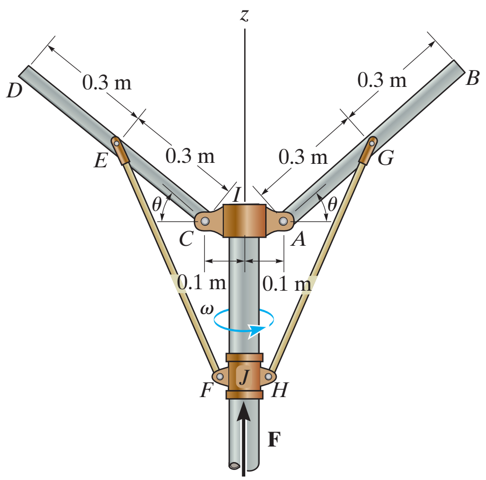
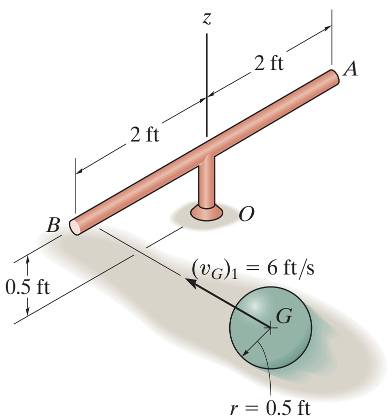
19–42. The vertical shaft is rotating with an angular velocity of 3 rad/s when $\theta = 0°$. If a force **F** is applied to the collar so that $\theta = 90°$, determine the angular velocity of the shaft. Also, find the work done by force **F**. Neglect the mass of rods GH and EF and the collars I and J. The rods AB and CD each have a mass of 10 kg.



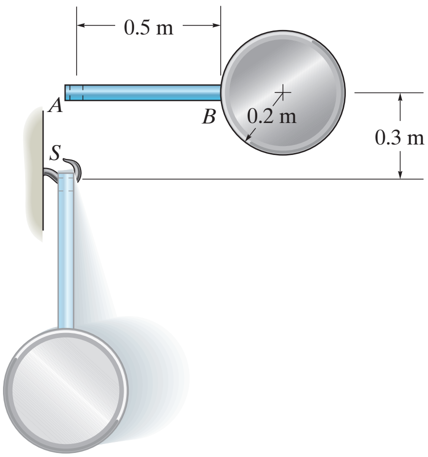
Prob. 19–42

19–43. The mass center of the 3-lb ball has a velocity of $(v_G)_1 = 6$ ft/s when it strikes the end of the smooth 5-lb slender bar which is at rest. Determine the angular velocity of the bar about the z axis just after impact if $e = 0.8$.



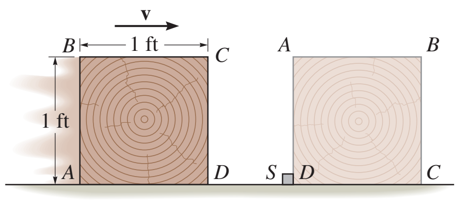
Prob. 19–43

***19–44.** The pendulum consists of a slender 2-kg rod AB and 5-kg disk. It is released from rest without rotating. When it falls 0.3 m, the end A strikes the hook S, which provides a permanent connection. Determine the angular velocity of the pendulum after it has rotated 90°. Treat the pendulum's weight during impact as a nonimpulsive force.



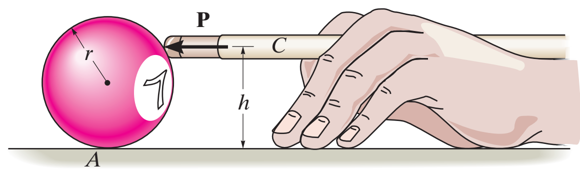
Prob. 19–44

19–45. The 10-lb block is sliding on the smooth surface when the corner D hits a stop block S. Determine the minimum velocity **v** the block should have which would allow it to tip over on its side and land in the position shown. Neglect the size of S. *Hint*: During impact consider the weight of the block to be nonimpulsive.



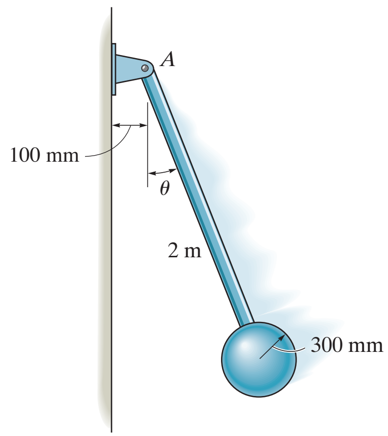
Prob. 19–45

19–46. Determine the height h at which a billiard ball of mass m must be struck so that no frictional force develops between it and the table at A. Assume that the cue C only exerts a horizontal force **P** on the ball.



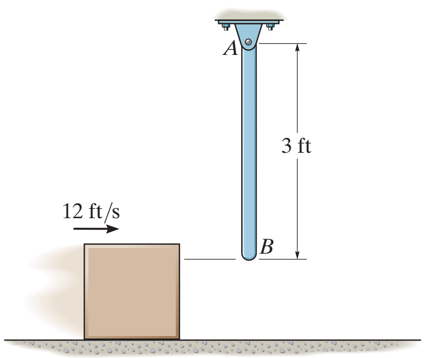
Prob. 19–46

19–47. The pendulum consists of a 15-kg solid ball and 6-kg rod. If it is released from rest when $\theta_1 = 90°$, determine the angle θ_2 after the ball strikes the wall, rebounds, and the pendulum swings up to the point of momentary rest. Take $e = 0.6$.



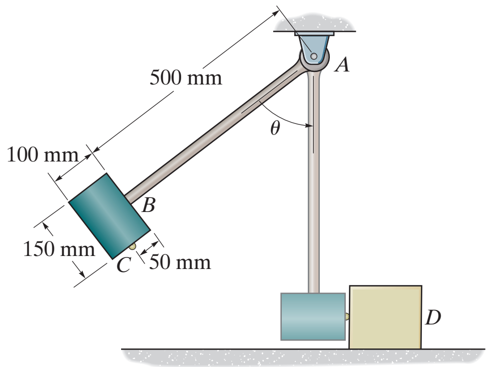
Prob. 19–47

***19–48.** The 4-lb rod AB is hanging in the vertical position. A 2-lb block, sliding on a smooth horizontal surface with a velocity of 12 ft/s, strikes the rod at its end B. Determine the velocity of the block immediately after the collision. The coefficient of restitution between the block and the rod at B is $e = 0.8$.



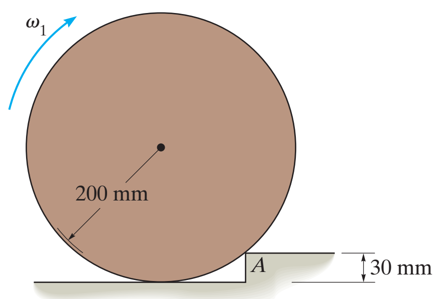
Prob. 19–48

19–49. The hammer consists of a 10-kg solid cylinder C and 6-kg uniform slender rod AB. If the hammer is released from rest when $\theta = 90°$ and strikes the 30-kg block D when $\theta = 0°$, determine the velocity of block D and the angular velocity of the hammer immediately after the impact. The coefficient of restitution between the hammer and the block is $e = 0.6$.



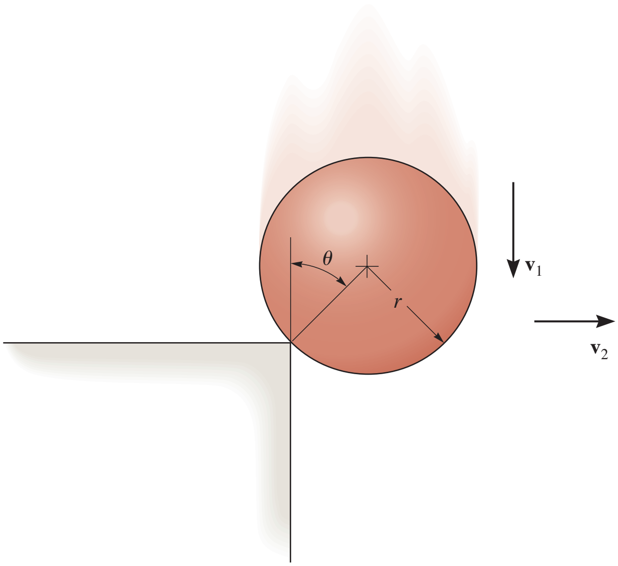
Prob. 19–49

19–50. The 20-kg disk strikes the step without rebounding. Determine the largest angular velocity ω_1 the disk can have and not lose contact with the step, A.



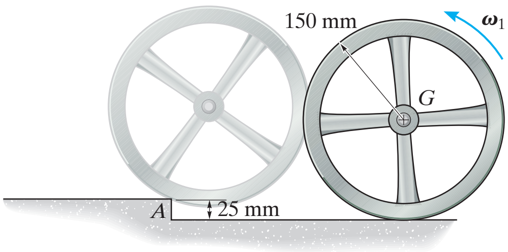
Prob. 19–50

19–51. The solid ball of mass m is dropped with a velocity $\mathbf{v}_1$ onto the edge of the rough step. If it rebounds horizontally off the step with a velocity $\mathbf{v}_2$, determine the angle θ at which contact occurs. Assume no slipping when the ball strikes the step. The coefficient of restitution is e.



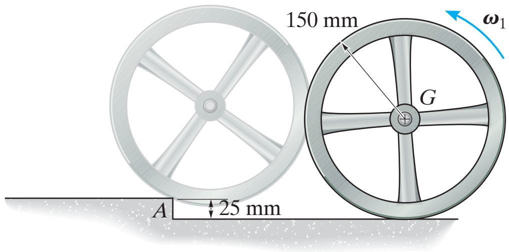
Prob. 19–51

***19–52.** The wheel has a mass of 50 kg and a radius of gyration of 125 mm about its center of mass G. Determine the minimum value of the angular velocity $\boldsymbol{\omega}_1$ of the wheel, so that it strikes the step at A without rebounding and then rolls over it without slipping.



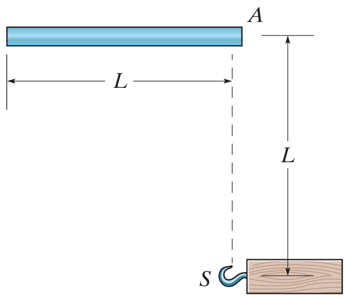
Prob. 19–52

19–53. The wheel has a mass of 50 kg and a radius of gyration of 125 mm about its center of mass G. If it rolls without slipping with an angular velocity of $\boldsymbol{\omega}_1 = 5$ rad/s before it strikes the step at A, determine its angular velocity after it rolls over the step. The wheel does not lose contact with the step when it strikes it.



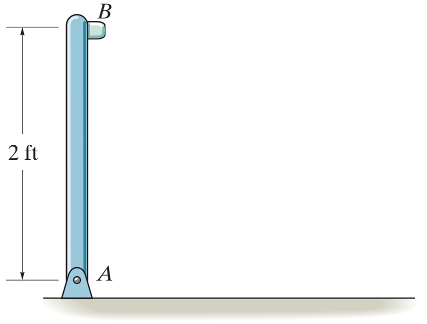
Prob. 19–53

19–54. The rod of mass m and length L is released from rest without rotating. When it falls a distance L, the end A strikes the hook S, which provides a permanent connection. Determine the angular velocity ω of the rod after it has rotated 90°. Treat the rod's weight during impact as a nonimpulsive force.



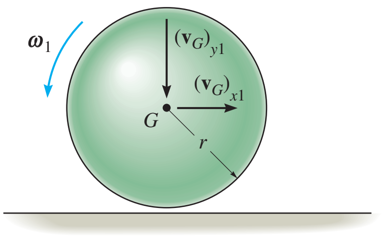
Prob. 19–54

19–55. The 15-lb rod AB is released from rest in the vertical position. If the coefficient or restitution between the floor and the cushion at B is $e = 0.7$, determine how high the end of the rod rebounds after impact with the floor.



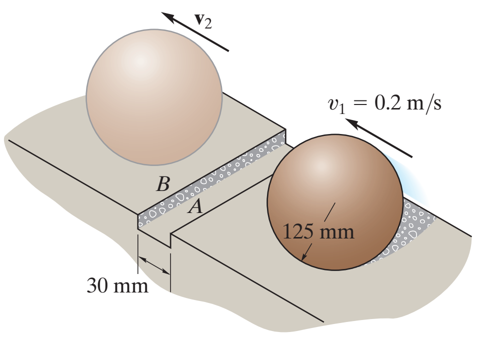
Prob. 19–55

19–57. A solid ball with a mass m is thrown on the ground such that at the instant of contact it has an angular velocity $\boldsymbol{\omega}_1$ and velocity components $(\mathbf{v}_G)_{x1}$ and $(\mathbf{v}_G)_{y1}$ as shown. If the ground is rough so no slipping occurs, determine the components of the velocity of its mass center just after impact. The coefficient of restitution is e.



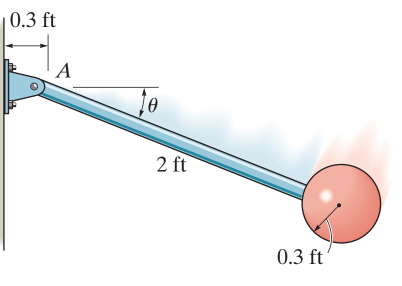
Prob. 19–57

***19–56.** A ball having a mass of 8 kg and initial speed of $v_1 = 0.2$ m/s rolls over a 30-mm-long depression. Assuming that the ball rolls off the edges of contact first A, then B, without slipping, determine its final velocity $\mathbf{v}_2$ when it reaches the other side.



Prob. 19–56

19–58. The pendulum consists of a 10-lb solid ball and 4-lb rod. If it is released from rest when $\theta_0 = 0°$, determine the angle θ_1 of rebound after the ball strikes the wall and the pendulum swings up to the point of momentary rest. Take $e = 0.6$.



Prob. 19–58

CONCEPTUAL PROBLEMS

C19–1. The soil compactor moves forward at constant velocity by supplying power to the rear wheels. Use appropriate numerical data for the wheel, roller, and body and calculate the angular momentum of this system about point *A* at the ground, point *B* on the rear axle, and point *G*, the center of gravity for the system.

Prob. C19–1 (© R.C. Hibbeler)

C19–2. The swing bridge opens and closes by turning 90° using a motor located under the center of the deck at *A* that applies a torque **M** to the bridge. If the bridge was supported at its end *B*, would the same torque open the bridge at the same time, or would it open slower or faster? Explain your answer using numerical values and an impulse and momentum analysis. Also, what are the benefits of making the bridge have the variable depth as shown?

Prob. C19–2 (© R.C. Hibbeler)

C19–3. Why is it necessary to have the tail blade *B* on the helicopter that spins perpendicular to the spin of the main blade *A*? Explain your answer using numerical values and an impulse and momentum analysis.

Prob. C19–3 (© R.C. Hibbeler)

C19–4. The amusement park ride consists of two gondolas *A* and *B*, and counterweights *C* and *D* that swing in opposite directions. Using realistic dimensions and mass, calculate the angular momentum of this system for any angular position of the gondolas. Explain through analysis why it is a good idea to design this system to have counterweights with each gondola.

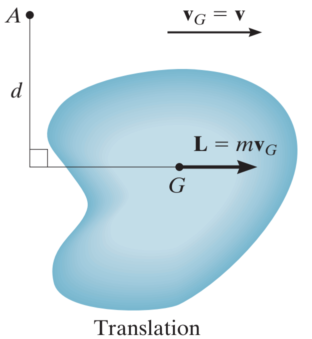
Prob. C19–4 (© R.C. Hibbeler)

CHAPTER REVIEW

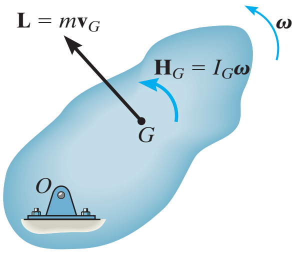
Linear and Angular Momentum

The linear and angular momentum of a rigid body can be referenced to its mass center G.

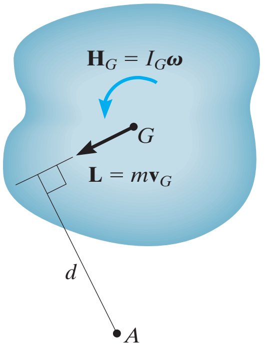
If the angular momentum is to be determined about an axis other than the one passing through the mass center, then the angular momentum is determined by summing vector $\mathbf{H}_G$ and the moment of vector $\mathbf{L}$ about this axis.



Translation



Rotation about a fixed axis



General plane motion

$$L = mv_G \qquad H_G = 0 \qquad H_A = (mv_G)d$$

$$L = mv_G \qquad H_G = I_G\omega \qquad H_O = I_O\omega$$

$$L = mv_G \qquad H_G = I_G\omega \qquad H_A = I_G\omega + (mv_G)d$$

Principle of Impulse and Momentum

The principles of linear and angular impulse and momentum are used to solve problems that involve force, velocity, and time. Before applying these equations, it is important to establish the x, y, z inertial coordinate system. The free-body diagram for the body should also be drawn in order to account for all of the forces and couple moments that produce impulses on the body.

$$m(v_{Gx})_1 + \Sigma \int_{t_1}^{t_2} F_x\, dt = m(v_{Gx})_2$$

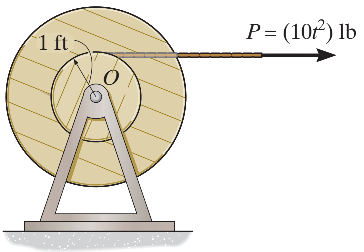
$$m(v_{Gy})_1 + \Sigma \int_{t_1}^{t_2} F_y\, dt = m(v_{Gy})_2$$

$$I_G\omega_1 + \Sigma \int_{t_1}^{t_2} M_G\, dt = I_G\omega_2$$

Conservation of Momentum Provided the sum of the linear impulses acting on a system of connected rigid bodies is zero in a particular direction, then the linear momentum for the system is conserved in this direction. Conservation of angular momentum occurs if the impulses pass through an axis or are parallel to it. Momentum is also conserved if the external forces are small and thereby create nonimpulsive forces on the system. A free-body diagram should accompany any application in order to classify the forces as impulsive or nonimpulsive and to determine an axis about which the angular momentum may be conserved.	$$\left(\sum \begin{matrix}\text{syst. linear}\\ \text{momentum}\end{matrix}\right)_1 = \left(\sum \begin{matrix}\text{syst. linear}\\ \text{momentum}\end{matrix}\right)_2$$ $$\left(\sum \begin{matrix}\text{syst. angular}\\ \text{momentum}\end{matrix}\right)_{O1} = \left(\sum \begin{matrix}\text{syst. angular}\\ \text{momentum}\end{matrix}\right)_{O2}$$
Eccentric Impact If the line of impact does not coincide with the line connecting the mass centers of two colliding bodies, then eccentric impact will occur. If the motion of the bodies just after the impact is to be determined, then it is necessary to consider a conservation of momentum equation for the system and use the coefficient of restitution equation.	$$e = \frac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}$$

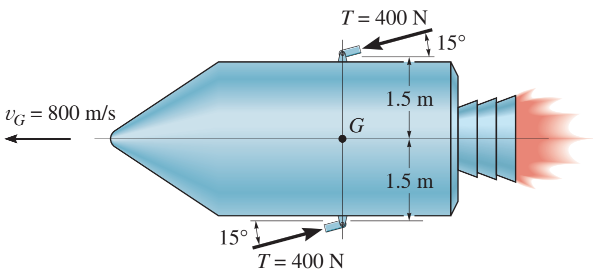
REVIEW PROBLEMS

R19–1. The cable is subjected to a force of $P = (10t^2)$ lb. where t is in seconds. Determine the angular velocity of the spool 3 s after **P** is applied, starting from rest. The spool has a weight of 150 lb and a radius of gyration of 1.25 ft about its center, O.



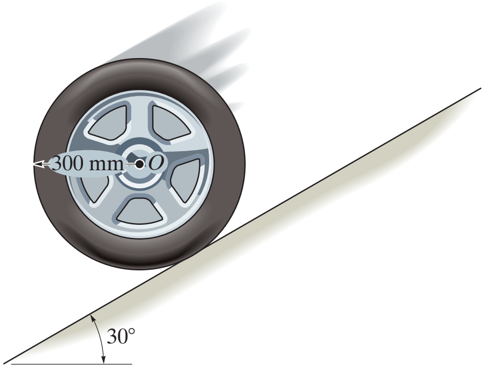
Prob. R19–1

R19–2. The space capsule has a mass of 1200 kg and a moment of inertia $I_G = 900\ \text{kg}\cdot\text{m}^2$ about an axis passing through G and directed perpendicular to the page. If it is traveling forward with a speed $v_G = 800\ \text{m/s}$ and executes a turn by means of two jets, which provide a constant thrust of 400 N for 0.3 s, determine the capsule's angular velocity just after the jets are turned off.



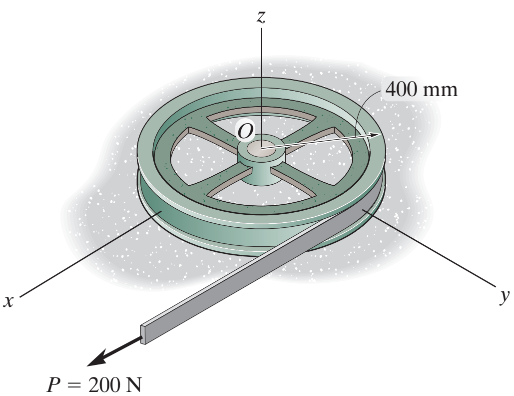
Prob. R19–2

R19–3. The tire has a mass of 9 kg and a radius of gyration $k_O = 225$ mm. If it is released from rest and rolls down the plane without slipping, determine the speed of its center O when $t = 3$ s.



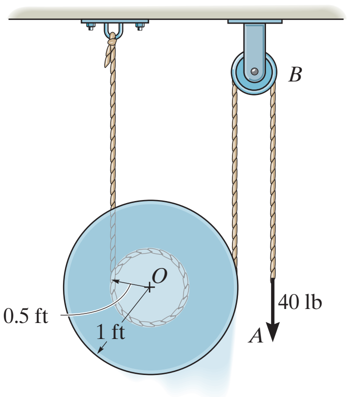
Prob. R19–3

R19–4. The wheel having a mass of 100 kg and a radius of gyration about the z axis of $k_z = 300$ mm, rests on the smooth horizontal plane. If the belt is subjected to a force of $P = 200$ N, determine the angular velocity of the wheel and the speed of its center of mass O, three seconds after the force is applied.



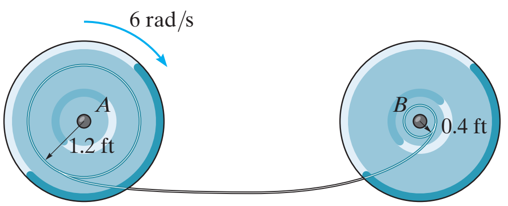
Prob. R19–4

R19–5. The spool has a weight of 30 lb and a radius of gyration $k_O = 0.65$ ft. If a force of 40 lb is applied to the cord at A, determine the angular velocity of the spool in $t = 3$ s starting from rest. Neglect the mass of the pulley and cord.



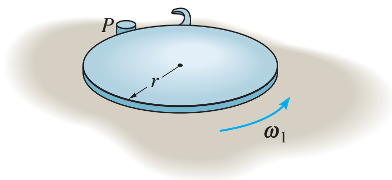
Prob. R19–5

R19–6. Spool B is at rest and spool A is rotating at 6 rad/s when the slack in the cord connecting them is taken up. If the cord does not stretch, determine the angular velocity of each spool immediately after the cord is jerked tight. The spools A and B have weights and radii of gyration $W_A = 30$ lb, $k_A = 0.8$ ft, $W_B = 15$ lb, $k_B = 0.6$ ft, respectively.



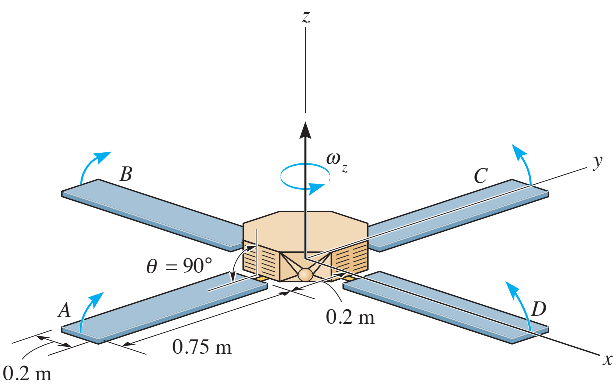
Prob. R19–6

R19–7. A thin disk of mass m has an angular velocity $\boldsymbol{\omega}_1$ while rotating on a smooth surface. Determine its new angular velocity just after the hook at its edge strikes the peg P and the disk starts to rotate about P without rebounding.



Prob. R19–7

R19–8. The space satellite has a mass of 125 kg and a moment of inertia $I_z = 0.940\ \text{kg}\cdot\text{m}^2$, excluding the four solar panels A, B, C, and D. Each solar panel has a mass of 20 kg and can be approximated as a thin plate. If the satellite is originally spinning about the z axis at a constant rate $\omega_z = 0.5$ rad/s when $\theta = 90°$, determine the rate of spin if all the panels are raised and reach the upward position, $\theta = 0°$, at the same instant.



Prob. R19–8

Chapter 20

(© Philippe Psaila/Science Source)

Design of industrial robots requires knowing the kinematics of their three-dimensional motions.

Three-Dimensional Kinematics of a Rigid Body

CHAPTER OBJECTIVES

- To analyze the kinematics of a body subjected to rotation about a fixed point and to general plane motion.
- To provide a relative-motion analysis of a rigid body using translating and rotating axes.

20.1 Rotation About a Fixed Point

When a rigid body rotates about a fixed point, the distance r from the point to a particle located on the body is the *same* for *any position* of the body. Thus, the path of motion for the particle lies on the *surface of a sphere* having a radius r and centered at the fixed point. Since motion along this path occurs only from a series of rotations made during a finite time interval, we will first develop a familiarity with some of the properties of rotational displacements.

The boom can rotate up and down, and because it is hinged at a point on the vertical axis about which it turns, it is subjected to rotation about a fixed point. (© R.C. Hibbeler)

Euler's Theorem. Euler's theorem states that two "component" rotations about different axes passing through a point are equivalent to a single resultant rotation about an axis passing through the point. If more than two rotations are applied, they can be combined into pairs, and each pair can be further reduced and combined into one rotation.

Finite Rotations. If component rotations used in Euler's theorem are *finite*, it is important that the *order* in which they are applied be maintained. To show this, consider the two finite rotations $\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2$ applied to the block in Fig. 20–1*a*. Each rotation has a magnitude of 90° and a direction defined by the right-hand rule, as indicated by the arrow. The final position of the block is shown at the right. When these two rotations are applied in the order $\boldsymbol{\theta}_2 + \boldsymbol{\theta}_1$, as shown in Fig. 20–1*b*, the final position of the block is *not* the same as it is in Fig. 20–1*a*. Because *finite rotations* do not obey the commutative law of addition ($\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2 \neq \boldsymbol{\theta}_2 + \boldsymbol{\theta}_1$), *they cannot be classified as vectors*. If smaller, yet finite, rotations had been used to illustrate this point, e.g., 10° instead of 90°, the *final position* of the block after each combination of rotations would also be different; however, in this case, the difference is only a small amount.

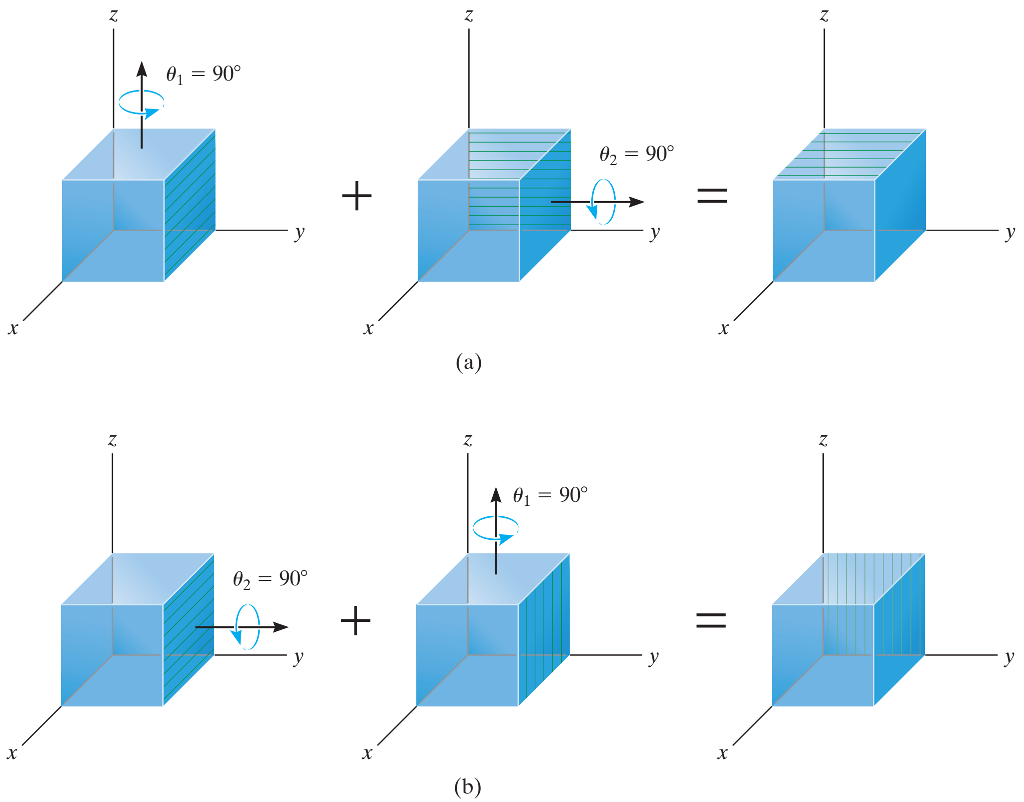


Fig. 20–1

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Infinitesimal Rotations. When defining the angular motions of a body subjected to three-dimensional motion, only rotations which are *infinitesimally small* will be considered. *Such rotations can be classified as vectors, since they can be added vectorially in any manner.* To show this, for purposes of simplicity let us consider the rigid body itself to be a sphere which is allowed to rotate about its central fixed point O, Fig. 20–2*a*. If we impose two infinitesimal rotations $d\boldsymbol{\theta}_1 + d\boldsymbol{\theta}_2$ on the body, it is seen that point P moves along the path $d\boldsymbol{\theta}_1 \times \mathbf{r} + d\boldsymbol{\theta}_2 \times \mathbf{r}$ and ends up at P'. Had the two successive rotations occurred in the order $d\boldsymbol{\theta}_2 + d\boldsymbol{\theta}_1$, then the resultant displacements of P would have been $d\boldsymbol{\theta}_2 \times \mathbf{r} + d\boldsymbol{\theta}_1 \times \mathbf{r}$. Since the vector cross product obeys the distributive law, by comparison $(d\boldsymbol{\theta}_1 + d\boldsymbol{\theta}_2) \times \mathbf{r} = (d\boldsymbol{\theta}_2 + d\boldsymbol{\theta}_1) \times \mathbf{r}$. Here infinitesimal rotations $d\boldsymbol{\theta}$ are vectors, since these quantities have both a magnitude and direction for which the order of (vector) addition is not important, i.e., $d\boldsymbol{\theta}_1 + d\boldsymbol{\theta}_2 = d\boldsymbol{\theta}_2 + d\boldsymbol{\theta}_1$. As a result, as shown in Fig. 20–2*a*, the two "component" rotations $d\boldsymbol{\theta}_1$ and $d\boldsymbol{\theta}_2$ are equivalent to a single resultant rotation $d\boldsymbol{\theta} = d\boldsymbol{\theta}_1 + d\boldsymbol{\theta}_2$, a consequence of Euler's theorem.

