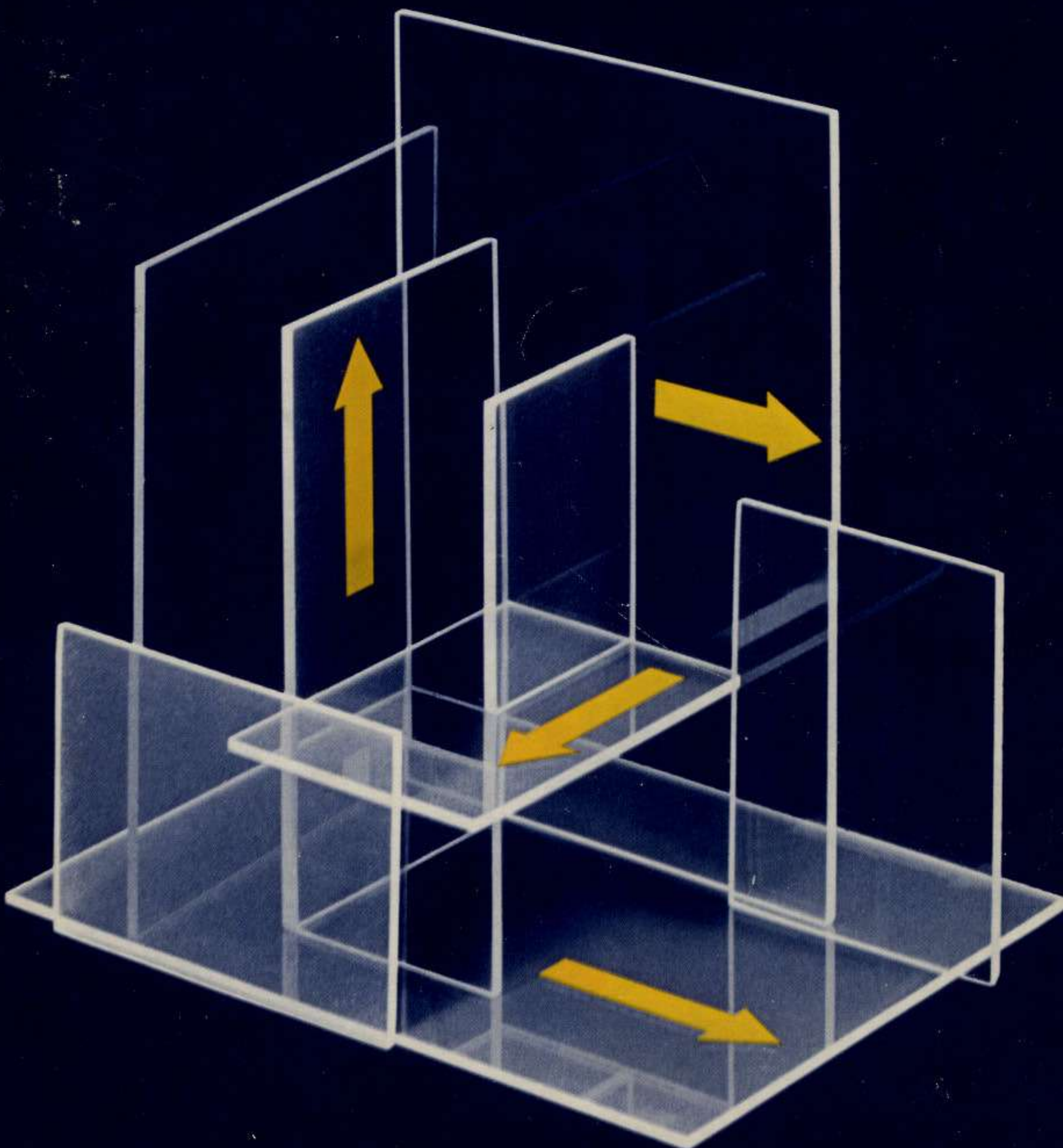
Third Edition

Vector Mechanics for Engineers

DYNAMICS

Ferdinand P. Beer and E. Russell Johnston, Jr.



Beer/Johnston Vector Mechanics for Engineers • DYNAMICS
Third Edition

McGraw-Hill

SI Prefixes

Multiplication Factor	Prefix†	Symbol
$1\,000\,000\,000\,000 = 10^{12}$	tera	T
$1\,000\,000\,000 = 10^{9}$	giga	G
$1\,000\,000 = 10^{6}$	mega	M
$1\,000 = 10^{3}$	kilo	k
$100 = 10^{2}$	hecto‡	h
$10 = 10^{1}$	deka‡	da
$0.1 = 10^{-1}$	deci‡	d
$0.01 = 10^{-2}$	centi‡	c
$0.001 = 10^{-3}$	milli	m
$0.000\,001 = 10^{-6}$	micro	μ
$0.000\,000\,001 = 10^{-9}$	nano	n
$0.000\,000\,000\,001 = 10^{-12}$	pico	p
$0.000\,000\,000\,000\,001 = 10^{-15}$	femto	f
$0.000\,000\,000\,000\,000\,001 = 10^{-18}$	atto	a

†The first syllable of every prefix is accented so that the prefix will retain its identity. Thus, the preferred pronunciation of kilometer places the accent on the first syllable, not the second.

‡The use of these prefixes should be avoided, except for the measurement of areas and volumes and for the nontechnical use of centimeter, as for body and clothing measurements.

Principal SI Units Used in Mechanics

Quantity	Unit	Symbol	Formula
Acceleration	Meter per second squared	. . .	m/s^2
Angle	Radian	rad	†
Angular acceleration	Radian per second squared	. . .	rad/s^2
Angular velocity	Radian per second	. . .	rad/s
Area	Square meter	. . .	m^2
Density	Kilogram per cubic meter	. . .	kg/m^3
Energy	Joule	J	$N \cdot m$
Force	Newton	N	$kg \cdot m/s^2$
Frequency	Hertz	Hz	s^{-1}
Impulse	Newton-second	. . .	$kg \cdot m/s$
Length	Meter	m	‡
Mass	Kilogram	kg	‡
Moment of a force	Newton-meter	. . .	$N \cdot m$
Power	Watt	W	J/s
Pressure	Pascal	Pa	N/m^2
Stress	Pascal	Pa	N/m^2
Time	Second	s	‡
Velocity	Meter per second	. . .	m/s
Volume, solids	Cubic meter	. . .	m^3
Liquids	Liter	l	$10^{-3}\ m^3$
Work	Joule	J	$N \cdot m$

†Supplementary unit (1 revolution $= 2\pi$ rad $= 360°$).

‡Base unit.

U.S. Customary Units and Their SI Equivalents

Quantity	U.S. Customary Unit	SI Equivalent
		$0.3048\ \mathrm{m/s^2}$
		$0.0254\ \mathrm{m/s^2}$
		$0.0929\ \mathrm{m^2}$
		$645.2\ \mathrm{mm^2}$
		$1.356\ \mathrm{J}$
		$4.448\ \mathrm{kN}$
		$4.448\ \mathrm{N}$
		$0.2780\ \mathrm{N}$
		$4.448\ \mathrm{N \cdot s}$
		$0.3048\ \mathrm{m}$
		$25.40\ \mathrm{mm}$
		$1.609\ \mathrm{km}$
		$28.35\ \mathrm{g}$
		$0.4536\ \mathrm{kg}$
		$14.59\ \mathrm{kg}$
		$907.2\ \mathrm{kg}$
		$1.356\ \mathrm{N \cdot m}$
		$0.1130\ \mathrm{N \cdot m}$
		$0.4162 \times 10^6\ \mathrm{mm^4}$
		$1.356\ \mathrm{kg \cdot m^2}$
		$4.448\ \mathrm{kg \cdot m/s}$
		$1.356\ \mathrm{W}$
		$745.7\ \mathrm{W}$
		$47.88\ \mathrm{Pa}$
		$6.895\ \mathrm{kPa}$
		$0.3048\ \mathrm{m/s}$
		$0.0254\ \mathrm{m/s}$
		$0.4470\ \mathrm{m/s}$
		$1.609\ \mathrm{km/h}$
		$0.02832\ \mathrm{m^3}$
		$16.39\ \mathrm{cm^3}$
		$3.785\ \mathrm{l}$
		$0.9464\ \mathrm{l}$
		$1.356\ \mathrm{J}$

U.S. Customary Units and Their SI Equivalents

Quantity	U.S. Customary Unit	SI Equivalent
Acceleration	ft/s^2	0.3048 m/s^2
	in./s^2	0.0254 m/s^2
Area	ft^2	0.0929 m^2
[illegible]	[illegible]	645.2 mm^2
[illegible]	[illegible] · lb	1.356 J
[illegible]	[illegible]	4.448 kN
[illegible]	lb	4.448 N
	oz	0.2780 N
Impulse	lb · s	4.448 N · s
Length	ft	0.3048 m
	in.	25.40 mm
	mi	1.609 km
Mass	oz mass	28.35 g
	lb mass	0.4536 kg
	slug	14.59 kg
	ton	907.2 kg
Moment of a force	lb · ft	1.356 N · m
	lb · in.	0.1130 N · m
Moment of inertia		
Of an area	in^4	0.4162×10^6 mm^4
Of a mass	lb · ft · s^2	1.356 kg · m^2
Momentum	lb · s	4.448 kg · m/s
Power	ft · lb/s	1.356 W
	hp	745.7 W
Pressure or stress	lb/ft^2	47.88 Pa
	lb/in^2 (psi)	6.895 kPa
Velocity	ft/s	0.3048 m/s
	in./s	0.0254 m/s
	mi/h (mph)	0.4470 m/s
	mi/h (mph)	1.609 km/h
Volume, solids	ft^3	0.02832 m^3
	in^3	16.39 cm^3
Liquids	gal	3.785 l
	qt	0.9464 l
Work	ft · lb	1.356 J

Vector Mechanics for Engineers DYNAMICS

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Vector Mechanics for Engineers
DYNAMICS

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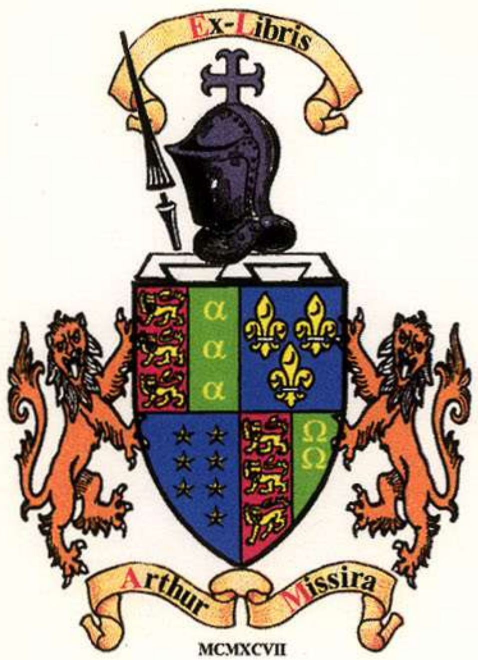
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Preface

The main objective of a first course in mechanics should be to develop in the engineering student the ability to analyze any problem in a simple and logical manner and to apply to its solution a few, well-understood, basic principles. It is hoped that this text, as well as the preceding volume, *Vector Mechanics for Engineers: Statics,* will help the instructor achieve this goal.†

Vector algebra was introduced at the beginning of the first volume and used in the presentation of the basic principles of statics, as well as in the solution of many problems, particularly three-dimensional problems. Similarly, the concept of vector differentiation will be introduced early in this volume, and vector analysis will be used throughout the presentation of dynamics. This approach results in a more concise derivation of the fundamental principles. It also makes it possible to analyze many problems in kinematics and kinetics which could not be solved by the standard scalar methods. The emphasis in this text, however, remains on the correct understanding of the principles of mechanics and on their application to the solution of engineering problems, and vector analysis is presented chiefly as a convenient tool.‡

One of the characteristics of the approach used in these volumes is that the mechanics of *particles* has been clearly separated from the mechanics of *rigid bodies*. This approach makes it possible to consider simple practical applications at an early stage and to postpone the introduction of more difficult concepts. In the volume on statics, the statics of particles was treated first, and the principle of equilibrium was immediately applied to practical situations involving only concurrent forces. The statics of rigid bodies was considered later, at which time the vector and scalar products of two vectors were introduced and used to define the moment of a force about a point and about an axis. In this volume, the same division is observed. The basic

† Both texts are also available in a single volume, *Vector Mechanics for Engineers: Statics and Dynamics,* third edition.

‡ In a parallel text, *Mechanics for Engineers: Dynamics,* third edition, the use of vector algebra is limited to the addition and subtraction of vectors, and vector differentiation is omitted.

concepts of force, mass, and acceleration, of work and energy, and of impulse and momentum are introduced and first applied to problems involving only particles. Thus the student may familiarize himself with the three basic methods used in dynamics and learn their respective advantages before facing the difficulties associated with the motion of rigid bodies.

Since this text is designed for a first course in dynamics, new concepts have been presented in simple terms and every step explained in detail. On the other hand, by discussing the broader aspects of the problems considered and by stressing methods of general applicability, a definite maturity of approach has been achieved. For example, the concept of potential energy is discussed in the general case of a conservative force. Also, the study of the plane motion of rigid bodies had been designed to lead naturally to the study of their general motion in space. This is true in kinematics as well as in kinetics, where the principle of equivalence of external and effective forces is applied directly to the analysis of plane motion, thus facilitating the transition to the study of three-dimensional motion.

The fact that mechanics is essentially a *deductive* science based on a few fundamental principles has been stressed. Derivations have been presented in their logical sequence and with all the rigor warranted at this level. However, the learning process being largely *inductive*, simple applications have been considered first. Thus the dynamics of particles precedes the dynamics of rigid bodies; and, in the latter, the fundamental principles of kinetics are first applied to the solution of two-dimensional problems, which can be more easily visualized by the student (Chaps. 16 and 17), while three-dimensional problems are postponed until Chap. 18.

The third edition of *Vector Mechanics for Engineers* retains the unified presentation of the principles of kinetics which characterized the second edition. The concepts of linear and angular momentum are introduced in Chap. 12 so that Newton's second law of motion may be presented, not only in its conventional form $\mathbf{F} = m\mathbf{a}$, but also as a law relating, respectively, the sum of the forces acting on a particle and the sum of their moments to the rates of change of the linear and angular momentum of the particle. This makes possible an earlier introduction of the principle of conservation of angular momentum and a more meaningful discussion of the motion of a particle under a central force (Sec. 12.8). More importantly, this approach may be readily extended to the study of the motion of a system of particles (Chap. 14) and leads to a more concise and unified treatment of the kinetics of rigid bodies in two and three dimensions (Chaps. 16 through 18).

Free-body diagrams were introduced early in statics. They were used not only to solve equilibrium problems but also to express the equivalence of two systems of forces or, more generally, of two systems of vectors. The advantage of this approach becomes apparent in the study of the dynamics of rigid bodies, where it is used to solve three-dimensional as well as two-dimensional problems. By placing the emphasis on "free-body-diagram equations" rather than on the standard algebraic equations of motion, a more intuitive and more complete understanding of the fundamental principles of dynamics may be achieved. This approach, which was first introduced in 1962 in the first edition of *Vector Mechanics for Engineers*, has now gained wide acceptance among mechanics teachers in this country. It is, therefore, used in preference to the method of dynamic equilibrium and to the equations of motion in the solution of all sample problems in this new edition.

Color has again been used in this edition to distinguish forces from other elements of the free-body diagrams. This makes it easier for the students to identify the forces acting on a given particle or rigid body and to follow the discussion of sample problems and other examples given in the text.

Because of the current trend among American engineers to adopt the international system of units (SI metric units), the SI units most frequently used in mechanics were introduced in Chap. 1 of *Statics*. They are discussed again in Chap. 12 of this volume and used throughout the text. Half the sample problems and problems to be assigned have been stated in these units, while the other half retain U.S. customary units. The authors believe that this approach will best serve the needs of the students, who will be entering the engineering profession during the period of transition from one system of units to the other. It also should be recognized that the passage from one system to the other entails more than the use of conversion factors. Since the SI system of units is an absolute system based on the units of time, length, and mass, whereas the U.S. customary system is a gravitational system based on the units of time, length, and force, different approaches are required for the solution of many problems. For example, when SI units are used, a body is generally specified by its mass expressed in kilograms; in most problems of statics it was necessary to determine the weight of the body in newtons, and an additional calculation was required for this purpose. On the other hand, when U.S. customary units are used, a body is specified by its weight in pounds and, in dynamics problems, an additional calculation will be required to determine its mass in slugs (or $\text{lb} \cdot \text{sec}^2/\text{ft}$). The authors, therefore, believe that problems assignments should include both types of units. A sufficient number of problems, however, have been provided so that, if so desired, two complete sets of assignments may be selected from problems stated in SI units only and two others from problems stated in U.S. customary units. Since the answers to all even-numbered problems stated in U.S. customary units have been given in both systems of units, teachers who wish to give special instruction to their students in the conversion of units may assign these problems and ask their students to use SI units in their solutions. This has been illustrated in two sample problems involving, respectively, the kinetics of particles (Sample Prob. 12.2) and the computation of mass moments of inertia (Sample Prob. 9.13 in Appendix B).

A number of optional sections have been included. These sections are indicated by asterisks and may thus easily be distinguished from those which form the core of the basic dynamics course. They may be omitted without prejudice to the understanding of the rest of the text. The topics covered in these additional sections include graphical methods for the solution of rectilinear-motion problems, the trajectory of a particle under a central force, the deflection of fluid streams, problems involving jet and rocket propulsion, the kinematics and kinetics of rigid bodies in three dimensions, damped mechanical vibrations, and electrical analogues. These topics will be found of particular interest when dynamics is taught in the junior year.

The material presented in this volume and most of the problems require no previous mathematical knowledge beyond algebra, trigonometry, elementary calculus, and the elements of vector algebra presented in Chaps. 2 and 3 of the volume on statics.† However, special problems have been included, which make use of a more advanced knowledge of calculus, and certain sections, such as Secs. 19.8 and 19.9 on damped vibrations, should be assigned only if the students possess the proper mathematical background.

The text has been divided into units, each consisting of one or several theory sections, one or several sample problems, and a large number of problems to be assigned. Each unit corresponds to a well-defined topic and generally may be covered in one lesson. In a number of cases, however, the instructor will find it desirable to devote more than one lesson to a given topic. The sample problems have been set up in much the same form that a student will use in solving the assigned problems. They thus serve the double purpose of amplifying the text and demonstrat-

† Some useful definitions and properties of vector algebra have been summarized in Appendix A at the end of this volume for the convenience of the reader. Also, Secs. 9.10 through 9.16 of the volume on statics, which deal with the moments of inertia of masses, have been reproduced in Appendix B.

ing the type of neat and orderly work that the student should cultivate in his own solutions. Most of the problems to be assigned are of a practical nature and should appeal to the engineering student. They are primarily designed, however, to illustrate the material presented in the text and to help the student understand the basic principles of mechanics. The problems have been grouped according to the portions of material they illustrate and have been arranged in order of increasing difficulty. Problems requiring special attention have been indicated by asterisks. Answers to all even-numbered problems are given at the end of the book.

The authors wish to acknowledge gratefully the many helpful comments and suggestions offered by the users of the previous editions of *Mechanics for Engineers* and of *Vector Mechanics for Engineers*.

FERDINAND P. BEER
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List of Symbols

$\mathbf{a}, a$	Acceleration
a	Constant; radius; distance; semimajor axis of ellipse
$\overline{\mathbf{a}}, \overline{a}$	Acceleration of mass center
$\mathbf{a}_{B/A}$	Acceleration of B relative to frame in translation with A
$\mathbf{a}_c$	Coriolis acceleration
$\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$	Reactions at supports and connections
$A, B, C, \ldots$	Points
A	Area
b	Width; distance; semiminor axis of ellipse
c	Constant; coefficient of viscous damping
C	Centroid; instantaneous center of rotation; capacitance
d	Distance
e	Coefficient of restitution; base of natural logarithms
E	Total mechanical energy; voltage
f	Frequency; scalar function
$\mathbf{F}$	Force; friction force
g	Acceleration of gravity
G	Center of gravity; mass center; constant of gravitation
h	Angular momentum per unit mass
$\mathbf{H}_O$	Angular momentum about point O
$\dot{\mathbf{H}}_G$	Rate of change of angular momentum $\mathbf{H}_G$ with respect to frame of fixed orientation
$(\dot{\mathbf{H}}_G)_{Gxyz}$	Rate of change of angular momentum $\mathbf{H}_G$ with respect to rotating frame $Gxyz$
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	Unit vectors along coordinate axes
$\mathbf{i}_n, \mathbf{i}_t$	Unit vectors along normal and tangent
$\mathbf{i}_r, \mathbf{i}_\theta$	Unit vectors in radial and transverse directions
i	Current
$I, I_x, \ldots$	Moment of inertia
$\overline{I}$	Centroidal moment of inertia
J	Polar moment of inertia
k	Spring constant
k_x, k_y, k_O	Radius of gyration

$\bar{k}$	Centroidal radius of gyration
l	Length
$\mathbf{L}$	Linear momentum
L	Length; inductance
m	Mass; mass per unit length
$\mathbf{M}$	Couple; moment
$\mathbf{M}_O$	Moment about point O
$\mathbf{M}_O^R$	Moment resultant about point O
M	Magnitude of couple or moment; mass of earth
M_{OL}	Moment about axis OL
n	Normal direction
$\mathbf{N}$	Normal component of reaction
O	Origin of coordinates
p	Circular frequency
$\mathbf{P}$	Force; vector
$\dot{\mathbf{P}}$	Rate of change of vector $\mathbf{P}$ with respect to frame of fixed orientation
$P_{xy}, \ldots$	Product of inertia
q	Mass rate of flow; electric charge
$\mathbf{Q}$	Force; vector
$\dot{\mathbf{Q}}$	Rate of change of vector $\mathbf{Q}$ with respect to frame of fixed orientation
$(\dot{\mathbf{Q}})_{Oxyz}$	Rate of change of vector $\mathbf{Q}$ with respect to frame $Oxyz$
$\mathbf{r}$	Position vector
r	Radius; distance; polar coordinate
$\mathbf{R}$	Resultant force; resultant vector; reaction
R	Radius of earth; resistance
$\mathbf{s}$	Position vector
s	Length of arc
t	Time; thickness; tangential direction
$\mathbf{T}$	Force
T	Tension; kinetic energy
$\mathbf{u}$	Velocity
u	Rectangular coordinate; variable
U	Work
$\mathbf{v}, v$	Velocity
v	Speed; rectangular coordinate
$\bar{\mathbf{v}}, \bar{v}$	Velocity of mass center
$\mathbf{v}_{B/A}$	Velocity of B relative to frame in translation with A.
$\mathbf{V}$	Vector product
V	Volume; potential energy
w	Load per unit length
$\mathbf{W}, W$	Weight; load
x, y, z	Rectangular coordinates; distances

$\dot{x}, \dot{y}, \dot{z}$	Time derivatives of coordinates x, y, z
$\bar{x}, \bar{y}, \bar{z}$	Rectangular coordinates of centroid, center of gravity, or mass center
$\boldsymbol{\alpha}, \alpha$	Angular acceleration
α, β, γ	Angles
γ	Specific weight
δ	Elongation
ε	Eccentricity of conic section or of orbit
$\boldsymbol{\lambda}$	Unit vector along a line
η	Efficiency
θ	Angular coordinate; Eulerian angle; angle; polar coordinate
μ	Coefficient of friction
ρ	Density; radius of curvature
τ	Period; periodic time
ϕ	Angle of friction; Eulerian angle; phase angle; angle
φ	Phase difference
ψ	Eulerian angle
$\boldsymbol{\omega}, \omega$	Angular velocity
ω	Circular frequency of forced vibration
$\boldsymbol{\Omega}$	Angular velocity of frame of reference

Kinematics of Particles

CHAPTER 11

11.1. Introduction to Dynamics. Chapters 1 to 10 were devoted to *statics*, i.e., to the analysis of bodies at rest. We shall now begin the study of *dynamics*, which is the part of mechanics dealing with the analysis of bodies in motion.

While the study of statics goes back to the time of the Greek philosophers, the first significant contribution to dynamics was made by Galileo (1564–1642). His experiments on uniformly accelerated bodies led Newton (1642–1727) to formulate his fundamental laws of motion.

Dynamics is divided into two parts: (1) *Kinematics*, which is the study of the geometry of motion; kinematics is used to relate displacement, velocity, acceleration, and time, without reference to the cause of the motion. (2) *Kinetics*, which is the study of the relation existing between the forces acting on a body, the mass of the body, and the motion of the body; kinetics is used to predict the motion caused by given forces or to determine the forces required to produce a given motion.

Chapters 11 to 14 are devoted to the *dynamics of particles*, and Chap. 11 more particularly to the *kinematics of particles*. The use of the word particles does not imply that we shall restrict our study to that of small corpuscles; it rather indicates that in these first chapters we shall study the motion of bodies—possibly as large as cars, rockets, or airplanes—without regard to their size. By saying that the bodies are analyzed as particles, we mean that only their motion as an entire unit will be considered; any rotation about their own mass center will be neglected. There are cases, however, when such a rotation is not negligible; the bodies, then, may not be considered as particles. The analysis of such motions will be carried out in later chapters dealing with the *dynamics of rigid bodies.*

RECTILINEAR MOTION OF PARTICLES

11.2. Position, Velocity, and Acceleration. A particle moving along a straight line is said to be in *rectilinear motion*. At any given instant t, the particle will occupy a certain position on the straight line. To define the position P of the particle, we choose a fixed origin O on the straight line and a positive direction along the line. We measure the distance x from O to P and record it with a plus or minus sign, according to whether P is reached from O by moving along the line in the positive or the negative direction. The distance x, with the appropriate sign, completely defines the position of the particle; it is called the *position coordinate* of the particle considered. For example, the position coordinate corresponding to P in Fig. 11.1a is $x = +5$ m, while the coordinate corresponding to P' in Fig. 11.1b is $x' = -2$ m.

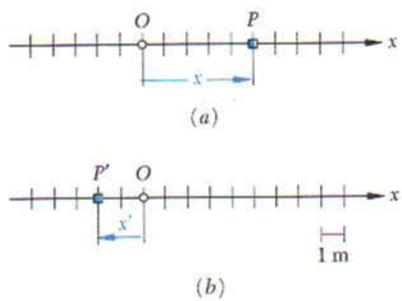


Fig. 11.1

When the position coordinate x of a particle is known for every value of time t, we say that the motion of the particle is known. The "timetable" of the motion may be given in the form of an equation in x and t, such as $x = 6t^2 - t^3$, or in the form of a graph of x vs. t as shown in Fig. 11.6. The units most generally used to measure the position coordinate x are the meter (m) in the SI system of units,† and the foot (ft) in the U.S. customary system of units. Time t will generally be measured in seconds (s).

† Cf. Sec. 1.3.

Consider the position P occupied by the particle at time t and the corresponding coordinate x (Fig. 11.2). Consider also the position P' occupied by the particle at a later time $t + \Delta t$; the position coordinate of P' may be obtained by adding to the coordinate x of P the small displacement Δx, which will be positive or negative according to whether P' is to the right or to the left of P. The *average velocity* of the particle over the time interval Δt is defined as the quotient of the displacement Δx and the time interval Δt,

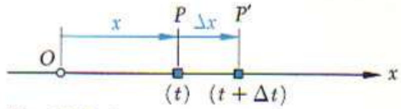


Fig. 11.2

$$\text{Average velocity} = \frac{\Delta x}{\Delta t}$$

If SI units are used, Δx is expressed in meters and Δt in seconds; the average velocity will thus be expressed in meters per second (m/s). If U.S. customary units are used, Δx is expressed in feet and Δt in seconds; the average velocity will then be expressed in feet per second (ft/s).

The *instantaneous velocity* v of the particle at the instant t is obtained from the average velocity by choosing shorter and shorter time intervals Δt and displacements Δx,

$$\text{Instantaneous velocity} = v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

The instantaneous velocity will also be expressed in m/s or ft/s. Observing that the limit of the quotient is equal, by definition, to the derivative of x with respect to t, we write

$$v = \frac{dx}{dt} \tag{11.1}$$

The velocity v is represented by an algebraic number which may be positive or negative.† A positive value of v indicates that x increases, i.e., that the particle moves in the positive direction (Fig. 11.3a); a negative value of v indicates that x decreases, i.e., that the particle moves in the negative direction (Fig. 11.3b). The magnitude of v is known as the *speed* of the particle.

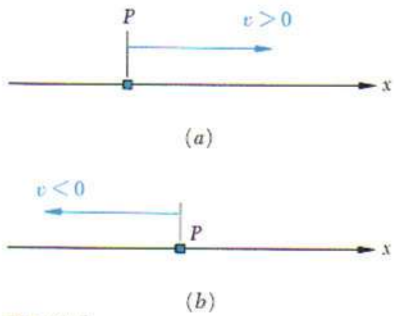


Fig. 11.3

†As we shall see in Sec. 11.9, the velocity is actually a vector quantity. However, since we are considering here the rectilinear motion of a particle, where the velocity of the particle has a known and fixed direction, we need only specify the sense and magnitude of the velocity; this may be conveniently done by using a scalar quantity with a plus or minus sign. The same remark will apply to the acceleration of a particle in rectilinear motion.

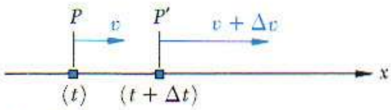


Fig. 11.4

Consider the velocity v of the particle at time t and also its velocity $v + \Delta v$ at a later time $t + \Delta t$ (Fig. 11.4). The *average acceleration* of the particle over the time interval Δt is defined as the quotient of Δv and Δt,

$$\text{Average acceleration} = \frac{\Delta v}{\Delta t}$$

If SI units are used, Δv is expressed in m/s and Δt in seconds; the average acceleration will thus be expressed in m/s^2. If U.S. customary units are used, Δv is expressed in ft/s and Δt in seconds; the average acceleration will then be expressed in ft/s^2.

The *instantaneous acceleration* a of the particle at the instant t is obtained from the average acceleration by choosing smaller and smaller values for Δt and Δv,

$$\text{Instantaneous acceleration} = a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

The instantaneous acceleration will also be expressed in m/s^2 or ft/s^2. The limit of the quotient is by definition the derivative of v with respect to t and measures the rate of change of the velocity. We write

$$a = \frac{dv}{dt} \tag{11.2}$$

or, substituting for v from (11.1),

$$a = \frac{d^2x}{dt^2} \tag{11.3}$$

The acceleration a is represented by an algebraic number which may be positive or negative.† A positive value of a indicates that the velocity (i.e., the algebraic number v) increases. This may mean that the particle is moving faster in the positive direction (Fig. 11.5a) or that it is moving more slowly in the negative direction (Fig. 11.5b); in both cases, Δv is positive. A negative value of a indicates that the velocity decreases; either the particle is moving more slowly in the positive direction (Fig. 11.5c), or it is moving faster in the negative direction (Fig. 11.5d).

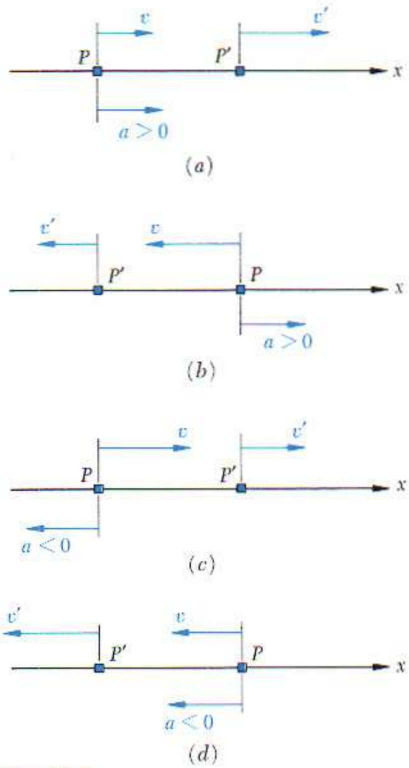


Fig. 11.5

The term *deceleration* is sometimes used to refer to a when the speed of the particle (i.e., the magnitude of v) decreases; the particle is then moving more slowly. For example, the particle of Fig. 11.5 is decelerated in parts b and c, while it is truly accelerated (i.e., moves faster) in parts a and d.

Another expression may be obtained for the acceleration by eliminating the differential dt in Eqs. (11.1) and (11.2). Solving

†See footnote, page 437.

(11.1) for dt, we obtain $dt = dx/v$; carrying into (11.2), we write

$$a = v\frac{dv}{dx} \tag{11.4}$$

Example. Consider a particle moving in a straight line, and assume that its position is defined by the equation

$$x = 6t^2 - t^3$$

where t is expressed in seconds and x in meters. The velocity v at any time t is obtained by differentiating x with respect to t,

$$v = \frac{dx}{dt} = 12t - 3t^2$$

The acceleration a is obtained by differentiating again with respect to t,

$$a = \frac{dv}{dt} = 12 - 6t$$

The position coordinate, the velocity, and the acceleration have been plotted against t in Fig. 11.6. The curves obtained are known as *motion curves*. It should be kept in mind, however, that the particle does not move along any of these curves; the particle moves in a straight line. Since the derivative of a function measures the slope of the corresponding curve, the slope of the x–t curve at any given time is equal to the value of v at that time and the slope of the v–t curve is equal to the value of a. Since $a = 0$ at $t = 2$ s, the slope of the v–t curve must be zero at $t = 2$ s; the velocity reaches a maximum at this instant. Also, since $v = 0$ at $t = 0$ and at $t = 4$ s, the tangent to the x–t curve must be horizontal for both of these values of t.

A study of the three motion curves of Fig. 11.6 shows that the motion of the particle from $t = 0$ to $t = \infty$ may be divided into four phases:

1. The particle starts from the origin, $x = 0$, with no velocity but with a positive acceleration. Under this acceleration, the particle gains a positive velocity and moves in the positive direction. From $t = 0$ to $t = 2$ s, x, v, and a are all positive.
2. At $t = 2$ s, the acceleration is zero; the velocity has reached its maximum value. From $t = 2$ s to $t = 4$ s, v is positive, but a is negative; the particle still moves in the positive direction but more and more slowly; the particle is decelerated.
3. At $t = 4$ s, the velocity is zero; the position coordinate x has reached its maximum value. From then on, both v and a are negative; the particle is accelerated and moves in the negative direction with increasing speed.
4. At $t = 6$ s, the particle passes through the origin; its coordinate x is then zero, while the total distance traveled since the beginning of the motion is 64 m. For values of t larger than 6 s, x, v, and a will all be negative. The particle keeps moving in the negative direction, away from O, faster and faster.

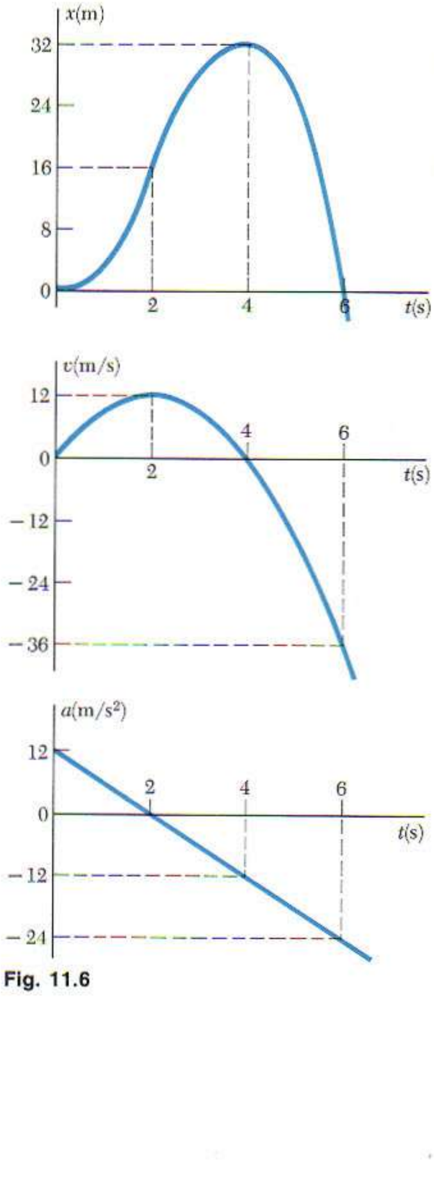


Fig. 11.6

11.3. Determination of the Motion of a Particle. We saw in the preceding section that the motion of a particle is said to be known if the position of the particle is known for every value of the time t. In practice, however, a motion is seldom defined by a relation between x and t. More often, the conditions of the motion will be specified by the type of acceleration that the particle possesses. For example, a freely falling body will have a constant acceleration, directed downward and equal to $9.81\ \mathrm{m/s^2}$ or $32.2\ \mathrm{ft/s^2}$; a mass attached to a spring which has been stretched will have an acceleration proportional to the instantaneous elongation of the spring measured from the equilibrium position; etc. In general, the acceleration of the particle may be expressed as a function of one or more of the variables x, v, and t. In order to determine the position coordinate x in terms of t, it will thus be necessary to perform two successive integrations.

We shall consider three common classes of motion:

1. $a = f(t)$. *The Acceleration Is a Given Function of* t. Solving (11.2) for dv and substituting $f(t)$ for a, we write

$$dv = a\,dt$$
$$dv = f(t)\,dt$$

Integrating both members, we obtain the equation

$$\int dv = \int f(t)\,dt$$

which defines v in terms of t. It should be noted, however, that an arbitrary constant will be introduced as a result of the integration. This is due to the fact that there are many motions which correspond to the given acceleration $a = f(t)$. In order to uniquely define the motion of the particle, it is necessary to specify the *initial conditions* of the motion, i.e., the value v_0 of the velocity and the value x_0 of the position coordinate at $t = 0$. Replacing the indefinite integrals by *definite integrals* with lower limits corresponding to the initial conditions $t = 0$ and $v = v_0$ and upper limits corresponding to $t = t$ and $v = v$, we write

$$\int_{v_0}^{v} dv = \int_0^t f(t)\,dt$$
$$v - v_0 = \int_0^t f(t)\,dt$$

which yields v in terms of t.

We shall now solve (11.1) for dx,

$$dx = v\,dt$$

and substitute for v the expression just obtained. Both members are then integrated, the left-hand member with respect to x from $x = x_0$ to $x = x$, and the right-hand member with respect to t from $t = 0$ to $t = t$. The position coordinate x is thus obtained in terms of t; the motion is completely determined.

Two important particular cases will be studied in greater detail in Secs. 11.4 and 11.5: the case when $a = 0$, corresponding to a *uniform motion*, and the case when $a =$ constant, corresponding to a *uniformly accelerated motion*.

2. $a = f(x)$. *The Acceleration Is a Given Function of* x. Rearranging Eq. (11.4) and substituting $f(x)$ for a, we write

$$v\,dv = a\,dx$$
$$v\,dv = f(x)\,dx$$

Since each member contains only one variable, we may integrate the equation. Denoting again by v_0 and x_0, respectively, the initial values of the velocity and of the position coordinate, we obtain

$$\int_{v_0}^{v} v\,dv = \int_{x_0}^{x} f(x)\,dx$$
$$\tfrac{1}{2}v^2 - \tfrac{1}{2}v_0^2 = \int_{x_0}^{x} f(x)\,dx$$

which yields v in terms of x. We now solve (11.1) for dt,

$$dt = \frac{dx}{v}$$

and substitute for v the expression just obtained. Both members may be integrated, and the desired relation between x and t is obtained.

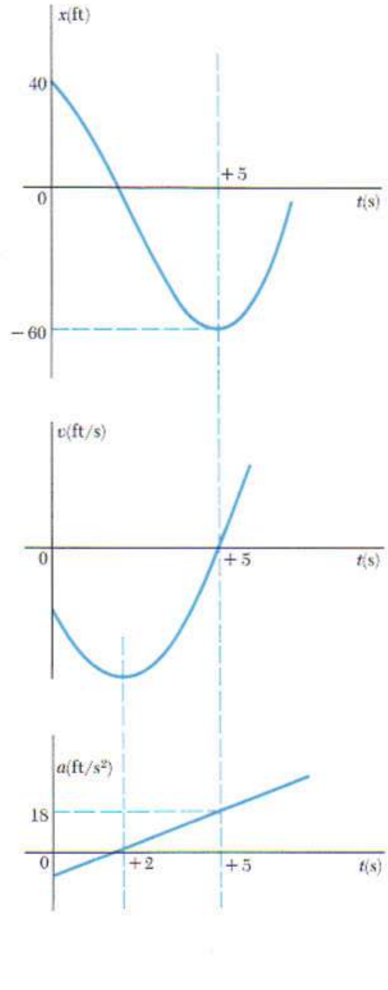
3. $a = f(v)$. *The Acceleration Is a Given Function of* v. We may then substitute $f(v)$ for a either in (11.2) or in (11.4) to obtain either of the following relations:

$$f(v) = \frac{dv}{dt} \qquad\qquad f(v) = v\frac{dv}{dx}$$
$$dt = \frac{dv}{f(v)} \qquad\qquad dx = \frac{v\,dv}{f(v)}$$

Integration of the first equation will yield a relation between v and t; integration of the second equation will yield a relation between v and x. Either of these relations may be used in conjunction with Eq. (11.1) to obtain the relation between x and t which characterizes the motion of the particle.

SAMPLE PROBLEM 11.1

The position of a particle which moves along a straight line is defined by the relation $x = t^3 - 6t^2 - 15t + 40$, where x is expressed in feet and t in seconds. Determine (*a*) the time at which the velocity will be zero, (*b*) the position and distance traveled by the particle at that time, (*c*) the acceleration of the particle at that time, (*d*) the distance traveled by the particle from $t = 4$ s to $t = 6$ s.



Solution. The equations of motion are

$$x = t^3 - 6t^2 - 15t + 40 \tag{1}$$

$$v = \frac{dx}{dt} = 3t^2 - 12t - 15 \tag{2}$$

$$a = \frac{dv}{dt} = 6t - 12 \tag{3}$$

a. Time at Which $v = 0$. We make $v = 0$ in (2),

$$3t^2 - 12t - 15 = 0 \qquad t = -1 \text{ s} \qquad \text{and} \qquad t = +5 \text{ s} \blacktriangleleft$$

Only the root $t = +5$ s corresponds to a time after the motion has begun: for $t < 5$ s, $v < 0$, the particle moves in the negative direction; for $t > 5$ s, $v > 0$, the particle moves in the positive direction.

b. Position and Distance Traveled When $v = 0$. Carrying $t = +5$ s into (1), we have

$$x_5 = (5)^3 - 6(5)^2 - 15(5) + 40 \qquad x_5 = -60 \text{ ft} \blacktriangleleft$$

The initial position at $t = 0$ was $x_0 = +40$ ft. Since $v \neq 0$ during the interval $t = 0$ to $t = 5$ s, we have

$$\text{Distance traveled} = x_5 - x_0 = -60 \text{ ft} - 40 \text{ ft} = -100 \text{ ft}$$

$$\text{Distance traveled} = 100 \text{ ft in the negative direction} \blacktriangleleft$$

c. Acceleration When $v = 0$. We carry $t = +5$ s into (3):

$$a_5 = 6(5) - 12 \qquad a_5 = +18 \text{ ft/s}^2 \blacktriangleleft$$

d. Distance Traveled from $t = 4$ s to $t = 6$ s. Since the particle moves in the negative direction from $t = 4$ s to $t = 5$ s and in the positive direction from $t = 5$ s to $t = 6$ s, we shall compute separately the distance traveled during each of these time intervals.

From $t = 4$ s to $t = 5$ s: $\quad x_5 = -60$ ft

$$x_4 = (4)^3 - 6(4)^2 - 15(4) + 40 = -52 \text{ ft}$$

$$\text{Distance traveled} = x_5 - x_4 = -60 \text{ ft} - (-52 \text{ ft}) = -8 \text{ ft}$$
$$= 8 \text{ ft in the negative direction}$$

From $t = 5$ s to $t = 6$ s: $\quad x_5 = -60$ ft

$$x_6 = (6)^3 - 6(6)^2 - 15(6) + 40 = -50 \text{ ft}$$

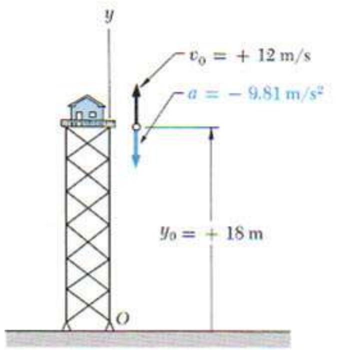
$$\text{Distance traveled} = x_6 - x_5 = -50 \text{ ft} - (-60 \text{ ft}) = +10 \text{ ft}$$
$$= 10 \text{ ft in the positive direction}$$

Total distance traveled from $t = 4$ s to $t = 6$ s is

$$8 \text{ ft} + 10 \text{ ft} = 18 \text{ ft} \blacktriangleleft$$

SAMPLE PROBLEM 11.2

A ball is thrown from the top of a tower 18 m high, with a velocity of 12 m/s directed vertically upward. Knowing that the acceleration of the ball is constant and equal to 9.81 m/s^2 downward, determine (*a*) the velocity v and elevation y of the ball above the ground at any time t, (*b*) the highest elevation reached by the ball and the corresponding value of t, (*c*) the time when the ball will hit the ground and the corresponding velocity. Draw the v–t and y–t curves.



a. Velocity and Elevation. The y axis measuring the position coordinate (or elevation) is chosen with its origin O on the ground and its positive sense upward. The value of the acceleration and the initial values of v and y are as indicated. Substituting for a in $a = dv/dt$ and noting that, at $t = 0$, $v_0 = +12$ m/s, we have

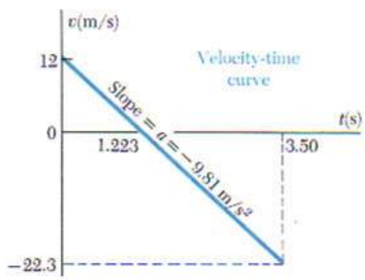
$$\frac{dv}{dt} = a = -9.81 \text{ m/s}^2$$

$$\int_{v_0=12}^{v} dv = -\int_0^t 9.81\, dt$$

$$[v]_{12}^{v} = -[9.81t]_0^t$$

$$v - 12 = -9.81t$$

$$v = 12 - 9.81t \quad (1)$$



Substituting for v in $v = dy/dt$ and noting that, at $t = 0$, $y_0 = 18$ m, we have

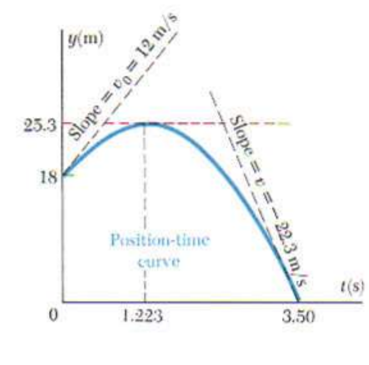
$$\frac{dy}{dt} = v = 12 - 9.81t$$

$$\int_{y_0=18}^{y} dy = \int_0^t (12 - 9.81t)\, dt$$

$$[y]_{18}^{y} = [12t - 4.90t^2]_0^t$$

$$y - 18 = 12t - 4.90t^2$$

$$y = 18 + 12t - 4.90t^2 \quad (2)$$



b. Highest Elevation. When the ball reaches its highest elevation, we have $v = 0$. Substituting into (1), we obtain

$$12 - 9.81t = 0 \qquad t = 1.223 \text{ s}$$

Carrying $t = 1.223$ s into (2), we have

$$y = 18 + 12(1.223) - 4.90(1.223)^2 \qquad y = 25.3 \text{ m}$$

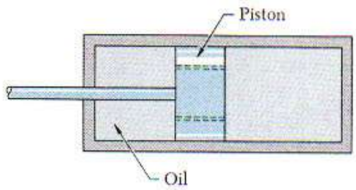
c. Ball Hits the Ground. When the ball hits the ground, we have $y = 0$. Substituting into (2), we obtain

$$18 + 12t - 4.90t^2 = 0 \qquad t = -1.05 \text{ s} \quad \text{and} \quad t = +3.50 \text{ s}$$

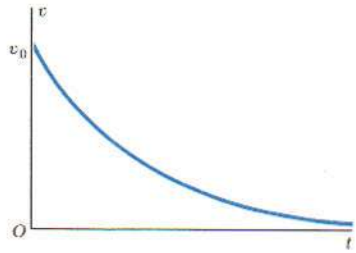
Only the root $t = +3.50$ s corresponds to a time after the motion has begun. Carrying this value of t into (1), we have

$$v = 12 - 9.81(3.50) = -22.3 \text{ m/s} \qquad v = 22.3 \text{ m/s} \downarrow$$

SAMPLE PROBLEM 11.3



The brake mechanism used to reduce recoil in certain types of guns consists essentially of a piston which is attached to the barrel and may move in a fixed cylinder filled with oil. As the barrel recoils with an initial velocity v_0, the piston moves and oil is forced through orifices in the piston, causing the piston and the barrel to decelerate at a rate proportional to their velocity, i.e., $a = -kv$. Express (*a*) v in terms of t, (*b*) x in terms of t, (*c*) v in terms of x. Draw the corresponding motion curves.



a. v in Terms of t. Substituting $-kv$ for a in the fundamental formula defining acceleration, $a = dv/dt$, we write

$$-kv = \frac{dv}{dt} \qquad \frac{dv}{v} = -k\,dt \qquad \int_{v_0}^{v} \frac{dv}{v} = -k\int_0^t dt$$

$$\ln\frac{v}{v_0} = -kt \qquad\qquad v = v_0 e^{-kt}$$

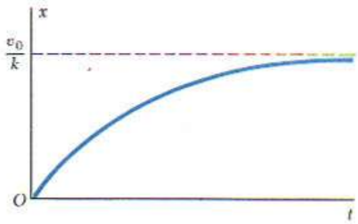
b. x in Terms of t. Substituting the expression just obtained for v into $v = dx/dt$, we write

$$v_0 e^{-kt} = \frac{dx}{dt}$$

$$\int_0^x dx = v_0 \int_0^t e^{-kt}\,dt$$

$$x = -\frac{v_0}{k}[e^{-kt}]_0^t = -\frac{v_0}{k}(e^{-kt} - 1)$$

$$x = \frac{v_0}{k}(1 - e^{-kt})$$



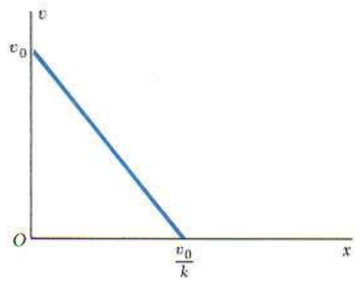
c. v in Terms of x. Substituting $-kv$ for a in $a = v\,dv/dx$, we write

$$-kv = v\frac{dv}{dx}$$

$$dv = -k\,dx$$

$$\int_{v_0}^{v} dv = -k\int_0^x dx$$

$$v - v_0 = -kx \qquad\qquad v = v_0 - kx$$



Check. Part *c* could have been solved by eliminating t from the answers obtained for parts *a* and *b*. This alternate method may be used as a check. From part *a* we obtain $e^{-kt} = v/v_0$; substituting in the answer of part *b*, we obtain

$$x = \frac{v_0}{k}(1 - e^{-kt}) = \frac{v_0}{k}\left(1 - \frac{v}{v_0}\right) \qquad v = v_0 - kx \qquad \text{(checks)}$$

PROBLEMS

11.1 The motion of a particle is defined by the relation $x = 2t^3 - 8t^2 + 5t + 9$, where x is expressed in inches and t in seconds. Determine the position, velocity, and acceleration when $t = 3$ s.

11.2 The motion of a particle is defined by the relation $x = 2t^3 - 9t^2 + 12$, where x is expressed in inches and t in seconds. Determine the time, position, and acceleration when $v = 0$.

11.3 The motion of a particle is defined by the relation $x = t^2 - 10t + 30$, where x is expressed in meters and t in seconds. Determine (*a*) when the velocity is zero, (*b*) the position and the total distance traveled when $t = 8$ s.

11.4 The motion of a particle is defined by the relation $x = \frac{1}{3}t^3 - 3t^2 + 8t + 2$, where x is expressed in meters and t in seconds. Determine (*a*) when the velocity is zero, (*b*) the position and the total distance traveled when the acceleration is zero.

11.5 The acceleration of a particle is directly proportional to the time t. At $t = 0$, the velocity of the particle is $v = -16$ m/s. Knowing that both the velocity and the position coordinate are zero when $t = 4$ s, write the equations of motion for the particle.

11.6 The acceleration of a particle is defined by the relation $a = -2\ \text{m/s}^2$. If $v = +8$ m/s and $x = 0$ when $t = 0$, determine the velocity, position, and total distance traveled when $t = 6$ s.

11.7 The acceleration of a particle is defined by the relation $a = kt^2$. (*a*) Knowing that $v = -250$ in./s when $t = 0$ and that $v = +250$ in./s when $t = 5$ s, determine the constant k. (*b*) Write the equations of motion knowing also that $x = 0$ when $t = 2$ s.

11.8 The acceleration of a particle is defined by the relation $a = 18 - 6t^2$. The particle starts at $t = 0$ with $v = 0$ and $x = 100$ in. Determine (*a*) the time when the velocity is again zero, (*b*) the position and velocity when $t = 4$ s, (*c*) the total distance traveled by the particle from $t = 0$ to $t = 4$ s.

11.9 The acceleration of a particle is defined by the relation $a = 21 - 12x^2$, where a is expressed in m/s^2 and x in meters. The particle starts with no initial velocity at the position $x = 0$. Determine (*a*) the velocity when $x = 1.5$ m, (*b*) the position where the velocity is again zero, (*c*) the position where the velocity is maximum.

11.10 The acceleration of an oscillating particle is defined by the relation $a = -kx$. Find the value of k such that $v = 10$ m/s when $x = 0$ and $x = 2$ m when $v = 0$.

11.11 The acceleration of a particle moving in a straight line is directed toward a fixed point O and is inversely proportional to the distance of the particle from O. At $t = 0$, the particle is 8 in. to the right of O, has a velocity of 16 in./s to the right, and has an acceleration of 12 in./s^2 to the left. Determine (*a*) the velocity of the particle when it is 12 in. away from O, (*b*) the position of the particle at which its velocity is zero.

11.12 The acceleration of a particle is defined by the relation $a = -kx^{-2}$. The particle starts with no initial velocity at $x = 12$ in., and it is observed that its velocity is 8 in./s when $x = 6$ in. Determine (*a*) the value of k, (*b*) the velocity of the particle when $x = 3$ in.

11.13 The acceleration of a particle is defined by the relation $a = -10v$, where a is expressed in m/s^2 and v in m/s. Knowing that at $t = 0$ the velocity is 30 m/s, determine (*a*) the distance the particle will travel before coming to rest, (*b*) the time required for the particle to come to rest, (*c*) the time required for the velocity of the particle to be reduced to 1 percent of its initial value.

11.14 The acceleration of a particle is defined by the relation $a = -0.0125v^2$, where a is the acceleration in m/s^2 and v is the velocity in m/s. If the particle is given an initial velocity v_0, find the distance it will travel (*a*) before its velocity drops to half the initial value, (*b*) before it comes to rest.

11.15 The acceleration of a particle falling through the atmosphere is defined by the relation $a = g(1 - k^2v^2)$. Knowing that the particle starts at $t = 0$ and $x = 0$ with no initial velocity, (*a*) show that the velocity at any time t is $v = (1/k) \tanh kgt$, (*b*) write an equation defining the velocity for any value of x. (*c*) Why is $v_t = 1/k$ called the terminal velocity?

11.16 It has been determined experimentally that the magnitude in ft/s^2 of the deceleration due to air resistance of a projectile is $0.001v^2$, where v is expressed in ft/s. If the projectile is released from rest and keeps pointing downward, determine its velocity after it has fallen 500 ft. (*Hint.* The total acceleration is $g - 0.001v^2$, where $g = 32.2$ ft/s^2.)

11.17 The acceleration of a particle is defined by the relation $a = -kv^{1.5}$. The particle starts at $t = 0$ and $x = 0$ with an initial velocity v_0. (*a*) Show that the velocity and position coordinate at any time t are related by the equation $x/t = \sqrt{v_0 v}$. (*b*) Determine the value of k, knowing that for $v_0 = 100$ ft/s the particle comes to rest after traveling 5 ft.

11.18 The acceleration of a particle is defined by the relation $a = k \sin(\pi t/T)$. Knowing that both the velocity and the position coordinate of the particle are zero when $t = 0$, determine (*a*) the equations of motion, (*b*) the maximum velocity, (*c*) the position at $t = 2T$, (*d*) the average velocity during the interval $t = 0$ to $t = 2T$.

11.19 The position of an oscillating particle is defined by the relation $x = A \sin(pt + \phi)$. Denoting the velocity and position coordinate when $t = 0$ by v_0 and x_0, respectively, show (*a*) that $\tan\phi = x_0 p/v_0$, (*b*) that the maximum value of the position coordinate is

$$A = \sqrt{x_0^2 + \left(\frac{v_0}{p}\right)^2}$$

11.20 The acceleration due to gravity at an altitude y above the surface of the earth may be expressed as

$$a = \frac{-32.2}{\left(1 + \dfrac{y}{20.9 \times 10^6}\right)^2}$$

where a is measured in ft/s^2 and y in feet. Using this expression, compute the height reached by a bullet fired vertically upward from the surface of the earth with the following initial velocities: (*a*) 1000 ft/s, (*b*) 10,000 ft/s, (*c*) 36,700 ft/s.

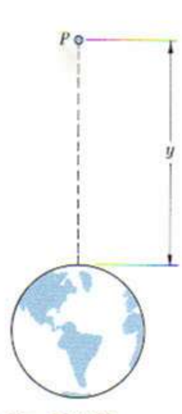


Fig. P11.20

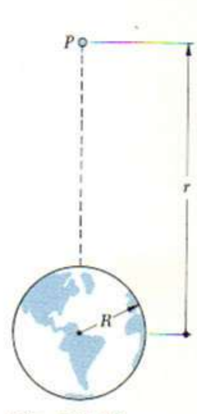


Fig. P11.21

11.21 The acceleration due to gravity of a particle falling toward the earth is $a = -gR^2/r^2$, where r is the distance from the *center* of the earth to the particle, R is the radius of the earth, and g is the acceleration due to gravity at the surface of the earth. Derive an expression for the *escape velocity*, i.e., for the minimum velocity with which a particle should be projected vertically upward from the surface of the earth if it is not to return to the earth. (*Hint.* $v = 0$ for $r = \infty$.)

***11.22** When a package is dropped on a rigid surface, the acceleration of its cushioned contents may be defined by the relation $a = -k \tan(\pi x/2L)$, where L is the distance through which the cushioning material can be compressed. Denoting by v_0 the velocity when $x = 0$, show that $v^2 = v_0^2 + (4kL/\pi) \ln \cos(\pi x/2L)$. If $k = 300$ m/s^2 and $L = 0.36$ m, compute the initial velocity v_0 for which the maximum value of the position coordinate x is (*a*) 0.18 m, (*b*) 0.36 m.

***11.23** Using the expression for the acceleration due to gravity given in Prob. 11.21, derive an expression for the time required for a particle to reach the surface of the earth if it is released with no velocity at a distance r_0 from the center of the earth.

11.4. Uniform Rectilinear Motion. This is a type of straight-line motion which is frequently encountered in practical applications. In this motion, the acceleration a of the particle is zero for every value of t. The velocity v is therefore constant, and Eq. (11.1) becomes

$$\frac{dx}{dt} = v = \text{constant}$$

The position coordinate x is obtained by integrating this equation. Denoting by x_0 the initial value of x, we write

$$\int_{x_0}^{x} dx = v \int_0^t dt$$

$$x - x_0 = vt$$

$$x = x_0 + vt \tag{11.5}$$

This equation may be used *only if the velocity of the particle is known to be constant.*

11.5. Uniformly Accelerated Rectilinear Motion. This is another common type of motion. In this motion, the acceleration a of the particle is constant, and Eq. (11.2) becomes

$$\frac{dv}{dt} = a = \text{constant}$$

The velocity v of the particle is obtained by integrating this equation,

$$\int_{v_0}^{v} dv = a \int_0^t dt$$

$$v - v_0 = at$$

$$v = v_0 + at \tag{11.6}$$

where v_0 is the initial velocity. Substituting for v into (11.1), we write

$$\frac{dx}{dt} = v_0 + at$$

Denoting by x_0 the initial value of x and integrating, we have

$$\int_{x_0}^{x} dx = \int_0^t (v_0 + at)\, dt$$

$$x - x_0 = v_0 t + \tfrac{1}{2}at^2$$

$$x = x_0 + v_0 t + \tfrac{1}{2}at^2 \tag{11.7}$$

We may also use Eq. (11.4) and write

$$v\frac{dv}{dx} = a = \text{constant}$$

$$v\, dv = a\, dx$$

Integrating both sides, we obtain

$$\int_{v_0}^{v} v\, dv = a\int_{x_0}^{x} dx$$

$$\tfrac{1}{2}(v^2 - v_0^2) = a(x - x_0)$$

$$v^2 = v_0^2 + 2a(x - x_0) \tag{11.8}$$

The three equations we have derived provide useful relations among position coordinate, velocity, and time in the case of a uniformly accelerated motion, as soon as appropriate values have been substituted for a, v_0, and x_0. The origin O of the x axis should first be defined and a positive direction chosen along the axis; this direction will be used to determine the signs of a, v_0, and x_0. Equation (11.6) relates v and t and should be used when the value of v corresponding to a given value of t is desired, or inversely. Equation (11.7) relates x and t; Eq. (11.8) relates v and x. An important application of uniformly accelerated motion is the motion of a *freely falling body*. The acceleration of a freely falling body (usually denoted by g) is equal to 9.81 m/s^2 or 32.2 ft/s^2.

