the moment of short-circuiting. The equivalent circuit in the q-axis remains the same, since no additional winding in this axis is present (in distinction to the rotor equipped with the damper winding – see further on). Electromagnetic force E'_d, being induced by a total flux linkage, also stays constant at the moment of fault, which allows us to calculate the first moment short-circuit current of the generator.

Knowing the above two transient parameters E'_d and X'_d of a synchronous generator we may readily obtain the *first moment a.c.* (*periodic*) and *d.c.* (*aperiodic*) *components* of the *short-circuit current*. Thus, after short-circuiting in Fig. 6.37(d), we have

$$I'_{d0} = \frac{E'_{d0}}{X'_d}, \tag{6.55a}$$

or as an instantaneous value:

$$i_d = \frac{E'_{d0}}{X'_d} \cos \omega t, \tag{6.55b}$$

where the initial, or switching, angle ψ_i is taken as zero.

The d-axis component of the terminal voltage prior to switching will then be

$$V_{d0} = E'_{d0} - X'_d I'_{d0}. \tag{6.56}$$

The steady-state s.c. current can be found from the circuit for the steady-state analysis, Fig. 6.36. After short-circuiting $V_d = 0$ we have

$$I_{d,\infty} = \frac{E_d}{X_d}. \tag{6.57}$$

Next we shall find the quadrature-axis component of the voltage. Consider the *phasor diagram* shown in Fig. 6.38, which is drawn for a round-pole generator with $X_d = X_q$. From this diagram we obtain

$$\tilde{E}_d = \tilde{V}_d + jX_d\tilde{I}_d = \tilde{V} - \tilde{V}_q + jX_d\tilde{I}_d. \tag{6.58a}$$

Since $\tilde{V}_q = -jX_q\tilde{I}_q = -jX_d\tilde{I}_q$ we have

$$\tilde{E}_d = \tilde{V} + jX_d\tilde{I}. \tag{6.58b}$$

(Here and further on the tilde sign $\sim$ stands for phasor quantities.)

Considering triangle ABC, we may determine angle δ, between E_d and V, which is known as the load angle:

$$\delta = \tan^{-1} \frac{X_d I \cos\varphi}{V + X_d I \sin\varphi}. \tag{6.59}$$

With this angle the V_{q0} component is

$$V_{q0} = V_0 \cos \delta_0. \tag{6.60a}$$

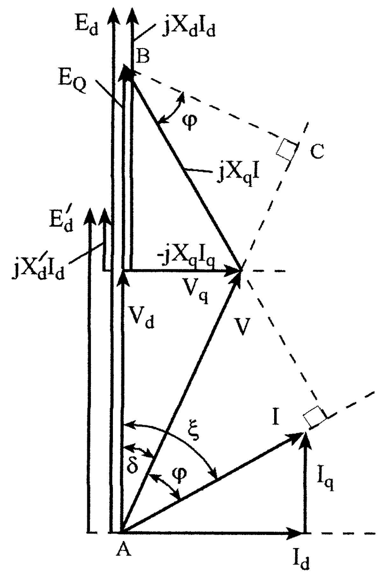


Figure 6.38 Phasor diagram for a round-pole generator.

and the q-axis component of the current is

$$I_{q0} = \frac{V_{q0}}{X_q}. \tag{6.60b}$$

The two components of the s.c. current (equations 6.55a and 6.60b) are actually the initial values of the two components of the s.c. current. To find the transient s.c. current we must solve the differential equations for each of the two components. To simplify the solution we will use the superposition principle and consider each of the currents as a sum of the current prior to switching, and a transient current due to switching. This current is found as a result of applying, to the passive circuit, the voltages equal and opposed to those which existed at the fault point prior to switching. In doing so we must remember that the two equivalent circuits in the d- and q-axes in transient behavior are not unconnected any more. Thus, any change of current in one will induce voltage in the second just like in transformer windings, Fig. 6.37(e). With the Laplace transform technique we may write

$$I'_d X'_d - sI'_q X_q = V_{q0},$$
$$-I'_q X_q - sI'_d X'_d = V_{d0}.$$

(Note that here, as in some technical books, the voltage induced by current I_q

is also assigned as V_q, although it is directed on the d-axis, and subsequently the voltage induced by current I_d is assigned as V_d.) Solving the above equations yields

$$I'_d = \frac{1}{X'_d}\left(- V_{d0}\frac{s}{s^2+1} - V_{q0}\frac{1}{s^2+1}\right)$$

$$I'_q = \frac{1}{X_q}\left(V_{q0}\frac{s}{s^2+1} - V_{d0}\frac{1}{s^2+1}\right).$$

Taking the inverse transform (using the table of the Laplace transform pairs) we may obtain

$$\begin{aligned} i'_d &= -\frac{V_{d0}}{X'_d}\cos\omega t - \frac{V_{q0}}{X'_d}\sin\omega t \\ i'_q &= \frac{V_{q0}}{X'_q}\cos\omega t - \frac{V_{d0}}{X_q}\sin\omega t. \end{aligned} \tag{6.61}$$

Next we can find the s.c. current in each of the three phases of a stator winding. Thus, for instance, for phase a, by substituting equation 6.61 in the first equation of 6.38a (note that in order to obtain the *instantaneous value of the short-circuit currents*, the argument ωt must be added to angle α under the cos and sin functions), and after algebraic simplifications we may obtain

$$\begin{aligned} i_a = \frac{E'_{dm,0}}{X'_d}\cos(\omega t+\alpha) + I_{d,2\omega}\cos(2\omega t+\alpha) + I_{da}\cos\alpha \\ + I_{q,2\omega}\sin(2\omega t+\alpha) + I_{qa}\sin\alpha, \end{aligned} \tag{6.62}$$

where

$$I_{d,2\omega} = \frac{V_{dm,0}(X'_d - X_q)}{2X'_dX_q} \quad \text{and} \quad I_{q,2\omega} = \frac{V_{qm,0}(X'_d - X_q)}{2X'_dX_q} \tag{6.63}$$

are the *double frequency component* amplitudes, and

$$I_{da} = -\frac{V_{dm,0}(X'_d + X_q)}{2X'_dX_q} \quad \text{and} \quad I_{qa} = -\frac{V_{qm,0}(X'_d - X_q)}{2X'_dX_q} \tag{6.64}$$

are the d.c., or *aperiodic components*, and α is the angle between the a-phase axis and d-axis. (Here the subscript "m" in EMF and voltages stands for the amplitude values.)

The appearance of the double frequency term in the stator transient current may be explained by considering the transient current in the rotor field winding. The sudden change of the stator currents in turn results in the appearance of a transient current in the rotor winding to keep the total magnetic flux constant. This current is of two components. As the three-phase stator current of basic frequency abruptly increases, the armature reaction on the d-axis also increases respectively. The transient current will appear in the field winding to compensate for the rise of this reaction. Since it is proportional to the change of the stator current, we may write its p.u. expression as

$$I_{fl,n} = \frac{X_{ad}}{X_{ad} + X_{rl}} (I'_d - I_{d,0}) = \frac{X_d - X'_d}{X_{ad}} \frac{V_{d,0}}{X'_d}. \tag{6.65}$$

The magnetic field produced by the d.c. component of the stator currents remains fixed in the air gap space and therefore is rotated with respect to the rotor with synchronous speed. This results in inducing an a.c. component in the field transient current. Since the field current, at the first moment, does not change, the initial value of an a.c. component is oppositely equal to the d.c. component. Subsequently, the resulting *current in the field winding* is

$$i_{fl} = I_{fl,f} + i_{fl,n} = I_{fl,0} - \frac{X_d - X'_d}{X_{ad}} \frac{V_{dm,0}}{X'_d} \cos \omega t, \tag{6.66}$$

where $I_{fl,0} = E_{d0}/X_{ad}$.

The a.c. term in the rotor produces a pulsating magnetic flux, which can be resolved into two components, having equal (half of the original) amplitudes, and revolving with synchronous speed in opposite directions. The component, which revolves in the direction opposite to that of the rotor, is actually fixed in the space and interferes, therefore, with the magnetic field of the d.c. component of the stator current, decreasing it slightly.

The component, which revolves in the same direction as the rotor, produces a magnetic field rotating in space with double the speed of the rotor and inducing in the stator windings the *double frequency component*. This component, however, is relatively small and when using engineering calculations is usually neglected.

The resistances of the generator windings, which have been neglected when determining the magnitudes of the transient currents, are responsible for decaying all these currents so that only the steady-state a.c. term in the stator and the d.c. term in the rotor remain invariable. The decay process is of an exponential form and the damping factors are mainly determined by the ratio of the resistance to the leakage inductance of the circuits. Since large synchronous machines have very small resistances compared to their considerable leakage reactances, their transient currents decrease very slowly and may predominantly determine the transients in a few seconds.

We may recognize two kinds of currents and the fields related to them, one of which adheres to the stator windings and the other to the rotor field winding. Each of them has a different damping factor or time constant, which primarily depends on the value of their resistances, related to the reactances. Thus, the stator resistances (including the external network) may be roughly estimated to be 10% of its leakage reactance, and the rotor circuit resistance may be about 1% of its leakage reactance.

The time constant of the rotor circuit is usually known and is given as a generator catalogue parameter. This time constant is related to the mutual flux linkage between the rotor and stator windings and is determined by an open-circuit test. This time constant determines the rate of increasing the field current (and therefore the generator open-circuit terminal voltage), when the constant voltage V_{fl} is suddenly applied to the field winding. Since the field winding is a simple LR circuit, the differential equation for the rotor circuit may be written as

$$\frac{d\lambda_{fl}}{dt} + I_{fl}R_{fl} = V_{fl},$$

or

$$\frac{1}{R_{fl}}\frac{d\lambda_{fl}}{dt} + I_{fl} = I_{fl,\infty}, \tag{6.67}$$

where $I_{fl,\infty} = V_{fl}/R_{fl}$ is a long-term (steady-state) field current. Since the open-circuit terminal voltage is approximately proportional to the field current: $E_d = X_{ad}I_{fl}$, where X_{ad} is the mutual reactance/inductance, we have

$$E_{d,\infty} = E_d + \frac{X_{ad}}{R_{fl}}\frac{d\lambda_{fl}}{dt}. \tag{6.68}$$

Using the relation between the transient EMF E'_d and λ_{fl}: $E'_d = (X_{ad}/[X_{fl} + X_{ad})]\lambda_{fl}$, equation 6.68 becomes

$$E_{d,\infty} = E_d + T_{d0}\frac{dE'_d}{dt}, \tag{6.69}$$

where

$$T_{d0} = \frac{X_{fl} + X_{ad}}{R_{fl}} \tag{6.70}$$

is the *open-circuit time constant* in p.u. (as the basic time is $t_b = 1/\omega$ s), or in seconds

$$T_{d0} = \frac{X_{fl} + X_{ad}}{\omega R_{fl}}. \tag{6.71}$$

Since $E_d = X_d I_d$ and $E'_d = X'_d I_d$, then $E'_d = (X'_d/X_d)E_d$, and substituting E'_d in equation 6.69 with $(X'_d/X_d)E_d$, we have

$$T'_d\frac{dE_d}{dt} + E_d = E_{d,\infty}, \tag{6.72}$$

where the *transient time constant* is

$$T'_d = \frac{X'_d}{X_d}T_{d0}. \tag{6.73}$$

For the short-circuit fault, remote from the generator, the transient time constant is given by the equation

$$T'_d = \frac{X'_d + X_F}{X_d + X_F} T_{d0}, \tag{6.74}$$

where X_F is the external reactance. If the system (external) impedance contains a relatively high resistance R_{ex}, the transient time constant is given by the extended relationship

$$T'_d = \frac{R_F^2 + (X'_d + X_F)(X_d + X_F)}{R_F^2 + (X_d + X_F)^2} T_{d0}. \tag{6.75}$$

The transient time constant T'_d lies in the range of 0.4 to 2 s for high power, high voltage turbogenerators and 0.7 to 2.55 s for salient-pole hydrogenerators. In low power generators T'_d may be less than 0.2 s.

The d.c. or aperiodical terms of both the stator and rotor windings are actually exponential functions and each of them decays with an appropriate time constant, which is determined by the parameters of that winding to which they are linked. Thus, the d.c. term of the rotor current and, adherent to it, the a.c. transient term of the stator current, decay at the rate of the above transient time constant T'_d, which is determined primarily by the time constant of the rotor winding, T_{d0}.

The d.c. (aperiodic) and double-frequency components in the stator currents die out with the *armature time constant* T_a. Although the initial value of the d.c. components in different phases is determined by the switching moment, or by the initial phase angle of the prior to switching phase currents, the total MMF, produced by these currents, is stationary in space and of a magnitude which is independent of the initial phase angle. This stationary MMF reacts with the rotating rotor alternately on the d- and q-axes. Therefore, the inductance associated with the d.c. component may be regarded as a sort of average of X'_d and X_q, More precisely, by observing equation 6.64 we may conclude that the reactance, which determines the d.c. (aperiodic) term, is $X_2 = X'_d X_q/(X'_d + X_q)$ (also known as the negative-sequence of a synchronous machine – see further on). Thus, by using this reactance, the stator winding transient time constant may be determined as:

in p.u.

$$T_a = \frac{2X'_d X_q}{R_a(X'_d + X_q)} = \frac{X_2}{R_a}\ \text{pu} \tag{6.76}$$

or in seconds

$$T_a = \frac{2X'_d X_q}{R_a \omega (X'_d + X_q)} = \frac{X_2}{R_a \omega}\ \text{s} \tag{6.77}$$

For high-voltage generators, T_a is in the range of 0.07 to 0.5 s and for low-voltage generators its value lies in the range of 0.01 to 0.1 s. If the short-circuit occurs at a distance from the generator, then the time constant is given as

$$T_a = \frac{X_2 + X_{ex}}{R_a}. \tag{6.78}$$

The rotor rotation relative to the fixed MMF of the d.c. component in the stator causes the a.c. component of the fundamental frequency in the field current to appear. This a.c. component in the field current, as has been previously mentioned, is responsible for the double-frequency component of the armature short-circuit current. The time constant T_a therefore also applies to the a.c. component in the rotor current and to the double-frequency component of the stator current.

The a.c. component of s.c. current, which appears at the first moment of switching (equation 6.62) differs from the steady-state s.c. current, which can be approximately determined by the saturated reactance X_d, or by using the linearization method (see section 6.5.2), as E_d/X_d (equation 6.57). This difference may then be expressed as

$$\Delta I'_d = \frac{E'_{d0}}{X'_d} - \frac{E_d}{X_d} = I'_{d0} - I_{d,\infty}. \tag{6.79}$$

The first component is a *transient current*, which decays at the rate of the transient time constant; therefore, the difference, $\Delta I'_d$, also decays at the same rate, so that the short-circuit a.c. current at fundamental frequency falls off from its initial value I'_{d0} to its final value of the steady-state short-circuit current with the time constant T'_d.

In conclusion, the *total transient response current* of a synchronous generator to the short-circuit fault at its terminal (phase *a*) and *total rotor field current*

$$\begin{aligned} i_a = {} & \left[\left(\frac{E'_{d0m}}{X'_d} - \frac{E_{dm}}{X_d}\right) e^{-t/T'_d} + \frac{E_{dm}}{X_d}\right] \cos(\omega t + \alpha) \\ & - \frac{V_{d0m}(X'_d + X_q)}{2X'_d X_q} e^{-t/T_a} \cos\alpha + \frac{V_{d0m}(X'_d - X_q)}{2X'_d X_q} e^{-t/T_a} \cos(2\omega t + \alpha) \\ & + \frac{V_{q0m}(X'_d + X_q)}{2X'_d X_q} e^{-t/T_a} \sin\alpha + \frac{V_{q0m}(X'_d - X_q)}{2X'_d X_q} e^{-t/T_a} \sin(2\omega t + \alpha), \end{aligned} \tag{6.80}$$

$$i_{fl} = I_{fl0} + \frac{V_{dm,0}}{X'_d} \frac{X_d - X'_d}{X_{ad}} e^{-t/T'_d} - \frac{V_{dm,0}}{X'_d} \frac{X_d - X'_d}{X_{ad}} e^{-t/T_a} \cos\omega t. \tag{6.81}$$

Both currents and their components are plotted in Fig. 6.39.

The initial value of the aperiodic term may be obtained by combining two components: of axes d and q

$$A = \frac{V_{d0m}\cos\alpha + V_{q0m}\sin\alpha}{2X'_d X_q/(X'_d + X_q)}. \tag{6.82}$$

(b) *Transient effects of the damper windings: subtransient EMF, subtransient reactance and time constant*

Nowadays synchronous machines are usually equipped with **damper windings**, which consist of short-circuited turns, or bars of copper strip set in poles. The

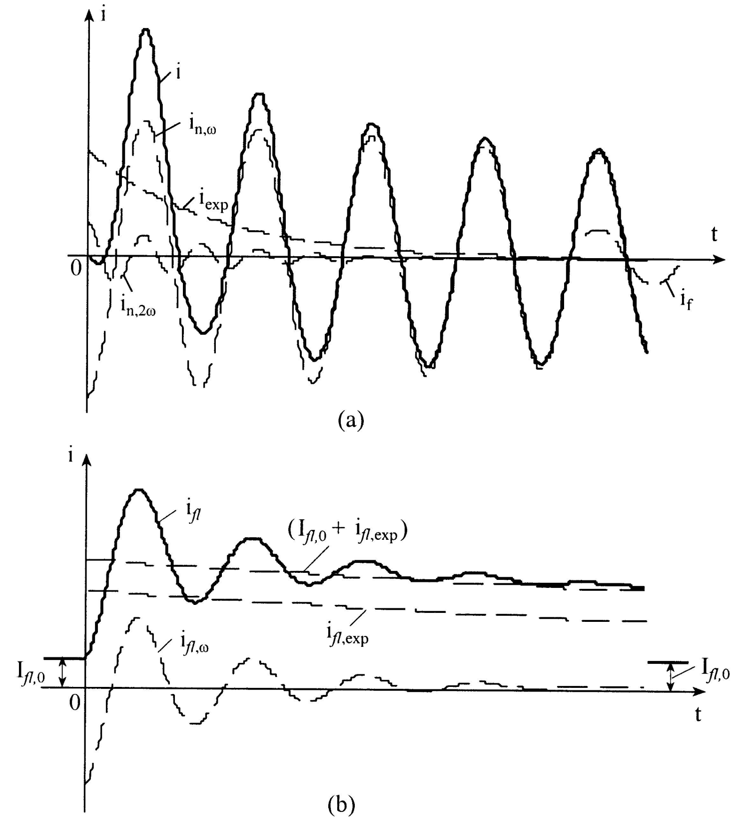


Figure 6.39 The short-circuit currents of a synchronous generator at sudden fault at its terminals: stator current (a) and rotor current (b).

reason for using the damper windings is to aid in starting and to reduce (to damp) mechanical oscillatory tendencies, which may arise under different faults, and thereby to increase the dynamic stability of the generator. The damper windings are placed in both axes, *d* and *q*, as can be seen from Fig. 6.40.

The damper winding does not change in principle the nature of the transients. Its influence results in increasing the short-circuit current magnitudes and in the appearance of an additional component on the *q*-axis, which is a subtransient EMF E''_q. Presenting an additional winding on a rotor makes the straightforward analysis of the generator transients even more complicated. However, analyzing the generator equivalent circuit in both axes, shown in Figure 6.41, will allow us to get the final results in a much easier way.

As can be seen, the equivalent circuit in the *d*-axis differs from those in Fig. 6.37 by an additional mutual winding (damper winding). By illuminating the mutual inductance (in a similar way as for a three-winding power transformer) we may obtain the circuit shown in (b), and by applying the Thévenin

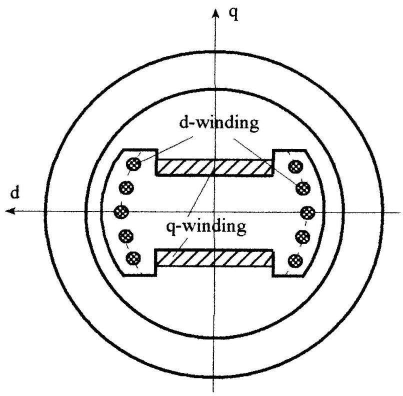


Figure 6.40 The damper windings in axes *d* and *q*.

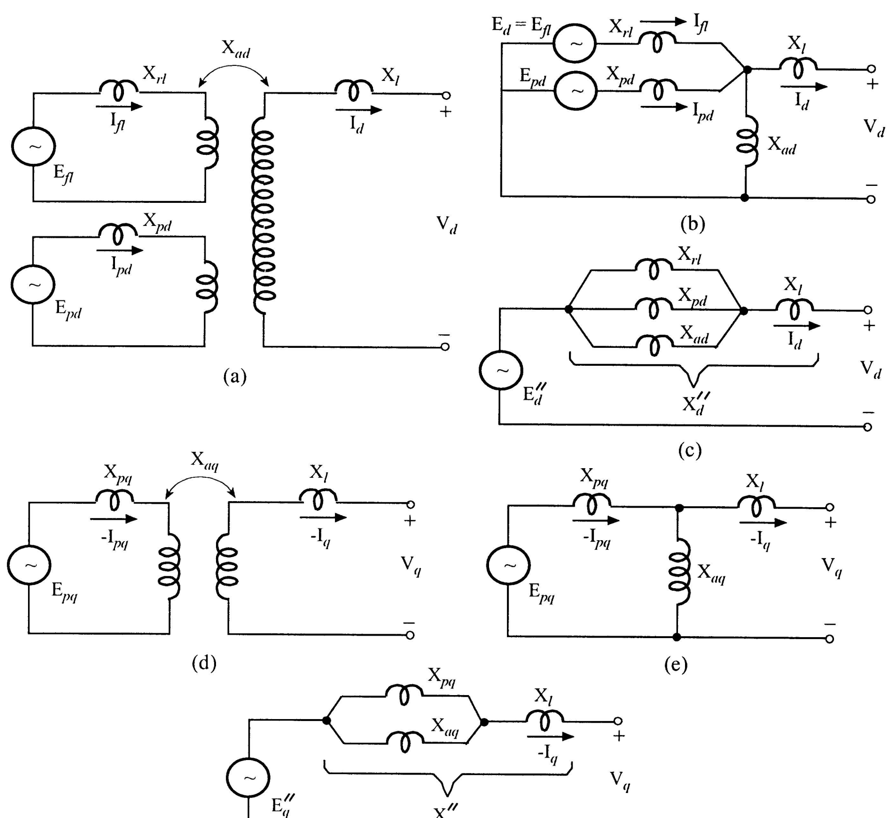


Figure 6.41. An equivalent circuit of a synchronous machine having damper windings and its simplification: in the *d*-axis (a, b and c) and in the *q*-axis (d, e and f).

theorem the one shown in (c). In the resulting circuit E''_d is called a *subtransient EMF* and X''_d is a *subtransient reactance*:

$$E''_d = \frac{E_d/X_{fl} + E_{pd}/X_{pd}}{1/X_{fl} + 1/X_{pd} + 1/X_{ad}} \tag{6.83}$$

and

$$X''_d = \frac{1}{1/X_{rl} + 1/X_{pd} + 1/X_{ad}} + X_l. \tag{6.84}$$

The subtransient EMF E''_d may also be determined by using the known terminal voltage and load current prior to short-circuiting:

$$E''_{d0} = V_{d0} + X''_d I_{d0}. \tag{6.85a}$$

The phasor diagram for a synchronous generator having damper windings is shown in Fig. 6.42.

For the generators having $X''_d = X''_q$, the initial subtransient EMF can be easily found from the simplified *phasor diagram of a synchronous generator with a damper winding*, shown in Fig. 6.42(b). Thus,

$$E''_0 = \sqrt{(V_0 \cos\varphi_0)^2 + (V_0 \sin\varphi_0 + X''I_0)^2}, \tag{6.85b}$$

or approximately as a projection on V_0,

$$E''_0 \cong V_0 + X''I_0 \sin\varphi_0. \tag{6.85c}$$

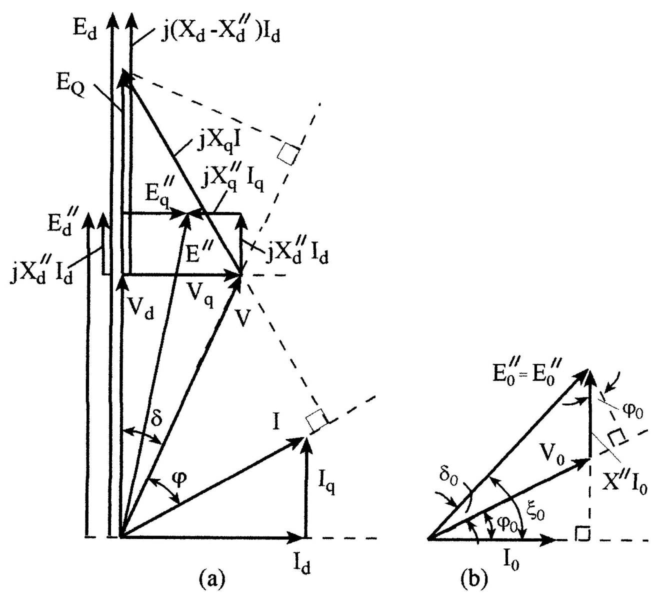


Figure 6.42 Phasor diagram for a synchronous generator with a damper winding (a) and a simplified phasor diagram for the generator having $X''_d = X''_q$ (b).

The subtransient parameters on the q-axis may be obtained by using the equivalent circuit in Fig. 6.41 (d, e and f). After simplification we have

$$E''_q = \frac{E_{pq}/X_{pq}}{1/X_{pq} + 1/X_{aq}} \tag{6.86}$$

and

$$X''_q = \frac{1}{1/X_{pq} + 1/X_{aq}} + X_l. \tag{6.87}$$

Similar to equation 6.85a we have

$$E''_{q0} = V_{q0} - X''_q I_{q0}. \tag{6.88}$$

The *subtransient time constants* are found as

$$T''_d = \frac{X''_d}{X'_d} T''_{d0} \tag{6.89}$$

and usually

$$T''_q \cong T''_d, \tag{6.90}$$

where T''_{d0} is a subtransient open-circuit d-axis time constant of a generator having a damper winding. The subtransient time constant is relatively small, $T''_d < T'_d$, and is in the range of 20 to 50 ms.

For the generator with damper windings the magnitudes of the fundamental frequency subtransient currents at the first moment of the fault are given by expressions

$$I''_{d0} = \frac{E''_{d0}}{X''_d}, \quad I''_{q0} = \frac{E''_{q0}}{X''_q} \tag{6.91}$$

and

$$I''_0 = \sqrt{I''^2_{d0} + I''^2_{q0}}. \tag{6.92}$$

Similar to equation 6.80, the *total short-circuit current versus time for a generator with damper windings is*

$$\begin{aligned} i_a = {} & \left[\left(\frac{E''_{d0m}}{X''_d} - \frac{E'_{d0m}}{X'_d}\right) e^{-t/T''_d} + \left(\frac{E'_{d0m}}{X'_d} - \frac{E_{dm}}{X_d}\right) e^{-t/T'_d} + \frac{E_{dm}}{X_d}\right] \cos(\omega t + \alpha) \\ & - \frac{V_{d0m}(X''_d + X''_q)}{2X''_d X''_q} e^{-t/T_a} \cos\alpha + \frac{V_{d0m}(X''_d - X''_q)}{2X''_d X''_q} e^{-t/T_a} \cos(2\omega t + \alpha) \\ & - \frac{E''_{q0m}}{X''_q} e^{-t/T''_q} \sin(\omega t + \alpha) + \frac{V_{q0m}(X''_d + X''_q)}{2X''_d X''_q} e^{-t/T_a} \sin\alpha \\ & + \frac{V_{q0m}(X''_d - X''_q)}{2X'_d X_q} e^{-t/T_a} \sin(2\omega t + \alpha). \end{aligned} \tag{6.93}$$

When the s.c. fault occurs after some external reactance X_F in all the previous expressions, this reactance must be added to the generator reactances on both axes.

As previously mentioned, this short-circuit current differs from that of a generator without damper windings (equation 6.80) by the presence of the subtransient term E''_{d0}/X''_d and the term on the q-axis E''_{q0}/X''_q. Both of these terms, however, decay very fast, at the rate of the time constants T''_d and T''_q.

After these two components die out, the instantaneous s.c. current is practically similar to those of a generator without damper windings. (Precisely speaking the damper winding also influences the transient process after decaying the subtransient currents: as an additional short-circuited winding on the d-axis it results in an increase in the aperiodic component in the field current to a slightly higher value than in the first moment of the fault. However, in practice this phenomenon is usually neglected. It should also be noted that for turbogenerators having $X''_d \cong X''_q$, the double frequency component in the s.c. currents is practically absent.)

In conclusion, the change of the r.m.s. or amplitude values, the *envelope curve* of an a.c. short-circuit (fundamental frequency) versus time is plotted in Fig. 6.43. As can be seen, this curve consists of three stages of the transient process: subtransient, transient and steady-state. The *subtransient stage* is given by the difference

$$\Delta I''_{d0} = I''_{d0} - I'_{d0},$$

and the *transient stage* is given by the difference

$$\Delta I'_{d0} = I'_{d0} - I'_{d\infty}.$$

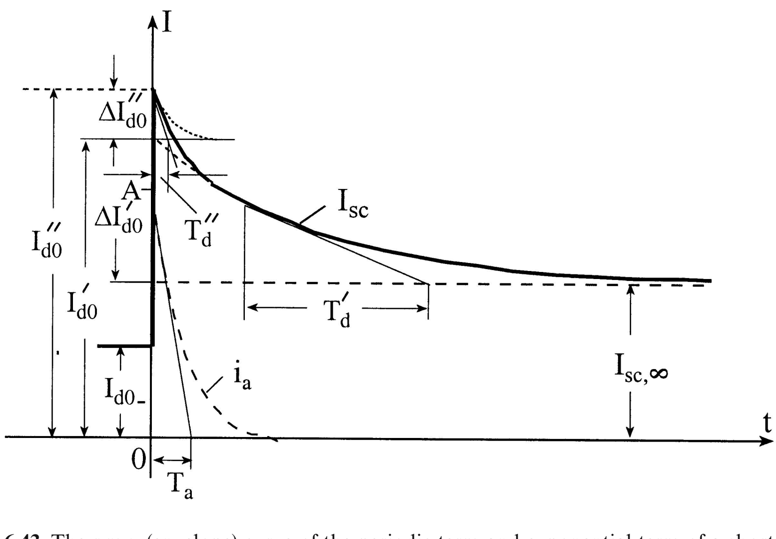


Figure 6.43 The r.m.s. (envelope) curve of the periodic term and exponential term of a short-circuit current for a generator having a damper winding.

The aperiodic (or exponential) component, which is decaying from its initial value A with the time constant T_a, is also plotted in Fig. 6.43. The curves in Fig. 6.43 can be used for experimentally determining the generator time constants, as shown in the figure.

The decaying process of the a.c. term of the short-circuit current can also be explained by increasing the generator reactances gradually from X''_d to X'_d and to X_d (remember that $X''_d < X'_d < X_d$) during the transient process. This phenomenon is opposed to the one in the first moment of short-circuiting: the armature reaction flux Φ_{ad} being suddenly increased, is opposed to the damper windings flux and is forced out of the poles to some extent, thereby increasing the reluctance and yielding a reduced synchronous reactance X''_d. As the time of the transients progresses, Φ_{ad} moves back on through the pole, which yields a relatively low reluctance path, and therefore the reactance will increase.

(c) *Transient behavior of a synchronous generator with AVR*

If the generator is equipped with AVR (automatic voltage regulator), the voltage supplied to the field winding does not keep constant, but is increased at the moment of short-circuiting. It can be approximately assumed that this rise of the supplied voltage is exponential. Hence, equation 6.72 may now be written in the form

$$T'_d \frac{dE_d}{dt} + E_d = E_{d,\max} - (E_{d,\max} - E_{d0\_})e^{-t/T_{ff}}. \tag{6.94}$$

Here on the right side of the differential equation is an exponential function having its prior to switching value $E_{d0\_}$ and the final, steady-state value $E_{d,\max}$; T_{ff} is the time constant of the supplied voltage circuit (exciter and/or power supply circuit). The range of this constant is 0.4 to 1 s.

The natural solution of this equation as we already know is a simple exponent

$$E_{d,n} = Ae^{-t/T'_d}. \tag{6.95}$$

The forced solution should be of the same form as the forced function

$$E_{d,f} = B + Ce^{-t/T_{ff}}. \tag{6.96}$$

The integration constants A, B and C might be found by applying the known quantities: prior to switching ($t = 0_-$) value $E_{d0\_}$, the initial ($t = 0_+$) value E_{d0_+} [$E_{d0_+} = E'_{d0}(X_d/X'_d)$] and the final or steady-state value $E_{d,\max}$ (in accordance with the known $I_{ff,\max}$). Omitting all the algebraic calculations we may obtain the integration constants as

$$B = E_{d,\max}, \quad C = \Delta E_d \frac{T_{ff}}{T'_d - T_{ff}}, \tag{6.97a}$$

where $\Delta E_d = E_{d,\max} - E_{d0\_}$ and

$$A = -(E_{d,\max} - E_{d0_+}) - \Delta E_d \frac{T_{ff}}{T'_d - T_{ff}}. \tag{6.97b}$$

Thus, we finally have

$$E_d(t) = E_{d,f} + E_{d,n} = E_{d,\max} + \Delta E_d \frac{T_{ff} e^{-t/T_{ff}}}{T'_d - T_{ff}}$$
$$- \left[E_{d,\max} - E_{d0_+} + \Delta E_d \frac{T_{ff}}{T'_d - T_{ff}} \right] e^{-t/T'_d}, \tag{6.98}$$

which results in E_{d0_+} at $t = 0$ and in $E_{d,\max}$ at $t \to \infty$, i.e. in accordance with the given initial and steady-state conditions.

For the generator without AVR the time constant T_{ff} should be infinite, i.e., $T_{ff} = \infty$, and the differential equation 6.94 turns into

$$T'_d \frac{dE_d}{dt} + E_d = E_{d0_-}.$$

The solution of this equation is

$$E_d(t) = (E_{d0_+} - E_{d0_-}) e^{-t/T'_d} + E_{d0_-}. \tag{6.99}$$

By rearranging the terms in equation 6.98 and after performing the appropriate algebraic calculations, we may obtain

$$E_d(t) = [(E_{d0_+} - E_{d0_-}) e^{-t/T'_d} + E_{d0_-}] + (E_{d,\max} - E_{d0_-}) F(t)$$
$$= E_{d(\text{without AVR})} + \Delta E_{d(\text{with AVR})}, \tag{6.100a}$$

where

$$F(t) = 1 - \frac{T'_d e^{-t/T'_d} - T_{ff} e^{-t/T_{ff}}}{T'_d - T_{ff}}. \tag{6.100b}$$

The above expression clearly shows that due to AVR the EMF of the generator increases gradually during the transients by $\Delta E_{d(\text{with AVR})}$ relatively to the EMF of the generator without AVR. This in turn results in increasing the transient a.c. term of the short-circuit current. Dividing equation 6.100a by X_d and taking into consideration that $E_d = (X_d/X'_d)E'_d$ and $E_{d0_-}/X_d = I_\infty$, we have

$$I'(t) = [(I'_d - I_\infty) e^{-t/T'_d} + I_\infty] + \Delta I_\infty F(t), \tag{6.101}$$

where $\Delta I_\infty = I_{\infty,\max} - I_\infty$ and $I_{\infty,\max}$ is the steady-state short-circuit current due to the maximal increased by AVR field current. The term in the brackets on the right hand side in equation 6.101 is the transient a.c. component of the short-circuit current without AVR, and ΔI_∞ is its increase due to AVR. The short-circuit current, given by equation 6.101, which is actually the r.m.s. envelope curve of an a.c. component, is plotted in Fig. 6.44 for two cases: with and without AVR. Note that, in the very beginning of the transients, the two curves are practically the same, which means that the subtransient current is not influenced by AVR. This current decays before the AVR has had time to affect the generator EMF.

As was earlier shown, operating regime of the AVR depends on the fault

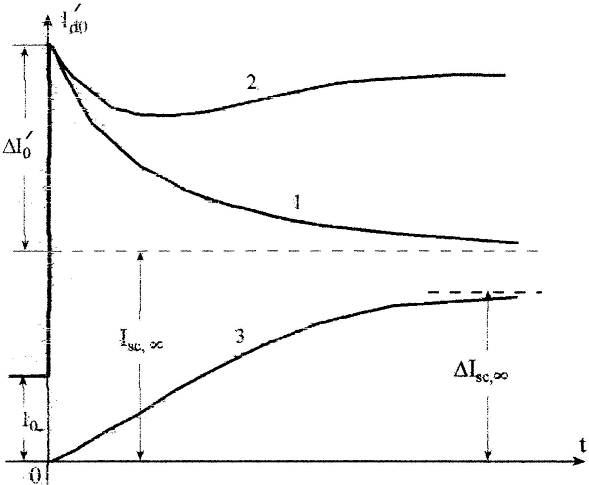


Figure 6.44 An envelope curve of the transient current for two cases: 1, without AVR; 2, with AVR and 3, the increase on the current due to AVR.

point location. Namely, if the external reactance is lower than the critical one. i.e., $X_F < X_{cr}$, then the AVR operates under the condition of maximal field current. If, however, $X_F > X_{cr}$, then the AVR operates under the condition of nominal/rated terminal voltage, $V = V_{nom}$. Hence, in the first case the short-circuit current varies in accordance with equation 6.101, but in the second case the current is limited by the value of $V_{nom}/(X'_d + X_F)$.

(d) *Peak values of a short-circuit current*

For some conditions (such as to check the dynamic stability of electric equipment under short-circuit conditions or for the proper design of relay protection), it is necessary to know the maximum instantaneous or *peak value* of the *short-circuit current*. In highly inductive circuits the peak current appears nearly half a period after the occurrence of the short-circuit.

Neglecting the double-frequency term, we shall take into consideration the subtransient a.c. term I''_d and the exponential term A (equation 6.82), so that the peak value will be

$$i_{pk} = \sqrt{2}I''_d + Ae^{-0.01/T_a}, \tag{6.102}$$

or by approximating $A \cong \sqrt{2}I''_d$, and with $T_a = 0.05$ (or $X/R = 1.5$) we have

$$i_{pk} = \sqrt{2}I''_d(1 + e^{-0.01/T_a}) = \sqrt{2}k_{pk}I''_d \cong \sqrt{2}\cdot 1.8 I''_d, \tag{6.103}$$

where, as previously (equation 6.20a),

$$k_{pk} = 1 + e^{-0.01/T_a}$$

is the *peak coefficient*.

Furthermore, because of very small damping, the exponential term approaches unity and the maximal peak is simply

$$i_{pk} = 2\sqrt{2}I''_d \tag{6.104}$$

The *r.m.s. value* of the *peak current* can be calculated in accordance with equation 6.22a

$$I_{pk} = I''_d\sqrt{1 + 2(k_{pk} - 1)^2},$$

and for $T_a = 0.05$ ($k_{pk} = 1.8$) we have

$$I_{pk} = 1.52I''_d.$$

As can be seen, the peak coefficient depends on the value of the time constant. Thus, for $T_a = 0.008$ ($X/R = 2.5$) the peak coefficient decreases to 1.3 and

$$i_{pk} = \sqrt{2}\cdot 1.3I''_d \quad \text{and} \quad I_{pk} = 1.1I''_d.$$

However, the aperiodic component should be taken into consideration for the periods of less than $t = 0.15$ s after the short-circuit fault occurs.

Example 6.8

For a synchronous generator having the following p.u. parameters: $X_\ell = 0.1$, $X_d = 1.2$, $X'_d = 0.25$, $X_q = 0.6$, $R = 0.005$ and $T_{d0} = 8.5$ s: a) find all the components of the transient short-circuit current at $t = 0$, b) write the expression of the s.c. current in phase *a* and in the field (rotor) winding versus time, c) write the expression of the s.c. current envelope and plot the phase *a* current and the envelope curve and d) calculate the peak value of the s.c. current. Prior to short-circuiting the generator has been operated at the rated voltage $V = 1$ and 0.8 of the rated current with PF = 0.85 ($\varphi = 31.8^\circ$) and $f = 50$ Hz (the AVR is absent).

Solution

a) To find E_d and E'_d we must calculate the power angle δ (see the phasor diagram in Fig. 6.38). For this purpose we first calculate the angle ξ:

$$\xi = \tan^{-1}\frac{V\sin\varphi + X_q I}{V\cos\varphi} = \tan^{-1}\frac{1\cdot\sin 31.8^\circ + 0.6\cdot 0.8}{1\cdot 0.85} = 49.8^\circ$$

and $\delta = \xi - \varphi = 49.8^\circ - 31.8^\circ = 18.0^\circ$.

Now we may find

$$E_d = V\cos\delta + X_d I\sin\xi = 1\cdot\cos 18.0^\circ + 1.2\cdot 0.8\sin 49.8^\circ = 1.70$$

$$E'_d = V\cos\delta + X'_d I\sin\xi = 1\cdot\cos 18.0^\circ + 0.25\cdot 0.8\sin 49.8^\circ = 1.1.$$

Then the s.c. current components at $t = 0$ are:

a.c. (or periodic)

$$I'_{d0} = \frac{E'_{d0}}{X'_d} = \frac{1.1}{0.25} = 4.4, \quad I_\infty = \frac{E_{d0}}{X_d} = \frac{1.70}{1.2} = 1.42,$$

and

$$\Delta I_d = I'_{d0} - I_\infty = 4.4 - 1.42 = 2.98;$$

d.c. (or aperiodic)

$$I_{da} = \frac{V_{d0}(X_q + X'_d)}{2X_q X'_d} = \frac{0.951\cdot(0.6 + 0.25)}{2\cdot 0.6\cdot 0.25} = 2.69$$

$$I_{qa} = \frac{V_{q0}(X_q + X'_d)}{2X_q X'_d} = \frac{0.310\cdot(0.6 + 0.25)}{2\cdot 0.6\cdot 0.25} = 0.878,$$

where

$$V_{d0} = V \cos\delta = 1\cdot\cos 18.0^\circ = 0.951$$

$$V_{q0} = X_q I_q = X_q I \cos\xi = 0.6\cdot 0.8 \cos 49.8^\circ = 0.310;$$

2ω (or double frequency)

$$I_{d,2\omega} = \frac{V_{d0}(X_q - X'_d)}{2X_q X'_d} = \frac{0.951\cdot(0.6 - 0.25)}{2\cdot 0.6\cdot 0.25} = 1.11$$

$$I_{q,2\omega} = \frac{V_{q0}(X_q - X'_d)}{2X_q X'_d} = \frac{0.310\cdot(0.6 - 0.25)}{2\cdot 0.6\cdot 0.25} = 0.362.$$

b) The time constants are:

$$T_a = \frac{X_2}{\omega R} = \frac{0.353}{314\cdot 0.005} = 0.225,$$

where

$$X_2 = \frac{2X'_d X_q}{X'_d + X_q} = \frac{2\cdot 0.25\cdot 0.6}{0.25 + 0.6} = 0.353,$$

and

$$T'_d = T_{d0}\frac{X'_d}{X_d} = 8.5\cdot\frac{0.25}{1.2} = 1.77.$$

The s.c. current of phase a versus time for the initial angle $\alpha = 30^\circ$ will be

$$\begin{aligned} i_a(t) = \sqrt{2}[&(1.42 + 2.98e^{-t/1.77})\cos(\omega t + 30^\circ) - 2.69e^{-t/0.225}\cos 30^\circ \\ &- 1.11e^{-t/0.225}\cos(2\omega t + 30^\circ) + 0.878e^{-t/0.225}\sin 30^\circ \\ &- 0.362e^{-t/0.225}\sin(2\omega t + 30^\circ)], \end{aligned}$$

or, after simplification,

$$i_a(t) = (2.01 + 4.21\,e^{-t/1.77})\cos(\omega t + 30^\circ)$$
$$- 2.67\,e^{-t/0.225} - 1.65e^{-t/0.225}\cos(2\omega t + 11.9^\circ).$$

Note that at $t = 0$, the d- and q-components are

$$I_{d0} = 4.4 - 2.69 - 1.11 = 0.60, \quad I_{q0} = 0.878 - 0.362 = 0.516,$$

and $I_0 = \sqrt{0.60^2 + 0.516^2} = 0.79 \cong 0.8$, as it is given.

The rotor winding current, i.e., the field current, is calculated with equation 6.81:

$$i_{fl}(t) = I_{fl0} + i_{fl,a0} + i_{fl,\omega 0} = 1.55 + 4.65\,e^{-t/1.77} - 4.65\,e^{-t/0.225}\cos\omega t,$$

where

$$I_{fl0} = \frac{E_{d0}}{x_{ad}} = \frac{1.70}{1.1} = 1.55, \quad X_{ad} = X_d - X_\ell = 1.2 - 0.1 = 1.1$$

and

$$I_{fl,a0} = -I_{fl,\omega 0} = \frac{\sqrt{2}V_{d0}}{X'_d}\frac{X_d - X'_d}{X_{ad}} = \frac{\sqrt{2}\cdot 0.951}{0.25}\cdot\frac{1.2 - 0.25}{1.1} = 4.65.$$

c) The envelope curve may be obtained (by neglecting the 2ω-component) as

$$I_d(t) = \sqrt{2}(1.42 + 2.98\,e^{-t/1.77} - 2.69\,e^{-t/0.225}), \quad I_q(t) = \sqrt{2}\cdot 0.878\,e^{-t/0.225}.$$

The phase a s.c. current versus time as well as the envelope curve are given in Fig. 6.45.

d) The peak value of the s.c. current (which arises after about $t = 0.01$ s) is

$$I_{pq} \cong 2.01 + 4.21^{-0.01/1.77} + 2.67\,e^{-0.01/0.225} = 8.8,$$

or by using the approximate formula 6.103

$$i_{pk} = \sqrt{2}\cdot 1.8\cdot 4.4 = 11.2.$$

The difference in the above results is because the approximate formula is given for a no-loaded generator and therefore the initial value of an aperiodic component is as high as the subtransient s.c. current. However, in our example the generator prior to switching was operated under load and the aperiodic component is much lower.

Example 6.9

A turbogenerator is connected to the system through a power transformer. The parameters of these two apparatuses are: 1) turbo-generator – 125 MVA, 15.8 kV, PF = 0.8, and $X_\ell = 0.1$, $X_d = X_q = 1.35$, $X'_d = 0.2$, $X''_d = 0.13$, $X''_q = 0.15$, $T_a = 0.1$ s, $T'_{d0} = 11.45$ s, $T''_{d0} = 0.25$ s, $T''_{q0} = 0.55$ s; 2) transformer – 120 MVA, 242/15.8 kV, $v_{sc} = 11.5\%$. For the three-pole short-circuit fault at the secondary of the transformer, find the subtransient s.c. current as its r.m.s. value versus

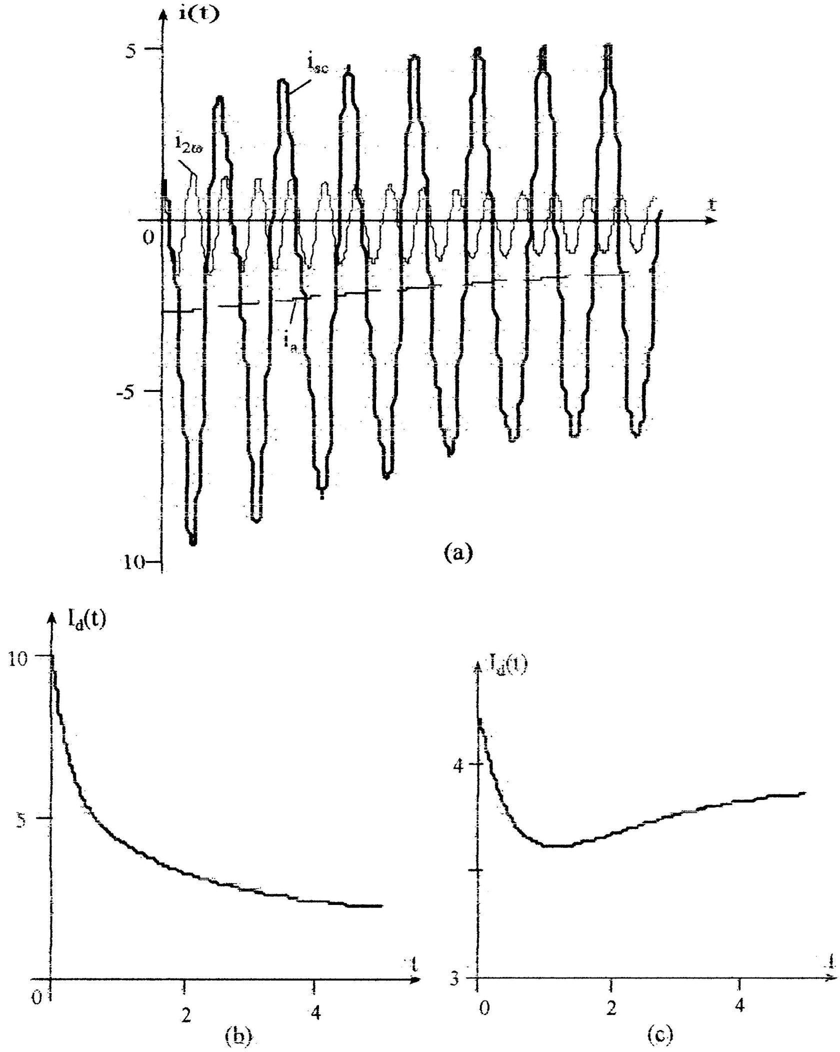


Figure 6.45 Short-circuit current of phase *a* versus time (a) and envelope curve of the short-circuit current without AVR (b) and with AVR (see Example 6.10) (c).

time (an envelope curve). The prior to switching operating conditions of the generator were as follows: $P = 100$ MVA, PF $= 0.8$, $V_t = 16.5$ kV (without AVR).

Solution

The rated and load currents of the generator are

$$I_r = \frac{125}{\sqrt{3} \cdot 15.8} = 4.6\ \text{kA}, \quad I_{ld} = \frac{100}{\sqrt{3} \cdot 15.8} = 3.5\ \text{kA}.$$

Thus, the generator current, prior to switching, in p.u. is

$$I_0 = \frac{3.5}{4.6} = 0.76,$$

and the operating voltage in p.u. is

$$V_0 = \frac{16.6}{15.8} = 1.05.$$

In accordance with the phasor diagram (see Fig. 6.42)

$$E_{d0} = \sqrt{(1.05\cdot 0.8)^2 + (1.05\cdot 0.6 + 1.35\cdot 0.76)^2} = 1.85,$$

where $\cos\varphi_0 = 0.8$ and $\sin\varphi_0 = 0.6$.

The angle between the current I_0 and the EMF E_{d0} is

$$\xi = \tan^{-1} = \frac{1.05\cdot 0.6 + 1.35\cdot 0.76}{1.05\cdot 0.8} = \tan^{-1} 1.97 = 63.1^\circ.$$

Thus,

$$\cos\xi = 0.452 \quad \text{and} \quad \sin\xi = 0.883.$$

The power angle is

$$\delta_0 = 63.1^\circ - \cos^{-1} 0.8 = 26.2^\circ,$$

and $\cos\delta_0 = 0.897$, $\sin\delta_0 = 0.442$.

The d- and q-components of the initial current and voltage can now be calculated as

$$I_{d0} = 0.76\cdot 0.883 = 0.671 \qquad I_{q0} = 0.76\cdot 0.452 = 0.344$$
$$V_{d0} = 1.05\cdot 0.897 = 0.942 \qquad V_{q0} = 1.05\cdot 0.442 = 0.464.$$

The d- and q-components of the subtransient EMF are

$$E''_{d0} = V_{d0} + X''_d I_{d0} = 0.942 + 0.13\cdot 0.671 = 1.03$$
$$E''_{q0} = V_{q0} - X''_q I_{q0} = 0.464 - 0.15\cdot 0.344 = 0.412.$$

For further calculation we need to know the p.u. reactance of the transformer referred to the generator power $X_T = 0.115(125/120) = 0.12$. The first moment subtransient current components, therefore, will be

$$I''_{d0} = \frac{E''_{d0}}{X''_d + X_T} = \frac{1.03}{0.13 + 0.12} = 4.12$$
$$I''_{q0} = \frac{E''_{q0}}{X''_q + X_T} = \frac{0.412}{0.15 + 0.12} = 1.53,$$

and

$$I''_0 = \sqrt{I''^2_{d0} + I''^2_{q0}} = \sqrt{4.12^2 + 1.53^2} = 4.40.$$

The steady-state s.c. current will be

$$I_\infty = \frac{E_{d0}}{X_d + X_T} = \frac{1.85}{1.35 + 0.12} = 1.26.$$

To find the transient current we must first determine the transient EMF

$$E'_{d0} = V_{d0} + X'_d I_{d0} = 0.942 + 0.2 \cdot 0.671 = 1.08.$$

Hence, the transient current is

$$I'_{d0} = \frac{E'_{d0}}{X'_d + X_T} = \frac{1.08}{0.2 + 0.12} = 3.38,$$

and the subtransient and transient stages are given by

$$\Delta I''_d = 4.12 - 3.38 = 0.74, \quad \Delta I' = 3.38 - 1.26 = 2.12.$$

The aperiodic components (at $t = 0$) are

$$I_{da0} = \frac{V_{dF0}(X''_d + X''_q + 2X_T)}{2(X''_d + X_T)(X''_q + X_T)} = \frac{0.861 \cdot (0.12 + 0.15 + 0.24)}{2 \cdot 0.25 \cdot 0.27} = 3.32$$

$$I_{qa0} = \frac{V_{qF0}(X''_d + X''_q + 2X_T)}{2(X''_d + X_T)(X''_q + X_T)} = \frac{0.50 \cdot (0.12 + 0.15 + 0.24)}{2 \cdot 0.25 \cdot 0.27} = 1.92,$$

where the voltages at the fault point are

$$V_{dF0} = V_{d0} - X_T I_{d0} = 0.942 - 0.12 \cdot 0.671 = 0.861$$

$$V_{qF0} = V_{q0} + X_T I_{q0} = 0.464 + 0.12 \cdot 0.344 = 0.50.$$

Note that since the subtransient reactances in the d- and q-axes are almost equal, the double-frequency terms are neglected.

The subtransient and transient time constants are

$$T''_d = \frac{X''_d + X_T}{X'_d + X_T} T''_{d0} = \frac{0.13 + 0.12}{0.2 + 0.12} 0.25 = 0.195 \text{ s}$$

$$T''_q = \frac{X''_q + X_T}{X_q + X_T} T''_{q0} = \frac{0.15 + 0.12}{1.35 + 0.12} 0.55 = 0.101 \text{ s},$$

and

$$T'_d = \frac{X'_d + X_T}{X_d + X_T} T'_{d0} = \frac{0.2 + 0.12}{1.35 + 0.12} 11.45 = 2.49 \text{ s}.$$

Thus, the r.m.s. value of the a.c. component versus time (the envelope curve) is

$$I_d = 0.74 e^{-t/0.195} + 2.12 e^{-t/2.49} + 1.26, \quad I_q = -1.53 e^{-t/0.101}.$$

The aperiodic terms in both axes are

$$I_{da} = -3.32 e^{-t/0.1}, \quad I_{qa} = 1.92 e^{-t/0.1}.$$

The initial value of the subtransient current is

$$I_0'' = \sqrt{4.12^2 + 1.53^2} = 4.4.$$

The initial value of the entire current is

$$I_0 = \sqrt{(I_{d0}'' - I_{da0})^2 + (-I_{q0} + I_{qa0})^2} = \sqrt{(4.12 - 3.32)^2 + (-1.53 + 1.92)^2} \cong 0.88.$$

(Note that this value varies slightly from the actual initial current because we neglected the double-frequency terms.)

Example 6.10

For the generator of Example 6.8 find the a.c. component of the short-circuit current (an envelope curve) if the generator is equipped with an AVR, having $I_{fl,\max} = 4.3$ and $T_{ff} = 0.55$ s.

Solution

Since the subtransient current decays very fast, practically before the AVR substantially affects the field current increasing, we shall take into consideration only the transient and steady-state currents. The steady-state s.c. current under the maximal field current will be

$$I_{\infty,\max} = \frac{I_{fl,\max}}{X_{ad}} = \frac{4.3}{1.1} = 3.91,$$

where $X_{ad} = X_d - X_\ell = 1.2 - 0.1 = 1.1$.
Thus,

$$\Delta I_0' = I_0' - I_\infty = 4.4 - 1.42 = 2.98, \quad \Delta I_\infty = I_{\infty,\max} - I_\infty = 3.91 - 1.42 = 2.49.$$

The increasing function will be

$$F(t) = 1 - \frac{T_d' e^{-t/T_d'} - T_{ff} e^{-t/T_{ff}}}{T_d' - T_{ff}} = 1 - \frac{1.77 e^{-t/1.77} - 0.55 e^{-t/0.55}}{1.77 - 0.55}$$
$$= 1 + 0.451 e^{-t/0.55} - 1.45 e^{-t/1.77}.$$

From Example 6.8 we have

$$I_{d(\text{without AVR})} = 2.98 e^{-t/1.77} + 1.42.$$

Therefore, we may now obtain

$$I_d(t) = I_{d(\text{without AVR})} + \Delta I_\infty F(t) = 3.91 + 1.12 e^{-t/0.55} - 0.63 e^{-t/1.77},$$

which at $t = 0$ again gives 4.4. This curve is also shown in Fig. 6.45.

6.6 SHORT-CIRCUIT ANALYSIS IN INTERCONNECTED (LARGE) NETWORKS

In general an electric system is supplied by a number of generators of different designs and different ratings, which are interconnected in complicated networks.

In practice the operation of a single synchronous generator in an isolated system, as has been discussed so far, is limited. The short-circuit analysis in interconnected systems is very complicated. The short-circuit currents of each of the generators are dependent on each other. The operation conditions of the AVR of each of the generators will depend on the distance to the location point of the fault. The mechanical oscillations of some of the generators will almost always follow the short-circuit faults. All this makes the precise calculation extremely complicated, if not impossible.

Therefore, in practical calculations it is common to make a few additional simplifications:

1) Each of the generators has a round (cylindrical) rotor, which allows us to neglect the double-frequency component and operate with only one current, alleviating the need of dividing the current and voltage onto two axes;

2) The periodic a.c. and aperiodic exponential terms are of the same form and obey the same law of behavior, as for a single generator;

3) All the generators operate under a constant speed of rotation.

These assumptions allow us to obtain the final results of a short-circuit fault in an interconnected system relatively easily and with accuracy, satisfying the practical needs. We shall now illustrate our study with the following examples.

Example 6.11

Find the subtransient s.c. current and its peak value in the network, shown in Fig. 6.46(a), if the three-pole fault occurs at the secondary (2) terminals of transformer T_1, point F. The two transformers T_1 and T_2 are identical and the circuit breaker Br is closed.

