

M. Nelkon & P. Parker

Advanced Level Physics

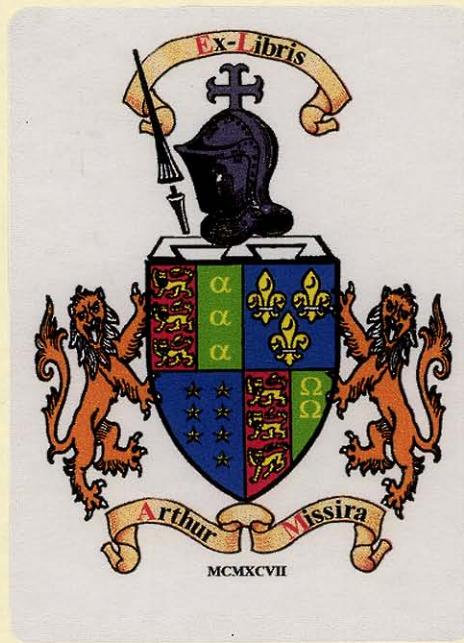
Third Edition With SI Units

M. Nelkon & P. Parker

Advanced Level Physics

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Advanced Level Physics

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Advanced Level Physics

Third edition with SI units

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Preface to Third Edition

This edition covers the new syllabus of the examining boards and is written in SI units to conform to their use in all future Advanced level examinations.

The main change in the text has been in the sections in Electricity on magnetic and electric fields and their associated phenomena. In the treatment, (i) magnetic flux density or induction **B** and electric intensity **E** have been used in preference to **H** and **D**—this follows the recommendations of the 1966 report of the Association for Science Education, (ii) magnetic flux density has been defined from the relation $F = BIl$. A new section on electromagnetic waves has been added.

Other changes are as follows: *Waves*. This has now been treated generally. *Optics*. The sections on interference and diffraction have been expanded. *Sound*. An account of recording on tape and film has been added. *Heat*. The joule has been used as the unit of heat and van der Waals' equation has been discussed. *Properties of Matter*. The repulsive and attractive forces between molecules have been emphasised. *Mechanics*. There are further discussions on angular momentum and the dynamics of a rigid body. Throughout the text, worked examples in SI units have been given to assist the student and the exercises at the end of chapters contain recent questions from examining boards. It is hoped that this SI edition will continue to assist students of Advanced level standard.

We are very much indebted to M. V. Detheridge, Woodhouse Grammar School, London, for his valuable co-operation in the writing and the preparation of the new electricity sections and for his generous assistance with this edition. We are also grateful to Rev. M. D. Phillips, O.S.B., Ampleforth College, York; S. S. Alexander, Woodhouse Grammar School, London; C. A. Boyle, William Ellis School, London; S. Freake, Queen's College, Cambridge, and R. P. T. Hills, St. John's College, Cambridge, for reading parts of the work; and to Prof. M. L. McGlashan, Exeter University, and M. Sayer, Chetham's Hospital School, Manchester, for advice on SI units.

I am grateful to the following for permission to include photographs in this book. To the Head of the Physics Department, the City University, London, for Newton rings, Fresnel biprism interference bands, Diffraction rings and Diffraction bands; to the late Sir J. J. Thomson for Positive Rays photographs; to the National Chemical Laboratory, for X-Ray diffraction rings; to Lord Blackett of the Imperial College of Science and Technology, for Transmutation of Nitrogen; to Professor George Thomson and the Science Museum, for

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Thanks are due to the following Examining Boards for their kind permission to translate numerical quantities in past questions to SI units; the translation is the sole responsibility of the author:

London University School Examinations (*L.*),
Oxford and Cambridge Schools Examination Board (*O. & C.*),
Joint Matriculation Board (*N.*),
Cambridge Local Examinations Syndicate (*C.*),
Oxford Delegacy of Local Examinations (*O.*).

1970

M.N.

Preface to First Edition

This text-book is designed for Advanced Level students of Physics, and covers Mechanics and Properties of Matter, Heat, Optics, and Sound. Electricity and Atomic Physics to that standard. It is based on the experience gained over many years of teaching and lecturing to a wide variety of students in schools and polytechnics.

In the treatment, an Ordinary Level knowledge of the subject is assumed. We have aimed at presenting the physical aspect of topics as much as possible, and then at providing the mathematical arguments and formulae necessary for a thorough understanding. Historical details have also been given to provide a balanced perspective of the subject. As a help to the student, numerous worked examples from past examination papers have been included in the text.

It is possible here to mention only a few points borne in mind by the authors. In Mechanics and Properties of Matter, the theory of dimensions has been utilized where the mathematics is difficult, as in the subject of viscosity, and the "excess pressure" formula has been extensively used in the treatment of surface tension. In Heat, the kinetic theory of gases has been fully discussed, and the experiments of Joule and Andrews have been presented in detail. The constant value of $n \sin i$ has been emphasized in refraction at plane surfaces in Optics, there is a full treatment of optical instruments, and accounts of interference, diffraction and polarization. In Sound, the physical principles of stationary waves, and their applications to pipes and strings, have been given prominence. Finally, in Electricity the electron and ion have been used extensively to produce explanations of phenomena in electrostatics, electromagnetism, electrolysis and atomic physics; the concept of e.m.f. has been linked at the outset with energy; and there are accounts of measurements and instruments.

We acknowledge our gratitude to the following for their kindness in reading sections of the work before the complete volume was compiled: Mr. J. H. Avery, Stockport Grammar School; Dr. J. Duffey, formerly of Watford Technical College; Mr. J. Newton, The City University, London; Mr. A. W. K. Ingram, Lawrence Sheriff School, Rugby; Mr. O. C. Gay, College of Technology, Hull; Mr. T. N. Littledale, Gunnersbury Grammar School; Mr. C. R. Ensor, Downside School, Bath; Mr. L. S. Powell, Garnett College, London; Dr. D. W. Stops, The City University, London; and Professor H. T. Flint, formerly of London University.

Preface to Second Edition

In this edition I have added an introduction to Atomic Structure, which covers the Advanced level syllabus on this topic. I am particularly indebted to Mr. J. Yarwood, M.Sc., F.Inst.P., head of the physics and mathematics department, Regent Street Polytechnic, London, for reading this section and for valuable advice, and to Prof. L. Pincherle, Bedford College, London University, for his kind assistance in parts of the text.

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PART ONE

Mechanics and Properties of Matter

chapter one

Dynamics

Motion in a Straight Line. Velocity

If a car travels steadily in a constant direction and covers a distance s in a time t , then its *velocity* in that direction = s/t . If the car does not travel steadily, then s/t is its average velocity, and

$$\text{distance } s = \text{average velocity} \times t.$$

We are here concerned with motion in a constant direction. The term 'displacement' is given to the distance moved in a constant direction, for example, from L to C in Fig. 1.1 (i). Velocity may therefore be defined as the *rate of change of displacement*.

Velocity can be expressed in *centimetres per second* (cm/s or cm s^{-1}) or *metres per second* (m/s or m s^{-1}) or *kilometres per hour* (km/h or km h^{-1}). By calculations, $36 \text{ km h}^{-1} = 10 \text{ m s}^{-1}$. It should be noted that complete information is provided for a velocity by stating its direction in addition to its magnitude, as explained shortly.

If an object moving in a straight line travels equal distances in equal times, no matter how small these distances may be, the object is said to be moving with *uniform* velocity. The velocity of a falling stone increases continuously, and so is a *non-uniform* velocity.

If, at any point of a journey, Δs is the small change in displacement in a small time Δt , the velocity v is given by $v = \Delta s/\Delta t$. In the limit, using calculus notation,

$$v = \frac{ds}{dt}.$$

Vectors

Displacement and *velocity* are examples of a class of quantities called *vectors* which have both magnitude and direction. They may therefore be represented to scale by a line drawn in a particular direction. Thus

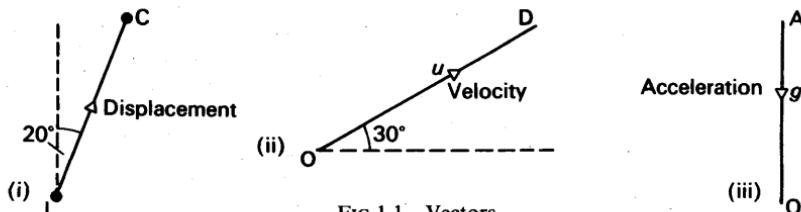


FIG 1.1 Vectors

Cambridge is 80 km from London in a direction 20° E. of N. We can therefore represent the displacement between the cities in magnitude

and direction by a straight line LC 4 cm long 20° E. of N., where 1 cm represents 20 km, Fig. 1.1 (i). Similarly, we can represent the velocity u of a ball initially thrown at an angle of 30° to the horizontal by a straight line OD drawn to scale in the direction of the velocity u , the arrow on the line showing the direction, Fig. 1.1 (ii). The acceleration due to gravity, g , is always represented by a straight line AO to scale drawn vertically downwards, since this is the direction of the acceleration, Fig. 1.1 (iii). We shall see later that 'force' and 'momentum' are other examples of vectors.

Speed and Velocity

A car moving along a winding road or a circular track at 80 km h^{-1} is said to have a *speed* of 80 km h^{-1} . 'Speed' is a quantity which has no direction but only magnitude, like 'mass' or 'density' or 'temperature'. These quantities are called *scalars*.

The distinction between speed and velocity can be made clear by reference to a car moving round a circular track at 80 km h^{-1} say, Fig. 1.2. At every point on the track the *speed* is the same—it is 80 km h^{-1} .

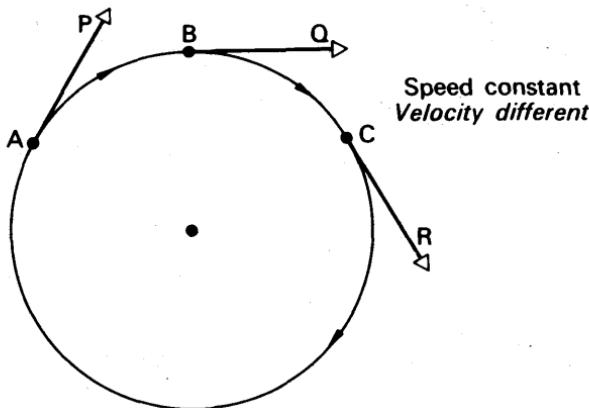


FIG. 1.2. Velocity and speed

At every point, however, the *velocity* is different. At A, B or C, for example, the velocity is in the direction of the particular tangent, AP, BQ or CR, so that even though the magnitudes are the same, the three velocities are all different because they point in different directions. Generally, vector quantities can be represented by a line drawn in the direction of the vector and whose length represents its magnitude.

Distance-Time Curve

When the displacement, or distance, s of a moving car from some fixed point is plotted against the time t , a *distance-time* ($s-t$) curve of

the motion is obtained. The velocity of the car at any instant is given by the change in distance per second at that instant. At E, for example, if the change in distance s is Δs and this change is made in a time Δt ,

$$\text{velocity at E} = \frac{\Delta s}{\Delta t}$$

In the limit, then, when Δt approaches zero, the velocity at E becomes equal to the *gradient of the tangent to the curve at E*. Using calculus notation, $\Delta s/\Delta t$ then becomes equal to ds/dt (p. 1).

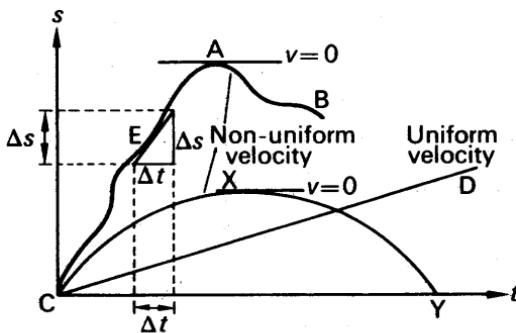


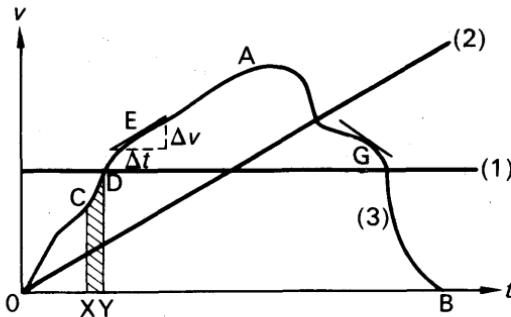
FIG. 1.3 Displacement (s)–time (t) curves

If the distance-time curve is a straight line CD, the gradient is constant at all points; it therefore follows that the car is moving with a *uniform* velocity, Fig. 1.3. If the distance-time curve is a curve CAB, the gradient varies at different points. The car then moves with non-uniform velocity. We may deduce that the velocity is zero at the instant corresponding to A, since the gradient at A to the curve CAB is zero.

When a ball is thrown upwards, the height s reached at any instant t is given by $s = ut - \frac{1}{2}gt^2$, where u is the initial velocity and g is the constant equal to the acceleration due to gravity (p. 8). The graph of s against t is represented by the parabolic curve CX \bar{Y} in Fig. 1.3; the gradient at X is zero, illustrating that the velocity of the ball at its maximum height is zero.

Velocity-Time Curves

When the velocity of a moving train is plotted against the time, a ‘velocity-time (v - t) curve’ is obtained. Useful information can be deduced from this curve, as we shall see shortly. If the velocity is uniform, the velocity-time graph is a straight line parallel to the time-axis, as shown by line (1) in Fig. 1.4. If the train accelerates uniformly from rest, the velocity-time graph is a straight line, line (2), inclined to the time-axis. If the acceleration is not uniform, the velocity-time graph is curved.

FIG. 1.4 Velocity (v)–time (t) curves

In Fig. 1.4, the velocity-time graph OAB represents the velocity of a train starting from rest which reaches a maximum velocity at A, and then comes to rest at the time corresponding to B; the acceleration and retardation are both not uniform in this case.

Acceleration is the ‘rate of change of velocity’, i.e. the change of velocity per second. *The acceleration of the train at any instant is given by the gradient to the velocity-time graph* at that instant, as at E. At the peak point A of the curve OAB the gradient is zero, i.e., the acceleration is then zero. At any point, such as G, between A, B the gradient to the curve is negative, i.e., the train undergoes retardation.

The gradient to the curve at any point such as E is given by :

$$\frac{\text{velocity change}}{\text{time}} = \frac{\Delta v}{\Delta t}$$

where Δv represents a small change in v in a small time Δt . In the limit, the ratio $\Delta v/\Delta t$ becomes dv/dt , using calculus notation.

Area Between Velocity-Time Graph and Time-Axis

Consider again the velocity-time graph OAB, and suppose the velocity increases in a very small time-interval XY from a value represented by XC to a value represented by YD, Fig. 1.4. Since the small distance travelled = average velocity \times time XY, the distance travelled is represented by the *area* between the curve CD and the time-axis, shown shaded in Fig. 1.4. By considering every small time-interval between OB in the same way, it follows that *the total distance travelled by the train in the time OB is given by the area between the velocity-time graph and the time-axis*. This result applies to any velocity-time graph, whatever its shape.

Fig. 1.5 illustrates the velocity-time graph AB of an object moving with uniform acceleration a from an initial velocity u . From above, the distance s travelled in a time t or OC is equivalent to the area OABC. The area OADC = $u \cdot t$. The area of the triangle ABD =

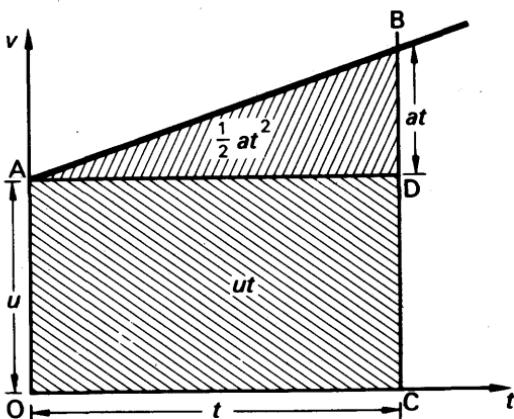


FIG. 1.5 Uniform acceleration

$\frac{1}{2}AD \cdot BD = \frac{1}{2}t \cdot BD$. Now BD = the increase in velocity in a time t = at . Hence area of triangle ABD = $\frac{1}{2}t \cdot at = \frac{1}{2}at^2$

$$\therefore \text{total area } OABC = s = ut + \frac{1}{2}at^2.$$

This result is also deduced on p. 6.

Acceleration

The *acceleration* of a moving object at an instant is the *rate of change of its velocity* at that instant. In the case of a train accelerating steadily from 36 km h^{-1} (10 m s^{-1}) to 54 km h^{-1} (15 m s^{-1}) in 10 second, the uniform acceleration

$$= (54 - 36) \text{ km h}^{-1} \div 10 \text{ seconds} = 1.8 \text{ km h}^{-1} \text{ per second,}$$

or

$$(15 - 10) \text{ m s}^{-1} \div 10 \text{ seconds} = 0.5 \text{ m s}^{-1} \text{ per second.}$$

Since the time element (second) is repeated twice in the latter case, the acceleration is usually given as 0.5 m s^{-2} . Another unit of acceleration is ' cm s^{-2} '. In terms of the calculus, the acceleration a of a moving object is given by

$$a = \frac{dv}{dt}$$

where dv/dt is the velocity change per second.

Distance Travelled with Uniform Acceleration. Equations of Motion

If the velocity changes by equal amounts in equal times, no matter how small the time-intervals may be, the acceleration is said to be *uniform*. Suppose that the velocity of an object moving in a straight

line with uniform acceleration a increases from a value u to a value v in a time t . Then, from the definition of acceleration,

$$a = \frac{v-u}{t},$$

from which

$$v = u + at \quad \quad (1)$$

Suppose an object with a velocity u accelerates with a uniform acceleration a for a time t and attains a velocity v . The distance s travelled by the object in the time t is given by

$$s = \text{average velocity} \times t$$

$$= \frac{1}{2}(u+v) \times t$$

But

$$v = u + at$$

$$\therefore s = \frac{1}{2}(u + u + at)t$$

$$\therefore s = ut + \frac{1}{2}at^2 \quad \quad (2)$$

If we eliminate t by substituting $t = (v-u)/a$ from (1) in (2), we obtain, on simplifying,

$$v^2 = u^2 + 2as \quad \quad (3)$$

Equations (1), (2), (3) are the equations of motion of an object moving in a straight line with uniform acceleration. When an object undergoes a uniform *retardation*, for example when brakes are applied to a car, a has a *negative* value.

EXAMPLES

1. A car moving with a velocity of 54 km h^{-1} accelerates uniformly at the rate of 2 m s^{-2} . Calculate the distance travelled from the place where acceleration began to that where the velocity reaches 72 km h^{-1} , and the time taken to cover this distance.

(i) $54 \text{ km h}^{-1} = 15 \text{ m s}^{-1}$, $72 \text{ km h}^{-1} = 20 \text{ m s}^{-1}$, acceleration $a = 2 \text{ m s}^{-2}$.

Using

$$v^2 = u^2 + 2as,$$

$$\therefore 20^2 = 15^2 + 2 \times 2 \times s$$

$$\therefore s = \frac{20^2 - 15^2}{2 \times 2} = 43\frac{3}{4} \text{ m.}$$

(ii) Using

$$v = u + at$$

$$\therefore 20 = 15 + 2t$$

$$\therefore t = \frac{20 - 15}{2} = 2.5 \text{ s.}$$

2. A train travelling at 72 km h^{-1} undergoes a uniform retardation of 2 m s^{-2} when brakes are applied. Find the time taken to come to rest and the distance travelled from the place where the brakes were applied.

$$(i) 72 \text{ km h}^{-1} = 20 \text{ m s}^{-1}, \text{ and } a = -2 \text{ m s}^{-2}, v = 0.$$

Using

$$v = u + at$$

$$\therefore 0 = 20 - 2t$$

$$\therefore t = 10 \text{ s}$$

$$(ii) \text{ The distance, } s = ut + \frac{1}{2}at^2.$$

$$= 20 \times 10 - \frac{1}{2} \times 2 \times 10^2 = 100 \text{ m.}$$

Motion Under Gravity

When an object falls to the ground under the action of gravity, experiment shows that the object has a constant or uniform acceleration of about 980 cm s^{-2} , while it is falling (see p. 49). In SI units this is 9.8 m s^{-2} or 10 m s^{-2} approximately. The numerical value of this acceleration is usually denoted by the symbol g . Suppose that an object is dropped from a height of 20 m above the ground. Then the initial velocity $u = 0$, and the acceleration $a = g = 10 \text{ m s}^{-2}$ (approx). Substituting in $s = ut + \frac{1}{2}at^2$, the distance fallen s in metres is calculated from

$$s = \frac{1}{2}gt^2 = 5t^2.$$

When the object reaches the ground, $s = 20 \text{ m}$.

$$\therefore 20 = 5t^2, \text{ or } t = 2 \text{ s}$$

Thus the object takes 2 seconds to reach the ground.

If a cricket-ball is thrown vertically upwards, it slows down owing to the attraction of the earth. The ball is thus retarded. The magnitude of the retardation is 9.8 m s^{-2} , or g . Mathematically, a retardation can be regarded as a negative acceleration in the direction along which the object is moving; and hence $a = -9.8 \text{ m s}^{-2}$ in this case.

Suppose the ball was thrown straight up with an initial velocity, u , of 30 m s^{-1} . The time taken to reach the top of its motion can be obtained from the equation $v = u + at$. The velocity, v , at the top is zero; and since $u = 30 \text{ m}$ and $a = -9.8$ or 10 m s^{-2} (approx), we have

$$0 = 30 - 10t.$$

$$\therefore t = \frac{30}{10} = 3 \text{ s.}$$

The highest distance reached is thus given by

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ &= 30 \times 3 - 5 \times 3^2 = 45 \text{ m.} \end{aligned}$$

Resultant Components

If a boy is running along the deck of a ship in a direction OA, and the

ship is moving in a different direction OB, the boy will move relatively to the sea along a direction OC, between OA and OB, Fig. 1.6 (i). Now in one second the boat moves from O to B, where OB represents the velocity of the boat, a vector quantity, in magnitude and direction. The boy moves from O to A in the same time, where OA represents the velocity of the boy in magnitude and direction. Thus in one second the net effect relative to the sea is that the boy moves from O to C. It can now be seen that if lines OA, OB are drawn to represent in magnitude and direction the respective velocities of the boy and the ship, the magnitude and direction of the *resultant* velocity of the boy is represented by the diagonal OC of the completed parallelogram having OA, OB as two of its sides; OACB is known as a *parallelogram of velocities*. Conversely, a velocity represented completely by OC can be regarded as having an 'effective part', or *component* represented by OA, and another component represented by OB.

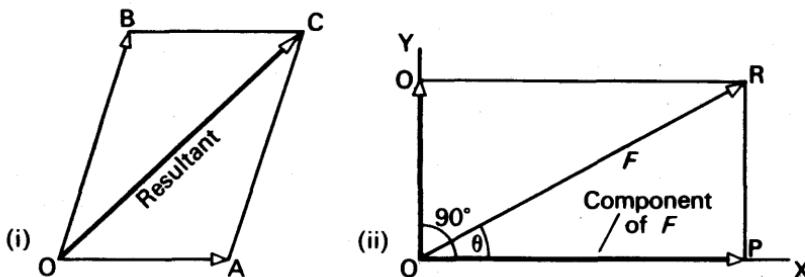


FIG. 1.6. Resultant and component.

In practice, we often require to find the component of a vector quantity in a certain direction. Suppose OR represents the vector F , and OX is the direction, Fig. 1.6 (ii). If we complete the parallelogram OQRP by drawing a perpendicular RP from R to OX, and a perpendicular RQ from R to OY, where OY is perpendicular to OX, we can see that OP, OQ represent the components of F along OX, OY respectively. Now the component OQ has no effect in a perpendicular direction; consequently OP represents the total effect of F along the direction OX. OP is called the 'resolved component' in this direction. If θ is the angle ROX, then, since triangle OPR has a right angle at P,

$$OP = OR \cos \theta = F \cos \theta \quad . \quad . \quad . \quad (4)$$

Components of g

The acceleration due to gravity, g , acts vertically downwards. In free fall, an object has an acceleration g . An object sliding freely down an inclined plane, however, has an acceleration due to gravity equal to the component of g down the plane. If it is inclined at 60° to the vertical, the acceleration down the plane is then $g \cos 60^\circ$ or $9.8 \cos 60^\circ \text{ m s}^{-2}$, which is 4.9 m s^{-2} .

Consider an object O thrown forward from the top of a cliff OA

with a horizontal velocity u of 15 m s^{-1} . Fig. 1.7. Since u is horizontal, it has no component in a *vertical* direction. Similarly, since g acts vertically, it has no component in a *horizontal* direction.

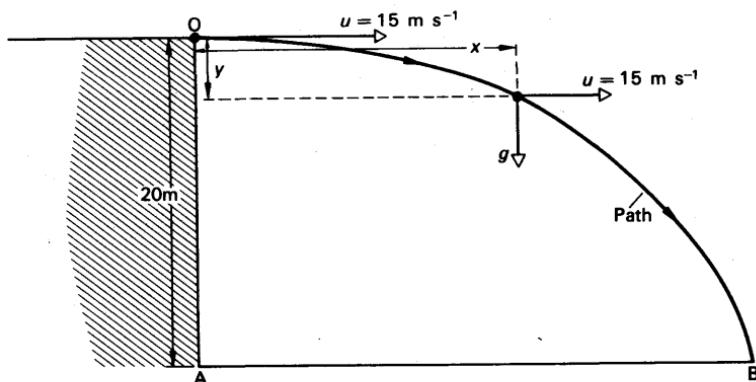


FIG. 1.7 Motion under gravity

We may thus treat vertical and horizontal motion independently. Consider the vertical motion from O. If OA is 20 m, the ball has an initial vertical velocity of zero and a vertical acceleration of g , which is 9.8 m s^{-2} (10 m s^{-2} approximately). Thus, from $s = ut + \frac{1}{2}at^2$, the time t to reach the bottom of the cliff is given, using $g = 10 \text{ m s}^{-2}$, by

$$20 = \frac{1}{2} \cdot 10 \cdot t^2 = 5t^2, \text{ or } t = 2 \text{ s.}$$

So far as the horizontal motion is concerned, the ball continues to move forward with a constant velocity of 15 m s^{-1} since g has no component horizontally. In 2 seconds, therefore,

$$\text{horizontal distance AB} = \text{distance from cliff} = 15 \times 2 = 30 \text{ m.}$$

Generally, in a time t the ball falls a vertical distance, y say, from O given by $y = \frac{1}{2}gt^2$. In the same time the ball travels a horizontal distance, x say, from O given by $x = ut$, where u is the velocity of 15 m s^{-1} . If t is eliminated by using $t = x/u$ in $y = \frac{1}{2}gt^2$, we obtain $y = gx^2/2u$. This is the equation of a *parabola*. It is the path OB in Fig. 1.7.

Addition of Vectors

Suppose a ship is travelling due east at 30 km h^{-1} and a boy runs across the deck in a north-west direction at 6 km h^{-1} , Fig. 1.8 (i). We

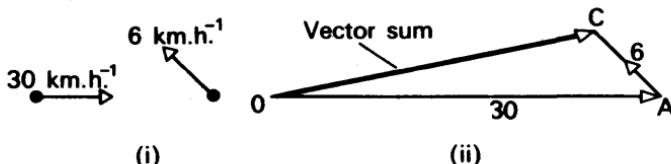


FIG. 1.8 Addition of vectors

can find the velocity and direction of the boy relative to the sea by adding the two velocities. Since velocity is a vector quantity, we draw a line OA to represent 30 km h^{-1} in magnitude and direction, and then, from the end of A , draw a line AC to represent 6 km h^{-1} in magnitude and direction, Fig. 1.8 (ii). The sum, or resultant, of the velocities is now represented by the line OC in magnitude and direction, because a distance moved in one second by the ship (represented by OA) together with a distance moved in one second by the boy (represented by AC) is equivalent to a movement of the boy from O to C relative to the sea.

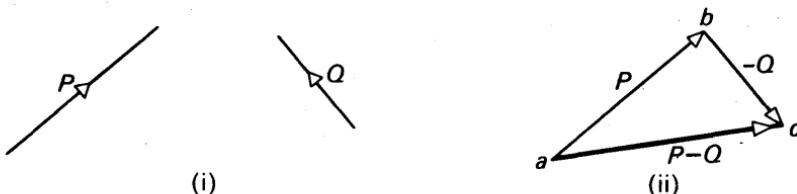


FIG. 1.9 Subtraction of velocities

In other words, the difference between the vectors \vec{P} , \vec{Q} in Fig. 1.9 (i) is the *sum* of the vectors \vec{P} and $(-\vec{Q})$. Now $(-\vec{Q})$ is a vector drawn exactly equal and opposite to the vector \vec{Q} . We therefore draw ab to represent \vec{P} completely, and then draw bc to represent $(-\vec{Q})$ completely, Fig. 1.9 (ii). Then $\vec{P} + (-\vec{Q}) =$ the vector represented by $ac = \vec{P} - \vec{Q}$.

Relative Velocity and Relative Acceleration

If a car A travelling at 50 km h^{-1} is moving in the same direction as another car B travelling at 60 km h^{-1} , the *relative velocity* of B to A = $60 - 50 = 10 \text{ km h}^{-1}$. If, however, the cars are travelling in opposite directions, the relative velocity of B to A = $60 - (-50) = 110 \text{ km h}^{-1}$.

Suppose that a car X is travelling with a velocity v along a road 30° east of north, and a car Y is travelling with a velocity u along a road due east, Fig. 1.10 (i). Since 'velocity' has direction as well as magnitude, i.e., 'velocity' is a vector quantity (p. 1), we cannot subtract u and v numerically to find the relative velocity. We must adopt a method which takes into account the direction as well as the magnitude of the velocities, i.e., a vector subtraction is required.

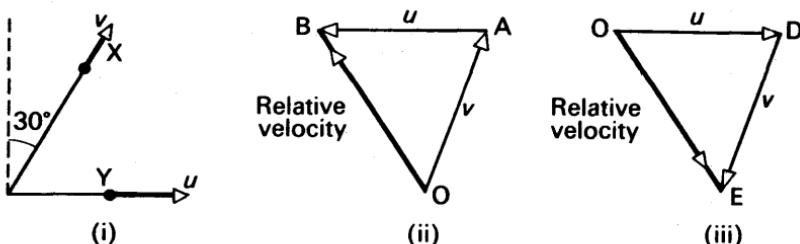


FIG. 1.10. Relative velocity.

The velocity of X relative to Y = $\vec{v} - \vec{u} = \vec{v} + (-\vec{u})$. Suppose OA represents the velocity, v , of X in magnitude and direction, Fig. 1.10 (ii). Since Y is travelling due east, a velocity AB numerically equal to u but in the due west direction represents the vector $(-\vec{u})$. The vector sum of OA and AB is OB from p. 0, which therefore represents in magnitude and direction the velocity of X relative to Y. By drawing an accurate diagram of the two velocities, OB can be found.

The velocity of Y relative to X = $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$, and can be found by a similar method. In this case, OD represents the velocity, u , of Y in magnitude and direction, while DE represents the vector $(-\vec{v})$, which it is drawn numerically equal to v but in the *opposite* direction, Fig. 1.10 (iii). The vector sum of OD and DE is OE, which therefore represents the velocity of Y relative to X in magnitude and direction.

When two objects P, Q are each accelerating, the acceleration of P relative to Q = acceleration of P – acceleration of Q. Since ‘acceleration’ is a vector quantity, the relative acceleration must be found by vector subtraction, as for the case of relative velocity.

EXAMPLE

Explain the difference between a scalar and a vector quantity.

What is meant by the relative velocity of one body with respect to another? Two ships are 10 km apart on a line running S. to N. The one farther north is steaming W. at 20 km h^{-1} . The other is steaming N. at 20 km h^{-1} . What is their distance of closest approach and how long do they take to reach it? (C.)

Suppose the two ships are at X, Y, moving with velocities u , v respectively, each 20 km h^{-1} Fig. 1.11 (i). The velocity of Y relative to X = $\vec{v} - \vec{u} = \vec{v} + (-\vec{u})$. We therefore draw OA to represent \vec{v} (20) and add to it AB, which represents $(-\vec{u})$, Fig. 1.11 (ii). The relative velocity is then represented by OB.

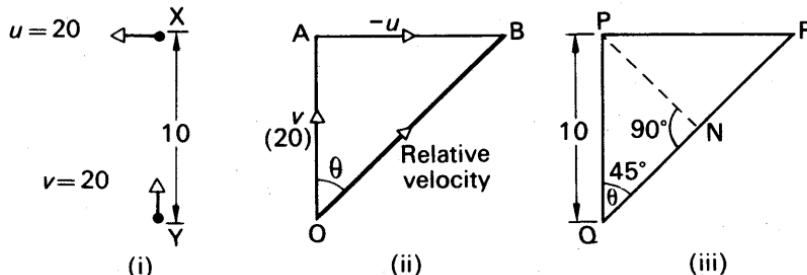


FIG. 1.11 Example

Since OAB is a right-angled triangle,

$$OB = \sqrt{OA^2 + AB^2} = \sqrt{20^2 + 20^2} = 28.28 = 28.3 \text{ km h}^{-1} \quad (i)$$

$$\text{Also, } \tan \theta = \frac{AB}{OA} = \frac{20}{20} = 1, \text{ i.e., } \theta = 45^\circ \quad \quad (ii)$$

Thus the ship Y will move along a direction QR relative to the ship X, where QR is at 45° to PQ, the north-south direction, Fig. 1.11(iii). If $PQ = 10$ km, the distance of closest approach is PN, where PN is the perpendicular from P to QR.

$$\therefore PN = PQ \sin 45^\circ = 10 \sin 45^\circ = 7.07 \text{ km.}$$

The distance QN = $10 \cos 45^\circ = 7.07$ km. Since, from (i), the relative velocity is 28.28 km h^{-1} , it follows that

$$\text{time to reach N} = \frac{7.07}{28.28} = \frac{1}{4} \text{ hour.}$$

LAWS OF MOTION. FORCE AND MOMENTUM

Newton's Laws of Motion

In 1686 SIR ISAAC NEWTON published a work called *Principia*, in which he expounded the Laws of Mechanics. He formulated in the book three 'laws of motion':

Law I. Every body continues in its state of rest or uniform motion in a straight line, unless impressed forces act on it.

Law II. The change of momentum per unit time is proportional to the impressed force, and takes place in the direction of the straight line along which the force acts.

Law III. Action and reaction are always equal and opposite.

These laws cannot be proved in a formal way; we believe they are correct because all the theoretical results obtained by assuming their truth agree with the experimental observations, as for example in astronomy (p. 58).

Inertia. Mass

Newton's first law expresses the idea of **inertia**. The inertia of a body is its reluctance to start moving, and its reluctance to stop once it has begun moving. Thus an object at rest begins to move only when it is pushed or pulled, i.e., when a force acts on it. An object O moving in a

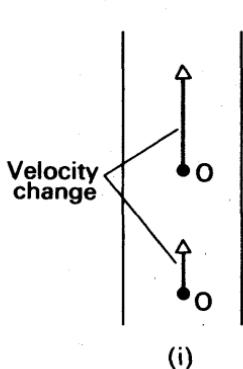
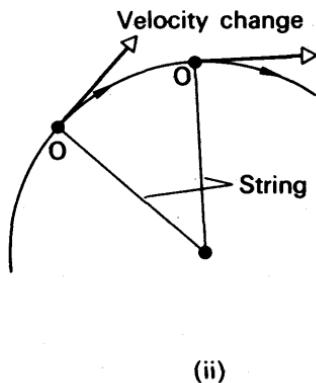


FIG. 1.12 Velocity changes



straight line with constant velocity will change its direction or move faster only if a new force acts on it. Fig. 1.12 (i). This can be demonstrated by a puck moving on a cushion of gas on a smooth level sheet of glass. As the puck slides over the glass, photographs taken at successive equal times by a stroboscopic method show that the motion is practically that of uniform velocity. Passengers in a bus or car are jerked forward when the vehicle stops suddenly. They continue in their state of motion until brought to rest by friction or collision. The use of safety belts reduces the shock.

Fig. 1.12 (ii) illustrates a velocity change when an object O is whirled at constant speed by a string. This time the magnitude of the velocity v is constant but its direction changes.

'Mass' is a measure of the inertia of a body. If an object changes its direction or its velocity slightly when a large force acts on it, its inertial mass is high. The mass of an object is constant all over the world; it is the same on the earth as on the moon. Mass is measured in kilogrammes (kg) or grammes (g) by means of a chemical balance, where it is compared with standard masses based on the International Prototype Kilogramme (see also p. 14).

Force. The newton

When an object X is moving it is said to have an amount of *momentum* given, by definition, by

$$\text{momentum} = \text{mass of } X \times \text{velocity} \quad (1)$$

Thus an object of mass 20 kg moving with a velocity of 10 m s^{-1} has a momentum of 200 kg m s^{-1} . If another object collides with X its velocity alters, and thus the momentum of X alters. From Newton's second law, a force acts on X which is equal to the change in momentum per second.

Thus if F is the magnitude of a force acting on a constant mass m ,

$$F \propto m \times \text{change of velocity per second}$$

$$\therefore F \propto ma,$$

where a is the *acceleration* produced by the force, by definition of a .

$$\therefore F = kma \quad (2)$$

where k is a constant.

With SI units, the **newton** (N) is the unit of force. It is defined as the force which gives a mass of 1 kilogramme an acceleration of 1 metre s^{-2} . Substituting $F = 1\text{N}$, $m = 1 \text{ kg}$ and $a = 1 \text{ m s}^{-2}$ in the expression for F in (i), we obtain $k = 1$. Hence, with units as stated, $k = 1$.

$$\therefore F = ma,$$

which is a standard equation in dynamics. Thus if a mass of 200 g is acted upon by a force F which produces an acceleration a of 4 m s^{-2} , then, since $m = 200 \text{ g} = 0.2 \text{ kg}$,

$$F = ma = 0.2(\text{kg}) \times 4(\text{m s}^{-2}) = 0.8 \text{ N.}$$

C.g.s. units of force

The *dyne* is the unit of force in the centimetre-gramme-second system; it is defined as the force acting on a mass of 1 gramme which gives it an acceleration of 1 cm s^{-2} . The equation $F = ma$ also applies when m is in grammes, a is in cm s^{-2} , and F is in dynes. Thus if a force of 10000 dynes acts on a mass of 200 g, the acceleration a is given by

$$F = 10000 = 200 \times a, \text{ or } a = 50 \text{ cm s}^{-2}.$$

Suppose $m = 1 \text{ kg} = 1000 \text{ g}$, $a = 1 \text{ m s}^{-2} = 100 \text{ cm s}^{-2}$. Then, the force F is given by

$$F = ma = 1000 \times 100 \text{ dynes} = 10^5 \text{ dynes.}$$

But the force acting on a mass of 1 kg which gives it an acceleration of 1 m s^{-2} is the *newton*, N. Hence

$$1 \text{ N} = 10^5 \text{ dynes}$$

Weight. Relation between newton, kgf and dyne, gf

The *weight* of an object is defined as the *force* acting on it due to gravity; the weight of an object can hence be measured by attaching it to a spring-balance and noting the extension, as the latter is proportional to the force acting on it (p. 50).

Suppose the weight of an object of mass m is denoted by W . If the object is released so that it falls to the ground, its acceleration is g . Now $F = ma$. Consequently the force acting on it, i.e., its weight, is given by

$$W = mg.$$

If the mass is 1 kg, then, since $g = 9.8 \text{ m s}^{-2}$, the weight $W = 1 \times 9.8 = 9.8 \text{ N}$ (newton). The force due to gravity on a mass of 1 kg where g has the value 9.80665 m s^{-2} is called a 1 *kilogramme force* or 1 kgf (this is roughly equal to 1 kilogramme weight or 1 kg wt, which depends on the value of g and thus varies from place to place). Hence it follows that

$$1 \text{ kgf} = 9.8 \text{ N} = 10 \text{ N approximately.}$$

A weight of 5 kgf is thus about 50 N. Further, $1 \text{ N} = \frac{1}{10} \text{ kgf approx} = 100 \text{ gf}$. The weight of an apple is about 1 newton.

The weight of a mass of 1 gramme is called *gramme-force* (1 gf); it was formerly called '1 gramme wt'. From $F = ma$, it follows that

$$1 \text{ gf} = 1 \times 980 = 980 \text{ dynes.}$$

since $g = 980 \text{ cm s}^{-2}$ (approx).

The reader should note carefully the difference between the 'kilogramme' and the 'kilogramme force'; the former is a *mass* and is therefore constant all over the universe, whereas the kilogramme force is a *force* whose magnitude depends on the value of g . The acceleration due to gravity, g , depends on the distance of the place considered from the centre of the earth; it is slightly greater at the poles than at the

equator, since the earth is not perfectly spherical (see p. 41). It therefore follows that the weight of an object differs in different parts of the world. On the moon, which is smaller than the earth and has a smaller density, an object would weigh about one-sixth of its weight on the earth.

The relation $F = ma$ can be verified by using a ticker-tape and timer to measure the acceleration of a moving trolley. Details are given in a more basic text, such as *Fundamentals of Physics* (Chatto and Windus) by the author.

The following examples illustrate the application of $F = ma$. It should be carefully noted that (i) F represents the *resultant* force on the object of mass m , (ii) F must be expressed in the appropriate units of a 'force' and m in the corresponding units of a 'mass'.

EXAMPLES

1. A force of 20 kgf pulls a sledge of mass 50 kg and overcomes a constant frictional force of 4 kgf. What is the acceleration of the sledge?

$$\text{Resultant force, } F = 20 \text{ kgf} - 4 \text{ kgf} = 16 \text{ kgf.}$$

To change this to units of newtons, use $1 \text{ kgf} = 9.8 \text{ N} = 10 \text{ N approx.}$

$$\therefore 16 \text{ kgf} = 160 \text{ N approx.}$$

From $F = ma$,

$$\therefore 160 = 50 \times a$$

$$\therefore a = 3.2 \text{ m s}^{-2}.$$

2. An object of mass 2.00 kg is attached to the hook of a spring-balance, and the latter is suspended vertically from the roof of a lift. What is the reading on the spring-balance when the lift is (i) ascending with an acceleration of 20 cm s^{-2} , (ii) descending with an acceleration of 10 cm s^{-2} , (iii) ascending with a uniform velocity of 15 cm s^{-1} .

Suppose T is the tension (force) in the spring-balance in kgf.

(i) The object is acted upon two forces: (a) The tension T kgf in the spring-balance, which acts upwards, (b) its weight, 2 kgf, which acts downwards. Since the object moves upwards, T is greater than 2 kgf. Hence the net force, F , acting on the object $= (T - 2) \text{ kgf} = (T - 2) \times 10 \text{ N, approx. Now}$

$$F = ma,$$

where a is the acceleration in m s^{-2} .

$$\therefore (T - 2) \times 10 = 2 \times a = 2 \times 0.2$$

$$\therefore T = 2.04 \text{ kgf} \quad \quad (1)$$

(ii) When the lift descends with an acceleration of 10 cm s^{-2} or 0.1 m s^{-2} , the weight, 2 kgf, is now greater than T_1 kgf, the tension in the spring-balance.

$$\therefore \text{resultant force} = (2 - T_1) \text{ kgf} = (2 - T_1) \times 10 \text{ N approx.}$$

$$\therefore F = (2 - T_1) \times 10 = ma = 2 \times 0.1$$

$$\therefore T_1 = 2 - 0.02 = 1.98 \text{ kgf.}$$

(iii) When the lift moves with constant velocity, the acceleration is zero. In this case the reading on the spring-balance is exactly equal to the weight, 2 kgf.

Linear Momentum

Newton defined the force acting on an object as the rate of change of its momentum, the momentum being the product of its mass and velocity (p. 13). *Momentum is thus a vector quantity.* Suppose that the mass of an object is m , its initial velocity is u , and its final velocity due to a force F acting on it for a time t is v . Then

$$\text{change of momentum} = mv - mu,$$

and hence

$$F = \frac{mv - mu}{t}$$

$$\therefore Ft = mv - mu = \text{momentum change} \quad . . . \quad (1)$$

The quantity Ft (force \times time) is known as the *impulse* of the force on the object, and from (1) it follows that the units of momentum are the same as those of Pt , i.e., *newton second* ($N\ s$). From 'mass \times velocity', alternative units are ' kg m s^{-1} '.

Force and momentum change

A person of mass 50 kg who is jumping from a height of 5 metres will land on the ground with a velocity $= \sqrt{2gh} = \sqrt{2 \times 10 \times 5} = 10 \text{ m s}^{-1}$, assuming $g = 980 \text{ cm s}^{-2} = 10 \text{ m s}^{-2}$ approx. If he does not flex his knees on landing, he will be brought to rest very quickly, say in $\frac{1}{10}$ th second. The force F acting is then given by

$$F = \frac{\text{momentum change}}{\text{time}}$$

$$= \frac{50 \times 10}{\frac{1}{10}} = 5000 \text{ N} = 500 \text{ kgf (approx.)}$$

This is a force of about 10 times the person's weight and this large force has a severe effect on the body.

Suppose, however, that the person flexes his knees and is brought to rest much more slowly on landing, say in 1 second. Then, from above, the force F now acting is 10 times less than before, or 50 kgf (approx.). Consequently, much less damage is done to the person on landing.

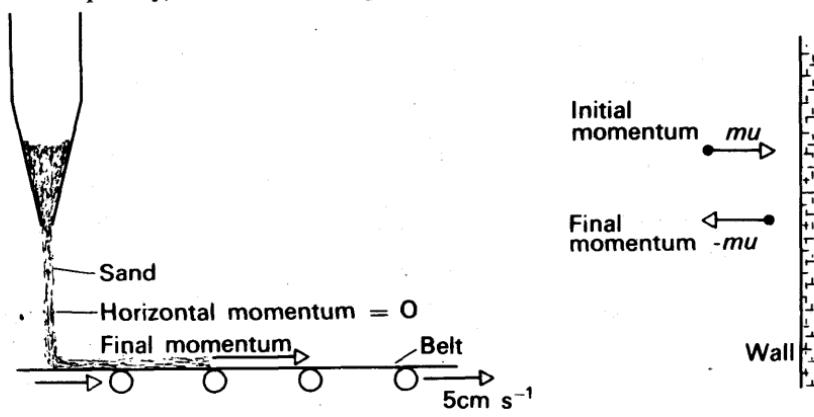


FIG. 1.13 Linear momentum

Suppose sand is allowed to fall vertically at a steady rate of 100 g s^{-1} on to a horizontal conveyor belt moving at a steady velocity of 5 cm s^{-1} . Fig. 1.13 (i). The initial horizontal velocity of the sand is zero. The final horizontal velocity is 5 cm s^{-1} . Now

$$\text{mass} = 100 \text{ g} = 0.1 \text{ kg}, \text{velocity} = 5 \text{ cm s}^{-1} = 5 \times 10^{-2} \text{ m s}^{-1}$$

$$\therefore \text{momentum change per second} = 0.1 \times 5 \times 10^{-2} = 5 \times 10^{-3} \text{ newton} \\ = \text{force on belt}$$

Observe that this is a case where the mass changes with time and the velocity gained is constant. In terms of the calculus, the force is the rate of change of momentum mv , which is $v \times dm/dt$, and dm/dt is 100 g s^{-1} in this numerical example.

Consider a molecule of mass m in a gas, which strikes the wall of a vessel repeatedly with a velocity u and rebounds with a velocity $-u$. Fig. 1.13 (ii). Since momentum is a vector quantity, the momentum change = final momentum - initial momentum = $mu - (-mu) = 2mu$. If the containing vessel is a cube of side l , the molecule repeatedly takes a time $2l/u$ to make an impact with the same side.

$$\therefore \text{average force on wall due to molecule}$$

$$= \frac{\text{momentum change}}{\text{time}} \\ = \frac{2mu}{2l/u} = \frac{mu^2}{l}.$$

The total gas pressure is the average force per unit area on the walls of the container due to all the numerous gas molecules.

EXAMPLES

1. A hose ejects water at a speed of 20 cm s^{-1} through a hole of area 100 cm^2 . If the water strikes a wall normally, calculate the force on the wall in newton, assuming the velocity of the water normal to the wall is zero after collision.

The volume of water per second striking the wall = $100 \times 20 = 2000 \text{ cm}^3$.

$$\therefore \text{mass per second striking wall} = 2000 \text{ g s}^{-1} = 2 \text{ kg s}^{-1}.$$

Velocity change of water on striking wall = $20 - 0 = 20 \text{ cm s}^{-1} = 0.2 \text{ m s}^{-1}$.

$$\therefore \text{momentum change per second} = 2 (\text{kg s}^{-1}) \times 0.2 (\text{m s}^{-1}) = 0.4 \text{ newton.}$$

2. Sand drops vertically at the rate of 2 kg s^{-1} on to a conveyor belt moving horizontally with a velocity of 0.1 m s^{-1} . Calculate (i) the extra power needed to keep the belt moving, (ii) the rate of change of kinetic energy of the sand. Why is the power twice as great as the rate of change of kinetic energy?

(i) Force required to keep belt moving = rate of increase of horizontal momentum of sand = mass per second $(dm/dt) \times$ velocity change = $2 \times 0.1 = 0.2 \text{ newton.}$

$$\therefore \text{power} = \text{work done per second} = \text{force} \times \text{rate of displacement} \\ = \text{force} \times \text{velocity} = 0.2 \times 0.1 = 0.02 \text{ watt (p. 25).}$$

(ii) Kinetic energy of sand = $\frac{1}{2}mv^2$.

$$\therefore \text{rate of change of energy} = \frac{1}{2}v^2 \times \frac{dm}{dt}, \text{ since } v \text{ is constant,}$$

$$= \frac{1}{2} \times 0.1^2 \times 2 = 0.01 \text{ watt.}$$

Thus the power supplied is twice as great as the rate of change of kinetic energy. The extra power is due to the fact that the sand does not immediately assume the velocity of the belt, so that the belt at first moves relative to the sand. The extra power is needed to overcome the friction between the sand and belt.

Conservation of Linear Momentum

We now consider what happens to the linear momentum of objects which *collide* with each other.

Experimentally, this can be investigated by several methods:

1. Trolleys in collision, with ticker-tapes attached to measure velocities.
2. Linear Air-track, using perspex models in collision and stroboscopic photography for measuring velocities.

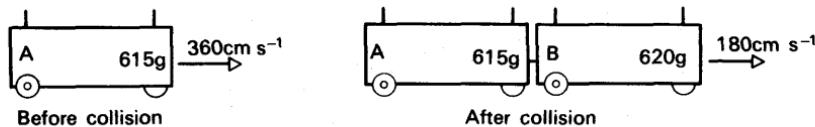


FIG. 1.14 Linear momentum experiment

As an illustration of the experimental results, the following measurements were taken in trolley collisions (Fig. 1.14):

Before collision.

Mass of trolley A = 615 g; initial velocity = 360 cm s^{-1} .

After collision.

A and B coalesced and both moved with velocity of 180 cm s^{-1} .

Thus the total linear momentum of A and B before collision = $0.615 (\text{kg}) \times 3.6 (\text{m s}^{-1}) + 0 = 2.20 \text{ kg m s}^{-1}$ (approx). The total momentum of A and B after collision = $1.235 \times 1.8 = 2.20 \text{ kg m s}^{-1}$ (approx).

Within the limits of experimental accuracy, it follows that *the total momentum of A and B before collision = the total momentum after collision*. Similar results are obtained if A and B are moving with different speeds after collision, or in opposite directions before collision.

Principle of Conservation of Linear Momentum

These experimental results can be shown to follow from Newton's second and third laws of motion (p. 12).

Suppose that a moving object A, of mass m_1 and velocity u_1 , collides

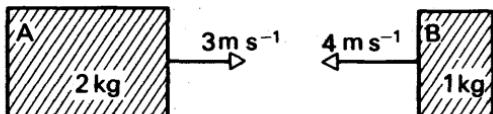
with another object B, of mass m_2 and velocity u_2 , moving in the same direction, Fig. 1.15. By Newton's law of action and reaction, the force F exerted by A on B is equal and opposite to that exerted by B on A. Moreover, the time t during which the force acted on B is equal to the time during which the force of reaction acted on A. Thus the magnitude of the impulse, Ft , on B is equal and opposite to the magnitude of the impulse on A. From equation (1), p. 16, the impulse is equal to the change of momentum. It therefore follows that the change in the total momentum of the two objects is zero, i.e., the total momentum of the two objects is constant although a collision had occurred. Thus if A moves with a reduced velocity v_1 after collision, and B then moves with an increased velocity v_2 ,

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2.$$

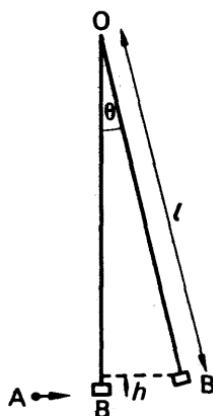
The principle of the conservation of linear momentum states that, if no external forces act on a system of colliding objects, the total momentum of the objects remains constant.

EXAMPLES

1. An object A of mass 2 kg is moving with a velocity of 3 m s^{-1} and collides head on with an object B of mass 1 kg moving in the opposite direction with a velocity of 4 m s^{-1} . Fig. 1.16 (i). After collision both objects coalesce, so that they move with a common velocity v . Calculate v .



(i)



(ii)

FIG. 1.16 Examples

Total momentum before collision of A and B in the direction of A

$$= 2 \times 3 - 1 \times 4 = 2 \text{ kg m s}^{-1}.$$

Note that momentum is a vector and the momentum of B is of opposite sign to A.

After collision, momentum of A and B in the direction of A = $2v + 1v = 3v$.

$$\therefore 3v = 2$$

$$\therefore v = \frac{2}{3} \text{ m s}^{-1}$$

2. What is understood by (a) the principle of the *conservation of energy*, (b) the principle of the *conservation of momentum*?

A bullet of mass 20 g travelling horizontally at 100 m s^{-1} , embeds itself in the centre of a block of wood of mass 1 kg which is suspended by light vertical strings 1 m in length. Calculate the maximum inclination of the strings to the vertical.

Describe in detail how the experiment might be carried out and used to determine the velocity of the bullet just before the impact of the block. (N.)

Second part. Suppose A is the bullet, B is the block suspended from a point O, and θ is the maximum inclination to the vertical, Fig. 1.16(ii). If $v \text{ cm s}^{-1}$ is the common velocity of block and bullet when the latter is brought to rest relative to the block, then, from the principle of the conservation of momentum, since $20 \text{ g} = 0.02 \text{ kg}$,

$$(1+0.02)v = 0.02 \times 100$$

$$\therefore v = \frac{2}{1.02} = \frac{100}{51} \text{ m s}^{-1}$$

The vertical height risen by block and bullet is given by $v^2 = 2gh$, where $g = 9.8 \text{ m s}^{-2}$ and $h = l - l \cos \theta = l(1 - \cos \theta)$.

$$\therefore v^2 = 2gl(1 - \cos \theta)$$

$$\therefore \left(\frac{100}{51}\right)^2 = 2 \times 9.8 \times 1(1 - \cos \theta)$$

$$\therefore 1 - \cos \theta = \left(\frac{100}{51}\right)^2 \times \frac{1}{2 \times 9.8} = 0.1962$$

$$\therefore \cos \theta = 0.8038, \text{ or } \theta = 37^\circ \text{ (approx.)}$$

The velocity, v , of the bullet can be determined by applying the conservation of momentum principle.

Thus $mv = (m+M)V$, where m is the mass of the bullet, M is the mass of the block, and V is the common velocity. Then $v = (m+M)V/m$. The quantities m and M can be found by weighing. V is calculated from the horizontal displacement a of the block, since (i) $V^2 = 2gh$ and (ii) $h(2l-h) = a^2$ from the geometry of the circle, so that, to a good approximation, $2h = a^2/l$.

Inelastic and elastic collisions

In collisions, the total momentum of the colliding objects is always conserved. Usually, however, their total kinetic energy is not conserved. Some of it is changed to heat or sound energy, which is not recoverable. Such collisions are said to be *inelastic*. If the total kinetic energy is conserved, the collision is said to be *elastic*. The collision between two smooth billiard balls is approximately elastic. Many atomic collisions are elastic. Electrons may make elastic or inelastic collisions

with atoms of a gas. As proved on p. 28, the kinetic energy of a mass m moving with a velocity v has kinetic energy equal to $\frac{1}{2}mv^2$.

As an illustration of the mechanics associated with elastic collisions, consider a sphere A of mass m and velocity v incident on a stationary sphere B of equal mass m . (Fig. 1.17 (i). Suppose the collision is elastic, and after collision let A move with a velocity v_1 at an angle of 60° to its original direction and B move with a velocity v_2 at an angle θ to the direction of v .

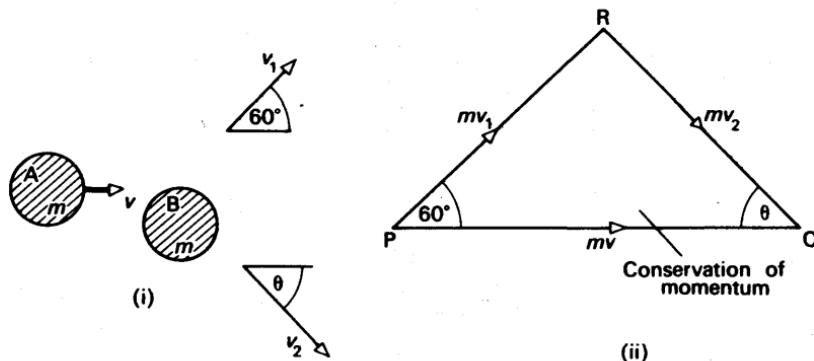


FIG. 1.17 Conservation of momentum

Since momentum is a vector (p. 17), we may represent the momentum mv of A by the line PQ drawn in the direction of v . Fig. 1.17 (ii). Likewise, PR represents the momentum mv_1 of A after collision. Since momentum is conserved, the vector RQ must represent the momentum mv_2 of B after collision, that is,

$$\vec{mv} = \vec{mv}_1 + \vec{mv}_2.$$

Hence

$$\vec{v} = \vec{v}_1 + \vec{v}_2,$$

or PQ represents v in magnitude, PR represents v_1 and RQ represents v_2 . But if the collision is elastic,

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2$$

$$\therefore v^2 = v_1^2 + v_2^2.$$

Consequently, triangle PRQ is a right-angled triangle with angle R equal to 90° .

$$\therefore v_1 = v \cos 60^\circ = \frac{v}{2}.$$

Also, $\theta = 90^\circ - 60^\circ = 30^\circ$, and $v_2 = v \cos 30^\circ = \frac{\sqrt{3}v}{2}$.

Coefficient of restitution

In practice, colliding objects do not stick together and kinetic energy is always lost. If a ball X moving with velocity u_1 collides head-on with a ball Y moving with a velocity u_2 in the same direction, then Y will move faster with a velocity v_1 say and X may then have a reduced velocity v_2 in the same direction. The coefficient of restitution, e , between X and Y is defined as the ratio:

$$\frac{\text{velocity of separation}}{\text{velocity of approach}} \quad \text{or} \quad \frac{v_2 - v_1}{u_1 - u_2}$$

The coefficient of restitution is approximately constant between two given materials. It varies from $e = 0$, when objects stick together and the collision is completely inelastic, to $e = 1$, when objects are very hard and the collision is practically elastic. Thus, from above, if $u_1 = 4 \text{ m s}^{-1}$, $u_2 = 1 \text{ m s}^{-1}$ and $e = 0.8$, then velocity of separation, $v_2 - v_1 = 0.8 \times (4 - 1) = 2.4 \text{ m s}^{-1}$.

Momentum and Explosive forces

There are numerous cases where momentum changes are produced by explosive forces. An example is a bullet of mass $m = 50 \text{ g}$ say, fired from a rifle of mass $M = 2 \text{ kg}$ with a velocity v of 100 m s^{-1} . Initially, the total momentum of the bullet and rifle is zero. From the principle of the conservation of linear momentum, when the bullet is fired the total momentum of bullet and rifle is still zero, since no external force has acted on them. Thus if V is the velocity of the rifle,

$$mv(\text{bullet}) + MV(\text{rifle}) = 0$$

$$\therefore MV = -mv, \quad \text{or} \quad V = -\frac{m}{M}v.$$

The momentum of the rifle is thus *equal and opposite* to that of the bullet. Further, $V/v = -m/M$. Since $m/M = 50/2000 = 1/40$, it follows that $V = -v/40 = 2.5 \text{ m s}^{-1}$. This means that the rifle moves back or *recoils* with a velocity only about $\frac{1}{40}$ th that of the bullet.

If it is preferred, one may also say that the explosive force produces the same numerical momentum change in the bullet as in the rifle. Thus $mv = MV$, where V is the velocity of the rifle in the *opposite* direction to that of the bullet. The joule (J) is the unit of energy (p. 24).

The kinetic energy, E_1 , of the bullet = $\frac{1}{2}mv^2 = \frac{1}{2} \cdot 0.05 \cdot 100^2 = 250\text{J}$

The kinetic energy, E_2 , of the rifle = $\frac{1}{2}MV^2 = \frac{1}{2} \cdot 2 \cdot 2.5^2 = 6.25\text{J}$

Thus the total kinetic energy produced by the explosion = 256.25J . The kinetic energy E_1 of the bullet is thus $250/256.25$, or about 98% of the total energy. This is explained by the fact that the kinetic energy depends on the *square* of the velocity. The high velocity of the bullet thus more than compensates for its small mass relative to that of the rifle. See also p. 26.

Rocket

Consider a rocket moving in outer space where no external forces act on it. Suppose its mass is M and its velocity is v at a particular instant. Fig. 1.18 (i). When a mass m of fuel is ejected, the mass of the rocket becomes $(M - m)$ and its velocity increases to $(v + \Delta v)$. Fig. 1.18 (ii).

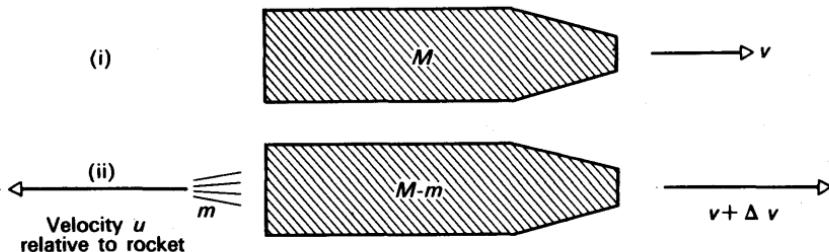


FIG. 1.18. Motion of rocket

Suppose the fuel is always ejected at a constant speed u relative to the rocket. Then the velocity of the mass $m = v + \frac{\Delta v}{2} - u$ in the direction of the rocket, since the initial velocity of the rocket is v and the final velocity is $v + \Delta v$, an average of $v + \Delta v/2$.

We now apply the principle of the conservation of momentum to the rocket and fuel. Initially, before m of fuel was ejected, momentum of rocket and fuel inside rocket = Mv .

After m is ejected, momentum of rocket = $(M-m)(v+\Delta v)$

$$\text{and momentum of fuel} = m\left(v + \frac{\Delta v}{2} - u\right).$$

$$\therefore (M-m)(v+\Delta v) + m\left(v + \frac{\Delta v}{2} - u\right) = Mv.$$

Neglecting the product of $m \cdot \Delta v$, then, after simplification,

$$M \cdot \Delta v - mu = 0,$$

$$\therefore \frac{m}{M} = \frac{\Delta v}{u}.$$

Now

$$m = \text{mass of fuel ejected} = -\Delta M,$$

$$\therefore -\frac{\Delta M}{M} = \frac{\Delta v}{u}.$$

Integrating between limits of M , M_0 and v , v_0 respectively

$$\therefore \int_{M_0}^M -\frac{\Delta M}{M} = \frac{1}{u} \int_{v_0}^v \Delta v.$$

$$\therefore -\log_e \frac{M}{M_0} = \frac{v - v_0}{u}.$$

$$\therefore M = M_0 e^{-(v-v_0)/u} \quad \quad (1)$$

or

$$v = v_0 - u \log_e(M/M_0) \quad \quad (2)$$

When the mass M decreases to $M_0/2$

$$v = v_0 + u \log_e 2.$$

Motion of centre of mass

If two particles, masses m_1 and m_2 , are distances x_1 , x_2 respectively from a given axis, their *centre of mass* is at a distance x from the axis given by $m_1x_1 + m_2x_2 = (m_1 + m_2)x$. See p. 104. Since velocity, $v = dx/dt$ generally, the velocity \bar{v} of the centre of mass in the particular direction is given by $m_1v_1 + m_2v_2 = (m_1 + m_2)\bar{v}$, where v_1, v_2 are the respective velocities of m_1, m_2 . The quantity $(m_1v_1 + m_2v_2)$ represents the total momentum of the two particles. The quantity $(m_1 + m_2)\bar{v} = M\bar{v}$, where M is the total mass of the particles. Thus we can imagine that the total mass of the particles is concentrated at the centre of mass while they move, and that the velocity \bar{v} of the centre of mass is always given by *total momentum* = $M\bar{v}$.

If *internal forces* act on the particles while moving, then, since action and reaction are equal and opposite, their resultant on the whole body is zero. Consequently the total momentum is unchanged and hence the velocity or motion of their centre of mass is unaffected. If an *external force*, however, acts on the particles, the total momentum is changed. The motion of their centre of mass now follows a path which is due to the external force.

We can apply this to the case of a shell fired from a gun. The centre of mass of the shell follows at first a parabolic path. This is due to the external force of gravity, its weight. If the shell explodes in mid-air, the fragments fly off in different directions. But the numerous internal forces which occur in the explosion have zero resultant, since action and reaction are equal and opposite and the forces can all be paired. Consequently *the centre of mass of all the fragments continues to follow the same parabolic path*. As soon as one fragment reaches the ground, an external force now acts on the system of particles. A different parabolic path is then followed by the centre of mass.

If a bullet is fired in a horizontal direction from a rifle, where is their centre of mass while the bullet and rifle are both moving?

Work

When an engine pulls a train with a constant force of 50 units through a distance of 20 units in its own direction, the engine is said by definition to do an amount of *work* equal to 50×20 or 1000 units, the product of the force and the distance. Thus if W is the amount of work,

$$W = \text{force} \times \text{distance moved in direction of force.}$$

Work is a *scalar* quantity; it has no property of direction but only magnitude. When the force is one newton and the distance moved is one metre, then the work done is one *joule*. Thus a force of 50 N moving through a distance of 10 m does 50×10 or 500 joule of work. Note this is also a measure of the *energy* transferred to the object.

The force to raise steadily a mass of 1 kg is 1 kilogram force (1 kgf), which is about 10 N (see p. 14). Thus if the mass of 1 kg is raised vertically through 1 m, then, approximately, work done = $10(N) \times 1(m) = 10$ joule.

The c.g.s. unit of work is the *erg*; it is the work done when a force of 1 dyne moves through 1 cm. Since $1 \text{ N} = 10^5 \text{ dynes}$ and $1 \text{ m} = 100 \text{ cm}$, then 1 N moving through 1 m does an amount of work = $10^5 \text{ (dyne)} \times 100 \text{ (cm)} = 10^7 \text{ ergs} = 1 \text{ joule}$, by definition of the joule (p. 24).

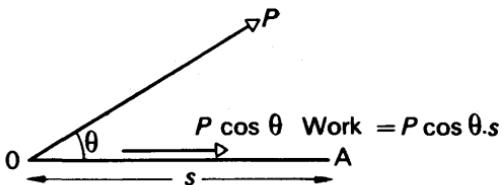


FIG. 1.19 Work

Before leaving the topic of 'work', the reader should note carefully that we have assumed the force to move an object in its own direction. Suppose, however, that a force P pulls an object a distance s along a line OA acting at an angle θ to it, Fig. 1.19. The component of P along OA is $P \cos \theta$ (p. 8), and this is the effective part of P pulling along the direction OA. The component of P along a direction perpendicular to OA has no effect along OA. Consequently

$$\text{work done} = P \cos \theta \times s.$$

In general, the work done by a force is equal to the product of the force and the displacement of its point of application in the direction of the force.

Power

When an engine does work quickly, it is said to be operating at a high *power*; if it does work slowly it is said to be operating at a low power. 'Power' is defined as the *work done per second*, i.e.,

$$\text{power} = \frac{\text{work done}}{\text{time taken}}.$$

The practical unit of power, the SI unit, is 'joule per second' or *watt* (W); the watt is defined as the rate of working at 1 joule per second.

$$1 \text{ horse-power (hp)} = 746 \text{ W} = \frac{3}{4} \text{ kW (approx)},$$

where $1 \text{ kW} = 1 \text{ kilowatt}$ of 1000 watt. Thus a small motor of $\frac{1}{6}$ hp in a vacuum carpet cleaner has a power of about 125 W.

Kinetic Energy

An object is said to possess *energy* if it can do work. When an object possesses energy because it is moving, the energy is said to be *kinetic*, e.g., a flying stone can disrupt a window. Suppose that an object of mass m is moving with a velocity u , and is gradually brought to rest in a distance s by a constant force F acting against it. The kinetic energy originally possessed by the object is equal to the work done against F , and hence

$$\text{kinetic energy} = F \times s.$$

But $F = ma$, where a is the retardation of the object. Hence $F \times s = mas$. From $v^2 = u^2 + 2as$ (see p. 6), we have, since $v = 0$ and a is negative in this case,

$$0 = u^2 - 2as, \text{ i.e., } as = \frac{u^2}{2}.$$

$$\therefore \text{kinetic energy} = mas = \frac{1}{2}mu^2.$$

When m is in kg and u is in m s^{-1} , then $\frac{1}{2}mu^2$ is in *joule*. Thus a car of mass 1000 kg, moving with a velocity of 36 km h^{-1} or 10 m s^{-1} , has an amount W of kinetic energy given by

$$W = \frac{1}{2}mu^2 = \frac{1}{2} \times 1000 \times 10^2 = 50000 \text{ J}$$

Kinetic Energies due to Explosive Forces

Suppose that, due to an explosion or nuclear reaction, a particle of mass m breaks away from the total mass concerned and moves with velocity v , and a mass M is left which moves with velocity V in the opposite direction. Then

$$\frac{\text{kinetic energy, } E_1, \text{ of mass } m}{\text{kinetic energy, } E_2, \text{ of mass } M} = \frac{\frac{1}{2}mv^2}{\frac{1}{2}MV^2} = \frac{mv^2}{MV^2} \quad . \quad (1)$$

Now from the principle of the conservation of linear momentum, $mv = MV$. Thus $v = MV/m$. Substituting for v in (1).

$$\therefore \frac{E_1}{E_2} = \frac{mM^2V^2}{m^2MV^2} = \frac{M}{m} = \frac{1/m}{1/M}.$$

Hence the energy is *inversely-proportional* to the masses of the particles, that is, the smaller mass, m say, has the larger energy. Thus if E is the total energy of the two masses, the energy of the smaller mass = $ME/(M+m)$. An α -particle has a mass of 4 units and a radium nucleus a mass of 228 units. If disintegration of a thorium nucleus, mass 232, produces an α -particle and radium nucleus, and a release of energy of 4.05 MeV, where $1 \text{ MeV} = 1.6 \times 10^{-13} \text{ J}$, then

$$\text{energy of } \alpha\text{-particle} = \frac{228}{(4+228)} \times 4.05 = 3.98 \text{ MeV.}$$

The α -particle thus travels a relatively long distance before coming to rest compared to the radium nucleus.

Potential Energy

A weight held stationary above the ground has energy, because, when released, it can raise another object attached to it by a rope passing over a pulley, for example. A coiled spring also has energy, which is released gradually as the spring uncoils. The energy of the weight or spring is called *potential energy*, because it arises from the position or arrangement of the body and not from its motion. In the case of the

weight, the energy given to it is equal to the work done by the person or machine which raises it steadily to that position against the force of attraction of the earth. In the case of the spring, the energy is equal to the work done in displacing the molecules from their normal equilibrium positions against the forces of attraction of the surrounding molecules.

If the mass of an object is m , and the object is held stationary at a height h above the ground, the energy released when the object falls to the ground is equal to the work done

$$= \text{force} \times \text{distance} = \text{weight of object} \times h.$$

Suppose the weight is 5 kgf and h is 4 metre. Then, since 1 kgf = 9.8 N = 10 N approx, then

$$\begin{aligned}\text{potential energy P.E.} &= 50 (\text{N}) \times 4 (\text{m}) = 200 \text{ J} \\ &\quad (\text{more accurately, P.E.} = 192 \text{ J}).\end{aligned}$$

Generally, at a height of h ,

$$\text{potential energy} = mgh,$$

where m is in kg, h is in metre, $g = 9.8$.

EXAMPLE

Define *work*, *kinetic energy*, *potential energy*. Give one example of each of the following: (a) the conversion into kinetic energy of the work done on a body and (b) the conversion into potential energy of the work done on a body.

A rectangular block of mass 10 g rests on a rough plane which is inclined to the horizontal at an angle $\sin^{-1}(0.05)$. A force of 0.03 newton, acting in a direction parallel to a line of greatest slope, is applied to the block so that it moves up the plane. When the block has travelled a distance of 110 cm from its initial position, the applied force is removed. The block moves on and comes to rest again after travelling a further 25 cm. Calculate (i) the work done by the applied force, (ii) the gain in potential energy of the block and (iii) the value of the coefficient of sliding friction between the block and the surface of the inclined plane. How would the coefficient of sliding friction be measured if the angle of the slope could be altered? (O. and C.)

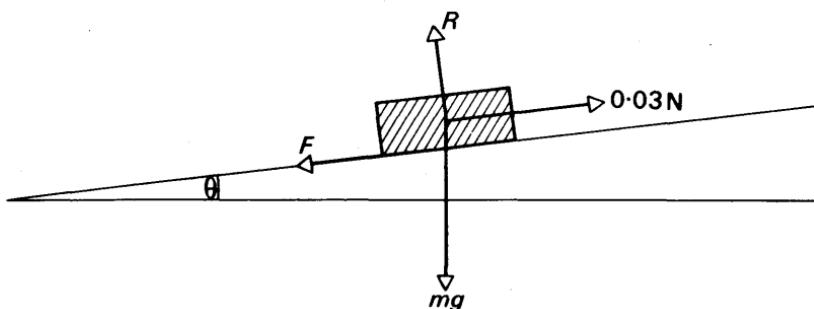


FIG. 1.20 Example

- (i) Force = 0.03 newton; distance = 110 cm = 1.1 m.

$$\therefore \text{work} = 0.03 \times 1.1 = 0.033 \text{ J.}$$

- (ii) Gain in P.E. = wt \times height moved = $0.01 \text{ kgf} \times 1.35 \sin \theta \text{ m}$,
 $= 0.01 \times 9.8 \text{ newton} \times 1.35 \times 0.05 \text{ m} = 0.0066 \text{ J (approx.)}$
- (iii) Work done against frictional force F = work done by force – gain in P.E.
 $= 0.033 - 0.0066 = 0.0264 \text{ J.}$
 $\therefore F \times 1.35 = 0.0264.$

$$\therefore F = \frac{0.0264}{1.35} \text{ newton.}$$

Normal reaction,

$$R = mg \cos \theta = mg \text{ (approx.), since } \theta \text{ is so small}$$

$$\therefore \mu = \frac{F}{R} = \frac{0.0264}{1.35 \times 0.01 \times 9.8} = 0.2 \text{ (approx.)}$$

Conservative Forces

If a ball of weight W is raised steadily from the ground to a point X at a height h above the ground, the work done is $W.h$. The potential energy, P.E., of the ball is thus $W.h$. Now whatever route is taken from ground level to X, the work done is the same—if a *longer* path is chosen, for example, the component of the weight in the particular direction must then be overcome and so the force required to move the ball is correspondingly smaller. The P.E. of the ball at X is thus independent of the route to X. This implies that if the ball is taken in a closed path round to X again, *the total work done is zero*. Work has been expended on one part of the closed path, and regained on the remaining part.

When the work done in moving round a closed path in a field to the original point is zero, the forces in the field are called *conservative forces*. The earth's gravitational field is an example of a field containing conservative forces, as we now show.

Suppose the ball falls from a place Y at a height h to another X at a height of x above the ground. Fig. 1.21. Then, if W is the weight of the ball and m its mass,

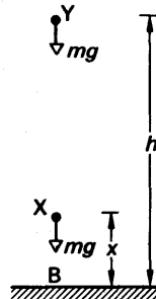


FIG. 1.21.
Mechanical
energy

$$\text{P.E. at X} = Wx = mgx$$

$$\text{and K.E. at X} = \frac{1}{2}mv^2 = \frac{1}{2}m \cdot 2g(h-x) = mg(h-x),$$

using $v^2 = 2as = 2g(h-x)$. Hence

$$\text{P.E. + K.E.} = mgx + mg(h-x) = mgh.$$

Thus at any point such as X, the total mechanical energy of the falling ball is equal to the original energy. The mechanical energy is hence constant or conserved. This is the case for a conservative field.

Non-Conservative forces. Principle of Conservation of Energy

The work done in taking a mass m round a closed path in the conservative earth's gravitational field is zero. Fig. 1.22 (i). If the work done in taking an object round a closed path to its original position is

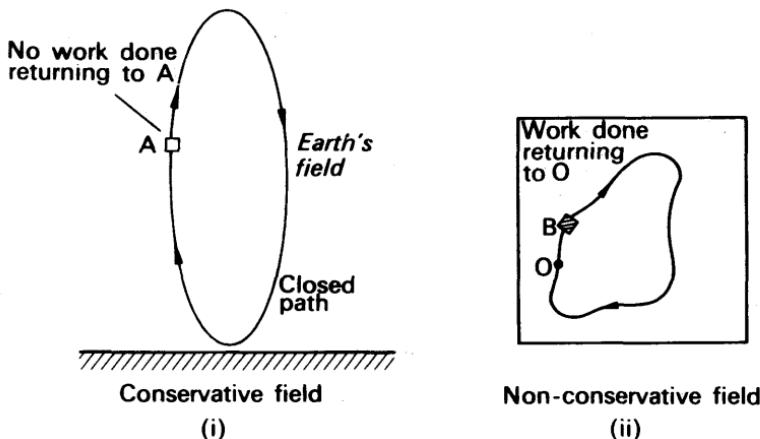


FIG. 1.22 Non-conservative and conservative fields

not zero, the forces in the field are said to be *non-conservative*. This is the case, for example, when a wooden block B is pushed round a closed path on a rough table to its initial position O. Work is therefore done against friction, both as A moves away from O and as it returns. In a conservative field, however, work is done during part of the path and regained for the remaining part.

When a body falls in the earth's gravitational field, a small part of the energy is used up in overcoming the resistance of the air. This energy is dissipated or lost as heat—it is not regained in moving the body back to its original position. This resistance is another example of the action of a non-conservative force.

Although energy may be transformed from one form to another, as in the last example from mechanical energy to heat, *the total energy in a given system is always constant*. If an electric motor is supplied with 1000 joule of energy, 850 joule of mechanical energy, 140 joule of heat energy and 10 joule of sound energy may be produced. This is called the *Principle of the Conservation of Energy* and is one of the key principles in science.

Mass and Energy

Newton said that the 'mass' of an object was 'a measure of the quantity of matter' in it. In 1905, Einstein showed from his Special Theory of Relativity that energy is released from an object when its mass decreases. His mass-energy relation states that if the mass decreases by Δm kg, the energy released in joule, ΔW , is given by

$$\Delta W = \Delta m \cdot c^2,$$

where c is the numerical value of the speed of light in m s^{-1} , which is 3×10^8 . Experiments in Radioactivity on nuclear reactions showed that Einstein's relation was true. Thus mass is a form of energy.

Einstein's relation shows that even if a small change in mass occurs, a

relatively large amount of energy is produced. Thus if $\Delta m = 1$ milligramme $= 10^{-6}$ kg, the energy ΔW released

$$= \Delta m \cdot c^2 = 10^{-6} \times (3 \times 10^8)^2 = 9 \times 10^{10} \text{ J.}$$

This energy will keep 250000 100-W lamps burning for about an hour. In practice, significant mass changes occur only in nuclear reactions.

The internal energy of a body of mass m may be considered as $E_{\text{int}} = mc^2$, where m is its rest mass. In nuclear reactions where two particles collide, a change occurs in their total kinetic energy and in their total mass. The increase in total kinetic energy is accompanied by an equal decrease in internal energy, $\Delta m \cdot c^2$. Thus the total energy, kinetic plus internal, remains constant.

Before Einstein's mass-energy relation was known, two independent laws of science were:

(1) *The Principle of the Conservation of Mass* (the total mass of a given system of objects is constant even though collisions or other actions took place between them);

(2) *The Principle of the Conservation of Energy* (the total energy of a given system is constant). From Einstein's relation, however, the two laws can be combined into one, namely, the Principle of the Conservation of Energy.

The summary below may assist the reader; it refers to the units of some of the quantities encountered, and their relations.

Quantity	SI	C.G.S.	Relations
Force (vector)	newton (N)	dyne	$10^5 \text{ dyne} = 1 \text{ N}$ $1 \text{ kgf} = 9.8 \text{ N} (\text{approx}, 10 \text{ N})$ $1 \text{ gf} = 0.0098 \text{ N}$ (approx, 0.01 N)
Mass (scalar)	kilogramme (kg)	gramme (g)	$1000 \text{ g} = 1 \text{ kg}$
Momentum (vector)	newton second (Ns)	dyne second	$10^5 \text{ dyn s} = 1 \text{ N s}$
Energy (scalar)	joule (J)	erg	$10^7 \text{ erg} = 1 \text{ J}$
Power (scalar)	watt (W)	erg s^{-1}	$1 \text{ W} = 1 \text{ J s}^{-1}$ $1 \text{ h.p.} = 746 \text{ W}$

Dimensions

By the *dimensions* of a physical quantity we mean the way it is related to the fundamental quantities mass, length and time; these are usually denoted by M, L, and T respectively. An area, length \times breadth, has dimensions $L \times L$ or L^2 ; a volume has dimensions L^3 ; density, which is mass/volume, has dimensions M/L^3 or ML^{-3} ; relative density has no dimensions, since it is the ratio of similar quantities, in this case two masses (p. 114); an angle has no dimensions, since it is the ratio of two lengths.

As an area has dimensions L^2 , the unit may be written in terms of the metre as ' m^2 '. Similarly, the dimensions of a volume are L^3 and hence

the unit is ' m^3 '. Density has dimensions ML^{-3} . The density of mercury is thus written as ' 13600 kg m^{-3} '. If some physical quantity has dimensions $ML^{-1}T^{-1}$, its unit may be written as ' $\text{kg m}^{-1}\text{s}^{-1}$ '.

The following are the dimensions of some quantities in Mechanics:

Velocity. Since velocity = $\frac{\text{distance}}{\text{time}}$, its dimensions are L/T or LT^{-1} .

Acceleration. The dimensions are those of velocity/time, i.e., L/T^2 or LT^{-2} .

Force. Since force = mass \times acceleration, its dimensions are MLT^{-2} .

Work or Energy. Since work = force \times distance, its dimensions are ML^2T^{-2} .

EXAMPLE

In the gas equation $(p + \frac{a}{V^2})(V - b) = RT$, what are the dimensions of the constants a and b ?

p represents pressure, V represents volume. The quantity a/V^2 must represent a pressure since it is added to p . The dimensions of p = [force]/[area] = $MLT^{-2}/L^2 = ML^{-1}T^{-2}$; the dimensions of $V = L^3$. Hence

$$\frac{[a]}{L^6} = ML^{-1}T^{-2}, \text{ or } [a] = ML^5T^{-2}.$$

The constant b must represent a volume since it is subtracted from V . Hence

$$[b] = L^3.$$

Application of Dimensions. Simple Pendulum

If a small mass is suspended from a long thread so as to form a simple pendulum, we may reasonably suppose that the period, T , of the oscillations depends only on the mass m , the length l of the thread, and the acceleration, g , due to gravity at the place concerned. Suppose then that

$$T = km^x l^y g^z \quad \dots \quad \dots \quad \dots \quad (i)$$

where x, y, z, k are unknown numbers. The dimensions of g are LT^{-2} from above. Now the dimensions of both sides of (i) must be the same.

$$\therefore T = M^x L^y (LT^{-2})^z.$$

Equating the indices of M, L, T on both sides, we have

$$x = 0,$$

$$y + z = 0,$$

and

$$-2z = 1.$$

$$\therefore z = -\frac{1}{2}, y = \frac{1}{2}, x = 0.$$

Thus, from (i), the period T is given by

$$T = kl^{\frac{1}{2}}g^{-\frac{1}{2}},$$

$$T = k \sqrt{\frac{l}{g}}.$$

or

We cannot find the magnitude of k by the method of dimensions, since it is a number. A complete mathematical investigation shows that $k = 2\pi$ in this case, and hence $T = 2\pi\sqrt{l/g}$. (See also p. 48).

Velocity of Transverse Wave in a String

As another illustration of the use of dimensions, consider a wave set up in a stretched string by plucking it. The velocity, V , of the wave depends on the tension, F , in the string, its length l , and its mass m , and we can therefore suppose that

$$V = kF^x l^y m^z, \quad \dots \quad \dots \quad \dots \quad \dots \quad (i)$$

where x , y , z are numbers we hope to find by dimensions and k is a constant.

The dimensions of velocity, V , are LT^{-1} , the dimensions of tension, F , are MLT^{-2} , the dimension of length, l , is L , and the dimension of mass, m , is M . From (i), it follows that

$$LT^{-1} \equiv (MLT^{-2})^x \times L^y \times M^z.$$

Equating powers of M , L , and T on both sides,

$$\therefore 0 = x + z, \quad \dots \quad \dots \quad \dots \quad \dots \quad (i)$$

$$1 = x + y, \quad \dots \quad \dots \quad \dots \quad \dots \quad (ii)$$

and

$$-1 = -2x, \quad \dots \quad \dots \quad \dots \quad \dots \quad (iii)$$

$$\therefore x = \frac{1}{2}, z = -\frac{1}{2}, y = \frac{1}{2}.$$

$$\therefore V = k \cdot F^{\frac{1}{2}} l^{\frac{1}{2}} m^{-\frac{1}{2}},$$

$$\text{or } V = k \sqrt{\frac{Fl}{m}} = k \sqrt{\frac{F}{m/l}} = k \sqrt{\frac{\text{Tension}}{\text{mass per unit length}}}$$

A complete mathematical investigation shows that $k = 1$.

The method of dimensions can thus be used to find the relation between quantities when the mathematics is too difficult. It has been extensively used in hydrodynamics, for example. See also pp. 176, 181.

EXERCISES 1

(Assume $g = 10 \text{ m s}^{-2}$, unless otherwise given)

What are the missing words in the statements 1–10?

1. The dimensions of velocity are ...
2. The dimensions of force are ...
3. Using ‘vector’ or ‘scalar’, (i) mass is a ... (ii) force is a ... (iii) energy is a ... (iv) momentum is a ...
4. Linear momentum is defined as ...
5. An ‘elastic’ collision is one in which the ... and the ... are conserved.
6. When two objects collide, their ... is constant provided no ... forces act.
7. One newton \times one metre = ...

8. 1 kilogram force = ... newton, approx.
9. The momentum of two different bodies must be added by a ... method.
10. Force is the ... of change of momentum.

Which of the following answers, A, B, C, D or E, do you consider is the correct one in the statements 11–14?

11. When water from a hosepipe is incident horizontally on a wall, the force on the wall is calculated from *A* speed of water, *B* mass × velocity, *C* mass per second × velocity, *D* energy of water, *E* momentum change.
12. When a ball of mass 2 kg moving with a velocity of 10 m s^{-1} collides head-on with a ball of mass 3 kg and both move together after collision, the common velocity is *A* 5 m s^{-1} and energy is lost, *B* 4 m s^{-1} and energy is lost, *C* 2 m s^{-1} and energy is gained, *D* 6 m s^{-1} and momentum is gained, *E* 6 m s^{-1} and energy is conserved.
13. An object of mass 2 kg moving with a velocity of 4 m s^{-1} has a kinetic energy of *A* 8 joule, *B* 16 erg, *C* 4000 erg, *D* 16 joule, *E* 40000 joule.
14. The dimensions of work are *A* ML^2T^{-2} and it is a scalar, *B* ML^2T^{-2} and it is a vector, *C* MLT^{-1} and it is a scalar, *D* ML^2T and it is a scalar, *E* MLT and it is a vector.
15. A car moving with a velocity of 36 km h^{-1} accelerates uniformly at 1 m s^{-2} until it reaches a velocity of 54 km h^{-1} . Calculate (i) the time taken, (ii) the distance travelled during the acceleration, (iii) the velocity reached 100 m from the place where the acceleration began.
16. A ball of mass 100 g is thrown vertically upwards with an initial speed of 72 km h^{-1} . Calculate (i) the time taken to return to the thrower, (ii) the maximum height reached, (iii) the kinetic and potential energies of the ball half-way up.
17. The velocity of a ship A relative to a ship B is 10.0 km h^{-1} in a direction N. 45° E. If the velocity of B is 20.0 km h^{-1} in a direction N. 60° W., find the actual velocity of A in magnitude and direction.
18. Calculate the energy of (i) a 2 kg object moving with a velocity of 10 m s^{-1} , (ii) a 10 kg object held stationary 5 m above the ground.
19. A 4 kg ball moving with a velocity of 10.0 m s^{-1} collides with a 16 kg ball moving with a velocity of 4.0 m s^{-1} (i) in the same direction, (ii) in the opposite direction. Calculate the velocity of the balls in each case if they coalesce on impact, and the loss of energy resulting from the impact. State the principle used to calculate the velocity.
20. A ship X moves due north at 30.0 km h^{-1} ; a ship Y moves N. 60° W. at 20.0 km h^{-1} . Find the velocity of Y relative to X in magnitude and direction. If Y is 10 km due east of X at this instant, find the closest distance of approach of the two ships.
21. Two buckets of mass 6 kg are each attached to one end of a long inextensible string passing over a fixed pulley. If a 2 kg mass of putty is dropped from a height of 5 m into one bucket, calculate (i) the initial velocity of the system, (ii) the acceleration of the system, (iii) the loss of energy of the 2 kg mass due to the impact.

22. A bullet of mass 25 g and travelling horizontally at a speed of 200 m s^{-1} imbeds itself in a wooden block of mass 5 kg suspended by cords 3 m long. How far will the block swing from its position of rest before beginning to return? Describe a suitable method of suspending the block for this experiment and explain briefly the principles used in the solution of the problem. (L.)

23. State the principle of the conservation of linear momentum and show how it follows from Newton's laws of motion.

A stationary radioactive nucleus of mass 210 units disintegrates into an alpha particle of mass 4 units and a residual nucleus of mass 206 units. If the kinetic energy of the alpha particle is E , calculate the kinetic energy of the residual nucleus. (N.)

24. Define linear momentum and state the principle of conservation of linear momentum. Explain briefly how you would attempt to verify this principle by experiment.

Sand is deposited at a uniform rate of 20 kilogramme per second and with negligible kinetic energy on to an empty conveyor belt moving horizontally at a constant speed of 10 metre per minute. Find (a) the force required to maintain constant velocity, (b) the power required to maintain constant velocity, and (c) the rate of change of kinetic energy of the moving sand. Why are the latter two quantities unequal? (O. & C.)

25. What do you understand by the *conservation of energy*? Illustrate your answer by reference to the energy changes occurring (a) in a body whilst falling to and on reaching the ground, (b) in an X-ray tube.

The constant force resisting the motion of a car, of mass 1500 kg, is equal to one-fifteenth of its weight. If, when travelling at 48 km per hour, the car is brought to rest in a distance of 50 m by applying the brakes, find the additional retarding force due to the brakes (assumed constant) and the heat developed in the brakes. (N.)

26. Define *uniform acceleration*. State, for each case, one set of conditions sufficient for a body to describe (a) a parabola, (b) a circle.

A projectile is fired from ground level, with velocity 500 m s^{-1} at 30° to the horizontal. Find its horizontal range, the greatest vertical height to which it rises, and the time to reach the greatest height. What is the least speed with which it could be projected in order to achieve the same horizontal range? (The resistance of the air to the motion of the projectile may be neglected.) (O.)

27. Define *momentum* and state the *law of conservation of linear momentum*.

Discuss the conservation of linear momentum in the following cases (a) a freely falling body strikes the ground without rebounding, (b) during free flight an explosive charge separates an earth satellite from its propulsion unit, (c) a billiard ball bounces off the perfectly elastic cushion of a billiard table.

A bullet of mass 10 g travelling horizontally with a velocity of 300 m s^{-1} strikes a block of wood of mass 290 g which rests on a rough horizontal floor. After impact the block and bullet move together and come to rest when the block has travelled a distance of 15 m. Calculate the coefficient of sliding friction between the block and the floor. (O. & C.)

28. Explain the distinction between *fundamental* and *derived* units, using two examples of each.

Derive the dimensions of (a) *the moment of a couple* and *work*, and comment on the results, (b) the constants a and b in van der Waals' equation $(p + a/v^2)(v - b) = rT$ for unit mass of a gas. (N.)

29. Explain what is meant by the relative velocity of one moving object with respect to another.

A ship *A* is moving eastward with a speed of 15 km h^{-1} and another ship *B*, at a given instant 10 km east of *A*, is moving southwards with a speed of 20 km h^{-1} . How long after this instant will the ships be nearest to each other, how far apart will they be then, and in what direction will *B* be sighted from *A*? (C.)

30. Define *momentum* and state the *law of conservation of linear momentum*.

Outline an experiment to demonstrate momentum conservation and discuss the accuracy which could be achieved.

Show that in a collision between two moving bodies in which no external act, the conservation of linear momentum may be deduced directly from Newton's laws of motion.

A small spherical body slides with velocity *v* and without rolling on a smooth horizontal table and collides with an identical sphere which is initially at rest on the table. After the collision the two spheres slide without rolling away from the point of impact, the velocity of the first sphere being in a direction at 30° to its previous velocity. Assuming that energy is conserved, and that there are no horizontal external forces acting, calculate the speed and direction of travel of the target sphere away from the point of impact. (O. & C.)

31. Answer the following questions making particular reference to the physical principles concerned (a) explained why the load on the back wheels of a motor car increases when the vehicle is accelerating, (b) the diagram, Fig. 1.23, shows a painter in a crate which hangs alongside a building. When the painter who weighs 100 kgf pulls on the rope the force he exerts on the floor of the crate is 45 kgf. If the crate weighs 25 kgf find the acceleration. (N.)

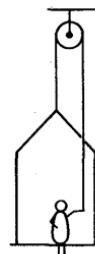


FIG. 1.23

32. Derive an expression for the kinetic energy of a moving body.

A vehicle of mass 2000 kg travelling at 10 m s^{-1} on a horizontal surface is brought to rest in a distance of 12.5 m by the action of its brakes. Calculate the average retarding force. What horse-power must the engine develop in order to take the vehicle up an incline of 1 in 10 at a constant speed of 10 m s^{-1} if the frictional resistance is equal to 20 kgf? (L.)

33. Explain what is meant by the principle of conservation of energy for a system of particles not acted upon by any external forces. What modifications are introduced when external forces are operative?

A bobsleigh is travelling at 10 m s^{-1} when it starts ascending an incline of 1 in 100. If it comes to rest after travelling 150 m up the slope, calculate the proportion of the energy lost in friction and deduce the coefficient of friction between the runners and the snow. (O. & C.)

34. State Newton's Laws of Motion and deduce from them the relation between the distance travelled and the time for the case of a body acted upon by a constant force. Explain the units in which the various quantities are measured.

A fire engine pumps water at such a rate that the velocity of the water leaving the nozzle is 15 m s^{-1} . If the jet be directed perpendicularly on to a wall and the rebound of the water be neglected, calculate the pressure on the wall (1 m^3 water weighs 1000 kg). (O. & C.)

chapter two

Circular motion. S.H.M. Gravitation

Angular Velocity

IN the previous chapter we discussed the motion of an object moving in a straight line. There are numerous cases of objects moving in a

curve about some fixed point. The earth and the moon revolve continuously round the sun, for example, and the rim of the balance-wheel of a watch moves to-and-fro in a circular path about the fixed axis of the wheel. In this chapter we shall study the motion of an object moving in a circle with a *uniform speed* round a fixed point O as centre, Fig. 2.1.

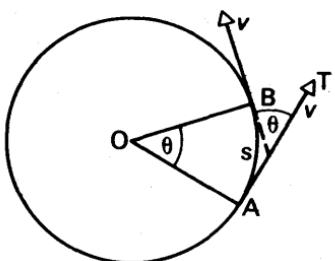


FIG. 2.1 Circular motion

an angle θ , its *angular velocity*, ω , about O is defined as the *change of the angle per second*. Thus if t is the time taken by the object to move from A to B,

$$\omega = \frac{\theta}{t} \quad (1)$$

Angular velocity is usually expressed in 'radian per second' (rad s^{-1}). From (1),

$$\theta = \omega t \quad (2)$$

which is analogous to the formula 'distance = uniform velocity \times time' for motion in a straight line. It will be noted that the time T to describe the circle once, known as the *period* of the motion, is given by

$$T = \frac{2\pi}{\omega} \quad (3)$$

since 2π radians = 360° by definition.

If s is the length of the arc AB, then $s/r = \theta$, by definition of an angle in radians.

$$\therefore s = r\theta.$$

Dividing by t , the time taken to move from A to B,

$$\therefore \frac{s}{t} = \frac{r\theta}{t}$$

But s/t = the *velocity*, v , of the rotating object, and θ/t is the angular velocity.

$$\therefore v = r\omega \quad (4)$$

Acceleration in a circle

When a stone is attached to a string and whirled round at constant speed in a circle, one can feel the force in the string needed to keep the stone moving. The presence of the force, called a *centripetal force*, implies that the stone has an acceleration. And since the force acts towards the centre of the circle, the direction of the acceleration, which is a vector quantity, is also towards the centre.

To obtain an expression for the acceleration towards the centre, consider an object moving with a constant speed v round a circle of radius r . Fig. 2.2 (i). At A, its velocity v_A is in the direction of the tangent AC; a short time δt later at B, its velocity v_B is in the direction of the tangent BD. Since their directions are different, the velocity v_B is different from the velocity v_A , although their magnitudes are both equal to v . Thus a velocity change or acceleration has occurred from A to B.

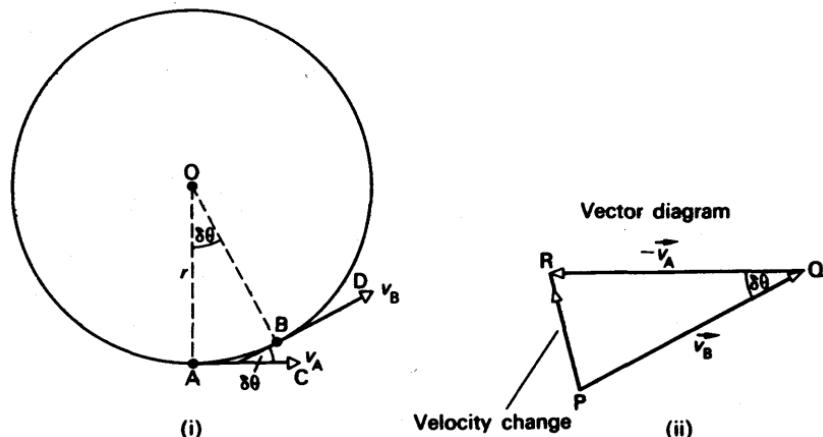


FIG. 2.2 Acceleration in circle

The velocity change from A to B = $\vec{v}_B - \vec{v}_A = \vec{v}_B + (-\vec{v}_A)$. The arrows denote vector quantities. In Fig. 2.2(ii), PQ is drawn to represent v_B in magnitude (v) and direction (BD); QR is drawn to represent $(-\vec{v}_A)$ in magnitude (v) and direction (CA). Then, as shown on p. 11,

$$\text{velocity change} = \vec{v}_B + (-\vec{v}_A) = \vec{PR}.$$

When δt is small, the angle AOB or $\delta\theta$ is small. Thus angle PQR, equal to $\delta\theta$, is small. PR then points towards O, the centre of the circle. *The velocity change or acceleration is thus directed towards the centre.*

The magnitude of the acceleration, a , is given by

$$a = \frac{\text{velocity change}}{\text{time}} = \frac{\vec{PR}}{\delta t}.$$

$$= \frac{v \cdot \delta\theta}{\delta t}.$$

since $PR = v \cdot \delta\theta$. In the limit, when δt approaches zero, $\delta\theta/\delta t = d\theta/dt = \omega$, the angular velocity. But $v = r\omega$ (p. 36). Hence, since $a = v\omega$,

$$a = \frac{v^2}{r} \quad \text{or} \quad r\omega^2.$$

Thus an object moving in a circle of radius r with a constant speed v has a constant acceleration towards the centre equal to v^2/r or $r\omega^2$.

Centripetal forces

The force F required to keep an object of mass m moving in a circle of radius $r = ma = mv^2/r$. It is called a *centripetal force* and acts *towards the centre* of the circle. When a stone A is whirled in a horizontal circle of centre O by means of a string, the tension T provides the centripetal force. Fig. 2.3 (i). For a racing car moving round a circular track, the friction at the wheels provides the centripetal force. Planets such as P, moving in a circular orbit round the sun S, have a centripetal force due to gravitational attraction between S and P (p. 59). Fig. 2.3 (ii).

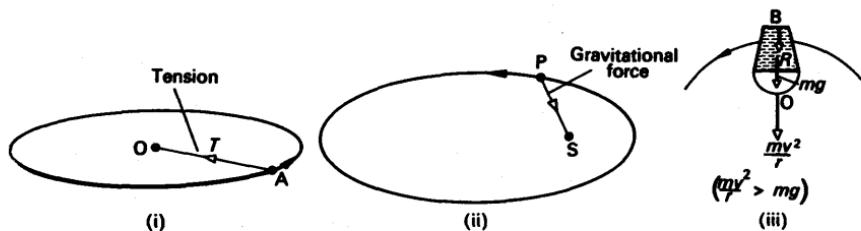


FIG. 2.3 Centripetal forces

If some water is placed in a bucket B attached to the end of a string, the bucket can be whirled in a vertical plane without any water falling out. When the bucket is vertically above the point of support O, the weight mg of the water is less than the required force mv^2/r towards the centre and so the water stays in. Fig. 2.3 (iii). The reaction R of the bucket base on the water provides the rest of the force. If the bucket is whirled slowly and $mg > mv^2/r$, part of the weight provides the force mv^2/r . The rest of the weight causes the water to accelerate downward and hence to leave the bucket.

Centrifuges

Centrifuges are used to separate particles in suspension from the less dense liquid in which they are contained. This mixture is poured into a tube in the centrifuge, which is then whirled at high speed in a horizontal circle.

The pressure gradient due to the surrounding liquid at a particular distance, r say, from the centre provides a centripetal force of $mr\omega^2$ for a small volume of liquid of mass m , where ω is the angular velocity.

If the volume of liquid is replaced by an equal volume of particles of smaller mass m' than the liquid, the centripetal force acting on the particles at the same place is then greater than that required by $(m - m')r\omega^2$. The net force urges the particles towards the centre in spiral paths, and here they collect. Thus when the centrifuge is stopped, and the container or tube assumes a vertical position, the suspension is found at the top of the tube and clear liquid at the bottom. For the same reason, cream is separated from the denser milk by spinning the mixture in a vessel. The cream spirals towards the centre and collects here.

Motion of Bicycle Rider Round Circular Track

When a person on a bicycle rides round a circular racing track, the frictional force F at the ground provides the inward force towards the centre or centripetal force. Fig. 2.4. This produces a moment about his centre of gravity G which is counterbalanced, when he leans inwards, by the moment of the normal reaction R . Thus provided no skidding occurs, $F \cdot h = R \cdot a = mg \cdot a$, since $R = mg$ for no vertical motion.

$$\therefore \frac{a}{h} = \tan \theta = \frac{F}{mg},$$

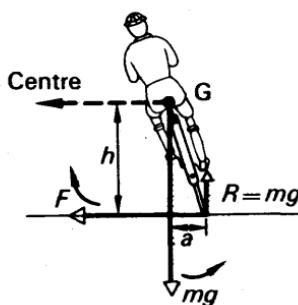


FIG. 2.4 Rider on circular track

where θ is the angle of inclination to the vertical. Now $F = mv^2/r$.

$$\therefore \tan \theta = \frac{v^2}{rg}.$$

When F is greater than the limiting friction, skidding occurs. In this case $F > \mu mg$, or $mg \tan \theta > \mu mg$. Thus $\tan \theta > \mu$ is the condition for skidding.

Motion of Car (or Train) Round Circular Track

Suppose a car (or train) is moving with a velocity v round a horizontal circular track of radius r , and let R_1 , R_2 be the respective normal re-

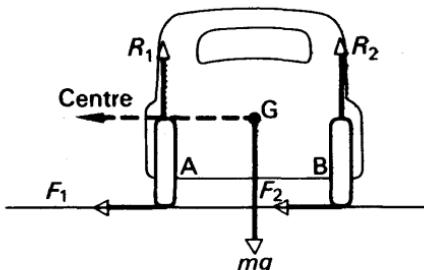


FIG. 2.5 Car on circular track

actions at the wheels A, B, and F_1, F_2 the corresponding frictional forces, Fig. 2.5. Then, for circular motion we have

$$F_1 + F_2 = \frac{mv^2}{r}, \quad \dots \quad \dots \quad \dots \quad (i)$$

and vertically $R_1 + R_2 = mg, \quad \dots \quad \dots \quad \dots \quad (ii)$

Also, taking moments about G,

$$(F_1 + F_2)h + R_1a - R_2a = 0 \quad \dots \quad \dots \quad \dots \quad (iii)$$

where $2a$ is the distance between the wheels, assuming G is mid-way between the wheels, and h is the height of G above the ground. From these three equations, we find

$$R_2 = \frac{1}{2}m\left(g + \frac{v^2h}{ra}\right)$$

and, vertically, $R_1 = \frac{1}{2}m\left(g - \frac{v^2h}{ra}\right).$

R_2 never vanishes since it always has a positive value. But if $v^2 = arg/h$, $R_1 = 0$, and the car is about to overturn outwards. R_1 will be positive if $v^2 < arg/h$.

Motion of Car (or Train) Round Banked Track

Suppose a car (or train) is moving round a banked track in a circular path of horizontal radius r , Fig. 2.6. If the only forces at the wheels

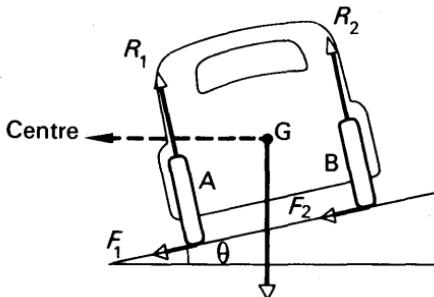


FIG. 2.6 Car on banked track

A, B are the normal reactions R_1, R_2 respectively, that is, there is no side-slip or strain at the wheels, the force towards the centre of the track is $(R_1 + R_2) \sin \theta$, where θ is the angle of inclination of the plane to the horizontal.

$$\therefore (R_1 + R_2) \sin \theta = \frac{mv^2}{r} \quad \dots \quad \dots \quad \dots \quad (i)$$

For vertical equilibrium, $(R_1 + R_2) \cos \theta = mg \quad \dots \quad \dots \quad \dots \quad (ii)$

Dividing (i) by (ii), $\therefore \tan \theta = \frac{v^2}{rg} \quad \dots \quad \dots \quad \dots \quad (iii)$

Thus for a given velocity v and radius r , the angle of inclination of the track for no side-slip must be $\tan^{-1}(v^2/rg)$. As the speed v increases, the angle θ increases, from (iii). A racing-track is made saucer-shaped because at higher speeds the cars can move towards a part of the track which is steeper and sufficient to prevent side-slip. The outer rail of a curved railway track is raised about the inner rail so that the force towards the centre is largely provided by the component of the reaction at the wheels. It is desirable to bank a road at corners for the same reason as a racing track is banked.

Thrust at Ground

Suppose now that the car (or train) is moving at such a speed that the frictional forces at A, B are F_1, F_2 respectively, each acting towards the centre of the track. Resolving horizontally,

$$\therefore (R_1 + R_2) \sin \theta + (F_1 + F_2) \cos \theta = \frac{mv^2}{r} \quad \quad (i)$$

Resolving vertically,

$$\therefore (R_1 + R_2) \cos \theta - (F_1 + F_2) \sin \theta = mg \quad \quad (ii)$$

Solving, we find

$$F_1 + F_2 = m \left(\frac{v^2}{r} \cos \theta - g \sin \theta \right) \quad \quad (iii)$$

If $\frac{v^2}{r} \cos \theta > g \sin \theta$, then $(F_1 + F_2)$ is positive; and in this case both the thrusts on the wheels at the ground are towards the centre of the track.

If $\frac{v^2}{r} \cos \theta < g \sin \theta$, then $(F_1 + F_2)$ is negative. In this case the forces F_1 and F_2 act outwards away from the centre of the track.

For stability, we have, by moments about G,

$$(F_1 + F_2)h + R_1a - R_2a = 0$$

$$\therefore (F_1 + F_2) \frac{h}{a} = R_2 - R_1.$$

From (iii), $\therefore \frac{mh}{a} \left(\frac{v^2}{r} \cos \theta - g \sin \theta \right) = R_2 - R_1 \quad \quad (iv)$

The reactions R_1, R_2 can be calculated by finding $(R_1 + R_2)$ from equations (i), (ii), and combining the result with equation (iv). This is left as an exercise to the student.

Variation of g with latitude

The acceleration due to gravity, g , varies over the earth's surface. This is due to two main causes. Firstly, the earth is elliptical, with the polar radius, b , 6.357×10^6 metre and the equatorial radius, a , 6.378×10^6 metre, and hence g is greater at the poles than at the equator, where the body is further away from the centre of the earth. Secondly, the earth rotates about the polar axis, AB. Fig. 2.7. We shall consider this effect in more detail, and suppose the earth is a perfect sphere.

In general, an object of mass m suspended by a spring-balance at a

point on the earth would be acted on by an upward force $T = mg'$, where g' is the observed or apparent acceleration due to gravity. There would also be a downward attractive force mg towards the centre of the earth, where g is the acceleration in the absence of rotation.

(1) At the poles, A or B, there is no rotation. Hence $mg - T = 0$, or $mg = T = mg'$. Thus $g' = g$.

(2) At the equator, C or D, there is a resultant force $mr\omega^2$ towards the centre where r is the earth's radius. Since OD is the vertical, we have

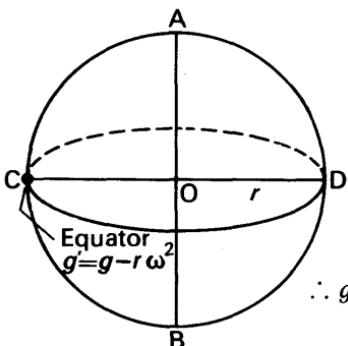


FIG. 2.7 Variation of g

$$mg - T = mr\omega^2.$$

$$\therefore T = mg - mr\omega^2 = mg'$$

$$\therefore g' = g - r\omega^2.$$

The radius r of the earth is about 6.37×10^6 m, and $\omega = [2\pi/(24 \times 3600)]$ radian per second.

$$\therefore g - g' = r\omega^2 = \frac{6.37 \times 10^6 \times (2\pi)^2}{(24 \times 3600)^2} = 0.034.$$

Latest figures give g , at the pole, 9.832 m s^{-2} , and g' , at the equator, 9.780 m s^{-2} , a difference of 0.052 m s^{-2} .

The earth's rotation accounts for 0.034 m s^{-2} .

EXAMPLE

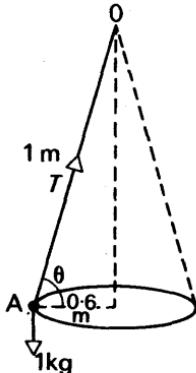


FIG. 2.8 Example

Explain the action of a centrifuge when used to hasten the deposition of a sediment from a liquid.

A pendulum bob of mass 1 kg is attached to a string 1 m long and made to revolve in a horizontal circle of radius 60 cm. Find the period of the motion and the tension of the string. (C.)

First part. See text, p. 38.

Second part. Suppose A is the bob, and OA is the string, Fig. 2.8. If T is the tension in newton, and θ is the angle of inclination of OA to the horizontal, then, for motion in the circle of radius $r = 60 \text{ cm} = 0.6 \text{ m}$,

$$T \cos \theta = \frac{mv^2}{r} = \frac{mv^2}{0.6} \quad \quad (i)$$

Since the bob A does not move in a vertical direction, then

$$T \sin \theta = mg \quad \quad (ii)$$

Now $\cos \theta = \frac{60}{100} = \frac{3}{5}$; hence $\sin \theta = \frac{4}{5}$.

From (ii),

$$\therefore T = \frac{mg}{\sin \theta} = \frac{1 \times 9.8}{4/5} = 12.25 \text{ newton.}$$

From (i)

$$v = \sqrt{\frac{0.6T \cos \theta}{m}}$$

$$= \sqrt{\frac{0.6 \times 12.25 \times 3}{1 \times 5}} = 2.1 \text{ m s}^{-1}$$

$$\therefore \text{angular velocity, } \omega = \frac{v}{r} = \frac{2.1}{0.6} = \frac{7}{2} \text{ rad s}^{-1}$$

$$\therefore \text{period, } T = \frac{2\pi}{\omega} = \frac{2\pi}{7/2} = \frac{4\pi}{7} \text{ second.}$$

$$\therefore T = 1.8 \text{ second.}$$

SIMPLE HARMONIC MOTION

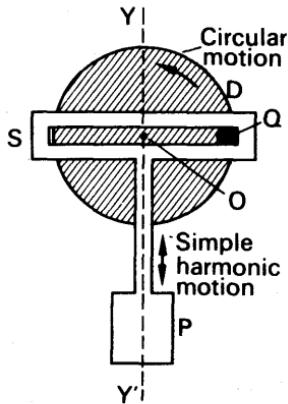


FIG. 2.9

Simple harmonic motion

motion drives P up and down. Any horizontal component of the motion merely causes Q to move along the slot S. Thus the simple harmonic motion of P is the *projection* on the vertical line YY' of the circular motion of Q.