It is important to keep in mind that the three equations above may be used *only when the acceleration of the particle is known to be constant.* If the acceleration of the particle is variable, its motion should be determined from the fundamental equations (11.1) to (11.4), according to the methods outlined in Sec. 11.3.

11.6. Motion of Several Particles. When several particles move independently along the same line, independent equations of motion may be written for each particle. Whenever possible, time should be recorded from the same initial instant for all particles, and displacements should be measured from the same origin and in the same direction. In other words, a single clock and a single measuring tape should be used.

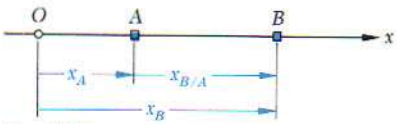


Fig. 11.7

Relative Motion of Two Particles. Consider two particles A and B moving along the same straight line (Fig. 11.7). If the position coordinates x_A and x_B are measured from the same origin, the difference $x_B - x_A$ defines the *relative position coordinate of B with respect to A* and is denoted by $x_{B/A}$. We write

$$x_{B/A} = x_B - x_A \qquad \text{or} \qquad x_B = x_A + x_{B/A} \tag{11.9}$$

A positive sign for $x_{B/A}$ means that B is to the right of A, a negative sign that B is to the left of A, regardless of the position of A and B with respect to the origin.

The rate of change of $x_{B/A}$ is known as the *relative velocity of B with respect to A* and is denoted by $v_{B/A}$. Differentiating (11.9), we write

$$v_{B/A} = v_B - v_A \qquad \text{or} \qquad v_B = v_A + v_{B/A} \tag{11.10}$$

A positive sign for $v_{B/A}$ means that B is *observed from A* to move in the positive direction; a negative sign, that it is observed to move in the negative direction.

The rate of change of $v_{B/A}$ is known as the *relative acceleration of B with respect to A* and is denoted by $a_{B/A}$. Differentiating (11.10), we obtain

$$a_{B/A} = a_B - a_A \qquad \text{or} \qquad a_B = a_A + a_{B/A} \tag{11.11}$$

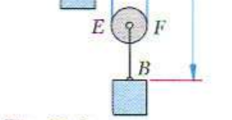


Fig. 11.8

Dependent Motions. Sometimes, the position of a particle will depend upon the position of another or of several other particles. The motions are then said to be dependent. For example, the position of block B in Fig. 11.8 depends upon the position of block A. Since the rope $ACDEFG$ is of constant length, and since the lengths of the portions of rope CD and EF wrapped around the pulleys remain constant, it follows that the sum of the lengths of the segments AC, DE, and FG is constant. Observing that the length of the segment AC differs from x_A only by a constant, and that, similarly, the lengths of the segments DE and FG differ from x_B only by a constant, we write

$$x_A + 2x_B = \text{constant}$$

Since only one of the two coordinates x_A and x_B may be chosen arbitrarily, we say that the system shown in Fig. 11.8 has *one degree of freedom.* From the relation between the position coordinates x_A and x_B, it follows that if x_A is given an increment Δx_A, i.e., if block A is lowered by an amount Δx_A, the coordinate x_B will receive an increment $\Delta x_B = -\frac{1}{2}\Delta x_A$, that is, block B will rise by half the same amount; this may easily be checked directly from Fig. 11.8.

In the case of the three blocks of Fig. 11.9, we may again observe that the length of the rope which passes over the pulleys is constant, and thus that the following relation must be satisfied by the position coordinates of the three blocks:

$$2x_A + 2x_B + x_C = \text{constant}$$

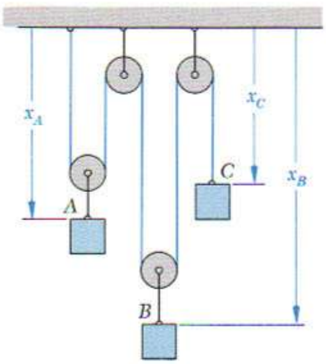


Fig. 11.9

Since two of the coordinates may be chosen arbitrarily, we say that the system shown in Fig. 11.9 has *two degrees of freedom.*

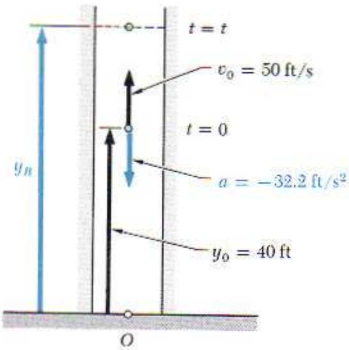
When the relation existing between the position coordinates of several particles is *linear,* a similar relation holds between the velocities and between the accelerations of the particles. In the case of the blocks of Fig. 11.9, for instance, we differentiate twice the equation obtained and write

$$2\frac{dx_A}{dt} + 2\frac{dx_B}{dt} + \frac{dx_C}{dt} = 0 \qquad \text{or} \qquad 2v_A + 2v_B + v_C = 0$$

$$2\frac{dv_A}{dt} + 2\frac{dv_B}{dt} + \frac{dv_C}{dt} = 0 \qquad \text{or} \qquad 2a_A + 2a_B + a_C = 0$$

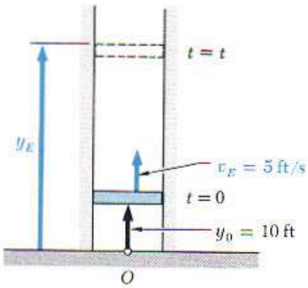
SAMPLE PROBLEM 11.4

A ball is thrown vertically upward from the 40-ft level in an elevator shaft, with an initial velocity of 50 ft/s. At the same instant an open-platform elevator passes the 10-ft level, moving upward with a constant velocity of 5 ft/s. Determine (*a*) when and where the ball will hit the elevator, (*b*) the relative velocity of the ball with respect to the elevator when the ball hits the elevator.



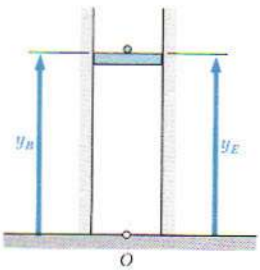
Motion of Ball. Since the ball has a constant acceleration, its motion is *uniformly accelerated*. Placing the origin O of the y axis at ground level and choosing its positive direction upward, we find that the initial position is $y_0 = +40$ ft, the initial velocity is $v_0 = +50$ ft/s, and the acceleration is $a = -32.2$ ft/s^2. Substituting these values in the equations for uniformly accelerated motion, we write

$$v_B = v_0 + at \qquad v_B = 50 - 32.2t \qquad (1)$$
$$y_B = y_0 + v_0 t + \tfrac{1}{2}at^2 \qquad y_B = 40 + 50t - 16.1t^2 \qquad (2)$$



Motion of Elevator. Since the elevator has a constant velocity, its motion is *uniform*. Again placing the origin O at the ground level and choosing the positive direction upward, we note that $y_0 = +10$ ft and write

$$v_E = +5 \text{ ft/s} \qquad (3)$$
$$y_E = y_0 + v_E t \qquad y_E = 10 + 5t \qquad (4)$$



Ball Hits Elevator. We first note that the same time t and the same origin O were used in writing the equations of motion of both the ball and the elevator. We see from the figure that, when the ball hits the elevator,

$$y_E = y_B \qquad (5)$$

Substituting for y_E and y_B from (2) and (4) into (5), we have

$$10 + 5t = 40 + 50t - 16.1t^2$$
$$t = -0.56 \text{ s} \quad \text{and} \quad t = +3.35 \text{ s} \blacktriangleleft$$

Only the root $t = 3.35$ s corresponds to a time after the motion has begun. Substituting this value into (4), we have

$$y_E = 10 + 5(3.35) = 26.7 \text{ ft}$$
$$\text{Elevation from ground} = 26.7 \text{ ft} \blacktriangleleft$$

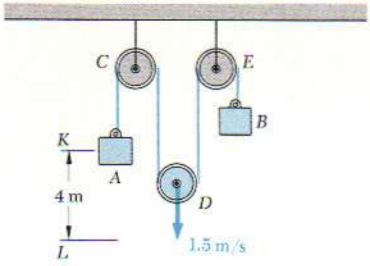
The relative velocity of the ball with respect to the elevator is

$$v_{B/E} = v_B - v_E = (50 - 32.2t) - 5 = 45 - 32.2t$$

When the ball hits the elevator at time $t = 3.35$ s, we have

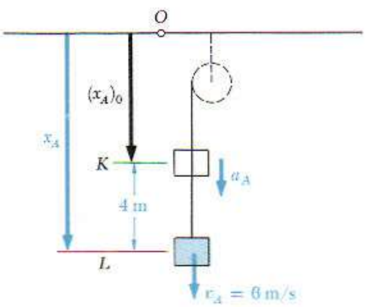
$$v_{B/E} = 45 - 32.2(3.35) \qquad v_{B/E} = -62.9 \text{ ft/s} \blacktriangleleft$$

The negative sign means that the ball is observed from the elevator to be moving in the negative sense (downward).



SAMPLE PROBLEM 11.5

Two blocks A and B are connected by a cord passing over three pulleys C, D, and E as shown. Pulleys C and E are fixed, while D is pulled downward with a constant velocity of 1.5 m/s. At $t = 0$, block A starts moving downward from the position K with a constant acceleration and no initial velocity. Knowing that the velocity of block A is 6 m/s as it passes through point L, determine the change in elevation, the velocity, and the acceleration of block B when A passes through L.

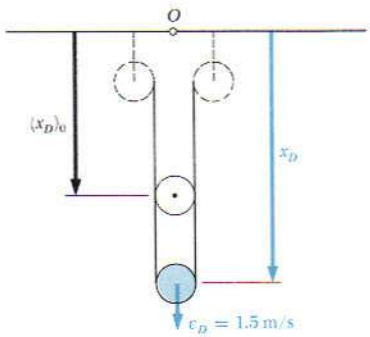


Motion of Block A. We place the origin O at the horizontal surface and choose the positive direction downward. We observe that when $t = 0$, block A is at position K and $(v_A)_0 = 0$. Since $v_A = 6$ m/s and $x_A - (x_A)_0 = 4$ m when the block passes through L, we write

$$v_A^2 = (v_A)_0^2 + 2a_A[x_A - (x_A)_0] \qquad (6)^2 = 0 + 2a_A(4)$$
$$a_A = 4.50 \text{ m/s}^2$$

The time at which block A reaches point L is obtained by writing

$$v_A = (v_A)_0 + a_A t \qquad 6 = 0 + 4.50t \qquad t = 1.333 \text{ s}$$



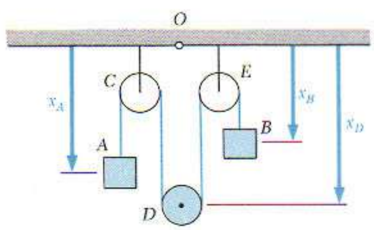
Motion of Pulley D. Recalling that the positive direction is downward, we write

$$a_D = 0 \qquad v_D = 1.5 \text{ m/s} \qquad x_D = (x_D)_0 + v_D t = (x_D)_0 + 1.5t$$

When block A reaches L, at $t = 1.333$ s, we have

$$x_D = (x_D)_0 + 1.5(1.333) = (x_D)_0 + 2$$

Thus, $$x_D - (x_D)_0 = 2 \text{ m}$$



Motion of Block B. We note that the total length of cord $ACDEB$ differs from the quantity $(x_A + 2x_D + x_B)$ only by a constant. Since the cord length is constant during the motion, this quantity must also remain constant. Thus considering the times $t = 0$ and $t = 1.333$ s, we write

$$x_A + 2x_D + x_B = (x_A)_0 + 2(x_D)_0 + (x_B)_0 \qquad (1)$$
$$[x_A - (x_A)_0] + 2[x_D - (x_D)_0] + [x_B - (x_B)_0] = 0 \qquad (2)$$

But we know that $x_A - (x_A)_0 = 4$ m and $x_D - (x_D)_0 = 2$ m; substituting these values in (2), we find

$$4 + 2(2) + [x_B - (x_B)_0] = 0 \qquad x_B - (x_B)_0 = -8 \text{ m}$$

Thus: Change in elevation of $B = 8$ m $\uparrow$ ◄

Differentiating (1) twice, we obtain equations relating the velocities and the accelerations of A, B, and D. Substituting for the velocities and accelerations of A and D at $t = 1.333$ s, we have

$$v_A + 2v_D + v_B = 0: \qquad 6 + 2(1.5) + v_B = 0$$
$$v_B = -9 \text{ m/s} \qquad v_B = 9 \text{ m/s} \uparrow \blacktriangleleft$$
$$a_A + 2a_D + a_B = 0: \qquad 4.50 + 2(0) + a_B = 0$$
$$a_B = -4.50 \text{ m/s}^2 \qquad a_B = 4.50 \text{ m/s}^2 \uparrow \blacktriangleleft$$

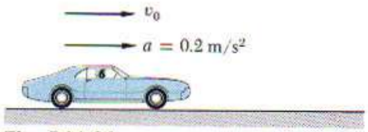


Fig. P11.24

PROBLEMS

11.24 An automobile travels 240 m in 30 s while being accelerated at a constant rate of $0.2\ m/s^2$. Determine (*a*) its initial velocity, (*b*) its final velocity, (*c*) the distance traveled during the first 10 s.

11.25 A stone is released from an elevator moving up at a speed of 5 m/s and reaches the bottom of the shaft in 3 s. (*a*) How high was the elevator when the stone was released? (*b*) With what speed does the stone strike the bottom of the shaft?

11.26 A stone is thrown vertically upward from a point on a bridge located 135 ft above the water. Knowing that it strikes the water 4 s after release, determine (*a*) the speed with which the stone was thrown upward, (*b*) the speed with which the stone strikes the water.

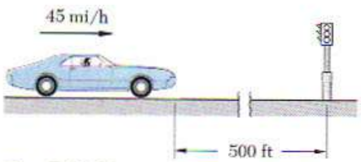


Fig. P11.27

11.27 A motorist is traveling at 45 mi/h when he observes that a traffic light 800 ft ahead of him turns red. The traffic light is timed to stay red for 15 s. If the motorist wishes to pass the light without stopping just as it turns green again, determine (*a*) the required uniform deceleration of the car, (*b*) the speed of the car as it passes the light.

11.28 Automobile A starts from O and accelerates at the constant rate of $4\ ft/s^2$. A short time later it is passed by truck B which is traveling in the opposite direction at a constant speed of 45 ft/s. Knowing that truck B passes point O, 25 s after automobile A started from there, determine when and where the vehicles passed each other.

Fig. P11.28

11.29 An open-platform elevator is moving down a mine shaft at a constant velocity v_e when the elevator platform hits and dislodges a stone. Assuming that the stone starts falling with no initial velocity, (*a*) show that the stone will hit the platform with a relative velocity of magnitude v_e. (*b*) If $v_e = 16\ ft/s$, determine when and where the stone will hit the elevator platform.

11.30 Two automobiles A and B are traveling in the same direction in adjacent highway lanes. Automobile B is stopped when it is passed by A, which travels at a constant speed of 36 km/h. Two seconds later automobile B starts and accelerates at a constant rate of $1.5\ \text{m/s}^2$. Determine (*a*) when and where B will overtake A, (*b*) the speed of B at that time.

11.31 Drops of water are observed to drip from a faucet at uniform intervals of time. As any drop B begins to fall freely, the preceding drop A has already fallen 0.3 m. Determine the distance drop A will have fallen by the time the distance between A and B will have increased to 0.9 m.

11.32 The elevator shown in the figure moves upward at the constant velocity of 18 ft/s. Determine (*a*) the velocity of the cable C, (*b*) the velocity of the counterweight W, (*c*) the relative velocity of the cable C with respect to the elevator, (*d*) the relative velocity of the counterweight W with respect to the elevator.

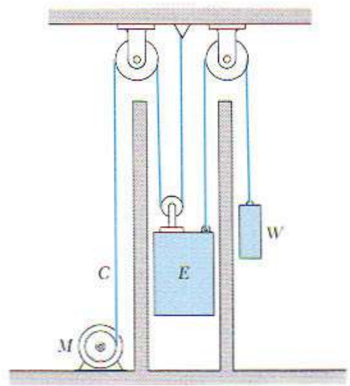


Fig. P11.32 and P11.33

11.33 The elevator shown starts from rest and moves upward with a constant acceleration. If the counterweight W moves through 24 ft in 4 s, determine (*a*) the accelerations of the elevator and the cable C, (*b*) the velocity of the elevator after 4 s.

11.34 The slider block A moves to the left at a constant velocity of 300 mm/s. Determine (*a*) the velocity of block B, (*b*) the velocities of portions C and D of the cable, (*c*) the relative velocity of A with respect to B, (*d*) the relative velocity of portion C of the cable with respect to portion D.

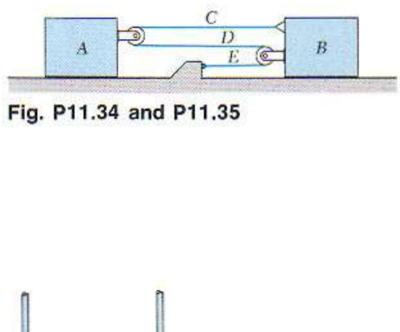


Fig. P11.34 and P11.35

11.35 The slider block B starts from rest and moves to the right with a constant acceleration. After 4 s the relative velocity of A with respect to B is 60 mm/s. Determine (*a*) the accelerations of A and B, (*b*) the velocity and position of B after 3 s.

11.36 Collars A and B start from rest and move with the following accelerations: $a_A = 3\ \text{in./s}^2$ upward and $a_B = 6t\ \text{in./s}^2$ downward. Determine (*a*) the time at which the velocity of block C is again zero, (*b*) the distance through which block C will have moved at that time.

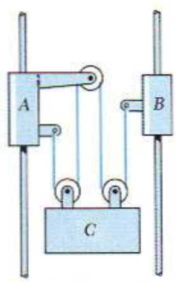


Fig. P11.36 and P11.37

11.37 (*a*) Choosing the positive sense upward, express the velocity of block C in terms of the velocities of collars A and B. (*b*) Knowing that both collars start from rest and move upward with the accelerations $a_A = 4\ \text{in./s}^2$ and $a_B = 3\ \text{in./s}^2$, determine the velocity of block C at $t = 4$ s and the distance through which it will have moved at that time.

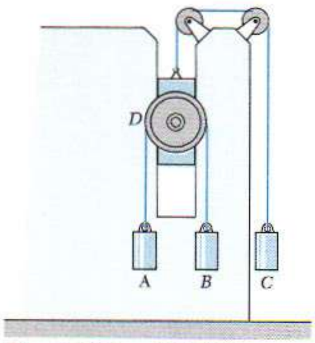


Fig. P11.38 and P11.40

11.38 The three blocks shown move with constant velocities. Find the velocity of each block, knowing that the relative velocity of C with respect to A is 200 mm/s upward and that the relative velocity of B with respect to C is 120 mm/s downward.

11.39 The three blocks of Fig. 11.9 move with constant velocities. Find the velocity of each block, knowing that C is observed from B to move downward with a relative velocity of 180 mm/s and A is observed from B to move downward with a relative velocity of 160 mm/s.

***11.40** The three blocks shown are equally spaced horizontally and move vertically with constant velocities. Knowing that initially they are at the same level and that the relative velocity of A with respect to B is 160 mm/s downward, determine the velocity of each block so that the three blocks will remain aligned during their motion.

*11.7. Graphical Solution of Rectilinear-Motion Problems.

It was observed in Sec. 11.2 that the fundamental formulas

$$v = \frac{dx}{dt} \qquad \text{and} \qquad a = \frac{dv}{dt}$$

have a geometrical significance. The first formula expresses that the velocity at any instant is equal to the slope of the x–t curve at the same instant (Fig. 11.10). The second formula expresses that the acceleration is equal to the slope of the v–t curve. These

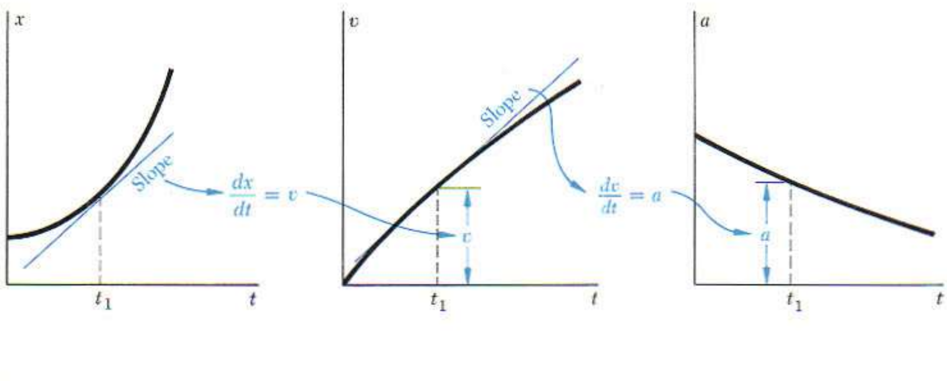


Fig. 11.10

two properties may be used to derive graphically the v–t and a–t curves of a motion when the x–t curve is known.

Integrating the two fundamental formulas from a time t_1 to a time t_2, we write

$$x_2 - x_1 = \int_{t_1}^{t_2} v\,dt \qquad \text{and} \qquad v_2 - v_1 = \int_{t_1}^{t_2} a\,dt \quad (11.12)$$

The first formula expresses that the area measured under the v–t curve from t_1 to t_2 is equal to the change in x during that time interval (Fig. 11.11). The second formula expresses similarly that the area measured under the a–t curve from t_1 to t_2 is equal to the change in v during that time interval. These two properties may be used to determine graphically the x–t curve of a motion when its v–t curve or its a–t curve is known (see Sample Prob. 11.6).

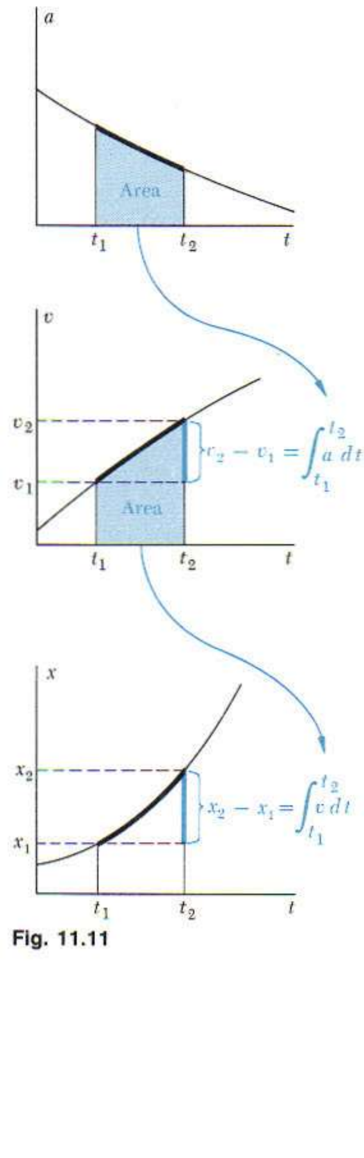
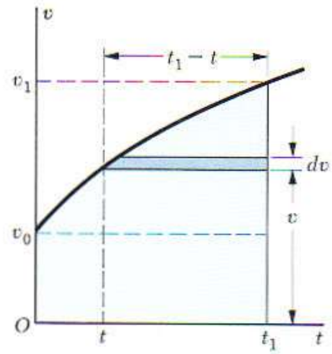


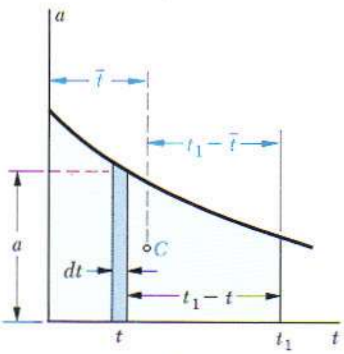
Fig. 11.11

Graphical solutions are particularly useful when the motion considered is defined from experimental data and when x, v, and a are not analytical functions of t. They may also be used to advantage when the motion consists of distinct parts and when its analysis requires writing a different equation for each of its parts. When using a graphical solution, however, one should be careful to note (1) that the area under the v–t curve measures the *change in* x, not x itself, and, similarly, that the area under the a–t curve measures the change in v; (2) that, while an area above the t axis corresponds to an increase in x or v, an area located below the t axis measures a decrease in x or v.

It will be useful to remember, in drawing motion curves, that, if the velocity is constant, it will be represented by a horizontal straight line; the position coordinate x will then be a linear function of t and will be represented by an oblique straight line. If the acceleration is constant and different from zero, it will be represented by a horizontal straight line; v will then be a linear function of t, represented by an oblique straight line; and x will be expressed as a second-degree polynomial in t, represented by a parabola. If the acceleration is a linear function of t, the velocity and the position coordinate will be equal, respectively, to second-degree and third-degree polynomials; a is then represented by an oblique straight line, v by a parabola, and x by a cubic. In general, if the acceleration is a polynomial of degree n in t, the velocity will be a polynomial of degree $n + 1$ and the position coordinate a polynomial of degree $n + 2$; these polynomials are represented by motion curves of a corresponding degree.



(a)



(b)

Fig. 11.12

***11.8. Other Graphical Methods.** An alternate graphical solution may be used to determine directly from the a–t curve the position of a particle at a given instant. Denoting respectively by x_0 and v_0 the values of x and v at $t = 0$, by x_1 and v_1 their values at $t = t_1$, and observing that the area under the v–t curve may be divided into a rectangle of area $v_0 t_1$ and horizontal differential elements of area $(t_1 - t)\,dv$ (Fig. 11.12a), we write

$$x_1 - x_0 = \text{area under } v\text{–}t \text{ curve} = v_0 t_1 + \int_{v_0}^{v_1} (t_1 - t)\,dv$$

Substituting $dv = a\,dt$ in the integral, we obtain

$$x_1 - x_0 = v_0 t_1 + \int_0^{t_1} (t_1 - t) a\,dt$$

Referring to Fig. 11.12b, we note that the integral represents the first moment of the area under the a–t curve with respect to the line $t = t_1$ bounding the area on the right. This method of solution is known, therefore, as the *moment-area method.* If the abscissa $\bar{t}$ of the centroid C of the area is known, the position coordinate x_1 may be obtained by writing

$$x_1 = x_0 + v_0 t_1 + (\text{area under } a\text{–}t \text{ curve})(t_1 - \bar{t}) \quad (11.13)$$

If the area under the a–t curve is a composite area, the last term in (11.13) may be obtained by multiplying each component area by the distance from its centroid to the line $t = t_1$. Areas above the t axis should be considered as positive and areas below the t axis as negative.

Another type of motion curve, the v–x curve, is sometimes used. If such a curve has been plotted (Fig. 11.13), the acceleration a may be obtained at any time by drawing the normal to the curve and *measuring the subnormal BC.* Indeed, observing that the angle between AC and AB is equal to the angle θ between the horizontal and the tangent at A (the slope of which is $\tan\theta = dv/dx$), we write

$$BC = AB \tan\theta = v\frac{dv}{dx}$$

and thus, recalling formula (11.4),

$$BC = a$$

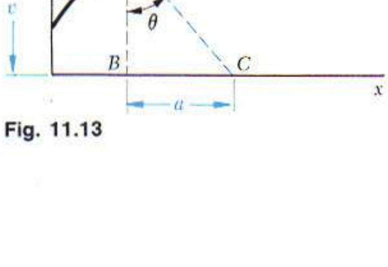
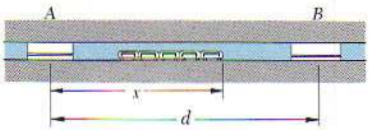
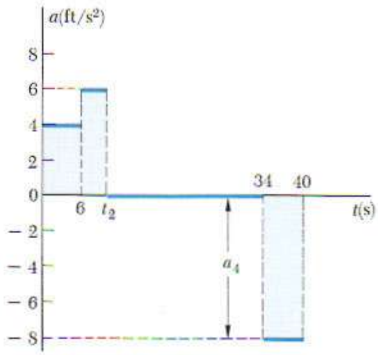


Fig. 11.13

SAMPLE PROBLEM 11.6



A subway train leaves station A; it gains speed at the rate of $4\ \mathrm{ft/s^2}$ for 6 s, and then at the rate of $6\ \mathrm{ft/s^2}$ until it has reached the speed of 48 ft/s. The train maintains the same speed until it approaches station B; brakes are then applied, giving the train a constant deceleration and bringing it to a stop in 6 s. The total running time from A to B is 40 s. Draw the a–t, v–t, and x–t curves, and determine the distance between stations A and B.



Acceleration-Time Curve. Since the acceleration is either constant or zero, the a–t curve is made of horizontal straight-line segments. The values of t_2 and a_4 are determined as follows:

$0 < t < 6$: Change in v = area under a–t curve

$$v_6 - 0 = (6\ \mathrm{s})(4\ \mathrm{ft/s^2}) = 24\ \mathrm{ft/s}$$

$6 < t < t_2$: Since the velocity increases from 24 to 48 ft/s,

Change in v = area under a–t curve

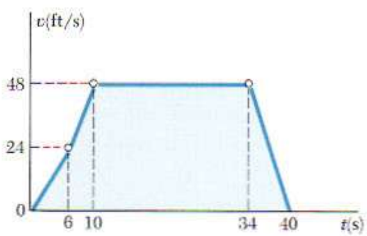
$$48 - 24 = (t_2 - 6)(6\ \mathrm{ft/s^2}) \qquad t_2 = 10\ \mathrm{s}$$

$t_2 < t < 34$: Since the velocity is constant, the acceleration is zero.

$34 < t < 40$: Change in v = area under a–t curve

$$0 - 48 = (6\ \mathrm{s})a_4 \qquad a_4 = -8\ \mathrm{ft/s^2}$$

The acceleration being negative, the corresponding area is below the t axis; this area represents a decrease in velocity.



Velocity-Time Curve. Since the acceleration is either constant or zero, the v–t curve is made of segments of straight line connecting the points determined above.

Change in x = area under v–t curve

$0 < t < 6$: $\quad x_6 - 0 = \frac{1}{2}(6)(24) = 72\ \mathrm{ft}$

$6 < t < 10$: $\quad x_{10} - x_6 = \frac{1}{2}(4)(24 + 48) = 144\ \mathrm{ft}$

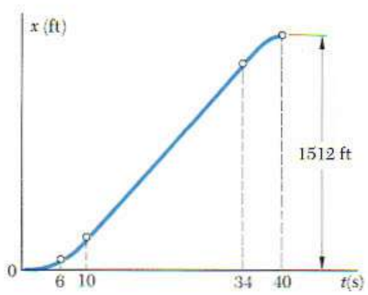
$10 < t < 34$: $\quad x_{34} - x_{10} = (24)(48) = 1152\ \mathrm{ft}$

$34 < t < 40$: $\quad x_{40} - x_{34} = \frac{1}{2}(6)(48) = 144\ \mathrm{ft}$

Adding the changes in x, we obtain the distance from A to B:

$$d = x_{40} - 0 = 1512\ \mathrm{ft}$$

$d = 1512\ \mathrm{ft}$ ◄



Position-Time Curve. The points determined above should be joined by three arcs of parabola and one segment of straight line. The construction of the x–t curve will be performed more easily and more accurately if we keep in mind that for any value of t the slope of the tangent to the x–t curve is equal to the value of v at that instant.

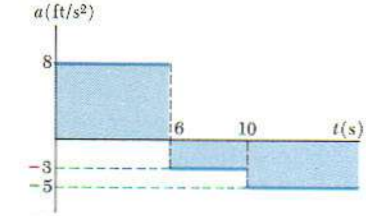


Fig. P11.41

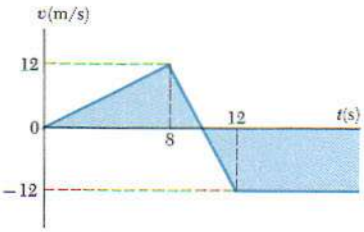


Fig. P11.43

PROBLEMS

11.41 A particle moves in a straight line with the acceleration shown in the figure. Knowing that it starts from the origin with $v_0 = -16$ ft/s, (*a*) plot the v–t and x–t curves for $0 < t < 16$ s, (*b*) determine its velocity, its position, and the total distance traveled after 12 s.

11.42 For the particle and motion of Prob. 11.41, plot the v–t and x–t curves for $0 < t < 16$ s and determine (*a*) the maximum value of the velocity of the particle, (*b*) the maximum value of its position coordinate.

11.43 A particle moves in a straight line with the velocity shown in the figure. Knowing that $x = -12$ m at $t = 0$, draw the a–t and x–t curves for $0 < t < 16$ s and determine (*a*) the total distance traveled by the particle after 12 s, (*b*) the two values of t for which the particle passes through the origin.

11.44 For the particle and motion of Prob. 11.43, plot the a–t and x–t curves for $0 < t < 16$ s and determine (*a*) the maximum value of the position coordinate of the particle, (*b*) the values of t for which the particle is at a distance of 15 m from the origin.

11.45 A series of city traffic signals is timed so that an automobile traveling at a constant speed of 25 mi/h will reach each signal just as it turns green. A motorist misses a signal and is stopped at signal A. Knowing that the next signal B is 750 ft ahead and that the maximum acceleration of his automobile is 6 ft/s^2, determine what the motorist should do to keep his maximum speed as small as possible, yet reach signal B just as it turns green. What is the maximum speed reached?

11.46 A bus starts from rest at point A and accelerates at the rate of 0.9 m/s^2 until it reaches a speed of 7.2 m/s. It then proceeds at 7.2 m/s until the brakes are applied; it comes to rest at point B, 18 m beyond the point where the brakes were applied. Assuming uniform deceleration and knowing that the distance between A and B is 90 m, determine the time required for the bus to travel from A to B.

11.47 The firing of a howitzer causes the barrel to recoil 800 mm before a braking mechanism brings it to rest. From a high-speed photographic record, it is found that the maximum value of the recoil velocity is 5.4 m/s and that this is reached 0.02 s after firing. Assuming that the recoil period consists of two phases during which the acceleration has, respectively, a constant positive value a_1 and a constant negative value a_2, determine (*a*) the values of a_1 and a_2, (*b*) the position of the barrel 0.02 s after firing, (*c*) the time at which the velocity of the barrel is zero.

11.48 A motorist is traveling at 45 mi/h when he observes that a traffic signal 1200 ft ahead of him turns red. He knows that the signal is timed to stay red for 24 s. What should he do to pass the signal at 45 mi/h just as it turns green again? Draw the v–t curve, selecting the solution which calls for the smallest possible deceleration and acceleration, and determine (*a*) the common value of the deceleration and acceleration in ft/s^2, (*b*) the minimum speed reached in mi/h.

11.49 A policeman on a motorcycle is escorting a motorcade which is traveling at 54 km/h. The policeman suddenly decides to take a new position in the motorcade, 70 m ahead. Assuming that he accelerates and decelerates at the rate of 2.5 m/s^2 and that he does not exceed at any time a speed of 72 km/h, draw the a–t and v–t curves for his motion and determine (*a*) the shortest time in which he can occupy his new position in the motorcade, (*b*) the distance he will travel in that time.

11.50 A freight elevator moving upward with a constant velocity of 5 m/s passes a passenger elevator which is stopped. Three seconds later, the passenger elevator starts upward with an acceleration of 1.25 m/s^2. When the passenger elevator has reached a velocity of 10 m/s, it proceeds at constant speed. Draw the v–t and y–t curves, and from them determine the time and distance required by the passenger elevator to overtake the freight elevator.

11.51 A car and a truck are both traveling at the constant speed of 35 mi/h; the car is 40 ft behind the truck. The driver of the car wants to pass the truck, i.e., he wishes to place his car at B, 40 ft in front of the truck, and then resume the speed of 35 mi/h. The maximum acceleration of the car is 5 ft/s^2 and the maximum deceleration obtained by applying the brakes is 20 ft/s^2. What is the shortest time in which the driver of the car can complete the passing operation if he does not at any time exceed a speed of 50 mi/h? Draw the v–t curve.

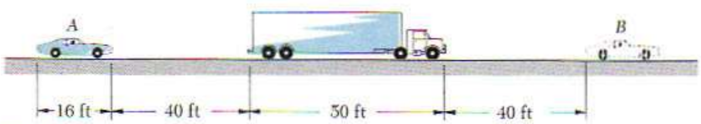


Fig. P11.51

11.52 Solve Prob. 11.51 assuming that the driver of the car does not pay any attention to the speed limit while passing and concentrates on reaching position B and resuming a speed of 35 mph in the shortest possible time. What is the maximum speed reached? Draw the v–t curve.

11.53 A car and a truck are both traveling at the constant speed of 60 mi/h; the car is 30 ft behind the truck. The truck driver suddenly applies his brakes, causing the truck to decelerate at the constant rate of 9 ft/s^2. Two seconds later the driver of the car applies his brakes and just manages to avoid a rear-end collision. Determine the constant rate at which the car decelerated.

11.54 Two cars are traveling toward each other on a single-lane road at 16 and 12 m/s, respectively. When 120 m apart, both drivers realize the situation and apply their brakes. They succeed in stopping simultaneously, and just short of colliding. Assuming a constant deceleration for each car, determine (*a*) the time required for the cars to stop, (*b*) the deceleration of each car, and (*c*) the distance traveled by each car while slowing down.

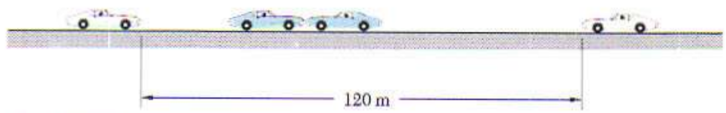


Fig. P11.54

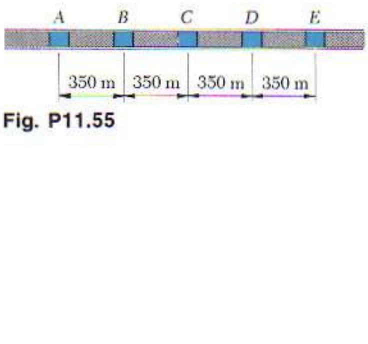


Fig. P11.55

11.55 An express subway train and a train making local stops run on parallel tracks between stations A and E, which are 1400 m apart. The local train makes stops of 30-s duration at each of the stations B, C, and D; the express train proceeds to station E without any intermediate stop. Each train accelerates at a rate of 1.25 m/s^2 until it reaches a speed of 12.5 m/s; it then proceeds at that constant speed. As the train approaches its next stop, the brakes are applied, providing a constant deceleration of 1.5 m/s^2. If the express train leaves station A 4 min after the local train has left A, determine (*a*) which of the two trains will arrive at station E first, (*b*) how much later the other train will arrive at station E.

11.56 The acceleration of a particle varies uniformly from $a = 75$ in./s^2 at $t = 0$, to $a = -75$ in./s^2 at $t = 8$ s. Knowing that $x = 0$ and $v = 0$ when $t = 0$, determine (*a*) the maximum velocity of the particle, (*b*) its position at $t = 8$ s, (*c*) its *average* velocity over the interval $0 < t < 8$ s. Draw the a–t, v–t, and x–t curves for the motion.

11.57 The rate of change of acceleration is known as the *jerk;* large or abrupt rates of change of acceleration cause discomfort to elevator passengers. If the jerk, or rate of change of the acceleration, of an elevator is limited to ± 0.5 m/s^2 per second, determine the shortest time required for an elevator, starting from rest, to rise 8 m and stop.

11.58 The acceleration record shown was obtained for a truck traveling on a straight highway. Knowing that the initial velocity of the truck was 18 km/h, determine the velocity and distance traveled when (*a*) $t = 4$ s, (*b*) $t = 6$ s.

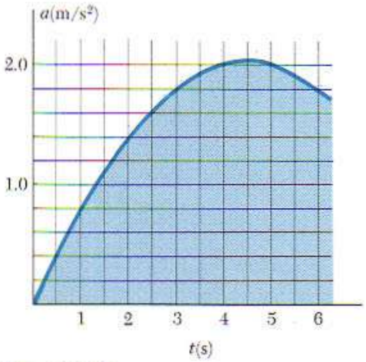


Fig. P11.58

11.59 A training airplane lands on an aircraft carrier and is brought to rest in 4 s by the arresting gear of the carrier. An accelerometer attached to the airplane provides the acceleration record shown. Determine by approximate means (*a*) the initial velocity of the airplane relative to the deck, (*b*) the distance the airplane travels along the deck before coming to rest.

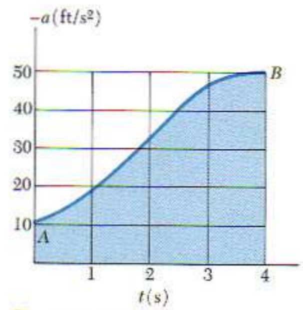


Fig. P11.59

11.60 The v–x curve shown was obtained experimentally during the motion of the bed of an industrial planer. Determine by approximate means the acceleration (*a*) when $x = 3$ in., (*b*) when $v = 40$ in./s.

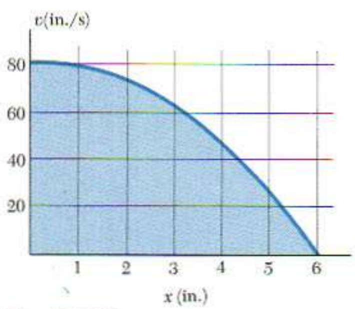


Fig. P11.60

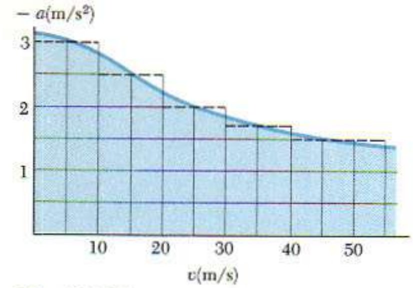


Fig. P11.61

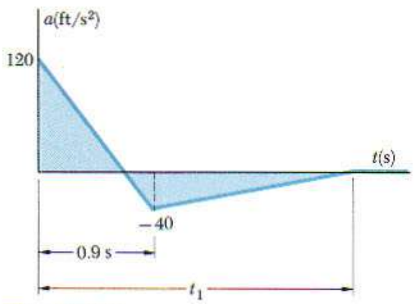


Fig. P11.64

11.61 The maximum possible acceleration of a passenger train under emergency conditions was determined experimentally; the results are shown (solid curve) in the figure. If the brakes are applied when the train is traveling at 90 km/h, determine by approximate means (*a*) the time required for the train to come to rest, (*b*) the distance traveled in that time.

11.62 Using the method of Sec. 11.8, derive the formula $x = x_0 + v_0t + \frac{1}{2}at^2$, for the position coordinate of a particle in uniformly accelerated rectilinear motion.

11.63 Using the method of Sec. 11.8, obtain an approximate solution for Prob. 11.59, assuming that the a–t curve is a straight line from point A to point B.

11.64 The acceleration of an object subjected to the pressure wave of a large explosion is defined approximately by the curve shown. The object is initially at rest and is again at rest at time t_1. Using the method of Sec. 11.8, determine (*a*) the time t_1, (*b*) the distance through which the object is moved by the pressure wave.

11.65 Using the method of Sec. 11.8, determine the position of the particle of Prob. 11.41 when $t = 12$ s.

11.66 For the particle of Prob. 11.43, draw the a–t curve and, using the method of Sec. 11.8, determine (*a*) the position of the particle when $t = 14$ s, (*b*) the maximum value of its position coordinate.

CURVILINEAR MOTION OF PARTICLES

11.9. Position Vector, Velocity, and Acceleration. When a particle moves along a curve other than a straight line, we say that the particle is in *curvilinear motion*. To define the position P occupied by the particle at a given time t, we select a fixed reference system, such as the x, y, z axes shown in Fig. 11.14*a*, and draw the vector $\mathbf{r}$ joining the origin O and point P. Since the vector $\mathbf{r}$ is characterized by its magnitude r and its direction with respect to the reference axes, it completely defines the position of the particle with respect to those axes; the vector $\mathbf{r}$ is referred to as the *position vector* of the particle at time t.

Consider now the vector $\mathbf{r}'$ defining the position P' occupied by the same particle at a later time $t + \Delta t$. The vector $\Delta\mathbf{r}$ joining P and P' represents the change in the position vector during the time interval Δt since, as we may easily check from Fig. 11.14*a*, the vector $\mathbf{r}'$ is obtained by adding the vectors $\mathbf{r}$ and $\Delta\mathbf{r}$ according to the triangle rule. We note that $\Delta\mathbf{r}$ represents a change in *direction* as well as a change in *magnitude* of the position vector $\mathbf{r}$. The *average velocity* of the particle over the time interval Δt is defined as the quotient of $\Delta\mathbf{r}$ and Δt. Since $\Delta\mathbf{r}$ is a vector and Δt a scalar, the quotient $\Delta\mathbf{r}/\Delta t$ is a vector attached at P, of the same direction as $\Delta\mathbf{r}$, and of magnitude equal to the magnitude of $\Delta\mathbf{r}$ divided by Δt (Fig. 11.14*b*).

The *instantaneous velocity* of the particle at time t is obtained by choosing shorter and shorter time intervals Δt and, correspondingly, shorter and shorter vector increments $\Delta\mathbf{r}$. The instantaneous velocity is thus represented by the vector

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{r}}{\Delta t} \tag{11.14}$$

As Δt and $\Delta\mathbf{r}$ become shorter, the points P and P' get closer; the vector $\mathbf{v}$ obtained at the limit must therefore be tangent to the path of the particle (Fig. 11.14*c*).

Since the position vector $\mathbf{r}$ depends upon the time t, we may refer to it as a *vector function* of the scalar variable t and denote it by $\mathbf{r}(t)$. Extending the concept of derivative of a scalar function introduced in elementary calculus, we shall refer to the limit of the quotient $\Delta\mathbf{r}/\Delta t$ as the *derivative* of the vector function $\mathbf{r}(t)$. We write

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \tag{11.15}$$

The magnitude v of the vector $\mathbf{v}$ is called the *speed* of the particle. It may be obtained by substituting for the vector $\Delta\mathbf{r}$ in formula (11.14) its magnitude represented by the straight-line segment PP'. But the length of the segment PP' approaches the length Δs of the arc PP' as Δt decreases (Fig. 11.14*a*), and we may write

$$v = \lim_{\Delta t \to 0} \frac{PP'}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

$$v = \frac{ds}{dt} \tag{11.16}$$

The speed v may thus be obtained by differentiating with respect to t the length s of the arc described by the particle.

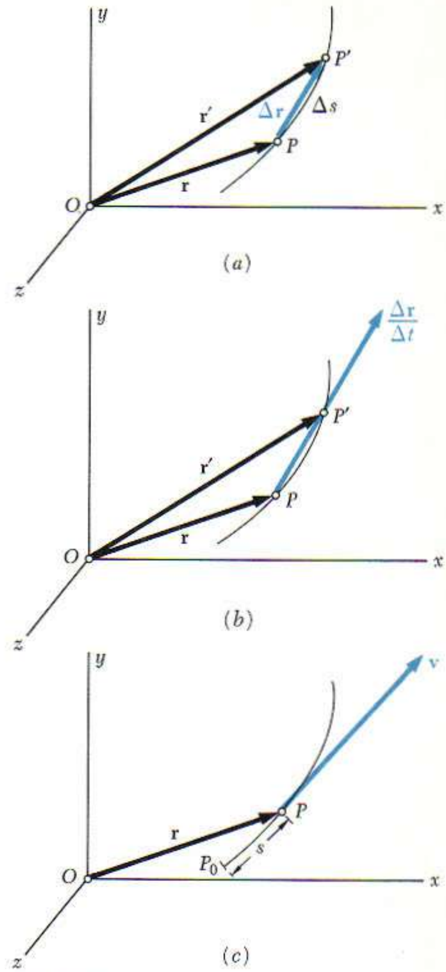
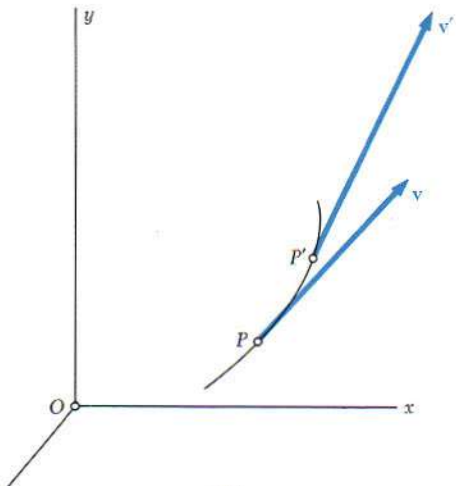
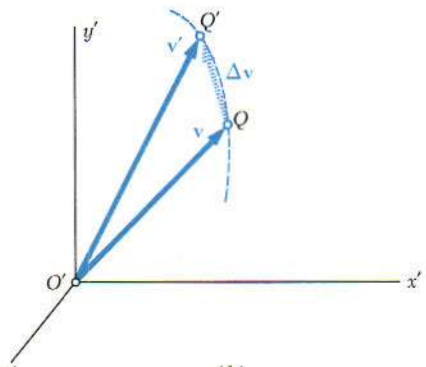


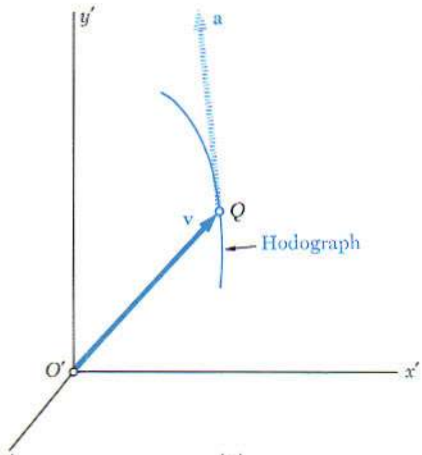
Fig. 11.14



(a)



(b)



(c)

Consider the velocity $\mathbf{v}$ of the particle at time t and also its velocity $\mathbf{v}'$ at a later time $t + \Delta t$ (Fig. 11.15*a*). Let us draw both vectors $\mathbf{v}$ and $\mathbf{v}'$ from the same origin O' (Fig. 11.15*b*). The vector $\Delta\mathbf{v}$ joining Q and Q' represents the change in the velocity of the particle during the time interval Δt, since the vector $\mathbf{v}'$ may be obtained by adding the vectors $\mathbf{v}$ and $\Delta\mathbf{v}$. We should note that $\Delta\mathbf{v}$ represents a change in the *direction* of the velocity as well as a change in *speed*. The *average acceleration* of the particle over the time interval Δt is defined as the quotient of $\Delta\mathbf{v}$ and Δt. Since $\Delta\mathbf{v}$ is a vector and Δt a scalar, the quotient $\Delta\mathbf{v}/\Delta t$ is a vector of the same direction as $\Delta\mathbf{v}$.

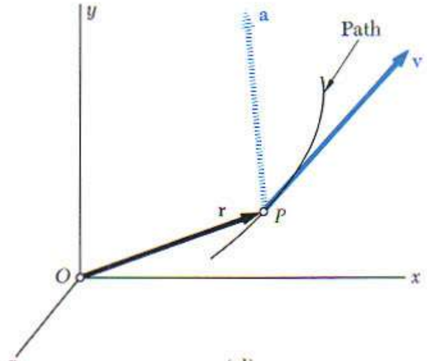
The *instantaneous acceleration* of the particle at time t is obtained by choosing smaller and smaller values for Δt and $\Delta\mathbf{v}$. The instantaneous acceleration is thus represented by the vector

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{v}}{\Delta t} \qquad (11.17)$$

Noting that the velocity $\mathbf{v}$ is a vector function $\mathbf{v}(t)$ of the time t, we may refer to the limit of the quotient $\Delta\mathbf{v}/\Delta t$ as the derivative of $\mathbf{v}$ with respect to t. We write

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \qquad (11.18)$$

We observe that the acceleration $\mathbf{a}$ is tangent to the curve described by the tip Q of the vector $\mathbf{v}$ when the latter is drawn from a fixed origin O' (Fig. 11.15*c*) and that, in general, the acceleration is *not* tangent to the path of the particle (Fig. 11.15*d*). The curve described by the tip of $\mathbf{v}$ and shown in Fig. 11.15*c* is called the *hodograph* of the motion.



(d)

Fig. 11.15

11.10. Derivatives of Vector Functions. We saw in the preceding section that the velocity $\mathbf{v}$ of a particle in curvilinear motion may be represented by the derivative of the vector function $\mathbf{r}(t)$ characterizing the position of the particle. Similarly, the acceleration $\mathbf{a}$ of the particle may be represented by the derivative of the vector function $\mathbf{v}(t)$. In this section, we shall give a formal definition of the derivative of a vector function and establish a few rules governing the differentiation of sums and products of vector functions.

Let $\mathbf{P}(u)$ be a vector function of the scalar variable u. By that we mean that the scalar u completely defines the magnitude and direction of the vector $\mathbf{P}$. If the vector $\mathbf{P}$ is drawn from a fixed origin O and the scalar u is allowed to vary, the tip of $\mathbf{P}$ will describe a given curve in space. Consider the vectors $\mathbf{P}$ corresponding respectively to the values u and $u + \Delta u$ of the scalar variable (Fig. 11.16*a*). Let $\Delta\mathbf{P}$ be the vector joining the tips of the two given vectors; we write

$$\Delta\mathbf{P} = \mathbf{P}(u + \Delta u) - \mathbf{P}(u)$$

Dividing through by Δu and letting Δu approach zero, *we define the derivative of the vector function* $\mathbf{P}(u)$:

$$\frac{d\mathbf{P}}{du} = \lim_{\Delta u \to 0} \frac{\Delta\mathbf{P}}{\Delta u} = \lim_{\Delta u \to 0} \frac{\mathbf{P}(u + \Delta u) - \mathbf{P}(u)}{\Delta u} \tag{11.19}$$

As Δu approaches zero, the line of action of $\Delta\mathbf{P}$ becomes tangent to the curve of Fig. 11.16*a*. Thus, the derivative $d\mathbf{P}/du$ of the vector function $\mathbf{P}(u)$ *is tangent to the curve described by the tip of* $\mathbf{P}(u)$ (Fig. 11.16*b*).

We shall now show that the standard rules for the differentiation of the sums and products of scalar functions may be extended to vector functions. Consider first the *sum of two vector functions* $\mathbf{P}(u)$ and $\mathbf{Q}(u)$ of the same scalar variable u. According to the definition given in (11.19), the derivative of the vector $\mathbf{P} + \mathbf{Q}$ is

$$\frac{d(\mathbf{P} + \mathbf{Q})}{du} = \lim_{\Delta u \to 0} \frac{\Delta(\mathbf{P} + \mathbf{Q})}{\Delta u} = \lim_{\Delta u \to 0} \left(\frac{\Delta\mathbf{P}}{\Delta u} + \frac{\Delta\mathbf{Q}}{\Delta u}\right)$$

or, since the limit of a sum is equal to the sum of the limits of its terms,

$$\frac{d(\mathbf{P} + \mathbf{Q})}{du} = \lim_{\Delta u \to 0} \frac{\Delta\mathbf{P}}{\Delta u} + \lim_{\Delta u \to 0} \frac{\Delta\mathbf{Q}}{\Delta u}$$

$$\frac{d(\mathbf{P} + \mathbf{Q})}{du} = \frac{d\mathbf{P}}{du} + \frac{d\mathbf{Q}}{du} \tag{11.20}$$

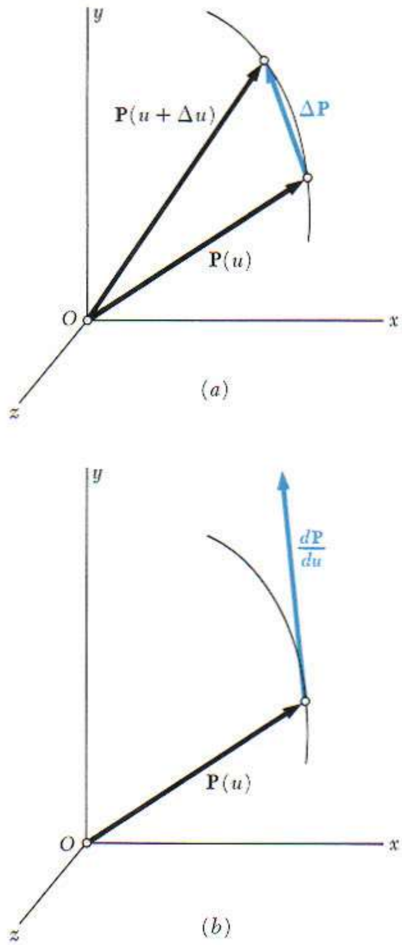


Fig. 11.16

Next, we shall consider the *product of a scalar function* $f(u)$ *and of a vector function* $\mathbf{P}(u)$ of the same scalar variable u. The derivative of the vector $f\mathbf{P}$ is

$$\frac{d(f\mathbf{P})}{du} = \lim_{\Delta u \to 0} \frac{(f + \Delta f)(\mathbf{P} + \Delta \mathbf{P}) - f\mathbf{P}}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \left(\frac{\Delta f}{\Delta u}\mathbf{P} + f\frac{\Delta \mathbf{P}}{\Delta u} \right)$$

or, recalling the properties of the limits of sums and products,

$$\frac{d(f\mathbf{P})}{du} = \frac{df}{du}\mathbf{P} + f\frac{d\mathbf{P}}{du} \tag{11.21}$$

The derivatives of the *scalar product* and of the *vector product* of two vector functions $\mathbf{P}(u)$ and $\mathbf{Q}(u)$ may be obtained in a similar way. We have

$$\frac{d(\mathbf{P} \cdot \mathbf{Q})}{du} = \frac{d\mathbf{P}}{du} \cdot \mathbf{Q} + \mathbf{P} \cdot \frac{d\mathbf{Q}}{du} \tag{11.22}$$

$$\frac{d(\mathbf{P} \times \mathbf{Q})}{du} = \frac{d\mathbf{P}}{du} \times \mathbf{Q} + \mathbf{P} \times \frac{d\mathbf{Q}}{du} \tag{11.23}†$$

We shall use the properties established above to determine the *rectangular components of the derivative of a vector function* $\mathbf{P}(u)$. Resolving $\mathbf{P}$ into components along fixed rectangular axes x, y, z, we write

$$\mathbf{P} = P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k} \tag{11.24}$$

where P_x, P_y, P_z are the rectangular scalar components of the vector $\mathbf{P}$, and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ the unit vectors corresponding respectively to the x, y, and z axes (Sec. 2.11). By (11.20), the derivative of $\mathbf{P}$ is equal to the sum of the derivatives of the terms in the right-hand member. Since each of these terms is the product of a scalar and a vector function, we should use (11.21). But the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ have a constant magnitude (equal to 1) and fixed directions. Their derivatives are therefore zero, and we write

$$\frac{d\mathbf{P}}{du} = \frac{dP_x}{du}\mathbf{i} + \frac{dP_y}{du}\mathbf{j} + \frac{dP_z}{du}\mathbf{k} \tag{11.25}$$

† Since the vector product is not commutative (Sec. 3.3), the order of the factors in (11.23) must be maintained.

Noting that the coefficients of the unit vectors are, by definition, the scalar components of the vector $d\mathbf{P}/du$, we conclude that *the rectangular scalar components of the derivative* $d\mathbf{P}/du$ *of the vector function* $\mathbf{P}(u)$ *are obtained by differentiating the corresponding scalar components of* $\mathbf{P}$.

