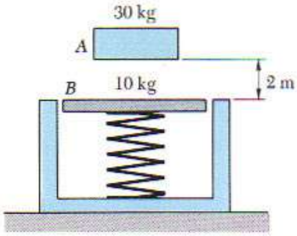
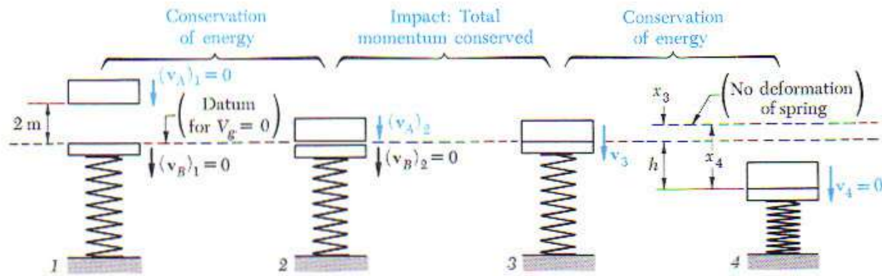


SAMPLE PROBLEM 13.16

A 30-kg block is dropped from a height of 2 m onto the 10-kg pan of a spring scale. Assuming the impact to be perfectly plastic, determine the maximum deflection of the pan. The constant of the spring is $k = 20$ kN/m.

Solution. The impact between the block and the pan *must* be treated separately; therefore we divide the solution into three parts.



Conservation of Energy. Block: $W_A = (30 \text{ kg})(9.81 \text{ m/s}^2) = 294$ N

$$T_1 = \tfrac{1}{2}m_A(v_A)_1^2 = 0 \qquad V_1 = W_A y = (294 \text{ N})(2 \text{ m}) = 588 \text{ J}$$
$$T_2 = \tfrac{1}{2}m_A(v_A)_2^2 = \tfrac{1}{2}(30 \text{ kg})(v_A)_2^2 \qquad V_2 = 0$$
$$T_1 + V_1 = T_2 + V_2: \qquad 0 + 588 \text{ J} = \tfrac{1}{2}(30 \text{ kg})(v_A)_2^2 + 0$$
$$(v_A)_2 = +6.26 \text{ m/s} \qquad (\mathbf{v}_A)_2 = 6.26 \text{ m/s} \downarrow$$

Impact: Conservation of Momentum. Since the impact is perfectly plastic, $e = 0$; the block and pan move together after the impact.

$$m_A(v_A)_2 + m_B(v_B)_2 = (m_A + m_B)v_3$$
$$(30 \text{ kg})(6.26 \text{ m/s}) + 0 = (30 \text{ kg} + 10 \text{ kg})v_3$$
$$v_3 = +4.70 \text{ m/s} \qquad \mathbf{v}_3 = 4.70 \text{ m/s} \downarrow$$

Conservation of Energy. Initially the spring supports the weight W_B of the pan; thus the initial deflection of the spring is

$$x_3 = \frac{W_B}{k} = \frac{(10 \text{ kg})(9.81 \text{ m/s}^2)}{20 \times 10^3 \text{ N/m}} = \frac{98.1 \text{ N}}{20 \times 10^3 \text{ N/m}} = 4.91 \times 10^{-3} \text{ m}$$

Denoting by x_4 the total maximum deflection of the spring, we write

$$T_3 = \tfrac{1}{2}(m_A + m_B)v_3^2 = \tfrac{1}{2}(30 \text{ kg} + 10 \text{ kg})(4.70 \text{ m/s})^2 = 442 \text{ J}$$
$$V_3 = V_g + V_e = 0 + \tfrac{1}{2}kx_3^2 = \tfrac{1}{2}(20 \times 10^3)(4.91 \times 10^{-3})^2 = 0.241 \text{ J}$$
$$T_4 = 0$$
$$V_4 = V_g + V_e = (W_A + W_B)(-h) + \tfrac{1}{2}kx_4^2 = -(392)h + \tfrac{1}{2}(20 \times 10^3)x_4^2$$

Noting that the displacement of the pan is $h = x_4 - x_3$, we write

$$T_3 + V_3 = T_4 + V_4:$$
$$442 + 0.241 = 0 - 392(x_4 - 4.91 \times 10^{-3}) + \tfrac{1}{2}(20 \times 10^3)x_4^2$$
$$x_4 = 0.230 \text{ m} \qquad h = x_4 - x_3 = 0.230 \text{ m} - 4.91 \times 10^{-3} \text{ m}$$
$$h = 0.225 \text{ m} \qquad h = 225 \text{ mm} \blacktriangleleft$$

PROBLEMS

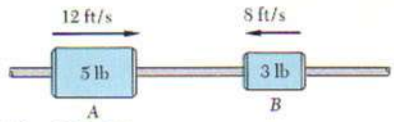


Fig. P13.134

13.134 The coefficient of restitution between the two collars is known to be 0.75; determine (*a*) their velocities after impact, (*b*) the energy loss during the impact.

13.135 Solve Prob. 13.134, assuming that the velocity of collar *B* is 4 ft/s to the right.

13.136 Two steel blocks slide without friction on a horizontal surface; immediately before impact their velocities are as shown. Knowing that $e = 0.75$, determine (*a*) their velocities after impact, (*b*) the energy loss during impact.

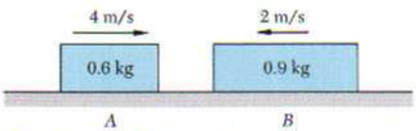


Fig. P13.136 and P13.137

13.137 The velocities of two steel blocks before impact are as shown. If after the impact the velocity of block *B* is observed to be 2.5 m/s to the right, determine the coefficient of restitution between the two blocks.

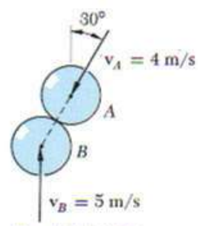


Fig. P13.138

13.138 As ball *A* is falling, a juggler tosses an identical ball *B* which strikes ball *A*. The line of impact forms an angle of 30° with the vertical. Assuming the balls frictionless and $e = 0.8$, determine the velocity of each ball immediately after impact.

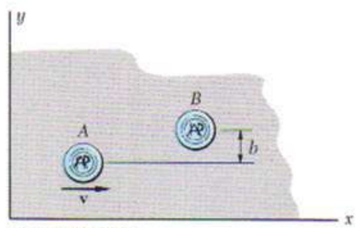


Fig. P13.139

13.139 Two identical pucks *A* and *B*, of 80-mm diameter, may move freely on a hockey rink. Puck *B* is at rest and puck *A* has an initial velocity **v** as shown. (*a*) Knowing that $b = 40$ mm and $e = 0.80$, and assuming no friction, determine the velocity of each puck after the impact. (*b*) Show that if $e = 1$, the final velocities of the pucks form a right angle for all values of *b*.

13.140 Assuming perfectly elastic impact, determine the velocity imparted to a quarter-dollar coin which is at rest and is struck squarely by (*a*) a dime moving with a velocity $\mathbf{v}_0$, (*b*) a half-dollar moving with a velocity $\mathbf{v}_0$. (*Masses:* half-dollar, 192.9 grains; quarter dollar, 96.45 grains; dime, 38.58 grains.)

13.141 A dime which is at rest on a *rough* surface is struck squarely by a half dollar moving to the right. After the impact, each coin slides and comes to rest; the dime slides 19.2 in. to the right, and the half dollar slides 3.8 in. to the right. Assuming the coefficient of friction is the same for each coin, determine the value of the coefficient of restitution between the coins. (See Prob. 13.140 for the mass of United States coins.)

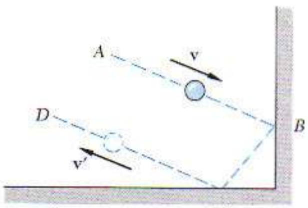


Fig. P13.142

13.142 A ball is thrown into a 90° corner with an initial velocity $\mathbf{v}$. Denoting the coefficient of restitution by e and assuming no friction, show that the final velocity $\mathbf{v}'$ is of magnitude ev and that the initial and final paths AB and CD are parallel.

13.143 A steel ball falling vertically strikes a rigid plate A and rebounds horizontally as shown. Denoting by e the coefficient of restitution and assuming no friction, determine (*a*) the required angle θ, (*b*) the magnitude of the velocity $\mathbf{v}_1$.

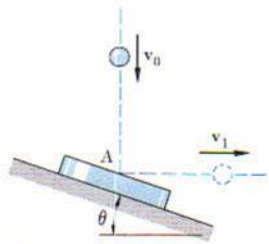


Fig. P13.143

13.144 A ball is dropped from a height $h_0 = 36$ in. onto a frictionless floor. Knowing that for the first bounce $h_1 = 32$ in. and $d_1 = 16$ in., determine (*a*) the coefficient of restitution, (*b*) the height and length of the second bounce.

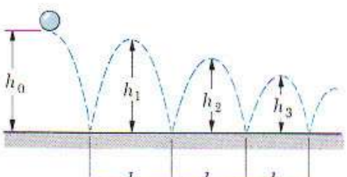


Fig. P13.144, P13.145, and P13.146

13.145 A ball is dropped onto a frictionless floor and allowed to bounce several times as shown. Derive an expression for the coefficient of restitution in terms of (*a*) the heights of two successive bounces h_n and h_{n+1}, (*b*) the lengths of two successive bounces d_n and d_{n+1}, (*c*) the durations of two successive bounces t_n and t_{n+1}.

13.146 A ball is dropped onto a frictionless floor and bounces as shown. The lengths of the first two bounces are measured and found to be $d_1 = 14.5$ in. and $d_2 = 12.8$ in. Determine (*a*) the coefficient of restitution, (*b*) the expected length d_3 of the third bounce.

13.147 A ball is dropped from a height h above the landing and bounces down a flight of stairs. Denoting by e the coefficient of restitution, determine the value of h for which the ball will bounce to the same height above each step.

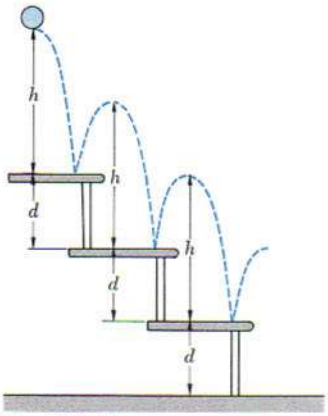


Fig. P13.147

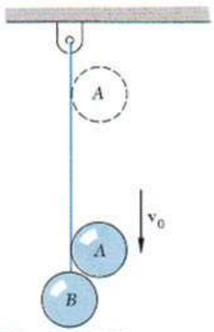


Fig. P13.148

∗13.148 Ball B is suspended by an inextensible cord. An identical ball A is released from rest when it is just touching the cord and acquires a velocity $\mathbf{v}_0$ before striking ball B. Assuming $e = 1$ and no friction, determine the velocity of each ball immediately after impact.

∗13.149 A 2-kg sphere moving to the left with a velocity of 10 m/s strikes the frictionless, inclined surface of a 5-kg block which is at rest. The block rests on rollers and may move freely in the horizontal direction. Knowing that $e = 0.75$, determine the velocities of the block and of the sphere immediately after impact.

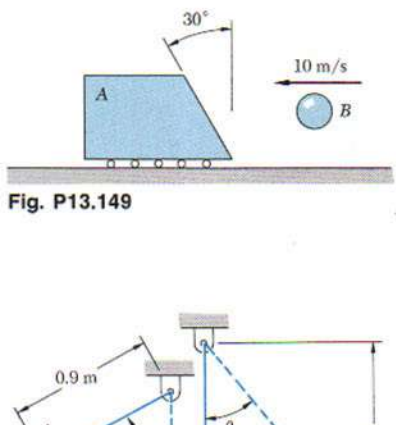


Fig. P13.149

13.150 The 4.5-kg sphere A strikes the 1.5-kg sphere B. Knowing that $e = 0.90$, determine the angle θ_A at which A must be released if the maximum angle θ_B reached by B is to be 90°.

13.151 The 4.5-kg sphere A is released from rest when $\theta_A = 60°$ and strikes the 1.5-kg sphere B. Knowing that $e = 0.90$, determine (*a*) the highest position to which sphere B will rise, (*b*) the maximum tension which will occur in the cord holding B.

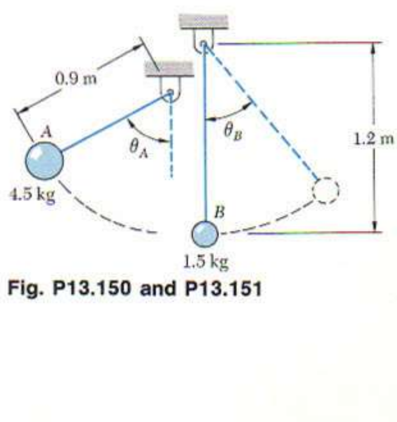


Fig. P13.150 and P13.151

13.152 Block A is released when $\theta_A = 90°$ and slides without friction until it strikes ball B. Knowing that $e = 0.90$, determine (*a*) the velocity of B immediately after impact, (*b*) the maximum tension in the cord holding B, (*c*) the maximum height to which ball B will rise.

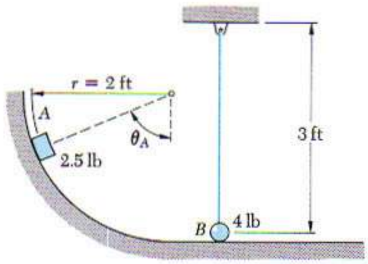


Fig. P13.152

13.153 What should be the value of the angle θ_A in Prob. 13.152 if the maximum angle between the cord holding ball B and the vertical is to be 45°?

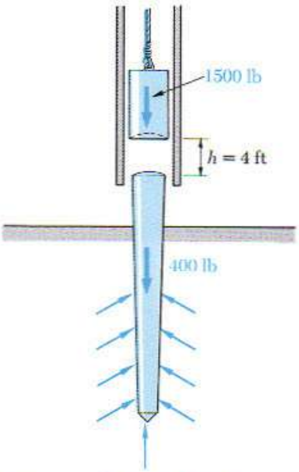


Fig. P13.154 and P13.155

13.154 It is desired to drive the 400-lb pile into the ground until the resistance to its penetration is 24,000 lb. Each blow of the 1500-lb hammer is the result of a 4-ft free fall onto the top of the pile. Determine how far the pile will be driven into the ground by a single blow when the 24,000-lb resistance is achieved. Assume that the impact is perfectly plastic.

13.155 The 1500-lb hammer of a drop-hammer pile driver falls from a height of 4 ft onto the top of a 400-lb pile. The pile is driven 4 in. into the ground. Assuming perfectly plastic impact, determine the average resistance of the ground to penetration.

13.156 Cylinder A is dropped 2 m onto cylinder B, which is resting on a spring of constant $k = 3$ kN/m. Assuming a perfectly plastic impact, determine (*a*) the maximum deflection of cylinder B, (*b*) the energy loss during the impact.

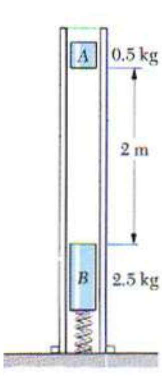


Fig. P13.156

13.157 The efficiency η of a drop-hammer pile driver may be defined as the ratio of the kinetic energy available after impact to the kinetic energy immediately before impact. Denoting by r the ratio of the pile mass m_p to the hammer mass m_h, and assuming perfectly plastic impact, show that $\eta = 1/(1 + r)$.

13.158 A bumper is designed to protect a 1600-kg automobile from damage when it hits a rigid wall at speeds up to 12 km/h. Assuming perfectly plastic impact, determine (*a*) the energy absorbed by the bumper during the impact, (*b*) the speed at which the automobile can hit another 1600-kg automobile without incurring any damage, if the other automobile is at rest and is similarly protected.

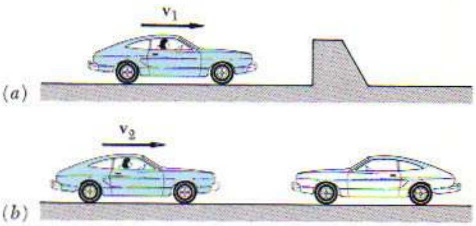


Fig. P13.158

13.159 Solve Prob. 13.158, assuming a coefficient of restitution $e = 0.50$. Show that the answer to part b is independent of e.

13.160 A small rivet connecting two pieces of sheet metal is being clinched by hammering. Determine the energy absorbed by the rivet under each blow, knowing that the head of the hammer weighs 1.5 lb and that it strikes the rivet with a velocity of 20 ft/s. Assume that the anvil is supported by springs and (*a*) is infinite in weight (rigid support), (*b*) weighs 10 lb.

***13.161** A ball of mass m_A moving to the right with a velocity $\mathbf{v}_A$ strikes a second ball of mass m_B which is at rest. Derive an expression for the kinetic-energy loss during impact. Assume that the balls strike each other squarely, and denote the coefficient of restitution by e.

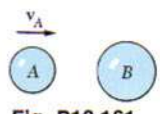


Fig. P13.161

REVIEW PROBLEMS

13.162 Collar B has an initial velocity of 2 m/s. It strikes collar A causing a series of impacts involving the collars and the fixed support at C. Assuming $e = 1$ for all impacts and neglecting friction, determine (*a*) the number of impacts which will occur, (*b*) the final velocity of B, (*c*) the final position of A.

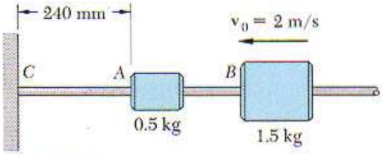


Fig. P13.162

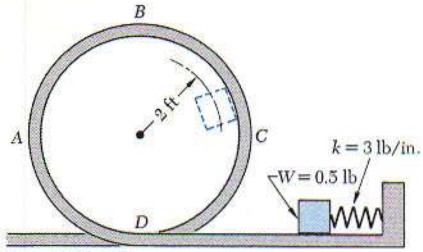


Fig. P13.163

13.163 The 0.5-lb pellet is released when the spring is compressed 6 in. and travels without friction around the vertical loop *ABCD*. Determine the force exerted by the loop on the pellet (*a*) at point *A*, (*b*) at point *B*, (*c*) at point *C*.

13.164 In Prob. 13.163, determine the smallest allowable deflection of the spring if the pellet is to travel around the entire loop without leaving the track.

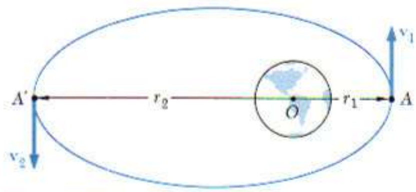


Fig. P13.165

13.165 Show that the values v_1 and v_2 of the speed of a satellite at the perigee A and the apogee A' of an elliptic orbit are defined by the relations

$$v_1^2 = \frac{2GM}{r_1 + r_2}\frac{r_2}{r_1} \qquad v_2^2 = \frac{2GM}{r_1 + r_2}\frac{r_1}{r_2}$$

13.166 The 20-Mg truck and the 40-Mg railroad flatcar are both at rest with their brakes released. An engine bumps the flatcar and causes the flatcar alone to start moving with a velocity of 1 m/s to the right. Assuming $e = 1$ between the truck and the ends of the flatcar, determine the velocities of the truck and of the flatcar after end A strikes the truck. Describe the subsequent motion of the system. Neglect the effect of friction.

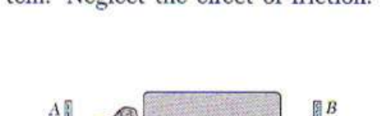


Fig. P13.166

13.167 An elevator travels upward at a constant speed of 2 m/s. A boy riding the elevator throws a 0.8-kg stone upward with a speed of 4 m/s *relative* to the elevator. Determine (*a*) the work done by the boy in throwing the stone, (*b*) the difference in the values of the kinetic energy of the stone before and after it was thrown. (*c*) Why are the values obtained in parts *a* and *b* not the same?

13.168 Two blocks are joined by an inextensible cable as shown. If the system is released from rest, determine the velocity of block A after it has moved 2 m. Assume that μ equals 0.25 between block A and the plane and neglect the mass and friction of the pulleys.

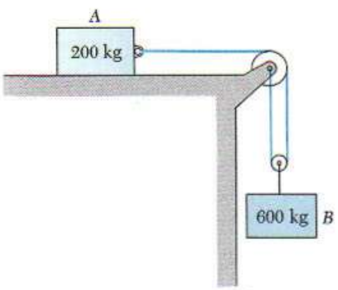


Fig. P13.168 and P13.169

13.169 Knowing that the system is released from rest, determine the additional mass that must be added to block B if its velocity is to be 4 m/s half a second after release. Assume that $\mu = 0.25$ between block A and the plane and neglect the mass and friction of the pulley.

13.170 A steel ball is dropped from A, strikes a rigid, frictionless steel plate at B, and bounces to point C. Knowing that the coefficient of restitution is 0.80, determine the distance d.

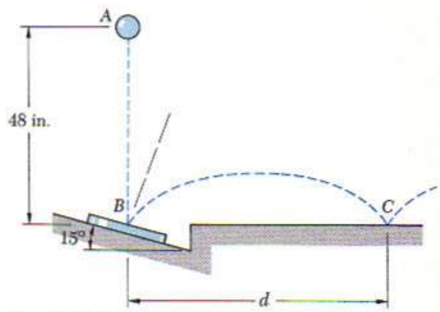


Fig. P13.170

13.171 Two portions AB and BC of the same elastic cord are connected as shown. The portion of cord BC supports a load W while, initially, the portion AB is under no tension. Determine the maximum tension which will develop in the entire cord after the stick DE suddenly breaks. (Assume that the tensions in AB and BC are instantaneously equalized after the stick breaks and that the elongation of the cord is small compared to L.)

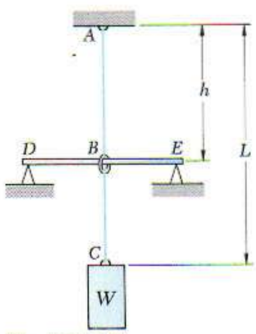


Fig. P13.171

13.172 A 5-kg collar slides without friction along a rod which forms an angle of 30° with the vertical. The spring is unstretched when the collar is at A. If the collar is released from rest at A, determine the value of the spring constant k for which the collar has zero velocity at B.

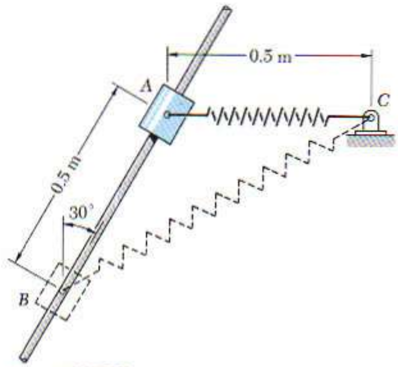


Fig. P13.172

13.173 In Prob. 13.172, determine the value of the spring constant k for which the velocity of the collar at B is 1.5 m/s.

Systems of Particles

CHAPTER

14

14.1. Application of Newton's Laws to the Motion of a System of Particles. Effective Forces. We shall be concerned in this chapter with the application of Newton's laws of motion to a *system of particles*, i.e., to a large number of particles considered together. The results obtained will enable us to analyze the effect of streams of particles on vanes or ducts and will provide us with the basic principles underlying the theory of jet and rocket propulsion (Secs. 14.9 through 14.11). Since a rigid body may be assumed to consist of a very large number of particles, the principles developed in this chapter will also provide us with a basis for the study of the kinetics of rigid bodies.

In order to derive the equations of motion for a system of n particles, we shall begin by writing Newton's second law for each individual particle of the system. Consider the particle P_i, where $1 \leq i \leq n$. Let m_i be the mass of P_i and $\mathbf{a}_i$ its acceleration with respect to the newtonian frame of reference $Oxyz$. We shall denote by $\mathbf{f}_{ij}$ the force exerted on P_i by another particle P_j of the system (Fig. 14.1); this force is called an *internal force*. The resultant of the internal forces exerted on P_i by all the other particles of the system is thus $\sum_{j=1}^{n} \mathbf{f}_{ij}$ (where $\mathbf{f}_{ii}$ has no meaning and is assumed equal to zero). Denoting, on the other hand, by $\mathbf{F}_i$

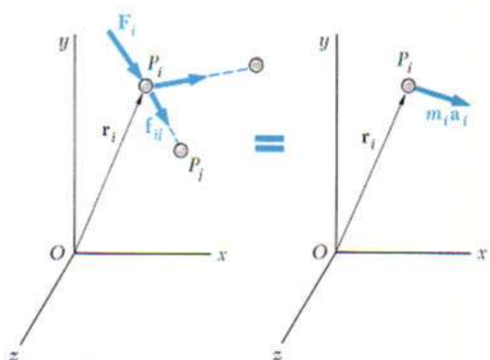


Fig. 14.1

the resultant of all the *external forces* acting on P_i, we write Newton's second law for the particle P_i as follows:

$$\mathbf{F}_i + \sum_{j=1}^{n} \mathbf{f}_{ij} = m_i \mathbf{a}_i \tag{14.1}$$

Denoting by $\mathbf{r}_i$ the position vector of P_i and taking the moments about O of the various terms in Eq. (14.1), we also write

$$\mathbf{r}_i \times \mathbf{F}_i + \sum_{j=1}^{n} (\mathbf{r}_i \times \mathbf{f}_{ij}) = \mathbf{r}_i \times m_i \mathbf{a}_i \tag{14.2}$$

Repeating this procedure for each particle P_i of the system, we obtain n equations of the type (14.1) and n equations of the type (14.2), where i takes successively the values $1, 2, \ldots, n$. The vectors $m_i\mathbf{a}_i$ are referred to as the *effective forces* of the particles. Thus the equations obtained express the fact that the external forces $\mathbf{F}_i$ and the internal forces $\mathbf{f}_{ij}$ acting on the various particles form a system equivalent to the system of the effective forces $m_i\mathbf{a}_i$ (i.e., one system may be replaced by the other) (Fig. 14.2).

Before proceeding further with our derivation, let us examine the internal forces $\mathbf{f}_{ij}$. We note that these forces occur in pairs $\mathbf{f}_{ij}$, $\mathbf{f}_{ji}$, where $\mathbf{f}_{ij}$ represents the force exerted by the particle P_j on the particle P_i and $\mathbf{f}_{ji}$ the force exerted by P_i on P_j (Fig. 14.2). Now,

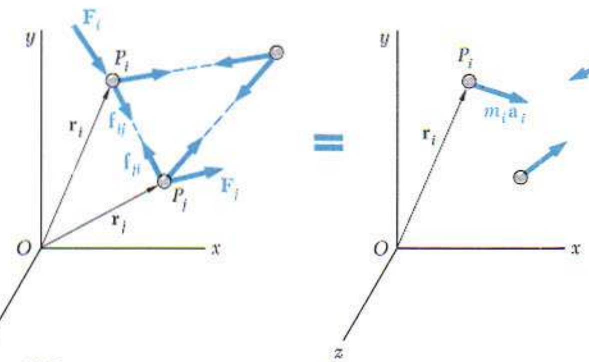


Fig. 14.2

according to Newton's third law (Sec. 6.1), as extended by Newton's law of gravitation to particles acting at a distance (Sec. 12.9), the forces $\mathbf{f}_{ij}$ and $\mathbf{f}_{ji}$ are equal and opposite and have the same line of action. Their sum is therefore $\mathbf{f}_{ij} + \mathbf{f}_{ji} = 0$ and the sum of their moments about O is

$$\mathbf{r}_i \times \mathbf{f}_{ij} + \mathbf{r}_j \times \mathbf{f}_{ji} = \mathbf{r}_i \times (\mathbf{f}_{ij} + \mathbf{f}_{ji}) + (\mathbf{r}_j - \mathbf{r}_i) \times \mathbf{f}_{ji} = 0$$

since the vectors $\mathbf{r}_j - \mathbf{r}_i$ and $\mathbf{f}_{ji}$ in the last term are collinear.

Adding all the internal forces of the system, and summing their moments about O, we obtain the equations

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{f}_{ij}=0 \qquad \sum_{i=1}^{n}\sum_{j=1}^{n}(\mathbf{r}_i\times\mathbf{f}_{ij})=0 \tag{14.3}$$

which express the fact that the resultant and the moment resultant of the internal forces of the system are zero.

Returning now to the n equations (14.1), where $i = 1, 2, \ldots, n$, we add them member by member. Taking into account the first of Eqs. (14.3), we obtain

$$\sum_{i=1}^{n}\mathbf{F}_i=\sum_{i=1}^{n}m_i\mathbf{a}_i \tag{14.4}$$

Proceeding similarly with Eqs. (14.2), and taking into account the second of Eqs. (14.3), we have

$$\sum_{i=1}^{n}(\mathbf{r}_i\times\mathbf{F}_i)=\sum_{i=1}^{n}(\mathbf{r}_i\times m_i\mathbf{a}_i) \tag{14.5}$$

Equations (14.4) and (14.5) express the fact that the system of the external forces $\mathbf{F}_i$ and the system of the effective forces $m_i\mathbf{a}_i$ have the same resultant and the same moment resultant. Referring to the definition given in Sec. 3.18 for two equipollent systems of vectors, we may therefore state that *the system of the external forces acting on the particles and the system of the effective forces of the particles are equipollent*† (Fig. 14.3).

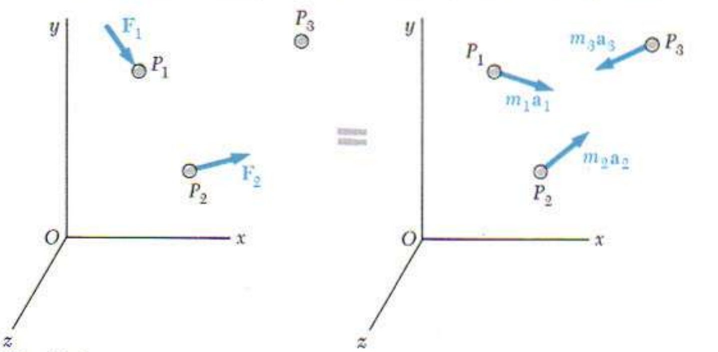


Fig. 14.3

† The result just obtained is often referred to as *D'Alembert's principle*, after the French mathematician Jean le Rond d'Alembert (1717–1783). However, d'Alembert's original statement refers to the motion of a system of connected bodies, with $\mathbf{f}_{ij}$ representing constraint forces which, if applied by themselves, will not cause the system to move. Since, as it will now be shown, this is in general not the case for the internal forces acting on a system of free particles, we shall postpone the consideration of D'Alembert's principle until the study of the motion of rigid bodies (Chap. 16).

We may note that Eqs. (14.3) express the fact that the system of the internal forces $\mathbf{f}_{ij}$ is equipollent to zero. It does *not* follow, however, that the internal forces have no effect on the particles under consideration. Indeed, the gravitational forces that the sun and the planets exert on each other are internal to the solar system and equipollent to zero. Yet these forces are alone responsible for the motion of the planets about the sun.

Similarly, it does not follow from Eqs. (14.4) and (14.5) that two systems of external forces which have the same resultant and the same moment resultant will have the same effect on a given system of particles. Clearly, the systems shown in Figs. 14.4*a* and 14.4*b* have the same resultant and the same moment resultant; yet the first system accelerates particle A and leaves particle B unaffected, while the second accelerates B and does not affect A.

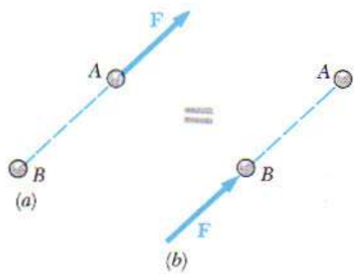


Fig. 14.4

It is important to recall that, when we stated in Sec. 3.18 that two equipollent systems of forces acting on a rigid body are also equivalent, we specifically noted that this property could *not* be extended to a system of forces acting on a set of independent particles such as those considered in this chapter.

In order to avoid any confusion, we shall use gray equals signs to connect equipollent systems of vectors, such as those shown in Figs. 14.3 and 14.4. These signs will indicate that the two systems of vectors have the same resultant and the same moment resultant. Blue equals signs will continue to be used to indicate that two systems of vectors are equivalent, i.e., that one system may actually be replaced by the other (Fig. 14.2).

14.2. Linear and Angular Momentum of a System of Particles. Equations (14.4) and (14.5), obtained in the preceding section for the motion of a system of particles, may be expressed in a more condensed form if we introduce the linear and the angular momentum of the system of particles. Defining the linear momentum $\mathbf{L}$ of the system of particles as the sum of the linear momenta of the various particles of the system (Sec. 12.2), we write

$$\mathbf{L} = \sum_{i=1}^{n} m_i \mathbf{v}_i \tag{14.6}$$

Defining the angular momentum $\mathbf{H}_O$ about O of the system of particles in a similar way (Sec. 12.6), we have

$$\mathbf{H}_O = \sum_{i=1}^{n} (\mathbf{r}_i \times m_i \mathbf{v}_i) \tag{14.7}$$

Differentiating both members of Eqs. (14.6) and (14.7) with respect to t, we write

$$\dot{\mathbf{L}} = \sum_{i=1}^{n} m_i \dot{\mathbf{v}}_i = \sum_{i=1}^{n} m_i \mathbf{a}_i \tag{14.8}$$

and

$$\dot{\mathbf{H}}_O = \sum_{i=1}^{n} (\dot{\mathbf{r}}_i \times m_i \mathbf{v}_i) + \sum_{i=1}^{n} (\mathbf{r}_i \times m_i \dot{\mathbf{v}}_i)$$

$$= \sum_{i=1}^{n} (\mathbf{v}_i \times m_i \mathbf{v}_i) + \sum_{i=1}^{n} (\mathbf{r}_i \times m_i \mathbf{a}_i)$$

which reduces to

$$\dot{\mathbf{H}}_O = \sum_{i=1}^{n} (\mathbf{r}_i \times m_i \mathbf{a}_i) \tag{14.9}$$

since the vectors $\mathbf{v}_i$ and $m_i \mathbf{v}_i$ are collinear.

We observe that the right-hand members of Eqs. (14.8) and (14.9) are, respectively, identical with the right-hand members of Eqs. (14.4) and (14.5). It follows that the left-hand members of these equations are respectively equal. Recalling that the left-hand member of Eq. (14.5) represents the sum of the moments $\mathbf{M}_O$ about O of the external forces acting on the particles of the system, and omitting the subscript i from the sums, we write

$$\Sigma \mathbf{F} = \dot{\mathbf{L}} \tag{14.10}$$

$$\Sigma \mathbf{M}_O = \dot{\mathbf{H}}_O \tag{14.11}$$

These equations express that *the resultant and the moment resultant about the fixed point O of the external forces are respectively equal to the rates of change of the linear momentum and of the angular momentum about O of the system of particles.*

14.3. Motion of the Mass Center of a System of Particles. Equation (14.10) may be written in an alternate form if the *mass center* of the system of particles is considered. The mass center of the system is the point G defined by the position vector $\bar{\mathbf{r}}$ which satisfies the relation

$$m\bar{\mathbf{r}} = \sum_{i=1}^{n} m_i \mathbf{r}_i \tag{14.12}$$

where m represents the total mass $\sum_{i=1}^{n} m_i$ of the particles. Resolving the position vectors $\bar{\mathbf{r}}$ and $\mathbf{r}_i$ into rectangular components, we obtain the following three scalar equations, which may be used to determine the coordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ of the mass center:

$$m\bar{x} = \sum_{i=1}^{n} m_i x_i \qquad m\bar{y} = \sum_{i=1}^{n} m_i y_i \qquad m\bar{z} = \sum_{i=1}^{n} m_i z_i \tag{14.12'}$$

Since $m_i g$ represents the weight of the particle P_i, and mg the total weight of the particles, we note that G is also the center of gravity of the system of particles. However, in order to avoid any confusion, we shall call G the *mass center* of the system of particles when discussing properties of the system associated with the *mass* of the particles, and we shall refer to it as the *center of gravity* of the system when considering properties associated with the *weight* of the particles. Particles located outside the gravitational field of the earth, for example, have a mass but no weight. We may then properly refer to their mass center, but obviously not to their center of gravity.†

Differentiating both members of Eq. (14.12) with respect to t, we write

$$m\dot{\bar{\mathbf{r}}} = \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i$$

or

$$m\bar{\mathbf{v}} = \sum_{i=1}^{n} m_i \mathbf{v}_i \tag{14.13}$$

† It may also be pointed out that the mass center and the center of gravity of a system of particles do not exactly coincide, since the weights of the particles are directed toward the center of the earth and thus do not truly form a system of parallel forces.

where $\overline{\mathbf{v}}$ represents the velocity of the mass center G of the system of particles. But the right-hand member of Eq. (14.13) is, by definition, the linear momentum $\mathbf{L}$ of the system (Sec. 14.2). We have therefore

$$\mathbf{L} = m\overline{\mathbf{v}} \tag{14.14}$$

and, differentiating both members with respect to t,

$$\dot{\mathbf{L}} = m\overline{\mathbf{a}} \tag{14.15}$$

where $\overline{\mathbf{a}}$ represents the acceleration of the mass center G. Substituting for $\dot{\mathbf{L}}$ from (14.15) into (14.10), we write the equation

$$\Sigma\mathbf{F} = m\overline{\mathbf{a}} \tag{14.16}$$

which defines the motion of the mass center G of the system of particles.

We note that Eq. (14.16) is identical with the equation we would obtain for a particle of mass m equal to the total mass of the particles of the system, acted upon by all the external forces. We state therefore: *The mass center of a system of particles moves as if the entire mass of the system and all the external forces were concentrated at that point.*

This principle is best illustrated by the motion of an exploding shell. We know that, if the resistance of the air is neglected, a shell may be assumed to travel along a parabolic path. After the shell has exploded, the mass center G of the fragments of shell will continue to travel along the same path. Indeed, point G must move as if the mass and the weight of all fragments were concentrated at G; it must move, therefore, as if the shell had not exploded.

It should be noted that the preceding derivation does not involve the moments of the external forces. Therefore, *it would be wrong to assume* that the external forces are equipollent to a vector $m\overline{\mathbf{a}}$ attached at the mass center G. This is, in general, not the case, since, as we shall see in the next section, the sum of the moments about G of the external forces is, in general, not equal to zero.

14.4. Angular Momentum of a System of Particles about Its Mass Center. In some applications (for example, in the analysis of the motion of a rigid body) it is convenient to consider the motion of the particles of the system with respect to a centroidal frame of reference $Gx'y'z'$ which translates with respect to the newtonian frame of reference $Oxyz$ (Fig. 14.5).

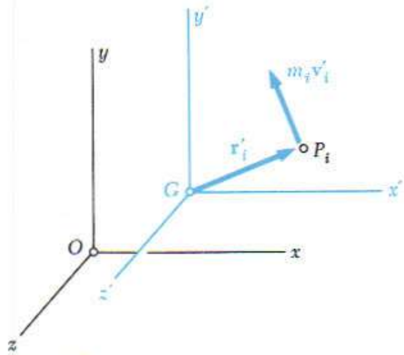


Fig. 14.5

While such a frame is not, in general, a newtonian frame of reference, we shall see that the fundamental relation (14.11) still holds when the frame $Oxyz$ is replaced by $Gx'y'z'$.

Denoting respectively by $\mathbf{r}'_i$ and $\mathbf{v}'_i$ the position vector and the velocity of the particle P_i relative to the moving frame of reference $Gx'y'z'$, we define the *angular momentum* $\mathbf{H}'_G$ of the system of particles *about the mass center* G as follows:

$$\mathbf{H}'_G = \sum_{i=1}^{n} (\mathbf{r}'_i \times m_i\mathbf{v}'_i) \tag{14.17}$$

We now differentiate both members of Eq. (14.17) with respect to t. This operation being similar to that performed in Sec. 14.2 on Eq. (14.7), we write immediately

$$\dot{\mathbf{H}}'_G = \sum_{i=1}^{n} (\mathbf{r}'_i \times m_i\mathbf{a}'_i) \tag{14.18}$$

where $\mathbf{a}'_i$ denotes the acceleration of P_i relative to the moving frame of reference. Referring to Sec. 11.12, we write

$$\mathbf{a}_i = \bar{\mathbf{a}} + \mathbf{a}'_i$$

where $\mathbf{a}_i$ and $\bar{\mathbf{a}}$ denote, respectively, the accelerations of P_i and G relative to the frame $Oxyz$. Solving for $\mathbf{a}'_i$ and substituting into (14.18), we have

$$\dot{\mathbf{H}}'_G = \sum_{i=1}^{n} (\mathbf{r}'_i \times m_i\mathbf{a}_i) - \left(\sum_{i=1}^{n} m_i\mathbf{r}'_i\right) \times \bar{\mathbf{a}} \tag{14.19}$$

But, by (14.12), the second sum in Eq. (14.19) is equal to $m\bar{\mathbf{r}}'$ and, thus, to zero, since the position vector $\bar{\mathbf{r}}'$ of G relative to the frame $Gx'y'z'$ is clearly zero. On the other hand, since $\mathbf{a}_i$ represents the acceleration of P_i relative to a newtonian frame, we may use Eq. (14.1) and replace $m_i\mathbf{a}_i$ by the sum of the internal forces $\mathbf{f}_{ij}$ and of the resultant $\mathbf{F}_i$ of the external forces acting on P_i. But a reasoning similar to that used in Sec. 14.1 shows that the moment resultant about G of the internal forces $\mathbf{f}_{ij}$ of the entire system is zero. The first sum in Eq. (14.19) reduces therefore to the moment resultant about G of the external forces acting on the particles of the system, and we write

$$\Sigma\mathbf{M}_G = \dot{\mathbf{H}}'_G \tag{14.20}$$

which expresses that *the moment resultant about G of the exter-*

nal forces is equal to the rate of change of the angular momentum about G of the system of particles.

It should be noted that, in Eq. (14.17), we defined the angular momentum $\mathbf{H}'_G$ as the sum of the moments about G of the momenta of the particles $m_i\mathbf{v}'_i$ *in their motion relative to the centroidal frame of reference* $Gx'y'z'$. We may sometimes want to compute the sum $\mathbf{H}_G$ of the moments about G of the momenta of the particles $m_i\mathbf{v}_i$ *in their absolute motion*, i.e., in their motion as observed from the newtonian frame of reference $Oxyz$ (Fig. 14.6):

$$\mathbf{H}_G = \sum_{i=1}^{n} (\mathbf{r}'_i \times m_i\mathbf{v}_i) \qquad (14.21)$$

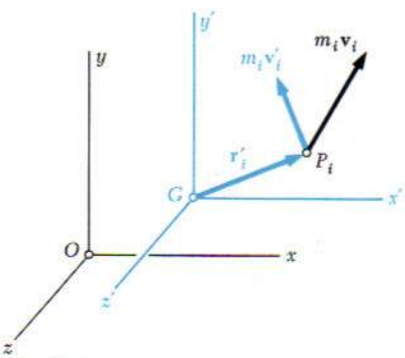


Fig. 14.6

Remarkably, the angular momenta $\mathbf{H}'_G$ and $\mathbf{H}_G$ are identically equal. This may be verified by referring to Sec. 11.12 and writing

$$\mathbf{v}_i = \overline{\mathbf{v}} + \mathbf{v}'_i \qquad (14.22)$$

Substituting for $\mathbf{v}_i$ from (14.22) into Eq. (14.21), we have

$$\mathbf{H}_G = \left(\sum_{i=1}^{n} m_i\mathbf{r}'_i\right) \times \overline{\mathbf{v}} + \sum_{i=1}^{n} (\mathbf{r}'_i \times m_i\mathbf{v}'_i)$$

But, as observed earlier, the first sum is equal to zero. Thus $\mathbf{H}_G$ reduces to the second sum which, by definition, is equal to $\mathbf{H}'_G$.†

†Note that this property is peculiar to the centroidal frame $Gx'y'z'$ and does not hold, in general, for other frames of reference (see Prob. 14.19).

Taking advantage of the property we have just established, we shall simplify our notation by dropping the prime (′) from Eq. (14.20). We therefore write

$$\Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G \tag{14.23}$$

where it is understood that the angular momentum $\mathbf{H}_G$ may be computed by forming the moments about G of the momenta of the particles in their motion with respect to either the newtonian frame $Oxyz$ or the centroidal frame $Gx'y'z'$:

$$\mathbf{H}_G = \sum_{i=1}^{n} (\mathbf{r}'_i \times m_i \mathbf{v}_i) = \sum_{i=1}^{n} (\mathbf{r}'_i \times m_i \mathbf{v}'_i) \tag{14.24}$$

14.5. Conservation of Momentum for a System of Particles. If no external force acts on the particles of a system, the left-hand members of Eqs. (14.10) and (14.11) are equal to zero and these equations reduce to $\dot{\mathbf{L}} = 0$ and $\dot{\mathbf{H}}_O = 0$. We conclude that

$$\mathbf{L} = \text{constant} \qquad \mathbf{H}_O = \text{constant} \tag{14.25}$$

The equations obtained express that the linear momentum of the system of particles and its angular momentum about the fixed point O are conserved.

In some applications, such as problems involving central forces, the moment about a fixed point O of each of the external forces may be zero, without any of the forces being zero. In such cases, the second of Eqs. (14.25) still holds; the angular momentum of the system of particles about O is conserved.

The concept of conservation of momentum may also be applied to the analysis of the motion of the mass center G of a system of particles and to the analysis of the motion of the system about G. For example, if the sum of the external forces is zero, the first of Eqs. (14.25) applies. Recalling Eq. (14.14), we write

$$\overline{\mathbf{v}} = \text{constant} \tag{14.26}$$

which expresses that the mass center G of the system moves in a straight line and at a constant speed. On the other hand, if the sum of the moments about G of the external forces is zero, it follows from Eq. (14.23) that the angular momentum of the system about its mass center is conserved:

$$\mathbf{H}_G = \text{constant} \tag{14.27}$$

SAMPLE PROBLEM 14.1

A 200-kg space vehicle is observed at $t = 0$ to pass through the origin of a newtonian reference frame $Oxyz$ with the velocity $\mathbf{v}_0 = (150 \text{ m/s})\mathbf{i}$ relative to the frame. Following the detonation of explosive charges, the vehicle separates into three parts, A, B, and C, of mass 100 kg, 60 kg, and 40 kg, respectively. Knowing that, at $t = 2.5$ s, the positions of parts A and B are observed to be $A(555, -180, 240)$ and $B(255, 0, -120)$, where the coordinates are expressed in meters, determine the position of part C at that time.

Solution. Since there is no external force, the mass center G of the system moves with the constant velocity $\mathbf{v}_0 = (150 \text{ m/s})\mathbf{i}$. At $t = 2.5$ s, its position is

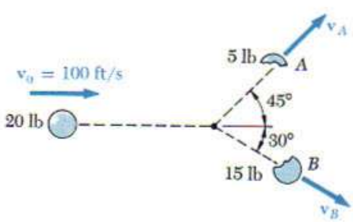
$$\bar{\mathbf{r}} = \mathbf{v}_0 t = (150 \text{ m/s})\mathbf{i}(2.5 \text{ s}) = (375 \text{ m})\mathbf{i}$$

Recalling Eq. (14.12), we write

$$m\bar{\mathbf{r}} = m_A\mathbf{r}_A + m_B\mathbf{r}_B + m_C\mathbf{r}_C$$

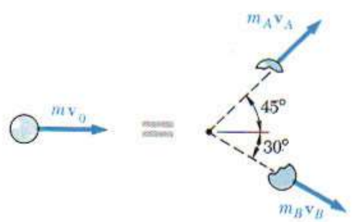
$$(200 \text{ kg})(375 \text{ m})\mathbf{i} = (100 \text{ kg})[(555 \text{ m})\mathbf{i} - (180 \text{ m})\mathbf{j} + (240 \text{ m})\mathbf{k}] + (60 \text{ kg})[(255 \text{ m})\mathbf{i} - (120 \text{ m})\mathbf{k}] + (40 \text{ kg})\mathbf{r}_C$$

$$\mathbf{r}_C = (105 \text{ m})\mathbf{i} + (450 \text{ m})\mathbf{j} - (420 \text{ m})\mathbf{k} \quad ◄$$



SAMPLE PROBLEM 14.2

A 20-lb projectile is moving with a velocity of 100 ft/sec when it explodes into two fragments A and B, weighing 5 lb and 15 lb, respectively. Knowing that immediately after the explosion the fragments travel in the directions shown, determine the velocity of each fragment.



Solution. Since there is no external force, the linear momentum of the system is conserved, and we write

$$m_A\mathbf{v}_A + m_B\mathbf{v}_B = m\mathbf{v}_0$$

$$(5/g)\mathbf{v}_A + (15/g)\mathbf{v}_B = (20/g)\mathbf{v}_0$$

$\xrightarrow{+}$ x components: $\quad 5v_A \cos 45° + 15v_B \cos 30° = 20(100)$

$+\uparrow y$ components: $\quad 5v_A \sin 45° - 15v_B \sin 30° = 0$

Solving simultaneously the two equations for v_A and v_B, we have

$$v_A = 207 \text{ ft/s} \qquad v_B = 97.6 \text{ ft/s}$$

$$\mathbf{v}_A = 207 \text{ ft/s} \measuredangle 45° \qquad \mathbf{v}_B = 97.6 \text{ ft/s} \searrow 30° \quad ◄$$

PROBLEMS

14.1 Two men dive horizontally and to the right off the end of a 300-lb boat. The boat is initially at rest, and each man weighs 150 lb. If each man dives so that his relative horizontal velocity with respect to the boat is 12 ft/s, determine (*a*) the velocity of the boat after the men dive simultaneously, (*b*) the velocity of the boat after one man dives and the velocity of the boat after the second man dives.

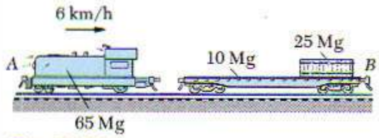


Fig. P14.2

14.2 A 65-Mg engine coasting at 6 km/h strikes, and is automatically coupled with, a 10-Mg flat car which carries a 25-Mg load. The load is *not* securely fastened to the car but may slide along the floor ($\mu = 0.20$). Knowing that the car was at rest with its brakes released and that the coupling takes place instantaneously, determine the velocity of the engine (*a*) immediately after the coupling, (*b*) after the load has slid to a stop relative to the car.

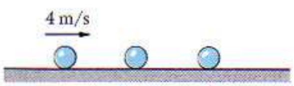


Fig. P14.3

14.3 Two identical balls B and C are at rest when ball B is struck by a ball A of the same mass, moving with a velocity of 4 m/s. This causes a series of collisions between the various balls. Knowing that $e = 0.40$, determine the velocity of each ball after *all* collisions have taken place.

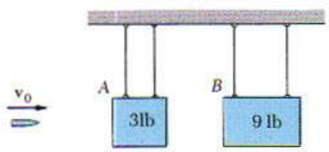


Fig. P14.4

14.4 A $\frac{3}{4}$-oz bullet is fired in a horizontal direction through block A and becomes embedded in block B. The bullet causes A and B to start moving with velocities of 8 and 6 ft/s, respectively. Determine (*a*) the initial velocity $\mathbf{v}_0$ of the bullet, (*b*) the velocity of the bullet as it travels from block A to block B.

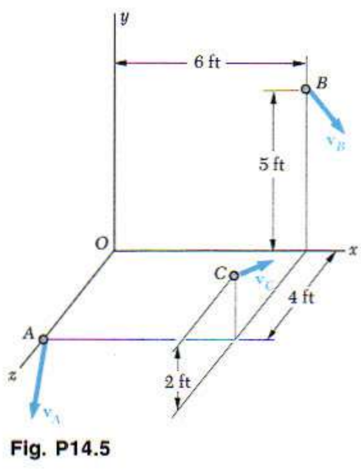


Fig. P14.5

14.5 A system consists of three particles A, B, and C. We know that $W_A = 2$ lb, $W_B = 3$ lb, and $W_C = 4$ lb and that the velocities of the particles expressed in feet per second are, respectively, $\mathbf{v}_A = -10\mathbf{j} + 5\mathbf{k}$, $\mathbf{v}_B = 8\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}$, and $\mathbf{v}_C = v_x\mathbf{i} + v_y\mathbf{j} + 10\mathbf{k}$. Determine (*a*) the components v_x and v_y of the velocity of particle C for which the angular momentum $\mathbf{H}_O$ of the system about O is parallel to the z axis, (*b*) the corresponding value of $\mathbf{H}_O$.

14.6 For the system of particles of Prob. 14.5, determine (*a*) the components v_x and v_y of the velocity of particle C for which the angular momentum $\mathbf{H}_O$ of the system about O is parallel to the x axis, (*b*) the corresponding value of $\mathbf{H}_O$.

14.7 A system consists of three particles A, B, and C. We know that $m_A = 1$ kg, $m_B = 2$ kg, and $m_C = 3$ kg and that the velocities of the particles expressed in meters per second are, respectively, $\mathbf{v}_A = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{v}_B = 4\mathbf{i} + 3\mathbf{j}$, and $\mathbf{v}_C = 2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$. (*a*) Determine the angular momentum $\mathbf{H}_O$ of the system about O. (*b*) Using the result of part *a* and the answers to Prob. 14.8, check that the relation given in Prob. 14.17 is satisfied.

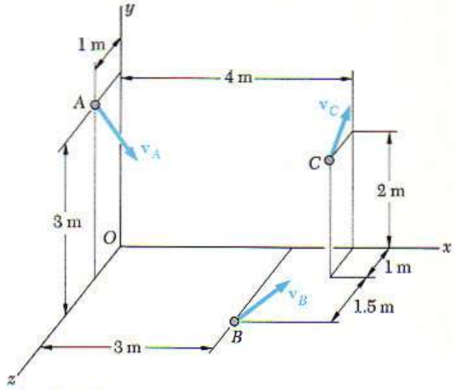


Fig. P14.7

14.8 For the system of particles of Prob. 14.7, determine (*a*) the position vector $\bar{\mathbf{r}}$ of the mass center G of the system, (*b*) the linear momentum $m\bar{\mathbf{v}}$ of the system, (*c*) the angular momentum $\mathbf{H}_G$ of the system about G.

14.9 A 240-kg space vehicle traveling with the velocity $\mathbf{v}_0 = (500 \text{ m/s})\mathbf{k}$ passes through the origin O at $t = 0$. Explosive charges then separate the vehicle into three parts, A, B, and C, of mass 40 kg, 80 kg, and 120 kg, respectively. Knowing that at $t = 3$ s the positions of parts B and C are observed to be $B(375, 825, 2025)$ and $C(-300, -600, 1200)$, where the coordinates are expressed in meters, determine the corresponding position of part A. Neglect the effect of gravity.

14.10 Two 30-lb cannon balls are chained together and fired horizontally with a velocity of 500 ft/s from the top of a 45-ft wall. The chain breaks during the flight of the cannon balls and one of them strikes the ground at $t = 1.5$ s, at a distance of 720 ft from the foot of the wall, and 21 ft to the right of the line of fire. Determine the position of the other cannon ball at that instant. Neglect the resistance of the air.

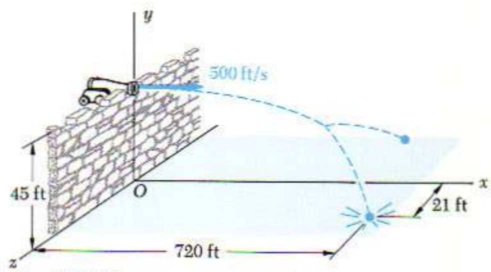


Fig. P14.10

14.11 Solve Prob. 14.10, if the cannon ball which first strikes the ground weighs 24 lb and the other 36 lb. Assume that the time of flight and the point of impact of the first cannon ball remain the same.

14.12 A 10-kg projectile is passing through the origin O with a velocity $\mathbf{v}_0 = (60 \text{ m/s})\mathbf{i}$ when it explodes into two fragments, A and B, of mass 4 kg and 6 kg, respectively. Knowing that, 2 s later, the position of the first fragment is $A(150 \text{ m}, 12 \text{ m}, -24 \text{ m})$, determine the position of fragment B at the same instant. Assume $g = 9.81 \text{ m/s}^2$ and neglect the resistance of the air.

14.13 An archer hits a game bird flying in a horizontal straight line 30 ft above the ground with a 500-grain wooden arrow [1 grain = (1/7000) lb]. Knowing that the arrow strikes the bird from behind with a velocity of 350 ft/s at an angle of 30° with the vertical, and that the bird falls to the ground in 1.5 s and 48 ft beyond the point where it was hit, determine (*a*) the weight of the bird, (*b*) the speed at which it was flying when it was hit.

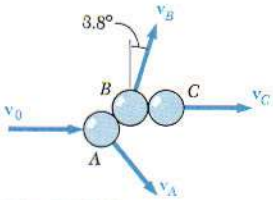


Fig. P14.14

14.14 In a game of billiards, ball A is moving with the velocity $\mathbf{v}_0 = (10 \text{ ft/s})\mathbf{i}$ when it strikes balls B and C which are at rest side by side. After the collision, A is observed to move with the velocity $\mathbf{v}_A = (3.92 \text{ ft/s})\mathbf{i} - (4.56 \text{ ft/s})\mathbf{j}$, while B and C move in the directions shown. Determine the magnitudes of the velocities $\mathbf{v}_B$ and $\mathbf{v}_C$.

14.15 A 5-kg object is falling vertically when, at point D, it explodes into three fragments A, B, and C, weighing, respectively, 1.5 kg, 2.5 kg, and 1 kg. Immediately after the explosion the velocity of each fragment is directed as shown and the speed of fragment A is observed to be 70 m/s. Determine the velocity of the 5-kg object immediately before the explosion.

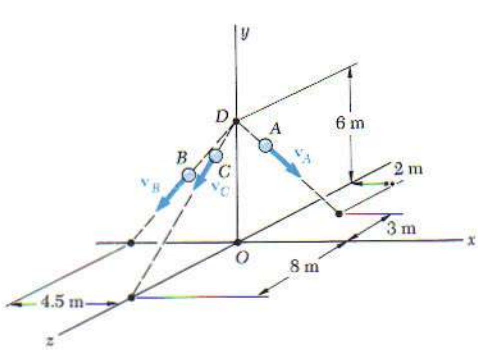


Fig. P14.15

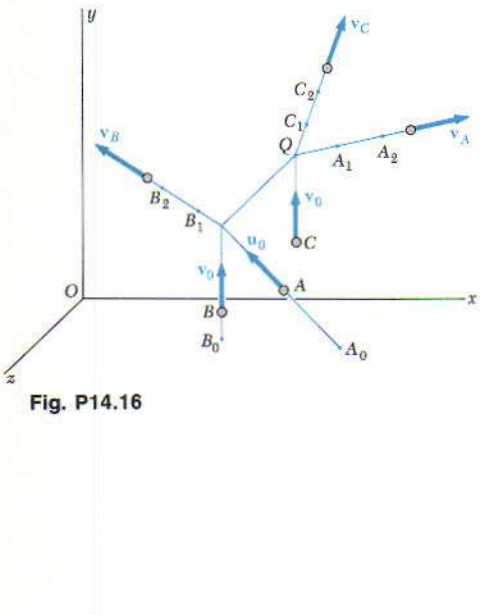


Fig. P14.16

14.16 In a scattering experiment, an alpha particle A is projected with the velocity $\mathbf{u}_0 = -(600 \text{ m/s})\mathbf{i} + (750 \text{ m/s})\mathbf{j} - (800 \text{ m/s})\mathbf{k}$ into a stream of oxygen nuclei moving with the common velocity $\mathbf{v}_0 = (600 \text{ m/s})\mathbf{j}$. After colliding successively with the nuclei B and C, particle A is observed to move along the path defined by the points $A_1(280, 240, 120)$, $A_2(360, 320, 160)$, while nuclei B and C are observed to move along paths defined, respectively, by $B_1(147, 220, 130)$, $B_2(114, 290, 120)$ and by $C_1(240, 232, 90)$, $C_2(240, 280, 75)$. All paths are along straight lines and all coordinates are expressed in millimeters. Knowing that the mass of an oxygen nucleus is four times that of an alpha particle, determine the speed of each of the three particles after the collisions.

14.17 Derive the relation

$$\mathbf{H}_O = \bar{\mathbf{r}} \times m\bar{\mathbf{v}} + \mathbf{H}_G$$

between the angular momenta $\mathbf{H}_O$ and $\mathbf{H}_G$ defined in Eqs. (14.7) and (14.24), respectively. The vectors $\bar{\mathbf{r}}$ and $\bar{\mathbf{v}}$ define, respectively, the position and velocity of the mass center G of the system of particles relative to the newtonian frame of reference $Oxyz$, and m represents the total mass of the system.

14.18 Show that Eq. (14.23) may be derived directly from Eq. (14.11) by substituting for $\mathbf{H}_O$ the expression given in Prob. 14.17.

14.19 Consider the frame of reference $Ax'y'z'$ in translation with respect to the newtonian frame of reference $Oxyz$. We define the angular momentum $\mathbf{H}'_A$ of a system of n particles about A as the sum

$$\mathbf{H}'_A = \sum_{i=1}^{n} \mathbf{r}'_i \times m_i \mathbf{v}'_i \tag{1}$$

of the moments about A of the momenta $m_i\mathbf{v}'_i$ of the particles in their motion relative to the frame $Ax'y'z'$. Denoting by $\mathbf{H}_A$ the sum

$$\mathbf{H}_A = \sum_{i=1}^{n} \mathbf{r}'_i \times m_i \mathbf{v}_i \tag{2}$$

of the moments about A of the momenta $m_i\mathbf{v}_i$ of the particles in their motion relative to the newtonian frame $Oxyz$, show that $\mathbf{H}_A = \mathbf{H}'_A$ at a given instant if, and only if, one of the following conditions is satisfied at that instant: (*a*) A has zero velocity with respect to the frame $Oxyz$, (*b*) A coincides with the mass center G of the system, (*c*) the velocity $\mathbf{v}_A$ relative to $Oxyz$ is directed along the line AG.

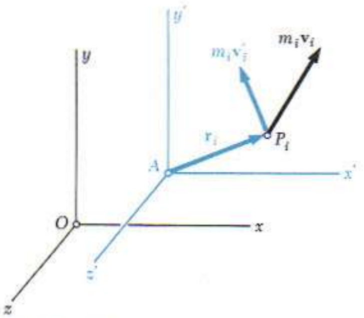


Fig. P14.19

14.20 Show that the relation

$$\Sigma \mathbf{M}_A = \dot{\mathbf{H}}'_A$$

where $\mathbf{H}'_A$ is defined by Eq. (1) of Prob. 14.19 and where $\Sigma\mathbf{M}_A$ represents the sum of the moments about A of the external forces acting on the system of particles, is valid if, and only if, one of the following conditions is satisfied: (*a*) the frame $Ax'y'z'$ is itself a newtonian frame of reference, (*b*) A coincides with the mass center G, (*c*) the acceleration $\mathbf{a}_A$ of A relative to $Oxyz$ is directed along the line AG.

14.6. Kinetic Energy of a System of Particles. The kinetic energy T of a system of particles is defined as the sum of the kinetic energies of the various particles of the system. Referring to Sec. 13.3, we therefore write

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2 \tag{14.28}$$

Using a Centroidal Frame of Reference. It is often convenient, when computing the kinetic energy of a system comprising a large number of particles (as in the case of a rigid body), to consider separately the motion of the mass center G of the system and the motion of the system relative to a moving frame of reference attached to G.

Let P_i be a particle of the system, $\mathbf{v}_i$ its velocity relative to the newtonian frame of reference $Oxyz$, and $\mathbf{v}'_i$ its velocity relative to the moving frame $Gx'y'z'$ which is in translation with respect to $Oxyz$ (Fig. 14.7). We recall from the preceding section that

$$\mathbf{v}_i = \overline{\mathbf{v}} + \mathbf{v}'_i \tag{14.22}$$

where $\overline{\mathbf{v}}$ denotes the velocity of the mass center G relative to the newtonian frame $Oxyz$. Observing that v_i^2 is equal to the scalar

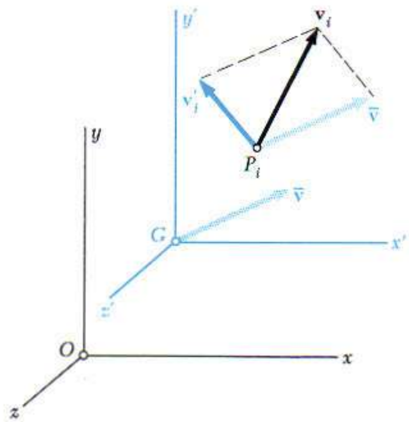


Fig. 14.7

product $\mathbf{v}_i \cdot \mathbf{v}_i$, we express as follows the kinetic energy T of the system relative to the newtonian frame $Oxyz$:

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2 = \frac{1}{2}\sum_{i=1}^{n} (m_i \mathbf{v}_i \cdot \mathbf{v}_i)$$

or, substituting for $\mathbf{v}_i$ from (14.22),

$$T = \frac{1}{2}\sum_{i=1}^{n} [m_i(\overline{\mathbf{v}} + \mathbf{v}'_i) \cdot (\overline{\mathbf{v}} + \mathbf{v}'_i)]$$

$$= \frac{1}{2}\left(\sum_{i=1}^{n} m_i\right)\bar{v}^2 + \overline{\mathbf{v}} \cdot \sum_{i=1}^{n} m_i\mathbf{v}'_i + \frac{1}{2}\sum_{i=1}^{n} m_i v_i'^2$$

The first sum represents the total mass m of the system. Recalling Eq. (14.13), we note that the second sum is equal to $m\overline{\mathbf{v}}'$ and thus to zero, since $\overline{\mathbf{v}}'$, which represents the velocity of G relative to the frame $Gx'y'z'$, is clearly zero. We therefore write

$$T = \tfrac{1}{2}m\bar{v}^2 + \frac{1}{2}\sum_{i=1}^{n} m_i v_i'^2 \tag{14.29}$$

This equation shows that the kinetic energy T of a system of particles may be obtained *by adding the kinetic energy of the mass center* G (assuming the entire mass concentrated at G) *and the kinetic energy of the system in its motion relative to the frame* $Gx'y'z'$.

14.7. Work-Energy Principle. Conservation of Energy for a System of Particles. The principle of work and energy may be applied to each particle P_i of a system of particles. We write

$$T_1 + U_{1\to2} = T_2 \tag{14.30}$$

for each particle P_i, where $U_{1\to2}$ represents the work done by the internal forces $\mathbf{f}_{ij}$ and the resultant external force $\mathbf{F}_i$ acting on P_i. Adding the kinetic energies of the various particles of the system, and considering the work of all the forces involved, we may apply Eq. (14.30) to the entire system. The quantities T_1 and T_2 now represent the kinetic energy of the entire system and may be computed from either Eq. (14.28) or Eq. (14.29). The quantity $U_{1\to2}$ represents the work of all the forces acting on the particles of the system. We should note that, while the internal forces $\mathbf{f}_{ij}$ and $\mathbf{f}_{ji}$ are equal and opposite, the work of these forces, in general, will not cancel out, since the particles P_i and P_j on which they act will, in general, undergo different displacements. Therefore, in computing $U_{1\to2}$, *we should consider the work of the internal forces* $\mathbf{f}_{ij}$ *as well as the work of the external forces* $\mathbf{F}_i$.

If all the forces acting on the particles of the system are conservative, Eq. (14.30) may be replaced by

$$T_1 + V_1 = T_2 + V_2 \qquad (14.31)$$

where V represents the potential energy associated with the internal and external forces acting on the particles of the system. Equation (14.31) expresses the principle of *conservation of energy* for the system of particles.

14.8. Principle of Impulse and Momentum for a System of Particles. Integrating Eqs. (14.10) and (14.11) in t from a time t_1 to a time t_2, we write

$$\sum \int_{t_1}^{t_2} \mathbf{F}\, dt = \mathbf{L}_2 - \mathbf{L}_1 \qquad (14.32)$$

$$\sum \int_{t_1}^{t_2} \mathbf{M}_O\, dt = (\mathbf{H}_O)_2 - (\mathbf{H}_O)_1 \qquad (14.33)$$

Recalling the definition of the linear impulse of a force given in Sec. 13.10, we observe that the integrals in Eq. (14.32) represent the linear impulses of the external forces acting on the particles of the system. We shall refer in a similar way to the integrals in Eq. (14.33) as the *angular impulses* about O of the external forces. Thus, Eq. (14.32) expresses that the sum of the linear impulses of the external forces acting on the system is equal to the change in linear momentum of the system. Similarly, Eq. (14.33) expresses that the sum of the angular impulses about O of the external forces is equal to the change in angular momentum about O of the system.

In order to understand the physical significance of Eqs. (14.32) and (14.33), we shall rearrange the terms in these equations and write

$$\mathbf{L}_1 + \sum \int_{t_1}^{t_2} \mathbf{F}\, dt = \mathbf{L}_2 \qquad (14.34)$$

$$(\mathbf{H}_O)_1 + \sum \int_{t_1}^{t_2} \mathbf{M}_O\, dt = (\mathbf{H}_O)_2 \qquad (14.35)$$

We have sketched in parts a and c of Fig. 14.8 the momenta of the particles of the system at times t_1 and t_2, respectively, and we have shown in part b of the same figure a vector equal to the sum of the linear impulses of the external forces and a couple of moment equal to the sum of the angular impulses about O of the external forces. For simplicity, the particles have been assumed to move in the plane of the figure, but the present discussion remains valid in the case of particles moving in space. Recalling from Eq. (14.6) that $\mathbf{L}$, by definition, is the resultant of the momenta $m_i\mathbf{v}_i$, we note that Eq. (14.34) expresses that the resultant of the vectors shown in parts a and b of Fig. 14.8 is equal to

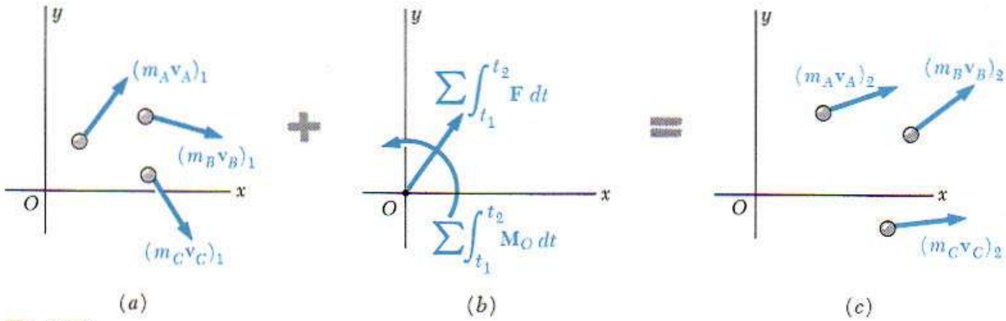


Fig. 14.8

the resultant of the vectors shown in part c of the same figure. Recalling from Eq. (14.7) that $\mathbf{H}_O$ is the moment resultant of the momenta $m_i\mathbf{v}_i$, we note that Eq. (14.35) similarly expresses that the moment resultant of the vectors in parts a and b of Fig. 14.8 is equal to the moment resultant of the vectors in part c. Together, Eqs. (14.34) and (14.35) thus express that *the momenta of the particles at time t_1 and the impulses of the external forces from t_1 to t_2 form a system of vectors equipollent to the system of the momenta of the particles at time t_2*. This has been indicated in Fig. 14.8 by the use of gray plus and equals signs.