An everyday example of an opposite conversion of motion occurs in car engines. Here the to-and-fro or 'reciprocating' motion of the piston engine is changed to a regular circular motion by connecting rods and shafts so that the wheels are turned.

Formulae in Simple Harmonic Motion

Consider an object moving round a circle of radius r and centre Z with a uniform angular velocity ω , Fig. 2.10. If CZF is a fixed diameter, the foot of the perpendicular from the moving object to this diameter moves from Z to C, back to Z and across to F, and then returns to Z, while the object moves once round the circle from O in an anti-clockwise direction. The to-and-fro motion along CZF of the foot of the perpendicular is defined as *simple harmonic motion*.

Suppose the object moving round the circle is at A at some instant, where angle OZA = θ , and suppose the foot of the perpendicular from A to CZ is M. The acceleration of the object at A is $\omega^2 r$, and this

When the bob of a pendulum moves to-and-fro through a small angle, the bob is said to be moving with *simple harmonic motion*. The prongs of a sounding tuning fork, and the layers of air near it, are moving with simple harmonic motion, and light waves can be considered due to simple harmonic variations.

Simple harmonic motion is closely associated with circular motion. An example is shown in Fig. 2.9. This illustrates an arrangement used to convert the circular motion of a disc D into the to-and-fro or simple harmonic motion of a piston P. The disc is driven about its axle O by a peg Q fixed near its rim. The vertical motion of P is the projection on the vertical line YY' of the circular motion of Q.

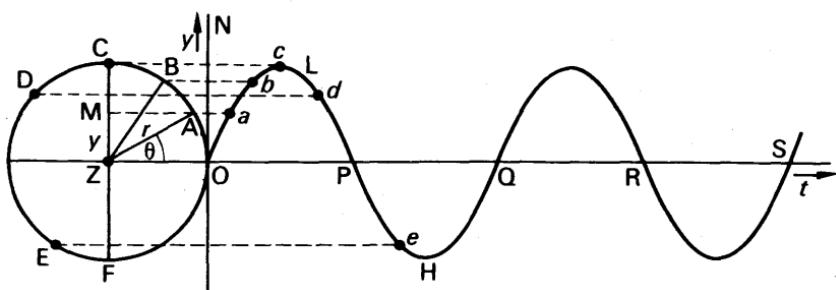


FIG. 2.10 Simple harmonic curve

acceleration is directed along the radius AZ (see p. 37). Hence the acceleration of M towards Z

$$= \omega^2 r \cos \angle AZC = \omega^2 r \sin \theta.$$

But $r \sin \theta = MZ = y$ say.

$$\therefore \text{acceleration of } M \text{ towards } Z = \omega^2 y.$$

Now ω^2 is a constant.

$$\therefore \text{acceleration of } M \text{ towards } Z \propto \text{distance of } M \text{ from } Z.$$

If we wish to express mathematically that the acceleration is always directed towards Z, we must say

$$\text{acceleration towards } Z = -\omega^2 y \quad \quad (1)$$

The minus indicates, of course, that the object begins to retard as it passes the centre, Z, of its motion. If the minus were omitted from equation (1) the latter would imply that the acceleration increases as y increases, and the object would then never return to its original position.

We can now form a definition of simple harmonic motion. It is the motion of a particle whose acceleration is always (i) directed towards a fixed point, (ii) directly proportional to its distance from that point.

Period, Amplitude. Sine Curve

The time taken for the foot of the perpendicular to move from C to F and back to C is known as the *period* (T) of the simple harmonic motion. In this time, the object moving round the circle goes exactly once round the circle from C; and since ω is the angular velocity and 2π radians (360°) is the angle described, the period T is given by

$$T = \frac{2\pi}{\omega} \quad \quad (1)$$

The distance ZC, or ZF, is the maximum distance from Z of the foot of the perpendicular, and is known as the *amplitude* of the motion. It is equal to r , the radius of the circle.

We have now to consider the variation with time, t , of the distance,

y , from Z of the foot of the perpendicular. The distance $y = ZM = r \sin \theta$. But $\theta = \omega t$, where ω is the angular velocity.

$$\therefore y = r \sin \omega t \quad \quad (2)$$

The graph of y v. t is shown in Fig. 2.10, where ON represents the y -axis and OS the t -axis; since the angular velocity of the object moving round the circle is constant, θ is proportional to the time t . Thus as the foot of the perpendicular along CZF moves from Z to C and back to Z, the graph OLP is traced out; as the foot moves from Z to F and returns to Z, the graph PHQ is traced out. The graph is a *sine curve*. The complete set of values of y from O to Q is known as a cycle. The number of cycles per second is called the *frequency*. The unit '1 cycle per second' is called '1 hertz (Hz)'. The mains frequency in Great Britain is 50 Hz or 50 cycles per second.

Velocity during S.H.M.

Suppose the object moving round the circle is at A at some instant, Fig. 2.10. The velocity of the object is $r\omega$, where r is the radius of the circle, and it is directed along the tangent at A. Consequently the velocity parallel to the diameter FC at this instant = $r\omega \cos \theta$, by resolving.

$$\therefore \text{velocity, } v, \text{ of M along FC} = r\omega \cos \theta.$$

But

$$y = r \sin \theta$$

$$\therefore \cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - y^2/r^2} = \frac{1}{r} \sqrt{r^2 - y^2}$$

$$\therefore v = \omega \sqrt{r^2 - y^2} \quad \quad (1)$$

This is the expression for the velocity of an object moving with simple harmonic motion. The maximum velocity, v_m , corresponds to $y = 0$, and hence

$$v_m = \omega r. \quad \quad (2)$$

Summarising our results:

- (1) If the acceleration a of an object = $-\omega^2 y$, where y is the distance or displacement of the object from a fixed point, the motion is simple harmonic motion.
- (2) The *period*, T , of the motion = $2\pi/\omega$, where T is the time to make a complete to-and-fro movement or cycle. The *frequency*, f , = $1/T$ and its unit is 'Hz'.
- (3) The amplitude, r , of the motion is the maximum distance on either side of the centre of oscillation.
- (4) The velocity at any instant, v , = $\omega \sqrt{r^2 - y^2}$; the maximum velocity = ωr . Fig. 2.11 (i) shows a graph of the variation of v and acceleration a with displacement y , which are respectively an ellipse and a straight line.

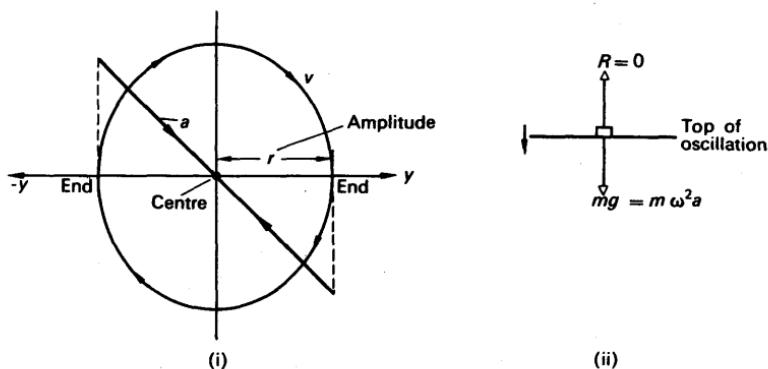


FIG. 2.11 Simple harmonic motion

S.H.M. and g

If a small coin is placed on a horizontal platform connected to a vibrator, and the amplitude is kept constant as the frequency is increased from zero, the coin will be heard 'chattering' at a particular frequency f_0 . At this stage the reaction of the table with the coin becomes zero at some part of every cycle, so that it loses contact periodically with the surface. Fig. 2.11 (ii).

The maximum acceleration in S.H.M. occurs at the end of the oscillation because the acceleration is directly proportional to the displacement. Thus maximum acceleration = $\omega^2 a$, where a is the amplitude and ω is $2\pi f_0$.

The coin will lose contact with the table when it is moving *down* with acceleration g (Fig. 2.11 (ii)). Suppose the amplitude is 8.0 cm. Then

$$(2\pi f_0)^2 a = g$$

$$\therefore 4\pi^2 f_0^2 \times 0.08 = 9.8$$

$$\therefore f_0 = \sqrt{\frac{9.8}{4\pi^2 \times 0.08}} = 1.8 \text{ Hz.}$$

Damping of S.H.M.

In practice, simple harmonic variations of a pendulum, for example, will die away as the energy is dissipated by viscous forces due to the air. The oscillation is then said to be *damped*. In the absence of any damping forces the oscillations are said to be *free*.

A simple experiment to investigate the effect of damping is illustrated in Fig. 2.12 (i). A suitable weight A is suspended from a helical spring S, a pointer P is attached to S, and a vertical scale R is set up behind P. The weight A is then set pulled down and released. The period, and the time taken for the oscillations to die away, are noted.

As shown in Fig. 2.12 (ii), A is now fully immersed in a damping medium, such as a light oil, water or glycerine. A is then set oscillating,

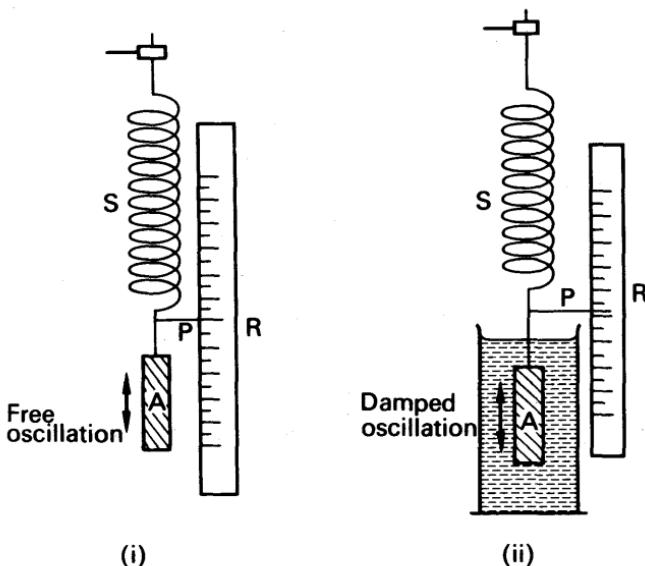


FIG. 2.12 Experiment on damped oscillations

and the time for oscillations to die away is noted. It is shorter than before and least for the case of glycerine. The decreasing amplitude in successive oscillations may also be noted from the upward limit of travel of P and the results plotted.

Fig. 2.13 (i), (ii) shows how damping produces an exponential fall in the amplitude with time.

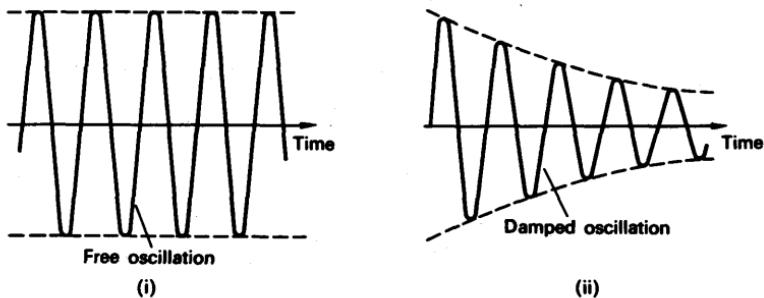


FIG. 2.13 Free and damped oscillations

The experiment works best for a period of about $\frac{1}{2}$ -second and a weight which is long and thin so that the damping is produced by non-turbulent fluid flow over the vertical sides. During the whole cycle, A must be totally immersed in the fluid.

EXAMPLE

A steel strip, clamped at one end, vibrates with a frequency of 20 Hz and an amplitude of 5 mm at the free end, where a small mass of 2 g is positioned.

Find (a) the velocity of the end when passing through the zero position, (b) the acceleration at maximum displacement, (c) the maximum kinetic and potential energy of the mass.

Suppose $y = r \sin \omega t$ represents the vibration of the strip where r is the amplitude.

(a) The velocity, $v, = \omega \sqrt{r^2 - y^2}$ (p. 45). When the end of the strip passes through the zero position $y = 0$; and the maximum speed, v_m , is given by

$$v_m = \omega r.$$

Now $\omega = 2\pi f = 2\pi \times 20$, and $r = 0.005$ m.

$$\therefore v_m = 2\pi \times 20 \times 0.005 = 0.628 \text{ m s}^{-1}.$$

(b) The acceleration $= -\omega^2 y = -\omega^2 r$ at the maximum displacement.

$$\begin{aligned}\therefore \text{acceleration} &= (2\pi \times 20)^2 \times 0.005 \\ &= 79 \text{ m s}^{-2}.\end{aligned}$$

(c) $m = 2 \text{ g} = 2 \times 10^{-3} \text{ kg}$, $v_m = 0.628 \text{ m s}^{-1}$.

$$\therefore \text{maximum K.E.} = \frac{1}{2}mv_m^2 = \frac{1}{2} \times (2 \times 10^{-3}) \times 0.628^2 = 3.9 \times 10^{-4} \text{ J (approx.)}.$$

Maximum P.E. ($v = 0$) = Maximum K.E. = 3.9×10^{-4} J.

Simple Pendulum

We shall now study some cases of simple harmonic motion. Consider a *simple pendulum*, which consists of a small mass m attached to the end of a length l of wire, Fig. 2.14. If the other end of the wire is attached to a fixed point P and the mass is displaced slightly, it oscillates to-and-fro along the arc of a circle of centre P. We shall now show that the motion of the mass about its original position O is simple harmonic motion.

Suppose that the vibrating mass is at B at some instant, where $OB = y$ and angle $OPB = \theta$. At B, the force pulling the mass towards O is directed along the tangent at B, and is equal to $mg \sin \theta$. The tension, T , in the wire has no component in this direction, since PB is perpendicular to the tangent at B. Thus, since force = mass \times acceleration (p. 13),

$$-mg \sin \theta = ma,$$

where a is the acceleration along the arc OB; the minus indicates that the force is towards O, while the displacement, y , is measured along the arc from O in the opposite direction. When θ is small, $\sin \theta = \theta$ in radians; also $\theta = y/l$. Hence,

$$-mg\theta = -mg\frac{y}{l} = ma$$

$$\therefore a = -\frac{g}{l}y = -\omega^2 y,$$

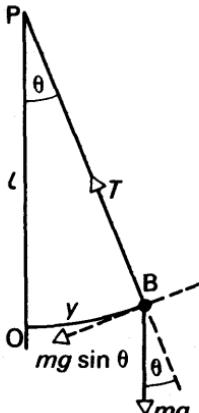


FIG. 2.14
Simple pendulum

where $\omega^2 = g/l$. Since the acceleration is proportional to the distance y from a fixed point, the motion of the vibrating mass is simple harmonic motion (p. 50). Further, from p. 50, the period $T = 2\pi/\omega$.

$$\therefore T = \frac{2\pi}{\sqrt{g/l}} = 2\pi \sqrt{\frac{1}{g}} \quad \quad (1)$$

At a given place on the earth, where g is constant, the formula shows that the period T depends only on the length, l , of the pendulum. Moreover, the period remains constant even when the amplitude of the vibrations diminish owing to the resistance of the air. This result was first obtained by Galileo, who noticed a swinging lantern one day, and timed the oscillations by his pulse. He found that the period remained constant although the swings gradually diminished in amplitude.

Determination of g by Simple Pendulum

The acceleration due to gravity, g , can be found by measuring the period, T , of a simple pendulum corresponding to a few different lengths, l , from 80 cm to 180 cm for example. To perform the experiment

accurately : (i) Fifty oscillations should be timed, (ii) a small angle of swing is essential, less than 10° , (iii) a small sphere should be tied to the end of a thread to act as the mass, and its radius added to the length of the thread to determine l .

A graph of l against T^2 is now plotted from the results, and a straight line AB, which should pass through the origin, is then drawn to lie evenly between the points, Fig. 2.15.

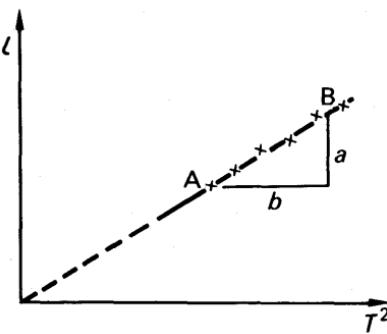


FIG. 2.15 Graph of l v. T^2

Now

$$T = 2\pi \sqrt{\frac{l}{g}}$$

$$\therefore T^2 = \frac{4\pi^2 l}{g}$$

$$\therefore g = 4\pi^2 \times \frac{l}{T^2} \quad \quad (1)$$

The gradient a/b of the line AB is the magnitude of l/T^2 ; and by substituting in (1), g can then be calculated.

If the pendulum is suspended from the ceiling of a very tall room and the string and bob reaches nearly to the floor, then one may proceed to find g by (i) measuring the period T_1 , (ii) cutting off a measured length a of the string and determining the new period T_2 with the

shortened string. Then, if h is the height of the ceiling above the bob initially, $T_1 = 2\pi\sqrt{h/g}$ and $T_2 = 2\pi\sqrt{(h-a)/g}$. Thus

$$h = \frac{gT_1^2}{4\pi^2} \quad \text{and} \quad h-a = \frac{gT_2^2}{4\pi^2}.$$

$$\therefore a = \frac{g}{4\pi^2}(T_1^2 - T_2^2).$$

$$\therefore g = \frac{4\pi^2 a}{T_1^2 - T_2^2}.$$

Thus g can be calculated from a , T_1 and T_2 . Alternatively, the period T can be measured for several lengths a . Then, since $T = 2\pi\sqrt{(h-a)/g}$,

$$h-a = \frac{g}{4\pi^2}T^2.$$

A graph of a v. T^2 is thus a straight line whose gradient is $g/4\pi^2$. Hence g can be found. The intercept on the axis of a , when $T^2 = 0$, is h , the height of the ceiling above the bob initially.

The Spiral Spring or Elastic Thread

When a weight is suspended from the end of a spring or an elastic thread, experiment shows that the extension of the spring, i.e., the increase in length, is proportional to the weight, provided that the elastic limit of the spring is not exceeded (see p. 181).

Generally, then, the tension (force), T , in a spring is proportional to the extension x produced, i.e., $T = kx$, where k is a constant of the spring.

Consider a spring or an elastic thread PA of length l suspended from a fixed point P, Fig. 2.16. When a mass m is placed on it, the spring stretches to O by a length e given by

$$mg = ke, \quad \dots \quad (i)$$

since the tension in the spring is then mg . If the mass is pulled down a little and then released, it vibrates up-and-down above and below O. Suppose at an instant that B is at a distance x below O. The tension T of the spring at B is then equal to $k(e+x)$, and hence the force towards O = $k(e+x)-mg$. Since force = mass \times acceleration,

$$\therefore -[k(e+x)-mg] = ma,$$

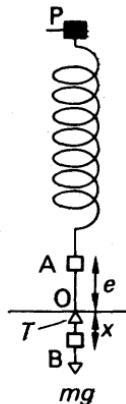


FIG. 2.16
Spiral spring

the minus indicates that the net force is upward at this instant, whereas the displacement x is measured from O in the opposite direction at the same instant. From this equation,

$$-ke - kx + mg = ma.$$

But, from (i),

$$mg = ke,$$

$$\therefore -kx = ma,$$

$$\therefore a = -\frac{k}{m}x = -\omega^2 x,$$

where $\omega^2 = k/m$. Thus the motion is simple harmonic about O, and the period T is given by

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}} \quad (1)$$

Also, since $mg = ke$, it follows that $m/k = e/g$.

$$\therefore T = 2\pi \sqrt{\frac{e}{g}} \quad (2)$$

From (1), it follows that $T^2 = 4\pi^2 m/k$. Consequently a graph of T^2 v. m should be a straight line passing through the origin. In practice, when the load m is varied and the corresponding period T is measured, a straight line graph is obtained when T^2 is plotted against m , thus verifying indirectly that the motion of the load was simple harmonic. The graph does not pass through the origin, however, owing to the mass and the movement of the various parts of the spring. This has not been taken into account in the foregoing theory and we shall now show how g may be found in this case.

Determination of g by Spiral Spring

The mass s of a vibrating spring is taken into account, in addition to the mass m suspended at the end, theory beyond the scope of this book then shows that the period of vibration, T , is given by

$$T = 2\pi \sqrt{\frac{m+\lambda s}{k}} \quad (i)$$

where λ is approximately $\frac{1}{3}$ and k is the elastic constant of the spring. Squaring (i) and re-arranging,

$$\therefore \frac{k}{4\pi^2} T^2 = m + \lambda s \quad (ii)$$

Thus, since λ , k , s are constants, a graph of T^2 v. m should be a straight line when m is varied and T observed. A straight line graph verifies indirectly that the motion of the mass at the end of the spring is simple harmonic. Further, the magnitude of $k/4\pi^2$ can be found from the slope of the line, and hence k can be calculated.

If a mass M is placed on the end of the spring, producing a steady extension e less than the elastic limit, then $Mg = ke$.

$$\therefore g = \frac{e}{M} \times k \quad (iii)$$

By attaching different masses to the spring, and measuring the corresponding extension, the magnitude of e/M can be found by plotting e v. M and measuring the slope of the line. This is called the 'static' experiment on the spring. From the magnitude of k obtained in the 'dynamic' experiment when the period was determined for different loads, the value of g can be found by substituting the magnitudes of e/M and k in (iii).

Oscillations of a Liquid in a U-Tube

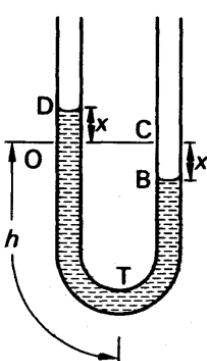


FIG. 2.17
S.H.M. of liquid

If the liquid on one side of a U-tube T is depressed by blowing gently down that side, the levels of the liquid will oscillate for a short time about their respective initial positions O, C, before finally coming to rest, Fig. 2.17.

The period of oscillation can be found by supposing that the level of the liquid on the left side of T is at D at some instant, at a height x above its original (undisturbed) position O. The level B of the liquid on the other side is then at a depth x below its original position C, and hence the excess pressure on the whole liquid, as shown on p. 110,

$$\begin{aligned} &= \text{excess height} \times \text{density of liquid} \times g \\ &= 2x\rho g. \end{aligned}$$

Now pressure = force per unit area.

\therefore force on liquid = pressure \times area of cross-section of the tube

$$= 2x\rho g \times A,$$

where A is the cross-sectional area of the tube.

This force causes the liquid to accelerate. The mass of liquid in the U-tube = volume \times density = $2hA\rho$, where $2h$ is the total length of the liquid in T. Now the acceleration, a , towards O or C is given by $\text{force} = \text{mass} \times a$.

$$\therefore -2x\rho g A = 2hA\rho a.$$

The minus indicates that the force towards O is opposite to the displacement measured from O at that instant.

$$\therefore a = -\frac{g}{h}x = -\omega^2 x,$$

where $\omega^2 = \frac{g}{h}$. The motion of the liquid about O (or C) is thus simple harmonic, and the period T is given by

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{h}{g}}.$$

P.E. and K.E. exchanges in oscillating systems

We can now make a general point about *oscillations* and *oscillating systems*. As an illustration, suppose that one end of a spring S of negligible mass is attached to a smooth object A, and that S and A are laid on a horizontal smooth table. If the free end of S is attached to the table and A is pulled slightly to extend the spring and then released, the system vibrates with simple harmonic motion. This is the case discussed on p. 50, without taking gravity into account. The centre of oscillation O is the position of the end of the spring corresponding

to its natural length, that is, when the spring is neither extended or compressed. If the spring extension obeys the law $force = kx$, where k is a constant, and m is the mass of A, then, as on p. 51, it can easily be shown that the period T of oscillation is given by:

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}}.$$

The energy of the stretched spring is *potential energy*, P.E.—its molecules are continually displaced or compressed relative to their normal distance apart. The P.E. for an extension $x = \int F \cdot dx = \int kx \cdot dx = \frac{1}{2}kx^2$.

The energy of the mass is *kinetic energy*, K.E., or $\frac{1}{2}mv^2$, where v is the velocity. Now from $x = a \sin \omega t$, $v = dx/dt = \omega a \cos \omega t$.

$$\therefore \text{total energy of spring plus mass} = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \\ = \frac{1}{2}ka^2 \sin^2 \omega t + \frac{1}{2}m\omega^2 a^2 \cos^2 \omega t.$$

But $\omega^2 = k/m$, or $k = m\omega^2$.

$$\therefore \text{total energy} = \frac{1}{2}m\omega^2 a^2 (\sin^2 \omega t + \cos^2 \omega t) = \frac{1}{2}m\omega^2 a^2 = \text{constant.}$$

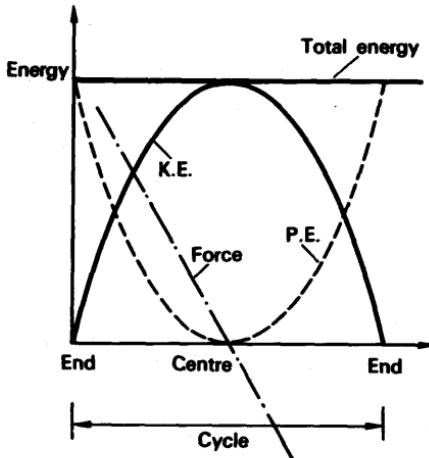


FIG. 2.18 Energy of S.H.M.

Thus the total energy of the vibrating mass and spring is constant. When the K.E. of the mass is a maximum (energy $= \frac{1}{2}m\omega^2 a^2$ and mass passing through the centre of oscillation), the P.E. of the spring is then zero ($x = 0$). Conversely, when the P.E. of the spring is a maximum (energy $= \frac{1}{2}ka^2 = \frac{1}{2}m\omega^2 a^2$ and mass at end of the oscillation), the K.E. of the mass is zero ($v = 0$). Fig. 2.18 shows the variation of P.E. and K.E. with displacement x ; the force F extending the spring, also shown, is directly proportional to the displacement from the centre of oscillation.

The constant interchange of energy between potential and kinetic energies is essential for producing and maintaining oscillations, whatever their nature. In the case of the oscillating bob of a simple pendulum,

for example, the bob loses kinetic energy after passing through the middle of the swing, and then stores the energy as potential energy as it rises to the top of the swing. The reverse occurs as it swings back. In the case of oscillating layers of air when a sound wave passes, kinetic energy of the moving air molecules is converted to potential energy when the air is compressed. In the case of electrical oscillations, a coil L and a capacitor C in the circuit constantly exchange energy; this is stored alternately in the magnetic field of L and the electric field of C .

EXAMPLES

1. Define *simple harmonic motion* and state the relation between displacement from its mean position and the restoring force when a body executes simple harmonic motion.

A body is supported by a spiral spring and causes a stretch of 1.5 cm in the spring. If the mass is now set in vertical oscillation of small amplitude, what is the periodic time of the oscillation? (L .)

First part. Simple harmonic motion is the motion of an object whose acceleration is proportional to its distance from a fixed point and is always directed towards that point. The relation is: Restoring force = $-k \times$ distance from fixed point, where k is a constant.

Second part. Let m be the mass of the body in kg. Then, since 1.5 cm = 0.015 m

$$mg = k \times 0.015 \quad \dots \quad (i)$$

where k is a constant of the spring in $N\ m^{-1}$. Suppose the vibrating body is x m below its original position at some instant and is moving downwards. Then since the extension is $(x + 0.015)$ m, the net downward force

$$\begin{aligned} &= mg - k(x + 0.015) \\ &= mg - k \times 0.015 - kx = -kx \end{aligned}$$

from (i). Now mass \times acceleration = force.

$$\therefore m \times \text{acceleration} = -kx$$

$$\therefore \text{acceleration} = -\frac{k}{m}x.$$

But, from (i),

$$\frac{k}{m} = \frac{g}{0.015}$$

$$\therefore \text{acceleration} = \frac{g}{0.015}x = -\omega^2 x,$$

where $\omega^2 = g/0.015$.

$$\begin{aligned} \therefore \text{period } T &= \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{0.015}{g}} = 2\pi \sqrt{\frac{0.015}{9.8}} \\ &= 0.25 \text{ second.} \end{aligned}$$

2. A small bob of mass 20 g oscillates as a simple pendulum, with amplitude 5 cm and period 2 seconds. Find the velocity of the bob and the tension in the supporting thread, when the velocity of the bob is a maximum.

First part. See text.

Second part. The velocity, v , of the bob is a maximum when it passes through its original position. With the usual notation (see p. 45), the maximum velocity v_m is given by

$$v_m = \omega r,$$

where r is the amplitude of 0.05 m. Since $T = 2\pi/\omega$,

$$\therefore \omega = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi \quad \quad (i)$$

$$\therefore v_m = \omega r = \pi \times 0.05 = 0.16 \text{ m s}^{-1}.$$

Suppose P is the tension in the thread. The net force towards the centre of the circle along which the bob moves is then given by $(P - mg)$. The acceleration towards the centre of the circle, which is the point of suspension, is v_m^2/l , where l is the length of the pendulum.

$$\therefore P - mg = \frac{mv_m^2}{l}$$

$$\therefore P = mg + \frac{mv_m^2}{l}$$

Now

$$T = 2\pi \sqrt{\frac{l}{g}}$$

$$\therefore l = \frac{gT^2}{4\pi^2} = \frac{g \times 4}{4\pi^2} = \frac{g}{\pi^2}$$

Since $m = 0.02 \text{ kg}$, $g = 9.8 \text{ m s}^{-2}$, it follows from above that

$$P = 0.02 \times 9.8 + \frac{0.02 \times (0.05\pi)^2 \times \pi^2}{9.8}$$

$$= 19.65 \times 10^{-2} \text{ newton}$$

Waves. Wave equation

Waves and their properties can be demonstrated by producing them on the surface of water, as in a ripple tank. As the wave travels outwards from the centre of disturbance, it reaches more distant particles of water at a later time. Thus the particles of water vibrate out of *phase* with each other while the wave travels. It should be noted that the vibrating particles are the origin of the wave. Their mean position remains the same as the wave travels, but like the simple harmonic oscillators previously discussed, they store and release *energy* which is handed on from one part of the medium to another. The wave shows the energy travelling through the medium.

If the displacement y of a vibrating particle P is represented by $y = a \sin \omega t$, the displacement of a neighbouring particle Q can be represented by $y = a \sin (\omega t + \phi)$. ϕ is called the *phase angle* between the two vibrations. If $\phi = \pi/2$ or 90° , the vibration of Q is $y = a \sin (\omega t + \pi/2)$. In this case, $y = 0$ when $t = 0$ for P , but $y = a \sin \pi/2 = a$ when $t = 0$ for Q . Comparing the two simple harmonic variations, it can be seen that Q *leads* on P by a quarter of a period.

If the wave is 'frozen' at different times, the displacements of the various particles will vary according to their position or distance x

from some chosen origin such as the centre of disturbance. Now the wavelength, λ , of a wave is the distance between successive crests or troughs. At these points the phase difference is 2π . Consequently the phase angle for a distance x is $(x/\lambda) \times 2\pi$ or $2\pi x/\lambda$. The wave equation, which takes x into account as well as the time t , can thus be written as:

$$y = a \sin \left(2\pi \frac{t}{T} - 2\pi \frac{x}{\lambda} \right) = a \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda} \right) \quad . . . \quad (1)$$

Other forms of the wave equation may be used. The velocity v of a wave is the distance travelled by the disturbance in 1 second. If the frequency of the oscillations is f , then f waves travel outwards in 1 second. Each wave occupies a length λ . Hence $v = f\lambda$. Further, the period T is the time for 1 oscillation. Thus $f = 1/T$ and hence $v = f\lambda = \lambda/T$. Substituting for T in (1), the wave equation may also be written as:

$$y = a \sin \frac{2\pi}{\lambda} (vt - x) \quad . . . \quad (2)$$

The wave equation in (1) or (2) is a *progressive wave*. The energy of the wave travels outwards through the medium as time goes on.

Longitudinal and transverse waves

Waves can be classified according to the direction of their vibrations. A *longitudinal wave* is one produced by *vibrations parallel* to the direction of travel of the wave. An example is a sound wave. The layers of air are always vibrating in a direction parallel to the direction of travel of the wave. A longitudinal wave can be seen travelling in a 'Slinky' coil when one end is fixed and the other is pulled to-and-fro in the direction of the coil.

A *transverse wave* is one produced by *vibrations perpendicular* to the direction of travel of the wave. Light waves are transverse waves. The wave along a bowed string of a violin is a transverse wave.

Velocity of waves

There are various types of waves. A longitudinal wave such as a sound wave is a *mechanical wave*. The speed v with which the energy travels depends on the restoring stress after particles in the medium are strained from their original position. Thus v depends on the *modulus of elasticity* of the medium. It also depends on the inertia of the particles, of which the mass per unit volume or density ρ is a measure.

By dimensions, as well as rigorously, it can be shown that

$$v = \sqrt{\frac{\text{modulus of elasticity}}{\rho}}$$

For a solid, the modulus is Young's modulus, E . Thus $v = \sqrt{E/\rho}$. For a liquid or gas, the modulus is the bulk modulus, k . Hence $v = \sqrt{k/\rho}$. In air, $k = \gamma p$, where γ is the ratio of the principal specific heats of air and p is the atmospheric pressure. Thus $v = \sqrt{\gamma p/\rho}$ (p. 163).

When a taut string is plucked or bowed, the velocity of the transverse wave along it is given by $v = \sqrt{T/m}$, where T is the tension and m is the

mass per unit length of the string. In this case T provides the restoring force acting on the displaced particles of string and m is a measure of their inertia.

Electromagnetic waves, which are due to electric and magnetic vibrations, form an important group of waves in nature. Radio waves, infra-red, visible and ultra-violet light, X-rays, and γ -rays are all electromagnetic waves, ranging from long wavelength such as 1000 metres (radio waves) to short wavelengths such as 10^{-8} m (γ -waves). Unlike the mechanical waves, no material medium is needed to carry the waves. The speed of all electromagnetic waves in a vacuum is the same, about 3×10^8 metre per second. The speed varies with wavelength in material media and this explains why dispersion (separation of colours) of white light is produced by glass.

Stationary waves

The equation $y = a \sin 2\pi(t/T - x/\lambda)$ represents a progressive wave travelling in the x -direction. A wave of the same amplitude and frequency travelling in the *opposite direction* is represented by the same form of equation but with $-x$ in place of x , that is, by $y = a \sin 2\pi(t/T + x/\lambda)$.

The *principle of superposition* states that the combined effect or resultant of two waves in a medium can be obtained by adding the displacements at each point due to the respective waves. Thus if the displacement due to one wave is represented by y_1 , and that due to the other wave by y_2 , the resultant displacement y is given by

$$\begin{aligned} y &= y_1 + y_2 = a \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) + a \sin 2\pi\left(\frac{t}{T} + \frac{x}{\lambda}\right) \\ &= 2a \sin 2\pi \frac{t}{T} \cdot \cos 2\pi \frac{x}{\lambda} = A \sin 2\pi \frac{t}{T} \end{aligned}$$

where $A = 2a \cos 2\pi x/\lambda$.

A represents the amplitude at different points in the medium. When $x = 0$, $y = A$; when $x = \lambda/4$, $A = 0$; when $x = \lambda/2$, $y = -A$; when $x = -3\lambda/4$, $y = 0$. Thus at some points called *antinodes*, A , the amplitude of vibration is a maximum. At points half-way between the antinodes called *nodes*, N , the amplitude is zero, that is, there is no vibration here. Fig. 2.19 (i). This type of wave, which stays in one place in a medium, is called a *stationary* or *standing wave*. Stationary waves may be produced which are either longitudinal or transverse.

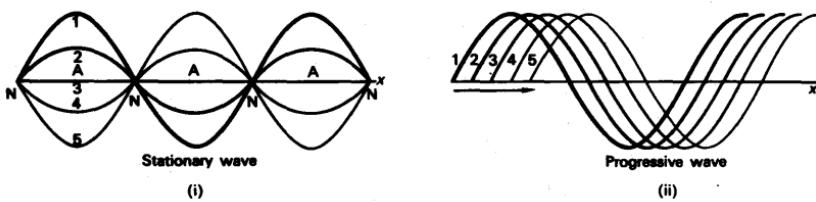


FIG. 2.19 Stationary and progressive waves

Unlike the progressive wave, where the energy travels outwards through the medium, Fig. 2.19 (ii), the energy of the stationary wave remains stored in one part of the medium. Stationary waves are produced in musical instruments when they are played. Stationary radio waves are also produced in receiving aerials. Stationary waves, due to electron motion, are believed to be present around the nucleus of atoms.

Interference. Diffraction

A stationary wave is a special case of *interference* between two waves. Another example occurs when two tuning forks of nearly equal frequency are sounded together. A periodic variation of loud sounds called 'beats' is then heard. They are due to the periodic variation of the amplitude of the resultant wave. If two very close coherent sources of light are obtained, interference between the two waves may produce bright and dark bands.

Diffraction is the name given to the interference between waves coming from coherent sources on the same undivided wavefront. The effect is pronounced when a wave is incident on a narrow opening whose width is of comparable order to the wavelength. The wave now spreads out or is 'diffracted' after passing through the slit. If the width of the slit, however, is large compared with the wavelength, the wave passes straight through the opening without any noticeable diffraction. This is why visible light, which has wavelengths of the order of 6×10^{-7} m, passes straight through wide openings and produces sharp shadows; whereas sound, which has wavelengths over a million times longer and of the order of say 0.5 m, can be heard round corners.

Further details of wave phenomena are discussed in the Sound and Optics sections of the book.

GRAVITATION

Kepler's Laws

The motion of the planets in the heavens had excited the interest of the earliest scientists, and Babylonian and Greek astronomers were able to predict their movements fairly accurately. It was considered for some time that the earth was the centre of the universe, but about 1542 COPERNICUS suggested that the planets revolved round the sun as centre. A great advance was made by KEPLER about 1609. He had studied for many years the records of observations on the planets made by TYCHO BRAHE, and he enunciated three laws known by his name. These state:

- (1) The planets describe ellipses about the sun as one focus.
- (2) The line joining the sun and the planet sweeps out equal areas in equal times.

(3) The squares of the periods of revolution of the planets are proportional to the cubes of their mean distances from the sun.

The third law was announced by Kepler in 1619.

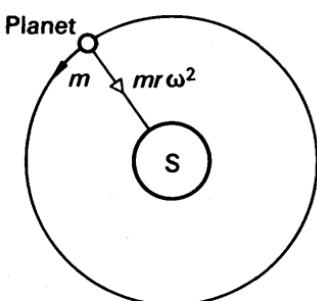
Newton's Law of Gravitation

About 1666, at the early age of 24, NEWTON discovered a universal law known as the *law of gravitation*.

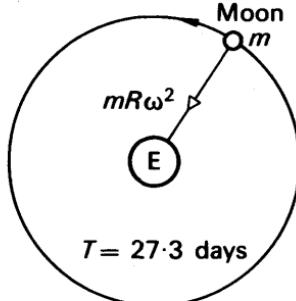
He was led to this discovery by considering the motion of a planet moving in a circle round the sun S as centre. Fig. 2.20 (i). The force acting on the planet of mass m is $mr\omega^2$, where r is the radius of the circle and ω is the angular velocity of the motion (p. 38). Since $\omega = 2\pi/T$, where T is the period of the motion,

$$\text{force on planet} = mr \left(\frac{2\pi}{T} \right)^2 = \frac{4\pi^2 mr}{T^2}.$$

This is equal to the force of attraction of the sun on the planet. Assuming an inverse-square law, then, if k is a constant,



(i)



(ii).

FIG. 2.20 Satellites

$$\text{force on planet} = \frac{km}{r^2}.$$

$$\therefore \frac{km}{r^2} = \frac{4\pi^2 mr}{T^2}$$

$$\therefore T^2 = \frac{4\pi^2}{k} r^3$$

$$\therefore T^2 \propto r^3,$$

since k, π are constants.

Now Kepler had announced that the squares of the periods of revolution of the planets are proportional to the cubes of their mean distances from the sun (see above). Newton thus suspected that the force between the sun and the planet was inversely proportional to the square of the distance between them. The great scientist now proceeded to test the inverse-square law by applying it to the case of the moon's motion

round the earth. Fig. 2.20(ii). The moon has a period of revolution, T , about the earth of approximately 27.3 days, and the force on it = $mR\omega^2$, where R is the radius of the moon's orbit and m is its mass.

$$\therefore \text{force} = mR\left(\frac{2\pi}{T}\right)^2 = \frac{4\pi^2 mR}{T^2}.$$

If the planet were at the earth's surface, the force of attraction on it due to the earth would be mg , where g is the acceleration due to gravity. Fig. 2.20 (ii). Assuming that the force of attraction varies as the inverse square of the distance between the earth and the moon,

$$\therefore \frac{4\pi^2 mR}{T^2} : mg = \frac{1}{R^2} : \frac{1}{r^2},$$

where r is the radius of the earth.

$$\begin{aligned} \therefore \frac{4\pi^2 R}{T^2 g} &= \frac{r^2}{R^2}, \\ \therefore g &= \frac{4\pi^2 R^3}{r^2 T^2}. \end{aligned} \quad (1)$$

Newton substituted the then known values of R , r , and T , but was disappointed to find that the answer for g was not near to the observed value, 9.8 m s^{-2} . Some years later, he heard of a new estimate of the radius of the moon's orbit, and on substituting its value he found that the result for g was close to 9.8 m s^{-2} . Newton saw that a universal law could be formulated for the attraction between any two particles of matter. He suggested that: *The force of attraction between two given masses is inversely proportional to the square of their distance apart.*

Gravitational Constant, G, and its Determination

From Newton's law, it follows that the force of attraction, F , between two masses m , M at a distance r apart is given by $F \propto \frac{mM}{r^2}$.

$$\therefore F = G \frac{mM}{r^2}, \quad (2)$$

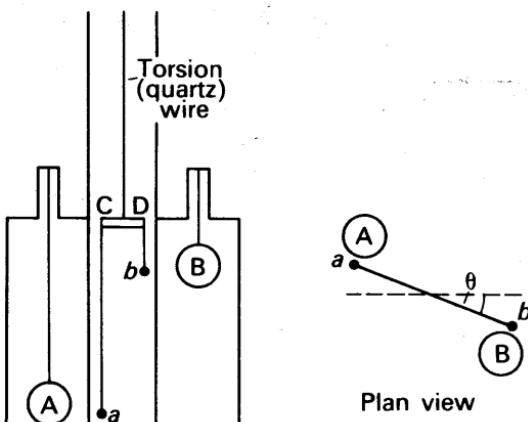
where G is a universal constant known as the *gravitational constant*. This expression for F is *Newton's law of gravitation*.

From (2), it follows that G can be expressed in ' $\text{N m}^2 \text{ kg}^{-2}$ '. The dimensions of G are given by

$$[G] = \frac{\text{MLT}^{-2} \times \text{L}^2}{\text{M}^2} = \text{M}^{-1} \text{L}^3 \text{T}^{-2}.$$

Thus the unit of G may also be expressed as $\text{m}^3 \text{ kg}^{-1} \text{ s}^{-2}$.

A celebrated experiment to measure G was carried out by C. V. Boys in 1895, using a method similar to one of the earliest determinations of G by CAVENDISH in 1798. Two identical balls, a , b , of gold,

FIG. 2.21 Experiment on G

5 mm in diameter, were suspended by a long and a short fine quartz fibre respectively from the ends, C, D, of a highly-polished bar CD, Fig. 2.21. Two large identical lead spheres, A, B, 115 mm in diameter, were brought into position near a , b respectively. As a result of the attraction between the masses, two equal but opposite forces acted on CD. The bar was thus deflected, and the angle of deflection, θ , was measured by a lamp and scale method by light reflected from CD. The high sensitivity of the quartz fibres enabled the small deflection to be measured accurately, and the small size of the apparatus allowed it to be screened considerably from air convection currents.

Calculation for G

Suppose d is the distance between a , A, or b , B, when the deflection is θ . Then if m , M are the respective masses of a , A,

$$\text{torque of couple on } CD = G \frac{mM}{d^2} \times CD.$$

But

$$\text{torque of couple} = c\theta,$$

where c is the torque in the torsion wire per unit radian of twist (p. 192).

$$\begin{aligned} \therefore G \frac{mM}{d^2} \times CD &= c\theta \\ \therefore G &= \frac{c\theta d^2}{mM \times CD} \end{aligned} \quad (1)$$

The constant c was determined by allowing CD to oscillate through a small angle and then observing its period of oscillation, T , which was of the order of 3 minutes. If I is the known moment of inertia of the system about the torsion wire, then (see p. 75),

$$T = 2\pi \sqrt{\frac{I}{c}}$$

The constant c can now be calculated, and by substitution in (i), G can be determined. Accurate experiments showed that $G = 6.66 \times 10^{-11}$ N m² kg⁻² and Heyl, in 1942, found G to be 6.67×10^{-11} N m² kg⁻².

Mass and Density of Earth

At the earth's surface the force of attraction on a mass m is mg , where g is the acceleration due to gravity. Now it can be shown that it is legitimate in calculations to assume that the mass, M , of the earth is concentrated at its centre, if it is a sphere. Assuming that the earth is spherical and of radius r , it then follows that the force of attraction of the earth on the mass m is GmM/r^2 .

$$\therefore G\frac{mM}{r^2} = mg.$$

$$\therefore g = \frac{GM}{r^2}.$$

$$\therefore M = \frac{gr^2}{G}.$$

Now, $g = 9.8$ m s⁻², $r = 6.4 \times 10^6$ m, $G = 6.7 \times 10^{-11}$ N m² kg⁻².

$$\therefore M = \frac{9.8 \times (6.4 \times 10^6)^2}{6.7 \times 10^{-11}} = 6.0 \times 10^{24}$$
 kg.

The volume of a sphere is $4\pi r^3/3$, where r is its radius. Thus the density, ρ , of the earth is approximately given by

$$\rho = \frac{M}{V} = \frac{gr^2}{4\pi r^3 G/3} = \frac{3g}{4\pi r G}.$$

By substituting known values of g , G , and r , the mean density of the earth is found to be about 5500 kg m⁻³. The density may approach a value of 10000 kg m⁻³ towards the interior.

It is now believed that gravitational force travels with the speed of light. Thus if the gravitational force between the sun and earth were suddenly to disappear by the vanishing of the sun, it would take about 7 minutes for the effect to be experienced on the earth. The earth would then fly off along a tangent to its original curved path.

Gravitational and inertial mass

The mass m of an object appearing in the expression $F = ma$, force = mass \times acceleration, is the *inertial mass*, as stated on p. 13. It is a measure of the reluctance of the object to move when forces act on it. It appears in $F = ma$ from Newton's second law of motion.

The 'mass' of the same object concerned in Newton's theory of gravitational attraction can be distinguished from the inertial mass. This is called the *gravitational mass*. If it is given the symbol m_g , then $F_g = GMm_g/r^2$, where F_g is the gravitational force, M is the mass of the earth and r its radius. Now $GM/r^2 = g$, the acceleration due to gravity (see above). Thus $F_g = m_g g = W$, the weight of the object.

In the simple pendulum theory on p. 48, we can derive the period T using $W = \text{weight} = m_g g$ in place of the symbols adopted there.

Thus

$$-m_g \frac{y}{l} = ma,$$

or

$$a = -\frac{m_g g}{ml} \cdot y = -\omega^2 y.$$

$$\therefore T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{ml}{m_g g}}.$$

Experiments show that to a high degree of accuracy, $T = 2\pi\sqrt{l/g}$ no matter what mass is used, that is, the period depends only on l and g . Thus $m = m_g$, or the gravitational mass is equal to the inertial mass to the best of our present knowledge.

Mass of Sun

The mass M_s of the sun can be found from the period of a satellite and its distance from the sun. Consider the case of the earth. Its period T is about 365 days or $365 \times 24 \times 3600$ seconds. Its distance r_s from the centre of the sun is about 1.5×10^{11} m. If the mass of the earth is m , then, for circular motion round the sun,

$$\frac{GM_s m}{r_s^2} = mr_s \omega^2 = \frac{mr_s 4\pi^2}{T^2},$$

$$\therefore M_s = \frac{4\pi^2 r_s^3}{GT^2} = \frac{4\pi^2 \times (1.5 \times 10^{11})^3}{6.7 \times 10^{-11} \times (365 \times 24 \times 3600)^2} = 2 \times 10^{30} \text{ kg.}$$

Orbits round the earth

Satellites can be launched from the earth's surface to circle the earth. They are kept in their orbit by the gravitational attraction of the earth.

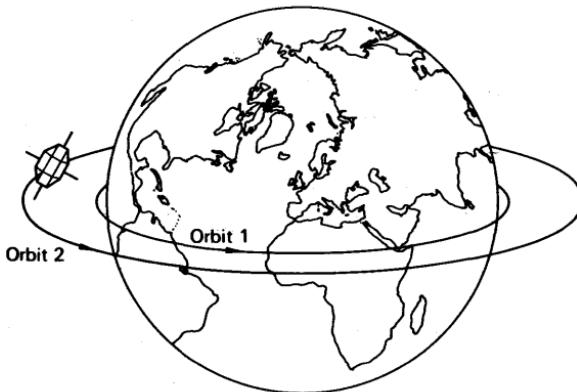


FIG. 2.22 Orbits round earth

Consider a satellite of mass m which just circles the earth of mass M

close to its surface in an orbit 1. Fig. 2.22 (i). Then, if r is the radius of the earth,

$$\frac{mv^2}{r} = G \frac{Mm}{r^2} = mg,$$

where g is the acceleration due to gravity at the earth's surface and v is the velocity of m in its orbit. Thus $v^2 = rg$, and hence, using $r = 6.4 \times 10^6$ m and $g = 9.8$ m s $^{-2}$,

$$\begin{aligned} v &= \sqrt{rg} = \sqrt{6.4 \times 10^6 \times 9.8} = 8 \times 10^3 \text{ m s}^{-1} \text{ (approx)}, \\ &= 8 \text{ km s}^{-1}. \end{aligned}$$

The velocity v in the orbit is thus about 8 km s $^{-1}$. In practice, the satellite is carried by a rocket to the height of the orbit and then given an impulse, by firing jets, to deflect it in a direction parallel to the tangent of the orbit (see p. 66). Its velocity is boosted to 8 km s $^{-1}$ so that it stays in the orbit. The period in orbit

$$\begin{aligned} &= \frac{\text{circumference of earth}}{v} = \frac{2\pi \times 6.4 \times 10^6 \text{ m}}{8 \times 10^3 \text{ m s}^{-1}} \\ &= 5000 \text{ seconds (approx)} = 83 \text{ min.} \end{aligned}$$

Parking Orbits

Consider now a satellite of mass m circling the earth in the plane of the equator in an orbit 2 concentric with the earth. Fig. 2.22 (ii). Suppose the direction of rotation as the same as the earth and the orbit is at a distance R from the centre of the earth. Then if v is the velocity in orbit,

$$\frac{mv^2}{R} = \frac{GMm}{R^2}.$$

But $GM = gr^2$, where r is the radius of the earth.

$$\begin{aligned} \therefore \frac{mv^2}{R} &= \frac{mgr^2}{R^2} \\ \therefore v^2 &= \frac{gr^2}{R}. \end{aligned}$$

If T is the period of the satellite in its orbit, then $v = 2\pi R/T$.

$$\begin{aligned} \therefore \frac{4\pi^2 R^2}{T^2} &= \frac{gr^2}{R} \\ \therefore T^2 &= \frac{4\pi^2 R^3}{gr^2}. \quad \quad (i) \end{aligned}$$

If the period of the satellite in its orbit is exactly equal to the period of the earth as it turns about its axis, which is 24 hours, *the satellite will stay over the same place on the earth while the earth rotates*. This

is sometimes called a 'parking orbit'. Relay satellites can be placed in parking orbits, so that television programmes can be transmitted continuously from one part of the world to another. *Syncom* was a satellite used for transmission of the Tokio Olympic Games in 1964.

Since $T = 24$ hours, the radius R can be found from (i). Thus from

$$R = \sqrt[3]{\frac{T^2 gr^2}{4\pi^2}} \quad \text{and} \quad g = 9.8 \text{ m s}^{-2}, r = 6.4 \times 10^6 \text{ m},$$

$$\therefore R = \sqrt[3]{\frac{(24 \times 3600)^2 \times 9.8 \times (6.4 \times 10^6)^2}{4\pi^2}} = 42400 \text{ km}$$

The height above the earth's surface of the parking orbit

$$= R - r = 42400 - 6400 = 36000 \text{ km.}$$

In the orbit, the velocity of the satellite

$$= \frac{2\pi R}{T} = \frac{2\pi \times 42400}{24 \times 3600 \text{ seconds}} = 3.1 \text{ km s}^{-1}.$$

Weightlessness

When a rocket is fired to launch a spacecraft and astronaut into orbit round the earth, the initial acceleration must be very high owing to the large initial thrust required. This acceleration, a , is of the order of $15g$, where g is the gravitational acceleration at the earth's surface.

Suppose S is the reaction of the couch to which the astronaut is initially strapped. Fig. 2.23 (i). Then, from $F = ma$, $S - mg = ma = m \cdot 15g$, where m is the mass of the astronaut. Thus $S = 16mg$. This force is 16 times the weight of the astronaut and thus, initially, he experiences a large force.

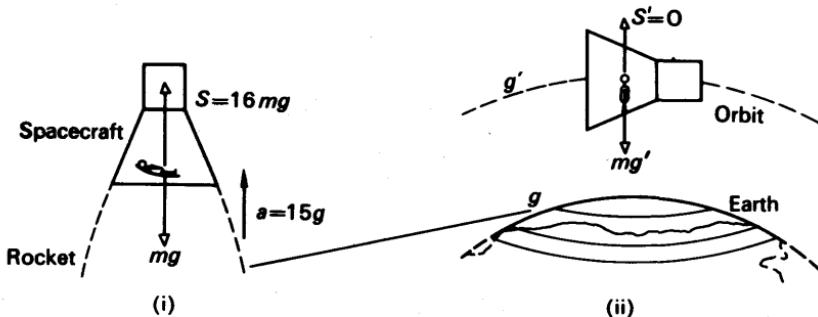


FIG. 2.23 Weight and weightlessness

In orbit, however, the state of affairs is different. This time the acceleration of the spacecraft and astronaut are both g' in magnitude,

where g' is the acceleration due to gravity outside the spacecraft at the particular height of the orbit. Fig. 2.23 (ii). If S' is the reaction of the surface of the spacecraft in contact with the astronaut, then, for circular motion,

$$F = mg' - S' = ma = mg'.$$

Thus $S' = 0$. Consequently the astronaut becomes 'weightless'; he experiences no reaction at the floor when he walks about, for example. At the earth's surface we feel the reaction at the ground and are thus conscious of our weight. Inside a lift which is falling fast, the reaction at our feet diminishes. If the lift falls freely, the acceleration of objects inside is the same as that outside and hence the reaction on them is zero. This produces the sensation of 'weightlessness'. In orbit, as in Fig. 2.23 (ii), objects inside a spacecraft are also in 'free fall' because they have the same acceleration g' as the spacecraft. Consequently the sensation of weightlessness is experienced.

EXAMPLE

A satellite is to be put into orbit 500 km above the earth's surface. If its vertical velocity after launching is 2000 m s^{-1} at this height, calculate the magnitude and direction of the impulse required to put the satellite directly into orbit, if its mass is 50 kg. Assume $g = 10 \text{ m s}^{-2}$; radius of earth, $R = 6400 \text{ km}$.

Suppose u is the velocity required for orbit, radius r . Then, with usual notation,

$$\frac{mu^2}{r} = \frac{GmM}{r^2} = \frac{gR^2m}{r^2}, \text{ as } \frac{GM}{R^2} = g.$$

$$\therefore u^2 = \frac{gR^2}{r}.$$

Now $R = 6400 \text{ km}$, $r = 6900 \text{ km}$, $g = 10 \text{ m s}^{-2}$.

$$\therefore u^2 = \frac{10 \times (6400 \times 10^3)^2}{6900 \times 10^3}.$$

$$\therefore u = 7700 \text{ m s}^{-1} (\text{approx.})$$

At this height, vertical momentum

$$U_Y = mv = 50 \times 2000 = 100000 \text{ kg m s}^{-1}.$$

Fig. 2.24.

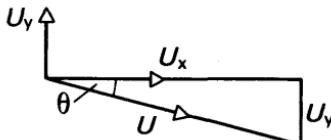


FIG. 2.24 Example

Horizontal momentum required $U_X = mu = 50 \times 7700 = 385000 \text{ kg m s}^{-1}$.

$$\therefore \text{impulse needed, } U, = \sqrt{U_Y^2 + U_X^2} = \sqrt{100000^2 + 385000^2}$$

$$= 40 \times 10^5 \text{ kg m s}^{-1} \quad \quad (1)$$

Direction. The angle θ made by the total impulse with the horizontal or orbit tangent is given by $\tan \theta = U_Y/U_X = 100000/385000 = 0.260$. Thus $\theta = 14.6^\circ$.

Magnitudes of acceleration due to gravity

(i) *Above the earth's surface.* Consider an object of mass m in an orbit of radius R from the centre, where $R > r$, the radius of the earth. Then, if g' is the acceleration due to gravity at this place,

$$mg' = \frac{GmM}{R^2} \quad \quad (i)$$

But, if g is the acceleration due to gravity at the earth's surface,

$$mg = \frac{GmM}{r^2} \quad \dots \quad (\text{ii})$$

Dividing (i) by (ii), $\therefore \frac{g'}{g} = \frac{r^2}{R^2}$, or $g' = \frac{r^2}{R^2} \cdot g$.

Thus above the earth's surface, the acceleration due to gravity g' varies *inversely as the square of the distance* from the centre. Fig. 2.25.

For a height h above the earth, $R = r + h$.

$$\begin{aligned}\therefore g' &= \frac{r^2}{(r+h)^2} \cdot g = \frac{1}{\left(1+\frac{h}{r}\right)^2} \cdot g. \\ &= \left(1+\frac{h}{r}\right)^{-2} \cdot g = \left(1-\frac{2h}{r}\right)g,\end{aligned}$$

since powers of $(h/r)^2$ and higher can be neglected when h is small compared with r .

$\therefore g - g' = \text{reduction in acceleration due to gravity.}$

$$= \frac{2h}{r} \cdot g \quad \dots \quad (1)$$

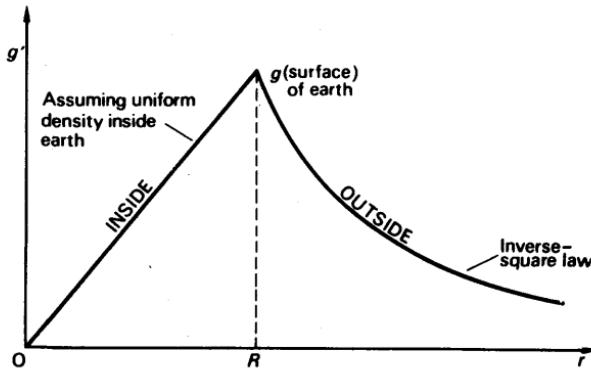


FIG. 2.25 Variation of g

(ii) *Below the earth's surface.* Consider an object of mass m at a point below the earth's surface. If its distance from the centre is b , the 'effective' mass M' of the earth which attracts it is that contained in a sphere of radius b . Assuming a constant density, then, since the mass of a sphere is proportional to radius^3 ,

$$M' = \frac{b^3}{r^3} M,$$

where M is the mass of the earth. Suppose g'' is the acceleration due to gravity at the radius b . Then, from above,

$$mg'' = \frac{GmM'}{b^2} = \frac{GmMb}{r^2}$$

Since $GM/r^2 = g$, it follows by substitution that

$$g'' = \frac{b}{r}g$$

Thus assuming a uniform density of core, which is not the case in practice, the acceleration due to gravity g'' is directly proportional to the distance from the centre. Fig. 2.25.

If the depth below the earth's surface is h , then $b = r - h$.

$$\therefore g'' = \left(\frac{r-h}{r} \right) g = \left(1 - \frac{h}{r} \right) g$$

$$\therefore g - g'' = \frac{h}{r}g \quad \quad (2)$$

Comparing (1) and (2), it can be seen that the acceleration at a distance h below the earth's surface is *greater* than at the same distance h above the earth's surface.

Potential

The *potential*, V , at a point due to the gravitational field of the earth is defined as numerically equal to the work done in taking a unit mass from infinity to that point. This is analogous to 'electric potential'. The potential at infinity is conventionally taken as *zero*.

For a point outside the earth, assumed spherical, we can imagine the whole mass M of the earth concentrated at its centre. The force of attraction on a unit mass outside the earth is thus GM/r^2 , where r is the distance from the centre. The work done by the gravitational force in moving a distance δr towards the earth = force \times distance = $GM \cdot \delta r/r^2$. Hence the potential at a point distant a from the centre is given by

$$V_a = \int_{\infty}^a \frac{GM}{r^2} dr = -\frac{GM}{a} \quad \quad (1)$$

if the potential at infinity is taken as zero by convention. The negative sign indicates that the potential at infinity (zero) is *higher* than the potential close to the earth.

On the earth's surface, of radius r , we therefore obtain

$$V = -\frac{GM}{r} \quad \quad (2)$$

Velocity of Escape. Suppose a rocket of mass m is fired from the earth's surface Q so that it just escapes from the gravitational influence of the earth. Then work done = $m \times$ potential difference between infinity and Q .

$$= m \times \frac{GM}{r}$$

$$\therefore \text{kinetic energy of rocket} = \frac{1}{2}mv^2 = m \times \frac{GM}{r}$$

$$\therefore v = \sqrt{\frac{2GM}{r}} = \text{velocity of escape.}$$

Now

$$GM/r^2 = g.$$

$$\therefore v = \sqrt{2gr}.$$

$$\therefore v = \sqrt{2 \times 9.8 \times 6.4 \times 10^6} = 11 \times 10^3 \text{ m s}^{-1} = 11 \text{ km s}^{-1} (\text{approx}).$$

With an initial velocity, then, of about 11 km s^{-1} , a rocket will completely escape from the gravitational attraction of the earth. It can be made to travel towards the moon, for example, so that eventually it comes under the gravitational attraction of this planet. At present, 'soft' landings on the moon have been made by firing retarding retro rockets.

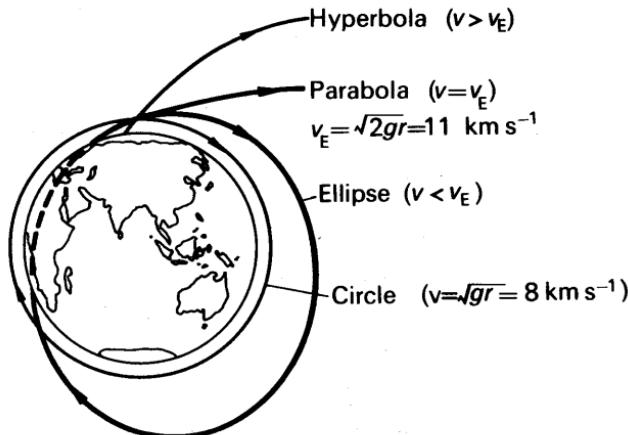


FIG. 2.26 Orbits

Summarising, with a velocity of about 8 km s^{-1} , a satellite can describe a circular orbit close to the earth's surface (p. 64). With a velocity greater than 8 km s^{-1} but less than 11 km s^{-1} , a satellite describes an elliptical orbit round the earth. Its maximum and minimum height in the orbit depends on its particular velocity. Fig. 2.26 illustrates the possible orbits of a satellite launched from the earth.

The molecules of air at normal temperatures and pressures have an average velocity of the order of 480 m s^{-1} or 0.48 km s^{-1} which is much less than the velocity of escape. Many molecules move with higher velocity than 0.48 km s^{-1} but gravitational attraction keeps the atmosphere round the earth. The gravitational attraction of the moon is much less than that of the earth and this accounts for the lack of atmosphere round the moon.

EXERCISES 2

(Assume $g = 10 \text{ m s}^{-2}$)

What are the missing words in the statements 1–6?

1. The force towards the centre in circular motion is called the ... force.
2. In simple harmonic motion, the maximum kinetic energy occurs at the ... of the oscillation.

3. The constant of gravitation G is related to g by . . .
4. In simple harmonic motion, the maximum potential energy occurs at the . . . of the oscillation.
5. Outside the earth, the acceleration due to gravity is proportional to . . . from the centre.
6. A satellite in orbit in an equatorial plane round the earth will stay at the same place above the earth if its period is . . . hours.

Which of the following answers, A, B, C, D or E, do you consider is the correct one in the statements 7–10?

7. The earth retains its atmosphere because *A* the earth is spherical, *B* the velocity of escape is greater than the mean speed of molecules, *C* the constant of gravitation is a universal constant, *D* the velocity of escape is less than the mean speed of molecules, *E* gases are lighter than solids.
8. In simple harmonic motion, the moving object has *A* only kinetic energy, *B* mean kinetic energy greater than the mean potential energy, *C* total energy equal to the sum of the maximum kinetic energy and maximum potential energy, *D* mean kinetic energy equal to the mean potential energy, *E* minimum potential energy at the centre of oscillation.
9. If r is the radius of the earth and g is the acceleration at its surface, then the acceleration g' at an orbit distance R from the centre of the earth is given by *A* $g'/g = r/R$, *B* $g'/g = r^2/R^2$, *C* $g'/g = R^2/r^2$, *D* $g'/g = (R-r)^2/r^2$, *E* $g'/g = (R-r)/r$.
10. When water in a bucket is whirled fast overhead, the water does not fall out at the top of the motion because *A* the centripetal force on the water is greater than the weight of water, *B* the force on the water is opposite to gravity, *C* the reaction of the bucket on the water is zero, *D* the centripetal force on the water is less than the weight of water, *E* atmospheric pressure counteracts the weight.

Circular Motion

11. An object of mass 4 kg moves round a circle of radius 6 m with a constant speed of 12 m s^{-1} . Calculate (i) the angular velocity, (ii) the force towards the centre.
12. An object of mass 10 kg is whirled round a horizontal circle of radius 4 m by a revolving string inclined to the vertical. If the uniform speed of the object is 5 m s^{-1} , calculate (i) the tension in the string in kgf, (ii) the angle of inclination of the string to the vertical.
13. A racing-car of 1000 kg moves round a banked track at a constant speed of 108 km h^{-1} . Assuming the total reaction at the wheels is normal to the track, and the horizontal radius of the track is 100 m, calculate the angle of inclination of the track to the horizontal and the reaction at the wheels.
14. An object of mass 8.0 kg is whirled round in a vertical circle of radius 2 m with a constant velocity of 6 m s^{-1} . Calculate the maximum and minimum tensions in the string.
15. Define the terms (a) *acceleration*, and (b) *force*. Show that the acceleration of a body moving in a circular path of radius r with uniform speed v is v^2/r , and draw a diagram to show the direction of the acceleration.

A small body of mass m is attached to one end of a light inelastic string of

length l . The other end of the string is fixed. The string is initially held taut and horizontal, and the body is then released. Find the values of the following quantities when the string reaches the vertical position: (a) the kinetic energy of the body, (b) the velocity of the body, (c) the acceleration of the body, and (d) the tension in the string. (O. & C.)

16. Explain what is meant by *angular velocity*. Derive an expression for the force required to make a particle of mass m move in a circle of radius r with uniform angular velocity w .

A stone of mass 500 g is attached to a string of length 50 cm which will break if the tension in it exceeds 2.0 kgf. The stone is whirled in a vertical circle, the axis of rotation being at a height of 100 cm above the ground. The angular speed is very slowly increased until the string breaks. In what position is this break most likely to occur, and at what angular speed? Where will the stone hit the ground? (C.)

Simple Harmonic Motion

17. An object moving with simple harmonic motion has an amplitude of 2 cm and a frequency of 20 Hz. Calculate (i) the period of oscillation, (ii) the acceleration at the middle and end of an oscillation, (iii) the velocities at the corresponding instants.

18. Calculate the length in centimetres of a simple pendulum which has a period of 2 seconds. If the amplitude of swing is 2 cm, calculate the velocity and acceleration of the bob (i) at the end of a swing, (ii) at the middle, (iii) 1 cm from the centre of oscillation.

19. Define *simple harmonic motion*. An elastic string is extended 1 cm when a small weight is attached at the lower end. If the weight is pulled down $\frac{1}{4}$ cm and then released, show that it moves with simple harmonic motion, and find the period.

20. A uniform wooden rod floats upright in water with a length of 30 cm immersed. If the rod is depressed slightly and then released, prove that its motion is simple harmonic and calculate the period.

21. A simple pendulum, has a period of 4.2 seconds. When the pendulum is shortened by 1 m, the period is 3.7 seconds. From these measurements, calculate the acceleration due to gravity and the original length of the pendulum.

22. What is *simple harmonic motion*? Show how it is related to the uniform motion of a particle with velocity v in a circle of radius r .

A steel strip, clamped at one end, vibrates with a frequency of 50 Hz and an amplitude of 8 mm at the free end. Find (a) the velocity of the end when passing through the zero position, (b) the acceleration at the maximum displacement.

23. Explain what is meant by *simple harmonic motion*.

Show that the vertical oscillations of a mass suspended by a light helical spring are simple harmonic and describe an experiment with the spring to determine the acceleration due to gravity.

A small mass rests on a horizontal platform which vibrates vertically in simple harmonic motion with a period of 0.50 second. Find the maximum amplitude of the motion which will allow the mass to remain in contact with the platform throughout the motion. (L.)

24. Define simple harmonic motion and state a formula for its period. Show that under suitable conditions the motion of a simple pendulum is simple harmonic and hence obtain an expression for its period.

If a pendulum bob is suspended from an inaccessible point, by a string whose length may be varied, describe how to determine (a) the acceleration due to gravity, (b) the height of the point of suspension above the floor.

How and why does the value of the acceleration due to gravity at the poles differ from its value at the equator? (L.)

25. Derive an expression for the time period of vertical oscillations of small amplitude of a mass suspended from the free end of a light helical spring.

What deformation of the wire of the spring occurs when the mass moves? (N.)

26. Give two practical examples of oscillatory motion which approximate to simple harmonic motion. What conditions must be satisfied if the approximations are to be good ones.

A point mass moves with simple harmonic motion. Draw on the same axes sketch graphs to show the variation with position of (a) the potential energy, (b) the kinetic energy, and (c) the total energy of the particle.

A particle rests on a horizontal platform which is moving vertically in simple harmonic motion with an amplitude of 10 cm. Above a certain frequency, the thrust between the particle and the platform would become zero at some point in the motion. What is this frequency, and at what point in the motion does the thrust become zero at this frequency? (C.)

27. In what circumstances will a particle execute simple harmonic motion? Show how simple harmonic motion can be considered to be the projection on the diameter of a circle of the motion of a particle describing the circle with uniform speed.

The balance wheel of a watch vibrates with an angular amplitude of π radians and a period of 0.5 second. Calculate (a) the maximum angular speed, (b) the angular speed when the displacement is $\pi/2$, and (c) the angular acceleration when the displacement is $\pi/4$. If the radius of the wheel is r , calculate the maximum radial force acting on a small dust particle of mass m situated on the rim of the wheel. (O. & C.)

28. Prove that the bob of a simple pendulum may move with simple harmonic motion, and find an expression for its period.

Describe with full details how you would perform an experiment, based on the expression derived, to measure the value of the acceleration due to gravity. What factors would influence your choice of (a) the length of the pendulum, (b) the material of the bob, and (c) the number of swings to be timed? (O. & C.)

29. Define *simple harmonic motion* and show that the free oscillations of a simple pendulum are simple harmonic for small amplitudes.

Explain what is meant by damping of oscillations and describe an experiment to illustrate the effects of damping on the motion of a simple pendulum. Briefly discuss the difficulties you would encounter and indicate qualitatively the results you would expect to observe. (O. & C.)

30. What is meant by simple harmonic motion? Obtain an expression for the kinetic energy of a body of mass m , which is performing S.H.M. of amplitude a and period $2\pi/\omega$, when its displacement from the origin is x .

Describe an experiment, or experiments, to verify that a mass oscillating at the end of a helical spring moves with simple harmonic motion. (C.)

31. State the dynamical condition under which a particle will describe simple harmonic motion. Show that it is approximately fulfilled in the case of the bob of a simple pendulum, and derive, from first principles, an expression for the period of the pendulum.

Explain how it can be demonstrated from observations on simple pendulums, that the weight of a body at a given place is proportional to its mass. (O. & C.)

32. Define *simple harmonic motion*. Show that a heavy body supported by a light spiral spring executes simple harmonic motion when displaced vertically from its equilibrium position by an amount which does not exceed a certain value and then released. How would you determine experimentally the maximum amplitude for simple harmonic motion?

A spiral spring gives a displacement of 5 cm for a load of 500 g. Find the maximum displacement produced when a mass of 80 g is dropped from a height of 10 cm on to a light pan attached to the spring. (N.)

Gravitation

33. Calculate the force of attraction between two small objects of mass 5 and 8 kg respectively which are 10 cm apart. ($G = 6.7 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$)

34. If the acceleration due to gravity is 9.8 m s^{-2} and the radius of the earth is 6400 km, calculate a value for the mass of the earth. ($G = 6.7 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$.) Give the theory.

35. Assuming that the mean density of the earth is 5500 kg m^{-3} , that the constant of gravitation is $6.7 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$, and that the radius of the earth is 6400 km, find a value for the acceleration due to gravity at the earth's surface. Derive the formula used.

36. How do you account for the sensation of 'weightlessness' experienced by the occupant of a space capsule (a) in a circular orbit round the earth, (b) in outer space? Give one other instance in which an object would be 'weightless'. (N.)

37. State Newton's law of universal gravitation. Distinguish between the gravitational constant (G) and the acceleration due to gravity (g) and show the relation between them.

Describe an experiment by which the value of g may be determined. Indicate the measurements taken and how to calculate the result. Derive any formula used. (L.)

38. State *Newton's law of gravitation*. What experimental evidence is there for the validity of this law?

A binary star consists of two dense spherical masses of 10^{30} kg and $2 \times 10^{30} \text{ kg}$ whose centres are 10^7 km apart and which rotate together with a uniform angular velocity ω about an axis which intersects the line joining their centres. Assuming that the only forces acting on the stars arise from their mutual gravitational attraction and that each mass may be taken to act at its centre, show that the axis of rotation passes through the centre of mass of the system and find the value of ω . ($G = 6.7 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$.) (O. & C.)

39. Assuming that the planets are moving in circular orbits, apply Kepler's laws to show that the acceleration of a planet is inversely proportional to the square of its distance from the sun. Explain the significance of this and show clearly how it leads to Newton's law of universal gravitation.

Obtain the value of g from the motion of the moon, assuming that its period of rotation round the earth is 27 days 8 hours and that the radius of its orbit is 60.1 times the radius of the earth. (Radius of earth = $6.36 \times 10^6 \text{ m}$.) (N.)

40. Explain what is meant by the *gravitation constant* (G), and describe an accurate laboratory method of measuring it. Give an outline of the theory of your method.

Assuming that the earth is a sphere of radius 6370 km and that $G = 6.66 \times 10^{-11}$ N m² kg⁻², calculate the mean density of the earth. (O. & C.)

41. Assuming the earth to be perfectly spherical, give sketch graphs to show how (a) the acceleration due to gravity, (b) the gravitational potential due to the earth's mass, vary with distance from the surface of the earth for points external to it. If any other assumption has been made, state what it is.

Explain why, even if the earth were a perfect sphere, the period of oscillation of a simple pendulum at the poles would not be the same as at the equator. Still assuming the earth to be perfectly spherical, discuss whether the velocity required to project a body vertically upwards, so that it rises to a given height, depends on the position on the earth from which it is projected. (C.)

42. Explain what is meant by the *constant of gravitation*. Describe a laboratory experiment to determine it, showing how the result is obtained from the observations.

A proposed communication satellite would revolve round the earth in a circular orbit in the equatorial plane, at a height of 35880 km above the earth's surface. Find the period of revolution of the satellite in hours, and comment on the result. (Radius of earth = 6370 km, mass of earth = 5.98×10^{24} kg constant of gravitation = 6.66×10^{-11} N m² kg⁻².) (N.)

chapter three

Rotation of Rigid Bodies

So far in this book we have considered the equations of motion and other dynamical formulae associated with a particle. In practice, however, an object is made of millions of particles, each at different places, and we need now to consider the motion of moving objects.

Moment of Inertia, I

Suppose a rigid object is rotating about a fixed axis O, and a particle A of the object makes an angle θ with a fixed line OY in space at some instant, Fig. 3.1. The angular velocity, $d\theta/dt$ or ω , of every particle about O is the same, since we are dealing with a rigid body, and the velocity v_1 of A at this instant is given by $r_1\omega$, where $r_1 = OA$. Thus the kinetic energy of A = $\frac{1}{2}m_1v_1^2 = \frac{1}{2}m_1r_1^2\omega^2$. Similarly, the kinetic energy of another particle of the body = $\frac{1}{2}m_2r_2^2\omega^2$, where r_2 is its distance from O and m_2 is its mass. In this way we see that the kinetic energy, K.E., of the whole object is given by

$$\begin{aligned}\text{K.E.} &= \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 + \frac{1}{2}m_3r_3^2\omega^2 + \dots \\ &= \frac{1}{2}\omega^2(m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \dots) \\ &= \frac{1}{2}\omega^2(\Sigma mr^2),\end{aligned}$$

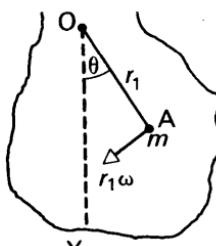


FIG. 3.1
Rotating rigid body

where Σmr^2 represents the sum of the magnitudes of ' mr^2 ' for all the particles of the object. We shall see shortly how the quantity Σmr^2 can be calculated for a particular object. The magnitude of Σmr^2 is known as the *moment of inertia* of the object about the axis concerned, and we shall denote it by the symbol I . Thus

$$\text{Kinetic energy, K.E.} = \frac{1}{2}I\omega^2. \quad . \quad . \quad . \quad (1)$$

The units of I are kg metre^2 (kg m^2). The unit of ω is 'radian s^{-1} ' (rad s^{-1}). Thus if $I = 2 \text{ kg m}^2$ and $\omega = 3 \text{ rad s}^{-1}$, then

$$\text{K.E.} = \frac{1}{2}I\omega^2 = \frac{1}{2} \times 2 \times 3^2 \text{ joule} = 9 \text{ J.}$$

The kinetic energy of a particle of mass m moving with a velocity v is $\frac{1}{2}mv^2$. It will thus be noted that the formula for the kinetic energy of a rotating object is similar to that of a moving particle, the mass m being replaced by the moment of inertia I and the velocity v being replaced by the angular velocity ω . As we shall require values of I , the moment of inertia of several objects about a particular axis will first be calculated.

Moment of Inertia of Uniform Rod

(1) *About axis through middle.* The moment of inertia of a small element δx about an axis PQ through its centre O perpendicular to the length $= \left(\frac{\delta x}{l}M\right)x^2$, where l is the length of the rod, M is its mass, and x is the distance of the small element from O, Fig. 3.2.

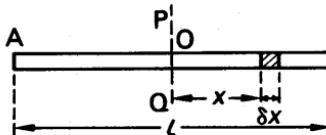


FIG. 3.2 Moment of inertia—uniform rod

$$\begin{aligned}\therefore \text{moment of inertia, } I, &= 2 \int_0^{l/2} \left(\frac{\delta x}{l} M \right) x^2 dx \\ &= \frac{2M}{l} \int_0^{l/2} x^2 dx = \frac{Ml^2}{12} \quad (1)\end{aligned}$$

Thus if the mass of the rod is 60 g and its length is 20 cm, $M = 6 \times 10^{-2}$ kg, $l = 0.2$ m, and $I = 6 \times 10^{-2} \times 0.2^2 / 12 = 2 \times 10^{-4}$ kg m².

(2) *About the axis through one end, A.* In this case, measuring distances x from A instead of O,

$$\text{moment of inertia, } I, = \int_0^l \left(\frac{\delta x}{l} M \right) \times x^2 dx = \frac{Ml^2}{3} \quad (2)$$

Moment of Inertia of Ring

Every element of the ring is the same distance from the centre. Hence the moment of inertia about an axis through the centre perpendicular to the plane of the ring $= Ma^2$, where M is the mass of the ring and a is its radius.

Moment of Inertia of Circular Disc

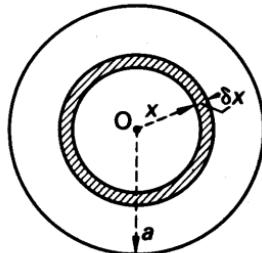


FIG. 3.3
Moment of inertia—disc

Consider the moment of inertia of a circular disc about an axis through its centre perpendicular to its plane, Fig. 3.3. If we take a small ring of the disc enclosed between radii x and $x + \delta x$, its mass $= \frac{2\pi x \delta x}{\pi a^2} M$, where a is the radius of the disc and M is its mass. Each element of the ring is distant x from the centre, and hence the moment of inertia of the ring

$$\text{about the axis through O} = \left(\frac{2\pi x \delta x}{\pi a^2} M \right) \times x^2$$

$$\therefore \text{moment of inertia of whole disc} = \int_0^a \frac{2\pi x dx}{\pi a^2} M \times x^2 \\ = \frac{Ma^2}{2} \quad (1)$$

Thus if the disc weighs 60 g and has a radius of 10 cm, $M = 60 \text{ g} = 6 \times 10^{-2} \text{ kg}$, $a = 0.1 \text{ m}$, so that $I = 6 \times 10^{-2} \times 0.1^2 / 2 = 3 \times 10^{-4} \text{ kg m}^2$.

Moment of Inertia of Cylinder

If a cylinder is *solid*, its moment of inertia about the axis of symmetry is the sum of the moments of inertia of discs into which we may imagine the cylinder cut. The moment of inertia of each disc = $\frac{1}{2}$ mass $\times a^2$, where a is the radius; and hence, if M is the mass of the cylinder,

$$\text{moment of inertia of solid cylinder} = \frac{1}{2} Ma^2 \quad (i)$$

If a cylinder is *hollow*, its moment of inertia about the axis of symmetry is the sum of the moments of inertia of the curved surface and that of the two ends, assuming the cylinder is closed at both ends. Suppose a is the radius, h is the height of the cylinder, and σ is the mass per unit area of the surface. Then

$$\text{mass of curved surface} = 2\pi ah\sigma,$$

$$\text{and } \text{moment of inertia about axis} = \text{mass} \times a^2 = 2\pi a^3 h\sigma,$$

since we can imagine the surface cut into rings.

The moment of inertia of one end of the cylinder = mass $\times a^2 / 2 = \pi a^2 \sigma \times a^2 / 2 = \pi a^4 \sigma / 2$. Hence the moment of inertia of both ends = $\pi a^4 \sigma$.

$$\therefore \text{moment inertia of cylinder, } I, = 2\pi a^3 h\sigma + \pi a^4 \sigma.$$

$$\text{The mass of the cylinder, } M, = 2\pi ah\sigma + 2\pi a^2 \sigma$$

$$\begin{aligned} \therefore I &= \frac{2\pi a^3 h\sigma + \pi a^4 \sigma}{2\pi ah\sigma + 2\pi a^2 \sigma} M. \\ &= \frac{2a^2 h + a^3}{2h + 2a} M. \\ &= \frac{1}{2} Ma^2 + \frac{a^2 h}{2h + 2a} M \end{aligned} \quad (ii)$$

If a hollow and a solid cylinder have the same mass M and the same radius and height, it can be seen from (i) and (ii) that the moment of inertia of the hollow cylinder is greater than that of the solid cylinder about the axis of symmetry. This is because the mass is distributed on the average at a greater distance from the axis in the former case.

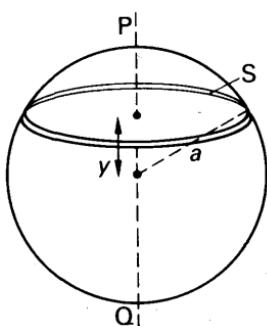


FIG. 3.4
Moment of inertia—sphere

Moment of Inertia of Sphere

The moment of inertia of a sphere about an axis PQ through its centre can be found by cutting thin discs such as S perpendicular to the axis, Fig. 3.4. The volume of the disc, of thickness δy and distance y from the centre,

$$= \pi r^2 \delta y = \pi(a^2 - y^2) \delta y.$$

$$\therefore \text{mass } M' \text{ of disc} = \frac{\pi(a^2 - y^2)\delta y}{4\pi a^3/3} M$$

$$= \frac{3M}{4a^3}(a^2 - y^2)\delta y,$$

where M is the mass of the sphere and a is its radius, since the volume of the sphere $= 4\pi a^3/3$. Now the moment of inertia of the disc about PQ

$$= M' \times \frac{\text{radius}^2}{2} = \frac{3M}{4a^3}(a^2 - y^2)\delta y \times \frac{(a^2 - y^2)}{2}$$

$$\therefore \text{moment of inertia of sphere} = \frac{3M}{8a^3} \int_{-a}^{+a} (a^4 - 2a^2y^2 + y^4)dy \\ = \frac{2}{5}Ma^2 \quad \quad (1)$$

Thus if the sphere has mass 4 kg and a radius of 0.2 m, the moment of inertia $= \frac{2}{5} \times 4 \times 0.2^2 = 0.064 \text{ kg m}^2$.

Radius of Gyration

The moment of inertia of an object about an axis, Σmr^2 , is sometimes written as Mk^2 , where M is the mass of the object and k is a quantity called the *radius of gyration* about the axis.

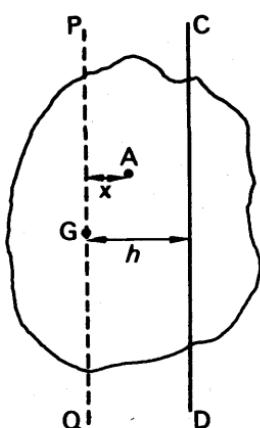
For example, the moment of inertia of a rod about an axis through one end $= Ml^2/3$ (p. 76) $= M(l/\sqrt{3})^2$. Thus the radius of gyration, $k = l/\sqrt{3} = 0.58l$. The moment of inertia of a sphere about its centre $= \frac{2}{5}Ma^2 = M \times (\sqrt{\frac{2}{5}}a)^2$. Thus the radius of gyration, $k = \sqrt{\frac{2}{5}}a = 0.63a$ in this case.

Relation Between Moment of Inertia About C.G. and Parallel Axis.

Suppose I is the moment of inertia of a body about an axis CD and I_G is the moment of inertia about a parallel axis PQ through the centre of gravity, G, distant h from the axis CD, Fig. 3.5. If A is a particle of mass m whose distance from PQ is x , its moment of inertia about CD $= m(h-x)^2$

$$\therefore I = \Sigma m(h-x)^2 = \Sigma mh^2 + \Sigma mx^2 - \Sigma 2mhx.$$

FIG. 3.5
Theorem of parallel axes



Now $\Sigma mh^2 = h^2 \times \Sigma m = Mh^2$, where M is the total mass of the object, and $\Sigma mx^2 = I_G$, the moment of inertia through the centre of gravity.

Also,

$$\Sigma 2mhx = 2h\Sigma mx = 0,$$

since Σmx , the sum of the moments about the centre of gravity, is zero; this follows because the moment of the resultant (the weight) about G is zero.

$$\therefore I = I_G + Mh^2 \quad \quad (1)$$

From this result, it follows that the moment of inertia, I , of a disc of radius a and mass M about an axis through a point on its circumference $= I_G + Ma^2$, since $h = a$ = radius of disc in this case. But I_G = moment of inertia about the centre $= Ma^2/2$ (p.77).

$$\therefore \text{moment of inertia, } I, = \frac{Ma^2}{2} + Ma^2 = \frac{3Ma^2}{2}.$$

Similarly the moment of inertia of a sphere of radius a and mass M about an axis through a point on its circumference $= I_G + Ma^2 = 2Ma^2/5 + Ma^2 = 7Ma^2/5$, since I_G , the moment of inertia about an axis through its centre, is $2Ma^2/5$.

Relation Between Moments of Inertia about Perpendicular Axes

Suppose OX, OY are any two perpendicular axes and OZ is an axis perpendicular to OX and OY, Fig. 3.6 (i). The moment of inertia, I , of

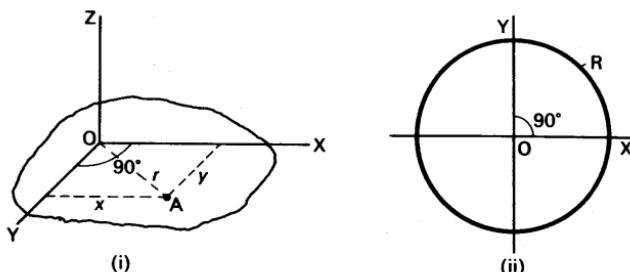


FIG. 3.6 Theorem of perpendicular axes

a body about the axis OZ $= \Sigma mr^2$, where r is the distance of a particle A from OZ and m is its mass. But $r^2 = x^2 + y^2$, where x, y are the distances of A from the axis OY, OX respectively.

$$\begin{aligned} \therefore I &= \Sigma m(x^2 + y^2) = \Sigma mx^2 + \Sigma my^2. \\ \therefore I &= I_y + I_x \quad \quad (1) \end{aligned}$$

where I_y, I_x are the moments of inertia about OX, OY respectively.

As a simple application, consider a ring R and two perpendicular axes OX, OY in its plane, Fig. 3.6 (ii). Then from the above result,

$I_y + I_x = I = \text{moment of inertia through O perpendicular to ring.}$

$$\therefore I_y + I_x = Ma^2.$$

But $I_y = I_x$, by symmetry.

$$\therefore I_x + I_x = Ma^2,$$

$$\therefore I_x = \frac{Ma^2}{2}.$$

This is the moment of inertia of the ring about any diameter in its plane.

In the same way, the moment of inertia, I , of a disc about a diameter in its plane is given by

$$I + I = \frac{Ma^2}{2},$$

since the moments of inertia, I , about the two perpendicular diameters are the same and $Ma^2/2$ is the moment of inertia of the disc about an axis perpendicular to its plane.

$$\therefore I = \frac{Ma^2}{4}.$$

Couple on a Rigid Body

Consider a rigid body rotating about a fixed axis O with an angular velocity ω at some instant. Fig. 3.7.

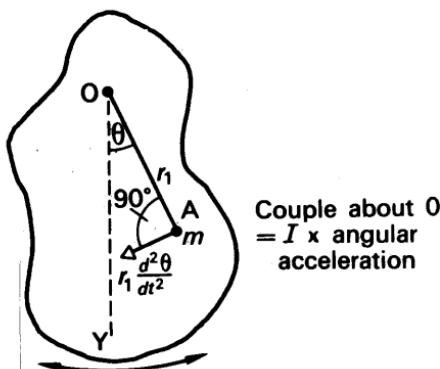


FIG. 3.7 Couple on rigid body

The force acting on the particle A = $m_1 \times$ acceleration = $m_1 \times \frac{d}{dt}(r_1 \omega) = m_1 \times r_1 \frac{d\omega}{dt} = m_1 r_1 \frac{d^2\theta}{dt^2}$, since $\omega = \frac{d\theta}{dt}$. The moment of this

force about the axis O = force \times perpendicular distance from O = $m_1 r_1 \frac{d^2\theta}{dt^2} \times r_1$, since the force acts perpendicularly to the line OA.

$$\therefore \text{moment or torque} = m_1 r_1 \frac{d^2\theta}{dt^2}.$$

\therefore total moment of all forces on body about O, or torque,

$$= m_1 r_1 \frac{d^2\theta}{dt^2} + m_2 r_2 \frac{d^2\theta}{dt^2} + m_3 r_3 \frac{d^2\theta}{dt^2} + \dots$$

$$= (\Sigma mr^2) \times \frac{d^2\theta}{dt^2},$$

since the angular acceleration, $d^2\theta/dt^2$, about O is the same for all particles.

$$\therefore \text{total torque about O} = I \frac{d^2\theta}{dt^2}, \quad . . . \quad (1)$$

where $I = \Sigma mr^2$ = moment of inertia about O. The moment about O is produced by external forces which together act as a couple of torque C say. Thus, for any rotating rigid body,

$$\text{Couple, } C = I \frac{d^2\theta}{dt^2}.$$

This result is analogous to the case of a particle of mass m which undergoes an acceleration a when a force F acts on it. Here $F = ma$. In place of F we have a couple C for a rotating rigid object; in place of m we have the moment of inertia I ; and in place of linear acceleration a , we have angular acceleration $d^2\theta/dt^2(d\omega/dt)$.

EXAMPLES

1. A heavy flywheel of mass 15 kg and radius 20 cm is mounted on a horizontal axle of radius 1 cm and negligible mass compared with the flywheel. Neglecting friction, find (i) the angular acceleration if a force of 4 kgf is applied tangentially to the axle, (ii) the angular velocity of the flywheel after 10 seconds.

$$(i) \quad \text{Moment of inertia} = \frac{Ma^2}{2} = \frac{15 \times 0.2^2}{2} = 0.3 \text{ kg m}^2.$$

$$\text{Couple } C = 4 \times 9.8 \text{ (N)} \times 0.01 \text{ (m)} = 0.4 \text{ N m approx.}$$

$$\therefore \text{angular acceleration} = \frac{0.4}{0.3} = 1.3 \text{ rad s}^{-2}.$$

(ii) After 10 seconds, angular velocity = angular acceleration \times time.

$$= 1.3 \times 10 = 13 \text{ rad s}^{-1}.$$

2. The moment of inertia of a solid flywheel about its axis is 0.1 kg m². It is set in rotation by applying a tangential force of 2 kgf with a rope wound round the circumference, the radius of the wheel being 10 cm. Calculate the

angular acceleration of the flywheel. What would be the acceleration if a mass of 2 kg were hung from the end of the rope? (O. & C.)

Couple $C = I \frac{d^2\theta}{dt^2}$ = moment of inertia \times angular acceleration.

$$\text{Now } C = 2 \times 9.8 \times 0.1 \text{ N m.}$$

$$\therefore \text{angular acceleration} = \frac{2 \times 9.8 \times 0.1}{0.1} \\ = 19.6 \text{ rad s}^{-2}.$$

If a mass of 2 kg is hung from the end of the rope, it moves down with an acceleration a . Fig. 3.8. In this case, if T is the tension in the rope,

$$mg - T = ma \quad (1)$$

$$\text{For the flywheel, } T \cdot r = \text{couple} = I \frac{d^2\theta}{dt^2} \quad (2)$$

where r is the radius of the flywheel. Now the mass of 2 kg descends a distance given by $r\theta$, where θ is the angle the flywheel has turned. Hence the acceleration $a = rd^2\theta/dt^2$. Substituting in (1),

$$\therefore mg - T = mr \frac{d^2\theta}{dt^2} \\ \therefore mgr - T \cdot r = mr^2 \frac{d^2\theta}{dt^2} \quad (3)$$

Adding (2) and (3),

$$\therefore mgr = (I + mr^2) \frac{d^2\theta}{dt^2} \\ \therefore \frac{d^2\theta}{dt^2} = \frac{mgr}{I + mr^2} = \frac{2 \times 10 \times 0.1}{0.1 + 2 \times 0.1^2} \\ = 16.7 \text{ rad s}^{-2}.$$

using $g = 10 \text{ m s}^{-2}$.

Angular Momentum and Conservation

In linear or straight-line motion, an important property of a moving object is its linear momentum (p. 18). When an object spins or rotates about an axis, its *angular momentum* plays an important part in its motion.

Consider a particle A of a rigid object rotating about an axis O. Fig. 3.9. The momentum of A = mass \times velocity = $m_1 v = m_1 r_1 \omega$. The 'angular momentum' of A about O is defined as the *moment of the momentum* about O. Its magnitude is thus $m_1 v \times p$, where p is the perpendicular distance from O to the direction of v . Thus angular momentum of A = $m_1 v p = m_1 r_1 \omega \times r_1 = m_1 r_1^2 \omega$.

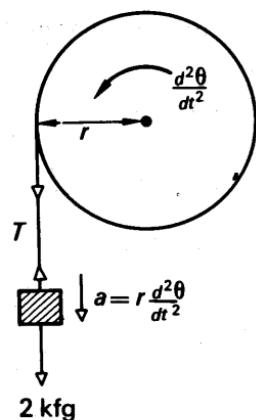


FIG. 3.8 Example

$$\therefore \text{total angular momentum of whole body} = \sum m_1 r_1^2 \omega = \omega \sum m_1 r_1^2 \\ = I\omega,$$

where I is the moment of inertia of the body about O.

Angular momentum is analogous to 'linear momentum', mv , in the dynamics of a moving particle. In place of m we have I , the moment of inertia; in place of v we have ω , the angular velocity.

Further, the *conservation of angular momentum*, which corresponds to the conservation of linear momentum, states that *the angular momentum about an axis of a given rotating body or system of bodies is constant, if no external couple acts about that axis*. Thus when a high diver jumps from a diving board, his moment of inertia, I , can be decreased by curling his body more, in which case his angular velocity ω is increased. Fig. 3.9 (ii). He may then be able to turn more somersaults before striking the water. Similarly, a dancer on skates can spin faster by folding her arms.

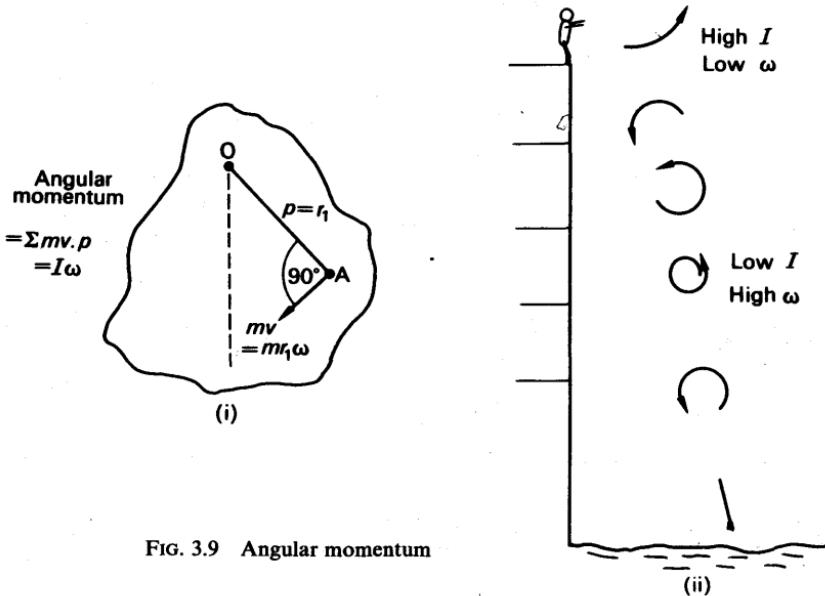


FIG. 3.9 Angular momentum

The earth is an object which rotates about an axis passing through its geographic north and south poles with a period of 1 day. If it is struck by meteorites, then, since action and reaction are equal, no external couple acts on the earth and meteorites. Their total angular momentum is thus conserved. Neglecting the angular momentum of the meteorites about the earth's axis before collision compared with that of the earth, then

$$\begin{aligned} \text{angular momentum of earth plus meteorites after collision} &= \\ \text{angular momentum of earth before collision.} & \end{aligned}$$

Since the effective mass of the earth has increased after collision the moment of inertia has increased. Hence the earth will slow up slightly.

Similarly, if a mass is dropped gently on to a turntable rotating freely at a steady speed, the conservation of angular momentum leads to a reduction in the speed of the table.

Angular momentum, and the principle of the conservation of angular momentum, have wide applications in physics. They are used in connection with enormous rotating masses such as the earth, as well as minute spinning particles such as electrons, neutrons and protons found inside atoms.

Experiment on Conservation of Angular Momentum

A simple experiment on the principle of the conservation of angular momentum is illustrated below.

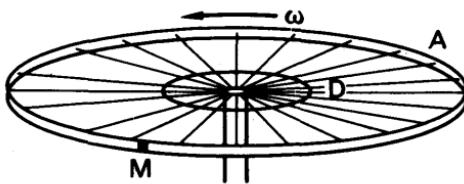


FIG. 3.10 Conservation of angular momentum

Briefly, in Fig. 3.10 (i) a bicycle wheel A without a tyre is set rotating in a horizontal plane and the time for three complete revolutions is obtained with the aid of a white tape marker M on the rim. A ring D of known moment of inertia, I , is then gently placed on the wheel concentric with it, by 'dropping' it from a small height. The time for the next three revolutions is then determined. This is repeated with several more rings of greater known moment of inertia.

If the principle of conservation of angular momentum is true, then $I_0\omega_0 = (I_0 + I_1)\omega_1$, where I_0 is the moment of inertia of the wheel alone, ω_0 is the angular frequency of the wheel alone, and ω_1 is the angular frequency with a ring. Thus if t_0 , t_1 are the respective times for three revolutions,

$$\frac{I_0 + I_1}{t_1} = \frac{I_0}{t_0}$$

$$\therefore \frac{I_1}{I_0} + 1 = \frac{t_1}{t_0}$$

Thus a graph of t_1/t_0 v. I_1 should be a straight line. Within the limits of experimental error, this is found to be the case.

EXAMPLE

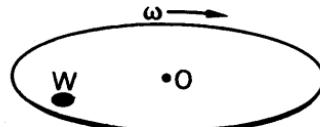
Consider a disc of mass 100 g and radius 10 cm is rotating freely about axis O through its centre at 40 r.p.m. Fig. 3.11. Then, about O,

$$\text{moment of inertia } I = \frac{Ma^2}{2} = \frac{1}{2} \times 0.1 \text{ (kg)} \times 0.1^2 \text{ (m}^2\text{)} = 5 \times 10^{-4} \text{ kg m}^2,$$

and angular momentum = $I\omega = 5 \times 10^{-4}\omega$,
where ω is the angular velocity corresponding to 40 r.p.m.

Suppose some wax W of mass m 20 g is dropped gently on to the disc at a distance r of 8 cm from the centre O. The disc then slows down to another speed, corresponding to an angular velocity ω_1 say. The total angular momentum about O of disc plus wax

$$= I\omega_1 + mr^2\omega_1 = 5 \times 10^{-4}\omega_1 + 0.02 \times 0.08^2 \cdot \omega_1 \\ = 6.28 \times 10^{-4}\omega_1.$$



From the conservation of angular momentum for the disc and wax about O

FIG. 3.11 Example

$$6.28 \times 10^{-4}\omega_1 = 5 \times 10^{-4}\omega.$$

$$\therefore \frac{\omega_1}{\omega} = \frac{500}{628} = \frac{n}{40},$$

where n is the r.p.m. of the disc.

$$\therefore n = \frac{500}{628} \times 40 = 32 \text{ (approx).}$$

Kepler's law and angular momentum

Consider a planet moving in an orbit round the sun S. Fig. 3.12.

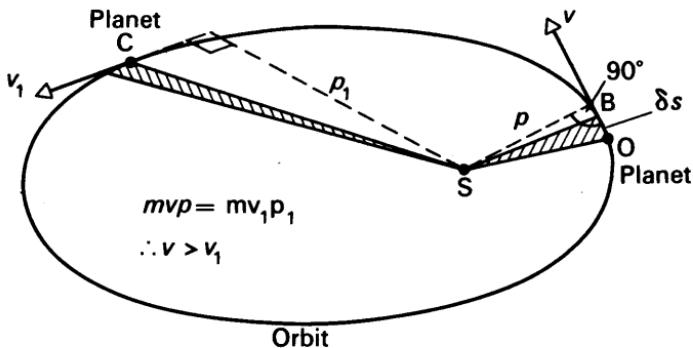


FIG. 3.12 Angular momentum and planets

At an instant when the planet is at O, its velocity v is along the tangent to the orbit at O. Suppose the planet moves a very small distance δs from O to B in a small time δt , so that the velocity $v = \delta s / \delta t$ and its direction is practically along OB. Then, if the conservation of angular momentum is obeyed,

$$mv \times p = \text{constant},$$

where m is the mass of the planet and p is the perpendicular from S to OB produced.

$$\therefore \frac{m \cdot \delta s \cdot p}{\delta t} = \text{constant}.$$

But the area δA of the triangle SBO = $\frac{1}{2}$ base \times height = $\delta s \times p/2$.

$$\therefore m \cdot 2 \frac{\delta A}{\delta t} = \text{constant}$$

$$\therefore \frac{\delta A}{\delta t} = \text{constant},$$

since $2m$ is constant. Thus if the conservation of angular momentum is true, the area swept out per second by the radius SO is constant while the planet O moves in its orbit. In other words, equal areas are swept out in equal times. *But this is Kepler's second law*, which has been observed to be true for centuries (see p. 58). Consequently, the principle of the conservation of angular momentum has stood the test of time. From the equality of the angular momentum values at O and C, where p is less than p_1 , it follows that v is greater than v_1 . Thus the planet speeds up on approaching S.

The force on O is always one of attraction towards S. It is described as a *central force*. Thus the force has no moment about O and hence the angular momentum of the planet about S is conserved.

Kinetic Energy of a Rolling Object

When an object such as a cylinder or ball rolls on a plane, the object is rotating as well as moving bodily along the plane; therefore it has rotational energy as well as translational energy.

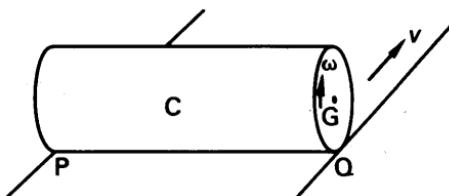


FIG. 3.13 Rolling object

Consider a cylinder C rolling along a plane without slipping, Fig. 3.13. At any instant the line of contact, PQ, with the plane is at rest, and we can consider the whole of the cylinder to be rotating about this axis. Hence the energy of the cylinder = $\frac{1}{2}I_1\omega^2$, where I_1 is the moment of inertia about PQ and ω is the angular velocity.

But if I is the moment of inertia about a parallel axis through the centre of gravity of the cylinder, M is the mass of the cylinder and a its radius, then

$$I_1 = I + Ma^2,$$

from the result on p. 79.

$$\begin{aligned}\therefore \text{energy of cylinder} &= \frac{1}{2}(I + Ma^2)\omega^2 \\ &= \frac{1}{2}I\omega^2 + \frac{1}{2}Ma^2\omega^2 \\ \therefore \text{Energy} &= \frac{1}{2}I\omega^2 + \frac{1}{2}Mv^2 \quad . \quad . \quad . \quad (1)\end{aligned}$$

since, by considering the distance rolled and the angle then turned, $v = a\omega$ = velocity of centre of gravity. This energy formula is true for any moving object.

As an application of the energy formula, suppose a *ring* rolls along a plane. The moment of inertia about the centre of gravity, its centre, $= Ma^2$ (p. 76); also, the angular velocity, ω , about its centre $= v/a$, where v is the velocity of the centre of gravity.

$$\begin{aligned}\therefore \text{kinetic energy of ring} &= \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \\ &= \frac{1}{2}Mv^2 + \frac{1}{2}Ma^2 \times \left(\frac{v}{a}\right)^2 = Mv^2.\end{aligned}$$

By similar reasoning, the kinetic energy of a sphere rolling down a plane

$$\begin{aligned}&= \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \\ &= \frac{1}{2}Mv^2 + \frac{1}{2} \times \frac{2}{5}Ma^2 \times \left(\frac{v}{a}\right)^2 = \frac{7}{10}Mv^2,\end{aligned}$$

since $I = 2Ma^2/5$ (p. 78).

Acceleration of Rolling Object

We can now deduce the acceleration of a rolling object down an inclined plane.

As an illustration, suppose a solid cylinder rolls down a plane. Then

$$\text{kinetic energy} = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2.$$

But moment of inertia, I , about an axis through the centre of gravity parallel to the plane $= \frac{1}{2}Ma^2$, and $\omega = v/a$, where a is the radius.

$$\therefore \text{kinetic energy} = \frac{1}{2}Mv^2 + \frac{1}{2}Mv^2 = \frac{3}{4}Mv^2.$$

If the cylinder rolls from *rest* through a distance s , the loss of potential energy $= Mgs \sin \alpha$, where α is the inclination of the plane to the horizontal.

$$\therefore \frac{3}{4}Mv^2 = Mgs \sin \alpha$$

$$\therefore v^2 = \frac{4g}{3}s \sin \alpha$$

But $v^2 = 2as$, where a is the linear acceleration.

$$\therefore 2as = \frac{4g}{3}s \sin \alpha$$

$$\therefore a = \frac{2g}{3} \sin \alpha \quad \quad (i)$$

The acceleration if sliding, and no rolling, took place down the plane is $g \sin \alpha$. The cylinder has thus a smaller acceleration when rolling.

The time t taken to move through a distance s from rest is given by $s = \frac{1}{2}at^2$. Thus, from (i),

$$s = \frac{1}{2}gt^2 \sin \alpha,$$

$$\text{or } t = \sqrt{\frac{3s}{g \sin \alpha}}.$$

If the cylinder is *hollow*, instead of solid as assumed, the moment of inertia about an axis through the centre of gravity parallel to the plane is greater than that for a solid cylinder, assuming the same mass and dimensions (p. 88). The time taken for a hollow cylinder to roll a given distance from rest on the plane is then greater than that taken by the solid cylinder, from reasoning similar to that above; and thus if no other means were available, a time test on an inclined plane will distinguish between a solid and a hollow cylinder of the same dimensions and mass. If a torsion wire is available, however, the cylinders can be suspended in turn, and the period of torsional oscillations determined. The cylinder of larger moment of inertia, the hollow cylinder, will have a greater period, as explained on p. 89.

Measurement of Moment of Inertia of Flywheel

The moment of inertia of a flywheel W about a horizontal axle A can be determined by tying one end of some string to a pin on the axle, winding the string round the axle, and attaching a mass M to the other end of the string, Fig. 3.14. The length of string is such that M reaches the floor, when released, at the same instant as the string is completely unwound from the axle.

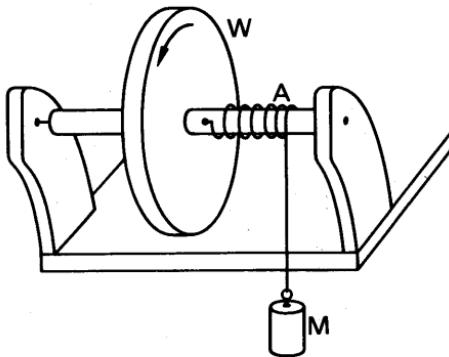


FIG. 3.14 Moment of inertia of flywheel

M is released, and the number of revolutions, n , made by the wheel W up to the occasion when M strikes the ground is noted. The further number of revolutions n_1 made by W until it comes finally to rest, and the time t taken, are also observed by means of a chalk-mark on W.

Now the loss in potential energy of M = gain in kinetic energy of M + gain in kinetic energy of flywheel + work done against friction.

$$\therefore Mgh = \frac{1}{2}Mr^2\omega^2 + \frac{1}{2}I\omega^2 + nf, \quad (i)$$

where h is the distance M has fallen, r is the radius of the axle, ω is the angular velocity, I is the moment of inertia, and f is the energy per turn expended against friction. Since the energy of rotation of the flywheel

when the mass M reaches the ground = work done against friction in n_1 revolutions, then

$$\frac{1}{2}I\omega^2 = n_1 f.$$

$$\therefore f = \frac{\frac{1}{2}I\omega^2}{n_1}.$$

Substituting for f in (i),

$$\therefore Mgh = \frac{1}{2}Mr^2\omega^2 + \frac{1}{2}I\omega^2 \left(1 + \frac{n}{n_1}\right) \quad \quad (\text{ii})$$

Since the angular velocity of the wheel when M reaches the ground is ω , and the final angular velocity of the wheel is zero after a time t , the average angular velocity $= \omega/2 = 2\pi n_1/t$. Thus $\omega = 4\pi n_1/t$. Knowing ω and the magnitude of the other quantities in (ii), the moment of inertia I of the flywheel can be calculated.

Period of Oscillation of Rigid Body

On p. 81 we showed that the moment of the forces acting on rotating objects $= Id\omega/dt = Id^2\theta/dt^2$, where I is the moment of inertia about the axis concerned and $d^2\theta/dt^2$ is the angular acceleration about the axis. Consider a rigid body oscillating about a fixed axis O, Fig. 3.15. The moment of the weight mg (the only external force) about O is $mgh \sin \theta$, or $mgh\theta$ if θ is small, where h is the distance of the centre of gravity from O.

$$\therefore I \frac{d^2\theta}{dt^2} = -mgh\theta,$$

the minus indicating that the moment due to the weight always *opposes* the growth of the angle θ .

$$\therefore \frac{d^2\theta}{dt^2} = \frac{-mgh}{I}\theta = -\omega^2\theta,$$

where $\omega^2 = mgh/I$.

\therefore the motion is simple harmonic motion (p. 44),

$$\text{and period, } T, = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{mgh/I}} = 2\pi \sqrt{\frac{I}{mgh}} \quad \quad (1)$$

If $I = mk_1^2$, where k_1 is the radius of gyration about O,

$$T = 2\pi \sqrt{\frac{mk_1^2}{mgh}} = 2\pi \sqrt{\frac{k_1^2}{gh}} \quad \quad (2)$$

A *ring* of mass m and radius a will thus oscillate about an axis through a point O on its circumference normal to the plane of the ring with a period T given by

$$T = 2\pi \sqrt{\frac{I_0}{mga}}.$$

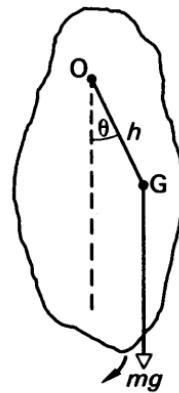


FIG. 3.15
Compound pendulum

But $I_0 = I_G + ma^2$ (theorem of parallel axes) $= ma^2 + ma^2 = 2ma^2$.

$$\therefore T = 2\pi \sqrt{\frac{2ma^2}{mga}} = 2\pi \sqrt{\frac{2a}{g}}$$

Thus if $a = 0.5$ m, $g = 9.8$ m s⁻²,

$$T = 2\pi \sqrt{\frac{2 \times 0.5}{9.8}} = 2.0 \text{ seconds (approx).}$$

Measurement of Moment of Inertia of Plate

The moment of inertia of a circular disc or other plate about an axis perpendicular to its plane, for example, can be measured by means of torsional oscillations. The plate is suspended horizontally from a vertical torsion wire, and the period T_1 of torsional oscillations is measured. Then, from (1),

$$T_1 = 2\pi \sqrt{\frac{I_1}{c}}, \quad \dots \quad \dots \quad \dots \quad (i)$$

where I_1 is the moment of inertia and c is the constant (opposing couple per unit radian) of the wire (p. 164). A ring or annulus of known moment of inertia I_2 is now placed on the plate concentric with the axis, and the new period T_2 is observed. Then

$$T_2 = 2\pi \sqrt{\frac{I_1 + I_2}{c}} \quad \dots \quad \dots \quad \dots \quad (ii)$$

By squaring (i) and (ii), and then eliminating c , we obtain

$$I_1 = \frac{T_1^2}{T_2^2 - T_1^2} \cdot I_2.$$

Thus knowing T_1 , T_2 , and I_2 , the moment of inertia I_1 can be calculated.