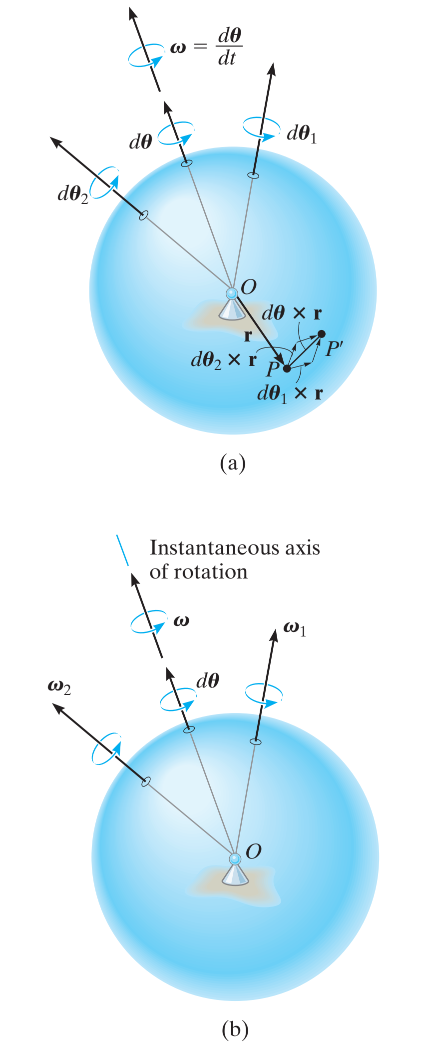

Fig. 20–2

Angular Velocity. If the body is subjected to an angular rotation $d\boldsymbol{\theta}$ about a fixed point, the angular velocity of the body is defined by the time derivative,

$$\boxed{\boldsymbol{\omega} = \dot{\boldsymbol{\theta}}} \tag{20–1}$$

The line specifying the direction of $\boldsymbol{\omega}$, which is collinear with $d\boldsymbol{\theta}$, is referred to as the *instantaneous axis of rotation*, Fig. 20–2*b*. In general, this axis changes direction during each instant of time. Since $d\boldsymbol{\theta}$ is a vector quantity, so too is $\boldsymbol{\omega}$, and it follows from vector addition that if the body is subjected to two component angular motions, $\boldsymbol{\omega}_1 = \dot{\boldsymbol{\theta}}_1$ and $\boldsymbol{\omega}_2 = \dot{\boldsymbol{\theta}}_2$, the resultant angular velocity is $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$.

Angular Acceleration. The body's angular acceleration is determined from the time derivative of its angular velocity, i.e.,

$$\boxed{\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}} \tag{20–2}$$

For motion about a fixed point, $\boldsymbol{\alpha}$ must account for a change in *both* the magnitude and direction of $\boldsymbol{\omega}$, so that, in general, $\boldsymbol{\alpha}$ is not directed along the instantaneous axis of rotation, Fig. 20–3.

As the direction of the instantaneous axis of rotation (or the line of action of $\boldsymbol{\omega}$) changes in space, the locus of the axis generates a fixed *space cone*, Fig. 20–4. If the change in the direction of this axis is viewed with respect to the rotating body, the locus of the axis generates a *body cone*.

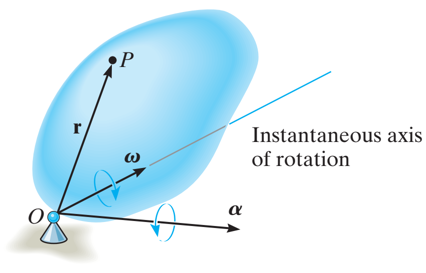

Fig. 20–3

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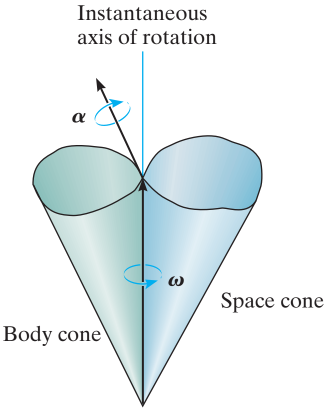
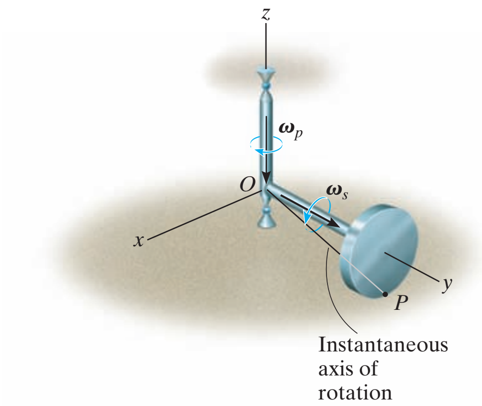
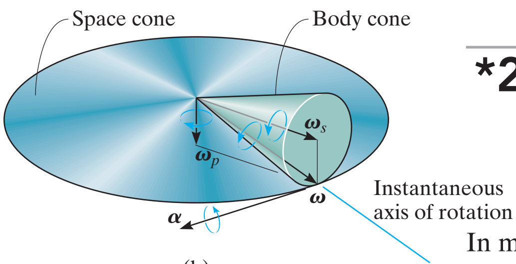


Fig. 20–4



(a)



(b)

Fig. 20–5

At any given instant, these cones meet along the instantaneous axis of rotation, and when the body is in motion, the body cone appears to roll either on the inside or the outside surface of the fixed space cone. Provided the paths defined by the open ends of the cones are described by the head of the $\boldsymbol{\omega}$ vector, then $\boldsymbol{\alpha}$ must act tangent to these paths at any given instant, since the time rate of change of $\boldsymbol{\omega}$ is equal to $\boldsymbol{\alpha}$. Fig. 20–4.

To illustrate this concept, consider the disk in Fig. 20–5*a* that spins about the rod at $\boldsymbol{\omega}_s$, while the rod and disk precess about the vertical axis at $\boldsymbol{\omega}_p$. The resultant angular velocity of the disk is therefore $\boldsymbol{\omega} = \boldsymbol{\omega}_s + \boldsymbol{\omega}_p$. Since both point O and the contact point P have zero velocity, then all points on a line between these points must have zero velocity. Thus, both $\boldsymbol{\omega}$ and the instantaneous axis of rotation are along OP. Therefore, as the disk rotates, this axis appears to move along the surface of the fixed space cone shown in Fig. 20–5*b*. If the axis is observed from the rotating disk, the axis then appears to move on the surface of the body cone. At any instant, though, these two cones meet each other along the axis OP. If $\boldsymbol{\omega}$ has a constant magnitude, then $\boldsymbol{\alpha}$ indicates only the change in the direction of $\boldsymbol{\omega}$, which is tangent to the cones at the tip of $\boldsymbol{\omega}$ as shown in Fig. 20–5*b*.

Velocity. Once $\boldsymbol{\omega}$ is specified, the velocity of any point on a body rotating about a fixed point can be determined using the same methods as for a body rotating about a fixed axis. Hence, by the cross product,

$$\boxed{\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}} \tag{20–3}$$

Here $\mathbf{r}$ defines the position of the point measured from the fixed point O, Fig. 20–3.

Acceleration. If $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are known at a given instant, the acceleration of a point can be obtained from the time derivative of Eq. 20–3, which yields

$$\boxed{\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})} \tag{20–4}$$

*20.2 The Time Derivative of a Vector Measured from Either a Fixed or Translating-Rotating System

In many types of problems involving the motion of a body about a fixed point, the angular velocity $\boldsymbol{\omega}$ is specified in terms of its components. Then, if the angular acceleration $\boldsymbol{\alpha}$ of such a body is to be determined, it is often easier to compute the time derivative of $\boldsymbol{\omega}$ using a coordinate system that has a *rotation* defined by one or more of the components of $\boldsymbol{\omega}$. For example, in the case of the disk in Fig. 20–5*a*, where $\boldsymbol{\omega} = \boldsymbol{\omega}_s + \boldsymbol{\omega}_p$, the x, y, z axes can be given an angular velocity of $\boldsymbol{\omega}_p$. For this reason, and for other uses later, an equation will now be derived, which relates the time derivative of any vector $\mathbf{A}$ defined from a translating-rotating reference to its time derivative defined from a fixed reference.

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Consider the x, y, z axes of the moving frame of reference to be rotating with an angular velocity $\mathbf{\Omega}$, which is measured from the fixed X, Y, Z axes, Fig. 20–6*a*. In the following discussion, it will be convenient to express vector $\mathbf{A}$ in terms of its $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ components, which define the directions of the moving axes. Hence,

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$$

In general, the time derivative of $\mathbf{A}$ must account for the change in both its magnitude and direction. However, if this derivative is taken *with respect to the moving frame of reference*, only the change in the magnitudes of the components of $\mathbf{A}$ must be accounted for, since the directions of the components do not change with respect to the moving reference. Hence,

$$(\dot{\mathbf{A}})_{xyz} = \dot{A}_x\mathbf{i} + \dot{A}_y\mathbf{j} + \dot{A}_z\mathbf{k} \qquad (20\text{–}5)$$

When the time derivative of $\mathbf{A}$ is taken *with respect to the fixed frame of reference*, the *directions* of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ change only on account of the *rotation* $\mathbf{\Omega}$ of the axes and not their translation. Hence, in general,

$$\dot{\mathbf{A}} = \dot{A}_x\mathbf{i} + \dot{A}_y\mathbf{j} + \dot{A}_z\mathbf{k} + A_x\dot{\mathbf{i}} + A_y\dot{\mathbf{j}} + A_z\dot{\mathbf{k}}$$

The time derivatives of the unit vectors will now be considered. For example, $\dot{\mathbf{i}} = d\mathbf{i}/dt$ represents only the change in the *direction* of $\mathbf{i}$ with respect to time, since $\mathbf{i}$ always has a magnitude of 1 unit. As shown in Fig. 20–6*b*, the change, $d\mathbf{i}$, is *tangent to the path* described by the arrowhead of $\mathbf{i}$ as $\mathbf{i}$ swings due to the rotation $\mathbf{\Omega}$. Accounting for both the magnitude and direction of $d\mathbf{i}$, we can therefore define $\dot{\mathbf{i}}$ using the cross product, $\dot{\mathbf{i}} = \mathbf{\Omega} \times \mathbf{i}$. In general, then

$$\dot{\mathbf{i}} = \mathbf{\Omega} \times \mathbf{i} \qquad \dot{\mathbf{j}} = \mathbf{\Omega} \times \mathbf{j} \qquad \dot{\mathbf{k}} = \mathbf{\Omega} \times \mathbf{k}$$

These formulations were also developed in Sec. 16.8, regarding planar motion of the axes. Substituting these results into the above equation and using Eq. 20–5 yields

$$\boxed{\dot{\mathbf{A}} = (\dot{\mathbf{A}})_{xyz} + \mathbf{\Omega} \times \mathbf{A}} \qquad (20\text{–}6)$$

This result is important, and will be used throughout Sec. 20.4 and Chapter 21. It states that the time derivative of *any vector* $\mathbf{A}$ as observed from the fixed X, Y, Z frame of reference is equal to the time rate of change of $\mathbf{A}$ as observed from the x, y, z translating-rotating frame of reference, Eq. 20–5, plus $\mathbf{\Omega} \times \mathbf{A}$, the change of $\mathbf{A}$ caused by the rotation of the x, y, z frame. As a result, Eq. 20–6 should always be used whenever $\mathbf{\Omega}$ produces a change in the direction of $\mathbf{A}$ as seen from the X, Y, Z reference. If this change does not occur, i.e., $\mathbf{\Omega} = \mathbf{0}$, then $\dot{\mathbf{A}} = (\dot{\mathbf{A}})_{xyz}$, and so the time rate of change of $\mathbf{A}$ as observed from both coordinate systems will be the *same*.

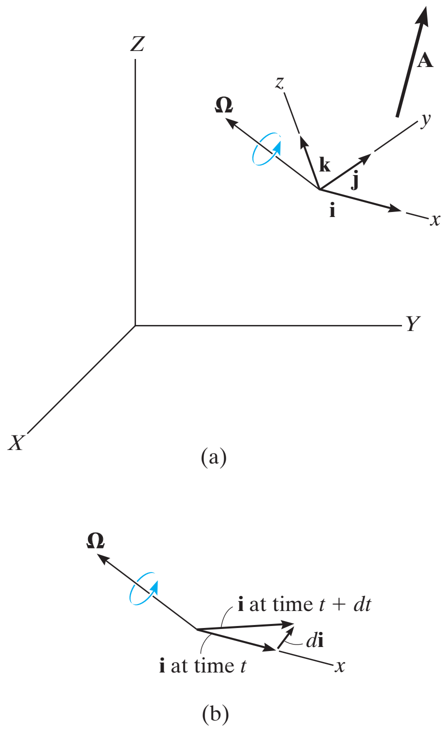

Fig. 20–6

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EXAMPLE 20.1

The disk shown in Fig. 20–7 spins about its axle with a constant angular velocity $\omega_s = 3\ \text{rad/s}$, while the horizontal platform on which the disk is mounted rotates about the vertical axis at a constant rate $\omega_p = 1\ \text{rad/s}$. Determine the angular acceleration of the disk and the velocity and acceleration of point A on the disk when it is in the position shown.

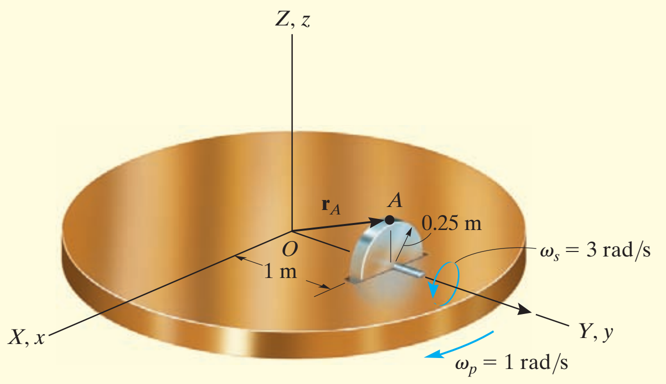


Fig. 20–7

SOLUTION

Point O represents a fixed point of rotation for the disk if one considers a hypothetical extension of the disk to this point. To determine the velocity and acceleration of point A, it is first necessary to determine the angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$ of the disk, since these vectors are used in Eqs. 20–3 and 20–4.

Angular Velocity. The angular velocity, which is measured from X, Y, Z, is simply the vector addition of its two component motions. Thus,

$$\boldsymbol{\omega} = \boldsymbol{\omega}_s + \boldsymbol{\omega}_p = \{3\mathbf{j} - 1\mathbf{k}\}\ \text{rad/s}$$

Angular Acceleration. Since the magnitude of $\boldsymbol{\omega}$ is constant, only a change in its direction, as seen from the fixed reference, creates the angular acceleration $\boldsymbol{\alpha}$ of the disk. One way to obtain $\boldsymbol{\alpha}$ is to compute the time derivative of *each of the two components* of $\boldsymbol{\omega}$ using Eq. 20–6. At the instant shown in Fig. 20–7, imagine the fixed X, Y, Z and a rotating x, y, z frame to be coincident. If the rotating x, y, z frame is chosen to have an angular velocity of $\boldsymbol{\Omega} = \boldsymbol{\omega}_p = \{-1\mathbf{k}\}$ rad/s, then $\boldsymbol{\omega}_s$ will *always* be directed along the y (not Y) axis, and the time rate of change of $\boldsymbol{\omega}_s$ *as seen from* x, y, z is *zero*; i.e., $(\dot{\boldsymbol{\omega}}_s)_{xyz} = \mathbf{0}$ (the magnitude and direction of $\boldsymbol{\omega}_s$ is constant). Thus,

$$\dot{\boldsymbol{\omega}}_s = (\dot{\boldsymbol{\omega}}_s)_{xyz} + \boldsymbol{\omega}_p \times \boldsymbol{\omega}_s = \mathbf{0} + (-1\mathbf{k}) \times (3\mathbf{j}) = \{3\mathbf{i}\} \text{ rad/s}^2$$

By the same choice of axes rotation, $\boldsymbol{\Omega} = \boldsymbol{\omega}_p$, or even with $\boldsymbol{\Omega} = \mathbf{0}$, the time derivative $(\dot{\boldsymbol{\omega}}_p)_{xyz} = \mathbf{0}$, since $\boldsymbol{\omega}_p$ has a constant magnitude and direction with respect to x, y, z. Hence,

$$\dot{\boldsymbol{\omega}}_p = (\dot{\boldsymbol{\omega}}_p)_{xyz} + \boldsymbol{\omega}_p \times \boldsymbol{\omega}_p = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

The angular acceleration of the disk is therefore

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_s + \dot{\boldsymbol{\omega}}_p = \{3\mathbf{i}\} \text{ rad/s}^2 \qquad \textit{Ans.}$$

Velocity and Acceleration. Since $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ have now been determined, the velocity and acceleration of point A can be found using Eqs. 20–3 and 20–4. Realizing that $\mathbf{r}_A = \{1\mathbf{j} + 0.25\mathbf{k}\}$ m, Fig. 20–7, we have

$$\mathbf{v}_A = \boldsymbol{\omega} \times \mathbf{r}_A = (3\mathbf{j} - 1\mathbf{k}) \times (1\mathbf{j} + 0.25\mathbf{k}) = \{1.75\mathbf{i}\} \text{ m/s} \qquad \textit{Ans.}$$

$$\begin{aligned} \mathbf{a}_A &= \boldsymbol{\alpha} \times \mathbf{r}_A + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_A) \\ &= (3\mathbf{i}) \times (1\mathbf{j} + 0.25\mathbf{k}) + (3\mathbf{j} - 1\mathbf{k}) \times [(3\mathbf{j} - 1\mathbf{k}) \times (1\mathbf{j} + 0.25\mathbf{k})] \\ &= \{-2.50\mathbf{j} - 2.25\mathbf{k}\} \text{ m/s}^2 \qquad \textit{Ans.} \end{aligned}$$

EXAMPLE 20.2

At the instant $\theta = 60°$, the gyrotop in Fig. 20–8 has three components of angular motion directed as shown and having magnitudes defined as:

Spin: $\omega_s = 10$ rad/s, increasing at the rate of 6 rad/s^2

Nutation: $\omega_n = 3$ rad/s, increasing at the rate of 2 rad/s^2

Precession: $\omega_p = 5$ rad/s, increasing at the rate of 4 rad/s^2

Determine the angular velocity and angular acceleration of the top.

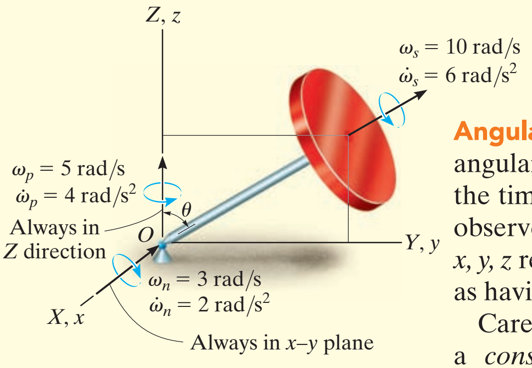


Fig. 20–8

SOLUTION

Angular Velocity. The top rotates about the fixed point O. If the fixed and rotating frames are coincident at the instant shown, then the angular velocity can be expressed in terms of **i**, **j**, **k** components, with reference to the x, y, z frame; i.e.,

$$\begin{aligned}\boldsymbol{\omega} &= -\omega_n\mathbf{i} + \omega_s \sin\theta\mathbf{j} + (\omega_p + \omega_s\cos\theta)\mathbf{k} \\ &= -3\mathbf{i} + 10\sin 60°\mathbf{j} + (5 + 10\cos 60°)\mathbf{k} \\ &= \{-3\mathbf{i} + 8.66\mathbf{j} + 10\mathbf{k}\}\ \text{rad/s} \qquad \textit{Ans.}\end{aligned}$$

Angular Acceleration. As in the solution of Example 20.1, the angular acceleration $\boldsymbol{\alpha}$ will be determined by investigating separately the time rate of change of *each of the angular velocity components* as observed from the fixed X, Y, Z reference. We will choose an $\boldsymbol{\Omega}$ for the x, y, z reference so that the component of $\boldsymbol{\omega}$ being considered is viewed as having a *constant direction* when observed from x, y, z.

Careful examination of the motion of the top reveals that $\boldsymbol{\omega}_s$ has a *constant direction* relative to x, y, z if these axes rotate at $\boldsymbol{\Omega} = \boldsymbol{\omega}_n + \boldsymbol{\omega}_p$. Thus,

$$\begin{aligned}\dot{\boldsymbol{\omega}}_s &= (\dot{\boldsymbol{\omega}}_s)_{xyz} + (\boldsymbol{\omega}_n + \boldsymbol{\omega}_p) \times \boldsymbol{\omega}_s \\ &= (6\sin 60°\mathbf{j} + 6\cos 60°\mathbf{k}) + (-3\mathbf{i} + 5\mathbf{k}) \times (10\sin 60°\mathbf{j} + 10\cos 60°\mathbf{k}) \\ &= \{-43.30\mathbf{i} + 20.20\mathbf{j} - 22.98\mathbf{k}\}\ \text{rad/s}^2\end{aligned}$$

Since $\boldsymbol{\omega}_n$ *always* lies in the fixed X–Y plane, this vector has a *constant direction* if the motion is viewed from axes x, y, z having a rotation of $\boldsymbol{\Omega} = \boldsymbol{\omega}_p$ (not $\boldsymbol{\Omega} = \boldsymbol{\omega}_s + \boldsymbol{\omega}_p$). Thus,

$$\dot{\boldsymbol{\omega}}_n = (\dot{\boldsymbol{\omega}}_n)_{xyz} + \boldsymbol{\omega}_p \times \boldsymbol{\omega}_n = -2\mathbf{i} + (5\mathbf{k}) \times (-3\mathbf{i}) = \{-2\mathbf{i} - 15\mathbf{j}\}\ \text{rad/s}^2$$

Finally, the component $\boldsymbol{\omega}_p$ is *always directed* along the Z axis so that here it is not necessary to think of x, y, z as rotating, i.e., $\boldsymbol{\Omega} = \mathbf{0}$. Expressing the data in terms of the **i**, **j**, **k** components, we therefore have

$$\dot{\boldsymbol{\omega}}_p = (\dot{\boldsymbol{\omega}}_p)_{xyz} + \mathbf{0} \times \boldsymbol{\omega}_p = \{4\mathbf{k}\}\ \text{rad/s}^2$$

Thus, the angular acceleration of the top is

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}_s + \dot{\boldsymbol{\omega}}_n + \dot{\boldsymbol{\omega}}_p = \{-45.3\mathbf{i} + 5.20\mathbf{j} - 19.0\mathbf{k}\}\ \text{rad/s}^2 \qquad \textit{Ans.}$$

20.3 General Motion

Shown in Fig. 20–9 is a rigid body subjected to general motion in three dimensions for which the angular velocity is $\boldsymbol{\omega}$ and the angular acceleration is $\boldsymbol{\alpha}$. If point A has a known motion of $\mathbf{v}_A$ and $\mathbf{a}_A$, the motion of any other point B can be determined by using a relative-motion analysis. In this section a *translating coordinate system* will be used to define the relative motion, and in the next section a reference that is both rotating and translating will be considered.

If the origin of the translating coordinate system x, y, z ($\boldsymbol{\Omega} = \mathbf{0}$) is located at the "base point" A, then, at the instant shown, the motion of the body can be regarded as the sum of an instantaneous translation of the body having a motion of $\mathbf{v}_A$, and $\mathbf{a}_A$, and a rotation of the body about an instantaneous axis passing through point A. Since the body is rigid, the motion of point B measured by an observer located at A is therefore the same as *the rotation of the body about a fixed point*. This relative motion occurs about the instantaneous axis of rotation and is defined by $\mathbf{v}_{B/A} = \boldsymbol{\omega} \times \mathbf{r}_{B/A}$, Eq. 20–3, and $\mathbf{a}_{B/A} = \boldsymbol{\alpha} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A})$, Eq. 20–4. For translating axes, the relative motions are related to absolute motions by $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$ and $\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$, Eqs. 16–15 and 16–17, so that the absolute velocity and acceleration of point B can be determined from the equations

$$\boxed{\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}} \tag{20–7}$$

and

$$\boxed{\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A})} \tag{20–8}$$

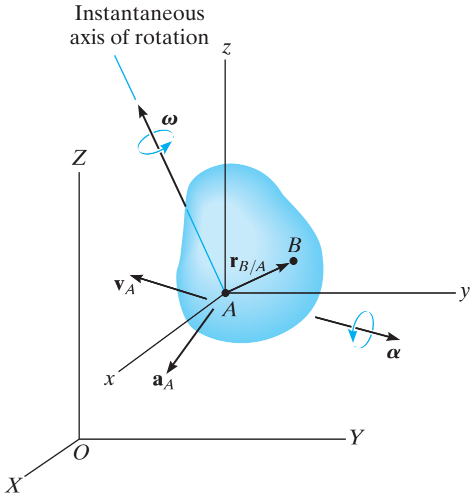


Fig. 20–9

These two equations are essentially the same as those describing the general plane motion of a rigid body, Eqs. 16–16 and 16–18. However, difficulty in application arises for three-dimensional motion, because $\boldsymbol{\alpha}$ now measures the change in *both* the magnitude and direction of $\boldsymbol{\omega}$.

Although this may be the case, a direct solution for $\mathbf{v}_B$ and $\mathbf{a}_B$ can be obtained by noting that $\mathbf{v}_{B/A} = \mathbf{v}_B - \mathbf{v}_A$, and so Eq. 20–7 becomes $\mathbf{v}_{B/A} = \boldsymbol{\omega} \times \mathbf{r}_{B/A}$. The cross product indicates that $\mathbf{v}_{B/A}$ is *perpendicular* to $\mathbf{r}_{B/A}$, and so, as noted by Eq. C–14 of Appendix C, we require

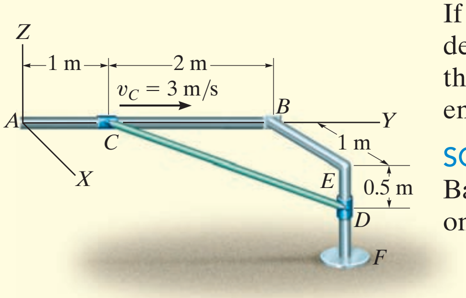
$$\mathbf{r}_{B/A} \cdot \mathbf{v}_{B/A} = 0 \tag{20–9}$$

Taking the time derivative, we have

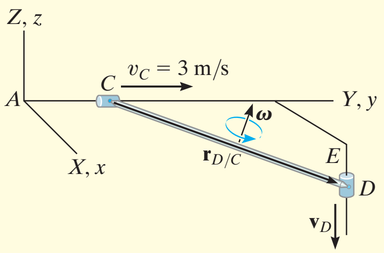
$$\mathbf{v}_{B/A} \cdot \mathbf{v}_{B/A} + \mathbf{r}_{B/A} \cdot \mathbf{a}_{B/A} = 0 \tag{20–10}$$

Solution II of the following example illustrates application of this idea.

EXAMPLE 20.3



(a)



(b)

Fig. 20–10

If the collar at C in Fig. 20–10a moves toward B with a speed of 3 m/s, determine the velocity of the collar at D and the angular velocity of the bar at the instant shown. The bar is connected to the collars at its end points by ball-and-socket joints.

SOLUTION I

Bar CD is subjected to general motion. Why? The velocity of point D on the bar can be related to the velocity of point C by the equation

$$\mathbf{v}_D = \mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{r}_{D/C}$$

The fixed and translating frames of reference are assumed to coincide at the instant considered, Fig. 20–10b. We have

$$\mathbf{v}_D = -v_D\mathbf{k} \qquad \mathbf{v}_C = \{3\mathbf{j}\} \text{ m/s}$$

$$\mathbf{r}_{D/C} = \{1\mathbf{i} + 2\mathbf{j} - 0.5\mathbf{k}\} \text{ m} \qquad \boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$$

Substituting into the above equation we get

$$-v_D\mathbf{k} = 3\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 1 & 2 & -0.5 \end{vmatrix}$$

Expanding and equating the respective $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components yields

$$-0.5\omega_y - 2\omega_z = 0 \tag{1}$$

$$0.5\omega_x + 1\omega_z + 3 = 0 \tag{2}$$

$$2\omega_x - 1\omega_y + v_D = 0 \tag{3}$$

These equations contain four unknowns.* A fourth equation can be written if the direction of $\boldsymbol{\omega}$ is specified. In particular, any component of $\boldsymbol{\omega}$ acting along the bar's axis has no effect on moving the collars. This is because the bar is *free to rotate* about its axis. Therefore, if $\boldsymbol{\omega}$ is specified as acting *perpendicular* to the axis of the bar, then $\boldsymbol{\omega}$ must have a *unique magnitude* to satisfy the above equations. Hence,

$$\boldsymbol{\omega} \cdot \mathbf{r}_{D/C} = (\omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}) \cdot (1\mathbf{i} + 2\mathbf{j} - 0.5\mathbf{k}) = 0$$

$$1\omega_x + 2\omega_y - 0.5\omega_z = 0 \tag{4}$$

*Although this is the case, the magnitude of $\mathbf{v}_D$ can be obtained. For example, solve Eqs. 1 and 2 for ω_y and ω_x in terms of ω_z and substitute this into Eq. 3. Then ω_z will cancel out, which will allow a solution for v_D.

Solving Eqs. 1 through 4 simultaneously yields

$$\omega_x = -4.86\ \text{rad/s} \quad \omega_y = 2.29\ \text{rad/s} \quad \omega_z = -0.571\ \text{rad/s},$$
$$v_D = 12.0\ \text{m/s, so that} \qquad \omega = 5.40\ \text{rad/s} \qquad \textit{Ans.}$$

SOLUTION II

Applying Eq. 20–9, $\mathbf{v}_{D/C} = \mathbf{v}_D - \mathbf{v}_C = -v_D\mathbf{k} - 3\mathbf{j}$, so that

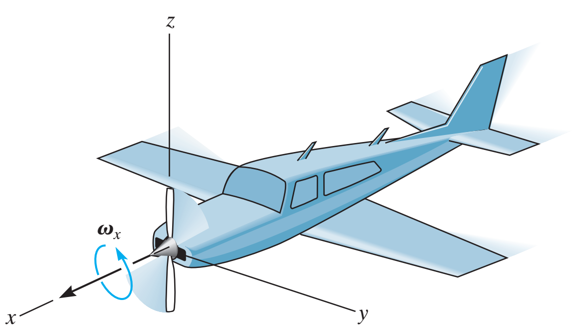
$$\mathbf{r}_{D/C} \cdot \mathbf{v}_{D/C} = (1\mathbf{i} + 2\mathbf{j} - 0.5\mathbf{k}) \cdot (-v_D\mathbf{k} - 3\mathbf{j}) = 0$$
$$(1)(0) + (2)(-3) + (-0.5)(-v_D) = 0$$
$$v_D = 12\ \text{m/s} \qquad \textit{Ans.}$$

Since $\boldsymbol{\omega}$ is *perpendicular* to $\mathbf{r}_{D/C}$ then $\mathbf{v}_{D/C} = \boldsymbol{\omega} \times \mathbf{r}_{D/C}$ or

$$v_{D/C} = \omega\, r_{D/C}$$
$$\sqrt{(-12)^2 + (-3)^2} = \omega\sqrt{(1)^2 + (2)^2 + (-0.5)^2}$$
$$\omega = 5.40\ \text{rad/s} \qquad \textit{Ans.}$$

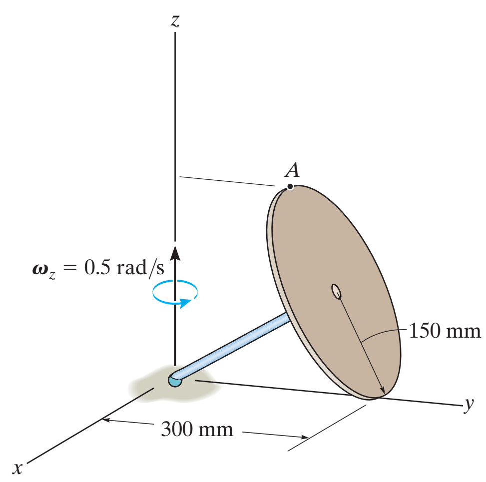
PROBLEMS

20–1. The propeller of an airplane is rotating at a constant speed $\omega_x\,\mathbf{i}$, while the plane is undergoing a turn at a constant rate ω_t. Determine the angular acceleration of the propeller if (a) the turn is horizontal, i.e., $\omega_t\mathbf{k}$, and (b) the turn is vertical, downward, i.e., $\omega_t\,\mathbf{j}$.



Prob. 20–1

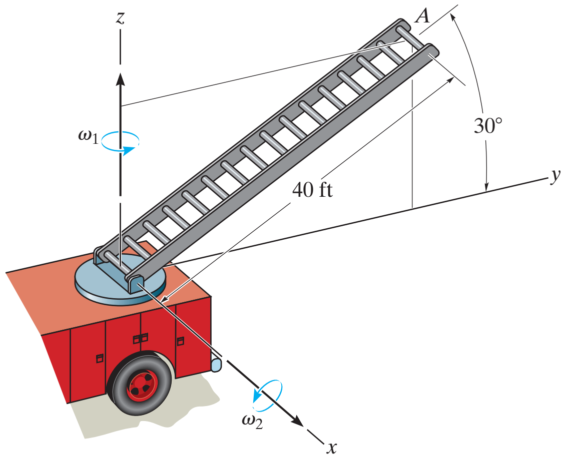
20–2. The disk rotates about the z axis at a constant rate $\omega_z = 0.5$ rad/s without slipping on the horizontal plane. Determine the velocity and the acceleration of point A on the disk.