If no external force acts on the particles of the system, the integrals in Eqs. (14.34) and (14.35) are zero, and these equations reduce to

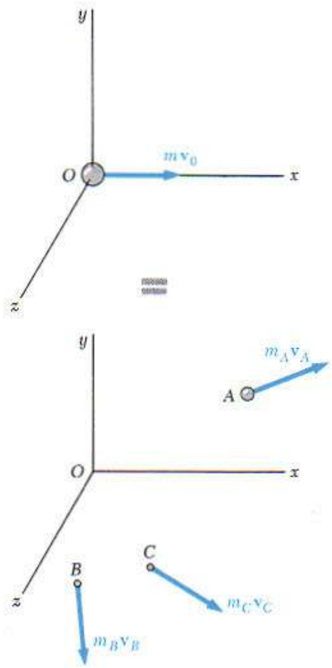
$$\mathbf{L}_1 = \mathbf{L}_2 \tag{14.36}$$

$$(\mathbf{H}_O)_1 = (\mathbf{H}_O)_2 \tag{14.37}$$

We thus check the result obtained in Sec. 14.5: If no external force acts on the particles of a system, the linear momentum and the angular momentum about O of the system of particles are conserved. The system of the initial momenta is equipollent to the system of the final momenta, and it follows that the angular momentum of the system of particles about *any* fixed point is conserved.

SAMPLE PROBLEM 14.3

For the 200-kg space vehicle of Sample Prob. 14.1, it is known that, at $t = 2.5$ s, the velocity of part A is $\mathbf{v}_A = (270 \text{ m/s})\mathbf{i} - (120 \text{ m/s})\mathbf{j} + (160 \text{ m/s})\mathbf{k}$ and the velocity of part B is parallel to the xz plane. Determine the velocity of part C.



Solution. Since there is no external force, the initial momentum $m\mathbf{v}_0$ is equipollent to the system of the final momenta. Equating first the sums of the vectors in both parts of the adjoining sketch, and then the sums of their moments about O, we write

$$\mathbf{L}_1 = \mathbf{L}_2: \qquad m\mathbf{v}_0 = m_A\mathbf{v}_A + m_B\mathbf{v}_B + m_C\mathbf{v}_C \qquad (1)$$

$$(\mathbf{H}_O)_1 = (\mathbf{H}_O)_2: \qquad 0 = \mathbf{r}_A \times m_A\mathbf{v}_A + \mathbf{r}_B \times m_B\mathbf{v}_B + \mathbf{r}_C \times m_C\mathbf{v}_C \qquad (2)$$

Recalling from Sample Prob. 14.1 that $\mathbf{v}_0 = (150 \text{ m/s})\mathbf{i}$,

$$m_A = 100 \text{ kg} \qquad m_B = 60 \text{ kg} \qquad m_C = 40 \text{ kg}$$

$$\mathbf{r}_A = (555 \text{ m})\mathbf{i} - (180 \text{ m})\mathbf{j} + (240 \text{ m})\mathbf{k}$$

$$\mathbf{r}_B = (255 \text{ m})\mathbf{i} - (120 \text{ m})\mathbf{k}$$

$$\mathbf{r}_C = (105 \text{ m})\mathbf{i} + (450 \text{ m})\mathbf{j} - (420 \text{ m})\mathbf{k}$$

and using the information given in the statement of this problem, we rewrite Eqs. (1) and (2) as follows:

$$200(150\mathbf{i}) = 100(270\mathbf{i} - 120\mathbf{j} + 160\mathbf{k}) + 60[(v_B)_x\mathbf{i} + (v_B)_z\mathbf{k}] + 40[(v_C)_x\mathbf{i} + (v_C)_y\mathbf{j} + (v_C)_z\mathbf{k}] \qquad (1')$$

$$0 = 100\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 555 & -180 & 240 \\ 270 & -120 & 160 \end{vmatrix} + 60\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 255 & 0 & -120 \\ (v_B)_x & 0 & (v_B)_z \end{vmatrix} + 40\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 105 & 450 & -420 \\ (v_C)_x & (v_C)_y & (v_C)_z \end{vmatrix} \qquad (2')$$

Equating to zero the coefficient of $\mathbf{j}$ in (1′) and the coefficients of $\mathbf{i}$ and $\mathbf{k}$ in (2′), we write, after reductions, the three scalar equations

$$(v_C)_y - 300 = 0$$

$$450(v_C)_z + 420(v_C)_y = 0$$

$$105(v_C)_y - 450(v_C)_x - 45\,000 = 0$$

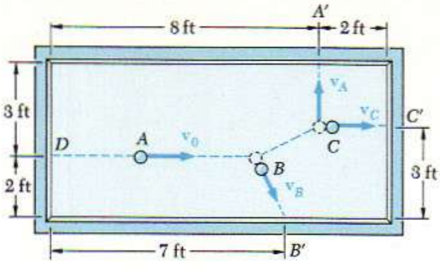
which yield, respectively,

$$(v_C)_y = 300 \qquad (v_C)_z = -280 \qquad (v_C)_x = -30$$

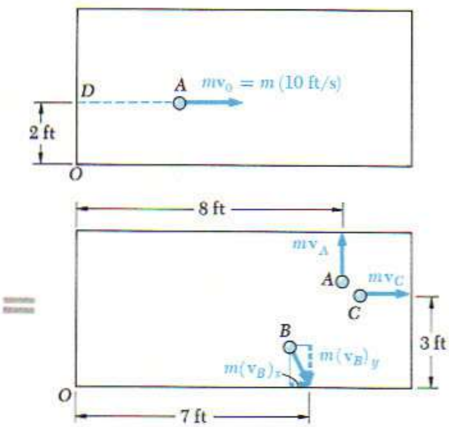
The velocity of part C is thus

$$\mathbf{v}_C = -(30 \text{ m/s})\mathbf{i} + (300 \text{ m/s})\mathbf{j} - (280 \text{ m/s})\mathbf{k} \quad ◄$$

SAMPLE PROBLEM 14.4



In a game of billiards, ball A is given an initial velocity $\mathbf{v}_0$ of magnitude $v_0 = 10$ ft/s along line DA parallel to the axis of the table. It hits ball B and then ball C, which are both at rest. Knowing that A and C hit the sides of the table squarely at points A' and C', respectively, that B hits the side obliquely at B', and assuming frictionless surfaces and perfectly elastic impacts, determine the velocities $\mathbf{v}_A$, $\mathbf{v}_B$, and $\mathbf{v}_C$ with which the balls hit the sides of the table. (*Remark.* In this sample problem and in several of the problems which follow, the billiard balls are assumed to be particles moving freely in a horizontal plane, rather than the rolling and sliding spheres they actually are.)



Solution. *Conservation of Momentum.* Since there is no external force, the initial momentum $m\mathbf{v}_0$ is equipollent to the system of momenta after the two collisions (and before any of the balls hits the side of the table). Referring to the adjoining sketch, we write

$\rightarrow x$ components: $\quad m(10\text{ ft/s}) = m(v_B)_x + mv_C \qquad (1)$

$+\uparrow y$ components: $\quad 0 = mv_A - m(v_B)_y \qquad (2)$

$+\circlearrowleft$ moments about O: $\quad -(2\text{ ft})m(10\text{ ft/s}) = (8\text{ ft})mv_A - (7\text{ ft})m(v_B)_y - (3\text{ ft})mv_C \qquad (3)$

Solving the three equations for v_A, $(v_B)_x$, and $(v_B)_y$ in terms of v_C:

$$v_A = (v_B)_y = 3v_C - 20 \qquad (v_B)_x = 10 - v_C \qquad (4)$$

Conservation of Energy. Since the surfaces are frictionless and the impacts are perfectly elastic, the initial kinetic energy $\frac{1}{2}mv_0^2$ is equal to the final kinetic energy of the system:

$$\tfrac{1}{2}mv_0^2 = \tfrac{1}{2}m_Av_A^2 + \tfrac{1}{2}m_Bv_B^2 + \tfrac{1}{2}m_Cv_C^2$$

$$v_A^2 + (v_B)_x^2 + (v_B)_y^2 + v_C^2 = (10\text{ ft/s})^2 \qquad (5)$$

Substituting for v_A, $(v_B)_x$, and $(v_B)_y$ from (4) into (5), we have

$$2(3v_C - 20)^2 + (10 - v_C)^2 + v_C^2 = 100$$

$$20v_C^2 - 260v_C + 800 = 0$$

Solving for v_C, we find $v_C = 5$ ft/s and $v_C = 8$ ft/s. Since only the second root yields a positive value for v_A after substitution into Eqs. (4), we conclude that $v_C = 8$ ft/s and

$$v_A = (v_B)_y = 3(8) - 20 = 4\text{ ft/s} \qquad (v_B)_x = 10 - 8 = 2\text{ ft/s}$$

$\mathbf{v}_A = 4\text{ ft/s}\uparrow \qquad \mathbf{v}_B = 4.47\text{ ft/s} \measuredangle 63.4^\circ \qquad \mathbf{v}_C = 8\text{ ft/s}\rightarrow$ ◄

PROBLEMS

14.21 In Prob. 14.13, determine the amount of energy lost as the arrow hits the game bird.

14.22 In Prob. 14.14, determine the percentage of the initial kinetic energy lost due to the impacts among the three balls.

14.23 In Prob. 14.15, determine the work done by the internal forces during the explosion.

14.24 In Prob. 14.16, determine the percentage of the initial kinetic energy lost due to the collisions between the alpha particle and the two oxygen nuclei and check that, taking into account the numerical accuracy of the given data and of the calculations, the result obtained suggests conservation of energy.

14.25 A 5-lb weight slides without friction on the xy plane. At $t = 0$ it passes through the origin with a velocity $\mathbf{v}_0 = (20\text{ ft/s})\mathbf{i}$. Internal springs then separate the weight into the three parts shown. Knowing that, at $t = 3\text{ s}$, $\mathbf{r}_A = (42\text{ ft})\mathbf{i} + (27\text{ ft})\mathbf{j}$ and $\mathbf{r}_B = (60\text{ ft})\mathbf{i} - (6\text{ ft})\mathbf{j}$, that $\mathbf{v}_A = (14\text{ ft/s})\mathbf{i} + (9\text{ ft/s})\mathbf{j}$, and that $\mathbf{v}_B$ is parallel to the x axis, determine the corresponding position and velocity of part C.

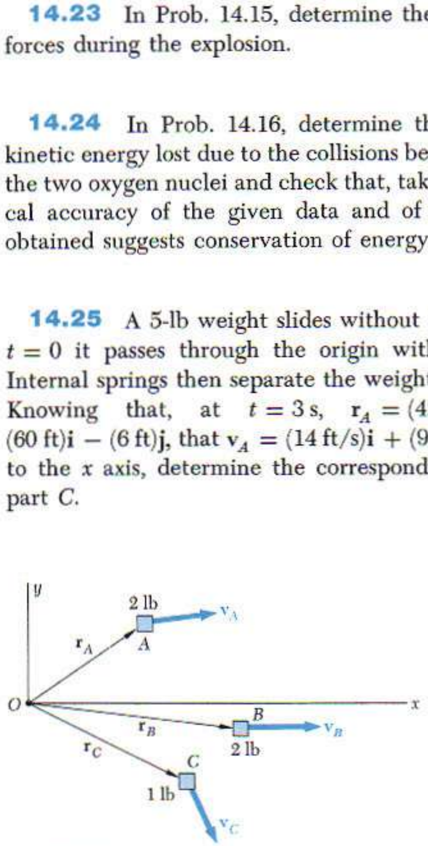


Fig. P14.25

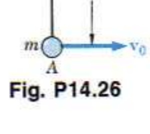


Fig. P14.26

14.26 Two small spheres A and B, respectively of mass m and $3m$, are connected by a rigid rod of length l and negligible mass. The two spheres are resting on a horizontal, frictionless surface when A is suddenly given the velocity $\mathbf{v}_0 = v_0\mathbf{i}$. Determine (a) the linear momentum of the system and its angular momentum about its mass center G, (b) the velocities of A and B after the rod AB has rotated through 90°, (c) the velocities of A and B after the rod AB has rotated through 180°.

14.27 A 240-kg space vehicle traveling with the velocity $\mathbf{v}_0 = (500 \text{ m/s})\mathbf{k}$ passes through the origin O at $t = 0$. Explosive charges then separate the vehicle into three parts, A, B, and C, of mass 40 kg, 80 kg, and 120 kg, respectively. Knowing that at $t = 3$ s the positions of the three parts are, respectively, $A(150, 150, 1350)$, $B(375, 825, 2025)$, and $C(-300, -600, 1200)$, where the coordinates are expressed in meters, that the velocity of C is $\mathbf{v}_C = -(100 \text{ m/s})\mathbf{i} - (200 \text{ m/s})\mathbf{j} + (400 \text{ m/s})\mathbf{k}$, and that the y component of the velocity of B is $+350$ m/s, determine the velocity of part A.

14.28 In the scattering experiment of Prob. 14.16, it is known that the alpha particle is projected from $A_0(300, 0, 300)$ and that it collides with the oxygen nucleus C at $Q(240, 200, 100)$, where all coordinates are expressed in millimeters. Determine the coordinates of point B_0 where the original path of nucleus B intersects the xz plane. (*Hint.* Express that the angular momentum of the three particles about Q is conserved.)

14.29 In a game of billiards, ball A is moving with the velocity $\mathbf{v}_0 = v_0\mathbf{i}$ when it strikes balls B and C which are at rest side by side. After the collision, the three balls are observed to move in the directions shown. Assuming frictionless surfaces and perfectly elastic impacts (i.e., conservation of energy), determine the magnitudes of the velocities $\mathbf{v}_A$, $\mathbf{v}_B$, and $\mathbf{v}_C$ in terms of v_0 and θ.

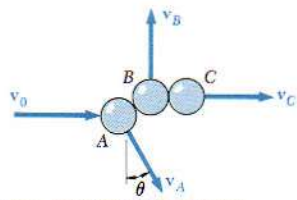


Fig. P14.29 and P14.30

14.30 In a game of billiards, ball A is moving with the velocity $\mathbf{v}_0 = (3 \text{ m/s})\mathbf{i}$ when it strikes balls B and C which are at rest side by side. After the collision, the three balls are observed to move in the directions shown, with $\theta = 30°$. Assuming frictionless surfaces and perfectly elastic impacts (i.e., conservation of energy), determine the magnitudes of the velocities $\mathbf{v}_A$, $\mathbf{v}_B$, and $\mathbf{v}_C$.

14.31 In a scattering experiment, an alpha particle A is projected with the velocity $\mathbf{u}_0 = (960 \text{ m/s})\mathbf{i} + (1200 \text{ m/s})\mathbf{j} + (1280 \text{ m/s})\mathbf{k}$ into a stream of oxygen nuclei moving with the common velocity $\mathbf{v}_0 = v_0\mathbf{j}$. After colliding successively with the nuclei B and C, particle A is observed to move in the direction defined by the unit vector $\boldsymbol{\lambda}_A = -0.463\mathbf{i} + 0.853\mathbf{j} - 0.241\mathbf{k}$, while nuclei B and C are observed to move in directions defined, respectively, by $\boldsymbol{\lambda}_B = 0.939\mathbf{j} + 0.344\mathbf{k}$ and $\boldsymbol{\lambda}_C = 0.628\mathbf{i} + 0.778\mathbf{j}$. Knowing that the mass of an oxygen nucleus is four times that of an alpha particle, and assuming conservation of energy, determine (*a*) the speed v_0 of the oxygen nuclei before the collisions, (*b*) the speed of each of the three particles after the collisions.

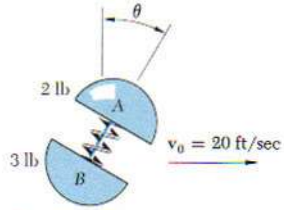


Fig. P14.32

14.32 When the cord connecting particles A and B is severed, the compressed spring causes the particles to fly apart (the spring is not connected to the particles). The potential energy of the compressed spring is known to be 20 ft · lb and the assembly has an initial velocity $\mathbf{v}_0$ as shown. If the cord is severed when $\theta = 30°$, determine the resulting velocity of each particle.

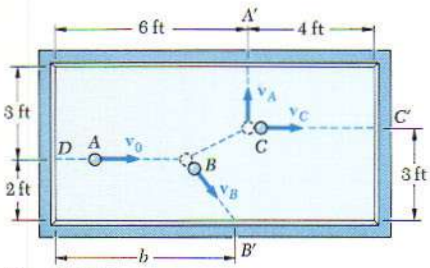


Fig. P14.33

14.33 In a game of billiards, ball A is given an initial velocity $\mathbf{v}_0$ along line DA parallel to the axis of the table. It hits ball B and then ball C, which are at rest. Knowing that A and C hit the sides of the table squarely at points A' and C', respectively, with velocities of magnitude $v_A = 4$ ft/s and $v_C = 6$ ft/s, and assuming frictionless surfaces and perfectly elastic impacts (i.e., conservation of energy), determine (*a*) the initial velocity $\mathbf{v}_0$ of ball A, (*b*) the velocity $\mathbf{v}_B$ of ball B, (*c*) the point B' where B hits the side of the table.

14.34 Solve Prob. 14.33 if $v_A = 6$ ft/s and $v_C = 4$ ft/s.

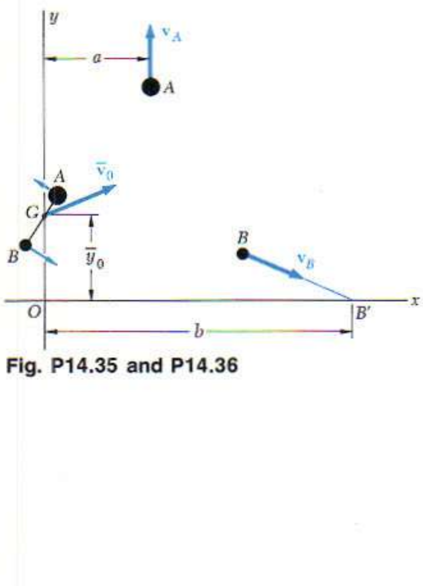


Fig. P14.35 and P14.36

14.35 Two small disks A and B, of mass 2 kg and 1 kg, respectively, may slide on a horizontal and frictionless surface. They are connected by a cord of negligible mass and spin about their mass center G. At $t = 0$, the coordinates of G are $\bar{x}_0 = 0$, $\bar{y}_0 = 1.6$ m, and its velocity is $\bar{\mathbf{v}}_0 = (1.5\text{ m/s})\mathbf{i} + (1.2\text{ m/s})\mathbf{j}$. Shortly thereafter, the cord breaks and disk A is observed to move along a path parallel to the y axis at a distance $a = 1.96$ m from that axis. Knowing that, initially, the angular momentum of the two disks about G was $3\text{ kg} \cdot \text{m}^2/\text{s}$ counterclockwise and that their kinetic energy relative to a centroidal frame was 18.75 J, determine (*a*) the velocities of A and B after the cord breaks, (*b*) the abscissa b of the point B' where the path of B intersects the x axis.

14.36 Two small disks A and B, of mass 2 kg and 1 kg, respectively, may slide on a horizontal and frictionless surface. They are connected by a cord of negligible mass and spin about their mass center G. At $t = 0$, G is moving with the velocity $\bar{\mathbf{v}}_0$ and its coordinates are $\bar{x}_0 = 0$, $\bar{y}_0 = 1.89$ m. Shortly thereafter, the cord breaks and disk A is observed to move with the velocity $\mathbf{v}_A = (5\text{ m/s})\mathbf{j}$ in a straight line and at a distance $a = 2.56$ m from the y axis, while B moves with the velocity $\mathbf{v}_B = (7.2\text{ m/s})\mathbf{i} - (4.6\text{ m/s})\mathbf{j}$ along a path intersecting the x axis at a distance $b = 7.48$ m from the origin O. Determine (*a*) the initial velocity $\bar{\mathbf{v}}_0$ of the mass center G of the two disks, (*b*) the angular momentum $\mathbf{H}_G$ of the system about G and its kinetic energy relative to a centroidal frame before the cord broke, (*c*) the length of the cord initially connecting the two disks, (*d*) the rate in rad/s at which the disks were spinning about G.

***14.9. Variable Systems of Particles.** All the systems of particles considered so far consisted of well-defined particles. These systems did not gain or lose any particles during their motion. In a large number of engineering applications, however, it is necessary to consider *variable systems of particles,* i.e., systems which are continuously gaining or losing particles, or doing both at the same time. Consider, for example, a hydraulic turbine. Its analysis involves the determination of the forces exerted by a stream of water on rotating blades, and we note that the particles of water in contact with the blades form an everchanging system which continuously acquires and loses particles. Rockets furnish another example of variable systems, since their propulsion depends upon the continuous ejection of fuel particles.

We recall that all the kinetics principles established so far were derived for constant systems of particles, which neither gain nor lose particles. We must therefore find a way to reduce the analysis of a variable system of particles to that of an auxiliary constant system. The procedure to follow is indicated in Secs. 14.10 and 14.11 for two broad categories of applications.

***14.10. Steady Stream of Particles.** Consider a steady stream of particles, such as a stream of water diverted by a fixed vane or a flow of air through a duct or through a blower. In order to determine the resultant of the forces exerted on the particles in contact with the vane, duct, or blower, we isolate these particles and denote by S the system thus defined (Fig. 14.9). We observe that S is a variable system of particles, since it continuously gains particles flowing in and loses an equal number of particles flowing out. Therefore, the kinetics principles that have been established so far cannot be directly applied to S.

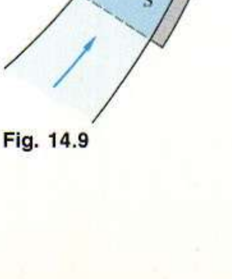


Fig. 14.9

However, we may easily define an auxiliary system of particles which does remain constant for a short interval of time Δt. Consider at time t the system S *plus* the particles which will enter S during the interval of time Δt (Fig. 14.10*a*). Next, consider at time $t + \Delta t$ the system S *plus* the particles which have left S during the interval Δt (Fig. 14.10*c*). Clearly, *the same particles are involved in both cases,* and we may apply to these particles the principle of impulse and momentum. Since the total mass m of the system S remains constant, the particles entering the system and those leaving the system in the time Δt must have the same mass Δm. Denoting by $\mathbf{v}_A$ and $\mathbf{v}_B$, respectively, the velocities of the particles entering S at A and leaving S at B, we represent the momentum of the particles entering S by $(\Delta m)\mathbf{v}_A$ (Fig. 14.10*a*) and the momentum of the particles leaving S by $(\Delta m)\mathbf{v}_B$ (Fig. 14.10*c*). We also represent the momenta $m_i\mathbf{v}_i$ of the particles forming S and the impulses of the forces exerted on S by the appropriate vectors, and indicate by gray plus and equals signs that the system of the momenta and impulses in parts a and b of Fig. 14.10 is equipollent to the system of the momenta in part c of the same figure.

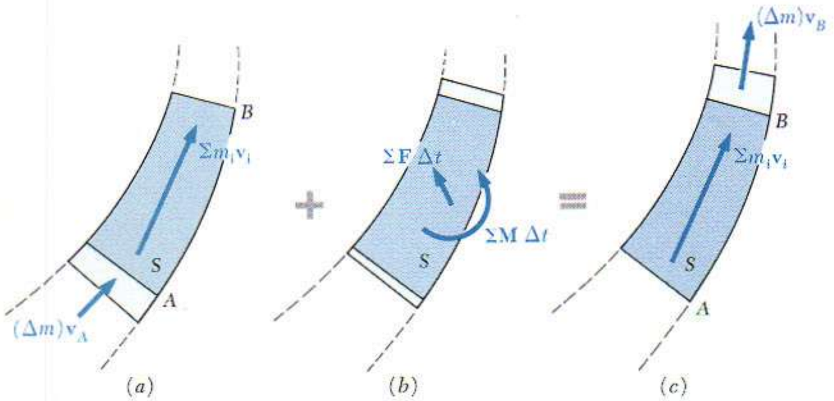


Fig. 14.10

Since the resultant $\Sigma m_i\mathbf{v}_i$ of the momenta of the particles of S is found on both sides of the equals sign, it may be omitted. We conclude that *the system formed by the momentum* $(\Delta m)\mathbf{v}_A$ *of the particles entering* S *in the time* Δt *and the impulses of the forces exerted on* S *during that time is equipollent to the momentum* $(\Delta m)\mathbf{v}_B$ *of the particles leaving* S *in the same time* Δt. We may therefore write

$$(\Delta m)\mathbf{v}_A + \Sigma\mathbf{F}\,\Delta t = (\Delta m)\mathbf{v}_B \tag{14.38}$$

A similar equation may be obtained by taking the moments of the vectors involved (see Sample Prob. 14.5). Dividing all terms of Eq. (14.38) by Δt and letting Δt approach zero, we obtain at the limit

$$\Sigma \mathbf{F} = \frac{dm}{dt}(\mathbf{v}_B - \mathbf{v}_A) \tag{14.39}$$

where $\mathbf{v}_B - \mathbf{v}_A$ represents the difference between the *vectors* $\mathbf{v}_B$ and $\mathbf{v}_A$.

If SI units are used, dm/dt is expressed in kg/s and the velocities in m/s; we check that both members of Eq. (14.39) are expressed in the same units (newtons). If U.S. customary units are used, dm/dt must be expressed in slugs/s and the velocities in ft/s; we check again that both members of the equation are expressed in the same units (pounds).†

The principle we have established may be used to analyze a large number of engineering applications. Some of the most common are indicated below.

Fluid Stream Diverted by a Vane. If the vane is fixed, the method of analysis given above may be applied directly to find the force $\mathbf{F}$ exerted by the vane on the stream. We note that $\mathbf{F}$ is the only force which needs to be considered since the pressure in the stream is constant (atmospheric pressure). The force exerted by the stream on the vane will be equal and opposite to $\mathbf{F}$. If the vane moves with a constant velocity, the stream is not steady. However, it will appear steady to an observer moving with the vane. We should therefore choose a system of axes moving with the vane. Since this system of axes is not accelerated, Eq. (14.38) may still be used, but $\mathbf{v}_A$ and $\mathbf{v}_B$ must be replaced by the *relative velocities* of the stream with respect to the vane (see Sample Prob. 14.6).

Fluid Flowing through a Pipe. The force exerted by the fluid on a pipe transition such as a bend or a contraction may be determined by considering the system of particles S in contact with the transition. Since, in general, the pressure in the flow will vary, we should also consider the forces exerted on S by the adjoining portions of the fluid.

† It is often convenient to express the mass rate of flow dm/dt as the product ρQ, where ρ is the density of the stream (mass per unit volume) and Q its volume rate of flow (volume per unit time). If SI units are used, ρ is expressed in kg/m^3 (for instance, $\rho = 1000$ kg/m^3 for water) and Q in m^3/s. However, if U.S. customary units are used, ρ will generally have to be computed from the corresponding specific weight γ (weight per unit volume), $\rho = \gamma/g$. Since γ is expressed in lb/ft^3 (for instance, $\gamma = 62.4$ lb/ft^3 for water), ρ is obtained in slugs/ft^3. The volume rate of flow Q is expressed in ft^3/s.

Jet Engine. In a jet engine, air enters with no velocity through the front of the engine and leaves through the rear with a high velocity. The energy required to accelerate the air particles is obtained by burning fuel. While the exhaust gases contain burned fuel, the mass of the fuel is small compared with the mass of the air flowing through the engine and usually may be neglected. Thus, the analysis of a jet engine reduces to that of an air stream. This stream may be considered as a steady stream if all velocities are measured with respect to the airplane. The air stream shall be assumed, therefore, to enter the engine with a velocity $\mathbf{v}$ of magnitude equal to the speed of the airplane and to leave with a velocity $\mathbf{u}$ equal to the relative velocity of the exhaust gases (Fig. 14.11). Since the intake and exhaust

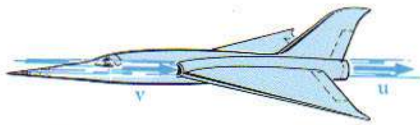


Fig. 14.11

pressures are nearly atmospheric, the only external force which needs to be considered is the force exerted by the engine on the air stream. This force is equal and opposite to the thrust.†

Fan. We consider the system of particles S shown in Fig. 14.12. The velocity $\mathbf{v}_A$ of the particles entering the system is assumed equal to zero, and the velocity $\mathbf{v}_B$ of the particles leaving the system is the velocity of the *slipstream.* The rate of flow may be obtained by multiplying v_B by the cross-sectional

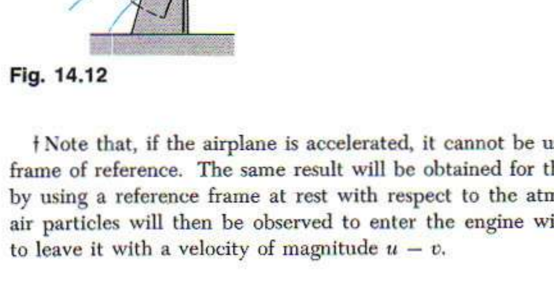


Fig. 14.12

† Note that, if the airplane is accelerated, it cannot be used as a newtonian frame of reference. The same result will be obtained for the thrust, however, by using a reference frame at rest with respect to the atmosphere, since the air particles will then be observed to enter the engine with no velocity and to leave it with a velocity of magnitude $u - v$.

area of the slipstream. Since the pressure all around S is atmospheric, the only external force acting on S is the thrust of the fan.

Airplane Propeller. In order to obtain a steady stream of air, velocities should be measured with respect to the airplane. Thus, the air particles will be assumed to enter the system with a velocity $\mathbf{v}$ of magnitude equal to the speed of the airplane and to leave with a velocity $\mathbf{u}$ equal to the relative velocity of the slipstream.

***14.11. Systems Gaining or Losing Mass.** We shall now analyze a different type of variable system of particles, namely, a system which gains mass by continuously absorbing particles or loses mass by continuously expelling particles. Consider the system S shown in Fig. 14.13. Its mass, equal to m at the instant t, increases by Δm in the interval of time Δt. In order to apply the principle of impulse and momentum to the analysis of this system, we must consider at time t the system

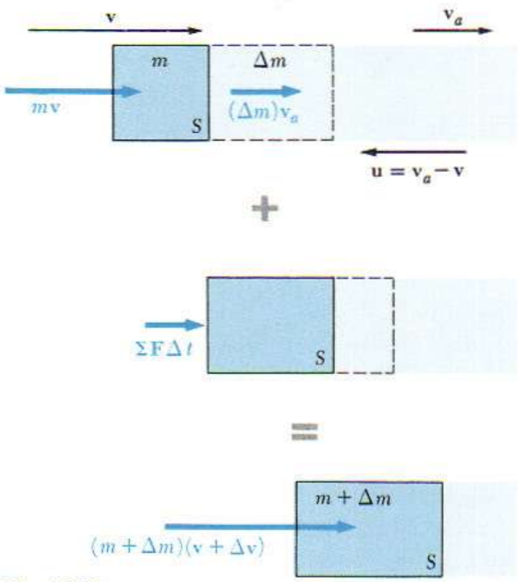


Fig. 14.13

S *plus* the particles of mass Δm which S absorbs during the time interval Δt. The velocity of S at time t is denoted by $\mathbf{v}$, and its velocity at time $t + \Delta t$ is denoted by $\mathbf{v} + \Delta\mathbf{v}$, while the absolute velocity of the particles which are absorbed is denoted by $\mathbf{v}_a$. Applying the principle of impulse and momentum, we write

$$m\mathbf{v} + (\Delta m)\mathbf{v}_a + \Sigma\mathbf{F}\,\Delta t = (m + \Delta m)(\mathbf{v} + \Delta\mathbf{v})$$

Solving for the sum $\Sigma \mathbf{F}\,\Delta t$ of the impulses of the external forces acting on S (excluding the forces exerted by the particles being absorbed), we have

$$\Sigma \mathbf{F}\,\Delta t = m\,\Delta \mathbf{v} + \Delta m(\mathbf{v} - \mathbf{v}_a) + (\Delta m)(\Delta \mathbf{v}) \qquad (14.40)$$

Introducing the *relative velocity* $\mathbf{u}$ with respect to S of the particles which are absorbed, we write $\mathbf{u} = \mathbf{v}_a - \mathbf{v}$ and note, since $v_a < v$, that the relative velocity $\mathbf{u}$ is directed to the left, as shown in Fig. 14.13. Neglecting the last term in Eq. (14.40), which is of the second order, we write

$$\Sigma \mathbf{F}\,\Delta t = m\,\Delta \mathbf{v} - (\Delta m)\mathbf{u}$$

Dividing through by Δt and letting Δt approach zero, we have at the limit†

$$\Sigma \mathbf{F} = m\frac{d\mathbf{v}}{dt} - \frac{dm}{dt}\mathbf{u} \qquad (14.41)$$

Rearranging the terms, we obtain the equation

$$\Sigma \mathbf{F} + \frac{dm}{dt}\mathbf{u} = m\frac{d\mathbf{v}}{dt} \qquad (14.42)$$

which shows that the action on S of the particles being absorbed is equivalent to a thrust of magnitude $(dm/dt)u$ which tends to slow down the motion of S, since the relative velocity $\mathbf{u}$ of the particles is directed to the left. If SI units are used, dm/dt is expressed in kg/s, the relative velocity u in m/s, and the corresponding thrust in newtons. If U.S. customary units are used, dm/dt must be expressed in slugs/s and u in ft/s; the corresponding thrust will then be expressed in pounds.‡

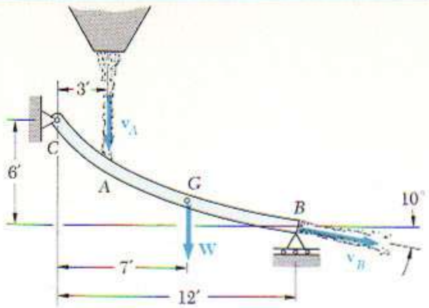
The equations obtained may also be used to determine the motion of a system S losing mass. In this case, the rate of change of mass is negative, and the action on S of the particles being expelled is equivalent to a thrust in the direction of $-\mathbf{u}$, that is, in the direction opposite to that in which the particles are being expelled. A *rocket* represents a typical case of a system continuously losing mass (see Sample Prob. 14.7).

† When the absolute velocity $\mathbf{v}_a$ of the particles absorbed is zero, we have $\mathbf{u} = -\mathbf{v}$, and formula (14.41) becomes

$$\Sigma \mathbf{F} = \frac{d}{dt}(m\mathbf{v})$$

Comparing the formula obtained to Eq. (12.3) of Sec. 12.2, we observe that Newton's second law may be applied to a system gaining mass, *provided that the particles absorbed are initially at rest.* It may also be applied to a system losing mass, *provided that the velocity of the particles expelled is zero* with respect to the frame of reference selected.

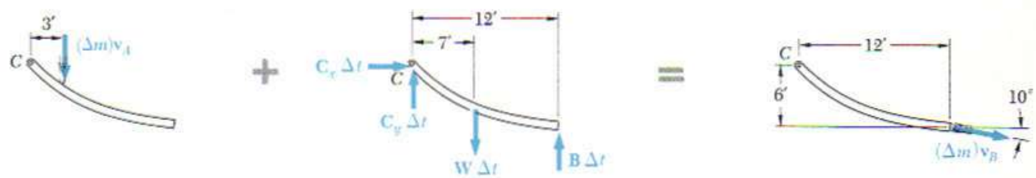
‡ See footnote on page 637.



SAMPLE PROBLEM 14.5

Grain falls from a hopper onto a chute CB at the rate of 240 lb/s. It hits the chute at A with a velocity of 20 ft/s and leaves at B with a velocity of 15 ft/s, forming an angle of 10° with the horizontal. Knowing that the combined weight of the chute and of the grain it supports is a force $\mathbf{W}$ of magnitude 600 lb applied at G, determine the reaction at the roller-support B and the components of the reaction at the hinge C.

Solution. We apply the principle of impulse and momentum for the time interval Δt to the system consisting of the chute, the grain it supports, and the amount of grain which hits the chute in the interval Δt. Since the chute does not move, it has no momentum. We also note that the sum $\Sigma m_i \mathbf{v}_i$ of the momenta of the particles supported by the chute is the same at t and $t + \Delta t$ and thus may be omitted.



Since the system formed by the momentum $(\Delta m)\mathbf{v}_A$ and the impulses is equipollent to the momentum $(\Delta m)\mathbf{v}_B$, we write

$\xrightarrow{+}$ x components: $$C_x\,\Delta t = (\Delta m)v_B \cos 10° \qquad (1)$$

$+\uparrow y$ components: $$-(\Delta m)v_A + C_y\,\Delta t - W\,\Delta t + B\,\Delta t = -(\Delta m)v_B \sin 10° \qquad (2)$$

$+\circlearrowleft$ moments about C: $$-3(\Delta m)v_A - 7(W\,\Delta t) + 12(B\,\Delta t) = 6(\Delta m)v_B \cos 10° - 12(\Delta m)v_B \sin 10° \qquad (3)$$

Using the given data, $W = 600$ lb, $v_A = 20$ ft/s, $v_B = 15$ ft/s, $\Delta m/\Delta t = 240/32.2 = 7.45$ slugs/s, and solving Eq. (3) for B and Eq. (1) for C_x:

$$12B = 7(600) + 3(7.45)(20) + 6(7.45)(15)(\cos 10° - 2\sin 10°)$$

$$12B = 5075 \qquad B = 423 \text{ lb} \qquad \mathbf{B} = 423 \text{ lb}\uparrow$$

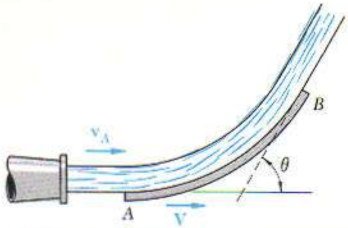
$$C_x = (7.45)(15)\cos 10° = 110.1 \text{ lb} \qquad \mathbf{C}_x = 110.1 \text{ lb}\rightarrow$$

Substituting for B and solving Eq. (2) for C_y:

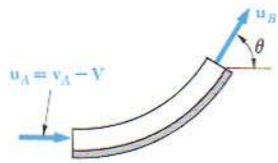
$$C_y = 600 - 423 + (7.45)(20 - 15\sin 10°) = 307 \text{ lb}$$

$$\mathbf{C}_y = 307 \text{ lb}\uparrow$$

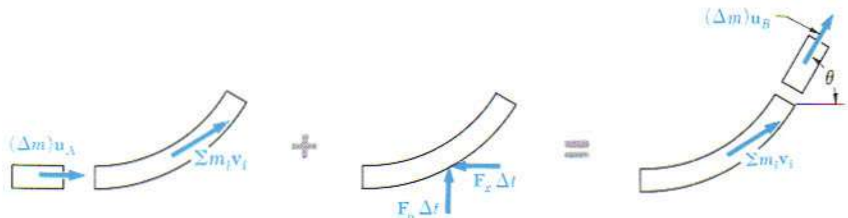
SAMPLE PROBLEM 14.6



A nozzle discharges a stream of water of cross-sectional area A with a velocity $\mathbf{v}_A$. The stream is deflected by a *single* blade which moves to the right with a constant velocity $\mathbf{V}$. Assuming that the water moves along the blade at constant speed, determine (*a*) the components of the force $\mathbf{F}$ exerted by the blade on the stream, (*b*) the velocity $\mathbf{V}$ for which maximum power is developed.



a. **Components of Force Exerted on Stream.** We choose a coordinate system which moves with the blade at a constant velocity $\mathbf{V}$. The particles of water strike the blade with a relative velocity $\mathbf{u}_A = \mathbf{v}_A - \mathbf{V}$ and leave the blade with a relative velocity $\mathbf{u}_B$. Since the particles move along the blade at a constant speed, the relative velocities $\mathbf{u}_A$ and $\mathbf{u}_B$ have the same magnitude u. Denoting the density of water by ρ, the mass of the particles striking the blade during the time interval Δt is $\Delta m = A\rho(v_A - V)\,\Delta t$; an equal mass of particles leaves the blade during Δt. We apply the principle of impulse and momentum to the system formed by the particles in contact with the blade and by those striking the blade in the time Δt.



Recalling that $\mathbf{u}_A$ and $\mathbf{u}_B$ have the same magnitude u, and omitting the momentum $\Sigma m_i\mathbf{v}_i$ which appears on both sides, we write

$\xrightarrow{+} x$ components: $\quad (\Delta m)u - F_x\,\Delta t = (\Delta m)u\cos\theta$

$+\uparrow y$ components: $\quad +F_y\,\Delta t = (\Delta m)u\sin\theta$

Substituting $\Delta m = A\rho(v_A - V)\,\Delta t$ and $u = v_A - V$, we obtain

$$\mathbf{F}_x = A\rho(v_A - V)^2(1 - \cos\theta) \leftarrow \qquad \mathbf{F}_y = A\rho(v_A - V)^2\sin\theta \uparrow \quad ◄$$

b. **Velocity of Blade for Maximum Power.** The power is obtained by multiplying the velocity V of the blade by the component F_x of the force exerted by the stream on the blade.

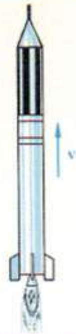
$$\text{Power} = F_x V = A\rho(v_A - V)^2(1 - \cos\theta)V$$

Differentiating the power with respect to V and setting the derivative equal to zero, we obtain

$$\frac{d(\text{power})}{dV} = A\rho(v_A^2 - 4v_A V + 3V^2)(1 - \cos\theta) = 0$$

$$V = v_A \qquad V = \tfrac{1}{3}v_A \quad \text{For maximum power } \mathbf{V} = \tfrac{1}{3}v_A \rightarrow \quad ◄$$

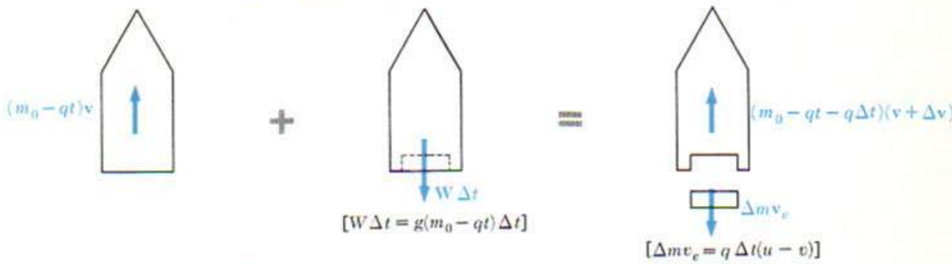
Note. These results are valid only when a *single* blade deflects the stream. Different results are obtained when a series of blades deflects the stream, as in a Pelton-wheel turbine. (See Prob. 14.90.)



SAMPLE PROBLEM 14.7

A rocket of initial mass m_0 (including shell and fuel) is fired vertically at time $t = 0$. The fuel is consumed at a constant rate $q = dm/dt$ and is expelled at a constant speed u relative to the rocket. Derive an expression for the velocity of the rocket at time t, neglecting the resistance of the air.

Solution. At time t, the mass of the rocket shell and remaining fuel is $m = m_0 - qt$, and the velocity is $\mathbf{v}$. During the time interval Δt, a mass of fuel $\Delta m = q\,\Delta t$ is expelled with a speed u relative to the rocket. Denoting by $\mathbf{v}_e$ the absolute velocity of the expelled fuel, we apply the principle of impulse and momentum between time t and time $t + \Delta t$.



We write

$$(m_0 - qt)v - g(m_0 - qt)\,\Delta t = (m_0 - qt - q\,\Delta t)(v + \Delta v) - q\,\Delta t(u - v)$$

Dividing through by Δt, and letting Δt approach zero, we obtain

$$-g(m_0 - qt) = (m_0 - qt)\frac{dv}{dt} - qu$$

Separating variables and integrating from $t = 0$, $v = 0$ to $t = t$, $v = v$,

$$dv = \left(\frac{qu}{m_0 - qt} - g\right)dt \qquad \int_0^v dv = \int_0^t \left(\frac{qu}{m_0 - qt} - g\right)dt$$

$$v = [-u \ln(m_0 - qt) - gt]_0^t \qquad v = u \ln \frac{m_0}{m_0 - qt} - gt \quad \blacktriangleleft$$

Remark. The mass remaining at time t_f, after all the fuel has been expended, is equal to the mass of the rocket shell $m_s = m_0 - qt_f$, and the maximum velocity attained by the rocket is $v_m = u \ln(m_0/m_s) - gt_f$. Assuming that the fuel is expelled in a relatively short period of time, the term gt_f is small and we have $v_m \approx u \ln(m_0/m_s)$. In order to escape the gravitational field of the earth, a rocket must reach a velocity of 11 180 m/s. Assuming $u = 2200$ m/s and $v_m = 11\,180$ m/s, we obtain $m_0/m_s = 161$. Thus, to project each kilogram of the rocket shell into space, it is necessary to consume more than 161 kg of fuel if a propellant yielding $u = 2200$ m/s is used.

PROBLEMS

Note. In the following problems use $\rho = 1000\ \text{kg/m}^3$ for the density of water in SI units, and $\gamma = 62.4\ \text{lb/ft}^3$ for its specific weight in U.S. customary units. (See footnote on page 637.)

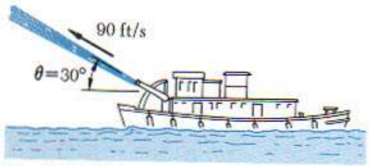


Fig. P14.37

14.37 A hose discharges 2000 gal/min from the stern of a 20-ton fireboat. If the velocity of the water stream is 90 ft/s, determine the reaction on the boat.

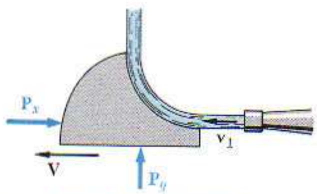


Fig. P14.38 and P14.39

14.38 A stream of water of cross-sectional area A and velocity $\mathbf{v}_1$ strikes the curved surface of a block which is held motionless ($V = 0$) by the forces $\mathbf{P}_x$ and $\mathbf{P}_y$. Determine the magnitudes of $\mathbf{P}_x$ and $\mathbf{P}_y$ when $A = 500\ \text{mm}^2$ and $v_1 = 40\ \text{m/s}$.

14.39 A stream of water of cross-sectional area A and velocity $\mathbf{v}_1$ strikes the curved surface of a block which moves to the left with a velocity $\mathbf{V}$. Determine the magnitudes of the forces $\mathbf{P}_x$ and $\mathbf{P}_y$ required to hold the block when $A = 3\ \text{in}^2$, $v_1 = 90\ \text{ft/s}$, and $V = 25\ \text{ft/s}$.

14.40 Sand is discharged at the rate m (kg/s) from a conveyor belt moving with a velocity $\mathbf{v}_0$. The sand is deflected by a plate at B so that it falls in a vertical stream. After falling a distance h, the sand is again deflected as shown by a curved plate at C. Neglecting the friction between the sand and the plates, determine the force required to hold each plate in the position shown.

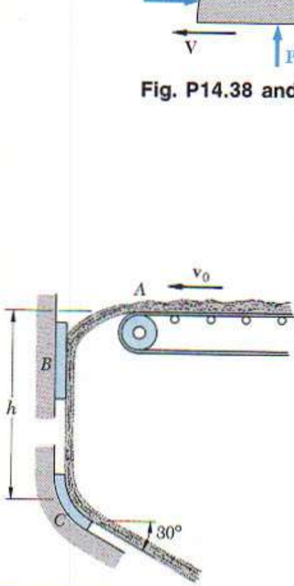


Fig. P14.40

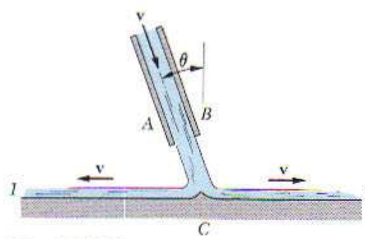


Fig. P14.41

14.41 Water flows in a continuous sheet from between two plates A and B with a velocity $\mathbf{v}$. The stream is split into two equal streams 1 and 2 by a vane attached to plate C. Denoting the total rate of flow by Q, determine the force exerted by the stream on plate C.

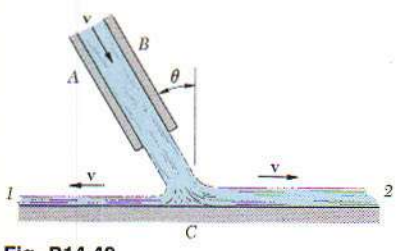


Fig. P14.42

14.42 Water flows in a continuous sheet from between two plates A and B with a velocity $\mathbf{v}$. The stream is split into two parts by a smooth horizontal plate C. Denoting the total rate of flow by Q, determine the rate of flow of each of the resulting streams. (*Hint.* The plate C can exert only a vertical force on the water.)

14.43 A stream of water of cross-sectional area A and velocity $\mathbf{v}_A$ is deflected by a vane AB in the shape of an arc of circle of radius R. Knowing that the vane is welded to a fixed support at A, determine the components of the force-couple system exerted by the support on the vane.

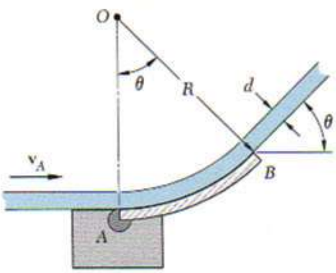


Fig. P14.43

14.44 The nozzle shown discharges 250 gal/min of water with a velocity $\mathbf{v}_A$ of 120 ft/s. The stream is deflected by the fixed vane AB. Determine the force-couple system which must be applied at C in order to hold the vane in place ($1\text{ ft}^3 = 7.48\text{ gal}$).

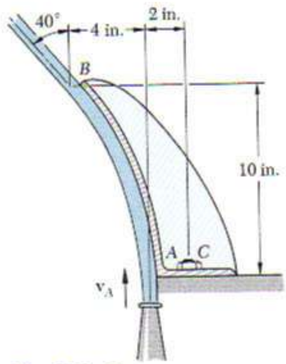


Fig. P14.44

14.45 Knowing that the blade AB of Sample Prob. 14.6 is in the shape of an arc of circle, show that the resultant force $\mathbf{F}$ exerted by the blade on the stream is applied at the midpoint C of the arc AB. (*Hint.* First show that the line of action of $\mathbf{F}$ must pass through the center O of the circle.)

14.46 The stream of water shown flows at the rate of $0.9\text{ m}^3/\text{min}$ and moves with a velocity of magnitude 30 m/s at both A and B. The vane is supported by a pin connection at C and by a load cell at D which can exert only a horizontal force. Neglecting the weight of the vane, determine the reactions at C and D.

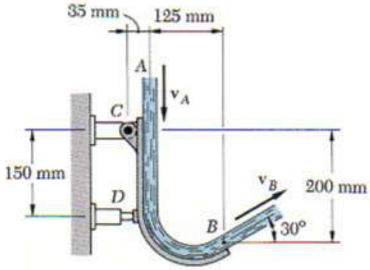


Fig. P14.46

14.47 The nozzle shown discharges water at the rate of $1.2\ m^3/min$. Knowing that at both A and B the stream of water moves with a velocity of magnitude 25 m/s and neglecting the weight of the vane, determine the components of the reactions at C and D.

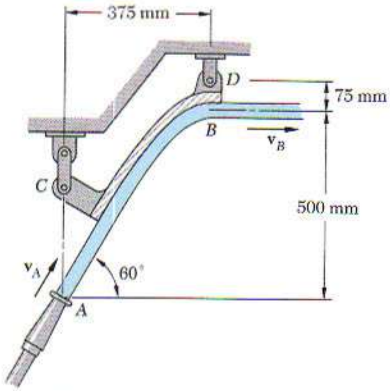


Fig. P14.47

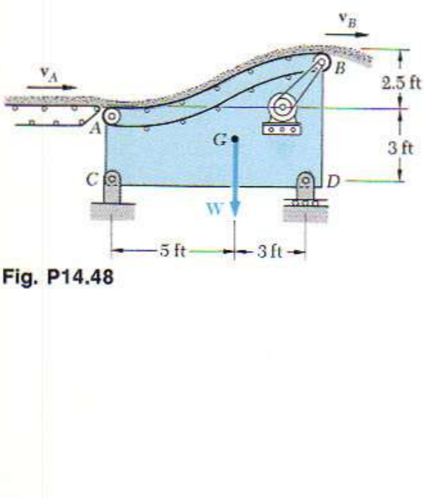


Fig. P14.48

14.48 The final component of a conveyor system receives sand at the rate of 180 lb/s at A and discharges it at B. The sand is moving horizontally at A and B with a velocity of magnitude $v_A = v_B =$ 12 ft/s. Knowing that the combined weight of the component and of the sand it supports is $W = 800$ lb, determine the reactions at C and D.

14.49 Solve Prob. 14.48, assuming the velocity of the belt of the final component of the conveyor system is increased in such a way that, while the sand is still received with a velocity $\mathbf{v}_A$ of 12 ft/s, it is discharged with a velocity $\mathbf{v}_B$ of 24 ft/s.

14.50 A jet airplane scoops in air at the rate of 250 lb/s and discharges it with a velocity of 2200 ft/s relative to the airplane. If the speed of the airplane is 600 mi/h, determine (*a*) the propulsive force developed, (*b*) the horsepower actually used to propel the airplane, (*c*) the horsepower developed by the engine.

14.51 The total drag due to air friction on a jet airplane traveling at 1000 km/h is 16 kN. Knowing that the exhaust velocity is 600 m/s relative to the airplane, determine the mass of air which must pass through the engine per second to maintain the speed of 1000 km/h in level flight.

14.52 While cruising in horizontal flight at a speed of 800 km/h, a 9000-kg jet airplane scoops in air at the rate of 70 kg/s and discharges it with a velocity of 600 m/s relative to the airplane. (*a*) Determine the total drag due to air friction. (*b*) Assuming that the drag is proportional to the square of the speed, determine the horizontal cruising speed if the flow of air through the jet is increased by 10 percent, i.e., to 77 kg/s.

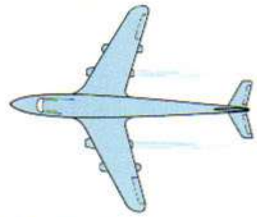


Fig. P14.53

14.53 The cruising speed of a jet airliner is 600 mi/h. Each of the four engines discharges air with a velocity of 2000 ft/s relative to the plane. Assuming that the drag due to air resistance is proportional to the square of the speed, determine the speed of the airliner when only two of the engines are in operation.

Fig. P14.54

14.54 For use in shallow water the pleasure boat shown is powered by a water jet. Water enters the engine through orifices located in the bow and is discharged through a horizontal pipe at the stern. Knowing that the water is discharged at the rate of 10 m^3/min with a velocity of 15 m/s relative to the boat, determine the propulsive force developed when the speed of the boat is (*a*) 6 m/s, (*b*) zero.

14.55 In order to shorten the distance required for landing, a jet airplane is equipped with movable vanes which partially reverse the direction of the air discharged by each of its engines. Each engine scoops in air at the rate of 200 lb/s and discharges it with a velocity of 2000 ft/s relative to the engine. At an instant when the speed of the airplane is 120 mi/h, determine the reversed thrust provided by each of the engines.

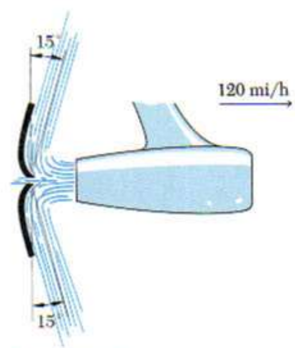


Fig. P14.55

14.56 An unloaded helicopter of weight 5000 lb produces a slipstream of 38-ft diameter. Assuming that air weighs 0.076 lb/ft^3, determine the vertical component of the velocity of the air in the slipstream when the helicopter is hovering in midair.

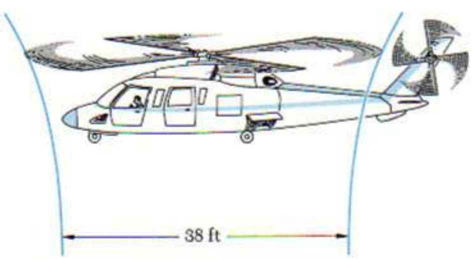


Fig. P14.56 and P14.57

14.57 The helicopter shown weighs 5000 lb and can produce a maximum downward air speed of 60 ft/s in the 38-ft diameter slipstream. Determine the maximum load which the helicopter can carry while hovering in midair.

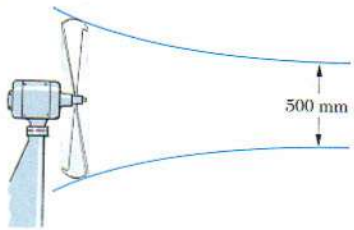


Fig. P14.58

14.58 The slipstream of a fan has a diameter of 500 mm and a velocity of 10 m/s relative to the fan. Assuming $\rho = 1.21\ \mathrm{kg/m^3}$ for air and neglecting the velocity of approach of the air, determine the force required to hold the fan motionless.

14.59 The propeller of an airplane produces a thrust of 4000 N when the airplane is at rest on the ground and has a slipstream of 2-m diameter. Assuming $\rho = 1.21\ \mathrm{kg/m^3}$ for air, determine (*a*) the speed of the air in the slipstream, (*b*) the volume of air passing through the propeller per second, (*c*) the kinetic energy imparted per second to the air of the slipstream.

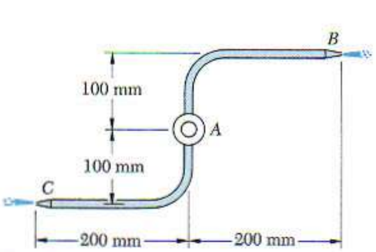


Fig. P14.60

14.60 Each arm of the sprinkler shown discharges water at the rate of 10 liters per minute with a velocity of 12 m/s relative to the arm. Neglecting the effect of friction, determine (*a*) the constant rate at which the sprinkler will rotate, (*b*) the couple **M** which must be applied to the sprinkler to hold it stationary.

14.61 A circular reentrant orifice (also called Borda's mouthpiece) of diameter D is placed at a depth h below the surface of a tank. Knowing that the speed of the issuing stream is $v = \sqrt{2gh}$ and assuming that the speed of approach v_1 is zero, show that the diameter of the stream is $d = D/\sqrt{2}$. (*Hint.* Consider the section of water indicated, and note that P is equal to the pressure at a depth h multiplied by the area of the orifice.)

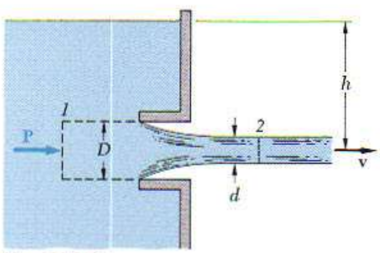


Fig. P14.61

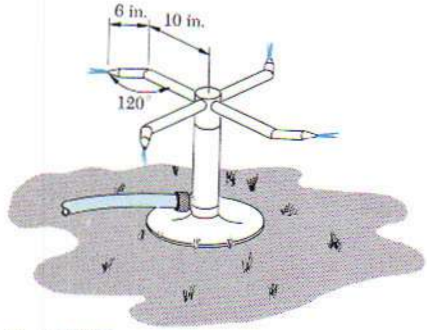


Fig. P14.62

14.62 A garden sprinkler has four rotating arms, each of which consists of two horizontal straight sections of pipe forming an angle of 120°. Each arm discharges water at the rate of 3 gal/min with a velocity of 48 ft/s relative to the arm. Knowing that the friction between the moving and stationary parts of the sprinkler is equivalent to a couple of magnitude $M = 0.200\ \mathrm{lb \cdot ft}$, determine the constant rate at which the sprinkler rotates ($1\ \mathrm{ft^3} = 7.48$ gal).

14.63 Each of the two conveyor belts shown discharges sand at a constant rate of 5 lb/s. The sand falls through a height h and is deflected by a stationary vane. Knowing that the velocity of the sand is horizontal as it leaves the vane, determine the force **P** required to hold the vane when (*a*) $h = 6$ ft, (*b*) $h = 12$ ft.

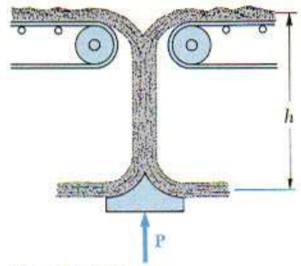


Fig. P14.63

14.64 Gravel falls with practically zero velocity onto a conveyor belt at the constant rate $q = dm/dt$. A force **P** is applied to the belt to maintain a constant speed v. Derive an expression for the angle θ for which the force **P** is zero.

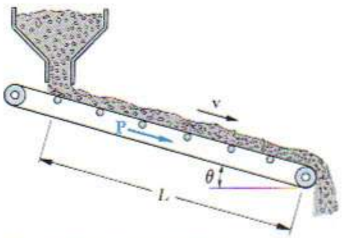


Fig. P14.64 and P14.65

14.65 Gravel falls with practically zero velocity onto a conveyor belt at the constant rate $q = dm/dt$. (*a*) Determine the magnitude of the force **P** required to maintain a constant belt speed v, when $\theta = 0$. (*b*) Show that the kinetic energy acquired by the gravel in a given time interval is equal to half the work done in that interval by the force **P**. Explain what happens to the other half of the work done by **P**.

Fig. P14.66

14.66 A chain of mass m per unit length and total length l lies in a pile on the floor. At time $t = 0$ a force **P** is applied and the chain is raised with a constant velocity **v**. Express the required magnitude of the force **P** as a function of the time t.

14.67 A chain of length l and mass m per unit length falls through a small hole in a plate. Initially, when y is very small, the chain is at rest. In each case shown, determine (*a*) the acceleration of the first link A as a function of y, (*b*) the velocity of the chain as the last link passes through the hole. In case *1* assume that the individual links are at rest until they fall through the hole; in case *2* assume that at any instant all links have the same speed. Ignore the effect of friction.

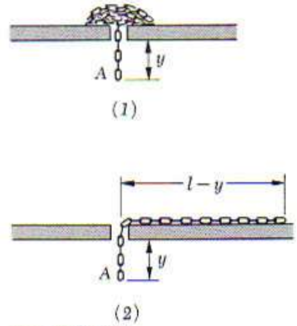


Fig. P14.67

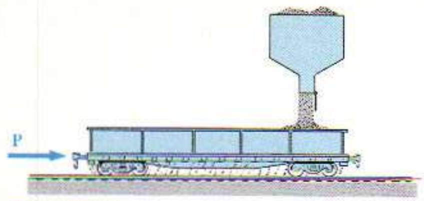


Fig. P14.68

14.68 A moving railroad car, of mass m_0 when empty, is loaded by dropping sand vertically into it from a stationary chute at the rate $q = dm/dt$. At the same time, however, sand is leaking out through the floor of the car at the lesser rate q'. Determine the magnitude of the horizontal force $\mathbf{P}$ required to keep the car moving at a constant speed v while being loaded.

14.69 For the car and loading conditions of Prob. 14.68, express as a function of t the magnitude of the horizontal force $\mathbf{P}$ required to keep the car moving with a constant acceleration $\mathbf{a}$ while being loaded. Denote by v_0 the speed of the car at $t = 0$, when the loading operation begins.

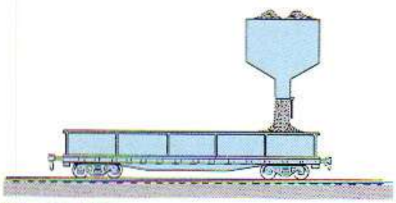


Fig. P14.70

14.70 A railroad car, of mass m_0 when empty and moving freely on a horizontal track, is loaded by dropping sand vertically into it from a stationary chute at the rate $q = dm/dt$. Determine the velocity and acceleration of the car as functions of t. Denote by v_0 the speed of the car at $t = 0$, when the loading operation begins.

14.71 If the car of Prob. 14.68 moves freely ($\mathbf{P} = 0$), determine its velocity and acceleration as functions of t. Denote by v_0 the speed of the car at $t = 0$, when the loading operation begins.

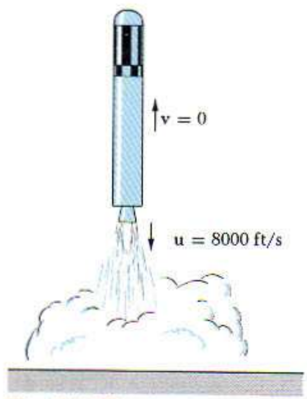


Fig. P14.72

14.72 A test rocket is designed to hover motionless above the ground. The shell of the rocket weighs 2500 lb, and the initial fuel load is 7500 lb. The fuel is burned and ejected with a velocity of 8000 ft/s. Determine the required rate of fuel consumption (a) when the rocket is fired, (b) as the last particle of fuel is being consumed.

14.73 The main engine installation of a space shuttle consists of three identical rocket engines which are required to provide a total thrust of 6000 kN. Knowing that the hydrogen-oxygen propellent is burned and ejected with a velocity of 3900 m/s, determine the required total rate of fuel consumption.

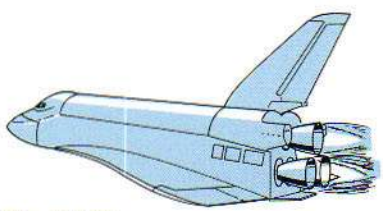


Fig. P14.73

14.74 A space vehicle describing a circular orbit at a speed of 15,000 mi/h releases a capsule which has a gross weight of 1000 lb, including 750 lb of fuel. If the fuel is consumed at the constant rate of 30 lb/s and is ejected with a relative velocity of 8000 ft/s, determine the tangential acceleration of the capsule (*a*) as the engine is fired, (*b*) as the last particle of fuel is being consumed.

Fig. P14.74

14.75 A rocket of gross mass 1000 kg, including 900 kg of fuel, is fired vertically when $t = 0$. Knowing that fuel is consumed at the rate of 10 kg/s and ejected with a relative velocity of 3500 m/s, determine the acceleration and velocity of the rocket when (*a*) $t = 0$, (*b*) $t = 45$ s, (*c*) $t = 90$ s.

14.76 A space tug describing a low-level circular orbit is to be transferred to a high-level orbit. The maneuver is started by firing the rocket engines to increase the speed of the tug from 7370 to 9850 m/s. The initial mass of the tug, fuel, and payload is 14.1 Mg. Knowing that the hydrogen-oxygen propellent is consumed at the rate of 20 kg/s and is ejected with a velocity of 3750 m/s, determine (*a*) the mass of fuel which must be expended to initiate the maneuver, (*b*) the time interval for which the engines must be fired.

Fig. P14.77

14.77 The rocket of Prob. 14.75 is redesigned as a two-stage rocket consisting of rockets *A* and *B*, each of gross mass 500 kg, including 450 kg of fuel. The fuel is again consumed at the rate of 10 kg/s and is ejected with a relative velocity of 3500 m/s. Knowing that, when rocket *A* expells its last particle of fuel, its shell is released and rocket *B* is fired, determine (*a*) the speed when rocket *A* is released, (*b*) the maximum speed attained by rocket *B*.

14.78 A spacecraft is launched vertically by a two-stage rocket. When the speed is 10,000 mi/h the first-stage-rocket casing is released and the second-stage rocket is fired. Fuel is consumed at the rate of 200 lb/s and ejected with a relative velocity of 8000 ft/s. Knowing that the combined weight of the second-stage rocket and spacecraft is 20,000 lb, including 17,000 lb of fuel, determine the maximum speed which can be attained by the spacecraft.

14.79 For the rocket of Sample Prob. 14.7, derive an expression for the height of the rocket as a function of the time *t*.

14.80 Determine the distance between the spacecraft and the first-stage-rocket casing of Prob. 14.78 as the last particle of fuel is being expelled by the second-stage rocket.

14.81 Determine the distance between the capsule and the space vehicle of Prob. 14.74 as the last particle of fuel is being ejected by the rocket of the capsule. Both the capsule and the space vehicle may be considered to move in a straight line during the time interval considered.

14.82 In a jet airplane, the kinetic energy imparted to the exhaust gases is wasted as far as propelling the airplane is concerned. The useful power is equal to the product of the force available to propel the airplane and the speed of the airplane. If v is the speed of the airplane and u is the relative speed of the expelled gases, show that the efficiency is $\eta = 2v/(u + v)$. Explain why $\eta = 1$ when $u = v$.

14.83 In a rocket, the kinetic energy imparted to the consumed and ejected fuel is wasted as far as propelling the rocket is concerned. The useful power is equal to the product of the force available to propel the rocket and the speed of the rocket. If v is the speed of the rocket and u is the relative speed of the expelled fuel, show that the efficiency is $\eta = 2uv/(u^2 + v^2)$. Explain why $\eta = 1$ when $u = v$.

REVIEW PROBLEMS

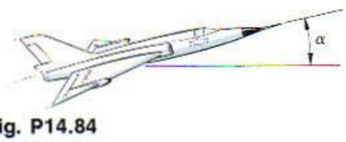


Fig. P14.84

14.84 A 9000-kg jet airplane maintains a constant speed of 900 km/h while climbing at an angle $\alpha = 5°$. The airplane scoops in air at the rate of 80 kg/s and discharges it with a velocity of 700 m/s relative to the airplane. If the pilot changes to a horizontal flight and the same engine conditions are maintained, determine (*a*) the initial acceleration of the plane, (*b*) the maximum horizontal speed attained. Assume that the drag due to air friction is proportional to the square of the speed.

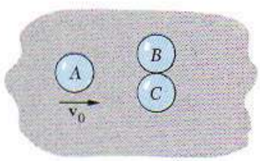


Fig. P14.85

14.85 Three identical balls A, B, and C may roll freely on a horizontal surface. Balls B and C are at rest and in contact when struck by ball A, which was moving to the right with a velocity $\mathbf{v}_0$. Assuming $e = 1$ and no friction, determine the final velocity of ball A if (*a*) the path of A is perfectly centered and A strikes B and C simultaneously, (*b*) the path of A is not perfectly centered and A strikes B slightly before it strikes C.

14.86 A 1-oz bullet is fired with a velocity of 1600 ft/s into block A, which weighs 10 lb. The coefficient of friction between block A and the cart BC is 0.50. Knowing that the cart weighs 8 lb and can roll freely, determine (*a*) the final velocity of the cart and block, (*b*) the final position of the block on the cart.

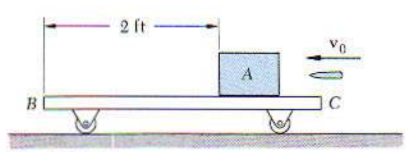


Fig. P14.86

14.87 The ends of a chain of mass m per unit length lie in piles at A and at C; when released, the chain moves over the pulley at B. Determine the required initial speed v for which the chain will move at a constant speed. Neglect axle friction.

