 - R_1^2)}dA$$

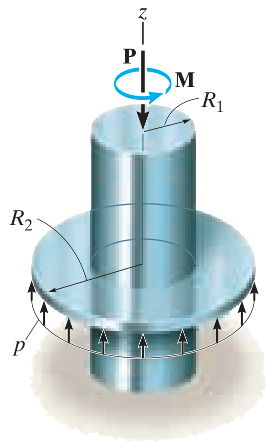


Fig. 8–21 (Repeated)

The normal force does not create a moment about the z axis of the shaft; however, the frictional force does; namely, $dM = r\,dF$. Integration is needed to compute the applied moment **M** needed to overcome all the frictional forces. Therefore, for impending rotational motion,

$$\Sigma M_z = 0; \qquad M - \int_A r\,dF = 0$$

Substituting for dF and dA and integrating over the entire bearing area yields

$$M = \int_{R_1}^{R_2}\int_0^{2\pi} r\left[\frac{\mu_s P}{\pi(R_2^2 - R_1^2)}\right](r\,d\theta\,dr) = \frac{\mu_s P}{\pi(R_2^2 - R_1^2)}\int_{R_1}^{R_2} r^2\,dr\int_0^{2\pi} d\theta$$

or

$$M = \frac{2}{3}\mu_s P\left(\frac{R_2^3 - R_1^3}{R_2^2 - R_1^2}\right) \qquad (8\text{–}7)$$

The motor that turns the disk of this sanding machine develops a torque that must overcome the frictional forces acting on the disk. (© Russell C. Hibbeler)

The moment developed at the end of the shaft, when it is *rotating* at constant speed, can be found by substituting μ_k for μ_s in Eq. 8–7.

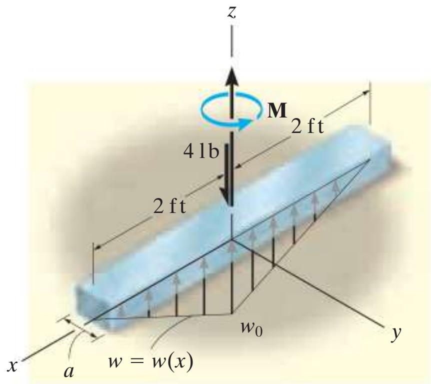
In the case of a pivot bearing, Fig. 8–20*a*, then $R_2 = R$ and $R_1 = 0$, and Eq. 8–7 reduces to

$$M = \frac{2}{3}\mu_s PR \qquad (8\text{–}8)$$

Remember that Eqs. 8–7 and 8–8 apply only for bearing surfaces subjected to *constant pressure*. If the pressure is not uniform, a variation of the pressure as a function of the bearing area must be determined before integrating to obtain the moment. The following example illustrates this concept.

EXAMPLE 8.9

The uniform bar shown in Fig. 8–22*a* has a weight of 4 lb. If it is assumed that the normal pressure acting at the contacting surface varies linearly along the length of the bar as shown, determine the couple moment **M** required to rotate the bar. Assume that the bar's width is negligible in comparison to its length. The coefficient of static friction is equal to $\mu_s = 0.3$.



(a)

SOLUTION

A free-body diagram of the bar is shown in Fig. 8–22*b*. The intensity w_0 of the distributed load at the center ($x = 0$) is determined from vertical force equilibrium, Fig. 8–22*a*.

$$+\uparrow \Sigma F_z = 0; \qquad -4\text{ lb} + 2\left[\frac{1}{2}\left(2\text{ ft}\right)w_0\right] = 0 \qquad w_0 = 2\text{ lb/ft}$$

Since $w = 0$ at $x = 2$ ft, the distributed load expressed as a function of x is

$$w = (2\text{ lb/ft})\left(1 - \frac{x}{2\text{ ft}}\right) = 2 - x$$

The magnitude of the normal force acting on a differential segment of area having a length dx is therefore

$$dN = w\,dx = (2 - x)dx$$

The magnitude of the frictional force acting on the same element of area is

$$dF = \mu_s\,dN = 0.3(2 - x)dx$$

Hence, the moment created by this force about the z axis is

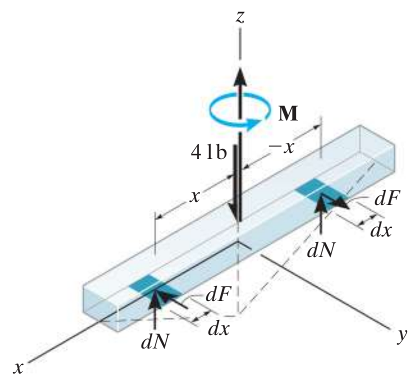
$$dM = x\,dF = 0.3(2x - x^2)dx$$

The summation of moments about the z axis of the bar is determined by integration, which yields

$$\Sigma M_z = 0; \qquad M - 2\int_0^2 (0.3)(2x - x^2)\,dx = 0$$

$$M = 0.6\left(x^2 - \frac{x^3}{3}\right)\Big|_0^2$$

$$M = 0.8\text{ lb}\cdot\text{ft} \qquad \textit{Ans.}$$



(b)

Fig. 8–22

8.7 Frictional Forces on Journal Bearings

Unwinding the cable from this spool requires overcoming friction from the supporting shaft. (© Russell C. Hibbeler)

When a shaft or axle is subjected to lateral loads, a ***journal bearing*** is commonly used for support. Provided the bearing is not lubricated, or is only partially lubricated, a reasonable analysis of the frictional resistance on the bearing can be based on the laws of dry friction.

Frictional Analysis. A typical journal-bearing support is shown in Fig. 8–23*a*. As the shaft rotates, the contact point moves up the wall of the bearing to some point A where slipping occurs. If the vertical load acting at the end of the shaft is $\mathbf{P}$, then the bearing reactive force $\mathbf{R}$ acting at A will be equal and opposite to $\mathbf{P}$, Fig. 8–23*b*. The moment needed to maintain constant rotation of the shaft can be found by summing moments about the z axis of the shaft; i.e.,

$$\Sigma M_z = 0; \qquad M - (R \sin \phi_k)r = 0$$

or

$$M = Rr \sin \phi_k \tag{8–9}$$

where ϕ_k is the angle of kinetic friction defined by $\tan \phi_k = F/N = \mu_k N/N = \mu_k$. In Fig. 8–23*c*, it is seen that $r \sin \phi_k = r_f$. The dashed circle with radius r_f is called the ***friction circle***, and as the shaft rotates, the reaction $\mathbf{R}$ will always be tangent to it. If the bearing is partially lubricated, μ_k is small, and therefore $\sin \phi_k \approx \tan \phi_k \approx \mu_k$. Under these conditions, a reasonable *approximation* to the moment needed to overcome the frictional resistance becomes

$$M \approx Rr\mu_k \tag{8–10}$$

Notice that to minimize friction the bearing radius r should be as small as possible. In practice, however, this type of journal bearing is not suitable for long service since friction between the shaft and bearing will eventually wear down the surfaces. Instead, designers will incorporate "ball bearings" or "rollers" in journal bearings to minimize frictional losses.

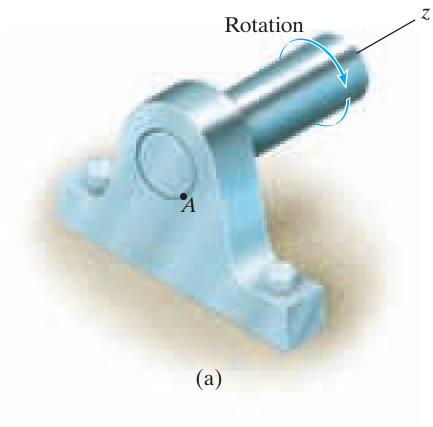
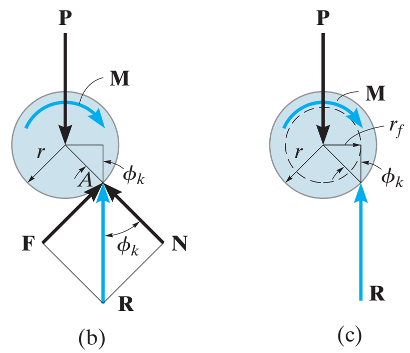
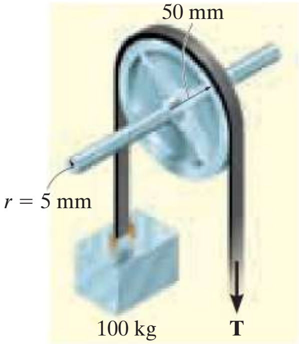


Fig. 8–23



EXAMPLE 8.10

The 100-mm-diameter pulley shown in Fig. 8–24*a* fits loosely on a 10-mm-diameter shaft for which the coefficient of static friction is $\mu_s = 0.4$. Determine the minimum tension T in the belt needed to (a) raise the 100-kg block and (b) lower the block. Assume that no slipping occurs between the belt and pulley and neglect the weight of the pulley.



(a)

SOLUTION

Part (a). A free-body diagram of the pulley is shown in Fig. 8–24*b*. When the pulley is subjected to belt tensions of 981 N each, it makes contact with the shaft at point P_1. As the tension T is *increased*, the contact point will move around the shaft to point P_2 before motion impends. From the figure, the friction circle has a radius $r_f = r \sin \phi_s$. Using the simplification that $\sin \phi_s \approx \tan \phi_s \approx \mu_s$ then $r_f \approx r\mu_s = (5 \text{ mm})(0.4) = 2 \text{ mm}$, so that summing moments about P_2 gives

$$\circlearrowleft + \Sigma M_{P_2} = 0; \qquad 981 \text{ N}(52 \text{ mm}) - T(48 \text{ mm}) = 0$$

$$T = 1063 \text{ N} = 1.06 \text{ k N} \qquad \textit{Ans.}$$

If a more exact analysis is used, then $\phi_s = \tan^{-1} 0.4 = 21.8°$. Thus, the radius of the friction circle would be $r_f = r \sin \phi_s = 5 \sin 21.8° = 1.86 \text{ mm}$. Therefore,

$$\circlearrowleft + \Sigma M_{P_2} = 0;$$

$$981 \text{ N}(50 \text{ mm} + 1.86 \text{ mm}) - T(50 \text{ mm} - 1.86 \text{ mm}) = 0$$

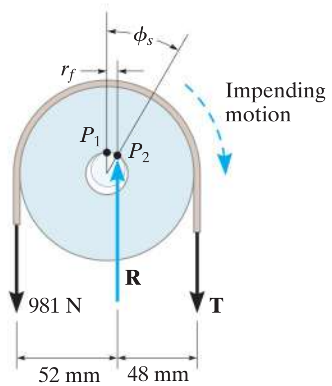
$$T = 1057 \text{ N} = 1.06 \text{ kN} \qquad \textit{Ans.}$$

Part (b). When the block is lowered, the resultant force **R** acting on the shaft passes through point as shown in Fig. 8–24*c*. Summing moments about this point yields

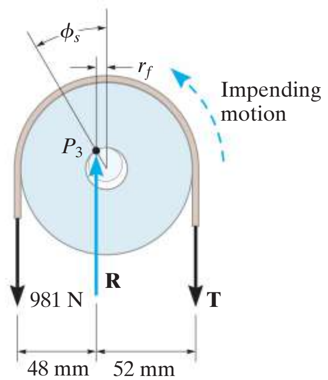
$$\circlearrowleft + \Sigma M_{P_3} = 0; \qquad 981 \text{ N}(48 \text{ mm}) - T(52 \text{ mm}) = 0$$

$$T = 906 \text{ N} \qquad \textit{Ans.}$$

NOTE: Using the approximate analysis, the difference between raising and lowering the block is thus 157 N.



(b)



(c)

Fig. 8–24

*8.8 Rolling Resistance

When a *rigid* cylinder rolls at constant velocity along a *rigid* surface, the normal force exerted by the surface on the cylinder acts perpendicular to the tangent at the point of contact, as shown in Fig. 8–25*a*. Actually, however, no materials are perfectly rigid, and therefore the reaction of the surface on the cylinder consists of a distribution of normal pressure. For example, consider the cylinder to be made of a very hard material, and the surface on which it rolls to be relatively soft. Due to its weight, the cylinder compresses the surface underneath it, Fig. 8–25*b*. As the cylinder rolls, the surface material in front of the cylinder *retards* the motion since it is being *deformed*, whereas the material in the rear is *restored* from the deformed state and therefore tends to *push* the cylinder forward. The normal pressures acting on the cylinder in this manner are represented in Fig. 8–25*b* by their resultant forces $\mathbf{N}_d$ and $\mathbf{N}_r$. The magnitude of the force of *deformation*, $\mathbf{N}_d$, and its horizontal component is *always greater* than that of *restoration*, $\mathbf{N}_r$, and consequently a horizontal driving force $\mathbf{P}$ must be applied to the cylinder to maintain the motion. Fig. 8–25*b*.*

Rolling resistance is caused primarily by this effect, although it is also, to a lesser degree, the result of surface adhesion and relative micro-sliding between the surfaces of contact. Because the actual force $\mathbf{P}$ needed to overcome these effects is difficult to determine, a simplified method will be developed here to explain one way engineers have analyzed this phenomenon. To do this, we will consider the resultant of the *entire* normal pressure, $\mathbf{N} = \mathbf{N}_d + \mathbf{N}_r$, acting on the cylinder, Fig. 8–25*c*. As shown in Fig. 8–25*d*, this force acts at an angle θ with the vertical. To keep the cylinder in equilibrium, i.e., rolling at a constant rate, it is necessary that $\mathbf{N}$ be *concurrent* with the driving force $\mathbf{P}$ and the weight $\mathbf{W}$. Summing moments about point A gives $Wa = P(r\cos\theta)$. Since the deformations are generally very small in relation to the cylinder's radius, $\cos\theta \approx 1$; hence,

$$Wa \approx Pr$$

or

$$\boxed{P \approx \frac{Wa}{r}} \qquad (8\text{–}11)$$

The distance a is termed the ***coefficient of rolling resistance***, which has the dimension of length. For instance, $a \approx 0.5$ mm for a wheel rolling on a rail, both of which are made of mild steel. For hardened steel ball

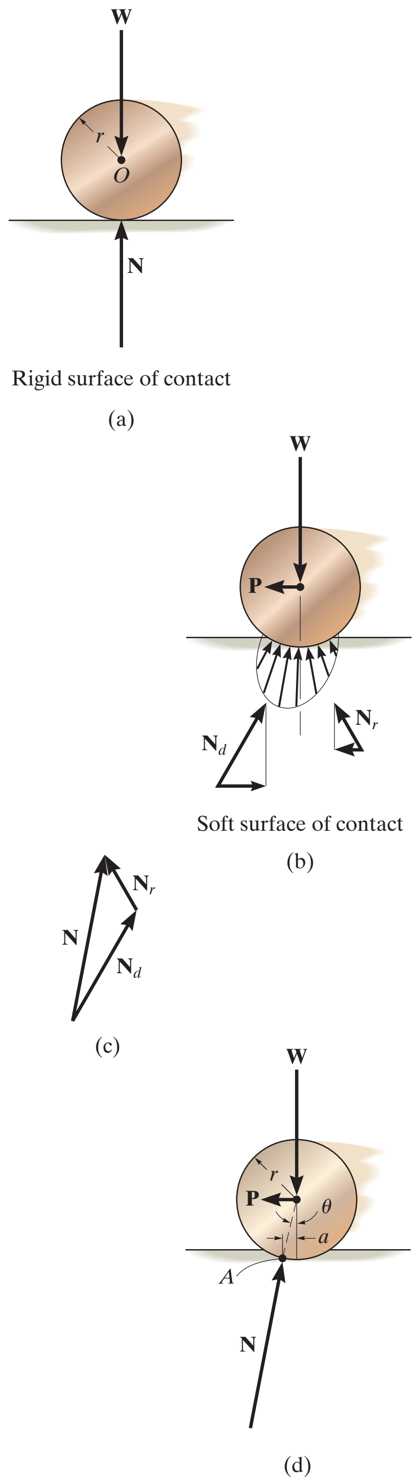
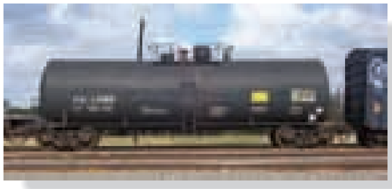


Fig. 8–25

*Actually, the deformation force $\mathbf{N}_d$ causes *energy* to be stored in the material as its magnitude is increased, whereas the restoration force $\mathbf{N}_r$, as its magnitude is decreased, allows some of this energy to be released. The remaining energy is *lost* since it is used to heat up the surface, and if the cylinder's weight is very large, it accounts for permanent deformation of the surface. Work must be done by the horizontal force $\mathbf{P}$ to make up for this loss.

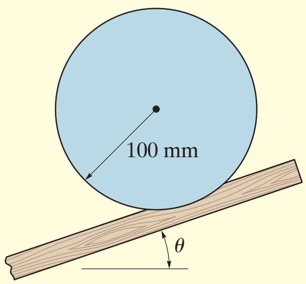
bearings on steel, $a \approx 0.1$ mm. Experimentally, though, this factor is difficult to measure, since it depends on such parameters as the rate of rotation of the cylinder, the elastic properties of the contacting surfaces, and the surface finish. For this reason, little reliance is placed on the data for determining a. The analysis presented here does, however, indicate why a heavy load (W) offers greater resistance to motion (P) than a light load under the same conditions. Furthermore, since Wa/r is generally very small compared to $\mu_k W$, the force needed to *roll* a cylinder over the surface will be much less than that needed to *slide* it across the surface. It is for this reason that a roller or ball bearings are often used to minimize the frictional resistance between moving parts.



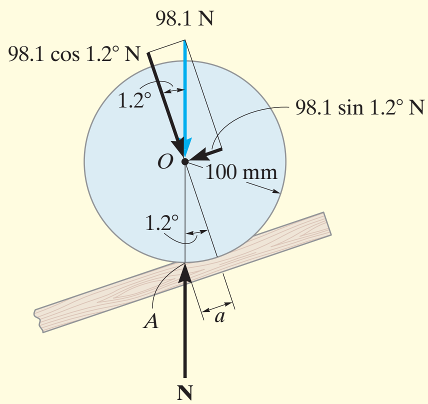
Rolling resistance of railroad wheels on the rails is small since steel is very stiff. By comparison, the rolling resistance of the wheels of a tractor in a wet field is very large. (© Russell C. Hibbeler)

EXAMPLE 8.11

A 10-kg steel wheel shown in Fig. 8–26*a* has a radius of 100 mm and rests on an inclined plane made of soft wood. If θ is increased so that the wheel begins to roll down the incline with constant velocity when $\theta = 1.2°$, determine the coefficient of rolling resistance.



(a)



(b)

Fig. 8–26

SOLUTION

As shown on the free-body diagram, Fig. 8–26*b*, when the wheel has impending motion, the normal reaction **N** acts at point A defined by the dimension a. Resolving the weight into components parallel and perpendicular to the incline, and summing moments about point A, yields

$$\circlearrowleft +\Sigma M_A = 0;$$

$$-(98.1 \cos 1.2° \text{ N})(a) + (98.1 \sin 1.2° \text{ N})(100 \cos 1.2° \text{ mm}) = 0$$

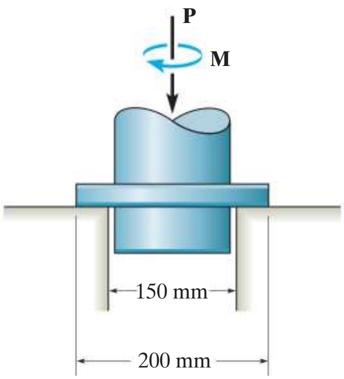
Solving, we obtain

$$a = 2.09 \text{ mm} \qquad \textit{Ans.}$$

PROBLEMS

8–107. The collar bearing uniformly supports an axial force of $P = 5$ kN. If the coefficient of static friction is $\mu_s = 0.3$, determine the smallest torque M required to overcome friction.

***8–108.** The collar bearing uniformly supports an axial force of $P = 8$ kN. If a torque of $M = 200\ \text{N}\cdot\text{m}$ is applied to the shaft and causes it to rotate at constant velocity, determine the coefficient of kinetic friction at the surface of contact.

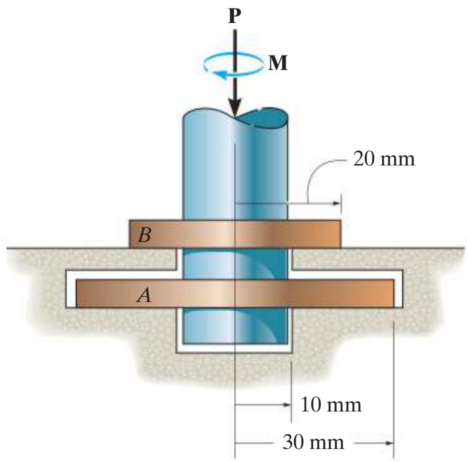


Probs. 8–107/108

8–109. The floor-polishing machine rotates at a constant angular velocity. If it has a weight of 80 lb, determine the couple forces F the operator must apply to the handles to hold the machine stationary. The coefficient of kinetic friction between the floor and brush is $\mu_k = 0.3$. Assume the brush exerts a uniform pressure on the floor.

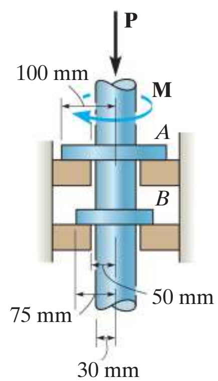
Prob. 8–109

8–110. The *double-collar bearing* is subjected to an axial force $P = 4$ kN. Assuming that collar A supports $0.75P$ and collar B supports $0.25P$, both with a uniform distribution of pressure, determine the maximum frictional moment M that may be resisted by the bearing. Take $\mu_s = 0.2$ for both collars.



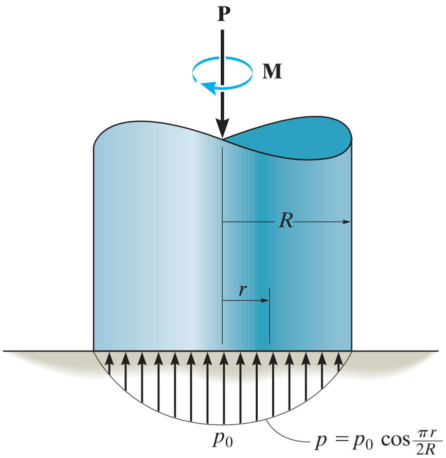
Prob. 8–110

8–111. The *double-collar bearing* is subjected to an axial force $P = 16$ kN. Assuming that collar A supports $0.75P$ and collar B supports $0.25P$, both with a uniform distribution of pressure, determine the smallest torque **M** that must be applied to overcome friction. Take $\mu_s = 0.2$ for both collars.



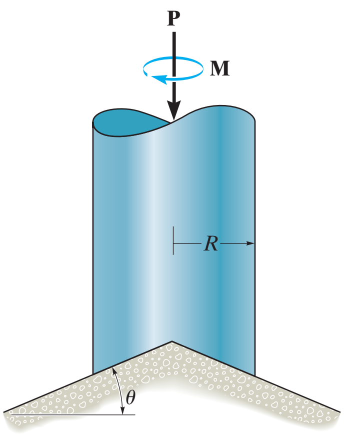
Prob. 8–111

***8–112.** The pivot bearing is subjected to a pressure distribution at its surface of contact which varies as shown. If the coefficient of static friction is μ, determine the torque M required to overcome friction if the shaft supports an axial force $\mathbf{P}$.



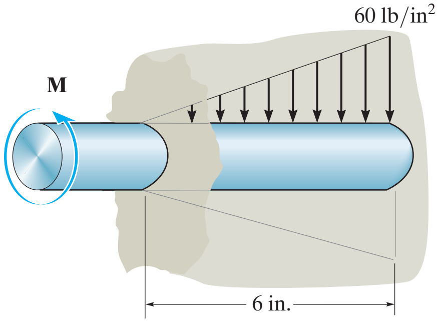
Prob. 8–112

8–113. The conical bearing is subjected to a constant pressure distribution at its surface of contact. If the coefficient of static friction is μ_s, determine the torque M required to overcome friction if the shaft supports an axial force $\mathbf{P}$.



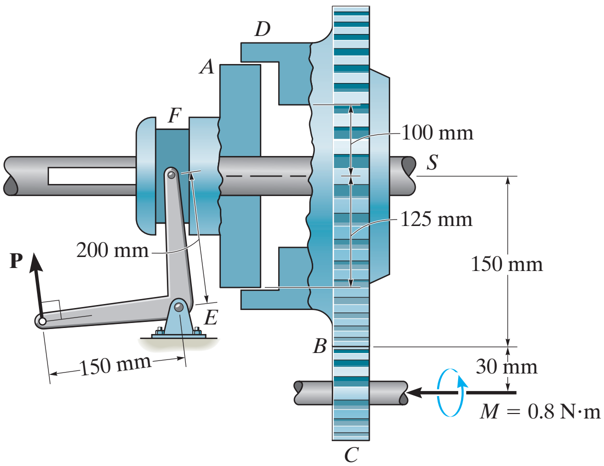
Prob. 8–113

8–114. The 4-in.-diameter shaft is held in the hole such that the normal pressure acting around the shaft varies linearly with its depth as shown. Determine the frictional torque that must be overcome to rotate the shaft. Take $\mu_s = 0.2$.



Prob. 8–114

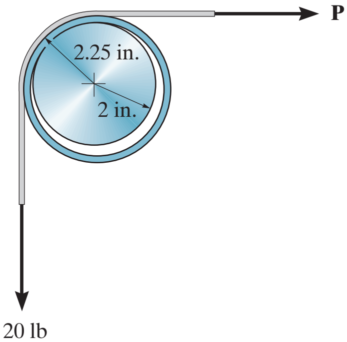
8–115. The plate clutch consists of a flat plate A that slides over the rotating shaft S. The shaft is fixed to the driving plate gear B. If the gear C, which is in mesh with B, is subjected to a torque of $M = 0.8\ \text{N}\cdot\text{m}$, determine the smallest force P, that must be applied via the control arm, to stop the rotation. The coefficient of static friction between the plates A and D is $\mu_s = 0.4$. Assume the bearing pressure between A and D to be uniform.



Prob. 8–115

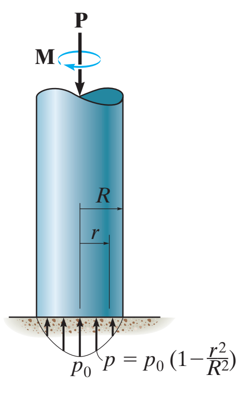
*8–116. The collar fits *loosely* around a fixed shaft that has a radius of 2 in. If the coefficient of kinetic friction between the shaft and the collar is $\mu_k = 0.3$, determine the force P on the horizontal segment of the belt so that the collar rotates *counterclockwise* with a constant angular velocity. Assume that the belt does not slip on the collar; rather, the collar slips on the shaft. Neglect the weight and thickness of the belt and collar. The radius, measured from the center of the collar to the mean thickness of the belt, is 2.25 in.

8–117. The collar fits *loosely* around a fixed shaft that has a radius of 2 in. If the coefficient of kinetic friction between the shaft and the collar is $\mu_k = 0.3$, determine the force P on the horizontal segment of the belt so that the collar rotates *clockwise* with a constant angular velocity. Assume that the belt does not slip on the collar; rather, the collar slips on the shaft. Neglect the weight and thickness of the belt and collar. The radius, measured from the center of the collar to the mean thickness of the belt, is 2.25 in.



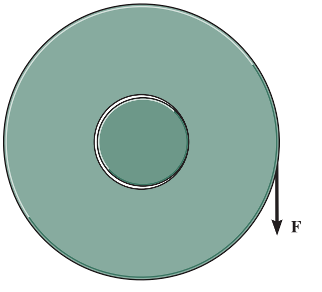
Probs. 8–116/117

8–118. The pivot bearing is subjected to a parabolic pressure distribution at its surface of contact. If the coefficient of static friction is μ_k, determine the torque M required to overcome friction and turn the shaft if it supports an axial force **P**.



Prob. 8–118

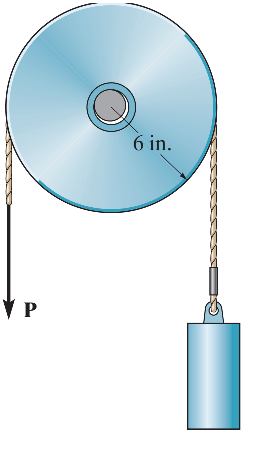
8–119. A disk having an outer diameter of 120 mm fits loosely over a fixed shaft having a diameter of 30 mm. If the coefficient of static friction between the disk and the shaft is $\mu_s = 0.15$ and the disk has a mass of 50 kg, determine the smallest vertical force **F** acting on the rim which must be applied to the disk to cause it to slip over the shaft.



Prob. 8–119

*8–120. The 4-lb pulley has a diameter of 1 ft and the axle has a diameter of 1 in. If the coefficient of kinetic friction between the axle and the pulley is $\mu_k = 0.20$, determine the vertical force P on the rope required to lift the 20-lb block at constant velocity.

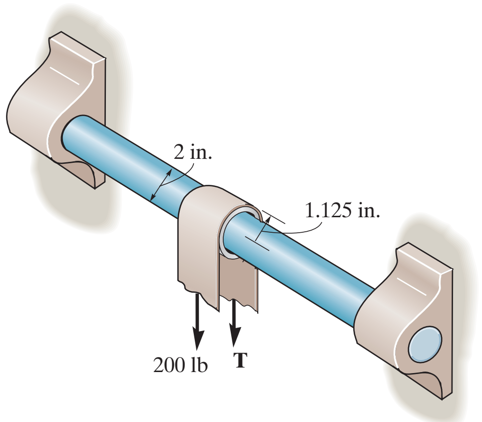
8–121. Solve Prob. 8–120 if the force **P** is applied horizontally to the left.



Probs. 8–120/121

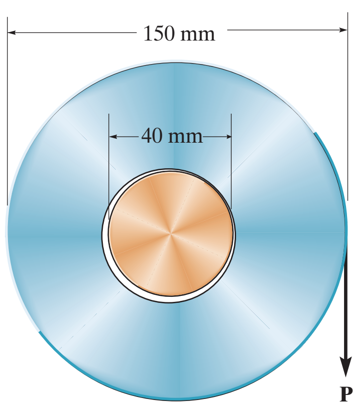
8–122. Determine the tension **T** in the belt needed to overcome the tension of 200 lb created on the other side. Also, what are the normal and frictional components of force developed on the collar bushing? The coefficient of static friction is $\mu_s = 0.21$.

8–123. If a tension force $T = 215$ lb is required to pull the 200-lb force around the collar bushing, determine the coefficient of static friction at the contacting surface. The belt does not slip on the collar.



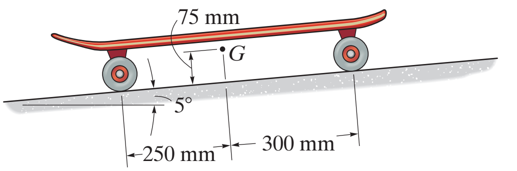
Probs. 8–122/123

***8–124.** The uniform disk fits loosely over a fixed shaft having a diameter of 40 mm. If the coefficient of static friction between the disk and the shaft is $\mu_s = 0.15$, determine the smallest vertical force P, acting on the rim, which must be applied to the disk to cause it to slip on the shaft. The disk has a mass of 20 kg.



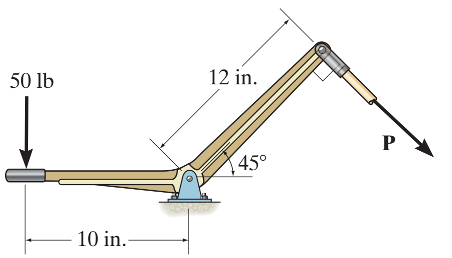
Prob. 8–124

8–125. The 5-kg skateboard rolls down the 5° slope at constant speed. If the coefficient of kinetic friction between the 12.5-mm-diameter axles and the wheels is $\mu_k = 0.3$, determine the radius of the wheels. Neglect rolling resistance of the wheels on the surface. The center of mass for the skateboard is at G.



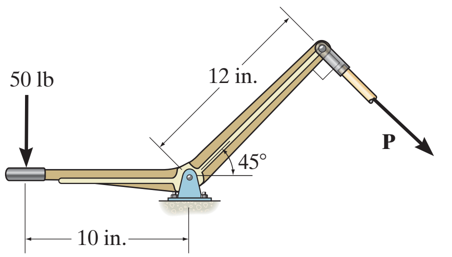
Prob. 8–125

8–126. The bell crank fits loosely into a 0.5-in-diameter pin. Determine the required force P which is just sufficient to rotate the bell crank clockwise. The coefficient of static friction between the pin and the bell crank is $\mu_s = 0.3$.



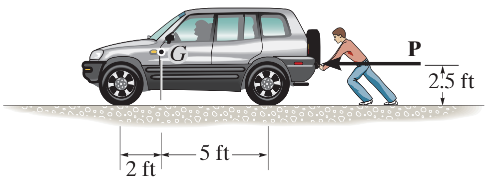
Prob. 8–126

8–127. The bell crank fits loosely into a 0.5-in-diameter pin. If P = 41 lb, the bell crank is then on the verge of rotating counterclockwise. Determine the coefficient of static friction between the pin and the bell crank.



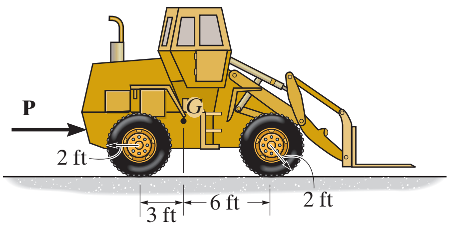
Prob. 8–127

***8–128.** The vehicle has a weight of 2600 lb and center of gravity at G. Determine the horizontal force **P** that must be applied to overcome the rolling resistance of the wheels. The coefficient of rolling resistance is 0.5 in. The tires have a diameter of 2.75 ft.



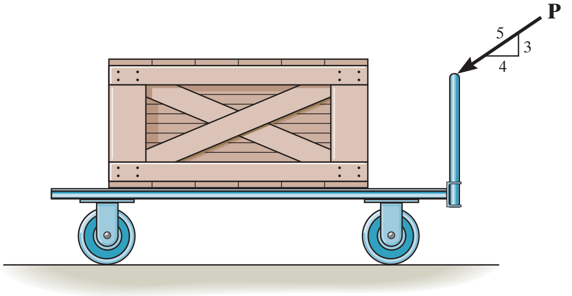
Prob. 8–128

8–129. The tractor has a weight of 16 000 lb and the coefficient of rolling resistance is $a = 2$ in. Determine the force **P** needed to overcome rolling resistance at all four wheels and push it forward.



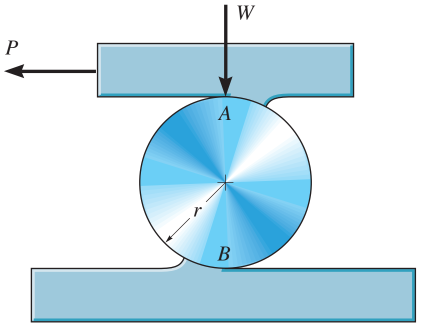
Prob. 8–129

8–130. The handcart has wheels with a diameter of 6 in. If a crate having a weight of 1500 lb is placed on the cart, determine the force P that must be applied to the handle to overcome the rolling resistance. The coefficient of rolling resistance is 0.04 in. Neglect the weight of the cart.



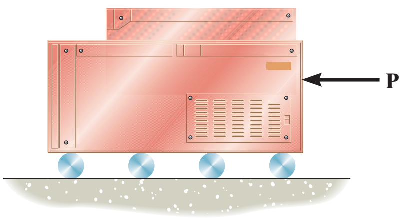
Prob. 8–130

8–131. The cylinder is subjected to a load that has a weight W. If the coefficients of rolling resistance for the cylinder's top and bottom surfaces are a_A and a_B, respectively, show that a horizontal force having a magnitude of $P = [W(a_A + a_B)]/2r$ is required to move the load and thereby roll the cylinder forward. Neglect the weight of the cylinder.



Prob. 8–131

***8–132.** The 1.4-Mg machine is to be moved over a level surface using a series of rollers for which the coefficient of rolling resistance is 0.5 mm at the ground and 0.2 mm at the bottom surface of the machine. Determine the appropriate diameter of the rollers so that the machine can be pushed forward with a horizontal force of $P = 250$ N. *Hint*: Use the result of Prob. 8–131.

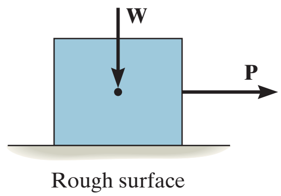


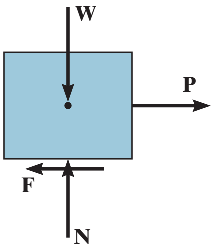
Prob. 8–132

CHAPTER REVIEW

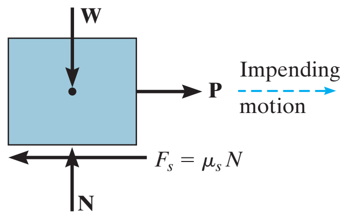
Dry Friction

Frictional forces exist between two rough surfaces of contact. These forces act on a body so as to oppose its motion or tendency of motion.

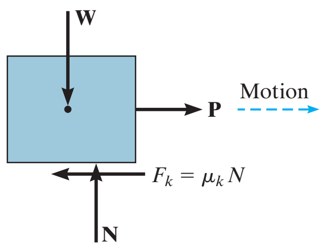




A static frictional force approaches a maximum value of $F_s = \mu_s N$, where μ_s is the *coefficient of static friction*. In this case, motion between the contacting surfaces is *impending*.

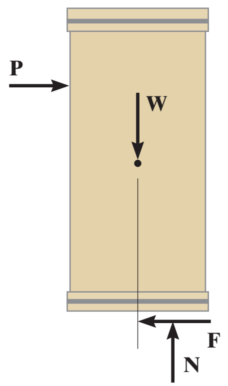


If slipping occurs, then the friction force remains essentially constant and equal to $F_k = \mu_k N$. Here μ_k is the *coefficient of kinetic friction*.

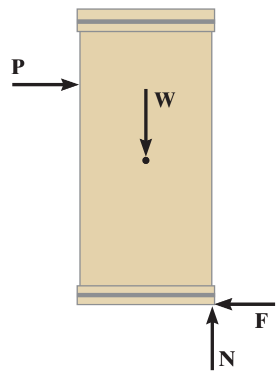


The solution of a problem involving friction requires first drawing the free-body diagram of the body. If the unknowns cannot be determined strictly from the equations of equilibrium, and the possibility of slipping occurs, then the friction equation should be applied at the appropriate points of contact in order to complete the solution.

It may also be possible for slender objects, like crates, to tip over, and this situation should also be investigated.



Impending slipping
$F = \mu_s N$



Tipping

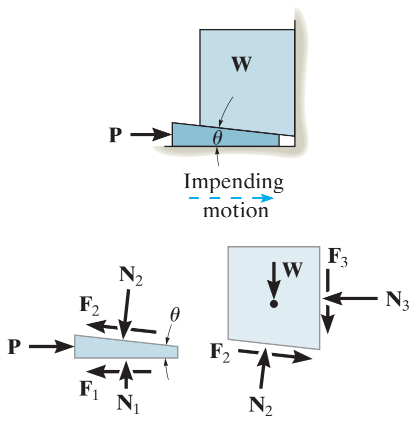
Wedges

Wedges are inclined planes used to increase the application of a force. The two force equilibrium equations are used to relate the forces acting on the wedge.

An applied force **P** must push on the wedge to move it to the right.

If the coefficients of friction between the surfaces are large enough, then **P** can be removed, and the wedge will be self-locking and remain in place.

$$\Sigma F_x = 0$$
$$\Sigma F_y = 0$$



Screws

Square-threaded screws are used to move heavy loads. They represent an inclined plane, wrapped around a cylinder.

The moment needed to turn a screw depends upon the coefficient of friction and the screw's lead angle θ.

If the coefficient of friction between the surfaces is large enough, then the screw will support the load without tending to turn, i.e., it will be self-locking.

$$M = rW\tan(\theta + \phi_s)$$

Upward Impending Screw Motion

$$M' = rW\tan(\theta - \phi_s)$$

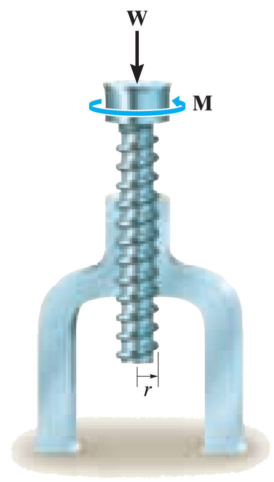
Downward Impending Screw Motion

$$\theta > \phi_s$$

$$M'' = rW\tan(\phi_s - \theta)$$

Downward Screw Motion

$$\phi_s > \theta$$

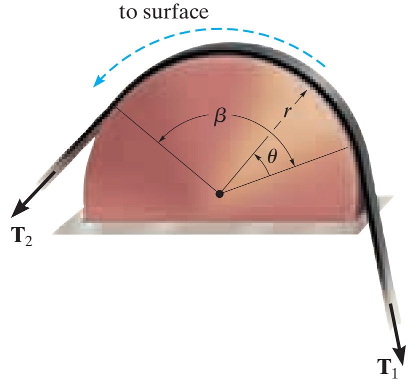


Flat Belts

The force needed to move a flat belt over a rough curved surface depends only on the angle of belt contact, β, and the coefficient of friction.

$$T_2 = T_1 e^{\mu\beta}$$

$$T_2 > T_1$$

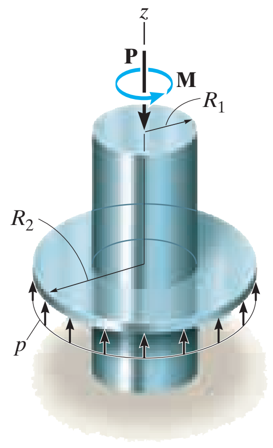


Collar Bearings and Disks

The frictional analysis of a collar bearing or disk requires looking at a differential element of the contact area. The normal force acting on this element is determined from force equilibrium along the shaft, and the moment needed to turn the shaft at a constant rate is determined from moment equilibrium about the shaft's axis.

If the pressure on the surface of a collar bearing is uniform, then integration gives the result shown.

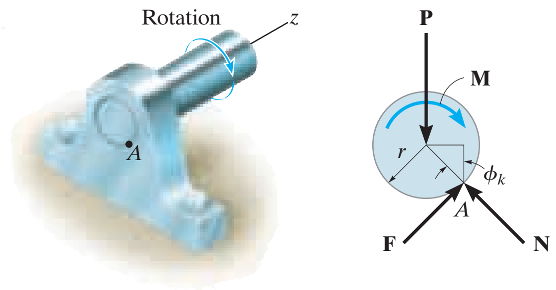
$$M = \frac{2}{3}\mu_s P\left(\frac{R_2^3 - R_1^3}{R_2^2 - R_1^2}\right)$$



Journal Bearings

When a moment is applied to a shaft in a nonlubricated or partially lubricated journal bearing, the shaft will tend to roll up the side of the bearing until slipping occurs. This defines the radius of a friction circle, and from it the moment needed to turn the shaft can be determined.

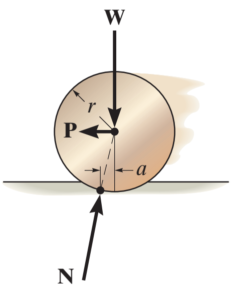
$$M = Rr\sin\phi_k$$



Rolling Resistance

The resistance of a wheel to rolling over a surface is caused by localized *deformation* of the two materials in contact. This causes the resultant normal force acting on the rolling body to be inclined so that it provides a component that acts in the opposite direction of the applied force **P** causing the motion. This effect is characterized using the *coefficient of rolling resistance*, *a*, which is determined from experiment.

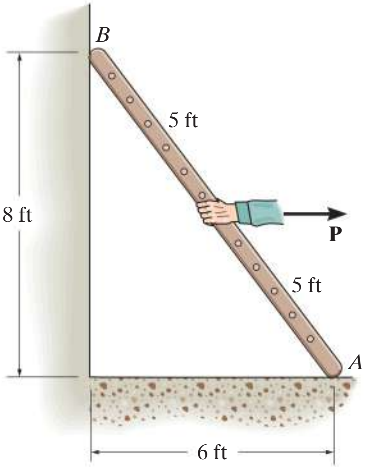
$$P \approx \frac{Wa}{r}$$



REVIEW PROBLEMS

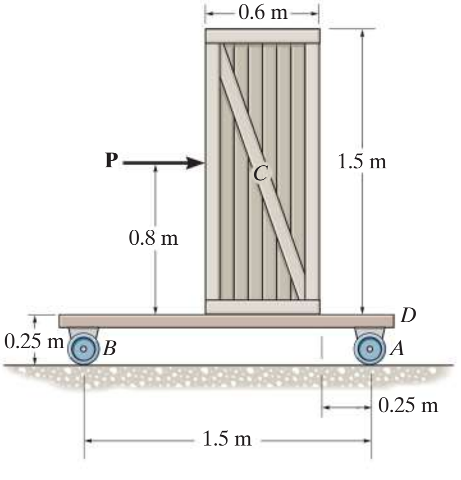
All problem solutions must include FBDs.

R8–1. The uniform 20-lb ladder rests on the rough floor for which the coefficient of static friction is $\mu_s = 0.4$ and against the smooth wall at B. Determine the horizontal force P the man must exert on the ladder in order to cause it to move.



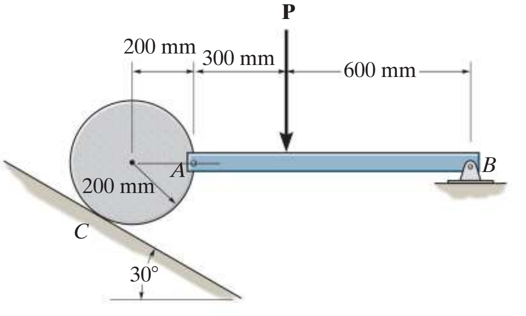
Prob. R8–1

R8–2. The uniform 60-kg crate C rests uniformly on a 10-kg dolly D. If the front casters of the dolly at A are locked to prevent rolling while the casters at B are free to roll, determine the maximum force **P** that may be applied without causing motion of the crate. The coefficient of static friction between the casters and the floor is $\mu_f = 0.35$ and between the dolly and the crate, $\mu_d = 0.5$.



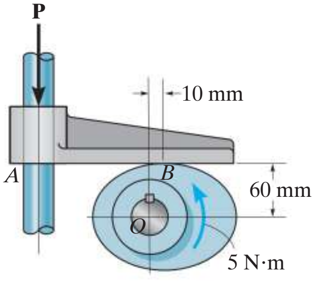
Prob. R8–2

R8–3. A 35-kg disk rests on an inclined surface for which $\mu_s = 0.2$. Determine the maximum vertical force **P** that may be applied to bar AB without causing the disk to slip at C. Neglect the mass of the bar.



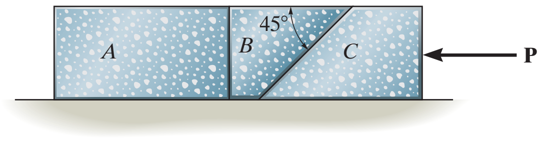
Prob. R8–3

R8–4. The cam is subjected to a couple moment of 5 N·m. Determine the minimum force **P** that should be applied to the follower in order to hold the cam in the position shown. The coefficient of static friction between the cam and the follower is $\mu = 0.4$. The guide at **A** is smooth.



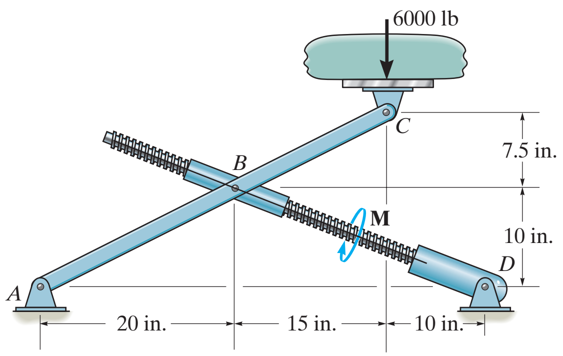
Prob. R8–4

R8–5. The three stone blocks have weights of $W_A = 600$ lb, $W_B = 150$ lb, and $W_C = 500$ lb. Determine the smallest horizontal force P that must be applied to block C in order to move this block. The coefficient of static friction between the blocks is $\mu_s = 0.3$, and between the floor and each block $\mu_s' = 0.5$.



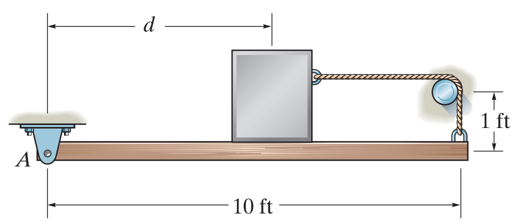
Prob. R8–5

R8–6. The jacking mechanism consists of a link that has a square-threaded screw with a mean diameter of 0.5 in. and a lead of 0.20 in., and the coefficient of static friction is $\mu_s = 0.4$. Determine the torque M that should be applied to the screw to start lifting the 6000-lb load acting at the end of member ABC.



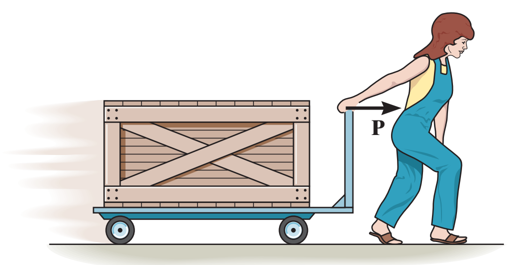
Prob. R8–6

R8–7. The uniform 50-lb beam is supported by the rope that is attached to the end of the beam, wraps over the rough peg, and is then connected to the 100-lb block. If the coefficient of static friction between the beam and the block, and between the rope and the peg, is $\mu_s = 0.4$, determine the maximum distance that the block can be placed from A and still remain in equilibrium. Assume the block will not tip.



Prob. R8–7

R8–8. The hand cart has wheels with a diameter of 80 mm. If a crate having a mass of 500 kg is placed on the cart so that each wheel carries an equal load, determine the horizontal force P that must be applied to the handle to overcome the rolling resistance. The coefficient of rolling resistance is 2 mm. Neglect the mass of the cart.



Prob. R8–8

Chapter 9

(© Heather Reeder/Shutterstock)

When a tank of any shape is designed, it is important to be able to determine its center of gravity, calculate its volume and surface area, and determine the forces of the liquids they contain. These topics will be covered in this chapter.

Center of Gravity and Centroid

CHAPTER OBJECTIVES

- To discuss the concept of the center of gravity, center of mass, and the centroid.
- To show how to determine the location of the center of gravity and centroid for a body of arbitrary shape and one composed of composite parts.
- To use the theorems of Pappus and Guldinus for finding the surface area and volume for a body having axial symmetry.
- To present a method for finding the resultant of a general distributed loading and to show how it applies to finding the resultant force of a pressure loading caused by a fluid.

9.1 Center of Gravity, Center of Mass, and the Centroid of a Body

Knowing the resultant or total weight of a body and its location is important when considering the effect this force produces on the body. The point of location is called the center of gravity, and in this section we will show how to find it for an irregularly shaped body. We will then extend this method to show how to find the body's center of mass, and its geometric center or centroid.

Center of Gravity. A body is composed of an infinite number of particles of differential size, and so if the body is located within a gravitational field, then each of these particles will have a weight dW. These weights will form a parallel force system, and the resultant of this system is the total weight of the body, which passes through a single point called the ***center of gravity***, G*.

*In a strict sense this is true as long as the gravity field is assumed to have the same magnitude and direction everywhere. Although the actual force of gravity is directed toward the center of the earth, and this force varies with its distance from the center, for most engineering applications we can assume the gravity field has the same magnitude and direction everywhere.

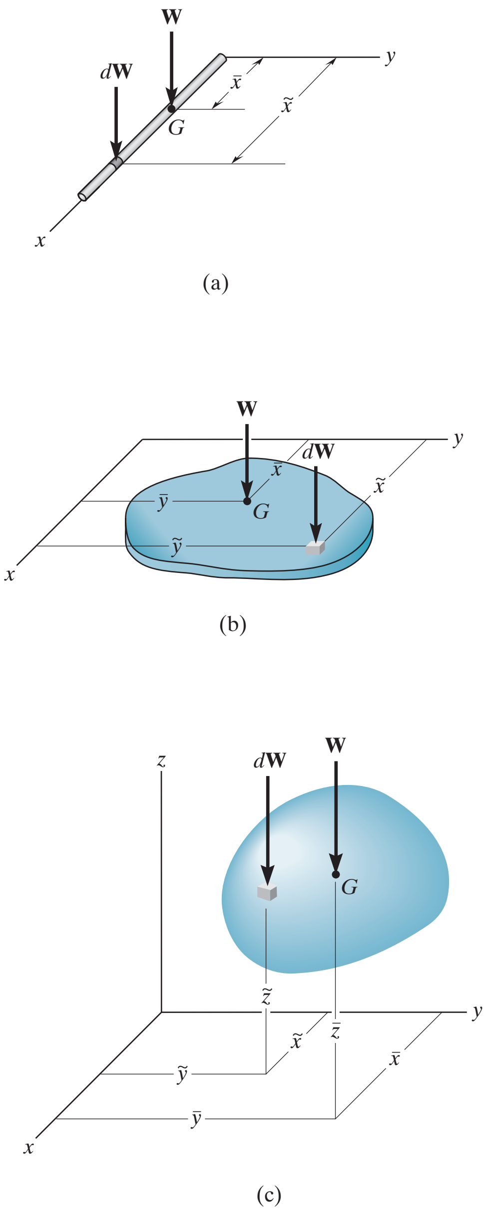


Fig. 9–1

To show how to determine the location of the center of gravity, consider the rod in Fig. 9–1*a*, where the segment having the weight dW is located at the arbitrary position $\tilde{x}$. Using the methods outlined in Sec. 4.8, the total weight of the rod is the sum of the weights of all of its particles, that is

$$+\downarrow F_R = \Sigma F_z; \qquad W = \int dW$$

The location of the center of gravity, measured from the y axis, is determined by equating the moment of W about the y axis, Fig. 9–1*b*, to the sum of the moments of the weights of all its particles about this same axis. Therefore,

$$(M_R)_y = \Sigma M_y; \qquad \bar{x}W = \int \tilde{x}dW$$

$$\bar{x} = \frac{\int \tilde{x}\,dW}{\int dW}$$

In a similar manner, if the body represents a plate, Fig. 9–1*b*, then a moment balance about the x and y axes would be required to determine the location $(\bar{x}, \bar{y})$ of point G. Finally we can generalize this idea to a three-dimensional body, Fig. 9–1*c*, and perform a moment balance about all three axes to locate G for any rotated position of the axes. This results in the following equations.

$$\bar{x} = \frac{\int \tilde{x}\,dW}{\int dW} \qquad \bar{y} = \frac{\int \tilde{y}\,dW}{\int dW} \qquad \bar{z} = \frac{\int \tilde{z}\,dW}{\int dW} \tag{9–1}$$

where

$\bar{x}, \bar{y}, \bar{z}$ are the coordinates of the center of gravity G.
$\tilde{x}, \tilde{y}, \tilde{z}$ are the coordinates of an arbitrary particle in the body.