Prob. 20–2

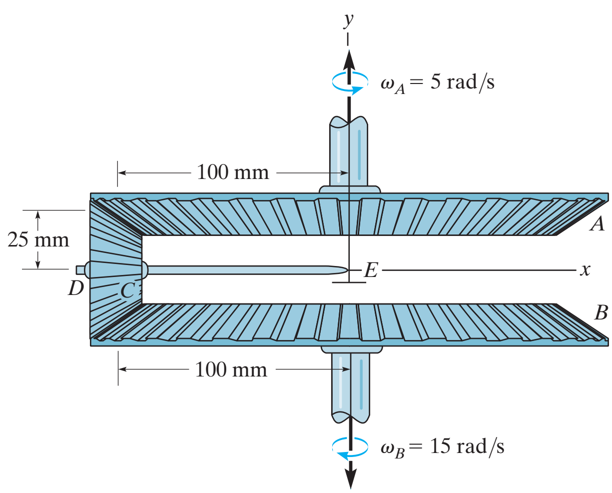
20–3. The ladder of the fire truck rotates around the z axis with an angular velocity $\omega_1 = 0.15$ rad/s, which is increasing at 0.8 rad/s^2. At the same instant it is rotating upward at a constant rate $\omega_2 = 0.6$ rad/s. Determine the velocity and acceleration of point A located at the top of the ladder at this instant.

***20–4.** The ladder of the fire truck rotates around the z axis with an angular velocity of $\omega_1 = 0.15$ rad/s, which is increasing at 0.2 rad/s^2. At the same instant it is rotating upward at $\omega_2 = 0.6$ rad/s while increasing at 0.4 rad/s^2. Determine the velocity and acceleration of point A located at the top of the ladder at this instant.



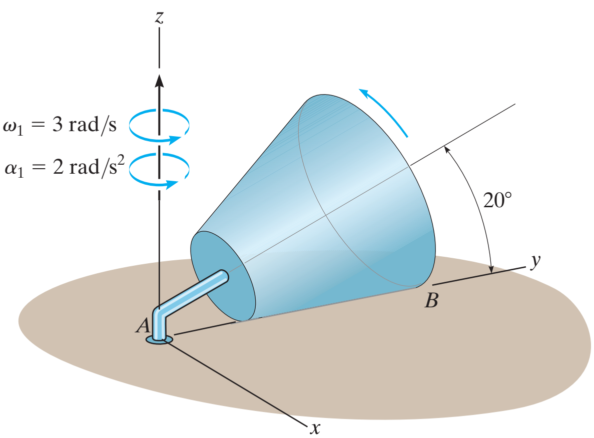
Probs. 20–3/4

20–5. If the plate gears A and B are rotating with the angular velocities shown, determine the angular velocity of gear C about the shaft DE. What is the angular velocity of DE about the y axis?



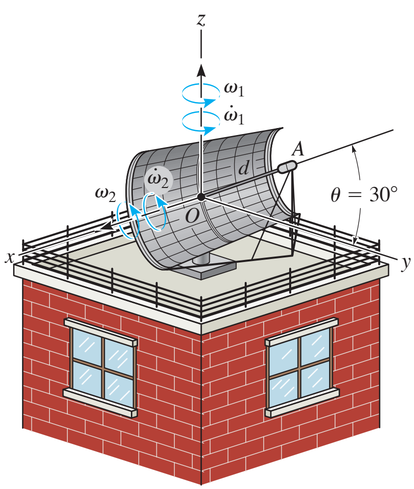
Prob. 20–5

20–6. The conical spool rolls on the plane without slipping. If the axle has an angular velocity of $\omega_1 = 3$ rad/s and an angular acceleration of $\alpha_1 = 2$ rad/s^2 at the instant shown, determine the angular velocity and angular acceleration of the spool at this instant.



Prob. 20–6

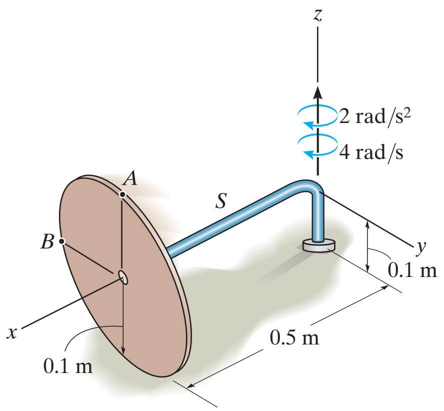
20–7. At a given instant, the antenna has an angular motion $\omega_1 = 3$ rad/s and $\dot{\omega}_1 = 2$ rad/s^2 about the z axis. At this same instant $\theta = 30°$, the angular motion about the x axis is $\omega_2 = 1.5$ rad/s, and $\dot{\omega}_2 = 4$ rad/s^2. Determine the velocity and acceleration of the signal horn A at this instant. The distance from O to A is $d = 3$ ft.



Prob. 20–7

*20–8. The disk rotates about the shaft S, while the shaft is turning about the z axis at a rate of $\omega_z = 4$ rad/s, which is increasing at 2 rad/s^2. Determine the velocity and acceleration of point A on the disk at the instant shown. No slipping occurs.

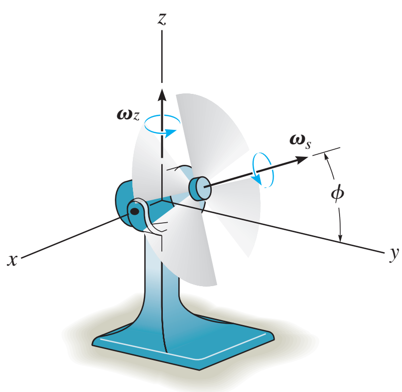
20–9. The disk rotates about the shaft S, while the shaft is turning about the z axis at a rate of $\omega_z = 4$ rad/s, which is increasing at 2 rad/s^2. Determine the velocity and acceleration of point B on the disk at the instant shown. No slipping occurs.



Probs. 20–8/9

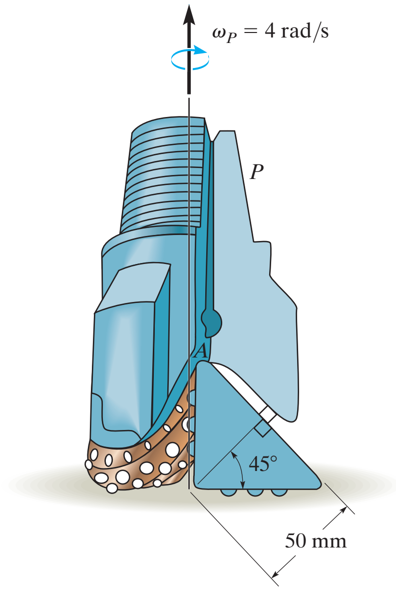
20–10. The electric fan is mounted on a swivel support such that the fan rotates about the z axis at a constant rate of $\omega_z = 1$ rad/s and the fan blade is spinning at a constant rate $\omega_s = 60$ rad/s. If $\phi = 45°$ for the motion, determine the angular velocity and the angular acceleration of the blade.

20–11. The electric fan is mounted on a swivel support such that the fan rotates about the z axis at a constant rate of $\omega_z = 1$ rad/s and the fan blade is spinning at a constant rate $\omega_s = 60$ rad/s. If at the instant $\phi = 45°$, $\dot{\phi} = 2$ rad/s for the motion, determine the angular velocity and the angular acceleration of the blade.



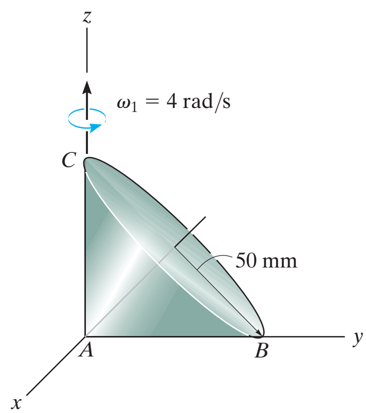
Probs. 20–10/11

*20–12. The drill pipe P turns at a constant angular rate $\omega_P = 4$ rad/s. Determine the angular velocity and angular acceleration of the conical rock bit, which rolls without slipping. Also, what are the velocity and acceleration of point A?



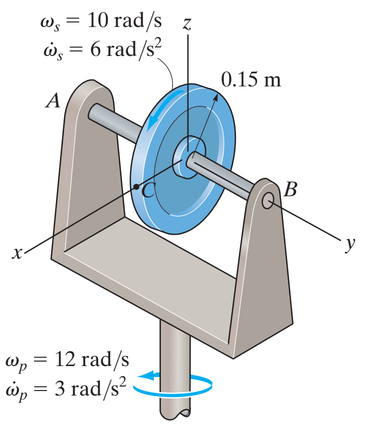
Prob. 20–12

20–13. The right circular cone rotates about the z axis at a constant rate of $\omega_1 = 4$ rad/s without slipping on the horizontal plane. Determine the magnitudes of the velocity and acceleration of points B and C.



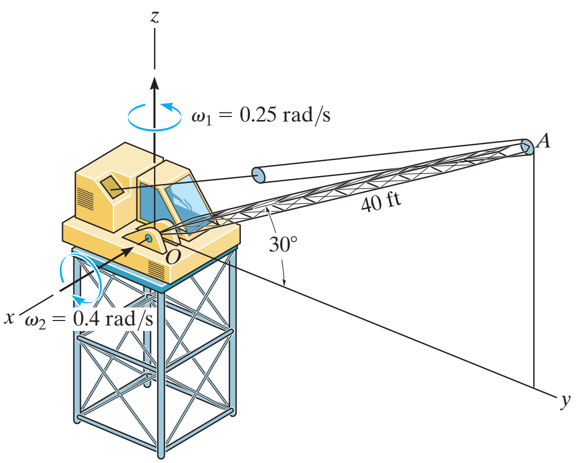
Prob. 20–13

20–14. The wheel is spinning about shaft AB with an angular velocity of $\omega_s = 10\ \text{rad/s}$, which is increasing at a constant rate of $\dot{\omega}_s = 6\ \text{rad/s}^2$, while the frame precesses about the z axis with an angular velocity of $\omega_p = 12\ \text{rad/s}$, which is increasing at a constant rate of $\dot{\omega}_p = 3\ \text{rad/s}^2$. Determine the velocity and acceleration of point C located on the rim of the wheel at this instant.



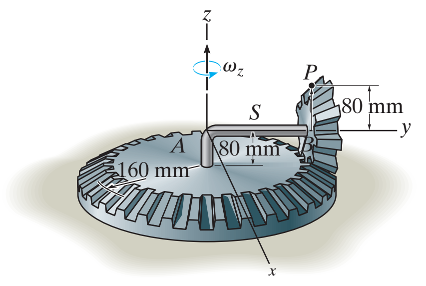
Prob. 20–14

20–15. At the instant shown, the tower crane rotates about the z axis with an angular velocity $\omega_1 = 0.25\ \text{rad/s}$, which is increasing at $0.6\ \text{rad/s}^2$. The boom OA rotates downward with an angular velocity $\omega_2 = 0.4\ \text{rad/s}$, which is increasing at $0.8\ \text{rad/s}^2$. Determine the velocity and acceleration of point A located at the end of the boom at this instant.



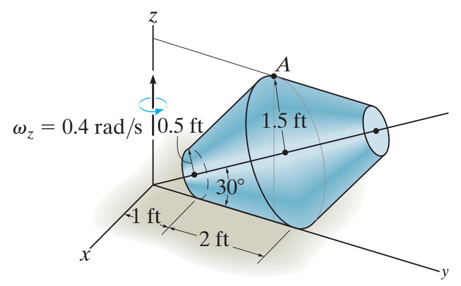
Prob. 20–15

***20–16.** Gear A is fixed while gear B is free to rotate on the shaft S. If the shaft is turning about the z axis at $\omega_z = 5\ \text{rad/s}$, while increasing at $2\ \text{rad/s}^2$, determine the velocity and acceleration of point P at the instant shown. The face of gear B lies in a vertical plane.



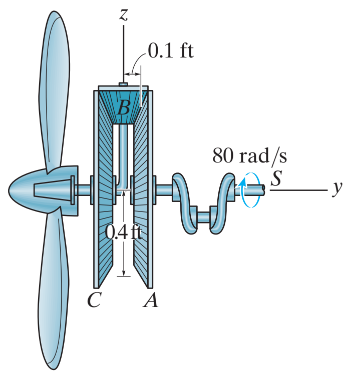
Prob. 20–16

20–17. The truncated double cone rotates about the z axis at $\omega_z = 0.4\ \text{rad/s}$ without slipping on the horizontal plane. If at this same instant ω_z is increasing at $\dot{\omega}_z = 0.5\ \text{rad/s}^2$, determine the velocity and acceleration of point A on the cone.



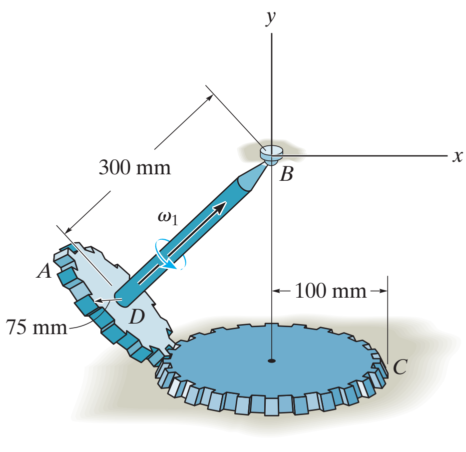
Prob. 20–17

20–18. Gear A is fixed to the crankshaft S, while gear C is fixed. Gear B and the propeller are free to rotate. The crankshaft is turning at $80\ \text{rad/s}$ about its axis. Determine the magnitudes of the angular velocity of the propeller and the angular acceleration of gear B.



Prob. 20–18

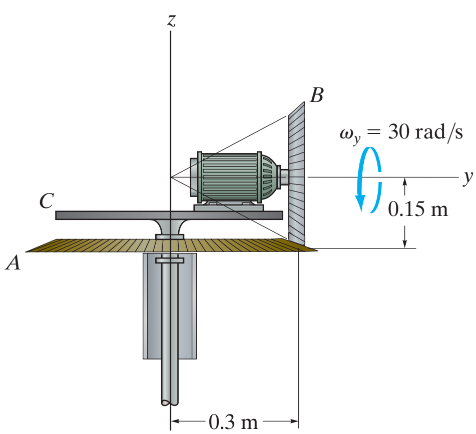
20–19. Shaft BD is connected to a ball-and-socket joint at B, and a beveled gear A is attached to its other end. The gear is in mesh with a fixed gear C. If the shaft and gear A are *spinning* with a constant angular velocity $\omega_1 = 8$ rad/s, determine the angular velocity and angular acceleration of gear A.



Prob. 20–19

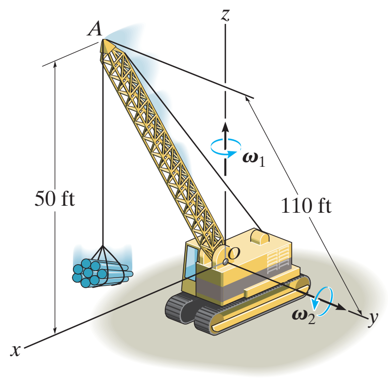
***20–20.** Gear B is driven by a motor mounted on turntable C. If gear A is held fixed, and the motor shaft rotates with a constant angular velocity of $\omega_y = 30$ rad/s, determine the angular velocity and angular acceleration of gear B.

20–21. Gear B is driven by a motor mounted on turntable C. If gear A and the motor shaft rotate with constant angular speeds of $\omega_A = \{10\mathbf{k}\}$ rad/s and $\omega_y = \{30\mathbf{j}\}$ rad/s, respectively, determine the angular velocity and angular acceleration of gear B.



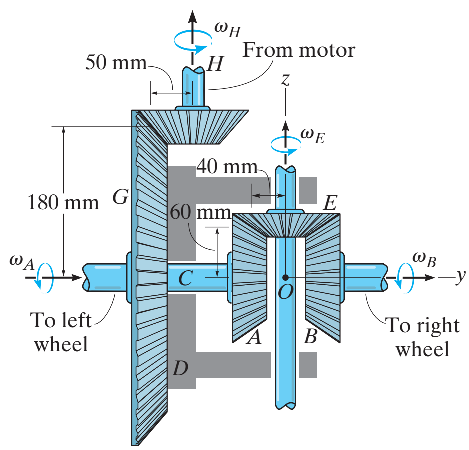
Probs. 20–20/21

20–22. The crane boom OA rotates about the z axis with a constant angular velocity of $\omega_1 = 0.15$ rad/s, while it is rotating downward with a constant angular velocity of $\omega_2 = 0.2$ rad/s. Determine the velocity and acceleration of point A located at the end of the boom at the instant shown.



Prob. 20–22

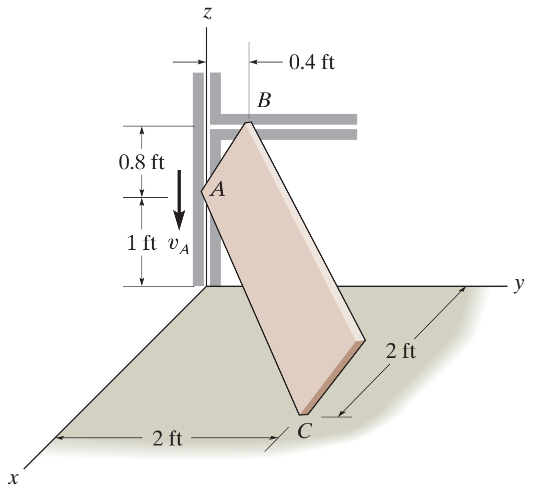
20–23. The differential of an automobile allows the two rear wheels to rotate at different speeds when the automobile travels along a curve. For operation, the rear axles are attached to the wheels at one end and have beveled gears A and B on their other ends. The differential case D is placed over the left axle but can rotate about C independent of the axle. The case supports a pinion gear E on a shaft, which meshes with gears A and B. Finally, a ring gear G is *fixed* to the differential case so that the case rotates with the ring gear when the latter is driven by the drive pinion H. This gear, like the differential case, is free to rotate about the left wheel axle. If the drive pinion is turning at $\omega_H = 100$ rad/s and the pinion gear E is spinning about its shaft at $\omega_E = 30$ rad/s, determine the angular velocity, ω_A and ω_B, of each axle.



Prob. 20–23

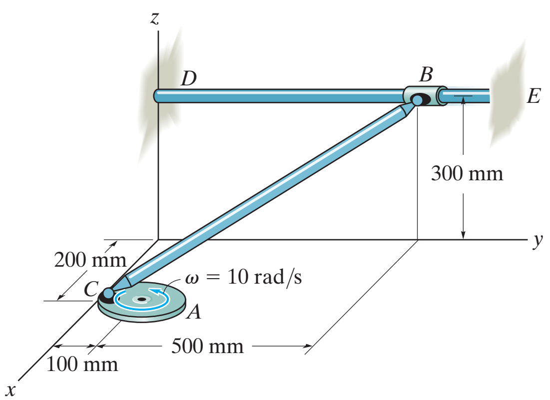
20

***20–24.** The end C of the plate rests on the horizontal plane, while end points A and B are restricted to move along the grooved slots. If at the instant shown A is moving downward with a constant velocity of $v_A = 4$ ft/s, determine the angular velocity of the plate and the velocities of points B and C.



Prob. 20–24

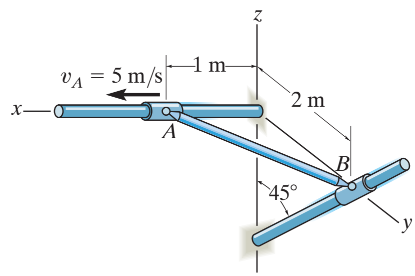
20–25. Disk A rotates at a constant angular velocity of 10 rad/s. If rod BC is joined to the disk and a collar by ball-and-socket joints, determine the velocity of collar B at the instant shown. Also, what is the rod's angular velocity $\boldsymbol{\omega}_{BC}$ if it is directed perpendicular to the axis of the rod?



Prob. 20–25

20–26. Rod AB is attached to collars at its ends by using ball-and-socket joints. If collar A moves along the fixed rod at $v_A = 5$ m/s, determine the angular velocity of the rod and the velocity of collar B at the instant shown. Assume that the rod's angular velocity is directed perpendicular to the axis of the rod.

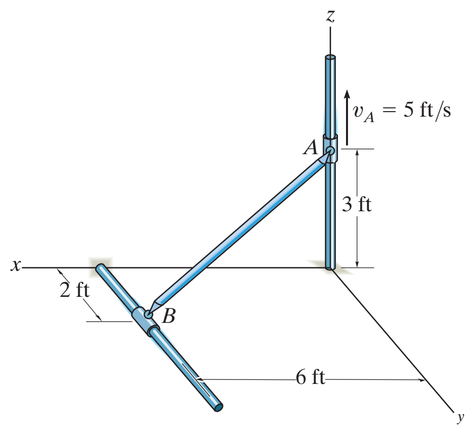
20–27. Rod AB is attached to collars at its ends by using ball-and-socket joints. If collar A moves along the fixed rod with a velocity of $v_A = 5$ m/s and has an acceleration $a_A = 2$ m/s^2 at the instant shown, determine the angular acceleration of the rod and the acceleration of collar B at this instant. Assume that the rod's angular velocity and angular acceleration are directed perpendicular to the axis of the rod.



Probs. 20–26/27

***20–28.** If the rod is attached with ball-and-socket joints to smooth collars A and B at its end points, determine the velocity of B at the instant shown if A is moving upward at a constant speed of $v_A = 5$ ft/s. Also, determine the angular velocity of the rod if it is directed perpendicular to the axis of the rod.

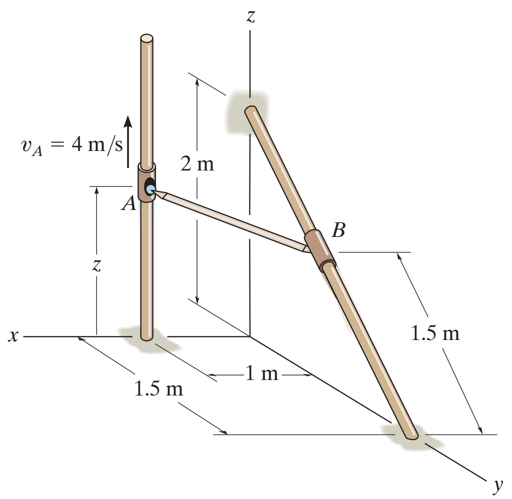
20–29. If the collar at A in Prob. 20–28 is moving upward with an acceleration of $\mathbf{a}_A = \{-2\mathbf{k}\}$ ft/s^2, at the instant its speed is $v_A = 5$ ft/s, determine the acceleration of the collar at B at this instant.



Probs. 20–28/29

20

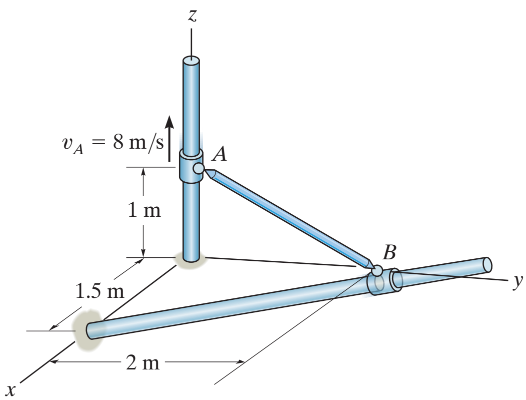
20–30. Rod AB is attached to collars at its ends by ball-and-socket joints. If collar A has a speed $v_A = 4$ m/s, determine the speed of collar B at the instant $z = 2$ m. Assume the angular velocity of the rod is directed perpendicular to the rod.



Prob. 20–30

20–31. The rod is attached to smooth collars A and B at its ends using ball-and-socket joints. Determine the speed of B at the instant shown if A is moving at $v_A = 8$ m/s. Also, determine the angular velocity of the rod if it is directed perpendicular to the axis of the rod.

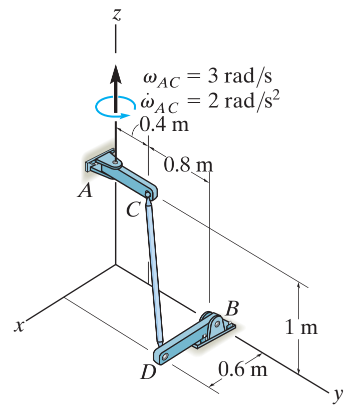
***20–32.** If the collar A in Prob. 20–31 has a deceleration of $\mathbf{a}_A = \{-5\mathbf{k}\}$ m/s^2, at the instant shown, determine the acceleration of collar B at this instant.



Probs. 20–31/32

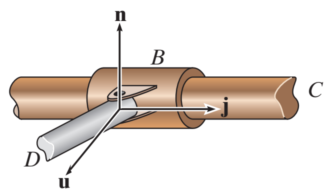
20–33. Rod CD is attached to the rotating arms using ball-and-socket joints. If AC has the motion shown, determine the angular velocity of link BD at the instant shown.

20–34. Rod CD is attached to the rotating arms using ball-and-socket joints. If AC has the motion shown, determine the angular acceleration of link BD at this instant.



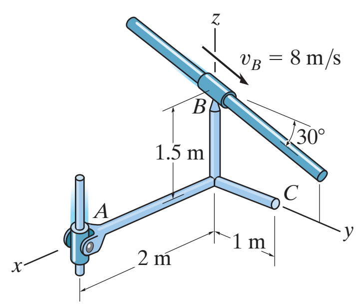
Probs. 20–33/34

20–35. Solve Prob. 20–28 if the connection at B consists of a pin as shown in the figure below, rather than a ball-and-socket joint. *Hint*: The constraint allows rotation of the rod both along the bar ($\mathbf{j}$ direction) and along the axis of the pin ($\mathbf{n}$ direction). Since there is no rotational component in the $\mathbf{u}$ direction, i.e., perpendicular to $\mathbf{n}$ and $\mathbf{j}$ where $\mathbf{u} = \mathbf{j} \times \mathbf{n}$, an additional equation for solution can be obtained from $\boldsymbol{\omega} \cdot \mathbf{u} = 0$. The vector $\mathbf{n}$ is in the same direction as $\mathbf{r}_{D/B} \times \mathbf{r}_{C/B}$.



Prob. 20–35

***20–36.** Member ABC is pin connected at A and has a ball-and-socket joint at B. If the collar at B is moving along the inclined rod at $v_B = 8$ m/s, determine the velocity of point C at the instant shown. *Hint*: See Prob. 20–35.



Prob. 20–36

*20.4 Relative-Motion Analysis Using Translating and Rotating Axes

The most general way to analyze the three-dimensional motion of a rigid body requires the use of x, y, z axes that both translate and rotate relative to a second frame X, Y, Z. This analysis also provides a means to determine the motions of two points A and B located on separate members of a mechanism, and the relative motion of one particle with respect to another when one or both particles are moving along *curved paths*.

As shown in Fig. 20–11, the locations of points A and B are specified relative to the X, Y, Z frame of reference by position vectors $\mathbf{r}_A$ and $\mathbf{r}_B$. The base point A represents the origin of the x, y, z coordinate system, which is translating and rotating with respect to X, Y, Z. At the instant considered, the velocity and acceleration of point A are $\mathbf{v}_A$ and $\mathbf{a}_A$, and the angular velocity and angular acceleration of the x, y, z axes are $\mathbf{\Omega}$ and $\dot{\mathbf{\Omega}} = d\mathbf{\Omega}/dt$. All these vectors are *measured* with respect to the X, Y, Z frame of reference, although they can be expressed in Cartesian component form along either set of axes.

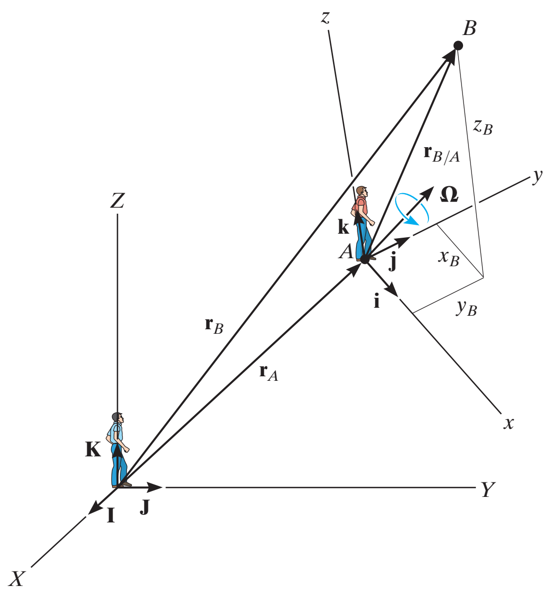


Fig. 20–11

Position. If the position of "B with respect to A" is specified by the *relative-position vector* $\mathbf{r}_{B/A}$, Fig. 20–11, then, by vector addition,

$$\boxed{\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}} \qquad (20\text{–}11)$$

where

$\mathbf{r}_B$ = position of B
$\mathbf{r}_A$ = position of the origin A
$\mathbf{r}_{B/A}$ = position of "B with respect to A"

Velocity. The velocity of point B measured from X, Y, Z can be determined by taking the time derivative of Eq. 20–11,

$$\dot{\mathbf{r}}_B = \dot{\mathbf{r}}_A + \dot{\mathbf{r}}_{B/A}$$

The first two terms represent $\mathbf{v}_B$ and $\mathbf{v}_A$. The last term must be evaluated by applying Eq. 20–6, since $\mathbf{r}_{B/A}$ is measured with respect to a rotating reference. Hence,

$$\dot{\mathbf{r}}_{B/A} = (\dot{\mathbf{r}}_{B/A})_{xyz} + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} = (\mathbf{v}_{B/A})_{xyz} + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} \qquad (20\text{–}12)$$

Therefore,

$$\boxed{\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} + (\mathbf{v}_{B/A})_{xyz}} \qquad (20\text{–}13)$$

where

$\mathbf{v}_B$ = velocity of B
$\mathbf{v}_A$ = velocity of the origin A of the x, y, z frame of reference
$(\mathbf{v}_{B/A})_{xyz}$ = velocity of "B with respect to A" as measured by an observer attached to the rotating x, y, z frame of reference
$\boldsymbol{\Omega}$ = angular velocity of the x, y, z frame of reference
$\mathbf{r}_{B/A}$ = position of "B with respect to A"

Acceleration. The acceleration of point B measured from X, Y, Z is determined by taking the time derivative of Eq. 20–13.

$$\dot{\mathbf{v}}_B = \dot{\mathbf{v}}_A + \dot{\mathbf{\Omega}} \times \mathbf{r}_{B/A} + \mathbf{\Omega} \times \dot{\mathbf{r}}_{B/A} + \frac{d}{dt}(\mathbf{v}_{B/A})_{xyz}$$

The time derivatives defined in the first and second terms represent $\mathbf{a}_B$ and $\mathbf{a}_A$, respectively. The fourth term can be evaluated using Eq. 20–12, and the last term is evaluated by applying Eq. 20–6, which yields

$$\frac{d}{dt}(\mathbf{v}_{B/A})_{xyz} = (\dot{\mathbf{v}}_{B/A})_{xyz} + \mathbf{\Omega} \times (\mathbf{v}_{B/A})_{xyz} = (\mathbf{a}_{B/A})_{xyz} + \mathbf{\Omega} \times (\mathbf{v}_{B/A})_{xyz}$$

Here $(\mathbf{a}_{B/A})_{xyz}$ is the acceleration of B with respect to A measured from x, y, z. Substituting this result and Eq. 20–12 into the above equation and simplifying, we have

$$\boxed{\mathbf{a}_B = \mathbf{a}_A + \dot{\mathbf{\Omega}} \times \mathbf{r}_{B/A} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}_{B/A}) + 2\mathbf{\Omega} \times (\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz}} \tag{20–14}$$

where

$\mathbf{a}_B$ = acceleration of B

$\mathbf{a}_A$ = acceleration of the origin A of the x, y, z frame of reference

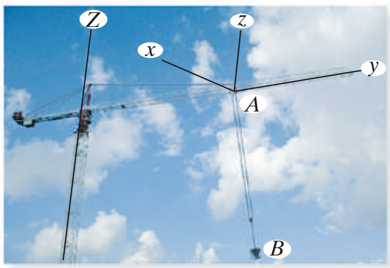
$(\mathbf{a}_{B/A})_{xyz}, (\mathbf{v}_{B/A})_{xyz}$ = relative acceleration and relative velocity of "B with respect to A" as measured by an observer attached to the rotating x, y, z frame of reference

$\dot{\mathbf{\Omega}}, \mathbf{\Omega}$ = angular acceleration and angular velocity of the x, y, z frame of reference

$\mathbf{r}_{B/A}$ = position of "B with respect to A"

Equations 20–13 and 20–14 are identical to those used in Sec. 16.8 for analyzing relative plane motion.* In that case, however, application is simplified since $\mathbf{\Omega}$ and $\dot{\mathbf{\Omega}}$ have a *constant direction* which is always perpendicular to the plane of motion. For three-dimensional motion, $\dot{\mathbf{\Omega}}$ must be computed by using Eq. 20–6, since $\dot{\mathbf{\Omega}}$ depends on the change in *both* the magnitude and direction of $\mathbf{\Omega}$.