Solution

Let the basic power be 600 MVA. Then the p.u. reactances (see Fig. 6.46(b)) are calculated as

$$X_1 = 0.20\frac{600}{380} = 0.32, \quad X_2 = 0.093\frac{600}{160} = 0.35,$$

$$X_3 = 0.38\cdot 200\frac{600}{230^2} = 0.86,$$

$$X_4 = X_5 = 0.5(0.181 + 0.123 - 0.058)\frac{600}{60} = 1.23, \quad X_6 = X_7 \cong 0,$$

$$X_8 = X_9 = 0.5(0.181 - 0.123 + 0.058)\frac{600}{60} = 0.58.$$

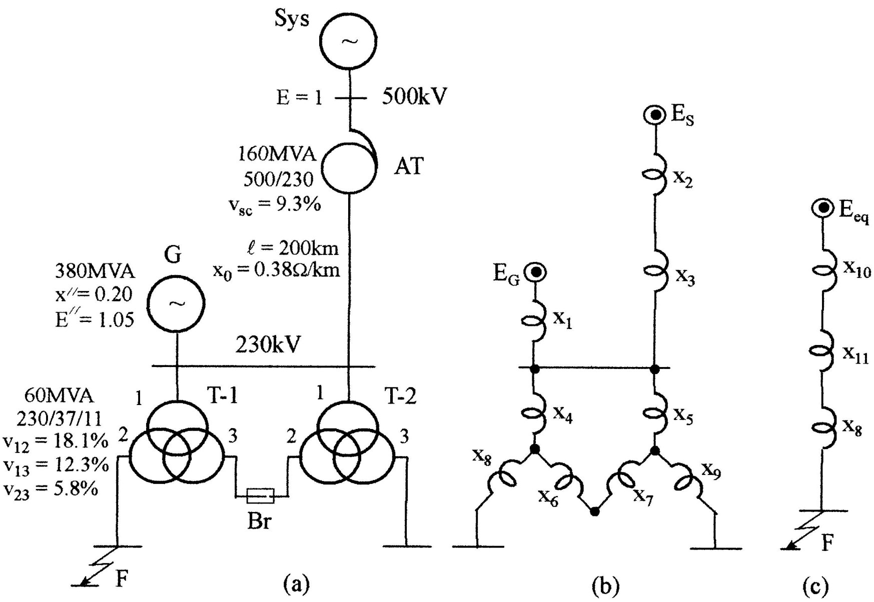


Figure 6.46 A network diagram for Example 6.11 (a) and its simplification (b) and (c).

To simplify the network, we find

$$X_{10} = 0.32//(0.35 + 0.86) = 0.25, \quad X_{11} = X_4/2 = 0.62,$$

$$X_{eq} = 0.25 + 0.62 + 0.58 = 1.45,$$

and

$$E''_{eq} = \frac{1.05 \cdot 1.21 + 1 \cdot 0.32}{1.21 + 0.32} = 1.04.$$

Therefore, the subtransient current will be

$$I'' = \frac{E''_{eq}}{X_{eq}} = \frac{1.04}{1.45} = 0.72, \quad \text{or in amperes} \quad I_{sc} = 0.72 \frac{600}{\sqrt{3} \cdot 37} = 6.74 \text{ kA}.$$

The peak current (equation 6.103) will be

$$i_{pk} = \sqrt{2} \cdot 1.8 I_{sc} = 2.6 \cdot 6.74 = 17.2 \text{ kA}.$$

Example 6.12

The power network, shown in Fig. 6.47(a), consists of two identical generators G_1 and G_2, two identical transformers T_1 and T_2 and a power station G_3, which are connected by a 161 kV transmission line. All the circuit parameters are given in Fig. 6.47(a). If a three-pole fault occurs at point F, find the s.c. current at the moments of 0.1 s and 1 s. All the generators are equipped with AVR and the circuit breaker Br is opened.

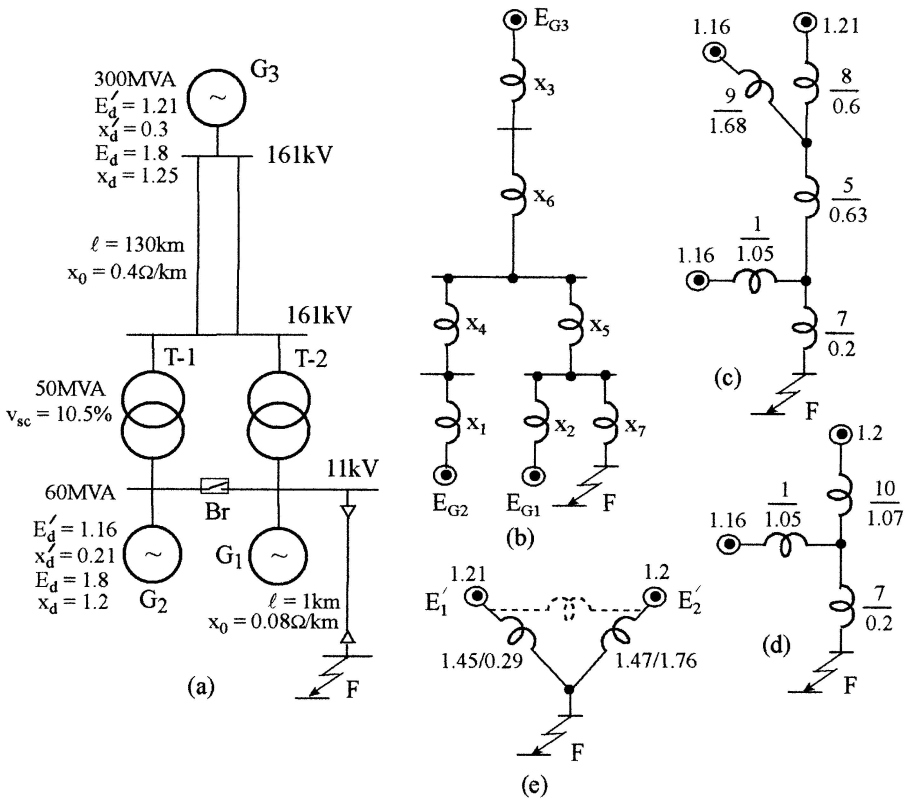


Figure 6. 47 A network for Example 6.12 (a) and the stages of its simplification (b)–(e).

Solution

Since at the time of 0.1 s the subtransient currents are already decayed, we shall represent the generators by their transient parameters. By choosing the basic power of 300 MVA, the circuit reactances are calculated as follows (see Fig. 6.47(b)):

$$X_1 = X_2 = 0.21\,\frac{300}{60} = 1.05, \quad X_3 = 0.3, \quad X_4 = X_5 = 0.105\,\frac{300}{50} = 0.63,$$

$$X_6 = \frac{1}{2}\,0.4\cdot 130\,\frac{300}{161^2} = 0.3, \quad X_7 = 0.08\cdot 1\cdot\frac{300}{11^2} = 0.2.$$

The obvious simplification of the circuit is then performed in three stages, as shown in Fig. 6.47(c), (d) and (e). Generator G_1 is relatively "close" to the fault point; therefore it is treated separately, as shown in Fig. 6.47(e). First we find the s.c. current flowing from this generator. The transfer reactance between the generator and the fault point is found by transforming the Y-configuration in

Fig. 6.47(d) to the Δ-configuration in Fig. 6.47(e):

$$X'_{trs1} = 1.05 + 0.2 + \frac{1.05 \cdot 0.2}{1.07} = 1.45,$$

and referred to the rated power of the generator $X^{(n)}_{trs1} = 1.45(60/300) = 0.29$. Hence, the transient current is

$$I'_{01} = \frac{E'_1}{X_{trs1}} = \frac{1.16}{0.29} = 4.0.$$

Next we shall calculate the steady-state s.c. current. To do this, the generators must be represented by their synchronous reactances, X_d, and steady-state EMF, E_d. Performing the same steps of circuit simplification we will obtain

$$X^{(n)}_{trs1} = 1.37 \quad \text{and} \quad I_{\infty,1} = \frac{E_{d1}}{X^{(n)}_{trs1}} = \frac{1.8}{1.37} = 1.31.$$

Suppose that the maximal field current and time constant of the generators are: $I_{fl,\max} = 4.7$, $T_{ff} = 0.55$ s and $T'_{d0} = 8.5$ s. Then we may find the maximal steady-state s.c. current (equation 6.53)

$$I_{\infty,1\max} = \frac{I_{fl,\max}}{X_{ad}} = \frac{4.7}{1.37 - 0.1} = 3.7,$$

where $X_{ad} = X_d - X_\ell$. Then

$$\Delta I'_{01} = I'_{01} - I_{\infty,1} = 4 - 1.31 = 2.69$$

$$\Delta I'_{\infty,1} = I'_{\infty,1\max} - I_{\infty,1} = 3.62 - 1.31 = 2.31.$$

The transient time constant is

$$T'_{d1} = T'_{d0} \frac{X'_{trs1}}{X_{trs1}} = 8.5 \frac{0.29}{1.37} = 1.8.$$

Now we may find the increasing function (equation 6.100b)

$$F(t) = 1 - \frac{1.8e^{-t/1.8} - 0.55e^{-t/0.55}}{1.8 - 0.55} = 1 + 0.44e^{-t/0.55} - 1.44e^{-t/1.8}.$$

The r.m.s. short-circuit current versus time without AVR is

$$I_{sc1(\text{without AVR})} = 2.69e^{-t/1.8} + 1.31,$$

and the total current due to the AVR action will be

$$\begin{aligned} I_{sc1}(t) &= I_{sc1(\text{without AVR})} + \Delta I_{\infty,1} F(t) \\ &= 2.69e^{-t/1.8} + 1.3 + 2.31(1 + 0.44e^{-t/0.55} - 1.44e^{-t/1.8}) \\ &= 3.7 + 1.05e^{-t/0.55} - 0.75e^{-t/1.8.} \end{aligned}$$

Thus, the s.c. current of generator G_1 is:

at $t = 0.1$ $\quad I_{sc1}(0.1) = 3.9 \quad$ or $\quad 3.9 \dfrac{60}{\sqrt{3 \cdot 11}} = 12.3$ kA,

at $t = 1.0$ $\quad I_{sc1}(1.0) = 3.4 \quad$ or $\quad 3.4 \dfrac{60}{\sqrt{3 \cdot 11}} = 10.7$ kA.

(Note that without AVR the s.c. current at 1 s would be 2.8 or 8.8 kA, i.e., less than with the AVR.) Next we find the s.c. current flowing from the power station and generator 2. The equivalent EMF in Fig. 6.47(d) is

$$E'_2 = \frac{1.16 \cdot 0.6 + 1.21 \cdot 1.68}{1.68 + 0.6} = 1.2.$$

The transfer reactance in Fig. 6.47(e) is

$$X^{(n)}_{trs2} = 1.07 + 0.2 + \frac{1.07 \cdot 0.2}{1.05} = 1.47,$$

which as referred to the rated power will be

$$X^{(n)}_{trs2} = 1.47 \frac{360}{300} = 1.76.$$

Hence, the transient current flowing from the rest of the network will be

$$I'_{02} = \frac{1.2}{1.76} = 0.674.$$

To determine in which regime the AVR is operated, we shall calculate the terminal voltage of the equivalent generator in Fig. 4.67(d):

$$V_{ter2} = X_{F2} I'_{02} = 1.55 \cdot 0.674 = 1.05 > 1,$$

where $X_{F2} = X_{trs2} - X'_d \cong 1.76 - 0.21 = 1.55$.

Since $V_{ter2} > 1$, the generators of the power station and G_2 operate under constant voltage and, therefore, the s.c. current flowing from the rest of the network is almost constant. Its ampere value is

$$I_{sc2} = 0.674 \frac{360}{\sqrt{3 \cdot 11}} = 12.7 \text{ kA}.$$

Thus, the total s.c. currents are

$$I_{sc2}(0.1) = 12.3 + 12.7 = 25.0 \text{ kA}, \quad I_{sc2}(1.0) = 10.7 + 12.7 = 23.4 \text{ kA}.$$

6.6.1 Simple computation of short-circuit currents

The simplest calculation of short-circuit transients is based on the assumption that the fault circuit is connected to a system of infinite power. In this case the inner impedance of such a system is taken as zero and its voltage is unity. The change of the short-circuit current in this case is only due to the aperiodic component and can be approximated by using the peak coefficient. The periodic component of the short-circuit current, therefore, may be found with just the total reactance between the fault point and the system on the equivalent circuit

$$I_{sc} = \frac{1}{X_{tot}}. \tag{6.105a}$$

The elements of the equivalent circuit are usually transformers, cables and/or transmission lines. The short-circuit currents in such a calculation become a little bit larger than in reality. However, because of its simplicity, this way of calculating is widely used for a quick estimation of the s.c. currents and the results might be appropriate for solving some of the practical problems. This method is also used when the system configuration and its parameters are unknown.

Up to this point in our transient analysis, power circuits, which have been under consideration, consisted primarily of pure reactances, i.e., their very small resistances have been neglected. It can be shown that if $R \leq (1/3)X$, then neglecting such resistances results in increasing the periodic component of the s.c. current only at a rate of less than 5%, which anyway is within the accuracy of engineering calculations.

However, in the distribution networks the value of the resistances might be much higher. In such cases the resistances should be taken into consideration:

1) by the correction of the time constant of the aperiodic component:

$$T_a = \frac{X}{\omega R},$$

and respectively of the peak coefficient

$$k_{pk} = \sqrt{2}(1 - e^{-0.01/T_a}),$$

2) if the ratio of $R/X \geq 1/3$, by the replacement of X_{tot} with $Z_{tot} = \sqrt{R_{tot}^2 + X_{tot}^2}$ in the formula

$$I_{sc} = \frac{1}{Z_{tot}}. \tag{6.105b}$$

Finally, the influence of the load, such as big motors and high power composed loads, can be considered by equivalent parameters

$$X''_{Ld} = 0.35, \quad E''_{Ld} = 0.8. \tag{6.106}$$

A very rough approximation of the initial value of a subtransient s.c. can be made by

$$I''_{sc} = \frac{V_{F0}}{X''_{tot}}, \tag{6.107}$$

where V_{F0} is the voltage prior to switching at the fault point F, and (if the generator and/or system are represented by their subtransient reactances) X''_{tot} is the total subtransient reactance of the circuit up to the fault point.

6.6.2 Short-circuit power

The product of the initial subtransient s.c. current I''_{sc} and the rated voltage with the factor $\sqrt{3}$ gives the short-circuit power:

$$S''_{sc} = \sqrt{3} V_r I''_{sc}. \tag{6.108}$$

This power is used for characterizing the rate of the fault disturbance, which includes both the s.c. current and the voltage at the fault point. The s.c. power is primarily used for determining the breaking capacity, which is given in MVA, and is included in the information which manufacturers of circuit breakers are required to provide.

Sometimes the short-circuit power is given for the s.c. current at the switching instant, i.e., at the moment that the circuit breaker opens its contacts, rather than at $t = 0$, and which is called the breaking current.

Example 6.13

In the network shown in Fig. 6.48(a), find the peak and r.m.s value of the s.c. current when the three-pole fault occurs at points F_1 and F_2.

Solution

Assuming $S_B = 100$ MVA, the p.u. reactances will be as shown in Fig. 6.48(b):

$$X_1 = 0.4 \cdot 140 \frac{100}{161^2} = 0.22, \quad X_2 = 0.105 \frac{100}{50} = 0.21, \quad X_3 = 0.04 \frac{5.24}{0.5} = 0.42,$$

where

$$I_B = \frac{100}{\sqrt{3} \cdot 11} = 5.24 \text{ kA} \quad \text{and} \quad X_4 = 0.08 \cdot 2 \frac{100}{11^2} = 1.45.$$

The s.c current at point F_1 will be

$$I_{sc1} = \frac{1}{(0.22 + 0.21)} = 2.32, \quad \text{or} \quad I_{sc1} = 2.32 \cdot 5.24 = 12.2 \text{ kA}.$$

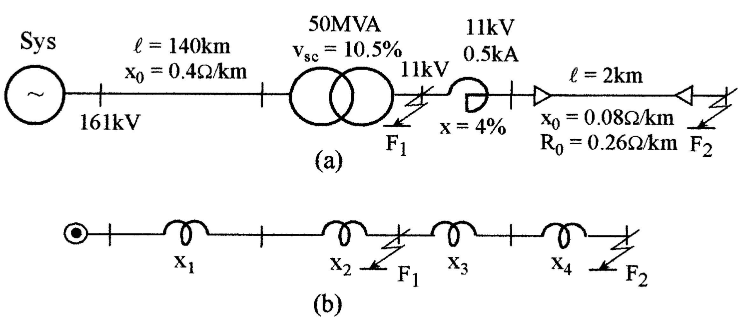


Figure 6.48 A circuit diagram for Example 6.13 (a) and its equivalent circuit (b).

Assuming $T_a = 0.05$ s ($k_{pk} = 1.8$), we have (see equation 6.103)

$$i_{pk} = 1.8 \cdot \sqrt{2} \cdot 12.2 = 31.1 \text{ kA},$$

and r.m.s. value is

$$I_{pk} = \sqrt{1 + 2(1.8 - 1)^2} I_{sc} = 1.52 \cdot 12.2 = 18.5 \text{ kA}.$$

For the short-circuiting at point F_2 we have

$$X_{tot} = 0.43 + 0.42 + 1.45 = 2.23,$$

and

$$I_{sc2} = \frac{1}{2.33} = 0.429, \quad \text{or} \quad I_{sc2} = 0.429 \cdot 5.24 = 2.45 \text{ kA}.$$

The peak values with a 1.8 peak coefficient will be

$$i_{pk} = 1.8 \cdot \sqrt{2} \cdot 2.45 = 6.25 \text{ kA} \quad \text{and} \quad I_{pk} = 1.52 \cdot 2.45 = 3.72 \text{ kA}.$$

However, the resistance of the cable is relatively high:

$$R_{tot} = 0.260 \cdot 2 \frac{100}{11^2} = 0.43,$$

and by taking it into consideration we can calculate the s.c. current more precisely. Thus, the time constant of the aperiodic component will be

$$T_a = \frac{X_{tot}}{\omega R_{tot}} = \frac{2.33}{314 \cdot 0.43} = 0.02,$$

and

$$k_{pk} = 1 + e^{-0.01/0.02} = 1.6.$$

Hence,

$$i_{pk} = 1.6 \cdot \sqrt{2} \cdot 2.45 = 5.54 \text{ kA},$$

and

$$I_{pk} = \sqrt{1 + 2(1.6 - 1)^2} \cdot 2.45 = 3.21 \text{ kA}.$$

If we now consider that the transformer is connected straight to the system (by neglecting the transmission line), the s.c. current at point F_1 will increase by 35%, but at point F_2 only by 7%.

Example 6.14

The power network shown in Fig. 6.49a includes a generator, synchronous condenser (SC) and three compound loads. Taking into consideration SC and all the loads, (a) find the first moment s.c. current and its peak value and (b) calculate the short-circuit power.

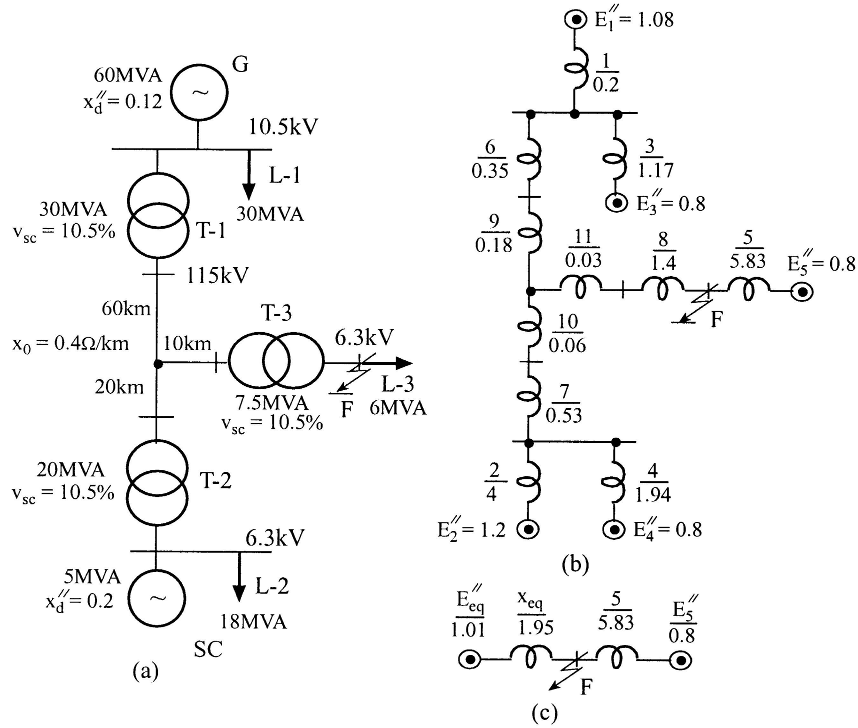


Figure 6.49 A network diagram for Example 6.14 (a), its equivalent circuit (b) and simplified circuit (c).

Solution

Assuming $S_B = 100$ MVA, the p.u. reactances of all the circuit elements are calculated and shown in Fig. 6.49(b). The loads are represented by $X_{ld}'' = 0.35$ and $E_{ld}'' = 0.8$. The given circuit is then simplified in a few obvious steps:

$$X_{12} = 0.2//1.17 = 0.18, \quad E_6 = \frac{1.08 \cdot 1.17 + 0.8 \cdot 0.2}{1.17 + 0.2} = 1.04,$$

$$X_{13} = 0.18 + 0.35 + 0.18 = 0.71, \quad X_{14} = 1.94//4 = 1.31,$$

$$E_7 = \frac{1.2 \cdot 1.94 + 0.8 \cdot 4}{1.94 + 4} = 0.93, \quad X_{15} = 1.31 + 0.53 + 0.06 = 1.9,$$

$$X_{16} = 1.9//0.71 = 0.52,$$

$$X_{eq} = 0.52 + 0.03 + 1.4 = 1.95, \quad E_{eq} = \frac{1.04 \cdot 1.9 + 0.93 \cdot 0.71}{1.9 + 0.71} = 1.01.$$

The s.c. current flowing from the system through transformer T_3 is

$$I_S'' = \frac{1.01}{1.95} = 0.52 \quad \text{or in amperes} \quad I_S'' = 0.52 \cdot 9.2 = 4.8 \text{ kA},$$

where $I_B = (100/\sqrt{3}\cdot 6.3) = 9.2$ kA is the basic current.

The s.c. current flowing from the load is

$$I''_{Ld} = \frac{0.8}{5.83}\,9.2 = 1.26 \text{ kA}.$$

Thus, the total short-circuit current is

$$I''_{sc} = I''_S + I''_{Ld} = 4.8 + 1.26 = 6.06 \text{ kA}.$$

The total peak current will be

$$i_{pk} = 1.8\sqrt{2}\cdot 4.8 + \sqrt{2}\cdot 1.26 = 14.0 \text{ kA}.$$

Note that the s.c. current from the load can be calculated straightforwardly, without referring its parameters to the basic quantities. Indeed,

$$I''_{Ld} = \frac{0.8}{0.35}\,I_r = 2.29\,\frac{6}{\sqrt{3}\cdot 6.3} = 1.26 \text{ kA},$$

which is the same as it was calculated previously.

(b) With the first moment s.c. current the short-circuit power at the load L-3 bus is

$$S''_{sc} = \sqrt{3}V_{L3}I''_{sc} = \sqrt{3}\cdot 6.3\cdot 6.06 = 66.1 \text{ MVA}.$$

6.7 METHOD OF SYMMETRICAL COMPONENTS FOR UNBALANCED FAULT ANALYSIS

Earlier we mentioned that truly balanced three-phase systems exist only in theory. Actually many real systems are very nearly balanced and for practical purposes can be analyzed as balanced systems, i.e., on per-phase basis. However, sometimes the degree of unbalance cannot be neglected. Such cases may occur during emergency conditions like unsymmetrical faults (one- or two-phase short-circuiting), unbalanced loads, open conductors, unsymmetrical operation of rotating machines, etc. Of course straightforward methods for the application of Kirchhoff's laws might be used for such three-phase circuit analysis. However, such cases may be calculated without difficulty by an indirect method in which the unbalanced or unsymmetrical system is replaced by equivalent component systems, each of which is symmetrical and balanced. The calculation of the currents and voltages in these symmetrical systems is a simple process (since it can be provided on a per-phase basis), and the superposition or vector addition of these currents and voltages is then easily carried out to obtain the actual results for the original unbalanced system.

This method, called the **method of symmetrical components**, was proposed by Charles L. Fortescue in 1913[(*)] and was developed by C.F. Wagner and R.D.

[(*)]This method was published by Fortescue, C.L. (1918), "Method of symmetrical co-ordinates applied to the solution of polyphase networks", *AIEE Transactions*, 37.

Evans later to apply it to the analysis of unsymmetrical faults in three-phase systems. Today, the symmetrical component method is widely used in studying unbalanced systems. Many electrical devices have been developed to operate on the basis of the concept of symmetrical components. In this section we shall briefly introduce this method followed by a few examples of its application.

6.7.1 Principle of symmetrical components

(a) *Positive-, negative- and zero-sequence systems*

Any unbalanced (unsymmetrical) three-phase system of phasors can be resolved into three balanced systems of phasors: (1) positive-sequence system, (2) negative-sequence system, and (3) zero-sequence system, as shown in Fig. 6.50, as an example of a set of three unbalanced voltages.

The **positive-sequence system** is represented by a balanced system of phasors

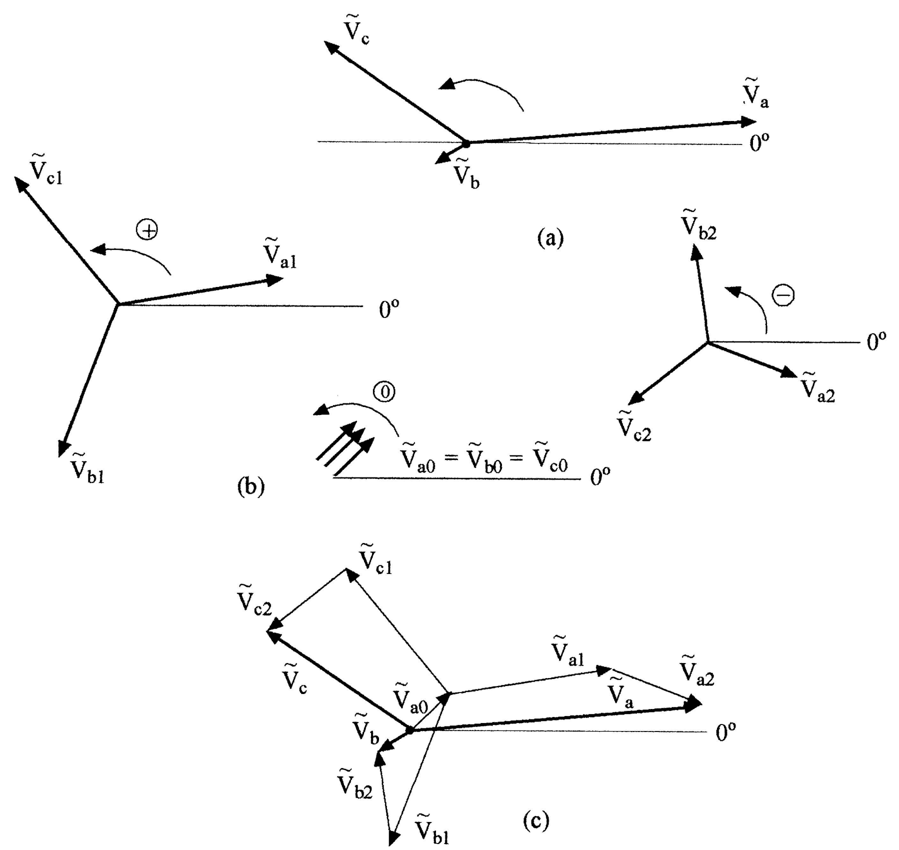


Figure 6.50 The symmetrical components of three unbalanced voltages: given system of unbalanced phasors (a); positive (+), negative (−) and zero (0) sequence components (b) and the graphical addition of the symmetrical components to obtain the given set of unbalanced phasors (c).

having the same phase sequence as the original unbalanced system. This set consists of three-phase currents and three-phase line-to-neutral voltages supplied by the power system generator and therefore of **positive** or counterclockwise **phase rotation**. Thus, the phasors of the positive-sequence system are equal in magnitude and displaced from each other by 120°, as shown by set "+" in Fig. 6.50(b).

The **negative-sequence system** is represented by a balanced system of phasors having the opposite phase sequence from the original system and, therefore, a **negative phase rotation**. The phasors of the negative-sequence system are also equal in magnitude and displaced from each other by 120°, set "−" in Fig. 6.50(b). Thus, if a positive sequence is *abc*, a negative sequence will be *acb*.

The **zero-sequence system** is represented by three single phasors that are equal in magnitude and are in phase, as shown by set "0" in Fig. 6.50(b). Note that the zero-sequence system is also a set of rotating phasors.

Using subscripts 0, 1 and 2 to denote the zero, positive and negative sequences we may write

$$\begin{aligned}\tilde{V}_a &= \tilde{V}_{a1} + \tilde{V}_{a2} + \tilde{V}_{a0}\\ \tilde{V}_b &= \tilde{V}_{b1} + \tilde{V}_{b2} + \tilde{V}_{b0}\\ \tilde{V}_c &= \tilde{V}_{c1} + \tilde{V}_{c2} + \tilde{V}_{c0},\end{aligned} \tag{6.109}$$

i.e., three voltage phasors $\tilde{V}_a$, $\tilde{V}_b$, $\tilde{V}_c$ of an unbalanced set can be expressed in terms of their symmetrical components as shown in Fig. 6.50(c).

With a **unit phasor operator** a ($a = \angle 120°$; $a^2 = \angle 240°$; $a^3 = 1$; $a^{-1} = \angle -120°$, etc.) the positive-sequence set can be designated

$$\tilde{V}_{a1} = V_1 \angle \psi_{a1}, \quad \tilde{V}_{b1} = a^2 \tilde{V}_{a1}, \quad \tilde{V}_{c1} = a\tilde{V}_{a1} \tag{6.109a}$$

$$\tilde{V}_{a2} = V_2 \angle \psi_{a2}, \quad \tilde{V}_{b2} = a\tilde{V}_{a2}, \quad \tilde{V}_{c2} = a^2 \tilde{V}_{a2} \tag{6.109b}$$

$$\tilde{V}_{a0} = V_0 \angle \psi_{a0}, \quad \tilde{V}_{b0} = \tilde{V}_{a0}, \quad \tilde{V}_{c0} = \tilde{V}_{a0}. \tag{6.109c}$$

Substituting the above equations into equation 6.109, the phase voltages can be expressed in terms of the *sequence voltages* as

$$\begin{aligned}\tilde{V}_a &= \tilde{V}_{a0} + \tilde{V}_{a1} + \tilde{V}_{a2}\\ \tilde{V}_b &= \tilde{V}_{a0} + a^2\tilde{V}_{a1} + a\tilde{V}_{a2}\\ \tilde{V}_c &= \tilde{V}_{a0} + a\tilde{V}_{a1} + a^2\tilde{V}_{a2},\end{aligned} \tag{6.110a}$$

and in matrix form as

$$\begin{bmatrix}\tilde{V}_a\\ \tilde{V}_b\\ \tilde{V}_c\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\ 1 & a^2 & a\\ 1 & a & a^2\end{bmatrix}\begin{bmatrix}\tilde{V}_{a0}\\ \tilde{V}_{a1}\\ \tilde{V}_{a2}\end{bmatrix} \tag{6.110b}$$

or

$$[V_{abc}] = [a][\tilde{V}_{012}], \tag{6.110c}$$

where

$$[\tilde{V}_{abc}] = \begin{bmatrix} \tilde{V}_a \\ \tilde{V}_b \\ \tilde{V}_c \end{bmatrix}, \quad [\tilde{V}_{012}] = \begin{bmatrix} \tilde{V}_{a0} \\ \tilde{V}_{a1} \\ \tilde{V}_{a2} \end{bmatrix}$$

and the *operator matrix* **a** is

$$\mathbf{a} = [a] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}.$$

To shorten the writing of the symmetrical components, later on we will ignore the subscript "a" for phase a, which means that the symmetrical components $\tilde{V}_0$, $\tilde{V}_1$ and $\tilde{V}_2$ and $\tilde{I}_0$, $\tilde{I}_1$ and $\tilde{I}_2$ belong to the phase a voltages and currents.

Equations 6.110 are also known as the **synthesis equations** since they synthesize the set of unbalanced phasors from three sets of symmetrical components. These equations may be solved to find the symmetrical components of a known three-phase system of unbalanced voltages or currents:

$$[\tilde{V}_{012}] = [a]^{-1}[\tilde{V}_{abc}] \tag{6.111}$$

or

$$\begin{bmatrix} \tilde{V}_{a0} \\ \tilde{V}_{a1} \\ \tilde{V}_{a2} \end{bmatrix} = \frac{1}{\det[a]} \begin{bmatrix} a^4 - a^2 & a - a^2 & a - a^2 \\ a - a^2 & a^2 - 1 & 1 - a \\ a - a^2 & 1 - a & a^2 - 1 \end{bmatrix} \begin{bmatrix} \tilde{V}_a \\ \tilde{V}_b \\ \tilde{V}_c \end{bmatrix}. \tag{6.112}$$

By performing the appropriate computations with the phasor operators in equation 6.112, as with complex numbers, we may simplify the inverse of matrix $[a]$ as follows

$$[a]^{-1} = \frac{1}{3\sqrt{3}j} \begin{bmatrix} \sqrt{3}j & \sqrt{3}j & \sqrt{3}j \\ \sqrt{3}j & \sqrt{3}\angle -150^\circ & \sqrt{3}\angle -30^\circ \\ \sqrt{3}j & \sqrt{3}\angle -30^\circ & \sqrt{3}\angle -150^\circ \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}$$

where $\det[a] = 3(a - a^2) = 3\sqrt{3}j$. Therefore,

$$\begin{bmatrix} \tilde{V}_{a0} \\ \tilde{V}_{a1} \\ \tilde{V}_{a2} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} \tilde{V}_a \\ \tilde{V}_b \\ \tilde{V}_c \end{bmatrix}, \tag{6.113}$$

i.e. the sequence voltages can be expressed in terms of phase voltages as

$$\tilde{V}_{a0} = \frac{1}{3}(\tilde{V}_a + \tilde{V}_b + \tilde{V}_c) \tag{6.114a}$$

$$\tilde{V}_{a1} = \frac{1}{3}(\tilde{V}_a + a\tilde{V}_b + a^2\tilde{V}_c) \tag{6.114b}$$

$$\tilde{V}_{a2} = \frac{1}{3}(\tilde{V}_a + a^2\tilde{V}_b + a\tilde{V}_c). \tag{6.114c}$$

These equations are also known as the **analysis equations**.

Of course, the synthesis and analysis equations can also be used for current phasors

$$[\tilde{I}_{abc}] = [a][\tilde{I}_{012}] \tag{6.115a}$$

$$[\tilde{I}_{012}] = [a]^{-1}[\tilde{I}_{abc}]. \tag{6.115b}$$

Example 6.15

Determine the symmetrical components for the line voltages $\tilde{V}_a = 220\angle 0^\circ$ V, $\tilde{V}_b = 200\angle -150^\circ$ V, and $\tilde{V}_c = 180\angle 120^\circ$ V, Fig. 6.51(a), and construct their phasor diagram.

Solution

In accordance with equation 6.114a we have

$$\tilde{V}_{a0} = \tilde{V}_{b0} = \tilde{V}_{c0} = \frac{1}{3}(\tilde{V}_a + \tilde{V}_b + \tilde{V}_c)$$

$$= \frac{1}{3}(220\angle 0^\circ + 200\angle -150^\circ + 180\angle 120^\circ)$$

$$= -14.3 + j18.6 = 23.5\angle 127.7^\circ \text{ V}.$$

Applying equation 6.114b, the positive components are

$$\tilde{V}_{a1} = \frac{1}{3}(\tilde{V}_a + a\tilde{V}_b + a^2\tilde{V}_c)$$

$$= \frac{1}{3}[220\angle 0^\circ + (1\angle 120^\circ)(200\angle -150^\circ) + (1\angle 240^\circ)(180\angle 120^\circ)]$$

$$= 191.1 - j33.3 = 194.0\angle -9.9^\circ \text{ V},$$

and

$$\tilde{V}_{b1} = a^2\tilde{V}_{a1} = (1\angle 240^\circ)(194.0\angle -9.9^\circ) = 194.0\angle -129.9^\circ \text{ V}$$

$$\tilde{V}_{c1} = a\tilde{V}_{a1} = (1\angle 120^\circ)(194.0\angle -9.9^\circ) = 194.0\angle 110.1^\circ \text{ V}.$$

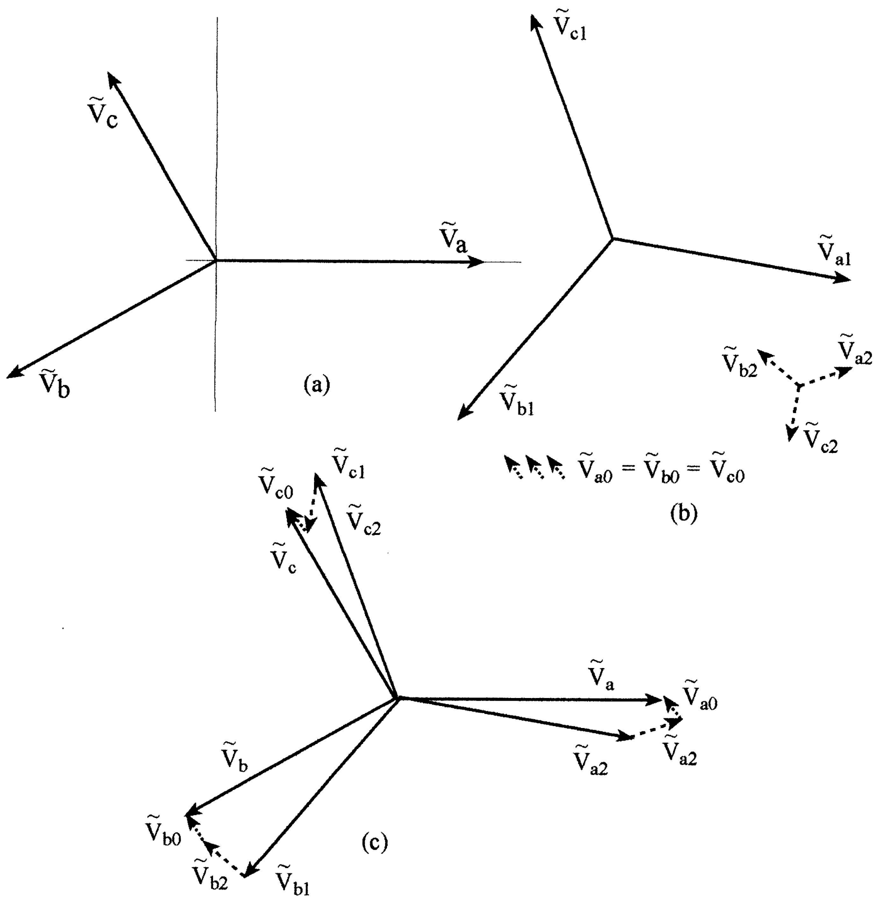


Figure 6.51 Three unbalanced voltages (a), their symmetrical components (b) and the original phasor system as composed of the symmetrical components (c).

Applying equation 6.114c, the negative components are

$$\tilde{V}_{a2} = \frac{1}{3}(\tilde{V}_a + a^2\tilde{V}_b + a\tilde{V}_c)$$

$$= \frac{1}{3}[220\angle 0^\circ + (1\angle 240^\circ)(200\angle -150^\circ) + (1\angle 120^\circ)(180\angle 120^\circ)]$$

$$= 43.3 + j14.8 = 45.8\angle 18.9^\circ \text{ V},$$

and

$$\tilde{V}_{b2} = a\tilde{V}_{a2} = (1\angle 120^\circ)(45.8\angle 18.9^\circ) = 45.8\angle 138.9^\circ \text{ V},$$

$$\tilde{V}_{c2} = a^2\tilde{V}_{a2} = (1\angle 240^\circ)(45.8\angle 18.9^\circ) = 45.8\angle -101.1^\circ \text{ V}.$$

Using the above results the phasor diagrams for positive and negative symmetrical components are constructed in Fig. 6.51(b). The diagram in Fig. 6.51(c) shows that the original phasor system is obtained when the symmetrical components are compounded either numerically or also graphically.

In the general case of an unsymmetrical three-phase, three-wire system, i.e. when the neutral line is absent, the vector sum of three line currents is also always (like sum of line voltages) zero. Therefore, the zero-sequence components for these unbalanced currents as well as for line voltages are zero. Furthermore, we may conclude that in a four-wire system, since the neutral-wire current in every case is the sum of line currents, the zero-sequence components, equation 6.114a, are equal to one-third of this current.

In the next example we shall show the resolving of an unbalanced set of phase voltages into symmetrical components.

Example 6.16

A synchronous generator, which is connected to an infinite busbar system Fig. 6.52(a), is subjected to a short-circuit line-to-line fault at its terminals. The generator's p.u. short-circuit currents are found (see further on) to be $\tilde{I}_a = -2.1$, $\tilde{I}_b = 3.37\angle -71.8^\circ$ and $\tilde{I}_c = 3.37\angle 71.8^\circ$. Find the symmetrical components of these currents and construct their phasor diagram.

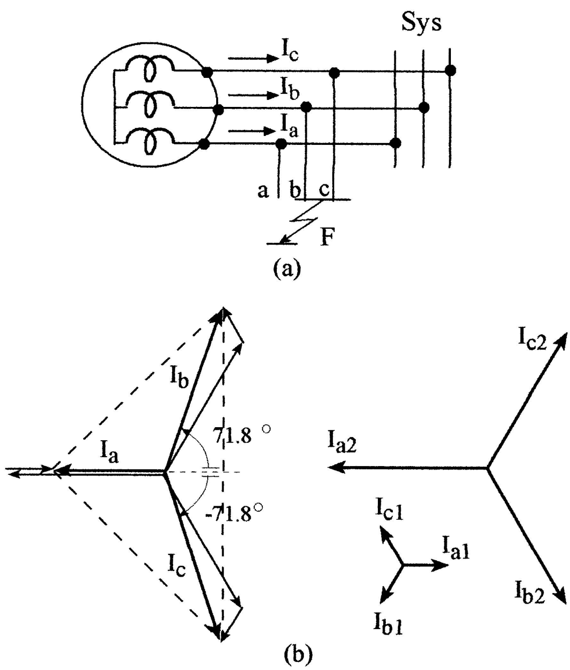


Figure 6.52 A circuit diagram for Example 6.15 (a) and the phasor diagram of the current sequences (b).

Solution

In accordance with equation 4.114a we obtain

$$\tilde{I}_{a0} = \tilde{I}_{b0} = \tilde{I}_{c0} = \frac{1}{3}(-2.1 + 3.37\angle -71.8^\circ + 3.37\angle 71.8^\circ) = 0.$$

This result should be expected since the sum of three-phase system currents (without a neutral line) is always zero (as the sum of three vectors, which form a triangle, see Fig. 6.52(b)).

In accordance with equation 4.114b we obtain the positive components:

$$\tilde{I}_{a1} = \frac{1}{3}[-2.1 + (1\angle 120^\circ)(3.37\angle -71.8^\circ) + (1\angle -120^\circ)(3.37\angle 71.8^\circ)] = 0.8$$

$$\tilde{I}_{b1} = a^2\tilde{I}_{a1} = 0.8\angle -120^\circ, \quad \tilde{I}_{c1} = a\tilde{I}_{a1} = 0.8\angle 120^\circ,$$

and in accordance with equation 4.114c, the negative components are

$$\tilde{I}_{a2} = \frac{1}{3}[-2.1 + (1\angle -120^\circ)(3.37\angle -71.8^\circ)$$

$$+ (1\angle 120^\circ)(3.37\angle 71.8^\circ)] = -2.9$$

$$\tilde{I}_{b2} = a\tilde{I}_{a2} = -2.9\angle 120^\circ, \quad \tilde{I}_{c2} = a^2\tilde{I}_{a2} = -2.9\angle -120^\circ.$$

In checking the results, we have $\tilde{I}_a = \tilde{I}_{a1} + \tilde{I}_{a2} = 0.8 - 2.9 = -2.1$. The phasor diagram of the currents is shown in Fig. 6.52(b).

If the set of line voltages is balanced, it is obvious that the negative-sequence for these voltages is zero; hence, the negative-sequence for the phase voltages will also be zero. That is, the set of unbalanced phase voltages, forming a balanced set of line voltages, resolves into positive- and zero-components, as shown, for example, in Fig. 6.53. As can be seen the negative-sequence voltage, $\tilde{V}_2$, is absent and each of the phase voltages is equal to the sum of the positive- and zero-sequences. Their values can then be easily found:

$$V_{a,ph} = V_1 + V_0$$

$$V_{b,ph} = V_{c,ph} = \sqrt{(V/2)^2 - [V/(2\cdot\sqrt{3}) - V_0]^2},$$

where V is the given line voltage.

(b) *Sequence impedances*

Consider first the circuit of Fig. 6.54(a), which represents a three-phase, three-wire, Y-connected, generally unbalanced system, i.e., $Z_a \neq Z_b \neq Z_c$. The matrix equation for phase voltages, across these three impedances, will be

$$\begin{bmatrix} \tilde{V}_a \\ \tilde{V}_b \\ \tilde{V}_c \end{bmatrix} = \begin{bmatrix} Z_a & 0 & 0 \\ 0 & Z_b & 0 \\ 0 & 0 & Z_c \end{bmatrix} \begin{bmatrix} \tilde{I}_a \\ \tilde{I}_b \\ \tilde{I}_c \end{bmatrix} \tag{6.116a}$$

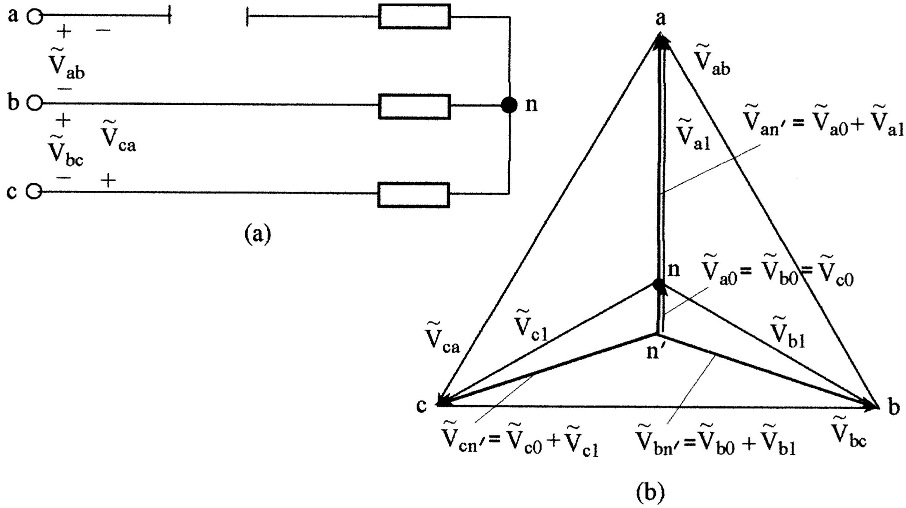


Figure 6.53 A faulted network with balanced line voltages but unbalanced phase voltages (a) and the phasor diagram of the symmetrical components (b).

or

$$[\tilde{V}_{abc}] = [Z_{abc}][\tilde{I}_{abc}]. \tag{6.116b}$$

Here both the voltages and currents are unsymmetrical. Multiplying both sides of equation 6.116b by $[a]^{-1}$ and also substituting equation 6.115a, we obtain

$$[a]^{-1}[\tilde{V}_{abc}] = [a]^{-1}[Z_{abc}][a][\tilde{I}_0\tilde{I}_1\tilde{I}_2],$$

or, with equation 6.111,

$$[\tilde{V}_{012}] = [Z_{012}][\tilde{I}_{012}], \tag{6.117}$$

where the *matrix transformation* is defined as

$$[Z_{012}] = [a]^{-1}[Z_{abc}][a]. \tag{6.118a}$$

Performing the matrix multiplication and upon simplification this transformation results in a *sequence impedance matrix* of an unbalanced load

$$[Z_{012}] = \begin{bmatrix} Z_{00} & Z_{01} & Z_{02} \\ Z_{10} & Z_{11} & Z_{12} \\ Z_{20} & Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} Z_0 & Z_2 & Z_1 \\ Z_1 & Z_0 & Z_2 \\ Z_2 & Z_1 & Z_0 \end{bmatrix} \tag{6.118b}$$

where by definition the **zero-sequence impedance** is

$$Z_0 = \frac{1}{3}(Z_a + Z_b + Z_c) \quad (Z_{00} = Z_{11} = Z_{22} = Z_0), \tag{6.119a}$$

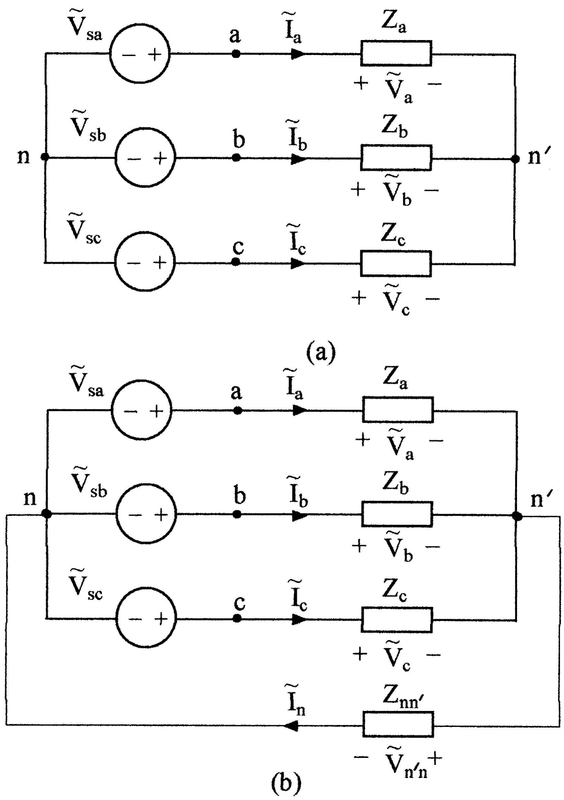


Figure 6.54 Three-phase, three-wire unbalanced system ($\tilde{I}_n = 0$) (a), three-phase, four-wire unbalanced system ($\tilde{I}_n \neq 0$) (b).

the **positive-sequence impedance** is

$$Z_1 = \frac{1}{3}(Z_a + aZ_b + a^2 Z_c) \quad (Z_{02} = Z_{10} = Z_{21} = Z_1) \tag{6.119b}$$

and the **negative-sequence impedance** is

$$Z_2 = \frac{1}{3}(Z_a + a^2 Z_b + aZ_c) \quad (Z_{01} = Z_{12} = Z_{20} = Z_2). \tag{6.119c}$$

These component impedances have little physical significance but they are useful in a general mathematical formulation of symmetrical-component theory. It should be noted in this respect that the real parts of the component impedances may possess negative signs even though the real parts of Z_a, Z_b and Z_c are all positive.

Providing the matrix multiplication in equation 6.117 yields

$$\begin{aligned}\tilde{V}_0 &= Z_0\tilde{I}_0 + Z_2\tilde{I}_1 + Z_1\tilde{I}_2\\ \tilde{V}_1 &= Z_1\tilde{I}_0 + Z_0\tilde{I}_1 + Z_2\tilde{I}_2\\ \tilde{V}_2 &= Z_2\tilde{I}_0 + Z_1\tilde{I}_1 + Z_0\tilde{I}_2.\end{aligned} \tag{6.120}$$

Note that the sum of the sequence indexes (0, 1, or 2) of Z and $\tilde{I}$ in the voltage drops $Z_i\tilde{I}_j$ in each of these equations gives the index of the voltage-sequence to which these voltage drops belong. Therewith, $2+1=3$ is considered as 0 $(3-3=0)$ and $2+2=4$ is considered as 1 $(4-3=1)$, since there are only three sequences (0, 1 and 2). This simple rule is known as the **sequence rule**.

Recall that the above symmetrical components (equation 6.120) are of the a-phase voltage, i.e.,

$$\tilde{V}_a = \tilde{V}_0 + \tilde{V}_1 + \tilde{V}_2.$$

Applying, for example, the second equation of equation 6.120 to phase b and making appropriate substitutions we may write

$$\begin{aligned}\tilde{V}_b &= Z_{b1}\tilde{I}_{b0} + Z_{b0}\tilde{I}_{b1} + Z_{b2}\tilde{I}_{b2}\\ &= (Z_{a1}\angle -120^\circ)\tilde{I}_{a0} + Z_{a0}(I_{a1}\angle -120^\circ) + (Z_{a2}\angle 120^\circ)(I_{a2}\angle 120^\circ)\\ &= (Z_1\tilde{I}_0 + Z_0\tilde{I}_1 + Z_2\tilde{I}_2)\angle -120^\circ = \tilde{V}_{a1}\angle -120^\circ.\end{aligned}$$

This result shows that $\tilde{V}_{b1}$ is equal in magnitude to $\tilde{V}_{a1}$ and 120° displaced behind $\tilde{V}_{a1}$, as, of course, it should be for a positive-sequence system. An opportunity is given to the reader to check in the above manner that

$$\tilde{V}_{c1} = \tilde{V}_{a1}\angle 120^\circ \quad \text{and} \quad \tilde{V}_{c0} = \tilde{V}_{b0} = \tilde{V}_{a0}.$$

The sequence currents can be found by solving equation 6.117, i.e.,

$$[\tilde{I}_{012}] = [Y_{012}][\tilde{V}_{012}], \tag{6.121}$$

where $[Y_{012}]$ is the associated *sequence admittance matrix*

$$[Y_{012}] = [Z_{012}]^{-1}.$$

This sequence admittance matrix may be found in the same manner as the impedance sequence matrix (equation 6.118a), i.e.,

$$[Y_{012}] = [a^{-1}][Y_{abc}][a]. \tag{6.122}$$

Indeed, applying the reversal rule to find the inverse of the product of the matrixes, we obtain

$$[Z_{012}]^{-1} = ([a]^{-1}[Z_{abc}][a])^{-1} = [a]^{-1}[Z_{abc}]^{-1}[a],$$

where

$$[Z_{abc}]^{-1} = [Y_{abc}] = \begin{bmatrix} Y_a & 0 & 0\\ 0 & Y_b & 0\\ 0 & 0 & Y_c\end{bmatrix}$$

and $Y_a = 1/Z_a,\ Y_b = 1/Z_b,\ Y_c = 1/Z_c$.

Therefore, similar to equation 6.119 we observe that:

the **zero-sequence admittance** is

$$Y_0 = \frac{1}{3}(Y_a + Y_b + Y_c), \tag{6.123a}$$

the **positive-sequence admittance** is

$$Y_1 = \frac{1}{3}(Y_a + aY_b + a^2 Y_c), \tag{6.123b}$$

and the **negative-sequence admittance** is

$$Y_2 = \frac{1}{3}(Y_a + a^2 Y_b + aY_c). \tag{6.123c}$$

When the applied voltage sequence-components are known, the sequence-components of the a-phase current may be readily found according to equation 6.121. Thus,

$$\begin{aligned} \tilde{I}_0 &= Y_0\tilde{V}_0 + Y_2\tilde{V}_1 + Y_1\tilde{V}_2 \\ \tilde{I}_1 &= Y_1\tilde{V}_0 + Y_0\tilde{V}_1 + Y_2\tilde{V}_2 \\ \tilde{I}_2 &= Y_2\tilde{V}_0 + Y_1\tilde{V}_1 + Y_0\tilde{V}_2. \end{aligned} \tag{6.124}$$

Example 6.17

Let the line-to-line voltages and the phase impedances of the Y-connected, three-wire, load, as shown in Fig. 6.54(a), be as follows:

$$\tilde{V}_{ab} = 200\angle 0^\circ\,\mathrm{V}, \quad \tilde{V}_{bc} = 141.4\angle -135^\circ\,\mathrm{V}, \quad \tilde{V}_{ca} = 141.4\angle 135^\circ\,\mathrm{V}$$
$$Z_a = 6\,\Omega, \quad Z_b = 6\angle -30^\circ\,\Omega, \quad Z_c = j12\,\Omega.$$

Find the a-phase current symmetrical components.