Rate of Change of a Vector. When the vector $\mathbf{P}$ is a function of the time t, its derivative $d\mathbf{P}/dt$ represents the *rate of change* of $\mathbf{P}$ with respect to the frame $Oxyz$. Resolving $\mathbf{P}$ into rectangular components, we have, by (11.25),

$$\frac{d\mathbf{P}}{dt} = \frac{dP_x}{dt}\mathbf{i} + \frac{dP_y}{dt}\mathbf{j} + \frac{dP_z}{dt}\mathbf{k}$$

or, using dots to indicate differentiation with respect to t,

$$\dot{\mathbf{P}} = \dot{P}_x\mathbf{i} + \dot{P}_y\mathbf{j} + \dot{P}_z\mathbf{k} \tag{11.25'}$$

As we shall see in Sec. 15.10, the rate of change of a vector, as observed from a *moving frame of reference,* is, in general, different from its rate of change as observed from a fixed frame of reference. However, if the moving frame $O'x'y'z'$ is in *translation,* i.e., if its axes remain parallel to the corresponding axes of the fixed frame $Oxyz$ (Fig. 11.17), the same unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are

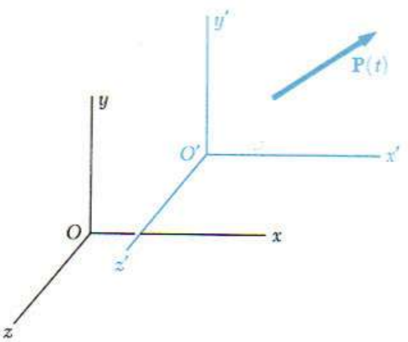


Fig. 11.17

used in both frames, and the vector $\mathbf{P}$ has, at any given instant, the same components P_x, P_y, P_z in both frames. It follows from (11.25′) that the rate of change $\dot{\mathbf{P}}$ is the same with respect to the frames $Oxyz$ and $O'x'y'z'$. We state, therefore: *The rate of change of a vector is the same with respect to a fixed frame and with respect to a frame in translation.* This property will greatly simplify our work, since we shall deal mainly with frames in translation.

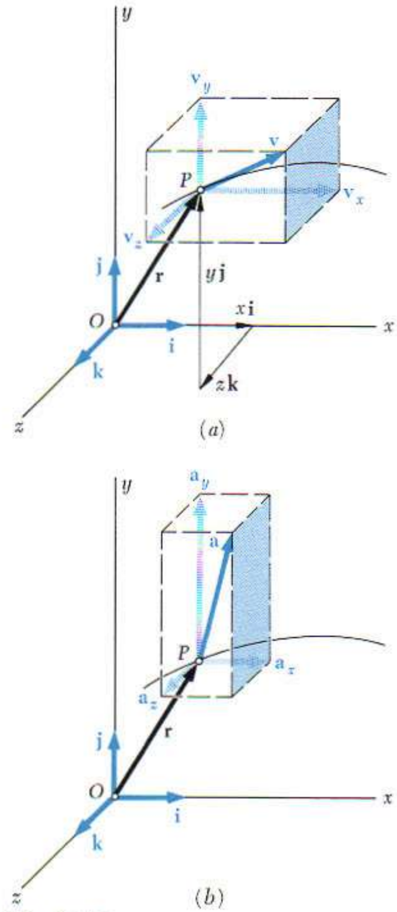


Fig. 11.18

11.11. Rectangular Components of Velocity and Acceleration. When the position of a particle P is defined at any instant by its rectangular coordinates x, y, and z, it is convenient to resolve the velocity $\mathbf{v}$ and the acceleration $\mathbf{a}$ of the particle into rectangular components (Fig. 11.18).

Resolving the position vector $\mathbf{r}$ of the particle into rectangular components, we write

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{11.26}$$

where the coordinates x, y, z are functions of t. Differentiating twice, we obtain

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \tag{11.27}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k} \tag{11.28}$$

where $\dot{x}$, $\dot{y}$, $\dot{z}$ and $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ represent, respectively, the first and second derivatives of x, y, and z with respect to t. It follows from (11.27) and (11.28) that the scalar components of the velocity and acceleration are

$$v_x = \dot{x} \qquad v_y = \dot{y} \qquad v_z = \dot{z} \tag{11.29}$$

$$a_x = \ddot{x} \qquad a_y = \ddot{y} \qquad a_z = \ddot{z} \tag{11.30}$$

A positive value for v_x indicates that the vector component $\mathbf{v}_x$ is directed to the right, a negative value that it is directed to the left; the sense of each of the other vector components may be determined in a similar way from the sign of the corresponding scalar component. If desired, the magnitudes and directions of the velocity and acceleration may be obtained from their scalar components by the methods of Secs. 2.6 and 2.11.

The use of rectangular components to describe the position, the velocity, and the acceleration of a particle is particularly effective when the component a_x of the acceleration depends only upon t, x, and/or v_x, and when, similarly, a_y depends only upon t, y, and/or v_y, and a_z upon t, z, and/or v_z. Equations (11.30) may then be integrated independently, and so may Eqs. (11.29). In other words, the motion of the particle in the x direction, its motion in the y direction, and its motion in the z direction may be considered separately.

In the case of the *motion of a projectile,* for example, it may be shown (see Sec. 12.4) that the components of the acceleration are

$$a_x = \ddot{x} = 0 \qquad a_y = \ddot{y} = -g \qquad a_z = \ddot{z} = 0$$

if the resistance of the air is neglected. Denoting by x_0, y_0, z_0 the coordinates of the gun, and by $(v_x)_0$, $(v_y)_0$, $(v_z)_0$ the components of the initial velocity $\mathbf{v}_0$ of the projectile, we integrate twice in t and obtain

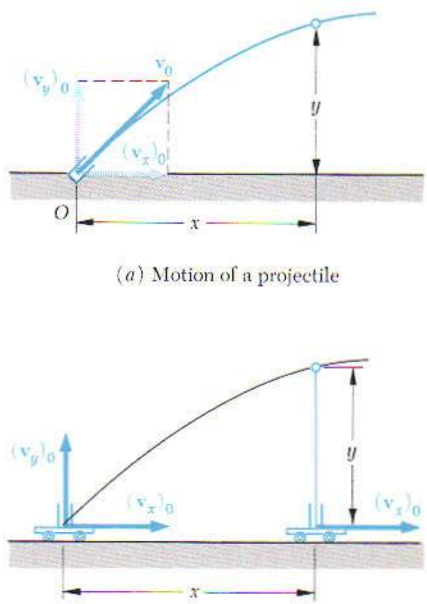
$$\begin{array}{lll} v_x = \dot{x} = (v_x)_0 & v_y = \dot{y} = (v_y)_0 - gt & v_z = \dot{z} = (v_z)_0 \\ x = x_0 + (v_x)_0 t & y = y_0 + (v_y)_0 t - \frac{1}{2}gt^2 & z = z_0 + (v_z)_0 t \end{array}$$

If the projectile is fired in the xy plane from the origin O, we have $x_0 = y_0 = z_0 = 0$ and $(v_z)_0 = 0$, and the equations of motion reduce to

$$\begin{array}{lll} v_x = (v_x)_0 & v_y = (v_y)_0 - gt & v_z = 0 \\ x = (v_x)_0 t & y = (v_y)_0 t - \frac{1}{2}gt^2 & z = 0 \end{array}$$

These equations show that the projectile remains in the xy plane and that its motion in the horizontal direction is uniform, while its motion in the vertical direction is uniformly accelerated. The motion of a projectile may thus be replaced by two independent rectilinear motions, which are easily visualized if we assume that the projectile is fired vertically with an initial velocity $(\mathbf{v}_y)_0$ from a platform moving with a constant horizontal velocity $(\mathbf{v}_x)_0$ (Fig. 11.19). The coordinate x of the projectile is equal at any instant to the distance traveled by the platform, while its coordinate y may be computed as if the projectile were moving along a vertical line.

It may be observed that the equations defining the coordinates x and y of a projectile at any instant are the parametric equations of a parabola. Thus, the trajectory of a projectile is *parabolic*. This result, however, ceases to be valid when the resistance of the air or the variation with altitude of the acceleration of gravity is taken into account.



(a) Motion of a projectile

(b) Equivalent rectilinear motions

Fig. 11.19

11.12. Motion Relative to a Frame in Translation.

In the preceding section, a single frame of reference was used to describe the motion of a particle. In most cases this frame was attached to the earth and was considered as fixed. We shall now analyze situations in which it is convenient to use simultaneously several frames of reference. If one of the frames is attached to the earth, we shall call it a *fixed frame of reference* and the other frames will be referred to as *moving frames of reference*. It should be understood, however, that the selection of a fixed frame of reference is purely arbitrary. Any frame may be designated as "fixed"; all other frames not rigidly attached to this frame will then be described as "moving."

Consider two particles A and B moving in space (Fig. 11.20); the vectors $\mathbf{r}_A$ and $\mathbf{r}_B$ define their positions at any given instant with respect to the fixed frame of reference $Oxyz$. Consider now

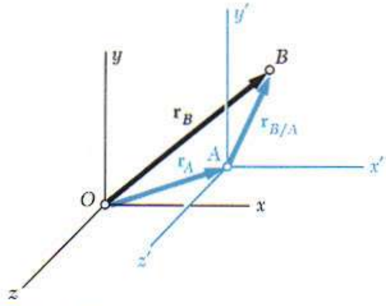


Fig. 11.20

a system of axes x', y', z' centered at A and parallel to the x, y, z axes. While the origin of these axes moves, their orientation remains the same; the frame of reference $Ax'y'z'$ is in *translation* with respect to $Oxyz$. The vector $\mathbf{r}_{B/A}$ joining A and B defines *the position of B relative to the moving frame* $Ax'y'z'$ (or, for short, *the position of B relative to A*).

We note from Fig. 11.20 that the position vector $\mathbf{r}_B$ of particle B is the sum of the position vector $\mathbf{r}_A$ of particle A and of the position vector $\mathbf{r}_{B/A}$ of B relative to A; we write

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A} \tag{11.31}$$

Differentiating (11.31) with respect to t within the fixed frame of reference, and using dots to indicate time derivatives, we have

$$\dot{\mathbf{r}}_B = \dot{\mathbf{r}}_A + \dot{\mathbf{r}}_{B/A} \tag{11.32}$$

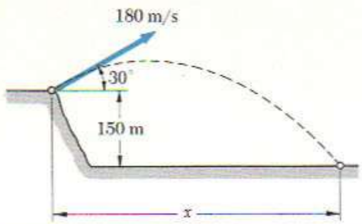
The derivatives $\dot{\mathbf{r}}_A$ and $\dot{\mathbf{r}}_B$ represent, respectively, the velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ of the particles A and B. The derivative $\dot{\mathbf{r}}_{B/A}$ represents the rate of change of $\mathbf{r}_{B/A}$ with respect to the frame $Ax'y'z'$, as well as with respect to the fixed frame, since $Ax'y'z'$ is in translation (Sec. 11.10). This derivative, therefore, defines *the velocity* $\mathbf{v}_{B/A}$ *of B relative to the frame* $Ax'y'z'$ (or, for short, *the velocity* $\mathbf{v}_{B/A}$ *of B relative to A*). We write

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \tag{11.33}$$

Differentiating Eq. (11.33) with respect to t, and using the derivative $\dot{\mathbf{v}}_{B/A}$ to define *the acceleration* $\mathbf{a}_{B/A}$ *of B relative to the frame* $Ax'y'z'$ (or, for short, *the acceleration of* $\mathbf{a}_{B/A}$ *of B relative to A*), we write

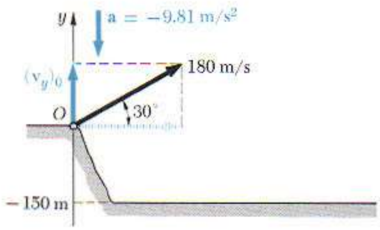
$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \tag{11.34}$$

The motion of B with respect to the fixed frame $Oxyz$ is referred to as the *absolute motion of B*. The equations derived in this section show that *the absolute motion of B may be obtained by combining the motion of A and the relative motion of B with respect to the moving frame attached to A*. Equation (11.33), for example, expresses that the absolute velocity $\mathbf{v}_B$ of particle B may be obtained by adding vectorially the velocity of A and the velocity of B relative to the frame $Ax'y'z'$. Equation (11.34) expresses a similar property in terms of the accelerations. We should keep in mind, however, that *the frame $Ax'y'z'$ is in translation;* i.e., while it moves with A, it maintains the same orientation. As we shall see later (Sec. 15.14), different relations must be used in the case of a rotating frame of reference.



SAMPLE PROBLEM 11.7

A projectile is fired from the edge of a 150-m cliff with an initial velocity of 180 m/s, at an angle of 30° with the horizontal. Neglecting air resistance, find (*a*) the horizontal distance from the gun to the point where the projectile strikes the ground, (*b*) the greatest elevation above the ground reached by the projectile.



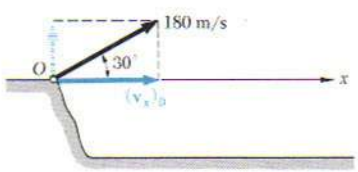
Solution. We shall consider separately the vertical and the horizontal motion.

Vertical Motion. Uniformly accelerated motion. Choosing the positive sense of the y axis upward and placing the origin O at the gun, we have

$$(v_y)_0 = (180 \text{ m/s}) \sin 30° = +90 \text{ m/s}$$
$$a = -9.81 \text{ m/s}^2$$

Substituting into the equations of uniformly accelerated motion, we have

$$v_y = (v_y)_0 + at \qquad v_y = 90 - 9.81t \tag{1}$$
$$y = (v_y)_0 t + \tfrac{1}{2}at^2 \qquad y = 90t - 4.90t^2 \tag{2}$$
$$v_y^2 = (v_y)_0^2 + 2ay \qquad v_y^2 = 8100 - 19.62y \tag{3}$$



Horizontal Motion. Uniform motion. Choosing the positive sense of the x axis to the right, we have

$$(v_x)_0 = (180 \text{ m/s}) \cos 30° = +155.9 \text{ m/s}$$

Substituting into the equation of uniform motion, we obtain

$$x = (v_x)_0 t \qquad x = 155.9t \tag{4}$$

a. Horizontal Distance. When the projectile strikes the ground, we have

$$y = -150 \text{ m}$$

Carrying this value into Eq. (2) for the vertical motion, we write

$$-150 = 90t - 4.90t^2 \qquad t^2 - 18.37t - 30.6 = 0 \qquad t = 19.91 \text{ s}$$

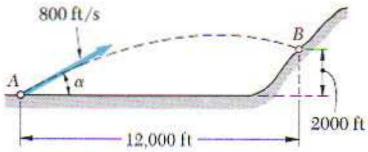
Carrying $t = 19.91$ s into Eq. (4) for the horizontal motion, we obtain

$$x = 155.9(19.91) \qquad x = 3100 \text{ m} \blacktriangleleft$$

b. Greatest Elevation. When the projectile reaches its greatest elevation, we have $v_y = 0$; carrying this value into Eq. (3) for the vertical motion, we write

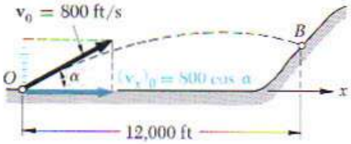
$$0 = 8100 - 19.62y \qquad y = 413 \text{ m}$$

$$\text{Greatest elevation above ground} = 150 \text{ m} + 413 \text{ m}$$
$$= 563 \text{ m} \blacktriangleleft$$



SAMPLE PROBLEM 11.8

A projectile is fired with an initial velocity of 800 ft/s at a target B located 2000 ft above the gun A and at a horizontal distance of 12,000 ft. Neglecting air resistance, determine the value of the firing angle α.



Solution. We shall consider separately the horizontal and the vertical motion.

Horizontal Motion. Placing the origin of coordinates at the gun, we have

$$(v_x)_0 = 800 \cos \alpha$$

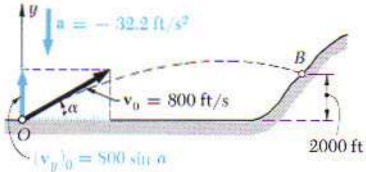
Substituting into the equation of uniform horizontal motion, we obtain

$$x = (v_x)_0 t \qquad x = (800 \cos \alpha)t$$

The time required for the projectile to move through a horizontal distance of 12,000 ft is obtained by making x equal to 12,000 ft.

$$12{,}000 = (800 \cos \alpha)t$$

$$t = \frac{12{,}000}{800 \cos \alpha} = \frac{15}{\cos \alpha}$$



Vertical Motion

$$(v_y)_0 = 800 \sin \alpha \qquad a = -32.2 \text{ ft/s}^2$$

Substituting into the equation of uniformly accelerated vertical motion, we obtain

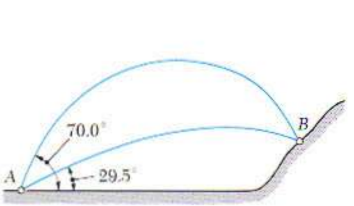
$$y = (v_y)_0 t + \tfrac{1}{2}at^2 \qquad y = (800 \sin \alpha)t - 16.1t^2$$

Projectile Hits Target. When $x = 12{,}000$ ft, we must have $y = 2000$ ft. Substituting for y and making t equal to the value found above, we write

$$2000 = 800 \sin \alpha \frac{15}{\cos \alpha} - 16.1\left(\frac{15}{\cos \alpha}\right)^2$$

Since $1/\cos^2 \alpha = \sec^2 \alpha = 1 + \tan^2 \alpha$, we have

$$2000 = 800(15) \tan \alpha - 16.1(15^2)(1 + \tan^2 \alpha)$$
$$3622 \tan^2 \alpha - 12{,}000 \tan \alpha + 5622 = 0$$

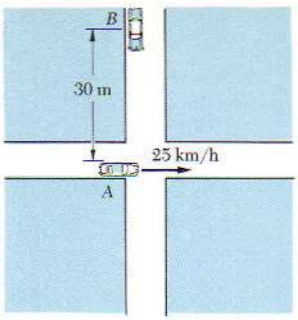


Solving this quadratic equation for $\tan \alpha$, we have

$$\tan \alpha = 0.565 \qquad \text{and} \qquad \tan \alpha = 2.75$$

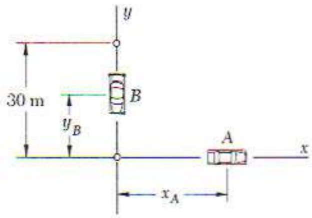
$$\alpha = 29.5° \qquad \text{and} \qquad \alpha = 70.0° \quad ◄$$

The target will be hit if either of these two firing angles is used (see figure).



SAMPLE PROBLEM 11.9

Automobile A is traveling east at the constant speed of 25 km/h. As automobile A crosses the intersection shown, automobile B starts from rest 30 m north of the intersection and moves south with a constant acceleration of 1.2 m/s^2. Determine the position, velocity, and acceleration of B relative to A five seconds after A crosses the intersection.



Solution. We choose x and y axes with origin at the intersection of the two streets and with positive senses directed respectively east and north.

Motion of Automobile A. First the speed is expressed in m/s:

$$25\text{ km/h} = \frac{25\text{ km}}{1\text{ h}} = \frac{25\,000\text{ m}}{3600\text{ s}} = 6.94\text{ m/s}$$

Noting that the motion of A is uniform, we write, for any time t,

$$\begin{aligned} a_A &= 0 \\ v_A &= +6.94\text{ m/s} \\ x_A &= (x_A)_0 + v_A t = 0 + 6.94t \end{aligned}$$

For $t = 5$ s, we have

$$\begin{aligned} a_A &= 0 & \mathbf{a}_A &= 0 \\ v_A &= +6.94\text{ m/s} & \mathbf{v}_A &= 6.94\text{ m/s} \rightarrow \\ x_A &= +(6.94\text{ m/s})(5\text{ s}) = +34.7\text{ m} & \mathbf{r}_A &= 34.7\text{ m} \rightarrow \end{aligned}$$

Motion of Automobile B. We note that the motion of B is uniformly accelerated, and write

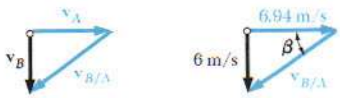
$$\begin{aligned} a_B &= -1.2\text{ m/s}^2 \\ v_B &= (v_B)_0 + at = 0 - 1.2t \\ y_B &= (y_B)_0 + (v_B)_0 t + \tfrac{1}{2}a_B t^2 = 30 + 0 - \tfrac{1}{2}(1.2)t^2 \end{aligned}$$

For $t = 5$ s, we have

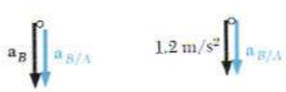
$$\begin{aligned} a_B &= -1.2\text{ m/s}^2 & \mathbf{a}_B &= 1.2\text{ m/s}^2 \downarrow \\ v_B &= -(1.2\text{ m/s})(5\text{ s}) = -6\text{ m/s} & \mathbf{v}_B &= 6\text{ m/s} \downarrow \\ y_B &= 30 - \tfrac{1}{2}(1.2\text{ m/s})(5\text{ s})^2 = +15\text{ m} & \mathbf{r}_B &= 15\text{ m} \uparrow \end{aligned}$$

Motion of B Relative to A. We draw the triangle corresponding to the vector equation $\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}$ and obtain the magnitude and direction of the position vector of B relative to A.

$r_{B/A} = 37.8\text{ m} \qquad \alpha = 23.4° \qquad \mathbf{r}_{B/A} = 37.8\text{ m} \searrow 23.4°$ ◄



Proceeding in a similar fashion, we find the velocity and acceleration of B relative to A.



$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

$v_{B/A} = 9.17\text{ m/s} \qquad \beta = 40.8° \qquad \mathbf{v}_{B/A} = 9.17\text{ m/s} \swarrow 40.8°$ ◄

$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \qquad \mathbf{a}_{B/A} = 1.2\text{ m/s}^2 \downarrow$ ◄

PROBLEMS

Note. Neglect air resistance in problems concerning projectiles.

11.67 The motion of a particle is defined by the equations $x = \frac{1}{3}t^3 - 2t^2$ and $y = \frac{1}{2}t^2 - 2t$, where x and y are expressed in meters and t in seconds. Determine the velocity and acceleration when (*a*) $t = 1$ s, (*b*) $t = 3$ s.

11.68 In Prob. 11.67, determine (*a*) the time at which the value of the y coordinate is minimum, (*b*) the corresponding velocity and acceleration of the particle.

11.69 The motion of a particle is defined by the equations $x = e^{t/2}$ and $y = e^{-t/2}$, where x and y are expressed in feet and t in seconds. Show that the path of the particle is a rectangular hyperbola and determine the velocity and acceleration when (*a*) $t = 0$, (*b*) $t = 1$ s.

11.70 The motion of a particle is defined by the equations $x = 5(1 - e^{-t})$ and $y = 5t/(t + 1)$, where x and y are expressed in feet and t in seconds. Determine the velocity and acceleration when $t = 1$ s.

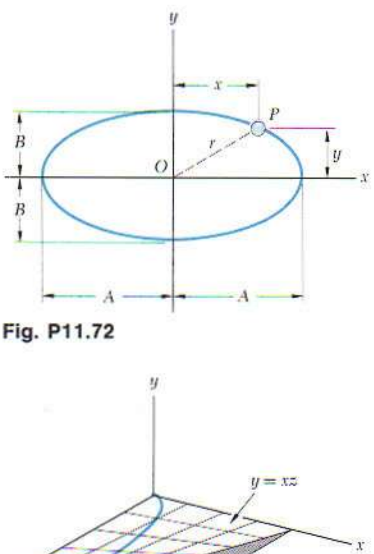


Fig. P11.72

11.71 The motion of a vibrating particle is defined by the position vector $\mathbf{r} = (100 \sin \pi t)\mathbf{i} + (25 \cos 2\pi t)\mathbf{j}$, where r is expressed in millimeters and t in seconds. (*a*) Determine the velocity and acceleration when $t = 1$ s. (*b*) Show that the path of the particle is parabolic.

11.72 A particle moves in an elliptic path defined by the position vector $\mathbf{r} = (A \cos pt)\mathbf{i} + (B \sin pt)\mathbf{j}$. Show that the acceleration (*a*) is directed toward the origin, (*b*) is proportional to the distance from the origin to the particle.

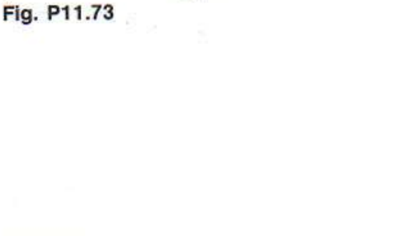


Fig. P11.73

11.73 The three-dimensional motion of a particle is defined by the position vector $\mathbf{r} = At\mathbf{i} + ABt^3\mathbf{j} + Bt^2\mathbf{k}$, where r is expressed in feet and t in seconds. Show that the space curve described by the particle lies on the hyperbolic paraboloid $y = xz$. For $A = B = 1$, determine the magnitudes of the velocity and acceleration when (*a*) $t = 0$, (*b*) $t = 2$ s.

11.74 The three-dimensional motion of a particle is defined by the position vector $\mathbf{r} = (R \sin pt)\mathbf{i} + ct\mathbf{j} + (R \cos pt)\mathbf{k}$. Determine the magnitudes of the velocity and acceleration of the particle. (The space curve described by the particle is a helix.)

11.75 A man standing on a bridge 20 m above the water throws a stone in a horizontal direction. Knowing that the stone hits the water 30 m from a point on the water directly below the man, determine (*a*) the initial velocity of the stone, (*b*) the distance at which the stone would hit the water if it were thrown with the same velocity from a bridge 5 m lower.

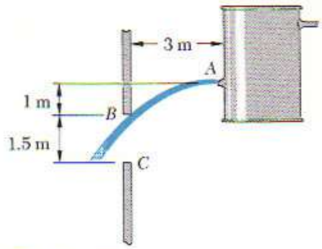


Fig. P11.76

11.76 Water issues at A from a pressure tank with a horizontal velocity $\mathbf{v}_0$. For what range of values v_0 will the water enter the opening BC?

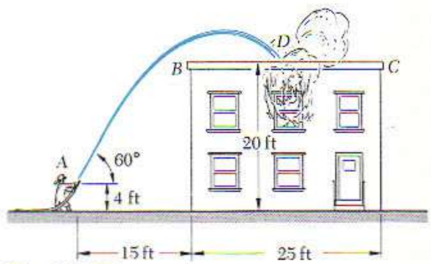


Fig. P11.77

11.77 A nozzle at A discharges water with an initial velocity of 40 ft/s at an angle of 60° with the horizontal. Determine where the stream of water strikes the roof. Check that the stream will clear the edge of the roof.

11.78 In Prob. 11.77, determine the largest and smallest initial velocity for which the water will fall on the roof.

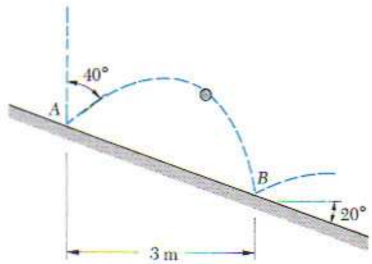


Fig. P11.79

11.79 A ball is dropped vertically onto a 20° incline at A; the direction of rebound forms an angle of 40° with the vertical. Knowing that the ball next strikes the incline at B, determine (*a*) the velocity of rebound at A, (*b*) the time required for the ball to travel from A to B.

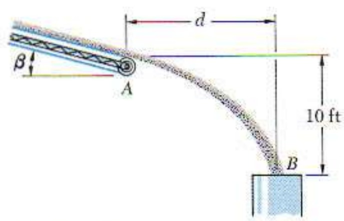


Fig. P11.80

11.80 Sand is discharged at A from a conveyor belt and falls into a collection pipe at B. Knowing that the conveyor belt forms an angle $\beta = 15°$ with the horizontal and moves at a constant speed of 20 ft/s, determine what the distance d should be so that the sand will hit the center of the pipe.

11.81 The conveyor belt moves at a constant speed of 12 ft/s. Knowing that $d = 8$ ft, determine the angle β for which the sand reaches the center of the pipe B.

11.82 A projectile is fired with an initial velocity of 210 m/s. Find the angle at which it should be fired if it is to hit a target located at a distance of 3600 m on the same level.

11.83 A boy can throw a baseball a maximum distance of 30 m in New York, where $g = 9.81\ \mathrm{m/s^2}$. How far could he throw the baseball (*a*) in Singapore, where $g = 9.78\ \mathrm{m/s^2}$? (*b*) On the moon, where $g = 1.618\ \mathrm{m/s^2}$?

11.84 If the maximum horizontal range of a given gun is R, determine the firing angle which should be used to hit a target located at a distance $\frac{1}{2}R$ on the same level.

11.85 A projectile is fired with an initial velocity $\mathbf{v}_0$ at an angle α with the horizontal. Determine (*a*) the maximum height h reached by the projectile, (*b*) the horizontal range R of the projectile, (*c*) the maximum horizontal range R and the corresponding firing angle α.

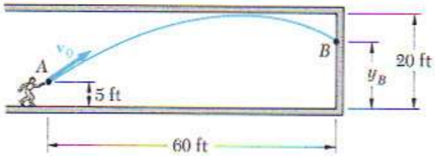


Fig. P11.86

11.86 A player throws a ball with an initial velocity $\mathbf{v}_0$ of 50 ft/s from a point A located 5 ft above the floor. Knowing that the ceiling of the gymnasium is 20 ft high, determine the highest point B at which the ball can strike the wall 60 ft away.

11.87 A fire nozzle discharges water with an initial velocity $\mathbf{v}_0$ of 80 ft/s. Knowing that the nozzle is located 100 ft from a building, determine (*a*) the maximum height h that can be reached by the water, (*b*) the corresponding angle α.

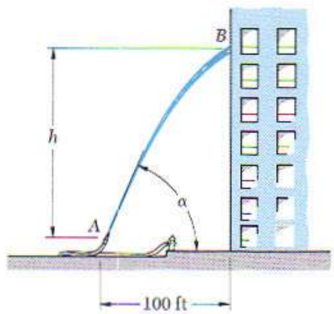


Fig. P11.87

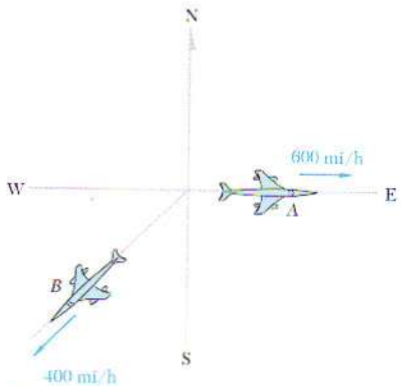


Fig. P11.88

11.88 Two airplanes A and B are each flying at a constant altitude; plane A is flying due east at a constant speed of 600 mi/h while plane B is flying southwest at a constant speed of 400 mi/h. Determine the change in position of plane B relative to plane A which takes place during a 1.5-min interval.

11.89 Instruments in an airplane indicate that, with respect to the air, the plane is moving east at a speed of 350 mi/h. At the same time ground-based radar indicates the plane to be moving at a speed of 325 mi/h in a direction 8° north of east. Determine the magnitude and direction of the velocity of the air.

11.90 As he passes a pole, a man riding in a truck tries to hit the pole by throwing a stone with a horizontal velocity of 20 m/s relative to the truck. Knowing that the speed of the truck is 40 km/h, determine (*a*) the direction in which he must throw the stone, (*b*) the horizontal velocity of the stone with respect to the ground.

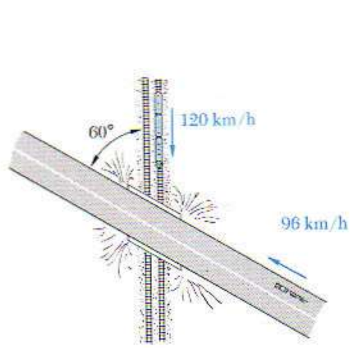


Fig. P11.91

11.91 An automobile and a train travel at the constant speeds shown. Three seconds after the train passes under the highway bridge the automobile crosses the bridge. Determine (*a*) the velocity of the train relative to the automobile, (*b*) the change in position of the train relative to the automobile during a 4-s interval, (*c*) the distance between the train and the automobile 5 s after the automobile has crossed the bridge.

11.92 During a rainstorm the paths of the raindrops appear to form an angle of 30° with the vertical when observed from a side window of a train moving at a speed of 15 km/h. A short time later, after the speed of the train has increased to 30 km/h, the angle between the vertical and the paths of the drops appears to be 45°. If the train were stopped, at what angle and with what velocity would the drops be observed to fall?

11.93 As the speed of the train of Prob. 11.92 increases, the angle between the vertical and the paths of the drops becomes equal to 60°. Determine the speed of the train at that time.

11.94 As observed from a ship moving due south at 10 mi/h, the wind appears to blow from the east. After the ship has changed course, and as it is moving due west at 10 mi/h, the wind appears to blow from the northeast. Assuming that the wind velocity is constant during the period of observation, determine the magnitude and direction of the true wind velocity.

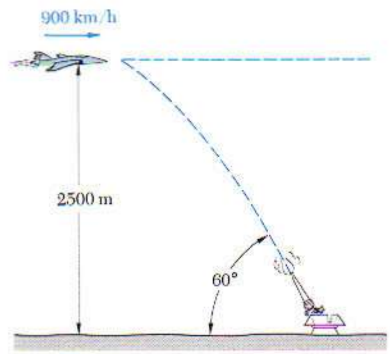


Fig. P11.95

11.95 An airplane is flying horizontally at an altitude of 2500 m and at a constant speed of 900 km/h on a path which passes directly over an antiaircraft gun. The gun fires a shell with a muzzle velocity of 500 m/s and hits the airplane. Knowing that the firing angle of the gun is 60°, determine the velocity and acceleration of the shell relative to the airplane at the time of impact.

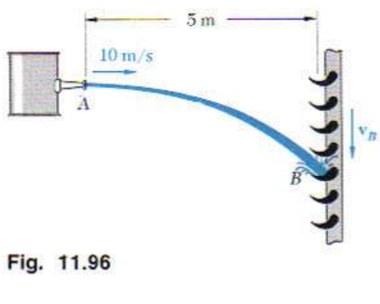


Fig. 11.96

11.96 Water is discharged at A with an initial velocity of 10 m/s and strikes a series of vanes at B. Knowing that the vanes move downward with a constant speed of 3 m/s, determine the velocity and acceleration of the water relative to the vane at B.

11.13. Tangential and Normal Components. We saw in Sec. 11.9 that the velocity of a particle is a vector tangent to the path of the particle but that, in general, the acceleration is not tangent to the path. It is sometimes convenient to resolve the acceleration into components directed, respectively, along the tangent and the normal to the path of the particle.

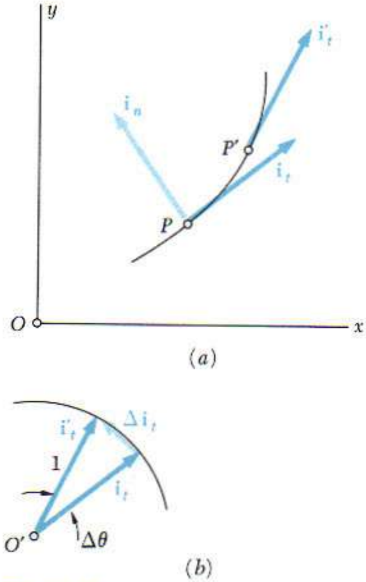


Fig. 11.21

Plane Motion of a Particle. We shall first consider a particle which moves along a curve contained in the plane of the figure. Let P be the position of the particle at a given instant. We attach at P a unit vector $\mathbf{i}_t$ tangent to the path of the particle and pointing toward the direction of motion (Fig. 11.21*a*). Let $\mathbf{i}'_t$ be the unit vector corresponding to the position P' of the particle at a later instant. Drawing both vectors from the same origin O', we define the vector $\Delta\mathbf{i}_t = \mathbf{i}'_t - \mathbf{i}_t$ (Fig. 11.21*b*). Since $\mathbf{i}_t$ and $\mathbf{i}'_t$ are of unit length, their tips lie on a circle of radius 1. Denoting by $\Delta\theta$ the angle formed by $\mathbf{i}_t$ and $\mathbf{i}'_t$, we find that the magnitude of $\Delta\mathbf{i}_t$ is $2 \sin(\Delta\theta/2)$. Considering now the vector $\Delta\mathbf{i}_t/\Delta\theta$, we note that, as $\Delta\theta$ approaches zero, this vector becomes tangent to the unit circle of Fig. 11.21*b*, i.e., perpendicular to $\mathbf{i}_t$, and that its magnitude approaches

$$\lim_{\Delta\theta \to 0} \frac{2 \sin(\Delta\theta/2)}{\Delta\theta} = \lim_{\Delta\theta \to 0} \frac{\sin(\Delta\theta/2)}{\Delta\theta/2} = 1$$

Thus, the vector obtained at the limit is a unit vector along the normal to the path of the particle, in the direction toward which $\mathbf{i}_t$ turns. Denoting this vector by $\mathbf{i}_n$, we write

$$\mathbf{i}_n = \lim_{\Delta\theta \to 0} \frac{\Delta\mathbf{i}_t}{\Delta\theta} \qquad \mathbf{i}_n = \frac{d\mathbf{i}_t}{d\theta} \tag{11.35}$$

Since the velocity $\mathbf{v}$ of the particle is tangent to the path, we may express it as the product of the scalar v and the unit vector $\mathbf{i}_t$. We have

$$\mathbf{v} = v\mathbf{i}_t \tag{11.36}$$

To obtain the acceleration of the particle, we shall differentiate (11.36) with respect to t. Applying the rule for the differentiation of the product of a scalar and a vector function (Sec. 11.10), we write

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{dv}{dt}\mathbf{i}_t + v\frac{d\mathbf{i}_t}{dt} \tag{11.37}$$

But

$$\frac{d\mathbf{i}_t}{dt} = \frac{d\mathbf{i}_t}{d\theta}\frac{d\theta}{ds}\frac{ds}{dt}$$

i_n $\frac{1}{\rho}$ v

Recalling from (11.16) that $ds/dt = v$, from (11.35) that $d\mathbf{i}_t/d\theta = \mathbf{i}_n$, and from elementary calculus that $d\theta/ds$ is equal to $1/\rho$, where ρ is the radius of curvature of the path at P (Fig. 11.22), we have

$$\omega = \frac{d\mathbf{i}_t}{dt} = \frac{v}{\rho}\mathbf{i}_n \quad \therefore v = \omega\rho \tag{11.38}$$

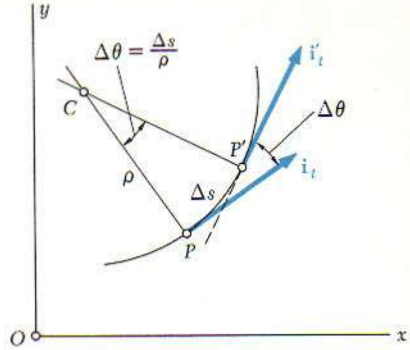


Fig. 11.22

Substituting into (11.37), we obtain

$$\mathbf{a} = \frac{dv}{dt}\mathbf{i}_t + \frac{v^2}{\rho}\mathbf{i}_n \tag{11.39}$$

Thus, the scalar components of the acceleration are

$$a_t = \frac{dv}{dt} \qquad a_n = \frac{v^2}{\rho} \tag{11.40}$$

The relations obtained express that the *tangential component* of the acceleration is equal to the *rate of change of the speed of the particle*, while the *normal component* is equal to the *square of the speed divided by the radius of curvature of the path at P.* Depending upon whether the speed of the particle increases or decreases, a_t is positive or negative, and the vector component $\mathbf{a}_t$ points in the direction of motion or against the direction of motion. The vector component $\mathbf{a}_n$, on the other hand, *is always directed toward the center of curvature C of the path* (Fig. 11.23).

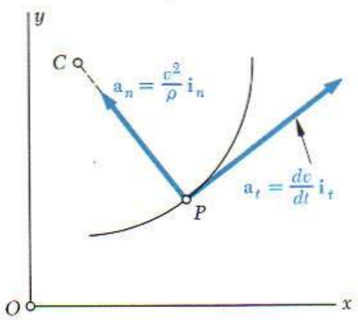


Fig. 11.23

It appears from the above that the tangential component of the acceleration reflects a change in the speed of the particle, while its normal component reflects a change in the direction of motion of the particle. The acceleration of a particle will be zero only if both its components are zero. Thus, the acceleration of a particle moving with constant speed along a curve will not be zero, unless the particle happens to pass through a point of inflection of the curve (where the radius of curvature is infinite) or unless the curve is a straight line.

The fact that the normal component of the acceleration depends upon the radius of curvature of the path followed by the particle is taken into account in the design of structures or mechanisms as widely different as airplane wings, railroad tracks, and cams. In order to avoid sudden changes in the acceleration of the air particles flowing past a wing, wing profiles are designed without any sudden change in curvature. Similar care is taken in designing railroad curves, to avoid sudden changes in the acceleration of the cars (which would be hard on the equipment and unpleasant for the passengers). A straight section of track, for instance, is never directly followed by a circular section. Special transition sections are used, to help pass smoothly from the infinite radius of curvature of the straight section to the finite radius of the circular track. Likewise, in the design of high-speed cams, abrupt changes in acceleration are avoided by using transition curves which produce a continuous change in acceleration.

Motion of a Particle in Space. The relations (11.39) and (11.40) still hold in the case of a particle moving along a space curve. However, since there is an infinite number of straight lines which are perpendicular to the tangent at a given point P of a space curve, it is then necessary to define more precisely the direction of the unit vector $\mathbf{i}_n$.

Let us consider again the unit vectors $\mathbf{i}_t$ and $\mathbf{i}'_t$ tangent to the path of the particle at two neighboring points P and P' (Fig. 11.24*a*) and the vector $\Delta\mathbf{i}_t$ representing the difference between $\mathbf{i}_t$ and $\mathbf{i}'_t$ (Fig. 11.24*b*). Let us now imagine a plane through P (Fig. 11.24*a*), parallel to the plane defined by the vectors $\mathbf{i}_t$, $\mathbf{i}'_t$, and $\Delta\mathbf{i}_t$ (Fig. 11.24*b*). This plane contains the tangent to the curve at P and is parallel to the tangent at P'. If we let P' approach P, we obtain at the limit the plane which fits the curve most closely in the neighborhood of P. This plane is called the *osculating plane* at P.† It follows from this definition that the osculating plane contains the unit vector $\mathbf{i}_n$, since this vector represents the limit of the vector $\Delta\mathbf{i}_t/\Delta\theta$. The normal defined by $\mathbf{i}_n$ is thus contained in the osculating plane; it is called the *principal normal* at P. The unit vector $\mathbf{i}_b = \mathbf{i}_t \times \mathbf{i}_n$ which completes the right-handed triad $\mathbf{i}_t$, $\mathbf{i}_n$, $\mathbf{i}_b$ (Fig. 11.24*c*) defines the *binormal* at P. The binormal is thus perpendicular to the osculating plane. We conclude that, as stated in (11.39), the acceleration of the particle at P may be resolved into two components, one along the tangent, the other along the principal normal at P. The acceleration has no component along the binormal.

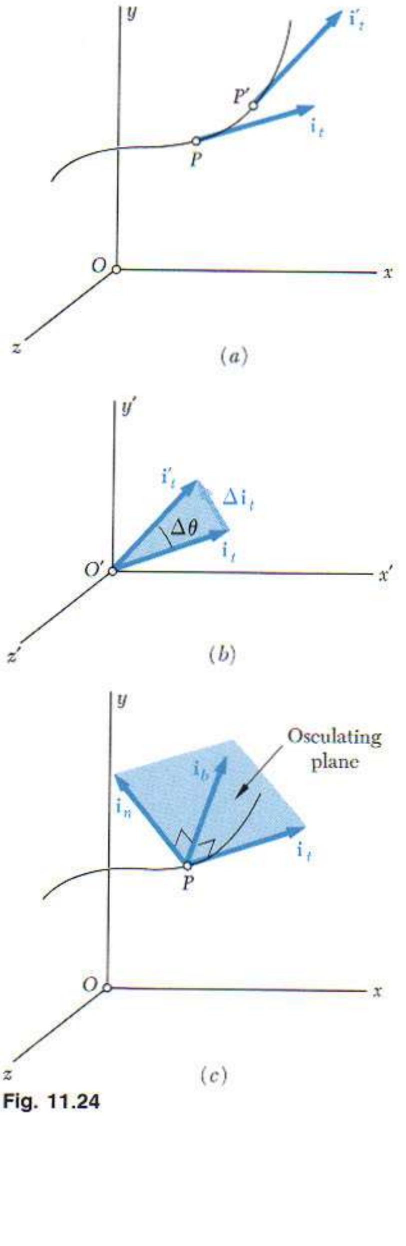


Fig. 11.24

†From the Latin *osculari*, to embrace.

11.14. Radial and Transverse Components.

In certain problems of plane motion, the position of the particle P is defined by its polar coordinates r and θ (Fig. 11.25*a*). It is then convenient to resolve the velocity and acceleration of the particle into components parallel and perpendicular, respectively, to the line OP. These components are called *radial* and *transverse components*.

We attach at P two unit vectors, $\mathbf{i}_r$ and $\mathbf{i}_\theta$ (Fig. 11.25*b*). The vector $\mathbf{i}_r$ is directed along OP and the vector $\mathbf{i}_\theta$ is obtained by rotating $\mathbf{i}_r$ through 90° counterclockwise. The unit vector $\mathbf{i}_r$ defines the *radial* direction, i.e., the direction in which P would move if r were increased and θ kept constant; the unit vector $\mathbf{i}_\theta$ defines the *transverse* direction, i.e., the direction in which P would move if θ were increased and r kept constant. A derivation similar to the one we used in Sec. 11.13 to determine the derivative of the unit vector $\mathbf{i}_t$ leads to the relations

$$\frac{d\mathbf{i}_r}{d\theta} = \mathbf{i}_\theta \qquad \frac{d\mathbf{i}_\theta}{d\theta} = -\mathbf{i}_r \tag{11.41}$$

where $-\mathbf{i}_r$ denotes a unit vector of sense opposite to that of $\mathbf{i}_r$ (Fig. 11.25*c*).

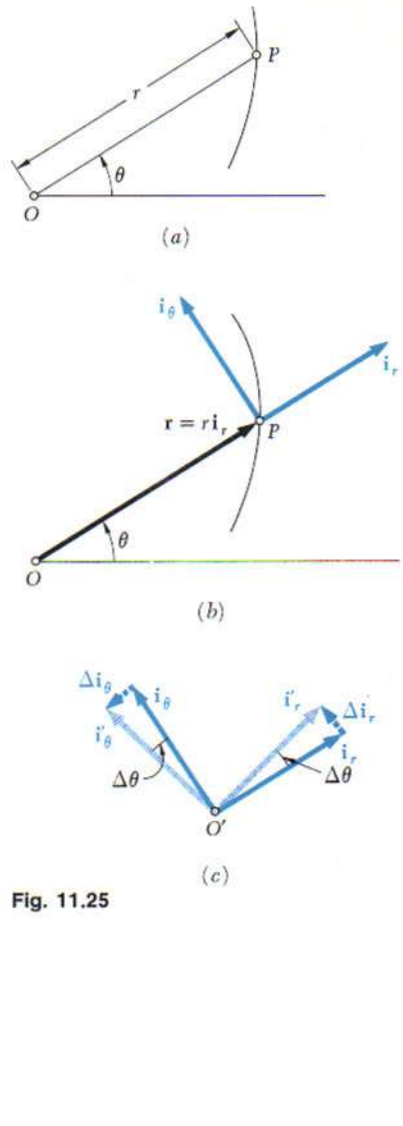


Fig. 11.25

Expressing the position vector $\mathbf{r}$ of the particle P as the product of the scalar r and the unit vector $\mathbf{i}_r$, and differentiating with respect to t, we write

$$\mathbf{r} = r\mathbf{i}_r \tag{11.42}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{i}_r + r\frac{d\mathbf{i}_r}{dt}$$

$$= \frac{dr}{dt}\mathbf{i}_r + r\frac{d\theta}{dt}\frac{d\mathbf{i}_r}{d\theta}$$

Recalling the first of the relations (11.41), and using dots to indicate time derivatives, we have

$$\mathbf{v} = \dot{r}\mathbf{i}_r + r\dot{\theta}\mathbf{i}_\theta \tag{11.43}$$

Differentiating again with respect to t, we write

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{r}\mathbf{i}_r + \dot{r}\frac{d\mathbf{i}_r}{dt} + \dot{r}\dot{\theta}\mathbf{i}_\theta + r\ddot{\theta}\mathbf{i}_\theta + r\dot{\theta}\frac{d\mathbf{i}_\theta}{dt}$$

or, since $d\mathbf{i}_r/dt = \dot{\theta}\mathbf{i}_\theta$ and $d\mathbf{i}_\theta/dt = -\dot{\theta}\mathbf{i}_r$,

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{i}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{i}_\theta \tag{11.44}$$

The scalar components of the velocity and acceleration in the radial and transverse directions are therefore

$$v_r = \dot{r} \qquad v_\theta = r\dot{\theta} \tag{11.45}$$

$$a_r = \ddot{r} - r\dot{\theta}^2 \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \tag{11.46}$$

It is important to note that a_r is *not* equal to the time derivative of v_r, and that a_θ is *not* equal to the time derivative of v_θ.

In the case of a particle moving along a circle of center O, we have $r =$ constant, $\dot{r} = \ddot{r} = 0$, and the formulas (11.43) and (11.44) reduce, respectively, to

$$\mathbf{v} = r\dot{\theta}\mathbf{i}_\theta \qquad \mathbf{a} = -r\dot{\theta}^2\mathbf{i}_r + r\ddot{\theta}\mathbf{i}_\theta \tag{11.47}$$

Extension to the Motion of a Particle in Space: Cylindrical Coordinates. The position of a particle P in space is sometimes defined by its cylindrical coordinates R, θ, and z (Fig. 11.26*a*). It is then convenient to use the unit vectors $\mathbf{i}_R$, $\mathbf{i}_\theta$, and $\mathbf{k}$

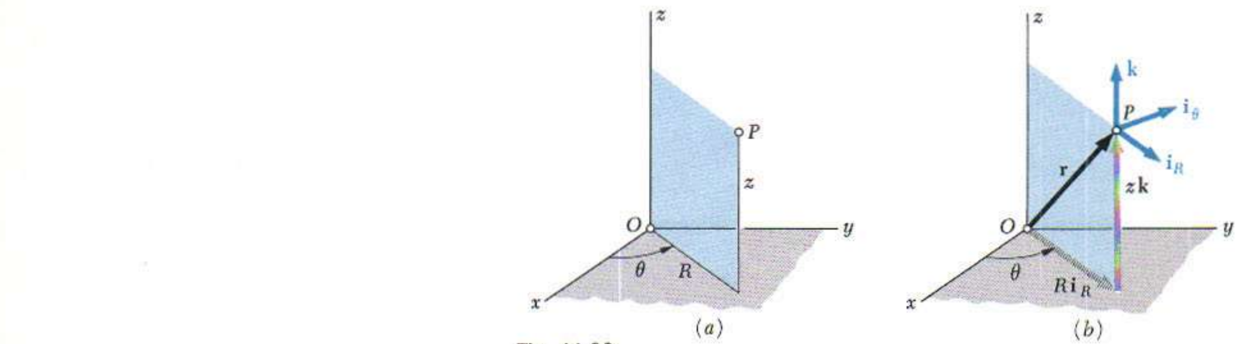


Fig. 11.26

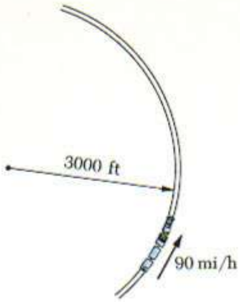
shown in Fig. 11.26*b*. Resolving the position vector $\mathbf{r}$ of the particle P into components along the unit vectors, we write

$$\mathbf{r} = R\mathbf{i}_R + z\mathbf{k} \tag{11.48}$$

Observing that $\mathbf{i}_R$ and $\mathbf{i}_\theta$ define, respectively, the radial and transverse direction in the horizontal xy plane, and that the vector $\mathbf{k}$, which defines the *axial* direction, is constant in direction as well as in magnitude, we easily verify that

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{R}\mathbf{i}_R + R\dot{\theta}\mathbf{i}_\theta + \dot{z}\mathbf{k} \tag{11.49}$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = (\ddot{R} - R\dot{\theta}^2)\mathbf{i}_R + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\mathbf{i}_\theta + \ddot{z}\mathbf{k} \tag{11.50}$$



SAMPLE PROBLEM 11.10

A train is traveling on a curved section of track of radius 3000 ft at the speed of 90 mi/h. The brakes are suddenly applied, causing the train to slow down at a constant rate; after 6 s, the speed has been reduced to 60 mi/h. Determine the acceleration of a car immediately after the brakes have been applied.

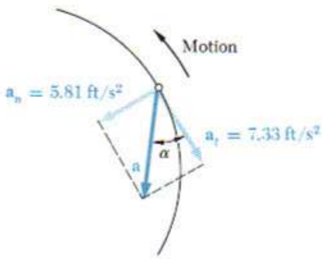
Tangential Component of Acceleration. First the speeds are expressed in ft/s.

$$90 \text{ mi/h} = \left(90 \frac{\text{mi}}{\text{h}}\right)\left(\frac{5280 \text{ ft}}{1 \text{ mi}}\right)\left(\frac{1 \text{ h}}{3600 \text{ s}}\right) = 132 \text{ ft/s}$$

$$60 \text{ mi/h} = 88 \text{ ft/s}$$

Since the train slows down at a constant rate, we have

$$a_t = \text{average } a_t = \frac{\Delta v}{\Delta t} = \frac{88 \text{ ft/s} - 132 \text{ ft/s}}{6 \text{ s}} = -7.33 \text{ ft/s}^2$$



Normal Component of Acceleration. Immediately after the brakes have been applied, the speed is still 132 ft/s, and we have

$$a_n = \frac{v^2}{\rho} = \frac{(132 \text{ ft/s})^2}{3000 \text{ ft}} = 5.81 \text{ ft/s}^2$$

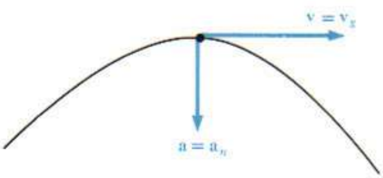
Magnitude and Direction of Acceleration. The magnitude and direction of the resultant **a** of the components $\mathbf{a}_n$ and $\mathbf{a}_t$ are

$$\tan \alpha = \frac{a_n}{a_t} = \frac{5.81 \text{ ft/s}^2}{7.33 \text{ ft/s}^2} \qquad \alpha = 38.4° \quad ◄$$

$$a = \frac{a_n}{\sin \alpha} = \frac{5.81 \text{ ft/s}^2}{\sin 38.4°} \qquad a = 9.35 \text{ ft/s}^2 \quad ◄$$

SAMPLE PROBLEM 11.11

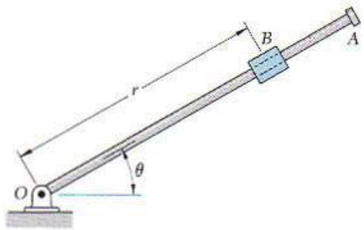
Determine the minimum radius of curvature of the trajectory described by the projectile considered in Sample Prob. 11.7.



Solution. Since $a_n = v^2/\rho$, we have $\rho = v^2/a_n$. The radius will be small when v is small or when a_n is large. The speed v is minimum at the top of the trajectory since $v_y = 0$ at that point; a_n is maximum at that same point, since the direction of the vertical coincides with the direction of the normal. Therefore, the minimum radius of curvature occurs at the top of the trajectory. At this point, we have

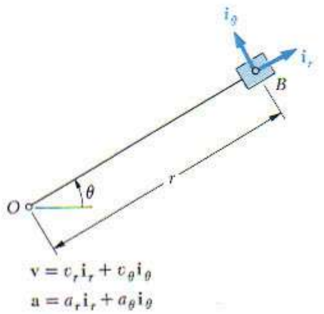
$$v = v_x = 155.9 \text{ m/s} \qquad a_n = a = 9.81 \text{ m/s}^2$$

$$\rho = \frac{v^2}{a_n} = \frac{(155.9 \text{ m/s})^2}{9.81 \text{ m/s}^2} \qquad \rho = 2480 \text{ m} \quad ◄$$



SAMPLE PROBLEM 11.12

The rotation of the 3-ft arm OA about O is defined by the relation $\theta = 0.15t^2$, where θ is expressed in radians and t in seconds. Block B slides along the arm in such a way that its distance from O is $r = 3 - 0.40t^2$, where r is expressed in feet and t in seconds. Determine the total velocity and the total acceleration of block B after the arm OA has rotated through 30°.



Solution. We first find the time t at which $\theta = 30°$. Substituting $\theta = 30° = 0.524$ rad into the expression for θ, we obtain

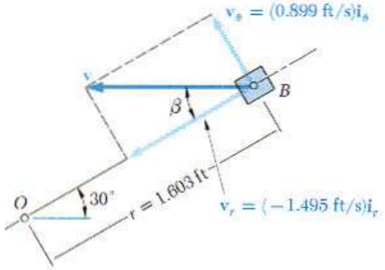
$$\theta = 0.15t^2 \qquad 0.524 = 0.15t^2 \qquad t = 1.869 \text{ s}$$

Equations of Motion. Substituting $t = 1.869$ s in the expressions for r, θ, and their first and second derivatives, we have

$$\begin{aligned} r &= 3 - 0.40t^2 = 1.603 \text{ ft} & \theta &= 0.15t^2 = 0.524 \text{ rad} \\ \dot{r} &= -0.80t = -1.495 \text{ ft/s} & \dot{\theta} &= 0.30t = 0.561 \text{ rad/s} \\ \ddot{r} &= -0.80 = -0.800 \text{ ft/s}^2 & \ddot{\theta} &= 0.30 = 0.300 \text{ rad/s}^2 \end{aligned}$$

Velocity of B. Using Eqs. (11.45), we obtain the values of v_r and v_θ when $t = 1.869$ s.

$$\begin{aligned} v_r &= \dot{r} = -1.495 \text{ ft/s} \\ v_\theta &= r\dot{\theta} = 1.603(0.561) = 0.899 \text{ ft/s} \end{aligned}$$



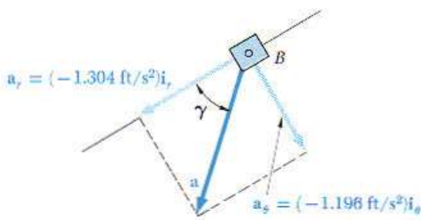
Solving the right triangle shown, we obtain the magnitude and direction of the velocity,

$$v = 1.744 \text{ ft/s} \qquad \beta = 31.0° \quad ◄$$

Acceleration of B. Using Eqs. (11.46), we obtain

$$\begin{aligned} a_r &= \ddot{r} - r\dot{\theta}^2 \\ &= -0.800 - 1.603(0.561)^2 = -1.304 \text{ ft/s}^2 \\ a_\theta &= r\ddot{\theta} + 2\dot{r}\dot{\theta} \\ &= 1.603(0.300) + 2(-1.495)(0.561) = -1.196 \text{ ft/s}^2 \end{aligned}$$

$$a = 1.770 \text{ ft/s} \qquad \gamma = 42.5° \quad ◄$$



PROBLEMS

11.97 An automobile travels at a constant speed on a highway curve of 1000-m radius. If the normal component of the acceleration is not to exceed $1.2\ m/s^2$, determine the maximum allowable speed.

11.98 A car goes around a highway curve of 300-m radius at a speed of 90 km/h. (*a*) What is the normal component of its acceleration? (*b*) At what speed is the normal component of the acceleration one-half as large as that found in part *a*?

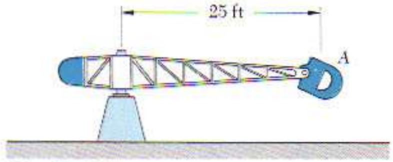


Fig. P11.99

11.99 Determine the peripheral speed of the centrifuge test cab A for which the normal component of the acceleration is $10g$.

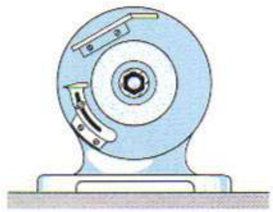


Fig. P11.100

11.100 A small grinding wheel has a 5-in. diameter and is attached to the shaft of an electric motor which has a rated speed of 3600 rpm. Determine the normal component of the acceleration of a point on the circumference of the wheel when the wheel is rotating at the rated speed.

11.101 A motorist starts from rest on a curve of 400-ft radius and accelerates at the uniform rate of $3\ ft/s^2$. Determine the distance that his automobile will travel before the magnitude of its total acceleration is $6\ ft/s^2$.