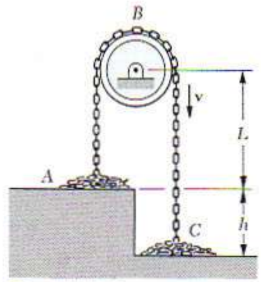


Fig. P14.87

14.88 Two railroad freight cars move with a velocity $\mathbf{v}$ through a switchyard. Car B hits a third car C, which was at rest with its brakes released, and it automatically couples with C. Knowing that all three cars have the same mass, determine their common velocity after they are all coupled together, as well as the percentage of their total initial kinetic energy which is absorbed by each coupling mechanism, assuming (*a*) that cars A and B were originally coupled, (*b*) that cars A and B were moving a few feet apart and that the coupling operation between B and C is completed before A hits B and becomes coupled with it.

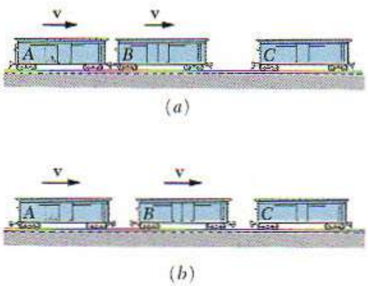


Fig. P14.88

14.89 A 5-kg sphere is moving with a velocity of 60 m/s when it explodes into two fragments. Immediately after the explosion the fragments are observed to travel in the directions shown and the speed of fragment A is observed to be 90 m/s. Determine (*a*) the mass of fragment A, (*b*) the speed of fragment B.

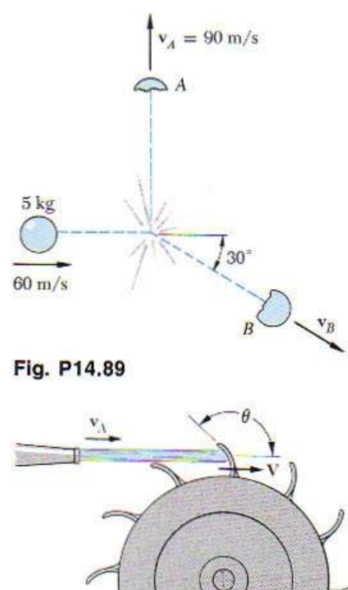


Fig. P14.89

14.90 In a Pelton-wheel turbine, a stream of water is deflected by a series of blades so that the rate at which water is deflected by the blades is equal to the rate at which water issues from the nozzle $(\Delta m/\Delta t = A\rho v_A)$. Using the same notation as in Sample Prob. 14.6, (*a*) determine the velocity $\mathbf{V}$ of the blades for which maximum power is developed, (*b*) derive an expression for the maximum power, (*c*) derive an expression for the mechanical efficiency.

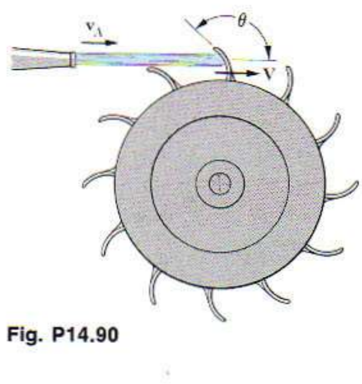


Fig. P14.90

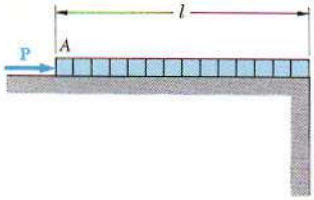


Fig. P14.91

14.91 A large number of small blocks of total mass m are at rest on a table when a constant force **P** is applied to block A. Knowing that the blocks are in contact with each other but not connected, determine the speed of block A after half of the blocks have been pushed off the table, (a) neglecting the effect of friction, (b) assuming a coefficient of friction μ between the table and the blocks.

14.92 A jet of water having a cross-sectional area $A = 600\ \text{mm}^2$ and moving with a velocity of magnitude $v_A = v_B = 20$ m/s is deflected by the two vanes shown, which are welded to a vertical plate. Knowing that the combined mass of the plate and vanes is 5 kg, determine the reactions at C and D.

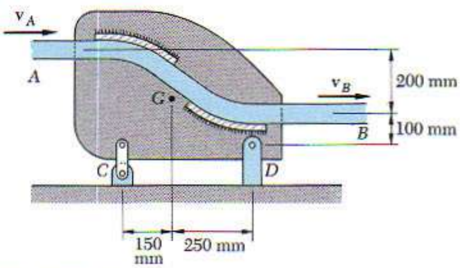


Fig. P14.92

14.93 A space vehicle equipped with a retrorocket, which may expel fuel with a relative velocity **u**, is moving with a velocity $\mathbf{v}_0$. Denoting by m_s the net mass of the vehicle and by m_f the mass of the unexpended fuel, determine the minimum ratio m_f/m_s for which the velocity of the vehicle can be reduced to zero.

14.94 The jet engine shown scoops in air at A at the rate of 165 lb/s and discharges it at B with a velocity of 2500 ft/s relative to the airplane. Determine the magnitude and line of action of the propulsive thrust developed by the engine when the speed of the airplane is (a) 300 mi/h, (b) 600 mi/h.

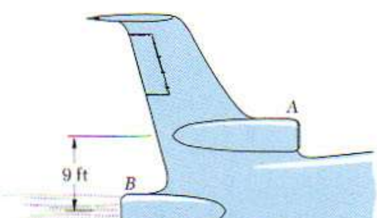


Fig. P14.94

14.95 Solve Prob. 14.94, including the effect of the fuel which is consumed by the engine at the rate of 3 lb/s.

Kinematics of Rigid Bodies

CHAPTER **15**

15.1. Introduction. In this chapter, we shall study the kinematics of *rigid bodies*. We shall investigate the relations existing between the time, the positions, the velocities, and the accelerations of the various particles forming a rigid body. As we shall see, the various types of rigid-body motion may be conveniently grouped as follows:

1. *Translation.* A motion is said to be a translation if any straight line inside the body keeps the same direction during the motion. It may also be observed that in a translation all the particles forming the body move along parallel paths. If these paths are straight lines, the motion is said to be a *rectilinear translation* (Fig. 15.1); if the paths are curved lines, the motion is a *curvilinear translation* (Fig. 15.2).

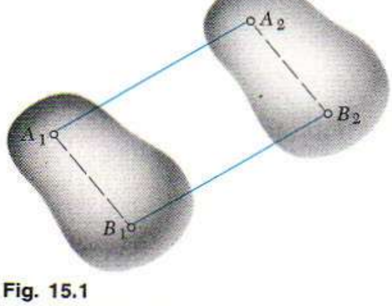


Fig. 15.1

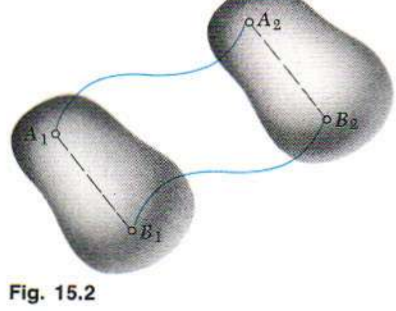


Fig. 15.2

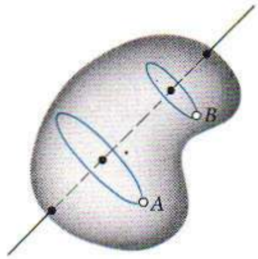
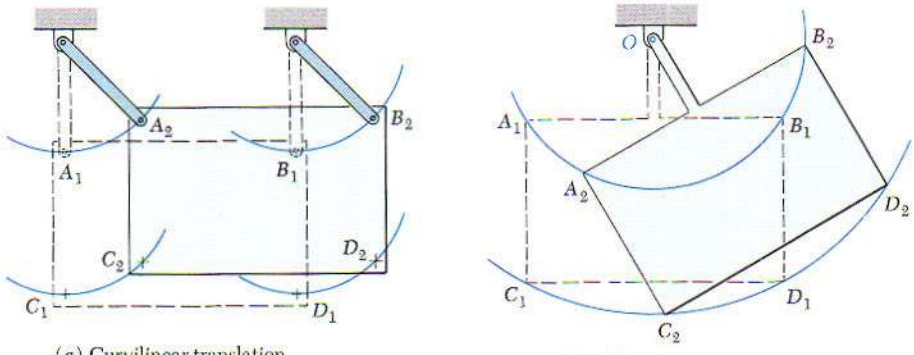


Fig. 15.3

2. *Rotation about a Fixed Axis.* In this motion, the particles forming the rigid body move in parallel planes along circles centered on the same fixed axis (Fig. 15.3). If this axis, called the *axis of rotation,* intersects the rigid body, the particles located on the axis have zero velocity and zero acceleration.

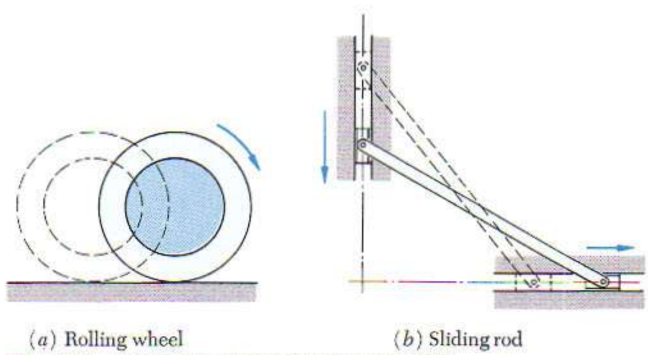
Rotation should not be confused with certain types of curvilinear translation. For example, the plate shown in Fig. 15.4*a* is in curvilinear translation, with all its particles moving along *parallel* circles, while the plate shown in Fig. 15.4*b* is in rotation, with all its particles moving along *concentric* circles. In the first case, any given straight line drawn on the plate will maintain the same direction, while, in the second case, point O remains fixed.



(*a*) Curvilinear translation
(*b*) Rotation
Fig. 15.4

Because each particle moves in a given plane, the rotation of a body about a fixed axis is said to be a *plane motion.*

3. *General Plane Motion.* There are many other types of plane motion, i.e., motions in which all the particles of the body move in parallel planes. Any plane motion which is neither a rotation nor a translation is referred to as a general plane motion. Two examples of general plane motion are given in Fig. 15.5.


(*a*) Rolling wheel
(*b*) Sliding rod
Fig. 15.5 Examples of general plane motion

4. *Motion about a Fixed Point.* This is the three-dimensional motion of a rigid body attached at a fixed point O. An example of motion about a fixed point is provided by the motion of a top on a rough floor (Fig. 15.6).
5. *General Motion.* Any motion of a rigid body which does not fall in any of the above categories is referred to as a general motion.

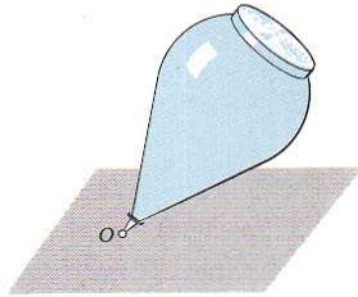


Fig. 15.6

15.2. Translation. Consider a rigid body in translation (either rectilinear or curvilinear translation), and let A and B be any two of its particles (Fig. 15.7*a*). Denoting respectively by $\mathbf{r}_A$ and $\mathbf{r}_B$ the position vectors of A and B with respect to a fixed frame of reference, and by $\mathbf{r}_{B/A}$ the vector joining A and B, we write

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A} \tag{15.1}$$

Let us differentiate this relation with respect to t. We note that, from the very definition of a translation, the vector $\mathbf{r}_{B/A}$ must maintain a constant direction; its magnitude must also be constant, since A and B belong to the same rigid body. Thus, the derivative of $\mathbf{r}_{B/A}$ is zero and we have

$$\mathbf{v}_B = \mathbf{v}_A \tag{15.2}$$

Differentiating once more, we write

$$\mathbf{a}_B = \mathbf{a}_A \tag{15.3}$$

Thus, *when a rigid body is in translation, all the points of the body have the same velocity and the same acceleration at any given instant* (Fig. 15.7*b* and *c*). In the case of curvilinear translation, the velocity and acceleration change in direction as well as in magnitude at every instant. In the case of rectilinear

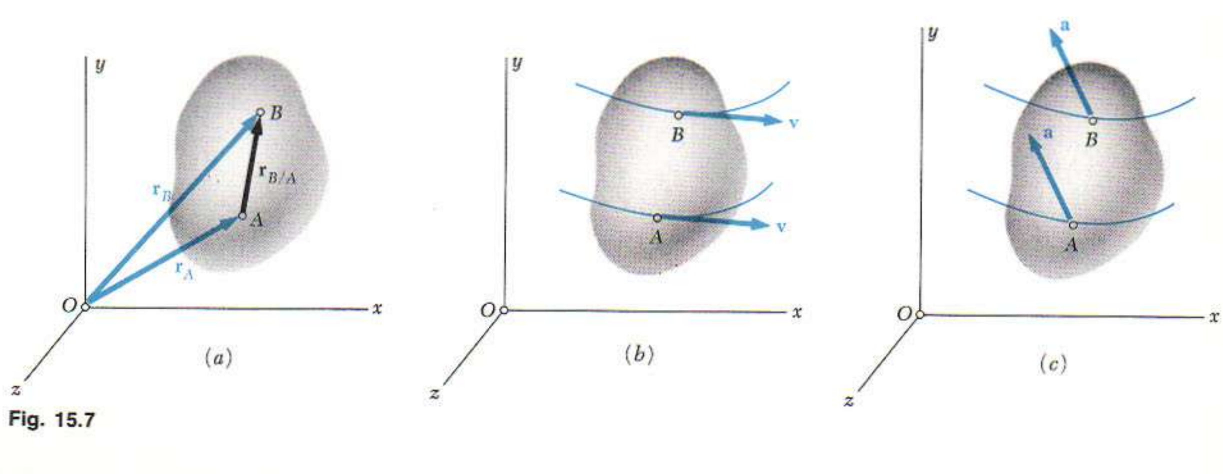


Fig. 15.7

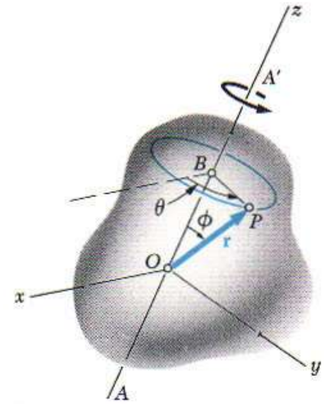


Fig. 15.8

translation, all particles of the body move along parallel straight lines, and their velocity and acceleration keep the same direction during the entire motion.

15.3. Rotation about a Fixed Axis. Consider a rigid body which rotates about a fixed axis AA'. Let P be a point of the body and $\mathbf{r}$ its position vector with respect to a fixed frame of reference. For convenience, we shall assume that the frame is centered at point O on AA' and that the z axis coincides with AA' (Fig. 15.8). Let B be the projection of P on AA'; since P must remain at a constant distance from B, it will describe a circle of center B and of radius $r \sin \phi$, where ϕ denotes the angle formed by $\mathbf{r}$ and AA'.

The position of P and of the entire body is completely defined by the angle θ the line BP forms with the zx plane. The angle θ is known as the *angular coordinate* of the body. The angular coordinate is defined as positive when counterclockwise as viewed from A' and will be expressed in radians (rad) or, occasionally, in degrees (°) or revolutions (rev). We recall that

$$1 \text{ rev} = 2\pi \text{ rad} = 360°$$

We recall from Sec. 11.9 that the velocity $\mathbf{v} = d\mathbf{r}/dt$ of a particle P is a vector tangent to the path of P and of magnitude $v = ds/dt$. Observing that the length Δs of the arc described by P when the body rotates through $\Delta\theta$ is

$$\Delta s = (BP)\,\Delta\theta = (r \sin \phi)\,\Delta\theta$$

and dividing both members by Δt, we obtain at the limit, as Δt approaches zero,

$$v = \frac{ds}{dt} = r\dot{\theta} \sin \phi \tag{15.4}$$

where $\dot{\theta}$ denotes the time derivative of θ. (Note that, while the angle θ depends upon the position of P within the body, the rate of change $\dot{\theta}$ is itself independent of P.) We conclude that the velocity $\mathbf{v}$ of P is a vector perpendicular to the plane containing AA' and $\mathbf{r}$, and of magnitude v defined by (15.4). But this is precisely the result we would obtain if we drew along AA' a vector $\boldsymbol{\omega} = \dot{\theta}\mathbf{k}$ and formed the vector product $\boldsymbol{\omega} \times \mathbf{r}$ (Fig. 15.9). We thus write

Fig. 15.9

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r} \tag{15.5}$$

The vector

$$\boldsymbol{\omega} = \omega\mathbf{k} = \dot{\theta}\mathbf{k} \tag{15.6}$$

is called the *angular velocity* of the body. It is directed along the axis of rotation, it is equal in magnitude to the rate of change $\dot{\theta}$ of the angular coordinate, and its sense may be obtained by the right-hand rule (Sec. 3.5) from the sense of rotation of the body.†

We shall now determine the acceleration $\mathbf{a}$ of the particle P. Differentiating (15.5) and recalling the rule for the differentiation of a vector product (Sec. 11.10), we write

$$\begin{aligned}\mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\boldsymbol{\omega} \times \mathbf{r}) \\ &= \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt} \\ &= \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v} \qquad (15.7)\end{aligned}$$

The vector $d\boldsymbol{\omega}/dt$ is denoted by $\boldsymbol{\alpha}$ and called the *angular acceleration* of the body. Substituting also for $\mathbf{v}$ from (15.5), we have

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \qquad (15.8)$$

Differentiating (15.6), and recalling that $\mathbf{k}$ is constant in magnitude and direction, we have

$$\boldsymbol{\alpha} = \alpha\mathbf{k} = \dot{\omega}\mathbf{k} = \ddot{\theta}\mathbf{k} \qquad (15.9)$$

Thus, the angular acceleration of a body rotating about a fixed axis is a vector directed along the axis of rotation, and equal in magnitude to the rate of change $\dot{\omega}$ of the angular velocity. Returning to (15.8), we note that the acceleration of P is the sum of two vectors. The first vector is equal to the vector product $\boldsymbol{\alpha} \times \mathbf{r}$; it is tangent to the circle described by P and represents, therefore, the tangential component of the acceleration. The second vector is equal to the *vector triple product* $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ obtained by forming the vector product of $\boldsymbol{\omega}$ and $\boldsymbol{\omega} \times \mathbf{r}$; since $\boldsymbol{\omega} \times \mathbf{r}$ is tangent to the circle described by P, the vector triple product is directed toward the center B of the circle and represents, therefore, the normal component of the acceleration.

Rotation of a Representative Slab. The rotation of a rigid body about a fixed axis may be defined by the motion of a representative slab in a reference plane perpendicular to the axis of rotation. Let us choose the xy plane as the reference plane and assume that it coincides with the plane of the figure, with the

†It will be shown in Sec. 15.12 in the more general case of a rigid body rotating simultaneously about axes having different directions, that angular velocities obey the parallelogram law of addition and, thus, are actually vector quantities.

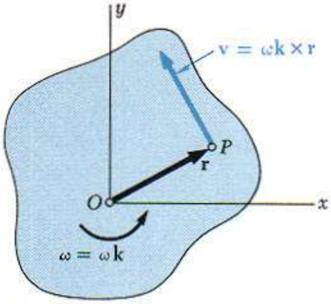


Fig. 15.10

z axis pointing out of the paper (Fig. 15.10). Recalling from (15.6) that $\boldsymbol{\omega} = \omega\mathbf{k}$, we note that a positive value of the scalar ω corresponds to a counterclockwise rotation of the representative slab, and a negative value to a clockwise rotation. Substituting $\omega\mathbf{k}$ for $\boldsymbol{\omega}$ into Eq. (15.5), we express the velocity of any given point P of the slab as

$$\mathbf{v} = \omega\mathbf{k} \times \mathbf{r} \tag{15.10}$$

Since the vectors $\mathbf{k}$ and $\mathbf{r}$ are mutually perpendicular, the magnitude of the velocity $\mathbf{v}$ is

$$v = r\omega \tag{15.10$'$}$$

and its direction may be obtained by rotating $\mathbf{r}$ through 90° in the sense of rotation of the slab.

Substituting $\boldsymbol{\omega} = \omega\mathbf{k}$ and $\boldsymbol{\alpha} = \alpha\mathbf{k}$ into Eq. (15.8), and observing that cross-multiplying $\mathbf{r}$ twice by $\mathbf{k}$ results in a 180° rotation of the vector $\mathbf{r}$, we express the acceleration of point P as

$$\mathbf{a} = \alpha\mathbf{k} \times \mathbf{r} - \omega^2\mathbf{r} \tag{15.11}$$

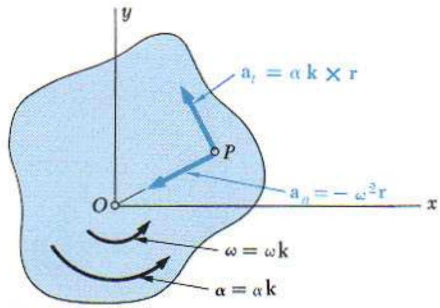


Fig. 15.11

Resolving $\mathbf{a}$ into tangential and normal components (Fig. 15.11), we write

$$\begin{aligned} \mathbf{a}_t &= \alpha\mathbf{k} \times \mathbf{r} \qquad & a_t &= r\alpha \\ \mathbf{a}_n &= -\omega^2\mathbf{r} \qquad & a_n &= r\omega^2 \end{aligned} \tag{15.11$'$}$$

The tangential component $\mathbf{a}_t$ points in the counterclockwise direction if the scalar α is positive, and in the clockwise direction if α is negative. The normal component $\mathbf{a}_n$ always points in the direction opposite to that of $\mathbf{r}$, i.e., toward O.

15.4. Equations Defining the Rotation of a Rigid Body about a Fixed Axis. The motion of a rigid body rotating about a fixed axis AA' is said to be *known* when its angular coordinate θ may be expressed as a known function of t.

In practice, however, the rotation of a rigid body is seldom defined by a relation between θ and t. More often, the conditions of motion will be specified by the type of angular acceleration that the body possesses. For example, α may be given as a function of t, or as a function of θ, or as a function of ω. Recalling the relations (15.6) and (15.9), we write

$$\omega = \frac{d\theta}{dt} \tag{15.12}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \tag{15.13}$$

or, solving (15.12) for dt and substituting into (15.13),

$$\alpha = \omega \frac{d\omega}{d\theta} \tag{15.14}$$

Since these equations are similar to those obtained in Chap. 11 for the rectilinear motion of a particle, their integration may be performed by following the procedure outlined in Sec. 11.3.

Two particular cases of rotation are frequently encountered:

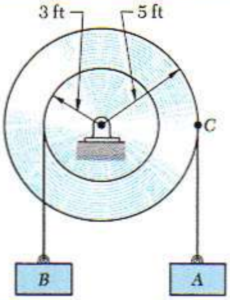
1. *Uniform Rotation.* This case is characterized by the fact that the angular acceleration is zero. The angular velocity is thus constant, and the angular coordinate is given by the formula

$$\theta = \theta_0 + \omega t \tag{15.15}$$

2. *Uniformly Accelerated Rotation.* In this case, the angular acceleration is constant. The following formulas relating angular velocity, angular coordinate, and time may then be derived in a manner similar to that described in Sec. 11.5. The similitude between the formulas derived here and those obtained for the rectilinear uniformly accelerated motion of a particle is easily noted.

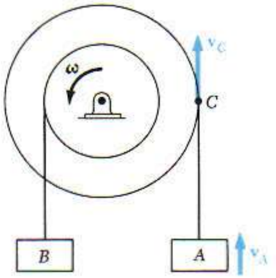
$$\begin{aligned} \omega &= \omega_0 + \alpha t \\ \theta &= \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2 \\ \omega^2 &= \omega_0^2 + 2\alpha(\theta - \theta_0) \end{aligned} \tag{15.16}$$

It should be emphasized that formula (15.15) may be used only when $\alpha = 0$, and formulas (15.16) only when $\alpha =$ constant. In any other case, the general formulas (15.12) to (15.14) should be used.



SAMPLE PROBLEM 15.1

A pulley and two loads are connected by inextensible cords as shown. Load A has a constant acceleration of 10 ft/s^2 and an initial velocity of 15 ft/s, both directed upward. Determine (*a*) the number of revolutions executed by the pulley in 3 s, (*b*) the velocity and position of load B after 3 s, (*c*) the acceleration of point C on the rim of the pulley at $t = 0$.

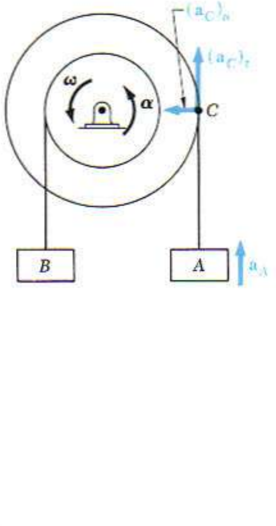


a. Motion of Pulley. Since the cord connecting the pulley to load A is inextensible, the velocity of C is equal to the velocity of A and the tangential component of the acceleration of C is equal to the acceleration of A.

$$(\mathbf{v}_C)_0 = (\mathbf{v}_A)_0 = 15 \text{ ft/s} \uparrow \qquad (\mathbf{a}_C)_t = \mathbf{a}_A = 10 \text{ ft/s}^2 \uparrow$$

Noting that the distance from C to the center of the pulley is 5 ft, we write

$$(v_C)_0 = r\omega_0 \qquad 15 \text{ ft/s} = (5 \text{ ft})\omega_0 \qquad \boldsymbol{\omega}_0 = 3 \text{ rad/s} \circlearrowleft$$
$$(a_C)_t = r\alpha \qquad 10 \text{ ft/s}^2 = (5 \text{ ft})\alpha \qquad \boldsymbol{\alpha} = 2 \text{ rad/s}^2 \circlearrowleft$$



From the equations for uniformly accelerated motion, we obtain, for $t = 3$ s,

$$\omega = \omega_0 + \alpha t = 3 \text{ rad/s} + (2 \text{ rad/s}^2)(3 \text{ s}) = 9 \text{ rad/s}$$
$$\boldsymbol{\omega} = 9 \text{ rad/s} \circlearrowleft$$
$$\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2 = (3 \text{ rad/s})(3 \text{ s}) + \tfrac{1}{2}(2 \text{ rad/s}^2)(3 \text{ s})^2$$
$$\theta = 18 \text{ rad}$$

$$\text{Number of revolutions} = (18 \text{ rad})\left(\frac{1 \text{ rev}}{2\pi \text{ rad}}\right) = 2.86 \text{ rev} \blacktriangleleft$$

b. Motion of Load B. Using the following relations between the linear and angular motion, with $r = 3$ ft, we write

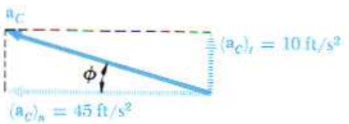
$$v_B = r\omega = (3 \text{ ft})(9 \text{ rad/s}) \qquad \mathbf{v}_B = 27 \text{ ft/s} \downarrow \blacktriangleleft$$
$$s_B = r\theta = (3 \text{ ft})(18 \text{ rad}) \qquad s_B = 54 \text{ ft} \downarrow \blacktriangleleft$$

c. Acceleration of Point C at t = 0. The tangential component of the acceleration is

$$(\mathbf{a}_C)_t = \mathbf{a}_A = 10 \text{ ft/s}^2 \uparrow$$

Since, at $t = 0$, $\omega_0 = 3$ rad/s, the normal component of the acceleration is

$$(a_C)_n = r_C\omega_0^2 = (5 \text{ ft})(3 \text{ rad/s})^2 \qquad (\mathbf{a}_C)_n = 45 \text{ ft/s}^2 \leftarrow$$



The magnitude and direction of the total acceleration are obtained by writing

$$\tan \phi = (10 \text{ ft/s}^2)/(45 \text{ ft/s}^2) \qquad \phi = 12.5^\circ$$
$$a_C \sin 12.5^\circ = 10 \text{ ft/s}^2 \qquad a_C = 46.1 \text{ ft/s}^2$$
$$\mathbf{a}_C = 46.1 \text{ ft/s}^2 \nwarrow 12.5^\circ \blacktriangleleft$$

PROBLEMS

15.1 The motion of a cam is defined by the relation $\theta = t^3 - 2t^2 - 4t + 10$, where θ is expressed in radians and t in seconds. Determine the angular coordinate, the angular velocity, and the angular acceleration of the cam when (*a*) $t = 0$, (*b*) $t = 3$ s.

15.2 The rotor of a steam turbine is rotating at a speed of 7200 rpm when the steam supply is suddenly cut off. It is observed that 5 min are required for the rotor to come to rest. Assuming uniformly accelerated motion, determine (*a*) the angular acceleration, (*b*) the total number of revolutions that the rotor executes before coming to rest.

15.3 A small grinding wheel is attached to the shaft of an electric motor which has a rated speed of 1800 rpm. When the power is turned on, the unit reaches its rated speed in 4 s, and when the power is turned off, the unit coasts to rest in 50 s. Assuming uniformly accelerated motion, determine the number of revolutions that the motor executes (*a*) in reaching its rated speed, (*b*) in coasting to rest.

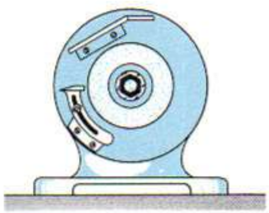


Fig. P15.3

15.4 The rotor of an electric motor has a speed of 1200 rpm when the power is cut off. The rotor is then observed to come to rest after executing 520 revolutions. Assuming uniformly accelerated motion, determine (*a*) the angular acceleration, (*b*) the time required for the rotor to come to rest.

15.5 The assembly shown consists of the straight rod ABC which passes through and is welded to the rectangular plate $DEFH$. The assembly rotates about the axis AC with a constant angular velocity of 18 rad/s. Knowing that the motion when viewed from C is counterclockwise, determine the velocity and acceleration of corner F.

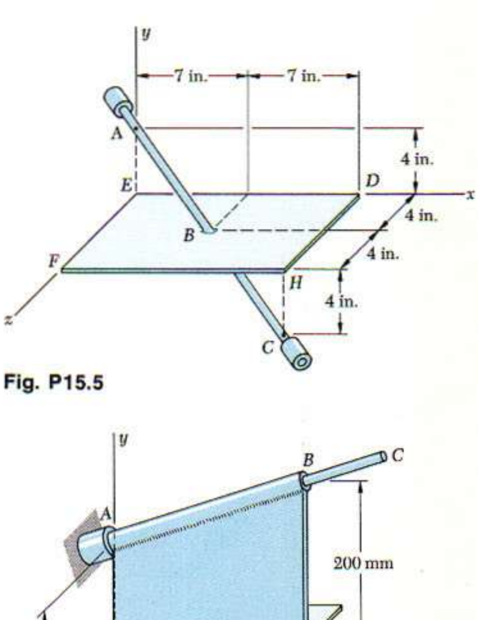


Fig. P15.5

15.6 In Prob. 15.5, assuming that the angular velocity is 18 rad/s and decreases at the rate of 45 rad/s², determine the velocity and acceleration of corner H.

15.7 The assembly shown rotates about the rod AC with a constant angular velocity of 5 rad/s. Knowing that at the instant considered, the velocity of corner D is downward, determine the velocity and acceleration of corner D.

15.8 In Prob. 15.7, determine the velocity and acceleration of corner E, assuming that the angular velocity is 5 rad/s and increases at the rate of 25 rad/s².

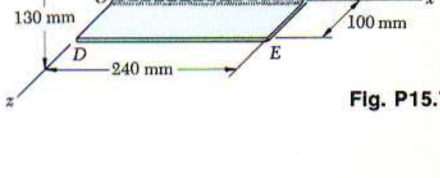


Fig. P15.7

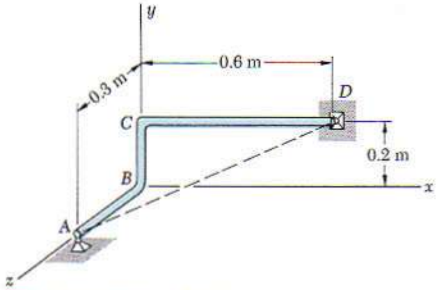


Fig. P15.9 and P15.10

15.9 The rod $ABCD$ has been bent as shown and may rotate about the line joining points A and D. Knowing that the rod starts from rest in the position shown with a constant angular acceleration of 14 rad/s^2 and that the initial acceleration of point B is upward, determine the initial acceleration of point C.

15.10 The bent rod $ABCD$ rotates about the line joining points A and D. At the instant shown, the angular velocity of the rod is 7 rad/s and the angular acceleration is 21 rad/s^2, both counterclockwise when viewed from end A of line AD. Determine the velocity and acceleration of point C.

15.11 The earth makes one complete revolution on its axis in 23.93 h. Knowing that the mean radius of the earth is 3960 mi, determine the linear velocity and acceleration of a point on the surface of the earth (*a*) at the equator, (*b*) at Philadelphia, latitude 40° north, (*c*) at the North Pole.

15.12 The earth makes one complete revolution about the sun in 365.24 days. Assuming that the orbit of the earth is circular and has a radius of 93,000,000 mi, determine the velocity and acceleration of the earth.

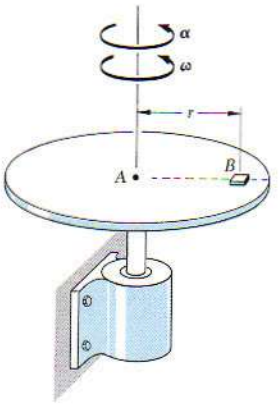


Fig. P15.13 and P15.14

15.13 A small block B rests on a horizontal plate which rotates about a fixed vertical axis. If the plate is initially at rest at $t = 0$ and is accelerated at the constant rate α, derive an expression (*a*) for the total acceleration of the block at time t, (*b*) for the angle between the total acceleration and the radius AB at time t.

15.14 It is known that the static-friction force between block B and the plate will be exceeded and that the block will start sliding on the plate when the total acceleration of the block reaches 5 m/s^2. If the plate starts from rest at $t = 0$ and is accelerated at the constant rate of 6 rad/s^2, determine the time t and the angular velocity of the plate when the block starts sliding, assuming $r = 100$ mm.

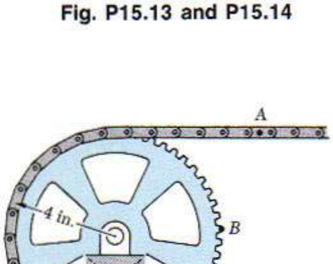


Fig. P15.15 and P15.16

15.15 The sprocket wheel and chain are initially at rest. If the acceleration of point A of the chain has a constant magnitude of 5 in./s^2 and is directed to the left, determine (*a*) the angular velocity of the wheel after it has completed three revolutions, (*b*) the time required for the wheel to reach an angular velocity of 100 rpm.

15.16 At the instant shown the velocity of point A is 8 in./s directed to the right and its acceleration is 12 in./s^2 directed to the left. Determine (*a*) the angular velocity and angular acceleration of the sprocket wheel, (*b*) the total acceleration of sprocket B.

15.17 The friction wheel B executes 100 revolutions about its fixed shaft during the time interval t, while its angular velocity is being increased uniformly from 200 to 600 rpm. Knowing that wheel B rolls without slipping on the inside rim of wheel A, determine (*a*) the angular acceleration of wheel A, (*b*) the time interval t.

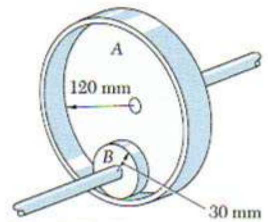


Fig. P15.17

15.18 Ring C has an inside diameter of 120 mm and hangs from the 40-mm-diameter shaft which rotates with a constant angular velocity of 30 rad/s. Knowing that no slipping occurs between the shaft and the ring, determine (*a*) the angular velocity of the ring, (*b*) the acceleration of the points of B and C which are in contact.

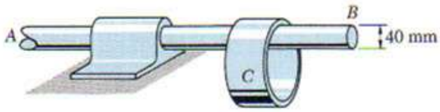


Fig. P15.18

15.19 The system shown starts from rest at $t = 0$ and accelerates uniformly. Knowing that at $t = 4$ s the velocity of the load is 4.8 m/s downward, determine (*a*) the angular acceleration of gear A, (*b*) the number of revolutions executed by gear A during the 4-s interval.

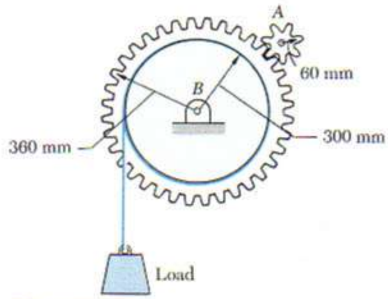


Fig. P15.19

15.20 The two pulleys shown may be operated with the V belt in any of three positions. If the angular acceleration of shaft A is 6 rad/s^2 and if the system is initially at rest, determine the time required for shaft B to reach a speed of 400 rpm with the belt in each of the three positions.

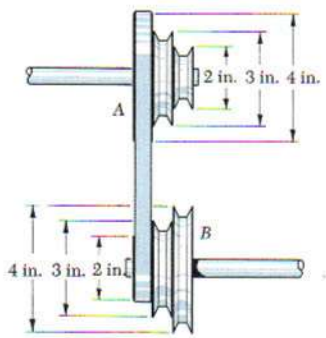


Fig. P15.20

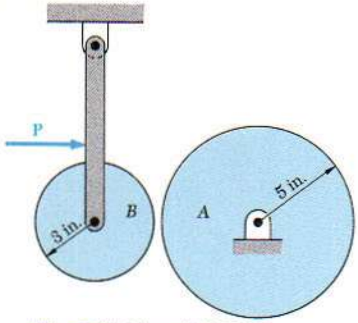


Fig. P15.21 and P15.22

15.21 The two friction wheels A and B are to be brought together. Wheel A has an initial angular velocity of 600 rpm clockwise and will coast to rest in 40 s, while wheel B is initially at rest and is given a constant counterclockwise angular acceleration of 2 rad/s^2. Determine (a) at what time the wheels may be brought together if they are not to slip, (b) the angular velocity of each wheel as contact is made.

15.22 Two friction wheels A and B are both rotating freely at 300 rpm clockwise when they are brought into contact. After 6 s of slippage, during which each wheel has a constant angular acceleration, wheel A reaches a final angular velocity of 60 rpm clockwise. Determine (a) the angular acceleration of each wheel during the period of slippage, (b) the time at which the angular velocity of wheel B is equal to zero.

***15.23** The motion of the circular plate of Prob. 15.13 is defined by the relation $\theta = \theta_0 \sin(2\pi t/T)$, where θ is expressed in radians and t in seconds. Derive expressions (a) for the magnitude of the total acceleration of B, (b) for the values of θ at which the total acceleration of B reaches its maximum and minimum values, and for the corresponding values of the total acceleration of B.

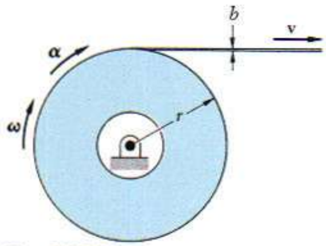


Fig. P15.24

***15.24** In a continuous printing process, paper is drawn into the presses at a constant speed v. Denoting by r the radius of paper on the roll at any given time and by b the thickness of the paper, derive an expression for the angular acceleration of the paper roll.

15.5. General Plane Motion.

As indicated in Sec. 15.1, we understand by general plane motion a plane motion which is neither a translation nor a rotation. As we shall presently see, however, *a general plane motion may always be considered as the sum of a translation and a rotation.*

Consider, for example, a wheel rolling on a straight track (Fig. 15.12). Over a certain interval of time, two given points A and

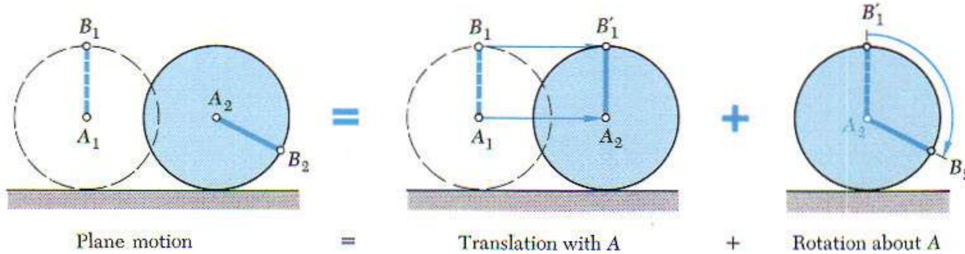


Fig. 15.12

B will have moved, respectively, from A_1 to A_2 and from B_1 to B_2. The same result could be obtained through a translation which would bring A and B into A_2 and B'_1 (the line AB remaining vertical), followed by a rotation about A bringing B into B_2. Although the original rolling motion differs from the combination of translation and rotation when these motions are taken in succession, the original motion may be completely duplicated by a combination of simultaneous translation and rotation.

Another example of plane motion is given in Fig. 15.13, which represents a rod whose extremities slide, respectively, along a horizontal and a vertical track. This motion may be replaced by a translation in a horizontal direction and a rotation about A (Fig. 15.13a) or by a translation in a vertical direction and a rotation about B (Fig. 15.13b).

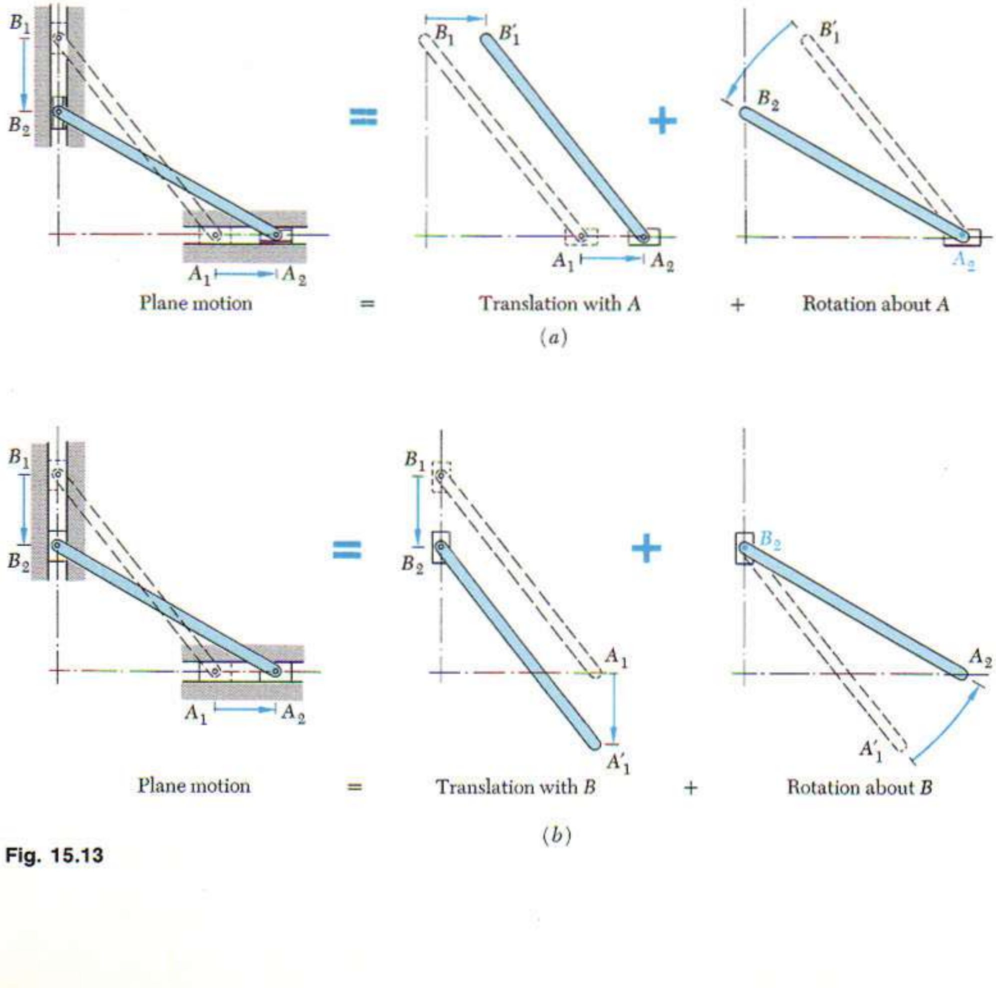


Fig. 15.13

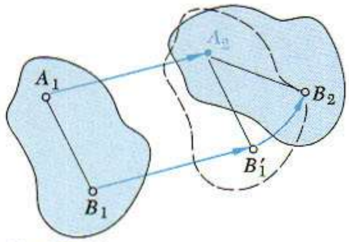


Fig. 15.14

In general, we shall consider a small displacement which brings two particles A and B of a representative slab, respectively, from A_1 and B_1 into A_2 and B_2 (Fig. 15.14). This displacement may be divided into two parts, one in which the particles move into A_2 and B_1' while the line AB maintains the same direction, the other in which B moves into B_2 while A remains fixed. Clearly, the first part of the motion is a translation and the second part a rotation about A.

Recalling from Sec. 11.12 the definition of the "relative motion" of a particle with respect to a moving frame of reference—as opposed to its "absolute motion" with respect to a fixed frame of reference—we may restate as follows the result obtained above: Given two particles A and B of a rigid slab in plane motion, the relative motion of B with respect to a frame attached to A and of fixed orientation is a rotation. To an observer moving with A, but not rotating, particle B will appear to describe an arc of circle centered at A.

15.6. Absolute and Relative Velocity in Plane Motion. We saw in the preceding section that any plane motion of a slab may be replaced by a translation defined by the motion of an arbitrary reference point A, and by a rotation about A. The absolute velocity $\mathbf{v}_B$ of a particle B of the slab is obtained from the relative-velocity formula derived in Sec. 11.12,

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \tag{15.17}$$

where the right-hand member represents a vector sum. The velocity $\mathbf{v}_A$ corresponds to the translation of the slab with A, while the relative velocity $\mathbf{v}_{B/A}$ is associated with the rotation of the slab about A and is measured with respect to axes centered at A and of fixed orientation (Fig. 15.15). Denoting by $\mathbf{r}_{B/A}$ the position vector of B relative to A, and by $\omega\mathbf{k}$ the angular velocity of the slab with respect to axes of fixed orientation, we have from (15.10) and (15.10′)

$$\mathbf{v}_{B/A} = \omega\mathbf{k} \times \mathbf{r}_{B/A} \qquad v_{B/A} = r\omega \tag{15.18}$$

where r is the distance from A to B. Substituting for $\mathbf{v}_{B/A}$ from (15.18) into (15.17), we may also write

$$\mathbf{v}_B = \mathbf{v}_A + \omega\mathbf{k} \times \mathbf{r}_{B/A} \tag{15.17′}$$

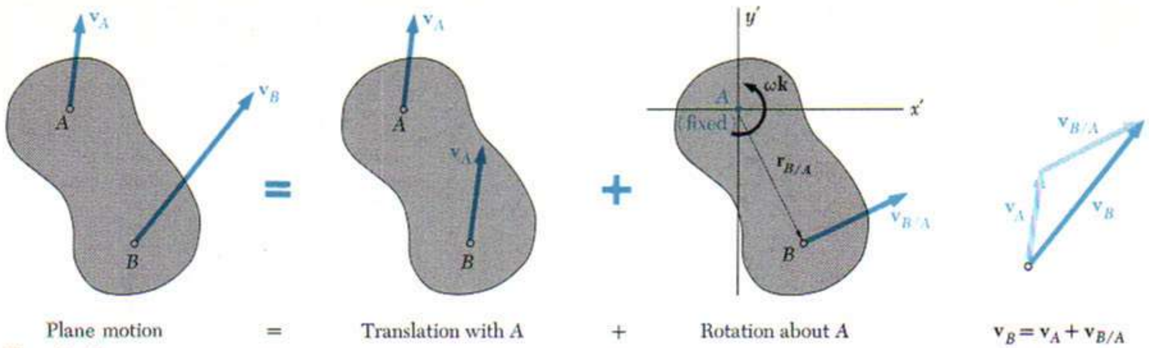


Fig. 15.15

As an example, we shall consider again the rod AB of Fig. 15.13. Assuming that the velocity $\mathbf{v}_A$ of end A is known, we propose to find the velocity $\mathbf{v}_B$ of end B and the angular velocity ω of the rod, in terms of the velocity $\mathbf{v}_A$, the length l, and the angle θ. Choosing A as reference point, we express that the given motion is equivalent to a translation with A and a rotation about A (Fig. 15.16). The absolute velocity of B must therefore be equal to the vector sum

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \tag{15.17}$$

We note that, while the direction of $\mathbf{v}_{B/A}$ is known, its magnitude $l\omega$ is unknown. However, this is compensated by the fact that the direction of $\mathbf{v}_B$ is known. We may therefore complete the diagram of Fig. 15.16. Solving for the magnitudes v_B and ω, we write

$$v_B = v_A \tan\theta \qquad \omega = \frac{v_{B/A}}{l} = \frac{v_A}{l\cos\theta} \tag{15.19}$$

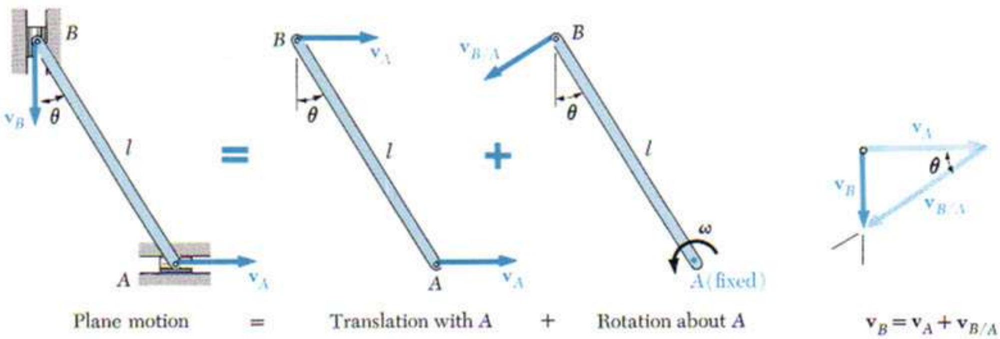


Fig. 15.16

The same result may be obtained by using B as a point of reference. Resolving the given motion into a translation with B and a rotation about B (Fig. 15.17), we write the equation

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B} \tag{15.20}$$

which is represented graphically in Fig. 15.17. We note that $\mathbf{v}_{A/B}$ and $\mathbf{v}_{B/A}$ have the same magnitude $l\omega$ but opposite sense. The sense of the relative velocity depends, therefore, upon the point of reference which has been selected and should be carefully ascertained from the appropriate diagram (Fig. 15.16 or 15.17).

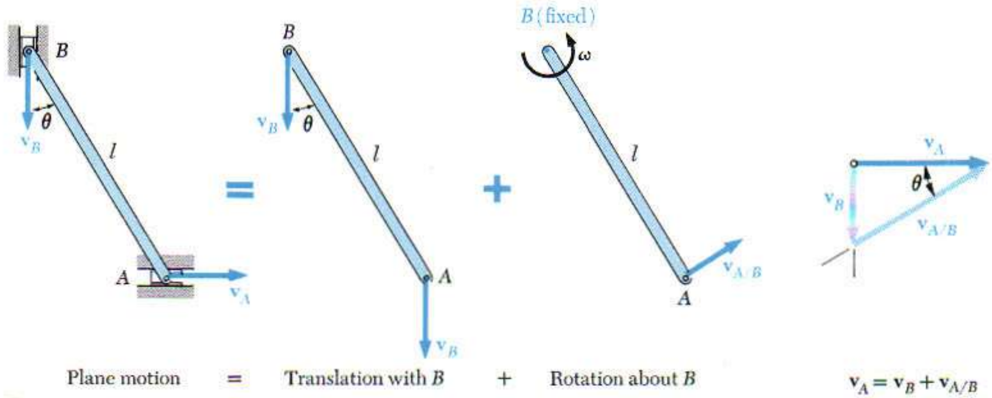
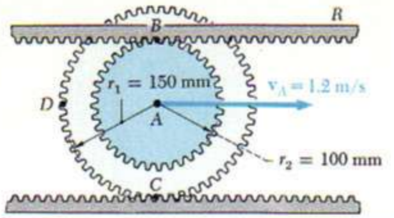


Fig. 15.17

Finally, we observe that the angular velocity ω of the rod in its rotation about B is the same as in its rotation about A. It is measured in both cases by the rate of change of the angle θ. This result is quite general; we should therefore bear in mind that *the angular velocity* ω *of a rigid body in plane motion is independent of the reference point.*

Most mechanisms consist, not of one, but of *several* moving parts. When the various parts of a mechanism are pin-connected, its analysis may be carried out by considering each part as a rigid body, while keeping in mind that the points where two parts are connected must have the same absolute velocity (see Sample Prob. 15.3). A similar analysis may be used when gears are involved, since the teeth in contact must also have the same absolute velocity. However, when a mechanism contains parts which slide on each other, the relative velocity of the parts in contact must be taken into account (see Secs. 15.10 and 15.11).

SAMPLE PROBLEM 15.2



The double gear shown rolls on the stationary lower rack; the velocity of its center A is 1.2 m/s directed to the right. Determine (*a*) the angular velocity of the gear, (*b*) the velocities of the upper rack R and of point D of the gear.

a. Angular Velocity of the Gear. Since the gear rolls on the lower rack, its center A moves through a distance equal to the outer circumference $2\pi r_1$ for each full revolution of the gear. Noting that 1 rev = 2π rad, and that when A moves to the right ($x_A > 0$) the gear rotates clockwise ($\theta < 0$), we write

$$\frac{x_A}{2\pi r_1} = -\frac{\theta}{2\pi} \qquad x_A = -r_1\theta$$

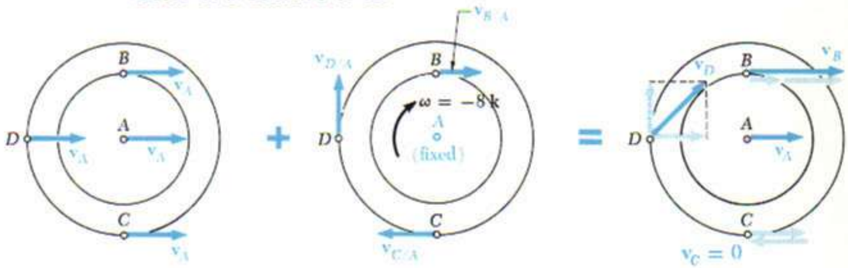
Differentiating with respect to the time t and substituting the known values $v_A = 1.2$ m/s and $r_1 = 150$ mm $= 0.150$ m, we obtain

$$v_A = -r_1\omega \qquad 1.2 \text{ m/s} = -(0.150 \text{ m})\omega \qquad \omega = -8 \text{ rad/s}$$

$$\boldsymbol{\omega} = \omega\mathbf{k} = -(8 \text{ rad/s})\mathbf{k} \quad ◀$$

where $\mathbf{k}$ is a unit vector pointing out of the paper.

b. Velocities. The rolling motion is resolved into two component motions: a translation with the center A and a rotation about the center A. In the translation, all points of the gear move with the same velocity $\mathbf{v}_A$. In the rotation, each point P of the gear moves about A with a relative velocity $\mathbf{v}_{P/A} = \omega\mathbf{k} \times \mathbf{r}_{P/A}$, where $\mathbf{r}_{P/A}$ is the position vector of P relative to A.

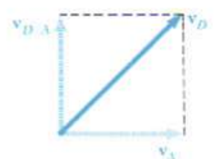


Translation + Rotation = Rolling motion

Velocity of Upper Rack. The velocity of the upper rack is equal to the velocity of point B; we write

$$\begin{aligned}\mathbf{v}_R = \mathbf{v}_B &= \mathbf{v}_A + \mathbf{v}_{B/A} = \mathbf{v}_A + \omega\mathbf{k} \times \mathbf{r}_{B/A}\\ &= (1.2 \text{ m/s})\mathbf{i} - (8 \text{ rad/s})\mathbf{k} \times (0.100 \text{ m})\mathbf{j}\\ &= (1.2 \text{ m/s})\mathbf{i} + (0.8 \text{ m/s})\mathbf{i} = (2 \text{ m/s})\mathbf{i}\end{aligned}$$

$$\mathbf{v}_R = 2 \text{ m/s} \rightarrow \quad ◀$$

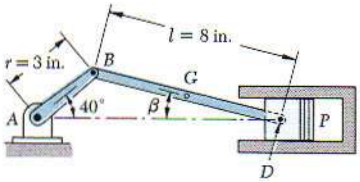


Velocity of Point D:

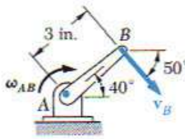
$$\begin{aligned}\mathbf{v}_D &= \mathbf{v}_A + \mathbf{v}_{D/A} = \mathbf{v}_A + \omega\mathbf{k} \times \mathbf{r}_{D/A}\\ &= (1.2 \text{ m/s})\mathbf{i} - (8 \text{ rad/s})\mathbf{k} \times (-0.150 \text{ m})\mathbf{i}\\ &= (1.2 \text{ m/s})\mathbf{i} + (1.2 \text{ m/s})\mathbf{j}\end{aligned}$$

$$\mathbf{v}_D = 1.697 \text{ m/s} \measuredangle 45° \quad ◀$$

SAMPLE PROBLEM 15.3



In the engine system shown, the crank AB has a constant clockwise angular velocity of 2000 rpm. For the crank position indicated, determine (*a*) the angular velocity of the connecting rod BD, (*b*) the velocity of the piston P.



Motion of Crank AB. The crank AB rotates about point A. Expressing ω_{AB} in rad/s and writing $v_B = r\omega_{AB}$, we obtain

$$\omega_{AB} = \left(2000\,\frac{\text{rev}}{\text{min}}\right)\left(\frac{1\text{ min}}{60\text{ s}}\right)\left(\frac{2\pi\text{ rad}}{1\text{ rev}}\right) = 209\text{ rad/s}$$

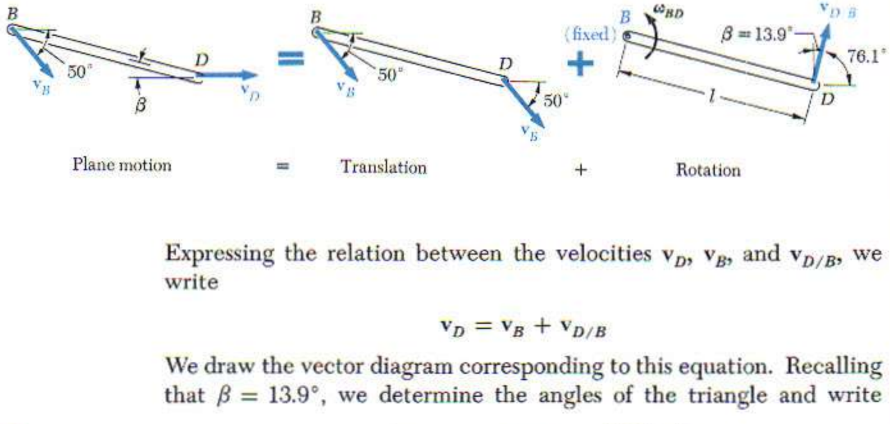
$$v_B = (AB)\omega_{AB} = (3\text{ in.})(209\text{ rad/s}) = 627\text{ in./s}$$

$$\mathbf{v}_B = 627\text{ in./s} \searrow 50^\circ$$

Motion of Connecting Rod BD. We consider this motion as a general plane motion. Using the law of sines, we compute the angle β between the connecting rod and the horizontal,

$$\frac{\sin 40^\circ}{8\text{ in.}} = \frac{\sin\beta}{3\text{ in.}} \qquad \beta = 13.9^\circ$$

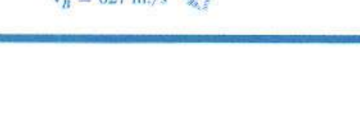
The velocity $\mathbf{v}_D$ of the point D where the rod is attached to the piston must be horizontal, while the velocity of point B is equal to the velocity $\mathbf{v}_B$ obtained above. Resolving the motion of BD into a translation with B and a rotation about B, we obtain



Expressing the relation between the velocities $\mathbf{v}_D$, $\mathbf{v}_B$, and $\mathbf{v}_{D/B}$, we write

$$\mathbf{v}_D = \mathbf{v}_B + \mathbf{v}_{D/B}$$

We draw the vector diagram corresponding to this equation. Recalling that $\beta = 13.9^\circ$, we determine the angles of the triangle and write



$$\frac{v_D}{\sin 53.9^\circ} = \frac{v_{D/B}}{\sin 50^\circ} = \frac{627\text{ in./s}}{\sin 76.1^\circ}$$

$$v_{D/B} = 495\text{ in./s} \qquad \mathbf{v}_{D/B} = 495\text{ in./s} \angle 76.1^\circ$$

$$v_D = 522\text{ in./s} = 43.5\text{ ft/s} \qquad \mathbf{v}_D = 43.5\text{ ft/s} \rightarrow$$

$$\mathbf{v}_P = \mathbf{v}_D = 43.5\text{ ft/s} \rightarrow \quad ◄$$

Since $v_{D/B} = l\omega_{BD}$, we have

$$495\text{ in./s} = (8\text{ in.})\omega_{BD} \qquad \omega_{BD} = 61.9\text{ rad/s} \circlearrowright \quad ◄$$

PROBLEMS

15.25 An automobile travels to the right at a constant speed of 50 km/h. (*a*) If the diameter of a wheel is 610 mm, determine the velocity of points *B*, *C*, *D*, and *E* on the rim of the wheel. (*b*) Solve part *a* assuming that the diameter of the wheel is reduced to 560 mm.

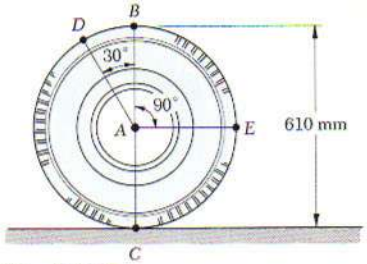


Fig. P15.25

15.26 Collar *B* moves with a constant velocity of 25 in./s to the left. At the instant when $\theta = 30°$, determine (*a*) the angular velocity of rod *AB*, (*b*) the velocity of collar *A*.

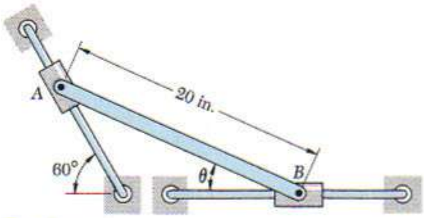


Fig. P15.26

15.27 Solve Prob. 15.26, assuming that $\theta = 45°$.

15.28 The plate shown moves in the *xy* plane. Knowing that $(v_A)_x = 80$ mm/s, $(v_B)_y = 200$ mm/s, and $(v_C)_y = -40$ mm/s, determine (*a*) the angular velocity of the plate, (*b*) the velocity of point *A*.

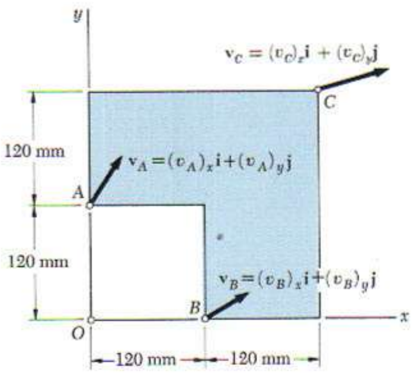


Fig. P15.28

15.29 In Prob. 15.28, determine the equation of the locus of the points of the plate for which the magnitude of the velocity is 100 mm/s.

15.30 In Prob. 15.28, determine (*a*) the velocity of point *B*, (*b*) the point of the plate with zero velocity.

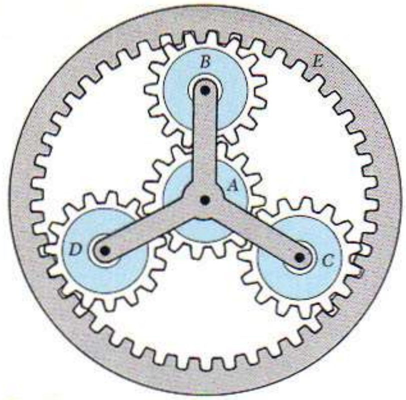


Fig. P15.31

15.31 In the planetary gear system shown, the radius of gears A, B, C, and D is a and the radius of the outer gear E is $3a$. Knowing that the angular velocity of gear A is ω_A clockwise and that the outer gear E is stationary, determine (*a*) the angular velocity of the spider connecting the planetary gears, (*b*) the angular velocity of each planetary gear.

15.32 Two rollers A and B of radius r are joined by a link AB and roll along a horizontal surface. A drum C of radius $2r$ is placed on the rollers as shown. If the link moves to the right with a constant velocity $\mathbf{v}$, determine (*a*) the angular velocity of the rollers and of the drum, (*b*) the velocity of points D, E, and F of the drum.

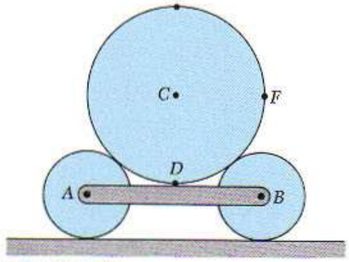


Fig. P15.32

15.33 Gear A rotates clockwise with a constant angular velocity of 60 rpm. Knowing that at the same time the arm AB rotates counterclockwise with a constant angular velocity of 30 rpm, determine the angular velocity of gear B.

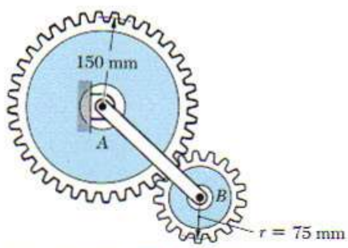


Fig. P15.33 and P15.34

15.34 Arm AB rotates with an angular velocity of 120 rpm clockwise. If the motion of gear B is to be a curvilinear translation, determine (*a*) the required angular velocity of gear A, (*b*) the corresponding velocity of the center of gear B.

15.35 Crank AB has a constant angular velocity of 12 rad/s clockwise. Determine the angular velocity of rod BD and the velocity of collar D when (a) $\theta = 0$, (b) $\theta = 90°$, (c) $\theta = 180°$.

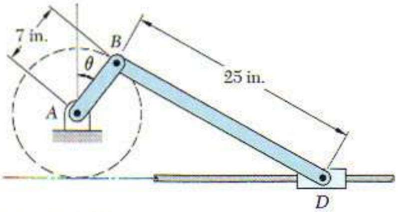


Fig. P15.35

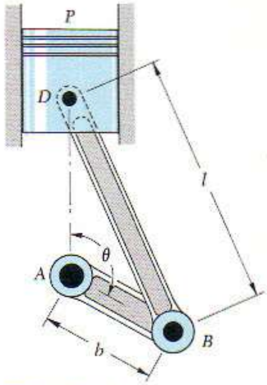


Fig. P15.36

15.36 In the engine system shown, $l = 160$ mm and $b = 60$ mm; the crank AB rotates with a constant angular velocity of 1000 rpm clockwise. Determine the velocity of the piston P and the angular velocity of the connecting rod for the position corresponding to (a) $\theta = 0$, (b) $\theta = 90°$, (c) $\theta = 180°$.

15.37 Solve Prob. 15.36 for the position corresponding to $\theta = 60°$.

15.38 Solve Prob. 15.35 for the position corresponding to $\theta = 120°$.

15.39 through 15.42 In the position shown, bar AB has a constant angular velocity of 3 rad/s counterclockwise. Determine the angular velocity of bars BD and DE.

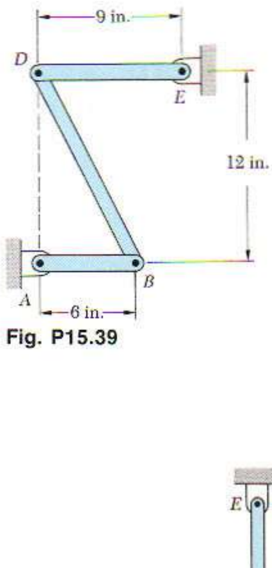


Fig. P15.39

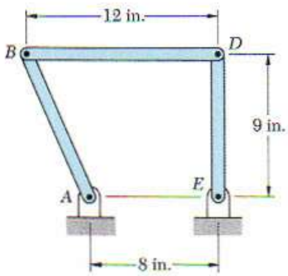


Fig. P15.40

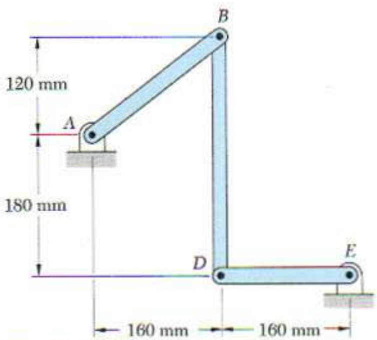


Fig. P15.41

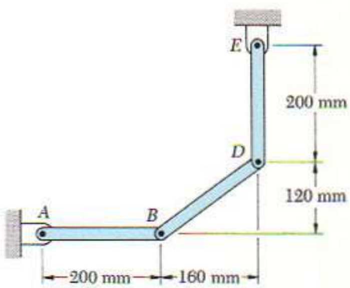


Fig. P15.42

15.43 Two gears, each of 12-in. diameter, are connected by an 18-in. rod AC. Knowing that the center of gear B has a constant velocity of 30 in./s to the right, determine the velocity of the center of gear A and the angular velocity of the connecting rod (*a*) when $\beta = 0$, (*b*) when $\beta = 60°$.

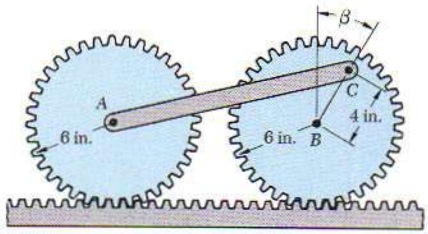


Fig. P15.43

15.44 Solve Prob. 15.43, assuming (*a*) $\beta = 180°$, (*b*) $\beta = 30°$.

15.45 Two collars C and D move along the vertical rod shown. Knowing that the velocity of collar D is 0.210 m/s downward, determine (*a*) the velocity of collar C, (*b*) the angular velocity of member AB.

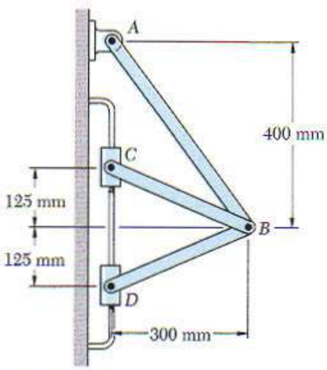


Fig. P15.45

***15.46** Prove for any given position of the mechanism of Prob. 15.45 that the ratio of the magnitudes of the velocities of collars C and D is equal to the ratio of the distances AC and AD.

***15.47** Assuming that the crank AB of Prob. 15.36 rotates with a constant clockwise angular velocity ω and that $\theta = 0$ at $t = 0$, derive an expression for the velocity of the piston P in terms of the time t.

15.7. Instantaneous Center of Rotation in Plane Motion. Consider the general plane motion of a slab. We shall show that at any given instant the velocities of the various particles of the slab are the same as if the slab were rotating about a certain axis perpendicular to the plane of the slab, called the *instantaneous axis of rotation*. This axis intersects the plane of the slab at a point C, called the *instantaneous center of rotation* of the slab.