Compound Pendulum. Since $I = I_G + mh^2 = mk^2 + mh^2$, where I_G is the moment of inertia about the centre of gravity, h is the distance of the axis O from the centre of gravity, and k is the radius of gyration about the centre of gravity, then, from previous,

$$T = 2\pi \sqrt{\frac{I}{mgh}} = 2\pi \sqrt{\frac{mk^2 + mh^2}{mgh}}.$$

$$\therefore T = 2\pi \sqrt{\frac{k^2 + h^2}{gh}}.$$

Hence

$$T = 2\pi \sqrt{\frac{l}{g}},$$

where

$$l = \frac{k^2 + h^2}{h} \quad \dots \quad \dots \quad \dots \quad (i)$$

Thus $(k^2 + h^2)/h$ is the length, l , of the *equivalent simple pendulum*.

From (i),

$$h^2 - lh + k^2 = 0.$$

$$\therefore h_1 + h_2 = l, \text{ and } h_1 h_2 = k^2,$$

where h_1 and h_2 are the roots of the equation.

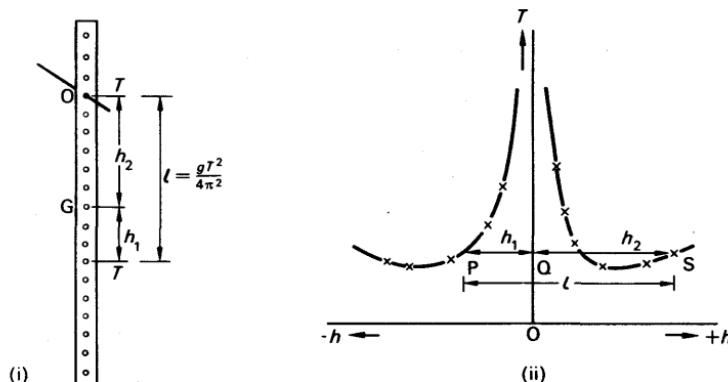


FIG. 3.16 Compound pendulum experiment

By timing the period of vibration, T , of a long rod about a series of axes at varying distances h on either side of the centre of gravity, and then plotting a graph of T v. h , two different values of h giving the same period can be obtained, Fig. 3.16 (i), (ii). Suppose h_1, h_2 are the two values. Then from the result just obtained, $h_1 + h_2 = l$, the length of the equivalent simple pendulum. Thus, since $T = 2\pi\sqrt{l/g}$,

$$g = \frac{4\pi^2 l}{T^2} = \frac{4\pi^2(h_1 + h_2)}{T^2}.$$

In Fig. 3.16 (ii), $PQ + QS = h_1 + h_2 = l$.

Kater's Pendulum. The acceleration due to gravity was first measured by the simple pendulum method, and calculated from the relation $g = 4\pi^2l/T^2$, with the usual notation. The length l , the distance from the point of suspension to the centre of gravity of the bob, however, cannot be determined with very great accuracy.

In 1817 Captain Kater designed a reversible pendulum, with knife-edges for the suspension; it was a compound pendulum. Now it has just been shown that the same period is obtained between two non-symmetrical points on a compound pendulum when their distance apart is l , the length of the equivalent simple pendulum. Thus if T is the period about either knife-edge when this occurs, $g = 4\pi^2l/T^2$, where l is now the distance between the knife-edges. The pendulum is made geometrically symmetrical about the mid-point, with a brass bob at one end and a wooden bob of the same size at the other. A movable large and small weight are placed between the knife-edges, which are about one metre apart. The period is then slightly greater than 2 seconds.

To find g , the pendulum is set up in front of an accurate seconds clock, with the bob of the clock and that of the Kater pendulum in line with each other, and both sighted through a telescope. The large weight on the pendulum is moved until the period is nearly the same about either knife-edge, and the small weight is used as a fine adjustment. When the periods are the same, the distance l between the knife-edges is measured very accurately by a comparator method with a microscope and standard metre. Thus knowing T and l , g can be calculated from $g = 4\pi^2l/T^2$.

For details of the experiment the reader should consult *Advanced Practical Physics for Students* by Worsnop and Flint (Methuen).

Summary

The following table compares the translational (linear) motion of a small mass m with the rotational motion of a large object of moment of inertia I .

Linear Motion	Rotational Motion
1. Velocity, v	Velocity $v = r\omega$
2. Momentum = mv	Angular momentum = $I\omega$
3. Energy = $\frac{1}{2}mv^2$	Rotational energy = $\frac{1}{2}I\omega^2$
4. Force, $F, = ma$	Torque, $C, = I \times \text{ang. accn. } (d^2\theta/dt^2)$
5. Simple pendulum: $T = 2\pi\sqrt{\frac{I}{g}}$	Compound pendulum: $T = 2\pi\sqrt{\frac{I}{mgh}}$
6. Motion down inclined plane – energy equation: $\frac{1}{2}mv^2 = mgh \sin \theta$	Rotating without slipping down inclined plane – energy equation: $\frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = Mgh \sin \theta$
7. Conservation of linear momentum on collision, if no external forces	Conservation of angular momentum on collision, if no external couple

EXAMPLE

What is meant by the *moment of inertia* of an object about an axis?

Describe and give the theory of an experiment to determine the moment of inertia of a flywheel mounted on a horizontal axle.

A uniform circular disc of mass 20 kg and radius 15 cm is mounted on a horizontal cylindrical axle of radius 1.5 cm and negligible mass. Neglecting frictional losses in the bearings, calculate (a) the angular velocity acquired from rest by the application for 12 seconds of a force of 2.0 kgf tangential to the axle, (b) the kinetic energy of the disc at the end of this period, (c) the time required to bring the disc to rest if a braking force of 0.1 kgf were applied tangentially to its rim. (L.)

$$\text{Moment of inertia of disc, } I, = \frac{1}{2}Ma^2 = \frac{1}{2} \times 20 \text{ (kg)} \times 0.15^2 \text{ (m}^2) = 0.225 \text{ kg m}^2.$$

(a) Torque due to 2 kgf tangential to axle

$$= 2 \times 9.8 \text{ (N)} \times 0.015 \text{ (m)} = 0.294 \text{ N m.}$$

$$\therefore \text{angular acceleration} = \frac{\text{torque}}{I} = \frac{0.294}{0.225} \text{ rad s}^{-2}.$$

$$\therefore \text{after 12 seconds, angular velocity} = \frac{12 \times 0.294}{0.225} = 15.7 \text{ rad s}^{-1}.$$

$$(b) \text{K.E. of disc after 12 seconds} = \frac{1}{2}I\omega^2$$

$$= \frac{1}{2} \times 0.225 \times 15.7^2 = 27.8 \text{ J.}$$

$$(c) \text{Decelerating torque} = 0.1 \times 9.8 \text{ (N)} \times 0.15 \text{ (m).}$$

$$\therefore \text{angular deceleration} = \frac{\text{torque}}{I} = \frac{0.1 \times 9.8 \times 0.15}{0.225} \text{ rad s}^{-2}.$$

$$\therefore \text{time to bring disc to rest} = \frac{\text{initial angular velocity}}{\text{angular deceleration}}$$

$$= \frac{15.7 \times 0.225}{0.1 \times 9.8 \times 0.15} = 24 \text{ seconds.}$$

EXERCISES 3

(Assume $g = 10 \text{ m s}^{-2}$ unless otherwise stated)*What are the missing words in the statements 1–4?*

1. The kinetic energy of an object rotating about an axis is calculated from ...
2. The angular momentum of the object is calculated from ...
3. ' $I \times$ angular acceleration' is equal to the ... on the object.
4. The period of oscillation of an object about an axis is calculated from ...

Which of the following answers, A, B, C, D or E, do you consider is the correct one in the statements 5–8?

5. When a sphere of moment of inertia I about its centre of gravity, and mass m , rolls from rest down an inclined plane without slipping, its kinetic energy is calculated from $A \frac{1}{2}I\omega^2$, $B \frac{1}{2}mv^2$, $C I\omega + mv$, $D \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$, $E I\omega$.
6. If a hoop of radius a oscillates about an axis through its circumference perpendicular to its plane, the period is $A 2\pi\sqrt{a/g}$, $B 2\pi\sqrt{2a/g}$, $C 2\pi\sqrt{g/a}$, $D 2\pi\sqrt{g/2a}$, $E a/2$.
7. Planets moving in orbit round the sun A increase in velocity at points near the sun because their angular momentum is constant, B increase in velocity near the sun because their energy is constant, C decrease in velocity near the sun owing to the increased attraction, D sweep out equal mass in equal times because their energy is constant, E always have circular orbits.
8. If a constant couple of 500 newton metre turns a wheel of moment of inertia 100 kg m^2 about an axis through its centre, the angular velocity gained in two seconds is $A 5 \text{ rad s}^{-1}$, $B 100 \text{ m s}^{-1}$, $C 200 \text{ m s}^{-1}$, $D 2 \text{ m s}^{-1}$, $E 10 \text{ rad s}^{-1}$.
9. A uniform rod has a mass of 60 g and a length 20 cm . Calculate the moment of inertia about an axis perpendicular to its length (i) through its centre, (ii) through one end. Prove the formulae used.
10. What is the *Theorem of Parallel Axes*? A uniform disc has a mass of 4 kg and a radius of 2 m . Calculate the moment of inertia about an axis perpendicular to its plane (i) through its centre, (ii) through a point of its circumference.
11. What is the *Theorem of Perpendicular Axes*? A ring has a radius of 20 cm and a mass of 100 g . Calculate the moment of inertia about an axis (i) perpendicular to its plane through its centre, (ii) perpendicular to its plane passing through a point on its circumference, (iii) in its plane passing through the centre.
12. What is the formula for the kinetic energy of (i) a particle, (ii) a rigid body rotating about an axis through its centre of gravity, (iii) a rigid body rotating about an axis through any point? Calculate the kinetic energy of a disc of mass 5 kg and radius 1 m rolling along a plane with a uniform velocity of 2 m s^{-1} .
13. A sphere rolls down a plane inclined at 30° to the horizontal. Find the acceleration and velocity of the sphere after it has moved 5.0 m from rest along the plane, assuming the moment of inertia of a sphere about a diameter is $2Ma^2/5$, where M is the mass and a is the radius.
14. A uniform rod of length 3.0 m is suspended at one end so that it can move about an axis perpendicular to its length, and is held inclined at 60° to the vertical and then released. Calculate the angular velocity of the rod when (i) it is inclined at 30° to the vertical, (ii) reaches the vertical.

15. Define the *moment of inertia* of a rigid object about an axis.

A ring of radius 2·0 m oscillates about an axis on its circumference which is perpendicular to the plane of the ring. Calculate the period of oscillation. Give an explanation of any formula used.

16. A flywheel with an axle 1·0 cm in diameter is mounted in frictionless bearings and set in motion by applying a steady tension of 200 gf to a thin thread wound tightly round the axle. The moment of inertia of the system about its axis of rotation is 5.0×10^{-4} kg m². Calculate (a) the angular acceleration of the flywheel when 1 m of thread has been pulled off the axle, (b) the constant retarding couple which must then be applied to bring the flywheel to rest in one complete turn, the tension in the thread having been completely removed. (N.)

17. Define the moment of inertia of a body about a given axis. Describe how the moment of inertia of a flywheel can be determined experimentally.

A horizontal disc rotating freely about a vertical axis makes 100 r.p.m. A small piece of wax of mass 10 g falls vertically on to the disc and adheres to it at a distance of 9 cm from the axis. If the number of revolutions per minute is thereby reduced to 90, calculate the moment of inertia of the disc. (N.)

18. Describe an experiment using a bar pendulum to determine the acceleration due to gravity. Show how the result is calculated from the observations.

A uniform disc of diameter 12·0 cm and mass 810 g is suspended with its plane horizontal by a torsion wire and allowed to perform small torsional oscillations about a vertical axis through its centre. The disc is then replaced by a uniform sphere which is allowed to oscillate similarly about a diameter. If the period of oscillation of the sphere is 1·66 times that of the disc, determine the moment of inertia of the sphere about a diameter. (L.)

19. A plane sheet of metal of uniform thickness and of irregular shape is pierced with a number of small holes, irregularly distributed, so that it can be pivoted about an axis through any one of them to swing in its own plane. Describe how you would proceed in order to find the value of the moment of inertia of the sheet about an axis through its centre of gravity normal to its plane.

A rigid bar pendulum is pivoted at a distance h from its centre of gravity. When a piece of lead of negligible size and of mass M equal to that of the pendulum, is attached at the centre of gravity the periodic time of the pendulum is reduced to 0·80 of its former value. Find an expression for the moment of inertia of the pendulum about the pivot. (L.)

20. Define *moment of inertia* and derive an expression for the kinetic energy of a rigid body of moment of inertia I about a given axis when it is rotating about that axis with a uniform angular velocity ω .

Give two examples of physical phenomena in which moment of inertia is a necessary concept for a theoretical description, in each case showing how the concept is applied.

A uniform spherical ball starts from rest and rolls freely without slipping down an inclined plane at 10° to the horizontal along a line of greatest slope. Calculate its velocity after it has travelled 5 m. (M.I. of sphere about a diameter = $2Mr^2/5$.) (O. & C.)

21. Explain the meaning of the term *moment of inertia*. Describe in detail how you would find experimentally the moment of inertia of a bicycle wheel about the central line of its hub.

A uniform cylinder 20 cm long, suspended by a steel wire attached to its mid-point so that its long axis is horizontal, is found to oscillate with a period of

2 seconds when the wire is twisted and released. When a small thin disc, of mass 10 g, is attached to each end the period is found to be 2.3 seconds. Calculate the moment of inertia of the cylinder about the axis of oscillation. (N.)

22. What is meant by 'moment of inertia'? Explain the importance of this concept in dealing with problems concerning rotating bodies.

Describe, with practical details, how you would determine whether a given cylindrical body were hollow or not without damaging it. (C.)

23. Define *moment of inertia*, and find an expression for the kinetic energy of a rigid body rotating about a fixed axis.

A sphere, starting from rest, rolls (without slipping) down a rough plane inclined to the horizontal at an angle of 30° , and it is found to travel a distance of 13.5 m in the first 3 seconds of its motion. Assuming that F , the frictional resistance to the motion, is independent of the speed, calculate the ratio of F to the *weight* of the sphere. (For a sphere of mass m and radius r , the moment of inertia about a diameter is $\frac{2}{5}mr^2$.) (O. & C.)

chapter four

Static Bodies. Fluids

STATIC BODIES

Statics

1. STATICS is a subject which concerns the *equilibrium* of forces, such as the forces which act on a bridge. In Fig. 4.1 (i), for example, the joint O of a light bridge is in equilibrium under the action of the two forces P, Q acting in the girders meeting at O and the reaction S of the masonry at O.

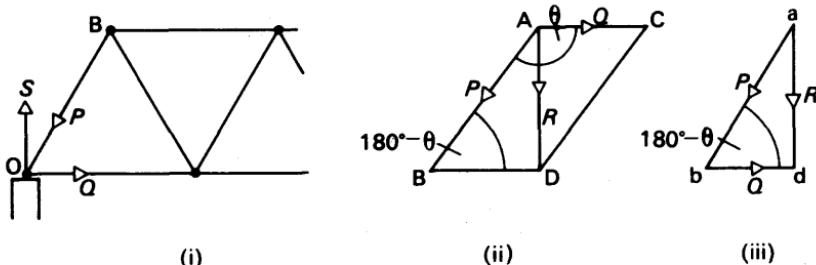


FIG. 4.1 Equilibrium of forces

Parallelogram of Forces

A force is a vector quantity, i.e., it can be represented in magnitude and direction by a straight line (p. 1). If AB, AC represent the forces P, Q respectively at the joint O, their *resultant*, R , is represented in magnitude and direction by the diagonal AD of the parallelogram ABDC which has AB, AC as two of its adjacent sides, Fig. 4.1 (ii). This is known as the *parallelogram of forces*, and is exactly analogous to the parallelogram of velocities discussed on p. 8. Alternatively, a line ab may be drawn to represent the vector P and bd to represent Q , in which case ad represents the resultant R .

By trigonometry for triangle ABD, we have

$$AD^2 = BA^2 + BD^2 - 2BA \cdot BD \cos ABD.$$

$$\therefore R^2 = P^2 + Q^2 + 2PQ \cos \theta,$$

where $\theta = \text{angle } BAC$; the angle between the forces $P, Q, = 180^\circ - \text{angle } ABD$. This formula enables R to be calculated when P, Q

and the angle between them are known. The angle BAD, or α say, between the resultant R and the force P can then be found from the relation

$$\frac{R}{\sin \theta} = \frac{Q}{\sin \alpha},$$

applying the sine rule to triangle ABD and noting that angle ABD = $180^\circ - \theta$.

Resolved component. On p. 8 we saw that the effective part, or resolved component, of a vector quantity X in a direction θ inclined to it is given by $X \cos \theta$. Thus the resolved component of a force P in a direction making an angle of 30° with it is $P \cos 30^\circ$; in a perpendicular direction to the latter the resolved component is $P \cos 60^\circ$, or $P \sin 30^\circ$. In Fig. 4.1 (i), the downward component of the force P on the joint of O is given by $P \cos BOS$.

Forces in Equilibrium. Triangle of Forces

Since the joint O is in equilibrium, Fig. 4.1 (i), the resultant of the forces P, Q in the rods meeting at this joint is equal and opposite to the reaction S at O. Now the diagonal AD of the parallelogram ABDC in Fig. 4.1 (ii) represents the resultant R of P, Q since ABDC is the parallelogram of forces for P, Q ; and hence DA represents the force S . Consequently the sides of the triangle ABD represent the three forces

at O in magnitude and direction: This result can be generalised as follows. *If three forces are in equilibrium, they can be represented by the three sides of a triangle taken in order.* This theorem in Statics is known as the *triangle of forces*. In Fig. 4.1 (ii), AB, BD, DA, in this order, represent, P, Q, S respectively in Fig. 4.1 (i)

We can derive another relation between forces in equilibrium. Suppose X, Y are the respective algebraic sums of the resolved components in two perpendicular directions of three

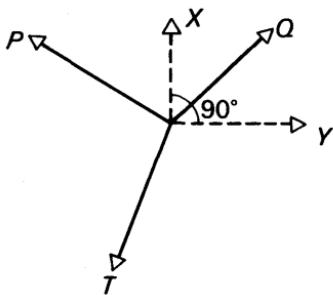


Fig. 4.2 Resolution of forces

forces P, Q, T in equilibrium, Fig. 4.2. Then, since X, Y can each be represented by the sides of a *rectangle* drawn to scale, their resultant R is given by

$$R^2 = X^2 + Y^2 \quad \quad (i)$$

Now if the forces are in equilibrium, R is zero. It then follows from (i) that X must be zero and Y must be zero. Thus *if forces are in equilibrium the algebraic sum of their resolved components in any two perpendicular directions is respectively zero.* This result applies to any number of forces in equilibrium.

EXAMPLE

State what is meant by *scalar* and *vector* quantities, giving examples of each.

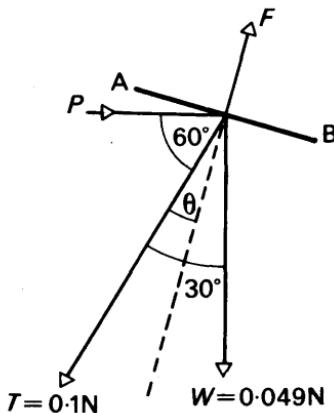


FIG. 4.3 Example

Explain how a flat kite can be flown in a wind that is blowing horizontally. The line makes an angle of 30° with the vertical and is under a tension of 0.1 newton; the mass of the kite is 5 g. What angle will the plane of the kite make with the vertical, and what force will the wind exert on it? (O. & C.)

Second part. When the kite AB is inclined to the horizontal, the wind blowing horizontally exerts an upward force F normal to AB, Fig. 4.3. For equilibrium of the kite, F must be equal and opposite to the resultant of the tension T , 0.1 newton, and the weight W , 0.005×9.8 or 0.049 N. By drawing the parallelogram of forces for the resultant of T and W , F and the angle θ between F and T can be found. θ is nearly 10° , and F is about 0.148 N. The angle between AB and the vertical = $60^\circ + \theta = 70^\circ$ (approx.). Also,

since F is the component of the horizontal force P of the wind.

$$P \cos(60^\circ + \theta) = F.$$

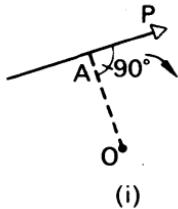
$$\therefore P = \frac{F}{\cos(60^\circ + \theta)} = \frac{0.148}{\cos 70^\circ} \\ = 0.43 \text{ N.}$$

Moments

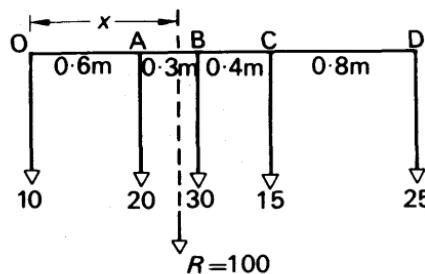
When the steering-wheel of a car is turned, the applied force is said to exert a *moment*, or turning-effect, about the axle attached to the wheel. The magnitude of the moment of a force P about a point O is defined as the product of the force P and the perpendicular distance OA for all the forces in Fig. 4.4 (ii), we have

$$\text{moment} = P \times AO.$$

The magnitude of the moment is expressed in newton metre (N m) when



(i)



(ii)

FIG. 4.4 Parallel forces

P is in newtons and AO is in metres. We shall take an anticlockwise moment as positive in sign and a clockwise moment as negative in sign.

Parallel Forces

If a rod carries loads of 10, 20, 30, 15, and 25 N at O, A, B, C, D respectively, the resultant R of the weights, which are parallel forces, for all the forces in Fig. 4.4 (ii), we have

$$\text{resultant, } R, = 10 + 20 + 30 + 15 + 25 = 100 \text{ N.}$$

Experiment and theory show that *the moment of the resultant of a number of forces about any point is equal to the algebraic sum of the moments of the individual forces about the same point*. This result enables us to find where the resultant R acts. Taking moments about O for all the forces in Fig. 4.4 (ii), we have

$$(20 \times 0.6) + (30 \times 0.9) + (15 \times 1.3) + (25 \times 2.1),$$

because the distances between the forces are 0·6 m, 0·3 m, 0·4 m, 0·8 m, as shown. If x m is the distance of the line of action of R from O, the moment of R about O = $R \times x = 100 \times x$.

$$\therefore 100x = (20 \times 0.6) + (30 \times 0.9) + (15 \times 1.3) + (25 \times 2.1),$$

from which $x = 1.1$ m.

Equilibrium of Parallel Forces

The resultant of a number of forces *in equilibrium* is zero; and the moment of the resultant about any point is hence zero. It therefore follows that the algebraic sum of the moments of all the forces about any point is zero when those forces are in equilibrium. This means that the total clockwise moment of the forces about any point = the total anticlockwise moment of the remaining forces about the same point.

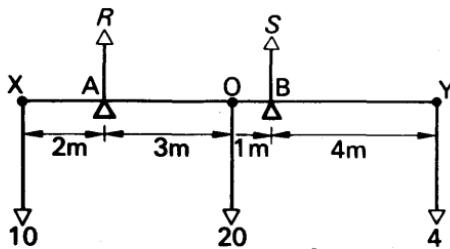


FIG. 4.5 Example

As a simple example of the equilibrium of parallel forces, suppose a light beam XY rests on supports, A, B, and has loads of 10, 20, and 4 N concentrated at X, O, Y respectively, Fig. 4.5. Let R, S be the reactions at A, B respectively. Then, for equilibrium in a vertical direction,

To find R , we take moments about a suitable point such as B, in which case the moment of S is zero. Then, for the remaining four forces,

$$+10 \cdot 6 + 20 \cdot 1 - R \cdot 4 - 4 \cdot 4 = 0,$$

from which $R = 16$ N. From (i), it follows that $S = 34 - 16 = 18$ N.

Equilibrium of Three Coplanar Forces

If any object is in equilibrium under the action of *three* forces, the resultant of two of the forces must be equal and opposite to the third force. Thus the line of action of the third force must pass through the point of intersection of the lines of action of the other two forces.

As an example of calculating unknown forces in this case, suppose that a 12 m ladder of 200 kgf is placed at an angle of 60° to the horizontal, with one end B leaning against a smooth wall and the other end

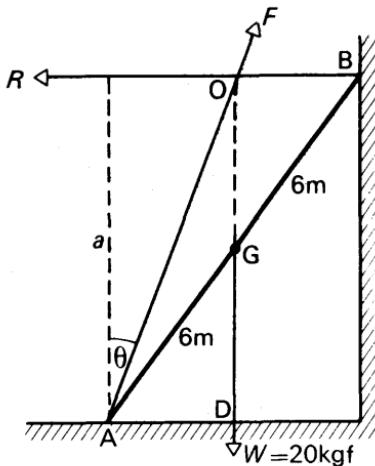


FIG. 4.6 Triangle of forces

A on the ground, Fig. 4.6. The force R at B on the ladder is called the *reaction* of the wall, and if the latter is smooth, R acts perpendicularly to the wall. Assuming the weight, W , of the ladder acts at its mid-point G, the forces W and R meet at O, as shown. Consequently the frictional force F at A passes through O.

The *triangle of forces* can be used to find the unknown forces R , F . Since DA is parallel to R , AO is parallel to F , and OD is parallel to W , the triangle of forces is represented by AOD. By means of a scale drawing R and F can be found, since

$$\frac{W(20)}{OD} = \frac{F}{AO} = \frac{R}{DA}.$$

A quicker method is to take moments about A for all the forces. The algebraic sum of the moments is zero about any point since the object is in equilibrium, and hence

$$R \cdot a - W \cdot AD = 0,$$

where a is the perpendicular from A to R. (F has zero moment about A.) But $a = 12 \sin 60^\circ$, and $AD = 6 \cos 60^\circ$.

$$\therefore R \times 12 \sin 60^\circ - 20 \times 6 \cos 60^\circ = 0.$$

$$\therefore R = 10 \frac{\cos 60^\circ}{\sin 60^\circ} = 5.8 \text{ kgf.}$$

Suppose θ is the angle F makes with the vertical.

Resolving the forces vertically, $F \cos \theta = W = 20 \text{ kgf.}$

Resolving horizontally, $F \sin \theta = R = 5.8 \text{ kgf.}$

$$\therefore F^2 \cos^2 \theta + F^2 \sin^2 \theta = F^2 = 20^2 + 5.8^2.$$

$$\therefore F = \sqrt{20^2 + 5.8^2} = 20.8 \text{ kgf.}$$

Couples and Torque

There are many examples in practice where two forces, acting together, exert a moment or turning-effect on some object. As a very simple case, suppose two strings are tied to a wheel at X, Y, and two *equal and opposite forces*, F , are exerted tangentially to the wheel, Fig. 4.7 (i). If the wheel is pivoted at its centre, O, it begins to rotate about O in an anticlockwise direction.

Two equal and opposite forces whose lines of action do not coincide are said to constitute a *couple* in Mechanics. The two forces always have a turning-effect, or moment, called a *torque*, which is defined by

$$\text{torque} = \text{one force} \times \text{perpendicular distance between forces} \quad (1)$$

Since XY is perpendicular to each of the forces F in Fig. 46 (i), the

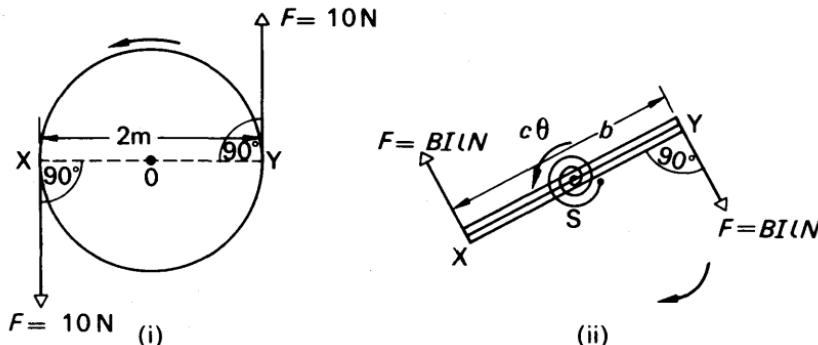


FIG. 4.7 Couple and torque

moment of the couple acting on the wheel $= F \times XY = F \times \text{diameter of wheel}$. Thus if $F = 10$ newton and the diameter is 2 metre, the moment of the couple or torque $= 20$ newton metre (N m).

In the theory of the *moving-coil electrical instrument*, we meet a case where a coil rotates when a current I is passed into it and comes to rest

after deflection through an angle θ . Fig. 4.7 (ii). The forces F on the two sides X and Y of the coil are both equal to $BILN$, where B is the strength of the magnetic field, l is the length of the coil and N is the number of turns (see Electricity section, chapter 35). Thus the coil is deflected by a couple. The moment or torque of the deflecting couple = $F \times b$, where $b = XY = \text{breadth of coil}$. Hence

$$\text{torque} = BILN \times b = BANI,$$

where $A = lb = \text{area of coil}$. The opposing couple, due to the spring S, is $c\theta$, where c is its elastic constant (p.164). Thus, for equilibrium, $BANI = c\theta$.

Work Done by a Couple

Suppose two equal and opposite forces F act tangentially to a wheel W, and rotate it through an angle θ while the forces keep tangentially to the wheel, Fig. 4.8. The moment of the couple is then constant.

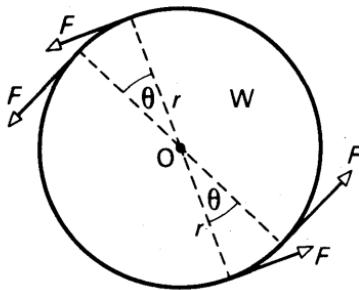


FIG. 4.8 Work done by couple

The work done by each force = $F \times \text{distance} = F \times r\theta$, since $r\theta$ is the distance moved by a point on the rim if θ is in radians.

$$\therefore \text{total work done by couple} = Fr\theta + Fr\theta = 2Fr\theta.$$

$$\text{But } \text{moment of couple} = F \times 2r = 2Fr$$

$$\therefore \text{work done by couple} = \text{torque or moment of couple} \times \theta$$

Although we have chosen a simple case, the result for the work done by a couple is always given by *torque* \times *angle of rotation*. In the formula, it should be carefully noted that θ is in radians. Thus suppose $F = 100 \text{ gf} = 0.1 \text{ kgf} = 0.1 \times 9.8 \text{ newton}$, $r = 4 \text{ cm} = 0.04 \text{ metre}$, and the wheel makes 5 revolutions while the moment of the couple is kept constant. Then

$$\text{torque or moment of couple} = 0.1 \times 9.8 \times 0.08 \text{ newton metre},$$

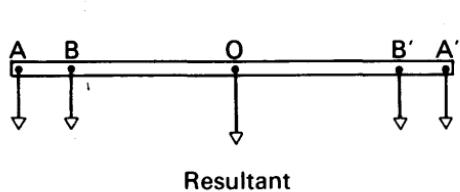
$$\text{and } \text{angle of rotation} = 2\pi \times 5 \text{ radian.}$$

$$\therefore \text{work done} = 0.1 \times 9.8 \times 0.08 \times 2\pi \times 5 = 2.5 \text{ J}$$

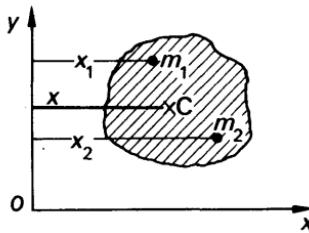
Centre of Gravity

Every particle is attracted towards the centre of the earth by the force of gravity, and the *centre of gravity* of a body is the point where the *resultant* force of attraction or *weight* of the body acts. In the simple case of a ruler, the centre of gravity is the point of support when the ruler is balanced. A similar method can be used to find roughly the centre of gravity of a flat plate. A more accurate method consists of suspending the object in turn from two points on it, so that it hangs freely in each case, and finding the point of intersection of a plumb-line, suspended in turn from each point of suspension. This experiment is described in elementary books.

An object can be considered to consist of many small particles. The forces on the particles due to the attraction of the earth are all parallel since they act vertically, and hence their resultant is the sum of all the forces. The resultant is the *weight* of the whole object, of course. In the case of a rod of uniform cross-sectional area, the weight of a particle A at one end, and that of a corresponding particle A' at the other end, have a resultant which acts at the mid-point O of the rod, Fig. 4.9 (i).



(i)



(ii)

FIG. 4.9 Centre of gravity and mass

Similarly, the resultant of the weight of a particle B, and that of a corresponding particle at B', have a resultant acting at O. In this way, i.e., by symmetry, it follows that the resultant of the weights of all the particles of the rod acts at O. Hence the centre of gravity of a uniform rod is at its mid-point.

The centre of gravity, C.G., of the curved surface of a hollow cylinder acts at the midpoint of the cylinder axis. This is also the position of the C.G. of a uniform solid cylinder. The C.G. of a triangular plate or lamina is two-thirds of the distance along a median from corresponding point of the triangle. The C.G. of a uniform right solid cone is three-quarters along the axis from the apex.

Centre of Mass

The 'centre of mass' of an object is the point where its total mass acts or appears to act. Fig. 4.9 (ii) illustrates how the position of the centre of mass of an object may be calculated, using axes Ox , Oy .

If m_1 is the mass of a small part of the object and x_1 is the perpendicular distance to Oy , then m_1x_1 represents a product similar to the

moment of a weight at m_1 about Oy . Likewise, m_2x_2 is a 'moment' about Oy , where m_2 is another small part of the object. The sum of the total 'moments' about Oy of all the parts of the object can be written Σmx . The total mass = $\Sigma m = M$ say. The distance \bar{x} of the centre of mass C from Oy is then given by

$$\bar{x} = \frac{\Sigma mx}{M}.$$

Similarly, the distance \bar{y} of the centre of mass C from Ox is given by

$$\bar{y} = \frac{\Sigma my}{M}.$$

If the earth's field is uniform at all parts of the body, then the weight of a small mass m of it is typically mg . Thus, by moments, the distance of the centre of gravity from Oy is given by

$$\frac{\Sigma mg \times x}{\Sigma mg} = \frac{\Sigma mx}{\Sigma m} = \frac{\Sigma mx}{M}.$$

The acceleration due to gravity, g , cancels in numerator and denominator. It therefore follows that the centre of mass coincides with the centre of gravity. However, if the earth's field is not uniform at all parts of the object, the weight of a small mass m_1 is then m_1g_1 say and the weight of another small mass m_2 is m_2g_2 . Clearly, the centre of gravity does not now coincide with the centre of mass. A very long or large object has different values of g at various parts of it.

EXAMPLE

What is meant by (a) the centre of mass of a body, (b) the centre of gravity of a body?

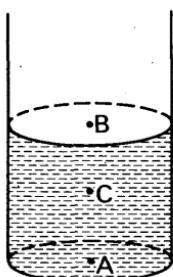


FIG. 4.10 Example

A cylindrical can is made of a material of mass 10 g cm^{-2} and has no lid. The diameter of the can is 25 cm and its height 50 cm. Find the position of the centre of mass when the can is half full of water. (C.)

The area of the base = $\pi r^2 = \pi \times (25/2)^2 \text{ cm}^2$; hence the mass is $\pi \times (25/2)^2 \times 10 \text{ g}$, and acts at A, the centre of the base, Fig. 4.10.

The mass of the curved surface of the centre = $2\pi rh \times 10 \text{ g} = 2\pi \times (25/2) \times 50 \times 10 \text{ g}$, and acts at B, half-way along the axis.

The mass of water = $\pi r^2 h \text{ g} = \pi \times (25/2)^2 \times 25 \text{ g}$, and acts at C, the mid-point of AB.

Thus the resultant mass in gramme

$$\begin{aligned} &= \frac{\pi \times 625 \times 10}{4} + \frac{2\pi \times 25 \times 50 \times 10}{2} + \frac{\pi \times 625 \times 25}{4} \\ &= \pi \times 625 \times 28\frac{3}{4}. \end{aligned}$$

Taking moments about A,

$$\therefore \pi \times 625 \times 28\frac{3}{4} \times x = (\pi \times 12500) \times AB + \left(\pi \times \frac{625 \times 25}{4} \right) \times AC$$

where x is the distance of the centre of mass from A.

$$\therefore 28\frac{3}{4}x = 20 \times 25 + \frac{25}{4} \times 12\frac{1}{2}$$

$$\therefore x = 20 \text{ (approx).}$$

\therefore centre of mass is 20 cm from the base.

Types of Equilibrium

If a marble A is placed on the curved surface of a bowl S, it rolls

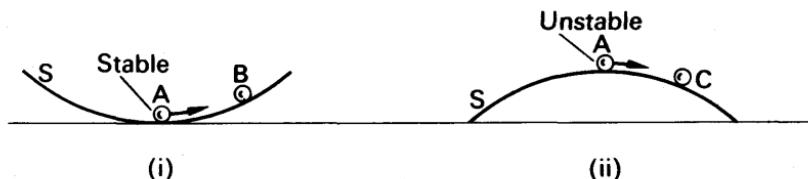


FIG. 4.11 Stable and unstable equilibrium

down and settles in equilibrium at the lowest point. Fig. 4.11 (i). Its potential energy is then a minimum. This is the case for objects in any field, gravitational, magnetic or electrical. The equilibrium position corresponds to minimum potential energy.

If the marble A is disturbed and displaced to B, its energy increases. When it is released, the marble rolls back to A. Thus the marble at A is said to be in *stable equilibrium*. Note that the centre of gravity of A is raised on displacement to B. On this account the forces in the field return the marble from B to A, where its potential energy is lower.

Suppose now that the bowl S is inverted and the marble is placed at its top point at A. Fig. 4.11 (ii). If A is displaced slightly to C, its potential energy and centre of gravity are then lowered. A now continues to move further away from B under the action of the forces in the field. Thus in Fig. 4.10 (ii), A is said to be in *unstable equilibrium*.

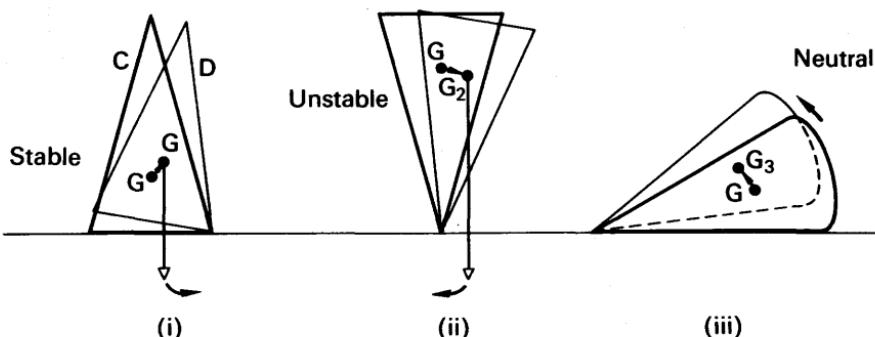


FIG. 4.12 Movement of C.G.

Fig. 4.12 (i) shows a cone C with its base on a horizontal surface. If it is slightly displaced to D, its centre of gravity G rises to G_1 . As previously explained, D returns to C when the cone is released, so that the equilibrium is stable. In Fig. 4.12 (ii), the cone is balanced on its apex. When it is slightly displaced, the centre of gravity, G_1 , is lowered to G_2 . This is unstable equilibrium. Fig. 4.12 (iii) illustrates the case of the cone resting on its curved surface. If it is slightly displaced, the centre of gravity G remains at the same height G_3 . The cone hence remains in its displaced position. This is called *neutral equilibrium*.

EXAMPLE

A rectangular beam of thickness a is balanced on the curved surface of a rough cylinder of radius r . Show that the beam is stable if r is greater than $a/2$.

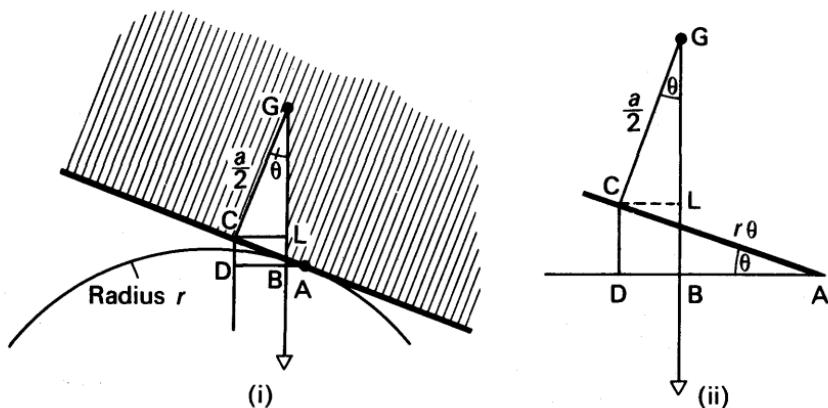


FIG. 4.13 Example

Suppose the beam is tilted through a small angle θ . The point of contact C then moves to A, the radius of the cylinder moves through an angle θ , and the vertical GB through the centre of gravity G of the beam makes an angle θ with CG. (Fig. 4.13 (i). As shown in the exaggerated sketch in Fig. 4.13 (ii), AC = $r\theta$.

The beam is in stable equilibrium if the vertical through G lies to the left of A, since a restoring moment is then exerted. Thus for stable equilibrium, AD must be greater than DB, where CD is the vertical through C.

$$\text{Now } AD = r\theta \cos \theta, DB = CL = \frac{a}{2} \sin \theta.$$

$$\therefore r\theta \cos \theta > \frac{a}{2} \sin \theta.$$

When θ is very small, $\cos \theta \rightarrow 1$, $\sin \theta \rightarrow \theta$.

$$\therefore r\theta > \frac{a}{2}\theta.$$

$$\therefore r > \frac{a}{2}.$$

Common Balance

The common balance is basically a lever whose two arms are equal, Fig. 4.14. The fulcrum, about which the beam and pointer tilt, is an agate wedge resting on an agate plate; agate wedges, B, at the ends of the beam, support the scale-pans. The centre of gravity of the beam and pointer is vertically below the fulcrum, to make the arrangement stable. The weights placed on the two scale-pans are equal when there is a 'balance'.

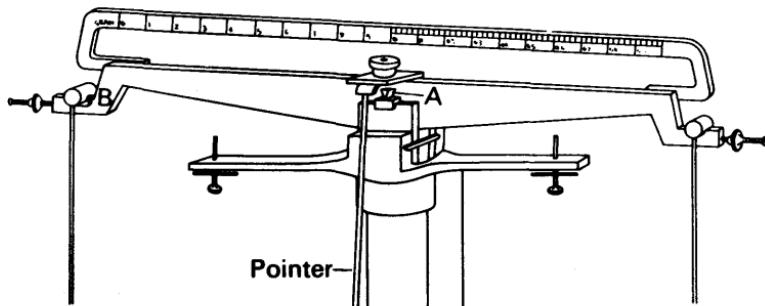


FIG. 4.14 Common balance

On rare occasions the arms of the balance are slightly unequal. The mass W of an object is then determined by finding the respective masses W_1, W_2 required to balance it on each scale-pan. Suppose a, b are the lengths of the respective arms. Then, taking moments,

$$\therefore W_1 \cdot a = W \cdot b, \text{ and } W \cdot a = W_2 \cdot b.$$

$$\therefore \frac{W}{W_1} = \frac{a}{b} = \frac{W_2}{W}$$

$$\therefore W^2 = W_1 W_2$$

$$\therefore W = \sqrt{W_1 W_2}.$$

Thus W can be found from the two masses W_1, W_2 .

Sensitivity of a Balance

A balance is said to be very *sensitive* if a small difference in weights on the scale-pans causes a large deflection of the beam. To investigate the factors which affect the sensitivity of a balance, suppose a weight W_1 is placed on the left scale-pan and a slightly smaller weight W_2 is placed on the right scale-pan, Fig. 4.15. The beam AOB will then be inclined at some angle θ to the horizontal, where O is the fulcrum.

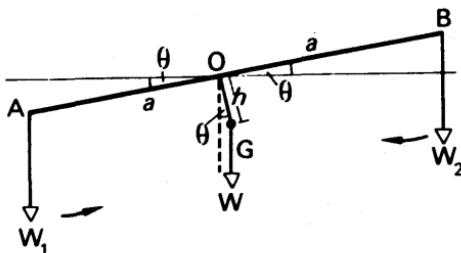


FIG. 4.15 Theory of balance

The weight W of the beam and pointer acts at G , at a distance h below O . Suppose $AO = OB = a$. Then, taking moments about O ,

$$W_1 a \cos \theta = Wh \sin \theta + W_2 a \cos \theta$$

$$\therefore (W_1 - W_2)a \cos \theta = Wh \sin \theta$$

$$\therefore \tan \theta = \frac{(W_1 - W_2)a}{Wh}.$$

Thus for a given value of $(W_1 - W_2)$, the difference of the weights on the scale-pans, θ will increase when a increases and W, h both decrease. In theory, then, a sensitive balance must be light and have long arms, and the centre of gravity of its beam and pointer must be very close to the fulcrum. Now a light beam will not be rigid. Further, a beam with long arms will take a long time to settle down when it is deflected. A compromise must therefore be made between the requirements of sensitivity and those of design.

If the knife-edges of the scale-pan and beam are in the same plane, corresponding to A, B and O in Fig. 4.15, then the weights W_1, W_2 on them always have the same perpendicular distance from O, irrespective of the inclination of the beam. In this case the net moment about O is $(W_1 - W_2)a \cos \theta$. Thus the moment depends on the difference, $W_1 - W_2$, of the weights and not on their actual values. Hence the sensitivity is independent of the actual load value over a considerable range.

When the knife-edge of the beam is below the knife-edges of the two scale-pans, the sensitivity increases with the load; the reverse is the case if the knife-edge of the beam is above those of the scale-pans.

Buoyancy Correction in Weighing

In very accurate weighing, a correction must be made for the buoyancy of the air. Suppose the body weighed has a density ρ and a mass m . From Archimedes principle (p. 114), the upthrust due to the air of density σ is equal to the weight of air displaced by the body, and hence the net downward force = $\left(m - \frac{m}{\rho} \cdot \sigma\right)g$, since the volume of the

body is m/ρ . Similarly, if the weights restoring a balance have a total mass m_1 and a density ρ_1 , the net downward force = $\left(m_1 - \frac{m_1}{\rho_1} \cdot \sigma\right)g$. Since there is a balance,

$$m - \frac{m\sigma}{\rho} = m_1 - \frac{m_1\sigma}{\rho_1}$$

$$\therefore m = m_1 \frac{\left(1 - \frac{\sigma}{\rho_1}\right)}{1 - \frac{\sigma}{\rho}}$$

Thus knowing the density of air, σ , and the densities ρ, ρ_1 , the true mass m can be found in terms of m_1 . The pressure and temperature of air, which may vary from day to day, affects the magnitude of its density σ , from the gas laws; the humidity of the air is also taken into account in very accurate weighing, as the density of moist air differs from that of dry air.

FLUIDS

Pressure

Liquids and gases are called *fluids*. Unlike solid objects, fluids can flow.

If a piece of cork is pushed below the surface of a pool of water and then released, the cork rises to the surface again. The liquid thus exerts an upward force on the cork and this is due to the *pressure* exerted on the cork by the surrounding liquid. Gases also exert pressures. For example, when a thin closed metal can is evacuated, it

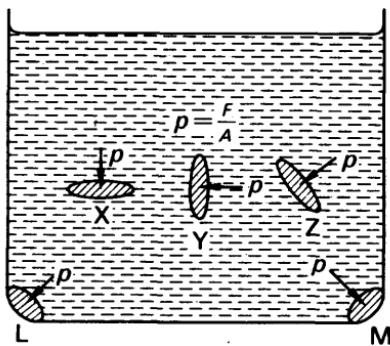


FIG. 4.16 Pressure in liquid

usually collapses with a loud explosion. The surrounding air now exerts a pressure on the outside which is no longer counter-balanced by the pressure inside, and hence there is a resultant force.

Pressure is defined as the *average force per unit area* at the particular region of liquid or gas. In Fig. 4.16, for example, X represents a small

horizontal area, Y a small vertical area and Z a small inclined area, all inside a vessel containing a liquid. The pressure p acts normally to the planes of X, Y or Z. In each case

$$\text{average pressure, } p, = \frac{F}{A},$$

where F is the normal force due to the liquid on an area A of X, Y or Z. Similarly, the pressure p on the sides L or M of the curved vessel act normally to L and M have magnitude F/A . In the limit, when the area is very small, $p = dF/dA$.

At a given point in a liquid, the pressure can act in any direction. Thus *pressure is a scalar*, not a vector. The direction of the force on a particular surface is normal to the surface.

Formula for Pressure

Observation shows that the pressure increases with the depth, h , below the liquid surface and with its density ρ .

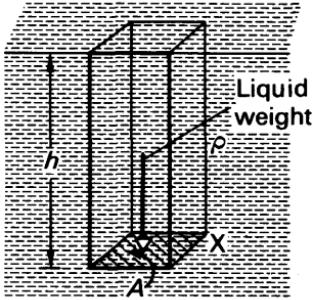


FIG. 4.17 Pressure and depth

To obtain a formula for the pressure, p , suppose that a horizontal plate X of area A is placed at a depth h below the liquid surface, Fig. 4.17. By drawing vertical lines from points on the perimeter of X, we can see that the force on X due to the liquid is equal to the weight of liquid of height h and uniform cross-section A. Since the volume of this liquid is Ah , the mass of the liquid = $Ah \times \rho$.

$$\therefore \text{weight} = Ah\rho g \text{ newton},$$

where g is 9.8, h is in m, A is in m^2 , and ρ is in kg m^{-3} .

$$\therefore \text{pressure, } p, \text{ on } X = \frac{\text{force}}{\text{area}} = \frac{Ah\rho g}{A}$$

$$\therefore p = h\rho g \dots \dots \dots \quad (1)$$

When h , ρ , g have the units already mentioned, the pressure p is in newton m^{-2} (N m^{-2}).

1 bar = 10^6 dyne cm^{-2} . To change 10^6 dyne cm^{-2} to N m^{-2} , we may proceed as follows:

$$10^6 \frac{\text{dyne}}{\text{cm}^2} = 10^6 \frac{\text{dyne}}{\text{N}} \cdot \frac{\text{N}}{\text{m}^2} \cdot \frac{\text{m}^2}{\text{cm}^2} = 10^6 \times \frac{1}{10^5} \cdot \frac{\text{N}}{\text{m}^2} \cdot 10^4 = 10^5 \frac{\text{N}}{\text{m}^2}$$

$$\therefore 1 \text{ bar} = 10^5 \text{ N m}^{-2} \dots \dots \dots \quad (2)$$

Pressure is often expressed in terms of that due to a height of mercury (Hg). One unit is the *torr* (after Torricelli):

$$1 \text{ torr} = 1 \text{ mmHg} = 133.3 \text{ N m}^{-2} (\text{approx}).$$

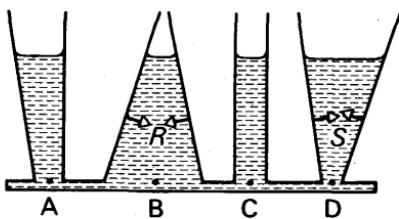
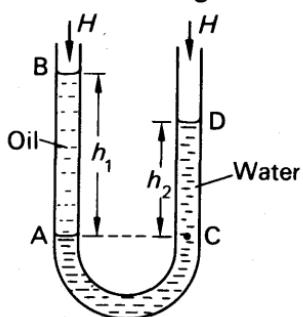


FIG. 4.18 Pressure and cross-section

From $p = h\rho g$ it follows that *the pressure in a liquid is the same at all points on the same horizontal level in it*. Experiment also gives the same result. Thus a liquid filling the vessel shown in Fig. 4.18 rises to the same height in each section if ABCD is horizontal. The cross-sectional area of B is greater than that of D; but the force on B is the sum of the weight of water above it together with the downward

component of reaction R of the sides of the vessel, whereas the force on D is the weight of water above it minus the upward component of the reaction S of the sides of the vessel. It will thus be noted that the pressure in a vessel is independent of the cross-sectional area of the vessel.

FIG. 4.19
Comparison of densities

surface of separation, A, of the water and oil, while the water on the right side of the tube will then reach some level D at a height h_2 above the level of A, Fig. 4.19.

Since the pressure in the water at A is equal to the pressure at C on the same horizontal level, it follows that

$$H + h_1 \rho_1 g = H + h_2 \rho_2 g,$$

where H is the atmospheric pressure, and ρ_1 , ρ_2 are the respective densities of oil and water. Simplifying,

$$h_1 \rho_1 = h_2 \rho_2$$

$$\therefore \rho_1 = \rho_2 \times \frac{h_2}{h_1}$$

Since ρ_2 (water) = 1000 kg m^{-3} , and h_2 , h_1 can be measured, the density ρ_1 of the oil can be found.

Atmospheric Pressure

The pressure of the atmosphere was first measured by Galileo, who observed the height of a water column in a tube placed in a deep well.

About 1640 TORRICELLI thought of the idea of using mercury instead of water, to obtain a much shorter column. He completely filled a glass tube about a metre long with mercury, and then inverted it in a vessel D containing the liquid, taking care that no air entered the tube. He observed that the mercury in the tube fell to a level A about 76 cm or 0.76 m above the level of the mercury in D, Fig. 4.20. Since there was

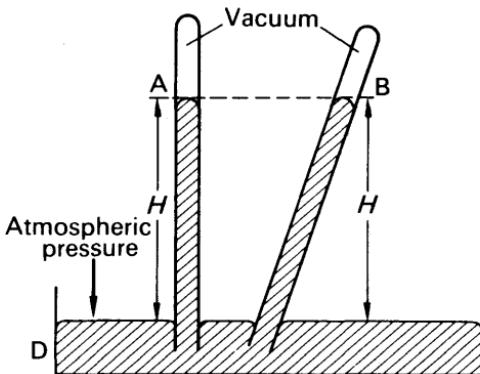


FIG. 4.20 Atmospheric pressure

no air originally in the tube, there must be a vacuum above the mercury at A, and it is called a *Torricellian vacuum*. This was the first occasion in the history of science that a vacuum had been created.

If the tube in Fig. 4.20 is inclined to the vertical, the mercury ascends the tube to a level B at the same vertical height H above the level of the mercury in D as A.

The pressure on the surface of the mercury in D is atmospheric pressure; and since the pressure is transmitted through the liquid, the atmospheric pressure supports the column of mercury in the tube. Suppose A is at a height H above the level of the mercury in D. Now the pressure, p , at the bottom of a column of liquid of height H and density ρ is given by $p = H\rho g$ (p. 110). Thus if $H = 760 \text{ mm} = 0.76 \text{ m}$ and $\rho = 13600 \text{ kg m}^{-3}$,

$$p = H\rho g = 0.76 \times 13600 \times 9.8 = 1.013 \times 10^5 \text{ newton metre}^{-2}.$$

The pressure at the bottom of a column of mercury 76 cm high for a particular mercury density and value of g is known as *standard pressure* or *one atmosphere*. By definition, 1 atmosphere $= 1.01325 \times 10^5 \text{ N m}^{-2}$. *Standard temperature and pressure* (S.T.P.) is 0°C and 76 cm Hg pressure.

A *bar* is the name given to a pressure of one million (10^6) dyne cm^{-2} , and is thus very nearly equal to one atmosphere. 1 bar $= 10^5 \text{ newton m}^{-2}$ (p. 110).

Fortin's Barometer

A *barometer* is an instrument for measuring the pressure of the atmosphere, which is required in weather-forecasting, for example. The most accurate form of barometer is due to FORTIN, and like the simple arrangement already described, it consists basically of a barometer tube containing mercury, with a vacuum at the top, Fig. 4.21. One end of the tube dips into a pool of mercury contained in a washleather bag B. A brass scale C graduated in centimetres and millimetres is fixed at the top of the barometer. The zero of the scale corresponds to the tip of an ivory tooth P, and hence, before the level of the top of the mercury is read from the scales, the screw S is adjusted until the level of the mercury in B just reaches the tip of P. A vernier scale V can be moved by a screw D until the bottom of it just reaches the top of the mercury in the tube, and the reading of the height of the mercury is taken from C and V. Torricelli was the first person to observe the variation of the barometric height as the weather changed.

'Correction' to the Barometric Height

For comparison purposes, the pressure read on a barometer is often 'reduced' or 'corrected' to the magnitude the pressure would have at 0°C and at sea-level, latitude 45° . Suppose the 'reduced' pressure is H_o cm of mercury, and the observed pressure is H_t cm of mercury, corresponding to a temperature of $t^\circ\text{C}$. Then, since pressure = $h\rho g$ (p. 110),

$$H_o \rho_o g = H_t \rho_t g'$$

where g is the acceleration due to gravity at sea-level, latitude 45° , and g' is the acceleration at the latitude of the place where the barometer was read.

$$\therefore H_o = H_t \times \frac{\rho_t}{\rho_o} \times \frac{g}{g'}$$

The magnitude of g'/g can be obtained from standard tables. The ratio ρ_t/ρ_o of the densities = $1/(1 + \gamma t)$, where γ is the absolute or true cubic expansivity of mercury. Further, the observed height H_t , on the brass scale requires correction for the expansion of brass from the temperature at which it was correctly calibrated. If the latter is 0°C , then the corrected height is $H_t(1 + \alpha t)$, where α is the mean linear

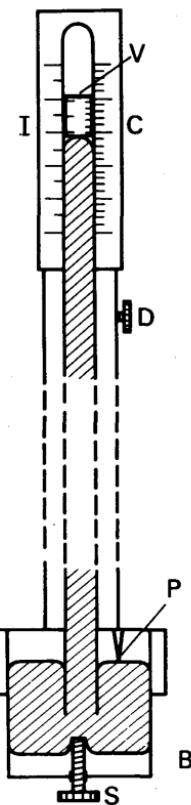


FIG. 4.21
Fortin barometer

expansivity of brass. Thus, finally, the 'corrected' height H is given by

$$H_o = H_t \cdot \frac{1 + \alpha t}{1 + \gamma t} \cdot \frac{g'}{g}$$

For further accuracy, a correction must be made for the surface tension of mercury (p. 132).

Variation of atmospheric pressure with height

The density of a liquid varies very slightly with pressure. The density of a gas, however, varies appreciably with pressure. Thus at sea-level the density of the atmosphere is about 1.2 kg m^{-3} ; at 1000 m above sea-level the density is about 1.1 kg m^{-3} ; and at 5000 m above sea-level it is about 0.7 kg m^{-3} . Normal atmospheric pressure is the pressure at the base of a column of mercury 760 mm high, a liquid which has a density of about 13600 kg m^{-3} . Suppose air has a constant density of about 1.2 kg m^{-3} . Then the height of an air column of this density which has a pressure equal to normal atmospheric pressure

$$= \frac{760}{1000} \times \frac{13600}{1.2} \text{ m} = 8.4 \text{ km.}$$

In fact, the air 'thins' the higher one goes, as explained above. The height of the air is thus much greater than 8.4 km.

Density, Relative Density

As we have seen, the pressure in a fluid depends on the density of the fluid.

The *density* of a substance is defined as its *mass per unit volume*. Thus

$$\text{density, } \rho, = \frac{\text{mass of substance}}{\text{volume of substance}} \quad . . . \quad (47)$$

The density of copper is about 9.0 g cm^{-3} or $9 \times 10^3 \text{ kg m}^{-3}$; the density of aluminium is 2.7 g cm^{-3} or $2.7 \times 10^3 \text{ kg m}^{-3}$; the density of water at 4°C is 1 g cm^{-3} or 1000 kg m^{-3} .

Substances which float on water have a density less than 1000 kg m^{-3} (p. 117). For example, ice has a density of about 900 kg m^{-3} ; cork has a density of about 250 kg m^{-3} . Steel, of density 8500 kg m^{-3} , will float on mercury, whose density is about 13600 kg m^{-3} at 0°C .

The density of a substance is often expressed relative to the density of water. This is called the *relative density* or *specific gravity* of the substance. It is a ratio or number, and has no units. The relative density of mercury is 13.6. Thus the density of mercury is 13.6 times the density of water, 1000 kg m^{-3} , and is hence 13600 kg m^{-3} . Copper has a relative density of 9.0 and hence a density of 9000 kg m^{-3} .

Archimedes' Principle

An object immersed in a fluid experiences a resultant upward force owing to the pressure of fluid on it. This upward force is called the *upthrust* of the fluid on the object. ARCHIMEDES stated that *the upthrust is equal to the weight of fluid displaced by the object*, and this is known as

Archimedes' Principle. Thus if an iron cube of volume 400 cm^3 is totally immersed in water of density 1 g cm^{-3} , the upthrust on the cube = $400 \times 1 = 400 \text{ gf}$. If the same cube is totally immersed in oil of density 0.8 g cm^{-3} , the upthrust on it = $400 \times 0.8 = 320 \text{ gf}$.

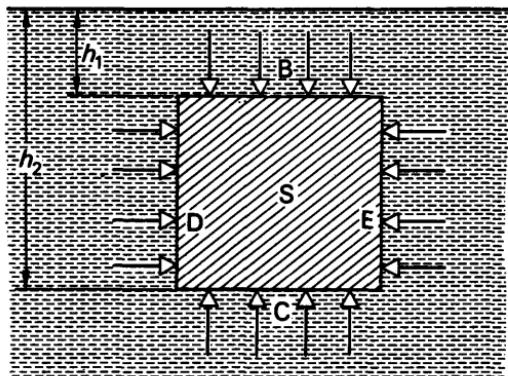


FIG. 4.22 Archimedes' Principle

Fig. 4.22 shows why Archimedes' Principle is true. If S is a solid immersed in a liquid, the pressure on the lower surface C is greater than on the upper surface B , since the pressure at the greater depth h_2 is more than that at h_1 . The pressure on the remaining surfaces D and E act as shown. The *force* on each of the four surfaces is calculated by summing the values of *pressure* \times *area* over every part, remembering that vector addition is needed to sum forces. With a simple *rectangular-shaped solid* and the sides, D , E vertical, it can be seen that (i) the resultant horizontal force is zero, (ii) the upward force on C = pressure \times area $A = h_2 \rho g A$, where ρ is the liquid density and the downward force on B = pressure \times area $A = h_1 \rho g A$. Thus

$$\text{resultant force on solid} = \text{upward force (upthrust)} = (h_2 - h_1) \rho g A.$$

But $(h_2 - h_1)A = \text{volume of solid, } V,$

$$\therefore \text{upthrust} = V \rho g = mg, \text{ where } m = V \rho.$$

$$\therefore \text{upthrust} = \text{weight of liquid displaced}.$$

With a solid of irregular shape, taking into account horizontal and vertical components of forces, the same result is obtained. The upthrust is the weight of *liquid* displaced whatever the nature of the object immersed, or whether it is hollow or not. This is due primarily to the fact that the pressure on the object depends on the liquid in which it is placed.

Density or Relative Density measurement by Archimedes' Principle

The upthrust on an object immersed in water, for example, is the difference between (i) its weight in air when attached to a spring-balance and (ii) the reduced reading on the spring-balance or 'weight'

when it is totally immersed in the liquid. Suppose the upthrust is found to be 100 gf. Then, from Archimedes' Principle, the object displaces 100 gf of water. But the density of water is 1 g cm^{-3} . Hence the volume of the object = 100 cm^3 , which is numerically equal to the difference in weighings in (i) and (ii).

The density or relative density of a *solid* such as brass or iron can thus be determined by (1) weighing it in air, m_0 gf say, (2) weighing it when it is totally immersed in water, m_1 gf say. Then

$$\text{upthrust} = m_0 - m_1 = \text{wt. of water displaced.}$$

$$\therefore \text{relative density of solid} = \frac{m_0}{m_0 - m_1},$$

$$\text{and density of solid, } \rho, = \frac{m_0}{m_0 - m_1} \times \text{density of water.}$$

The density or relative density of a *liquid* can be found by weighing a solid in air (m_0), then weighing it totally immersed in the liquid (m_1), and finally weighing it totally immersed in water (m_2).

Now $m_0 - m_2$ = upthrust in water = weight of water displaced,
and $m_0 - m_1$ = upthrust in liquid = weight of liquid displaced.

$$\therefore \frac{m_0 - m_1}{m_0 - m_2} = \text{relative density of liquid,}$$

$$\text{or } \frac{m_0 - m_1}{m_0 - m_2} \times \text{density of water} = \text{density of liquid.}$$

Density of Copper Sulphate crystals

If a solid dissolves in water, such as a copper sulphate crystal for example, its density can be found by totally immersing it in a liquid in which it is insoluble. Copper sulphate can be weighed in paraffin oil, for example. Suppose the apparent weight is m_1 , and the weight in air is m_0 . Then

$$m_0 - m_1 = \text{upthrust in liquid} = V\rho,$$

where V is the volume of the solid and ρ is the density of the liquid.

$$\therefore V = \frac{m_0 - m_1}{\rho}.$$

$$\therefore \text{density of solid} = \frac{\text{mass}}{\text{volume}} = \frac{m_0}{V} = \frac{m_0}{m_0 - m_1} \cdot \rho.$$

The density, ρ , of the liquid can be found by means of a density bottle, for example. Thus knowing m_0 and m_1 , the density of the solid can be calculated.

Density of Cork

If a solid floats in water, cork for example, its density can be found by attaching a brass weight or 'sinker' to it so that both solids become totally immersed in water. The apparent weight (m_1) of the sinker and cork together is then obtained. Suppose m_2 is the weight of the sinker in air, m_3 is the weight of the sinker alone in water, and m_0 is the weight of the cork in air.

Then

$$m_2 - m_3 = \text{upthrust on sinker in water.}$$

$$\therefore m_0 + m_2 - m_1 - (m_2 - m_3) = \text{upthrust on cork in water}$$

$$= m_0 - m_1 + m_3$$

$$\therefore \text{relative density of cork} = \frac{m_0}{m_0 - m_1 + m_3}.$$

Flotation

When an object *floats* in a liquid, the upthrust on the object must be equal to its weight for equilibrium. Cork has a density of about 0.25 g cm^{-3} , so that 100 cm^3 of cork has a mass of 25 g. In water, then, cork sinks until the upthrust is 25 gf. Now from Archimedes' Principle, 25 gf is the weight of water displaced. Thus the cork sinks until 25 cm^3 of its 100 cm^3 volume is immersed. The fraction of the volume immersed is hence equal to the relative density.

Ice has a density of about 0.9 g cm^{-3} . A block of ice therefore floats in water with about $\frac{9}{10}$ ths of it immersed.

Hydrometer

Hydrometers use the principle of flotation to measure density or relative density. Fig. 4.23. Since they have a constant weight, the upthrust when they float in a liquid is always the same. Thus in a liquid of density 1.0 g cm^{-3} , a hydrometer of 20 gf will sink until 20 cm^3 is immersed. In a liquid of 2.0 g cm^{-3} , it will sink until only 10 cm^3 is immersed. The density or relative density readings hence increase in a *downward* direction, as shown in Fig. 4.23.

Practical hydrometers have a weighted end M for stability, a wide bulb to produce sufficient upthrust to counterbalance the weight, and a narrow stem BL for sensitivity. If V is the whole volume of the hydrometer in Fig. 4.23, a is the area of the stem and y is the length not immersed in a liquid of density ρ , then

$$\text{upthrust} = \text{wt. of liquid displaced} = (V - ay)\rho = w,$$

where w is the weight of the hydrometer.

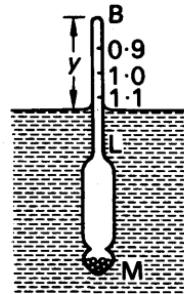


FIG. 4.23
Hydrometer

EXAMPLE

An ice cube of mass 50.0 g floats on the surface of a strong brine solution of volume 200.0 cm^3 inside a measuring cylinder. Calculate the level of the liquid in the measuring cylinder (i) before and (ii) after all the ice is melted. (iii) What happens to the level if the brine is replaced by 200.0 cm^3 water and 50.0 g of ice is again added? (Assume density of ice, brine = $900, 1100 \text{ kg m}^{-3}$ or $0.9, 1.1 \text{ g cm}^{-3}$.)

(i) Floating ice displaces 50 g of brine since upthrust is 50 gf.

$$\therefore \text{volume displaced} = \frac{\text{mass}}{\text{density}} = \frac{50}{1.1} = 45.5 \text{ cm}^3.$$

$$\therefore \text{level on measuring cylinder} = 245.5 \text{ cm}^3.$$

(ii) 50 g of ice forms 50 g of water when all of it is melted.

\therefore level on measuring cylinder falls to 250.0 cm³.

(iii) Water. Initially, volume of water displaced = 50 cm³, since upthrust = 50 g.

\therefore level on cylinder = 250.0 cm³.

If 1 g of ice melts, volume displaced is 1 cm³ less. But volume of water formed is 1 cm³. Thus the net change in water level is zero. Hence the water level remains unchanged as the ice melts.

Fluids in Motion. Streamlines and velocity

A stream or river flows slowly when it runs through open country and faster through narrow openings or constrictions. As shown shortly, this is due to the fact that water is practically an incompressible fluid, that is, changes of pressure cause practically no change in fluid density at various parts.

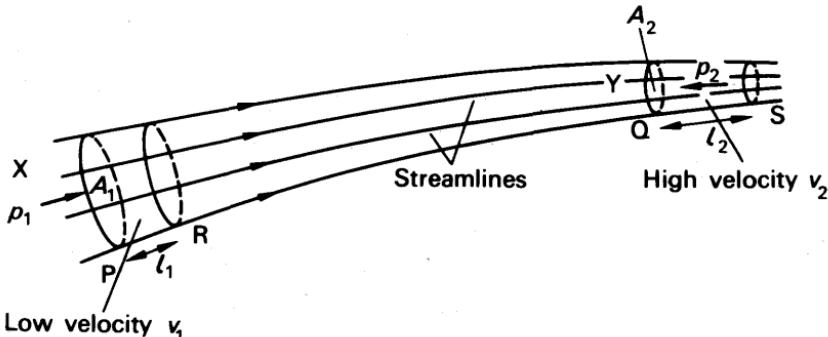


FIG. 4.24 Bernoulli's theorem

Fig. 4.24 shows a tube of water flowing steadily between X and Y, where X has a bigger cross-sectional area A_1 than the part Y, of cross-sectional area A_2 . The *streamlines* of the flow represent the directions of the velocities of the particles of the fluid and the flow is uniform or laminar (p. 204). Assuming the liquid is incompressible, then, if it moves from PQ to RS, the volume of liquid between P and R is equal to the volume between Q and S. Thus $A_1 l_1 = A_2 l_2$, where l_1 is PR and l_2 is QS, or $l_2/l_1 = A_1/A_2$. Hence l_2 is greater than l_1 . Consequently the *velocity* of the liquid at the narrow part of the tube, where, it should be noted, the streamlines are closer together, is greater than at the wider part Y, where the streamlines are further apart. For the same reason, slow-running water from a tap can be made into a fast jet by placing a finger over the tap to narrow the exit.

Pressure and velocity. Bernoulli's Principle

About 1740, Bernoulli obtained a relation between the pressure and velocity at different parts of a moving incompressible fluid. If the viscosity is negligibly small, there are no frictional forces to overcome (p. 174). In this case the work done by the pressure difference per unit

volume of a fluid flowing along a pipe steadily is equal to the gain of kinetic energy per unit volume plus the gain in potential energy per unit volume.

Now the work done by a pressure in moving a fluid through a distance = force \times distance moved = (pressure \times area) \times distance moved = pressure \times volume moved, assuming the area is constant at a particular place for a short time of flow. At the beginning of the pipe where the pressure is p_1 , the work done per unit volume on the fluid is thus p_1 ; at the other end, the work done per unit volume by the fluid is likewise p_2 . Hence the net work done on the fluid per unit volume = $p_1 - p_2$. The kinetic energy per unit volume = $\frac{1}{2}$ mass per unit volume \times velocity 2 . = $\frac{1}{2}\rho \times$ velocity 2 , where ρ is the density of the fluid. Thus if v_2 and v_1 are the final and initial velocities respectively at the end and the beginning of the pipe, the kinetic energy gained per unit volume = $\frac{1}{2}\rho(v_2 - v_1)^2$. Further, if h_2 and h_1 are the respective heights measured from a fixed level at the end and beginning of the pipe, the potential energy gained per unit volume = mass per unit volume $\times g \times (h_2 - h_1)$ = $\rho g(h_2 - h_1)$.

Thus, from the conservation of energy,

$$\begin{aligned} p_1 - p_2 &= \frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(h_2 - h_1) \\ \therefore p_1 + \frac{1}{2}\rho v_1^2 + \rho gh_1 &= p_2 + \frac{1}{2}\rho v_2^2 + \rho gh_2 \\ \therefore p + \frac{1}{2}\rho v^2 + \rho gh &= \text{constant}, \end{aligned}$$

where p is the pressure at any part and v is the velocity there. Hence it can be said that, for streamline motion of an incompressible non-viscous fluid,

the sum of the pressure at any part plus the kinetic energy per unit volume plus the potential energy per unit volume there is always constant.

This is known as *Bernoulli's principle*.

Bernoulli's principle shows that at points in a moving fluid where the potential energy change ρgh is very small, or zero as in flow through a horizontal pipe, the pressure is low where the velocity is high; conversely, the pressure is high where the velocity is low. The principle has wide applications.

EXAMPLE

As a numerical illustration of the previous analysis, suppose the area of cross-section A_1 of X in Fig. 4.25 is 4 cm^2 , the area A_2 of Y is 1 cm^2 , and water flows past each section in laminar flow at the rate of $400 \text{ cm}^3 \text{ s}^{-1}$. Then

$$\text{at X, speed } v_1 \text{ of water} = \frac{\text{vol. per second}}{\text{area}} = \frac{400 \text{ cm}^3 \text{ s}^{-1}}{4 \text{ cm}^2} = 100 \text{ cm s}^{-1} = 1 \text{ m s}^{-1};$$

$$\text{at Y, speed } v_2 \text{ of water} = 400 \text{ cm s}^{-1} = 4 \text{ m s}^{-1}.$$

The density of water, $\rho = 1000 \text{ kg m}^{-3}$.

$$\therefore p = \frac{1}{2}\rho(v_2^2 - v_1^2) = \frac{1}{2} \times 1000 \times (4^2 - 1^2) = 7.5 \times 10^3 \text{ newton m}^{-2}.$$

If h is in metres, $\rho = 1000 \text{ kg m}^{-3}$ for water, $g = 9.8 \text{ m s}^{-2}$, then, from $h\rho g$,

$$h = \frac{7.5 \times 10^3}{1000 \times 9.8} = 0.77 \text{ m (approx.)}$$

The pressure head h is thus equivalent to 0.77 m of water.

Applications of Bernoulli's Principle

1. A suction effect is experienced by a person standing close to the platform at a station when a fast train passes. The fast-moving air between the person and train produces a decrease in pressure and the excess air pressure on the other side pushes the person towards the train.

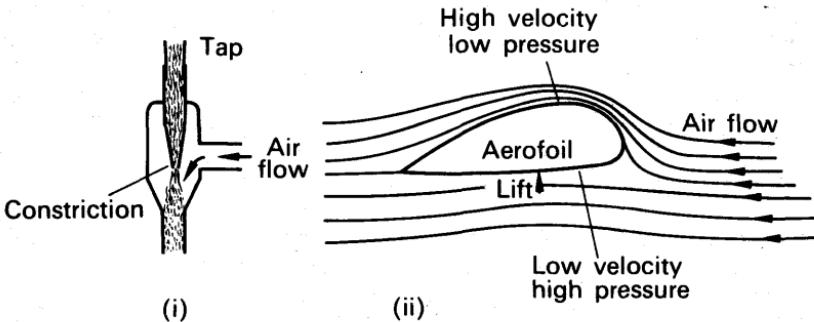


FIG. 4.25 Fluid velocity and pressure

2. *Filter pump*. A filter pump has a narrow section in the middle, so that a jet of water from the tap flows faster here. Fig. 4.25 (i). This causes a drop in pressure near it and air therefore flows in from the side tube to which a vessel is connected. The air and water together are expelled through the bottom of the filter pump.

3. *Aerofoil lift*. The curved shape of an aerofoil creates a fast flow of air over its top surface than the lower one. Fig. 4.25 (ii). This is shown by the closeness of the streamlines above the aerofoil compared with those below. From Bernoulli's principle, the pressure of the air below is greater than that above, and this produces the lift on the aerofoil.

4. *Flow of liquid from wide tank*. Suppose a liquid flows through a hole H at the bottom of a wide tank, as shown in Fig. 4.26. Assuming negligible viscosity and streamline flow at a small distance from the hole, which is an approximation, Bernoulli's theorem can be applied.

At the top X of the liquid in the tank, the pressure is atmospheric, say B , the height measured from a fixed level such as the hole H is h , and the kinetic energy is negligible if the tank is wide so that the level falls very slowly. At the bottom, Y, near H, the pressure is again B , the height above H is now zero, and the kinetic energy is $\frac{1}{2}\rho v^2$, where ρ is the density and v is the

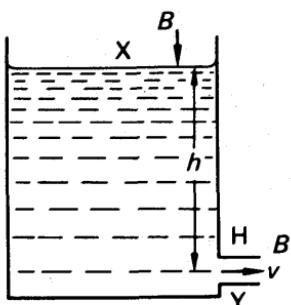


FIG. 4.26 Torricelli's theorem

velocity of emergence of the liquid. Thus, from Bernoulli's Principle,

$$B + \rho hg = B + \frac{1}{2} \rho v^2$$

$$\therefore v^2 = 2gh$$

Thus the velocity of the emerging liquid is the same as that which would be obtained if it fell freely through a height h , and this is known as *Torricelli's theorem*. In practice the velocity is less than that given by $\sqrt{2gh}$ owing to viscous forces, and the lack of streamline flow must also be taken into account.

EXERCISES 4

What are the missing words in the statements 1–6?

1. In SI units, the moment or torque of a couple is measured in ...
2. In stable equilibrium, when an object is slightly displaced its centre of gravity ...
3. When an object is in equilibrium under the action of three non-parallel forces, the three forces must ... one point.
4. The component of a force F in a direction inclined to it at an angle θ is ...
5. The sensitivity of a beam balance depends on the depth of the ... below the fulcrum.
6. When an object floats, the weight of fluid displaced is equal to the ...
7. In laminar flow of non-viscous fluid along a pipe, at regions of high pressure the ... is low.

Which of the following answers, A, B, C, D or E, do you consider is the correct one in the statements 8–10?

8. If a cone is balanced on its apex on a horizontal table and then slightly displaced, the potential energy of the cone is then *A* increased, *B* decreased, *C* constant, *D* a minimum, *E* a maximum.
9. If a hydrometer of mass 20 g and volume 30 cm^3 has a graduated stem of 1 cm^2 , and floats in water, the exposed length of stem is *A* 30 cm, *B* 25 cm, *C* 20 cm, *D* 10 cm, *E* 1 cm.
10. In laminar flow of a non-viscous fluid along a horizontal pipe, the work per second done by the pressure at any section is equal to *A* the pressure, *B* the volume per second there, *C* pressure \times volume per second there, *D* pressure \times volume, *E* pressure \times area of cross-section.
11. A flat plate is cut in the shape of a square of side 20·0 cm, with an equilateral triangle of side 20·0 cm adjacent to the square. Calculate the distance of the centre of mass from the apex of the triangle.
12. The foot of a uniform ladder is on a rough horizontal ground, and the top rests against a smooth vertical wall. The weight of the ladder is 40 kgf, and a man weighing 80 kgf stands on the ladder one-quarter of its length from the bottom. If the inclination of the ladder to the horizontal is 30° , find the reaction at the wall and the total force at the ground.

13. A rectangular plate ABCD has two forces of 10 kgf acting along AB and DC in opposite directions. If AB = 3 m, BC = 5 m, what is the moment of the couple acting on the plate? What forces acting along BC and AD respectively are required to keep the plate in equilibrium?

14. A hollow metal cylinder 2 m tall has a base of diameter 35 cm and is filled with water to a height of (i) 1 m, (ii) 50 cm. Calculate the distance of the centre of gravity in metre from the base in each case if the cylinder has no top.
(Metal weighs 20 kg m^{-2} of surface. Assume $\pi = 22/7$.)

15. A trap-door 120 cm by 120 cm is kept horizontal by a string attached to the mid-point of the side opposite to that containing the hinge. The other end of the string is tied to a point 90 cm vertically above the hinge. If the trap-door weight is 5 kgf, calculate the tension in the string and the reaction at the hinge.

16. Two smooth inclined planes are arranged with their lower edges in contact; the angles of inclination of the plane to the horizontal are 30° , 60° respectively, and the surfaces of the planes are perpendicular to each other. If a uniform rod rests in the principal section of the planes with one end on each plane, find the angle of inclination of the rod to the horizontal.

17. Describe and give the theory of an accurate beam balance. Point out the factors which influence the sensitivity of the balance. Why is it necessary, in very accurate weighing, to take into account the pressure, temperature, and humidity of the atmosphere? (O. & C.)

18. Summarise the various conditions which are being satisfied when a body remains in equilibrium under the action of three non-parallel forces.

A wireless aerial attached to the top of a mast 20 m high exerts a horizontal force upon it of 60 kgf. The mast is supported by a stay-wire running to the ground from a point 6 m below the top of the mast, and inclined at 60° to the horizontal. Assuming that the action of the ground on the mast can be regarded as a single force, draw a diagram of the forces acting on the mast, and determine by measurement or by calculation the force in the stay-wire. (C.)

19. The beam of a balance has mass 150 g and its moment of inertia is 5×10^{-4} kg m². Each arm of the balance is 10 cm long. When set swinging the beam makes one complete oscillation in 6 seconds. How far is the centre of gravity of the beam below its point of support, and through what angle would the beam be deflected by a weight of 1 milligram placed in one of the scale pans? (C.)

20. Under what conditions is a body said to be in equilibrium? What is meant by (a) *stable equilibrium* and (b) *unstable equilibrium*? Give one example of each.

A pair of railway carriage wheels, each of radius r , are joined by a thin axle; the mass of the whole is m . A light arm of length l ($< r$) is attached perpendicularly to the axle and the free end of the arm carries a point mass M . The wheels rest, with the axle horizontal, on rails which are laid down a slope inclined at an angle ϕ to the horizontal. Show that, provided that ϕ is not too large and that the wheels do not slip on the rails, there are two values of the angle θ that the arm makes with the horizontal when the system is in equilibrium, and find these values of θ . Discuss whether, in each case, the equilibrium is stable or unstable. (O. & C.)

21. Give a labelled diagram to show the structure of a beam balance. Show if the knife-edges are collinear the sensitivity is independent of the load. Discuss other factors which then determine the sensitivity.

A body is weighed at a place on the equator, both with a beam balance and a very sensitive spring balance, with identical results. If the observations are

repeated at a place near one of the poles, using the same two instruments, discuss whether identical results will again be obtained. (L.)

22. Three forces in one plane act on a rigid body. What are the conditions for equilibrium?

The plane of a kite of mass 6 kg is inclined to the horizon at 60° . The resultant thrust of the air on the kite acts at a point 25 cm above its centre of gravity, and the string is attached at a point 30 cm above the centre of gravity. Find the thrust of the air on the kite, and the tension in the string. (C.)

23. In what circumstances is a physical system in equilibrium? Distinguish between stable, unstable and neutral equilibria.

Discuss the stability of the equilibrium of a uniform rough plank of thickness t , balanced horizontally on a rough cylindrical-fixed log of radius r , it being assumed that the axes of plank and log lie in perpendicular directions. (N.)

24. State the conditions of equilibrium for a body subjected to a system of coplanar parallel forces and briefly describe an experiment which you could carry out to verify these conditions.

Show how the equilibrium of a beam balance is achieved and discuss the factors which determine its sensitivity. Explain how the sensitivity of a given balance may be altered and why, for a particular adjustment, the sensitivity may be practically independent of the mass in the balance pans. Why is it inconvenient in practice to attempt to increase the sensitivity of a given balance beyond a certain limit? (O. & C.)

Fluids

25. An alloy of mass 588 g and volume 100 cm^3 is made of iron of relative density 8·0 and aluminium of relative density 2·7. Calculate the proportion (i) by volume, (ii) by mass of the constituents of the alloy.

26. A string supports a solid iron object of mass 180 g totally immersed in a liquid of density 800 kg m^{-3} . Calculate the tension in the string if the density of iron is 8000 kg m^{-3} .

27. A hydrometer floats in water with 6·0 cm of its graduated stem unimmersed, and in oil of relative density 0·8 with 4·0 cm of the stem unimmersed. What is the length of stem unimmersed when the hydrometer is placed in a liquid of relative density 0·9?

28. An alloy of mass 170 g has an apparent weight of 95 gf in a liquid of density $1\cdot5 \text{ g cm}^{-3}$. If the two constituents of the alloy have relative densities of 4·0 and 3·0 respectively, calculate the proportion by volume of the constituents in the alloy.

29. State the principle of Archimedes and use it to derive an expression for the resultant force experienced by a body of weight W and density σ when it is totally immersed in a fluid of density ρ .

A solid weighs 237·5 g in air and 12·5 g when totally immersed in a liquid of relative density 0·9. Calculate (a) the specific gravity of the solid, (b) the relative density of a liquid in which the solid would float with one-fifth of its volume exposed above the liquid surface. (L.)

30. Distinguish between *mass* and *weight*. Define *density*.

Describe and explain how you would proceed to find an accurate value for the density of gold, the specimen available being a wedding ring of pure gold.

What will be the reading of (a) a mercury barometer, (b) a water barometer, when the atmospheric pressure is 10^5 N m^{-2} ? The density of mercury may be taken as 13600 kg m^{-3} and the pressure of saturated water vapour at room temperature as 13 mm of mercury. (L.)

31. Describe an experiment which demonstrates the difference between laminar and turbulent flow in a fluid.

A straight pipe of uniform radius R is joined, in the same straight line, to a narrower pipe of uniform radius r . Water (which may be assumed to be incompressible) flows from the wider into the narrower pipe. The velocity of flow in the wider pipe is V and in the narrower pipe is v . By equating work done against fluid pressures with change of kinetic energy of the water, show that the hydrostatic pressure is lower where the velocity of flow is higher.

Describe and explain one practical consequence or application of this difference in pressures. (O. & C.)

32. Describe some form of barometer used for the accurate measurement of atmospheric pressure, and point out the corrections to be applied to the observation.

Obtain an expression for the correction to be applied to the reading of a mercurial barometer when the reading is made at a temperature other than 0°C . (L.)

33. State the principle of Archimedes, and discuss its application to the determination of specific gravities by means of a common hydrometer. Why is this method essentially less accurate than the specific gravity bottle?

A common hydrometer is graduated to read specific gravities from 0.8 to 1.0. In order to extend its range a small weight is attached to the stem, above the liquid, so that the instrument reads 0.8 when floating in water. What will be the specific gravity of the liquid corresponding to the graduation 1.0? (O. & C.)

34. A hydrometer consists of a bulb of volume V and a uniform stem of volume v per cm of its length. It floats upright in water so that the bulb is just completely immersed. Explain for what density range this hydrometer may be used and how you would determine the density of such liquids. Describe the graph which would be obtained by plotting the reciprocal of the density against the length of the stem immersed.

A hydrometer such as that described sinks to the mark 3 on the stem, which is graduated in cm, when it is placed in a liquid of density 0.95 g cm^{-3} . If the volume per cm of the stem is 0.1 cm^3 , find the volume of the bulb. (L.)

35. A straight rod of length l , small cross sectional area a and of material density ρ is supported by a thread attached to its upper end. Initially the rod hangs in a vertical position over a liquid of density σ and then is lowered until it is partially submerged. Derive and discuss the equilibrium conditions of the rod neglecting surface tension. (N.)

chapter five

Surface Tension

Intermolecular Forces

THE forces which exist between molecules can explain many of the bulk properties of solids, liquids and gases. These intermolecular forces arise from two main causes:

(1) The *potential energy* of the molecules, which is due to interactions with surrounding molecules (this is principally electrical, not gravitational, in origin).

(2) The *thermal energy* of the molecules—this is the kinetic energy of the molecules and depends on the temperature of the substance concerned.

We shall see later that the particular state or phase in which matter appears—that is, solid, liquid or gas—and the properties it then has, are determined by the relative magnitudes of these two energies.

Potential energy and Force

In bulk, matter consists of numerous molecules. To simplify the situation, Fig. 5.1 shows the variation of the potential energy V between two molecules at a distance r apart.

Along the part BCD of the curve, the potential energy V is negative. Along the part AB, the potential energy V is positive. The force between the molecules is always given by $F = -dV/dr$ = —potential gradient. Along CD the force is *attractive* and it decreases with distance r according to an inverse-power of r . Along ABC, the force is *repulsive*. Fig. 5.1 shows the variation of F with r .

At C, the minimum potential energy point of the curve, the molecules would be at their normal distance apart in the absence of thermal energy. The equilibrium distance OM, r_0 , is of the order 2 or 3×10^{-10} m (2 or 3 Å) for a solid. At this distance apart, the attractive and repulsive forces balance each other. If the molecules are closer, ($r < r_0$), they would repel each other. If they are further apart, ($r > r_0$), they attract each other.

Phases or States of Matter

The molecules in a *solid* are said to be in a ‘condensed’ phase or state. Their thermal energy is then relatively low compared with their potential energy V and the molecules are ‘bound’ to each other. They may now vibrate about C, the minimum of the curve in Fig. 5.1.

When the thermal energy increases by an amount corresponding to CC' in Fig. 5.1, the molecule can then oscillate between the limits

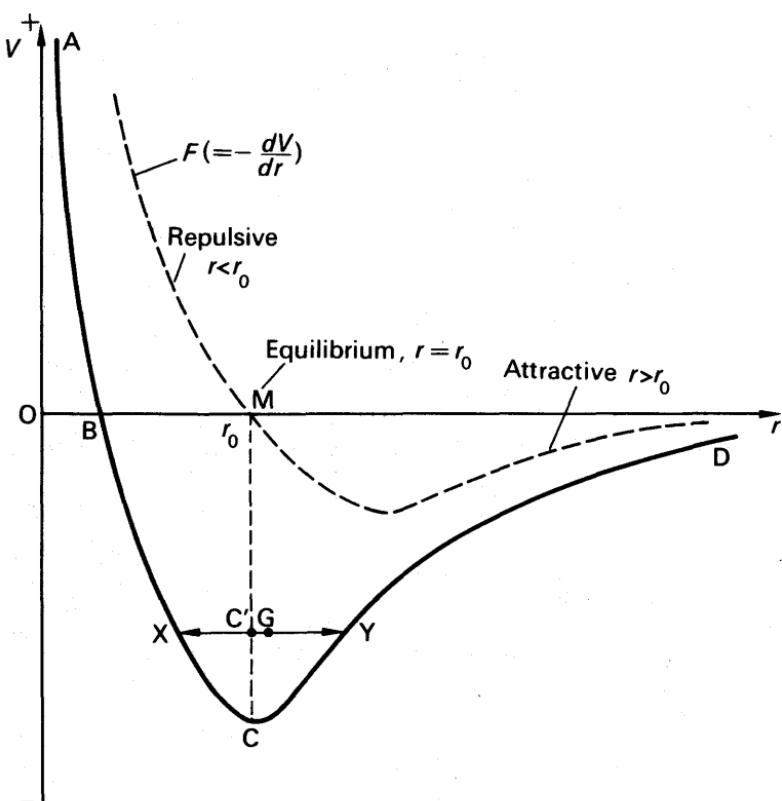


FIG. 5.1 Molecular potential energy and force

corresponding to X and Y. From the graph of force, F , it can be seen that when the molecule is on the left of C' it experiences a greater force towards it than when on the right. Consequently the molecule returns quicker to C'. Thus the *mean position* G is on the right of C'. This corresponds to a mean separation of molecules which is *greater* than r_0 . Thus the solid expands when its thermal energy is increased.

As the thermal energy increases further, at some particular temperature the molecules are able to move comparatively freely relative to neighbouring molecules. The solid then loses its rigid form and becomes a *liquid*. The molecules in the liquid constantly exchange places with other molecules, whereas in a solid the neighbours of a particular molecule remain unchanged. Further, the molecules of a liquid have translational as well as vibrational energy, that is, they move about constantly through the liquid, whereas molecules of a solid have vibrational energy only.

As the temperature of the liquid rises, the thermal energy of the molecules further increases. The average distance between the molecules then also increases and so their mean potential energy approaches zero, as can be seen from Fig. 5.1. At some stage the increased thermal

energy enables the molecules to completely break the bonds of attraction which keep them in a liquid state. The molecules then have little or no interaction and now form a *gas*. At normal pressures the forces of attraction between the gas molecules are comparatively very small and the molecules move about freely inside the volume they occupy. Gas molecules which are monatomic such as helium have translational energy only. Gas molecules such as oxygen or carbon dioxide, with two or more atoms, have rotational and vibrational energies in addition to translational energy.

Gases

At normal pressure, permanent gases such as air or oxygen obey Boyle's law, $pV = \text{constant}$, to a very good approximation. Now in the absence of attractive forces between the molecules, and assuming their actual volume is negligibly small, the kinetic theory of gases shows that Boyle's law is obeyed by this ideal gas. Consequently, the attractive forces between the gas molecules at normal pressure are unimportant. They increase appreciably when the gas is at high pressure as the molecules are then on the average very much closer.

In the bulk of the gas, the resultant force of attraction between a particular molecule and those all round it is zero when averaged over a period. Molecules which strike the wall of the containing vessel, however, are retarded by an unbalanced force due to molecules behind them. The observed pressure p of a gas is thus *less* than the pressure in the ideal case, when the attractive forces due to molecules is zero.

Van der Waals derived an expression for this pressure 'defect'. He considered that it was proportional to the product of the number of molecules per second striking unit area of the wall and the number per unit volume behind them, since this is a measure of the force of attraction. For a given volume of gas, both these numbers are proportional to the density of the gas. Consequently the pressure defect, p_1 say, is proportional to $\rho \times \rho$ or ρ^2 . For a fixed mass of gas, $\rho \propto 1/V$, where V is the volume. Thus $p_1 = a/V^2$, where a is a constant for the particular gas. Taking into account the attractive forces between the molecules, it follows that, if p is the observed pressure, the gas pressure in the bulk of the gas = $p + a/V^2$.

The attraction of the walls on the molecules arriving there is to increase their velocity from v say to $v + \Delta v$. Immediately after rebounding from the walls, however, the force of attraction decreases the velocity to v again. Thus the attraction of the walls has no net effect on the momentum change due to collision. Likewise, the increase in momentum of the walls due to their attraction by the molecules arriving is lost after the molecules rebound.

The effect of the volume actually occupied by all the molecules is represented by a constant b , so that the volume of the space in which they move is not V but $(V - b)$. The magnitude of b is not the actual volume of the molecules, as if they were swept into one corner of the space, since they are in constant motion. b has been estimated to be about four times the actual volume.

Surface Tension.

We now consider in detail a phenomenon of a liquid surface called *surface tension*. As we shall soon show, surface tension is due to intermolecular attraction.

It is a well-known fact that some insects, for example a water-carrier, are able to walk across a water surface; that a drop of water may remain suspended for some time from a tap before falling, as if the water particles were held together in a bag; that mercury gathers into small droplets when spilt; and that a dry steel needle may be made, with

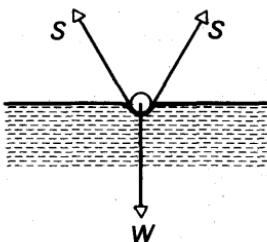


FIG. 5.2 Needle floating on water

care, to float on water, Fig. 5.2. These observations suggest that *the surface of a liquid acts like an elastic skin covering the liquid or is in a state of tension*. Thus forces S in the liquid support the weight W of the needle, as shown in Fig. 5.2.

Energy of Liquid Surface. Molecular theory

The fact that a liquid surface is in a state of tension can be explained by the intermolecular forces discussed on p. 125. In the bulk of the liquid, which begins only a few molecular diameters downwards from the surface, a particular molecule such as A is surrounded by an equal number of molecules on all sides. This can be seen by drawing a sphere round A. Fig. 5.3. The average distance apart of the molecules is such that the attractive forces balance the repulsive forces (p. 145). Thus the average intermolecular force between A and the surrounding molecules is zero. Fig. 5.3.

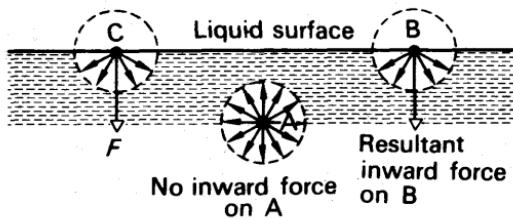


FIG. 5.3 Molecular forces in liquid

Consider now a molecule such as C or B in the surface of the liquid. There are very few molecules on the vapour side above C or B compared with the liquid below, as shown by drawing a sphere round C or B. Thus if C is displaced very slightly upward, a resultant attractive force F on C, due to the large number of molecules below C, now has to be overcome. It follows that if all the molecules in the surface were removed to infinity, a definite amount of work is needed. *Consequently molecules in the surface have potential energy.* A molecule in the bulk of the liquid forms bonds with more neighbours than one in the surface. Thus bonds must be broken, i.e. work must be done, to bring a molecule into the surface. Molecules in the surface of the liquid hence have more potential energy than those in the bulk.

Surface area. Shape of drop

The potential energy of any system in stable equilibrium is a minimum. Thus under surface tension forces, the area of a liquid surface will have the least number of molecules in it, that is, the surface area of a given volume of liquid is a minimum. Mathematically, it can be shown that the shape of a given volume of liquid with a minimum surface area is a *sphere*.

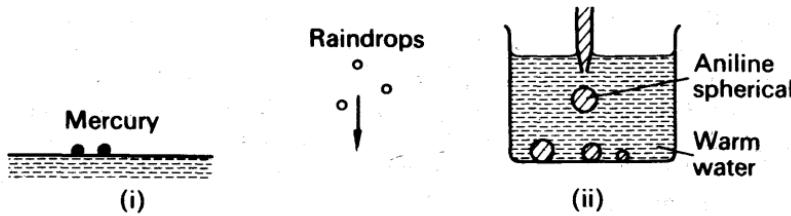


FIG. 5.4 Liquid drops

This is why raindrops, and small droplets of mercury, are approximately spherical in shape. Fig. 5.4 (i). To eliminate completely the effect of gravitational forces, Plateau placed a drop of oil in a mixture of alcohol and water of the same density. In this case the weight of the drop is counterbalanced by the upthrust of the surrounding liquid. He then observed that the drop was a perfect sphere. Plateau's 'spherule' experiment can be carried out by warming water in a beaker and then carefully introducing aniline with the aid of a pipette. Fig. 5.4 (ii). At room temperature the density of aniline is slightly greater than water. At a higher temperature the densities of the two liquids are roughly the same and the aniline is then seen to form *spheres*, which rise and fall in the liquid.

A soap bubble is spherical because its weight is extremely small and the liquid shape is then mainly due to surface tension forces. Although the density of mercury is high, small drops of mercury are spherical. The ratio of surface area ($4\pi r^2$) to weight (or volume, $4\pi r^3/3$) of a sphere is proportional to the ratio r^2/r^3 , or to $1/r$. Thus the smaller

the radius, the greater is the influence of surface tension forces compared to the weight. Large mercury drops, however, are flattened on top. This time the effect of gravity is relatively greater. The shape of the drop conforms to the principle that the sum of the gravitational potential energy and the surface energy must be a minimum, and so the centre of gravity moves down as much as possible.

Lead shot is manufactured by spraying lead from the top of a tall tower. As they fall, the small drops form spheres under the action of surface tension forces.

Surface tension definition. Units, dimensions

Since the surface of a liquid acts like an elastic skin, the surface is in a state of tension. A blown-up football bladder has a surface in a state of tension. This is a very rough analogy because the surface tension of a bladder increases as the surface area increases, whereas the surface tension of a liquid is independent of surface area. Any line in the bladder surface is then acted on by two equal and opposite forces, and if the bladder is cut with a knife the rubber is drawn away from the incision by the two forces present.

R. C. Brown and others have pointed out that molecules in the surface of a liquid have probably a less dense packing than those in the bulk of the liquid, as there are fewer molecules in the surface when its area is a minimum. The average separation between molecules in the surface are then slightly greater than those inside. On average, then, the force between neighbouring molecules in the surface are attractive (see p. 125). This would explain the existence of surface tension.

The *surface tension*, γ , of a liquid, sometimes called the *coefficient of surface tension*, is defined as the *force per unit length acting in the surface at right angles to one side of a line drawn in the surface*. In Fig. 5.5 AB represents a line 1 m long. The unit of γ is *newton metre⁻¹* ($N\ m^{-1}$).

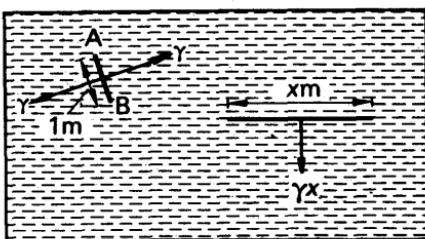


FIG. 5.5 Surface tension

The 'magnitude' of γ depends on the temperature of the liquid and on the medium on the other side of the surface. For water at 20°C in contact with air, $\gamma = 7.26 \times 10^{-2}$ newton metre⁻¹. For mercury at 20°C in contact with air, $\gamma = 46.5 \times 10^{-2}$ N m⁻¹. The surface tension of a water-oil (olive-oil) boundary is 2.06×10^{-2} N m⁻¹, and for a mercury-water boundary it is 42.7×10^{-2} N m⁻¹.

Since surface tension γ is a 'force per unit length', the dimensions of

$$\text{surface tension} = \frac{\text{dimensions of force}}{\text{dimensions of length}} = \frac{MLT^{-2}}{L}$$

$$= MT^{-2}.$$

We shall see later that surface tension can be defined also in terms of surface energy (p. 146).

Some surface tension phenomena

The effect of surface tension forces in a soap film can be demonstrated by placing a thread B carefully on a soap film formed in a metal ring A, Fig. 5.6 (i). The surface tension forces on both sides of the thread counterbalance, as shown in Fig. 5.6 (i). If the film enclosed by the thread is pierced, however, the thread is pulled out into a circle by the surface tension forces F at the junction of the air and soap-film, Fig. 5.6 (ii). Observe that the film has contracted to a minimum area.

Another demonstration of surface tension forces can be made by sprinkling light dust or lycopodium powder over the surface of water contained in a dish. If the middle of the water is touched with the end of a glass rod which had previously been dipped into soap solution, the powder is carried away to the sides by the water. The explanation lies in the fact that the surface tension of water is greater than that of a

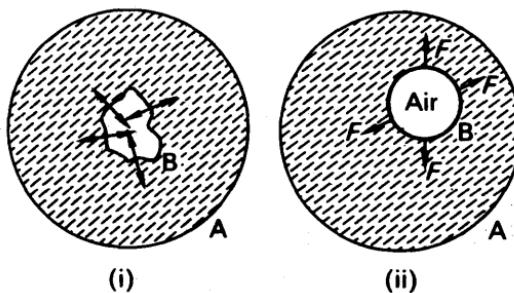


FIG. 5.6 Contraction of surface

soap-film (p. 136). The resultant force at the place where the rod touched the water is hence *away from* the rod, and thus the powder moves away from the centre towards the sides of the vessel.

A toy duck moves by itself across the surface of water when it has a small bag of camphor attached to its base. The camphor lowers the surface tension of the water in contact with it, and the duck is urged across the water by the resultant force on it.

Capillarity

When a capillary tube is immersed in water, and then placed vertically with one end in the liquid, observation shows that the water rises in the tube to a height above the surface. The narrower the tube,

the greater is the height to which the water rises, Fig. 5.7 (i). See also p. 140). This phenomenon is known as *capillarity*, and it occurs when blotting-paper is used to dry ink. The liquid rises up the pores of the paper when it is pressed on the ink.

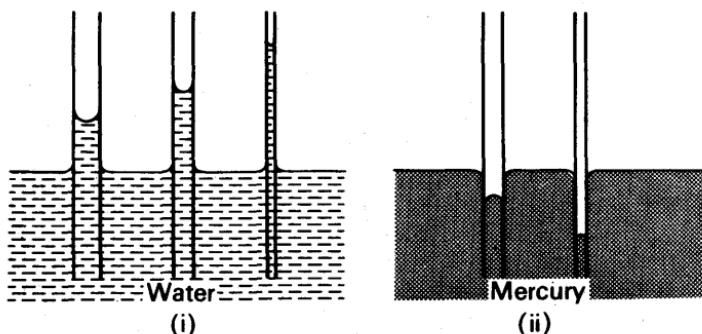


FIG. 5.7 Capillary rise and fall

When a capillary tube is placed inside mercury, however, the liquid is depressed *below* the outside level, Fig. 5.7 (ii). The depression increases as the diameter of the capillary tube decreases. See also p. 141.

Angle of Contact

In the case of water in a glass capillary tube, observation of the meniscus shows that it is hemispherical if the glass is clean, that is, the glass surface is tangential to the meniscus where the water touches it. In other cases where liquids rise in a capillary tube, the tangent BN to the liquid surface where it touches the glass may make an acute angle θ with the glass, Fig. 5.8 (i). The angle θ is known as the *angle of contact* between the liquid and the glass, and is always measured *through the liquid*. The angle of contact between two given surfaces varies largely with their freshness and cleanliness. The angle of contact between water and very clean glass is zero, but when the glass is not clean the angle of contact may be about 8° for example. The angle of contact between alcohol and very clean glass is zero.

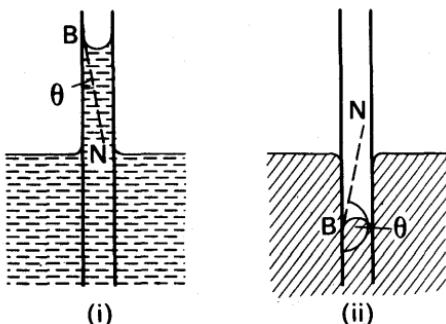


FIG. 5.8 Angle of contact

When a capillary tube is placed inside mercury, observation shows that the surface of the liquid is depressed in the tube and is convex upwards. Fig. 5.8 (ii). The tangent BN to the mercury at the point B where the liquid touches the glass thus makes an obtuse angle, θ , with the glass when measured through the liquid. We shall see later (p. 160) that a liquid will rise in a capillary tube if the angle of contact is acute, and that a liquid will be depressed in the tube if the angle of contact is obtuse. For the same reason, clean water spreads over, or 'wets', a clean glass surface when spilt on it, Fig. 5.9 (i); the angle of contact is zero. On the other hand, mercury gathers itself into small pools or globules when spilt on glass, and does not 'wet' glass, Fig. 5.9 (ii). The angle of contact is obtuse.



FIG. 5.9 Water and mercury on glass

The difference in behaviour of water and mercury on clean glass can be explained in terms of the attraction between the molecules of these substances. It appears that the force of *cohesion* between two molecules of water is less than the force of *adhesion* between a molecule of water and a molecule of glass; and thus water spreads over glass. On the other hand, the force of cohesion between two molecules of mercury is greater than the force of adhesion between a molecule of mercury and a molecule of glass; and thus mercury gathers in pools when spilt on glass.

Angle of Contact measurement

The angle of contact can be found by means of the method outlined in Fig. 5.10 (i), (ii).

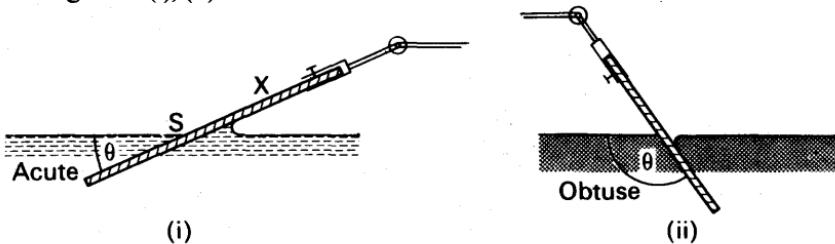


FIG. 5.10 Angle of contact measurement

A plate X of the solid is placed at varying angles to liquid until the surface S appears to be *plane* at X. The angle θ made with the liquid surface is then the angle of contact. For an obtuse angle of contact, a similar method can be adopted. In the case of mercury and glass, a thin plane mirror enables the liquid surface to be seen by reflection. For a freshly-formed mercury drop in contact with a clean glass plate, the angle of contact is 137° .

Measurement of Surface Tension by Capillary Tube Method

Theory. Suppose γ is the magnitude of the surface tension of a liquid such as water, which rises up a clean glass capillary tube and has an angle of contact zero. Fig. 5.11 shows a section of the meniscus M at B, which is a hemisphere. Since the glass AB is a tangent to the liquid, the surface tension forces, which act along the boundary of the liquid with the air, act vertically downwards on the glass. By the law of action and reaction, the glass exerts an equal force in an upward direction on the liquid. Now surface tension, γ , is the force per unit length acting in the surface of the liquid, and the length of liquid in contact with the glass is $2\pi r$, where r is the radius of the capillary tube.

$$\therefore 2\pi r \times \gamma = \text{upward force on liquid} \quad (1)$$

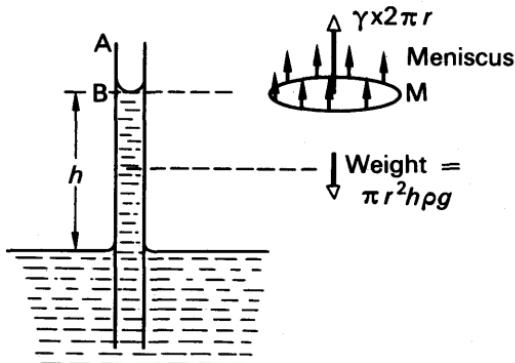


FIG. 5.11 Rise in capillary tube—theory

If γ is in newton metre⁻¹ and r is in metres, then the upward force is in newtons.

This force supports the weight of a column of height h above the outside level of liquid. The volume of the liquid = $\pi r^2 h$, and thus the mass, m , of the liquid column = volume × density = $\pi r^2 h \rho$, where ρ is the density. The weight of the liquid = mg = $\pi r^2 h \rho g$.

If ρ is in kg m⁻³, r and h in metres, and $g = 9.8 \text{ m s}^{-2}$, then $\pi r^2 h \rho g$ is in newtons.

From (1), it now follows that

$$\therefore 2\pi r \gamma = \pi r^2 h \rho g$$

$$\therefore \gamma = \frac{r h \rho g}{2} \quad (2)$$

If $r = 0.2 \text{ mm} = 0.2 \times 10^{-3} \text{ m}$, $h = 6.6 \text{ cm}$ for water = $6.6 \times 10^{-2} \text{ m}$, and $\rho = 1 \text{ g cm}^{-3} = 1000 \text{ kg m}^{-3}$, then

$$\gamma = \frac{0.2 \times 10^{-3} \times 6.6 \times 10^{-2} \times 1000 \times 9.8}{2} = 6.5 \times 10^{-2} \text{ N m}^{-1}$$

In deriving this formula for γ it should be noted that we have (i) assumed the glass to be a tangent to the liquid surface meeting it, (ii) neglected the weight of the small amount of liquid above the bottom of the meniscus at B, Fig. 5.11.

Experiment. In the experiment, the capillary tube C is supported in a beaker Y, and a pin P, bent at right angles at two places, is attached to C by a rubber band, Fig. 5.12. P is adjusted until its point just touches the horizontal level of the liquid in the beaker. A travelling microscope is now focussed on to the meniscus M in C, and then it is focussed on to the point of P, the beaker being removed for this observation. In this way the height h of M above the level in the beaker is determined. The

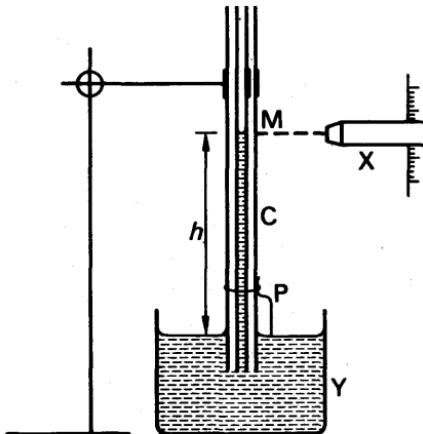


FIG. 5.12 Surface tension by capillary rise

radius of the capillary at M can be found by cutting the tube at this place and measuring the diameter by the travelling microscope; or by measuring the length, l , and mass, m , of a mercury thread drawn into the tube, and calculating the radius, r , from the relation $r = \sqrt{m/\pi l \rho}$, where ρ is the density of mercury. The surface tension γ is then calculated from the formula $\gamma = rhp\gamma/2$. Its magnitude for water at 15°C is 7.33×10^{-2} newton metre $^{-1}$.