Center of Mass of a Body. In order to study the *dynamic response* or accelerated motion of a body, it becomes important to locate the body's ***center of mass*** C_m, Fig. 9–2. This location can be determined by substituting $dW = g\,dm$ into Eqs. 9–1. Provided g is constant, it cancels out, and so

$$\bar{x} = \frac{\int \tilde{x}\,dm}{\int dm} \qquad \bar{y} = \frac{\int \tilde{y}\,dm}{\int dm} \qquad \bar{z} = \frac{\int \tilde{z}\,dm}{\int dm} \tag{9–2}$$

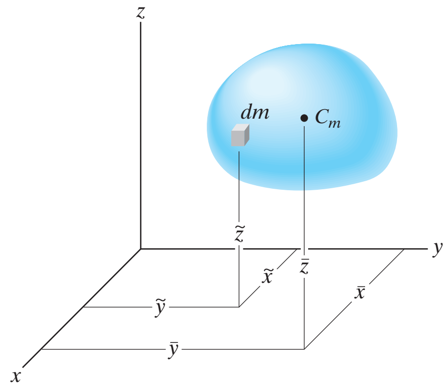


Fig. 9–2

Centroid of a Volume. If the body in Fig. 9–3 is made from a *homogeneous material*, then its density ρ (rho) will be *constant*. Therefore, a differential element of volume dV has a mass $dm = \rho\,dV$. Substituting this into Eqs. 9–2 and canceling out ρ, we obtain formulas that locate the ***centroid*** C or geometric center of the body; namely

$$\bar{x} = \frac{\int_V \tilde{x}\,dV}{\int_V dV} \qquad \bar{y} = \frac{\int_V \tilde{y}\,dV}{\int_V dV} \qquad \bar{z} = \frac{\int_V \tilde{z}\,dV}{\int_V dV} \tag{9–3}$$

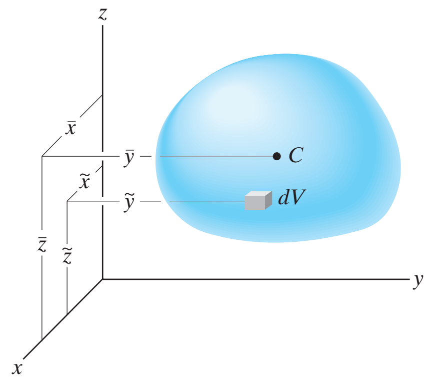


Fig. 9–3

These equations represent a balance of the moments of the volume of the body. Therefore, if the volume possesses two planes of symmetry, then its centroid must lie along the line of intersection of these two planes. For example, the cone in Fig. 9–4 has a centroid that lies on the y axis so that $\bar{x} = \bar{z} = 0$. The location $\bar{y}$ can be found using a single integration by choosing a differential element represented by a *thin disk* having a thickness dy and radius $r = z$. Its volume is $dV = \pi r^2\,dy = \pi z^2\,dy$ and its centroid is at $\tilde{x} = 0, \tilde{y} = y, \tilde{z} = 0$.

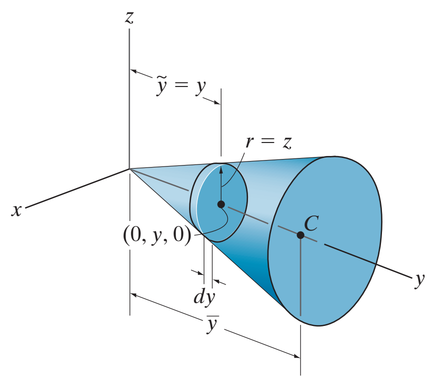


Fig. 9–4

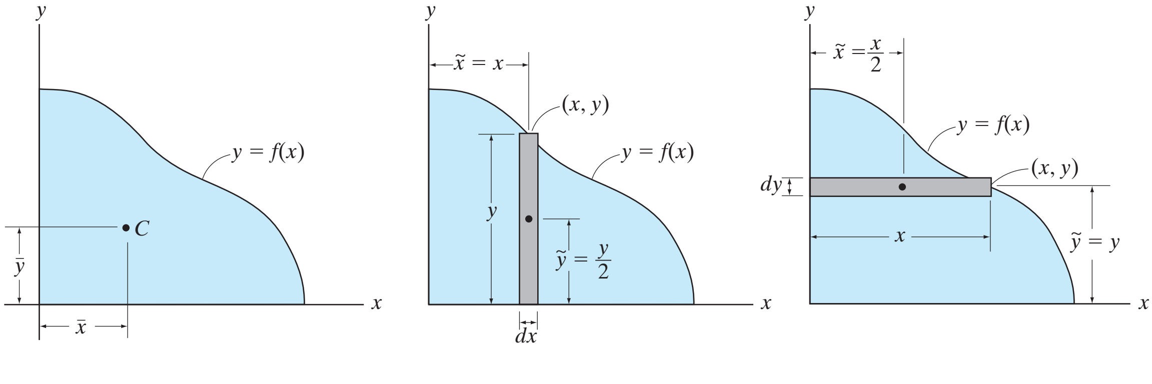


Fig. 9–5

Integration must be used to determine the location of the center of gravity of this lamp post due to the curvature of the member. (© Russell C. Hibbeler)

Centroid of an Area.

If an area lies in the x–y plane and is bounded by the curve $y = f(x)$, as shown in Fig. 9–5*a*, then its centroid will be in this plane and can be determined from integrals similar to Eqs. 9–3, namely,

$$\bar{x} = \frac{\int_A \tilde{x}\, dA}{\int_A dA} \qquad \bar{y} = \frac{\int_A \tilde{y}\, dA}{\int_A dA} \tag{9–4}$$

These integrals can be evaluated by performing a *single integration* if we use a *rectangular strip* for the differential area element. For example, if a vertical strip is used, Fig. 9–5*b*, the area of the element is $dA = y\,dx$, and its centroid is located at $\tilde{x} = x$ and $\tilde{y} = y/2$. If we consider a horizontal strip, Fig. 9–5*c*, then $dA = x\,dy$, and its centroid is located at $\tilde{x} = x/2$ and $\tilde{y} = y$.

Centroid of a Line.

If a line segment (or rod) lies within the x–y plane and it can be described by a thin curve $y = f(x)$, Fig. 9–6*a*, then its centroid is determined from

$$\bar{x} = \frac{\int_L \tilde{x}\, dL}{\int_L dL} \qquad \bar{y} = \frac{\int_L \tilde{y}\, dL}{\int_L dL} \tag{9–5}$$

Here, the length of the differential element is given by the Pythagorean theorem, $dL = \sqrt{(dx)^2 + (dy)^2}$, which can also be written in the form

$$dL = \sqrt{\left(\frac{dx}{dx}\right)^2 dx^2 + \left(\frac{dy}{dx}\right)^2 dx^2}$$

$$= \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right) dx$$

or

$$dL = \sqrt{\left(\frac{dx}{dy}\right)^2 dy^2 + \left(\frac{dy}{dy}\right)^2 dy^2}$$

$$= \left(\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\right) dy$$

Either one of these expressions can be used; however, for application, the one that will result in a simpler integration should be selected. For example, consider the rod in Fig. 9–6*b*, defined by $y = 2x^2$. The length of the element is $dL = \sqrt{1 + (dy/dx)^2}\, dx$, and since $dy/dx = 4x$, then $dL = \sqrt{1 + (4x)^2}\, dx$. The centroid for this element is located at $\tilde{x} = x$ and $\tilde{y} = y$.

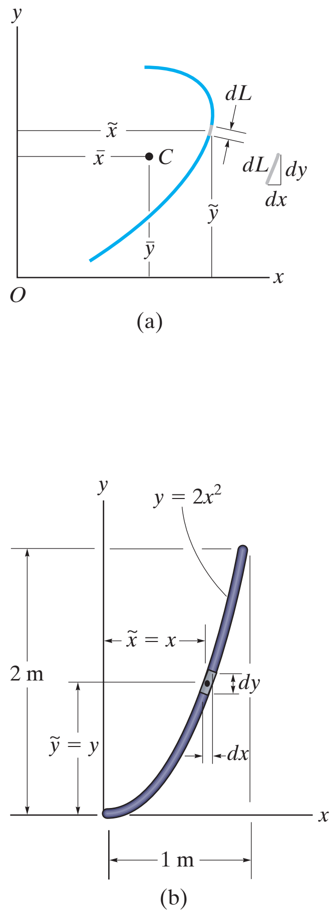


Fig. 9–6

Important Points

- The centroid represents the geometric center of a body. This point coincides with the center of mass or the center of gravity only if the material composing the body is uniform or homogeneous.
- Formulas used to locate the center of gravity or the centroid simply represent a balance between the sum of moments of all the parts of the system and the moment of the "resultant" for the system.
- In some cases the centroid is located at a point that is not on the object, as in the case of a ring, where the centroid is at its center. Also, this point will lie on any axis of symmetry for the body, Fig. 9–7.

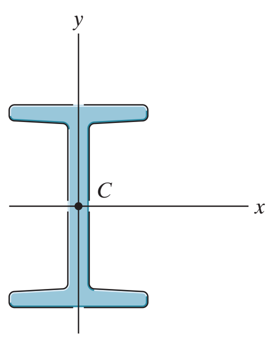


Fig. 9–7

Procedure for Analysis

The center of gravity or centroid of an object or shape can be determined by single integrations using the following procedure.

Differential Element.

- Select an appropriate coordinate system, specify the coordinate axes, and then choose a differential element for integration.
- For lines the element is represented by a differential line segment of length dL.
- For areas the element is generally a rectangle of area dA, having a finite length and differential width.
- For volumes the element can be a circular disk of volume dV, having a finite radius and differential thickness.
- Locate the element so that it touches the arbitrary point (x, y, z) on the curve that defines the boundary of the shape.

Size and Moment Arms.

- Express the length dL, area dA, or volume dV of the element in terms of the coordinates describing the curve.
- Express the moment arms $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ for the centroid or center of gravity of the element in terms of the coordinates describing the curve.

Integrations.

- Substitute the formulations for $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ and dL, dA, or dV into the appropriate equations (Eqs. 9–1 through 9–5).
- Express the function in the integrand in terms of the *same variable as the differential thickness of the element*.
- The limits of the integral are defined from the two extreme locations of the element's differential thickness, so that when the elements are "summed" or the integration performed, the entire region is covered.

EXAMPLE 9.1

Locate the centroid of the rod bent into the shape of a parabolic arc as shown in Fig. 9–8.

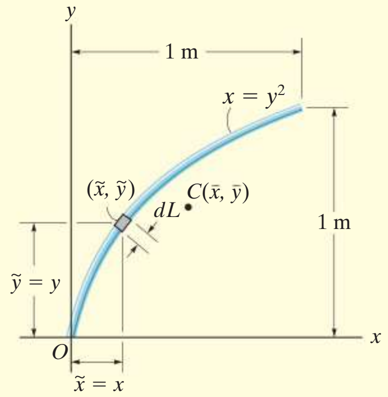


Fig. 9–8

SOLUTION

Differential Element. The differential element is shown in Fig. 9–8. It is located on the curve at the *arbitrary point* (x, y).

Area and Moment Arms. The differential element of length dL can be expressed in terms of the differentials dx and dy using the Pythagorean theorem.

$$dL = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$$

Since $x = y^2$, then $dx/dy = 2y$. Therefore, expressing dL in terms of y and dy, we have

$$dL = \sqrt{(2y)^2 + 1}\, dy$$

As shown in Fig. 9–8, the centroid of the element is located at $\tilde{x} = x$, $\tilde{y} = y$.

Integrations. Applying Eq. 9–5 and using the integration formula to evaluate the integrals, we get

$$\bar{x} = \frac{\int_L \tilde{x}\, dL}{\int_L dL} = \frac{\int_0^{1\text{ m}} x\sqrt{4y^2 + 1}\, dy}{\int_0^{1\text{ m}} \sqrt{4y^2 + 1}\, dy} = \frac{\int_0^{1\text{ m}} y^2\sqrt{4y^2 + 1}\, dy}{\int_0^{1\text{ m}} \sqrt{4y^2 + 1}\, dy}$$

$$= \frac{0.6063}{1.479} = 0.410 \text{ m} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\int_L \tilde{y}\, dL}{\int_L dL} = \frac{\int_0^{1\text{ m}} y\sqrt{4y^2 + 1}\, dy}{\int_0^{1\text{ m}} \sqrt{4y^2 + 1}\, dy} = \frac{0.8484}{1.479} = 0.574 \text{ m} \qquad \textit{Ans.}$$

NOTE: These results for C seem reasonable when they are plotted on Fig. 9–8.

9

EXAMPLE 9.2

Locate the centroid of the circular wire segment shown in Fig. 9–9.

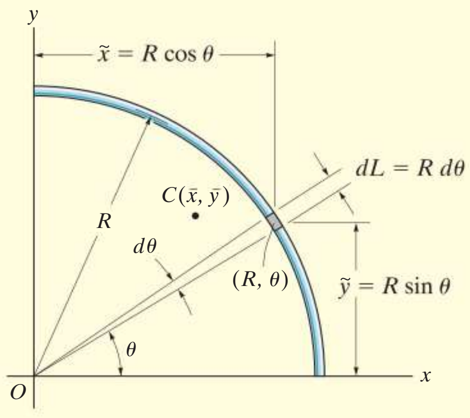


Fig. 9–9

SOLUTION

Polar coordinates will be used to solve this problem since the arc is circular.

Differential Element. A differential circular arc is selected as shown in the figure. This element lies on the curve at (R, θ).

Length and Moment Arm. The length of the differential element is $dL = R\,d\theta$, and its centroid is located at $\tilde{x} = R\cos\theta$ and $\tilde{y} = R\sin\theta$.

Integrations. Applying Eqs. 9–5 and integrating with respect to θ, we obtain

$$\bar{x} = \frac{\int_L \tilde{x}\,dL}{\int_L dL} = \frac{\int_0^{\pi/2} (R\cos\theta)R\,d\theta}{\int_0^{\pi/2} R\,d\theta} = \frac{R^2\int_0^{\pi/2}\cos\theta\,d\theta}{R\int_0^{\pi/2} d\theta} = \frac{2R}{\pi} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\int_L \tilde{y}\,dL}{\int_L dL} = \frac{\int_0^{\pi/2} (R\sin\theta)R\,d\theta}{\int_0^{\pi/2} R\,d\theta} = \frac{R^2\int_0^{\pi/2}\sin\theta\,d\theta}{R\int_0^{\pi/2} d\theta} = \frac{2R}{\pi} \qquad \textit{Ans.}$$

NOTE: As expected, the two coordinates are numerically the same due to the symmetry of the wire.

EXAMPLE 9.3

Determine the distance $\bar{y}$ measured from the x axis to the centroid of the area of the triangle shown in Fig. 9–10.

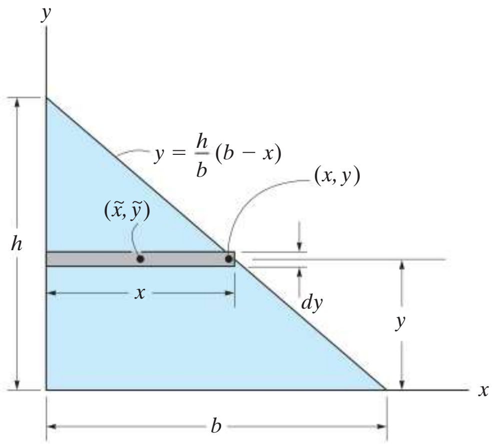


Fig. 9–10

SOLUTION

Differential Element. Consider a rectangular element having a thickness dy, and located in an arbitrary position so that it intersects the boundary at (x, y), Fig. 9–10.

Area and Moment Arms. The area of the element is $dA = x\,dy = \dfrac{b}{h}(h - y)\,dy$, and its centroid is located a distance $\tilde{y} = y$ from the x axis.

Integration. Applying the second of Eqs. 9–4 and integrating with respect to y yields

$$\bar{y} = \frac{\displaystyle\int_A \tilde{y}\,dA}{\displaystyle\int_A dA} = \frac{\displaystyle\int_0^h y\left[\frac{b}{h}(h - y)\,dy\right]}{\displaystyle\int_0^h \frac{b}{h}(h - y)\,dy} = \frac{\frac{1}{6}bh^2}{\frac{1}{2}bh}$$

$$= \frac{h}{3} \qquad \textit{Ans.}$$

NOTE: This result is valid for any shape of triangle. It states that the centroid is located at one-third the height, measured from the base of the triangle.

EXAMPLE 9.4

Locate the centroid for the area of a quarter circle shown in Fig. 9–11.

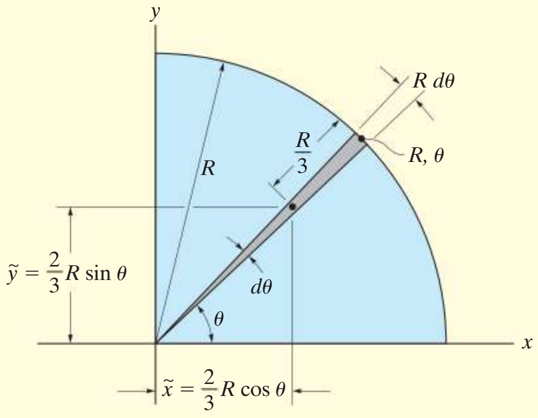


Fig. 9–11

SOLUTION

Differential Element. Polar coordinates will be used, since the boundary is circular. We choose the element in the shape of a *triangle*, Fig. 9–11. (Actually the shape is a circular sector; however, neglecting higher-order differentials, the element becomes triangular.) The element intersects the curve at point (R, θ).

Area and Moment Arms. The area of the element is

$$dA = \tfrac{1}{2}(R)(R\,d\theta) = \frac{R^2}{2}d\theta$$

and using the results of Example 9.3, the centroid of the (triangular) element is located at $\tilde{x} = \frac{2}{3}R\cos\theta$, $\tilde{y} = \frac{2}{3}R\sin\theta$.

Integrations. Applying Eqs. 9–4 and integrating with respect to θ, we obtain

$$\bar{x} = \frac{\int_A \tilde{x}\,dA}{\int_A dA} = \frac{\int_0^{\pi/2}\left(\frac{2}{3}R\cos\theta\right)\frac{R^2}{2}d\theta}{\int_0^{\pi/2}\frac{R^2}{2}d\theta} = \frac{\left(\frac{2}{3}R\right)\int_0^{\pi/2}\cos\theta\,d\theta}{\int_0^{\pi/2}d\theta} = \frac{4R}{3\pi} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^{\pi/2}\left(\frac{2}{3}R\sin\theta\right)\frac{R^2}{2}d\theta}{\int_0^{\pi/2}\frac{R^2}{2}d\theta} = \frac{\left(\frac{2}{3}R\right)\int_0^{\pi/2}\sin\theta\,d\theta}{\int_0^{\pi/2}d\theta} = \frac{4R}{3\pi} \qquad \textit{Ans.}$$

EXAMPLE 9.5

Locate the centroid of the area shown in Fig. 9–12*a*.

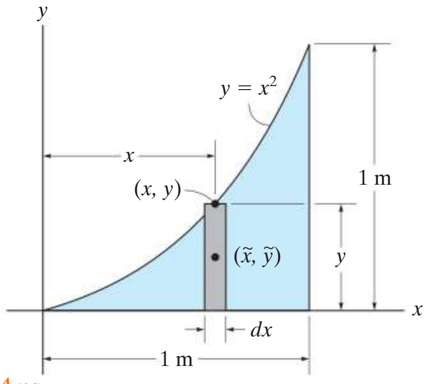
SOLUTION I

Differential Element. A differential element of thickness dx is shown in Fig. 9–12*a*. The element intersects the curve at the *arbitrary point* (x, y), and so it has a height y.

Area and Moment Arms. The area of the element is $dA = y\,dx$, and its centroid is located at $\tilde{x} = x$, $\tilde{y} = y/2$.

Integrations. Applying Eqs. 9–4 and integrating with respect to x yields

$$\bar{x} = \frac{\int_A \tilde{x}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} xy\,dx}{\int_0^{1\text{ m}} y\,dx} = \frac{\int_0^{1\text{ m}} x^3\,dx}{\int_0^{1\text{ m}} x^2\,dx} = \frac{0.250}{0.333} = 0.75\text{ m} \qquad \textit{Ans.}$$



(a)

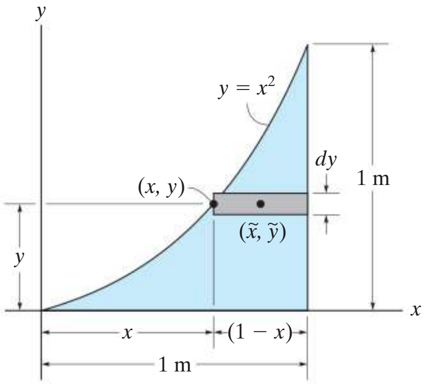
$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} (y/2)y\,dx}{\int_0^{1\text{ m}} y\,dx} = \frac{\int_0^{1\text{ m}} (x^2/2)x^2\,dx}{\int_0^{1\text{ m}} x^2\,dx} = \frac{0.100}{0.333} = 0.3\text{ m} \qquad \textit{Ans.}$$

SOLUTION II

Differential Element. The differential element of thickness dy is shown in Fig. 9–12*b*. The element intersects the curve at the *arbitrary point* (x, y), and so it has a length $(1 - x)$.

Area and Moment Arms. The area of the element is $dA = (1 - x)\,dy$, and its centroid is located at

$$\tilde{x} = x + \left(\frac{1-x}{2}\right) = \frac{1+x}{2},\ \tilde{y} = y$$



(b)

Fig. 9–12

Integrations. Applying Eqs. 9–4 and integrating with respect to y, we obtain

$$\bar{x} = \frac{\int_A \tilde{x}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} [(1+x)/2](1-x)\,dy}{\int_0^{1\text{ m}} (1-x)\,dy} = \frac{\frac{1}{2}\int_0^{1\text{ m}} (1-y)\,dy}{\int_0^{1\text{ m}} (1-\sqrt{y})\,dy} = \frac{0.250}{0.333} = 0.75\text{ m} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ m}} y(1-x)\,dy}{\int_0^{1\text{ m}} (1-x)\,dy} = \frac{\int_0^{1\text{ m}} (y - y^{3/2})\,dy}{\int_0^{1\text{ m}} (1-\sqrt{y})\,dy} = \frac{0.100}{0.333} = 0.3\text{ m} \qquad \textit{Ans.}$$

NOTE: Plot these results and notice that they seem reasonable. Also, for this problem, elements of thickness dx offer a simpler solution.

Locate the centroid of the semi-elliptical area shown in Fig. 9–13*a*.

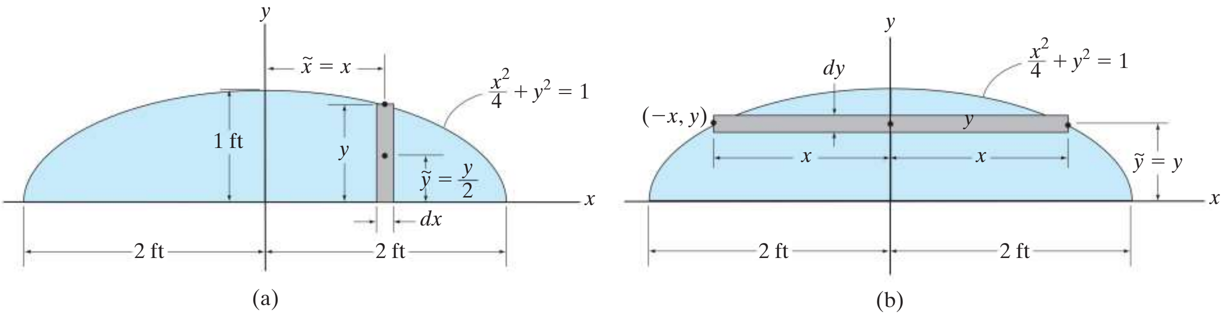


Fig. 9–13

SOLUTION I

Differential Element. The rectangular differential element parallel to the y axis shown shaded in Fig. 9–13*a* will be considered. This element has a thickness of dx and a height of y.

Area and Moment Arms. Thus, the area is $dA = y\,dx$, and its centroid is located at $\tilde{x} = x$ and $\tilde{y} = y/2$.

Integration. Since the area is symmetrical about the y axis,

$$\bar{x} = 0 \qquad \textit{Ans.}$$

Applying the second of Eqs. 9–4 with $y = \sqrt{1 - \dfrac{x^2}{4}}$, we have

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_{-2\text{ ft}}^{2\text{ ft}} \frac{y}{2}(y\,dx)}{\int_{-2\text{ ft}}^{2\text{ ft}} y\,dx} = \frac{\frac{1}{2}\int_{-2\text{ ft}}^{2\text{ ft}}\left(1 - \frac{x^2}{4}\right)dx}{\int_{-2\text{ ft}}^{2\text{ ft}}\sqrt{1 - \frac{x^2}{4}}\,dx} = \frac{4/3}{\pi} = 0.424\text{ ft} \qquad \textit{Ans.}$$

SOLUTION II

Differential Element. The shaded rectangular differential element of thickness dy and width $2x$, parallel to the x axis, will be considered, Fig. 9–13*b*.

Area and Moment Arms. The area is $dA = 2x\,dy$, and its centroid is at $\tilde{x} = 0$ and $\tilde{y} = y$.

Integration. Applying the second of Eqs. 9–4, with $x = 2\sqrt{1 - y^2}$, we have

$$\bar{y} = \frac{\int_A \tilde{y}\,dA}{\int_A dA} = \frac{\int_0^{1\text{ ft}} y(2x\,dy)}{\int_0^{1\text{ ft}} 2x\,dy} = \frac{\int_0^{1\text{ ft}} 4y\sqrt{1 - y^2}\,dy}{\int_0^{1\text{ ft}} 4\sqrt{1 - y^2}\,dy} = \frac{4/3}{\pi}\text{ ft} = 0.424\text{ ft} \qquad \textit{Ans.}$$

EXAMPLE 9.7

Locate the $\bar{y}$ centroid for the paraboloid of revolution, shown in Fig. 9–14.

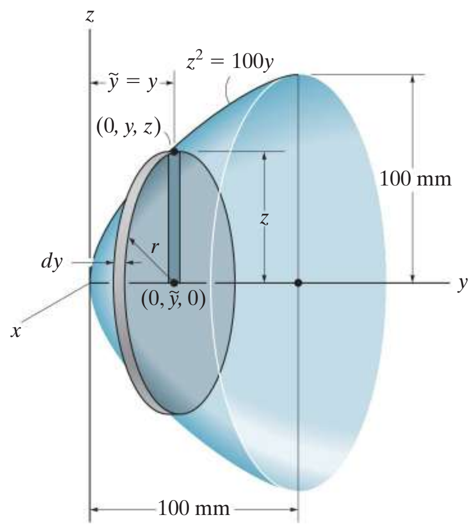


Fig. 9–14

SOLUTION

Differential Element. An element having the shape of a *thin disk* is chosen. This element has a thickness dy, it intersects the generating curve at the *arbitrary point* $(0, y, z)$, and so its radius is $r = z$.

Volume and Moment Arm. The volume of the element is $dV = (\pi z^2)\, dy$, and its centroid is located at $\tilde{y} = y$.

Integration. Applying the second of Eqs. 9–3 and integrating with respect to y yields.

$$\bar{y} = \frac{\int_V \tilde{y}\, dV}{\int_V dV} = \frac{\int_0^{100\text{ mm}} y(\pi z^2)\, dy}{\int_0^{100\text{ mm}} (\pi z^2)\, dy} = \frac{100\pi \int_0^{100\text{ mm}} y^2\, dy}{100\pi \int_0^{100\text{ mm}} y\, dy} = 66.7\text{ mm} \qquad \textit{Ans.}$$

Determine the location of the center of mass of the cylinder shown in Fig. 9–15 if its density varies directly with the distance from its base, i.e., $\rho = 200z \text{ kg/m}^3$.

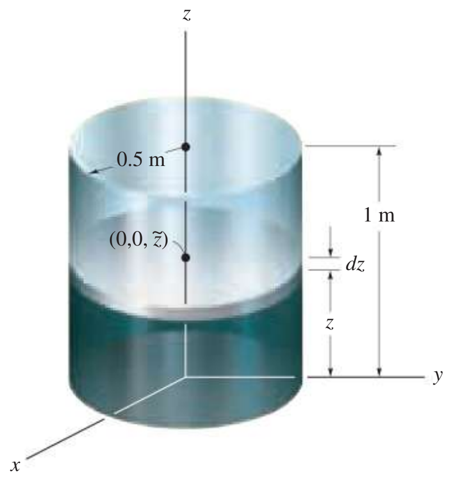


Fig. 9–15

SOLUTION

For reasons of material symmetry,

$$\bar{x} = \bar{y} = 0 \qquad \textit{Ans.}$$

Differential Element. A disk element of radius 0.5 m and thickness dz is chosen for integration, Fig. 9–15, since the *density of the entire element is constant* for a given value of z. The element is located along the z axis at the *arbitrary point* $(0, 0, z)$.

Volume and Moment Arm. The volume of the element is $dV = \pi(0.5)^2\, dz$, and its centroid is located at $\tilde{z} = z$.

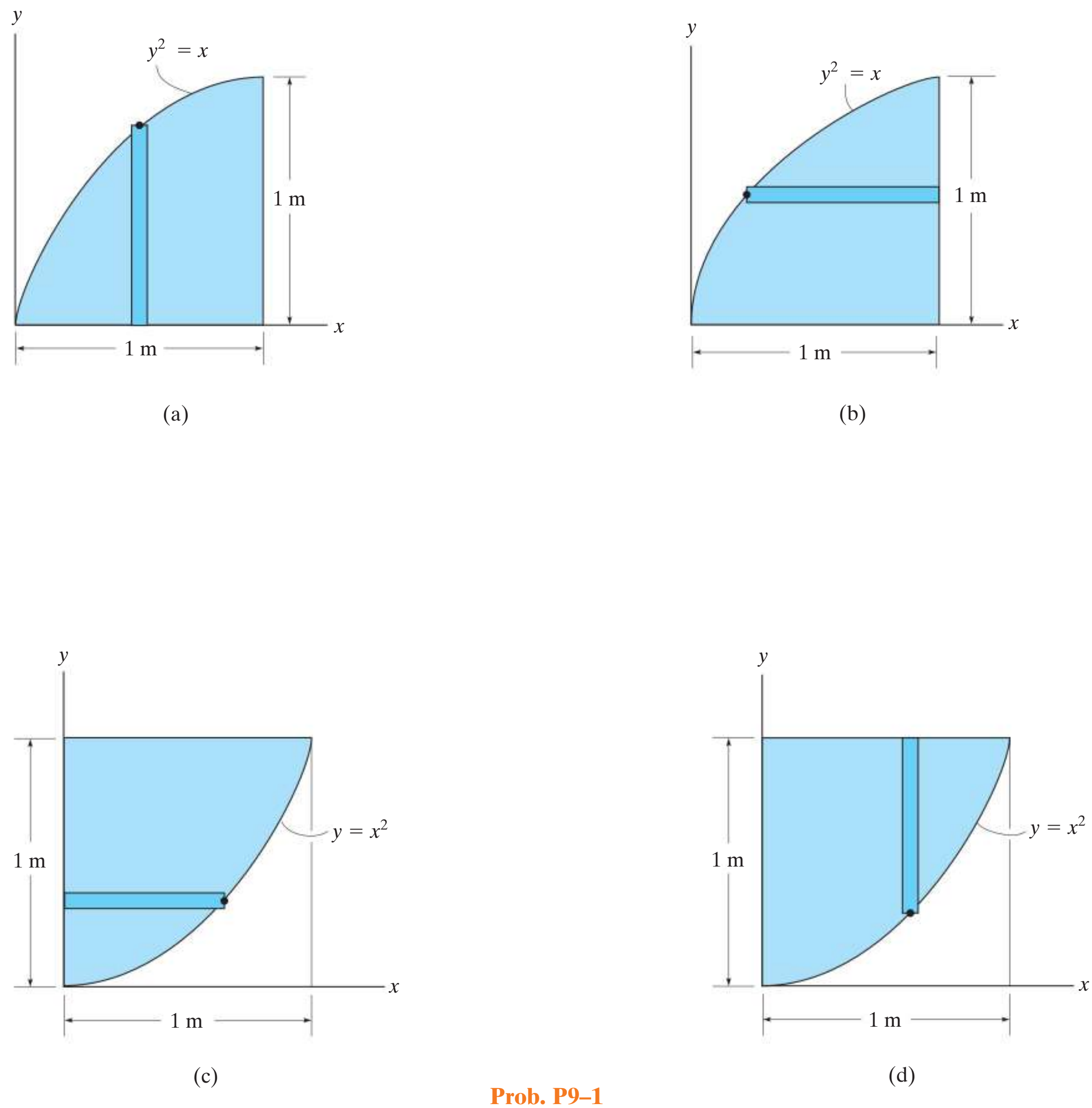
Integrations. Using the third of Eqs. 9–2 with $dm = \rho\, dV$ and integrating with respect to z, noting that $\rho = 200z$, we have

$$\bar{z} = \frac{\displaystyle\int_V \tilde{z}\rho\, dV}{\displaystyle\int_V \rho\, dV} = \frac{\displaystyle\int_0^{1\text{ m}} z(200z)\left[\pi(0.5)^2\, dz\right]}{\displaystyle\int_0^{1\text{ m}} (200z)\pi(0.5)^2\, dz}$$

$$= \frac{\displaystyle\int_0^{1\text{ m}} z^2\, dz}{\displaystyle\int_0^{1\text{ m}} z\, dz} = 0.667 \text{ m} \qquad \textit{Ans.}$$

PRELIMINARY PROBLEM

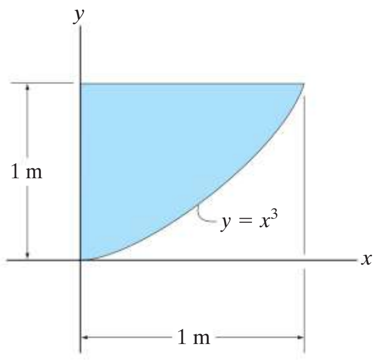
P9–1. In each case, use the element shown and specify $\tilde{x}$, $\tilde{y}$, and dA.



Prob. P9–1

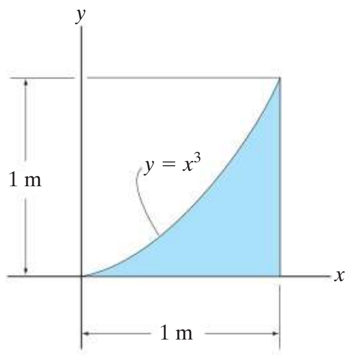
FUNDAMENTAL PROBLEMS

F9–1. Determine the centroid $(\bar{x}, \bar{y})$ of the shaded area.



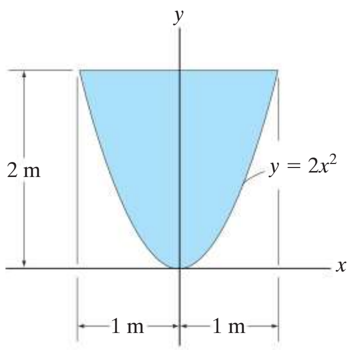
Prob. F9–1

F9–2. Determine the centroid $(\bar{x}, \bar{y})$ of the shaded area.



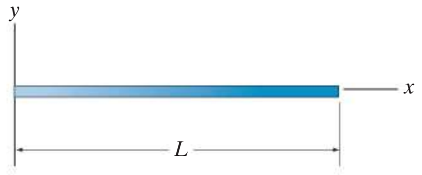
Prob. F9–2

F9–3. Determine the centroid $\bar{y}$ of the shaded area.



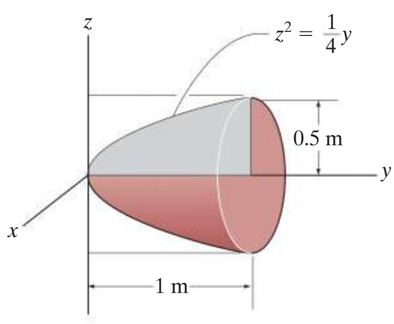
Prob. F9–3

F9–4. Locate the center of mass $\bar{x}$ of the straight rod if its mass per unit length is given by $m = m_0(1 + x^2/L^2)$.



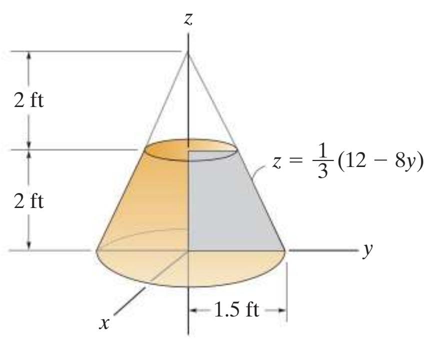
Prob. F9–4

F9–5. Locate the centroid $\bar{y}$ of the homogeneous solid formed by revolving the shaded area about the y axis.



Prob. F9–5

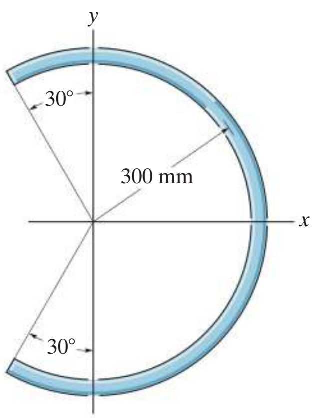
F9–6. Locate the centroid $\bar{z}$ of the homogeneous solid formed by revolving the shaded area about the z axis.



Prob. F9–6

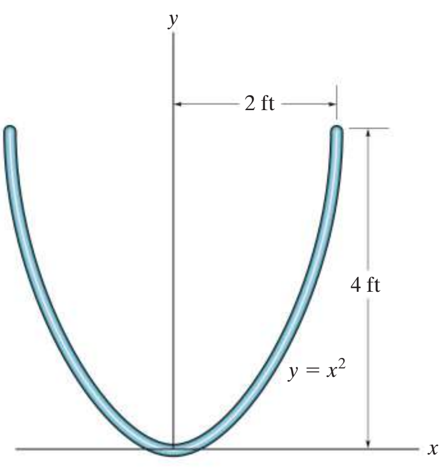
PROBLEMS

9–1. Locate the center of mass of the homogeneous rod bent into the shape of a circular arc.



Prob. 9–1

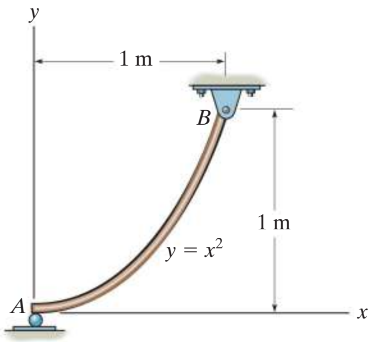
9–2. Determine the location $(\bar{x}, \bar{y})$ of the centroid of the wire.



Prob. 9–2

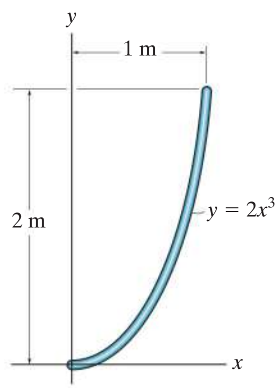
9–3. Locate the center of gravity $\bar{x}$ of the homogeneous rod. If the rod has a weight per unit length of 100 N/m, determine the vertical reaction at A and the x and y components of reaction at the pin B.

***9–4.** Locate the center of gravity $\bar{y}$ of the homogeneous rod.



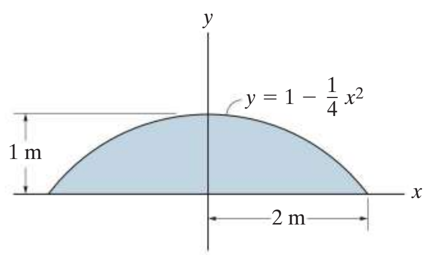
Probs. 9–3/4

9–5. Determine the distance $\bar{y}$ to the center of gravity of the homogeneous rod.



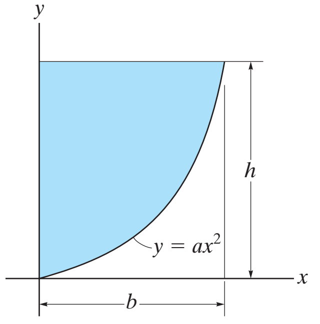
Prob. 9–5

9–6. Locate the centroid $\bar{y}$ of the area.



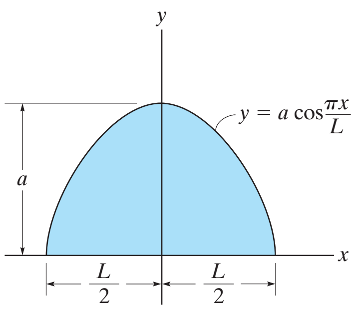
Prob. 9–6

9–7. Locate the centroid $\bar{x}$ of the parabolic area.



Prob. 9–7

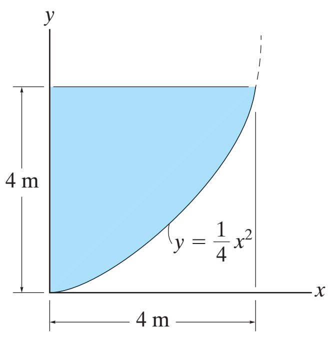
***9–8.** Locate the centroid of the shaded area.



Prob. 9–8

9–9. Locate the centroid $\bar{x}$ of the shaded area.

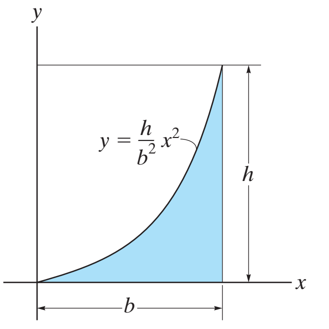
9–10. Locate the centroid $\bar{y}$ of the shaded area.



Probs. 9–9/10

9–11. Locate the centroid $\bar{x}$ of the area.

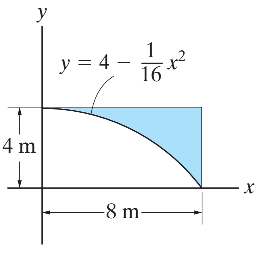
***9–12.** Locate the centroid $\bar{y}$ of the area.



Probs. 9–11/12

9–13. Locate the centroid $\bar{x}$ of the area.

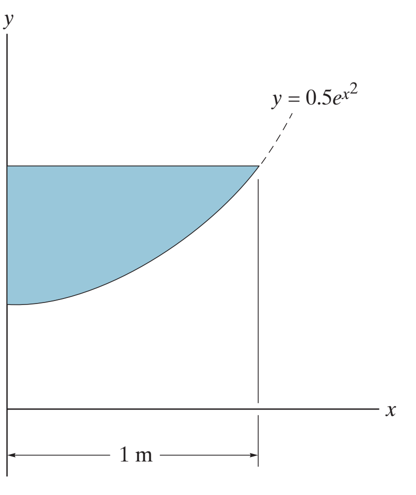
9–14. Locate the centroid $\bar{y}$ of the area.



Probs. 9–13/14

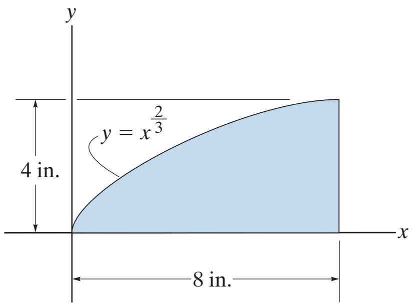
9–15. Locate the centroid $\bar{x}$ of the shaded area. Solve the problem by evaluating the integrals using Simpson's rule.

***9–16.** Locate the centroid $\bar{y}$ of the shaded area. Solve the problem by evaluating the integrals using Simpson's rule.



Probs. 9–15/16

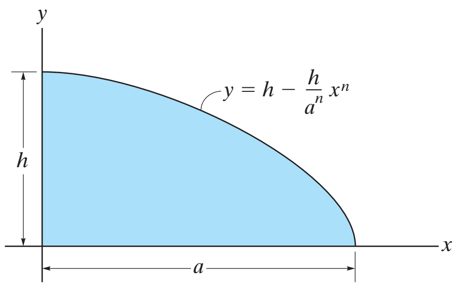
9–17. Locate the centroid $\bar{y}$ of the area.



Prob. 9–17

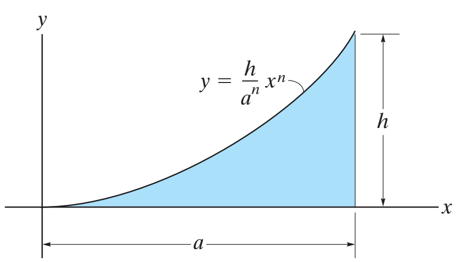
9–18. Locate the centroid $\bar{x}$ of the area.

9–19. Locate the centroid $\bar{y}$ of the area.



Probs. 9–18/19

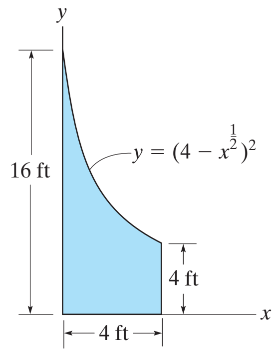
***9–20.** Locate the centroid $\bar{y}$ of the shaded area.



Prob. 9–20

9–21. Locate the centroid $\bar{x}$ of the shaded area.

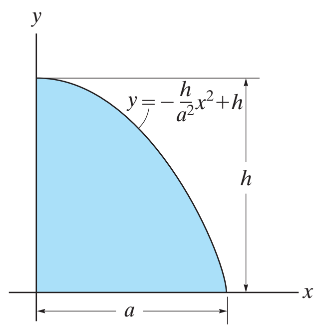
9–22. Locate the centroid $\bar{y}$ of the shaded area.



Probs. 9–21/22

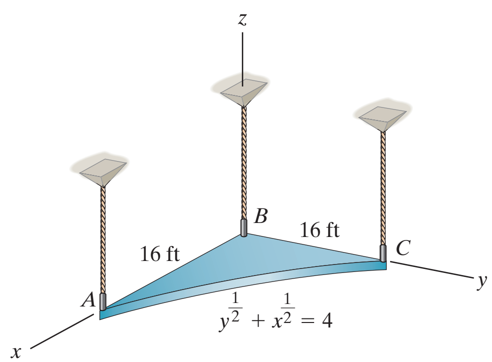
9–23. Locate the centroid $\bar{x}$ of the shaded area.

***9–24.** Locate the centroid $\bar{y}$ of the shaded area.



Probs. 9–23/24

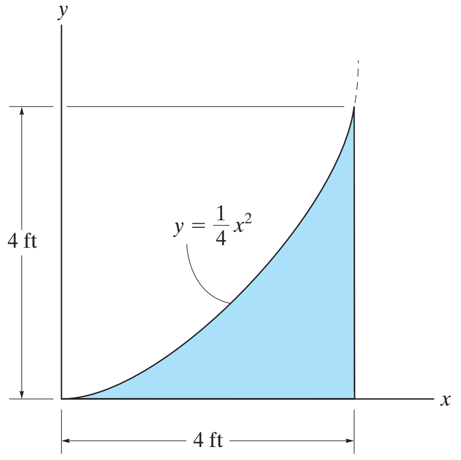
9–25. The plate has a thickness of 0.25 ft and a specific weight of $\gamma = 180\ \text{lb/ft}^3$. Determine the location of its center of gravity. Also, find the tension in each of the cords used to support it.



Prob. 9–25

9–26. Locate the centroid $\bar{x}$ of the shaded area.

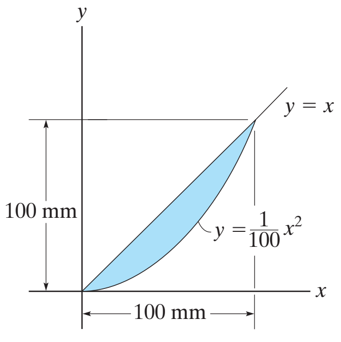
9–27. Locate the centroid $\bar{y}$ of the shaded area.



Probs. 9–26/27

***9–28.** Locate the centroid $\bar{x}$ of the shaded area.

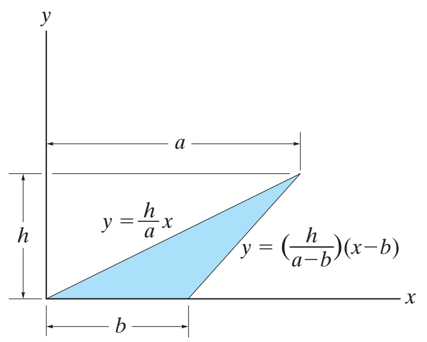
9–29. Locate the centroid $\bar{y}$ of the shaded area.



Probs. 9–28/29

9–30. Locate the centroid $\bar{x}$ of the shaded area.

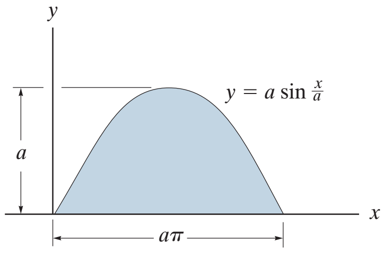
9–31. Locate the centroid $\bar{y}$ of the shaded area.



Probs. 9–30/31

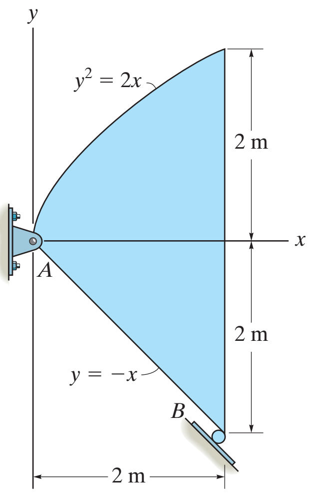
***9–32.** Locate the centroid $\bar{x}$ of the area.

9–33. Locate the centroid $\bar{y}$ of the area.



Probs. 9–32/33

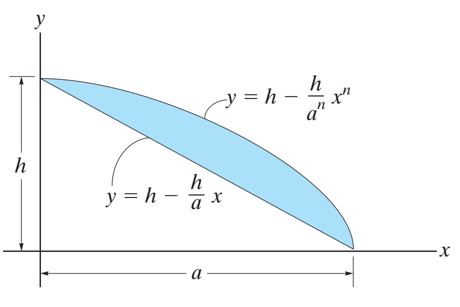
9–34. The steel plate is 0.3 m thick and has a density of 7850 kg/m^3. Determine the location of its center of mass. Also find the reactions at the pin and roller support.



Prob. 9–34

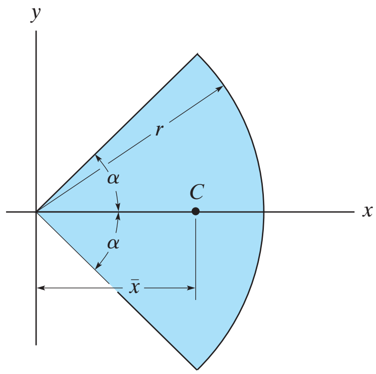
9–35. Locate the centroid $\bar{x}$ of the shaded area.

***9–36.** Locate the centroid $\bar{y}$ of the shaded area.



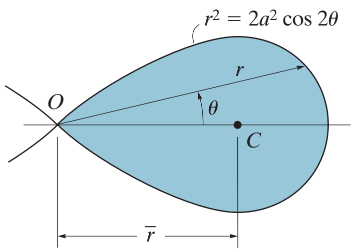
Probs. 9–35/36

9–37. Locate the centroid $\bar{x}$ of the circular sector.



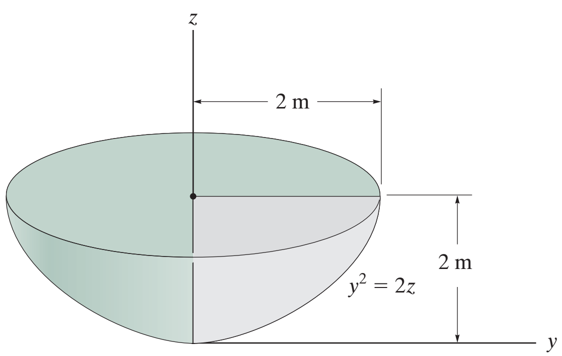
Prob. 9–37

9–38. Determine the location $\bar{r}$ of the centroid C for the loop of the lemniscate, $r^2 = 2a^2\cos 2\theta$, $(-45^\circ \le \theta \le 45^\circ)$.



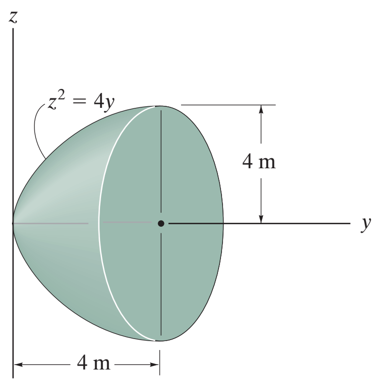
Prob. 9–38

9–39. Locate the center of gravity of the volume. The material is homogeneous.



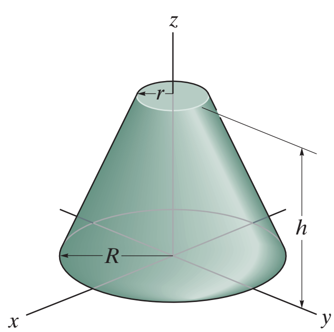
Prob. 9–39

***9–40.** Locate the centroid $\bar{y}$ of the paraboloid.



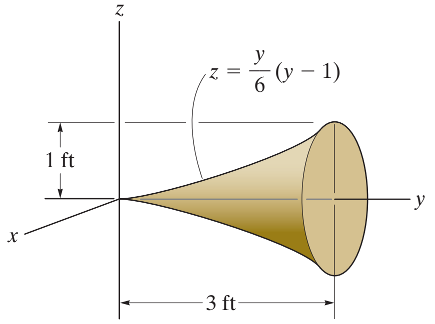
Prob. 9–40

9–41. Locate the centroid $\bar{z}$ of the frustum of the right-circular cone.



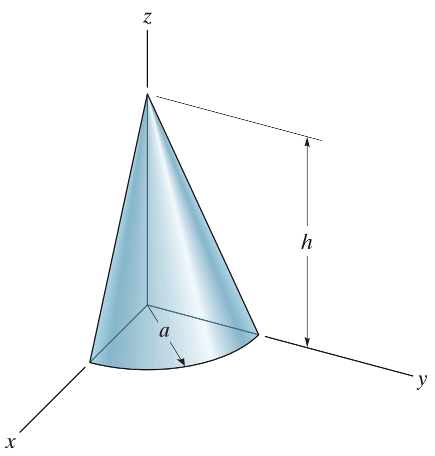
Prob. 9–41

9–42. Determine the centroid $\bar{y}$ of the solid.



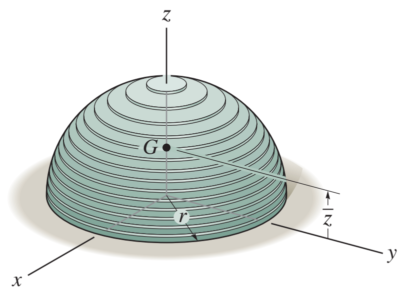
Prob. 9–42

9–43. Locate the centroid of the quarter-cone.



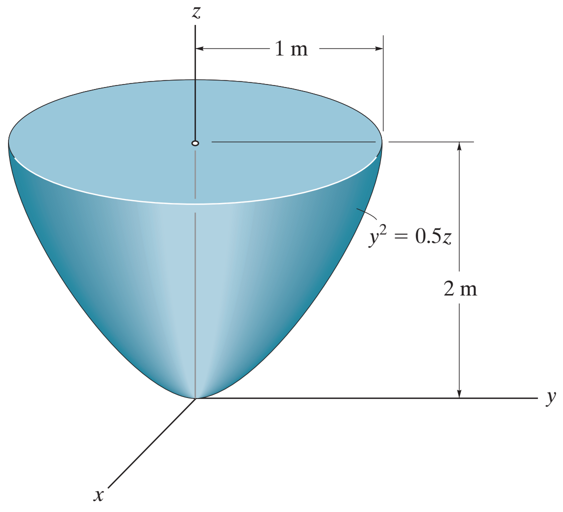
Prob. 9–43

***9–44.** The hemisphere of radius r is made from a stack of very thin plates such that the density varies with height, $\rho = kz$, where k is a constant. Determine its mass and the distance $\bar{z}$ to the center of mass G.



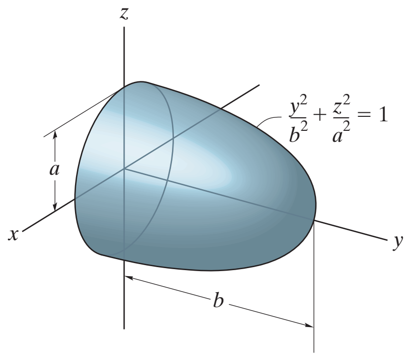
Prob. 9–44

9–45. Locate the centroid $\bar{z}$ of the volume.