To prove our statement, we first recall that the plane motion of a slab may always be replaced by a translation defined by the motion of an arbitrary reference point A, and by a rotation about A. As far as the velocities are concerned, the translation is characterized by the velocity $\mathbf{v}_A$ of the reference point A and the rotation is characterized by the angular velocity $\boldsymbol{\omega}$ of the slab (which is independent of the choice of A). Thus, the velocity $\mathbf{v}_A$ of point A and the angular velocity $\boldsymbol{\omega}$ of the slab define completely the velocities of all the other particles of the slab (Fig. 15.18*a*). Now let us assume that $\mathbf{v}_A$ and $\boldsymbol{\omega}$ are known and

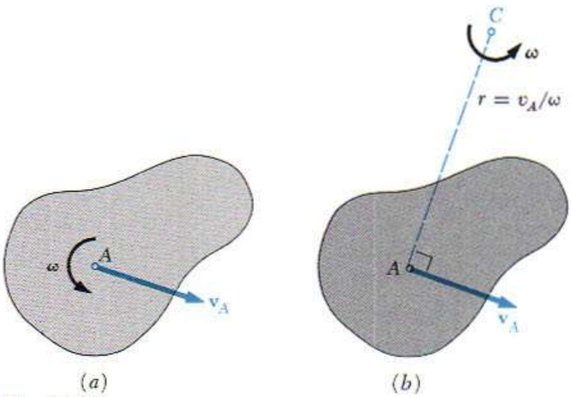


Fig. 15.18

that they are both different from zero. (If $\mathbf{v}_A = 0$, point A is itself the instantaneous center of rotation, and if $\boldsymbol{\omega} = 0$, all the particles have the same velocity $\mathbf{v}_A$.) These velocities could be obtained by letting the slab rotate with the angular velocity $\boldsymbol{\omega}$ about a point C located on the perpendicular to $\mathbf{v}_A$ at a distance $r = v_A/\omega$ from A as shown in Fig. 15.18*b*. We check that the velocity of A would be perpendicular to AC and that its magnitude would be $r\omega = (v_A/\omega)\omega = v_A$. Thus the velocities of all the other particles of the slab would be the same as originally defined. Therefore, *as far as the velocities are concerned, the slab seems to rotate about the instantaneous center* C at the instant considered.

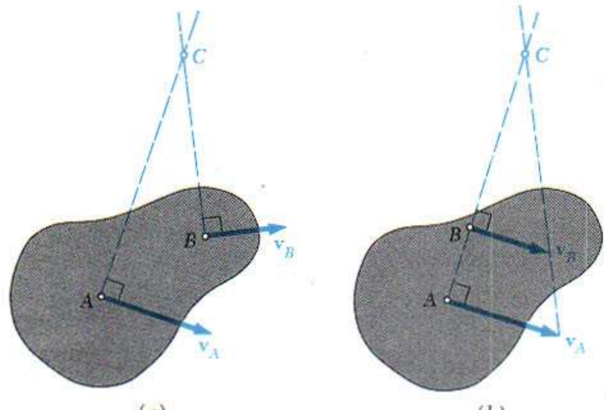


Fig. 15.19

The position of the instantaneous center may be defined in two other ways. If the directions of the velocities of two particles A and B of the slab are known, and if they are different, the instantaneous center C is obtained by drawing the perpendicular to $\mathbf{v}_A$ through A and the perpendicular to $\mathbf{v}_B$ through B and determining the point in which these two lines intersect (Fig. 15.19*a*). If the velocities $\mathbf{v}_A$ and $\mathbf{v}_B$ of two particles A and B are perpendicular to the line AB, and if their magnitudes are known, the instantaneous center may be found by intersecting the line AB with the line joining the extremities of the vectors $\mathbf{v}_A$ and $\mathbf{v}_B$ (Fig. 15.19*b*). Note that, if $\mathbf{v}_A$ and $\mathbf{v}_B$ were parallel in Fig. 15.19*a*, or if $\mathbf{v}_A$ and $\mathbf{v}_B$ had the same magnitude in Fig. 15.19*b*, the instantaneous center C would be at an infinite distance and $\boldsymbol{\omega}$ would be zero; all points of the slab would have the same velocity.

To see how the concept of instantaneous center of rotation may be put to use, let us consider again the rod of Sec. 15.6. Drawing the perpendicular to $\mathbf{v}_A$ through A and the perpendicular to $\mathbf{v}_B$ through B (Fig. 15.20), we obtain the instantaneous center C. At the instant considered, the velocities of all the particles of the rod are thus the same as if the rod rotated about C. Now, if the magnitude v_A of the velocity of A is known, the magnitude ω of the angular velocity of the rod may be obtained by writing

Fig. 15.20

$$\omega = \frac{v_A}{AC} = \frac{v_A}{l\cos\theta}$$

The magnitude of the velocity of B may then be obtained by writing

$$v_B = (BC)\omega = l \sin\theta \frac{v_A}{l\cos\theta} = v_A \tan\theta$$

Note that only *absolute* velocities are involved in the computation.

The instantaneous center of a slab in plane motion may be located either on the slab or outside the slab. If it is located on the slab, the particle C coinciding with the instantaneous center at a given instant t must have zero velocity at that instant. However, it should be noted that the instantaneous center of rotation is valid only at a given instant. Thus, the particle C of the slab which coincides with the instantaneous center at time t will generally not coincide with the instantaneous center at time $t + \Delta t$; while its velocity is zero at time t, it will probably be different from zero at time $t + \Delta t$. This means that, in general, the particle C *does not have zero acceleration*, and therefore that the *accelerations* of the various particles of the slab *cannot* be determined as if the slab were rotating about C.

As the motion of the slab proceeds, the instantaneous center moves in space. But it was just pointed out that the position of the instantaneous center on the slab keeps changing. Thus, the instantaneous center describes one curve in space, called the *space centrode*, and another curve on the slab, called the *body centrode* (Fig. 15.21). It may be shown that, at any instant, these two curves are tangent at C and that, as the slab moves, the body centrode appears to *roll* on the space centrode.

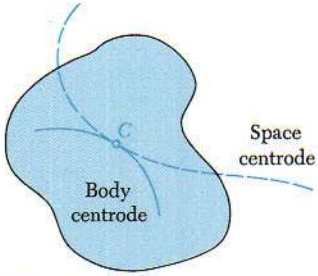
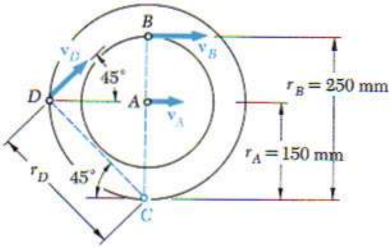


Fig. 15.21

SAMPLE PROBLEM 15.4

Solve Sample Prob. 15.2, using the method of the instantaneous center of rotation.



a. Angular Velocity of the Gear. Since the gear rolls on the stationary lower rack, the point of contact C of the gear with the rack has no velocity; point C is therefore the instantaneous center of rotation. We write

$$v_A = r_A\omega \qquad 1.2\text{ m/s} = (0.150\text{ m})\omega \qquad \omega = 8\text{ rad/s} \circlearrowright \quad ◄$$

b. Velocities. All points of the gear seem to rotate about the instantaneous center as far as velocities are concerned.

Velocity of Upper Rack. Recalling that $v_R = v_B$, we write

$$v_R = v_B = r_B\omega \qquad v_R = (0.250\text{ m})(8\text{ rad/s}) = 2\text{ m/s}$$

$$\mathbf{v}_R = 2\text{ m/s} \rightarrow \quad ◄$$

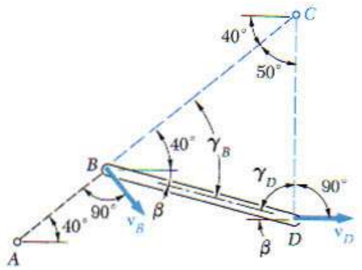
Velocity of Point D. Since $r_D = (0.150\text{ m})\sqrt{2} = 0.212$ m, we write

$$v_D = r_D\omega \qquad v_D = (0.212\text{ m})(8\text{ rad/s}) = 1.696\text{ m/s}$$

$$\mathbf{v}_D = 1.696\text{ m/s} \measuredangle 45° \quad ◄$$

SAMPLE PROBLEM 15.5

Solve Sample Prob. 15.3, using the method of the instantaneous center of rotation.



Motion of Crank AB. Referring to Sample Prob. 15.3, we obtain the velocity of point B; $\mathbf{v}_B = 627$ in./s $\searrow$ 50°.

Motion of the Connecting Rod BD. We first locate the instantaneous center C by drawing lines perpendicular to the absolute velocities $\mathbf{v}_B$ and $\mathbf{v}_D$. Recalling from Sample Prob. 15.3 that $\beta = 13.9°$ and that $BD = 8$ in., we solve the triangle BCD.

$$\gamma_B = 40° + \beta = 53.9° \qquad \gamma_D = 90° - \beta = 76.1°$$

$$\frac{BC}{\sin 76.1°} = \frac{CD}{\sin 53.9°} = \frac{8\text{ in.}}{\sin 50°}$$

$$BC = 10.14\text{ in.} \qquad CD = 8.44\text{ in.}$$

Since the connecting rod BD seems to rotate about point C, we write

$$v_B = (BC)\omega_{BD}$$

$$627\text{ in./s} = (10.14\text{ in.})\omega_{BD}$$

$$\omega_{BD} = 61.9\text{ rad/s} \circlearrowleft \quad ◄$$

$$v_D = (CD)\omega_{BD} = (8.44\text{ in.})(61.9\text{ rad/s})$$

$$= 522\text{ in./s} = 43.5\text{ ft/s}$$

$$\mathbf{v}_P = \mathbf{v}_D = 43.5\text{ ft/s} \rightarrow \quad ◄$$

PROBLEMS

15.48 A helicopter moves horizontally in the x direction at a speed of 120 mi/h. Knowing that the main blades rotate clockwise at an angular velocity of 180 rpm, determine the instantaneous axis of rotation of the main blades.

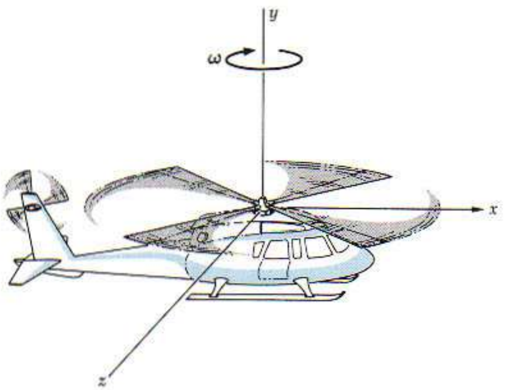


Fig. P15.48

15.49 Denoting by $\mathbf{r}_A$ the position vector of a point A of a rigid slab which moves in plane motion, show that the position vector $\mathbf{r}_C$ of the instantaneous center of rotation is

$$\mathbf{r}_C = \mathbf{r}_A + \frac{\boldsymbol{\omega} \times \mathbf{v}_A}{\omega^2}$$

where $\boldsymbol{\omega}$ is the angular velocity of the slab and $\mathbf{v}_A$ the velocity of point A.

15.50 A drum, of radius 4.5 in., is mounted on a cylinder, of radius 6 in. A cord is wound around the drum, and its extremity D is pulled to the left at a constant velocity of 3 in./s, causing the cylinder to roll without sliding. Determine (*a*) the angular velocity of the cylinder, (*b*) the velocity of the center of the cylinder, (*c*) the length of cord which is wound or unwound per second.

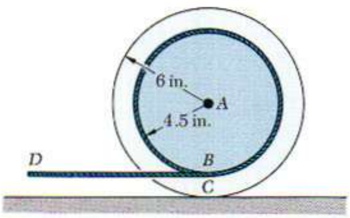


Fig. P15.50

15.51 Solve Sample Prob. 15.2, assuming that the lower rack is not stationary but moves to the left with a velocity of 0.6 m/s.

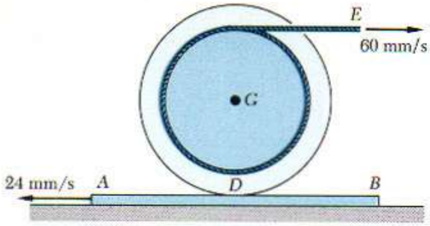


Fig. P15.52

15.52 A double pulley rolls without sliding on the plate AB, which moves to the left at a constant speed of 24 mm/s. The 60-mm-radius inner pulley is rigidly attached to the 80-mm-radius outer pulley. Knowing that cord E is pulled at a constant speed of 60 mm/s as shown, determine (*a*) the angular velocity of the pulley, (*b*) the velocity of the center G of the pulley.

15.53 Knowing that at the instant shown the velocity of collar D is 20 in./s upward, determine (*a*) the angular velocity of rod AD, (*b*) the velocity of point B, (*c*) the velocity of point A.

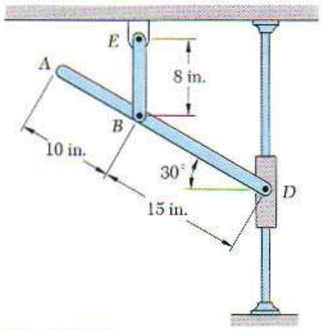


Fig. P15.53

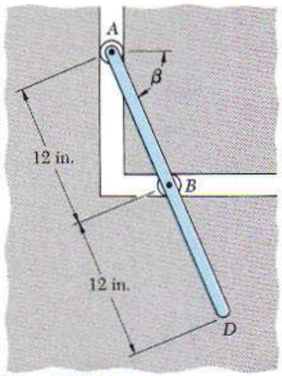


Fig. P15.54

15.54 The rod ABD is guided by wheels which roll in the tracks shown. Knowing that $\beta = 60°$ and that the velocity of A is 24 in./s downward, determine (*a*) the angular velocity of the rod, (*b*) the velocity of point D.

15.55 Solve Prob. 15.54, assuming that $\beta = 30°$.

15.56 Knowing that at the instant shown the angular velocity of crank AB is 3 rad/s clockwise, determine (*a*) the angular velocity of link BD, (*b*) the velocity of collar D, (*c*) the velocity of the midpoint of link BD.

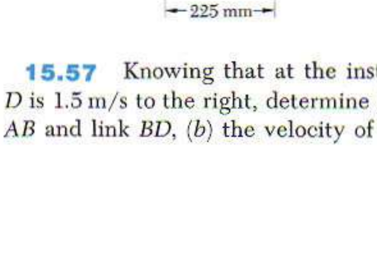


Fig. P15.56 and P15.57

15.57 Knowing that at the instant shown the velocity of collar D is 1.5 m/s to the right, determine (*a*) the angular velocities of crank AB and link BD, (*b*) the velocity of the midpoint of link BD.

15.58 Collar A slides downward with a constant velocity $\mathbf{v}_A$. Determine the angle θ corresponding to the position of rod AB for which the velocity of B is horizontal.

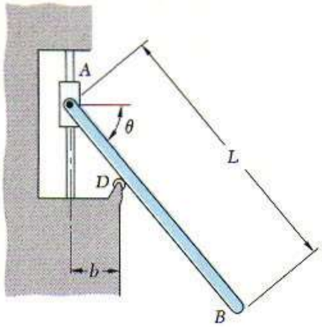


Fig. P15.58 and P15.60

15.59 Two rods AB and BD are connected to three collars as shown. Knowing that collar A moves downward with a constant velocity of 120 mm/s, determine at the instant shown (*a*) the angular velocity of each rod, (*b*) the velocity of collar D.

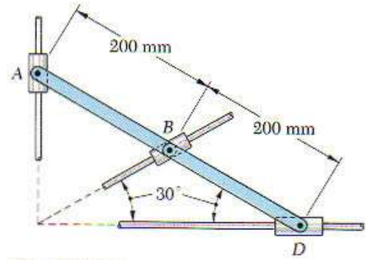


Fig. P15.59

15.60 Collar A slides downward with a constant speed of 16 in./s. Knowing that $b = 2$ in., $L = 10$ in., and $\theta = 60°$, determine (*a*) the angular velocity of rod AB, (*b*) the velocity of B.

15.61 The rectangular plate is supported by two 6-in. links as shown. Knowing that at the instant shown the angular velocity of link AB is 4 rad/s clockwise, determine (*a*) the angular velocity of the plate, (*b*) the velocity of the center of the plate, (*c*) the velocity of corner F.

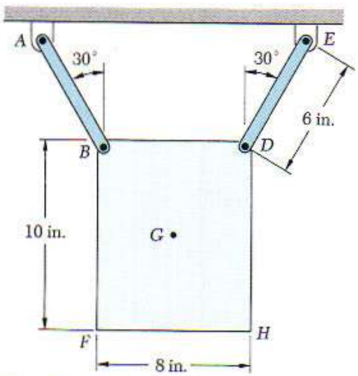


Fig. P15.61 and P15.62

15.62 Knowing that, at the instant shown, the angular velocity of link AB is 4 rad/s clockwise, determine (*a*) the angular velocity of the plate, (*b*) the points of the plate for which the magnitude of the velocity is equal to or less than 6 in./s.

15.63 At the instant shown, the velocity of the center of the gear is 200 mm/s to the right. Determine (*a*) the velocity of point *B*, (*b*) the velocity of collar *D*.

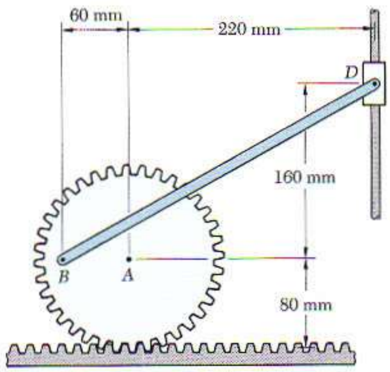


Fig. P15.63 and P15.64

15.64 At the instant shown, the velocity of collar *D* is 360 mm/s downward. Determine (*a*) the angular velocity of rod *BD*, (*b*) the velocity of the center of the gear.

15.65 Describe the space centrode and the body centrode of gear *A* of Prob. 15.63 as the gear rolls on the horizontal rack.

15.66 Describe the space centrode and the body centrode of rod *ABD* of Prob. 15.54 as point *A* moves downward. (*Note.* The body centrode need not lie on a physical portion of the rod.)

15.67 Using the method of Sec. 15.7, solve Prob. 15.35.

15.68 Using the method of Sec. 15.7, solve Prob. 15.36.

15.69 Using the method of Sec. 15.7, solve Prob. 15.39.

15.70 Using the method of Sec. 15.7, solve Prob. 15.40.

15.71 Using the method of Sec. 15.7, solve Prob. 15.41.

15.72 Using the method of Sec. 15.7, solve Prob. 15.42.

15.73 Using the method of Sec. 15.7, solve Prob. 15.43.

15.74 Using the method of Sec. 15.7, solve Prob. 15.32.

15.75 Using the method of Sec. 15.7, solve Prob. 15.33.

15.8. Absolute and Relative Acceleration in Plane Motion. We saw in Sec. 15.5 that any plane motion may be replaced by a translation defined by the motion of an arbitrary reference point A, and by a rotation about A. This property was used in Sec. 15.6 to determine the velocity of the various points of a moving slab. We shall now use the same property to determine the acceleration of the points of the slab.

We first recall that the absolute acceleration $\mathbf{a}_B$ of a particle of the slab may be obtained from the relative-acceleration formula derived in Sec. 11.12,

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A} \tag{15.21}$$

where the right-hand member represents a vector sum. The acceleration $\mathbf{a}_A$ corresponds to the translation of the slab with A, while the relative acceleration $\mathbf{a}_{B/A}$ is associated with the rotation of the slab about A and is measured with respect to axes centered at A and of fixed orientation. We recall from Sec. 15.3 that the relative acceleration $\mathbf{a}_{B/A}$ may be resolved into two components, a *tangential component* $(\mathbf{a}_{B/A})_t$ perpendicular to the line AB, and a *normal component* $(\mathbf{a}_{B/A})_n$ directed toward A (Fig. 15.22). Denoting by $\mathbf{r}_{B/A}$ the position vector of B relative to A and, respectively, by $\omega\mathbf{k}$ and $\alpha\mathbf{k}$ the angular velocity and angular acceleration of the slab with respect to axes of fixed orientation, we have

$$\begin{aligned} (\mathbf{a}_{B/A})_t &= \alpha\mathbf{k} \times \mathbf{r}_{B/A} \qquad & (a_{B/A})_t &= r\alpha \\ (\mathbf{a}_{B/A})_n &= -\omega^2\mathbf{r}_{B/A} \qquad & (a_{B/A})_n &= r\omega^2 \end{aligned} \tag{15.22}$$

where r is the distance from A to B. Substituting into (15.21) the expressions obtained for the tangential and normal components of $\mathbf{a}_{B/A}$, we may also write

$$\mathbf{a}_B = \mathbf{a}_A + \alpha\mathbf{k} \times \mathbf{r}_{B/A} - \omega^2\mathbf{r}_{B/A} \tag{15.21'}$$

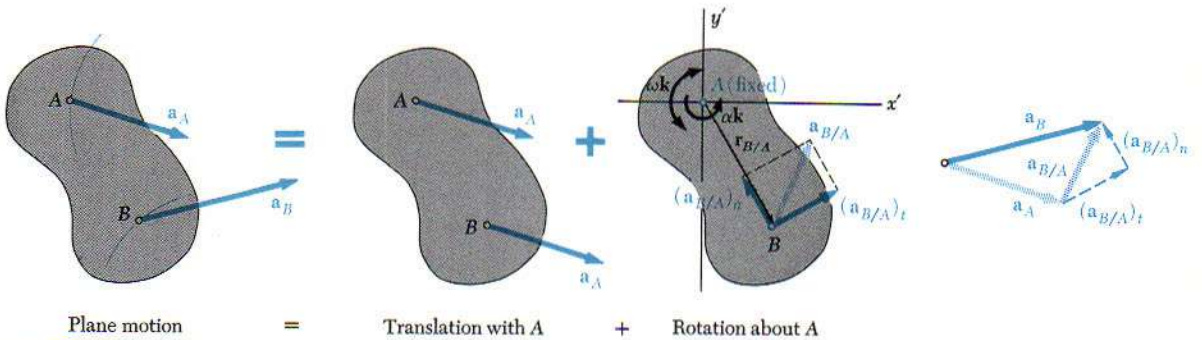


Fig. 15.22

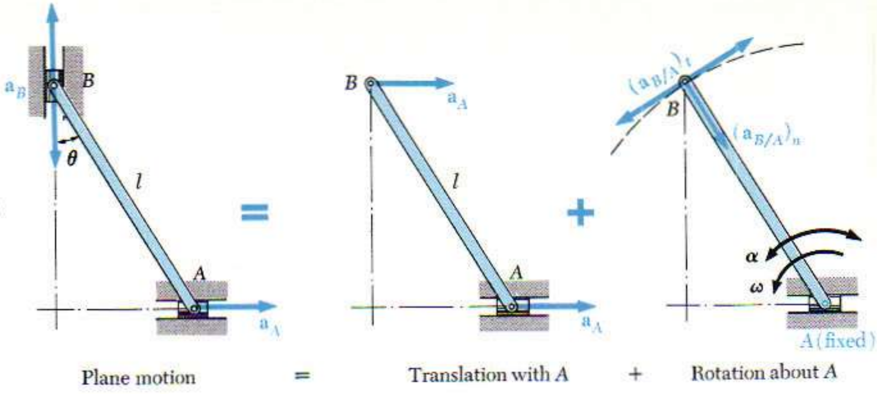


Fig. 15.23

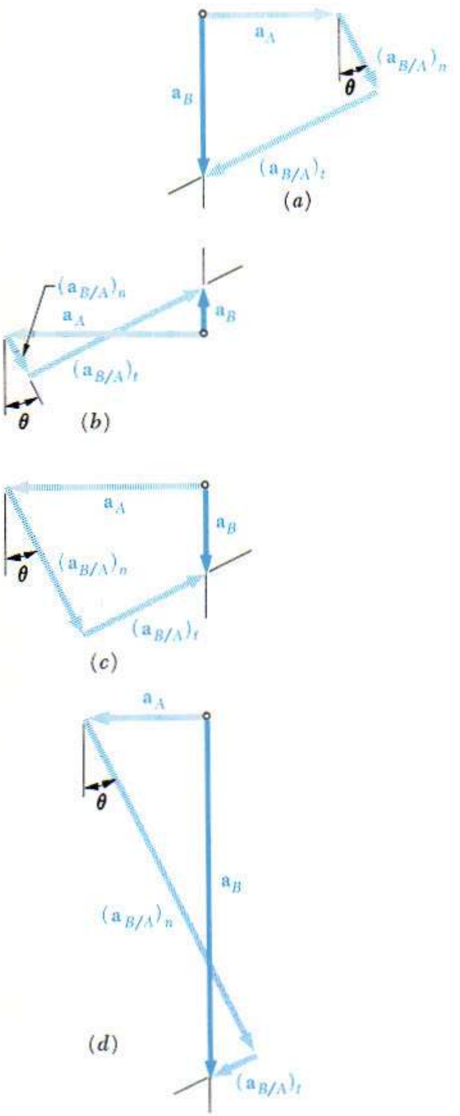


Fig. 15.24

As an example, we shall consider again the rod AB whose extremities slide, respectively, along a horizontal and a vertical track (Fig. 15.23). Assuming that the velocity $\mathbf{v}_A$ and the acceleration $\mathbf{a}_A$ of A are known, we propose to determine the acceleration $\mathbf{a}_B$ of B and the angular acceleration α of the rod. Choosing A as a reference point, we express that the given motion is equivalent to a translation with A and a rotation about A. The absolute acceleration of B must be equal to the sum

$$\begin{aligned} \mathbf{a}_B &= \mathbf{a}_A + \mathbf{a}_{B/A} \\ &= \mathbf{a}_A + (\mathbf{a}_{B/A})_n + (\mathbf{a}_{B/A})_t \end{aligned} \tag{15.23}$$

where $(\mathbf{a}_{B/A})_n$ has the magnitude $l\omega^2$ and is *directed toward* A, while $(\mathbf{a}_{B/A})_t$ has the magnitude $l\alpha$ and is perpendicular to AB. There is no way of telling at the present time whether the tangential component $(\mathbf{a}_{B/A})_t$ is directed to the left or to the right, and the student should not rely on his "intuition" in this matter. We shall therefore indicate both possible directions for this component in Fig. 15.23. Similarly, we indicate both possible senses for $\mathbf{a}_B$, since we do not know whether point B is accelerated upward or downward.

Equation (15.23) has been expressed geometrically in Fig. 15.24. Four different vector polygons may be obtained, depending upon the sense of $\mathbf{a}_A$ and the relative magnitude of a_A and $(a_{B/A})_n$. If we are to determine a_B and α from one of these diagrams, we must know not only a_A and θ but also ω. The angular velocity of the rod, therefore, should be separately determined by one of the methods indicated in Secs. 15.6 and 15.7. The values of a_B and α may then be obtained by considering successively the x and y components of the vectors shown in Fig. 15.24. In the case of polygon a, for example, we write

$\overset{+}{\rightarrow} x$ components: $\quad 0 = a_A + l\omega^2 \sin\theta - l\alpha\cos\theta$

$+\uparrow y$ components: $\quad -a_B = -l\omega^2\cos\theta - l\alpha\sin\theta$

and solve for a_B and α. The two unknowns may also be obtained by direct measurement on the vector polygon. In that case, care should be taken to draw first the known vectors $\mathbf{a}_A$ and $(\mathbf{a}_{B/A})_n$.

It is quite evident that the determination of accelerations is considerably more involved than the determination of velocities. Yet, in the example considered here, the extremities A and B of the rod were moving along straight tracks, and the diagrams drawn were relatively simple. If A and B had moved along curved tracks, the accelerations $\mathbf{a}_A$ and $\mathbf{a}_B$ should have been resolved into normal and tangential components and the solution of the problem would have involved six different vectors.

When a mechanism consists of several moving parts which are pin-connected, its analysis may be carried out by considering each part as a rigid body, while keeping in mind that the points where two parts are connected must have the same absolute acceleration (see Sample Prob. 15.7). In the case of meshed gears, the tangential components of the accelerations of the teeth in contact are equal, but their normal components are different.

***15.9. Analysis of Plane Motion in Terms of a Parameter.** In the case of certain mechanisms, it is possible to express the coordinates x and y of all the significant points of the mechanism by means of simple analytic expressions containing a single parameter. It may be advantageous in such a case to determine directly the absolute velocity and the absolute acceleration of the various points of the mechanism, since the components of the velocity and of the acceleration of a given point may be obtained by differentiating the coordinates x and y of that point.

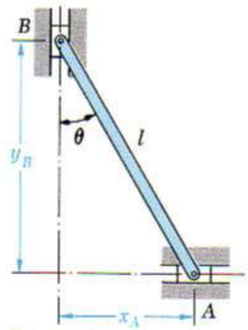


Fig. 15.25

Let us consider again the rod AB whose extremities slide, respectively, in a horizontal and a vertical track (Fig. 15.25). The coordinates x_A and y_B of the extremities of the rod may be expressed in terms of the angle θ the rod forms with the vertical,

$$x_A = l \sin\theta \qquad y_B = l\cos\theta \tag{15.24}$$

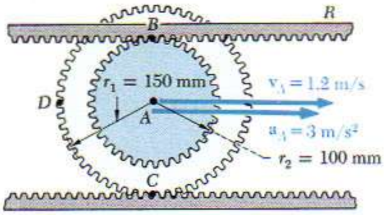
Differentiating Eqs. (15.24) twice with respect to t, we write

$$v_A = \dot{x}_A = l\dot{\theta}\cos\theta \qquad v_B = \dot{y}_B = -l\dot{\theta}\sin\theta$$
$$a_A = \ddot{x}_A = -l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta$$
$$a_B = \ddot{y}_B = -l\dot{\theta}^2\cos\theta - l\ddot{\theta}\sin\theta$$

Recalling that $\dot{\theta} = \omega$ and $\ddot{\theta} = \alpha$, we obtain

$$v_A = l\omega\cos\theta \qquad v_B = -l\omega\sin\theta \tag{15.25}$$
$$a_A = -l\omega^2\sin\theta + l\alpha\cos\theta \qquad a_B = -l\omega^2\cos\theta - l\alpha\sin\theta \tag{15.26}$$

We note that a positive sign for v_A or a_A indicates that the velocity $\mathbf{v}_A$ or the acceleration $\mathbf{a}_A$ is directed to the right; a positive sign for v_B or a_B indicates that $\mathbf{v}_B$ or $\mathbf{a}_B$ is directed upward. Equations (15.25) may be used, for example, to determine v_B and ω when v_A and θ are known. Substituting for ω in (15.26), we may then determine a_B and α if a_A is known.



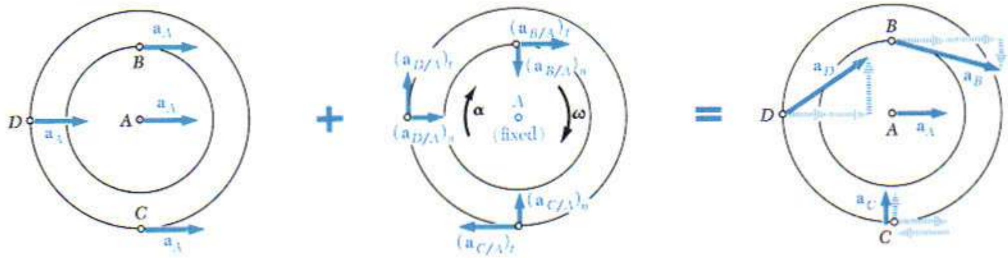
SAMPLE PROBLEM 15.6

The center of the double gear of Sample Prob. 15.2 has a velocity of 1.2 m/s to the right and an acceleration of 3 m/s² to the right. Determine (*a*) the angular acceleration of the gear, (*b*) the acceleration of points *B*, *C*, and *D* of the gear.

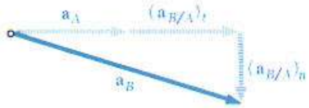
a. Angular Acceleration of the Gear. In Sample Prob. 15.2, we found that $x_A = -r_1\theta$ and $v_A = -r_1\omega$. Differentiating the latter with respect to time, we obtain $a_A = -r_1\alpha$.

$$v_A = -r_1\omega \qquad 1.2\text{ m/s} = -(0.150\text{ m})\omega \qquad \omega = -8\text{ rad/s}$$
$$a_A = -r_1\alpha \qquad 3\text{ m/s}^2 = -(0.150\text{ m})\alpha \qquad \alpha = -20\text{ rad/s}^2$$
$$\boldsymbol{\alpha} = \alpha\mathbf{k} = -(20\text{ rad/s}^2)\mathbf{k} \blacktriangleleft$$

b. Accelerations. The rolling motion of the gear is resolved into a translation with *A* and a rotation about *A*.



Translation + Rotation = Rolling motion



Acceleration of Point B. Adding vectorially the accelerations corresponding to the translation and to the rotation, we obtain

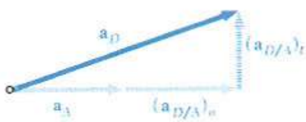
$$\begin{aligned}\mathbf{a}_B &= \mathbf{a}_A + \mathbf{a}_{B/A} = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n\\ &= \mathbf{a}_A + \alpha\mathbf{k} \times \mathbf{r}_{B/A} - \omega^2\mathbf{r}_{B/A}\\ &= (3\text{ m/s}^2)\mathbf{i} - (20\text{ rad/s}^2)\mathbf{k} \times (0.100\text{ m})\mathbf{j} - (8\text{ rad/s})^2(0.100\text{ m})\mathbf{j}\\ &= (3\text{ m/s}^2)\mathbf{i} + (2\text{ m/s}^2)\mathbf{i} - (6.40\text{ m/s}^2)\mathbf{j}\end{aligned}$$

$$\mathbf{a}_B = 8.12\text{ m/s}^2 \searrow 52.0° \blacktriangleleft$$

Acceleration of Point C

$$\begin{aligned}\mathbf{a}_C &= \mathbf{a}_A + \mathbf{a}_{C/A} = \mathbf{a}_A + \alpha\mathbf{k} \times \mathbf{r}_{C/A} - \omega^2\mathbf{r}_{C/A}\\ &= (3\text{ m/s}^2)\mathbf{i} - (20\text{ rad/s}^2)\mathbf{k} \times (-0.150\text{ m})\mathbf{j} - (8\text{ rad/s})^2(-0.150\text{ m})\mathbf{j}\\ &= (3\text{ m/s}^2)\mathbf{i} - (3\text{ m/s}^2)\mathbf{i} + (9.60\text{ m/s}^2)\mathbf{j}\end{aligned}$$

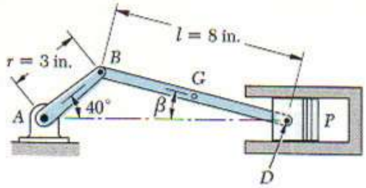
$$\mathbf{a}_C = 9.60\text{ m/s}^2 \uparrow \blacktriangleleft$$



Acceleration of Point D

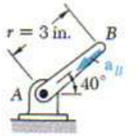
$$\begin{aligned}\mathbf{a}_D &= \mathbf{a}_A + \mathbf{a}_{D/A} = \mathbf{a}_A + \alpha\mathbf{k} \times \mathbf{r}_{D/A} - \omega^2\mathbf{r}_{D/A}\\ &= (3\text{ m/s}^2)\mathbf{i} - (20\text{ rad/s}^2)\mathbf{k} \times (-0.150\text{ m})\mathbf{i} - (8\text{ rad/s})^2(-0.150\text{ m})\mathbf{i}\\ &= (3\text{ m/s}^2)\mathbf{i} + (3\text{ m/s}^2)\mathbf{j} + (9.60\text{ m/s}^2)\mathbf{i}\end{aligned}$$

$$\mathbf{a}_D = 12.95\text{ m/s}^2 \angle 13.4° \blacktriangleleft$$



SAMPLE PROBLEM 15.7

Crank AB of the engine system of Sample Prob. 15.3 has a constant clockwise angular velocity of 2000 rpm. For the crank position shown, determine the angular acceleration of the connecting rod BD and the acceleration of point D.



Motion of Crank AB. Since the crank rotates about A with constant $\omega_{AB} = 2000$ rpm $= 209$ rad/s, we have $\alpha_{AB} = 0$. The acceleration of B is therefore directed toward A and has a magnitude

$$a_B = r\omega_{AB}^2 = (\tfrac{3}{12}\text{ ft})(209\text{ rad/s})^2 = 10{,}920\text{ ft/s}^2$$

$$\mathbf{a}_B = 10{,}920\text{ ft/s}^2 \measuredangle 40°$$

Motion of the Connecting Rod BD. The angular velocity $\boldsymbol{\omega}_{BD}$ and the value of β were obtained in Sample Prob. 15.3.

$$\boldsymbol{\omega}_{BD} = 61.9\text{ rad/s} \circlearrowleft \qquad \beta = 13.9°$$

The motion of BD is resolved into a translation with B and a rotation about B. The relative acceleration $\mathbf{a}_{D/B}$ is resolved into normal and tangential components.

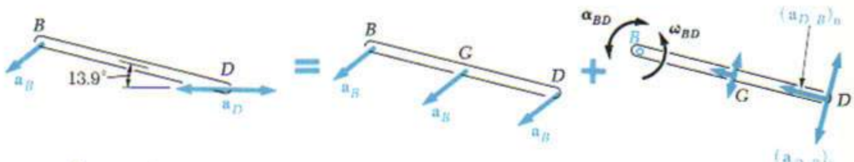
$$(a_{D/B})_n = (BD)\omega_{BD}^2 = (\tfrac{8}{12}\text{ ft})(61.9\text{ rad/s})^2 = 2550\text{ ft/s}^2$$

$$(\mathbf{a}_{D/B})_n = 2550\text{ ft/s}^2 \measuredangle 13.9°$$

$$(a_{D/B})_t = (BD)\alpha_{BD} = (\tfrac{8}{12})\alpha_{BD} = 0.667\alpha_{BD}$$

$$(\mathbf{a}_{D/B})_t = 0.667\alpha_{BD} \measuredangle 76.1°$$

While $(\mathbf{a}_{B/D})_t$ must be perpendicular to BD, its sense is not known.

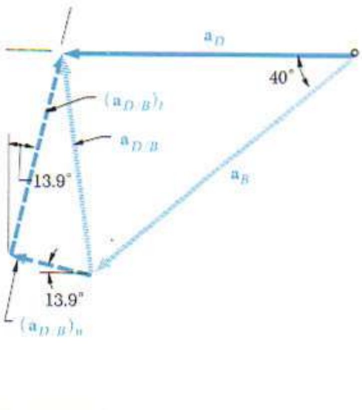


Plane motion = Translation + Rotation

Noting that the acceleration $\mathbf{a}_D$ must be horizontal, we write

$$\mathbf{a}_D = \mathbf{a}_B + \mathbf{a}_{D/B} = \mathbf{a}_B + (\mathbf{a}_{D/B})_n + (\mathbf{a}_{D/B})_t$$

$$[a_D \leftrightarrow] = [10{,}920 \measuredangle 40°] + [2560 \measuredangle 13.9°] + [0.667\alpha_{BD} \measuredangle 76.1°]$$



Equating x and y components, we obtain the following scalar equations:

$\xrightarrow{+} x$ components:

$$-a_D = -10{,}920\cos 40° - 2550\cos 13.9° + 0.667\alpha_{BD}\sin 13.9°$$

$+\uparrow y$ components:

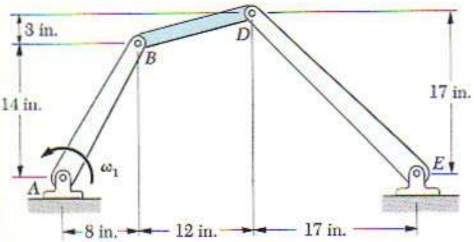
$$0 = -10{,}920\sin 40° + 2550\sin 13.9° + 0.667\alpha_{BD}\cos 13.9°$$

Solving the equations simultaneously, we obtain $\alpha_{BD} = +9890$ rad/s^2 and $a_D = +9260$ ft/s^2. The positive signs indicate that the senses shown on the vector polygon are correct; we write

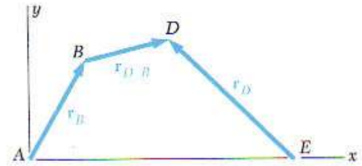
$$\boldsymbol{\alpha}_{BD} = 9890\text{ rad/s}^2 \circlearrowleft \blacktriangleleft$$

$$\mathbf{a}_D = 9260\text{ ft/s}^2 \leftarrow \blacktriangleleft$$

SAMPLE PROBLEM 15.8



The linkage *ABDE* moves in the vertical plane. Knowing that in the position shown crank *AB* has a constant angular velocity $\boldsymbol{\omega}_1$ of 20 rad/s counterclockwise, determine the angular velocities and angular accelerations of the connecting rod *BD* and of the crank *DE*.



$\mathbf{r}_B = 8\mathbf{i} + 14\mathbf{j}$
$\mathbf{r}_D = -17\mathbf{i} + 17\mathbf{j}$
$\mathbf{r}_{D/B} = 12\mathbf{i} + 3\mathbf{j}$

Solution. While this problem could be solved by the method used in Sample Prob. 15.7, we shall make full use of the vector approach in the present case. The position vectors $\mathbf{r}_B$, $\mathbf{r}_D$, and $\mathbf{r}_{D/B}$ are chosen as shown in the sketch.

Velocities. Since the motion of each element of the linkage is contained in the plane of the figure, we have

$$\boldsymbol{\omega}_{AB} = \omega_{AB}\mathbf{k} = (20 \text{ rad/s})\mathbf{k} \qquad \boldsymbol{\omega}_{BD} = \omega_{BD}\mathbf{k} \qquad \boldsymbol{\omega}_{DE} = \omega_{DE}\mathbf{k}$$

where $\mathbf{k}$ is a unit vector pointing out of the paper. We now write

$$\begin{aligned}
\mathbf{v}_D &= \mathbf{v}_B + \mathbf{v}_{D/B} \\
\omega_{DE}\mathbf{k} \times \mathbf{r}_D &= \omega_{AB}\mathbf{k} \times \mathbf{r}_B + \omega_{BD}\mathbf{k} \times \mathbf{r}_{D/B} \\
\omega_{DE}\mathbf{k} \times (-17\mathbf{i} + 17\mathbf{j}) &= 20\mathbf{k} \times (8\mathbf{i} + 14\mathbf{j}) + \omega_{BD}\mathbf{k} \times (12\mathbf{i} + 3\mathbf{j}) \\
-17\omega_{DE}\mathbf{j} - 17\omega_{DE}\mathbf{i} &= 160\mathbf{j} - 280\mathbf{i} + 12\omega_{BD}\mathbf{j} - 3\omega_{BD}\mathbf{i}
\end{aligned}$$

Equating the coefficients of the unit vectors $\mathbf{i}$ and $\mathbf{j}$, we obtain the following two scalar equations:

$$\begin{aligned}
-17\omega_{DE} &= -280 - 3\omega_{BD} \\
-17\omega_{DE} &= +160 + 12\omega_{BD}
\end{aligned}$$

$$\boldsymbol{\omega}_{BD} = -(29.3 \text{ rad/s})\mathbf{k} \qquad \boldsymbol{\omega}_{DE} = (11.29 \text{ rad/s})\mathbf{k} \quad ◄$$

Accelerations. Noting that at the instant considered crank *AB* has a constant angular velocity, we write

$$\boldsymbol{\alpha}_{AB} = 0 \qquad \boldsymbol{\alpha}_{BD} = \alpha_{BD}\mathbf{k} \qquad \boldsymbol{\alpha}_{DE} = \alpha_{DE}\mathbf{k}$$

$$\mathbf{a}_D = \mathbf{a}_B + \mathbf{a}_{D/B} \tag{1}$$

Each term of Eq. (1) is evaluated separately:

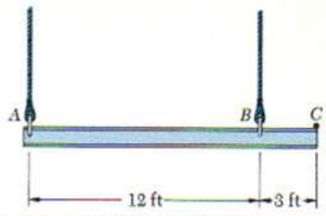
$$\begin{aligned}
\mathbf{a}_D &= \alpha_{DE}\mathbf{k} \times \mathbf{r}_D - \omega_{DE}^2\mathbf{r}_D \\
&= \alpha_{DE}\mathbf{k} \times (-17\mathbf{i} + 17\mathbf{j}) - (11.29)^2(-17\mathbf{i} + 17\mathbf{j}) \\
&= -17\alpha_{DE}\mathbf{j} - 17\alpha_{DE}\mathbf{i} + 2170\mathbf{i} - 2170\mathbf{j} \\
\mathbf{a}_B &= \alpha_{AB}\mathbf{k} \times \mathbf{r}_B - \omega_{AB}^2\mathbf{r}_B = 0 - (20)^2(8\mathbf{i} + 14\mathbf{j}) \\
&= -3200\mathbf{i} - 5600\mathbf{j} \\
\mathbf{a}_{D/B} &= \alpha_{BD}\mathbf{k} \times \mathbf{r}_{D/B} - \omega_{BD}^2\mathbf{r}_{D/B} \\
&= \alpha_{BD}\mathbf{k} \times (12\mathbf{i} + 3\mathbf{j}) - (29.3)^2(12\mathbf{i} + 3\mathbf{j}) \\
&= 12\alpha_{BD}\mathbf{j} - 3\alpha_{BD}\mathbf{i} - 10{,}320\mathbf{i} - 2580\mathbf{j}
\end{aligned}$$

Substituting into Eq. (1) and equating the coefficients of $\mathbf{i}$ and $\mathbf{j}$, we obtain

$$\begin{aligned}
-17\alpha_{DE} + 3\alpha_{BD} &= -15{,}690 \\
-17\alpha_{DE} - 12\alpha_{BD} &= -6010
\end{aligned}$$

$$\boldsymbol{\alpha}_{BD} = -(645 \text{ rad/s}^2)\mathbf{k} \qquad \boldsymbol{\alpha}_{DE} = (809 \text{ rad/s}^2)\mathbf{k} \quad ◄$$

PROBLEMS



Figs. P15.76 and P15.77

15.76 A 15-ft steel beam is lowered by means of two cables unwinding at the same speed from overhead cranes. As the beam approaches the ground, the crane operators apply brakes to slow down the unwinding motion. At the instant considered the deceleration of the cable attached at A is 13 ft/s^2, while that of the cable attached at B is 7 ft/s^2. Determine (*a*) the angular acceleration of the beam, (*b*) the acceleration of point C.

15.77 The acceleration of point C is 5 ft/s^2 downward and the angular acceleration of the beam is 2 rad/s^2 clockwise. Knowing that the angular velocity of the beam is zero at the instant considered, determine the acceleration of each cable.

15.78 A 600-mm rod rests on a smooth horizontal table. A force **P** applied as shown produces the following accelerations: $\mathbf{a}_A = 0.8\ \text{m/s}^2$ to the right, $\alpha = 2\ \text{rad/s}^2$ clockwise as viewed from above. Determine the acceleration (*a*) of point B, (*b*) of point G.

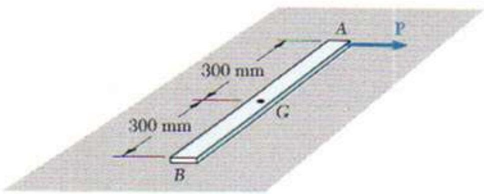


Fig. P15.78

15.79 In Prob. 15.78, determine the point of the rod which (*a*) has no acceleration, (*b*) has an acceleration of 0.350 m/s^2 to the right.

15.80 Determine the accelerations of points C and D of the 610-mm-diameter wheel of Prob. 15.25, knowing that the automobile moves at a constant speed of 50 km/h.

15.81 Determine the accelerations of points B and E of the wheel of Prob. 15.25, knowing that the automobile moves at a constant speed of 50 km/h and assuming the diameter of the wheel is reduced to 560 mm.

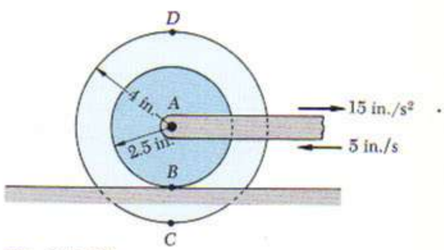


Fig. P15.82

15.82 The flanged wheel rolls without slipping on the horizontal rail. If at a given instant the velocity and acceleration of the center of the wheel are as shown, determine the acceleration (*a*) of point B, (*b*) of point C, (*c*) of point D.

15.83 The moving carriage is supported by two casters A and C, each of $\frac{1}{2}$-in. diameter, and by a $\frac{1}{2}$-in.-diameter ball B. If at a given instant the velocity and acceleration of the carriage are as shown, determine (*a*) the angular accelerations of the ball and of each caster, (*b*) the accelerations of the center of the ball and of each caster.

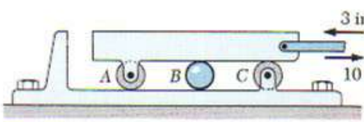


Fig. P15.83

15.84 and 15.85 At the instant shown, the disk rotates with a constant angular velocity ω_0 clockwise. Determine the angular velocities and the angular accelerations of the rods AB and BC.

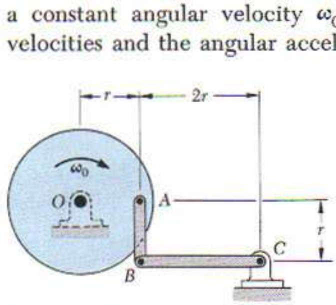


Fig. P15.84

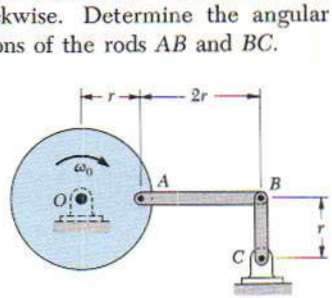


Fig. P15.85

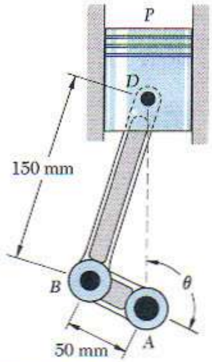


Fig. P15.86

15.86 Crank AB rotates about A with a constant angular velocity of 900 rpm clockwise. Determine the acceleration of the piston P when (a) $\theta = 90°$, (b) $\theta = 180°$.

15.87 Solve Prob. 15.86 when (a) $\theta = 0$, (b) $\theta = 270°$.

15.88 Arm AB rotates with a constant angular velocity of 120 rpm clockwise. Knowing that gear A does not rotate, determine the acceleration of the tooth of gear B which is in contact with gear A.

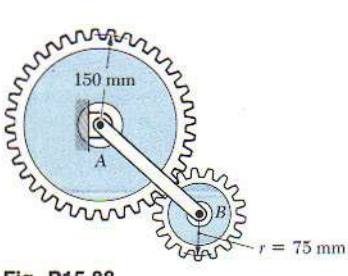


Fig. P15.88

15.89 and 15.90 For the linkage indicated, determine the angular acceleration (a) of bar BD, (b) of bar DE.

15.89 Linkage of Prob. 15.41.

15.90 Linkage of Prob. 15.40.

15.91 and 15.92 The end A of the rod AB moves downward with a constant velocity of 9 in./s. For the position shown, determine (a) the angular acceleration of the rod, (b) the acceleration of the midpoint G of the rod.

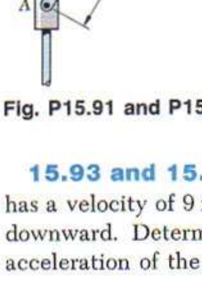


Fig. P15.91 and P15.93

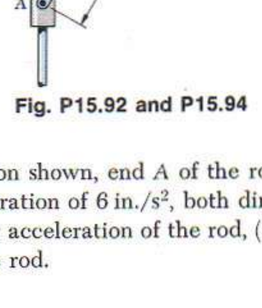


Fig. P15.92 and P15.94

15.93 and 15.94 In the position shown, end A of the rod AB has a velocity of 9 in./s and an acceleration of 6 in./s^2, both directed downward. Determine (a) the angular acceleration of the rod, (b) the acceleration of the midpoint G of the rod.

15.95 In the position shown, point A of bracket $ABCD$ has a velocity of magnitude $v_A = 250$ mm/s with $dv_A/dt = 0$. Determine (*a*) the angular acceleration of the bracket, (*b*) the acceleration of point C.

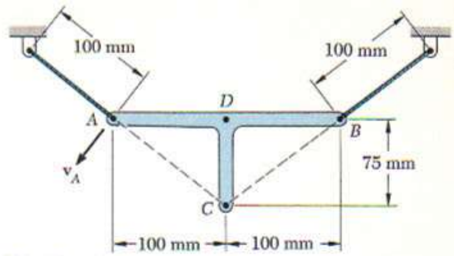


Fig. P15.95

15.96 In Prob. 15.95, determine the acceleration of point D.

15.97 Show that the acceleration of the instantaneous center of rotation of the slab of Prob. 15.49 is zero if, and only if,

$$\mathbf{a}_A = \frac{\alpha}{\omega}\mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{v}_A$$

where $\boldsymbol{\alpha} = \alpha\mathbf{k}$ is the angular acceleration of the slab.

***15.98** Rod AB slides with its ends in contact with the floor and the inclined plane. Using the method of Sec. 15.9, derive an expression for the angular velocity of the rod in terms of v_B, θ, l, and β.

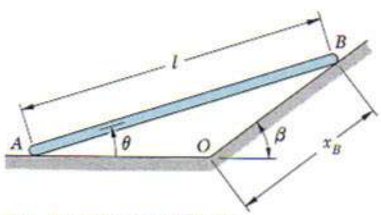


Fig. P15.98 and P15.99

***15.99** Derive an expression for the angular acceleration of the rod AB in terms of v_B, θ, l, and β, knowing that the acceleration of point B is zero.

***15.100** The drive disk of the Scotch crosshead mechanism shown has an angular velocity $\boldsymbol{\omega}$ and an angular acceleration $\boldsymbol{\alpha}$, both directed clockwise. Using the method of Sec. 15.9, derive an expression (*a*) for the velocity of point B, (*b*) for the acceleration of point B.

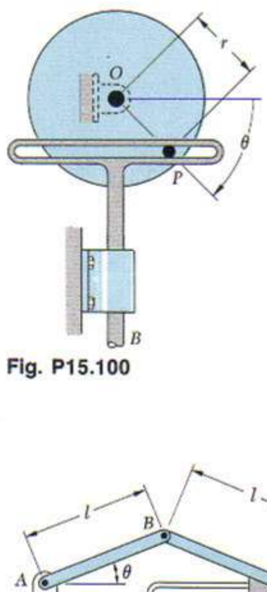


Fig. P15.100

***15.101** A disk of radius r rolls to the right with a constant velocity $\mathbf{v}$. Denoting by P the point of the rim in contact with the ground at $t = 0$, derive expressions for the horizontal and vertical components of the velocity of P at any time t. (The curve described by point P is called a *cycloid*.)

***15.102** Knowing that rod AB rotates with an angular velocity $\boldsymbol{\omega}$ and with an angular acceleration $\boldsymbol{\alpha}$, both counterclockwise, derive expressions for the velocity and acceleration of collar D.

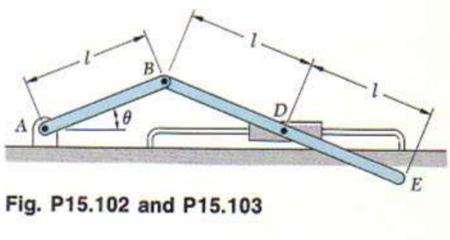


Fig. P15.102 and P15.103

***15.103** Knowing that rod AB rotates with an angular velocity $\boldsymbol{\omega}$ and an angular acceleration $\boldsymbol{\alpha}$, both counterclockwise, derive expressions for the components of the velocity and acceleration of point E.

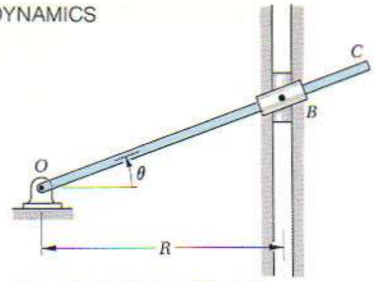


Fig. P15.104 and P15.105

$*$15.104 Collar B slides along rod OC and is attached to a sliding block which moves in a vertical slot. Knowing that rod OC rotates with an angular velocity ω and with an angular acceleration $\boldsymbol{\alpha}$, both counterclockwise, derive expressions for the velocity and acceleration of collar B.

$*$15.105 Collar B slides along rod OC and is attached to a sliding block which moves upward with a constant velocity $\mathbf{v}$ in a vertical slot. Using the method of Sec. 15.9, derive an expression (a) for the angular velocity of rod OC, (b) for the angular acceleration of rod OC.

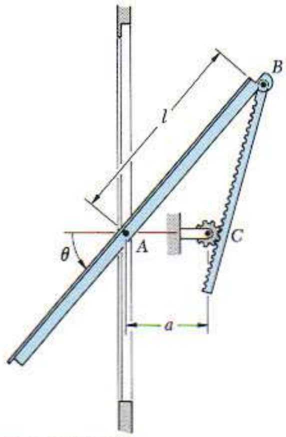


Fig. P15.106

$*$15.106 The position of a factory window is controlled by the rack and pinion shown. Knowing that the pinion C has a radius r and rotates counterclockwise at a constant rate ω, derive an expression for the angular velocity of the window.

$*$15.107 The crank AB of Prob. 15.36 rotates with a constant clockwise angular velocity ω, and $\theta = 0$ at $t = 0$. Using the method of Sec. 15.9, derive an expression for the velocity of the piston P in terms of the time t.

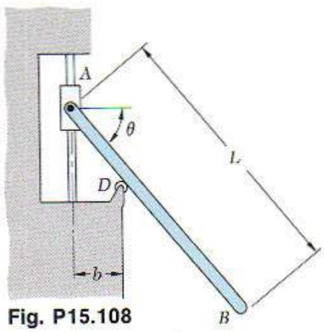


Fig. P15.108

$*$15.108 Collar A slides upward with a constant velocity $\mathbf{v}_A$. Using the method of Sec. 15.9, derive an expression for (a) the angular velocity of rod AB, (b) the components of the velocity of point B.

$*$15.109 In Prob. 15.108, derive an expression for the angular acceleration of rod AB.

15.10. Rate of Change of a Vector with Respect to a Rotating Frame.

We saw in Sec. 11.10 that the rate of change of a vector is the same with respect to a fixed frame and with respect to a frame in translation. In this section, we shall compare the rates of change of a vector $\mathbf{Q}$ with respect to a fixed frame and with respect to a rotating frame of reference.† We shall also learn to determine the rate of change of $\mathbf{Q}$ with respect to one frame of reference when $\mathbf{Q}$ is defined by its components in another frame.

Consider two frames of reference centered at O, a fixed frame $OXYZ$ and a frame $Oxyz$ which rotates about the fixed axis OA; let $\boldsymbol{\Omega}$ denote the angular velocity of the frame $Oxyz$ at a given

† It is recalled that the selection of a fixed frame of reference is arbitrary. Any frame may be designated as "fixed"; all others will then be considered as moving.

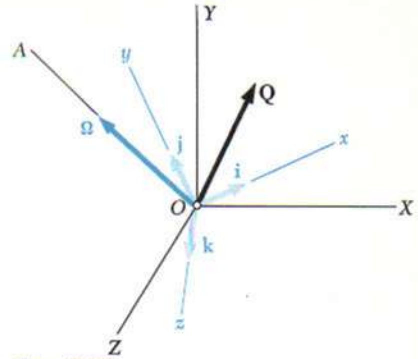


Fig. 15.26

instant (Fig. 15.26). Consider now a vector function $\mathbf{Q}(t)$ represented by the vector $\mathbf{Q}$ attached at O; as the time t varies, both the direction and the magnitude of $\mathbf{Q}$ change. Since the variation of $\mathbf{Q}$ is viewed differently by an observer using $OXYZ$ as a frame of reference and by an observer using $Oxyz$, we should expect the rate of change of $\mathbf{Q}$ to depend upon the frame of reference which has been selected. Therefore, we shall denote by $(\dot{\mathbf{Q}})_{OXYZ}$ the rate of change of $\mathbf{Q}$ with respect to the fixed frame $OXYZ$, and by $(\dot{\mathbf{Q}})_{Oxyz}$ its rate of change with respect to the rotating frame $Oxyz$. We propose to determine the relationship existing between these two rates of change.

Let us first resolve the vector $\mathbf{Q}$ into components along the x, y, and z axes of the rotating frame. Denoting by $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ the corresponding unit vectors, we write

$$\mathbf{Q} = Q_x\mathbf{i} + Q_y\mathbf{j} + Q_z\mathbf{k} \tag{15.27}$$

Differentiating (15.27) with respect to t and considering the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ as fixed, we obtain the rate of change of $\mathbf{Q}$ *with respect to the rotating frame Oxyz*:

$$(\dot{\mathbf{Q}})_{Oxyz} = \dot{Q}_x\mathbf{i} + \dot{Q}_y\mathbf{j} + \dot{Q}_z\mathbf{k} \tag{15.28}$$

To obtain the rate of change of $\mathbf{Q}$ *with respect to the fixed frame OXYZ*, we must consider the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ as variable when differentiating (15.27). We therefore write

$$(\dot{\mathbf{Q}})_{OXYZ} = \dot{Q}_x\mathbf{i} + \dot{Q}_y\mathbf{j} + \dot{Q}_z\mathbf{k} + Q_x\frac{d\mathbf{i}}{dt} + Q_y\frac{d\mathbf{j}}{dt} + Q_z\frac{d\mathbf{k}}{dt} \tag{15.29}$$

Recalling (15.28), we observe that the sum of the first three terms in the right-hand member of (15.29) represents the rate of change $(\dot{\mathbf{Q}})_{Oxyz}$. We note, on the other hand, that the rate of change $(\dot{\mathbf{Q}})_{OXYZ}$ would reduce to the last three terms in (15.29) if the vector $\mathbf{Q}$ were fixed within the frame $Oxyz$, since $(\dot{\mathbf{Q}})_{Oxyz}$ would then be zero. But, in that case, $(\dot{\mathbf{Q}})_{OXYZ}$ would represent the velocity of a particle located at the tip of $\mathbf{Q}$ and belonging to a body rigidly attached to the frame $Oxyz$. Thus, the last three terms in (15.29) represent the velocity of that particle; since the frame $Oxyz$ has an angular velocity $\boldsymbol{\Omega}$ with respect to $OXYZ$ at the instant considered, we write, by (15.5),

$$Q_x\frac{d\mathbf{i}}{dt} + Q_y\frac{d\mathbf{j}}{dt} + Q_z\frac{d\mathbf{k}}{dt} = \boldsymbol{\Omega} \times \mathbf{Q} \tag{15.30}$$

Substituting from (15.28) and (15.30) into (15.29), we obtain the fundamental relation

$$(\dot{\mathbf{Q}})_{OXYZ} = (\dot{\mathbf{Q}})_{Oxyz} + \boldsymbol{\Omega} \times \mathbf{Q} \tag{15.31}$$

We conclude that the rate of change of the vector $\mathbf{Q}$ with respect to the fixed frame $OXYZ$ is made of two parts: The first part represents the rate of change of $\mathbf{Q}$ with respect to the rotating frame $Oxyz$; the second part, $\mathbf{\Omega} \times \mathbf{Q}$, is induced by the rotation of the frame $Oxyz$.

The use of relation (15.31) simplifies the determination of the rate of change of a vector $\mathbf{Q}$ with respect to a fixed frame of reference $OXYZ$ when the vector $\mathbf{Q}$ is defined by its components along the axes of a rotating frame $Oxyz$, since this relation does not require the separate computation of the derivatives of the unit vectors defining the orientation of the rotating frame.

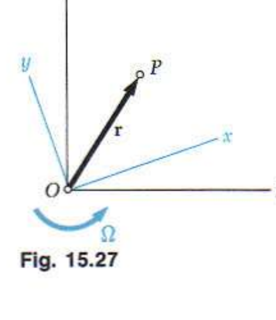


Fig. 15.27

15.11. Plane Motion of a Particle Relative to a Rotating Frame. Coriolis Acceleration. Consider two frames of reference, both centered at O and both in the plane of the figure, a fixed frame OXY, and a rotating frame Oxy (Fig. 15.27). Let P be a particle moving in the plane of the figure. While the position vector $\mathbf{r}$ of P is the same in both frames, its rate of change depends upon the frame of reference which has been selected.

The absolute velocity $\mathbf{v}_P$ of the particle is defined as the velocity observed from the fixed frame OXY and is equal to the rate of change $(\dot{\mathbf{r}})_{OXY}$ of $\mathbf{r}$ with respect to that frame. We may, however, express $\mathbf{v}_P$ in terms of the rate of change $(\dot{\mathbf{r}})_{Oxy}$ observed from the rotating frame if we make use of Eq. (15.31). Denoting by $\mathbf{\Omega}$ the angular velocity of the frame Oxy with respect to OXY at the instant considered, we write

$$\mathbf{v}_P = (\dot{\mathbf{r}})_{OXY} = \mathbf{\Omega} \times \mathbf{r} + (\dot{\mathbf{r}})_{Oxy} \tag{15.32}$$

But $(\dot{\mathbf{r}})_{Oxy}$ defines the velocity $\mathbf{v}_{P/F}$ of the particle P relative to the frame Oxy. If we imagine that a rigid slab has been attached to the rotating frame, $\mathbf{v}_{P/F}$ will represent the velocity of P along the path that it describes on that slab (Fig. 15.28). On the other hand, the term $\mathbf{\Omega} \times \mathbf{r}$ in (15.32) will represent the velocity $\mathbf{v}_{P'}$ of the point P' of the slab—or rotating frame—which coincides with P at the instant considered. Thus, we have

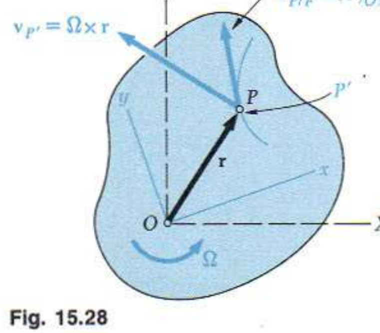


Fig. 15.28

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/F} \tag{15.33}$$

where $\mathbf{v}_P$ = absolute velocity of particle P
$\mathbf{v}_{P'}$ = velocity of point P' of moving frame coinciding with P
$\mathbf{v}_{P/F}$ = velocity of P relative to moving frame

The absolute acceleration $\mathbf{a}_P$ of the particle is defined as the rate of change of $\mathbf{v}_P$ with respect to the fixed frame OXY. Computing the rates of change with respect to OXY of the terms in (15.32), we write

$$\mathbf{a}_P = \dot{\mathbf{v}}_P = \dot{\mathbf{\Omega}} \times \mathbf{r} + \mathbf{\Omega} \times \dot{\mathbf{r}} + \frac{d}{dt}[(\dot{\mathbf{r}})_{Oxy}] \qquad (15.34)$$

where all derivatives are defined with respect to OXY, except where indicated otherwise. Referring to Eq. (15.31), we note that the last term in (15.34) may be expressed as

$$\frac{d}{dt}[(\dot{\mathbf{r}})_{Oxy}] = (\ddot{\mathbf{r}})_{Oxy} + \mathbf{\Omega} \times (\dot{\mathbf{r}})_{Oxy}$$

On the other hand, $\dot{\mathbf{r}}$ represents the velocity $\mathbf{v}_P$ and may be replaced by the right-hand member of Eq. (15.32). After completing these two substitutions into (15.34), we write

$$\mathbf{a}_P = \dot{\mathbf{\Omega}} \times \mathbf{r} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) + 2\mathbf{\Omega} \times (\dot{\mathbf{r}})_{Oxy} + (\ddot{\mathbf{r}})_{Oxy} \qquad (15.35)$$

Referring to the expression (15.8) obtained in Sec. 15.3 for the acceleration of a particle in a rigid body rotating about a fixed axis, we note that the sum of the first two terms represents the acceleration $\mathbf{a}_{P'}$ of the point P' of the rotating frame which coincides with P at the instant considered. On the other hand, the last term defines the acceleration $\mathbf{a}_{P/\mathcal{F}}$ of P relative to the rotating frame. If it were not for the third term, which has not been accounted for, a relation similar to (15.33) could be written for the accelerations, and $\mathbf{a}_P$ could be expressed as the sum of $\mathbf{a}_{P'}$ and $\mathbf{a}_{P/\mathcal{F}}$. However, it is clear that *such a relation would be incorrect* and that we must include the additional term. This term, which we shall denote by $\mathbf{a}_c$, is called the *complementary acceleration*, or *Coriolis acceleration*, after the French mathematician De Coriolis (1792–1843). We write

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/\mathcal{F}} + \mathbf{a}_c \qquad (15.36)$$

where $\mathbf{a}_P$ = absolute acceleration of particle P
$\mathbf{a}_{P'}$ = acceleration of point P' of moving frame coinciding with P
$\mathbf{a}_{P/\mathcal{F}}$ = acceleration of P relative to moving frame
$\mathbf{a}_c = 2\mathbf{\Omega} \times (\dot{\mathbf{r}})_{Oxy} = 2\mathbf{\Omega} \times \mathbf{v}_{P/\mathcal{F}}$
= complementary, or Coriolis, acceleration

We note that, since point P' moves in a circle about the origin O, its acceleration $\mathbf{a}_{P'}$ has, in general, two components: a component $(\mathbf{a}_{P'})_t$ tangent to the circle, and a component $(\mathbf{a}_{P'})_n$ directed toward O. Similarly, the acceleration $\mathbf{a}_{P/F}$ generally has two components: a component $(\mathbf{a}_{P/F})_t$ tangent to the path that P describes on the rotating slab, and a component $(\mathbf{a}_{P/F})_n$ directed toward the center of curvature of that path. We further note that, since the vector $\boldsymbol{\Omega}$ is perpendicular to the plane of motion, and thus to $\mathbf{v}_{P/F}$, the magnitude of the Coriolis acceleration $\mathbf{a}_c = 2\boldsymbol{\Omega} \times \mathbf{v}_{P/F}$ is equal to $2\Omega v_{P/F}$, and its direction may be obtained by rotating the vector $\mathbf{v}_{P/F}$ through 90° in the sense of rotation of the moving frame (Fig. 15.29). The Coriolis acceleration reduces to zero when either $\boldsymbol{\Omega}$ or $\mathbf{v}_{P/F}$ is zero.