Measurement of Surface Tension by Microscope Slide

Besides the capillary tube method, the surface tension of water can be measured by weighing a microscope slide in air, and then lowering it until it just meets the surface of water, Fig. 5.13. The surface tension force acts vertically downward round the boundary of the slide, and pulls the slide down. If a and b are the length and thickness of the slide,

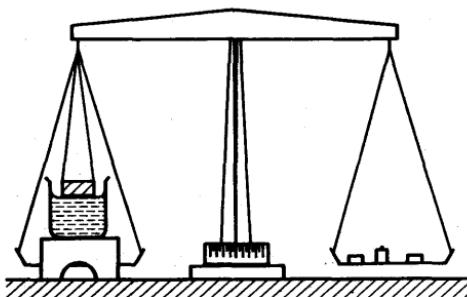


FIG. 5.13 Surface tension by microscope slide

then, since γ is the force per unit length in the liquid surface and $(2a + 2b)$ is the length of the boundary of the slide, the downward force = $\gamma(2a + 2b)$. If the mass required to counterbalance the force is m , then

$$\gamma(2a + 2b) = mg,$$

$$\therefore \gamma = \frac{mg}{2a + 2b}.$$

If $m = 0.88$ gramme, $a = 6.0$ cm, $b = 0.2$ cm, then:

$$\gamma = \frac{0.88 \times 10^{-3} \text{ (kg)} \times 9.8 \text{ (m s}^{-2})}{2 \times (6 + 0.2) \times 10^{-2} \text{ (m)}} = 7.0 \times 10^{-2} \text{ N m}^{-1}.$$

Surface Tension of a Soap Solution

The surface tension of a soap solution can be found by a similar method. A soap-film is formed in a three-sided metal frame ABCD, and the apparent weight is found, Fig. 5.14. When the film is broken by piercing it, the decrease in the apparent weight, mg , is equal to the

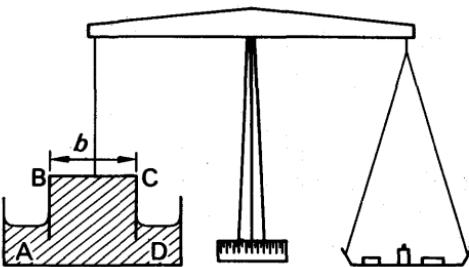


FIG. 5.14 Surface tension of soap film

surface tension force acting downwards when the film existed. This is equal to $2\gamma b$, where $b = BC$, since the film has two sides.

$$\therefore 2\gamma b = mg,$$

$$\therefore \gamma = \frac{mg}{2b}.$$

It will be noted that the surface tension forces on the sides AB, CD of the frame act horizontally, and their resultant is zero.

A soap film can be supported in a vertical rectangular frame but a film of water can not. This is due to the fact that the soap drains downward in a vertical film, so that the top of the film has a lower concentration of soap than the bottom. The surface tension at the top is thus greater than at the bottom (soap diminishes the surface tension of pure water). The upward pull on the film by the top bar is hence greater than the downward pull on the film by the lower bar. The net upward pull supports the weight of the film. In the case of pure water, however, the surface tension would be the same at the top and bottom, and hence there is no net force in this case to support a water film in a rectangular frame.

Pressure Difference in a Bubble or Curved Liquid Surface

As we shall see presently, the magnitude of the curvature of a liquid, or of a bubble formed in a liquid, is related to the surface tension of the liquid.

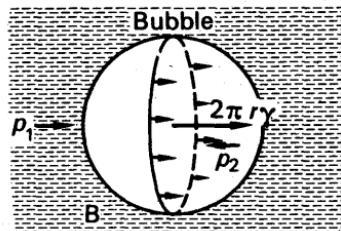


FIG. 5.15 Excess pressure in bubble

Consider a bubble formed inside a liquid, Fig. 5.15. If we consider the equilibrium of *one half*, B, of the bubble, we can see that the surface tension force on B plus the force on B due to the external pressure p_1 = the force on B due to the internal pressure p_2 inside the bubble. The force on B due to the pressure p_1 is given by $\pi r^2 \times p_1$, since πr^2 is the area of the circular face of B and pressure is 'force per unit area'; the force on B due to the pressure p_2

is given similarly by $\pi r^2 \times p_2$. The surface tension force acts round the *circumference* of the bubble, which has a length $2\pi r$; thus the force is $2\pi r\gamma$. It follows that

$$2\pi r\gamma + \pi r^2 p_1 = \pi r^2 p_2.$$

Simplifying,

$$\therefore 2\gamma = r(p_2 - p_1),$$

or

$$p_2 - p_1 = \frac{2\gamma}{r}.$$

Now $(p_2 - p_1)$ is the excess pressure, p , in the bubble over the outside pressure.

$$\therefore \text{excess pressure, } p, = \frac{2\gamma}{r} \quad (1)$$

Although we considered a bubble, the same formula for the excess pressure holds for any curved liquid surface or meniscus, where r is its radius of curvature and γ is its surface tension, provided the angle of contact is zero. If the angle of contact is θ , the formula is modified by replacing γ by $\gamma \cos \theta$. Thus, in general,

$$\text{excess pressure, } p, = \frac{2\gamma \cos \theta}{r} \quad (2)$$

Excess Pressure in Soap Bubble

A soap bubble has two liquid surfaces in contact with air, one inside the bubble and the other outside the bubble. The force on one half, B, of the bubble due to surface tension forces is thus $\gamma \times 2\pi r \times 2$, i.e., $\gamma \times 4\pi r$, Fig. 5.16. For the equilibrium of B, it follows that

$$4\pi r\gamma + \pi r^2 p_1 = \pi r^2 p_2,$$

where p_2, p_1 are the pressures inside and outside the bubble respectively. Simplifying,

$$\therefore p_2 - p_1 = \frac{4\gamma}{r},$$

$$\therefore \text{excess pressure } p = \frac{4\gamma}{r} \quad (3)$$

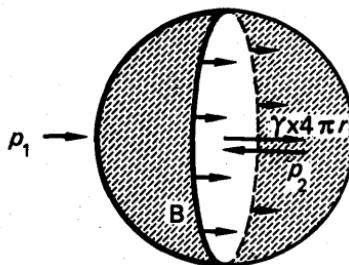


FIG. 5.16 Bubble in water

This result for excess pressure should be compared with the result obtained for a bubble formed inside a liquid, equation (1).

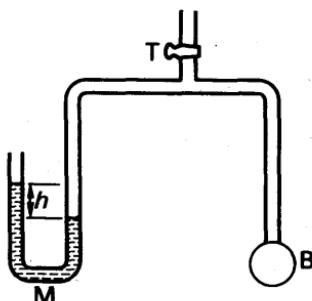
If γ for a soap solution is $25 \times 10^{-3} \text{ N m}^{-1}$, the excess pressure inside a bubble of radius 0.5 cm or $0.5 \times 10^{-2} \text{ m}$ is hence given by:

$$p = \frac{4 \times 25 \times 10^{-3}}{0.5 \times 10^{-2}} = 20 \text{ N m}^{-2}.$$

Two soap-bubbles of unequal size can be blown on the ends of a tube, communication between them being prevented by a closed tap in the middle. If the tap is opened, the *smaller* bubble is observed to collapse gradually and the size of the larger bubble increases. This can be explained from our formula $p = 4\gamma/r$, which shows that the pressure of air inside the smaller bubble is greater than that inside the larger bubble. Consequently air flows from the smaller to the larger bubble when communication is made between the bubbles, and the smaller bubble thus gradually collapses.

Since the excess pressure in a bubble is inversely-proportional to the radius, the pressure needed to form a very small bubble is high. This explains why one needs to blow hard to start a balloon growing. Once the balloon has grown, less air pressure is needed to make it expand more.

Surface Tension of Soap-Bubble

FIG. 5.17
Surface tension of soap-bubble

The surface tension of a soap solution can be measured by blowing a small soap-bubble at the end B of a tube connected to a manometer M, Fig. 5.17. The tap T is then closed, the diameter d of the bubble is measured by a travelling microscope, and the difference in levels h of the liquid in the manometer is observed with the same instrument. The excess pressure, p , in the bubble = $h\rho g$, where ρ is the density of the liquid in M.

$$\therefore h\rho g = \frac{4\gamma}{r} = \frac{4\gamma}{d/2}.$$

$$\therefore \gamma = \frac{h\rho gd}{8}.$$

Rise or Fall of Liquids in Capillary Tubes

From our knowledge of the angle of contact and the excess pressure on one side of a curved liquid surface, we can deduce that some liquids will rise in a capillary tube, whereas others will be depressed.

Suppose the tube A is placed in water, for example, Fig. 5.18 (i). At first the liquid surface becomes concave upwards in the tube, because the angle of contact with the glass is zero. Consequently the pressure on the air side, X, of the curved surface is greater than the pressure on the liquid side Y by $2\gamma/r$, where γ is the surface tension and, r is the radius of curvature of the tube. But the pressure at X is atmospheric, H . Hence the pressure at Y must be less than atmospheric by $2\gamma/r$. Fig. 5.18 (i) is therefore impossible because it shows the pressure at Y equal to the atmospheric pressure. Thus, as shown in Fig. 5.18 (ii), the liquid ascends the tube to a height h such that the pressure at N is less than at M by $2\gamma/r$, Fig. 5.18 (ii). A similar argument shows that a liquid rises in a capillary tube when the angle of contact is acute.

The angle of contact between mercury and glass is obtuse (p. 133). Thus when a capillary tube is placed in mercury the liquid first curves downwards. The pressure inside the liquid just below the curved surface is now greater than the pressure on the other side, which is atmospheric, and the mercury therefore moves down the tube until the excess pressure = $2\gamma \cos \theta/r$, with the usual notation. A liquid thus falls in a capillary tube if the angle of contact is obtuse.

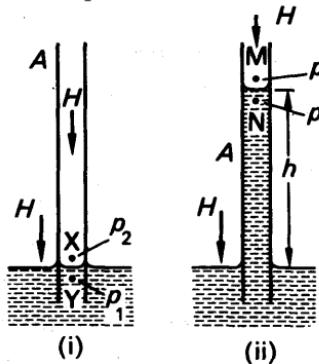


FIG. 5.18
Capillary rise by excess pressure

Capillary Rise and Fall by Pressure Method

We shall now calculate the capillary rise of water by the excess pressure formula $p = 2\gamma/r$, or $p = 2\gamma \cos \theta/r$.

In the case of a capillary tube dipping into water, the angle of contact is practically zero, Fig. 5.19 (i). Thus if p_2 is the pressure of the atmosphere, and p_1 is the pressure in the liquid, we have

$$p_2 - p_1 = \frac{2\gamma}{r}$$

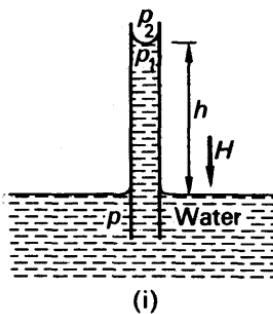
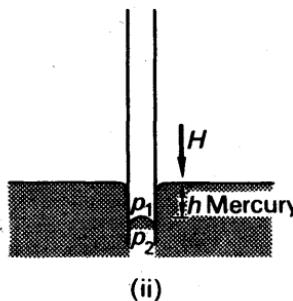


FIG. 5.19
Excess
pressure
application



Now if H is the atmospheric pressure, h is the height of the liquid in the tube and ρ its density,

$$p_2 = H \text{ and } p_1 = H - h\rho g,$$

$$\therefore H - (H - h\rho g) = \frac{2\gamma}{r},$$

$$\therefore h\rho g = \frac{2\gamma}{r},$$

$$\therefore h = \frac{2\gamma}{r\rho g} \quad \quad (\text{i})$$

The formula shows that h increases as r decreases, i.e., the narrower the tube, the greater is the height to which the water rises (see Fig. 5.7 (i), p. 132).

If the height l of the tube above the water is less than the calculated value of h in the above formula, the water surface at the top of the tube now meets it at an *acute angle of contact* θ . The radius of the meniscus is therefore $r/\cos \theta$, and $l\rho g = 2\gamma/(r/\cos \theta)$, or

$$l = \frac{2\gamma \cos \theta}{r\rho g} \quad \quad (\text{ii})$$

Dividing (ii) by (i), it follows that

$$\cos \theta = \frac{l}{h}.$$

Thus suppose water rises to a height of 10 cm in a capillary tube when it is placed in a beaker of water. If the tube is pushed down until the top is only 5 cm above the outside water surface, then $\cos \theta = \frac{5}{10} = 0.5$. Thus $\theta = 60^\circ$. The meniscus now makes an angle of contact of 60° with the glass. As the tube is pushed down further, the angle of contact increases beyond 60° . When the top of the tube is level with the water in the beaker, the meniscus in the tube becomes plane. (See Example 2, p. 141.)

With Mercury in Glass

Suppose that the depression of the mercury inside a tube of radius r is h , Fig. 5.19 (ii). The pressure p_2 below the curved surface of the mercury is then greater than the (atmospheric) pressure p_1 outside the curved surface; and, from our general result,

$$p_2 - p_1 = \frac{2\gamma \cos \theta}{r},$$

where θ is the supplement of the obtuse angle of contact of mercury with glass, that is, θ is an acute angle and its cosine is positive. But $p_1 = H$ and $p_2 = H + h\rho g$, where H is the atmospheric pressure.

$$\therefore (H + h\rho g) - H = \frac{2\gamma \cos \theta}{r}.$$

$$\therefore h \rho g = \frac{2\gamma \cos \theta}{r}$$

$$\therefore h = \frac{2\gamma \cos \theta}{r \rho g} \quad \quad (1)$$

The height of depression, h , thus increases as the radius r of the tube decreases. See Fig. 5.7 (ii), p. 132.

EXAMPLES

1. Define surface tension of a liquid and describe a method of finding this quantity for alcohol.

If water rises in a capillary tube 5.8 cm above the free surface of the outer liquid, what will happen to the mercury level in the same tube when it is placed in a dish of mercury? Illustrate this by the aid of a diagram. Calculate the difference in level between the mercury surfaces inside the tube and outside. (S.T. of water = 75×10^{-3} N m $^{-1}$. S.T. of mercury = 547×10^{-3} N m $^{-1}$. Angle of contact of mercury with clean glass = 130° . Density of mercury = 13600 kg m^{-3} .) (L.)

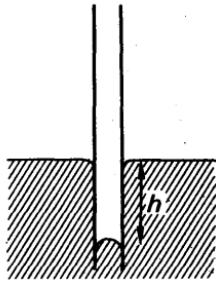


FIG. 5.20 Example

For water, $h = 5.8 \text{ cm} = 5.8 \times 10^{-2} \text{ m}$, $\gamma = 75 \times 10^{-3} \text{ newton m}^{-1}$, $\rho = 1000 \text{ kg m}^{-3}$, $g = 9.8 \text{ m s}^{-2}$.

From $\gamma = rh \rho g / 2$,

$$\therefore 75 \times 10^{-3} = r \times 5.8 \times 10^{-2} \times 1000 \times 9.8 / 2 \quad (\text{r in metre}).$$

For mercury, $\rho = 13.6 \times 10^3 \text{ kg m}^{-3}$, $\gamma = 547 \times 10^{-3} \text{ newton m}^{-1}$.

$$\begin{aligned} \therefore h &= \frac{2\gamma \cos 50^\circ}{r \rho g} \\ &= \frac{2 \times 547 \times 10^{-3} \cos 50^\circ \times 5.8 \times 10^{-2} \times 1000 \times 9.8}{13.6 \times 10^3 \times 9.8 \times 75 \times 10^{-3} \times 2} \\ &= 0.02 \text{ m} = 2 \text{ cm}. \end{aligned}$$

2. On what grounds would you anticipate some connection between the surface tension of a liquid and its latent heat of vaporization?

A vertical capillary tube 10 cm long tapers uniformly from an internal diameter of 1 mm at the lower end to 0.5 mm at the upper end. The lower end is just touching the surface of a pool of liquid of surface tension 6×10^{-2} N m $^{-1}$, density 1200 kg m^{-3} and zero angle of contact with the tube. Calculate the capillary rise, justifying your method. Explain what will happen to the meniscus if the tube is slowly lowered vertically until the upper end is level with the surface of the pool. (O. & C.)

Suppose S is the meniscus at a height h cm above the liquid surface. The tube tapers uniformly and the change in radius for a height of 10 cm is (0.05 – 0.025) or 0.025 cm, so that the change in radius per cm height is 0.0025 cm. Thus at a height h cm, radius of meniscus S is given by

$$r = (0.05 - 0.0025 h) \times 10^{-2} \text{ m}$$

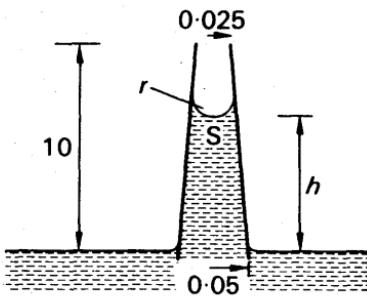


FIG. 5.21 Example

The pressure above S is atmospheric, A. The pressure below S is $(A - h\rho g)$.

$$\therefore \text{pressure difference} = (h \times 10^{-2})\rho g = \frac{2y}{r} = \frac{200y}{0.05 - 0.0025h}$$

$$\therefore 0.05h - 0.0025h^2 = \frac{200y}{\rho g} = \frac{200 \times 6 \times 10^{-2}}{10^{-2} \times 1200 \times 9.8} = 0.102.$$

$$\therefore h^2 - 20h = -40 \text{ (approx.)}$$

$$\therefore (h-10)^2 = 100-40 = 60.$$

$$\therefore h = 10 - \sqrt{60} = 2.2 \text{ cm.}$$

If the tube is slowly lowered the meniscus reaches the top at some stage. On further lowering the tube the angle of contact changes from zero to an acute angle. When the upper end is level with liquid surface the meniscus becomes plane.

3. 'The surface tension of water is 7.5×10^{-2} newton m⁻¹ and the angle of contact of water with glass is zero.' Explain what these statements mean. Describe an experiment to determine either (a) the surface tension of water, or (b) the angle of contact between paraffin wax and water.

A glass U-tube is inverted with the open ends of the straight limbs, of diameters respectively 0.500 mm and 1.00 mm, below the surface of water in a beaker. The air pressure in the upper part is increased until the meniscus in one limb is level with the water outside. Find the height of water in the other limb. (The density of water may be taken as 1000 kg m⁻³.) (L.)

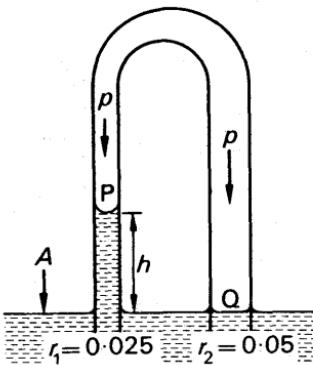


FIG. 5.22 Example (radii in cm)

Suppose p is the air pressure inside the U-table when the meniscus Q is level with the water outside and P is the other meniscus at a height h . Let A be the atmospheric pressure. Then, if r_1 is the radius at P,

$$p - (A - h\rho g) = \frac{2\gamma}{r_1} \quad (i)$$

since the pressure in the liquid below P is $(A - h\rho g)$.

The pressure in the liquid below Q = A . Hence, for Q,

$$p - A = \frac{2\gamma}{r_2} \quad (ii)$$

where r_2 is the radius.

From (i) and (ii), it follows that

$$h\rho g = \frac{2\gamma}{r_1} - \frac{2\gamma}{r_2}$$

$$\therefore h = \frac{1}{\rho g} \left[\frac{2\gamma}{r_1} - \frac{2\gamma}{r_2} \right]$$

$$= \frac{1}{9800} \left[\frac{2 \times 0.075}{0.25 \times 10^{-3}} - \frac{2 \times 0.075}{0.5 \times 10^{-3}} \right]$$

$$= 3.1 \times 10^{-2} \text{ m (approx.)}$$

Effects of surface tension in measurements

When a hydrometer is used to measure relative density or density, the surface tension produces a downward force F on the hydrometer. If r is the radius of the stem and the angle of contact is zero, then $F = 2\pi r\gamma$. For a narrow stem, the error produced in reading the relative density from the graduations is small.

Another case of an undesirable surface tension effect occurs in measurements of the height of liquid columns in glass tubes. The height of mercury in a barometer, for example, is depressed by surface tension (p. 114). If the tubes are wide, surface tension forces can be neglected. If they are narrow, the forces must be taken into account. As an illustration, consider an inverted U-tube dipping into two liquids B and C. Fig. 5.23. These can be drawn up into the tubes to heights

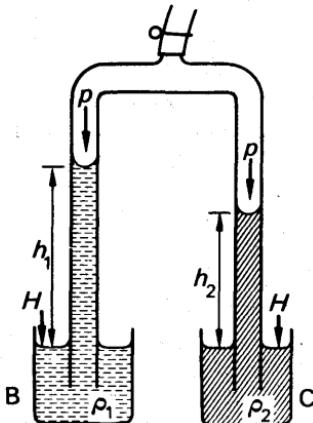


FIG. 5.23 Comparison of densities

h_1, h_2 respectively above the outside level. In the absence of surface tension forces, $p + h_1\rho_1g = \text{atmospheric pressure } H$, where p is the air pressure at the top of the tubes = $p + h_2\rho_2g$. Thus $h_1\rho_1 = h_2\rho_2$, or $h_1/h_2 = \rho_2/\rho_1$. Thus the liquid densities may be compared from the ratio of the heights of the liquid columns.

To take account of surface tension, we proceed as follows. Using the notation on p. 137.

$$p - p_1 = \frac{2\gamma_1}{r_1},$$

where p is the air pressure at the top of the tubes, p_1 is the pressure in the liquid near the meniscus of the tube in B, γ_1 is the surface tension of the liquid, and r_1 is the radius. But, from hydrostatics, $p_1 = H - h_1\rho_1g$.

$$\therefore p - (H - h_1\rho_1g) = \frac{2\gamma_1}{r_1}$$

$$\therefore H - p = h_1\rho_1g - \frac{2\gamma_1}{r_1} \quad \text{(i)}$$

If γ_2 is the surface tension of the liquid in C, and r_2 is the radius of the tube in the liquid, then, by similar reasoning,

$$H - p = h_2\rho_2g - \frac{2\gamma_2}{r_2} \quad \text{(ii)}$$

From (i) and (ii),

$$\therefore h_2\rho_2g - \frac{2\gamma_2}{r_2} = h_1\rho_1g - \frac{2\gamma_1}{r_1}.$$

Re-arranging,

$$\therefore h_2 = \frac{\rho_1 h_1}{\rho_2} - \frac{2}{\rho_2 g} \left(\frac{\gamma_1 - \gamma_2}{r_1 - r_2} \right),$$

which is an equation of the form $y = mx + c$, where c is a constant, $h_2 = y$, $h_1 = x$, and $\rho_1/\rho_2 = m$. Thus by taking different values of h_2 and h_1 , and plotting h_2 against h_1 , a straight-line graph is obtained whose slope is equal to ρ_1/ρ_2 , the ratio of the densities. In this way the effect of the surface tension can be eliminated.

Variation of Surface Tension with Temperature. Jaeger's Method

By forming a bubble inside a liquid, and measuring the excess pressure, JAEGER was able to determine the variation of the surface tension of a liquid with temperature. One form of the apparatus is shown in Fig. 5.24 (i). A capillary or drawn-out tubing A is connected to a vessel W containing a funnel C, so that air is driven slowly through A when water enters W through C, so that air is driven slowly through A when water enters W through C. The capillary A is placed inside a beaker containing the liquid L, and a bubble forms slowly at the end of A when air is passed through it at a slow rate.

Fig. 5.24. (ii) shows the bubble at three possible stages of growth. The radius grows from that at *a* to a hemispherical shape at *b*. Here its pressure is larger since the radius is smaller. If we consider the bubble growing to *c*, the radius of *c* would be greater than that of *b* and hence it cannot contain the increasing pressure. The downward force on the bubble due to the pressure, in fact, would be greater than

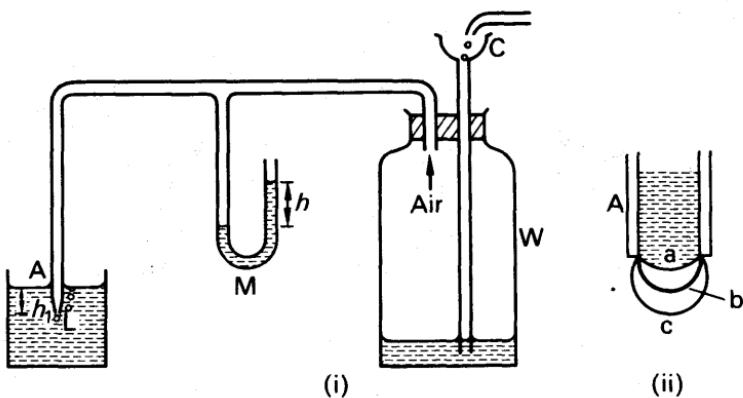


FIG. 5.24 Jaeger's method

the upward force due to surface tension. Hence *the bubble becomes unstable and breaks away from A when its radius is the same as that of A.* Thus as the bubble grows the pressure in it increases to a maximum, and then decreases as the bubble breaks away. The maximum pressure is observed from a manometer M containing a light oil of density ρ , and a series of observations are taken as several bubbles grow.

The maximum pressure inside the bubble = $H + h\rho g$ where h is the maximum difference in levels in the manometer M, and H is the atmospheric pressure. The pressure outside the bubble = $H + h_1\rho_1 g$, where h_1 is the depth of the orifice of A below the level of the liquid L, and ρ_1 is the latter's density.

$$\therefore \text{excess pressure} = (H + h\rho g) - (H + h_1\rho_1 g) = h\rho g - h_1\rho_1 g.$$

$$\text{But} \quad \text{excess pressure} = \frac{2\gamma}{r},$$

where r is the radius of the orifice of A (p. 158).

$$\therefore \frac{2\gamma}{r} = h\rho g - h_1\rho_1 g,$$

$$\therefore \gamma = \frac{rg}{2}(h\rho - h_1\rho_1).$$

By adding warm liquid to the vessel containing L, the variation of the surface tension with temperature can be determined. Experiment shows that the surface tension of liquids, and water in particular, decreases with increasing temperature along a fairly smooth curve. Various formulae relating the surface tension to temperature have been proposed, but none has been found to be completely satisfactory. The decrease of surface tension with temperature may be attributed to the greater average separation of the molecules at higher temperature. The force of attraction between molecules is then reduced, and hence the surface energy is reduced, as can be seen from the potential energy curve on p. 126.

Surface Tension and Surface Energy

We now consider the surface energy of a liquid and its relation to its surface tension γ . Consider a film of liquid stretched across a horizontal frame ABCD, Fig. 5.25. Since γ is the force per unit length, the force on the rod BC of length $l = \gamma \times 2l$, because there are two surfaces to the film.

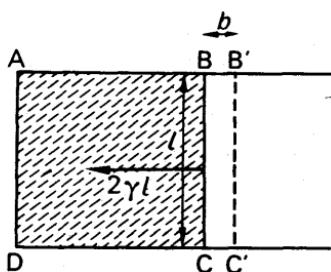


FIG. 5.25
Surface energy and work

Suppose the rod is now moved a distance b from BC to B'C' against the surface tension forces, so that the surface area of the film increases. The temperature of the film then usually decreases, in which case the surface tension alters (p. 145). If the surface area increases under *isothermal* (constant temperature) conditions, however, the surface tension is constant; and we can then say that, if γ is the surface tension at that temperature,

$$\begin{aligned}\text{work done in enlarging surface area} &= \text{force} \times \text{distance}, \\ &= 2\gamma l \times b = \gamma \times 2lb.\end{aligned}$$

But $2lb$ is the total increase in surface area of the film.

$$\therefore \text{work done per unit area in enlarging area} = \gamma.$$

Thus the surface tension, γ , can be defined as *the work done per unit area in increasing the surface area of a liquid under isothermal conditions*. This is also called the *free surface energy*.

Surface energy and Latent heat

Inside a liquid molecules move about in all directions, continually breaking and reforming bonds with neighbours. If a molecule in the surface passes into the vapour outside, a definite amount of energy is needed to permanently break the bonds with molecules in the liquid. This amount of energy is the work done in overcoming the inward force on a molecule in the surface, discussed on p. 129. Thus the energy needed to evaporate a liquid is related to its surface energy or surface tension. The latent heat of vaporisation, which is the energy needed to change liquid to vapour at the boiling point, is therefore related to surface energy.

Surface energy

As we have seen, when the surface area of a liquid is increased, the surface energy is increased. The molecules which then reach the surface are slowed up by the inward force, so the average translational kinetic energy of all the liquid molecules is reduced. On this account the liquid cools while the surface is increased, and heat flows in from the surroundings to restore the temperature.

The increase in the *total surface energy per unit area E* is thus given by

$$E = \gamma + H \quad \quad (1)$$

where H is the heat per unit area from the surroundings. Advanced theory shows that $H = -\theta \left(\frac{dy}{d\theta} \right)$, where θ is the absolute temperature and $dy/d\theta$ is the corresponding gradient of the γ v. θ graph, the variation of surface tension with temperature. Thus

$$E = \gamma - \theta \frac{dy}{d\theta} \quad \quad (2)$$

In practice, since γ decreases with rising temperature, $dy/d\theta$ is negative, and E is thus greater than γ . At 15°C, for example, $\gamma = 74 \times 10^{-3}$ N m⁻¹, $dy/d\theta = -0.15 \times 10^{-3}$ N m⁻¹ K⁻¹, $\theta = 288$ K. Thus, from (2),

$$E = (74 + 288 \times 0.15) \times 10^{-3} = 0.117 \text{ N m}^{-1} = 0.117 \text{ J m}^{-2}.$$

The variation of E with temperature is shown in Fig. 5.26, together with the similar variation of L , the latent heat of vaporisation (see p. 146). Both vanish at the critical temperature, since no liquid exists above the critical temperature whatever the pressure.

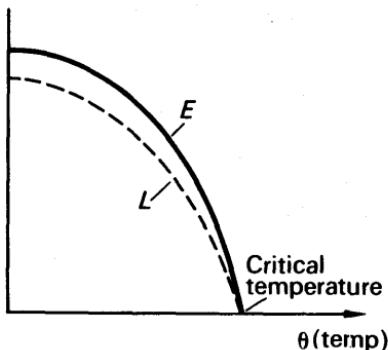


FIG. 5.26 Variation of E and L with temperature

EXAMPLES

1. A soap bubble in a vacuum has a radius of 3 cm and another soap bubble in the vacuum has a radius of 6 cm. If the two bubbles coalesce under isothermal conditions, calculate the radius of the bubble formed.

Since the bubbles coalesce under isothermal conditions, the surface tension γ is constant. Suppose R is the radius in cm, $R \times 10^{-2}$ m, of the bubble formed.

Then $\text{work done} = \gamma \times \text{surface area} = \gamma \times 8\pi R^2 \times 10^{-4}$

But $\text{original work done} = (\gamma \times 8\pi \cdot 3^2 \times \gamma \times 8\pi \cdot 6^2) \times 10^{-4}$

$$\therefore \gamma \times 8\pi R^2 = \gamma \times 8\pi \cdot 3^2 + \gamma \cdot 8\pi \cdot 6^2.$$

$$\therefore R^2 = 3^2 + 6^2.$$

$$\therefore R = \sqrt{3^2 + 6^2} = 6.7 \text{ cm.}$$

2. (i) Calculate the work done against surface tension forces in blowing a soap bubble of 1 cm diameter if the surface tension of soap solution is 2.5×10^{-2} N m⁻¹. (ii) Find the work required to break up a drop of water of radius 0.5 cm into drops of water each of radii 1 mm. (Surface tension of water = 7×10^{-2} N m⁻¹.)

(i) The original surface area of the bubble is zero, and the final surface area = $2 \times 4\pi r^2$ (two surfaces of bubble) = $(2 \times 4\pi \times 0.5^2) \times 10^{-4} = 2\pi \times 10^{-4} \text{ m}^2$.

$$\therefore \text{work done} = \gamma \times \text{increase in surface area.}$$

$$= 2.5 \times 10^{-2} \times 2\pi \times 10^{-4} = 1.57 \times 10^{-5} \text{ J.}$$

(ii) Since volume of a drop = $\frac{4}{3}\pi r^3$,

$$\text{number of drops formed} = \frac{\frac{4}{3}\pi \times 0.5^3}{\frac{4}{3}\pi \times 0.1^3} = 125.$$

\therefore final total surface area of drops

$$= 125 \times 4\pi r^2 = 125 \times 4\pi \times 0.1^2 \times 10^{-4},$$

$$= 5\pi \times 10^{-4} \text{ m}^2.$$

But original surface area of drop = $4\pi \times 0.5^2 \times 10^{-4} = \pi \times 10^{-4} \text{ m}^2$.

$$\therefore \text{work done} = \gamma \times \text{change in surface area,}$$

$$= 7 \times 10^{-2} \times (5\pi - \pi) \times 10^{-4} = 8.8 \times 10^{-5} \text{ J.}$$

EXERCISES 5

What are the missing words in the statements 1–8?

1. The units of surface tension are ...
2. The dimensions of surface tension are ...
3. Small drops of mercury are spherical because the surface area is a ...
4. The excess pressure in a soap-bubble is given by ...
5. The excess pressure at the meniscus of water in a capillary tube is ...
6. A liquid will not 'wet' the surface of a solid if the angle of contact is ...
7. Surface tension may be defined as the 'force ...'
8. Surface tension may also be defined as the '... per unit area'.

Which of the following answers, A, B, C, D or E, do you consider is the correct one in the statements 9–12?

9. A molecule of a liquid which reaches the surface from the interior gains energy because *A* it reaches the surface with higher speed than when inside the liquid, *B* it overcomes a force of repulsion on molecules at the surface, *C* it overcomes a force of attraction on molecules at the surface, *D* its temperature increases, *E* the gravitational potential energy due to the earth is then higher.

10. If a section of a soap bubble through its centre is considered, the force on one half due to surface tension is *A* $2\pi r\gamma$, *B* $4\pi r\gamma$, *C* $\pi r^2\gamma$, *D* $2\gamma/r$, *E* $2\pi r^2\gamma$.

11. If water has a surface tension of $7 \times 10^{-2} \text{ N m}^{-1}$ and an angle of contact with water of zero, it rises in a capillary of diameter 0.5 mm to a height of *A* 70 cm, *B* 7.0 cm, *C* 6.2 cm, *D* 5.7 cm, *E* 0.5 cm.

12. In an experiment to measure the surface tension of a liquid by rise in a capillary tube which tapers, the necessary radius *r* would be best obtained *A* by cutting the tube at the position of the meniscus and measuring the diameter here directly, *B* by drawing up a thread of mercury of length *l* and using 'mass =

$\pi r^2 l \rho'$, C by measuring the diameter of the lower end of the tube with a travelling microscope, D by measuring the upper end of the tube with a travelling microscope, E by finding the average of the two measurements in C and D.

13. Define *surface tension*. A rectangular plate of dimensions 6 cm by 4 cm and thickness 2 mm is placed with its largest face flat on the surface of water. Calculate the force due to surface tension on the plate. What is the downward force due to surface tension if the plate is placed vertical and its longest side just touches the water? (Surface tension of water = $7.0 \times 10^{-2} \text{ N m}^{-1}$.)

14. What are the *dimensions* of surface tension? A capillary tube of 0.4 mm diameter is placed vertically inside (i) water of surface tension $6.5 \times 10^{-2} \text{ N m}^{-1}$ and zero angle of contact, (ii) a liquid of density 800 kg m^{-3} , surface tension $5.0 \times 10^{-2} \text{ N m}^{-1}$ and angle of contact 30° . Calculate the height to which the liquid rises in the capillary in each case.

15. Define the *angle of contact*. What do you know about the angle of contact of a liquid which (i) wets glass, (ii) does not wet glass?

A capillary tube is immersed in water of surface tension $7.0 \times 10^{-2} \text{ N m}^{-1}$ and rises 6.2 cm. By what depth will mercury be depressed if the same capillary is immersed in it? (Surface tension of mercury = 0.54 N m^{-1} ; angle of contact between mercury and glass = 140° ; density of mercury = 13600 kg m^{-3} .)

16. (i) A soap-bubble has a diameter of 4 mm. Calculate the pressure inside it if the atmospheric pressure is 10^5 N m^{-2} . (Surface tension of soap solution = $2.8 \times 10^{-2} \text{ N m}^{-1}$.) (ii) Calculate the radius of a bubble formed in water if the pressure outside it is $1.000 \times 10^5 \text{ N m}^{-2}$ and the pressure inside it is $1.001 \times 10^5 \text{ N m}^{-2}$. (Surface tension of water = $7.0 \times 10^{-2} \text{ N m}^{-1}$.)

17. Define *surface tension* of a liquid. State the units in which it is usually expressed and give its dimensions in mass, length, and time.

Derive an expression for the difference between the pressure inside and outside a spherical soap bubble. Describe a method of determining surface tension, based on the difference of pressure on the two sides of a curved liquid surface or film. (L.)

18. Explain briefly (a) the approximately spherical shape of a rain drop, (b) the movement of tiny particles of camphor on water, (c) the possibility of floating a needle on water, (d) why a column of water will remain in an open vertical capillary tube after the lower end has been dipped in water and withdrawn. (N.)

19. Define the terms surface tension, angle of contact. Describe a method for measuring the surface tension of a liquid which wets glass. List the principal sources of error and state what steps you would take to minimize them.

A glass tube whose inside diameter is 1 mm is dipped vertically into a vessel containing mercury with its lower end 1 cm below the surface. To what height will the mercury rise in the tube if the air pressure inside it is $3 \times 10^3 \text{ N m}^{-2}$ below atmospheric pressure? Describe the effect of allowing the pressure in the tube to increase gradually to atmospheric pressure. (Surface tension of mercury = 0.5 N m^{-1} , angle of contact with glass = 180° , density of mercury = 13600 kg m^{-3} , $g = 9.81 \text{ m s}^{-2}$.) (O. & C.)

20. Explain how to measure the surface tension of a soap film.

The diameters of the arms of a U-tube are respectively 1 cm and 1 mm. A liquid of surface tension $7.0 \times 10^{-2} \text{ N m}^{-1}$ is poured into the tube which is placed vertically. Find the difference in levels in the two arms. The density may be taken as 1000 kg m^{-3} and the contact angle zero. (L.)

21. Explain what is meant by surface tension, and show how its existence is accounted for by molecular theory.

Find an expression for the excess pressure inside a soap-bubble of radius R and surface tension T . Hence find the work done by the pressure in increasing the radius of the bubble from a to b . Find also the increase in surface area of the bubble, and in the light of this discuss the significance of your result. (C.)

22. A clean glass capillary tube, of internal diameter 0·04 cm, is held vertically with its lower end below the surface of clean water in a beaker, and with 10 cm of the tube above the surface. To what height will the water rise in the tube? What will happen if the tube is now depressed until only 5 cm of its length is above the surface? The surface tension of water is $7\cdot2 \times 10^{-2}$ N m⁻¹.

Describe, and give the theory of some method, other than that of the rise in a capillary tube, of measuring surface tension. (O. & C.)

23. Explain (a) in terms of molecular forces why the water is drawn up above the horizontal liquid level round a steel needle which is held vertically and partly immersed in water, (b) why, in certain circumstances, a steel needle will rest on a water surface. In each case show the relevant forces on a diagram. (N.)

24. The force between two molecules may be regarded as an attractive force which increases as their separation decreases and a repulsive force which is only important at small separations and which there varies very rapidly. Draw sketch graphs (a) for force-separation, (b) for potential-energy separation. On each graph mark the equilibrium distance and on (b) indicate the energy which would be needed to separate two molecules initially at the equilibrium distance.

With the help of your graphs discuss briefly the resulting motion if the molecules are displaced from the equilibrium position. (N.)

25. Explain briefly the meaning of *surface tension* and *angle of contact*.

Account for the following: (a) A small needle may be placed on the surface of water in a beaker so that it 'floats', and (b) if a small quantity of detergent is added to the water the needle sinks.

A solid glass cylinder of length l , radius r and density σ is suspended with its axis vertical from one arm of a balance so that it is partly immersed in a liquid of density ρ . The surface tension of the liquid is γ and its angle of contact with the glass is α . If W_1 is the weight required to achieve a balance when the cylinder is in air and W_2 is the weight required to balance the cylinder when it is partly immersed with a length h ($< l$) below the free surface of the liquid, derive an expression for the value of $W_1 - W_2$. If this method were used to measure the surface tension of a liquid, why would the result probably be less accurate than that obtained from a similar experiment using a thin glass plate? (O. & C.)

26. Explain in terms of molecular forces why some liquids spread over a solid surface whilst others do not.

A glass capillary tube of uniform bore of diameter 0·050 cm is held vertically with its lower end in water. Calculate the capillary rise. Describe and explain what happens if the tube is lowered so that 4·0 cm protrudes above the water surface. Assume that the surface tension of water is $7\cdot0 \times 10^{-2}$ N m⁻¹. (N.)

27. Define *surface tension*. Describe how the surface tension of water at room temperature may be determined by using a capillary tube. Derive the formula used to calculate the result.

A hydrometer has a cylindrical glass stem of diameter 0·50 cm. It floats in water of density 1000 kg m⁻³ and surface tension $7\cdot2 \times 10^{-2}$ N m⁻¹. A drop of

liquid detergent added to the water reduces the surface tension to 5.0×10^{-2} N m⁻¹. What will be the change in length of the exposed portion of the glass stem? Assume that the relevant angle of contact is always zero. (N.)

28. The lower end of a vertical clean glass capillary tube is just immersed in water. Why does water rise up the tube?

A vertical capillary tube of internal radius r m has its lower end dipping in water of surface tension T newton m⁻¹. Assuming the angle of contact between water and glass to be zero, obtain from first principles an expression for the pressure excess which must be applied to the upper end of the tube in order just to keep the water levels inside and outside the tube the same.

A capillary of internal diameter 0.7 mm is set upright in a beaker of water with one end below the surface; air is forced slowly through the tube from the upper end, which is also connected to a U-tube manometer containing a liquid of density 800 kg m⁻³. The difference in levels on the manometer is found to build up to 9.1 cm, drop to 4.0 cm, build up to 9.1 cm again, and so on. Estimate (a) the depth of the open end of the capillary below the free surface of the water in the beaker, (b) the surface tension of water. [State clearly any assumptions you have made in arriving at these estimates.] (O.)

29. It is sometimes stated that, in virtue of its surface tension, the surface of a liquid behaves as if it were a stretched rubber membrane. To what extent do you think this analogy is justified?

Explain why the pressure inside a spherical soap bubble is greater than that outside. How would you investigate experimentally the relation between the excess pressure and the radius of the bubble? Show on a sketch graph the form of the variation you would expect to obtain.

If olive oil is sprayed on to the surface of a beaker of hot water, it remains as separated droplets on the water surface; as the water cools, the oil forms a continuous thin film on the surface. Suggest a reason for this phenomenon. (C.)

30. Describe the capillary tube method of measuring the surface tension of a liquid.

An inverted U-tube (Hare's apparatus) for measuring the specific gravity of a liquid was constructed of glass tubing of internal diameter about 2 mm. The following observations of the heights of balanced columns of water and another liquid were obtained:

Height of water (cm)	2.8	4.2	5.4	6.9	8.5	9.8	11.6
Height of liquid (cm)	2.0	3.8	5.3	7.0	9.1	10.7	13.0

Plot the above results, explain why the graph does not pass through the origin, and deduce from the graph an accurate value for the specific gravity of the liquid. (N.)

31. How does simple molecular theory account for surface tension? Illustrate your account by explaining the rise of water up a glass capillary.

A light wire frame in the form of a square of side 5 cm hangs vertically in water with one side in the water-surface. What additional force is necessary to pull the frame clear of the water? Explain why, if the experiment is performed with soap-solution, as the force is increased a vertical film is formed, whereas with pure water no such effect occurs. (Surface tension of water is 7.4×10^{-2} N m⁻¹.) (O. & C.)

32. Define *surface tension* and state the effect on the surface tension of water of raising its temperature.

Describe an experiment to measure the surface tension of water over the range of temperatures from 20°C to 70°C. Why is the usual capillary rise method unsuitable for this purpose?

Two unequal soap bubbles are formed one on each end of a tube closed in the middle by a tap. State and explain what happens when the tap is opened to put the two bubbles into connection. Give a diagram showing the bubbles when equilibrium has been reached. (L.)

chapter six

Elasticity

Elasticity

A bridge, when used by traffic during the day, is subjected to loads of varying magnitude. Before a steel bridge is erected, therefore, samples of the steel are sent to a research laboratory, where they undergo tests to find out whether the steel can withstand the loads to which it is likely to be subjected.

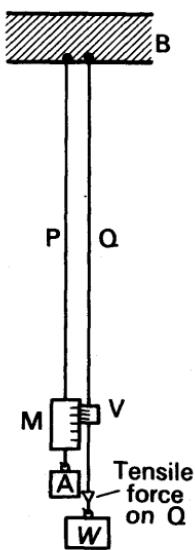


FIG. 6.1
Tensile force

Fig. 6.1 illustrates a simple laboratory method of discovering useful information about the property of steel we are discussing. Two long thin steel wires, P, Q, are suspended beside each other from a rigid support B, such as a girder at the top of the ceiling. The wire P is kept taut by a weight A attached to its end and carries a scale M graduated in centimetres. The wire Q carries a vernier scale V which is alongside the scale M.

When a load W such as 1 kgf is attached to the end of Q, the wire increases in length by an amount which can be read from the change in the reading on the vernier V. If the load is taken off and the reading on V returns to its original value, the wire is said to be **elastic** for loads from zero to 1 kgf, a term adopted by analogy with an elastic thread. When the load W is increased to 2 kgf the extension (increase in length) is obtained from V again; and if the reading on V returns to origin value when the load is removed the wire is said to be elastic at least for loads from zero to 2 kgf.

The extension of a thin wire such as Q for increasing loads may be found by experiments to be as follows:

W (kgf)	0	1	2	3	4	5	6	7	8
Extension (mm.)	0	0.14	0.28	0.42	0.56	0.70	0.85	1.01	1.19

Elastic Limit

When the extension, e , is plotted against the load, W , a graph is obtained which is a *straight line* OA, followed by a curve ABY rising slowly

at first and then very sharply, Fig. 6.2. Up to about 5 kgf, then, the results in the table show that the extension increased by 0·14 mm for every kgf which is added to the wire. Further, the wire returned to its

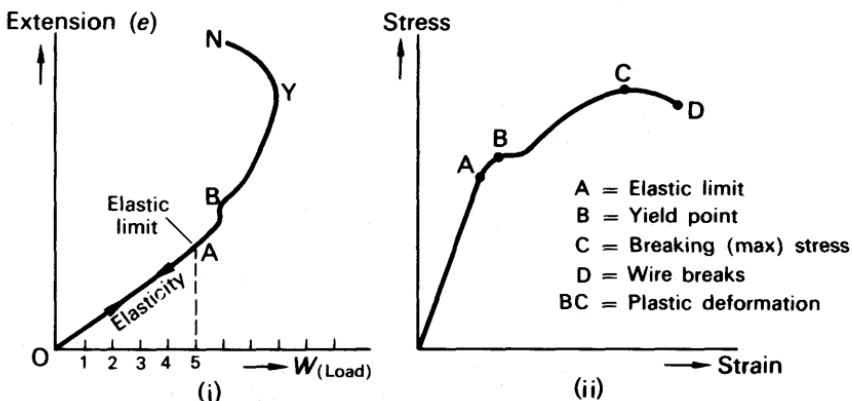


FIG. 6.2 Extension v. Load

original length when the load was removed. For loads greater than about 5 kgf, however, the extension increases relatively more and more, and the wire now no longer returns to its original length when it is unloaded. The wire is thus permanently strained, and A corresponds to its *elastic limit*.

Hooke's Law

From the straight line graph OA, we deduce that *the extension is proportional to the load or tension in the wire when the elastic limit is not exceeded*. This is known as *Hooke's law*, after ROBERT HOOKE, founder of the Royal Society, who discovered the relation in 1676. The law shows that when a molecule of a solid is displaced farther from its mean position, the restoring force is proportional to its displacement (see p. 126). One may therefore conclude that the molecules of a solid are undergoing simple harmonic motion (p. 44).

The measurements also show that it would be dangerous to load the wire with weights greater than 5 kilogrammes, the elastic limit, because the wire then suffers a permanent strain. Similar experiments in the research laboratory enable scientists to find the maximum load which a steel bridge, for example, should carry for safety. Rubber samples are also subjected to similar experiments, to find the maximum safe tension in rubber belts used in machinery.

Yield Point. Ductile and Brittle Substances. Breaking Stress

Careful experiments show that, for mild steel and iron for example, the molecules of the wire begin to 'slide' across each other soon after the load exceeds the elastic limit, that is, the material becomes *plastic*. This is indicated by the slight 'kink' at B beyond A in Fig. 6.2 (i), and it is called the *yield point* of the wire. The change from an elastic to

a plastic stage is shown by a sudden increase in the extension, and as the load is increased further the extension increases rapidly along the curve YN and the wire then snaps. The *breaking stress* of the wire is the corresponding force per unit area of cross-section of the wire. Substances such as those just described, which elongate considerably and undergo plastic deformation until they break, are known as *ductile* substances. Lead, copper and wrought iron are ductile. Other substances, however, break just after the elastic limit is reached; they are known as *brittle* substances. Glass and high carbon steels are brittle.

Brass, bronze, and many alloys appear to have no yield point. These materials increase in length beyond the elastic limit as the load is increased without the appearance of a plastic stage.

The strength and ductility of a metal, its ability to flow, are dependent on defects in the metal crystal lattice. Such defects may consist of a missing atom at a site or a *dislocation* at a plane of atoms. Plastic deformation is the result of the 'slip' of atomic planes. The latter is due to the movement of dislocations, which spreads across the crystal.

Tensile Stress and Tensile Strain. Young's Modulus

We have now to consider the technical terms used in the subject of elasticity of wires. When a force or tension F is applied to the end of a wire of cross-sectional area A , Fig. 6.3,

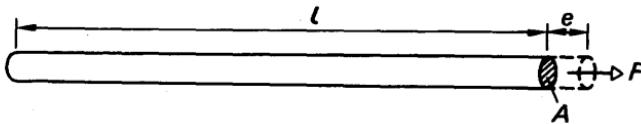


FIG. 6.3 Tensile stress and tensile strain

$$\text{the tensile stress} = \text{force per unit area} = \frac{F}{A} \quad . \quad (1)$$

If the extension of the wire is e , and its original length is l ,

$$\text{the tensile strain} = \text{extension per unit length} = \frac{e}{l} \quad . \quad (2)$$

Suppose 2 kgf is attached to the end of a wire of length 2 metres of diameter 0·64 mm, and the extension is 0·60 mm. Then

$$F = 2 \text{ kgf} = 2 \times 9\cdot8 \text{ N}, A = \pi \times 0\cdot032^2 \text{ cm}^2 = \pi \times 0\cdot032^2 \times 10^{-4} \text{ m}^2.$$

$$\therefore \text{tensile stress} = \frac{2 \times 9\cdot8}{\pi \times 0\cdot032^2 \times 10^{-4}} \text{ N m}^{-2},$$

$$\text{and} \quad \text{tensile strain} = \frac{0\cdot6 \times 10^{-3} \text{ metre}}{2 \text{ metre}} = 0\cdot3 \times 10^{-3}.$$

It will be noted that 'stress' has units such as 'newton m⁻²'; 'strain' has no units because it is the ratio of two lengths.

A *modulus of elasticity* of the wire, called **Young's modulus (E)**, is defined as the ratio

$$E = \frac{\text{tensile stress}}{\text{tensile strain}} \quad \quad (3)$$

Thus

$$E = \frac{F/A}{e/l}.$$

Using the above figures,

$$\begin{aligned} E &= \frac{2 \times 9.8 / (\pi \times 0.032^2 \times 10^{-4})}{0.3 \times 10^{-3}}, \\ &= \frac{2 \times 9.8}{\pi \times 0.032^2 \times 10^{-4} \times 0.3 \times 10^{-3}}, \\ &= 2.0 \times 10^{11} \text{ N m}^{-2}. \end{aligned}$$

It should be noted that Young's modulus, E , is calculated from the ratio stress:strain only when the wire is under 'elastic' conditions, that is, the load does not then exceed the elastic limit (p. 154). Fig. 6.2 (ii) shows the general stress-strain diagram for a ductile material.

Dimensions of Young's Modulus

As stated before, the 'strain' of a wire has no dimensions of mass, length, or time, since, by definition, it is the ratio of two lengths. Now

$$\begin{aligned} \text{dimensions of stress} &= \frac{\text{dimensions of force}}{\text{dimensions of area}} \\ &= \frac{MLT^{-2}}{L^2} \\ &= ML^{-1}T^{-2}. \end{aligned}$$

\therefore dimensions of Young's modulus, E ,

$$\begin{aligned} &= \frac{\text{dimensions of stress}}{\text{dimensions of strain}} \\ &= ML^{-1}T^{-2}. \end{aligned}$$

Determination of Young's Modulus

The magnitude of Young's modulus for a material in the form of a wire can be found with the apparatus illustrated in Fig. 6.1, p. 153, to which the reader should now refer. The following practical points should be specially noted:

- (1) The wire is made *thin* so that a moderate load of several kilograms produces a large tensile stress. The wire is also made *long* so that a measurable extension is produced.
- (2) The use of two wires, P, Q, of the same material and length, eliminates the correction for (i) the yielding of the support when loads are added to Q, (ii) changes of temperature.
- (3) Both wires should be free of kinks, otherwise the increase in length cannot be accurately measured. The wires are straightened by attaching weights to their ends, as shown in Fig. 6.1.

(4) A vernier scale is necessary to measure the extension of the wire since this is always small. The 'original length' of the wire is measured from the top B to the vernier V by a ruler, since an error of 1 millimetre is negligible compared with an original length of several metres. For very accurate work, the extension can be measured by using a spirit level between the two wires, and adjusting a vernier screw to restore the spirit level to its original reading after a load is added.

(5) The diameter of the wire must be found by a micrometer screw gauge at several places, and the average value then calculated. The area of cross-section, $A = \pi r^2$, where r is the radius.

(6) The readings on the vernier are also taken when the load is gradually removed in steps of 1 kilogramme; they should be very nearly the same as the readings on the vernier when the weights were added, showing that the elastic limit was not exceeded. Suppose the reading on V for loads, W , of 1 to 6 kilogramme are a, b, c, d, e, f , as follows :

W (kgf)	1	2	3	4	5	6
Reading on V	a	b	c	d	e	f

The average extension for 3 kilogramme is found by taking the average of $(d - a)$, $(e - b)$, and $(f - c)$. Young's modulus can then be calculated from the relation stress/strain, where the stress = $3 \times 9.8/\pi r^2$, and the strain = average extension/original length of wire (p. 155).

Magnitude of Young's Modulus

Mild steel (0.2% carbon) has a Young's modulus value of about 2.0×10^{11} newton m^{-2} , copper has a value about 1.2×10^{11} newton m^{-2} ; and brass a value about 1.0×10^{11} newton m^{-2} .

The breaking stress (tenacity) of cast-iron metal is about 1.5×10^8 newton m^{-2} ; the breaking stress of mild steel metal is about 4.5×10^8 newton m^{-2} .

At Royal Ordnance and other Ministry of Supply factories, tensile testing is carried out by placing a sample of the material in a machine known as an extensometer, which applies stresses of increasing value along the length of the sample and automatically measures the slight increase in length. When the elastic limit is reached, the pointer on the dial of the machine flickers, and soon after the yield point is reached the sample becomes thin at some point and then breaks. A graph showing the load v. extension is recorded automatically by a moving pen while the sample is undergoing test.

EXAMPLE

Find the maximum load in kgf which may be placed on a steel wire of diameter 0.10 cm if the permitted strain must not exceed $\frac{1}{1000}$ and Young's modulus for steel is 2.0×10^{11} N m^{-2} .

We have $\frac{\text{max. stress}}{\text{max. strain}} = 2 \times 10^{11}$.

$$\therefore \text{max. stress} = \frac{1}{1000} \times 2 \times 10^{11} = 2 \times 10^8 \text{ N m}^{-2}.$$

$$\text{Now area of cross-section in } \text{m}^2 = \frac{\pi d^2}{4} = \frac{\pi \times 0.1^2 \times 10^{-4}}{4}$$

and

$$\text{stress} = \frac{\text{load } F}{\text{area}}$$

$$\therefore F = \text{stress} \times \text{area} = 2 \times 10^8 \times \frac{\pi \times 0.1^2 \times 10^{-4}}{4} \text{ newton}$$

$$= 157 \text{ newton} = 15.7 \text{ kgf (approx.)}$$

since 10 newtons = 1 kgf (approx.).

Force in Bar Due to Contraction or Expansion

When a bar is heated, and then prevented from contracting as it cools, a considerable force is exerted at the ends of the bar. We can derive a formula for the force if we consider a bar of Young's modulus E , a cross-sectional area A , a linear expansivity of magnitude α , and a decrease in temperature of $t^\circ\text{C}$. Then, if the original length of the bar is l , the decrease in length e if the bar were free to contract = $\alpha l t$.

Now

$$E = \frac{F/A}{e/l}.$$

$$\therefore F = \frac{EAe}{l} = \frac{EA\alpha lt}{l}.$$

$$\therefore F = EA\alpha t.$$

As an illustration, suppose a steel rod of cross-sectional area 2.0 cm^2 is heated to 100°C , and then prevented from contracting when it is cooled to 10°C . The linear expansivity of steel = $12 \times 10^{-6} \text{ K}^{-1}$ and Young's modulus = $2.0 \times 10^{11} \text{ newton m}^{-2}$. Then

$$A = 2 \text{ cm}^2 = 2 \times 10^{-4} \text{ m}^2, t = 90 \text{ deg C.}$$

$$\therefore F = EA\alpha t = 2 \times 10^{11} \times 2 \times 10^{-4} \times 12 \times 10^{-6} \times 90 \text{ newton (N)},$$

$$= 43200 \text{ N} = \frac{43200}{9.8} \text{ kgf} = 4400 \text{ kgf.}$$

Energy Stored in a Wire

Suppose that a wire has an original length l and is stretched by a length e when a force F is applied at one end. If the elastic limit is not exceeded, the extension is directly proportional to the applied load (p. 154). Consequently the force in the wire has increased in magnitude from zero to F , and hence the average force in the wire while stretching was $F/2$. Now

$$\text{work done} = \text{force} \times \text{distance.}$$

$$\therefore \text{work} = \text{average force} \times \text{extension}$$

$$= \frac{1}{2}Fe \quad \dots \dots \dots \quad (1)$$

This is the amount of energy stored in the wire. The formula $\frac{1}{2}Fe$ gives the energy in joule when F is in newton and e is in metre.

Further, since $F = EAe/l$,

$$\text{energy} = \frac{1}{2}EA\frac{e^2}{l}.$$

As an illustration, suppose $E = 2.0 \times 10^{11}$ newton m^{-2} , $A = 3 \times 10^{-2} \text{ cm}^2 = 3 \times 10^{-6} \text{ m}^2$, $e = 1 \text{ mm} = 1 \times 10^{-3} \text{ m}$, $l = 400 \text{ cm} = 4 \text{ m}$. Then

$$\begin{aligned}\text{energy stored} &= \frac{1}{2}EA\frac{e^2}{l} = \frac{1}{2} \times \frac{2 \times 10^{11} \times 3 \times 10^{-6} \times (1 \times 10^{-3})^2}{4} \text{ joule}, \\ &= 0.075 \text{ J}.\end{aligned}$$

The volume of the wire = Al . Thus, from (1),

$$\text{energy per unit volume} = \frac{\frac{1}{2}Fe}{Al} = \frac{1}{2} \frac{F}{A} \times \frac{e}{l}$$

But F/A = stress, e/l = strain,

$$\therefore \text{energy per unit volume} = \frac{1}{2} \text{ stress} \times \text{strain} \quad . \quad (2)$$

Graph of F v. e and energy

The energy stored in the wire when it is stretched can also be found from the graph of F v. e . Fig. 6.4. Suppose the wire extension is e_1 when a load F_1 is applied, and the extension increases to e_2 when the load increases to F_2 . If F is the load between F_1 and F_2 at some stage, and Δx is the small extension which then occurs, then

$$\text{energy stored} = \text{work done} = F \cdot \Delta x.$$

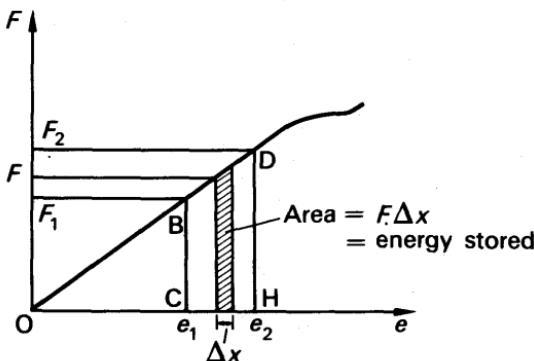


FIG. 6.4 Energy in stretched wire

Now $F \cdot \Delta x$ is represented by the small area between the axis of e and the graph, shown shaded in Fig. 6.4. Thus the total work done between e_1 and e_2 is represented by the area CBDH.

If the extension occurs on the straight part of the curve, when Hooke's law is obeyed, then CBDH is a trapezium. The area of a trapezium = half the sum of the parallel sides \times perpendicular distance between them = $\frac{1}{2}(BC + DH) \times CH = \frac{1}{2}(F_1 + F_2)(e_2 - e_1)$.

$$\therefore \text{energy stored} = \text{average force} \times \text{increase in length.}$$

If the extension occurs beyond the elastic limit, for example, along the curved part of the graph in Fig. 6.4, the energy expended can be obtained from the area between the curve and the axis of e .

EXAMPLES

1. A 20 kg weight is suspended from a length of copper wire 1 mm in radius. If the wire breaks suddenly, does its temperature increase or decrease? Calculate the change in temperature; Young's modulus for copper = $12 \times 10^{10} \text{ N m}^{-2}$; density of copper = 9000 kg m^{-3} ; specific heat capacity of copper = $0.42 \text{ J g}^{-1} \text{ K}^{-1}$. (C.S.)

When the wire is stretched, it gains potential energy equal to the work done on it. When the wire is suddenly broken, this potential energy is released as the molecules return to their original position. The energy is converted into heat and thus the temperature rises.

$$\begin{aligned}\text{Gain in potential energy of molecules} &= \text{work done in stretching wire} \\ &= \frac{1}{2} \text{force} (F) \times \text{extension} (e).\end{aligned}$$

With the usual notation, $F = EA\frac{e}{l}$

$$\therefore e = \frac{Fl}{E \cdot A} = \frac{(20 \times 9.8) \times l}{12 \times 10^{10} \times \pi \times (10^{-3})^2} \text{ m} = 5.2 \times 10^{-4}l \text{ m},$$

$$\therefore \text{potential energy gained} = \frac{1}{2} \times 20 \times 9.8 \times 5.2 \times 10^{-4}l = 5.1 \times 10^{-2}l \text{ J}$$

$$\begin{aligned}\text{Heat capacity of wire} &= \text{mass} \times \text{specific heat capacity} \\ &= \pi \times (10^{-3})^2 \times 9000l \times (0.42 \times 1000) = 11.9l \text{ J K}^{-1} \\ \therefore \text{temperature rise} &= \frac{\text{potential energy}}{\text{heat capacity}} = \frac{5.1 \times 10^{-2}l}{11.9l} \\ &= 4.3 \times 10^{-3} \text{ deg C}.\end{aligned}$$

2. Define *stress* and *strain*. Describe the behaviour of a copper wire when it is subjected to an increasing longitudinal stress. Draw a stress-strain diagram and mark on it the elastic region, yield point and breaking stress.

A wire of length 5 m, of uniform circular cross-section of radius 1 mm is extended by 1.5 mm when subjected to a uniform tension of 100 newton. Calculate from first principles the strain energy per unit volume assuming that deformation obeys Hooke's law.

Show how the stress-strain diagram may be used to calculate the work done in producing a given strain, when the material is stretched beyond the Hooke's law region. (O. & C.)

$$\text{Strain energy} = \frac{1}{2} \text{tension} \times \text{extension}$$

$$\text{Tension} = 100 \text{ newton. Extension} = 1.5 \times 10^{-3} \text{ m.}$$

$$\therefore \text{energy} = \frac{1}{2} \times 100 \times 1.5 \times 10^{-3} = 0.075 \text{ J.}$$

Volume of wire = length × area = $5 \times \pi \times 1 \times 10^{-6} \text{ m}^3$.

$$\therefore \text{energy per unit volume} = \frac{0.075}{5 \times \pi \times 1 \times 10^{-6}} = 4.7 \times 10^3 \text{ J m}^{-3} (\text{approx.})$$

Bulk Modulus

When a gas or a liquid is subjected to an increased pressure the substance contracts. A change in bulk thus occurs, and the *bulk strain* is defined by:

$$\text{strain} = \frac{\text{change in volume}}{\text{original volume}}$$

The *bulk stress* on the substance is the increased force per unit area, by definition, and the bulk modulus, K , is given by:

$$K = \frac{\text{bulk stress}}{\text{bulk strain}}$$

$$= \frac{\text{increase in force per unit area}}{\text{change in volume/original volume}}$$

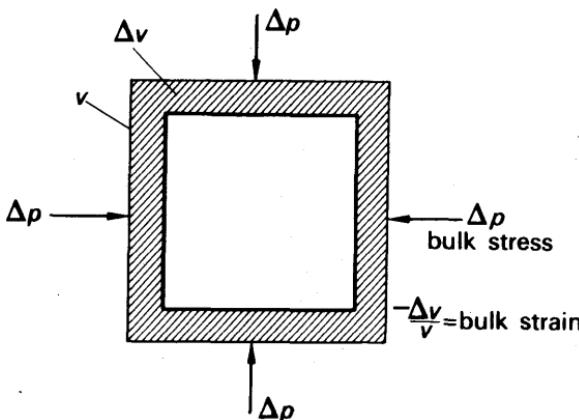


FIG. 6.5 Bulk stress and bulk strain

If the original volume of the substance is v , the change in volume may be denoted by $-\Delta v$ when the pressure increases by a small amount Δp ; the minus indicates that the volume decreases. Thus (Fig. 6.5)

$$K = -\frac{\Delta p}{\Delta v/v}$$

When δp and δv become very small, then, in the limit,

$$K = -v \frac{dp}{dv} \quad \quad (1)$$

The bulk modulus of water is about $2 \times 10^9 \text{ N m}^{-2}$ for pressures in the range 1–25 atmospheres; the bulk modulus of mercury is about

$27 \times 10^9 \text{ N m}^{-2}$. The bulk modulus of gases depends on the pressure, as now explained. Generally, since the volume change is relatively large, the bulk modulus of a gas is low compared with that of a liquid.

Bulk Modulus of a Gas

If the pressure, p , and volume, v , of a gas change under conditions such that

$$pv = \text{constant},$$

which is Boyle's law, the changes are said to be *isothermal* ones. In this case, by differentiating the product pv with respect to v , we have

$$p + v \frac{dp}{dv} = 0.$$

$$\therefore p = -v \frac{dp}{dv}.$$

But the bulk modulus, K , of the gas is equal to $-v \frac{dp}{dv}$ by definition (see p. 161).

$$\therefore K = p \quad \quad (2)$$

Thus the *isothermal bulk modulus is equal to the pressure*.

When the pressure, p , and volume, v , of a gas change under conditions such that

$$pv^\gamma = \text{constant},$$

where $\gamma = c_p/c_V$ = the ratio of the specific heat capacities of the gas, the changes are said to be *adiabatic* ones. This equation is the one obeyed by local values of pressure and volume in air when a sound wave travels through it. Differentiating both sides with respect to v ,

$$\therefore p \times \gamma v^{\gamma-1} + v^\gamma \frac{dp}{dv} = 0,$$

$$\therefore \gamma p = -v \frac{dp}{dv},$$

$$\therefore \text{adiabatic bulk modulus} = \gamma p \quad \quad (3)$$

For air at normal pressure, $K = 10^5 \text{ newton m}^{-2}$ isothermally and $1.4 \times 10^5 \text{ newton m}^{-2}$ adiabatically. The values of K are of the order 10^5 times smaller than liquids as gases are much more compressible.

Velocity of Sound

The velocity of sound waves through any material depends on (i) its density ρ , (ii) its modulus of elasticity, E . Thus if V is the velocity, we may say that

$$V = kE^x \rho^y \quad \quad (i),$$

where k is a constant and x, y are indices we can find by the theory of dimensions (p. 34).

The units of velocity, V , are LT^{-1} ; the units of density ρ are ML^{-3} ; and the units of modulus of elasticity, E , are $\text{ML}^{-1}\text{T}^{-2}$ (see p. 156). Equating the dimensions on both sides of (i),

$$\therefore \text{LT}^{-1} = (\text{ML}^{-1}\text{T}^{-2})^x \times (\text{ML}^{-3})^y.$$

Equating the indices of M, L, T on both sides, we have

$$\begin{aligned}0 &= x + y, \\1 &= -x - 3y, \\-1 &= -2x.\end{aligned}$$

Solving, we find $x = \frac{1}{2}$, $y = -\frac{1}{2}$. Thus $V = kE^{\frac{1}{2}}\rho^{-\frac{1}{2}}$. A rigid investigation shows $k = 1$, and thus

$$V = E^{\frac{1}{2}}\rho^{-\frac{1}{2}} = \sqrt{\frac{E}{\rho}}.$$

In the case of a solid, E is Young's modulus. In the case of air and other gases, and of liquids, E is replaced by the bulk modulus K . Laplace showed that the adiabatic bulk modulus must be used in the case of a gas, and since this is γp , the velocity of sound in a gas is given by the expression

$$V = \sqrt{\frac{\gamma p}{\rho}}.$$

Modulus of Rigidity

So far we have considered the strain in one direction, or tensile strain, to which Young's modulus is applicable and the strain in bulk or volume, to which the bulk modulus is applicable.

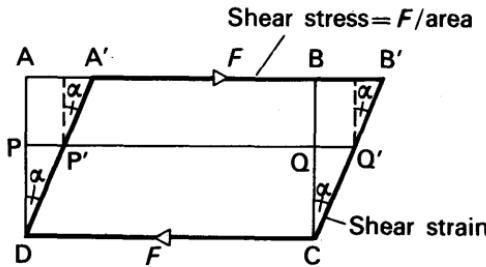


FIG. 6.6 Shear stress and shear strain

Consider a block of material ABCD, such as pitch or plastic for convenience. Fig. 6.6. Suppose the lower plane CD is fixed, and a stress parallel to CD is applied by a force F to the upper side AB. The block then changes its shape and takes up a position A'B'CD. It can now be seen that planes in the material parallel to DC are displaced relative to each other. The plane AB, for example, which was originally directly opposite the plane PQ, is displaced to A'B' and PQ is displaced to P'Q'. The *angular displacement* α is defined as the *shear strain*. α is the angular displacement between any two planes, for example, between CD and P'Q'.

No volume change occurs in Fig. 6.6. Further, since the force along CD is F in magnitude, it forms a *couple* with the force F applied to the upper side AB. The *shear stress* is defined as the 'shear force per unit area' on the face AB (or CD), as in Young's modulus or the bulk

modulus. Unlike the case for these modulii, however, the shear stress has a turning or 'displacement' effect owing to the couple present. The solid does not collapse because in a strained equilibrium position such as A'B'CD in Fig. 6.6, the external couple acting on the solid due to the forces F is balanced by an opposing couple due to stresses inside the material.

If the elastic limit is not exceeded when a shear stress is applied, that is, the solid recovers its original shape when the stress is removed, the *modulus of rigidity*, G , is defined by:

$$G = \frac{\text{shear stress (force per unit area)}}{\text{shear strain (angular displacement, } \alpha\text{)}}.$$

Shear strain has no units; shear stress has units of newton m^{-2} . The modulus of rigidity of copper is $4.8 \times 10^{10} \text{ N m}^{-2}$; for phosphor-bronze it is $4.4 \times 10^{10} \text{ N m}^{-2}$, and for quartz fibre it is $3.0 \times 10^{10} \text{ N m}^{-2}$.

If a spiral spring is stretched, all parts of the spiral become twisted. The applied force has thus developed a 'torsional' or shear strain. The extension of the spring hence depends on its modulus of rigidity, in addition to its dimensions.

Torsion wire

In sensitive current-measuring instruments, a very weak control is needed for the rotation of the instrument coil. This may be provided by using a long elastic or *torsion wire* of phosphor bronze in place of a spring. The coil is suspended from the lower end of the wire and when it rotates through an angle θ , the wire sets up a weak opposing couple equal to $c\theta$, where c is the elastic constant of the wire. Quartz fibres are very fine but comparatively strong, and have elastic properties. They are also used for sensitive control (see p. 61).

The magnitude of c , the elastic constant, can be derived as follows. Consider a wire of radius a , length l , modulus of rigidity G , fixed at the upper end and twisted by a couple of moment C at the other end. If we take a section of the cylindrical wire between radii r and $r + \delta r$, then a 'slice' of the material ODBX has been sheared through an angle α to a position ODB₁X where X is the centre of the lower end of the wire. Fig. 6.7. From the definition of modulus of rigidity, $G = \text{torsional stress} \div \text{torsional strain} = F/A \div \alpha$, where F is the tangential force applied over an area A .

Now $A = \text{area of circular annulus at lower end} = 2\pi r \cdot \delta r$.

$$\therefore F = GA\alpha = G \cdot 2\pi r \cdot \delta r \cdot \alpha.$$

From Fig. 6.7, it follows that BB₁ = $l\alpha$, and BB₁ = $r\theta$.

$$\therefore l\alpha = r\theta, \text{ or } \alpha = r\theta/l.$$

$$\therefore F = \frac{G \cdot 2\pi r \cdot \delta r \cdot r\theta}{l} = \frac{2\pi G\theta r^2 \cdot \delta r}{l}.$$

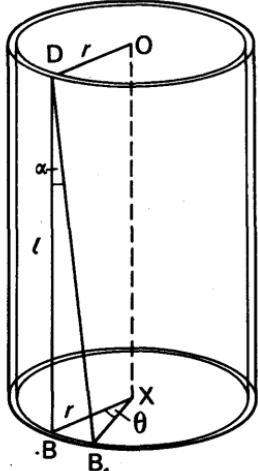


FIG. 6.7
Shear (Torsion) in wire

\therefore moment of F about axis OX of wire = $F \cdot r$

$$= \frac{2\pi G\theta}{l} \cdot r^3 \cdot \delta r.$$

\therefore total moment, or couple torque C ,

$$= \int_0^a \frac{2\pi G\theta}{l} \cdot r^3 dr = \frac{2\pi G\theta}{l} \cdot \frac{a^4}{4}$$

$$\therefore C = \frac{\pi G a^4 \theta}{2l} \quad \text{(i)}$$

If the wire is a hollow cylinder of radii a, b respectively, the limits of integration are altered accordingly, and

$$\text{moment of couple} = \int_a^b \frac{2\pi G\theta}{l} \cdot r^3 dr = \frac{\pi G(b^4 - a^4)\theta}{2l}.$$

Determinations of modulus of rigidity. Dynamical method. One method of measuring the modulus of rigidity of a wire E is to clamp it vertically at one end, attach a horizontal disc D of known moment of inertia, I , at the other end, and then time the horizontal torsional oscillations of D. Fig. 6.8 (i). On p. 90, it was shown that the period of oscillation, $T = 2\pi\sqrt{I/c}$, where c is the opposing couple per unit angle of twist. Thus, with our previous notation, as $\theta = 1$,

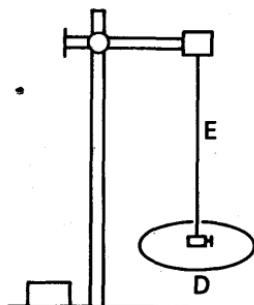
$$c = \frac{\pi G a^4}{2l}.$$

$$\therefore T = 2\pi \sqrt{\frac{2I}{\pi G a^4}}$$

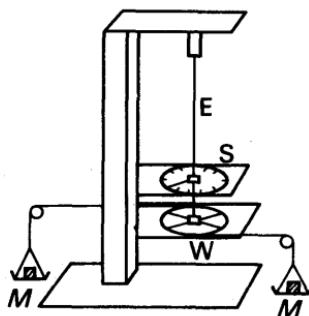
or

$$G = \frac{8\pi I l}{a^4 T^2}$$

Hence G can be evaluated from measurements of l, a, I, T .



(i)



(ii)

FIG. 6.8 Modulus of rigidity measurement

Statical method. The modulus of rigidity, G , of the wire E can also be found by measuring the steady deflection θ at the lower end on a scale S graduated in degrees when a couple is applied round a wheel W. Fig. 6.8 (ii). If M is the mass

in each scale-pan, and d is the diameter of W , the moment of the couple on the wire = $Mgd = \pi Ga^4\theta/2l$. The angle θ in radians, and a, l , are known, and hence G can be evaluated.

Poisson's Ratio

When a rubber cord is extended its diameter usually decreases at the same time. *Poisson's ratio*, σ , is the name given to the ratio

$$\frac{\text{lateral contraction/original diameter}}{\text{longitudinal extension/original length}} \quad (1)$$

and is a constant for a given material. If the original length of a rubber strip is 100 cm and it is stretched to 102 cm, the fractional longitudinal extension = 2/100. If the original diameter of the cord is 0.5 cm and it decreases to 0.495 cm, the fractional lateral contraction = 0.005/0.5 = 1/100. Thus, from the definition of Poisson's ratio,

$$\sigma = \frac{1/100}{2/100} = \frac{1}{2}.$$

When the *volume* of a strip of material remains *constant* while an extension and a lateral contraction takes place, it can easily be shown that Poisson's ratio is 0.5 in this case. Thus suppose that the length of the strip is l and the radius is r .

Then

$$\text{volume}, V = \pi r^2 l.$$

By differentiating both sides, noting that V is a constant and that we have a product of variables on the right side,

$$\therefore 0 = \pi r^2 \times \delta l + l \times 2\pi r \delta r.$$

$$\therefore r \delta l = -2l \delta r.$$

$$\therefore -\frac{\delta r/r}{\delta l/l} = \frac{1}{2}.$$

But $-\delta r/r$ is the lateral contraction in radius/original radius, and $\delta l/l$ is the longitudinal extension/original length.

$$\therefore \text{Poisson's ratio, } \sigma, = \frac{1}{2}.$$

Experiments show that σ is 0.48 for rubber, 0.29 for steel, 0.27 for iron, and 0.26 for copper. Thus the three metals increase in volume when stretched, whereas rubber remains almost unchanged in volume.

Summary

The three modulii of elasticity are compared in the table below:

	Young's modulus, E	Modulus of Rigidity, G	Bulk modulus, K
1. Definition :			Definition :
	tensile stress tensile strain	shear stress shear strain	pressure change $-\Delta v/v$
2. Relates to change in length ('tensile')	Relates to change in shape ('shear')	Relates to change in volume ('bulk')	

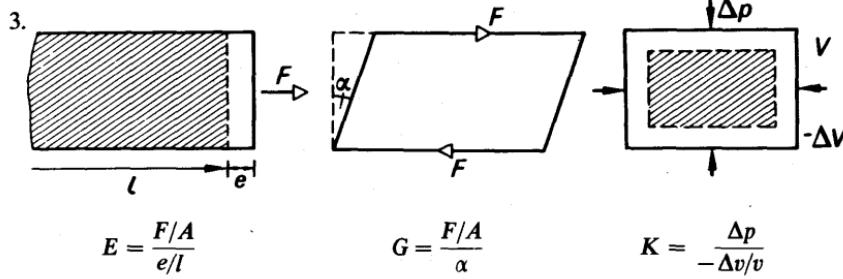


FIG. 6.9

4.	Applies only to solids	Applies to solids and liquids	Applies to all materials. Low value for gases
5.	Used in stretching wires, bending beams, linear expansion and contraction with temperature change	Used in torsion wires, helical springs	Used in velocity of sound formula for all materials. In gas, $K = p$ (isothermal) or γp (adiabatic)

EXERCISES 6

(Assume $g = 9.8 \text{ m s}^{-2}$ unless otherwise stated)

What are the missing words in the statements 1–6?

- When a weight is attached to a suspended long wire, it produces a ... strain.
- The units of Young's modulus are ...
- In measuring Young's modulus, the ... must not be exceeded.
- The energy gained by a wire when stretched = ... × extension.
- Bulk stress is defined as the ... change.
- When a wire is twisted, a ... strain is produced.

Which of the following answers, A, B, C, D or E, do you consider is the correct one in the statements 9–12?

- If a metal bar, coefficient of linear expansion α , Young's modulus E , area of cross-section A and length l , is heated through $t^\circ\text{C}$ when clamped at both ends, the force in the bar is calculated from A $EAlt$, B EAt/α , C $EA\alpha t$, D $E^2 A\alpha l$, E Atl .
- When a spiral spring is stretched by a weight attached to it, the strain is A tensile, B shear, C bulk, D elastic, E plastic.
- The energy in a stretched wire is A $\frac{1}{2}$ load × extension, B load × extension, C stress × extension, D load × strain, E $\frac{1}{2}$ load × strain.
- In an experiment to measure Young's modulus, the wire is thin and long so that A very heavy weights can be attached, B the wire can be suspended from the ceiling, C another identical wire can be arranged parallel to it, D the stress is large and the extension is measurable for laboratory loads, E a micrometer gauge can be used for accurate measurement.

11. Define *tensile stress*, *tensile strain*, *Young's modulus*. What are the units and dimensions of each?

A load of 2 kgf is applied to the ends of a wire 4 m long, and produces an extension of 0·24 mm. If the diameter of the wire is 2 mm, calculate the stress on the wire, its strain, and the value of Young's modulus.

12. What load in kilogrammes must be applied to a steel wire 6 m long and diameter 1·6 mm to produce an extension of 1 mm? (Young's modulus for steel = $2 \times 10^{11} \text{ N m}^{-2}$.)

13. Find the extension produced in a copper wire of length 2 m and diameter 3 mm when a load of 3 kgf is applied. (Young's modulus for copper = $1 \cdot 1 \times 10^{11} \text{ N m}^{-2}$.)

14. What is meant by (i) elastic limit, (ii) Hooke's law, (iii) yield point, (iv) perfectly elastic? Draw sketches of stress *v.* strain to illustrate your answers.

15. 'In an experiment to determine Young's modulus, the strain should not exceed 1 in 1000.' Explain why this limitation is necessary and describe an experiment to determine Young's modulus for the material of a metal wire.

In such an experiment, a brass wire of diameter 0·0950 cm is used. If Young's modulus for brass is $9 \cdot 86 \times 10^{10} \text{ newton m}^{-2}$, find in kilogram force the greatest permissible load. (*L.*)

16. Define *stress* and *strain*, and explain why these quantities are useful in studying the elastic behaviour of a material.

State one advantage and one disadvantage in using a long wire rather than a short stout bar when measuring Young's modulus by direct stretching.

Calculate the minimum tension with which platinum wire of diameter 0·1 mm must be mounted between two points in a stout invar frame if the wire is to remain taut when the temperature rises 100K. Platinum has coefficient of linear expansion $9 \times 10^{-6} \text{ K}^{-1}$ and Young's modulus $17 \times 10^{10} \text{ N m}^{-2}$. The thermal expansion of invar may be neglected. (*O. & C.*)

17. Explain the terms *stress*, *strain*, *modulus of elasticity* and *elastic limit*. Derive an expression in terms of the tensile force and extension for the energy stored in a stretched rubber cord which obeys Hooke's law.

The rubber cord of a catapult has a cross-sectional area $1 \cdot 0 \text{ mm}^2$ and a total unstretched length 10·0 cm. It is stretched to 12·0 cm and then released to project a missile of mass 5·0 g. From energy considerations, or otherwise, calculate the velocity of projection, taking Young's modulus for the rubber as $5 \cdot 0 \times 10^8 \text{ N m}^{-2}$. State the assumptions made in your calculation.

18. State Hooke's law, and describe in detail how it may be verified experimentally for copper wire. A copper wire, 200 cm long and 1·22 mm diameter, is fixed horizontally to two rigid supports 200 cm long. Find the mass in grams of the load which, when suspended at the mid-point of the wire, produces a sag of 2 cm at that point. Young's modulus for copper = $12 \cdot 3 \times 10^{10} \text{ N m}^{-2}$. (*L.*)

19. Distinguish between Young's modulus, the bulk modulus and the shear modulus of a material. Describe a method for measuring Young's modulus. Discuss the probable sources of error and assess the magnitude of the contribution from each.

A piece of copper wire has twice the radius of a piece of steel wire. Young's modulus for steel is twice that for the copper. One end of the copper wire is joined to one end of the steel wire so that both can be subjected to the same longitudinal force. By what fraction of its length will the steel have stretched when the length of the copper has increased by 1%? (*O. & C.*)

20. In an experiment to measure Young's modulus for steel a wire is suspended vertically and loaded at the free end. In such an experiment, (a) why is the wire long and thin, (b) why is a second steel wire suspended adjacent to the first?

Sketch the graph you would expect to obtain in such an experiment showing the relation between the applied load and the extension of the wire. Show how it is possible to use the graph to determine (a) Young's modulus for the wire, (b) the work done in stretching the wire.

If Young's modulus for steel is $2.00 \times 10^{11} \text{ N m}^{-2}$, calculate the work done in stretching a steel wire 100 cm in length and of cross-sectional area 0.030 cm^2 when a load of 10 kgf is slowly applied without the elastic limit being reached. (N.)

21. Describe the changes which take place when a wire is subjected to a steadily increasing tension. Include in your description a sketch graph of tension against extension for (a) a ductile material such as drawn copper and (b) a brittle one such as cast iron.

Show that the energy stored in a rod of length L when it is extended by a length l is $\frac{1}{2}El^2/L^2$ per unit volume where E is Young's modulus of the material.

A railway track uses long welded steel rails which are prevented from expanding by friction in the clamps. If the cross-sectional area of each rail is 75 cm^2 what is the elastic energy stored per kilometre of track when its temperature is raised by 10°C ? (Coefficient of thermal expansion of steel = $1.2 \times 10^{-5} \text{ K}^{-1}$; Young's modulus for steel = $2 \times 10^{11} \text{ N m}^{-2}$.) (O. & C.)

22. What is meant by saying that a substance is 'elastic'?

A vertical brass rod of circular section is loaded by placing a 5 kg weight on top of it. If its length is 50 cm, its radius of cross-section 1 cm, and the Young's modulus of the material $3.5 \times 10^{10} \text{ N m}^{-2}$, find (a) the contraction of the rod, (b) the energy stored in it. (C.)

23. Give a short account of what happens when a copper wire is stretched under a gradually increasing load. What is meant by *modulus of elasticity*, *elastic limit*, *perfectly elastic*?

When a rubber cord is stretched the change in volume is very small compared with the change in shape. What will be the numerical value of Poisson's ratio for rubber, i.e., the ratio of the fractional decrease in diameter of the stretched cord to its fractional increase in length? (L.)

24. Describe an accurate method of determining Young's modulus for a wire. Detail the precautions necessary to avoid error, and estimate the accuracy attainable.

A steel tyre is heated and slipped on to a wheel of radius 40 cm which it fits exactly at a temperature $t^\circ\text{C}$. What is the maximum value of t if the tyre is not to be stretched beyond its elastic limit when it has cooled to air temperature (17°C)? What will then be the tension in the tyre, assuming it to be 4 cm wide and 3 mm thick? The value of Young's modulus for steel is $1.96 \times 10^{11} \text{ N m}^{-2}$, its coefficient of linear expansion is $1.1 \times 10^{-5} \text{ K}^{-1}$, and its elastic limit occurs for a tension of $2.75 \times 10^8 \text{ N m}^{-2}$. The wheel may be assumed to be at air temperature throughout, and to be incompressible. (O. & C.)

25. State Hooke's law and describe, with the help of a rough graph, the behaviour of a copper wire which hangs vertically and is loaded with a gradually increasing load until it finally breaks. Describe the effect of gradually reducing the load to zero (a) before, (b) after the elastic limit has been reached.

A uniform steel wire of density 7800 kg m^{-3} weighs 16 g and is 250 cm long. It lengthens by 1.2 mm when stretched by a force of 8 kgf. Calculate (a) the value of Young's modulus for the steel, (b) the energy stored in the wire. (N.)

26. Describe an experimental method for the determination of (a) Young's modulus, (b) the elastic limit, of a metal in the form of a thin wire.

A steel rod of mass 97.5 g and of length 50 cm is heated to 200°C and its ends securely clamped. Calculate the tension in the rod when its temperature is reduced to 0°C , explaining how the calculation is made. (Young's modulus for steel = $2.0 \times 10^{11} \text{ N m}^{-2}$; linear expansivity = $1.1 \times 10^{-5} \text{ K}^{-1}$; density of steel = 7800 kg m^{-3} .) (L.)

27. What do you understand by Hooke's law of elasticity? Describe how you would verify it in any particular case.

A wire of radius 0.2 mm is extended by 0.1% of its length when it supports a load of 1 kg; calculate Young's modulus for the material of the wire. (L.)

28. Define Young's modulus of elasticity. Describe an accurate method of determining it. The rubber cord of a catapult is pulled back until its original length has been doubled. Assuming that the cross-section of the cord is 2 mm square, and that Young's modulus for rubber is 10^7 N m^{-2} calculate the tension in the cord. If the two arms of the catapult are 6 cm apart, and the unstretched length of the cord is 8 cm what is the stretching force? (O. & C.)

29. Define *Young's modulus of elasticity* and *coefficient of linear expansion*. State units in which each may be expressed and describe an experimental determination of Young's modulus.

For steel, Young's modulus is $1.8 \times 10^{11} \text{ N m}^{-2}$ and the coefficient of expansion $1.1 \times 10^{-5} \text{ K}^{-1}$. A steel wire 1 mm in diameter is stretched between two supports when its temperature is 200°C . By how much will the force the wire exerts on the supports increase when it cools to 20°C , if they do not yield? Express the answer in terms of the weight of a kilogramme. (L.)

30. Define *elastic limit* and *Young's modulus* and describe how you would find the values for a copper wire.

What stress would cause a wire to increase in length by one-tenth of one per cent if Young's modulus for the wire is $12 \times 10^{10} \text{ N m}^{-2}$? What load in kg wt. would produce this stress if the diameter of the wire is 0.56 mm? (L.)

chapter seven

Solid Friction. Viscosity

SOLID FRICTION

Static Friction

WHEN a person walks along a road, he or she is prevented from slipping by the force of friction at the ground. In the absence of friction, for example on an icy surface, the person's shoe would slip when placed on the ground. The frictional force always *opposes* the motion of the shoe.

The frictional force between the surface of a table and a block of wood A can be investigated by attaching one end of a string to A and the other to a scale-pan S, Fig. 7.1. The string passes over a fixed

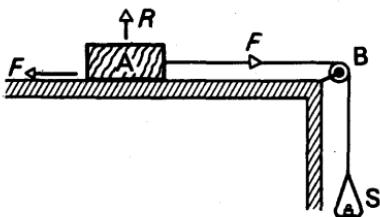


FIG. 7.1 Solid friction

grooved wheel B. When small weights are added to S, the block does not move. The frictional force between the block and table is thus equal to the total weight on S together with the weight of S. When more weights are added, A does not move, showing that the frictional force has increased, but as the weight is increased further, A suddenly begins to slip. The frictional force now present between the surfaces is called the *limiting frictional force*, and we are said to have reached *limiting friction*. The limiting frictional force is the maximum frictional force between the surfaces.

Coefficient of Static Friction

The normal reaction, R , of the table on A is equal to the weight of A. By placing various weights on A to alter the magnitude of R , we can find how the limiting frictional force F varies with R by the experiment just described. The results show that, approximately,

$$\frac{\text{limiting frictional force } (F)}{\text{normal reaction } (R)} = \mu, \text{ a constant,}$$

and μ is known as the *coefficient of static friction* between the two surfaces. The magnitude of μ depends on the nature of the two surfaces; for example, it is about 0.2 to 0.5 for wood on wood, and about 0.2 to 0.6 for wood on metals. Experiment also shows that the limiting frictional force is the same if the block A in Fig. 7.1 is turned on one side so that its surface area of contact with the table decreases, and thus the limiting frictional force is independent of the area of contact when the normal reaction is the same.

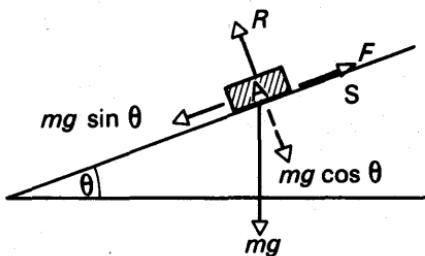


FIG. 7.2 Coefficient by inclined plane

The coefficient of static friction, μ , can also be found by placing the block A on the surface S, and then gently tilting S until A is on the point of slipping down the plane, Fig. 7.2. The static frictional force F is then equal to $mg \sin \theta$, where θ is the angle of inclination of the plane to the horizontal; the normal reaction R is equal to $mg \cos \theta$.

$$\therefore \mu = \frac{F}{R} = \frac{mg \sin \theta}{mg \cos \theta} = \tan \theta,$$

and hence μ can be found by measuring θ .

Kinetic Friction. Coefficient of Kinetic (Dynamic) Friction

When brakes are applied to a bicycle, a frictional force is exerted between the moving wheels and brake blocks. In contrast to the case of static friction, when one of the objects is just on the point of slipping, the frictional force between the moving wheel and brake blocks is called a *kinetic (or dynamic) frictional force*. Kinetic friction thus occurs between two surfaces which have relative motion.

The *coefficient of kinetic (dynamic) friction*, μ' , between two surfaces is defined by the relation

$$\mu' = \frac{F'}{R},$$

where F' is the frictional force when the object moves with a uniform velocity and R is the normal reaction between the surfaces. The coefficient of kinetic friction between a block A and a table can be found by the apparatus shown in Fig. 7.1. Weights are added to the scale-pan, and each time A is given a slight push. At one stage A continues to move with a constant velocity, and the kinetic frictional

force F' is then equal to the total weight in the scale-pan together with the latter's weight. On dividing F' by the weight of A, the coefficient can be calculated. Experiment shows that, when weights are placed on A to vary the normal reaction R , the magnitude of the ratio F'/R is approximately constant. Results also show that the coefficient of kinetic friction between two given surfaces is less than the coefficient of static friction between the same surfaces, and than the coefficient of kinetic friction between two given surfaces is approximately independent of their relative velocity.

Laws of Solid Friction

Experimental results on solid friction are summarised in the *laws of friction*, which state:

- (1) The frictional force between two surfaces opposes their relative motion.
- (2) The frictional force is independent of the area of contact of the given surfaces when the normal reaction is constant.
- (3) The limiting frictional force is proportional to the normal reaction for the case of static friction. The frictional force is proportional to the normal reaction for the case of kinetic (dynamic) friction, and is independent of the relative velocity of the surfaces.

Theory of Solid Friction

The laws of solid friction were known hundreds of years ago, but they have been explained only in comparatively recent years, mainly by F. P. Bowden and collaborators. Sensitive methods, based on electrical conductivity measurements, reveal that the true area of contact between two surfaces is extremely small, perhaps one ten-thousandth of the area actually placed together for steel surfaces. This is explained by photographs which show that some of the atoms of a metal project slightly above the surface, making a number of crests or 'humps'. As Bowden has stated: 'The finest mirror, which is flat to a millionth of a centimetre, would to anyone of atomic size look rather like the South Downs—valley and rolling hills a hundred or more atoms high.' Two metal surfaces thus rest on each others projections when placed one on the other.

Since the area of actual contact is extremely small, the pressures at the points of contact are very high, perhaps 1000 million kgf per m^2 for steel surfaces. The projections merge a little under the high pressure, producing adhesion or 'welding' at the points, and a force which opposes motion is therefore obtained. This explains Law 1 of the laws of solid friction. When one of the objects is turned over, so that a smaller or larger surface is presented to the other object, measurements show that the small area of actual contact remains constant. Thus the frictional force is independent of the area of the surfaces, which explains Law 2. When the load increases the tiny projections are further squeezed

by the enormous pressures until the new area of contact becomes big enough to support the load. The greater the load, the greater is the area of actual contact, and the frictional force is thus approximately proportional to the load, which explains Law 3.

VISCOSITY

If we move through a pool of water we experience a resistance to our motion. This shows that there is a *frictional force* in liquids. We say this is due to the **viscosity** of the liquid. If the frictional force is comparatively low, as in water, the viscosity of the liquid is low; if the frictional force is large, as in glue or glycerine, the viscosity of the liquid is high. We can compare roughly the viscosity of two liquids by filling two measuring cylinders with each of them, and allowing identical small steel ball-bearings to fall through each liquid. The sphere falls more slowly through the liquid of higher viscosity.

As we shall see later, the viscosity of a lubricating oil is one of the factors which decide whether it is suitable for use in an engine. The Ministry of Aircraft Production, for example, listed viscosity values to which lubricating oils for aero-engines must conform. The subject of viscosity has thus considerable practical importance.

Newton's Formula. Coefficient of Viscosity

When water flows slowly and steadily through a pipe, the layer A of the liquid in contact with the pipe is practically stationary, but the central part C of the water is moving relatively fast, Fig. 7.3. At other layers between A and C, such as B, the water has a velocity less than at C, the magnitude of the velocities being represented by the length of the arrowed lines in Fig. 7.3. Now as in the case of two solid surfaces

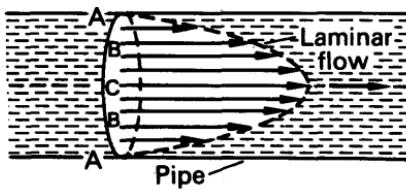


FIG. 7.3 Laminar (uniform) flow through pipe

moving over each other, a frictional force is exerted between two liquid layers when they move over each other. Thus because the velocities of neighbouring layers are different, as shown in Fig. 7.3, a frictional force occurs between the various layers of a liquid when flowing through a pipe.

The basic formula for the frictional force, F , in a liquid was first suggested by NEWTON. He saw that the larger the *area* of the surface of liquid considered, the greater was the frictional force F . He also stated that F was directly proportional to the *velocity gradient* at the part of the liquid considered. This is the case for most common liquids,

called *Newtonian liquids*. If v_1 , v_2 are the velocities of C, B respectively in Fig. 7.3, and h is their distance apart, the velocity gradient between the liquids is defined as $(v_1 - v_2)/h$. The velocity gradient can thus be expressed in $(\text{m/s})/\text{m}$, or as ' s^{-1} '.

Thus if A is the area of the liquid surface considered, the frictional force F on the surface is given by

$$F \propto A \times \text{velocity gradient},$$

or
$$F = \eta A \times \text{velocity gradient}, \quad \quad (1)$$

where η is a constant of the liquid known as the *coefficient of viscosity*. This expression for the frictional force in a liquid should be contrasted with the case of solid friction, in which the frictional force is independent of the area of contact and of the relative velocity between the solid surfaces concerned (p. 173).

Definition, Units, and Dimensions of Coefficient of Viscosity

The magnitude of η is given by

$$\eta = \frac{F}{A \times \text{velocity gradient}}.$$

The unit of F is a newton, the unit of A is m^2 , and the unit of velocity gradient is 1 m/s per m . Thus η may be defined as *the frictional force per unit area of a liquid when it is in a region of unit velocity gradient*.

The 'unit velocity gradient' = 1 m s^{-1} change per m. Since the 'm' cancels, the 'unit velocity gradient' = 1 per second. From $\eta = F/(A \times \text{velocity gradient})$, it follows that η may be expressed in units of *newton s m⁻² (N s m⁻²)*, or 'dekapoise'.

The coefficient of viscosity of water at 10°C is $1.3 \times 10^{-3} \text{ N s m}^{-2}$. Since $F = \eta A \times \text{velocity gradient}$, the frictional force on an area of 10 cm^2 in water at 10°C between two layers of water 0.1 cm apart which move with a relative velocity of 2 cm s^{-1} is found as follows:

Coefficient of viscosity $\eta = 1.3 \times 10^{-3} \text{ newton m}^{-2}$, $A = 10 \times 10^{-4} \text{ m}^2$, velocity gradient = $2 \times 10^{-2} \text{ m s}^{-1} \div 0.1 \times 10^{-2} \text{ m} = 2/0.1 \text{ s}^{-1}$.

$$\therefore F = 1.3 \times 10^{-3} \times 10 \times 10^{-4} \times 2/0.1 = 2.6 \times 10^{-5} \text{ newton.}$$

Dimensions. The dimensions of a force, F , (= mass \times acceleration = mass \times velocity change/time) are MLT^{-2} . See p. 31. The dimensions of an area, A , are L^2 . The dimensions of velocity gradient

$$= \frac{\text{velocity change}}{\text{distance}} = \frac{\text{L}}{\text{T}} \div \text{L} = \frac{1}{\text{T}}$$

Now

$$\eta = \frac{F}{A \times \text{velocity gradient}},$$

$$\begin{aligned} \therefore \text{dimensions of } \eta &= \frac{\text{MLT}^{-2}}{\text{L}^2 \times 1/\text{T}} \\ &= \text{ML}^{-1}\text{T}^{-1}. \end{aligned}$$

Thus η may be expressed in units ' $\text{kg m}^{-1} \text{s}^{-1}$ '.

Steady Flow of Liquid Through Pipe. Poiseuille's Formula

The steady flow of liquid through a pipe was first investigated thoroughly by POISEUILLE in 1844, who derived an expression for the volume of liquid issuing per second from the pipe. The proof of the formula is given on p. 208, but we can derive most of the formula by the *method of dimensions* (p. 31).

The volume of liquid issuing per second from the pipe depends on (i) the coefficient of viscosity, η , (ii) the radius, a , of the pipe, (iii), the *pressure gradient*, g , set up along the pipe. The pressure gradient = p/l , where p is the pressure difference between the ends of the pipe and l is its length. Thus x , y , z being indices which require to be found, suppose

$$\text{volume per second} = k\eta^x a^y g^z \quad \quad (1)$$

Now the dimensions of volume per second are $L^3 T^{-1}$; the dimensions of η are $ML^{-1}T^{-1}$, see p. 205; the dimension of a is L ; and the dimensions of g are

$$\frac{[\text{pressure}]}{[\text{length}]}, \text{ or } \frac{[\text{force}]}{[\text{area}][\text{length}]}, \text{ or } \frac{MLT^{-2}}{L^2 \times L}, \text{ which is } ML^{-2}T^{-2}.$$

Thus from (i), equating dimensions on both sides,

$$L^3 T^{-1} \equiv (ML^{-1}T^{-1})^x L^y (ML^{-2}T^{-2})^z.$$

Equating the respective indices of M, L, T on both sides, we have

$$\begin{aligned} x+z &= 0, \\ -x+y-2z &= 3, \\ x+2z &= 1. \end{aligned}$$

Solving, we obtain $x = -1$, $z = 1$, $y = 4$. Hence, from (1),

$$\text{volume per second} = k \frac{a^4 g}{\eta} = k \frac{pa^4}{l\eta}.$$

We cannot obtain the numerical factor k from the method of dimensions. As shown on p. 209, the factor of $\pi/8$ enters into the formula, which is:

$$\text{Volume per second} = \frac{\pi pa^4}{8\eta l} \quad \quad (2)$$

EXAMPLE

Explain as fully as you can the phenomenon of viscosity, using the viscosity of a gas as the basis of discussion. Show by the method of dimensions how the volume of liquid flowing in unit time along a uniform tube depends on the radius of the tube, the coefficient of viscosity of the liquid, and the pressure gradient along the tube.

The water supply to a certain house consists of a horizontal water main 20 cm in diameter and 5 km long to which is joined a horizontal pipe 15 mm in diameter and 10 m long leading into the house. When water is being drawn by this house

only, what fraction of the total pressure drop along the pipe appears between the ends of the narrow pipe? Assume that the rate of flow of the water is very small. (O. & C.)

$$\text{Volume per second} = \frac{\pi p a^4}{8\eta l}, \text{ with usual notation.}$$

$$\text{Thus volume per second} = \frac{\pi p_1 \cdot 0.1^4}{8\eta \cdot 5 \times 10^3} = \frac{\pi p_2 \cdot 0.0075^4}{8\eta \cdot 10}$$

where p_1, p_2 are the respective pressures in the two pipes, since the volume per second is the same.

$$\therefore \frac{p_1}{p_2} = \frac{0.0075^4}{0.1} \times \frac{5 \times 10^3}{10} = \frac{1}{63} \text{ (approx.).}$$

$$\therefore p_1 = \frac{1}{64} \times \text{total pressure} = 0.016 \times \text{total pressure.}$$

Turbulent Motion

Poiseuille's formula holds as long as the velocity of each layer of the liquid is parallel to the axis of the pipe and the flow pattern has been developed. As the pressure difference between the ends of the pipe is increased, a critical velocity is reached at some stage, and the motion of the liquid changes from an orderly to a *turbulent* one. Poiseuille's formula does not apply to turbulent motion.

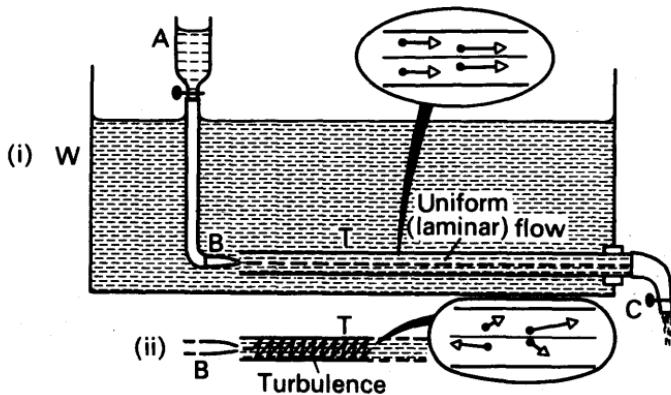


FIG. 7.4 Laminar and turbulent flow

The onset of turbulence was first demonstrated by O. REYNOLDS in 1883, and was shown by placing a horizontal tube T, about 0.5 cm in diameter, at the bottom of a tank W of water, Fig. 7.4 (i). The flow of water along T is controlled by a clip C on rubber tubing connected to T. A drawn-out glass jet B, attached to a reservoir A containing coloured water, is placed at one end of T, and at low velocities of flow a thin coloured stream of water is observed flowing along the middle of T. As the rate of flow of the water along T is increased, a stage is reached

when the colouring in T begins to spread out and fill the whole of the tube, Fig. 7.4 (ii). The critical velocity has now been exceeded, and turbulence has begun.

Fig. 7.4 shows diagrammatically in inset: (i) laminar or uniform flow—here particles of liquid at the same distance from the axis always have equal velocities directed parallel to the axis, (ii) turbulence—here particles at the same distance from the axis have different velocities, and these vary in magnitude and direction with time.

Analogy with Ohm's Law

For orderly flow along a pipe, Poiseuille's formula in equation (2) states :

$$\begin{aligned}\text{Volume per second flowing} &= \frac{\pi p a^4}{8\eta l}, \\ &= \frac{p \times \pi a^2}{8\pi\eta \times \frac{l}{\pi a^2}}.\end{aligned}$$

Now $p \times \pi a^2 = \text{excess pressure} \times \text{area of cross-section of liquid} = \text{excess force } F \text{ on liquid}$, and $\frac{l}{\pi a^2} = l/A$, where A is the area of cross-section.

$$\therefore \text{volume per second flowing} = \frac{F}{8\pi\eta \times \frac{l}{A}}. \quad (i)$$

The volume of liquid per second is analogous to electric current (I) if we compare the case of electricity flowing along a conductor, and the excess force F is analogous to the potential difference (V) along the conductor. Also, the resistance R of the conductor $= \rho l/A$, where ρ is its resistivity, l is its length, and A is the cross-sectional area. Since, from Ohm's law, $I = V/R$, it follows from (i) that

$$8\pi\eta \text{ is analogous to } \rho, \text{ the resistivity};$$

that is, the coefficient of viscosity η is a measure of the 'resistivity' of a liquid in orderly flow.

Proof of Poiseuille's Formula. Suppose a pipe of radius a has a liquid flowing steadily along it. Consider a cylinder of the liquid of radius r having the same axis as the pipe, where r is less than a . Then the force on this cylinder due to the excess pressure $p = p \times \pi r^2$. We can imagine the cylinder to be made up of cylindrical shells; the force on the cylinder due to viscosity is the algebraic sum of the viscous forces on these shells. The force on one shell is given by $\eta A dv/dr$, where dv/dr is the corresponding velocity gradient and A is the surface area of the shell. And although dv/dr changes as we proceed from the narrowest shell outwards, the forces on the neighbouring shells cancel each other out, by the law of action and reaction, leaving a net force of $\eta A dv/dr$, where dv/dr is the velocity gradient at the surface of the cylinder. The viscous force on the cylinder, and the force on it due to the excess pressure p , are together zero since there is no acceleration of the liquid, i.e., we have orderly or laminar flow.

$$\begin{aligned}\therefore \eta A \frac{dv}{dr} + \pi r^2 p &= 0. \\ \therefore \eta \cdot 2\pi r l \frac{dv}{dr} + \pi r^2 p &= 0, \text{ since } A = 2\pi r l. \\ \therefore \frac{dv}{dr} &= -\frac{pr}{2\eta l} \\ \therefore v &= -\frac{p}{4\eta l} r^2 + c,\end{aligned}$$

where c is a constant. Since $v = 0$ when $r = a$, at the surface of the tube, $c = pa^2/4\eta l$.

$$\therefore v = \frac{p}{4\eta l} (a^2 - r^2) \quad \quad (i)$$

Consider a cylindrical shell of the liquid between radii r and $(r + \delta r)$. The liquid in this shell has a velocity v given by the expression in (i), and the volume per second of liquid flowing along this shell $= v \times$ cross-sectional area of shell, since v is the distance travelled in one second, $= v \times 2\pi r \cdot \delta r$.

$$\begin{aligned}\therefore \text{total volume of liquid per second along tube} &= \int_0^a v \cdot 2\pi r \cdot dr \\ &= \int_0^a \frac{p}{4\eta l} (a^2 - r^2) \cdot 2\pi r \cdot dr \\ &= \frac{\pi p a^4}{8\eta l}.\end{aligned}$$

Determination of Viscosity by Poiseuille's Formula

The viscosity of a liquid such as water can be measured by connecting one end of a capillary tube T to a constant pressure apparatus A, which provides a steady flow of liquid, Fig. 7.5. By means of a beaker B

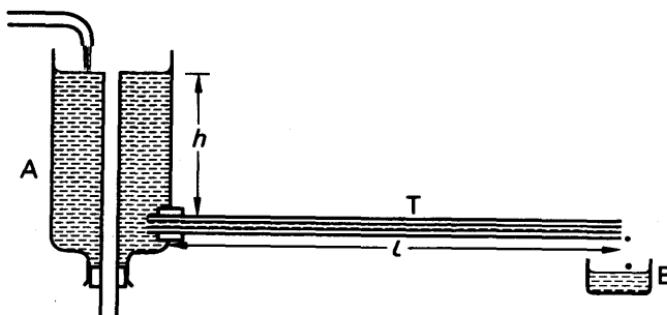


FIG. 7.5 Absolute measurement of viscosity

and a stop-clock, the volume of water per second flowing through the tube can be measured. The pressure difference between the ends of T is $hp\rho g$, where h is the pressure head, ρ is the density of the liquid, and g is 9.8 m s^{-2} .

$$\therefore \text{volume per second} = \frac{\pi p a^4}{8\eta l} = \frac{\pi h p \rho a^4}{8\eta l},$$

where l is the length of T and a is its radius. The radius of the tube can be measured by means of a mercury thread or by a microscope. The coefficient of viscosity η can then be calculated, since all the other quantities in the above equation are known.

Comparison of Viscosities. Ostwald Viscometer

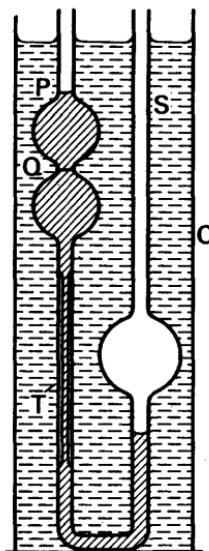


FIG. 7.6
Ostwald viscometer

An Ostwald viscometer, which contains a vertical capillary tube T , is widely used for comparing the viscosities of two liquids, Fig. 7.6. The liquid is introduced at S , drawn by suction above P , and the time t_1 taken for the liquid level to fall between the fixed marks P , Q is observed. The experiment is then repeated with the *same volume* of a second liquid, and the time t_2 for the liquid level to fall from P to Q is noted.

Suppose the liquids have respective densities ρ_1 , ρ_2 . Then, since the average head h of liquid forcing it through T is the same in each case, the pressure excess between the ends of T = $h\rho_1g$, $h\rho_2g$ respectively. If the volume between the marks P , Q is V , then, from Poiseuille's formula, we have

$$\frac{V}{t_1} = \frac{\pi(h\rho_1g)a^4}{8\eta_1 l} \quad (i)$$

where a is the radius of T , η_1 is the coefficient of viscosity of the liquid, and l is the length of T . Similarly, for the second liquid,

$$\frac{V}{t_2} = \frac{\pi(h\rho_2g)a^4}{8\eta_2 l} \quad (ii)$$

Dividing (ii) by (i),

$$\begin{aligned} \therefore \frac{t_1}{t_2} &= \frac{\eta_1 \rho_2}{\eta_2 \rho_1} \\ \therefore \frac{\eta_1}{\eta_2} &= \frac{t_1}{t_2} \cdot \frac{\rho_1}{\rho_2} \end{aligned} \quad (iii)$$

Thus knowing t_1 , t_2 and the densities ρ_1 , ρ_2 , the coefficients of viscosity can be compared. Further, if a pure liquid of a known viscosity is used, the viscometer can be used to measure the coefficient of viscosity of a liquid. Since the viscosity varies with temperature, the viscometer should be used in a cylinder C and surrounded by water at a constant temperature, Fig. 7.6. The arrangement can then also be used to investigate the variation of viscosity with temperature. In very accurate work a small correction is required in equation (iii). BARR, an authority on viscosity, estimates that nearly 90% of petroleum oil is tested by an Ostwald viscometer.