Solution

The phase admittances are

$$Y_a = 0.1667\,\mathrm{S}, \quad Y_b = 0.1667\angle 30^\circ\,\mathrm{S}, \quad Y_c = -j0.08333\,\mathrm{S}.$$

Then, employing equation 6.123, the sequence-component admittances are

$$Y_0 = \frac{1}{3}(0.1667 + 0.1443 + j0.08333 - j0.08333) = 0.1037 \text{ S}$$

$$Y_1 = \frac{1}{3}[0.1667 + (1\angle 120^\circ)(0.1667\angle 30^\circ) + (1\angle 240^\circ)(0.08333\angle -90^\circ)]$$

$$= 0.04485\angle 111.71^\circ \text{ S}$$

$$Y_2 = \frac{1}{3}[0.1667 + (1\angle 240^\circ)(0.1667\angle 30^\circ) + (1\angle 120^\circ)(0.08333\angle -90^\circ)]$$

$$= 0.08987\angle -27.63 \text{ S}.$$

Resolving the above line-to-line voltages into symmetrical components yields

$$\tilde{V}_{ab0} = \frac{1}{3}[200\angle 0^\circ + 141.4\angle -135^\circ + 141.4\angle 135^\circ] = 0$$

$$\tilde{V}_{ab1} = \frac{1}{3}[200\angle 0^\circ + 141.4(\angle -135^\circ + \angle 120^\circ) + 141.4(\angle 135^\circ + \angle 240^\circ)]$$

$$= 157.7 \text{ V}$$

$$\tilde{V}_{ab2} = \frac{1}{3}[200\angle 0^\circ + 141.4(\angle -135^\circ + \angle 240^\circ) + 141.4(\angle 135^\circ + \angle 120^\circ)]$$

$$= 42.3 \text{ V}.$$

The positive- and negative-components of the phase voltages are

$$\tilde{V}_{a1} = \tilde{V}_1 = \frac{157.7}{\sqrt{3}}\angle -30^\circ = 91.1\angle -30^\circ \text{ V}$$

$$\tilde{V}_{a2} = \tilde{V}_2 = \frac{42.3}{\sqrt{3}}\angle 30^\circ = 24.4\angle 30^\circ \text{ V}.$$

Note that even if $\tilde{I}_0 = 0$ (since the neutral wire is absent) $\tilde{V}_0$ will possess a finite value, which may be calculated in accordance with the first equation of 6.124:

$$\tilde{V}_0 = \frac{1}{Y_0}(-Y_2\tilde{V}_1 - Y_1\tilde{V}_2)$$

$$= \frac{1}{0.1037}[-(0.08987\angle -27.63^\circ)(91.1\angle -30^\circ)$$

$$-(0.04485\angle 111.71^\circ)(24.4\angle 30^\circ)] = 69.08\angle 119.47^\circ \text{ V}.$$

Now, the positive- and negative-sequence currents may be calculated in accordance with equation 6.124

$$\tilde{I}_1 = (0.4485\angle 111.71^\circ)(69.08\angle 119.47^\circ) + 0.1037(91.1\angle -30^\circ)$$

$$+(0.08987\angle-27.63^\circ)(24.4\angle 30^\circ) = 8.42 - j7.04 = 10.98\angle-39.9^\circ \text{ A}$$

$$\tilde{I}_2 = (0.08987\angle-37.63^\circ)(69.08\angle 119.47^\circ) + (0.04485\angle 111.71^\circ)(91.1\angle-30^\circ)$$
$$+0.1037(24.4\angle 30^\circ) = 2.58 - j11.51 = 11.80\angle 77.36^\circ \text{ A.}$$

Note that for a balanced load, i.e., $Z_a = Z_b = Z_c = Z_L$, the positive- and negative-sequence impedances are zero and $Z_0 = Z_L$. Thus,

$$\begin{bmatrix}\tilde{V}_0\\ \tilde{V}_1\\ \tilde{V}_2\end{bmatrix} = \begin{bmatrix}Z_L & 0 & 0\\ 0 & Z_L & 0\\ 0 & 0 & Z_L\end{bmatrix}\begin{bmatrix}\tilde{I}_0\\ \tilde{I}_1\\ \tilde{I}_2\end{bmatrix}. \tag{6.125a}$$

This matrix equation indicates that there is no mutual coupling among the three sequences and it can be separated into three independent equations

$$\tilde{V}_0 = Z_L\tilde{I}_0, \quad \tilde{V}_1 = Z_L\tilde{I}_1, \quad \tilde{V}_2 = Z_L\tilde{I}_2. \tag{6.125b}$$

Consider next the circuit of Fig. 6.54(b), in which, for more generality, a neutral wire is represented by the impedance $Z_{nn'}$. In this case the matrix equation for phase voltages (equation 6.116a) shall be written as

$$\begin{bmatrix}\tilde{V}_a\\ \tilde{V}_b\\ \tilde{V}_c\end{bmatrix} = \begin{bmatrix}Z_a & 0 & 0\\ 0 & Z_b & 0\\ 0 & 0 & Z_c\end{bmatrix}\begin{bmatrix}\tilde{I}_a\\ \tilde{I}_b\\ \tilde{I}_c\end{bmatrix} + \begin{bmatrix}Z_{nn'}\\ Z_{nn'}\\ Z_{nn'}\end{bmatrix}[\tilde{I}_n].$$

Substituting equations 6.110c and 6.115a into this equation, and since $\tilde{I}_n = 3\tilde{I}_0$, we obtain

$$[a][\tilde{V}_{012}] = [Z_{abc}][a][\tilde{I}_{012}] + [Z_{nn'}][3\tilde{I}_0].$$

Performing the matrix multiplication and upon simplification this equation becomes

$$[a]\begin{bmatrix}\tilde{V}_0\\ \tilde{V}_1\\ \tilde{V}_2\end{bmatrix} = \begin{bmatrix}Z_a + 3Z_{nn'} & Z_a & Z_a\\ Z_b + 3Z_{nn'} & a^2Z_b & aZ_b\\ Z_c + 3Z_{nn'} & aZ_c & a^2Z_c\end{bmatrix}\begin{bmatrix}\tilde{I}_0\\ \tilde{I}_1\\ \tilde{I}_2\end{bmatrix}.$$

Multiplying both sides of this equation by $[a]^{-1}$ yields

$$[\tilde{V}_{012}] = [Z'_{012}][\tilde{I}_0\tilde{I}_1\tilde{I}_2],$$

where the sequence-impedance matrix can be expressed as

$$[Z'_{012}] = \begin{bmatrix}Z_0 + 3Z_{nn'} & Z_2 & Z_1\\ Z_1 & Z_0 & Z_2\\ Z_2 & Z_1 & Z_0\end{bmatrix} \tag{6.126a}$$

and

$$Z_{00} = Z_0 + 3Z_{nn'}. \tag{6.126b}$$

Note again that for a balanced load, i.e., $Z_a = Z_b = Z_c = Z_L$, the positive- and negative-sequence impedances are zero, $Z_1 = Z_2 = 0$, and $Z_0 = Z_L$. Thus,

$$\begin{bmatrix} \tilde{V}_0 \\ \tilde{V}_1 \\ \tilde{V}_2 \end{bmatrix} = \begin{bmatrix} Z_L = 3Z_{nn'} & 0 & 0 \\ 0 & Z_L & 0 \\ 0 & 0 & Z_L \end{bmatrix} \begin{bmatrix} \tilde{I}_0 \\ \tilde{I}_1 \\ \tilde{I}_2 \end{bmatrix}. \tag{6.127a}$$

This matrix equation may also be separated into three independent equations

$$\tilde{V}_0 = (Z_L + 3Z_{nn'})\tilde{I}_0, \quad \tilde{V}_1 = Z_L\tilde{I}_1, \quad \tilde{V}_2 = Z_L\tilde{I}_2. \tag{6.127b}$$

In accordance with these equations three sequence networks of the balanced load may be drawn, as shown in Fig 6.55. Therefore, the positive- and negative-network impedances for a balanced load are equal to each other and simply equal the load phase impedance, but the zero-sequence network includes, in addition, the triplicate neutral line impedance. It is important to mention that with $Z_{nn'} = \infty$, i.e., for a three-wire system, the zero-sequence current $\tilde{I}_0$ is zero ($\tilde{I}_0 = 0$). It follows from the first equation in 6.127b.

It is worthwhile to note that the **impedances of sequence networks**, Fig 6.55, are not the same as the **sequence impedances** (equation 6.119) in equation 6.117. To make this sentence clearer we shall consider the sequence networks' impedances as the ratio of the voltage and the current of the same sequence, which

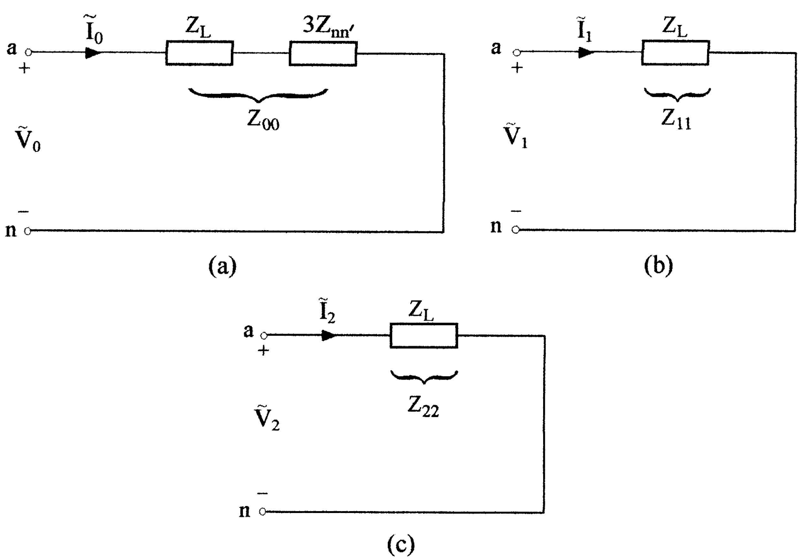


Figure 6.55 *Sequence networks*: zero-sequence network (a), positive sequence network (b) and negative-sequence network (c).

also are called the **impedances to positive-**, **negative-**, and **zero-sequence currents**. Thus, if a balanced system of positive-sequence voltages is applied to a balanced three-phase network, then the currents are also balanced and of positive-sequence. So, the ratio of the positive-sequence phase voltages and the appropriate positive-sequence phase currents gives the *positive-sequence network impedance*

$$Z_{11}=\frac{\tilde{V}_{a1}}{\tilde{I}_{a1}}=\frac{\tilde{V}_{b1}}{\tilde{I}_{b1}}=\frac{\tilde{V}_{c1}}{\tilde{I}_{c1}}. \tag{6.128a}$$

In a similar manner we shall define the *negative-* and *zero-sequence network impedances*

$$Z_{22}=\frac{\tilde{V}_{a2}}{\tilde{I}_{a2}}=\frac{\tilde{V}_{b2}}{\tilde{I}_{b2}}=\frac{\tilde{V}_{c2}}{\tilde{I}_{c2}}, \tag{6.128b}$$

and, since $\tilde{V}_{a0}=\tilde{V}_{b0}=\tilde{V}_{c0}=\tilde{V}_0$ and $\tilde{I}_{a0}=\tilde{I}_{b0}=\tilde{I}_{c0}=\tilde{I}_0$.

$$Z_{00}=\frac{\tilde{V}_0}{\tilde{I}_0}. \tag{6.128c}$$

In other words, positive-sequence currents flowing in a balanced network produce **only** positive-sequence voltage drops, negative-sequence currents will produce **only** negative-sequence voltage drops and zero-sequence currents will produce only zero-sequence voltage drops, as follows from equations 6.125a and 6.127a.

Therefore, as has already been mentioned, for a balanced load the three-sequence networks, Fig 6.55, can be separated and treated independently. It is important to recall at this point that the zero-sequence system is not a three-phase system but a single-phase system, i.e. the zero-sequence currents and voltages are equal in magnitude and are in phase at any point in all the phases of the system. Thus, the zero-sequence currents can only exist in a circuit if there is a complete path for their flow.

Figure 6.56 shows zero-sequence networks for Y- and Δ-connected three-phase loads. As can be seen from Fig 6.56(a), in a Y-connected load with an open neutral wire, there is no return path to zero-sequence currents, hence the zero-sequence impedance is infinite (in a zero-sequence network drawing this infinite impedance is indicated by an open circuit). In the circuit of Fig 6.56(b) the fourth wire, connecting the neutrals, provides a return path for the zero-sequence currents, so that their sum, $3\tilde{I}_0$, flows through this wire. If the neutral wire impedance is $Z_{nn'}$, the zero-sequence voltage drop of $3Z_{nn'}\tilde{I}_0$ will be produced, across this impedance, by a triple zero-sequence current $3\tilde{I}_0$. For this reason an impedance of $3Z_{nn'}$ should be inserted in the zero-sequence network, as shown in Fig 6.56(b). This result is in full agreement with those achieved previously (equation 6.126b) by the mathematical treatment. Note that in the particular case of $Z_{nn'}=0$ (solidly grounded neutrals) no potential difference exists between neutral points *n*-*n*′, so they should be short-circuited. A

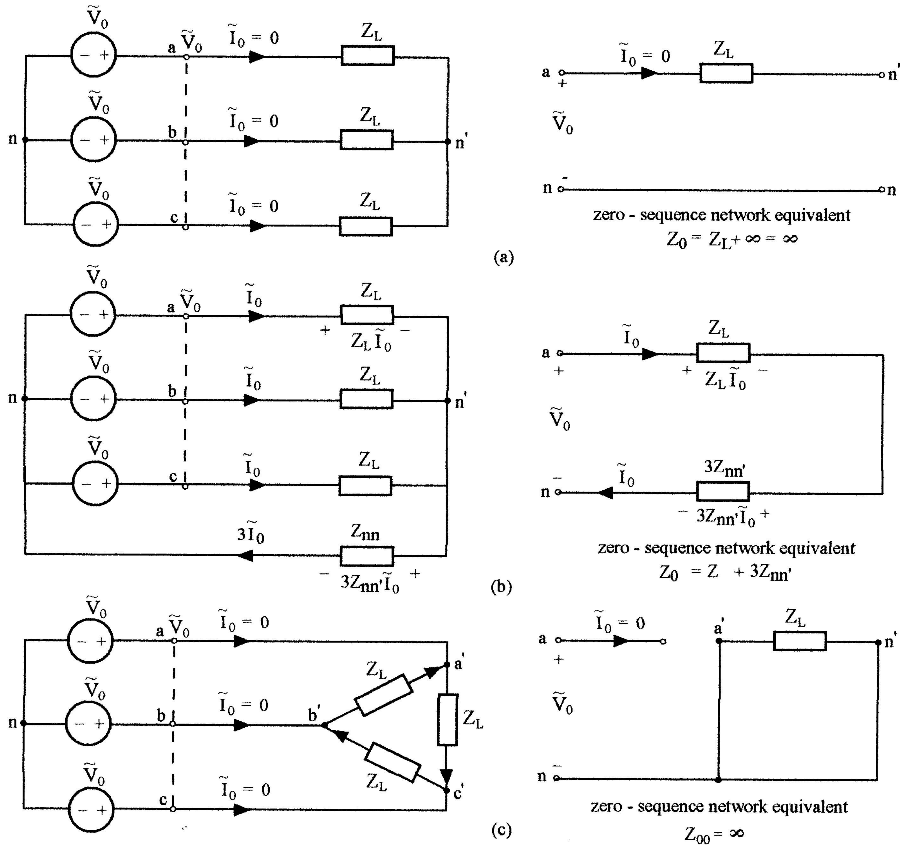


Figure 6.56 Zero-sequence network equivalent for a Y-connected three-wire load (a), Y-connected four-wire load (b) and Δ-connected load (c).

Δ-connected load, as shown in Fig 6.56(c), provides no path for zero-sequence currents flowing in the line wires. Therefore, the zero-sequence impedance, as seen from the source terminals, is infinite (open circuit). However it is possible to have zero-sequence currents circulating within the delta circuit, if zero-sequence voltages are applied independently, or by induction, in the delta circuit.

Consider, for example, the network shown in Fig. 6.57(a), in which the single-pole short-circuit to earth occurs on the transmission line between transformers T_1 and T_2. The arrows in each of the generator and transformer windings show the circulation paths of the zero-sequence currents. In accordance with these possible zero-sequence currents flowing, the zero-sequence equivalent circuit is formed and shown in Fig. 6.57(b).

Consider next, a more general case representing a circuit with unequal mutual impedances (e.g. transmission lines, transformers and tree-core cables). We also assume that there is mutual coupling between the phase branches and the neutral

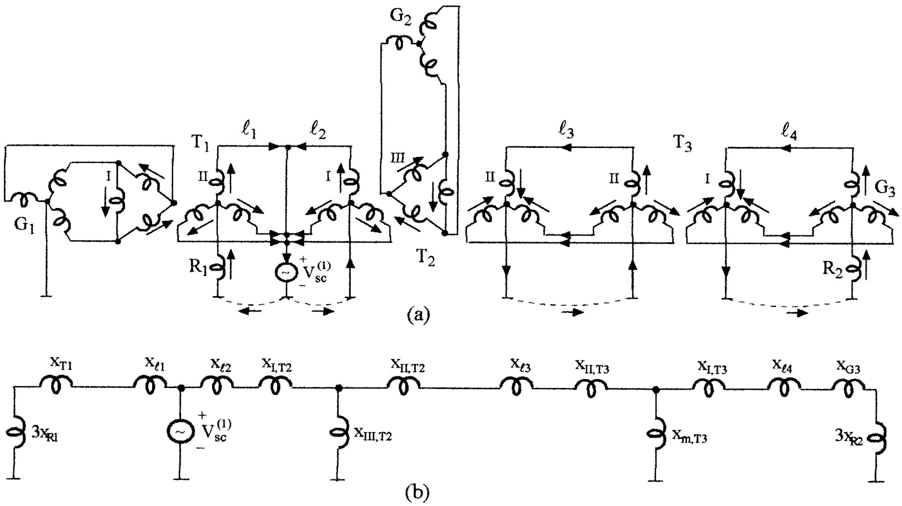


Figure 6.57 A network with a single-pole short-circuit to earth (a) and its zero-sequence equivalent (b).

line (e.g., as in transmission lines with overhead ground wire), as shown in Fig 6.58.

The KVL equation for phase a may be written as

$$\tilde{V}_a = Z_{aa}\tilde{I}_a + Z_{ab}\tilde{I}_b + Z_{ac}\tilde{I}_c - Z_{an}\tilde{I}_n + \tilde{V}_{nn'},$$

where $\tilde{V}_{nn'} = -Z_{na}\tilde{I}_a - Z_{nb}\tilde{I}_b - Z_{nc}\tilde{I}_c + Z_{nn'}\tilde{I}_n$ and, as shown earlier, $\tilde{I}_n = 3\tilde{I}_0$.

For three phases, in matrix form, these equations can be expressed as

$$\begin{bmatrix}\tilde{V}_a\\ \tilde{V}_b\\ \tilde{V}_c\end{bmatrix} = \begin{bmatrix}Z_{aa} & Z_{ab} & Z_{ac}\\ Z_{ba} & Z_{bb} & Z_{bc}\\ Z_{ca} & Z_{cb} & Z_{cc}\end{bmatrix}\begin{bmatrix}\tilde{I}_a\\ \tilde{I}_b\\ \tilde{I}_c\end{bmatrix} - \begin{bmatrix}Z_{na} & Z_{nb} & Z_{nc}\\ Z_{na} & Z_{nb} & Z_{nc}\\ Z_{na} & Z_{nb} & Z_{nc}\end{bmatrix}\begin{bmatrix}\tilde{I}_a\\ \tilde{I}_b\\ \tilde{I}_c\end{bmatrix}$$

$$+\begin{bmatrix}3(Z_{nn'} - Z_{an}) & 0 & 0\\ 3(Z_{nn'} - Z_{bn}) & 0 & 0\\ 3(Z_{nn'} - Z_{cn}) & 0 & 0\end{bmatrix}\begin{bmatrix}\tilde{I}_0\\ \tilde{I}_1\\ \tilde{I}_2\end{bmatrix} \tag{129a}$$

where Z_{aa}, Z_{bb}, Z_{cc} and $Z_{nn'}$ are the phase and neutral self-impedances, and $Z_{ab} = Z_{ba}$, $Z_{bc} = Z_{cb}$, $Z_{ca} = Z_{ac}$ and $Z_{an} = Z_{na}$, $Z_{bn} = Z_{nb}$, $Z_{cn} = Z_{nc}$ are the *phase-phase* and *phase-neutral mutual impedances*. In reduced notation equation 6.129a may be written as

$$[\tilde{V}_{abc}] = [Z_{abc}][\tilde{I}_{abc}] - [Z_{nabc}][\tilde{I}_{abc}] + 3[Z_{nn'}][\tilde{I}_{012}]. \tag{6.129b}$$

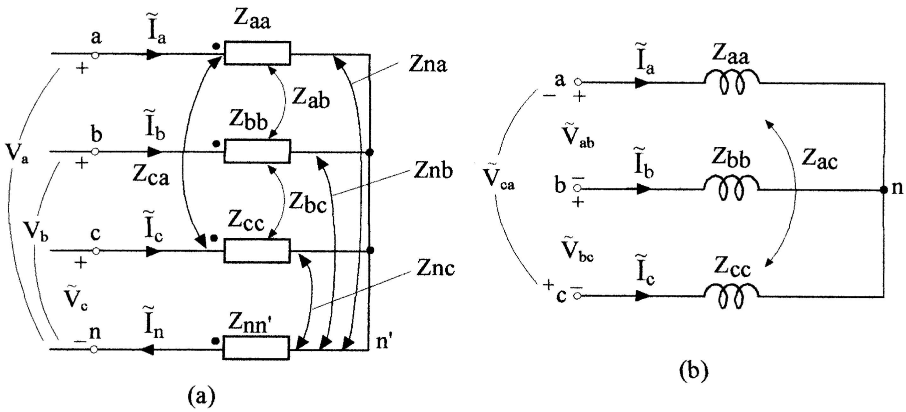


Figure 6.58 Mutually coupled three-phase network (a) and network for Example 6.18 (b).

Substituting equations 6.110c and 6.115a into this equation we obtain

$$[a][\tilde{V}_{012}] = [Z_{abc}][a][\tilde{I}_{012}] - [Z_{nabc}[a][\tilde{I}_{012}] + 3[Z_{nn'}][\tilde{I}_{012}],$$

or

$$[a][\tilde{V}_{012}] = (\{[Z_{abc}] - [Z_{nabc}]\}[a] + 3[Z_{nn'}])[\tilde{I}_{012}] = [Z_{nabc}]_{eq}[\tilde{I}_{012}]. \tag{6.130}$$

Solving this equation for $[\tilde{V}_{012}]$ yields

$$[\tilde{V}_{012}] = [a]^{-1}(\{[Z_{abc}] - [Z_{nabc}]\}[a] + 3[Z_{nn'}])[\tilde{I}_{012}], \tag{6.131a}$$

or

$$[\tilde{V}_{012}] = [Z_{012}^{(M)}][\tilde{I}_{012}]. \tag{6.131b}$$

Here $[Z_{012}^{(M)}]$ is the *sequence impedance matrix* of a three-phase load with *mutual coupling*. Performing all the matrix operations on the right side of equation 6.131a and after simplification the sequence impedance matrix can be expressed as

$$[Z_{012}^{(M)}]$$

$$= \begin{bmatrix} (Z_0 + 2Z_{M0} + 3Z_{nn'} - 6Z_{n0}) & (Z_2 - Z_{M2} - 3Z_{n2}) & (Z_1 - Z_{M1} - 3Z_{n1}) \\ (Z_1 - Z_{M1} - 6Z_{n1}) & (Z_0 - Z_{M0}) & (Z_2 + 2Z_{M2}) \\ (Z_2 - Z_{M2} - 6Z_{n2}) & (Z_1 + 2Z_{M1}) & (Z_0 - Z_{M0}) \end{bmatrix} \tag{6.132}$$

where, in addition to equation 6.119, we defined *sequence mutual impedances*:

$$Z_{M0} = \frac{1}{3}(Z_{bc} + Z_{ca} + Z_{ab}) \tag{6.133a}$$

as a zero-sequence mutual phase impedance,

$$Z_{M1} = \frac{1}{3}(Z_{bc} + aZ_{ca} + a^2 Z_{ab}) \tag{6.133b}$$

as a positive-sequence mutual phase impedance,

$$Z_{M2} = \frac{1}{3}(Z_{bc} + a^2 Z_{ca} + aZ_{ab}) \tag{6.133c}$$

as a negative-sequence mutual phase impedance,

$$Z_{n0} = \frac{1}{3}(Z_{nc} + Z_{nb} + Z_{nc}) \tag{6.133d}$$

as a zero-sequence mutual neutral impedance,

$$Z_{n1} = \frac{1}{3}(Z_{na} + aZ_{nb} + a^2 Z_{nc}) \tag{6.133e}$$

as a positive-sequence mutual neutral impedance,

$$Z_{n2} = \frac{1}{3}(Z_{na} + a^2 Z_{nb} + aZ_{nc}) \tag{6.133f}$$

as a negative-sequence mutual neutral impedance, and $Z_{nn'}$ as the impedance of a neutral line.

If neither self- nor mutual-impedances are equal, the application of equation 6.131b will show that there is a mutual coupling among three sequences and, therefore, the sequence networks cannot be separated. The sequence currents can be found by solving equation 6.131b:

$$[\tilde{I}_{012}] = [Y_{012}^{(M)}][\tilde{V}_{012}], \tag{6.134}$$

where $[Y_{012}^{(M)}]$ is the associated *sequence admittance matrix*

$$[Y_{012}^{(M)}] = [Z_{012}^{(M)}]^{-1} = \begin{bmatrix} Y_{00} & Y_{01} & Y_{02} \\ Y_{10} & Y_{11} & Y_{12} \\ Y_{20} & Y_{21} & Y_{22} \end{bmatrix}. \tag{6.135}$$

If a balanced voltage is applied to an unbalanced load (as frequently happens), then the symmetrical component voltage matrix $[\tilde{V}_{012}]$ reduces to only a positive-sequence component $\tilde{V}_1$. Indeed, if the applied voltage is balanced, then

$$\tilde{V}_b = a^2 \tilde{V}_a, \quad \tilde{V}_c = a\tilde{V}_a.$$

Substituting this in equation 6.113 yields

$$\begin{bmatrix}\tilde{V}_0\\ \tilde{V}_1\\ \tilde{V}_2\end{bmatrix}=\frac{1}{3}\begin{bmatrix}1&1&1\\1&a&a^2\\1&a^2&a\end{bmatrix}\begin{bmatrix}\tilde{V}_a\\ a^2\bar{V}_a\\ a\tilde{V}_a\end{bmatrix}=\begin{bmatrix}0\\ \tilde{V}_a\\ 0\end{bmatrix}.$$

In other words, the three-phase balanced system consists only of the positive-sequence components, $\tilde{V}_1=\tilde{V}_a$. However, if the system of line-to-line voltages is balanced, then in general zero-sequence voltages may also be present. Thus, the current sequences can be expressed as

$$\begin{bmatrix}I_{a0}\\ I_{a1}\\ I_{a2}\end{bmatrix}=\begin{bmatrix}Y_{00}&Y_{01}&Y_{02}\\Y_{10}&Y_{11}&Y_{12}\\Y_{20}&Y_{21}&Y_{22}\end{bmatrix}\begin{bmatrix}\tilde{V}_0\\ \tilde{V}_1\\ 0\end{bmatrix}. \tag{6.136}$$

However, if only a few mutual inductances are present, the solution may be simplified, as can be seen from the following example.

Example 6.18

Consider a particular case of a mutually coupled three-phase network as shown in Fig. 6.58(b). Let the self-impedances be $Z_{aa}=j1\,\Omega$, $Z_{bb}=2\,\Omega$, $Z_{cc}=j3\,\Omega$ and only the mutual-impedances be $Z_{ac}=Z_{ca}=-j0.5\,\Omega$. Find the current $\tilde{I}_a$, if the system of line-to-line voltages is balanced and given as $\tilde{V}_{ab}=100\angle 0°$, $\tilde{V}_{bc}=100\angle -120°$, $\tilde{V}_{ca}=100\angle 120°$ V.

Solution

The impedance matrix is

$$[Z_{abc}]=\begin{bmatrix}j1&0&-j0.5\\0&2&0\\-j0.5&0&j3\end{bmatrix}$$

and we first calculate $[Y_{abc}]$:

$$[Y_{abc}]=[Z_{abc}]^{-1}=\frac{1}{-5.5}\begin{bmatrix}j6&0&j1\\0&-2.75&0\\j1&0&j2\end{bmatrix}.$$

Then, as in equation 4.132 and taking into consideration that $Y_{n1}=Y_{n2}=Y_{n0}=Y_{nn'}=0$, we may obtain the sequence admittance matrix as

$$[Y_{012}]=\begin{bmatrix}Y_0+2Y_{M0}&Y_2-Y_{M2}&Y_1-Y_{M1}\\Y_1-Y_{M1}&Y_0-Y_{M0}&Y_2+2Y_{M2}\\Y_2-Y_{M2}&Y_1+2Y_{M1}&Y_0-Y_{M0}\end{bmatrix}=\begin{bmatrix}Y_{00}&Y_{01}&Y_{02}\\Y_{10}&Y_{11}&Y_{12}\\Y_{20}&Y_{21}&Y_{22}\end{bmatrix},$$

where

$$Y_0 = \frac{1}{3(-5.5)}(j6 - 2.75 + j2) = 0.1667 - j0.4849 = 0.5128\angle -71.03^\circ \text{ S}$$

$$Y_1 = \frac{1}{3(-5.5)}(j6 - 2.75\angle 120^\circ + j2\angle 240)$$

$$= -0.1883 - j0.1587 = -0.2463\angle 40.12^\circ \text{ S}$$

$$Y_3 = \frac{1}{3(-5.5)}(j6 - 2.75\angle 240^\circ + j2\angle 120^\circ)$$

$$= 0.0216 - j0.4474 = 0.4479\angle 87.23^\circ \text{ S}$$

and as in equations 4.133

$$Y_{M0} = \frac{1}{3(-5.5)}(0 + j1 + 0) = -j0.06061 \text{ S}$$

$$Y_{M1} = \frac{1}{3(-5.5)}(0 + j1\angle 120^\circ + 0) = 0.06061\angle 30^\circ \text{ S}$$

$$Y_{M2} = \frac{1}{3(-5.5)}(0 + j1\angle 240^\circ + 0) = 0.06061\angle 150^\circ \text{ S.}$$

Now the elements of matrix $[Y_{012}]$ are

$$Y_{00} = Y_0 + 2Y_{M0} = 0.6286\angle -74.62^\circ \text{ S}$$

$$Y_{11} = Y_{22} = Y_0 - Y_{M0} = 0.4559\angle -68.55^\circ \text{ S}$$

$$Y_{10} = Y_{02} = Y_1 - Y_{M1} = -0.3061\angle 38.13^\circ \text{ S}$$

$$Y_{01} = Y_{20} = Y_1 - Y_{M2} = 0.4834\angle -81.18^\circ \text{ S}$$

$$Y_{21} = Y_1 + 2Y_{M1} = -0.1287\angle 49.66^\circ \text{ S.}$$

Next we determine the sequence-voltages:

$$\tilde{V}_1 = \frac{100}{\sqrt{3}}\angle -30^\circ = 57.4\angle -30^\circ \text{ V}, \quad \tilde{V}_2 = 0.$$

Since $\tilde{I}_0 = 0$, in accordance with the first equation in 4.136 we may calculate $\tilde{V}_0$

$$\tilde{V}_0 = -\frac{Y_{01}}{Y_{00}}\tilde{V}_1 = -\frac{0.4834\angle -81.18^\circ}{0.6286\angle -74.62^\circ}57.74\angle -30^\circ = -44.4\angle -36.56^\circ \text{ V.}$$

With the two other equations in 6.136 we have

$$\tilde{I}_1 = Y_{10}\tilde{V}_0 + Y_{11}\tilde{V}_1 = 9.67 - j25.66 \text{ A}$$

$$\tilde{I}_2 = Y_{20}\tilde{V}_0 + Y_{21}\tilde{V}_1 = 2.99 + j16.50 \text{ A}$$

and

$$\tilde{I}_a = \tilde{I}_1 + \tilde{I}_2 = 12.66 - j9.16 = 15.63\angle -35.89^\circ \text{ A}.$$

When the load is balanced, i.e., the mutual impedances are equal to each other: $Z_{bc} = Z_{ca} = Z_{ab} = Z_M$ and $Z_{na} = Z_{nb} = Z_{nc} = Z_{np}$ and so the self-impedances: $Z_{aa} = Z_{bb} = Z_{cc} = Z_L$, the sequence impedance matrix (equation 6.132) simplifies to

$$[Z_{012}^{(M)}] = \begin{bmatrix} (Z_L + 2Z_M + 3Z_{nn'} - 6Z_{n0}) & 0 & 0 \\ 0 & (Z_L - Z_M) & 0 \\ 0 & 0 & (Z_L - Z_M) \end{bmatrix}. \tag{6.137}$$

As can be seen, there is no mutual coupling among the three sequences in this case either, and the sequence circuit impedances are

$$Z_{00} = Z_L + 2Z_M + 3Z_{nn'} - 6Z_{np}, \quad Z_{11} = Z_{22} = Z_L - Z_M. \tag{6.138}$$

Thus,

$$\tilde{V}_0 = Z_{00}\tilde{I}_0, \quad \tilde{V}_1 = Z_{11}\tilde{I}_1, \quad \tilde{V}_2 = Z_{22}\tilde{I}_2, \tag{6.139}$$

The *degree of current or voltage unbalances* is usually estimated as:

for the **zero sequence**

$$m_{0i} = \frac{I_{a0}}{I_{a1}}, \quad m_{0v} = \frac{V_{a0}}{V_{a1}}; \tag{6.140a}$$

for the **negative sequence**

$$m_{2i} = \frac{I_{a2}}{I_{a1}}, \quad m_{2v} = \frac{V_{a2}}{V_{a1}}. \tag{6.140b}$$

Example 6.19

The Y-connected load, having self- and mutual-impedances $Z_L = 1 + j22\,\Omega$ and $Z_M = j6\,\Omega$; and the self- and mutual-impedance of a neutral line $Z_{nn'} = 2 + j18\ \Omega$ and $Z_{np} = j2\,\Omega$ is supplied by an unbalanced three-phase system with the phase voltages being $\tilde{V}_a = 100\angle -30^\circ$, $\tilde{V}_b = 150\angle 180^\circ$ and $\tilde{V}_c = 75\angle 60^\circ$ V. Calculate the current in each branch of the load.

Solution

The first step is to calculate the sequence impedance matrix (in accordance with equation 6.137):

$$\begin{aligned} Z_{01} &= Z_L + 2Z_M + 3Z_{nn'} - 6Z_{np} \\ &= 1 + j22 + 2\cdot j6 + 3(2 + j8) - 6\cdot j2 = 7 + j46 = 46.5\angle 81.3^\circ \text{ V} \end{aligned}$$

and

$$Z_{11} = Z_{22} = Z_L - Z_M = 1 + j22 - j6 = 1 + j16 = 16.03\angle 86.4^\circ \text{ V}.$$

Next we shall calculate the phase-sequence components of the unbalanced voltages

$$\tilde{V}_0 = \frac{1}{3}(100\angle -30^\circ + 150\angle 180^\circ + 75\angle 60^\circ) = 10.0\angle 150^\circ \text{ V}$$

$$\tilde{V}_1 = \frac{1}{3}(100\angle -30^\circ + (1\angle 120^\circ)(150\angle 180^\circ) + (1\angle 240^\circ)(75\angle 60^\circ)]$$

$$= 105\angle -51^\circ \text{ V}$$

$$\tilde{V}_2 = \frac{1}{3}(100\angle -30^\circ + (1\angle 240^\circ)(150\angle 180^\circ) + (1\angle 120^\circ)(75\angle 60^\circ)]$$

$$= 39\angle 43^\circ \text{ V}.$$

Since there is no mutual coupling along the three sequences, the second step is to calculate the phase-sequence components of the current $\tilde{I}_a$. Thus,

$$\tilde{I}_{a0} = \tilde{I}_0 = \frac{\tilde{V}_0}{Z_{00}} = \frac{10.0\angle 150^\circ}{46.5\angle 81.3^\circ} = 0.215\angle 68.7^\circ \text{ A}$$

$$\tilde{I}_{a1} = \tilde{I}_1 = \frac{\tilde{V}_1}{Z_{11}} = \frac{105\angle -51^\circ}{16.03\angle 86.4^\circ} = 6.55\angle -137.4^\circ \text{ A}$$

$$\tilde{I}_{a2} = \tilde{I}_2 = \frac{\tilde{V}_2}{Z_{22}} = \frac{39.0\angle 43^\circ}{16.03\angle 86.4^\circ} = 2.43\angle -43.4^\circ \text{ A}.$$

Therefore,

$$\tilde{I}_a = \tilde{I}_{a0} + \tilde{I}_{a1} + \tilde{I}_{a2} = 0.215\angle 68.7^\circ + 6.55\angle -137.4^\circ + 2.43\angle -43.4^\circ$$

$$= 6.60\angle -116.7 \text{ A}$$

$$\tilde{I}_b = \tilde{I}_{a0} + a^2\tilde{I}_{a1} + a\tilde{I}_{a2}$$

$$= 0.215\angle 68.7^\circ + (1\angle 240^\circ)(6.55\angle -137.4^\circ) + (1\angle 120^\circ)(2.43\angle -43.4^\circ)$$

$$= 8.98\angle 85^\circ \text{ A}$$

$$\tilde{I}_c = \tilde{I}_{a0} + a\tilde{I}_{a1} + a^2\tilde{I}_{a2}$$

$$= 0.215\angle 68.7^\circ + (1\angle 120^\circ)(6.55\angle -137.4^\circ) + (1\angle 240^\circ)(2.43\angle -43.4^\circ)$$

$$= 4.69\angle -31.5^\circ \text{ A}.$$

In the above treatment of three-phase loads, and the development of the phase-sequence networks' equivalent, it was derived that the values of these network impedances are the same for currents of positive-, and negative-sequences. In practice, such a result is quite in order in the case of "static" circuits, such as transformers, transmission lines and the like, in which the mutual inductances between the circuits of different phases are bilateral. The phase sequence, positive or negative, of the currents flowing in static circuits does not change the impedances, so the same values of impedances in both the positive-, and negative-sequence networks are used.

With rotating machinery, e.g. alternators, induction motors, synchronous motors, etc., the impedance will have different values for currents of positive and negative phase-sequences. Indeed, the negative-network impedance, Z_{22}, can be determined by applying the negative-sequence voltages and measuring the negative-sequence currents, when the machine is run at specified speed and direction. Since the negative phase-sequence field (also called the *backward field*) rotates in the direction opposite to the positive phase-sequence field (also called the *forward field*), it will also rotate opposite to the rotor. Thus, for instance, for asynchronous machines the difference in speed between the backward field and rotor is $n_s + n$, where n_s and n are rotating speeds of the field and rotor respectively. This results in a slip for the backward field $s_2 = 2 - s$. Since the regular slip s (i.e. slip for the forward field) is very small ($s = 0.02$–0.05), the slip s_2 equals approximately 2, so that it is much larger than s. As a result the negative-sequence currents will be larger than the positive-sequence currents and, therefore, the impedance to currents of negative phase-sequence, Z_{22}, will be lower than that to currents of positive phase-sequence, Z_{11}. To develop the mathematical representation of rotating machine symmetrical-component impedances we shall assume that the mutual inductances between the phases of these machines are not bilateral, as shown in Fig 6.59. Thus, two different values Z_p and Z_q are the **mutual impedances of rotating machines** (clockwise and counterclockwise respectively) between the phases. The impedance matrix in

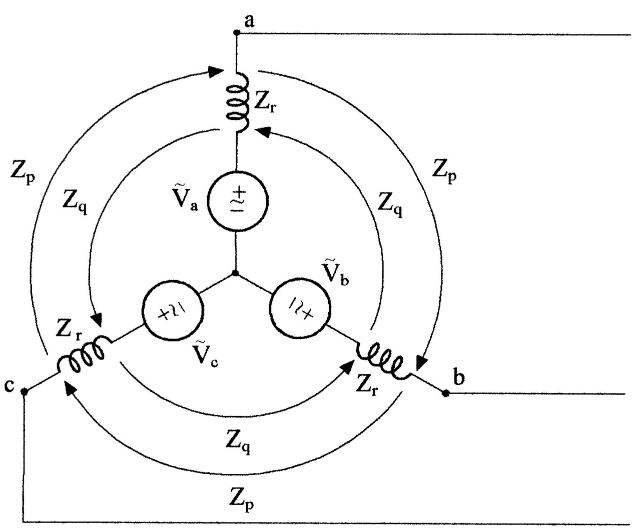


Figure 6.59 Equivalent circuit of a rotating machine.

this case is of a circular form:

$$[Z_{rpq}] = \begin{bmatrix} Z_r & Z_p & Z_q \\ Z_q & Z_r & Z_p \\ Z_p & Z_q & Z_r \end{bmatrix} \tag{6.141}$$

where Z_r is the self-impedance of each phase and $Z_p \neq Z_q$. Applying the matrix transformation of the form of equation 6.118a yields

$$[Z^{(r)}_{012}] = \begin{bmatrix} Z_{00} & 0 & 0 \\ 0 & Z_{11} & 0 \\ 0 & 0 & Z_{22} \end{bmatrix} \tag{6.142}$$

where

$$\begin{aligned} Z_{00} &= Z_r + Z_p + Z_q \\ Z_{11} &= Z_r + a^2 Z_p + aZ_q \\ Z_{22} &= Z_r + aZ_p + a^2 Z_q \end{aligned} \tag{6.143}$$

are the *zero-*, *positive- and negative-sequence impedances of the machine*. Thus, the sequence matrix equation for a rotating machine will be

$$[\tilde{V}_{012}] = [Z^{(r)}_{012}][\tilde{I}_{012}]. \tag{6.144}$$

Since the matrix in equation 6.142 is diagonal, also in this case, this matrix equation may be separated into three independent equations, each for each sequence:

$$\tilde{V}_0 = Z_0 \tilde{I}_0, \quad \tilde{V}_1 = Z_1 \tilde{I}_1, \quad \tilde{V}_2 = Z_2 \tilde{I}_2. \tag{6.145}$$

(For the sake of simplicity here and further on single subscripts, 0, 1 and 2, are used to indicate sequence-network impedances.) However, in distinction to the "static" load (see equations 6.125, 6.127 and 6.139) here the positive- and negative-network impedances are unequal, with the negative-network impedance lower than the positive-network impedance, $|Z_2| < |Z_1|$(*).

Example 6.20

A three-phase, Y-connected, induction motor, having the positive- and negative-sequence network impedances: $Z_1 = 3.6 + j3.6\ \Omega$ and $Z_2 = 0.15 + j0.5\ \Omega$, is supplied from an unsymmetrical three-wire system. The line voltages being $V_{ab} = V_{ca} = 365\,\text{V}$ and $V_{bc} = 312\,\text{V}$, calculate the current in each phase of the motor.

Solution

The first step is to calculate the phase voltages. Drawing the triangle of line

(*)There is more about symmetrical components in Gönen, T. (1988) *Electric Power Transmission System Engineering*. Wiley & Sons, New York, Chichester, Brisbane, Toronto, Singapore.

voltages as shown in Fig 6.60(a), we assumed that the neutral point n is located at the midpoint of the line voltage V_{bc} (it was already shown that the location of a neutral point does not influence the positive- and negative-sequence voltages, but only the zero-sequence; however the zero-sequence currents anyway are zero, since $Z_{nn'} = \infty$). Choosing $\tilde{V}_a$ as a reference phasor, we have

$$\tilde{V}_b = -j156\ \mathrm{V}, \quad \tilde{V}_c = j156\ \mathrm{V}$$

and

$$\tilde{V}_a = \sqrt{365^2 - 156^2} = 330\ \mathrm{V}.$$

The next step is to calculate the positive- and negative-sequence components of the phase voltages

$$\tilde{V}_1 = \frac{1}{3}(\tilde{V}_a + a\tilde{V}_b + a^2\tilde{V}_c) = \frac{1}{3}[330 + j156(-a + a^2)] = 200\ \mathrm{V}$$

and

$$\tilde{V}_2 = \frac{1}{3}(\tilde{V}_a + a^2\tilde{V}_b + a\tilde{V}_c) = \frac{1}{3}[330 + j156(a - a^2)] = 20\ \mathrm{V}$$

Now, from positive- and negative-sequence circuits, Fig 6.60(b), we obtain

$$\tilde{I}_1 = \frac{\tilde{V}_1}{Z_1} = \frac{200}{3.6\sqrt{2}\angle 45^\circ} = 39.3\angle -45^\circ\ \mathrm{A}$$

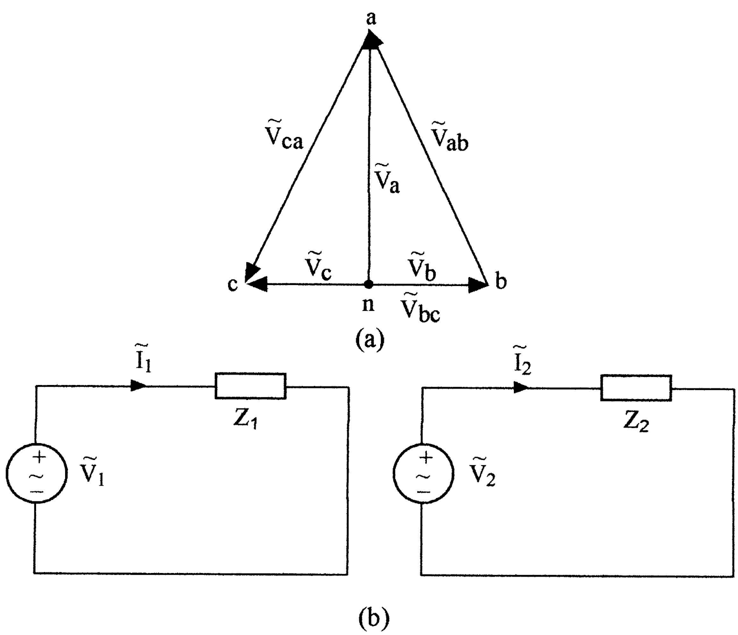


Figure 6.60 Phasor diagram (a) and sequence network for Example 6.20 (b).

$$\tilde{I}_2 = \frac{\tilde{V}_2}{Z_2} = \frac{20}{0.522\angle 73.3^\circ} = 38.3\angle -73.3^\circ \text{ A}.$$

Finally,

$$\tilde{I}_a = \tilde{I}_1 + \tilde{I}_2 = 38.8 - j64.5 = 75.3\angle -60^\circ \text{ A}$$
$$\tilde{I}_b = a^2\tilde{I}_1 + a\tilde{I}_2 = -11.7 - j17.7 = 21.2\angle 123.5^\circ \text{ A}$$
$$\tilde{I}_c = a\tilde{I}_1 + a^2\tilde{I}_2 = -27.1 - j46.8 = 54.1\angle 120^\circ \text{ A}.$$

6.7.2 Using symmetrical components for unbalanced three-phase system analysis

As we have already mentioned, the symmetrical components method is very useful for analyzing and solving the unbalanced faults of power systems. To illustrate this let us consider the most frequently occurring **single line-to-ground fault**, which occurs when one conductor contacts the ground or the neutral wire. Fig. 6.61 shows the general representation of a single line-to-ground fault at a fault point F with fault impedance Z_F. Usually, the fault impedance Z_F is ignored in fault studies. In general the voltage-current sequences' relationship for an unbalanced system is given by the matrix equation 6.117:

$$\tilde{\mathbf{V}}_{012} = \mathbf{Z}_{012}\tilde{\mathbf{I}}_{012}. \tag{6.146}$$

Here the elements of the sequence-impedance matrix usually are known. However, neither the voltage nor current symmetrical components are known. The remaining equations, called constraint equations, may be obtained using the relationship between the symmetrical components in accordance with a kind of unsymmetrical fault. Thus for the fault under consideration we have

$$\tilde{I}_b = \tilde{I}_c = 0.$$

Then, using equation 6.115b, we have

$$\tilde{I}_0 = \tilde{I}_1 = \tilde{I}_2 = \tfrac{1}{3}\tilde{I}_a. \tag{6.147}$$

Now, the current-sequence matrix can be written in terms, for instance, of $\tilde{I}_0$ as

$$\mathbf{I}_{012} = \begin{bmatrix} \tilde{I}_0 \\ \tilde{I}_1 \\ \tilde{I}_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tilde{\mathbf{I}}_0 = \mathbf{C}\tilde{\mathbf{I}}', \tag{6.148a}$$

where

$$\tilde{\mathbf{I}}' = [\tilde{I}_0] = \tilde{I}_0 \tag{6.148b}$$

(note that in this particular case matrix $\tilde{\mathbf{I}}'$ is reduced to just a scalar I_0), and

$$\mathbf{C} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{6.148c}$$

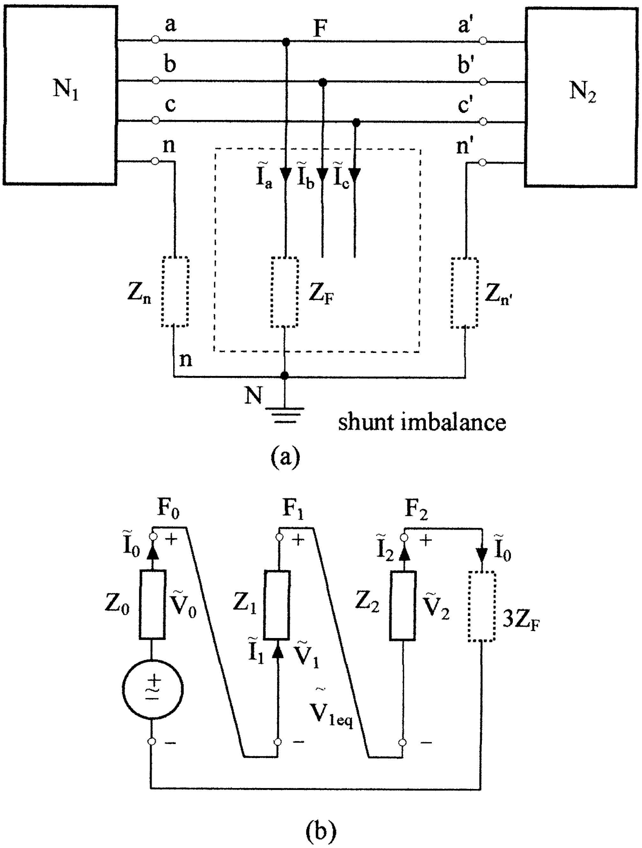


Figure 6.61 Single line-to-ground fault (a) and interconnection of sequence networks (b).

Then equation 6.148a may be written in the general form as

$$\tilde{\mathbf{I}} = \mathbf{C}\tilde{\mathbf{I}}', \tag{6.149}$$

where matrix **C** is called a **constraint matrix**. This *matrix transformation* is similar to those we used in mesh and nodal analysis, when a new set of variables, currents and voltages were chosen, for some reason, instead of a previous one. The new set of voltages' matrix is then given as

$$\tilde{\mathbf{V}}' = \mathbf{C}^{\mathrm{T}}\tilde{\mathbf{V}}. \tag{6.150}$$

(This also follows from the fact that the power volt-amperes of the network calculated in terms of the old voltage and current matrixes ($\mathbf{V}^{\mathrm{T}}\mathbf{I}^*$) must be the same as when calculated in terms of the new voltage and current matrixes ($\mathbf{V}'^{\mathrm{T}}\mathbf{I}'^*$). Then the corresponding impedance matrix is given by

$$\mathbf{Z}' = \mathbf{C}^{\mathrm{T}}\mathbf{Z}\mathbf{C}, \tag{6.151}$$

and the equation in terms of the new set of variables is denoted by

$$\mathbf{V}' = \mathbf{Z}'\mathbf{I}' \tag{6.152}$$

Continuing with the above example, we apply equation 6.150 to yield

$$\mathbf{V}' = \tilde{\mathbf{V}}'_{012} = \mathbf{C}^{\mathrm{T}}\mathbf{V}_{012} = [1 \quad 1 \quad 1]\begin{bmatrix} 0 \\ \tilde{V}_1 \\ 0 \end{bmatrix} = \tilde{V}_1, \tag{6.153}$$

where the symmetrical components of the applied voltages consist only of a positive sequence since the three-phase sources of the network N_1 and/or network N_2 in Fig. 6.61(a), which actually represent the power system generators, are symmetrical. The *transformed impedance matrix* (equation 6.151) is

$$\tilde{\mathbf{Z}}'_{012} = [1 \quad 1 \quad 1][Z_{012}]\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{6.154}$$

The sequence impedances of matrix $\mathbf{Z}_{012}$ are viewed from the fault point F and, since the system generally is balanced and consists of the rotating loads as well as the static ones, this impedance matrix $[Z_{012}]$ is diagonal with unequal positive- and negative-sequence impedances

$$Z_{012} = \begin{bmatrix} Z_0 & 0 & 0 \\ 0 & Z_1 & 0 \\ 0 & 0 & Z_2 \end{bmatrix}. \tag{6.155}$$

Substituting this matrix into equation 6.154 and performing the multiplication we easily obtain

$$\mathbf{Z}' = [Z_0 + Z_1 + Z_2]. \tag{6.156}$$

Substituting equations 6.153, 6.156 and 6.148b into equation 6.152 yields

$$\tilde{V}_1 = [Z_0 + Z_1 + Z_2]\tilde{I}_0. \tag{6.157}$$

Thus the matrix equation 6.152 in this case becomes the single scalar equation. Then

$$\tilde{I}_0 = \frac{\tilde{V}_1}{Z_0 + Z_1 + Z_2}. \tag{6.158}$$

These equations 6.157 and 6.158 are appropriate for Fig. 6.61(b), where, to meet this relationship, the symmetrical component networks have to be connected in series. The fault current of phase a, therefore, is (equation 6.147)

$$\tilde{I}^{(1)}_{sc} = 3\tilde{I}_0. \tag{6.159}$$

Recall that the superscript indicates the following kind of faults: (1) a single-pole ground fault, (2) two-pole fault and (1,1) two-pole-ground fault. The numerical examples follow.

Example 6.21

The faulted network and all the parameters are given in Fig. 6.62(a). Form the sequence networks and calculate the steady-state single-pole-to-ground short-circuit current.

Solution

The p.u. reactances referred to $S_B = 120$ MVA and to the average basic voltages are shown in Fig. 6.62(b), where the three sequence networks are also given.

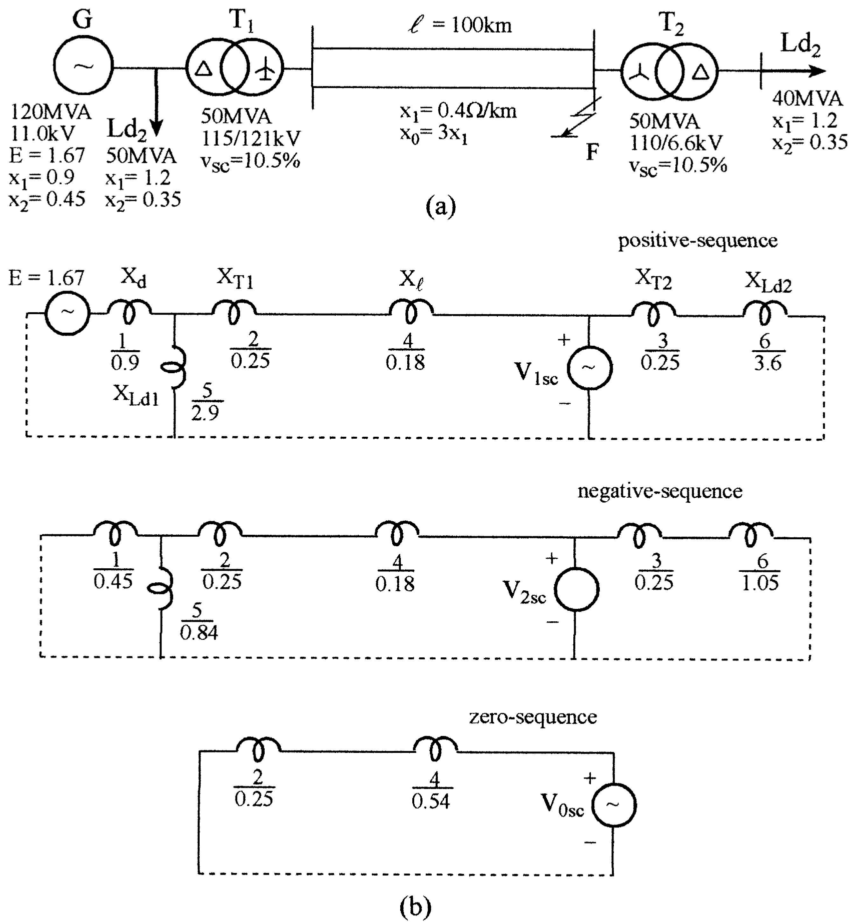


Figure 6.62 A given network for Example 6.21 (a) and the sequence networks (b).

By simplifying the positive-sequence network we have

$$X_{1,7} = 0.9//2.9 = 0.69, \quad X_{1,8} = 0.25 + 3.6 = 3.85,$$

$$X_{1,9} = 0.69 + 0.25 + 0.18 = 1.12,$$

and the equivalent reactance is

$$X_{1,eq} = 1.12//3.85 = 0.87.$$

The equivalent EMF is found as follows

$$E_2 = \frac{1.67 \cdot 2.9}{0.9 + 2.9} = 1.27 \quad \text{and} \quad E_{1eq} = \frac{1.27 \cdot 3.85}{1.12 + 3.85} = 0.98.$$

The simplification of the negative-sequence network gives

$$X_{2,7} = 0.45//0.84 = 0.29, \quad X_{2,8} = 0.25 + 1.05 = 1.3,$$

$$X_{2,9} = 0.29 + 0.25 + 0.18 = 0.72,$$

and

$$X_{2,eq} = 0.72//1.3 = 0.46.$$

Finally, the simplification of the zero-sequence network results in

$$X_{0,eq} = 0.25 + 0.54 = 0.79.$$

The zero sequence current will be (equation 6.158)

$$I_0^{(1)} = \frac{0.98}{0.87 + 0.46 + 0.79} = 0.45.$$

And the short-circuit current in a single-pole-ground fault, therefore, is (equation 6.159)

$$I_{sc}^{(1)} = 3I_0^{(1)} = 3 \cdot 0.45 = 1.35,$$

or

$$I_{sc}^{(1)} = 1.35 \frac{120}{\sqrt{3} \cdot 115} = 0.81.$$

Example 6.22

Consider the low power system shown in Fig. 6.61(a) and assume that there is a single-line-to-ground solid (i.e., $Z_F = 0$) fault involving phase a at the end of a transmission line, i.e. at point a'. Let the network N_1 represent a generator having phase voltages 240 V and the sequence impedances $Z_{s1} = j4\,\Omega$, $Z_{s2} = j2\,\Omega$ and $Z_{s0} = j1\,\Omega$; and the network N_2 represents a load with $Z_{L1} = 22.22\angle 25.84^\circ = 20 + j9.69\,\Omega$ (which corresponds to 0.9 PF), $Z_{L2} = (8 + j5)\,\Omega$ and $Z_{L0} = (2.5 + j1)\,\Omega$. The transmission line sequence impedances are $Z_{l1} = Z_{l2} = j1\ \Omega$ and $Z_{l0} = j1.5\ \Omega$. Also assume that the neutral wire/ground impedances are $Z_n = Z_{n'} = 0.5\,\Omega$, and the fault impedance Z_F is zero. At the fault point F, determine: (a) the sequence and phase currents and (b) the sequence and phase voltages.