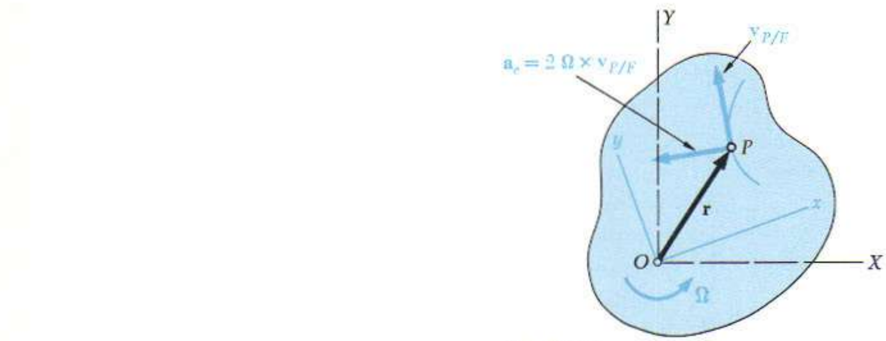


Fig. 15.29

The following example will help in understanding the physical meaning of the Coriolis acceleration. Consider a collar P which is made to slide at a constant relative speed u along a rod OB rotating at a constant angular velocity $\boldsymbol{\omega}$ about O (Fig. 15.30a). According to formula (15.36), the absolute acceleration of P may be obtained by adding vectorially the acceleration $\mathbf{a}_A$ of the point A of the rod coinciding with P, the relative acceleration $\mathbf{a}_{P/OB}$ of P with respect to the rod, and the Coriolis acceleration $\mathbf{a}_c$. Since the angular velocity $\boldsymbol{\omega}$ of the rod is constant, $\mathbf{a}_A$ reduces to its normal component $(\mathbf{a}_A)_n$ of magnitude $r\omega^2$; and since u is constant, the relative acceleration $\mathbf{a}_{P/OB}$ is zero. According to the definition given above, the Coriolis acceleration is a vector perpendicular to OB, of magnitude $2\omega u$, and directed as shown in the figure. The acceleration of the collar P consists, therefore, of the two vectors shown in Fig. 15.30*a*. Note that the result obtained may be checked by applying the relation (11.44).

To understand better the significance of the Coriolis acceleration, we shall consider the absolute velocity of P at time t and at time $t + \Delta t$ (Fig. 15.30*b*). At time t, the velocity may be resolved into its components $\mathbf{u}$ and $\mathbf{v}_A$, and at time $t + \Delta t$ into its components $\mathbf{u}'$ and $\mathbf{v}_{A'}$. Drawing these components from the same origin (Fig. 15.30*c*), we note that the change in velocity during the time Δt may be represented by the sum of three vectors $\overrightarrow{RR'}$, $\overrightarrow{TT''}$, and $\overrightarrow{T''T'}$. The vector $\overrightarrow{TT''}$ measures the change in direction of the velocity $\mathbf{v}_A$, and the quotient $\overrightarrow{TT''}/\Delta t$ represents the acceleration $\mathbf{a}_A$ when Δt approaches zero. We check that the direction of $\overrightarrow{TT''}$ is that of $\mathbf{a}_A$ when Δt approaches zero and that

$$\lim_{\Delta t \to 0} \frac{TT''}{\Delta t} = \lim_{\Delta t \to 0} v_A \frac{\Delta\theta}{\Delta t} = r\omega\omega = r\omega^2 = a_A$$

The vector $\overrightarrow{RR'}$ measures the change in direction of $\mathbf{u}$ due to the rotation of the rod; the vector $\overrightarrow{T''T'}$ measures the change in magnitude of $\mathbf{v}_A$ due to the motion of P on the rod. The vectors $\overrightarrow{RR'}$ and $\overrightarrow{T''T'}$ result from the *combined effect* of the relative motion of P and of the rotation of the rod; they would vanish if *either* of these two motions stopped. We may easily verify that the sum of these two vectors defines the Coriolis acceleration. Their direction is that of $\mathbf{a}_c$ when Δt approaches zero and, since $RR' = u\,\Delta\theta$ and $T''T' = v_{A'} - v_A = (r + \Delta r)\omega - r\omega = \omega\,\Delta r$, we check that

$$\lim_{\Delta t \to 0} \left(\frac{RR'}{\Delta t} + \frac{T''T'}{\Delta t} \right) = \lim_{\Delta t \to 0} \left(u \frac{\Delta\theta}{\Delta t} + \omega \frac{\Delta r}{\Delta t} \right)$$
$$= u\omega + \omega u = 2\omega u = a_c$$

Formulas (15.33) and (15.36) may be used to analyze the motion of mechanisms which contain parts sliding on each other. They make it possible, for example, to relate the absolute and relative motions of sliding pins and collars (see Sample Probs. 15.9 and 15.10). The concept of Coriolis acceleration is also very useful in the study of long-range projectiles and of other bodies whose motions are appreciably affected by the rotation of the earth. As was pointed out in Sec. 12.1, a system of axes attached to the earth does not truly constitute a newtonian frame of reference; such a system of axes should actually be considered as rotating. The formulas derived in this section will therefore facilitate the study of the motion of bodies with respect to axes attached to the earth.

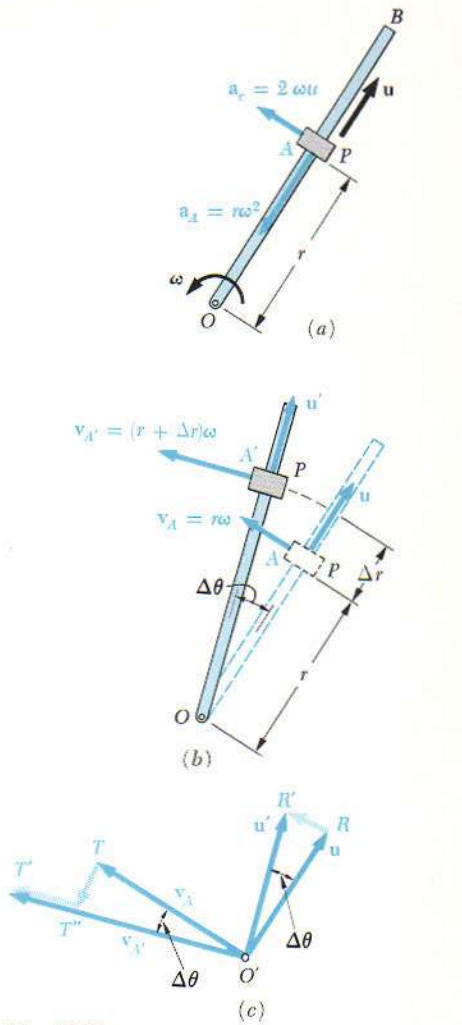
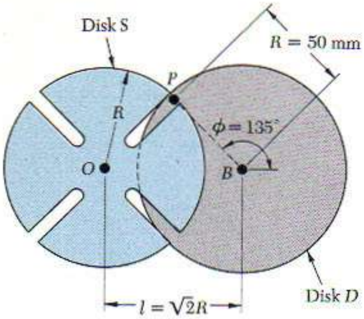


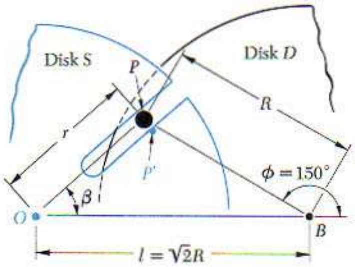
Fig. 15.30

SAMPLE PROBLEM 15.9



The Geneva mechanism shown is used in many counting instruments and in other applications where an intermittent rotary motion is required. Disk D rotates with a constant counterclockwise angular velocity ω_D of 10 rad/s. A pin P is attached to disk D and slides along one of several slots cut in disk S. It is desirable that the angular velocity of disk S be zero as the pin enters and leaves each slot; in the case of four slots, this will occur if the distance between the centers of the disks is $l = \sqrt{2}\,R$.

At the instant when $\phi = 150°$, determine (*a*) the angular velocity of disk S, (*b*) the velocity of pin P relative to disk S.



Solution. We solve triangle OPB, which corresponds to the position $\phi = 150°$. Using the law of cosines, we write

$$r^2 = R^2 + l^2 - 2Rl\cos 30° = 0.551R^2 \qquad r = 0.742R = 37.1 \text{ mm}$$

From the law of sines

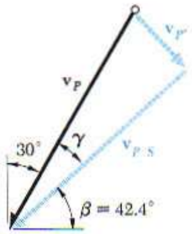
$$\frac{\sin\beta}{R} = \frac{\sin 30°}{r} \qquad \sin\beta = \frac{\sin 30°}{0.742} \qquad \beta = 42.4°$$

Since pin P is attached to disk D, and since disk D rotates about point B, the magnitude of the absolute velocity of P is

$$v_P = R\omega_D = (50 \text{ mm})(10 \text{ rad/s}) = 500 \text{ mm/s}$$
$$\mathbf{v}_P = 500 \text{ mm/s} \measuredangle 60°$$

We consider now the motion of pin P along the slot in disk S. Denoting by P' the point of disk S which coincides with P at the instant considered, we write

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/S}$$



Noting that $\mathbf{v}_{P'}$ is perpendicular to the radius OP and that $\mathbf{v}_{P/S}$ is directed along the slot, we draw the velocity triangle corresponding to the above equation. From the triangle, we compute

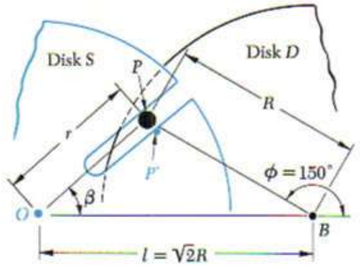
$$\gamma = 90° - 42.4° - 30° = 17.6°$$
$$v_{P'} = v_P \sin\gamma = (500 \text{ mm/s}) \sin 17.6°$$
$$\mathbf{v}_{P'} = 151.2 \text{ mm/s} \measuredangle 42.4°$$
$$v_{P/S} = v_P \cos\gamma = (500 \text{ mm/s}) \cos 17.6°$$
$$\mathbf{v}_{P/S} = 477 \text{ mm/s} \measuredangle 42.4° \quad ◄$$

Since $\mathbf{v}_{P'}$ is perpendicular to the radius OP, we write

$$v_{P'} = r\omega_S \qquad 151.2 \text{ mm/s} = (37.1 \text{ mm})\omega_S$$
$$\omega_S = 4.08 \text{ rad/s} \circlearrowright \quad ◄$$

SAMPLE PROBLEM 15.10

In the Geneva mechanism of Sample Prob. 15.9, disk D rotates with a constant counterclockwise angular velocity ω_D of 10 rad/s. At the instant when $\phi = 150°$, determine the angular acceleration of disk S.



Solution. Referring to Sample Prob. 15.9, we obtain the angular velocity of disk S and the velocity of the pin relative to disk S.

$$\omega_S = 4.08 \text{ rad/s} \curvearrowright$$

$$\beta = 42.4° \qquad \mathbf{v}_{P/S} = 477 \text{ mm/s} \angle 42.4°$$

Since pin P moves with respect to the rotating disk S, we write

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/S} + \mathbf{a}_c \tag{1}$$

Each term of this vector equation is investigated separately.

Absolute Acceleration $\mathbf{a}_P$. Since disk D rotates with constant ω, the absolute acceleration $\mathbf{a}_P$ is directed toward B.

$$a_P = R\omega_D^2 = (50 \text{ mm})(10 \text{ rad/s})^2 = 5000 \text{ mm/s}^2$$

$$\mathbf{a}_P = 5000 \text{ mm/s}^2 \angle 30°$$

Acceleration $\mathbf{a}_{P'}$ *of the Coinciding Point* P'. The acceleration $\mathbf{a}_{P'}$ of the point P' of disk S which coincides with P at the instant considered is resolved into normal and tangential components. (We recall from Sample Prob. 15.9 that $r = 37.1$ mm.)

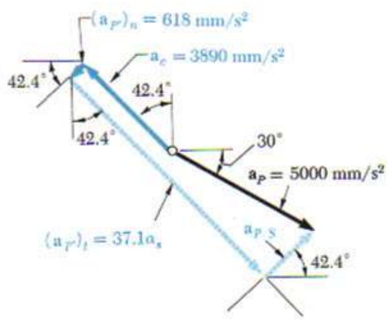
$$(a_{P'})_n = r\omega_S^2 = (37.1 \text{ mm})(4.08 \text{ rad/s})^2 = 618 \text{ mm/s}^2$$

$$(\mathbf{a}_{P'})_n = 618 \text{ mm/s}^2 \angle 42.4°$$

$$(a_{P'})_t = r\alpha_S = 37.1\alpha_S \qquad (\mathbf{a}_{P'})_t = 37.1\alpha_S \angle 42.4°$$

Relative Acceleration $\mathbf{a}_{P/S}$. Since the pin P moves in a straight slot cut in disk S, the relative acceleration $\mathbf{a}_{P/S}$ must be parallel to the slot; i.e., its direction must be $\angle 42.4°$.

Coriolis Acceleration $\mathbf{a}_c$. Rotating the relative velocity $\mathbf{v}_{P/S}$ through 90° in the sense of ω_S, we obtain the direction of the Coriolis component of the acceleration.



$$a_c = 2\omega_S v_{P/S} = 2(4.08 \text{ rad/s})(477 \text{ mm/s}) = 3890 \text{ mm/s}^2$$

$$\mathbf{a}_c = 3890 \text{ mm/s}^2 \angle 42.4°$$

We rewrite Eq. (1) and substitute the accelerations found above.

$$\mathbf{a}_P = (\mathbf{a}_{P'})_n + (\mathbf{a}_{P'})_t + \mathbf{a}_{P/S} + \mathbf{a}_c$$

$$[5000 \angle 30°] = [618 \angle 42.4°] + [37.1\alpha_S \angle 42.4°] + [a_{P/S} \angle 42.4°] + [3890 \angle 42.4°]$$

Equating components in a direction perpendicular to the slot:

$$5000 \cos 17.6° = 37.1\alpha_S - 3890$$

$$\alpha_S = 233 \text{ rad/s}^2 \curvearrowright \quad ◄$$

PROBLEMS

15.110 and 15.111 Two rotating rods are connected by a slider block P. The rod attached at B rotates with a constant clockwise angular velocity ω_B. For the given data, determine for the position shown (*a*) the angular velocity of the rod attached at A, (*b*) the relative velocity of the slider block P with respect to the rod on which it slides.

15.110 $b = 10$ in., $\omega_B = 5$ rad/s.

15.111 $b = 200$ mm, $\omega_B = 9$ rad/s.

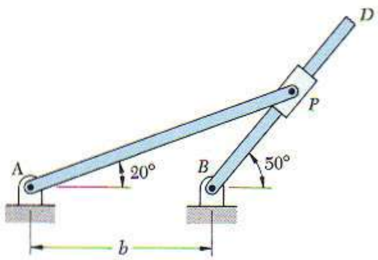


Fig. P15.110 and P15.112

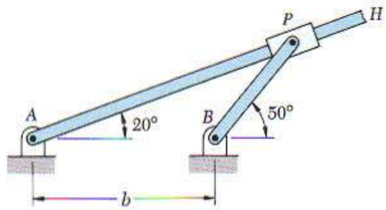


Fig. P15.111 and P15.113

15.112 and 15.113 Two rotating rods are connected by a slider block P. The velocity $\mathbf{v}_0$ of the slider block relative to the rod on which it slides is constant and is directed outward. For the given data, determine the angular velocity of each rod for the position shown.

15.112 $b = 200$ mm, $v_0 = 300$ mm/s.

15.113 $b = 10$ in., $v_0 = 15$ in./s.

15.114 Two rods AH and BD pass through smooth holes drilled in a hexagonal block. (The holes are drilled in different planes so that the rods will not hit each other.) Knowing that rod AH rotates counterclockwise at the rate ω, determine the angular velocity of rod BD and the relative velocity of the block with respect to each rod when (*a*) $\theta = 30°$, (*b*) $\theta = 15°$.

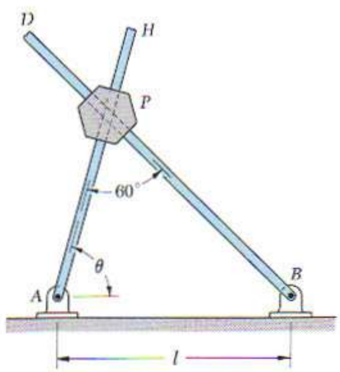


Fig. P15.114

15.115 Solve Prob. 15.114 when (*a*) $\theta = 90°$, (*b*) $\theta = 60°$.

15.116 Four pins slide in four separate slots cut in a circular plate as shown. When the plate is at rest, each pin has a velocity directed as shown and of the same constant magnitude u. If each pin maintains the same velocity in relation to the plate when the plate rotates about O with a constant *clockwise* angular velocity ω, determine the acceleration of each pin.

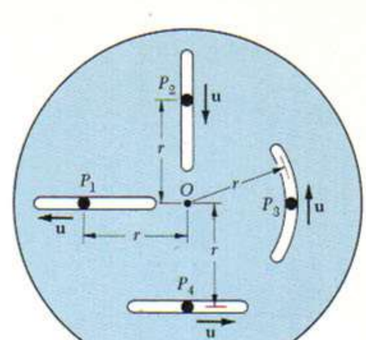


Fig. P15.116

15.117 Solve Prob. 15.116, assuming that the plate rotates about O with a constant *counterclockwise* angular velocity ω.

15.118 At the instant shown the length of the boom is being decreased at the constant rate of 150 mm/s and the boom is being lowered at the constant rate of 0.08 rad/s. Knowing that $\theta = 30°$, determine (*a*) the velocity of point B, (*b*) the acceleration of point B.

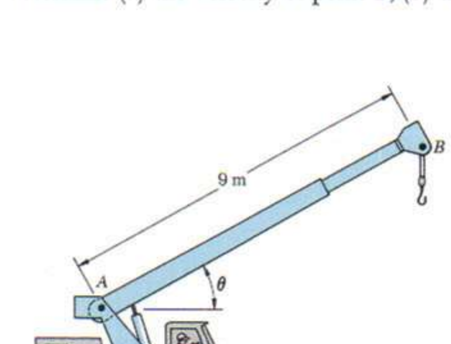


Fig. P15.118

Fig. P15.119

15.119 Water flows through a straight pipe OB which rotates counterclockwise with an angular velocity of 120 rpm. If the velocity of the water relative to the pipe is 20 ft/s, determine the total acceleration (*a*) of the particle of water P_1, (*b*) of the particle of water P_2.

15.120 Pin P slides in the circular slot cut in the plate $ABDE$ at a constant relative speed $u = 0.5$ m/s as the plate rotates about A at the constant rate $\omega = 6$ rad/s. Determine the acceleration of the pin as it passes through (*a*) point B, (*b*) point D, (*c*) point E.

15.121 Solve Prob. 15.120, assuming that at the instant considered the angular velocity ω is being decreased at the rate of 10 rad/s^2 and that the relative velocity **u** is being decreased at the rate of 3 m/s^2.

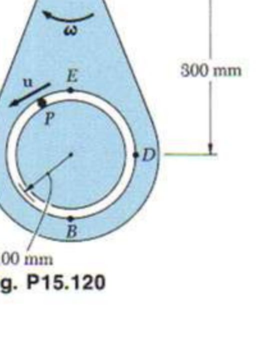


Fig. P15.120

15.122 The cage of a mine elevator moves downward with a constant speed of 40 ft/s. Determine the magnitude and direction of the Coriolis acceleration of the cage if the elevator is located (*a*) at the equator, (*b*) at latitude 40° north, (*c*) at latitude 40° south. (*Hint.* In parts *b* and *c* consider separately the components of the motion parallel and perpendicular to the plane of the equator.)

15.123 A train crosses the parallel 50° north, traveling due north at a constant speed v. Determine the speed of the train if the Coriolis component of its acceleration is $0.01\ \text{ft/s}^2$. (See hint of Prob. 15.122.)

15.124 In Prob. 15.110, determine the angular acceleration of the rod attached at A.

15.125 In Prob. 15.111, determine the angular acceleration of the rod attached at A.

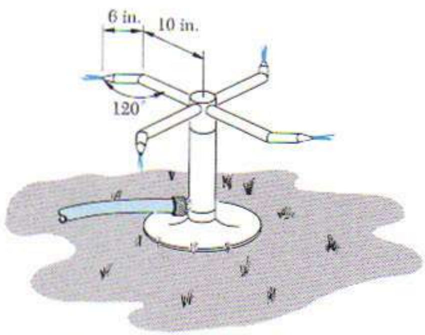


Fig. P15.126

15.126 A garden sprinkler has four rotating arms, each of which consists of two horizontal straight sections of pipe forming an angle of 120°. The sprinkler when operating rotates with a constant angular velocity of 180 rpm. If the velocity of the water relative to the pipe sections is 12 ft/s, determine the magnitude of the total acceleration of a particle of water as it passes the midpoint of (*a*) the 10-in. section of pipe, (*b*) the 6-in. section of pipe.

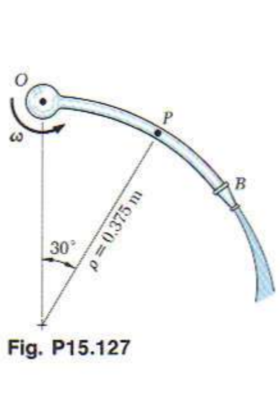


Fig. P15.127

15.127 Water flows through the curved pipe OB, which has a constant radius of 0.375 m and which rotates with a constant counterclockwise angular velocity of 120 rpm. If the velocity of the water relative to the pipe is 12 m/s, determine the total acceleration of the particle of water P.

15.128 The disk shown rotates with a constant clockwise angular velocity of 12 rad/s. At the instant shown, determine (*a*) the angular velocity and angular acceleration of rod *BD*, (*b*) the velocity and acceleration of the point of the rod in contact with collar *E*.

15.129 Solve Prob. 15.128, assuming that the disk rotates with a constant counterclockwise angular velocity of 12 rad/s.

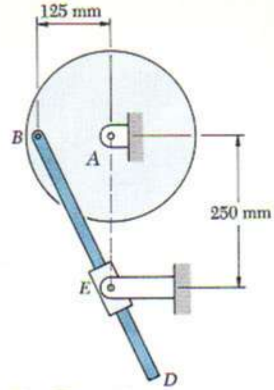


Fig. P15.128

*15.12. Motion about a Fixed Point.

We have studied in Sec. 15.3 the motion of a rigid body constrained to rotate about a fixed axis. We shall now consider the more general case of the motion of a rigid body which has a fixed point O.

First, we shall prove that *the most general displacement of a rigid body with a fixed point O is equivalent to a rotation of the body about an axis through O.*† Instead of considering the rigid body itself, we may detach a sphere of center O from the body and analyze the motion of that sphere. Clearly, the motion of the sphere completely characterizes the motion of the given body. Since three points define the position of a solid in space, the center O and two points A and B on the surface of the sphere will define the position of the sphere and, thus, the position of the body. Let A_1 and B_1 characterize the position of the sphere at one instant, and A_2 and B_2 its position at a later instant (Fig. 15.31*a*). Since the sphere is rigid, the lengths of the arcs of great circle A_1B_1 and A_2B_2 must be equal, but, except for this requirement, the positions of A_1, A_2, B_1, and B_2 are arbitrary. We propose to prove that the points A and B may be brought, respectively, from A_1 and B_1 into A_2 and B_2 by a single rotation of the sphere about an axis.

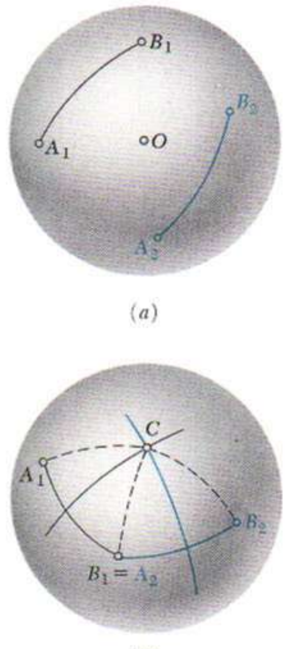


Fig. 15.31

For convenience, and without loss of generality, we may select point B so that its initial position coincides with the final position of A; thus, $B_1 = A_2$ (Fig. 15.31*b*). We draw the arcs of great circle A_1A_2, A_2B_2 and the arcs bisecting, respectively, A_1A_2 and A_2B_2. Let C be the point of intersection of these last two arcs; we complete the construction by drawing A_1C, A_2C, and B_2C. As pointed out above, $A_1B_1 = A_2B_2$ on account of the rigidity of the sphere; on the other hand, since C is by construction equidistant from A_1, A_2, and B_2, we have $A_1C = A_2C = B_2C$. As a result, the spherical triangles A_1CA_2 and B_1CB_2 are congruent and the angles A_1CA_2 and B_1CB_2 are equal. Denoting by θ the common value of these angles, we conclude that the sphere may be brought from its initial position into its final position by a single rotation through θ about the axis OC.

It follows that the motion during a time interval Δt of a rigid body with a fixed point O may be considered as a rotation

†This is known as Euler's theorem.

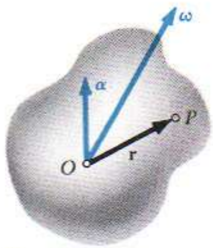


Fig. 15.32

through $\Delta\theta$ about a certain axis. Drawing along that axis a vector of magnitude $\Delta\theta/\Delta t$ and letting Δt approach zero, we obtain at the limit the *instantaneous axis of rotation* and the angular velocity $\boldsymbol{\omega}$ of the body at the instant considered (Fig. 15.32). The velocity of a particle P of the body may then be obtained, as in Sec. 15.3, by forming the vector product of $\boldsymbol{\omega}$ and of the position vector $\mathbf{r}$ of the particle:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r} \tag{15.37}$$

The acceleration of the particle is obtained by differentiating (15.37) with respect to t. As in Sec. 15.3 we have

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{15.38}$$

where the angular acceleration $\boldsymbol{\alpha}$ is defined as the derivative

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt} \tag{15.39}$$

of the angular velocity $\boldsymbol{\omega}$.

In the case of the motion of a rigid body with a fixed point, the direction of $\boldsymbol{\omega}$ and of the instantaneous axis of rotation changes from one instant to the next. The angular acceleration $\boldsymbol{\alpha}$, therefore, reflects the change in direction of $\boldsymbol{\omega}$ as well as its change in magnitude and, in general, *is not directed along the instantaneous axis of rotation*. While the particles of the body located on the instantaneous axis of rotation have zero velocity at the instant considered, they do not have zero acceleration. Also, the accelerations of the various particles of the body *cannot* be determined as if the body were rotating permanently about the instantaneous axis.

Recalling the definition of the velocity of a particle with position vector $\mathbf{r}$, we note that the angular acceleration $\boldsymbol{\alpha}$, as expressed in (15.39), represents the velocity of the tip of the vector $\boldsymbol{\omega}$. This property may be useful in the determination of the angular acceleration of a rigid body. For example, it follows that the vector $\boldsymbol{\alpha}$ is tangent to the curve described in space by the tip of the vector $\boldsymbol{\omega}$.

We should note that the vector $\boldsymbol{\omega}$ moves within the body, as well as in space. It thus generates two cones, respectively called the *body cone* and the *space cone* (Fig. 15.33).† It may be shown that, at any given instant, the two cones are tangent along the instantaneous axis of rotation and that, as the body moves, the body cone appears to *roll* on the space cone.

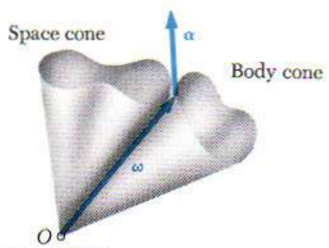


Fig. 15.33

† It is recalled that a *cone* is, by definition, a surface generated by a straight line passing through a fixed point. In general, the cones considered here *will not be circular cones*.

Before concluding our analysis of the motion of a rigid body with a fixed point, we should prove that angular velocities are actually vectors. As it was indicated in Sec. 2.2, some quantities, such as the *finite rotations* of a rigid body, have magnitude and direction, but do not obey the parallelogram law of addition; these quantities cannot be considered as vectors. We shall see presently that angular velocities (and also *infinitesimal rotations*) *do obey* the parallelogram law and, thus, are truly vector quantities.

Consider a rigid body with a fixed point O which, at a given instant, rotates simultaneously about the axes OA and OB with angular velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ (Fig. 15.34*a*). We know that this motion must be equivalent at the instant considered to a single rotation of angular velocity $\boldsymbol{\omega}$. We propose to show that

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 \tag{15.40}$$

i.e., that the resulting angular velocity may be obtained by adding $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ by the parallelogram law (Fig. 15.34*b*).

Consider a particle P of the body, defined by the position vector $\mathbf{r}$. Denoting respectively by $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}$ the velocity of P when the body rotates about OA only, about OB only, and about both axes simultaneously, we write

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \qquad \mathbf{v}_1 = \boldsymbol{\omega}_1 \times \mathbf{r} \qquad \mathbf{v}_2 = \boldsymbol{\omega}_2 \times \mathbf{r} \tag{15.41}$$

But the vectorial character of *linear* velocities is well established (since they represent the derivatives of position vectors). We have therefore

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$$

where the plus sign indicates vector addition. Substituting from (15.41), we write

$$\boldsymbol{\omega} \times \mathbf{r} = \boldsymbol{\omega}_1 \times \mathbf{r} + \boldsymbol{\omega}_2 \times \mathbf{r}$$
$$\boldsymbol{\omega} \times \mathbf{r} = (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2) \times \mathbf{r}$$

where the plus sign still indicates vector addition. Since the relation obtained holds for an arbitrary $\mathbf{r}$, we conclude that (15.40) must be true.

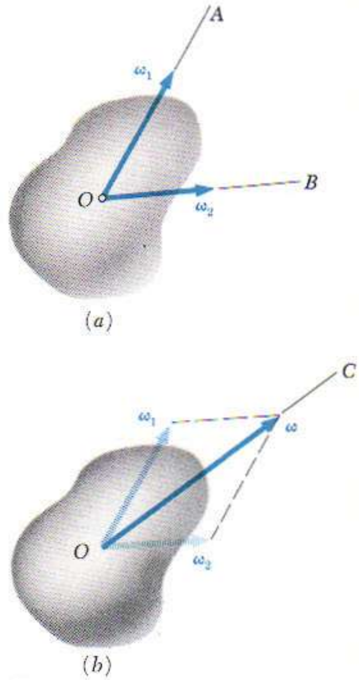


Fig. 15.34

*15.13. General Motion.

We shall now consider the most general motion of a rigid body in space. Let A and B be two particles of the body. We recall from Sec. 11.12 that the velocity of B with respect to the fixed frame of reference $OXYZ$ may be expressed as

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \tag{15.42}$$

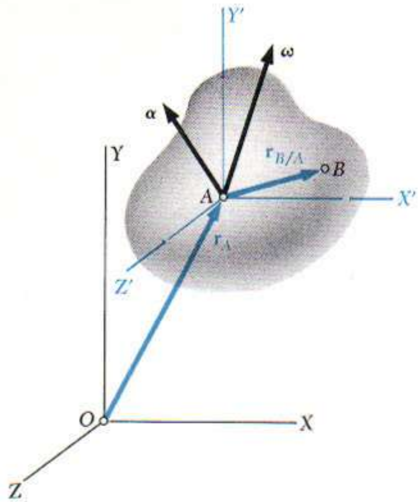


Fig. 15.35

where $\mathbf{v}_{B/A}$ is the velocity of B relative to a frame $AX'Y'Z'$ attached to A and of fixed orientation (Fig. 15.35). Since A is fixed in this frame, the motion of the body relative to $AX'Y'Z'$ is the motion of a body with a fixed point. Therefore, the relative velocity $\mathbf{v}_{B/A}$ may be obtained from (15.37), after $\mathbf{r}$ has been replaced by the position vector $\mathbf{r}_{B/A}$ of B relative to A. Substituting for $\mathbf{v}_{B/A}$ into (15.42), we write

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A} \tag{15.43}$$

where $\boldsymbol{\omega}$ is the angular velocity of the body at the instant considered.

The acceleration of B is obtained by a similar reasoning. We first write

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

and, recalling Eq. (15.38),

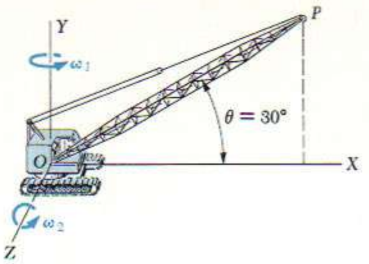
$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A}) \tag{15.44}$$

where $\boldsymbol{\alpha}$ is the angular acceleration of the body at the instant considered.

Equations (15.43) and (15.44) show that *the most general motion of a rigid body is equivalent, at any given instant, to the sum of a translation,* in which all the particles of the body have the same velocity and acceleration as a reference particle A, *and of a motion in which particle A is assumed to be fixed.*†

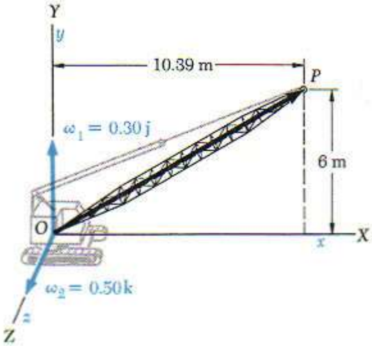
It may easily be shown, by solving (15.43) and (15.44) for $\mathbf{v}_A$ and $\mathbf{a}_A$, that the motion of the body with respect to a frame attached to B would be characterized by the same vectors $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ as its motion relative to $AX'Y'Z'$. Thus, the angular velocity and angular acceleration of a rigid body at a given instant are independent of the choice of the reference point. On the other hand, one should keep in mind that, whether it is attached to A or to B, the moving frame should maintain a fixed orientation; i.e., it should remain parallel to the fixed reference frame $OXYZ$ throughout the motion of the rigid body. In many problems it is found more convenient to use a moving frame which is allowed to rotate as well as to translate. The use of such moving frames will be discussed in Secs. 15.14 and 15.15.

†It is recalled from Sec. 15.12 that, in general, the vectors $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are not collinear, and that the accelerations of the particles of the body in their motion relative to the frame $AX'Y'Z'$ cannot be determined as if the body were rotating permanently about the instantaneous axis through A.



SAMPLE PROBLEM 15.11

The crane shown rotates with a constant angular velocity $\boldsymbol{\omega}_1$ of 0.30 rad/s. Simultaneously, the boom is being raised with a constant angular velocity $\boldsymbol{\omega}_2$ of 0.50 rad/s relative to the cab. Knowing that the length of the boom OP is $l = 12$ m, determine (*a*) the angular velocity $\boldsymbol{\omega}$ of the boom, (*b*) the angular acceleration $\boldsymbol{\alpha}$ of the boom, (*c*) the velocity $\mathbf{v}$ of the tip of the boom, (*d*) the acceleration $\mathbf{a}$ of the tip of the boom.



a. **Angular Velocity of Boom.** Adding the angular velocity $\boldsymbol{\omega}_1$ of the cab and the angular velocity $\boldsymbol{\omega}_2$ of the boom relative to the cab, we obtain the angular velocity $\boldsymbol{\omega}$ of the boom at the instant considered:

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 \qquad \boldsymbol{\omega} = (0.30 \text{ rad/s})\mathbf{j} + (0.50 \text{ rad/s})\mathbf{k} \quad ◀$$

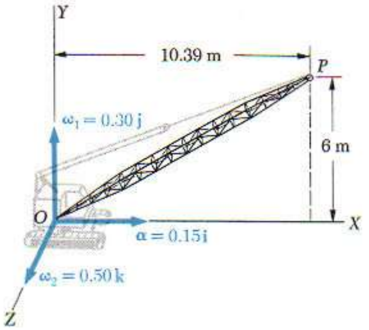
b. **Angular Acceleration of Boom.** The angular acceleration $\boldsymbol{\alpha}$ of the boom is obtained by differentiating $\boldsymbol{\omega}$. Since the vector $\boldsymbol{\omega}_1$ is constant in magnitude and direction, we have

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_1 + \dot{\boldsymbol{\omega}}_2 = 0 + \dot{\boldsymbol{\omega}}_2$$

where the rate of change $\dot{\boldsymbol{\omega}}_2$ is to be computed with respect to the fixed frame $OXYZ$. However, it is more convenient to use a frame $Oxyz$ attached to the cab and rotating with it, since the vector $\boldsymbol{\omega}_2$ also rotates with the cab and, therefore, has zero rate of change with respect to that frame. Using Eq. (15.31) with $\mathbf{Q} = \boldsymbol{\omega}_2$ and $\boldsymbol{\Omega} = \boldsymbol{\omega}_1$, we write

$$\begin{aligned}
(\dot{\mathbf{Q}})_{OXYZ} &= (\dot{\mathbf{Q}})_{Oxyz} + \boldsymbol{\Omega} \times \mathbf{Q} \\
(\dot{\boldsymbol{\omega}}_2)_{OXYZ} &= (\dot{\boldsymbol{\omega}}_2)_{Oxyz} + \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2 \\
\boldsymbol{\alpha} = (\dot{\boldsymbol{\omega}}_2)_{OXYZ} &= 0 + (0.30 \text{ rad/s})\mathbf{j} \times (0.50 \text{ rad/s})\mathbf{k}
\end{aligned}$$

$$\boldsymbol{\alpha} = (0.15 \text{ rad/s}^2)\mathbf{i} \quad ◀$$



c. **Velocity of Tip of Boom.** Noting that the position vector of point P is $\mathbf{r} = (10.39 \text{ m})\mathbf{i} + (6 \text{ m})\mathbf{j}$ and using the expression found for $\boldsymbol{\omega}$ in part *a*, we write

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0.30 \text{ rad/s} & 0.50 \text{ rad/s} \\ 10.39 \text{ m} & 6 \text{ m} & 0 \end{vmatrix}$$

$$\mathbf{v} = -(3 \text{ m/s})\mathbf{i} + (5.20 \text{ m/s})\mathbf{j} - (3.12 \text{ m/s})\mathbf{k} \quad ◀$$

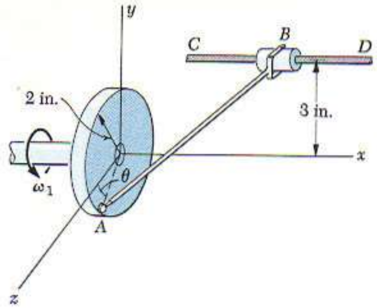
d. **Acceleration of Tip of Boom.** Recalling that $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, we write

$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times \mathbf{v}$$

$$\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.15 & 0 & 0 \\ 10.39 & 6 & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0.30 & 0.50 \\ -3 & 5.20 & -3.12 \end{vmatrix}$$

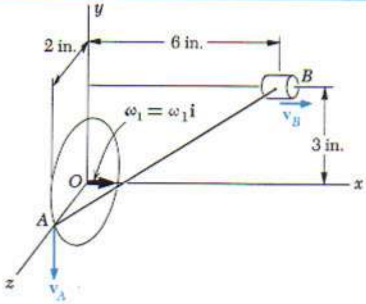
$$= 0.90\mathbf{k} - 0.94\mathbf{i} - 2.60\mathbf{i} - 1.50\mathbf{j} + 0.90\mathbf{k}$$

$$\mathbf{a} = -(3.54 \text{ m/s}^2)\mathbf{i} - (1.50 \text{ m/s}^2)\mathbf{j} + (1.80 \text{ m/s}^2)\mathbf{k} \quad ◀$$



SAMPLE PROBLEM 15.12

The rod AB, of length 7 in., is attached to the disk by a ball-and-socket connection and to the collar B by a clevis. The disk rotates in the yz plane at a constant rate $\omega_1 = 12$ rad/s, while the collar is free to slide along the horizontal rod CD. For the position $\theta = 0$, determine (*a*) the velocity of the collar, (*b*) the angular velocity of the rod.



$\boldsymbol{\omega}_1 = 12\mathbf{i}$
$\mathbf{r}_A = 2\mathbf{k}$
$\mathbf{r}_B = 6\mathbf{i} + 3\mathbf{j}$
$\mathbf{r}_{B/A} = 6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

a. Velocity of Collar. Since point A is attached to the disk and since collar B moves parallel to the x axis, we have

$$\mathbf{v}_A = \boldsymbol{\omega}_1 \times \mathbf{r}_A = 12\mathbf{i} \times 2\mathbf{k} = -24\mathbf{j} \qquad \mathbf{v}_B = v_B\mathbf{i}$$

Denoting by $\boldsymbol{\omega}$ the angular velocity of the rod, we write

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$$

$$v_B\mathbf{i} = -24\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 6 & 3 & -2 \end{vmatrix}$$

$$v_B\mathbf{i} = -24\mathbf{j} + (-2\omega_y - 3\omega_z)\mathbf{i} + (6\omega_z + 2\omega_x)\mathbf{j} + (3\omega_x - 6\omega_y)\mathbf{k}$$

Equating the coefficients of the unit vectors, we obtain

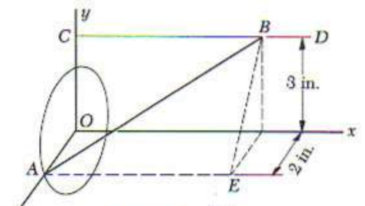
$$v_B = \qquad -2\omega_y \quad -3\omega_z \tag{1}$$
$$24 = 2\omega_x \qquad\qquad +6\omega_z \tag{2}$$
$$0 = 3\omega_x \quad -6\omega_y \tag{3}$$

Multiplying Eqs. (1), (2), (3), respectively, by 6, 3, -2 and adding, we write

$$6v_B + 72 = 0 \qquad v_B = -12 \qquad \mathbf{v}_B = -(12 \text{ in./s})\mathbf{i} \;◄$$

b. Angular Velocity of Rod AB. We note that the angular velocity cannot be determined from Eqs. (1), (2), and (3), since the determinant formed by the coefficients of ω_x, ω_y, and ω_z is zero. We must therefore obtain an additional equation by considering the constraint imposed by the clevis at B.

The collar-clevis connection at B permits rotation of AB about the rod CD and also about an axis perpendicular to the plane containing AB and CD. It prevents rotation of AB about the axis EB, which is perpendicular to CD and lies in the plane containing AB and CD. Thus the projection of $\boldsymbol{\omega}$ on $\mathbf{r}_{E/B}$ must be zero and we write†



$\mathbf{r}_{E/B} = -3\mathbf{j} + 2\mathbf{k}$

$$\boldsymbol{\omega} \cdot \mathbf{r}_{E/B} = 0 \qquad (\omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}) \cdot (-3\mathbf{j} + 2\mathbf{k}) = 0$$
$$-3\omega_y + 2\omega_z = 0 \tag{4}$$

Solving Eqs. (1) through (4) simultaneously, we obtain

$$v_B = -12 \qquad \omega_x = 3.69 \qquad \omega_y = 1.846 \qquad \omega_z = 2.77$$
$$\boldsymbol{\omega} = (3.69 \text{ rad/s})\mathbf{i} + (1.846 \text{ rad/s})\mathbf{j} + (2.77 \text{ rad/s})\mathbf{k} \;◄$$

†We could also note that the direction of EB is that of the vector triple product $\mathbf{r}_{B/C} \times (\mathbf{r}_{B/C} \times \mathbf{r}_{B/A})$ and write $\boldsymbol{\omega} \cdot [\mathbf{r}_{B/C} \times (\mathbf{r}_{B/C} \times \mathbf{r}_{B/A})] = 0$. This formulation would be particularly useful if the rod CD were skew.

PROBLEMS

15.130 The rigid body shown rotates about the origin of coordinates with an angular velocity $\boldsymbol{\omega}$. Denoting the velocity of point A by $\mathbf{v}_A = (v_A)_x\mathbf{i} + (v_A)_y\mathbf{j} + (v_A)_z\mathbf{k}$, and knowing that $(v_A)_x = 40$ mm/s and $(v_A)_y = -200$ mm/s, determine (*a*) the velocity component $(v_A)_z$, (*b*) the velocity of point B.

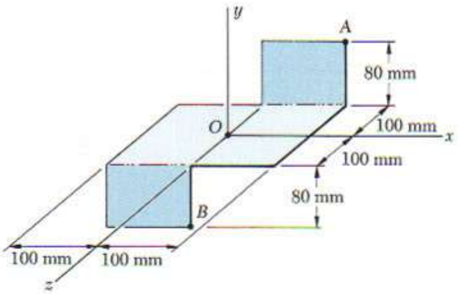


Fig. P15.130 and P15.131

15.131 The rigid body shown rotates about the origin of coordinates with an angular velocity $\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$. Knowing that $(v_A)_y = 400$ mm/s, $(v_B)_y = -300$ mm/s, and $\omega_y = 2$ rad/s, determine (*a*) the angular velocity of the body, (*b*) the velocities of points A and B.

15.132 The circular plate and rod are rigidly connected and rotate about the ball-and-socket joint O with an angular velocity $\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$. Knowing that $\mathbf{v}_A = -(27 \text{ in./s})\mathbf{i} + (18 \text{ in./s})\mathbf{j} + (v_A)_z\mathbf{k}$ and $\omega_y = 4$ rad/s, determine (*a*) the angular velocity of the assembly, (*b*) the velocity of point B.

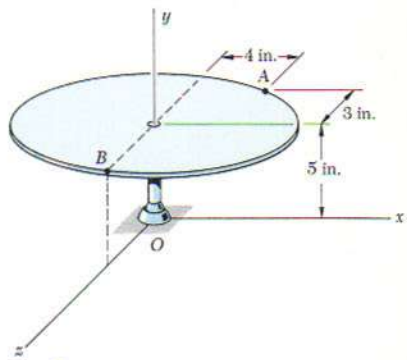


Fig. P15.132

15.133 Solve Prob. 15.132, assuming that $\omega_y = 0$.

15.134 The rotor of an electric motor rotates at the constant rate $\omega_1 = 3600$ rpm. Determine the angular acceleration of the rotor as the motor is rotated about the y axis with a constant angular velocity of 6 rpm clockwise when viewed from the positive y axis.

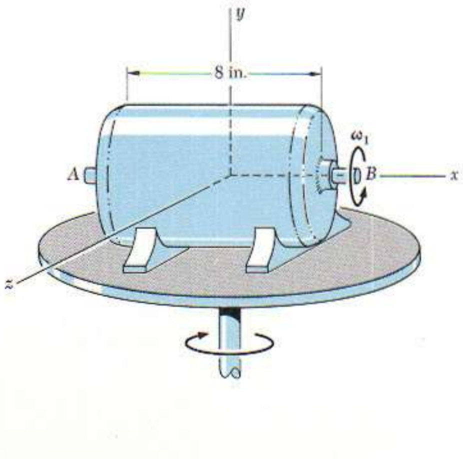


Fig. P15.134

15.135 The propeller of a small airplane rotates at a constant rate of 2200 rpm in a clockwise sense when viewed by the pilot. Knowing that the airplane is turning left along a horizontal circular path of radius 1000 ft, and that the speed of the airplane is 150 mi/h, determine the angular acceleration of the propeller at the instant the airplane is moving due south.

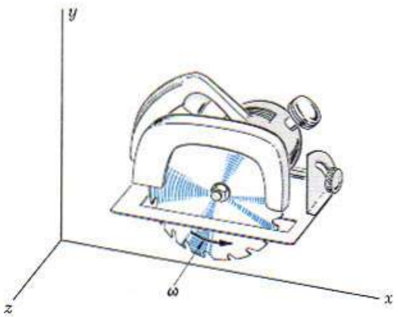


Fig. P15.136

15.136 The blade of a portable saw rotates at a constant rate $\omega = 1800$ rpm as shown. Determine the angular acceleration of the blade as a man rotates the saw about the y axis with an angular velocity of 3 rad/s and an angular acceleration of 5 rad/s^2, both clockwise when viewed from above.

15.137 Knowing that the turbine rotor shown rotates at a constant rate $\omega_1 = 10{,}000$ rpm, determine the angular acceleration of the rotor if the turbine housing has a constant angular velocity of 3 rad/s clockwise as viewed from (*a*) the positive y axis, (*b*) the positive z axis.

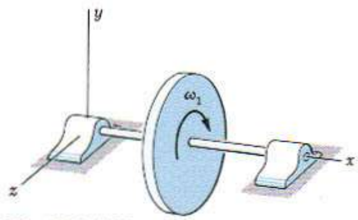


Fig. P15.137

15.138 In the gear system shown, gear A is free to rotate about the horizontal rod OA. Assuming that gear B is fixed and that shaft OC rotates with a constant angular velocity $\boldsymbol{\omega}_1$, determine (*a*) the angular velocity of gear A, (*b*) the angular acceleration of gear A.

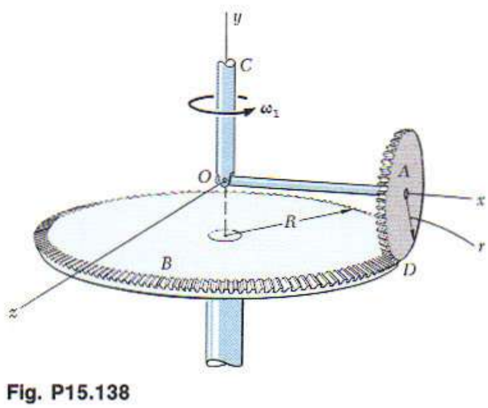


Fig. P15.138

15.139 Solve Prob. 15.138, assuming that shaft OC and gear B rotate with constant angular velocities $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$, respectively, both counterclockwise as viewed from the positive y axis.

15.140 Two shafts AC and CF, which lie in the vertical xy plane, are connected by a universal joint at C. Shaft CF rotates with a constant angular velocity ω_1 as shown. At a time when the arm of the crosspiece attached to shaft CF is horizontal, determine the angular velocity of shaft AC.

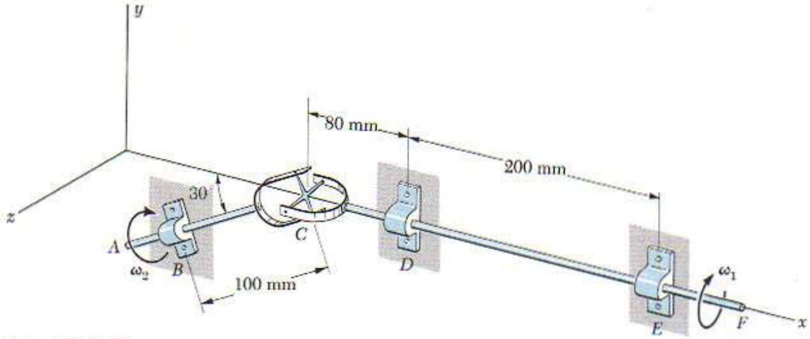


Fig. P15.140

15.141 Solve Prob. 15.140, assuming that the arm of the crosspiece attached to shaft CF is vertical.

15.142 The radar antenna shown rotates with a constant angular velocity ω_1 of 1.5 rad/s about the y axis. At the instant shown the antenna is also rotating about the z axis with an angular velocity ω_2 of 2 rad/s and an angular acceleration α_2 of 2.5 rad/s^2. Determine (a) the angular acceleration of the antenna, (b) the accelerations of points A and B.

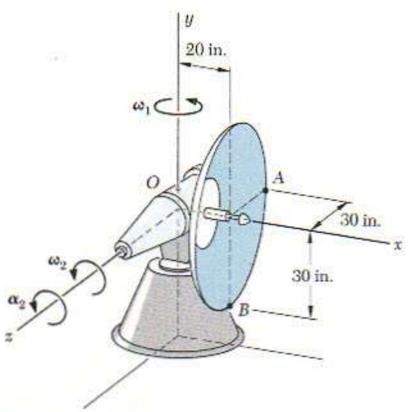


Fig. P15.142

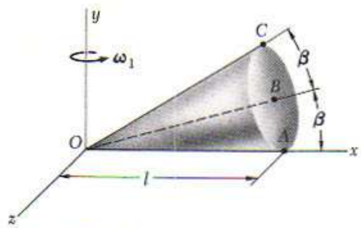


Fig. P15.143

15.143 The cone shown rolls on the zx plane with its apex at the origin of coordinates. Denoting by ω_1 the constant angular velocity of the axis OB of the cone about the y axis, determine (*a*) the rate of spin of the cone about the axis OB, (*b*) the total angular velocity of the cone, (*c*) the angular acceleration of the cone.

15.144 A rod of length $OP = 500$ mm is mounted on a bracket as shown. At the instant considered the angle β is being increased at the constant rate $d\beta/dt = 4$ rad/s and the elevation angle γ is being increased at the constant rate $d\gamma/dt = 1.6$ rad/s. For the position $\beta = 0$ and $\gamma = 30°$, determine (*a*) the angular velocity of the rod, (*b*) the angular acceleration of the rod, (*c*) the velocity and acceleration of point P.

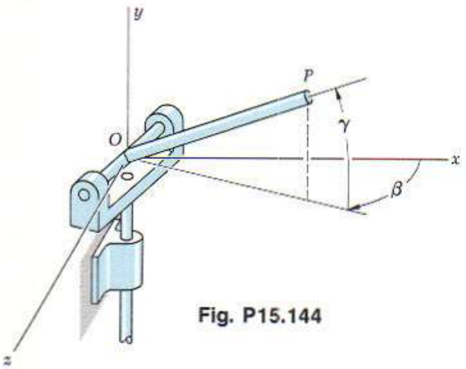


Fig. P15.144

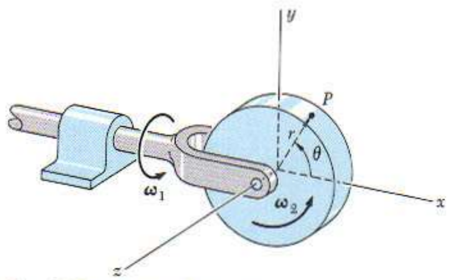


Fig. P15.145 and P15.146

15.145 A disk of radius r spins at the constant rate ω_2 about an axle held by a fork-ended horizontal rod which rotates at the constant rate ω_1. Determine the acceleration of point P for an arbitrary value of the angle θ.

15.146 A disk of radius r spins at the constant rate ω_2 about an axle held by a fork-ended horizontal rod which rotates at the constant rate ω_1. Determine (*a*) the angular acceleration of the disk, (*b*) the acceleration of point P on the rim of the disk when $\theta = 0$, (*c*) the acceleration of P when $\theta = 90°$.

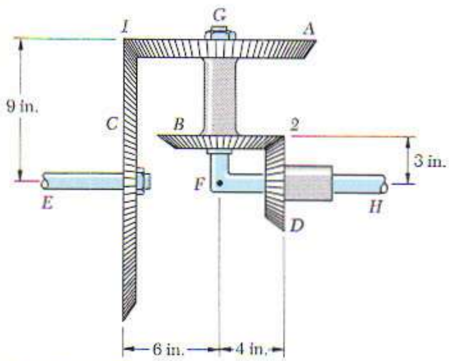


Fig. P15.147

15.147 In the planetary gear system shown, gears A and B are rigidly connected to each other and rotate as a unit about shaft FG. Gears C and D rotate with constant angular velocities of 15 rad/s and 30 rad/s, respectively (both counterclockwise when viewed from the right). Choosing the x axis to the right, the y axis upward, and the z axis pointing out of the plane of the figure, determine (*a*) the common angular velocity of gears A and B, (*b*) the angular velocity of shaft FH, which is rigidly attached to FC.

15.148 In Prob. 15.147, determine (*a*) the common angular acceleration of gears A and B, (*b*) the acceleration of the tooth of gear B which is in contact with gear D at point 2.

15.149 Three rods are welded together to form the corner assembly shown which is attached to a fixed ball-and-socket joint at O. The end of rod OA moves on the inclined plane D which is perpendicular to the xy plane. The end of rod OB moves on the horizontal plane E which coincides with the zx plane. Knowing that at the instant shown $\mathbf{v}_B = (1.6 \text{ m/s})\mathbf{k}$, determine (a) the angular velocity of the assembly, (b) the velocity of point C.

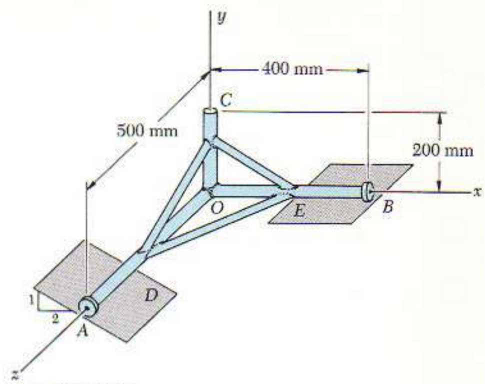


Fig. P15.149

15.150 In Prob. 15.149 the speed of point B is known to be constant. For the position shown, determine (a) the angular acceleration of the assembly, (b) the acceleration of point C.

15.151 In Prob. 15.149 the speed of point B is being decreased at the rate of 0.8 m/s^2. For the position shown, determine (a) the angular acceleration of the assembly, (b) the acceleration of point C.

15.152 Rod AB, of length 220 mm, is connected by ball-and-socket joints to collars A and B, which slide along the two rods shown. Knowing that collar A moves downward with a constant speed of 63 mm/s, determine the velocity of collar B when $c = 120$ mm.

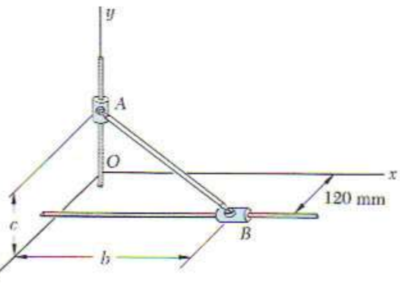


Fig. P15.152

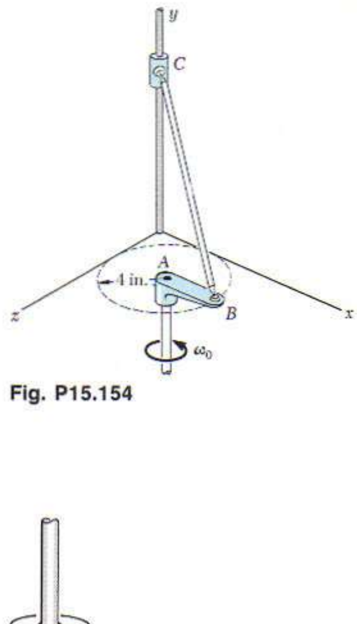


Fig. P15.154

15.153 Solve Prob. 15.152 when $c = 40$ mm.

15.154 Rod BC, of length 21 in., is connected by ball-and-socket joints to the collar C and to the rotating arm AB. Knowing that arm AB rotates in the zx plane at the constant rate $\omega_0 = 38$ rad/s, determine the velocity of collar C.

15.155 In Prob. 15.152, the ball-and-socket joint between the rod and collar A is replaced by the clevis connection shown. Determine (a) the angular velocity of the rod, (b) the velocity of collar B.

15.156 In Prob. 15.154, the ball-and-socket joint between the rod and collar C is replaced by the clevis connection shown. Determine (a) the angular velocity of the rod, (b) the velocity of collar C.

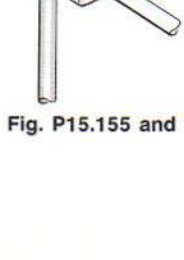


Fig. P15.155 and P15.156

15.157 In the linkage shown, crank BC rotates in the yz plane while crank ED rotates in a plane parallel to the xy plane. Knowing that in the position shown crank BC has an angular velocity ω_1 of 10 rad/s and no angular acceleration, determine the corresponding angular velocity ω_2 of crank ED.

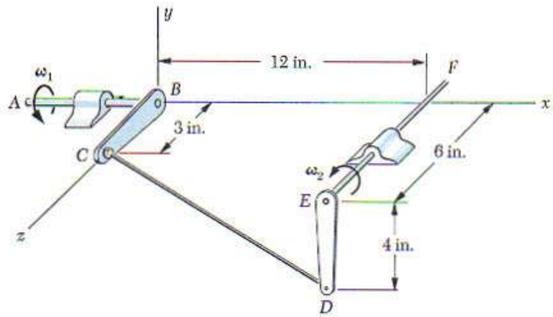


Fig. P15.157

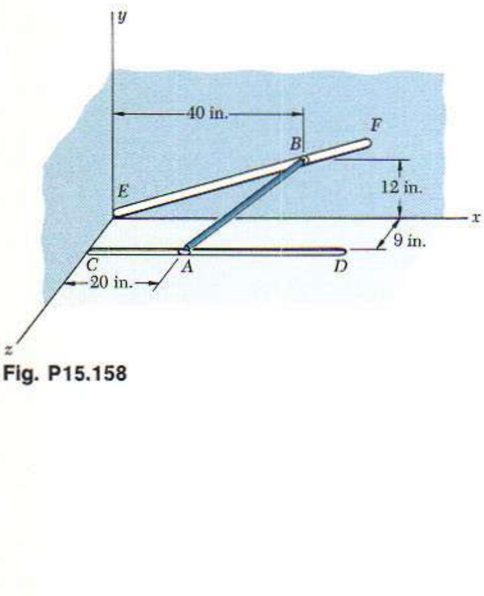


Fig. P15.158

15.158 Rod AB has a length of 25 in. and is guided by pins sliding in the slots CD and EF, which lie in the zx and xy planes, respectively. Knowing that in the position shown end A moves to the left along slot CD with a speed of 17 in./s, determine the velocity of end B of the rod.

***15.159** In Prob. 15.152, determine the acceleration of collar B when $c = 40$ mm.

***15.160** In Prob. 15.152, determine the acceleration of collar B when $c = 120$ mm.

***15.161** In Prob. 15.157, determine the angular acceleration of crank ED.

***15.162** In Prob. 15.154, determine the acceleration of collar C.

***15.14. Three-dimensional Motion of a Particle Relative to a Rotating Frame. Coriolis Acceleration.** We saw in Sec. 15.10 that, given a vector function $\mathbf{Q}(t)$ and two frames of reference centered at O—a fixed frame $OXYZ$ and a rotating frame $Oxyz$—the rates of change of $\mathbf{Q}$ with respect to the two frames satisfy the relation

$$(\dot{\mathbf{Q}})_{OXYZ} = (\dot{\mathbf{Q}})_{Oxyz} + \boldsymbol{\Omega} \times \mathbf{Q} \qquad (15.31)$$

We had assumed at the time that the frame $Oxyz$ was constrained to rotate about a fixed axis OA. However, the derivation given in

Sec. 15.10 remains valid when the frame $Oxyz$ is constrained only to have a fixed point O. Under this more general assumption, the axis OA represents the *instantaneous* axis of rotation of the frame $Oxyz$ (Sec. 15.12), and the vector $\mathbf{\Omega}$ its angular velocity at the instant considered (Fig. 15.36).

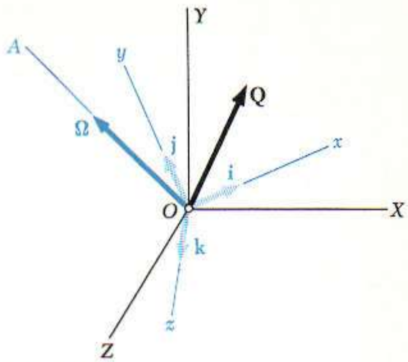


Fig. 15.36

We shall now consider the three-dimensional motion of a particle P relative to a rotating frame $Oxyz$ constrained to have a fixed origin O. Let $\mathbf{r}$ be the position vector of P at a given instant, and $\mathbf{\Omega}$ the angular velocity of the frame $Oxyz$ with respect to the fixed frame $OXYZ$ at the same instant (Fig. 15.37). The derivations given in Sec. 15.11 for the two-dimensional motion of a particle may readily be extended to the three-dimensional case, and we may express the absolute velocity $\mathbf{v}_P$ of P (i.e., its velocity with respect to the fixed frame $OXYZ$) as

$$\mathbf{v}_P = \mathbf{\Omega} \times \mathbf{r} + (\dot{\mathbf{r}})_{Oxyz} \tag{15.45}$$

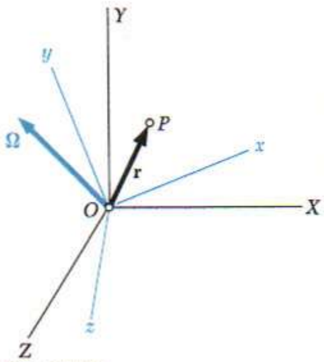


Fig. 15.37

This relation may be written in the alternate form

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/\mathcal{F}} \tag{15.46}$$

where $\mathbf{v}_P$ = absolute velocity of particle P
$\mathbf{v}_{P'}$ = velocity of point P' of moving frame coinciding with P
$\mathbf{v}_{P/\mathcal{F}}$ = velocity of P relative to moving frame

The absolute acceleration $\mathbf{a}_P$ of P may be expressed as

$$\mathbf{a}_P = \dot{\mathbf{\Omega}} \times \mathbf{r} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) + 2\mathbf{\Omega} \times (\dot{\mathbf{r}})_{Oxyz} + (\ddot{\mathbf{r}})_{Oxyz} \tag{15.47}$$

We may also use the alternate form

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/\mathcal{F}} + \mathbf{a}_c \tag{15.48}$$

where $\mathbf{a}_P$ = absolute acceleration of particle P
$\mathbf{a}_{P'}$ = acceleration of point P' of moving frame coinciding with P
$\mathbf{a}_{P/\mathcal{F}}$ = acceleration of P relative to moving frame
$\mathbf{a}_c = 2\mathbf{\Omega} \times (\dot{\mathbf{r}})_{Oxyz} = 2\mathbf{\Omega} \times \mathbf{v}_{P/\mathcal{F}}$
= complementary, or Coriolis, acceleration

We note that the Coriolis acceleration is perpendicular to the vectors $\mathbf{\Omega}$ and $\mathbf{v}_{P/\mathcal{F}}$. However, since these vectors are usually not perpendicular to each other, the magnitude of $\mathbf{a}_c$, in general, is *not* equal to $2\Omega v_{P/F}$, as was the case for the plane motion of a particle. We further note that the Coriolis acceleration reduces to zero when the vectors $\mathbf{\Omega}$ and $\mathbf{v}_{P/F}$ are parallel, or when either of them is zero.

Rotating frames of reference are particularly useful in the study of the three-dimensional motion of rigid bodies. If a rigid body has a fixed point O, as was the case for the crane of Sample Prob. 15.11, we may use a frame $Oxyz$ which is neither fixed nor rigidly attached to the rigid body. Denoting by $\mathbf{\Omega}$ the angular velocity of the frame $Oxyz$, we then resolve the angular velocity $\boldsymbol{\omega}$ of the body into the components $\mathbf{\Omega}$ and $\boldsymbol{\omega}_{B/F}$, where the second component represents the angular velocity of the body relative to the frame $Oxyz$ (see Sample Prob. 15.14). An appropriate choice of the rotating frame will often lead to a simpler analysis of the motion of the rigid body than would be possible with axes of fixed orientation. This is especially true in the case of the general three-dimensional motion of a rigid body, i.e., when the rigid body under consideration has no fixed point (see Sample Prob. 15.15).

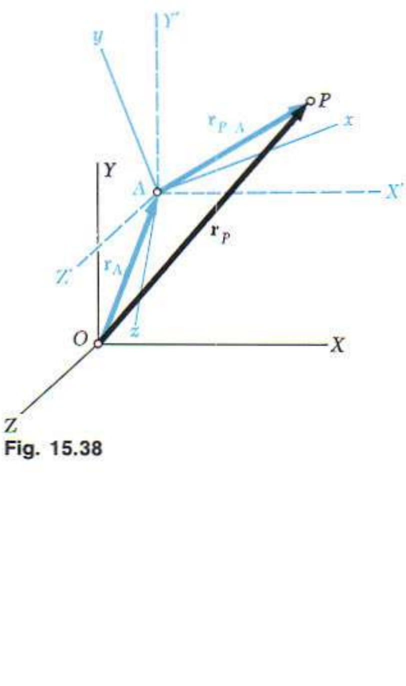


Fig. 15.38

*15.15. Frame of Reference in General Motion.

Consider a fixed frame of reference $OXYZ$ and a frame $Axyz$ which moves in a known, but arbitrary, fashion with respect to $OXYZ$ (Fig. 15.38). Let P be a particle moving in space. The position of P is defined at any instant by the vector $\mathbf{r}_P$ in the fixed frame, and by the vector $\mathbf{r}_{P/A}$ in the moving frame. Denoting by $\mathbf{r}_A$ the position vector of A in the fixed frame, we have

$$\mathbf{r}_P = \mathbf{r}_A + \mathbf{r}_{P/A} \tag{15.49}$$

The absolute velocity $\mathbf{v}_P$ of the particle is obtained by writing

$$\mathbf{v}_P = \dot{\mathbf{r}}_P = \dot{\mathbf{r}}_A + \dot{\mathbf{r}}_{P/A} \tag{15.50}$$

where the derivatives are defined with respect to the fixed frame $OXYZ$. Thus, the first term in the right-hand member of (15.50) represents the velocity $\mathbf{v}_A$ of the origin A of the moving axes. On the other hand, since the rate of change of a vector is the same with respect to a fixed frame and with respect to a frame in translation (Sec. 11.10), the second term may be regarded as the velocity $\mathbf{v}_{P/A}$ of P relative to the frame $AX'Y'Z'$ of the same orientation as $OXYZ$ and the same origin as $Axyz$. We therefore have

$$\mathbf{v}_P = \mathbf{v}_A + \mathbf{v}_{P/A} \tag{15.51}$$

But the velocity $\mathbf{v}_{P/A}$ of P relative to $AX'Y'Z'$ may be obtained from (15.45) by substituting $\mathbf{r}_{P/A}$ for $\mathbf{r}$ in that equation. We write

$$\mathbf{v}_P = \mathbf{v}_A + \mathbf{\Omega} \times \mathbf{r}_{P/A} + (\dot{\mathbf{r}}_{P/A})_{Axyz} \tag{15.52}$$

where $\mathbf{\Omega}$ is the angular velocity of the frame $Axyz$ at the instant considered.

The absolute acceleration $\mathbf{a}_P$ of the particle is obtained by differentiating (15.51) and writing

$$\mathbf{a}_P = \dot{\mathbf{v}}_P = \dot{\mathbf{v}}_A + \dot{\mathbf{v}}_{P/A} \tag{15.53}$$

where the derivatives are defined with respect to either of the frames $OXYZ$ or $AX'Y'Z'$. Thus, the first term in the right-hand member of (15.53) represents the acceleration $\mathbf{a}_A$ of the origin A of the moving axes, and the second term the acceleration $\mathbf{a}_{P/A}$ of P relative to the frame $AX'Y'Z'$. This acceleration may be obtained from (15.47) by substituting $\mathbf{r}_{P/A}$ for $\mathbf{r}$. We therefore write

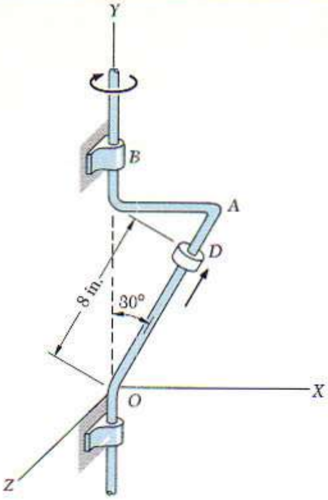
$$\begin{aligned}\mathbf{a}_P = \mathbf{a}_A + \dot{\mathbf{\Omega}} \times \mathbf{r}_{P/A} &+ \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{P/A}) \\ &+ 2\mathbf{\Omega} \times (\dot{\mathbf{r}}_{P/A})_{Axyz} + (\ddot{\mathbf{r}}_{P/A})_{Axyz}\end{aligned} \tag{15.54}$$

Formulas (15.52) and (15.54) make it possible to determine the velocity and acceleration of a given particle with respect to a fixed frame of reference, when the motion of the particle is known with respect to a moving frame. These formulas become more significant, and considerably easier to remember, if we note that the sum of the first two terms in (15.52) represents the velocity of the point P' of the moving frame which coincides with P at the instant considered, and that the sum of the first three terms in (15.54) represents the acceleration of the same point. Thus, the relations (15.46) and (15.48) of the preceding section are still valid in the case of a reference frame in general motion, and we write

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/\mathcal{F}} \tag{15.46}$$
$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/\mathcal{F}} + \mathbf{a}_c \tag{15.48}$$

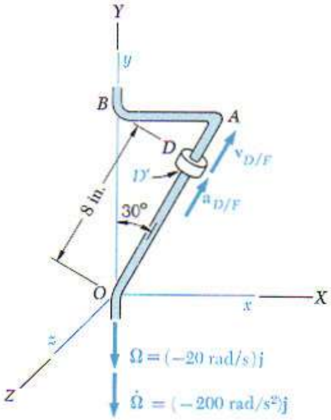
where the various vectors involved have been defined in Sec. 15.14.