Experiment shows that the viscosity coefficient of a liquid diminishes as its temperature rises. Thus for water, η at 15°C is $1.1 \times 10^{-3} \text{ N s m}^{-2}$, at 30°C it is $0.8 \times 10^{-3} \text{ N s m}^{-2}$ and at 50°C it is $0.6 \times 10^{-3} \text{ N s m}^{-2}$. Lubricating oils for motor engines which have the same coefficient of viscosity in summer and winter are known as 'viscostatic' oils.

Stokes' Law. Terminal Velocity

When a small object, such as a steel ball-bearing, is dropped into a viscous liquid like glycerine it accelerates at first, but its velocity soon reaches a steady value known as the *terminal velocity*. In this case the viscous force acting upwards, and the upthrust due to the liquid on the object, are together equal to its weight acting downwards, so that the resultant force on the object is zero. An object dropped from an aeroplane at first increases its speed v , but soon reaches its terminal speed. Fig. 7.7 shows that variation of v with time as the terminal velocity v_0 is reached.

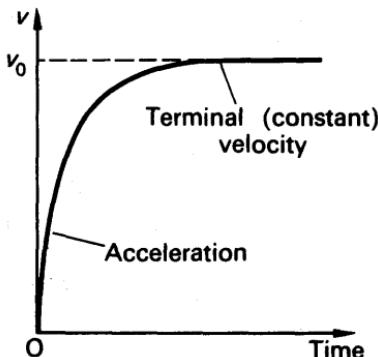


FIG. 7.7 Motion of falling sphere

Suppose a sphere of radius a is dropped into a viscous liquid of coefficient of viscosity η , and its velocity at an instant is v . The frictional force, F , can be partly found by the method of dimensions. Thus suppose $F = ka^x\eta^y v^z$, where k is a constant. The dimensions of F are MLT^{-2} ; the dimension of a is L ; the dimensions of η are $\text{ML}^{-1}\text{T}^{-1}$; and the dimensions of v are LT^{-1} .

$$\therefore \text{MLT}^{-2} \equiv \text{L}^x \times (\text{ML}^{-1}\text{T}^{-1})^y \times (\text{LT}^{-1})^z.$$

Equating indices of M, L, T on both sides,

$$\therefore y = 1,$$

$$x - y + z = 1,$$

$$-y - z = -2.$$

Hence $z = 1$, $x = 1$, $y = 1$. Consequently $F = k\eta av$. In 1850 STOKES showed mathematically that the constant k was 6π , and he arrived at the formula

$$F = 6\pi a\eta v \quad \quad (1)$$

Comparison of Viscosities of Viscous Liquids

Stokes' formula can be used to compare the coefficients of viscosity

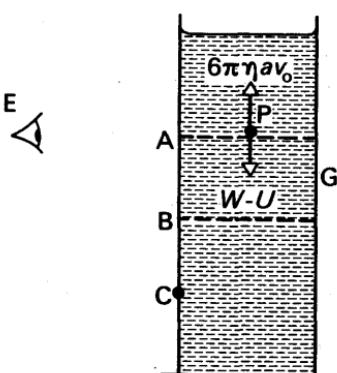


FIG. 7.8 Stokes' law

of very viscous liquids such as glycerine or treacle. A tall glass vessel G is filled with the liquid, and a small ball-bearing P is dropped gently into the liquid so that it falls along the axis of G, Fig. 7.8. Towards the middle of the liquid P reaches its terminal velocity v_0 , which is measured by timing its fall through a distance AB or BC.

The upthrust, U , on P due to the liquid = $4\pi a^3 \sigma g/3$, where a is the radius of P and σ is the density of the liquid. The weight, W , of P is $4\pi a^3 \rho g/3$, where ρ is density of the bearing's material. The net downward force is

thus $4\pi a^3 g(\rho - \sigma)/3$. When the opposing frictional force grows to this magnitude, the resultant force on the bearing is zero. Thus for the terminal velocity v_0 , we have

$$\begin{aligned} 6\pi\eta av_0 &= \frac{4}{3}\pi a^3 g(\rho - \sigma), \\ \therefore \eta &= \frac{2ga^2(\rho - \sigma)}{9v_0} \end{aligned} \quad \quad (i)$$

When the experiment is repeated with a liquid of coefficient of viscosity η_1 and density σ_1 , using the same ball-bearing, then

$$\eta_1 = \frac{2ga^2(\rho - \sigma_1)}{9v_1} \quad \quad (ii)$$

where v_1 is the new terminal velocity. Dividing (i) by (ii),

$$\therefore \frac{\eta}{\eta_1} = \frac{v_1(\rho - \sigma)}{v_0(\rho - \sigma_1)} \quad \quad (iii)$$

Thus knowing v_1 , v , ρ , σ_1 , σ , the coefficients of viscosity can be compared. In very accurate work a correction to (iii) is required for the effect of the walls of the vessel containing the liquid.

Molecular theory of viscosity

Viscous forces are detected in gases as well as in liquids. Thus if a disc is spun round in a gas close to a suspended stationary disc, the latter rotates in the same direction. The gas hence transmits frictional forces. The flow of gas through pipes, particularly in long pipes as in transmission of natural gas from the North Sea area, is affected by the viscosity of the gas.

The viscosity of gases is explained by the transfer of momentum which

takes place between neighbouring layers of the gas as it flows in a particular direction. Fast-moving molecules in a layer X cross with their own velocity to a layer Y say where molecules are moving with a slower velocity. Fig. 7.9. Molecules in Y likewise move to X. The net

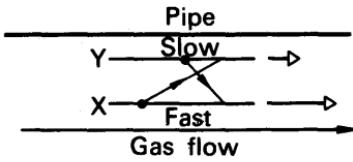


FIG. 7.9 Viscosity of gas-momentum effect

effect is an increase in momentum in Y and a corresponding decrease in X, although on the average the total number of molecules in the two layers is unchanged. Thus the layer Y speeds up and the layer X slows down, that is, a *force* acts on the layers of the gas while they move. This is the viscous force. We consider the movement of molecules in more detail shortly.

Although there is transfer of momentum as in the gas, the viscosity of a *liquid* is mainly due to the molecular attraction between molecules in neighbouring layers. Energy is needed to drag one layer over the other against the force of attraction. Thus a shear stress is required to make the liquid move in laminar flow.

EXERCISE 7

What are the missing words in the statements 1–6?

1. The coefficient of dynamic (kinetic) friction is the ratio ...
2. The coefficient of friction between two given surfaces is ... of the area in contact.
3. In orderly or laminar flow of liquids in a pipe, the volume per second flowing past any section is given by the formula ...
4. The dimensions of coefficient of viscosity are ...
5. When a small sphere of radius a falls through a liquid with a constant velocity v , the frictional force is given by the formula ...
6. In comparing the viscosities of water and alcohol by an Ostwald viscometer, the same liquid ... must be used.

Which of the following answers, A, B, C, D or E, do you consider is the correct one in the statements 7–10?

7. In orderly or laminar flow of a liquid through a pipe, A tensile forces act on the layers and the volume per second V is proportional to the pressure at one end, B shear forces act on the layers and V is proportional to the pressure at one end, C shear forces act on the layers and V is proportional to the pressure difference between the ends, D bulk forces act throughout the liquid, E V is directly proportional to a^4 and to the coefficient of viscosity.

8. When a small steel sphere is dropped gently down the axis of a wide jar of glycerine, the sphere *A* travels with constant velocity throughout its motion, *B* accelerates at first and then reaches a constant velocity, *C* decelerates at first and then reaches a constant velocity, *D* accelerates throughout its motion, *E* slowly comes to rest.

9. When a gas flows steadily along a pipe, the viscous forces in it are due to *A* transfer of energy from one layer to another, *B* the uniform speed of the molecules, *C* the varying density along the pipe, *D* the transfer of momentum from one layer to another, *E* the varying pressure at a given section of the pipe.

10. A pipe P has twice the diameter of a pipe Q, and P has a liquid X flowing along it which has twice the viscosity of a liquid Y flowing through Q. If the flow is orderly or laminar in each, and the volume per second in P and Q is the same, the pressure difference at the ends of P compared to that of Q is *A* 1:8, *B* 1:4, *C* 8:1, *D* 4:1, *E* 1:1.

Solid Friction

11. State the laws of solid friction. Describe an experiment to determine the coefficient of dynamic (or sliding) friction between two surfaces.

A horizontal circular turntable rotates about its centre at the uniform rate of 120 revolutions per minute. Find the greatest distance from the centre at which a small body will remain stationary relative to the turntable, if the coefficient of static friction between the turntable and the body is 0.80. (*L*.)

12. State (a) the laws of solid friction, (b) the triangle law for forces in equilibrium. Describe an experiment to determine the coefficient of sliding (dynamic) friction between two wooden surfaces.

A block of wood of mass 150 g rests on an inclined plane. If the coefficient of static friction between the surfaces in contact is 0.30, find (a) the greatest angle to which the plane may be tilted without the block slipping, (b) the force parallel to the plane necessary to prevent slipping when the angle of the plane with the horizontal is 30° , showing that this direction of the force is the one for which the force required to prevent slipping is a minimum. (*L*.)

13. Distinguish between *static* and *sliding* (kinetic) friction and define the coefficient of sliding friction.

How would you investigate the laws of sliding friction between wood and iron?

An iron block, of mass 10 kg, rests on a wooden plane inclined at 30° to the horizontal. It is found that the least force parallel to the plane which causes the block to slide up the plane is 10 kgf. Calculate the coefficient of sliding friction between wood and iron. (*N*.)

14. Give an account of the factors which determine the force of friction (i) between solids, (ii) in liquids.

A block of mass 12 kg is drawn along a horizontal surface by a steadily applied force of 4 kg weight acting in the direction of motion. Find the kinetic energy acquired by the block at the end of 10 seconds and compare it with the total work done on the block in the same time. (Coefficient of friction = 0.28). (*L*.)

15. State the laws of solid friction.

Describe experiments to verify these laws, and to determine the coefficient of static friction, for two wooden surfaces.

A small coin is placed on a gramophone turntable at a distance of 7.0 cm from the axis of rotation. When the rate of rotation is gradually increased from zero

the coin begins to slide outwards when the rate reaches 60 revolutions per minute. Calculate the rate of rotation for which sliding would commence if (a) the coin were placed 12·0 cm from the axis, (b) the coin were placed in the original position with another similar coin stuck on top of it. (L.)

16. Define *coefficient of sliding friction*, *coefficient of viscosity*. Contrast the laws of solid friction with those which govern the flow of liquids through tubes.

Sketch the apparatus you would employ to determine the coefficient of sliding friction between a wood block and a board and show how you would deduce the coefficient from a suitable graph. (L.)

Viscosity

17. Define *coefficient of viscosity* of a fluid.

When the flow is orderly the volume V of liquid which flows in time t through a tube of radius r and length l when a pressure difference p is maintained between its ends is given by the equation $\frac{V}{t} = \frac{\pi p r^4}{8 l \eta}$ where η is the coefficient of viscosity

of the liquid. Describe an experiment based on this equation either (a) to determine the value of η for a liquid, or (b) to compare the values of η for two liquids, pointing out the precautions which must be taken in the experiment chosen to obtain an accurate result.

Water flows steadily through a horizontal tube which consists of two parts joined end to end; one part is 21 cm long and has a diameter of 0·225 cm and the other is 7·0 cm long and has a diameter of 0·075 cm. If the pressure difference between the ends of the tube is 14 cm of water find the pressure difference between the ends of each part. (L.)

18. The dimensions of *energy*, and also those of *moment of a force* are found to be 1 in *mass*, 2 in *length* and -2 in *time*. Explain and justify this statement.

(a) A sphere of radius a moving through a fluid of density ρ with *high* velocity V experiences a retarding force F given by $F = k \cdot a^x \cdot \rho^y \cdot V^z$, where k is a non-dimensional coefficient. Use the method of dimensions to find the values of x , y and z .

(b) A sphere of radius 2 cm and mass 100 g, falling vertically through air of density $1\cdot2 \text{ kg m}^{-3}$, at a place where the acceleration due to gravity is $9\cdot81 \text{ m s}^{-2}$, attains a steady velocity of 30 m s^{-1} . Explain why a constant velocity is reached and use the data to find the value of k in this case. (O. & C.)

19. Mass, length and time are *fundamental* units, whereas acceleration, force and energy are *derived* units. Explain the distinction between these two types of unit. Define each of the three derived units and apply your definition in each case to deduce its dimensions.

An incompressible fluid of viscosity η flows along a straight tube of length l and uniform circular cross-section of radius r . Provided the pressure difference p between the ends of the tube is not too great the velocity u of fluid flow along the axis of the tube is found to be directly proportional to p . Apply the method of dimensions to deduce this result assuming u depends only on r , l , η and p .

How may the viscosity of an ideal gas be accounted for by elementary molecular theory? (O. & C.)

20. Define *coefficient of viscosity*. Describe an experiment to compare the coefficients of viscosity of water and benzene at room temperature.

A small metal sphere is released from rest in a tall wide vessel of liquid. Discuss the forces acting on the sphere (a) at the moment of release, (b) soon after release, (c) after the terminal velocity has been attained.

Castor oil at 20°C has a coefficient of viscosity 2.42 N s m^{-2} and a density 940 kg m^{-3} . Calculate the terminal velocity of a steel ball of radius 2.0 mm falling under gravity in the oil, taking the density of steel as 7800 kg m^{-3} . (L.)

21. Define *coefficient of viscosity*. What are its dimensions?

By the method of dimensions, deduce how the rate of flow of a viscous liquid through a narrow tube depends upon the viscosity, the radius of the tube, and the pressure difference per unit length. Explain how you would use your results to compare the coefficients of viscosity of alcohol and water. (C.)

2 . Define *coefficient of viscosity*. For orderly flow of a given liquid through a capillary tube of length l , radius r , the volume of liquid issuing per second is proportional to pr^4/l where p is the pressure difference between the ends of the tube. How would you verify this relation experimentally for water at room temperature? How would you detect the onset of turbulence? (N.)

23. The viscous force acting on a small sphere of radius a moving slowly through a liquid of viscosity η with velocity v is given by the expression $6\pi\eta av$. Sketch the general shape of the velocity-time graph for a particle falling from rest through a viscous fluid, and explain the form of the graph. List the observations you would make to determine the coefficient of viscosity of the fluid from the motion of the particle.

Some particles of sand are sprinkled on to the surface of the water in a beaker filled to a depth of 10 cm. Estimate the least time for which grains of diameter 0.10 mm remain in suspension in the water, stating any assumptions made.

[Viscosity of water = $1.1 \times 10^{-3} \text{ N s m}^{-2}$; density of sand = 2200 kg m^{-3} .] (C.)

24. Define *coefficient of friction* and *coefficient of viscosity*.

Describe how you would (a) measure the coefficient of sliding friction between iron and wood, and (b) compare the viscosities of water and paraffin oil. (L.)

PART TWO

Heat



chapter eight

Introduction

Temperature

WE are interested in heat because it is the commonest form of energy, and because changes of temperature have great effects on our personal comfort, and on the properties of substances, such as water, which we use every day. *Temperature* is a scientific quantity which corresponds to primary sensations—hotness and coldness. These sensations are not reliable enough for scientific work, because they depend on

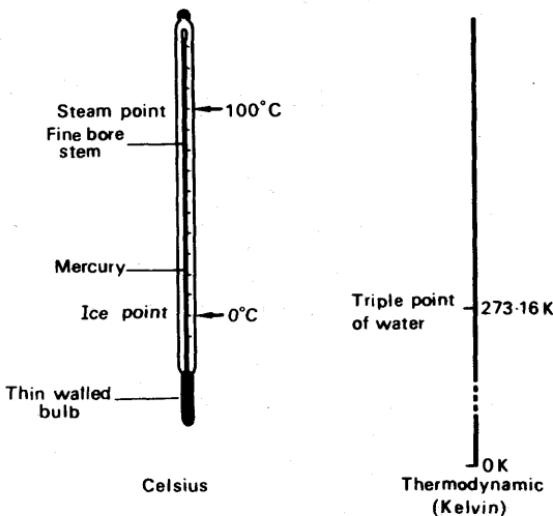


FIG. 8.1. Mercury-in-glass thermometer (left); °C and K scales.

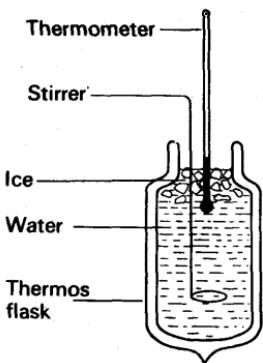
contrast—the air in a thick-walled barn or church feels cool on a summer's day, but warm on a winter's day, although a thermometer may show that it has a lower temperature in the winter. A thermometer, such as the familiar mercury-in-glass instrument (Fig. 8.1), is a device whose readings depend on hotness or coldness, and which we choose to consider more reliable than our senses. We are justified in considering it more reliable because different thermometers of the same type agree with one another better than different people do.

The temperature of a body, then, is its degree of hotness, as measured on a thermometer. The thermometer was invented in Italy about 1630: it consisted of an open-ended tube, with a bulb full of water at its lower end. The water rose in the tube when the bulb was warmed, and fell when it was cooled.

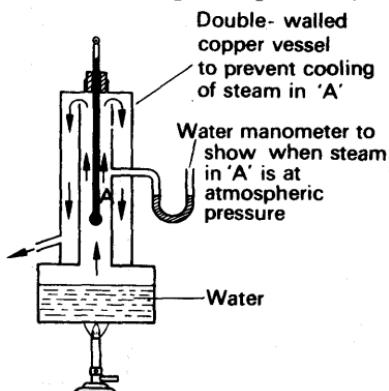
As a liquid for use in thermometers, water soon gave way to linseed oil or alcohol, and by about 1660 thermometer-makers had begun to seal the top of the tube. Early thermometers had no definite scale, like that of a modern thermometer; some of them were used for showing the temperatures of greenhouses, and were mounted on wooden backboards, which were carved with grapes and peaches for example, to indicate the correct temperatures for growing the different fruits. The thermometer as we know it to-day, containing pure mercury and graduated according to a universal scale, was developed by Fahrenheit in 1724.

Temperature Scales. Celsius

When a mercury thermometer is to be graduated, it is placed first in melting ice, and then in steam from boiling water (Fig. 8.2). The temperature of the steam depends on the atmospheric pressure, as we



(a) Ice point



(b) Steam point

FIG. 8.2. Determination of fixed points.

shall see in Chapter 12; for calibrating thermometers, an atmospheric pressure of 76 cm mercury is chosen. In both steam and ice, when the level of the mercury has become steady, it is marked on the glass: the level in ice is called the *lower fixed point*, or ice-point, and the level in steam is called the *upper fixed point*, or steam-point. The distance between the fixed points is called the *fundamental interval* of the thermometer. For scientific work, the fundamental interval is divided into 100 equal parts (Fig. 8.1). This division was first proposed by Celsius in 1742, and the graduations are called degrees Centigrade or, in modern nomenclature, degrees Celsius ($^{\circ}\text{C}$); the ice-point is 0°C and the steam-point 100°C .

Thermodynamic Scale

The *thermodynamic scale* of temperature is adopted as the SI temperature scale. On this scale, the *kelvin* is the unit of temperature. It is defined as $1/273.16$ of the *thermodynamic temperature of the triple point of water* (p. 319).

The symbol for temperature is 'K' without a degree sign. Thus the triple point of water, T_{tr} , is 273.16 K exactly. The absolute zero, 0 K,

is -273.15°C . To a good approximation, $0^{\circ}\text{C} = 273 \text{ K}$ and $100^{\circ}\text{C} = 373 \text{ K}$. (Fig. 8.1.)

The temperature change or interval of one degree Celsius, 1 degC or 1°C , is exactly the same as the temperature interval of one degree on the thermodynamic scale. On this account the interval 'degC or $^{\circ}\text{C}$ ' is written 'K' in SI units. Similarly, 'per degC or per $^{\circ}\text{C}$ ' is written ' K^{-1} ' in SI units. For example, the linear expansivity (formerly, linear coefficient of expansion) of steel is written ' $12 \times 10^{-6} \text{ K}^{-1}$ ' in SI units, in place of ' 12×10^{-6} per degC'. The use of ' K^{-1} ' occurs frequently in units throughout the subject and should be noted by the reader.

Types of Thermometer

The *mercury-in-glass* thermometer depends on the change in volume of the mercury with hotness; it is cheap and simple, but is not reliable enough for accurate work (Chapter 14). Other types of thermometer depend on the change, with hotness, of the pressure of a gas at constant volume or the electrical resistance of a metal (Fig. 8.3). Another type of

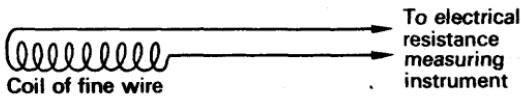


FIG. 8.3. A resistance thermometer; the wire is usually of platinum.

thermometer depends on the electromotive force change with temperature of two metals joined together. Fig. 8.4 (a) shows two wires, one of copper and one of iron, soldered together at A. The ends of the wires are joined to a galvanometer G. When the junction A is heated, a current flows which deflects the galvanometer. The current usually increases with the temperature difference between the hot and cold ends of the wires. For temperature measurement two junctions are used, as in Fig. 8.4 (b); the second one, called the cold junction, is maintained at 0°C by ice-water.

Each of these quantities—e.m.f., resistance, pressure—gives its own temperature scale, and the different scales agree only at the fixed points, where their readings are *defined* as 0°C and 100°C . (When we speak of

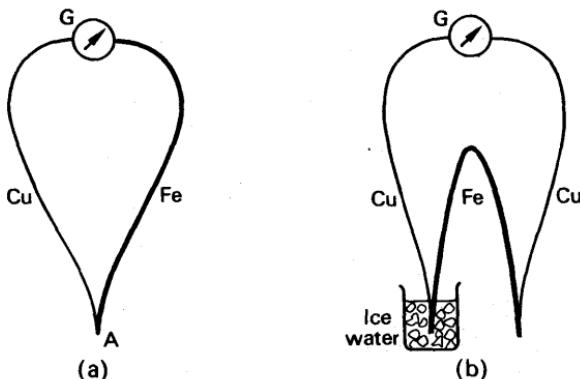


FIG. 8.4. Thermojunctions or thermocouples.

a temperature *scale*, we refer to the quantity used to define it; the difference between °C and K is only a difference in the *graduation* of a given scale.) If, for example, a given platinum wire has resistances R_0 and R_{100} at the ice- and steam-point respectively, then its fundamental interval is $R_{100} - R_0$. And if it has a resistance R at an unknown temperature, the value of that temperature, t_p , on the platinum resistance Celsius scale, is given by

$$t_p = \frac{R - R_0}{R_{100} - R_0} \times 100(\text{°C}).$$

The platinum-resistance scale differs appreciably from the mercury-in-glass scale, as the following table shows :

Mercury-in-glass	0	50	100	200	300	°C
Platinum-resistance	0	50.25	100	197	291	°C

We shall discuss temperature scales again later (p. 366); here we wish only to point out that they differ from one another, that no one of them is any more 'true' than any other, and that our choice of which to adopt is arbitrary, though it may be decided by convenience.

Effects of Temperature

Most bodies, when they are made hotter, become larger; their increase in size is called thermal expansion. Thermal expansion may be useful, as in a thermometer, or it may be a nuisance, as in bridges and railway lines. If the thermal expansion of a solid or liquid is resisted, great forces are set up: that is why gaps are left between railway lines, and why beer-bottles are never filled quite full. If the thermal expansion of a gas is resisted, however, the forces set up are not so great; the pressure of the gas increases, but not catastrophically. The increase of pressure is, in fact, made use of in most forms of engine; it is also made use of in accurate thermometry.

Besides causing a change in size or pressure, a change of temperature

may cause a change of state—from solid to liquid, liquid to gas, or vice versa. If we heat some crystals of lead acetate in a crucible, and measure their temperature, with a thermometer reading to 300°C, we find that the crystals warm steadily up to 75°C and then start to melt. Their temperature does not rise further until they have all melted (Fig. 8.5). After it has melted, the lead acetate warms up to 280°C, and

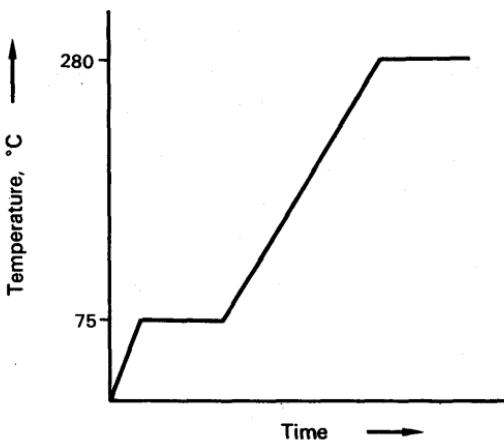


FIG. 8.5. Warming curve of lead acetate.

then keeps a constant temperature until it has all boiled away. We call 280°C the boiling-point of lead acetate; likewise we call 75°C the melting- (or freezing-) point of lead acetate.

HEAT AND ENERGY

Heat and Temperature

If we run hot water into a lukewarm bath, we make it hotter; if we run cold water in, we make it cooler. The hot water, we say, gives heat to the cooler bath-water; but the cold water takes heat from the warmer bath-water. The quantity of heat which we can get from hot water depends on both the mass of water and on its temperature: a bucket-full at 80°C will warm the bath more than a cup-full at 100°C. Roughly speaking, temperature is analogous to electrical potential, and heat is analogous to quantity of electricity. We can perceive temperature changes, and whenever the temperature of a body rises, that body has gained heat. The converse is not always true; when a body is melting or boiling, it is absorbing heat from the flame beneath it, but its temperature is not rising.

Latent Heat and Specific Heat

The heat which a body absorbs, in melting or boiling, it gives out again in freezing or condensing; such heat is called *latent*, or hidden, *heat*, because it does not show itself by a change in temperature. When a body absorbs heat without changing its state, its temperature rises, and the heat absorbed was first called 'sensible heat'.

The term 'latent heat' was used by Black (1728-99); he and a Swede, Wilcke, discovered latent heats independently at about the same date—Black by hanging a bucket of ice in a warm room, Wilcke by pouring boiling water on to snow.

Also independently, Black and Wilcke studied what we now call *specific heats*; the name is due to Wilcke. In his experiments Wilcke dropped various hot bodies into cold water, and measured the temperature rises which they caused. In this way he showed that a given mass of glass, for example, gave out only one-fifth as much heat as an equal mass of water, in cooling through the same temperature range. He therefore said that the specific heat of glass was 0.2.

In the seventeenth and eighteenth centuries the nature of heat was disputed; some thought of heat as the motion of the particles of a body, others thought of it as a fluid, filling the body's pores. Measurements of heat were all relative, and no unit of the quantity of heat was defined. In the nineteenth century, however, the increasing technical importance of heat made a unit of it essential. The units of heat chosen were:

- (i) the *calorie* (cal): this is the amount of heat required to warm 1 gramme (g) of water through 1 deg C (see also p. 194);
- (ii) the *British Thermal Unit* (Btu): this is the amount of heat required to warm 1 lb of water through 1 deg F.

Heat and Energy

Steam-engines became common in the early part of the eighteenth century; but they were not thought of as heat-engines until the latter part of that century. Consequently the early engines were wasteful of fuel, squandering useful heat in warming and cooling the cylinder at every stroke of the piston. Watt reduced this waste of heat by his invention of the separate condenser in 1769. Trevithick, about 1800, devised an engine which was driven by steam which entered the cylinder at a pressure above atmospheric, and therefore at a temperature above 100°C (p. 304). In this engine, the steam came out of the exhaust at a temperature no higher than in earlier engines, so that a greater fraction of the heat which it carried from the boiler was used in the engine.

The idea of heat as a form of energy was developed particularly by Benjamin Thompson (1753–1814); he was an American who, after adventures in Europe, became a Count of the Holy Roman Empire, and war minister of Bavaria. He is now generally known as Count Rumford. While supervising his arsenal, he noticed the great amount of heat which was liberated in the boring of cannon. The idea common at the time was that this heat was a fluid, pressed out of the chips of metal as they were bored out of the barrel. To measure the heat produced, Rumford used a blunt borer, and surrounded it and the end of the cannon with a wooden box, filled with water (Fig. 8.6). From the weight of water, and the rate at which its temperature rose, he concluded that the boring operation liberated heat at the same rate as 'nine wax candles, burning with a clear flame'. He showed that the amount of heat liberated was in no way connected with the mass of metal bored away, and concluded that it depended only on the work done against friction. It followed that **heat was a form of energy**.

Rumford published the results of his experiments in 1798. No similar experiments were made until 1840, when Joule began his study of heat and other forms of energy. Joule measured the work done, and the heat produced, when water was churned, in an apparatus which we shall describe on p. 197. He also measured the work done and heat produced when oil was churned, when air was compressed, when water was forced through fine tubes, and when cast iron bevel wheels were rotated one against the other. Always, within the limits of experimental error, he found that the heat liberated was proportional to the mechanical work done, and that the ratio of the two was the same in all types of experiment. His last experiments, made in 1878, showed that about 772 ft-lbf of work were equivalent to one British thermal unit of heat. This ratio Joule called the *mechanical equivalent of heat*. The metric unit of work or energy is the *joule*, J. Since experiment shows that heat is a form of energy, *the joule is now the scientific unit of heat*. 'Heat per second' is expressed in 'joules per second' or *watts*, W.

Today, from definition, about 4.2 J = 1 calorie (more accurately, 4.187 J = 1 calorie). Calories are units still used by chemists, for example. Values in calories met can be converted to joules approximately by multiplying them by 4.2. See p. 199.

In other experiments, Joule measured the heat liberated by an electric

current in flowing through a resistance; at the same time he measured the work done in driving the dynamo which generated the current. He obtained about the same ratio for work done to heat liberated as in his direct experiments. This work linked the ideas of heat, mechanical, and electrical energy. He also showed that the heat produced by a current is related to the chemical energy used up.

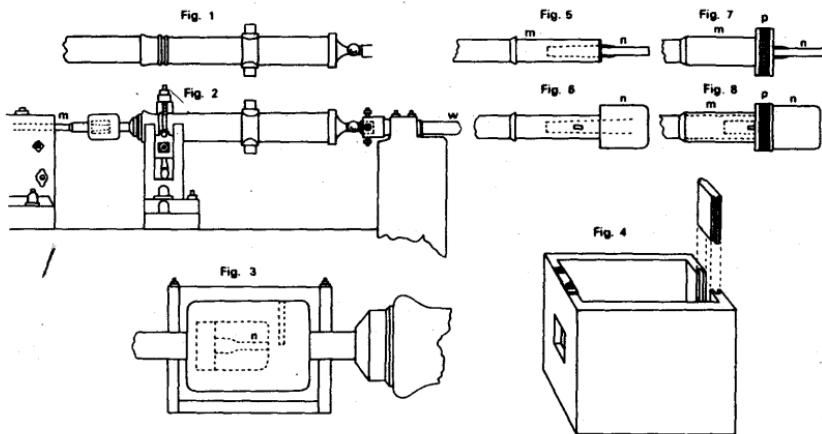


FIG. 8.6. Rumford's apparatus for converting work into heat.

Fig. 1 shows the cannon. Fig. 2 shows the complete apparatus; *w* is connected to machinery driven round by horses, *m* is joined to a blunt borer *n* in a cylinder shown enlarged in Fig. 3 and Fig. 4. Figs. 5, 6, 7, 8 show further details of *m* and *n*.

The Conservation of Energy

As a result of all his experiments, Joule developed the idea that energy in any one form could be converted into any other. There might be a loss of useful energy in the process—for example, some of the heat from the furnace of a steam-engine is lost up the chimney, and some more down the exhaust—but no energy is destroyed. The work done by the engine added to the heat lost as described and the heat developed as friction, it is equal to the heat provided by the fuel burnt. The idea underlying this statement is called the *Principle of the Conservation of Energy*. It implies that, if we start with a given amount of energy in any one form, we can convert it in turn into all other forms; we may not always be able to convert it completely, but if we keep an accurate balance-sheet we shall find that the total amount of energy, expressed in any one form—say heat or work—is always the same, and is equal to the original amount.

The conservation of energy applies to living organisms—plants and animals—as well as to inanimate systems. For example, we may put a man or a mouse into a box or a room, give him a treadmill to work, and feed him. His food is his fuel; if we burn a sample of it, we can measure its chemical energy, in heat units. And if we now add up the

heat value of the work which the man does, and the heat which his body gives off, we find that their total is equal to the chemical energy of the food which the man eats. Because food is the source of man's energy, food values are commonly expressed in *kilocalories*, which is the heat required to warm 1 kilogramme of water through 1 deg C. A man needs about 3000 kilocalories per day.

Muscles are unique in their capacity to turn chemical energy directly into mechanical energy. When a muscle is stimulated, complex phosphates in its tissues break down; in doing so, they cause the muscle fibres to swell and shorten. Thus, via the bones and joints, the muscle does external work. When the muscle is recovering after contraction, the phosphates are built up again by a series of reactions, involving the oxidation of sugars. The sugars and oxygen are brought to the muscle in the arterial blood; the waste products of the reactions, water and carbon dioxide, are carried away in the venous blood.¹ Recently physiologists have found evidence that muscles may also convert mechanical energy into chemical.² For example, when we walk downstairs, gravity does work on our leg-muscles; some of this appears as heat, but some, it now seems, is used in reversing the chemical actions of muscle activity.

All the energy by which we live comes from the sun. The sun's ultra-violet rays are absorbed in the green matter of plants, and make them grow; the animals eat the plants, and we eat them—we are all vegetarians at one remove. The plants and trees of an earlier age decayed, were buried, and turned into coal. Even water-power comes

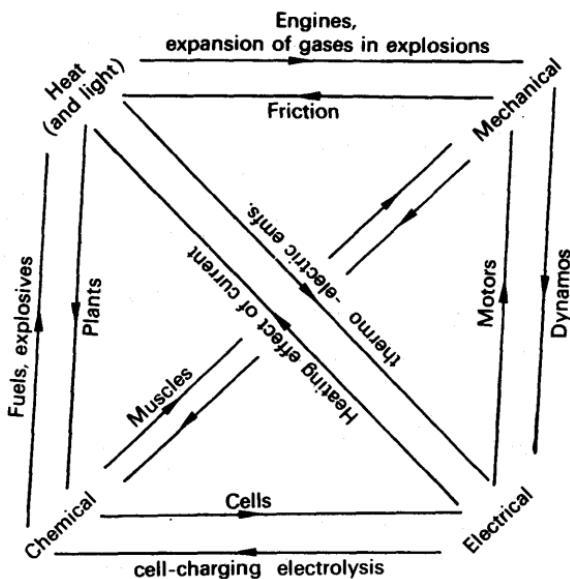


FIG. 8.7. Forms of energy, and their interconversions.

¹ DAVSON, *General Physiology*, Chap. XVII (Churchill).

² ABBOTT, AUBERT and HILL, *Jour. Physiology*, Vol. III.

from the sun—we would have no lakes if the sun did not evaporate the sea and provide the rainfall which fills the lakes. The relationship between all the principal forms of energy are summarized in Fig. 8.7.

Joule's Historic Experiments

About 1847 Joule measured the mechanical equivalent of heat by an apparatus of the form shown in Fig. 8.8. C is a copper cylinder, about 30 cm in diameter, containing water. The water is churned by paddles P, and prevented from whirling round *en masse* by baffles B. The paddles are connected by a coupling

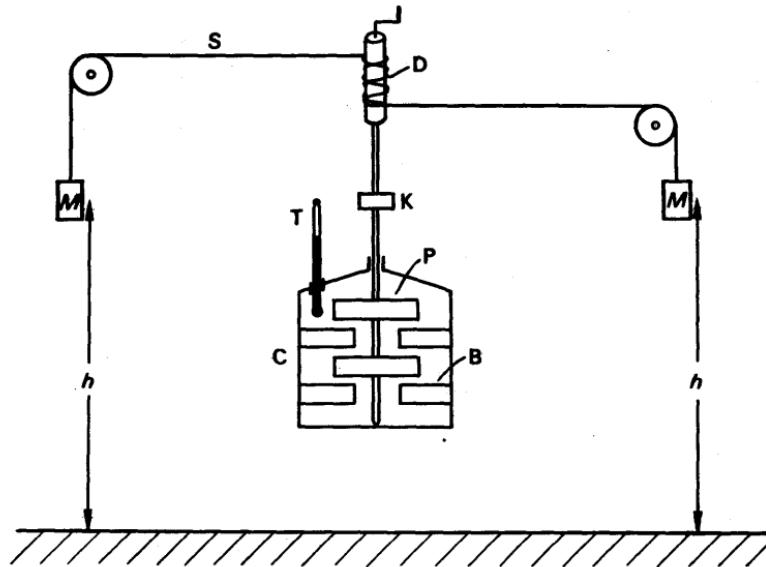


FIG. 8.8. Joule's apparatus for mechanical equivalent.

K to a drum D, which is rotated by strings S attached to lead weights M. A thermometer T shows the temperature of the water.

Joule would start an experiment by allowing the weights M to fall to the ground and turn the paddles. He would then break the coupling K, and re-wind the weights without disturbing the paddles. In this way he would make the weights fall twenty times or more, in a single experiment, and so increase the work done on the water and consequently the temperature rise.

Suppose n is the number of falls; then the work W' done on the water = $2nMgh$, where M is the mass of one weight, g is the acceleration of gravity, and h is the height of the fall. The heat Q gained by the cylinder and the water is $(mc_w + C)\theta$, where m is the mass of water, c_w the heat capacity of the cylinder and paddles, and θ is their rise in temperature. The rise θ includes the correction for heat losses. The mechanical equivalent is given by W'/Q

$$= \frac{2nMgh}{(mc_w + C)\theta} \quad (1)$$

An experiment of this kind takes a long time—about half an hour—because a great deal of work, by everyday standards, must be done to produce a measurable amount of heat. The cooling correction is therefore relatively great.

Many people refused to accept Joule's work at first, because of the very small temperature differences on which it rested. Nevertheless, Joule's final result differs only by about one part in 400 from the value given by the best modern

experiments. In calculating his final result, Joule made corrections for the kinetic energy of the weights as they struck the floor, the work done against friction in the pulleys and the bearings of the paddle wheel, and the energy stored by the stretching of the strings; he even estimated the energy in the hum which the strings emitted, but found it was negligible.

Joule's Large-scale Experiments

In his last experiments, about 1878, Joule rotated the paddles with an engine,

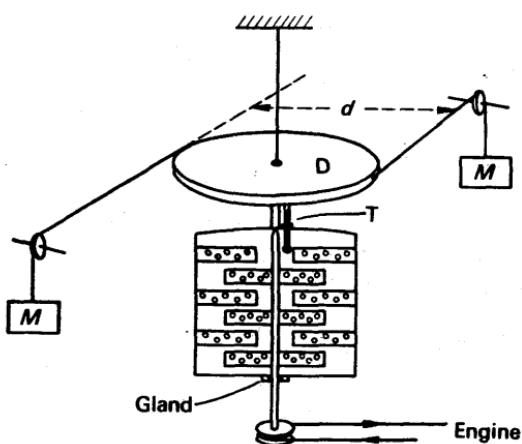


FIG. 8.9. Joule's final apparatus, also Rowland's.
is therefore

$$T = Mgd,$$

where d is the diameter of the wheel. Now the work done by a couple is equal to the product of its moment and the angle θ in radians through which it turns. Hence if the paddles make n revolutions, the work done on the water, since $\theta = 2\pi n$, is

$$\begin{aligned} W' &= 2\pi n T \\ &= 2\pi n Mgd. \end{aligned}$$

The number of revolutions was measured on a revolution counter attached to the paddle spindle. If θ is the rise in temperature measured by the thermometer T , corrected for cooling, the heat developed is

$$Q = (mc_w + C)\theta,$$

in our previous notation. Hence the mechanical equivalent

$$\frac{W'}{Q} = \frac{2\pi n Mgd}{(mc_w + C)\theta}.$$

The term 'mechanical equivalent of heat', used in the past, has no meaning nowadays because heat is measured in joules in SI units (p. 194). Experiments in which mechanical energy is converted to heat energy are now regarded as experiments which measure the specific heat capacity of the heated substance in 'joule per kilogramme (or gramme) per deg K' ($J \text{ kg}^{-1} \text{ K}^{-1}$ or $J \text{ g}^{-1} \text{ K}^{-1}$). The specific heat capacity of water may be found by the energy conversion method described on p. 206.

chapter nine

Calorimetry

Calorimetry is the measurement of heat; here we shall be concerned with the measurement of specific heat capacities and specific latent heats.

Heat (Thermal) Capacity, Specific Heat Capacity

The *heat capacity* of a body, such as a lump of metal, is the quantity of heat required to raise its temperature by 1 degree. It is expressed in *joules per deg K* (J K^{-1}).

The *specific heat capacity* of a substance is the heat required to warm unit mass of it through 1 degree; it is the heat capacity per unit mass of the substance. Specific heat capacities are expressed in *joule per kilogramme per deg K* ($\text{J kg}^{-1} \text{K}^{-1}$) or in *joule per gramme per deg K* ($\text{J g}^{-1} \text{K}^{-1}$). The specific heat of water, c_w , is about $4.2 \text{ J g}^{-1} \text{K}^{-1}$, or $4200 \text{ J kg}^{-1} \text{K}^{-1}$, or $4.2 \text{ kJ kg}^{-1} \text{K}^{-1}$, where $1 \text{ kJ} = 1 \text{ kilojoule} = 1000 \text{ J}$. Formerly, specific heat capacities were expressed in calories per g per deg C—the values in joule are about 4.2 times as great.

From the definition of specific heat capacity, it follows that

$$\text{heat capacity, } C = \text{mass} \times \text{specific heat capacity.}$$

The specific heat capacity of copper, for example, is about $0.4 \text{ J g}^{-1} \text{K}^{-1}$ or $400 \text{ J kg}^{-1} \text{K}^{-1}$. Hence the heat capacity of 5 kg of copper = $5 \times 400 = 2000 \text{ J K}^{-1} = 2 \text{ kJ K}^{-1}$.

SPECIFIC HEAT CAPACITIES

Substance	Sp. Ht.: $\text{J kg}^{-1} \text{K}^{-1}$	Substance	Sp. Ht.: $\text{J kg}^{-1} \text{K}^{-1}$
Aluminium . .	0.91×10^3	Ice . .	2.1×10^3
Brass . .	0.38	Paraffin wax . .	2.9
Copper . .	0.39	Quartz . .	0.7
Iron . .	0.47	Rubber . .	1.7
Lead . .	0.13	Stone . .	0.9
Mercury . .	0.14	Wood . .	1.7
Nickel . .	0.46	Alcohol . .	2.5
Platinum . .	0.13	Brine (25% by wt.)	3.4
Silver . .	0.24	Carbon tetrachloride . .	0.84
Solder . .	0.18	Ether . .	2.4
Steel . .	0.45	Glycerine . .	2.5
Ebonite . .	1.7	Paraffin oil . .	2.1
Glass . .	0.7	Turpentine . .	1.76

MEASUREMENT OF SPECIFIC HEAT CAPACITY

Method of Mixtures

A common way of measuring specific heat capacities is the method of mixtures, used by Wilcke (p. 193). Fig. 9.1 shows how we may

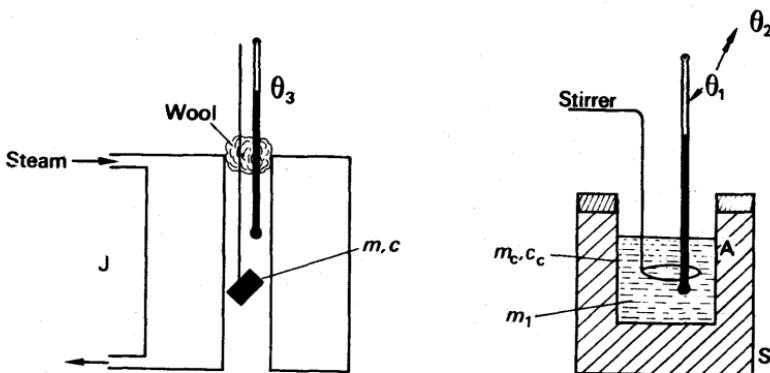


FIG. 9.1. Specific heat capacity by mixtures.

apply it to a solid, such as a metal. We weigh the specimen (m g) and hang it on a thread in a steam jacket, J , fitted with a thermometer. The jacket is plugged with cotton wool to prevent loss of heat by convection. While the solid is warming, we weigh a thin-walled copper vessel A called a calorimeter (m_c g), then run about 50 cm^3 of water into it, and by subtraction find the mass m_1 of this water. We put the calorimeter into a draught-shield S , and take the temperature, θ_1 , of the water in it. After we have given the specimen time to warm up—say half an hour—we read its temperature, θ_3 ; then we slide the calorimeter under the jacket, and drop the specimen into it. After stirring the mixture, we measure its final temperature, θ_2 . If no heat leaves the calorimeter by radiation, conduction, or convection, after the hot specimen has been dropped into the calorimeter, we have:

heat lost by solid in cooling from θ_3 to θ_2 = heat gained by water and calorimeter in warming from θ_1 to θ_2 .

(The heat gained by the thermometer and stirrer may be neglected if high accuracy is not required.)

Therefore, if c is the specific heat of the solid, c_w that of water and c_c that of the calorimeter :

$$\begin{aligned} mc(\theta_3 - \theta_2) &= m_1 c_w (\theta_2 - \theta_1) + m_c c_c (\theta_2 - \theta_1) \\ &= (m_1 c_w + m_c c_c)(\theta_2 - \theta_1), \\ \text{whence } c &= \frac{(m_1 c_w + m_c c_c)(\theta_2 - \theta_1)}{m(\theta_3 - \theta_2)}. \end{aligned} \quad (1)$$

Liquids

The specific heat capacity of a liquid can be found by putting some in a calorimeter and dropping a hot solid, of known specific heat capacity c , into it. If m_l , c_l are the mass and specific heat capacity of the

liquid, then the product $m_l c_l$ replaces m in equation (1), from which c_l can be calculated.

Calculations

As an illustration of a specific heat capacity determination, suppose a metal of mass 200 g at 100°C is dropped into 80 g of water at 15°C contained in a calorimeter of mass 120 g and specific heat capacity 0.4 kJ kg⁻¹ K⁻¹. The final temperature reached is 35°C. Then:

$$\text{heat capacity of calorimeter} = 120 \times 0.4 = 48 \text{ J K}^{-1}$$

$$\text{heat capacity of water} = 80 \times 4.2 = 336 \text{ J K}^{-1}$$

$$\therefore \text{heat gained by water + cal.} = (336 + 48) \times (35 - 15) \text{ J}$$

$$\text{and heat lost by hot metal} = 0.2 \times c \times (100 - 35) \text{ J}$$

$$\therefore 0.2 \times c \times 65 = 384 \times 20$$

$$\therefore c = \frac{384 \times 20}{0.2 \times 65} = 590 \text{ J kg}^{-1} \text{ K}^{-1} (\text{approx.})$$

Heat Losses

In a calorimetric experiment, some heat is always lost by leakage. Leakage of heat cannot be prevented, as leakage of electricity can, by insulation, because even the best insulator of heat still has appreciable conductivity (p. 333).

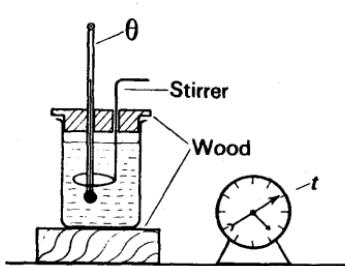
When convection is prevented, gases are the best thermal insulators. Hence calorimeters are often surrounded with a shield S, as in Fig. 9.1, and the heat loss due to conduction is made small by packing S with insulating material or by supporting the calorimeter on an insulating ring, or on threads. The loss by radiation is small at small excess temperatures over the surroundings. In some simple calorimetric experiments the final temperature of the mixture is reached quickly, so that the time for leakage is small. The total loss of heat is therefore negligible in laboratory experiments on the specific heats of metals, but not on the specific heat capacities of bad conductors, such as rubber, which give up their heat slowly. When great accuracy is required, the loss of heat by leakage is always taken into account.

Newton's Law of Cooling

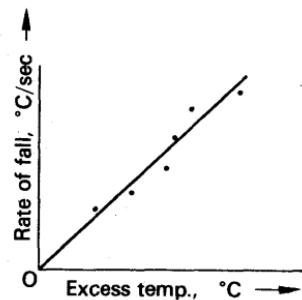
Newton was the first person to investigate the heat lost by a body in air. He found that *the rate of loss of heat is proportional to the excess temperature over the surroundings*. This result, called *Newton's law of cooling*, is approximately true in still air only for a temperature excess of about 20°C or 30°C; but it is true for all excess temperatures in conditions of forced convection of the air, i.e. in a draught. With natural convection Dulong and Petit found that the rate of loss of heat was proportional to $\theta^{5/4}$, where θ is the excess temperature, and this appears to be true for higher excess temperatures, such as from 50°C to 300°C. At low excess temperatures, however, less than 1°C,

G. T. P. Tarrant has pointed out that radiation, not convection, is the major contributing factor to the rate of cooling of an object.

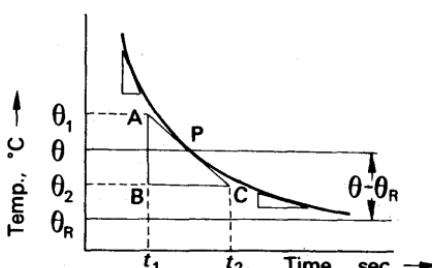
To demonstrate Newton's law of cooling, we plot a temperature (θ)-time (t) cooling curve for hot water in a calorimeter placed in a draught (Fig. 9.2 (a)). If θ_R is the room temperature, then the excess temperature of the water is $(\theta - \theta_R)$. At various temperatures, such as θ in Fig. 9.2 (b), we drew tangents such as APC to the curve. The slope of the tangent, in degrees per second, gives us the rate of fall of temperature, when the water is at the temperature θ :



(a) Apparatus



(c) Treatment of results



(b) Results

FIG. 9.2. Newton's law of cooling.

$$\text{rate of fall} = \frac{AB}{BC} = \frac{\theta_1 - \theta_2}{t_2 - t_1}$$

We then plot these rates against the excess temperature, $\theta - \theta_R$, as in Fig. 9.2 (c), and find a straight line passing through the origin. Since the heat lost per second by the water and calorimeter is proportional to the rate of fall of the temperature, Newton's law is thus verified.

Heat Loss and Temperature Fall

Besides the excess temperature, the rate of heat loss depends on the exposed area of the calorimeter, and on the nature of its surface: a dull surface loses heat a little faster than a shiny one, because it is a better radiator (p. 343). This can be shown by doing a cooling experiment twice, with equal masses of water, but once with the calorimeter polished, and once after it has been blackened in a candle-flame. In

general, for any body with a uniform surface at a uniform temperature θ , we may write, if Newton's law is true,

$$\text{heat lost/second} = \frac{dQ}{dt} = kS(\theta - \theta_R) \quad (2)$$

where S is the area of the body's surface, θ_R is the temperature of its surroundings, k is a constant depending on the nature of the surface, and Q denotes the heat lost from the body.

When a body loses heat Q , its temperature θ falls; if m is its mass, and c its specific heat capacity, then its rate of fall of temperature, $d\theta/dt$, is given by

$$\frac{dQ}{dt} = -mc \frac{d\theta}{dt}.$$

Now the mass of a body is proportional to its volume. The rate of heat loss, however, is proportional to the surface area of the body. The rate of fall of temperature is therefore proportional to the ratio of surface to volume of the body. For bodies of similar shape, the ratio of surface to volume is inversely proportional to any linear dimension. If the bodies have surfaces of similar nature, therefore, the rate of fall of temperature is inversely proportional to the linear dimension: a small body cools faster than a large one. This is a fact of daily experience: a small coal which falls out of the fire can be picked up sooner than a large one; a tiny baby should be more thoroughly wrapped up than a grown man. In calorimetry by the method of mixtures, the fact that a small body cools faster than a large one means that, the larger the specimen, the less serious is the heat loss in transferring it from its heating place to the calorimeter. It also means that the larger the scale of the whole apparatus, the less serious are the errors due to loss of heat from the calorimeter.

Correction for Heat Losses in Calorimetry

Newton's law of cooling enables us to estimate the heat lost in an experiment on the method of mixtures.

In doing the experiment, we take the temperature of the mixture at half-minute intervals, and plot it against time, as in Fig. 9.3. The broken line shows how we would expect the temperature to rise if no heat were lost; we have therefore to estimate the difference, p , between the plateau of this imaginary curve, and the crest of the experimental curve, C . p is known as the 'cooling correction'.

We start by drawing an ordinate CN through the crest, and another LM through any convenient point L further along the curve; OM should be not less than

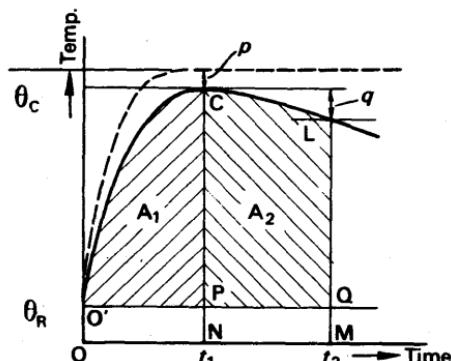


FIG. 9.3. Cooling correction.

twice ON—the greater it is, the more accurate the correction. We next draw an abscissa O'PQ through the room temperature, θ_R ; and by counting the squares of the graph paper, we measure the areas O'CP (A_1), PCLQ (A_2). Then, if q is the fall in temperature from C to L:

$$\frac{p}{q} = \frac{\text{O}'\text{CP}}{\text{PCLQ}} = \frac{A_1}{A_2}, \quad (3)$$

or

$$p = q \frac{A_1}{A_2}.$$

Before establishing this equation let us see how to use it. Suppose m_1, c , are the mass and specific heat capacity of the specimen; m, c_w are the mass and specific heat capacity of water; and C the heat capacity of the calorimeter. Then the heat which these lose to their surroundings is the heat which would have raised their temperature by p . Thus

$$\text{heat lost} = (m_1 c + m c_w + C)p.$$

Let θ_1 denote the initial temperature of the specimen, θ_c the highest temperature of the mixture; and θ_2 the original temperature of the water and calorimeter. Then we have:

$$\text{heat given out} = \text{heat taken in} + \text{heat lost}.$$

$$\therefore m_1 c(\theta_1 - \theta_c) = (m c_w + C)(\theta_c - \theta_2) + (m_1 c + m c_w + C)p,$$

$$\text{from which } m_1 c(\theta_1 - \overline{\theta_c + p}) = (m c_w + C)(\overline{\theta_c + p} - \theta_2).$$

To correct for the heat losses we must therefore add the correction p to the crest temperature θ on each side of the heat balance equation. In equation (1), p. 200, p must be added to θ_2 in both numerator and denominator.

Theory of the Correction. To establish equation (3), we write down the expression for the heat lost per second from the calorimeter, assuming Newton's law of cooling:

$$\frac{dQ}{dt} = kS(\theta - \theta_R). \quad (4)$$

where k is a constant, and S the exposed area of the calorimeter. Between times $t = 0$ and $t = t_1$, the total heat lost is

$$\begin{aligned} Q_1 &= \int_0^{t_1} kS(\theta - \theta_R)dt \\ &= kS \int_0^{t_1} (\theta - \theta_R)dt \\ &= kS \times \text{area O}'\text{CP} = kSA_1. \end{aligned}$$

This is the heat which, if it had not been lost, would have warmed the calorimeter and contents by p degrees. Therefore

$$(m_1 c + m c_w + C)p = kSA_1. \quad (5)$$

Similarly the heat lost between t_1 and t_2 is given by $Q_2 = kSA_2$, and since this loss caused a fall in temperature of q , we have, by the argument above

$$(m_1c + mc_w + C)q = kSA_2 \quad \quad (6)$$

On dividing equation (5) by equation (6), we find

$$\frac{p}{q} = \frac{A_1}{A_2}, \text{ or } p = q \frac{A_1}{A_2}.$$

Specific Heat Capacity of Liquid by Cooling

Specific heat capacities of liquids which react with water are often measured by the so-called *method of cooling*. The cooling curve of a calorimeter is plotted, first when it contained a known volume of hot water, and then when it contains an *equal* volume of hot liquid (Fig. 9.4). The volumes are made equal so as to make the temperature distribution,

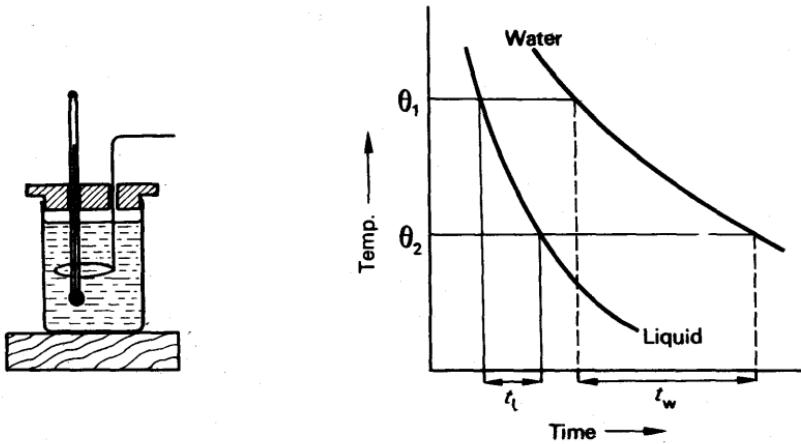


FIG. 9.4. Specific heat capacity by cooling.

over the surface of the calorimeter, the same in each experiment. From the curves, the respective times t_l and t_w are found which the calorimeter and contents take to cool from θ_1 to θ_2 . Whatever the contents of the calorimeter, it gives off heat at a rate which depends only on its excess temperature, since the area and nature of its surface are constant. Therefore, at each temperature between θ_1 and θ_2 , the calorimeter gives off heat at the same rate whatever its contents. Thus the average rate at which it loses heat, over the whole range, is the same with water and with liquid. Consequently

$$\frac{(m_1c + C)(\theta_1 - \theta_2)}{t_l} = \frac{(mc_w + C)(\theta_1 - \theta_2)}{t_w}$$

where m_1, c , are the mass and specific heat capacity of the liquid, m, c_w that of water, and C is the heat capacity of the calorimeter. Thus

$$\frac{m_1c + C}{t_l} = \frac{mc_w + C}{t_w}$$

from which c can be calculated.

Specific Heat Capacity by Electrical Method

The simplest way to measure the specific heat capacity of a liquid in the laboratory is by electrical heating, as illustrated in Fig. 9.5. In this case, the energy supplied = IVt joules, where I is the current in amperes the coil R of resistance wire, V is the potential difference across it in volts and t is the time in seconds for which current flows.

The coil may be immersed in a suitable oil of mass m in a calorimeter of heat capacity C .

We pass a steady current I through the coil, and measure the potential difference V across it. Stirring continuously, we plot the temperature of the oil against the time. After a time t long enough to give several degrees rise, we switch off the current and plot the cooling curve. If θ is the corrected rise in temperature, we have

$$IVt = (mc + C)\theta,$$

whence we can calculate c in $\text{J kg}^{-1} \text{K}^{-1}$.

Specific Heat Capacity of Water by Continuous Flow Method

In 1899, Callendar and Barnes devised a method for specific heat capacity in which only steady temperatures are measured. They used platinum resistance thermometers, which are more accurate than mercury ones but take more time to read. In the measurement of steady temperatures, however, this is no drawback. As we shall see shortly *the heat capacity of the apparatus is not required*, which is a great advantage of the method.

Fig. 9.6 shows Callendar and Barnes' apparatus. Water from the constant-head tank K flows through the glass tube U , and can be collected as it flows out. It is heated by the spiral resistance wire R , which carries a steady electric current I . Its temperature, as it enters

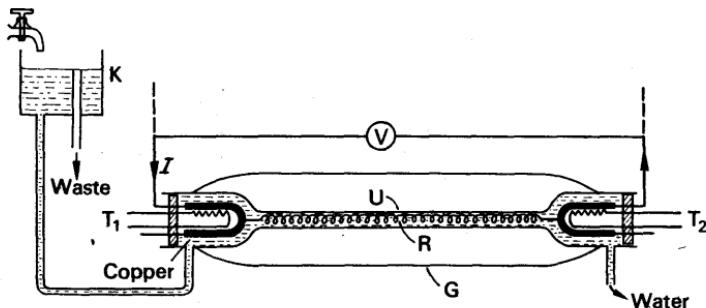


FIG. 9.6. Callendar and Barnes' apparatus (contracted several times in length relative to diameters).

and leaves, is measured by the thermometers T_1 and T_2 . (In a simplified laboratory experiment, these may be mercury thermometers.) Surrounding the apparatus is a glass jacket G , which is evacuated, so that heat cannot escape from the water by conduction or convection.

When the apparatus is running, it settles down eventually to a steady state, in which the heat supplied by the current is all carried away by the water. *None is then taken in warming the apparatus, because every part of it is at a constant temperature.* The mass of water m , which flows out of the tube in t seconds, is then measured. If the water enters at a temperature θ_1 and leaves at θ_2 , then if c_w is its mean specific heat capacity,

$$\text{heat gained by water} = Q = mc_w(\theta_2 - \theta_1) \text{ joules.}$$

The energy which liberates this heat is electrical. To find it, the current I , and the potential difference across the wire V , are measured with a potentiometer. If I and V are in amperes and volts respectively, then, in t seconds :

$$\text{energy supplied to wire} = IVt \text{ joules.}$$

$$\therefore mc_w(\theta_2 - \theta_1) = IVt$$

$$\therefore c_w = \frac{IVt}{m(\theta_2 - \theta_1)}$$

To get the highest accuracy from this experiment, the small heat losses due to radiation, and conduction along the glass, must be allowed for. These are determined by the temperatures θ_1 and θ_2 . For a given pair of values of θ_1 and θ_2 , and constant-temperature surroundings (not shown), let the heat lost per second be h . Then, in t seconds,

$$\text{heat supplied by heating coil} = mc_w(\theta_2 - \theta_1) + ht,$$

$$\therefore IVt = mc_w(\theta_2 - \theta_1) + ht \quad . \quad (1)$$

To allow for the loss h , the rate of flow of water is changed, to about half or twice its previous value. The current and voltage are then adjusted to bring θ_2 back to its original value. The inflow temperature, θ_1 , is fixed by the temperature of the water in the tank. If I' , V' , are the new values of I , V , and m' is the new mass of water flowing in the same t seconds, then :

$$I'V't = m'c_w(\theta_2 - \theta_1) + ht \quad . \quad (2)$$

On subtracting equation (2) from equation (1), we find

$$(IV - I'V')t = (m - m')c_w(\theta_2 - \theta_1),$$

$$\therefore c_w = \frac{(IV - I'V')t}{(m - m')(\theta_2 - \theta_1)} \quad . \quad (3)$$

When the temperature rise, $\theta_2 - \theta_1$, is made small, for example, $\theta_1 = 20.0^\circ\text{C}$, $\theta_2 = 22.0^\circ\text{C}$, then c_w may be considered as the specific heat at 21.0°C , the mean temperature. If the inlet water temperature is now raised to say $\theta_1 = 40.0^\circ\text{C}$ and θ_2 is then 42.0°C , c_w is now the

specific heat at 41·0°C. In this way it was found that *the specific heat capacity of water varied with temperature*. The continuous flow method can be used to find the variation in specific heat capacity of any liquid in the same way.

The '15°C-calorie' was defined as the heat required to raise the temperature of 1 gramme of water from 14·5°C to 15·5°C. The table shows the relative variation of the specific heat capacity of water, taking the 15°C-calorie as 1·0000 in magnitude.

SPECIFIC HEAT CAPACITY OF WATER

Temperature (°C)	5	15	25	40	70	100
c_w	1·0047	1·0000	0·9980	0·9973	1·0000	1·0057

Nernst's Method

Modern methods of measuring specific heat capacities use electrical heating. Nernst's method for the specific heat capacity of a metal is shown in Fig. 9.7. The metal S has a heating coil R of insulated platinum wire wound round the outside, and is covered with silver foil F to minimize heat loss by radiation. It is suspended by the leads to the coil in a glass vessel, which is then evacuated, to prevent losses by convection and conduction. The resistance of the coil is measured, and from it the temperature is calculated.

A steady current I , at a known potential difference V , is then passed through the coil for t seconds. After the current has been switched off, the resistance of the coil is again measured, to find the rise in temperature of the specimen. Resistance measurements are made at intervals, and enable the cooling curve to be plotted. If m is the mass of the specimen, C the heat capacity of the coil and foil, and θ the corrected rise in temperature, then the specific heat capacity c of the specimen is given by

$$IVt = (mc + C)\theta.$$

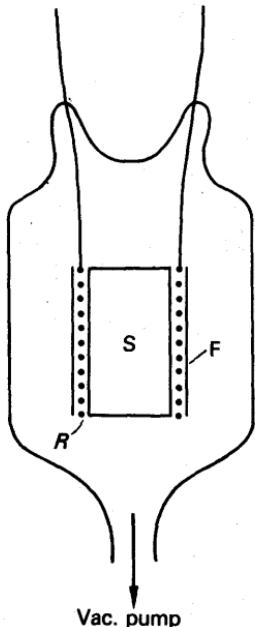


FIG. 9.7. Nernst's calorimeter.

SPECIFIC LATENT HEAT

Fusion

The *specific latent heat of fusion* of a solid is the heat required to convert unit mass of it, at its melting-point, into liquid at the same temperature. It is expressed in joules per kilogramme ($J\ kg^{-1}$). High values can be more conveniently expressed in $kJ\ kg^{-1}$.

MELTING-POINTS AND SPECIFIC LATENT HEATS OF FUSION

Substance	M.P. °C	S.L.H. J kg^{-1}	Substance	M.P. °C	S.L.H. J kg^{-1}
Aluminium	658	393×10^3	Acetic acid	. .	$17.5 \quad 184 \times 10^3$
Antimony	630	163	Beeswax	. .	62 176
Bismuth	271	58	Brass	. .	900 —
Copper	1083	180	Naphthalene	. .	80 146
Gold	1063	67	Paradichlorbenzene	. .	53 —
Iron	1530	205	(non-flam.)		
Lead	327	25	Ice	. .	0 333
Mercury	—38.9	12.5	Paraffin wax	. .	50–60 —
Nickel	1452	272	Solder (soft)	. .	c. 180 50–85
Platinum	1773	113	Solder (hard)	. .	c. 900 —
Sulphur	113–119	38	Steel	. .	1400 —
Silver	960.8	109	Glass	. .	300–400 —
Sodium	97.5	113	Quartz (fused)	. .	1,700 —
Tin	232	58	Hypo	. .	48.2 c. 170
Tungsten	3380	—			
Zinc	419	109			

Ice is one of the substances whose specific latent heat of fusion we are likely to have to measure. To do so, place warm water, at a temperature θ_1 , a few degrees above room temperature, inside a calorimeter. Then add small lumps of ice, dried by blotting paper, until the temperature reaches a value θ_2 as much below room temperature as θ_1 was above. In this case a 'cooling correction' is not necessary. Weigh the mixture, to find the mass m of ice which has been added. Then the specific latent heat l is given by :

$$\left. \begin{aligned} \text{heat given by calorimeter and water in cooling} \end{aligned} \right\} = \left\{ \begin{aligned} \text{heat used in melting ice} \end{aligned} \right\} + \left\{ \begin{aligned} \text{heat used in warming melted ice from } 0^\circ\text{C to } \theta_2 \end{aligned} \right\}$$

$$\therefore (m_1 c_w + C)(\theta_1 - \theta_2) = ml + mc_w(\theta_2 - 0),$$

where m_1 = mass of water and c_w = specific heat capacity, C = thermal capacity of calorimeter, and θ_1 = initial temperature.

$$\text{Hence } l = \frac{(m_1 c_w + C)(\theta_1 - \theta_2)}{m} - c_w \theta_2.$$

A modern electrical method, similar to Nernst's for specific heat capacities, gives

$$l = 334 \text{ kJ kg}^{-1} \quad \text{or} \quad 334 \text{ J g}^{-1}.$$

Bunsen's Ice Calorimeter

Bunsen's ice calorimeter is a device for measuring a quantity of heat by using it to melt ice.

When ice turns to water, it shrinks; the volume of 1 g of ice at 0°C is 1.0908 cm^3 , whereas that of 1 g of water at 0°C is 1.0001 cm^3 (p. 296). Thus the melting of 1 g of ice causes a contraction of 0.0907 cm^3 .

In the Bunsen calorimeter, the contraction due to the melting is measured, and from it the mass of ice melted is calculated. The apparatus is shown in Fig. 9.8. It consists of a test-tube T fused into a wider tube Y. The wider tube leads to a capillary C, and is filled with mercury from X to Y. The space above Y is filled with water from which all dissolved air has been boiled.

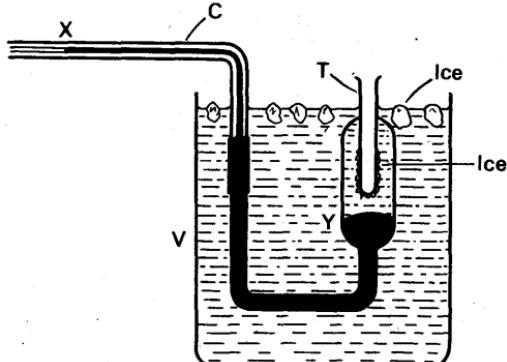


FIG. 9.8. Bunsen's ice calorimeter.

Except for the capillary, the whole apparatus is placed in ice-water in the vessel V, and, after some time, it all settles down to 0°C. A little ether is then poured into T, and air is blown through it via a thin tube; the ether evaporates and cools the tube T, so that ice forms on the outside of it. A pad of cotton wool is then dropped to the bottom of T and the apparatus is left for some more time, to allow the newly formed ice to settle down to 0°C.

When the apparatus is ready for use, the end of the mercury thread in C is observed by a travelling microscope. If the specific heat capacity of a solid is to be measured, the specimen is weighed (m g) and left to come to room temperature θ . The solid is then gently dropped into the tube T. As it cools, it melts ice, and causes the mercury thread to run back along the capillary. When the thread has ceased to move, its end is again observed. If it has moved through l cm, and the cross-section of the capillary is a cm 2 , then the contraction is al cm 3 . The mass of ice melted is therefore $al/0.0907$ g, and the heat absorbed is $334 al/0.0907$ joules. This heat is given out by m g of solid cooling from θ to 0°C; the specific heat capacity c of the solid is therefore given by

$$mc\theta = \frac{334 al}{0.0907}$$

In practice, the cross-section is not measured, and the instrument is calibrated by dropping into it a solid of known mass, m_1 , and specific heat capacity, c_1 . If the room temperature is constant, then

$$\frac{mc}{m_1 c_1} = \frac{l}{l_1},$$

where l_1 is the displacement of the mercury in the calibration experiment. Thus c can be found.

Advantages of the Ice Calorimeter

The advantages of the ice calorimeter are:

- (i) no correction for heat capacity of the container: the specimen tube starts at 0°C and finishes at 0°C—all the heat from the specimen is used to melt ice, at constant temperature;

- (ii) no heat losses from the apparatus—it is surrounded by a bath at the same temperature as itself, and therefore neither loses heat to the outside, nor gains any from it;
- (iii) no loss of heat from the specimen before it enters the calorimeter—the specimen starts at room temperature, and therefore gives up no heat until it enters the specimen tube (contrast the method of mixtures, in which the specimen is heated to 100°C or so): this is a great advantage when the specimen is small;
- (iv) easy, and therefore accurate, thermometry—the only temperature to be measured is the room temperature, which is constant and can be determined at leisure.

An advantage sometimes asserted is that specimens can be added one after another, without having to re-set the apparatus. That is true, because each specimen comes to 0°C in turn, and then behaves simply like part of the apparatus, taking no heat from any following specimen. But it does not mean that the calorimeter has the advantage of speed—the time taken to set it up would be enough for half a dozen measurements by the method of mixtures. A disadvantage of this calorimeter is that it never settles down completely—the mercury is always slowly creeping along the capillary, and the creep during an experiment must be estimated and allowed for.

The calorimeter was devised in 1871; it is rarely used nowadays, because electrical methods of calorimetry are more convenient and accurate. However, it has been used for measuring the specific heat capacities of rare earths of which only small specimens were available.

Evaporation

The *specific latent heat of evaporation* of a liquid is the heat required to convert unit mass of it, at its boiling-point, into vapour at the same temperature. It is expressed in joule per kilogramme ($J\ kg^{-1}$), or, with high values, in $\text{kJ}\ \text{kg}^{-1}$.

BOILING-POINTS AND SPECIFIC LATENT HEATS OF EVAPORATION

Substance	B.P. °C	S.L.H. $\text{J}\ \text{kg}^{-1}$
Aluminium	1800	—
Acetone	56.7	—
Alcohol (ethyl)	78.3	867×10^3
Alcohol (methyl)	64.7	1120
Benzene	80.2	389
Carbon disulphide	46.2	351
Carbon tetrachloride (non-flam.)	76.7	193
Ether	34.6	370
Glycerine	290	—
Turpentine	161	—
Mercury	357	272
Platinum	3910	—
Sodium	877	—
Sulphur	444.6	—

To find the specific latent heat of evaporation of water, we pass steam into a calorimeter with water (Fig. 9.9). On its way the steam passes through a vessel, T in the figure, which traps any water carried over by the steam and is called a steam-trap. The mass m of condensed steam is found by weighing. If θ_1 and θ_2 are the initial and final temperatures of the water, the specific latent heat l is given by:

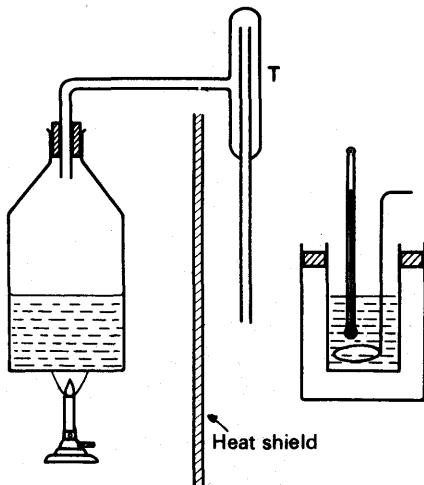


FIG. 9.9. Latent heat of evaporation of water.

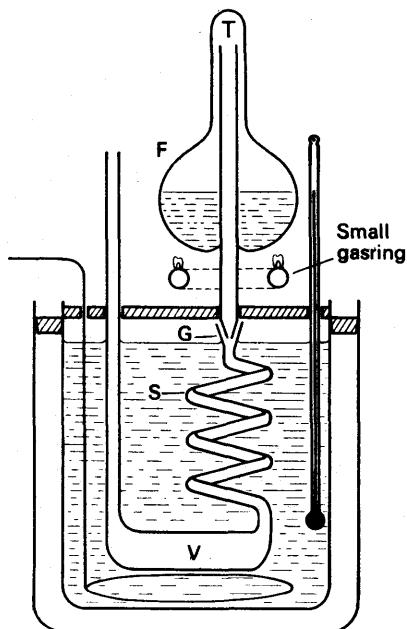


FIG. 9.10. Berthelot's apparatus for latent heat of evaporation.

$$\left. \begin{array}{l} \text{heat given by steam} \\ \text{condensing} \end{array} \right\} + \left. \begin{array}{l} \text{heat given by con-} \\ \text{densed water cooling} \\ \text{from } 100^\circ\text{C to } \theta_2 \end{array} \right\} = \left. \begin{array}{l} \text{heat taken by} \\ \text{calorimeter and} \\ \text{water} \end{array} \right\}$$

$$ml + mc_w(100 - \theta_2) = (m_1 c_w + C)(\theta_2 - \theta_1)$$

where $m_1 c_w$ and C have their usual meanings.

$$\text{Hence } l = \frac{(m_1 c_w + C)(\theta_2 - \theta_1)}{m} - c_w(100 - \theta_2).$$

The accepted value of the specific latent heat of evaporation of water is about $l = 2260 \text{ kJ kg}^{-1}$ or 2260 J g^{-1} .

Berthelot's Apparatus. An apparatus suitable for use with liquids other than water was devised by Berthelot in 1877 (see Fig. 9.10). The liquid is boiled in the flask F, and its vapour passes out through the tube T. This fits with a ground joint G into the glass spiral S, which is surrounded by water in a calorimeter. The vapour condenses in the

spiral, and collects in the vessel V, where it can afterwards be weighed.
Let

θ_b = boiling-point of liquid.

c = specific heat capacity of liquid.

m = mass of liquid condensed.

θ_1 = initial temperature of water.

θ_2 = final temperature of water, *corrected for cooling*.

m_1 = mass of water of specific heat c_w .

C = thermal capacity of calorimeter + glassware below joint.

Then $ml + mc(\theta_b - \theta_2) = (m_1 c_w + C)(\theta_2 - \theta_1)$,

$$\text{whence } l = \frac{(m_1 c_w + C)(\theta_2 - \theta_1)}{m} - c(\theta_b - \theta_2).$$

Electrical Method for Specific Latent Heat

A modern electrical method for the specific latent heat of evaporation of water is illustrated in Fig. 9.11 below. The liquid is heated in a vacuum-jacketed vessel U by the heating coil R. Its vapour passes down

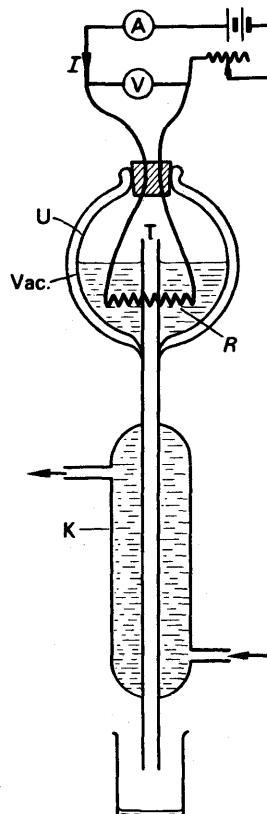


FIG. 9.11. Electrical method for latent heat of evaporation.

the tube T, and is condensed by cold water flowing through the jacket K. When the apparatus has reached its steady state, the liquid is at its boiling-point, and the heat supplied by the coil is used in evaporating the liquid, and in offsetting the losses. The liquid emerging from the condenser is then collected for a measured time, and weighed.

If I and V are the current through the coil, and the potential difference across it, the electrical energy supplied in t seconds is IVt . And if h is the heat lost from the vessel per second, and m the mass of liquid collected in t seconds, then

$$IVt = ml + ht \quad \quad (1)$$

The heat losses h are determined by the temperature of the vessel, which is fixed at the boiling-point of the liquid. Therefore they may be eliminated by a second experiment with a different rate of evaporation (cf. Callendar and Barnes, p. 206). If $I'V'$ are the new current and potential difference, and if m' grammes are evaporated in t seconds, then

$$I'V't = m'l + ht.$$

Hence by subtraction from equation (1)

$$l = \frac{(IV - I'V')t}{(m - m')}.$$

EXAMPLES

1. An electric kettle has a 750 W–240 V heater and is used on a 200 V mains. If the heat capacity of the kettle is 400 J K^{-1} and the initial water temperature is 20°C , how long will it take to boil 500 g of water, assuming the resistance of the heater is unaltered on changing to the new mains.

Firstly, find the new power absorbed on the 200 V mains. Since the resistance R is constant and $P = V^2/R$, it follows that $P \propto V^2$.

$$\therefore \text{new power} = \left(\frac{200}{240}\right)^2 \times 750 \text{ W} = 520 \text{ W (approx.)}$$

$$\therefore \text{heat supplied to water} = 520 \text{ J per second} \quad \quad (1)$$

Secondly, assuming 100°C is the boiling point and $4.2 \text{ kJ kg}^{-1} \text{ K}^{-1}$ ($4.2 \text{ J g}^{-1} \text{ K}^{-1}$) is the specific heat capacity of water,

$$\begin{aligned} \text{heat gained by water and kettle} &= 500 \times 4.2 \times (100 - 20) + 400 \times (100 - 20) \\ &= (500 \times 4.2 + 400)(100 - 20) \\ &= 610 \times 80 \text{ J.} \end{aligned}$$

From (1), \therefore time, $t = \frac{2500 \times 80}{520} = 385$ seconds (approx.) = 6.4 min.

2. Water flows at the rate of 150.0 g min^{-1} through a tube and is heated by a heater dissipating 25.2 W . The inflow and outflow water temperatures are 15.2°C and 17.4°C respectively. When the rate of flow is increased to 231.8 g min^{-1} and the rate of heating to 37.8 W , the inflow and outflow temperatures are unaltered. Find (i) the specific heat capacity of water, (ii) the rate of loss of heat from the tube.

Suppose c_w is the specific heat of water in $\text{J g}^{-1} \text{ K}^{-1}$ and h is the heat lost in J s^{-1} . Then, since $1 \text{ W} = 1 \text{ J per second}$,

$$25.2 = \frac{150.0}{60} c_w (17.4 - 15.2) + h \quad (1)$$

and $37.8 = \frac{231.8}{60} c_w (17.4 - 15.2) + h \quad (2)$

Subtracting (1) from (2),

$$\begin{aligned} \therefore 37.8 - 25.2 &= \frac{231.8 - 150.0}{60} c_w (17.4 - 15.2) \\ \therefore c_w &= \frac{12.6 \times 60}{81.8} = 4.2 \text{ J g}^{-1} \text{ K}^{-1} = 4.2 \text{ kJ kg}^{-1} \text{ K}^{-1}. \end{aligned}$$

Substituting for c_w in (1),

$$\therefore h = 25.2 - \frac{150}{60} \times 4.2 \times 2.2 = 2.1 \text{ J s}^{-1}.$$

3. Define latent heat. Describe the measurement of the specific latent heat of evaporation of water under school laboratory conditions.

A copper calorimeter of mass 70.5 g contains 100.0 g of water at 19.5°C. Naphthalene (M.P. 79.9°C) is melted in a test tube, cooled to 80.0°C, and then poured into the calorimeter. If the highest temperature reached by the water after stirring is 28.7°C and the final mass of the calorimeter and its contents is 188.3 g calculate the latent heat of fusion of naphthalene. (Specific heat capacity of copper 0.4, of naphthalene 1.3 $\text{kJ kg}^{-1} \text{ K}^{-1}$) (L .)

First part. The specific latent heat of a substance is the heat required to change unit mass of the solid at the melting-point to liquid at the same temperature (fusion), or the heat required to change unit mass of the liquid at the boiling-point to vapour at the same temperature (vaporization).

The measurement of the specific latent heat of evaporation of water requires the following, among other points: (i) use of a steam trap, (ii) a rise in temperature of the water in the calorimeter of about 10°C, (iii) a 'correction' to 100°C as the steam temperature, if the barometric pressure is not 76 cm mercury, (iv) a cooling correction.

Second part. The mass of naphthalene = 188.3 - 170.5 = 17.8 g.

Heat lost by naphthalene = heat gained by water and calorimeter.

$$\therefore 17.8l + 17.8 \times 1.3 \times (79.9 - 28.7)$$

$$= 100 \times 4.2 \times (28.7 - 19.5) + 70.5 \times 0.4 \times (28.7 - 19.5).$$

Solving,

$$\therefore l = 164 \text{ J g}^{-1} (\text{approx.}) = 164 \text{ kJ kg}^{-1}.$$

4. In an X-ray tube, 10^{18} electrons per second arrive with a speed of $2 \times 10^6 \text{ m s}^{-1}$ at a metal target of mass 200 g and specific heat capacity $0.5 \text{ J g}^{-1} \text{ K}^{-1}$. If the mass of an electron is $9.1 \times 10^{-31} \text{ kg}$, and assuming 98% of the incident energy is converted into heat, find how long the target will take to rise in temperature by 50°C assuming no heat losses.

The kinetic energy of a moving object is $\frac{1}{2}mv^2$ joules, where m is the mass in kg and v is the speed in m s^{-1} . Assuming the initial speed is zero,

$$\begin{aligned} \therefore \text{energy per second of incident electrons} &= \frac{1}{2} \times (10^{18} \times 9.1 \times 10^{-31}) \times (2 \times 10^6)^2 \text{ J} \\ &= 1.8 \text{ J (approx.)}. \end{aligned}$$

$$\text{Heat gained by target} = 200 \times 0.5 \times 50 \text{ J} = 50000 \text{ J}$$

$$\therefore \text{time} = \frac{50000}{1.8} = 2780 \text{ seconds (approx.)} = 46.3 \text{ min.}$$

EXERCISES 9

1. Describe, with the aid of a labelled diagram, how you would find the specific heat capacity of a liquid by the method of continuous flow.

Discuss the advantages and disadvantages of the method compared with the method of mixtures.

The temperature of 50 g of a liquid contained in a calorimeter is raised from 15.0°C (room temperature) to 45.0°C in 530 seconds by an electric heater dissipating 10.0 watts. When 100 g of liquid is used and the same change in temperature occurs in the same time, the power of the heater is 16.1 watts. Calculate the specific heat capacity of the liquid. (N.)

2. Distinguish between *specific* heat capacity and *latent* heat capacity. With what physical changes is each associated? Describe the processes involved in terms of simple molecular theory.

A thin-walled tube containing 5 cm³ of ether is surrounded by a jacket of water calibrated so that changes in the volume of the water can be read off. The whole apparatus is cooled down to 0°C and all the ether is then evaporated by blowing a rapid stream of air pre-cooled to 0°C through it. The change of volume as ice forms in the water is 0.35 cm³. Calculate the specific latent heat of evaporation of the ether.

(Use the following values: specific latent heat of fusion of ice = 334 J g⁻¹. Densities at 0°C: water, 1.000 g cm⁻³; ice, 0.917 g cm⁻³; ether, 0.736 g cm⁻³.) (O. & C.)

3. Give an account of an electrical method of finding the specific latent heat of vaporisation of a liquid boiling at about 60°C. Point out any causes of inaccuracy and explain how to reduce their effect.

Ice at 0°C is added to 200 g of water initially at 70°C in a vacuum flask. When 50 g of ice has been added and has all melted the temperature of the flask and contents is 40°C. When a further 80 g of ice has been added and has all melted the temperature of the whole becomes 10°C. Calculate the specific latent heat of fusion of ice, neglecting any heat lost to the surroundings.

In the above experiment the flask is well shaken before taking each temperature reading. Why is this necessary? (C.)

4. The specific heat capacity of gallium metal is 0.33 kJ kg⁻¹ K⁻¹. Explain carefully how this result may be determined experimentally. Indicate the sources of error in your method and estimate the accuracy which could be achieved.

[Melting point of gallium = 30°C.]

A ball of gallium is released from a stationary balloon, falls freely under gravity and on striking the ground it just melts. Calculate the height of the balloon assuming that the temperature of the gallium just before impact is 1°C and that all the energy gained during its free fall is used to heat the gallium on impact. Why are the conditions specified in this problem unrealistic?

[Specific latent heat of fusion of gallium = 79 kJ kg⁻¹.] (O. & C.)

5. Explain what is meant by the specific latent heat of vaporization of a liquid, and describe an experiment for an accurate determination of this quantity for carbon tetrachloride, which boils at 77°C.

A thermally insulated vessel connected to a vacuum pump contains 100 g of water at a temperature of 0°C. As air and water vapour are exhausted from the vessel, it is observed that the water remaining in the vessel freezes. Explain why this happens, and find the mass of water which is converted into ice.

[Specific latent heat of vaporization of water at 0°C = 2520 kJ kg⁻¹; specific latent heat of fusion of ice at 0°C = 336 kJ kg⁻¹.] (C.)

6. Discuss the nature of the heat energy (a) of a solid, (b) of a gas, (c) of the sun during the transmission of this energy to the earth.

265430 joules of heat are produced when a vehicle of total mass 1270 kg is brought to rest on a level road. Calculate the speed of the vehicle in km per hr just before the brakes are applied. (L.)

7. State *Newton's law of cooling* and describe how to obtain observations and how to use them in order to test the validity of the law.

A solid of mass 250 g in a vessel of thermal capacity 67.2 J K^{-1} is heated to a few degrees above its melting point and allowed to cool in steady conditions until solid again. Sketch the graph of its temperature plotted against time. If this graph shows that immediately before solidification starts the rate of cooling is $3.2 \text{ deg C min}^{-1}$, while immediately after solidification it is $4.7 \text{ deg C min}^{-1}$, calculate the specific heat capacity of the solid taken by the solidifying process. (The specific latent heat of fusion of the substance may be taken as 146.6 kJ kg^{-1} and its specific heat capacity in the liquid state as $1.22 \text{ kJ kg}^{-1} \text{ K}^{-1}$.) (L.)

8. What do you understand by the *specific heat capacity* of a substance? Describe how you would measure the specific heat capacity of a sample of rock, describing the precautions that you would take to obtain an accurate result.

A room is heated during the day by a 1 kW electric fire. The fire is to be replaced by an electric storage heater consisting of a cube of concrete which is heated overnight and is allowed to cool during the day, giving up its heat to the room. Estimate the length of an edge of the cube if the heat it gives out in cooling from 70°C to 30°C is the same as that given out by the electric fire in 8 hours.

[Density of concrete = 2700 kg m^{-3} ; specific heat capacity of concrete = $0.85 \text{ kJ kg}^{-1} \text{ K}^{-1}$.] (O. & C.)

9. An experiment was performed to determine the specific latent heat of vaporization of a volatile liquid at the prevailing boiling point by the method of electrical heating. The results are summarized in the following table:

Rate of supply of energy to boiling liquid (watt)	Mass of liquid vaporized in 200 seconds (g)
10	1.6
20	6.4
30	11.2
40	16.0

Use the data to plot a graph (using the graph paper available) and hence determine the latent heat of vaporization of the liquid and the rate of loss of heat from the calorimeter containing the boiling liquid.

Draw a labelled diagram of a suitable apparatus for use in the experiment and indicate how the above results would have been obtained.

Give two advantages of this method over the method of mixtures. (N.)

10. In 1948 the International Conference on Weights and Measures recommended that the calorie should no longer be regarded as the basic unit of heat, but that it should be replaced by the joule. Discuss the reasons for, and the advantages and possible disadvantages of, this recommendation.

Give a labelled diagram of a continuous flow calorimeter suitable for the determination of the specific heat capacity of a liquid. What measurements would you make in such a determination, and how would the result be obtained from them? State the precautions which you would adopt to ensure an accurate result. (C.)

11. Give an account of a method of determining the specific latent heat of evaporation of water, pointing out the ways in which the method you describe achieves, or fails to achieve, high accuracy.

A 600 watt electric heater is used to raise the temperature of a certain mass of water from room temperature to 80°C. Alternatively, by passing steam from a boiler into the same initial mass of water at the same initial temperature the same temperature rise is obtained in the same time. If 16 g of water were being evaporated every minute in the boiler, find the specific latent heat of steam, assuming that there were no heat losses. (O. & C.)

12. 300 g of a certain metal of density about 19 g cm⁻³ is available in the form of a coarse powder, together with a calorimeter of heat capacity about 33.6 J K⁻¹ and volume about 160 cm³, and a 50°C thermometer reading to $\frac{1}{5}$ deg.

Using this and other necessary apparatus, how would you verify, by the method of mixtures, that the specific heat of the metal is 0.13 kJ kg⁻¹ K⁻¹?

In the experiment you describe why is it (a) unnecessary to apply a correction for heat exchange with the surroundings, (b) necessary to decide on a suitable maximum temperature of the mixture? How would you ensure that such a temperature is realized? (N.)

13. Describe an *electrical* method for the determination of the *specific latent heat of steam*. State the probable sources of error in the experiment and suggest how they may be minimised.

In one method for storing solar energy, Glauber's salt can be allowed to warm up to 45°C in the sun's rays during the day and the stored heat is used during the night, the salt cooling down to 25°C. Glauber's salt melts at 32°C. Calculate the mass of salt needed to store 1 million joules.

(Specific heat capacity of Glauber's salt, solid = 0.11 kJ kg⁻¹ K⁻¹; specific heat capacity of Glauber's salt, molten = 0.16 kJ kg⁻¹ K⁻¹. Specific latent heat of fusion = 14 kJ kg⁻¹.) (O. & C.)

14. What is meant by the *specific heat capacity* of a substance? Give a brief account of two methods, one in each case, which may be used to find the specific heat capacity of each of the following: (a) a specimen of a metal in the form of a block a few cm in linear dimensions, and (b) a liquid which is available in large quantities. Indicate whether the methods you describe involve a cooling correction.

An electrical fuse consists of a piece of lead wire 1.5 mm in diameter and 5 cm long, which has a resistance of 6.5×10^{-3} ohm. Owing to a fault a constant current of 800 A passes through the fuse. If the wire is initially at 10°C and melts at 330°C, find the time interval before it starts to melt, assuming that its specific heat capacity and its electrical resistance are constant and that there are no heat losses.

[Specific heat capacity of lead = 0.13 kJ kg⁻¹ K⁻¹. Density of lead = 11000 kg m⁻³.] (O. & C.)

15. Describe the determination of the latent heat of fusion of ice by the method of mixtures and, in particular, show how allowance is made for heat interchange with the surroundings.

A calorimeter of heat capacity 84 J K⁻¹ contains 980 g of water supercooled to -4°C. Taking the latent heat of fusion of ice at 0°C as 336 kJ kg⁻¹, find the amount of ice formed when the water suddenly freezes. Calculate also the specific latent heat of fusion at -4°C if the specific heat capacity of ice is 2.1 kJ kg⁻¹ K⁻¹. (N.)

16. State Newton's law of cooling, and describe an experiment by which you would verify it. A calorimeter containing first 40 and then 100 g of water is heated and suspended in the same constant-temperature enclosure. It is found

that the times taken to cool from 50° to 40°C in the two cases are 15 and 33 minutes respectively. Calculate the heat capacity of the calorimeter. (O. & C.)

17. Oil at 15.6°C enters a long glass tube containing an electrically heated platinum wire and leaves it at 17.4°C, the rate of flow being 25 cm³ per min and the rate of supply of energy 1.34 watts. On changing the rate of flow to 15 cm³ per min and the power to 0.76 watt the temperature again rises from 15.6° to 17.4°C. Calculate the mean specific heat capacity of the oil between these temperatures. Assume that the density of the oil is 870 kg m⁻³. (N.)

18. In the absence of bearing friction a winding engine would raise a cage weighing 1000 kg at 10 m s⁻¹, but this is reduced by friction to 9 m s⁻¹. How much oil, initially at 20°C, is required per second to keep the temperature of the bearings down to 70°C? (Specific heat capacity of oil = 2.1 kJ kg⁻¹ K⁻¹; g = 9.81 m s⁻². (O. & C.)

19. A heating coil is embedded in a copper cylinder which also carries a thermocouple. The whole is thermally equivalent to 25 g of copper. The cylinder is suspended in liquid air until the thermocouple reading is constant. The cylinder is taken out and rapidly transferred into a beaker of water at 0°C. A coating of ice forms on the cylinder and when its temperature is again constant it is taken out of the water and suspended in a space maintained at 0°C. The heating coil is switched on at a steady energy dissipation of 24 watts. After 1 minute 5 seconds the whole of the ice has just melted. What is the temperature of the liquid air?

What assumptions were made in carrying out the calculations? (Mean specific heat capacity of copper is 0.336 kJ kg⁻¹ K⁻¹.) (L.)

20. Describe a continuous flow method of measuring the specific heat capacity of a liquid. Explain the advantages of the method.

Use the following data to calculate the specific heat capacity of the liquid flowing through a continuous flow calorimeter: *Experiment I.* Current 2.0 amp, applied p.d. 3.0 volt, rate of flow of liquid 30 g min⁻¹, rise in temperature of liquid 2.7°C. *Experiment II.* Current 2.5 amp, applied p.d. 3.75 volts, rate of flow of liquid 48 g min⁻¹, rise in temperature of liquid 2.7°C. (L.)

chapter ten

Gases

IN this chapter we shall be concerned with the relationship between the temperature, pressure and volume of a gas. Unlike the case of a solid or liquid this can be expressed in very simple laws, called the Gas Laws, and reduced to a simple equation, called the Equation of State. We shall also deal in this chapter with the specific heat capacities of gases.

THE GAS LAWS AND THE EQUATION OF STATE

Pressure and Volume: Boyle's Law

In 1660 Robert Boyle—whose epitaph reads ‘Father of Chemistry, and Nephew of the Earl of Cork’—published the results of his experiments on the natural spring of air. In the vigorous language of the seventeenth century, he meant what we now tamely call the relationship between the pressure of air and its volume. Similar results were published in 1676 by Mariotte, who had not heard of Boyle’s work. Boyle trapped air in the closed limb of a U-tube, with mercury (Fig. 10.1 (a)). He first adjusted the amount of mercury until its level was the same in each limb, so that the trapped air was at atmospheric pressure. He next

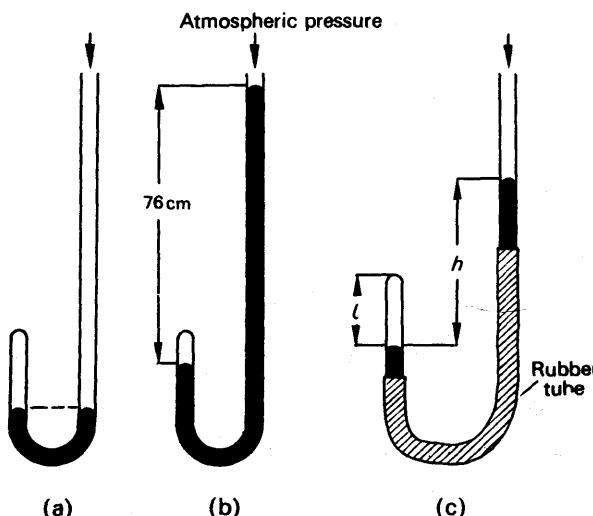


FIG. 10.1. Boyle's law apparatus.

poured in more mercury, until he had halved the volume of the trapped air (Fig. 10.1 (b)). Then 'not without delight and satisfaction' he found that the mercury in the open limb stood 740 mm above the mercury in the closed limb. Since he knew that the height of the barometer was about 740 mm of mercury, he realized that to halve the volume of his air he had had to double the pressure on it.

We can repeat Boyle's experiment with the apparatus shown in Fig. 10.1 (c); its form makes the pouring of mercury unnecessary. We set the open limb of the tube at various heights above and below the closed limb and measure the difference in level, h , of the mercury. When the mercury in the open limb is below that in the closed, we reckon h as negative. At each value of h we measure the corresponding length l of the air column in the closed limb. To find the pressure of the air we add the difference in level h to the height of the barometer, H ; their sum gives the pressure p of the air in the closed limb:

$$p = g\rho(H + h)$$

where g is the acceleration of gravity and ρ is the density of mercury.

If S is the area of cross-section of the closed limb, the volume of the trapped air is

$$V = lS.$$

To interpret our measurements we may either plot $H + h$, which is a measure of p against $1/l$ or tabulate the product $(H + h)l$. We find that the plot is a straight line, and therefore

$$(H + h) \propto \frac{1}{l} \quad \quad (1)$$

Alternatively, we find

$$(H + h)l = \text{constant}, \quad \quad (2)$$

which means the same as (1).

Since, g , ρ , and S are constants, the relationships (1) and (2) give

$$p \propto \frac{1}{V}$$

or

$$pV = \text{constant}.$$

A little later in this chapter we shall see that the pressure of a gas depends on its temperature as well as its volume. To express the results of the above experiments, therefore, we say that *the pressure of a given mass of gas, at constant temperature, is inversely proportional to its volume*. This is *Boyle's Law*.

Mixture of Gases: Dalton's Law

Fig. 10.2 shows an apparatus with which we can study the pressure of a mixture of gases. A is a bulb, of volume V_1 , containing air at atmospheric pressure, p_1 . C is another bulb, of volume V_2 , containing

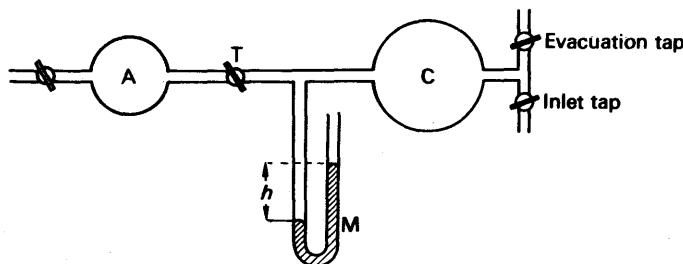


FIG. 10.2. Apparatus for demonstrating Dalton's law of partial pressures.

carbon dioxide at a pressure p_2 . The pressure p_2 is measured on the manometer M; in millimetres of mercury it is

$$p_2 = h + H,$$

where H is the height of the barometer. (In the same units, the air pressure, $p_1 = H$.)

When the bulbs are connected by opening the tap T, the gases mix, and reach the same pressure, p ; this pressure is given by the new height of the manometer. Its value is found to be given by

$$p = p_1 \frac{V_1}{V_1 + V_2} + p_2 \frac{V_2}{V_1 + V_2}.$$

Now the quantity $p_1 V_1 / (V_1 + V_2)$ is the pressure which the air originally in A would have, if it expanded to occupy A and C; for, if we denote this pressure by p' , then $p'(V_1 + V_2) = p_1 V_1$. Similarly $p_2 V_2 / (V_1 + V_2)$ is the pressure which the carbon dioxide originally in C would have, if it expanded to occupy A and C. Thus the total pressure of the mixture is the sum of the pressures which the individual gases exert, when they have expanded to fill the vessel containing the mixture.

The pressure of an individual gas in a mixture is called its *partial pressure*: it is the pressure which would be observed if that gas alone occupied the volume of the mixture, and had the same temperature as the mixture. The experiment described shows that *the pressure of a mixture of gases is the sum of the partial pressures of its constituents*. This statement was first made by Dalton, in 1801, and is called *Dalton's Law of Partial Pressures*.

Volume and Temperature: Charles's Law

Measurements of the change in volume of a gas with temperature, at constant pressure, were published by Charles in 1787 and by Gay-Lussac in 1802. Fig. 10.3 shows an apparatus which we may use for repeating their experiments. Air is trapped by mercury in the closed limb C of the tube AC; a scale engraved upon C enables us to measure the length of the air column, l . The tube is surrounded by a water-bath W, which we can heat by passing in steam. After making the temperature uniform by stirring, we level the mercury in the limbs A and C, by pouring mercury in at A, or running it off at B. The air

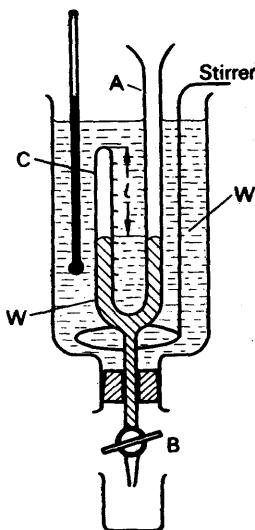


FIG. 10.3. Charles' law experiment.

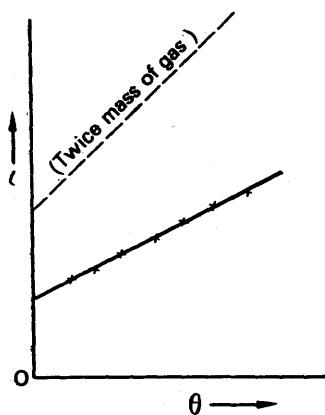


FIG. 10.4. Results of experiment.

in C is then always at atmospheric (constant) pressure. We measure the length l and plot it against the temperature, θ (Fig. 10.4).

If S is the cross-section of the tube, the volume of the trapped air is

$$V = lS.$$

The cross-section S , and the distance between the divisions on which we read l , both increase with the temperature θ . But their increases are very small compared with the expansion of the gas, and therefore we may say that the volume of the gas is proportional to the scale-reading of l . The graph then shows that the volume of the gas, at constant pressure, increases uniformly with its temperature. A similar result is obtained with twice the mass of gas, as indicated in Fig. 10.4.

Expansivity of Gas (Volume Coefficient)

We express the rate at which the volume of a gas increases with temperature by defining a quantity called its *expansivity at constant pressure*, α_p or *volume coefficient*:

$$\alpha_p = \frac{\text{volume at } \theta^\circ\text{C} - \text{volume at } 0^\circ\text{C}}{\text{volume at } 0^\circ\text{C}} \times \frac{1}{\theta}.$$

Thus, if V is the volume at $\theta^\circ\text{C}$, and V_0 the volume at 0°C , then

$$\alpha_p = \frac{V - V_0}{V_0 \theta},$$

whence

$$V - V_0 = V_0 \alpha_p \theta,$$

or

$$V = V_0(1 + \alpha_p \theta).$$

The expansivity α_p has the dimensions

$$\frac{[\text{volume}]}{[\text{volume}] \times [\text{temperature}]} = \frac{1}{[\text{temperature}]}$$

Its value is about $\frac{1}{273}$ when the temperature is measured in °C, and we therefore say that

$$\alpha_p = \frac{1}{273} \text{ per deg C, or } \frac{1}{273} \text{ K}^{-1}.$$

Charles, and Gay-Lussac, found that α_p had the same value, $\frac{1}{273}$, for all gases. This observation is now called Charles's or Gay-Lussac's Law: *The volume of a given mass of any gas, at constant pressure, increases by $\frac{1}{273}$ of its value at 0°C, for every degree Centigrade rise in temperature.*

Absolute Temperature

Charles's Law shows that, if we plot the volume V of a given mass of any gas at constant pressure against its temperature θ , we shall get a straight line graph A as shown in Fig. 10.5. If we produce this line back-

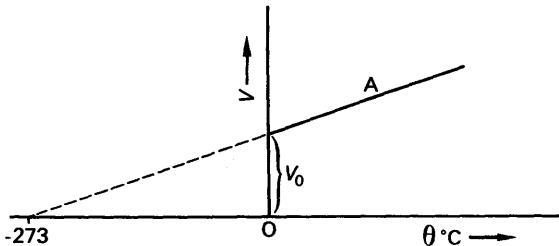


FIG. 10.5. Absolute zero.

wards, it will meet the temperature axis at -273°C . This temperature is called the *absolute zero*. If a gas is cooled, it liquefies before it reaches -273°C , and Charles's Law no longer holds; but that fact does not affect the form of the relationship between the volume and temperature at higher temperatures. We may express this relationship by writing

$$V \propto (273 + \theta).$$

The quantity $(273 + \theta)$ is called the *absolute temperature* of the gas, and is denoted by T . The idea of absolute temperature was developed by Lord Kelvin, and absolute temperatures are hence expressed in degrees Kelvin:

$$T \text{ K} = (273 + \theta) \text{ }^{\circ}\text{C}.$$

From Charles's Law, we see that the volume of a given mass of gas at constant pressure is proportional to its absolute temperature, since

$$V \propto (273 + \theta) \propto T.$$

Thus if a given mass of gas has a volume V_1 at θ_1 °C, and is heated at constant pressure to θ_2 °C, its new volume is given by

$$\frac{V_1}{V_2} = \frac{273 + \theta_1}{273 + \theta_2} = \frac{T_1}{T_2}.$$

Pressure and Temperature

The effect of temperature on the pressure of a gas, at constant volume, was investigated by Amontons in 1702. His work was forgotten, however, and was re-discovered only after the work of Gay-Lussac and Charles on the effect of temperature on volume.

An apparatus for measuring the pressure of a constant volume of gas at various known temperatures is shown in Fig. 10.6 (a). The bulb B

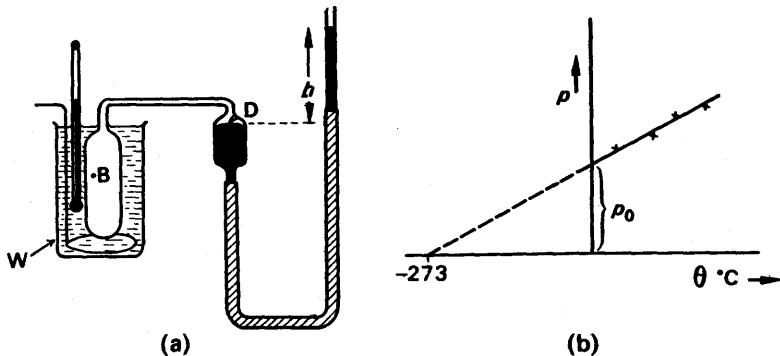


FIG. 10.6. Pressure and temperature.

contains air, which can be brought to any temperature θ by heating the water in the surrounding bath W. When the temperature is steady, the mercury in the closed limb of the tube is brought to a fixed level D, so that the volume of the air is fixed. The difference in level, h , of the mercury in the open and closed limbs is then added to the height of the barometer, H , to give the pressure p of the gas in cm of mercury. If p , $(h+H)$, is plotted against the temperature, the plot is a straight line (Fig. 10.6 (b)).

The coefficient of pressure increase at constant volume, α_v , known as the *pressure coefficient*, is given by

$$\alpha_v = \frac{p - p_0}{p_0 \theta},$$

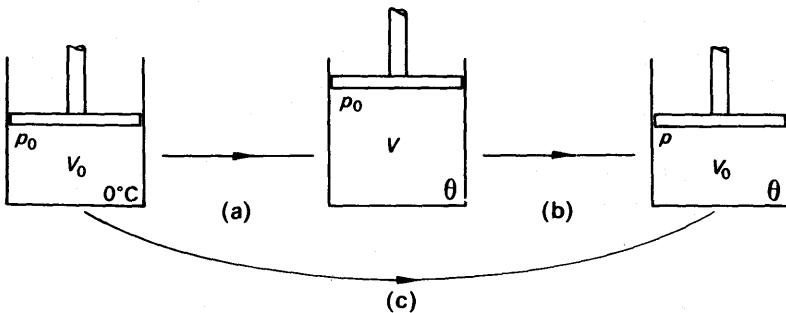
where p_0 is the pressure at 0°C . The coefficient α_v , which expresses the change of pressure with temperature, at constant volume, has practically the same value for all gases: $\frac{1}{273} \text{ K}^{-1}$. It is thus numerically equal to the expansivity, α_p . We may therefore say that, at constant volume, the pressure of a given mass of gas is proportional to its absolute temperature T , since

$$p \propto (273 + \theta).$$

$$\therefore \frac{p_1}{p_2} = \frac{273 + \theta_1}{273 + \theta_2} = \frac{T_1}{T_2}.$$

Equality of Pressure and Volume Coefficients

If a gas obeys Boyle's Law, its coefficient of pressure change at constant volume, α_v , and of volume change at constant pressure, α_p ,

FIG. 10.7. Showing that $\alpha_V = \alpha_p$.

must be equal. For let us suppose that a given mass of gas is warmed at constant pressure, p_0 , from 0°C to $\theta^\circ\text{C}$ (Fig. 10.7 (a)). Its volume expands from V_0 to V , where

$$V = V_0(1 + \alpha_p \theta).$$

Now let us suppose that it is compressed, at constant temperature, until its volume returns to V_0 (Fig. 10.7 (b)). Then its pressure rises to p , where

$$\begin{aligned} pV_0 &= p_0V \\ &= p_0V_0(1 + \alpha_p \theta) \\ p &= p_0(1 + \alpha_p \theta) \end{aligned} \quad \quad (3)$$

or
The condition of the gas is now the same as if it had been warmed at constant volume from 0°C to $\theta^\circ\text{C}$ (Fig. 10.7 (c)). Therefore

$$p = p_0(1 + \alpha_V \theta);$$

and, by equation (3), it follows that

$$\alpha_V = \alpha_p.$$

We shall see later that gases do not obey Boyle's law exactly, although at moderate pressures they do so very nearly. The difference between α_p and α_V provides a sensitive test for departures from Boyle's Law.

The Equation of State

Fig. 10.8 illustrates the argument by which we may find the general relationship between pressure, volume and temperature of a given mass of gas. This relationship is called the *equation of state*.

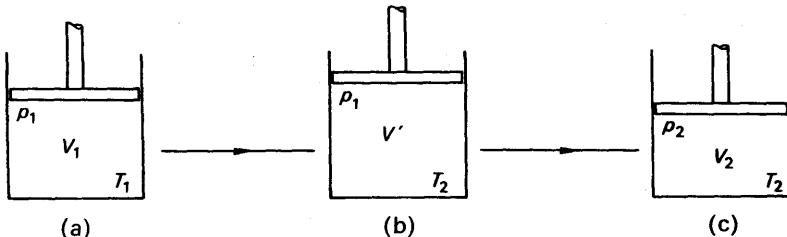


FIG. 10.8. Changing temperature and pressure of a gas.

At (a) we have the gas occupying a volume V_1 at a pressure p_1 , and an absolute temperature T_1 . We wish to calculate its volume V_2 at an absolute temperature T_2 and pressure p_2 , as at (c). We proceed via (b), raising the temperature of T_2 while keeping the pressure constant at p_1 . If V' is the volume of the gas at (b), then, by Charles's law:

$$\frac{V'}{V_1} = \frac{T_2}{T_1} \quad \dots \quad (4)$$

We proceed now to (c), by increasing the pressure to p_2 , while keeping the temperature constant at T_2 . By Boyle's law,

$$\frac{V_2}{V'} = \frac{p_1}{p_2} \quad \dots \quad (5)$$

Eliminating V' between equations (4) and (5), we find

$$\frac{V_2}{V_1} = \frac{T_2}{T_1} \cdot \frac{p_1}{p_2}$$

or $\frac{p_2 V_2}{T_2} = \frac{p_1 V_1}{T_1}$

In general therefore,

$$\frac{pV}{T} = R \quad \dots \quad (6)$$

where R is a constant. This equation is often given in the form

$$pV = RT \quad \dots \quad (7)$$

Equation (7) is the *equation of state* for a perfect gas. The value of the constant R depends on the nature of the gas—air, hydrogen, etc.—and on the mass of the gas concerned. If we consider *unit mass* of a gas, we can denote its volume by V and write

$$pV = RT \quad \dots \quad (8)$$

R is called the *gas-constant* for unit mass of the gas. If ρ is the density of the gas, at absolute temperature T and pressure p , then

$$\rho = \frac{1}{V}$$

and equation (8) becomes $\frac{p}{\rho} = RT \quad \dots \quad (9)$

The volume V of an arbitrary mass M of the gas, at absolute temperature T and pressure p , is

$$V = MV; \quad \dots$$

therefore, by (8) $pV = MRT; \quad \dots \quad (10)$

and, by (7), $R = MR.$

Magnitude of the Gas Constant

To calculate the constant R for a gas, we need to know the density of the gas at a given temperature and pressure. Very often, in dealing with gases, we specify the pressure not in newton per metre² ($N\ m^{-2}$) but simply in millimetres of mercury. 1 mm mercury pressure is called 1 torr. We do so because we are concerned only with relative values. A pressure of 760 mm of mercury, which is about the average pressure of the atmosphere, is sometimes called 'standard' or 'normal' pressure. A temperature of 0°C, or 273 K, is likewise called standard or normal temperature. The conditions 273 K and 760 mm pressure are together called standard temperature and pressure (s.t.p.). A pressure of 760 mm mercury is given, in newton per metre², by

$$\begin{aligned} p &= g\rho H \\ &= 9.8 \times 13600 \times 0.76 = 1.013 \times 10^5 \text{ N m}^{-2}, \end{aligned}$$

since ρ = mercury density = 13600 kg m⁻³; g = 9.8 m s⁻²; H = 0.76 m.