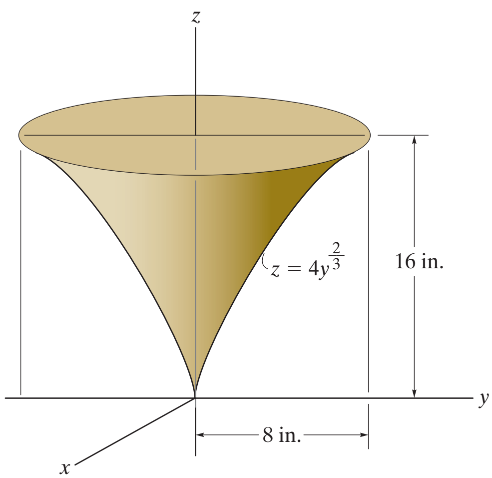
Prob. 9–45

9–46. Locate the centroid of the ellipsoid of revolution.



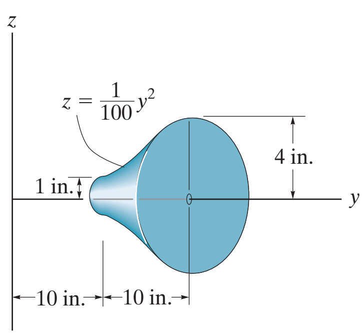
Prob. 9–46

9–47. Locate the center of gravity $\bar{z}$ of the solid.



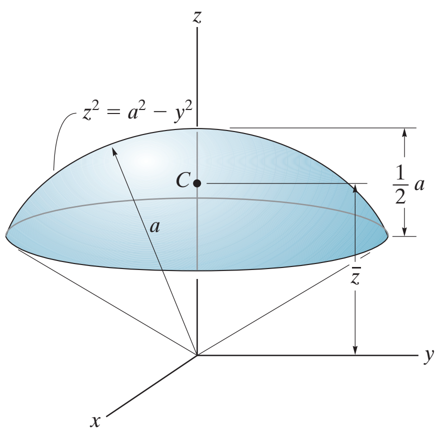
Prob. 9–47

***9–48.** Locate the center of gravity $\bar{y}$ of the volume. The material is homogeneous.



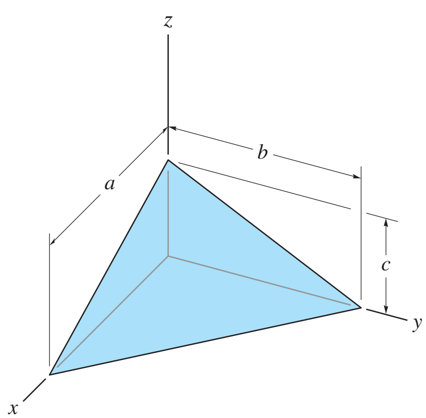
Prob. 9–48

9–49. Locate the centroid $\bar{z}$ of the spherical segment.



Prob. 9–49

9–50. Determine the location $\bar{z}$ of the centroid for the tetrahedron. *Suggestion:* Use a triangular "plate" element parallel to the x–y plane and of thickness dz.



Prob. 9–50

9.2 Composite Bodies

A ***composite body*** consists of a series of connected "simpler" shaped bodies, which may be rectangular, triangular, semicircular, etc. Such a body can often be sectioned or divided into its composite parts and, provided the *weight* and location of the center of gravity of each of these parts are known, we can then eliminate the need for integration to determine the center of gravity for the entire body. The method for doing this follows the same procedure outlined in Sec. 9.1. Formulas analogous to Eqs. 9–1 result; however, rather than account for an infinite number of differential weights, we have instead a finite number of weights. Therefore,

A stress analysis of this angle requires that the centroid of its cross-sectional area be located. (© Russell C. Hibbeler)

$$\bar{x} = \frac{\Sigma \tilde{x} W}{\Sigma W} \qquad \bar{y} = \frac{\Sigma \tilde{y} W}{\Sigma W} \qquad \bar{z} = \frac{\Sigma \tilde{z} W}{\Sigma W} \tag{9–6}$$

Here

$\bar{x}, \bar{y}, \bar{z}$ represent the coordinates of the center of gravity G of the composite body.

$\tilde{x}, \tilde{y}, \tilde{z}$ represent the coordinates of the center of gravity of each composite part of the body.

ΣW is the sum of the weights of all the composite parts of the body, or simply the total weight of the body.

When the body has a *constant density or specific weight*, the center of gravity *coincides* with the centroid of the body. The centroid for composite lines, areas, and volumes can be found using relations analogous to Eqs. 9–6; however, the W's are replaced by L's, A's, and V's, respectively. Centroids for common shapes of lines, areas, shells, and volumes that often make up a composite body are given in the table on the inside back cover.

In order to determine the force required to tip over this concrete barrier it is first necessary to determine the location of its center of gravity G. Due to symmetry, G will lie on the vertical axis of symmetry. (© Russell C. Hibbeler)

Procedure for Analysis

The location of the center of gravity of a body or the centroid of a composite geometrical object represented by a line, area, or volume can be determined using the following procedure.

Composite Parts.

- Using a sketch, divide the body or object into a finite number of composite parts that have simpler shapes.
- If a composite body has a *hole*, or a geometric region having no material, then consider the composite body without the hole and consider the hole as an *additional* composite part having *negative* weight or size.

Moment Arms.

- Establish the coordinate axes on the sketch and determine the coordinates $\tilde{x}, \tilde{y}, \tilde{z}$ of the center of gravity or centroid of each part.

Summations.

- Determine $\bar{x}, \bar{y}, \bar{z}$ by applying the center of gravity equations, Eqs. 9–6, or the analogous centroid equations.
- If an object is *symmetrical* about an axis, the centroid of the object lies on this axis.

If desired, the calculations can be arranged in tabular form, as indicated in the following three examples.

The center of gravity of this water tank can be determined by dividing it into composite parts and applying Eqs. 9–6. (© Russell C. Hibbeler)

EXAMPLE 9.9

Locate the centroid of the wire shown in Fig. 9–16*a*.

SOLUTION

Composite Parts. The wire is divided into three segments as shown in Fig. 9–16*b*.

Moment Arms. The location of the centroid for each segment is determined and indicated in the figure. In particular, the centroid of segment (1) is determined either by integration or by using the table on the inside back cover.

Summations. For convenience, the calculations can be tabulated as follows:

Segment	L (mm)	$\tilde{x}$ (mm)	$\tilde{y}$ (mm)	$\tilde{z}$ (mm)	$\tilde{x}L$ (mm^2)	$\tilde{y}L$ (mm^2)	$\tilde{z}L$ (mm^2)
1	$\pi(60) = 188.5$	60	-38.2	0	11 310	-7200	0
2	40	0	20	0	0	800	0
3	20	0	40	-10	0	800	-200
	$\Sigma L = 248.5$				$\Sigma\tilde{x}L = 11\,310$	$\Sigma\tilde{y}L = -5600$	$\Sigma\tilde{z}L = -200$

Thus,

$$\bar{x} = \frac{\Sigma\tilde{x}L}{\Sigma L} = \frac{11\,310}{248.5} = 45.5 \text{ mm} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\Sigma\tilde{y}L}{\Sigma L} = \frac{-5600}{248.5} = -22.5 \text{ mm} \qquad \textit{Ans.}$$

$$\bar{z} = \frac{\Sigma\tilde{z}L}{\Sigma L} = \frac{-200}{248.5} = -0.805 \text{ mm} \qquad \textit{Ans.}$$

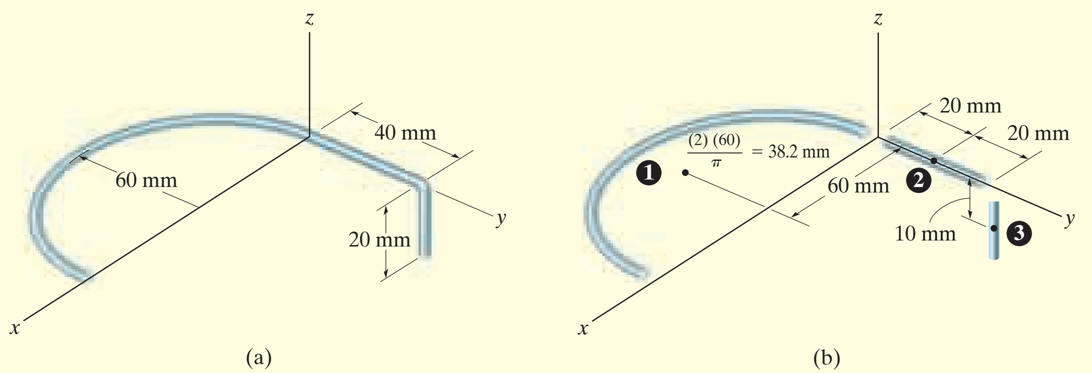
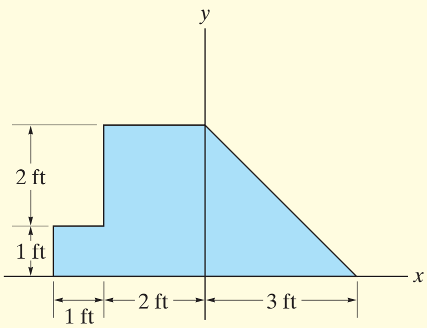


Fig. 9–16

EXAMPLE 9.10

Locate the centroid of the plate area shown in Fig. 9–17*a*.



(a)

Fig. 9–17

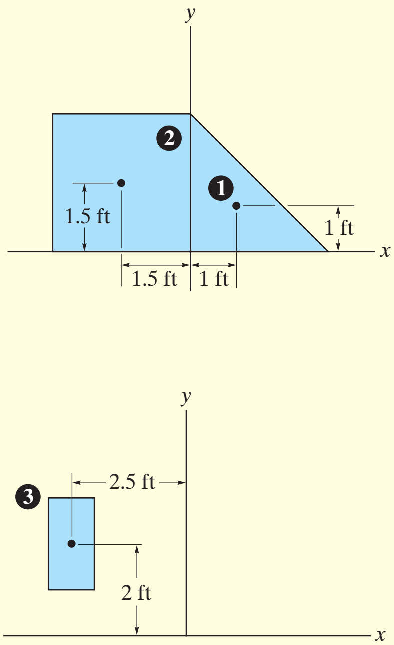
SOLUTION

Composite Parts. The plate is divided into three segments as shown in Fig. 9–17*b*. Here the area of the small rectangle 3 is considered "negative" since it must be subtracted from the larger one 2.

Moment Arms. The centroid of each segment is located as indicated in the figure. Note that the $\tilde{x}$ coordinates of 2 and 3 are *negative*.

Summations. Taking the data from Fig. 9–17*b*, the calculations are tabulated as follows:

Segment	A (ft^2)	$\tilde{x}$ (ft)	$\tilde{y}$ (ft)	$\tilde{x}A$ (ft^3)	$\tilde{y}A$ (ft^3)
1	$\frac{1}{2}(3)(3) = 4.5$	1	1	4.5	4.5
2	$(3)(3) = 9$	-1.5	1.5	-13.5	13.5
3	$-(2)(1) = -2$	-2.5	2	5	-4
	$\Sigma A = 11.5$			$\Sigma \tilde{x}A = -4$	$\Sigma \tilde{y}A = 14$



(b)

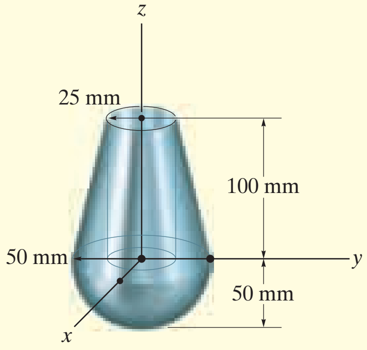
Thus,

$$\bar{x} = \frac{\Sigma \tilde{x}A}{\Sigma A} = \frac{-4}{11.5} = -0.348 \text{ ft} \qquad \textit{Ans.}$$

$$\bar{y} = \frac{\Sigma \tilde{y}A}{\Sigma A} = \frac{14}{11.5} = 1.22 \text{ ft} \qquad \textit{Ans.}$$

NOTE: If these results are plotted in Fig. 9–17*a*, the location of point *C* seems reasonable.

EXAMPLE 9.11



(a)

Fig. 9–18

Locate the center of mass of the assembly shown in Fig. 9–18*a*. The conical frustum has a density of $\rho_c = 8\ \text{Mg/m}^3$, and the hemisphere has a density of $\rho_h = 4\ \text{Mg/m}^3$. There is a 25-mm-radius cylindrical hole in the center of the frustum.

SOLUTION

Composite Parts. The assembly can be thought of as consisting of four segments as shown in Fig. 9–18*b*. For the calculations, (3) and (4) must be considered as "negative" segments in order that the four segments, when added together, yield the total composite shape shown in Fig. 9–18*a*.

Moment Arm. Using the table on the inside back cover, the computations for the centroid $\tilde{z}$ of each piece are shown in the figure.

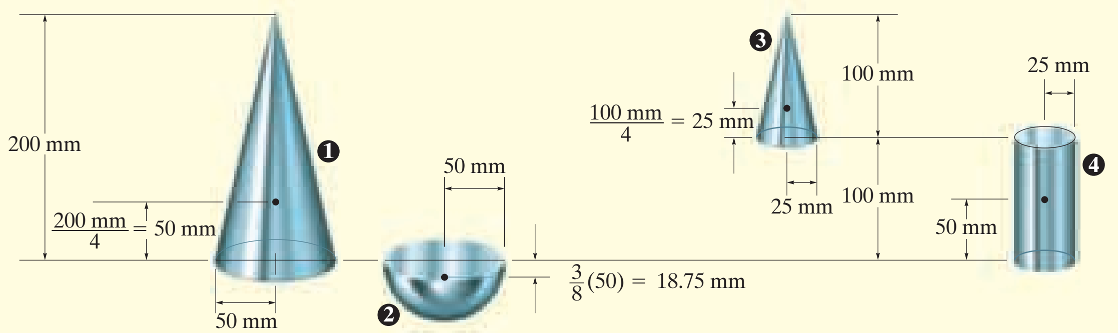
Summations. Because of *symmetry*, note that

$$\bar{x} = \bar{y} = 0 \qquad \textit{Ans.}$$

Since $W = mg$, and g is constant, the third of Eqs. 9–6 becomes $\bar{z} = \Sigma\tilde{z}m/\Sigma m$. The mass of each piece can be computed from $m = \rho V$ and used for the calculations. Also, $1\ \text{Mg/m}^3 = 10^{-6}\ \text{kg/mm}^3$, so that

Segment	m (kg)	$\tilde{z}$ (mm)	$\tilde{z}m$ (kg · mm)
1	$8(10^{-6})\left(\frac{1}{3}\right)\pi(50)^2(200) = 4.189$	50	209.440
2	$4(10^{-6})\left(\frac{2}{3}\right)\pi(50)^3 = 1.047$	−18.75	−19.635
3	$-8(10^{-6})\left(\frac{1}{3}\right)\pi(25)^2(100) = -0.524$	100 + 25 = 125	−65.450
4	$-8(10^{-6})\pi(25)^2(100) = -1.571$	50	−78.540
	$\Sigma m = 3.142$		$\Sigma\tilde{z}m = 45.815$

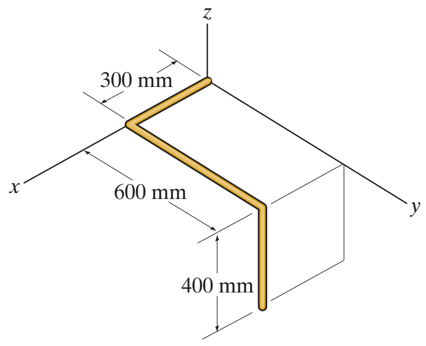
Thus, $$\bar{z} = \frac{\Sigma\tilde{z}m}{\Sigma m} = \frac{45.815}{3.142} = 14.6\ \text{mm} \qquad \textit{Ans.}$$



(b)

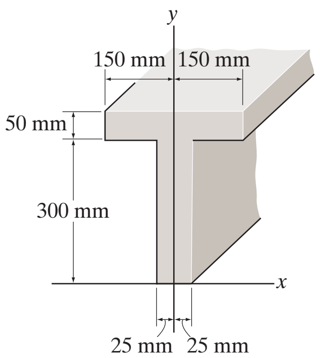
FUNDAMENTAL PROBLEMS

F9–7. Locate the centroid $(\bar{x}, \bar{y}, \bar{z})$ of the wire bent in the shape shown.



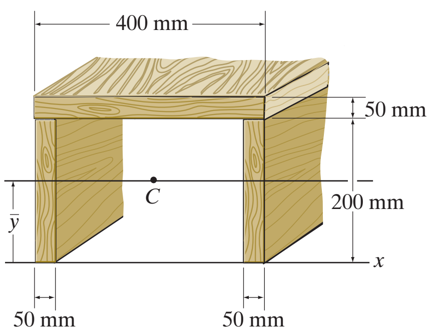
Prob. F9–7

F9–8. Locate the centroid $\bar{y}$ of the beam's cross-sectional area.



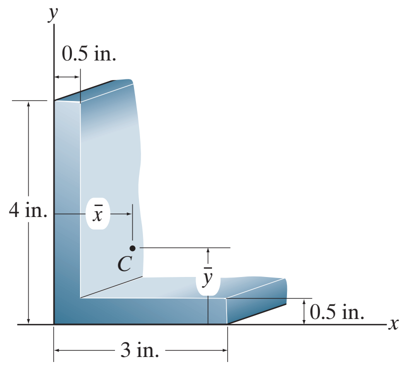
Prob. F9–8

F9–9. Locate the centroid $\bar{y}$ of the beam's cross-sectional area.



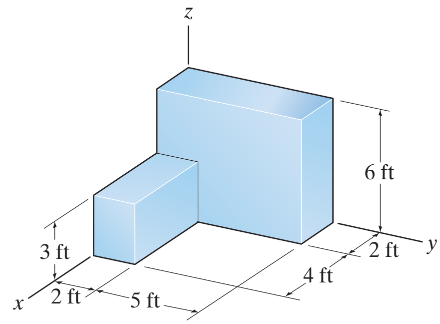
Prob. F9–9

F9–10. Locate the centroid $(\bar{x}, \bar{y})$ of the cross-sectional area.



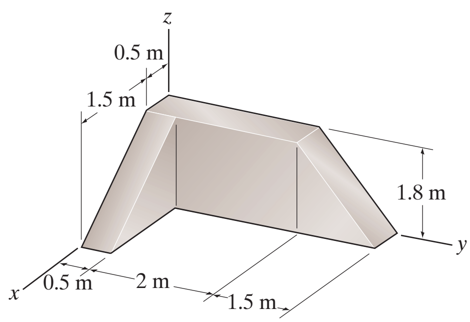
Prob. F9–10

F9–11. Locate the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of the homogeneous solid block.



Prob. F9–11

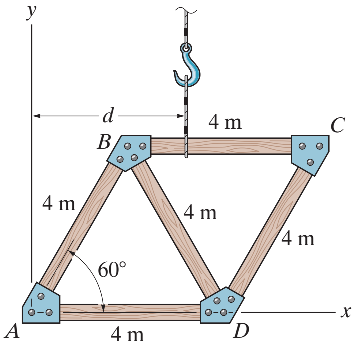
F9–12. Determine the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of the homogeneous solid block.



Prob. F9–12

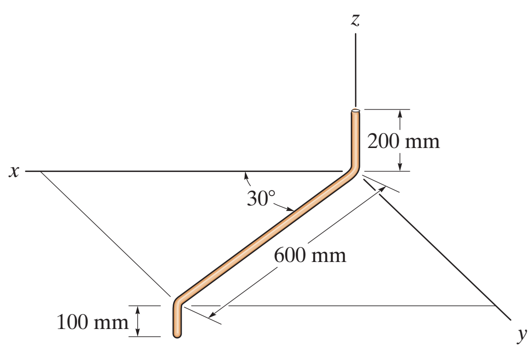
PROBLEMS

9–51. The truss is made from five members, each having a length of 4 m and a mass of 7 kg/m. If the mass of the gusset plates at the joints and the thickness of the members can be neglected, determine the distance d to where the hoisting cable must be attached, so that the truss does not tip (rotate) when it is lifted.



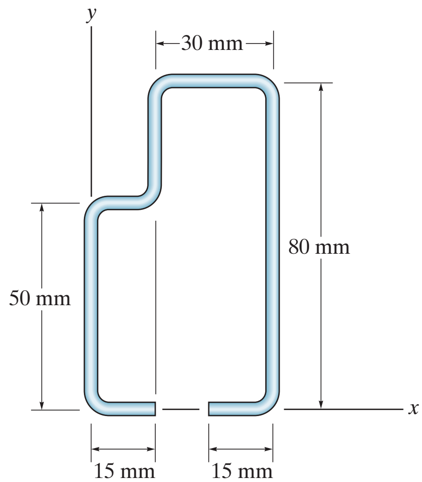
Prob. 9–51

***9–52.** Determine the location $(\bar{x}, \bar{y}, \bar{z})$ of the centroid of the homogeneous rod.



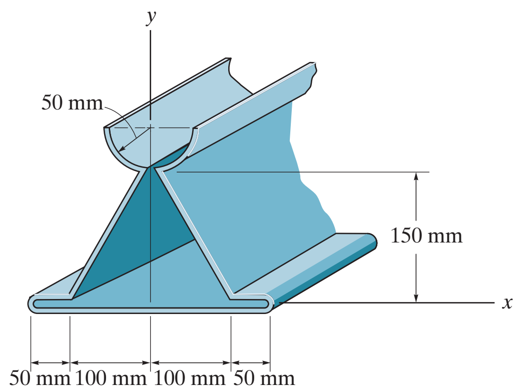
Prob. 9–52

9–53. A rack is made from roll-formed sheet steel and has the cross section shown. Determine the location $(\bar{x}, \bar{y})$ of the centroid of the cross section. The dimensions are indicated at the center thickness of each segment.



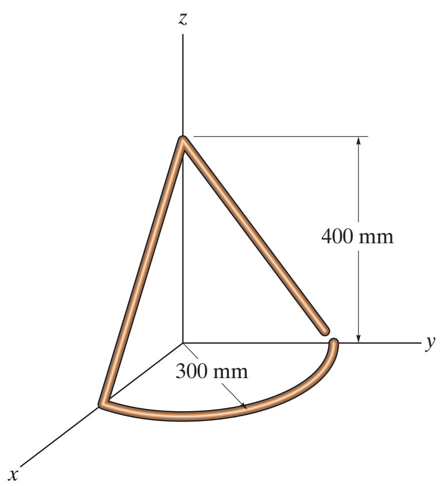
Prob. 9–53

9–54. Locate the centroid $(\bar{x}, \bar{y})$ of the metal cross section. Neglect the thickness of the material and slight bends at the corners.



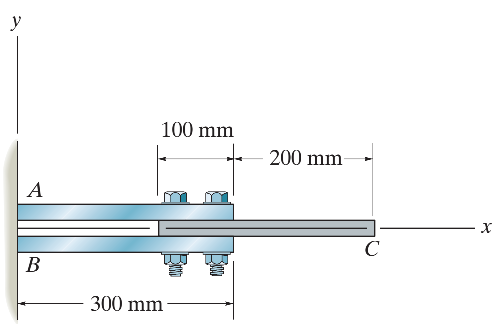
Prob. 9–54

9–55. Locate the center of gravity $(\bar{x}, \bar{y}, \bar{z})$ of the homogeneous wire.



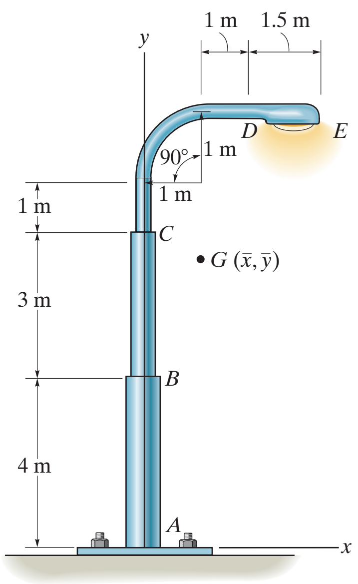
Prob. 9–55

***9–56.** The steel and aluminum plate assembly is bolted together and fastened to the wall. Each plate has a constant width in the z direction of 200 mm and thickness of 20 mm. If the density of A and B is $\rho_s = 7.85\ \mathrm{Mg/m^3}$, and for C, $\rho_{al} = 2.71\ \mathrm{Mg/m^3}$, determine the location $\bar{x}$ of the center of mass. Neglect the size of the bolts.



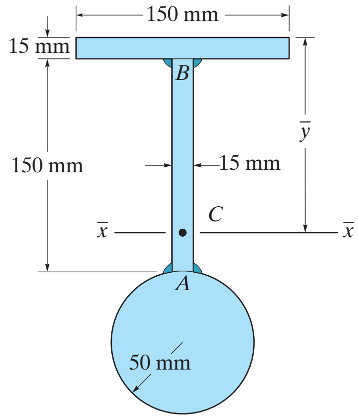
Prob. 9–56

9–57. Locate the center of gravity $G(\bar{x}, \bar{y})$ of the streetlight. Neglect the thickness of each segment. The mass per unit length of each segment is as follows: $\rho_{AB} = 12\ \mathrm{kg/m}$, $\rho_{BC} = 8\ \mathrm{kg/m}$, $\rho_{CD} = 5\ \mathrm{kg/m}$, and $\rho_{DE} = 2\ \mathrm{kg/m}$.



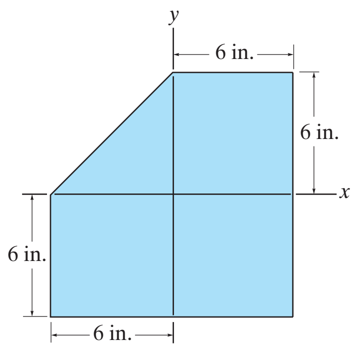
Prob. 9–57

9–58. Determine the location $\bar{y}$ of the centroidal axis $\bar{x}$–$\bar{x}$ of the beam's cross-sectional area. Neglect the size of the corner welds at A and B for the calculation.



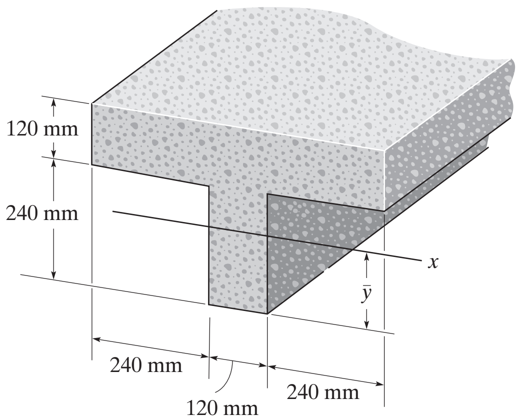
Prob. 9–58

9–59. Locate the centroid $(\bar{x}, \bar{y})$ of the shaded area.



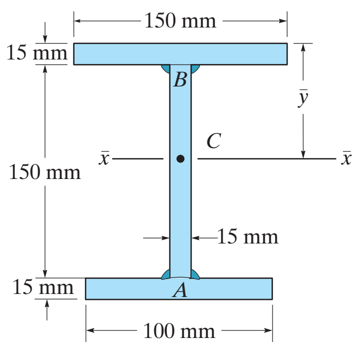
Prob. 9–59

***9–60.** Locate the centroid $\bar{y}$ for the beam's cross-sectional area.



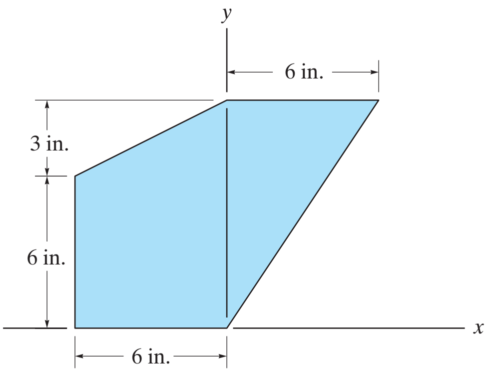
Prob. 9–60

9–61. Determine the location $\bar{y}$ of the centroid C of the beam having the cross-sectional area shown.



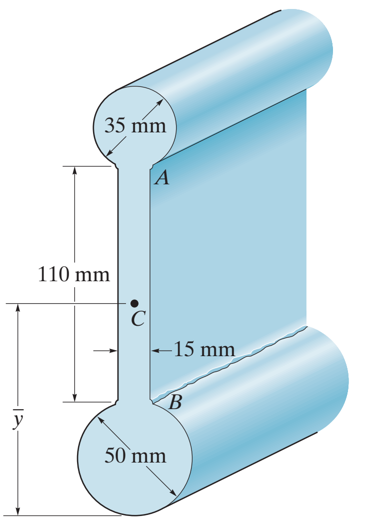
Prob. 9–61

9–62. Locate the centroid $(\bar{x}, \bar{y})$ of the shaded area.



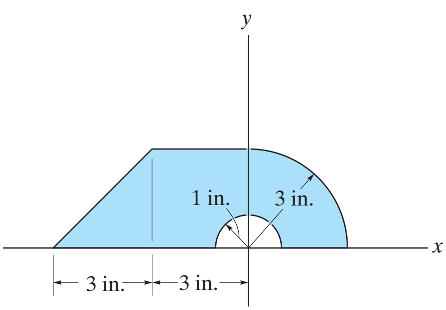
Prob. 9–62

9–63. Determine the location $\bar{y}$ of the centroid of the beam's cross-sectional area. Neglect the size of the corner welds at A and B for the calculation.



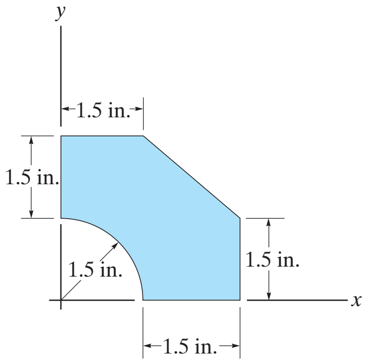
Prob. 9–63

***9–64.** Locate the centroid $(\bar{x}, \bar{y})$ of the shaded area.



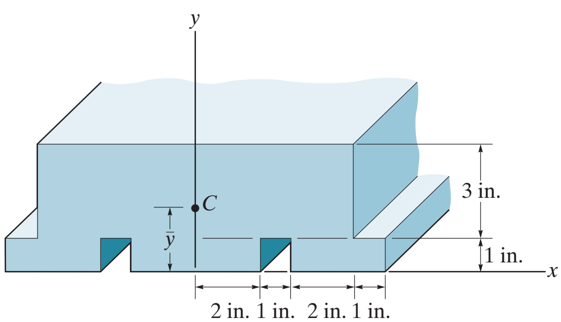
Prob. 9–64

9–65. Determine the location $(\bar{x}, \bar{y})$ of the centroid C of the area.



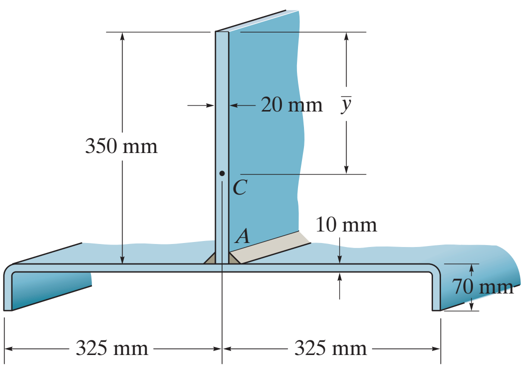
Prob. 9–65

9–66. Determine the location $\bar{y}$ of the centroid C for a beam having the cross-sectional area shown. The beam is symmetric with respect to the y axis.



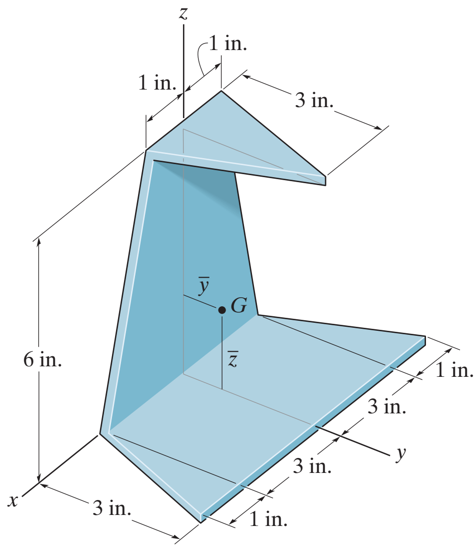
Prob. 9–66

9–67. Locate the centroid $\bar{y}$ of the cross-sectional area of the beam constructed from a channel and a plate. Assume all corners are square and neglect the size of the weld at A.



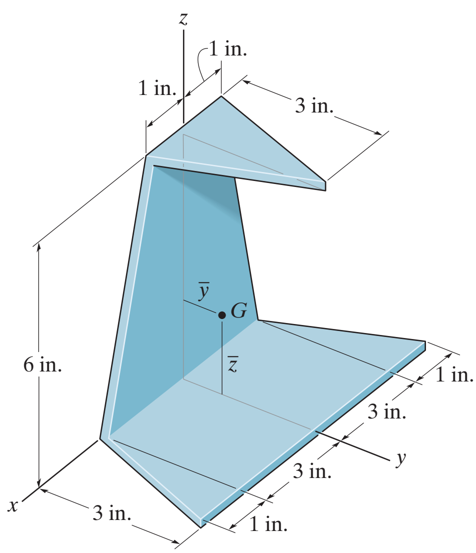
Prob. 9–67

***9–68.** A triangular plate made of homogeneous material has a constant thickness that is very small. If it is folded over as shown, determine the location $\bar{y}$ of the plate's center of gravity G.



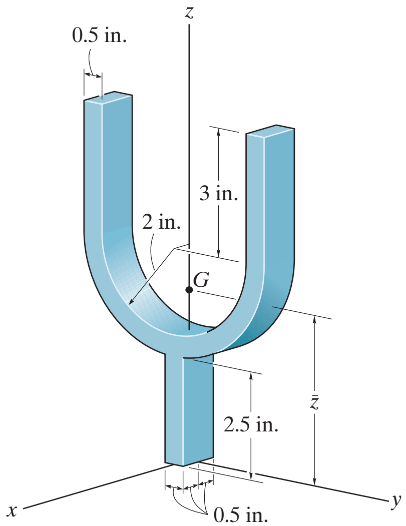
Prob. 9–68

9–69. A triangular plate made of homogeneous material has a constant thickness that is very small. If it is folded over as shown, determine the location $\bar{z}$ of the plate's center of gravity G.



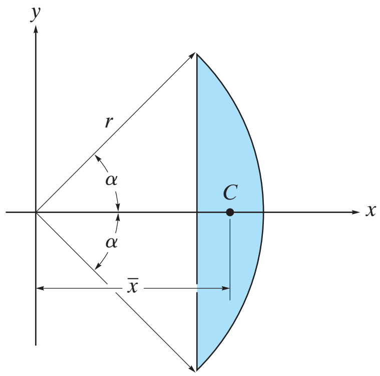
Prob. 9–69

9–70. Locate the center of mass $\bar{z}$ of the forked level which is made from a homogeneous material and has the dimensions shown.



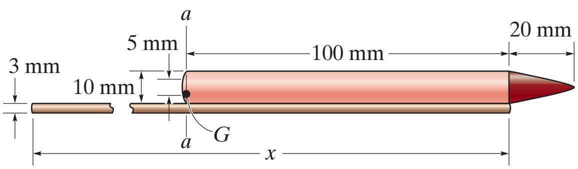
Prob. 9–70

9–71. Determine the location $\bar{x}$ of the centroid C of the shaded area that is part of a circle having a radius r.



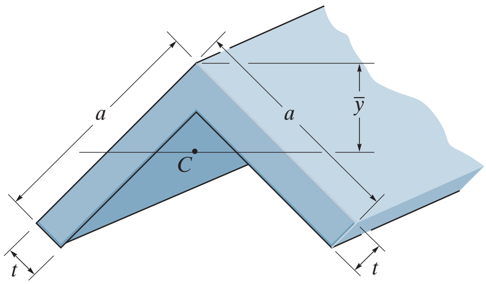
Prob. 9–71

***9–72.** A toy skyrocket consists of a solid conical top, $\rho_t = 600\ \text{kg/m}^3$, a hollow cylinder, $\rho_c = 400\ \text{kg/m}^3$, and a stick having a circular cross section, $\rho_s = 300\ \text{kg/m}^3$. Determine the length of the stick, x, so that the center of gravity G of the skyrocket is located along line aa.



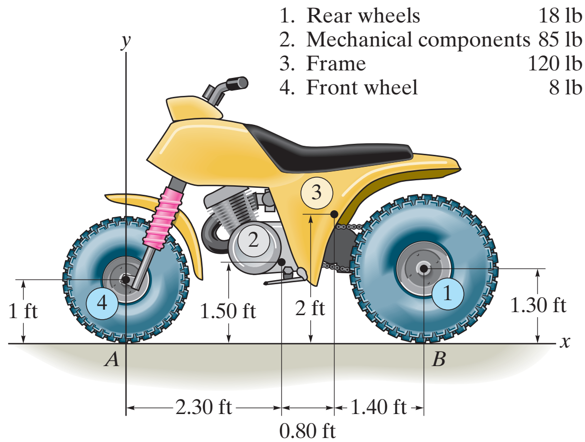
Prob. 9–72

9–73. Locate the centroid $\bar{y}$ for the cross-sectional area of the angle.



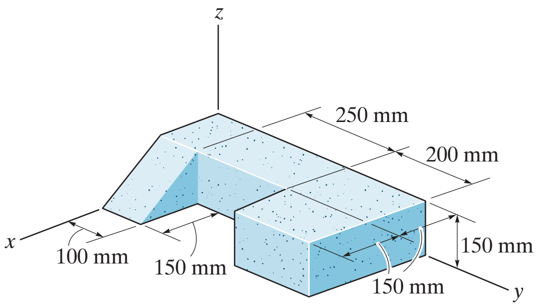
Prob. 9–73

9–74. Determine the location $(\bar{x}, \bar{y})$ of the center of gravity of the three-wheeler. The location of the center of gravity of each component and its weight are tabulated in the figure. If the three-wheeler is symmetrical with respect to the x–y plane, determine the normal reaction each of its wheels exerts on the ground.



Prob. 9–74

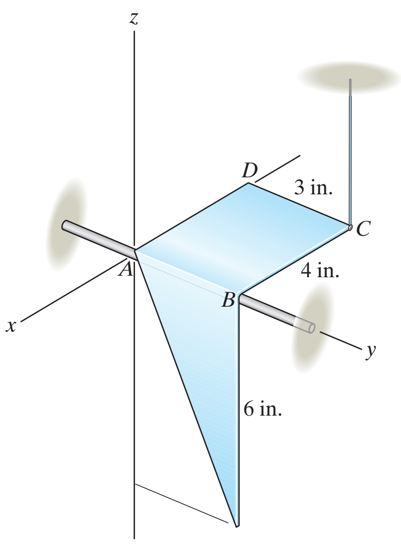
9–75. Locate the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of the homogeneous block assembly.



Prob. 9–75

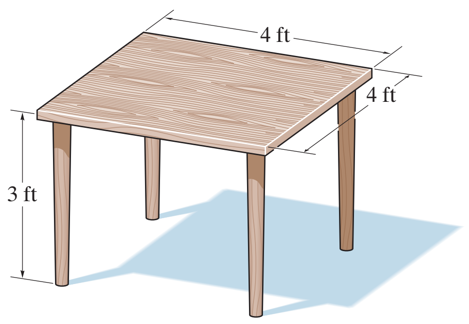
***9–76.** The sheet metal part has the dimensions shown. Determine the location $(\bar{x}, \bar{y}, \bar{z})$ of its centroid.

9–77. The sheet metal part has a weight per unit area of $2\ \text{lb/ft}^2$ and is supported by the smooth rod and the cord at C. If the cord is cut, the part will rotate about the y axis until it reaches equilibrium. Determine the equilibrium angle of tilt, measured downward from the negative x axis, that AD makes with the $-x$ axis.



Probs. 9–76/77

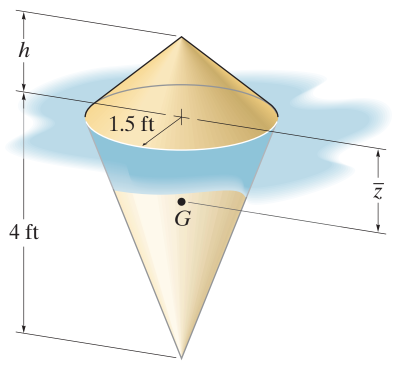
9–78. The wooden table is made from a square board having a weight of 15 lb. Each of the legs weighs 2 lb and is 3 ft long. Determine how high its center of gravity is from the floor. Also, what is the angle, measured from the horizontal, through which its top surface can be tilted on two of its legs before it begins to overturn? Neglect the thickness of each leg.



Prob. 9–78

9–79. The buoy is made from two homogeneous cones each having a radius of 1.5 ft. If $h = 1.2$ ft, find the distance $\bar{z}$ to the buoy's center of gravity G.

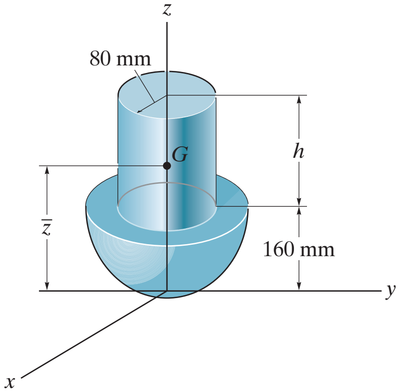
***9–80.** The buoy is made from two homogeneous cones each having a radius of 1.5 ft. If it is required that the buoy's center of gravity G be located at $\bar{z} = 0.5$ ft, determine the height h of the top cone.



Probs. 9–79/80

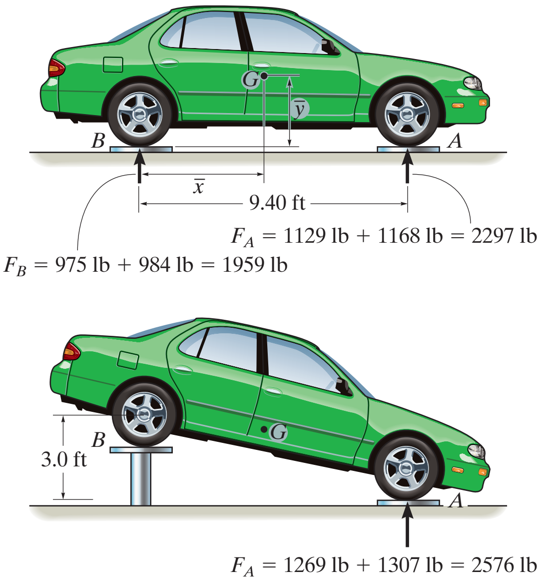
9–81. The assembly is made from a steel hemisphere, $\rho_{st} = 7.80 \text{ Mg/m}^3$, and an aluminum cylinder, $\rho_{al} = 2.70 \text{ Mg/m}^3$. Determine the mass center of the assembly if the height of the cylinder is $h = 200$ mm.

9–82. The assembly is made from a steel hemisphere, $\rho_{st} = 7.80 \text{ Mg/m}^3$, and an aluminum cylinder, $\rho_{al} = 2.70 \text{ Mg/m}^3$. Determine the height h of the cylinder so that the mass center of the assembly is located at $\bar{z} = 160$ mm.



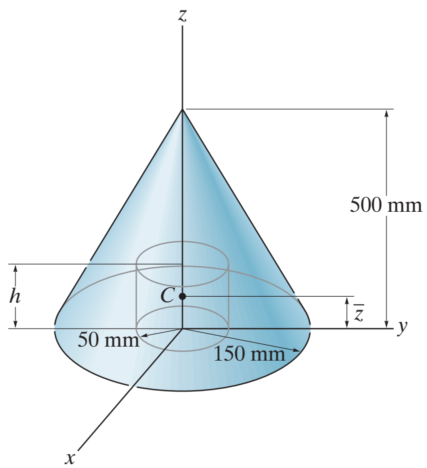
Probs. 9–81/82

9–83. The car rests on four scales and in this position the scale readings of both the front and rear tires are shown by F_A and F_B. When the rear wheels are elevated to a height of 3 ft above the front scales, the new readings of the front wheels are also recorded. Use this data to compute the location $\bar{x}$ and $\bar{y}$ to the center of gravity G of the car. The tires each have a diameter of 1.98 ft.



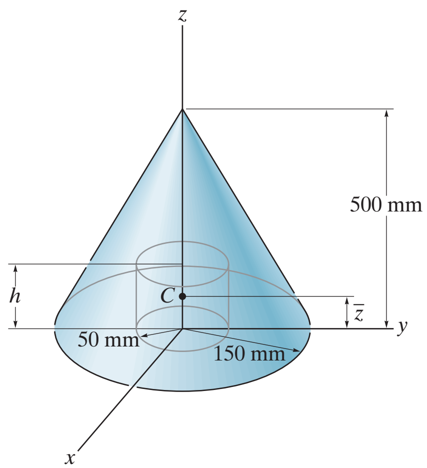
Prob. 9–83

***9–84.** Determine the distance h to which a 100-mm-diameter hole must be bored into the base of the cone so that the center of mass of the resulting shape is located at $\bar{z} = 115$ mm. The material has a density of 8 Mg/m^3.



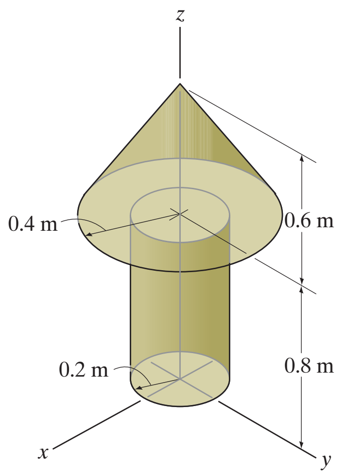
Prob. 9–84

9–85. Determine the distance $\bar{z}$ to the centroid of the shape that consists of a cone with a hole of height $h = 50$ mm bored into its base.



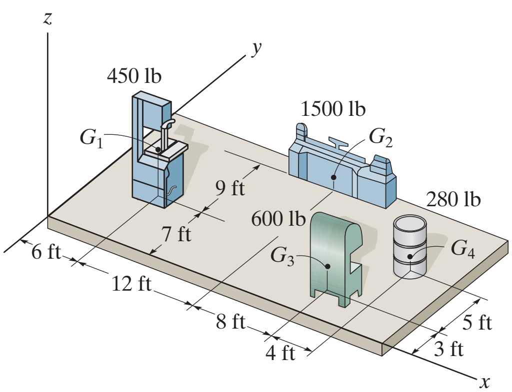
Prob. 9–85

9–86. Locate the center of mass $\bar{z}$ of the assembly. The cylinder and the cone are made from materials having densities of 5 Mg/m^3 and 9 Mg/m^3, respectively.



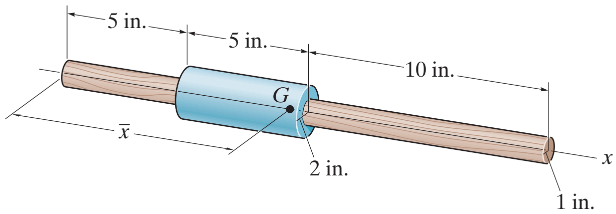
Prob. 9–86

9–87. Major floor loadings in a shop are caused by the weights of the objects shown. Each force acts through its respective center of gravity G. Locate the center of gravity $(\bar{x}, \bar{y})$ of all these components.



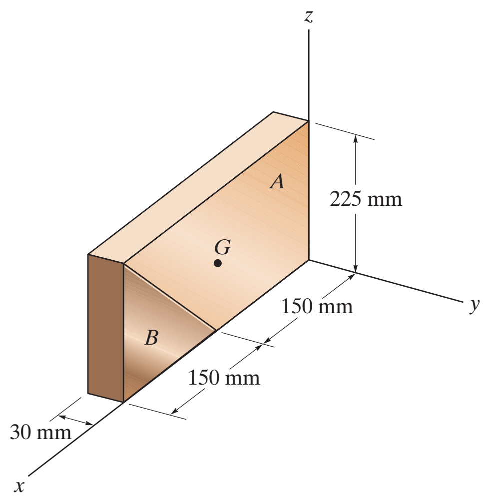
Prob. 9–87

***9–88.** The assembly consists of a 20-in. wooden dowel rod and a tight-fitting steel collar. Determine the distance $\bar{x}$ to its center of gravity if the specific weights of the materials are $\gamma_w = 150\ \text{lb/ft}^3$ and $\gamma_{st} = 490\ \text{lb/ft}^3$. The radii of the dowel and collar are shown.



Prob. 9–88

9–89. The composite plate is made from both steel (A) and brass (B) segments. Determine the mass and location $(\bar{x}, \bar{y}, \bar{z})$ of its mass center G. Take $\rho_{st} = 7.85\ \text{Mg/m}^3$ and $\rho_{br} = 8.74\ \text{Mg/m}^3$.



Prob. 9–89

*9.3 Theorems of Pappus and Guldinus

The two ***theorems of Pappus and Guldinus*** are used to find the surface area and volume of any body of revolution. They were first developed by Pappus of Alexandria during the fourth century A.D. and then restated at a later time by the Swiss mathematician Paul Guldin or Guldinus (1577–1643).

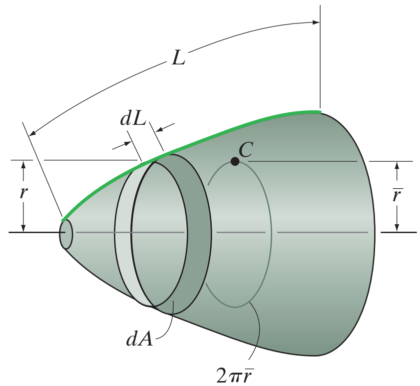


Fig. 9–19

Surface Area. If we revolve a *plane curve* about an axis that does not intersect the curve we will generate a *surface area of revolution*. For example, the surface area in Fig. 9–19 is formed by revolving the curve of length L about the horizontal axis. To determine this surface area, we will first consider the differential line element of length dL. If this element is revolved 2π radians about the axis, a ring having a surface area of $dA = 2\pi r\, dL$ will be generated. Thus, the surface area of the entire body is $A = 2\pi \int r\, dL$. Since $\int r\, dL = \bar{r}L$ (Eq. 9–5), then $A = 2\pi\bar{r}L$. If the curve is revolved only through an angle θ (radians), then

$$\boxed{A = \theta\bar{r}L} \qquad (9\text{–}7)$$

where

A = surface area of revolution

θ = angle of revolution measured in radians, $\theta \leq 2\pi$

$\bar{r}$ = perpendicular distance from the axis of revolution to the centroid of the generating curve

L = length of the generating curve

Therefore the first theorem of Pappus and Guldinus states that *the area of a surface of revolution equals the product of the length of the generating curve and the distance traveled by the centroid of the curve in generating the surface area.*

The amount of material used on this storage building can be estimated by using the first theorem of Pappus and Guldinus to determine its surface area. (© Russell C. Hibbeler)

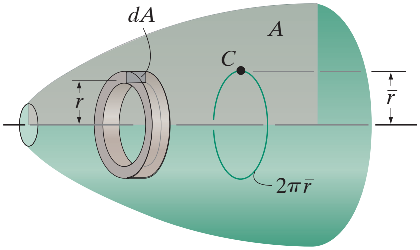


Fig. 9–20

Volume. A *volume* can be generated by revolving a *plane area* about an axis that does not intersect the area. For example, if we revolve the shaded area A in Fig. 9–20 about the horizontal axis, it generates the volume shown. This volume can be determined by first revolving the differential element of area dA 2π radians about the axis, so that a ring having the volume $dV = 2\pi r\,dA$ is generated. The entire volume is then $V = 2\pi \int r\,dA$. However, $\int r\,dA = \bar{r}A$, Eq. 9–4, so that $V = 2\pi\bar{r}A$. If the area is only revolved through an angle θ (radians), then

$$\boxed{V = \theta\bar{r}A} \tag{9–8}$$

where

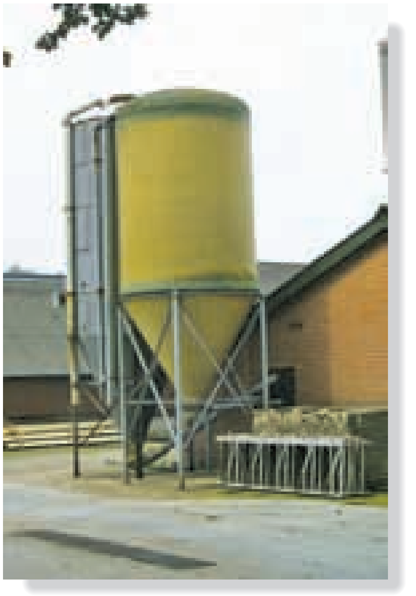
V = volume of revolution

θ = angle of revolution measured in radians, $\theta \leq 2\pi$

$\bar{r}$ = perpendicular distance from the axis of revolution to the centroid of the generating area

A = generating area

Therefore the second theorem of Pappus and Guldinus states that *the volume of a body of revolution equals the product of the generating area and the distance traveled by the centroid of the area in generating the volume.*



The volume of fertilizer contained within this silo can be determined using the second theorem of Pappus and Guldinus. (© Russell C. Hibbeler)

Composite Shapes. We may also apply the above two theorems to lines or areas that are composed of a series of composite parts. In this case the total surface area or volume generated is the addition of the surface areas or volumes generated by each of the composite parts. If the perpendicular distance from the axis of revolution to the centroid of each composite part is $\tilde{r}$, then

$$A = \theta\Sigma(\tilde{r}L) \tag{9–9}$$

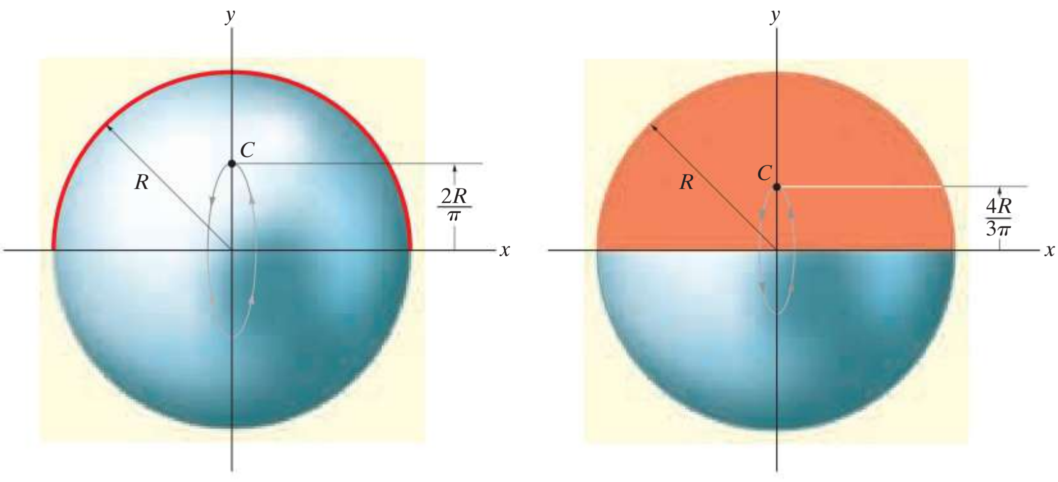
and

$$V = \theta\Sigma(\tilde{r}A) \tag{9–10}$$

Application of the above theorems is illustrated numerically in the following examples.

EXAMPLE 9.12

Show that the surface area of a sphere is $A = 4\pi R^2$ and its volume is $V = \frac{4}{3}\pi R^3$.