Solution

(a) Figure 6.63(a) shows the corresponding positive-, negative- and zero-sequence networks which are interconnected in series. To reduce them we first find the equivalent impedances:

$$Z_{1eq} = \frac{(Z_{s1} + Z_{l1})Z_{L1}}{Z_{s1} + Z_{l1} + Z_{L1}} = \frac{(j4 + j1)(20 + j9.69)}{20 + j14.69} = 4.47\angle 79.54^\circ\ \Omega$$

$$Z_{2eq} = \frac{(Z_{s2} + Z_{l2})Z_{L2}}{Z_{s2} + Z_{l2} + Z_{L2}} = \frac{(j2 + j1)(8 + j5)}{8 + j8} = 2.50\angle 77.01^\circ\ \Omega$$

$$Z_{0eq} = \frac{(Z_{s0} + Z_{l0} + 3Z_n)(Z_{L0} + 3Z_{n'})}{Z_{s0} + Z_{l0} + Z_{L0} + 3(Z_n - Z_{n'})} = \frac{(j1 + j1.5 + 1.5)(2.5 + j1 + 1.5)}{5.5 + j3.5}$$

$$= 1.84\angle 40.61^\circ\ \Omega,$$

and the equivalent voltage source seen at the fault point (which is the Thévenin

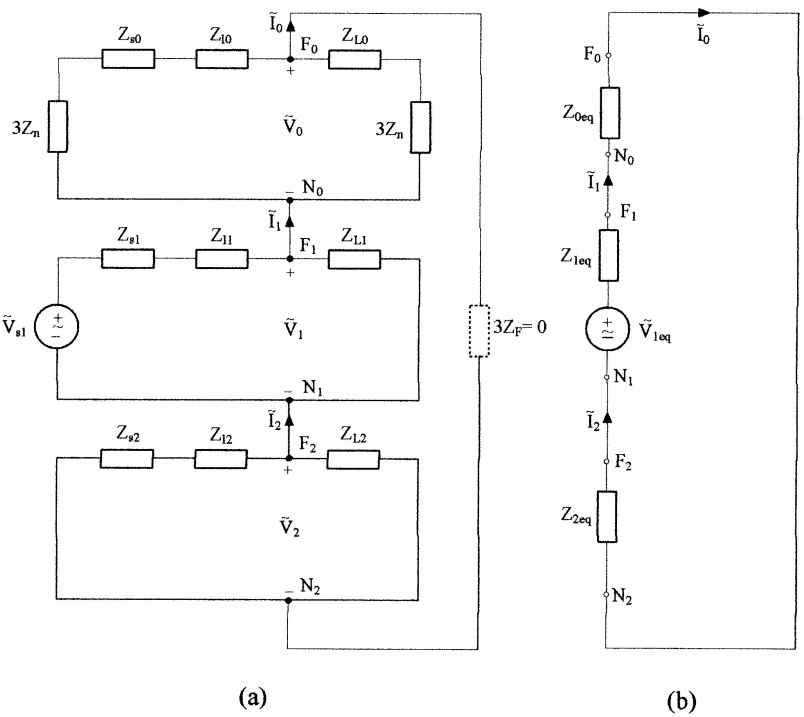


Figure 6.63 Interconnection of zero-, positive- and negative-sequence networks (a) and an equivalent network for Example 6.22 (b).

voltage)

$$\tilde{V}_{1eq} = V_{s1}\frac{Z_{L1}}{Z_{s1}+Z_{l1}+Z_{L1}} = 240\,\frac{20+j9.69}{20+j14.69} = 215.0\angle-10.46^\circ\ \mathrm{V}.$$

The resulting equivalent sequence network interconnection is shown in Fig. 6.63(b). Thus, the sequence currents of phase a are (equation 6.158)

$$\tilde{I}_0 = \tilde{I}_1 = \tilde{I}_2 = \frac{V_{1eq}}{Z_{0eq}+Z_{1eq}+Z_{2eq}} = 25.3\angle-81.42^\circ\ \mathrm{A},$$

and the phase currents (Fig. 6.61a)

$$\begin{bmatrix}\tilde{I}_{a'F}\\ \tilde{I}_{b'F}\\ \tilde{I}_{c'F}\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\ 1 & a^2 & a\\ 1 & a & a^2\end{bmatrix}\begin{bmatrix}25.3\angle-81.4^\circ\\ 25.3\angle-81.4^\circ\\ 25.3\angle-81.4^\circ\end{bmatrix} = \begin{bmatrix}75.8\angle-81.4^\circ\\ 0\\ 0\end{bmatrix}\ \mathrm{A}.$$

(b) The sequence voltages are (in matrix representation)

$$\tilde{\mathbf{V}}_{F,012} = \tilde{\mathbf{V}}_{s,012} - \mathbf{Z}_{012}\tilde{\mathbf{I}}_{012},$$

or

$$\begin{bmatrix}\tilde{V}_0\\ \tilde{V}_1\\ \tilde{V}_2\end{bmatrix} = \begin{bmatrix}0\\ 215.0\angle-10.46^\circ\\ 0\end{bmatrix} - \begin{bmatrix}1.84\angle40.61^\circ & 0 & 0\\ 0 & 4.48\angle79.54^\circ & 0\\ 0 & 0 & 2.50\angle77.01^\circ\end{bmatrix}$$

$$\times\begin{bmatrix}25.3\angle-81.42^\circ\\ 25.3\angle-81.42^\circ\\ 25.3\angle-81.42^\circ\end{bmatrix} = \begin{bmatrix}-35.3+j30.44\\ 98.3-j35.30\\ -63.0+j4.86\end{bmatrix} = \begin{bmatrix}46.6\angle139.19^\circ\\ 104.4\angle-19.76^\circ\\ 63.2\angle175.59^\circ\end{bmatrix}\ \mathrm{V},$$

and the phase voltages are

$$\begin{bmatrix}\tilde{V}_{a'F}\\ \tilde{V}_{b'F}\\ \tilde{V}_{c'F}\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\ 1 & a^2 & a\\ 1 & a & a^2\end{bmatrix}\begin{bmatrix}46.6\angle139.19^\circ\\ 104.4\angle-19.76^\circ\\ 63.2\angle175.59^\circ\end{bmatrix}$$

$$= \begin{bmatrix}0.02+j0.00\\ -87.7-j94.0\\ -18.1+j185.4\end{bmatrix} = \begin{bmatrix}\approx 0\\ 129\angle-133.0^\circ\\ 186\angle95.6^\circ\end{bmatrix}\ \mathrm{V}.$$

For the **line-to line fault**, shown in Fig. 6.64(a), if, for example, the fault occurs on phases b and c, the constraint equations are

$$\tilde{I}_a = 0 \quad \text{and} \quad \tilde{I}_b = -\tilde{I}_c. \tag{6.160}$$

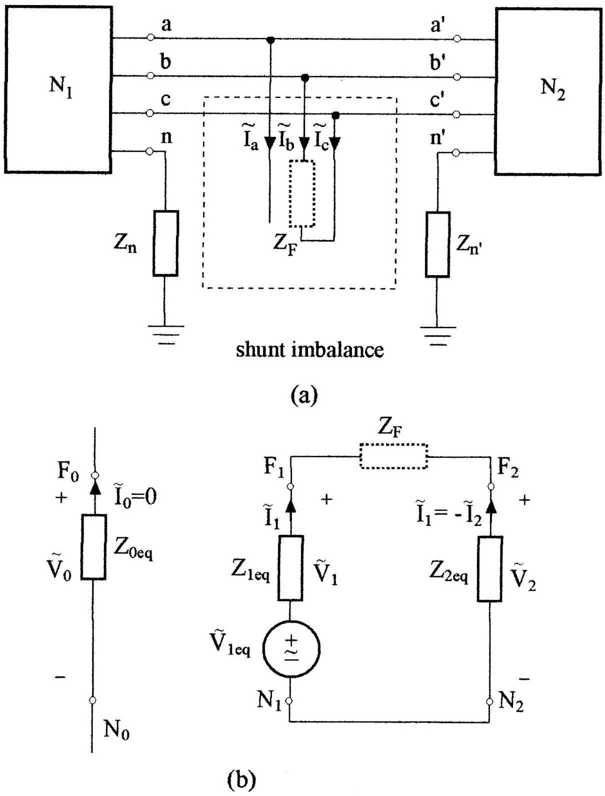


Figure 6.64 Line-to-line fault (a) and the interconnection of the sequence networks (b).

Applying now equation 6.115b we have

$$\begin{bmatrix} \tilde{I}_0 \\ \tilde{I}_1 \\ \tilde{I}_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} 0 \\ \tilde{I}_b \\ -\tilde{I}_b \end{bmatrix} = \frac{j}{\sqrt{3}} \begin{bmatrix} 0 \\ \tilde{I}_b \\ -\tilde{I}_b \end{bmatrix}$$

or

$$\tilde{I}_0 = 0 \quad \text{and} \quad \tilde{I}'_1 = -\tilde{I}'_2 = \frac{j}{\sqrt{3}} I_b. \tag{6.161}$$

Note that the absence of I_0 can also be recognized from the fact that the zero-sequence current fault path in the circuit of Fig. 6.64(a) is open, which is indicated in Fig. 6.64(b) by ignoring the zero-sequence network.

Hence the constraint matrix in terms of $\tilde{I}_1$ would be written as

$$\mathbf{C} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}. \tag{6.162}$$

Substituting this constraint equation into equations 6.150 and 6.151 and remembering that only the positive-sequence source voltage is non-zero, we easily obtain

$$[\tilde{V}_{012}] = \tilde{V}_1 \quad \text{and} \quad [Z_{012}] = [Z_1 + Z_2].$$

Thus, with equation 6.152 we have

$$\tilde{V}_1 = [Z_1 + Z_2]\tilde{I}_1 \tag{6.163a}$$

and

$$\tilde{I}'_1 = \frac{\tilde{V}'_1}{Z_1 + Z_2}. \tag{6.163b}$$

These equations are appropriate for Fig. 6.64(b), where only two symmetrical component networks, positive and negative, are connected in series.

In accordance with equation 6.161 the short-circuit current in phase b is

$$\tilde{I}_{sc,b} = -j\sqrt{3}I'_1. \tag{6.163c}$$

Since at the fault point the voltage is zero we have

$$\tilde{V}_{F,b} = \tilde{V}_{F,c},$$

which gives

$$\tilde{V}_F = \tilde{V}_{F,b} - \tilde{V}_{F,c} = 0.$$

Then

$$\tilde{V}_{F1} = \frac{1}{3}(\tilde{V}_{F,a} + a\tilde{V}_{F,b} + a^2\tilde{V}_{F,c}) = \frac{1}{3}[\tilde{V}_{F,a} + (a + a^2)\tilde{V}_{F,b}]$$

$$\tilde{V}_{F2} = \frac{1}{3}(\tilde{V}_{F,a} + a^2\tilde{V}_{F,b} + a\tilde{V}_{F,c}) = \frac{1}{3}[\tilde{V}_{F,a} + (a + a^2)\tilde{V}_{F,b}],$$

and

$$\tilde{V}_{F1} = \tilde{V}_{F2}. \tag{6.163d}$$

Example 6.23

Repeat Example 6.22 assuming that there is a line-to-line fault, involving phases b and c at the end of the transmission line, i.e. at points b' and c'.

Solution

(a) In accordance with Fig. 6.64(b), where $\tilde{V}_{s1} \equiv \tilde{V}_{1eq}$, $Z_1 \equiv Z_{1eq}$ and $Z_2 \equiv Z_{2eq}$ and with the other data of the previous example, the sequence currents are

$$\tilde{I}_0 = 0$$

$$\tilde{I}_1 = -\tilde{I}_2 = \frac{\tilde{V}_{1eq}}{Z_{1eq} + Z_{2eq}} = \frac{215\angle -10.46^\circ}{4.48\angle 79.54^\circ + 2.50\angle 77.01^\circ}$$

$$= 30.8\angle -89.1^\circ \text{ A},$$

and the phase currents are

$$\begin{bmatrix} \tilde{I}_{a'F} \\ \tilde{I}_{b'F} \\ \tilde{I}_{c'F} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0 \\ 30.8\angle -89.1^\circ \\ 30.8\angle 90.9^\circ \end{bmatrix} = \begin{bmatrix} 0 \\ 53.3\angle -179.1^\circ \\ 53.3\angle 0.9^\circ \end{bmatrix} \text{ A}.$$

(b) The sequence and phase voltages are

$$\begin{bmatrix} \tilde{V}_0 \\ \tilde{V}_1 \\ \tilde{V}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 215\angle -10.46^\circ \\ 0 \end{bmatrix} - \begin{bmatrix} 1.84\angle 40.61^\circ & 0 & 0 \\ 0 & 4.48\angle 79.54^\circ & 0 \\ 0 & 0 & 2.50\angle 77.01^\circ \end{bmatrix}$$

$$\times \begin{bmatrix} 0 \\ 30.8\angle -89.1^\circ \\ 30.8\angle 90.9^\circ \end{bmatrix} = \begin{bmatrix} 0 \\ 77.1\angle -12.07^\circ \\ 77.0\angle -12.09^\circ \end{bmatrix} \text{ V},$$

i.e., $\tilde{V}_1 \approx \tilde{V}_2$, as can also be seen from Fig. 6.64(b), since $Z_F = 0$, and

$$\begin{bmatrix} \tilde{V}_{a'F} \\ \tilde{V}_{b'F} \\ \tilde{V}_{c'F} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 0 \\ 77.06\angle -12.1^\circ \\ 77.06\angle -12.1^\circ \end{bmatrix} = \begin{bmatrix} 154.1\angle -12.1^\circ \\ 77.06\angle 107.9^\circ \\ 77.06\angle 107.9^\circ \end{bmatrix} \text{ V}.$$

Example 6.24

Repeat Example 6.21 and find the line-to-line short-circuit current.

Solution

The resulting sequence-network in this kind of fault is formed by a series connection of positive- and negative-sequences. Therefore, the positive-sequence current is

$$I_1 = \frac{0.98}{0.87 + 0.46} = 0.71,$$

and the short-circuit current (equation 6.163c) is

$$I_{sc}^{(2)} = \sqrt{3}\cdot 0.71 = 1.24,$$

or (in amperes)

$$I_{sc}^{(2)} = 1.24\,\frac{120}{\sqrt{3}\cdot 115} = 0.74\ \text{kA}.$$

Example 6.25

In the power network shown in Fig. 6.65(a) a power station is connected through an equivalent reactance to an infinite busbar. Assume that there is a line-to-line fault involving phases b and c and find 1) the steady-state value of a short-circuit current and 2) the currents flowing from the generator.

Solution

In accordance with the sequence networks shown in Fig. 6.65(b) we find that

$$X_{1eq} = 1.23//0.25 = 0.21 \quad \text{and} \quad E_{1eq} = \frac{1.7\cdot 0.25 + 1\cdot 1.23}{1.23 + 0.25} = 1.12,$$

$$X_{2eq} = 0.17//0.25 = 0.1.$$

Thus, the positive- and negative-sequence of the fault current are

$$\tilde{I}_1 = -\tilde{I}_2 = \frac{1.12}{j(0.21 + 0.1)} = -j3.61.$$

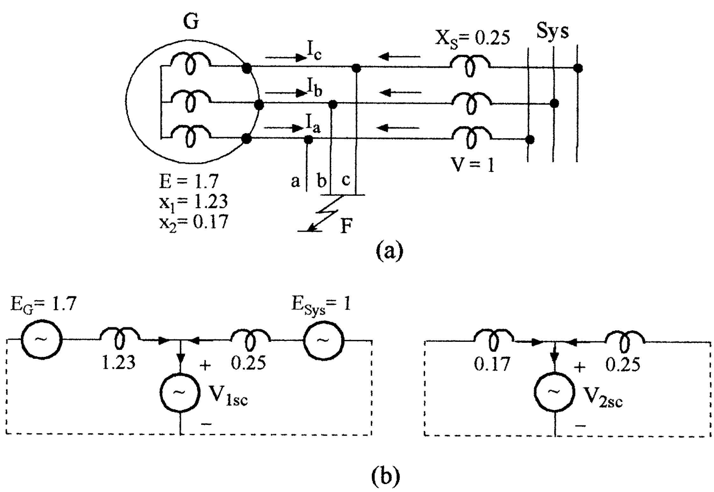


Figure 6.65 A network diagram for Example 6.25 (a) and the sequence networks (b).

The phase currents of the fault point will be

$$\tilde{I}_{sc,b} = -j\sqrt{3}(-j3.61) = -6.25 \quad \text{and} \quad I_{sc,c} = 6.25.$$

2) The sequence voltages of the fault point will then be (equation 6.163d)

$$\tilde{V}_{1F} = \tilde{V}_{2F} = -\tilde{I}_2(jX_{2eq}) = -j3.61(j0.1) = 0.361.$$

We can now find the generator current sequence

$$\tilde{I}_{1G} = \frac{1.7 - 0.361}{j1.23} = -j1.09, \quad \tilde{I}_{2G} = -\frac{0.361}{j0.17} = j2.12,$$

and the phase currents are

$$\begin{aligned}
\tilde{I}_{aG} &= -j1.09 + j2.12 = j1.03 \\
\tilde{I}_{bG} &= a^2(-j1.09) + a(j2.12) = -2.78 - j0.51 \quad \text{or} \quad I_{bG} = 2.83 \\
\tilde{I}_{cG} &= a(-j1.09) + a^2(j2.12) = -2.78 - j0.515 \quad \text{or} \quad I_{cG} = 2.83.
\end{aligned}$$

Note that the current in the non-faulted phase is about 40% of the current in the faulted phases. This means that the short-circuit current flows not only through the faulted phases, but also through a non-faulted phase.

Finally, consider the **double line-to-ground fault** on a transmission system, as shown in Fig. 6.66(a). This fault occurs when two conductors are connected through ground, Z_G, or directly, to the neutral of a three-phase grounded, or four-wire, system. If the fault is between phases b and c then

$$\tilde{I}_a = \tilde{I}_0 + \tilde{I}_1 + \tilde{I}_2 = 0, \tag{6.164}$$

and the current-sequence matrix could be written in terms, for instance, of $\tilde{I}_0$ and $\tilde{I}_1$

$$\tilde{I}_{012} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \tilde{I}_0 \\ \tilde{I}_1 \end{bmatrix}, \quad \text{i.e., } \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

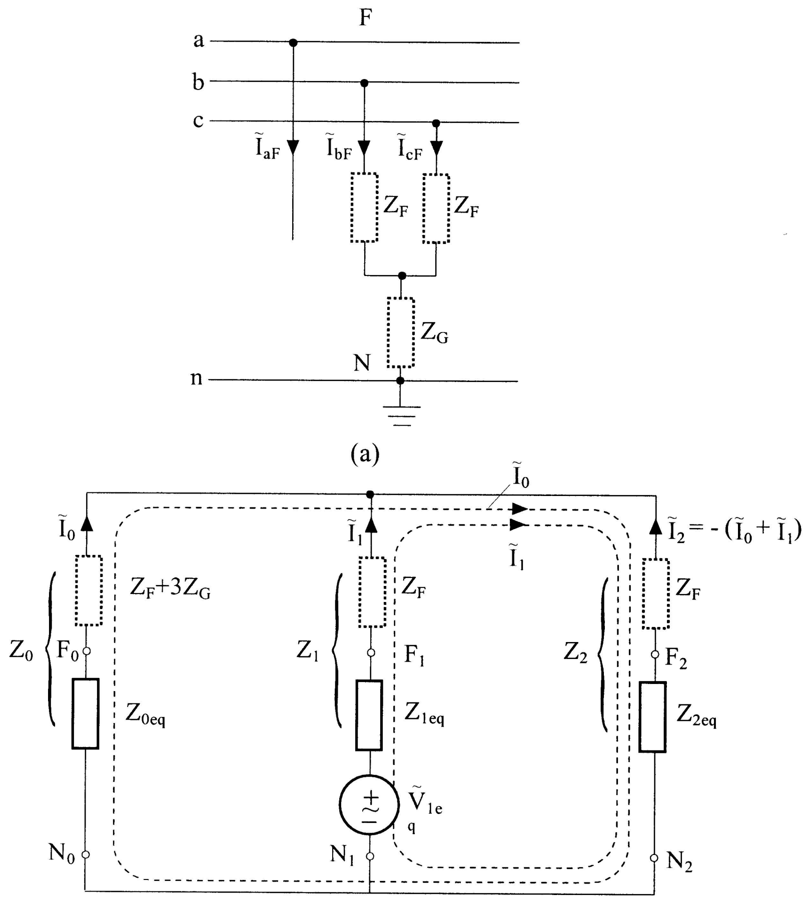
By determining $\mathbf{V}'$ (equation 6.150) and $\mathbf{Z}'$ (equation 6.151) and substituting them into equation 6.152 we obtain

$$\begin{bmatrix} 0 \\ \tilde{V}_1 \end{bmatrix} = \begin{bmatrix} Z_0 + Z_2 & Z_2 \\ Z_2 & Z_1 + Z_2 \end{bmatrix} \begin{bmatrix} \tilde{I}_0 \\ \tilde{I}_1 \end{bmatrix}$$

and

$$\tilde{I}_1 = \frac{\begin{vmatrix} Z_0 Z_2 & 0 \\ Z_2 & \tilde{V} \end{vmatrix}}{(Z_0 + Z_2)^2 - Z_2^2} = \frac{\tilde{V}_1}{Z_1 + Z_2 Z_0/(Z_2 + Z_0)} \tag{6.165}$$

which is interpreted as for the equivalent circuit shown in Fig. 6.66(b). Note



(b)

Figure 6.66 Double line-to-line ground fault: general representation (a) and interconnection of sequence networks (b).

that if $Z_G = \infty$, i.e., there is a line-to-line fault only, this circuit will reduce to the circuit in Fig. 6.64(b).

The faults, considered above, are commonly called **shunt faults**. A variety of series imbalances that occur in a power system are called **series faults**. A common one is a **broken or open conductor fault**, as shown in Fig. 6.67(a). The constraint equation for this fault is

$$\tilde{I}_a = \tilde{I}_0 + \tilde{I}_1 + \tilde{I}_2 = 0, \tag{6.166}$$

or

$$\tilde{I}_2 = -\tilde{I}_0 - \tilde{I}_1.$$

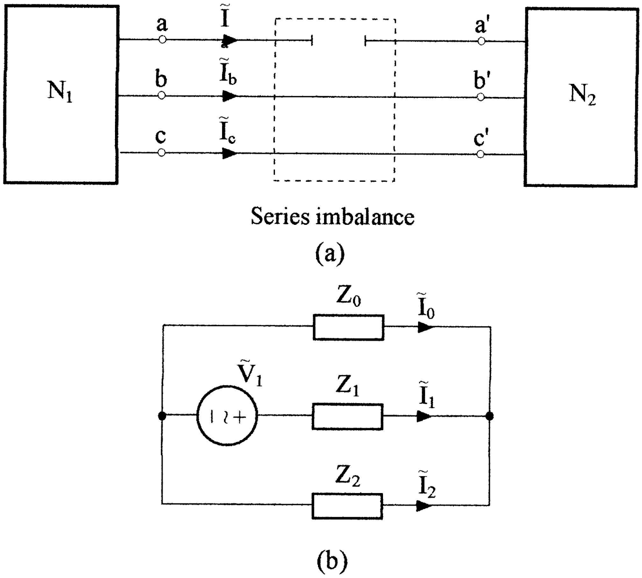


Figure 6.67 Single-phase open-fault (a) and interconnection of sequence networks (b).

Hence the constraint matrix in terms of $\tilde{I}_0$ and $\tilde{I}_1$ may be determined from the equation

$$\begin{bmatrix} \tilde{I}_0 \\ \tilde{I}_1 \\ \tilde{I}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \tilde{I}_0 \\ \tilde{I}_1 \end{bmatrix}$$

i.e.,

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}. \tag{6.167}$$

With this constraint matrix the transformed voltage-sequence (equation 6.150) and impedance-sequence (equation 6.151) matrixes can be expressed as

$$\mathbf{V}' = \begin{bmatrix} 0 \\ \tilde{V}_1 \end{bmatrix}$$

and

$$\mathbf{Z}' = \begin{bmatrix} Z_0 + Z_2 & Z_2 \\ Z_2 & Z_1 + Z_2 \end{bmatrix}.$$

Substituting these expressions into equation 6.152 yields

$$\begin{bmatrix} Z_0 + Z_2 & Z_2 \\ Z_2 & Z_1 + Z_2 \end{bmatrix} \begin{bmatrix} \tilde{I}_0 \\ \tilde{I}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{V}_1 \end{bmatrix} \tag{6.168}$$

and solving it for $\tilde{I}_1$ (using, for instance, Kramer's rule) we obtain

$$\tilde{I}_1 = \frac{\tilde{V}_1(Z_0 + Z_2)}{Z_1 Z_0 + Z_1 Z_2 + Z_0 Z_2} = \frac{\tilde{V}_1}{Z_1 + Z_0 Z_2/(Z_0 + Z_2)}. \tag{6.169}$$

This result could be obtained straightforwardly from the parallel interconnection of the three sequence-networks as shown in Fig. 6.67(b). Note that this kind of sequence-network interconnection is actually the same as for a double line-to-ground fault. The difference, however, is that here the interconnection circuit refers to the line currents, whereas in the double line-to-ground case it refers to the fault currents.

Example 6.27

An induction motor is supplied by a three-phase three-wire balanced system. With the fault being the phase a conductor open, Fig. 6.68(a), find the line currents of the remaining phases $\tilde{I}_b$ and $\tilde{I}_c$ and phase voltages across the load $\tilde{V}_{a'n'}$, $\tilde{V}_{b'n'}$ and $\tilde{V}_{c'n'}$. Also find the voltages $\tilde{V}_{aa'}$ and $\tilde{V}_{nn'}$. The supplied line voltage is 400 V and the motor impedances are: positive-sequence $Z_{M1} = 3.6 + j3.6\,\Omega$ and negative-sequence $Z_{M2} = 0.15 + j0.5\,\Omega$. The line sequence impedances are $Z_{l1} = Z_{l2} = 0.1 + j0.1\ \Omega$. (The system impedances might be neglected, being relatively very small.)

Solution

Since the neutral wire is absent, $Z_{nn'} = \infty$, only two sequence networks (positive- and negative-sequence networks) are connected in parallel, as shown in Fig. 6.68(b). Thus, the positive-sequence current is found as

$$\tilde{I}_1 = \frac{\tilde{V}_{s1}}{Z_1 + Z_2} = \frac{231}{3.7 + j3.7 + 0.25 + j0.6} = \frac{231}{5.84\angle 47.4^\circ} = 39.6\angle -47.4^\circ \text{ A},$$

where $\tilde{V}_{s1} = 400/\sqrt{3} = 231$ V.

Therefore,

$$\tilde{I}_b = a^2\tilde{I}_1 + a\tilde{I}_2 = (a^2 - a)\tilde{I}_1 = -j\sqrt{3}\tilde{I}_1 = 68.5\angle -137.4^\circ \text{ A}$$
$$\tilde{I}_c = -\tilde{I}_b = 68.5\angle 42.6^\circ \text{ A}.$$

The phase voltages are found as

$$\tilde{V}_{a'n'} = Z_{M1}\tilde{I}_1 + Z_{M2}\tilde{I}_2 = (Z_{1M} - Z_{2M})\tilde{I}_1$$
$$= (3.45 + j3.10)(39.6\angle -47.4^\circ) = 183.7\angle -5.4^\circ \text{ V}$$
$$\tilde{V}_{b'n'} = a^2 Z_{M1}\tilde{I}_1 + aZ_{M2}\tilde{I}_2 = (a^2 Z_{M1} - aZ_{M2})\tilde{I}_1$$

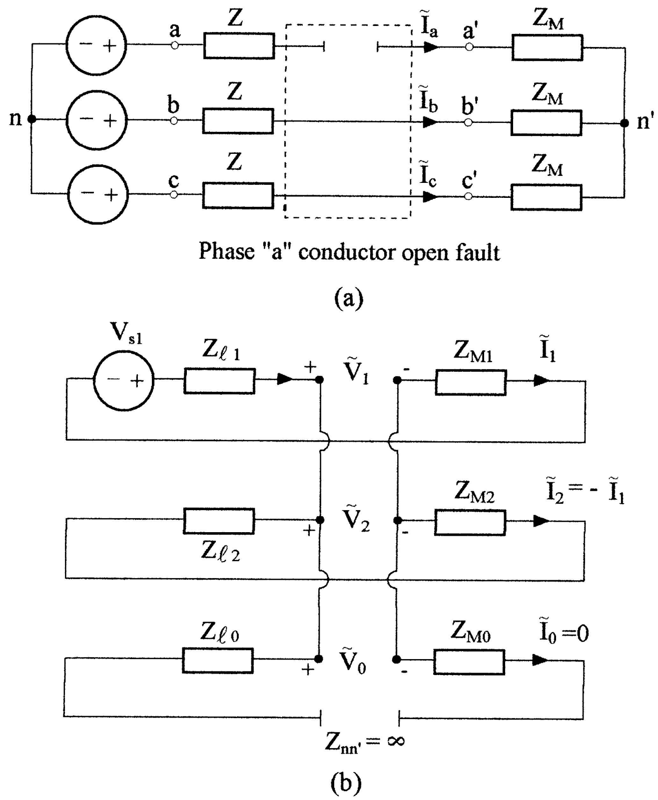


Figure 6.68 An open-fault circuit (a) and the interconnection of sequence networks (b) for Example 6.27.

$$= (5.09\angle -75^\circ - 0.522\angle 193.3^\circ)(39.6\angle -47.4^\circ) = 203\angle -116.6^\circ \text{ V}$$

$$\tilde{V}_{c'n'} = (aZ_{M1} - a^2 Z_{M2})\tilde{I}_1 = 220\angle 114.8^\circ \text{ V}.$$

To find the fault voltage $\tilde{V}_{aa'}$ we shall first analyze the series unbalanced voltages:

$$\begin{aligned} \tilde{V}_{aa'} &= \tilde{V}_0 + \tilde{V}_1 + \tilde{V}_2 \\ \tilde{V}_{bb'} &= \tilde{V}_0 + a^2\tilde{V}_1 + a\tilde{V}_2 \\ \tilde{V}_{cc'} &= \tilde{V}_0 + a\tilde{V}_1 + a^2\tilde{V}_2. \end{aligned} \tag{6.170}$$

The constraint voltage equations are

$$\tilde{V}_{bb'} = 0 \quad \tilde{V}_{cc'} = 0. \tag{6.171}$$

Solving equation 6.170 with equation 6.171 yields

$$\tilde{V}_1 = \tilde{V}_2 = \tilde{V}_0. \tag{6.172}$$

The second step is to determine $\tilde{V}_2$ in accordance with the negative-sequence network, which is the part of the equivalent circuit shown in Fig. 6.68(b). Thus,

$$\tilde{V}_2 = -Z_2\tilde{I}_2 = -(0.25 + j0.6)(-39.6\angle -47.5^\circ)$$
$$= 0.65\angle 67.4^\circ \cdot 39.6\angle -47.5^\circ = 25.7\angle 19.9^\circ \text{ V}.$$

Therefore, with the first equation of (1.170) we have

$$\tilde{V}_{aa'} = 3\tilde{V}_2 = 77.1\angle 19.9^\circ \text{ V}.$$

Since the neutral line is open the potential difference between neutral points n and n' equals zero-sequence voltage. Thus,

$$\tilde{V}_{nn'} = \tilde{V}_0 = 25.7\angle 19.9^\circ \text{ V}.$$

In the final example of this section let us consider the influence of AVR on the unsymmetrical faults.

Example 6.28

A two-pole-ground-fault occurs in the network shown in Fig. 6.69. Find at $t = 0.5$ s the short-circuit currents at the fault point F. The generators and transformers are identical and both generators are equipped with an AVR.

Solution

The network reactances, referred to the basic power $S_B = 100$ MVA, are shown in Fig. 6.69(b) and (c). Note that the reactances of the high voltage winding of the transformers are not taken into account due to the symmetrical properties of the network relative to the fault point position.

By simplification of the positive-sequence circuit, we have

$$X_{1eq} = X_{2eq} = \frac{0.24 + 0.06}{2} + 0.23 = 0.38,$$

and for the zero-sequence circuit

$$X_{11} = 0.06//(2\cdot 0.12 + 0.06) = 0.5, \quad X_{12} = 0.05 + 0.8 = 0.85$$

and

$$X_{0eq} = 0.85//0.5 = 0.31.$$

In accordance with equation 6.165 we may calculate the positive-sequence of the short-circuit current

$$\tilde{I}_1 = \frac{1.15}{j(0.38 + 0.38//0.31)} = -j2.1.$$

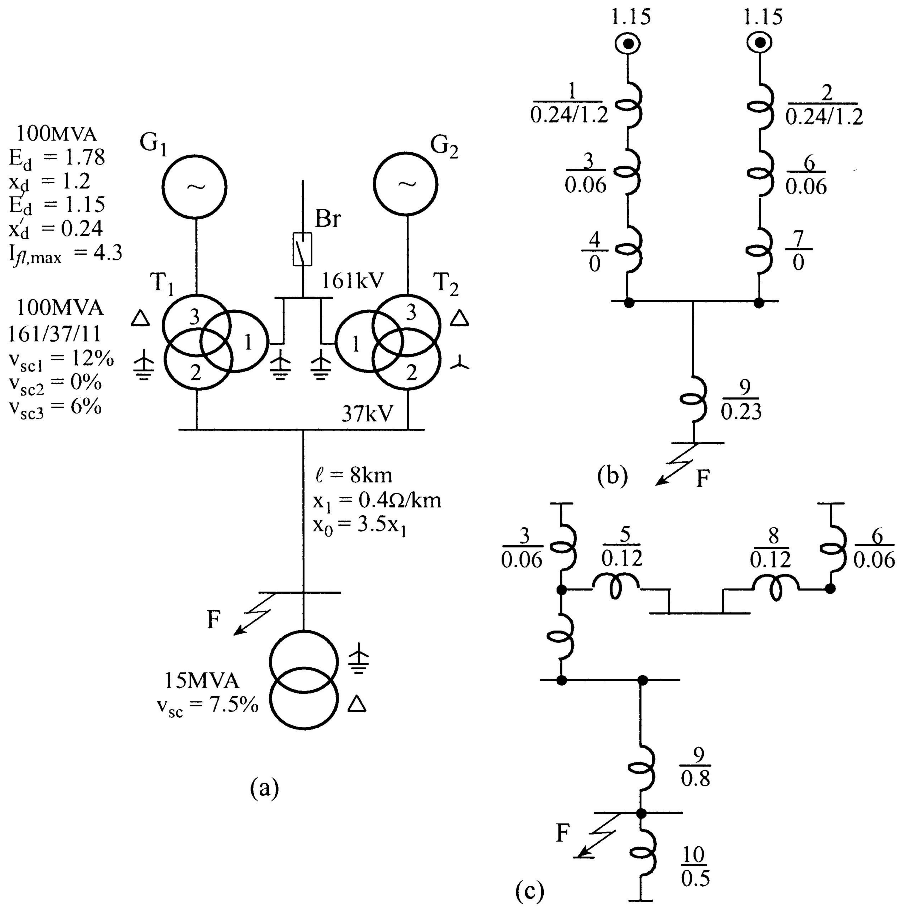


Figure 6.69 A network diagram for Example 6.28 (a), positive- and negative-sequence circuit (b) and zero-sequence circuit (c).

In accordance with Fig. 6.66(b) we have

$$\tilde{I}_2 = -\tilde{I}_1 \frac{X_{0eq}}{X_{2eq} + X_{0eq}} = j2.1 \frac{0.31}{0.38 + 0.31} = j0.94,$$

and

$$\tilde{I}_0 = -(\tilde{I}_1 + \tilde{I}_2) = -(-j2.1 + j0.94) = j1.16.$$

Thus, the first moment short-circuit current (phase b) is

$$\tilde{I}_{sc} = a^2\tilde{I}_1 + a\tilde{I}_2 + \tilde{I}_0 = (-0.5 - j0.866)(-j2.1) + (-0.5 + j0.866)(j0.94) + j1.16$$
$$= -2.63 + j1.74 \quad \text{or} \quad I_{sc} = 3.15.$$

Performing the same calculation for the steady-state s.c. yields

$$X_{1eq} = X_{2eq} = \frac{1.2 + 0.06}{2} + 0.23 = 0.86.$$

The zero-sequence resistances do not change, therefore

$$X_{0eq} = 0.31.$$

The positive-sequence of the steady-state s.c. current can now be calculated as

$$\tilde{I}_{1,\infty} = \frac{1.8}{j(0.86 + 0.86//0.31)} = -j1.65.$$

The negative- and zero-sequences of the s.c. current are

$$\tilde{I}_{2,\infty} = -\left(-j1.65\frac{0.31}{0.86 + 0.31}\right) = j0.44,$$

$$\tilde{I}_{0,\infty} = -(-j1.65 + j0.44) = j1.21.$$

The short-circuit current (phase b) is then found as

$$I_{\infty} = 2.57.$$

With the maximal field current, $I_{fl,\max} = 4.3$, we have

$$I_{\infty,\max} = \frac{4.3}{1.2 - 0.1} = 3.91,$$

and

$$\Delta I'_0 = 3.15 - 2.57 = 0.58, \quad \Delta I_{\infty} = 3.91 - 2.57 = 1.34.$$

Suppose that the transient time constants (see example 6.12) are $T'_d = 1.8$ s and $T_{ff} = 0.55$ s. Then the s.c. current at $t = 0.5$ s will be

$$I_{sc}^{(1,1)}(0.5) = 0.58e^{-0.5/1.8} + 2.57 + 0.08 \cdot 1.34 = 3.1,$$

where (see Example 6.12) $F(0.5) = 1 + 0.44e^{-0.5/0.55} - 1.44e^{-0.5/1.8} = 0.08$. As can be seen, the s.c. current, due to AVR action, has almost not changed.

6.7.3 Power in terms of symmetrical components

In general, the three-phase complex power of an unbalanced three-phase system can be expressed as the sum of three complex powers of each phase

$$\bar{S}_{3ph} = P_{3ph} + jQ_{3ph} = \bar{S}_a + \bar{S}_b + \bar{S}_c = \tilde{V}_a\tilde{I}_a^* + \tilde{V}_b\tilde{I}_b^* + \tilde{V}_c\tilde{I}_c^*. \tag{6.173}$$

The above in matrix notation will be

$$\bar{S}_{3ph} = [\tilde{V}_a \quad \tilde{V}_b \quad \tilde{V}_c]\begin{bmatrix}\tilde{I}_a \\ \tilde{I}_b \\ \tilde{I}_c\end{bmatrix}^* = \begin{bmatrix}\tilde{V}_a \\ \tilde{V}_b \\ \tilde{V}_c\end{bmatrix}^T \begin{bmatrix}\tilde{I}_a \\ \tilde{I}_b \\ \tilde{I}_c\end{bmatrix}^* \tag{6.174}$$

or

$$\bar{\mathbf{S}}_{3ph} = \tilde{\mathbf{V}}_{abc}^{T} \tilde{\mathbf{I}}_{abc}^{*}.$$

Using the matrix transformations

$$\tilde{\mathbf{V}}_{abc} = \mathbf{a}\tilde{\mathbf{V}}_{012}, \quad \tilde{\mathbf{I}}_{abc} = \mathbf{a}\tilde{\mathbf{I}}_{012}.$$

we may write

$$\tilde{\mathbf{V}}_{abc}^{T} = \tilde{\mathbf{V}}_{012}^{T}\,\mathbf{a}^{T}, \quad \tilde{\mathbf{I}}_{abc}^{*} = \mathbf{a}^{*}\tilde{\mathbf{I}}_{012}^{*}.$$

Substituting these equations into equation 6.174 we obtain

$$\bar{\mathbf{S}}_{3ph} = \tilde{\mathbf{V}}_{012}^{T}\,\mathbf{a}^{T}\mathbf{a}^{*}\tilde{\mathbf{I}}_{012}^{*},$$

where

$$\mathbf{a}^{T}\mathbf{a}^{*} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} = 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore

$$\bar{\mathbf{S}}_{3ph} = 3\tilde{\mathbf{V}}_{012}^{T}\tilde{\mathbf{I}}_{012}^{*} = 3[\tilde{V}_0 \quad \tilde{V}_1 \quad \tilde{V}_2] \begin{bmatrix} \tilde{I}_0 \\ \tilde{I}_1 \\ \tilde{I}_2 \end{bmatrix}^{*} \tag{6.175a}$$

or

$$\bar{S}_{3ph} = 3(\tilde{V}_0\tilde{I}_0^{*} + \tilde{V}_1\tilde{I}_1^{*} + \tilde{V}_2\tilde{I}_2^{*}). \tag{6.175b}$$

This significant result means that there are no **cross terms** (e.g., $\tilde{V}_0 I_1^{*}$ or $\tilde{V}_1 I_2^{*}$) in the expression of a power (equation 6.175). In other words, there is no **coupling** of power among three sequences. It is also important to mention that the symmetrical components of three-phase voltages and currents belong to the same phase, i.e., in equation 6.175 all the sequence components are of phase *a* (the subscript of phase *a* here is just ignored).

Example 6.29

For the motor operated under unbalanced conditions of Example 6.27 determine the power delivered to the motor. Perform the calculations in two ways: (a) using the symmetrical components of currents and voltages; and (b) straightforwardly by calculating each phase power.

Solution

(a) First we shall calculate the sequence voltages across the motor. Since the zero-sequence current is zero, the zero-sequence voltage is also zero. The positive-sequence voltage may be calculated as

$$\begin{aligned}\tilde{V}_{1M} &= Z_{1M}\tilde{I}_1 = (\sqrt{2}\cdot 3.6\angle 45^\circ)(39.6\angle -47.4^\circ)\\ &= 202\angle -2.4^\circ \text{ V},\end{aligned}$$

and similarly, the negative-sequence voltage is

$$\begin{aligned}\tilde{V}_{2M} &= Z_{2M}\tilde{I}_2 = (0.15 + j0.5)(-39.6\angle -47.4^\circ)\\ &= 20.7\angle -154.1^\circ \text{ V}.\end{aligned}$$

Therefore, in accordance with equation 6.175b, we have

$$\begin{aligned}\bar{S}_M &= 3[(202\angle -2.4^\circ)(39.6\angle 47.4^\circ) + (20.7\angle -154.1)(-39.6\angle 47.4^\circ)]\\ &= (24.0\angle 45^\circ + 2.46\angle 73.3^\circ)\cdot 10^3 = 17.68 + j19.33 \text{ kW}.\end{aligned}$$

(b) In accordance with equation 6.173 and substituting the results of the previous example, we have

$$\begin{aligned}\bar{S}_M &= 0 + (203\angle -116.6^\circ)(68.5\angle 137.4^\circ) + (220\angle 114.8)(68.5\angle -42.6^\circ)\\ &= (13.91\angle 20.8^\circ + 15.07\angle 72.2^\circ)\cdot 10^3 = 17.61 + j19.29 \text{ kW}.\end{aligned}$$

Note that the minor differences (less than 0.5%) in results (a) and (b) are due to rounding off the calculated numbers.

6.8 TRANSIENT OVERVOLTAGES IN POWER SYSTEMS

Transients occurring in a power system, primarily as a result of switching and lightning strokes, cause overvoltages whose peak values can be much in excess of the normal operating voltage. The first kind of overvoltage, caused by switching, is considered an inner overvoltage, and the second kind, which is caused by lightning, is considered an outer overvoltage. Estimation and/or calculation of such overvoltages is of importance in the design of a power system, particularly in consideration of the insulation requirements and the protective equipment for the lines, transformers, generators etc.

Until recently outer overvoltages have been largely determined by the insulation requirements. However, with much higher operating voltages now in use of 500 kV and 750 kV, and the projected range likely to be 1000–1500 kV, the inner overvoltages due to switching have become the major consideration.

The outer overvoltages appear on an overhead conductor of transmission lines, caused by a lightning stroke, which can be as high as 200 kA (although an average value is in the order of 20 kA). When such a current stroke arrives on an overhead conductor, two equal current surges propagate in both directions away from the point of impact. The magnitude of each voltage surge is estimated therefore as $(1/2)Z_c i_{peak}$, where Z_c is the conductor surge impedance, usually of the value of 350 Ω to 400 Ω. Thus, the average voltage surges on a 400Ω transmission line will have a peak of $(400/2)\cdot 20\cdot 10^3 = 4000$ kV. A detailed analysis of the traveling current and voltage waves, and the different methods of calculating the overvoltage in transmission lines, is given in the next chapter. We will now continue with our consideration of inner overvoltages.

6.8.1 Switching surges

From our previous considerations (see section 2.7.4) we know that when a.c. circuits are to be interrupted, as in the case of a switching short-circuit in any line (Fig. 6.70(a)), the arc between the circuit-breaker contacts occurs and when breaking the arc the recovery voltage suddenly appears across the open gap.

In our previous analysis of this circuit, however, we had assumed an instantaneous switching, i.e. that the air gap resistance was increased from zero to infinity in zero time. Omitting the detailed analysis of the *phenomenon of the burning arc* caused by an interrupted a.c. current (which is beyond the scope of this book), we may perform our analysis under the assumption that the arc has a constant length and possesses a rectangular volt-ampere characteristic, shown in Fig. 6.71.

However, during the *quashing period of the arc*, the arc voltage does not

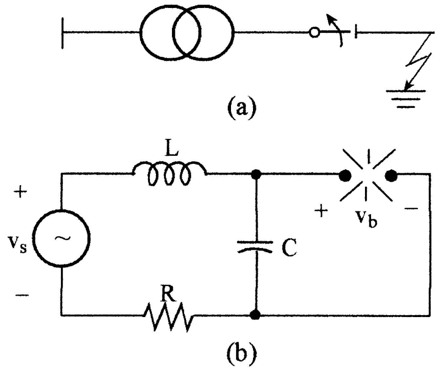


Figure 6.70 Switching of an s.c. fault: a network diagram (a) and an equivalent circuit (b).

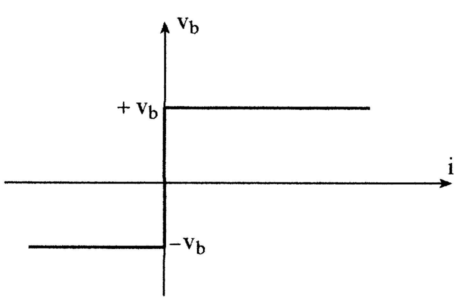


Figure 6.71 A rectangular characteristic of a burning a.c. arc.

remain constant, but gradually increases. After the interruption in such circuits, like Fig. 6.70, transient oscillations occur, which have been analyzed in sections 2.7.3 and 2.7.4. With equations 2.62, 2.63 and 2.95, 2.96 they are

$$i_n = I_n e^{-\alpha t}\sin(\omega_n t + \beta), \quad v_{C,n} \cong V_{C,n} e^{-\alpha t}\sin(\omega_n t + \beta - 90^\circ), \qquad (6.176a)$$

where $\omega_n = 1/\sqrt{LC}$, $\alpha = R/2L$, $\tan\beta = (\omega/\omega_n)\tan\psi_i$,

$$V_{C,n} \cong V_S\sqrt{\left(\frac{\omega_n}{\omega}\right)^2 \sin^2\psi_i + \cos^2\psi_i}, \quad I_n = \sqrt{\frac{C}{L}}\,V_{C,n}. \qquad (6.176b)$$

Such oscillations will occur, after the circuit-breaker contacts start to move, at every zero passage of the current. At a few of the first passages, however, since the *restriking voltage* is higher than the electric strength of the arc, the arc will reignite, Fig. 6.72. As can be seen this happens four times at the reversal of the current during the gradual separation of the contacts. The last time, however, the transient voltage of the *restriking oscillation* does not succeed in igniting the arc again and the circuit is ultimately switched. The number of reignitions depends on the speed of the contact's separation and the electric strength of the arc, which in turn depends on the deionization process, i.e. on the diffusion and recombination of the ions and the temperature of the arc and the electrodes(*).

To analyze the overvoltages under the influence of a burning arc, we shall derive the differential equations for the circuit, shown in Fig. 6.73, in which for simplicity sake the relatively small resistances are neglected. Thus, with Kirchhoff's two laws we have

$$L\frac{di}{dt} + v_B = v_S, \quad v_B = v_C = \frac{1}{C}\int i_C, \quad i = i_C + i_B, \qquad (6.177)$$

where v_B is the voltage across and i_B is the current through the arc.

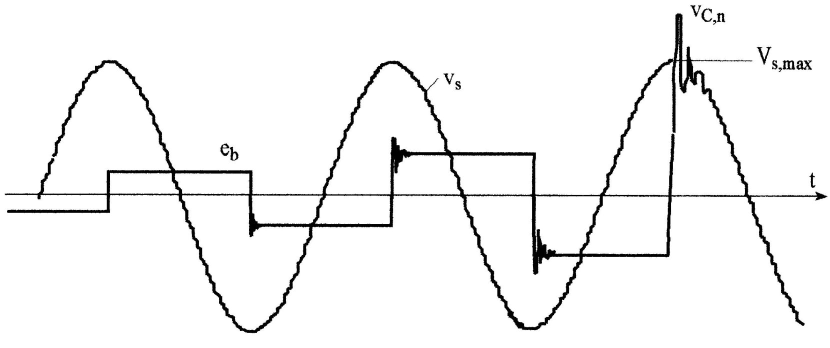


Figure 6.72 Transient oscillations during the contact separation.

(*)For a more detailed analysis of the restriking voltage after interruption see in R. Rudenberg (1969), *Transient Performance of Electric Power Systems*, MIT Press.

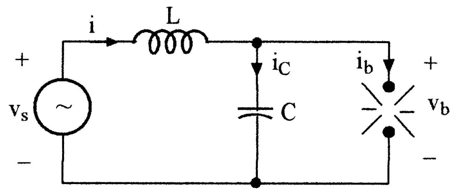


Figure 6.73 An equivalent circuit for analyzing the influence of a burning arc.

Or, substituting the third equation into the first one and expressing the capacitance current by the second yields

$$LC\frac{d^2 v_B}{dt^2} + v_B + L\frac{di_B}{dt} = v_s. \tag{6.178}$$

Since the arc has a resistive characteristic, the voltage and current will have the same form, and it is suitable to assume for the voltage (and current) the form of the exponent

$$v_B = v_{B,0}e^{(t-t_0)/T_B} = v_{B,0}e^{-t'/T_B}, \tag{6.179}$$

where t_0 is the time at which the quenching starts and T_B is the *quenching time constant*, i.e. of the deionization process. This time constant is different for different types of quenching agents: for air it is about 10^{-3} s, for gases, as in oil breakers, 10^{-4} s and for pure hydrogen 10^{-5} s.

The voltage change in accordance with equation 6.179 is shown in Fig. 6.74. Before the time t_0 the arc voltage within every half period will be nearly constant, as in Fig. 6.71, and after t_0 it will rise according to equation 6.179, which means that with arc quenching its electric strength will increase. As can be seen, the *quenching curves of the current* (starting at t_0) depend on the capacitance in parallel to the arc: from $t' = 0$ when $C = \infty$, to $t' = \pi/\omega$ when $C = 0$ (see further on).

Substituting equation 6.179 into equation 6.178 yields

$$L\frac{di_B}{dt} = v_s - \frac{1}{\omega_n^2 T_B^2} v_{B,0}e^{t'/T_B} - v_{B,0}e^{t'/T_B}, \tag{6.180}$$

and by straightforward integration we have

$$i_B = i_{B,0} - v_{B,0}\frac{T_B}{L}\left(1 + \frac{1}{\omega_n^2 T_B^2}\right)(e^{t'/T_B} - 1), \tag{6.181}$$

where $i_{B,0}$ is the current at t_0, Fig. 6.74. This current can be found by integrating only the first equation in 6.177 (the capacitance prior to interruption does not act), and the solution is

$$i_B = \frac{V_s}{\omega L}\sin(\omega t + \theta) + \frac{v_B}{\omega L}(\pi/2 - \omega t). \tag{6.182}$$

This current consists of two components: the first represents the steady-state

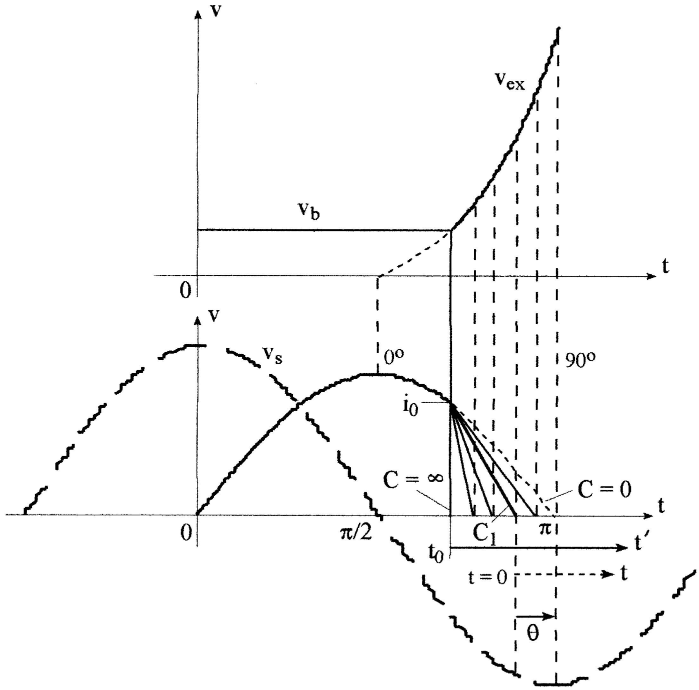


Figure 6.74 Exponentially increasing quenching voltage and current curve.

inductive current in the circuit with a closed circuit breaker and lags behind the applied voltage by 90°; the second component represents the linearly changing arc current. Note that the arc voltage, which opposes the current flow and thereby changes the phase of the current, brings it more into phase with the supply voltage, so that it will be less than 90°.

The displacement angle θ may be found, using the condition that in the quasi-steady-state regime the current must pass each half period through zero. Thus,

$$\sin\theta = \frac{\pi}{2}\frac{v_B}{V_s}. \tag{6.183a}$$

Knowing θ, the initial current i_0 can be found from its expression 6.182, or from the plot in Fig. 6.74, and may be approximated as

$$i_0 = I\sin\theta. \tag{6.183b}$$

In accordance with equation 6.181, under the effect of an exponentially increasing quenching voltage, the current will decrease exponentially passing through zero. This will happen when $i_B = 0$, then in accordance with equation 6.181 and since $i_{B,0} = i_0$ we have

$$e^{t'/T_B} = \frac{i_0 L}{v_{B,0} T_B (1 + 1/\omega_n^2 T_B^2)} + 1. \tag{6.184}$$

The amplitude of the current prior to interruption can be determined approximately as

$$I \cong \frac{V_s}{\omega L}, \tag{6.185a}$$

then

$$i_0 L = \frac{i_0 V_s}{I\omega}. \tag{6.185b}$$

Solving equation 6.184 for t' by using equation 6.185b, we have

$$t' = T_B \ln\left[1 + \frac{i_0}{I}\frac{V_s}{v_{B,0}}\frac{1}{\omega T_B(1 + 1/\omega_n^2 T_B^2)}\right]. \tag{6.186}$$

Due to the relatively small capacitance, the natural frequency is very high so that the quantity $1/\omega_n^2 T_B^2$ might be approximated as a unity. Then, with time constant $T_B \cong 10^{-4}$ and with the most common ratios $v_{B,0}/V_s \cong 1/20$ and $i_0/I \cong 1/5$, we have

$$t' = T_B \ln\left(1 + \frac{1}{5}\cdot 20\cdot\frac{1}{377\cdot 10^{-4}\cdot 2}\right) \cong 4T_B.$$

Thus, the approximate quenching time is about four times the quenching time constant.

The *extinction voltage* in p.u. at the moment that the current attains zero is determined by substituting condition 6.184 into equation 6.179, and using equation 6.185b

$$v_{B,pu}|_{i=0} = v_{ex,pu} = \frac{v_{B,0}}{V_s} + \frac{i_0/I}{\omega T_B(1 + 1/\omega_n^2 T_B^2)}. \tag{6.187a}$$

Now, equations 6.186 and 6.187a are giving better insight into the part of the capacitance, in parallel to the arc. Thus, without the capacitance, the natural frequency would be $\omega_n \to \infty$ and the quenching time will be maximal, while the p.u. extinction voltage will reach the value:

$$v_{ex,pu} = \frac{v_{B,0}}{V_s} + \frac{i_0/I}{\omega T_B}. \tag{6.187b}$$

With the previously used data, this voltage will be

$$v_{ex,pu} = 1/20 + \frac{1/5}{377\cdot 10^{-4}} \cong 5.4,$$

On the other hand, by using a very large capacitance the natural frequency approaches zero, so that the second term under the logarithm in equation 6.186 disappears and the quenching time therefore reduces to zero. The physical explanation of this result is that the current shifts instantaneously from the arc to the capacitance. The extinction p.u. voltage in this case with a moderate capacitance, giving a natural frequency of 10^3 Hz, will be much lower:

$$v_{ex,pu} = \frac{1}{20} + \frac{1/5}{377\cdot 10^{-4}[1 + 1/(2\pi\cdot 10^3\cdot 10^{-4})^2]} \cong 1.5.$$

Our next goal is to derive the actual values of the restriking voltage, in accordance with (6.176) and applying the switching laws. The charging current, prior to the passage of the arc current through zero, is

$$i_C = C\frac{d}{dt}(v_{B,0}e^{t'/T_B}) = \frac{C}{T_B}v_{B,0}e^{t'/T_B}, \tag{6.188}$$

which increases exponentially as does the voltage. At the end of the quenching period this current, by substituting equation 6.184, is

$$i_C(t') = \frac{C}{T_B}v_{B,0} + \frac{i_0}{1 + (\omega_n T_B)^2}. \tag{6.189}$$

If the extinction voltage, at the moment of passing the current zero, has risen to a value above the burning voltage of the arc, there is an expectation that with the reversal of the current the electric strength of the arc will withstand the appearing restriking voltage. For the sake of simplicity, we shall neglect the small burning voltage $v_{B,0}$ in equations 6.187 and 6.189, and also the very small current of fundamental frequency through the capacitance in the steady state. Note also that at the moment of the appearance of the restriking voltage the arc is extinct and $i_C = i_L$.

The initial conditions with equations 6.187a and 6.189, at $t = 0$, will now be

$$\begin{aligned} V_{C,n}\cos\beta' &= v_C(0) - v_{C,f}(0) = \frac{V_s(i_0/I)}{\omega T_B(1 + 1/\omega_n^2 T_B^2)} - V_s(-\cos(-\theta)) \\ I_n \sin\beta' &= i_C(0) - i_{C,f}(0) = \frac{i_0}{1 + \omega_n^2 T_B^2} - 0, \quad \text{where } \beta' = 90^\circ - \beta. \end{aligned} \tag{6.190}$$

(Note that at $t = 0$, the moment of passing the current zero, the forced capacitance voltage, i.e., the applied voltage v_s, has the initial angle $-\theta$, as shown in Fig. 6.74, for instance, for capacitance C_1.)

Dividing the first equation in 6.190 by the second one and noting that $I_n/V_{C,n} = \sqrt{C/L}$ (see equation 6.176b), we obtain for the initial phase angle β' of the restriking oscillations

$$\cot\beta' = \sqrt{\frac{C}{L}}\left[\frac{V_s(1 + \omega_n^2 T_B^2)}{\omega I T_B(1 + 1/\omega_n^2 T_B^2)} + \frac{V_s\cos\theta(1 + \omega_n^2 T_B^2)}{i_0}\right]. \tag{6.191}$$

Using equations 6.183b and 6.185a, and substituting the natural frequency from equation 6.176b, simplifies the above expression to

$$\cot\beta' = \omega_n T_B + (1 + \omega_n^2 T_B^2)\frac{\omega}{\omega_n}\cot\theta, \tag{6.192}$$

where θ is related to the end of the quenching period, Fig. 6.74.

Note that for instantaneous switching, i.e. with $T_B = 0$, this expression reduces to the previous one, as in equation 6.176. However, by comparing equation 6.192 with those in 6.176a one should take into consideration that here θ comes instead of ψ_i and the current is taken as a basic phasor, which means that β' must be replaced by $(90^\circ - \beta)$.