11.102 A motorist enters a curve of 500-ft radius at a speed of 45 mi/h. As he applies his brakes, he decreases his speed at a constant rate of $5\ ft/s^2$. Determine the magnitude of the total acceleration of the automobile when its speed is 40 mi/h.

11.103 The speed of a racing car is increased at a constant rate from 90 km/h to 126 km/h over a distance of 150 m along a curve of 250-m radius. Determine the magnitude of the total acceleration of the car after it has traveled 100 m along the curve.

11.104 A monorail train is traveling at a speed of 144 km/h along a curve of 1000-m radius. Determine the maximum rate at which the speed may be decreased if the total acceleration of the train is not to exceed $2\ m/s^2$.

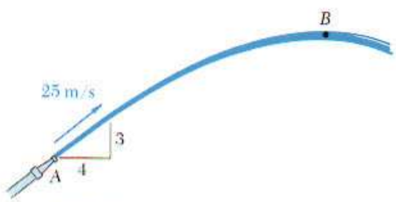


Fig. P11.105

11.105 A nozzle discharges a stream of water in the direction shown with an initial velocity of 25 m/s. Determine the radius of curvature of the stream (*a*) as it leaves the nozzle, (*b*) at the maximum height of the stream.

11.106 Determine the radius of curvature of the trajectory described by the projectile of Sample Prob. 11.7 as the projectile leaves the gun.

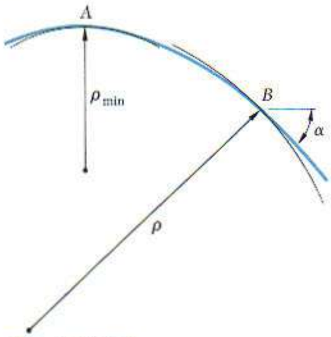


Fig. P11.107

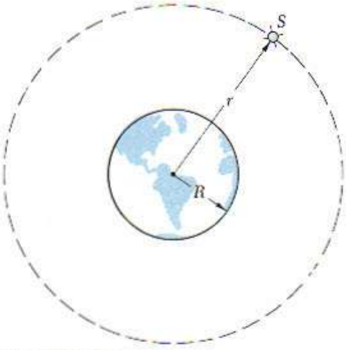


Fig. P11.111

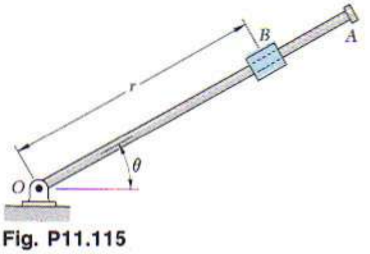


Fig. P11.115

11.107 (*a*) Show that the radius of curvature of the trajectory of a projectile reaches its minimum value at the highest point A of the trajectory. (*b*) Denoting by α the angle formed by the trajectory and the horizontal at a given point B, show that the radius of curvature of the trajectory at B is $\rho = \rho_{\text{min}}/\cos^3 \alpha$.

11.108 For each of the two firing angles obtained in Sample Prob. 11.8, determine the radius of curvature of the trajectory described by the projectile as it leaves the gun.

***11.109** Determine the radius of curvature of the path described by the particle of Prob. 11.73 when (*a*) $t = 0$, (*b*) $t = 2$ s.

***11.110** Determine the radius of curvature of the helix of Prob. 11.74.

11.111 A satellite will travel indefinitely in a circular orbit around the earth if the normal component of its acceleration is equal to $g(R/r)^2$, where $g = 32.2$ ft/s^2, R = radius of the earth = 3960 mi, and r = distance from the center of the earth to the satellite. Determine the height above the surface of the earth at which a satellite will travel indefinitely around the earth at a speed of 15,000 mi/h.

11.112 Determine the speed of an earth satellite traveling in a circular orbit 300 mi above the surface of the earth. (See information given in Prob. 11.111.)

11.113 Assuming the orbit of the moon to be a circle of radius 239,000 mi, determine the speed of the moon relative to the earth. (See information given in Prob. 11.111.)

11.114 Show that the speed of an earth satellite traveling in a circular orbit is inversely proportional to the square root of the radius of its orbit. Also, determine the minimum time in which a satellite can circle the earth. (See information given in Prob. 11.111.)

11.115 The two-dimensional motion of a particle is defined by the relations $r = 60t^2 - 20t^3$ and $\theta = 2t^2$, where r is expressed in millimeters, t in seconds, and θ in radians. Determine the velocity and acceleration of the particle when (*a*) $t = 0$, (*b*) $t = 1$ s.

11.116 The particle of Prob. 11.115 is at the origin at $t = 0$. Determine its velocity and acceleration as it returns to the origin.

11.117 The two-dimensional motion of a particle is defined by the relations $r = 2b \sin \omega t$ and $\theta = \omega t$, where b and ω are constants. Determine (*a*) the velocity and acceleration of the particle at any instant, (*b*) the radius of curvature of its path. What conclusion can you draw regarding the path of the particle?

11.118 As circle B rolls on the fixed circle A, point P describes a cardioid defined by the relations $r = 2b(1 + \cos 2\pi t)$ and $\theta = 2\pi t$. Determine the velocity and acceleration of P when (a) $t = 0.25$, (b) $t = 0.50$.

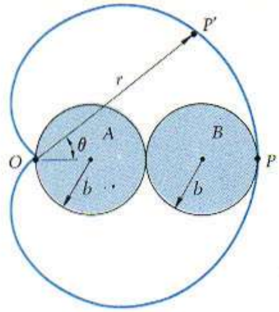


Fig. P11.118

11.119 A wire OA connects the collar A and a reel located at O. Knowing that the collar moves to the right with a constant speed v_0, determine $d\theta/dt$ in terms of v_0, b, and θ.

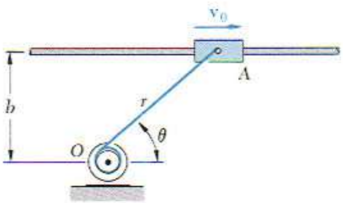


Fig. P11.119

11.120 A rocket is fired vertically from a launching pad at B. Its flight is tracked by radar from point A. Determine the velocity of the rocket in terms of b, θ, and $\dot{\theta}$.

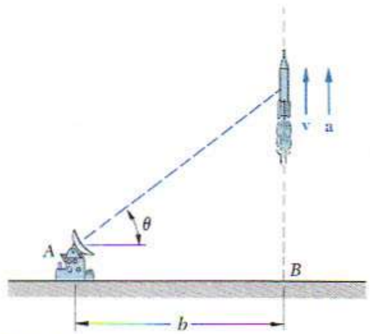


Fig. P11.120

11.121 Determine the acceleration of the rocket of Prob. 11.120 in terms of b, θ, $\dot{\theta}$, and $\ddot{\theta}$.

11.122 As the rod OA rotates, the pin P moves along the parabola BCD. Knowing that the equation of the parabola is $r = 2b/(1 + \cos\theta)$ and that $\theta = kt$, determine the velocity and acceleration of P when (a) $\theta = 0$, (b) $\theta = 90°$.

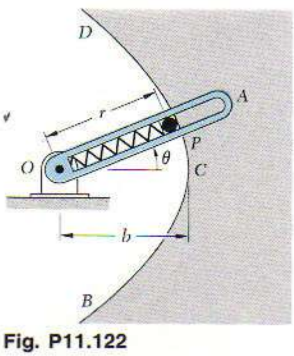


Fig. P11.122

11.123 The pin at B is free to slide along the circular slot and along the rotating rod OC. If pin B slides counterclockwise around the circular slot at a constant speed v_0, determine the rate $d\theta/dt$ at which rod OC rotates and the radial component v_r of the velocity of the pin B (*a*) when $\phi = 0°$, (*b*) when $\phi = 90°$.

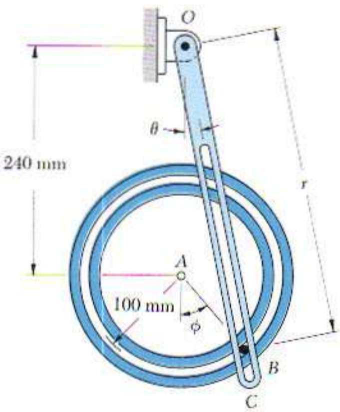


Fig. P11.123

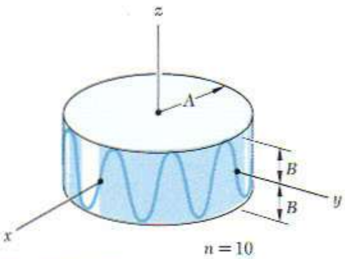


Fig. P11.124

11.124 The motion of a particle on the surface of a right circular cylinder is defined by the relations $R = A$, $\theta = 2\pi t$, and $z = B \sin 2\pi nt$, where A and B are constants and n is an integer. Determine the magnitudes of the velocity and acceleration of the particle at any time t.

11.125 For the case when $n = 1$ in Prob. 11.124, (*a*) show that the path of the particle is contained in a plane, (*b*) determine the maximum and minimum radii of curvature of the path.

11.126 The motion of a particle on the surface of a right circular cone is defined by the relations $R = ht \tan \beta$, $\theta = 2\pi t$, and $z = ht$, where β is the apex angle of the cone and h is the distance the particle rises in one passage around the cone. Determine the magnitudes of the velocity and acceleration at any time t.

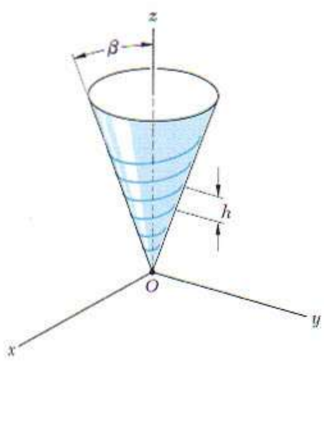


Fig. P11.126

11.127 The three-dimensional motion of a particle is defined by the relations $R = A$, $\theta = 2\pi t$, and $z = A \sin^2 2\pi t$. Determine the magnitudes of the velocity and acceleration at any time t.

***11.128** For the helix of Prob. 11.74, determine the angle that the osculating plane forms with the y axis.

***11.129** Determine the direction of the binormal of the path described by the particle of Prob. 11.73 when (*a*) $t = 0$, (*b*) $t = 2$ s.

***11.130** The position vector of a particle is defined by the relation

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

where x, y, z are known functions of the time t, and **i**, **j**, **k** are unit vectors along fixed rectangular axes. Express in terms of the functions x, y, z and their first and second derivatives (*a*) the tangential component of the acceleration of the particle, (*b*) the normal component of its acceleration, (*c*) the radius of curvature of the path described by the particle.

***11.131** For the particle of Prob. 11.130, express the direction cosines of (*a*) the tangent, (*b*) the binormal, (*c*) the principal normal of the path described by the particle, in terms of the functions x, y, z and their first and second derivatives.

REVIEW PROBLEMS

11.132 The a–t curve shown was obtained during the motion of a test sled. Knowing that the sled started from rest at $t = 0$, determine the velocity and position of the sled at $t = 0.08$ s.

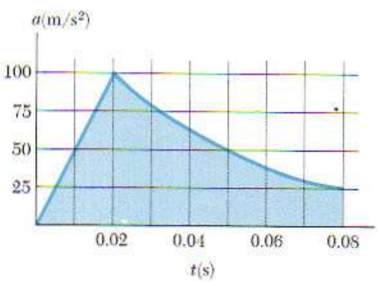


Fig. P11.132

11.133 An experimental ion-propulsion engine is capable of giving a space vehicle a constant acceleration of 0.01 ft/s^2. If the engine is placed in operation when the speed of the vehicle is 21,000 mi/h, determine the time required to bring the speed of the vehicle to 22,000 mi/h. Assume that the vehicle is moving in a straight line, far from the sun or any planet.

11.134 The velocity of a particle is given by the relation $v = 100 - 10x$, where v is expressed in meters per second and x in meters. Knowing that $x = 0$ at $t = 0$, determine (*a*) the distance traveled before the particle comes to rest, (*b*) the time t when $x = 5$ m, (*c*) the acceleration at $t = 0$.

11.135 A nozzle discharges a stream of water with an initial velocity $\mathbf{v}_0$ of 50 ft/s into the end of a horizontal pipe of inside diameter $d = 5$ ft. Determine the largest distance x that the stream can reach.

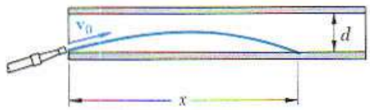


Fig. P11.135

11.136 The magnitude in m/s^2 of the deceleration due to air resistance of the nose cone of a small experimental rocket is known to be $6 \times 10^{-4}\, v^2$, where v is expressed in m/s. If the nose cone is projected vertically from the ground with an initial velocity of 100 m/s, determine the maximum height that it will reach.

11.137 Determine the velocity of the nose cone of Prob. 11.136 when it returns to the ground.

11.138 Standing on the side of a hill, an archer shoots an arrow with an initial velocity of 250 ft/s at an angle $\alpha = 15°$ with the horizontal. Determine the horizontal distance d traveled by the arrow before it strikes the ground at B.

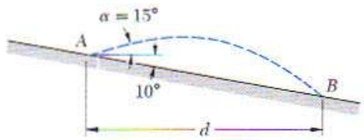


Fig. P11.138

11.139 In Prob. 11.138, determine the radius of curvature of the trajectory (*a*) immediately after the arrow has been shot, (*b*) as the arrow passes through its point of maximum elevation.

11.140 A train starts at a station and accelerates uniformly at a rate of $0.6\ \text{m/s}^2$ until it reaches a speed of 24 m/s; it then proceeds at the constant speed of 24 m/s. Determine the time and the distance traveled if its average velocity is (*a*) 16 m/s, (*b*) 22 m/s.

11.141 A man jumps from a 20-ft cliff with no initial velocity. (*a*) How long does it take him to reach the ground, and with what velocity does he hit the ground? (*b*) If this takes place on the moon, where $g = 5.31\ \text{ft/s}^2$, what are the values obtained for the time and velocity? (*c*) If a motion picture is taken on the earth, but if the scene is supposed to take place on the moon, how many frames per second should be used so that the scene would appear realistic when projected at the standard speed of 24 frames per second?

11.142 Drops of water fall down a mine shaft at the uniform rate of one drop per second. A mine elevator moving up the shaft at 30 ft/s is struck by a drop of water when it is 300 ft below ground level. When and where will the next drop of water strike the elevator?

11.143 Knowing that block B moves downward with a constant velocity of 180 mm/s, determine (*a*) the velocity of block A, (*b*) the velocity of pulley D.

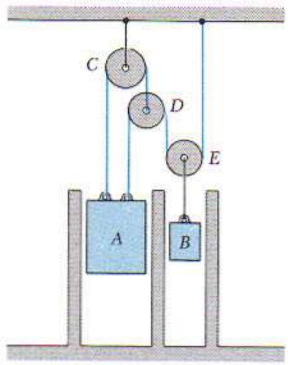


Fig. P11.143

CHAPTER 12

Kinetics of Particles: Newton's Second Law

12.1. Newton's Second Law of Motion. Newton's first and third laws of motion were used extensively in statics to study bodies at rest and the forces acting upon them. These two laws are also used in dynamics; in fact, they are sufficient for the study of the motion of bodies which have no acceleration. However, when bodies are accelerated, i.e., when the magnitude or the direction of their velocity changes, it is necessary to use the second law of motion in order to relate the motion of the body with the forces acting on it. This law may be stated as follows:

If the resultant force acting on a particle is not zero, the particle will have an acceleration proportional to the magnitude of the resultant and in the direction of this resultant force.

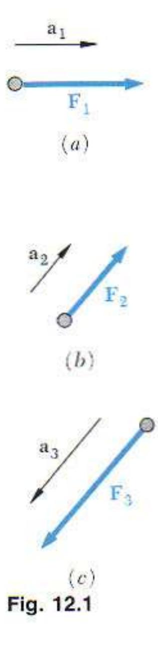


Fig. 12.1

Newton's second law of motion may best be understood if we imagine the following experiment: A particle is subjected to a force $\mathbf{F}_1$ of constant direction and constant magnitude F_1. Under the action of that force, the particle will be observed to move in a straight line and *in the direction of the force* (Fig. 12.1*a*). By determining the position of the particle at various instants, we find that its acceleration has a constant magnitude a_1. If the experiment is repeated with forces $\mathbf{F}_2$, $\mathbf{F}_3$, etc., of different magnitude or direction (Fig. 12.1*b* and *c*), we find each time that the particle moves in the direction of the force acting on it and that the magnitudes a_1, a_2, a_3, etc., of the accelerations are proportional to the magnitudes F_1, F_2, F_3, etc., of the corresponding forces,

$$\frac{F_1}{a_1} = \frac{F_2}{a_2} = \frac{F_3}{a_3} = \cdots = \text{constant}$$

The constant value obtained for the ratio of the magnitudes of the forces and accelerations is a characteristic of the particle under consideration. It is called the *mass* of the particle and is denoted by m. When a particle of mass m is acted upon by a force $\mathbf{F}$, the force $\mathbf{F}$ and the acceleration $\mathbf{a}$ of the particle must therefore satisfy the relation

$$\mathbf{F} = m\mathbf{a} \tag{12.1}$$

This relation provides a complete formulation of Newton's second law; it expresses not only that the magnitudes of $\mathbf{F}$ and $\mathbf{a}$ are proportional, but also (since m is a positive scalar) that the vectors $\mathbf{F}$ and $\mathbf{a}$ have the same direction (Fig. 12.2). We should note that Eq. (12.1) still holds when $\mathbf{F}$ is not constant but varies with t in magnitude or direction. The magnitudes of $\mathbf{F}$ and $\mathbf{a}$ remain proportional, and the two vectors have the same direction at any given instant. However, they will not, in general, be tangent to the path of the particle.

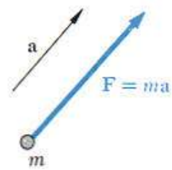


Fig. 12.2

When a particle is subjected simultaneously to several forces, Eq. (12.1) should be replaced by

$$\Sigma\mathbf{F} = m\mathbf{a} \tag{12.2}$$

where $\Sigma\mathbf{F}$ represents the sum, or resultant, of all the forces acting on the particle.

It should be noted that the system of axes with respect to which the acceleration $\mathbf{a}$ is determined is not arbitrary. These axes must have a constant orientation with respect to the stars, and their origin must either be attached to the sun† or move with a constant velocity with respect to the sun. Such a system of axes is called a *newtonian frame of reference*.‡ A system of axes attached to the earth does *not* constitute a newtonian frame of reference, since the earth rotates with respect to the stars and is accelerated with respect to the sun. However, in most

† More accurately, to the mass center of the solar system.

‡ Since the stars are not actually fixed, a more rigorous definition of a newtonian frame of reference (also called *inertial system*) is *one with respect to which Eq.* (12.2) *holds*.

engineering applications, the acceleration **a** may be determined with respect to axes attached to the earth and Eqs. (12.1) and (12.2) used without any appreciable error. On the other hand, these equations do not hold if **a** represents a relative acceleration measured with respect to moving axes, such as axes attached to an accelerated car or to a rotating piece of machinery.

We may observe that, if the resultant $\Sigma\mathbf{F}$ of the forces acting on the particle is zero, it follows from Eq. (12.2) that the acceleration **a** of the particle is also zero. If the particle is initially at rest ($\mathbf{v}_0 = 0$) with respect to the newtonian frame of reference used, it will thus remain at rest ($\mathbf{v} = 0$). If originally moving with a velocity $\mathbf{v}_0$, the particle will maintain a constant velocity $\mathbf{v} = \mathbf{v}_0$; that is, it will move with the constant speed v_0 in a straight line. This, we recall, is the statement of Newton's first law (Sec. 2.9). Thus, Newton's first law is a particular case of Newton's second law and may be omitted from the fundamental principles of mechanics.

12.2. Linear Momentum of a Particle. Rate of Change of Linear Momentum. Replacing the acceleration **a** by the derivative $d\mathbf{v}/dt$ in Eq. (12.2), we write

$$\Sigma\mathbf{F} = m\frac{d\mathbf{v}}{dt}$$

or, since the mass m of the particle is constant,

$$\Sigma\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) \tag{12.3}$$

Fig. 12.3

The vector $m\mathbf{v}$ is called the *linear momentum*, or simply the *momentum*, of the particle. It has the same direction as the velocity of the particle and its magnitude is equal to the product of the mass m and the speed v of the particle (Fig. 12.3). Equation (12.3) expresses that *the resultant of the forces acting on the particle is equal to the rate of change of the linear momentum of the particle*. It is in this form that the second law of motion was originally stated by Newton. Denoting by $\mathbf{L}$ the linear momentum of the particle,

$$\mathbf{L} = m\mathbf{v} \tag{12.4}$$

and by $\dot{\mathbf{L}}$ its derivative with respect to t, we may write Eq. (12.3) in the alternate form

$$\Sigma\mathbf{F} = \dot{\mathbf{L}} \tag{12.5}$$

It should be noted that the mass m of the particle is assumed constant in Eqs. (12.3), (12.4), and (12.5). Equations (12.3) or (12.5), therefore, should not be used to solve problems involving the motion of bodies which gain or lose mass, such as rockets. Problems of that type will be considered in Sec. 14.11.†

It follows from Eq. (12.3) that the rate of change of the linear momentum $m\mathbf{v}$ is zero when $\Sigma\mathbf{F} = 0$. Thus, *if the resultant force acting on a particle is zero, the linear momentum of the particle remains constant, both in magnitude and direction.* This is the principle of *conservation of linear momentum* for a particle, which we may recognize as just an alternate statement of Newton's first law (Sec. 2.9).

12.3. Systems of Units. In using the fundamental equation $\mathbf{F} = m\mathbf{a}$, the units of force, mass, length, and time cannot be chosen arbitrarily. If they are, the magnitude of the force $\mathbf{F}$ required to give an acceleration $\mathbf{a}$ to the mass m will *not* be numerically equal to the product ma; it will only be proportional to this product. Thus, we may choose three of the four units arbitrarily but must choose the fourth unit so that the equation $\mathbf{F} = m\mathbf{a}$ is satisfied. The units are then said to form a system of consistent kinetic units.

Two systems of consistent kinetic units are currently used by American engineers, the International System of Units (SI units‡), and the U.S. customary units. Since both systems have been discussed in detail in Sec. 1.3, we shall describe them only briefly in this section.

International System of Units (SI Units). In this system, the base units are the units of length, mass, and time, and are called, respectively, the *meter* (m), the *kilogram* (kg), and the *second* (s). All three are arbitrarily defined (Sec. 1.3). The unit of force is a derived unit. It is called the *newton* (N) and is defined as the force which gives an acceleration of 1 m/s² to a mass of 1 kg (Fig. 12.4). From Eq. (12.1) we write

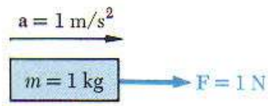


Fig. 12.4

$$1\ \mathrm{N} = (1\ \mathrm{kg})(1\ \mathrm{m/s^2}) = 1\ \mathrm{kg \cdot m/s^2}$$

The SI units are said to form an *absolute* system of units. This means that the three base units chosen are independent of the location where measurements are made. The meter, the kilogram, and the second may be used anywhere on the earth; they

† On the other hand, Eqs. (12.3) and (12.5) do hold in *relativistic mechanics,* where the mass m of the particle is assumed to vary with the speed of the particle.

‡ SI stands for *Système International d'Unités* (French).

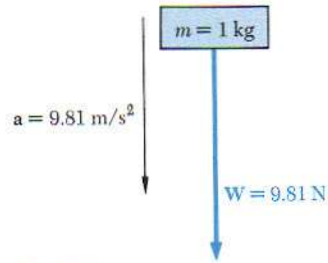


Fig. 12.5

may even be used on another planet. They will always have the same significance.

Like any other force, the *weight* **W** of a body should be expressed in newtons. Since a body subjected to its own weight acquires an acceleration equal to the acceleration of gravity g, it follows from Newton's second law that the magnitude W of the weight of a body of mass m is

$$W = mg \tag{12.6}$$

Recalling that $g = 9.81 \text{ m/s}^2$, we find that the weight of a body of mass 1 kg (Fig. 12.5) is

$$W = (1 \text{ kg})(9.81 \text{ m/s}^2) = 9.81 \text{ N}$$

Multiples and submultiples of the units of length, mass, and force are frequently used in engineering practice. They are, respectively, the *kilometer* (km) and the *millimeter* (mm); the *megagram*† (Mg) and the *gram* (g); and the *kilonewton* (kN). By definition

$$\begin{aligned} 1 \text{ km} &= 1000 \text{ m} & 1 \text{ mm} &= 0.001 \text{ m} \\ 1 \text{ Mg} &= 1000 \text{ kg} & 1 \text{ g} &= 0.001 \text{ kg} \end{aligned}$$
$$1 \text{ kN} = 1000 \text{ N}$$

The conversion of these units to meters, kilograms, and newtons, respectively, can be effected by simply moving the decimal point three places to the right or to the left.

Units other than the units of mass, length, and time may all be expressed in terms of these three base units. For example, the unit of linear momentum may be obtained by recalling the definition of linear momentum and writing

$$mv = (\text{kg})(\text{m/s}) = \text{kg} \cdot \text{m/s}$$

U.S. Customary Units. Most practicing American engineers still commonly use a system in which the base units are the units of length, force, and time. These units are, respectively, the *foot* (ft), the *pound* (lb), and the *second* (s). The second is the same as the corresponding SI unit. The foot is defined as 0.3048 m. The pound is defined as the *weight* of a platinum standard, called the *standard pound* and kept at the National Bureau of Standards in Washington, the mass of which is 0.453 592 43 kg. Since the weight of a body depends upon the gravitational attraction of the earth, which varies with location, it is specified that the standard pound should be placed at sea level and at the latitude of 45° to properly define a force of 1 lb. Clearly the U.S. cus-

† Also known as a *metric ton*.

tomary units do not form an absolute system of units. Because of their dependence upon the gravitational attraction of the earth, they are said to form a *gravitational* system of units.

While the standard pound also serves as the unit of mass in commercial transactions in the United States, it cannot be so used in engineering computations since such a unit would not be consistent with the base units defined in the preceding paragraph. Indeed, when acted upon by a force of 1 lb, that is, when subjected to its own weight, the standard pound receives the acceleration of gravity, $g = 32.2 \text{ ft/s}^2$ (Fig. 12.6), not the unit acceleration required by Eq. (12.1). The unit of mass consistent with the foot, the pound, and the second is the mass which receives an acceleration of 1 ft/s^2 when a force of 1 lb is applied to it (Fig. 12.7). This unit, sometimes called a *slug*, can be derived from the equation $F = ma$ after substituting 1 lb and 1 ft/s^2 for F and a, respectively. We write

$$F = ma \qquad 1 \text{ lb} = (1 \text{ slug})(1 \text{ ft/s}^2)$$

and obtain

$$1 \text{ slug} = \frac{1 \text{ lb}}{1 \text{ ft/s}^2} = 1 \text{ lb} \cdot \text{s}^2/\text{ft}$$

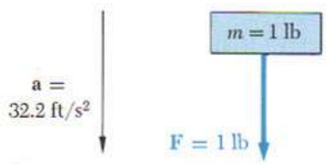


Fig. 12.6

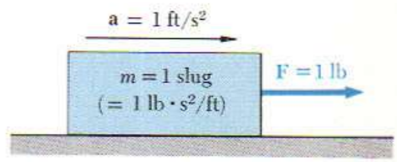


Fig. 12.7

Comparing Figs. 12.6 and 12.7, we conclude that the slug is a mass 32.2 times larger than the mass of the standard pound.

The fact that bodies are characterized in the U.S. customary system of units by their weight in pounds, rather than by their mass in slugs, was a convenience in the study of statics, where we were dealing constantly with weights and other forces and only seldom with masses. However, in the study of kinetics, where forces, masses, and accelerations are involved, we repeatedly shall have to express the mass m in slugs of a body, the weight W of which has been given in pounds. Recalling Eq. (12.6), we shall write

$$m = \frac{W}{g} \tag{12.7}$$

where g is the acceleration of gravity ($g = 32.2 \text{ ft/s}^2$).

Units other than the units of force, length, and time may all be expressed in terms of these three base units. For example, the unit of linear momentum may be obtained by recalling the definition of linear momentum and writing

$$mv = (\text{lb} \cdot \text{s}^2/\text{ft})(\text{ft/s}) = \text{lb} \cdot \text{s}$$

The conversion from U.S. customary units to SI units, and vice versa, has been discussed in Sec. 1.4. We shall recall the conversion factors obtained respectively for the units of length, force, and mass:

Length:	$1 \text{ ft} = 0.3048 \text{ m}$
Force:	$1 \text{ lb} = 4.448 \text{ N}$
Mass:	$1 \text{ slug} = 1 \text{ lb} \cdot \text{s}^2/\text{ft} = 14.59 \text{ kg}$

Although it cannot be used as a consistent unit of mass, we also recall that the mass of the standard pound is, by definition,

$$1 \text{ pound-mass} = 0.4536 \text{ kg}$$

This constant may be used to determine the *mass* in SI units (kilograms) of a body which has been characterized by its *weight* in U.S. customary units (pounds).

12.4. Equations of Motion. Consider a particle of mass m acted upon by several forces. We recall from Sec. 12.1 that Newton's second law may be expressed by writing the equation

$$\Sigma \mathbf{F} = m\mathbf{a} \tag{12.2}$$

Fig. 12.8

which relates the forces acting on the particle and the vector $m\mathbf{a}$ (Fig. 12.8). In order to solve problems involving the motion of a particle, however, it will be found more convenient to replace Eq. (12.2) by equivalent equations involving scalar quantities.

Rectangular Components. Resolving each force $\mathbf{F}$ and the acceleration $\mathbf{a}$ into rectangular components, we write

$$\Sigma(F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}) = m(a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})$$

from which it follows that

$$\Sigma F_x = ma_x \qquad \Sigma F_y = ma_y \qquad \Sigma F_z = ma_z \tag{12.8}$$

Recalling from Sec. 11.11 that the components of the acceleration are equal to the second derivatives of the coordinates of the particle, we have

$$\Sigma F_x = m\ddot{x} \qquad \Sigma F_y = m\ddot{y} \qquad \Sigma F_z = m\ddot{z} \tag{12.8'}$$

Consider, as an example, the motion of a projectile. If the resistance of the air is neglected, the only force acting on the projectile after it has been fired is its weight $\mathbf{W} = -W\mathbf{j}$. The equations defining the motion of the projectile are therefore

$$m\ddot{x} = 0 \qquad m\ddot{y} = -W \qquad m\ddot{z} = 0$$

and the components of the acceleration of the projectile are

$$\ddot{x} = 0 \qquad \ddot{y} = -\frac{W}{m} = -g \qquad \ddot{z} = 0$$

where g is 9.81 m/s² or 32.2 ft/s². The equations obtained may be integrated independently, as was shown in Sec. 11.11, to obtain the velocity and displacement of the projectile at any instant.

Tangential and Normal Components. Resolving the forces and the acceleration of the particle into components along the tangent to the path (in the direction of motion) and the normal (toward the inside of the path) (Fig. 12.9), and substituting into Eq. (12.2), we obtain the two scalar equations

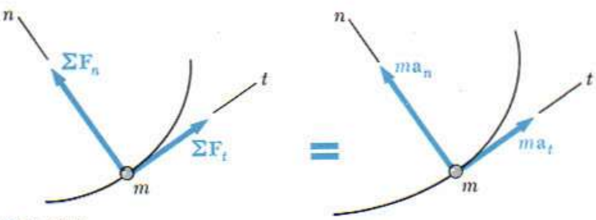


Fig. 12.9

$$\Sigma F_t = ma_t \qquad \Sigma F_n = ma_n \tag{12.9}$$

Substituting for a_t and a_n from Eqs. (11.40), we have

$$\Sigma F_t = m\frac{dv}{dt} \qquad \Sigma F_n = m\frac{v^2}{\rho} \tag{12.9'}$$

The equations obtained may be solved for two unknowns.

12.5. Dynamic Equilibrium. Returning to Eq. (12.2) and transposing the right-hand member, we write Newton's second law in the alternate form

$$\Sigma \mathbf{F} - m\mathbf{a} = 0 \tag{12.10}$$

which expresses that, if we add the vector $-m\mathbf{a}$ to the forces acting on the particle, *we obtain a system of vectors equivalent to zero* (Fig. 12.10). The vector $-m\mathbf{a}$, of magnitude ma and of *direction opposite* to that of the acceleration, is called an *inertia vector*. The particle may thus be considered to be in equilibrium

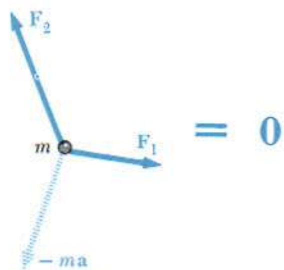


Fig. 12.10

under the given forces and the inertia vector. The particle is said to be in *dynamic equilibrium*, and the problem under consideration may be solved by the methods developed earlier in statics.

In the case of coplanar forces, we may draw in tip-to-tail fashion all the vectors shown in Fig. 12.10, *including the inertia vector*, to form a closed-vector polygon. Or we may write that the sums of the components of all the vectors in Fig. 12.10, including again the inertia vector, are zero. Using rectangular components, we therefore write

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \textbf{\textit{including inertia vector}} \quad (12.11)$$

When tangential and normal components are used, it is more convenient to represent the inertia vector by its two components $-m\mathbf{a}_t$ and $-m\mathbf{a}_n$ in the sketch itself (Fig. 12.11). The tangential component of the inertia vector provides a measure of the resistance the particle offers to a change in speed, while its normal component (also called *centrifugal force*) represents the tendency of the particle to leave its curved path. We should note that either of these two components may be zero under special conditions: (1) if the particle starts from rest, its initial velocity is zero and the normal component of the inertia vector is zero at $t = 0$; (2) if the particle moves at constant speed along its path, the tangential component of the inertia vector is zero and only its normal component needs to be considered.

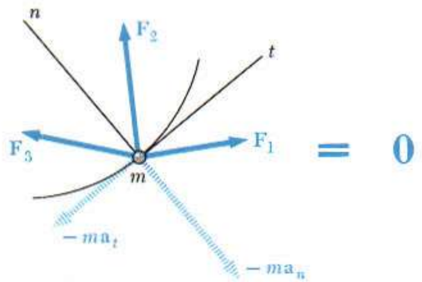
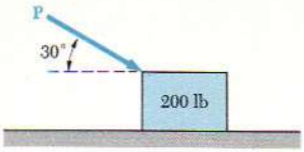


Fig. 12.11

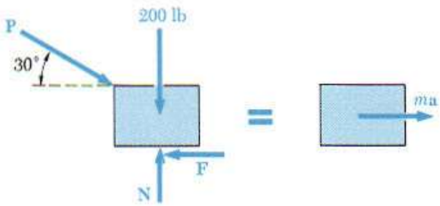
Because they measure the resistance that particles offer when we try to set them in motion or when we try to change the conditions of their motion, inertia vectors are often called *inertia forces*. The inertia forces, however, are not forces like the forces found in statics, which are either contact forces or gravitational forces (weights). Many people, therefore, object to the use of the word "force" when referring to the vector $-m\mathbf{a}$ or even avoid altogether the concept of dynamic equilibrium. Others point out that inertia forces and actual forces, such as gravitational forces, affect our senses in the same way and cannot be distinguished by physical measurements. A man riding in an elevator which is accelerated upward will have the feeling that his weight has suddenly increased; and no measurement made within the elevator could establish whether the elevator is truly accelerated or whether the force of attraction exerted by the earth has suddenly increased.

Sample problems have been solved in this text by the direct application of Newton's second law, as illustrated in Figs. 12.8 and 12.9, rather than by the method of dynamic equilibrium.

SAMPLE PROBLEM 12.1



A 200-lb block rests on a horizontal plane. Find the magnitude of the force **P** required to give the block an acceleration of $10\ \text{ft/s}^2$ to the right. The coefficient of friction between the block and the plane is $\mu = 0.25$.



Solution. The mass of the block is

$$m = \frac{W}{g} = \frac{200\ \text{lb}}{32.2\ \text{ft/s}^2} = 6.21\ \text{lb}\cdot\text{s}^2/\text{ft}$$

We note that $F = \mu N = 0.25N$ and that $a = 10\ \text{ft/s}^2$. Expressing that the forces acting on the block are equivalent to the vector $m\mathbf{a}$, we write

$$\xrightarrow{+} \Sigma F_x = ma: \qquad \begin{aligned} P\cos 30^\circ - 0.25N &= (6.21\ \text{lb}\cdot\text{s}^2/\text{ft})(10\ \text{ft/s}^2) \\ P\cos 30^\circ - 0.25N &= 62.1\ \text{lb} \end{aligned} \tag{1}$$

$$+\uparrow \Sigma F_y = 0: \qquad N - P\sin 30^\circ - 200\ \text{lb} = 0 \tag{2}$$

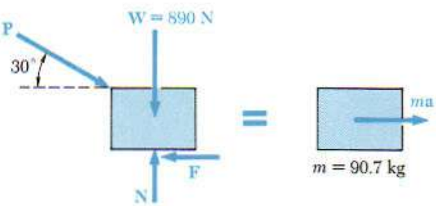
Solving (2) for N and carrying the result into (1), we obtain

$$N = P\sin 30^\circ + 200\ \text{lb}$$

$$P\cos 30^\circ - 0.25(P\sin 30^\circ + 200\ \text{lb}) = 62.1\ \text{lb} \qquad P = 151\ \text{lb} \blacktriangleleft$$

SAMPLE PROBLEM 12.2

Solve Sample Prob. 12.1 using SI units.



Solution. Using the conversion factors given in Sec. 12.3, we write

$$a = (10\ \text{ft/s}^2)(0.3048\ \text{m/ft}) = 3.05\ \text{m/s}^2$$
$$W = (200\ \text{lb})(4.448\ \text{N/lb}) = 890\ \text{N}$$

Recalling that, by definition, 1 lb is the weight of a mass of 0.4536 kg, we find that the mass of the 200-lb block is

$$m = 200(0.4536\ \text{kg}) = 90.7\ \text{kg}$$

Noting that $F = \mu N = 0.25N$ and expressing that the forces acting on the block are equivalent to the vector $m\mathbf{a}$, we write

$$\xrightarrow{+} \Sigma F_x = ma: \qquad P\cos 30^\circ - 0.25N = (90.7\ \text{kg})(3.05\ \text{m/s}^2)$$

$$P\cos 30^\circ - 0.25N = 277\ \text{N} \tag{1}$$

$$+\uparrow \Sigma F_y = 0: \qquad N - P\sin 30^\circ - 890\ \text{N} = 0 \tag{2}$$

Solving (2) for N and carrying the result into (1), we obtain

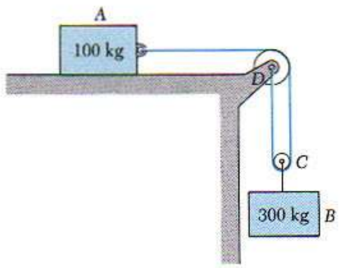
$$N = P\sin 30^\circ + 890\ \text{N}$$

$$P\cos 30^\circ - 0.25(P\sin 30^\circ + 890\ \text{N}) = 277\ \text{N} \qquad P = 674\ \text{N} \blacktriangleleft$$

or, in U.S. customary units,

$$P = (674\ \text{N}) \div (4.448\ \text{N/lb}) \qquad P = 151\ \text{lb} \blacktriangleleft$$

SAMPLE PROBLEM 12.3



The two blocks shown start from rest. The horizontal plane and the pulley are frictionless, and the pulley is assumed to be of negligible mass. Determine the acceleration of each block and the tension in each cord.

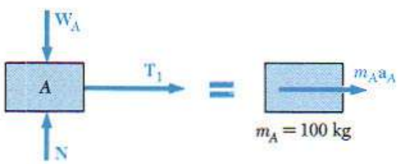
Solution. We denote by T_1 the tension in cord ACD and by T_2 the tension in cord BC. We note that if block A moves through s_A, block B moves through

$$s_B = \tfrac{1}{2} s_A$$

Differentiating twice with respect to t, we have

$$a_B = \tfrac{1}{2} a_A \qquad (1)$$

We shall apply Newton's second law successively to block A, block B, and pulley C.

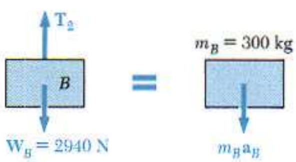


Block A

$$\xrightarrow{+} \Sigma F_x = m_A a_A: \qquad T_1 = 100 a_A \qquad (2)$$

Block B. Observing that the weight of block B is

$$W_B = m_B g = (300 \text{ kg})(9.81 \text{ m/s}^2) = 2940 \text{ N}$$

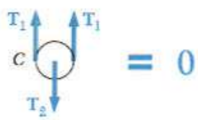


we write

$$+\!\downarrow \Sigma F_y = m_B a_B: \qquad 2940 - T_2 = 300 a_B$$

or, substituting for a_B from (1),

$$2940 - T_2 = 300(\tfrac{1}{2} a_A)$$
$$T_2 = 2940 - 150 a_A \qquad (3)$$



Pulley C. Since m_C is assumed to be zero, we have

$$+\!\downarrow \Sigma F_y = m_C a_C = 0: \qquad T_2 - 2T_1 = 0 \qquad (4)$$

Substituting for T_1 and T_2 from (2) and (3), respectively, into (4), we write

$$2940 - 150 a_A - 2(100 a_A) = 0$$
$$2940 - 350 a_A = 0 \qquad a_A = 8.40 \text{ m/s}^2 \quad ◄$$

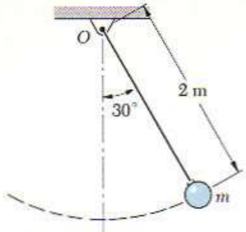
Substituting the value obtained for a_A into (1) and (2), we have

$$a_B = \tfrac{1}{2} a_A = \tfrac{1}{2}(8.40 \text{ m/s}^2) \qquad a_B = 4.20 \text{ m/s}^2 \quad ◄$$
$$T_1 = 100 a_A = (100 \text{ kg})(8.40 \text{ m/s}^2) \qquad T_1 = 840 \text{ N} \quad ◄$$

Recalling (4), we write

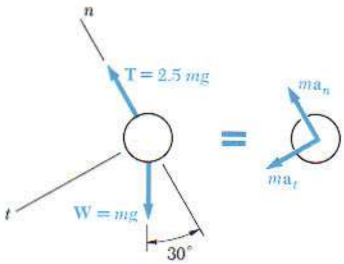
$$T_2 = 2T_1 \qquad T_2 = 2(840 \text{ N}) \qquad T_2 = 1680 \text{ N} \quad ◄$$

We note that the value obtained for T_2 is *not* equal to the weight of block B.



SAMPLE PROBLEM 12.4

The bob of a 2-m pendulum describes an arc of circle in a vertical plane. If the tension in the cord is 2.5 times the weight of the bob for the position shown, find the velocity and acceleration of the bob in that position.



Solution. The weight of the bob is $W = mg$; the tension in the cord is thus $2.5\,mg$. Recalling that $\mathbf{a}_n$ is directed toward O and assuming $\mathbf{a}_t$ as shown, we apply Newton's second law and obtain

$$+\swarrow \Sigma F_t = ma_t: \qquad mg \sin 30^\circ = ma_t$$

$$a_t = g \sin 30^\circ = +4.90\ \text{m/s}^2 \qquad \mathbf{a}_t = 4.90\ \text{m/s}^2 \swarrow \blacktriangleleft$$

$$+\nwarrow \Sigma F_n = ma_n: \qquad 2.5\,mg - mg \cos 30^\circ = ma_n$$

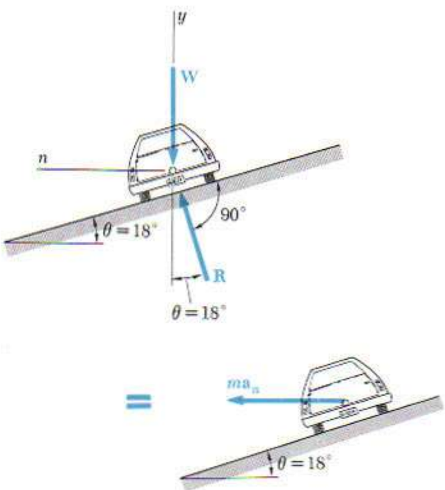
$$a_n = 1.634\,g = +16.03\ \text{m/s}^2 \qquad \mathbf{a}_n = 16.03\ \text{m/s}^2 \nwarrow \blacktriangleleft$$

Since $a_n = v^2/\rho$, we have $v^2 = \rho a_n = (2\ \text{m})(16.03\ \text{m/s}^2)$

$$v = \pm 5.66\ \text{m/s} \qquad \mathbf{v} = 5.66\ \text{m/s} \ \text{(up or down)} \blacktriangleleft$$

SAMPLE PROBLEM 12.5

Determine the rated speed of a highway curve of radius $\rho = 400$ ft banked through an angle $\theta = 18°$. The rated speed of a banked curved road is the speed at which a car should travel if no lateral friction force is to be exerted on its wheels.



Solution. The car travels in a *horizontal* circular path of radius ρ. The normal component $\mathbf{a}_n$ of the acceleration is directed toward the center of the path; its magnitude is $a_n = v^2/\rho$, where v is the speed of the car in ft/s. The mass m of the car is W/g, where W is the weight of the car. Since no lateral friction force is to be exerted on the car, the reaction $\mathbf{R}$ of the road is shown perpendicular to the roadway. Applying Newton's second law, we write

$$+\uparrow \Sigma F_y = 0: \qquad R \cos\theta - W = 0 \qquad R = \frac{W}{\cos\theta} \tag{1}$$

$$\overset{+}{\leftarrow} \Sigma F_n = ma_n: \qquad R \sin\theta = \frac{W}{g} a_n \tag{2}$$

Substituting for R from (1) into (2), and recalling that $a_n = v^2/\rho$:

$$\frac{W}{\cos\theta} \sin\theta = \frac{W}{g}\frac{v^2}{\rho} \qquad v^2 = g\rho \tan\theta$$

Substituting the given data, $\rho = 400$ ft and $\theta = 18°$, into this equation, we obtain

$$v^2 = (32.2\ \text{ft/s}^2)(400\ \text{ft}) \tan 18^\circ$$
$$v = 64.7\ \text{ft/s} \qquad v = 44.1\ \text{mi/h} \blacktriangleleft$$

PROBLEMS

12.1 The value of g at any latitude ϕ may be obtained from the formula

$$g = 9.7807(1 + 0.0053 \sin^2 \phi) \qquad \text{m/s}^2$$

Determine to four significant figures the weight in newtons and the mass in kilograms, at the latitudes of 0°, 45°, and 90°, of a silver bar whose mass is officially defined as 10 kg.

12.2 The acceleration due to gravity on the moon is 5.31 ft/s^2. Determine the weight in pounds, the mass in pounds, and the mass in lb · s^2/ft, on the moon, of a silver bar whose mass is officially defined as 100.00 lb.

12.3 A 100-kg satellite has been placed in a circular orbit 2000 km above the surface of the earth. The acceleration of gravity at this elevation is 5.68 m/s^2. Determine the linear momentum of the satellite, knowing that its orbital speed is 24 800 km/h.

12.4 Two boxes are weighed on the scales shown: scale *a* is a lever scale; scale *b* is a spring scale. The scales are attached to the roof of an elevator. When the elevator is at rest, each scale indicates a load of 20 lb. If the spring scale indicates a load of 18 lb, determine the acceleration of the elevator and the load indicated by the lever scale.

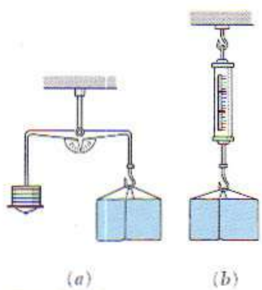


Fig. P12.4

12.5 A motorist traveling at a speed of 45 mi/h suddenly applies his brakes and comes to a stop after skidding 150 ft. Determine (*a*) the time required for the car to stop, (*b*) the coefficient of friction between the tires and the pavement.

12.6 An automobile skids 90 ft on a level road before coming to a stop. If the coefficient of friction between the tires and the pavement is 0.75, determine (*a*) the speed of the automobile before the brakes were applied, (*b*) the time required for the automobile to come to a stop.

12.7 A truck is proceeding up a long 3-percent grade at a constant speed of 60 km/h. If the driver does not change the setting of his throttle or shift gears, what will be the acceleration of the truck as it starts moving on the level section of the road?

Fig. P12.7

12.8 A 5-kg package is projected down the incline with an initial velocity of 4 m/s. Knowing that the coefficient of friction between the package and the incline is 0.35, determine (*a*) the velocity of the package after 3 m of motion, (*b*) the distance *d* at which the package comes to rest.

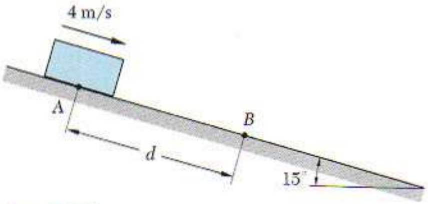


Fig. P12.8

12.9 The 3-kg collar was moving down the rod with a velocity of 3 m/s when a force **P** was applied to the horizontal cable. Assuming negligible friction between the collar and the rod, determine the magnitude of the force **P** if the collar stopped after moving 1 m more down the rod.

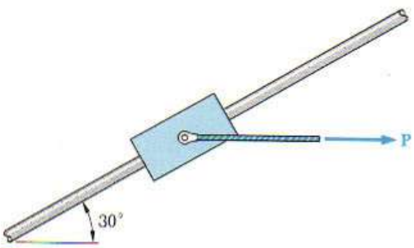


Fig. P12.9

12.10 Solve Prob. 12.9, assuming a coefficient of friction of 0.20 between the collar and the rod.

12.11 The subway train shown travels at a speed of 30 mi/h. Determine the force in each coupling when the brakes are applied, knowing that the braking force is 5000 lb on each car.

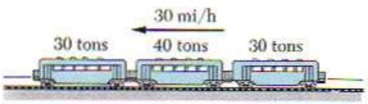


Fig. P12.11

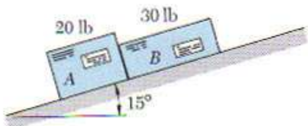


Fig. P12.12

12.12 Two packages are placed on an incline as shown. The coefficient of friction is 0.25 between the incline and package A, and 0.15 between the incline and package B. Knowing that the packages are in contact when released, determine (a) the acceleration of each package, (b) the force exerted by package A on package B.

12.13 Solve Prob. 12.12, assuming the positions of the packages are reversed so that package A is to the right of package B.

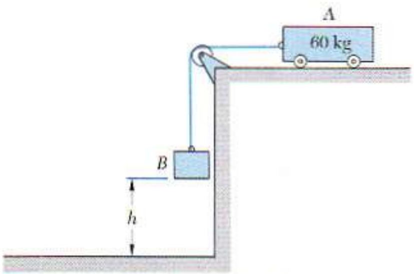


Fig. P12.14, P12.15, and P12.16

12.14 When the system shown is released from rest, the acceleration of block B is observed to be 3 m/s^2 downward. Neglecting the effect of friction, determine (a) the tension in the cable, (b) the mass of block B.

12.15 The system shown is released from rest when $h = 1.4$ m. (a) Determine the mass of block B, knowing that it strikes the ground with a speed of 3 m/s. (b) Attempt to solve part a, assuming the final speed to be 6 m/s; explain the difficulty encountered.

12.16 The system shown is released from rest. Knowing that the mass of block B is 30 kg, determine how far the cart will move before it reaches a speed of 2.5 m/s, (a) if the pulley may be considered as weightless and frictionless, (b) if the pulley "freezes" on its shaft and the cable must slip, with $\mu = 0.10$, over the pulley.

12.17 Each of the systems shown is initially at rest. Assuming the pulleys to be weightless and neglecting axle friction, determine for each system (a) the acceleration of block A, (b) the velocity of block A after 4 s, (c) the velocity of block A after it has moved 10 ft.

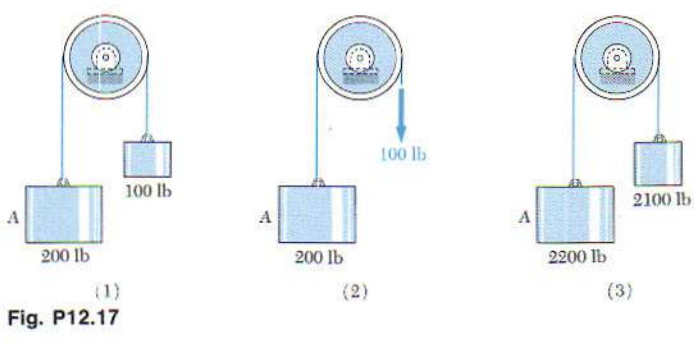


Fig. P12.17

12.18 The 100-kg block A is connected to a 25-kg counterweight B by the cable arrangement shown. If the system is released from rest, determine (*a*) the tension in the cable, (*b*) the velocity of B after 3 s, (*c*) the velocity of A after it has moved 1.2 m.

12.19 Block A is observed to move with an acceleration of 0.9 m/s^2 directed upward. Determine (*a*) the mass of block B, (*b*) the corresponding tension in the cable.

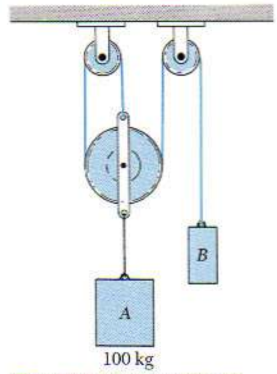


Fig. P12.18 and P12.19

12.20 The system shown is initially at rest. Neglecting the effect of friction, determine (*a*) the force **P** required if the velocity of collar B is to be 12 ft/s after it has moved 18 in. to the right, (*b*) the corresponding tension in the cable.

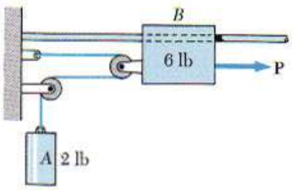


Fig. P12.20 and P12.21

12.21 A force **P** of magnitude 15 lb is applied to collar B, which is observed to move 3 ft in 0.5 s after starting from rest. Neglecting the effect of friction in the pulleys, determine the friction force that the rod exerts on collar B.

12.22 Neglecting the effect of friction, determine (*a*) the acceleration of each block, (*b*) the tension in the cable.

Fig. P12.22

12.23 The *rimpull* of a truck is defined as the tractive force between the rubber tires of the driving wheels and the ground. For a truck used to haul earth at a construction site, the rimpull actually utilized by the average driver in each of the first five forward gears and the maximum speed attained in each gear are as follows:

Gear	Max v (mi/h)	Average rimpull (lb)
1st	3	6000
2d	6	3800
3d	9	2800
4th	15	2000
5th	27	1500

Knowing that a truck (and load) weighs 44,000 lb and has a rolling resistance of 60 lb/ton for the unpaved surface encountered, determine the time required for the truck to attain a speed of 27 mi/h. Neglect the time needed to shift gears.

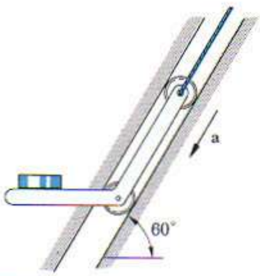


Fig. P12.24

12.24 In a manufacturing process, disks are moved from one elevation to another by the lifting arm shown; the coefficient of friction between a disk and the arm is 0.20. Determine the magnitude of the acceleration for which the disks slide on the arm, assuming the acceleration is directed (*a*) downward as shown, (*b*) upward.

12.25 The coefficient of friction between the load and the flat-bed trailer shown is 0.40. Knowing that the forward speed of the truck is 50 km/h, determine the shortest distance in which the truck can be brought to a stop if the load is not to shift.

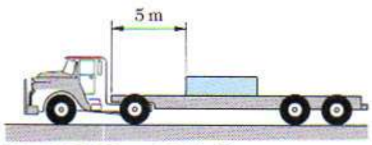


Fig. P12.25 and P12.26

12.26 The coefficient of friction between the load and the flat-bed trailer is 0.40. While traveling at 100 km/h, the driver makes an emergency stop and the truck skids to rest in 90 m. Determine the velocity of the load relative to the trailer as it reaches the forward edge of the trailer.

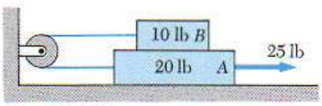


Fig. P12.27

12.27 Knowing that the coefficient of friction is 0.30 at all surfaces of contact, determine (*a*) the acceleration of plate *A*, (*b*) the tension in the cable. (Neglect bearing friction in the pulley.)

12.28 Solve Prob. 12.27, assuming that the 25-lb force is applied to plate *B*.

12.29 A 30-kg crate rests on a 20-kg cart; the coefficient of static friction between the crate and the cart is 0.25. If the crate is not to slip with respect to the cart, determine (*a*) the maximum allowable magnitude of **P**, (*b*) the corresponding acceleration of the cart.

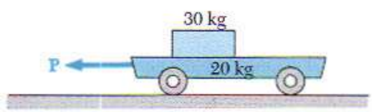


Fig. P12.29 and P12.30

12.30 The coefficients of friction between the 30-kg crate and the 20-kg cart are $\mu_s = 0.25$ and $\mu_k = 0.20$. If a force **P** of magnitude 150 N is applied to the cart, determine the acceleration (*a*) of the cart, (*b*) of the crate, (*c*) of the crate with respect to the cart.

12.31 The force exerted by a magnet on a small steel block varies inversely as the square of the distance between the block and the magnet. When the block is 250 mm from the magnet, the magnetic force is 1.5 N. The coefficient of friction between the steel block and the horizontal surface is 0.50. If the block is released from the position shown, determine its velocity when it is 100 mm from the magnet.

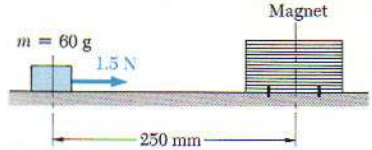


Fig. P12.31

12.32 A constant force **P** is applied to a piston and rod of total mass m in order to make them move in a cylinder filled with oil. As the piston moves, the oil is forced through orifices in the piston and exerts on the piston an additional force of magnitude kv, proportional to the speed v of the piston and in a direction opposite to its motion. Express the acceleration and velocity of the piston as a function of the time t, assuming that the piston starts from rest at time $t = 0$.

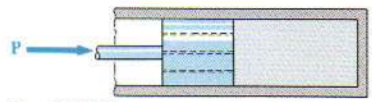


Fig. P12.32

12.33 A ship of total mass m is anchored in the middle of a river which is flowing with a constant velocity $\mathbf{v}_0$. The horizontal component of the force exerted on the ship by the anchor chain is $\mathbf{T}_0$. If the anchor chain suddenly breaks, determine the time required for the ship to attain a velocity equal to $\frac{1}{2}\mathbf{v}_0$. Assume that the frictional resistance of the water is proportional to the velocity of the ship relative to the water.

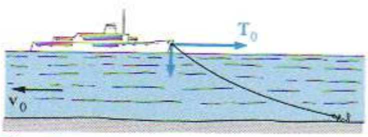


Fig. P12.33

12.34 A spring AB of constant k is attached to a support at A and to a collar of mass m. The unstretched length of the spring is l. Neglecting friction between the collar and the horizontal rod, express the acceleration of the collar as a function of the distance x.

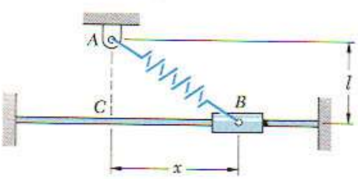


Fig. P12.34

12.35 Knowing that blocks B and C strike the ground simultaneously and exactly 1 s after the system is released from rest, determine W_B and W_C in terms of W_A.

12.36 Determine the acceleration of each block when $W_A = 10$ lb, $W_B = 30$ lb, and $W_C = 20$ lb. Which block strikes the ground first?

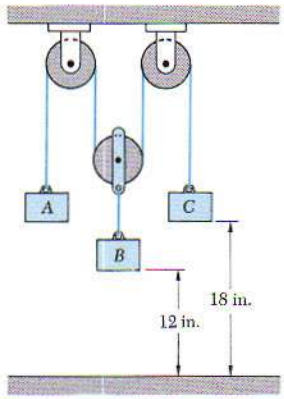


Fig. P12.35, P12.36, and P12.37

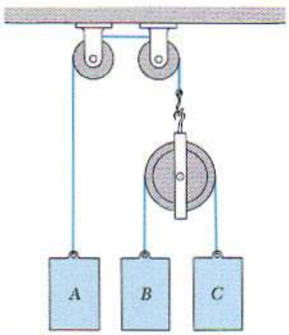


Fig. P12.38

12.37 In the system shown, $W_A = 10$ lb and $W_C = 20$ lb. Determine the required weight W_B if block B is not to move when the system is released from rest.