At s.t.p. the density of hydrogen is about 0.09 g/litre, or 0.09 kg m⁻³. The gas-constant for unit mass of hydrogen, from (9), is therefore

$$\begin{aligned} R &= \frac{p}{\rho T} = \frac{1.013 \times 10^5}{0.09 \times 273} \\ &= 4.16 \times 10^3, \text{ in the appropriate units.} \end{aligned}$$

We will now discuss the units in which R is expressed.

The Gas-constant Units: Work done in Expansion

The gas-constant R for an *arbitrary mass* of gas is defined by the equation

$$pV = RT$$

or

$$R = \frac{pV}{T}.$$

Its unit is therefore that of

$$\frac{\text{pressure} \times \text{volume}}{\text{temperature}}.$$

In SI units, the pressure is in N m⁻², the volume in m³ and the temperature in K. If we are given values of p , V and T and work out the value of R , we express it in the corresponding units. The constant per unit mass, R , for 1 kg has the unit of

$$\begin{aligned} \frac{\text{pressure} \times \text{volume}}{\text{temperature} \times \text{mass}} &= \frac{\text{N m}^{-2} \times \text{m}^3}{\text{K} \times \text{kg}} = \text{N m kg}^{-1} \text{ K}^{-1} \\ &= \text{J kg}^{-1} \text{ K}^{-1}. \end{aligned}$$

since 1 newton × 1 metre = 1 joule. The gas constant may thus be expressed in the same units as specific heat capacity (p. 199).

The gas constant depends on the mass of gas. For 1 kg, the unit is

$\text{J kg}^{-1} \text{ K}^{-1}$. For 1 mole, the unit is $\text{J mol}^{-1} \text{ K}^{-1}$; for 1 kmol, the unit is $\text{J kmol}^{-1} \text{ K}^{-1}$.

Work Done. The product of pressure and volume has the dimensions of *work*. To see this, let us imagine some gas, at a pressure p , in a cylinder fitted with a piston (Fig. 10.9).

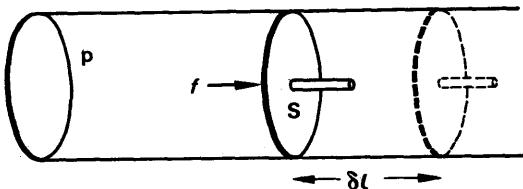


FIG. 10.9. Work done in expansion.

If the piston has an area S , the force on it is

$$f = pS.$$

If we allow the piston to move outwards a distance δl , the gas will expand, and its pressure will fall. But by making the distance very short, we can make the fall in pressure so small that we may consider the pressure constant. The force f is then constant, and the work done is

$$\delta W = f \cdot \delta l = pS \cdot \delta l.$$

The product $S \cdot \delta l$ is the increase in volume, δV , of the gas, so that

$$\delta W = p \cdot \delta V \quad \quad (11)$$

The product of pressure and volume, in general, therefore represents work. If the pressure p is in newton m^{-2} , and the area S is in m^2 , the force f is in newtons. And if the movement δl is in m, the work $f \cdot \delta l$ is in newton \times metre or *joule* (J). The increase of volume, δV , is in m^3 . Thus the product of pressure in N m^{-2} , and volume in m^3 , represents work in joules.

Consequently, if we express the pressure of a gas in newton m^{-2} and its volume in m^3 , the gas-constant R , $= pV/T$, is in joule (J) per degree. The constant for 1 kg, R , is in joule per kg per degree. The value of R for hydrogen, which we calculated on p. 228, is

$$R = 4.16 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1} = 4.16 \text{ kJ kg}^{-1} \text{ K}^{-1}.$$

Avogadro's Hypothesis: Molecular Weight

Amedeo Avogadro, with one simple-looking idea, illuminated chemistry as Newton illuminated mechanics. In 1811 he suggested that chemically active gases, such as oxygen, existed not as single atoms, but as pairs: he proposed to distinguish between an atom, O, and a molecule, O_2 . Ampere proposed the same distinction, independently, in 1814. Avogadro also put forward another idea, now called *Avogadro's hypothesis*: that equal volumes of all gases, at the same temperature and pressure, contained equal numbers of molecules. The number of molecules in 1 cm^3 of gas at s.t.p. is called Loschmidt's number; it is 2.69×10^{19} .

Avogadro's hypothesis became accepted in the middle of the nineteenth century. Because molecules could not be observed, their masses could not be measured directly; but they could be compared, by chemical methods. The molecular mass of a substance, μ , was at first defined as the ratio of the mass of its molecule, m , to the mass of a hydrogen atom. Later, for the convenience of chemists, it was defined as the ratio of the molecular mass to the mass of an imaginary atom, this atom having $\frac{1}{12}$ th the mass of a carbon atom ^{12}C :

$$\mu = \frac{\text{mass of molecule}}{\frac{1}{12} \text{ mass of C-atom}} = \frac{m}{\frac{1}{12} m_{\text{C}}} = \frac{12m}{m_{\text{C}}}.$$

On this scale, the mass of a hydrogen atom is 1.008 times the mass of the imaginary atom. And since the hydrogen molecule contains two atoms, its molar mass is

$$\mu_{\text{H}_2} = 2.016.$$

The unit of molecular mass, $m_{\text{C}}/12$, is also the unit of atomic mass; its value is 1.66×10^{-24} g.

The Mole: Molar Gas-constant

The amount of a substance which contains as many elementary units as there are atoms in 0.012 kg (12 g) of carbon-12 is called a *mole*, symbol 'mol'. The number of molecules in a mole, N_A , is given, if m is the mass of a molecule in grammes, by

$$\mu = N_A m,$$

whence

$$N_A = \frac{\mu}{m} = \frac{12m/m_{\text{C}}}{m} \\ = \frac{12}{m_{\text{C}}}.$$

The number of molecules per mole is thus the same for all substances. It is called Avogadro's constant, and is equal to $6.02 \times 10^{23} \text{ mol}^{-1}$.

From Avogadro's hypothesis, it follows that the mole of all gases, at the same temperature and pressure, occupy equal volumes. Experiment confirms this; at s.t.p. 1 mole of any gas occupies 22.4 litres. Consequently, if we denote by V the volume of 1 mole, then the ratio $\frac{pV}{T}$ is the same for all gases. We call it the *molar gas constant*, R , and

$$R = \frac{pV}{T}.$$

At s.t.p. $V = 22.4 \text{ litres} = 22.4 \times 10^{-3} \text{ m}^3$

$$p = 760 \text{ mm mercury} = 1.013 \times 10^5 \text{ N m}^{-2} (\text{p. 228})$$

$$T = 273 \text{ K.}$$

$$\therefore R = \frac{1.013 \times 10^5 \times 22.4 \times 10^{-3}}{273}$$

$$= 8.31 \text{ J mol}^{-1} \text{ K}^{-1}.$$

The value of R is the same for the moles of all substances. If μ g is the molar mass of a gas, the constant for 1 g is thus

$$R = \frac{R}{\mu} \quad \quad (12)$$

KINETIC THEORY

The kinetic theory of matter, which regards all bodies as assemblies of particles in motion—either vibrating or flying about—is an old one. Lucretius described it in the first century A.D. and Gassendi and Hooke revived it in the seventeenth century. In 1738 D. Bernoulli applied it in detail to a gas, and from it deduced Boyle's law, which was already known from experiment. Another century passed, however, before the kinetic view of a gas was fully developed—mainly by Clausius (1822–88), Boltzmann (1844–1906), and Maxwell (1831–79).

In the kinetic theory of gases, we seek to explain the behaviour of gases by considering the motion of their molecules. In particular, we suppose that the pressure of a gas is due to the molecules bombarding the walls of its container. Whenever a molecule bounces off a wall, its momentum at right-angles to the wall is reversed; the force which it exerts on the wall is equal to the rate of change of its momentum. The average force exerted by the gas on the whole of its container is the average rate at which the momentum of its molecules is changed by collision with the walls.

To find the pressure of the gas we must find this force, and then divide it by the area of the walls. The following assumptions are made to simplify the calculation:

- (a) The attraction between the molecules is negligible.
- (b) The volume of the molecules is negligible compared with the volume occupied by the gas.
- (c) The molecules are like perfectly elastic spheres.
- (d) The duration of a collision is negligible compared with the time between collisions.

Calculation of Pressure

Consider for convenience a cube of side l containing N molecules of gas each of mass m . Fig. 10.10. A typical molecule will have a velocity c at any instant and this will have components of u , v , w respectively in the direction of the three perpendicular axes Ox , Oy , Oz as shown. Thus $c^2 = u^2 + v^2 + w^2$.

Consider the force exerted on the face X of the cube due to the component u . Just before impact, the momentum of the molecule due to u is mu . After impact, the momentum is $-mu$, since the momentum reverses. Thus

$$\text{momentum change on impact} = mu - (-mu) = 2mu.$$

The time taken for the molecule to move across the cube to the opposite face and back to X is $2l/u$. Hence

$$\text{momentum change per second} = \frac{\text{momentum change}}{\text{time}}$$

$$= \frac{2mu}{2l/u} = \frac{mu^2}{l}$$

$$\therefore \text{force on } X = \frac{mu^2}{l}$$

$$\therefore \text{pressure on } X = \frac{\text{force}}{\text{area}} = \frac{mu^2}{l \times l^2} = \frac{mu^2}{l^3} \quad (\text{i})$$

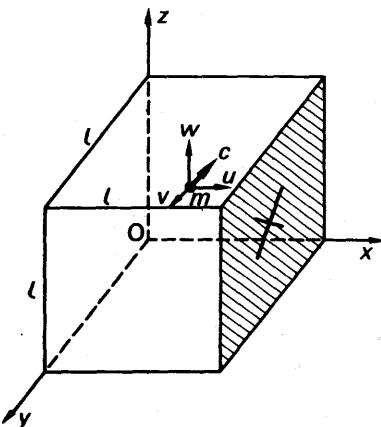


FIG. 10.10. Calculation of gas pressure.

We now take account of the N molecules in the cube. Each has a different velocity and hence a component of different magnitude in the direction Ox. If these are represented by $u_1, u_2, u_3, \dots, u_N$, it follows from (i) that the total pressure on X, p , is given by

$$\begin{aligned} p &= \frac{mu_1^2}{l^3} + \frac{mu_2^2}{l^3} + \frac{mu_3^2}{l^3} + \dots + \frac{mu_N^2}{l^3} \\ &= \frac{m}{l^3}(u_1^2 + u_2^2 + u_3^2 + \dots + u_N^2) \quad (\text{ii}) \end{aligned}$$

Let the symbol $\bar{u^2}$ represent the average or mean value of all the squares of the components in the Ox direction, that is,

$$\bar{u^2} = \frac{u_1^2 + u_2^2 + u_3^2 + \dots + u_N^2}{N}.$$

$$\text{Then } N\bar{u^2} = u_1^2 + u_2^2 + u_3^2 + \dots + u_N^2.$$

Hence, from (ii),

$$p = \frac{Nm\bar{u^2}}{l^3} \quad \quad (\text{iii})$$

Now with a large number of molecules of varying speed in random motion, the mean square of the component speed in any one of the three axes is the same.

$$\therefore \bar{u^2} = \bar{v^2} = \bar{w^2}.$$

But, for each molecule, $c^2 = u^2 + v^2 + w^2$, so that the mean square $\bar{c^2}$ is given by $\bar{c^2} = \bar{u^2} + \bar{v^2} + \bar{w^2}$.

$$\therefore \bar{u^2} = \frac{1}{3}\bar{c^2}.$$

Hence, from (iii),

$$p = \frac{1}{3} \frac{Nm\bar{c^2}}{l^3}.$$

The number of molecules per unit volume, n , = N/l^3 . Thus we may write

$$p = \frac{1}{3} n m c^2 \quad \quad (13)$$

If n is in molecules per metre³, m is in kilogramme and c in metre per second, then the pressure p is in newton per metre² ($N\text{ m}^{-2}$).

It should be carefully noted that the pressure p of the gas depends on the 'mean square' of the speed. This is because (a) the momentum change at a wall is proportional to u , as previously explained, and (b) the time interval before this change is repeated is inversely-proportional to u . Thus the rate of change of momentum is proportional to $u \div 1/u$ or to u^2 . Further, the mean-square speed is not equal to the square of the average speed. As an example, let us suppose that the speeds of six molecules are, 1, 2, 3, 4, 5, 6 units. Their mean speed is

$$\bar{c} = \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = 3.5,$$

and its square is $(\bar{c})^2 = 3.5^2 = 12.25$.

Their mean square speed, however is

$$\bar{c^2} = \frac{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2}{6} = \frac{91}{6} = 15.2.$$

This differs by about 25 per cent from the square of the mean speed.

In our calculation, we assumed that molecules of a gas do not collide with other molecules as they move to-and-fro across the cube. If,

however, we assume that their collisions are perfectly elastic, both the kinetic energy and the momentum are conserved in them. The average momentum with which all the molecules strike the walls is then not changed by their collisions with one another; what one loses, another gains. The important effect of collisions between molecules is to distribute their individual speeds; on the average, the fast ones lose speed to the slow. We suppose, then, that different molecules have different speeds, and that the speeds of individual molecules vary with time, as they make collisions with one another; but we also suppose that the average speed of all the molecules is constant. These assumptions are justified by the fact that the kinetic theory leads to conclusions which agree with experiment.

Root-mean-square (R.M.S.) Speed

In equation (13) the factor nm is the product of the molecules per unit volume and the mass of one molecule. It is therefore the total mass of the gas per unit volume: its density ρ . Thus the equation gives

$$p = \frac{1}{3}\rho\bar{c}^2 \quad \dots \quad (14)$$

or

$$\frac{p}{\rho} = \frac{1}{3}\bar{c}^2 \quad \dots \quad (15)$$

If we substitute known values of p and ρ in equation (15), we can find \bar{c}^2 . For hydrogen at s.t.p.,

$$\rho = 0.09 \text{ kg m}^{-3}$$

The pressure in newton per m^2 , is $p = g\rho H$, where g = acceleration of gravity = 9.81 m s^{-2} , ρ = density of mercury = 13600 kg m^{-3} , H = barometer height = $760 \text{ mm} = 0.76 \text{ m}$.

$$\begin{aligned} \therefore \bar{c}^2 &= \frac{3p}{\rho} = \frac{3 \times 9.81 \times 13600 \times 0.76}{9 \times 10^{-2}} \\ &= 3.37 \times 10^6 \text{ m}^2 \text{ s}^{-2}. \end{aligned}$$

The square root of \bar{c}^2 is called the *root-mean-square speed*; it is of the same magnitude as the average speed, but not quite equal to it. Its value is

$$\sqrt{\bar{c}^2} = \sqrt{3.37} \times 10^3 = 1840 \text{ m s}^{-1} (\text{approx.})$$

$$= 1.84 \text{ km s}^{-1}$$

Molecular speeds were first calculated in this way by Joule in 1848; they turn out to have a magnitude which is high, but reasonable. The value is reasonable because it has the same order of magnitude as the speed of sound (1.30 km s^{-1} in hydrogen at 0°C); the speed of sound is the speed with which the molecules of a gas pass on a disturbance from one to another, and this we may expect to be of the same magnitude as the speeds of their natural motion.

Introduction of the Temperature

Let us consider a volume V of gas, containing N molecules. The number of molecules per unit volume is

$$n = \frac{N}{V}$$

and therefore the pressure of the gas, by equation (13) is

$$p = \frac{1}{3}nmc^2 = \frac{1}{3}\frac{N}{V}mc^2$$

$$\therefore pV = \frac{1}{3}Nm\bar{c}^2. \quad . \quad . \quad . \quad . \quad (16)$$

Equation (16) reminds us of the equation combining Boyle's and Charles's laws:

$$pV = RT.$$

We can therefore make the kinetic theory consistent with the observed behaviour of a gas, if we write

$$\frac{1}{3}Nm\bar{c}^2 = RT \quad . \quad . \quad . \quad . \quad (17)$$

Essentially, we are here assuming that the mean square speed of the molecules, \bar{c}^2 , is proportional to the absolute temperature of the gas. This is a reasonable assumption, because we have learnt that heat is a form of energy; and the kinetic energy of a molecule, due to its random motion within its container, is proportional to the square of its speed. When we heat a gas, we expect to speed-up its molecules. See p. 237.

The kinetic energy of a molecule moving with a speed c is $\frac{1}{2}mc^2$; the average kinetic energy of translation of the random motion of the molecule of a gas is therefore $\frac{1}{2}m\bar{c}^2$. To relate this to the temperature, we put equation (17) into the form

$$RT = \frac{1}{3}Nm\bar{c}^2 = \frac{2}{3}N(\frac{1}{2}m\bar{c}^2),$$

$$\text{whence } \frac{1}{2}m\bar{c}^2 = \frac{3}{2}\frac{R}{N}T \quad . \quad . \quad . \quad . \quad (18)$$

Thus, the average kinetic energy of translation of a molecule is proportional to the absolute temperature of the gas.

The ratio R/N in equation (18) is a universal constant. To see that it is, we have only to consider a mole. We have already seen that, for a mole, the gas constant R , and number of molecules N , are universal constants. If our arbitrary mass of gas is x moles, then $R = xR$, and $N = xN$; therefore

$$\frac{R}{N} = \frac{R}{N} = k.$$

The constant k , the gas constant per molecule, is also a universal constant; it is often called *Boltzmann's constant*. In terms of k equation (18) becomes

$$\frac{1}{2}m\bar{c}^2 = \frac{3}{2}kT \quad . \quad . \quad . \quad . \quad (19)$$

Boltzmann's constant is usually given in joules per degree, since it relates energy to temperature:

$$k = \frac{\frac{1}{2}mc^2}{\frac{3}{2}T}$$

Its value is $k = 1.38 \times 10^{-23} \text{ J K}^{-1}$.

Diffusion: Graham's Law

When a gas passes through a porous plug, a cotton-wool wad, for example, it is said to 'diffuse'. Diffusion differs from the flow of a gas through a wide tube, in that it is not a motion of the gas in bulk, but is a result of the motion of its individual molecules.

Fig. 10.11 shows an apparatus devised by Graham (1805–69) to compare the rates of diffusion of different gases. D is a glass tube, closed with a plug P of plaster of Paris. It is first filled with mercury, and inverted over mercury in a bowl. Hydrogen is then passed into it until the mercury levels are the same on each side; the hydrogen is then at atmospheric pressure. The volume of hydrogen, V_H , is proportional to the length of the tube above the mercury. The apparatus is now left; hydrogen diffuses out through P, and air diffuses in. Ultimately no hydrogen remains in the tube D. The tube is then adjusted until the level of mercury is again the same on each side, so that the air within it is at atmospheric pressure. The volume of air, V_A , is proportional to the new length of the tube above the mercury.

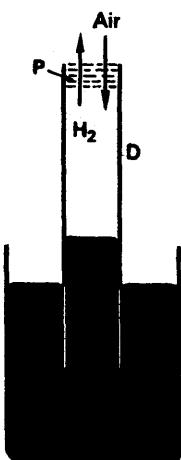


FIG. 10.11. Graham's apparatus for diffusion.

The volumes V_A and V_H are, respectively, the volumes of air and hydrogen which diffused through the plug in the same time. Therefore the rates of diffusion of the gases air and hydrogen are proportional to the volumes V_A and V_H :

$$\frac{\text{rate of diffusion of air}}{\text{rate of diffusion of hydrogen}} = \frac{V_A}{V_H}$$

Graham found in his experiments that the volumes were inversely proportional to the square roots of the densities of the gases, ρ :

$$\frac{V_A}{V_H} = \sqrt{\frac{\rho_H}{\rho_A}}$$

thus
$$\frac{\text{rate of diffusion of air}}{\text{rate of diffusion of hydrogen}} = \sqrt{\frac{\rho_H}{\rho_A}}$$

In general:
$$\text{rate of diffusion} \propto \frac{1}{\sqrt{\rho}}$$

and in words: *the rate of diffusion of a gas is inversely proportional to the square root of its density*. This is *Graham's Law*.

Graham's law of diffusion is readily explained by the kinetic theory. At the same Kelvin temperature T , the mean kinetic energies of the molecules of different gases are equal, since

$$\frac{1}{2}mc^2 = \frac{3}{2}kT$$

and k is a universal constant. Therefore, if the subscripts A and H denote air and hydrogen respectively,

$$\frac{1}{2}m_A c_A^2 = \frac{1}{2}m_H c_H^2,$$

whence

$$\frac{c_A^2}{c_H^2} = \frac{m_H}{m_A}.$$

At a given temperature and pressure, the density of a gas, ρ , is proportional to the mass of its molecule, m , since equal volumes contain equal numbers of molecules:

Therefore

$$\frac{m_H}{m_A} = \frac{\rho_H}{\rho_A},$$

whence

$$\frac{c_A^2}{c_H^2} = \frac{\rho_H}{\rho_A}.$$

$$\therefore \frac{\sqrt{c_A^2}}{\sqrt{c_H^2}} = \frac{\sqrt{\rho_H}}{\sqrt{\rho_A}} \quad \quad (20)$$

The average speed of the molecules of a gas is roughly equal to—and strictly proportional to—the square root of its mean square speed. Equation (20) therefore shows that the average molecular speeds are inversely proportional to the square roots of the densities of the gases. And so it explains why the rates of diffusion—which depend on the molecular speeds—are also inversely proportional to the square roots of the densities.

Thermal Agitations and Internal Energy

The random motion of the molecules of a gas, whose kinetic energy depends upon the temperature, is often called the *thermal agitation* of the molecules. And the kinetic energy of the thermal agitation is called the *internal energy* of the gas. We must appreciate that this energy is quite independent of any motion of the gas in bulk: when a cylinder of oxygen is being carried by an express train, its kinetic energy as a whole is greater than when it is standing on the platform; but the random motion of the molecules within the cylinder is unchanged—and so is the temperature of the gas. The same is true of a liquid; in a water-churning experiment to convert mechanical energy into heat, baffles must be used to prevent the water from acquiring any mass-motion—all the work done must be converted into random motion, if it is to appear as heat. Likewise, the internal energy of a solid is the kinetic energy of its atoms' vibrations about their mean positions: throwing a lump of metal through the air does not raise its temperature, but hitting it with a hammer does.

The internal energy of a gas depends on the number of atoms in its molecule. A gas whose molecules consist of single atoms is said to be monatomic: for example, chemically inert gases and metallic vapours, Hg, Na, He, Ne, A. A gas

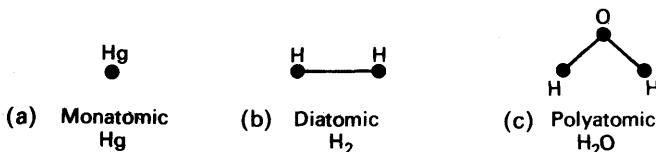


FIG. 10.12. Types of gas molecule.

with two atoms to the molecule is said to be diatomic: O₂, H₂, N₂, Cl₂, CO. And a gas with more than two atoms to the molecule is said to be polyatomic: H₂O, O₃, H₂S, CO₂, CH₄. The molecules of a monatomic gas we may regard as points, but those of a diatomic gas we must regard as 'dumb-bells', and those of a polyatomic gas as more complicated structures (Fig. 10.12). A molecule which extends appreciably in space—a diatomic or polyatomic molecule—has an appreciable moment of inertia. It may therefore have kinetic energy of rotation, as well as of translation. A monatomic molecule, however, must have a much smaller moment of inertia than a diatomic or polyatomic; its kinetic energy of rotation can therefore be neglected.

Fig. 10.12A shows a monatomic molecule whose velocity c has been resolved into its components u , v , w along the x , y , z axes:

$$c^2 = u^2 + v^2 + w^2.$$

FIG. 10.12A. Components of velocity.

The x , y , z axes are called the molecules' degrees of freedom: they are three directions such that the motion of the molecule along any one is independent of its motion along the others.

If we average the speed c , and the components u , v , w , over all the molecules in a gas, we have

$$\bar{c}^2 = \bar{u}^2 + \bar{v}^2 + \bar{w}^2.$$

And since the molecules do not pile up in any corner of the vessel containing the gas, their average velocities in all directions must be the same. We may therefore write

$$\begin{aligned} \bar{u}^2 &= \bar{v}^2 = \bar{w}^2, \\ \text{whence } \bar{c}^2 &= 3\bar{u}^2 = 3\bar{v}^2 = 3\bar{w}^2, \\ \text{or } \bar{u}^2 &= \bar{v}^2 = \bar{w}^2 = \frac{1}{3}\bar{c}^2. \end{aligned}$$

The average kinetic energy of a molecule of the gas is given by equation (19):

$$\frac{1}{2}mc^2 = \frac{3}{2}kT.$$

Therefore the average kinetic energy of a monatomic molecule, in each degree of freedom, is

$$\frac{1}{2}m\bar{u}^2 = \frac{1}{2}m\bar{v}^2 = \frac{1}{2}m\bar{w}^2 = \frac{1}{2}kT.$$

Thus the molecule has kinetic energy $\frac{1}{2}kT$ per degree of freedom.

Rotational Energy

Let us now consider a diatomic or polyatomic gas. When two of its molecules collide, they will, in general, tend to rotate, as well as to deflect each other. In some collisions, energy will be transferred from the translations of the molecules

to their rotations; in others, from the rotations to the translations. We may assume, then, that the internal energy of the gas is shared between the rotations and translations of its molecules.

To discuss the kinetic energy of rotation, we must first extend the idea of degrees of freedom to it. A diatomic molecule can have kinetic energy of rotation about any axis at right-angles to its own. Its motion about any such axis can be resolved into motions about two such axes at right-angles to each other (Fig. 10.13 (a)). Motions about these axes are independent of each other, and a diatomic

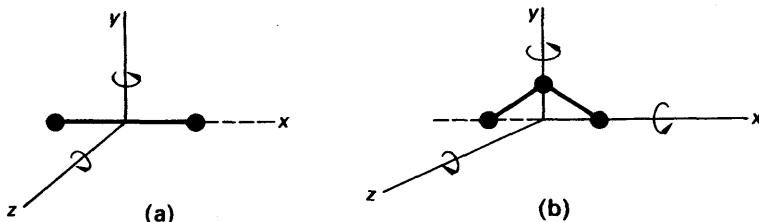


FIG. 10.13. Rotation of molecules.

molecule therefore has two degrees of rotational freedom. A polyatomic molecule, unless it happens to consist of molecules all in a straight line, has no axis about which its moment of inertia is negligible. It can therefore have kinetic energy of rotation about three mutually perpendicular axes (Fig. 10.13 (b)). It has three degrees of rotational freedom.

We have seen that the internal energy of a gas is shared between the translations and rotations of its molecules. Maxwell assumed that the average kinetic energy of a molecule, in each degree of freedom, *rotational as well as translational*, was $\frac{1}{2}kT$. This assumption is called the *principle of equipartition of energy*; experiment shows, as we shall find, that it is true at room temperature and above. At very low temperatures, when the gas is near liquefaction, it fails. At ordinary temperatures, then, we have:

$$\text{average K.E. of monatomic molecule} = \frac{3}{2}kT \text{ (trans.)};$$

$$\text{average K.E. of diatomic molecule} = \frac{3}{2}kT \text{ (trans.)} + \frac{2}{2}kT \text{ (rot.)} = \frac{5}{2}kT;$$

$$\text{average K.E. of polyatomic molecule} = \frac{3}{2}kT \text{ (trans.)} + \frac{3}{2}kT \text{ (rot.)} = \frac{6}{2}kT.$$

Internal Energy of any Gas

From the average kinetic energy of its molecules, we can find the internal energy of a mass M of a gas. The number of molecules in this mass is, if m is the mass of one molecule,

$$N = \frac{M}{m}.$$

Its internal energy, U , is the total kinetic energy of its molecules' random motions; thus

$$U = N \times \text{average K.E. of molecule.}$$

For a monatomic gas, therefore,

$$U = \frac{3}{2}NkT \text{ (monatomic).}$$

The constant k is the gas-constant per molecule; the product Nk is therefore the gas-constant R for the mass M of the gas. Thus

$$U = \frac{3}{2}RT \text{ (monatomic).}$$

In particular, if R is the gas-constant per kg, the internal energy per kg is

$$U = \frac{3}{2}RT \text{ (monatomic)} \quad (21)$$

Similarly, for a diatomic gas,

$$U = \frac{5}{2}NkT = \frac{5}{2}RT \quad (22)$$

$$U = \frac{5}{2}RT$$

And for a polyatomic gas,

$$U = \frac{6}{2}NkT = \frac{6}{2}RT \quad (23)$$

$$U = \frac{6}{2}RT$$

Internal Energy and Volume

In our simple account of the kinetic theory of gases, we have implicitly assumed that the molecules of a gas do not attract one another. If they did, any molecule approaching the boundary of the gas would be pulled towards the body of it, as is a molecule of water approaching the surface (see Chapter 6, *Surface Tension*, p. 128). The attractions of the molecules would thus reduce the pressure of the gas.

Since the molecules of a substance are presumably the same whether it is liquid or gas, the molecules of a gas must attract one another somewhat. But except for brief instants when they collide, the molecules of a gas are much further apart than those of a liquid. In 1 cubic centimetre of gas at s.t.p. there are 2.69×10^{19} molecules, and in 1 cubic centimetre of water there are 3.33×10^{22} ; there are a thousand times as many molecules in the liquid, and so the molecules in the gas are ten times further apart. We may therefore expect that the mutual attraction of the molecules of a gas, for most purposes, can be neglected, as experiment, in fact, shows.

The experiment consists in allowing a gas to expand without doing external work; that is, to expand into a vacuum. Then, if the molecules attract one another, work is done against their attractions, as they move further apart. But if the molecular attractions are negligible, the work done is also negligible. If any work is done against the molecular attractions, it will be done at the expense of the molecular kinetic energies; as the molecules move apart, they will exert retarding forces on one another. Thus the internal energy of the gas, and therefore its temperature, will fall.

The expansion of a gas into a vacuum is called a 'free expansion'. If a gas does not cool when it makes a free expansion, then the mutual attractions of its molecules are negligible.

Joule's Experiments

Experiments on the free expansion of a gas were made in 1807 by Gay-Lussac; they showed no fall in temperature. Joule repeated these experiments with a better vacuum in 1845; he got the same negative result, and the greater accuracy of his experiments made them more trustworthy. Joule used two forms of apparatus, as shown in Fig. 10.14 (a) and (b). Each consisted of a cylinder of air, R, at 22 atmospheres, connected by a stop-cock S to an evacuated cylinder E. In the apparatus (a) both cylinders stood in the same tin can C, which contained $16\frac{1}{2}$ lb water. In (b) the cylinders stood in different cans, and the stop-cock in a third, also containing water. When the stop-cock was opened, gas expanded from R to E. With the apparatus (a) Joule found, after stirring the water, that its temperature was unchanged. The expanding gas had therefore neither liberated heat nor absorbed it. With the apparatus (b) Joule found that heat was absorbed from the water round R, and given to the water round S and E; the heat given out was equal to the heat

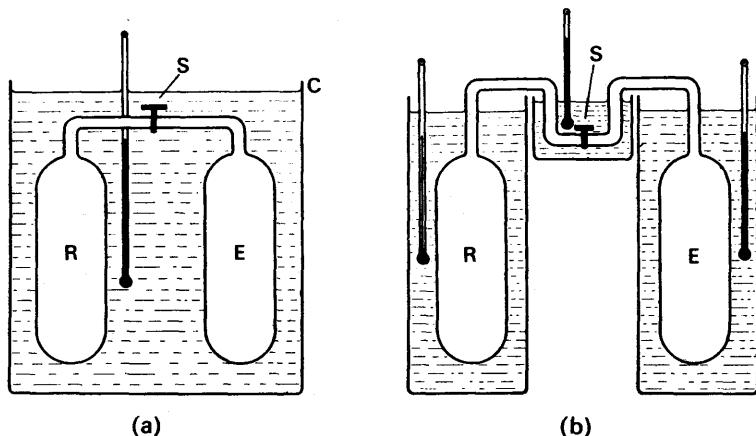


FIG. 10.14. Joule's experiment on internal energy.

taken in. The heat taken in represented the work done by the gas from R, expanding against the rising pressure of the gas in E and in the pipe beyond S. The heat given out represented the work done on the gas in E and S by the gas flowing in, against the rising pressure. The equality of the two showed that the total mass of gas neither gained nor lost energy in making its free expansion. Joule's experiments, therefore, showed that *the internal energy of a gas is independent of its volume*.

From Joule's results we may argue back to show that the mutual attractions of the molecules of the gas are negligible; in practice, however, it is the property of the bulk gas which is important—the fact that its internal energy does not depend on its volume.

Joule's experiments, though more reliable than Gay-Lussac's, were crude; with so much water, a small amount of heat would not produce a measurable temperature rise. Between 1852 and 1862, Joule worked with William Thomson, later Lord Kelvin, on more delicate experi-

ments. They found that most gases, in expanding from high pressure to low, do lose a little of their internal energy. The loss represents work done against the molecular attractions, which are therefore not quite negligible.

If the internal energy of a gas is independent of its volume, it is determined only by the temperature of the gas. The simple expression for the pressure, $p = \frac{1}{3}\rho c^2$, then holds; and the gas obeys Boyle's and Charles's laws. Its pressure coefficient, α_p , is equal to its volume coefficient, α_v . Such a gas is called an ideal, or perfect, gas. All gases, when far from liquefaction, behave for most practical purposes as though they were ideal.

Van der Waals' Equation

In deriving the ideal gas equation $pV = RT$ from the kinetic theory of gases, a number of assumptions were made. These are listed on p. 231. Van der Waals modified the ideal gas equation to take account that two of these assumptions may not be valid. Thus, as explained on p. 127, to which the student should refer:

- (1) *The volume of the molecules may not be negligible in relation to the volume V occupied by the gas.*
- (2) *The attractive forces between the molecules may not be negligible.*

Molecules have a particular diameter or volume because repulsive forces occur when they approach very closely and hence they can not be compressed indefinitely. The volume of the space inside a container occupied by the molecules is thus not V but $(V - b)$, where b is a factor depending on the actual volume of the molecules.

If the attractive forces between molecules are not negligible, the molecules approaching the container walls are attracted by the molecules behind them. This reduces the momentum of the approaching molecules and hence the pressure. The observed pressure p is thus less than the ideal gas pressure, where there are no molecular forces, by a pressure p' . Hence we write $(p + p')$ in place of p in the ideal gas equation. As explained on p. 127, the 'pressure defect' $p' \propto \rho^2$, where ρ is the density of the gas, or $p = a/V^2$, where a is a constant. Thus *van der Waals' equation* for real gases is:

$$\left(p + \frac{a}{V^2} \right) (V - b) = RT.$$

At high pressures, when the molecules are relatively numerous and close together, the volume factor b and pressure 'defect' a/V^2 both become important. Conversely, at low pressures, where the molecules are relatively few and far apart on the average, a gas behaves like an ideal gas and obeys the equation $pV = RT$.

Critical Phenomena

A graph of pressure p v. volume V at constant temperature is called an *isotherm*. Fig. 10.15 (i) shows a number of isotherms for an ideal gas, which obeys the perfect gas law $pV = RT$. Fig. 10.15 (ii) shows a

number of isotherms for a gas which obeys van der Waals' equation, $(p + a/V^2)(V - b) = RT$.

At high temperatures the isotherms are similar. As the temperature is lowered, however, the isotherms in Fig. 10.15 (ii) change in shape. One curve has a point of inflection at C, which corresponds to the *critical point* of a real gas. The isotherms thus approximate to those obtained by Andrews in his experiments on actual gases such as carbon dioxide, described on p. 314.

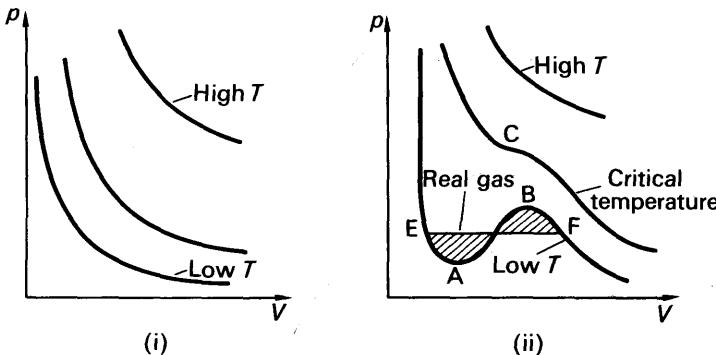


FIG. 10.15. Isotherms for ideal and van der Waals gases.

Below this temperature, however, isotherms such as EABF are obtained by using van der Waals' equation. These are unlike the isotherms obtained with real gases, because in the region AB the pressure increases with the volume, which is impossible. However, an actual isotherm in this region corresponds to a straight line EF, as shown. Here the liquid and vapour are in equilibrium (see p. 317) and the line EF is drawn to make the shaded areas above and below it equal. Thus van der Waals' equation roughly fits the isotherms of actual gases above the critical temperature but below the critical temperature it must be modified considerably. Many other gas equations have been suggested for real gases but quantitative agreement is generally poor.

SPECIFIC HEAT CAPACITIES

Specific Heat Capacities at Constant Volume and Constant Pressure

When we warm a gas, we may let it expand or not, as we please. If we do not let it expand—if we warm it in a closed vessel—then it does no external work, and all the heat we give it goes to increase its internal energy. *The heat required to warm unit mass of a gas through one degree, when its volume is kept constant, is called the specific heat capacity of the gas at constant volume.* It is denoted by c_V , and is generally expressed in $\text{J kg}^{-1} \text{K}^{-1}$.

If we allow a gas to expand as we warm it, then it does external work. The heat we give the gas appears partly as an increase to its internal energy—and hence its temperature—and partly as the heat equivalent of the work done. The work done depends on the increase in volume of the gas, which in turn depends on the way in which we allow the

gas to expand. We can get an important theoretical result by supposing that the pressure is constant, and defining the corresponding specific heat capacity. *The specific heat capacity of a gas at constant pressure is the heat required to warm unit mass of it by one degree, when its pressure is kept constant.* It is denoted by c_p , and is expressed in the same units as c_v .

Specific Heat Capacities: their Difference

Any number of heat capacities can be defined for a gas, according to the mass and the conditions imposed upon its pressure and volume. For unit mass, 1 kg or 1 g, of a gas, the heat capacities at constant pressure c_p , and at constant volume c_v , are the *specific heat capacities*.

Fig. 10.16 shows how we can find a relationship between the specific heat capacities of a gas. We first consider 1 kg of the gas warmed through

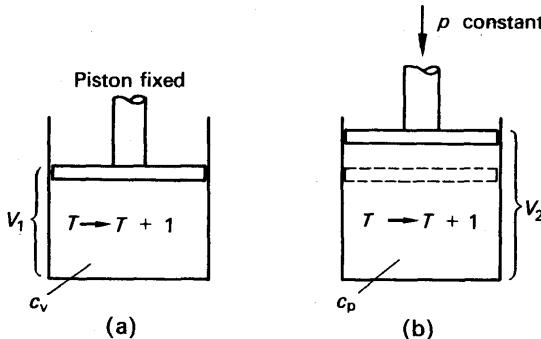


FIG. 10.16. Specific heat capacity at constant volume and pressure.

1°C at constant volume, (a). The heat required is c_v joules, and goes wholly to increase the internal energy.

We next consider 1 kg warmed through 1°C at constant pressure, (b). It expands from V_1 to V_2 , and does an amount of external work given by

$$W = p(V_2 - V_1) \quad (\text{equation (11), p. 229}).$$

The work W is in joules if p is in newton m^{-2} , and the volumes in m^3 . Thus the amount of heat in joules required for this work is

$$W = p(V_2 - V_1).$$

Further, since the temperature rise of the gas is 1°C, and the internal energy of the gas is independent of volume, the rise in internal energy is c_v , the specific heat at constant volume. Hence, from $\delta Q = \delta U + p \cdot \delta V$, the total amount of heat required to warm the gas at constant pressure is therefore

$$c_p = c_v + p(V_2 - V_1) \quad \quad (24)$$

We can simplify the last term of this expression by using the equation of state for unit mass:

$$pV = RT,$$

where T is the absolute temperature of the gas, and R is the gas-

constant for unit mass of it in J K^{-1} . If T_1 is the absolute temperature before warming, then

$$pV_1 = RT_1 \quad \dots \quad \dots \quad \dots \quad (25)$$

The absolute temperature after warming is $T_1 + 1$; therefore

$$pV_2 = R(T_1 + 1), \quad \dots \quad \dots \quad \dots \quad (26)$$

and on subtracting (25) from (26) we find

$$p(V_2 - V_1) = R$$

Equation (24) now gives $c_p = c_v + R$

or

$$c_p - c_v = R \quad \dots \quad \dots \quad \dots \quad (27)$$

Equation (27) was first derived by Robert Mayer in 1842. He used it, before Joule had done his water-churning experiments, to derive a relation between heat and mechanical energy.

Ratio of Specific Heat Capacities

We have seen that the internal energy of a gas, at a given temperature, depends on the number of atoms in its molecule. For a monatomic gas its value in joules per kg is

$$U = \frac{3}{2}RT \quad \dots \quad \dots \quad \dots \quad (i)$$

where U is the internal energy of the gas, R is the gas constant in $\text{J kg}^{-1} \text{K}^{-1}$ and T is the absolute temperature of the gas.

The heat required to increase the internal energy of 1 kg of a monatomic gas, when it is warmed through 1 degree, is therefore $\frac{3}{2}R$ joule. But this is the specific heat capacity at constant volume, and so

$$c_v = \frac{3}{2}R$$

The specific heat capacity of a monatomic gas at constant pressure is therefore

$$\begin{aligned} c_p &= c_v + R = \frac{3}{2}R + R \\ &= \frac{5}{2}R \end{aligned}$$

Let us now divide c_p by c_v ; their quotient is called the ratio of the specific heat capacities, and is denoted by γ .

For a monatomic gas, its value is

$$\begin{aligned} \gamma &= \frac{c_p}{c_v} = \frac{\frac{5}{2}R}{\frac{3}{2}R} \\ &= \frac{5}{3} = 1.667. \end{aligned}$$

Similarly, for a diatomic molecule,

$$U = \frac{5}{2}RT \quad \dots \quad \dots \quad \dots \quad (ii)$$

This was shown on p. 240.

Hence

$$c_V = \frac{5}{2}R$$

and

$$c_p = c_V + R = \frac{7}{2}R$$

Hence

$$\gamma = \frac{c_p}{c_V} = \frac{7}{5} = 1.40.$$

And for a polyatomic molecule,

$$U = \frac{6}{2}RT, \quad \text{(iii)}$$

$$c_V = \frac{6}{2}R$$

and

$$\gamma = \frac{c_p}{c_V} = \frac{8}{6} = 1.33.$$

In general, if the molecules of a gas have f degrees of freedom, the average kinetic energy of a molecule is $f \times \frac{1}{2}kT$ (p. 238).

$$\therefore U = \frac{f}{2}RT,$$

$$c_V = \frac{f}{2}R,$$

$$c_p = c_V + R = \left(\frac{f}{2} + 1 \right)R,$$

$$\text{and } \gamma = \frac{c_p}{c_V} = \frac{\frac{f}{2} + 1}{\frac{f}{2}} = 1 + \frac{2}{f} \quad \text{(iv)}$$

The ratio of the specific heat capacities of a gas thus gives us a measure of the number of atoms in its molecule, at least when that number is less than three. This ratio is fairly easy to measure, as we shall see later in this chapter. The poor agreement between the observed and theoretical values of γ for some of the polyatomic gases shows that, in its application to such gases, the theory is oversimplified.

Measurement of c_V .

Fig. 10.17 shows an apparatus for measuring the specific heat capacity of a gas at constant volume, called a *differential steam calorimeter*. The calorimeter consists of two copper globes, A and B, as nearly identical as they can be made. They hang from the beam of a balance, and are surrounded by a chest C into which steam can be admitted at D. The sphere B is evacuated and A is filled with the gas whose specific heat is required. By filling A to a high pressure, the mass of gas can be made great enough to be accurately measurable on the balance. Let its value be M . Steam is now admitted to the chest, and condenses on both globes until they reach the temperature of the

steam. This will generally be about 100°C, but we shall denote it by θ_1 . The balance measures the excess steam condensed on A, over that condensed on B; let its mass by M_s . If the globes are identical, their heat capacities are equal, and the masses of steam required to warm the globes alone are equal. The excess steam condensed on A is then the mass required to warm the gas within it. Therefore if θ_R is the room temperature, and l the specific latent heat of evaporation of water, the specific heat capacity c_v for the gas is given by

$$Mc_v(\theta_1 - \theta_R) = M_s l;$$

whence

$$c_v = \frac{M_s l}{M(\theta_1 - \theta_R)} \quad \quad (28)$$

The calorimeter is called 'differential' for the reason that it measures the difference in mass of the steam on the two globes. This is an important feature of it, because the heat capacities of the globes may be much greater than that of the gas. If a single globe were used, two measurements would have to be made, one with and one without the

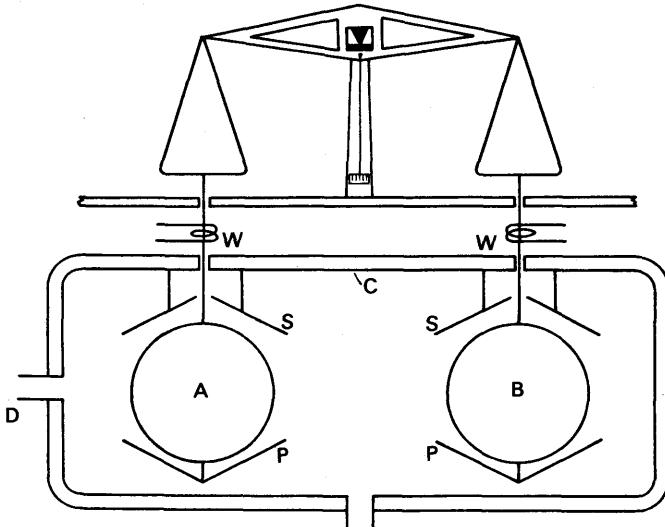


FIG. 10.17. Joly's differential steam calorimeter.

gas. The mass of steam condensed by the gas would then appear as the difference of two nearly equal masses, and could not be determined accurately. In practice the globes are not identical, and a control experiment with both evacuated is made to find the difference in mass of the steam condensed on them. This appears as a small correction to M_s in equation (28).

A small correction has also to be made to the result of the experiment, because the volume of the gas is not quite constant: the globe A expands when it is warmed.

The figure shows a few of the practical refinements of the apparatus. S, S are shields to prevent drops of moisture, condensed on the roof of

the chest, from falling on to the globes. P, P are pans to catch any drops which, having condensed on A or B, might fall off. W, W are platinum wires heated by an electric current, which prevent drops forming in the holes through which the suspension wires pass out of the chest.

Measurement of c_p .

The method of mixtures was used to determine the specific heat capacity of a gas at constant pressure by Regnault. Regnault was one of the greatest experimenters of the nineteenth century—the reader who sees pictures of his apparatus in other books should remember that he worked before Bunsen had invented his famous burner—but his method for c_p is now outmoded. We shall describe here only a continuous flow method, similar to Callendar and Barnes' for the specific heat capacity of water. It is due to Swann (1909).

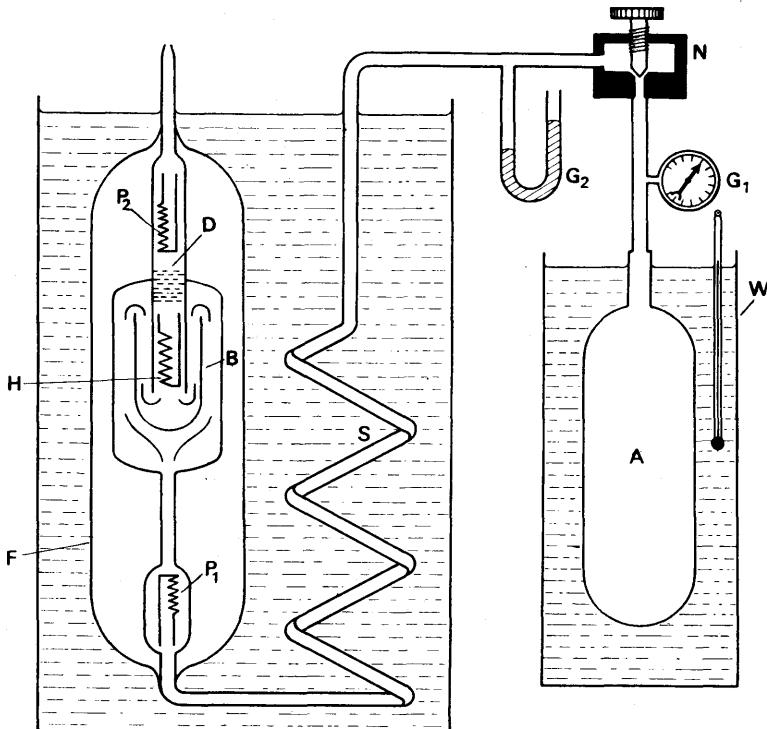


FIG. 10.18. Constant flow calorimeter for c_p .

Gas from a cylinder, A in Fig. 10.18, flows out through a needle valve N, which reduces its pressure to a little above atmospheric. If the pressure in the cylinder is high, it will fall slowly during an experiment, and the pressure of the emerging gas will be almost constant. Manometers G_1 , G_2 indicate the pressure of the gas in the cylinder, and of the gas emerging. The gas passes through a coiled tube S, in a water bath,

which brings it to a uniform temperature. It then flows past a platinum resistance thermometer, P_1 , which measures the temperature, θ_1 °C. From there it goes to a heating coil H, past a baffle B which enables it to receive any heat that escapes from the neighbourhood of the coil. Beyond the coil it passes through copper gauze D, which mixes the stream of gas and so brings it to a uniform temperature. This temperature, θ_2 , is measured by the platinum resistance thermometer P_2 . A vacuum jacket F makes the heat losses very small.

If M is the mass of gas flowing through the apparatus in t seconds, then the heat received by it is $Mc_p(\theta_2 - \theta_1)$. If the heat losses are negligible, the heat supplied by the coil in t seconds is IVt joule, where I is the current through it in ampere and V the potential difference across it in volt. Then

$$IVt = Mc_p(\theta_2 - \theta_1).$$

The mass of gas, M , is found from the fall of pressure in the cylinder. If v is the volume of the cylinder, and ρ_1 the density of the gas at the initial pressure p_1 , then the mass initially in the cylinder is

$$M_1 = \rho_1 v.$$

And if, after t seconds, the pressure has fallen to p_2 , and the density to ρ_2 , the mass remaining in the cylinder is

$$M_2 = \rho_2 v.$$

The mass of gas which has escaped in then

$$M = M_1 - M_2 = (\rho_1 - \rho_2)v.$$

The densities ρ_1 and ρ_2 can be readily calculated from the density ρ_0 at s.t.p.: if θ_3 is the temperature of the cylinder, and the pressures are measured in mm mercury, then

$$\frac{p_1}{\rho_1(273 + \theta_3)} = \frac{p_2}{\rho_2(273 + \theta_3)} = \frac{760}{273\rho_0}.$$

The cylinder temperature θ_3 is kept constant by the water bath W.

CHANGES OF PRESSURE, VOLUME AND TEMPERATURE

In, for example, a steam engine or motor, gases expand and are compressed, cool and are heated, in ways more complicated than those which we have already described. We shall now consider some of these ways.

Isothermal Changes

We have seen that the pressure p , and volume V of a given mass of gas are related by the equation

$$pV = RT,$$

where T is the absolute temperature of the gas, and R is a constant.

If the temperature is constant the curve of pressure against volume is a rectangular hyperbola,

$$pV = \text{constant},$$

representing Boyle's law. Such a curve is called an *isothermal* for the given mass of the given gas, at the temperature T . Fig. 10.19 shows a family of isothermals, for 1 g of air at different temperatures. When a gas expands, or is compressed, at *constant temperature*, its pressure and volume vary along the appropriate isothermal, and the gas is said to undergo an *isothermal compression or expansion*.

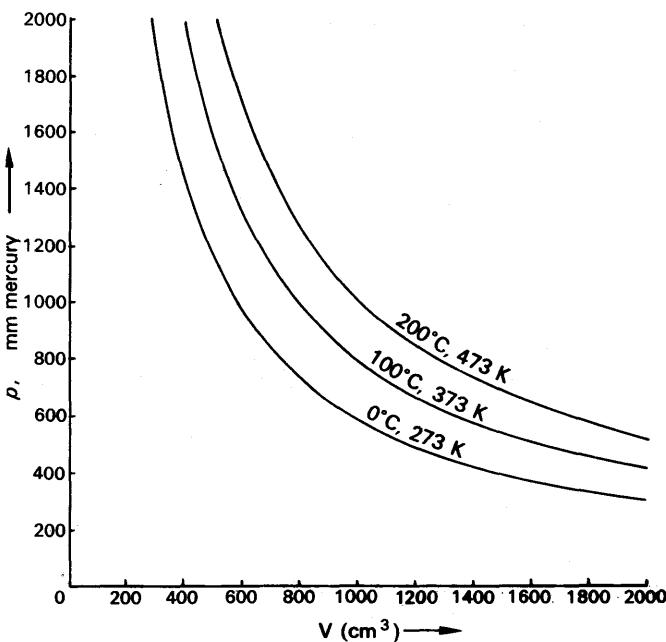


FIG. 10.19. Isotherms for 1 g air.

When a gas expands, it does work—for example, in driving a piston (Fig. 10.9, p. 229). The molecules of the gas bombard the piston, and if the piston moves they give up some of their kinetic energy to it; when a molecule bounces off a *moving* piston, it does so with a velocity less in magnitude than that with which it struck. The change in velocity is small, because the piston moves much more slowly than the molecule; but there are many molecules striking the piston at any instant, and their total loss of kinetic energy is equal to the work done in driving the piston forward.

The work done by a gas in expanding, therefore, is done at the expense of its internal energy. The temperature of the gas will consequently fall during expansion, unless heat is supplied to it. For an isothermal expansion, the gas must be held in a thin-walled, highly conducting vessel, surrounded by a constant temperature bath. And the expansion must take place slowly, so that heat can pass into the gas to maintain its temperature at every instant during the expansion.

External Work done in Expansion

The heat taken in when a gas expands isothermally is the heat equivalent of the mechanical work done. If the volume of the gas increases by a small amount δV , at the pressure p , then the work done is

$$\delta W = p\delta V \text{ (equation (11), p. 229).}$$

In an expansion from V_1 to V_2 , therefore, the work done is

$$W = \int dW = \int_{V_1}^{V_2} pdV.$$

By the gas equation, $p = \frac{RT}{V}$,

whence $W = \int_{V_1}^{V_2} pdV = \int_{V_1}^{V_2} RT \frac{dV}{V}$

or $W = RT \log_e \left(\frac{V_2}{V_1} \right)$.

The heat required, Q , is therefore

$$Q = W = RT \log_e \left(\frac{V_2}{V_1} \right),$$

where W is in joules if R is in $\text{J kg}^{-1} \text{K}^{-1}$.

Now let us consider an isothermal compression. When a gas is compressed, work is done on it by the compressing agent. To keep its temperature constant, therefore, heat must be withdrawn from the gas, to prevent the work done from increasing its internal energy. The gas must again be held in a thin well-conducting vessel, surrounded by a constant-temperature bath; and it must be compressed slowly.

The conditions for an isothermal compression or expansion of a gas are difficult to realize; heat cannot flow through the walls of the vessel unless there is at least a small difference of temperature across them, and therefore the temperature of the gas is bound to rise a little in compression, or to fall a little in expansion.

Reversible isothermal change

Suppose a gas expands isothermally from p_1, V_1, T to p_2, V_2, T . If the change can be reversed so that the state of the gas is returned from p_2, V_2, T to p_1, V_1, T through exactly the same values of pressure and volume at every stage, then the isothermal change is said to be *reversible*. A reversible isothermal change is an ideal one. It requires conditions such as a light frictionless piston, so that the pressure inside and outside the gas can always be equalised and no work is done against friction; very slow expansion, so that no eddies are produced in the gas to dissipate the energy; and a constant temperature reservoir with very thin good-conducting walls, as we have seen. In a reversible isothermal change, $pV = \text{constant} = RT$.

Equation for Reversible Adiabatic change

Let us now consider a change of volume in which the conditions are at the opposite extreme from isothermal; no heat is allowed to enter or leave the gas.

An expansion or contraction in which *no heat* enters or leaves the gas is called an *adiabatic expansion or contraction*. In an adiabatic expansion, the external work is done wholly at the expense of the internal energy of the gas, and the gas therefore cools. In an adiabatic compression, all the work done on the gas by the compressing agent appears as an increase in its internal energy and therefore as a rise in its temperature. We have already discussed a reversible isothermal change. A *reversible adiabatic change* is an adiabatic change which can be exactly reversed in the sense explained on p. 251. As noted there, a reversible change is an ideal case.

The curve relating pressure and volume for a given mass of a given gas for adiabatic changes is called an '*adiabatic*'. In Fig. 10.20, the heavy curve is an adiabatic for 1 g of air; it is steeper, at any point, than the isothermal through that point. The curve AB is the isothermal for the temperature $T_0 = 373$ K, which cuts the adiabatic at the point $p_0 V_0$. If the gas is adiabatically compressed from V_0 to V_1 , its

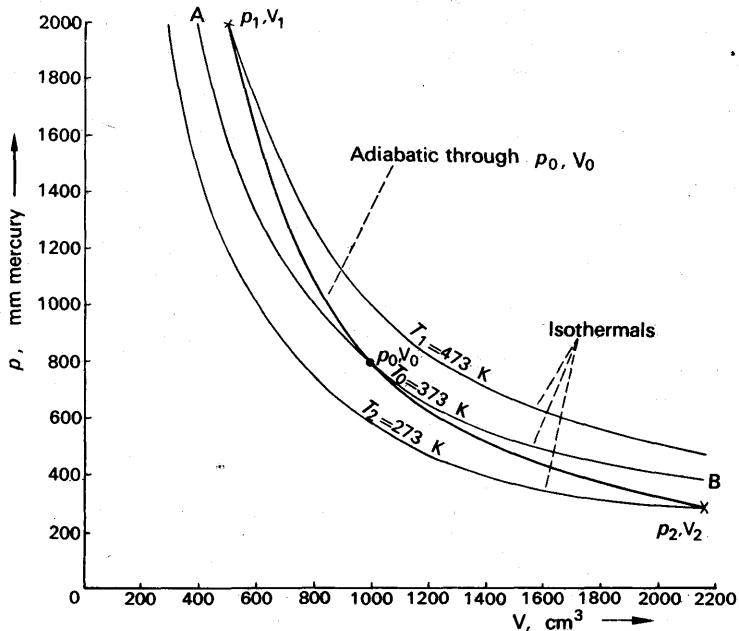


FIG. 10.20. Relationship between adiabatic and isothermals.

temperature rises to some value T_1 . Its representative point p_1, V_1 now lies on the isothermal for T_1 , since $p_1 V_1 = RT_1$. Similarly, if the gas is expanded adiabatically to V_2 , it cools to T_2 and its representative point p_2, V_2 lies on the isothermal for T_2 . Thus the adiabatic through

any point—such as p_0 , V_0 —is steeper than the isothermal. We will find its equation shortly.

The condition for an adiabatic change is that no heat must enter or leave the gas. The gas must therefore be held in a thick-walled, badly conducting vessel; and the change of volume must take place rapidly, to give as little time as possible for heat to escape. However, in a rapid compression, for example, eddies may be formed, so that some of the work done appears as kinetic energy of the gas in bulk, instead of as random kinetic energy of its molecules. All the work done then does not go to increase the internal energy of the gas, and the temperature rise is less than in a truly adiabatic compression. If the compression is made slowly, then more heat leaks out; since no vessel has perfectly insulating walls.

Perfectly adiabatic changes are therefore impossible; and so, we have seen, are perfectly isothermal ones. Any practical expansion or compression of a gas must lie between isothermal and adiabatic. It may lie anywhere between them, but if it approximates to isothermal, the curve representing it will always be a little steeper than the ideal (Fig. 10.21); if it approximates to adiabatic, the curve representing it will never be quite as steep as the ideal.

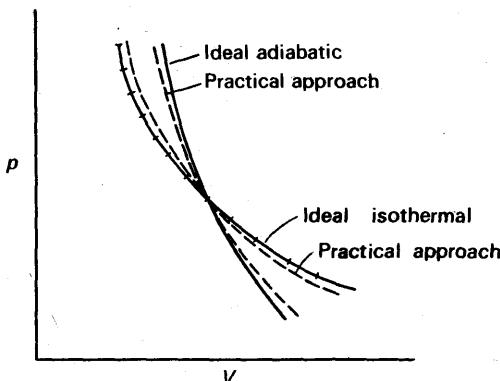


FIG. 10.21. Ideal and real p - V curves for a gas.

Equation of Reversible Adiabatic

Before considering adiabatic changes in particular, let us first consider a change of volume and temperature which takes place in an arbitrary manner. For simplicity, we consider a unit mass of the gas, and we suppose that its volume expands from V to $V + \delta V$, and that an amount of heat δQ is supplied to it. In general, the internal energy of the gas will increase by an amount δU . And the gas will do an amount of external work equal to $p\delta V$, where p is its pressure. The heat supplied is equal to the increase in internal energy, plus the external work done:

$$\delta Q = \delta U + p\delta V \quad \quad (29)$$

The increase in internal energy represents a temperature rise, δT . We have seen already that the internal energy is independent of the volume,

and is related to the temperature by the specific heat capacity at constant volume, c_V (p. 244). Therefore

$$\delta U = c_V \delta T.$$

Equation (29) becomes

$$\delta Q = c_V \delta T + p \delta V \quad \quad (30)$$

Equation (30) is the fundamental equation for any change in the state of unit mass of a gas.

For a reversible isothermal change, $\delta T = 0$, and $\delta Q = p \delta V$.

For a reversible adiabatic change, $\delta Q = 0$ and therefore

$$c_V \delta T + p \delta V = 0 \quad \quad (31)$$

To eliminate δT we use the general equation, relating pressure, volume and temperature:

$$pV = RT,$$

where R is the gas constant for unit mass. Since both pressure and volume may change, when we differentiate this to find δT we must write

$$p \delta V + V \delta p = R \delta T,$$

whence

$$\delta T = \frac{p \delta V + V \delta p}{R}.$$

Therefore, by equation (31),

$$c_V \frac{p \delta V + V \delta p}{R} + p \delta V = 0$$

or

$$c_V(p \delta V + V \delta p) + Rp \delta V = 0.$$

Now we have seen, on p. 245, that

$$R = c_p - c_V;$$

therefore

$$c_V(p \delta V + V \delta p) + (c_p - c_V)p \delta V = 0.$$

Hence

$$c_V V \delta p + c_p p \delta V = 0$$

or

$$V \delta p + \frac{c_p}{c_V} p \delta V = 0$$

or

$$V \delta p + \gamma p \delta V = 0 \left(\text{where } \gamma = \frac{c_p}{c_V} \right).$$

Therefore

$$\frac{\delta p}{p} + \gamma \frac{\delta V}{V} = 0.$$

Integrating, we find

$$\int \frac{dp}{p} + \gamma \int \frac{dV}{V} = 0$$

or

$$\log_e p + \gamma \log_e V = A,$$

where A is a constant.

Therefore,

$$pV^\gamma = C,$$

where C is also a constant. This is the equation of a reversible adiabatic; the value of C can be found from the initial pressure and volume of the gas.

If we have a mass M of the gas, its volume at any temperature and pressure is

$$V = MV,$$

where V is the volume of unit mass at the same temperature and pressure. Therefore for any mass of gas, the equation of an adiabatic change is

$$pV^\gamma = \text{constant} \quad \quad (32)$$

Equation for Temperature Change in an Adiabatic

If we wish to introduce the temperature, T , into equation (32), we use the general gas equation

$$pV = RT.$$

Thus

$$p = \frac{RT}{V}$$

and $pV^\gamma = \frac{RT}{V} \cdot V^\gamma = RTV^{\gamma-1}.$

Thus equation (32) becomes

$$RTV^{\gamma-1} = \text{constant},$$

and since R is a constant for a given mass of gas, the equation for an adiabatic temperature change becomes

$$TV^{\gamma-1} = \text{constant}.$$

Measurement of γ .

In books on Sound, it is shown that sound waves are propagated through a gas by rapid compressions and rarefactions; these changes in pressure and volume are adiabatic. In consequence, the velocity of sound in a gas depends upon the ratio of the specific heat capacities of the gas, γ ; and the value of γ can be found from measurements of the velocity of sound in the gas. This is the most convenient way of measuring γ .

A direct measurement of γ can be made by the method of Clément and Désormes (1819). A large vessel—such as a carboy—contains the gas, which in a teaching experiment is usually air (Fig. 10.22). The

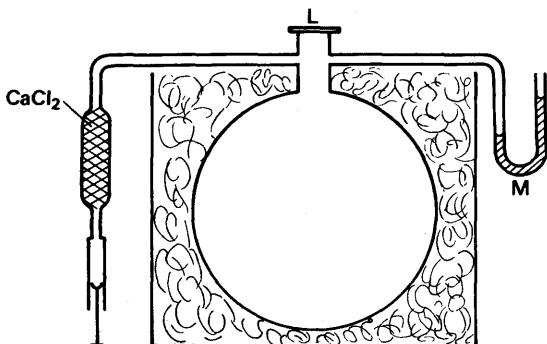


FIG. 10.22. Clément and Désormes experiment.

carboy is well lagged to minimize the exchange of heat with its surroundings. It is attached to a manometer M, and, via a drying-tube, to a bicycle pump. Its mouth has a large and well-fitting, flap-like, lid, L. Air is blown in until its pressure is a little above atmospheric, and time is allowed for the gas to settle down to room temperature. When it has done so the manometer reading becomes steady, and the pressure p_1 of the gas is recorded. The flap-valve is now sharply opened and closed. The gas makes an adiabatic expansion, and its pressure p_2 is immediately read. With the flap still closed, the gas is then left; it gradually returns to room temperature, absolute temperature T_1 , at constant volume, and its pressure rises to p_3 .

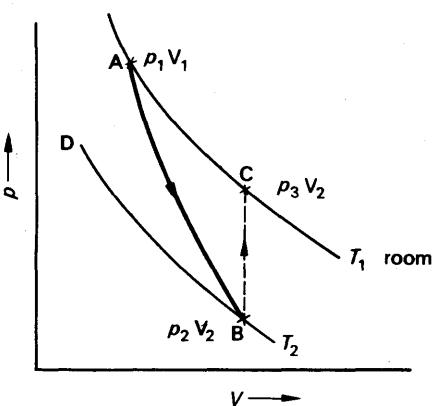


FIG. 10.23. Theory of Clément and Désormes. The gas starts at point A with pressure p_1 and volume V_1 on the isothermal for temperature T_1 . It undergoes an adiabatic expansion to point B with pressure p_2 and volume V_2 . The gas then returns to room temperature T_1 at volume V_2 along the isothermal T_1 room, reaching point C. The final state is at pressure p_3 and volume V_2 (point D).

$$p_1 V_1^\gamma = p_2 V_2^\gamma,$$

$$\text{or } \frac{p_1}{p_2} = \left(\frac{V_2}{V_1}\right)^\gamma. \quad (33)$$

After the gas has returned to room temperature, its representative point C lies on the same isothermal as A; therefore

$$p_3 V_2 = p_1 V_1,$$

$$\text{or } \frac{p_1}{p_3} = \frac{V_2}{V_1}.$$

From equation (33), therefore,

$$\frac{p_1}{p_2} = \left(\frac{p_1}{p_3}\right)^\gamma$$

$$\text{whence } \log_e p_1 - \log_e p_2 = \gamma(\log_e p_1 - \log_e p_3)$$

$$\text{and } \gamma = \frac{\log_e p_1 - \log_e p_2}{\log_e p_1 - \log_e p_3}.$$

If h_1 is the difference in levels of M corresponding to the pressure

p_1 , and h_2 is the final difference in levels, corresponding to the pressure p_3 , it can be shown that, to a good approximation, the formula for γ reduces to

$$\gamma = \frac{h_1}{h_1 - h_2}.$$

A light oil is used in the manometer M.

VACUUM PUMPS AND GAUGES

The Filter Pump

The simplest pump for evacuating a vessel is the filter pump, so-called because it is used for speeding-up filtration (Fig. 10.24). It consists of a nozzle N surrounded by a chamber C; water rushes from the nozzle and out of the chamber at the bottom. The layer of air around the jet is dragged along with it, and carried out of the chamber. The lowest pressure which this pump can produce is the saturation vapour-pressure of water, about 15 mm mercury at 18°C. (Chapter 14). It cannot produce what we would nowadays call a 'good vacuum'.

The Piston Pump

A piston-type air pump is similar to the common water pump, but more accurately made. Its plunger has a greased leather washer, W in Fig. 10.25, and its valves F_1 and F_2 are flaps of oiled silk.

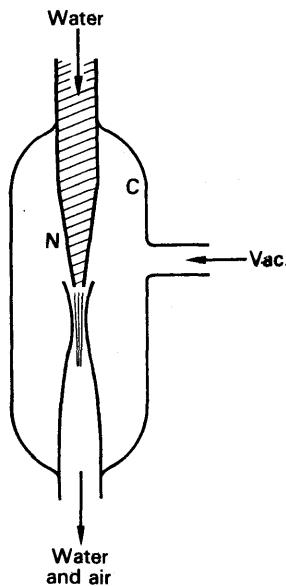


FIG. 10.24. Filter pump.

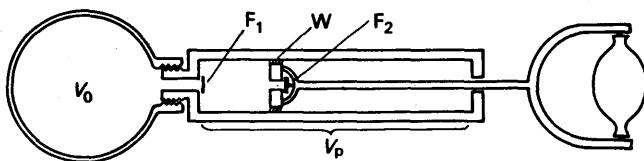


FIG. 10.25. Piston-type vacuum pump.

To develop a simple theory of the pump, we first assume that, when the piston is pushed right in, there is no space between it and the bottom of the barrel. We suppose that the pump is connected to a vessel of volume V_0 , that the piston displaces a volume V_p , and that the pressure in the vessel is p_0 when the piston is right in, at the start of the evacuation. When the piston is pulled right out, the volume of the air originally in the vessel increases to $V_0 + V_p$. Since the action

is slow, we may assume that the expansion is isothermal; the pressure p_1 after expansion is therefore given by

$$p_1(V_0 + V_p) = p_0 V_0,$$

whence

$$p_1 = \left(\frac{V_0}{V_0 + V_p} \right) p_0.$$

When the piston is pushed in again, the valve F_1 closes, and the air in the vessel remains at the pressure p_1 . The second out-stroke then reduces the pressure to

$$p_2 = \left(\frac{V_0}{V_0 + V_p} \right) p_1 = \left(\frac{V_0}{V_0 + V_p} \right)^2 p_0.$$

Similarly, after n strokes, the pressure is reduced to

$$p_n = \left(\frac{V_0}{V_0 + V_p} \right)^n p_0.$$

According to this theory, the final pressure tends to zero as the number of strokes tends to infinity. In practice, however, a pump has a limiting pressure. This is due to the fact that the piston can never in practice be brought right down to the valve F_1 , so that there is a residual volume, or dead-space, v , between the piston and the bottom of the barrel. Air can escape through the valve F_2 only when the pressure in the volume v is greater than atmospheric. And air can pass from the vessel through F_1 only when the pressure in the barrel is less than the pressure in the vessel. Thus the limiting pressure, p_∞ , is the pressure which $v \text{ cm}^3$ of gas, at atmospheric pressure, exert when expanded to $V_p \text{ cm}^3$. That is to say

$$p_\infty V_p = p_{atmos.} v,$$

whence

$$p_\infty = \frac{v}{V_p} p_{atmos.}$$

The ratio v/V_p may be about $1/1000$, so that p_∞ is about 1 mm mercury. Piston pumps of more elaborate designs can give better vacua than this, about 0.01 mm mercury; they have all been made obsolete, however, by the rotary pump.

Rotary Pumps

Fig. 10.26 illustrates one form of rotary vacuum pump. It consists of a rotor, R, which turns eccentrically in a casing, C, being a close fit at the ends and along the line A. The rotor carries two scraping blades B_1 , B_2 , separated by a strong spring. The vessel to be evacuated is connected to the inlet port I, and the outlet port O is fitted with a valve N. As the rotor turns, the volume V_1 increases, so that air expands from the vessel into the pump. When the blades are in the positions shown, the air in the space V_2 is being compressed; when its pressure rises to atmospheric, the valve N opens, and the air passes out through O. As the blade B_2 crosses the seat of the valve N, the valve closes

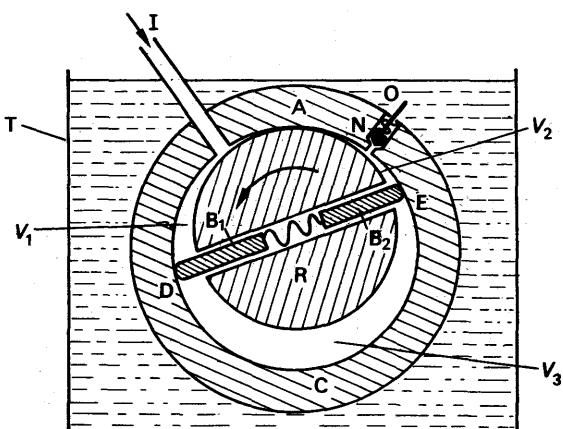


FIG. 10.26. Rotary vacuum pump.

because the air in V_3 is below atmospheric pressure. Thus atmospheric air cannot blow back into the pump. The lines of contact, A, D, E are made airtight by a film of oil. All the working parts of the pump are enclosed in a tank T containing oil. When the pump is at rest, the oil seeps back through the outlet valve and fills the working space; but the first revolutions of the pump sweep out the excess oil, and leave just the necessary film over the metal surfaces.

A single rotary pump will give an ultimate pressure of about 0.01 mm mercury. Very often two pumps are housed in the same tank of oil, and driven off the same shaft; they are connected in cascade, and may give an ultimate pressure of less than 0.001 mm mercury.

The MacLeod Gauge

The MacLeod Gauge is a gauge used for measuring pressures below a few mm mercury; these are pressures which cannot be measured accurately on a U-tube manometer. It consists of a bulb B, connected to a mercury reservoir M and terminated in a capillary tube T (Fig. 10.27 (a)); just below the bulb a branch-tube, P, leads to the vacuum system under test. It also carries a branch-tube, D, which is a capillary of the same bore as T. A millimetre scale, S, lies underneath T and D.

To measure the pressure in a vacuum system, the reservoir M is lowered until the mercury falls below the branch-point C. The air in B is then at the unknown pressure, p . The reservoir is now slowly raised. As soon as the mercury closes the branch at C, the air in B starts to be compressed. M is raised until the mercury in B just reaches the foot of the capillary T. The height h of the mercury in D, above that in B, is then measured. The purpose of having equal bores for T and D is to equalize surface tension effects in each.

If the pressure p is expressed in mm mercury then the pressure of the air trapped in T is $p + h$. The volume of this air is the volume v of the capillary T. At the moment when the mercury passed the point C, this air had the pressure p , and the volume $V + v$, where V is the

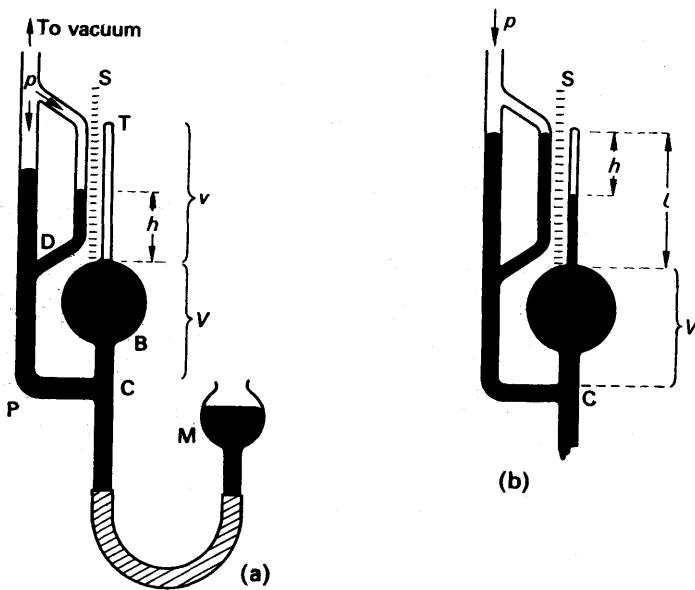


FIG. 10.27. MacLeod gauge.

volume between C and the base of T. The compression is slow enough to be isothermal, so that

$$p(V+v) = (p+h)v$$

Hence

$$pV = hv,$$

and

$$p = \frac{v}{V}h.$$

Another way of using the gauge is to raise the reservoir M until the mercury in D is level with the top of T (Fig. 10.27 (b)). Then if s is the cross-section of T, the volume of trapped air is hs . And if l is the whole length of T, its volume is ls . Therefore, as before,

$$p(V+ls) = (p+h)hs,$$

or

$$p(V+ls-hs) = h^2s,$$

whence

$$p = \frac{h^2s}{V+(l-h)s}.$$

The term $(l-h)s$ in the denominator is usually negligible compared with V , so that

$$p \simeq \frac{h^2s}{V}.$$

Because p is proportional to h^2 , the gauge can cover a wider range of pressures when it is used in this way, than when it is used in the way first described. But for the same reason it is less accurate. In practice

the second way is generally chosen, and the scale S is calibrated to read the pressure p directly.

EXAMPLES

1. The density of a gas is 10775 kg m^{-3} at 27°C and 10^5 N m^{-2} pressure and its specific heat capacity at constant pressure is $0.846 \text{ kJ kg}^{-1} \text{ K}^{-1}$. Find the ratio of its specific heat capacity at constant pressure to that at constant volume.

The gas constant per kg of gas is given by

$$R = \frac{pV}{T} = \frac{10^5 \times 1}{1.775 \times 300} \text{ J kg}^{-1} \text{ K}^{-1}$$

since $V = 1 \text{ m}^3/1.775$, $T = 273 + 27 = 300 \text{ K}$. Converting J to kJ,

$$\therefore R = \frac{10^5 \times 1}{1.775 \times 300 \times 1000} \text{ kJ kg}^{-1} \text{ K}^{-1}.$$

Now

$$c_p - c_v = R$$

$$\therefore 0.846 - c_v = \frac{10^5 \times 1}{1.775 \times 300 \times 1000} = 0.188$$

$$\therefore c_v = 0.846 - 0.188 = 0.658 \text{ kJ kg}^{-1} \text{ K}^{-1}$$

$$\therefore \gamma = \frac{c_p}{c_v} = \frac{0.846}{0.658} = 1.29.$$

This value for γ suggests that the gas is polyatomic (see p. 246).

2. An ideal gas at 17°C has a pressure of 760 mm mercury, and is compressed (i) isothermally, (ii) adiabatically until its volume is halved, in each case reversibly. Calculate in each case the final pressure and temperature of the gas, assuming $c_p = 2.1$, $c_v = 1.5 \text{ kJ kg}^{-1} \text{ K}^{-1}$.

- (i) Isothermally, $pV = \text{constant}$.

$$\therefore p \times \frac{V}{2} = 760 \times V$$

$$\therefore p = 1520 \text{ mm mercury.}$$

The temperature is constant at 17°C .

- (ii) Adiabatically, $pV^\gamma = \text{constant}$, and $\gamma = 2.1/1.5 = 1.4$.

$$\therefore p \times \left(\frac{V}{2}\right)^{1.4} = 760 \times V^{1.4}$$

$$\therefore p = 760 \times 2^{1.4} = 2010 \text{ mm mercury.}$$

Since $TV^{\gamma-1} = \text{constant}$,

$$\therefore T \times \left(\frac{V}{2}\right)^{0.4} = (273 + 17) \times V^{0.4}$$

$$\therefore T = 290 \times 2^{0.4} = 383 \text{ K.}$$

$$\therefore \text{temperature} = 110^\circ\text{C}.$$

3. State the laws of gases usually associated with the names of Boyle, Charles, Dalton and Graham. Two gas containers with volumes of 100 cm^3 and 1000 cm^3 respectively are connected by a tube of negligible volume, and contain air at a

pressure of 1000 mm mercury. If the temperature of both vessels is originally 0°C, how much air will pass through the connecting tube when the temperature of the smaller is raised to 100°C? Give your answer in cm³, measured at 0°C and 760 mm mercury. (L.)

First part. Boyle, Charles, Dalton, Graham, see text.

Second part. The pressure is 1000 mm mercury when the temperature is 0°C (273 K). Let the density of air under these conditions be ρ_1 . Let the volumes of the large and small vessels be V and V^1 ; then the mass of air in the two vessels is

$$M = (V + V^1)\rho_1 = (1000 + 100)\rho_1 = 1100 \rho_1 \quad . . . \quad (i)$$

When the smaller vessel is heated, the pressure throughout the system rises to p , say. Let ρ_2 be the density of the air in the smaller vessel; then, by equation (9), p. 227:

$$\frac{p}{\rho_2} = R \times 373; \frac{1000}{\rho_1} = R \times 273 \quad . . . \quad (ii)$$

$$\therefore \frac{\rho_2}{\rho_1} = \frac{273}{373} \times \frac{p}{1000}$$

$$\therefore \rho_2 = \frac{273}{373} \times \frac{p}{1000} \rho_1$$

In the larger vessel, the temperature of the air does not change; therefore the density of the air in the larger vessel, ρ_3 , is

$$\rho_3 = \frac{p}{1000} \rho_1$$

The total mass of air, which is unchanged, is therefore

$$\begin{aligned} M &= V\rho_3 + V^1\rho_2 = 1000\rho_3 + 100\rho_2 \\ &= 1000 \frac{p\rho_1}{1000} + 100 \frac{273}{373} \frac{p\rho_1}{1000} \\ &= 1 + \left(\frac{273}{3730} \right) p\rho_1. \end{aligned}$$

Hence, by equation (i),

$$1100 \rho_1 = \left(1 + \frac{273}{3730} \right) p\rho_1,$$

and
$$p = \frac{3730 \times 1100}{4003} = 1025 \text{ mm mercury.}$$

The mass which flows out of the smaller vessel is

$$\begin{aligned} m &= V^1(\rho_1 - \rho_2) = 100 \rho_1 \left(1 - \frac{\rho_2}{\rho_1} \right) \\ &= 100 \rho_1 \left(1 - \frac{273}{373} \times \frac{p}{1000} \right) \\ &= 100 \rho_1 \left(1 - \frac{273}{373} \times \frac{1025}{1000} \right) \quad . . . \quad (iii) \end{aligned}$$

The volume of this mass, at 0°C and 760 mm mercury, is

$$V = \frac{m}{\rho_4},$$

where ρ_4 is the density of air at this temperature and pressure.

From the equation of state,

$$\frac{760}{\rho_4} \leq R \times 273;$$

therefore $\frac{\rho_1}{\rho_4} = \frac{1000}{760}$,

or $\frac{1}{\rho_4} = \frac{100}{76\rho_1}$.

Hence, by (iii), $V = \frac{m}{\rho_4} = \frac{100 \times 100}{76} \left(1 - \frac{273}{373} \times \frac{1025}{1000}\right)$
 $= 33 \text{ cm}^3$.

4. Distinguish between *isothermal* and *adiabatic* changes. Show that for an ideal gas the curves relating pressure and volume for an adiabatic change have a greater slope than those for an isothermal change, at the same pressure.

A quantity of oxygen is compressed isothermally until its pressure is doubled. It is then allowed to expand adiabatically until its original volume is restored. Find the final pressure in terms of the initial pressure. (The ratio of the specific heat capacities of oxygen is to be taken at 1.40.) (*L*)

First part. An isothermal change is one made at constant temperature; an adiabatic change is one made at constant heat, that is, no heat enters or leaves the system concerned.

For a reversible isothermal change, $pV = \kappa$, or $p = \kappa/V$. By differentiation, the slope, $dp/dV = -\kappa/V^2 = -pV/V^2 = -p/V$.

For a reversible adiabatic change, $pV^\gamma = c$, or $p = c/V^\gamma$. By differentiation, we find the slope, $dp/dV = -\gamma p/V$.

$$\therefore \text{ratio of adiabatic slope to isothermal slope} = \gamma.$$

Since γ is always greater than 1, the adiabatic slope is greater than the isothermal slope.

Second part. Let p_0, V_0 = the original pressure and volume of the oxygen.

Since $pV = \text{constant}$ for an isothermal change,

$$\therefore \text{new volume} = \frac{V_0}{2} \text{ when new pressure is } 2p_0.$$

Suppose the gas expands adiabatically to its volume V_0 , when the pressure is p .

Then $p \times V_0^{1.4} = 2p_0 \times \left(\frac{V_0}{2}\right)^{1.4}$

$$\therefore p = 2p_0 \times \left(\frac{1}{2}\right)^{1.4} = 0.8 p_0.$$

5. Derive an expression for the difference between the specific heat capacities of an ideal gas and discuss the significance of the ratio of these two specific heat capacities for real gases.

Assuming that the ratio of the specific heat capacities of hydrogen is 1.41 and that its density at s.t.p. is 0.0900 kg m^{-3} , find a value for its specific heat capacity at constant volume in $\text{J kg}^{-1} \text{ K}^{-1}$.

What explanation can you suggest for the small difference between the specific heat capacities of a solid? (Standard atmospheric pressure = $1.013 \times 10^5 \text{ N m}^{-2}$.) (*N*)

First part. The expression required is $c_p - c_v = R$, discussed on p. 245. The

ratio, γ , of these two specific heats = $1 + \frac{2}{n}$ on the kinetic theory of gases, where n is the number of degrees of freedom of the molecules. For a monatomic gas $n = 3$, so that $\gamma = 1.66$; for a diatomic gas $n = 5$ usually, so that $\gamma = 1.4$; for triatomic gases γ is less than 1.4, e.g. 1.29. Thus γ gives information about the number of atoms in the molecule of the gas.

Second part. $c_p - c_v = R$, where R may be in $\text{J kg}^{-1} \text{K}^{-1}$ and c_p, c_v are in the same units. Since 0.09 kg occupies 1 m³ and $p = 1.013 \times 10^5$ newton m⁻², then, from $pV = RT$,

$$\begin{aligned} R &= \frac{pV}{T} = \frac{(1.013 \times 10^5) \times 1}{273 \times 0.09} \text{ J kg}^{-1} \text{ K}^{-1} \\ &= 4.12 \text{ kJ kg}^{-1} \text{ K}^{-1} \end{aligned}$$

$$\therefore c_p - c_v = 4.12 \quad \quad (i)$$

$$\text{But} \quad \frac{c_p}{c_v} = 1.41 \quad \quad (ii)$$

$$\therefore c_p = 1.41 c_v. \text{ Substituting for } c_p \text{ in (i),}$$

$$\therefore 1.41 c_v - c_v = 4.12 = 0.41 c_v.$$

$$\therefore c_v = \frac{4.12}{0.41} = 10.0 \text{ kJ kg}^{-1} \text{ K}^{-1}.$$

Third part. The difference in the specific heat capacities of a solid is proportional to the external work done in expansion. But the expansion of a solid is small. Consequently the difference in specific heat capacities of the solid is small.

EXERCISES 10

Gas Laws—Specific Heat Capacities of Gases

1. State *Boyle's law* and *Charles' law*, and show how they lead to the gas equation $PV = RT$. Describe an experiment you would perform to measure the thermal expansion coefficient of dry air.

What volume of liquid oxygen (density 1140 kg m⁻³) may be made by liquefying completely the contents of a cylinder of gaseous oxygen containing 100 litres of oxygen at 120 atmospheres pressure and 20°C? Assume that oxygen behaves as an ideal gas in this latter region of pressure and temperature.

[1 atmosphere = 1.01×10^5 newton metre⁻²; gas constant = 8.31 joule mol⁻¹ K⁻¹; molecular weight of oxygen = 32.0.] (O. & C.)

2. Give brief accounts of experiments which illustrate the relationship between the volume of a fixed mass of gas and (a) the pressure it exerts at a fixed temperature, (b) the temperature on a centigrade mercury thermometer at a fixed pressure. State the two 'laws' which summarise the results.

A gas cylinder contains 6400 g of oxygen at a pressure of 5 atmospheres. An exactly similar cylinder contains 4200 g of nitrogen at the same temperature. What is the pressure on the nitrogen? (Molecular weights: oxygen = 32, nitrogen = 28; assume that each behaves as a perfect gas.) (O. & C.)

3. State Boyle's Law.

Describe how you would verify the law for dry air over a range of pressures from 0.5 to 1.5 atmospheres. Would the form of the apparatus you describe be suitable if the working range of pressure was 0.5 to 10 atmospheres? Give reasons for your answer.

Two glass vessels of equal volume are joined by a tube, the volume of which may be neglected. The whole is sealed and contains air at S.T.P. If one vessel is

placed in boiling water at 100°C and the other is placed in melting ice, what will be the resultant pressure of the air? (N.)

4. The formula $pv = mrT$ is often used to describe the relationship between the pressure p , volume v , and temperature T of a mass m of a gas, r being a constant. Referring in particular to the experimental evidence how do you justify (a) the use of this formula, (b) the usual method of calculating T from the temperature t of the gas on the centigrade (Celsius) scale?

Two vessels each of capacity 1.00 litre are connected by a tube of negligible volume. Together they contain 0.342 g of helium at a pressure of 80 cm of mercury and temperature 27°C. Calculate (i) a value for the constant r for helium, (ii) the pressure developed in the apparatus if one vessel is cooled to 0°C and the other heated to 100°C, assuming that the capacity of each vessel is unchanged. (N.)

5. State Boyle's law and Charles' law and show how they may be combined to give the equation of state of an ideal gas.

Two glass bulbs of equal volume are joined by a narrow tube and are filled with a gas at s.t.p. When one bulb is kept in melting ice and the other is placed in a hot bath, the new pressure is 87.76 cm mercury. Calculate the temperature of the bath. (L.)

6. Describe experiments in which the relation between the pressure and volume of a gas has been investigated at constant temperature over a wide range of pressure. Sketch the form of the isothermal curves obtained.

Explain briefly how far van der Waals' equation accounts for the form of these isothermals. (L.)

7. Describe, with a diagram, the essential features of an experiment to study the departure of a real gas from ideal gas behaviour.

Give freehand, labelled sketches of the graphs you would expect to obtain on plotting (a) pressure P against volume V , (b) PV against P for such a gas at its critical temperature and at one temperature above and one below the critical temperature.

Explain van der Waals' attempt to produce an equation of state which would describe the behaviour of real gases.

Show that van der Waals' equation is consistent with the statement that all gases approach ideal gas behaviour at low pressures. (O. & C.)

8. State Boyle's Law and describe how you would attempt to discover whether air shows any deviations from the law. Draw an approximate set of curves to show the way in which a gas deviates from Boyle's Law in the region close to where the gas liquefies.

Two one-litre flasks are joined by a closed tap and the whole is held at a constant temperature of 50°C. One flask is evacuated and the other contains air, water vapour, and a small quantity of liquid water. The total pressure in the latter flask is 200 mm Hg. The tap is then opened, and the system is allowed to reach equilibrium, when some liquid water remains. Assuming that air obeys Boyle's law, find the final pressure in the flasks.

[Vapour pressure of water at 50°C = 93 mmHg.] (O. & C.)

9. Explain why the specific heat capacity of a gas is greater if it is allowed to expand while being heated than if the volume is kept constant. Discuss whether it is possible for the specific heat capacity of a gas to be zero.

When 1 g of water at 100°C is converted into steam at the same temperature 2264 J must be supplied. How much of this energy is used in forcing back the atmosphere? Explain what happens to the remainder of the energy. [1 g of water 100°C occupies 1 cm³. 1 g of steam at 100°C and 76 cm of mercury occupies 1601 cm³. Density of mercury = 13600 kg m⁻³.] (C.)

10. Define heat capacity and specific heat capacity.

Describe an experiment to determine the specific heat capacity of a gas either at constant volume or at constant pressure. Point out likely sources of error and indicate how they may be minimized.

Explain why it is necessary to specify the condition of constant pressure or constant volume. (L.)

11. Explain why, when quoting the specific heat of a gas, it is necessary to specify the conditions under which the change of temperature occurs. What conditions are normally specified?

A vessel of capacity 10 litres contains 130 g of a gas at 20°C and 10 atmospheres pressure. 8000 joule of heat energy are suddenly released in the gas and raise the pressure to 14 atmospheres. Assuming no loss of heat to the vessel, and ideal gas behaviour, calculate the specific heat of the gas under these conditions.

In a second experiment the same mass of gas, under the same initial conditions, is heated through the same rise in temperature while it is allowed to expand slowly so that the pressure remains constant. What fraction of the heat energy supplied in this case is used in doing external work? Take 1 atmosphere = 10^5 newton metre $^{-2}$. (O. & C.)

12. The two specific heat capacities in $\text{kJ kg}^{-1} \text{K}^{-1}$ units for argon are 0.521 and 0.313 and for air are 1.012 and 0.722. Explain these statements and discuss their significance in relation to (a) the atomicity of the molecules of the two gases, (b) the relative values of the adiabatic elasticities of argon and air.

Describe an experiment to verify one of the above values of specific heat capacities. (L.)

13. A litre of air, initially at 20°C and at 76.0 cm of mercury pressure, is heated at constant pressure until its volume is doubled. Find (a) the final temperature, (b) the external work done by the air in expanding, (c) the quantity of heat supplied.

[Assume that the density of air at s.t.p. is 1.293 kg m^{-3} and that the specific heat capacity of air at constant volume is $0.714 \text{ kJ kg}^{-1} \text{ K}^{-1}$.] (L.)

14. Describe an experiment to determine the specific heat capacity of water, at about 15°C, deriving from first principles any equations used.

Deduce an expression for the difference between the specific heat capacities of an ideal gas. If the specific heat capacity of air at constant pressure is $1.013 \text{ kJ kg}^{-1} \text{ K}^{-1}$ and the density at s.t.p. is 1.29 kg m^{-3} estimate a value for the specific heat capacity of air at constant volume. [Assume the density of mercury at 0°C to be 13600 kg m^{-3} .] (L.)

15. Distinguish between an isothermal change and an adiabatic change. In each instance state, for a reversible change of an ideal gas, the relation between pressure and volume.

A mass of air occupying initially a volume 2000 cm^3 at a pressure of 76.0 cm of mercury and a temperature of 20.0°C is expanded adiabatically and reversibly to twice its volume, and then compressed isothermally and reversibly to a volume of 3000 cm^3 . Find the final temperature and pressure, assuming the ratio of the specific heat capacities of air to be 1.40. (L.)

16. Explain why the specific heat of a gas at constant pressure is different from that at constant volume.

The density of an ideal gas is 1.60 kg m^{-3} at 27°C and 1.00×10^5 newton metre $^{-2}$ pressure and its specific heat capacity at constant volume is $0.312 \text{ kJ kg}^{-1} \text{ K}^{-1}$. Find the ratio of the specific heat capacity at constant pressure to that at constant volume. Point out any significance to be attached to the result. (N.)

17. Explain the meaning of the terms isothermal, adiabatic. What is the importance of the ratio of the specific heat capacity of an ideal gas?

Air initially at 27°C and at 75 cm of mercury pressure is compressed isothermally until its volume is halved. It is then expanded adiabatically until its original volume is recovered. Assuming the changes to be reversible, find the final pressure and temperature.

[Take the ratio of the specific heat capacities of air as 1·40.] (L.)

Kinetic Theory of Gases

18. Explain what is meant by the *root mean square velocity* of the molecules of a gas. Use the concepts of the elementary kinetic theory of gases to derive an expression for the root mean square velocity of the molecules in terms of the pressure and density of the gas.

Assuming the density of nitrogen at s.t.p. to be 1·251 kg m⁻³, find the root mean square velocity of nitrogen molecules at 127°C. (L.)

19. State the postulates on which the simple kinetic theory of gases is based. What modifications are made to the postulates in dealing with real gases? How are these modifications represented in van der Waals' equation? (N.)

20. State the assumptions that are made in the kinetic theory of gases and derive an expression for the pressure exerted by a gas which conforms to these assumptions, in terms of its density (p) and the mean square velocity (c^2) of its molecules.

Show (a) how *temperature* may be interpreted in terms of the theory, (b) how the theory accounts for Dalton's law of partial pressures. (L.)

21. Calculate the pressure in mm of mercury exerted by hydrogen gas if the number of molecules per cm³ is $6\cdot80 \times 10^{15}$ and the root mean square speed of the molecules is $1\cdot90 \times 10^3$ m s⁻¹. Comment on the effect of a pressure of this magnitude (a) above the mercury in a barometer tube; (b) in a cathode ray tube. (Avogadro's Number = $6\cdot02 \times 10^{23}$. Molecular weight of hydrogen = 2·02.) (N.)

22. Use a simple treatment of the kinetic theory of gases, stating any assumptions you make, to derive an expression for the pressure exerted by a gas on the walls of its container. Thence deduce a value for the root mean square speed of thermal agitation of the molecules of helium in a vessel at 0°C. (Density of helium at s.t.p. = 0·1785 kg m⁻³; 1 atmosphere = $1\cdot013 \times 10^5$ N m⁻².)

If the total translational kinetic energy of all the molecules of helium in the vessel is 5×10^{-6} joule, what is the temperature in another vessel which contains twice the mass of helium and in which the total kinetic energy is 10^{-5} joule? (Assume that helium behaves as a perfect gas.) (O. & C.)

23. Explain the meaning of the terms *ideal gas* and *molecule*.

What properties of a gas such as carbon dioxide distinguish it from an ideal gas and how may these differences from 'ideal' be demonstrated experimentally?

According to simple kinetic theory the pressure exerted by a gas of density ρ is $\frac{1}{3} \rho c^2$ where c^2 is the mean square molecular velocity. Show how this relation may be correlated with the equation of state for an ideal gas, $PV = RT$, explaining clearly what further assumptions you have to make. (O. & C.)

24. Derive an expression for the pressure (p) exerted by a mass of gas in terms of its molecular velocities, stating the assumptions made. What further assumption regarding the absolute temperature (T) of the gas is necessary to show that the expression is consistent with the equation $pv = kT$ where v is the volume of the mass of gas and k a constant? Which of the assumptions referred to did van der Waals modify to bring the equation $pv = kT$ more closely in agreement with the behaviour of real gases and what equation did he deduce? (L.)

25. Explain the following in terms of the simple kinetic theory without mathematical treatment:

- (a) A gas fills any container in which it is placed and exerts a pressure on the walls of the container.
- (b) The pressure of a gas rises if its temperature is increased without the mass and volume being changed.
- (c) The temperature of a gas rises if it is compressed in a vessel from which heat cannot escape.
- (d) The pressure in an oxygen cylinder falls continuously as the gas is taken from it, while the pressure in a cylinder containing chlorine remains constant until very nearly all the chlorine has been used. The contents of the cylinder are kept at room temperature in both cases.

[Critical temperature of chlorine = 146°C.] (C.)

26. (a) State the assumptions of the kinetic theory of gases. How does the theory represent the temperature of a gas, and how does it account for the fact that a gas exerts a pressure on the walls of its container? (The derivation of an expression for the pressure is NOT required.)

- (b) Near 0 K the specific heat capacity of silver, c , is not constant, but obeys the relation

$$c = \alpha T^3 + \beta T,$$

where T is the absolute temperature and α and β are constants typical of silver given by $\alpha = 15.12 \times 10^{-7} \text{ kJ kg}^{-1} \text{ deg}^{-4}$ and $\beta = 5.88 \times 10^{-6} \text{ kJ kg}^{-1} \text{ deg}^{-1}$. By means of a graph, or otherwise, find the heat required to raise the temperature of 5.0 g of silver from 1 K to 20 K. (C.)

27. Explain in terms of the kinetic theory what happens to the energy supplied to a gas when it is heated (a) at constant volume, (b) at constant pressure.

Deduce the total kinetic energy of the molecules in 1 g of an ideal gas at 0°C if its specific heat capacity at constant volume is $0.60 \text{ kJ kg}^{-1} \text{ K}^{-1}$.

An iron rod 1 metre long is heated without being allowed to expand lengthwise. When the temperature has been raised by 50°C the rod exerts a force of 1.2×10^4 newtons on the walls preventing its expansion. How much work could be obtained if it were possible to maintain it at this temperature and allow it to expand gradually until free from stress? [Linear expansivity of iron = $1.0 \times 10^{-5} \text{ K}^{-1}$.] (C.)

chapter eleven

Thermal expansion

IN this chapter we shall discuss the thermal expansion of solids and liquids.

SOLIDS

Linear Expansion

Most solids increase in length when they are warmed. Fig. 11.1 shows a simple apparatus with which we can measure the linear expansion of a metal tube A. We first measure the length of the tube, l_1 , at room temperature, θ_1 ; then we screw the spherometer S against the end of the tube and take its reading, S_1 . We next heat the tube by passing steam through it. At intervals we readjust the spherometer; when its reading becomes constant, the temperature of the rod is steady. We measure the temperature θ_2 on the thermometer B, and take the new reading of the spherometer, S_2 . The expansion of the tube is

$$e = S_2 - S_1.$$

The increase in length, λ , of unit length of the material for one degree temperature rise is then given by

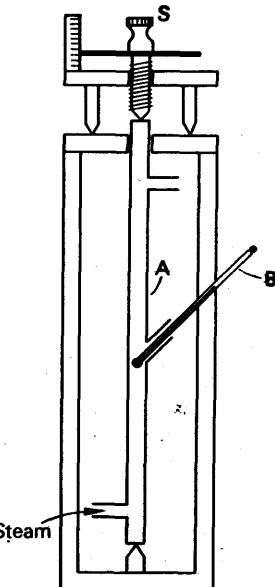


FIG. 11.1. Linear expansivity..

$$\lambda = \frac{\text{expansion}}{\text{original length} \times \text{temperature rise}} = \frac{e}{l_1(\theta_2 - \theta_1)}.$$

The quantity λ is called the *mean linear expansivity* of the metal, over the range θ_1 to θ_2 . If this range is not too great—say less than 100°C —the quantity λ may, to a first approximation, be taken as constant.

The linear expansivity of a solid, like the pressure and volume coefficients of a gas, has the unit deg C^{-1} or K^{-1} in SI units; its dimensions are

$$[\lambda] = \frac{[\text{length}]}{[\text{length}] \times [\text{temp.}]} = [\text{temp.}]^{-1}.$$

From the definition of λ , we can estimate the new length of a rod, l_2 , at a temperature θ_2 from the equation

$$l_2 = l_1 \{1 + \lambda(\theta_2 - \theta_1)\}, \quad \quad (1)$$

where l_1 is the length of the rod at the temperature θ_1 , and λ is the mean value over a range which includes θ_2 and θ_1 .

For accurate work, however, the length of a solid at a temperature θ must be represented by an equation of the form

$$l = l_0(1 + a\theta + b\theta^2 + c\theta^3 + \dots), \quad \quad (2)$$

where l_0 is the length at 0°C , and a, b, c are constants. The constant a is of the same order of magnitude as the mean coefficient λ ; the other constants are smaller.

MEAN LINEAR EXPANSIVITY

(Near room temperature)

Substance	λ, K^{-1}		Substance	λ, K^{-1}
	$\times 10^{-6}$			$\times 10^{-6}$
Copper	17		Bakelite	22
Iron	12		Brick	9.5
Brass	19		Glass (soda)	8.5
Nickel	13		Quartz (fused) (0 – 30°C)	0.42
Platinum	9		Pine—across grain	c. 0.34
Invar (36% nickel-steel)	c. 0.1		Pyrex	3

When a solid is subjected to small changes of temperature, about a mean value θ , its linear expansivity λ_θ in the neighbourhood of θ may be defined by the equation

$$\lambda_\theta = \frac{1}{l} \frac{dl}{d\theta},$$

where l is the length of the bar at the temperature θ . The following table shows how the coefficient varies with temperature.

VALUES OF λ_θ COPPER

θ	-87	0	100	400	600	${}^\circ\text{C}$
λ_θ	14.1	16.1	16.9	19.3	20.9	$\times 10^{-6} \text{ K}^{-1}$

Accurate Measurement of Expansion

An instrument for accurately measuring the length of a bar, at a controlled temperature, is called a *comparator* (Fig. 11.2). It consists of two microscopes, M_1 , M_2 , rigidly attached to steel or concrete pillars P_1 , P_2 . Between the pillars are rails R_1 , R_2 , carrying water-baths such as B . One of these baths contains the bar under test, X , which has scratches near its ends; the scratches are nominally a metre apart. Another water bath contains a substandard metre. The eyepieces of the microscopes are fitted with cross-webs carried on micrometer screws, m_1 , m_2 .

First the substandard metre is run under the microscopes, and the temperature of its bath is adjusted to that at which the bar was calibrated (usually about 18°C).

When the temperature of the bar is steady, the eyepiece webs are adjusted to intersect the scratches on its ends (Fig. 11.2 (b)), and their micrometers are read. The distance between the cross-webs is then 1 metre.

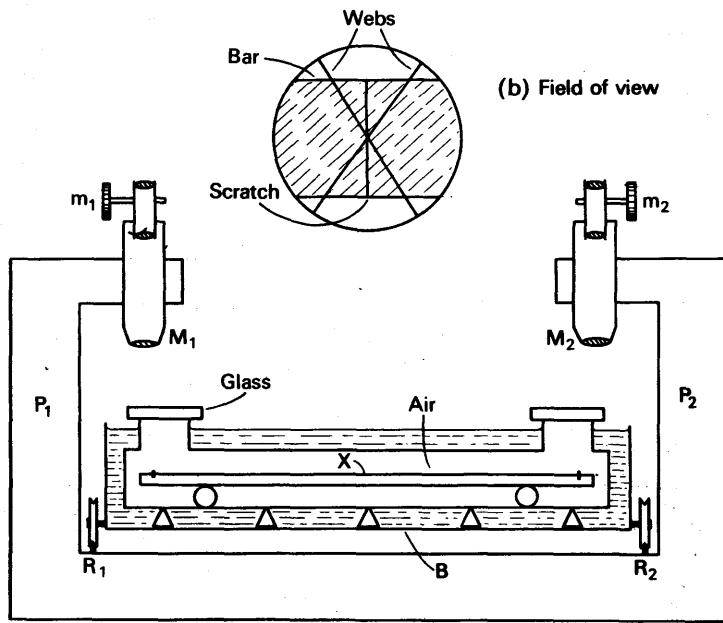


FIG. 11.2. The comparator.

The substandard is now removed, and the unknown bar put in its place; the temperature of the bar is brought to 0°C by filling its bath with ice-water. When the temperature of the bar is steady, the eyepiece webs are re-adjusted to intersect the scratches on its ends, and their micrometers are read. If the right-hand web has been shifted x mm to the right, and the left-hand y mm, also to the right, then the length of the bar at 0°C is

$$l_0 = 1 \text{ metre} + (x - y) \text{ mm.}$$

The bath is now warmed to say 10°C, and the length of the bar again measured. In this way the length can be measured at small intervals of temperature, and the mean linear expansivity, or the coefficients a, b, c in equation (2), can be determined.

Expansion at High Temperatures

Fig. 11.3 illustrates the principle of a method for measuring the expansion of a solid at high temperatures. A is a tube of fused silica,

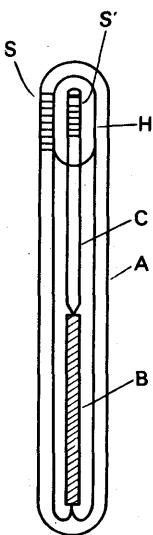


FIG. 11.3. Apparatus for measuring expansion at high temperatures.

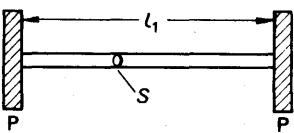


FIG. 11.4. Force in bar.

The force which opposes the expansion is the force which would compress a bar of natural length l_2 by the amount e . Its magnitude F depends on the cross-section of the bar, A , and the Young's modulus of its material, E :

$$\frac{F}{A} = E \frac{e}{l_2}$$

To a very good approximation we may replace l_2 by l_1 , because their difference is small compared with either of them. Thus

$$\begin{aligned} \frac{F}{A} &= E \frac{e}{l_1} = E \frac{\lambda l_1 (\theta_2 - \theta_1)}{l_1} \\ &= E \lambda (\theta_2 - \theta_1), \\ \therefore F &= EA \lambda (\theta_2 - \theta_1). \end{aligned}$$

For steel, $\lambda = 12 \times 10^{-6} \text{ K}^{-1}$ and $E = 2 \times 10^{11} \text{ newton per m}^2$. If the temperature difference, $\theta_2 - \theta_1$, is 100°C , then, for a cross-sectional area S of 4 cm^2 or $4 \times 10^{-4} \text{ m}^2$,

$$\begin{aligned} F &= 2 \times 10^{11} \times 12 \times 10^{-6} \times 4 \times 10^{-4} \times 100 \text{ newton} \\ &= 9.6 \times 10^4 \text{ newton}. \end{aligned}$$

On converting to kgf we find F is nearly 10000 kgf.

having a scale S engraved on the edge of an opening, H , in its side. B is the specimen, and C is a rod of fused silica with a second scale S' engraved on it.

The thermal expansion of fused silica is much less than that of most solids, over a given temperature range, and has been accurately measured by a method depending on optical interference. When the apparatus shown is placed in a high-temperature bath, the rod C rises by an amount equal to the difference in expansion of the specimen, and an equal length of fused silica. Its rise is measured by observing the displacement of the scale S' relative to S through a microscope.

Force set up when Expansion is Resisted

Consider a metal rod between two supports P , which we suppose are immovable (Fig. 11.4). Let l_1 be the distance between the supports, and θ_1 the temperature at which the rod just fits between them. If the rod is heated to θ_2 , it will try to expand to a greater length l_2 , but will not be able to do so.

The value of l_2 would be

$$l_2 = l_1 \{1 + \lambda(\theta_2 - \theta_1)\},$$

and the expansion would be

$$e = l_2 - l_1 = l_1 \lambda (\theta_2 - \theta_1),$$

where λ is the mean linear expansivity of the rod.

Expansion of a Measuring Scale

A scale, such as a metre rule, expands with rise in temperature; its readings, therefore, are correct at one temperature, θ_1 say. When the temperature of the scale is greater than θ_1 , the distance between any two of its divisions increases, and its reading is therefore too low (Fig. 11.5); when the scale is below θ_1 , its reading is too high. Let us suppose that, at θ_1 , the distance between any two points P, Q, on the scale is l_1 cm. At θ_2 it is

$$l_2 = l_1 \{1 + \lambda(\theta_2 - \theta_1)\},$$

where λ is the mean linear expansivity of the material of this scale. According to the divisions on the scale, however, the distance between P and Q will still be l_1 cm. Thus

$$\text{true distance at } \theta_2 = \text{scale value} \times \{1 + \lambda(\theta_2 - \theta_1)\} \quad . \quad (3)$$

If a sheet of material with a hole in it is warmed, it expands, and the hole expands with it. In Fig. 11.6, A represents a hole in a plate, and A' represents a plug, of the same material, that fits the hole. If A and A' are at the same temperature, then A' will fit A, whatever the value of that temperature; for we can always imagine A' to have just been cut out, without loss of material. It follows that the expansion of the hole A, in every direction, is the same as the expansion of the solid plug A'.

Differential Expansion

The difference in the expansions of different materials is used in practical arrangements discussed shortly. Fig. 11.7 (A) shows two rods AB, AB', of different metals, rigidly connected at A. If l_1 , l'_1 , are their lengths at a temperature θ_1 , their difference is

$$d_1 = BB' = l_1 - l'_1$$

If λ and λ' are the mean linear expansivities of the materials of the rods, then the lengths of the rods at θ_2 are

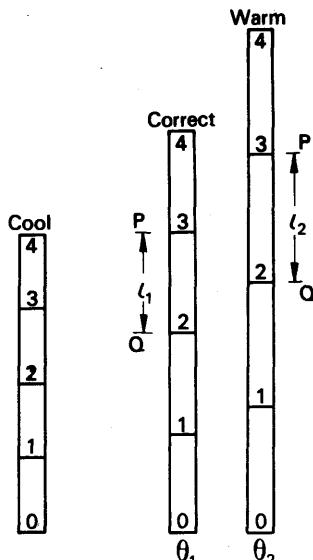


FIG. 11.5. Expansion of a measuring scale.

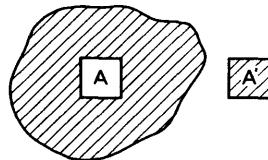


FIG. 11.6. Expansion of a hole.