(a) (b)

Fig. 9–21

SOLUTION

Surface Area. The surface area of the sphere in Fig. 9–21*a* is generated by revolving a semicircular *arc* about the x axis. Using the table on the inside back cover, it is seen that the centroid of this arc is located at a distance $\bar{r} = 2R/\pi$ from the axis of revolution (x axis). Since the centroid moves through an angle of $\theta = 2\pi$ rad to generate the sphere, then applying Eq. 9–7 we have

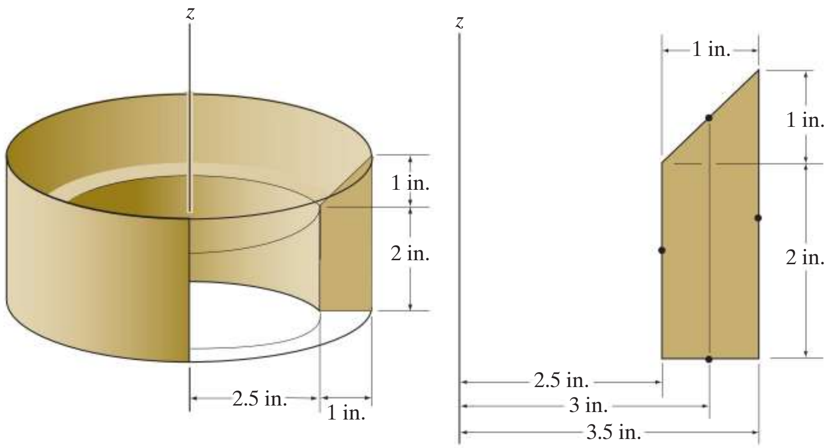
$$A = \theta \bar{r} L; \qquad A = 2\pi\left(\frac{2R}{\pi}\right)\pi R = 4\pi R^2 \qquad \textit{Ans.}$$

Volume. The volume of the sphere is generated by revolving the semicircular *area* in Fig. 9–21*b* about the x axis. Using the table on the inside back cover to locate the centroid of the area, i.e., $\bar{r} = 4R/3\pi$, and applying Eq. 9–8, we have

$$V = \theta \bar{r} A; \qquad V = 2\pi\left(\frac{4R}{3\pi}\right)\left(\frac{1}{2}\pi R^2\right) = \frac{4}{3}\pi R^3 \qquad \textit{Ans.}$$

EXAMPLE 9.13

Determine the surface area and volume of the full solid in Fig. 9–22*a*.



(a) (b)

Fig. 9–22

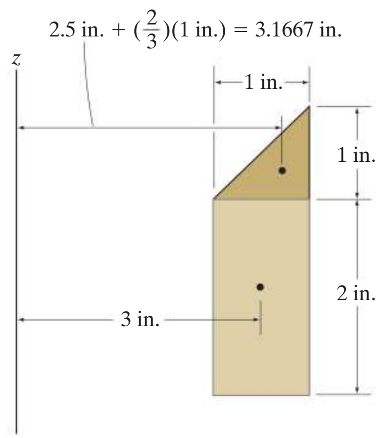
SOLUTION

Surface Area. The surface area is generated by revolving the four line segments shown in Fig. 9–22*b* 2π radians about the z axis. The distances from the centroid of each segment to the z axis are also shown in the figure. Applying Eq. 9–7 yields

$$A = 2\pi\Sigma\tilde{r}L = 2\pi\left[(2.5 \text{ in.})(2 \text{ in.}) + (3 \text{ in.})\left(\sqrt{(1 \text{ in.})^2 + (1 \text{ in.})^2}\right) + (3.5 \text{ in.})(3 \text{ in.}) + (3 \text{ in.})(1 \text{ in.})\right]$$
$$= 143 \text{ in}^2 \qquad \textit{Ans.}$$

Volume. The volume of the solid is generated by revolving the two area segments shown in Fig. 9–22*c* 2π radians about the z axis. The distances from the centroid of each segment to the z axis are also shown in the figure. Applying Eq. 9–10, we have

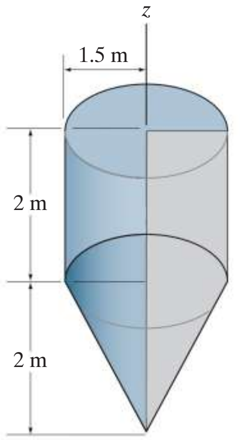
$$V = 2\pi\Sigma\tilde{r}A = 2\pi\left\{(3.1667 \text{ in.})\left[\frac{1}{2}(1 \text{ in.})(1 \text{ in.})\right] + (3 \text{ in.})[(2 \text{ in.})(1 \text{ in.})]\right\}$$
$$= 47.6 \text{ in}^3 \qquad \textit{Ans.}$$



(c)

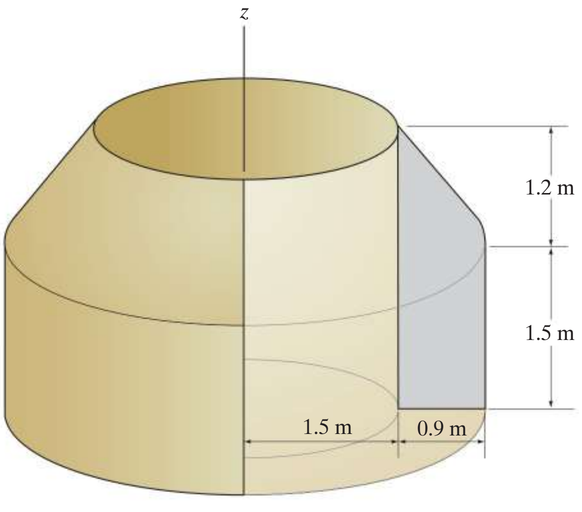
FUNDAMENTAL PROBLEMS

F9–13. Determine the surface area and volume of the solid formed by revolving the shaded area 360° about the z axis.



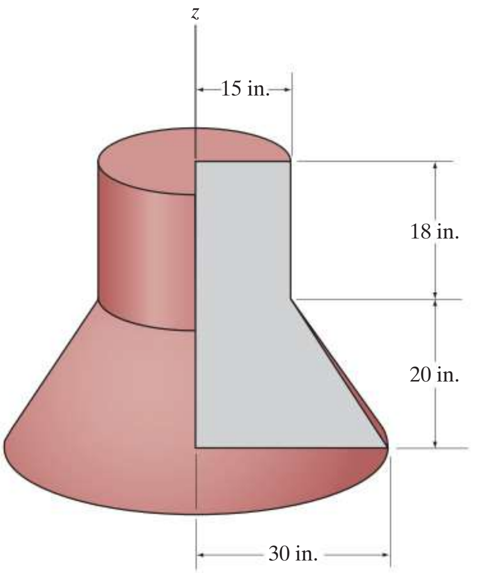
Prob. F9–13

F9–14. Determine the surface area and volume of the solid formed by revolving the shaded area 360° about the z axis.



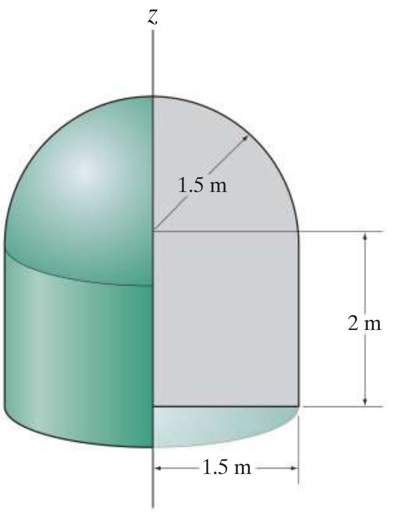
Prob. F9–14

F9–15. Determine the surface area and volume of the solid formed by revolving the shaded area 360° about the z axis.



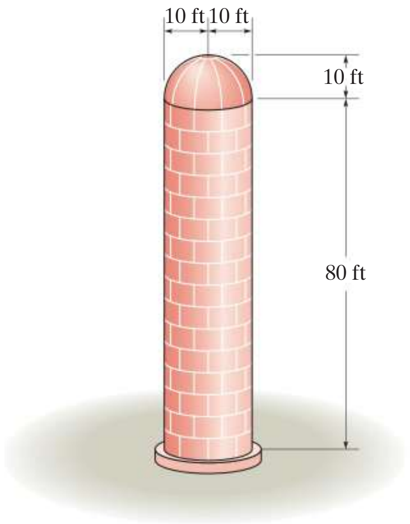
Prob. F9–15

F9–16. Determine the surface area and volume of the solid formed by revolving the shaded area 360° about the z axis.



Prob. F9–16

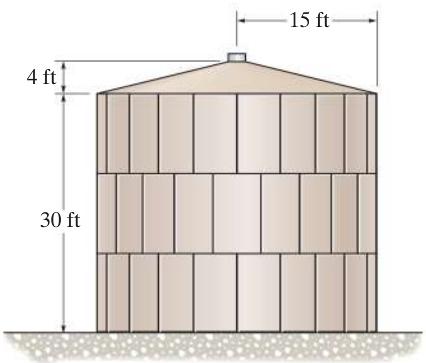
9–90. Determine the volume of the silo which consists of a cylinder and hemispherical cap. Neglect the thickness of the plates.



Prob. 9–90

9–91. Determine the outside surface area of the storage tank.

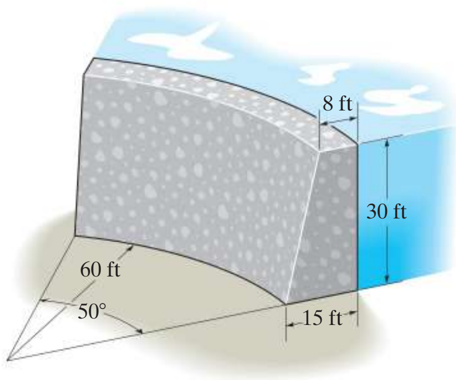
***9–92.** Determine the volume of the storage tank.



Probs. 9–91/92

9–93. Determine the surface area of the concrete seawall, excluding its bottom.

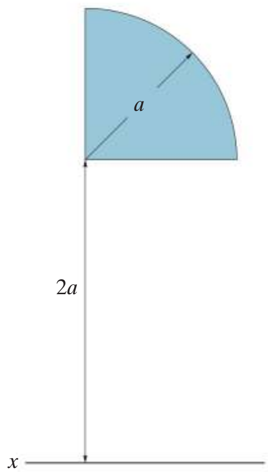
9–94. A circular seawall is made of concrete. Determine the total weight of the wall if the concrete has a specific weight of $\gamma_c = 150 \text{ lb/ft}^3$.



Probs. 9–93/94

9–95. A ring is generated by rotating the quarter circular area about the x axis. Determine its volume.

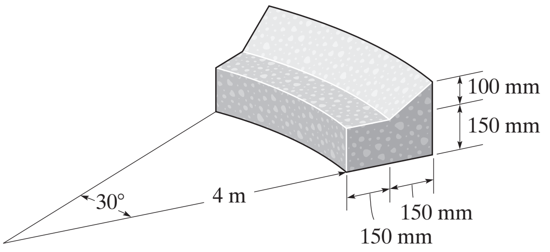
***9–96.** A ring is generated by rotating the quarter circular area about the x axis. Determine its surface area.



Probs. 9–95/96

9–97. Determine the volume of concrete needed to construct the curb.

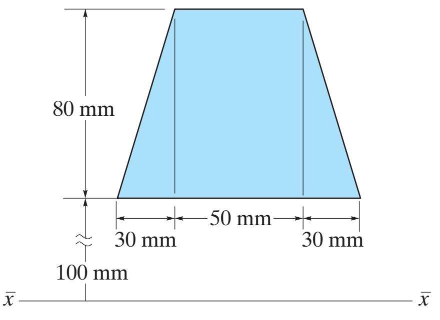
9–98. Determine the surface area of the curb. Do not include the area of the ends in the calculation.



Probs. 9–97/98

9–99. A ring is formed by rotating the area 360° about the $\overline{x}-\overline{x}$ axes. Determine its surface area.

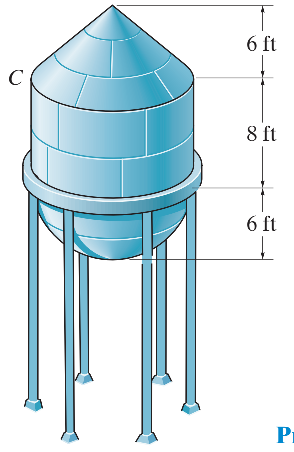
***9–100.** A ring is formed by rotating the area 360° about the $\overline{x}-\overline{x}$ axes. Determine its volume.



Probs. 9–99/100

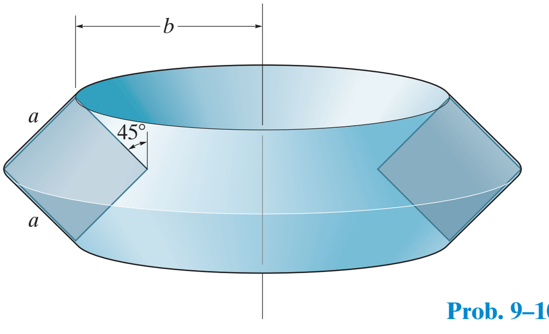
9–101. The water-supply tank has a hemispherical bottom and cylindrical sides. Determine the weight of water in the tank when it is filled to the top at C. Take $\gamma_w = 62.4\ \text{lb/ft}^3$.

9–102. Determine the number of gallons of paint needed to paint the outside surface of the water-supply tank, which consists of a hemispherical bottom, cylindrical sides, and conical top. Each gallon of paint can cover 250 ft².



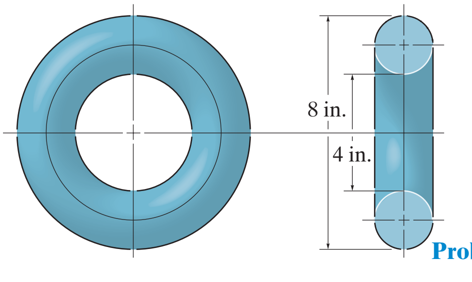
Probs. 9–101/102

9–103. Determine the surface area and the volume of the ring formed by rotating the square about the vertical axis.



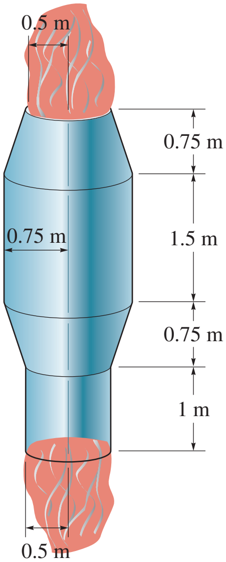
Prob. 9–103

***9–104.** Determine the surface area of the ring. The cross section is circular as shown.



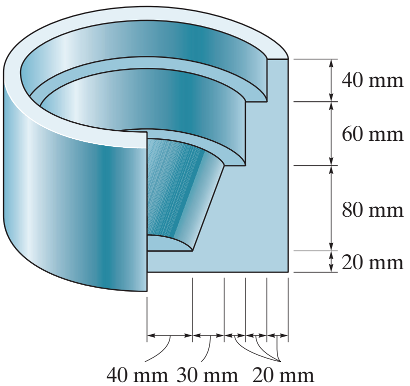
Prob. 9–104

9–105. The heat exchanger radiates thermal energy at the rate of 2500 kJ/h for each square meter of its surface area. Determine how many joules (J) are radiated within a 5-hour period.



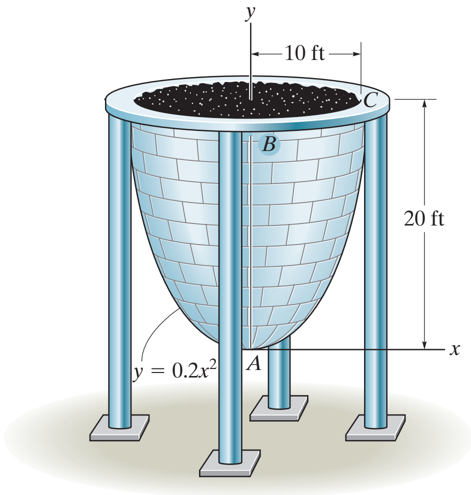
Prob. 9–105

9–106. Determine the interior surface area of the brake piston. It consists of a full circular part. Its cross section is shown in the figure.



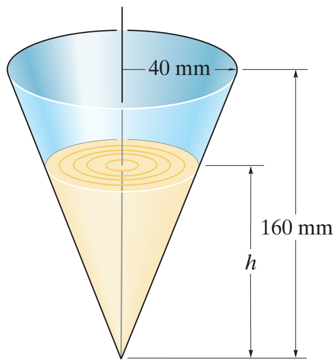
Prob. 9–106

9–107. The suspension bunker is made from plates which are curved to the natural shape which a completely flexible membrane would take if subjected to a full load of coal. This curve may be approximated by a parabola, $y = 0.2x^2$. Determine the weight of coal which the bunker would contain when completely filled. Coal has a specific weight of $\gamma = 50\ \text{lb/ft}^3$, and assume there is a 20% loss in volume due to air voids. Solve the problem by integration to determine the cross-sectional area of ABC; then use the second theorem of Pappus–Guldinus to find the volume.



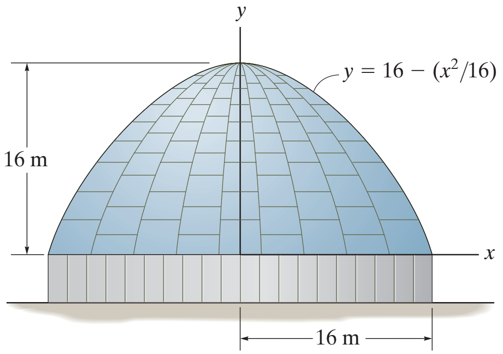
Prob. 9–107

***9–108.** Determine the height h to which liquid should be poured into the cup so that it contacts three-fourths the surface area on the inside of the cup. Neglect the cup's thickness for the calculation.



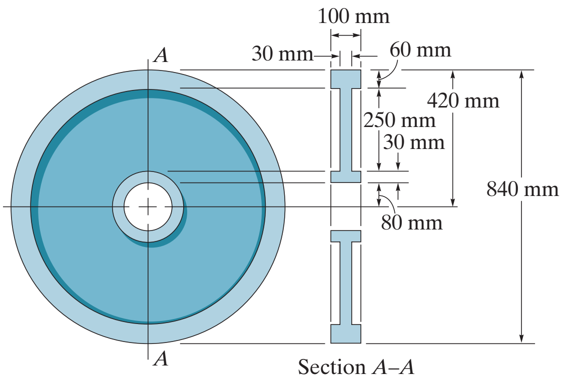
Prob. 9–108

9–109. Determine the surface area of the roof of the structure if it is formed by rotating the parabola about the y axis.



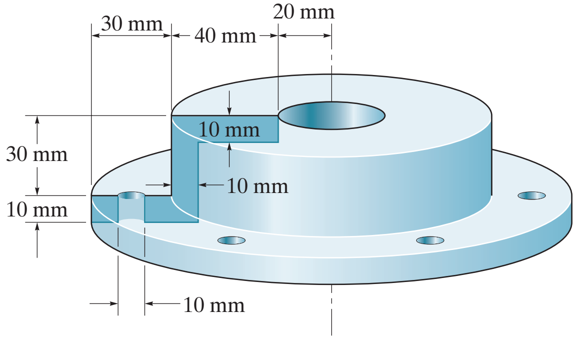
Prob. 9–109

9–110. A steel wheel has a diameter of 840 mm and a cross section as shown in the figure. Determine the total mass of the wheel if $\rho = 5\ \mathrm{Mg/m^3}$.



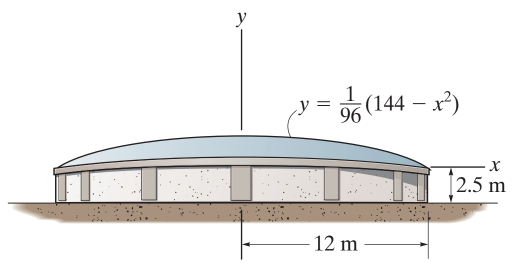
Prob. 9–110

9–111. Half the cross section of the steel housing is shown in the figure. There are six 10-mm-diameter bolt holes around its rim. Determine its mass. The density of steel is $7.85\ \mathrm{Mg/m^3}$. The housing is a full circular part.



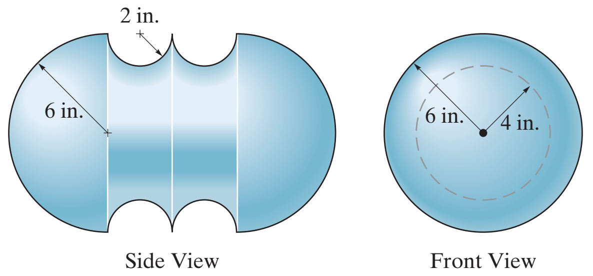
Prob. 9–111

***9–112.** The water tank has a paraboloid-shaped roof. If one liter of paint can cover $3\ \mathrm{m^2}$ of the tank, determine the number of liters required to coat the roof.



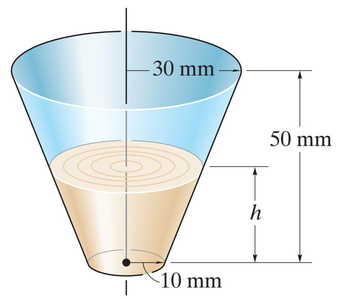
Prob. 9–112

9–113. Determine the volume of material needed to make the casting.



Prob. 9–113

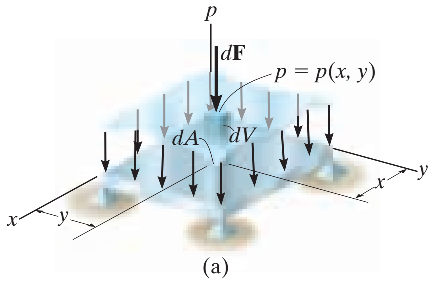
9–114. Determine the height h to which liquid should be poured into the cup so that it contacts half the surface area on the inside of the cup. Neglect the cup's thickness for the calculation.



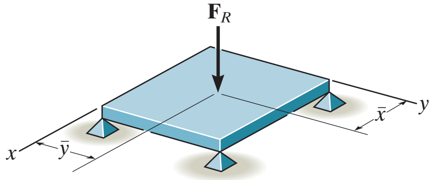
Prob. 9–114

*9.4 Resultant of a General Distributed Loading

In Sec. 4.9, we discussed the method used to simplify a two-dimensional distributed loading to a single resultant force acting at a specific point. In this section we will generalize this method to include flat surfaces that have an arbitrary shape and are subjected to a variable load distribution. Consider, for example, the flat plate shown in Fig. 9–23*a*, which is subjected to the loading defined by $p = p(x, y)$ Pa, where 1 Pa (pascal) $= 1\ \text{N/m}^2$. Knowing this function, we can determine the resultant force $\mathbf{F}_R$ acting on the plate and its location $(\bar{x}, \bar{y})$, Fig. 9–23*b*.



(a)



(b)

Fig. 9–23

Magnitude of Resultant Force. The force $d\mathbf{F}$ acting on the differential area dA m^2 of the plate, located at the arbitrary point (x, y), has a magnitude of $dF = [p(x, y)\ \text{N/m}^2](dA\ \text{m}^2) = [p(x, y)\, dA]$ N. Notice that $p(x, y)\, dA = dV$, the colored differential *volume element* shown in Fig. 9–23*a*. The *magnitude* of $\mathbf{F}_R$ is the sum of the differential forces acting over the plate's *entire surface area A*. Thus:

$$F_R = \Sigma F; \qquad \boxed{F_R = \int_A p(x, y)\, dA = \int_V dV = V} \qquad (9\text{–}11)$$

This result indicates that the *magnitude of the resultant force is equal to the total volume under the distributed-loading diagram*.

The resultant of a wind loading that is distributed on the front or side walls of this building must be calculated using integration in order to design the framework that holds the building together. (© Russell C. Hibbeler)

Location of Resultant Force. The location $(\bar{x}, \bar{y})$ of $\mathbf{F}_R$ is determined by setting the moments of $\mathbf{F}_R$ equal to the moments of all the differential forces $d\mathbf{F}$ about the respective y and x axes: From Figs. 9–23*a* and 9–23*b*, using Eq. 9–11, this results in

$$\boxed{\bar{x} = \frac{\displaystyle\int_A x p(x, y)\, dA}{\displaystyle\int_A p(x, y)\, dA} = \frac{\displaystyle\int_V x\, dV}{\displaystyle\int_V dV} \qquad \bar{y} = \frac{\displaystyle\int_A y p(x, y)\, dA}{\displaystyle\int_A p(x, y)\, dA} = \frac{\displaystyle\int_V y\, dV}{\displaystyle\int_V dV}} \qquad (9\text{–}12)$$

Hence, the *line of action of the resultant force passes through the geometric center or centroid of the volume under the distributed-loading diagram*.

*9.5 Fluid Pressure

According to Pascal's law, a fluid at rest creates a pressure p at a point that is the *same* in *all* directions. The magnitude of p, measured as a force per unit area, depends on the specific weight γ or mass density ρ of the fluid and the depth z of the point from the fluid surface.* The relationship can be expressed mathematically as

$$\boxed{p = \gamma z = \rho g z} \tag{9–13}$$

where g is the acceleration due to gravity. This equation is valid only for fluids that are assumed *incompressible*, as in the case of most liquids. Gases are compressible fluids, and since their density changes significantly with both pressure and temperature, Eq. 9–13 cannot be used.

To illustrate how Eq. 9–13 is applied, consider the submerged plate shown in Fig. 9–24. Three points on the plate have been specified. Since point B is at depth z_1 from the liquid surface, the *pressure* at this point has a magnitude $p_1 = \gamma z_1$. Likewise, points C and D are both at depth z_2; hence, $p_2 = \gamma z_2$. In all cases, the pressure acts *normal* to the surface area dA located at the specified point.

Using Eq. 9–13 and the results of Sec. 9.4, it is possible to determine the resultant force caused by a liquid and specify its location on the surface of a submerged plate. Three different shapes of plates will now be considered.

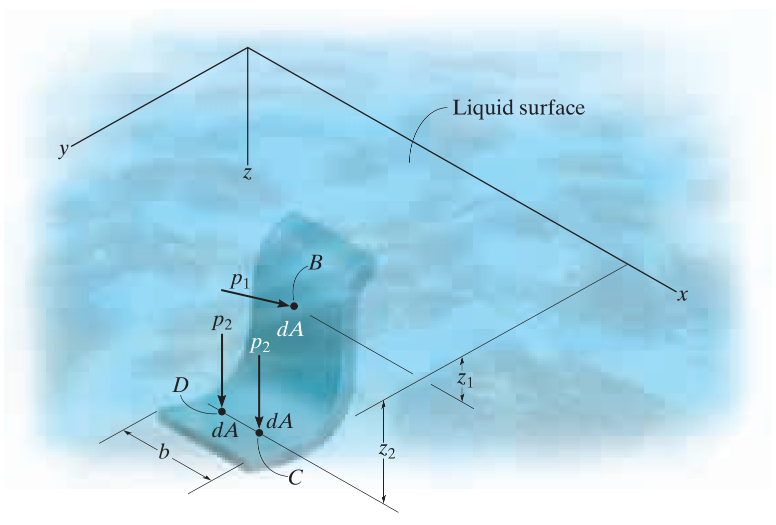


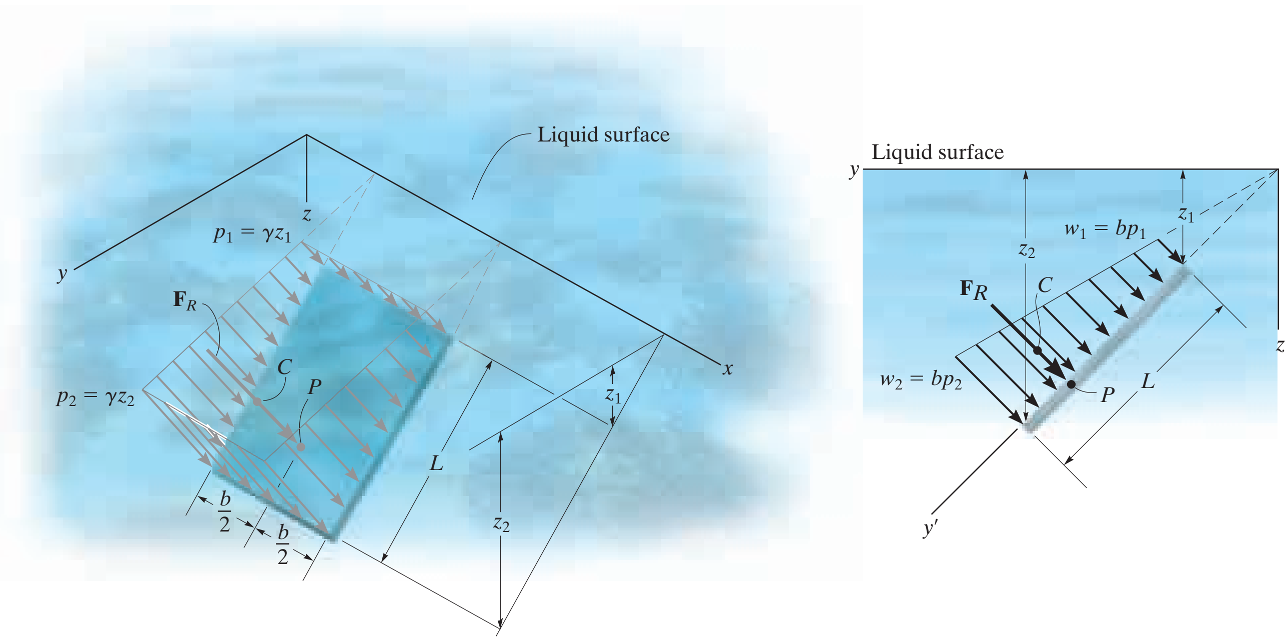
Fig. 9–24

*In particular, for water $\gamma = 62.4\ \text{lb/ft}^3$, or $\gamma = \rho g = 9810\ \text{N/m}^3$ since $\rho = 1000\ \text{kg/m}^3$ and $g = 9.81\ \text{m/s}^2$.

Flat Plate of Constant Width. A flat rectangular plate of constant width, which is submerged in a liquid having a specific weight γ, is shown in Fig. 9–25*a*. Since pressure varies linearly with depth, Eq. 9–13, the distribution of pressure over the plate's surface is represented by a trapezoidal volume having an intensity of $p_1 = \gamma z_1$ at depth z_1 and $p_2 = \gamma z_2$ at depth z_2. As noted in Sec. 9.4, the magnitude of the *resultant force* $\mathbf{F}_R$ is equal to the *volume* of this loading diagram and $\mathbf{F}_R$ has a *line of action* that passes through the volume's centroid C. Hence, $\mathbf{F}_R$ does *not* act at the centroid of the plate; rather, it acts at point P, called the *center of pressure*.

Since the plate has a *constant width*, the loading distribution may also be viewed in two dimensions, Fig. 9–25*b*. Here the loading intensity is measured as force/length and varies linearly from $w_1 = bp_1 = b\gamma z_1$ to $w_2 = bp_2 = b\gamma z_2$. The magnitude of $\mathbf{F}_R$ in this case equals the trapezoidal *area*, and $\mathbf{F}_R$ has a *line of action* that passes through the area's *centroid* C. For numerical applications, the area and location of the centroid for a trapezoid are tabulated on the inside back cover.

The walls of the tank must be designed to support the pressure loading of the liquid that is contained within it. (© Russell C. Hibbeler)



(a) (b)

Fig. 9–25

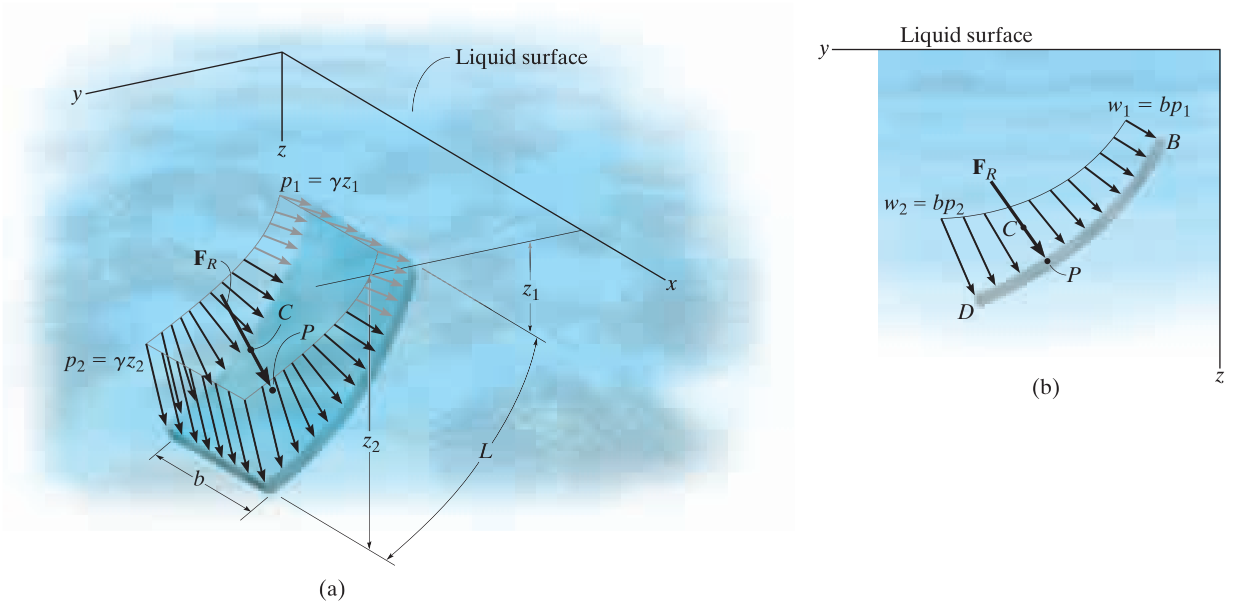
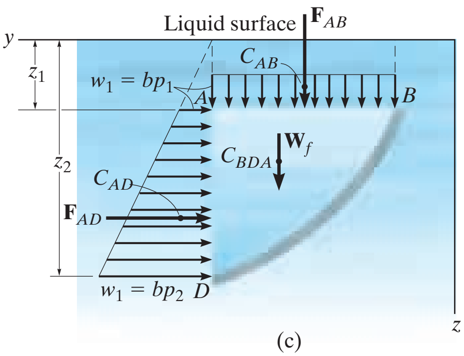


Fig. 9–26



Curved Plate of Constant Width. When a submerged plate of constant width is curved, the pressure acting normal to the plate continually changes both its magnitude and direction, and therefore calculation of the magnitude of $\mathbf{F}_R$ and its location P is more difficult than for a flat plate. Three- and two-dimensional views of the loading distribution are shown in Figs. 9–26*a* and 9–26*b*, respectively. Although integration can be used to solve this problem, a simpler method exists. This method requires separate calculations for the horizontal and vertical *components* of $\mathbf{F}_R$.

For example, the distributed loading acting on the plate can be represented by the *equivalent loading* shown in Fig. 9–26*c*. Here the plate supports the weight of liquid W_f contained within the block BDA. This force has a magnitude $W_f = (\gamma b)(\text{area}_{BDA})$ and acts through the centroid of BDA. In addition, there are the pressure distributions caused by the liquid acting along the vertical and horizontal sides of the block. Along the vertical side AD, the force $\mathbf{F}_{AD}$ has a magnitude equal to the area of the trapezoid. It acts through the centroid C_{AD} of this area. The distributed loading along the horizontal side AB is *constant* since all points lying in this plane are at the same depth from the surface of the liquid. The magnitude of $\mathbf{F}_{AB}$ is simply the area of the rectangle. This force acts through the centroid C_{AB} or at the midpoint of AB. Summing these three forces yields $\mathbf{F}_R = \Sigma\mathbf{F} = \mathbf{F}_{AD} + \mathbf{F}_{AB} + \mathbf{W}_f$. Finally, the location of the center of pressure P on the plate is determined by applying $M_R = \Sigma M$, which states that the moment of the resultant force about a convenient reference point such as D or B, in Fig. 9–26*b*, is equal to the sum of the moments of the three forces in Fig. 9–26*c* about this same point.

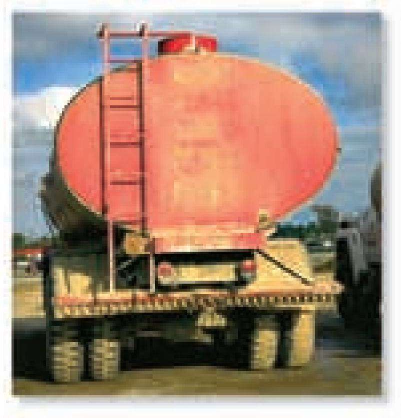
Flat Plate of Variable Width. The pressure distribution acting on the surface of a submerged plate having a variable width is shown in Fig. 9–27. If we consider the force $d\mathbf{F}$ acting on the differential area strip dA, parallel to the x axis, then its magnitude is $dF = p\,dA$. Since the depth of dA is z, the pressure on the element is $p = \gamma z$. Therefore, $dF = (\gamma z)dA$ and so the resultant force becomes

$$F_R = \int dF = \gamma \int z\,dA$$

If the depth to the centroid C' of the area is $\bar{z}$, Fig. 9–27, then, $\int z\,dA = \bar{z}A$. Substituting, we have

$$F_R = \gamma \bar{z} A \tag{9–14}$$

In other words, *the magnitude of the resultant force acting on any flat plate is equal to the product of the area A of the plate and the pressure $p = \gamma\bar{z}$ at the depth of the area's centroid C'.* As discussed in Sec. 9.4, this force is also equivalent to the volume under the pressure distribution. Realize that its line of action passes through the centroid C of this *volume* and intersects the plate at the center of pressure P, Fig. 9–27. Notice that the location of C' does not coincide with the location of P.



The resultant force of the water pressure and its location on the elliptical back plate of this tank truck must be determined by integration. (© Russell C. Hibbeler)

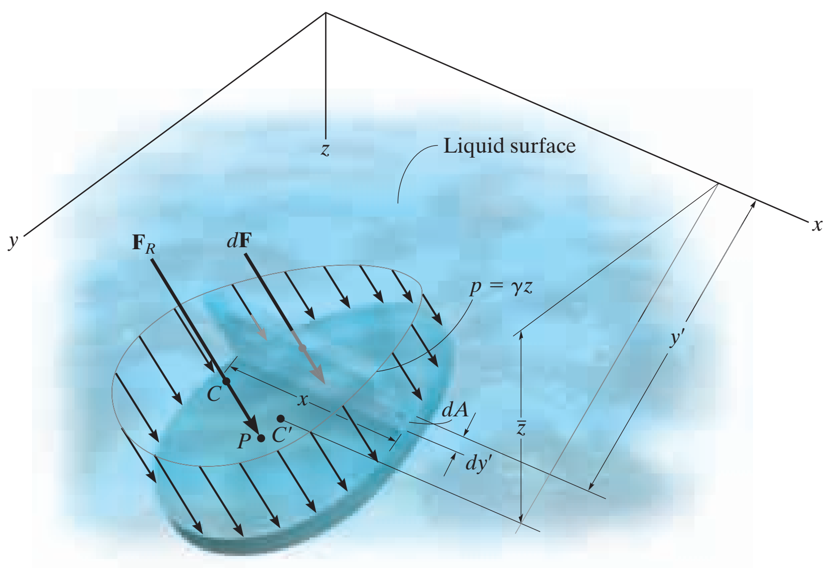


Fig. 9–27

EXAMPLE 9.14

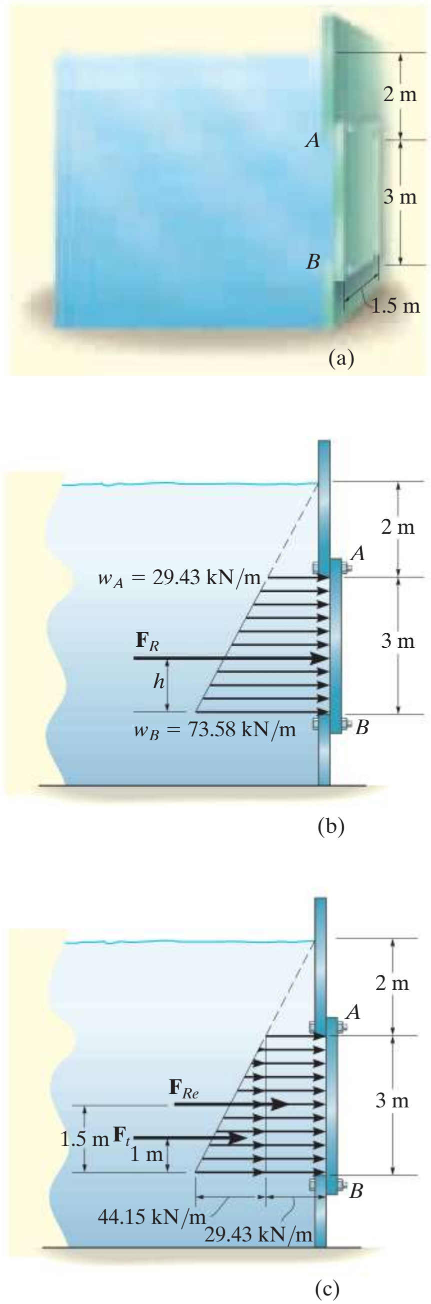


Fig. 9–28

Determine the magnitude and location of the resultant hydrostatic force acting on the submerged rectangular plate AB shown in Fig. 9–28a. The plate has a width of 1.5 m; $\rho_w = 1000\ \text{kg/m}^3$.

SOLUTION I

The water pressures at depths A and B are

$$p_A = \rho_w g z_A = (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(2\ \text{m}) = 19.62\ \text{kPa}$$

$$p_B = \rho_w g z_B = (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(5\ \text{m}) = 49.05\ \text{kPa}$$

Since the plate has a constant width, the pressure loading can be viewed in two dimensions, as shown in Fig. 9–28b. The intensities of the load at A and B are

$$w_A = bp_A = (1.5\ \text{m})(19.62\ \text{kPa}) = 29.43\ \text{kN/m}$$

$$w_B = bp_B = (1.5\ \text{m})(49.05\ \text{kPa}) = 73.58\ \text{kN/m}$$

From the table on the inside back cover, the magnitude of the resultant force $\mathbf{F}_R$ created by this distributed load is

$$F_R = \text{area of a trapezoid} = \tfrac{1}{2}(3)(29.4 + 73.6) = 154.5\ \text{kN} \qquad \textit{Ans.}$$

This force acts through the centroid of this area,

$$h = \frac{1}{3}\left(\frac{2(29.43) + 73.58}{29.43 + 73.58}\right)(3) = 1.29\ \text{m} \qquad \textit{Ans.}$$

measured upward from B, Fig. 9–31b.

SOLUTION II

The same results can be obtained by considering two components of $\mathbf{F}_R$, defined by the triangle and rectangle shown in Fig. 9–28c. Each force acts through its associated centroid and has a magnitude of

$$F_{Re} = (29.43\ \text{kN/m})(3\ \text{m}) = 88.3\ \text{kN}$$

$$F_t = \tfrac{1}{2}(44.15\ \text{kN/m})(3\ \text{m}) = 66.2\ \text{kN}$$

Hence,

$$F_R = F_{Re} + F_t = 88.3 + 66.2 = 154.5\ \text{kN} \qquad \textit{Ans.}$$

The location of $\mathbf{F}_R$ is determined by summing moments about B, Figs. 9–28b and c, i.e.,

$$\circlearrowleft + (M_R)_B = \Sigma M_B;\quad (154.5)h = 88.3(1.5) + 66.2(1)$$

$$h = 1.29\ \text{m} \qquad \textit{Ans.}$$

NOTE: Using Eq. 9–14, the resultant force can be calculated as $F_R = \gamma \bar{z} A = (9810\ \text{N/m}^3)(3.5\ \text{m})(3\ \text{m})(1.5\ \text{m}) = 154.5\ \text{kN}$.

EXAMPLE 9.15

Determine the magnitude of the resultant hydrostatic force acting on the surface of a seawall shaped in the form of a parabola, as shown in Fig. 9–29*a*. The wall is 5 m long; $\rho_w = 1020\ \text{kg/m}^3$.

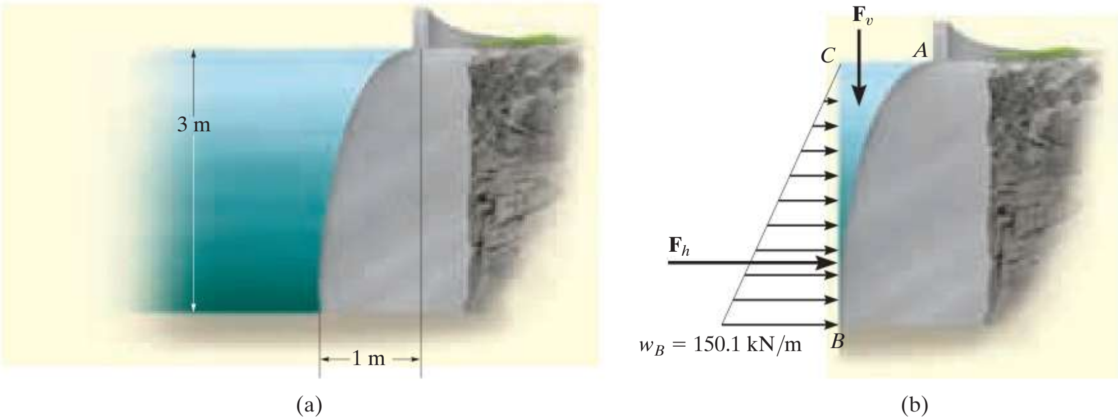


Fig. 9–29

SOLUTION

The horizontal and vertical components of the resultant force will be calculated, Fig. 9–29*b*. Since

$$p_B = \rho_w g z_B = (1020\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(3\ \text{m}) = 30.02\ \text{kPa}$$

then

$$w_B = bp_B = 5\ \text{m}(30.02\ \text{kPa}) = 150.1\ \text{kN/m}$$

Thus,

$$F_h = \tfrac{1}{2}(3\ \text{m})(150.1\ \text{kN/m}) = 225.1\ \text{kN}$$

The area of the parabolic section ABC can be determined using the formula for a parabolic area $A = \frac{1}{3}ab$. Hence, the weight of water within this 5-m-long region is

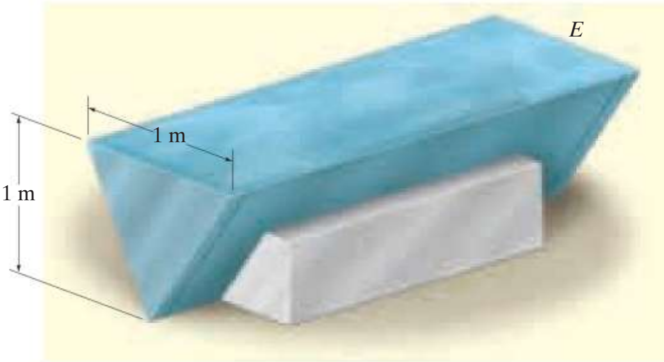
$$F_v = (\rho_w g b)(\text{area}_{ABC})$$
$$= (1020\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(5\ \text{m})\left[\tfrac{1}{3}(1\ \text{m})(3\ \text{m})\right] = 50.0\ \text{kN}$$

The resultant force is therefore

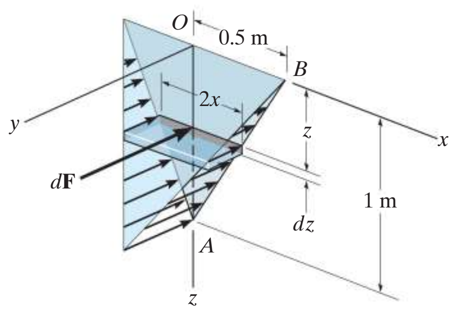
$$F_R = \sqrt{F_h^2 + F_v^2} = \sqrt{(225.1\ \text{kN})^2 + (50.0\ \text{kN})^2}$$
$$= 231\ \text{kN} \qquad \textit{Ans.}$$

EXAMPLE 9.16

Determine the magnitude and location of the resultant force acting on the triangular end plates of the water trough shown in Fig. 9–30*a*; $\rho_w = 1000\ \text{kg/m}^3$.



(a)



(b)

Fig. 9–30

SOLUTION

The pressure distribution acting on the end plate E is shown in Fig. 9–30*b*. The magnitude of the resultant force is equal to the volume of this loading distribution. We will solve the problem by integration. Choosing the differential volume element shown in the figure, we have

$$dF = dV = p\,dA = \rho_w g z(2x\,dz) = 19\,620zx\,dz$$

The equation of line AB is

$$x = 0.5(1 - z)$$

Hence, substituting and integrating with respect to z from $z = 0$ to $z = 1$ m yields

$$F = V = \int_V dV = \int_0^{1\text{ m}} (19\,620)z[0.5(1 - z)]\,dz$$

$$= 9810\int_0^{1\text{ m}} (z - z^2)\,dz = 1635\text{ N} = 1.64\text{ kN} \qquad \textit{Ans.}$$

This resultant passes through the *centroid of the volume*. Because of symmetry,

$$\bar{x} = 0 \qquad \textit{Ans.}$$

Since $\tilde{z} = z$ for the volume element, then

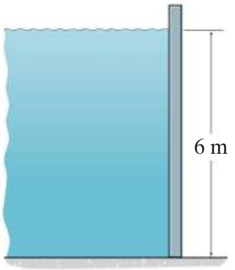
$$\bar{z} = \frac{\int_V \tilde{z}\,dV}{\int_V dV} = \frac{\int_0^{1\text{ m}} z(19\,620)z[0.5(1 - z)]\,dz}{1635} = \frac{9810\int_0^{1\text{ m}} (z^2 - z^3)\,dz}{1635}$$

$$= 0.5\text{ m} \qquad \textit{Ans.}$$

NOTE: We can also determine the resultant force by applying Eq. 9–14, $F_R = \gamma\bar{z}A = \left(9810\ \text{N/m}^3\right)\left(\frac{1}{3}\right)(1\text{ m})\left[\frac{1}{2}(1\text{ m})(1\text{ m})\right] = 1.64\text{ kN}$.

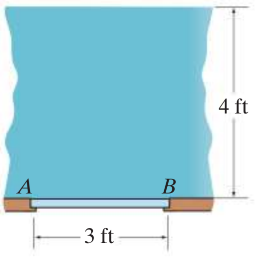
FUNDAMENTAL PROBLEMS

F9–17. Determine the magnitude of the hydrostatic force acting per meter length of the wall. Water has a density of $\rho = 1\ \mathrm{Mg/m^3}$.



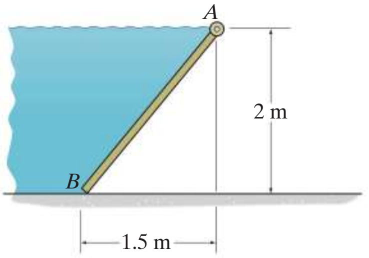
Prob. F9–17

F9–18. Determine the magnitude of the hydrostatic force acting on gate AB, which has a width of 4 ft. The specific weight of water is $\gamma = 62.4\ \mathrm{lb/ft^3}$.



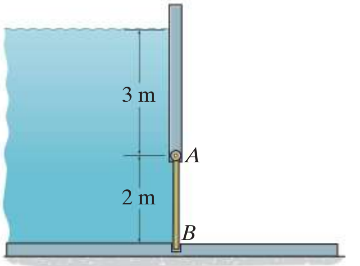
Prob. F9–18

F9–19. Determine the magnitude of the hydrostatic force acting on gate AB, which has a width of 1.5 m. Water has a density of $\rho = 1\ \mathrm{Mg/m^3}$.



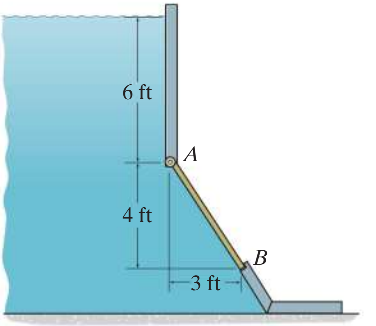
Prob. F9–19

F9–20. Determine the magnitude of the hydrostatic force acting on gate AB, which has a width of 2 m. Water has a density of $\rho = 1\ \mathrm{Mg/m^3}$.



Prob. F9–20

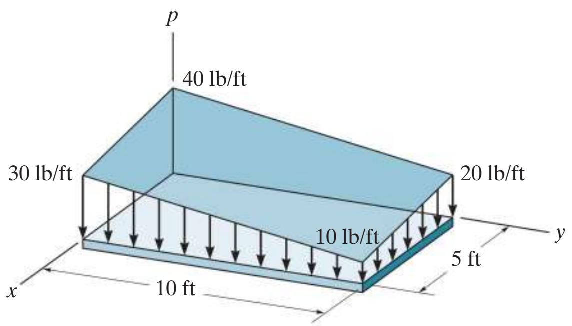
F9–21. Determine the magnitude of the hydrostatic force acting on gate AB, which has a width of 2 ft. The specific weight of water is $\gamma = 62.4\ \mathrm{lb/ft^3}$.



Prob. F9–21

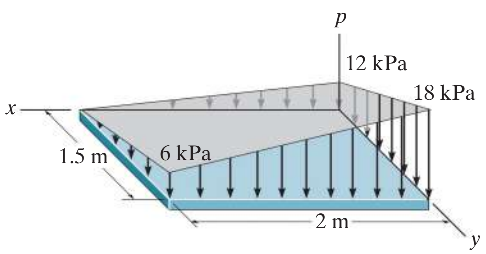
PROBLEMS

9–115. The pressure loading on the plate varies uniformly along each of its edges. Determine the magnitude of the resultant force and the coordinates $(\bar{x}, \bar{y})$ of the point where the line of action of the force intersects the plate. *Hint*: The equation defining the boundary of the load has the form $p = ax + by + c$, where the constants a, b, and c have to be determined.



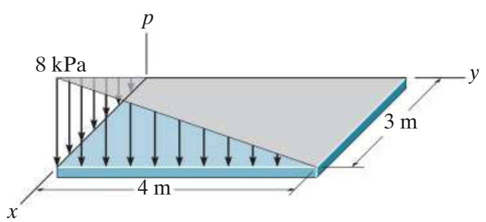
Prob. 9–115

***9–116.** The load over the plate varies linearly along the sides of the plate such that $p = (12 - 6x + 4y)$ kPa. Determine the magnitude of the resultant force and the coordinates $(\bar{x}, \bar{y})$ of the point where the line of action of the force intersects the plate.



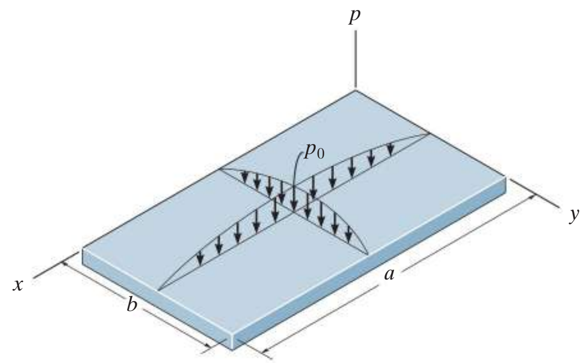
Prob. 9–116

9–117. The load over the plate varies linearly along the sides of the plate such that $p = \frac{2}{3}[x(4 - y)]$ kPa. Determine the resultant force and its position $(\bar{x}, \bar{y})$ on the plate.



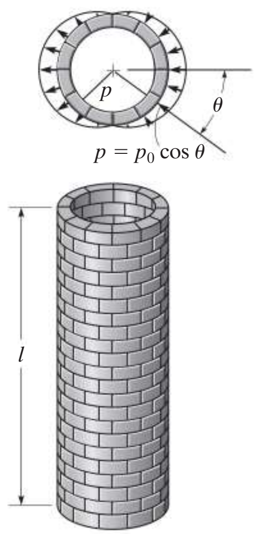
Prob. 9–117

9–118. The rectangular plate is subjected to a distributed load over its *entire surface*. The load is defined by the expression $p = p_0 \sin(\pi x/a)\sin(\pi y/b)$, where p_0 represents the pressure acting at the center of the plate. Determine the magnitude and location of the resultant force acting on the plate.