12.38 Determine the acceleration of each block when $m_A = 15$ kg, $m_B = 10$ kg, and $m_C = 5$ kg.

12.39 Knowing that $\mu = 0.30$, determine the acceleration of each block when $m_A = m_B = m_C$.

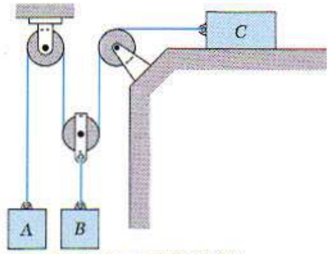


Fig. P12.39 and P12.40

12.40 Knowing that $\mu = 0.50$, determine the acceleration of each block when $m_A = 5$ kg, $m_B = 20$ kg, and $m_C = 15$ kg.

12.41 A small ball of mass $m = 5$ kg is attached to a cord of length $L = 2$ m and is made to revolve in a horizontal circle at a constant speed v_0. Knowing that the cord forms an angle $\theta = 40°$ with the vertical, determine (*a*) the tension in the cord, (*b*) the speed v_0 of the ball.

12.42 A small ball of mass $m = 5$ kg is made to revolve in a horizontal circle as shown. Knowing that the maximum allowable tension in the cord is 100 N, determine (*a*) the maximum allowable velocity if $L = 2$ m, (*b*) the corresponding value of the angle θ.

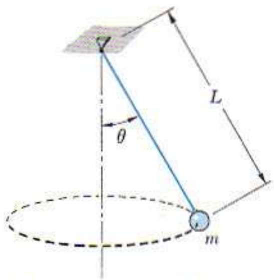


Fig. P12.41 and P12.42

12.43 Two wires AC and BC are each tied to a sphere at C. The sphere is made to revolve in a horizontal circle at a constant speed v. Determine the range of values of the speed v for which *both* wires are taut.

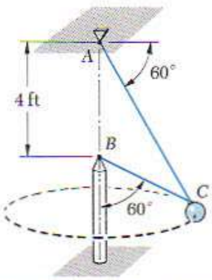


Fig. P12.43 and P12.44

12.44 Two wires AC and BC are each tied to a 10-lb sphere. The sphere is made to revolve in a horizontal circle at a constant speed v. Determine (*a*) the speed for which the tension is the same in both wires, (*b*) the corresponding tension.

12.45 A 3-kg ball is swung in a vertical circle at the end of a cord of length $l = 0.8$ m. Knowing that when $\theta = 60°$ the tension in the cord is 25 N, determine the instantaneous velocity and acceleration of the ball.

12.46 A ball of weight W is released with no velocity from position A and oscillates in a vertical plane at the end of a cord of length l. Determine (*a*) the tangential component of the acceleration in position B in terms of the angle θ, (*b*) the velocity in position B in terms of θ, θ_0, and l, (*c*) the tension in the cord in terms of W and θ_0 when the ball passes through its lowest position C, (*d*) the value of θ_0 if the tension in the cord is $T = 2W$ when the ball passes through position C.

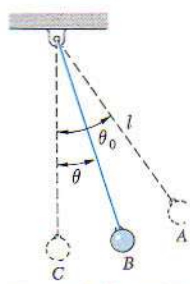


Fig. P12.45 and P12.46

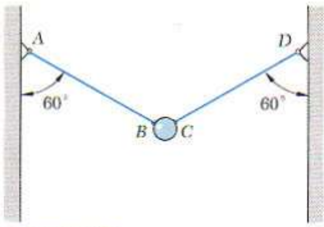


Fig. P12.47

12.47 A small sphere of weight W is held as shown by two wires AB and CD. Wire AB is then cut. Determine (a) the tension in wire CD before AB was cut, (b) the tension in wire CD and the acceleration of the sphere just after AB has been cut.

12.48 A man swings a bucket full of water in a vertical plane in a circle of radius 0.75 m. What is the smallest velocity that the bucket should have at the top of the circle if no water is to be spilled?

12.49 A 175-lb pilot flies a small plane in a vertical loop of 400-ft radius. Determine the speed of the plane at points A and B, knowing that at point A the pilot experiences weightlessness and that at point B the pilot's apparent weight is 600 lb.

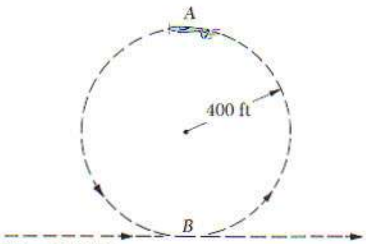


Fig. P12.49

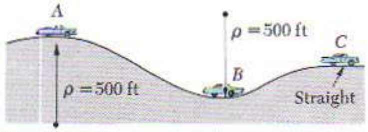


Fig. P12.50

12.50 Three automobiles are proceeding at a speed of 50 mi/h along the road shown. Knowing that the coefficient of friction between the tires and the road is 0.60, determine the tangential deceleration of each automobile if its brakes are suddenly applied and the wheels skid.

Fig. P12.51

12.51 The rod OAB rotates in a vertical plane at a constant rate such that the speed of collar C is 1.5 m/s. The collar is free to slide on the rod between two stops A and B. Knowing that the distance between the stops is only slightly larger than the collar and neglecting the effect of friction, determine the range of values of θ for which the collar is in contact with stop A.

12.52 Express the minimum and maximum safe speeds, with respect to skidding, of a car traveling on a banked road, in terms of the radius r of the curve, the banking angle θ, and the friction angle ϕ between the tires and the pavement.

12.53 A man on a motorcycle takes a turn on a flat unbanked road at 72 km/h. If the radius of the turn is 50 m, determine the minimum value of the coefficient of friction between the tires and the road which will ensure no skidding.

12.54 What angle of banking should be given to the road in Prob. 12.53 if the man on the motorcycle is to be able to take the turn at 72 km/h with a coefficient of friction $\mu = 0.30$?

12.55 A stunt driver proposes to drive a small automobile on the vertical wall of a circular pit of radius 40 ft. Knowing that the coefficient of friction between the tires and the wall is 0.65, determine the minimum speed at which the stunt can be performed.

12.56 The assembly shown rotates about a vertical axis at a constant rate. Knowing that the coefficient of friction between the small block A and the cylindrical wall is 0.20, determine the lowest speed v for which the block will remain in contact with the wall.

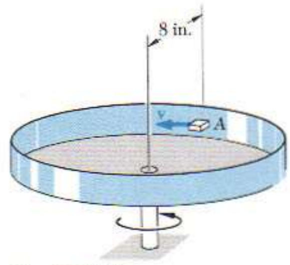


Fig. P12.56

12.57 A small ball rolls at a speed v_0 along a horizontal circle inside the circular cone shown. Express the speed v_0 in terms of the height y of the path above the apex of the cone.

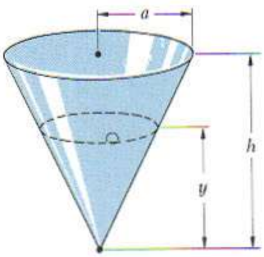


Fig. P12.57

12.58 A small ball rolls at a speed v_0 along a horizontal circle inside a bowl as shown. The inside surface of the bowl is a surface of revolution obtained by rotating the curve OA about the y axis. Determine the required equation of the curve OA if the speed v_0 of the ball is to be proportional to the distance x from the y axis to the ball.

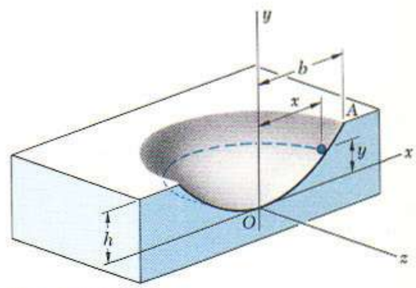


Fig. P12.58

12.59 Assuming that the equation of the curve OA in Prob. 12.58 is $y = kx^n$, where n is an arbitrary positive number, express the speed v_0 in terms of the height y of the path above the origin.

12.60 In the cathode-ray tube shown, electrons emitted by the cathode and attracted by the anode pass through a small hole in the anode and keep traveling in a straight line with a speed v_0 until they strike the screen at A. However, if a difference of potential V is established between the two parallel plates, each electron will be subjected to a force $\mathbf{F}$ perpendicular to the plates while it travels between the plates and will strike the screen at point B at a distance δ from A. The magnitude of the force $\mathbf{F}$ is $F = eV/d$, where $-e$ is the charge of the electron and d is the distance between the plates. Derive an expression for the deflection δ in terms of V, v_0, the charge $-e$ of the electron, its mass m, and the dimensions d, l, and L.

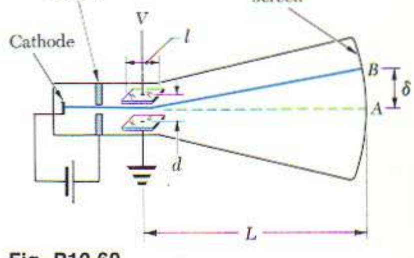


Fig. P12.60

12.61 A manufacturer wishes to design a new cathode-ray tube which will be only half as long as his current model. If the size of the screen is to remain the same, how should the length l of the plates be modified if all the other characteristics of the circuit are to remain unchanged? (See Prob. 12.60 for description of cathode-ray tube.)

12.62 In Prob. 12.60, determine the smallest allowable value of the ratio d/l in terms of e, m, v_0, and V if the electrons are not to strike the positive plate.

12.63 A cathode-ray tube emitting electrons with a velocity $\mathbf{v}_0$ is placed as shown between the poles of a large electromagnet which creates a uniform magnetic field of strength $\mathbf{B}$. Determine the coordinates of the point where the electron beam strikes the tube screen when no difference of potential exists between the plates. It is known that an electron (mass m and charge $-e$) traveling with a velocity $\mathbf{v}$ at a right angle to the lines of force of a magnetic field of strength $\mathbf{B}$ is subjected to a force $\mathbf{F} = e\mathbf{B} \times \mathbf{v}$.

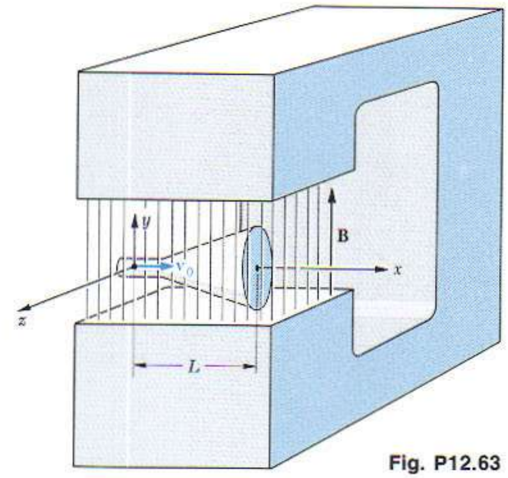


Fig. P12.63

12.6. Angular Momentum of a Particle. Rate of Change of Angular Momentum.

Consider a particle P of mass m moving with respect to a newtonian frame of reference $Oxyz$. As we saw in Sec. 12.2, the linear momentum of the particle at a given instant is defined as the vector $m\mathbf{v}$ obtained by multiplying the velocity $\mathbf{v}$ of the particle by its mass m. The moment about O of the vector $m\mathbf{v}$ is called the *moment of momentum*, or the *angular momentum*, of the particle about O at that instant and is denoted by $\mathbf{H}_O$. Recalling the definition of the moment of a vector (Sec. 3.5), and denoting by $\mathbf{r}$ the position vector of P, we write

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v} \tag{12.12}$$

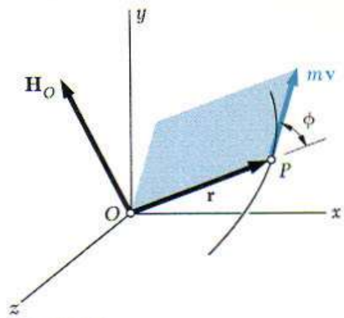


Fig. 12.12

and note that $\mathbf{H}_O$ is a vector perpendicular to the plane containing $\mathbf{r}$ and $m\mathbf{v}$, and of magnitude

$$H_O = rmv \sin \phi \tag{12.13}$$

where ϕ is the angle between $\mathbf{r}$ and $m\mathbf{v}$ (Fig. 12.12). The sense of $\mathbf{H}_O$ may be determined from the sense of $m\mathbf{v}$ by applying the right-hand rule. The unit of angular momentum is obtained by multiplying the units of length and of linear momentum (Sec. 12.3). With SI units we have

$$(\text{m})(\text{kg} \cdot \text{m/s}) = \text{kg} \cdot \text{m}^2/\text{s}$$

while, with U.S. customary units, we write

$$(\text{ft})(\text{lb} \cdot \text{s}) = \text{ft} \cdot \text{lb} \cdot \text{s}$$

Resolving the vectors $\mathbf{r}$ and $m\mathbf{v}$ into components, and applying formula (3.10), we write

$$\mathbf{H}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ mv_x & mv_y & mv_z \end{vmatrix} \tag{12.14}$$

The components of $\mathbf{H}_O$, which also represent the moments of the linear momentum $m\mathbf{v}$ about the coordinate axes, may be obtained by expanding the determinant in (12.14). We have

$$\begin{aligned} H_x &= m(yv_z - zv_y) \\ H_y &= m(zv_x - xv_z) \\ H_z &= m(xv_y - yv_x) \end{aligned} \tag{12.15}$$

In the case of a particle moving in the xy plane, we have $z = v_z = 0$ and the components H_x and H_y reduce to zero. The angular momentum is thus perpendicular to the xy plane; it is then completely defined by the scalar

$$H_O = H_z = m(xv_y - yv_x) \tag{12.16}$$

which will be positive or negative, according to the sense in which the particle is observed to move from O. If polar coordinates are used, we resolve the linear momentum of the particle into radial and transverse components (Fig. 12.13) and write

$$H_O = rmv \sin \phi = rmv_\theta \tag{12.17}$$

or, recalling from (11.45) that $v_\theta = r\dot{\theta}$,

$$H_O = mr^2\dot{\theta} \tag{12.18}$$

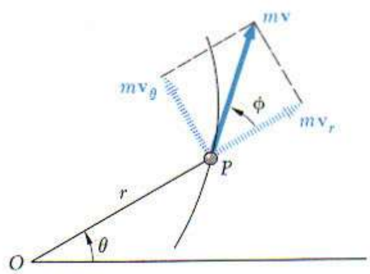


Fig. 12.13

We shall now compute the derivative with respect to t of the angular momentum $\mathbf{H}_O$ of a particle P moving in space. Differentiating both members of Eq. (12.12), and recalling the rule for the differentiation of a vector product (Sec. 11.10), we write

$$\dot{\mathbf{H}}_O = \dot{\mathbf{r}} \times m\mathbf{v} + \mathbf{r} \times m\dot{\mathbf{v}} = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times m\mathbf{a}$$

Since the vectors $\mathbf{v}$ and $m\mathbf{v}$ are collinear, the first term of the expression obtained is zero; and, by Newton's second law, $m\mathbf{a}$ is equal to the sum $\Sigma\mathbf{F}$ of the forces acting on P. Noting that $\mathbf{r} \times \Sigma\mathbf{F}$ represents the sum $\Sigma\mathbf{M}_O$ of the moments about O of these forces, we write

$$\Sigma\mathbf{M}_O = \dot{\mathbf{H}}_O \tag{12.19}$$

Equation (12.19), which results directly from Newton's second law, expresses that *the sum of the moments about O of the forces acting on the particle is equal to the rate of change of the moment of momentum, or angular momentum, of the particle about O.*

12.7. Equations of Motion in Terms of Radial and Transverse Components. Consider a particle P, of polar coordinates r and θ, which moves in a plane under the action of several forces. Resolving the forces and the acceleration of the particle into radial and transverse components (Fig. 12.14), and substituting into Eq. (12.2), we obtain the two scalar equations

$$\Sigma F_r = ma_r \qquad \Sigma F_\theta = ma_\theta \tag{12.20}$$

Substituting for a_r and a_θ from Eqs. (11.46), we have

$$\Sigma F_r = m(\ddot{r} - r\dot{\theta}^2) \tag{12.21}$$

$$\Sigma F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \tag{12.22}$$

The equations obtained may be solved for two unknowns.

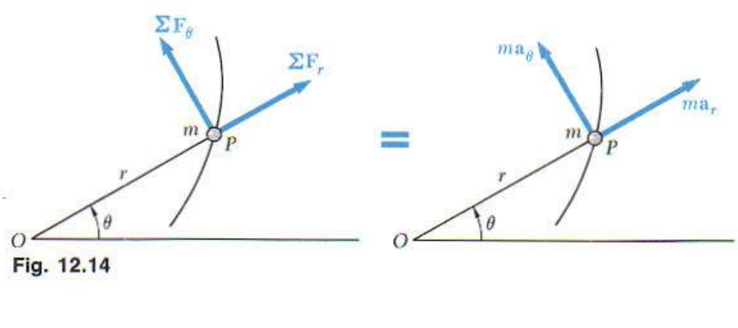


Fig. 12.14

Equation (12.22) could have been derived from Eq. (12.19). Recalling (12.18) and noting that $\Sigma M_O = r\Sigma F_\theta$, Eq. (12.19) yields

$$r\Sigma F_\theta = \frac{d}{dt}(mr^2\dot{\theta})$$
$$= m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta})$$

and, after dividing both members by r,

$$\Sigma F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \qquad (12.22)$$

12.8. Motion under a Central Force. Conservation of Angular Momentum. When the only force acting on a particle P is a force $\mathbf{F}$ directed toward or away from a fixed point O, the particle is said to be moving *under a central force*, and the point O is referred to as the *center of force* (Fig. 12.15). Since the line of action of $\mathbf{F}$ passes through O, we must have $\Sigma \mathbf{M}_O = 0$ at any given instant. Substituting into Eq. (12.19), we therefore obtain

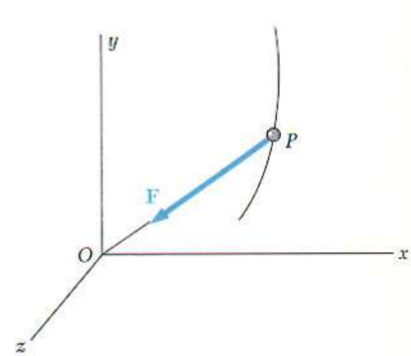


Fig. 12.15

$$\dot{\mathbf{H}}_O = 0$$

for all values of t or, integrating in t,

$$\mathbf{H}_O = \text{constant} \qquad (12.23)$$

We thus conclude that *the angular momentum of a particle moving under a central force is constant, both in magnitude and direction.*

Recalling the definition of the angular momentum of a particle (Sec. 12.6), we write

$$\mathbf{r} \times m\mathbf{v} = \mathbf{H}_O = \text{constant} \qquad (12.24)$$

from which it follows that the position vector $\mathbf{r}$ of the particle P must be perpendicular to the constant vector $\mathbf{H}_O$. Thus, a particle under a central force moves in a fixed plane perpendicular to $\mathbf{H}_O$. The vector $\mathbf{H}_O$ and the fixed plane are defined by the initial position vector $\mathbf{r}_0$ and the initial velocity $\mathbf{v}_0$ of the particle. For convenience, we shall assume that the plane of the figure coincides with the fixed plane of motion (Fig. 12.16).

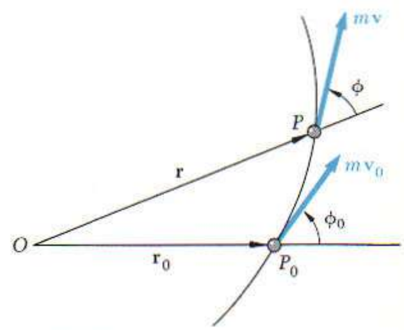


Fig. 12.16

Since the magnitude H_O of the angular momentum of the particle P is constant, the right-hand member in Eq. (12.13) must be constant. We therefore write

$$rmv \sin\phi = r_0 m v_0 \sin\phi_0 \qquad (12.25)$$

This relation applies to the motion of any particle under a central force. Since the gravitational force exerted by the sun on a planet is a central force directed toward the center of the sun, Eq. (12.25) is fundamental to the study of planetary motion. For a similar reason, it is also fundamental to the study of the motion of space vehicles in orbit about the earth.

Recalling Eq. (12.18), we may alternatively express the fact that the magnitude H_O of the angular momentum of the particle P is constant by writing

$$mr^2\dot{\theta} = H_O = \text{constant} \tag{12.26}$$

or, dividing by m and denoting by h the angular momentum per unit mass H_O/m,

$$r^2\dot{\theta} = h \tag{12.27}$$

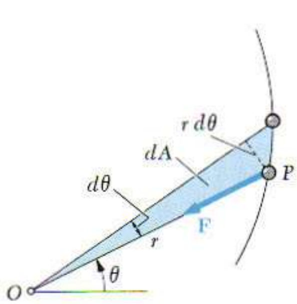


Fig. 12.17

Equation (12.27) may be given an interesting geometric interpretation. Observing from Fig. 12.17 that the radius vector OP sweeps an infinitesimal area $dA = \frac{1}{2}r^2\,d\theta$ as it rotates through an angle $d\theta$, and defining the *areal velocity* of the particle as the quotient dA/dt, we note that the left-hand member of Eq. (12.27) represents twice the areal velocity of the particle. We thus conclude that, *when a particle moves under a central force, its areal velocity is constant.*

12.9. Newton's Law of Gravitation. As we saw in the preceding section, the gravitational force exerted by the sun on a planet, or by the earth on an orbiting satellite, is an important example of a central force. In this section we shall learn how to determine the magnitude of a gravitational force.

In his *law of universal gravitation*, Newton states that two particles at a distance r from each other and, respectively, of mass M and m attract each other with equal and opposite forces $\mathbf{F}$ and $-\mathbf{F}$ directed along the line joining the particles (Fig. 12.18). The common magnitude F of the two forces is

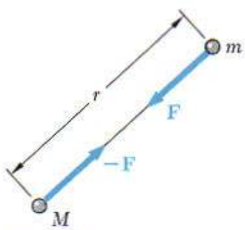


Fig. 12.18

$$F = G\frac{Mm}{r^2} \tag{12.28}$$

where G is a universal constant, called the *constant of gravitation*. Experiments show that the value of G is $(6.673 \pm 0.003) \times 10^{-11}\ \text{m}^3/\text{kg}\cdot\text{s}^2$ in SI units, or approximately $3.44 \times 10^{-8}\ \text{ft}^4/\text{lb}\cdot\text{s}^4$ in U.S. customary units. While gravitational forces exist between any pair of bodies, their effect is appreciable only when one of the bodies has a very large mass. The effect of gravitational forces is apparent in the case of the motion of a planet about the sun, of satellites orbiting about the earth, or of bodies falling on the surface of the earth.

Since the force exerted by the earth on a body of mass m located on or near its surface is defined as the weight $\mathbf{W}$ of the body, we may substitute the magnitude $W = mg$ of the weight for F, and the radius R of the earth for r, in Eq. (12.28). We obtain

$$W = mg = \frac{GM}{R^2}m \qquad \text{or} \qquad g = \frac{GM}{R^2} \tag{12.29}$$

where M is the mass of the earth. Since the earth is not truly spherical, the distance R from the center of the earth depends upon the point selected on its surface, and the values of W and g will thus vary with the altitude and latitude of the point considered. Another reason for the variation of W and g with the latitude is that a system of axes attached to the earth does not constitute a newtonian frame of reference (see Sec. 12.1). A more accurate definition of the weight of a body should therefore include a component representing the centrifugal force due to the rotation of the earth. Values of g at sea level vary from 9.781 m/s^2 or 32.09 ft/s^2 at the equator to 9.833 m/s^2 or 32.26 ft/s^2 at the poles.†

The force exerted by the earth on a body of mass m located in space at a distance r from its center may be found from Eq. (12.28). The computations will be somewhat simplified if we note that, according to Eq. (12.29), the product of the constant of gravitation G and of the mass M of the earth may be expressed as

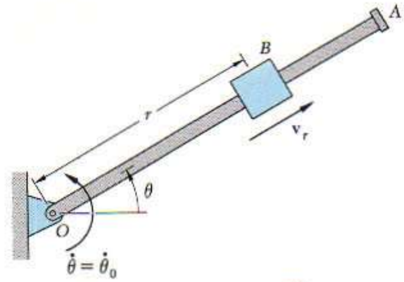
$$GM = gR^2 \tag{12.30}$$

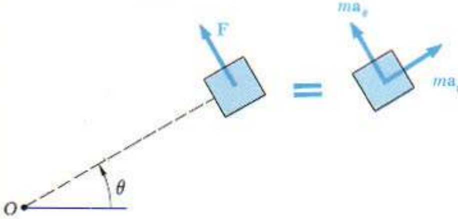
where g and the radius R of the earth will be given their average values $g = 9.81 \text{ m/s}^2$ and $R = 6.37 \times 10^6 \text{ m}$ in SI units,‡ or $g = 32.2 \text{ ft/s}^2$ and $R = (3960 \text{ mi})(5280 \text{ ft/mi})$ in U.S. customary units.

The discovery of the law of universal gravitation has often been attributed to the fact that Newton, after observing an apple falling from a tree, had reflected that the earth must attract an apple and the moon in much the same way. While it is doubtful that this incident actually took place, it may be said that Newton would not have formulated his law if he had not first perceived that the acceleration of a falling body must have the same cause as the acceleration which keeps the moon in its orbit. This basic concept of continuity of the gravitational attraction is more easily understood now, when the gap between the apple and the moon is being filled with long-range ballistic missiles and artificial earth satellites.

† A formula expressing g in terms of the latitude ϕ was given in Prob. 12.1.

‡ The value of R is easily found if one recalls that the circumference of the earth is $2\pi R = 40 \times 10^6$ m.





SAMPLE PROBLEM 12.6

A block B of mass m may slide freely on a frictionless arm OA which rotates in a horizontal plane at a constant rate $\dot{\theta}_0$. Knowing that B is released at a distance r_0 from O, express as a function of r, (*a*) the component v_r of the velocity of B along OA, (*b*) the magnitude of the horizontal force $\mathbf{F}$ exerted on B by the arm OA.

Solution. Since all other forces are perpendicular to the plane of the figure, the only force shown acting on B is the force $\mathbf{F}$ perpendicular to OA.

Equations of Motion. Using radial and transverse components:

$$+\nearrow \Sigma F_r = ma_r: \qquad 0 = m(\ddot{r} - r\dot{\theta}^2) \tag{1}$$

$$+\nwarrow \Sigma F_\theta = ma_\theta: \qquad F = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) \tag{2}$$

a. Component v_r of Velocity. Since $v_r = \dot{r}$, we have

$$\ddot{r} = \dot{v}_r = \frac{dv_r}{dt} = \frac{dv_r}{dr}\frac{dr}{dt} = v_r\frac{dv_r}{dr}$$

Substituting for $\ddot{r}$ into (1), recalling that $\dot{\theta} = \dot{\theta}_0$, and separating the variables:

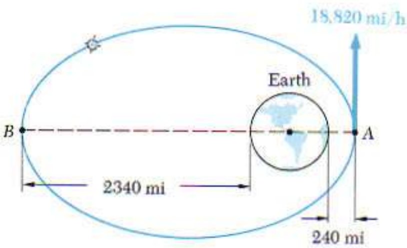
$$v_r\,dv_r = \dot{\theta}_0^2 r\,dr$$

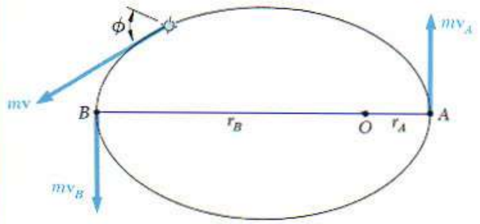
Multiplying by 2, and integrating from 0 to v_r and from r_0 to r:

$$v_r^2 = \dot{\theta}_0^2(r^2 - r_0^2) \qquad v_r = \dot{\theta}_0(r^2 - r_0^2)^{1/2} \blacktriangleleft$$

b. Horizontal Force F. Making $\dot{\theta} = \dot{\theta}_0$, $\ddot{\theta} = 0$, $\dot{r} = v_r$ in Eq. (2), and substituting for v_r the expression obtained in part *a*:

$$F = 2m\dot{\theta}_0(r^2 - r_0^2)^{1/2}\dot{\theta}_0 \qquad F = 2m\dot{\theta}_0^2(r^2 - r_0^2)^{1/2} \blacktriangleleft$$





SAMPLE PROBLEM 12.7

A satellite is launched in a direction parallel to the surface of the earth with a velocity of 18,820 mi/h from an altitude of 240 mi. Determine the velocity of the satellite as it reaches its maximum altitude of 2340 mi. It is recalled that the radius of the earth is 3960 mi.

Solution. Since the satellite is moving under a central force directed toward the center O of the earth, its angular momentum $\mathbf{H}_O$ is constant. From Eq. (12.13) we have

$$rmv \sin\phi = H_O = \text{constant}$$

which shows that v is minimum at B, where both r and $\sin\phi$ are maximum. Expressing conservation of angular momentum between A and B:

$$r_A m v_A = r_B m v_B$$

$$v_B = v_A\frac{r_A}{r_B} = (18{,}820\text{ mi/h})\frac{3960\text{ mi} + 240\text{ mi}}{3960\text{ mi} + 2340\text{ mi}}$$

$$v_B = 12{,}550\text{ mi/h} \blacktriangleleft$$

PROBLEMS

12.64 The two-dimensional motion of particle B is defined by the relations $r = t^2 - \frac{1}{3}t^3$ and $\theta = 2t^2$, where r is expressed in meters, t in seconds, and θ in radians. If the particle has a mass of 2 kg and moves in a horizontal plane, determine the radial and transverse components of the force acting on the particle when (*a*) $t = 0$, (*b*) $t = 1$ s.

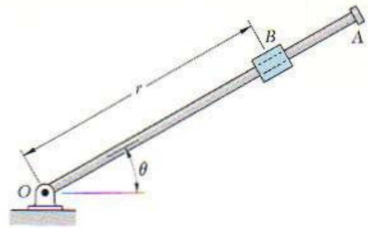


Fig. P12.64 and P12.66

12.65 For the motion defined in Prob. 12.64, determine the radial and transverse components of the force acting on the 2-kg particle as it returns to the origin at $t = 3$ s.

12.66 The two-dimensional motion of a particle B is defined by the relations $r = 10(1 + \cos 2\pi t)$ and $\theta = 2\pi t$, where r is expressed in inches, t in seconds, and θ in radians. If the particle weighs 2 lb and moves in a horizontal plane, determine the radial and transverse components of the force acting on the particle when (*a*) $t = 0$, (*b*) $t = 0.25$ s.

12.67 A block B of mass m may slide on the frictionless arm OA which rotates in a *horizontal* plane at a constant rate $\dot{\theta}_0$. As the arm rotates, the cord wraps around a *fixed* drum of radius b and pulls the block toward O with a speed $b\dot{\theta}_0$. Express as a function of m, r, b, and $\dot{\theta}_0$, (*a*) the tension T in the cord, (*b*) the magnitude of the horizontal force $\mathbf{Q}$ exerted on B by the arm OA.

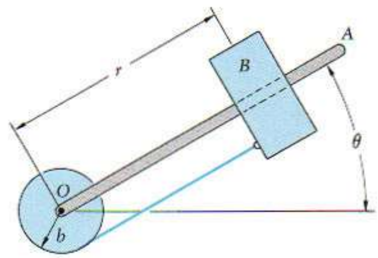


Fig. P12.67

12.68 Solve Prob. 12.67, knowing that the weight of the block is 3 lb and that $r = 2$ ft, $b = 3$ in., and $\dot{\theta}_0 = 8$ rad/s.

12.69 Slider C has a mass of 250 g and oscillates in the radial slot in arm AB as the arm rotates in a horizontal plane at a constant rate $\dot{\theta} = 12$ rad/s. In the position shown, it is known that the slider is moving outward along the slot at the speed of 1.5 m/s and that the spring is compressed and exerts a force of 10 N on the slider. Neglecting the effect of friction, determine (*a*) the components of the acceleration of the slider, (*b*) the horizontal force exerted on the slider by the arm AB.

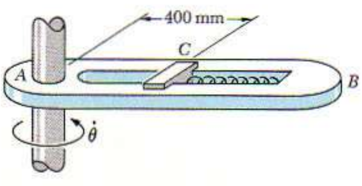


Fig. P12.69

12.70 While aiming at a moving target, a man rotates his rifle clockwise in a horizontal plane at the rate of 15° per second. Assuming that he can maintain the motion as the rifle is fired, determine the horizontal force exerted by the barrel on a 45-g bullet just before it leaves the barrel with a muzzle velocity of 550 m/s.

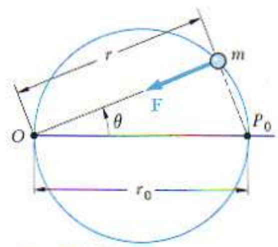


Fig. P12.71

12.71 A particle moves under a central force in a circular path of diameter r_0 which passes through the center of force O. Show that its speed is $v = v_0/\cos^2\theta$, where v_0 is the speed of the particle at point P_0 directly across the circle from O. [*Hint.* Use Eq. (12.27) with $r = r_0 \cos\theta$].

12.72 A particle moves under a central force in a path defined by the equation $r = r_0/\cos n\theta$, where n is a positive constant. Using Eq. (12.27) show that the radial and transverse components of the velocity are $v_r = nv_0 \sin n\theta$ and $v_\theta = v_0 \cos n\theta$, where v_0 is the velocity of the particle for $\theta = 0$. What is the motion of the particle when $n = 0$ and when $n = 1$?

12.73 For the particle and motion of Prob. 12.72, show that the radial and transverse components of the acceleration are $a_r = (n^2 - 1)(v_0^2/r_0)\cos^3 n\theta$ and $a_\theta = 0$.

12.74 If a particle of mass m is attached to the end of a very light circular rod as shown in (*1*), the rod exerts on the mass a force $\mathbf{F}$ of magnitude $F = kr$ directed toward the origin O, as shown in (*2*). The path of the particle is observed to be an ellipse with semiaxes $a = 6$ in. and $b = 2$ in. (*a*) Knowing that the speed of the particle at A is 8 in./s, determine the speed at B. (*b*) Further knowing that the constant k/m is equal to $16\ \text{s}^{-2}$, determine the radius of curvature of the path at A and at B.

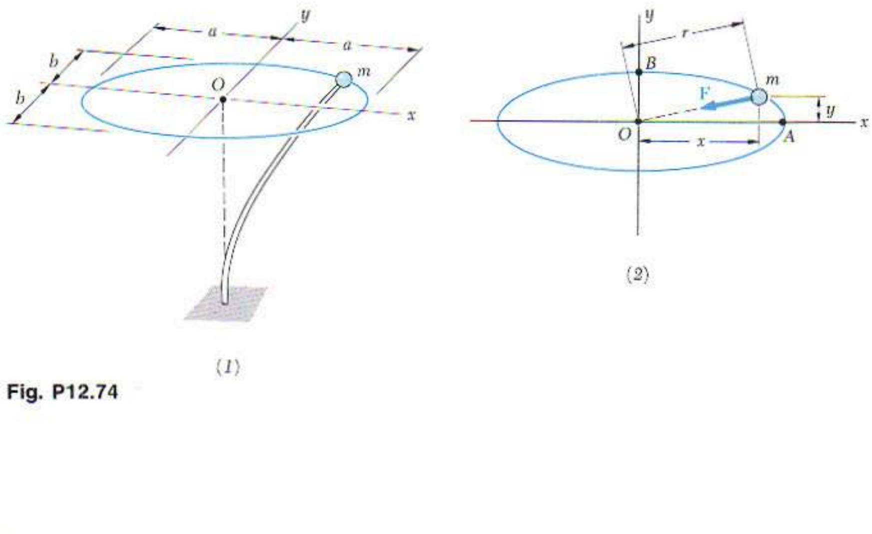


Fig. P12.74

12.75 Show that the radius r of the moon's orbit may be determined from the radius R of the earth, the acceleration of gravity g at the surface of the earth, and the time τ required by the moon to revolve once around the earth. Compute r knowing that $\tau = 27.3$ days.

12.76 Determine the mass of the earth from Newton's law of gravitation, knowing that it takes 94.14 min for a satellite to describe a circular orbit 300 mi above the surface of the earth.

12.77 Two solid steel spheres, each of radius 100 mm, are placed so that their surfaces are in contact. (*a*) Determine the force of gravitational attraction between the spheres, knowing that the density of steel is 7850 kg/m^3. (*b*) If the spheres are moved 2 mm apart and released with zero velocity, determine the approximate time required for their gravitational attraction to bring them back into contact. (*Hint.* Assume the gravitational forces to remain constant.)

12.78 Communication satellites have been placed in a geosynchronous orbit, i.e., in a circular orbit such that they complete one full revolution about the earth in one sidereal day (23 h 56 min), and thus appear stationary with respect to the ground. Determine (*a*) the altitude of the satellites above the surface of the earth, (*b*) the velocity with which they describe their orbit. Give the answers in both SI and U.S. customary units.

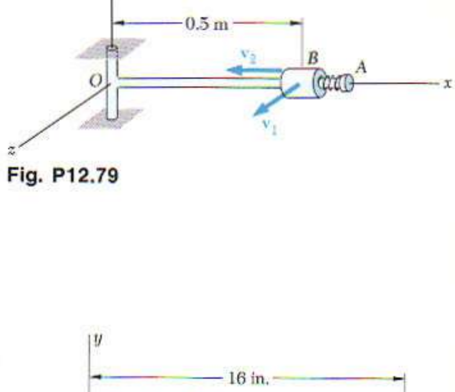


Fig. P12.79

12.79 Collar B may slide freely on rod OA, which in turn may rotate freely in the horizontal plane. The collar is describing a circle of radius 0.5 m with a speed $v_1 = 0.28$ m/s when a spring located between A and B is released, projecting the collar along the rod with an initial relative speed $v_2 = 0.96$ m/s. Neglecting the mass of the rod, determine the minimum distance between the collar and point O in the ensuing motion.

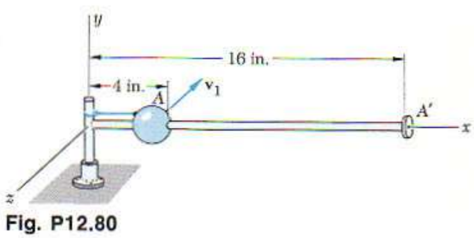


Fig. P12.80

12.80 A heavy ball is mounted on a horizontal rod which rotates freely about a vertical shaft. In the position shown, the speed of the ball is $v_1 = 30$ in./s and the ball is held by a cord attached to the shaft. The cord is suddenly cut and the ball moves to position A' as the rod rotates. Neglecting the mass of the rod, determine (*a*) the speed of the ball in position A', (*b*) the path (on the xz plane) of the ball as it moves from A to A'.

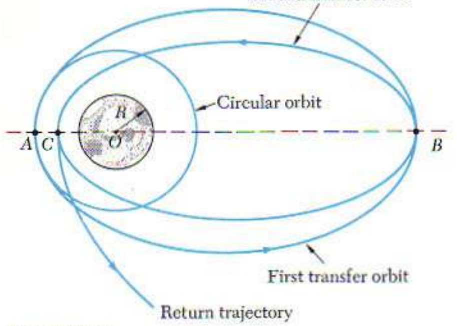


Fig. P12.81

12.81 Plans for an unmanned landing mission on the planet Mars call for the earth-return vehicle to first describe a circular orbit about the planet at an altitude $d_A = 2200$ km with a velocity of 2771 m/s. As it passes through point A, the vehicle will be inserted into an elliptic transfer orbit by firing its engine and increasing its speed by $\Delta v_A = 1046$ m/s. As it passes through point B, at an altitude $d_B = 100\,000$ km, the vehicle will be inserted into a second transfer orbit located in a slightly different plane, by changing the direction of its velocity and reducing its speed by $\Delta v_B = -22$ m/s. Finally, as the vehicle passes through point C, at an altitude $d_C = 1000$ km, its speed will be increased by $\Delta v_C = 660$ m/s to insert it into its return trajectory. Knowing that the radius of the planet Mars is $R = 3400$ km, determine the velocity of the vehicle after the last maneuver has been completed.

12.82 A space tug describes a circular orbit of 6000-mi radius around the earth. In order to transfer it to a larger circular orbit of 24,000-mi radius, the tug is first placed on an elliptic path AB by firing its engine as it passes through A, thus increasing its velocity by 3810 mi/h. By how much should the tug's velocity be increased as it reaches B to insert it into the larger circular orbit?

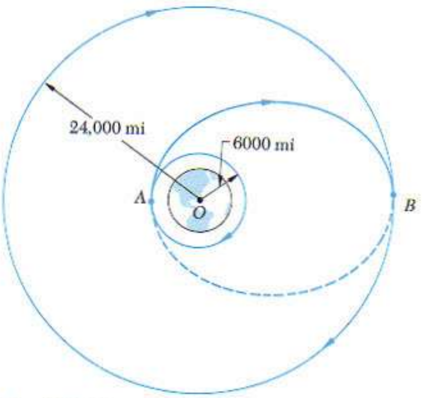


Fig. P12.82

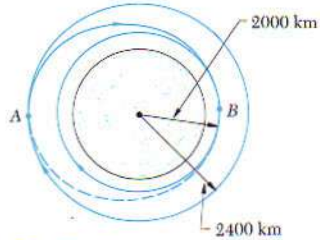


Fig. P12.83

12.83 An Apollo spacecraft describes a circular orbit of 2400-km radius around the moon with a velocity of 5140 km/h. In order to transfer it to a smaller circular orbit of 2000-km radius, the spacecraft is first placed on an elliptic path AB by reducing its velocity to 4900 km/h as it passes through A. Determine (a) the velocity of the spacecraft as it approaches B on the elliptic path, (b) the value to which its velocity must be reduced at B to insert it into the smaller circular orbit.

12.84 Solve Prob. 12.83, assuming that the Apollo spacecraft is to be transferred from the orbit of 2400-km radius to a circular orbit of 1800-km radius and that its velocity is reduced to 4760 km/h as it passes through A.

12.85 A 3-oz ball slides on a smooth horizontal table at the end of a string which passes through a small hole in the table at O. When the length of string above the table is $r_1 = 15$ in., the speed of the ball is $v_1 = 8$ ft/s. Knowing that the breaking strength of the string is 3.00 lb, determine (a) the smallest distance r_2 which can be achieved by slowly drawing the string through the hole, (b) the corresponding speed v_2.

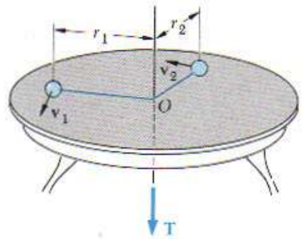


Fig. P12.85

12.86 A small ball swings in a horizontal circle at the end of a cord of length l_1 which forms an angle θ_1 with the vertical. The cord is then slowly drawn through the support at O until the free end is l_2. (a) Derive a relation between l_1, l_2, θ_1, and θ_2. (b) If the ball is set in motion so that, initially, $l_1 = 600$ mm and $\theta_1 = 30°$, determine the length l_2 for which $\theta_2 = 60°$.

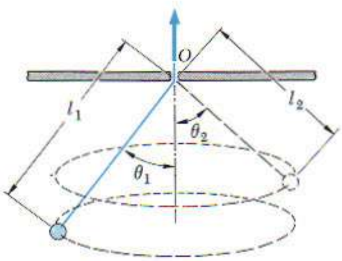


Fig. P12.86

*12.10. Trajectory of a Particle under a Central Force.

Consider a particle P moving under a central force $\mathbf{F}$. We propose to obtain the differential equation which defines its trajectory.

Assuming that the force $\mathbf{F}$ is directed toward the center of force O, we note that ΣF_r and ΣF_θ reduce respectively to $-F$ and zero in Eqs. (12.21) and (12.22). We therefore write

$$m(\ddot{r} - r\dot{\theta}^2) = -F \qquad (12.31)$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \qquad (12.32)$$

These equations define the motion of P. We shall, however, replace Eq. (12.32) by Eq. (12.27), which is more convenient to use and which is equivalent to Eq. (12.32), as we may easily check by differentiating it with respect to t. We write

$$r^2\dot{\theta} = h \qquad \text{or} \qquad r^2\frac{d\theta}{dt} = h \qquad (12.33)$$

Equation (12.33) may be used to eliminate the independent variable t from Eq. (12.31). Solving Eq. (12.33) for $\dot{\theta}$ or $d\theta/dt$, we have

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{h}{r^2} \tag{12.34}$$

from which it follows that

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{h}{r^2}\frac{dr}{d\theta} = -h\frac{d}{d\theta}\left(\frac{1}{r}\right) \tag{12.35}$$

$$\ddot{r} = \frac{d\dot{r}}{dt} = \frac{d\dot{r}}{d\theta}\frac{d\theta}{dt} = \frac{h}{r^2}\frac{d\dot{r}}{d\theta}$$

or, substituting for $\dot{r}$ from (12.35),

$$\ddot{r} = \frac{h}{r^2}\frac{d}{d\theta}\left[-h\frac{d}{d\theta}\left(\frac{1}{r}\right)\right]$$

$$\ddot{r} = -\frac{h^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) \tag{12.36}$$

Substituting for $\dot{\theta}$ and $\ddot{r}$ from (12.34) and (12.36), respectively, into Eq. (12.31), and introducing the function $u = 1/r$, we obtain after reductions

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{mh^2u^2} \tag{12.37}$$

In deriving Eq. (12.37), the force $\mathbf{F}$ was assumed directed toward O. The magnitude F should therefore be positive if $\mathbf{F}$ is actually directed toward O (attractive force) and negative if $\mathbf{F}$ is directed away from O (repulsive force). If F is a known function of r and thus of u, Eq. (12.37) is a differential equation in u and θ. This differential equation defines the trajectory followed by the particle under the central force $\mathbf{F}$. The equation of the trajectory will be obtained by solving the differential equation (12.37) for u as a function of θ and determining the constants of integration from the initial conditions.

***12.11. Application to Space Mechanics.** After the last stage of their launching rockets has burned out, earth satellites and other space vehicles are subjected only to the gravitational pull of the earth. Their motion may therefore be determined from Eqs. (12.33) and (12.37), which govern the motion of a particle under a central force, after F has been replaced by the expression obtained for the force of gravitational attraction.† Setting in Eq. (12.37)

$$F = \frac{GMm}{r^2} = GMmu^2$$

where M = mass of earth
m = mass of space vehicle
r = distance from center of earth to vehicle
$u = 1/r$

we obtain the differential equation

$$\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2} \tag{12.38}$$

where the right-hand member is observed to be a constant.

The solution of the differential equation (12.38) is obtained by adding the particular solution $u = GM/h^2$ to the general solution $u = C \cos(\theta - \theta_0)$ of the corresponding homogeneous equation (i.e., the equation obtained by setting the right-hand member equal to zero). Choosing the polar axis so that $\theta_0 = 0$, we write

$$\frac{1}{r} = u = \frac{GM}{h^2} + C \cos\theta \tag{12.39}$$

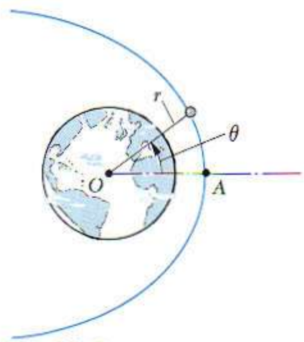


Fig. 12.19

Equation (12.39) is the equation of a *conic section* (ellipse, parabola, or hyperbola) in the polar coordinates r and θ. The origin O of the coordinates, which is located at the center of the earth, is a *focus* of this conic section, and the polar axis is one of its axes of symmetry (Fig. 12.19).

The ratio of the constants C and GM/h^2 defines the *eccentricity* ε of the conic section; setting

$$\varepsilon = \frac{C}{GM/h^2} = \frac{Ch^2}{GM} \tag{12.40}$$

we may write Eq. (12.39) in the form

$$\frac{1}{r} = \frac{GM}{h^2}(1 + \varepsilon \cos\theta) \tag{12.39'}$$

†It is assumed that the space vehicles considered here are attracted only by the earth and that their mass is negligible compared to the mass of the earth. If a vehicle moves very far from the earth, its path may be affected by the attraction of the sun, the moon, or another planet.

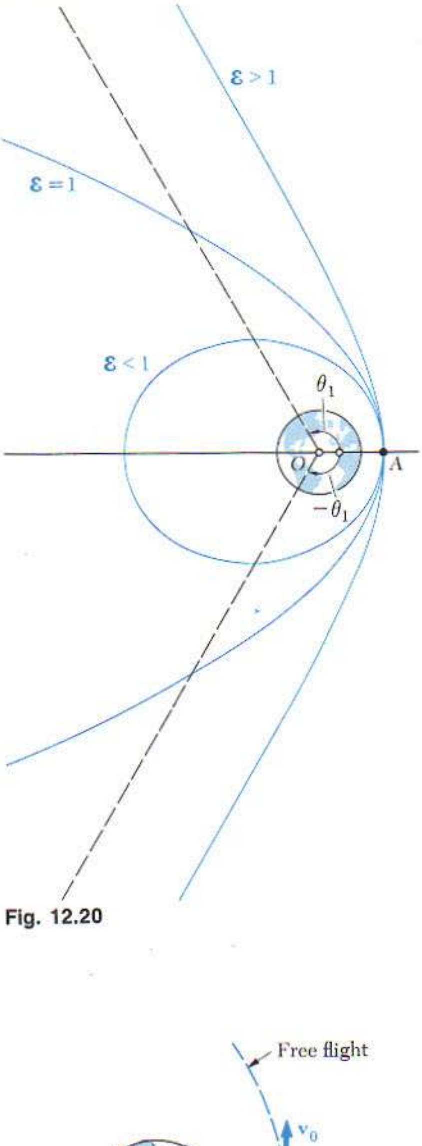


Fig. 12.20

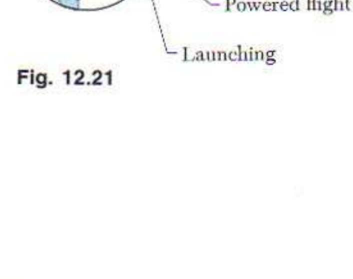


Fig. 12.21

Three cases may be distinguished:

1. $\varepsilon > 1$, or $C > GM/h^2$: There are two values θ_1 and $-\theta_1$ of the polar angle, defined by $\cos\theta_1 = -GM/Ch^2$, for which the right-hand member of Eq. (12.39) becomes zero. For both of these values, the radius vector r becomes infinite; the conic section is a *hyperbola* (Fig. 12.20).
2. $\varepsilon = 1$, or $C = GM/h^2$: The radius vector becomes infinite for $\theta = 180°$; the conic section is a *parabola*.
3. $\varepsilon < 1$, or $C < GM/h^2$: The radius vector remains finite for every value of θ; the conic section is an *ellipse*. In the particular case when $\varepsilon = C = 0$, the length of the radius vector is constant; the conic section is a circle.

We shall see now how the constants C and GM/h^2 which characterize the trajectory of a space vehicle may be determined from the position and the velocity of the space vehicle at the beginning of its free flight. We shall assume, as it is generally the case, that the powered phase of its flight has been programmed in such a way that, as the last stage of the launching rocket burns out, the vehicle has a velocity parallel to the surface of the earth (Fig. 12.21). In other words, we shall assume that the space vehicle begins its free flight at the vertex A of its trajectory.†

Denoting respectively by r_0 and v_0 the radius vector and speed of the vehicle at the beginning of its free flight, we observe, since the velocity reduces to its transverse component, that $v_0 = r_0\dot{\theta}_0$. Recalling Eq. (12.27), we express the angular momentum per unit mass h as

$$h = r_0^2\dot{\theta}_0 = r_0 v_0 \tag{12.41}$$

The value obtained for h may be used to determine the constant GM/h^2. We also note that the computation of this constant will be simplified if we use the relation indicated in Sec. 12.9,

$$GM = gR^2 \tag{12.30}$$

where R is the radius of the earth ($R = 6.37 \times 10^6$ m or 3960 mi) and g the acceleration of gravity at the surface of the earth.

The constant C will be determined by setting $\theta = 0$, $r = r_0$ in Eq. (12.39); we obtain

$$C = \frac{1}{r_0} - \frac{GM}{h^2} \tag{12.42}$$

† Problems involving oblique launchings will be considered in Sec. 13.9.

Substituting for h from (12.41), we may then easily express C in terms of r_0 and v_0.

Let us now determine the initial conditions corresponding to each of the three fundamental trajectories indicated above. Considering first the parabolic trajectory, we set C equal to GM/h^2 in Eq. (12.42) and eliminate h between Eqs. (12.41) and (12.42). Solving for v_0, we obtain

$$v_0 = \sqrt{\frac{2GM}{r_0}}$$

We may easily check that a larger value of the initial velocity corresponds to a hyperbolic trajectory, and a smaller value to an elliptic orbit. Since the value of v_0 obtained for the parabolic trajectory is the smallest value for which the space vehicle does not return to its starting point, it is called the *escape velocity*. We write therefore

$$v_{\text{esc}} = \sqrt{\frac{2GM}{r_0}} \qquad \text{or} \qquad v_{\text{esc}} = \sqrt{\frac{2gR^2}{r_0}} \tag{12.43}$$

if we make use of Eq. (12.30). We note that the trajectory will be (1) hyperbolic if $v_0 > v_{\text{esc}}$; (2) parabolic if $v_0 = v_{\text{esc}}$; (3) elliptic if $v_0 < v_{\text{esc}}$.

Among the various possible elliptic orbits, one is of special interest, the *circular orbit*, which is obtained when $C = 0$. The value of the initial velocity corresponding to a circular orbit is easily found to be

$$v_{\text{circ}} = \sqrt{\frac{GM}{r_0}} \qquad \text{or} \qquad v_{\text{circ}} = \sqrt{\frac{gR^2}{r_0}} \tag{12.44}$$

if Eq. (12.30) is taken into account. We may note from Fig. 12.22 that, for values of v_0 comprised between v_{circ} and v_{esc}, point A where free flight begins is the point of the orbit closest to the earth; this point is called the *perigee*, while point A', which is farthest away from the earth, is known as the *apogee*. For values of v_0 smaller than v_{circ}, point A becomes the apogee, while point A'', on the other side of the orbit, becomes the perigee. For values of v_0 much smaller than v_{circ}, the trajectory of the space vehicle intersects the surface of the earth; in such a case, the vehicle does not go into orbit.

Ballistic missiles, which are designed to hit the surface of the earth, also travel along elliptic trajectories. In fact, we should now realize that any object projected in vacuum with an initial

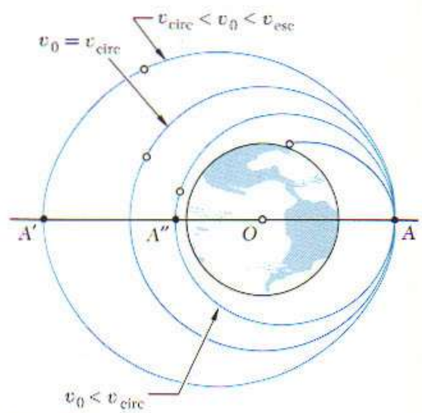


Fig. 12.22

velocity v_0 smaller than v_{esc} will move along an elliptic path. It is only when the distances involved are small that the gravitational field of the earth may be assumed uniform, and that the elliptic path may be approximated by a parabolic path, as was done earlier (Sec. 11.11) in the case of conventional projectiles.

Periodic Time. An important characteristic of the motion of an earth satellite is the time required by the satellite to describe its orbit. This time is known as the *periodic time* of the satellite and is denoted by τ. We first observe, in view of the definition of the areal velocity (Sec. 12.8), that τ may be obtained by dividing the area inside the orbit by the areal velocity. Since the area of an ellipse is equal to πab, where a and b denote, respectively, the semimajor and semiminor axes, and since the areal velocity is equal to $h/2$, we write

$$\tau = \frac{2\pi ab}{h} \tag{12.45}$$

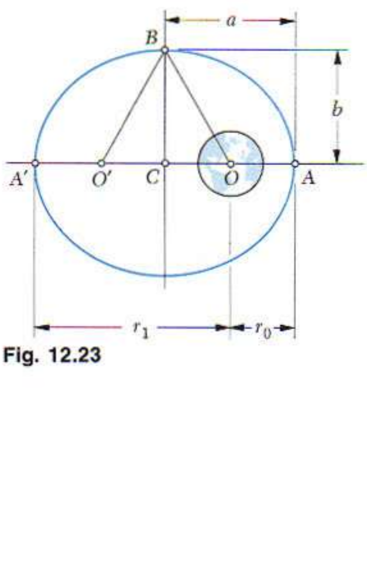


Fig. 12.23

While h may be readily determined from r_0 and v_0 in the case of a satellite launched in a direction parallel to the surface of the earth, the semiaxes a and b are not directly related to the initial conditions. Since, on the other hand, the values r_0 and r_1 of r corresponding to the perigee and apogee of the orbit may easily be determined from Eq. (12.39), we shall express the semiaxes a and b in terms of r_0 and r_1.

Consider the elliptic orbit shown in Fig. 12.23. The earth's center is located at O and coincides with one of the two foci of the ellipse, while the points A and A' represent, respectively, the perigee and apogee of the orbit. We easily check that

$$r_0 + r_1 = 2a$$

and thus

$$a = \tfrac{1}{2}(r_0 + r_1) \tag{12.46}$$

Recalling that the sum of the distances from each of the foci to any point of the ellipse is constant, we write

$$O'B + BO = O'A + OA = 2a \qquad \text{or} \qquad BO = a$$

On the other hand, we have $CO = a - r_0$. We may therefore write

$$b^2 = (BC)^2 = (BO)^2 - (CO)^2 = a^2 - (a - r_0)^2$$
$$b^2 = r_0(2a - r_0) = r_0 r_1$$

and thus

$$b = \sqrt{r_0 r_1} \tag{12.47}$$

Formulas (12.46) and (12.47) indicate that the semimajor and semiminor axes of the orbit are respectively equal to the arithmetic and geometric means of the maximum and minimum values of the radius vector. Once r_0 and r_1 have been determined, the lengths of the semiaxes may thus be easily computed and substituted for a and b in formula (12.45).

***12.12. Kepler's Laws of Planetary Motion.** The equations governing the motion of an earth satellite may be used to describe the motion of the moon around the earth. In that case, however, the mass of the moon is not negligible compared to the mass of the earth, and the results obtained are not entirely accurate.

The theory developed in the preceding sections may also be applied to the study of the motion of the planets around the sun. While another error is introduced by neglecting the forces exerted by the planets on each other, the approximation obtained is excellent. Indeed, the properties expressed by Eq. (12.39), where M now represents the mass of the sun, and by Eq. (12.33) had been discovered by the German astronomer Johann Kepler (1571–1630) from astronomical observations of the motion of the planets, even before Newton had formulated his fundamental theory.

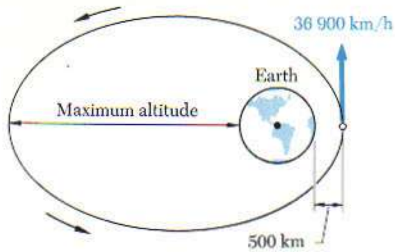
Kepler's three *laws of planetary motion* may be stated as follows:

1. Each planet describes an ellipse, with the sun located at one of its foci.
2. The radius vector drawn from the sun to a planet sweeps equal areas in equal times.
3. The squares of the periodic times of the planets are proportional to the cubes of the semimajor axes of their orbits.