We may note that, if the reference frame $Axyz$ is in translation, the velocity and acceleration of the point P' of the frame which coincides with P become respectively equal to the velocity and acceleration of the origin A of the frame. On the other hand, since the frame maintains a fixed orientation, $\mathbf{a}_c$ is zero, and the relations (15.46) and (15.48) reduce, respectively, to the relations (11.33) and (11.34) derived in Sec. 11.12.



SAMPLE PROBLEM 15.13

The bent rod OAB rotates about the vertical OB. At the instant considered, its angular velocity and angular acceleration are, respectively, 20 rad/s and 200 rad/s^2, both clockwise when viewed from the positive Y axis. The collar D moves along the rod and, at the instant considered, $OD = 8$ in., and the velocity and acceleration of the collar relative to the rod are, respectively, 50 in./s and 600 in./s^2, both upward. Determine (*a*) the velocity of the collar, (*b*) the acceleration of the collar.



Frames of Reference. The frame $OXYZ$ is fixed. We attach the rotating frame $Oxyz$ to the bent rod. Its angular velocity and angular acceleration relative to $OXYZ$, therefore, are $\boldsymbol{\Omega} = (-20 \text{ rad/s})\mathbf{j}$ and $\dot{\boldsymbol{\Omega}} = (-200 \text{ rad/s}^2)\mathbf{j}$, respectively. The position vector of D is

$$\mathbf{r} = (8 \text{ in.})(\sin 30°\mathbf{i} + \cos 30°\mathbf{j}) = (4 \text{ in.})\mathbf{i} + (6.93 \text{ in.})\mathbf{j}$$

a. **Velocity $\mathbf{v}_D$.** Denoting by D' the point of the rod which coincides with D, we write from Eq. (15.46)

$$\mathbf{v}_D = \mathbf{v}_{D'} + \mathbf{v}_{D/F} \tag{1}$$

where

$$\mathbf{v}_{D'} = \boldsymbol{\Omega} \times \mathbf{r} = (-20 \text{ rad/s})\mathbf{j} \times [(4 \text{ in.})\mathbf{i} + (6.93 \text{ in.})\mathbf{j}] = (80 \text{ in./s})\mathbf{k}$$
$$\mathbf{v}_{D/F} = (50 \text{ in./s})(\sin 30°\mathbf{i} + \cos 30°\mathbf{j}) = (25 \text{ in./s})\mathbf{i} + (43.3 \text{ in./s})\mathbf{j}$$

Substituting the values obtained for $\mathbf{v}_{D'}$ and $\mathbf{v}_{D/F}$ into (1), we find

$$\mathbf{v}_D = (25 \text{ in./s})\mathbf{i} + (43.3 \text{ in./s})\mathbf{j} + (80 \text{ in./s})\mathbf{k} \quad ◄$$

b. **Acceleration $\mathbf{v}_D$.** From Eq. (15.48) we write

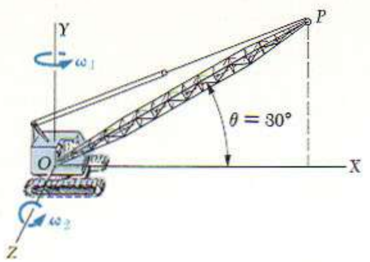
$$\mathbf{a}_D = \mathbf{a}_{D'} + \mathbf{a}_{D/F} + \mathbf{a}_c \tag{2}$$

where

$$\begin{aligned}
\mathbf{a}_{D'} &= \dot{\boldsymbol{\Omega}} \times \mathbf{r} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \\
&= (-200 \text{ rad/s}^2)\mathbf{j} \times [(4 \text{ in.})\mathbf{i} + (6.93 \text{ in.})\mathbf{j}] \\
&\qquad - (20 \text{ rad/s})\mathbf{j} \times (80 \text{ in./s})\mathbf{k} \\
&= +(800 \text{ in./s}^2)\mathbf{k} - (1600 \text{ in./s}^2)\mathbf{i} \\
\mathbf{a}_{D/F} &= (600 \text{ in./s}^2)(\sin 30°\mathbf{i} + \cos 30°\mathbf{j}) = (300 \text{ in./s}^2)\mathbf{i} + (520 \text{ in./s}^2)\mathbf{j} \\
\mathbf{a}_c &= 2\boldsymbol{\Omega} \times \mathbf{v}_{D/F} \\
&= 2(-20 \text{ rad/s})\mathbf{j} \times [(25 \text{ in./s})\mathbf{i} + (43.3 \text{ in./s})\mathbf{j}] = (1000 \text{ in./s}^2)\mathbf{k}
\end{aligned}$$

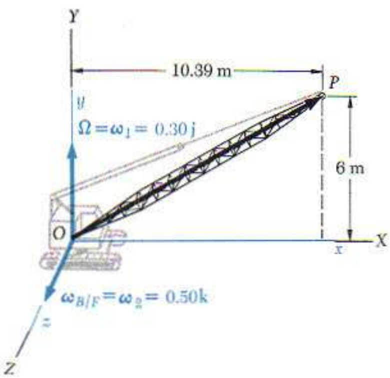
Substituting the values obtained for $\mathbf{a}_{D'}$, $\mathbf{a}_{D/F}$, and $\mathbf{a}_c$ into (2):

$$\mathbf{a}_D = -(1300 \text{ in./s}^2)\mathbf{i} + (520 \text{ in./s}^2)\mathbf{j} + (1800 \text{ in./s}^2)\mathbf{k} \quad ◄$$



SAMPLE PROBLEM 15.14

The crane shown rotates with a constant angular velocity ω_1 of 0.30 rad/s. Simultaneously, the boom is being raised with a constant angular velocity ω_2 of 0.50 rad/s relative to the cab. Knowing that the length of the boom OP is $l = 12$ m, determine (*a*) the velocity of the tip of the boom, (*b*) the acceleration of the tip of the boom.



Frames of Reference. The frame $OXYZ$ is fixed. We attach the rotating frame $Oxyz$ to the cab. Its angular velocity with respect to the frame $OXYZ$, therefore, is $\boldsymbol{\Omega} = \boldsymbol{\omega}_1 = (0.30 \text{ rad/s})\mathbf{j}$. The angular velocity of the boom relative to the cab and the rotating frame $Oxyz$ is $\boldsymbol{\omega}_{B/F} = \boldsymbol{\omega}_2 = (0.50 \text{ rad/s})\mathbf{k}$.

a. **Velocity $\mathbf{v}_P$.** From Eq. (15.46) we write

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/F} \tag{1}$$

where $\mathbf{v}_{P'}$ is the velocity of the point P' of the frame $Oxyz$ which coincides with P,

$$\mathbf{v}_{P'} = \boldsymbol{\Omega} \times \mathbf{r} = (0.30 \text{ rad/s})\mathbf{j} \times [(10.39 \text{ m})\mathbf{i} + (6 \text{ m})\mathbf{j}] = -(3.12 \text{ m/s})\mathbf{k}$$

and where $\mathbf{v}_{P/F}$ is the velocity of P relative to the rotating frame $Oxyz$. But the angular velocity of the boom relative to $Oxyz$ was found to be $\boldsymbol{\omega}_{B/F} = (0.50 \text{ rad/s})\mathbf{k}$. The velocity of its tip P relative to $Oxyz$ is therefore

$$\begin{aligned}\mathbf{v}_{P/F} = \boldsymbol{\omega}_{B/F} \times \mathbf{r} &= (0.50 \text{ rad/s})\mathbf{k} \times [(10.39 \text{ m})\mathbf{i} + (6 \text{ m})\mathbf{j}] \\ &= -(3 \text{ m/s})\mathbf{i} + (5.20 \text{ m/s})\mathbf{j}\end{aligned}$$

Substituting the values obtained for $\mathbf{v}_{P'}$ and $\mathbf{v}_{P/F}$ into (1), we find

$$\mathbf{v}_P = -(3 \text{ m/s})\mathbf{i} + (5.20 \text{ m/s})\mathbf{j} - (3.12 \text{ m/s})\mathbf{k} \quad ◄$$

b. **Acceleration $\mathbf{a}_P$.** From Eq. (15.48) we write

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/F} + \mathbf{a}_c \tag{2}$$

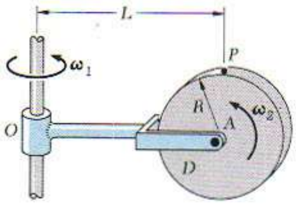
Since $\boldsymbol{\Omega}$ and $\boldsymbol{\omega}_{B/F}$ are both constant, we have

$$\begin{aligned}\mathbf{a}_{P'} &= \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = (0.30 \text{ rad/s})\mathbf{j} \times (-3.12 \text{ m/s})\mathbf{k} = -(0.94 \text{ m/s}^2)\mathbf{i} \\ \mathbf{a}_{P/F} &= \boldsymbol{\omega}_{B/F} \times (\boldsymbol{\omega}_{B/F} \times \mathbf{r}) \\ &= (0.50 \text{ rad/s})\mathbf{k} \times [-(3 \text{ m/s})\mathbf{i} + (5.20 \text{ m/s})\mathbf{j}] \\ &= -(1.50 \text{ m/s}^2)\mathbf{j} - (2.60 \text{ m/s}^2)\mathbf{i} \\ \mathbf{a}_c &= 2\boldsymbol{\Omega} \times \mathbf{v}_{P/F} \\ &= 2(0.30 \text{ rad/s})\mathbf{j} \times [-(3 \text{ m/s})\mathbf{i} + (5.20 \text{ m/s})\mathbf{j}] = (1.80 \text{ m/s}^2)\mathbf{k}\end{aligned}$$

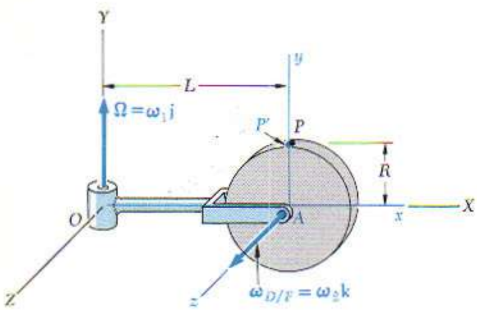
Substituting for $\mathbf{a}_{P'}$, $\mathbf{a}_{P/F}$, and $\mathbf{a}_c$ into (2), we find

$$\mathbf{a}_P = -(3.54 \text{ m/s}^2)\mathbf{i} - (1.50 \text{ m/s}^2)\mathbf{j} + (1.80 \text{ m/s}^2)\mathbf{k} \quad ◄$$

SAMPLE PROBLEM 15.15



Disk D, of radius R, is pinned to end A of the arm OA of length L located in the plane of the disk. The arm rotates about a vertical axis through O at the constant rate ω_1, and the disk rotates about A at the constant rate ω_2. Determine (a) the velocity of point P located directly above A, (b) the acceleration of P, (c) the angular velocity and angular acceleration of the disk.



Frames of Reference. The frame $OXYZ$ is fixed. We attach the moving frame $Axyz$ to the arm OA. Its angular velocity with respect to the frame $OXYZ$, therefore, is $\boldsymbol{\Omega} = \omega_1 \mathbf{j}$. The angular velocity of disk D relative to the frame $Axyz$ is $\boldsymbol{\omega}_{D/F} = \omega_2 \mathbf{k}$. The position vector of P relative to O is $\mathbf{r} = L\mathbf{i} + R\mathbf{j}$ and its position vector relative to A is $\mathbf{r}_{P/A} = R\mathbf{j}$.

a. **Velocity $\mathbf{v}_P$.** Denoting by P' the point of the frame $Axyz$ which coincides with P, we write from Eq. (15.46)

$$\mathbf{v}_P = \mathbf{v}_{P'} + \mathbf{v}_{P/F} \tag{1}$$

where $\mathbf{v}_{P'} = \boldsymbol{\Omega} \times \mathbf{r} = \omega_1 \mathbf{j} \times (L\mathbf{i} + R\mathbf{j}) = -\omega_1 L\mathbf{k}$

$$\mathbf{v}_{P/F} = \boldsymbol{\omega}_{D/F} \times \mathbf{r}_{P/A} = \omega_2 \mathbf{k} \times R\mathbf{j} = -\omega_2 R\mathbf{i}$$

Substituting the values obtained for $\mathbf{v}_{P'}$ and $\mathbf{v}_{P/F}$ into (1), we find

$$\mathbf{v}_P = -\omega_2 R\mathbf{i} - \omega_1 L\mathbf{k} \quad ◀$$

b. **Acceleration $\mathbf{a}_P$.** From Eq. (15.48) we write

$$\mathbf{a}_P = \mathbf{a}_{P'} + \mathbf{a}_{P/F} + \mathbf{a}_c \tag{2}$$

Since $\boldsymbol{\Omega}$ and $\boldsymbol{\omega}_{D/F}$ are both constant, we have

$$\mathbf{a}_{P'} = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = \omega_1 \mathbf{j} \times (-\omega_1 L\mathbf{k}) = -\omega_1^2 L\mathbf{i}$$
$$\mathbf{a}_{P/F} = \boldsymbol{\omega}_{D/F} \times (\boldsymbol{\omega}_{D/F} \times \mathbf{r}_{P/A}) = \omega_2 \mathbf{k} \times (-\omega_2 R\mathbf{i}) = -\omega_2^2 R\mathbf{j}$$
$$\mathbf{a}_c = 2\boldsymbol{\Omega} \times \mathbf{v}_{P/F} = 2\omega_1 \mathbf{j} \times (-\omega_2 R\mathbf{i}) = 2\omega_1\omega_2 R\mathbf{k}$$

Substituting the values obtained into (2), we find

$$\mathbf{a}_P = -\omega_1^2 L\mathbf{i} - \omega_2^2 R\mathbf{j} + 2\omega_1\omega_2 R\mathbf{k} \quad ◀$$

c. **Angular Velocity and Angular Acceleration of Disk.**

$$\boldsymbol{\omega} = \boldsymbol{\Omega} + \boldsymbol{\omega}_{D/F} \qquad \boldsymbol{\omega} = \omega_1 \mathbf{j} + \omega_2 \mathbf{k} \quad ◀$$

Using Eq. (15.31) with $\mathbf{Q} = \boldsymbol{\omega}$, we write

$$\boldsymbol{\alpha} = (\dot{\boldsymbol{\omega}})_{OXYZ} = (\dot{\boldsymbol{\omega}})_{Axyz} + \boldsymbol{\Omega} \times \boldsymbol{\omega}$$
$$= 0 + \omega_1 \mathbf{j} \times (\omega_1 \mathbf{j} + \omega_2 \mathbf{k})$$

$$\boldsymbol{\alpha} = \omega_1\omega_2 \mathbf{i} \quad ◀$$

PROBLEMS

15.163 The bent rod ABC rotates at a constant rate $\omega_1 = 8$ rad/s. Knowing that the collar D moves downward along the rod at a constant relative speed $u = 780$ mm/s, determine for the position shown (*a*) the velocity of D, (*b*) the acceleration of D.

15.164 Solve Prob. 15.163, assuming $\omega_1 = 6$ rad/s and $u = 650$ mm/s.

15.165 The bent rod ABC rotates at a constant rate ω_1. Knowing that the collar D moves downward along the rod at a constant relative speed u, determine for the position shown (*a*) the velocity of D, (*b*) the acceleration of D.

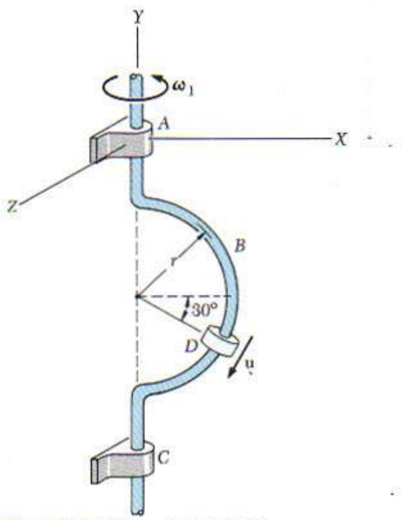


Fig. P15.165 and P15.167

15.166 Solve Prob. 15.165, assuming that $\omega_1 = 9$ rad/s, $u = 40$ in./s, and $r = 6$ in.

15.167 At the instant shown the magnitude of the angular velocity $\boldsymbol{\omega}_1$ of the bent rod ABC is 9 rad/s and is increasing at the rate of 20 rad/s^2, while the relative speed u of collar D is 40 in./s and is increasing at the rate of 100 in./s^2. Knowing that $r = 6$ in., determine the acceleration of D.

15.168 Solve Prob. 15.163, assuming that at the instant shown the angular velocity $\boldsymbol{\omega}_1$ of the rod is 8 rad/s and is decreasing at the rate of 18 rad/s^2, while the relative speed u of the collar is 780 mm/s and is decreasing at the rate of 2.6 m/s^2.

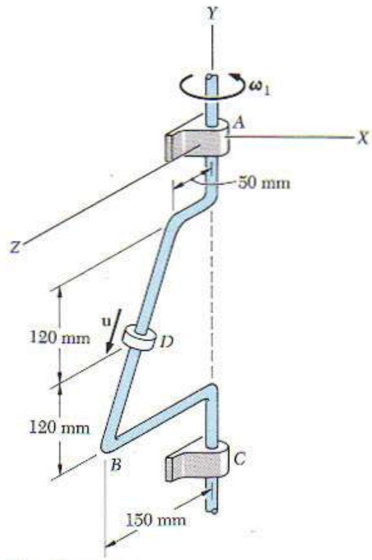


Fig. P15.163

15.169 The cab of the backhoe shown rotates with the constant angular velocity $\boldsymbol{\omega}_1 = (0.4 \text{ rad/s})\mathbf{j}$ about the Y axis. The arm OA is fixed with respect to the cab, while the arm AB rotates about the horizontal axle A at the constant rate $\omega_2 = d\beta/dt = 0.6$ rad/s. Knowing that $\beta = 30°$, determine (*a*) the angular velocity and angular acceleration of AB, (*b*) the velocity and acceleration of point B.

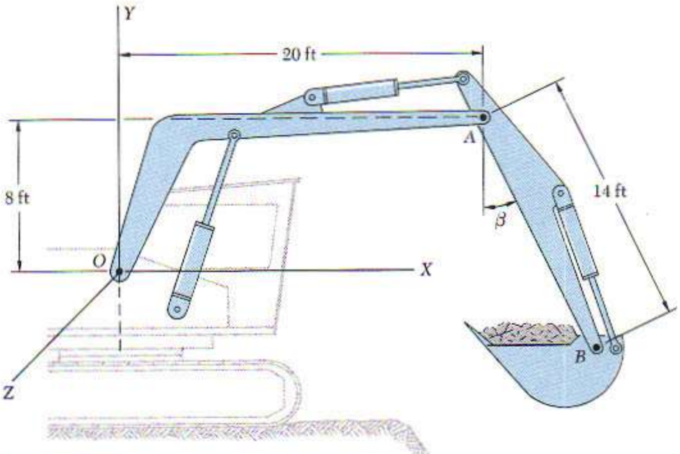


Fig. P15.169

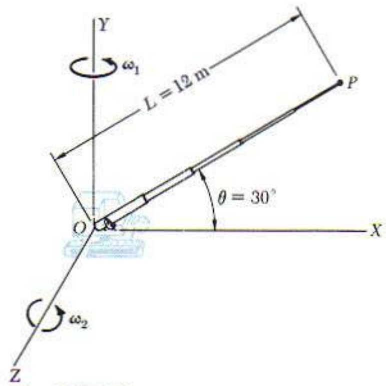


Fig. P15.170

15.170 Solve Sample Prob. 15.14, assuming that the crane has a telescoping boom as shown and that the length of the boom is being increased at the rate $dL/dt = 1.5$ m/s.

15.171 Solve Prob. 15.169, assuming that $\beta = 30°$ and that arms OA and AB rotate as a rigid body with respect to the cab with a constant angular velocity $(0.6 \text{ rad/s})\mathbf{k}$.

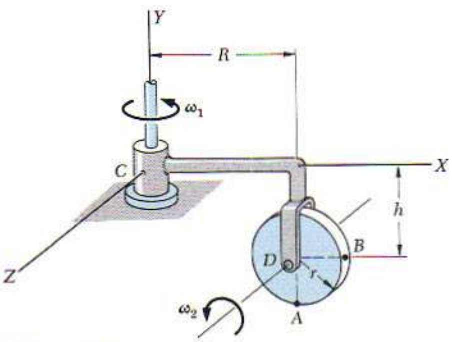


Fig. P15.172

15.172 A disk of radius r rotates at a constant rate ω_2 with respect to the arm CD, which itself rotates at a constant rate ω_1 about the Y axis. Determine (*a*) the angular velocity and angular acceleration of the disk, (*b*) the velocity and acceleration of point B on the rim of the disk.

15.173 In Prob. 15.172, determine the velocity and acceleration of point A on the rim of the disk.

15.174 The 40-ft blades of the experimental wind-turbine generator rotate at a constant rate $\omega = 30$ rpm. Knowing that at the instant shown the entire unit is being rotated about the Y axis at a constant rate $\Omega = 0.1$ rad/s, determine (*a*) the angular acceleration of the blades, (*b*) the velocity and acceleration of blade tip B.

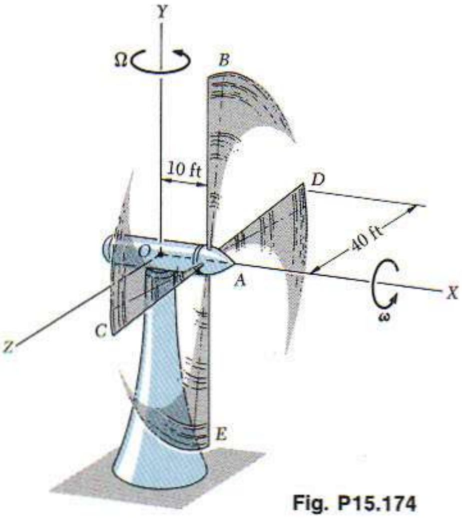


Fig. P15.174

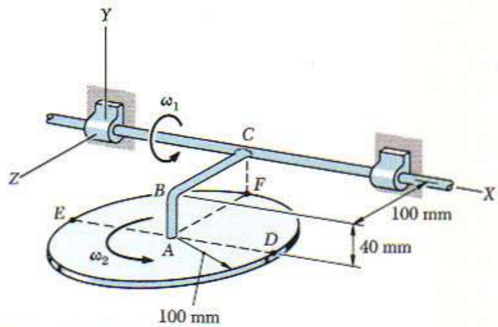


Fig. P15.176

15.175 In Prob. 15.174, determine the velocity and acceleration of (*a*) blade tip C, (*b*) blade tip E.

15.176 A disk of radius 100 mm rotates at a constant rate $\omega_2 = 20$ rad/s with respect to the arm ABC, which itself rotates at a constant rate $\omega_1 = 10$ rad/s about the X axis. Determine (*a*) the angular acceleration of the disk, (*b*) the velocity and acceleration of point D on the rim of the disk.

15.177 In Prob. 15.176, determine the acceleration (*a*) of point E, (*b*) of point F.

15.178 through 15.180 Two collars A and B are connected by a 15-in. rod AB as shown. Knowing that collar A moves downward at a constant speed of 18 in./s, determine the velocities and accelerations of collars A and B for the constant rate of rotation indicated.

15.178 $\omega_1 = 10$ rad/s, $\omega_2 = \omega_3 = 0$.

15.179 $\omega_2 = 10$ rad/s, $\omega_1 = \omega_3 = 0$.

15.180 $\omega_3 = 10$ rad/s, $\omega_1 = \omega_2 = 0$.

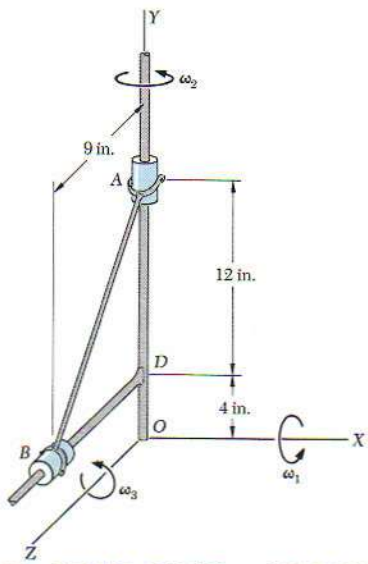


Fig. P15.178, P15.179, and P15.180

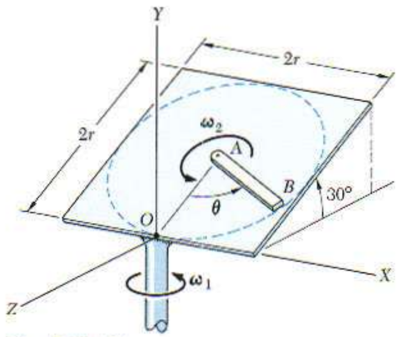


Fig. P15.181

15.181 A square plate of side $2r$ is welded to a vertical shaft which rotates with a constant angular velocity $\boldsymbol{\omega}_1$. At the same time, rod AB of length r rotates about the center of the plate with a constant angular velocity $\boldsymbol{\omega}_2$ with respect to the plate. For the position of the plate shown, determine the acceleration of end B of the rod if (*a*) $\theta = 0$, (*b*) $\theta = 90°$, (*c*) $\theta = 180°$.

15.182 Solve Prob. 15.181, assuming $\omega_1 = 2$ rad/s, $\omega_2 = 3$ rad/s, and $r = 100$ mm.

15.183 In Prob. 15.181, the plate rotates at a constant rate $\omega_1 = 2$ rad/s. At the same time, the magnitude ω_2 is 3 rad/s and is increasing at the rate $\alpha_2 = 5$ rad/s^2. Knowing that $r = 100$ mm, determine the acceleration of end B of the rod if $\theta = 90°$.

15.184 In Prob. 15.181, the magnitude of the angular velocity of the plate is $\omega_1 = 2$ rad/s and is increasing at the rate $\alpha_1 = 8$ rad/s^2. At the same time, the rod AB rotates with respect to the plate at the constant rate $\omega_2 = 3$ rad/s. Knowing that $r = 100$ mm, determine the acceleration of end B of the rod if $\theta = 90°$.

REVIEW PROBLEMS

15.185 It takes 0.8 s for the turntable of a 33-rpm record player to reach full speed after being started. Assuming uniformly accelerated motion, determine (*a*) the angular acceleration of the turntable, (*b*) the normal and tangential components of the acceleration of a point on the rim of the 12-in.-diameter turntable just before the speed of 33 rpm is reached, (*c*) the total acceleration of the same point at that time.

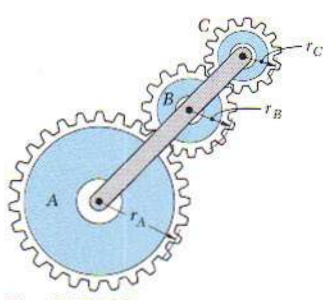


Fig. P15.186

15.186 Three gears A, B, and C are pinned at their centers to rod ABC. Knowing that $r_A = 3r_B = 3r_C$ and that gear A does not rotate, determine the angular velocity of gears B and C when the rod ABC rotates clockwise with a constant angular velocity of 10 rpm.

15.187 In Prob. 15.186 it is known that $r_A = 12$ in., $r_B = r_C = 4$ in. Determine the acceleration of the tooth of gear C which is in contact with gear B.

15.188 Three links AB, BC, and BD are connected by a pin B as shown. Knowing that at the instant shown point D has a velocity of 200 mm/s to the right and no acceleration, determine (*a*) the angular acceleration of each link, (*b*) the accelerations of points A and B.

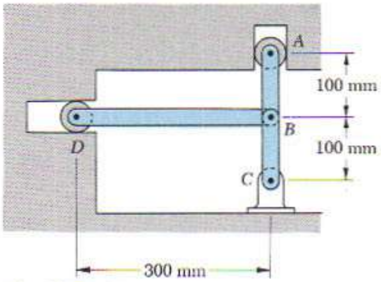


Fig. P15.188

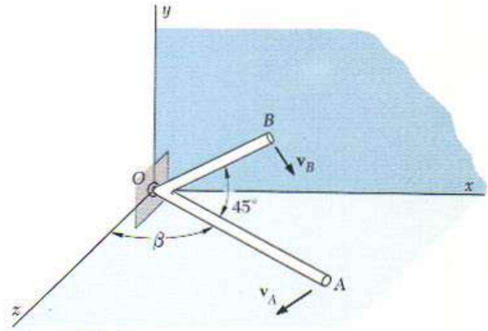


Fig. P15.189

15.189 The bent rod AOB is attached to a fixed ball-and-socket joint at O. The lengths of portions OA and OB are 200 mm and 120 mm, respectively, and the angle formed by the two portions is 45°. As portion OA moves on the horizontal surface, portion OB moves on the vertical wall. Knowing that end A moves at a constant speed of 600 mm/s, determine, at the instant when $\beta = 60°$, (*a*) the angular velocity of the rod, (*b*) the velocity of point B.

15.190 Water flows through the sprinkler arm ABC with a velocity of 16 ft/s relative to the arm. Knowing that the angular velocity of the arm is 90 rpm counterclockwise, determine at the instant shown the total acceleration (*a*) of the particle of water P_1, (*b*) of the particle of water P_2.

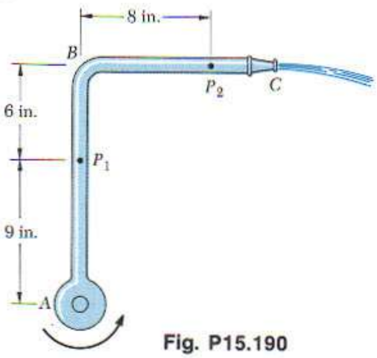


Fig. P15.190

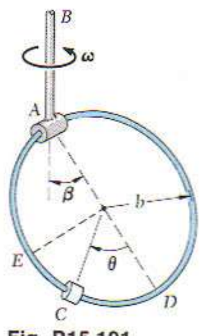


Fig. P15.191

15.191 A thin ring of radius b is attached to a vertical shaft AB which rotates with a constant angular velocity ω. Collar C moves at a constant speed u relative to the ring. For the position $\beta = 30°$, determine the velocity and acceleration of the collar when (*a*) $\theta = 0$, (*b*) $\theta = 90°$.

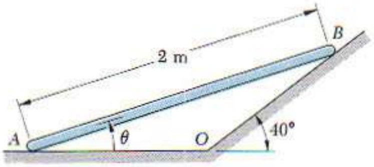


Fig. P15.192

15.192 Rod AB is 2 m long and slides with its ends in contact with the floor and the inclined plane. End A moves with a constant velocity of 6 m/s to the right. At the instant when $\theta = 25°$, determine (*a*) the angular velocity and angular acceleration of the rod, (*b*) the velocity and acceleration of end B.

15.193 Gear A rolls on the fixed gear B and rotates about the axle AD which is rigidly attached at D to the vertical shaft DE. Knowing that shaft DE rotates with a constant angular velocity $\boldsymbol{\omega}_1$, determine (*a*) the rate of spin of gear A about the axle AD, (*b*) the angular acceleration of gear A, (*c*) the acceleration of tooth C of gear A.

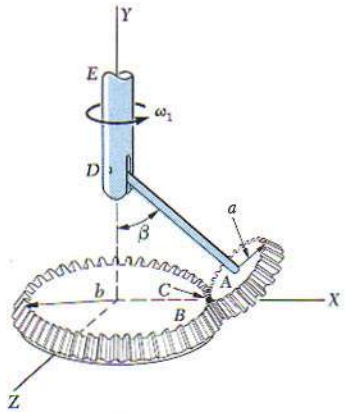


Fig. P15.193

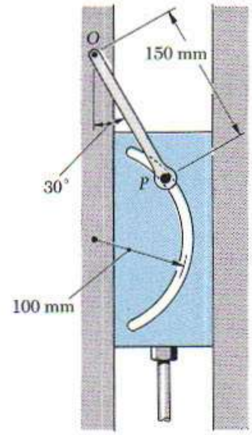


Fig. P15.194

15.194 At the instant shown, the slotted plate slides with a velocity of 0.5 m/s upward and has an acceleration of 2 m/s^2 downward. Determine the angular velocity and the angular acceleration of rod OP.

15.195 Solve Prob. 15.193, assuming $\omega_1 = 90$ rpm, $a = 60$ mm, $b = 160$ mm, and $\beta = 30°$.

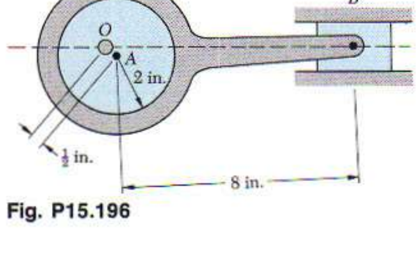


Fig. P15.196

15.196 The eccentric shown consists of a disk of 2-in. radius which revolves about a shaft O located $\frac{1}{2}$ in. from the center of the disk A. Assuming that the disk rotates about O with a constant angular velocity of 1800 rpm clockwise, determine the velocity and acceleration of block B when point A is directly below the shaft O.

Plane Motion of Rigid Bodies: Forces and Accelerations

CHAPTER

16

16.1 Introduction. In this chapter and in Chaps. 17 and 18, we shall study the *kinetics of rigid bodies*, i.e., the relations existing between the forces acting on a rigid body, the shape and mass of the body, and the motion produced. In Chaps. 12 and 13, we studied similar relations, assuming then that the body could be considered as a particle, i.e., that its mass could be concentrated in one point and that all forces acted at that point. We shall now take the shape of the body into account, as well as the exact location of the points of application of the forces. Besides, we shall be concerned not only with the motion of the body as a whole but also with the motion of the body about its mass center.

Our approach will be to consider rigid bodies as made of large numbers of particles and to use the results obtained in Chap. 14 for the motion of systems of particles. In this chapter, we shall use specifically Eq. (14.16), $\Sigma\mathbf{F} = m\bar{\mathbf{a}}$, which relates the resultant of the external forces and the acceleration of the mass center G of the system of particles, and Eq. (14.23), $\Sigma\mathbf{M}_G = \dot{\mathbf{H}}_G$, which relates the moment resultant of the external forces and the angular momentum of the system of particles about G.

Except for Sec. 16.2, which applies to the most general case of the motion of a rigid body, the results derived in this chapter will be limited in two ways: (1) They will be restricted to the *plane motion* of rigid bodies, i.e., to a motion in which each particle of the body remains at a constant distance from a fixed reference plane. (2) The rigid bodies considered will consist only of plane slabs and of bodies which are symmetrical with respect to the reference plane.† The study of the plane motion of nonsymmetrical three-dimensional bodies and, more generally, the motion of rigid bodies in three-dimensional space will be postponed until Chap. 18.

16.2. Equations of Motion for a Rigid Body. Consider a rigid body acted upon by several external forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, etc. (Fig. 16.1). We may assume the body to be made of a large number n of particles of mass Δm_i $(i = 1,2, \ldots ,n)$ and apply the results obtained in Chap. 14 for a system of particles (Fig. 16.2).

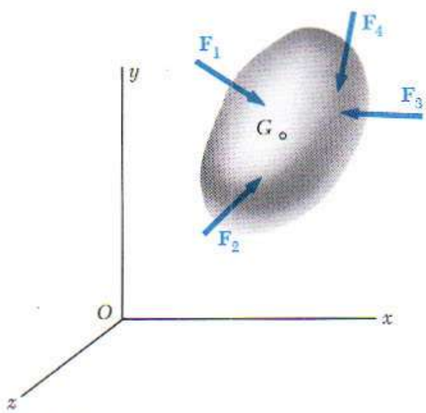


Fig. 16.1

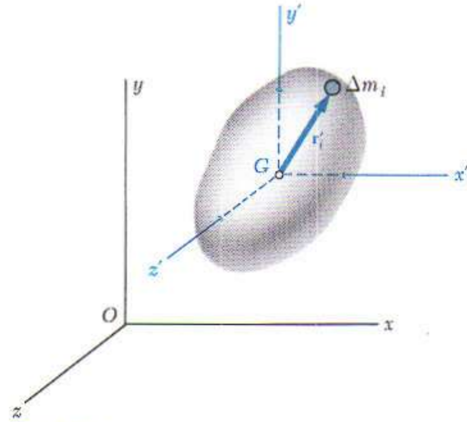


Fig. 16.2

Considering first the motion of the mass center G of the body with respect to the newtonian frame of reference $Oxyz$, we recall Eq. (14.16) and write

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}} \tag{16.1}$$

where m is the mass of the body and $\bar{\mathbf{a}}$ the acceleration of the mass center G. Turning now to the motion of the body relative to the centroidal frame of reference $Gx'y'z'$, we recall Eq. (14.23) and write

$$\Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G \tag{16.2}$$

where $\dot{\mathbf{H}}_G$ represents the rate of change of $\mathbf{H}_G$, the angular momentum about G of the system of particles forming the rigid body. In the following we shall simply refer to $\mathbf{H}_G$ as the *angular momentum of the rigid body about its mass center G*. Together Eqs. (16.1) and (16.2) express that *the system of the external forces*

†Or, more generally, bodies which have a principal centroidal axis of inertia perpendicular to the reference plane.

is equipollent to the system consisting of the vector $m\bar{\mathbf{a}}$ *attached at G and the couple of moment* $\dot{\mathbf{H}}_G$ (Fig. 16.3).†

Equations (16.1) and (16.2) apply in the most general case of the motion of a rigid body. In the rest of this chapter, however, we shall limit our analysis to the *plane motion* of rigid bodies, i.e., to a motion in which each particle remains at a constant distance from a fixed reference plane, and we shall assume that the rigid bodies considered consist only of plane slabs and of bodies which are symmetrical with respect to the reference plane. Further study of the plane motion of nonsymmetrical three-dimensional bodies and of the motion of rigid bodies in three-dimensional space will be postponed until Chap. 18.

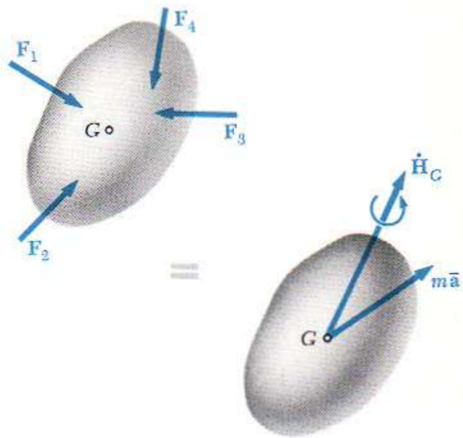


Fig. 16.3

16.3. Angular Momentum of a Rigid Body in Plane Motion. Consider a rigid slab in plane motion. Assuming the slab to be made of a large number n of particles P_i of mass Δm_i and recalling Eq. (14.24) of Sec. 14.4, we note that the angular momentum $\mathbf{H}_G$ of the slab about its mass center G may be computed by taking the moments about G of the momenta of the particles of the slab in their motion with respect to either of the frames Oxy or $Gx'y'$. Choosing the latter course, we write

$$\mathbf{H}_G = \sum_{i=1}^{n} (\mathbf{r}'_i \times \mathbf{v}'_i \, \Delta m_i) \tag{16.3}$$

where $\mathbf{r}'_i$ and $\mathbf{v}'_i \, \Delta m_i$ denote, respectively, the position vector and the linear momentum of the particle P_i relative to the centroidal frame of reference $Gx'y'$ (Fig. 16.4). But, since the particle belongs to the slab, we have $\mathbf{v}'_i = \boldsymbol{\omega} \times \mathbf{r}'_i$, where $\boldsymbol{\omega}$ is the angular velocity of the slab at the instant considered. We write

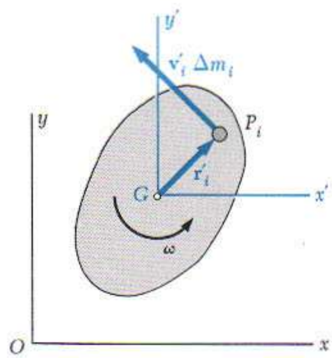


Fig. 16.4

$$\mathbf{H}_G = \sum_{i=1}^{n} [\mathbf{r}'_i \times (\boldsymbol{\omega} \times \mathbf{r}'_i) \, \Delta m_i]$$

Referring to Fig. 16.4, we easily verify that the expression obtained represents a vector of the same direction as $\boldsymbol{\omega}$ (i.e., perpendicular to the slab) and of magnitude equal to $\omega \Sigma r_i'^2 \, \Delta m_i$. Recalling that the sum $\Sigma r_i'^2 \, \Delta m_i$ represents the moment of inertia $\bar{I}$ of the slab about a centroidal axis perpendicular to the slab, we conclude that the angular momentum $\mathbf{H}_G$ of the slab about its mass center is

$$\mathbf{H}_G = \bar{I}\boldsymbol{\omega} \tag{16.4}$$

† Since the systems involved act on a rigid body, we could conclude at this point, by referring to Sec. 3.18, that the two systems are *equivalent* as well as equipollent and use blue rather than gray equals signs in Fig. 16.3. However, by postponing this conclusion, we shall be able to arrive at it independently (Secs. 16.4 and 18.5), thus eliminating the necessity of including the principle of transmissibility among the axioms of mechanics (Sec. 16.5).

Differentiating both members of Eq. (16.4) we obtain

$$\dot{\mathbf{H}}_G = \bar{I}\dot{\boldsymbol{\omega}} = \bar{I}\boldsymbol{\alpha} \tag{16.5}$$

Thus the rate of change of the angular momentum of the slab is represented by a vector of the same direction as $\boldsymbol{\alpha}$, (i.e., perpendicular to the slab) and of magnitude $\bar{I}\alpha$.

It should be kept in mind that the results obtained in this section have been derived for a rigid slab in plane motion. As we shall see in Chap. 18, they remain valid in the case of the plane motion of rigid bodies which are symmetrical with respect to the reference plane.† However, they do not apply in the case of nonsymmetrical bodies or in the case of three-dimensional motion.

16.4. Plane Motion of a Rigid Body. D'Alembert's Principle. Consider a rigid slab of mass m moving under the action of several external forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, etc., contained in the plane of the slab (Fig. 16.5). Substituting for $\dot{\mathbf{H}}_G$ from Eq. (16.5) into Eq. (16.2), and writing the fundamental equations of motion (16.1) and (16.2) in scalar form, we have

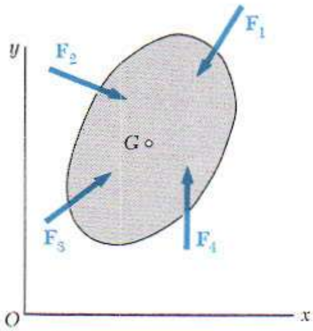


Fig. 16.5

$$\Sigma F_x = m\bar{a}_x \qquad \Sigma F_y = m\bar{a}_y \qquad \Sigma M_G = \bar{I}\alpha \tag{16.6}$$

Equations (16.6) show that the acceleration of the mass center G of the slab and its angular acceleration $\boldsymbol{\alpha}$ may easily be obtained, once the resultant of the external forces acting on the slab and their moment resultant about G have been determined. Given appropriate initial conditions, the coordinates $\bar{x}$ and $\bar{y}$ of the mass center and the angular coordinate θ of the slab may then be obtained at any instant t by integration. Thus *the motion of the slab is completely defined by the resultant and moment resultant about G of the external forces acting on it.*

This property, which will be extended in Chap. 18 to the case of the three-dimensional motion of a rigid body, is characteristic of the motion of a rigid body. Indeed, as we saw in Chap. 14, the motion of a system of particles which are not rigidly connected will in general depend upon the specific external forces, as well as upon the internal forces, acting on the various particles.

Since the motion of a rigid body depends only upon the resultant and moment resultant of the external forces acting on it, it follows that *two systems of forces which are equipollent*, i.e., which have the same resultant and the same moment resultant,

†Or, more generally, bodies which have a principal centroidal axis of inertia perpendicular to the reference plane.

are also equivalent; i.e., they have exactly the same effect on a given rigid body.†

Consider in particular the system of the external forces acting on a rigid body (Fig. 16.6*a*) and the system of the effective forces associated with the particles forming the rigid body (Fig. 16.6*b*). It was shown in Sec. 14.1 that the two systems thus defined are equipollent. But since the particles considered now form a rigid body, it follows from the above discussion that the two systems are also equivalent. We may thus state that *the external forces acting on a rigid body are equivalent to the effective forces of the various particles forming the body.* This statement is referred to as *D'Alembert's principle,* after the French mathematician Jean le Rond d'Alembert (1717–1783), even though D'Alembert's original statement was written in a somewhat different form.

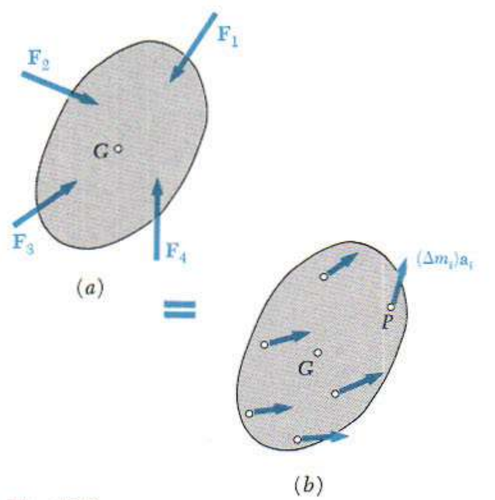


Fig. 16.6

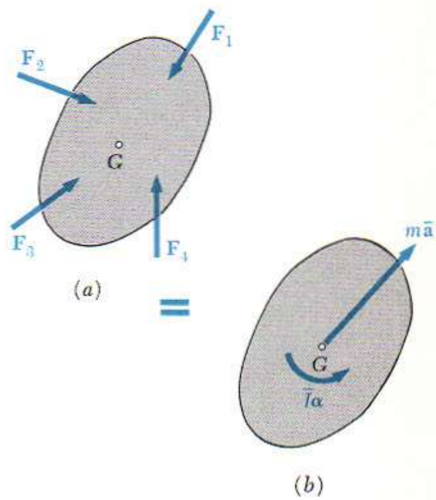


Fig. 16.7

The significance of D'Alembert's principle has been emphasized by the use of a blue equals sign in Fig. 16.6 and also in Fig. 16.7, where, using results obtained earlier in this section, the effective forces have been replaced by a vector $m\bar{\mathbf{a}}$ attached at the mass center G of the slab and a couple of moment $\bar{I}\boldsymbol{\alpha}$.

†This result has already been derived in Sec. 3.18 from the principle of transmissibility (Sec. 3.2). The present derivation, however, is independent of that principle and will make possible its elimination from the axioms of mechanics (Sec. 16.5).

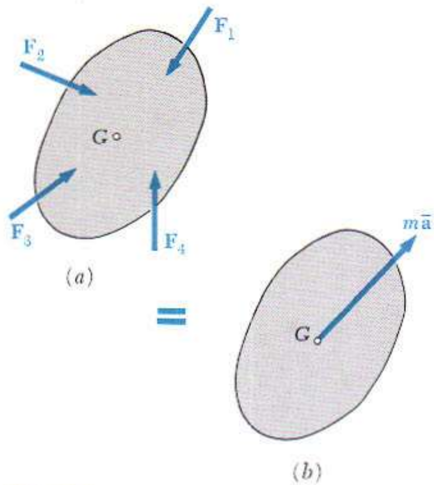


Fig. 16.8

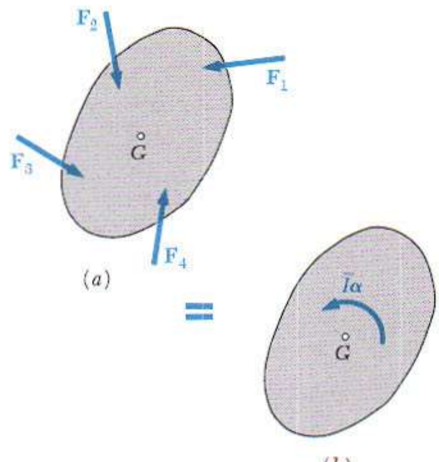


Fig. 16.9

Translation. In the particular case of a body in translation, the angular acceleration of the body is identically equal to zero and its effective forces reduce to the vector $m\overline{\mathbf{a}}$ attached at G (Fig. 16.8). Thus, the resultant of the external forces acting on a rigid body in translation passes through the mass center of the body and is equal to $m\overline{\mathbf{a}}$.

Centroidal Rotation. When a slab, or, more generally, a body symmetrical with respect to the reference plane, rotates about a fixed axis perpendicular to the reference plane and passing through its mass center G, we say that the body is in *centroidal rotation*. Since the acceleration $\overline{\mathbf{a}}$ is identically equal to zero, the effective forces of the body reduce to the couple $\bar{I}\boldsymbol{\alpha}$ (Fig. 16.9). Thus, the external forces acting on a body in centroidal rotation are equivalent to a couple of moment $\bar{I}\boldsymbol{\alpha}$.

General Plane Motion. Comparing Fig. 16.7 with Figs. 16.8 and 16.9, we observe that, from the point of view of *kinetics*, the most general plane motion of a rigid body symmetrical with respect to the reference plane may be replaced by the sum of a translation and a centroidal rotation. We should note that this statement is more restrictive than the similar statement made earlier from the point of view of *kinematics* (Sec. 15.5), since we now require that the mass center of the body be selected as the reference point.

Fig. 16.7 (*repeated*)

Referring to Eqs. (16.6), we observe that the first two equations are identical with the equations of motion of a particle of mass m acted upon by the given forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, etc. We thus check that *the mass center G of a rigid body in plane motion moves as if the entire mass of the body were concentrated at that point, and as if all the external forces acted on it.* We recall that this result has already been obtained in Sec. 14.3 in the general case of a system of particles, the particles being not necessarily rigidly connected. We also note, as we did in Sec. 14.3, that the system of the external forces does not, in general, reduce to a single vector $m\bar{\mathbf{a}}$ attached at G. Therefore, in the general case of the plane motion of a rigid body, *the resultant of the external forces acting on the body does not pass through the mass center of the body.*

Finally, we may observe that the last of Eqs. (16.6) would still be valid if the rigid body, while subjected to the same applied forces, were constrained to rotate about a fixed axis through G. Thus, *a rigid body in plane motion rotates about its mass center as if this point were fixed.*

*** 16.5. A Remark on the Axioms of the Mechanics of Rigid Bodies.** The fact that two equipollent systems of external forces acting on a rigid body are also equivalent, i.e., have the same effect on that rigid body, has already been established in Sec. 3.18. But there it was derived from the *principle of transmissibility,* one of the axioms used in our study of the statics of rigid bodies. It should be noted that this axiom has not been used in the present chapter, because Newton's second and third laws of motion make its use unnecessary in the study of the dynamics of rigid bodies.

In fact, the principle of transmissibility may now be *derived* from the other axioms used in the study of mechanics. This principle stated, without proof, (Sec. 3.2) that the conditions of equilibrium or motion of a rigid body remain unchanged if a force $\mathbf{F}$ acting at a given point of the rigid body is replaced by a force $\mathbf{F}'$ of the same magnitude and same direction, but acting at a different point, provided that the two forces have the same line of action. But, since $\mathbf{F}$ and $\mathbf{F}'$ have the same moment about any given point, it is clear that they form two equipollent systems of external forces. Thus, we may now *prove,* as a result of what we established in the preceding section, that $\mathbf{F}$ and $\mathbf{F}'$ have the same effect on the rigid body (Fig. 3.3).

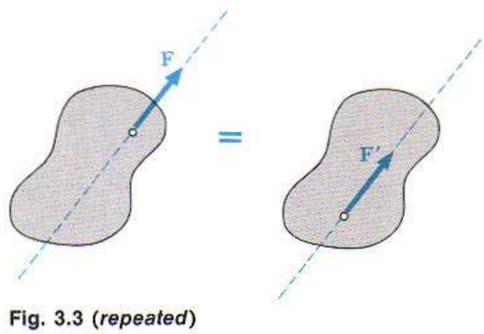


Fig. 3.3 (*repeated*)

The principle of transmissibility may therefore be removed from the list of axioms required for the study of the mechanics of rigid bodies. These axioms are reduced to the parallelogram law of addition of vectors and to Newton's laws of motion.

16.6. Solution of Problems Involving the Motion of a Rigid Body. We saw in Sec. 16.4 that, when a rigid body is in plane motion, there exists a fundamental relation between the forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, etc., acting on the body, the acceleration $\bar{\mathbf{a}}$ of its mass center, and the angular acceleration $\boldsymbol{\alpha}$ of the body. This relation, which is represented in Fig. 16.7, may be used to determine the acceleration $\bar{\mathbf{a}}$ and the angular acceleration $\boldsymbol{\alpha}$ produced by a given system of forces acting on a rigid body or, conversely,

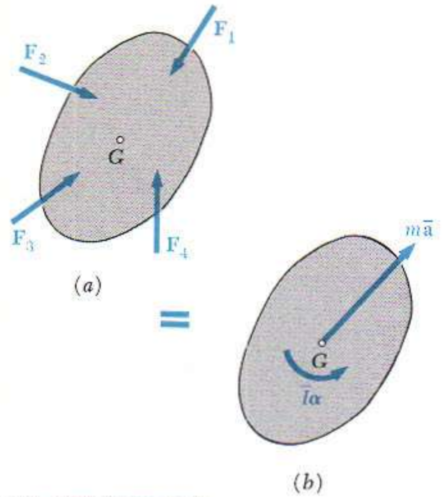


Fig. 16.7 (*repeated*)

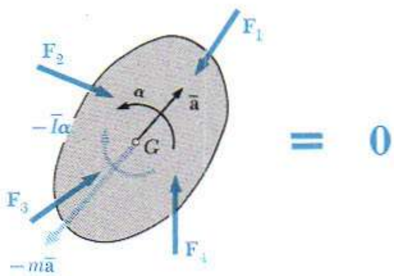


Fig. 16.10

to determine the forces which produce a given motion of the rigid body.

While the three algebraic equations (16.6) may be used to solve problems of plane motion,† our experience in statics suggests that the solution of many problems involving rigid bodies could be simplified by an appropriate choice of the point about which the moments of the forces are computed. It is therefore preferable to remember the relation existing between the forces and the accelerations in the vectorial form shown in Fig. 16.7, and to derive from this fundamental relation the component or moment equations which fit best the solution of the problem under consideration.

The fundamental relation shown in Fig. 16.7 may be presented in an alternate form if we add to the external forces an inertia vector $-m\bar{\mathbf{a}}$ of sense opposite to that of $\bar{\mathbf{a}}$, attached at G, and an inertia couple $-\bar{I}\boldsymbol{\alpha}$ of moment equal in magnitude to $\bar{I}\alpha$ and of sense opposite to that of $\boldsymbol{\alpha}$ (Fig. 16.10). The system obtained is equivalent to zero, and the rigid body is said to be in dynamic equilibrium.

Whether the principle of equivalence of external and effective forces is directly applied, as in Fig. 16.7, or whether the concept of dynamic equilibrium is introduced, as in Fig. 16.10, the use of free-body diagrams showing vectorially the relationship existing between the forces applied on the rigid body and the resulting linear and angular accelerations presents considerable advantages over the blind application of the formulas (16.6). These advantages may be summarized as follows:

1. First of all, a much clearer understanding of the effect of the forces on the motion of the body will result from the use of a pictorial representation.
2. This approach makes it possible to divide the solution of a dynamics problem into two parts: In the first part, the analysis of the kinematic and kinetic characteristics of the problem leads to the free-body diagrams of Fig. 16.7 or 16.10; in the second part, the diagram obtained is used to analyze by the methods of Chap. 3 the various forces and vectors involved.
3. A unified approach is provided for the analysis of the plane motion of a rigid body, regardless of the particular type of motion involved. While the kinematics of the various motions

† We recall that the last of Eqs. (16.6) is valid only in the case of the plane motion of a rigid body symmetrical with respect to the reference plane. In all other cases, the methods of Chap. 18 should be used.

considered may vary from one case to the other, the approach to the kinetics of the motion is consistently the same. In every case we shall draw a diagram showing the external forces, the vector $m\bar{\mathbf{a}}$ associated with the motion of G, and the couple $\bar{I}\boldsymbol{\alpha}$ associated with the rotation of the body about G.

4. The resolution of the plane motion of a rigid body into a translation and a centroidal rotation, which is used here, is a basic concept which may be applied effectively throughout the study of mechanics. We shall use it again in Chap. 17 with the method of work and energy and the method of impulse and momentum.
5. As we shall see in Chap. 18, this approach may be extended to the study of the general three-dimensional motion of a rigid body. The motion of the body will again be resolved into a translation and a rotation about the mass center, and free-body diagrams will be used to indicate the relationship existing between the external forces and the rates of change of the linear and angular momentum of the body.

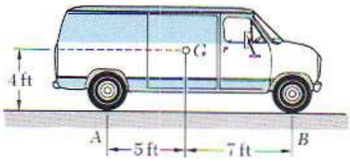
16.7. Systems of Rigid Bodies. The method described in the preceding section may also be used in problems involving the plane motion of several connected rigid bodies. A diagram similar to Fig. 16.7 or Fig. 16.10 may be drawn for each part of the system. The equations of motion obtained from these diagrams are solved simultaneously.

In some cases, as in Sample Prob. 16.3, a single diagram may be drawn for the entire system. This diagram should include all the external forces, as well as the vectors $m\bar{\mathbf{a}}$ and the couples $\bar{I}\boldsymbol{\alpha}$ associated with the various parts of the system. However, internal forces, such as the forces exerted by connecting cables, may be omitted since they occur in pairs of equal and opposite forces and are thus equipollent to zero. The equations obtained by expressing that the system of the external forces is equipollent to the system of the effective forces may be solved for the remaining unknowns.†

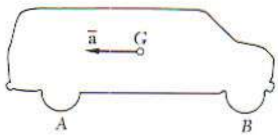
This second approach may not be used in problems involving more than three unknowns, since only three equations of motion are available when a single diagram is used. We shall not elaborate upon this point, since the discussion involved would be completely similar to that given in Sec. 6.11 in the case of the equilibrium of a system of rigid bodies.

† Note that we cannot speak of *equivalent* systems since we are not dealing with a single rigid body.

SAMPLE PROBLEM 16.1



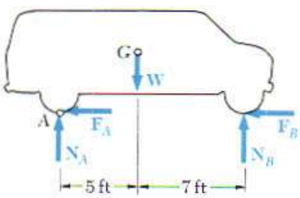
When the forward speed of the truck shown was 30 ft/s, the brakes were suddenly applied, causing all four wheels to stop rotating. It was observed that the truck skidded to rest in 20 ft. Determine the magnitude of the normal reaction and of the friction force at each wheel as the truck skidded to rest.



Kinematics of Motion. Choosing the positive sense to the right and using the equations of uniformly accelerated motion, we write

$$\bar{v}_0 = +30 \text{ ft/s} \qquad \bar{v}^2 = \bar{v}_0^2 + 2\bar{a}\bar{x} \qquad 0 = (30)^2 + 2\bar{a}(20)$$
$$\bar{a} = -22.5 \text{ ft/s}^2 \qquad \bar{\mathbf{a}} = 22.5 \text{ ft/s}^2 \leftarrow$$

Equations of Motion. The external forces consist of the weight $\mathbf{W}$ of the truck and of the normal reactions and friction forces at the wheels. (The vectors $\mathbf{N}_A$ and $\mathbf{F}_A$ represent the sum of the reactions at the rear wheels, while $\mathbf{N}_B$ and $\mathbf{F}_B$ represent the sum of the reactions at the front wheels.) Since the truck is in translation, the effective forces reduce to the vector $m\bar{\mathbf{a}}$ attached at G. Three equations of motion are obtained by expressing that the system of the external forces is equivalent to the system of the effective forces.

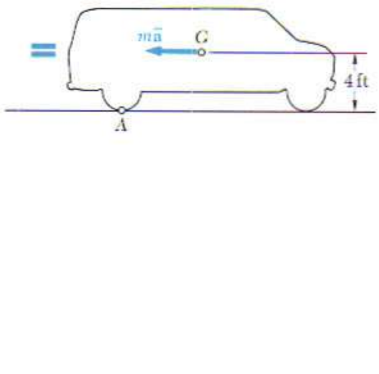


$$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \qquad N_A + N_B - W = 0$$

Since $F_A = \mu N_A$ and $F_B = \mu N_B$, we find

$$F_A + F_B = \mu(N_A + N_B) = \mu W$$

$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \qquad -(F_A + F_B) = -m\bar{a}$$
$$-\mu W = -\frac{W}{32.2 \text{ ft/s}^2}(22.5 \text{ ft/s}^2)$$
$$\mu = 0.699$$

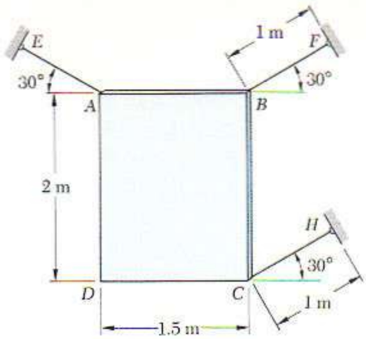


$$+\circlearrowleft \Sigma M_A = \Sigma(M_A)_{\text{eff}}: \qquad -W(5 \text{ ft}) + N_B(12 \text{ ft}) = m\bar{a}(4 \text{ ft})$$
$$-W(5 \text{ ft}) + N_B(12 \text{ ft}) = \frac{W}{32.2 \text{ ft/s}^2}(22.5 \text{ ft/s}^2)(4 \text{ ft})$$
$$N_B = 0.650W$$
$$F_B = \mu N_B = (0.699)(0.650W) \qquad F_B = 0.454W$$

$$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \qquad N_A + N_B - W = 0$$
$$N_A + 0.650W - W = 0$$
$$N_A = 0.350W$$
$$F_A = \mu N_A = (0.699)(0.350W) \qquad F_A = 0.245W$$

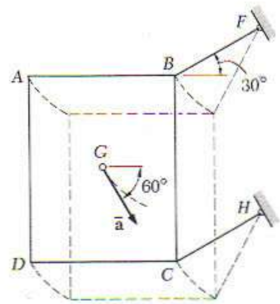
Reactions at Each Wheel. Recalling that the values computed above represent the sum of the reactions at the two front wheels or the two rear wheels, we obtain the magnitude of the reactions at each wheel by writing

$$N_{\text{front}} = \tfrac{1}{2}N_B = 0.325W \qquad N_{\text{rear}} = \tfrac{1}{2}N_A = 0.175W \blacktriangleleft$$
$$F_{\text{front}} = \tfrac{1}{2}F_B = 0.227W \qquad F_{\text{rear}} = \tfrac{1}{2}F_A = 0.122W \blacktriangleleft$$



SAMPLE PROBLEM 16.2

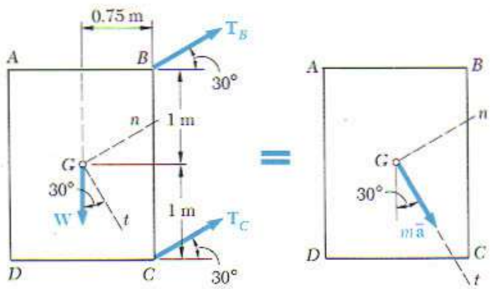
The thin plate $ABCD$ has a mass of 50 kg and is held in position by the three inextensible wires AE, BF, and CH. Wire AE is then cut. Determine both (*a*) the acceleration of the plate, (*b*) the tension in wires BF and CH immediately after wire AE has been cut.



Motion of Plate. After wire AE has been cut, we observe that corners B and C move along parallel circles of radius 1 m centered, respectively, at F and H. The motion of the plate is thus a curvilinear translation; the particles forming the plate move along parallel circles of radius 1 m.

At the instant wire AE is cut, the velocity of the plate is zero; the acceleration $\bar{\mathbf{a}}$ of the mass center G is thus tangent to the circular path which will be described by G.

Equations of Motion. The external forces consist of the weight $\mathbf{W}$ and of the forces $\mathbf{T}_B$ and $\mathbf{T}_C$ exerted by the wires. Since the plate is in translation, the effective forces reduce to the vector $m\bar{\mathbf{a}}$ attached at G and directed along the t axis. Expressing that the system of the external forces is equivalent to the system of the effective forces, we write



$$+\searrow \Sigma F_t = \Sigma(F_t)_{\text{eff}}: \qquad W\cos 30° = m\bar{a}$$
$$mg\cos 30° = m\bar{a} \tag{1}$$
$$\bar{a} = g\cos 30° = (9.81\text{ m/s}^2)\cos 30°$$
$$\bar{\mathbf{a}} = 8.50\text{ m/s}^2 \measuredangle 60° \blacktriangleleft$$

$$+\nearrow \Sigma F_n = \Sigma(F_n)_{\text{eff}}: \qquad T_B + T_C - W\sin 30° = 0 \tag{2}$$

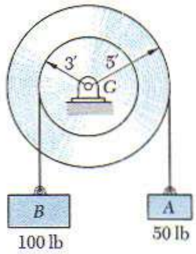
$$+\circlearrowleft \Sigma M_G = \Sigma(M_G)_{\text{eff}}: \qquad (T_B\sin 30°)(0.75\text{ m}) - (T_B\cos 30°)(1\text{ m})$$
$$+ (T_C\sin 30°)(0.75\text{ m}) + (T_C\cos 30°)(1\text{ m}) = 0$$
$$-0.491T_B + 1.241T_C = 0$$
$$T_C = 0.396T_B \tag{3}$$

Substituting for T_C from (3) into (2), we write

$$T_B + 0.396T_B - W\sin 30° = 0$$
$$T_B = 0.358W$$
$$T_C = 0.396(0.358W) = 0.1418W$$

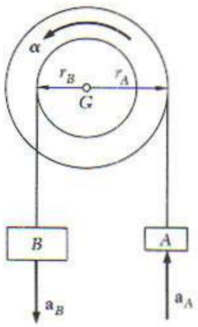
Noting that $W = mg = (50\text{ kg})(9.81\text{ m/s}^2) = 491\text{ N}$, we have

$$T_B = 175.8\text{ N} \qquad T_C = 69.6\text{ N} \blacktriangleleft$$



SAMPLE PROBLEM 16.3

A pulley weighing 120 lb and having a radius of gyration of 4 ft is connected to two blocks as shown. Assuming no axle friction, determine the angular acceleration of the pulley.



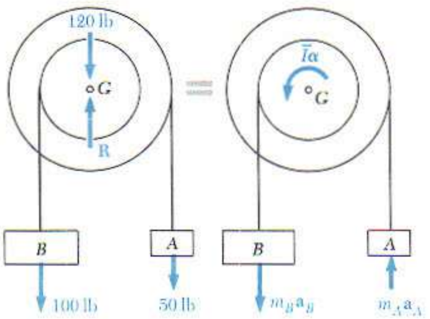
Sense of Motion. Although an arbitrary sense of motion may be assumed (since no friction forces are involved) and later checked by the sign of the answer, we may prefer first to determine the actual sense of rotation of the pulley. We first find the weight of block B required to maintain the equilibrium of the pulley when it is acted upon by the 50-lb block A. We write

$$+\circlearrowleft \Sigma M_G = 0: \qquad W_B(3\text{ ft}) - (50\text{ lb})(5\text{ ft}) = 0 \qquad W_B = 83.3\text{ lb}$$

Since block B actually weighs 100 lb, the pulley will rotate counterclockwise.

Kinematics of Motion. Assuming $\boldsymbol{\alpha}$ counterclockwise and noting that $a_A = r_A\alpha$ and $a_B = r_B\alpha$, we obtain

$$\mathbf{a}_A = 5\alpha \uparrow \qquad \mathbf{a}_B = 3\alpha \downarrow$$



Equations of Motion. A single system consisting of the pulley and the two blocks is considered. Forces external to this system consist of the weights of the pulley and the two blocks and of the reaction at G. (The forces exerted by the cables on the pulley and on the blocks are internal to the system considered and cancel out.) Since the motion of the pulley is a centroidal rotation and the motion of each block is a translation, the effective forces reduce to the couple $\bar{I}\boldsymbol{\alpha}$ and the two vectors $m\mathbf{a}_A$ and $m\mathbf{a}_B$. The centroidal moment of inertia of the pulley is

$$\bar{I} = mk^2 = \frac{W}{g}k^2 = \frac{120\text{ lb}}{32.2\text{ ft/s}^2}(4\text{ ft})^2 = 59.6\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

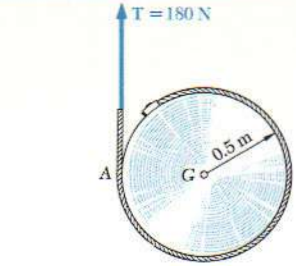
Since the system of the external forces is equipollent to the system of the effective forces, we write

$$+\circlearrowleft \Sigma M_G = \Sigma(M_G)_{\text{eff}}:$$

$$(100\text{ lb})(3\text{ ft}) - (50\text{ lb})(5\text{ ft}) = +\bar{I}\alpha + m_B a_B(3\text{ ft}) + m_A a_A(5\text{ ft})$$

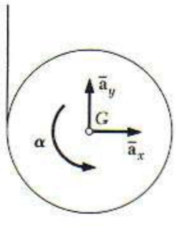
$$(100)(3) - (50)(5) = +59.6\alpha + \frac{100}{32.2}(3\alpha)(3) + \frac{50}{32.2}(5\alpha)(5)$$

$$\alpha = +0.396\text{ rad/s}^2 \qquad \boldsymbol{\alpha} = 0.396\text{ rad/s}^2 \circlearrowleft$$



SAMPLE PROBLEM 16.4

A cord is wrapped around a homogeneous disk of radius $r = 0.5$ m and mass $m = 15$ kg. If the cord is pulled upward with a force **T** of magnitude 180 N, determine (*a*) the acceleration of the center of the disk, (*b*) the angular acceleration of the disk, (*c*) the acceleration of the cord.