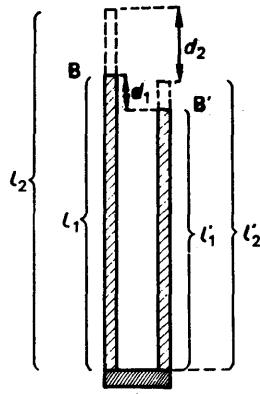


FIG. 11.7A. Differential expansion.

$$l_2 = l_1 \{1 + \lambda(\theta_2 - \theta_1)\},$$

$$l'_2 = l'_1 \{1 + \lambda'(\theta_2 - \theta_1)\}.$$

The distance between their ends is now

$$d_2 = l_2 - l'_2 = l_1 - l'_1 + (l_1\lambda - l'_1\lambda')(\theta_2 - \theta_1),$$

or $d_2 = d_1 + (l_1\lambda - l'_1\lambda')(\theta_2 - \theta_1) \quad \dots \quad (4)$

By a suitable choice of lengths and materials, the distance BB' can be made to vary with temperature in any one of the following ways:

(1) The bar AB' is made of invar, a nickel-steel whose linear expansivity is very small (p. 270). The point B' then does not move with changes in temperature. In equation (4) we neglect λ' , and find:

$$d_2 = d_1 + l_1\lambda(\theta_2 - \theta_1).$$

Thus the short distance BB' expands by the same amount as the long distance AB. Consequently, the relative expansion of BB' with temperature is much greater than that of a bar of length d_1 .

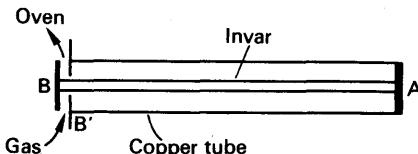


FIG. 11.7B. Thermostat principle.

(2) AB is made of invar, so that λ is negligible. The point B then does not move, and

the distance BB' shrinks rapidly as the temperature rises. This principle is used in the thermostats used to maintain gas ovens at constant temperatures (Fig. 11.7 (B)).

(3) The lengths and materials are chosen so that

$$\frac{l_1}{l'_1} = \frac{\lambda'}{\lambda},$$

or

$$l_1\lambda = l'_1\lambda'.$$

Then, by equation (4), d_1 or BB' does not change with temperature. This principle is used in compensating clock pendula for temperature changes (p. 276).

Bimetal Strip

Fig. 11.8 (a) shows two strips of different metals, welded together along B, called a bimetal strip. The metal M_1 has a greater linear expansivity than the metal M_2 . Therefore, when the strip is heated, M_1 will expand more than M_2 , and the strip will curl with M_1 on the outside. The reverse is true when the strip is cooled, as M_1 then shrinks more than M_2 (see Fig. 11.8(a)).

Bi-metal strips are used in electrical thermostats for ovens, irons, laboratories, etc. The strip carries a contact, K in Fig. 11.8 (b), which presses against another contact K' on the end of an adjusting screw S. When the strip warms, it tends to curl away from K'; the temperature at which the contact is broken can be set by turning the screw. When

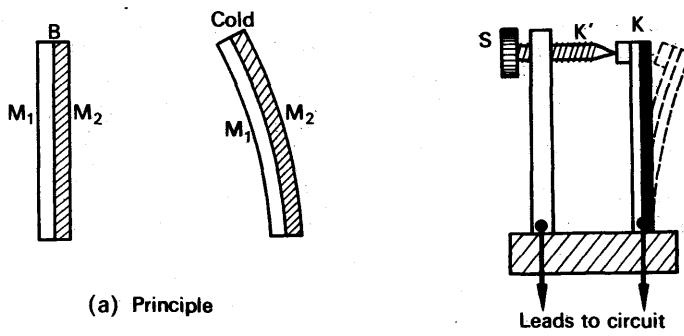


FIG. 11.8. Principle of gas thermostat.

the contacts open, they switch off the heating current. If the heating current is too great to be controlled by the contacts KK', it is switched off by a relay, which is controlled by the contacts on the bimetal.

Let us now estimate the deflection of a heated bimetal strip. We suppose that the component strips have the same thickness d , and the same length l at a temperature θ_1 (Fig. 11.9 (a)). When they are heated, they are distorted, but to a first approximation we may assume that the mid-line of each, shown dotted, has the length which it would naturally have (Fig. 11.9 (b)). The difference in length of the mid-lines, p , is then the difference in their expansions. At a temperature θ_2 ,

$$\begin{aligned} p &= l\lambda(\theta_2 - \theta_1) - l\lambda'(\theta_2 - \theta_1) \\ &= l(\lambda - \lambda')(\theta_2 - \theta_1). \end{aligned}$$

The difference is taken up by the curvature of the strip. If α is the angle through which it bends, then, from the figure,

$$\alpha = \frac{p}{d} = \frac{l}{d}(\lambda - \lambda')(\theta_2 - \theta_1).$$

(The expansion of d is negligible, to a very good approximation.)

To find the radius r of the arc formed by the strip, we assume that the length of the arc is l ; this we may do because the expansions in length are all small. Then

$$\alpha = \frac{l}{r};$$

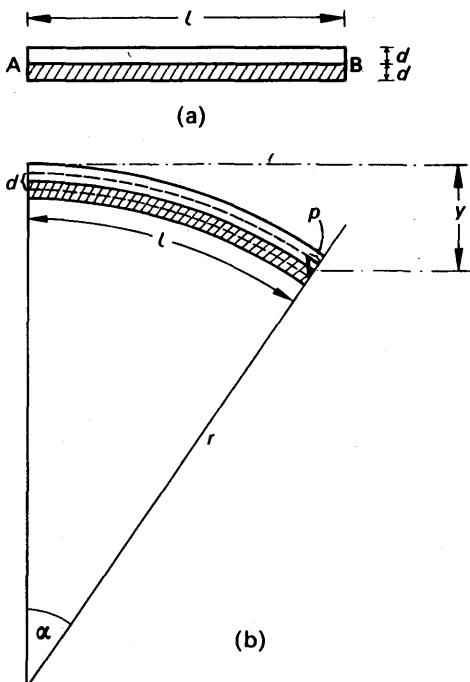


FIG. 11.9. Expansion of bimetal strip (exaggerated).

and, by the above equation,

$$\frac{l}{r} = \frac{l}{d}(\lambda - \lambda')(\theta_2 - \theta_1),$$

whence

$$r = \frac{d}{(\lambda - \lambda')(\theta_2 - \theta_1)}.$$

The deflection y of the end of the strip is given by the approximate equation

$$2ry = l^2,$$

from the geometry of the circle.

Thus

$$y = \frac{l^2}{2r} = \frac{l^2(\lambda - \lambda')(\theta_2 - \theta_1)}{2d}.$$

Temperature Compensation

The rate of a clock varies considerably with temperature, unless arrangements are made to prevent its doing so. If the clock is governed by a pendulum, the length of the pendulum increases with temperature, its period therefore also increases, and the clock loses. A clock governed by a balance-wheel and hair-spring also loses as the temperature rises. For, as the temperature rises, the spring becomes less stiff, and the period of the balance-wheel increases. Also the spokes of the wheel expand a little, increasing the moment of inertia of the wheel, and thus further increasing its period.

A balance-wheel clock may be compensated against temperature changes by making the circumference of the wheel in the form of two or three bimetal strips, as shown in Fig. 11.10. The strips carry small weights W , to give the wheel the necessary moment of inertia.

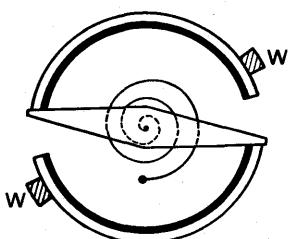


FIG. 11.10. Bimetal balance-wheel.

As the temperature rises, the strips curl inwards, and bring the weights nearer to the axle; thus the moment of inertia of the wheel decreases. In a correctly designed timing-system, the decrease in moment of inertia just offsets the decrease in stiffness of the spring, and then the period of the balance-wheel does not change with temperature.

Many modern watches are not compensated. Their balance-wheels are made of invar, which, as we have seen, has a very small coefficient of expansion. Another nickel-steel, of slightly different composition, is used for their hair-springs. This alloy changes its elasticity very little with temperature, and is called *elinvar*. The combination of invar balance-wheel and elinvar hair-spring gives a rate which is nearly enough independent of temperature for everyday purposes.

Pendula

To compensate a pendulum clock against changes of temperature, the pendulum must be so made that its effective length remains con-

stant. Fig. 11.11 (a) shows one way of doing this, in the so-called grid-iron pendulum (Harrison, 1761). Brass and steel rods are arranged so that the expansion of the brass rods raises the bob B of the pendulum, while the expansion of the steel rods lowers it. As explained on p. 274, the expansions can be made to cancel if

$$\frac{l_B}{l_S} = \frac{\lambda_S}{\lambda_B};$$

here l_B and l_S are the total lengths of brass and steel respectively, and λ_B , λ_S are their linear expansivities.

Fig. 11.11 (b) shows the same principle applied to a pendulum with a wooden rod and a cylindrical metal bob. To a first approximation the effective length of the pendulum, l , is the distance from its support to the centre of gravity, G, of the bob. The condition for constant period is then

$$\frac{\text{length of rod}}{\frac{1}{2} \text{length of bob}} = \frac{\lambda_{\text{wood}}}{\lambda_{\text{metal}}}.$$

The bob may be made of lead or zinc, either of which is much more expandable than is wood, along its grain.

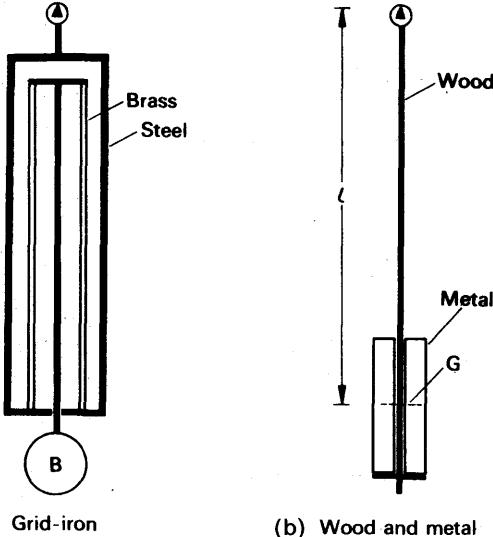


FIG. 11.11. Compensated pendula.

Metal-Glass Seals

In radio valves and many other pieces of physical apparatus, it is necessary to seal metal cones into glass tubes, with a vacuum-tight joint. The seal must be made at about 400°C ; when the glass is soft; as it cools to room temperature, the glass will crack unless the glass and metal contract at the same rate. This condition requires that the metal and the glass have the same linear expansivity at every temperature between room temperature and the melting-point of

glass. It is satisfied nearly enough by platinum and soda glass (mean $\lambda = 9$ and 8.5×10^{-6} per deg C, respectively), and by tungsten and some types of hard glass similar to pyrex (mean linear expansivity $\lambda = 3-4 \times 10^{-6}$ per deg C).

Modern seals through soft glass are not made with platinum, but with a wire of nickel-iron alloy, which has about the same linear coefficient as the glass. The wire has a thin coating of copper, which adheres to glass more firmly than the alloy. Also, being soft, the copper takes up small differences in expansion between the alloy and the glass.

In transmitting valves, and large vacuum plants, glass and metal tubes several centimetres in diameter must be joined end-to-end. The metal tubes are made of copper, chamfered to a fine taper at the end where the joint is to be made. The glass is sealed on to the edge of the chamfer; the copper there is thin enough to distort, with the difference in contraction, without cracking the glass.

Superficial Expansion

The increase in area of a body with temperature change is called the *superficial expansion* of the body. A rectangular plate, of sides a, b , at a given temperature, has an area

$$S_1 = ab.$$

If its temperature is increased by θ its sides become $a(1 + \lambda\theta), b(1 + \lambda\theta)$ where λ is its mean linear expansivity. Thus its area becomes

$$\begin{aligned} S_2 &= a(1 + \lambda\theta) b(1 + \lambda\theta) \\ &= S_1(1 + \lambda\theta)^2 \\ &= S_1(1 + 2\lambda\theta + \lambda^2\theta^2). \end{aligned}$$

In this expression, the term $\lambda^2\theta^2$ is small compared with $2\lambda\theta$; if λ is of the order of 10^{-5} , and θ of the order of 100, then $\lambda\theta \approx 10^{-3}$, and $\lambda^2\theta^2 \approx 10^{-6}$. Therefore we may neglect $\lambda^2\theta^2$ and write

$$S_2 = S_1(1 + 2\lambda\theta).$$

The superficial expansivity of the material of the plate is defined as

$$\frac{\text{increase of area}}{\text{original area} \times \text{temp. rise}} = \frac{S_2 - S_1}{S_1\theta}.$$

Its value is hence equal to 2λ , twice the linear expansivity. A hole in a plate changes its area by the same amount as would a plug that fitted the hole.

Cubic Expansivity

Cubic expansivity is expansion in volume. Consider a rectangular block of sides a, b, c , and therefore of volume

$$V_1 = abc.$$

If the block is raised in temperature by θ its sides expand, and its volume becomes

$$\begin{aligned}V_2 &= a(1 + \lambda\theta) b(1 + \lambda\theta) c(1 + \lambda\theta) = abc(1 + \lambda\theta)^3 \\&= abc(1 + 3\lambda\theta + 3\lambda^2\theta^2 + \lambda^3\theta^3).\end{aligned}$$

Since $\lambda^2\theta^2$ and $\lambda^3\theta^3$ are small compared with $\lambda\theta$, we may in practice neglect them. We then have

$$V_2 = abc(1 + 3\lambda\theta) = V_1(1 + 3\lambda\theta).$$

The cubic expansivity of the solid is defined as

$$\begin{aligned}\gamma &= \frac{\text{increase in volume}}{\text{original volume} \times \text{temp. rise}} \\&= \frac{V_2 - V_1}{V_1\theta} \\&= 3\lambda, \text{ to a very good approximation.}\end{aligned}$$

Thus the cubic expansivity is *three* times the linear expansivity.

By imagining a block cut out of a larger block, we can see that the cubical expansion of a hollow vessel is the same as that of a solid plug which would fit into it.

LIQUIDS

Cubic Expansivity

The temperature of a liquid determines its volume, but its vessel determines its shape. The only expansivity which we can define for a liquid is therefore its cubic expansivity, γ . Most liquids, like most solids, do not expand uniformly, and γ is not constant over a wide range of temperature. Over a given range θ_1 to θ_2 , the *mean coefficient* γ is defined as

$$\gamma = \frac{V_2 - V_1}{V_1(\theta_2 - \theta_1)},$$

where V_1 and V_2 are the volumes of a given mass of liquid at the temperatures θ_1 and θ_2 .

MEAN EXPANSIVITIES OF LIQUIDS (Near room temperature)

Liquid	$\gamma (\text{K}^{-1})$	Liquid	$\gamma (\text{K}^{-1})$
	$\times 10^{-4}$		$\times 10^{-4}$
Alcohol (methyl)	12.2	Water:	0.53
Alcohol (ethyl)	11.0	10–20°C	1.50
Aniline	8.5	20–40°C	3.02
Ether (ethyl)	16.3	40–60°C	4.58
Glycerine	5.3	60–80°C	5.87
Olive oil	7.0	Mercury:	1.81
Paraffin oil	9.0	0–100°C	1.82
Toluene	10.9	0–300°C	1.87

True and Apparent Expansion: Change of Density

If we try to find the cubic expansivity of a liquid by warming it in a vessel, the vessel also expands. The expansion which we observe is the difference between the increases in volume of the liquid and the vessel. This is true whether we start with the vessel full, and catch the overflow, or observe the creep of the liquid up the vessel (Fig. 11.12). The expansion we observe we call the *apparent expansion*; it is always less than the true expansion of the liquid.

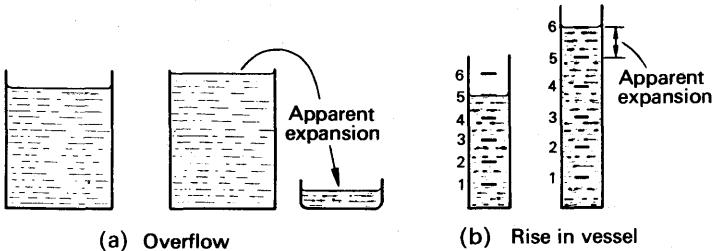


FIG. 11.12. Apparent expansion of a liquid.

Most methods of measuring the expansion of a liquid, whether true or apparent, depend on the change in density of the liquid when it expands. We therefore consider this change, before describing the measurements in detail. The mean true or absolute expansivity of a liquid γ , is defined in the same way as the mean cubic expansivity of a solid:

$$\gamma = \frac{\text{increase in volume}}{\text{initial volume} \times \text{temperature rise}}$$

Thus, if V_1 and V_2 are the volumes of unit mass of the liquid at θ_1 and θ_2 , then

$$V_2 = V_1 \{1 + \gamma(\theta_2 - \theta_1)\}.$$

The densities of the liquid at the two temperatures are

$$\rho_1 = \frac{1}{V_1},$$

$$\rho_2 = \frac{1}{V_2};$$

so that

$$\frac{1}{\rho_2} = \frac{1}{\rho_1} \{1 + \gamma(\theta_2 - \theta_1)\}$$

or

$$\rho_2 = \frac{\rho_1}{1 + \gamma(\theta_2 - \theta_1)} \quad \quad (5)$$

Measurement of True (Absolute) Expansivity

The first measurement of the true expansion of a liquid was made by Dulong and Petit in 1817. A simple form of their apparatus is

shown in Fig. 11.13. It consists of a glass tube ABCD, a foot or two high, containing mercury, and surrounded by glass jackets XY and YZ. The jacket X contains ice-water, and steam is passed through the jacket Y.

For the mercury to be in equilibrium, its hydrostatic pressure at B must equal its hydrostatic pressure at C. Let h_0 be the height of the mercury in the limb at 0°C and ρ_0 its density; and let h and ρ be the corresponding quantities at the temperature θ of the steam.

Then

$$g\rho_0 h_0 = g\rho h,$$

where g is the acceleration of gravity.

Hence

$$\frac{\rho}{\rho_0} = \frac{h_0}{h}.$$

But, by equation (5),

$$\frac{\rho}{\rho_0} = \frac{1}{1 + \gamma\theta}.$$

$$\therefore \frac{h_0}{h} = \frac{1}{1 + \gamma\theta}$$

or

$$h_0 + h_0\gamma\theta = h.$$

$$\therefore \gamma = \frac{h - h_0}{h_0\theta}.$$

The height $h - h_0$ is measured with a cathetometer (a travelling telescope on a vertical column).

This simple apparatus is inaccurate because:

- (i) the expansion of CD throws BC out of the horizontal;
- (ii) the wide separation of A and D makes the measurement of $(h - h_0)$ inaccurate;
- (iii) surface tension causes a difference of pressure across each free surface of mercury; and these do not cancel one another, because the surface tensions are different at the temperatures of the hot and cold columns.

Regnault got round these difficulties with the apparatus shown, somewhat simplified, in Fig. 11.14. The points B and G are fixed at the same horizontal level, the join DE is made of flexible iron tubing, and

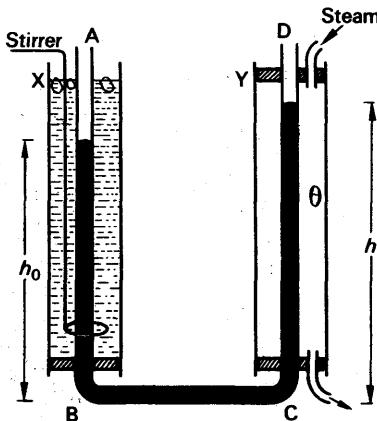


FIG. 11.13. Apparatus for true (absolute) expansivity of mercury.

the difference in height between its ends, h_2 , is measured. The parts AB, GH are at room temperature θ_1 ; and to a fair approximation, the average temperature of DE is also θ_1 . Suppose θ is the steam temperature.

If the density of mercury at θ_1 , θ is ρ_1 , ρ , respectively then equating the pressures on both sides at the horizontal level of E, we have

$$g\rho_1 h_1 + g\rho_0 h_0 + g\rho_1 h_2 = g\rho h + g\rho_1 h_3.$$

Therefore*

$$\frac{\rho_0 h_1}{1+\gamma\theta_1} + \rho_0 h_0 + \frac{\rho_0 h_2}{1+\gamma\theta_1} = \frac{\rho_0 h}{1+\gamma\theta} + \frac{\rho_0 h_3}{1+\gamma\theta_1}. \quad (6)$$

The uncertainty in the temperature of DE is not important, since the height h_2 is very small. Equation (6) gives

$$h_0 + \frac{h_2}{1+\gamma\theta_1} = \frac{h}{1+\gamma\theta} + \frac{h_3 - h_1}{1+\gamma\theta_1}. \quad . . . \quad (7)$$

Equation (7) can be solved for γ : the quantities which need to be known accurately are h_0 , h , and the difference $h_3 - h_1$. This difference can be measured accurately, because AB and GH are close together; and because they are at the same temperature there is no error due to surface tension. The heights h_0 and h are 1 or 2 metres, and so are easy to measure accurately.

Callender and Moss used six pairs of hot and cold columns, each 2 metres long, to increase the difference in level of the liquid. In this way they avoided the complication due to density change of the liquid under high pressure. All the hot columns were beside one another in a hot oil bath kept at a constant high temperature, while the cold columns were similarly placed in a bath of melting ice. Platinum resistance thermometers were used to measure the temperatures, and a cathetometer to measure the heights and difference in level.

* Strictly, equation (6) should be written

$$\frac{\rho_0 h_1}{1+\gamma_1\theta_1} + \rho_0 h_0 + \frac{\rho_0 h_2}{1+\gamma_1\theta_1} = \frac{\rho_0 h}{1+\gamma\theta} + \frac{\rho_0 h_3}{1+\gamma_1\theta_1},$$

where γ_1 is the mean coefficient between 0°C and θ_1 , and γ is that between 0°C and θ . This equation can be solved for γ by successive approximations. (See Roberts-Miller, *Heat and Thermodynamics*, Chapter X. Blackie.)

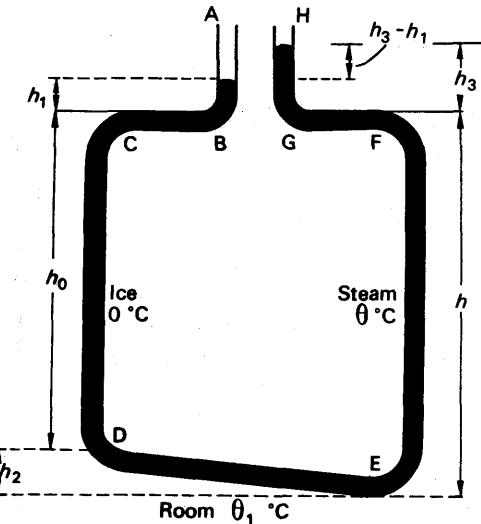


FIG. 11.14. Modified apparatus for true expansivity

Apparent Expansion: Weight Thermometer Method

The method of balancing columns for the absolute expansivity of a liquid is slow and awkward; it has only been applied to mercury. Routine measurements are more conveniently made by measuring the apparent expansion; from this, as we shall see, the absolute expansion can be calculated.

A *weight thermometer* is a bulb of fused quartz fitted with a fine stem (Fig. 11.15). It is filled with liquid at a low temperature, and then warmed; the liquid expands, and from the mass which flows out the apparent expansion of the liquid can be found. The weight thermometer is filled by warming it to expel air, and then dipping the stem into the liquid. The process has to be repeated many times; and for accurate work the liquid in the thermometer should be boiled at intervals during the filling, to expel dissolved air. In a laboratory experiment, a glass density bottle may be used, filled in the usual way.

The weight thermometer must be filled at a temperature slightly below the lower limit, θ_1 , of the range over which the expansion is to be measured. It is then kept in a bath at θ_1 , until no more liquid flows out. Next it is weighed, and from its known mass when empty the mass of liquid in it is found. Let this be m_1 . The weight thermometer is then placed in a bath at the higher temperature of the range, θ_2 , and the mass remaining in it, m_2 , is found by weighing.

If V_1 and V_2 are the volumes of the weight thermometer at θ_1 and θ_2 , then

$$V_2 = V_1 \{1 + 3\lambda(\theta_2 - \theta_1)\},$$

where λ is the linear expansivity of quartz, the material of the weight thermometer. Now, if ρ denotes the density of the liquid, whose cubic expansivity is γ , we have

$$m_1 = V_1 \rho_1,$$

$$m_2 = V_2 \rho_2,$$

and

$$\frac{\rho_2}{\rho_1} = \frac{1}{1 + \gamma(\theta_2 - \theta_1)}.$$

Therefore

$$\begin{aligned} \frac{m_2}{m_1} &= \frac{V_2 \rho_2}{V_1 \rho_1} \\ &= \frac{1 + 3\lambda(\theta_2 - \theta_1)}{1 + \gamma(\theta_2 - \theta_1)}. \end{aligned}$$

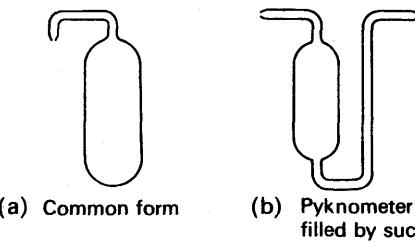


FIG. 11.15. Weight thermometers.

Hence $m_2 + m_2\gamma(\theta_2 - \theta_1) = m_1 + 3\lambda m_1(\theta_2 - \theta_1)$

or $m_2\gamma(\theta_2 - \theta_1) = m_1 - m_2 + 3\lambda m_1(\theta_2 - \theta_1)$

and $\gamma = \frac{m_1 - m_2}{m_2(\theta_2 - \theta_1)} + 3\lambda \frac{m_1}{m_2}$ (11)

If we ignore the expansion of the solid, we obtain the apparent expansivity of the liquid, γ_a , relative to the solid.

Thus $\gamma_a = \frac{m_1 - m_2}{m_2(\theta_2 - \theta_1)}$ (12)

The expression for the apparent expansivity may be expressed in words:

$$\gamma_a = \frac{\text{mass expelled}}{\text{mass left in} \times \text{temp. rise}}$$

Attention is often drawn to the fact that the mass in the denominator is the mass at the higher temperature, and not the lower.

From equation (11),

$$\gamma = \gamma_a + \frac{m_1}{m_2}\gamma_g$$
 (13a)

where $\gamma_g = 3\lambda$ = the cubical coefficient of the solid. Since m_1 and m_2 are nearly equal, then, to a good approximation,

$$\gamma = \gamma_a + \gamma_g$$
 (13b)

Thus after the apparent coefficient has been calculated, the absolute coefficient is given to a good approximation by

absolute expansivity = apparent expansivity + cubic expansivity of vessel.

Coefficient of Weight-thermometer Material

The cubic expansivity of a glass weight-thermometer may not be accurately known, because glasses differ considerably in their physical properties. Also the expansivity may be changed when the glass is heated in the blowing of the bulb. The expansivity can conveniently be measured, however, by using the weight thermometer to find the apparent value for mercury; and then subtracting the value found from the known value of the absolute expansivity of mercury.

Expansion of a Powder

The weight thermometer can be used to find the cubic expansivity of a granular or powdery solid, such as sand. The procedure is the same as in finding the relative density of the solid, but is gone through at two known temperatures. From the change in relative density, and a knowledge of the expansions of the liquid and the weight thermometer, the change in absolute density of the powder can be found, and hence its cubic expansivity.

The Dilatometer

A dilatometer is an instrument for rapidly—but roughly—measuring the expansion of a liquid. It consists of a glass bulb B, with a graduated stem S (Fig. 11.16). The volume V_b of the bulb, up to the zero of S, is known, and S is graduated in cubic millimetres or other small units. The volume of the bulb, and the value of one scale division, vary with temperature; the dilatometer therefore measures apparent expansion.

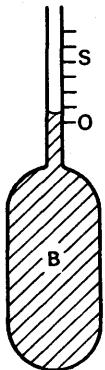


FIG. 11.16. Dilatometer.

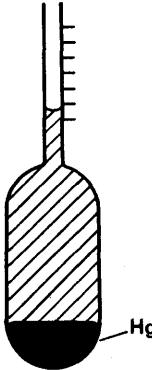


FIG. 11.17. Compensated dilatometer.

The dilatometer is filled with the liquid under test to a point just above the zero of the stem, at a temperature θ_1 . The volume V_1 of the liquid is found by adding V_b to the stem-reading v_1 . Next the dilatometer is warmed to θ_2 , and the liquid rises to v_2 . Then $(v_2 - v_1)$ is the apparent expansion of the liquid, and hence

$$\gamma_a = \frac{v_2 - v_1}{V_1(\theta_2 - \theta_1)}.$$

If γ_g is the cubical coefficient of the glass, then

$$\gamma = \gamma_a + \gamma_g.$$

For demonstration work, a dilatometer can be compensated so that it shows roughly the true expansion of a liquid. Mercury is introduced into the bulb, until it occupies 1/7th of the bulb's volume (Fig. 11.17). The expansion of the mercury is then about equal to the expansion of the glass, so that the free volume in the bulb is roughly constant. The cubic expansivities of mercury and glass are given respectively by

$$\gamma_{Hg} = 18.1 \times 10^{-5} \text{ K}^{-1}$$

$$\text{and } \gamma_g = 3\lambda_g = 3 \times 8.5 \times 10^{-6} = 2.55 \times 10^{-5} \text{ K}^{-1}.$$

$$\text{Thus } \frac{\gamma_{Hg}}{\gamma_g} = \frac{18.1}{2.55} = 7.1.$$

Thus the expansion of the mercury offsets that of the glass, within about $1\frac{1}{2}$ per cent.

The space above the mercury, whose volume is constant, is filled

with the liquid to be examined. When the bulb is warmed, the movement of the liquid up the stem shows the liquid's true expansion. This device may be used to show the anomalous expansion of water (p. 288).

Correction of the Barometer

The hydrostatic pressure of a column of mercury, such as that in a barometer, depends on its density as well as its height. When we speak of a pressure of 760 mm mercury, therefore, we must specify the temperature of the mercury; we choose 0°C. In practice barometers are generally warmer than that, and their readings must therefore be

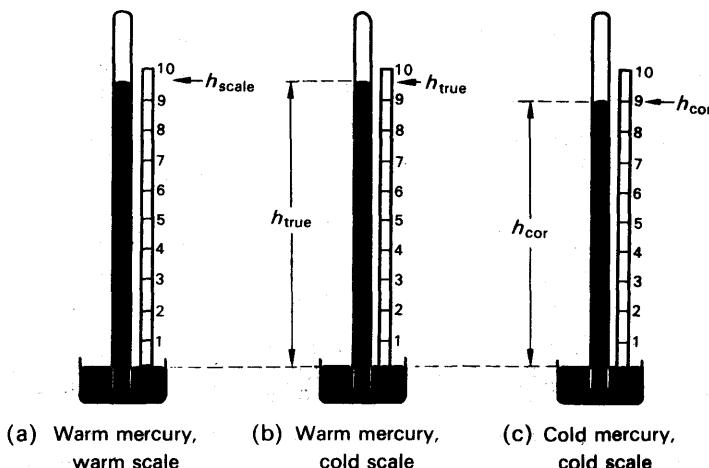


FIG. 11.18. Reduction of barometer height to 0°C.

reduced to what they would be at 0°C. Also we must allow for the expansion of the scale with which the height is measured.

The scale of a barometer may be calibrated at 0°C. At any higher temperature, θ , the height which it indicates, h_{scale} , is less than the true height, h_{true} , of the mercury meniscus above the free surface in the reservoir (Fig. 11.18 (a) (b)). The true height is given by equation (3) of p. 273.

$$h_{true} = h_{scale} (1 + \lambda \theta) \quad \quad (14)$$

where λ is the linear expansivity of the scale. If ρ_θ is the density of mercury at θ , then the pressure of the atmosphere is

$$p = g \rho_\theta h_{true}.$$

The height of a mercury column at 0°C which would exert the same pressure is called the *corrected height*, $h_{cor.}$ (Fig. 11.18 (b) (c)). It is given by

$$p = g \rho_0 h_{cor.},$$

where ρ_0 is the density of mercury at 0°C.

Therefore

$$\rho_\theta h_{\text{true}} = \rho_0 h_{\text{cor.}}$$

If γ is the coefficient of expansion of mercury, then

$$\rho_\theta = \frac{\rho_0}{1 + \gamma\theta}.$$

Hence $\frac{\rho_0}{1 + \gamma\theta} h_{\text{true}} = \rho_0 h_{\text{cor.}}$

and $h_{\text{cor.}} = \frac{h_{\text{true}}}{1 + \gamma\theta}.$

Therefore, by equation (13),

$$h_{\text{cor.}} = \frac{h_{\text{scale}} (1 + \lambda\theta)}{1 + \gamma\theta}.$$

Let us write this as

$$h_{\text{cor.}} = h_{\text{scale}} (1 + \lambda\theta)(1 + \gamma\theta)^{-1}.$$

Then, if we ignore $\gamma^2\theta^2$ and higher terms, we may write

$$h_{\text{cor.}} = h_{\text{scale}} (1 + \lambda\theta)(1 - \gamma\theta).$$

Hence $h_{\text{cor.}} = h_{\text{scale}} (1 + \lambda\theta - \gamma\theta + \gamma\lambda\theta^2),$

and ignoring $\gamma\lambda\theta^2$ we find

$$h_{\text{cor.}} = h_{\text{scale}} \{1 + (\lambda - \gamma)\theta\}.$$

The coefficient γ is greater than λ , and the corrected height is less than the scale height. It is convenient therefore, to write

$$h_{\text{cor.}} = h_{\text{scale}} \{1 - (\gamma - \lambda)\theta\}.$$

The reader should notice that the correction depends on the difference between the cubic expansivity of the mercury, and the linear expansivity of the scale; as in Dulong and Petit's experiment, there is no question of apparent expansion.

The Anomalous Expansion of Water

If we nearly fill a tall jar with water, and float lumps of ice on it, the water at the base of the jar does not cool below 4°C, although the water at the top soon reaches 0°C. At 4°C water has its greatest density; as it cools to this temperature it sinks to the bottom. When the water at the top cools below 4°C, it becomes less dense than the water below, and stays on the top. Convection ceases, and the water near the bottom of the jar can lose heat only by conduction. Since water is a bad conductor, the loss by conduction is extremely small, and in practice the water at the bottom does not cool below 4°C. The same happens in a pond, cooled at the top by cold air. Ice forms at the surface, but a little below it the water remains at 4°C, and life in the pond survives. It

could not if the water contracted in volume continuously to 0°C ; for then convection would always carry the coldest water to the bottom, and the pond would freeze solid. In fact, lakes and rivers, unless they are extremely shallow, never do freeze solid; even in arctic climates, they take only a crust of ice.

Fig. 11.19 shows how the volume of 1 g of water varies with temperature. The decrease from 0°C to 4°C is called *anomalous expansion*; it can be shown with a compensated dilatometer (p. 285). Water also has the unusual property of expanding when it freezes, hence burst

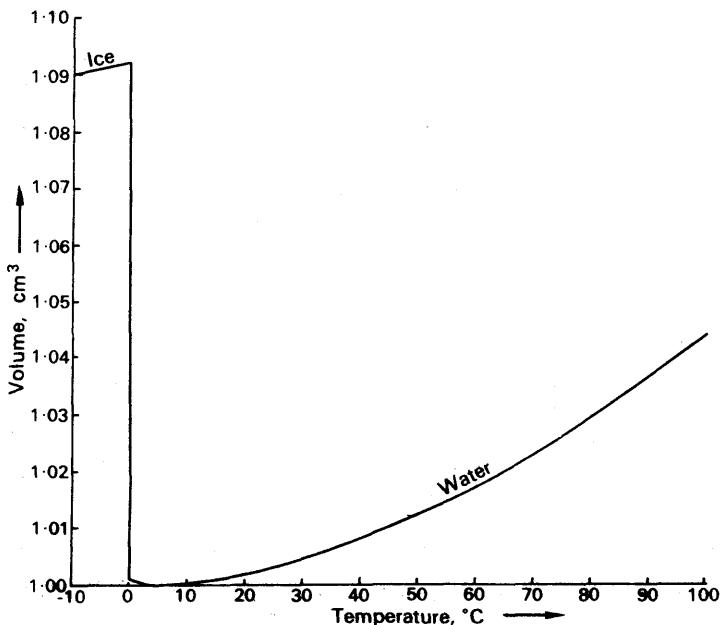


FIG. 11.19. Volume variation of 1 g of water.

pipes. As the figure shows, the expansion is about 9 per cent. If ice were not less dense than water, it would sink, and so, despite the anomalous expansion of water, lakes and rivers would freeze solid in winter.

The following table gives the densities of ice and water at various temperatures. Accurate experiments show that the temperature of maximum density is 3.98°C .

DENSITY OF ICE AND WATER

Temperature °C .	0	2	4	6	8
$\rho_{\text{water}} \text{ kg m}^{-3}$.	999.84	999.94	999.97	999.94	999.85
$\rho_{\text{ice}} \text{ kg m}^{-3}$.	916.0 (volume of 1 g ice at 0°C = 1.092 cm^3)				
Temperature °C .	10	20	40	60	80
$\rho_{\text{water}} \text{ kg m}^{-3}$.	999.70	998.20	992.2	983.2	971.8
					958.4

The temperature of maximum density was first measured in 1804, by Hope, whose apparatus is shown in Fig. 11.20. Both vessels are made of metal; the inner one contains water, and the outer one contains a mixture of ice and salt, which has a temperature below 0°C . After several hours, the water at the top cools to 0°C —it may even freeze—but the water at the bottom does not fall below 4°C .

The contraction of water from 0 to 4°C is explained by supposing that the molecules form clusters, such as H_4O_2 , H_6O_3 . At first, the contraction due to the formation of these more than offsets the expansion due to the rise of temperature; but above 4°C the expansion prevails. The metal antimony behaves like water: it expands on freezing, and contracts at first when warmed above its melting-point. Because it expands on freezing, it makes sharp castings, and is used as a constituent of type-metal.

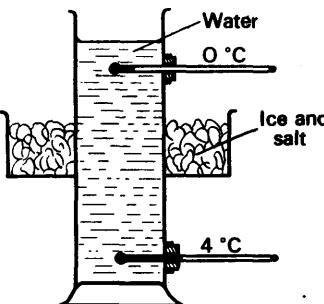


FIG. 11.20. Hope's experiment.

EXAMPLES

1. Describe how you would use a specific gravity bottle to find the coefficient of expansion of paraffin oil relative to glass between 0° and 50°C .

A specific gravity bottle contains 44.25 g of a liquid at 0°C and 42.02 g at 50°C . Assuming that the linear expansivity of the glass is 0.00001 K^{-1} , (a) compare the densities of the liquid at 0° and 50°C , (b) deduce the real expansivity of the liquid. Prove any formula employed. (N.)

First part. See text.

Second part. The apparent expansivity of the liquid, γ_a , is given by

$$\gamma_a = \frac{\text{mass expelled}}{\text{mass left} \times \text{temp. rise}} = \frac{44.25 - 42.02}{42.02 \times 50}.$$

$$\therefore \gamma_a = \frac{2.23}{42.02 \times 50} = 0.00106 \text{ K}^{-1}.$$

Now cubic expansivity, γ_g , of glass = $3 \times 0.00001 = 0.00003 \text{ K}^{-1}$.

$$\therefore \text{real coefficient, } \gamma = \gamma_a + \gamma_g = 0.00106 + 0.00003 = 0.00109 \text{ K}^{-1}.$$

Also, if ρ_0 , ρ_t are the densities at 0°C , 50°C respectively,

$$\begin{aligned} \frac{\rho_0}{\rho_t} &= 1 + \gamma t \\ &= 1 + 0.00109 \times 50 \\ &= 1.055. \end{aligned}$$

2. Define the linear expansivity of a solid, and show how it is related to the cubic expansivity. Describe an accurate method of determining the coefficient of expansion of a solid.

In order to make connexion to the carbon anode of a transmitting valve it is

required to thread a copper wire (2 mm in diameter at 20°C) through a smaller hole in the carbon block. If the process can just be carried out by immersing both specimens in dry ice (solid CO₂) at -80°C and the coefficients of linear expansion of copper and carbon are 17×10^{-6} and $5 \times 10^{-6} \text{ K}^{-1}$, calculate the size of the hole at 20°C. (O. & C.)

First part. The linear expansivity is the increase in length per unit length per degree rise in temperature. It is one-third of the cubic expansivity, proved on p. 279. An accurate method of measuring the linear expansivity is the comparator method, described on p. 270.

Second part. Suppose r cm is the diameter of the hole at 20°C. Then, from the formula

$$l_2 = l_1[1 + \lambda(\theta_2 - \theta_1)], \text{ where } (\theta_2 - \theta_1) = -80 - 20 = -100^\circ\text{C},$$

$$\text{diameter at } -80^\circ\text{C} = r(1 - 5 \times 10^{-6} \times 100) = 0.9995 r \text{ cm.}$$

Similarly, the diameter of the copper wire at -80°C

$$= 0.2(1 - 17 \times 10^{-6} \times 100) = 0.19966 \text{ cm.}$$

$$\therefore 0.9995 r = 0.19966$$

$$\therefore r = \frac{0.19966}{0.9995} = 0.19976 \text{ cm} = 1.998 \text{ mm.}$$

3. Describe in detail how the cubic expansivity of a liquid may be determined by the weight thermometer method.

The height of the mercury column in a barometer provided with a brass scale correct at 0°C is observed to be 749.0 mm on an occasion when the temperature is 15°C. Find (a) the true height of the column at 15°C, (b) the height of a column of mercury at 0°C which would exert an equal pressure. Assume that the cubic expansivities of brass and of mercury are respectively 0.000054 and 0.000181 K⁻¹. (N.)

First part. The weight thermometer method gives the apparent expansivity of the liquid, which is calculated from :

$$\frac{\text{mass expelled}}{\text{mass left} \times \text{temperature rise}}$$

The true expansivity is obtained by adding the cubic expansivity of the container's material to the apparent expansivity.

Second part. (a) The linear expansivity of brass, λ ,

$$= \frac{1}{3} \times \text{volume expansivity}$$

$$= \frac{1}{3} \times 0.000054 = 0.000018 \text{ K}^{-1}.$$

$$\therefore \text{true height, } h_{\theta}, = 749.0 (1 + \lambda\theta) = 749.0 (1 + 0.000018 \times 15)$$

$$= 749.2 \text{ mm.}$$

(b) Suppose h_0 is the required height at 0°C. Then, since pressure = $h\rho g$,

$$h_0 \rho_0 g = h_{\theta} \rho_{\theta} g$$

$$\therefore h_0 = h_{\theta} \frac{\rho_{\theta}}{\rho_0}.$$

But

$$\frac{\rho_\theta}{\rho_0} = \frac{1}{1+\gamma\theta} = \frac{1}{1+0.000181 \times 15} = \frac{1}{1.0027}$$

$$\therefore h_0 = 749.0 \times \frac{1.00027}{1.0027}$$

$$= 749 [1 - (0.0027 - 0.00027)]$$

$$= 747.2 \text{ mm.}$$

EXERCISES 11

1. Describe and explain how the absolute expansivity of a liquid may be determined without a previous knowledge of any other expansivity.

Aniline is a liquid which does not mix with water, and when a small quantity of it is poured into a beaker of water at 20°C it sinks to the bottom, the densities of the two liquids at 20°C being 1021 and 998 kg m⁻³ respectively. To what temperature must the beaker and its contents be uniformly heated so that the aniline will form a globule which just floats in the water? (The mean absolute expansivity of aniline and water over the temperature range concerned are 0.00085 and 0.00045 K⁻¹, respectively.) (L.)

2. Define the *linear expansivity* of a solid, and describe a method by which it may be measured.

Show how the superficial expansivity can be derived from this value.

A 'thermal tap' used in certain apparatus consists of a silica rod which fits tightly inside an aluminium tube whose internal diameter is 8 mm at 0°C. When the temperature is raised, the fit is no longer exact. Calculate what change in temperature is necessary to produce a channel whose cross-section is equal to that of a tube of 1 mm internal diameter. [Linear expansivity of silica = 8×10^{-6} K⁻¹. Linear expansivity of aluminium = 26×10^{-6} K⁻¹.] (O. & C.)

3. Define the *cubic expansivity* of a liquid. Find an expression for the variation of the density of a liquid with temperature in terms of its expansivity.

Describe, without experimental details, how the cubic expansivity of a liquid may be determined by the use of balanced columns.

A certain Fortin barometer has its pointers, body and scales made from brass. When it is at 0°C it records a barometric pressure of 760 mm Hg. What will it read when its temperature is increased to 20°C if the pressure of the atmosphere remains unchanged?

[Cubic expansivity of mercury = 1.8×10^{-4} K⁻¹; linear expansivity of brass = 2×10^{-5} K⁻¹.] (O. & C.)

4. Describe an experiment to determine the absolute (real) expansivity of paraffin between 0°C and 50°C, using a weight thermometer made of glass of known cubic expansivity. Derive the formula used to calculate the result.

A glass capillary tube with a uniform bore contains a thread of mercury 100.0 cm long, the temperature being 0°C. When the temperature of the tube and mercury is raised to 100°C the thread is increased in length by 1.65 cm. If the mean coefficient of absolute expansion of mercury between 0°C and 100°C is 0.000182 K⁻¹, calculate a value for the linear expansivity of glass (N.).

5. Define the linear and cubic expansivity, and derive the relation between them for a particular substance. Describe, and give the theory of, a method for finding *directly* the absolute coefficient of expansion of a liquid. The bulb of a mercury-in-glass thermometer has a volume of 0.5 cm³ and the distance between progressive degree marks is 2 mm. If the linear expansivity of glass is 10^{-5} K⁻¹,

and the cubic expansivity of mercury is $1.8 \times 10^{-4} \text{ K}^{-1}$, find the cross-sectional area of the bore of the stem. (C.)

6. Define the terms 'apparent' and 'absolute' expansivity of a liquid, and show how the former is found by means of a weight thermometer. A litre flask, which is correctly calibrated at 4°C , is filled to the mark with water at 80°C . What is the weight of water in the flask? [Linear expansivity of the glass of the flask = $8.5 \times 10^{-6} \text{ K}^{-1}$; mean cubic expansivity of water = $5.0 \times 10^{-4} \text{ K}^{-1}$.] (O. & C.)

7. Describe how to measure the apparent expansivity of a liquid using a weight thermometer. Show how the result can be calculated from the observations.

A specific gravity bottle of volume 50.0 cm^3 at 0°C is filled with glycerine at 20°C . What mass of glycerine is contained in the bottle if the density of glycerine at 0°C is 1.26 g per cm^3 , and its real expansivity is $5.2 \times 10^{-4} \text{ K}^{-1}$? Assume that the linear expansivity of the glass is $8 \times 10^{-6} \text{ K}^{-1}$. (N.)

8. Describe in detail how a reliable value for the expansivity of mercury may be found by a method independent of the expansion of the containing vessel. Give the necessary theory.

A silica bulb of negligible expansivity holds 340.0 g of mercury at 0°C when full. Some steel balls are introduced and the remaining space is occupied at 0°C by 255.0 g of mercury. On heating the bulb and its contents to 100°C , 4.800 g of mercury overflow. Find the linear expansivity of the steel.

[Assume that the expansivity of mercury is $180 \times 10^{-6} \text{ K}^{-1}$.] (L.)

9. Describe an accurate method for determining the linear expansivity of a solid in the form of a rod. The pendulum of a clock is made of brass whose linear expansivity is 1.9×10^{-5} per deg C. If the clock keeps correct time at 15°C , how many seconds per day will it lose at 20°C ? (O. & C.)

10. A steel wire 8 metres long and 4 mm in diameter is fixed to two rigid supports. Calculate the increase in tension when the temperature falls 10°C . [Linear expansivity of steel = $12 \times 10^{-6} \text{ K}^{-1}$, Young's modulus for steel = $2 \times 10^{11} \text{ N m}^{-2}$.] (O. & C.)

11. How can you show that the density of water does not fall steadily as the temperature is raised from 0°C to 100°C ? What does your experiment indicate about the expansion of water? What importance has this result in nature? (C.)

12. Why is mercury used as a thermometric fluid? Compare the advantages and disadvantages of the use of a mercury-in-glass thermometer and a platinum resistance thermometer to determine the temperature of a liquid at about 300°C .

A dilatometer having a glass bulb and a tube of uniform bore contains 150 g of mercury which extends into the tube at 0°C . How far will the meniscus rise up the tube when the temperature is raised to 100°C if the area of cross-section of the bore is 0.8 mm^2 at 0°C ? Assume that the density of mercury at 0°C is 13.6 g cm^{-3} , that the expansivity of mercury is $1.82 \times 10^{-4} \text{ K}^{-1}$, and that the linear expansivity of glass is $1.1 \times 10^{-5} \text{ K}^{-1}$. (N.)

13. Describe in detail how you would determine the linear expansivity of a metal rod or tube. Indicate the chief sources of error and discuss the accuracy you would expect to obtain.

A steel cylinder has an aluminium alloy piston and, at a temperature of 20°C when the internal diameter of the cylinder is exactly 10 cm , there is an all-round clearance of 0.05 mm between the piston and the cylinder wall. At what temperature will the fit be perfect? (The linear expansivity of steel and the aluminium alloy are 1.2×10^{-5} and $1.6 \times 10^{-5} \text{ K}^{-1}$ respectively.) (O. & C.)

14. Explain the statement: *the absolute expansivity of mercury is $1.81 \times 10^{-4} \text{ K}^{-1}$* . Describe an experiment to test the accuracy of this value. Why is a knowledge of it important?

Calculate the volume at 0°C required in a thermometer to give a degree of length 0.15 cm on the stem, the diameter of the bore being 0.24 mm . What would be the volume of this mercury at 100°C ? [The linear expansivity of the glass may be taken as $8.5 \times 10^{-6} \text{ K}^{-1}$.] (L.)

chapter twelve

Changes of State. Vapours

SOLID TO LIQUID: FUSION

The Solid State

Substances exist in the solid, liquid or gaseous state. In the *solid state*, a body has a regular, geometrical structure. Sometimes this structure gives the body a regular outward form, as in a crystal of alum; sometimes, as in a strand of wool, it does not. But X-rays can reveal to us the arrangement of the individual atoms or molecules in a solid; and whether the solid is wool or alum, we find that its atoms or molecules are arranged in a regular pattern. This pattern we call a *space-lattice*; its form may be simple, as in metals, or complicated, as in wool, proteins, and other chemically complex substances.

We consider that the atoms or molecules of a solid are vibrating about their mean positions in its space-lattice. And we consider that the kinetic energy of their vibrations increases with the temperature of the solid: its increase is the heat energy supplied to cause the rise in temperature. When the temperature reaches the melting-point, the solid liquefies. Lindemann has suggested that, at the melting-point, the atoms or molecules vibrate so violently that they collide with one another. The attractive forces between them can then no longer hold them in their pattern, the space-lattice collapses, and the solid melts. The work necessary to overcome the forces between the atoms or molecules of the solid, that is, to break-up the space-lattice, is the latent heat of melting or fusion.

The Liquid State

In the liquid state, a body has no form, but a fixed volume. It adapts itself to the shape of its vessel, but does not expand to fill it. We consider that its molecules still dart about at random, as in the gaseous state, and we consider that their average kinetic energy rises with the liquid's temperature. But we think that they are now close enough together to attract one another—by forces of a more-or-less gravitational nature. Any molecule approaching the surface of the liquid experiences a resultant force opposing its escape (see p. 128, Surface Tension) Nevertheless, some molecules do escape, as is shown by the fact that the liquid evaporates: even in cold weather, a pool of water does not last for ever. The molecules which escape are the fastest, for they have the greatest kinetic energy, and therefore the greatest chance of overcoming the attraction of the others. Since the fastest escape, the slower, which remain, begin to predominate: the average kinetic energy of the molecules falls, and the liquid cools. The faster a liquid evaporates, the colder it feels on the hand—petrol feels colder than water, water feels

colder than paraffin. To keep a liquid at constant temperature as it evaporates, heat must be supplied to it; the heat required is the latent heat of evaporation.

Melting and Freezing

When a solid changes to a liquid, we say it undergoes a *change of state* or *phase*. Pure crystalline solids melt and freeze sharply. If, for example, paradichlorbenzene is warmed in a test tube until it melts, and then allowed to cool, its temperature falls as shown in Fig. 12.1 (a).

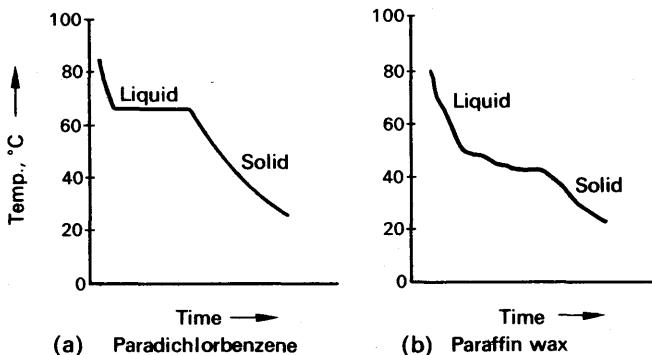


FIG. 12.1. Cooling curves, showing freezing.

A well-defined plateau in the cooling curve indicates the freezing (or melting) point. While the substance is freezing, it is evolving its latent heat of fusion, which compensates for the heat lost by cooling, and its temperature does not fall. An impure substance such as paraffin wax, on the other hand, has no definite plateau on its cooling curve; it is a mixture of several waxes, which freeze out from the liquid at slightly different temperatures (Fig. 12.1 (b)).

Supercooling

If we try to find the melting-point of hypo from its cooling curve, we generally fail; the liquid goes on cooling down to room temperature. But if we drop a crystal of solid hypo into the liquid the temperature rises to the melting-point of hypo, and the hypo starts to freeze. While the hypo is freezing, its temperature stays constant at the melting-point; when all the hypo has frozen, its temperature starts to fall again (Fig. 12.2).

The cooling of a liquid below its freezing-point is called *supercooling*; the

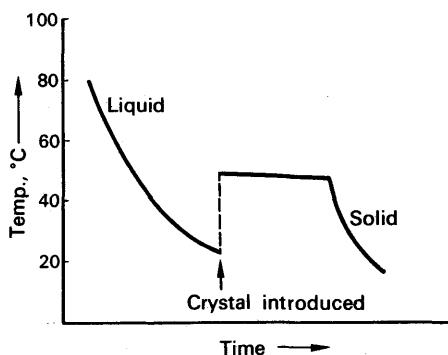


FIG. 12.2. Cooling curve of hypo.

molecules of the liquid lose their kinetic energy as it cools, but do not take up the rigid geometric pattern of the solid. Shaking or stirring the liquid, or dropping grit or dust into it, may cause it to solidify; but dropping in a crystal of its own solid is more likely to make it solidify. As soon as the substance begins to solidify, it returns to its melting (or freezing) point. *The melting-point is the only temperature at which solid and liquid can be in equilibrium.*

No one has succeeded in warming a solid above its melting-point—or, if he has, he has failed to report his success. We may therefore suppose that to superheat a solid is not possible; and we need not be surprised. For the melting-point of a solid is the temperature at which its atoms or molecules have enough kinetic energy to break up its crystal lattice: as soon as the molecules are moving fast enough, they burst from their pattern. On the other hand, when a liquid cools to its melting-point, there is no particular reason why its molecules should spontaneously arrange themselves. They may readily do so, however, around a crystal in which their characteristic pattern is already set up.

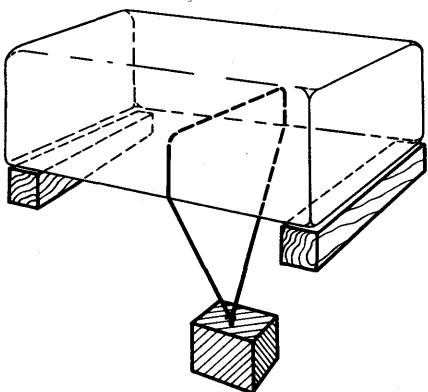
Pressure and Melting

The melting-point of a solid is affected by increase of pressure. If we run a copper wire over a block of ice, and hang a heavy weight from it, as in Fig. 12.3, we find that the wire slowly works through the block. It does not cut its way through, for the ice freezes up behind it; the pressure of the wire makes the ice under it melt, and above the wire, where the pressure is released, the ice freezes again. The freezing again after melting by pressure is called *regelation*.

FIG. 12.3. Melting of ice under pressure.

This experiment shows that increasing the pressure on ice makes it melt more readily; that is to say, *it lowers the melting-point* of the ice. We can understand this effect when we remember that ice shrinks when it melts (see p. 209); pressure encourages shrinking, and therefore melting.

The fall in the melting-point of ice with increase in pressure is small: 0.0072°C per atmosphere. It is interesting, because it explains the slipperiness of ice; skates for example, are hollow ground, so that the pressure on the line of contact is very high, and gives rise to a lubricating film of water. Ice which is much colder than 0°C is not slippery, because to bring its melting-point down to its actual temperature would require a greater pressure than can be realized. Most substances swell on melting; an increase of pressure opposes the melting of such substances, and raises their melting-point.



Freezing of Solutions

Water containing a dissolved substance freezes below 0°C. The depression of the freezing-point increases at first with the concentration, but eventually reaches a maximum. The lowest freezing-point of common salt solution is -22°C, when the solution contains about one-quarter of its weight of salt. When a solution does freeze, pure ice separates out; an easy way of preparing pure water is therefore to freeze it, remove the ice, and then melt the ice. The water which is mixed with the ice in determining the ice-point of a thermometer must be pure, or its temperature will not be 0°C.

When ice and salt are mixed, the mixture cools below 0°C, but remains liquid. As the proportion of salt is increased, the temperature of the mixture falls, until it reaches a minimum at -22°C. A mixture of ice and salt provides a simple means of reaching temperatures below 0°C, and is called a 'freezing mixture'.

The phenomena of the freezing of solutions are important in chemistry, and particularly in metallurgy. We shall give a brief explanation of them later in this chapter.

LIQUID TO GAS: EVAPORATION

Evaporation differs from melting in that it takes place at all temperatures; as long as the weather is dry, a puddle will always clear up. In cold weather the puddle lasts longer than in warm, as the rate of evaporation falls rapidly with the temperature.

Solids as well as liquids evaporate. Tungsten evaporates from the filament of an electric lamp, and blackens its bulb; the blackening can be particularly well seen on the headlamp bulb of a bicycle dynamo set, if it has been frequently over-run through riding down-hill. The rate of evaporation of a solid is negligible at temperatures well below its melting-point, as we may see from the fact that bars of metal do not gradually disappear.

Saturated and Unsaturated Vapours

Fig. 12.4 (a) shows an apparatus with which we can study vapours and their pressures. A is a glass tube, about a metre long, dipping in a mercury trough and backed by a scale S. Its upper end carries a bulb B, which is fitted with three taps T, of which T_1 and T_2 should be as close together as possible. Above T_1 is a funnel F. With T_1 closed but T_2 open, we evacuate the bulb and tube through T_3 , with a rotary pump. If the apparatus is clean, the mercury in A rises to the barometer height H . Meanwhile we put some ether in the funnel F. When the apparatus is evacuated, we close T_3 and T_2 . We now open and close T_1 , so that a little ether flows into the space C. Lastly, we open T_2 , so that the ether evaporates into the bulb B. As it does so, the mercury in A falls, showing that the ether-vapour is exerting a pressure (Fig. 12.4 (b)). If h is the new height of the mercury in A, then the pressure of the vapour in mm of mercury is equal to $H - h$.

By closing T_1 , opening and closing T_2 , and then opening T_1 again,

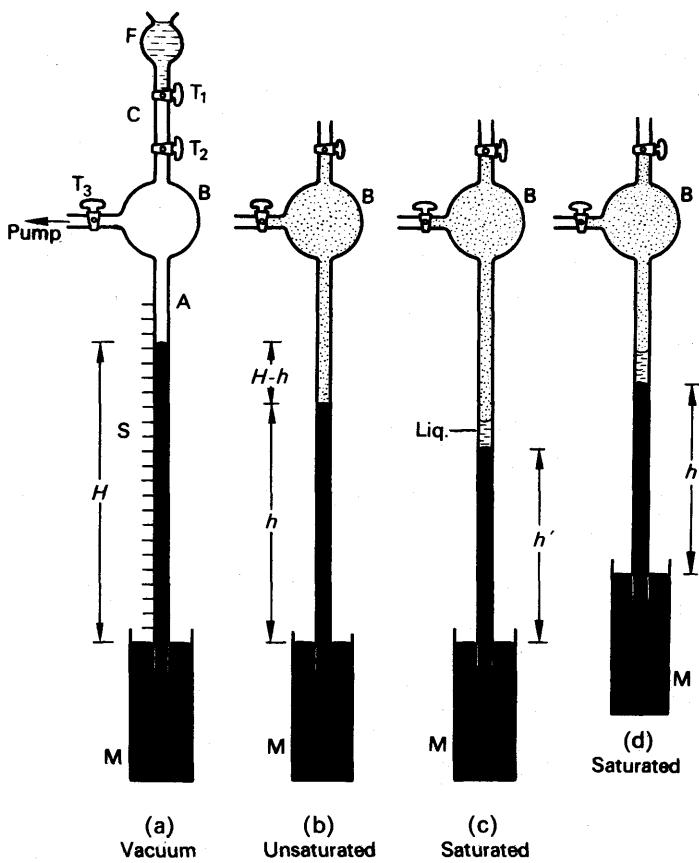


FIG. 12.4. Apparatus for studying vapours.

we can introduce more ether into the space B. At first, we find that, with each introduction, the pressure of the vapour, $H - h$, increases. But we reach a point at which the introduction of more ether does not increase the pressure, the height of the mercury column remains constant at h' . At this point we notice that liquid ether appears above the mercury in A (Fig. 12.4 (c)). We say that the vapour in B is now *saturated*; a *saturated vapour* is one that is in contact with its own liquid.

Before the liquid appeared in the above experiment, the pressure of the vapour could be increased by introducing more ether, and we say that the vapour in B was then *unsaturated*.

Behaviour of Saturated Vapour

To find out more about the saturated vapour, we may try to expand or compress it. We can try to compress it by raising the mercury reservoir M. But when we do, we find that the height h' does not change: the pressure of the vapour, $H - h'$, is therefore constant (Fig. 12.4 (d)). The only change we notice is an increase in the volume of liquid above

the mercury. We conclude, therefore, that reducing the volume of a saturated vapour does not increase its pressure, but merely makes some of it condense to liquid.

Similarly, if we lower the reservoir M, to increase the volume of the vapour, we do not decrease its pressure. Its pressure stays constant, but the volume of liquid above the mercury now decreases; liquid evaporates, and keeps the vapour saturated. If we increase the volume of the vapour until all the liquid has evaporated, then the pressure of the vapour begins to fall, because it becomes unsaturated (see Fig. 12.5 (a)).

Effect of Temperature: Validity of Gas-laws for Vapours

We cannot heat the apparatus of Fig. 12.4 through any great rise of temperature. But we can warm it with our hands, or by pointing an electric fire at it. If we do warm it, we find that the ether above the mercury evaporates further, and the pressure of the vapour increases. Experiments which we shall describe later show that the pressure of a saturated vapour rises, with the temperature, at a rate much greater than that given by Charles's law. Its rise is roughly exponential.

To Boyle's law, saturated vapours are indifferent: *their pressure is independent of their volume*. Unsaturated vapours obey Boyle's law roughly, as they also obey roughly Charles's law. Fig. 12.5 (a). Vapours, saturated and unsaturated, are gases in that they spread throughout their vessels; but we find it convenient to distinguish them by name from gases such as air, which obey Charles's and Boyle's laws closely. We shall elaborate this distinction later.

PROPERTIES OF SATURATED VAPOURS

Temperature °C	Water Pressure, mm mercury	Mercury	Ethyl Ether
		Pressure, mm mercury	Pressure, mm mercury
-20	0.784 (ice)		
-10	1.96 (ice)		112
0	4.56	0.00016	185
10	9.20	0.00043	291
20	17.5	0.0011	440
30	31.7	0.0026	
34.6			760
40	55.1	0.0057	921
50	92.3	0.0122	
60	149	0.0246	1734
70	234		
80	355	0.0885	2974
90	526		
100	760	0.276	4855
150	3569	2.88	
200	11647	17.8	
250	29770	75.8	
300	67620	249	
356.7		760	

Fig. 12.5 (b) shows the effect of heating a saturated vapour. More and more of the liquid evaporates, and the pressure rises very rapidly. As soon as all the liquid has evaporated, however, the vapour becomes unsaturated, and its pressure rises more sedately.

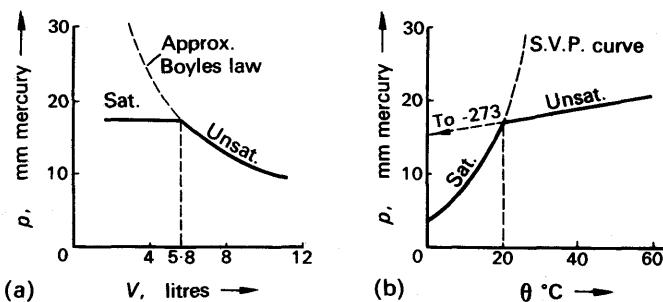


FIG. 12.5. Effect of volume and temperature on pressure of water vapour.

Fig. 12.6 (a) shows isothermals for a given mass of liquid and vapour at two temperatures, $\theta_1 = 10^\circ\text{C}$, and $\theta_2 = 21^\circ\text{C}$. The temperatures are chosen so that the saturated vapour pressure at θ_2 is double that at θ_1 . The absolute temperatures are $T_1 = 273 + \theta_1 = 283\text{ K}$, and $T_2 = 273 + \theta_2 = 294\text{ K}$. Because the saturated vapour pressure rises

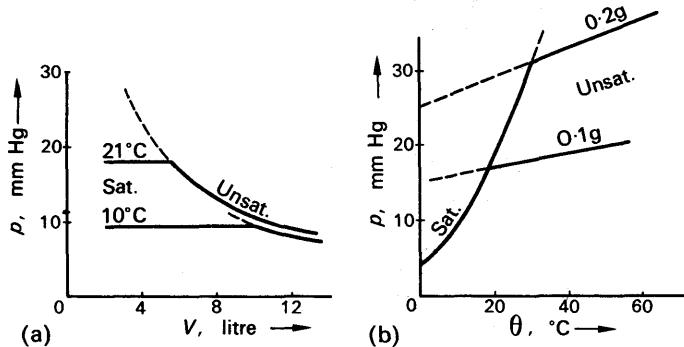


FIG. 12.6. Relationship between pressure, temperature, total mass, and volume, for water-vapour and liquid.

so rapidly with temperature, the absolute temperature T_2 is not nearly double the absolute temperature T_1 . Consequently the isothermals for the unsaturated vapour are fairly close together, as shown; and the transition from saturated to unsaturated vapour occurs at a smaller volume at the higher temperature.

Fig. 12.6 (b) shows pressure-temperature curves for a vapour, initially in contact with different amounts of liquid, in equal total volumes. The more liquid present, the greater is the density of the vapour when it becomes unsaturated, and therefore the higher the pressure and temperature at which it does so.

Kinetic Theory of Saturation

Let us consider a vapour in contact with its liquid, in an otherwise empty vessel which is closed by a piston (Fig. 12.7). The molecules of the vapour, we suppose, are rushing randomly about, like the molecules of a gas, with kinetic energies whose average value is determined by the temperature of the vapour. They bombard the walls of the vessel, giving rise to the pressure of the vapour, and they also bombard the surface of the liquid.

The molecules of the liquid, we further suppose, are also rushing about with kinetic energies determined by the temperature of the liquid. The fastest of them escape from the surface of the liquid. At the surface, therefore, there are molecules leaving the liquid, and molecules arriving from the vapour. To complete our picture of the conditions at the surface, we suppose that the vapour molecules bombarding it are not reflected—as they are at the walls of the vessel—but are absorbed into the liquid. We may expect them to be, because we consider that molecules near the surface of a liquid are attracted towards the body of the liquid.

We shall assume that the liquid and vapour have the same temperature. Then the proportions of liquid and vapour will not change, if the temperature and the total volume are kept constant. Therefore, at the surface of the liquid, molecules must be arriving and departing at the same rate, and hence evaporation from the liquid is balanced by condensation from the vapour. This state of affairs is called a *dynamic equilibrium*. In terms of it, we can explain the behaviour of a saturated vapour.

The rate at which molecules leave unit area of the liquid depends simply on their average kinetic energy, and therefore on the temperature. The rate at which molecules strike unit area of the liquid, from the vapour, likewise depends on the temperature; but it also depends on the concentration of the molecules in the vapour, that is to say, on the density of the vapour. The density and temperature of the vapour also determine its pressure; the rate of bombardment therefore depends on the pressure of the vapour.

Now let us suppose that we decrease the volume of the vessel in Fig. 12.7 by pushing in the piston. Then we momentarily increase the density of the vapour, and hence the number of its molecules striking the liquid surface per second. The rate of condensation thus becomes greater than the rate of evaporation, and the liquid grows at the expense of the vapour. As the vapour condenses its density falls, and so does the rate of condensation. The dynamic equilibrium is restored when the rate of condensation, and the density of the vapour, have returned to their original values. The pressure of the vapour will then also have returned to its original value. *Thus the pressure of a saturated vapour is independent of its volume.* The proportion of liquid to vapour, however, increases as the volume decreases.

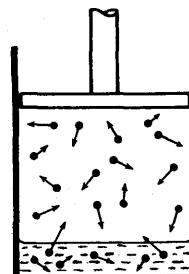


FIG. 12.7. Dynamic equilibrium.

Let us now suppose that we warm the vessel in Fig. 12.7, but keep the piston fixed. Then we increase the rate of evaporation from the liquid, and increase the proportion of vapour in the mixture. Since the volume is constant, the pressure of the vapour rises, and increases the rate at which molecules bombard the liquid. Thus the dynamic equilibrium is restored, at a higher pressure of vapour. The increase of pressure with temperature is rapid, because the rate of evaporation of the liquid increases rapidly—almost exponentially—with the temperature. A small rise in temperature causes a large increase in the proportion of molecules in the liquid moving fast enough to escape from it.

Boiling

A liquid boils when its saturated vapour pressure is equal to the atmospheric pressure. To see that this is true, we take a closed J-shaped tube, with water trapped in its closed limb (Fig. 12.8 (a)). We heat the tube in a beaker of water, and watch the water in the J-tube. It remains trapped as at (a) until the water in the beaker is boiling. Then the water in the J-tube comes to the same level in each limb, showing that the pressure of the vapour in the closed limb is equal to the pressure of the air outside (Fig. 12.8 (b)).

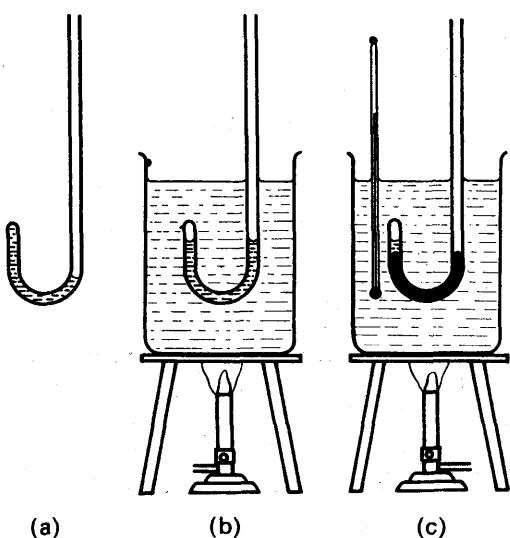


FIG. 12.8. Use of J-tube for boiling-point.

of which only a small quantity can be had. A few drops of the liquid are imprisoned by mercury in the closed limb of the tube, all entrapped air having been shaken out (Fig. 12.8 (c)). The tube is then heated in a bath, and the temperature observed at which the mercury comes to the same level in both limbs. The bath is warmed a little further, and then a second observation made as the bath cools; the mean of the two observations is taken as the boiling-point of the liquid.

Boiling differs from evaporation in that a liquid evaporates from its surface alone, but it boils throughout its volume. If we ignore the small hydrostatic pressure of the liquid itself, we may say that the pressure throughout a vessel of liquid is the atmospheric pressure. Therefore, when the saturated vapour pressure is equal to the atmospheric pressure, a bubble of vapour can form anywhere in the liquid.

The J-tube gives a simple means of measuring the boiling-point of a liquid which is inflammable, or which has a poisonous vapour, or

The J-tube gives a simple means of measuring the boiling-point of a liquid which is inflammable, or which has a poisonous vapour, or

Generally the bottom of the liquid is the hottest part of it, and bubbles form there and rise through the liquid to the surface. Just before the liquid boils, its bottom part may be at the boiling-point, and its upper part below. Bubbles of vapour then form at the bottom, rise in to the colder liquid, and then collapse. The collapsing gives rise to the singing of a kettle about to boil.

Further Consideration of Boiling

The account of boiling which we have just given is crude because in it we ignored the effect of surface-tension. Because of surface tension, a bubble can exist in a liquid only if there is an excess pressure inside it. If γ is the surface-tension of the liquid, and r the radius of the bubble, the excess pressure is $2\gamma/r$. If the bubble is formed at a depth h below the surface of the liquid, as in Fig. 12.9, the external pressure acting on it is

$$P = P_a + h\rho g,$$

where P_a = atmospheric pressure, ρ = density of liquid, g = acceleration of gravity.

Therefore a bubble, of radius r , can form at a depth h only if its vapour pressure, p , satisfies the equation

$$\begin{aligned} p &= P + \frac{2\gamma}{r} \\ &= P_a + h\rho g + \frac{2\gamma}{r}. \end{aligned}$$

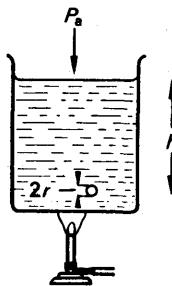


FIG. 12.9. Boiling.

If the radius r is small, the term $2\gamma/r$ is great, and the bubble cannot form unless the vapour pressure is considerably above atmospheric. In fact the equation shows that a bubble can never start from zero radius, because it would require an infinite vapour pressure to do so. Bubbles actually form on roughnesses in the vessel, or specks of solid suspended in the liquid. Very clean water in a very smooth beaker may not boil until it is well above 100°C; its bubbles then grow violently, and the liquid 'bumps' in the beaker. A piece of broken pipe-clay prevents bumping, by presenting fine points for the bubbles to form on.

Thus the temperature of a boiling liquid is not definite—it depends on the conditions of boiling. But the temperature of the vapour is definite. The vapour escaping is in equilibrium with the liquid at the surface, and is at atmospheric pressure. Its temperature, therefore, is the temperature at which the saturated vapour pressure is equal to the atmospheric pressure. This idea is important in defining the upper fixed point of the temperature scale (p. 190). We say that the upper fixed point is the temperature of the steam from water boiling under a pressure of 760 mm mercury. We must not refer to the temperature of the water, and we must specify the atmospheric pressure because as we have just seen, it determines the temperature of the steam.

TEMPERATURE OF SATURATED STEAM AT PRESSURES
NEAR NORMAL ATMOSPHERIC

Barometer height, H, mm .	680	690	700	710	720	730	740	750	751
Temperature, θ , °C .	96.910	97.312	709	98.102	490	874	99.254	.629	.666
Barometer height, H, mm .	752	753	754	755	756	757	758	759	760
Temperature, θ , °C .	.704	.741	.778	.815	.852	.889	.926	.963	.037
Barometer height, H, mm .	762	763	764	765	766	767	768	769	780
Temperature, θ , °C .	.074	.110	.147	.184	.220	.257	.294	.330	.367
									.729

In general, around $H = 760$ mm, the temperature is given by

$$\theta = 100 + 0.036(H - 760) - 2.3 \times 10^{-5}(H - 760)^2.$$

Variation of Saturated Vapour Pressure with Temperature

We can now see how the relationship between the pressure of a saturated vapour and its temperature can be measured. We must apply various known air pressures to the liquid, heat the liquid, and measure the temperature of its vapour. Fig. 12.10 shows a suitable apparatus, due to Regnault. The flask F contains the liquid, water in

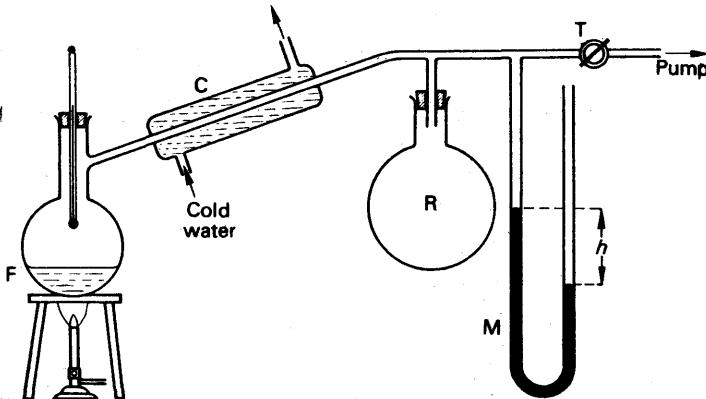


FIG. 12.10. Apparatus for variation of S.V.P. with temperature.

a laboratory experiment, and the flask R is an air reservoir. The pressure of the air in R is shown by the mercury manometer M; if its height is h , the pressure in mm mercury is

$$p = H - h,$$

where H is the barometer height.

We first withdraw some air from R through the tap T, with a filter pump, until p is about 700 mm. We then close T and heat the water gently. The water vapour condenses in the condenser, and runs back to the flask. After a few minutes the water boils steadily. The temperature of the vapour, θ , and the pressure, p , become constant and we record their values. We next remove the flame from the flask F, and let the apparatus cool for a minute or two. Then we withdraw some more air from R, close T again, and repeat the observations.

If we wish to find the saturated vapour pressure when it is above

atmospheric, that is to say, when the temperature is above the normal boiling-point of the liquid, air is pumped into the reservoir R—with a bicycle pump—instead of drawing it out. The manometer M then shows the excess pressure, and

$$p = H + h.$$

With simple glass apparatus we cannot go far in this direction.

Boiling Point of a Solution

At a given pressure, the boiling-point of water containing a dissolved substance is higher than that of pure water. The temperature of the steam evolved from the solution, however, is the temperature of saturated steam at the prevailing pressure. Traces of dissolved substances in the water therefore do not affect the steam point in thermometry.

Since a liquid boils when its saturated vapour pressure is equal to the atmospheric pressure, we must conclude that dissolving a substance in water lowers its saturated vapour pressure, at a given temperature (Fig. 12.11). We may explain this by supposing that the molecules of the dissolved substance, which do not evaporate, hinder the escape of the molecules of the water.

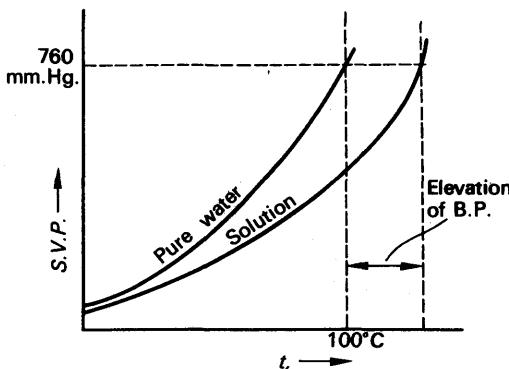


FIG. 12.11. Effect of solute on saturated vapour pressure.

The lowering of the vapour pressure of water by a dissolved solid gives striking support to the kinetic theory of evaporation. For measurements of the vapour pressure show that reduction does not depend on the nature of the solute; it depends only on the number of dissolved particles in the solution expressed as a fraction of the total number of particles (solute plus water molecules). In fact, if there are n solute particles to every $100 - n$ water molecules, then

$$\frac{\Delta p}{p} = \frac{n}{100},$$

where p is the saturated vapour pressure of water, and Δp is the lowering by solution. Thus the lowering simply depends on the number of particles hindering evaporation.

Effect of Altitude on Boiling Point

The pressure of the atmosphere decreases with increasing height above the earth's surface, because the thickness, and therefore the weight, of the belt of air above the observer decreases. The rate of fall in pressure is almost uniform over fairly small heights—about 85 mm mercury per km. But at great altitudes the rate of fall diminishes. At the height of Everest, 9000 m, the atmospheric pressure is about 280 mm of mercury. On account of the fall in atmospheric pressure, the boiling point of water falls with increasing height. Cooking-pots for use in high mountainous districts, such as the Andes, are therefore fitted with clamped lids. As the water boils, the steam accumulates in the pot, and its pressure rises above atmospheric. At about 760 mm mercury a safety valve opens, so that the pressure does not rise above that value, and the cooking is done at 100°C.

The fall in the boiling-point with atmospheric pressure gives a simple way of determining one's height above sea-level. One observes the steam point with a thermometer and hypsometer (p. 190). Knowing how the steam point falls with pressure, and how atmospheric pressure falls with increasing height, one can then find one's altitude. The hypsometer was, in fact, devised for this purpose, and takes its name from it; *hypso* is Greek for height. Hypsometers have been carried up Himalayan peaks; and one was found by Scott and his companions in Amundsen's abandoned tent at the South Pole.

Variation of Latent Heat with Temperature

When we speak of the latent heat of evaporation of a liquid, we usually mean the heat required to vaporize unit mass of it at its normal boiling-point, that is to say, under normal atmospheric pressure. But since evaporation takes place at all temperatures, the latent heat has a value for every temperature. Regnault measured the latent heat of steam over a range of temperatures, by boiling water at controlled pressures, as in measuring its saturated vapour pressure. His apparatus was in principle similar to Berthelot's (Fig. 9.10); but he connected the outlet tube to an air reservoir, manometer, and pump, as in Fig. 12.10. Modern measurements give, approximately,

$$l = 2520 - 2.5\theta$$

where l is the specific latent heat in kJ kg^{-1} at $\theta^\circ\text{C}$.

Internal and External Latent Heats

The volume of 1 g of steam at 100°C is 1672 cm³. Therefore when 1 g of water turns into steam, it expands by 1671 cm³; in doing so, it does work against the atmospheric pressure. The heat equivalent of this work is that part of the latent heat which must be supplied to the water to make it overcome atmospheric pressure as it evaporates; it is called the 'external latent heat'. The rest of the specific latent heat—the internal part—is the equivalent of the work done in separating the molecules, against their mutual attractions.

The work done, W , in the expansion of 1 g from water to steam is the product of the atmospheric pressure p and the increase in volume ΔV :

$$W = p \cdot \Delta V.$$

Normal atmospheric pressure corresponds to a barometer height H 760 mm. Hence, as on p. 228,

$$\begin{aligned} p &= g\rho H = 9.81 \times 13600 \times 0.76 \\ &= 1.013 \times 10^5 \text{ N m}^{-2} \end{aligned}$$

and $W = p \cdot \Delta V = 1.013 \times 10^5 \times 1671 \times 10^{-6}$ joule.

The external specific latent heat in joules is therefore

$$\begin{aligned} l_{ex} &= 1.013 \times 10^5 \times 1671 \times 10^{-6} \\ &= 170 \text{ J g}^{-1} = 170 \text{ kJ kg}^{-1}. \end{aligned}$$

This result shows that the external part of the specific latent heat is much less than the internal part. Since the total specific latent heat l is 2270 joule g^{-1} , the internal part is

$$\begin{aligned} l_{in} &= l - l_{ex} = 2270 - 170 \\ &= 2100 \text{ J g}^{-1} = 2100 \text{ kJ kg}^{-1}. \end{aligned}$$

Density of a Saturated Vapour

In any experiment to measure the density of a saturated vapour, the vessel containing the vapour must also contain some liquid, to ensure that the vapour is saturated. The problem is therefore to find how much of the total mass is vapour, and how much is liquid. Fig. 12.12 shows one method of solving this problem, due to Cailletet and Mathias.

A, B are two glass tubes, which have been calibrated with volume scales, and then evacuated. Known masses m_1 , m_2 of liquid are introduced into the tubes, which are then sealed off. The tubes are warmed to the same temperature in a bath, and the volumes of liquid V_{l_1} , V_{l_2} and of vapour V_{v_1} , V_{v_2} are observed. Then if ρ_l and ρ_v are the densities of liquid and vapour respectively:

$$m_1 = \rho_l V_{l_1} + \rho_v V_{v_1}$$

$$m_2 = \rho_l V_{l_2} + \rho_v V_{v_2}.$$

From these equations ρ_l can be eliminated, and ρ_v found. The equations can also, of course, be made to give ρ_l ; this method is useful for finding the density of a liquid 'gas'—e.g. liquid oxygen (p. 324).

Density of an Unsaturated Vapour

We have seen that the molecular weight of a gas, μ , is given very nearly by $\mu = 2\Delta$, where Δ is the density of the gas relative to that of hydrogen (p. 230).

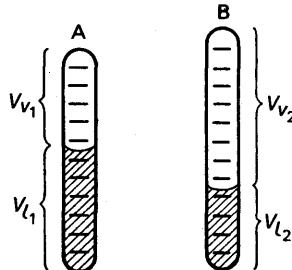


FIG. 12.12. Determination of density of a saturated vapour.

The proof that $\mu = 2\Delta$ depends on Avogadro's principle, which says that equal volumes of all gases at the same temperature and pressure contain equal numbers of molecules. This principle is true only of those gases which we normally call 'perfect'—which obey Boyle's and Charles's laws accurately (p. 225). It is not true of saturated vapours, but it is roughly true of vapours which are far from saturation. To find the molecular weight of a substance which is liquid at room temperature, therefore, we must vaporize it, and measure the density of its vapour when it is as far from saturation as we can conveniently get it.

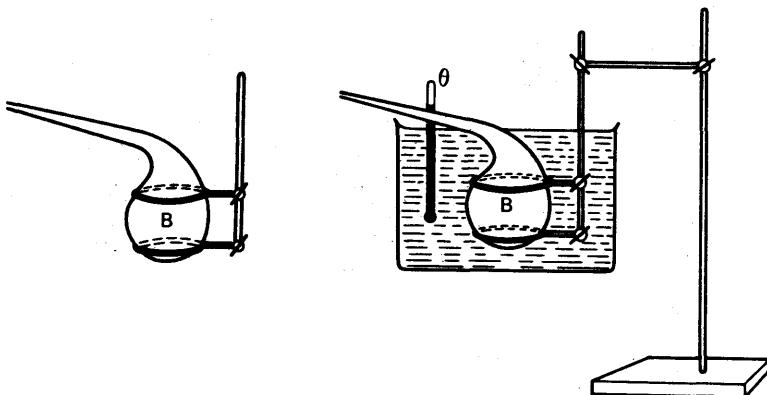


FIG. 12.13. Determination of density of an unsaturated vapour.

Several methods have been devised for doing this, one of which, Dumas's of 1827, is illustrated in Fig. 12.13. A glass bulb B, with a long thin stem, is weighed and then partly filled with liquid by warming and dipping. The amount of liquid introduced must be great enough to ensure that all the air in the bulb will be driven out by vapour when the liquid evaporates. The liquid is made to evaporate by plunging the bulb into a bath at a temperature θ about 40°C above its boiling-point. It then evaporates rapidly, and its vapour sweeps the air out of the bulb. When vapour has stopped coming out, the stem is sealed with a flame: the bulb now contains nothing but the vapour, at the temperature θ and under atmospheric pressure.

The bulb is removed, allowed to cool, dried, and weighed. The stem is then broken at the tip under water; since nearly all the vapour has condensed, at the room temperature, water rushes in and fills the bulb.

Let m_1 = mass of bulb full of air, at room temperature.

m_2 = mass of bulb full of vapour.

m_3 = mass of bulb full of water.

Since the mass of air in the bulb is negligible compared with that of water, we have, *numerically*,

$$\text{Volume of bulb in cm}^3, V_1, = m_3 - m_1 \text{ in g.}$$

The mass of air in the bulb at room temperature is

$$m_a = V\rho_a$$

where ρ_a is the density of air at room temperature and atmospheric pressure. The mass of the bulb itself is

$$m_b = m_1 - m_a$$

and the mass of vapour which filled it when hot is

$$m_v = m_2 - m_b$$

This mass of vapour occupied the volume V_1 at the temperature θ ; its density was therefore

$$\rho_v = \frac{m_v}{V_1}$$

Since the temperature was well above the boiling-point, the vapour was far from saturated; Boyle's and Charles's laws can therefore be used to reduce its density to s.t.p., for comparison with that of hydrogen.

WATER-VAPOUR IN THE ATMOSPHERE: HYGROMETRY

The water vapour in the atmosphere is important because it affects our comfort. Except in cold weather, we sweat continuously: the water in the sweat evaporates, draws its latent heat of evaporation from the skin, and so keeps us cool. Beads of sweat appear only when the water cannot evaporate as fast as it reaches the surface of the skin; we then feel uncomfortably hot.

On the other hand, if water evaporates from the skin too rapidly, the skin feels parched and hard; around the mucous membranes—at the mouth and nose—it tends to crack.

The rate at which water evaporates, from the skin or anywhere else, depends on the pressure of the water vapour surrounding it. If the water vapour above the skin is far from saturated, evaporation is swift. If the vapour is already saturated, water reaching the skin comes immediately into dynamic equilibrium with it; individual molecules are exchanged between liquid and vapour, but no mass of liquid is lost, and water accumulates.

The Partial Pressure of Atmospheric Water

The atmosphere contains other gases besides water-vapour, such as oxygen and nitrogen. In speaking of the water-vapour, therefore, we must refer to its 'partial pressure', as explained on p. 222.

Water-vapour in the atmosphere is also important because it affects the weather. Let us suppose that the atmosphere has a temperature of 20°C—a warm day—and that the water vapour in it has a partial pressure of 12 mm mercury. It will have a density of about 12 mg per litre. The density of *saturated* water vapour at 20°C is 17.3 mg per litre, and its pressure 17.5 mm mercury. The water vapour in the atmosphere is therefore not saturated.

Now let us suppose that the atmosphere cools to 14°C, without changing its composition. The 6°C fall in temperature will hardly affect the density of the water vapour, but it will bring the atmosphere to

saturation. For the pressure of saturated water vapour at 14°C is 12 mm mercury, and its density about 12 mg per litre. If the atmosphere cools any further, water vapour will condense out of it, forming drops of liquid water—that is, of fog or cloud.

Relative Humidity

The dampness of the atmosphere, besides affecting the weather and our comfort, is important also in storage and manufacture of many substances—tobacco and cotton, for example. From what we have said already, we can see that the important factor is not the actual proportion of water vapour in the atmosphere, but its nearness to saturation. In the above example, the density of the vapour remained almost constant, but we would have felt the atmosphere becoming much damper as it cooled from 20°C to 14°C.

The dampness of the atmosphere is expressed by its *relative humidity*, *R.H.*, which is defined as follows:

$$R.H. = \frac{\text{mass of water-vapour in a given volume of atmosphere}}{\text{mass of an equal volume of saturated water-vapour at the same temperature}} \quad (1)$$

In other words,

$$R.H. = \frac{\text{density of water-vapour in atmosphere}}{\text{density of saturated water-vapour at the same temperature}}$$

Because an unsaturated vapour roughly obeys Boyle's law, its density is roughly proportional to its pressure; the relative humidity as defined above is therefore roughly given by

$$R.H. = \frac{\text{partial pressure of water-vapour present}}{\text{S.V.P. at temperature of atmosphere}}$$

where S.V.P. stands for 'saturated vapour pressure'.

Before describing the methods of measurement, we must warn the reader against thinking that the atmosphere 'takes up' water vapour. The atmosphere is not a sponge. Water-vapour exists in it in its own right; and our knowledge of vapours makes us feel sure that, if we could live in an atmosphere of water-vapour alone, we would have just the same experiences of humidity as we now have in our happily richer surroundings.

Dew-point

In the evening, the earth cools more rapidly, by radiation, than the air above it. Then, on smooth surfaces such as metals, we often find a thin film of moisture. The surface has cooled to such a temperature that the water vapour in contact with it has become saturated, and has begun to condense. No fog has formed because the atmosphere in general is warmer than the cold solid, and the vapour in it is not saturated. The temperature of a cold surface on which dew just appears is called the *dew-point*: it is the temperature at which the saturated vapour-pressure of water is equal to the partial pressure of the water-vapour present in the atmosphere.

If we know the dew-point, we can find the corresponding pressure of saturated water-vapour, p_1 , from tables. From the same tables we can find the pressure of saturated water-vapour at the temperature of the atmosphere. If this is p_2 , then, from p. 310, the relative humidity, R.H., is given by

$$R.H. = \frac{p_1}{p_2} \quad \quad (2)$$

Dew-point Hygrometers

Hygrometry is the measurement of relative humidity, and a *hygrometer* is an instrument for measuring it. Many forms of dew-point hygrometer have been devised; we shall describe only the one due, like so much apparatus in Heat, to Regnault. It consists of two glass tubes, A, B, with silver-plated thimbles C, D cemented on to their lower ends. C contains ether, into which dips a thermometer T and a glass tube R (Fig. 12.14). To use the hygrometer, we first place a sheet of glass between it and ourselves, to prevent our breath from adding to the humidity of the air around it. We then gently blow air through the ether, by means of a scent-spray bulb connected through a rubber tube to R. (Alternatively, we may pass in a gentle stream of coal-gas, and burn it at the end of a long tube connected to the outlet S.) The gas or air passing through the ether carries away its vapour, and makes it evaporate continuously. In doing so it cools, since it must provide its latent heat of evaporation from its own heat content. When the ether has cooled to the dew-point, a film of mist appears on the thimble C; the thimble D enables us to notice this more sharply, by contrast. At the moment when the dew appears, the thermometer T is read. The flow of air or gas through the ether is stopped, and the ether allowed to warm up. The temperature at which the dew vanishes is noted, and the mean of this and the temperature at which it appeared is taken as the dew-point, θ_1 say. By so doing we correct reasonably well for any difference of temperature between the ether and the outer surface of the tube. Lastly, we take the temperature of the room, θ_2 , often from a thermometer (not shown) in B, and look up the saturation pressures at θ_1 , θ_2 respectively. These are given in the following table. Then

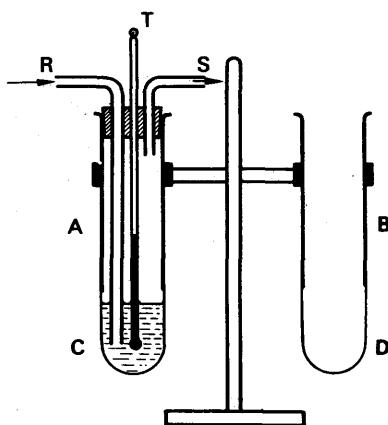


FIG. 12.14. Regnault's hygrometer.

$$\text{relative humidity} = \frac{\text{S.V.P. at } \theta_1}{\text{S.V.P. at } \theta_2} \times 100 \text{ per cent.}$$

PRESSURE AND DENSITY OF SATURATED WATER VAPOUR

$\theta^{\circ}\text{C}$	0	2	4	6	8	10	12	14	16	18	20
$p, \text{mm mercury}$	4.58	5.29	6.10	7.01	8.04	9.21	10.5	12.0	13.6	15.5	17.5
$\rho, \text{mg/litre (or g m}^{-3}\text{)}$	4.84	5.54	6.33	7.22	8.21	9.33	10.6	12.0	13.5	15.2	17.1
$\theta^{\circ}\text{C}$	22	24	26	28	30	32	34	36	38	40	
$p, \text{mm mercury}$	19.8	22.3	25.1	28.3	31.7	35.5	39.8	44.4	49.5	55.1	
$\rho, \text{mg/litre (or g m}^{-3}\text{)}$	19.2	21.5	24.1	26.9	30.0	33.5	37.2	41.3	45.8	51.1	

The Wet-and-dry-bulb Hygrometer

A piece of wet cloth feels cold, because the moisture evaporating from it takes latent heat, and cools the remaining liquid. This effect is used in the *wet-and-dry-bulb hygrometer*. A piece of muslin is tied round the bulb of a thermometer, and allowed to dip into a small

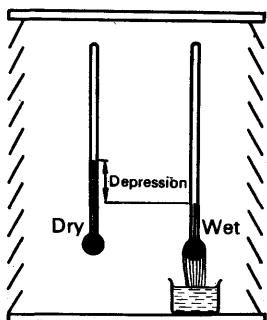


FIG. 12.15. Wet-and-dry-bulb hygrometer.

jar of water. It is mounted, with a second, dry-bulb, thermometer, in a louvred draught-shield (Fig. 12.15). The rate at which water evaporates from the muslin increases as the relative humidity of the atmosphere falls; the cooling of the wet bulb therefore also increases. The greater the difference in reading of the two thermometers, the less is the relative humidity. By calibration against a chemical hygrometer, tables and charts have been prepared which give the relative humidity in terms of the thermometer readings.

WET-AND-DRY-BULB HYGROMETER TABLE
(Percentage relative humidity)

Dry-bulb reading $^{\circ}\text{C}$	Difference (depression of wet-bulb), $^{\circ}\text{C}$									
	1	2	3	4	5	6	8	10	12	14
0	82	65	48	31%						
5	85	72	58	45	32%					
10	88	76	65	54	44	34%				
15	90	80	71	61	52	44	27	12%		
20	91	83	74	66	59	51	37	24	12%	
25	92	84	77	70	63	57	44	33	22	12%
30	86	79	73	67	61	50	39	30	21	21%

The wet-and-dry-bulb hygrometer is not very reliable when used in a simple screen. It is more accurate if a steady stream of air is driven past it by a fan, or by whirling the thermometers around in a frame like a football-fan's rattle. The hygrometer is then said to be ventilated.

The Hair Hygrometer

Human hair expands in length in damp air. A hair hygrometer is one consisting of a bundle of hairs fixed to a spring at one end, and

wrapped round a spindle at the other. The expansion of the hair turns the spindle and moves a pointer over a scale, which is directly calibrated in relative humidities. Such instruments need to be recalibrated frequently, because the hair shows elastic fatigue.

THE BEHAVIOUR OF REAL GASES; CRITICAL PHENOMENA

A perfect, or ideal, gas is one which obeys Boyle's and Charles's law exactly, and whose internal energy is independent of its volume. No such gas exists, but at room temperature, and under moderate pressures, many gases approach the ideal closely enough for most purposes.

We shall consider now the departures of gases from perfection; in doing so we shall come to appreciate better the relationship between liquid, vapour, and gas, and we shall see how gases such as air can be liquefied.

Departures from Boyle's Law

In 1847 Regnault measured the volumes of various gases at pressures of several atmospheres. Using the apparatus of Fig. 12.16, he found that, to halve the volume of the gas, he did not have quite to double the pressure on it. The product pV , therefore, instead of being constant, decreased slightly with the pressure. He found one exception to this

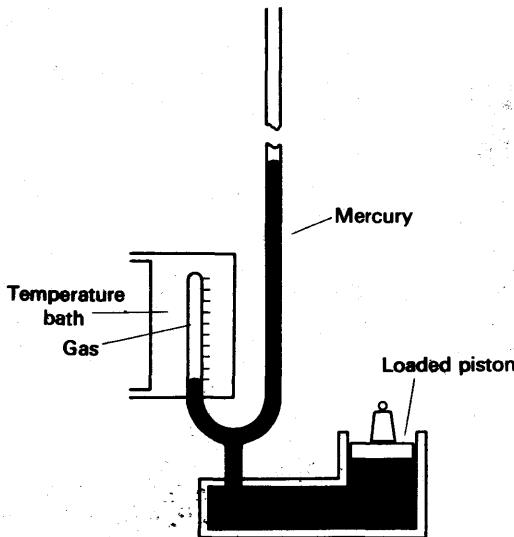


FIG. 12.16. Regnault's apparatus for isotherms at high pressure.

rule: hydrogen. By compressing the gases further, Regnault found the variation of pV with p at constant temperature, and obtained results which are represented by the early parts of the curves in Fig. 12.17.

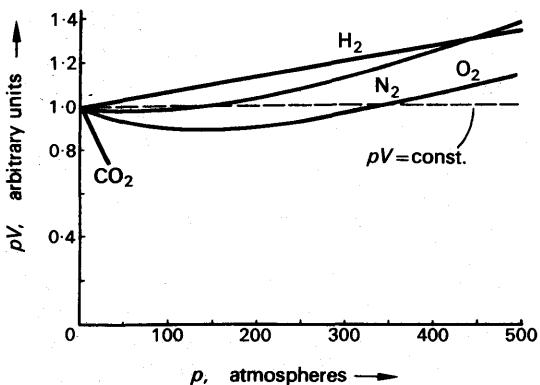


FIG. 12.17. Isotherms for various gases, at room temperature and high pressure.

The complete curves in the figure show some of the results obtained by Amagat in 1892. Amagat's apparatus for nitrogen is shown in Fig. 12.18. To get high pressures, he put the apparatus at the bottom of a coal-mine, and made the manometer tube out of rifle barrels, screwed together and standing up the shaft. He reached a pressure of 3000 atmospheres.

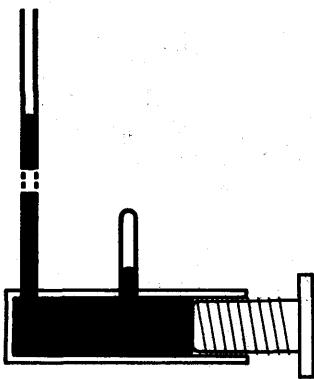


FIG. 12.18. Amagat's apparatus for isotherms at high pressure.

Having found the volume-pressure relationship for nitrogen, Amagat used it to measure high pressures in the laboratory, without having to resort to the mine. His method was similar to that of Andrews, which we are about to describe; by means of it he found the pressure-volume relationships for other gases.

Andrews' Work on Carbon Dioxide

In 1863 Andrews made experiments on carbon dioxide which have become classics. Fig. 12.19 shows his apparatus. In the glass tube A he trapped carbon dioxide above the pellet of mercury X. To do this, he started with the tube open at both ends and passed the gas through it for a long time. Then he sealed the end of the capillary. He introduced the mercury pellet by warming the tube, and allowing it to cool with the open end dipping into mercury. Similarly, he trapped nitrogen in the tube B.

Andrews then fitted the tubes into the copper casing C, which contained water. By turning the screws S, he forced water into the lower parts of the tubes A and B, and drove the mercury upwards. The wide parts of the tubes were under the same pressure inside and out, and so were under no stress. The capillary extensions were strong enough to

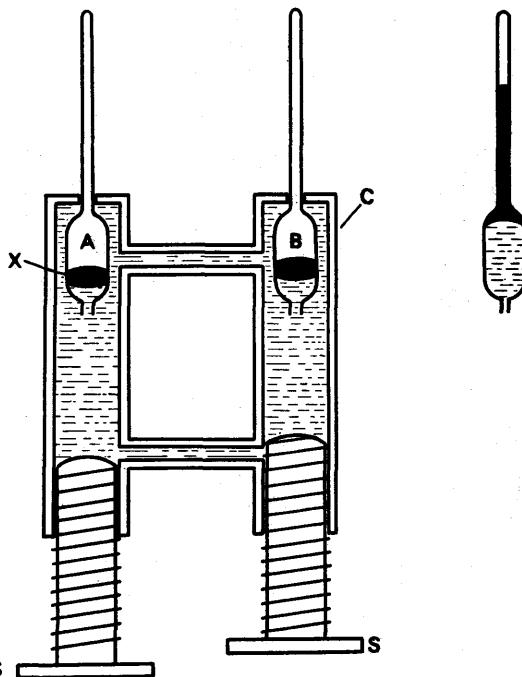


FIG. 12.19. Andrews's apparatus for isotherms of CO_2 at high pressures.

withstand hundreds of atmospheres. Andrews actually reached 108 atmospheres.

When the screws S were turned far into the casing, the gases were forced into the capillaries, as shown on the right of the figure, and greatly compressed. From the known volumes of the wide parts of the tubes, and the calibrations of the capillaries, Andrews determined the

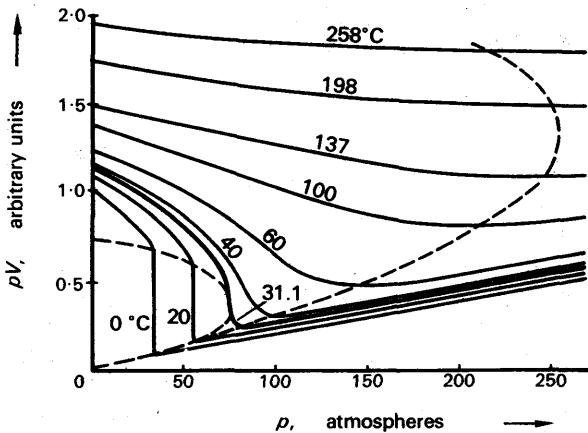


FIG. 12.20. Isotherms for CO_2 , as pV/p curves, at various temperatures. The small dotted loop passes through the ends of the vertical parts; the large dotted loop is the locus of the minima of pV .

volumes of the gases. He estimated the pressure from the compression of the nitrogen, assuming that it obeyed Boyle's law.

For work above and below room temperature, Andrews surrounded the capillary part of A with a water bath, which he maintained at a constant temperature between about 10°C and 50°C.

Fig. 12.20 shows some of Andrews's results, corrected for the departure of nitrogen from Boyle's law; it also shows the results of similar experiments over a wider range of temperature, by Amagat in 1893.

Critical Temperature

Before we can interpret Andrews's results for carbon dioxide, we must describe a simple experiment, made by Cagniard de la Tour in

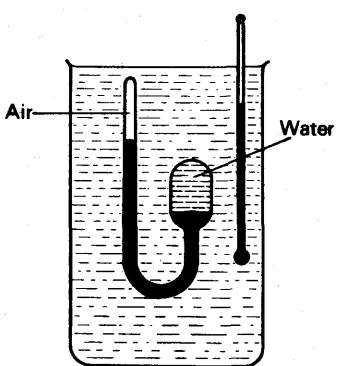


FIG. 12.21. Cagniard de la Tour's experiment.

1822. De la Tour made a tube of strong glass, as shown in Fig. 12.21. In the bulb he had water, round the bend mercury, and at the top—where the tube was sealed off—air. He heated the tube in a bath to over 300°C. The expansion of the liquids was taken up by the compression of the air, from which de la Tour estimated the pressure; it went beyond 100 atmospheres. Above about 100°C he observed what we would expect; that a meniscus formed in bulb, showing that steam was present as well as water. But above about 300°C he noticed that the

meniscus vanished: that there was no observable distinction between liquid and vapour. The temperature at which the meniscus vanished he called the *critical temperature*.

If we consider the nature of a saturated vapour, the phenomenon of the critical temperature need not surprise us. For as its temperature rises a saturated vapour becomes denser, whereas a liquid becomes less dense. The critical temperature is, we may suppose, the temperature at which liquid and saturated vapour have the same density. Fig. 12.22 supports this view: it shows the results of measurements made on liquid oxygen by the method of Cailletet and Mathias (p. 307).

Behaviour of Carbon Dioxide near the Critical Point

Now let us turn to Andrews' isothermals for carbon dioxide. These are shown again, this time as a simple pressure-volume diagram, in Fig. 12.23. Let us consider the one for 21.5°C, ABCD. Andrews noticed that, when the pressure reached the value corresponding to B, a meniscus appeared above the mercury in the capillary containing the carbon dioxide. He concluded that the liquid had begun to form. From B to C, he found no change in pressure as the screws were turned, but simply a decrease in the volume of the carbon dioxide. At the same time the meniscus moved upwards, suggesting that the proportion of

liquid was increasing. At C the meniscus disappeared at the top of the tube, suggesting that the carbon dioxide had become wholly liquid. Beyond C the pressure rose very rapidly; this confirmed the idea that the carbon dioxide was wholly liquid, since liquids are almost incompressible.

Thus the part CBA of the isothermal for 21.5°C is a curve of volume

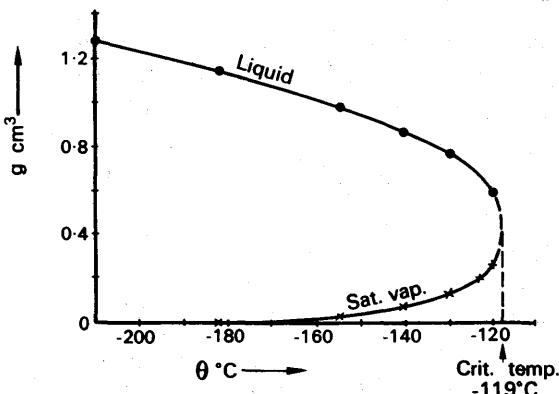


FIG. 12.22. Densities of liquid oxygen, and its saturated vapour.

against pressure for a liquid and vapour, showing saturation at B; it is like the isothermal for water given in Fig. 12.5 (a), p. 300. And the curve GFE is another such isothermal, for the lower temperature 13.1°C; the two curves are like the two in Fig. 12.6 (a), p. 300.

The isothermal for 31.1°C has no extended plateau; it merely shows

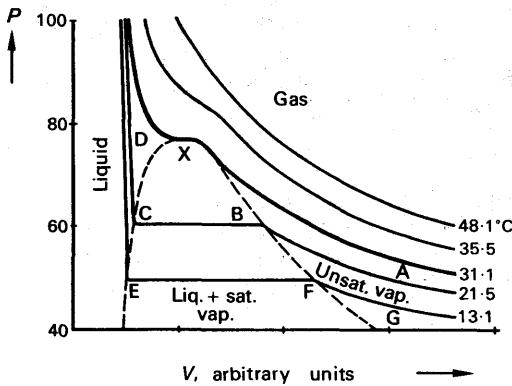


FIG. 12.23. Andrews's isothermals for CO_2 .

a point of inflection at X. At that temperature, Andrews observed no meniscus; he concluded that it was the critical temperature. The isothermals for temperature above 31.1°C never become horizontal, and show no breaks such as B or F. At temperatures above the critical, no transition from gas to liquid can be observed.

The isothermal for 48.1°C conforms fairly well to Boyle's law; even

when the gas is highly compressed its behaviour is not far from ideal.

The point X in Fig. 12.23 is called the critical point. The pressure and volume (of unit mass) corresponding to it are called the *critical pressure* and *volume*; the reciprocal of the critical volume is the critical density.

CRITICAL CONSTANTS OF GASES AND BOILING POINTS
(At atmospheric pressure)

Substance	Critical			Boiling-point °C
	Temperature °C	Pressure, atmospheres	Density kg m ⁻³	
Argon . . .	−122	48	0.53×10^3	−186
Neon . . .	−229	27	0.48	−246
Helium . . .	−268	2.26	0.069	−269
Chlorine . . .	146	76	0.57	−34
Hydrogen . . .	−240	12.8	0.031	−253
Nitrogen . . .	−146	33	0.31	−196
Oxygen . . .	−118	50	0.43	−183
Air . . .	−140	39	0.35	
Ammonia . . .	130	115	0.24	−33.5
Carbon dioxide . . .	31.1	73	0.46	−78.2
Ethylen . . .	10	52	0.22	−102.7
Freon, CCl_2F_2 . . .	112	40	0.56	
Sulphur dioxide . . .	155	79	0.52	−10.8
Water . . .	374	219	0.4	100

The above account of the phenomena near the critical point is over-simple, and may create the impression that these phenomena are fully understood. They are not; but this is not the place to say much about the matter. We may just point out that, even at temperature well above the critical, and when no meniscus can be seen, considerable differences of density can be found in a so-called gas. They have been shown by including, in a sealed tube of liquid and vapour, a number of small glass balls of different densities. When the tube was heated above the critical temperature, each ball floated at a point where the substance had a density equal to that of the ball.

Gases and Vapours

A gas above its critical temperature cannot be liquefied. Early attempts to liquefy gases such as air, by compression without cooling, failed; and the gases were wrongly called 'permanent' gases. We still, for convenience, refer to a gas as a vapour when it is below its critical temperature, and as a gas when it is above it. But the distinction is not the same as that between an ideal gas and one which is far from ideal. For a gas which is near its critical point, though it may be a little above its critical temperature, does not obey Boyle's law, as Fig. 12.23 shows. On the other hand, a vapour which is far from saturation obeys Boyle's law fairly well.

Refrigeration

The action of a refrigerator depends on the absorption of its latent heat by a liquid—the working substance—in evaporating. The working substance must be one whose vapour has a critical temperature above

normal atmospheric temperatures, so that it can be liquefied by compression alone. Common working substances are ammonia, carbon dioxide, sulphur dioxide, and specially developed compounds such as the two varieties of Freon: CCl_2F_2 , and $\text{C}_2\text{Cl}_2\text{F}_4$. The working substance is compressed by a pump, P, in Fig. 12.24, and passed through a metal pipe C; there the heat of compression is carried away by circulating water, and the substance liquefies. The liquid passes to a reservoir R. From the reservoir, liquid escapes through a throttle valve V into the coil D, which is connected to the low pressure side of the pump. The coil D lies round the walls of the space to be cooled (not shown).

When the liquid escapes from the reservoir, it starts to evaporate, because of the low pressure. It draws its latent heat from its own heat content, and cools. Not all of the liquid evaporates as it emerges, and the mixture of cool liquid and vapour passes round the metal coil D. If the atmosphere in the chamber containing D is warmer than the liquid, the liquid evaporates further. The latent heat which it requires is furnished by the surroundings of D, which are therefore cooled.

THE EQUILIBRIUM OF SOLID, LIQUID, VAPOUR

We have pointed out that solids as well as liquids evaporate (p. 297). A solid thus has saturated vapour over it, just as a liquid has, and the pressure of the saturated vapour depends on the temperature. The table on p. 299 shows the pressure of saturated water vapour over ice, at -10°C and -20°C .

The Triple Point

In Fig. 12.25, the curve AP relates the saturated vapour pressure of ice to its temperature; at any point on the curve, ice and water-vapour are in equilibrium. BP is the saturated-vapour-pressure curve of water; at any point on it, water and water-vapour are in equilibrium. CP is the curve relating the melting-point of ice with the pressure: at any point on it, ice and water are in equilibrium. The three curves meet at the point P, whose co-ordinates are $p = 4.6$ mm mercury, $\theta = 0.01^\circ\text{C}$. These are the only conditions in which ice, water, and water-vapour can exist together: if either the temperature or pressure is altered, at least one phase vanishes. If, when the pressure and temperature are altered, their new values happen to lie on one of the curves, then the two corresponding phases survive, liquid and solid along PC, for example. But if the new conditions lie in one of the three sectors of the diagram—say in PAC—then the only phase which survives is the one corresponding to that sector: solid, in PAC. The point P is called the *triple point*.

The curve AP, which gives the saturated vapour pressure of ice, is steeper at P than the curve BP for water. It is steeper because a solid evaporates less readily than a liquid—molecules escape from it less easily. Therefore the saturated-vapour-pressure of the solid falls more rapidly with the temperature.