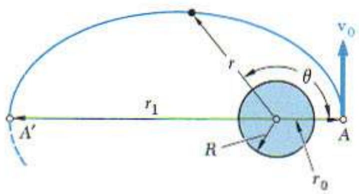
The first law states a particular case of the result established in Sec. 12.11, while the second law expresses that the areal velocity of each planet is constant (see Sec. 12.8). Kepler's third law may also be derived from the results obtained in Sec. 12.11.†

† See Prob. 12.104

SAMPLE PROBLEM 12.8



A satellite is launched in a direction parallel to the surface of the earth with a velocity of 36 900 km/h from an altitude of 500 km. Determine (*a*) the maximum altitude reached by the satellite, (*b*) the periodic time of the satellite.



a. Maximum Altitude. After launching, the satellite is subjected only to the gravitational attraction of the earth; its motion is thus governed by Eq. (12.39).

$$\frac{1}{r} = \frac{GM}{h^2} + C\cos\theta \tag{1}$$

Since the radial component of the velocity is zero at the point of launching A, we have $h = r_0 v_0$. Recalling that the radius of the earth is $R = 6370$ km, we compute

$$r_0 = 6370 \text{ km} + 500 \text{ km} = 6870 \text{ km} = 6.87 \times 10^6 \text{ m}$$

$$v_0 = 36\,900 \text{ km/h} = \frac{3.69 \times 10^7 \text{ m}}{3.6 \times 10^3 \text{ s}} = 1.025 \times 10^4 \text{ m/s}$$

$$h = r_0 v_0 = (6.87 \times 10^6 \text{ m})(1.025 \times 10^4 \text{ m/s}) = 7.04 \times 10^{10} \text{ m}^2\text{/s}$$

$$h^2 = 4.96 \times 10^{21} \text{ m}^4\text{/s}^2$$

Since $GM = gR^2$, where R is the radius of the earth, we have

$$GM = gR^2 = (9.81 \text{ m/s}^2)(6.37 \times 10^6 \text{ m})^2 = 3.98 \times 10^{14} \text{ m}^3\text{/s}^2$$

$$\frac{GM}{h^2} = \frac{3.98 \times 10^{14} \text{ m}^3\text{/s}^2}{4.96 \times 10^{21} \text{ m}^4\text{/s}^2} = 8.03 \times 10^{-8} \text{ m}^{-1}$$

Substituting this value into (1), we obtain

$$\frac{1}{r} = 8.03 \times 10^{-8} + C\cos\theta \tag{2}$$

Noting that at point A we have $\theta = 0$ and $r = r_0 = 6.87 \times 10^6$ m, we compute the constant C.

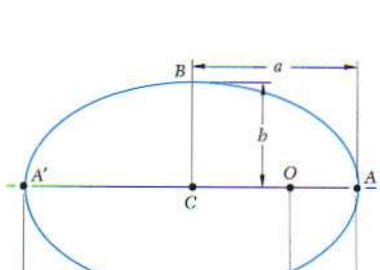
$$\frac{1}{6.87 \times 10^6 \text{ m}} = 8.03 \times 10^{-8} + C\cos 0^\circ \qquad C = 6.53 \times 10^{-8} \text{ m}^{-1}$$

At A', the point on the orbit farthest from the earth, we have $\theta = 180^\circ$. Using (2), we compute the corresponding distance r_1.

$$\frac{1}{r_1} = 8.03 \times 10^{-8} + (6.53 \times 10^{-8})\cos 180^\circ$$

$$r_1 = 0.667 \times 10^8 \text{ m} = 66\,700 \text{ km}$$

Maximum altitude = 66 700 km − 6370 km = 60 300 km ◄



b. Periodic Time. Since A and A' are the perigee and apogee, respectively, of the elliptic orbit, we use Eqs. (12.46) and (12.47) and compute the semimajor and semiminor axes of the orbit.

$$a = \tfrac{1}{2}(r_0 + r_1) = \tfrac{1}{2}(6.87 + 66.7)(10^6) \text{ m} = 36.8 \times 10^6 \text{ m}$$

$$b = \sqrt{r_0 r_1} = \sqrt{(6.87)(66.7)} \times 10^6 \text{ m} = 21.4 \times 10^6 \text{ m}$$

$$\tau = \frac{2\pi ab}{h} = \frac{2\pi(36.8 \times 10^6 \text{ m})(21.4 \times 10^6 \text{ m})}{7.04 \times 10^{10} \text{ m}^2\text{/s}}$$

$$\tau = 7.03 \times 10^4 \text{ s} = 1171 \text{ min} = 19 \text{ h } 31 \text{ min} \quad ◄$$

PROBLEMS

12.87 A spacecraft is describing a circular orbit at an altitude of 240 mi above the surface of the earth when its engine is fired and its speed increased by 4000 ft/s. Determine the maximum altitude reached by the spacecraft.

12.88 A space tug is used to place communication satellites into a geosynchronous orbit (see Prob. 12.78) at an altitude of 22,230 mi above the surface of the earth. Knowing that the tug initially describes a circular orbit at an altitude of 220 mi, determine (*a*) the increase in speed required at A to insert the tug into an elliptic transfer orbit, (*b*) the increase in speed required at B to insert the tug into the geosynchronous orbit.

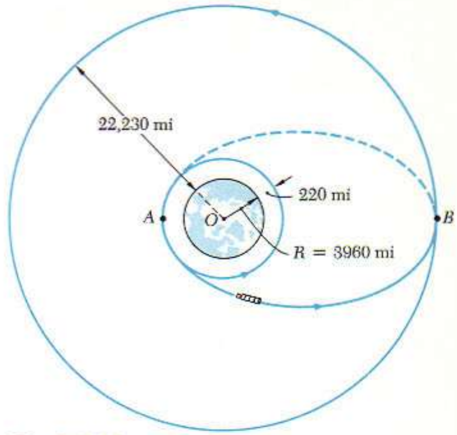


Fig. P12.88

12.89 Plans for an unmanned landing mission on the planet Mars call for the earth-return vehicle to first describe a circular orbit about the planet. As it passes through point A, the vehicle will be inserted into an elliptic transfer orbit by firing its engine and increasing its speed by Δv_A. As it passes through point B, the vehicle will be inserted into a second transfer orbit located in a slightly different plane, by changing the direction of its velocity and by reducing its speed by Δv_B. Finally, as the vehicle passes through point C, its speed will be increased by Δv_C to insert it into its return trajectory. Knowing that the radius of the planet Mars is $R = 3400$ km, that its mass is 0.108 times the mass of the earth, and that the altitudes of points A and B are, respectively, $d_A = 2500$ km and $d_B = 90\,000$ km, determine the increase in speed Δv_A required at point A to insert the vehicle into its first transfer orbit.

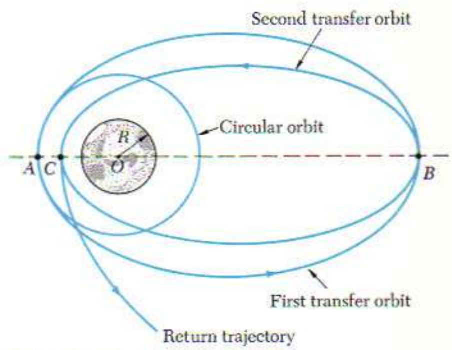


Fig. P12.89

12.90 For the vehicle of Prob. 12.89, it is known that the altitudes of points A, B, and C are, respectively, $d_A = 2500$ km, $d_B = 90\,000$ km, and $d_C = 1000$ km. Determine the change in speed Δv_B required at point B to insert the vehicle into its second transfer orbit.

12.91 For the vehicle of Prob. 12.89, it is known that the altitudes of points B and C are, respectively, $d_B = 90\,000$ km and $d_C = 1000$ km. Determine the minimum increase in speed Δv_C required at point C to insert the vehicle into an escape trajectory.

12.92 For the vehicle of Prob. 12.89, it is known that the altitude of point B is $d_B = 90\,000$ km. If, for a given mission, the speed of the vehicle is 215 m/s immediately after its insertion into the second transfer orbit, determine (*a*) the altitude of point C, (*b*) the speed of the vehicle as it approaches point C, (*c*) the eccentricity of the return trajectory if the speed of the vehicle is increased at C by $\Delta v_C = 630$ m/s.

12.93 After completing their moon-exploration mission, the two astronauts forming the crew of an Apollo lunar excursion module (LEM) prepare to rejoin the command module which is orbiting the moon at an altitude of 85 mi. They fire the LEM's engine, bring it along a curved path to a point A, 5 mi above the moon's surface, and shut off the engine. Knowing that the LEM is moving at that time in a direction parallel to the moon's surface and that it will coast along an elliptic path to a rendezvous at B with the command module, determine (*a*) the speed of the LEM at engine shutoff, (*b*) the relative velocity with which the command module will approach the LEM at B. (The radius of the moon is 1080 miles and its mass is 0.01230 times the mass of the earth.)

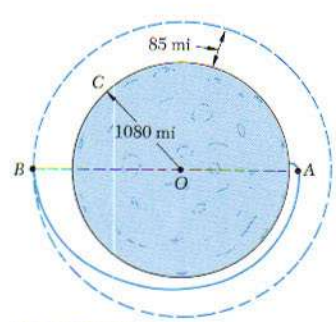


Fig. P12.93

12.94 Solve Prob. 12.93, assuming that the Apollo command module is orbiting the moon at an altitude of 55 mi.

12.95 Referring to Prob. 12.89, determine the time required for the vehicle to describe its first transfer orbit form A to B.

12.96 Referring to Probs. 12.89 and 12.90, determine the time required for the vehicle to describe its second transfer orbit from B to C.

12.97 Determine the time required for the LEM of Prob. 12.93 to travel from A to B.

12.98 Determine the time required for the space tug of Prob. 12.88 to travel from A to B.

12.99 Determine the approximate time required for an object to fall to the surface of the earth after being released with no velocity from a distance equal to the radius of the orbit of the moon, namely, 239,000 mi. (*Hint.* Assume that the object is given a very small initial velocity in a transverse direction, say, $v_\theta = 1$ ft/s, and determine the periodic time τ of the object on the resulting orbit. An examination of the orbit will show that the time of fall must be approximately equal to $\frac{1}{2}\tau$.)

12.100 A spacecraft describes a circular orbit at an altitude of 3200 km above the earth's surface. Preparatory to reentry it reduces its speed to a value $v_0 = 5400$ m/s, thus placing itself on an elliptic trajectory. Determine the value of θ defining the point B where splashdown will occur. (*Hint.* Point A is the apogee of the elliptic trajectory.)

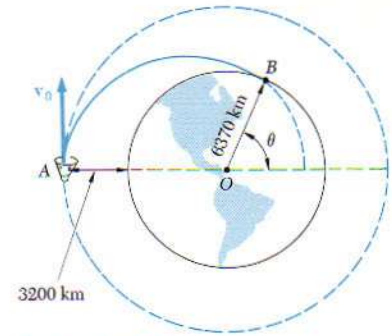


Fig. P12.100 and P12.101

12.101 A spacecraft describes a circular orbit at an altitude of 3200 km above the earth's surface. Preparatory to reentry it places itself on an elliptic trajectory by reducing its speed to a value v_0. Determine v_0 so that splashdown will occur at a point B corresponding to $\theta = 120°$. (See hint of Prob. 12.100.)

12.102 Upon the LEM's return to the command module, the Apollo spacecraft of Prob. 12.93 is turned around so that the LEM faces to the rear. After completing a full orbit, i.e., as the craft passes again through B, the LEM is cast adrift and crashes on the moon's surface at point C. Determine the velocity of the LEM relative to the command module as it is cast adrift, knowing that the angle BOC is 90°. (*Hint.* Point B is the apogee of the elliptic crash trajectory.)

12.103 Upon the LEM's return to the command module, the Apollo spacecraft of Prob. 12.93 is turned around so that the LEM faces to the rear. After completing a full orbit, i.e., as the craft passes again through B, the LEM is cast adrift with a velocity of 600 ft/s relative to the command module. Determine the point C where the LEM will crash on the moon's surface. (See hint of Prob. 12.102.)

12.104 Derive Kepler's third law of planetary motion from Eqs. (12.39) and (12.45).

12.105 (*a*) Express the eccentricity ε of the elliptic orbit described by a satellite about the earth (or any other planet) in terms of the distances r_0 and r_1 corresponding, respectively, to the perigee and apogee of the orbit. (*b*) Use the result obtained in part *a* to determine the eccentricities of the two transfer orbits described in Probs. 12.89 and 12.90.

12.106 Two space stations S_1 and S_2 are describing coplanar circular counterclockwise orbits of radius r_0 and $8r_0$, respectively, around the earth. It is desired to send a vehicle from S_1 to S_2. The vehicle is to be launched in a direction tangent to the orbit of S_1 and is to reach S_2 with a velocity tangent to the orbit of S_2. After a short powered phase, the vehicle will travel in free flight from S_1 to S_2. (*a*) Determine the launching velocity (velocity of the vehicle relative to S_1) in terms of the velocity $\mathbf{v}_0$ of S_1. (*b*) Determine the angle θ defining the required position of S_2 relative to S_1 at the time of launching.

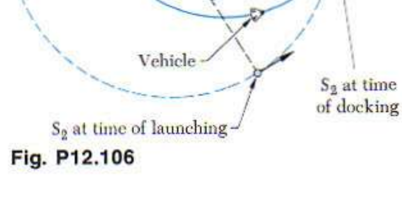


Fig. P12.106

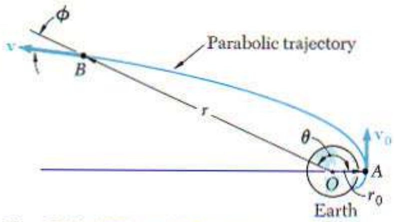


Fig. P12.107 and P12.108

∗12.107 A space vehicle is inserted at point A, at a distance r_0 from the center O of the earth, into a parabolic trajectory. (*a*) For any position B of the vehicle on its trajectory, express the distance r from O to B and the time t elapsed since the insertion of the vehicle into its trajectory in terms of θ, r_0, g, and the radius R of the earth. (*b*) Use the result obtained in part *a*, assuming $r_0 = 4300$ mi, to determine the time required for the space vehicle to reach a distance r equal to the radius of the orbit of the moon (239,000 mi).

∗12.108 A space vehicle is inserted at point A, at a distance r_0 from the center O of the earth, into a parabolic trajectory. (*a*) For any position B of the vehicle on its trajectory, express in terms of θ, r_0, g, and the radius R of the earth (*a*) the magnitude of the velocity $\mathbf{v}$ of the vehicle, (*b*) the angle ϕ that $\mathbf{v}$ forms with the line OB.

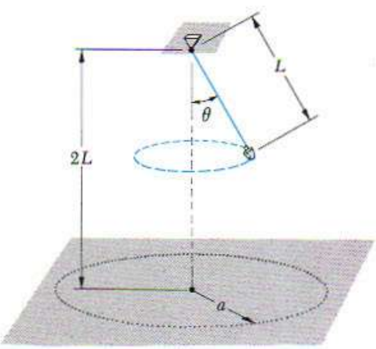


Fig. P12.109

REVIEW PROBLEMS

12.109 A bucket is attached to a rope of length $L = 1.2$ m and is made to revolve in a horizontal circle. Drops of water leaking from the bucket fall and strike the floor along the perimeter of a circle of radius a. Determine the radius a when $\theta = 30°$.

12.110 Determine the radius a in Prob. 12.109, assuming that the speed of the bucket is 5 m/s. (The angle θ is not 30° in this case.)

12.111 Determine the required tension T if the acceleration of the 500-lb cylinder is to be (*a*) 6 ft/s^2 upward, (*b*) 6 ft/s^2 downward.

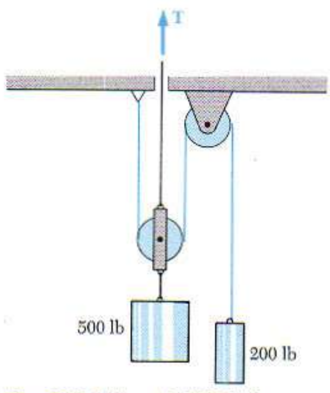


Fig. P12.111 and P12.112

12.112 Determine the acceleration of the 200-lb cylinder if (*a*) $T = 300$ lb, (*b*) $T = 800$ lb.

12.113 A series of small packages, being moved by a conveyor belt at a constant speed v, passes over an idler roller as shown. Knowing that the coefficient of friction between the packages and the belt is 0.75, determine the maximum value of v for which the packages do not slip with respect to the belt.

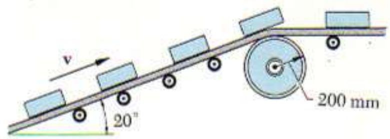


Fig. P12.113

12.114 A 4-kg collar slides without friction along a rod which forms an angle of 30° with the vertical. The spring, of constant $k = 150$ N/m, is unstretched when the collar is at A. Determine the initial acceleration of the collar if it is released from rest at point B.

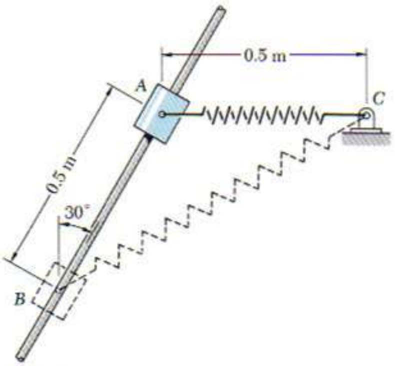


Fig. P12.114

12.115 (*a*) Express the rated speed of a banked road in terms of the radius r of the curve and the banking angle θ. (*b*) What is the apparent weight of an automobile traveling at the rated speed? (See Sample Prob. 12.5 for the definition of rated speed.)

12.116 Denoting by v_t the terminal speed of an object dropped from a great height, determine the distance the object will fall before its speed reaches the value $\frac{1}{2}v_t$. Assume that the frictional resistance of the air is proportional to the square of the speed of the object.

12.117 A spacecraft is describing a circular orbit of radius r_0 with a speed v_0 around an unspecified celestial body of center O, when its engine is suddenly fired, increasing the speed of the spacecraft from v_0 to αv_0, where $1 < \alpha^2 < 2$. Show that the maximum distance r_{max} from O reached by the spacecraft depends only upon r_0 and α, and express the ratio r_{max}/r_0 as a function of α.

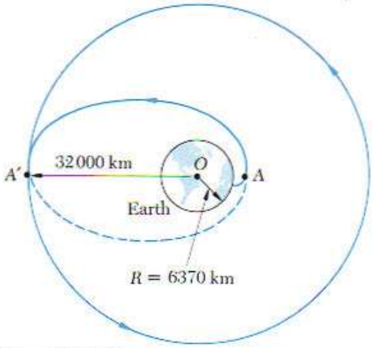


Fig. P12.118

12.118 In order to place a satellite in a circular orbit of radius 32×10^3 km around the earth, the satellite is first projected horizontally from A at an altitude of 500 km into an elliptic path whose apogee A' is at a distance of 32×10^3 km from the center of the earth. Auxiliary rockets are fired as the satellite reaches A' in order to place it in its final orbit. Determine (a) the initial velocity of the satellite at A, (b) the increase in velocity resulting from the firing of the rockets at A'.

12.119 Two packages are placed on a conveyor belt which is at rest. The coefficient of friction is 0.20 between the belt and package A, and 0.10 between the belt and package B. If the belt is suddenly started to the right and slipping occurs between the belt and the packages, determine (a) the acceleration of the packages, (b) the force exerted by package A on package B.

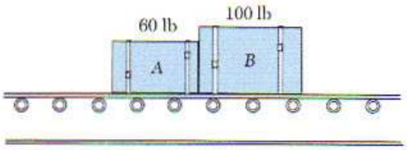


Fig. P12.119

12.120 Two plates A and B, each of mass 50 kg, are placed as shown on a 15° incline. The coefficient of friction between A and B is 0.10; the coefficient of friction between A and the incline is 0.20. (a) If the plates are released from rest, determine the acceleration of each plate. (b) Solve part a assuming that plates A and B are welded together and act as a single rigid body.

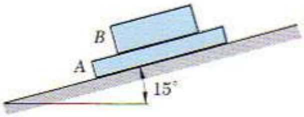


Fig. P12.120

Kinetics of Particles: Energy and Momentum Methods

CHAPTER **13**

13.1. Introduction. In the preceding chapter, most problems dealing with the motion of particles were solved through the use of the fundamental equation of motion $\mathbf{F} = m\mathbf{a}$. Given a particle acted upon by a force $\mathbf{F}$, we could solve this equation for the acceleration $\mathbf{a}$; then, by applying the principles of kinematics, we could determine from $\mathbf{a}$ the velocity and position of the particle at any time.

If the equation $\mathbf{F} = m\mathbf{a}$ and the principles of kinematics are combined, two additional methods of analysis may be obtained, the *method of work and energy* and the *method of impulse and momentum*. The advantage of these methods lies in the fact that they make the determination of the acceleration unnecessary. Indeed, the method of work and energy relates directly force, mass, velocity, and displacement, while the method of impulse and momentum relates force, mass, velocity, and time.

The method of work and energy will be considered first. It is based on two important concepts, the concept of the *work of a force* and the concept of the *kinetic energy of a particle*. These concepts are defined in the following sections.

13.2. Work of a Force. We shall first define the terms *displacement* and *work* as they are used in mechanics.† Consider a particle which moves from a point A to a neighboring point A'

† The definition of work was given in Sec. 10.1, and the basic properties of the work of a force were outlined in Secs. 10.1 and 10.5. For convenience, we repeat here the portions of this material which relate to the kinetics of particles.

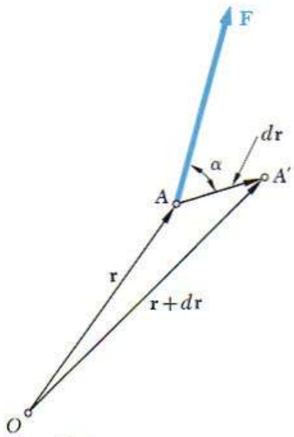


Fig. 13.1

(Fig. 13.1). If $\mathbf{r}$ denotes the position vector corresponding to point A, the small vector joining A and A' may be denoted by the differential $d\mathbf{r}$; the vector $d\mathbf{r}$ is called the *displacement* of the particle. Now, let us assume that a force $\mathbf{F}$ is acting on the particle. The *work of the force* $\mathbf{F}$ *corresponding to the displacement* $d\mathbf{r}$ is defined as the quantity

$$dU = \mathbf{F} \cdot d\mathbf{r} \tag{13.1}$$

obtained by forming the scalar product of the force $\mathbf{F}$ and of the displacement $d\mathbf{r}$. Denoting respectively by F and ds the magnitudes of the force and of the displacement, and by α the angle formed by $\mathbf{F}$ and $d\mathbf{r}$, and recalling the definition of the scalar product of two vectors (Sec. 3.8), we write

$$dU = F\,ds\cos\alpha \tag{13.1$'$}$$

Using formula (3.30), we may also express the work dU in terms of the rectangular components of the force and of the displacement:

$$dU = F_x\,dx + F_y\,dy + F_z\,dz \tag{13.1$''$}$$

Being a *scalar quantity*, work has a magnitude and a sign, but no direction. We also note that work should be expressed in units obtained by multiplying units of length by units of force. Thus, if U.S. customary units are used, work should be expressed in ft · lb or in · lb. If SI units are used, work should be expressed in N · m. The unit of work N · m is called a *joule* (J).† Recalling the conversion factors indicated in Sec. 12.3, we write

$$1\ \text{ft}\cdot\text{lb} = (1\ \text{ft})(1\ \text{lb}) = (0.3048\ \text{m})(4.448\ \text{N}) = 1.356\ \text{J}$$

It follows from (13.1′) that the work dU is positive if the angle α is acute, and negative if α is obtuse. Three particular cases are of special interest. If the force $\mathbf{F}$ has the same direction as $d\mathbf{r}$, the work dU reduces to $F\,ds$. If $\mathbf{F}$ has a direction opposite to that of $d\mathbf{r}$, the work is $dU = -F\,ds$. Finally, if $\mathbf{F}$ is perpendicular to $d\mathbf{r}$, the work dU is zero.

The work of $\mathbf{F}$ during a *finite* displacement of the particle from A_1 to A_2 (Fig. 13.2*a*) is obtained by integrating Eq. (13.1) along the path described by the particle. This work, denoted by $U_{1\to 2}$, is

† The joule (J) is the SI unit of *energy*, whether in mechanical form (work, potential energy, kinetic energy) or in chemical, electrical, or thermal form. We should note that, even though N · m = J, the moment of a force must be expressed in N · m, and not in joules, since the moment of a force is not a form of energy.

$$U_{1\to 2} = \int_{A_1}^{A_2} \mathbf{F} \cdot d\mathbf{r} \qquad (13.2)$$

Using the alternate expression (13.1′) for the elementary work dU, and observing that $F \cos \alpha$ represents the tangential component F_t of the force, we may also express the work $U_{1\to 2}$ as

$$U_{1\to 2} = \int_{s_1}^{s_2} (F \cos \alpha)\, ds = \int_{s_1}^{s_2} F_t\, ds \qquad (13.2')$$

where the variable of integration s measures the distance traveled by the particle along the path. The work $U_{1\to 2}$ is represented by the area under the curve obtained by plotting $F_t = F \cos \alpha$ against s (Fig. 13.2*b*).

When the force $\mathbf{F}$ is defined by its rectangular components, the expression (13.1″) may be used for the elementary work. We write then

$$U_{1\to 2} = \int_{A_1}^{A_2} (F_x\, dx + F_y\, dy + F_z\, dz) \qquad (13.2'')$$

where the integration is to be performed along the path described by the particle.

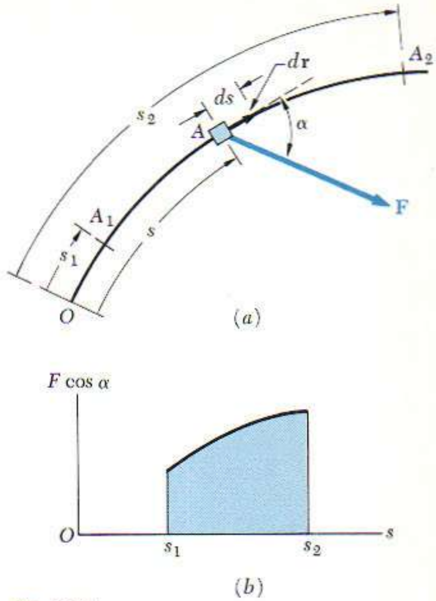


Fig. 13.2

Work of a Constant Force in Rectilinear Motion. When a particle moving in a straight line is acted upon by a force $\mathbf{F}$ of constant magnitude and of constant direction (Fig. 13.3), formula (13.2′) yields

$$U_{1\to 2} = (F \cos \alpha)\, \Delta x \qquad (13.3)$$

where α = angle the force forms with direction of motion

Δx = displacement from A_1 to A_2

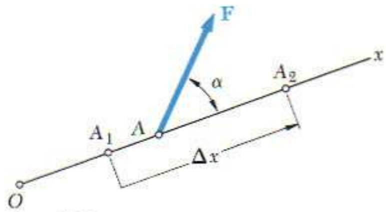


Fig. 13.3

Work of a Weight. The work of the weight $\mathbf{W}$ of a body is obtained by substituting the components of $\mathbf{W}$ into (13.1″) and (13.2″). With the y axis chosen upward (Fig. 13.4), we have $F_x = 0$, $F_y = -W$, $F_z = 0$, and we write

$$dU = -W\, dy$$

$$U_{1\to 2} = -\int_{y_1}^{y_2} W\, dy = Wy_1 - Wy_2 \qquad (13.4)$$

or

$$U_{1\to 2} = -W(y_2 - y_1) = -W\,\Delta y \qquad (13.4')$$

where Δy is the vertical displacement from A_1 to A_2. The work of the weight $\mathbf{W}$ is thus equal to *the product of W and of the vertical displacement of the center of gravity of the body.* The work is *positive* when $\Delta y < 0$, that is, *when the body moves down.*

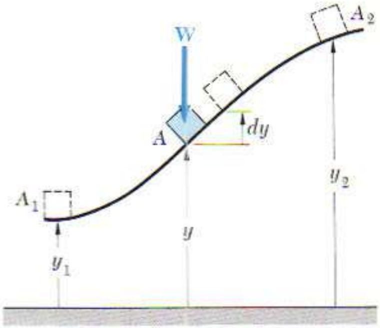


Fig. 13.4

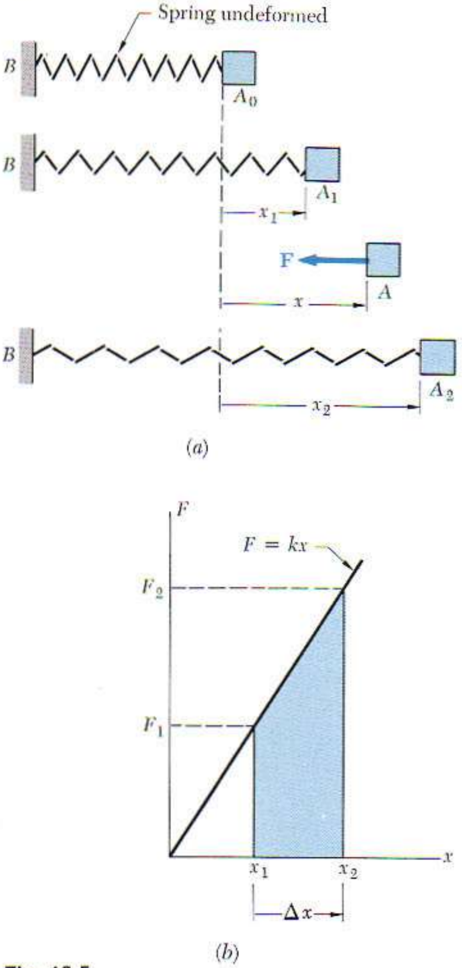


Fig. 13.5

Work of the Force Exerted by a Spring. Consider a body A attached to a fixed point B by a spring; it is assumed that the spring is undeformed when the body is at A_0 (Fig. 13.5*a*). Experimental evidence shows that the magnitude of the force $\mathbf{F}$ exerted by the spring on body A is proportional to the deflection x of the spring measured from the position A_0. We have

$$F = kx \tag{13.5}$$

where k is the *spring constant,* expressed in N/m or kN/m if SI units are used and in lb/ft or lb/in. if U.S. customary units are used.†

The work of the force $\mathbf{F}$ exerted by the spring during a finite displacement of the body from $A_1(x = x_1)$ to $A_2(x = x_2)$ is obtained by writing

$$dU = -F\,dx = -kx\,dx$$

$$U_{1\to2} = -\int_{x_1}^{x_2} kx\,dx = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2 \tag{13.6}$$

Care should be taken to express k and x in consistent units. For example, if U.S. customary units are used, k should be expressed in lb/ft and x in feet, or k in lb/in. and x in inches; in the first case, the work is obtained in ft · lb, in the second case, in in · lb. We note that the work of the force $\mathbf{F}$ exerted by the spring on the body is *positive* when $x_2 < x_1$, i.e., *when the spring is returning to its undeformed position.*

Since Eq. (13.5) is the equation of a straight line of slope k passing through the origin, the work $U_{1\to2}$ of $\mathbf{F}$ during the displacement from A_1 to A_2 may be obtained by evaluating the area of the trapezoid shown in Fig. 13.5*b*. This is done by computing F_1 and F_2 and multiplying the base Δx of the trapezoid by its mean height $\frac{1}{2}(F_1 + F_2)$. Since the work of the force $\mathbf{F}$ exerted by the spring is positive for a negative value of Δx, we write

$$U_{1\to2} = -\tfrac{1}{2}(F_1 + F_2)\,\Delta x \tag{13.6'}$$

Formula (13.6′) is usually more convenient to use than (13.6) and affords fewer chances of confusing the units involved.

† The relation $F = kx$ is correct under static conditions only. Under dynamic conditions, formula (13.5) should be modified to take the inertia of the spring into account. However, the error introduced by using the relation $F = kx$ in the solution of kinetics problems is small if the mass of the spring is small compared with the other masses in motion.

Work of a Gravitational Force. We saw in Sec. 12.9 that two particles at distance r from each other and, respectively, of mass M and m, attract each other with equal and opposite forces $\mathbf{F}$ and $-\mathbf{F}$ directed along the line joining the particles, and of magnitude

$$F = G\frac{Mm}{r^2}$$

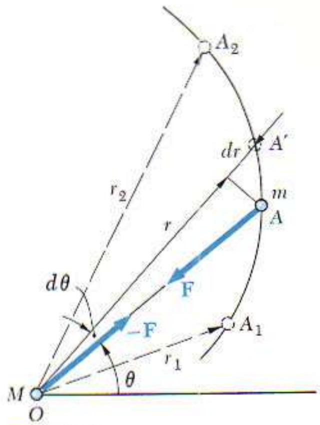


Fig. 13.6

Let us assume that the particle M occupies a fixed position O while the particle m moves along the path shown in Fig. 13.6. The work of the force $\mathbf{F}$ exerted on the particle m during an infinitesimal displacement of the particle from A to A' may be obtained by multiplying the magnitude F of the force by the radial component dr of the displacement. Since $\mathbf{F}$ is directed toward O, the work is negative and we write

$$dU = -F\,dr = -G\frac{Mm}{r^2}\,dr$$

The work of the gravitational force $\mathbf{F}$ during a finite displacement from $A_1(r = r_1)$ to $A_2(r = r_2)$ is therefore

$$U_{1\to 2} = -\int_{r_1}^{r_2} \frac{GMm}{r^2}\,dr = \frac{GMm}{r_2} - \frac{GMm}{r_1} \qquad (13.7)$$

The formula obtained may be used to determine the work of the force exerted by the earth on a body of mass m at a distance r from the center of the earth, when r is larger than the radius R of the earth. The letter M represents then the mass of the earth; recalling the first of the relations (12.29), we may thus replace the product GMm in Eq. (13.7) by WR^2, where R is the radius of the earth ($R = 6.37 \times 10^6$ m or 3960 mi) and W the value of the weight of the body at the surface of the earth.

A number of forces frequently encountered in problems of kinetics *do no work.* They are forces applied to fixed points ($ds = 0$) or acting in a direction perpendicular to the displacement ($\cos\alpha = 0$). Among the forces which do no work are the following: the reaction at a frictionless pin when the body supported rotates about the pin, the reaction at a frictionless surface when the body in contact moves along the surface, the reaction at a roller moving along its track, and the weight of a body when its center of gravity moves horizontally.

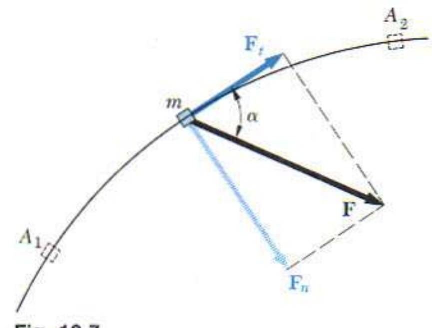


Fig. 13.7

13.3. Kinetic Energy of a Particle. Principle of Work and Energy. Consider a particle of mass m acted upon by a force $\mathbf{F}$ and moving along a path which is either rectilinear or curved (Fig. 13.7). Expressing Newton's second law in terms of the tangential components of the force and of the acceleration (see Sec. 12.4), we write

$$F_t = ma_t \qquad \text{or} \qquad F_t = m\frac{dv}{dt}$$

where v is the speed of the particle. Recalling from Sec. 11.9 that $v = ds/dt$, we obtain

$$F_t = m\frac{dv}{ds}\frac{ds}{dt} = mv\frac{dv}{ds}$$

$$F_t\,ds = mv\,dv$$

Integrating from A_1, where $s = s_1$ and $v = v_1$, to A_2, where $s = s_2$ and $v = v_2$, we write

$$\int_{s_1}^{s_2} F_t\,ds = m\int_{v_1}^{v_2} v\,dv = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 \qquad (13.8)$$

The left-hand member of Eq. (13.8) represents the work $U_{1\to2}$ of the force $\mathbf{F}$ exerted on the particle during the displacement from A_1 to A_2; as indicated in Sec. 13.2, the work $U_{1\to2}$ is a scalar quantity. The expression $\frac{1}{2}mv^2$ is also a scalar quantity; it is defined as the kinetic energy of the particle and is denoted by T. We write

$$T = \tfrac{1}{2}mv^2 \qquad (13.9)$$

Substituting into (13.8), we have

$$U_{1\to2} = T_2 - T_1 \qquad (13.10)$$

which expresses that, when a particle moves from A_1 to A_2 under the action of a force **F**, *the work of the force* **F** *is equal to the change in kinetic energy of the particle.* This is known as the *principle of work and energy.* Rearranging the terms in (13.10), we write

$$T_1 + U_{1\to 2} = T_2 \tag{13.11}$$

Thus, *the kinetic energy of the particle at* A_2 *may be obtained by adding to its kinetic energy at* A_1 *the work done during the displacement from* A_1 *to* A_2 *by the force* **F** *exerted on the particle.* As Newton's second law from which it is derived, the principle of work and energy applies only with respect to a newtonian frame of reference (Sec. 12.1). The speed v used to determine the kinetic energy T should therefore be measured with respect to a newtonian frame of reference.

Since both work and kinetic energy are scalar quantities, their sum may be computed as an ordinary algebraic sum, the work $U_{1\to 2}$ being considered as positive or negative according to the direction of **F**. When several forces act on the particle, the expression $U_{1\to 2}$ represents the total work of the forces acting on the particle; it is obtained by adding algebraically the work of the various forces.

As noted above, the kinetic energy of a particle is a scalar quantity. It further appears from the definition $T = \frac{1}{2}mv^2$ that the kinetic energy is always positive, regardless of the direction of motion of the particle. Considering the particular case when $v_1 = 0$, $v_2 = v$, and substituting $T_1 = 0$, $T_2 = T$ into (13.10), we observe that the work done by the forces acting on the particle is equal to T. Thus, the kinetic energy of a particle moving with a speed v represents the work which must be done to bring the particle from rest to the speed v. Substituting $T_1 = T$ and $T_2 = 0$ into (13.10), we also note that, when a particle moving with a speed v is brought to rest, the work done by the forces acting on the particle is $-T$. Assuming that no energy is dissipated into heat, we conclude that the work done by the forces exerted *by the particle* on the bodies which cause it to come to rest is equal to T. Thus, the kinetic energy of a particle also represents *the capacity to do work associated with the speed of the particle.*

The kinetic energy is measured in the same units as work, i.e., in joules if SI units are used, and in ft · lb if U.S. customary units are used. We check that, in SI units,

$$T = \tfrac{1}{2}mv^2 = \text{kg}(\text{m/s})^2 = (\text{kg}\cdot\text{m/s}^2)\text{m} = \text{N}\cdot\text{m} = \text{J}$$

while, in customary units,

$$T = \tfrac{1}{2}mv^2 = (\text{lb}\cdot\text{s}^2/\text{ft})(\text{ft/s})^2 = \text{lb}\cdot\text{ft}$$

13.4. Applications of the Principle of Work and Energy. The application of the principle of work and energy greatly simplifies the solution of many problems involving forces, displacements, and velocities. Consider, for example, the pendulum OA consisting of a bob A of weight W attached to a cord of length l (Fig. 13.8*a*). The pendulum is released with no initial velocity from a horizontal position OA_1 and allowed to swing in a vertical plane. We wish to determine the speed of the bob as it passes through A_2, directly under O.

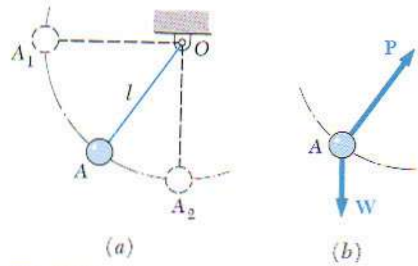


Fig. 13.8

We first determine the work done during the displacement from A_1 to A_2 by the forces acting on the bob. We draw a free-body diagram of the bob, showing all the *actual* forces acting on it, i.e., the weight **W** and the force **P** exerted by the cord (Fig. 13.8*b*). (An inertia vector is not an actual force and *should not* be included in the free-body diagram.) We note that the force **P** does no work, since it is normal to the path; the only force which does work is thus the weight **W**. The work of **W** is obtained by multiplying its magnitude W by the vertical displacement l (Sec. 13.2); since the displacement is downward, the work is positive. We therefore write $U_{1\to2} = Wl$.

Considering, now, the kinetic energy of the bob, we find $T_1 = 0$ at A_1 and $T_2 = \frac{1}{2}(W/g)v_2^2$ at A_2. We may now apply the principle of work and energy; recalling formula (13.11), we write

$$T_1 + U_{1\to2} = T_2 \qquad 0 + Wl = \frac{1}{2}\frac{W}{g}v_2^2$$

Solving for v_2, we find $v_2 = \sqrt{2gl}$. We note that the speed obtained is that of a body falling freely from a height l.

The example we have considered illustrates the following advantages of the method of work and energy:

1. In order to find the speed at A_2, there is no need to determine the acceleration in an intermediate position A and to integrate the expression obtained from A_1 to A_2.
2. All quantities involved are scalars and may be added directly, without using x and y components.
3. Forces which do no work are eliminated from the solution of the problem.

What is an advantage in one problem, however, may become a disadvantage in another. It is evident, for instance, that the method of work and energy cannot be used to directly determine an acceleration. We also note that it should be supplemented by the direct application of Newton's second law in order to determine a force which is normal to the path of the particle, since such a force does no work. Suppose, for example, that we wish to determine the tension in the cord of the pendulum of Fig. 13.8a as the bob passes through A_2. We draw a free-body diagram of the bob in that position (Fig. 13.9) and express

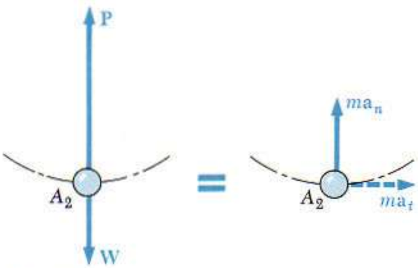


Fig. 13.9

Newton's second law in terms of tangential and normal components. The equations $\Sigma F_t = ma_t$ and $\Sigma F_n = ma_n$ yield, respectively, $a_t = 0$ and

$$P - W = ma_n = \frac{W}{g}\frac{v^2}{l}$$

But the speed at A_2 was determined earlier by the method of work and energy. Substituting $v^2 = 2gl$ and solving for P, we write

$$P = W + \frac{W}{g}\frac{2gl}{l} = 3W$$

When a problem involves two particles or more, each particle may be considered separately and the principle of work and energy may be applied to each particle. Adding the kinetic energies of the various particles, and considering the work of all the forces acting on them, we may also write a single equation of work and energy for all the particles involved. We have

$$T_1 + U_{1\to 2} = T_2 \tag{13.11}$$

where T represents the arithmetic sum of the kinetic energies of the particles involved (all terms are positive) and $U_{1\to 2}$ the work of all the forces acting on the particles, *including the forces of action and reaction exerted by the particles on each other.* In problems involving bodies connected by *inextensible cords or links,* however, the work of the forces exerted by a given cord or link on the two bodies it connects cancels out since the points of application of these forces move through equal distances (see Sample Prob. 13.2).†

13.5. Power and Efficiency. *Power* is defined as the time rate at which work is done. In the selection of a motor or engine, power is a much more important criterion than the actual amount of work to be performed. A small motor or a large power plant may both be used to do a given amount of work; but the small motor may require a month to do the work done by the power plant in a matter of minutes. If ΔU is the work done during the time interval Δt, then the average power during this time interval is

$$\text{Average power} = \frac{\Delta U}{\Delta t}$$

Letting Δt approach zero, we obtain at the limit

$$\text{Power} = \frac{dU}{dt} \tag{13.12}$$

Substituting the scalar product $\mathbf{F} \cdot d\mathbf{r}$ for dU, we may also write

$$\text{Power} = \frac{dU}{dt} = \frac{\mathbf{F} \cdot d\mathbf{r}}{dt}$$

and, recalling that $d\mathbf{r}/dt$ represents the velocity $\mathbf{v}$ of the point of application of $\mathbf{F}$,

$$\text{Power} = \mathbf{F} \cdot \mathbf{v} \tag{13.13}$$

†The application of the method of work and energy to a system of particles is discussed in detail in Chap. 14.

Since power was defined as the time rate at which work is done, it should be expressed in units obtained by dividing units of work by the unit of time. Thus, if SI units are used, power should be expressed in J/s; this unit is called a *watt* (W). We have

$$1\ \text{W} = 1\ \text{J/s} = 1\ \text{N} \cdot \text{m/s}$$

If U.S. customary units are used, power should be expressed in ft · lb/s or in *horsepower* (hp), with the latter defined as

$$1\ \text{hp} = 550\ \text{ft} \cdot \text{lb/s}$$

Recalling from Sec. 13.2 that $1\ \text{ft} \cdot \text{lb} = 1.356\ \text{J}$, we verify that

$$1\ \text{ft} \cdot \text{lb/s} = 1.356\ \text{J/s} = 1.356\ \text{W}$$
$$1\ \text{hp} = 550(1.356\ \text{W}) = 746\ \text{W} = 0.746\ \text{kW}$$

The *mechanical efficiency* of a machine was defined in Sec. 10.4 as the ratio of the output work to the input work:

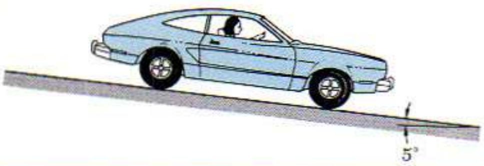
$$\eta = \frac{\text{output work}}{\text{input work}} \tag{13.14}$$

This definition is based on the assumption that work is done at a constant rate. The ratio of the output to the input work is therefore equal to the ratio of the rates at which output and input work are done, and we have

$$\eta = \frac{\text{power output}}{\text{power input}} \tag{13.15}$$

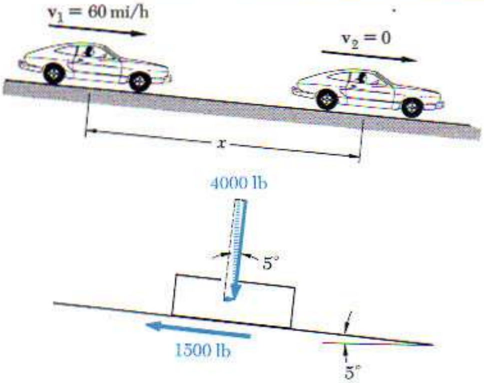
Because of energy losses due to friction, the output work is always smaller than the input work, and, consequently, the power output is always smaller than the power input. The mechanical efficiency of a machine, therefore, is always less than 1.

When a machine is used to transform mechanical energy into electric energy, or thermal energy into mechanical energy, its *overall efficiency* may be obtained from formula (13.15). The overall efficiency of a machine is always less than 1; it provides a measure of all the various energy losses involved (losses of electric or thermal energy as well as frictional losses). We should note that it is necessary, before using formula (13.15), to express the power output and the power input in the same units.



SAMPLE PROBLEM 13.1

An automobile weighing 4000 lb is driven down a 5° incline at a speed of 60 mi/h when the brakes are applied, causing a constant total braking force (applied by the road on the tires) of 1500 lb. Determine the distance traveled by the automobile as it comes to a stop.



Solution. ***Kinetic Energy***

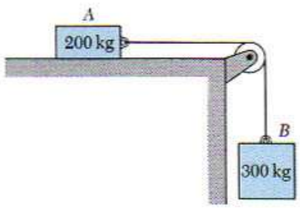
Position *1*:
$$v_1 = \left(60\,\frac{\text{mi}}{\text{h}}\right)\left(\frac{5280\text{ ft}}{1\text{ mi}}\right)\left(\frac{1\text{ h}}{3600\text{ s}}\right) = 88\text{ ft/s}$$
$$T_1 = \tfrac{1}{2}mv_1^2 = \tfrac{1}{2}(4000/32.2)(88)^2 = 481{,}000\text{ ft}\cdot\text{lb}$$

Position *2*:
$$v_2 = 0 \qquad T_2 = 0$$

Work $U_{1\to 2} = -1500x + (4000\sin 5°)x = -1151x$

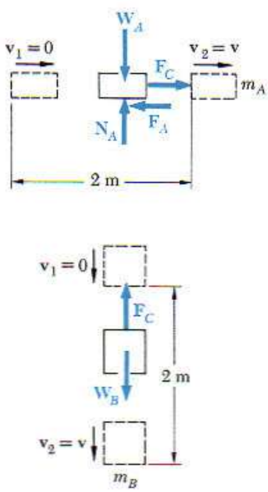
Principle of Work and Energy

$$T_1 + U_{1\to 2} = T_2$$
$$481{,}000 - 1151x = 0 \qquad x = 418\text{ ft} ◄$$



SAMPLE PROBLEM 13.2

Two blocks are joined by an inextensible cable as shown. If the system is released from rest, determine the velocity of block *A* after it has moved 2 m. Assume that μ equals 0.25 between block *A* and the plane and that the pulley is weightless and frictionless.



Solution. ***Work and Energy for Block A.*** We denote by $\mathbf{F}_A$ the friction force, by $\mathbf{F}_C$ the force exerted by the cable, and write

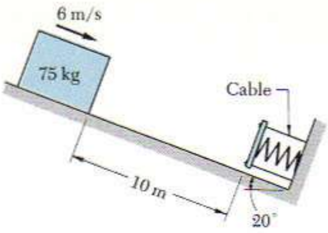
$$m_A = 200\text{ kg} \qquad W_A = (200\text{ kg})(9.81\text{ m/s}^2) = 1962\text{ N}$$
$$F_A = \mu N_A = \mu W_A = 0.25(1962\text{ N}) = 490\text{ N}$$
$$T_1 + U_{1\to 2} = T_2: \qquad 0 + F_C(2\text{ m}) - F_A(2\text{ m}) = \tfrac{1}{2}m_A v^2$$
$$F_C(2\text{ m}) - (490\text{ N})(2\text{ m}) = \tfrac{1}{2}(200\text{ kg})v^2 \qquad (1)$$

Work and Energy for Block B. We write

$$m_B = 300\text{ kg} \qquad W_B = (300\text{ kg})(9.81\text{ m/s}^2) = 2940\text{ N}$$
$$T_1 + U_{1\to 2} = T_2: \qquad 0 + W_B(2\text{ m}) - F_C(2\text{ m}) = \tfrac{1}{2}m_B v^2$$
$$(2940\text{ N})(2\text{ m}) - F_C(2\text{ m}) = \tfrac{1}{2}(300\text{ kg})v^2 \qquad (2)$$

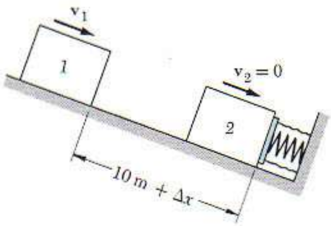
Adding the left-hand and right-hand members of (1) and (2), we observe that the work of the forces exerted by the cable on *A* and *B* cancels out:

$$(2940\text{ N})(2\text{ m}) - (490\text{ N})(2\text{ m}) = \tfrac{1}{2}(200\text{ kg} + 300\text{ kg})v^2$$
$$4900\text{ J} = \tfrac{1}{2}(500\text{ kg})v^2 \qquad v = 4.43\text{ m/s} ◄$$



SAMPLE PROBLEM 13.3

A spring is used to stop a 75-kg package which is moving down a 20° incline. The spring has a constant $k = 25\ \text{kN/m}$, and is held by cables so that it is initially compressed 100 mm. If the velocity of the package is 6 m/s when it is 10 m from the spring, determine the maximum additional deformation of the spring in bringing the package to rest. Assume $\mu = 0.20$.



Kinetic Energy

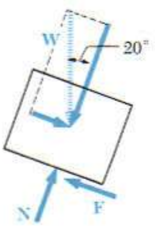
Position *1*: $\quad v_1 = 6\ \text{m/s}$

$$T_1 = \tfrac{1}{2}mv_1^2 = \tfrac{1}{2}(75\ \text{kg})(6\ \text{m/s})^2 = 1350\ \text{N}\cdot\text{m} = 1350\ \text{J}$$

Position 2 (maximum spring deformation):

$$v_2 = 0 \qquad T_2 = 0$$

Work. We assume that when the package is brought to rest the additional deflection of the spring is Δx. The component of the weight parallel to the plane and the friction force act through the entire displacement, i.e., through $10\ \text{m} + \Delta x$. The total work done by these forces is

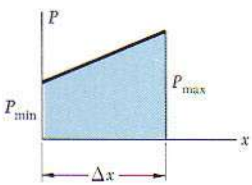


$$\begin{aligned} U_{1\to2} &= W_t(10\ \text{m} + \Delta x) - F(10\ \text{m} + \Delta x) \\ &= (W \sin 20°)(10\ \text{m} + \Delta x) - 0.20(W \cos 20°)(10\ \text{m} + \Delta x) \\ &= 0.1541W(10\ \text{m} + \Delta x) \end{aligned}$$

or, since $W = mg = (75\ \text{kg})(9.81\ \text{m/s}^2) = 736\ \text{N}$,

$$U_{1\to2} = 0.1541(736\ \text{N})(10\ \text{m} + \Delta x) = 1134\ \text{J} + (113.4\ \text{N})\,\Delta x$$

In addition, during the compression of the spring, the variable force **P** exerted by the spring does an amount of negative work equal to the area under the force-deflection curve of the spring force.



$$P_{\text{min}} = kx = (25\ \text{kN/m})(100\ \text{mm}) = (25\,000\ \text{N/m})(0.100\ \text{m}) = 2500\ \text{N}$$

$$\begin{aligned} P_{\text{max}} &= P_{\text{min}} + k\,\Delta x \\ &= 2500\ \text{N} + (25\,000\ \text{N/m})\,\Delta x \end{aligned}$$

$$U_{1\to2} = -\tfrac{1}{2}(P_{\text{min}} + P_{\text{max}})\,\Delta x = -(2500\ \text{N})\,\Delta x - (12\,500\ \text{N/m})(\Delta x)^2$$

The total work is thus

$$U_{1\to2} = 1134\ \text{J} + (113.4\ \text{N})\,\Delta x - (2500\ \text{N})\,\Delta x - (12\,500\ \text{N/m})(\Delta x)^2$$

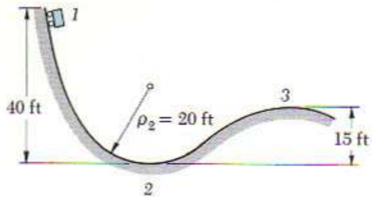
Principle of Work and Energy

$$T_1 + U_{1\to2} = T_2$$

$$1350 + 1134 + 113.4\,\Delta x - 2500\,\Delta x - 12\,500(\Delta x)^2 = 0$$

$$(\Delta x)^2 + 0.1909\,\Delta x - 0.1987 = 0$$

$$\Delta x = 0.360\ \text{m} \qquad \Delta x = 360\ \text{mm} \blacktriangleleft$$



SAMPLE PROBLEM 13.4

A 2000-lb car starts from rest at point *1* and moves without friction down the track shown. (*a*) Determine the force exerted by the track on the car at point *2*, where the radius of curvature of the track is 20 ft. (*b*) Determine the minimum safe value of the radius of curvature at point *3*.

a. **Force Exerted by the Track at Point 2.** The principle of work and energy is used to determine the velocity of the car as it passes through point *2*.

Kinetic Energy: $\quad T_1 = 0 \qquad T_2 = \frac{1}{2} m v_2^2 = \frac{1}{2}\frac{W}{g}v_2^2$

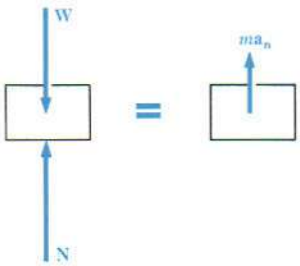
Work. The only force which does work is the weight **W**. Since the vertical displacement from point *1* to point *2* is 40 ft downward, the work of the weight is

$$U_{1\to2} = +W(40\text{ ft})$$

Principle of Work and Energy

$$T_1 + U_{1\to2} = T_2 \qquad 0 + W(40\text{ ft}) = \frac{1}{2}\frac{W}{g}v_2^2$$

$$v_2^2 = 80g = 80(32.2) \qquad v_2 = 50.8\text{ ft/s}$$



Newton's Second Law at Point 2. The acceleration $\mathbf{a}_n$ of the car at point *2* has a magnitude $a_n = v_2^2/\rho$ and is directed upward. Since the external forces acting on the car are **W** and **N**, we write

$$+\uparrow\Sigma F_n = ma_n: \qquad -W + N = ma_n$$

$$= \frac{W}{g}\frac{v_2^2}{\rho}$$

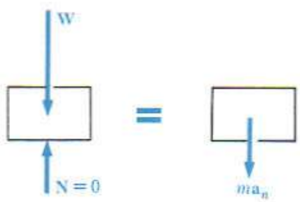
$$= \frac{W}{g}\frac{80g}{20}$$

$$N = 5W \qquad \mathbf{N} = 10{,}000\text{ lb}\uparrow \blacktriangleleft$$

b. **Minimum Value of ρ at Point 3.** *Principle of Work and Energy.* Applying the principle of work and energy between point *1* and point *3*, we obtain

$$T_1 + U_{1\to3} = T_3 \qquad 0 + W(25\text{ ft}) = \frac{1}{2}\frac{W}{g}v_3^2$$

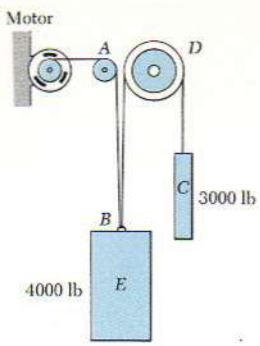
$$v_3^2 = 50g = 50(32.2) \qquad v_3 = 40.1\text{ ft/s}$$



Newton's Second Law at Point 3. The minimum safe value of ρ occurs when $\mathbf{N} = 0$. In this case, the acceleration $\mathbf{a}_n$, of magnitude $a_n = v_3^2/\rho$, is directed downward, and we write

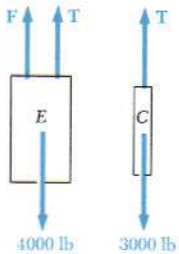
$$+\downarrow\Sigma F_n = ma_n: \qquad W = \frac{W}{g}\frac{v_3^2}{\rho}$$

$$= \frac{W}{g}\frac{50g}{\rho} \qquad \rho = 50\text{ ft} \blacktriangleleft$$



SAMPLE PROBLEM 13.5

The elevator shown weighs 4000 lb when fully loaded. It is connected to a 3000-lb counterweight C and is powered by an electric motor. Determine the power required when the elevator (*a*) is moving upward at a constant speed of 20 ft/s, (*b*) has an instantaneous velocity of 20 ft/s upward and an upward acceleration of 3 ft/s^2.



Solution. Since **F** and **v** have the same direction, the power is equal to Fv. We must first determine the force **F** exerted by cable AB on the elevator in each of the two given situations.

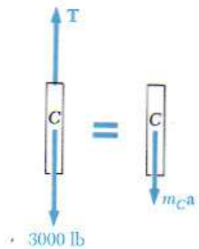
***Force* F.** The forces acting on the elevator and on the counterweight are shown in the adjoining sketches.

a. Uniform Motion. We have $a = 0$; both bodies are in equilibrium.

Free Body C: $+\uparrow \Sigma F_y = 0$: $\quad T - 3000 \text{ lb} = 0$

Free Body E: $+\uparrow \Sigma F_y = 0$: $\quad F + T - 4000 \text{ lb} = 0$

Eliminating T, we find: $\quad F = 1000 \text{ lb}$

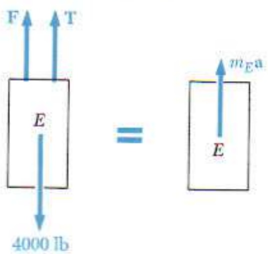


b. Accelerated Motion. We have $a = 3 \text{ ft/s}^2$. The equations of motion are

Free Body C: $+\downarrow \Sigma F_y = m_C a$: $\quad 3000 - T = \dfrac{3000}{32.2}3$

Free Body E: $+\uparrow \Sigma F_y = m_E a$: $\quad F + T - 4000 = \dfrac{4000}{32.2}3$

Eliminating T: $\quad F = 1000 + \dfrac{7000}{32.2}3 = 1652 \text{ lb}$



Power. Substituting the given values of v and the values found for F into the expression for the power, we have

a. $Fv = (1000 \text{ lb})(20 \text{ ft/s}) = 20{,}000 \text{ ft} \cdot \text{lb/s}$

$$\text{Power} = (20{,}000 \text{ ft} \cdot \text{lb/s})\frac{1 \text{ hp}}{550 \text{ ft} \cdot \text{lb/s}} = 36.4 \text{ hp} \quad ◄$$

b. $Fv = (1652 \text{ lb})(20 \text{ ft/s}) = 33{,}040 \text{ ft} \cdot \text{lb/s}$

$$\text{Power} = \frac{33{,}040}{550} = 60.1 \text{ hp} \quad ◄$$

PROBLEMS

13.1 A stone which weighs 8 lb is dropped from a height h and strikes the ground with a velocity of 75 ft/s. (*a*) Find the kinetic energy of the stone as it strikes the ground and the height h from which it was dropped. (*b*) Solve part *a*, assuming that the same stone is dropped on the moon. (Acceleration of gravity on the moon = 5.31 ft/s^2.)

13.2 A 100-kg satellite was placed in a circular orbit 2000 km above the surface of the earth. At this elevation the acceleration of gravity is 5.68 m/s^2. Determine the kinetic energy of the satellite, knowing that its orbital speed is 24.8×10^3 km/h.