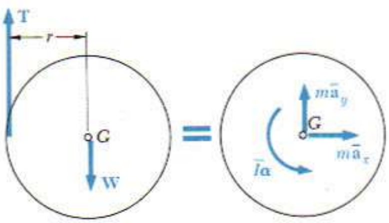


Equations of Motion. We assume that the components $\bar{\mathbf{a}}_x$ and $\bar{\mathbf{a}}_y$ of the acceleration of the center are directed, respectively, to the right and upward and that the angular acceleration of the disk is counterclockwise. The external forces acting on the disk consist of the weight **W** and the force **T** exerted by the cord. This system is equivalent to the system of the effective forces, which consists of a vector of components $m\bar{\mathbf{a}}_x$ and $m\bar{\mathbf{a}}_y$ attached at G and a couple $\bar{I}\boldsymbol{\alpha}$. We write

$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \qquad 0 = m\bar{a}_x \qquad\qquad \bar{\mathbf{a}}_x = 0 \blacktriangleleft$$

$$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \qquad T - W = m\bar{a}_y$$

$$\bar{a}_y = \frac{T - W}{m}$$



Since $T = 180$ N, $m = 15$ kg, and $W = (15\text{ kg})(9.81\text{ m/s}^2) = 147.1$ N, we have

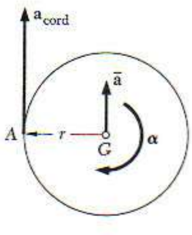
$$\bar{a}_y = \frac{180\text{ N} - 147.1\text{ N}}{15\text{ kg}} = +2.19\text{ m/s}^2 \qquad \bar{\mathbf{a}}_y = 2.19\text{ m/s}^2\uparrow \blacktriangleleft$$

$$+\circlearrowleft \Sigma M_G = \Sigma(M_G)_{\text{eff}}: \qquad -Tr = \bar{I}\alpha$$

$$-Tr = (\tfrac{1}{2}mr^2)\alpha$$

$$\alpha = -\frac{2T}{mr} = -\frac{2(180\text{ N})}{(15\text{ kg})(0.5\text{ m})} = -48.0\text{ rad/s}^2$$

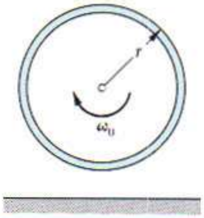
$$\boldsymbol{\alpha} = 48.0\text{ rad/s}^2 \circlearrowright \blacktriangleleft$$



Acceleration of Cord. Since the acceleration of the cord is equal to the tangential component of the acceleration of point A on the disk, we write

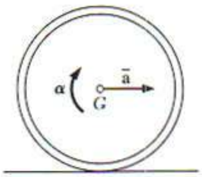
$$\mathbf{a}_{\text{cord}} = (\mathbf{a}_A)_t = \bar{\mathbf{a}} + (\mathbf{a}_{A/G})_t$$
$$= [2.19\text{ m/s}^2\uparrow] + [(0.5\text{ m})(48\text{ rad/s}^2)\uparrow]$$

$$\mathbf{a}_{\text{cord}} = 26.2\text{ m/s}^2\uparrow \blacktriangleleft$$



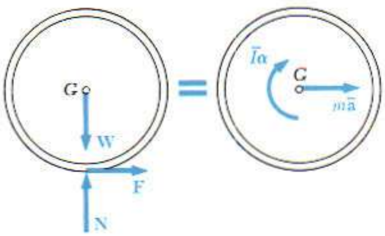
SAMPLE PROBLEM 16.5

A hoop of radius r and mass m is placed on a horizontal surface with no linear velocity but with a clockwise angular velocity ω_0. Denoting by μ the coefficient of friction between the hoop and the floor, determine (*a*) the time t_1 at which the hoop will start rolling without sliding, (*b*) the linear and angular velocities of the hoop at time t_1.



Solution. Since the entire mass is located at a distance r from the center of the hoop, we write $\bar{I} = mr^2$.

Equations of Motion. The positive sense is chosen to the right for $\bar{\mathbf{a}}$ and clockwise for $\boldsymbol{\alpha}$. The external forces acting on the hoop consist of the weight **W**, the normal reaction **N**, and the friction force **F**. While the hoop is sliding, the magnitude of the friction force is $F = \mu N$. The effective forces consist of the vector $m\bar{\mathbf{a}}$ attached at G and the couple $\bar{I}\boldsymbol{\alpha}$. Expressing that the system of the external forces is equivalent to the system of the effective forces, we write



$$+\uparrow \Sigma F_y = \Sigma (F_y)_{\text{eff}}: \quad N - W = 0$$

$$N = W = mg \qquad F = \mu N = \mu mg$$

$$\overset{+}{\rightarrow} \Sigma F_x = \Sigma (F_x)_{\text{eff}}: \qquad F = m\bar{a} \qquad \mu mg = m\bar{a} \qquad \bar{a} = +\mu g$$

$$+\curvearrowright \Sigma M_G = \Sigma (M_G)_{\text{eff}}: \qquad -Fr = \bar{I}\alpha$$

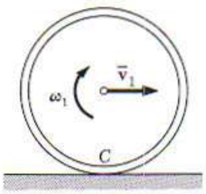
$$-(\mu mg)r = (mr^2)\alpha \qquad \alpha = -\frac{\mu g}{r}$$

Kinematics of Motion. As long as the hoop both rolls and slides, its linear and angular motions are uniformly accelerated.

$$t = 0,\ \bar{v}_0 = 0 \qquad \bar{v} = \bar{v}_0 + \bar{a}t = 0 + \mu g t \tag{1}$$

$$t = 0,\ \omega = \omega_0 \qquad \omega = \omega_0 + \alpha t = \omega_0 + \left(-\frac{\mu g}{r}\right)t \tag{2}$$

The hoop will start rolling without sliding when the velocity $\mathbf{v}_C$ of the point of contact is zero. At that time, $t = t_1$, point C becomes the instantaneous center of rotation, and we have



$$\bar{v}_1 = r\omega_1$$

$$\mu g t_1 = r\left(\omega_0 - \frac{\mu g}{r} t_1\right) \qquad t_1 = \frac{r\omega_0}{2\mu g} \blacktriangleleft$$

Substituting for t_1 into (1), we have

$$\bar{v}_1 = \mu g t_1 = \mu g \frac{r\omega_0}{2\mu g} \qquad \bar{v}_1 = \tfrac{1}{2}r\omega_0 \qquad \bar{\mathbf{v}}_1 = \tfrac{1}{2}r\omega_0 \rightarrow \blacktriangleleft$$

$$\bar{v}_1 = r\omega_1 \qquad \tfrac{1}{2}r\omega_0 = r\omega_1 \qquad \omega_1 = \tfrac{1}{2}\omega_0 \qquad \omega_1 = \tfrac{1}{2}\omega_0 \curvearrowright \blacktriangleleft$$

PROBLEMS

16.1 A 6-ft board is placed in a truck so that one end rests against a block on the floor while the other end rests against a vertical wall. Determine the maximum possible uniform acceleration of the truck if the board is to remain in the position shown.

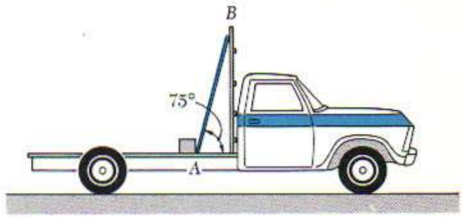


Fig. P16.1

16.2 A uniform rod ABC of mass 8 kg is connected to two collars of negligible mass which slide on smooth horizontal rods located in the same vertical plane. If a force **P** of magnitude 40 N is applied at C, determine (a) the acceleration of the rod, (b) the reactions at B and C.

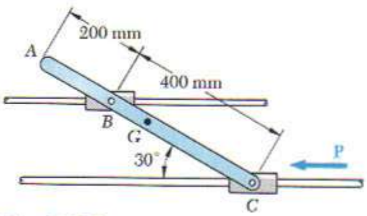


Fig. P16.2

16.3 In Prob. 16.2, determine (a) the required magnitude of **P** if the reaction at B is to be 45 N upward, (b) the corresponding acceleration of the rod.

16.4 The motion of a 3-lb semicircular rod is guided by two small wheels which roll freely in a vertical slot. Knowing that the acceleration of the rod is $a = \frac{1}{4}g$ upward, determine (a) the magnitude of the force **P**, (b) the reactions at A and B.

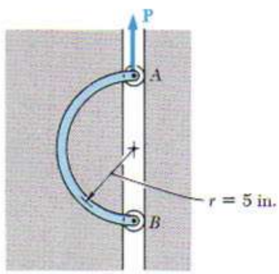


Fig. P16.4

16.5 Cylindrical cans are transported from one elevation to another by the moving horizontal arms shown. Assuming that $\mu = 0.20$ between the cans and the arms, determine (a) the magnitude of the upward acceleration **a** for which the cans slide on the horizontal arms, (b) the smallest ratio h/d for which the cans tip before they slide.

16.6 Solve Prob. 16.5, assuming that the acceleration **a** of the horizontal arms is directed downward.

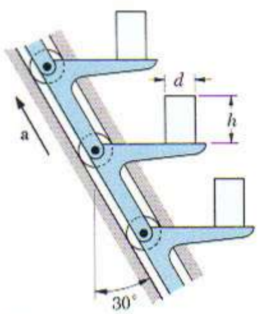


Fig. P16.5

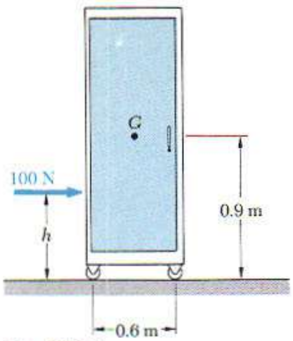


Fig. P16.7

16.7 A 20-kg cabinet is mounted on casters which allow it to move freely ($\mu = 0$) on the floor. If a 100-N force is applied as shown, determine (*a*) the acceleration of the cabinet, (*b*) the range of values of h for which the cabinet will not tip.

16.8 Solve Prob. 16.7, assuming that the casters are locked and slide along the rough floor ($\mu = 0.25$).

16.9 Determine the distance through which the truck of Sample Prob. 16.1 will skid if (*a*) the rear-wheel brakes fail to operate, (*b*) the front-wheel brakes fail to operate.

16.10 A 600-kg fork-lift truck carries the 300-kg crate at the height shown. The truck is moving to the left when the brakes are applied causing a deceleration of 3 m/s^2. Knowing that the coefficient of friction between the crate and the fork lift is 0.5, determine the vertical component of the reaction (*a*) at each of the two wheels A (one wheel on each side of the truck), (*b*) at the single steerable wheel B.

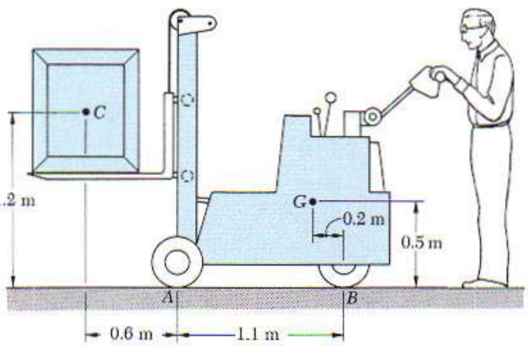


Fig. P16.10

16.11 In Prob. 16.10, determine the maximum deceleration of the truck if the crate is not to slide forward and if the truck is not to tip forward.

16.12 Knowing that the coefficient of friction between the tires and road is 0.80 for the car shown, determine the maximum possible acceleration on a level road, assuming (*a*) four-wheel drive, (*b*) conventional rear-wheel drive, (*c*) front-wheel drive.

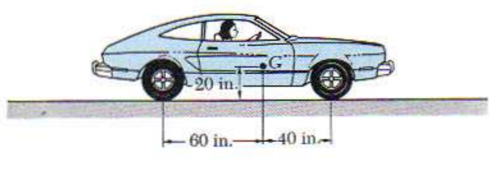


Fig. P16.12

16.13 A man rides a bicycle at a speed of 30 km/h. The distance between axles is 1050 mm, and the mass center of the man and bicycle is located 650 mm behind the front axle and 1000 mm above the ground. If the man applies the brakes on the front wheel only, determine the shortest distance in which he can stop without being thrown over the front wheel.

16.14 The total mass of the loading car and its load is 2500 kg. Neglecting the mass and friction of the wheels, determine (*a*) the minimum tension T in the cable for which the upper wheels are lifted from the track, (*b*) the corresponding acceleration of the car.

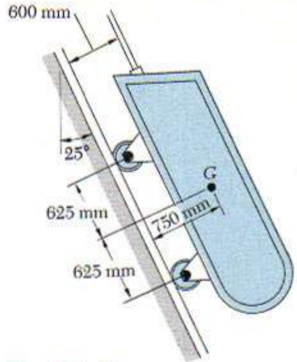


Fig. P16.14

16.15 A 200-lb rectangular panel is suspended from two skids which may slide with no friction on the inclined track shown. If the panel is released from rest, determine (*a*) the acceleration of the panel, (*b*) the reaction at B.

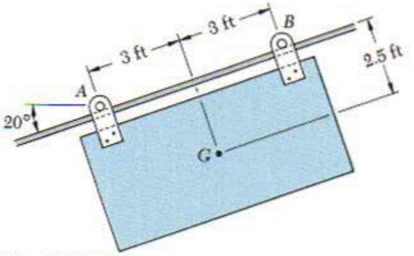


Fig. P16.15

16.16 Solve Prob. 16.15, assuming that the coefficient of friction between each skid and the track is 0.10.

16.17 The 200-kg fire door is supported by wheels B and C which may roll freely on the horizontal track. The 40-kg counterweight A is connected to the door by the cable shown. If the system is released from rest, determine (*a*) the acceleration of the door, (*b*) the reactions at B and C.

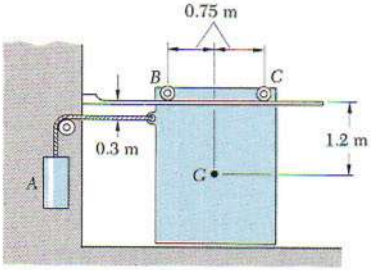


Fig. P16.17

16.18 Two uniform rods AB and CD, each of mass 2.5 kg, are welded together and are attached to two links CE and DF. Neglecting the mass of the links, determine the force in each link immediately after the system is released from rest in the position shown.

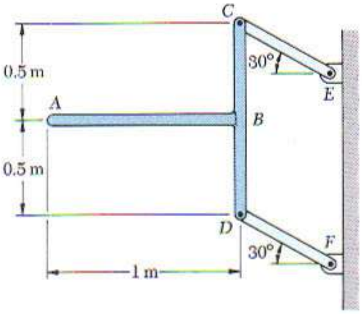


Fig. P16.18

16.19 The retractable shelf shown is supported by two identical linkage-and-spring systems; only one of the systems is shown. A 40-lb machine is placed on the shelf so that half of its weight is supported by the system shown. If the springs are removed and the system is released from rest, determine (*a*) the acceleration of the machine, (*b*) the tension in link AB. Neglect the weight of the shelf and links.

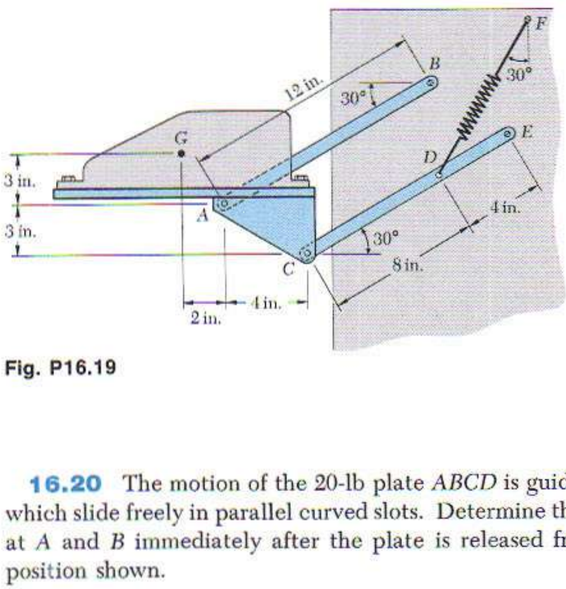


Fig. P16.19

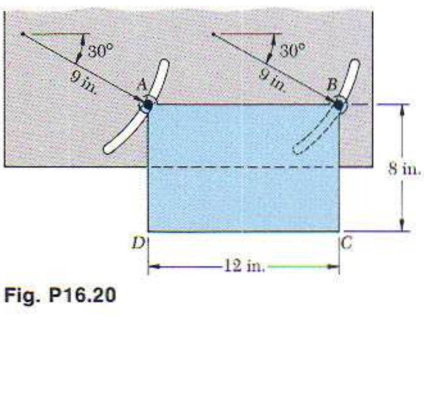


Fig. P16.20

16.20 The motion of the 20-lb plate $ABCD$ is guided by two pins which slide freely in parallel curved slots. Determine the pin reactions at A and B immediately after the plate is released from rest in the position shown.

16.21 The cranks AB and CD rotate at a constant speed of 240 rpm. For the position $\phi = 30°$, determine the horizontal components of the forces exerted on the 5-kg uniform connecting rod BC by the pins B and C.

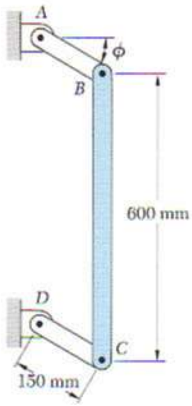


Fig. P16.21

16.22 The control rod AC is guided by two pins which slide freely in parallel curved slots of radius 200 mm. The rod has a mass of 10 kg, and its mass center is located at point G. Knowing that for the position shown the *vertical* component of the velocity of C is 1.25 m/s upward and the *vertical* component of the acceleration of C is 5 m/s² upward, determine the magnitude of the force **P**.

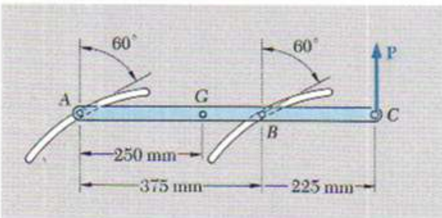


Fig. P16.22

16.23 Assuming that the plate of Prob. 16.20 has acquired a velocity of 4 ft/s in the position shown, determine (*a*) the acceleration of the plate, (*b*) the pin reactions at A and B.

***16.24** A 12-kg block is placed on a 3-kg platform BD which is held in the position shown by three wires. Determine the accelerations of the block and of the platform immediately after wire AB has been cut. Assume (*a*) that the block is rigidly attached to BD, (*b*) that $\mu = 0$ between the block and BD.

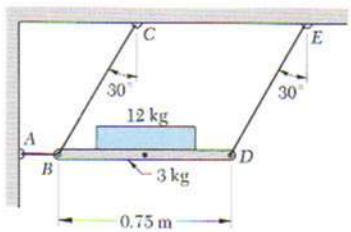


Fig. P16.24 and P16.25

***16.25** The coefficient of friction between the 12-kg block and the 3-kg platform BD is 0.50. Determine the accelerations of the block and of the platform immediately after wire AB has been cut.

***16.26** Draw the shear and bending moment diagrams for the horizontal rod AB of Prob. 16.18.

***16.27** Draw the shear and bending-moment diagrams for the rod BC of Prob. 16.21.

16.28 For a rigid slab in translation, show that the system of the effective forces consists of vectors $(\Delta m_i)\bar{\mathbf{a}}$ attached to the various particles of the slab, where $\bar{\mathbf{a}}$ is the acceleration of the mass center G of the slab. Further show, by computing their sum and the sum of their moments about G, that the effective forces reduce to a single vector $m\bar{\mathbf{a}}$ attached at G.

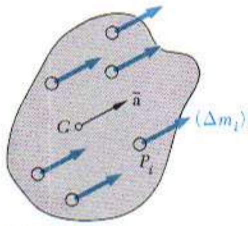


Fig. P16.28

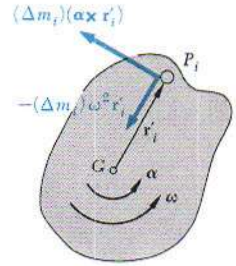


Fig. P16.29

16.29 For a rigid slab in centroidal rotation, show that the system of the effective forces consists of vectors $-(\Delta m_i)\omega^2\mathbf{r}'_i$ and $(\Delta m_i)(\boldsymbol{\alpha} \times \mathbf{r}'_i)$ attached to the various particles P_i of the slab, where $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are the angular velocity and angular acceleration of the slab, and where $\mathbf{r}'_i$ denotes the position vector of the particle P_i relative to the mass center G of the slab. Further show, by computing their sum and the sum of their moments about G, that the effective forces reduce to a couple $\bar{I}\boldsymbol{\alpha}$.

16.30 A turbine-generator unit is shut off when its rotor is rotating at 3600 rpm; it is observed that the rotor coasts to rest in 7.10 min. Knowing that the 1850-kg rotor has a radius of gyration of 234 mm, determine the average magnitude of the couple due to bearing friction.

16.31 An electric motor is rotating at 1200 rpm when the load and power are cut off. The rotor weighs 180 lb and has a radius of gyration of 8 in. If the kinetic friction of the rotor produces a couple of moment 15 lb·in., how many revolutions will the rotor execute before stopping?

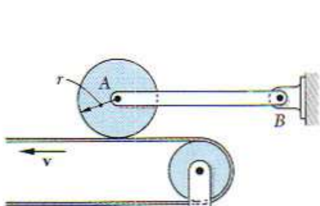


Fig. P16.32 and P16.33

16.32 Disk A weighs 12 lb and is at rest when it is placed in contact with a conveyor belt moving at a constant speed. The link AB connecting the center of the disk to the support at B is of negligible weight. Knowing that $r = 6$ in. and $\mu = 0.35$, determine the angular acceleration of the disk while slipping occurs.

16.33 The uniform disk A is at rest when it is placed in contact with a conveyor belt moving at a constant speed. Neglecting the weight of the link AB, derive an expression for the angular acceleration of the disk while slipping occurs.

16.34 Each of the double pulleys shown has a mass moment of inertia of $10\ \text{kg}\cdot\text{m}^2$ and is initially at rest. The outside radius is 400 mm, and the inner radius is 200 mm. Determine (*a*) the angular acceleration of each pulley, (*b*) the angular velocity of each pulley at $t = 2$ s, (*c*) the angular velocity of each pulley after point *A* on the cord has moved 2 m.

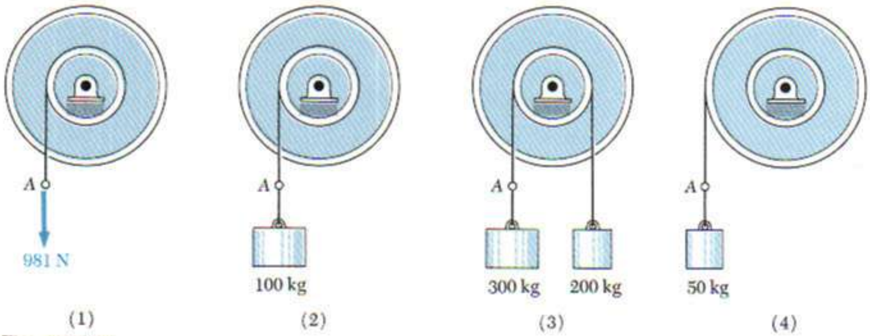


Fig. P16.34

16.35 Solve Prob. 12.17*a* assuming that each pulley is of 8-in. radius and has a centroidal mass moment of inertia $0.25\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$.

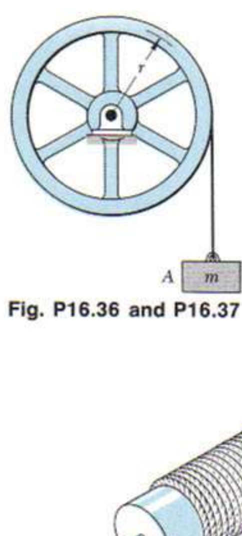


Fig. P16.36 and P16.37

16.36 The flywheel shown weighs 250 lb and has a radius of gyration of 15 in. A block *A* of weight 30 lb is attached to a wire wrapped around the rim of radius $r = 20$ in. The system is released from rest. Neglecting the effect of friction, determine (*a*) the acceleration of block *A*, (*b*) the speed of block *A* after it has moved 6 ft.

16.37 In order to determine the mass moment of inertia of a flywheel of radius $r = 600$ mm a block of mass 12 kg is attached to a cord which is wrapped around the rim of the flywheel. The block is released from rest and is observed to fall 3 m in 4.6 s. To eliminate bearing friction from the computation, a second block of mass 24 kg is used and is observed to fall 3 m in 3.1 s. Assuming that the moment of the couple due to friction is constant, determine the mass moment of inertia of the flywheel.

16.38 A rope of total mass 10 kg and total length 20 m is wrapped around the drum of a hoist as shown. The mass of the drum and shaft is 18 kg, and they have a combined radius of gyration of 200 mm. Knowing that the system is released from rest when a length $h = 5$ m hangs from the drum, determine the initial angular acceleration of the drum.

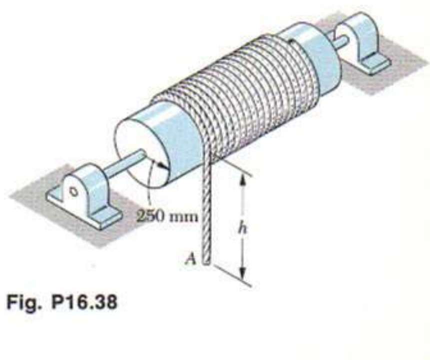


Fig. P16.38

16.39 The flywheel shown consists of a 3-ft-diameter disk which weighs 300 lb. The coefficient of friction between the band and the flywheel is 0.30. If the initial angular velocity of the flywheel is 300 rpm clockwise, determine the magnitude of the force **P** required sto stop the flywheel in 20 revolutions.

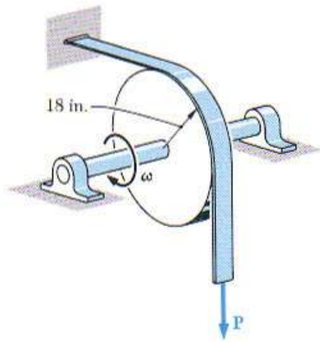


Fig. P16.39

16.40 Solve Prob. 16.39 assuming that the initial angular velocity of the flywheel is 300 rpm counterclockwise.

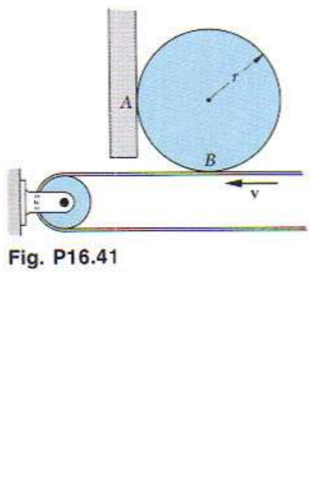


Fig. P16.41

16.41 A cylinder of radius r and mass m is placed with no initial velocity on a belt as shown. Denoting by μ the coefficient of friction at A and at B and assuming that $\mu < 1$, determine the angular acceleration $\boldsymbol{\alpha}$ of the cylinder.

16.42 Shaft A and friction disk B have a combined mass of 15 kg and a combined radius of gyration of 150 mm. Shaft D and friction wheel C rotate with a constant angular velocity of 1000 rpm. Disk B is at rest when it is brought into contact with the rotating wheel. Knowing that disk B accelerates uniformly for 12 s before acquiring its final angular velocity, determine the magnitude of the friction force between the disk and the wheel.

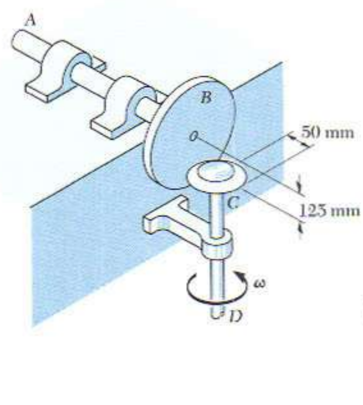


Fig. P16.42

16.43 Each of the gears A and B weighs 4 lb and has a radius of gyration of 3 in., while gear C weighs 20 lb and has a radius of gyration of 9 in. If a couple $\mathbf{M}$ of constant magnitude 60 lb·in. is applied to gear C, determine (a) the angular acceleration of gear A, (b) the time required for the angular velocity of gear A to increase from 150 to 500 rpm.

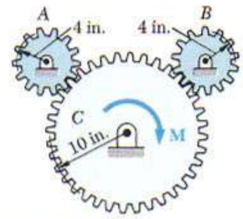


Fig. P16.43

16.44 Disk A is of mass 5 kg and has an initial angular velocity of 300 rpm clockwise. Disk B is of mass 1.8 kg and is at rest when it is placed in contact with disk A. Knowing that $\mu = 0.30$ between the disks and neglecting bearing friction, determine (a) the angular acceleration of each disk, (b) the reaction at the support C.

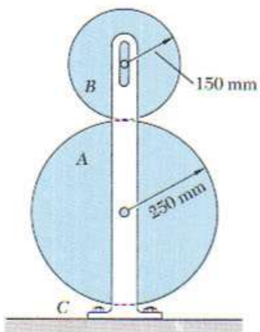


Fig. P16.44

16.45 In Prob. 16.44, (a) determine the final angular velocity of each disk, (b) show that the final angular velocities are independent of μ.

16.46 The two friction disks A and B are brought together by applying the 8-lb force shown. Disk A weighs 6 lb and had an initial angular velocity of 1200 rpm clockwise; disk B weighs 15 lb and was initially at rest. Knowing that $\mu = 0.30$ between the disks and neglecting bearing friction, determine (a) the angular acceleration of each disk, (b) the final angular velocity of each disk.

16.47 Solve Prob. 16.46, assuming that, initially, disk A was at rest and disk B had an angular velocity of 1200 rpm clockwise.

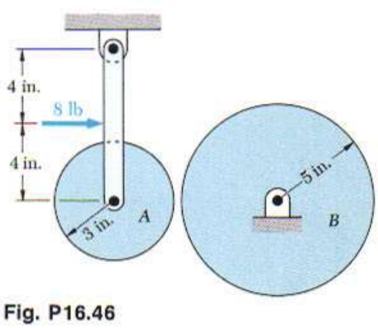


Fig. P16.46

16.48 A coder C, used to record in digital form the rotation of a shaft S, is connected to the shaft by means of the gear train shown, which consists of four gears of the same thickness and of the same material. Two of the gears have a radius r and the other two a radius nr. Let I_R denote the ratio M/α of the moment M of the couple applied to the shaft S and of the resulting angular acceleration α of S. (I_R is sometimes called the "reflected moment of inertia" of the coder and gear train.) Determine I_R in terms of the gear ratio n, the moment of inertia I_0 of the first gear, and the moment of inertia I_C of the coder. Neglect the moments of inertia of the shafts.

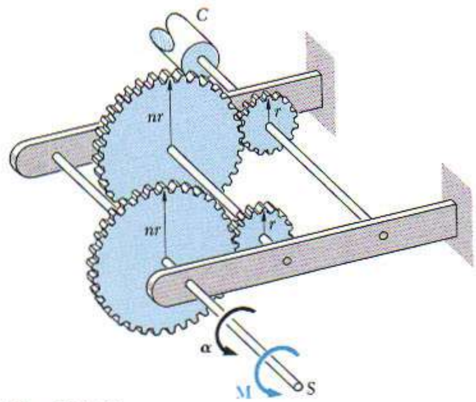


Fig. P16.48

16.49 A 6-kg bar is held between four disks as shown. Each disk has a mass of 3 kg and a diameter of 200 mm. The disks may rotate freely, and the normal reaction exerted by each disk on the bar is sufficient to prevent slipping. If the bar is released from rest, determine (*a*) its acceleration immediately after release, (*b*) its velocity after it has dropped 0.75 m

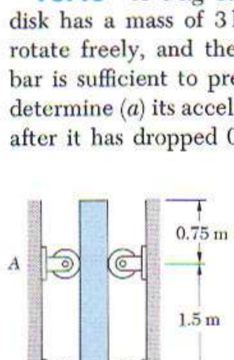


Fig. P16.49

16.50 Show that the system of the effective forces for a rigid slab in plane motion reduces to a single vector, and express the distance from the mass center G of the slab to the line of action of this vector in terms of the centroidal radius of gyration $\bar{k}$ of the slab, the magnitude $\bar{a}$ of the acceleration of G, and the angular acceleration α.

16.51 For a rigid slab in plane motion, show that the system of the effective forces consists of vectors $(\Delta m_i)\bar{\mathbf{a}}$, $-(\Delta m_i)\omega^2\mathbf{r}_i'$, and $(\Delta m_i)(\boldsymbol{\alpha} \times \mathbf{r}_i')$ attached to the various particles P_i of the slab, where $\bar{\mathbf{a}}$ is the acceleration of the mass center G of the slab, $\boldsymbol{\omega}$ the angular velocity of the slab, $\boldsymbol{\alpha}$ its angular acceleration, and where $\mathbf{r}_i'$ denotes the position vector of the particle P_i relative to G. Further show, by computing their sum and the sum of their moments about G, that the effective forces reduce to a vector $m\bar{\mathbf{a}}$ attached at G and a couple $\bar{I}\boldsymbol{\alpha}$.

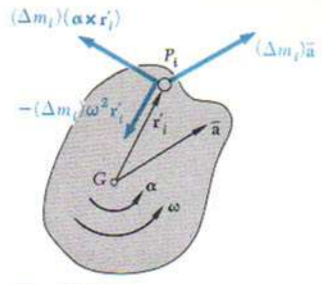


Fig. P16.51

16.52 The uniform slender rod AB weighs 8 lb and is at rest on a frictionless horizontal surface. A force $\mathbf{P}$ of magnitude 2 lb is applied at A in a horizontal direction perpendicular to the rod. Determine (*a*) the angular acceleration of the rod, (*b*) the acceleration of the center of the rod, (*c*) the point of the rod which has no acceleration.

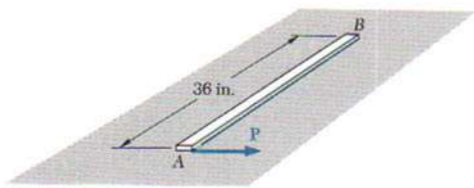


Fig. P16.52

16.53 In Prob. 16.52, determine the point of the rod AB at which the force $\mathbf{P}$ should be applied if the acceleration of point B is to be zero. Knowing that the magnitude of $\mathbf{P}$ is 2 lb, determine the corresponding angular acceleration of the rod and the acceleration of the center of the rod.

16.54 A 50-kg space satellite has a radius of gyration of 450 mm with respect to the y axis, and is symmetrical with respect to the zx plane. The orientation of the satellite is changed by firing four small rockets A, B, C, and D which are equally spaced around the perimeter of the satellite. While being fired, each rocket produces a thrust $\mathbf{T}$ of magnitude 10 N directed as shown. Determine the angular acceleration of the satellite and the acceleration of its mass center G (*a*) when all four rockets are fired, (*b*) when all rockets except rocket D are fired.

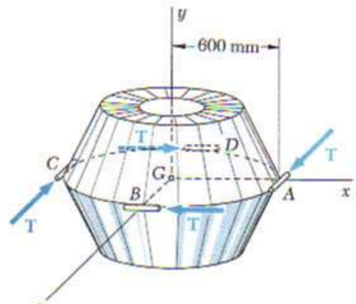


Fig. P16.54

16.55 Solve Prob. 16.54 assuming that only rocket A is fired.

Fig. P16.56 and P16.57

16.56 A 15-ft beam weighing 500 lb is lowered from a considerable height by means of two cables unwinding from overhead cranes. As the beam approaches the ground, the crane operators apply brakes to slow the unwinding motion. The deceleration of cable A is 20 ft/s^2, while that of cable B is 2 ft/s^2. Determine the tension in each cable.

16.57 A 15-ft beam weighing 500 lb is lowered from a considerable height by means of two cables unwinding from overhead cranes. As the beam approaches the ground, the crane operators apply brakes to slow the unwinding motion. Determine the acceleration of each cable at that instant, knowing that $T_A = 360$ lb and $T_B = 320$ lb.

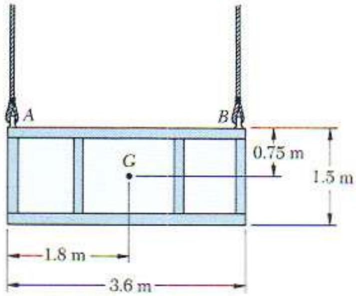


Fig. P16.58 and P16.59

16.58 The 180-kg crate shown is being lowered by means of two overhead cranes. Knowing that at the instant shown the deceleration of cable A is 7 m/s^2, while that of cable B is 1 m/s^2, determine the tension in each cable.

16.59 The 180-kg crate is being lowered by means of two overhead cranes. As the crate approaches the ground, the crane operators apply brakes to slow the motion. Determine the acceleration of each cable at that instant, knowing that $T_A = 1450$ N and $T_B = 1200$ N.

16.60 Solve Sample Prob. 16.4, assuming that the disk rests flat on a frictionless horizontal surface and that the cord is pulled horizontally with a force of magnitude 180 N.

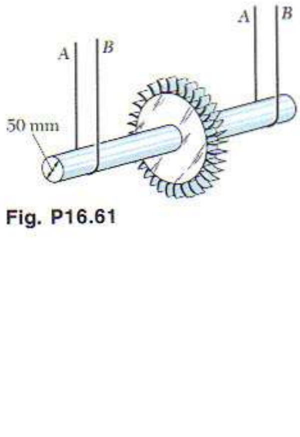


Fig. P16.61

16.61 A turbine disk and shaft have a combined mass of 100 kg and a centroidal radius of gyration of 50 mm. The unit is lifted by two ropes looped around the shaft as shown. Knowing that for each rope $T_A = 270$ N and $T_B = 320$ N, determine (*a*) the angular acceleration of the unit, (*b*) the acceleration of its mass center.

16.62 By pulling on the cord of a yo-yo just fast enough, a man manages to make the yo-yo spin counterclockwise, while remaining at a constant height above the floor. Denoting the weight of the yo-yo by W, the radius of the inner drum on which the cord is wound by r, and the radius of gyration of the yo-yo by $\bar{k}$, determine (*a*) the tension in the cord, (*b*) the angular acceleration of the yo-yo.

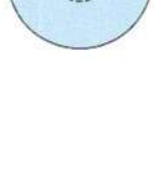


Fig. P16.62

16.63 The 80-lb crate shown rests on four casters which allow it to move without friction in any horizontal direction. A 20-lb horizontal force is applied at the midpoint A of edge CE. Knowing that the force is perpendicular to side $BCDE$, determine the angular acceleration of the crate and the acceleration of point A.

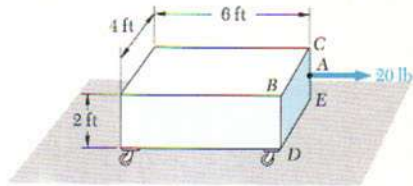


Fig. P16.63

16.64 and 16.65 A uniform slender bar AB of mass m is suspended from two *springs* as shown. If spring BC breaks, determine at that instant (*a*) the angular acceleration of the bar, (*b*) the acceleration of point A, (*c*) the acceleration of point B.

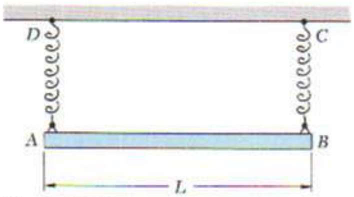


Fig. P16.64

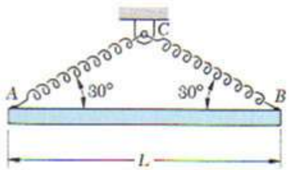


Fig. P16.65

16.66 A sphere of mass m and radius r is projected along a rough horizontal surface with a linear velocity $\mathbf{v}_0$ and with $\omega_0 = 0$. The sphere will decelerate and then reach a uniform motion. Denoting by μ the coefficient of friction, determine (*a*) the linear and angular acceleration of the sphere before it reaches a uniform motion, (*b*) the time required for the motion to become uniform, (*c*) the distance traveled before the motion becomes uniform, (*d*) the final linear and angular velocities of the sphere.

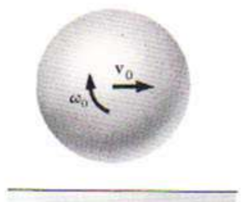


Fig. P16.66

16.67 Solve Prob. 16.66, assuming that the sphere is replaced by a uniform disk of radius r and mass m.

16.68 A heavy square plate of weight W, suspended from four vertical wires, supports a small block E of much smaller weight w. The coefficient of friction between E and the plate is denoted by μ. If the coordinates of E are $x = \frac{1}{4}L$ and $z = \frac{1}{4}L$, derive an expression for the magnitude of the force $\mathbf{P}$ required to cause E to slip with respect to the plate. (*Hint.* Neglect w in all equations containing W.)

***16.69** A square plate of weight $W = 20$ lb and side $L = 3$ ft is suspended from four wires and supports a block E of much smaller weight w. The coefficient of friction between E and the plate is 0.50. If a force $\mathbf{P}$ of magnitude 10 lb is applied as shown, determine the area of the plate where E should be placed if it is not to slip with respect to the plate. (*Hint.* Neglect w in all equations containing W.)

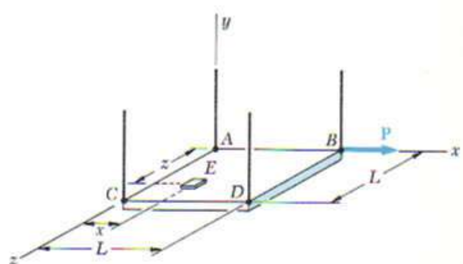


Fig. P16.68 and P16.69

16.8. Constrained Plane Motion. Most engineering applications deal with rigid bodies which are moving under given constraints. Cranks, for example, are constrained to rotate about a fixed axis, wheels roll without sliding, connecting rods must describe certain prescribed motions. In all such cases, definite relations exist between the components of the acceleration $\bar{\mathbf{a}}$ of the mass center G of the body considered and its angular acceleration $\boldsymbol{\alpha}$; the corresponding motion is said to be a *constrained motion.*

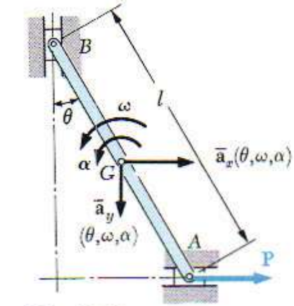


Fig. 16.11

The solution of a problem involving a constrained plane motion calls first for a *kinematic analysis* of the problem. Consider, for example, a slender rod AB of length l and mass m whose extremities are connected to blocks of negligible mass which slide along horizontal and vertical frictionless tracks. The rod is pulled by a force $\mathbf{P}$ applied at A (Fig. 16.11). We know from Sec. 15.8 that the acceleration $\bar{\mathbf{a}}$ of the mass center G of the rod may be determined at any given instant from the position of the rod, its angular velocity, and its angular acceleration at that instant. Suppose, for instance, that the values of θ, ω, and α are known at a given instant and that we wish to determine the corresponding value of the force $\mathbf{P}$, as well as the reactions at A and B. We should first *determine the components $\bar{a}_x$ and $\bar{a}_y$ of the acceleration of the mass center G* by the method of Sec. 15.8. We next apply D'Alembert's principle (Fig. 16.12), using the expressions obtained for $\bar{a}_x$ and $\bar{a}_y$. The unknown forces $\mathbf{P}$, $\mathbf{N}_A$, and $\mathbf{N}_B$ may then be determined by writing and solving the appropriate equations.

Suppose now that the applied force $\mathbf{P}$, the angle θ, and the angular velocity ω of the rod are known at a given instant and that we wish to find the angular acceleration α of the rod and the components $\bar{a}_x$ and $\bar{a}_y$ of the acceleration of its mass center at that instant, as well as the reactions at A and B. The prelimi-

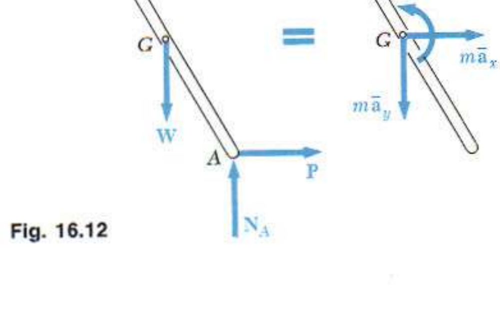


Fig. 16.12

nary kinematic study of the problem will have for its object *to express the components $\bar{a}_x$ and $\bar{a}_y$ of the acceleration of G in terms of the angular acceleration α of the rod.* This will be done by first expressing the acceleration of a suitable reference point such as A in terms of the angular acceleration α. The components $\bar{a}_x$ and $\bar{a}_y$ of the acceleration of G may then be determined in terms of α, and the expressions obtained carried into Fig. 16.12. Three equations may then be derived in terms of α, N_A, and N_B, and solved for the three unknowns (see Sample Prob. 16.10). Note that the method of dynamic equilibrium may also be used to carry out the solution of the two types of problems we have considered (Fig. 16.13).

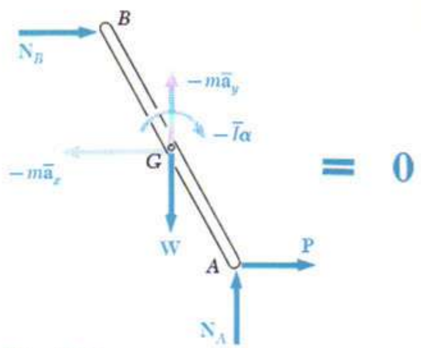


Fig. 16.13

When a mechanism consists of *several moving parts,* the approach just described may be used with each part of the mechanism. The procedure required to determine the various unknowns is then similar to the procedure followed in the case of the equilibrium of a system of connected rigid bodies (Sec. 6.11).

We have analyzed earlier two particular cases of constrained plane motion, the translation of a rigid body, in which the angular acceleration of the body is constrained to be zero, and the centroidal rotation, in which the acceleration $\bar{\mathbf{a}}$ of the mass center of the body is constrained to be zero. Two other particular cases of constrained plane motion are of special interest, the *noncentroidal rotation* of a rigid body and the *rolling motion* of a disk or wheel. These two cases should be analyzed by one of the general methods described above. However, in view of the range of their applications, they deserve a few special comments.

Noncentroidal Rotation. This is the motion of a rigid body constrained to rotate about a fixed axis which does not pass through its mass center. Such a motion is called a *noncentroidal rotation.* The mass center G of the body moves along a circle of radius $\bar{r}$ centered at the point O, where the axis of rotation intersects the plane of reference (Fig. 16.14). Denoting, respectively, by $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ the angular velocity and the angular acceleration of the line OG, we obtain the following expressions for the tangential and normal components of the acceleration of G:

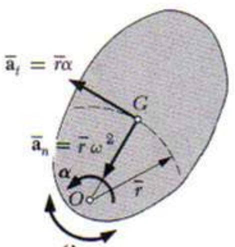


Fig. 16.14

$$\bar{a}_t = \bar{r}\alpha \qquad \bar{a}_n = \bar{r}\omega^2 \tag{16.7}$$

Since line OG belongs to the body, its angular velocity $\boldsymbol{\omega}$ and its angular acceleration $\boldsymbol{\alpha}$ also represent the angular velocity and the angular acceleration of the body in its motion relative to G. Equations (16.7) define, therefore, the kinematic relation existing between the motion of the mass center G and the motion of the body about G. They should be used to eliminate $\bar{a}_t$ and $\bar{a}_n$ from the equations obtained by applying D'Alembert's principle (Fig. 16.15) or the method of dynamic equilibrium (Fig. 16.16).

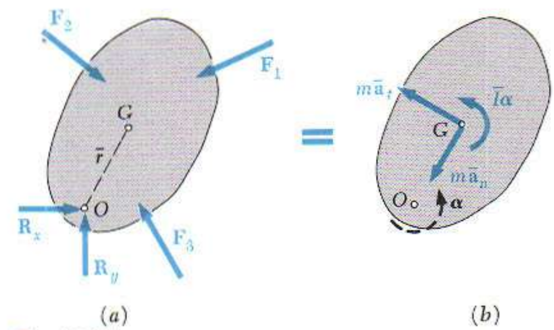


Fig. 16.15

An interesting relation may be obtained by equating the moments about the fixed point O of the forces and vectors shown respectively in parts a and b of Fig. 16.15. We write

$$+\circlearrowleft \Sigma M_O = \bar{I}\alpha + (m\bar{r}\alpha)\bar{r} = (\bar{I} + m\bar{r}^2)\alpha$$

But, according to the parallel-axis theorem, we have $\bar{I} + m\bar{r}^2 = I_O$, where I_O denotes the moment of inertia of the rigid body about the fixed axis. We write, therefore,

$$\Sigma M_O = I_O\alpha \qquad (16.8)$$

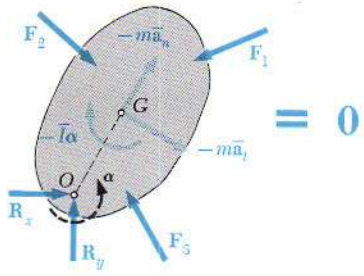


Fig. 16.16

While formula (16.8) expresses an important relation between the sum of the moments of the external forces about the fixed point O and the product $I_O\alpha$, it should be clearly understood that this formula *does not mean* that the system of the external forces is equivalent to a couple of moment $I_O\alpha$. The system of the effective forces, and thus the system of the external forces, reduces to a couple only when O coincides with G, that is, *only when the rotation is centroidal* (Sec. 16.4). In the more general case of noncentroidal rotation, the system of the external forces does not reduce to a couple.

A particular case of noncentroidal rotation is of special interest: the case of *uniform rotation*, in which the angular velocity $\boldsymbol{\omega}$ is constant. Since $\boldsymbol{\alpha}$ is zero, the inertia couple in Fig. 16.16 vanishes and the inertia vector reduces to its normal component. This component (also called *centrifugal force*) represents the tendency of the rigid body to break away from the axis of rotation.

Rolling Motion. Another important case of plane motion is the motion of a disk or wheel rolling on a plane surface. If the disk is constrained to roll without sliding, the acceleration $\bar{\mathbf{a}}$ of its mass center G and its angular acceleration $\boldsymbol{\alpha}$ are not independent. Assuming the disk to be balanced, so that its mass center and its geometric center coincide, we first write that the distance $\bar{x}$ traveled by G during a rotation θ of the disk is $\bar{x} = r\theta$, where r is the radius of the disk. Differentiating this relation twice, we write

$$\bar{a} = r\alpha \tag{16.9}$$

Recalling that the system of the effective forces in plane motion reduces to a vector $m\bar{\mathbf{a}}$ and a couple $\bar{I}\boldsymbol{\alpha}$, we find that, in the particular case of the rolling motion of a balanced disk, the effective forces reduce to a vector of magnitude $m\bar{r}\alpha$ attached at G and to a couple of magnitude $\bar{I}\alpha$. We may thus express that the external forces are equivalent to the vector and couple shown in Fig. 16.17.

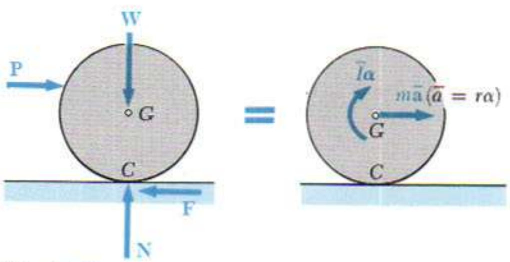


Fig. 16.17

When a disk *rolls without sliding*, there is no relative motion between the point of the disk which is in contact with the ground and the ground itself. As far as the computation of the friction force $\mathbf{F}$ is concerned, a rolling disk may thus be compared with a block at rest on a surface. The magnitude F of the friction force may have any value, as long as it does not exceed the maximum value $F_m = \mu_s N$, where μ_s is the coefficient of static friction and N the magnitude of the normal force. In the case of a rolling disk, the magnitude F of the friction force should therefore be determined independently of N by solving the equation obtained from Fig. 16.17.

When *sliding is impending*, the friction force reaches its maximum value $F_m = \mu_s N$ and may be obtained from N.

When the disk *rotates and slides* at the same time, a relative motion exists between the point of the disk which is in contact with the ground and the ground itself, and the force of friction has the magnitude $F_k = \mu_k N$, where μ_k is the coefficient of kinetic friction. In this case, however, the motion of the mass center G of the disk and the rotation of the disk about G are independent, and $\bar{a}$ is not equal to $r\alpha$.

These three different cases may be summarized as follows:

Rolling, no sliding:	$F \leq \mu_s N$	$\bar{a} = r\alpha$
Rolling, sliding impending:	$F = \mu_s N$	$\bar{a} = r\alpha$
Rotating and sliding:	$F = \mu_k N$	$\bar{a}$ *and* α *independent*

When it is not known whether a disk slides or not, it should first be assumed that the disk rolls without sliding. If F is found smaller than, or equal to, $\mu_s N$, the assumption is proved correct. If F is found larger than $\mu_s N$, the assumption is incorrect and the problem should be started again, assuming rotating and sliding.

When a disk is *unbalanced*, i.e., when its mass center G does not coincide with its geometric center O, the relation (16.9) does not hold between $\bar{a}$ and α. A similar relation will hold, however, between the magnitude a_O of the acceleration of the geometric center and the angular acceleration α,

$$a_O = r\alpha \tag{16.10}$$

To determine $\bar{a}$ in terms of the angular acceleration α and the angular velocity ω of the disk, we may use the relative-acceleration formula,

$$\begin{aligned}\bar{\mathbf{a}} = \mathbf{a}_G &= \mathbf{a}_O + \mathbf{a}_{G/O} \\ &= \mathbf{a}_O + (\mathbf{a}_{G/O})_t + (\mathbf{a}_{G/O})_n\end{aligned} \tag{16.11}$$

where the three component accelerations obtained have the directions indicated in Fig. 16.18 and the magnitudes $a_0 = r\alpha$, $(a_{G/O})_t = (OG)\alpha$, and $(a_{G/O})_n = (OG)\omega^2$.

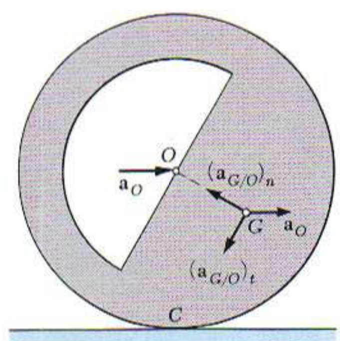
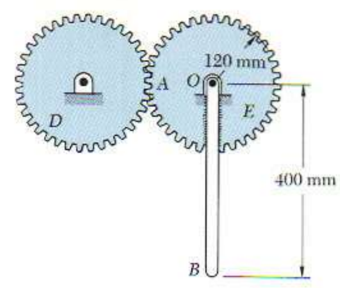
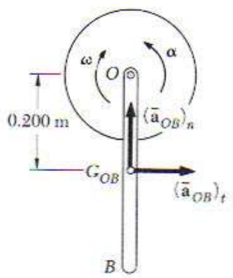


Fig. 16.18

SAMPLE PROBLEM 16.6



The portion AOB of a mechanism consists of a 400-mm steel rod OB welded to a gear E of radius 120 mm which may rotate about a horizontal shaft O. It is actuated by a gear D and, at the instant shown, has a clockwise angular velocity of 8 rad/s and a counterclockwise angular acceleration of 40 rad/s^2. Knowing that rod OB has a mass of 3 kg and gear E a mass of 4 kg and a radius of gyration of 85 mm, determine (*a*) the tangential force exerted by gear D on gear E, (*b*) the components of the reaction at shaft O.

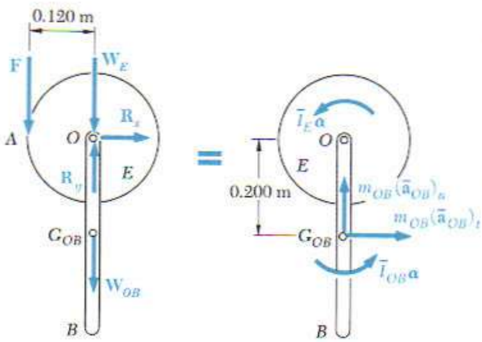


Solution. In determining the effective forces of the rigid body AOB we shall consider separately gear E and rod OB. Therefore, we shall first determine the components of the acceleration of the mass center G_{OB} of the rod:

$$(\bar{a}_{OB})_t = \bar{r}\alpha = (0.200\text{ m})(40\text{ rad/s}^2) = 8\text{ m/s}^2$$
$$(\bar{a}_{OB})_n = \bar{r}\omega^2 = (0.200\text{ m})(8\text{ rad/s})^2 = 12.8\text{ m/s}^2$$

Equations of Motion. Two sketches of the rigid body AOB have been drawn. The first shows the external forces consisting of the weight $\mathbf{W}_E$ of gear E, the weight $\mathbf{W}_{OB}$ of rod OB, the force $\mathbf{F}$ exerted by gear D, and the components $\mathbf{R}_x$ and $\mathbf{R}_y$ of the reaction at O. The magnitudes of the weights are, respectively,

$$W_E = m_E g = (4\text{ kg})(9.81\text{ m/s}^2) = 39.2\text{ N}$$
$$W_{OB} = m_{OB} g = (3\text{ kg})(9.81\text{ m/s}^2) = 29.4\text{ N}$$



The second sketch shows the effective forces, which consist of a couple $\bar{I}_E\boldsymbol{\alpha}$ (since gear E is in centroidal rotation) and of a couple and two vector components at the mass center of OB. Since the accelerations are known, we compute the magnitudes of these components and couples:

$$\bar{I}_E\alpha = m_E\bar{k}_E^2\alpha = (4\text{ kg})(0.085\text{ m})^2(40\text{ rad/s}^2) = 1.156\text{ N}\cdot\text{m}$$
$$m_{OB}(\bar{a}_{OB})_t = (3\text{ kg})(8\text{ m/s}^2) = 24.0\text{ N}$$
$$m_{OB}(\bar{a}_{OB})_n = (3\text{ kg})(12.8\text{ m/s}^2) = 38.4\text{ N}$$
$$\bar{I}_{OB}\alpha = (\tfrac{1}{12}m_{OB}L^2)\alpha = \tfrac{1}{12}(3\text{ kg})(0.400\text{ m})^2(40\text{ rad/s}^2) = 1.600\text{ N}\cdot\text{m}$$

Expressing that the system of the external forces is equivalent to the system of the effective forces, we write the following equations:

$+\circlearrowleft \Sigma M_O = \Sigma(M_O)_{\text{eff}}$:

$$F(0.120\text{ m}) = \bar{I}_E\alpha + m_{OB}(\bar{a}_{OB})_t(0.200\text{ m}) + \bar{I}_{OB}\alpha$$
$$F(0.120\text{ m}) = 1.156\text{ N}\cdot\text{m} + (24.0\text{ N})(0.200\text{ m}) + 1.600\text{ N}\cdot\text{m}$$

$F = 63.0\text{ N}$ $\qquad$ $\mathbf{F} = 63.0\text{ N}\downarrow$ ◄

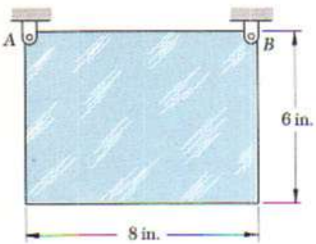
$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}$: $\qquad R_x = m_{OB}(\bar{a}_{OB})_t$

$R_x = 24.0\text{ N}$ $\qquad$ $\mathbf{R}_x = 24.0\text{ N}\rightarrow$ ◄

$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}$: $\qquad R_y - F - W_E - W_{OB} = m_{OB}(\bar{a}_{OB})_n$

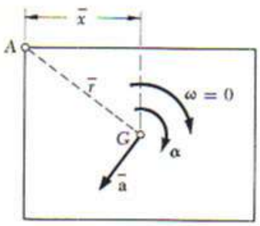
$$R_y - 63.0\text{ N} - 39.2\text{ N} - 29.4\text{ N} = 38.4\text{ N}$$

$R_y = 170.0\text{ N}$ $\qquad$ $\mathbf{R}_y = 170.0\text{ N}\uparrow$ ◄



SAMPLE PROBLEM 16.7

A rectangular plate, 6 by 8 in., weighs 60 lb and is suspended from two pins A and B. If pin B is suddenly removed, determine (*a*) the angular acceleration of the plate, (*b*) the components of the reactions at pin A, immediately after pin B has been removed.



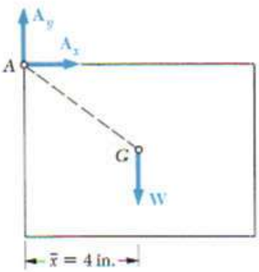
a. Angular Acceleration. We observe that as the plate rotates about point A, its mass center G describes a circle of radius $\bar{r}$ with center at A.

Since the plate is released from rest ($\omega = 0$), the normal component of the acceleration of G is zero. The magnitude of the acceleration $\bar{\mathbf{a}}$ of the mass center G is thus $\bar{a} = \bar{r}\alpha$. We draw the diagram shown to express that the external forces are equivalent to the effective forces:

$+\circlearrowright \Sigma M_A = \Sigma(M_A)_{\text{eff}}$: $\qquad W\bar{x} = (m\bar{a})\bar{r} + \bar{I}\alpha$

Since $\bar{a} = \bar{r}\alpha$, we have

$$W\bar{x} = m(\bar{r}\alpha)\bar{r} + \bar{I}\alpha \qquad \alpha = \frac{W\bar{x}}{\frac{W}{g}\bar{r}^2 + \bar{I}} \tag{1}$$

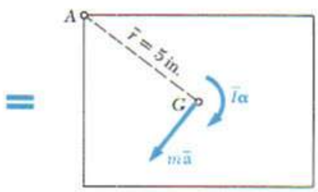


The centroidal moment of inertia of the plate is

$$\bar{I} = \frac{m}{12}(a^2 + b^2) = \frac{60\text{ lb}}{12(32.2\text{ ft/s}^2)}[(\tfrac{8}{12}\text{ ft})^2 + (\tfrac{6}{12}\text{ ft})^2]$$
$$= 0.1078\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

Substituting this value of $\bar{I}$ together with $W = 60$ lb, $\bar{r} = \frac{5}{12}$ ft, and $\bar{x} = \frac{4}{12}$ ft into Eq. (1), we obtain

$$\alpha = +46.4\text{ rad/s}^2 \qquad \boldsymbol{\alpha} = 46.4\text{ rad/s}^2 \circlearrowright$$ ◄



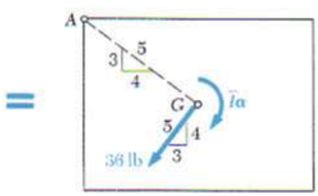
b. Reaction at A. Using the computed value of α, we determine the magnitude of the vector $m\bar{\mathbf{a}}$ attached at G,

$$m\bar{a} = m\bar{r}\alpha = \frac{60\text{ lb}}{32.2\text{ ft/s}^2}(\tfrac{5}{12}\text{ ft})(46.4\text{ rad/s}^2) = 36.0\text{ lb}$$

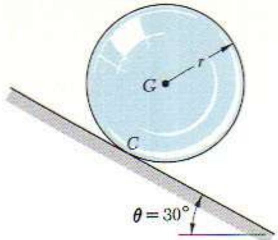
Showing this result on the diagram, we write the equations of motion

$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}$: $\qquad A_x = -\frac{3}{5}(36\text{ lb})$
$\qquad A_x = -21.6\text{ lb} \qquad \mathbf{A}_x = 21.6\text{ lb} \leftarrow$ ◄

$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}$: $\qquad A_y - 60\text{ lb} = -\frac{4}{5}(36\text{ lb})$
$\qquad A_y = +31.2\text{ lb} \qquad \mathbf{A}_y = 31.2\text{ lb}\uparrow$ ◄

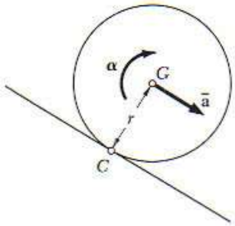


The couple $\bar{I}\boldsymbol{\alpha}$ is not involved in the last two equations; nevertheless, it should be indicated on the diagram.



SAMPLE PROBLEM 16.8

A sphere of radius r and weight W is released with no initial velocity on the incline and rolls without slipping. Determine (*a*) the minimum value of the coefficient of friction compatible with the rolling motion, (*b*) the velocity of the center G of the sphere after the sphere has rolled 10 ft, (*c*) the velocity of G if the sphere were to move 10 ft down a frictionless 30° incline.



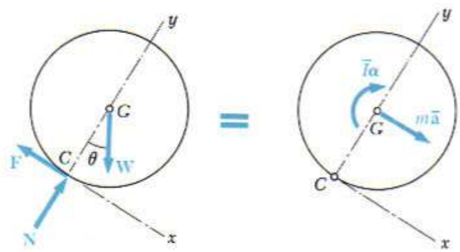
a. Minimum μ for Rolling Motion. The external forces **W**, **N**, and **F** form a system equivalent to the system of effective forces represented by the vector $m\bar{\mathbf{a}}$ and the couple $\bar{I}\boldsymbol{\alpha}$. Since the sphere rolls without sliding, we have $\bar{a} = r\alpha$.

$$+\circlearrowright \Sigma M_C = \Sigma(M_C)_{\text{eff}}: \qquad (W \sin\theta)r = (m\bar{a})r + \bar{I}\alpha$$
$$(W \sin\theta)r = (mr\alpha)r + \bar{I}\alpha$$

Noting that $m = W/g$ and $\bar{I} = \frac{2}{5}mr^2$, we write

$$(W \sin\theta)r = \left(\frac{W}{g}r\alpha\right)r + \frac{2}{5}\frac{W}{g}r^2\alpha \qquad \alpha = +\frac{5g\sin\theta}{7r}$$

$$\bar{a} = r\alpha = \frac{5g\sin\theta}{7} = \frac{5(32.2\text{ ft/s}^2)\sin 30^\circ}{7} = 11.50\text{ ft/s}^2$$



$$+\searrow \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \qquad W\sin\theta - F = m\bar{a}$$
$$W\sin\theta - F = \frac{W}{g}\frac{5g\sin\theta}{7}$$
$$F = +\tfrac{2}{7}W\sin\theta = \tfrac{2}{7}W\sin 30^\circ \qquad \mathbf{F} = 0.143W \measuredangle 30^\circ$$

$$+\nearrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \qquad N - W\cos\theta = 0$$
$$N = W\cos\theta = 0.866W \qquad \mathbf{N} = 0.866W \measuredangle 60^\circ$$

$$\mu_{\min} = \frac{F}{N} = \frac{0.143W}{0.866W} \qquad \mu_{\min} = 0.165 \blacktriangleleft$$

b. Velocity of Rolling Sphere. We have uniformly accelerated motion,

$$\bar{v}_0 = 0 \qquad \bar{a} = 11.50\text{ ft/s}^2 \qquad \bar{x} = 10\text{ ft} \qquad \bar{x}_0 = 0$$
$$\bar{v}^2 = \bar{v}_0^2 + 2\bar{a}(\bar{x} - \bar{x}_0) \qquad \bar{v}^2 = 0 + 2(11.50\text{ ft/s}^2)(10\text{ ft})$$
$$\bar{v} = 15.17\text{ ft/s} \qquad \bar{\mathbf{v}} = 15.17\text{ ft/s} \measuredangle 30^\circ \blacktriangleleft$$

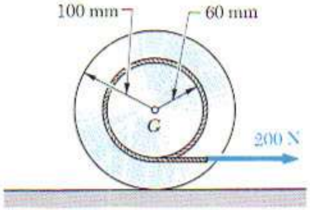
c. Velocity of Sliding Sphere. Assuming now no friction, we have $F = 0$ and obtain

$$+\circlearrowright \Sigma M_G = \Sigma(M_G)_{\text{eff}}: \qquad 0 = \bar{I}\alpha \qquad \alpha = 0$$

$$+\searrow \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \qquad W\sin 30^\circ = m\bar{a} \qquad 0.50W = \frac{W}{g}\bar{a}$$
$$\bar{a} = +16.1\text{ ft/s}^2 \qquad \bar{\mathbf{a}} = 16.1\text{ ft/s}^2 \measuredangle 30^\circ$$

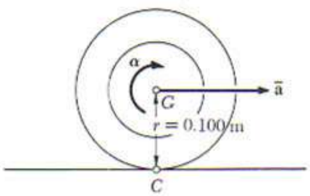
Substituting $\bar{a} = 16.1\text{ ft/s}^2$ into the equations for uniformly accelerated motion, we obtain

$$\bar{v}^2 = \bar{v}_0^2 + 2\bar{a}(\bar{x} - \bar{x}_0) \qquad \bar{v}^2 = 0 + 2(16.1\text{ ft/s}^2)(10\text{ ft})$$
$$\bar{v} = 17.94\text{ ft/s} \qquad \bar{\mathbf{v}} = 17.94\text{ ft/s} \measuredangle 30^\circ \blacktriangleleft$$



SAMPLE PROBLEM 16.9

A cord is wrapped around the inner drum of a wheel and pulled horizontally with a force of 200 N. The wheel has a mass of 50 kg and a radius of gyration of 70 mm. Knowing that $\mu_s = 0.20$ and $\mu_k = 0.15$, determine the acceleration of G and the angular acceleration of the wheel.