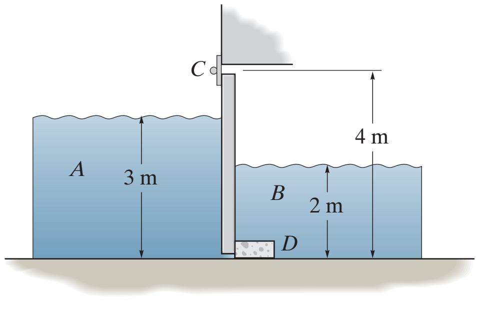
Prob. 9–118

9–119. A wind loading creates a positive pressure on one side of the chimney and a negative (suction) pressure on the other side, as shown. If this pressure loading acts uniformly along the chimney's length, determine the magnitude of the resultant force created by the wind.



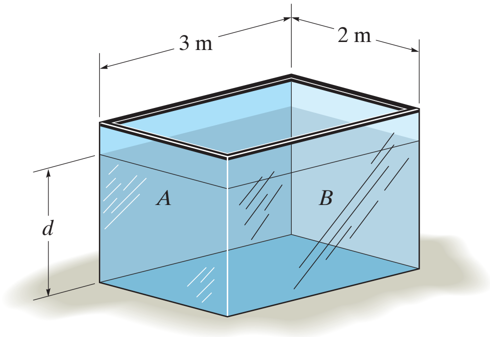
Prob. 9–119

***9–120.** When the tide water A subsides, the tide gate automatically swings open to drain the marsh B. For the condition of high tide shown, determine the horizontal reactions developed at the hinge C and stop block D. The length of the gate is 6 m and its height is 4 m. $\rho_w = 1.0\ \mathrm{Mg/m^3}$.



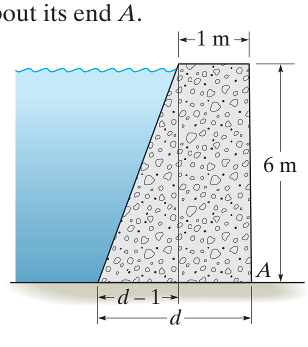
Prob. 9–120

9–121. The tank is filled with water to a depth of $d = 4$ m. Determine the resultant force the water exerts on side A and side B of the tank. If oil instead of water is placed in the tank, to what depth d should it reach so that it creates the same resultant forces? $\rho_o = 900\ \mathrm{kg/m^3}$ and $\rho_w = 1000\ \mathrm{kg/m^3}$.



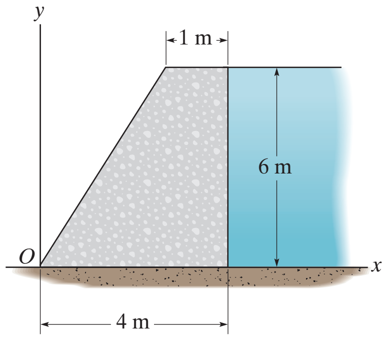
Prob. 9–121

9–122. The concrete "gravity" dam is held in place by its own weight. If the density of concrete is $\rho_c = 2.5\ \mathrm{Mg/m^3}$, and water has a density of $\rho_w = 1.0\ \mathrm{Mg/m^3}$, determine the smallest dimension d that will prevent the dam from overturning about its end A.



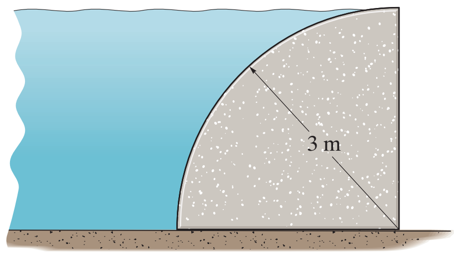
Prob. 9–122

9–123. The factor of safety for tipping of the concrete dam is defined as the ratio of the stabilizing moment due to the dam's weight divided by the overturning moment about O due to the water pressure. Determine this factor if the concrete has a density of $\rho_{\text{conc}} = 2.5\ \mathrm{Mg/m^3}$ and for water $\rho_w = 1\ \mathrm{Mg/m^3}$.



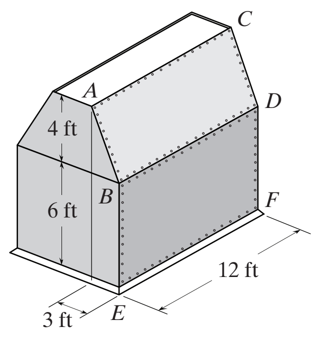
Prob. 9–123

***9–124.** The concrete dam in the shape of a quarter circle. Determine the magnitude of the resultant hydrostatic force that acts on the dam per meter of length. The density of water is $\rho_w = 1\ \mathrm{Mg/m^3}$.



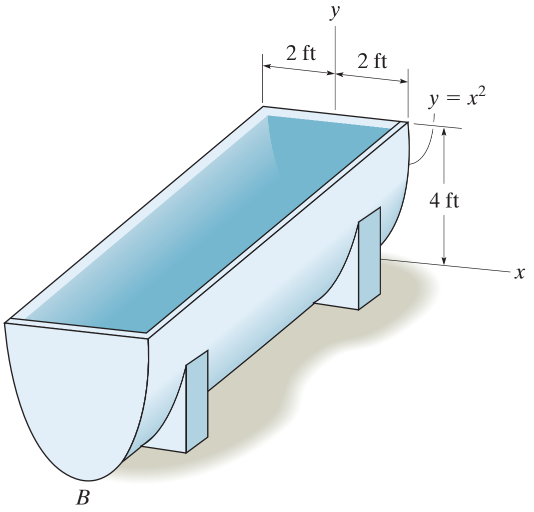
Prob. 9–124

9–125. The tank is used to store a liquid having a density of $80\ \mathrm{lb/ft^3}$. If it is filled to the top, determine the magnitude of force the liquid exerts on each of its two sides $ABDC$ and $BDFE$.



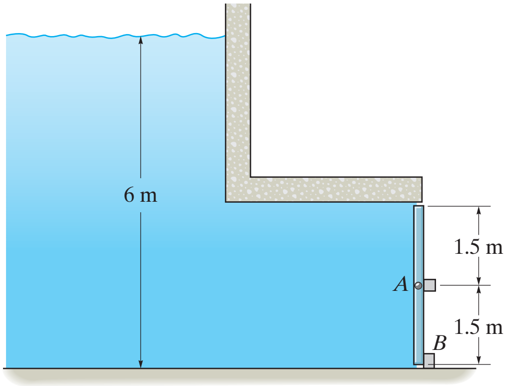
Prob. 9–125

9–126. The parabolic plate is subjected to a fluid pressure that varies linearly from 0 at its top to 100 lb/ft at its bottom B. Determine the magnitude of the resultant force and its location on the plate.



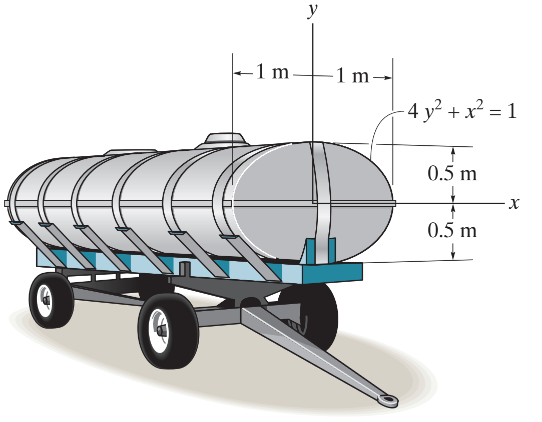
Prob. 9–126

9–127. The 2-m-wide rectangular gate is pinned at its center A and is prevented from rotating by the block at B. Determine the reactions at these supports due to hydrostatic pressure. $\rho_w = 1.0 \text{ Mg/m}^3$.



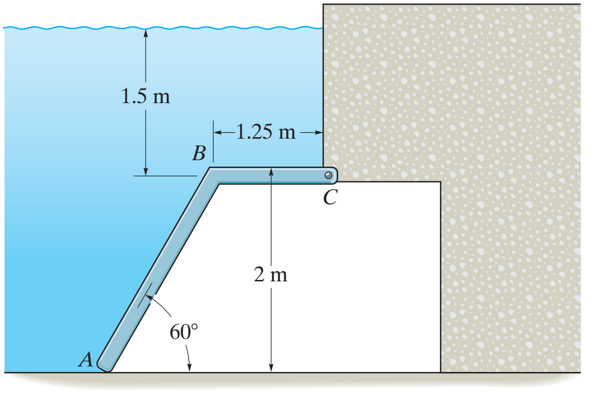
Prob. 9–127

***9–128.** The tank is filled with a liquid that has a density of 900 kg/m^3. Determine the resultant force that it exerts on the elliptical end plate, and the location of the center of pressure, measured from the x axis.



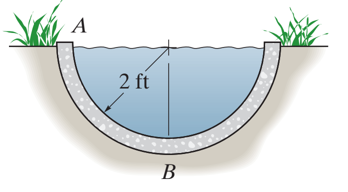
Prob. 9–128

9–129. Determine the magnitude of the resultant force acting on the gate ABC due to hydrostatic pressure. The gate has a width of 1.5 m. $\rho_w = 1.0 \text{ Mg/m}^3$.



Prob. 9–129

9–130. The semicircular drainage pipe is filled with water. Determine the resultant horizontal and vertical force components that the water exerts on the side AB of the pipe per foot of pipe length; $\gamma_w = 62.4 \text{ lb/ft}^3$.



Prob. 9–130

CHAPTER REVIEW

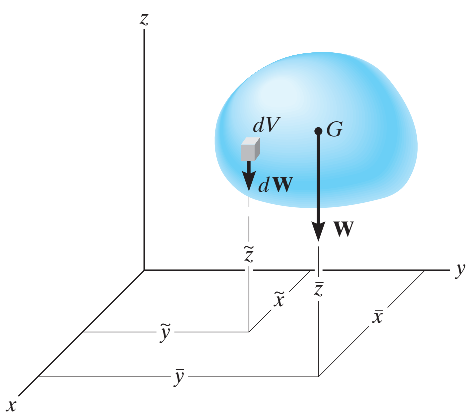
Center of Gravity and Centroid

The *center of gravity G* represents a point where the weight of the body can be considered concentrated. The distance from an axis to this point can be determined from a balance of moments, which requires that the moment of the weight of all the particles of the body about this axis must equal the moment of the entire weight of the body about the axis.

$$\bar{x} = \frac{\int \tilde{x}\, dW}{\int dW}$$

$$\bar{y} = \frac{\int \tilde{y}\, dW}{\int dW}$$

$$\bar{z} = \frac{\int \tilde{z}\, dW}{\int dW}$$



The center of mass will coincide with the center of gravity provided the acceleration of gravity is constant.

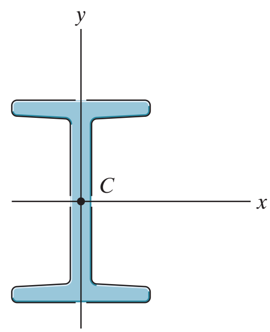
$$\bar{x} = \frac{\int_L \tilde{x}\, dL}{\int_L dL} \qquad \bar{y} = \frac{\int_L \tilde{y}\, dL}{\int_L dL} \qquad \bar{z} = \frac{\int_L \tilde{z}\, dL}{\int_L dL}$$

The *centroid* is the location of the geometric center for the body. It is determined in a similar manner, using a moment balance of geometric elements such as line, area, or volume segments. For bodies having a continuous shape, moments are summed (integrated) using differential elements.

$$\bar{x} = \frac{\int_A \tilde{x}\, dA}{\int_A dA} \qquad \bar{y} = \frac{\int_A \tilde{y}\, dA}{\int_A dA} \qquad \bar{z} = \frac{\int_A \tilde{z}\, dA}{\int_A dA}$$

$$\bar{x} = \frac{\int_V \tilde{x}\, dV}{\int_V dV} \qquad \bar{y} = \frac{\int_V \tilde{y}\, dV}{\int_A dV} \qquad \bar{z} = \frac{\int_V \tilde{z}\, dV}{\int_V dV}$$

The center of mass will coincide with the centroid provided the material is homogeneous, i.e., the density of the material is the same throughout. The centroid will always lie on an axis of symmetry.



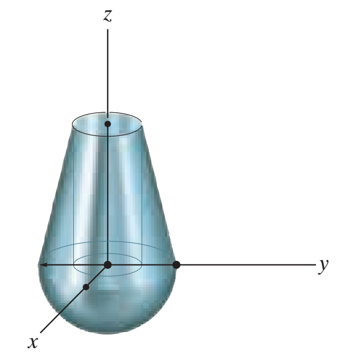
Composite Body

If the body is a composite of several shapes, each having a known location for its center of gravity or centroid, then the location of the center of gravity or centroid of the body can be determined from a discrete summation using its composite parts.

$$\bar{x} = \frac{\Sigma \tilde{x} W}{\Sigma W}$$

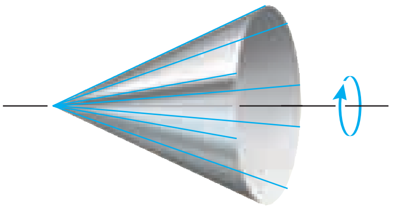
$$\bar{y} = \frac{\Sigma \tilde{y} W}{\Sigma W}$$

$$\bar{z} = \frac{\Sigma \tilde{z} W}{\Sigma W}$$



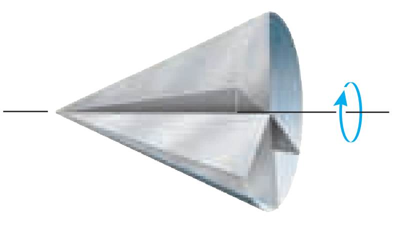
Theorems of Pappus and Guldinus

The theorems of Pappus and Guldinus can be used to determine the surface area and volume of a body of revolution.



The *surface area* equals the product of the length of the generating curve and the distance traveled by the centroid of the curve needed to generate the area.

$$A = \theta \bar{r} L$$



The *volume* of the body equals the product of the generating area and the distance traveled by the centroid of this area needed to generate the volume.

$$V = \theta \bar{r} A$$

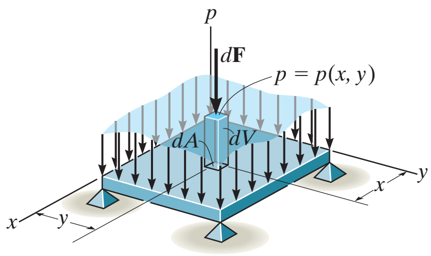
General Distributed Loading

The magnitude of the resultant force is equal to the total volume under the distributed-loading diagram. The line of action of the resultant force passes through the geometric center or centroid of this volume.

$$F_R = \int_A p(x, y)\, dA = \int_V dV$$

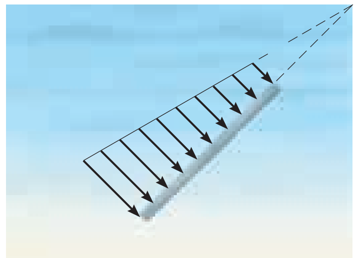
$$\bar{x} = \frac{\int_V x\, dV}{\int_V dV}$$

$$\bar{y} = \frac{\int_V y\, dV}{\int_V dV}$$

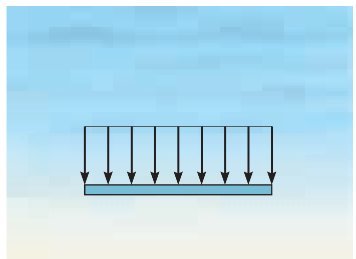


Fluid Pressure

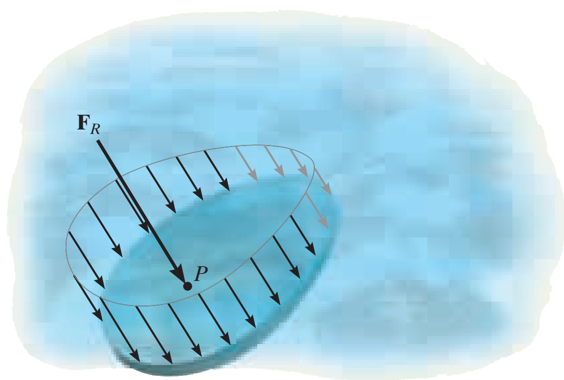
The pressure developed by a liquid at a point on a submerged surface depends upon the depth of the point and the density of the liquid in accordance with Pascal's law, $p = \rho g h = \gamma h$. This pressure will create a *linear distribution* of loading on a flat vertical or inclined surface.



If the surface is horizontal, then the loading will be *uniform*.



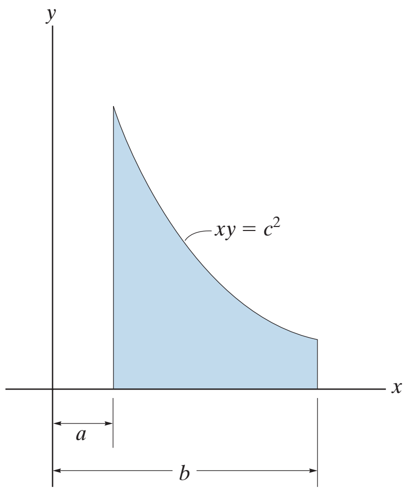
In any case, the resultants of these loadings can be determined by finding the volume under the loading curve or using $F_R = \gamma \bar{z} A$, where $\bar{z}$ is the depth to the centroid of the plate's area. The line of action of the resultant force passes through the centroid of the volume of the loading diagram and acts at a point P on the plate called the center of pressure.



REVIEW PROBLEMS

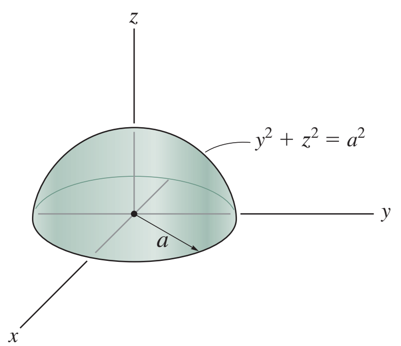
R9–1. Locate the centroid $\bar{x}$ of the area.

R9–2. Locate the centroid $\bar{y}$ of the area.



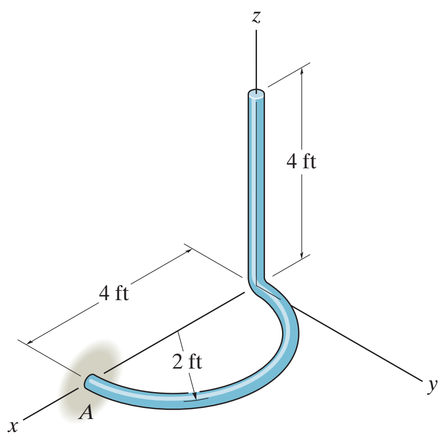
Probs. R9–1/2

R9–3. Locate the centroid $\bar{z}$ of the hemisphere.



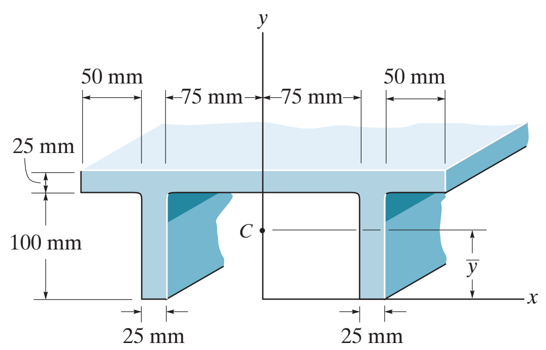
Prob. R9–3

R9–4. Locate the centroid of the rod.



Prob. R9–4

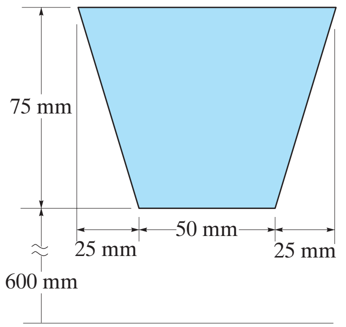
R9–5. Locate the centroid $\bar{y}$ of the beam's cross-sectional area.



Prob. R9–5

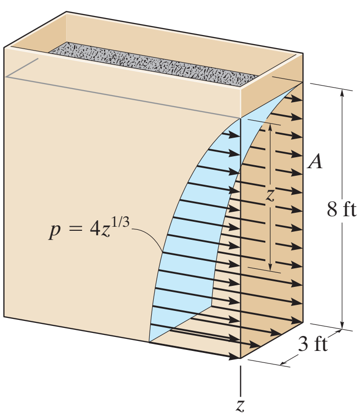
R9–6. A circular V-belt has an inner radius of 600 mm and a cross-sectional area as shown. Determine the surface area of the belt.

R9–7. A circular V-belt has an inner radius of 600 mm and a cross-sectional area as shown. Determine the volume of material required to make the belt.



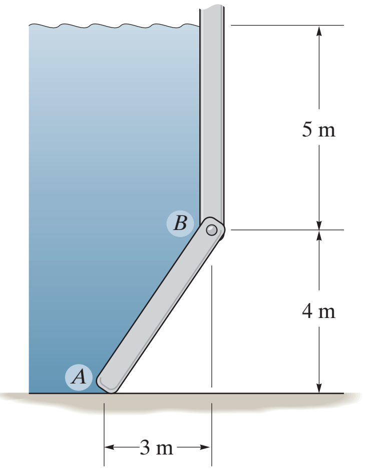
Probs. R9–6/7

R9–8. The rectangular bin is filled with coal, which creates a pressure distribution along wall A that varies as shown, i.e., $p = 4z^{1/3}$ lb/ft^2, where z is in feet. Determine the resultant force created by the coal, and its location, measured from the top surface of the coal.



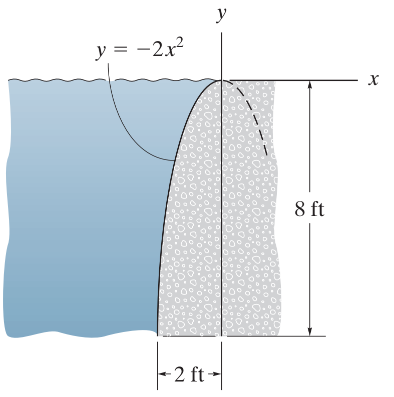
Prob. R9–8

R9–9. The gate AB is 8 m wide. Determine the horizontal and vertical components of force acting on the pin at B and the vertical reaction at the smooth support A; $\rho_w = 1.0$ Mg/m^3.



Prob. R9–9

R9–10. Determine the magnitude of the resultant hydrostatic force acting per foot of length on the seawall; $\gamma_w = 62.4$ lb/ft^3.



Prob. R9–10

Chapter 10

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The design of these structural members requires calculation of their cross-sectional moment of inertia. In this chapter we will discuss how this is done.

Moments of Inertia

CHAPTER OBJECTIVES

- To develop a method for determining the moment of inertia for an area.
- To introduce the product of inertia and show how to determine the maximum and minimum moments of inertia for an area.
- To discuss the mass moment of inertia.

10.1 Definition of Moments of Inertia for Areas

Whenever a distributed load acts perpendicular to an area and its intensity varies linearly, the calculation of the moment of the loading about an axis will involve an integral of the form $\int y^2 dA$. For example, consider the plate in Fig. 10–1, which is submerged in a fluid and subjected to the pressure p. As discussed in Sec. 9.5, this pressure varies linearly with depth, such that $p = \gamma y$, where γ is the specific weight of the fluid. Thus, the force acting on the differential area dA of the plate is $dF = p\,dA = (\gamma y)dA$. The *moment* of this force about the x axis is therefore $dM = y\,dF = \gamma y^2 dA$, and so integrating dM over the entire area of the plate yields $M = \gamma \int y^2 dA$. The integral $\int y^2 dA$ is sometimes referred to as the "second moment" of the area about an axis (the x axis), but more often it is called the ***moment of inertia of the area***. The word "inertia" is used here since the formulation is similar to the mass moment of inertia, $\int y^2 dm$, which is a dynamical property described in Sec. 10.8. Although for an area this integral has no physical meaning, it often arises in formulas used in fluid mechanics, mechanics of materials, structural mechanics, and mechanical design, and so the engineer needs to be familiar with the methods used to determine the moment of inertia.

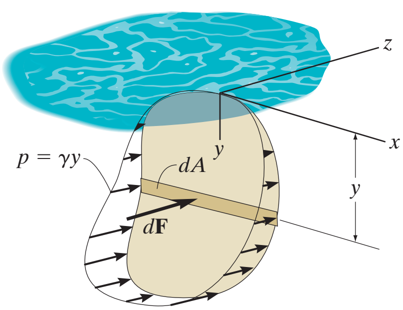


Fig. 10–1

Moment of Inertia. By definition, the moments of inertia of a differential area dA about the x and y axes are $dI_x = y^2\,dA$ and $dI_y = x^2\,dA$, respectively, Fig. 10–2. For the entire area A the ***moments of inertia*** are determined by integration; i.e.,

$$\boxed{\begin{aligned} I_x &= \int_A y^2\,dA \\ I_y &= \int_A x^2\,dA \end{aligned}} \qquad (10\text{–}1)$$

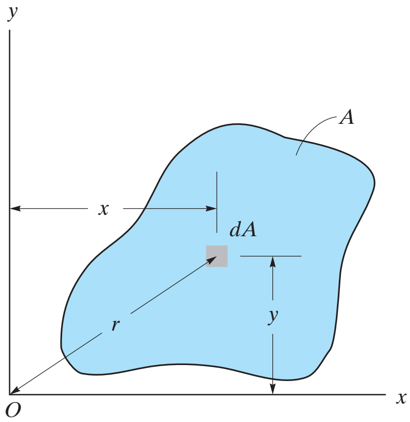


Fig. 10–2

We can also formulate this quantity for dA about the "pole" O or z axis, Fig. 10–2. This is referred to as the ***polar moment of inertia***. It is defined as $dJ_O = r^2\,dA$, where r is the perpendicular distance from the pole (z axis) to the element dA. For the entire area the *polar moment of inertia* is

$$\boxed{J_O = \int_A r^2\,dA = I_x + I_y} \qquad (10\text{–}2)$$

This relation between J_O and I_x, I_y is possible since $r^2 = x^2 + y^2$, Fig. 10–2.

From the above formulations it is seen that I_x, I_y, and J_O will *always* be *positive* since they involve the product of distance squared and area. Furthermore, the units for moment of inertia involve length raised to the fourth power, e.g., m^4, mm^4, or ft^4, $in.^4$.

10.2 Parallel-Axis Theorem for an Area

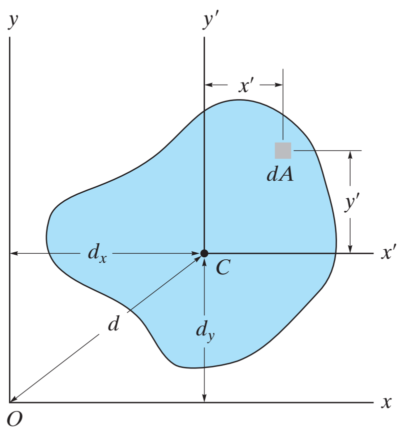


Fig. 10–3

The ***parallel-axis theorem*** can be used to find the moment of inertia of an area about *any axis* that is parallel to an axis passing through the centroid and about which the moment of inertia is known. To develop this theorem, we will consider finding the moment of inertia of the shaded area shown in Fig. 10–3 about the x axis. To start, we choose a differential element dA located at an arbitrary distance y' from the *centroidal* x' axis. If the distance between the parallel x and x' axis is d_y, then the moment of inertia of dA about the x axis is $dI_x = (y' + d_y)^2\,dA$. For the entire area,

$$\begin{aligned} I_x &= \int_A (y' + d_y)^2\,dA \\ &= \int_A y'^2\,dA + 2d_y \int_A y'\,dA + d_y^2 \int_A dA \end{aligned}$$

The first integral represents the moment of inertia of the area about the centroidal axis, $\bar{I}_{x'}$. The second integral is zero since the x' axis passes through the area's centroid C; i.e., $\int y'\,dA = \bar{y}'\int dA = 0$ since $\bar{y}' = 0$. Since the third integral represents the total area A, the final result is therefore

$$\boxed{I_x = \bar{I}_{x'} + Ad_y^2} \qquad (10\text{–}3)$$

A similar expression can be written for I_y; i.e.,

$$\boxed{I_y = \bar{I}_{y'} + Ad_x^2} \qquad (10\text{–}4)$$

And finally, for the polar moment of inertia, since $\bar{J}_C = \bar{I}_{x'} + \bar{I}_{y'}$ and $d^2 = d_x^2 + d_y^2$, we have

$$\boxed{J_O = \bar{J}_C + Ad^2} \qquad (10\text{–}5)$$

The form of each of these three equations states that *the moment of inertia for an area about an axis is equal to its moment of inertia about a parallel axis passing through the area's centroid plus the product of the area and the square of the perpendicular distance between the axes*.

In order to predict the strength and deflection of this beam, it is necessary to calculate the moment of inertia of the beam's cross-sectional area. (© Russell C. Hibbeler)

10.3 Radius of Gyration of an Area

The ***radius of gyration*** of an area about an axis has units of length and is a quantity that is often used for the design of columns in structural mechanics. Provided the areas and moments of inertia are *known*, the radii of gyration are determined from the formulas

$$k_x = \sqrt{\frac{I_x}{A}}$$

$$k_y = \sqrt{\frac{I_y}{A}} \qquad (10\text{–}6)$$

$$k_O = \sqrt{\frac{J_O}{A}}$$

The form of these equations is easily remembered since it is similar to that for finding the moment of inertia for a differential area about an axis. For example, $I_x = k_x^2 A$; whereas for a differential area, $dI_x = y^2\,dA$.

Important Points

- The moment of inertia is a geometric property of an area that is used to determine the strength of a structural member or the location of a resultant pressure force acting on a plate submerged in a fluid. It is sometimes referred to as the second moment of the area about an axis, because the distance from the axis to each area element is squared.
- If the moment of inertia of an area is known about its centroidal axis, then the moment of inertia about a corresponding parallel axis can be determined using the parallel-axis theorem.

Procedure for Analysis

In most cases the moment of inertia can be determined using a single integration. The following procedure shows two ways in which this can be done.

- If the curve defining the boundary of the area is expressed as $y = f(x)$, then select a rectangular differential element such that it has a finite length and differential width.
- The element should be located so that it intersects the curve at the *arbitrary point* (x, y).

Case 1.

- Orient the element so that its length is *parallel* to the axis about which the moment of inertia is computed. This situation occurs when the rectangular element shown in Fig. 10–4*a* is used to determine I_x for the area. Here the entire element is at a distance y from the x axis since it has a thickness dy. Thus $I_x = \int y^2 dA$. To find I_y, the element is oriented as shown in Fig. 10–4*b*. This element lies at the *same* distance x from the y axis so that $I_y = \int x^2 dA$.

Case 2.

- The length of the element can be oriented *perpendicular* to the axis about which the moment of inertia is computed; however, Eq. 10–1 *does not apply* since all points on the element will *not* lie at the same moment-arm distance from the axis. For example, if the rectangular element in Fig. 10–4*a* is used to determine I_y, it will first be necessary to calculate the moment of inertia of the *element* about an axis parallel to the y axis that passes through the element's centroid, and then determine the moment of inertia of the *element* about the y axis using the parallel-axis theorem. Integration of this result will yield I_y. See Examples 10.2 and 10.3.

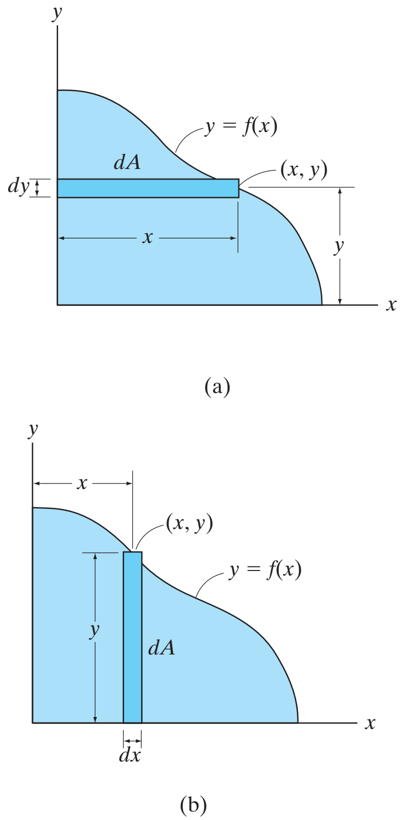


Fig. 10–4

EXAMPLE 10.1

Determine the moment of inertia for the rectangular area shown in Fig. 10–5 with respect to (a) the centroidal x' axis, (b) the axis x_b passing through the base of the rectangle, and (c) the pole or z' axis perpendicular to the x'–y' plane and passing through the centroid C.

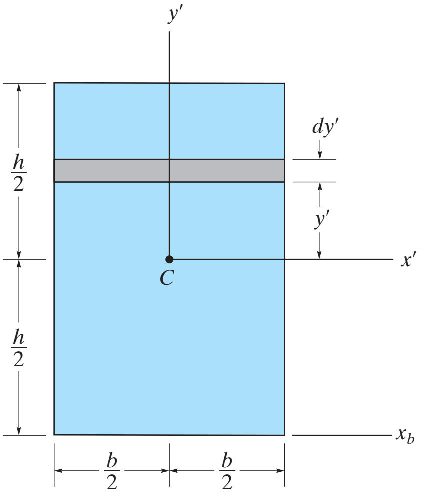


Fig. 10–5

SOLUTION (CASE 1)

Part (a). The differential element shown in Fig. 10–5 is chosen for integration. Because of its location and orientation, the *entire element* is at a distance y' from the x' axis. Here it is necessary to integrate from $y' = -h/2$ to $y' = h/2$. Since $dA = b\,dy'$, then

$$\bar{I}_{x'} = \int_A y'^2\,dA = \int_{-h/2}^{h/2} y'^2(b\,dy') = b\int_{-h/2}^{h/2} y'^2\,dy'$$

$$\bar{I}_{x'} = \frac{1}{12}bh^3 \qquad \textit{Ans.}$$

Part (b). The moment of inertia about an axis passing through the base of the rectangle can be obtained by using the above result of part (a) and applying the parallel-axis theorem, Eq. 10–3.

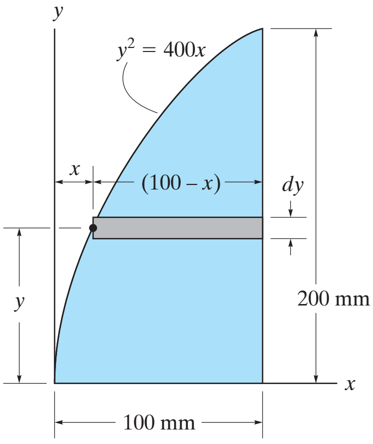
$$\begin{aligned} I_{x_b} &= \bar{I}_{x'} + A d_y^2 \\ &= \frac{1}{12}bh^3 + bh\left(\frac{h}{2}\right)^2 = \frac{1}{3}bh^3 \end{aligned} \qquad \textit{Ans.}$$

Part (c). To obtain the polar moment of inertia about point C, we must first obtain $\bar{I}_{y'}$, which may be found by interchanging the dimensions b and h in the result of part (a), i.e.,

$$\bar{I}_{y'} = \frac{1}{12}hb^3$$

Using Eq. 10–2, the polar moment of inertia about C is therefore

$$\bar{J}_C = \bar{I}_{x'} + \bar{I}_{y'} = \frac{1}{12}bh(h^2 + b^2) \qquad \textit{Ans.}$$



(a)

Determine the moment of inertia for the shaded area shown in Fig. 10–6*a* about the x axis.

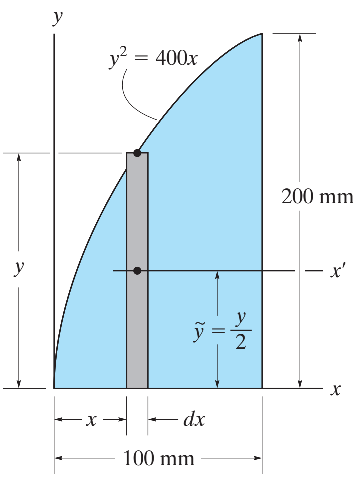
SOLUTION I (CASE 1)

A differential element of area that is *parallel* to the x axis, as shown in Fig. 10–6*a*, is chosen for integration. Since this element has a thickness dy and intersects the curve at the *arbitrary point* (x, y), its area is $dA = (100 - x)\,dy$. Furthermore, the element lies at the same distance y from the x axis. Hence, integrating with respect to y, from $y = 0$ to $y = 200$ mm, yields

$$I_x = \int_A y^2\,dA = \int_0^{200\text{ mm}} y^2(100 - x)\,dy$$

$$= \int_0^{200\text{ mm}} y^2\left(100 - \frac{y^2}{400}\right)dy = \int_0^{200\text{ mm}} \left(100y^2 - \frac{y^4}{400}\right)dy$$

$$= 107(10^6)\text{ mm}^4$$ *Ans.*



(b)

Fig. 10–6

SOLUTION II (CASE 2)

A differential element *parallel* to the y axis, as shown in Fig. 10–6*b*, is chosen for integration. It intersects the curve at the *arbitrary point* (x, y). In this case, all points of the element do *not* lie at the same distance from the x axis, and therefore the parallel-axis theorem must be used to determine the *moment of inertia of the element* with respect to this axis. For a rectangle having a base b and height h, the moment of inertia about its centroidal axis has been determined in part (a) of Example 10.1. There it was found that $\bar{I}_{x'} = \frac{1}{12}bh^3$. For the differential element shown in Fig. 10–6*b*, $b = dx$ and $h = y$, and thus $d\bar{I}_{x'} = \frac{1}{12}dx\,y^3$. Since the centroid of the element is $\tilde{y} = y/2$ from the x axis, the moment of inertia of the element about this axis is

$$dI_x = d\bar{I}_{x'} + dA\,\tilde{y}^2 = \frac{1}{12}dx\,y^3 + y\,dx\left(\frac{y}{2}\right)^2 = \frac{1}{3}y^3\,dx$$

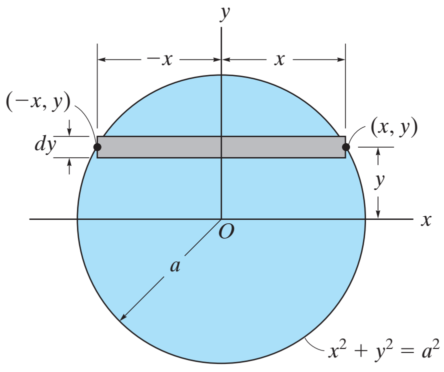
(This result can also be concluded from part (b) of Example 10.1.) Integrating with respect to x, from $x = 0$ to $x = 100$ mm, yields

$$I_x = \int dI_x = \int_0^{100\text{ mm}} \frac{1}{3}y^3\,dx = \int_0^{100\text{ mm}} \frac{1}{3}(400x)^{3/2}\,dx$$

$$= 107(10^6)\text{ mm}^4$$ *Ans.*

EXAMPLE 10.3

Determine the moment of inertia with respect to the x axis for the circular area shown in Fig. 10–7a.



(a)

SOLUTION I (CASE 1)

Using the differential element shown in Fig. 10–7a, since $dA = 2x\,dy$, we have

$$I_x = \int_A y^2\,dA = \int_A y^2(2x)\,dy$$

$$= \int_{-a}^{a} y^2\left(2\sqrt{a^2 - y^2}\right)dy = \frac{\pi a^4}{4} \qquad \textit{Ans.}$$

SOLUTION II (CASE 2)

When the differential element shown in Fig. 10–7b is chosen, the centroid for the element happens to lie on the x axis, and since $\bar{I}_{x'} = \frac{1}{12}bh^3$ for a rectangle, we have

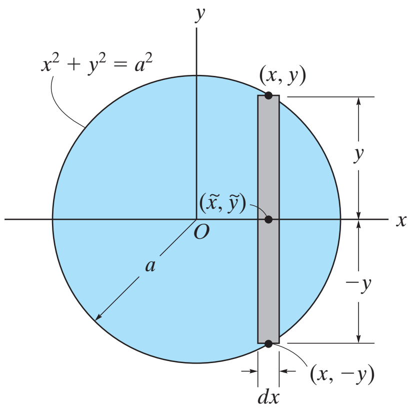
$$dI_x = \frac{1}{12}dx(2y)^3$$

$$= \frac{2}{3}y^3\,dx$$

Integrating with respect to x yields

$$I_x = \int_{-a}^{a}\frac{2}{3}(a^2 - x^2)^{3/2}\,dx = \frac{\pi a^4}{4} \qquad \textit{Ans.}$$

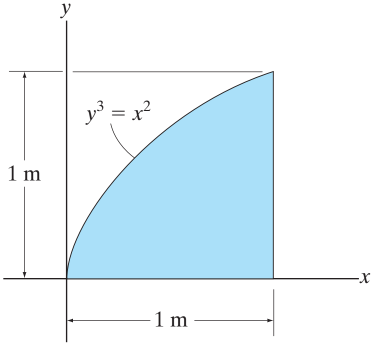
NOTE: By comparison, Solution I requires much less computation. Therefore, if an integral using a particular element appears difficult to evaluate, try solving the problem using an element oriented in the other direction.



(b)

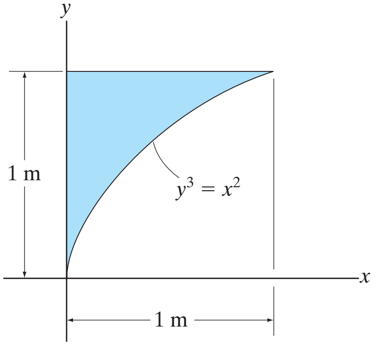
Fig. 10–7

F10–1. Determine the moment of inertia of the shaded area about the x axis.



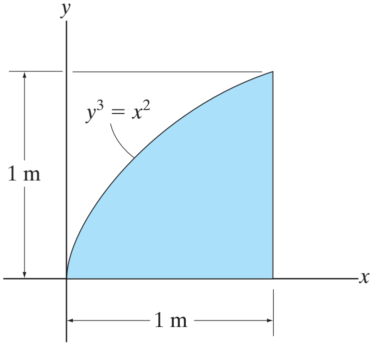
Prob. F10–1

F10–2. Determine the moment of inertia of the shaded area about the x axis.



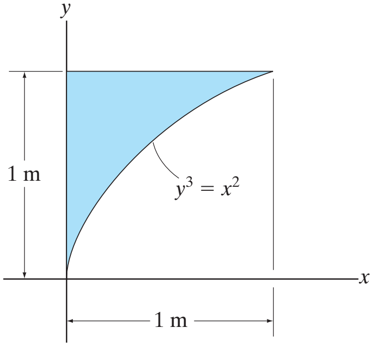
Prob. F10–2

F10–3. Determine the moment of inertia of the shaded area about the y axis.



Prob. F10–3

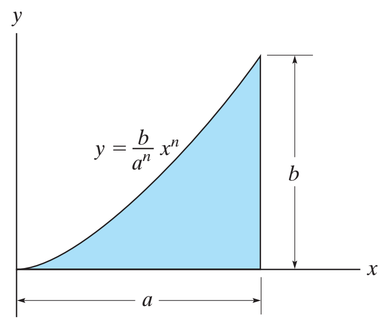
F10–4. Determine the moment of inertia of the shaded area about the y axis.



Prob. F10–4

10–1. Determine the moment of inertia about the x axis.

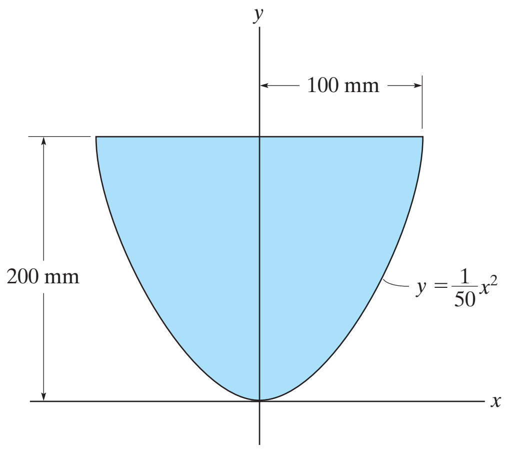
10–2. Determine the moment of inertia about the y axis.



Probs. 10–1/2

10–3. Determine the moment of inertia for the shaded area about the x axis.

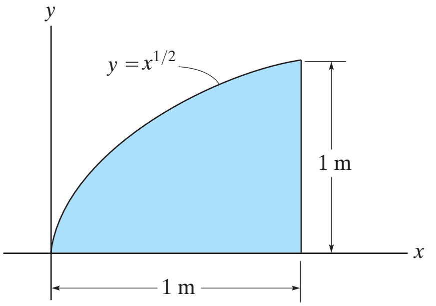
***10–4.** Determine the moment of inertia for the shaded area about the y axis.



Probs. 10–3/4

10–5. Determine the moment of inertia for the shaded area about the x axis.

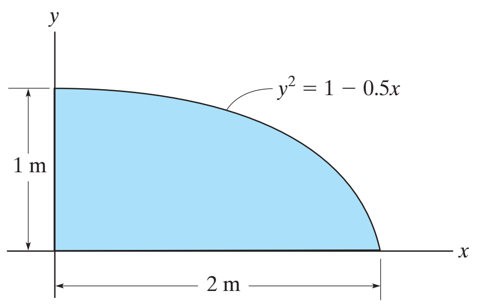
10–6. Determine the moment of inertia for the shaded area about the y axis.



Probs. 10–5/6

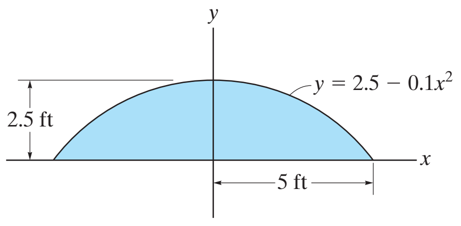
10–7. Determine the moment of inertia for the shaded area about the x axis.

***10–8.** Determine the moment of inertia for the shaded area about the y axis.



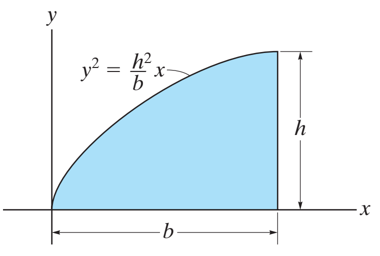
Probs. 10–7/8

10–9. Determine the moment of inertia of the area about the x axis. Solve the problem in two ways, using rectangular differential elements: (a) having a thickness dx and (b) having a thickness of dy.



Prob. 10–9

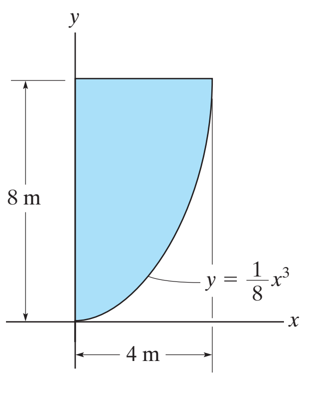
10–10. Determine the moment of inertia of the area about the x axis.



Prob. 10–10

10–11. Determine the moment of inertia for the shaded area about the x axis.

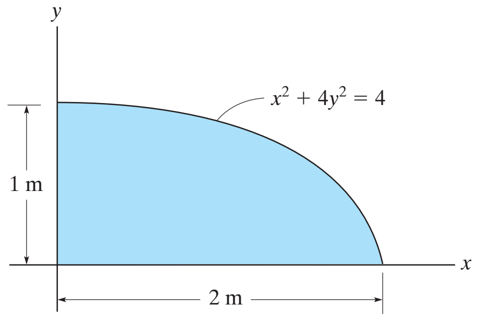
***10–12.** Determine the moment of inertia for the shaded area about the y axis.



Probs. 10–11/12

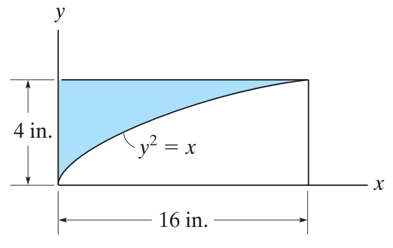
10–13. Determine the moment of inertia about the x axis.

10–14. Determine the moment of inertia about the y axis.



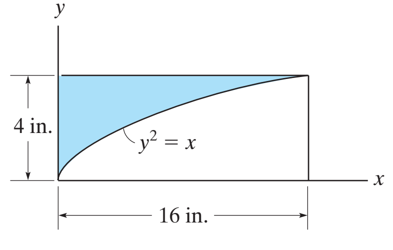
Probs. 10–13/14

10–15. Determine the moment of inertia for the shaded area about the x axis.



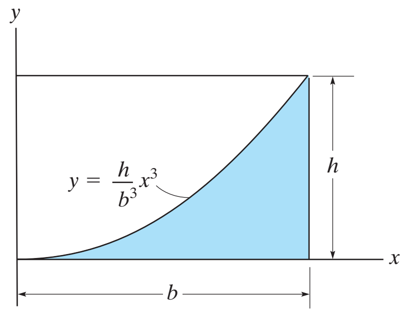
Prob. 10–15

***10–16.** Determine the moment of inertia for the shaded area about the y axis.



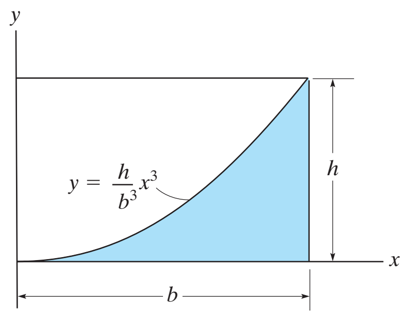
Prob. 10–16

10–17. Determine the moment of inertia for the shaded area about the x axis.



Prob. 10–17

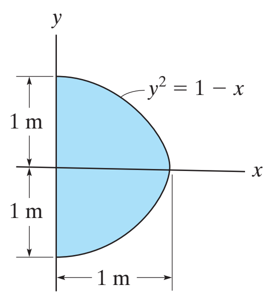
10–18. Determine the moment of inertia for the shaded area about the y axis.



Prob. 10–18

10–19. Determine the moment of inertia for the shaded area about the x axis.

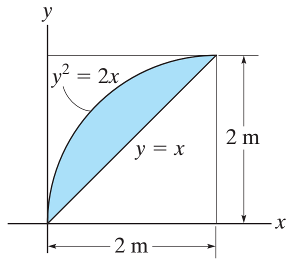
***10–20.** Determine the moment of inertia for the shaded area about the y axis.



Probs. 10–19/20

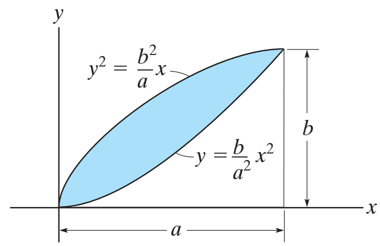
10–21. Determine the moment of inertia for the shaded area about the x axis.

10–22. Determine the moment of inertia for the shaded area about the y axis.



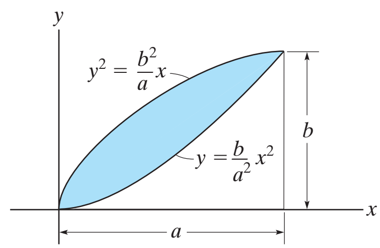
Probs. 10–21/22

10–23. Determine the moment of inertia for the shaded area about the x axis.



Prob. 10–23

***10–24.** Determine the moment of inertia for the shaded area about the y axis.



Prob. 10–24

10.4 Moments of Inertia for Composite Areas

A composite area consists of a series of connected "simpler" parts or shapes, such as rectangles, triangles, and circles. Provided the moment of inertia of each of these parts is known or can be determined about a common axis, then the moment of inertia for the composite area about this axis equals the *algebraic sum* of the moments of inertia of all its parts.

Procedure for Analysis

The moment of inertia for a composite area about a reference axis can be determined using the following procedure.

Composite Parts.

- Using a sketch, divide the area into its composite parts and indicate the perpendicular distance from the centroid of each part to the reference axis.

Parallel-Axis Theorem.

- If the centroidal axis for each part does not coincide with the reference axis, the parallel-axis theorem, $I = \bar{I} + Ad^2$, should be used to determine the moment of inertia of the part about the reference axis. For the calculation of $\bar{I}$, use the table on the inside back cover.

Summation.

- The moment of inertia of the entire area about the reference axis is determined by summing the results of its composite parts about this axis.
- If a composite part has an empty region (hole), its moment of inertia is found by subtracting the moment of inertia of this region from the moment of inertia of the entire part including the region.

For design or analysis of this T-beam, engineers must be able to locate the centroid of its cross-sectional area, and then find the moment of inertia of this area about the centroidal axis. (© Russell C. Hibbeler)

EXAMPLE 10.4

Determine the moment of inertia of the area shown in Fig. 10–8*a* about the *x* axis.

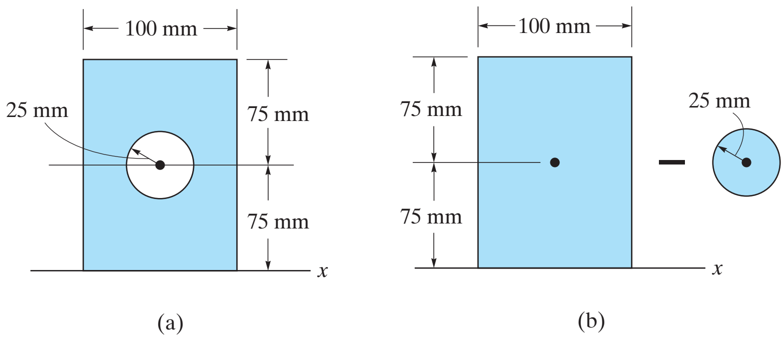


Fig. 10–8

SOLUTION

Composite Parts. The area can be obtained by *subtracting* the circle from the rectangle shown in Fig. 10–8*b*. The centroid of each area is located in the figure.

Parallel-Axis Theorem. The moments of inertia about the *x* axis are determined using the parallel-axis theorem and the geometric properties formulae for circular and rectangular areas $I_x = \frac{1}{4}\pi r^4$; $I_x = \frac{1}{12}bh^3$, found on the inside back cover.

Circle

$$I_x = \bar{I}_{x'} + A d_y^2$$

$$= \frac{1}{4}\pi(25)^4 + \pi(25)^2(75)^2 = 11.4(10^6)\text{ mm}^4$$

Rectangle

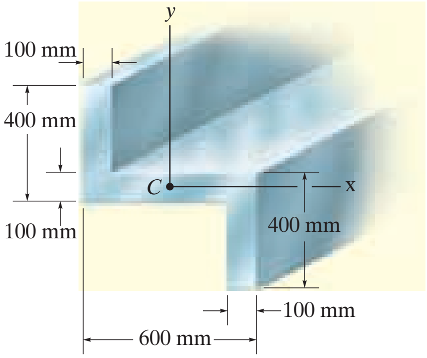
$$I_x = \bar{I}_{x'} + A d_y^2$$

$$= \frac{1}{12}(100)(150)^3 + (100)(150)(75)^2 = 112.5(10^6)\text{ mm}^4$$

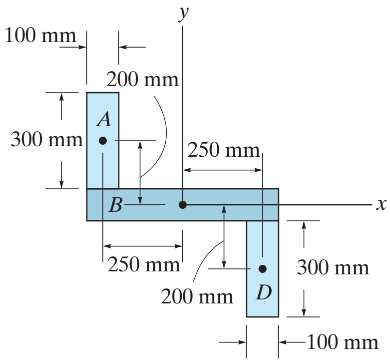
Summation. The moment of inertia for the area is therefore

$$I_x = -11.4(10^6) + 112.5(10^6)$$

$$= 101(10^6)\text{ mm}^4 \qquad \textit{Ans.}$$



(a)



(b)

Fig. 10–9

Determine the moments of inertia for the cross-sectional area of the member shown in Fig. 10–9*a* about the *x* and *y* centroidal axes.

SOLUTION

Composite Parts. The cross section can be subdivided into the three rectangular areas *A*, *B*, and *D* shown in Fig. 10–9*b*. For the calculation, the centroid of each of these rectangles is located in the figure.