*Refer to Sec. 16.8 for an interpretation of the terms.



Complicated spatial motion of the concrete bucket B occurs due to the rotation of the boom about the Z axis, motion of the carriage A along the boom, and extension and swinging of the cable AB. A translating-rotating x, y, z coordinate system can be established on the carriage, and a relative-motion analysis can then be applied to study this motion. (© R.C. Hibbeler)

Procedure for Analysis

Three-dimensional motion of particles or rigid bodies can be analyzed with Eqs. 20–13 and 20–14 by using the following procedure.

Coordinate Axes.

- Select the location and orientation of the X, Y, Z and x, y, z coordinate axes. Most often solutions can be easily obtained if at the instant considered:

  (1) the origins are *coincident*

  (2) the axes are collinear

  (3) the axes are parallel

- If several components of angular velocity are involved in a problem, the calculations will be reduced if the x, y, z axes are selected such that only one component of angular velocity is observed with respect to this frame $(\boldsymbol{\Omega}_{xyz})$ and the frame rotates with $\boldsymbol{\Omega}$ defined by the other components of angular velocity.

Kinematic Equations.

- After the origin of the moving reference, A, is defined and the moving point B is specified, Eqs. 20–13 and 20–14 should then be written in symbolic form as

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} + (\mathbf{v}_{B/A})_{xyz}$$

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{B/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{B/A}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz}$$

- If $\mathbf{r}_A$ and $\boldsymbol{\Omega}$ appear to *change direction* when observed from the fixed X, Y, Z reference then use a set of primed reference axes, x', y', z' having a rotation $\boldsymbol{\Omega}' = \boldsymbol{\Omega}$. Equation 20–6 is then used to determine $\dot{\boldsymbol{\Omega}}$ and the motion $\mathbf{v}_A$ and $\mathbf{a}_A$ of the origin of the moving x, y, z axes.

- If $\mathbf{r}_{B/A}$ and $\boldsymbol{\Omega}_{xyz}$ appear to change direction as observed from x, y, z, then use a set of double-primed reference axes x'', y'', z'' having $\boldsymbol{\Omega}'' = \boldsymbol{\Omega}_{xyz}$ and apply Eq. 20–6 to determine $\dot{\boldsymbol{\Omega}}_{xyz}$ and the relative motion $(\mathbf{v}_{B/A})_{xyz}$ and $(\mathbf{a}_{B/A})_{xyz}$.

- After the final forms of $\dot{\boldsymbol{\Omega}}$, $\mathbf{v}_A$, $\mathbf{a}_A$, $\dot{\boldsymbol{\Omega}}_{xyz}$, $(\mathbf{v}_{B/A})_{xyz}$, and $(\mathbf{a}_{B/A})_{xyz}$ are obtained, numerical problem data can be substituted and the kinematic terms evaluated. The components of all these vectors can be selected either along the X, Y, Z or along the x, y, z axes. The choice is arbitrary, provided a consistent set of unit vectors is used.

EXAMPLE 20.4

A motor and attached rod AB have the angular motions shown in Fig. 20–12. A collar C on the rod is located 0.25 m from A and is moving downward along the rod with a velocity of 3 m/s and an acceleration of 2 m/s^2. Determine the velocity and acceleration of C at this instant.

SOLUTION

Coordinate Axes.

The origin of the fixed X, Y, Z reference is chosen at the center of the platform, and the origin of the moving x, y, z frame at point A, Fig. 20–12. Since the collar is subjected to two components of angular motion, $\boldsymbol{\omega}_p$ and $\boldsymbol{\omega}_M$, it will be viewed as having an angular velocity of $\boldsymbol{\Omega}_{xyz} = \boldsymbol{\omega}_M$ in x, y, z. Therefore, the x, y, z axes will be attached to the platform so that $\boldsymbol{\Omega} = \boldsymbol{\omega}_p$.

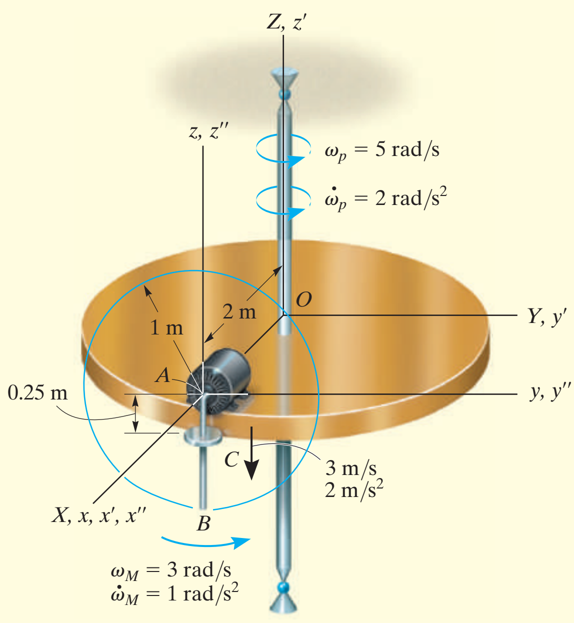


Fig. 20–12

Kinematic Equations. Equations 20–13 and 20–14, applied to points C and A, become

$$\mathbf{v}_C = \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{C/A} + (\mathbf{v}_{C/A})_{xyz}$$

$$\mathbf{a}_C = \mathbf{a}_A + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{C/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{C/A}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{C/A})_{xyz} + (\mathbf{a}_{C/A})_{xyz}$$

Motion of A. Here $\mathbf{r}_A$ changes direction relative to X, Y, Z. To find the time derivatives of $\mathbf{r}_A$ we will use a set of x', y', z' axes coincident with the X, Y, Z axes that rotate at $\boldsymbol{\Omega}' = \boldsymbol{\omega}_p$. Thus,

$\boldsymbol{\Omega} = \boldsymbol{\omega}_p = \{5\mathbf{k}\}$ rad/s ($\boldsymbol{\Omega}$ does not change direction relative to X, Y, Z.)

$$\dot{\boldsymbol{\Omega}} = \dot{\boldsymbol{\omega}}_p = \{2\mathbf{k}\}\ \text{rad/s}^2$$

$$\mathbf{r}_A = \{2\mathbf{i}\}\ \text{m}$$

$$\mathbf{v}_A = \dot{\mathbf{r}}_A = (\dot{\mathbf{r}}_A)_{x'y'z'} + \boldsymbol{\omega}_p \times \mathbf{r}_A = \mathbf{0} + 5\mathbf{k} \times 2\mathbf{i} = \{10\mathbf{j}\}\ \text{m/s}$$

$$\begin{aligned}\mathbf{a}_A = \ddot{\mathbf{r}}_A &= [(\ddot{\mathbf{r}}_A)_{x'y'z'} + \boldsymbol{\omega}_p \times (\dot{\mathbf{r}}_A)_{x'y'z'}] + \dot{\boldsymbol{\omega}}_p \times \mathbf{r}_A + \boldsymbol{\omega}_p \times \dot{\mathbf{r}}_A \\ &= [\mathbf{0} + \mathbf{0}] + 2\mathbf{k} \times 2\mathbf{i} + 5\mathbf{k} \times 10\mathbf{j} = \{-50\mathbf{i} + 4\mathbf{j}\}\ \text{m/s}^2\end{aligned}$$

Motion of C with Respect to A. Here $\mathbf{r}_{C/A}$ changes direction relative to x, y, z, and so to find its time derivatives use a set of x'', y'', z'' axes that rotate at $\boldsymbol{\Omega}'' = \boldsymbol{\Omega}_{xyz} = \boldsymbol{\omega}_M$. Thus,

$\boldsymbol{\Omega}_{xyz} = \boldsymbol{\omega}_M = \{3\mathbf{i}\}$ rad/s ($\boldsymbol{\Omega}_{xyz}$ does not change direction relative to x, y, z.)

$$\dot{\boldsymbol{\Omega}}_{xyz} = \dot{\boldsymbol{\omega}}_M = \{1\mathbf{i}\}\ \text{rad/s}^2$$

$$\mathbf{r}_{C/A} = \{-0.25\mathbf{k}\}\ \text{m}$$

$$\begin{aligned}(\mathbf{v}_{C/A})_{xyz} = (\dot{\mathbf{r}}_{C/A})_{xyz} &= (\dot{\mathbf{r}}_{C/A})_{x''y''z''} + \boldsymbol{\omega}_M \times \mathbf{r}_{C/A} \\ &= -3\mathbf{k} + [3\mathbf{i} \times (-0.25\mathbf{k})] = \{0.75\mathbf{j} - 3\mathbf{k}\}\ \text{m/s}\end{aligned}$$

$$\begin{aligned}(\mathbf{a}_{C/A})_{xyz} = (\ddot{\mathbf{r}}_{C/A})_{xyz} &= [(\ddot{\mathbf{r}}_{C/A})_{x''y''z''} + \boldsymbol{\omega}_M \times (\dot{\mathbf{r}}_{C/A})_{x''y''z''}] + \dot{\boldsymbol{\omega}}_M \times \mathbf{r}_{C/A} + \boldsymbol{\omega}_M \times (\dot{\mathbf{r}}_{C/A})_{xyz} \\ &= [-2\mathbf{k} + 3\mathbf{i} \times (-3\mathbf{k})] + (1\mathbf{i}) \times (-0.25\mathbf{k}) + (3\mathbf{i}) \times (0.75\mathbf{j} - 3\mathbf{k}) \\ &= \{18.25\mathbf{j} + 0.25\mathbf{k}\}\ \text{m/s}^2\end{aligned}$$

Motion of C.

$$\begin{aligned}\mathbf{v}_C &= \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{C/A} + (\mathbf{v}_{C/A})_{xyz} \\ &= 10\mathbf{j} + [5\mathbf{k} \times (-0.25\mathbf{k})] + (0.75\mathbf{j} - 3\mathbf{k}) \\ &= \{10.75\mathbf{j} - 3\mathbf{k}\}\ \text{m/s} \qquad \textit{Ans.}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_C &= \mathbf{a}_A + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{C/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{C/A}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{C/A})_{xyz} + (\mathbf{a}_{C/A})_{xyz} \\ &= (-50\mathbf{i} + 4\mathbf{j}) + [2\mathbf{k} \times (-0.25\mathbf{k})] + 5\mathbf{k} \times [5\mathbf{k} \times (-0.25\mathbf{k})] \\ &\quad + 2[5\mathbf{k} \times (0.75\mathbf{j} - 3\mathbf{k})] + (18.25\mathbf{j} + 0.25\mathbf{k}) \\ &= \{-57.5\mathbf{i} + 22.25\mathbf{j} + 0.25\mathbf{k}\}\ \text{m/s}^2 \qquad \textit{Ans.}\end{aligned}$$

EXAMPLE 20.5

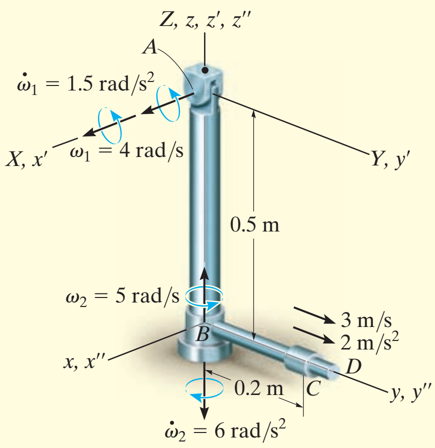


Fig. 20–13

The pendulum shown in Fig. 20–13 consists of two rods; AB is pin supported at A and swings only in the Y–Z plane, whereas a bearing at B allows the attached rod BD to spin about rod AB. At a given instant, the rods have the angular motions shown. Also, a collar C, located 0.2 m from B, has a velocity of 3 m/s and an acceleration of 2 m/s² along the rod. Determine the velocity and acceleration of the collar at this instant.

SOLUTION I

Coordinate Axes. The origin of the fixed X, Y, Z frame will be placed at A. Motion of the collar is conveniently observed from B, so the origin of the x, y, z frame is located at this point. We will choose $\boldsymbol{\Omega} = \boldsymbol{\omega}_1$ and $\boldsymbol{\Omega}_{xyz} = \boldsymbol{\omega}_2$.

Kinematic Equations.

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\Omega} \times \mathbf{r}_{C/B} + (\mathbf{v}_{C/B})_{xyz}$$
$$\mathbf{a}_C = \mathbf{a}_B + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{C/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{C/B}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{C/B})_{xyz} + (\mathbf{a}_{C/B})_{xyz}$$

Motion of B. To find the time derivatives of $\mathbf{r}_B$ let the x', y', z' axes rotate with $\boldsymbol{\Omega}' = \boldsymbol{\omega}_1$. Then

$$\boldsymbol{\Omega}' = \boldsymbol{\omega}_1 = \{4\mathbf{i}\}\ \text{rad/s} \qquad \dot{\boldsymbol{\Omega}}' = \dot{\boldsymbol{\omega}}_1 = \{1.5\mathbf{i}\}\ \text{rad/s}^2$$

$$\mathbf{r}_B = \{-0.5\mathbf{k}\}\ \text{m}$$

$$\mathbf{v}_B = \dot{\mathbf{r}}_B = (\dot{\mathbf{r}}_B)_{x'y'z'} + \boldsymbol{\omega}_1 \times \mathbf{r}_B = \mathbf{0} + 4\mathbf{i} \times (-0.5\mathbf{k}) = \{2\mathbf{j}\}\ \text{m/s}$$

$$\mathbf{a}_B = \ddot{\mathbf{r}}_B = [(\ddot{\mathbf{r}}_B)_{x'y'z'} + \boldsymbol{\omega}_1 \times (\dot{\mathbf{r}}_B)_{x'y'z'}] + \dot{\boldsymbol{\omega}}_1 \times \mathbf{r}_B + \boldsymbol{\omega}_1 \times \dot{\mathbf{r}}_B$$
$$= [\mathbf{0} + \mathbf{0}] + 1.5\mathbf{i} \times (-0.5\mathbf{k}) + 4\mathbf{i} \times 2\mathbf{j} = \{0.75\mathbf{j} + 8\mathbf{k}\}\ \text{m/s}^2$$

Motion of C with Respect to B. To find the time derivatives of $\mathbf{r}_{C/B}$ relative to x, y, z, let the x'', y'', z'' axes rotate with $\boldsymbol{\Omega}_{xyz} = \boldsymbol{\omega}_2$. Then

$$\boldsymbol{\Omega}_{xyz} = \boldsymbol{\omega}_2 = \{5\mathbf{k}\}\ \text{rad/s} \qquad \dot{\boldsymbol{\Omega}}_{xyz} = \dot{\boldsymbol{\omega}}_2 = \{-6\mathbf{k}\}\ \text{rad/s}^2$$

$$\mathbf{r}_{C/B} = \{0.2\mathbf{j}\}\ \text{m}$$

$$(\mathbf{v}_{C/B})_{xyz} = (\dot{\mathbf{r}}_{C/B})_{xyz} = (\dot{\mathbf{r}}_{C/B})_{x''y''z''} + \boldsymbol{\omega}_2 \times \mathbf{r}_{C/B} = 3\mathbf{j} + 5\mathbf{k} \times 0.2\mathbf{j} = \{-1\mathbf{i} + 3\mathbf{j}\}\ \text{m/s}$$

$$(\mathbf{a}_{C/B})_{xyz} = (\ddot{\mathbf{r}}_{C/B})_{xyz} = [(\ddot{\mathbf{r}}_{C/B})_{x''y''z''} + \boldsymbol{\omega}_2 \times (\dot{\mathbf{r}}_{C/B})_{x''y''z''}] + \dot{\boldsymbol{\omega}}_2 \times \mathbf{r}_{C/B} + \boldsymbol{\omega}_2 \times (\dot{\mathbf{r}}_{C/B})_{xyz}$$
$$= (2\mathbf{j} + 5\mathbf{k} \times 3\mathbf{j}) + (-6\mathbf{k} \times 0.2\mathbf{j}) + [5\mathbf{k} \times (-1\mathbf{i} + 3\mathbf{j})]$$
$$= \{-28.8\mathbf{i} - 3\mathbf{j}\}\ \text{m/s}^2$$

Motion of C.

$$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\Omega} \times \mathbf{r}_{C/B} + (\mathbf{v}_{C/B})_{xyz} = 2\mathbf{j} + 4\mathbf{i} \times 0.2\mathbf{j} + (-1\mathbf{i} + 3\mathbf{j})$$
$$= \{-1\mathbf{i} + 5\mathbf{j} + 0.8\mathbf{k}\}\ \text{m/s}$$ *Ans.*

$$\mathbf{a}_C = \mathbf{a}_B + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{C/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{C/B}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{C/B})_{xyz} + (\mathbf{a}_{C/B})_{xyz}$$
$$= (0.75\mathbf{j} + 8\mathbf{k}) + (1.5\mathbf{i} \times 0.2\mathbf{j}) + [4\mathbf{i} \times (4\mathbf{i} \times 0.2\mathbf{j})]$$
$$+ 2[4\mathbf{i} \times (-1\mathbf{i} + 3\mathbf{j})] + (-28.8\mathbf{i} - 3\mathbf{j})$$
$$= \{-28.8\mathbf{i} - 5.45\mathbf{j} + 32.3\mathbf{k}\}\ \text{m/s}^2$$ *Ans.*

SOLUTION II

Coordinate Axes. Here we will let the x, y, z axes rotate at

$$\boldsymbol{\Omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 = \{4\mathbf{i} + 5\mathbf{k}\} \text{ rad/s}$$

Then $\boldsymbol{\Omega}_{xyz} = \mathbf{0}$.

Motion of *B*. From the constraints of the problem $\boldsymbol{\omega}_1$ does not change direction relative to X, Y, Z; however, the direction of $\boldsymbol{\omega}_2$ is changed by $\boldsymbol{\omega}_1$. Thus, to obtain $\dot{\boldsymbol{\Omega}}$ consider x', y', z' axes coincident with the X, Y, Z axes at A, so that $\boldsymbol{\Omega}' = \boldsymbol{\omega}_1$. Then taking the derivative of the components of $\boldsymbol{\Omega}$,

$$\begin{aligned}\dot{\boldsymbol{\Omega}} &= \dot{\boldsymbol{\omega}}_1 + \dot{\boldsymbol{\omega}}_2 = [(\dot{\boldsymbol{\omega}}_1)_{x'y'z'} + \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_1] + [(\dot{\boldsymbol{\omega}}_2)_{x'y'z'} + \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2]\\ &= [1.5\mathbf{i} + \mathbf{0}] + [-6\mathbf{k} + 4\mathbf{i} \times 5\mathbf{k}] = \{1.5\mathbf{i} - 20\mathbf{j} - 6\mathbf{k}\} \text{ rad/s}^2\end{aligned}$$

Also, $\boldsymbol{\omega}_1$ changes the direction of $\mathbf{r}_B$ so that the time derivatives of $\mathbf{r}_B$ can be found using the primed axes defined above. Hence,

$$\begin{aligned}\mathbf{v}_B &= \dot{\mathbf{r}}_B = (\dot{\mathbf{r}}_B)_{x'y'z'} + \boldsymbol{\omega}_1 \times \mathbf{r}_B\\ &= \mathbf{0} + 4\mathbf{i} \times (-0.5\mathbf{k}) = \{2\mathbf{j}\} \text{ m/s}\\ \mathbf{a}_B &= \ddot{\mathbf{r}}_B = [(\ddot{\mathbf{r}}_B)_{x'y'z'} + \boldsymbol{\omega}_1 \times (\dot{\mathbf{r}}_B)_{x'y'z'}] + \dot{\boldsymbol{\omega}}_1 \times \mathbf{r}_B + \boldsymbol{\omega}_1 \times \dot{\mathbf{r}}_B\\ &= [\mathbf{0} + \mathbf{0}] + 1.5\mathbf{i} \times (-0.5\mathbf{k}) + 4\mathbf{i} \times 2\mathbf{j} = \{0.75\mathbf{j} + 8\mathbf{k}\} \text{ m/s}^2\end{aligned}$$

Motion of *C* with Respect to *B*.

$$\begin{aligned}\boldsymbol{\Omega}_{xyz} &= \mathbf{0}\\ \dot{\boldsymbol{\Omega}}_{xyz} &= \mathbf{0}\\ \mathbf{r}_{C/B} &= \{0.2\mathbf{j}\} \text{ m}\\ (\mathbf{v}_{C/B})_{xyz} &= \{3\mathbf{j}\} \text{ m/s}\\ (\mathbf{a}_{C/B})_{xyz} &= \{2\mathbf{j}\} \text{ m/s}^2\end{aligned}$$

Motion of *C*.

$$\begin{aligned}\mathbf{v}_C &= \mathbf{v}_B + \boldsymbol{\Omega} \times \mathbf{r}_{C/B} + (\mathbf{v}_{C/B})_{xyz}\\ &= 2\mathbf{j} + [(4\mathbf{i} + 5\mathbf{k}) \times (0.2\mathbf{j})] + 3\mathbf{j}\\ &= \{-1\mathbf{i} + 5\mathbf{j} + 0.8\mathbf{k}\} \text{ m/s} \qquad \textit{Ans.}\end{aligned}$$

$$\begin{aligned}\mathbf{a}_C &= \mathbf{a}_B + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{C/B} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{C/B}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{C/B})_{xyz} + (\mathbf{a}_{C/B})_{xyz}\\ &= (0.75\mathbf{j} + 8\mathbf{k}) + [(1.5\mathbf{i} - 20\mathbf{j} - 6\mathbf{k}) \times (0.2\mathbf{j})]\\ &\quad + (4\mathbf{i} + 5\mathbf{k}) \times [(4\mathbf{i} + 5\mathbf{k}) \times 0.2\mathbf{j}] + 2[(4\mathbf{i} + 5\mathbf{k}) \times 3\mathbf{j}] + 2\mathbf{j}\\ &= \{-28.8\mathbf{i} - 5.45\mathbf{j} + 32.3\mathbf{k}\} \text{ m/s}^2 \qquad \textit{Ans.}\end{aligned}$$

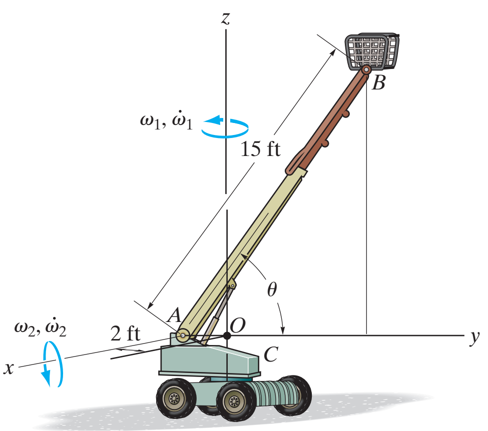
PROBLEMS

20–37. Solve Example 20.5 such that the x, y, z axes move with curvilinear translation, $\mathbf{\Omega} = \mathbf{0}$ in which case the collar appears to have both an angular velocity $\mathbf{\Omega}_{xyz} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$ and radial motion.

20–38. Solve Example 20.5 by fixing x, y, z axes to rod BD so that $\mathbf{\Omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$. In this case the collar appears only to move radially outward along BD; hence $\mathbf{\Omega}_{xyz} = \mathbf{0}$.

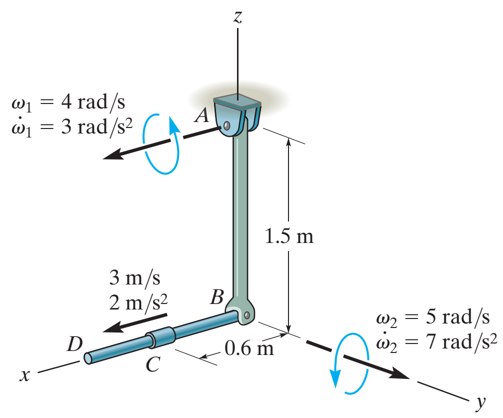
20–39. At the instant $\theta = 60°$, the telescopic boom AB of the construction lift is rotating with a constant angular velocity about the z axis of $\omega_1 = 0.5$ rad/s and about the pin at A with a constant angular speed of $\omega_2 = 0.25$ rad/s. Simultaneously, the boom is extending with a velocity of 1.5 ft/s, and it has an acceleration of 0.5 ft/s^2, both measured relative to the construction lift. Determine the velocity and acceleration of point B located at the end of the boom at this instant.

***20–40.** At the instant $\theta = 60°$, the construction lift is rotating about the z axis with an angular velocity of $\omega_1 = 0.5$ rad/s and an angular acceleration of $\dot{\omega}_1 = 0.25$ rad/s^2 while the telescopic boom AB rotates about the pin at A with an angular velocity of $\omega_2 = 0.25$ rad/s and angular acceleration of $\dot{\omega}_2 = 0.1$ rad/s^2. Simultaneously, the boom is extending with a velocity of 1.5 ft/s, and it has an acceleration of 0.5 ft/s^2, both measured relative to the frame. Determine the velocity and acceleration of point B located at the end of the boom at this instant.



Probs. 20–39/40

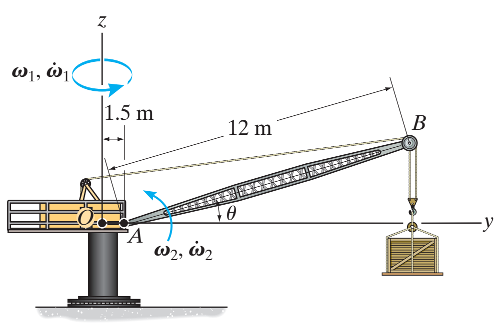
20–41. At the instant shown, the arm AB is rotating about the fixed pin A with an angular velocity $\omega_1 = 4$ rad/s and angular acceleration $\dot{\omega}_1 = 3$ rad/s^2. At this same instant, rod BD is rotating relative to rod AB with an angular velocity $\omega_2 = 5$ rad/s, which is increasing at $\dot{\omega}_2 = 7$ rad/s^2. Also, the collar C is moving along rod BD with a velocity of 3 m/s and an acceleration of 2 m/s^2, both measured relative to the rod. Determine the velocity and acceleration of the collar at this instant.



Prob. 20–41

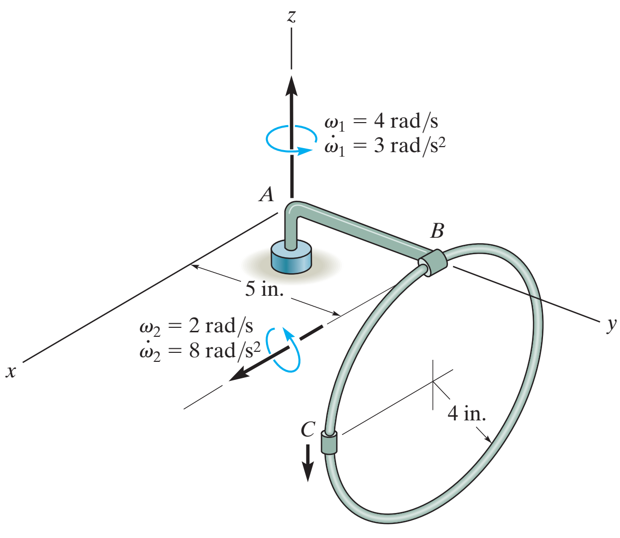
20–42. At the instant $\theta = 30°$, the frame of the crane and the boom AB rotate with a constant angular velocity of $\omega_1 = 1.5$ rad/s and $\omega_2 = 0.5$ rad/s, respectively. Determine the velocity and acceleration of point B at this instant.

20–43. At the instant $\theta = 30°$, the frame of the crane is rotating with an angular velocity of $\omega_1 = 1.5$ rad/s and angular acceleration of $\dot{\omega}_1 = 0.5$ rad/s^2, while the boom AB rotates with an angular velocity of $\omega_2 = 0.5$ rad/s and angular acceleration of $\dot{\omega}_2 = 0.25$ rad/s^2. Determine the velocity and acceleration of point B at this instant.



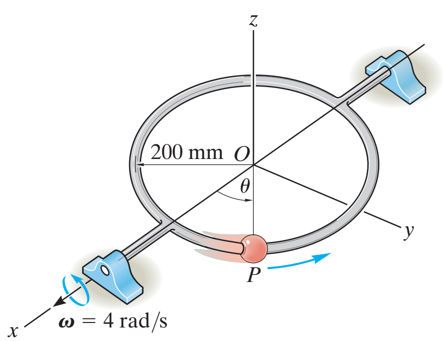
Probs. 20–42/43

***20–44.** At the instant shown, the rod AB is rotating about the z axis with an angular velocity $\omega_1 = 4$ rad/s and an angular acceleration $\dot{\omega}_1 = 3$ rad/s^2. At this same instant, the circular rod has an angular motion relative to the rod as shown. If the collar C is moving down around the circular rod with a speed of 3 in./s, which is increasing at 8 in./s^2, both measured relative to the rod, determine the collar's velocity and acceleration at this instant.



Prob. 20–44

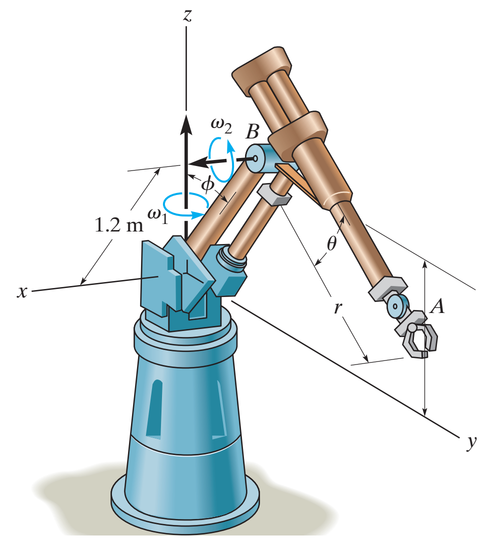
20–45. The particle P slides around the circular hoop with a constant angular velocity of $\dot{\theta} = 6$ rad/s, while the hoop rotates about the x axis at a constant rate of $\omega = 4$ rad/s. If at the instant shown the hoop is in the x–y plane and the angle $\theta = 45°$, determine the velocity and acceleration of the particle at this instant.



Prob. 20–45

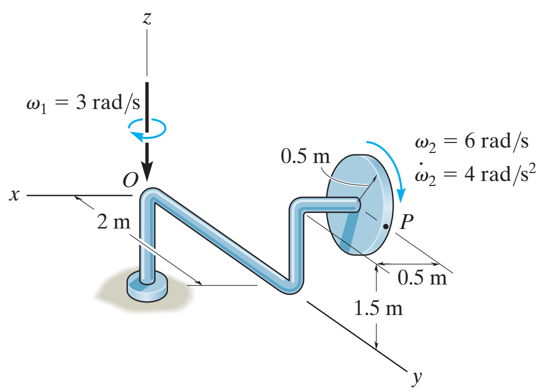
20–46. At the instant shown, the industrial manipulator is rotating about the z axis at $\omega_1 = 5$ rad/s, and about joint B at $\omega_2 = 2$ rad/s. Determine the velocity and acceleration of the grip A at this instant, when $\phi = 30°$, $\theta = 45°$, and $r = 1.6$ m.

20–47. At the instant shown, the industrial manipulator is rotating about the z axis at $\omega_1 = 5$ rad/s, and $\dot{\omega}_1 = 2$ rad/s^2; and about joint B at $\omega_2 = 2$ rad/s and $\dot{\omega}_2 = 3$ rad/s^2. Determine the velocity and acceleration of the grip A at this instant, when $\phi = 30°$, $\theta = 45°$, and $r = 1.6$ m.



Probs. 20–46/47

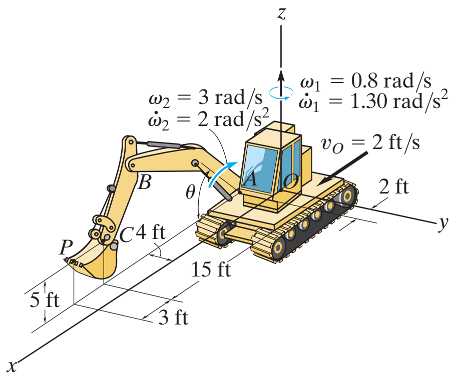
***20–48.** At the given instant, the rod is turning about the z axis with a constant angular velocity $\omega_1 = 3$ rad/s. At this same instant, the disk is spinning at $\omega_2 = 6$ rad/s when $\dot{\omega}_2 = 4$ rad/s^2, both measured *relative* to the rod. Determine the velocity and acceleration of point P on the disk at this instant.