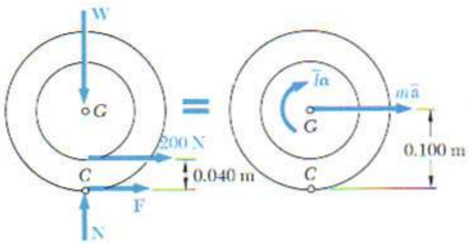
a. **Assume Rolling without Sliding.** In this case, we have

$$\bar{a} = r\alpha = (0.100\text{ m})\alpha$$

By comparing the friction force obtained with the maximum available friction force, we shall determine whether this assumption is justified. The moment of inertia of the wheel is

$$\bar{I} = m\bar{k}^2 = (50\text{ kg})(0.070\text{ m})^2 = 0.245\text{ kg}\cdot\text{m}^2$$

Equations of Motion



$+\circlearrowright\ \Sigma M_C = \Sigma(M_C)_{\text{eff}}$: $\quad (200\text{ N})(0.040\text{ m}) = m\bar{a}(0.100\text{ m}) + \bar{I}\alpha$

$$8.00\text{ N} = (50\text{ kg})(0.100\text{ m})\alpha(0.100\text{ m}) + (0.245\text{ kg}\cdot\text{m}^2)\alpha$$
$$\alpha = +10.74\text{ rad/s}^2$$
$$\bar{a} = r\alpha = (0.100\text{ m})(10.74\text{ rad/s}^2) = 1.074\text{ m/s}^2$$

$\xrightarrow{+}\ \Sigma F_x = \Sigma(F_x)_{\text{eff}}$: $\quad F + 200\text{ N} = m\bar{a}$

$$F + 200\text{ N} = (50\text{ kg})(1.074\text{ m/s}^2)$$
$$F = -146.3\text{ N} \qquad \mathbf{F} = 146.3\text{ N} \leftarrow$$

$+\uparrow\ \Sigma F_y = \Sigma(F_y)_{\text{eff}}$:

$$N - W = 0 \qquad N = W = mg = (50\text{ kg})(9.81\text{ m/s}^2) = 490.5\text{ N}$$
$$\mathbf{N} = 490.5\text{ N}\uparrow$$

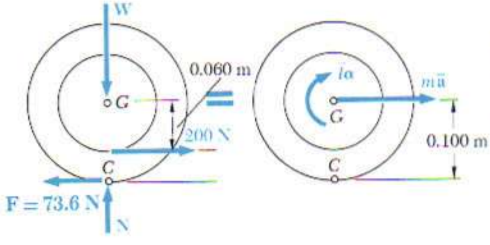
Maximum Available Friction Force

$$F_{\max} = \mu_s N = 0.20\,(490.5\text{ N}) = 98.1\text{ N}$$

Since $F > F_{\max}$, the assumed motion is impossible.

b. **Rotating and Sliding.** Since the wheel must rotate and slide at the same time, we draw a new diagram, where $\bar{\mathbf{a}}$ and $\boldsymbol{\alpha}$ are independent and where

$$F = F_k = \mu_k N = 0.15\,(490.5\text{ N}) = 73.6\text{ N}$$



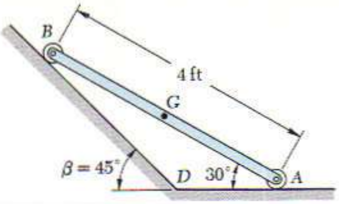
From the computation of part *a*, it appears that **F** should be directed to the left. We write the following equations of motion:

$\xrightarrow{+}\ \Sigma F_x = \Sigma(F_x)_{\text{eff}}$: $\quad 200\text{ N} - 73.6\text{ N} = (50\text{ kg})\bar{a}$

$$\bar{a} = +2.53\text{ m/s}^2 \qquad \bar{\mathbf{a}} = 2.53\text{ m/s}^2 \rightarrow \blacktriangleleft$$

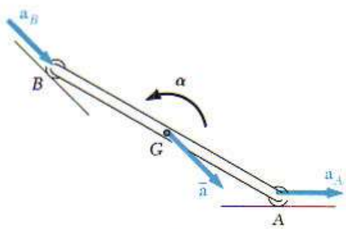
$+\circlearrowright\ \Sigma M_G = \Sigma(M_G)_{\text{eff}}$:

$$(73.6\text{ N})(0.100\text{ m}) - (200\text{ N})(0.060\text{ m}) = (0.245\text{ kg}\cdot\text{m}^2)\alpha$$
$$\alpha = -18.94\text{ rad/s}^2 \qquad \boldsymbol{\alpha} = 18.94\text{ rad/s}^2 \circlearrowleft \blacktriangleleft$$



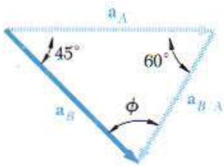
SAMPLE PROBLEM 16.10

The extremities of a 4-ft rod, weighing 50 lb, may move freely and with no friction along two straight tracks as shown. If the rod is released with no velocity from the position shown, determine (*a*) the angular acceleration of the rod, (*b*) the reactions at *A* and *B*.



Kinematics of Motion. Since the motion is constrained, the acceleration of *G* must be related to the angular acceleration $\boldsymbol{\alpha}$. To obtain this relation, we shall first determine the magnitude of the acceleration $\mathbf{a}_A$ of point *A* in terms of α; assuming $\boldsymbol{\alpha}$ directed counterclockwise and noting that $a_{B/A} = 4\alpha$, we write

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$
$$[a_B \searrow 45°] = [a_A \rightarrow] + [4\alpha \nearrow 60°]$$

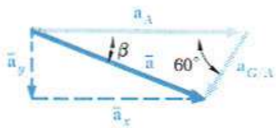


Noting that $\phi = 75°$ and using the law of sines, we obtain

$$a_A = 5.46\alpha \qquad a_B = 4.90\alpha$$

The acceleration of *G* is now obtained by writing

$$\bar{\mathbf{a}} = \mathbf{a}_G = \mathbf{a}_A + \mathbf{a}_{G/A}$$
$$\bar{\mathbf{a}} = [5.46\alpha \rightarrow] + [2\alpha \nearrow 60°]$$



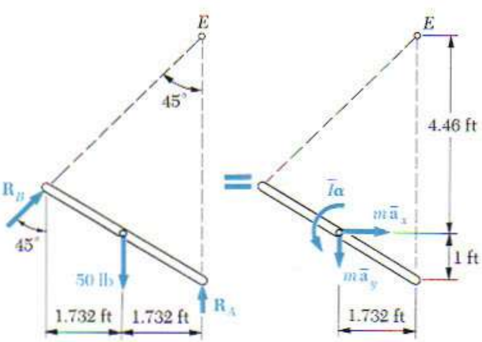
Resolving $\bar{\mathbf{a}}$ into *x* and *y* components, we obtain

$$\bar{a}_x = 5.46\alpha - 2\alpha\cos 60° = 4.46\alpha \qquad \bar{\mathbf{a}}_x = 4.46\alpha \rightarrow$$
$$\bar{a}_y = -2\alpha\sin 60° = -1.732\alpha \qquad \bar{\mathbf{a}}_y = 1.732\alpha \downarrow$$

Kinetics of Motion. We draw the two sketches shown to express that the system of external forces is equivalent to the system of effective forces represented by the vector of components $m\bar{\mathbf{a}}_x$ and $m\bar{\mathbf{a}}_y$ attached at *G* and the couple $\bar{I}\boldsymbol{\alpha}$. We compute the following magnitudes:

$$\bar{I} = \frac{1}{12}ml^2 = \frac{1}{12}\frac{50\text{ lb}}{32.2\text{ ft/s}^2}(4\text{ ft})^2 = 2.07\text{ lb}\cdot\text{ft}\cdot\text{s}^2 \qquad \bar{I}\alpha = 2.07\alpha$$

$$m\bar{a}_x = \frac{50}{32.2}(4.46\alpha) = 6.93\alpha \qquad m\bar{a}_y = -\frac{50}{32.2}(1.732\alpha) = -2.69\alpha$$



Equations of Motion

$+\circlearrowleft \Sigma M_E = \Sigma(M_E)_{\text{eff}}$:

$$(50)(1.732) = (6.93\alpha)(4.46) + (2.69\alpha)(1.732) + 2.07\alpha$$
$$\alpha = +2.30\text{ rad/s}^2 \qquad \boldsymbol{\alpha} = 2.30\text{ rad/s}^2 \circlearrowleft$$

$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}$:

$$R_B\sin 45° = (6.93)(2.30) = 15.94$$
$$R_B = 22.5\text{ lb} \qquad \mathbf{R}_B = 22.5\text{ lb} \angle 45°$$

$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}$:

$$R_A + R_B\cos 45° - 50 = -(2.69)(2.30)$$
$$R_A = -6.19 - 15.94 + 50 = 27.9\text{ lb} \qquad \mathbf{R}_A = 27.9\text{ lb}\uparrow$$

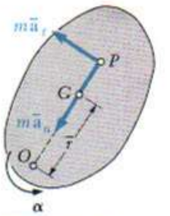


Fig. P16.70

PROBLEMS

16.70 Show that the couple $\bar{I}\boldsymbol{\alpha}$ of Fig. 16.15 may be eliminated by attaching the vectors $m\bar{\mathbf{a}}_t$ and $m\bar{\mathbf{a}}_n$ at a point P called the *center of percussion*, located on line OG at a distance $GP = \bar{k}^2/\bar{r}$ from the mass center of the body.

16.71 A uniform slender rod, of length $L = 900$ mm and mass $m = 4$ kg, is supported as shown. A horizontal force **P** of magnitude 75 N is applied at end B. For $\bar{r} = \frac{1}{4}L = 225$ mm, determine (*a*) the angular acceleration of the rod, (*b*) the components of the reaction at C.

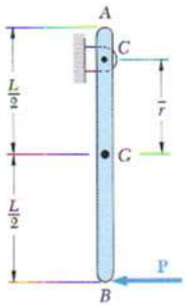


Fig. 16.71

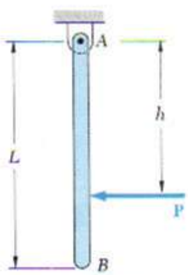


Fig. P16.73 and P16.74

16.72 In Prob. 16.71, determine (*a*) the distance $\bar{r}$ for which the horizontal component of the reaction at C is zero, (*b*) the corresponding angular acceleration of the rod.

16.73 A uniform slender rod, of length L and weight W, hangs freely from a hinge at A. If a horizontal force **P** is applied as shown, determine (*a*) the distance h for which the horizontal component of the reaction at A is zero, (*b*) the corresponding angular acceleration of the rod.

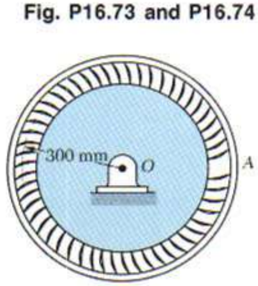


Fig. P16.75

16.74 A uniform slender rod, of length L and weight W, hangs freely from a hinge at A. If a force **P** is applied at B horizontally to the left ($h = L$), determine (*a*) the angular acceleration of the rod, (*b*) the components of the reaction at A.

16.75 A turbine disk of mass 75 kg rotates at a constant speed of 9600 rpm; the mass center of the disk coincides with the center of rotation O. Determine the reaction at O after a single vane at A, of mass 45 g, becomes loose and is thrown off.

16.76 A uniform slender rod of length l and mass m rotates about a vertical axis AA' at a constant angular velocity ω. Determine the tension in the rod at a distance x from the axis of rotation.

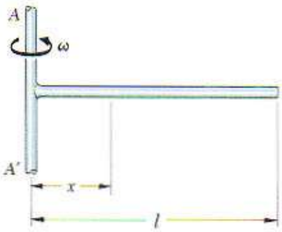


Fig. P16.76

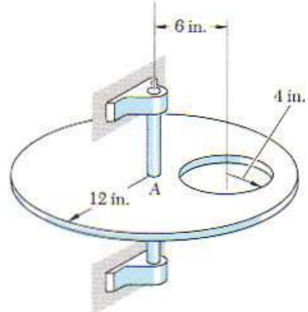


Fig. P16.77

16.77 An 8-in.-diameter hole is cut as shown in a thin disk of diameter 24 in. The disk rotates in a horizontal plane about its geometric center A at a constant angular velocity of 480 rpm. Knowing that the disk weighs 100 lb after the hole has been cut, determine the horizontal component of the force exerted by the shaft on the disk at A.

16.78 A large flywheel is mounted on a horizontal shaft and rotates at a constant rate of 1200 rpm. Experimental data show that the total force exerted by the flywheel on the shaft varies from 55 kN upward to 85 kN downward. Determine (*a*) the mass of the flywheel, (*b*) the distance from the center of the shaft to the mass center of the flywheel.

16.79 and 16.80 A uniform beam of length L and weight W is supported as shown. If the cable suddenly breaks, determine (*a*) the reaction at the pin support, (*b*) the acceleration of point B.

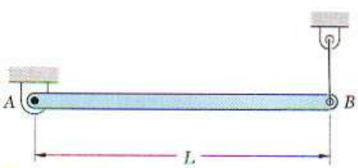


Fig. P16.79

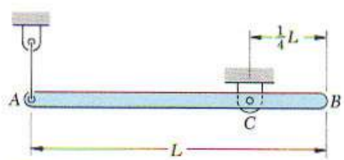


Fig. P16.80

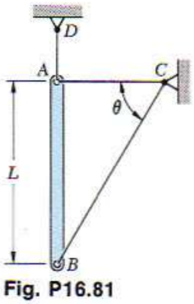


Fig. P16.81

16.81 A uniform slender rod AB, of length $L = 4$ ft and weight 10 lb, is held in the position shown by three wires. If $\theta = 60°$, determine the tension in wires AC and BC immediately after wire AD has been cut.

16.82 Two uniform rods, each of mass m, are attached as shown to small gears of negligible mass. If the rods are released from rest in the position shown, determine the angular acceleration of rod AB immediately after release, assuming (*a*) $\theta = 0$, (*b*) $\theta = 30°$.

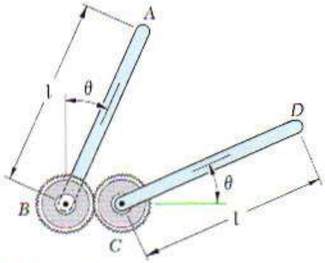


Fig. P16.82

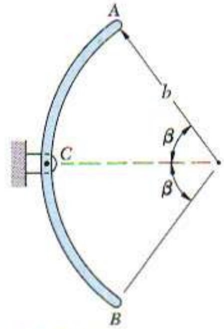


Fig. P16.83

16.83 A uniform rod AB is bent in the shape of an arc of circle. Determine the angular acceleration of the rod immediately after it is released from rest and show that it is independent of β.

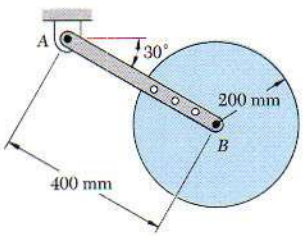


Fig. P16.84 and P16.85

16.84 A 2-kg slender rod is riveted to a 4-kg uniform disk as shown. The assembly swings freely in a vertical plane and, in the position shown, has an angular velocity of 4 rad/s clockwise. Determine (*a*) the angular acceleration of the assembly, (*b*) the components of the reaction at A.

16.85 A 2-kg slender rod is riveted to a 4-kg uniform disk as shown. The assembly rotates in a vertical plane under the combined effect of gravity and a couple **M** which is applied to rod AB. Knowing that at the instant shown the assembly has an angular velocity of 6 rad/s and an angular acceleration of 10 rad/s^2 both counterclockwise, determine (*a*) the magnitude of the couple **M**, (*b*) the components of the reaction at A.

16.86 After being released, the plate of Sample Prob. 16.7 is allowed to swing through 90°. Knowing that at that instant the angular velocity of the plate is 4.82 rad/s, determine (*a*) the angular acceleration of the plate, (*b*) the reaction at A.

16.87 Two uniform rods, AB of weight 12 lb and CD of weight 8 lb, are welded together to form the T-shaped assembly shown. The assembly rotates in a vertical plane about a horizontal shaft at E. Knowing that at the instant shown the assembly has an angular velocity of 12 rad/s and an angular acceleration of 36 rad/s^2, both clockwise, determine (*a*) the magnitude of the horizontal force **P**, (*b*) the components of the reaction at E.

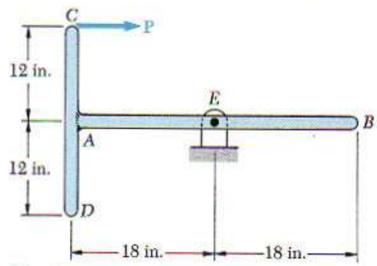


Fig. P16.87

16.88 The uniform rod AB of mass m is released from rest when $\beta = 60°$. Assuming that the friction between end A and the surface is large enough to prevent sliding, determine (*a*) the angular acceleration of the rod just after release, (*b*) the normal reaction and the friction force at A, (*c*) the minimum value of μ compatible with the described motion.

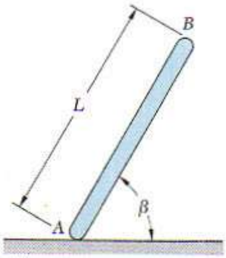


Fig. P16.88 and P16.89

***16.89** Knowing that the coefficient of friction between the rod and the floor is 0.30, determine the range of values of β for which the rod *will* slip immediately after being released from rest.

16.90 Derive the equation $\Sigma M_C = I_C\alpha$ for the rolling disk of Fig. 16.17, where ΣM_C represents the sum of the moments of the external forces about the instantaneous center C and I_C the moment of inertia of the disk about C.

16.91 Show that, in the case of an unbalanced disk, the equation derived in Prob. 16.90 is valid only when the mass center G, the geometric center O, and the instantaneous center C happen to lie in a straight line.

16.92 A homogeneous cylinder C and a section of pipe P are in contact when they are released from rest. Knowing that both the cylinder and the pipe roll without slipping, determine the clear distance between them after 2.5 s.

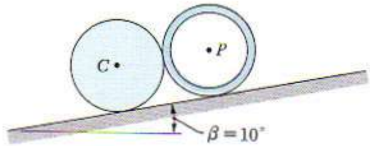


Fig. P16.92

16.93 A flywheel is rigidly attached to a shaft of 40-mm radius which may roll along parallel rails as shown. When released from rest, the system rolls a distance of 3 m in 30 s. Determine the centroidal radius of gyration of the system.

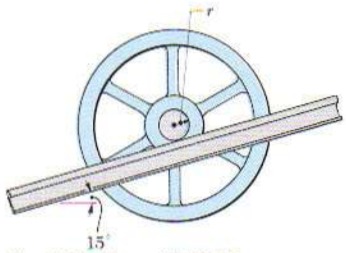


Fig. P16.93 and P16.94

16.94 A flywheel of centroidal radius of gyration $\bar{k} = 600$ mm is rigidly attached to a shaft of radius $r = 30$ mm which may roll along parallel rails. Knowing that the system is released from rest, determine the distance it will roll in 20 s.

16.95 through 16.98 A drum of 80-mm radius is attached to a disk of 160-mm radius. The disk and drum have a total mass of 5 kg and a radius of gyration of 120 mm. A cord is attached as shown and pulled with a force **P** of magnitude 20 N. Knowing that the disk rolls without sliding, determine (*a*) the angular acceleration of the disk and the acceleration of *G*, (*b*) the minimum value of the coefficient of friction compatible with this motion.

Fig. P16.95 and P16.99

Fig. P16.96 and P16.100

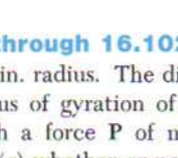


Fig. P16.97 and P16.101

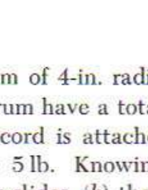


Fig. P16.98 and P16.102

16.99 through 16.102 A drum of 4-in. radius is attached to a disk of 8-in. radius. The disk and drum have a total weight of 10 lb and a radius of gyration of 6 in. A cord is attached as shown and pulled with a force **P** of magnitude 5 lb. Knowing that $\mu = 0.20$, determine (*a*) whether or not the disk slides, (*b*) the angular acceleration of the disk and the acceleration of *G*.

16.103 and 16.104 The 12-lb carriage is supported as shown by two uniform disks each of weight 8 lb and radius 3 in. Knowing that the disks roll without sliding, determine the acceleration of the carriage when a force of 4 lb is applied to it.

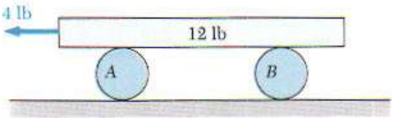


Fig. P16.103

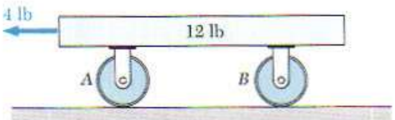


Fig. P16.104

16.105 A half section of pipe of mass m and radius r rests on a rough horizontal surface. A vertical force **P** is applied as shown. Assuming that the section rolls without sliding, derive an expression (*a*) for its angular acceleration, (*b*) for the minimum value of μ compatible with this motion. [*Hint.* Note that $OG = 2r/\pi$ and that, by the parallel-axis theorem, $\bar{I} = mr^2 - m(OG)^2$.]

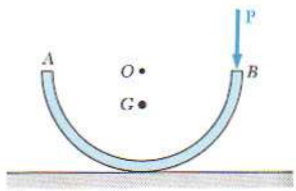


Fig. P16.105

16.106 A small block of mass m is attached at B to a hoop of mass m and radius r. Knowing that when the system is released from rest it starts to roll without sliding, determine (*a*) the angular acceleration of the hoop, (*b*) the acceleration of B.

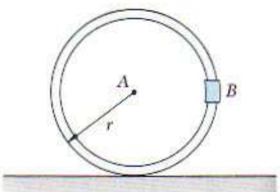


Fig. P16.106

16.107 Solve Prob. 16.105, assuming that the force **P** is applied at B and is directed horizontally to the right.

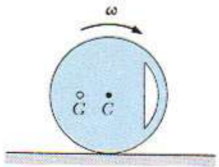


Fig. P16.108

16.108 The mass center G of a 10-lb wheel of radius $R = 12$ in. is located at a distance $r = 4$ in. from its geometric center C. The centroidal radius of gyration is $\bar{k} = 6$ in. As the wheel rolls without sliding, its angular velocity varies and it is observed that $\omega = 8$ rad/s in the position shown. Determine the corresponding angular acceleration of the wheel.

16.109 End A of the 100-lb beam AB moves along the frictionless floor, while end B is supported by a 4-ft cable. Knowing that at the instant shown end A is moving to the left with a constant velocity of 8 ft/s, determine (a) the magnitude of the force $\mathbf{P}$, (b) the corresponding tension in the cable.

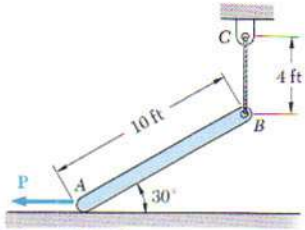


Fig. P16.109

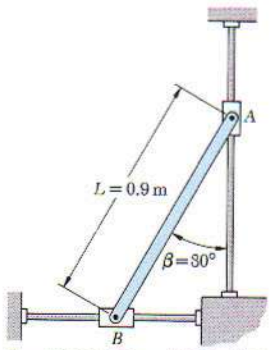


Fig. P16.110 and P16.112

16.110 Ends A and B of a 4-kg slender rod are attached to collars of negligible mass which slide without friction along the rods shown. A horizontal force $\mathbf{P}$ is applied to collar B, causing the rod to start from rest with a counterclockwise angular acceleration of 12 rad/s^2. Determine (a) the required magnitude of $\mathbf{P}$, (b) the reactions at A and B.

16.111 Solve Prob. 16.110, assuming that at the instant considered the angular velocity of the rod is 4 rad/s counterclockwise.

16.112 Ends A and B of a 4-kg slender rod are attached to collars of negligible mass which slide without friction along the rods shown. If the rod is released from rest in the position shown, determine (a) the angular acceleration of the rod, (b) the reactions at A and B.

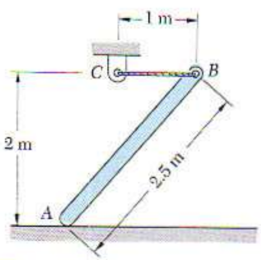


Fig. P16.113

16.113 The 50-kg uniform rod AB is released from rest in the position shown. Knowing that end A may slide freely on the frictionless floor, determine (a) the angular acceleration of the rod, (b) the tension in wire BC, (c) the reaction at A.

16.114 Rod AB weighs 3 lb and is released from rest in the position shown. Assuming that the ends of the rod slide without friction, determine (*a*) the angular acceleration of the rod, (*b*) the reactions at A and B.

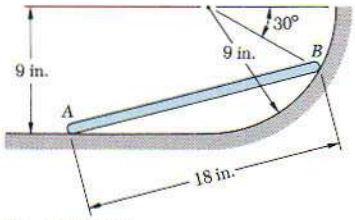


Fig. P16.114

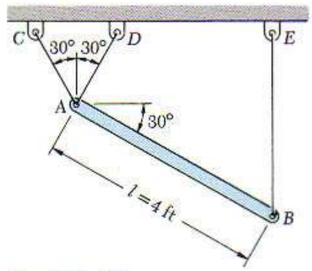


Fig. P16.115

16.115 The 12-lb uniform rod AB is held by the three wires shown. Determine the tension in wires AD and BE immediately after wire AC has been cut.

16.116 Show that, for a rigid slab in plane motion, the equation $\Sigma M_A = I_A\alpha$, where ΣM_A represents the sum of the moments of the external forces about point A and I_A the moment of inertia of the slab about the same point A, is verified *if and only if* one of the following conditions is satisfied: (*a*) A is the mass center of the slab, (*b*) A has zero acceleration, (*c*) the acceleration of A is directed along a line joining point A and the mass center G.

16.117 The 6-lb sliding block is connected to the rotating disk by the uniform rod AB which weighs 4 lb. Knowing that the disk has a constant angular velocity of 360 rpm, determine the forces exerted on the connecting rod at A and B when $\beta = 0$.

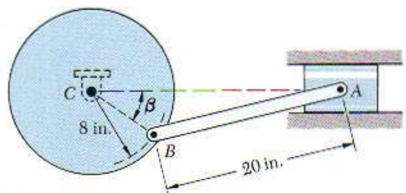


Fig. P16.117

16.118 Solve Prob. 16.117 when $\beta = 180°$.

16.119 Each of the bars shown is 600 mm long and has a mass of 4 kg. A horizontal and variable force $\mathbf{P}$ is applied at C, causing point C to move to the left with a constant speed of 10 m/s. Determine the force $\mathbf{P}$ for the position shown.

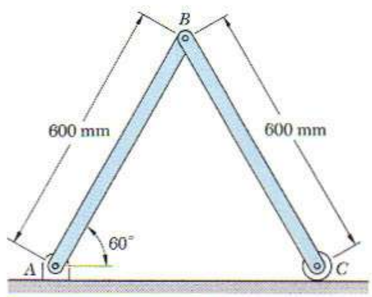


Fig. P16.119 and P16.120

16.120 The two bars AB and BC are released from rest in the position shown. Each bar is 600 mm long and has a mass of 4 kg. Determine (*a*) the angular acceleration of each bar, (*b*) the reactions at A and C.

16.121 and 16.122 Two rods AB and BC, of mass m per unit length, are connected as shown to a disk which is made to rotate in a vertical plane at a constant angular velocity ω_0. For the position shown, determine the components of the forces exerted at A and B on rod AB.

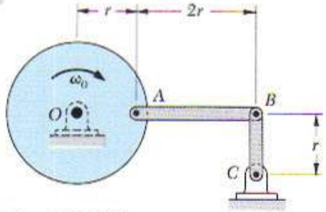


Fig. P16.121

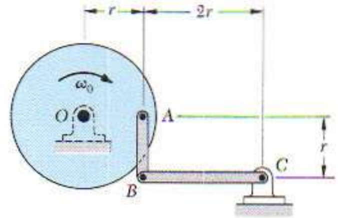


Fig. P16.122

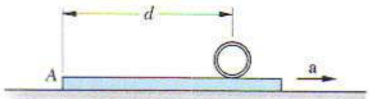


Fig. P16.123

16.123 A section of pipe rests on a plate. The plate is then given a constant acceleration **a** directed to the right. Assuming that the pipe rolls on the plate, determine (*a*) the acceleration of the pipe, (*b*) the distance through which the plate will move before the pipe reaches end A.

16.124 Solve Prob. 16.123, assuming that the pipe is replaced (1) by a solid cylinder, (2) by a sphere.

16.125 and 16.126 Gear C weighs 6 lb and has a centroidal radius of gyration of 3 in. The uniform bar AB weighs 5 lb and gear D is stationary. If the system is released from rest in the position shown, determine (*a*) the angular acceleration of gear C, (*b*) the acceleration of point B.

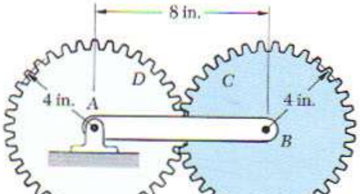


Fig. P16.125

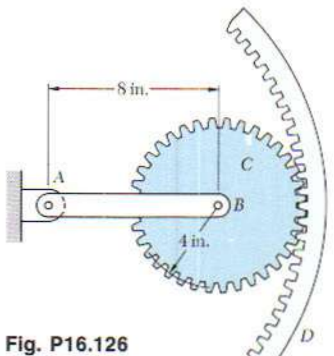


Fig. P16.126

***16.127** The disk shown rotates with a constant counterclockwise angular velocity of 12 rad/s. The uniform rod BD is 450 mm long and has a mass of 3 kg. Knowing that the system moves in a *horizontal* plane, determine the reaction at E.

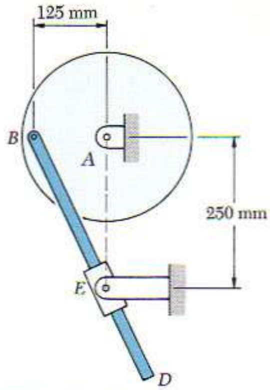


Fig. P16.127

***16.128** Solve Prob. 16.127, assuming that the disk rotates with a constant clockwise angular velocity of 12 rad/s.

***16.129** A uniform slender rod of length L and mass m is released from rest in the position shown. Derive an expression for (*a*) the angular acceleration of the rod, (*b*) the acceleration of end A, (*c*) the reaction at A, immediately after release. Neglect the mass and friction of the roller at A.

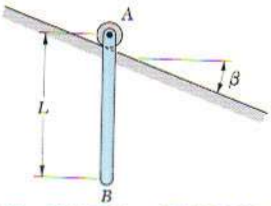


Fig. P16.129 and P16.130

***16.130** A uniform rod AB, of mass 3 kg and length $L = 1.2$ m, is released from rest in the position shown. Knowing that $\beta = 30°$, determine the values immediately after release of (*a*) the angular acceleration of the rod, (*b*) the acceleration of end A, (*c*) the reaction at A. Neglect the mass and friction of the roller at A.

***16.131** Each of the bars AB and BC is of length $L = 18$ in. and weight 3 lb. A couple $\mathbf{M}_0$ of magnitude 6 lb · ft is applied to bar BC. Determine the angular acceleration of each bar.

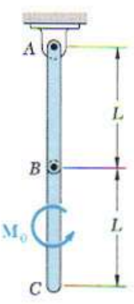


Fig. P16.131

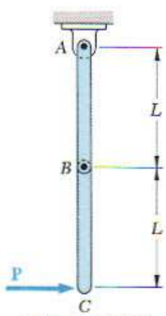


Fig. P16.132

***16.132** Each of the bars AB and BC is of length $L = 18$ in. and weight 3 lb. A horizontal force $\mathbf{P}$ of magnitude 4 lb is applied at C. Determine the angular acceleration of each bar.

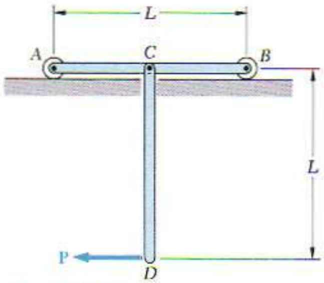


Fig. P16.133

∗16.133 Two uniform slender rods, each of mass m, are connected by a pin at C. Determine the acceleration of points C and D immediately after the horizontal force $\mathbf{P}$ has been applied at D.

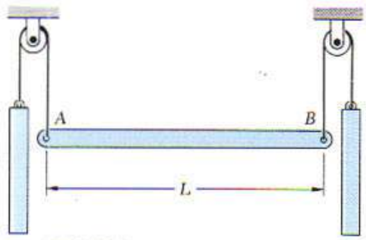


Fig. P16.134

∗16.134 The slender bar AB is of length L and mass m. It is held in equilibrium by two counterweights, each of mass $\frac{1}{2}m$. If the wire at B is cut, determine at that instant the acceleration of (*a*) point A, (*b*) point B.

∗16.135 (*a*) Determine the magnitude and the location of the maximum bending moment in the rod of Prob. 16.74. (*b*) Show that the answer to part *a* is independent of the weight W of the rod.

∗16.136 In Prob. 16.132 the pin at B is severely rusted and the bars rotate as a single rigid body. Determine the bending moment which occurs at B.

∗16.137 Draw the shear and bending-moment diagrams for the beam of Prob. 16.79 immediately after the cable at B breaks.

∗16.138 Draw the shear and bending-moment diagrams for the bar of Prob. 16.134 immediately after the wire at B has been cut.

REVIEW PROBLEMS

16.139 and 16.140 A uniform plate of mass m is suspended in each of the ways shown. For each case determine the acceleration of the center of the plate immediately after the connection at B has been released.

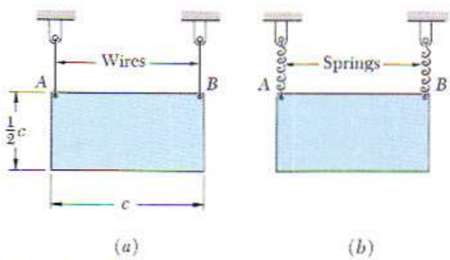


Fig. P16.139

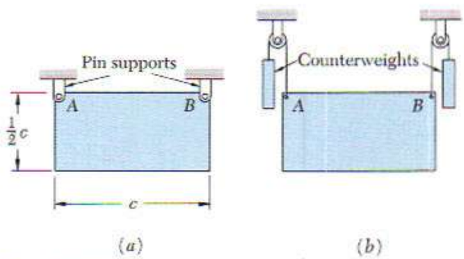


Fig. P16.140

16.141 The flanged wheel shown rolls to the right with a constant velocity of 1.5 m/s. The rod AB is 1.2 m long and has a mass of 5 kg. Knowing that point A slides without friction on the horizontal surface, determine the reaction at A (a) when $\beta = 0$, (b) when $\beta = 180°$.

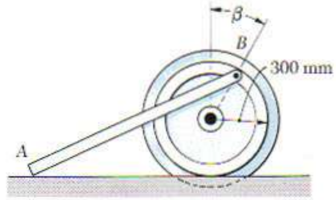


Fig. P16.141

16.142 A 15-lb uniform disk is suspended from a link AB of negligible weight. If a 10-lb force is applied at B, determine the acceleration of B (a) if the connection at B is a frictionless pin, (b) if the connection at B is "frozen" and the system rotates about A as a rigid body.

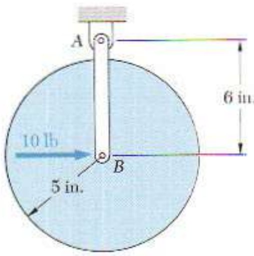


Fig. P16.142

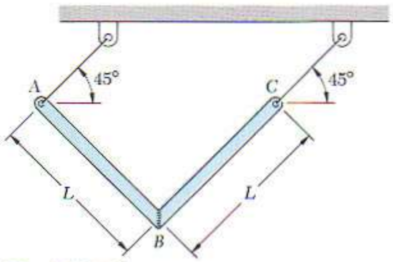


Fig. P16.143

16.143 Two uniform bars AB and BC, each of length $L = 10$ in., are welded together to form an L-shaped rigid body. Knowing that each bar weighs 3 lb, determine the tension in each wire immediately after the body has been released from rest.

16.144 A slender rod of mass m per unit length is placed inside a shallow drum of radius r which rotates at a constant angular velocity ω about a vertical shaft through O. (a) Determine the ratio L/r for which the maximum bending moment in the rod is as large as possible. (b) Derive an expression for the corresponding value of the maximum bending moment.

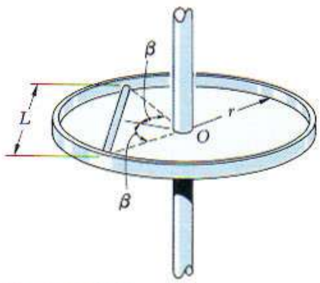


Fig. P16.144

16.145 A collar C of weight W_C is rigidly attached to a uniform slender rod AB of length L and weight W. If the rod is released from rest in the position shown, determine the ratio d/L for which the reaction at B is independent of W_C.

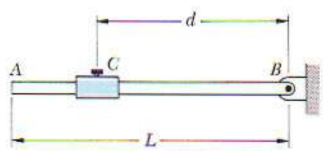


Fig. P16.145 and P16.146

16.146 A collar C of weight 2 lb is rigidly attached to a uniform slender rod AB of weight 12 lb and length $L = 20$ in. If the rod is released from rest in the position shown, determine the distance d for which the angular acceleration of the rod is maximum.

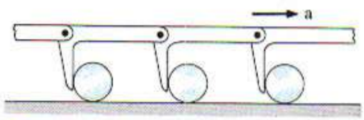


Fig. P16.147

16.147 Identical cylinders of mass m and radius r are pushed by a series of moving arms. Assuming the coefficient of friction between all surfaces to be $\mu < 1$, and denoting by a the magnitude of the acceleration of the arms, derive an expression for (*a*) the maximum allowable value of a if each cylinder is to roll without sliding, (*b*) the minimum allowable value of a if each cylinder is to move to the right without rotating.

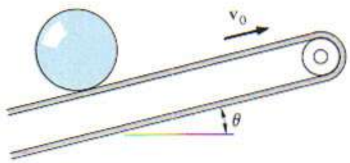


Fig. P16.148

16.148 A sphere of mass m and radius r is dropped with no initial velocity on a belt which moves with a constant velocity $\mathbf{v}_0$. At first the sphere will both rotate and slide on the belt. Denoting by μ the coefficient of friction between the sphere and the belt, determine the distance the sphere will move before it starts rolling without sliding.

16.149 A section of pipe, of mass 50 kg and radius 250 mm, rests on two corners as shown. Assuming that μ between the corners and the pipe is sufficient to prevent sliding, determine (*a*) the angular acceleration of the pipe just after corner B is removed, (*b*) the corresponding magnitude of the reaction at A.

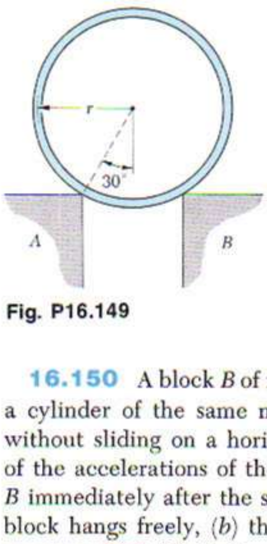


Fig. P16.149

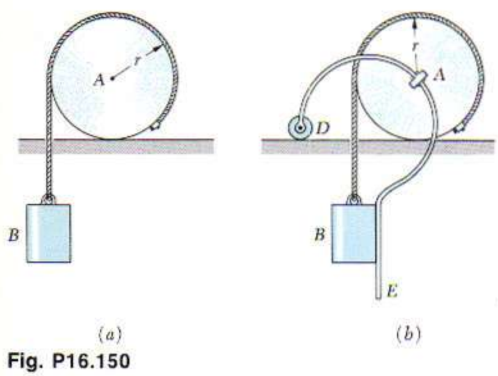


Fig. P16.150

16.150 A block B of mass m is attached to a cord wrapped around a cylinder of the same mass m and of radius r. The cylinder rolls without sliding on a horizontal surface. Determine the components of the accelerations of the center A of the cylinder and of the block B immediately after the system has been released from rest if (*a*) the block hangs freely, (*b*) the motion of the block is guided by a rigid member DAE, frictionless and of negligible mass, which is hinged to the cylinder at A.

Plane Motion of Rigid Bodies: Energy and Momentum Methods

CHAPTER

17

17.1 Principle of Work and Energy for a Rigid Body. In the first part of this chapter, the principle of work and energy will be used to analyze the plane motion of rigid bodies and of systems of rigid bodies. As was pointed out in Chap. 13, the method of work and energy is particularly well adapted to the solution of problems involving velocities and displacements. Its main advantage resides in the fact that the work of forces and the kinetic energy of particles are scalar quantities.

In order to apply the principle of work and energy to the analysis of the motion of a rigid body, we shall again assume that the rigid body is made of a large number n of particles of mass Δm_i. Recalling Eq. (14.30) of Sec. 14.7, we write

$$T_1 + U_{1\to 2} = T_2 \tag{17.1}$$

where T_1, T_2 = initial and final values of total kinetic energy of the particles forming the rigid body

$U_{1\to 2}$ = work of all forces acting on the various particles of the body

The total kinetic energy

$$T = \frac{1}{2}\sum_{i=1}^{n}(\Delta m_i)v_i^2 \tag{17.2}$$

is obtained by adding positive scalar quantities and is itself a positive scalar quantity. We shall see later how T may be determined for various types of motion of a rigid body.

The expression $U_{1\to2}$ in (17.1) represents the work of all the forces acting on the various particles of the body, whether these forces are internal or external. However, as we shall see presently, the total work of the internal forces holding together the particles of a rigid body is zero. Consider two particles A and B of a rigid body and the two equal and opposite forces $\mathbf{F}$ and $-\mathbf{F}$ they exert on each other (Fig. 17.1). While, in general, small displacements $d\mathbf{r}$ and $d\mathbf{r}'$ of the two particles are different, the components of these displacements along AB must be equal; otherwise, the particles would not remain at the same distance from each other, and the body would not be rigid. Therefore, the work of $\mathbf{F}$ is equal in magnitude and opposite in sign to the work of $-\mathbf{F}$, and their sum is zero. Thus, the total work of the internal forces acting on the particles of a rigid body is zero, and *the expression $U_{1\to2}$ in Eq. (17.1) reduces to the work of the external forces* acting on the body during the displacement considered.

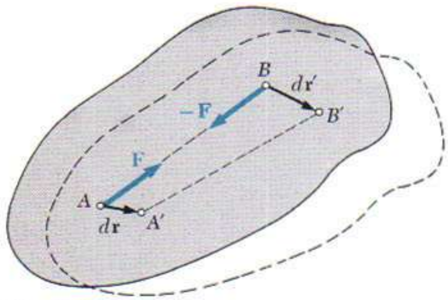


Fig. 17.1

17.2 Work of Forces Acting on a Rigid Body. We saw in Sec. 13.2 that the work of a force $\mathbf{F}$ during a displacement of its point of application from A_1 to A_2 is

$$U_{1\to2} = \int_{A_1}^{A_2} \mathbf{F}\cdot d\mathbf{r} \tag{17.3}$$

or

$$U_{1\to2} = \int_{s_1}^{s_2} (F\cos\alpha)\,ds \tag{17.3'}$$

where F is the magnitude of the force, α the angle it forms with the direction of motion of its point of application A, and s the variable of integration which measures the distance traveled by A along its path.

In computing the work of the external forces acting on a rigid body, it is often convenient to determine the work of a couple without considering separately the work of each of the two forces forming the couple. Consider the two forces $\mathbf{F}$ and $-\mathbf{F}$

forming a couple of moment $\mathbf{M}$ and acting on a rigid body (Fig. 17.2). Any small displacement of the rigid body bringing A and B, respectively, into A' and B'' may be divided into two parts, one in which points A and B undergo equal displacements $d\mathbf{r}_1$, the other in which A' remains fixed while B' moves into B'' through a displacement $d\mathbf{r}_2$ of magnitude $ds_2 = r\,d\theta$. In the first part of the motion, the work of $\mathbf{F}$ is equal in magnitude and opposite in sign to the work of $-\mathbf{F}$ and their sum is zero. In the second part of the motion, only force $\mathbf{F}$ works, and its work is $dU = F\,ds_2 = Fr\,d\theta$. But the product Fr is equal to the magnitude M of the moment of the couple. Thus, the work of a couple of moment $\mathbf{M}$ acting on a rigid body is

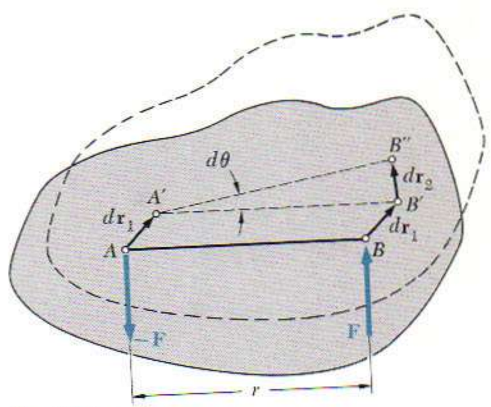


Fig. 17.2

$$dU = M\,d\theta \tag{17.4}$$

where $d\theta$ is the small angle expressed in radians through which the body rotates. We again note that work should be expressed in units obtained by multiplying units of force by units of length. The work of the couple during a finite rotation of the rigid body is obtained by integrating both members of (17.4) from the initial value θ_1 of the angle θ to its final value θ_2. We write

$$U_{1\to 2} = \int_{\theta_1}^{\theta_2} M\,d\theta \tag{17.5}$$

When the moment $\mathbf{M}$ *of the couple is constant,* formula (17.5) reduces to

$$U_{1\to 2} = M(\theta_2 - \theta_1) \tag{17.6}$$

It was pointed out in Sec. 13.2 that a number of forces encountered in problems of kinetics *do no work*. They are forces applied to fixed points or acting in a direction perpendicular to the displacement of their point of application. Among the forces which do no work the following have been listed: the reaction at a frictionless pin when the body supported rotates about the pin, the reaction at a frictionless surface when the body in contact moves along the surface, the weight of a body when its center of gravity moves horizontally. We should also indicate now that, *when a rigid body rolls without sliding on a fixed surface, the friction force* $\mathbf{F}$ *at the point of contact* C *does no work.* The velocity $\mathbf{v}_C$ of the point of contact C is zero, and the work of the friction force $\mathbf{F}$ during a small displacement of the rigid body is $dU = F\,ds_C = F(v_C\,dt) = 0$.

17.3 Kinetic Energy of a Rigid Body in Plane Motion. Consider a rigid body of mass m in plane motion. We recall from Sec. 14.6 that, if the absolute velocity $\mathbf{v}_i$ of each particle P_i of the body is expressed as the sum of the velocity $\overline{\mathbf{v}}$ of the mass center G of the body and of the velocity $\mathbf{v}'_i$ of the particle relative to a frame $Gx'y'$ attached to G and of fixed

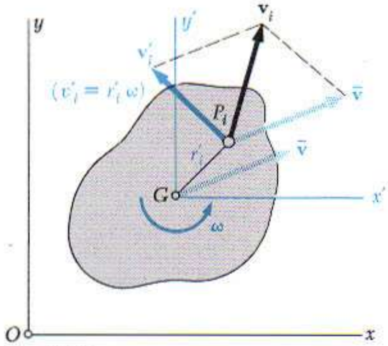


Fig. 17.3

orientation (Fig. 17.3), the kinetic energy of the system of particles forming the rigid body may be written in the form

$$T = \tfrac{1}{2}m\overline{v}^2 + \frac{1}{2}\sum_{i=1}^{n}(\Delta m_i)v_i'^2 \tag{17.7}$$

But the magnitude v'_i of the relative velocity of P_i is equal to the product $r'_i\omega$ of the distance r'_i of P_i from the axis through G perpendicular to the plane of motion and of the magnitude ω of the angular velocity of the body at the instant considered. Substituting into (17.7), we have

$$T = \tfrac{1}{2}m\overline{v}^2 + \frac{1}{2}\left(\sum_{i=1}^{n} r_i'^2\,\Delta m_i\right)\omega^2 \tag{17.8}$$

or, since the sum represents the moment of inertia $\overline{I}$ of the body about the axis through G,

$$T = \tfrac{1}{2}m\overline{v}^2 + \tfrac{1}{2}\overline{I}\omega^2 \tag{17.9}$$

We note that, in the particular case of a body in translation ($\omega = 0$), the expression obtained reduces to $\frac{1}{2}m\overline{v}^2$, while, in the case of a centroidal rotation ($\overline{v} = 0$), it reduces to $\frac{1}{2}\overline{I}\omega^2$. We conclude that the kinetic energy of a rigid body in plane motion may be separated into two parts: (1) the kinetic energy $\frac{1}{2}m\overline{v}^2$ associated with the motion of the mass center G of the body, and (2) the kinetic energy $\frac{1}{2}\overline{I}\omega^2$ associated with the rotation of the body about G.

Noncentroidal Rotation. The relation (17.9) is valid for any type of plane motion and may, therefore, be used to express the kinetic energy of a rigid body rotating with an angular velocity $\boldsymbol{\omega}$ about a fixed axis through O (Fig. 17.4). In that case, however, the kinetic energy of the body may be expressed more directly by noting that the speed v_i of the particle P_i is equal to the product $r_i\omega$ of the distance r_i of P_i from the fixed axis and of the magnitude ω of the angular velocity of the body at the instant considered. Substituting into (17.2), we write

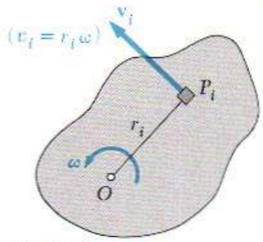


Fig. 17.4

$$T = \frac{1}{2}\sum_{i=1}^{n}(\Delta m_i)(r_i\omega)^2 = \frac{1}{2}\left(\sum_{i=1}^{n} r_i^2\,\Delta m_i\right)\omega^2$$

or, since the last sum represents the moment of inertia I_O of the body about the fixed axis through O,

$$T = \tfrac{1}{2}I_O\omega^2 \tag{17.10}$$

We note that the results obtained are not limited to the motion of plane slabs or to the motion of bodies which are symmetrical with respect to the reference plane. They may be applied to the study of the plane motion of any rigid body, regardless of its shape.

17.4 Systems of Rigid Bodies. When a problem involves several rigid bodies, each rigid body may be considered separately, and the principle of work and energy may be applied to each body. Adding the kinetic energies of all the particles and considering the work of all the forces involved, we may also write the equation of work and energy for the entire system. We have

$$T_1 + U_{1\to2} = T_2 \tag{17.11}$$

where T represents the arithmetic sum of the kinetic energies of the rigid bodies forming the system (all terms are positive) and $U_{1\to2}$ the work of all the forces acting on the various bodies, whether these forces are *internal* or *external* from the point of view of the system as a whole.

The method of work and energy is particularly useful in solving problems involving pin-connected members, or blocks and pulleys connected by inextensible cords, or meshed gears. In all these cases, the internal forces occur by pairs of equal and opposite forces, and the points of application of the forces in each pair *move through equal distances* during a small displacement of the system. As a result, the work of the internal forces is zero, and $U_{1\to2}$ reduces to the work of the *forces external to the system.*

17.5 Conservation of Energy. We saw in Sec. 13.6 that the work of conservative forces, such as the weight of a body or the force exerted by a spring, may be expressed as a change in potential energy. When a rigid body, or a system of rigid bodies, moves under the action of conservative forces, the principle of work and energy stated in Sec. 17.1 may be expressed in a modified form. Substituting for $U_{1 \rightarrow 2}$ from (13.19′) into (17.1), we write

$$T_1 + V_1 = T_2 + V_2 \tag{17.12}$$

Formula (17.12) indicates that, when a rigid body, or a system of rigid bodies, moves under the action of conservative forces, *the sum of the kinetic energy and of the potential energy of the system remains constant.* It should be noted that, in the case of the plane motion of a rigid body, the kinetic energy of the body should include both the *translational* term $\frac{1}{2}m\bar{v}^2$ and the *rotational* term $\frac{1}{2}\bar{I}\omega^2$.

As an example of application of the principle of conservation of energy, we shall consider a slender rod AB, of length l and mass m, whose extremities are connected to blocks of negligible mass sliding along horizontal and vertical tracks. We assume that the rod is released with no initial velocity from a horizontal position (Fig. 17.5*a*), and we wish to determine its angular velocity after it has rotated through an angle θ (Fig. 17.5*b*).

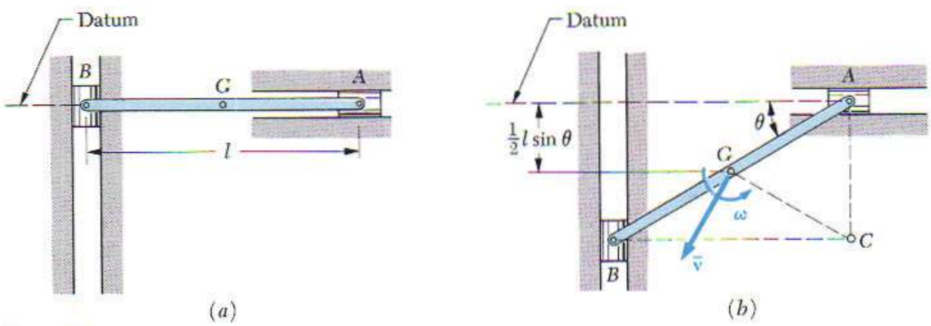


Fig. 17.5

Since the initial velocity is zero, we have $T_1 = 0$. Measuring the potential energy from the level of the horizontal track, we write $V_1 = 0$. After the rod has rotated through θ, the center of gravity G of the rod is at a distance $\frac{1}{2}l \sin\theta$ below the reference level and we have

$$V_2 = -\tfrac{1}{2}Wl\sin\theta = -\tfrac{1}{2}mgl\sin\theta$$

Observing that, in this position, the instantaneous center of the rod is located at C, and that $CG = \frac{1}{2}l$, we write $\bar{v}_2 = \frac{1}{2}l\omega$ and obtain

$$T_2 = \tfrac{1}{2}m\bar{v}_2^2 + \tfrac{1}{2}\bar{I}\omega_2^2 = \tfrac{1}{2}m(\tfrac{1}{2}l\omega)^2 + \tfrac{1}{2}(\tfrac{1}{12}ml^2)\omega^2$$
$$= \frac{1}{2}\frac{ml^2}{3}\omega^2$$

Applying the principle of conservation of energy, we write

$$T_1 + V_1 = T_2 + V_2$$
$$0 = \frac{1}{2}\frac{ml^2}{3}\omega^2 - \tfrac{1}{2}mgl\sin\theta$$
$$\omega = \left(\frac{3g}{l}\sin\theta\right)^{1/2}$$

We recall that the advantages of the method of work and energy, as well as its shortcomings, were indicated in Sec. 13.4. In this connection, we wish to mention that the method of work and energy must be supplemented by the application of D'Alembert's principle when reactions at fixed axles, at rollers, or at sliding blocks are to be determined. For example, in order to compute the reactions at the extremities A and B of the rod of Fig. 17.5b, a diagram should be drawn to express that the system of the external forces applied to the rod is equivalent to the vector $m\bar{\mathbf{a}}$ and the couple $\bar{I}\boldsymbol{\alpha}$. The angular velocity $\boldsymbol{\omega}$ of the rod, however, is determined by the method of work and energy before the equations of motion are solved for the reactions. The complete analysis of the motion of the rod and of the forces exerted on the rod requires, therefore, the combined use of the method of work and energy and of the principle of equivalence of the external and effective forces.

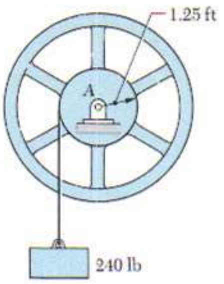
17.6. Power. *Power* was defined in Sec. 13.5 as the time rate at which work is done. In the case of a body acted upon by a force $\mathbf{F}$, and moving with a velocity $\mathbf{v}$, the power was expressed as follows:

$$\text{Power} = \frac{dU}{dt} = \mathbf{F}\cdot\mathbf{v} \qquad (13.13)$$

In the case of a rigid body rotating with an angular velocity $\boldsymbol{\omega}$ and acted upon by a couple of moment $\mathbf{M}$ parallel to the axis of rotation, we have, by (17.4),

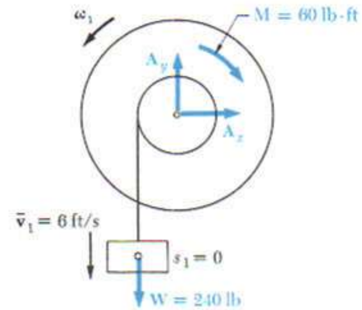
$$\text{Power} = \frac{dU}{dt} = \frac{M\,d\theta}{dt} = M\omega \qquad (17.13)$$

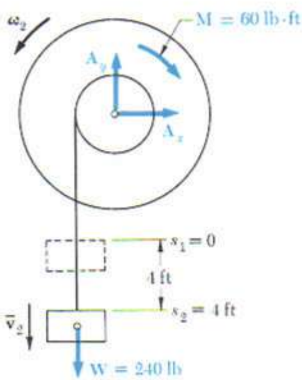
The various units used to measure power, such as the watt and the horsepower, were defined in Sec. 13.5.



SAMPLE PROBLEM 17.1

A 240-lb block is suspended from an inextensible cable which is wrapped around a drum of 1.25-ft radius rigidly attached to a flywheel. The drum and flywheel have a combined centroidal moment of inertia $\bar{I} = 10.5\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$. At the instant shown, the velocity of the block is 6 ft/s directed downward. Knowing that the bearing at A is poorly lubricated and that the bearing friction is equivalent to a couple $\mathbf{M}$ of magnitude 60 lb·ft, determine the velocity of the block after it has moved 4 ft downward.





Solution. We consider the system formed by the flywheel and the block. Since the cable is inextensible, the work done by the internal forces exerted by the cable cancels. The initial and final positions of the system and the external forces acting on the system are as shown.

Kinetic Energy. *Position 1.* We have

$$\bar{v}_1 = 6\ \text{ft/s} \qquad \omega_1 = \frac{\bar{v}_1}{r} = \frac{6\ \text{ft/s}}{1.25\ \text{ft}} = 4.80\ \text{rad/s}$$

$$\begin{aligned} T_1 &= \tfrac{1}{2}m\bar{v}_1^2 + \tfrac{1}{2}\bar{I}\omega_1^2 \\ &= \frac{1}{2}\frac{240\ \text{lb}}{32.2\ \text{ft/s}^2}(6\ \text{ft/s})^2 + \tfrac{1}{2}(10.5\ \text{lb}\cdot\text{ft}\cdot\text{s}^2)(4.80\ \text{rad/s})^2 \\ &= 255\ \text{ft}\cdot\text{lb} \end{aligned}$$

Position 2. Noting that $\omega_2 = \bar{v}_2/1.25$, we write

$$\begin{aligned} T_2 &= \tfrac{1}{2}m\bar{v}_2^2 + \tfrac{1}{2}\bar{I}\omega_2^2 \\ &= \frac{1}{2}\frac{240}{32.2}(\bar{v}_2)^2 + (\tfrac{1}{2})(10.5)\left(\frac{\bar{v}_2}{1.25}\right)^2 = 7.09\bar{v}_2^2 \end{aligned}$$

Work. During the motion, only the weight $\mathbf{W}$ of the block and the friction couple $\mathbf{M}$ do work. Noting that $\mathbf{W}$ does positive work and that the friction couple $\mathbf{M}$ does negative work, we write

$$s_1 = 0 \qquad s_2 = 4\ \text{ft}$$

$$\theta_1 = 0 \qquad \theta_2 = \frac{s_2}{r} = \frac{4\ \text{ft}}{1.25\ \text{ft}} = 3.20\ \text{rad}$$

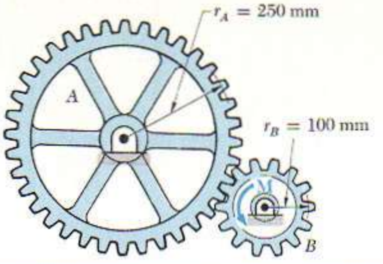
$$\begin{aligned} U_{1\to 2} &= W(s_2 - s_1) - M(\theta_2 - \theta_1) \\ &= (240\ \text{lb})(4\ \text{ft}) - (60\ \text{lb}\cdot\text{ft})(3.20\ \text{rad}) \\ &= 768\ \text{ft}\cdot\text{lb} \end{aligned}$$

Principle of Work and Energy

$$T_1 + U_{1\to 2} = T_2$$

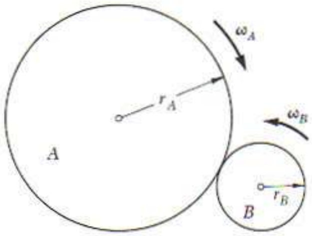
$$255\ \text{ft}\cdot\text{lb} + 768\ \text{ft}\cdot\text{lb} = 7.09\bar{v}_2^2$$

$$\bar{v}_2 = 12.01\ \text{ft/s} \qquad \bar{\mathbf{v}}_2 = 12.01\ \text{ft/s}\downarrow \blacktriangleleft$$



SAMPLE PROBLEM 17.2

Gear A has a mass of 10 kg and a radius of gyration of 200 mm, while gear B has a mass of 3 kg and a radius of gyration of 80 mm. The system is at rest when a couple $\mathbf{M}$ of magnitude 6 N·m is applied to gear B. Neglecting friction, determine (*a*) the number of revolutions executed by gear B before its angular velocity reaches 600 rpm, (*b*) the tangential force which gear B exerts on gear A.



Motion of Entire System. Noting that the peripheral speeds of the gears are equal, we write

$$r_A\omega_A = r_B\omega_B \qquad \omega_A = \omega_B\frac{r_B}{r_A} = \omega_B\frac{100 \text{ mm}}{250 \text{ mm}} = 0.40\omega_B$$

For $\omega_B = 600$ rpm, we have

$$\omega_B = 62.8 \text{ rad/s} \qquad \omega_A = 0.40\omega_B = 25.1 \text{ rad/s}$$
$$\bar{I}_A = m_A\bar{k}_A^2 = (10 \text{ kg})(0.200 \text{ m})^2 = 0.400 \text{ kg}\cdot\text{m}^2$$
$$\bar{I}_B = m_B\bar{k}_B^2 = (3 \text{ kg})(0.080 \text{ m})^2 = 0.0192 \text{ kg}\cdot\text{m}^2$$

Kinetic Energy. Since the system is initially at rest, $T_1 = 0$. Adding the kinetic energies of the two gears when $\omega_B = 600$ rpm, we obtain

$$\begin{aligned} T_2 &= \tfrac{1}{2}\bar{I}_A\omega_A^2 + \tfrac{1}{2}\bar{I}_B\omega_B^2 \\ &= \tfrac{1}{2}(0.400 \text{ kg}\cdot\text{m}^2)(25.1 \text{ rad/s})^2 + \tfrac{1}{2}(0.0192 \text{ kg}\cdot\text{m}^2)(62.8 \text{ rad/s})^2 \\ &= 163.9 \text{ J} \end{aligned}$$

Work. Denoting by θ_B the angular displacement of gear B, we have

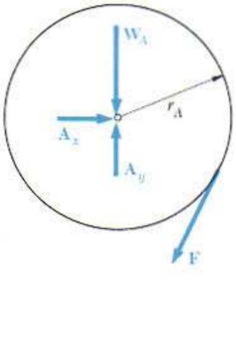
$$U_{1\to2} = M\theta_B = (6 \text{ N}\cdot\text{m})(\theta_B \text{ rad}) = (6\,\theta_B) \text{ J}$$

Principle of Work and Energy

$$T_1 + U_{1\to2} = T_2: \qquad 0 + (6\,\theta_B) \text{ J} = 163.9 \text{ J}$$
$$\theta_B = 27.32 \text{ rad} \qquad \theta_B = 4.35 \text{ rev} \blacktriangleleft$$

Motion of Gear A. *Kinetic Energy.* Initially, gear A is at rest, $T_1 = 0$. When $\omega_B = 600$ rpm, the kinetic energy of gear A is

$$T_2 = \tfrac{1}{2}\bar{I}_A\omega_A^2 = \tfrac{1}{2}(0.400 \text{ kg}\cdot\text{m}^2)(25.1 \text{ rad/s})^2 = 126.0 \text{ J}$$



Work. The forces acting on gear A are as shown. The tangential force $\mathbf{F}$ does work equal to the product of its magnitude and of the length $\theta_A r_A$ of the arc described by the point of contact. Since $\theta_A r_A = \theta_B r_B$, we have

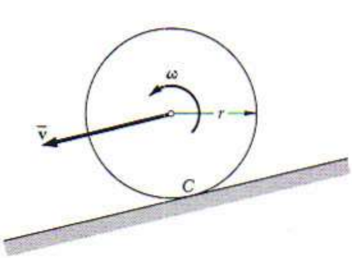
$$U_{1\to2} = F(\theta_B r_B) = F(27.3 \text{ rad})(0.100 \text{ m}) = F(2.73 \text{ m})$$

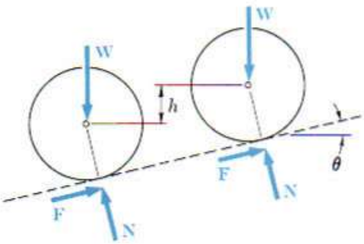
Principle of Work and Energy

$$T_1 + U_{1\to2} = T_2: \qquad 0 + F(2.73 \text{ m}) = 126.0 \text{ J}$$
$$F = +46.1 \text{ N} \qquad \mathbf{F} = 46.1 \text{ N} \swarrow \blacktriangleleft$$

SAMPLE PROBLEM 17.3

A sphere, a cylinder, and a hoop, each having the same mass and the same radius, are released from rest on an incline. Determine the velocity of each body after it has rolled through a distance corresponding to a change in elevation h.





Solution. We shall first solve the problem in general terms and then find particular results for each body. We denote the mass by m, the centroidal moment of inertia by $\bar{I}$, the weight by W, and the radius by r.

Since each body rolls, the instantaneous center of rotation is located at C and we write

$$\omega = \frac{\bar{v}}{r}$$

Kinetic Energy

$$T_1 = 0$$

$$T_2 = \tfrac{1}{2}m\bar{v}^2 + \tfrac{1}{2}\bar{I}\omega^2$$

$$= \tfrac{1}{2}m\bar{v}^2 + \tfrac{1}{2}\bar{I}\left(\frac{\bar{v}}{r}\right)^2 = \tfrac{1}{2}\left(m + \frac{\bar{I}}{r^2}\right)\bar{v}^2$$

Work. Since the friction force **F** in rolling motion does no work,

$$U_{1\to2} = Wh$$

Principle of Work and Energy

$$T_1 + U_{1\to2} = T_2$$

$$0 + Wh = \tfrac{1}{2}\left(m + \frac{\bar{I}}{r^2}\right)\bar{v}^2 \qquad \bar{v}^2 = \frac{2Wh}{m + \bar{I}/r^2}$$

Noting that $W = mg$, we rearrange the result and obtain

$$\bar{v}^2 = \frac{2gh}{1 + \bar{I}/mr^2}$$

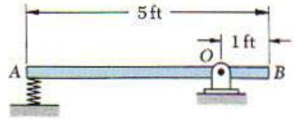
Velocities of Sphere, Cylinder, and Hoop. Introducing successively the particular expressions for $\bar{I}$, we obtain

Sphere:	$\bar{I} = \frac{2}{5}mr^2$	$\bar{v} = 0.845\sqrt{2gh}$ ◄
Cylinder:	$\bar{I} = \frac{1}{2}mr^2$	$\bar{v} = 0.816\sqrt{2gh}$ ◄
Hoop:	$\bar{I} = mr^2$	$\bar{v} = 0.707\sqrt{2gh}$ ◄

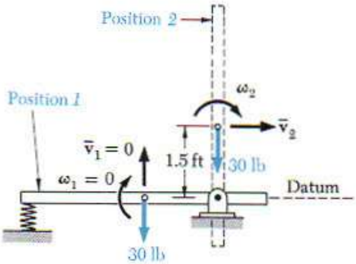
Remark. We may compare the results with the velocity attained by a frictionless block sliding through the same distance. The solution is identical to the above solution except that $\omega = 0$; we find $\bar{v} = \sqrt{2gh}$.

Comparing the results, we note that the velocity of the body is independent of both its mass and radius. However, the velocity does depend upon the quotient $\bar{I}/mr^2 = \bar{k}^2/r^2$, which measures the ratio of the rotational kinetic energy to the translational kinetic energy. Thus the hoop, which has the largest $\bar{k}$ for a given radius r, attains the smallest velocity, while the sliding block, which does not rotate, attains the largest velocity.

SAMPLE PROBLEM 17.4



A 30-lb slender rod AB is 5 ft long and is pivoted about a point O which is 1 ft from end B. The other end is pressed against a spring of constant $k = 1800$ lb/in. until the spring is compressed 1 in. The rod is then in a horizontal position. If the rod is released from this position, determine its angular velocity and the reaction at the pivot O as the rod passes through a vertical position.



Position 1. ***Potential Energy.*** Since the spring is compressed 1 in., we have $x_1 = 1$ in.

$$V_e = \tfrac{1}{2}kx_1^2 = \tfrac{1}{2}(1800 \text{ lb/in.})(1 \text{ in.})^2 = 900 \text{ in} \cdot \text{lb}$$

Choosing the datum as shown, we have $V_g = 0$; therefore,

$$V_1 = V_e + V_g = 900 \text{ in} \cdot \text{lb} = 75 \text{ ft} \cdot \text{lb}$$

Kinetic Energy. Since the velocity in position *1* is zero, we have $T_1 = 0$.

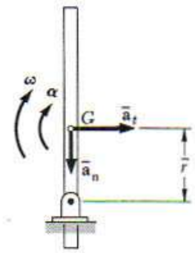
Position 2. ***Potential Energy.*** The elongation of the spring is zero, and we have $V_e = 0$. Since the center of gravity of the rod is now 1.5 ft above the datum,

$$V_g = (30 \text{ lb})(+1.5 \text{ ft}) = 45 \text{ ft} \cdot \text{lb}$$
$$V_2 = V_e + V_g = 45 \text{ ft} \cdot \text{lb}$$

Kinetic Energy. Denoting by $\boldsymbol{\omega}_2$ the angular velocity of the rod in position *2*, we note that the rod rotates about O and write $\bar{v}_2 = \bar{r}\omega_2 = 1.5\omega_2$.

$$\bar{I} = \tfrac{1}{12}ml^2 = \frac{1}{12}\frac{30 \text{ lb}}{32.2 \text{ ft/s}^2}(5 \text{ ft})^2 = 1.941 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

$$T_2 = \tfrac{1}{2}m\bar{v}_2^2 + \tfrac{1}{2}\bar{I}\omega_2^2 = \frac{1}{2}\frac{30}{32.2}(1.5\omega_2)^2 + \tfrac{1}{2}(1.941)\omega_2^2 = 2.019\omega_2^2$$



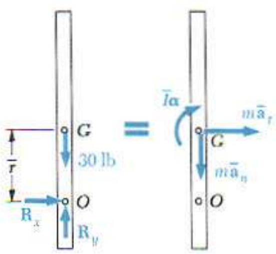
Conservation of Energy

$$T_1 + V_1 = T_2 + V_2: \qquad 0 + 75 \text{ ft} \cdot \text{lb} = 2.019\omega_2^2 + 45 \text{ ft} \cdot \text{lb}$$

$$\omega_2 = 3.86 \text{ rad/s} \curvearrowright \blacktriangleleft$$

Reaction in Position 2. Since $\omega_2 = 3.86$ rad/s, the components of the acceleration of G as the rod passes through position *2* are

$$\bar{a}_n = \bar{r}\omega_2^2 = (1.5 \text{ ft})(3.86 \text{ rad/s})^2 = 22.3 \text{ ft/s}^2 \qquad \bar{\mathbf{a}}_n = 22.3 \text{ ft/s}^2 \downarrow$$
$$\bar{a}_t = \bar{r}\alpha \qquad \bar{\mathbf{a}}_t = \bar{r}\alpha \rightarrow$$



We express that the system of external forces is equivalent to the system of effective forces represented by the vector of components $m\bar{\mathbf{a}}_t$ and $m\bar{\mathbf{a}}_n$ attached at G and the couple $\bar{I}\boldsymbol{\alpha}$.

$$+\curvearrowright \Sigma M_O = \Sigma(M_O)_{\text{eff}}: \qquad 0 = \bar{I}\alpha + m(\bar{r}\alpha)\bar{r} \qquad \alpha = 0$$
$$\xrightarrow{+} \Sigma F_x = \Sigma(F_x)_{\text{eff}}: \qquad R_x = m(\bar{r}\alpha) \qquad R_x = 0$$
$$+\uparrow \Sigma F_y = \Sigma(F_y)_{\text{eff}}: \qquad R_y - 30 \text{ lb} = -m\bar{a}_n$$

$$R_y - 30 \text{ lb} = -\frac{30 \text{ lb}}{32.2 \text{ ft/s}^2}(22.3 \text{ ft/s}^2)$$

$$R_y = +9.22 \text{ lb} \qquad \mathbf{R} = 9.22 \text{ lb} \uparrow \blacktriangleleft$$