Parallel-Axis Theorem. From the table on the inside back cover, or Example 10.1, the moment of inertia of a rectangle about its centroidal axis is $\bar{I} = \frac{1}{12}bh^3$. Hence, using the parallel-axis theorem for rectangles *A* and *D*, the calculations are as follows:

Rectangles A and D

$$I_x = \bar{I}_{x'} + A d_y^2 = \frac{1}{12}(100)(300)^3 + (100)(300)(200)^2$$
$$= 1.425(10^9)\ \text{mm}^4$$
$$I_y = \bar{I}_{y'} + A d_x^2 = \frac{1}{12}(300)(100)^3 + (100)(300)(250)^2$$
$$= 1.90(10^9)\ \text{mm}^4$$

Rectangle B

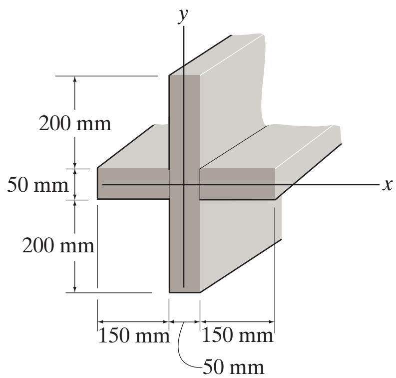
$$I_x = \frac{1}{12}(600)(100)^3 = 0.05(10^9)\ \text{mm}^4$$
$$I_y = \frac{1}{12}(100)(600)^3 = 1.80(10^9)\ \text{mm}^4$$

Summation. The moments of inertia for the entire cross section are thus

$$I_x = 2[1.425(10^9)] + 0.05(10^9)$$
$$= 2.90(10^9)\ \text{mm}^4 \qquad \textit{Ans.}$$
$$I_y = 2[1.90(10^9)] + 1.80(10^9)$$
$$= 5.60(10^9)\ \text{mm}^4 \qquad \textit{Ans.}$$

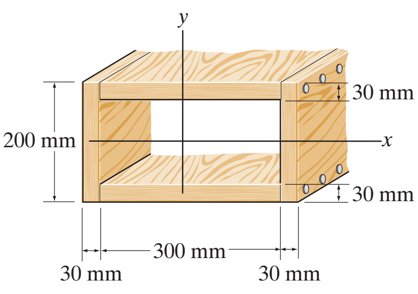
FUNDAMENTAL PROBLEMS

F10–5. Determine the moment of inertia of the beam's cross-sectional area about the centroidal x and y axes.



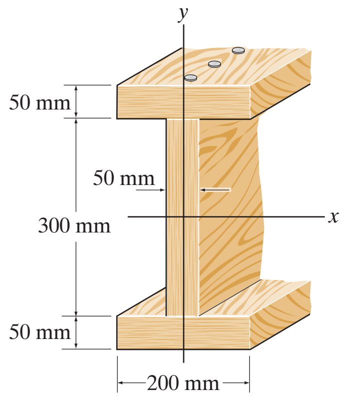
Prob. F10–5

F10–6. Determine the moment of inertia of the beam's cross-sectional area about the centroidal x and y axes.



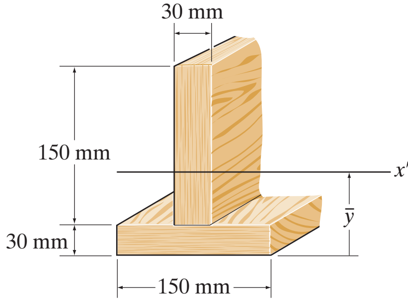
Prob. F10–6

F10–7. Determine the moment of inertia of the cross-sectional area of the channel with respect to the y axis.



Prob. F10–7

F10–8. Determine the moment of inertia of the cross-sectional area of the T-beam with respect to the x' axis passing through the centroid of the cross section.

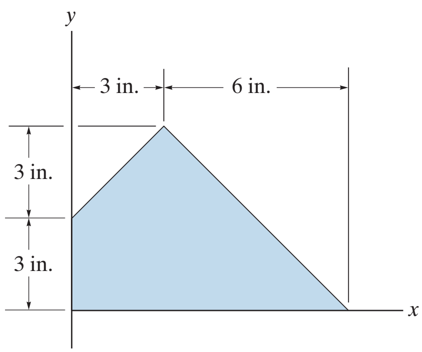


Prob. F10–8

PROBLEMS

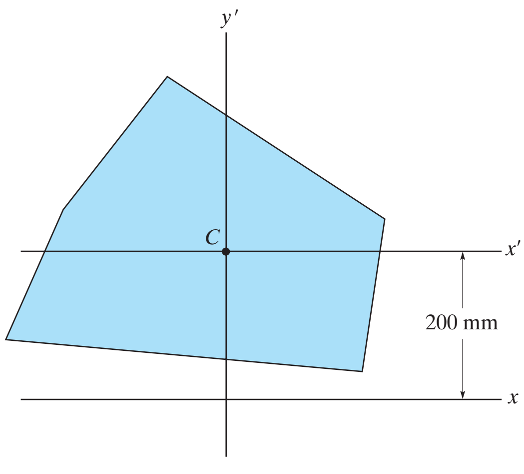
10–25. Determine the moment of inertia of the composite area about the x axis.

10–26. Determine the moment of inertia of the composite area about the y axis.



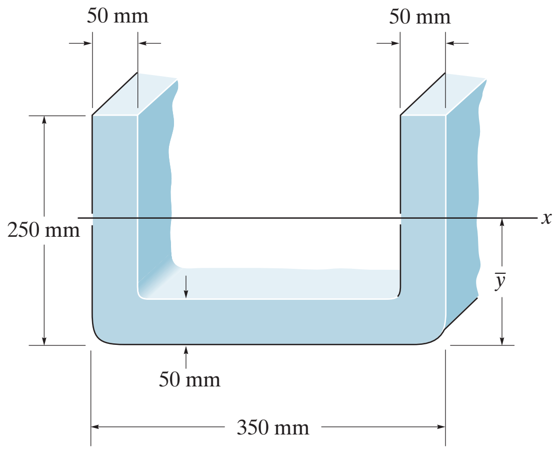
Probs. 10–25/26

10–27. The polar moment of inertia for the area is $\bar{J}_C = 642\ (10^6)\ \text{mm}^4$, about the z' axis passing through the centroid C. The moment of inertia about the y' axis is $264\ (10^6)\ \text{mm}^4$, and the moment of inertia about the x axis is $938\ (10^6)\ \text{mm}^4$. Determine the area A.



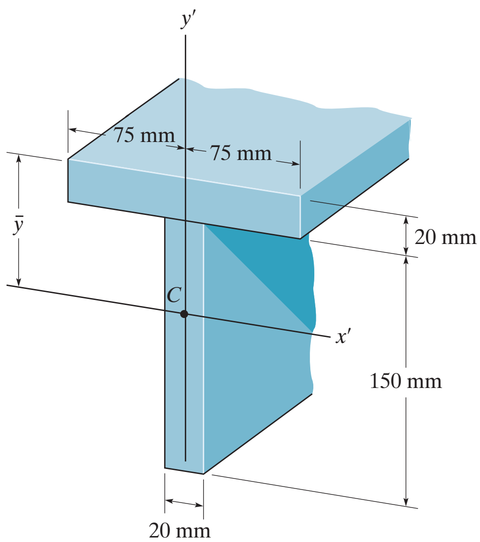
Prob. 10–27

***10–28.** Determine the location $\bar{y}$ of the centroid of the channel's cross-sectional area and then calculate the moment of inertia of the area about this axis.



Prob. 10–28

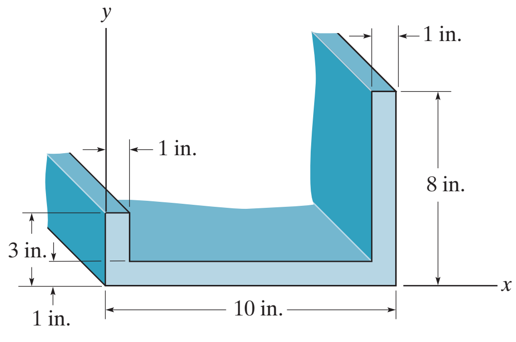
10–29. Determine $\bar{y}$, which locates the centroidal axis x' for the cross-sectional area of the T-beam, and then find the moments of inertia $I_{x'}$ and $I_{y'}$.



Prob. 10–29

10–30. Determine the moment of inertia for the beam's cross-sectional area about the x axis.

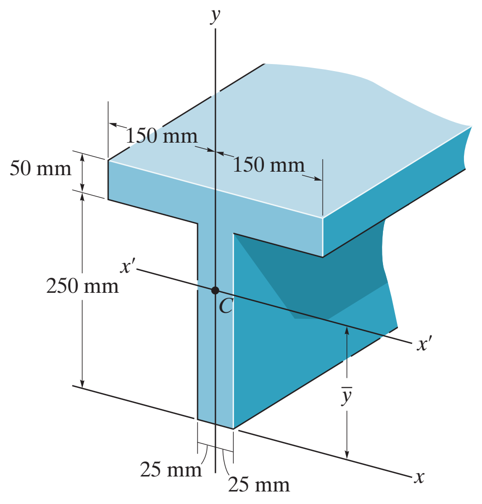
10–31. Determine the moment of inertia for the beam's cross-sectional area about the y axis.



Probs. 10–30/31

10–34. Determine the moment of inertia of the beam's cross-sectional area about the y axis.

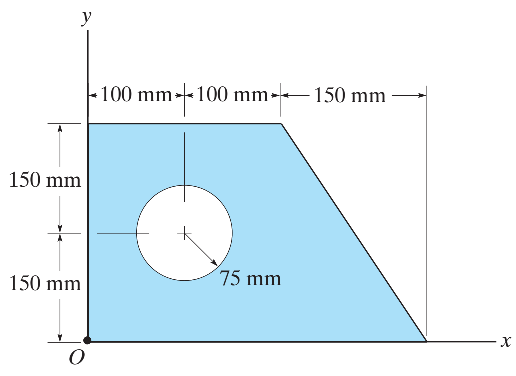
10–35. Determine $\bar{y}$, which locates the centroidal axis x' for the cross-sectional area of the T-beam, and then find the moment of inertia about the x' axis.



Probs. 10–34/35

***10–32.** Determine the moment of inertia I_x of the shaded area about the x axis.

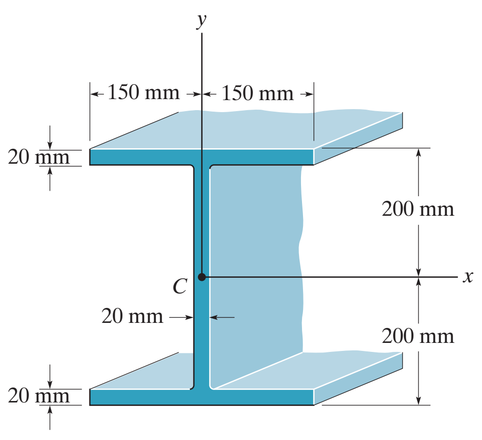
10–33. Determine the moment of inertia I_x of the shaded area about the y axis.



Probs. 10–32/33

***10–36.** Determine the moment of inertia about the x axis.

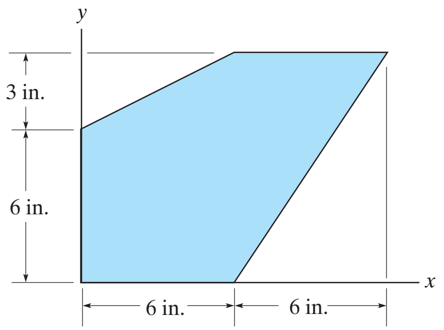
10–37. Determine the moment of inertia about the y axis.



Probs. 10–36/37

10–38. Determine the moment of inertia of the shaded area about the x axis.

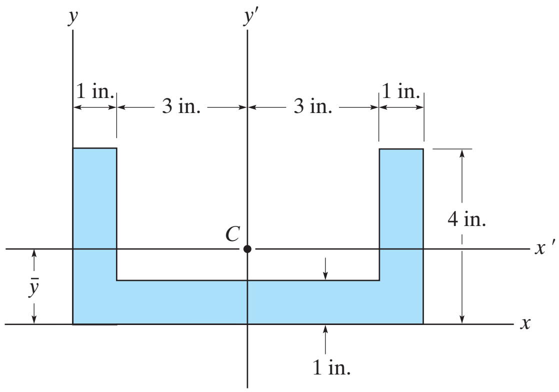
10–39. Determine the moment of inertia of the shaded area about the y axis.



Probs. 10–38/39

***10–40.** Determine the distance $\bar{y}$ to the centroid of the beam's cross-sectional area; then find the moment of inertia about the centroidal x' axis.

10–41. Determine the moment of inertia for the beam's cross-sectional area about the y axis.



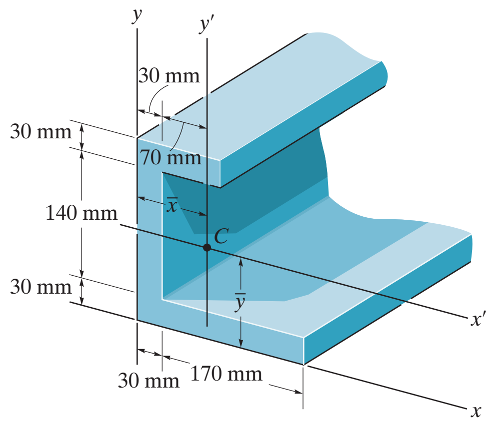
Probs. 10–40/41

10–42. Determine the moment of inertia of the beam's cross-sectional area about the x axis.

10–43. Determine the moment of inertia of the beam's cross-sectional area about the y axis.

***10–44.** Determine the distance $\bar{y}$ to the centroid C of the beam's cross-sectional area and then compute the moment of inertia $\bar{I}_{x'}$ about the x' axis.

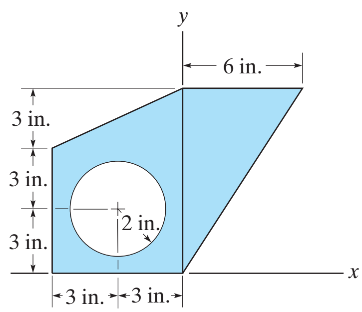
10–45. Determine the distance $\bar{x}$ to the centroid C of the beam's cross-sectional area and then compute the moment of inertia $\bar{I}_{y'}$ about the y' axis.



Probs. 10–42/43/44/45

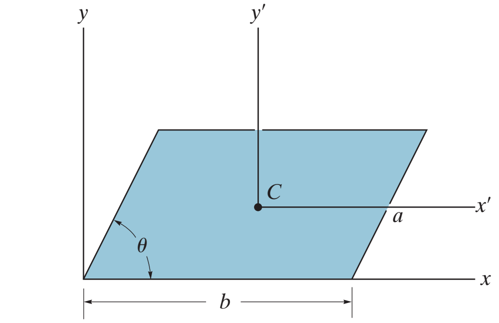
10–46. Determine the moment of inertia for the shaded area about the x axis.

10–47. Determine the moment of inertia for the shaded area about the y axis.



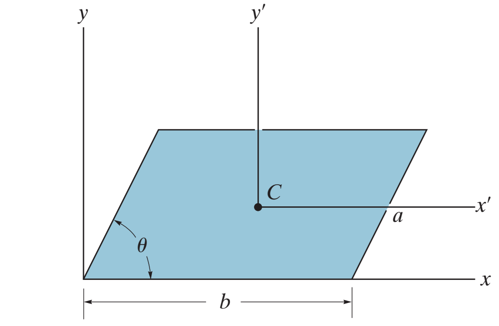
Probs. 10–46/47

***10–48.** Determine the moment of inertia of the parallelogram about the x' axis, which passes through the centroid C of the area.



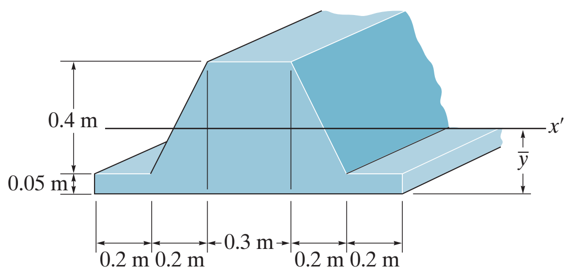
Prob. 10–48

10–49. Determine the moment of inertia of the parallelogram about the y' axis, which passes through the centroid C of the area.



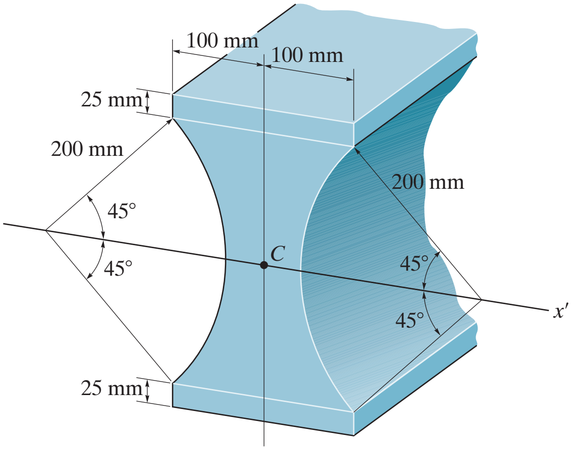
Prob. 10–49

10–50. Locate the centroid $\bar{y}$ of the cross section and determine the moment of inertia of the section about the x' axis.



Prob. 10–50

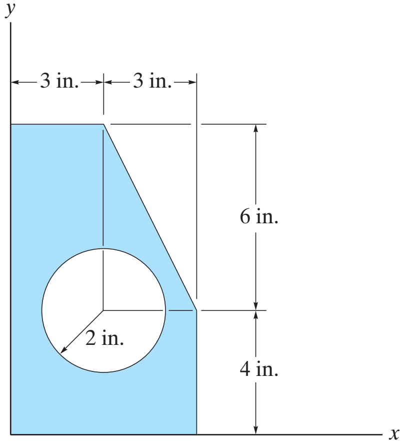
10–51. Determine the moment of inertia for the beam's cross-sectional area about the x' axis passing through the centroid C of the cross section.



Prob. 10–51

***10–52.** Determine the moment of inertia of the area about the x axis.

10–53. Determine the moment of inertia of the area about the y axis.



Probs. 10–52/53

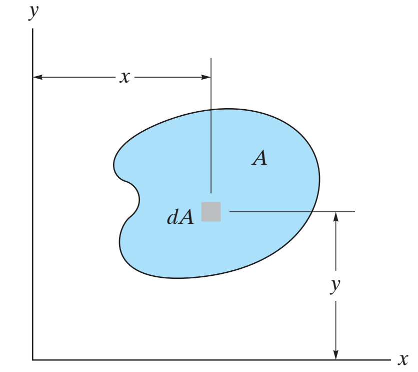


Fig. 10–10

The effectiveness of this beam to resist bending can be determined once its moments of inertia and its product of inertia are known. (© Russell C. Hibbeler)

*10.5 Product of Inertia for an Area

It will be shown in the next section that the property of an area, called the product of inertia, is required in order to determine the *maximum* and *minimum* moments of inertia for the area. These maximum and minimum values are important properties needed for designing structural and mechanical members such as beams, columns, and shafts.

The ***product of inertia*** of the area in Fig. 10–10 with respect to the x and y axes is defined as

$$\boxed{I_{xy} = \int_A xy\, dA} \qquad (10\text{–}7)$$

If the element of area chosen has a differential size in two directions, as shown in Fig. 10–10, a double integration must be performed to evaluate I_{xy}. Most often, however, it is easier to choose an element having a differential size or thickness in only one direction in which case the evaluation requires only a single integration (see Example 10.6).

Like the moment of inertia, the product of inertia has units of length raised to the fourth power, e.g., m^4, mm^4 or ft^4, in^4. However, since x or y may be negative, the product of inertia may either be positive, negative, or zero, depending on the location and orientation of the coordinate axes. For example, the product of inertia I_{xy} for an area will be *zero* if either the x or y axis is an axis of *symmetry* for the area, as in Fig. 10–11. Here every element dA located at point (x, y) has a corresponding element dA located at $(x, -y)$. Since the products of inertia for these elements are, respectively, $xy\, dA$ and $-xy\, dA$, the algebraic sum or integration of all the elements that are chosen in this way will cancel each other. Consequently, the product of inertia for the total area becomes zero. It also follows from the definition of I_{xy} that the "sign" of this quantity depends on the quadrant where the area is located. As shown in Fig. 10–12, if the area is rotated from one quadrant to another, the sign of I_{xy} will change.

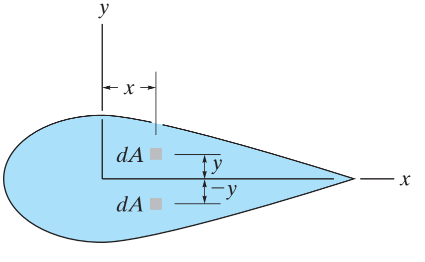


Fig. 10–11

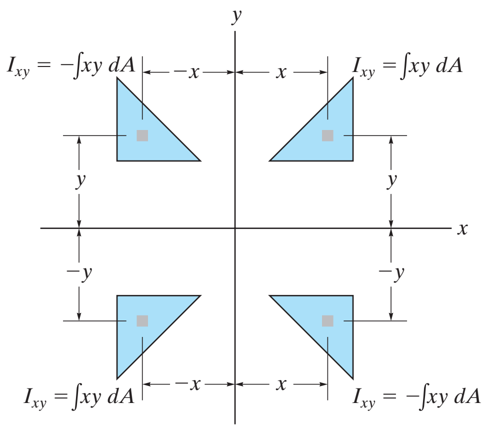


Fig. 10–12

Parallel-Axis Theorem. Consider the shaded area shown in Fig. 10–13, where x' and y' represent a set of axes passing through the *centroid* of the area, and x and y represent a corresponding set of parallel axes. Since the product of inertia of dA with respect to the x and y axes is $dI_{xy} = (x' + d_x)(y' + d_y)\, dA$, then for the entire area,

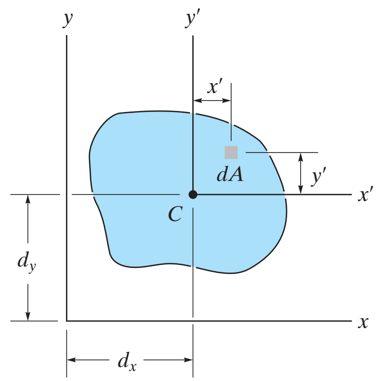


Fig. 10–13

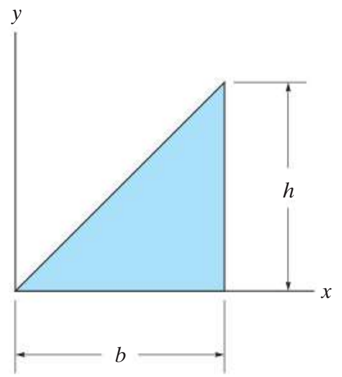
$$I_{xy} = \int_A (x' + d_x)(y' + d_y)\, dA$$

$$= \int_A x'y'\, dA + d_x \int_A y'\, dA + d_y \int_A x'\, dA + d_x d_y \int_A dA$$

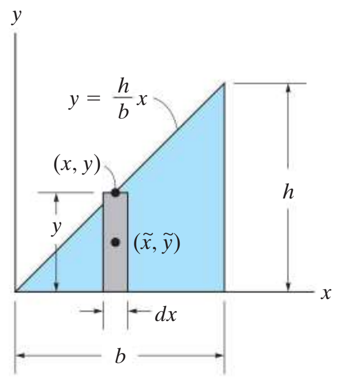
The first term on the right represents the product of inertia for the area with respect to the centroidal axes, $\bar{I}_{x'y'}$. The integrals in the second and third terms are zero since the moments of the area are taken about the centroidal axis. Realizing that the fourth integral represents the entire area A, the parallel-axis theorem for the product of inertia becomes

$$\boxed{I_{xy} = \bar{I}_{x'y'} + A\, d_x d_y} \qquad (10\text{–}8)$$

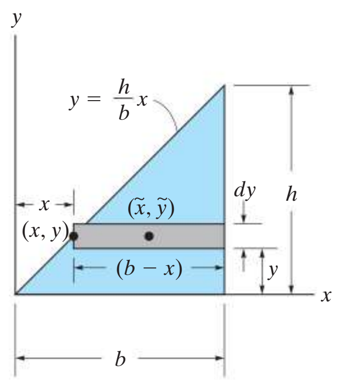
It is important that the *algebraic signs* for d_x and d_y be maintained when applying this equation.



(a)



(b)



(c)

Fig. 10–14

Determine the product of inertia I_{xy} for the triangle shown in Fig. 10–14*a*.

SOLUTION I

A differential element that has a thickness dx, as shown in Fig. 10–14*b*, has an area $dA = y\,dx$. The product of inertia of this element with respect to the x *and* y axes is determined using the parallel-axis theorem.

$$dI_{xy} = d\bar{I}_{x'y'} + dA\,\tilde{x}\,\tilde{y}$$

where $\tilde{x}$ and $\tilde{y}$ locate the *centroid* of the element or the origin of the x', y' axes. (See Fig. 10–13.) Since $d\bar{I}_{x'y'} = 0$, due to symmetry, and $\tilde{x} = x$, $\tilde{y} = y/2$, then

$$dI_{xy} = 0 + (y\,dx)x\left(\frac{y}{2}\right) = \left(\frac{h}{b}x\,dx\right)x\left(\frac{h}{2b}x\right)$$
$$= \frac{h^2}{2b^2}x^3\,dx$$

Integrating with respect to x from $x = 0$ to $x = b$ yields

$$I_{xy} = \frac{h^2}{2b^2}\int_0^b x^3\,dx = \frac{b^2h^2}{8} \qquad \textit{Ans.}$$

SOLUTION II

The differential element that has a thickness dy, as shown in Fig. 10–14*c*, can also be used. Its area is $dA = (b - x)\,dy$. The *centroid* is located at point $\tilde{x} = x + (b - x)/2 = (b + x)/2$, $\tilde{y} = y$, so the product of inertia of the element becomes

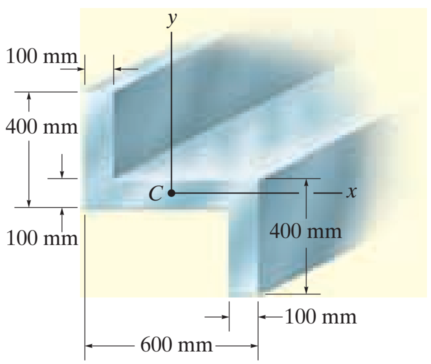
$$dI_{xy} = d\bar{I}_{x'y'} + dA\,\tilde{x}\,\tilde{y}$$
$$= 0 + (b - x)\,dy\left(\frac{b + x}{2}\right)y$$
$$= \left(b - \frac{b}{h}y\right)dy\left[\frac{b + (b/h)y}{2}\right]y = \frac{1}{2}y\left(b^2 - \frac{b^2}{h^2}y^2\right)dy$$

Integrating with respect to y from $y = 0$ to $y = h$ yields

$$I_{xy} = \frac{1}{2}\int_0^h y\left(b^2 - \frac{b^2}{h^2}y^2\right)dy = \frac{b^2h^2}{8} \qquad \textit{Ans.}$$

EXAMPLE 10.7

Determine the product of inertia for the cross-sectional area of the member shown in Fig. 10–15*a*, about the *x* and *y* centroidal axes.



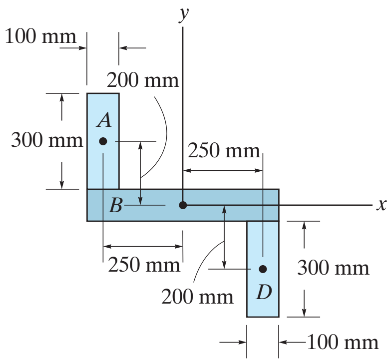


Fig. 10–15

SOLUTION

As in Example 10.5, the cross section can be subdivided into three composite rectangular areas *A*, *B*, and *D*, Fig. 10–15*b*. The coordinates for the centroid of each of these rectangles are shown in the figure. Due to symmetry, the product of inertia of *each rectangle* is *zero* about a set of x', y' axes that passes through the centroid of each rectangle. Using the parallel-axis theorem, we have

Rectangle A

$$I_{xy} = \bar{I}_{x'y'} + A d_x d_y$$
$$= 0 + (300)(100)(-250)(200) = -1.50(10^9)\text{ mm}^4$$

Rectangle B

$$I_{xy} = \bar{I}_{x'y'} + A d_x d_y$$
$$= 0 + 0 = 0$$

Rectangle D

$$I_{xy} = \bar{I}_{x'y'} + A d_x d_y$$
$$= 0 + (300)(100)(250)(-200) = -1.50(10^9)\text{ mm}^4$$

The product of inertia for the entire cross section is therefore

$$I_{xy} = -1.50(10^9) + 0 - 1.50(10^9) = -3.00(10^9)\text{ mm}^4 \qquad \textit{Ans.}$$

NOTE: This negative result is due to the fact that rectangles *A* and *D* have centroids located with negative *x* and negative *y* coordinates, respectively.

*10.6 Moments of Inertia for an Area about Inclined Axes

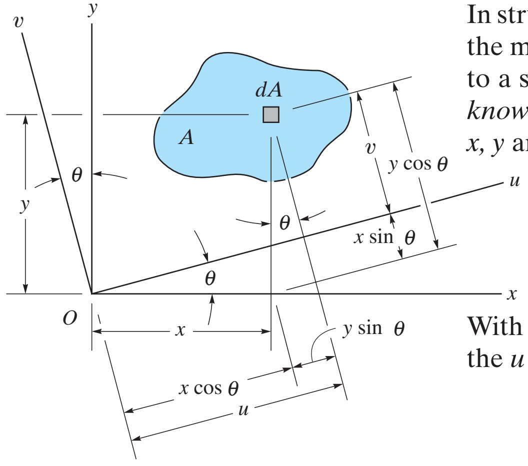


Fig. 10–16

In structural and mechanical design, it is sometimes necessary to calculate the moments and product of inertia I_u, I_v, and I_{uv} for an area with respect to a set of inclined u and v axes when the values for θ, I_x, I_y, and I_{xy} are *known*. To do this we will use *transformation equations* which relate the x, y and u, v coordinates. From Fig. 10–16, these equations are

$$u = x \cos\theta + y \sin\theta$$
$$v = y \cos\theta - x \sin\theta$$

With these equations, the moments and product of inertia of dA about the u and v axes become

$$dI_u = v^2\,dA = (y\cos\theta - x\sin\theta)^2\,dA$$
$$dI_v = u^2\,dA = (x\cos\theta + y\sin\theta)^2\,dA$$
$$dI_{uv} = uv\,dA = (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)\,dA$$

Expanding each expression and integrating, realizing that $I_x = \int y^2\,dA$, $I_y = \int x^2\,dA$, and $I_{xy} = \int xy\,dA$, we obtain

$$I_u = I_x\cos^2\theta + I_y\sin^2\theta - 2I_{xy}\sin\theta\cos\theta$$
$$I_v = I_x\sin^2\theta + I_y\cos^2\theta + 2I_{xy}\sin\theta\cos\theta$$
$$I_{uv} = I_x\sin\theta\cos\theta - I_y\sin\theta\cos\theta + I_{xy}(\cos^2\theta - \sin^2\theta)$$

Using the trigonometric identities $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$ we can simplify the above expressions, in which case

$$\boxed{\begin{aligned} I_u &= \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta \\ I_v &= \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta \\ I_{uv} &= \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta \end{aligned}} \qquad (10\text{–}9)$$

Notice that if the first and second equations are added together, we can show that the polar moment of inertia about the z axis passing through point O is, as expected, *independent* of the orientation of the u and v axes; i.e.,

$$J_O = I_u + I_v = I_x + I_y$$

Principal Moments of Inertia. Equations 10–9 show that I_u, I_v, and I_{uv} depend on the angle of inclination, θ, of the u, v axes. We will now determine the orientation of these axes about which the moments of inertia for the area are maximum and minimum. This particular set of axes is called the *principal axes* of the area, and the corresponding moments of inertia with respect to these axes are called the ***principal moments of inertia***. In general, there is a set of principal axes for every chosen origin O. However, for structural and mechanical design, the origin O is located at the centroid of the area.

The angle which defines the orientation of the principal axes can be found by differentiating the first of Eqs. 10–9 with respect to θ and setting the result equal to zero. Thus,

$$\frac{dI_u}{d\theta} = -2\left(\frac{I_x - I_y}{2}\right)\sin 2\theta - 2I_{xy}\cos 2\theta = 0$$

Therefore, at $\theta = \theta_p$,

$$\boxed{\tan 2\theta_p = \frac{-I_{xy}}{(I_x - I_y)/2}} \qquad (10\text{–}10)$$

The two roots θ_{p_1} and θ_{p_2} of this equation are 90° apart, and so they each specify the inclination of one of the principal axes. In order to substitute them into Eq. 10–9, we must first find the sine and cosine of $2\theta_{p_1}$ and $2\theta_{p_2}$. This can be done using these ratios from the triangles shown in Fig. 10–17, which are based on Eq. 10–10.

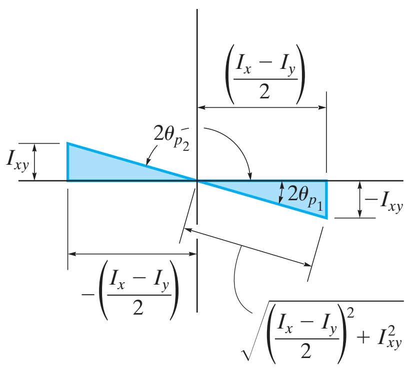
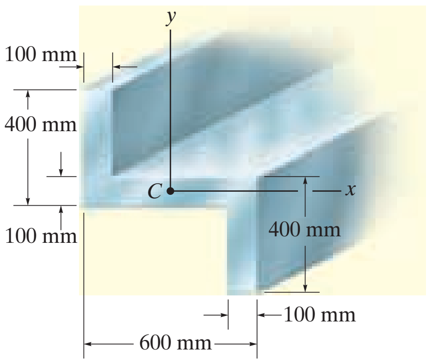


Fig. 10–17

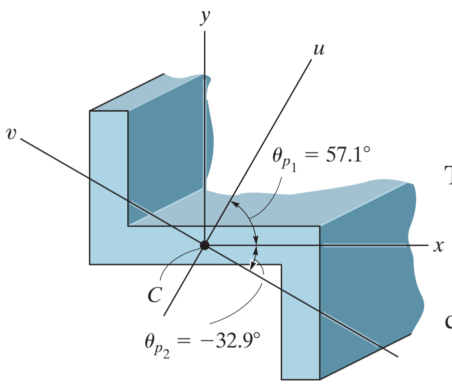
Substituting each of the sine and cosine ratios into the first or second of Eqs. 10–9 and simplifying, we obtain

$$\boxed{I_{\substack{\max\\ \min}} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}} \qquad (10\text{–}11)$$

Depending on the sign chosen, this result gives the maximum or minimum moment of inertia for the area. Furthermore, if the above trigonometric relations for θ_{p_1} and θ_{p_2} are substituted into the third of Eqs. 10–9, it can be shown that $I_{uv} = 0$; that is, the *product of inertia with respect to the principal axes is zero*. Since it was indicated in Sec. 10.6 that the product of inertia is zero with respect to any symmetrical axis, it therefore follows that *any symmetrical axis represents a principal axis of inertia for the area*.



(a)



(b)

Fig. 10–18

Determine the principal moments of inertia and the orientation of the principal axes for the cross-sectional area of the member shown in Fig. 10–18*a* with respect to an axis passing through the centroid.

SOLUTION

The moments and product of inertia of the cross section with respect to the x, y axes have been determined in Examples 10.5 and 10.7. The results are

$$I_x = 2.90(10^9)\text{ mm}^4 \quad I_y = 5.60(10^9)\text{ mm}^4 \quad I_{xy} = -3.00(10^9)\text{ mm}^4$$

Using Eq. 10–10, the angles of inclination of the principal axes u and v are

$$\tan 2\theta_p = \frac{-I_{xy}}{(I_x - I_y)/2} = \frac{-[-3.00(10^9)]}{[2.90(10^9) - 5.60(10^9)]/2} = -2.22$$

$$2\theta_p = -65.8° \text{ and } 114.2°$$

Thus, by inspection of Fig. 10–18*b*,

$$\theta_{p_2} = -32.9° \quad \text{and} \quad \theta_{p_1} = 57.1° \qquad \textit{Ans.}$$

The principal moments of inertia with respect to these axes are determined from Eq. 10–11. Hence,

$$I_{\substack{\max\\\min}} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

$$= \frac{2.90(10^9) + 5.60(10^9)}{2} \pm \sqrt{\left[\frac{2.90(10^9) - 5.60(10^9)}{2}\right]^2 + [-3.00(10^9)]^2}$$

$$I_{\substack{\max\\\min}} = 4.25(10^9) \pm 3.29(10^9)$$

or

$$I_{\max} = 7.54(10^9)\text{ mm}^4 \quad I_{\min} = 0.960(10^9)\text{ mm}^4 \qquad \textit{Ans.}$$

NOTE: The maximum moment of inertia, $I_{\max} = 7.54(10^9)\text{ mm}^4$, occurs with respect to the u axis since *by inspection* most of the cross-sectional area is farthest away from this axis. Or, stated in another manner, $I_{\max}$ occurs about the u axis since this axis is located within $\pm 45°$ of the y axis, which has the larger value of I ($I_y > I_x$). Also, this can be concluded by substituting the data with $\theta = 57.1°$ into the first of Eqs. 10–9 and solving for I_u.

*10.7 Mohr's Circle for Moments of Inertia

Equations 10–9 to 10–11 have a graphical solution that is convenient to use and generally easy to remember. Squaring the first and third of Eqs. 10–9 and adding, it is found that

$$\left(I_u - \frac{I_x + I_y}{2}\right)^2 + I_{uv}^2 = \left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2$$

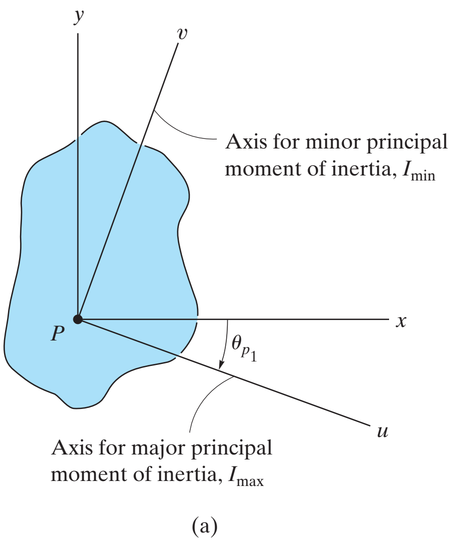
Here I_x, I_y, and I_{xy} are *known constants*. Thus, the above equation may be written in compact form as

$$(I_u - a)^2 + I_{uv}^2 = R^2$$

When this equation is plotted on a set of axes that represent the respective moment of inertia and the product of inertia, as shown in Fig. 10–19, the resulting graph represents a *circle* of radius

$$R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

and having its center located at point $(a, 0)$, where $a = (I_x + I_y)/2$. The circle so constructed is called ***Mohr's circle***, named after the German engineer Otto Mohr (1835–1918).



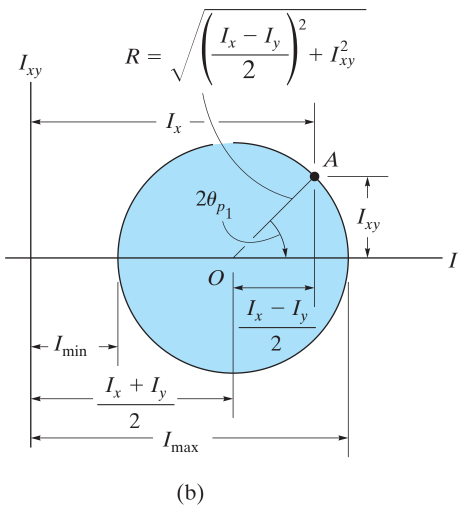
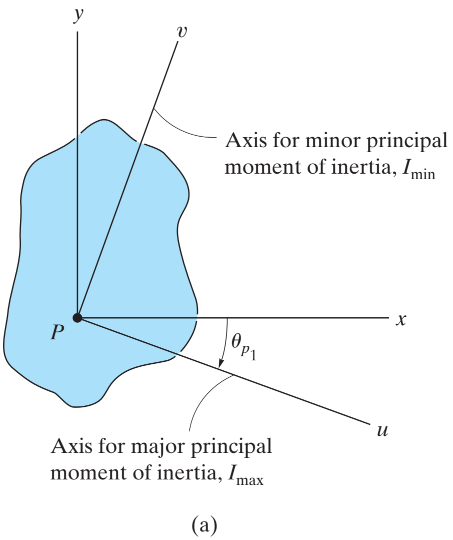


Fig. 10–19



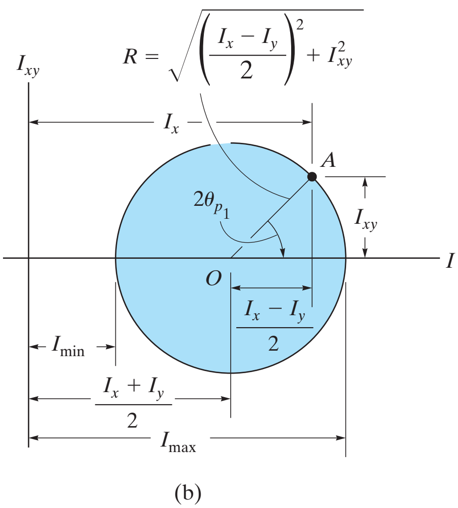


Fig. 10–19 (Repeated)

Procedure for Analysis

The main purpose in using Mohr's circle here is to have a convenient means for finding the principal moments of inertia for an area. The following procedure provides a method for doing this.

Determine I_x, I_y, and I_{xy}.

- Establish the x, y axes and determine I_x, I_y, and I_{xy}, Fig. 10–19*a*.

Construct the Circle.

- Construct a rectangular coordinate system such that the horizontal axis represents the moment of inertia I, and the vertical axis represents the product of inertia I_{xy}, Fig. 10–19*b*.
- Determine the center of the circle, O, which is located at a distance $(I_x + I_y)/2$ from the origin, and plot the reference point A having coordinates (I_x, I_{xy}). Remember, I_x is always positive, whereas I_{xy} can be either positive or negative.
- Connect the reference point A with the center of the circle and determine the distance OA by trigonometry. This distance represents the radius of the circle, Fig. 10–19*b*. Finally, draw the circle.

Principal Moments of Inertia.

- The points where the circle intersects the I axis give the values of the principal moments of inertia I_{min} and I_{max}. Notice that, as expected, the *product of inertia will be zero at these points*, Fig. 10–19*b*.

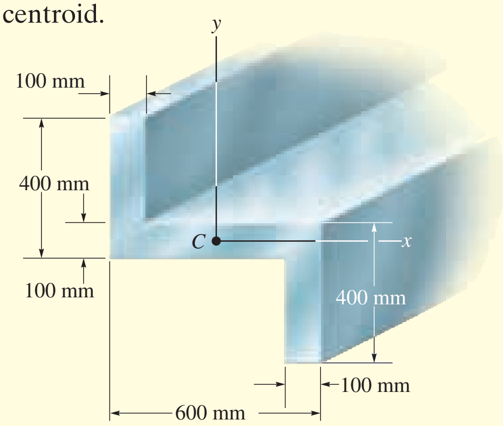
Principal Axes.

- To find the orientation of the major principal axis, use trigonometry to find the angle $2\theta_{p_1}$, *measured from the radius OA to the positive I axis*, Fig. 10–19*b*. This angle represents *twice* the angle from the x axis to the axis of maximum moment of inertia I_{max}, Fig. 10–19*a*. Both the angle on the circle, $2\theta_{p_1}$, and the angle θ_{p_1} *must be measured in the same sense*, as shown in Fig. 10–19. The axis for minimum moment of inertia I_{min} is perpendicular to the axis for I_{max}.

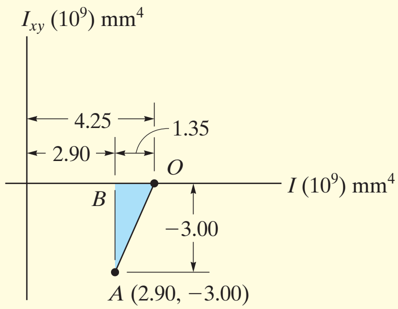
Using trigonometry, the above procedure can be verified to be in accordance with the equations developed in Sec. 10.6.

EXAMPLE 10.9

Using Mohr's circle, determine the principal moments of inertia and the orientation of the major principal axes for the cross-sectional area of the member shown in Fig. 10–20*a*, with respect to an axis passing through the centroid.



(a)



(b)

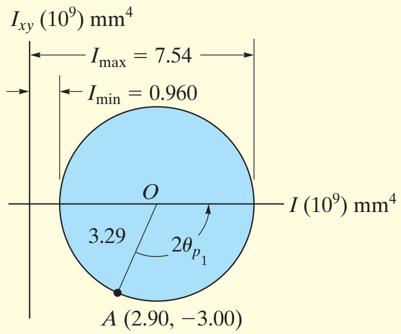
SOLUTION

Determine I_x, I_y, I_{xy}. The moments and product of inertia have been determined in Examples 10.5 and 10.7 with respect to the x, y axes shown in Fig. 10–20*a*. The results are $I_x = 2.90(10^9)\ \text{mm}^4$, $I_y = 5.60(10^9)\ \text{mm}^4$, and $I_{xy} = -3.00(10^9)\ \text{mm}^4$.

Construct the Circle. The I and I_{xy} axes are shown in Fig. 10–20*b*. The center of the circle, O, lies at a distance $(I_x + I_y)/2 = (2.90 + 5.60)/2 = 4.25$ from the origin. When the reference point $A(I_x, I_{xy})$ or $A(2.90, -3.00)$ is connected to point O, the radius OA is determined from the triangle OBA using the Pythagorean theorem.

$$OA = \sqrt{(1.35)^2 + (-3.00)^2} = 3.29$$

The circle is constructed in Fig. 10–20*c*.



(c)

Principal Moments of Inertia. The circle intersects the I axis at points (7.54, 0) and (0.960, 0). Hence,

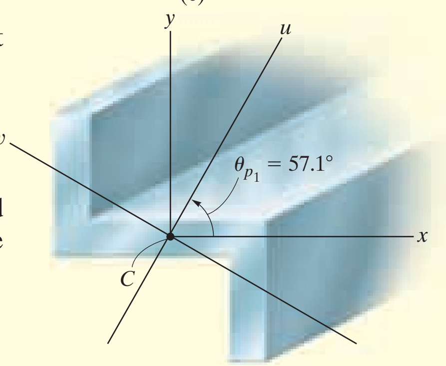
$$I_{\max} = (4.25 + 3.29)10^9 = 7.54(10^9)\ \text{mm}^4 \qquad \textit{Ans.}$$

$$I_{\min} = (4.25 - 3.29)10^9 = 0.960(10^9)\ \text{mm}^4 \qquad \textit{Ans.}$$

Principal Axes. As shown in Fig. 10–20*c*, the angle $2\theta_{p_1}$ is determined from the circle by measuring counterclockwise from OA to the direction of the *positive* I axis. Hence,

$$2\theta_{p_1} = 180^\circ - \sin^{-1}\left(\frac{|BA|}{|OA|}\right) = 180^\circ - \sin^{-1}\left(\frac{3.00}{3.29}\right) = 114.2^\circ$$

The principal axis for $I_{\max} = 7.54(10^9)\ \text{mm}^4$ is therefore oriented at an angle $\theta_{p_1} = 57.1^\circ$, measured *counterclockwise*, from the *positive* x axis to the *positive* u axis. The v axis is perpendicular to this axis. The results are shown in Fig. 10–20*d*.

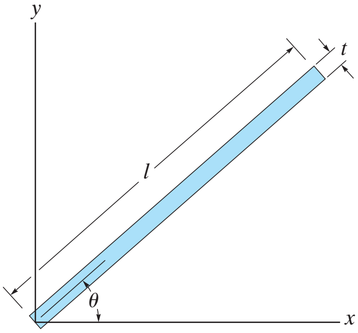


(d)

Fig. 10–20

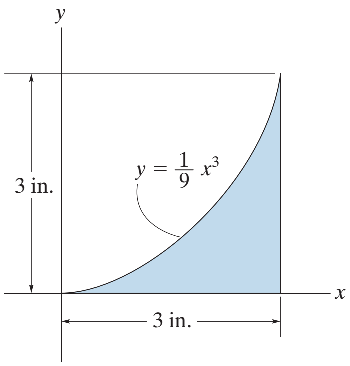
PROBLEMS

10–54. Determine the product of inertia of the thin strip of area with respect to the x and y axes. The strip is oriented at an angle θ from the x axis. Assume that $t \ll l$.



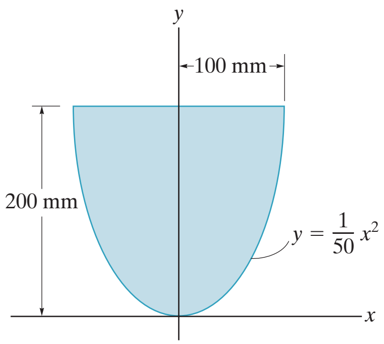
Prob. 10–54

10–55. Determine the product of inertia of the shaded area with respect to the x and y axes.



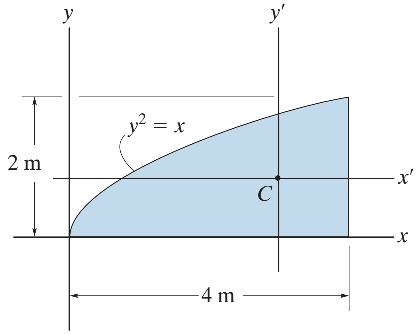
Prob. 10–55

***10–56.** Determine the product of inertia for the shaded portion of the parabola with respect to the x and y axes.



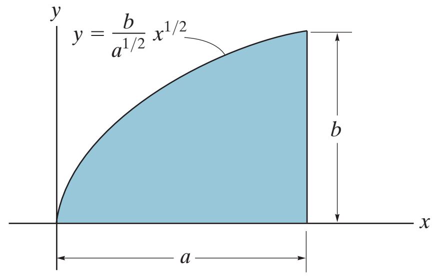
Prob. 10–56

10–57. Determine the product of inertia of the shaded area with respect to the x and y axes, and then use the parallel-axis theorem to find the product of inertia of the area with respect to the centroidal x' and y' axes.



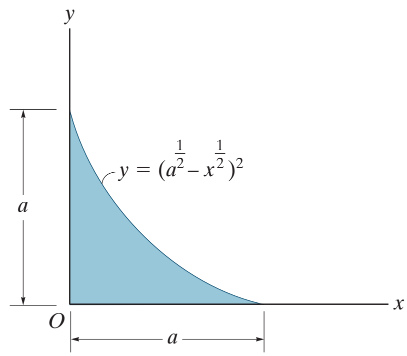
Prob. 10–57

10–58. Determine the product of inertia for the parabolic area with respect to the x and y axes.



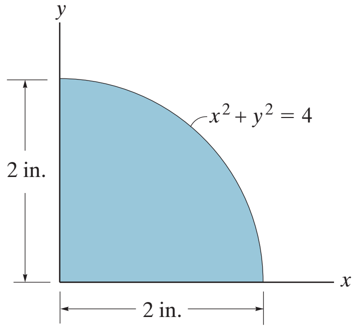
Prob. 10–58

10–59. Determine the product of inertia of the shaded area with respect to the x and y axes.



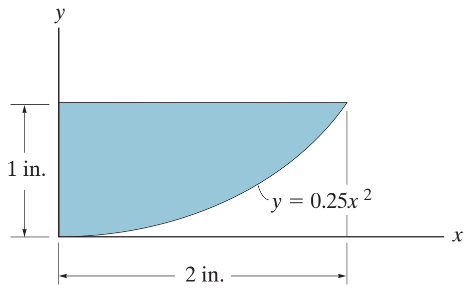
Prob. 10–59

***10–60.** Determine the product of inertia of the shaded area with respect to the x and y axes.



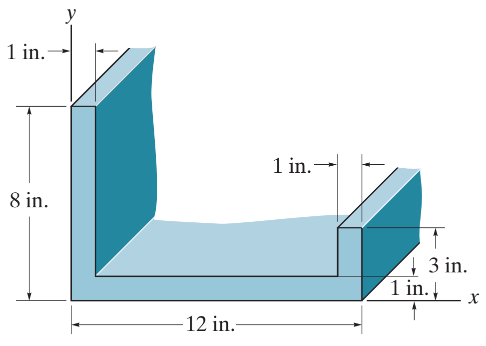
Prob. 10–60

10–61. Determine the product of inertia of the shaded area with respect to the x and y axes.



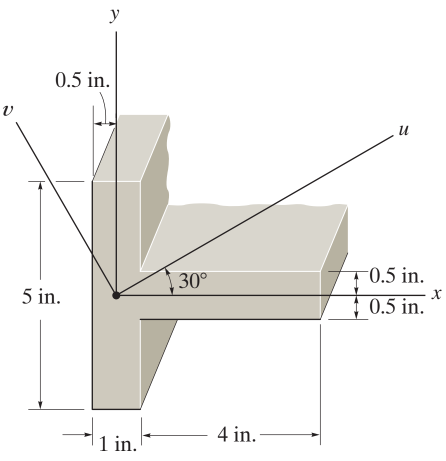
Prob. 10–61

10–62. Determine the product of inertia for the beam's cross-sectional area with respect to the x and y axes.



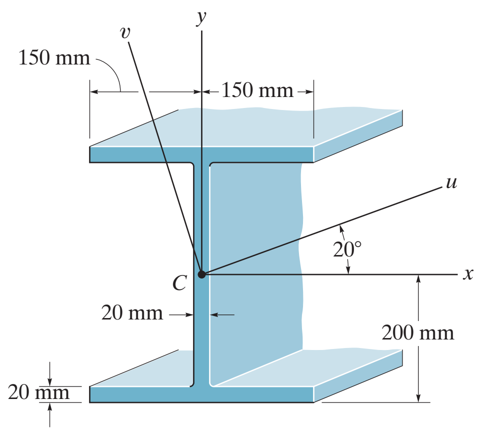
Prob. 10–62

10–63. Determine the moments of inertia of the shaded area with respect to the u and v axes.



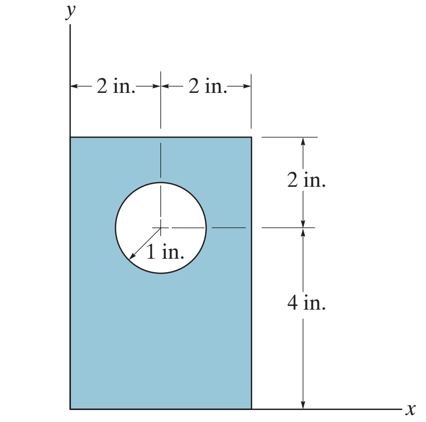
Prob. 10–63

***10–64.** Determine the product of inertia for the beam's cross-sectional area with respect to the u and v axes.



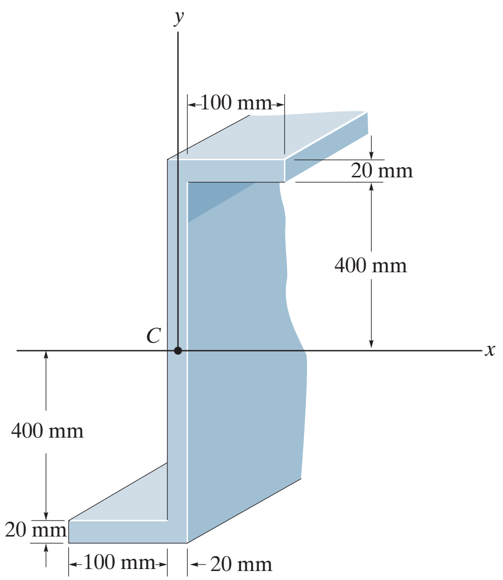
Prob. 10–64

10–65. Determine the product of inertia for the shaded area with respect to the x and y axes.