In power networks the value of ω/ω_n is always about 10^{-2} or lower, which in turn results in values of $\omega_n T_B$ greater than unity, and equation 6.192 then simplifies to

$$\cot \beta' = \omega_n T_B (1 + \omega T_B) \cot \theta. \tag{6.193}$$

Furthermore, if the product $\omega_n T_B$ is substantially larger than unity, as with most of the circuit breakers in practice, $\cot^2 \beta'$ will also be large compared to unity, so thus

$$1/\sin \beta' = \sqrt{1 + \cot^2 \beta'} \cong \cot \beta',$$

and with equation 6.193 for the current in equation 6.190 we have

$$I_n = i_0 \frac{\cot \beta'}{1 + \omega_n^2 T_B^2} \cong i_0 \left(\frac{1}{\omega_n T_B} + \frac{\omega}{\omega_n} \cot \theta \right). \tag{6.194}$$

Now the amplitude of the transient voltage in equation 6.190 with equations 6.194, 6.183b and 6.185a becomes

$$V_{C,n} = \sqrt{\frac{L}{C}} I_n = \sqrt{\frac{L}{C}} \frac{V_s}{\omega L} \sin \theta \left(\frac{1}{\omega_n T_B} + \frac{\omega}{\omega_n} \cot \theta \right),$$

or, after simplification

$$V_{C,n} = V_s \left(\frac{1}{\omega T_B} \sin \theta + \cos \theta \right). \tag{6.195}$$

Comparing these results with (6.176b), given for instantaneous interruption, we see that in the first term the reciprocal value of the quenching time constant T_B has taken the place of the natural frequency ω_n. Checking these results numerically we may obtain for a medium network frequency of 10 kHz, with instantaneous interruption, that the restriking voltage would be (wherein the insignificant term with $\cos \theta$ is omitted):

$$V_{C,n} = \frac{\omega_n}{\omega} V_s \sin \theta = \frac{2\pi \cdot 10 \cdot 10^3}{377} V_s \sin \theta = 167 V_s \sin \theta.$$

By gradual interruption with the quenching time constant of 10^{-4} s, an amplitude develops of only

$$V_{C,n} = \frac{1}{\omega T_B} V_s \sin \theta = \frac{1}{377 \cdot 10^{-4}} \sin \theta = 26.5 V_s \sin \theta.$$

Considering now the premature extinction by 10°, the interruption angle will be $\theta = 10^\circ$, as is often found with the interruption of s.c. currents, the p.u. restriking voltage amplitude will be

$$V_{C,n,pu} = \frac{V_{C,n}}{V_s} = 26.5 \sin 10^\circ = 4.6.$$

The restriking voltage will then be reduced by the damping effect at the rate of the damping coefficient α (equation 6.176a). Note that such restriking oscillations as shown in Fig. 6.75 start from the last extinction voltage v_{ex}.

Once again recall that the physical reason for the much smaller restriking voltage amplitude and the more favorable initial phase angle is the fact that, by increasing the arc voltage, the current is shifted away from the arc to the shunt capacitance before the final interruption is established. A resistance connected across the contacts of the circuit breaker can significantly increase the damping effect. With such a resistance the oscillation may be critically damped when $R_{dam} \geq \frac{1}{2}\sqrt{L/C}$, so that the severity of the transient will be reduced.

6.8.2 Multiple oscillations

This kind of oscillation will occur if the circuit breaker is located not at the place of the short-circuit, but rather at some distance away, as shown in Fig. 6.76. This may represent a case in which the circuit breaker is located in between a generator-fed bus and a current-limiting reactor.

The voltages across the two circuit meshes before the interruption are

$$V_1 = \frac{L_1}{L_1 + L_2} V_s, \quad V_2 = \frac{L_2}{L_1 + L_2} V_s.$$

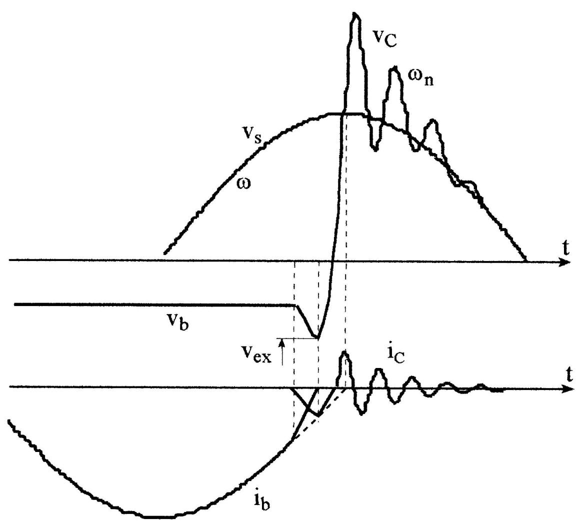


Figure 6.75 Restriking oscillations of the capacitance voltage and current.

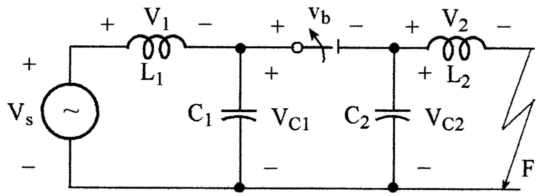


Figure 6.76 A case where a circuit breaker is connected between two meshes.

After the fault, from the instant at which the arc is finally extinct, the two circuits are separated and each oscillates at its own natural frequency.

$$\omega_{n1} = \frac{1}{\sqrt{L_1 C_1}} \quad \text{and} \quad \omega_{n2} = \frac{1}{\sqrt{L_2 C_2}}.$$

The voltage across the circuit breaker is then given by the difference between the two capacitive voltages

$$v_B = v_{C1} - v_{C2} = V_s \cos \omega t - V_1 e^{-t/T_1} \cos \omega_{n1} t - V_2 e^{-t/T_2} \cos \omega_{n2} t.$$

Figure 6.77 shows the above voltages after interruption at zero current. As can be seen from this figure, the restriking voltage across the circuit breaker has a more complicated form due to the summation of two oscillations at different frequencies.

Example 6.30

Determine the overvoltage surge set up on a 66 kV cable fed through a bulk-oil circuit breaker, when the breaker opens on a short-circuit fault. The network and breaker parameters are $R = 7.8\ \Omega$, $L = 6.5$ mH, $C = 0.16\ \mu$F and $T_B = 10^{-4}$ s. In order to increase the damping effect the shunt resistor R_{sh} is connected in parallel to the capacitance. What should its value be in order to damp the oscillations during 2 or 3 natural periods?

Solution

The natural frequency is

$$\omega_n = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{6.5 \cdot 10^{-3} \cdot 0.16 \cdot 10^{-6}}} = 3.1 \cdot 10^4 \text{ rad/s}.$$

The $\omega_n T_B$ product is

$$v = \omega_n T_B = 3.1 \cdot 10^4 \cdot 10^{-4} = 3.1.$$

With the assumption of a premature extinction by 10° as is often found with the interruption of short-circuit currents, we will have $\theta = 10^\circ$ and $i_0/I = \sin 10^\circ = 0.173$. The quenching time, with equation 6.186 and with the previously

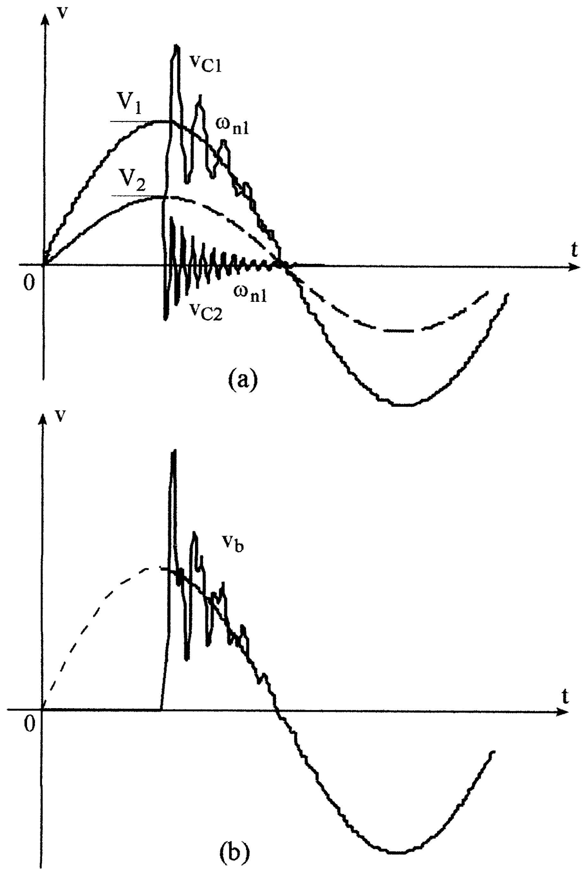


Figure 6.77 Development of the voltages across two capacitances (a) and the voltage across the circuit breaker (b).

assumed ratio $v_B/V_s = 1/120$, will be

$$t' = 10^{-4} \ln\left(1 + \frac{0.173 \cdot 20}{377 \cdot 10^{-4}(1 + 1/3.1^2)}\right) = 10^{-4} \ln 82.7 = 0.442 \text{ ms},$$

or $\omega t' = 377 \cdot 0.442 \cdot 10^{-3} = 0.167$ rad and the quenching angle will be $\theta = 0.167 \cdot 57.3 = 9.6^\circ$ (as about what was assumed). Next, we determine the initial angle β' (equation 6.193):

$$\cot \beta' = \nu(1 + \omega T_B) \cot \theta = 0.31(1 + 377 \cdot 10^{-4}) \cot 10^\circ = 18.2,$$

and $\beta' = 3.15^\circ$. The p.u. extinction voltage (equation 6.187a) is

$$v_{ex,pu} = \frac{1}{20} + \frac{0.173}{377 \cdot 10^{-4}(1 + 1/3.1^2)} = 4.1.$$

The amplitude of the transient voltage becomes (equation 6.195)

$$V_{C,n} = V_s\left(\frac{1}{377\cdot 10^{-4}}\sin 10^\circ + \cos 10^\circ\right) \cong 5.6V_s,$$

and the amplitude of the capacitance current is (equation 6.194)

$$I_{C,n} = i_0\frac{\cot\beta'}{1+\omega_n^2T_B^2} = i_0\frac{18.2}{1+3.1^2} = 1.71\cdot 0.173I = 0.3I.$$

Note that for the same circuit the amplitude of the voltage oscillation, in accordance with equation 6.176b, i.e., ignoring the arc and quenching time as happens with an instantaneous switching at the same premature angle of 10°, should be

$$\begin{aligned} V_{C,n} &= V_s\sqrt{\left(\frac{\omega_n}{\omega}\right)^2\sin^2\psi_i + \cos^2\psi_i} \\ &= V_s\sqrt{\left(\frac{3.1\cdot 10^4}{377}\right)^2\sin^2 10^\circ + \cos^2 10^\circ} = 14.2V_s, \end{aligned}$$

which is almost three times higher than in this example.

The desired damping may be derived by using a 2000 Ω resistor. Indeed, the reader may easily convince himself that in this case the damping coefficient is

$$\begin{aligned} \alpha &= \frac{R}{2L} + \frac{1}{2R_{sh}C} = \frac{7.8}{2\cdot 6.5\cdot 10^{-3}} + \frac{1}{2\cdot 2000\cdot 0.16\cdot 10^{-6}} \\ &= 4.62\cdot 10^3\ 1/\mathrm{s}, \quad \text{or} \quad \tau \cong 0.2\ \mathrm{ms}, \end{aligned}$$

which is about $2.3T_n$.

Finally, we have:

$$\begin{aligned} v_{C,n}(t) &= 5.6V_s e^{-2.16\cdot 10^3 t}\sin(3.1\cdot 10^4 t - 3.15^\circ) \\ i_{C,n}(t) &= 0.3Ie^{-9.6\cdot 10^3 t}\cos(3.1\cdot 10^4 t + 86.9^\circ). \end{aligned}$$

Here the initial angle β' is negative as the interruption is assumed to have occurred when the current changes from negative to positive values, as shown in Fig. 6.78.

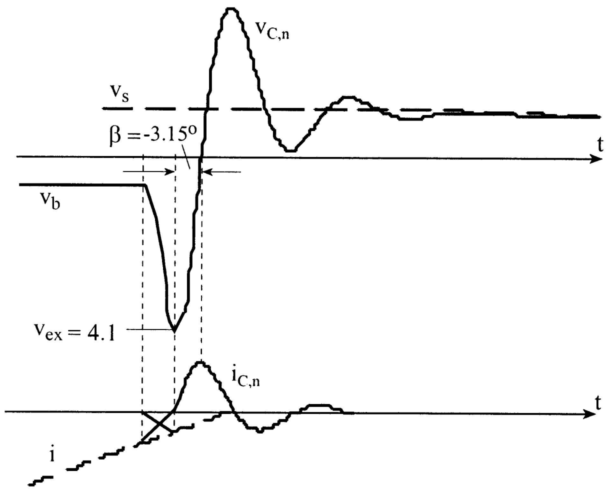


Figure 6.78 Restriking oscillation of the capacitance voltage and current.

Chapter #7

TRANSIENT BEHAVIOR OF TRANSMISSION LINES (TL)

7.1 INTRODUCTION

Transient phenomena in TL occur, like in networks with bulk parameters, when any change in their parameters, driving sources and/or configuration takes place. In general, the transients are caused by lightning, switching or faults in TL. Studies of transient disturbances on a transmission system have shown that changes are followed by traveling waves, which at first approximation can be treated as step front waves. For example, when the lightning's strike influences a line conductor, the induced voltage wave tends to divide into two halves, with the two halves *going in opposite directions*. When a voltage wave reaches a power transformer, for example, it causes a stress distribution, which is not uniform and may lead to the breakdown of the insulation system. Transient phenomena also occur in communication systems when signals of different forms are transmitted along the transmission line.

As the transmission line is a network with distributed parameters, its transient analysis, like the steady-state behavior, has to be based on partial differential equations.

7.2 THE DIFFERENTIAL EQUATIONS OF TL AND THEIR SOLUTION

Let R, G, L and C be the uniformly distributed parameters of the homogeneous line throughout its length (i.e. related to the unit of line length). Then we can represent the long line as a chain of an infinite number of incremental sections dx with the parameters: resistance Rdx, inductance Ldx, conductance Gdx and capacitance Cdx connected in series and parallel as shown in Fig. 7.1i. Let x be the distance from the sending-end to the considered section of the line; v and i be the voltage and the current at the beginning of section dx and $v + (\partial v/\partial x)dx$ and $i + (\partial i/\partial x)dx$ at the end of section dx.

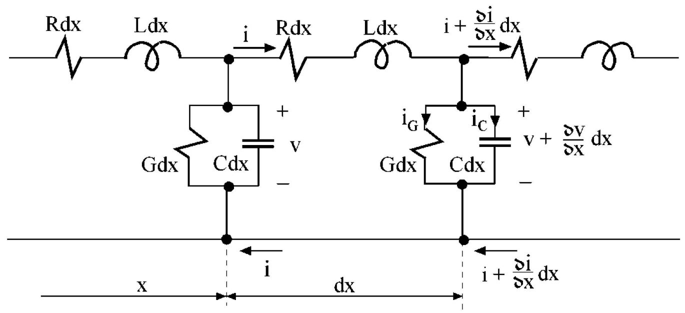


Figure 7.1i The incremental section of a transmission line.

Note that the voltage and current in a transmission line are functions of two variables x and t. We can now write two equations for this section by applying Kirchhoff's two laws:

$$v = \left(v + \frac{\partial v}{\partial x} dx \right) + Rdx\, i + Ldx \frac{\partial i}{\partial t}$$

$$i = \left(i + \frac{\partial i}{\partial x} dx \right) + Gdx \left(v + \frac{\partial v}{\partial x} dx \right) + Cdx \frac{\partial}{\partial t} \left(v + \frac{\partial v}{\partial x} dx \right).$$

Combining similar terms, dividing by dx and neglecting the quantities of second order infinitesimality, we obtain two *differential equations of partial derivatives*:

$$\begin{aligned} -\frac{\partial v}{\partial x} &= Ri + L\frac{\partial i}{\partial t}, \\ -\frac{\partial i}{\partial x} &= Gv + C\frac{\partial v}{\partial t}. \end{aligned} \tag{7.1}$$

Equations 7.1 are known in classical physics as the equations of telegraphy. They reduce to wave equations if R and G are set equal to zero. The solution of equation 7.1 with known initial and boundary (terminal) conditions allows for obtaining the line current and voltage in any point of the line as a function of time and distance from a terminal point.

The influence of resistance R and conductance G relative to L and C in transmission lines is negligible (especially for fast running processes like high frequency signals or transient phenomena). In addition, since the traveling time of waves is relatively small, the influence of losses is scarcely significant. So to simplify the analysis the line will be assumed to be loss-less. Therefore

$$\frac{\partial v}{\partial x} = -L\frac{\partial i}{\partial t} \tag{7.2a}$$

$$\frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t}. \tag{7.2b}$$

Note that the negative signs in equations 7.2 are due to the fact that both voltage v and current i decrease as x increases (the direction at which distance x advances along the line).

Taking the partial derivative of equation 7.2a with respect to x and the derivative of equation 7.2b with respect to t, we obtain

$$\frac{\partial^2 v}{\partial x^2} = -L\frac{\partial^2 i}{\partial x\,\partial t} \tag{7.3}$$

$$\frac{\partial^2 i}{\partial x\,\partial t} = -C\frac{\partial^2 v}{\partial t^2}. \tag{7.4}$$

Substituting equation 7.3 into equation 7.4, current i can be eliminated, so that

$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2}. \tag{7.5}$$

Similarly, voltage v can be eliminated, so that

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2}. \tag{7.6}$$

Equations 7.5 and 7.6 are known as wave equations – they are identical for both v and i. When one of these functions is found, the other can be found by applying either equation 7.2a or 7.2b.

The solution of the wave equation can be determined intuitively. Paying attention to the fact that the second derivatives of the voltage v and current i functions, with respect to t and x, have to be directly proportional to each other, means that the solution can be any function as long as both independent variables t and x appear in the form:

$$w_{1,2} = x \pm vt. \tag{7.7}$$

Therefore, usually the solution of equation 7.5 will be

$$v(x, t) = v_1 + v_2 = f_1(x - vt) + f_2(x + vt), \tag{7.8}$$

which satisfies equation 7.5.

In order to ensure this and determine the meaning of v, let us substitute one of the functions (equation 7.8), for example f_1, in equation 7.5. Its first derivative with respect to x is:

$$\frac{\partial v_1}{\partial x} = \frac{\partial f_1}{\partial w_1}\frac{\partial w_1}{\partial x} = \frac{\partial f_1}{\partial w_1}, \tag{7.9a}$$

and the second derivative is

$$\frac{\partial^2 v_1}{\partial x^2} = \frac{\partial^2 f_1}{\partial w_1^2}. \tag{7.9b}$$

The first derivative of equation 7.8 with respect to t is:

$$\frac{\partial v_1}{\partial t} = \frac{\partial f_1}{\partial w_1}\frac{\partial w_1}{\partial t} = \frac{\partial f_1}{\partial w_1}(-v), \tag{7.10a}$$

and the second derivative is

$$\frac{\partial^2 v_1}{\partial t^2} = v^2 \frac{\partial^2 f_1}{\partial w_1^2}. \tag{7.10b}$$

Substituting equations 7.9b and 7.10b in equation 7.5 yields

$$\frac{\partial^2 f_1}{\partial w_1^2} = LC\, v^2 \frac{\partial^2 f_1}{\partial w_1^2}.$$

This equation becomes an equality, if $LC\, v^2 = 1$, or

$$v = \frac{1}{\sqrt{LC}}\ (\mathrm{m/s}). \tag{7.11}$$

Hence, v having unit meters per second represents the velocity and, as will be shown in the following paragraph, it is the *velocity of the voltage and current wave propagation* along the line. Similarly, it can be shown that the second term (f_2) in equation 7.8 satisfies equation 7.5 with the same meaning of v.

Now the current function i may be found in accordance with equations 7.2a and 7.9a. Indeed, substituting first $\partial/\partial w_1$ (where f_1 is the first function of equation 7.8) into equation 7.2a for $\partial v/\partial x$ gives

$$\frac{\partial f_1}{\partial w_1} = -L\frac{\partial i_1}{\partial t},$$

and after integration, with respect to t

$$\int \frac{\partial f_1}{\partial w_1}\, dt = -L \int \frac{\partial i_1}{\partial t}\, dt$$

yields $[1/(-v)]f_1 = -Li_1$, since $\partial w_1/\partial t = -v = \text{const}$, or

$$i_1 = \frac{1}{vL} f_1(x - vt) = \frac{1}{Z_c} v_1, \tag{7.12}$$

where

$$Z_c = vL = \frac{L}{\sqrt{LC}} = \sqrt{\frac{L}{C}} \tag{7.13}$$

is the *characteristic impedance* of a loss-less transmission line.

Following the same steps, the second part of the current, i.e., i_2, may be obtained with only a difference in the sign. Indeed, after the integration of equation 7.13 for f_2 and i_2 we obtain $(1/v)f_2 = -Li_2$ or

$$i_2 = -\frac{1}{vL} f_2(x + vt) = -\frac{1}{Z_c} v_2. \tag{7.14}$$

Therefore, the entire current function is

$$i(x, t) = \frac{1}{Z_c} [f_1(x - vt) - f_2(x + vt)] = i_1 + i_2. \tag{7.15}$$

In conclusion, it must be mentioned that the actual shape of the voltage and current functions and their components f_1 and f_2 is defined by the initial and boundary (or terminal) conditions of a given problem, and also by the activating sources.

7.3 TRAVELING-WAVE PROPERTY IN A TRANSMISSION LINE

The behavior of the voltage and current functions of equations 7.8 and 7.15 can be understood by selecting some particular point on the wave (zero-crossing, maximum/minimum etc.) and checking (following) it for different instances of time. This result may be achieved by keeping the argument of v_1 (or i_1) constant, for example, for point A of $v_1 = 0$ in Fig. 7.1(a).

$$w_A = x - vt = \text{const.} \tag{7.16a}$$

This means that when t increases, x increases too, so $\Delta x = v\Delta t$ and this particular point A moves a distance of Δx, as shown in Fig. 7.1(a). Thus, the voltage function v_1, if plotted as a function of x for consecutive values of time as shown in Fig. 7.1(a) (bold line), appears to move in the positive $(+x)$ direction (broken line). Hence, v_1 and i_1 are said to be the **forward-traveling waves** v_f and i_f (or **incident waves**).

Similarly, checking v_2 (or i_2) and keeping

$$w_B = x + vt = \text{const.} \tag{7.16b}$$

causes x to decrease as t increases, i.e., $\Delta x = -v\Delta t$, which means that a particular point (for example, point B) on the v_2 wave shown in Fig. 7.1(c) appears to move in the negative $(-x)$ direction.

Hence, v_2 and i_2 are said to be the **backward-traveling waves** v_b and i_b (or **reflected waves**). In both cases, v represents the velocity of the voltage and current wave propagation, or simply the **velocity of propagation.**

In loss-less transmission lines the waves of voltage and current propagate without changing their shape. The measurement instrument, such as an oscilloscope, which is connected for example at the point x_1 of the line, will show the voltage wave as a function of time as shown in Fig. 7.1(b) (bold line). Note that the viewed curve is similar (although in a different scale) to the voltage distribution on the line, i.e., as a function of x. The oscilloscope connected at the next point x_2 will show the same curve (broken line) but with a time delay of $\Delta t =$

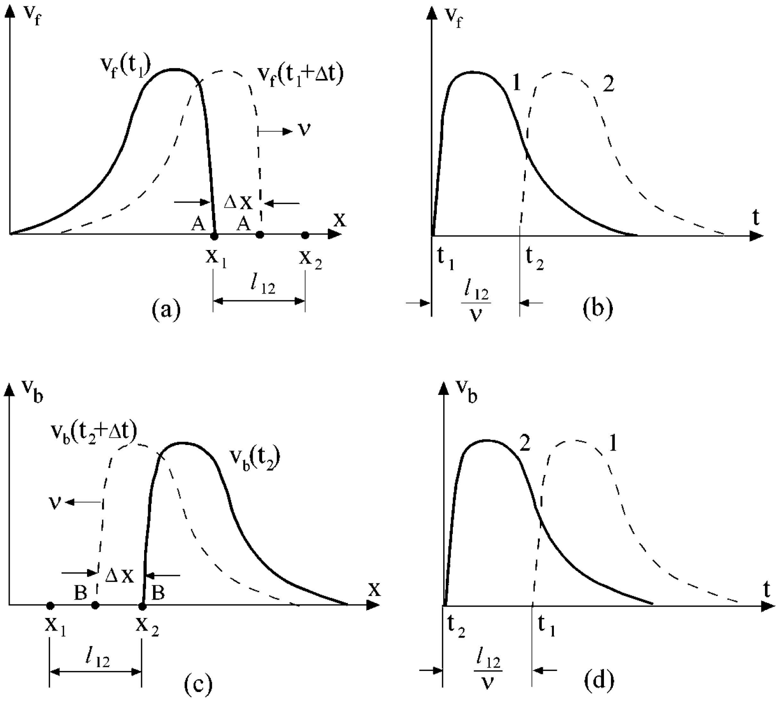


Figure 7.1 Traveling voltage wave as a function of distance x and as a function of time t: (a) and (b) forward-traveling wave, (c) and (d) backward-traveling wave.

$(x_2 - x_1)/v = \ell_{12}/v$, where ℓ_{12} is the distance between the points x_1 and x_2 (Fig. 7.1(b). The backward-traveling wave pattern for different values of time is shown in Fig. 7.1(d).

In conclusion, it must be mentioned that at any point on the line, including points of discontinuity (i.e., at the end of the line at the point of the connection of two different lines, etc.), the instantaneous voltage and current can be expressed as

$$v = v_f + v_b \tag{7.17a}$$

$$i = i_f + i_b, \tag{7.17b}$$

where the voltage and current traveling wave pair is connected by the characteristic impedance of the line Z_c:

$$i_f = \frac{v_f}{Z_c}, \quad i_b = -\frac{v_b}{Z_c}. \tag{7.18}$$

The negative sign in the relation between the voltage and current of the backward-traveling waves is important. It is not dependent on how either the coordinate system or the positive polarity/direction of voltage/current may be chosen. It is understandable in terms of the power shown that the power of a backward-traveling wave is always negative, which indicates a movement of energy in the negative direction of x, i.e. in the direction of travel of the $v_b(x, t)$ and $i_b(x, t)$ waves.

It can be shown that in the transient behavior of TL, like in the steady state regime, the power of a forward-traveling wave for example can be expressed in terms of energy content and wave propagation velocity

$$P_f = W_f v, \tag{7.19}$$

where $P_f = v_f i_f$ is the *power of the forward traveling wave.*

Indeed, the energies stored in electric (C) and magnetic (L) fields per unit length of TL are

$$W_e = \frac{1}{2} C v_f^2, \quad W_m = \frac{1}{2} L i_f^2. \tag{7.20}$$

Since the two components of energy storage are equal, the *total energy content stored per unit length* is

$$W_f = W_e + W_m = C v_f^2 = L i_f^2. \tag{7.21}$$

Therefore, the above power can be expressed as

$$P_f = v_f i_f = \frac{v_f^2}{Z_c} = \frac{C v_f^2}{\sqrt{LC}} = W_f v. \tag{7.22a}$$

Of course, the same result can be obtained for a backward-traveling wave

$$P_b = W_b v. \tag{7.22b}$$

Note that the *total transient power* is a sum of these two components, i.e., the forward- and backward-traveling waves:

$$P = vi = (v_f + v_b)(i_f - i_b) = v_f i_f + v_b i_b + (v_b i_f - v_f i_b) = P_f + P_b$$

since

$$(v_b i_f - v_f i_b) = \left(\frac{v_b v_f}{Z_c} - \frac{v_f v_b}{Z_c} \right) = 0.$$

Example 7.1

The surge voltage of 1000 kV caused by lightning propagates along the transmission line, having the distributed parameters $L = 1.34$ mH/km and $C = 8.6$ nF/km.

Determine: (a) The surge power in the line, (b) The surge current in the line.

Solution

(a) The total energy stored in an electromagnetic field per unit length of the line is

$$W = Cv^2 = 8.6\cdot 10^{-9}(1000\cdot 10^3)^2 = 8.6\cdot 10^3 \text{ J/km}$$

The surge velocity is

$$v = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1.34\cdot 10^{-3}\cdot 8.6\cdot 10^{-9}}} = 295\cdot 10^3 \text{ km/s}$$

Therefore, the surge power is

$$P = 8.6\cdot 10^3\cdot 295\cdot 10^3 \cong 2500 \text{ MW}.$$

(b) The characteristic impedance of the line is

$$Z_c = \sqrt{\frac{L}{C}} = \sqrt{\frac{1.34\cdot 10^{-3}}{8.6\cdot 10^{-9}}} = 395\ \Omega.$$

Therefore the surge current is

$$i = \frac{v}{Z_c} = \frac{1000\cdot 10^3}{395} = 2.53 \text{ kA}.$$

It should be noted that for the transient analysis of some problems, when the voltage and current change versus time is needed, it is more convenient to express the voltage and current-traveling waves in the form

$$v = v_f + v_b = \phi_1\left(t - \frac{x}{v}\right) + \phi_2\left(t + \frac{x}{v}\right) \tag{7.23a}$$

$$i = i_f - i_b = \frac{1}{Z_c}\left[\phi_1\left(t - \frac{x}{v}\right) - \phi_2\left(t - \frac{x}{v}\right)\right]. \tag{7.23b}$$

7.4 WAVE FORMATIONS IN TL AT THEIR CONNECTIONS

In practice, all kinds of transmission lines are necessarily terminated by sources or by loads. In addition, a lumped impedance or lumped admittance network may be inserted in tandem between sections of a line, or two (or more) different lines may be connected in a network junction. As has been already mentioned, to determine traveling-wave functions, the boundary conditions of line terminations must be taken into consideration. In other words, at any point of such non-uniformness or discontinuity, i.e., transition points, Ohm's and Kirchhoff's law equations must be obeyed in addition to traveling-wave equations.

Therefore, if the voltage and current at such a transition point are known, it can be written as

$$v_T = v_f + v_b \tag{7.24}$$

$$i_T = i_f + i_b. \tag{7.25}$$

Taking into consideration the relation between the voltage and current traveling waves (equation 7.18), according to equation 7.25 we obtain

$$Z_c i_T = v_f - v_b. \tag{7.26}$$

Adding equations 7.24 and 7.26 we have

$$v_T + Z_c i_T = 2v_f, \tag{7.27}$$

or the forward-traveling wave will be

$$v_f = \frac{1}{2}(v_T + Z_c i_T). \tag{7.28}$$

Similarly, by subtracting equation 7.26 from equation 7.24 we obtain

$$v_b = \frac{1}{2}(v_T - Z_c i_T), \tag{7.29}$$

also from equation 7.26.

$$v_b = v_f - Z_c i_T. \tag{7.30}$$

7.4.1 Connecting the TL to a d.c./a.c. voltage source

Consider the a.c. source, which is connected at $t = 0$ to a power transmission line. Such a source of industrial frequency of 50–60 Hz hardly changes during the time which is needed for a wave to propagate in hundreds of kilometres (note that the wave propagation time along a line of 1000 km length is about 30 μs as the period of the 50 Hz voltage source is $20 \cdot 10^3$ μs). Therefore, in the initial stage of wave formation, the a.c. source can be treated as a d.c. source.

Now consider the transmission line connecting at the time $t = 0$ to the d.c. source V_s having input impedance Z_s as shown in Fig. 7.2(a). As the line was not initially charged, no backward (reflection) waves exist at the first moment after the connection. Therefore equations 7.24 and 7.25 yield

$$v_T = v_f, \quad i_T = i_f. \tag{7.31}$$

Applying the boundary condition

$$v_T = V_s - Z_s i_T \tag{7.32}$$

and solving equations 7.28 and 7.32 with equation 7.31 yields

$$v_f = \frac{Z_c}{Z_s + Z_c} V_s = \rho_s V_s, \tag{7.33}$$

where

$$\rho_s = \frac{Z_c}{Z_s + Z_c} \tag{7.33a}$$

is the **source transmission coefficient**.

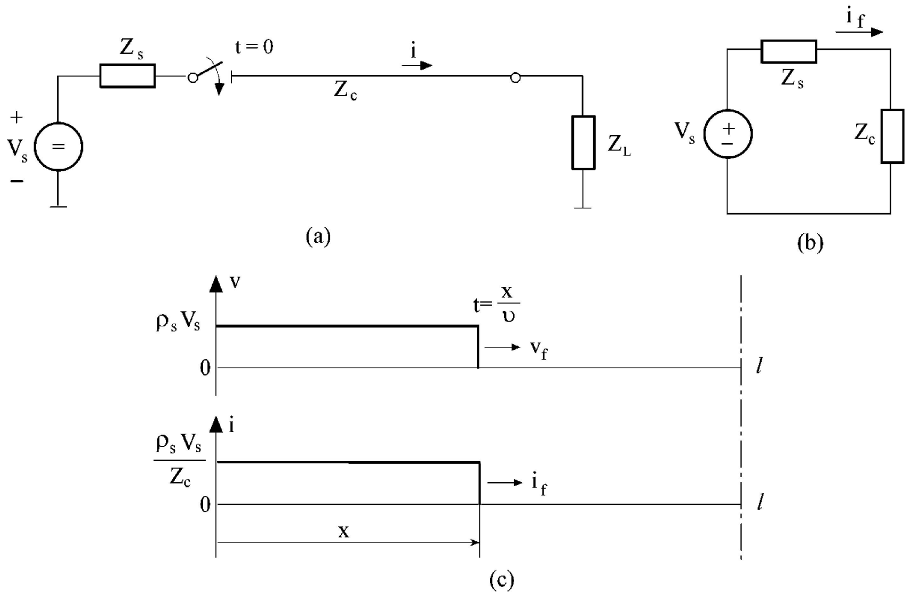


Figure 7.2 Waves traveling on a line by connecting to a source: circuit diagram (a), equivalent circuit (b), voltage and current distribution (c).

Note that the forward-traveling wave voltage (equation 7.33) might be determined in accordance with the equivalent circuit shown in Fig. 7.2(b) as the voltage across the characteristic impedance Z_c.

If the connecting source is ideal ($Z_s = 0$), the forward-traveling wave is simply $v_f = V_s$. Assuming at this point that the source input impedance is pure resistive, we can conclude that the voltage distribution along the line will be just the voltage at the sending-end of the line with the time delay $t = x/v$ as is shown in Fig. 7.2(c), i.e., a step function wave.

Disconnecting a transmission line from the source also causes step function waves to appear (Fig. 7.3). Assume that, at the disconnecting moment, the current in the line was I_s and the voltage was V_s. Since after the disconnection the current at the sending-end becomes zero $i_T = i_f + i_b + I_s = 0$ and noting that no reflection wave yet exists, we obtain

$$i_f = -I_s, \quad v_f = -Z_c I_s. \tag{7.34}$$

The voltage distribution will be a sum of the previous voltage V_s and the forward-traveling wave

$$v(x, t) = V_s - Z_c I_s \left(t - \frac{x}{v} \right), \tag{7.35}$$

and it will be positive if the current I_s is smaller than the charge current V_s/Z_c (i.e., $I_s < (V_s/Z_c)$) as shown in Fig. 7.3(b) or negative if vice versa.

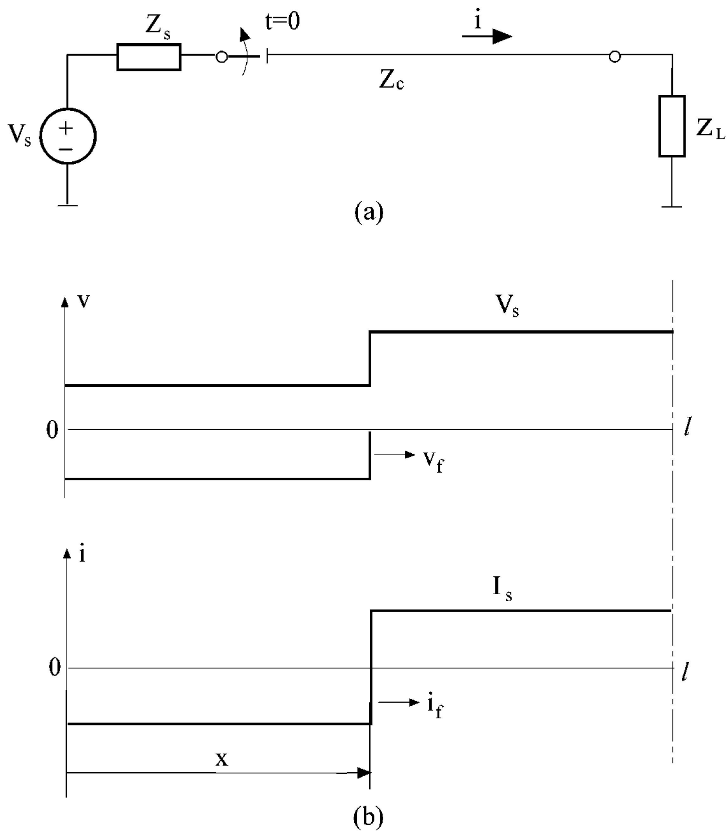


Figure 7.3 Waves traveling on line by disconnecting from a source: circuit diagram (a), voltage and current distribution (b).

7.4.2 Connecting the TL to load

Consider the transmission line shown in Fig. 7.4(a). After turning on the switch, the load impedance terminates the TL and the backward-traveling wave will appear. To determine it, Ohm's law must be obeyed for the receiving terminal of the line:

$$v_{\mathrm{T}} = Z_L i_{\mathrm{T}}. \tag{7.36}$$

Taking into consideration that the line was charged by voltage, say V_s, the equations 7.24 and 7.25 with equation 7.18 yield

$$v_{\mathrm{T}} = v_{\mathrm{b}} + V_s, \quad i_{\mathrm{T}} = i_{\mathrm{b}} = -\frac{v_{\mathrm{b}}}{Z_c}. \tag{7.37}$$

Solving equations 7.36 and 7.37 gives

$$v_{\mathrm{b}} = -\frac{Z_c}{Z_L + Z_c} V_s, \quad i_{\mathrm{b}} = \frac{V_s}{Z_L + Z_c}. \tag{7.38}$$

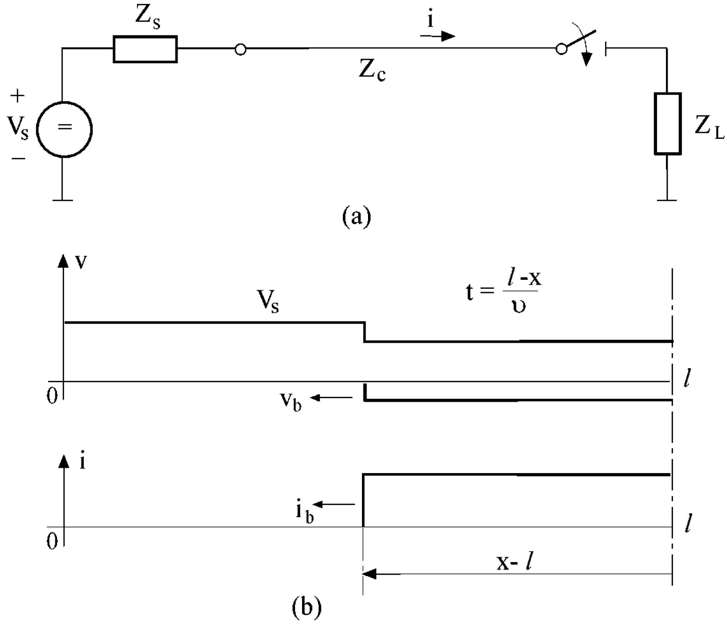


Figure 7.4 Traveling waves by load connection: circuit diagram (a): voltage and current distribution (b).

Assuming that load impedance Z_L is pure resistive ($Z_L = R_L$), the back-traveling wave will be a step-wave as shown in Fig. 7.4(b):

$$v_b(x, t) = \rho_\tau V_s\left(t + \frac{x - \ell}{v}\right), \tag{7.39}$$

where $\rho_\tau = -Z_c/(R_L + Z_c)$ is the **load transmission coefficient**.

The step-wave also appears at the time when the resistive load is disconnecting just as in disconnecting the source (Fig. 7.5(a)), $i_b = -I_L$ and $v_b = Z_c I_L$, where I_L is the load current in the line at the moment of disconnection.

The voltage distribution will be the sum of the previous voltage V_L and the backward-traveling wave:

$$v(x, t) = V_L + Z_c I_L\left(t + \frac{x - \ell}{v}\right), \tag{7.40}$$

as shown in Fig. 7.5(b). Note that disconnecting the load results in a voltage increase, which in power TL can be significant.

Using the above studied technique, situations that are more complicated can be solved as in the following example.

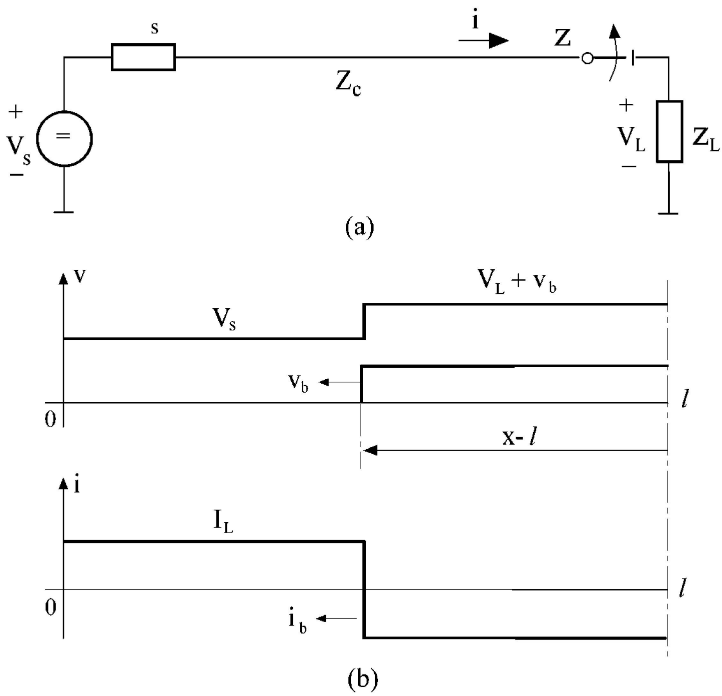


Figure 7.5 Traveling waves by load disconnection: circuit diagram (a); voltage and current distribution (b).

Example 7.2

Determine the voltage and current waves due to the connection of the resistive load $R_L = 300\,\Omega$ at the arbitrary point of the TL shown in Fig. 7.6(a). The characteristic impedance of TL is $400\,\Omega$ and it is charged with initial voltage $V_0 = 20$ kV and initial current $I_0 = 50$ A.

Solution

Since both directions of wave propagation (to the left and right from the connection point) are symmetrical, both current waves will be equal to each other, i.e., $i_f = i_b$ and $v_f = v_b$. Applying KCL and Ohm's law,

$$i_L = -(i_f + i_b) = -2i_f = -2i_b, \quad v_L = R_L i_L = V_0 + v_f = V_0 + v_b. \tag{7.41}$$

Solving equation 7.41 with the relation $v_f = Z_c i_f$, we obtain

$$i_b = i_f = -\frac{V_0}{2R_L + Z_c} = -\frac{20 \cdot 10^3}{600 + 400} = -20 \text{ A}$$

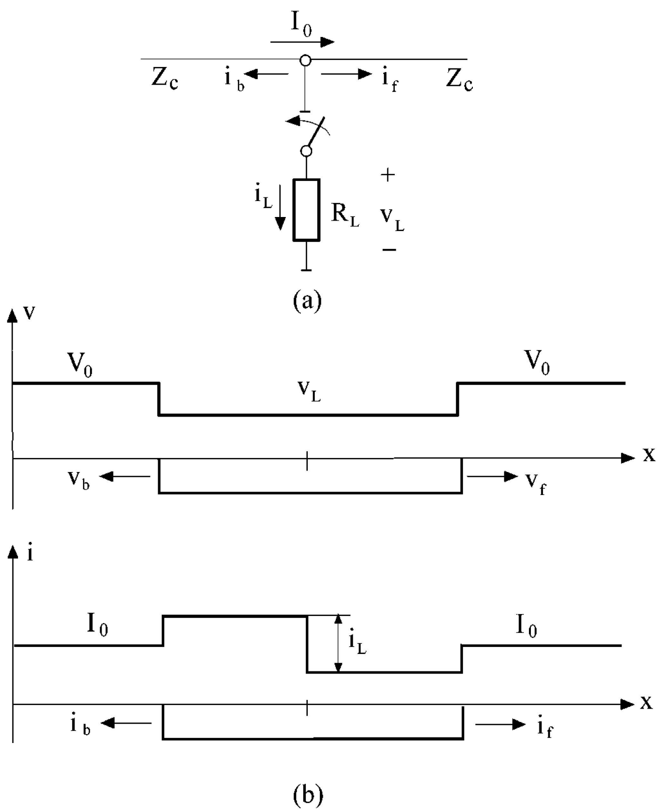


Figure 7.6 Traveling waves by load connection at an arbitrary point on the line: circuit diagram (a); voltage and current distribution (b).

and

$$v_{\mathrm{b}} = v_{\mathrm{f}} = Z_{\mathrm{c}} i_{\mathrm{f}} = -\frac{V_0 Z_{\mathrm{c}}}{2R_L + Z_{\mathrm{c}}} = -20\frac{400}{1000} = -8\ \mathrm{kV}.$$

The voltage and current distribution along the TL is shown in Fig. 7.6(b).

7.4.3 A common method of determining traveling waves by any kind of connection

Consider an active network connecting to the junction of two lines, as shown in Fig. 7.7(a). The forward-traveling wave will appear on the right line and the backward-traveling wave on the left line. Both current waves can be determined from the equivalent circuit in which the active network is presented by its *Thévenin equivalent* and the two lines by their characteristic impedances, as shown in Fig. 7.7(b). Note that the voltage source of the Thévenin equivalent is simply the voltage across the switch at the zero initial condition of the lines.

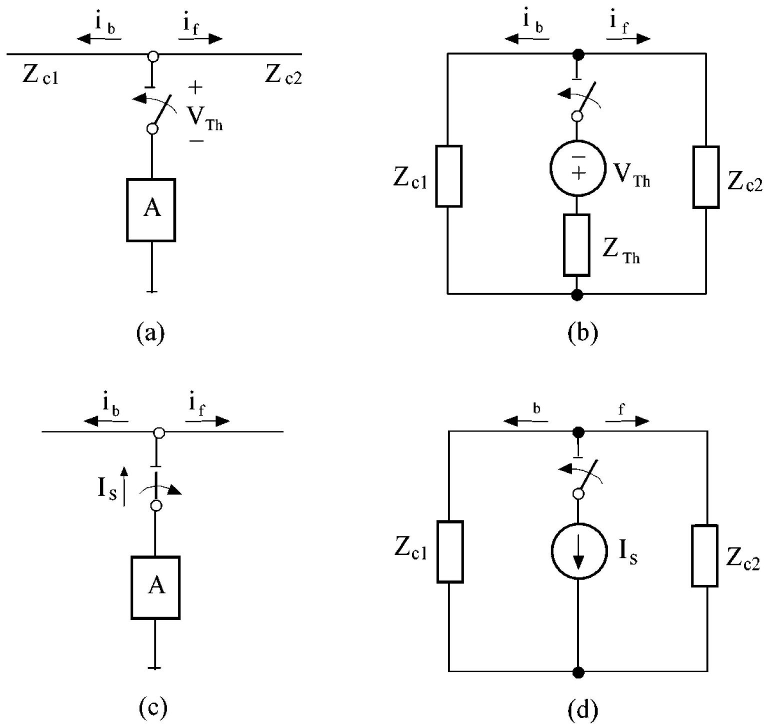


Figure 7.7 Active network connection and disconnection: circuit diagrams (a) and (c); the equivalent circuits for connection and disconnection (b) and (d).

If the lines were initially charged, the final voltage and current distribution will simply be the superposition of the initial values and the traveling waves.

If the switch is opening, i.e., the network is disconnecting, it has to be represented in the equivalent circuit by an ideal current source as shown in Fig. 7.7(c) and (d). Note that the value of the current source is equal to the current which flowed through the switch just before it opened.

As an example, consider the line connecting to the voltage source with inductive-resistive impedance, as shown in Fig. 7.8(a). Therefore, the equivalent circuit will simply be the series connection of the source and the characteristic impedance of the line as shown in Fig. 7.8b. The transient response of this circuit gives the forward-traveling current wave at the sending-end as

$$i_f = \frac{V_0}{R_0 + Z_c}\left(1 - e^{-\frac{t}{\tau}}\right),$$

where $\tau = L_0/(R_0 + Z_c)$ is the time constant of the circuit. The current distribution along the line will be

$$i_f(x, t) = \frac{V_0}{R_0 + Z_c}\left(1 - e^{-\frac{t - x/v}{\tau}}\right),$$

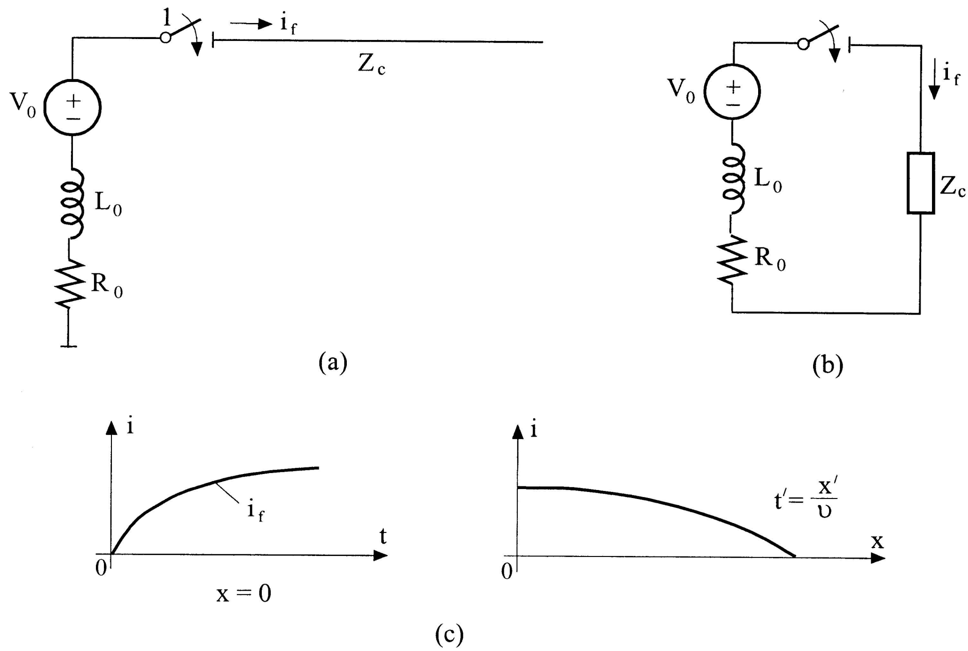


Figure 7.8 Voltage source with input impedance connecting to the TL: circuit diagram (a); the equivalent circuit (b); the current curve versus time, and current distribution along the line (c).

and is shown in Fig. 7.8(c), which corresponds to the moment of the arrival of the wave at point x'. The voltage wave is proportional to the current wave $v_f = Z_c i_f$.

7.5 WAVE REFLECTIONS IN TRANSMISSION LINES

Consider, at first, a step-function forward-traveling wave or **incident wave**. The moment that this wave reaches the receiving-end of the line (point 2) the **reflecting wave** will appear. In the general case of a line terminated in impedance Z_T, the boundary condition is simply Ohm's law(*)

$$v_T = Z_T i_T. \tag{7.42}$$

Let the front of the incident wave in expression 7.27 be V_0 ($v_f = V_0$), then we obtain

$$Z_T i_T + Z_c i_T = 2V_0, \tag{7.43}$$

(*)In cases where the line terminations consist of inductances or/and capacitances, expression (7.42) and subsequent ones are solved by means of the Laplace-transform method (see Chapter 3).

or

$$i_{\mathrm{T}} = \frac{2V_0}{Z_{\mathrm{T}} + Z_{\mathrm{c}}}. \tag{7.44}$$

Expression 7.44 shows that the current at the receiving-end of the line can be determined from the *equivalent lumped-impedance circuit* in which the line and its termination are represented by their impedances connected in series, while the circuit is activated by the double value of the incident wave, as shown in Fig. 7.9. The voltage at the receiving-end (equation 7.42) can be expressed as

$$v_{\mathrm{T}} = \frac{2Z_{\mathrm{T}}}{Z_{\mathrm{T}} + Z_{\mathrm{c}}} V_0 = \rho_{\mathrm{ref}} V_0, \tag{7.45}$$

where ρ_{ref} is the **refraction coefficient** or **transmission factor**

$$\rho_{\tau} = \frac{2Z_{\mathrm{T}}}{Z_{\mathrm{T}} + Z_{\mathrm{c}}}. \tag{7.46}$$

The reflecting or backward-traveling wave are easily obtained as

$$v_{\mathrm{b}} = v_{\mathrm{T}} - v_{\mathrm{f}} = \frac{2Z_{\mathrm{T}}}{Z_{\mathrm{T}} + Z_{\mathrm{c}}} V_0 - V_0 = \frac{Z_{\mathrm{T}} - Z_{\mathrm{c}}}{Z_{\mathrm{T}} + Z_{\mathrm{c}}} V_0 = \rho_{\mathrm{r}} V_0, \tag{7.47}$$

where ρ_{r} is the **receiving-end reflection coefficient**

$$\rho_{\mathrm{r}} = \frac{Z_{\mathrm{T}} - Z_{\mathrm{c}}}{Z_{\mathrm{T}} + Z_{\mathrm{c}}}. \tag{7.48}$$

When the reflecting wave arrives at the sending-end, it reflects again. The **sending-end reflection coefficient** will then be similar, i.e.,

$$\rho_{\mathrm{s}} = \frac{Z_{\mathrm{s}} - Z_{\mathrm{c}}}{Z_{\mathrm{s}} + Z_{\mathrm{c}}}, \tag{7.49}$$

where Z_{s} is the sending-end termination impedance or the generator input impedance.

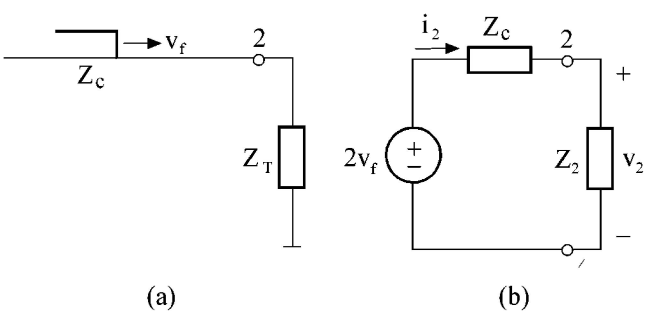


Figure 7.9 The incident wave arriving at the line termination: line diagram (a); equivalent circuit (b).

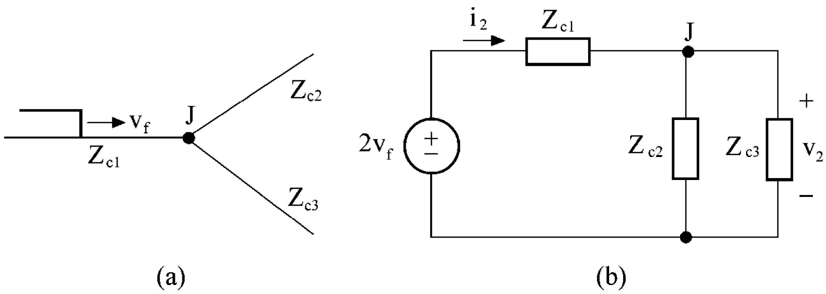


Figure 7.10 The junction of two outgoing lines: line diagram (a); equivalent circuit (b).

The concept of an equivalent circuit can be used for any kind of discontinuity. For example, in Fig. 7.10(a) the junction of three transmission lines is shown. The equivalent circuit is shown in Fig. 7.10(b) in which the two outgoing lines are represented by their characteristic impedances in parallel. The second example of an inductance connected between two transmission lines is shown in Figs. 7.11(a) and (b). Here the elements, which formed the junction of discontinuity, are represented in the equivalent circuit by their impedances in series.

In general, the equivalent circuit of any junction of discontinuity consists of lumped impedances, which represent the elements connected to the junction, and of the characteristic impedances of the lines. The circuit is driven by a voltage source of a double value of the incident wave voltage function.

Let us examine several particular kinds of TL terminations.

7.5.1 Line terminated in resistance

In this case the reflecting wave has the same shape as the incident wave, i.e. the shape of a step-function. (Note that the characteristic impedance of a loss-less line is also pure resistance.) The reflection wave is determined by the reflection

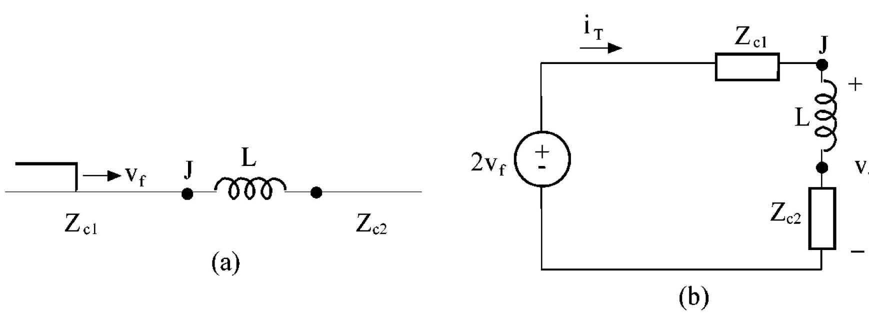


Figure 7.11 The connection of lumped-impedance in between two lines: line diagram (a); equivalent circuit (b).

coefficient (equation 7.47)

$$v_b = \rho_r V_f = \frac{R_T - Z_c}{R_T + Z_c} v_f. \tag{7.50}$$

The reflection coefficient ρ_r can be positive or negative, depending on the relative values of R_T and Z_c; it varies between ± 1 including zero, i.e., when $R_T = Z_c$ (**natural termination**).

The current reflected wave (equation 7.18) is

$$i_b = -(v_b/Z_c).$$

Then, the voltage v_T and current i_T at the receiving-end are simply the sum of both the incident and reflected waves (equation 7.17):

$$v_T = v_f + v_b, \quad i_T = i_f + i_b. \tag{7.51}$$

Figure 7.12 shows the analysis of traveling waves when the line is terminated in a resistance that is larger than the line characteristic impedance (i.e., $R_T > Z_c$). Thus, v_b is positive and i_b is negative. Therefore, the traveling wave arrival at the line termination results in increased voltage and reduced current, as shown in Fig. 7.12(b). The opposite case when $R_T < Z_c$ is shown in Fig. 7.13. Here the

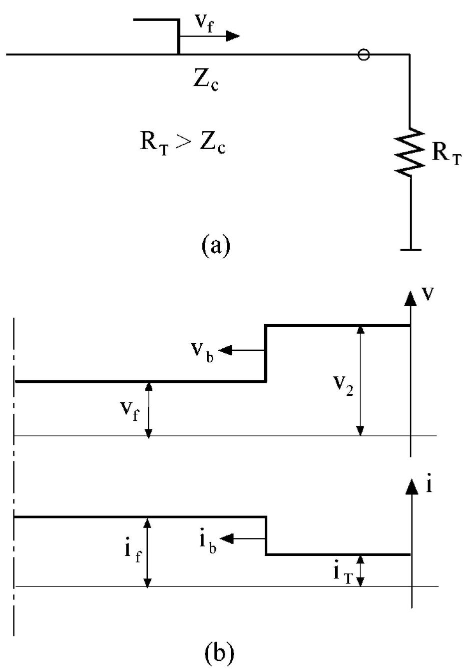


Figure 7.12 Traveling waves after arrival at termination in which $R_T > Z_c$: circuit diagram (a); voltage and current distributions (b).