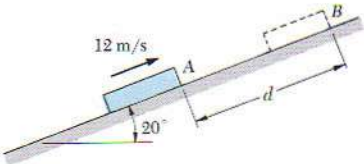


Fig. P13.3

13.3 A 20-kg package is projected up a 20° incline with an initial velocity of 12 m/s. The coefficient of friction between the incline and the package is 0.15. Determine (*a*) the maximum distance that the package will move up the incline, (*b*) the velocity of the package when it returns to its original position.

13.4 Using the method of work and energy, solve Prob. 12.8.

13.5 The conveyor belt shown moves at a constant speed v_0 and discharges packages on to the chute AB. The coefficient of friction between the packages and the chute is 0.50. Knowing that the packages must reach point B with a speed of 12 ft/s, determine the required speed v_0 of the conveyor belt.

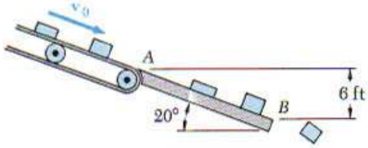


Fig. P13.5

13.6 Solve Prob. 13.5, assuming that the coefficient of friction between the packages and the chute is 0.30.

13.7 The 2-kg collar was moving down the rod with a velocity of 3 m/s when a force **P** was applied to the horizontal cable. Assuming negligible friction between the collar and the rod, determine the magnitude of the force **P** if the collar stopped after moving 1.2 m more down the rod.

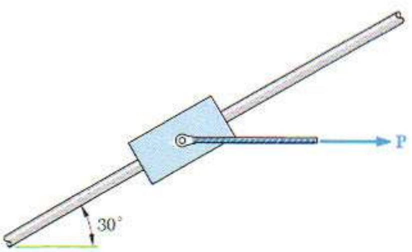


Fig. P13.7

13.8 Solve Prob. 13.7, assuming a coefficient of friction of 0.20 between the collar and the rod.

13.9 Knowing that the system shown is initially at rest and neglecting the effect of friction, determine the force **P** required if the velocity of collar *B* is to be 8 ft/s after it has moved 2.5 ft to the right.

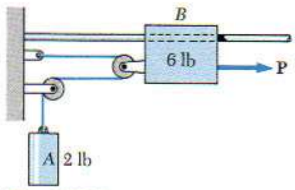


Fig. P13.9

13.10 The system shown is at rest when the 20-lb force is applied to block *A*. Neglecting the effect of friction, determine the velocity of block *A* after it has moved 9 ft.

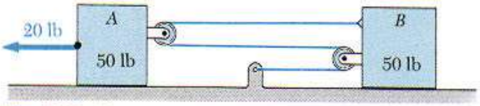


Fig. P13.10

13.11 Solve Prob. 13.10, assuming that the coefficient of friction between the blocks and the horizontal plane is 0.20.

13.12 Three 20-kg packages rest on a belt which passes over a pulley and is attached to a 40-kg block. Knowing that the coefficient of friction between the belt and the horizontal surface and also between the belt and the packages is 0.50, determine the speed of package *B* as it falls off the belt at *E*.

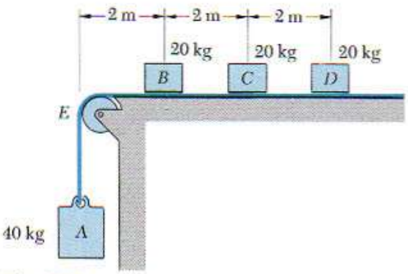


Fig. P13.12

13.13 In Prob. 13.12, determine the speed of package *C* as it falls off the belt at *E*.

13.14 Two cylinders are suspended from an inextensible cable as shown. If the system is released from rest, determine (*a*) the maximum velocity attained by the 10-lb cylinder, (*b*) the maximum height above the floor to which the 10-lb cylinder will rise.

13.15 Solve Prob. 13.14, assuming that the 20-lb cylinder is replaced by a 50-lb cylinder.

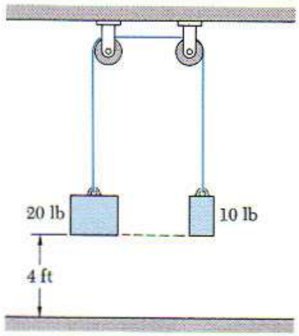


Fig. P13.14

13.16 Four packages weighing 50 lb each are placed as shown on a conveyor belt which is disengaged from its drive motor. Package *1* is just to the right of the horizontal portion of the belt. If the system is released from rest, determine the velocity of package *1* as it falls off the belt at point *A*. Assume that the weight of the belt and rollers is small compared to the weight of the packages.

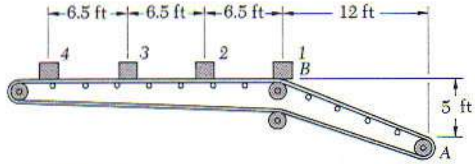


Fig. P13.16

13.17 In Prob. 13.16, determine the velocity of package *2* as it falls from the belt at *A*.

13.18 Using the method of work and energy, solve Prob. 12.16.

13.19 Using the method of work and energy, solve Prob. 12.15.

13.20 Using the method of work and energy, solve Prob. 12.18*c*.

13.21 Using the method of work and energy, solve Prob. 12.17*c*.

13.22 In order to protect it during shipping, a delicate instrument weighing 4 oz is packed in excelsior. From the static test of similar excelsior, the force-deflection curve shown was obtained. Determine the maximum height from which the package may be dropped if the force exerted on the instrument is not to exceed 12 lb.

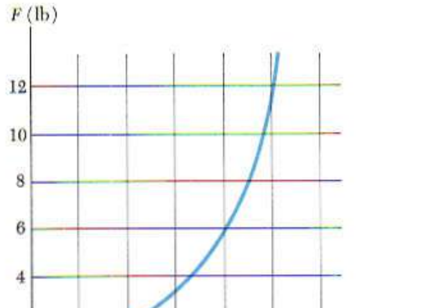


Fig. P13.22

13.23 A 5000-kg airplane lands on an aircraft carrier and is caught by an arresting cable which is characterized by the force-deflection diagram shown. Knowing that the landing speed of the plane is 144 km/h, determine (*a*) the distance required for the plane to come to rest, (*b*) the maximum rate of deceleration of the plane.

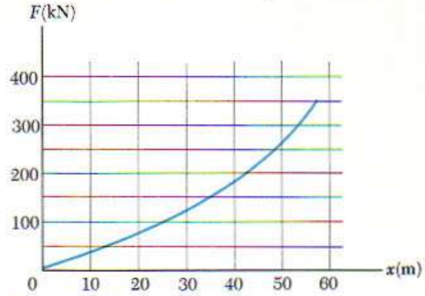


Fig. P13.23

13.24 A 2-kg block is at rest on a spring of constant 400 N/m. A 4-kg block is held above the 2-kg block so that it just touches it, and released. Determine (*a*) the maximum velocity attained by the blocks, (*b*) the maximum force exerted on the blocks.

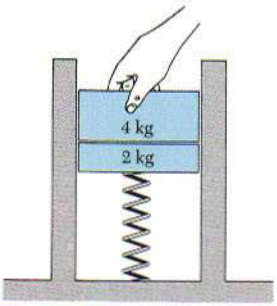


Fig. P13.24

13.25 As the bracket *ABC* is slowly rotated, the 6-kg block starts to slide toward the spring when $\theta = 15°$. The maximum deflection of the spring is observed to be 50 mm. Determine the values of the coefficients of static and kinetic friction.

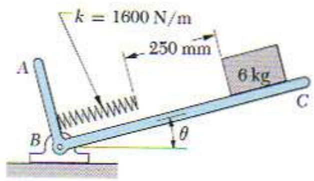


Fig. P13.25

13.26 A 15-lb plunger is released from rest in the position shown and is stopped by two nested springs; the stiffness of the outer spring is 20 lb/in. and the stiffness of the inner spring is 60 lb/in. If the maximum deflection of the outer spring is observed to be 5 in., determine the height *h* from which the plunger was released.

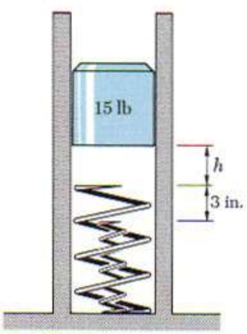


Fig. P13.26

13.27 A railroad car weighing 60,000 lb starts from rest and coasts down a 1-percent incline for a distance of 40 ft. It is stopped by a bumper having a spring constant of 7500 lb/in. (*a*) What is the speed of the car at the bottom of the incline? (*b*) How many inches will the spring be compressed?

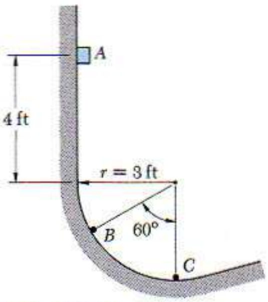


Fig. P13.28

13.28 A 0.5-lb pellet is released from rest at A and slides without friction along the surface shown. Determine the force exerted by the surface on the pellet as it passes (*a*) point B, (*b*) point C.

13.29 A roller coaster is released with no velocity at A and rolls down the track shown. The brakes are suddenly applied as the car passes through point B, causing the wheels of the car to slide on the track ($\mu = 0.30$). Assuming no energy loss between A and B and knowing that the radius of curvature of the track at B is 80 ft, determine the normal and tangential components of the acceleration of the car just after the brakes have been applied.

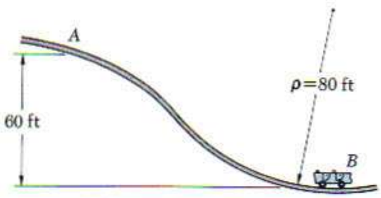


Fig. P13.29

13.30 A small package of mass m is projected into a vertical return loop at A with a velocity $\mathbf{v}_0$. The package travels without friction along a circle of radius r and is deposited on a horizontal surface at C. For each of the two loops shown, determine (*a*) the smallest velocity $\mathbf{v}_0$ for which the package will reach the horizontal surface at C, (*b*) the corresponding force exerted by the loop on the package as it passes point B.

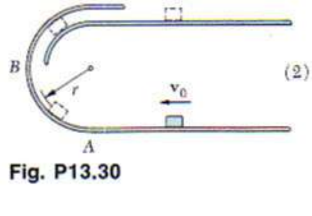


Fig. P13.30

13.31 In Prob. 13.30, it is desired to have the package deposited on the horizontal surface at C with a speed of 2 m/s. Knowing that $r = 0.6$ m, (*a*) show that this requirement cannot be fulfilled by the first loop, (*b*) determine the required initial velocity $\mathbf{v}_0$ when the second loop is used.

13.32 A 6-in.-diameter piston weighing 8 lb slides without friction in a cylinder. When the piston is at a distance $x = 10$ in. from the end of the cylinder, the pressure in the cylinder is atmospheric ($p_a = 14.7$ lb/in^2). If the pressure varies inversely as the volume, find the work done in moving the piston until $x = 4$ in.

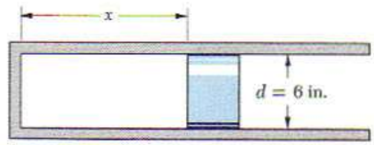


Fig. P13.32

13.33 The piston of Prob. 13.32 is moved to the left and released with no velocity when $x = 4$ in. Neglecting friction, determine (*a*) the maximum velocity attained by the piston, (*b*) the maximum value of the coordinate x.

13.34 An object is released with no velocity at an altitude equal to the radius of the earth. Neglecting air resistance, determine the velocity of the object as it strikes the earth. Give the answer in both SI and U.S. customary units.

13.35 A rocket is fired vertically from the ground. Knowing that at burnout the rocket is 80 km above the ground and has a velocity of 5000 m/s, determine the highest altitude it will reach.

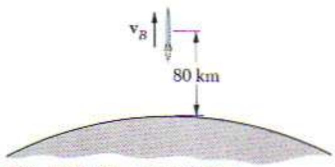


Fig. P13.35 and P13.36

13.36 A rocket is fired vertically from the ground. What should be its velocity $\mathbf{v}_B$ at burnout, 80 km above the ground, if it is to reach an altitude of 1000 km?

13.37 An object is released with no initial velocity at an altitude of 400 mi. (*a*) Neglecting air resistance, determine the velocity of the object as it strikes the ground. (*b*) What percent error is introduced by assuming a uniform gravitational field?

13.38 A 70-kg man and an 80-kg man run up a flight of stairs in 5 s. If the flight of stairs is 4 m high, determine the average power required by each man.

13.39 An industrial hoist can lift its maximum allowable load of 60,000 lb at the rate of 4 ft/min. Knowing that the hoist is run by a 15-hp motor, determine the overall efficiency of the hoist.

13.40 A 1500-kg automobile travels 200 m while being accelerated at a uniform rate from 50 to 75 km/h. During the entire motion, the automobile is traveling on a horizontal road, and the rolling resistance is equal to 2 percent of the weight of the automobile. Determine (*a*) the maximum power required, (*b*) the power required to maintain a constant speed of 75 km/h.

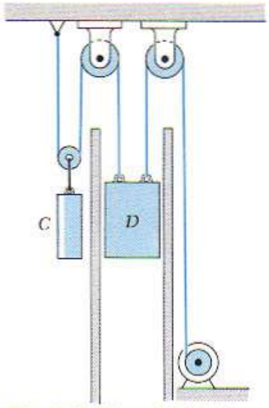


Fig. P13.42 and P13.43

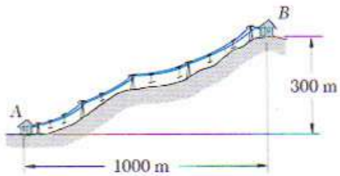


Fig. P13.44

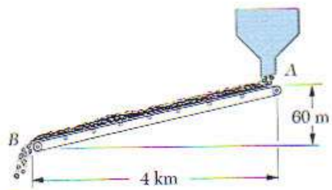


Fig. P13.45

13.41 A train of total weight 600 tons starts from rest and accelerates uniformly to a speed of 30 mi/h in 40 sec. After reaching this speed, the train travels with constant velocity. During the entire motion the train is traveling up a 2 percent grade, and the rolling resistance is 15 lb/ton. Determine the power required as a function of time.

13.42 The dumbwaiter D and its counterweight C weigh 750 lb each. Determine the power required when the dumbwaiter (a) is moving upward at a constant speed of 12 ft/s, (b) has an instantaneous velocity of 12 ft/s upward and an upward acceleration of 3 ft/s^2.

13.43 The dumbwaiter D and its counterweight C weigh 750 lb each. Knowing that the motor is delivering to the system 9 hp at the instant the speed of the dumbwaiter is 12 ft/s upward, determine the acceleration of the dumbwaiter.

13.44 A chair-lift is designed to transport 900 skiers per hour from the base A to the summit B. The average mass of a skier is 75 kg, and the average speed of the lift is 80 m/min. Determine (a) the average power required, (b) the required capacity of the motor if the mechanical efficiency is 85 percent and if a 300-percent overload is to be allowed.

13.45 Crushed stone is moved from a quarry at A to a construction site at B at the rate of 2000 Mg per 8-h period. An electric generator is attached to the system in order to maintain a constant belt speed. Knowing that the efficiency of the belt-generator system is 0.65, determine the average power developed by the generator (a) if the belt speed is 0.75 m/s, (b) if the belt speed is 2 m/s.

13.46 The fluid transmission of a 15-ton truck permits the engine to deliver an essentially constant power of 60 hp to the driving wheels. Determine the time required and the distance traveled as the speed of the truck is increased (a) from 15 to 30 mi/h, (b) from 30 to 45 mi/h.

13.47 The fluid transmission of a truck of mass m permits the engine to deliver an essentially constant power P to the driving wheels. Determine the time elapsed and the distance traveled as the speed is increased from v_0 to v_1.

13.48 The frictional resistance of a ship is known to vary directly as the 1.75 power of the speed v of the ship. A single tugboat at full power can tow the ship at a constant speed of 5 km/h by exerting a constant force of 200 kN. Determine (a) the power developed by the tugboat, (b) the maximum speed at which two tugboats, capable of delivering the same power, can tow the ship.

13.49 Determine the speed at which the single tugboat of Prob. 13.48 will tow the ship if the tugboat is developing half of its maximum power.

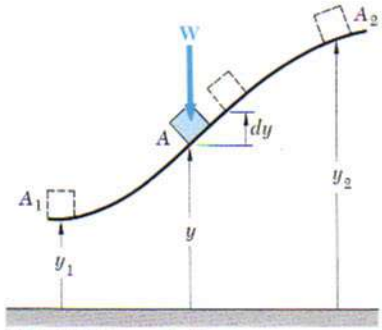


Fig. 13.4 (*repeated*)

13.6. Potential Energy.† Let us consider again a body of weight **W** which moves along a curved path from a point A_1 of elevation y_1 to a point A_2 of elevation y_2 (Fig. 13.4). We recall from Sec. 13.2 that the work of the weight **W** during this displacement is

$$U_{1\to 2} = Wy_1 - Wy_2 \tag{13.4}$$

The work of **W** may thus be obtained by subtracting the value of the function Wy corresponding to the second position of the body from its value corresponding to the first position. The work of **W** is independent of the actual path followed; it depends only upon the initial and final values of the function Wy. This function is called the *potential energy* of the body with respect to the *force of gravity* **W** and is denoted by V_g. We write

$$U_{1\to 2} = (V_g)_1 - (V_g)_2 \qquad \text{with } V_g = Wy \tag{13.16}$$

We note that if $(V_g)_2 > (V_g)_1$, i.e., *if the potential energy increases* during the displacement (as in the case considered here), *the work* $U_{1\to 2}$ *is negative.* If, on the other hand, the work of **W** is positive, the potential energy decreases. Therefore, the potential energy V_g of the body provides a measure of the work which may be done by its weight **W**. Since only the *change* in potential energy, and not the actual value of V_g, is involved in formula (13.16), an arbitrary constant may be added to the expression obtained for V_g. In other words, the level, or datum, from which the elevation y is measured may be chosen arbitrarily. Note that potential energy is expressed in the same units as work, i.e., in joules if SI units are used, and in ft · lb or in · lb if U.S. customary units are used.

It should be noted that the expression just obtained for the potential energy of a body with respect to gravity is valid only as long as the weight **W** of the body may be assumed to remain constant, i.e., as long as the displacements of the body are small compared to the radius of the earth. In the case of a space vehicle, however, we should take into consideration the variation of the force of gravity with the distance r from the center of

†Some of the material in this section has already been considered in Sec. 10.6.

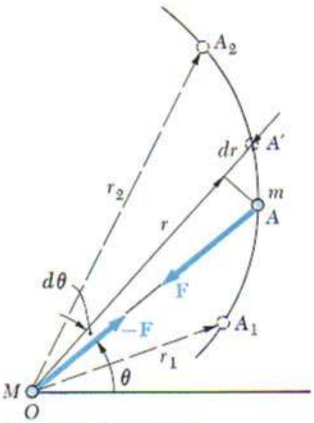


Fig. 13.6 (*repeated*)

the earth. Using the expression obtained in Sec. 13.2 for the work of a gravitational force, we write (Fig. 13.6)

$$U_{1\to 2} = \frac{GMm}{r_2} - \frac{GMm}{r_1} \tag{13.7}$$

The work of the force of gravity may therefore be obtained by subtracting the value of the function $-GMm/r$ corresponding to the second position of the body from its value corresponding to the first position. Thus, the expression which should be used for the potential energy V_g when the variation in the force of gravity cannot be neglected is

$$V_g = -\frac{GMm}{r} \tag{13.17}$$

Taking the first of the relations (12.29) into account, we write V_g in the alternate form

$$V_g = -\frac{WR^2}{r} \tag{13.17$'$}$$

where R is the radius of the earth and W the value of the weight of the body at the surface of the earth. When either of the relations (13.17) and (13.17′) is used to express V_g, the distance r should, of course, be measured from the center of the earth.† Note that V_g is always negative and that it approaches zero for very large values of r.

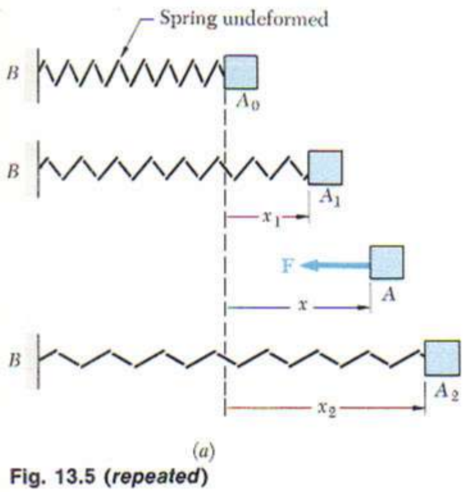
Consider, now, a body attached to a spring and moving from a position A_1, corresponding to a deflection x_1 of the spring, to a position A_2, corresponding to a deflection x_2 (Fig. 13.5). We recall from Sec. 13.2 that the work of the force $\mathbf{F}$ exerted by the spring on the body is

$$U_{1\to 2} = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2 \tag{13.6}$$

The work of the elastic force is thus obtained by subtracting the value of the function $\frac{1}{2}kx^2$ corresponding to the second position of the body from its value corresponding to the first position. This function is denoted by V_e and is called the *potential energy* of the body with respect to the *elastic force* $\mathbf{F}$. We write

$$U_{1\to 2} = (V_e)_1 - (V_e)_2 \qquad \text{with } V_e = \tfrac{1}{2}kx^2 \tag{13.18}$$

and observe that, during the displacement considered, the work of the force $\mathbf{F}$ exerted by the spring on the body is negative



(*a*)
Fig. 13.5 (*repeated*)

†The expressions given for V_g in (13.17) and (13.17′) are valid only when $r \geqq R$, that is, when the body considered is above the surface of the earth.

and the potential energy V_e increases. We should note that the expression obtained for V_e is valid only if the deflection of the spring is measured from its undeformed position. On the other hand, formula (13.18) may be used even when the spring is rotated about its fixed end (Fig. 13.10*a*). The work of the elastic

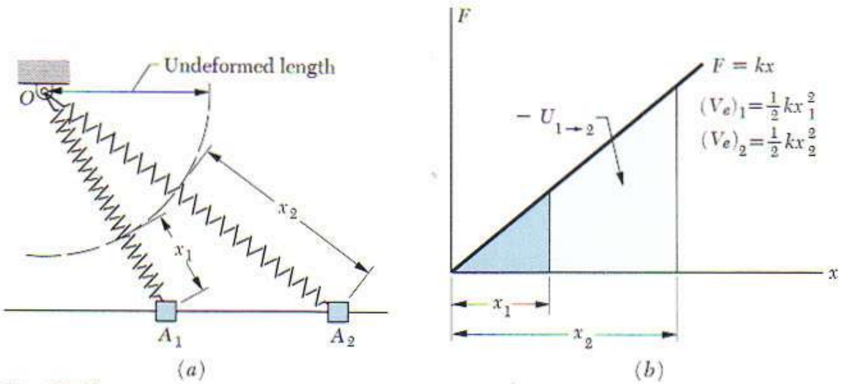


Fig. 13.10

force depends only upon the initial and final deflections of the spring (Fig. 13.10*b*).

The concept of potential energy may be used when forces other than gravity forces and elastic forces are involved. Indeed, it remains valid as long as the work of the force considered is independent of the path followed by its point of application as this point moves from a given position A_1 to a given position A_2. Such forces are said to be *conservative forces;* the general properties of conservative forces are studied in the following section.

***13.7. Conservative Forces.** As indicated in the preceding section, a force **F** acting on a particle A is said to be conservative *if its work U_{1-2} is independent of the path followed by the particle A as it moves from A_1 to A_2* (Fig. 13.11*a*). We may then write

$$U_{1-2} = V(x_1,y_1,z_1) - V(x_2,y_2,z_2) \tag{13.19}$$

or, for short,

$$U_{1-2} = V_1 - V_2 \tag{13.19'}$$

The function $V(x,y,z)$ is called the potential energy, or *potential function,* of **F**.

We note that, if A_2 is chosen to coincide with A_1, i.e., if the particle describes a closed path (Fig. 13.11*b*), we have $V_1 = V_2$

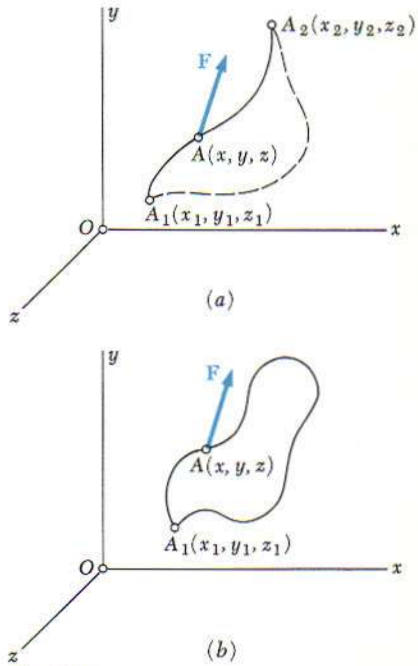


Fig. 13.11

and the work is zero. We may thus write for any conservative force $\mathbf{F}$

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0 \tag{13.20}$$

where the circle on the integral sign indicates that the path is closed.

Let us now apply (13.19) between two neighboring points $A(x,y,z)$ and $A'(x + dx, y + dy, z + dz)$. The elementary work dU corresponding to the displacement $d\mathbf{r}$ from A to A' is

$$dU = V(x,y,z) - V(x + dx, y + dy, z + dz)$$

or,

$$dU = -dV(x,y,z) \tag{13.21}$$

Thus, the elementary work of a conservative force is an *exact differential.*

Substituting for dU in (13.21) the expression obtained in (13.1″), and recalling the definition of the differential of a function of several variables, we write

$$F_x\,dx + F_y\,dy + F_z\,dz = -\left(\frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \frac{\partial V}{\partial z}\,dz\right)$$

from which it follows that

$$F_x = -\frac{\partial V}{\partial x} \qquad F_y = -\frac{\partial V}{\partial y} \qquad F_z = -\frac{\partial V}{\partial z} \tag{13.22}$$

It is clear that the components of $\mathbf{F}$ must be functions of the coordinates x, y, z. Thus, a *necessary* condition for a conservative force is that it depend only upon the position of its point of application. The relations (13.22) may be expressed more concisely if we write

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} = -\left(\frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}\right)$$

The vector in parentheses is known as the *gradient of the scalar function* V and is denoted by **grad** V. We thus write for any conservative force

$$\mathbf{F} = -\mathbf{grad}\, V \tag{13.23}$$

The relations (13.19) to (13.23) were shown to be satisfied by any conservative force. It may also be shown that if a force $\mathbf{F}$ satisfies one of these relations, $\mathbf{F}$ must be a conservative force.

13.8. Conservation of Energy. We saw in the preceding two sections that the work of a conservative force, such as the weight of a particle or the force exerted by a spring, may be expressed as a change in potential energy. When a particle moves under the action of conservative forces, the principle of work and energy stated in Sec. 13.3 may be expressed in a modified form. Substituting for $U_{1\to 2}$ from (13.19′) into (13.10), we write

$$V_1 - V_2 = T_2 - T_1$$

$$\boxed{T_1 + V_1 = T_2 + V_2} \tag{13.24}$$

Formula (13.24) indicates that, when a particle moves under the action of conservative forces, *the sum of the kinetic energy and of the potential energy of the particle remains constant.* The sum $T + V$ is called the *total mechanical energy* of the particle and is denoted by E.

Consider, for example, the pendulum analyzed in Sec. 13.4, which is released with no velocity from A_1 and allowed to swing in a vertical plane (Fig. 13.12). Measuring the potential energy from the level of A_2, we have, at A_1,

$$T_1 = 0 \qquad V_1 = Wl \qquad T_1 + V_1 = Wl$$

Recalling that, at A_2, the speed of the pendulum is $v_2 = \sqrt{2gl}$, we have

$$T_2 = \tfrac{1}{2} m v_2^2 = \frac{1}{2}\frac{W}{g}(2\,gl) = Wl \qquad V_2 = 0$$

$$T_2 + V_2 = Wl$$

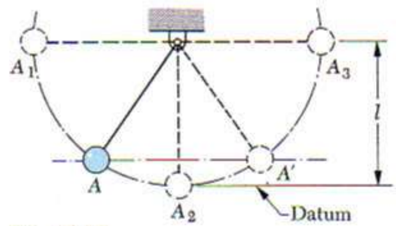


Fig. 13.12

We thus check that the total mechanical energy $E = T + V$ of the pendulum is the same at A_1 and A_2. While the energy is entirely potential at A_1, it becomes entirely kinetic at A_2 and, as the pendulum keeps swinging to the right, the kinetic energy is transformed back into potential energy. At A_3, we shall have $T_3 = 0$ and $V_3 = Wl$.

Since the total mechanical energy of the pendulum remains constant and since its potential energy depends only upon its elevation, the kinetic energy of the pendulum will have the same value at any two points located on the same level. Thus, the speed of the pendulum is the same at A and at A' (Fig. 13.12). This result may be extended to the case of a particle moving along any given path, regardless of the shape of the path, as long as the only forces acting on the particle are its weight and the normal reaction of the path. The particle of Fig. 13.13, for example, which slides in a vertical plane along a frictionless track, will have the same speed at A, A', and A''.

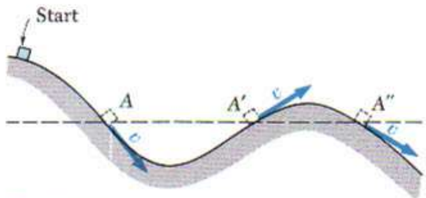


Fig. 13.13

While the weight of a particle and the force exerted by a spring are conservative forces, *friction forces are nonconservative forces*. In other words, *the work of a friction force cannot be expressed as a change in potential energy*. The work of a friction force depends upon the path followed by its point of application; and while the work $U_{1\to2}$ defined by (13.19) is positive or negative according to the sense of motion, *the work of a friction force is always negative*. It follows that, when a mechanical system involves friction, its total mechanical energy does not remain constant but decreases. The mechanical energy of the system, however, is not lost; it is transformed into heat, and the sum of the *mechanical energy* and of the *thermal energy* of the system remains constant.

Other forms of energy may also be involved in a system. For instance, a generator converts mechanical energy into *electric energy;* a gasoline engine converts *chemical energy* into mechanical energy; a nuclear reactor converts *mass* into thermal energy. If all forms of energy are considered, the energy of any system may be considered as constant and the principle of conservation of energy remains valid under all conditions.

13.9. Motion under a Conservative Central Force. Application to Space Mechanics. We saw in Sec. 12.8 that, when a particle P moves under a central force $\mathbf{F}$, the angular momentum $\mathbf{H}_O$ of the particle about the center of force O is constant. If the force $\mathbf{F}$ is also conservative, there exists a potential energy V associated with $\mathbf{F}$, and the total energy $E = T + V$ of the particle is constant (Sec. 13.8). Thus, when a particle moves under a conservative central force, both the principle of conservation of angular momentum and the principle of conservation of energy may be used to study its motion.

Consider, for example, a space vehicle moving under the earth's gravitational force. We shall assume that it begins its free flight at point P_0 at a distance r_0 from the center of the earth, with a velocity $\mathbf{v}_0$ forming an angle ϕ_0 with the radius vector OP_0 (Fig. 13.14). Let P be a point of the trajectory described by the vehicle; we denote by r the distance from O to P, by $\mathbf{v}$ the velocity of the vehicle at P, and by ϕ the angle formed by $\mathbf{v}$ and the radius vector OP. Applying the principle of conservation of angular momentum about O between P_0 and P (Sec. 12.8), we write

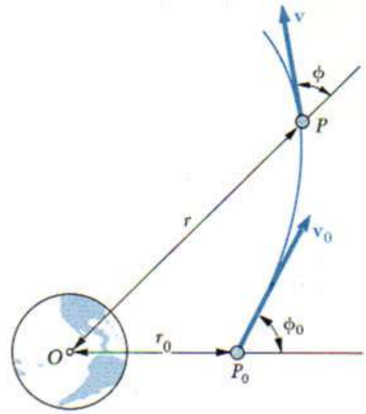


Fig. 13.14

$$r_0 m v_0 \sin \phi_0 = r m v \sin \phi \qquad (13.25)$$

Recalling expression (13.17) obtained for the potential energy due to a gravitational force, we apply the principle of conservation of energy between P_0 and P and write

$$T_0 + V_0 = T + V$$

$$\tfrac{1}{2} m v_0^2 - \frac{GMm}{r_0} = \tfrac{1}{2} m v^2 - \frac{GMm}{r} \qquad (13.26)$$

where M is the mass of the earth.

Equation (13.26) may be solved for the magnitude v of the velocity of the vehicle at P when the distance r from O to P is known; Eq. (13.25) may then be used to determine the angle ϕ that the velocity forms with the radius vector OP.

Equations (13.25) and (13.26) may also be used to determine the maximum and minimum values of r in the case of a satellite launched from P_0 in a direction forming an angle ϕ_0 with the vertical OP_0 (Fig. 13.15). The desired values of r are obtained by making $\phi = 90°$ in (13.25) and eliminating v between Eqs. (13.25) and (13.26).

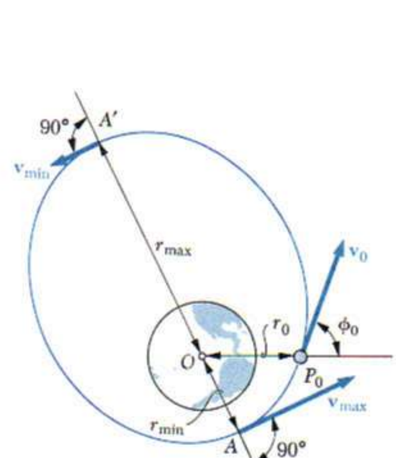
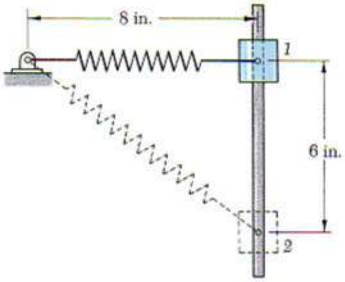


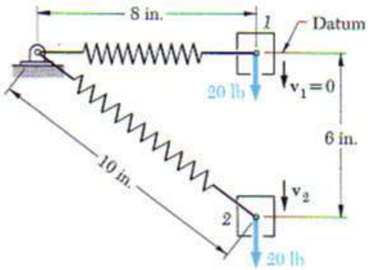
Fig. 13.15

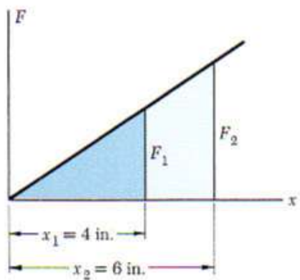
It should be noted that the application of the principles of conservation of energy and of conservation of angular momentum leads to a more fundamental formulation of the problems of space mechanics than the method indicated in Sec. 12.11. In all cases involving oblique launchings, it will also result in much simpler computations. And while the method of Sec. 12.11 must be used when the actual trajectory or the periodic time of a space vehicle is to be determined, the calculations will be simplified if the conservation principles are first used to compute the maximum and minimum values of the radius vector r.



SAMPLE PROBLEM 13.6

A 20-lb collar slides without friction along a vertical rod as shown. The spring attached to the collar has an undeformed length of 4 in. and a constant of 3 lb/in. If the collar is released from rest in position *1*, determine its velocity after it has moved 6 in. to position *2*.





Position *1*. ***Potential Energy.*** The elongation of the spring is $x_1 = 8\text{ in.} - 4\text{ in.} = 4\text{ in.}$, and we have

$$V_e = \tfrac{1}{2}kx_1^2 = \tfrac{1}{2}(3\text{ lb/in.})(4\text{ in.})^2 = 24\text{ in}\cdot\text{lb}$$

Choosing the datum as shown, we have $V_g = 0$. Therefore,

$$V_1 = V_e + V_g = 24\text{ in}\cdot\text{lb} = 2\text{ ft}\cdot\text{lb}$$

Kinetic Energy. Since the velocity in position *1* is zero, $T_1 = 0$.

Position *2*. ***Potential Energy.*** The elongation of the spring is $x_2 = 10\text{ in.} - 4\text{ in.} = 6\text{ in.}$, and we have

$$V_e = \tfrac{1}{2}kx_2^2 = \tfrac{1}{2}(3\text{ lb/in.})(6\text{ in.})^2 = 54\text{ in}\cdot\text{lb}$$
$$V_g = Wy = (20\text{ lb})(-6\text{ in.}) = -120\text{ in}\cdot\text{lb}$$

Therefore,

$$V_2 = V_e + V_g = 54 - 120 = -66\text{ in}\cdot\text{lb}$$
$$= -5.5\text{ ft}\cdot\text{lb}$$

Kinetic Energy

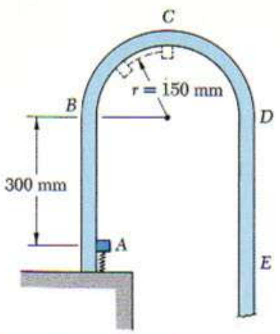
$$T_2 = \tfrac{1}{2}mv_2^2 = \frac{1}{2}\frac{20}{32.2}v_2^2 = 0.311v_2^2$$

Conservation of Energy. Applying the principle of conservation of energy between positions *1* and *2*, we write

$$T_1 + V_1 = T_2 + V_2$$
$$0 + 2\text{ ft}\cdot\text{lb} = 0.311v_2^2 - 5.5\text{ ft}\cdot\text{lb}$$
$$v_2 = \pm 4.91\text{ ft/s}$$

$\mathbf{v}_2 = 4.91\text{ ft/s}\downarrow$ ◄

SAMPLE PROBLEM 13.7



The 200-g pellet is released from rest at A when the spring is compressed 75 mm and travels around the loop $ABCDE$. Determine the smallest value of the spring constant for which the pellet will travel around the loop and will at all times remain in contact with the loop.

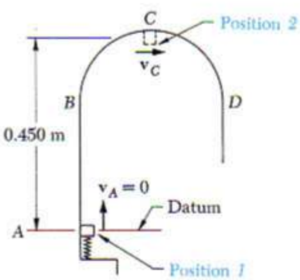
Required Speed at Point C. As the pellet passes through the highest point C, its potential energy with respect to gravity is maximum; thus, at the same point its kinetic energy and its speed are minimum. Since the pellet must remain in contact with the loop, the force $\mathbf{N}$ exerted on the pellet by the loop must be equal to, or greater than, zero. Setting $\mathbf{N} = 0$, we compute the smallest possible speed v_C.

$$+\downarrow \Sigma F_n = ma_n: \qquad W = ma_n \qquad mg = ma_n \qquad a_n = g$$

$$a_n = \frac{v_C^2}{r}: \qquad v_C^2 = ra_n = rg = (0.150\ \mathrm{m})(9.81\ \mathrm{m/s^2}) = 1.472\ \mathrm{m^2/s^2}$$

Position 1. ***Potential Energy.*** Since the spring is compressed 0.075 m from its undeformed position, we have

$$V_e = \tfrac{1}{2}kx^2 = \tfrac{1}{2}k(0.075\ \mathrm{m})^2 = (0.00281\ \mathrm{m^2})k$$



Choosing the datum at A, we have $V_g = 0$; therefore

$$V_1 = V_e + V_g = (0.00281\ \mathrm{m^2})k$$

Kinetic Energy. Since the pellet is released from rest, $v_A = 0$ and we have $T_1 = 0$.

Position 2. ***Potential Energy.*** The spring is now undeformed; thus $V_e = 0$. Since the pellet is 0.450 m above the datum, and since $W = (0.200\ \mathrm{kg})(9.81\ \mathrm{m/s^2}) = 1.962\ \mathrm{N}$, we have

$$V_g = Wy = (1.962\ \mathrm{N})(0.450\ \mathrm{m}) = 0.883\ \mathrm{N \cdot m} = 0.883\ \mathrm{J}$$
$$V_2 = V_e + V_g = 0.883\ \mathrm{J}$$

Kinetic Energy. Using the value of v_C^2 obtained above, we write

$$T_2 = \tfrac{1}{2}mv_C^2 = \tfrac{1}{2}(0.200\ \mathrm{kg})(1.472\ \mathrm{m^2/s^2}) = 0.1472\ \mathrm{N \cdot m} = 0.1472\ \mathrm{J}$$

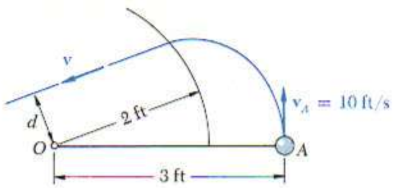
Conservation of Energy. Applying the principle of conservation of energy between positions 1 and 2, we write

$$T_1 + V_1 = T_2 + V_2$$
$$0 + (0.00281\ \mathrm{m^2})k = 0.1472\ \mathrm{J} + 0.883\ \mathrm{J}$$
$$k = 367\ \mathrm{J/m^2} = 367\ \mathrm{N/m}$$

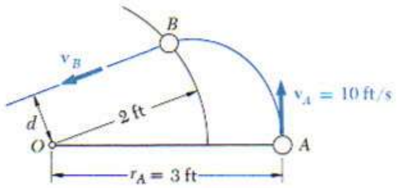
The required minimum value of k is therefore

$$k = 367\ \mathrm{N/m} \quad \blacktriangleleft$$

SAMPLE PROBLEM 13.8



A ball weighing 0.5 lb is attached to a fixed point O by means of an elastic cord of constant $k = 10$ lb/ft and of undeformed length equal to 2 ft. The ball slides on a horizontal frictionless surface. If the ball is placed at point A, 3 ft from O, and is given an initial velocity of 10 ft/s in a direction perpendicular to OA, determine (*a*) the speed of the ball after the cord has become slack, (*b*) the closest distance d that the ball will come to O.



Solution. The force exerted by the cord on the ball passes through the fixed point O, and its work may be expressed as a change in potential energy. It is therefore a conservative central force, and both the total energy of the ball and its angular momentum about O are conserved between points A and B. After the cord has become slack at B, the resultant force acting on the ball is zero. The ball, therefore, will move in a straight line at a constant speed v. The straight line is the line of action of $\mathbf{v}_B$ and the speed v is equal to v_B.

a. Conservation of Energy.

At point A: $$T_A = \tfrac{1}{2}mv_A^2 = \frac{1}{2}\frac{0.5\text{ lb}}{32.2\text{ ft/s}^2}(10\text{ ft/s})^2 = 0.776\text{ ft}\cdot\text{lb}$$

$$V_A = \tfrac{1}{2}kx_A^2 = \tfrac{1}{2}(10\text{ lb/ft})(3\text{ ft} - 2\text{ ft})^2 = 5\text{ ft}\cdot\text{lb}$$

At point B: $$T_B = \tfrac{1}{2}mv_B^2 = \frac{1}{2}\frac{0.5}{32.2}v_B^2 = 0.00776v_B^2$$

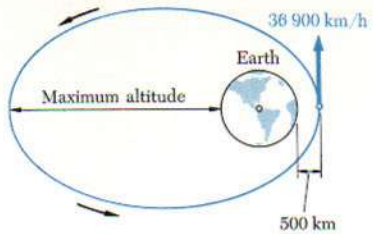
$$V_B = 0$$

Applying the principle of conservation of energy between points A and B, we write

$$T_A + V_A = T_B + V_B$$
$$0.776 + 5 = 0.00776v_B^2$$
$$v_B^2 = 744 \qquad v = v_B = 27.3\text{ ft/s}$$ ◄

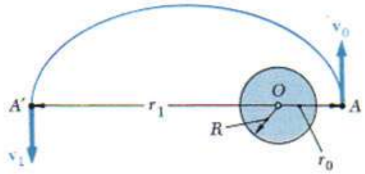
b. Conservation of Angular Momentum About O. Since r_A and d represent the perpendicular distances to $\mathbf{v}_A$ and $\mathbf{v}_B$, respectively, we write

$$r_A(mv_A) = d(mv_B)$$
$$(3\text{ ft})\left(\frac{0.5}{g}\right)(10\text{ ft/s}) = d\left(\frac{0.5}{g}\right)(27.3\text{ ft/s})$$
$$d = 1.099\text{ ft}$$ ◄



SAMPLE PROBLEM 13.9

A satellite is launched in a direction parallel to the surface of the earth with a velocity of 36 900 km/h from an altitude of 500 km. Determine (*a*) the maximum altitude reached by the satellite, (*b*) the maximum allowable error in the direction of launching if the satellite is to go into orbit and to come not closer than 200 km to the surface of the earth.



a. **Maximum Altitude.** We denote by A' the point of the orbit farthest from the earth and by r_1 the corresponding distance from the center of the earth. Since the satellite is in free flight between A and A', we apply the principle of conservation of energy.

$$T_A + V_A = T_{A'} + V_{A'}$$

$$\tfrac{1}{2}mv_0^2 - \frac{GMm}{r_0} = \tfrac{1}{2}mv_1^2 - \frac{GMm}{r_1} \qquad (1)$$

Since the only force acting on the satellite is the force of gravity, which is a central force, the angular momentum of the satellite about O is conserved. Considering points A and A', we write

$$r_0 m v_0 = r_1 m v_1 \qquad v_1 = v_0 \frac{r_0}{r_1} \qquad (2)$$

Substituting this expression for v_1 into Eq. (1) and dividing each term by the mass m, we obtain after rearranging the terms,

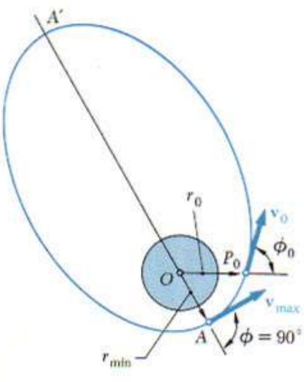
$$\tfrac{1}{2}v_0^2\left(1 - \frac{r_0^2}{r_1^2}\right) = \frac{GM}{r_0}\left(1 - \frac{r_0}{r_1}\right) \qquad 1 + \frac{r_0}{r_1} = \frac{2GM}{r_0 v_0^2} \qquad (3)$$

Recalling that the radius of the earth is $R = 6370$ km, we compute

$r_0 = 6370\text{ km} + 500\text{ km} = 6870\text{ km} = 6.87 \times 10^6\text{ m}$
$v_0 = 36\,900\text{ km/h} = (3.69 \times 10^7\text{ m})/(3.6 \times 10^3\text{ s}) = 1.025 \times 10^4\text{ m/s}$
$GM = gR^2 = (9.81\text{ m/s}^2)(6.37 \times 10^6\text{ m})^2 = 3.98 \times 10^{14}\text{ m}^3/\text{s}^2$

Substituting these values into (3), we obtain $r_1 = 66.8 \times 10^6$ m

Maximum altitude $= 66.8 \times 10^6\text{ m} - 6.37 \times 10^6\text{ m} = 60.4 \times 10^6\text{ m}$
$= 60\,400$ km ◄



b. **Allowable Error in Direction of Launching.** The satellite is launched from P_0 in a direction forming an angle ϕ_0 with the vertical OP_0. The value of ϕ_0 corresponding to $r_{min} = 6370\text{ km} + 200\text{ km} = 6570$ km is obtained by applying the principles of conservation of energy and of conservation of angular momentum between P_0 and A.

$$\tfrac{1}{2}mv_0^2 - \frac{GMm}{r_0} = \tfrac{1}{2}mv_{max}^2 - \frac{GMm}{r_{min}} \qquad (4)$$

$$r_0 m v_0 \sin \phi_0 = r_{min} m v_{max} \qquad (5)$$

Solving (5) for v_{max} and then substituting for v_{max} into (4), we may solve (4) for $\sin \phi_0$. Using the values of v_0 and GM computed in part *a* and noting that $r_0/r_{min} = 6870/6570 = 1.0457$, we find

$\sin \phi_0 = 0.9801 \qquad \phi_0 = 90° \pm 11.5° \qquad$ Allowable error $= \pm 11.5°$ ◄

PROBLEMS

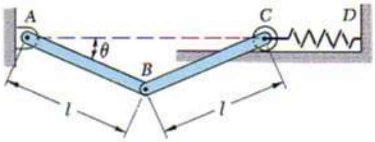


Fig. P13.50

13.50 The uniform rods AB and BC are each of mass m; the spring CD is of constant k and is unstretched when $\theta = 0$. Determine the potential energy of the system with respect to (a) the spring, (b) gravity. (Place datum at A.)

13.51 A slender rod AB of negligible mass is attached to blocks A and B, each of mass m. The constant of the spring is k and the spring is undeformed when AB is horizontal. Determine the potential energy of the system with respect to (a) the spring, (b) gravity. (Place datum at B.)

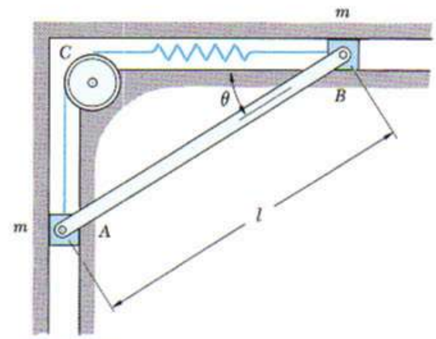


Fig. P13.51

13.52 Prove that a force $\mathbf{F}(x,y,z)$ is conservative if, and only if, the following relations are satisfied:

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} \qquad \frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y} \qquad \frac{\partial F_z}{\partial x} = \frac{\partial F_x}{\partial z}$$

13.53 The force $\mathbf{F} = (x\mathbf{i} + y\mathbf{j})/(x^2 + y^2)$ acts on the particle $P(x,y)$ which moves in the xy plane. (a) Using the first of the relations derived in Prob. 13.52, prove that $\mathbf{F}$ is a conservative force. (b) Determine the potential function $V(x,y)$ associated with $\mathbf{F}$.

13.54 The force $\mathbf{F} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/(x^2 + y^2 + z^2)^{3/2}$ acts on the particle $P(x,y,z)$ which moves in space. (a) Using the relations derived in Prob. 13.52, prove that $\mathbf{F}$ is a conservative force. (b) Determine the potential function $V(x,y,z)$ associated with $\mathbf{F}$.

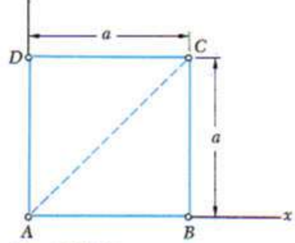


Fig. P13.55

13.55 The force $\mathbf{F} = x^2y\mathbf{i} + xy^2\mathbf{j}$ acts on the particle $P(x,y)$ which moves in the xy plane. Prove that $\mathbf{F}$ is a nonconservative force and determine the work of $\mathbf{F}$ as it moves from A to C along each of the paths ABC, ADC, and AC.

13.56 The spring AB is of constant 6 lb/in. and is attached to the 4-lb collar A which moves freely along the horizontal rod. The unstretched length of the spring is 10 in. If the collar is released from rest in the position shown, determine the maximum velocity attained by the collar.

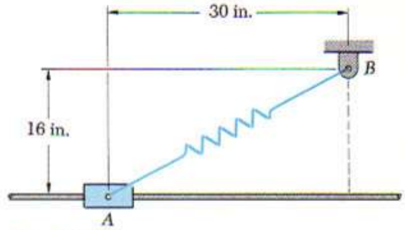


Fig. P13.56

13.57 In Prob. 13.56, determine the weight of the collar A for which the maximum velocity is 30 ft/s.

13.58 A collar of mass 1.5 kg is attached to a spring and slides without friction along a circular rod which lies in a *horizontal* plane. The spring is undeformed when the collar is at C and the constant of the spring is 400 N/m. If the collar is released from rest at B, determine the velocity of the collar as it passes through point C.

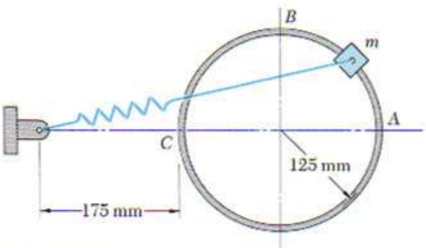


Fig. P13.58

13.59 Plunger A has a mass of 200 g and is to be shot to the right by the mechanism shown. The undeformed length of the spring is 180 mm and it is compressed to a length of 60 mm; it will expand to a length of 110 mm when the plunger is released. Knowing that a force of 36 N is required to hold the plunger in the position shown, determine the velocity attained by the plunger as it leaves the mechanism.

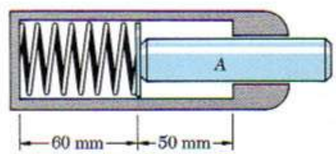


Fig. P13.59

13.60 The collar of Prob. 13.58 has a continuous, although nonuniform, motion along the rod. If the speed of the collar at A is to be half of its speed at C, determine (*a*) the required speed at C, (*b*) the corresponding speed at B.

13.61 The 2-lb collar slides without friction along the horizontal rod. Knowing that the constant of the spring is 3 lb/in. and that $v_0 = 12$ ft/s, determine the required spring tension in the position shown if the speed of the collar is to be 8 ft/s at point C.

13.62 The 2-lb collar slides without friction along the horizontal rod. Knowing that the spring has a constant $k = 3$ lb/in. and is unstretched in the position shown, determine the required speed v_0 if it is to reach point C.

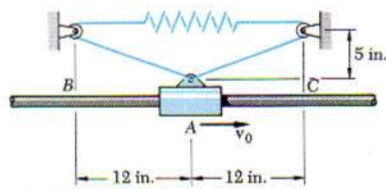


Fig. P13.61 and P13.62

13.63 The 50-kg block is released from rest when $\phi = 0$. If the speed of the block when $\phi = 90°$ is to be 2.5 m/s, determine the required value of the initial tension in the spring.

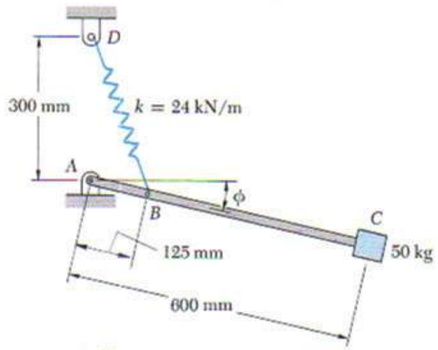


Fig. P13.63

13.64 A sling shot is made by stretching an elastic band between pins A and B located 100 mm apart in the same horizontal plane. The spring constant for the entire length of the elastic band is 600 N/m and the tension in the band is 40 N when it is stretched directly between A and B. Determine the maximum speed attained by a 50-g pellet which is placed at C and released.

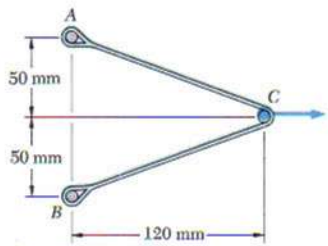


Fig. P13.64

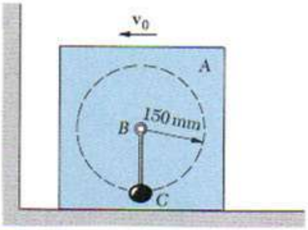


Fig. P13.65

13.65 The sphere C and the block A are both moving to the left with a velocity $\mathbf{v}_0$ when the block is suddenly stopped by the wall. Determine the smallest velocity $\mathbf{v}_0$ for which the sphere C will swing in a full circle about the pivot B (*a*) if BC is a slender rod of negligible weight, (*b*) if BC is a cord.

13.66 The collar of Prob. 13.58 is released from rest at point A. Determine the horizontal component of the force exerted by the rod on the collar as the collar passes through point B. Show that the force component is independent of the mass of the collar.

13.67 A 1.5-lb collar may slide without friction along the semicircular rod *BCD*. The spring is of constant 2 lb/in. and its undeformed length is 12 in. The collar is released from rest at *B*. As the collar passes through point *C*, determine (*a*) the speed of the collar, (*b*) the force exerted by the rod on the collar.

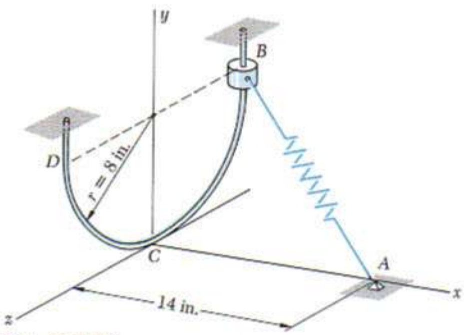


Fig. P13.67

13.68 A small block is released at *A* with zero velocity and moves along the frictionless guide to point *B* where it leaves the guide with a horizontal velocity. Knowing that $h = 8$ ft and $b = 3$ ft, determine (*a*) the speed of the block as it strikes the ground at *C*, (*b*) the corresponding distance *c*.

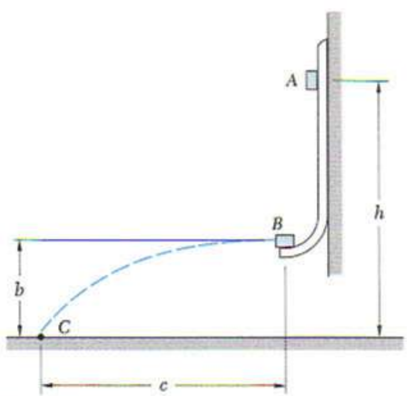


Fig. P13.68

13.69 Assuming a given height *h* in Prob. 13.68, (*a*) show that the speed at *C* is independent of the height *b*, (*b*) determine the height *b* for which the distance *c* is maximum and the corresponding value of *c*.

13.70 A ball of mass m attached to an inextensible cord rotates in a vertical circle of radius r. Show that the difference between the maximum value T_{max} of the tension in the cord and its minimum value T_{min} is independent of the speed v_0 of the ball as measured at the bottom of the circle, and determine $T_{max} - T_{min}$.

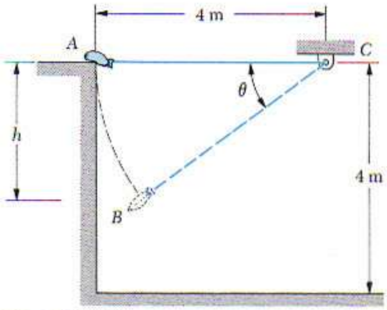


Fig. P13.71

13.71 A bag is gently pushed off the top of a wall at A and swings in a vertical plane at the end of a 4-m rope which can withstand a maximum tension equal to twice the weight of the bag. (*a*) Determine the difference in elevation h between point A and point B where the rope will break. (*b*) How far from the vertical wall will the bag strike the floor?

13.72 A delicate instrument weighing 12 lb is placed on a spring of length l so that its base is just touching the undeformed spring. The instrument is then inadvertently released from that position. Determine the maximum deflection x of the spring and the maximum force exerted by the spring if the constant of the spring is $k = 15$ lb/in.

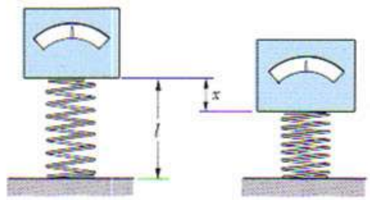


Fig. P13.72

13.73 Nonlinear springs are classified as hard or soft, depending upon the curvature of their force-deflection curves (see figure). Solve Prob. 13.72, assuming (*a*) that a hard spring is used, for which $F = 15x(1 + 0.1x^2)$, (*b*) that a soft spring is used, for which $F = 15x(1 - 0.1x^2)$.

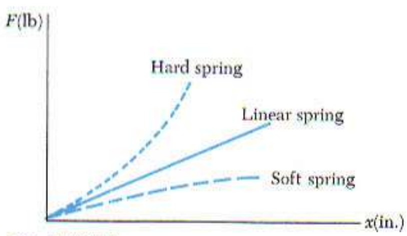


Fig. P13.73

13.74 Determine the escape velocity of a missile, i.e., the velocity with which it should be fired from the surface of the earth if it is to reach an infinite distance from the earth. Give the answer in both SI and U.S. customary units. Show that the result obtained is independent of the firing angle.

13.75 How much energy per kilogram should be imparted to a satellite in order to place it in a circular orbit at an altitude of (*a*) 500 km, (*b*) 5000 km?

13.76 A lunar excursion module (LEM) was used in the Apollo moon-landing missions to save fuel by making it unnecessary to launch the entire Apollo spacecraft from the moon's surface on its return trip to the earth. Check the effectiveness of this approach by computing the energy per pound required for a spacecraft to escape the gravitational field of the moon if the spacecraft starts (*a*) from the moon's surface, (*b*) from a circular orbit 60 mi above the moon's surface. Neglect the effect of the earth's gravitational field. (The radius of the moon is 1080 mi and its mass is 0.01230 times the mass of the earth.)

13.77 Show, by setting $r = R + y$ in formula (13.17′) and expanding in a power series in y/R, that the expression obtained in (13.16) for the potential energy V_g due to gravity is a first-order approximation for the expression given in (13.17′). Using the same expansion, derive a second-order approximation for V_g.