Prob. 20–48

20

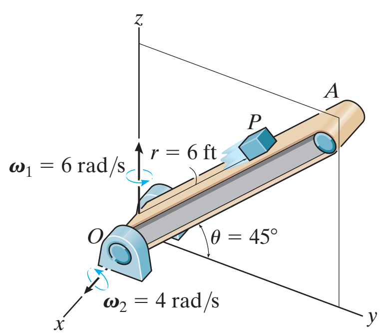
20–49. At the instant shown, the backhoe is traveling forward at a constant speed $v_O = 2$ ft/s, and the boom ABC is rotating about the z axis with an angular velocity $\omega_1 = 0.8$ rad/s and an angular acceleration $\dot{\omega}_1 = 1.30$ rad/s^2. At this same instant the boom is rotating with $\omega_2 = 3$ rad/s when $\dot{\omega}_2 = 2$ rad/s^2, both measured relative to the frame. Determine the velocity and acceleration of point P on the bucket at this instant.



Prob. 20–49

20–50. At the instant shown, the arm OA of the conveyor belt is rotating about the z axis with a constant angular velocity $\omega_1 = 6$ rad/s, while at the same instant the arm is rotating upward at a constant rate $\omega_2 = 4$ rad/s. If the conveyor is running at a constant rate $\dot{r} = 5$ ft/s, determine the velocity and acceleration of the package P at the instant shown. Neglect the size of the package.

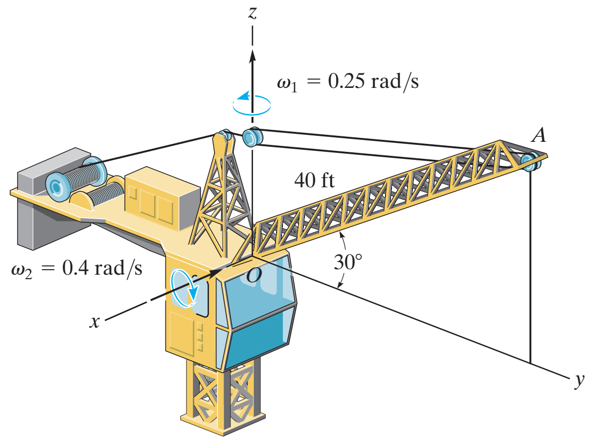
20–51. At the instant shown, the arm OA of the conveyor belt is rotating about the z axis with a constant angular velocity $\omega_1 = 6$ rad/s, while at the same instant the arm is rotating upward at a constant rate $\omega_2 = 4$ rad/s. If the conveyor is running at a rate $\dot{r} = 5$ ft/s, which is increasing at $\ddot{r} = 8$ ft/s^2, determine the velocity and acceleration of the package P at the instant shown. Neglect the size of the package.



Probs. 20–50/51

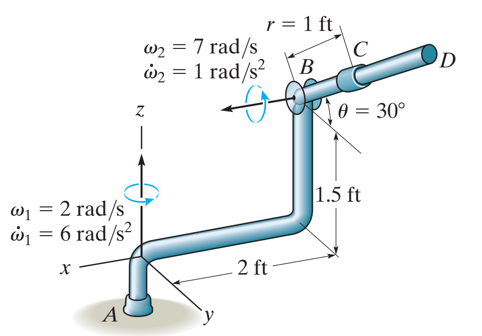
***20–52.** The crane is rotating about the z axis with a constant rate $\omega_1 = 0.25$ rad/s, while the boom OA is rotating downward with a constant rate $\omega_2 = 0.4$ rad/s. Compute the velocity and acceleration of point A located at the top of the boom at the instant shown.

20–53. Solve Prob. 20–52 if the angular motions are increasing at $\dot{\omega}_1 = 0.4$ rad/s^2 and $\dot{\omega}_2 = 0.8$ rad/s^2 at the instant shown.



Probs. 20–52/53

20–54. At the instant shown, the arm AB is rotating about the fixed bearing with an angular velocity $\omega_1 = 2$ rad/s and angular acceleration $\dot{\omega}_1 = 6$ rad/s^2. At the same instant, rod BD is rotating relative to rod AB at $\omega_2 = 7$ rad/s, which is increasing at $\dot{\omega}_2 = 1$ rad/s^2. Also, the collar C is moving along rod BD with a velocity $\dot{r} = 2$ ft/s and a *deceleration* $\ddot{r} = -0.5$ ft/s^2, both measured relative to the rod. Determine the velocity and acceleration of the collar at this instant.

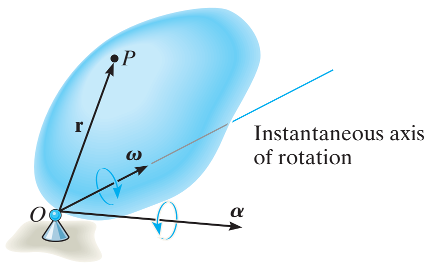


Prob. 20–54

CHAPTER REVIEW

Rotation About a Fixed Point

When a body rotates about a fixed point O, then points on the body follow a path that lies on the surface of a sphere centered at O.



Since the angular acceleration is a time rate of change in the angular velocity, then it is necessary to account for both the magnitude and directional changes of $\boldsymbol{\omega}$ when finding its time derivative. To do this, the angular velocity is often specified in terms of its component motions, such that the direction of some of these components will remain constant relative to rotating x, y, z axes. If this is the case, then the time derivative relative to the fixed axis can be determined using $\dot{\mathbf{A}} = (\dot{\mathbf{A}})_{xyz} + \boldsymbol{\Omega} \times \mathbf{A}$.

Once $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ are known, the velocity and acceleration of any point P in the body can then be determined.

$$\mathbf{v}_P = \boldsymbol{\omega} \times \mathbf{r}$$

$$\mathbf{a}_P = \boldsymbol{\alpha} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

General Motion

If the body undergoes general motion, then the motion of a point B on the body can be related to the motion of another point A using a relative motion analysis, with translating axes attached to A.

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$$

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A})$$

Relative Motion Analysis Using Translating and Rotating Axes

The motion of two points A and B on a body, a series of connected bodies, or each point located on two different paths, can be related using a relative motion analysis with rotating and translating axes at A.

When applying the equations, to find $\mathbf{v}_B$ and $\mathbf{a}_B$, it is important to account for both the magnitude and directional changes of $\mathbf{r}_A$, $\mathbf{r}_{B/A}$, $\boldsymbol{\Omega}$, and $\boldsymbol{\Omega}_{xyz}$ when taking their time derivatives to find $\mathbf{v}_A$, $\mathbf{a}_A$, $(\mathbf{v}_{B/A})_{xyz}$, $(\mathbf{a}_{B/A})_{xyz}$, $\dot{\boldsymbol{\Omega}}$, and $\dot{\boldsymbol{\Omega}}_{xyz}$. To do this properly, one must use Eq. 20–6.

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} + (\mathbf{v}_{B/A})_{xyz}$$

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{B/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{B/A}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz}$$

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Chapter 21

(© Derek Watt/Alamy)

The forces acting on each of these motorcycles can be determined using the equations of motion as discussed in this chapter.

Three-Dimensional Kinetics of a Rigid Body

CHAPTER OBJECTIVES

- To introduce the methods for finding the moments of inertia and products of inertia of a body about various axes.
- To show how to apply the principles of work and energy and linear and angular impulse and momentum to a rigid body having three-dimensional motion.
- To develop and apply the equations of motion in three dimensions.
- To study gyroscopic and torque-free motion.

*21.1 Moments and Products of Inertia

When studying the planar kinetics of a body, it was necessary to introduce the moment of inertia I_G, which was computed about an axis perpendicular to the plane of motion and passing through the body's mass center G. For the kinetic analysis of three-dimensional motion it will sometimes be necessary to calculate six inertial quantities. These terms, called the moments and products of inertia, describe in a particular way the distribution of mass for a body relative to a given coordinate system that has a specified orientation and point of origin.

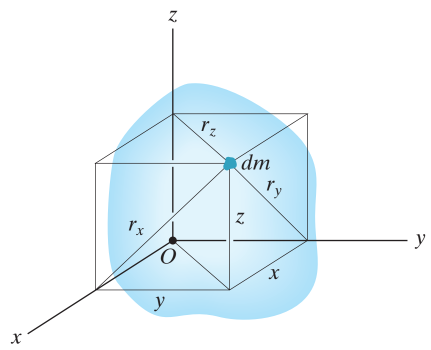

Fig. 21–1

Moment of Inertia. Consider the rigid body shown in Fig. 21–1. The *moment of inertia* for a differential element dm of the body about any one of the three coordinate axes is defined as the product of the mass of the element and the square of the shortest distance from the axis to the element. For example, as noted in the figure, $r_x = \sqrt{y^2 + z^2}$, so that the mass moment of inertia of the element about the x axis is

$$dI_{xx} = r_x^2\, dm = (y^2 + z^2)\, dm$$

The moment of inertia I_{xx} for the body can be determined by integrating this expression over the entire mass of the body. Hence, for each of the axes, we can write

$$\boxed{\begin{aligned} I_{xx} &= \int_m r_x^2 dm = \int_m (y^2 + z^2)\, dm \\ I_{yy} &= \int_m r_y^2 dm = \int_m (x^2 + z^2)\, dm \\ I_{zz} &= \int_m r_z^2 dm = \int_m (x^2 + y^2)\, dm \end{aligned}} \qquad (21\text{–}1)$$

Here it is seen that the moment of inertia is *always a positive quantity*, since it is the summation of the product of the mass dm, which is always positive, and the distances squared.

Product of Inertia. The *product of inertia* for a differential element dm with respect to a set of *two orthogonal planes* is defined as the product of the mass of the element and the perpendicular (or shortest) distances from the planes to the element. For example, this distance is x to the y–z plane and it is y to the x–z plane, Fig. 21–1. The product of inertia dI_{xy} for the element is therefore

$$dI_{xy} = xy\, dm$$

Note also that $dI_{yx} = dI_{xy}$. By integrating over the entire mass, the products of inertia of the body with respect to each combination of planes can be expressed as

$$\boxed{\begin{aligned} I_{xy} &= I_{yx} = \int_m xy\, dm \\ I_{yz} &= I_{zy} = \int_m yz\, dm \\ I_{xz} &= I_{zx} = \int_m xz\, dm \end{aligned}} \qquad (21\text{–}2)$$

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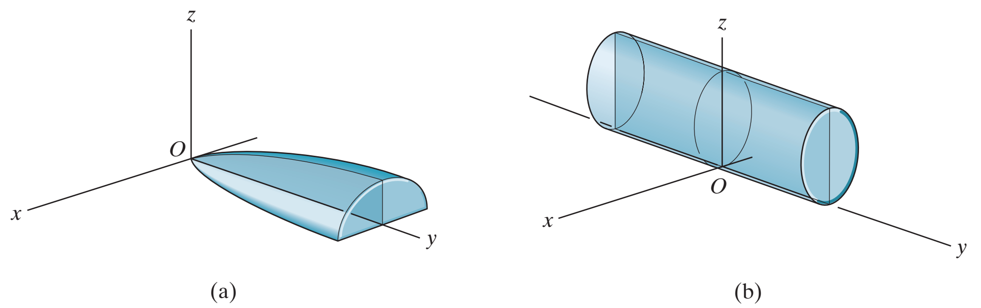


Fig. 21–2

Unlike the moment of inertia, which is always positive, the product of inertia may be positive, negative, or zero. The result depends on the algebraic signs of the two defining coordinates, which vary independently from one another. In particular, if either one or both of the orthogonal planes are *planes of symmetry* for the mass, the *product of inertia* with respect to these planes will be *zero*. In such cases, elements of mass will occur in *pairs* located on each side of the plane of symmetry. On one side of the plane the product of inertia for the element will be positive, while on the other side the product of inertia of the corresponding element will be negative, the sum therefore yielding zero. Examples of this are shown in Fig. 21–2. In the first case, Fig. 21–2*a*, the y–z plane is a plane of symmetry, and hence $I_{xy} = I_{xz} = 0$. Calculation of I_{yz} will yield a *positive* result, since all elements of mass are located using only positive y and z coordinates. For the cylinder, with the coordinate axes located as shown in Fig. 21–2*b*, the x–z and y–z planes are both planes of symmetry. Thus, $I_{xy} = I_{yz} = I_{zx} = 0$.

Parallel-Axis and Parallel-Plane Theorems. The techniques of integration used to determine the moment of inertia of a body were described in Sec. 17.1. Also discussed were methods to determine the moment of inertia of a composite body, i.e., a body that is composed of simpler segments, as tabulated on the inside back cover. In both of these cases the *parallel-axis theorem* is often used for the calculations. This theorem, which was developed in Sec. 17.1, allows us to transfer the moment of inertia of a body from an axis passing through its mass center G to a parallel axis passing through some other point. If G has coordinates x_G, y_G, z_G defined with respect to the x, y, z axes, Fig. 21–3, then the parallel-axis equations used to calculate the moments of inertia about the x, y, z axes are

$$\boxed{\begin{aligned} I_{xx} &= (I_{x'x'})_G + m(y_G^2 + z_G^2) \\ I_{yy} &= (I_{y'y'})_G + m(x_G^2 + z_G^2) \\ I_{zz} &= (I_{z'z'})_G + m(x_G^2 + y_G^2) \end{aligned}} \qquad (21\text{–}3)$$

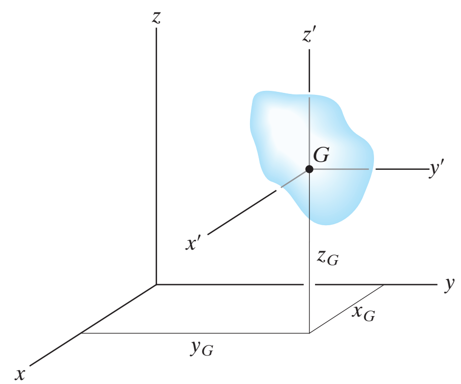


Fig. 21–3

21

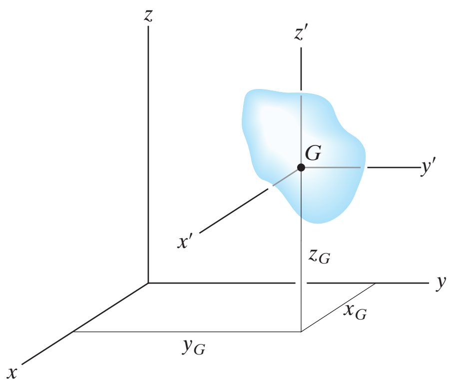


Fig. 21–3 (repeated)

The products of inertia of a composite body are computed in the same manner as the body's moments of inertia. Here, however, the *parallel-plane theorem* is important. This theorem is used to transfer the products of inertia of the body with respect to a set of three orthogonal planes passing through the body's mass center to a corresponding set of three parallel planes passing through some other point O. Defining the perpendicular distances between the planes as x_G, y_G, and z_G, Fig. 21–3, the parallel-plane equations can be written as

$$\boxed{\begin{aligned} I_{xy} &= (I_{x'y'})_G + m x_G y_G \\ I_{yz} &= (I_{y'z'})_G + m y_G z_G \\ I_{zx} &= (I_{z'x'})_G + m z_G x_G \end{aligned}} \qquad (21\text{–}4)$$

The derivation of these formulas is similar to that given for the parallel-axis equation, Sec. 17.1.

Inertia Tensor. The inertial properties of a body are therefore completely characterized by nine terms, six of which are independent of one another. This set of terms is defined using Eqs. 21–1 and 21–2 and can be written as

$$\begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix}$$

This array is called an *inertia tensor*.* It has a unique set of values for a body when it is determined for each location of the origin O and orientation of the coordinate axes.

In general, for point O we can specify a unique axes inclination for which the products of inertia for the body are zero when computed with respect to these axes. When this is done, the inertia tensor is said to be "diagonalized" and may be written in the simplified form

$$\begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}$$

Here $I_x = I_{xx}$, $I_y = I_{yy}$, and $I_z = I_{zz}$ are termed the *principal moments of inertia* for the body, which are computed with respect to the *principal axes of inertia*. Of these three principal moments of inertia, one will be a maximum and another a minimum of the body's moment of inertia.

The dynamics of the space shuttle while it orbits the earth can be predicted only if its moments and products of inertia are known relative to its mass center. (©Ablestock/Getty Images)

*The negative signs are here as a consequence of the development of angular momentum, Eqs. 21–10.

The mathematical determination of the directions of principal axes of inertia will not be discussed here (see Prob. 21–22). However, there are many cases in which the principal axes can be determined by inspection. From the previous discussion it was noted that if the coordinate axes are oriented such that *two* of the three orthogonal planes containing the axes are planes of *symmetry* for the body, then all the products of inertia for the body are zero with respect to these coordinate planes, and hence these coordinate axes are principal axes of inertia. For example, the x, y, z axes shown in Fig. 21–2b represent the principal axes of inertia for the cylinder at point O.

Moment of Inertia About an Arbitrary Axis. Consider the body shown in Fig. 21–4, where the nine elements of the inertia tensor have been determined with respect to the x, y, z axes having an origin at O. Here we wish to determine the moment of inertia of the body about the Oa axis, which has a direction defined by the unit vector $\mathbf{u}_a$. By definition $I_{Oa} = \int b^2\,dm$, where b is the *perpendicular distance* from dm to Oa. If the position of dm is located using $\mathbf{r}$, then $b = r\sin\theta$, which represents the *magnitude* of the cross product $\mathbf{u}_a \times \mathbf{r}$. Hence, the moment of inertia can be expressed as

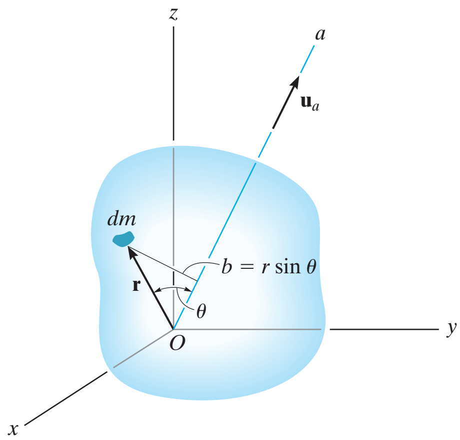


Fig. 21–4

$$I_{Oa} = \int_m |(\mathbf{u}_a \times \mathbf{r})|^2 dm = \int_m (\mathbf{u}_a \times \mathbf{r}) \cdot (\mathbf{u}_a \times \mathbf{r})dm$$

Provided $\mathbf{u}_a = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\mathbf{u}_a \times \mathbf{r} = (u_y z - u_z y)\mathbf{i} + (u_z x - u_x z)\mathbf{j} + (u_x y - u_y x)\mathbf{k}$. After substituting and performing the dot-product operation, the moment of inertia is

$$\begin{aligned} I_{Oa} &= \int_m [(u_y z - u_z y)^2 + (u_z x - u_x z)^2 + (u_x y - u_y x)^2]dm \\ &= u_x^2 \int_m (y^2 + z^2)dm + u_y^2 \int_m (z^2 + x^2)dm + u_z^2 \int_m (x^2 + y^2)\,dm \\ &\quad - 2u_x u_y \int_m xy\,dm - 2u_y u_z \int_m yz\,dm - 2u_z u_x \int_m zx\,dm \end{aligned}$$

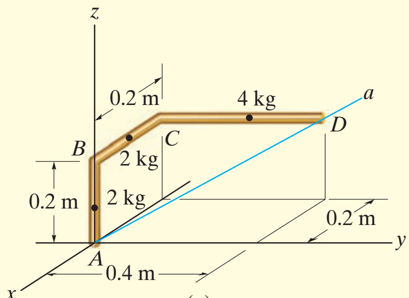
Recognizing the integrals to be the moments and products of inertia of the body, Eqs. 21–1 and 21–2, we have

$$\boxed{I_{Oa} = I_{xx}u_x^2 + I_{yy}u_y^2 + I_{zz}u_z^2 - 2I_{xy}u_x u_y - 2I_{yz}u_y u_z - 2I_{zx}u_z u_x} \qquad (21\text{–}5)$$

21

Thus, if the inertia tensor is specified for the x, y, z axes, the moment of inertia of the body about the inclined Oa axis can be found. For the calculation, the direction cosines u_x, u_y, u_z of the axes must be determined. These terms specify the cosines of the coordinate direction angles α, β, γ made between the positive Oa axis and the positive x, y, z axes, respectively (see Appendix B).

EXAMPLE 21.1



(a)

Determine the moment of inertia of the bent rod shown in Fig. 21–5*a* about the *Aa* axis. The mass of each of the three segments is given in the figure.

SOLUTION

Before applying Eq. 21–5, it is first necessary to determine the moments and products of inertia of the rod with respect to the *x, y, z* axes. This is done using the formula for the moment of inertia of a slender rod, $I = \frac{1}{12}ml^2$, and the parallel-axis and parallel-plane theorems, Eqs. 21–3 and 21–4. Dividing the rod into three parts and locating the mass center of each segment, Fig. 21–5*b*, we have

$$I_{xx} = \left[\tfrac{1}{12}(2)(0.2)^2 + 2(0.1)^2\right] + [0 + 2(0.2)^2] + \left[\tfrac{1}{12}(4)(0.4)^2 + 4((0.2)^2 + (0.2)^2)\right] = 0.480 \text{ kg}\cdot\text{m}^2$$

$$I_{yy} = \left[\tfrac{1}{12}(2)(0.2)^2 + 2(0.1)^2\right] + \left[\tfrac{1}{12}(2)(0.2)^2 + 2((-0.1)^2 + (0.2)^2)\right] + [0 + 4((-0.2)^2 + (0.2)^2)] = 0.453 \text{ kg}\cdot\text{m}^2$$

$$I_{zz} = \left[0 + 0\right] + \left[\tfrac{1}{12}(2)(0.2)^2 + 2(-0.1)^2\right] + \left[\tfrac{1}{12}(4)(0.4)^2 + 4((-0.2)^2 + (0.2)^2)\right] = 0.400 \text{ kg}\cdot\text{m}^2$$

$$I_{xy} = [0 + 0] + [0 + 0] + [0 + 4(-0.2)(0.2)] = -0.160 \text{ kg}\cdot\text{m}^2$$

$$I_{yz} = [0 + 0] + [0 + 0] + [0 + 4(0.2)(0.2)] = 0.160 \text{ kg}\cdot\text{m}^2$$

$$I_{zx} = [0 + 0] + [0 + 2(0.2)(-0.1)] + [0 + 4(0.2)(-0.2)] = -0.200 \text{ kg}\cdot\text{m}^2$$

The *Aa* axis is defined by the unit vector

$$\mathbf{u}_{Aa} = \frac{\mathbf{r}_D}{r_D} = \frac{-0.2\mathbf{i} + 0.4\mathbf{j} + 0.2\mathbf{k}}{\sqrt{(-0.2)^2 + (0.4)^2 + (0.2)^2}} = -0.408\mathbf{i} + 0.816\mathbf{j} + 0.408\mathbf{k}$$

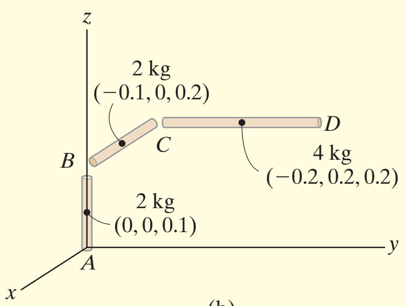
Thus,

$$u_x = -0.408 \quad u_y = 0.816 \quad u_z = 0.408$$

Substituting these results into Eq. 21–5 yields

$$\begin{aligned}
I_{Aa} &= I_{xx}u_x^2 + I_{yy}u_y^2 + I_{zz}u_z^2 - 2I_{xy}u_xu_y - 2I_{yz}u_yu_z - 2I_{zx}u_zu_x \\
&= 0.480(-0.408)^2 + (0.453)(0.816)^2 + 0.400(0.408)^2 \\
&\quad - 2(-0.160)(-0.408)(0.816) - 2(0.160)(0.816)(0.408) \\
&\quad - 2(-0.200)(0.408)(-0.408) \\
&= 0.169 \text{ kg}\cdot\text{m}^2
\end{aligned}$$

Ans.

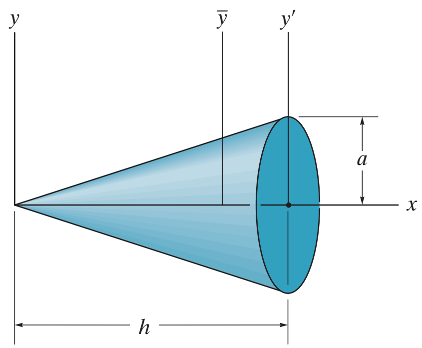


(b)

Fig. 21–5

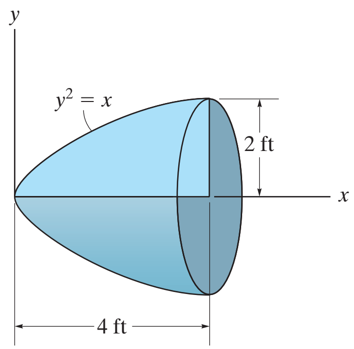
21–1. Show that the sum of the moments of inertia of a body, $I_{xx} + I_{yy} + I_{zz}$, is independent of the orientation of the x, y, z axes and thus depends only on the location of the origin.

21–2. Determine the moment of inertia of the cone with respect to a vertical $\bar{y}$ axis passing through the cone's center of mass. What is the moment of inertia about a parallel axis y' that passes through the diameter of the base of the cone? The cone has a mass m.



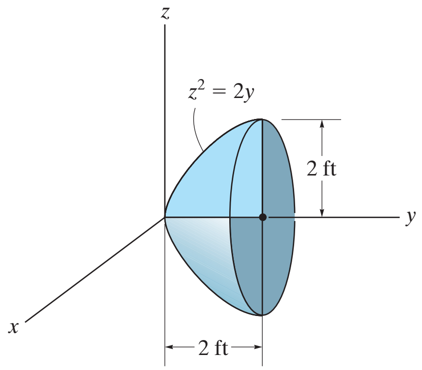
Prob. 21–2

21–3. Determine moment of inertia I_y of the solid formed by revolving the shaded area around the x axis. The density of the material is $\rho = 12\ \text{slug/ft}^3$.



Prob. 21–3

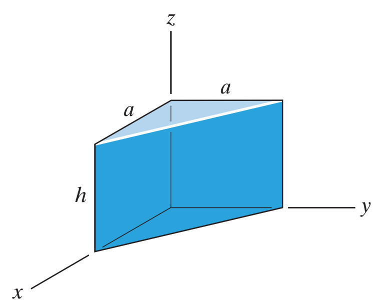
***21–4.** Determine the moments of inertia I_x and I_y of the paraboloid of revolution. The mass of the paraboloid is 20 slug.



Prob. 21–4

21–5. Determine by direct integration the product of inertia I_{yz} for the homogeneous prism. The density of the material is ρ. Express the result in terms of the total mass m of the prism.

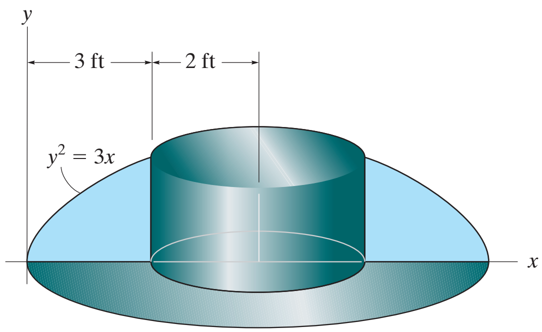
21–6. Determine by direct integration the product of inertia I_{xy} for the homogeneous prism. The density of the material is ρ. Express the result in terms of the total mass m of the prism.



Probs. 21–5/6

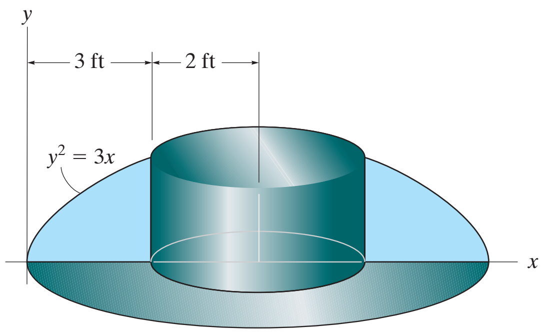
21

21–7. Determine the product of inertia I_{xy} of the object formed by revolving the shaded area about the line $x = 5$ ft. Express the result in terms of the density of the material, ρ.



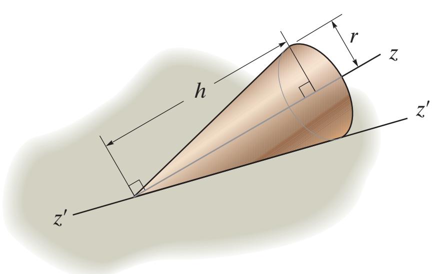
Prob. 21–7

***21–8.** Determine the moment of inertia I_y of the object formed by revolving the shaded area about the line $x = 5$ ft. Express the result in terms of the density of the material, ρ.



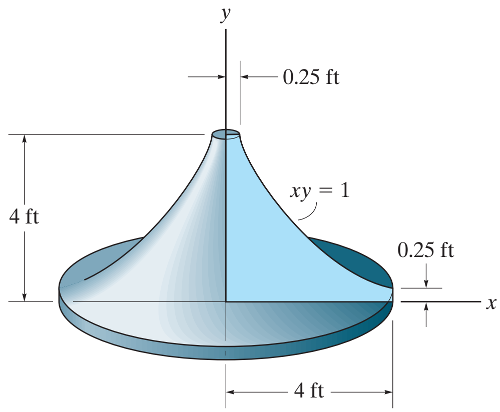
Prob. 21–8

21–9. Determine the moment of inertia of the cone about the z' axis. The weight of the cone is 15 lb, the *height* is $h = 1.5$ ft and the radius is $r = 0.5$ ft.



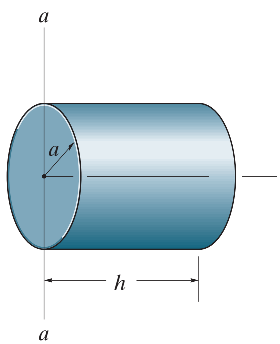
Prob. 21–9

21–10. Determine the radii of gyration k_x and k_y for the solid formed by revolving the shaded area about the y axis. The density of the material is ρ.



Prob. 21–10

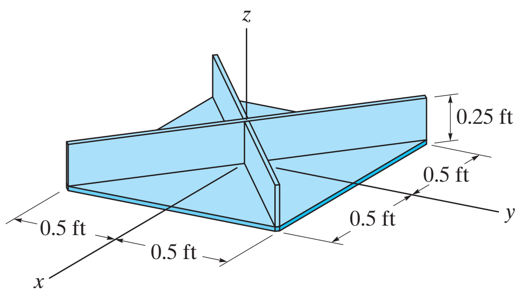
21–11. Determine the moment of inertia of the cylinder with respect to the a–a axis of the cylinder. The cylinder has a mass m.



Prob. 21–11

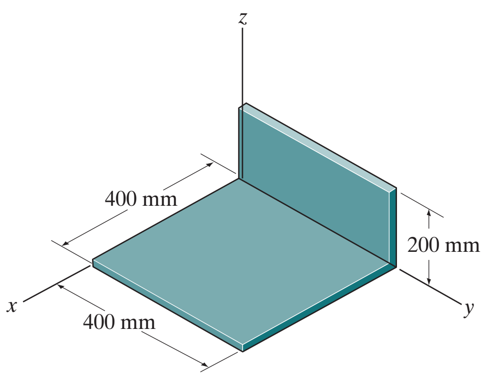
***21–12.** Determine the moment of inertia I_{xx} of the composite plate assembly. The plates have a specific weight of 6 lb/ft^2.

21–13. Determine the product of inertia I_{yz} of the composite plate assembly. The plates have a weight of 6 lb/ft^2.



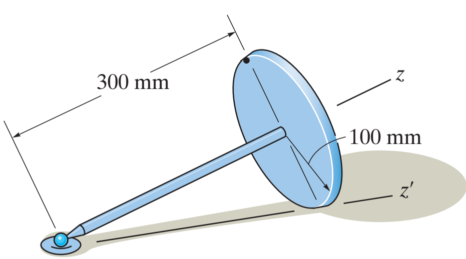
Probs. 21–12/13

21–14. Determine the products of inertia I_{xy}, I_{yz}, and I_{xz}, of the thin plate. The material has a density per unit area of 50 kg/m^2.



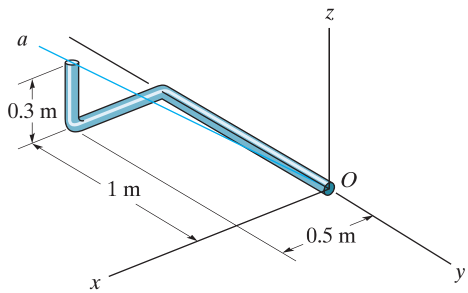
Prob. 21–14

21–15. Determine the moment of inertia of both the 1.5-kg rod and 4-kg disk about the z' axis.