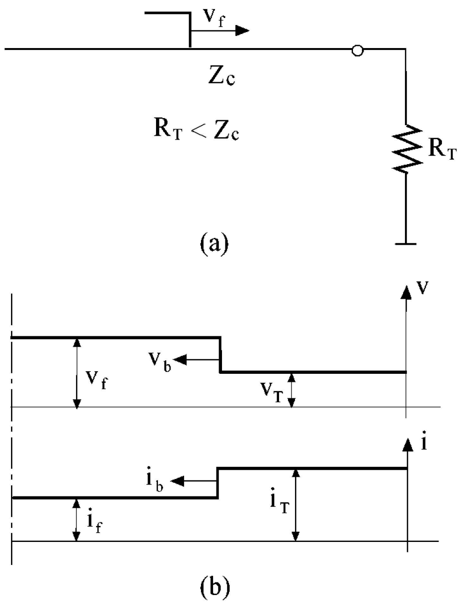


Figure 7.13 Traveling waves after arrival at termination in which $R_T < Z_c$: circuit diagram (a); voltage and current distributions (b).

traveling wave arrival at the line termination results in reduced voltage and increased currents as shown in Fig. 7.13(b).

Example 7.3

A line has a characteristic impedance of 400 Ω and a terminating resistance of 600 Ω. Assuming that the incident voltage wave is 100 kV, determine the following: (a) The reflection coefficient of the voltage wave; (b) The reflection coefficient of the current wave; (c) The backward-traveling voltage and current waves; (d) The voltage across and current through the resistor.

Solution

(a) $$\rho_{rv} = \frac{R_T - Z_c}{R_T + Z_c} = \frac{600 - 400}{600 + 400} = 0.2.$$

(b) $$\rho_{ri} = \frac{i_b}{i_f} = -\frac{v_b}{Z_c} \Big/ \frac{v_f}{Z_c} = -\frac{v_b}{v_f} = -\rho_{rv} = -0.2.$$

(c) $$v_b = \rho_{rv} v_f = 0.2 \cdot 100 = 20\ \text{kV}$$

$$i_b = -\frac{v_b}{Z_c} = -\frac{20 \cdot 10^3}{400} = -50\ \text{A}.$$

(d) $v_T = v_f + v_b = 100 + 20 = 120$ kV

$$i_T = \frac{v_T}{R_T} = \frac{120 \cdot 10^3}{600} = 200 \text{ A}.$$

7.5.2 Open- and short-circuit line termination

The boundary condition for the current in an open-circuit termination is $i_T = 0$. Therefore,

$$i_b = -i_f. \tag{7.52}$$

Using equation 7.18 yields

$$v_b = -Z_c i_b = Z_c i_f = v_f. \tag{7.53}$$

The same results, of course, can be obtained with a reflection coefficient. Since the *open-circuit termination* is an extreme termination in impedance $Z_T \to \infty$, the reflection coefficient is unity and $v_b = \rho_r v_f = v_f$. The total voltage at the open-end is $v_T = v_f + v_b = 2v_f$. Therefore, the voltage at the receiving-end is twice the forward voltage wave and this doubled value propagates on the line, as shown in Fig. 7.14(a).

The boundary condition for the voltage at the short-circuit termination is $v_T = 0$ and therefore

$$v_b = -v_f. \tag{7.54}$$

The *short-circuit termination* can be treated as the dual of the open-circuit termination. Therefore, the previous results for voltage traveling waves are now related to the current traveling waves and vice versa, concluding that the current at the short-circuited end of the line is twice the forward current wave as shown in Fig. 7.14(b).

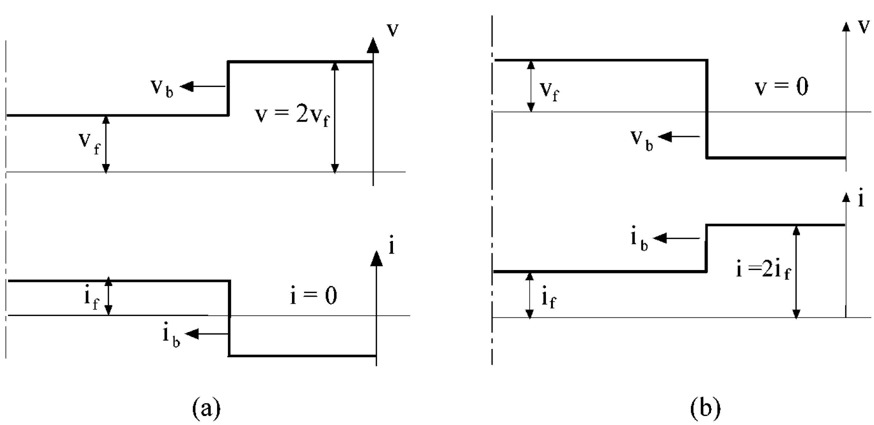


Figure 7.14 Traveling waves pattern for: open-circuit line termination (a); short-circuit line termination (b).

7.5.3 Junction of two lines

Applying the concept of an equivalent circuit to the junction of two lines (see Fig. 7.15(b)), we can conclude that this case is similar to the line terminated in resistance. Assume that $Z_{c1} > Z_{c2}$ where Z_{c1} and Z_{c2} are the characteristic impedances of the first and second lines, respectively. For example, it might represent the junction between an overhead line and an underground cable. If a voltage surge of a step function form approaches such a junction along the overhead line, the voltage at the junction decreases relative to the value of the increment wave. The voltage surge along the cable will be in accordance with the refraction coefficient (i.e. transmission factor):

$$v_{f2} = \rho_{ref} v_{f1} = \frac{2Z_{c2}}{Z_{c2} + Z_{c1}} v_{f1}. \tag{7.55}$$

The reflection, i.e. backward-traveling wave, in accordance with the reflection coefficient, is

$$v_{b1} = \rho_r v_{f1} = \frac{Z_{c2} - Z_{c1}}{Z_{c2} + Z_{c1}} v_{f1}. \tag{7.56}$$

Fig. 7.15(c) shows the waves occurring at the junction. It can be seen that the wave, which is refracted or transmitted to the cable, is equal to the sum of the forward and backward waves.

This property of cables to reduce the voltage surge is used in practice. When an overhead line is terminated by a transformer, the incident of a voltage surge on a transformer winding results in a very high voltage gradient at the winding turns nearest to the line conductor, and may lead to the breakdown of the

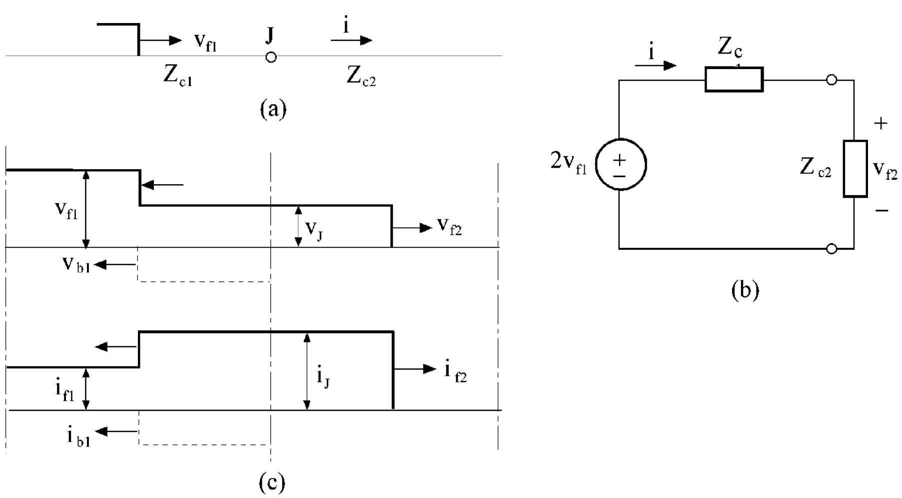


Figure 7.15 Traveling voltage and current waves at junction of two lines: circuit diagram (a); the equivalent circuit (b); the voltage and current distributions along two lines (c).

insulation. By putting in a short cable between the overhead line and the transformer, the magnitude of the voltage surge can be reduced before it reaches the transformer.

Example 7.4

The characteristic impedances of an overhead line and underground cable connected in series (Fig. 7.15(a)) are 400 Ω and 50 Ω respectively. The incident surge voltage of 800 kV rms is traveling on the overhead line toward the junction. Determine: (a) the surge voltage transmitted into the cable; (b) the surge current transmitted into the cable; (c) the surge voltage reflected back along the overhead line; (d) the power in the forward wave arriving at the junction and the transmitted wave power.

Solution

(a) $v_{f2} = \rho_{ref} v_{f1} = \dfrac{2Z_{c2}}{Z_{c2} + Z_{c1}} v_{f1} = \dfrac{2 \cdot 50}{50 + 400} 800 = 178 \text{ kV(rms)}.$

(b) $i_{f2} = \dfrac{v_{f2}}{Z_{c2}} = \dfrac{178}{50} = 3.56 \text{ kA(rms)}.$

(c) $v_{b1} = v_{f2} - v_{f1} = 178 - 800 = -622 \text{ kV(rms)}.$

(d) $P_{f1} = \dfrac{v_{f1}^2}{Z_{c1}} = \dfrac{800^2}{400} = 1600 \text{ MW}$

$$P_{f2} = \frac{v_{f2}^2}{Z_{c2}} = \frac{178^2}{50} = 634 \text{ MW}.$$

7.5.4 Capacitance connected at the junction of two lines

Figure 7.16(a) shows two lines connected in tandem and the *capacitance connected in parallel* to both lines at the junction J. Such a connection may represent two lines (incoming and outgoing) terminated in a transformer, since the behavior of a transformer at the first instance of wave arrival is as a capacitance. The equivalent circuit of the junction is shown in Fig. 7.16(b). Using the Laplace transform method, we obtain the expression of the voltage across the capacitance as

$$V_C(s) = \frac{2Z_{par}}{Z_{c1} + Z_{par}} V_{f1}(s), \tag{7.57}$$

where

$$Z_{par} = \frac{Z_{c2}(1/sC)}{Z_{c2} + 1/sC}$$

is the impedance of the parallel connection of C and Z_{c2}. Substituting Z_{par} into

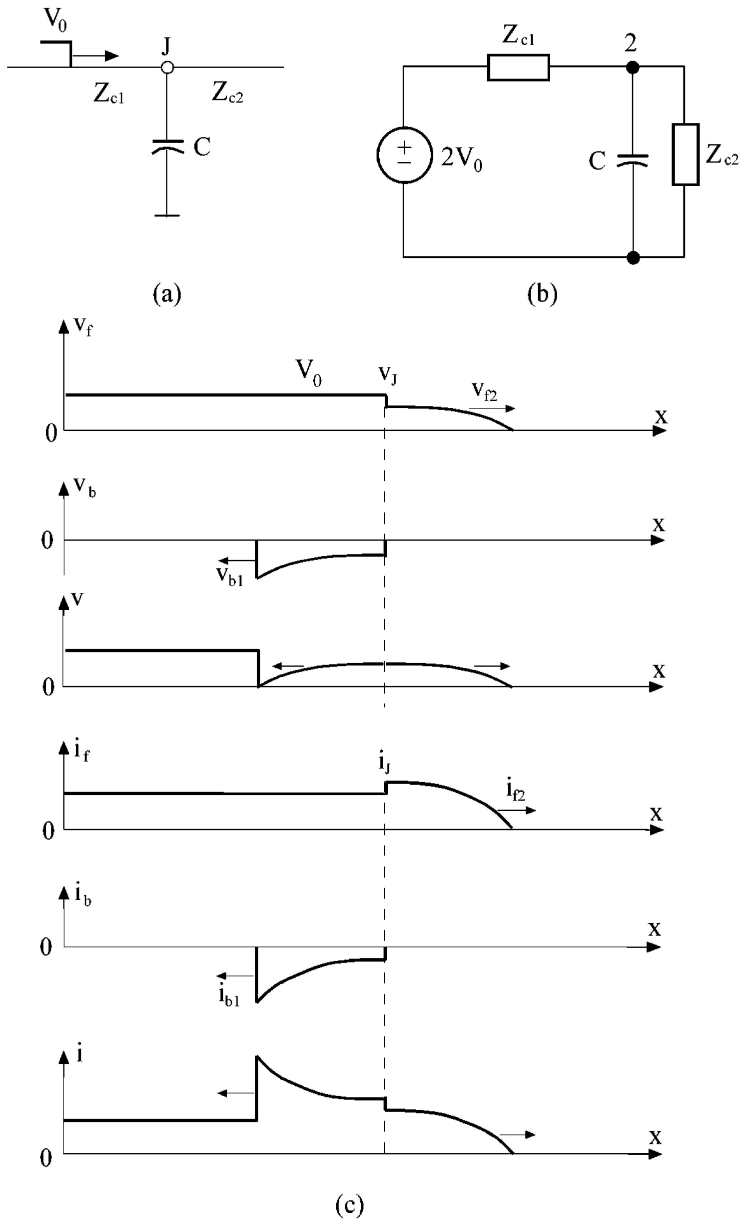


Figure 7.16 Traveling waves on two lines terminated by capacitance: circuit diagram (a); equivalent circuit (b); voltage and current waves on both lines (c).

equation 7.57 yields

$$V_C(s) = \frac{2Z_{c2}}{Z_{c1} + Z_2 + sCZ_{c1}Z_{c2}} V_{f1}(s), \tag{7.58}$$

or

$$V_C(s) = \rho_{ref} V_{f1}(s) \frac{1/T}{s + 1/T} \tag{7.59}$$

where $\rho_{ref} = 2Z_{c2}/(Z_{c1} + Z_{c2})$ is the transmission factor at infinite time ($t \to \infty$) after the capacitance is charged and $T = Z_{eq}C$ (where $Z_{eq} = Z_{c1}Z_{c2}/(Z_{c1} + Z_{c2})$ is the time constant of the circuit.

Considering the step-function incident wave and substituting its Laplace transform V_0/s into equation 7.59 gives

$$V_C(s) = \rho_{ref} V_0 \frac{1/T}{s(s + 1/T)}. \tag{7.60}$$

Taking the inverse Laplace transform, the time function of the capacitance voltage becomes

$$v_C(t) = \rho_{ref} V_0 (1 - e^{-(t/T)}). \tag{7.61}$$

Therefore the voltage distribution in the outgoing line will be

$$v_{f2}(x, t) = \rho_{ref} V_0 (1 - e^{-(t - x/v)/T}). \tag{7.62}$$

The reflected wave in the first line can be obtained as

$$v_{b1} = v_C - v_{f1} = (\rho_{ref} - 1)V_0 - \rho_{ref} V_0 e^{-(t/T)}.$$

Therefore the backward-traveling wave along the line is

$$v_{b1}(x, t) = \rho_r V_0 - \rho_{ref} V_0 e^{-(t - x/v)/T}. \tag{7.63}$$

The current traveling waves in both directions are the same shape as the voltage traveling waves and their value is related to them in the characteristic impedance of the corresponding line. The voltage and current distribution in both lines are shown in Fig. 7.16(c).

In most practical analyses, the *shape of the voltage/current waves caused by lightning is considered as an impulse* as shown in Fig. 7.17(a). In such an impulse, the voltage/current rises quickly to a maximum value and then decays slowly to zero. The crest time in which the voltage reaches its maximum value is about 1.2 μs.

When the first moment of the wave incident is of interest, the shape of the wave is simplified, having a constant value and ramped front as shown in Fig. 7.17(b). When the relatively long-term ($t \gg t_{cr}$) response is of interest, the voltage impulse of wave tail shape is considered, Fig. 7.17(c).

Now assume that the ramped front function incident wave reaches the junction of two lines with a *capacitance connected to the junction.* The ramped front

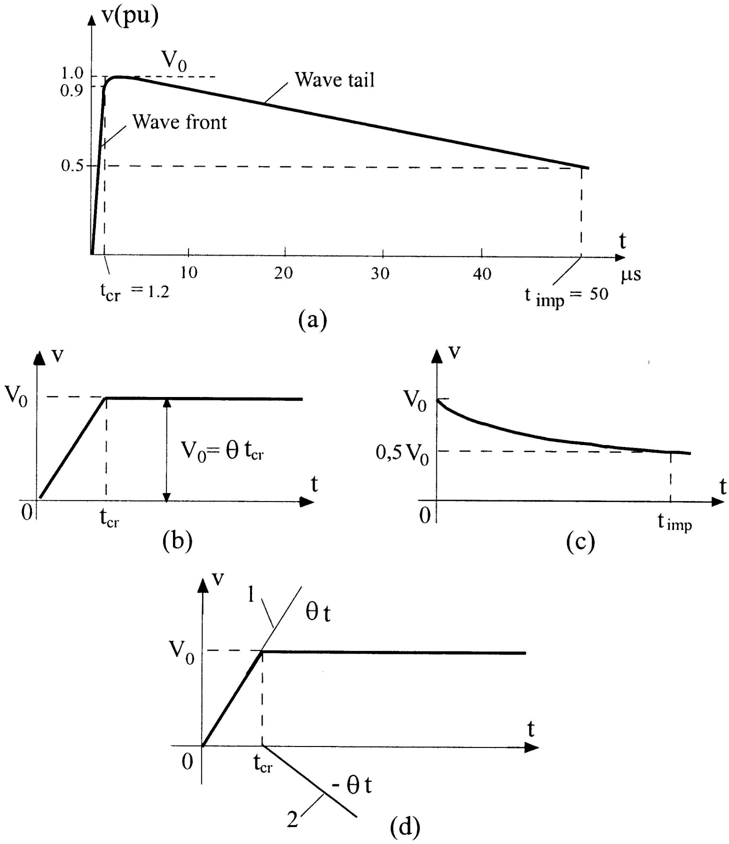


Figure 7.17 Impulse voltage waveform: standard impulse (a); ramped front function approximation (b); tail-shaped approximation (c); the superposition of two ramp functions (d).

function can be superposed from two ramp functions of slope $\theta = V_0/t_{\mathrm{cr}}$: one positive and one negative, while the letter is shifted by the crest time, t_{cr}:

$$v_{\mathrm{f}}(t) = \theta t - \theta(t - t_{\mathrm{cr}}), \tag{7.64}$$

as shown in Fig. 7.17(d). Substituting the Laplace transform of the ramp function $v_{\mathrm{r}} = \theta t \leftrightarrow \theta/s^2$ into equation 7.59 gives

$$V_{\mathrm{Cr}}(s) = \rho_\tau \theta \frac{1/T}{s^2(s + 1/T)}. \tag{7.65}$$

Taking the inverse Laplace transform, the capacitance voltage in the time domain for time less than t_{cr} becomes

$$v_{\mathrm{Cr}}(t) = \rho_\tau \theta [t - T(1 - e^{-(t/T)})], \quad t \le t_{\mathrm{cr}}. \tag{7.66}$$

In accordance with equation 7.64, the capacitance voltage for the time larger then t_{cr} will be

$$v_{Cr}(t) = \rho_\tau V_0 \left[1 + \frac{T}{t_{cr}} (e^{-(t/T)} - e^{-(t-t_{cr})/T} \right], \quad t \geq t_{cr}. \tag{7.66a}$$

The capacitance voltage change for both instances of time is shown in Fig. 7.18(a) and (b). First, notice that for $t \to \infty$ the capacitance voltage is $\rho_\tau V_0$ which is the same as in the previous case of the step function wave response (see equation 7.61). Secondly, as can also be seen from Fig. 7.18(b), the slope of the resulting voltage has changed relative to the slope of the incident voltage wave. In order to estimate this change, let us determine the equivalent slope as

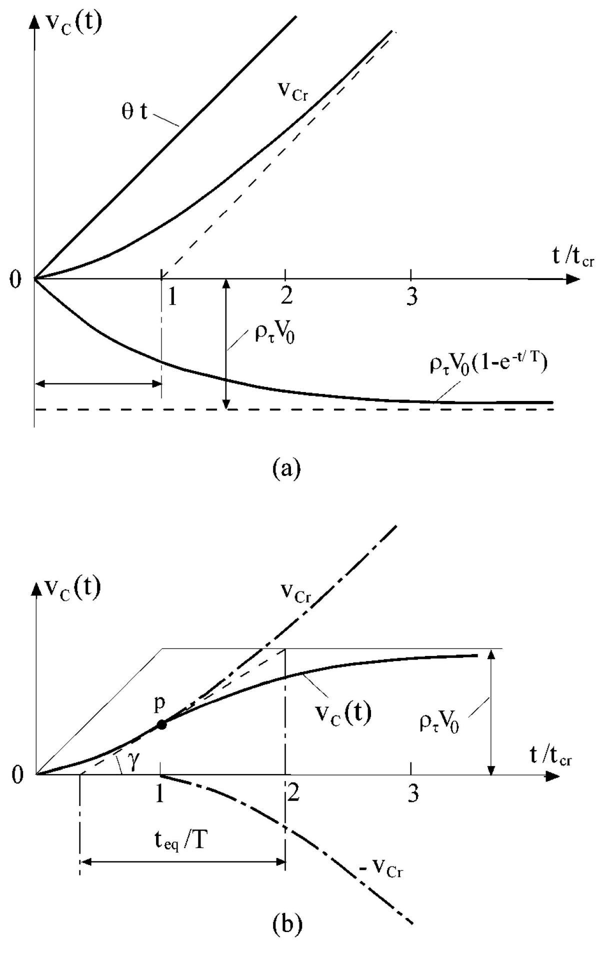


Figure 7.18 The voltage change across the capacitance as a response to: infinite ramp function incident wave (a); ramped front incident wave (b).

a $\tan\gamma$, where the angle γ is of a tangent drawn through a point p $(t = t_{cr})$:

$$\tan\gamma = \left[\frac{dv_C}{dt}\right]_{t=t_{cr}} = \rho_\tau V_0 \frac{1}{t_{cr}}(1 - e^{-t_{cr}/T}).$$

Therefore, the equivalent crest time of the capacitance voltage can be expressed as

$$t_{eq} = \frac{\rho_\tau V_0}{\tan\gamma} = \frac{t_{cr}}{1 - e^{-t_{cr}/T}}, \tag{7.67}$$

i.e., the bigger T is in relation to t_{cr}, the greater the change in the slope. (For $T \le 1/3 t_{cr}$ the change in slope is not significant.)

When the relatively long-term $(t \gg t_{cr})$ response is of interest, the voltage impulse of the shape shown in Fig. 7.17(c) is considered. In simple terms this is the decreasing exponential

$$v_f(t) = V_0 e^{-t/T_0}, \tag{7.68}$$

and T_0 is estimated in accordance with $T_0 = t_{imp}/0.7$ where t_{imp} is the time in which the maximum value of the impulse in Fig. 7.17(a) decreases by half.

Substituting the Laplace transform of an exponential into equation 7.59 yields

$$V_{C,\exp} = \rho_\tau V_0 \frac{1}{T(s + 1/T_0)(s + 1/T)}, \tag{7.69}$$

and with the inverse Laplace transform

$$v_{C,\exp}(t) = \rho_\tau V_0 \frac{T_0}{T_0 - T}(e^{-t/T_0} - e^{-t/T}). \tag{7.70}$$

The two exponential terms of equation 7.70 (see broken lines), the resulting voltage (1) and the voltage wave impulse (2), are shown in Fig. 7.19. Equating the derivative of equation 7.70 to zero yields the time in which the capacitance voltage reaches its maximum,

$$t_{(\max)} = \frac{T_0 T}{T_0 - T}\ln\frac{T}{T_0} = \alpha \ln\frac{T}{T_0}, \tag{7.71}$$

and the scaled value of the maximum voltage is

$$\frac{V_{2\max}}{\rho_\tau V_0} = \left(\frac{T}{T_0}\right)^{\alpha}, \tag{7.72}$$

where

$$\alpha = \frac{T/T_0}{1 - T/T_0},$$

i.e., the maximum voltage is dependent on the ratio T/T_0. If the ratio is $T/T_0 > 0.5$, then the capacitance results in reducing the maximum voltage by more than half.

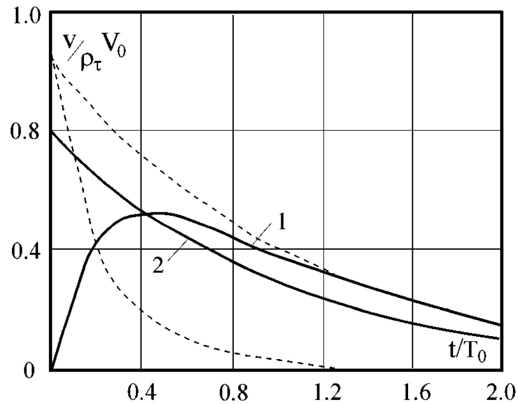


Figure 7.19 The voltage change across the capacitance as a response to the exponential function incident wave: curve 1, resulting voltage; curve 2, voltage incident wave. The broken lines are two exponential terms.

7.6 SUCCESSIVE REFLECTIONS OF WAVES

Consider the TL, which is terminated in the generator input impedance Z_s by the sending-end and in load impedance Z_T by the receiving-end (see the footnote in section 7.5 on p. 480). Neither of them is equal to the characteristic impedance Z_c, so in theory an infinite succession of reflected waves results.

The first forward-traveling (f.t.) wave (a step-function voltage source is assumed) will be (7.33)

$$v_{f1}(0, t) = \rho_\tau V_s = \frac{Z_c}{Z_s + Z_c} V_0 u(t), \tag{7.73}$$

where $u(t)$ is a unit function. The first backward-traveling (b.t.) wave appears after the first f.t. wave reaches the receiving end,

$$v_{b1}(\ell, t) = \rho_r v_{f1} = \rho_r \rho_\tau V_0 u(t - t_r), \tag{7.74}$$

where $t_r = \ell/v$ is the delay time in which the f.t. wave reaches the receiving-end of the line.

The second f.t. wave appears after the first b.t. wave reaches the sending-end of the line and it can be found in accordance with the sending-end reflection coefficient ρ_s (equation 7.49):

$$v_{f2}(0, t) = \rho_s \rho_r \rho_\tau V_0 u(t - 2t_r). \tag{7.75a}$$

In a similar way, the second b.t. wave becomes

$$v_{b2}(\ell, t) = \rho_s \rho_r^2 \rho_\tau V_0 u(t - 3t_r), \tag{7.75b}$$

or for kth incident $t > kt_r$

$$v_{f,k}(x,t) = (\rho_s\rho_r)^{k-1}\rho_\tau V_0 u\left(2(k-1)t_r - \frac{x}{v}\right)$$
$$v_{b,k}(x,t) = (\rho_s\rho_r)^{k-1}\rho_r\rho_\tau V_0 u\left(t - 2(k-1)t_r - \frac{x}{v}\right). \tag{7.76}$$

The current waves are simply related to the voltage waves by a characteristic impedance

$$i_{f,k} = \frac{v_{f,k}}{Z_c}, \quad i_{b,k} = -\frac{v_{b,k}}{Z_c}.$$

Thus, the complete response consists of an infinite series of voltage and current step-function waves which are added successively as the wave front travels from the source to its terminated end and back. Each of the forward- and backward-traveling wave series can be treated as infinitely decreasing geometric progressions having the ratio $\rho_s\rho_r$ (which is less than one) and the first terms $\rho_\tau V_0$ and $\rho_r\rho_\tau V_0$, respectively. Hence, the final value of the line voltage at $t \to \infty$ can be expressed as the sums of these two progressions,

$$v(x,t) = \frac{\rho_\tau V_0}{1-\rho_s\rho_r} + \frac{\rho_r\rho_\tau V_0}{1-\rho_s\rho_r} = V_0\frac{Z_T}{Z_s+Z_T}, \tag{7.77}$$

i.e., the steady-state voltage at the receiving-end of the line (note that the source is simply a d.c. quantity and the line is loss-less).

7.6.1 Lattice diagram

The voltage at a given point and time can be determined graphically with the help of the lattice diagram, suggested by Bewley(*). It gives a visual track representation of a traveling voltage or current wave as it reflects back and forth from the ends of the line, as shown in Fig. 7.20.

In the lattice diagram, the distance between the sending- and receiving-ends is represented by the horizontal line and time is represented by two vertical lines (t_r is the time for a wave to travel the line length). The diagonal zigzag line represents the wave as it travels back and forth between the ends or points of discontinuities: the line sloping to the right gives a forward-traveling wave in the increasing x direction, whereas the line sloping to the left gives the backward-traveling wave in the decreasing x direction. The slopes of the zigzag lines give the times corresponding to the distances traveled.

The value of each wave has been written above the corresponding line; each reflection is determined by multiplying the incident wave by the appropriate reflection coefficient ρ_r or ρ_τ. Of course, the same lattice diagram can also

(*) Bewley, L.V. (1951) *Traveling Waves on Transmission Systems.* Wiley, New York.

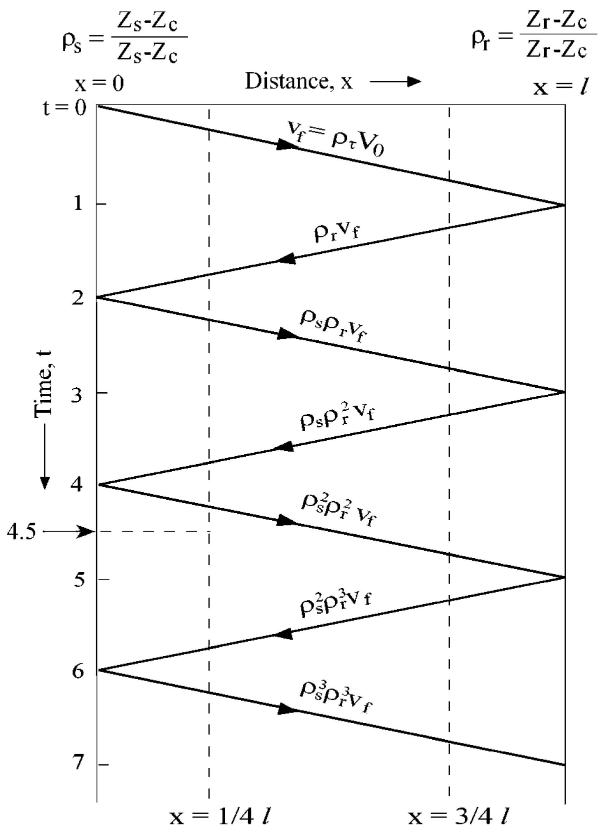


Figure 7.20 Lattice diagram for transmission line.

represent the current traveling waves. Nevertheless, the fact that the reflection coefficient for the current is always the negative of the reflection coefficient for the voltage should be taken into account. The voltage and current at a given point with the coordinates of time and distance along the line may then be determined by drawing a vertical line through this point and adding all the terms that are directly above that point corresponding to the intersections of the sloping lines with the given vertical line. For example, the voltage at $t = 4.5t_r$ and $x = (1/4)\ell$ is

$$v(0.25\ell, 4.5t_r) = \rho_\tau V_0(1 + \rho_r + \rho_s\rho_r + \rho_s\rho_r^2 + \rho_s^2\rho_r^2).$$

Example 7.5

Consider an underground cable line 1.6 km long with a characteristic impedance of 50 Ω and a wave propagation velocity of $1.6 \cdot 10^8$ m/s. The line is connected to the d.c. ideal $Z_s = 0$ voltage source $V_0 = 1000$ V and terminated in a 200 Ω resistor. (a) Determine the reflection coefficients at the sending- and receiving-ends; (b) Draw the appropriate lattice diagram for voltage and current; (c) Determine the value of voltage and current at $t = 5.5t_r$ and $x = (1/4)\ell$; (d) Plot the voltage and current versus time at line point $x = (1/2)\ell$.

Solution

(a) The reflection coefficients are

$$\rho_s = \frac{Z_s - Z_c}{Z_s + Z_c} = \frac{0 - 50}{0 + 50} = -1, \quad \rho_r = \frac{Z_T - Z_c}{Z_T + Z_c} = \frac{200 - 50}{200 + 50} = 0.6.$$

(b) The traveling time is

$$t_r = \frac{1.6 \cdot 10^3}{1.6 \cdot 10^8} = 10\ \mu\mathrm{s}.$$

The lattice diagram is shown in Fig. 7.21(a). The values of the voltage/current traveling waves are written above the arrows.

(c) From the lattice diagram the voltage at the point A is

$$v\left(\frac{1}{4}\ell, 5.5t_r\right) = 1000 + 600 - 600 - 360 + 360 = 1000\ \mathrm{V},$$

and the current is

$$i\left(\frac{1}{4}\ell, 5.5t_r\right) = \frac{1}{50}(1000 - 600 - 600 + 360 + 360) = 10.4\ \mathrm{A}.$$

(d) The plot of the voltage and current at the line point $x = 1/2\ell$ versus time is shown in Fig. 7.21(b) and (c).

7.6.2 Bergeron diagram

Another convenient way of graphically determining voltages and currents at the two ends of the line as a result of an incident wave reflection is by a diagram attributed to Bergeron(*). This method is based on a graphical solution of a system of two linear equations and can be explained by a single example of the equivalent circuit of Fig. 7.2(b) shown again in Fig. 7.22(a), where the source is designated by V_0 and its input impedance by R_s. The following two equations express the two voltage-current loci (see Fig. 7.22(b)), one for the source (generator) and the other for the load (which here is the characteristic impedance Z_c):

$$v = V_0 - R_s i \tag{7.78a}$$

(*) Bergeron, L. (1961) *Water Hammer in Hydraulics and Wave Surges in Electricity*. The American Society of Mechanical Engineers, New York.

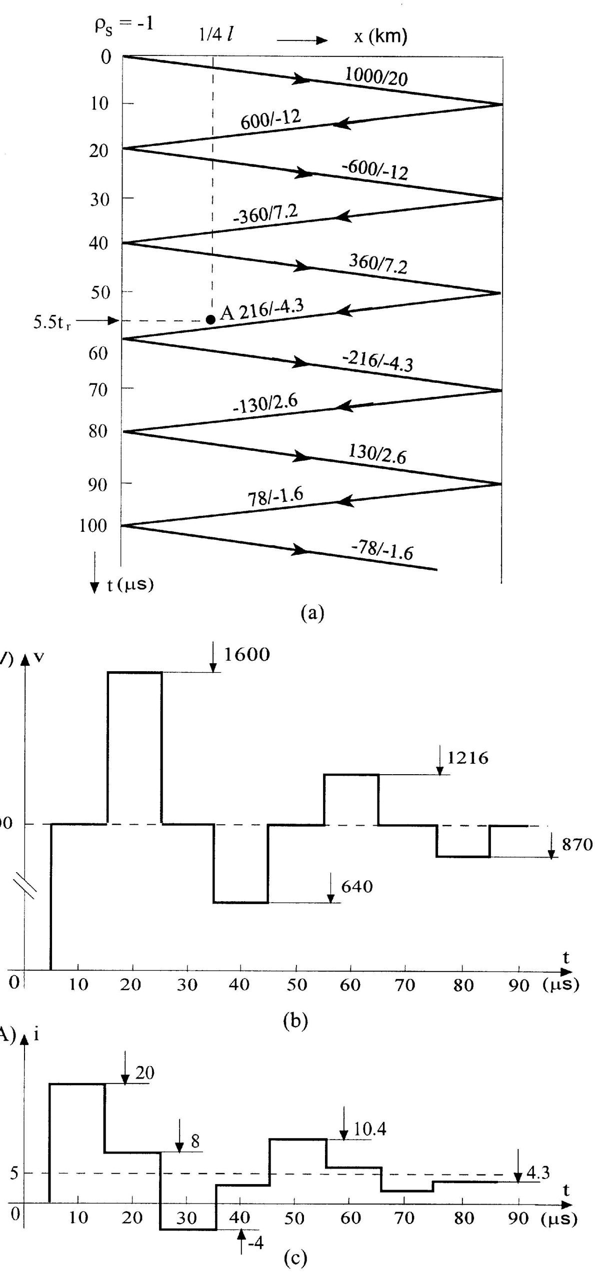


Figure 7.21 Lattice diagram (a) and voltage (b) and current (c) plots of Example 7.5.

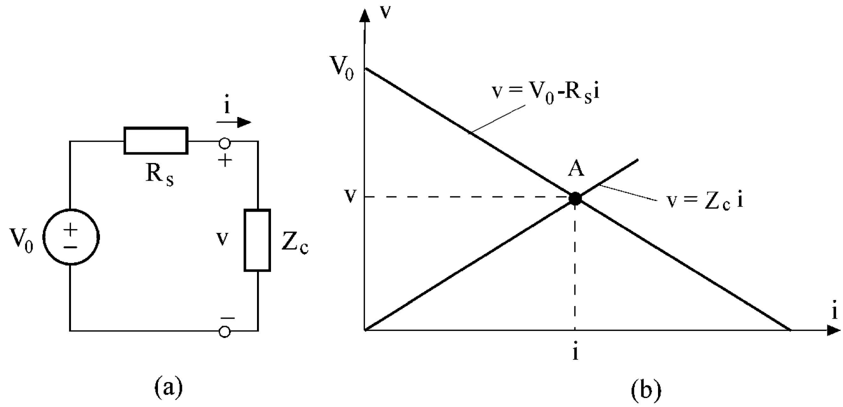


Figure 7.22 A graphical solution of series source-load circuit: the circuit diagram (a); the voltage-current loci (b).

and

$$v = Z_c i. \tag{7.78b}$$

Point A of their intersection gives the graphical solution of these two equations as shown in Fig. 7.22(b). Note that the slopes of these two lines are given by $-R_s$ and Z_c. It should also be noted that this graphical method is especially appropriate for nonlinear elements, like surge arresters, for example.

The complete solution for a line terminated in the voltage source (V_0, R_s) at the sending-end and in the resistor (R_T) at the receiving-end is given in Fig. 7.23(a). Point A_1 corresponds to the voltage and current at the sending-end after the source switching $t = 0_+$ (just like in Fig. 22(b)). Note that this voltage gives the first forward-traveling wave. The voltage and current at the receiving-end can be determined in accordance with the equivalent circuit similar to the one shown in Fig. 7.9(b) since the line is initially quiescent. The line drawing through point A_1 and sloping in accordance to $-Z_c$ represents the voltage-current locus of the source whose value is $2v_{f1}$ and whose input impedance is Z_c. (Note that in order to draw this locus it is not necessary to start with the point which lies on the ordinate axis, i.e. of $2v_{f1}$, and $i_{f1} = 0$, but it can be drawn through any point which belongs to this locus, for example, point A_1.) The intersection of this locus with the locus of the load resistor R_T, point B_1, yields the resultant voltage v_T and current i_T at the receiving-end immediately after the arrival of $2v_{f1}$. The first backward-traveling wave can then be obtained as

$$v_{b1} = v_{B1} - v_{f1}, \quad i_{b1} = i_{B1} - i_{f1}. \tag{7.79}$$

The equivalent circuit for calculating the next value of voltage and current at the sending-end is shown in Fig. 7.23(b). In accordance with the above, the locus, which determines the voltage and current at the sending-end, is parallel

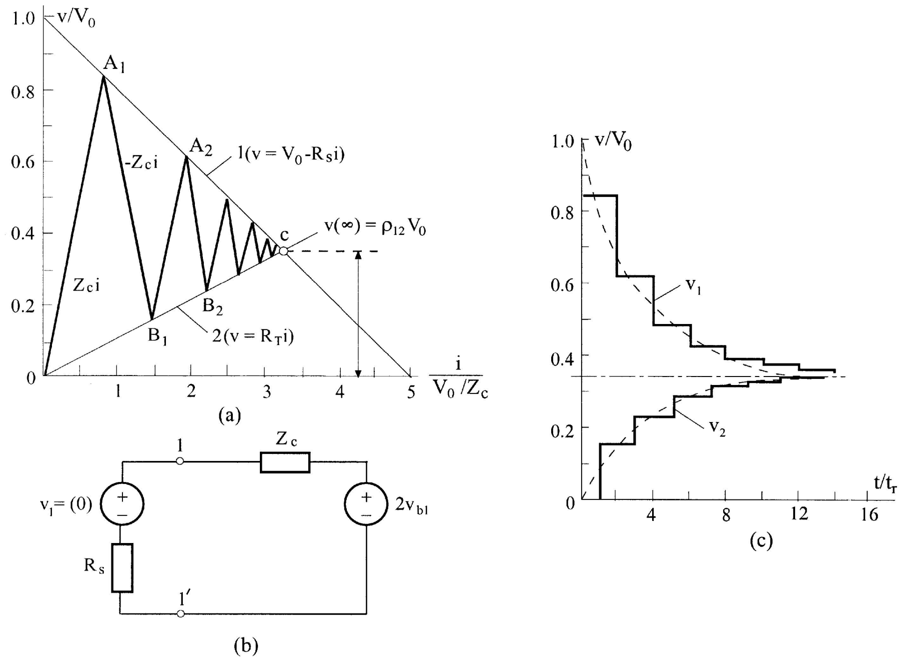


Figure 7.23 Bergeron diagram for TL with terminations $R_s = 0.2$, $R_T = 0.1Z_c$: Bergeron diagram (a); equivalent circuit for the sending-end (b); the plot of the sending- and receiving-end voltages versus time (c).

to the locus $v = Z_c i$ and passes through point B_1. (Note again that in order to draw this locus it is not necessary to start with the point on the ordinate axis, which is $2v_{b1}, i_{b1} = 0$.) The intersection of this locus with the sending-end voltage source locus, point A_2, gives the resultant voltage and current at $x = 0$ at the time of the arrival of v_{b1} and i_{b1}. The next forward-traveling wave will be

$$v_{f2} = v_{A2} - v_{b1}, \quad i_{f2} = i_{A2} - i_{b1}. \tag{7.80}$$

This process may be continued for any desired number of intervals. The intersection of two loci, point C, which represent both ends, yields the limiting values of line voltage and current as $t \to \infty$ (which is also in accordance with equation 7.77). The plot of both the sending- and receiving-end voltages versus time obtained with the help of the Bergeron diagram is shown in Fig. 7.23(c).

7.6.3 Nonlinear resistive terminations

The Bergeron diagram is the most suitable for reflecting wave determination when the transmission line is terminated in a nonlinear element. One example of such elements can be a surge arrester, which consists of an air gap and a nonlinear resistor. Fig. 7.24(a) shows the equivalent circuit of the line terminated in a surge arrester (SA), and Fig. 7.2(b) shows how the discharge voltage curve

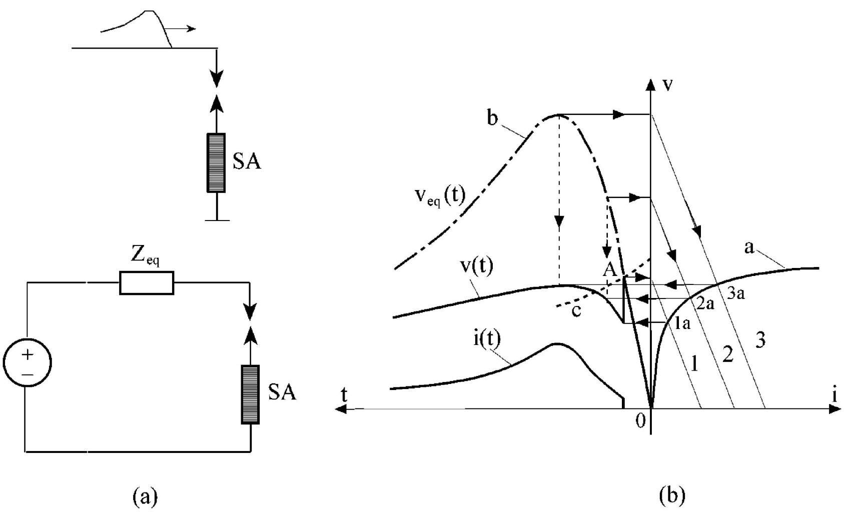


Figure 7.24 Transmission line terminated in a nonlinear resistor (surge arrester): equivalent circuit (a); arrester characteristic construction (b).

of the arrester can be built using the graphical solution. Since the line has been assumed to be initially quiescent, the voltage-current (v-i) locus of a nonlinear resistor (curve a in Fig. 7.24(b)) is drawn from the origin. In the left-half plane the equivalent surge voltage versus time (curve b) and volt-second characteristic of an arrester's air gap (curve c) are plotted. The intersection of these two curves, point A, determines the initial voltage, which activates the surge arrester (see the equivalent circuit (a) in Fig. 7.24)). The intersection of a sloped (in relation to Z_{eq}) line "1" and the v-i locus (curve a), point 1a, gives the voltage drop across the arrester, i.e., across the nonlinear resistor. The next voltage drops across the nonlinear resistor in accordance with the surge voltage change can be obtained in the same way (see points 2a, 3a and so on). Transferring these points to the left-half plane in accordance with the appropriate time results in the discharge voltage characteristic of an arrester: curve $v(t)$. Note that, because of nonlinearity, the voltage across the arrester hardly changes, being much lower than the surge voltage and thereby protecting the high voltage equipment.

7.7 LAPLACE TRANSFORM ANALYSIS OF TRANSIENTS IN TRANSMISSION LINES

As is known, any circuit equation, written in phasor notation, can be converted into a Laplace transform equation by simply replacing $j\omega$ with s. To use this procedure, let us first consider a transmission line activated at its sending-end by a sinusoidal voltage source. Applying the current and voltage phasors:

$\tilde{I} = I_m e^{j\psi_i}$ and $\tilde{V} = e^{j\psi_v}$, which become functions of x only (and since the multiplier $e^{j\omega t}$ is crossed throughout the equations), partial derivative equations 7.1 convert into ordinary differential equations:(*)

$$-\frac{dV}{dx} = (R + j\omega L)I = ZI \tag{7.81a}$$

$$-\frac{dI}{dx} = (G + j\omega C)V = YV, \tag{7.81b}$$

where $Z = R + j\omega L$ and $Y = G + j\omega C$ are the impedance and admittance per unit length, respectively. Differentiating equations 7.81 with respect to x gives

$$-\frac{d^2V}{dx^2} = Z\frac{dI}{dx}, \quad -\frac{d^2I}{dx^2} = Y\frac{dV}{dx},$$

and substituting the values of dI/dx and dV/dx according to equations 7.81, we obtain

$$\frac{d^2V}{dx^2} = ZYV \tag{7.82a}$$

$$\frac{d^2I}{dx^2} = ZYI. \tag{7.82b}$$

Two ordinary second-order differential equations 7.82, which define the current/voltage phasors change along the line, are similar (from a mathematical point of view). Therefore, it is sufficient to solve one of them, for example equation 7.82a for the voltage and realize the current from equation 7.81a.

The solution of the ordinary second-order differential equation 7.82a for the voltage is of the form

$$V(x) = A_1 e^{-\gamma x} + A_2 e^{\gamma x}, \tag{7.83a}$$

and for the current I, from equation 7.81a with equation 7.83, is

$$I = \frac{\gamma}{Z}(A_1 e^{-\gamma x} - A_2 e^{\gamma x}),$$

or

$$I = \frac{1}{Z_c}(A_1 e^{-\gamma x} - A_2 e^{\gamma x}). \tag{7.83b}$$

(*)For the simplification of formula, writing the superscript "$\sim$", for denoting phasors, is omitted throughout this chapter.

The exponent power coefficient γ is found as a root of the characteristic equation $s^2 = ZY$,

$$\pm\gamma = \pm\sqrt{ZY} = \pm\sqrt{(R + j\omega L)(G + j\omega C)}, \tag{7.84a}$$

which is a complex quantity, called the *propagation constant*, and

$$Z_c = \frac{Z}{\gamma} = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = |Z_c|e^{j\theta} \tag{7.84b}$$

is the *characteristic impedance* of the line with a magnitude $|Z_c|$ and argument (angle) θ:

$$|Z_c| = \sqrt[4]{\frac{R^2 + \omega^2 L^2}{G^2 + \omega^2 C^2}}, \quad \theta = \frac{1}{2}\tan^{-1}\frac{\omega(GL - RC)}{RG + \omega^2 LC}.$$

(Here, the resistivity (R) and conductivity (G) of a TL are taken into account. By neglecting R and G, argument θ turns into zero, and $|Z_c|$ turns into the previously obtained quantity, given by expression 7.13, i.e., $Z_c = \sqrt{L/C}$.)

In equations 7.83, A_1 and A_2 are the arbitrary constants, which have to be selected to conform to the boundary conditions, and they are complex quantities:

$$A_1 = |A_1|e^{j\psi_1}, \quad A_2 = |A_2|e^{j\psi_2}.$$

To solve equations 7.83 constants A_1 and A_2 shall be found from the known boundary conditions. Let V_1 and I_1 be the voltage and current of the sending-end ($x = 0$) of the line. According to (7.83) for $x = 0$ we have $V_1 = A_1 + A_2$ and $I_1 Z_c = A_1 - A_2$. Therefore,

$$A_1 = \frac{1}{2}(V_1 + Z_c I_1), \quad A_2 = \frac{1}{2}(V_1 - Z_c I_1). \tag{7.85}$$

Substituting equations 7.85 into 7.83, we obtain voltage V and current I in any point of the line at the distance of x from its sending-end

$$\begin{aligned} V(x) &= \frac{1}{2}(V_1 + Z_c I_1)e^{-\gamma x} + \frac{1}{2}(V_1 - Z_c I_1)e^{\gamma x} \\ I(x) &= \frac{1}{2}\left(\frac{V_1}{Z_c} + I_1\right)e^{-\gamma x} - \frac{1}{2}\left(\frac{V_1}{Z_c} - I_1\right)e^{\gamma x}. \end{aligned} \tag{7.86a}$$

Equations 7.86a are known as the equations of the transmission line in exponential form. Combining similar terms in equation 7.86a,

$$V(x) = \frac{e^{\gamma x} + e^{-\gamma x}}{2}V_1 - \frac{e^{\gamma x} - e^{-\gamma x}}{2}Z_c I_1$$

$$I(x) = -\frac{e^{\gamma x} - e^{-\gamma x}}{2}\frac{1}{Z_c}V_1 + \frac{e^{\gamma x} + e^{-\gamma x}}{2}I_1,$$

and using the hyperbolic functions, these equations can be written in *hyperbolic* *form*

$$\begin{aligned} V(x) &= (\cosh \gamma x)V_1 - (Z_c \sinh \gamma x)I_1 \\ I(x) &= \left(-\frac{1}{Z_c}\sinh \gamma x\right) V_1 + (\cosh \gamma x)I_1. \end{aligned} \tag{7.86b}$$

Now consider the case where the voltage V_2 and the current I_2 of the receiving-end of the line are known. Let x' be the distance between the receiving-end of the line and the observed point. Since $x = \ell - x'$ (ℓ is the length of the line), equations 7.83 will be

$$V = A_1 e^{-\gamma\ell} e^{\gamma x'} + A_2 e^{\gamma\ell} e^{-\gamma x'}, \quad I = \frac{1}{Z_c}(A_1 e^{-\gamma\ell} e^{\gamma x'} - A_2 e^{\gamma\ell} e^{-\gamma x'}).$$

Let $A_3 = A_1 e^{-\gamma\ell}$ and $A_4 = A_2 e^{\gamma\ell}$ be the new boundary constants. Omitting the prime-sign in x', but taking into consideration that the variable x is reckoned from the receiving-end, we obtain

$$V = A_3 e^{\gamma x} + A_4 e^{-\gamma x}, \quad I = \frac{1}{Z_c}(A_3 e^{\gamma x} - A_4 e^{-\gamma x}). \tag{7.87}$$

Substituting $x = 0$ allows for determining the boundary constants A_3 and A_4

$$A_3 = \frac{1}{2}(V_2 + Z_c I_2), \quad A_4 = \frac{1}{2}(V_2 - Z_c I_2).$$

Equations 7.87 with the above arbitrary constants give the equations of a transmission line when the receiving-end boundary conditions are known

$$\begin{aligned} V(x) &= \frac{1}{2}(V_2 + Z_c I_2)e^{\gamma x} + \frac{1}{2}(V_2 - Z_c I_2)e^{-\gamma x} \\ I(x) &= \frac{1}{2}\left(\frac{V_2}{Z_c} + I_2\right) e^{\gamma x} - \frac{1}{2}\left(\frac{V_2}{Z_c} - I_2\right) e^{-\gamma x}. \end{aligned} \tag{7.88a}$$

Combining the similar terms in equations 7.88a, the transmission line equations can be obtained in hyperbolic form

$$\begin{aligned} V(x) &= (\cosh \gamma x)V_2 + (Z_c \sinh \gamma x)I_2 \\ I(x) &= \left(\frac{1}{Z_c}\sinh \gamma x\right) V_2 + (cosh\, \gamma x)I_2. \end{aligned} \tag{7.88b}$$

For the sending-end of the line, i.e. when $x = \ell$, equations 7.88 become

$$\begin{aligned} V_1 &= (\cosh \gamma\ell)V_2 + (Z_c \sinh \gamma\ell)I_2 \\ I_1 &= \left(\frac{1}{Z_c}\sinh \gamma\ell\right) V_2 + (\cosh \gamma\ell)I_2. \end{aligned} \tag{7.89}$$

Equations 7.89 express the voltage and current phasors of the sending-end of the line in terms of the voltage and current phasors of the receiving-end.

7.7.1 Loss-less LC line

By replacing $j\omega$ with s the above-obtained phasor equations become the Laplace-transform equations. Thus, equations 7.89 become

$$
\begin{aligned}
V(x, s) &= V_2 \cosh \gamma\ell + Z_c I_2 \sinh \gamma\ell \\
I(x, s) &= \frac{V_2}{Z_c} \sinh \gamma\ell + I_2 \cosh \gamma\ell.
\end{aligned}
\tag{7.90}
$$

where γ is in accordance with equation 7.84 and after replacing $j\omega$ with s becomes

$$\gamma = \sqrt{(R + sL)(G + sC)}. \tag{7.91a}$$

For a loss-less line ($R = 0$ and $G = 0$) it turns to

$$\gamma = s\sqrt{LC} = s/v, \tag{7.91b}$$

and the phasor equations 7.90 become the Laplace equations:

$$
\begin{aligned}
V(x, s) &= V_2 \cosh st_r + Z_c I_2 \sinh st_r \\
I(x, s) &= \frac{V_2}{Z_c} \sinh st_r + I_2 \cosh st_r,
\end{aligned}
\tag{7.92}
$$

where $t_r = \ell/v$ is the wave traveling time along the line. In order to find the solution of (7.92), the boundary equations for voltage vs. current (or vice versa) have to be taken into account.

7.7.2 Line terminated in capacitance

Consider the loss-less line, terminated in the capacitance and connected to the ideal source d.c. as shown in Fig. 7.25(a). For this line the boundary equations are:

$$V(0, s) = V_0/s \qquad \text{for the sending-end} \tag{7.93a}$$

$$I(\ell, s) = sCV(\ell, s) \qquad \text{for the receiving-end.} \tag{7.93b}$$

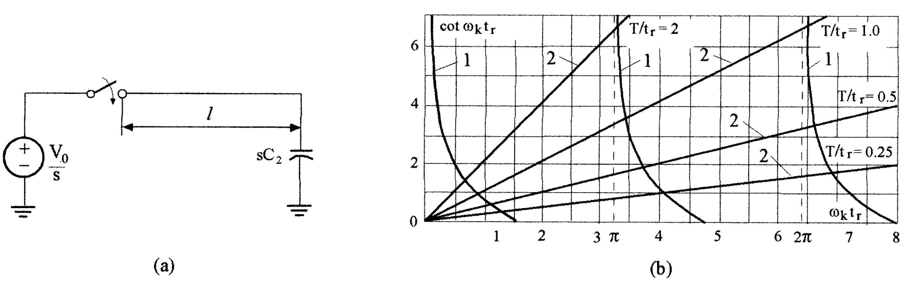


Figure 7.25 The circuit diagram of the TL terminated in capacitance (a) and the graphical solution of the characteristic equation (b) for a different ratio T/t_r.

Substituting these boundary conditions in the first equation 7.92 yields

$$V(\ell, s) = \frac{V_0}{s(\cosh st_r + sT \sinh st_r)} = \frac{V_0}{sF(s)} \tag{7.94}$$

where $T = C_2 Z_c$ is a time constant.

In order to obtain the receiving-end voltage in the time domain, we obtain the partial fraction expansions in the following form:

$$v(\ell, t) = \frac{V_0}{F(0)} + \sum_{-\infty}^{\infty} \frac{V_0}{s_k F'(s_k)} e^{s_k t}, \tag{7.95}$$

where s_k are roots (or the network poles) of the characteristic equation

$$\cosh st_r + sT \sinh st_r = 0. \tag{7.96}$$

Since the loss-less line only consists of inductances and capacitances, the roots of equation 7.96 are pure imaginary values ($s_k = \pm j\omega_k$). Therefore equation 7.96 changes into a trigonometrical equation

$$\cot \omega_k t_r = \frac{T}{t_r} \omega_k t_r. \tag{7.97}$$

The graphical solution of equation 7.97 is shown in Fig. 7.25(b). The intersection points of two plots, versus the variable $\omega_k t_r$, which represent two sides of equation 7.97, give the characteristic equation roots (the radiant frequency ω_k results in the quotient obtained by the division of the abscissa of the intersection points by t_r).

For any conjugate pair $\pm j\omega_k$ the sum of equation 7.95 consists of the kth cosine term

$$A_k e^{j\omega t} + \overset{*}{A}_k e^{-j\omega t} = 2A_k \cos \omega_k t, \tag{7.98}$$

where A_k and $\overset{*}{A}_k$ are also a conjugate pair. In order to determine A_k, the derivative of $F(s)$ has to be found

$$F'(s)\bigg|_{s=\pm j\omega} = \pm jt_r \left[\left(1 + \frac{T}{t_r}\right) \sin \omega_k t_r + \omega_k t_r \frac{T}{t_r} \cos \omega_k t_r \right]. \tag{7.99}$$

Then in accordance with equation 7.95 and using equation 7.99, we obtain

$$\begin{aligned} -A_k &= \frac{V_0}{\omega_k t_r \left[\left(1 + \dfrac{T}{t_r}\right) \sin \omega_k t_r + \omega_k t_r \dfrac{T}{t_r} \cos \omega_k t_r \right]} \\ &= \frac{V_0}{\dfrac{\omega_k t_r}{\sin \omega_k t_r} + \cos \omega_k t_r}. \end{aligned} \tag{7.100}$$

The first term of equation 7.95 is

$$\frac{V_0}{F(0)} = \left[\frac{V_0}{\cosh st_r + sT \sinh st_r} \right]_{s=0} = V_0. \tag{7.101}$$

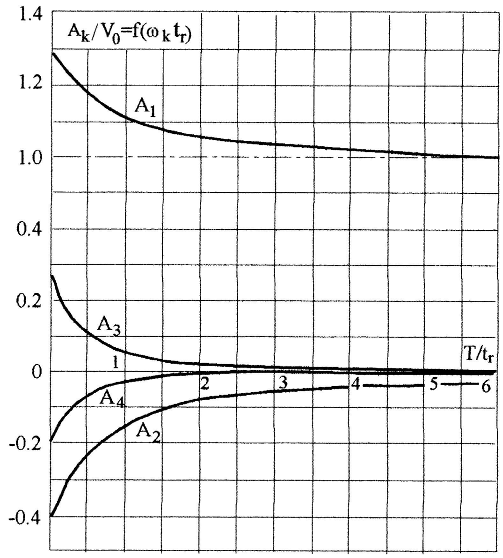


Figure 7.26 The dependence of magnitudes A_1–A_4 by the ratio T/t_r.