13.78 Show that the ratio of the potential and kinetic energies of an electron, as it enters the plates of the cathode-ray tube of Prob. 12.60, is equal to $d\delta/lL$. (Place the datum at the surface of the positive plate.)

13.79 In Sample Prob. 13.8, determine the required magnitude of $\mathbf{v}_A$ if the ball is to pass at a distance $d = 4$ in. from point O. Assume that the direction of $\mathbf{v}_A$ is not changed.

13.80 A 2-kg sphere is attached to an elastic cord of constant 150 N/m which is undeformed when the sphere is located at the origin O. Knowing that in the position shown $\mathbf{v}_A$ is perpendicular to OP and has a magnitude of 10 m/s, determine (*a*) the maximum distance from the origin attained by the sphere, (*b*) the corresponding speed of the sphere.

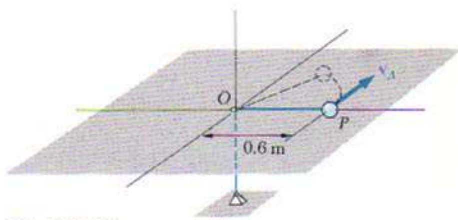


Fig. P13.80

13.81 In Prob. 13.80, determine the required initial speed v_A if the maximum distance from the origin attained by the sphere is to be 1.5 m.

13.82 A 1.5-lb block P rests on a frictionless horizontal table at a distance of 1 ft from a fixed pin O. The block is attached to pin O by an elastic cord of constant $k = 10$ lb/ft and of undeformed length 2 ft. If the block is set in motion to the right as shown, determine (*a*) the speed v_1 for which the distance from O to the block P will reach a maximum value of 3 ft, (*b*) the speed v_2 when $OP = 3$ ft, (*c*) the radius of curvature of the path of the block when $OP = 3$ ft.

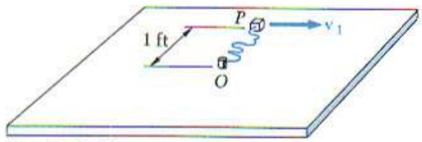


Fig. P13.82

13.83 Collar B weighs 10 lb and is attached to a spring of constant 50 lb/ft and of undeformed length equal to 18 in. The system is set in motion with $r = 12$ in., $v_\theta = 16$ ft/s, and $v_r = 0$. Neglecting the mass of the rod and the effect of friction, determine the radial and transverse components of the velocity of the collar when $r = 21$ in.

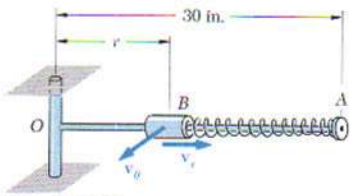


Fig. P13.83

13.84 For the motion described in Prob. 13.83, determine (*a*) the maximum distance between the origin and the collar, (*b*) the corresponding velocity.

13.85 In Sample Prob. 13.8, determine the smallest magnitude of $\mathbf{v}_A$ for which the elastic cord will remain taut at all times.

13.86 through 13.89 Using the principles of conservation of energy and conservation of angular momentum, solve the following problems:

13.86 Prob. 12.88.
13.87 Prob. 12.93.
13.88 Prob. 12.92.
13.89 Prob. 12.89.

13.90 A space shuttle is to rendezvous with an orbiting laboratory which circles the earth at the constant altitude of 240 mi. The shuttle has reached an altitude of 40 mi when its engine is shut off, and its velocity $\mathbf{v}_0$ forms an angle $\phi_0 = 45°$ with the vertical OB at that time. What magnitude should $\mathbf{v}_0$ have if the shuttle's trajectory is to be tangent at A to the orbit of the laboratory?

13.91 A space shuttle is to rendezvous with an orbiting laboratory which circles the earth at the constant altitude of 240 mi. The shuttle has reached an altitude of 40 mi and a velocity $\mathbf{v}_0$ of magnitude 12,000 ft/s when its engine is shut off. What is the angle ϕ_0 that $\mathbf{v}_0$ should form with the vertical OB if the shuttle's trajectory is to be tangent at A to the orbit of the laboratory?

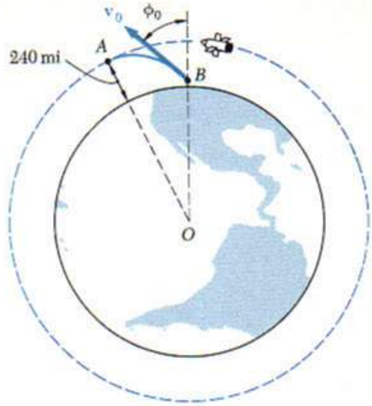


Fig. P13.90 and P13.91

13.92 Determine the magnitude and direction (angle ϕ formed with the vertical OB) of the velocity $\mathbf{v}_B$ of the spacecraft of Prob. 12.100 just before splashdown at B. Neglect the effect of the atmosphere.

13.93 To what value v_0 should the speed of the spacecraft of Prob. 12.101 be reduced preparatory to reentry if its velocity $\mathbf{v}_B$ just before splashdown at B is to form an angle $\phi = 30°$ with the vertical OB? Neglect the effect of the atmosphere.

13.94 Upon the LEM's return to the command module, the Apollo spacecraft of Prob. 12.93 is turned around so that the LEM faces to the rear. The LEM is then cast adrift with a velocity of 600 ft/s relative to the command module. Determine the magnitude and direction (angle ϕ formed with the vertical OC) of the velocity $\mathbf{v}_C$ of the LEM just before it crashes at C on the moon's surface.

13.95 At engine burnout a satellite has reached an altitude of 2400 km and has a velocity $\mathbf{v}_0$ of magnitude 8100 m/s forming an angle $\phi_0 = 76°$ with the vertical. Determine the maximum and minimum heights reached by the satellite.

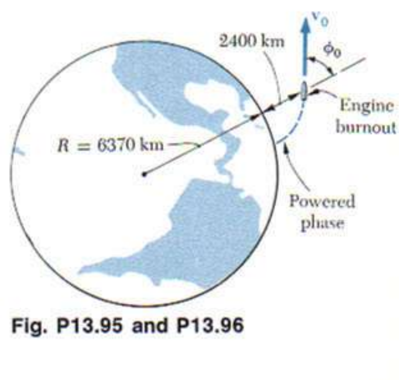


Fig. P13.95 and P13.96

13.96 At engine burnout a satellite has reached an altitude of 2400 km and has a velocity $\mathbf{v}_0$ of magnitude 8100 m/s. For what range of values of the angle ϕ_0, formed by $\mathbf{v}_0$ and the vertical, will the satellite go into a permanent orbit? (Assume that if the satellite gets closer than 300 km from the earth's surface, it will soon burn up.)

13.97 A satellite is projected into space with a velocity $\mathbf{v}_0$ at a distance r_0 from the center of the earth by the last stage of its launching rocket. The velocity $\mathbf{v}_0$ was designed to send the satellite into a circular orbit of radius r_0. However, owing to a malfunction of control, the satellite is not projected horizontally but at an angle α with the horizontal and, as a result, is propelled into an elliptic orbit. Determine the maximum and minimum values of the distance from the center of the earth to the satellite.

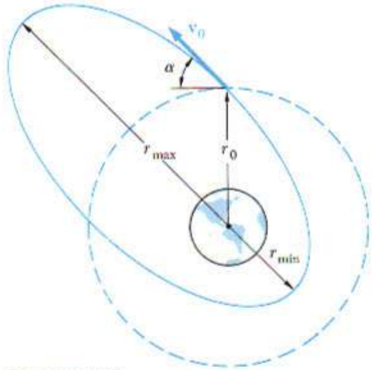


Fig. P13.97

***13.98** Using the answers obtained in Prob. 13.97, show that the intended circular orbit and the resulting elliptic orbit intersect at the ends of the minor axis of the elliptic orbit.

13.99 A spacecraft of mass m describes a circular orbit of radius r_1 around the earth. (*a*) Show that the additional energy ΔE which must be imparted to the spacecraft to transfer it to a circular orbit of larger radius r_2 is

$$\Delta E = \frac{GMm(r_2 - r_1)}{2r_1r_2}$$

where M is the mass of the earth. (*b*) Further show that, if the transfer from one circular orbit to the other is executed by placing the spacecraft on a transitional semielliptic path AB, the amounts of energy ΔE_A and ΔE_B which must be imparted at A and B are respectively proportional to r_2 and r_1:

$$\Delta E_A = \frac{r_2}{r_1 + r_2}\Delta E \qquad \Delta E_B = \frac{r_1}{r_1 + r_2}\Delta E$$

Fig. P13.99

13.100 Show that the total energy E of a satellite of mass m describing an elliptic orbit is $E = -GMm/(r_1 + r_2)$, where M is the mass of the earth, and r_1 and r_2 represent, respectively, the maximum and minimum distance of the orbit to the center of the earth. (It is recalled that the gravitational potential energy of a satellite was defined as being zero at an infinite distance from the earth.)

***13.101** (*a*) Express the angular momentum per unit mass, h, and the total energy per unit mass, E/m, of a space vehicle moving under the earth's gravitational force in terms of $r_{\min}$ and $v_{\max}$ (Fig. 13.15). (*b*) Eliminating $v_{\max}$ between the equations obtained, derive the formula

$$\frac{1}{r_{\min}} = \frac{GM}{h^2} + \sqrt{\left(\frac{GM}{h^2}\right)^2 + \frac{2(E/m)}{h^2}}$$

(*c*) Show that the constant C in Eq. (12.39) of Sec. 12.11 may be expressed as

$$C = \sqrt{\left(\frac{GM}{h^2}\right)^2 + \frac{2(E/m)}{h^2}}$$

(*d*) Further show that the trajectory of the vehicle is a hyperbola, an ellipse, or a parabola, depending on whether E is positive, negative or zero.

***13.102** In Prob. 13.90, determine the distance separating the two points located on the surface of the earth directly below points B and A where engine shut-off and rendezvous with the orbiting laboratory respectively take place. [*Hint.* Use Eq. (12.39) of Sec. 12.11, noting that point A corresponds to $\theta = 180°$.]

***13.103** A missile is fired from the ground with a velocity $\mathbf{v}_0$ of magnitude $v_0 = \sqrt{gR}$, forming an angle ϕ_0 with the vertical. (*a*) Express the maximum height d reached by the missile in terms of ϕ_0. (*b*) Show that the angle 2α subtending the trajectory BAC of the missile is equal to $2\phi_0$ and explain what happens when ϕ_0 approaches 90°. [*Hint.* Use Eq. (12.39) of Sec. 12.11 to solve part *b*, noting that $\theta = 180°$ for point A and $\theta = 180° - \alpha$ for point B.]

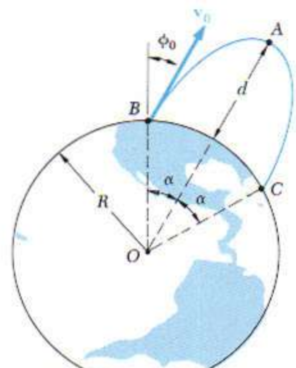


Fig. P13.103 and P13.104

***13.104** A missile is fired from the ground with an initial velocity $\mathbf{v}_0$ forming an angle ϕ_0 with the vertical. If the missile is to reach a maximum altitude equal to the radius of the earth, (*a*) show that the required angle ϕ_0 is defined by the relation

$$\sin \phi_0 = 2\sqrt{1 - \frac{1}{2}\left(\frac{v_{esc}}{v_0}\right)^2}$$

where v_{esc} is the escape velocity, (*b*) determine the maximum and minimum allowable values of v_0.

13.10. Principle of Impulse and Momentum.

A third basic method for the solution of problems dealing with the motion of particles will be considered now. This method is based on the principle of impulse and momentum and may be used to solve problems involving force, mass, velocity, and time. It is of particular interest in the solution of problems involving impulsive motion or impact (Secs. 13.11 and 13.12).

Consider a particle of mass m acted upon by a force $\mathbf{F}$. As we saw in Sec. 12.2, Newton's second law may be expressed in the form

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) \tag{13.27}$$

where $m\mathbf{v}$ is the linear momentum of the particle. Multiplying both sides of Eq. (13.27) by dt and integrating from a time t_1 to a time t_2, we write

$$\mathbf{F}\,dt = d(m\mathbf{v})$$

$$\int_{t_1}^{t_2} \mathbf{F}\,dt = m\mathbf{v}_2 - m\mathbf{v}_1$$

or, transposing the last term,

$$m\mathbf{v}_1 + \int_{t_1}^{t_2} \mathbf{F}\,dt = m\mathbf{v}_2 \tag{13.28}$$

The integral in Eq. (13.28) is a vector known as the *linear impulse*, or simply the *impulse*, of the force $\mathbf{F}$ during the interval of time considered. Resolving $\mathbf{F}$ into rectangular components, we write

$$\begin{aligned} \mathbf{Imp}_{1\to2} &= \int_{t_1}^{t_2} \mathbf{F}\,dt \\ &= \mathbf{i}\int_{t_1}^{t_2} F_x\,dt + \mathbf{j}\int_{t_1}^{t_2} F_y\,dt + \mathbf{k}\int_{t_1}^{t_2} F_z\,dt \end{aligned} \tag{13.29}$$

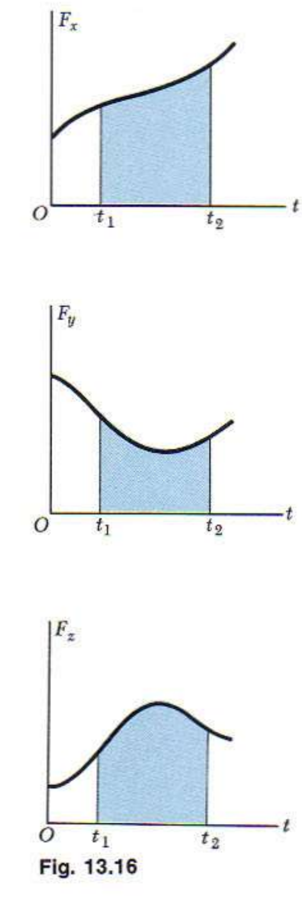


Fig. 13.16

and note that the components of the impulse of the force $\mathbf{F}$ are, respectively, equal to the areas under the curves obtained by plotting the components F_x, F_y, and F_z against t (Fig. 13.16). In the case of a force $\mathbf{F}$ of constant magnitude and direction, the impulse is represented by the vector $\mathbf{F}(t_2 - t_1)$, which has the same direction as $\mathbf{F}$.

If SI units are used, the magnitude of the impulse of a force is expressed in N · s. But, recalling the definition of the newton, we have

$$\mathrm{N \cdot s} = (\mathrm{kg \cdot m/s^2}) \cdot \mathrm{s} = \mathrm{kg \cdot m/s}$$

which is the unit obtained in Sec. 12.3 for the linear momentum of a particle. We thus check that Eq. (13.28) is dimensionally correct. If U.S. customary units are used, the impulse of a force is expressed in lb · s, which is also the unit obtained in Sec. 12.3 for the linear momentum of a particle.

Equation (13.28) expresses that, when a particle is acted upon by a force **F** during a given time interval, *the final momentum* $m\mathbf{v}_2$ *of the particle may be obtained by adding vectorially its initial momentum* $m\mathbf{v}_1$ *and the impulse of the force* **F** *during the time interval considered* (Fig. 13.17). We write

$$m\mathbf{v}_1 + \mathbf{Imp}_{1\rightarrow 2} = m\mathbf{v}_2 \tag{13.30}$$

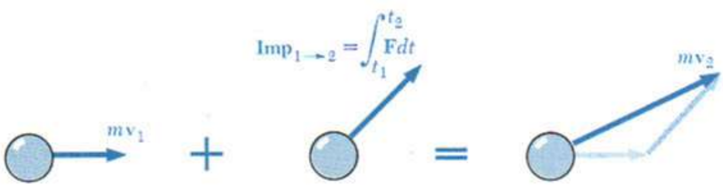


Fig. 13.17

We note that, while kinetic energy and work are scalar quantities, momentum and impulse are vector quantities. To obtain an analytic solution, it is thus necessary to replace Eq. (13.30) by the equivalent component equations

$$\begin{aligned}(mv_x)_1 + \int_{t_1}^{t_2} F_x\,dt &= (mv_x)_2\\ (mv_y)_1 + \int_{t_1}^{t_2} F_y\,dt &= (mv_y)_2\\ (mv_z)_1 + \int_{t_1}^{t_2} F_z\,dt &= (mv_z)_2\end{aligned} \tag{13.31}$$

When several forces act on a particle, the impulse of each of the forces must be considered. We have

$$m\mathbf{v}_1 + \Sigma\,\mathbf{Imp}_{1\rightarrow 2} = m\mathbf{v}_2 \tag{13.32}$$

Again, the equation obtained represents a relation between vector quantities; in the actual solution of a problem, it should be replaced by the corresponding component equations.

When a problem involves two particles or more, each particle may be considered separately and Eq. (13.32) may be written for each particle. We may also add vectorially the momenta of all the particles and the impulses of all the forces involved. We write then

$$\Sigma m\mathbf{v}_1 + \Sigma\,\mathbf{Imp}_{1\to 2} = \Sigma m\mathbf{v}_2 \tag{13.33}$$

Since the forces of action and reaction exerted by the particles on each other form pairs of equal and opposite forces, and since the time interval from t_1 to t_2 is common to all the forces involved, the impulses of the forces of action and reaction cancel out, and only the impulses of the external forces need be considered.†

If no external force is exerted on the particles or, more generally, if the sum of the external forces is zero, the second term in Eq. (13.33) vanishes, and Eq. (13.33) reduces to

$$\Sigma m\mathbf{v}_1 = \Sigma m\mathbf{v}_2 \tag{13.34}$$

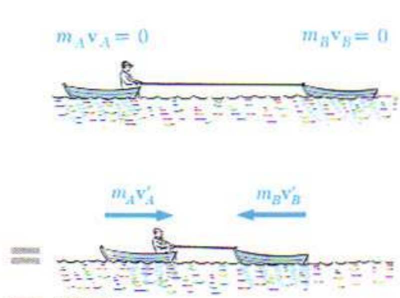


Fig. 13.18

which expresses that *the total momentum of the particles is conserved*. Consider, for example, two boats, of mass m_A and m_B, initially at rest, which are being pulled together (Fig. 13.18). If the resistance of the water is neglected, the only external forces acting on the boats are their weights and the buoyant forces exerted on them. Since these forces are balanced, we write

$$\Sigma m\mathbf{v}_1 = \Sigma m\mathbf{v}_2$$
$$0 = m_A\mathbf{v}'_A + m_B\mathbf{v}'_B$$

where $\mathbf{v}'_A$ and $\mathbf{v}'_B$ represent the velocities of the boats after a finite interval of time. The equation obtained indicates that the boats move in opposite directions (toward each other) with velocities inversely proportional to their masses.‡

† We should note the difference between this statement and the corresponding statement made in Sec. 13.4 regarding the work of the forces of action and reaction between several particles. While the sum of the impulses of these forces is always zero, the sum of their work is zero only under special circumstances, e.g., when the various bodies involved are connected by inextensible cords or links and are thus constrained to move through equal distances.

‡ The application of the method of impulse and momentum to a system of particles and the concept of conservation of momentum for a system of particles are discussed in detail in Chap. 14.

13.11. Impulsive Motion. In some problems, a very large force may act during a very short time interval on a particle and produce a definite change in momentum. Such a force is called an *impulsive force* and the resulting motion an *impulsive motion*. For example, when a baseball is struck, the contact between bat and ball takes place during a very short time interval Δt. But the average value of the force $\mathbf{F}$ exerted by the bat on the ball is very large, and the resulting impulse $\mathbf{F}\,\Delta t$ is large enough to change the sense of motion of the ball (Fig. 13.19).

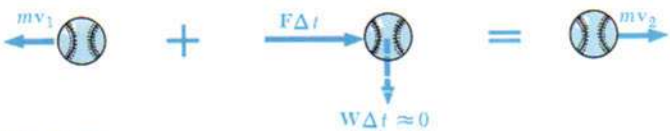


Fig. 13.19

When impulsive forces act on a particle, Eq. (13.32) becomes

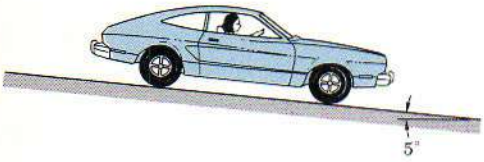
$$m\mathbf{v}_1 + \Sigma\mathbf{F}\,\Delta t = m\mathbf{v}_2 \tag{13.35}$$

Any force which is not an impulsive force may be neglected, since the corresponding impulse $\mathbf{F}\,\Delta t$ is very small. *Nonimpulsive forces* include the weight of the body, the force exerted by a spring, or any other force which is *known* to be small compared with an impulsive force. Unknown reactions may or may not be impulsive; their impulse should therefore be included in Eq. (13.35) as long as it has not been proved negligible. The impulse of the weight of the baseball considered above, for example, may be neglected. If the motion of the bat is analyzed, the impulse of the weight of the bat may also be neglected. The impulses of the reactions of the player's hands on the bat, however, should be included; these impulses will not be negligible if the ball is incorrectly hit.

In the case of the impulsive motion of several particles, Eq. (13.33) may be used. It reduces to

$$\Sigma m\mathbf{v}_1 + \Sigma\mathbf{F}\,\Delta t = \Sigma m\mathbf{v}_2 \tag{13.36}$$

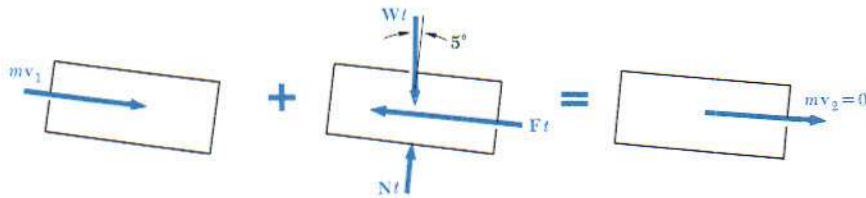
where the second term involves only impulsive, external forces. If all the external forces are nonimpulsive, the second term vanishes, and Eq. (13.36) reduces to Eq. (13.34); the total momentum of the particles is conserved.



SAMPLE PROBLEM 13.10

An automobile weighing 4000 lb is driven down a 5° incline at a speed of 60 mi/h when the brakes are applied, causing a constant total braking force (applied by the road on the tires) of 1500 lb. Determine the time required for the automobile to come to a stop.

Solution. We apply the principle of impulse and momentum. Since each force is constant in magnitude and direction, each corresponding impulse is equal to the product of the force and of the time interval t.

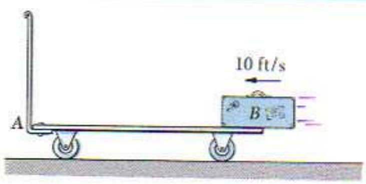


$$m\mathbf{v}_1 + \Sigma\, \mathbf{Imp}_{1\to 2} = m\mathbf{v}_2$$

$+\searrow x$ components: $\quad mv_1 + (W \sin 5°)t - Ft = 0$

$$(4000/32.2)(88 \text{ ft/s}) + (4000 \sin 5°)t - 1500t = 0$$

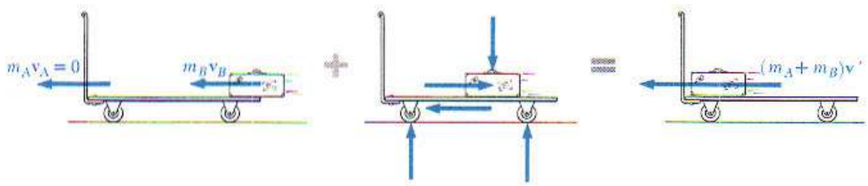
$$t = 9.49 \text{ s} \quad ◄$$



SAMPLE PROBLEM 13.11

An airline employee tosses a 30-lb suitcase with a horizontal velocity of 10 ft/s onto a 70-lb baggage carrier. Knowing that the carrier can roll freely and is initially at rest, determine the velocity of the carrier after the suitcase has slid to a relative stop on the carrier.

Solution. We apply the principle of impulse and momentum to the carrier-suitcase system. Since the impulses of the internal forces cancel out, and since there are no horizontal external forces, the total momentum of the carrier and suitcase is conserved.

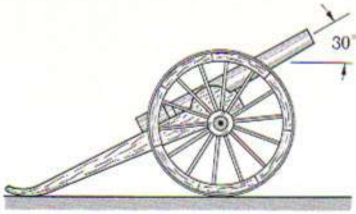


$$m_A\mathbf{v}_A + m_B\mathbf{v}_B = (m_A + m_B)\mathbf{v}'$$

$\xleftarrow{+}$ x components: $\quad 0 + \dfrac{30}{g}(10 \text{ ft/s}) = \dfrac{70 + 30}{g}v'$

$$v' = 3 \text{ ft/s} \quad ◄$$

SAMPLE PROBLEM 13.12



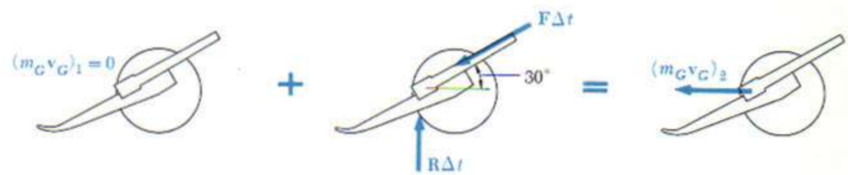
An old 2000-kg gun fires a 10-kg shell with an initial velocity of 600 m/s at an angle of 30°. The gun rests on a horizontal surface and is free to move horizontally. Assuming that the barrel of the gun is rigidly attached to the frame (no recoil mechanism) and that the shell leaves the barrel 6 ms after firing, determine the recoil velocity of the gun and the resultant **R** of the vertical impulsive forces exerted by the ground on the gun.

Solution. We first apply the principle of impulse and momentum to the shell and find the impulse $\mathbf{F}\,\Delta t$ exerted by the gun on the shell. We then apply it to the gun and find the final momentum of the gun and the impulse $\mathbf{R}\,\Delta t$ exerted by the ground on the gun. Since the time interval $\Delta t = 6 \text{ ms} = 0.006 \text{ s}$ is very short, we neglect all nonimpulsive forces.

Free Body: Shell

$$(m_S\mathbf{v}_S)_1 + \Sigma\,\mathbf{Imp}_{1\to2} = (m_S\mathbf{v}_S)_2$$
$$0 + F\,\Delta t = (10 \text{ kg})(600 \text{ m/s})$$
$$F\,\Delta t = 6000 \text{ kg}\cdot\text{m/s} = 6000 \text{ N}\cdot\text{s}$$

Free Body: Gun



$$(m_G\mathbf{v}_G)_1 + \Sigma\,\mathbf{Imp}_{1\to2} = (m_G\mathbf{v}_G)_2$$

$\xrightarrow{+}$ x components: $\quad 0 - (F\,\Delta t)\cos 30° = -m_G v_G$

$$0 - (6000 \text{ kg}\cdot\text{m/s})\cos 30° = -(2000 \text{ kg})v_G$$

$$v_G = +2.60 \text{ m/s} \qquad \mathbf{v}_G = 2.60 \text{ m/s} \leftarrow \blacktriangleleft$$

$+\uparrow y$ components: $\quad 0 + R\,\Delta t - (F\,\Delta t)\sin 30° = 0$

$$R\,\Delta t = (6000 \text{ N}\cdot\text{s})\sin 30° = 3000 \text{ N}\cdot\text{s}$$

$$R = \frac{3000 \text{ N}\cdot\text{s}}{0.006 \text{ s}} = +500\,000 \text{ N} \qquad \mathbf{R} = 500 \text{ kN} \uparrow \blacktriangleleft$$

The high value obtained for the magnitude of **R** stresses the need in modern guns for a recoil mechanism which allows the barrel to move and brings it to rest over a period of time substantially longer than Δt. Although the total vertical impulse remains the same, the longer time interval results in a smaller value for the magnitude of **R**.

PROBLEMS

13.105 A 2750-lb automobile is moving at a speed of 45 mi/h when the brakes are fully applied, causing all four wheels to skid. Determine the time required to stop the automobile (*a*) on concrete ($\mu = 0.80$), (*b*) on ice ($\mu = 0.10$).

13.106 A tugboat exerts a constant force of 25 tons on a 200,000-ton oil tanker. Neglecting the frictional resistance of the water, determine the time required to increase the speed of the tanker (*a*) from 1 mi/h to 2 mi/h, (*b*) from 2 mi/h to 3 mi/h.

13.107 A 3-lb particle is acted upon by a force $\mathbf{F}$ of magnitude $F = 14t^2$ (lb) which acts in the direction of the unit vector $\boldsymbol{\lambda} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$. Knowing that the velocity of the particle at $t = 0$ is $\mathbf{v} = (400\text{ ft/s})\mathbf{j} - (250\text{ ft/s})\mathbf{k}$, determine the velocity when $t = 3$ s.

13.108 A 2-kg particle is acted upon by the force, expressed in newtons, $\mathbf{F} = (8 - 6t)\mathbf{i} + (4 - t^2)\mathbf{j} + (4 + t)\mathbf{k}$. Knowing that the velocity of the particle is $\mathbf{v} = (150\text{ m/s})\mathbf{i} + (100\text{ m/s})\mathbf{j} - (250\text{ m/s})\mathbf{k}$ at $t = 0$, determine (*a*) the time at which the velocity of the particle is parallel to the *yz* plane, (*b*) the corresponding velocity of the particle.

13.109 and 13.110 The initial velocity of the 50-kg car is 5 m/s to the left. Determine the time *t* at which the car has (*a*) no velocity, (*b*) a velocity of 5 m/s to the right.

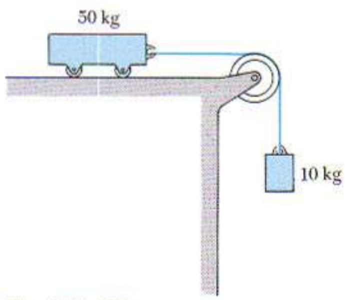


Fig. P13.109

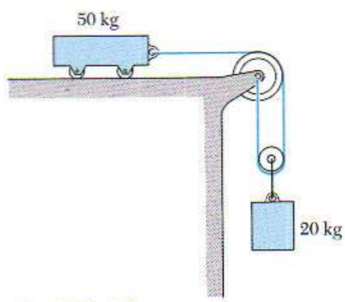


Fig. P13.110

13.111 Using the principle of impulse and momentum, solve Prob. 12.17*b*.

13.112 Using the principle of impulse and momentum, solve Prob. 12.18*b*.

13.113 A light train made of two cars travels at 100 km/h. The mass of car A is 15 Mg, and the mass of car B is 20 Mg. When the brakes are applied, a constant braking force of 25 kN is applied to each car. Determine (*a*) the time required for the train to stop after the brakes are applied, (*b*) the force in the coupling between the cars while the train is slowing down.

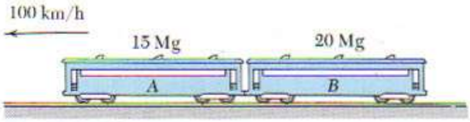


Fig. P13.113

13.114 Solve Prob. 13.113, assuming that a constant braking force of 25 kN is applied to car B but that the brakes on car A are not applied.

13.115 The 3-lb collar is initially at rest and is acted upon by the force **Q** which varies as shown. Knowing that $\mu = 0.25$, determine the velocity of the collar at (*a*) $t = 1$ s, (*b*) $t = 2$ s.

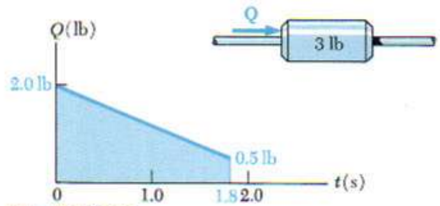


Fig. P13.115

13.116 In Prob. 13.115, determine (*a*) the maximum velocity reached by the collar and the corresponding time, (*b*) the time at which the collar comes to rest.

13.117 A 20-kg block is initially at rest and is subjected to a force **P** which varies as shown. Neglecting the effect of friction, determine (*a*) the maximum speed attained by the block, (*b*) the speed of the block at $t = 1.5$ s.

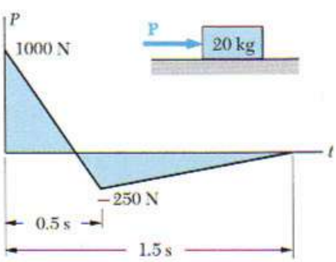


Fig. P13.117

13.118 Solve Prob. 13.117, assuming that $\mu = 0.25$ between the block and the surface.

13.119 A gun of mass 50 Mg is designed to fire a 250-kg shell with an initial velocity of 600 m/s. Determine the average force required to hold the gun motionless if the shell leaves the gun 0.02 s after being fired.

13.120 A 6000-kg plane lands on the deck of an aircraft carrier at a speed of 200 km/h relative to the carrier and is brought to a stop in 3.0 s. Determine the average horizontal force exerted by the carrier on the plane (*a*) if the carrier is at rest, (*b*) if the carrier is moving at a speed of 15 knots in the same direction as the airplane. (1 knot = 0.514 m/s.)

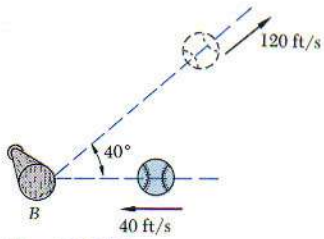


Fig. P13.121

13.121 A 4-oz baseball is pitched with a velocity of 40 ft/s toward a batter. After the ball is hit by the bat *B*, it has a velocity of 120 ft/s in the direction shown. If the bat and ball are in contact 0.025 s, determine the average impulsive force exerted on the ball during the impact.

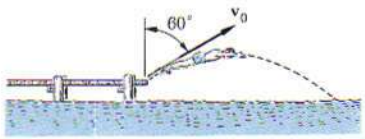


Fig. P13.122

13.122 A 160-lb man dives off the end of a pier with an initial velocity of 9 ft/s in the direction shown. Determine the horizontal and vertical components of the average force exerted on the pier during the 0.8 s that the man takes to leave the pier.

Fig. P13.123

13.123 A steel-jacketed bullet of mass 20 g is fired with a velocity of 600 m/s toward a steel plate; the bullet ricochets along the path *CD* with a velocity of 500 m/s. Knowing that the bullet caused a 50-mm scratch on the surface of the plate, determine the average impulsive force exerted on the bullet during its contact with the plate. (*Hint.* Assume an average speed of 550 m/s during contact.)

13.124 Determine the initial recoil velocity of an 8-lb rifle which fires a $\frac{3}{4}$-oz bullet with a velocity of 1600 ft/s.

13.125 A 2-oz rifle bullet is fired horizontally with a velocity of 1200 ft/s into an 8-lb block of wood which can move freely in the horizontal direction. Determine (*a*) the final velocity of the block, (*b*) the ratio of the final kinetic energy of the block and bullet to the initial kinetic energy of the bullet.

Fig. P13.126

13.126 Collars *A* and *B* are moved toward each other, thus compressing the spring, and are then released from rest. The spring is not attached to the collars. Neglecting the effect of friction and knowing that collar *B* is observed to move to the right with a velocity of 6 m/s, determine (*a*) the corresponding velocity of collar *A*, (*b*) the kinetic energy of each collar.

13.127 A barge is initially at rest and carries a 600-kg crate. The barge has a mass of 3000 kg and is equipped with a winch which is used to move the crate along the deck. Neglecting any friction between the crate and the barge, determine (*a*) the velocity of both the barge and the crate when the winch is drawing in rope at the rate of 1.5 m/s, (*b*) the final position of the barge after 12 m of rope has been drawn in by the winch. (*c*) Solve parts *a* and *b* assuming that $\mu = 0.30$ between the crate and the barge.

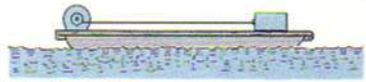


Fig. P13.127

13.128 A 60-ton railroad car is to be coupled to a second car which weighs 40 tons. If initially the speed of the 60-ton car is 1 mi/h and the 40-ton car is at rest, determine (*a*) the final speed of the coupled cars, (*b*) the average impulsive force acting on each car if the coupling is completed in 0.5 s.

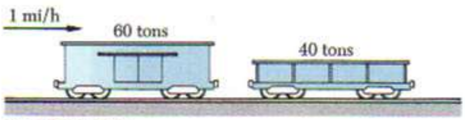


Fig. P13.128

13.129 Solve Prob. 13.128, assuming that, initially, the 60-ton car is at rest and the 40-ton car has a speed of 1 mi/h.

13.130 A 10-kg package is discharged from a conveyor belt with a velocity of 3 m/s and lands in a 25-kg cart. Knowing that the cart is initially at rest and may roll freely, determine the final velocity of the cart.

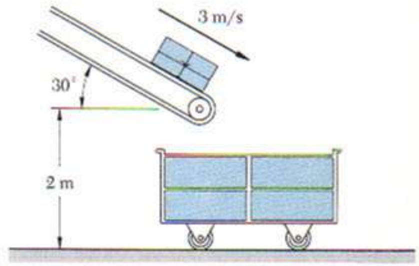


Fig. P13.130

13.131 Solve Prob. 13.130, assuming that the single 10-kg package is replaced by two 5-kg packages. The first 5-kg package comes to relative rest in the cart before the second package strikes the cart.

13.132 In order to test the resistance of a chain to impact, the chain is suspended from a 100-kg block supported by two columns. A rod attached to the last link of the chain is then hit by a 25-kg cylinder dropped from a 1.5-m height. Determine the initial impulse exerted on the chain, assuming that the impact is perfectly plastic and that the columns supporting the dead weight (*a*) are perfectly rigid, (*b*) are equivalent to two perfectly elastic springs. (*c*) Determine the energy absorbed by the chain in parts *a* and *b*.

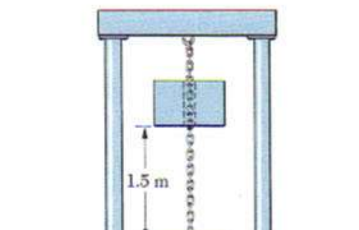
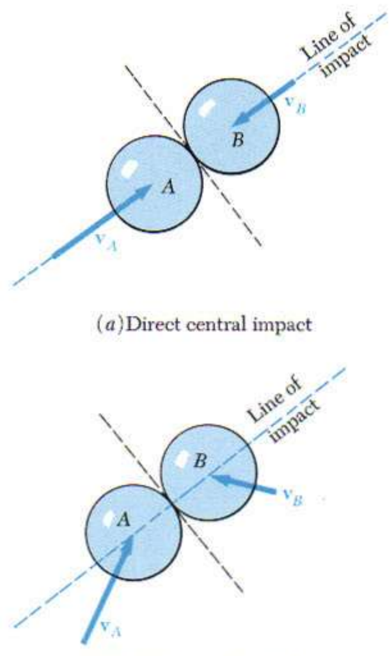


Fig. P13.132

13.133 A machine part is forged in a small drop forge. The hammer weighs 300 lb and is dropped from a height of 4 ft. Determine the initial impulse exerted on the machine part, assuming that the 800-lb anvil (*a*) is resting directly on hard ground, (*b*) is supported by springs.



(*a*) Direct central impact

(*b*) Oblique central impact

Fig. 13.20

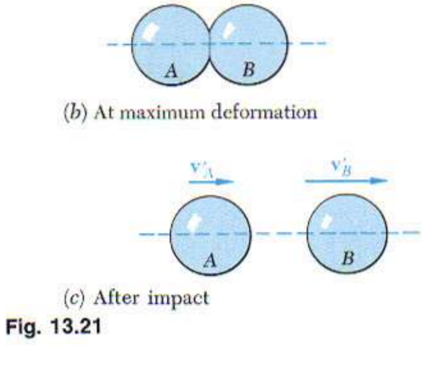
13.12. Impact.

A collision between two bodies which occurs in a very small interval of time, and during which the two bodies exert on each other relatively large forces, is called an *impact*. The common normal to the surfaces in contact during the impact is called the *line of impact*. If the mass centers of the two colliding bodies are located on this line, the impact is a *central impact*. Otherwise, the impact is said to be *eccentric*. We shall limit our present study to that of the central impact of two particles and postpone until later the analysis of the eccentric impact of two rigid bodies (Sec. 17.11).

If the velocities of the two particles are directed along the line of impact, the impact is said to be a *direct impact* (Fig. 13.20*a*). If, on the other hand, either or both particles move along a line other than the line of impact, the impact is said to be an *oblique impact* (Fig. 13.20*b*).

13.13. Direct Central Impact.

Consider two particles A and B, of mass m_A and m_B, which are moving in the same straight line and to the right with known velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ (Fig. 13.21*a*). If $\mathbf{v}_A$ is larger than $\mathbf{v}_B$, particle A will eventually strike particle B. Under the impact, the two particles will *deform* and, at the end of the period of deformation, they will have the same velocity $\mathbf{u}$ (Fig. 13.21*b*). A period of *restitution* will then take place, at the end of which, depending upon the magnitude of the impact forces and upon the materials involved, the two particles either will have regained their original shape or will stay permanently deformed. Our purpose here is to determine the velocities $\mathbf{v}'_A$ and $\mathbf{v}'_B$ of the particles at the end of the period of restitution (Fig. 13.21*c*).



(*a*) Before impact

(*b*) At maximum deformation

(*c*) After impact

Fig. 13.21

Considering first the two particles together, we note that there is no impulsive, external force. Thus, the total momentum of the two particles is conserved, and we write

$$m_A\mathbf{v}_A + m_B\mathbf{v}_B = m_A\mathbf{v}'_A + m_B\mathbf{v}'_B$$

Since all the velocities considered are directed along the same axis, we may replace the equation obtained by the following relation involving only scalar components:

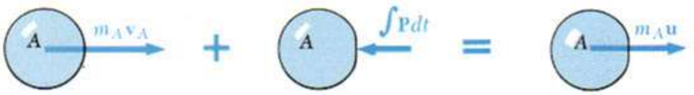
$$m_A v_A + m_B v_B = m_A v'_A + m_B v'_B \qquad (13.37)$$

A positive value for any of the scalar quantities v_A, v_B, v'_A, or v'_B means that the corresponding vector is directed to the right; a negative value indicates that the corresponding vector is directed to the left.

To obtain the velocities $\mathbf{v}'_A$ and $\mathbf{v}'_B$, it is necessary to establish a second relation between the scalars v'_A and v'_B. For this purpose, we shall consider now the motion of particle A during the period of deformation and apply the principle of impulse and momentum. Since the only impulsive force acting on A during this period is the force $\mathbf{P}$ exerted by B (Fig. 13.22*a*), we write, using again scalar components,

$$m_A v_A - \int P\,dt = m_A u \tag{13.38}$$

where the integral extends over the period of deformation.



(*a*) Period of deformation

(*b*) Period of restitution

Fig. 13.22

Considering now the motion of A during the period of restitution, and denoting by $\mathbf{R}$ the force exerted by B on A during this period (Fig. 13.22*b*), we write

$$m_A u - \int R\,dt = m_A v'_A \tag{13.39}$$

where the integral extends over the period of restitution.

In general, the force $\mathbf{R}$ exerted on A during the period of restitution differs from the force $\mathbf{P}$ exerted during the period of deformation, and the magnitude $\int R\,dt$ of its impulse is smaller than the magnitude $\int P\,dt$ of the impulse of $\mathbf{P}$. The ratio of the magnitudes of the impulses corresponding respectively to the period of restitution and to the period of deformation is called the *coefficient of restitution* and is denoted by e. We write

$$e = \frac{\int R\,dt}{\int P\,dt} \tag{13.40}$$

The value of the coefficient e is always between 0 and 1 and depends to a large extent on the two materials involved. However, it also varies considerably with the impact velocity and the shape and size of the two colliding bodies.

Solving Eqs. (13.38) and (13.39) for the two impulses and substituting into (13.40), we write

$$e = \frac{u - v'_A}{v_A - u} \tag{13.41}$$

A similar analysis of particle B leads to the relation

$$e = \frac{v'_B - u}{u - v_B} \tag{13.42}$$

Since the quotients in (13.41) and (13.42) are equal, they are also equal to the quotient obtained by adding, respectively, their numerators and their denominators. We have, therefore,

$$e = \frac{(u - v'_A) + (v'_B - u)}{(v_A - u) + (u - v_B)} = \frac{v'_B - v'_A}{v_A - v_B}$$

and

$$v'_B - v'_A = e(v_A - v_B) \tag{13.43}$$

Since $v'_B - v'_A$ represents the relative velocity of the two particles after impact and $v_A - v_B$ their relative velocity before impact, formula (13.43) expresses that *the relative velocity of the two particles after impact may be obtained by multiplying their relative velocity before impact by the coefficient of restitution.* This property is used to determine experimentally the value of the coefficient of restitution of two given materials.

The velocities of the two particles after impact may now be obtained by solving Eqs. (13.37) and (13.43) simultaneously for v'_A and v'_B. It is recalled that the derivation of Eqs. (13.37) and (13.43) was based on the assumption that particle B is located to the right of A, and that both particles are initially moving to the right. If particle B is initially moving to the left, the scalar v_B should be considered negative. The same sign convention holds for the velocities after impact: a positive sign for v'_A will indicate that particle A moves to the right after impact, and a negative sign that it moves to the left.

Two particular cases of impact are of special interest:

1. $e = 0$, *Perfectly Plastic Impact.* When $e = 0$, Eq. (13.43) yields $v'_B = v'_A$. There is no period of restitution, and both particles stay together after impact. Substituting $v'_B = v'_A = v'$ into Eq. (13.37), which expresses that the total momentum of the particles is conserved, we write

$$m_A v_A + m_B v_B = (m_A + m_B)v' \tag{13.44}$$

This equation may be solved for the common velocity v' of the two particles after impact.

2. $e = 1$, *Perfectly Elastic Impact.* When $e = 1$, Eq. (13.43) reduces to

$$v'_B - v'_A = v_A - v_B \tag{13.45}$$

which expresses that the relative velocities before and after impact are equal. The impulses received by each particle during the period of deformation and during the period of restitution are equal. The particles move away from each other after impact with the same velocity with which they approached each other before impact. The velocities v'_A and v'_B may be obtained by solving Eqs. (13.37) and (13.45) simultaneously.

It is worth noting that, *in the case of a perfectly elastic impact, the total energy of the two particles*, as well as their total momentum, *is conserved.* Equations (13.37) and (13.45) may be written as follows:

$$m_A(v_A - v'_A) = m_B(v'_B - v_B) \tag{13.37'}$$
$$v_A + v'_A = v_B + v'_B \tag{13.45'}$$

Multiplying (13.37′) and (13.45′) member by member, we have

$$m_A(v_A - v'_A)(v_A + v'_A) = m_B(v'_B - v_B)(v'_B + v_B)$$
$$m_A v_A^2 - m_A(v'_A)^2 = m_B(v'_B)^2 - m_B v_B^2$$

Rearranging the terms in the equation obtained, and multiplying by $\frac{1}{2}$, we write

$$\tfrac{1}{2}m_A v_A^2 + \tfrac{1}{2}m_B v_B^2 = \tfrac{1}{2}m_A(v'_A)^2 + \tfrac{1}{2}m_B(v'_B)^2$$

which expresses that the kinetic energy of the particles is conserved. It should be noted, however, that *in the general case of impact*, i.e., when e is not equal to 1, *the total energy of the particles is not conserved.* This may be shown in any given case by comparing the kinetic energies before and after impact. The lost kinetic energy is in part transformed into heat and in part spent in generating elastic waves within the two colliding bodies.

13.14. Oblique Central Impact.

Let us now consider the case when the velocities of the two colliding particles are *not* directed along the line of impact (Fig. 13.23). As indicated in Sec. 13.12, the impact is said to be *oblique*. Since the velocities $\mathbf{v}'_A$ and $\mathbf{v}'_B$ of the particles after impact are unknown in direction as well as in magnitude, their determination will require the use of four independent equations.

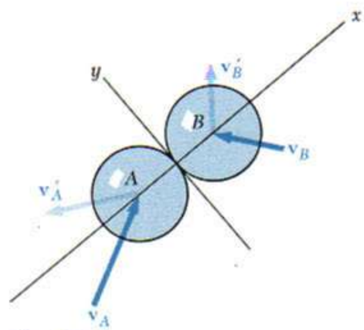


Fig. 13.23

We choose x and y axes, respectively, along the line of impact and along the common tangent to the surfaces in contact. Assuming that the particles are perfectly *smooth and frictionless,*

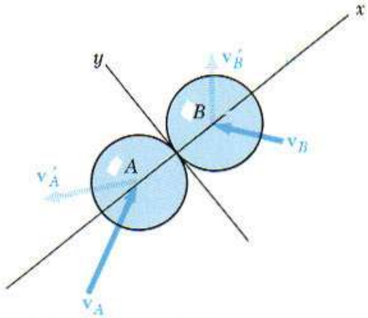


Fig. 13.23 (*repeated*)

we observe that the only impulsive forces acting on the particles during the impact are internal forces directed along the x axis. We may therefore express that:

1. The y component of the momentum of particle A is conserved.
2. The y component of the momentum of particle B is conserved.
3. The x component of the total momentum of the particles is conserved.
4. The x component of the relative velocity of the two particles after impact is obtained by multiplying the x component of their relative velocity before impact by the coefficient of restitution.

We thus obtain four independent equations which may be solved for the components of the velocities of A and B after impact. This method of solution is illustrated in Sample Prob. 13.15.

13.15. Problems Involving Energy and Momentum. We have now at our disposal three different methods for the solution of kinetics problems: the direct application of Newton's second law, $\Sigma\mathbf{F} = m\mathbf{a}$, the method of work and energy, and the method of impulse and momentum. To derive maximum benefit from these three methods, we should be able to choose the method best suited for the solution of a given problem. We should also be prepared to use different methods for solving the various parts of a problem when such a procedure seems advisable.

We have already seen that the method of work and energy is in many cases more expeditious than the direct application of Newton's second law. As indicated in Sec. 13.4, however, the method of work and energy has limitations, and it must sometimes be supplemented by the use of $\Sigma\mathbf{F} = m\mathbf{a}$. This is the case, for example, when we wish to determine an acceleration or a normal force.

There is generally no great advantage in using the method of impulse and momentum for the solution of problems involving no impulsive forces. It will usually be found that the equation $\Sigma\mathbf{F} = m\mathbf{a}$ yields a solution just as fast and that the method of work and energy, if it applies, is more rapid and more convenient. However, the method of impulse and momentum is the only practicable method in problems of impact. A solution based on the direct application of $\Sigma\mathbf{F} = m\mathbf{a}$ would be unwieldy, and the method of work and energy cannot be used since impact (unless perfectly elastic) involves a loss of mechanical energy.

Many problems involve only conservative forces, except for a short impact phase during which impulsive forces act. The solution of such problems may be divided into several parts. While the part corresponding to the impact phase calls for the use of the method of impulse and momentum and of the relation between relative velocities, the other parts may usually be solved by the method of work and energy. The use of the equation $\Sigma \mathbf{F} = m\mathbf{a}$ will be necessary, however, if the problem involves the determination of a normal force.

Consider, for example, a pendulum A, of mass m_A and length l, which is released with no velocity from a position A_1 (Fig. 13.24*a*). The pendulum swings freely in a vertical plane and hits a second pendulum B, of mass m_B and same length l, which is initially at rest. After the impact (with coefficient of restitution e), pendulum B swings through an angle θ that we wish to determine.

The solution of the problem may be divided into three parts:

1. *Pendulum A Swings from A_1 to A_2.* The principle of conservation of energy may be used to determine the velocity $(\mathbf{v}_A)_2$ of the pendulum at A_2 (Fig. 13.24*b*).
2. *Pendulum A Hits Pendulum B.* Using the fact that the total momentum of the two pendulums is conserved and the relation between their relative velocities, we determine the velocities $(\mathbf{v}_A)_3$ and $(\mathbf{v}_B)_3$ of the two pendulums after impact (Fig. 13.24*c*).
3. *Pendulum B Swings from B_3 to B_4.* Applying the principle of conservation of energy, we determine the maximum elevation y_4 reached by pendulum B (Fig. 13.24*d*). The angle θ may then be determined by trigonometry.

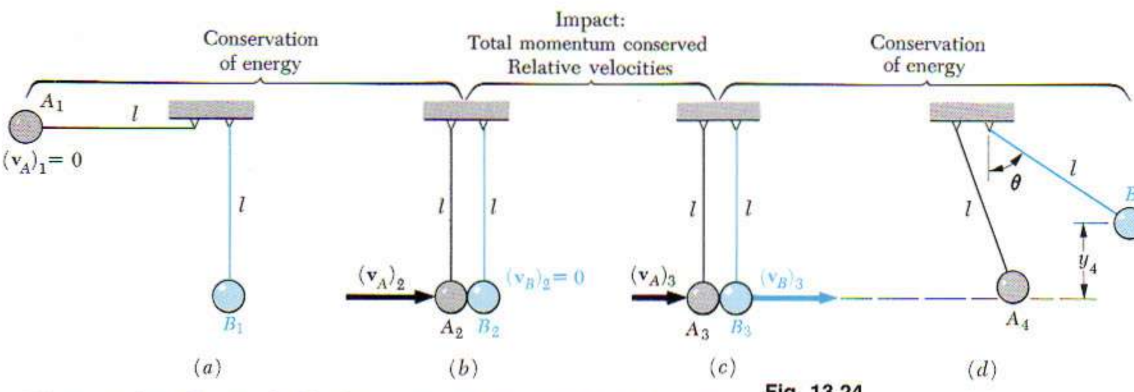


Fig. 13.24

We note that the method of solution just described should be supplemented by the use of $\Sigma \mathbf{F} = m\mathbf{a}$ if the tensions in the cords holding the pendulums are to be determined.

SAMPLE PROBLEM 13.13

A 20-Mg railroad car moving at a speed of 0.5 m/s to the right collides with a 35-Mg car which is at rest. If after the collision the 35-Mg car is observed to move to the right at a speed of 0.3 m/s, determine the coefficient of restitution between the two cars.

Solution. We express that the total momentum of the two cars is conserved.

$$m_A\mathbf{v}_A + m_B\mathbf{v}_B = m_A\mathbf{v}'_A + m_B\mathbf{v}'_B$$

$$(20\text{ Mg})(+0.5\text{ m/s}) + (35\text{ Mg})(0) = (20\text{ Mg})v'_A + (35\text{ Mg})(+0.3\text{ m/s})$$

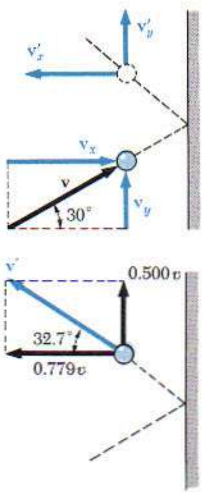
$$v'_A = -0.025\text{ m/s} \qquad \mathbf{v}'_A = 0.025\text{ m/s} \leftarrow$$

The coefficient of restitution is obtained by writing

$$e = \frac{v'_B - v'_A}{v_A - v_B} = \frac{+0.3 - (-0.025)}{+0.5 - 0} = \frac{0.325}{0.5} \qquad e = 0.65 \blacktriangleleft$$

SAMPLE PROBLEM 13.14

A ball is thrown against a frictionless vertical wall. Immediately before the ball strikes the wall, its velocity has a magnitude v and forms an angle of 30° with the horizontal. Knowing that $e = 0.90$, determine the magnitude and direction of the velocity of the ball as it rebounds from the wall.



Solution. We resolve the initial velocity of the ball into components

$$v_x = v\cos 30° = 0.866v \qquad v_y = v\sin 30° = 0.500v$$

Vertical Motion. Since the wall is frictionless, no vertical impulsive force acts on the ball during the time it is in contact with the wall. The vertical component of the momentum, and hence the vertical component of the velocity, of the ball is thus unchanged:

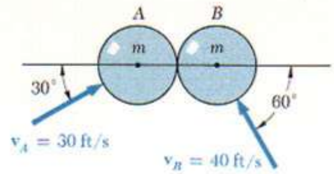
$$\mathbf{v}'_y = \mathbf{v}_y = 0.500v \uparrow$$

Horizontal Motion. Since the mass of the wall (and earth) is essentially infinite, there is no point in expressing that the total momentum of the ball and the wall is conserved. Using the relation between relative velocities, we write

$$0 - v'_x = e(v_x - 0)$$
$$v'_x = -0.90(0.866v) = -0.779v \qquad \mathbf{v}'_x = 0.779v \leftarrow$$

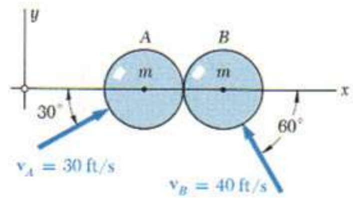
Resultant Motion. Adding vectorially the components $\mathbf{v}'_x$ and $\mathbf{v}'_y$,

$$\mathbf{v}' = 0.926v \nwarrow 32.7° \blacktriangleleft$$



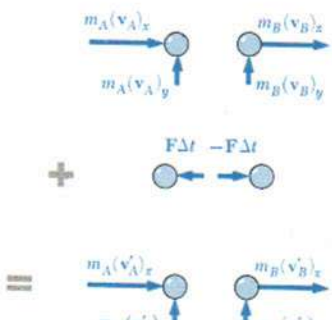
SAMPLE PROBLEM 13.15

The magnitude and direction of the velocities of two identical frictionless balls before they strike each other are as shown. Assuming $e = 0.90$, determine the magnitude and direction of the velocity of each ball after the impact.



Solution. The impulsive forces acting between the balls during the impact are directed along a line joining the centers of the balls called the *line of impact.* Choosing x and y axes, respectively, parallel and perpendicular to the line of impact and directed as shown, we write

$$\begin{aligned}(v_A)_x &= v_A \cos 30^\circ = +26.0 \text{ ft/s}\\ (v_A)_y &= v_A \sin 30^\circ = +15.0 \text{ ft/s}\\ (v_B)_x &= -v_B \cos 60^\circ = -20.0 \text{ ft/s}\\ (v_B)_y &= v_B \sin 60^\circ = +34.6 \text{ ft/s}\end{aligned}$$



Principle of Impulse and Momentum. In the adjoining sketches we show in turn the initial momenta, the impulsive reactions, and the final momenta.

Motion Perpendicular to the Line of Impact. Considering only the y components, we apply the principle of impulse and momentum to each ball *separately.* Since no vertical impulsive force acts during the impact, the vertical component of the momentum, and hence the vertical component of the velocity, of each ball is unchanged.

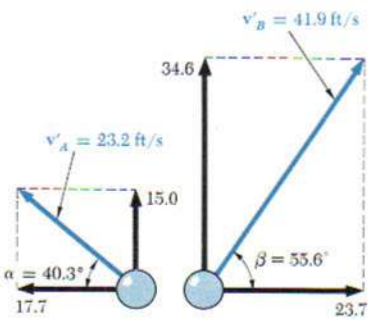
$$(\mathbf{v}'_A)_y = 15.0 \text{ ft/s} \uparrow \qquad (\mathbf{v}'_B)_y = 34.6 \text{ ft/s} \uparrow$$

Motion Parallel to the Line of Impact. In the x direction, we consider the two balls together and note that, by Newton's third law, the internal impulses are, respectively, $\mathbf{F}\,\Delta t$ and $-\mathbf{F}\,\Delta t$ and cancel. We thus write that the total momentum of the balls is conserved:

$$m_A(v_A)_x + m_B(v_B)_x = m_A(v'_A)_x + m_B(v'_B)_x$$
$$m(26.0) + m(-20.0) = m(v'_A)_x + m(v'_B)_x$$
$$(v'_A)_x + (v'_B)_x = 6.0 \quad (1)$$

Using the relation between relative velocities, we write

$$(v'_B)_x - (v'_A)_x = e[(v_A)_x - (v_B)_x]$$
$$(v'_B)_x - (v'_A)_x = (0.90)[26.0 - (-20.0)]$$
$$(v'_B)_x - (v'_A)_x = 41.4 \quad (2)$$



Solving Eqs. (1) and (2) simultaneously, we obtain

$$(v'_A)_x = -17.7 \qquad (v'_B)_x = +23.7$$
$$(\mathbf{v}'_A)_x = 17.7 \text{ ft/s} \leftarrow \qquad (\mathbf{v}'_B)_x = 23.7 \text{ ft/s} \rightarrow$$

Resultant Motion. Adding vectorially the velocity components of each ball, we obtain

$$\mathbf{v}'_A = 23.2 \text{ ft/s} \nwarrow 40.3^\circ \qquad \mathbf{v}'_B = 41.9 \text{ ft/s} \angle 55.6^\circ \blacktriangleleft$$