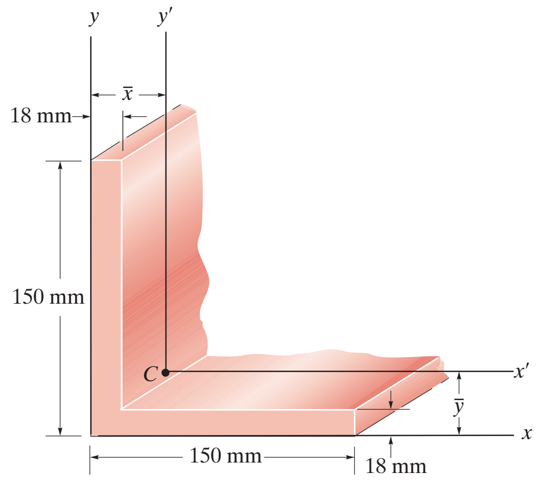
Prob. 10–65

10–66. Determine the product of inertia of the cross-sectional area with respect to the x and y axes.



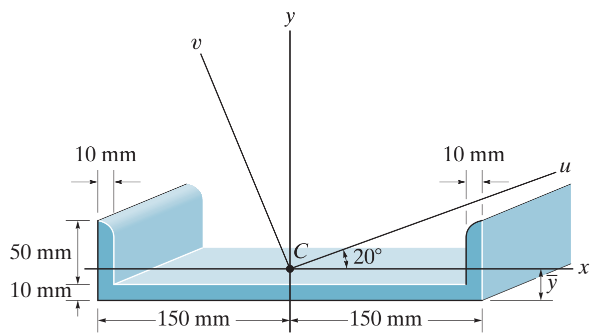
Prob. 10–66

10–67. Determine the location $(\bar{x}, \bar{y})$ to the centroid C of the angle's cross-sectional area, and then compute the product of inertia with respect to the x' and y' axes.



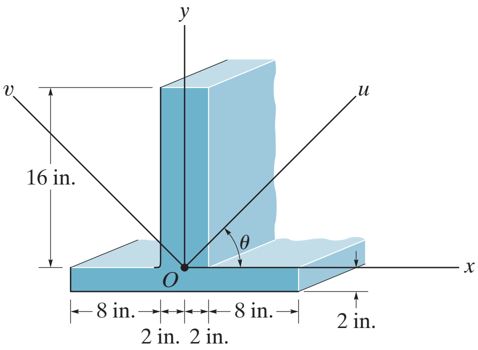
Prob. 10–67

***10–68.** Determine the distance $\bar{y}$ to the centroid of the area and then calculate the moments of inertia I_u and I_v of the channel's cross-sectional area. The u and v axes have their origin at the centroid C. For the calculation, assume all corners to be square.



Prob. 10–68

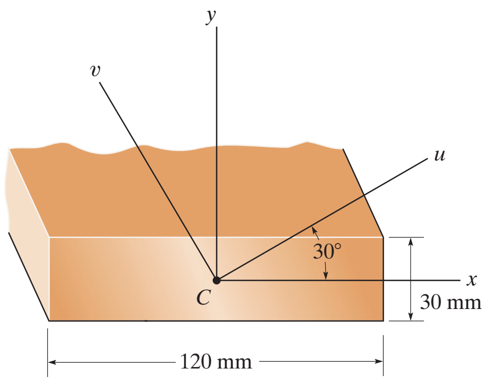
10–69. Determine the moments of inertia I_u, I_v and the product of inertia I_{uv} for the beam's cross-sectional area. Take $\theta = 45°$.



Prob. 10–69

10–70. Determine the moments of inertia I_u, I_v and the product of inertia I_{uv} for the rectangular area. The u and v axes pass through the centroid C.

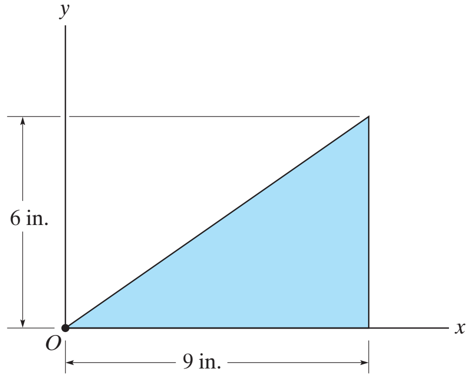
10–71. Solve Prob. 10–70 using Mohr's circle. *Hint*: To solve, find the coordinates of the point $P(I_u, I_{uv})$ on the circle, measured counterclockwise from the radial line OA. (See Fig. 10–19.) The point $Q(I_v, -I_{uv})$ is on the opposite side of the circle.



Probs. 10–70/71

***10–72.** Determine the directions of the principal axes having an origin at point O, and the principal moments of inertia for the triangular area about the axes.

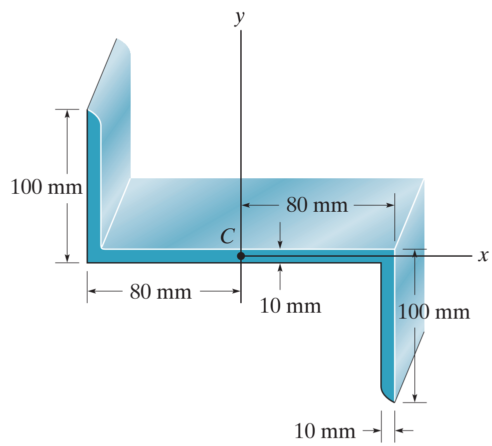
10–73. Solve Prob. 10–72 using Mohr's circle.



Probs. 10–72/73

10–74. Determine the orientation of the principal axes having an origin at point C, and the principal moments of inertia of the cross section about these axes.

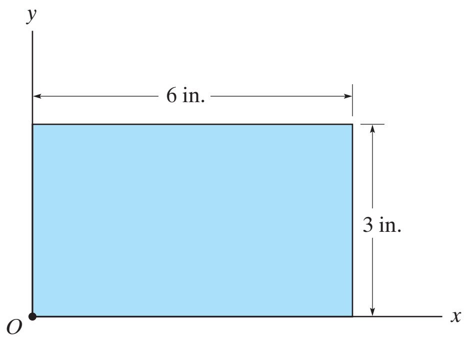
10–75. Solve Prob. 10–74 using Mohr's circle.



Probs. 10–74/75

***10–76.** Determine the orientation of the principal axes having an origin at point O, and the principal moments of inertia for the rectangular area about these axes.

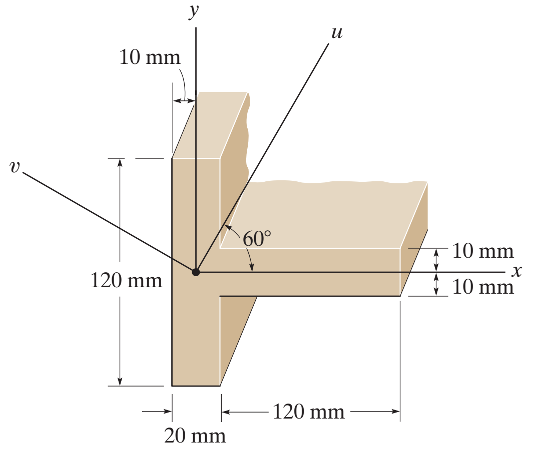
10–77. Solve Prob. 10–76 using Mohr's circle.



Probs. 10–76/77

***10–80.** Determine the moments and product of inertia for the shaded area with respect to the u and v axes.

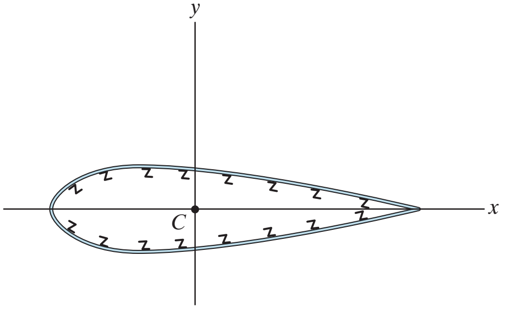
10–81. Solve Prob. 10–80 using Mohr's circle.



Probs. 10–80/81

10–78. The area of the cross section of an airplane wing has the following properties about the x and y axes passing through the centroid C: $\bar{I}_x = 450 \text{ in}^4$, $\bar{I}_y = 1730 \text{ in}^4$, $\bar{I}_{xy} = 138 \text{ in}^4$. Determine the orientation of the principal axes and the principal moments of inertia.

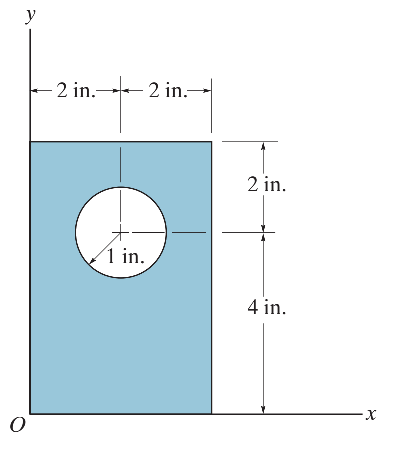
10–79. Solve Prob. 10–78 using Mohr's circle.



Probs. 10–78/79

10–82. Determine the directions of the principal axes with origin located at point O, and the principal moments of inertia for the area about these axes.

10–83. Solve Prob. 10–82 using Mohr's circle.



Probs. 10–82/83

10.8 Mass Moment of Inertia

The mass moment of inertia of a body is a measure of the body's resistance to angular acceleration. Since it is used in dynamics to study rotational motion, methods for its calculation will now be discussed.*

Consider the rigid body shown in Fig. 10–21. We define the *mass moment of inertia* of the body about the z axis as

$$I = \int_m r^2\, dm \tag{10–12}$$

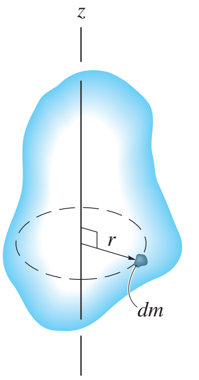


Fig. 10–21

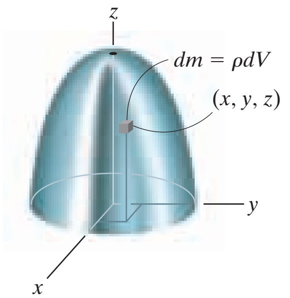
Here r is the perpendicular distance from the axis to the arbitrary element dm. Since the formulation involves r, the value of I is *unique* for each axis about which it is computed. The axis which is generally chosen, however, passes through the body's mass center G. Common units used for its measurement are $\text{kg}\cdot\text{m}^2$ or $\text{slug}\cdot\text{ft}^2$.

If the body consists of material having a density ρ, then $dm = \rho\, dV$, Fig. 10–22a. Substituting this into Eq. 10–12, the body's moment of inertia is then computed using *volume elements* for integration; i.e.,

$$I = \int_V r^2 \rho\, dV \tag{10–13}$$

For most applications, ρ will be a *constant*, and so this term may be factored out of the integral, and the integration is then purely a function of geometry.

$$I = \rho \int_V r^2\, dV \tag{10–14}$$



(a)

Fig. 10–22

*Another property of the body, which measures the symmetry of the body's mass with respect to a coordinate system, is the mass product of inertia. This property most often applies to the three-dimensional motion of a body and is discussed in *Engineering Mechanics: Dynamics* (Chapter 21).

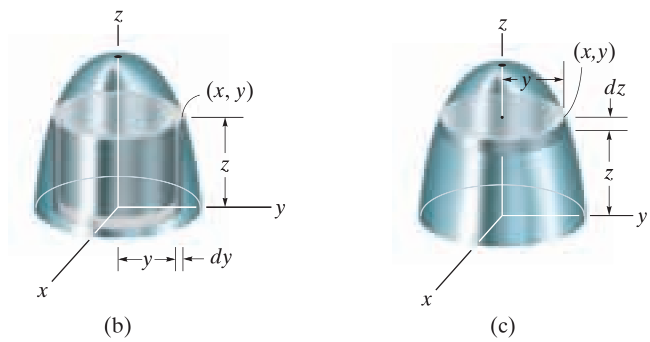


Fig. 10–22 (cont'd)

Procedure for Analysis

If a body is symmetrical with respect to an axis, as in Fig. 10–22, then its mass moment of inertia about the axis can be determined by using a single integration. Shell and disk elements are used for this purpose.

Shell Element.

- If a *shell element* having a height z, radius y, and thickness dy is chosen for integration, Fig. 10–22*b*, then its volume is $dV = (2\pi y)(z)\,dy$.
- This element can be used in Eq. 10–13 or 10–14 for determining the moment of inertia I_z of the body about the z axis since the *entire element*, due to its "thinness," lies at the *same* perpendicular distance $r = y$ from the z axis (see Example 10.10).

Disk Element.

- If a disk element having a radius y and a thickness dz is chosen for integration, Fig. 10–22*c*, then its volume is $dV = (\pi y^2)\,dz$.
- In this case the element is *finite* in the radial direction, and consequently its points *do not* all lie at the *same radial distance r* from the z axis. As a result, Eqs. 10–13 or 10–14 *cannot* be used to determine I_z. Instead, to perform the integration using this element, it is first necessary to determine the moment of inertia *of the element* about the z axis and then integrate this result (see Example 10.11).

EXAMPLE 10.10

Determine the mass moment of inertia of the cylinder shown in Fig. 10–23*a* about the z axis. The density of the material, ρ, is constant.

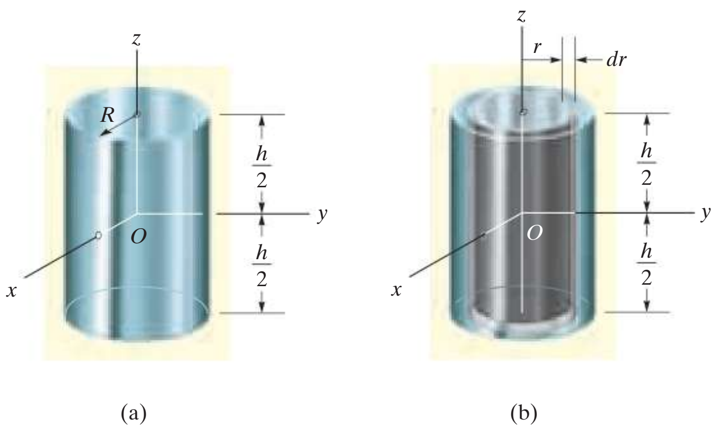


Fig. 10–23

SOLUTION

Shell Element. This problem will be solved using the *shell element* in Fig. 10–23*b* and thus only a single integration is required. The volume of the element is $dV = (2\pi r)(h)\,dr$, and so its mass is $dm = \rho\,dV = \rho(2\pi hr\,dr)$. Since the *entire element* lies at the same distance r from the z axis, the moment of inertia *of the element* is

$$dI_z = r^2\,dm = \rho 2\pi h r^3\,dr$$

Integrating over the entire cylinder yields

$$I_z = \int_m r^2\,dm = \rho 2\pi h \int_0^R r^3\,dr = \frac{\rho\pi}{2} R^4 h$$

Since the mass of the cylinder is

$$m = \int_m dm = \rho 2\pi h \int_0^R r\,dr = \rho\pi h R^2$$

then

$$I_z = \frac{1}{2} m R^2 \qquad \textit{Ans.}$$

If the density of the solid in Fig. 10–24*a* is 5 slug/ft^3, determine the mass moment of inertia about the *y* axis.

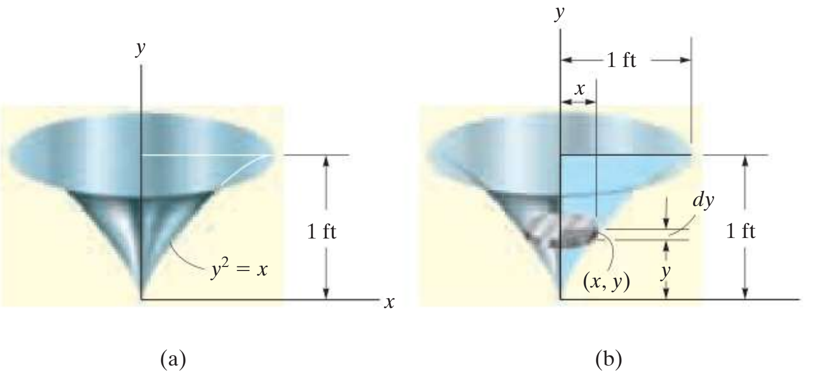


Fig. 10–24

SOLUTION

Disk Element. The moment of inertia will be determined using this *disk element*, as shown in Fig. 10–24*b*. Here the element intersects the curve at the arbitrary point (x, y) and has a mass

$$dm = \rho\, dV = \rho(\pi x^2)\, dy$$

Although all points on the element are *not* located at the same distance from the *y* axis, it is still possible to determine the moment of inertia dI_y *of the element* about the *y* axis. In the previous example it was shown that the moment of inertia of a homogeneous cylinder about its longitudinal axis is $I = \frac{1}{2}mR^2$, where m and R are the mass and radius of the cylinder. Since the height of the cylinder is not involved in this formula, we can also use this result for a disk. Thus, for the disk element in Fig. 10–24*b*, we have

$$dI_y = \frac{1}{2}(dm)x^2 = \frac{1}{2}[\rho(\pi x^2)\, dy]x^2$$

Substituting $x = y^2$, $\rho = 5$ slug/ft^3, and integrating with respect to y, from $y = 0$ to $y = 1$ ft, yields the moment of inertia for the entire solid.

$$I_y = \frac{5\pi}{2}\int_0^{1\text{ ft}} x^4\, dy = \frac{5\pi}{2}\int_0^{1\text{ ft}} y^8\, dy = 0.873 \text{ slug}\cdot\text{ft}^2 \qquad \textit{Ans.}$$

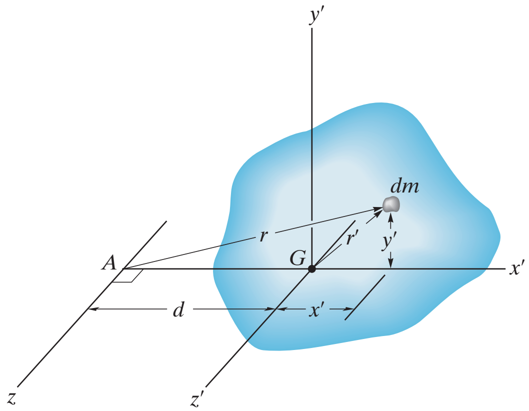


Fig. 10–25

Parallel-Axis Theorem. If the moment of inertia of the body about an axis passing through the body's mass center is known, then the moment of inertia about any other *parallel axis* can be determined by using the *parallel-axis theorem*. To derive this theorem, consider the body shown in Fig. 10–25. The z' axis passes through the mass center G, whereas the corresponding *parallel z axis* lies at a constant distance d away. Selecting the differential element of mass dm, which is located at point (x', y'), and using the Pythagorean theorem, $r^2 = (d + x')^2 + y'^2$, the moment of inertia of the body about the z axis is

$$I = \int_m r^2\, dm = \int_m [(d + x')^2 + y'^2]\, dm$$

$$= \int_m (x'^2 + y'^2)\, dm + 2d \int_m x'\, dm + d^2 \int_m dm$$

Since $r'^2 = x'^2 + y'^2$, the first integral represents I_G. The second integral is equal to *zero*, since the z' axis passes through the body's mass center, i.e., $\int x'\, dm = \bar{x} \int dm = 0$ since $\bar{x} = 0$. Finally, the third integral is the total mass m of the body. Hence, the moment of inertia about the z axis becomes

$$\boxed{I = I_G + md^2} \qquad (10\text{–}15)$$

where

I_G = moment of inertia about the z' axis passing through the mass center G

m = mass of the body

d = distance between the parallel axes

Radius of Gyration. Occasionally, the moment of inertia of a body about a specified axis is reported in handbooks using the *radius of gyration, k*. This value has units of length, and when it and the body's mass m are known, the moment of inertia can be determined from the equation

$$I = mk^2 \quad \text{or} \quad k = \sqrt{\frac{I}{m}} \tag{10–16}$$

Note the *similarity* between the definition of k in this formula and r in the equation $dI = r^2\,dm$, which defines the moment of inertia of a differential element of mass dm of the body about an axis.

Composite Bodies. If a body is constructed from a number of simple shapes such as disks, spheres, and rods, the moment of inertia of the body about any axis z can be determined by adding algebraically the moments of inertia of all the composite shapes calculated about the same axis. Algebraic addition is necessary since a composite part must be considered as a negative quantity if it has already been included within another part—as in the case of a "hole" subtracted from a solid plate. Also, the parallel-axis theorem is needed for the calculations if the center of mass of each composite part does not lie on the z axis. For calculations, a table of some simple shapes is given on the inside back cover.

This flywheel, which operates a metal cutter, has a large moment of inertia about its center. Once it begins rotating it is difficult to stop it and therefore a uniform motion can be effectively transferred to the cutting blade. (© Russell C. Hibbeler)

EXAMPLE 10.12

If the plate shown in Fig. 10–26*a* has a density of 8000 kg/m^3 and a thickness of 10 mm, determine its mass moment of inertia about an axis perpendicular to the page and passing through the pin at O.

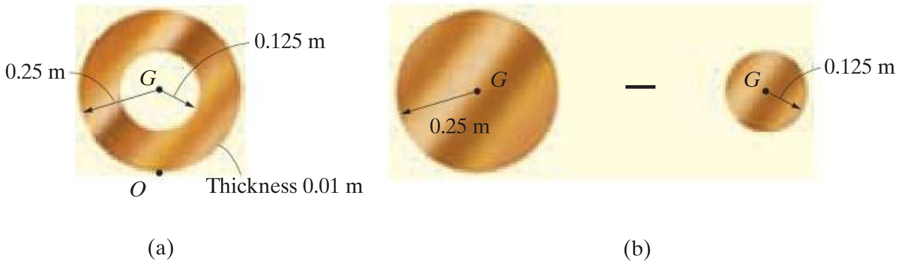


Fig. 10–26

SOLUTION

The plate consists of two composite parts, the 250-mm-radius disk *minus* a 125-mm-radius disk, Fig. 10–26*b*. The moment of inertia about O can be determined by finding the moment of inertia of each of these parts about O and then *algebraically* adding the results. The calculations are performed by using the parallel-axis theorem in conjunction with the mass moment of inertia formula for a circular disk, $I_G = \frac{1}{2}mr^2$, as found on the inside back cover.

Disk. The moment of inertia of a disk about an axis perpendicular to the plane of the disk and passing through G is $I_G = \frac{1}{2}mr^2$. The mass center of both disks is 0.25 m from point O. Thus,

$$m_d = \rho_d V_d = 8000 \text{ kg/m}^3 \,[\pi(0.25 \text{ m})^2(0.01 \text{ m})] = 15.71 \text{ kg}$$

$$\begin{aligned}(I_O)_d &= \tfrac{1}{2}m_d r_d^2 + m_d d^2 \\ &= \tfrac{1}{2}(15.71 \text{ kg})(0.25 \text{ m})^2 + (15.71 \text{ kg})(0.25 \text{ m})^2 \\ &= 1.473 \text{ kg}\cdot\text{m}^2\end{aligned}$$

Hole. For the smaller disk (hole), we have

$$m_h = \rho_h V_h = 8000 \text{ kg/m}^3 \,[\pi(0.125 \text{ m})^2(0.01 \text{ m})] = 3.93 \text{ kg}$$

$$\begin{aligned}(I_O)_h &= \tfrac{1}{2}m_h r_h^2 + m_h d^2 \\ &= \tfrac{1}{2}(3.93 \text{ kg})(0.125 \text{ m})^2 + (3.93 \text{ kg})(0.25 \text{ m})^2 \\ &= 0.276 \text{ kg}\cdot\text{m}^2\end{aligned}$$

The moment of inertia of the plate about the pin is therefore

$$\begin{aligned}I_O &= (I_O)_d - (I_O)_h \\ &= 1.473 \text{ kg}\cdot\text{m}^2 - 0.276 \text{ kg}\cdot\text{m}^2 \\ &= 1.20 \text{ kg}\cdot\text{m}^2 \qquad \textit{Ans.}\end{aligned}$$

EXAMPLE 10.13

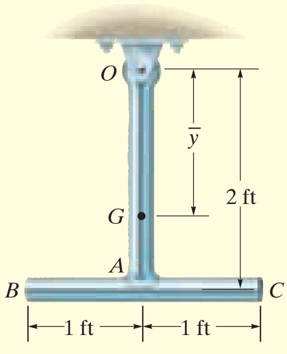


Fig. 10–27

The pendulum in Fig. 10–27 consists of two thin rods each having a weight of 10 lb. Determine the pendulum's mass moment of inertia about an axis passing through (a) the pin at O, and (b) the mass center G of the pendulum.

SOLUTION

Part (a). Using the table on the inside back cover, the moment of inertia of rod OA about an axis perpendicular to the page and passing through the end point O of the rod is $I_O = \frac{1}{3}ml^2$. Hence,

$$(I_{OA})_O = \frac{1}{3}ml^2 = \frac{1}{3}\left(\frac{10 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(2 \text{ ft})^2 = 0.414 \text{ slug}\cdot\text{ft}^2$$

Realize that this same value may be determined using $I_G = \frac{1}{12}ml^2$ and the parallel-axis theorem; i.e.,

$$(I_{OA})_O = \frac{1}{12}ml^2 + md^2 = \frac{1}{12}\left(\frac{10 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(2 \text{ ft})^2 + \frac{10 \text{ lb}}{32.2 \text{ ft/s}^2}(1 \text{ ft})^2$$
$$= 0.414 \text{ slug}\cdot\text{ft}^2$$

For rod BC we have

$$(I_{BC})_O = \frac{1}{12}ml^2 + md^2 = \frac{1}{12}\left(\frac{10 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(2 \text{ ft})^2 + \frac{10 \text{ lb}}{32.2 \text{ ft/s}^2}(2 \text{ ft})^2$$
$$= 1.346 \text{ slug}\cdot\text{ft}^2$$

The moment of inertia of the pendulum about O is therefore

$$I_O = 0.414 + 1.346 = 1.76 \text{ slug}\cdot\text{ft}^2 \qquad \textit{Ans.}$$

Part (b). The mass center G will be located relative to the pin at O. Assuming this distance to be $\bar{y}$, Fig. 10–27, and using the formula for determining the mass center, we have

$$\bar{y} = \frac{\Sigma \tilde{y} m}{\Sigma m} = \frac{1(10/32.2) + 2(10/32.2)}{(10/32.2) + (10/32.2)} = 1.50 \text{ ft}$$

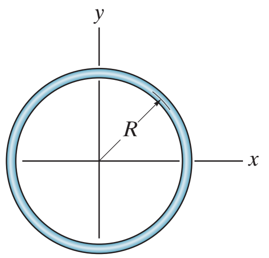
The moment of inertia I_G may be computed in the same manner as I_O, which requires successive applications of the parallel-axis theorem in order to transfer the moments of inertia of rods OA and BC to G. A more direct solution, however, involves applying the parallel-axis theorem using the result for I_O determined above; i.e.,

$$I_O = I_G + md^2; \qquad 1.76 \text{ slug}\cdot\text{ft}^2 = I_G + \left(\frac{20 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(1.50 \text{ ft})^2$$

$$I_G = 0.362 \text{ slug}\cdot\text{ft}^2 \qquad \textit{Ans.}$$

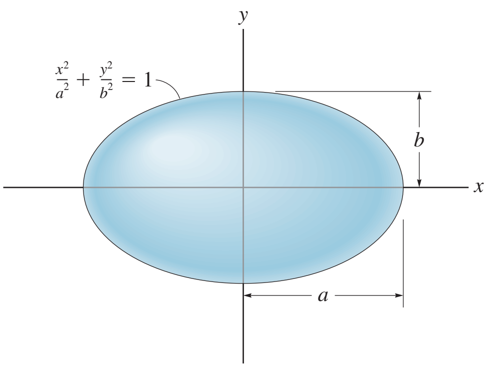
PROBLEMS

***10–84.** Determine the moment of inertia of the thin ring about the z axis. The ring has a mass m.



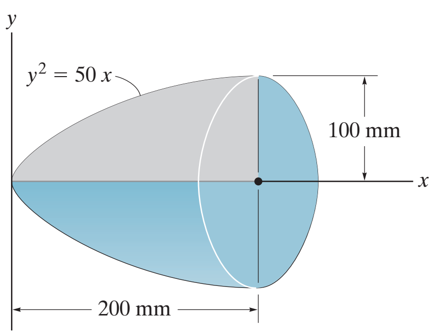
Prob. 10–84

10–85. Determine the moment of inertia of the ellipsoid with respect to the x axis and express the result in terms of the mass m of the ellipsoid. The material has a constant density ρ.



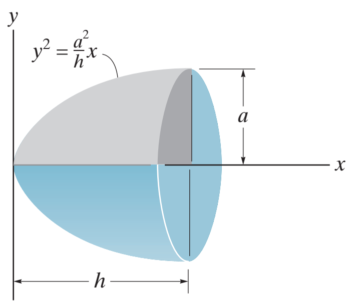
Prob. 10–85

10–86. Determine the radius of gyration k_x of the paraboloid. The density of the material is $\rho = 5\ \text{Mg/m}^3$.



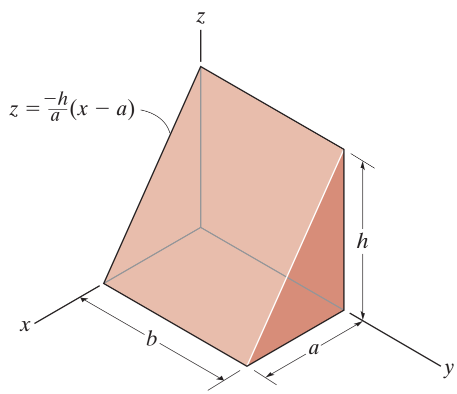
Prob. 10–86

10–87. The paraboloid is formed by revolving the shaded area around the x axis. Determine the moment of inertia about the x axis and express the result in terms of the total mass m of the paraboloid. The material has a constant density ρ.



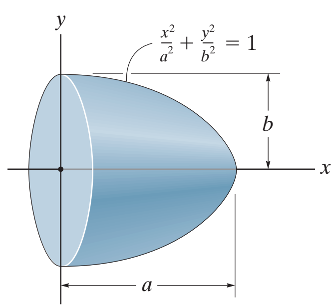
Prob. 10–87

***10–88.** Determine the moment of inertia of the homogenous triangular prism with respect to the y axis. Express the result in terms of the mass m of the prism. *Hint*: For integration, use thin plate elements parallel to the x–y plane having a thickness of dz.



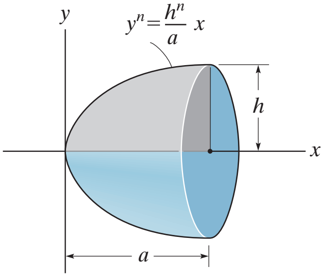
Prob. 10–88

10–89. Determine the moment of inertia of the semiellipsoid with respect to the x axis and express the result in terms of the mass m of the semiellipsoid. The material has a constant density ρ.



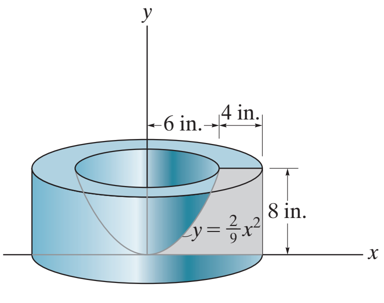
Prob. 10–89

10–90. Determine the radius of gyration k_x of the solid formed by revolving the shaded area about x axis. The density of the material is ρ.



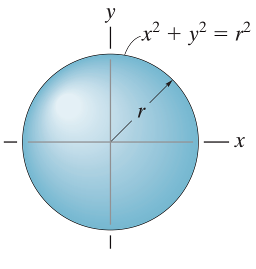
Prob. 10–90

10–91. The concrete shape is formed by rotating the shaded area about the y axis. Determine the moment of inertia I_y. The specific weight of concrete is $\gamma = 150 \text{ lb/ft}^3$.



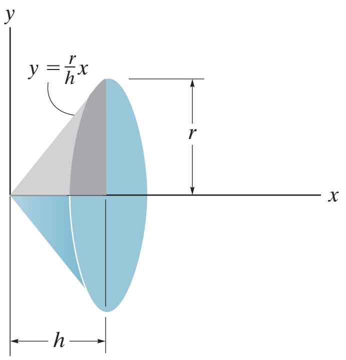
Prob. 10–91

***10–92.** Determine the moment of inertia I_x of the sphere and express the result in terms of the total mass m of the sphere. The sphere has a constant density ρ.



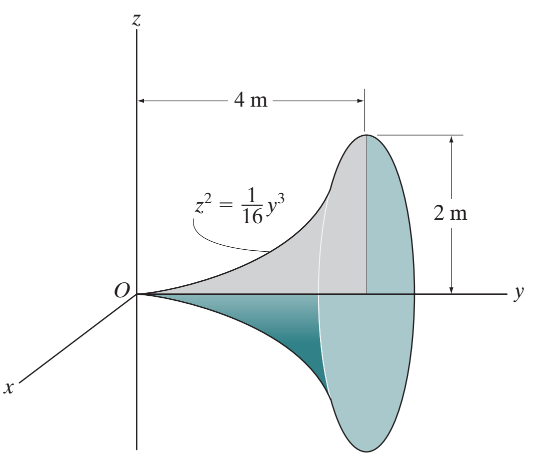
Prob. 10–92

10–93. The right circular cone is formed by revolving the shaded area around the x axis. Determine the moment of inertia I_x and express the result in terms of the total mass m of the cone. The cone has a constant density ρ.



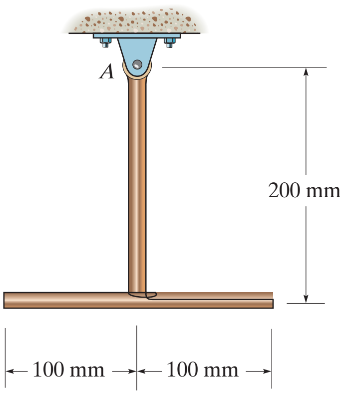
Prob. 10–93

10–94. Determine the mass moment of inertia I_y of the solid formed by revolving the shaded area around the y axis. The total mass of the solid is 1500 kg.



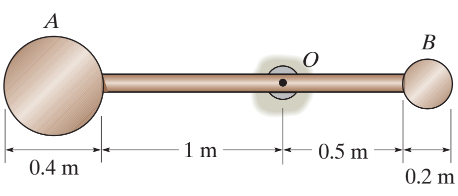
Prob. 10–94

10–95. The slender rods have a mass of 4 kg/m. Determine the moment of inertia of the assembly about an axis perpendicular to the page and passing through point A.



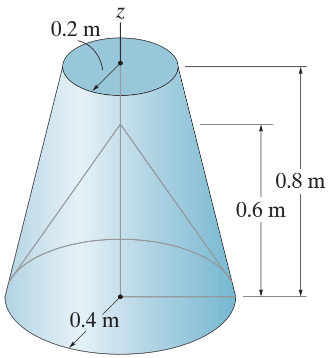
Prob. 10–95

*__10–96.__ The pendulum consists of a 8-kg circular disk A, a 2-kg circular disk B, and a 4-kg slender rod. Determine the radius of gyration of the pendulum about an axis perpendicular to the page and passing through point O.



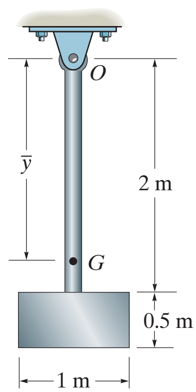
Prob. 10–96

10–97. Determine the moment of inertia I_z of the frustum of the cone which has a conical depression. The material has a density of 200 kg/m^3.



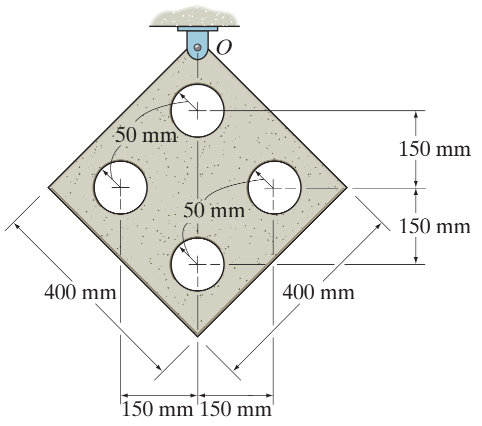
Prob. 10–97

10–98. The pendulum consists of the 3-kg slender rod and the 5-kg thin plate. Determine the location $\bar{y}$ of the center of mass G of the pendulum; then find the mass moment of inertia of the pendulum about an axis perpendicular to the page and passing through G.



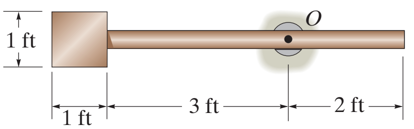
Prob. 10–98

10–99. Determine the mass moment of inertia of the thin plate about an axis perpendicular to the page and passing through point O. The material has a mass per unit area of $20\ \text{kg/m}^2$.



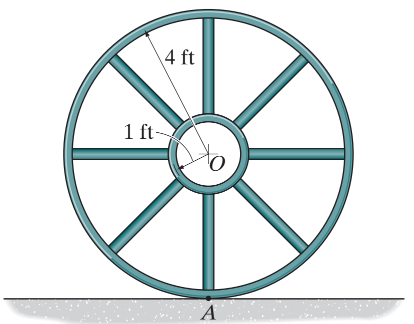
Prob. 10–99

***10–100.** The pendulum consists of a plate having a weight of 12 lb and a slender rod having a weight of 4 lb. Determine the radius of gyration of the pendulum about an axis perpendicular to the page and passing through point O.



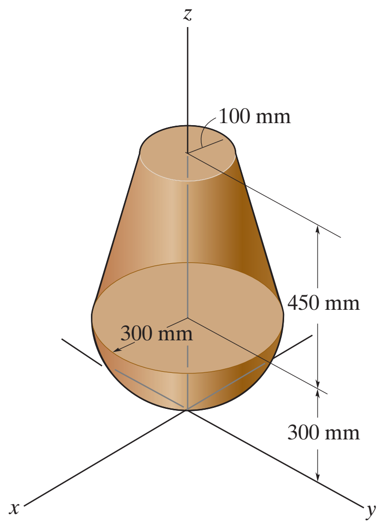
Prob. 10–100

10–101. If the large ring, small ring and each of the spokes weigh 100 lb, 15 lb, and 20 lb, respectively, determine the mass moment of inertia of the wheel about an axis perpendicular to the page and passing through point A.



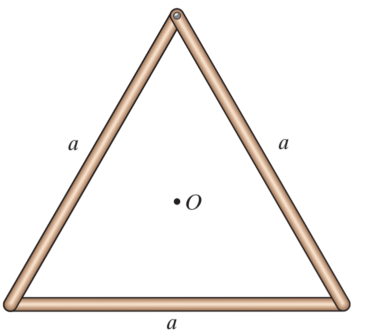
Prob. 10–101

10–102. Determine the mass moment of inertia of the assembly about the z axis. The density of the material is $7.85\ \text{Mg/m}^3$.



Prob. 10–102

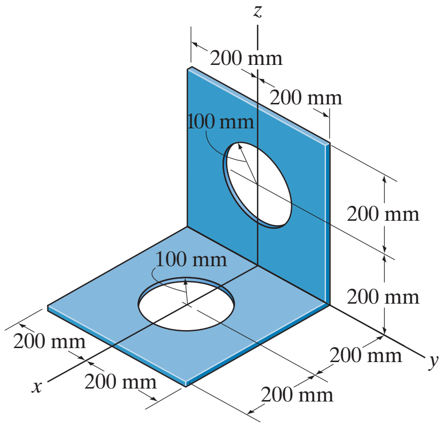
10–103. Each of the three slender rods has a mass m. Determine the moment of inertia of the assembly about an axis that is perpendicular to the page and passes through the center point O.



Prob. 10–103

***10–104.** The thin plate has a mass per unit area of $10\ \mathrm{kg/m^2}$. Determine its mass moment of inertia about the y axis.

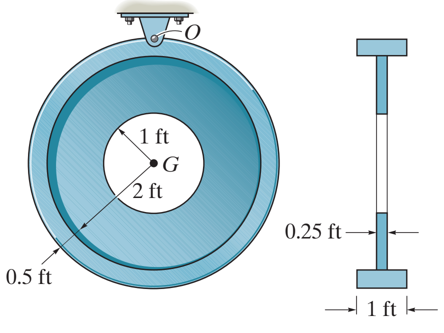
10–105. The thin plate has a mass per unit area of $10\ \mathrm{kg/m^2}$. Determine its mass moment of inertia about the z axis.



Probs. 10–104/105

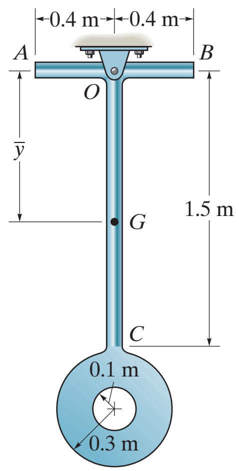
10–106. Determine the moment of inertia of the assembly about an axis that is perpendicular to the page and passes through the center of mass G. The material has a specific weight of $\gamma = 90\ \mathrm{lb/ft^3}$.

10–107. Determine the moment of inertia of the assembly about an axis that is perpendicular to the page and passes through point O. The material has a specific weight of $\gamma = 90\ \mathrm{lb/ft^3}$.



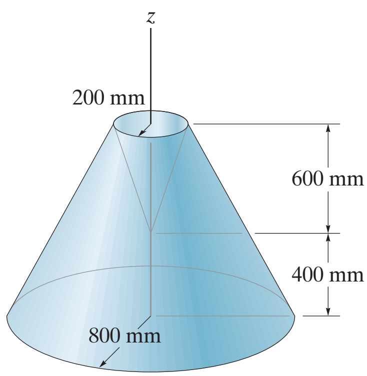
Probs. 10–106/107

***10–108.** The pendulum consists of two slender rods AB and OC which have a mass of $3\ \mathrm{kg/m}$. The thin plate has a mass of $12\ \mathrm{kg/m^2}$. Determine the location $\bar{y}$ of the center of mass G of the pendulum, then calculate the moment of inertia of the pendulum about an axis perpendicular to the page and passing through G.



Prob. 10–108

10–109. Determine the moment of inertia I_z of the frustum of the cone which has a conical depression. The material has a density of $200\ \mathrm{kg/m^3}$.



Prob. 10–109

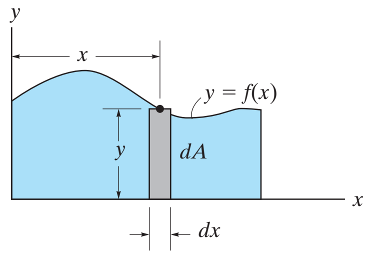
CHAPTER REVIEW

Area Moment of Inertia

The *area moment of inertia* represents the second moment of the area about an axis. It is frequently used in formulas related to the strength and stability of structural members or mechanical elements.

If the area shape is irregular but can be described mathematically, then a differential element must be selected and integration over the entire area must be performed to determine the moment of inertia.

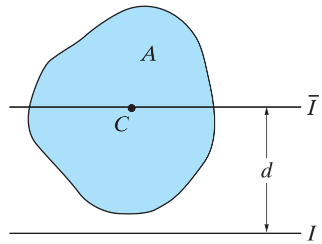
$$I_y = \int_A x^2\, dA$$



Parallel-Axis Theorem

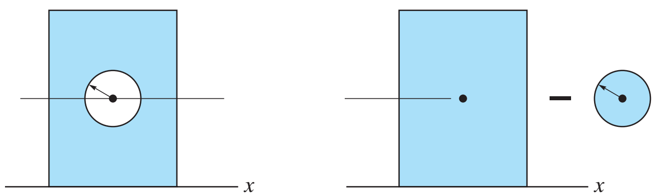
If the moment of inertia for an area is known about a centroidal axis, then its moment of inertia about a parallel axis can be determined using the parallel-axis theorem.

$$I = \bar{I} + Ad^2$$



Composite Area

If an area is a composite of common shapes, as found on the inside back cover, then its moment of inertia is equal to the algebraic sum of the moments of inertia of each of its parts.



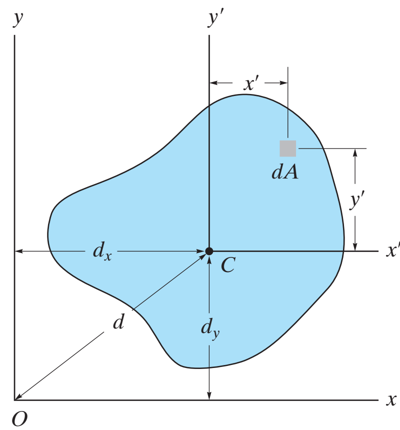
Product of Inertia

The *product of inertia* of an area is used in formulas to determine the orientation of an axis about which the moment of inertia for the area is a maximum or minimum.

$$I_{xy} = \int_A xy\, dA$$

If the product of inertia for an area is known with respect to its centroidal x', y' axes, then its value can be determined with respect to any x, y axes using the parallel-axis theorem for the product of inertia.

$$I_{xy} = \bar{I}_{x'y'} + Ad_x d_y$$



Principal Moments of Inertia

Provided the moments of inertia, I_x and I_y, and the product of inertia, I_{xy}, are known, then the transformation formulas, or Mohr's circle, can be used to determine the maximum and minimum or *principal moments of inertia* for the area, as well as finding the orientation of the principal axes of inertia.

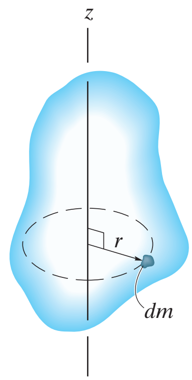
$$I_{\substack{\max \\ \min}} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

$$\tan 2\theta_p = \frac{-I_{xy}}{(I_x - I_y)/2}$$

Mass Moment of Inertia

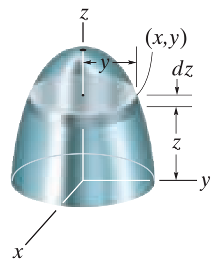
The *mass moment of inertia* is a property of a body that measures its resistance to a change in its rotation. It is defined as the "second moment" of the mass elements of the body about an axis.

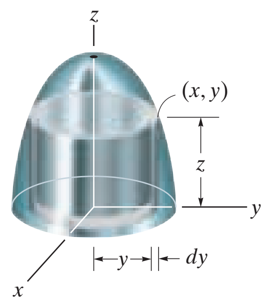
$$I = \int_m r^2\, dm$$



For homogeneous bodies having axial symmetry, the mass moment of inertia can be determined by a single integration, using a disk or shell element.

$$I = \rho \int_V r^2\, dV$$



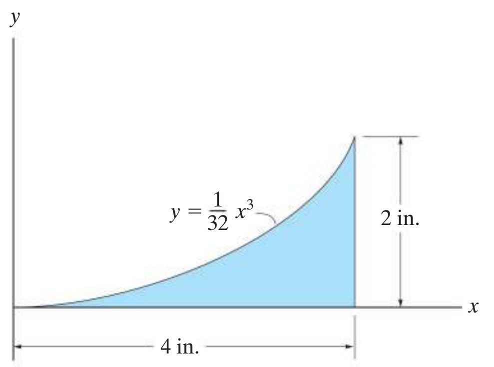


The mass moment of inertia of a composite body is determined by using tabular values of its composite shapes, found on the inside back cover, along with the parallel-axis theorem.

$$I = I_G + md^2$$

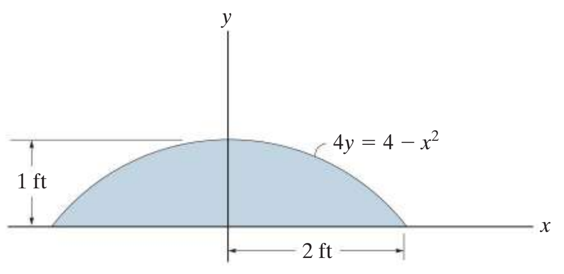
REVIEW PROBLEMS

R10–1. Determine the moment of inertia for the shaded area about the x axis.



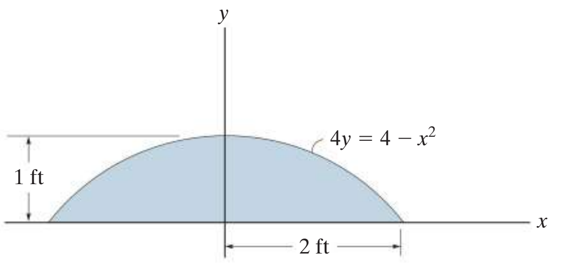
Prob. R10–1

R10–2. Determine the moment of inertia for the shaded area about the x axis.



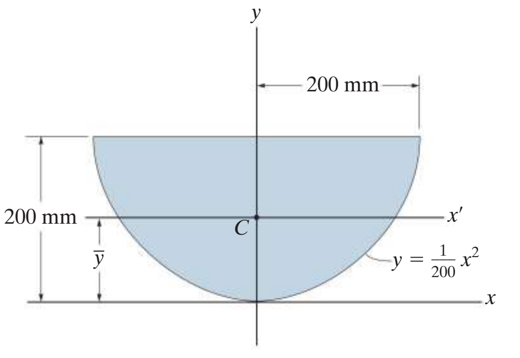
Prob. R10–2

R10–3. Determine the area moment of inertia of the shaded area about the y axis.



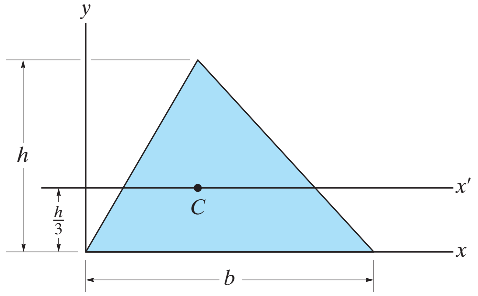
Prob. R10–3

R10–4. Determine the area moment of inertia of the area about the x axis. Then, using the parallel-axis theorem, find the area moment of inertia about the x' axis that passes through the centroid C of the area. $\bar{y} = 120$ mm.



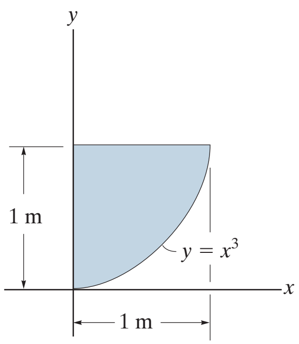
Prob. R10–4

R10–5. Determine the area moment of inertia of the triangular area about (a) the x axis, and (b) the centroidal x' axis.



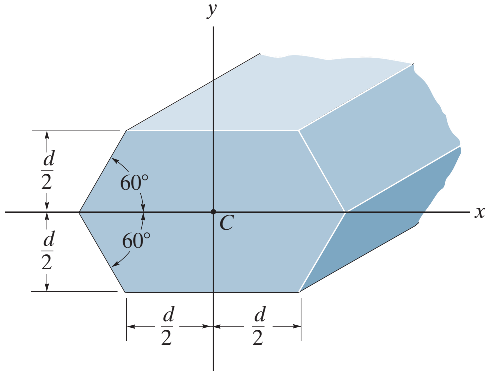
Prob. R10–5

R10–6. Determine the product of inertia of the shaded area with respect to the x and y axes.



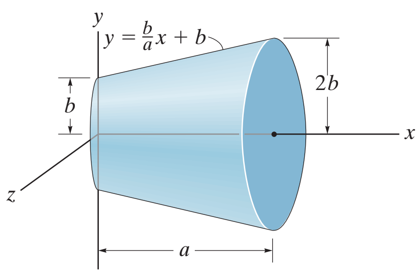
Prob. R10–6

R10–7. Determine the area moment of inertia of the beam's cross-sectional area about the x axis which passes through the centroid C.



Prob. R10–7

R10–8. Determine the mass moment of inertia I_x of the body and express the result in terms of the total mass m of the body. The density is constant.



Prob. R10–8

Chapter 11

(© John Kershaw/Alamy)

Equilibrium and stability of this scissors lift as a function of its position can be determined using the methods of work and energy, which are explained in this chapter.

Virtual Work

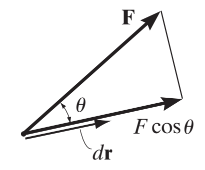
CHAPTER OBJECTIVES

- To introduce the principle of virtual work and show how it applies to finding the equilibrium configuration of a system of pin-connected members.
- To establish the potential-energy function and use the potential-energy method to investigate the type of equilibrium or stability of a rigid body or system of pin-connected members.

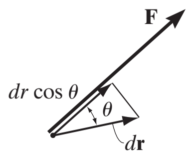
11.1 Definition of Work

The principle of virtual work was proposed by the Swiss mathematician Jean Bernoulli in the eighteenth century. It provides an alternative method for solving problems involving the equilibrium of a particle, a rigid body, or a system of connected rigid bodies. Before we discuss this principle, however, we must first define the work produced by a force and by a couple moment.

Work of a Force. A force does work when it undergoes a displacement in the direction of its line of action. Consider, for example, the force $\mathbf{F}$ in Fig. 11–1a that undergoes a differential displacement $d\mathbf{r}$. If θ is the angle between the force and the displacement, then the component of $\mathbf{F}$ in



(a)



(b)

Fig. 11–1

the direction of the displacement is $F \cos \theta$. And so the work produced by **F** is

$$dU = F\,dr \cos \theta$$

Notice that this expression is also the product of the force F and the component of displacement in the direction of the force, $dr \cos \theta$, Fig. 11–1*b*. If we use the definition of the dot product (Eq. 2–11) the work can also be written as

$$dU = \mathbf{F} \cdot d\mathbf{r}$$

As the above equations indicate, work is a ***scalar***, and like other scalar quantities, it has a magnitude that can either be *positive* or *negative*.

In the SI system, the unit of work is a ***joule*** (J), which is the work produced by a 1-N force that displaces through a distance of 1 m in the direction of the force ($1\text{ J} = 1\text{ N}\cdot\text{m}$). The unit of work in the FPS system is the foot-pound (ft · lb), which is the work produced by a 1-lb force that displaces through a distance of 1 ft in the direction of the force.

The moment of a force has this same combination of units; however, the concepts of moment and work are in no way related. A moment is a vector quantity, whereas work is a scalar.

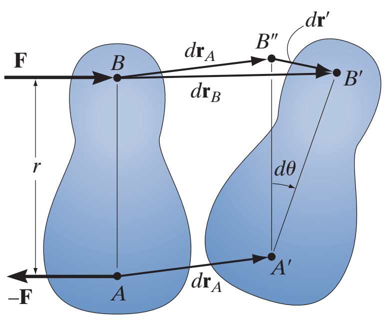


Fig. 11–2

Work of a Couple Moment. The rotation of a couple moment also produces work. Consider the rigid body in Fig. 11–2, which is acted upon by the couple forces **F** and –**F** that produce a couple moment **M** having a magnitude $M = Fr$. When the body undergoes the differential displacement shown, points A and B move $d\mathbf{r}_A$ and $d\mathbf{r}_B$ to their final positions A' and B', respectively. Since $d\mathbf{r}_B = d\mathbf{r}_A + d\mathbf{r}'$, this movement can be thought of as a *translation* $d\mathbf{r}_A$, where A and B move to A' and B'', and a *rotation* about A', where the body rotates through the angle $d\theta$ about A. The couple forces do no work during the translation $d\mathbf{r}_A$ because each force undergoes the same amount of displacement in opposite directions, thus canceling out the work. During rotation, however, **F** is displaced $dr' = r\,d\theta$, and so it does work $dU = F\,dr' = F\,r\,d\theta$. Since $M = Fr$, the work of the couple moment **M** is therefore

$$dU = M d\theta$$

If **M** and $d\theta$ have the same sense, the work is *positive*; however, if they have the opposite sense, the work will be *negative*.