Therefore the complete receiving-end voltage response is

$$v(\ell, t) = V_0\left(1 - \sum_1^{\infty} 2A_k \cos \omega_k t\right). \tag{7.102}$$

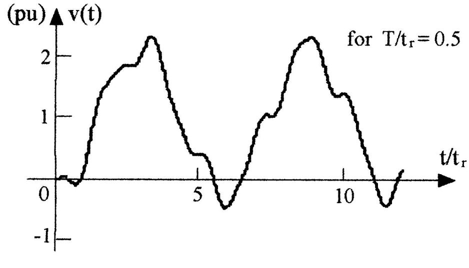
Figure 7.26 shows the dependency of the magnitude A_k by the ratio T/t_r. It can be concluded that for large ratios of T/t_r (i.e. the capacitance is relatively big and/or the line is short) the high harmonic magnitudes are negligibly small and the first magnitude A_1 approaches unity. This means that the line terminating in a big capacitance behaves as a lumped LC circuit. Two plots of $v(\ell, t)$ versus t/t_r for $T/t_r = 0.5$ and $T/t_r = 2$, are shown in Fig. 7.27. Note that for the ratio $T/t_r = 2$ the voltage response curve is very close to the sinusoidal function. The points of non-continuity of the first curve (in Fig. 7.27(a)) represent the arrival of the incident waves.

It should be emphasized that solution of equation 7.92 immediately gives the complete voltage compared to other techniques in which the voltage or current are determined by means of the sum of the traveling waves.

7.7.3 A solution as a sum of delayed waves

As a further example of using Laplace transform techniques, let us again consider the TL equations, for the phasor representation, given in exponential form (see equations 7.88a). These equations can be rewritten in the following form:

$$V(x) = \frac{V_2 + Z_c I_2}{2}\left(e^{\gamma x} + \rho_2(s) e^{-\gamma x}\right) \tag{7.103a}$$



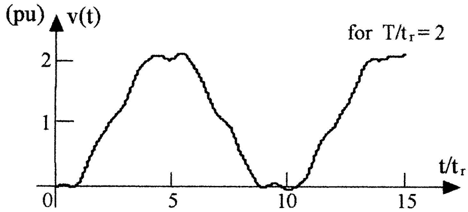


Figure 7.27 The receiving-end voltage curve versus the scaled time t/t_r: for the ratio $T/t_r = 0.5$ (a); for the ratio $T/t_r = 2$ (b).

$$I(x) = \frac{V_2 + Z_c I_2}{2Z_c}(e^{\gamma x} - \rho_2(s)e^{-\gamma x}), \tag{7.103b}$$

where $\rho_2 = (Z_2 - Z_c)/(Z_2 + Z_c)$ is the receiving-end reflection coefficient for the phasor quantities and $Z_2 = V_2/I_2$. Considering the voltage and current phasors as Laplace transform quantities (i.e., by replacing $j\omega$ by s) the above equations become the Laplace transform equations.

To simplify the following analysis we will assume, as before, that the TL is loss-less, therefore $\gamma = s\sqrt{LC} = s/v$ and equations 7.103 become

$$V(x, s) = \frac{V_2 + Z_c I_2}{2}(e^{sx/v} + \rho_2(s)e^{-sx/v}) \tag{7.104a}$$

$$I(x, s) = \frac{V_2 + Z_c I_2}{2Z_c}(e^{sx/v} - \rho_2(s)e^{-sx/v}). \tag{7.104b}$$

Now consider the boundary conditions

$$V_1(s) = \frac{V_0}{s} \quad (Z_1 = 0), \quad V_2(s) = Z_2(s)I_2(s). \tag{7.105}$$

Then equation 7.104a yields

$$V_1(s) = \frac{V_2(s) + Z_c V_2(s)/Z_2(s)}{2}(e^{s\ell/v} + \rho_2(s)e^{-s\ell/v}),$$

which allows the voltage transfer function to be determined:

$$H_{\mathrm{v}}(s) = \frac{V_2(s)}{V_1(s)} = \frac{2}{1 + Z_{\mathrm{c}}/Z_2} \frac{1}{e^{s\ell/v} + \rho_2(s)e^{-s\ell/v}}. \tag{7.106}$$

For the given boundary conditions, equation 7.105, and after substituting $I_2 = V_2/Z_2$ and $V_2 = V_1(s)H_{\mathrm{v}}(s)$ (from equation 7.106) into equation 7.104a, this equation can be expressed as

$$V(x, s) = \frac{V_0}{s} \frac{e^{s(x-\ell)/v} + \rho_2(s)e^{-s(x+\ell)/v}}{1 + \rho_2 e^{-2s\ell/v}}. \tag{7.107}$$

Assuming again that x is reckoned from the sending-end, i.e., $x' = \ell - x$, but omitting the prime-sign in x', after interchanging, we obtain

$$V(x, s) = \frac{V_0}{s} \frac{e^{-sx/v} + \rho_2(s)e^{-s(2\ell-x)/v}}{1 + \rho_2 e^{-2s\ell/v}}, \tag{7.108a}$$

and similarly for the current

$$I(x, s) = \frac{V_0}{sZ_{\mathrm{c}}} \frac{e^{-sx/v} + \rho_2(s)e^{-s(2\ell-x)/v}}{1 + \rho_2 e^{-2s\ell/v}}, \tag{7.108b}$$

where x is reckoned from the sending-end. To simplify the solution and better understand these techniques, consider an open-circuited receiving-end $Z_2 \to \infty$, i.e., $\rho_2(s) = 1$. Therefore, equation 7.108a yields

$$V(x, s) = \frac{V_0}{s} \frac{e^{-sx/v} + e^{-s(2\ell-x)/v}}{1 + e^{-2s\ell/v}}. \tag{7.109}$$

In order to find the inverse Laplace transform we note that the expression $1/(1 + e^{-2s\ell/v})$ can be treated as the sum of the infinitely decreasing geometric progression of the ratio $q = -e^{-2s\ell/v}$, i.e.,

$$\frac{1}{1 + e^{-2s\ell/v}} = 1 - e^{-2s\ell/v} + e^{-4s\ell/v} - e^{-6s\ell/v} + \cdots.$$

Then equation 7.109 becomes

$$V(s, x) = \frac{V_0}{s}(e^{-sx/v} + e^{-s(2\ell-x)/v} - e^{-s(2\ell+x)/v} - e^{-s(4\ell-x)/v} + e^{-s(4\ell+x)/v} + \cdots). \tag{7.110}$$

In accordance with the time-shift theorem, the time domain voltage is simply the infinite sum of the delayed step-functions of V_0

$$\begin{aligned} v(t, x) = V_0 \Bigg[& u\left(t - \frac{x}{v}\right) + u\left(t - \frac{2\ell - x}{v}\right) - u\left(t - \frac{2\ell + x}{v}\right) \\ & - u\left(t - \frac{4\ell - x}{v}\right) + \cdots \Bigg], \end{aligned} \tag{7.111a}$$

and similarly the time domain current is

$$i(t,x) = \frac{V_0}{Z_c}\left[u\left(t-\frac{x}{v}\right) - u\left(t-\frac{2\ell-x}{v}\right) - u\left(t-\frac{2\ell+x}{v}\right) + u\left(t-\frac{4\ell-x}{v}\right) + \cdots\right]. \tag{7.111b}$$

In equation 7.111 the terms $+u[t-(k2\ell-x)/v]$, $(k=0,1,2,\ldots)$ represent the unit step-functions delayed by the time $(k2\ell+x)/v$ in which the kth forward-traveling wave arrives the point x and the terms $-u[t-(k2\ell-x)/v]$ represent the unit step-functions delayed by the time $(k2\ell-x)/v$ in which the kth backward-traveling wave arrives at point x. Figure 7.28 shows the voltage and current changing in time at a half line distance.

As a next example, consider the line terminated in the capacitance. The Laplace transform reflection coefficient in this case is

$$\rho_2(s) = \frac{1/sC - Z_c}{1/sC + Z_c} = -\frac{s-\alpha}{s+\alpha}, \tag{7.112}$$

where $\alpha = 1/T = 1/CZ_c$. In accordance with equation 7.112 equation 7.108 yields

$$V(s,x) = \frac{V_0/s}{1-[(s-\alpha)/(s+\alpha)]e^{-2s\ell/v}}\left(e^{-sx/v} - \frac{s-\alpha}{s+\alpha}e^{-s(2\ell-x)/v}\right). \tag{7.113}$$

Again we will treat the expression

$$\frac{1}{1-[(s-\alpha)/(s+\alpha)]/e^{-2s\ell/v}}$$

as the sum of the infinitely decreasing geometric progression having the ratio $[(s-\alpha)/(s+\alpha)]e^{-2s\ell/v}$ (note that for $\mathrm{Re}[s]>0$ the magnitude $|(s-\alpha)/(s+\alpha)|<1$)

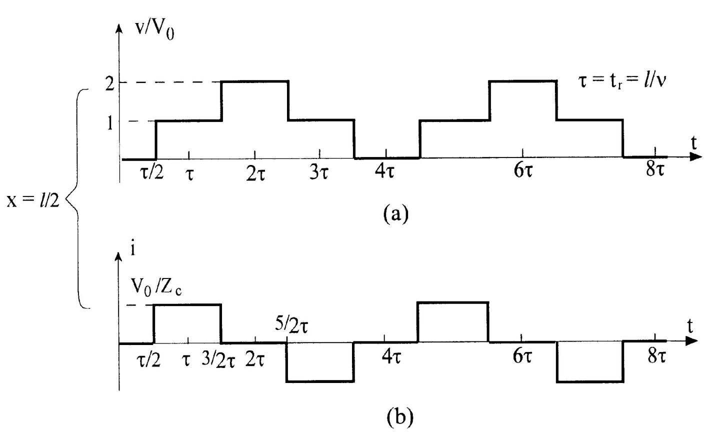


Figure 7.28 Voltage (a) and current (b) plots versus time at the line point $x=\ell/2$.

and the first member of unity. Therefore,

$$\frac{1}{1-[(s-\alpha)/(s+\alpha)]e^{-2s\ell/v}} = 1 + \frac{s-\alpha}{s+\alpha}e^{-2s\ell/v} + \left(\frac{s-\alpha}{s+\alpha}\right)^2 e^{-4s\ell/v} + \cdots$$

and

$$V(s,x) = \frac{V_0}{s}\Bigg[e^{-sx/v} - \frac{s-\alpha}{s+\alpha}e^{-s(2\ell-x)/v} + \frac{s-\alpha}{s+\alpha}e^{-s(2\ell+x)/v} - \left(\frac{s-\alpha}{s+\alpha}\right)^2 e^{-s(4\ell-x)/v}$$
$$+ \left(\frac{s-\alpha}{s+\alpha}\right)^2 e^{-s(4\ell+x)/v} - \left(\frac{s-\alpha}{s+\alpha}\right)^3 e^{-s(6\ell-x)/v} + \cdots\Bigg]. \tag{7.114}$$

The Laplace transform of the receiving-end voltage, i.e., voltage across the capacitance, becomes

$$V(s,\ell) = \frac{V_0}{s}\Bigg[e^{-s\ell/v} - \frac{s-\alpha}{s+\alpha}e^{-s\ell/v} + \frac{s-\alpha}{s+\alpha}e^{-3s\ell/v} - \left(\frac{s-\alpha}{s+\alpha}\right)^2 e^{-3s\ell/v}$$
$$+ \left(\frac{s-\alpha}{s+\alpha}\right)^2 e^{-5s\ell/v} - \left(\frac{s-\alpha}{s+\alpha}\right)^3 e^{-5s\ell/v} + \cdots\Bigg]. \tag{7.115}$$

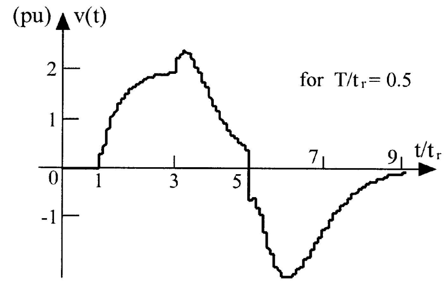
Note that the inverse Laplace transforms of the terms are

$$\frac{1}{s}\frac{s-\alpha}{s+\alpha} \leftrightarrow -1 + 2e^{-\alpha t}$$
$$\frac{1}{s}\left(\frac{s-\alpha}{s+\alpha}\right)^2 \leftrightarrow 1 - 4\alpha t e^{-\alpha t}$$
$$\frac{1}{s}\left(\frac{s-\alpha}{s+\alpha}\right)^3 \leftrightarrow -1 + 2(1 - 2\alpha t + 2\alpha^2 t^2)e^{-\alpha t}.$$
$$\vdots$$

and therefore with the time-shift theorem we obtain

$$v(t,\ell) = V_0\big(u(t-t_r) - (-1+2e^{-\alpha(t-t_r)})u(t-t_r) + (-1+2e^{-\alpha(t-3t_r)})u(t-3t_r)$$
$$-[1-4\alpha(t-3t_r)e^{-\alpha(t-3t_r)}]u(t-3t_r)$$
$$+\{-1+2[1-2\alpha(t-5t_r)+2\alpha^2(t-t_r)^2]e^{-\alpha(t-5t_r)}\}u(t-5t_r) - \cdots\big), \tag{7.116}$$

where $t = \ell/v$ is the wave traveling time along the line. The capacitance voltage plots versus t in accordance with equation 7.116, for the ratios $T/t_r = 0.5$ and $T/t_r = 0.2$ are shown in Fig. 7.29. (Note the similarity of the curves in Fig. 7.29 and Fig. 7.27.) However, since only a few first terms in equation 7.116 have



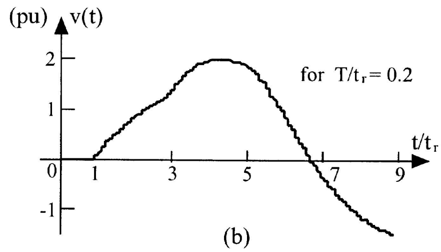


Figure 7.29 Voltage across the capacitance as versus t for the ratios $T/t_r = 0.5$ (a) and $T/t_r = 0.2$ (b).

been taken into consideration, the curves in Fig. 7.29 give just a rough approximation of the voltages after only a couple of reflections.

7.8 LINE WITH ONLY *LG* OR *CR* PARAMETERS

In some practical applications of electrical engineering techniques we consider networks in which the only significant parameters are L and G or C and R. An example of the former is ground rods used for grounding line towers and other power station (and substation) equipment. Under conditions of lightning impulse stress, the rods have to be treated as a network with distributed parameters as shown in Fig. 7.30(a). An example of the latter is an underground cable whose insulation is very good ($G = 0$) and inductance is negligible ($L = 0$) as shown in Fig. 7.30(b). The propagation constant and characteristic impedance in these cases are

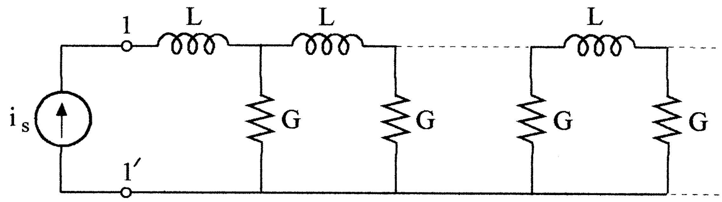
$$\gamma(s) = \sqrt{LGs} \quad \text{or} \quad \gamma(s) = \sqrt{CRs} \tag{7.117a}$$

$$Z_c = \sqrt{\frac{sL}{G}} \quad \text{or} \quad Z_c = \sqrt{\frac{R}{sC}}. \tag{7.117b}$$

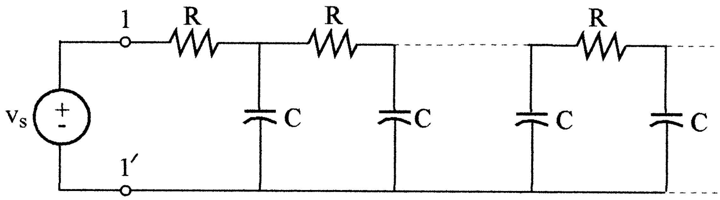
The differential equations 7.1 in such cases are simplified:

for the ground rod to

$$-\frac{\partial v}{\partial x} = L\frac{\partial i}{\partial t}, \quad -\frac{\partial i}{\partial x} = Gv \tag{7.118}$$



(a)



(b)

Figure 7.30 The equivalent circuits of: ground rod (a); underground cable (b).

and for the underground cable to

$$-\frac{\partial i}{\partial x} = C\frac{\partial v}{\partial t}, \quad -\frac{\partial v}{\partial x} = Ri. \tag{7.119}$$

Let us consider the case of an underground cable in more detail.

7.8.1 Underground cable

A transmission line, which behaves in accordance with differential equations 7.119, is an underground cable in which L and G can be neglected. Therefore its propagation constant and characteristic impedance in Laplace transform equations (see 7.84) are

$$\gamma = \sqrt{CRs} = \frac{\sqrt{s}}{a} \tag{7.120a}$$

$$Z_c = \sqrt{\frac{R}{C}}\frac{1}{\sqrt{s}}, \tag{7.120b}$$

where $a = 1/\sqrt{CR}$.

Applying the step voltage function $V_0u(t)$ at the sending-end of an infinite cable, and knowing that $V_1 = Z_cI_1$, equation 7.186a yields

$$V(x, s) = \frac{V_0}{s} e^{-\frac{x}{a}\sqrt{s}}. \tag{7.121}$$

Now the inverse Laplace transform gives

$$v(x, t) = V_0 \left(1 - \operatorname{erfc} \frac{x}{2a\sqrt{t}} \right) \tag{7.122}$$

where $\operatorname{erfc}(u) = \int_0^u e^{-\tau^2} d\tau$ is the error function. The Laplace transform of the current is

$$I(x, s) = \frac{V(x, s)}{Z_c} = V_0 \sqrt{\frac{C}{R}} \frac{1}{\sqrt{s}} e^{-\frac{x}{a}\sqrt{s}} \tag{7.123}$$

which gives the current in the time domain

$$i(x, t) = V_0 \sqrt{\frac{C}{R}} \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4a^2t}}. \tag{7.124}$$

Example 7.6

An underground cable ("very long") has distributed parameters $R = 1\ \Omega/\text{km}$ and $C = 0.1\ \mu\text{F/km}$. Assuming that at time $t = 0$ a step voltage source $v_s = 500u(t)$ V connects to the cable, find the voltage and current distribution along the cable line at $t_1 = 10\ \mu\text{s}$ and at $t_1 = 50\ \mu\text{s}$.

Solution

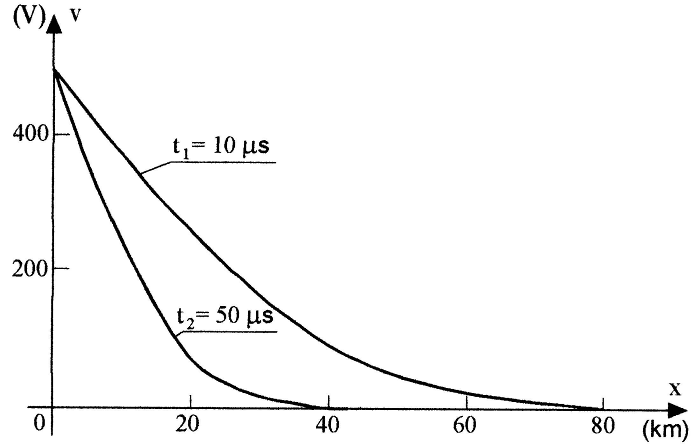
The parameters of the cable are $a = 1/\sqrt{CR} = 3.16 \cdot 10^3$ and $\sqrt{C/R} = 3.16 \cdot 10^{-4}$. In accordance with equation 7.122 the voltage distribution along the line for $t_1 = 10\ \mu\text{s}$ is

$$v(x) = 500 \left(1 - \operatorname{erfc} \frac{x}{20} \right) \text{V}.$$

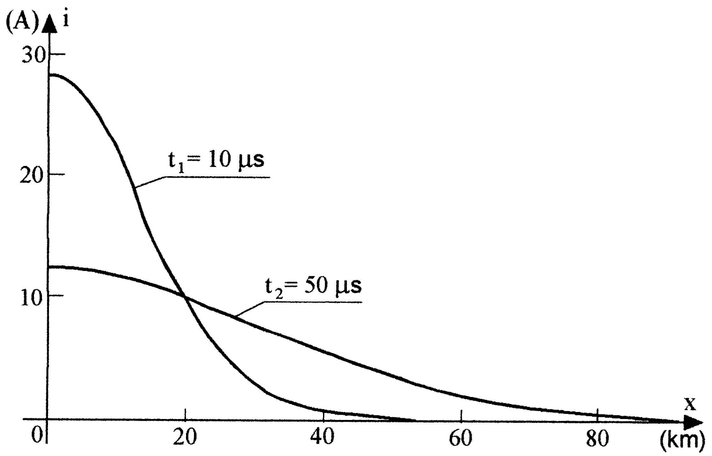
In accordance with equation 7.124 the current distribution along the line is

$$i(x) = 500 \cdot 3.16 \cdot 10^{-4} \frac{1}{\sqrt{\pi 10 \cdot 10^{-3}}} e^{-(x^2/400)} = 28.21 e^{-(x^2/400)}\ \text{A}.$$

Similarly we may calculate the voltage and current curves versus x for the second moment of time $t_{2.}$ The resulting curves for both moments of time are shown in Fig. 7.31(a) and (b).



(a)



(b)

Figure 7.31 The voltage (a) and current (b) distribution along the underground cable for two different moments of time.

In conclusion consider an underground cable of length ℓ having a short-circuited receiving-end and connecting to a voltage source of step function $v_1 = V_0 u(t)$. In accordance with equations 7.88b and 7.99 and assuming $Z_2 = 0$ the voltage Laplace transform at any point on the line is expressed in terms of V_1 (but x is reckoned from the receiving-end):

$$V(x, s) = V_1(s) \frac{\sinh \gamma x}{\sinh \gamma \ell}. \tag{7.125}$$

With the Laplace transform of the sending-end voltage source $V_1(s) = V_0/s$

equation 7.125 becomes

$$V_1(x,s) = V_0 \frac{1}{s} \frac{\sinh(x\sqrt{s}/a)}{\sinh(\ell\sqrt{s}/a)} = \frac{F_1(s)}{F_2(s)}. \tag{7.126}$$

Using the partial fraction expansion formula, we obtain

$$v(x,t) = V_0 \left[\frac{F_1(0)}{F_2(0)} + \sum_{k=1}^{\infty} \frac{\sinh \gamma x e^{s_k t}}{s_k F_2'(s)} \right], \tag{7.127}$$

where s_k are the roots of the characteristic equation $\sinh(\ell/a)\sqrt{s} = 0$.
Therefore,

$$\gamma_k \ell = \frac{\ell}{a}\sqrt{s_k} = jk\pi, \tag{7.128}$$

or

$$s_k = -\frac{k^2\pi^2 a^2}{\ell^2}. \tag{7.129}$$

Evaluating the terms of equation 7.127 yields

$$\frac{F_1(0)}{F_2(0)} = \lim_{s\to 0} \frac{\sinh \frac{x}{a}\sqrt{s}}{\sinh \frac{\ell}{a}\sqrt{s}} \to \left[\frac{\frac{x}{2a\sqrt{s}} \cosh \frac{x}{a}\sqrt{s}}{\frac{\ell}{2a\sqrt{s}} \cosh \frac{\ell}{a}\sqrt{s}} \right]_{s=0} = \frac{x}{\ell} \tag{7.130}$$

$$\begin{aligned} F_1(s_k) &= \sinh \frac{x}{\ell} jk\pi = j \sin \frac{k\pi x}{\ell} \\ F_2'(s_k) &= \frac{d}{ds} \sinh \frac{\ell}{a}\sqrt{s} \bigg|_{s=s_k} = \frac{\ell}{2a\sqrt{s}} \cosh \frac{\ell}{a}\sqrt{s} \bigg|_{s=s_k} = \frac{\ell^2 \cos k\pi}{2a^2 jk\pi}. \end{aligned} \tag{7.131}$$

Substituting equations 7.131, 7.130 and 7.129 into equation 7.127 yields

$$v(x,t) = V_0 \left[\frac{x}{\ell} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sin \frac{k\pi x}{\ell} e^{-(k\pi a/\ell)^2 t} \right]. \tag{7.132}$$

Note that equation 7.132 gives the voltage at the receiving-end $x = 0$ equal to 0 at any time and the voltage at the sending-end ($x = \ell$) equal to V_0. The time constants of the exponentials in equation 7.132 are proportional to the ratio $\ell^2/a^2 = \ell C \ell R = C_\ell R_\ell$, i.e., they are equal to the product of the complete capacitance and complete resistance of the cable.

Now, in accordance with the second equation of 7.119, we can obtain the current

$$i(x,t) = -\frac{1}{R}\frac{\partial v}{\partial x} = -\frac{V_0}{R\ell} \left[1 + 2 \sum_{k=1}^{\infty} (-1)^k \cos \frac{k\pi x}{\ell} e^{-(k\pi a/\ell)^2 t} \right]. \tag{7.133}$$

The voltage and current distributions along the cable are shown in Figs. 7.32(a) and (b). Other problems of the transient behavior of cables having different terminations and different sending-end conditions can be solved in a similar way.

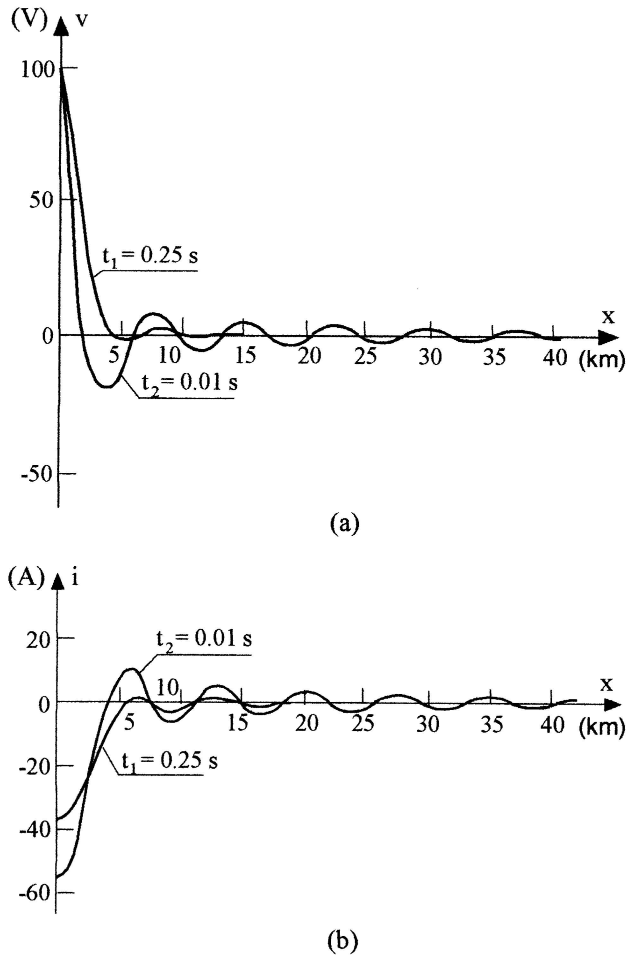


Figure 7.32 The voltage (a) and current (b) distribution along the short-circuited underground cable.

Chapter #8

STATIC AND DYNAMIC STABILITY OF POWER SYSTEMS

8.1 INTRODUCTION

Today's power systems are interconnected networks of transmission lines linking generators and loads into large integrated systems, some of which span entire countries and even continents. The main requirement for the reliable operation of such systems is to keep the synchronous generators running in parallel and with an adequate capacity to meet the load demand. When synchronous machines are electrically tied in parallel they must operate at the same frequency, i.e. they must all operate at the same speed (measured in electrical radians per second), which is called **being in synchronism.** If at any time a generator increases the speed and the rotor advances beyond a certain critical angle, counted between the rotor axis (usually the *d*-axis) and the system voltage phasor (the power angle δ), the magnetic coupling between the rotor and the stator fails. In such a situation the rotor rotates relatively to the field of the stator currents rather than being tied to this field, and pole slipping occurs, i.e., the generator **loses its synchronism** (falls out of step) with the rest of the system. Each time the generator speed changes, stability problems arise. The disturbance of the stability of the synchronous generators operating in parallel is one of the most arduous faults of power systems and may result in outage of entire regions.

8.2 DEFINITION OF STABILITY

Synchronous machines do not easily fall out of step under normal conditions. If a machine tends to speed up or slow down, synchronizing forces (see further on) tend to keep it in step. However, certain conditions may arise, in which the synchronizing forces for one or more generators may not be adequate and small impacts on the system may cause these generators to lose synchronism. On the other hand, if following an imbalance between the supply and demand created by a change in the load, in the generation or in the network conditions, all interconnected synchronous machines remain in synchronism adjusting themselves to a new state of operation, then the system is stable and the generators continue to operate at the same speed.

The perturbation could be of a major disturbance such as the loss of a generator, a fault or the loss of a line, or it could be of small, random load changes occurring under normal operation. The transients following system perturbations are oscillatory in nature, but if these oscillations are damped toward a new quiescent operating condition, we say that the system is stable. Thus, we may state that: **If the oscillatory response of a power system during the transient period is damped and the system settles in a finite time to a new steady-state condition, the system is stable.** Otherwise, the system is considered unstable.

The stability problems may be divided into two kinds: steady-state, or **static stability** and transient, or **dynamic stability**. The former is concerned with the effect of gradual infinitesimal power changes and is defined as the ability of a synchronous generator to reestablish its given state of operation after such changes. The latter, transient stability, deals with the effect of large, sudden disturbances such as line faults, the sudden switching of lines, the sudden application or removal of large loads, etc. The ability of the power system to retain synchronism when subject to such disturbances is considered as dynamic stability. Thus, the main criterion for stability in both regimes is that synchronous machines maintain synchronism at the end of the period of small as well as large disturbances.

8.3 STEADY-STATE STABILITY

Power systems form groups of synchronous generators (power station) interconnected by transmission lines. Experience in operating and theoretical study reveal that such transmission lines with synchronous machinery at both ends show that there are definite limits beyond which the operation becomes unstable, resulting in the loss of synchronism between the sending- and receiving-ends. This problem is termed the stability of the tie line, even though in reality it reflects the stability of two groups of machines. In order to understand this problem we shall introduce a transmission line power-transfer characteristic.

8.3.1 Power-transfer characteristic

Consider a group of synchronous generators, which is connected through a transmission line to a large system as shown in Fig. 8.1.

Here, the group of generators (power station) is represented by a single equivalent synchronous generator, operating with the phase EMF (along the quadrature axis) E_{ph}, and the system is represented by the **infinite bus**, whose voltage is kept constant regardless of any changes in the system behavior and is taken as the reference. The total reactance of the equivalent circuit is

$$X = X_d + X_{T1} + X_\ell + X_{T2}.$$

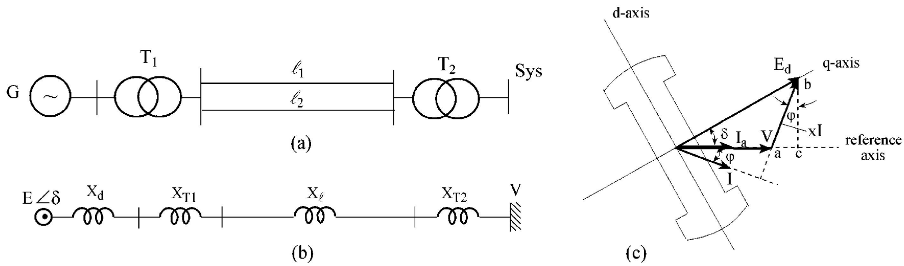


Figure 8.1 A group of generators connected to the system through a transmission line: one-line diagram (a), equivalent circuit (b) and the phasor diagram (c).

In accordance with the phasor diagram in Fig. 8.1(c) we have

$$\overline{bc} = E_d \sin\delta = \overline{ab}\cos\varphi = XI\cos\varphi,$$

where δ is the angle between the induced voltage of the generator E and the reference voltage V.

Thus,

$$I_a = I\cos\varphi = \frac{E_{ph}}{X}\sin\delta,$$

and the generator power transmitted by a transmission line is

$$P_e = 3V_{ph}I_a = 3\frac{V_{ph}E_{ph}}{X}\sin\delta = \frac{VE}{X}\sin\delta, \tag{8.1}$$

where E and V are the line voltages.

The physical significance of angle δ is understandable from the phasor diagram, in Fig. 8.1(c), where the rotor position, in relation to the phasors, is also shown by a thin line. Subsequently, we realize that this angle is not only the *electrical angle* between E and V, but it is also the *mechanical angle* between the rotor q-axis and the reference axes. At no-load operation the rotor q-axis and the reference axis coincide. With increasing shaft or input power, the rotor advances (in the direction of the rotation) by angle δ, which is therefore called the power angle. The relationship of the power, developed by the generator, versus δ is given in equation 8.1 and plotted in Fig. 8.2. This plot, assuming E and V are constant, is a pure sinusoid, having the amplitude of $P_{\max} = EV/X$, and is called the **power-transfer curve**, or **power-angle curve**. It should be noted that for motor action the rotor is retarded relative to the reference axis and δ becomes negative.

For the given constant values of the generator EMF E and the receiving-end voltage V, the load of the transmission line can be gradually increased until a condition is reached corresponding to point A in Fig. 8.2. At this point the power transmitted is maximum and corresponds to angle $\delta = 90^\circ$ (since the resistances are neglected) and represents the **static limit of stability** (i.e. for a gradually applied load) and any attempt to impose any additional load on the

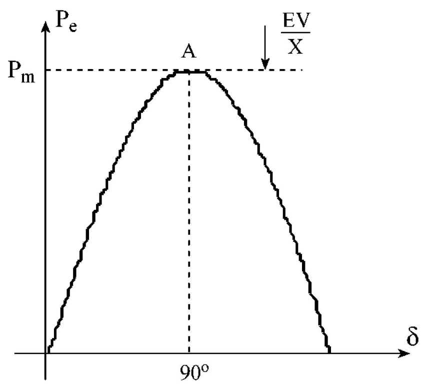


Figure 8.2 The power-angle curve.

line will result in the loss of the synchronism between the generator and the system. However, since today's generators are equipped with an AVR system, the terminal voltage of such generators is kept constant during the period of load changes. Hence, when increasing the load, the generator EMF will also increase and the operating power-angle curve will change, resulting in a higher limit of stability. This limit of stability may exceed the initial one by 30–40%.

Until now, system resistances have been neglected. However, sometimes transmission line resistances are relatively significant and should be considered. In this case, with the total impedance $Z = R + jX$, the line current is

$$\tilde{I} = \frac{\tilde{E} - \tilde{V}}{\sqrt{3}Z},$$

and the transmitted power will be

$$S = \sqrt{3}\overset{*}{E}\tilde{I} = \sqrt{3}\overset{*}{E}\frac{\tilde{E} - \tilde{V}}{\sqrt{3}Z}.$$

Substituting the polar forms of the quantities $\tilde{E} = E\angle\delta$, $\overset{*}{E} = E\angle-\delta$, $Z = z\angle\varphi$ in the above expressions after simplification, yields

$$S = \frac{E^2}{z}(\cos\varphi - j\sin\varphi) - \frac{EV}{z}[\cos(\delta+\varphi) - j\sin(\delta+\varphi)].$$

The real part of this expression gives the active power

$$P = \frac{E^2}{z}\cos\varphi - \frac{EV}{z}\cos(\delta+\varphi). \tag{8.2a}$$

For an easier comparison of this expression to the previous one (equation 8.1) we assign an additional angle $d = 90^\circ - \varphi$ to obtain

$$P = \frac{EV}{z}\sin(\delta - \alpha) + \frac{E^2}{z}\sin\alpha. \tag{8.2b}$$

This power-angle curve is shown in Fig. 8.3. The maximum power transferred, or static limit if stability, in this case is

$$P_{\max} = \frac{EV}{z} + \frac{E^2}{z} \sin\alpha,$$

which is higher than in the case where the resistances are neglected. The critical angle here is also larger than in the previous case, i.e., $\delta_{cr} = (90^\circ + \alpha) > 90^\circ$.

The power-angle characteristic of the turbine, which governs the generator, is a straight line, as shown in Fig. 8.4, since the power developed by the turbine does not depend on angle δ, which is a pure electrical parameter. At a steady-state operation the mechanical power P_m is equal to the electrical one and, as can be seen from Fig. 8.4, for the given mechanical power P_m, there are two points of equilibrium, *a* and *b*, on the intersection of the turbine and generator characteristics. This means that two steady-state regimes are possible at each

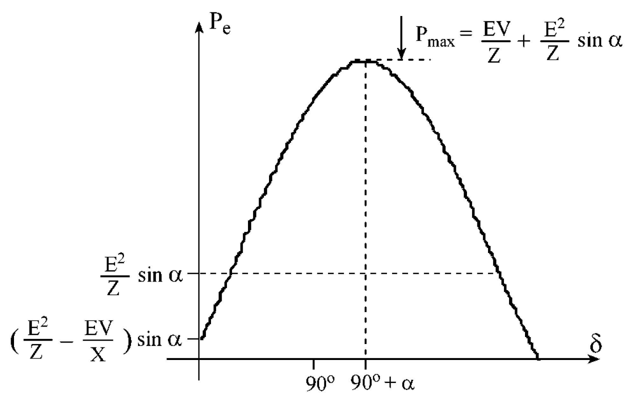


Figure 8.3 Power-angle curve for a system in which the resistances are considered.

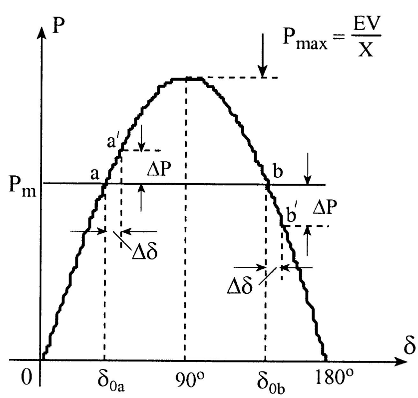


Figure 8.4 Steady-state stability of a synchronous generator.

of these points. However, the stable operation of the generator is possible only at point a. Indeed, assuming that angle δ is randomly increased by $\Delta\delta$, then the generator has to transmit power $P_e = P_m + \Delta P$ while the turbine power remains the same, P_m. The difference between P_m and P_e is an accelerated power P_a. Therefore, the acceleration power in this case becomes negative: $P_m - P_e = -\Delta P < 0$. This causes the generator to decelerate and return back to point a. With an analogous assumption for point b we realize that increasing δ by $\Delta\delta$, the generator power will be decreased and the acceleration power will, therefore, be positive. This causes the rotor of the generator to accelerate and δ will increase even more, actually up to 180°. The generator EMF E is at this position in the opposite phase with the system voltage, and this situation is equivalent to the short-circuit fault. The generator power drops to zero and it falls out of step.

It is self-evident that assuming a random decrease of δ at point a, we will arrive at the same conclusion, i.e., the generator returns back to point a. However, at point b, the generator, as previously, does not return to point b, but its rotor decelerates until it reaches point a. Hence, the stable operation of a generator in parallel with the system is possible only at point a, or in the increasing part of the power-angle curve, i.e., when $0 < \delta < 90°$. At angle $\delta = 90°$ $(\partial P/\partial\delta)_{90°} = 0$, i.e., the system is at the limit, and the operation at this point cannot be stable. The stability of the operation is often estimated by the assurance factor

$$k_{as} = \frac{P_{\max} - P_m}{P_m}. \tag{8.3}$$

The steady-state **stability limit** is the maximum power that can be transmitted in a network between sources and loads when the system is subject to small disturbances. The stability is assured if the generator operates within the "safe area" of the power characteristic, which is in about a 20% margin lower than the steady-state stability limit. It should be noted that, as already has been mentioned, this limit may be extended by the use of an automatic voltage regulator (AVR). Besides the AVR, by analyzing the system stability, the effects of machine inertia and governor action should be taken into consideration. These functions greatly increase the complexity of the analysis (this is beyond the scope of this text). For more details the reader is referred to the book by Anderson and Fouad[(*)].

Usually the normal operating load angle for modern machines is in the order of 60 electrical degrees, and for the limiting value of 90°, this leaves 30° for the "safe area", to cover the large disturbances in the transmission line (see further on).

In a system with several generators/power stations and loads the common procedure is to reduce the network to the simplest form in which only the

[(*)] P. M. Anderson and A. A. Fouad (1980) *Power System Control and Stability*, Iowa State University Press.

relevant generators are connected to each other and which then allows the transfer reactances to be calculated. The values of the load, the power angles and the voltage are then calculated for the given conditions, and the steady-state stability limit and the assurance factor are determined for each machine. If those stability criteria are satisfactory, the loading is increased and the process repeated. If the voltages change appreciably, the $P-V$, $Q-V$ characteristics of the load should be used with the redistribution of the power.

Example 8.1

A synchronous generator having a local load, represented by constant impedance, is connected to an infinite bus through a transformer and a double circuit transmission line. The direct axis generator synchronous reactance is 1.2 pu, the load impedance is $Z_{load} = 2\angle 36^\circ$ pu and the rest of the parameters are shown in Fig. 8.5. Check the steady-state stability of the given system, if the power transmitted to the systems is 0.5 pu and the generator terminal voltage is kept as 1.1 pu.

Solution

First we find the angle of the generator terminal voltage $V_T = V_2$. The power-angle equation is

$$P = \frac{V_T V}{X_{23}} \sin\delta_{23} \quad \text{or} \quad 0.5 = \frac{1.1 \cdot 1}{0.6} \sin\delta_{23},$$

where $X_{23} = 0.1 + 1.0/2 = 0.6$. Then

$$\sin\delta_{23} = 0.273 \quad \text{and} \quad \delta_{23} = 15.8^\circ.$$

The current is found as

$$\tilde{I}_{23} = \frac{\tilde{V}_2 - \tilde{V}_3}{X_{23}} = \frac{1.1\angle 15.8^\circ - 1}{0.6\angle 90^\circ} = 0.5 - j0.1 \text{ pu}.$$

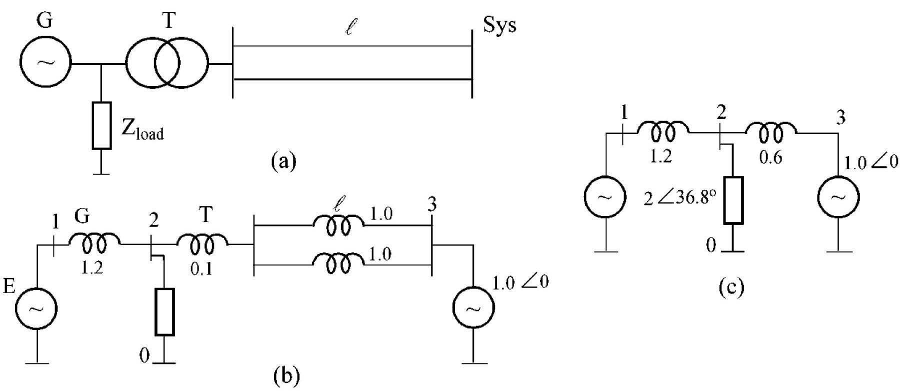


Figure 8.5 Network of Example 8.1: one-line diagram (a), equivalent circuit (b) and its simplification (c).

The load current is found as

$$\tilde{I}_{20} = \frac{1.1\angle 15.8^\circ}{2\angle 36.8^\circ} = 0.55\angle -21^\circ = 0.513 - j0.197 \text{ pu},$$

and the generator current is

$$\tilde{I}_G = \tilde{I}_{20} + \tilde{I}_{23} = 0.513 - j0.197 + 0.5 - j0.1 = 1.05\angle -16.13^\circ \text{ pu}.$$

Then, the internal generator voltage (EMF) is

$$E = 1.1\angle 15.8^\circ + (1.05\angle -16.3^\circ)(1.2\angle 90^\circ) = 1.41 + j1.51 = 2.07\angle 47.12^\circ \text{ pu}.$$

Hence, the angle between E and V is 47.12°. Since this angle is less than 90° with a safe area of about 40°, the system is stable. The active power produced by the generator is

$$P_G = EI\cos\varphi = 2.07\cdot 1.05\cos[47.12^\circ - (-16.3^\circ)] \cong 1.0 \text{ pu}.$$

8.3.2 Swing equation and criterion of stability

Assume that upon some change in the system operation the balance between the driving or mechanical input of the turbine P_m and the electrical power of the generator P_e is disturbed, so that $P_e < P_m$. Then, the additional kinetic energy will be stored by the rotated rotor, namely

$$J\frac{\omega_m^2 - \omega_{m0}^2}{2} = \int_0^t (P_m - P_e)dt = \int_0^t P_a dt, \tag{8.4}$$

where J is the moment of inertia (in kg/m²) of all the rotating masses attached to the shaft, $\omega_m = d\delta_m/dt$ is an angular velocity of the shaft/rotor (in mechanical rad/sec) and P_a is an acceleration power. (In our future study we shall distinguish between the *electrical angle* δ_e, or just δ and the *mechanical angle* δ_m, i.e., $\delta \equiv \delta_e = (p/2)\delta_m$, where p is the number of poles, or $\delta_e = p\delta_m$, where p is the pole pairs, as it is adopted in some technical books.)

By differentiation, equation 8.4, we have

$$J\omega_m\frac{d\omega_m}{dt} = P_a. \tag{8.5a}$$

The change in the angular velocity about its initial or rated value is $\omega_m = \omega_{m0} + (d\delta_m/dt)$, then $(d\omega_m/dt) = (d^2\delta_m/dt^2)$ and

$$J\omega_m\frac{d^2\delta_m}{dt^2} = P_a. \tag{8.5b}$$

This equation, which governs the motion of the rotor of a synchronous machine, represents the power-angle δ change versus time, expressing the accelerating power applied to the shaft, and is called a **swing equation**. Usually it is written in a slightly different form, namely

$$M\frac{d^2\delta_m}{dt^2} = P_m - P_e = P_a, \tag{8.6a}$$

where

$$M = J\omega_m \tag{8.6b}$$

is an *angular momentum*, or *moment of inertia* (in joule·s/rad).

For a generator connected to an infinite bus, with operation at P_0 and δ_0, and small changes in δ and in P (so that linearity may be assumed), we can write

$$M\frac{d^2\Delta\delta}{dt^2} = -\Delta P = -\left(\frac{\partial P}{\partial \delta}\right)_0 \Delta\delta \quad \text{or} \quad M\frac{d^2\Delta\delta}{dt^2} + \left(\frac{\partial P}{\partial \delta}\right)_0 \Delta\delta = 0. \tag{8.7}$$

The expression

$$\left.\frac{\partial P}{\partial \delta}\right|_{\delta=\delta_0} = P_m \cos\delta_0$$

is defined as the synchronizing power coefficient or just **synchronizing power** and is designated P_s, i.e., $P_s = (\partial P/\partial\delta)_0$. The characteristic equation of differential equation 8.7 is then

$$Ms^2 + P_s = 0, \tag{8.8}$$

which has two roots

$$s_{1,2} = \pm\sqrt{-\frac{P_s}{M}}. \tag{8.9}$$

If $(\partial P/\partial\delta)_0$ is positive, then both roots are imaginary numbers. In this case the solution of equation 8.7 is oscillatory undamped (since the resistances are neglected). Practically the oscillations decay and the stability is held (point *a* in Fig. 8.4). However, if $(\partial P/\partial\delta)_0$ is negative, both roots are real and one of them is positive, causing an unlimited increase in δ. In this case the stability is lost (point *b* in Fig. 8.4). If damping is present (i.e. the resistances are taken into consideration), equation 8.7 becomes

$$M\frac{d^2\Delta\delta}{dt^2} + K_d\frac{d\Delta\delta}{dt} + \left(\frac{\partial P}{\partial \delta}\right)_0 \Delta\delta = 0, \tag{8.10}$$

which results in a characteristic equation

$$Ms^2 + K_d s + P_s = 0. \tag{8.11}$$

Again, if $(\partial P/\partial\delta)_0$ is positive, the solution is a damped sinusoid and the operation is stable; in the opposite case, if $(\partial P/\partial\delta)_0$ is negative, the stability is lost, i.e., the sign of the sinchronizing power provides the *criterion of stability*. The derivative

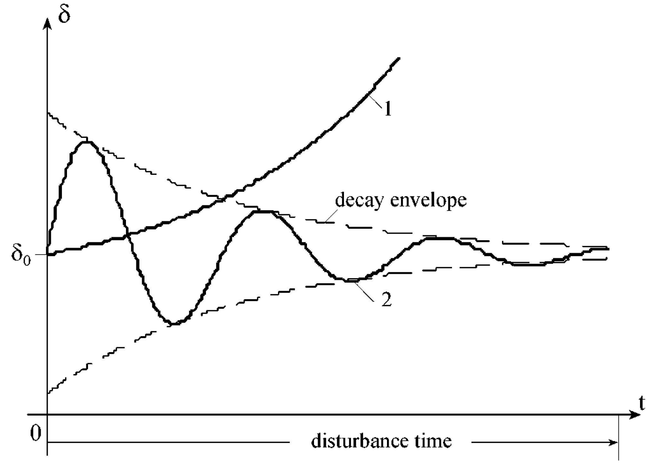


Figure 8.6 Two kinds of responses to a disturbance in the synchronous generator: 1) unstable, 2) stable.

$\partial P/\partial\delta$ is positive on the increasing branch of the power-angle curve and negative on the decreasing one, which confirms our previous conclusion that the operation at point a is stable, but at point b it is not. The two kinds of rotor motions, according to two kinds of solutions, are shown in Fig. 8.6.

For practical purposes (as is convenient in analyzing power systems) we shall normalize the swing equation. After dividing it by the rated power of the synchronous machine S_r, we have

$$\frac{M}{S_r}\frac{d^2\delta_m}{dt^2} = \frac{P_a}{S_r} = P_{an}, \tag{8.12}$$

where P_{an} is a normalized accelerating power in p.u. We shall next introduce the **inertia constant** H, which is one of the very important parameters of synchronous machines. It is defined as a quotient of the *kinetic energy* W_k, stored in the rotating rotor at rated angular velocity, and the rated power S_r:

$$H = \frac{W_k}{S_r} = \frac{1}{2}\frac{J\omega_{m,r}^2}{S_r} = \frac{1}{2}\frac{M}{S_r}\frac{\omega_{m,r}^2}{\omega_m}. \tag{8.13}$$

By using equation 8.13, equation 8.12 becomes

$$\frac{2H}{\omega_{m,r}}\frac{\omega_m}{\omega_{m,r}}\frac{d^2\delta_m}{dt^2} = P_{an}. \tag{8.14}$$

Since the change in angular velocity during the transient is relatively small, $(d\delta_m/dt) \ll \omega_{m,r}$ we may conclude that $\omega_m \cong \omega_{m,r}$(*) and equation 8.14 simplifies

(*)The angular frequency/velocity cannot change by a significant value before stability is lost. Thus for 60 Hz, $\omega_r = 377$ rad/s, and a 1% change in ω_m, i.e., 3.77 rad/s, will change the angle δ by 3.77 rad. Certainly, this would lead to a loss of synchronism.

to the swing equation in the form:

$$\frac{2H}{\omega_r}\frac{d^2\delta}{dt^2} = P_{an}. \tag{8.15}$$

Note that in equation 8.15 both angle δ and the angular velocity ω_r can be measured in electrical rad/s as well as in mechanical rad/s. We will treat this equation as written for electrical angle δ and electrical angular frequency.

The inertia constant H is somewhat similar to a per-unit quantity even though it is not a pure number. Since the quantities in the ratio, which express H (equation 8.13) do not have the same units, namely the kinetic energy is measured in MJ and the rated power in MVA, the unit of H is seconds. The value of H is usually in the range of 1–5 s (the smaller numbers are for small generators). The quantities of H given for a single generator may be modified for use in studies of a system with many generators by converting from the rating power S_r to the system base power (as in our previous study):

$$H_{sys} = H\frac{S_r}{S_b}. \tag{8.16}$$

The physical meaning of the inertia constant is that its value in seconds gives the time needed to accelerate the synchronous machine from zero speed to its rated value when the rated input power is applied.

As an example of using the swing equation, let us calculate the natural oscillations of a synchronous machine being subject to a small disturbance about the equilibrium point, like point a in Fig. 8.3. Assume that a small change in speed is given to the machine, i.e., $\omega = \omega_0 + \Delta\omega_0 u(t)$, where $\Delta\omega_0$ is the small change in speed and $u(t)$ is a unit step function. As a result of the change in speed, there will be a change in angle δ, i.e., $\delta = \delta_0 + \delta_\Delta$ and, in accordance with the power-angle curve, the electric power will be $P_e = P_{e0} + P_{e\Delta}$, while the mechanical power P_m remains constant and equal to P_{e0}. Then, the accelerating power $P_a = P_m - P_{e0} - P_{e\Delta} = -P_{e\Delta}$ and the *swing equation for the small changes becomes*

$$\frac{2H}{\omega_r}\frac{d^2\delta_\Delta}{dt^2} = -P_{e\Delta} = -P_s\delta_\Delta, \tag{8.17a}$$

or

$$\frac{2H}{\omega_r}\frac{d^2\delta_\Delta}{dt^2} + P_s\delta_\Delta = 0, \tag{8.17b}$$

where P_s is the synchronizing power, and as has been shown is

$$P_s = \left.\frac{\partial P}{\partial \delta}\right|_{\delta_0} = P_m \cos\delta_0.$$

The swing equation here is a second-order differential equation (when the damping is neglected) and the characteristic equation of which is

$$\frac{2H}{\omega_r} s^2 + P_s = 0. \tag{8.18}$$

The two roots of this equation are

$$s_{1,2} = \pm\sqrt{-P_s\omega_r/2H} = \pm j\omega_n, \tag{8.19}$$

where $\omega_n = \sqrt{P_s\omega_r/2H}$ is the *natural frequency* of the synchronous machine oscillations. Since the roots are imaginary numbers, the solution is pure sinusoid, i.e. an undamped oscillation:

$$\delta_\Delta(t) = A\sin(\omega_n t + \alpha).$$

To find the two unknown constants of integration, A and α, we shall determine two initial conditions which, obviously, are

$$\delta_\Delta(0) = 0, \quad \left.\frac{d\delta_\Delta}{dt}\right|_{t=0} = \omega_\Delta(0). \tag{8.20}$$

Thus,

$$A\sin(\omega_n + \alpha)|_{t=0} = 0 \quad \text{or} \quad \alpha = 0$$

and

$$\omega_n A\cos\omega_n t|_{t=0} = \omega_\Delta \quad \text{or} \quad A = \frac{\omega_\Delta}{\omega_n}.$$

Finally, we have

$$\delta_\Delta(t) = \frac{\omega_\Delta}{\omega_n}\sin\omega_n t. \tag{8.21}$$

Since the damping conditions are always present, these oscillations will decay (as shown in Fig. 8.6, curve 2) and the synchronous machine will return to point *a* of operation. The stability is held.

Example 8.2

A synchronous generator of reactance 1.25 pu is connected to an infinite bus bar system of $V = 1$ pu through a line and transformers of a total reactance of 0.5 pu. The generator's inertia constant is $H = 5$ s and EMF is 2.5 pu, and it operates at a load angle of 47°. Find the expression of the oscillations set up when the generator is subject to a sudden change of $+0.5\%$ of its speed.

Solution

The transmitted power of the generator is

$$P_e = \frac{EV}{X}\sin\delta = \frac{2.5\cdot 1}{1.25 + 0.5}\sin 47^\circ = 1.04 \text{ pu},$$

and the synchronized power is

$$P_s = \frac{2.5 \cdot 1}{1.25 + 0.5} \cos 47^\circ = 0.974 \text{ pu.}$$

Therefore, the angular frequency of oscillation is

$$\omega_n = \sqrt{0.974 \cdot (\pi \cdot 60)/5} = 6.06 \text{ rad/s,}$$

or

$$f_n = 6.06/2\pi = 0.96 \text{ Hz.}$$

The amplitude of oscillation will be

$$A = \frac{\omega_\Delta}{\omega_n} = \frac{0.005 \cdot 377}{6.06} = 0.31 \text{ rad.}$$

Thus

$$\delta_\Delta(t) = 0.31 \sin 6.06t \text{ rad.}$$

8.4 TRANSIENT STABILITY

Transient stability is concerned with the effect of large disturbances, which are usually due to faults, the most severe of which is a three-phase short-circuit (since by this kind of short-circuit the three-phase voltage may drop down to zero) and the most frequent is the single-line-to-earth fault. Some other kinds of such disturbances are line switching, sudden load changes, etc.

These kinds of disturbances are of a critical nature since they entail the sudden change of electrical output while the mechanical input from the turbine does not have time to change, during the relatively short period of fault, and remains practically constant. As a result, the rotor of the machine endeavors to gain speed and to store the excess energy. If the fault persists long enough the rotor angle will increase continuously and synchronism will be lost. Hence the time of operation of the protection devices and circuit breakers is of great importance.

The stability of the system may also be achieved using **autoreclosing circuit breakers.** The circuit breakers open when the fault is detected and automatically reclose after a prescribed period (usually less than 1 s). Due to the transitory nature of most faults (especially in the case of the single-line-to-earth fault) the circuit breaker often successfully recloses and the stability is held. However, if the fault persists, sometimes an autoreclosing is repeated. The length of the autoreclosing operation must be considered when the transient stability limits are calculated.

The transient stability of a power system is a function of the type and location of the disturbance to which the system is subjected. For instance, if two sections of a system are connected by a pair of lines, one of which is switched out, the power-angle characteristic is changed, having a lower power peak. The balance between the mechanical and electrical powers is disturbed, which causes transient stability problems. A more severe test of system stability is a short-circuit fault on one line followed by its being switched out.

One of the purposes of the analysis of the system transient stability is to determine a stability limit, usually in terms of a critical fault clearance and/or autoreclosing time t_{cr}. If, however, t_{cr} is given, being the minimal available time (as the fastest relay protection and circuit breakers have been anyway used) the system stability test in this case is an estimation of the maximum load which the system can carry without losing transient stability. In our next study using the following examples we shall illustrate how to check the transient stability of a system by solving the swing equation.

Therewith, some further assumptions and simplifications will be made. The cylindrical rotor machine is assumed and, therefore, the direct and quadrature axis reactances are assumed equal. Direct-axis transient reactance X'_d and transient EMF E'_d will be used for the machine representation. The input power for the first seconds following a disturbance remains constant. (This assumption is often made, as previously mentioned, considering that the mechanical system of governors, steam or hydraulic valves, and the like, is relatively sluggish with respect to the fast-changing electrical quantities. However, with improvements in mechanical equipment, such as fast valving, electronic regulators etc., the assumption of a constant input will not be valid and its appropriate change should be taken into consideration.)

Example 8.3

Assume that, at the sending-end of one of the transmission lines in the system shown in Fig. 8.7, a three-phase fault occurs. Develop and solve the swing equation of the system, if the fault reactance is 0.07 pu. The inertia constant of the generator is $H = 0.5$ s and the frequency $f = 60$ Hz. Initially the generator delivers a 0.8 pu power with a transient EMF of 1.22 pu.

Solution

The power-angle characteristic prior to the fault is

$$P_e = \frac{1.22 \cdot 1}{0.6} \sin\delta = 2.03 \sin\delta,$$

where $E'_d = 1.22$ and $X_{tot} = 0.6$ (see Fig. 8.7(b)).