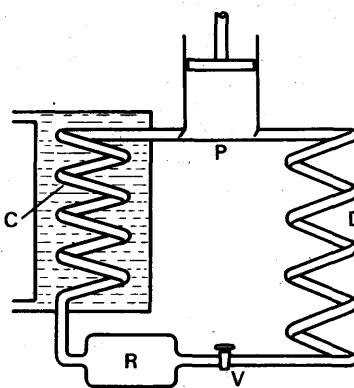


FIG. 12.24. Refrigeration.

Fig. 12.26 shows the triple point for carbon dioxide. Its co-ordinates are $p = 3800$ mm mercury, $\theta = -56.6^\circ\text{C}$. At atmospheric pressure, 760 mm mercury, therefore, solid carbon dioxide (CO_2) can be in equilibrium with its vapour, but not with liquid CO_2 . It is therefore dry, and in America it is called 'dry ice'; in England it is called 'carbon-dioxide snow'. At atmospheric pressure its temperature is -78.5°C , and it is much used as a coolant—in ice-cream trucks, for example.

Solid carbon dioxide is prepared by simply opening the valve of a cylinder containing carbon dioxide at high pressure. The gas rushes out, and does work

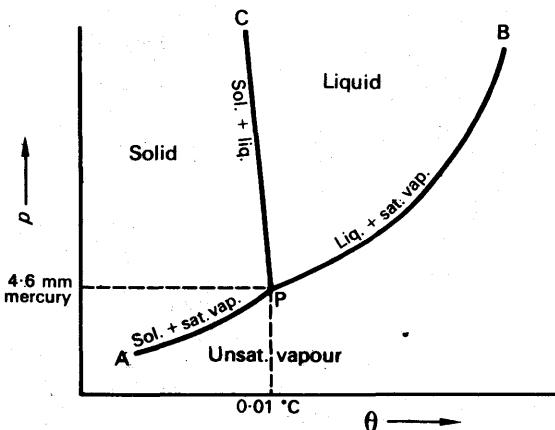


FIG. 12.25. Triple point for water (not to scale).

in acquiring kinetic energy of mass motion. Since the expansion is rapid, it is adiabatic, and the gas cools. As it does so, it goes over directly to the solid phase.

When solid carbon dioxide is warmed, it goes over directly into vapour. So, incidentally, do solid iodine and a few other substances, at atmospheric pressure. The change from solid to vapour is called sublimation. As the diagram shows, liquid CO_2 cannot exist at any temperature at all, if the pressure is below 3800 mm mercury (5.1 atmospheres).

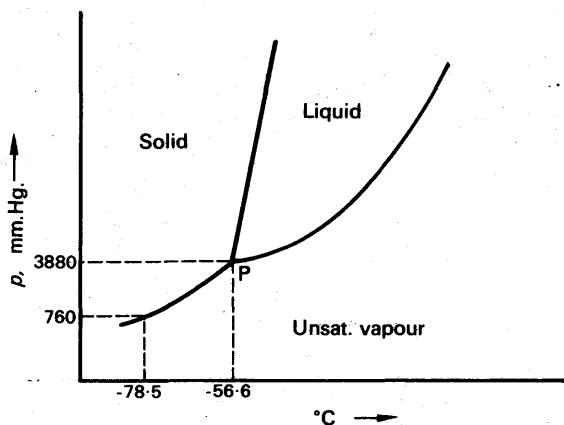


FIG. 12.26. Triple point for carbon dioxide (not to scale).

Freezing of Solutions

We have seen that dissolving a solid in water lowers its vapour pressure, and also its freezing-point (p. 297). To explain the lowering of the freezing-point, let us draw, as in Fig. 12.27, the curves of the saturated vapour pressures of ice,

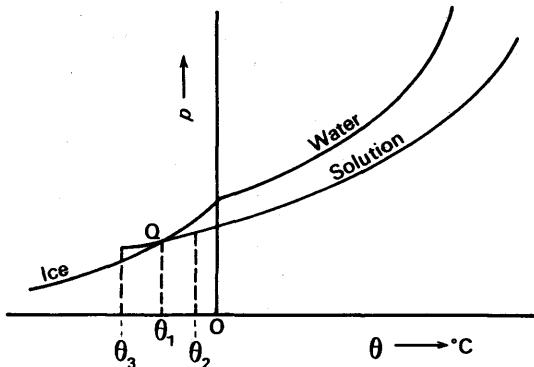


FIG. 12.27. Equilibrium between ice and a solution in water.

water and solution. We see that the curve for the solution cuts the ice curve at a point Q which corresponds to a temperature θ_1 , below 0°C . This is the only temperature at which ice and solution can be in equilibrium. At a higher temperature, θ_2 , ice has the higher vapour pressure: it therefore sublimes faster than water evaporates from the solution, and, on the whole, vapour from the ice condenses into the solution. At a lower temperature θ_3 , the solution has the higher vapour pressure; water therefore evaporates from it faster than the ice sublimes, and, on the whole, water from the solution condenses on the ice. Thus the temperature θ_1 is the freezing-point of the solution. It is the temperature at which solution and ice exchange water molecules one for one, and neither grows at the expense of the other.

We can now see why ice and salt, for example, form a freezing mixture. When salt is mixed with ice, it dissolves in the water clinging to the ice, and forms a solution. Since this is above its melting-point, being at 0°C , it has a lower saturation vapour pressure than the ice (Fig. 12.27). Therefore the ice sublimes and condenses in the solution. In effect, the ice becomes water. And in doing so it abstracts its latent heat of fusion from its surroundings. Thus the mixture changes from solid ice and salt to a liquid solution of salt, and its temperature falls below 0°C .

LIQUEFACTION OF GASES

If one of the so-called permanent gases, hydrogen or nitrogen, is to be liquefied, it must first be cooled below its critical temperature. There are three principal ways of doing this: (i) the gas may be passed through a cold bath containing a more easily liquefied gas, which is boiling at a reduced pressure and therefore has a very low temperature; (ii) the gas may be allowed to expand adiabatically and do work, losing its heat-energy in the process; (iii) the gas may be cooled by a method depending on the fact that, for a real gas, the internal energy is not independent of the volume.

The third of these is the commonest nowadays, and the only one

which we shall describe. First we must explain the phenomenon on which it depends.

The Joule-Kelvin Effect

We have already described, on p. 241, Joule's crude experiments on the expansion of a gas into a vacuum—a 'free expansion', as it is called. These experiments suggested that in such an expansion the

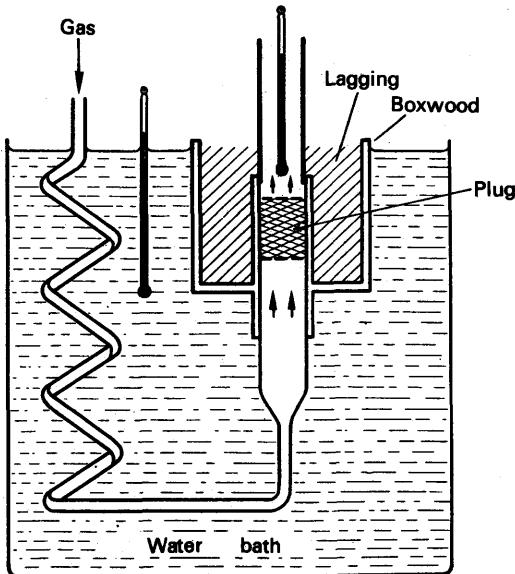


FIG. 12.28. Joule-Kelvin apparatus.

gas lost no internal energy, and therefore did no work. We concluded that the molecules of a gas had negligible attraction for one another, since otherwise work would have had to be done against their attractions whenever the gas expanded.

In 1852, Joule and Kelvin made more delicate experiments of essentially the same kind. They allowed a gas at high pressure to expand into a vacuum through a plug of cotton wool (Fig. 12.28). The plug prevented eddies from forming in the gas, so that the gas did not acquire any kinetic energy of motion in bulk. Neither did the gas do any external work, since it pushed back no piston. Nevertheless, Joule and Kelvin found that the gas was cooled slightly in its passage through the porous plug. Therefore work must have been done in separating its molecules; and this work must have been done at the expense of their kinetic energy, the heat-energy of the gas.

The magnitude of the cooling in the Joule-Kelvin effect depends on the temperature at which the gas enters the plug; for air at room temperature it has the order of 0.1°C per atmosphere pressure difference.

It is not essential for the gas to expand into a vacuum. Whenever a gas expands from high pressure to low, its volume increases, and some work is done against its inter-molecular attractions. If heat cannot

enter the gas, the work is done at the expense of the gas's internal energy, and the gas cools. The cooling is analogous to that which takes place in an adiabatic expansion, but in a normal adiabatic expansion most of the work is done externally against a piston (compare p. 251). An ideal gas would cool in an adiabatic expansion with external work, but not in a free expansion—it would show no Joule-Kelvin effect.

The Linde Process

The cooling of a gas in a free expansion is small, but Linde devised an ingenious arrangement for making it cumulative, and so producing a great temperature fall. His apparatus is shown diagrammatically in Fig. 12.29. When air is to be liquefied, it must first be freed of carbon dioxide and water, which would solidify and choke the pipes; both are removed by solid caustic soda in a vessel not shown in the figure. The pure air is compressed to about 150 atmospheres by the pump P, and the heat of compression is removed in the water-cooled copper coil C. The air then passes down the copper coil D, which runs within another copper coil E. It emerges through the nozzle N whose opening can be adjusted from outside. The nozzle lies inside a Dewar vessel or thermos flask F. The air expands on emerging, and is cooled by the Joule-Kelvin effect. It then passes upwards through the outer coil E, and as it does so, cools the incoming gas. The incoming gas is thus cooled before making its expansion, and after its expansion becomes cooler still. On escaping through E it cools the following gas yet further. Thus the cooling of the escaping gas continuously helps the cooling of the arriving gas, and the cooling is said to be *regenerative*. Eventually the gas emerging from the nozzle cools below the critical temperature; and since the actual pressure, 150 atmospheres, is well

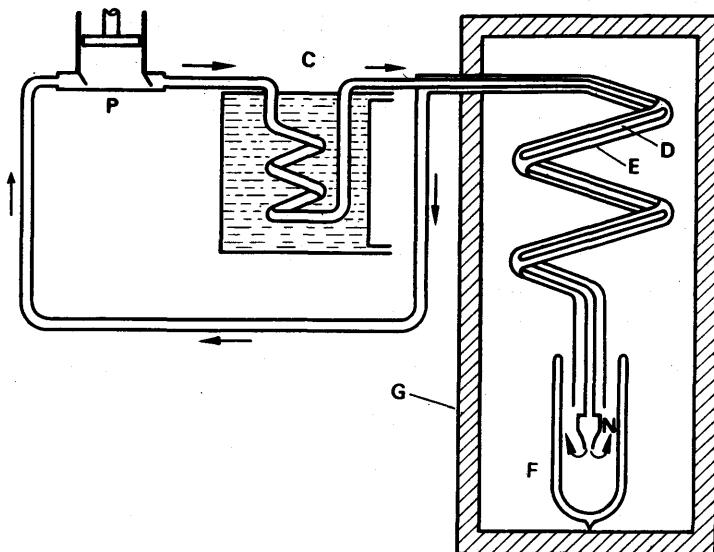


FIG. 12.29. Liquefaction of air.

above the critical pressure, 39 atmospheres, the gas liquefies and collects in the flask. The liquefier is heavily lagged with insulating material, G, to prevent heat coming in from the outside.

The reader should appreciate that the regenerative cooling takes place only in the double coil. At all stages in the process the air enters the inner coil at the temperature of the cooling water around C. But as time goes on it passes through ever-cooler gas coming up from the nozzle, until liquid begins to form and the system reaches equilibrium.

Liquid Nitrogen and Oxygen

As the table on p. 318 shows, the boiling-point of nitrogen, at atmospheric pressure, is -196°C , whereas that of oxygen is -183°C . When liquid air is exposed to the atmosphere, therefore, the nitrogen boils off faster and the proportion of liquid oxygen increases. The so-called liquid air sold commercially is mostly liquid oxygen. It is more dangerous than true liquid air, particularly if there is hydrogen about.

Hydrogen and Helium

At ordinary temperatures, the Joule-Kelvin effect is reversed for helium and hydrogen, that is, a free expansion causes warming. We cannot go into the explanation of that here, but we may say that it is connected with the fact that the pV/p curves of hydrogen and helium rise with increasing pressure, instead of falling at first (Fig. 12.17, p. 314).

These gases show a Joule-Kelvin cooling, however, if they are sufficiently cooled before the expansion. Hydrogen must be cooled below -83°C and helium below -240°C ; these temperatures are called the *inversion temperatures* of the gases.

Hydrogen can be cooled below its inversion temperature by passing it through a coil in liquid air before it enters the double coil of the liquefier. Helium must be passed through a coil in liquid hydrogen, boiling under reduced pressure.

EXAMPLES

1. Describe an experiment which demonstrates that the pressure of a vapour in equilibrium with its liquid depends on the temperature.

A narrow tube of uniform bore, closed at one end, has some air entrapped by a small quantity of water. If the pressure of the atmosphere is 760 mm of mercury, the equilibrium vapour pressure of water at 12°C and at 35°C is 10.5 mm of mercury and 42.0 mm of mercury respectively, and the length of the air column at 12°C is 10 cm, calculate its length at 35°C . (L.)

First part. See text.

Second part. For the given mass of air,

$$\frac{p_1 V_1}{T_1} = \frac{p_2 V_2}{T_2}$$

$$p_1 = 76 - 1.05 = 74.95 \text{ cm}, V_1 = 10, T = 273 + 12 = 285 \text{ K};$$

$$p_2 = 76 - 4.2 = 71.8 \text{ cm}, T_2 = 273 + 35 = 308 \text{ K};$$

$$\therefore \frac{74.95 \times 10}{285} = \frac{71.8 V_2}{308}$$

$$\therefore V_2 = \frac{74.95 \times 10 \times 308}{285 \times 71.8} = 11.3.$$

2. State Dalton's law of partial pressures; how is it explained on the kinetic theory? A closed vessel contains air, saturated water-vapour, and an excess of water. The total pressure in the vessel is 760 mm of mercury when the temperature is 25°C; what will it be when the temperature has been raised to 100°C? (Saturation vapour pressure of water at 25°C is 24 mm of mercury.) (C.)

First part. See text.

Second part. From Dalton's law, the pressure of the air at 25°C = 760 - 24 = 736 mm of mercury. Suppose the pressure is p mm at 100°C. Then, since pressure is proportional to absolute temperature for a fixed mass of air, we have

$$\frac{p}{736} = \frac{373}{298}$$

from which

$$p = 921 \text{ mm.}$$

Now the saturation vapour pressure of water at 100°C = 760 mm.

$$\therefore \text{total pressure in vessel} = 921 + 760 = 1681 \text{ mm mercury.}$$

3. Define *relative humidity* and *dew-point*. Describe an instrument with which the dew-point can be determined. The relative humidity in a closed room at 15°C is 60 per cent. If the temperature rises to 20°C, what will the relative humidity become? On what assumptions is your calculation based? (Saturation vapour pressure of water-vapour at 15°C = 12.67 mm of mercury, at 20°C = 17.36 mm.) (L.)

First part. See text.

Second part. Suppose p is the actual water vapour pressure in mm mercury in the air at 15°C.

Then

$$\frac{p}{12.67} = \text{relative humidity} = 60 \text{ per cent.}$$

$$\therefore p = \frac{60}{100} \times 12.67 = 7.60 \text{ mm.}$$

Assuming the pressure of the water-vapour is proportional to its absolute temperature, the pressure p_1 at 20°C is given by

$$\frac{p_1}{7.60} = \frac{273+20}{273+15}$$

$$\therefore p = \frac{7.60 \times 293}{288} = 7.73 \text{ mm.}$$

$$\therefore \text{relative humidity at } 20^\circ\text{C} = \frac{7.734}{17.36} \times 100 \text{ per cent} = 45 \text{ per cent.}$$

4. What is meant by *saturation pressure of water vapour, dew-point?* Describe briefly the principles underlying two different methods for the determination of the relative humidity in the laboratory.

A barometer tube dips into a mercury reservoir and encloses a mixture of air and saturated water vapour above the mercury column in the tube, the height of the column being 70 cm above the level in the reservoir. If the atmospheric pressure and the saturation pressure of water vapour are respectively 76 cm and 1 cm of mercury, determine the height of the column when the tube is depressed in the reservoir to reduce the air volume to half its initial value. (*L*)

First part. The saturation pressure of water vapour is the pressure of water vapour in contact with water in a closed space. The dew-point is the temperature at which the air is just saturated with the water-vapour present. The different methods for measuring relative humidity concern the dew-point (Regnault) hygrometer and the wet-and-dry bulb hygrometer, discussed on pp. 311-2.

Second part. We apply the gas laws to the *air only*, as the mass of the air remains constant. From Dalton's law,

$$\begin{aligned}\text{pressure of air} &= \text{total pressure} - \text{pressure of water-vapour} \\ &= (76 - 70) - 1 = 5 \text{ cm mercury.}\end{aligned}$$

The volume of the air changes from V , say, to $V/2$. Hence the new pressure, p , of the air is given, from Boyle's law, by

$$5 \times V = p \times \frac{V}{2}$$

$$\therefore p = 10 \text{ cm.}$$

$$\therefore \text{new total pressure of mixture of gases} = 10 + 1 = 11 \text{ cm.}$$

$$\therefore \text{new height of mercury column} = 76 - 11 = 65 \text{ cm.}$$

5. What is meant by the *relative humidity* of the air? Describe in detail a good method for finding it.

Air at 19.5°C has a relative humidity of 75 per cent. Calculate its dew-point and the mass of water vapour contained in 1 litre, being given that the boiling-points of water under pressure of 12 mm, 14 mm, 16 mm and 18 mm of mercury are 14°C, 16.45°C, 18.55°C and 20.45°C respectively. Assume that water vapour behaves as an ideal gas, that its density at s.t.p. is 0.00080 g per cm³. (*N*)

First part. The relative humidity of the air is the ratio of the mass of water-vapour in a given volume of the air to the mass of water-vapour required to saturate that volume. A good method for finding it is by the dew-point (Regnault) hygrometer, described on p. 311.

Second part.

$$\text{Relative humidity} = \frac{\text{s.v.p. of water at dew-point}}{\text{s.v.p. of water at } 19.5^\circ\text{C}} \times 100 \text{ per cent.}$$

$$\therefore 75 = \frac{\text{s.v.p. at dew-point}}{17 \text{ mm mercury}} \times 100$$

$$\therefore \text{s.v.p. at dew-point} = \frac{3}{4} \times 17 = 12.75 \text{ mm mercury.}$$

$$\begin{aligned}\therefore \text{dew-point} &= 14^\circ\text{C} + \frac{0.75}{2} \times 2.45^\circ\text{C} \\ &= 14.9^\circ\text{C},\end{aligned}$$

as s.v.p. at 14°C = 12 mm, at 16.45°C = 14 mm.

To find the mass of water-vapour in 1 litre. The pressure of water-vapour =

s.v.p. at dew-point = 12.75 mm = 1.275 cm mercury; the absolute temperature = $273 + 19.5 = 292.5$ K.

$$\therefore \text{vol. in cm}^3 \text{ at s.t.p.} = 1,000 \times \frac{1.275}{76} \times \frac{273}{292.5}$$

$$\therefore \text{mass of water-vapour} = 1,000 \times \frac{1.275}{76} \times \frac{273}{292.5} \times 0.0008$$

$$= 0.013 \text{ g.}$$

EXERCISES 12

1. Distinguish between a saturated and an unsaturated vapour. What is meant by *saturation vapour pressure*?

Describe an experiment to measure the saturation vapour pressure of water vapour for temperatures between 20°C and 100°C. Indicate graphically the results which would be obtained from such an experiment. (L.)

2. Explain concisely four of the following in terms of the simple kinetic theory of matter:

- (a) energy must be supplied to a liquid to convert it to a vapour without change of temperature;
- (b) when some water is introduced into an evacuated flask, some of the water at first evaporates, but subsequently, provided the temperature of the flask is kept constant, the volume of the water present remains unchanged;
- (c) gases are generally poor conductors of heat compared with solids;
- (d) when a gas, which is enclosed in a thermally insulated cylinder provided with a piston, is compressed by moving the piston, the temperature of the gas is raised;
- (e) water can be heated by stirring. (O. & C.)

3. Distinguish between a *saturated* and an *unsaturated* vapour. Describe an experiment to investigate the effect of pressure on the boiling point of water and draw a sketch graph to show the general nature of the results to be expected.

A column of air was sealed into a horizontal uniform-bore capillary tube by a water index. When the atmospheric pressure was 762.5 torr (mm of Hg) and the temperature was 20°C, the air-column was 15.6 cm long; with the tube immersed in a water bath at 50°C, it was 19.1 cm long, the atmospheric pressure remaining the same. If the s.v.p. of water at 20°C is 17.5 torr, deduce its value at 50°C. (O. & C.)

4. What is meant by *saturation vapour pressure*? Describe an experiment to investigate the variation of the saturation pressure of water vapour with temperature.

Sketch the isothermal curve relating pressure and volume (a) for a mass of dry air at room temperature, (b) for water vapour at 100°C. (L.)

5. Explain the physical principles of a domestic refrigerator employing an evaporating liquid and give a labelled diagram showing its essential components. How may the temperature of the main storage compartment be regulated and what factors determine the lowest attainable temperature?

A certain refrigerator converts water at 0°C into ice at a maximum rate of 5 g per minute when the exterior temperature is 15°C. Assuming that the rate at which heat leaks into the refrigerator from its surroundings is proportional to the temperature difference between the exterior and interior and is 2.5 watt deg C⁻¹, what is the maximum exterior temperature at which this refrigerator could just maintain a temperature of 0°C in the interior? [Specific latent heat of fusion of ice = 330 kJ kg⁻¹.] (O. & C.)

6. Use the simple kinetic theory of matter to answer the following questions:

- (a) How do gases conduct heat?
- (b) Why does a liquid tend to cool when it evaporates?

(c) Why does the boiling point of a liquid depend upon the external pressure?

Show that the pressure p , the density ρ , and the mean square molecular velocity c^2 of an ideal gas are related by $p = \frac{1}{3}\rho c^2$, stating any assumptions at the points where they become necessary in the proof. (O. & C.)

7. State *Boyle's law* and *Dalton's law of partial pressures*.

The space above the mercury in a Boyle's law apparatus contains air together with alcohol vapour and a little liquid alcohol. Describe how the saturation vapour pressure of alcohol at room temperature may be determined with this apparatus.

A mixture of air and saturated alcohol vapour in the presence of liquid alcohol exerts a pressure of 12.8 cm of mercury at 20°C. When the mixture is heated at constant volume to the boiling point of alcohol at standard pressure (i.e. 78°C), the vapour remaining saturated, the pressure becomes 86.0 cm of mercury. Find the saturation vapour pressure of alcohol at 20°C. (L.)

8. The saturation pressure of water vapour is 1.2 cm of mercury at 14°C and 2.4 cm of mercury at 25°C. Describe and explain the experiment you would perform to verify these data.

Explain qualitatively in terms of the kinetic theory of matter (a) what is meant by saturation vapour pressure, (b) the relative magnitudes of the saturation vapour pressures quoted.

Sketch a graph showing how the saturation pressure of water vapour varies between 0°C and 110°C. (N.)

9. Draw p against v curves for temperatures above, at, and below the critical temperature, for a real gas. Explain the significance of critical temperature.

Sketch graphs of pv against p at the same temperatures for a real gas. What would be the form of the pv against p curves for an ideal gas?

Describe briefly experiments which provide the data on which these curves for a real gas are based. (O. & C.)

10. Compare the properties of saturated and unsaturated vapours. By means of diagrams show how the pressure of (a) a gas, and (b) a vapour, vary with change (i) of volume at constant temperature, and (ii) of temperature at constant volume.

The saturation vapour pressure of ether vapour at 0°C is 185 mm of mercury and at 20°C it is 440 mm. The bulb of a constant volume gas thermometer contains dry air and sufficient ether for saturation. If the observed pressure in the bulb is 1000 mm at 20°C, what will it be at 0°C? (L.)

11. Explain the terms *relative humidity*, *dew-point*. Describe in detail an experiment to determine the dew-point.

Find the mass of air and water-vapour in a room of $20 \times 10 \times 5$ cubic metres capacity, the temperature being 20°C and the pressure 750 mm of mercury. Assume that the saturation pressure of water-vapour at the dew-point is 9.0 mm of mercury; that the density of dry air at s.t.p. is 1.30 kg m^{-3} ; that the density of water vapour is $\frac{5}{8}$ that of air under the same conditions of pressure and temperature. (L.)

12. Define *pressure of a saturated vapour*, *critical temperature*. Give an account of the isothermal curves for carbon dioxide at temperatures above and below its critical temperature.

Some liquid ether is sealed in a thick-walled glass tube, leaving a space containing only the vapour. Describe what is observed as the temperature of the tube and its contents is raised above 197°C, which is the critical temperature for

ether. (Assume that the vessel is strong enough to withstand the internal pressure.) (L.)

13. Define *dew-point* and explain what is meant by *relative humidity*. Describe how you would determine the dew-point of the atmosphere.

What is the relative humidity of an atmosphere whose temperature is 16.3°C if its dew-point is 12.5°C ? The following table gives the saturation pressure, p , of water-vapour in mm of mercury at various temperature, t .

$t^{\circ}\text{C}$	10	12	14	16	18	20
$p \text{ mm}$	9.20	10.51	11.98	13.62	15.46	17.51

(N.)

14. Give briefly the principles of one method each for liquefying (a) chlorine, (b) hydrogen.

Describe a suitable container for liquid air. Point out carefully the physical principles involved. (C.)

15. Give an account of three methods used to obtain temperatures below 0°C . Describe a method for liquefying air. How must the procedure be modified in order to liquefy hydrogen? (N.)

16. Describe in detail the method you would use to find (a) the melting-point of lead, and (b) the boiling-point of brine.

An alloy of copper and silver is made with different percentage composition. For each mix the melting-point is measured, with the following results:

M.P. deg C	960	810	760	740	760	790	900	1080
Per cent of copper in alloy	0	20	30	40	50	60	80	100

Plot a curve showing the relation between the melting-point in degrees Centigrade and the percentage of copper present. Comment on the curve. (L.)

chapter thirteen

Transfer of Heat

Conduction

If we put a poker into the fire, and hold on to it, then heat reaches us along the metal. We say the heat is *conducted*; and we soon find that some substances—metals—are good conductors, and others—such as wood or glass—are not. Good conductors feel cold to the touch on a cold day, because they rapidly conduct away the body's heat.

Temperature Distribution along a Conductor

In order to study conduction in more detail consider Fig. 13.1 (a), which shows a metal bar AB whose ends have been soldered into the

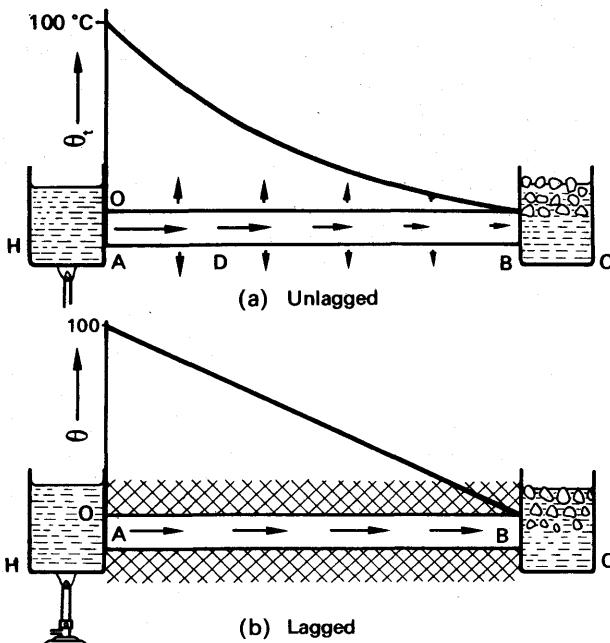


FIG. 13.1. Temperature fall along lagged and unlagged bars.

walls of two metal tanks, H, C; H contains boiling water, and C contains ice-water. Heat flows along the bar from A to B, and *when conditions are steady* the temperature θ of the bar is measured at points along its length; the measurements may be made with thermojunctions, not shown in the figure, which have been soldered to the rod. The curve in

the upper part of the figure shows how the temperature falls along the bar, less and less steeply from the hot end to the cold.

The figure 13.1 (b) shows how the temperature varies along the bar, if the bar is *well lagged* with a bad conductor, such as asbestos wool. It now falls uniformly from hot to cold.

The difference between the temperature distributions is due to the fact that, when the bar is unlagged, heat escapes from its sides, by convection in the surrounding air. Thus the heat flowing past D per second, is less than that entering the bar at A by the amount which escapes from the surface AD. The arrows in the figure represent the heat escaping per second from the surface of the bar, and the heat flowing per second along its length. The heat flowing per second along the length decreases from the hot end to the cold. But when the bar is lagged, the heat escaping from its sides is negligible, and the flow per second is constant along the length of the bar.

We thus see that the temperature gradient along a bar is greatest where the heat flow through it is greatest. We also see that the temperature gradient is uniform only when there is a negligible loss of heat from the sides of the bar.

Thermal Conductivity

Let us consider a very large thick bar, of which AB in Fig. 13.2 (i) is a part, and along which heat is flowing steadily. We suppose that the

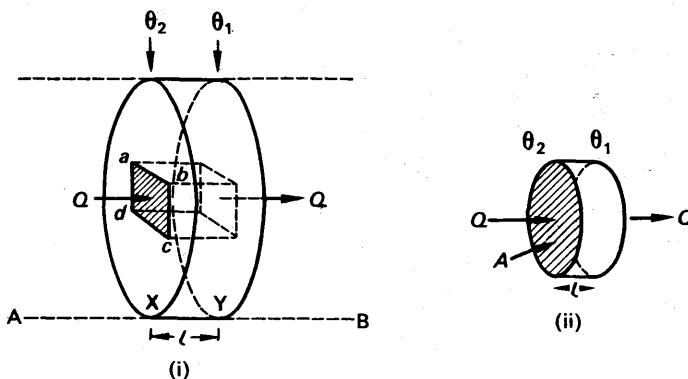


FIG. 13.2. Definition of thermal conductivity.

loss of heat from the sides of the bar is made negligible by lagging. XY is a slice of the bar, of thickness l , whose faces are at temperatures θ_2 and θ_1 . Then the *temperature gradient* over the slice is

$$\frac{\theta_2 - \theta_1}{l}.$$

We now consider an element $abcd$ of the slice of unit cross-sectional area, and we denote by Q the heat flowing through it per second. The value of Q depends on the temperature gradient, and, since some

substances are better conductors than others, it also depends on the material of the bar.

We therefore write

$$Q = k \frac{\theta_2 - \theta_1}{l}$$

where k is a factor depending on the material.

To a fair approximation the factor k is a constant for a given material; that is to say, it is independent of θ_2 , θ_1 , and l . It is called the *thermal conductivity* of the material concerned. To put its definition into words, we let $\theta_2 - \theta_1$ be 1°C, and l be 1 m, so that

$$Q = k.$$

We then say :

Consider a cube of material, whose faces are 1 m apart, and have a temperature difference of 1 deg C. If heat flows in the steady state through the cube at right angles to its faces, and none is lost from its sides, then the heat flow per unit area is numerically equal to the conductivity of the material.

This definition leads to a general equation for the flow of heat through any parallel-sided slab of the material, when no heat is lost from the sides of the slab. As in Fig. 13.2 (ii), we denote the cross-sectional area of the slab by A , its thickness by l , and the temperature of its faces by θ_1 and θ_2 . Then the heat Q flowing through it per second is

$$Q = \frac{kA(\theta_2 - \theta_1)}{l} \quad \quad (1)$$

A useful form of this equation is

$$\frac{Q}{A} = k \frac{\theta_2 - \theta_1}{l} \quad \quad (2)$$

or

$$\text{heat flow per } \text{m}^2 \text{ per second} = \text{conductivity} \times \text{temperature gradient.} \quad (2a)$$

In terms of the calculus, (2) may be re-written

$$\frac{1}{A} \frac{dQ}{dt} = -k \frac{d\theta}{dl} \quad \quad (3)$$

the temperature gradient being negative since θ diminishes as l increases.

Units and Magnitude of Conductivity

Equation (2) enables us to find the unit of thermal conductivity. We have

$$k = \frac{Q/A(\text{J m}^{-2} \text{s}^{-1})}{(\theta_2 - \theta_1)/l(\text{K m}^{-1})}$$

Thus the unit of thermal conductivity = $\text{J s}^{-1} \text{m}^{-1} \text{K}^{-1}$, or since $\text{joule second}^{-1} = \text{watt (W)}$, the unit of k is $\text{W m}^{-1} \text{K}^{-1}$.

THERMAL CONDUCTIVITIES

SOLIDS

(Mean values, c. 0–100°C)

Substance	k W m ⁻¹ K ⁻¹	Substance	k	Substance	k
Ag . .	420	Asbestos . .	0.13	Ice . .	2.1
Al . .	210	Brick . .	0.13	Marble . .	3.0
Cu . .	380	Cardboard . .	0.21	(white) Mica . .	0.76
Fe pure . .	76	Cork . .	0.42	Paraffin wax . .	0.25
wrought . .	59	Cotton . .	0.22	Silica (fused) . .	1.4
Hg . .	8	Cotton wool . .	0.025	Rubber (para) . .	0.19
Ni . .	87	Ebonite . .	0.17	Sand . .	0.054
Pb . .	35	Felt . .	0.038	Silk . .	0.092
Pt . .	71	Flannel . .	0.097	Slate . .	2.0
Brass . .	109	Glass . .	1.1	Wood . .	c. 0.21
Duralumin . .	130	(window)			
Steel . .	46	Graphite . .	130		

LIQUIDS AND GASES

Liquid	k	Gas	k
Alcohol (25°C) . .	0.18	Air (0°C) . .	0.024
Glycerine (20°C) . .	0.29	(100°C) . .	0.032
Olive oil (0°C) . .	0.17	CO ₂ (0°C) . .	0.015
Paraffin oil (0°C) . .	0.13	H ₂ (0°C) . .	0.17
Water (10°C) . .	0.62	N ₂ (0°C) . .	0.024
(80°C) . .	0.67	O ₂ (0°C) . .	0.024

To a rough approximation we may say that the conductivities of metals are about 1000 times as great as those of other solids, and of liquids; and they are about 10000 times as great as those of gases.

Effect of Thin Layer of Bad Conductor

Fig. 13.3 shows a lagged copper bar AB, whose ends are pressed against metal tanks at 0° and 100°C, but are separated from them by layers of dirt. The length of the bar is 10 cm or 0.1 m, and the dirt layers are 0.1 mm or 0.1×10^{-3} m thick. Assuming that the conductivity of dirt is 1/1000 that of copper, let us find the temperature of each end of the bar.

Suppose k = conductivity of copper,

A = cross-section of copper,

θ_2, θ_1 = temperature of hot and cold ends.

Since the bar is lagged, the heat flow per second Q is constant from end to end. Therefore,

$$Q = \frac{k}{1000} A \frac{100 - \theta_2}{0.1 \times 10^{-3}} = kA \frac{\theta_2 - \theta_1}{0.1} = \frac{k}{1000} A \frac{\theta_1 - 0}{0.1 \times 10^{-3}}$$

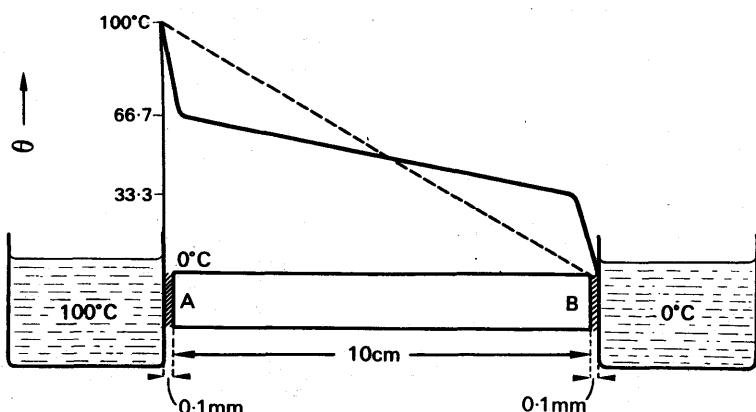


FIG. 13.3. Temperature gradients in good and bad conductors.

Dividing through k , these equations give

$$\frac{100 - \theta_2}{0.1} = \frac{\theta_2 - \theta_1}{0.1} = \frac{\theta_1}{0.1},$$

or

$$100 - \theta_2 = \theta_2 - \theta_1 = \theta_1,$$

whence

$$\theta_2 = 66.7^\circ\text{C},$$

$$\theta_1 = 33.3^\circ\text{C}.$$

Thus the total temperature drop, 100°C , is divided equally over the two thin layers of dirt and the long copper bar. The heavy lines in the figure show the temperature distribution; the broken line shows what it would be if there were no dirt.

Good and Bad Conductors

The foregoing example shows what a great effect a thin layer of a bad conductor may have on thermal conditions; 0.1 mm of dirt causes as great a temperature fall as 10 cm of copper. We can generalize this result with the help of equation (2a):

$$\text{heat flow}/\text{m}^2\text{ s} = \text{conductivity} \times \text{temperature gradient}.$$

The equation shows that, if the heat flow is uniform, the temperature gradient is inversely proportional to the conductivity. If the conductivity of dirt is $1/1000$ that of copper, the temperature gradient in it is 1000 times that in copper; thus 1 mm of dirt sets up the same temperature fall as 1 m of copper. In general terms we express this result by saying that the dirt prevents a good thermal contact, or that it provides a bad one. The reader who has already studied electricity will see an obvious analogy here. The flow of heat can, in fact, be treated mathematically in the same way as the flow of electricity; we may say that a dirt layer has a high thermal resistance, and hence causes a great temperature drop.

Boiler plates are made of steel, not copper, although copper is about

eight times as good a conductor of heat. The material of the plates makes no noticeable difference to the heat flow from the furnace outside the boiler to the water inside it, because there is always a layer of gas between the flame and the boiler-plate. This layer may be very thin, but its conductivity is about $1/10000$ that of steel; if the plate is a centimetre thick, and the gas-film $1/1000$ centimetre, then the temperature drop across the film is ten times that across the plate. Thus the rate at which heat flows into the boiler is determined mainly by the gas.

If the water in the boiler deposits scale on the plates, the rate of heat flow is further reduced. For scale is a bad conductor, and, though it may not be as bad a conductor as gas, it can build up a much thicker layer. Scale must therefore be prevented from forming, if possible; and if not, it must from time to time be removed.

Badly conducting materials are often called *insulators*. The importance of building dwelling-places from insulating materials hardly needs to be pointed out. Window-glass is a ten-times better conductor than brick, and it is also much thinner; a room with large windows therefore requires more heating in winter than one whose walls are more modestly pierced. Wood is as bad a conductor (or as good an insulator) as brick, but it also is thinner. Wooden houses therefore have double walls, with an air-space between them; air is an excellent insulator, and the walls prevent convection. In polar climates, wooden huts must not be built with steel bolts going right through them; otherwise the inside ends of the bolts grow icicles from the moisture in the explorer's breath.

Measurement of High Conductivity: Metals

When the thermal conductivity of a metal is to be measured, two conditions must usually be satisfied: heat must flow through the specimen at a measurable rate, and the temperature gradient along the specimen must be measurably steep. These conditions determine the form of the apparatus used.

When the conductor is a metal, it is easy to get a fast enough heat flow; the problem is to build up a temperature gradient. It is solved by

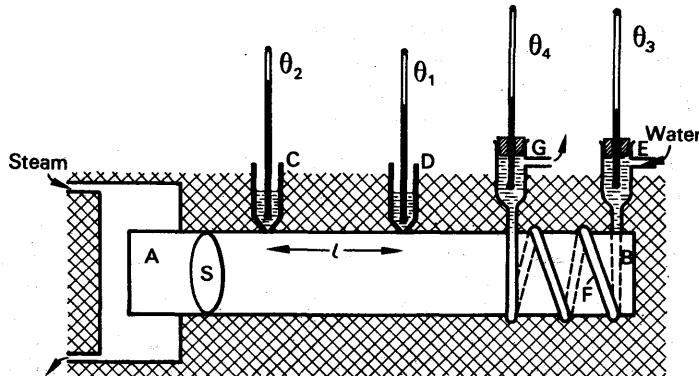


FIG. 13.4. Apparatus for thermal conductivity of a metal.

having as the specimen a bar long compared with its diameter. Fig. 13.4 shows the apparatus, which is due to Searle. AB is the specimen, about 4 cm diameter and 20 cm long. In one form of apparatus it is heated by steam at A, and cooled by circulating water at B. The whole apparatus is heavily lagged with felt. To measure the temperature gradient, thermometers are placed in the two mercury-filled cups C, D; the cups are made of copper, and are soldered to the specimen at a known distance apart. Alternatively, thermometers are placed in holes bored in the bar, which are filled with mercury. In this way errors due to bad thermal contact are avoided.

The cooling water flows in at E, round the copper coil F which is soldered to the specimen, and out at G. The water leaving at G is warmer than that coming in at E, so that the temperature falls continuously along the bar: if the water came in at G and out at E, it would tend to reverse the temperature gradient at the end of the bar, and might upset it as far back as D or C.

The whole apparatus is left running, with a steady flow of water, until all the temperatures have become constant: the temperature θ_2 and θ_1 , at C and D in the bar, and θ_4 and θ_3 of the water leaving and entering. The steady rate of flow of the cooling water is measured with a measuring cylinder and a stop-clock.

If A is the cross-sectional area of the bar and k its conductivity, then the heat flow per second through a section such as S is

$$Q = kA \frac{\theta_2 - \theta_1}{l}.$$

This heat is carried away by the cooling water; if a mass m of specific heat capacity c_w , flows through F in 1 second, the heat carried away is $mc_w(\theta_4 - \theta_3)$.

Therefore $kA \frac{\theta_2 - \theta_1}{l} = mc_w(\theta_4 - \theta_3).$

With this apparatus we can show that the conductivity k is a constant over small ranges of temperature. To do so we increase the flow of cooling water, and thus lower the outflow temperature θ_4 . The gradient in the bar then steepens, and $(\theta_2 - \theta_1)$ increases. When the new steady state has been reached, the conductivity k is measured as before. Within the limits of experimental error, it is found to be unchanged.

Measurement of Low Conductivity: Non-metallic Solids

In measuring the conductivity of a bad conductor, the difficulty is to get an adequate heat flow. The specimen is therefore made in the form of a thin disc, D, about 10 cm in diameter and a few millimetres thick (Fig. 13.5 (a)). It is heated by a steam-chest C, whose bottom is thick enough to contain a hole for a thermometer.

The specimen rests on a thick brass slab B, also containing a thermometer. The whole apparatus is hung in mid air by three strings attached to B.

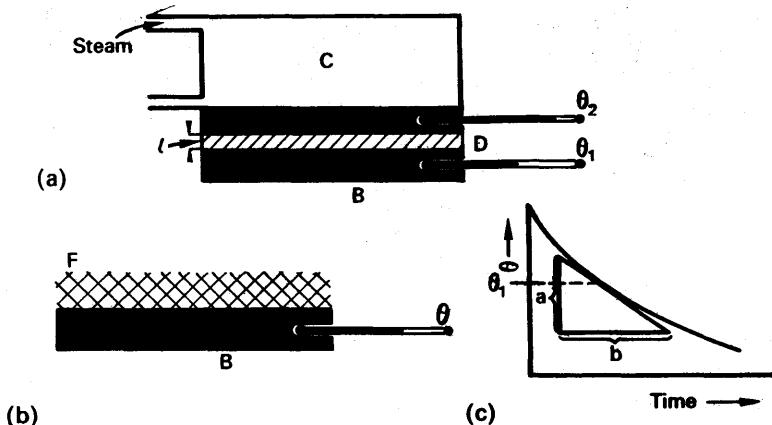


FIG. 13.5. Apparatus for thermal conductivity of a bad conductor.

To ensure good thermal contact, the adjoining faces of C, D and B must be flat and clean; those of C and B should be polished. A trace of vaseline smeared over each face improves the contact.

When the temperatures have become steady, the heat passing from C through D escapes from B by radiation and convection. Its rate of escape from B is roughly proportional to the excess temperature of B over the room (Newton's law). Thus B takes up a steady temperature θ_1 such that its rate of loss of heat to the outside is just equal to its gain through D. The rate of loss of heat from the sides of D is negligible, because their surface area is small.

This apparatus is derived from one due to Lees, and simplified for elementary work. If we use glass or ebonite for the specimen, the temperature θ_1 is generally about 70°C ; θ_2 is, of course, about 100°C . After these temperatures have become steady, and we have measured them, the problem is to find the rate of heat loss from B. To do this, we take away the specimen D and heat B directly from C until its temperature has risen by about 10°C . We then remove C, and cover the top part of B with a thick layer of felt F (Fig. 13.5 (b)). At intervals of a minute—or less—we measure the temperature of B, and afterwards plot it against the time (Fig. 13.5 (c)).

While the slab B is cooling it is losing heat by radiation and convection. It is doing so under the same conditions as in the first part of the experiment, because the felt prevents heat escaping from the top surface. Thus when the slab B passes through the temperature θ_1 , it is losing heat at the same rate as in the first part of the experiment. The heat which it loses is now drawn from its own heat content, whereas before it was supplied from C via D; that is why the temperature of B is now falling, whereas before it was steady. The rate at which B loses heat at the temperature θ_1 is given by:

$$\text{heat lost/second} = Mc \times \text{temperature fall/second},$$

where M, c are respectively the mass and specific heat capacity of the slab.

To find the rate of fall of temperature at θ_1 , we draw the tangent to the cooling curve at that point. If, as shown in Fig. 13.5 (c), its gradient at θ_1 would give a fall of a deg C in b seconds, then the rate of temperature fall is a/b deg C per second.

We then have, if A is the cross-sectional area of the specimen, l its thickness, and k its conductivity,

$$kA \frac{\theta_2 - \theta_1}{l} = Mc \frac{a}{b}$$

Thus k can be calculated.

Liquids

In finding the conductivity of a liquid, the liquid must be heated at the top and cooled at the bottom, to prevent convection. Lees' apparatus is therefore suitable. The liquid is held in a narrow glass ring, R, Fig. 13.6, sandwiched between the plates (not shown) of the Lees' disc apparatus. Let k_g , k_l be the conductivities of the glass and liquid

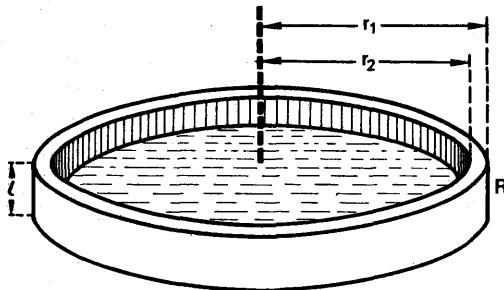


FIG. 13.6. Apparatus for finding thermal conductivity of a liquid.

respectively, and r_2 and r_1 the inner and outer radii of the ring. Then the downward heat flow per second is

$$Q = k_g \pi (r_1^2 - r_2^2) \frac{\theta_2 - \theta_1}{l} + k_l \pi r_2^2 \frac{\theta_2 - \theta_1}{l},$$

where θ_2 and θ_1 are the temperatures above and below the specimen and l is its thickness. The conductivity k_g need not be known; the heat flow through the ring may be determined in a preliminary experiment with the ring, but without the liquid.

Conduction through a Tube

The conductivity of glass tubing may be measured in the laboratory with the apparatus shown in Fig. 13.7 (a). The glass tube AB is surrounded by a steam jacket J, and water flows through it from A to B at a measured rate of m g/second. Thermometers measure the inflow and outflow temperatures of the water, θ_2 and θ_3 , which eventually become steady. In the steady state, the heat flowing through the walls of the tubing is equal to the heat carried away by the water, $mc_w(\theta_2 - \theta_3)$ joule/second.

To find the conductivity, we must know the area through which the heat flows. If r_1 , r_2 , are the inner and outer radii of the tube, and L is its length, then the areas of the inner and outer walls are $2\pi r_1 L$ and $2\pi r_2 L$ respectively (Fig. 13.7 (b)). If the tube is thin, we may take the area as constant and equal to its average value.

Thus

$$A = 2\pi L \frac{r_1 + r_2}{2}$$

At the entrance of the tube, the temperature gradient is $(\theta_1 - \theta_2)/(r_2 - r_1)$, where θ_1 is the temperature of the steam; at the exit end the gradient is $(\theta_1 - \theta_3)/(r_2 - r_1)$.

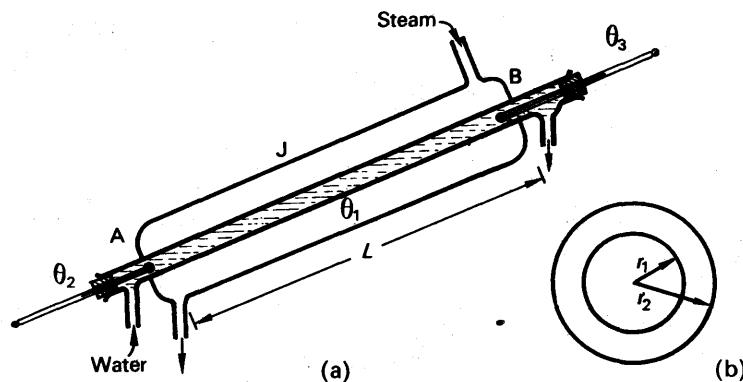


FIG. 13.7. Apparatus to measure conductivity of glass in form of tubing.

If θ_3 and θ_2 differ by not more than about 10°C , we may take the gradient as constant and equal to its average value:

$$\begin{aligned} \text{temperature-gradient} &= \frac{1}{2} \left(\frac{\theta_1 - \theta_2}{r_2 - r_1} + \frac{\theta_1 - \theta_3}{r_2 - r_1} \right) \\ &= \frac{\theta_1 - \frac{\theta_2 + \theta_3}{2}}{r_2 - r_1} \end{aligned}$$

The conductivity k therefore is given by

$$k \times 2\pi L \frac{r_1 + r_2}{2} \times \frac{\theta_1 - \frac{\theta_2 + \theta_3}{2}}{r_2 - r_1} = mc_w(\theta_3 - \theta_2) \quad . \quad (4)$$

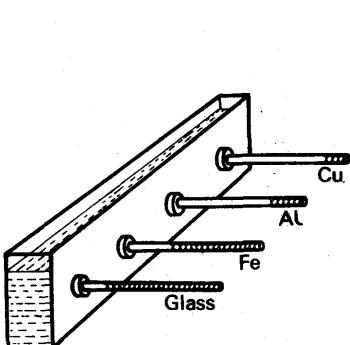
The conductivity of *rubber tubing* can be found by a modification of this method. A measured length of the tubing is submerged in a calorimeter of water, and steam passed through for a measured time t . The rise in temperature of the water must be corrected for cooling, as in the measurement of the specific heat of a bad conductor (p. 203). The heat flow through the rubber is given by the left-hand side of equation (4) with θ_2 and θ_3 standing for the initial and final temperature of the

water. If m is the mass of water, and C the heat capacity of the calorimeter, the right-hand side of the equation is

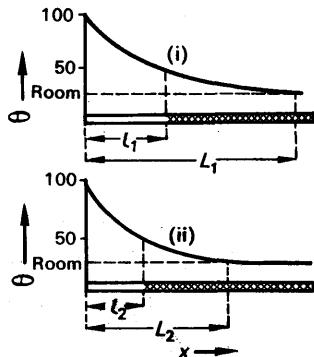
$$(mc_w + C)(\theta_3 - \theta_2)/t.$$

Comparison of Conductivities

Fig. 13.8 (a) shows an apparatus due to Ingenhousz for comparing the conductivities of solids. Metal, wood, glass, and other rods, of equal lengths and cross-sections, are stuck into a tank through corks. The rods are painted with the same paint, and coated with wax over



(a) Apparatus



(b) Theory

FIG. 13.8. Comparison of conductivities.

their whole projecting lengths. The tank is filled with water, leaks are stopped as well as possible, and the water is boiled. As the rods warm up, the wax melts off them. Eventually a steady state is reached, and the best conductor is the one from which wax has melted off the greatest length.

If the experiment is to give a quantitative comparison of the conductivities, the rods must be so long that the far end of each of them is at room temperature. Otherwise the following argument will not be true.

At the points where melted wax gives way to solid, each bar is at the melting-point of wax—let us call it 50°C. The temperature distributions along any two bars are therefore as shown in Fig. 13.8 (b); they have similar shapes, but at a given distance from the tank, (ii) is steeper than (i). If l_1 is the distance along (i) to the 50°C point, and l_2 the corresponding distance along (ii), then the curve (ii) is the same as curve (i) except that it is horizontally contracted in the ratio l_2/l_1 . Therefore the gradient of (ii), at any distance x , is steeper than that of (i) in the ratio l_1/l_2 . The temperature gradient at the tank end of each rod determines the rate at which heat flows into it from the hot water.

Now $Q \propto k \times \text{temp. grad. at hot end}$,

where k is the conductivity of the rod. Therefore, for rods (i) and (ii)

$$\begin{aligned} \frac{Q_1}{Q_2} &= \frac{k_1}{k_2} \times \frac{\text{temp. grad. at end of (i)}}{\text{temp. grad. at end of (ii)}} \\ &= \frac{k_1}{k_2} \times \frac{l_2}{l_1} \end{aligned} \quad (5)$$

The heat passing into a rod at the hot end escapes by convection from its sides. The lengths l_1 and l_2 respectively, of rods (i) and (ii), have the same average temperatures; 75°C. Over the lengths l_1 and l_2 , therefore, each rod loses heat at the same rate per unit area, since each has the same surface (p. 203). The heat lost from either rod per second, *between the 100°C and the 50°C points*, is therefore proportional to the area of the rod between those points. It is therefore proportional to the distance l between them.

Since the temperature curves differ only in scale, the distance to the 50°C point on either rod is proportional to the distance L to the point where the rod reaches room temperature (Fig. 13.8 (b)). Beyond this point, the bar loses no heat. Therefore, by the above argument, the distance L is proportional to the *total* heat lost per second by the bar. The distance L is therefore proportional to the total heat per second lost by the bar, and this heat is the heat Q entering the bar at the hot end.

Therefore

$$Q \propto L \propto l$$

or

$$\frac{Q_1}{Q_2} = \frac{L_1}{L_2} = \frac{l_1}{l_2}$$

But we have seen that

$$\frac{Q_1}{Q_2} = \frac{k_1 l_2}{k_2 l_1} \quad \quad (5)$$

Therefore

$$\frac{k_1 l_2}{k_2 l_1} = \frac{l_1}{l_2}$$

or

$$\frac{k_1}{k_2} = \frac{l_1^2}{l_2^2}$$

Thus the conductivity of a given rod is proportional to the square of the distance along it to the melting-point of the wax.

The Cracking of Glass

Glass is a bad conductor of heat. Therefore, when a piece of glass is heated in one place, the neighbouring parts of the glass do not at first warm up with it. Consequently they resist the expansion of the heated part, and the force set up cracks the glass (p. 262). To avoid cracking the glass, care must be taken to warm the whole region around the place to be made hot. Similarly, glass which has been heated must be made to cool slowly and uniformly by playing the flame over it now and then, for shorter and shorter times as it cools.

EXAMPLES

- Calculate the quantity of heat conducted through 2 m² of a brick wall 12 cm thick in 1 hour if the temperature on one side is 8°C and on the other side is 28°C. (Thermal conductivity of brick = 0.13 W m⁻¹ K⁻¹.)

$$\text{Temperature gradient} = \frac{28 - 8}{12 \times 10^{-2}} \text{ °C m}^{-1}$$

Since 1 hour = 3600 seconds,

$$\begin{aligned} \therefore Q &= kAt \times \text{temperature gradient} \\ &= 0.13 \times 2 \times 3600 \times \frac{28 - 8}{12 \times 10^{-2}} \text{ joules} \\ &= 156000 \text{ J.} \end{aligned}$$

2. Define *thermal conductivity*. Describe and give the theory of a method of measuring the thermal conductivity of copper.

A sheet of rubber and a sheet of cardboard, each 2 mm thick, are pressed together and their outer faces are maintained respectively at 0°C and 25°C. If the thermal conductivities of rubber and cardboard are respectively 0.13 and 0.05 W m⁻¹ K⁻¹, find the quantity of heat which flows in 1 hour across a piece of the composite sheet of area 100 cm². (L.)

First part. The thermal conductivity of a substance is the quantity of heat per second flowing in the steady state through opposite faces of a unit cube of the material when a temperature difference of 1 degree is maintained across these faces. The thermal conductivity of copper can be measured by Searle's method (p. 335).

Second part. We must first find the temperature, θ°C, of the junction of the rubber and cardboard. The temperature gradient across the rubber = (θ - 0)/2 × 10⁻³; the temperature gradient across the cardboard = (25 - θ)/2 × 10⁻³.

$$\therefore Q \text{ per second per m}^2 \text{ across rubber} = 0.13 \times (\theta - 0) / 2 \times 10^{-3}$$

$$\text{and } Q \text{ per second per m}^2 \text{ across cardboard} = 0.05 \times (25 - \theta) / 2 \times 10^{-3}.$$

But in the steady state the quantities of heat above are the same.

$$\therefore \frac{0.13(\theta - 0)}{2 \times 10^{-3}} = \frac{0.05(25 - \theta)}{2 \times 10^{-3}}$$

$$\therefore 13\theta = 125 - 5\theta$$

$$\therefore \theta = \frac{125}{18} = 7^\circ\text{C}.$$

$$\text{Now area} = 100 \text{ cm}^2 = 100 \times 10^{-4} \text{ m}^2.$$

$$\therefore Q \text{ through area in 1 hour (3600 seconds)}$$

$$= \frac{0.13 \times 100 \times 10^{-4} \times 7 \times 3600}{2 \times 10^{-3}} \\ = 16380 \text{ J.}$$

3. Define *thermal conductivity* and explain how you would measure its value for a poorly conducting solid.

In order to minimise heat losses from a glass container, the walls of the container are made of two sheets of glass, each 2 mm thick, placed 3 mm apart, the intervening space being filled with a poorly conducting solid. Calculate the ratio of the rate of conduction of heat per unit area through this composite wall to that which would have occurred had a single sheet of the same glass been used under the same internal and external temperature conditions. (Assume that the thermal conductivity of glass and the poorly conducting solid = 0.63 and 0.049 W m⁻¹ K⁻¹ respectively. (L.)

First part.

Second part. Let θ_1, θ_4 be the respective temperatures of the outer faces of the two glass sheets, and θ_2, θ_3 their respective junction temperatures with the solid between them. The thickness of glass = 2×10^{-3} m, that of the solid = 0.3×10^{-2} m. In the steady state, the quantity of heat per second per metre² is the same for each. Call this Q_1 . Then, for the first glass, since $Q_1 = k \times \text{temperature gradient}$,

$$Q_1 = 0.63 \times \frac{\theta_1 - \theta_2}{2 \times 10^{-3}}$$

$$\therefore \theta_1 - \theta_2 = Q_1 \times \frac{2 \times 10^{-3}}{0.63} = \frac{2}{630} Q_1 \quad \quad (i)$$

Similarly, for the solid,

$$\theta_2 - \theta_3 = Q_1 \times \frac{3 \times 10^{-3}}{0.049} = \frac{3}{49} Q_1 \quad \text{(ii)}$$

For the second glass,

$$\theta_3 - \theta_4 = Q_1 \times \frac{2 \times 10^{-3}}{0.63} = \frac{2}{630} Q_1 \quad \text{(iii)}$$

Adding (i), (ii), (iii) to eliminate θ_2 and θ_3 ,

$$\therefore \theta_1 - \theta_4 = Q_1 \left(\frac{4}{630} + \frac{3}{49} \right) \quad \text{(1)}$$

For a single sheet of glass and the same internal and external temperatures θ_1 and θ_4 respectively, the quantity of heat per second per metre², Q_2 say, is given, from (i), by

$$\theta_1 - \theta_4 = Q_2 \times \frac{2}{630} \quad \text{(2)}$$

Hence, from (1), $Q_1 \left(\frac{4}{630} + \frac{3}{49} \right) = Q_2 \times \frac{2}{630}$

Simplifying, $\therefore \frac{Q_1}{Q_2} = \frac{14}{298} = 0.05$ (approx.).

RADIATION

Radiation

All heat comes to us, directly or indirectly, from the sun. The heat which comes directly travels through 150 million km of space, mostly empty, and travels in straight lines, as does the light: the shade of a tree coincides with its shadow. Both heat and light travel with the same speed because they are cut off at the same instant in an eclipse. Since light is propagated by waves of some kind we conclude that the heat from the sun is propagated by similar waves, and we say it is 'radiated'.

As we show later, radiation is more copious from a dull black body than from a transparent or polished one. Black bodies are also better absorbers of radiation than polished or transparent ones, which either allow radiation to pass through themselves, or reflect it away from themselves. If we hold a piece of white card, with a patch of black drawing ink on it, in front of the fire, the black patch soon comes to feel warmer than its white surround.

Reflection and Refraction

If, with either a convex lens or a concave mirror, we focus the sun's light on our skin, we feel heat at the focal spot. The heat from the sun has therefore been reflected or refracted in the same way as the light.

If we wish to show the reflection of heat unaccompanied by light, we may use two searchlight mirrors, set up as in Fig. 13.9. At the focus of

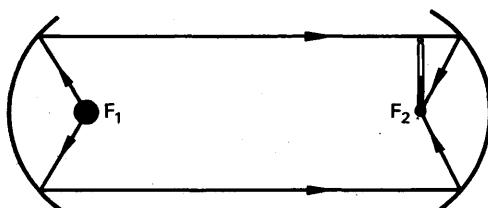


FIG. 13.9. Reflection of radiant heat.

one, F_1 , we put an iron ball heated to just below redness. At the focus of the other, F_2 , we put the bulb of a thermometer, which has been blackened with soot to make it a good absorber (p. 349). The mercury rises in the stem of the thermometer. If we move either the bulb or the ball away from the focus, the mercury falls back; the bulb has therefore been receiving heat from the ball, by reflection at the two mirrors. We can show that the foci of the mirrors are the same for heat as for light if we replace the ball and thermometer by a lamp and screen. (In practice we do this first, to save time in finding the foci for the main experiment.)

To show the refraction of heat apart from the refraction of light is more difficult. It was first done by the astronomer Herschel in 1800. Herschel passed a beam of sunlight through a prism, as shown diagrammatically in Fig. 13.10, and explored the spectrum with a sensitive

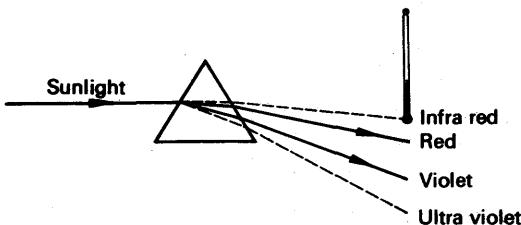


FIG. 13.10. Infra-red and ultra-violet (diagrammatic).

thermometer, whose bulb he had blackened. He found that in the visible part of the spectrum the mercury rose, showing that the light energy which it absorbed was converted into heat. But the mercury also rose when he carried the bulb into the darkened portion a little beyond the red of the visible spectrum; the sun's rays therefore carried energy which was not light.

Ultra-violet and Infra-red

The radiant energy which Herschel found beyond the red is now called *infra-red* radiation, because it is less refracted than the red. Radiant energy is also found beyond the violet and it is called *ultra-violet* radiation, because it is refracted more than the violet.

Ultra-violet radiation is absorbed by the human skin and causes sun-burn; more importantly, it stimulates the formation of vitamin D, which is necessary for the assimilation of calcium and the prevention of rickets. It is also absorbed by green plants; in them it enables water to combine with carbon dioxide to form carbohydrates. This process is called photo-synthesis; we have already, on p. 196, discussed its importance to animals and man. Ultra-violet radiation causes the emission of electrons from metals, as in photo-electric cells; and it excites a latent image on a photographic emulsion. It is harmful to the eyes.

Ultra-violet radiation is strongly absorbed by glass—spectacle-wearers do not sunburn round the eyes—but enough of it gets through

to affect a photographic film. It is transmitted with little absorption by quartz.

Infra-red radiation is transmitted by quartz, and rock-salt, but most of it is absorbed by glass. A little, that which lies near the visible red, passes fairly easily through glass—if it did not, Herschel would not have discovered it. When infra-red radiation falls on the skin, it gives the sensation of warmth. It is what we usually have in mind when we speak of heat radiation, and it is the main component of the radiation from a hot body; but it is in no essential way different from the other components, visible and ultra-violet radiation, as we shall now see.

Wavelengths of Radiation

In books on Optics, it is shown how the wavelength of light can be measured with a diffraction grating—a series of fine close lines ruled on glass. The wavelength ranges from 4000×10^{-10} m for the violet, to 7500×10^{-10} m for the red. The first accurate measurements of wavelength were published in 1868 by Angstrom, and in his honour a distance of 10^{-10} m is called an *Angstrom unit* (A.U.). The wavelengths of infra-red radiation can be measured with a grating made from fine wires stretched between two screws of close pitch. They range from 7500 A.U. to about 1 000 000 A.U. Often they are expressed in a longer unit than the Angstrom : this unit is the micron (μm), which is $1/1000$ mm. Thus

$$1\mu\text{m} = 10^{-6} \text{ m} = 10^4 \text{ A.U.}$$

We denote wavelength by the symbol λ ; its value for visible light ranges from $0.4 \mu\text{m}$ to $0.75 \mu\text{m}$, and for infra-red radiation from $0.75 \mu\text{m}$ to about $100 \mu\text{m}$.

We now consider that X-rays and radio waves also have the same nature as light, and that so do the γ -rays from radio-active substances.

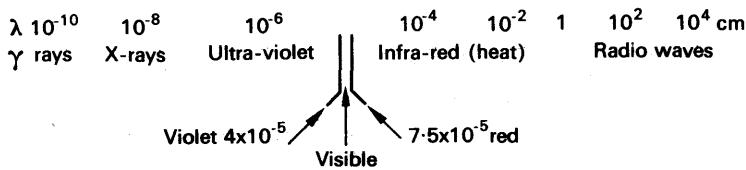


FIG. 13.11. The electromagnetic spectrum.

For reasons which we cannot here discuss, we consider all these waves to be due to oscillating electric and magnetic fields. Fig. 13.11 shows the range of their wavelengths : it is called a diagram of the *electromagnetic spectrum*.

Detection of Heat Radiation

A thermometer with a blackened bulb is a sluggish and insensitive detector of radiant heat. More satisfactory detectors, however, are



FIG. 13.12. Bolometer strip.

instrument was devised by Langley in 1881; it is called a bolometer, *bole* being Greek for a ray.

The other, commoner, type of radiation detector is called a thermopile (Nobili and Melloni, c. 1830). Its action depends on the electro-motive force, which appears between the junctions of two different

metals, when one junction is hot and the other cold. The modern thermopile is due to Coblenz (1913). It consists of many junctions between fine wires, as shown diagrammatically in Fig. 13.13; the wires are of silver and bismuth, 0·1 mm or less in diameter. Their junctions are attached to thin discs of tin, about 0·2 mm thick, and about 1 mm square. One set of discs is blackened and mounted behind a slit, through which radiation can fall on them; the junctions attached to them become the hot junctions of the thermopile. The other, cold, junctions are shielded from the radiation to

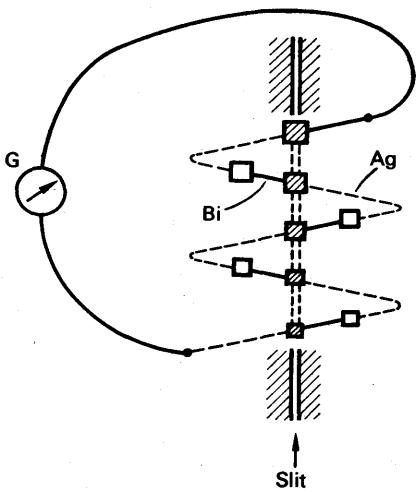


FIG. 13.13. Thermopile.

be measured; the discs attached to them help to keep them cool, by increasing their surface area.

Older types of thermopile are made from bars of metal about a millimetre thick. They are slow to warm up when radiation falls upon them, but are more rugged than the modern type.

When radiation falls on the blackened discs of a thermopile, it warms the junctions attached to them, and sets up an e.m.f. This e.m.f. can be measured with a potentiometer, or, for less accurate work, it can be used to deflect a galvanometer, G, connected directly to the ends of the thermopile (Fig. 13.13).

Reflection and Refraction observed with Thermopile: Inverse Square Law

With a thermopile and galvanometer, we can repeat Herschel's experiment more strikingly than with a thermometer. And with the

simple apparatus of Fig. 13.13 we can show that, when heat is reflected, the angle of reflection is equal to the angle of incidence. We can also show the first law of reflection; that the incident and reflected rays are in the same plane as the normal to the reflector at the point of incidence.

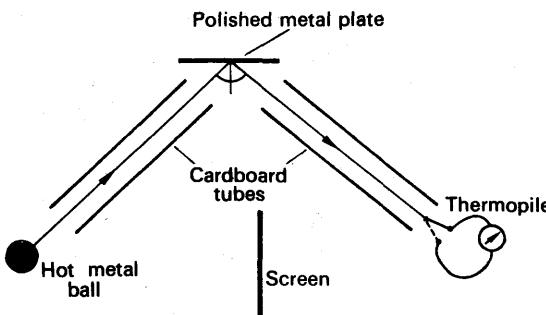


FIG. 13.14. Demonstration of reflection.

If heat is radiant energy, its intensity should fall off as the inverse square of the distance from a point source. We can check that it does so by setting up an electric lamp, with a compact filament, in a dark room preferably with black walls. When we put a thermopile at different distances from the lamp, the deflection of the galvanometer is found to be inversely proportional to the square of the distance.

If we wish to do this experiment with radiation that includes no visible light, we must modify it. Instead of the lamp, we use a large blackened tank of boiling water, A, and we fit the thermopile, B, with a conical mouthpiece, blackened on the inside. The blackening prevents any radiation from reaching the pile by reflection at the walls of the mouthpiece. We now find that the deflection of the galvanometer, G, does not vary with the distance of the pile from the tank, *provided that the tank occupies the whole field of view of the cone* (Fig. 13.15). The area S of the tank from which radiation can reach the thermopile is then proportional to the square of the distance d . And since the deflection is unchanged when the distance is altered, the total radiation from each element of S must therefore fall off as the inverse square of the distance d .

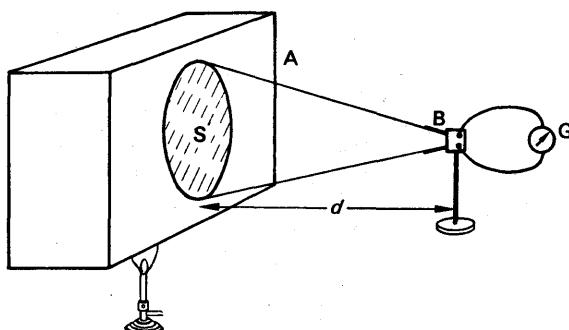


FIG. 13.15. Proof of inverse square law.

The Infra-red Spectrometer

Infra-red spectra are important in the study of molecular structure. They are observed with an infra-red spectrometer, whose principle is shown in Fig. 13.16. Since glass is opaque to the infra-red, the radiation is focused by concave mirrors instead of lenses; the mirrors are plated with copper or gold on their front surfaces. The source of light is a Nernst filament, a metal filament coated with alkaline-earth oxides, and heated electrically.

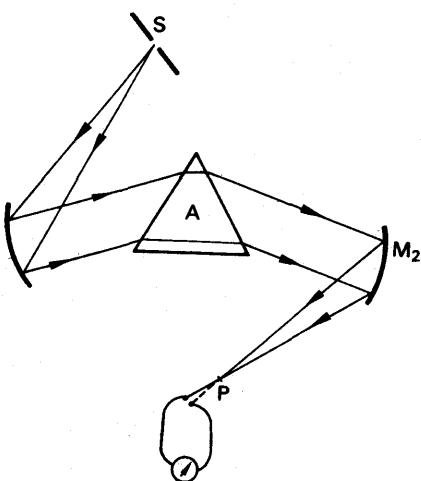


FIG. 13.16. Infra-red spectrometer.

places the telescope of an optical spectrometer. Rotating the prism brings different wavelengths on to the slit; the position of the prism is calibrated in wavelengths with the help of a grating.

To a fair approximation, the deflection of the galvanometer is proportional to the radiant power carried in the narrow band of wavelengths which fall on the thermopile. If an absorbing body, such as a solution of an organic compound, is placed between the source and the slit, it weakens the radiation passing through the spectrometer, in the wavelengths which it absorbs. These wavelengths are therefore shown by a fall in the galvanometer deflection.

Reflection, Transmission, Absorption

Measurements whose description is outside our scope give the amount of radiant energy approaching the earth from the sun. At the upper limit of our atmosphere, it is about $8.0 \text{ J cm}^{-2} \text{ min}^{-1}$.

At the surface of the earth it is always less than this because of absorption in the atmosphere. Even on a cloudless day it is less, because the ozone in the upper atmosphere absorbs much of the ultra-violet.

In Fig. 13.17, XY represents a body on which radiant energy is falling. The symbol I denotes the latter's intensity: to fix our ideas we may take

$$I = 4.0 \text{ joule per } \text{cm}^2 \text{ per minute}$$

$$= \frac{4.0}{60} \text{ joule per } \text{cm}^2 \text{ per second}$$

$$= 0.067 \text{ watt per } \text{cm}^2.$$

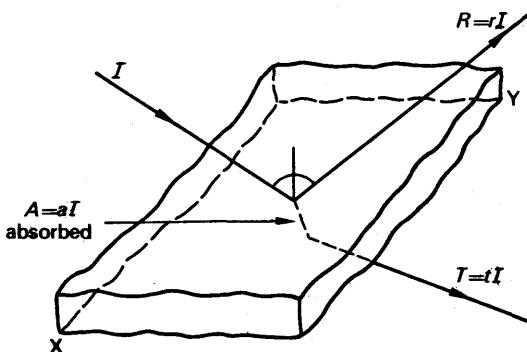


FIG. 13.17. Reflection, transmission, and absorption.

Some of this energy is reflected by the glass (R), some is absorbed (A), and some is transmitted (T). The total energy transmitted, absorbed and reflected per cm^2 per second is equal to the energy falling on the body over the same area and in the same time:

$$T + A + R = I.$$

If we denote by t , a , and r , the fractions of energy which are respectively transmitted, absorbed, and reflected by the body, then

$$tI + aI + rI = I$$

or

$$t + a + r = 1 \quad \quad (6)$$

This equation expresses common knowledge: if a body is transparent ($t \rightarrow 1$), it is not opaque, and it is not a good reflector ($a \rightarrow 0, r \rightarrow 0$). But also, if the body is a good absorber of radiation ($a \rightarrow 1$), it is not transparent, and its surface is dull ($t \rightarrow 0, r \rightarrow 0$). And if it is a good reflector ($r \rightarrow 1$), it is neither transparent nor a good absorber ($t \rightarrow 0, a \rightarrow 0$). The term opaque, as commonly used, simply means not transparent; we see that it does not necessarily mean absorbent.

Equation (6), as we have written it above, is over-simplified. For a body may transmit some wavelengths (colours, if visible) and absorb or reflect others. If we now let I denote the intensity of radiation of a particular wavelength λ , then by repeating the argument we get

$$t_\lambda + a_\lambda + r_\lambda = 1 \quad \quad (7)$$

where the coefficients t_λ , etc., all refer to the wavelength λ .

The truth of equation (7) is well shown by the metal gold, which reflects yellow light better than other colours. In thin films, gold is partly transparent, and the light which it transmits is green. Green is the colour complementary to yellow; gold removes the yellow from white light by reflection, and passes on the rest by transmission.

Radiation and Absorption

We have already pointed out that black surfaces are good absorbers and radiators of heat, and that polished surfaces are bad absorbers and radiators. This can be demonstrated by the apparatus in Fig. 13.18, in which is a cubical metal tank whose sides have a variety of finishes: dull black, dull white, highly polished. It contains boiling water, and, therefore, has a constant temperature. Facing it is a thermopile, P,

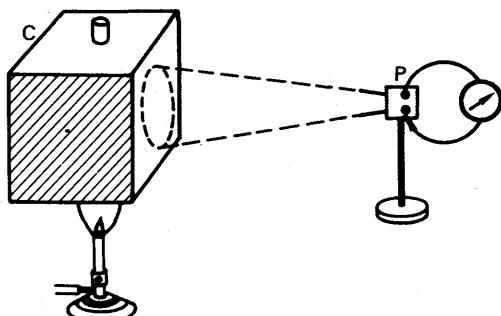


FIG. 13.18. Comparing radiators.

which is fitted with the blackened conical mouth-piece described on p. 347.

Provided that the face of the cube occupies the whole field of view of the cone, its distance from the thermopile does not matter (p. 347). The galvanometer deflection is greatest when the thermopile is facing the dull black surface of the cube, and least when it is facing the highly polished surface. The highly polished surface is therefore the worst radiator of all, and the dull black is the best.

This experiment was first done by Leslie in 1804. There were no thermopiles in those days, and Leslie detected the radiant heat with an instrument depending on the expansion of air, which we shall not describe. The tank with different surfaces is called *Leslie's cube*.

Leslie's cube can also be used in an experiment to compare the absorbing properties of surfaces, due to Ritchie (1833). A modern version of it is shown in Fig. 13.19. The cube C, full of boiling water, is placed between two copper plates, A, B, of which A is blackened and B is polished. The temperature difference between A and B is measured by making each of them one element in a thermo junction: they are joined by a constantan wire, XY, and connected to a galvanometer, by copper wires, AE, DB. If A is hotter than B, the junction, X, is hotter than the junction, Y, and a current flows through the galvanometer in one direction. If B is hotter than A, the current is reversed.

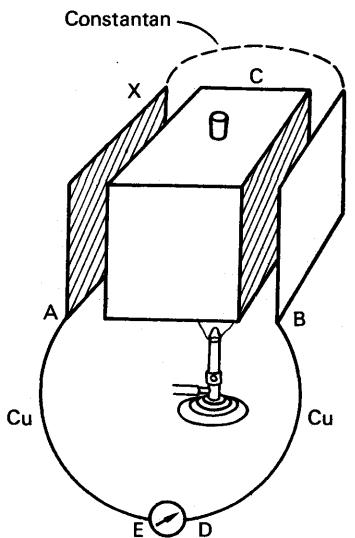


FIG. 13.19. Comparing absorbers.

The most suitable type of Leslie's cube is one which has two opposite faces similar—say grey—and the other two opposite faces very dissimilar—one black, one polished. At first the plates A, B are set

opposite similar faces. The blackened plate, A, then becomes the hotter, showing that it is the better absorber.

The cube is now turned so that the blackened plate, A, is opposite the polished face of the cube, while the polished plate, B, is opposite the blackened face of the cube. The galvanometer then shows no deflection; the plates thus reach the same temperature. It follows that the good radiating property of the blackened face of the cube, and the bad absorbing property of the polished plate, are just compensated by the good absorbing property of the blackened plate, and the bad radiating property of the polished face of the cube.

The Thermos Flask

A thermos flask—sometimes called a *Dewar flask* after its inventor (c. 1894)—is a device for reducing the transfer of heat to a minimum.

It consists of a double walled glass vessel, as shown in Fig. 13.20; the space between the walls is exhausted to as high a vacuum as possible, and the insides of the walls are silvered. Silvered surfaces are good reflectors, but bad absorbers and radiators. Heat therefore passes very slowly from the outer wall to the inner by radiation. If the vacuum is good, convection is almost inhibited—the goodness of the vacuum determines the goodness of the flask. Conduction through the glass is slight, because the conduction paths are long. In a good flask, the main cause of heat loss is conduction through the cork.

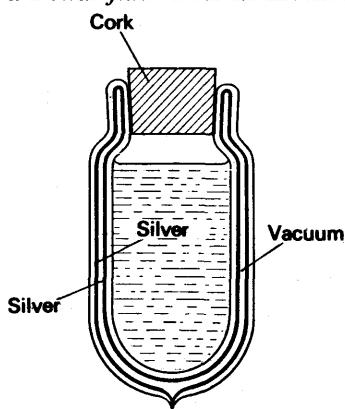


FIG. 13.20. A thermos flask.

The Black Body

The experiments described before lead us to the idea of a perfectly black body; one which absorbs all the radiation that falls upon it, and reflects and transmits none. The experiments also lead us to suppose that such a body would be the best possible radiator.

A perfectly black body can be very nearly realized—a good one can be made in half a minute, simply by punching a small hole in the lid of an empty tin. The hole looks almost black, although the shining tin is a good reflector. The hole looks black because the light which enters through it is reflected many times round the walls of the tin, before it meets the hole again (Fig. 13.21). At each reflection, about 80 per cent of the light energy is reflected, and 20 per cent is absorbed. After two reflections, 64 per cent of the

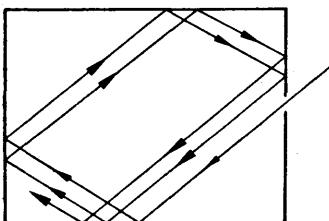


FIG. 13.21. Multiple reflections make a black body.

original light goes on to be reflected a third time; 36 per cent has been absorbed. After ten reflections, the fraction of the original energy which has been absorbed is 0.8^{10} , or 0.1.

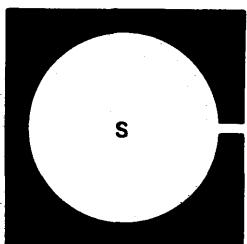


FIG. 13.22. A black body.

Any space which is almost wholly enclosed approximates to a black body. And, since a good absorber is also a good radiator, an almost closed space is the best radiator we can find.

A form of black-body which is used in radiation measurements is shown in Fig. 13.22. It consists of a porcelain sphere, S, with a small hole in it. The inside is blackened with soot to make it as good a radiator and as bad a reflector as possible. (The effect of multiple reflections is then to convert the body from nearly black to very nearly black indeed.)

The sphere is surrounded by a high-temperature bath of, for example, molten salt (the melting-point of common salt is 801°C).

The deepest recesses of a coal or wood fire are black bodies. Anyone who has looked into a fire knows that the deepest parts of it look brightest—they are radiating most power. Anyone who has looked into a fire also knows that, in the hottest part, no detail of the coals or wood can be seen. That is to say, the radiation from an almost enclosed space is uniform; its character does not vary with the nature of the surfaces of the space. This is so because the radiation coming out from any area is made up partly of the radiation emitted by that area, and partly of the radiation from other areas, reflected at the area in question. If the surface of the area is a good radiator, it is a bad reflector, and vice-versa. And if the hole in the body is small, the radiations from every area inside it are well mixed by reflection before they can escape; the intensity and quality of the radiation escaping thus does not depend on the particular surface from which it escapes.

When we speak of the *quality* of radiation we mean the relative intensities of the different wavelengths that it comprises; the proportion of red to blue, for example. The quality of the radiation from a perfectly black body depends only on its temperature. When the body is made hotter, its radiation becomes not only more intense, but also more nearly white; the proportion of blue to red in it increases. Because its quality is determined only by its temperature, black-body radiation is sometimes called ‘temperature radiation’.

Properties of Temperature Radiation

The quality of the radiation from a black body was examined by Lummer and Pringsheim in 1899. They used a black body represented by B in Fig. 13.23 and measured its temperature with a thermocouple; they took it to 2000°C . To measure the intensities of the various wavelengths, Lummer and Pringsheim used an infra-red spectrometer and a bolometer (p. 346) consisting of a single platinum strip.

The results of experiments such as these are shown in Fig. 13.24 (a). Each curve gives the relative intensities of the different wavelengths,

for a given temperature of the body. The curves show that, as the temperature rises, the intensity of every wavelength increases, but the intensities of the shorter wave lengths increase more rapidly. Thus the radiation becomes, as we have already observed, less red, that is to say, more nearly white. The curve for sunlight has its peak at about 5000 A.U., in the visible green; from the position of this peak we conclude that the surface temperature of the sun is about 6000 K. Stars which are hotter than the sun, such as Sirius and Vega, look blue, not white.

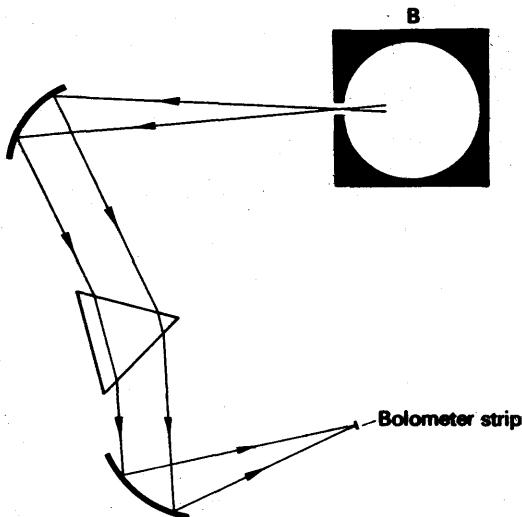


FIG. 13.23. Lummer and Pringsheim's apparatus for study of black body radiation (diagrammatic).

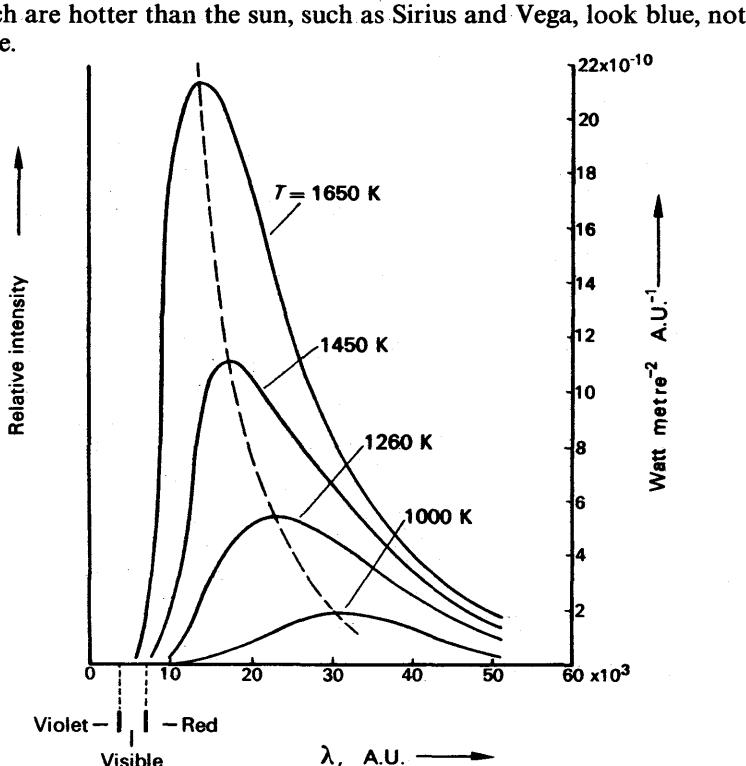


FIG. 13.24 (a). Distribution of intensity in black-body radiation.

because the peaks of their radiation curves lie further towards the visible blue than does the peak of sunlight.

The actual intensities of the radiations are shown on the right of the graph in Fig. 13.24 (a). To speak of the intensity of a single wavelength is meaningless because there is an infinite number of wavelengths, but the total intensity of the radiation is finite. The slit of the spectrometer always gathers a band of wavelengths—the narrower the slit the narrower the band—and we always speak of the intensity of a given band. We express it as follows ('s' represents 'second'):

$$\text{energy radiated } \text{m}^{-2} \text{s}^{-1}, \text{ in band } \lambda \text{ to } \lambda + \delta\lambda = E_\lambda \delta\lambda. \quad (8)$$

The quantity E_λ is called the *emissive power* of a black body for the wavelength λ and at the given temperature; its definition follows from equation (8):

$$E_\lambda = \frac{\text{energy radiated } \text{m}^{-2} \text{s}^{-1}, \text{ in band } \lambda \text{ to } \lambda + \delta\lambda}{\text{bandwidth, } \delta\lambda}.$$

The expression 'energy per second' can be replaced by the word 'power', whose unit is the watt. Thus

$$E_\lambda = \frac{\text{power radiated } \text{m}^{-2} \text{ in band } \lambda, \lambda + \delta\lambda}{\delta\lambda}.$$

In the figure E_λ is expressed in watt per m^2 per Angstrom unit. SI units may be 'watt per metre² per nanometre (10^{-9} m)'.

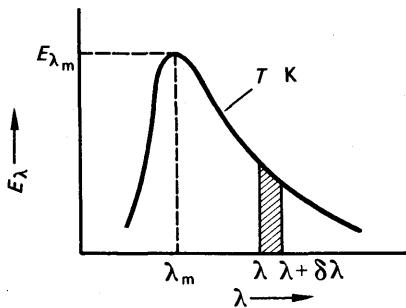


FIG. 13.24 (b). Definition of E , λ_m , and E_{λ_m} .

The quantity $E_\lambda \delta\lambda$ in equation (8) is the area beneath the radiation curve between the wavelengths λ and $\lambda + \delta\lambda$ (Fig. 13.24 (b)). Thus the energy radiated per cm^2 per second between those wavelengths is proportional to that area. Similarly the total radiation emitted per cm^2 per second over all wavelengths is proportional to the area under the whole curve.

Laws of Black Body Radiation

The curves of Fig. 13.24 (a) can be explained only by the quantum theory of radiation, which is outside our scope. Both theory and experiment lead to three generalizations, which together describe well the properties of black-body radiation:

(i) If λ_m is the wavelength of the peak of the curve for T K, then

$$\lambda_m T = \text{constant} \dots \dots \dots \quad (9)$$

The value of the constant is 2.9×10^{-3} m K. In Fig. 13.24 (a) the dotted line is the locus of the peaks of the curves for different temperatures.

(ii) If E_{λ_m} is the height of the peak of the curve for the temperature T K, then

$$E_{\lambda_m} \propto T^5 \quad \quad (10)$$

The relationships (10) and (9) are particular cases of a general law given by Wien in 1894; (9) is sometimes called *Wien's displacement law*.

(iii) If E is the total energy radiated per metre² per second at a temperature T , represented by the area under the curve, then

$$E = \sigma T^4,$$

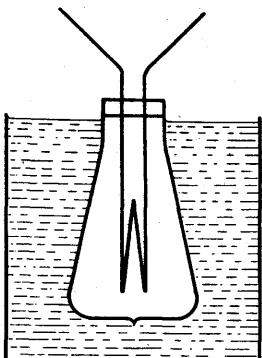
where σ is a constant. This result is called **Stefan's law**, and the constant σ is called **Stefan's constant**. Its value is

$$\sigma = 5.7 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}.$$

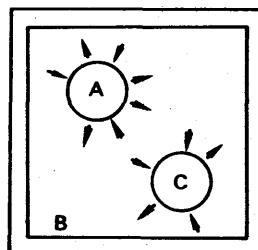
Prévost's Theory of Exchanges

In 1792 Prévost applied the idea of dynamic equilibrium to radiation. He asserted that a body radiates heat at a rate which depends only on its surface and its temperature, and that it absorbs heat at a rate depending on its surface and the temperature of its surroundings. When the temperature of a body is constant, the body is losing heat by radiation, and gaining it by absorption, at equal rates.

It is easy to think of experiments which seem to support Prévost's theory, and the reader will certainly grasp the general idea of it if he imagines hot pies and cold ice-creams put into the same cupboard. But in such experiments it is difficult to get rid of the possibility of convection. Let us rather take an old-fashioned, high vacuum, electric lamp, and put it in a can of water (Fig. 13.25 (a)). We can find the temperature of the lamp's filament by measuring its resistance. We find that, whatever the temperature of the water, the filament comes to that temperature, if we leave it long enough. When the water is cooler than the filament, the filament cools down; when the water is hotter, the filament warms up.



(a)



(b)

FIG. 13.25. Illustrating Prévost's theory of exchanges.

In the abstract language of theoretical physics, Prévost's theory is easy enough to discuss. If a hot body A (Fig. 13.25 (b)) is placed in an evacuated enclosure B, at a lower temperature than A, then A cools until it reaches the temperature of B. If a body C, cooler than B, is put in B, then C warms up to the temperature of B. We conclude that radiation from B falls on C, and therefore also on A, even though A

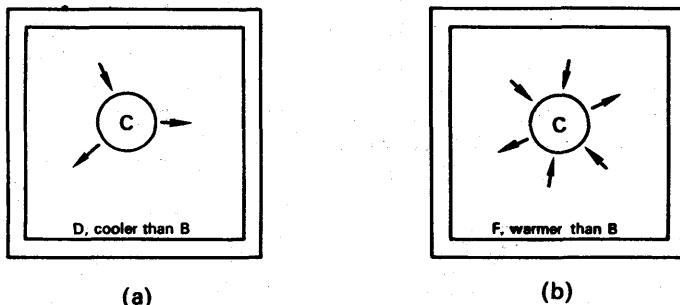


FIG. 13.26. Illustrating Prévost's theory.

is at a higher temperature. Thus A and C each come to equilibrium at the temperature of B when each is absorbing and emitting radiation at equal rates.

Now let us suppose that, after it has reached equilibrium with B, one of the bodies, say C, is transferred from B to a cooler evacuated enclosure D (Fig. 13.26 (a)). It loses heat and cools to the temperature of D. Therefore it is radiating heat. But if C is transferred from B to a warmer enclosure F, then C gains heat and warms up to the temperature of F (Fig. 13.26 (b)). It seems unreasonable to suppose that C stops radiating when it is transferred to F; it is more reasonable to suppose that it goes on radiating but, while it is cooler than F, it absorbs more than it radiates.

Emissivity

Let us consider a body B, in equilibrium with an enclosure A, at a temperature T K (Fig. 13.27). If the body is perfectly black, it emits radiation characteristics of the temperature T ; let us write the total intensity of this radiation over all wavelengths as E watts/m². Since the body is in equilibrium with the enclosure, it is absorbing as much as it radiates. And since it absorbs all the radiation that falls upon it, the energy falling on it per cm² per second must be equal to E . This conclusion need not surprise us, since the enclosure A is full of black body radiation characteristic of its temperature T .

Now let us consider, in the same enclosure, a body C which is not black. On each square metre of the body's surface, E watts of radiation fall (Fig. 13.27). Of this, let us suppose that the body absorbs a fraction a , that is to say, it absorbs aE watts per m². We may call a the total absorption factor of the body C, 'total' because it refers to the total radiation. The radiation which the body does not absorb, $(1-a)E$, it reflects or transmits.

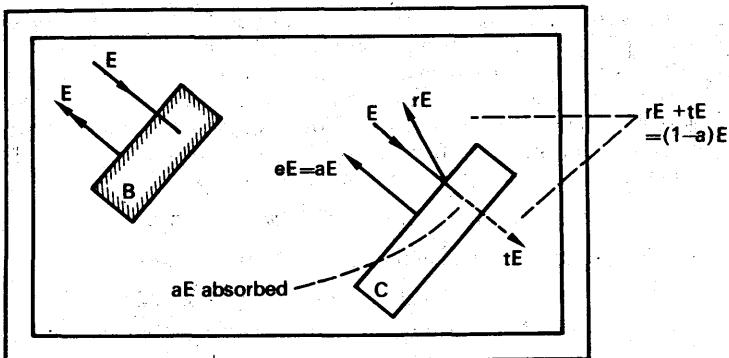


FIG. 13.27. Equilibria in an enclosure.

$$\text{Thus : } \left. \begin{array}{l} \text{power reflected or} \\ \text{transmitted/m}^2 \end{array} \right\} = E - aE.$$

For equilibrium, the total power leaving the body per \$\mathrm{m}^2\$ must be equal to the total power falling upon it, \$E \mathrm{W/m}^2\$. The power emitted by the body, which must be added to that reflected and transmitted, is therefore:

$$\text{total power radiated/m}^2 = aE \quad (12)$$

The ratio of the total power radiated per \$\mathrm{m}^2\$ by a given body, to that emitted by a black body at the same temperature, is called the *total emissivity* of the given body. Hence, by equation (12),

$$e = \frac{aE}{E} = a.$$

We have therefore shown that the total emissivity of a body is equal to its total absorption factor.

This is a formal expression of the results of Ritchie's experiment (p. 350). If we combine it with Stefan's law, we find that the total energy \$E\$ radiated per \$\mathrm{m}^2\$ per second by a body of emissivity \$e\$ at a temperature \$T\$ K is

$$E = eE = e\sigma T^4.$$

Spectral Emissivity; Kirchoff's Law

Most bodies are coloured; they transmit or reflect some wavelengths better than others. We have already seen that they must absorb these wavelengths weakly; we now see that, because they absorb them weakly, they must also

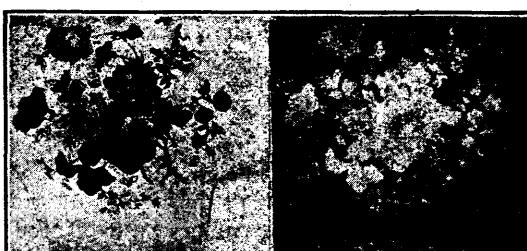


FIG. 13.28. Photographs showing how a piece of incandescent decorated crockery appears (a) by reflected light and (b) by its own emitted light.

radiate them weakly. To show this, we have only to repeat the foregoing argument, but restricting it to a narrow band of wavelengths between λ and $\lambda + \delta\lambda$. The energy falling per m^2 per second on the body, in this band, is $E_\lambda \delta\lambda$ where E_λ is the emissive power of a black-body in the neighbourhood of λ , at the temperature of the enclosure. If the body C absorbs a fraction a_λ of this, we call a_λ the spectral absorption factor of the body, for the wavelength λ . In equilibrium, the body emits as much radiation in the neighbourhood of λ as it absorbs; thus:

$$\text{energy radiated} = a_\lambda E_\lambda \delta\lambda \text{ watts per } m^2.$$

We define the spectral emissivity of the body e_λ , by the equation

$$\begin{aligned} e_\lambda &= \frac{\text{energy radiated by body in range } \lambda, \lambda + \delta\lambda}{\text{energy radiated in same range, by black body at same temperature}} \\ &= \frac{\text{energy radiated by body in range } \lambda, \lambda + \delta\lambda}{E_\lambda \delta\lambda} \\ &= \frac{a_\lambda E_\lambda \delta\lambda}{E_\lambda \delta\lambda}. \end{aligned}$$

Thus

$$e_\lambda = a_\lambda \quad \quad (13)$$

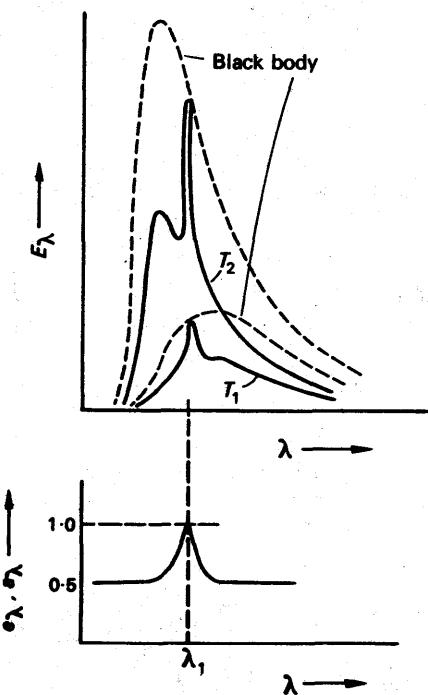


FIG. 13.29. Illustrating Kirchhoff's law of radiation.

tion factor of a coloured body may vary with wavelength, and how its emissive power E_λ does likewise. It is assumed that e_λ rises to unity at the wavelength λ_1 (which is not likely), and that it does not vary

Equation (13) expresses a law due to Kirchhoff:

The spectral emissivity of a body, for a given wavelength, is equal to its spectral absorption factor for the same wavelength.

Kirchhoff's law is not easy to demonstrate by experiment. One reads that a plate, which when cold shows a red pattern on a blue ground, glows blue on a red ground when heated in a furnace. But not all such plates do this, because the spectral emissivities of many coloured pigments vary with their temperature. However, Fig. 13.28 shows two photographs of a piece of pottery, one taken by reflected light at room temperature (left), the other by its own light when heated (right).

Fig. 13.29 illustrates Kirchhoff's law, by showing how the spectral emissivity and absorp-

with the temperature. A body for which e_λ is the same for all wavelengths, but is less than unity, is said to be 'grey'.

Absorption by Gases

An experiment which shows that, if a body radiates a given wavelength strongly, it also absorbs that wavelength strongly, can be made with sodium vapour. A sodium vapour lamp runs at about 220°C ; compared with the sun, or even an arc-lamp, it is cool. The experiment consists of passing sunlight or arc-light through a spectroscope, and observing its continuous spectrum. The sodium lamp is then placed in the path of the light, and a black line appears in the yellow. If the white light is now cut off, the line which looked black comes up brightly—it is the sodium yellow line.

The process of absorption by sodium vapour—or any other gas—is not, however, the same as the process of absorption by a solid. When a solid absorbs radiation, it turns it into heat—into the random kinetic energy of its molecules. It then re-radiates it in all wavelengths, but mostly in very long ones, because the solid is cool. When a vapour absorbs light of its characteristic wavelength, however, its atoms are excited; they then re-radiate the absorbed energy, in the same wavelength (5893 A.U. for sodium). But they re-radiate it in all directions, and therefore less of it passes on in the original direction than before (Fig. 13.30). Thus the yellow component of the original beam is weakened, but the yellow light radiated sideways by the sodium is strengthened. The sideways strengthening is hard to detect, but it was shown by R. W. Wood in 1906. He used mercury vapour instead of sodium. The phenomenon is called *optical resonance*, by analogy with resonance in sound.

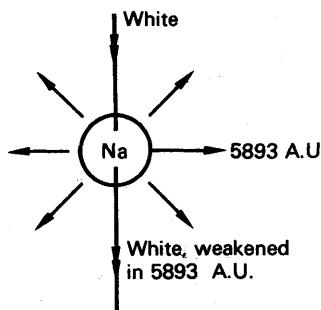


FIG. 13.30. Absorption by sodium vapour.

CONVECTION

Liquids—except mercury, which is a molten metal—are bad conductors of heat. If we hold a test-tube full of water by the bottom, we can boil the water near the top in a Bunsen flame, without any discomfort. But if we hold the tube at the top, and heat it at the bottom, then the top becomes unbearably hot long before the water boils. The heat is brought to the top by *convection*; the warm water at the bottom expands, becomes less dense, and rises; the cold water sinks to take its place. If we heat a beaker of water at one side, and drop in a crystal of potassium permanganate, we can see the currents of hot water rising, and cold descending. Central-heating systems rely on convection to bring hot water from the boiler, in the basement, to the so-called radiators, and to take the cooler water back to the boiler.

The radiators of a central-heating system are wrongly named; they are convectors. They warm the air around them, which rises, and gives way to cooler air from cooler parts of the room. Gases are even worse conductors of heat than liquids, and for most practical purposes we can neglect conduction through them altogether. Woollen clothes keep us warm because they contain pockets of air, which hardly conduct at all, and cannot convect because they cannot move. The wool fibres themselves are much better conductors than the air they imprison.

Convection by air is important in ventilation: the fire in a room maintains a draught of hot air up the chimney, and cool fresh air from outside comes in under the door. The draught also helps to keep the fire supplied with oxygen; factory chimneys are made tall to stimulate convection and increase the draught.

Forced and Free (Natural) Convection

A gas or a liquid may carry away heat from a hot body by convection. If the flow of liquid or gas is simply due to its being heated by the body, and hence rising, the convection is said to be *free*, or natural. But if the gas or liquid is flowing in a stream maintained by some other means, then the convection is said to be *forced*. Thus cooling one's porridge in the obvious way is an example of forced convection; it causes a more rapid loss of heat than does natural convection.

Critical Diameter of Pipes

Hot-water and steam pipes are often lagged with asbestos to reduce the loss of heat from them. The temperature drop across the lagging makes the outside cooler than the pipe, and so, by Newton's law, tends to reduce the rate at which heat escapes from it. However, the lagging increases the outside diameter of the whole, and so increases its area of contact with the atmosphere. The increase in area tends to make convection more vigorous, by enabling the pipe to heat a greater mass of air. If the diameter of the pipe is small, the increase in area may more than offset the reduction temperature of the outside, and so increase the rate of heat loss. Thus there is a critical diameter of pipe; if the diameter is less than the critical value, the pipe should not be lagged. The critical diameter depends on many factors, but is commonly of the order of 1 cm.

The Greenhouse

A greenhouse keeps plants warm by inhibiting convection. The glass allows radiant heat to reach the plants from the sun, but prevents the warm air in the greenhouse from escaping. In winter, when there is little sunshine, the heat is provided by hot water pipes. In summer the temperature is regulated by opening or closing the roof and windows, and so adjusting the loss of heat by convection.

EXERCISES 13

Conduction

1. Define the *thermal conductivity* of a substance. Describe how the thermal conductivity of a metal may be measured, pointing out the sources of error in the experiment.

A large hot-water tank has four steel legs in the form of cylindrical rods 2.5 cm in diameter and 15 cm long. The lower ends of the legs are in good thermal contact with the floor, which is at 20°C, and their upper ends can be taken to be at the temperature of the water in the tank. The tank and the legs are well lagged so that the only heat loss is through the legs. It is found that 22 watts are needed to maintain the tank at 60°C. What is the thermal conductivity of steel? When a sheet of asbestos 1.5 mm thick is placed between the lower end of each leg and the floor only 5 watts are needed to maintain the tank at 60°C. What is the thermal conductivity of asbestos? (O. & C.)

2. Define *thermal conductivity* and state a unit in which it is expressed.

Explain why, in an experiment to determine the thermal conductivity of copper using a Searle's arrangement, it is necessary (a) that the bar should be thick, of uniform cross-section and have its sides well lagged, (b) that the temperatures used in the calculation should be the steady values finally registered by the thermometers.

Straight metal bars X and Y of circular section and equal in length are joined end to end. The thermal conductivity of the material of X is twice that of the material of Y, and the uniform diameter of X is twice that of Y. The exposed ends of X and Y are maintained at 100°C and 0°C, respectively and the sides of the bars are ideally lagged. Ignoring the distortion of the heat flow at the junction, sketch a graph to illustrate how the temperature varies between the ends of the composite bar when conditions are steady. Explain the features of the graph and calculate the steady temperature of the junction. (N.)

3. Give a critical account of an experiment to determine the thermal conductivity of a material of low thermal conductivity such as cork. Why is it that most cellular materials, such as cotton wool, felt, etc., all have approximately the same thermal conductivity?

One face of a sheet of cork, 3 mm thick, is placed in contact with one face of a sheet of glass 5 mm thick, both sheets being 20 cm square. The outer faces of this square composite sheet are maintained at 100°C and 20°C, the glass being at the higher mean temperature. Find (a) the temperature of the glass-cork interface, and (b) the rate at which heat is conducted across the sheet, neglecting edge effects.

[Thermal conductivity of cork = $6.3 \times 10^{-2} \text{ W m}^{-1} \text{ K}^{-1}$, thermal conductivity of glass = $7.14 \times 10^{-1} \text{ W m}^{-1} \text{ K}^{-1}$.] (O. & C.)

4. Define *coefficient of thermal conductivity*. Describe a method of measuring this coefficient for a metal.

Assuming that the thermal insulation provided by a woollen glove is equivalent to a layer of quiescent air 3 mm thick, determine the heat loss per minute from a man's hand, surface area 200 cm² on a winter's day when the atmospheric air temperature is -3°C. The skin temperature is to be taken as 34°C and the thermal conductivity of air as $24 \times 10^{-3} \text{ W m}^{-1} \text{ K}^{-1}$. (L.)

5. Define *thermal conductivity*. Describe in detail a method of determining the thermal conductivity of cork in the form of a thin sheet.

The base and the vertical walls of an open thin-walled metal tank, filled with water maintained at 35°C, are lagged with a layer of cork of superficial area 2.00 m²

and 1.00 cm thick and the water surface is exposed. Heat is supplied electrically to the water at the rate of 250 watts. Find the mass of water that will evaporate per day, if the outside surface of the cork is at 15°C. [Assume that the thermal conductivity of cork is $5.0 \times 10^{-2} \text{ W m}^{-1} \text{ K}^{-1}$ and that the latent heat of vaporization of water at 35°C is 2520 J g^{-1} .] (L.)

6. Explain what is meant by the coefficient of thermal conductivity of a metal.

One end of a long uniform metal bar is heated in a steam chest and the other is kept cool by a current of water. Draw sketch graphs to show the variation of temperature along the bar when the steady state has been attained (a) when the bar is lagged so that no heat escapes from the sides, (b) when the bar is exposed to the air. Explain the shape of the graph in each case.

The surface temperatures of the glass in a window are 20°C for the side facing the room and 5°C for the outside. Compare the rate of flow of heat through (i) a window consisting of a single sheet of glass 5.0 mm thick, and (ii) a double-glazed window of the same area consisting of two sheets of glass each 2.5 mm thick separated by a layer of still air 5.0 mm thick. It may be assumed that the steady state has been attained.

[Use the following values of coefficient of thermal conductivity: glass: $1.0 \text{ W m}^{-1} \text{ K}^{-1}$; air: $2.5 \times 10^{-2} \text{ W m}^{-1} \text{ K}^{-1}$.] (C.)

7. Describe the construction of a Dewar (vacuum) vessel and explain the physical features which result in a reduction to a minimum of the heat exchange between the interior and exterior. Explain why such a vessel is equally suitable for thermally isolating a cold or a hot body.

A copper sphere of radius 0.5 cm is suspended in an evacuated enclosure by a copper wire of diameter 0.01 cm and length 3 cm. An insulated electrical heating coil in good thermal contact with the sphere is connected through the wall of the enclosure by two copper leads of negligible resistance each of diameter 0.02 cm and length 5 cm. What rate of heating in the coil is required to maintain the sphere at a temperature 50°C above that of the surroundings assuming that heat is lost only by conduction along the supports and along the electrical leads? When a steady state has been reached, the coil is disconnected from the electrical supply and the initial rate of fall of temperature of the sphere is found to be 0.013°C per second. Calculate the specific heat of copper, assuming that the electrical leads are still kept at the temperature of the surroundings at the points where they pass through the wall of the enclosure, that the temperature gradient in the sphere is negligible and given that the thermal capacity of the heating coil is equal to that of 1.5 g of copper. [Density of copper = 9.0 g cm^{-3} . Thermal conductivity of copper = $380 \text{ W m}^{-1} \text{ K}^{-1}$.] (O. & C.)

8. Define thermal conductivity.

A 'cold probe', i.e. an instrument to produce a low temperature at its extremity, consists of a solid copper rod 1 mm in diameter attached axially to a well-lagged

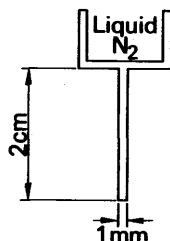


FIG. 13A.

copper reservoir which holds boiling liquid nitrogen, temperature 78 K (see Fig. 13A). The distance from the bottom of the reservoir to the tip of the probe is 2·0 cm. The curved surface of the rod is coated with a non-conducting material. Assuming that no heat can reach the copper except through its flat tip, calculate the maximum rate at which heat can be accepted there if the temperature is not to rise above -10°C.

(Thermal conductivity of copper = $385 \text{ W m}^{-1} \text{ K}^{-1}$.)

If you were given such a probe and told to use it to determine the mean conductivity of copper over this temperature range, how would you proceed? Describe the kind of apparatus you would use and specify (a) the quantities you would need to measure, (b) any data you would need to know. (O. & C.)

Radiation

9. A hot body, such as a wire heated by an electric current, can lose energy to its surroundings by various processes. Outline the nature of each of these processes.

A black body of temperature t is situated in a blackened enclosure maintained at a temperature of 10°C. When $t = 30^\circ\text{C}$ the net rate of loss of energy from the body is equal to 10 watts. What will the rate become when $t = 50^\circ\text{C}$ if the energy exchange takes place solely by the process of radiation? What percentage error is there in the answer obtained by basing the solution on Newton's law of cooling? (C.)

10. Explain what is meant by a *black body*. How do the total energy radiated by a black body and its distribution among the wavelengths in the spectrum depend upon the temperature of the radiator?

Describe the structure of an optical pyrometer and explain how it is used to measure the temperature of a furnace. (L.)

11. Explain what is meant by *black body radiation* and how it can be obtained in practice.

Give an account of Prévost's theory of exchanges and show how it can be used in conjunction with Stefan's law to obtain an expression for the net rate of loss of heat by a black body cooling in an evacuated enclosure.

Sketch the curves relating intensity of radiation and wavelength of radiation from a black body, for three different temperatures. (L.)

12. Explain what is meant by *Stefan's constant*. Defining any symbols used.

A sphere of radius 2·00 cm with a black surface is cooled and then suspended in a large evacuated enclosure the black walls of which are maintained at 27°C. If the rate of change of thermal energy of the sphere is $1\cdot848 \text{ J s}^{-1}$ when its temperature is -73°C, calculate a value for *Stefan's constant*. (N.)

13. What is Prévost's Theory of Exchanges? Describe some phenomenon of theoretical or practical importance to which it applies.

A metal sphere of 1 cm diameter, whose surface acts as a black body, is placed at the focus of a concave mirror with aperture of diameter 60 cm directed towards the sun. If the solar radiation falling normally on the earth is at the rate of $0\cdot14 \text{ watt cm}^{-2}$, Stefan's constant is taken as $6 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$ and the mean temperature of the surroundings is 27°C, calculate the maximum temperature which the sphere could theoretically attain, stating any assumptions you make. (O. & C.)

14. State Newton's law of cooling and Stefan's fourth power law. Describe an experiment to test the validity of *one* of these laws.

A sphere of copper cools at the rate of $10 \text{ deg C min}^{-1}$ when at a temperature of 70°C in an enclosure at 20°C. Calculate its rate of cooling when its temperature