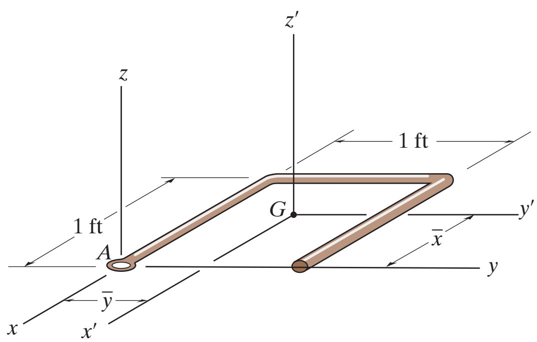
Prob. 21–15

***21–16.** The bent rod has a mass of 3 kg/m. Determine the moment of inertia of the rod about the O–a axis.



Prob. 21–16

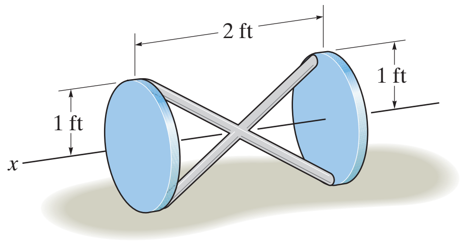
21–17. The bent rod has a weight of 1.5 lb/ft. Locate the center of gravity $G(\bar{x}, \bar{y})$ and determine the principal moments of inertia $I_{x'}$, $I_{y'}$, and $I_{z'}$ of the rod with respect to the x', y', z' axes.



Prob. 21–17

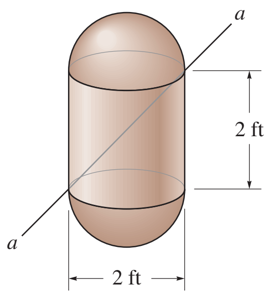
21

21–18. Determine the moment of inertia of the rod-and-disk assembly about the x axis. The disks each have a weight of 12 lb. The two rods each have a weight of 4 lb, and their ends extend to the rims of the disks.



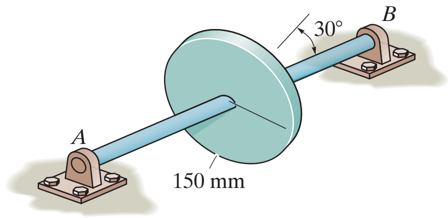
Prob. 21–18

21–19. Determine the moment of inertia of the composite body about the aa axis. The cylinder weighs 20 lb, and each hemisphere weighs 10 lb.



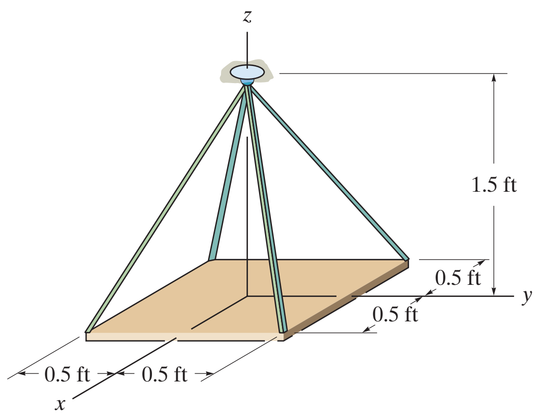
Prob. 21–19

***21–20.** Determine the moment of inertia of the disk about the axis of shaft AB. The disk has a mass of 15 kg.



Prob. 21–20

21–21. The thin plate has a weight of 5 lb and each of the four rods weighs 3 lb. Determine the moment of inertia of the assembly about the z axis.



Prob. 21–21

21.2 Angular Momentum

In this section we will develop the necessary equations used to determine the angular momentum of a rigid body about an arbitrary point. These equations will provide a means for developing both the principle of impulse and momentum and the equations of rotational motion for a rigid body.

Consider the rigid body in Fig. 21–6, which has a mass m and center of mass at G. The X, Y, Z coordinate system represents an inertial frame of reference, and hence, its axes are fixed or translate with a constant velocity. The angular momentum as measured from this reference will be determined relative to the arbitrary point A. The position vectors $\mathbf{r}_A$ and $\boldsymbol{\rho}_A$ are drawn from the origin of coordinates to point A and from A to the ith particle of the body. If the particle's mass is m_i, the angular momentum about point A is

$$(\mathbf{H}_A)_i = \boldsymbol{\rho}_A \times m_i \mathbf{v}_i$$

where $\mathbf{v}_i$ represents the particle's velocity measured from the X, Y, Z coordinate system. If the body has an angular velocity $\boldsymbol{\omega}$ at the instant considered, $\mathbf{v}_i$ may be related to the velocity of A by applying Eq. 20–7, i.e.,

$$\mathbf{v}_i = \mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_A$$

Thus,

$$\begin{aligned}(\mathbf{H}_A)_i &= \boldsymbol{\rho}_A \times m_i(\mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_A)\\ &= (\boldsymbol{\rho}_A m_i) \times \mathbf{v}_A + \boldsymbol{\rho}_A \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_A) m_i\end{aligned}$$

Summing the moments of all the particles of the body requires an integration. Since $m_i \rightarrow dm$, we have

$$\mathbf{H}_A = \left(\int_m \boldsymbol{\rho}_A\, dm\right) \times \mathbf{v}_A + \int_m \boldsymbol{\rho}_A \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_A)\, dm \qquad (21\text{–}6)$$

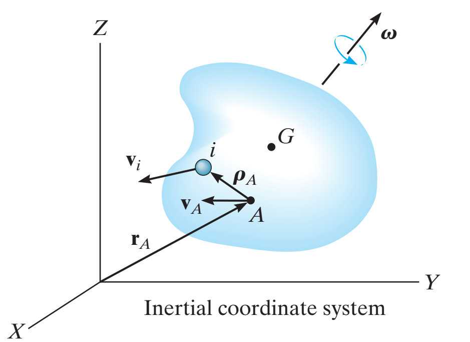
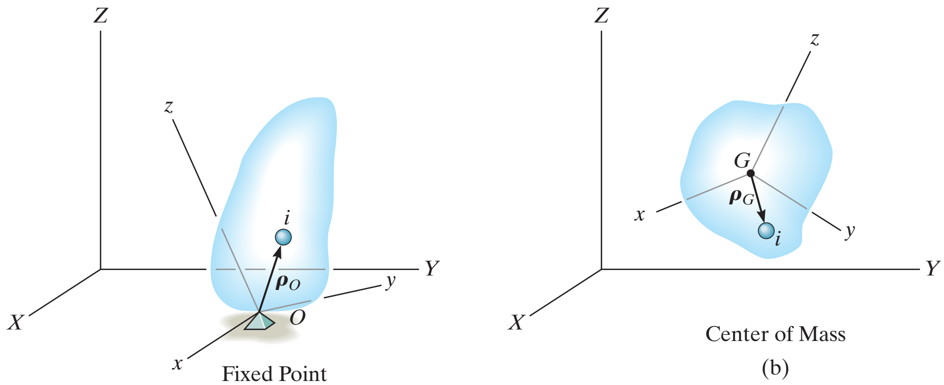


Fig. 21–6



Fixed Point
(a)

Center of Mass
(b)

Fig. 21–7

Fixed Point *O*. If A becomes a *fixed point O* in the body, Fig. 21–7*a*, then $\mathbf{v}_A = \mathbf{0}$ and Eq. 21–6 reduces to

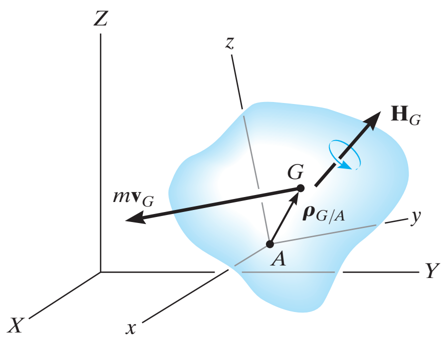
$$\boxed{\mathbf{H}_O = \int_m \boldsymbol{\rho}_O \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_O)\, dm} \qquad (21\text{–}7)$$

Center of Mass *G*. If A is located at the *center of mass G* of the body, Fig. 21–7*b*, then $\int_m \boldsymbol{\rho}_A\, dm = \mathbf{0}$ and

$$\boxed{\mathbf{H}_G = \int_m \boldsymbol{\rho}_G \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_G)\, dm} \qquad (21\text{–}8)$$

Arbitrary Point *A*. In general, A can be a point other than O or G, Fig. 21–7*c*, in which case Eq. 21–6 may nevertheless be simplified to the following form (see Prob. 21–23).

$$\boxed{\mathbf{H}_A = \boldsymbol{\rho}_{G/A} \times m\mathbf{v}_G + \mathbf{H}_G} \qquad (21\text{–}9)$$



Arbitrary Point
(c)

Here the angular momentum consists of two parts—the moment of the linear momentum $m\mathbf{v}_G$ of the body about point A added (vectorially) to the angular momentum $\mathbf{H}_G$. Equation 21–9 can also be used to determine the angular momentum of the body about a fixed point O. The results, of course, will be the same as those found using the more convenient Eq. 21–7.

Rectangular Components of *H*. To make practical use of Eqs. 21–7 through 21–9, the angular momentum must be expressed in terms of its scalar components. For this purpose, it is convenient to choose a second set of x, y, z axes having an arbitrary orientation relative to the X, Y, Z axes, Fig. 21–7, and for a general formulation, note that Eqs. 21–7 and 21–8 are both of the form

$$\mathbf{H} = \int_m \boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho})dm$$

Expressing $\mathbf{H}$, $\boldsymbol{\rho}$, and $\boldsymbol{\omega}$ in terms of x, y, z components, we have

$$H_x\mathbf{i} + H_y\mathbf{j} + H_z\mathbf{k} = \int_m (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times [(\omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}) \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})]dm$$

Expanding the cross products and combining terms yields

$$\begin{aligned} H_x\mathbf{i} + H_y\mathbf{j} + H_z\mathbf{k} = {} & \left[\omega_x \int_m (y^2 + z^2)dm - \omega_y \int_m xy\,dm - \omega_z \int_m xz\,dm\right]\mathbf{i} \\ & + \left[-\omega_x \int_m xy\,dm + \omega_y \int_m (x^2 + z^2)dm - \omega_z \int_m yz\,dm\right]\mathbf{j} \\ & + \left[-\omega_x \int_m zx\,dm - \omega_y \int_m yz\,dm + \omega_z \int_m (x^2 + y^2)dm\right]\mathbf{k} \end{aligned}$$

Equating the respective $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ components and recognizing that the integrals represent the moments and products of inertia, we obtain

$$\boxed{\begin{aligned} H_x &= I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z \\ H_y &= -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z \\ H_z &= -I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z \end{aligned}} \qquad (21\text{–}10)$$

These equations can be simplified further if the x, y, z coordinate axes are oriented such that they become *principal axes of inertia* for the body at the point. When these axes are used, the products of inertia $I_{xy} = I_{yz} = I_{zx} = 0$, and if the principal moments of inertia about the x, y, z axes are represented as $I_x = I_{xx}$, $I_y = I_{yy}$, and $I_z = I_{zz}$, the three components of angular momentum become

$$\boxed{H_x = I_x\omega_x \quad H_y = I_y\omega_y \quad H_z = I_z\omega_z} \qquad (21\text{–}11)$$

The motion of the astronaut is controlled by use of small directional jets attached to his or her space suit. The impulses these jets provide must be carefully specified in order to prevent tumbling and loss of orientation. (© NASA)

Principle of Impulse and Momentum. Now that the formulation of the angular momentum for a body has been developed, the *principle of impulse and momentum*, as discussed in Sec. 19.2, can be used to solve kinetic problems which involve *force, velocity, and time*. For this case, the following two vector equations are available:

$$\boxed{m(\mathbf{v}_G)_1 + \Sigma \int_{t_1}^{t_2} \mathbf{F}\, dt = m(\mathbf{v}_G)_2} \quad (21\text{–}12)$$

$$\boxed{(\mathbf{H}_O)_1 + \Sigma \int_{t_1}^{t_2} \mathbf{M}_O\, dt = (\mathbf{H}_O)_2} \quad (21\text{–}13)$$

In three dimensions each vector term can be represented by three scalar components, and therefore a total of *six scalar equations* can be written. Three equations relate the linear impulse and momentum in the x, y, z directions, and the other three equations relate the body's angular impulse and momentum about the x, y, z axes. Before applying Eqs. 21–12 and 21–13 to the solution of problems, the material in Secs. 19.2 and 19.3 should be reviewed.

21.3 Kinetic Energy

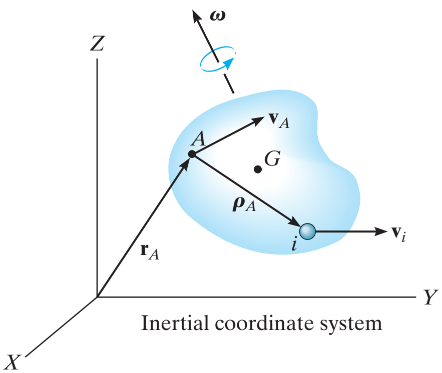


Fig. 21–8

In order to apply the principle of work and energy to solve problems involving general rigid body motion, it is first necessary to formulate expressions for the kinetic energy of the body. To do this, consider the rigid body shown in Fig. 21–8, which has a mass m and center of mass at G. The kinetic energy of the ith particle of the body having a mass m_i and velocity $\mathbf{v}_i$, measured relative to the inertial X, Y, Z frame of reference, is

$$T_i = \tfrac{1}{2} m_i v_i^2 = \tfrac{1}{2} m_i (\mathbf{v}_i \cdot \mathbf{v}_i)$$

Provided the velocity of an arbitrary point A in the body is known, $\mathbf{v}_i$ can be related to $\mathbf{v}_A$ by the equation $\mathbf{v}_i = \mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_A$, where $\boldsymbol{\omega}$ is the angular velocity of the body, measured from the X, Y, Z coordinate system, and $\boldsymbol{\rho}_A$ is a position vector extending from A to i. Using this expression, the kinetic energy for the particle can be written as

$$\begin{aligned} T_i &= \tfrac{1}{2} m_i (\mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_A) \cdot (\mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_A) \\ &= \tfrac{1}{2}(\mathbf{v}_A \cdot \mathbf{v}_A) m_i + \mathbf{v}_A \cdot (\boldsymbol{\omega} \times \boldsymbol{\rho}_A) m_i + \tfrac{1}{2}(\boldsymbol{\omega} \times \boldsymbol{\rho}_A) \cdot (\boldsymbol{\omega} \times \boldsymbol{\rho}_A) m_i \end{aligned}$$

The kinetic energy for the entire body is obtained by summing the kinetic energies of all the particles of the body. This requires an integration. Since $m_i \rightarrow dm$, we get

$$T = \tfrac{1}{2} m (\mathbf{v}_A \cdot \mathbf{v}_A) + \mathbf{v}_A \cdot \left(\boldsymbol{\omega} \times \int_m \boldsymbol{\rho}_A\, dm \right) + \tfrac{1}{2} \int_m (\boldsymbol{\omega} \times \boldsymbol{\rho}_A) \cdot (\boldsymbol{\omega} \times \boldsymbol{\rho}_A)\, dm$$

The last term on the right can be rewritten using the vector identity $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$, where $\mathbf{a} = \boldsymbol{\omega}$, $\mathbf{b} = \boldsymbol{\rho}_A$, and $\mathbf{c} = \boldsymbol{\omega} \times \boldsymbol{\rho}_A$. The final result is

$$T = \tfrac{1}{2}m(\mathbf{v}_A \cdot \mathbf{v}_A) + \mathbf{v}_A \cdot \left(\boldsymbol{\omega} \times \int_m \boldsymbol{\rho}_A dm\right) + \tfrac{1}{2}\boldsymbol{\omega} \cdot \int_m \boldsymbol{\rho}_A \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_A) dm \qquad (21\text{–}14)$$

This equation is rarely used because of the computations involving the integrals. Simplification occurs, however, if the reference point A is either a fixed point or the center of mass.

Fixed Point O. If A is a *fixed point O* in the body, Fig. 21–7*a*, then $\mathbf{v}_A = \mathbf{0}$, and using Eq. 21–7, we can express Eq. 21–14 as

$$T = \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{H}_O$$

If the x, y, z axes represent the principal axes of inertia for the body, then $\boldsymbol{\omega} = \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k}$ and $\mathbf{H}_O = I_x\omega_x \mathbf{i} + I_y\omega_y \mathbf{j} + I_z\omega_z \mathbf{k}$. Substituting into the above equation and performing the dot-product operations yields

$$\boxed{T = \tfrac{1}{2}I_x\omega_x^2 + \tfrac{1}{2}I_y\omega_y^2 + \tfrac{1}{2}I_z\omega_z^2} \qquad (21\text{–}15)$$

Center of Mass G. If A is located at the *center of mass G* of the body, Fig. 21–7*b*, then $\int \boldsymbol{\rho}_A \, dm = \mathbf{0}$ and, using Eq. 21–8, we can write Eq. 21–14 as

$$T = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}\boldsymbol{\omega} \cdot \mathbf{H}_G$$

In a manner similar to that for a fixed point, the last term on the right side may be represented in scalar form, in which case

$$\boxed{T = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}I_x\omega_x^2 + \tfrac{1}{2}I_y\omega_y^2 + \tfrac{1}{2}I_z\omega_z^2} \qquad (21\text{–}16)$$

Here it is seen that the kinetic energy consists of two parts; namely, the translational kinetic energy of the mass center, $\tfrac{1}{2}mv_G^2$, and the body's rotational kinetic energy.

Principle of Work and Energy. Having formulated the kinetic energy for a body, the *principle of work and energy* can be applied to solve kinetics problems which involve *force, velocity, and displacement*. For this case only one scalar equation can be written for each body, namely,

$$\boxed{T_1 + \Sigma U_{1\text{–}2} = T_2} \qquad (21\text{–}17)$$

Before applying this equation, the material in Chapter 18 should be reviewed.

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EXAMPLE 21.2

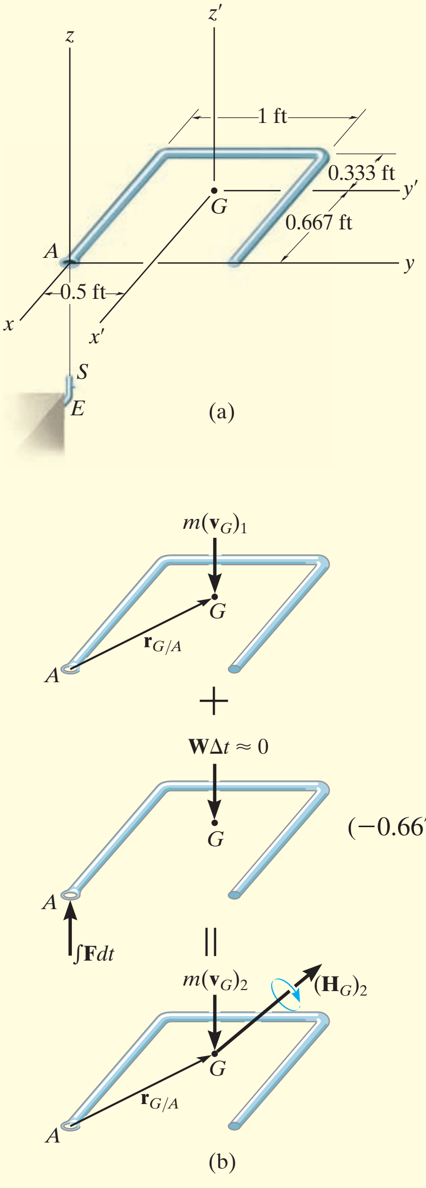


Fig. 21–9

The rod in Fig. 21–9*a* has a weight per unit length of 1.5 lb/ft. Determine its angular velocity just after the end A falls onto the hook at E. The hook provides a permanent connection for the rod due to the spring-lock mechanism S. Just before striking the hook the rod is falling downward with a speed $(v_G)_1 = 10$ ft/s.

SOLUTION

The principle of impulse and momentum will be used since impact occurs.

Impulse and Momentum Diagrams. Fig. 21–9*b*. During the short time Δt, the impulsive force $\mathbf{F}$ acting at A changes the momentum of the rod. (The impulse created by the rod's weight $\mathbf{W}$ during this time is small compared to $\int \mathbf{F}\,dt$, so that it can be neglected, i.e., the weight is a nonimpulsive force.) Hence, the angular momentum of the rod is *conserved* about point A since the moment of $\int \mathbf{F}\,dt$ about A is zero.

Conservation of Angular Momentum. Equation 21–9 must be used to find the angular momentum of the rod, since A does not become a *fixed point* until *after* the impulsive interaction with the hook. Thus, with reference to Fig. 21–9*b*, $(\mathbf{H}_A)_1 = (\mathbf{H}_A)_2$, or

$$\mathbf{r}_{G/A} \times m(\mathbf{v}_G)_1 = \mathbf{r}_{G/A} \times m(\mathbf{v}_G)_2 + (\mathbf{H}_G)_2 \qquad (1)$$

From Fig. 21–9*a*, $\mathbf{r}_{G/A} = \{-0.667\mathbf{i} + 0.5\mathbf{j}\}$ ft. Furthermore, the primed axes are principal axes of inertia for the rod because $I_{x'y'} = I_{x'z'} = I_{z'y'} = 0$. Hence, from Eqs. 21–11, $(\mathbf{H}_G)_2 = I_{x'}\omega_x\mathbf{i} + I_{y'}\omega_y\mathbf{j} + I_{z'}\omega_z\mathbf{k}$. The principal moments of inertia are $I_{x'} = 0.0272$ slug·ft², $I_{y'} = 0.0155$ slug·ft², $I_{z'} = 0.0427$ slug·ft² (see Prob. 21–17). Substituting into Eq. 1, we have

$$(-0.667\mathbf{i} + 0.5\mathbf{j}) \times \left[\left(\frac{4.5}{32.2}\right)(-10\mathbf{k})\right] = (-0.667\mathbf{i} + 0.5\mathbf{j}) \times \left[\left(\frac{4.5}{32.2}\right)(-v_G)_2\mathbf{k}\right] + 0.0272\omega_x\mathbf{i} + 0.0155\omega_y\mathbf{j} + 0.0427\omega_z\mathbf{k}$$

Expanding and equating the respective $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components yields

$$-0.699 = -0.0699(v_G)_2 + 0.0272\omega_x \qquad (2)$$

$$-0.932 = -0.0932(v_G)_2 + 0.0155\omega_y \qquad (3)$$

$$0 = 0.0427\omega_z \qquad (4)$$

Kinematics. There are four unknowns in the above equations; however, another equation may be obtained by relating $\boldsymbol{\omega}$ to $(\mathbf{v}_G)_2$ using *kinematics*. Since $\omega_z = 0$ (Eq. 4) and after impact the rod rotates about the fixed point A, Eq. 20–3 can be applied, in which case $(\mathbf{v}_G)_2 = \boldsymbol{\omega} \times \mathbf{r}_{G/A}$, or

$$-(v_G)_2\mathbf{k} = (\omega_x\mathbf{i} + \omega_y\mathbf{j}) \times (-0.667\mathbf{i} + 0.5\mathbf{j})$$

$$-(v_G)_2 = 0.5\omega_x + 0.667\omega_y \qquad (5)$$

Solving Eqs. 2, 3 and 5 simultaneously yields

$$(\mathbf{v}_G)_2 = \{-8.41\mathbf{k}\}\ \text{ft/s} \qquad \boldsymbol{\omega} = \{-4.09\mathbf{i} - 9.55\mathbf{j}\}\ \text{rad/s} \qquad \textit{Ans.}$$

A 5-N·m torque is applied to the vertical shaft CD shown in Fig. 21–10*a*, which allows the 10-kg gear A to turn freely about CE. Assuming that gear A starts from rest, determine the angular velocity of CD after it has turned two revolutions. Neglect the mass of shaft CD and axle CE and assume that gear A can be approximated by a thin disk. Gear B is fixed.

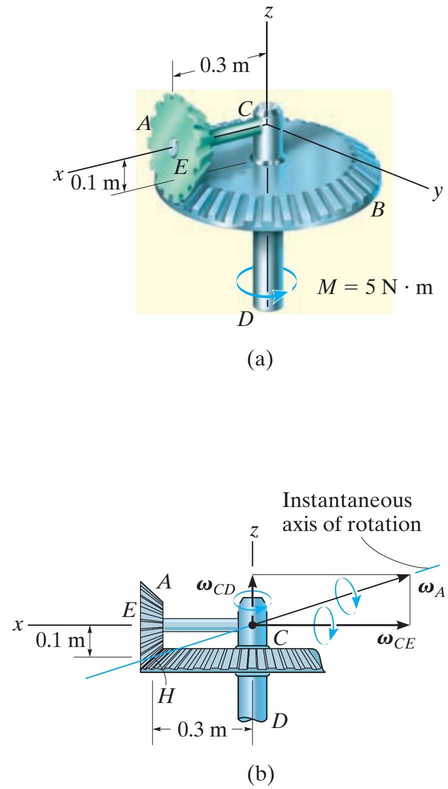


Fig. 21–10

SOLUTION

The principle of work and energy may be used for the solution. Why?

Work. If shaft CD, the axle CE, and gear A are considered as a system of connected bodies, only the applied torque $\mathbf{M}$ does work. For two revolutions of CD, this work is $\Sigma U_{1-2} = (5\ \text{N}\cdot\text{m})(4\pi\ \text{rad}) = 62.83\ \text{J}$.

Kinetic Energy. Since the gear is initially at rest, its initial kinetic energy is zero. A kinematic diagram for the gear is shown in Fig. 21–10*b*. If the angular velocity of CD is taken as $\boldsymbol{\omega}_{CD}$, then the angular velocity of gear A is $\boldsymbol{\omega}_A = \boldsymbol{\omega}_{CD} + \boldsymbol{\omega}_{CE}$. The gear may be imagined as a portion of a massless extended body which is rotating about the *fixed point C*. The instantaneous axis of rotation for this body is along line CH, because both points C and H on the body (gear) have zero velocity and must therefore lie on this axis. This requires that the components $\boldsymbol{\omega}_{CD}$ and $\boldsymbol{\omega}_{CE}$ be related by the equation $\omega_{CD}/0.1\ \text{m} = \omega_{CE}/0.3\ \text{m}$ or $\omega_{CE} = 3\omega_{CD}$. Thus,

$$\boldsymbol{\omega}_A = -\omega_{CE}\mathbf{i} + \omega_{CD}\mathbf{k} = -3\omega_{CD}\mathbf{i} + \omega_{CD}\mathbf{k} \qquad (1)$$

The x, y, z axes in Fig. 21–10*a* represent *principal axes of inertia* at C for the gear. Since point C is a fixed point of rotation, Eq. 21–15 may be applied to determine the kinetic energy, i.e.,

$$T = \tfrac{1}{2}I_x\omega_x^2 + \tfrac{1}{2}I_y\omega_y^2 + \tfrac{1}{2}I_z\omega_z^2 \qquad (2)$$

Using the parallel-axis theorem, the moments of inertia of the gear about point C are as follows:

$$I_x = \tfrac{1}{2}(10\ \text{kg})(0.1\ \text{m})^2 = 0.05\ \text{kg}\cdot\text{m}^2$$

$$I_y = I_z = \tfrac{1}{4}(10\ \text{kg})(0.1\ \text{m})^2 + 10\ \text{kg}(0.3\ \text{m})^2 = 0.925\ \text{kg}\cdot\text{m}^2$$

Since $\omega_x = -3\omega_{CD}$, $\omega_y = 0$, $\omega_z = \omega_{CD}$, Eq. 2 becomes

$$T_A = \tfrac{1}{2}(0.05)(-3\omega_{CD})^2 + 0 + \tfrac{1}{2}(0.925)(\omega_{CD})^2 = 0.6875\omega_{CD}^2$$

Principle of Work and Energy. Applying the principle of work and energy, we obtain

$$T_1 + \Sigma U_{1-2} = T_2$$

$$0 + 62.83 = 0.6875\omega_{CD}^2$$

$$\omega_{CD} = 9.56\ \text{rad/s} \qquad \textit{Ans.}$$

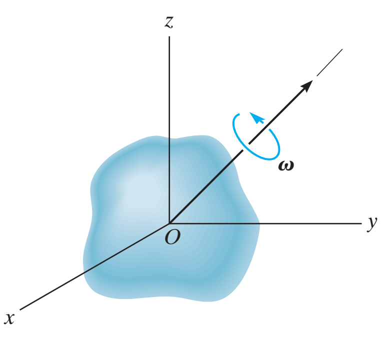
21

PROBLEMS

21–22. If a body contains *no planes of symmetry*, the principal moments of inertia can be determined mathematically. To show how this is done, consider the rigid body which is spinning with an angular velocity $\boldsymbol{\omega}$, directed along one of its principal axes of inertia. If the principal moment of inertia about this axis is I, the angular momentum can be expressed as $\mathbf{H} = I\boldsymbol{\omega} = I\omega_x\mathbf{i} + I\omega_y\mathbf{j} + I\omega_z\mathbf{k}$. The components of $\mathbf{H}$ may also be expressed by Eqs. 21–10, where the inertia tensor is assumed to be known. Equate the $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ components of both expressions for $\mathbf{H}$ and consider ω_x, ω_y, and ω_z to be unknown. The solution of these three equations is obtained provided the determinant of the coefficients is zero. Show that this determinant, when expanded, yields the cubic equation

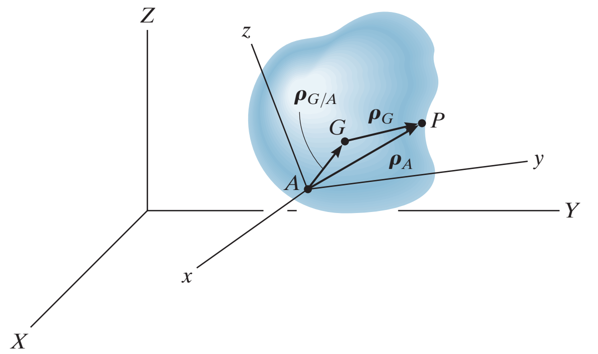
$$\begin{aligned} I^3 &- (I_{xx} + I_{yy} + I_{zz})I^2 \\ &+ (I_{xx}I_{yy} + I_{yy}I_{zz} + I_{zz}I_{xx} - I_{xy}^2 - I_{yz}^2 - I_{zx}^2)I \\ &- (I_{xx}I_{yy}I_{zz} - 2I_{xy}I_{yz}I_{zx} - I_{xx}I_{yz}^2 - I_{yy}I_{zx}^2 - I_{zz}I_{xy}^2) = 0 \end{aligned}$$

The three positive roots of I, obtained from the solution of this equation, represent the principal moments of inertia I_x, I_y, and I_z.



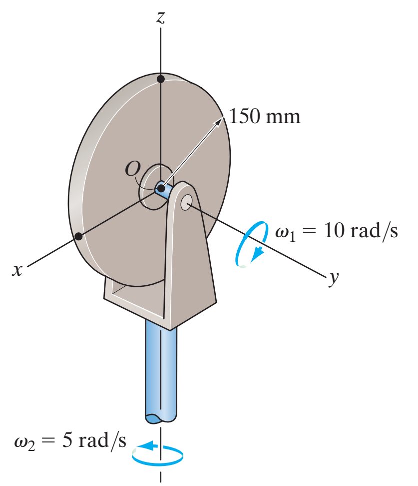
Prob. 21–22

21–23. Show that if the angular momentum of a body is determined with respect to an arbitrary point A, then $\mathbf{H}_A$ can be expressed by Eq. 21–9. This requires substituting $\boldsymbol{\rho}_A = \boldsymbol{\rho}_G + \boldsymbol{\rho}_{G/A}$ into Eq. 21–6 and expanding, noting that $\int \boldsymbol{\rho}_G \, dm = \mathbf{0}$ by definition of the mass center and $\mathbf{v}_G = \mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_{G/A}$.



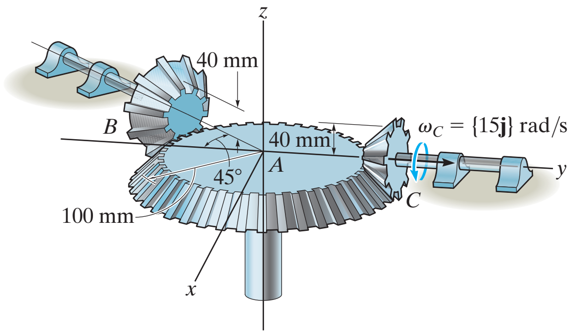
Prob. 21–23

***21–24.** The 15-kg circular disk spins about its axle with a constant angular velocity of $\omega_1 = 10$ rad/s. Simultaneously, the yoke is rotating with a constant angular velocity of $\omega_2 = 5$ rad/s. Determine the angular momentum of the disk about its center of mass O, and its kinetic energy.