Performing $Y \to \Delta$ transformation in the given circuit we obtain

$$Y_{12} = \frac{Y_1 Y_2}{Y_1 + Y_2 + Y_3} = \frac{(1/0.3)(1/0.3)}{1/0.3 + 1/0.3 + 1/0.07} = 0.528.$$

The electrical power output of the sending-end at the fault is

$$P_e = 1.22 \cdot 1 \cdot 0.528 \sin\delta = 0.644 \sin\delta.$$

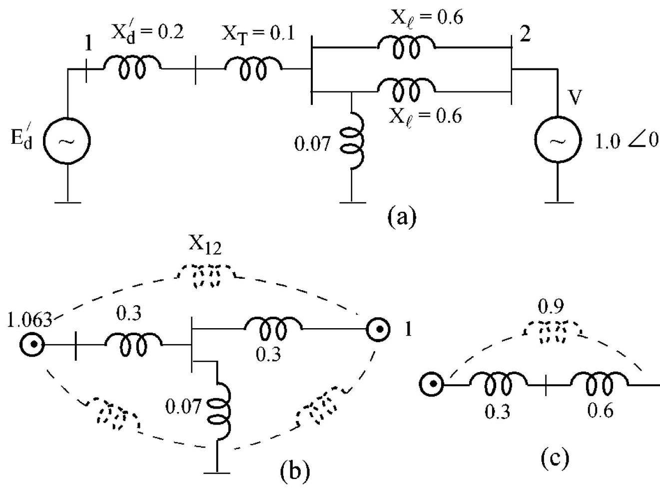


Figure 8.7 An equivalent circuit of the faulted network for Example 8.3 (a) and its simplifications (b) and(c).

Therefore, with equation 8.15 the swing equation is

$$\frac{d^2\delta}{dt^2} = \frac{\omega_r}{2H}(P_m - P_e),$$

or

$$\frac{d^2\delta}{dt^2} = \frac{377}{10}(0.8 - 0.644 \sin \delta) = 37.7(0.8 - 0.644 \sin \delta).$$

The solution of this nonlinear equation is obtained by the MATCAD program, which for the initial conditions of $\delta_0 = \sin^{-1}(0.8/2.03) = 23.2°$ (0.405 rad) and $\omega(0) = \omega_0 - \omega_r = 0$ is shown in Fig. 8.8. As can be seen, the power angle increases indefinitely and the system is unstable.

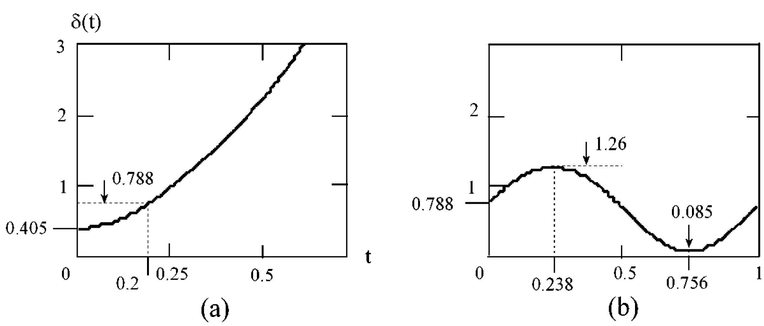


Figure 8.8 Angle-time curve for the faulted network of Example 8.3: at the first moment of the fault (a) and after clearing the fault (b).

Assume now that the fault is cleared in 0.2 s by opening the faulted line.

The equivalent circuit is now shown in Fig. 8.7(c), in which the total reactance is

$$X_{12} = 0.3 + 0.6 = 0.9\,\text{pu},$$

and the transient power will be

$$P_e = \frac{1.22}{0.9}\sin\delta = 1.36\sin\delta.$$

Thus, the swing equation is

$$\frac{d^2\delta}{dt^2} = 37.7(0.8 - 1.36\sin\delta).$$

The initial values of δ and ω are calculated with the previous solution for time 0.2 s, which gives $\delta_{0.2} = 0.788$ rad and $\omega_{0.2} = 3.56$, and the time solution is now shown in Fig. 8.9. (Note that for the new solution $\delta_{0.2} \equiv \delta_0$ and $\omega_{0.2} \equiv \omega_0$.) As can be seen the time change of the power angle is oscillatory and the first peak of about $\delta_{max} = 1.26$ rad is reached at $t = 0.238$ s after which δ is decreased until

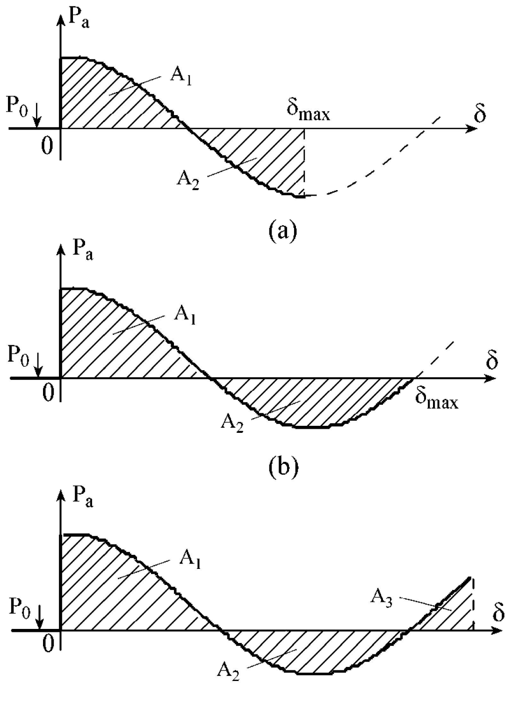


Figure 8.9 Equal-area criteria for a stable system (a), critical case (b) and for an unstable system (c).

it reaches a minimum value of about $\delta_{\min} = 0.085\,\mathsf{rad}$ at $t = 0.756\,\mathsf{s}$ and the oscillations of the rotor angle continue until they decay due to the damping effect. For the system under study and for the given fault the synchronism is not lost and the machine is stable.

8.4.1 Equal-area criterion

Consider once again the swing equation derived previously in the form

$$\frac{d^2\delta}{dt^2} = \frac{\omega_r}{2H} P_a. \tag{8.22}$$

Multiplying both sides by $2(d\delta/dt)$, we have

$$\left(2\frac{d\delta}{dt}\right)\frac{d^2\delta}{dt^2} = \frac{\omega_r}{2H} P_a \left(2\frac{d\delta}{dt}\right),$$

or

$$d\left[\left(\frac{d\delta}{dt}\right)^2\right] = \frac{\omega_r}{H} P_a d\delta.$$

Integrating both sides gives

$$\left(\frac{d\delta}{dt}\right)^2 = \frac{\omega_r}{H}\int_{\delta_0}^{\delta} P_a d\delta. \tag{8.23}$$

This equation determines the quadrature of the relative speed of the machine (with respect to a reference frame moving at a constant speed, for instance, like the infinite bus) as proportional to the integral of P_a versus δ. For a rotor that is accelerating, the condition of stability is that this speed becomes zero, or negative, causing the motor to slow down. In other words, the increasing of angle δ is restricted and after reaching some maximal value, $\delta_{\max}$, the angle decreases. Thus, we may conclude that $\delta_{\max}$ exists and it is given by the condition

$$\int_{\delta_0}^{\delta} P_a d\delta \leq 0. \tag{8.24}$$

In the opposite case $d\delta/dt$ does not become zero, the rotor will continue to move and synchronism is lost (the angle increases unlimitedly).

The integral of $P_a d\delta$ in equation 8.24 represents an area on the $P - \delta$ diagram. Hence, the criterion for stability is that the area between the $P - \delta$ curve and the line of the power input P_m (or P_0) must be zero. The difference between the $P - \delta$ curve and the input power, i.e., the accelerating power, might be also represented as a curve, $P_a(\delta)$, as shown in Fig. 8.9. Then the area under this curve must be zero, which again means that the positive and negative areas are equal. This is known as the **equal-area criterion.** Physically, this criterion means that the rotor must be able to return to the system all the energy gained from the turbine during the acceleration period. This is shown in Fig. 8.9. In figure (a) the positive area A_1 is equal to the negative area A_2 at the angle δ_{max}, at which the accelerating power is negative and the rotor slows down. Therefore, the system is stable and δ_{max} is the maximum rotor angle reached during the swing. In figure (b) the positive and negative areas are equal at the point where P_a reverses its sign, which means that it is a critical case: the oscillations will continue. However, due to the damping effect they will decay. The system is stable and angle δ_{max} is again the maximum rotor angle reached during the first swing.

If the accelerating power reverses it sign before the two areas A_1 and A_2 are equal, as in figure (c), angle δ continues to increase and synchronism is lost. The equal-area criterion is usually applied to the power-angle curve, where the electrical and mechanical powers are plotted as a function of δ. Note that the accelerating power curve could have discontinuities due to the switching of the network, faults occurring and the like.

A simple example of the equal-area criterion may be introduced by an examination of the system stability if one of the two parallel lines, which connect the generator to an infinite bus bar, is switched out (disconnected). The two power-angle curves pertaining to a normal (curve 1) and one line (curve 2) operation of the system are shown in Fig. 8.10.

The shaded area A_1 is proportional to the kinetic energy stored in the rotor, when the input power P_0 is larger than the electrical power delivered by the generator in accordance with curve 2, and in this case the rotor accelerates, Fig. 8.10(a). The shaded area A_2 represents the amount of energy, which the rotor returns to the system. Since these two areas are equal the rotor initially comes to rest at angle δ_{max} (point *c*) whereupon its speed is again synchronous. Having returned all of its extra kinetic energy back to the electrical circuit, the rotor continues to decelerate ($P_e > P_m$) falling through point *b* and back towards point *a*. Such oscillations will continue until completely damped at the new angle δ_1 ($\delta_1 > \delta_0$, point *b*). However, if the initial operating power P_0' and angle δ_0' are increased to such values that the area between δ_0' and δ_1' (A_1) is just

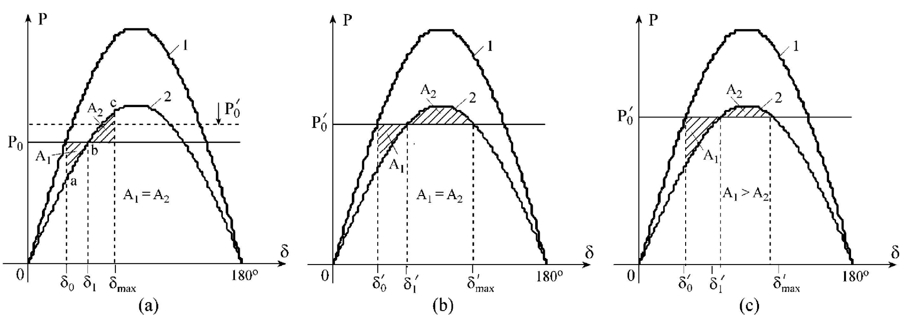


Figure 8.10 Power-angle curves for two lines in parallel (curve 1) and one line (curve 2) and equal area criterion: stable operation (a), critical operation (b) and nonstable operation (c).

equal to the available area between δ_1' and $\delta_{\max}'$, where $\delta_{\max} = 180^\circ - \delta_1'$, it will be the critical operation, Fig. 8.10(b). This would be the condition for maximum input ($P_{0,\max}$) power. If the input power is larger than $P_{0,\max}$, then the accelerating energy (A_1) will be bigger than the available decelerating energy ($A_1 > A_{2,avail}$). The excess kinetic energy will cause δ to continue increasing beyond $\delta_{\max}'$ and the energy would again be absorbed by the rotor (since P_e is now decreasing with an increase in δ, i.e., the slope is negative) and stability will be lost, Fig. 8.10(c). The coefficient

$$k = \frac{A_{2,avail} - A_1}{A_1}$$

is sometimes defined as the **transient stability security factor**. Notice that (as can be seen from Fig. 8.10) it is permissible for the rotor to oscillate past the point where $\delta = 90^\circ$, as long as the equal-area criterion is met.

As another example of using the equal-area criterion, let us consider the fault on one of two parallel lines as in Example 8.3. The power-angle curves pertaining to a fault on one of two parallel lines are shown in Fig. 8.11. The fault is cleared in a time corresponding to δ_1, and the shaded area δ_0 to δ_1, between the P_0-line and power-angle curve for the fault, A_1, indicates the energy stored. The rotor swings until it reaches δ_2 so that $A_1 = A_2$, where A_2 is the shaded area δ_1 to δ_2 between the P_0-line and the power-angle curve for one line after the faulted line has been switched out. Since δ_2 is less than the critical condition angle $\delta_{2\max}$ the system is stable. Critical conditions are reached when

$$\delta_2 = 180^\circ - \sin^{-1}(P_0/P_2). \tag{8.25}$$

The time corresponding to the critical clearing angle is called the **critical clearing time** for a particular (normally full-load) value of power input. This time is of great importance for system protection and to switchgear designers,

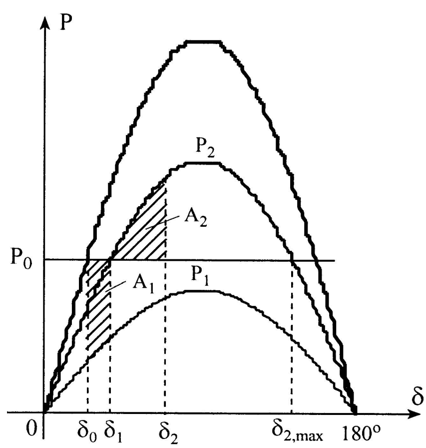


Figure 8.11 Equal-area criterion for the fault of one of two lines in parallel.

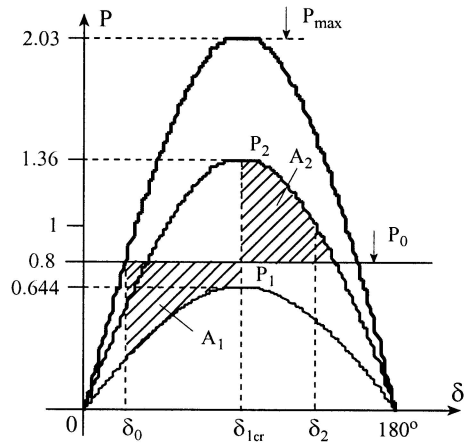


Figure 8.12 Application of the equal area criterion to a critically cleared system.

as it is the maximum time allowable for the equipment to operate without losing stability. The critical clearing angle for a fault on one of two parallel lines, for instance, may be determined as follows. With the equal-area criterion, as shown in Fig. 8.12, we have

$$\int_{\delta_0}^{\delta_{1cr}} (P_0 - P_1 \sin\delta)d\delta = -\int_{\delta_{1cr}}^{\delta_2} (P_0 - P_2 \sin\delta)d\delta,$$

and after integration

$$P_0(\delta_{1cr} - \delta_0) + P_1(\cos\delta_{1cr} - \cos\delta_0) = -P_0(\delta_2 - \delta_{1cr}) - P_2(\cos\delta_2 - \cos\delta_{1cr}),$$

from which the critical clearing angle is

$$\cos\delta_{1cr} = \frac{P_0(\delta_0 - \delta_2) + P_1\cos\delta_0 - P_2\cos\delta_2}{P_1 - P_2}. \tag{8.26}$$

where, with equation 8.25, $\delta_2 = 180^\circ - \sin^{-1}(P_0/P_2)$. Knowing a critical angle and swing frequency, the critical clearing time can be readily obtained.

Example 8.4

Apply the equal-area criterion to the system of Example 8.3.

Solution

We may calculate the critical clearing angle as follows. For this system we have $\delta_0 = 23.2^\circ$, $P_0 = 0.8$ pu, $P_1 = 0.644$ pu, $P_2 = 1.36$ pu and $\delta_2 = 180^\circ - \sin^{-1}(0.8/1.36) = 144.0^\circ$.

Calculation using equation 8.26 gives (note that electrical degrees must be expressed in radians)

$$\cos\delta_{1cr} = \frac{0.8\cdot(0.404 - 2.51) + 0.644\cos 23.2^\circ - 1.36\cos 144.0^\circ}{0.644 - 1.36} = -0.007,$$

or

$$\delta_{1cr} = \cos^{-1}(-0.007) = 90.4^\circ.$$

This situation is illustrated in Fig. 8.12.

8.5 REDUCTION TO A SIMPLE SYSTEM

When a number of generators are connected to the same bus bar, they can be represented by a single equivalent machine. E_{eq} and X_{eq} may be found as explained in section 6.2.2. The inertia constant H of the equivalent machine can then be evaluated by equating the stored energy of the equivalent machine to the total of the individual machines, which yields

$$H_{eq} = H_1\frac{S_1}{S_b} + H_2\frac{S_2}{S_b} + \cdots + H_n\frac{S_n}{S_b}, \tag{8.27}$$

where $S_1\cdots S_n$ are MVA powers of the generators and S_b is the base power. So, consider, for example, a power station, which consists of three generators of 60 MVA, 100 MVA and 300 MVA, having an H of 5 s, 6 s and 8 s respectively. Making the base power equal to 100 MVA, the inertia constant of an equivalent machine will be

$$H_{eq} = 5\frac{60}{100} + 6\frac{100}{100} + 8\frac{300}{100} = 33\ \mathrm{s}.$$

Consider two machines, having M_1 and M_2,which are connected through transformers' and lines' impedances/reactances. The equations of motion for small changes are

$$\begin{aligned} M_1\frac{d^2\Delta\delta_1}{dt^2} + \left(\frac{\partial P_1}{\partial\delta_{12}}\right)_0 \Delta\delta_{12} &= 0 \\ M_2\frac{d^2\Delta\delta_2}{dt^2} + \left(\frac{\partial P_2}{\partial\delta_{12}}\right)_0 \Delta\delta_{12} &= 0, \end{aligned} \tag{8.28}$$

where $\Delta\delta_{12} = \Delta\delta_1 - \Delta\delta_2$.

By subtracting these two equations we may obtain a single equation of the relative motion

$$\frac{d^2\Delta\delta_{12}}{dt} + \left[\frac{(\partial P_1/\partial\delta_{12})_0}{M_1} - \frac{(\partial P_2/\partial\delta_{12})_0}{M_2}\right]\Delta\delta_{12} = 0, \tag{8.29}$$

for which the characteristic equation has two roots

$$s_{1,2} = \pm j\sqrt{\frac{(\partial P_1/\partial \delta_{12})_0}{M_1} - \frac{(\partial P_2/\partial \delta_{12})_0}{M_2}}. \tag{8.30}$$

As was previously shown, if the quantity under the square root is positive, the stability of both generators is assured.

Consider now, two synchronous generators (or group of generators) connected by a reactance. In this case they may be reduced to one equivalent machine connected through the reactance to an infinite bus bar system. The transient equations of motion for the generators are

$$\frac{d^2\delta_1}{dt^2} = \frac{\Delta P_1}{M_1} \quad \text{and} \quad \frac{d^2\delta_2}{dt^2} = \frac{\Delta P_2}{M_2}. \tag{8.31}$$

Then the equation of motion for the two-machine system is

$$\frac{d^2\delta}{dt^2} = \frac{\Delta P_1}{M_1} - \frac{\Delta P_2}{M_2} = \left(\frac{1}{M_1} - \frac{1}{M_2}\right)(P_0 - P_{e,\max} \sin \delta), \tag{8.32}$$

where $\delta = \delta_1 - \delta_2$ is the relative angle between the machines and $d\delta/dt$ is the relative velocity of the two groups with respect to each other. Also note that $\Delta P_1 = -\Delta P_2 = P_0 - P_{e,\max} \sin \delta$, where P_0 is the input power and $P_{e,\max}$ is the maximum transmittable power. For a single generator of M_{eq} and the same input power connected to the infinite bus bar system we have

$$M_{eq}\frac{d^2\delta}{dt^2} = P_0 - P_{e,\max} \sin \delta. \tag{8.33}$$

Therefore, we may conclude that

$$M_{eq} = \frac{M_1 M_2}{M_1 + M_2}, \tag{8.34}$$

and that this equivalent generator has the same mechanical input as the actual machines and that the load angle δ in equation 8.33 is the angle between the rotors of the two machines.

The most useful method of network reduction is by nodal elimination, in which the network is finally represented by only the transfer reactances between the reduced nodes, as any shunt impedances at these nodes do not influence the power transferred.

The electrical network for the transient stability analysis will then obtain n generator buses, to which the voltages (i.e. the internal generator transient EMF's) behind their transient reactances are applied. The values of EMF's are determined, as in the one-machine system, from the pre-transient conditions.

Loads are represented by passive admittances i.e., $G_L = P_L/V_r^2$ and $B_L = Q_L/V_r^2$, which are connected at the load nodes. Note that such a representation is very simplified. Since a network fault usually causes a reduction in the voltages near the fault location, this will result in a decrease in the load power proportional to V^2. In the real system, however, the decrease in power is likely to be less than this, but to occur in a more complicated manner. Since the load usually contains a large proportion of non-static elements, such as induction motors, the nature of the load characteristics is such that beyond the critical point the motors will run down to a standstill and stall. For a more precise analysis of system stability, therefore, it is important to consider the actual load characteristic (see the next section), which makes such an analysis much more complicated.

Passive impedances are connected between various nodes of the network (representing transformers, lines, etc.), and the reference nodes of the active elements and loads are connected in a common reference bus.

Now, let us say that in the given network there are n generator, or active element, nodes and r (remaining) nodes with passive elements. Then the network admittance matrix $\mathbf{Y}$ may be partitioned as

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_{nn} \vdots \mathbf{Y}_{nr} \\ \cdots\cdots\cdots\cdots \\ \mathbf{Y}_{rn} \vdots \mathbf{Y}_{rr} \end{bmatrix}. \tag{8.35}$$

It can be shown that the matrix for the reduced network, which has only the active nodes, is

$$\mathbf{Y}_{red} = \mathbf{Y}_{nn} - \mathbf{Y}_{nr}\mathbf{Y}_{rr}^{-1}\mathbf{Y}_{rn}, \tag{8.36}$$

which is of dimension $(n \times n)$, where n is the number of generators.

The maximum powers transferable between the relevant generators, before and during a fault, can now be calculated from this reduced configuration of a network. In accordance with equation 8.2a (note that here $E \equiv E_i$, $V \equiv E_j$ and $(1/z)\cos\varphi = G$) we may write for the power of the i-ts generator

$$P_{ei} = E_i^2 G_{ii} + \sum_{\substack{j=1 \\ j \neq i}} E_i E_j Y_{ij} \cos(\delta_{ij} - \varphi_{ij}), \quad i = 1, 2, \ldots, n, \tag{8.37}$$

where $\mathbf{Y}_{ii} = Y_{ii} \angle -\varphi_{ii} = G_{ii} - jB_{ii}$ is the admittance for node i and $\mathbf{Y}_{ij} = Y_{ij} \angle -\varphi_{ij}$ is the negative of the transfer admittance between nodes i and j. The equations of motion are then given as

$$\begin{aligned} \frac{2H_i}{\omega_r}\frac{d\omega_i}{dt} + D_i\omega_i &= P_{mi} - \left[E_i^2 G_{ii} + \sum_{\substack{j=1 \\ j \neq i}} E_i E_j Y_{ij} \cos(\delta_{ij} - \varphi_{ij}) \right] \\ \frac{d\delta_i}{dt} &= \omega_i - \omega_r \qquad\qquad i = 1, 2, \ldots, n. \end{aligned} \tag{8.38}$$

The damping coefficient term $D\omega$, which represents the turbine damping, generator electrical damping and the damping effect of electrical loads, is frequently added in the swing equation. (Values of the damping coefficient used in the stability studies are in the range of 1–3 pu.)

8.6 STABILITY OF LOADS AND VOLTAGE COLLAPSE

If the load is purely static, i.e. represented by an impedance, the system will operate stably even at low voltages. However, in reality the load contains non-static elements such as induction motors. The nature of the load Q-V characteristics, when they include a large proportion of induction motors, is non-linear. Both characteristics P-V and Q-V for such a load are shown in Fig. 8.13. While the active power characteristic is almost always a straight line, the *reactive power Q-V characteristic* is a curve having a minimum or critical point and two branches with positive and negative slopes.

Beyond the critical point c even a very small decrease in voltage causes an increase in the reactive power Q, which in turn results in a decrease in the voltage (since the voltage drop depends on the power $\Delta V_{rct} = QX/V$) and so forth. This process of voltage collapse is mathematically defined as $dQ/dV \to \infty$, i.e., on the left branch of the Q-V curve. The physical explanation of the voltage collapse may be found in the behavior of induction motors. Beyond some critical voltage the motors will run down to a standstill or stall. In this situation induction motors consume pure reactive power, which at low voltage causes very large currents similar to short-circuit currents. Finally, this results in very low voltage.

In the power system the problem arises due to the relatively high impedance of the connection between the load and the feeding bus, which can be considered as an infinite bus bar. This happens when one line of two or more forming the load connection is suddenly lost. Consider a simple network, shown in Fig. 8.14, where the non-static load is supplied through a reactance from a constant voltage source E. In this circuit

$$E = \sqrt{\left(V + \frac{QX}{V}\right)^2 + \left(\frac{PX}{V}\right)^2}, \tag{8.39}$$

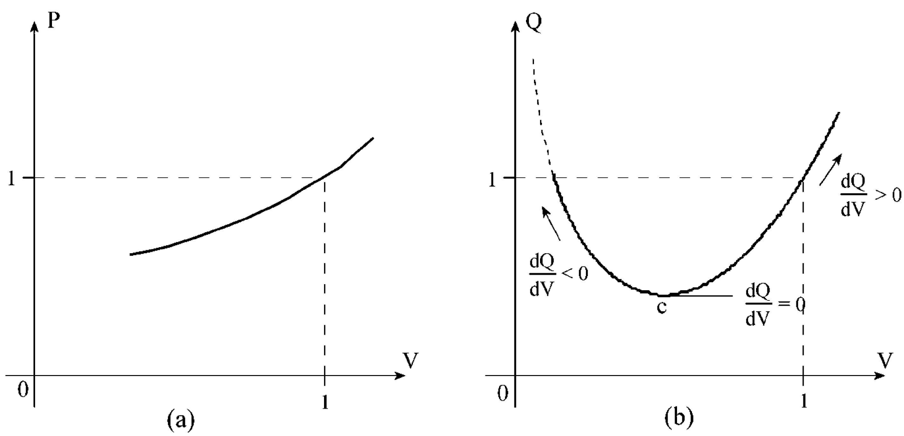


Figure 8.13 Non-static power characteristics: active power-voltage characteristic (a) and reactive power-voltage characteristic (b).

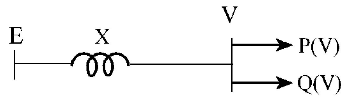


Figure 8.14 A network with a load dependent on voltage.

or if

$$\frac{PX}{V} \ll \frac{V^2 + QX}{V}$$

(which usually takes place)

$$E = V + \frac{QX}{V}. \tag{8.40}$$

From the system viewpoint, it is worthwhile to develop a voltage stability criterion with a dependency on E versus V, i.e., by using the *E-V curve*. This curve (equation 8.40) is plotted in Fig. 8.15, where the two components V and QX/V are also shown. Performing the differentiation of equation 8.40 with respect to V yields

$$\frac{dE}{dV} = 1 + \left(\frac{dQ}{dV} XV - QX\right)\frac{1}{V^2} = 1 + \left(\frac{dQ}{dV} - \frac{Q}{V}\right)\frac{X}{V}. \tag{8.41}$$

Here, when $dQ/dV \to -\infty$, the term in the parentheses and, therefore, the entire expression (8.41) approaches a negative infinity. Thus, we may conclude that if $dE/dV \to -\infty$, then the system is unstable and the *voltage collapse takes place.* When dQ/dV is positive, the term in the parentheses is also positive ($dQ/dV > Q/V$) and therefore $dE/dV > 0$, i.e. the system is stable, *which means*

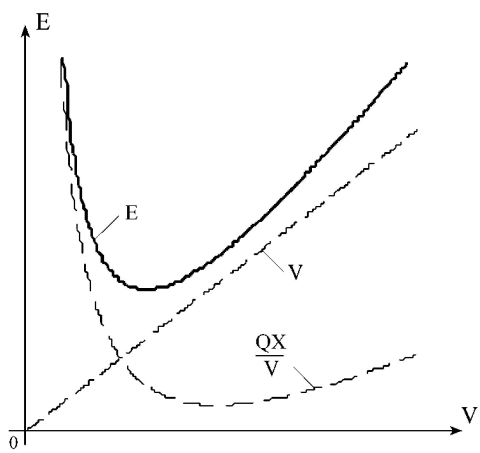


Figure 8.15 *E-V* curve and its two components.

that the sign of the derivative dE/dV provides the criterion of load stability. At the critical point on the E-V curve, i.e., where $dE/dV = 0$, we have

$$\frac{dQ}{dV} = \frac{Q}{V} - \frac{V}{X}, \tag{8.42}$$

or, since $QX/V = E - V$ we have

$$\frac{Q}{V} = \frac{E}{X} - \frac{V}{X}$$

and

$$\frac{dQ}{dV} = \frac{E}{X} - \frac{2V}{X}. \tag{8.43}$$

This means that at $dE/dV = 0$, the sign of dQ/dV is defined by the sign of $(E - 2V)$ and if it is negative, the system is unstable.

Example 8.5

Examine the voltage stability of a non-static load supplied from a 275 kV infinite bus bar through a line of reactance 50 Ω per phase. The load consists of a constant active power of 200 MW and 200 MVAr rating reactive power, which is related to the voltage by the equation (in pu) $Q = 5(V - 0.7)^2 + 0.8$.

Solution

With $S_b = 200$ MVA and $V_b = 275$ kV the pu value of the line reactance is

$$X = \frac{50 \cdot 200}{275^2} = 0.132 \text{ pu}.$$

The load voltage can then be found from the equation

$$E = V + \frac{QX}{V}.$$

Thus,

$$E = V + \frac{0.132}{V}\left[5(V - 0.7)^2 + 0.8\right].$$

Since $E = 1$ after simplification we have

$$V^2 - 1.159V + 0.258 = 0.$$

Thus, the roots are

$$V_{1,2} = 0.858;\ 0.301.$$

Taking the upper value (which is suitable to a physical reality) we obtain

$$Q = 5(0.858 - 0.7)^2 + 0.8 = 0.925,$$

and

$$\frac{dQ}{dV} = 10(V - 0.7) = 1.58.$$

Then,

$$\frac{dE}{dV} = 1 + \left(1.58 - \frac{0.925}{0.858}\right)\frac{0.132}{0.858} = 1.077.$$

Since the result is positive, the system is stable. (Note that $PX/V = 1 \cdot 0.132/0.858 = 0.15$, which is much less than $(V^2 + QX)/V = (0.858^2 + 0.924 \cdot 0.132)/0.858 \cong 1$.)

The reader can now convince himself that, if one of the two parallel lines is lost and the reactance changed to 0.264, dE/dV becomes negative and the system is unstable, i.e., the voltage collapses.

APPENDIX I

SOLVING EXAMPLE 5.6 USING THE MATHCAD PROGRAM[(*)]

Definition of the array first element subscript

$$\mathrm{ORIGIN} \equiv 1$$

Data:

$$C1 := 1 \quad C2 := 2 \quad L4 := 1 \quad G3 := 1 \quad R5 := 1$$

$$R6 := \frac{2}{7} \quad R7 := \frac{1}{3} \quad a := \frac{1}{(1 + R5 \cdot G3)}$$

Matrix **A**

$$\begin{bmatrix} \dfrac{-(1 + a \cdot R6 \cdot G3)}{R6 \cdot C1} & a \cdot \dfrac{G3}{C1} & \dfrac{a}{C1} \\ a \cdot \dfrac{G3}{C2} & \dfrac{-(1 + a \cdot R7 \cdot G3)}{R7 \cdot C2} & \dfrac{(-1 + a)}{C2} \\ \dfrac{a}{L4} & \dfrac{(1 - a)}{L4} & -a \cdot \dfrac{R5}{L4} \end{bmatrix}$$

$$= \begin{pmatrix} -4 & 0.5 & -0.5 \\ 0.25 & -1.75 & -0.25 \\ 0.5 & 0.5 & -0.5 \end{pmatrix}$$

The characteristic equation

$$\left| \begin{pmatrix} \lambda + 4 & -0.5 & 0.5 \\ -0.25 & \lambda + 1.75 & 0.25 \\ -0.5 & -0.5 & \lambda + 0.5 \end{pmatrix} \right| \rightarrow \lambda^3 + 6.25\lambda^2 + 10.125\lambda + 4.500$$

Finding the roots

$$\mathrm{coef} := \begin{pmatrix} 4.5 \\ 10.13 \\ 6.25 \\ 1 \end{pmatrix} \quad \mathrm{polyroots}\,(\mathrm{coef}) = \begin{pmatrix} -3.998 \\ -1.504 \\ -0.748 \end{pmatrix}$$

(*) See page 307.

Unity, initial, **b** and **w** matrixes

$$U := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad X0 := \begin{pmatrix} 0.5 \\ 1.5 \\ 1 \end{pmatrix} \quad b := \begin{pmatrix} 3.5 & 0 \\ 0 & 1.5 \\ 0 & 0 \end{pmatrix} \quad w := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The matrix for finding β coefficients

$$d := \begin{pmatrix} 1 & -0.75 & 0.5625 \\ 1 & -1.5 & 2.25 \\ 1 & -4 & 16 \end{pmatrix}$$

$$d^{-1} \cdot \begin{pmatrix} \exp(-0.75t) \\ \exp(-1.5t) \\ \exp(-4t) \end{pmatrix} \text{ float, } 4$$

$$\rightarrow \begin{pmatrix} 2.462 \exp(-0.75t) - 1.600 \exp(-1.5t) + 0.1385 \exp(-4t) \\ 2.256 \exp(-0.75t) - 2.533 \exp(-1.5t) + 0.2769 \cdot \exp(-4t) \\ 0.4103 \exp(-0.75t) - 0.5333 \exp(-1.5t) + 0.1231 \exp(-4t) \end{pmatrix}$$

The β-coefficients

$$\beta := \begin{pmatrix} 2.462 \exp(-0.75t) - 1.600 \exp(-1.5t) + 0.1385 \exp(-4t) \\ 2.256 \exp(-0.75t) - 2.534 \exp(-1.5t) + 0.2770 \cdot \exp(-4t) \\ 0.4103 \exp(-0.75t) - 0.5333 \exp(-1.5t) + 0.1231 \exp(-4t) \end{pmatrix}$$

Calculating exp(At)

$$A := \begin{pmatrix} -4 & 0.5 & -0.5 \\ 0.25 & -1.75 & -0.25 \\ 0.5 & 0.5 & -0.5 \end{pmatrix} \quad A^2 = \begin{pmatrix} 15.875 & -3.125 & 2.125 \\ -1.563 & 3.063 & 0.438 \\ -2.125 & -0.875 & -0.125 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -0.222 & 0 & 0.222 \\ 0 & -0.5 & 0.25 \\ -0.222 & -0.5 & -1.528 \end{pmatrix}$$

$$\exp(At) = (U \cdot \beta_1 + A \cdot \beta_2 + A^2 \cdot \beta_3)$$

Equation to calculate X_{nat}:

$$(U \cdot \beta_1 + A \cdot \beta_2 + A^2 \cdot \beta_3) \cdot X0$$

which results in

$$X_{nat} := \begin{pmatrix} -0.5101\exp(-0.75t) + 0.7695\exp(-1.5t) + 0.2466\exp(-4t) \\ -0.7.67\exp(-0.75t) + 2.303\exp(-1.5t) - 3.079\times10^{-2}\exp(-4t) \\ 2.565\exp(-0.75t) - 1.534\exp(-1.5t) - 3.074\times10^{-2}\exp(-4t) \end{pmatrix}$$

Equation to calculate X_{part}:

$$A^{-1}\cdot[(U\cdot\beta_1 + A\cdot\beta_2 + A^2\cdot\beta_3 - U)\cdot b\cdot w]$$

which results in

$$X_{part} := \begin{pmatrix} 0.5426\exp(-0.75t) - 0.5596\exp(-1.5t) - 0.7699\exp(-4t) + 0.7777 \\ 0.7500 + 0.8196\exp(-0.75t) - 1.670\exp(-1.5t) + 9.632\times10^{-2}\exp(-4t) \\ -2.738\exp(-0.75t) + 1.112\exp(-1.5t) + 9.59\times10^{-2}\exp(-4t) + 1.528 \end{pmatrix}$$

The total response $X(t) = X_{nat} + X_{part}$:

$$X(t) := \begin{pmatrix} 3.25\times10^{-2}\exp(-0.75t) + 0.2099\exp(-1.5t) - 0.5233\exp(-4t) + 0.7777 \\ 5.29\times10^{-2}\exp(-0.75t) + 0.633\exp(-1.5t) + 6.553\times10^{-2}\exp(-4t) + 0.7500 \\ -0.173\exp(-0.75t) - 0.422\exp(-1.5t) + 6.516\times10^{-2}\exp(-4t) + 1.528 \end{pmatrix}$$

APPENDIX II

THE CALCULATION OF THE p.u.VALUES FOR A GIVEN NETWORK[(*)]

The 1-line diagram of the network of example 6.2 is shown in Fig. AII-1. The parameters of the network elements are as follows.

Generators: G_1- $S_n = 470$ MVA, $V_n = 15.75$ kV, $E'_d = 1.25$, $X'_d = 0.3$,
$G_2 = G_3$- $S_n = 2 \times 118$ MVA, $V_n = 13.8$ kV, $E'_d = 1.33$, $X'_d = 0.38$.

System- $E_s = 1$, $X_s = 83\ \Omega$.

Transformers: T1- $S_n = 250$ MVA, $V_n = 242/13.8$ kV, $X_{s.c.} = 11\%$,
T2- $S_n = 120$ MVA, $V_n = 220/11.0$ kV, $X_{s.c.} = 12\%$.

Autotransformers:
AT1- $S_n = 480$ MVA, $V_n = 242/121/15.75$ kV, $X_{s.c.} = 13.5/12.5/18.8\%$,
AT2- $S_n = 360$ MVA, $V_n = 525/242/13.8$ kV, $X_{s.c.} = 8.4/28.4/19.0\%$.

Transmission lines: $\ell_1 = 120$ km, $\ell_2 = 95$ km, $\ell_3 = 80$ km, $x_0 = 0.4\ \Omega$/km.

Load Ld- $S_n = 250$ MVA, $X_n = 1.2$.

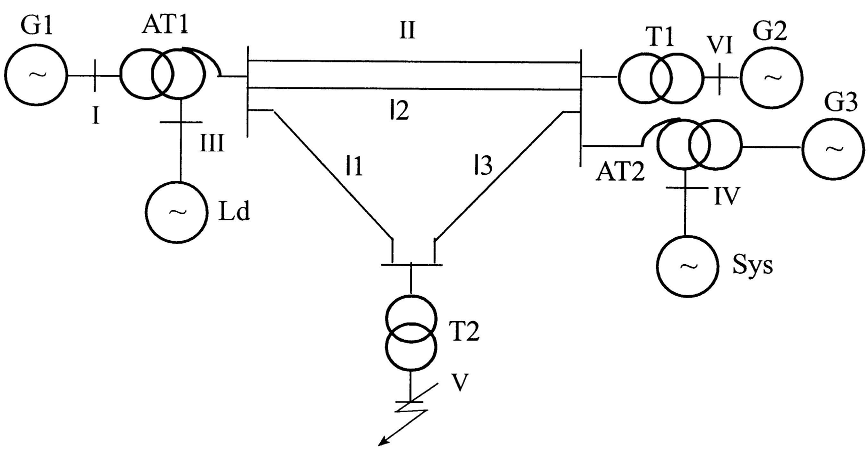


Fig. AII-1.

[(*)]See page 331.

The base values are chosen as:

$$S_b = 1000\ \text{MVA}, \quad V_{bI} = 15.75\ \text{kV (main level)}, \quad V_{bII} = 15.75\,\frac{242}{15.75} = 242\ \text{kV},$$

$$V_{bIII} = 121\ \text{kV}, \quad V_{bIV} = 242\,\frac{525}{242} = 242\ \text{kV}, \quad V_{bV} = 242\,\frac{11}{220} = 12.1\ \text{kV}.$$

The p.u.values of the network elements are (see Fig. 6.8a):

G_1- $\quad X_1 = 0.3\,\frac{1000}{470} = 0.64.$

G_2, G_3- $\quad X_2 = X_3 = 0.38\,\frac{1000}{2\cdot 118} = 1.61.$

Sys- $\quad X_4 = 83\,\frac{1000}{525^2} = 0.3.$

AT1- $\quad X_5 = \frac{1}{2}\cdot\frac{13.5 + 12.5 - 18.8}{100}\cdot\frac{1000}{480} = 0.075.$

$$X_6 = \frac{1}{2}\cdot\frac{13.5 - 12.5 + 18.8}{100}\cdot\frac{1000}{480} = 0.206.$$

$$X_7 = \frac{1}{2}\cdot\frac{-13.5 + 12.5 + 18.8}{100}\cdot\frac{1000}{480} = 0.185.$$

AT2- $\quad X_8 = \frac{1}{2}\cdot\frac{8.4 + 28.4 - 19.0}{100}\cdot\frac{1000}{360} = 0.247.$

$$X_9 = \frac{1}{2}\cdot\frac{8.4 - 28.4 - 19.0}{100}\cdot\frac{1000}{360} \cong 0.$$

$$X_{10} = \frac{1}{2}\cdot\frac{-8.4 + 28.4 + 19.0}{100}\cdot\frac{1000}{360} = 0.542.$$

T1- $\quad X_{11} = \frac{11}{100}\cdot\frac{1000}{250} = 0.44.$

T2- $\quad X_{12} = \frac{12}{100}\cdot\frac{1000}{120}\left(\frac{220}{242}\right)^2 = 0.83.$

$\ell 1$- $\quad X_{13} = 0.4\cdot 120\,\frac{1000}{242^2} = 0.82.$

$\ell 2$- $\quad X_{14} = \frac{1}{2}\,0.4\cdot 95\,\frac{1000}{242^2} = 0.32.$

$\ell 3$- $\quad X_{15} = 0.4\cdot 80\,\frac{1000}{242^2} = 0.54.$

Ld- $\quad X_{16} = 1.2\,\frac{1000}{250}\left(\frac{115}{121}\right)^2 = 4.34.$

APPENDIX III

AN EXAMPLE OF A SHORT-CIRCUIT FAULT CALCULATION IN A POWER NETWORK(*)

Find the short-circuit current at the fault point F using the linearization approach for two cases: a) the AVRs are not activated; b) the AVRs are activated.

The one-line diagram of the network is shown in Fig. AIII-1. The parameters of the network elements are as follows.

Turbo-generators: G_1 and G_2-	$S_n = 15$ MVA, $V_n = 6.3$ kV, $SCR = 0.68$, $I_f = 2.1$, $I_{f\,max} = 4$.
Hydro-generator: G_3-	$S = 60$ MVA, $V_n = 10.5$ kV, $SCR = 1$, $I_f = 1.75$, $I_{f\,max} = 3.1$.
Transformers: T1-	$S_n = 10$ MVA, $V_n = 6/37$ kV, $X_{s.c.} = 7.5\%$,
T2-	$S_n = 40.5$ MVA, $V_n = 121/37.5/10.5$ kV, $X_{s.c.1} = 11\%$, $X_{s.c.2} = 6\%$, $X_{s.c.3} = 0$.

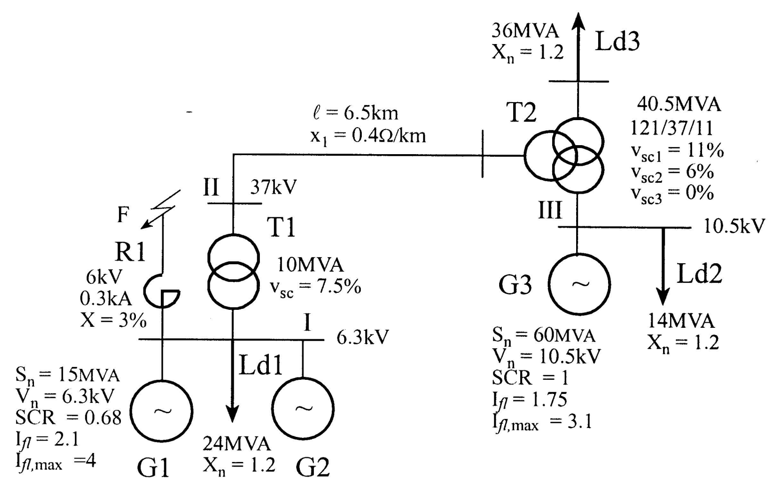


Fig. AIII-1.

(*)See page 370.

Transmission line- $\ell = 6.5\text{km}$, $x_0 = 0.4\ \Omega/\text{km}$.

Loads: Ld1- $S_n = 24$ MVA, $X_n = 1.2$,
Ld2- $S_n = 14$ MVA, $X_n = 1.2$,
Ld3- $S_n = 36$ MVA, $X_n = 1.2$.

Reactor R1- $I_n = 0.3$ kA, $V_n = 6.0$ kV, $X_{x.c.} = 4\%$.

The base values are chosen as:

$$S_b = 100\ \text{MVA}, \quad V_b = V_{level}, \quad I_{bI} = \frac{100}{\sqrt{3}\cdot 6.3} = 9.22\ \text{kA}.$$

a) The p.u. values of the network elements, if the AVRs are not activated, are (see Fig. AIII-2(a)):

G_1, G_2- $\quad E_1 = 0.2 + 0.8\cdot 2.1 = 1.88, \quad X_1 = \dfrac{1.88}{0.68\cdot 2.1}\dfrac{100}{30} = 4.4;$

G_3- $\quad E_2 = 0.2 + 0.8\cdot 1.75 = 1.6, \quad X_2 = \dfrac{1.6}{1\cdot 1.75}\dfrac{100}{60} = 1.52;$

T1- $\quad X_3 = \dfrac{7.5}{100}\dfrac{100}{10} = 0.75;$

T2- $\quad X_4 = \dfrac{11}{100}\cdot\dfrac{100}{40.5} = 0.27, \quad X_5 = \dfrac{6}{100}\cdot\dfrac{100}{40.5} = 0.15, \quad X_6 = 0;$

ℓ- $\quad X_7 = 0.4\cdot 6.5\,\dfrac{1000}{37^2} = 0.19;$

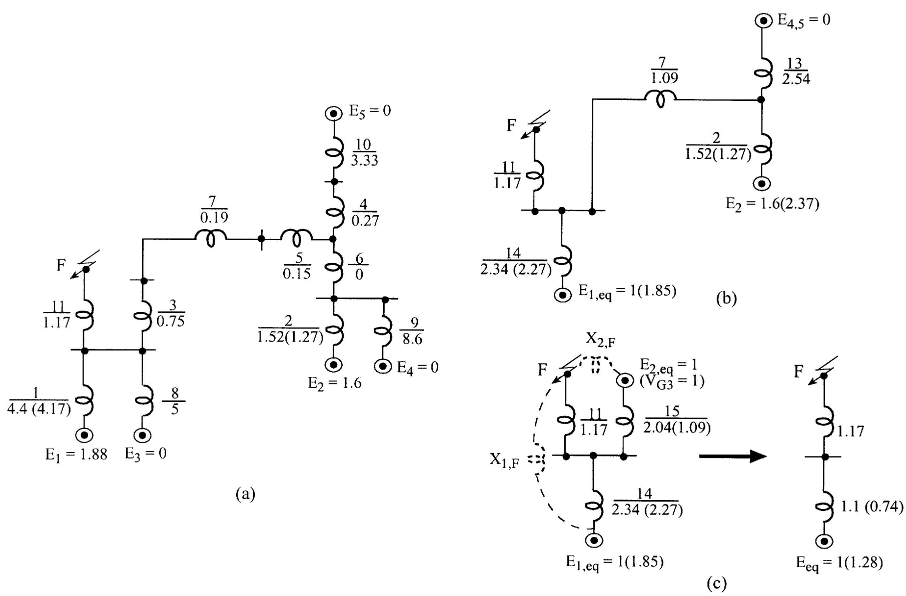


Fig. AIII-2.

Lds- $X_8 = 1.2\,\dfrac{100}{24} = 5, \quad X_9 = 1.2\,\dfrac{100}{14} = 8.6, \quad X_{10} = 1.2\,\dfrac{100}{36} = 3.33;$

R1- $X_{11} = \dfrac{4}{100}\cdot\dfrac{9.2}{0.3}\cdot\dfrac{6}{6.3} = 1.17 \quad \text{and} \quad E_3 = E_4 = E_5 = 0.$

The equivalent circuit of the network with the p.u. values of the network elements is shown in Fig. AIII.2(a). By simplifying this circuit we obtain:

$$X_{12} = 0.75 + 0.19 + 0.15 = 1.09, \quad X_{13} = (3.33 + 0.27)//8.6 = \frac{3.6\cdot 8.6}{3.6 + 8.6} = 2.54$$

and $E_{4,5} = 0$;

$$X_{14} = 5//4.4 = 2.34 \quad \text{and} \quad E_{1,eq} = \frac{1.88\cdot 5}{5 + 4.4} \cong 1 \quad \text{(Fig. AIII.2(b))}.$$

In the next step we have

$$X_{15} = X_{12} + X_2//X_{13} = 1.09 + \frac{1.52\cdot 2.54}{1.52 + 2.54} = 2.04,$$

$$E_{2,eq} = \frac{1.6/1.52}{1/1.52 + 1/2.54} \cong 1 \quad \text{(Fig. AIII.2(c))},$$

and finally

$$X_{eq} = 2.04//2.34 + 1.17 = 2.27 \quad \text{and} \quad E_{eq} = E_{1,eq}//E_{2,eq} = 1.$$

Thus, the short-circuit current is $I_{s.c.} = (1/2.27) = 0.441$, or in amperes $I_{s.c.} = 0.441\cdot 9.2 = 4.06$ kA.

The terminal voltage of G1 will be $V_1 = 1.17\cdot 0.441 = 0.516$ and the currents of each of the generators G1 and G2 will be

$$I_{G1} = \frac{E_1 - V_1}{X_1} = \frac{1.88 - 0.516}{4.4} = 0.31$$

and

$$I_{G2} = \frac{E_2 - V_2}{X_{15}} = \frac{1.6 - 0.775}{2.04} = 0.542,$$

where the voltage of V_2 can be found as

$$V_2 = V_1 + X_{12}I_{21} = 0.516 + 1.09\,\frac{1 - 0.516}{2.04} - 0.775.$$

To find the partial currents of each of the generators, which make up the fault current we shall first transfer the Y-connection of the circuit, Fig. AIII.2(c), into the Δ-connection (see the dash lines). Thus,

$$X_{1,F} = 1.17 + 2.34 + \frac{1.17 \cdot 2.34}{2.04} = 4.85 \quad \text{and} \quad I_{G1,F} = \frac{1}{4.85} = 0.206,$$

$$X_{1,F} = 1.17 + 2.04 + \frac{1.17 \cdot 2.04}{2.34} = 4.23 \quad \text{and} \quad I_{G2,F} = \frac{1}{4.23} = 0.236.$$

We may solve the above problem using the straightforward method. For this purpose we have to write and solve the matrix equation:

$$[Im] = [Zm]^{-1}[Em],$$

where the matrixes are: mesh-current, mesh-impedance and mesh-EMF matrixes.

The circuit in Fig. AIII.2(b), which has been obtained just after the trivial stages of simplification, is redrawn in a slightly different way in Fig. AIII.3(a). The four meshes of this circuit are chosen in such a way (note that the branches 0n–1n and 1n–2n form a tree of the circuit graph) that the mesh currents depict the short-circuit current ($I_3 \equiv I_{sc}$)and the first and the second generator currents ($I_1 \equiv I_{G1}$ and $I_2 \equiv I_{G2}$). The following solution is performed by the MATHCAD program.

$$\text{ORIGIN} := 1$$

The mesh matrixes are

$$\text{Zm} := \begin{pmatrix} 9.4 & 5 & -5 & 5 \\ 5 & 7.61 & -5 & 6.09 \\ -5 & -5 & 6.17 & -5 \\ 5 & 6.09 & -5 & 8.63 \end{pmatrix} \quad \text{Em} := \begin{pmatrix} 1.88 \\ 1.6 \\ 0 \\ 0 \end{pmatrix},$$

and the solution of the above matrix equation is

$$\text{Im} := \text{Zm}^{-1} \cdot \text{E}_\text{m}.$$

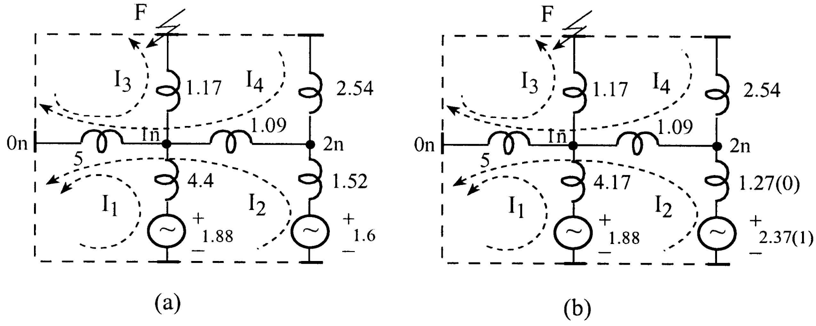


Fig. AIII-3.

Thus, the mesh currents are

$$\mathrm{Im} = \begin{pmatrix} 0.31 \\ 0.542 \\ 0.443 \\ -0.305 \end{pmatrix},$$

where the short-circuit current is $I_{s.c.} = 0.443$, and the generators currents are $I_{G1} = 0.31$ and $I_{G1} = 0.542$, i.e., as previously calculated.

To find the generators' partial currents of the short-circuit current, we shall find the determinant of the impedance matrix and its two cofactors:

$$\left| \begin{pmatrix} 9.4 & 5 & -5 & 5 \\ 5 & 7.61 & -5 & 6.09 \\ -5 & -5 & 6.17 & -5 \\ 5 & 6.09 & -5 & 8.63 \end{pmatrix} \right| = 377.932 \quad \left| \begin{pmatrix} 5 & -5 & 5 \\ 7.61 & -5 & 6.09 \\ 6.09 & -5 & 8.63 \end{pmatrix} \right| = 41.431$$

$$-\left| \begin{pmatrix} 9.4 & -5 & 5 \\ 5 & -5 & 6.09 \\ 5 & -5 & 8.63 \end{pmatrix} \right| = 55.88.$$

Now with equation 6.51 we have

$$I_{G1,F} = B_{F,1} E_1 = 0.109 \cdot 1.88 = 0.206, \quad \text{where } B_{F,1} = \frac{\Delta_{31}}{\Delta} = \frac{41.43}{377.9} = 0.109$$

and

$$I_{G2,F} = B_{F,2} E_2 = 0.148 \cdot 1.6 = 0.237, \quad \text{where } B_{F,2} = \frac{\Delta_{32}}{\Delta} = \frac{55.9}{377.9} = 0.148.$$

b) The p.u. values of the network elements, if the AVRs are activated, are (see the numbers in the parenthesis in Fig. AIII-2(a), (b) and (c)):

$$E_1 = 0.2 + 0.8 \cdot 4 = 3.4, \quad X_1 = \frac{4.17}{0.68 \cdot 4} \frac{100}{30} = 4.17 \quad \text{and} \quad X_{cr,1} = \frac{4.17}{3.4 - 1} = 1.74;$$

$$E_2 = 0.2 + 0.7 \cdot 3.1 = 2.37, \quad X_2 = \frac{2.37}{1 \cdot 3.1} \frac{100}{60} = 1.27 \quad \text{and} \quad X_{cr,2} = \frac{1.27}{2.73 - 1} = 0.93.$$

Thus, the critical currents are:

$$I_{cr,1} = \frac{1}{1.74} = 0.58 \quad \text{and} \quad I_{cr,2} = \frac{1}{0.93} = 1.08.$$

All other parameters are as in the previous calculation. By circuit simplification we assumed that generators G_1 and G_2 are operated in the maximal field regime (since they are relatively “close” to the fault point) and generator G_3 is operated in the nominal voltage regime (since it is relatively “far” from the fault point). In accordance with the circuit in Fig. AIII.2(c) we have

$$E_{eq} = E_{1,eq}//V_{G3} = \frac{1.85\cdot 1.09 + 1\cdot 2.27}{2.27 + 1.09} = 1.28, \quad \text{where } E_{1,eq} = \frac{3.4\cdot 5}{4.17 + 5} = 1.85,$$

and

$$X_{eq} = X_{14}//X_{12} + X_{11} = \frac{2.27\cdot 1.09}{2.27 + 1.09} + 1.17 = 1.91.$$

Thus, the short-circuit current is $I_{s.c.} = (1.28/1.91) = 0.67$, or in amperes $I_{s.c.} = 0.67\cdot 9.2 = 6.2\ \mathrm{kA}$.

The terminal voltage of G1 will be $V_{G1} = 0.67\cdot 1.17 = 0.78$ and the currents of each of the generators G1 and G2 will be

$$I_{G1} = \frac{E_1 - V_1}{X_1} = \frac{3.4 - 0.78}{4.17} = 0.628$$

and

$$I_{G2} = I_{L2} + I_{G2,F} = \frac{1}{2.54} + \frac{1 - 0.78}{1.09} = 0.596.$$

Since $I_{G1} > I_{cr1}$ and $I_{G2} \le I_{cr2}$, the assumption about their regimes was correct.

With the straightforward method, applied to the circuit in Fig. AIII.3(b), we can find the generators’ currents and therefore will know in which of the two regimes the generators are operated. Thus, by applying the MATHCAD program, we have:

The mesh-impedance and mesh-voltage matrixes are

$$\mathrm{Zm1} := \begin{pmatrix} 9.17 & 5 & -5 & 5 \\ 5 & 7.36 & -5 & 6.09 \\ -5 & -5 & 6.17 & -5 \\ 5 & 6.09 & -5 & 8.63 \end{pmatrix} \quad \mathrm{Em1} := \begin{pmatrix} 3.4 \\ 2.37 \\ 0 \\ 0 \end{pmatrix},$$

and the solution is

$$\mathrm{Im1} := \mathrm{Zm1}^{-1}\cdot \mathrm{Em1},$$

which gives

$$\mathrm{Im1} = \begin{pmatrix} 0.599 \\ 0.856 \\ 0.77 \\ -0.505 \end{pmatrix}.$$

Since the current of the first generator, $I_{G1} = 0.599$, is larger than the critical one, it is operated in the maximal field regime. However, since the current of the second generator, $I_{G2} = 0.856$, is smaller than the critical one, it is operated in the nominal (rated) voltage regime and its parameters should be changed to $V_{G2} = 1$ and $X_2 = 0$. (See the numbers in the parentheses in Fig. AIII.3(b).) Now the new mesh matrixes are

$$\text{Zm2} := \begin{pmatrix} 9.17 & 5 & -5 & 5 \\ 5 & 6.09 & -5 & 6.09 \\ -5 & -5 & 6.17 & -5 \\ 5 & 6.09 & -5 & 8.63 \end{pmatrix} \quad \text{Em2} := \begin{pmatrix} 3.4 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

and the solution is

$$\text{Im2} = \begin{pmatrix} 0.627 \\ 0.592 \\ 0.67 \\ -0.394 \end{pmatrix}.$$

Note that the short-circuit current, 0.67, and the currents of both generators, 0.627 and 0.592, are as previously calculated.

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