Virtual Work. The definitions of the work of a force and a couple have been presented in terms of *actual movements* expressed by differential displacements having magnitudes of dr and $d\theta$. Consider now an *imaginary* or ***virtual movement*** of a body in static equilibrium, which indicates a displacement or rotation that is *assumed* and *does not actually exist*. These movements are first-order differential quantities and will be denoted by the symbols δr and $\delta\theta$ (delta r and delta θ), respectively. The *virtual work* done by a force having a virtual displacement δr is

$$\boxed{\delta U = F\cos\theta\,\delta r} \tag{11–1}$$

Similarly, when a couple undergoes a virtual rotation $\delta\theta$ in the plane of the couple forces, the *virtual work* is

$$\boxed{\delta U = M\,\delta\theta} \tag{11–2}$$

11.2 Principle of Virtual Work

The ***principle of virtual*** work states that if a body is in equilibrium, then the algebraic sum of the virtual work done by all the forces and couple moments acting on the body is zero for any virtual displacement of the body. Thus,

$$\delta U = 0 \tag{11–3}$$

For example, consider the free-body diagram of the particle (ball) that rests on the floor, Fig. 11–3. If we "imagine" the ball to be displaced downwards a virtual amount δy, then the weight does positive virtual work, $W\,\delta y$, and the normal force does negative virtual work, $-N\,\delta y$. For equilibrium the total virtual work must be zero, so that $\delta U = W\,\delta y - N\,\delta y = (W - N)\,\delta y = 0$. Since $\delta y \neq 0$, then $N = W$ as required by applying $\Sigma F_y = 0$.

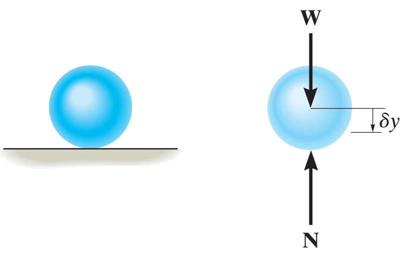


Fig. 11–3

In a similar manner, we can also apply the virtual-work equation $\delta U = 0$ to a rigid body subjected to a coplanar force system. Here, separate virtual translations in the x and y directions, and a virtual rotation about an axis perpendicular to the x–y plane that passes through an arbitrary point O, will correspond to the three equilibrium equations, $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M_O = 0$. When writing these equations, it is *not necessary* to include the work done by the *internal forces* acting within the body since a rigid body *does not deform* when subjected to an external loading, and furthermore, when the body moves through a virtual displacement, the internal forces occur in equal but opposite collinear pairs, so that the corresponding work done by each pair of forces will cancel.

To demonstrate an application, consider the simply supported beam in Fig. 11–4*a*. When the beam is given a virtual rotation $\delta\theta$ about point B, Fig. 11–4*b*, the only forces that do work are **P** and $\mathbf{A}_y$. Since $\delta y = l\,\delta\theta$ and $\delta y' = (l/2)\,\delta\theta$, the virtual work equation for this case is $\delta U = A_y(l\,\delta\theta) - P(l/2)\,\delta\theta = (A_y l - Pl/2)\,\delta\theta = 0$. Since $\delta\theta \neq 0$, then $A_y = P/2$. Excluding $\delta\theta$, notice that the terms in parentheses actually represent the application of $\Sigma M_B = 0$.

As seen from the above two examples, no added advantage is gained by solving particle and rigid-body equilibrium problems using the principle of virtual work. This is because for each application of the virtual-work equation, the virtual displacement, common to every term, factors out, leaving an equation that could have been obtained in a more *direct manner* by simply applying an equation of equilibrium.

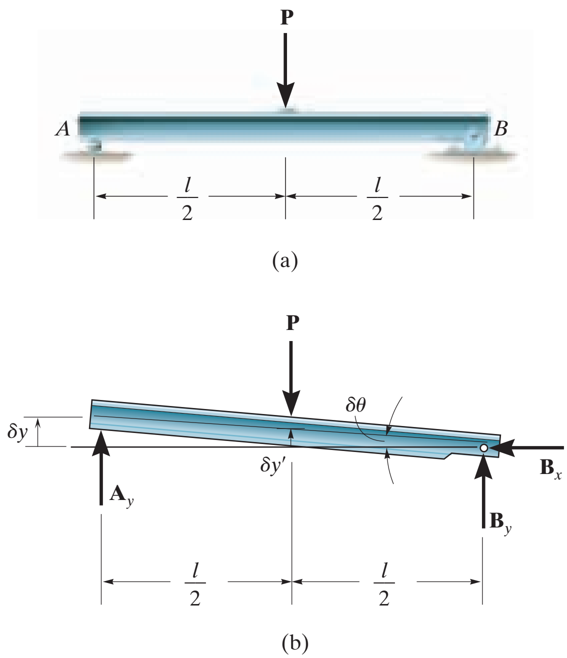


Fig. 11–4

11.3 Principle of Virtual Work for a System of Connected Rigid Bodies

The method of virtual work is particularly effective for solving equilibrium problems that involve a system of several *connected* rigid bodies, such as the ones shown in Fig. 11–5.

Each of these systems is said to have only one degree of freedom since the arrangement of the links can be completely specified using only one coordinate θ. In other words, with this single coordinate and the length of the members, we can locate the position of the forces **F** and **P**.

In this text, we will only consider the application of the principle of virtual work to systems containing one degree of freedom.* Because they are less complicated, they will serve as a way to approach the solution of more complex problems involving systems with many degrees of freedom. The procedure for solving problems involving a system of frictionless connected rigid bodies follows.

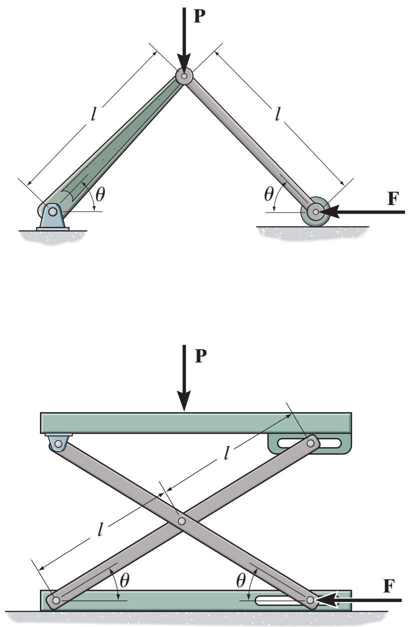


Fig. 11–5

Important Points

- A force does work when it moves through a displacement in the direction of the force. A couple moment does work when it moves through a collinear rotation. Specifically, positive work is done when the force or couple moment and its displacement have the same sense of direction.
- The principle of virtual work is generally used to determine the equilibrium configuration for a system of multiple connected members.
- A virtual displacement is imaginary; i.e., it does not really happen. It is a differential displacement that is given in the positive direction of a position coordinate.
- Forces or couple moments that do not virtually displace do no virtual work.

This scissors lift has one degree of freedom. Without the need for dismembering the mechanism, the force in the hydraulic cylinder AB required to provide the lift can be determined *directly* by using the principle of virtual work. (© Russell C. Hibbeler)

*This method of applying the principle of virtual work is sometimes called the *method of virtual displacements* because a virtual displacement is applied, resulting in the calculation of a real force. Although it is not used here, we can also apply the principle of virtual work as a *method of virtual forces*. This method is often used to apply a virtual force and then determine the displacements of points on deformable bodies. See R. C. Hibbeler, *Mechanics of Materials*, 8th edition, Pearson/Prentice Hall, 2011.

Procedure for Analysis

Free-Body Diagram.

- Draw the free-body diagram of the entire system of connected bodies and define the *coordinate* q.
- Sketch the "deflected position" of the system on the free-body diagram when the system undergoes a *positive virtual* displacement δq.

Virtual Displacements.

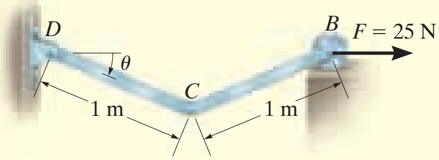
- Indicate *position coordinates* s, each measured from a *fixed point* on the free-body diagram. These coordinates are directed to the forces that do work.
- Each of these coordinate axes should be *parallel* to the line of action of the force to which it is directed, so that the virtual work along the coordinate axis can be calculated.
- Relate each of the position coordinates s to the coordinate q; then *differentiate* these expressions in order to express each virtual displacement δs in terms of δq.

Virtual-Work Equation.

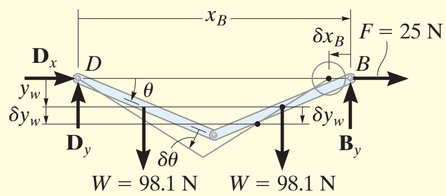
- Write the *virtual-work equation* for the system assuming that, whether possible or not, each position coordinate s undergoes a *positive* virtual displacement δs. If a force or couple moment is in the same direction as the positive virtual displacement, the work is positive. Otherwise, it is negative.
- Express the work of *each* force and couple moment in the equation in terms of δq.
- Factor out this common displacement from all the terms, and solve for the unknown force, couple moment, or equilibrium position q.

EXAMPLE 11.1

Determine the angle θ for equilibrium of the two-member linkage shown in Fig. 11–6*a*. Each member has a mass of 10 kg.



(a)



(b)

Fig. 11–6

SOLUTION

Free-Body Diagram. The system has only one degree of freedom since the location of both links can be specified by the single coordinate, $(q =)\ \theta$. As shown on the free-body diagram in Fig. 11–6*b*, when θ has a *positive* (clockwise) virtual rotation $\delta\theta$, only the force **F** and the two 98.1-N weights do work. (The reactive forces $\mathbf{D}_x$ and $\mathbf{D}_y$ are fixed, and $\mathbf{B}_y$ does not displace along its line of action.)

Virtual Displacements. If the origin of coordinates is established at the *fixed* pin support *D*, then the position of **F** and **W** can be specified by the *position coordinates* x_B and y_w. In order to determine the work, note that, as required, these coordinates are parallel to the lines of action of their associated forces. Expressing these position coordinates in terms of θ and taking the derivatives yields

$$x_B = 2(1 \cos\theta)\text{ m} \qquad \delta x_B = -2 \sin\theta\ \delta\theta \text{ m} \tag{1}$$

$$y_w = \tfrac{1}{2}(1 \sin\theta)\text{ m} \qquad \delta y_w = 0.5 \cos\theta\ \delta\theta \text{ m} \tag{2}$$

It is seen by the *signs* of these equations, and indicated in Fig. 11–6*b*, that an *increase* in θ (i.e., $\delta\theta$) causes a *decrease* in x_B and an *increase* in y_w.

Virtual-Work Equation. If the virtual displacements δx_B and δy_w were *both positive*, then the forces **W** and **F** would do positive work since the forces and their corresponding displacements would have the same sense. Hence, the virtual-work equation for the displacement $\delta\theta$ is

$$\delta U = 0; \qquad W\,\delta y_w + W\,\delta y_w + F\,\delta x_B = 0 \tag{3}$$

Substituting Eqs. 1 and 2 into Eq. 3 in order to relate the virtual displacements to the common virtual displacement $\delta\theta$ yields

$$98.1(0.5 \cos\theta\ \delta\theta) + 98.1(0.5 \cos\theta\ \delta\theta) + 25(-2 \sin\theta\ \delta\theta) = 0$$

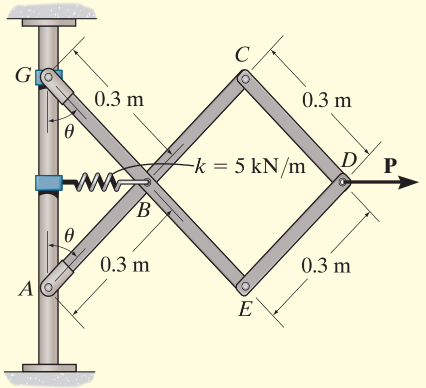
Notice that the "negative work" done by **F** (force in the opposite sense to displacement) has actually been *accounted for* in the above equation by the "negative sign" of Eq. 1. Factoring out the *common displacement* $\delta\theta$ and solving for θ, noting that $\delta\theta \neq 0$, yields

$$(98.1 \cos\theta - 50 \sin\theta)\ \delta\theta = 0$$

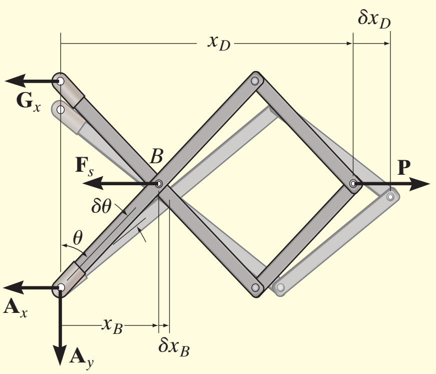
$$\theta = \tan^{-1}\frac{98.1}{50} = 63.0^\circ \qquad \textit{Ans.}$$

NOTE: If this problem had been solved using the equations of equilibrium, it would be necessary to dismember the links and apply three scalar equations to *each* link. The principle of virtual work, by means of calculus, has eliminated this task so that the answer is obtained directly.

EXAMPLE 11.2



(a)



(b)

Fig. 11–7

Determine the required force P in Fig. 11–7a needed to maintain equilibrium of the scissors linkage when $\theta = 60^\circ$. The spring is unstretched when $\theta = 30^\circ$. Neglect the mass of the links.

SOLUTION

Free-Body Diagram. Only $\mathbf{F}_s$ and $\mathbf{P}$ do work when θ undergoes a *positive* virtual displacement $\delta\theta$, Fig. 11–7b. For the arbitrary position θ, the spring is stretched (0.3 m) sin θ − (0.3 m) sin 30°, so that

$$F_s = ks = 5000 \text{ N/m}\,[(0.3 \text{ m}) \sin\theta - (0.3 \text{ m}) \sin 30^\circ]$$
$$= (1500 \sin\theta - 750) \text{ N}$$

Virtual Displacements. The position coordinates, x_B and x_D, measured from the *fixed point* A, are used to locate $\mathbf{F}_s$ and $\mathbf{P}$. These coordinates are parallel to the line of action of their corresponding forces. Expressing x_B and x_D in terms of the angle θ using trigonometry,

$$x_B = (0.3 \text{ m}) \sin\theta$$
$$x_D = 3[(0.3 \text{ m}) \sin\theta] = (0.9 \text{ m}) \sin\theta$$

Differentiating, we obtain the virtual displacements of points B and D.

$$\delta x_B = 0.3 \cos\theta\, \delta\theta \qquad (1)$$
$$\delta x_D = 0.9 \cos\theta\, \delta\theta \qquad (2)$$

Virtual-Work Equation. Force $\mathbf{P}$ does positive work since it acts in the positive sense of its virtual displacement. The spring force $\mathbf{F}_s$ does negative work since it acts opposite to its positive virtual displacement. Thus, the virtual-work equation becomes

$$\delta U = 0; \qquad -F_s\,\delta x_B + P\delta x_D = 0$$
$$-[1500 \sin\theta - 750]\,(0.3 \cos\theta\,\delta\theta) + P\,(0.9 \cos\theta\,\delta\theta) = 0$$
$$[0.9P + 225 - 450 \sin\theta] \cos\theta\,\delta\theta = 0$$

Since $\cos\theta\,\delta\theta \neq 0$, then this equation requires

$$P = 500 \sin\theta - 250$$

When $\theta = 60^\circ$,

$$P = 500 \sin 60^\circ - 250 = 183 \text{ N}$$ *Ans.*

EXAMPLE 11.3

If the box in Fig. 11–8*a* has a mass of 10 kg, determine the couple moment M needed to maintain equilibrium when $\theta = 60°$. Neglect the mass of the members.

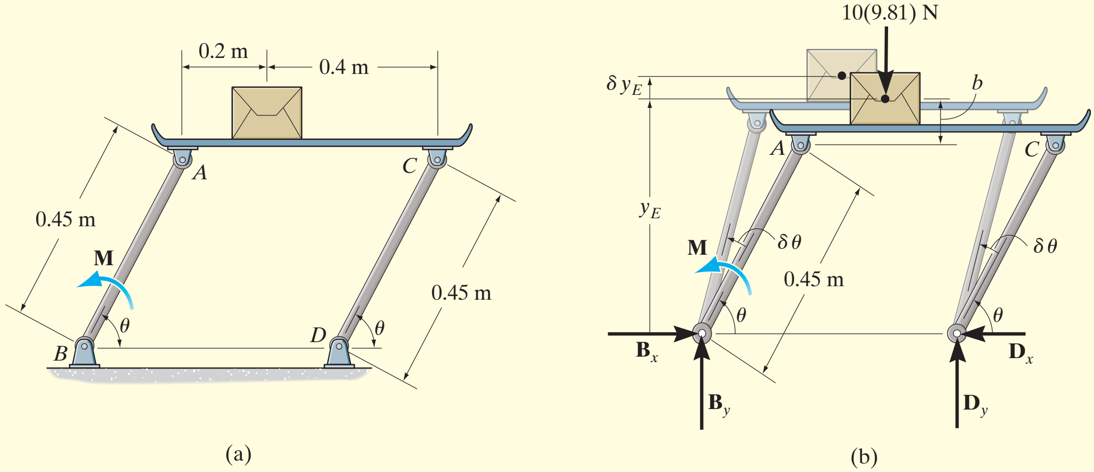


Fig. 11–8

SOLUTION

Free-Body Diagram. When θ undergoes a positive virtual displacement $\delta\theta$, only the couple moment **M** and the weight of the box do work, Fig. 11–8*b*.

Virtual Displacements. The position coordinate y_E, measured from the *fixed point B*, locates the weight, 10(9.81) N. Here,

$$y_E = (0.45 \text{ m}) \sin\theta + b$$

where b is a constant distance. Differentiating this equation, we obtain

$$\delta y_E = 0.45 \text{ m} \cos\theta \, \delta\theta \qquad (1)$$

Virtual-Work Equation. The virtual-work equation becomes

$$\delta U = 0; \qquad M\,\delta\theta - [10(9.81)\text{ N}]\delta y_E = 0$$

Substituting Eq. 1 into this equation

$$M\,\delta\theta - 10(9.81)\text{ N}(0.45\text{ m}\cos\theta\,\delta\theta) = 0$$
$$\delta\theta(M - 44.145\cos\theta) = 0$$

Since $\delta\theta \neq 0$, then

$$M - 44.145\cos\theta = 0$$

Since it is required that $\theta = 60°$, then

$$M = 44.145\cos 60° = 22.1\text{ N}\cdot\text{m} \qquad \textit{Ans.}$$

EXAMPLE 11.4

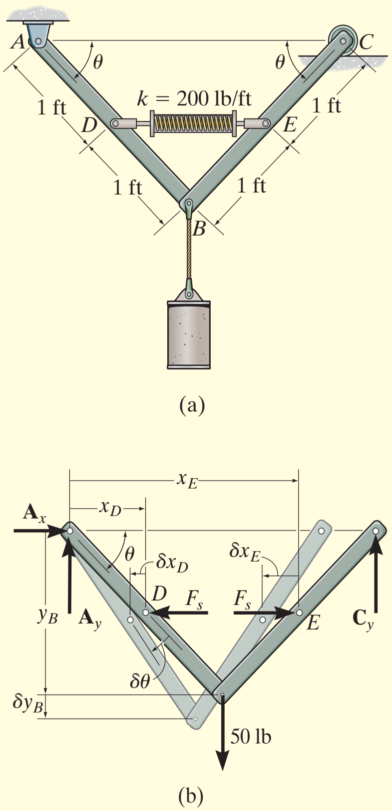


Fig. 11–9

The mechanism in Fig. 11–9*a* supports the 50-lb cylinder. Determine the angle θ for equilibrium if the spring has an unstretched length of 2 ft when $\theta = 0°$. Neglect the mass of the members.

SOLUTION

Free-Body Diagram. When the mechanism undergoes a positive virtual displacement $\delta\theta$, Fig. 11–9*b*, only $\mathbf{F}_s$ and the 50-lb force do work. Since the final length of the spring is 2(1 ft cos θ), then

$$F_s = ks = (200\ \text{lb/ft})(2\ \text{ft} - 2\ \text{ft}\cos\theta) = (400 - 400\cos\theta)\ \text{lb}$$

Virtual Displacements. The position coordinates x_D and x_E are established from the *fixed point* A to locate $\mathbf{F}_s$ at D and at E. The coordinate y_B, also measured from A, specifies the position of the 50-lb force at B. The coordinates can be expressed in terms of θ using trigonometry.

$$x_D = (1\ \text{ft})\cos\theta$$

$$x_E = 3[(1\ \text{ft})\cos\theta] = (3\ \text{ft})\cos\theta$$

$$y_B = (2\ \text{ft})\sin\theta$$

Differentiating, we obtain the virtual displacements of points D, E, and B as

$$\delta x_D = -1\sin\theta\,\delta\theta \qquad (1)$$

$$\delta x_E = -3\sin\theta\,\delta\theta \qquad (2)$$

$$\delta y_B = 2\cos\theta\,\delta\theta \qquad (3)$$

Virtual-Work Equation. The virtual-work equation is written as if all virtual displacements are positive, thus

$$\delta U = 0; \qquad F_s\,\delta x_E + 50\,\delta y_B - F_s\,\delta x_D = 0$$

$$(400 - 400\cos\theta)(-3\sin\theta\,\delta\theta) + 50(2\cos\theta\,\delta\theta)$$

$$- (400 - 400\cos\theta)(-1\sin\theta\,\delta\theta) = 0$$

$$\delta\theta(800\sin\theta\cos\theta - 800\sin\theta + 100\cos\theta) = 0$$

Since $\delta\theta \neq 0$, then

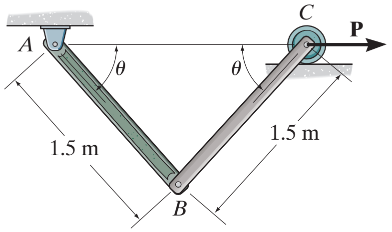
$$800\sin\theta\cos\theta - 800\sin\theta + 100\cos\theta = 0$$

Solving by trial and error,

$$\theta = 34.9° \qquad \textit{Ans.}$$

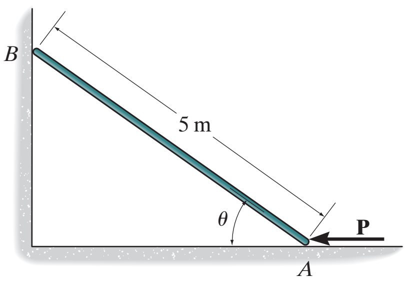
FUNDAMENTAL PROBLEMS

F11–1. Determine the required magnitude of force **P** to maintain equilibrium of the linkage at $\theta = 60°$. Each link has a mass of 20 kg.



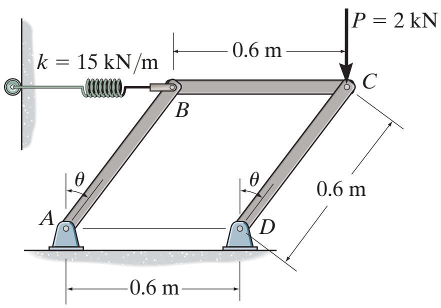
Prob. F11–1

F11–2. Determine the magnitude of force **P** required to hold the 50-kg smooth rod in equilibrium at $\theta = 60°$.



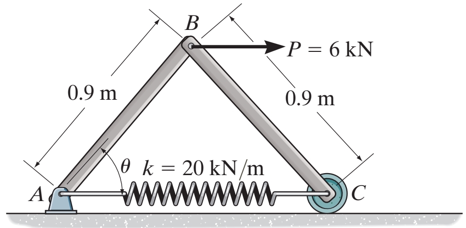
Prob. F11–2

F11–3. The linkage is subjected to a force of $P = 2$ kN. Determine the angle θ for equilibrium. The spring is unstretched when $\theta = 0°$. Neglect the mass of the links.



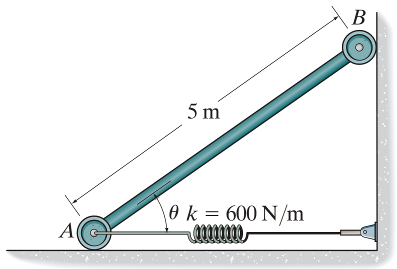
Prob. F11–3

F11–4. The linkage is subjected to a force of $P = 6$ kN. Determine the angle θ for equilibrium. The spring is unstretched at $\theta = 60°$. Neglect the mass of the links.



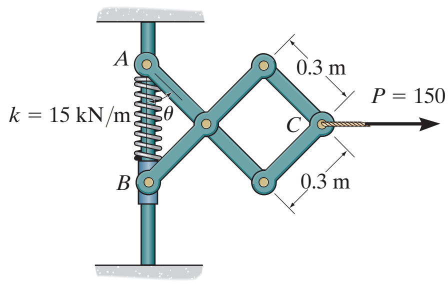
Prob. F11–4

F11–5. Determine the angle θ where the 50-kg bar is in equilibrium. The spring is unstretched at $\theta = 60°$.



Prob. F11–5

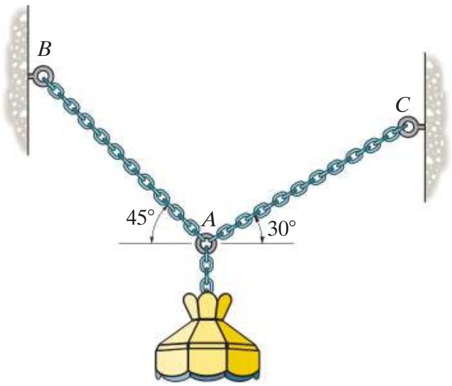
F11–6. The scissors linkage is subjected to a force of $P = 150$ N. Determine the angle θ for equilibrium. The spring is unstretched at $\theta = 0°$. Neglect the mass of the links.



Prob. F11–6

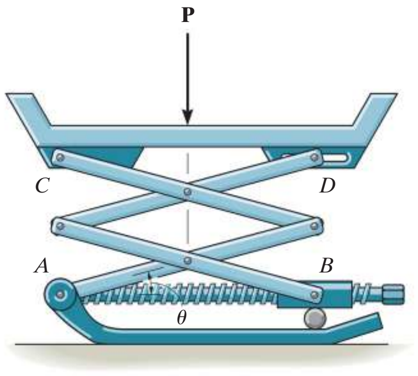
PROBLEMS

11–1. Use the method of virtual work to determine the tension in cable AC. The lamp weighs 10 lb.



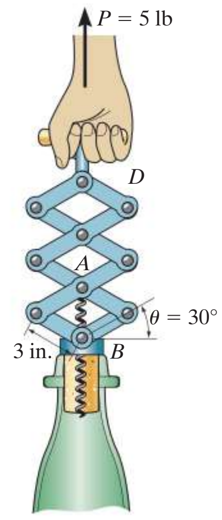
Prob. 11–1

11–2. The scissors jack supports a load **P**. Determine the axial force in the screw necessary for equilibrium when the jack is in the position θ. Each of the four links has a length L and is pin connected at its center. Points B and D can move horizontally.



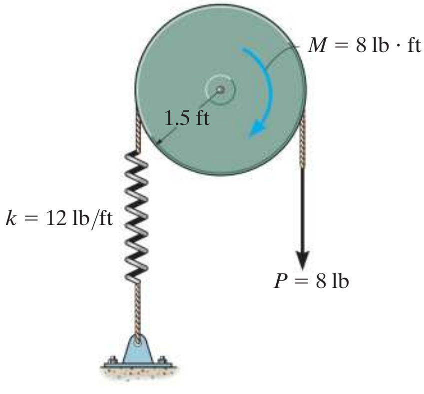
Prob. 11–2

11–3. If a force of $P = 5$ lb is applied to the handle of the mechanism, determine the force the screw exerts on the cork of the bottle. The screw is attached to the pin at A and passes through the collar that is attached to the bottle neck at B.



Prob. 11–3

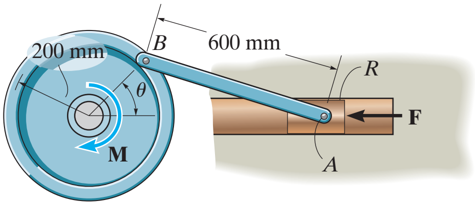
***11–4.** The disk has a weight of 10 lb and is subjected to a vertical force $P = 8$ lb and a couple moment $M = 8\text{ lb}\cdot\text{ft}$. Determine the disk's rotation θ if the end of the spring wraps around the periphery of the disk as the disk turns. The spring is originally unstretched.



Prob. 11–4

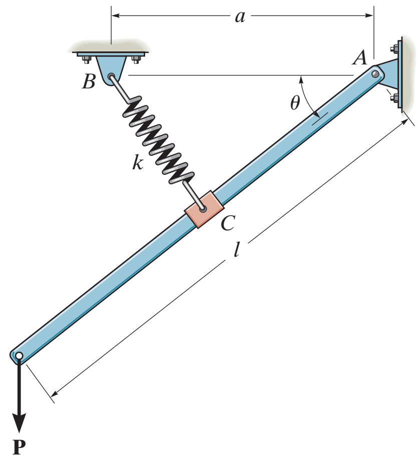
11–5. The punch press consists of the ram R, connecting rod AB, and a flywheel. If a torque of $M = 75\ \text{N}\cdot\text{m}$ is applied to the flywheel, determine the force $\mathbf{F}$ applied at the ram to hold the rod in the position $\theta = 60°$.

11–6. The flywheel is subjected to a torque of $M = 75\ \text{N}\cdot\text{m}$. Determine the horizontal compressive force F and plot the result of F (ordinate) versus the equilibrium position θ (abscissa) for $0° \leq \theta \leq 180°$.



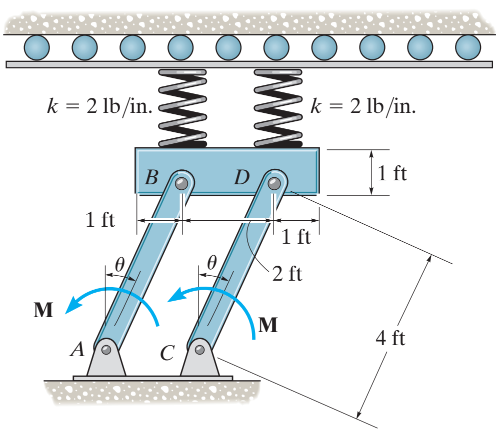
Probs. 11–5/6

***11–8.** The bar is supported by the spring and smooth collar that allows the spring to be always perpendicular to the bar for any angle θ. If the unstretched length of the spring is l_0, determine the force P needed to hold the bar in the equilibrium position θ. Neglect the weight of the bar.



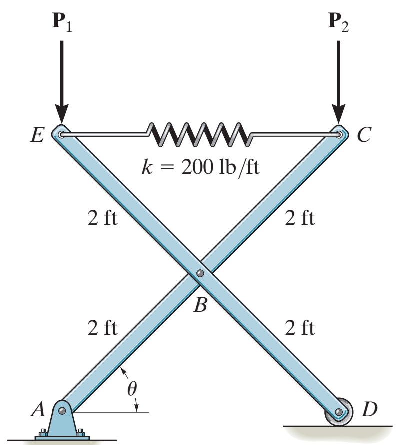
Prob. 11–8

11–7. When $\theta = 20°$, the 50-lb uniform block compresses the two vertical springs 4 in. If the uniform links AB and CD each weigh 10 lb, determine the magnitude of the applied couple moments $\mathbf{M}$ needed to maintain equilibrium when $\theta = 20°$.



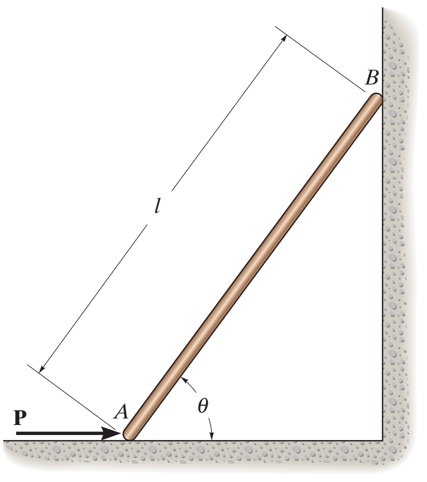
Prob. 11–7

11–9. The 4-ft members of the mechanism are pin connected at their centers. If vertical forces $P_1 = P_2 = 30$ lb act at C and E as shown, determine the angle θ for equilibrium. The spring is unstretched when $\theta = 45°$. Neglect the weight of the members.



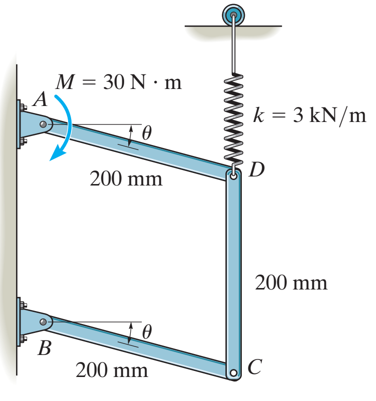
Prob. 11–9

11–10. The thin rod of weight W rests against the smooth wall and floor. Determine the magnitude of force **P** needed to hold it in equilibrium for a given angle θ.



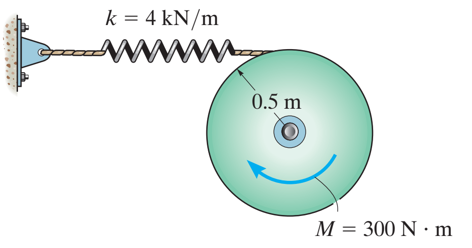
Prob. 11–10

11–11. If each of the three links of the mechanism have a mass of 4 kg, determine the angle θ for equilibrium. The spring, which always remains vertical, is unstretched when $\theta = 0°$.



Prob. 11–11

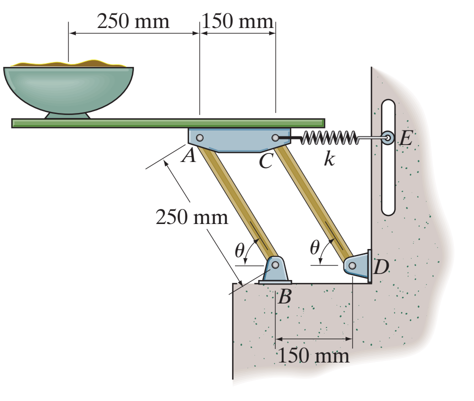
***11–12.** The disk is subjected to a couple moment M. Determine the disk's rotation θ required for equilibrium. The end of the spring wraps around the periphery of the disk as the disk turns. The spring is originally unstretched.



Prob. 11–12

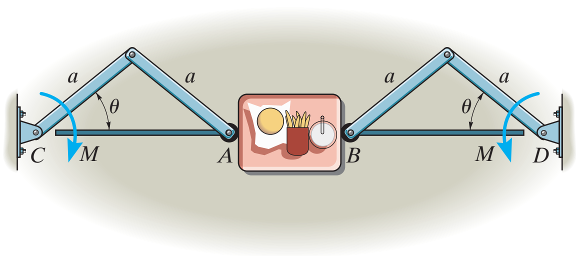
11–13. A 5-kg uniform serving table is supported on each side by pairs of two identical links, AB and CD, and springs CE. If the bowl has a mass of 1 kg, determine the angle θ where the table is in equilibrium. The springs each have a stiffness of $k = 200\,\text{N/m}$ and are unstretched when $\theta = 90°$. Neglect the mass of the links.

11–14. A 5-kg uniform serving table is supported on each side by two pairs of identical links, AB and CD, and springs CE. If the bowl has a mass of 1 kg and is in equilibrium when $\theta = 45°$, determine the stiffness k of each spring. The springs are unstretched when $\theta = 90°$. Neglect the mass of the links.



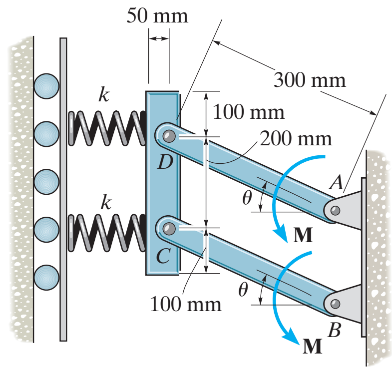
Probs. 11–13/14

11–15. The service window at a fast-food restaurant consists of glass doors that open and close automatically using a motor which supplies a torque **M** to each door. The far ends, A and B, move along the horizontal guides. If a food tray becomes stuck between the doors as shown, determine the horizontal force the doors exert on the tray at the position θ.



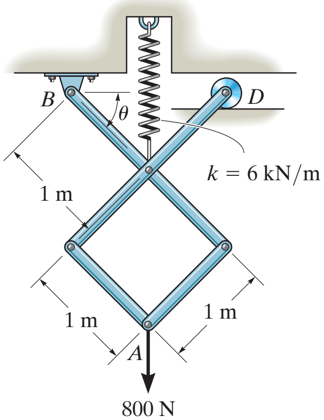
Prob. 11–15

11–17. When $\theta = 30°$, the 25-kg uniform block compresses the two horizontal springs 100 mm. Determine the magnitude of the applied couple moments **M** needed to maintain equilibrium. Take $k = 3\text{ kN/m}$ and neglect the mass of the links.



Prob. 11–17

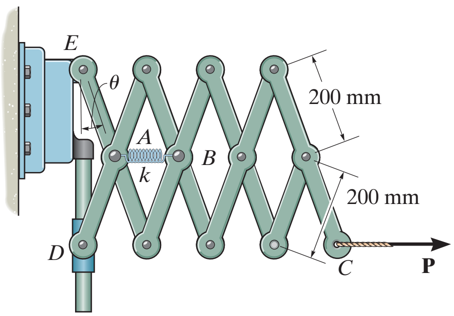
***11–16.** The members of the mechanism are pin connected. If a vertical force of 800 N acts at A, determine the angle θ for equilibrium. The spring is unstretched when $\theta = 0°$. Neglect the mass of the links.



Prob. 11–16

11–18. The "Nuremberg scissors" is subjected to a horizontal force of $P = 600$ N. Determine the angle θ for equilibrium. The spring has a stiffness of $k = 15\text{ kN/m}$ and is unstretched when $\theta = 15°$.

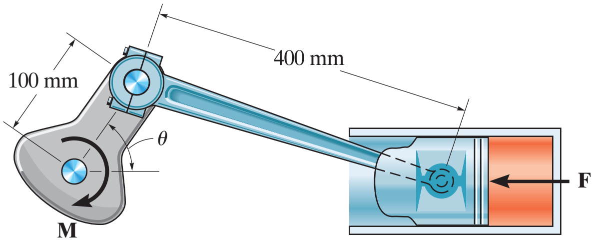
11–19. The "Nuremberg scissors" is subjected to a horizontal force of $P = 600$ N. Determine the stiffness k of the spring for equilibrium when $\theta = 60°$. The spring is unstretched when $\theta = 15°$.



Probs. 11–18/19

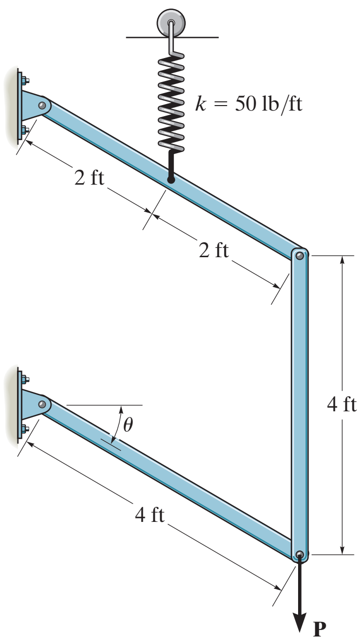
***11–20.** The crankshaft is subjected to a torque of $M = 50\ \text{N}\cdot\text{m}$. Determine the horizontal compressive force F applied to the piston for equilibrium when $\theta = 60^\circ$.

11–21. The crankshaft is subjected to a torque of $M = 50\ \text{N}\cdot\text{m}$. Determine the horizontal compressive force F and plot the result of F (ordinate) versus θ (abscissa) for $0^\circ \leq \theta \leq 90^\circ$.



Probs. 11–20/21

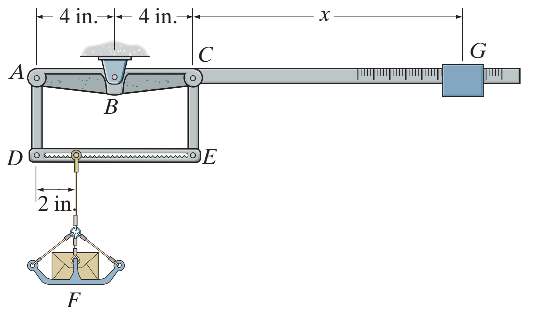
11–22. The spring is unstretched when $\theta = 0^\circ$. If $P = 8$ lb, determine the angle θ for equilibrium. Due to the roller guide, the spring always remains vertical. Neglect the weight of the links.



Prob. 11–22

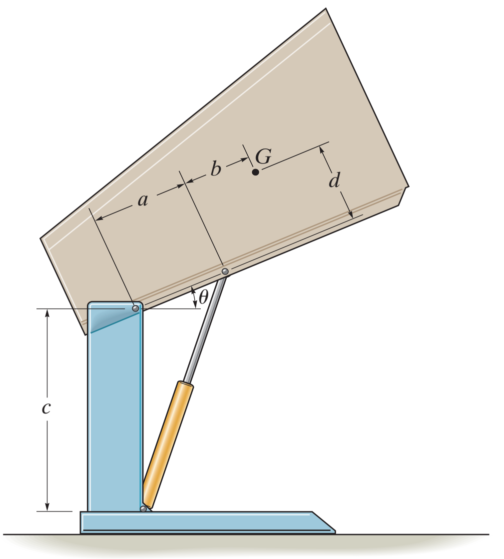
11–23. Determine the weight of block G required to balance the differential lever when the 20-lb load F is placed on the pan. The lever is in balance when the load and block are not on the lever. Take $x = 12$ in.

***11–24.** If the load F weighs 20 lb and the block G weighs 2 lb, determine its position x for equilibrium of the differential lever. The lever is in balance when the load and block are not on the lever.



Probs. 11–23/24

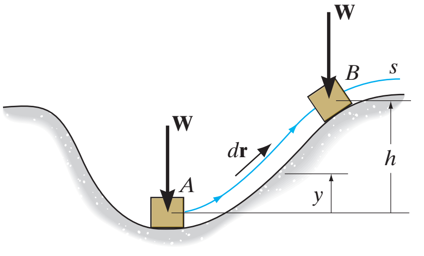
11–25. The dumpster has a weight W and a center of gravity at G. Determine the force in the hydraulic cylinder needed to hold it in the general position θ.



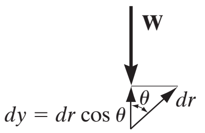
Prob. 11–25

*11.4 Conservative Forces

When a force does work that depends only upon the initial and final positions of the force, and it is *independent* of the path it travels, then the force is referred to as a ***conservative force***. The weight of a body and the force of a spring are two examples of conservative forces.



(a)



(b)

Fig. 11–10

Weight. Consider a block of weight $\mathbf{W}$ that travels along the path in Fig. 11–10*a*. When it is displaced up the path by an amount $d\mathbf{r}$, then the work is $dU = \mathbf{W}\cdot d\mathbf{r}$ or $dU = -W(dr\cos\theta) = -W\,dy$, as shown in Fig. 11–10*b*. In this case, the work is *negative* since $\mathbf{W}$ acts in the opposite sense of dy. Thus, if the block moves from A to B, through the vertical displacement h, the work is

$$U = -\int_0^h W\,dy = -Wh$$

The weight of a body is therefore a conservative force, since the work done by the weight depends only on the *vertical displacement* of the body, and is independent of the path along which the body travels.

Spring Force. Now consider the linearly elastic spring in Fig. 11–11, which undergoes a displacement ds. The work done by the spring force on the block is $dU = -F_s\,ds = -ks\,ds$. The work is *negative* because $\mathbf{F}_s$ acts in the opposite sense to that of ds. Thus, the work of $\mathbf{F}_s$ when the block is displaced from $s = s_1$ to $s = s_2$ is

$$U = -\int_{s_1}^{s_2} ks\,ds = -\left(\tfrac{1}{2}ks_2^2 - \tfrac{1}{2}ks_1^2\right)$$

Here the work depends only on the spring's initial and final positions, s_1 and s_2, measured from the spring's unstretched position. Since this result is independent of the path taken by the block as it moves, then a spring force is also a *conservative force*.

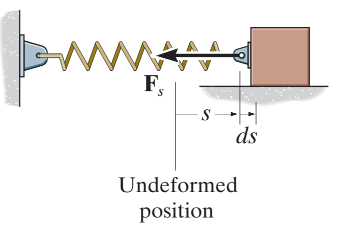


Fig. 11–11

Friction. In contrast to a conservative force, consider the force of *friction* exerted on a sliding body by a fixed surface. The work done by the frictional force depends on the path; the longer the path, the greater the work. Consequently, frictional forces are *nonconservative*, and most of the work done by them is dissipated from the body in the form of heat.

*11.5 Potential Energy

A conservative force can give the body the capacity to do work. This capacity, measured as ***potential energy***, depends on the location or "position" of the body measured relative to a fixed reference position or datum.

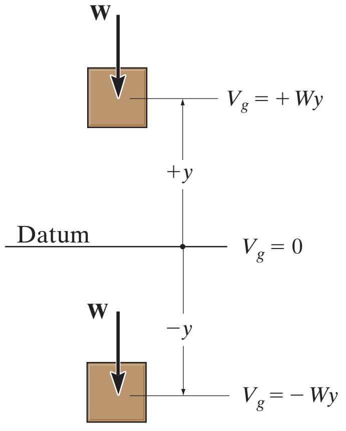


Fig. 11–12

Gravitational Potential Energy. If a body is located a distance y *above* a fixed horizontal reference or datum as in Fig. 11–12, the weight of the body has *positive* gravitational potential energy V_g since **W** has the capacity of doing positive work when the body is moved back down to the datum. Likewise, if the body is located a distance y *below* the datum, V_g is *negative* since the weight does negative work when the body is moved back up to the datum. At the datum, $V_g = 0$.

Measuring y as *positive upward*, the gravitational potential energy of the body's weight **W** is therefore

$$\boxed{V_g = Wy} \tag{11–4}$$

Elastic Potential Energy. When a spring is either elongated or compressed by an amount s from its unstretched position (the datum), the energy stored in the spring is called *elastic potential energy*. It is determined from

$$\boxed{V_e = \tfrac{1}{2}ks^2} \tag{11–5}$$

This energy is always a positive quantity since the spring force acting on the attached body does *positive* work on the body as the force returns the body to the spring's unstretched position, Fig. 11–13.

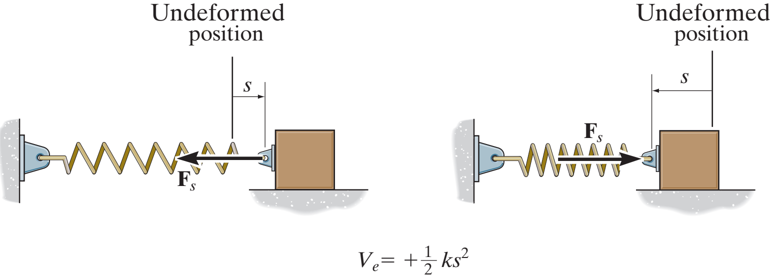


Fig. 11–13

Potential Function. In the general case, if a body is subjected to *both* gravitational and elastic forces, the *potential energy or potential function* V of the body can be expressed as the algebraic sum

$$\boxed{V = V_g + V_e} \tag{11–6}$$

where measurement of V depends on the location of the body with respect to a selected datum in accordance with Eqs. 11–4 and 11–5.

In particular, if a *system* of frictionless connected rigid bodies has a *single degree of freedom,* such that its vertical distance from the datum is defined by the coordinate q, then the potential function for the system can be expressed as $V = V(q)$. The work done by all the weight and spring forces acting on the system in moving it from q_1 to q_2, is measured by the *difference* in V; i.e.,

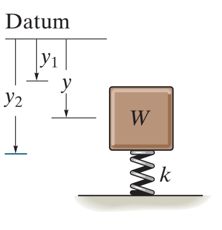
$$U_{1-2} = V(q_1) - V(q_2) \tag{11–7}$$

For example, the potential function for a system consisting of a block of weight $\mathbf{W}$ supported by a spring, as in Fig. 11–14, can be expressed in terms of the coordinate $(q =)\, y$, measured from a fixed datum located at the unstretched length of the spring. Here

$$\begin{aligned} V &= V_g + V_e \\ &= -Wy + \tfrac{1}{2}ky^2 \end{aligned} \tag{11–8}$$

If the block moves from y_1 to y_2, then applying Eq. 11–7 the work of $\mathbf{W}$ and $\mathbf{F}_s$ is

$$U_{1-2} = V(y_1) - V(y_2) = -W(y_1 - y_2) + \tfrac{1}{2}ky_1^2 - \tfrac{1}{2}ky_2^2$$



(a)

Fig. 11–14

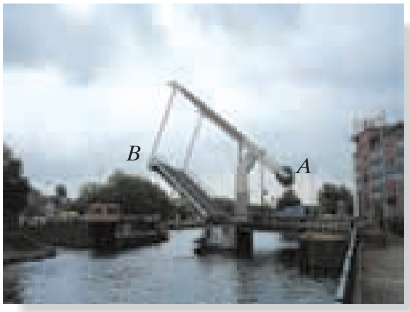
*11.6 Potential-Energy Criterion for Equilibrium

If a frictionless connected system has one degree of freedom, and its position is defined by the coordinate q, then if it displaces from q to $q + dq$, Eq. 11–7 becomes

$$dU = V(q) - V(q + dq)$$

or

$$dU = -dV$$



The counterweight at A balances the weight of the deck B of this simple lift bridge. By applying the method of potential energy we can analyze the equilibrium state of the bridge. (© Russell C. Hibbeler)

If the system is in equilibrium and undergoes a *virtual displacement* δq, rather than an actual displacement dq, then the above equation becomes $\delta U = -\delta V$. However, the principle of virtual work requires that $\delta U = 0$, and therefore, $\delta V = 0$, and so we can write $\delta V = (dV/dq)\delta q = 0$. Since $\delta q \neq 0$, this expression becomes

$$\boxed{\frac{dV}{dq} = 0} \tag{11–9}$$

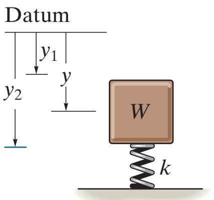
Hence, *when a frictionless connected system of rigid bodies is in equilibrium, the first derivative of its potential function is zero*. For example, using Eq. 11–8 we can determine the equilibrium position for the spring and block in Fig. 11–14*a*. We have

$$\frac{dV}{dy} = -W + ky = 0$$

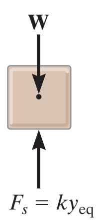
Hence, the equilibrium position $y = y_{eq}$ is

$$y_{eq} = \frac{W}{k}$$

Of course, this *same result* can be obtained by applying $\Sigma F_y = 0$ to the forces acting on the free-body diagram of the block, Fig. 11–14*b*.



(a)



(b)

Fig. 11–14 (cont'd)

*11.7 Stability of Equilibrium Configuration

The potential function V of a system can also be used to investigate the stability of the equilibrium configuration, which is classified as *stable*, *neutral*, or *unstable*.

Stable Equilibrium. A system is said to be in ***stable equilibrium*** if a system has a tendency to return to its original position when a small displacement is given to the system. The potential energy of the system in this case is at its *minimum*. A simple example is shown in Fig. 11–15*a*. When the disk is given a small displacement, its center of gravity G will always move (rotate) back to its equilibrium position, which is at the *lowest point* of its path. This is where the potential energy of the disk is at its *minimum*.

During high winds and when going around a curve, these sugar-cane trucks can become unstable and tip over since their center of gravity is high off the road when they are fully loaded. (© Russell C. Hibbeler)

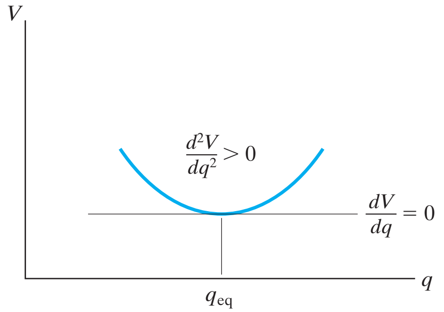
Neutral Equilibrium. A system is said to be in ***neutral equilibrium*** if the system still remains in equilibrium when the system is given a small displacement away from its original position. In this case, the potential energy of the system is *constant*. Neutral equilibrium is shown in Fig. 11–15*b*, where a disk is pinned at G. Each time the disk is rotated, a new equilibrium position is established and the potential energy remains unchanged.

Unstable Equilibrium. A system is said to be in ***unstable equilibrium*** if it has a tendency to be *displaced farther away* from its original equilibrium position when it is given a small displacement. The potential energy of the system in this case is a *maximum*. An unstable equilibrium position of the disk is shown in Fig. 11–15*c*. Here the disk will rotate away from its equilibrium position when its center of gravity is slightly displaced. At this *highest point*, its potential energy is at a *maximum*.

One-Degree-of-Freedom System. If a system has only one degree of freedom, and its position is defined by the coordinate q, then the potential function V for the system in terms of q can be plotted, Fig. 11–16.

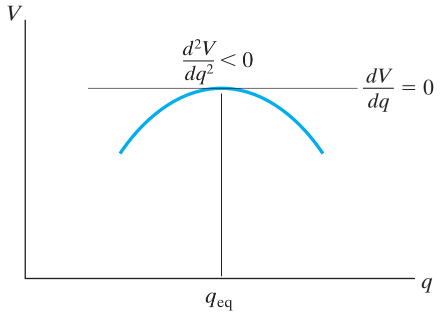
Stable equilibrium (a) Neutral equilibrium (b) Unstable equilibrium (c)

Fig. 11–15



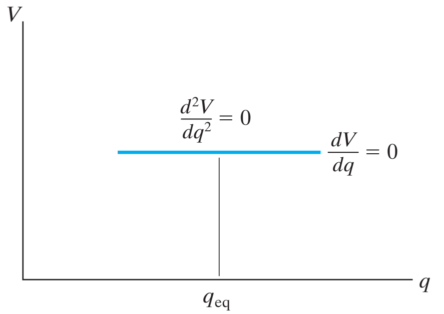
Stable equilibrium

(a)



Unstable equilibrium

(b)



Neutral equilibrium

(c)

Fig. 11–16

Provided the system is in *equilibrium*, then dV/dq, which represents the slope of this function, must be equal to zero. An investigation of stability at the equilibrium configuration therefore requires that the second derivative of the potential function be evaluated.

If d^2V/dq^2 is greater than zero, Fig. 11–16*a*, the potential energy of the system will be a *minimum*. This indicates that the equilibrium configuration is *stable*. Thus,

$$\frac{dV}{dq} = 0, \qquad \frac{d^2V}{dq^2} > 0 \qquad \text{stable equilibrium} \tag{11–10}$$

If d^2V/dq^2 is less than zero, Fig. 11–16*b*, the potential energy of the system will be a *maximum*. This indicates an *unstable* equilibrium configuration. Thus,

$$\frac{dV}{dq} = 0, \qquad \frac{d^2V}{dq^2} < 0 \qquad \text{unstable equilibrium} \tag{11–11}$$

Finally, if d^2V/dq^2 is equal to zero, it will be necessary to investigate the higher order derivatives to determine the stability. The equilibrium configuration will be *stable* if the first non-zero derivative is of an *even* order and it is *positive*. Likewise, the equilibrium will be *unstable* if this first non-zero derivative is odd or if it is even and negative. If all the higher order derivatives are *zero*, the system is said to be in *neutral equilibrium*, Fig. 11–16*c*. Thus,

$$\frac{dV}{dq} = \frac{d^2V}{dq^2} = \frac{d^3V}{dq^3} = \cdots = 0 \qquad \text{neutral equilibrium} \tag{11–12}$$

This condition occurs only if the potential-energy function for the system is constant at or around the neighborhood of q_{eq}.

Important Points

- A conservative force does work that is independent of the path through which the force moves. Examples include the weight and the spring force.
- Potential energy provides the body with the capacity to do work when the body moves relative to a fixed position or datum. Gravitational potential energy can be positive when the body is above a datum, and negative when the body is below the datum. Spring or elastic potential energy is always positive. It depends upon the stretch or compression of the spring.
- The sum of these two forms of potential energy represents the potential function. Equilibrium requires that the first derivative of the potential function be equal to zero. Stability at the equilibrium position is determined from the second derivative of the potential function.