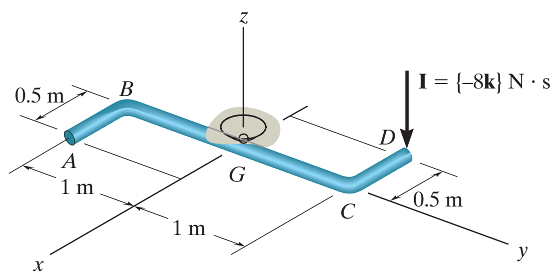
Prob. 21–24

21–25. The large gear has a mass of 5 kg and a radius of gyration of $k_z = 75$ mm. Gears B and C each have a mass of 200 g and a radius of gyration about the axis of their connecting shaft of 15 mm. If the gears are in mesh and C has an angular velocity of $\boldsymbol{\omega}_c = \{15\mathbf{j}\}$ rad/s, determine the total angular momentum for the system of three gears about point A.



Prob. 21–25

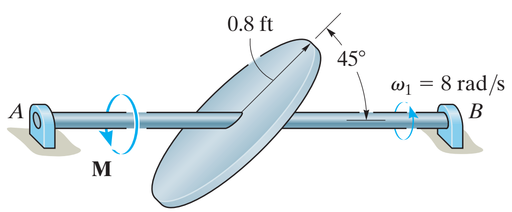
***21–28.** The rod assembly is supported at G by a ball-and-socket joint. Each segment has a mass of 0.5 kg/m. If the assembly is originally at rest and an impulse of $\mathbf{I} = \{-8\mathbf{k}\}$ N·s is applied at D, determine the angular velocity of the assembly just after the impact.



Prob. 21–28

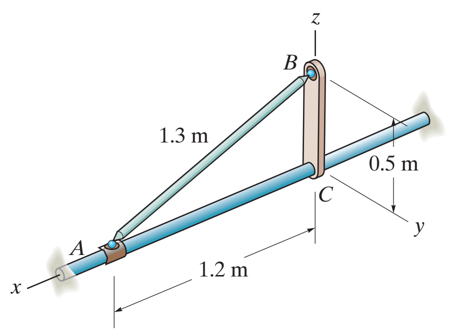
21–26. The circular disk has a weight of 15 lb and is mounted on the shaft AB at an angle of 45° with the horizontal. Determine the angular velocity of the shaft when $t = 3$ s if a constant torque $M = 2$ lb·ft is applied to the shaft. The shaft is originally spinning at $\omega_1 = 8$ rad/s when the torque is applied.

21–27. The circular disk has a weight of 15 lb and is mounted on the shaft AB at an angle of 45° with the horizontal. Determine the angular velocity of the shaft when $t = 2$ s if a torque $M = (4e^{0.1t})$ lb·ft, where t is in seconds, is applied to the shaft. The shaft is originally spinning at $\omega_1 = 8$ rad/s when the torque is applied.



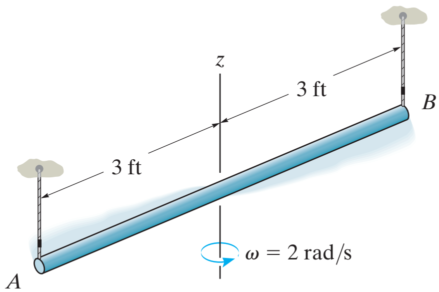
Probs. 21–26/27

21–29. The 4-lb rod AB is attached to the 1-lb collar at A and a 2-lb link BC using ball-and-socket joints. If the rod is released from rest in the position shown, determine the angular velocity of the link after it has rotated 180°.



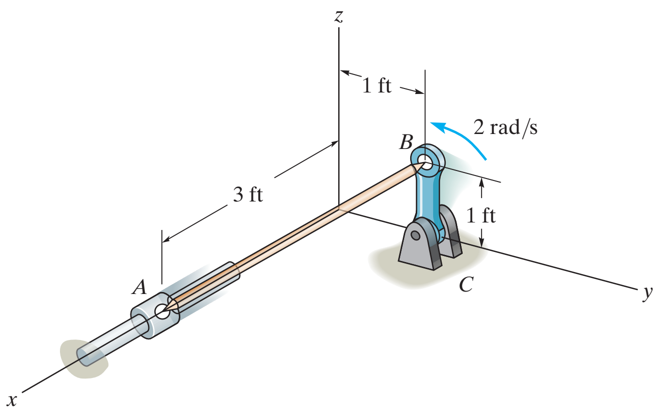
Prob. 21–29

21–30. The rod weighs 3 lb/ft and is suspended from parallel cords at A and B. If the rod has an angular velocity of 2 rad/s about the z axis at the instant shown, determine how high the center of the rod rises at the instant the rod momentarily stops swinging.



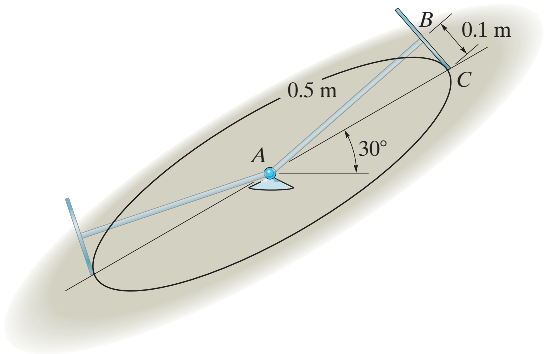
Prob. 21–30

21–31. The 4-lb rod AB is attached to the rod BC and collar A using ball-and-socket joints. If BC has a constant angular velocity of 2 rad/s, determine the kinetic energy of AB when it is in the position shown. Assume the angular velocity of AB is directed perpendicular to the axis of AB.



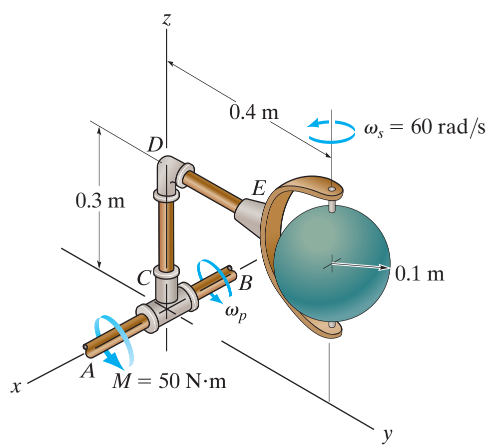
Prob. 21–31

***21–32.** The 2-kg thin disk is connected to the slender rod which is fixed to the ball-and-socket joint at A. If it is released from rest in the position shown, determine the spin of the disk about the rod when the disk reaches its lowest position. Neglect the mass of the rod. The disk rolls without slipping.



Prob. 21–32

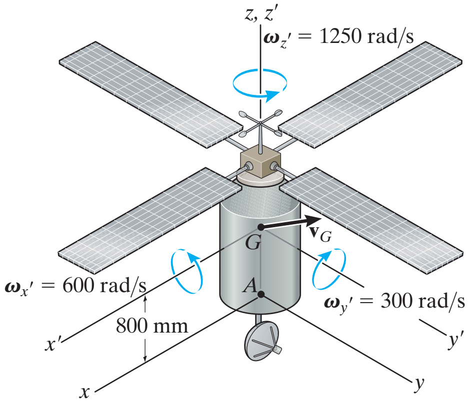
21–33. The 20-kg sphere rotates about the axle with a constant angular velocity of $\omega_s = 60$ rad/s. If shaft AB is subjected to a torque of $M = 50\text{ N}\cdot\text{m}$, causing it to rotate, determine the value of ω_p after the shaft has turned 90° from the position shown. Initially, $\omega_p = 0$. Neglect the mass of arm CDE.



Prob. 21–33

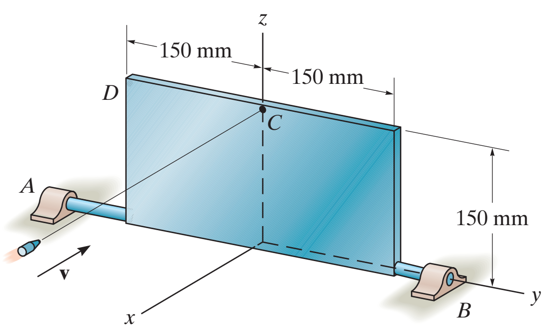
21–34. The 200-kg satellite has its center of mass at point G. Its radii of gyration about the z', x', y' axes are $k_{z'} = 300$ mm, $k_{x'} = k_{y'} = 500$ mm, respectively. At the instant shown, the satellite rotates about the x', y', and z' axes with the angular velocity shown, and its center of mass G has a velocity of $\mathbf{v}_G = \{-250\mathbf{i} + 200\mathbf{j} + 120\mathbf{k}\}$ m/s. Determine the angular momentum of the satellite about point A at this instant.

21–35. The 200-kg satellite has its center of mass at point G. Its radii of gyration about the z', x', y' axes are $k_{z'} = 300$ mm, $k_{x'} = k_{y'} = 500$ mm, respectively. At the instant shown, the satellite rotates about the x', y', and z' axes with the angular velocity shown, and its center of mass G has a velocity of $\mathbf{v}_G = \{-250\mathbf{i} + 200\mathbf{j} + 120\mathbf{k}\}$ m/s. Determine the kinetic energy of the satellite at this instant.



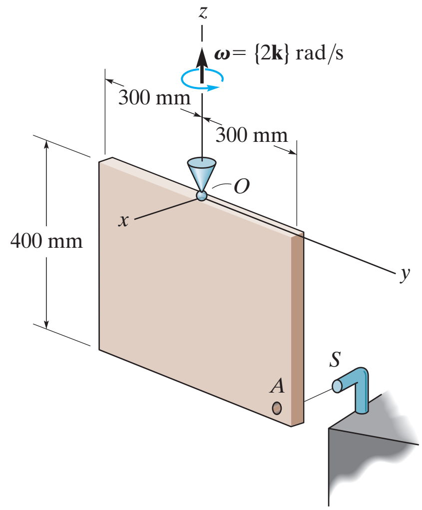
Probs. 21–34/35

***21–36.** The 15-kg rectangular plate is free to rotate about the y axis because of the bearing supports at A and B. When the plate is balanced in the vertical plane, a 3-g bullet is fired into it, perpendicular to its surface, with a velocity $\mathbf{v} = \{-2000\mathbf{i}\}$ m/s. Compute the angular velocity of the plate at the instant it has rotated 180°. If the bullet strikes corner D with the same velocity $\mathbf{v}$, instead of at C, does the angular velocity remain the same? Why or why not?



Prob. 21–36

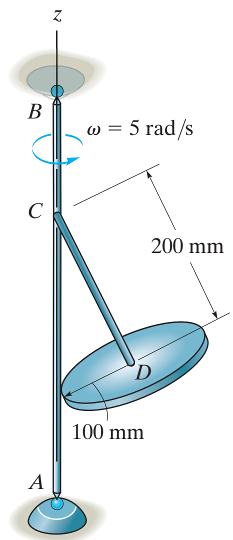
21–37. The 5-kg thin plate is suspended at O using a ball-and-socket joint. It is rotating with a constant angular velocity $\boldsymbol{\omega} = \{2\mathbf{k}\}$ rad/s when the corner A strikes the hook at S, which provides a permanent connection. Determine the angular velocity of the plate immediately after impact.



Prob. 21–37

21–38. Determine the kinetic energy of the 7-kg disk and 1.5-kg rod when the assembly is rotating about the z axis at $\omega = 5$ rad/s.

21–39. Determine the angular momentum $\mathbf{H}_z$ of the 7-kg disk and 1.5-kg rod when the assembly is rotating about the z axis at $\omega = 5$ rad/s.



Probs. 21–38/39

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*21.4 Equations of Motion

Having become familiar with the techniques used to describe both the inertial properties and the angular momentum of a body, we can now write the equations which describe the motion of the body in their most useful forms.

Equations of Translational Motion. The *translational motion* of a body is defined in terms of the acceleration of the body's mass center, which is measured from an inertial *X, Y, Z* reference. The equation of translational motion for the body can be written in vector form as

$$\Sigma \mathbf{F} = m\mathbf{a}_G \tag{21–18}$$

or by the three scalar equations

$$\boxed{\begin{aligned} \Sigma F_x &= m(a_G)_x \\ \Sigma F_y &= m(a_G)_y \\ \Sigma F_z &= m(a_G)_z \end{aligned}} \tag{21–19}$$

Here, $\Sigma \mathbf{F} = \Sigma F_x \mathbf{i} + \Sigma F_y \mathbf{j} + \Sigma F_z \mathbf{k}$ represents the sum of all the external forces acting on the body.

Equations of Rotational Motion. In Sec. 15.6, we developed Eq. 15–17, namely,

$$\Sigma \mathbf{M}_O = \dot{\mathbf{H}}_O \tag{21–20}$$

which states that the sum of the moments of all the external forces acting on a system of particles (contained in a rigid body) about a fixed point O is equal to the time rate of change of the total angular momentum of the body about point O. When moments of the external forces acting on the particles are summed about the system's *mass center G*, one again obtains the same simple form of Eq. 21–20, relating the moment summation $\Sigma \mathbf{M}_G$ to the angular momentum $\mathbf{H}_G$. To show this, consider the system of particles in Fig. 21–11, where *X, Y, Z* represents an inertial frame of reference and the *x, y, z* axes, with origin at *G*, *translate* with respect to this frame. In general, G is *accelerating*, so by definition the translating frame is *not* an inertial reference. The angular momentum of the ith particle with respect to this frame is, however,

$$(\mathbf{H}_i)_G = \mathbf{r}_{i/G} \times m_i \mathbf{v}_{i/G}$$

where $\mathbf{r}_{i/G}$ and $\mathbf{v}_{i/G}$ represent the position and velocity of the ith particle with respect to G. Taking the time derivative we have

$$(\dot{\mathbf{H}}_i)_G = \dot{\mathbf{r}}_{i/G} \times m_i \mathbf{v}_{i/G} + \mathbf{r}_{i/G} \times m_i \dot{\mathbf{v}}_{i/G}$$

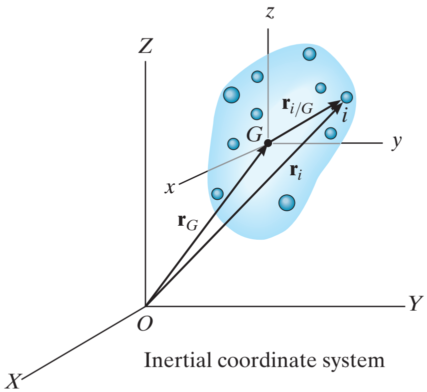


Fig. 21–11

By definition, $\mathbf{v}_{i/G} = \dot{\mathbf{r}}_{i/G}$. Thus, the first term on the right side is zero since the cross product of the same vectors is zero. Also, $\mathbf{a}_{i/G} = \dot{\mathbf{v}}_{i/G}$, so that

$$(\dot{\mathbf{H}}_i)_G = (\mathbf{r}_{i/G} \times m_i \mathbf{a}_{i/G})$$

Similar expressions can be written for the other particles of the body. When the results are summed, we get

$$\dot{\mathbf{H}}_G = \Sigma(\mathbf{r}_{i/G} \times m_i \mathbf{a}_{i/G})$$

Here $\dot{\mathbf{H}}_G$ is the time rate of change of the total angular momentum of the body computed about point G.

The relative acceleration for the ith particle is defined by the equation $\mathbf{a}_{i/G} = \mathbf{a}_i - \mathbf{a}_G$, where $\mathbf{a}_i$ and $\mathbf{a}_G$ represent, respectively, the accelerations of the ith particle and point G measured with respect to the *inertial frame of reference*. Substituting and expanding, using the distributive property of the vector cross product, yields

$$\dot{\mathbf{H}}_G = \Sigma(\mathbf{r}_{i/G} \times m_i \mathbf{a}_i) - (\Sigma m_i \mathbf{r}_{i/G}) \times \mathbf{a}_G$$

By definition of the mass center, the sum $(\Sigma m_i \mathbf{r}_{i/G}) = (\Sigma m_i)\bar{\mathbf{r}}$ is equal to zero, since the position vector $\bar{\mathbf{r}}$ relative to G is zero. Hence, the last term in the above equation is zero. Using the equation of motion, the product $m_i \mathbf{a}_i$ can be replaced by the resultant *external force* $\mathbf{F}_i$ acting on the ith particle. Denoting $\Sigma \mathbf{M}_G = \Sigma(\mathbf{r}_{i/G} \times \mathbf{F}_i)$, the final result can be written as

$$\Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G \tag{21–21}$$

The rotational equation of motion for the body will now be developed from either Eq. 21–20 or 21–21. In this regard, the scalar components of the angular momentum $\mathbf{H}_O$ or $\mathbf{H}_G$ are defined by Eqs. 21–10 or, if principal axes of inertia are used either at point O or G, by Eqs. 21–11. If these components are computed about x, y, z axes that are *rotating* with an angular velocity $\boldsymbol{\Omega}$ that is *different* from the body's angular velocity $\boldsymbol{\omega}$, then the time derivative $\dot{\mathbf{H}} = d\mathbf{H}/dt$, as used in Eqs. 21–20 and 21–21, must account for the rotation of the x, y, z axes as measured from the inertial X, Y, Z axes. This requires application of Eq. 20–6, in which case Eqs. 21–20 and 21–21 become

$$\begin{aligned} \Sigma \mathbf{M}_O &= (\dot{\mathbf{H}}_O)_{xyz} + \boldsymbol{\Omega} \times \mathbf{H}_O \\ \Sigma \mathbf{M}_G &= (\dot{\mathbf{H}}_G)_{xyz} + \boldsymbol{\Omega} \times \mathbf{H}_G \end{aligned} \tag{21–22}$$

Here $(\dot{\mathbf{H}})_{xyz}$ is the time rate of change of $\mathbf{H}$ measured from the x, y, z reference.

There are three ways in which one can define the motion of the x, y, z axes. Obviously, motion of this reference should be chosen so that it will yield the simplest set of moment equations for the solution of a particular problem.

x, y, z Axes Having Motion $\boldsymbol{\Omega} = \mathbf{0}$. If the body has general motion, the x, y, z axes can be chosen with origin at G, such that the axes only *translate* relative to the inertial X, Y, Z frame of reference. Doing this simplifies Eq. 21–22, since $\boldsymbol{\Omega} = \mathbf{0}$. However, the body may have a rotation $\boldsymbol{\omega}$ about these axes, and therefore the moments and products of inertia of the body would have to be expressed as *functions of time*. In most cases this would be a difficult task, so that such a choice of axes has restricted application.

x, y, z Axes Having Motion $\boldsymbol{\Omega} = \boldsymbol{\omega}$. The x, y, z axes can be chosen such that they are *fixed in and move with the body*. The moments and products of inertia of the body relative to these axes will then be *constant* during the motion. Since $\boldsymbol{\Omega} = \boldsymbol{\omega}$, Eqs. 21–22 become

$$
\begin{aligned}
\Sigma \mathbf{M}_O &= (\dot{\mathbf{H}}_O)_{xyz} + \boldsymbol{\omega} \times \mathbf{H}_O \\
\Sigma \mathbf{M}_G &= (\dot{\mathbf{H}}_G)_{xyz} + \boldsymbol{\omega} \times \mathbf{H}_G
\end{aligned}
\tag{21–23}
$$

We can express each of these vector equations as three scalar equations using Eqs. 21–10. Neglecting the subscripts O and G yields

$$
\begin{aligned}
\Sigma M_x &= I_{xx}\dot{\omega}_x - (I_{yy} - I_{zz})\omega_y\omega_z - I_{xy}(\dot{\omega}_y - \omega_z\omega_x) \\
&\quad - I_{yz}(\omega_y^2 - \omega_z^2) - I_{zx}(\dot{\omega}_z + \omega_x\omega_y) \\
\Sigma M_y &= I_{yy}\dot{\omega}_y - (I_{zz} - I_{xx})\omega_z\omega_x - I_{yz}(\dot{\omega}_z - \omega_x\omega_y) \\
&\quad - I_{zx}(\omega_z^2 - \omega_x^2) - I_{xy}(\dot{\omega}_x + \omega_y\omega_z) \\
\Sigma M_z &= I_{zz}\dot{\omega}_z - (I_{xx} - I_{yy})\omega_x\omega_y - I_{zx}(\dot{\omega}_x - \omega_y\omega_z) \\
&\quad - I_{xy}(\omega_x^2 - \omega_y^2) - I_{yz}(\dot{\omega}_y + \omega_z\omega_x)
\end{aligned}
\tag{21–24}
$$

If the x, y, z axes are chosen as *principal axes of inertia*, the products of inertia are zero, $I_{xx} = I_x$, etc., and the above equations become

$$
\boxed{
\begin{aligned}
\Sigma M_x &= I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z \\
\Sigma M_y &= I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x \\
\Sigma M_z &= I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y
\end{aligned}
}
\tag{21–25}
$$

This set of equations is known historically as the *Euler equations of motion*, named after the Swiss mathematician Leonhard Euler, who first developed them. They apply *only* for moments summed about either point O or G.

When applying these equations it should be realized that $\dot{\omega}_x$, $\dot{\omega}_y$, $\dot{\omega}_z$ represent the time derivatives of the magnitudes of the *x, y, z* components of $\boldsymbol{\omega}$ as observed from *x, y, z*. To determine these components, it is first necessary to find ω_x, ω_y, ω_z when the *x, y, z* axes are oriented in a *general position* and *then* take the time derivative of the magnitude of these components, i.e., $(\dot{\boldsymbol{\omega}})_{xyz}$. However, since the *x, y, z* axes are rotating at $\boldsymbol{\Omega} = \boldsymbol{\omega}$, then from Eq. 20–6, it should be noted that $\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{xyz} + \boldsymbol{\omega} \times \boldsymbol{\omega}$. Since $\boldsymbol{\omega} \times \boldsymbol{\omega} = \mathbf{0}$, then $\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{xyz}$. This important result indicates that the time derivative of $\boldsymbol{\omega}$ with respect to the fixed *X, Y, Z* axes, that is $\dot{\boldsymbol{\omega}}$, can also be used to obtain $(\dot{\boldsymbol{\omega}})_{xyz}$. Generally this is the easiest way to determine the result. See Example 21.5.

x, y, z Axes Having Motion $\boldsymbol{\Omega} \neq \boldsymbol{\omega}$. To simplify the calculations for the time derivative of $\boldsymbol{\omega}$, it is often convenient to choose the *x, y, z* axes having an angular velocity $\boldsymbol{\Omega}$ which is different from the angular velocity $\boldsymbol{\omega}$ of the body. This is particularly suitable for the analysis of spinning tops and gyroscopes which are *symmetrical* about their spinning axes.* When this is the case, the moments and products of inertia remain constant about the axis of spin.

Equations 21–22 are applicable for such a set of axes. Each of these two vector equations can be reduced to a set of three scalar equations which are derived in a manner similar to Eqs. 21–25,† i.e.,

$$\boxed{\begin{aligned}\Sigma M_x &= I_x\dot{\omega}_x - I_y\Omega_z\omega_y + I_z\Omega_y\omega_z\\ \Sigma M_y &= I_y\dot{\omega}_y - I_z\Omega_x\omega_z + I_x\Omega_z\omega_x\\ \Sigma M_z &= I_z\dot{\omega}_z - I_x\Omega_y\omega_x + I_y\Omega_x\omega_y\end{aligned}} \qquad (21\text{–}26)$$

Here Ω_x, Ω_y, Ω_z represent the *x, y, z* components of $\boldsymbol{\Omega}$, measured from the inertial frame of reference, and $\dot{\omega}_x$, $\dot{\omega}_y$, $\dot{\omega}_z$ must be determined relative to the *x, y, z* axes that have the rotation $\boldsymbol{\Omega}$. See Example 21.6.

Any one of these sets of moment equations, Eqs. 21–24, 21–25, or 21–26, represents a series of three first-order nonlinear differential equations. These equations are “coupled,” since the angular-velocity components are present in all the terms. Success in determining the solution for a particular problem therefore depends upon what is unknown in these equations. Difficulty certainly arises when one attempts to solve for the unknown components of $\boldsymbol{\omega}$ when the external moments are functions of time. Further complications can arise if the moment equations are coupled to the three scalar equations of translational motion, Eqs. 21–19. This can happen because of the existence of kinematic constraints which relate the rotation of the body to the translation of its mass center, as in the case of a hoop which rolls

*A detailed discussion of such devices is given in Sec. 21.5.
†See Prob. 21–42.

without slipping. Problems that require the simultaneous solution of differential equations are generally solved using numerical methods with the aid of a computer. In many engineering problems, however, we are given information about the motion of the body and are required to determine the applied moments acting on the body. Most of these problems have direct solutions, so that there is no need to resort to computer techniques.

Procedure for Analysis

Problems involving the three-dimensional motion of a rigid body can be solved using the following procedure.

Free-Body Diagram.

- Draw a *free-body diagram* of the body at the instant considered and specify the x, y, z coordinate system. The origin of this reference must be located either at the body's mass center G, or at point O, considered fixed in an inertial reference frame and located either in the body or on a massless extension of the body.
- Unknown reactive force components can be shown having a positive sense of direction.
- Depending on the nature of the problem, decide what type of rotational motion $\mathbf{\Omega}$ the x, y, z coordinate system should have, i.e., $\mathbf{\Omega} = \mathbf{0}$, $\mathbf{\Omega} = \boldsymbol{\omega}$, or $\mathbf{\Omega} \neq \boldsymbol{\omega}$. When choosing, keep in mind that the moment equations are simplified when the axes move in such a manner that they represent principal axes of inertia for the body at all times.
- Compute the necessary moments and products of inertia for the body relative to the x, y, z axes.

Kinematics.

- Determine the x, y, z components of the body's angular velocity and find the time derivatives of $\boldsymbol{\omega}$.
- Note that if $\mathbf{\Omega} = \boldsymbol{\omega}$, then $\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{xyz}$. Therefore we can either find the time derivative of $\boldsymbol{\omega}$ with respect to the X, Y, Z axes, $\dot{\boldsymbol{\omega}}$, and then determine its components $\dot{\omega}_x$, $\dot{\omega}_y$, $\dot{\omega}_z$, or we can find the components of $\boldsymbol{\omega}$ along the x, y, z axes, when the axes are oriented in a general position, and then take the time derivative of the magnitudes of these components, $(\dot{\boldsymbol{\omega}})_{xyz}$.

Equations of Motion.

- Apply either the two vector equations 21–18 and 21–22 or the six scalar component equations appropriate for the x, y, z coordinate axes chosen for the problem.

The gear shown in Fig. 21–12a has a mass of 10 kg and is mounted at an angle of 10° with the rotating shaft having negligible mass. If $I_z = 0.1\ \text{kg} \cdot \text{m}^2$, $I_x = I_y = 0.05\ \text{kg} \cdot \text{m}^2$, and the shaft is rotating with a constant angular velocity of $\omega = 30\ \text{rad/s}$, determine the components of reaction that the thrust bearing A and journal bearing B exert on the shaft at the instant shown.

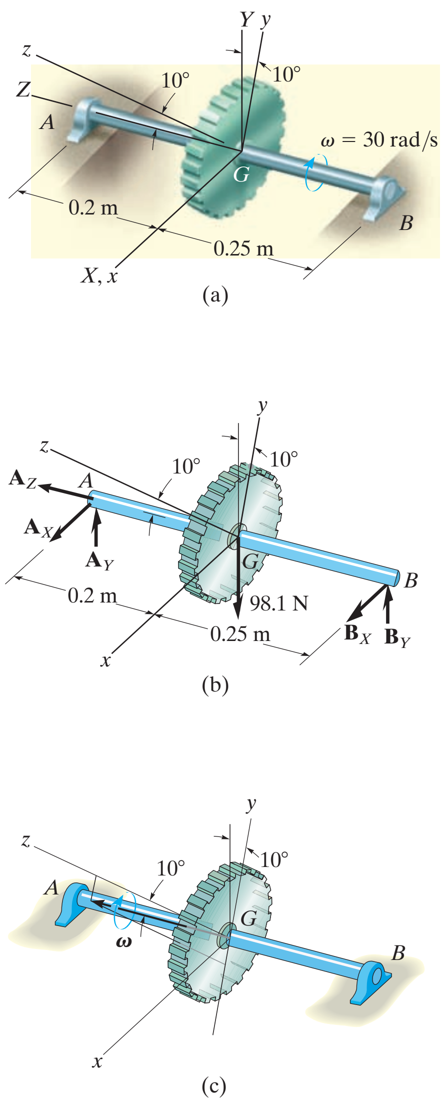


Fig. 21–12

SOLUTION

Free-Body Diagram. Fig. 21–12b. The origin of the x, y, z coordinate system is located at the gear's center of mass G, which is also a fixed point. The axes are fixed in and rotate with the gear so that these axes will then always represent the principal axes of inertia for the gear. Hence $\boldsymbol{\Omega} = \boldsymbol{\omega}$.

Kinematics. As shown in Fig. 21–12c, the angular velocity $\boldsymbol{\omega}$ of the gear is constant in magnitude and is always directed along the axis of the shaft AB. Since this vector is measured from the X, Y, Z inertial frame of reference, for any position of the x, y, z axes,

$$\omega_x = 0 \quad \omega_y = -30 \sin 10^\circ \quad \omega_z = 30 \cos 10^\circ$$

These components remain constant for any general orientation of the x, y, z axes, and so $\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$. Also note that since $\boldsymbol{\Omega} = \boldsymbol{\omega}$, then $\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{xyz}$. Therefore, we can find these time derivatives relative to the X, Y, Z axes. In this regard $\boldsymbol{\omega}$ has a constant magnitude and direction $(+Z)$ since $\dot{\boldsymbol{\omega}} = \mathbf{0}$, and so $\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$. Furthermore, since G is a fixed point, $(a_G)_x = (a_G)_y = (a_G)_z = 0$.

Equations of Motion. Applying Eqs. 21–25 ($\boldsymbol{\Omega} = \boldsymbol{\omega}$) yields

$$\Sigma M_x = I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z$$

$$-(A_Y)(0.2) + (B_Y)(0.25) = 0 - (0.05 - 0.1)(-30 \sin 10^\circ)(30 \cos 10^\circ)$$

$$-0.2A_Y + 0.25B_Y = -7.70 \qquad (1)$$

$$\Sigma M_y = I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x$$

$$A_X(0.2) \cos 10^\circ - B_X(0.25) \cos 10^\circ = 0 - 0$$

$$A_X = 1.25B_X \qquad (2)$$

$$\Sigma M_z = I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y$$

$$A_X(0.2) \sin 10^\circ - B_X(0.25) \sin 10^\circ = 0 - 0$$

$$A_X = 1.25B_X \text{ (check)}$$

Applying Eqs. 21–19, we have

$$\Sigma F_X = m(a_G)_X; \qquad A_X + B_X = 0 \qquad (3)$$

$$\Sigma F_Y = m(a_G)_Y; \qquad A_Y + B_Y - 98.1 = 0 \qquad (4)$$

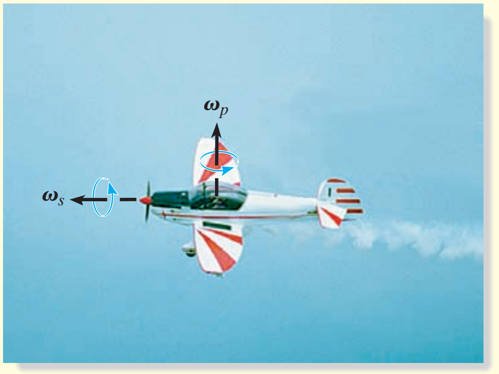
$$\Sigma F_Z = m(a_G)_Z; \qquad A_Z = 0 \qquad \textit{Ans.}$$

Solving Eqs. 1 through 4 simultaneously gives

$$A_X = B_X = 0 \quad A_Y = 71.6\ \text{N} \quad B_Y = 26.5\ \text{N} \qquad \textit{Ans.}$$

EXAMPLE 21.5

The airplane shown in Fig. 21–13*a* is in the process of making a steady *horizontal* turn at the rate of ω_p. During this motion, the propeller is spinning at the rate of ω_s. If the propeller has two blades, determine the moments which the propeller shaft exerts on the propeller at the instant the blades are in the vertical position. For simplicity, assume the blades to be a uniform slender bar having a moment of inertia I about an axis perpendicular to the blades passing through the center of the bar, and having zero moment of inertia about a longitudinal axis.



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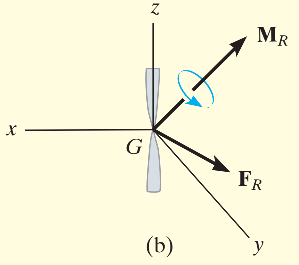
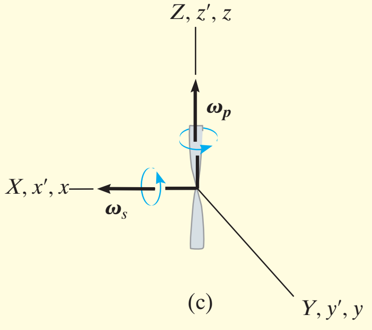
(a)



(b)



(c)

Fig. 21–13

SOLUTION

Free-Body Diagram. Fig. 21–13*b*. The reactions of the connecting shaft on the propeller are indicated by the resultants $\mathbf{F}_R$ and $\mathbf{M}_R$. (The propeller's weight is assumed to be negligible.) The x, y, z axes will be taken fixed to the propeller, since these axes always represent the principal axes of inertia for the propeller. Thus, $\boldsymbol{\Omega} = \boldsymbol{\omega}$. The moments of inertia I_x and I_y are equal ($I_x = I_y = I$) and $I_z = 0$.

Kinematics. The angular velocity of the propeller observed from the X, Y, Z axes, coincident with the x, y, z axes, Fig. 21–13*c*, is $\boldsymbol{\omega} = \boldsymbol{\omega}_s + \boldsymbol{\omega}_p = \omega_s\mathbf{i} + \omega_p\mathbf{k}$, so that the x, y, z components of $\boldsymbol{\omega}$ are

$$\omega_x = \omega_s \qquad \omega_y = 0 \qquad \omega_z = \omega_p$$

Since $\boldsymbol{\Omega} = \boldsymbol{\omega}$, then $\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{xyz}$. To find $\dot{\boldsymbol{\omega}}$, which is the time derivative with respect to the fixed X, Y, Z axes, we can use Eq. 20–6 since $\boldsymbol{\omega}$ changes direction relative to X, Y, Z. The time rate of change of each of these components $\dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_s + \dot{\boldsymbol{\omega}}_p$ relative to the X, Y, Z axes can be obtained by introducing a third coordinate system x', y', z', which has an angular velocity $\boldsymbol{\Omega}' = \boldsymbol{\omega}_p$ and is coincident with the X, Y, Z axes at the instant shown. Thus

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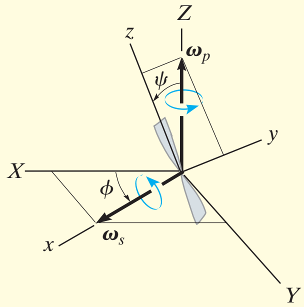
$$\begin{aligned}\dot{\boldsymbol{\omega}} &= (\dot{\boldsymbol{\omega}})_{x'y'z'} + \boldsymbol{\omega}_p \times \boldsymbol{\omega}\\ &= (\dot{\boldsymbol{\omega}}_s)_{x'y'z'} + (\dot{\boldsymbol{\omega}}_p)_{x'y'z'} + \boldsymbol{\omega}_p \times (\boldsymbol{\omega}_s + \boldsymbol{\omega}_p)\\ &= \mathbf{0} + \mathbf{0} + \boldsymbol{\omega}_p \times \boldsymbol{\omega}_s + \boldsymbol{\omega}_p \times \boldsymbol{\omega}_p\\ &= \mathbf{0} + \mathbf{0} + \omega_p\mathbf{k} \times \omega_s\mathbf{i} + \mathbf{0} = \omega_p\omega_s\mathbf{j}\end{aligned}$$

Since the X, Y, Z axes are coincident with the x, y, z axes at the instant shown, the components of $\dot{\boldsymbol{\omega}}$ along x, y, z are therefore

$$\dot{\omega}_x = 0 \qquad \dot{\omega}_y = \omega_p\omega_s \qquad \dot{\omega}_z = 0$$

These same results can also be determined by direct calculation of $(\dot{\boldsymbol{\omega}})_{xyz}$; however, this will involve a bit more work. To do this, it will be necessary to view the propeller (or the x, y, z axes) in some *general position* such as shown in Fig. 21–13*d*. Here the plane has turned through an angle ϕ (phi) and the propeller has turned through an angle ψ (psi) relative to the plane. Notice that $\boldsymbol{\omega}_p$ is always directed along the fixed Z axis and $\boldsymbol{\omega}_s$ follows the x axis. Thus the general components of $\boldsymbol{\omega}$ are



(d)

Fig. 21–13

$$\omega_x = \omega_s \qquad \omega_y = \omega_p \sin\psi \qquad \omega_z = \omega_p \cos\psi$$

Since ω_s and ω_p are constant, the time derivatives of these components become

$$\dot{\omega}_x = 0 \qquad \dot{\omega}_y = \omega_p \cos\psi\,\dot{\psi} \qquad \omega_z = -\omega_p \sin\psi\,\dot{\psi}$$

But $\phi = \psi = 0°$ and $\dot{\psi} = \omega_s$ at the instant considered. Thus,

$$\begin{aligned}&\omega_x = \omega_s \qquad \omega_y = 0 \qquad \omega_z = \omega_p\\ &\dot{\omega}_x = 0 \qquad \dot{\omega}_y = \omega_p\omega_s \qquad \dot{\omega}_z = 0\end{aligned}$$

which are the same results as those obtained previously.

Equations of Motion. Using Eqs. 21–25, we have

$$\Sigma M_x = I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z = I(0) - (I - 0)(0)\omega_p$$

$$M_x = 0 \qquad \textit{Ans.}$$

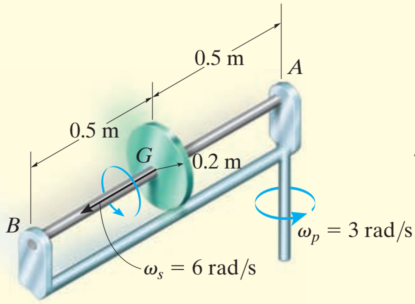
$$\Sigma M_y = I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x = I(\omega_p\omega_s) - (0 - I)\omega_p\omega_s$$

$$M_y = 2I\omega_p\omega_s \qquad \textit{Ans.}$$

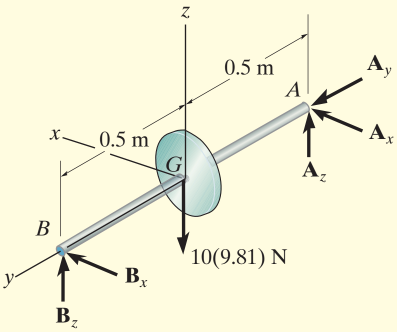
$$\Sigma M_z = I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y = 0(0) - (I - I)\omega_s(0)$$

$$M_z = 0 \qquad \textit{Ans.}$$

EXAMPLE 21.6



(a)



(b)

Fig. 21–14

The 10-kg flywheel (or thin disk) shown in Fig. 21–14*a* rotates (spins) about the shaft at a constant angular velocity of $\omega_s = 6$ rad/s. At the same time, the shaft rotates (precessing) about the bearing at A with an angular velocity of $\omega_p = 3$ rad/s. If A is a thrust bearing and B is a journal bearing, determine the components of force reaction at each of these supports due to the motion.

SOLUTION I

Free-Body Diagram. Fig. 21–14*b*. The origin of the x, y, z coordinate system is located at the center of mass G of the flywheel. Here we will let these coordinates have an angular velocity of $\mathbf{\Omega} = \boldsymbol{\omega}_p = \{3\mathbf{k}\}$ rad/s. Although the wheel spins relative to these axes, the moments of inertia *remain constant*,* i.e.,

$$I_x = I_z = \tfrac{1}{4}(10\text{ kg})(0.2\text{ m})^2 = 0.1\text{ kg}\cdot\text{m}^2$$
$$I_y = \tfrac{1}{2}(10\text{ kg})(0.2\text{ m})^2 = 0.2\text{ kg}\cdot\text{m}^2$$

Kinematics. From the coincident inertial X, Y, Z frame of reference, Fig. 21–14*c*, the flywheel has an angular velocity of $\boldsymbol{\omega} = \{6\mathbf{j} + 3\mathbf{k}\}$ rad/s, so that

$$\omega_x = 0 \quad \omega_y = 6\text{ rad/s} \quad \omega_z = 3\text{ rad/s}$$

The time derivative of $\boldsymbol{\omega}$ must be determined relative to the x, y, z axes. In this case both $\boldsymbol{\omega}_p$ and $\boldsymbol{\omega}_s$ do not change their magnitude or direction, and so

$$\dot{\omega}_x = 0 \quad \dot{\omega}_y = 0 \quad \dot{\omega}_z = 0$$

Equations of Motion. Applying Eqs. 21–26 ($\mathbf{\Omega} \neq \boldsymbol{\omega}$) yields

$$\Sigma M_x = I_x\dot{\omega}_x - I_y\Omega_z\omega_y + I_z\Omega_y\omega_z$$
$$-A_z(0.5) + B_z(0.5) = 0 - (0.2)(3)(6) + 0 = -3.6$$
$$\Sigma M_y = I_y\dot{\omega}_y - I_z\Omega_x\omega_z + I_x\Omega_z\omega_x$$
$$0 = 0 - 0 + 0$$
$$\Sigma M_z = I_z\dot{\omega}_z - I_x\Omega_y\omega_x + I_y\Omega_x\omega_y$$
$$A_x(0.5) - B_x(0.5) = 0 - 0 + 0$$

*This would not be true for the propeller in Example 21.5.

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Applying Eqs. 21–19, we have

$$\Sigma F_X = m(a_G)_X; \qquad A_x + B_x = 0$$

$$\Sigma F_Y = m(a_G)_Y; \qquad A_y = -10(0.5)(3)^2$$

$$\Sigma F_Z = m(a_G)_Z; \qquad A_z + B_z - 10(9.81) = 0$$

Solving these equations, we obtain

$$A_x = 0 \quad A_y = -45.0\text{ N} \quad A_z = 52.6\text{ N} \qquad \textit{Ans.}$$
$$B_x = 0 \qquad\qquad\qquad B_z = 45.4\text{ N} \qquad \textit{Ans.}$$

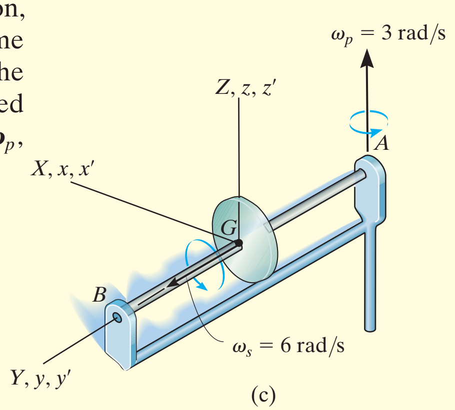
NOTE: If the precession $\boldsymbol{\omega}_p$ had not occurred, the z component of force at A and B would be equal to 49.05 N. In this case, however, the difference in these components is caused by the "gyroscopic moment" created whenever a spinning body precesses about another axis. We will study this effect in detail in the next section.

SOLUTION II

This example can also be solved using Euler's equations of motion, Eqs. 21–25. In this case $\boldsymbol{\Omega} = \boldsymbol{\omega} = \{6\mathbf{j} + 3\mathbf{k}\}$ rad/s, and the time derivative $(\dot{\boldsymbol{\omega}})_{xyz}$ can be conveniently obtained with reference to the fixed X, Y, Z axes since $\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{xyz}$. This calculation can be performed by choosing x', y', z' axes to have an angular velocity of $\boldsymbol{\Omega}' = \boldsymbol{\omega}_p$, Fig. 21–14*c*, so that

$$\dot{\boldsymbol{\omega}} = (\dot{\boldsymbol{\omega}})_{x'y'z'} + \boldsymbol{\omega}_p \times \boldsymbol{\omega} = \mathbf{0} + 3\mathbf{k} \times (6\mathbf{j} + 3\mathbf{k}) = \{-18\mathbf{i}\}\text{ rad/s}^2$$

$$\dot{\omega}_x = -18\text{ rad/s} \quad \dot{\omega}_y = 0 \quad \dot{\omega}_z = 0$$



(c)

Fig. 21–14

The moment equations then become

$$\Sigma M_x = I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z$$

$$-A_z(0.5) + B_z(0.5) = 0.1(-18) - (0.2 - 0.1)(6)(3) = -3.6$$

$$\Sigma M_y = I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x$$

$$0 = 0 - 0$$

$$\Sigma M_z = I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y$$

$$A_x(0.5) - B_x(0.5) = 0 - 0$$

The solution then proceeds as before.

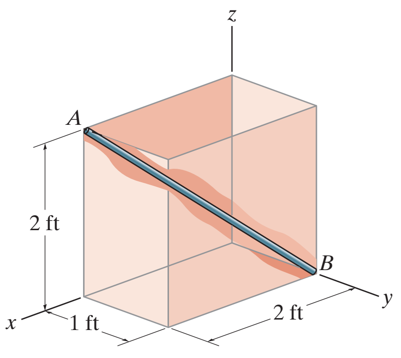
PROBLEMS

***21–40.** Derive the scalar form of the rotational equation of motion about the x axis if $\boldsymbol{\Omega} \neq \boldsymbol{\omega}$ and the moments and products of inertia of the body are *not constant* with respect to time.

21–41. Derive the scalar form of the rotational equation of motion about the x axis if $\boldsymbol{\Omega} \neq \boldsymbol{\omega}$ and the moments and products of inertia of the body are *constant* with respect to time.

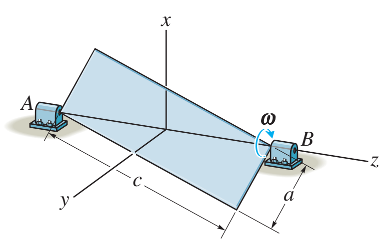
21–42. Derive the Euler equations of motion for $\boldsymbol{\Omega} \neq \boldsymbol{\omega}$, i.e., Eqs. 21–26.

21–43. The 4-lb bar rests along the smooth corners of an open box. At the instant shown, the box has a velocity $\mathbf{v} = \{3\mathbf{j}\}$ ft/s and an acceleration $\mathbf{a} = \{-6\mathbf{j}\}$ ft/s^2. Determine the x, y, z components of force which the corners exert on the bar.



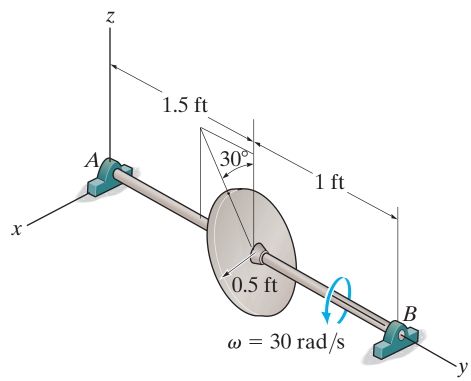
Prob. 21–43

***21–44.** The uniform plate has a mass of $m = 2$ kg and is given a rotation of $\omega = 4$ rad/s about its bearings at A and B. If $a = 0.2$ m and $c = 0.3$ m, determine the vertical reactions at the instant shown. Use the x, y, z axes shown and note that $I_{zx} = -\left(\dfrac{mac}{12}\right)\left(\dfrac{c^2 - a^2}{c^2 + a^2}\right)$.



Prob. 21–44

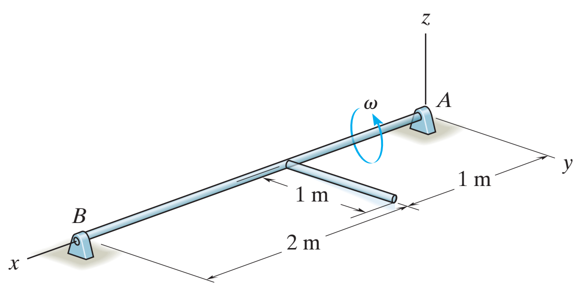
21–45. If the shaft AB is rotating with a constant angular velocity of $\omega = 30$ rad/s, determine the X, Y, Z components of reaction at the thrust bearing A and journal bearing B at the instant shown. The disk has a weight of 15 lb. Neglect the weight of the shaft AB.



Prob. 21–45

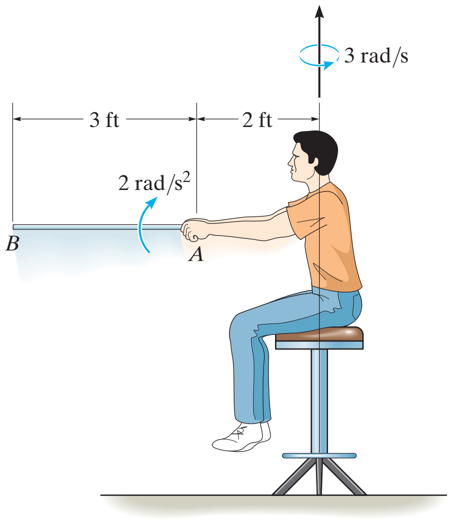
21–46. The assembly is supported by journal bearings at A and B, which develop only y and z force reactions on the shaft. If the shaft is rotating in the direction shown at $\boldsymbol{\omega} = \{2\mathbf{i}\}$ rad/s, determine the reactions at the bearings when the assembly is in the position shown. Also, what is the shaft's angular acceleration? The mass per unit length of each rod is 5 kg/m.

21–47. The assembly is supported by journal bearings at A and B, which develop only y and z force reactions on the shaft. If the shaft A is subjected to a couple moment $\mathbf{M} = \{40\mathbf{i}\}$ N·m, and at the instant shown the shaft has an angular velocity of $\boldsymbol{\omega} = \{2\mathbf{i}\}$ rad/s, determine the reactions at the bearings of the assembly at this instant. Also, what is the shaft's angular acceleration? The mass per unit length of each rod is 5 kg/m.



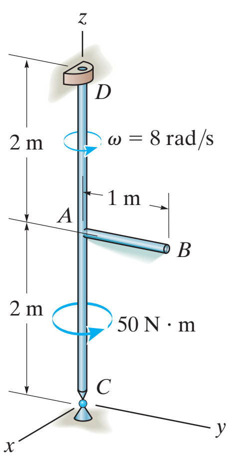
Probs. 21–46/47

***21–48.** The man sits on a swivel chair which is rotating with a constant angular velocity of 3 rad/s. He holds the uniform 5-lb rod AB horizontal. He suddenly gives it an angular acceleration of $2\ \text{rad/s}^2$, measured relative to him, as shown. Determine the required force and moment components at the grip, A, necessary to do this. Establish axes at the rod's center of mass G, with $+z$ upward, and $+y$ directed along the axis of the rod toward A.



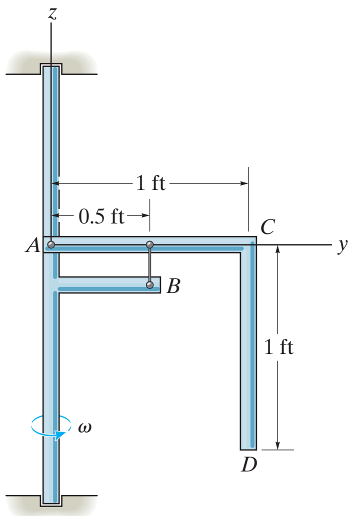
Prob. 21–48

21–49. The rod assembly is supported by a ball-and-socket joint at C and a journal bearing at D, which develops only x and y force reactions. The rods have a mass of 0.75 kg/m. Determine the angular acceleration of the rods and the components of reaction at the supports at the instant $\omega = 8$ rad/s as shown.



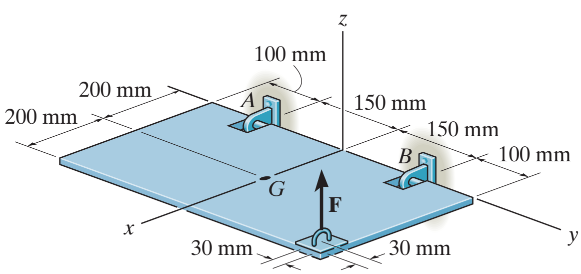
Prob. 21–49

21–50. The bent uniform rod ACD has a weight of 5 lb/ft and is supported at A by a pin and at B by a cord. If the vertical shaft rotates with a constant angular velocity $\omega = 20$ rad/s, determine the x, y, z components of force and moment developed at A and the tension in the cord.



Prob. 21–50

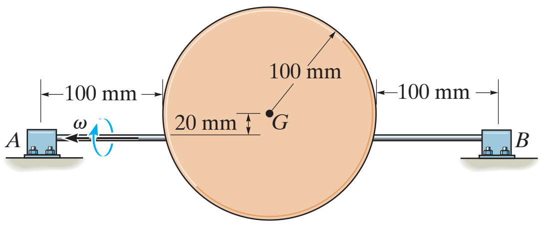
21–51. The uniform hatch door, having a mass of 15 kg and a mass center at G, is supported in the horizontal plane by bearings at A and B. If a vertical force $F = 300$ N is applied to the door as shown, determine the components of reaction at the bearings and the angular acceleration of the door. The bearing at A will resist a component of force in the y direction, whereas the bearing at B will not. For the calculation, assume the door to be a thin plate and neglect the size of each bearing. The door is originally at rest.



Prob. 21–51

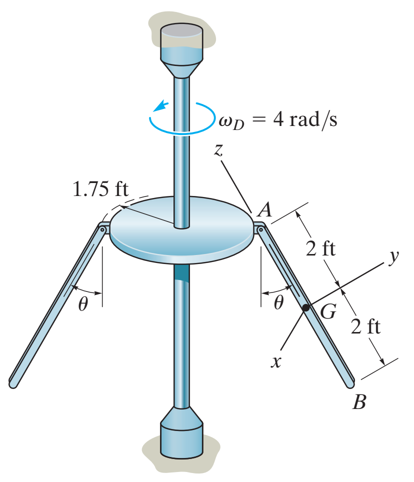
21

***21–52.** The 5-kg circular disk is mounted off center on a shaft which is supported by bearings at A and B. If the shaft is rotating at a constant rate of $\omega = 10$ rad/s, determine the vertical reactions at the bearings when the disk is in the position shown.



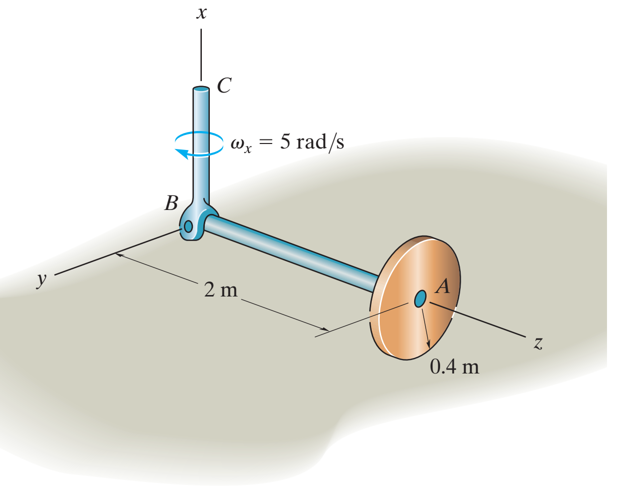
Prob. 21–52

21–53. Two uniform rods, each having a weight of 10 lb, are pin connected to the edge of a rotating disk. If the disk has a constant angular velocity $\omega_D = 4$ rad/s, determine the angle θ made by each rod during the motion, and the components of the force and moment developed at the pin A. *Suggestion*: Use the x, y, z axes oriented as shown.



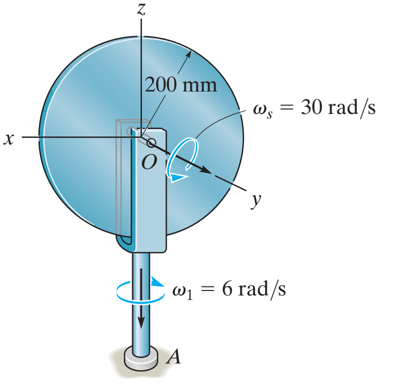
Prob. 21–53

21–54. The 10-kg disk turns around the shaft AB, while the shaft rotates about BC at a constant rate of $\omega_x = 5$ rad/s. If the disk does not slip, determine the normal and frictional force it exerts on the ground. Neglect the mass of shaft AB.



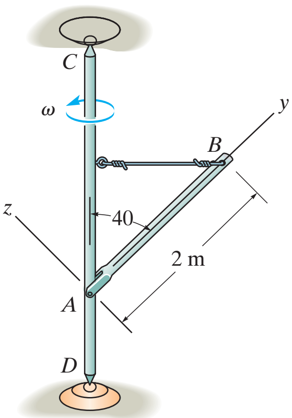
Prob. 21–54

21–55. The 20-kg disk is spinning on its axle at $\omega_s = 30$ rad/s, while the forked rod is turning at $\omega_1 = 6$ rad/s. Determine the x and z moment components the axle exerts on the disk during the motion.



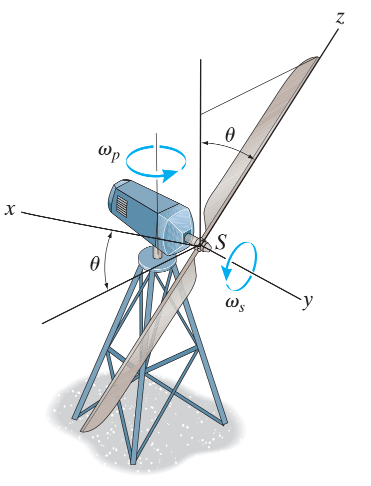
Prob. 21–55

***21–56.** The 4-kg slender rod AB is pinned at A and held at B by a cord. The axle CD is supported at its ends by ball-and-socket joints and is rotating with a constant angular velocity of 2 rad/s. Determine the tension developed in the cord and the magnitude of force developed at the pin A.



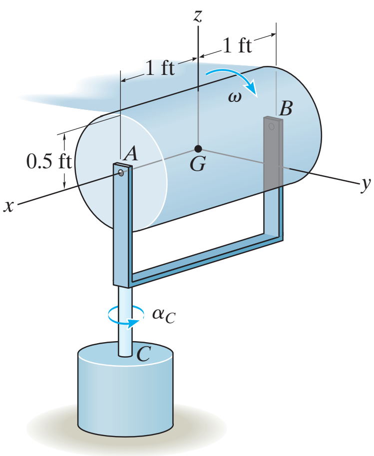
Prob. 21–56

21–57. The blades of a wind turbine spin about the shaft S with a constant angular speed of ω_s, while the frame precesses about the vertical axis with a constant angular speed of ω_p. Determine the x, y, and z components of moment that the shaft exerts on the blades as a function of θ. Consider each blade as a slender rod of mass m and length l.



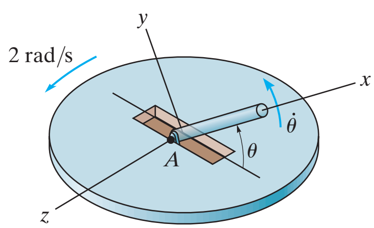
Prob. 21–57

21–58. The 15-lb cylinder is rotating about shaft AB with a constant angular speed $\omega = 4$ rad/s. If the supporting shaft at C, initially at rest, is given an angular acceleration $\alpha_C = 12\ \text{rad/s}^2$, determine the components of reaction at the bearings A and B. The bearing at A cannot support a force component along the x axis, whereas the bearing at B does.



Prob. 21–58

21–59. The *thin rod* has a mass of 0.8 kg and a total length of 150 mm. It is rotating about its midpoint at a constant rate $\dot{\theta} = 6$ rad/s, while the table to which its axle A is fastened is rotating at 2 rad/s. Determine the x, y, z moment components which the axle exerts on the rod when the rod is in any position θ.



Prob. 21–59

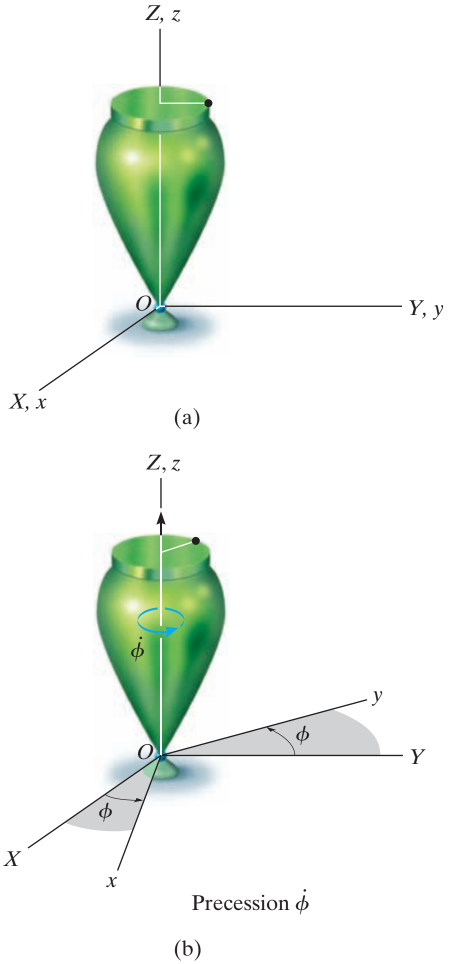
*21.5 Gyroscopic Motion

In this section we will develop the equations defining the motion of a body (top) which is symmetrical with respect to an axis and rotating about a fixed point. These equations also apply to the motion of a particularly interesting device, the gyroscope.

The body's motion will be analyzed using *Euler angles* ϕ, θ, ψ (phi, theta, psi). To illustrate how they define the position of a body, consider the top shown in Fig. 21–15*a*. To define its final position, Fig. 21–15*d*, a second set of *x, y, z* axes is fixed in the top. Starting with the *X, Y, Z* and *x, y, z* axes in coincidence, Fig. 21–15*a*, the final position of the top can be determined using the following three steps:

1. Rotate the top about the *Z* (or *z*) axis through an angle ϕ $(0 \leq \phi < 2\pi)$, Fig. 21–15*b*.
2. Rotate the top about the *x* axis through an angle θ $(0 \leq \theta \leq \pi)$, Fig. 21–15*c*.
3. Rotate the top about the *z* axis through an angle ψ $(0 \leq \psi < 2\pi)$ to obtain the final position, Fig. 21–15*d*.

The sequence of these three angles, ϕ, θ, then ψ, must be maintained, since finite rotations are *not vectors* (see Fig. 20–1). Although this is the case, the differential rotations $d\boldsymbol{\phi}$, $d\boldsymbol{\theta}$, and $d\boldsymbol{\psi}$ are vectors, and thus the angular velocity $\boldsymbol{\omega}$ of the top can be expressed in terms of the time derivatives of the Euler angles. The angular-velocity components $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$ are known as the *precession, nutation*, and *spin*, respectively.



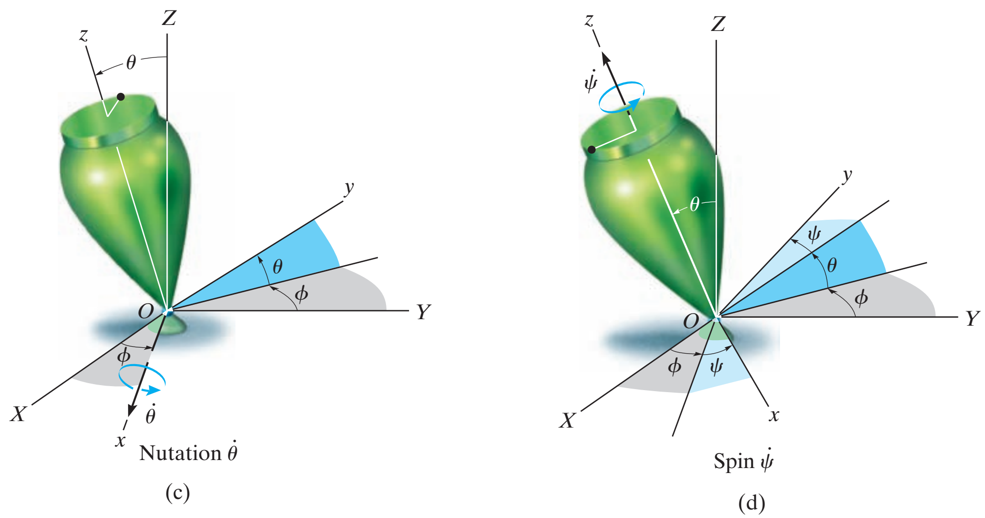


Fig. 21–15

21

Their positive directions are shown in Fig. 21–16. It is seen that these vectors are not all perpendicular to one another; however, $\boldsymbol{\omega}$ of the top can still be expressed in terms of these three components.

Since the body (top) is symmetric with respect to the z or spin axis, there is no need to attach the x, y, z axes to the top since the inertial properties of the top will remain constant with respect to this frame during the motion. Therefore $\boldsymbol{\Omega} = \boldsymbol{\omega}_p + \boldsymbol{\omega}_n$, Fig. 21–16. Hence, the angular velocity of the body is

$$\begin{aligned}\boldsymbol{\omega} &= \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}\\ &= \dot{\theta}\mathbf{i} + (\dot{\phi}\sin\theta)\mathbf{j} + (\dot{\phi}\cos\theta + \dot{\psi})\mathbf{k} \qquad (21\text{–}27)\end{aligned}$$

And the angular velocity of the axes is

$$\begin{aligned}\boldsymbol{\Omega} &= \Omega_x\mathbf{i} + \Omega_y\mathbf{j} + \Omega_z\mathbf{k}\\ &= \dot{\theta}\mathbf{i} + (\dot{\phi}\sin\theta)\mathbf{j} + (\dot{\phi}\cos\theta)\mathbf{k} \qquad (21\text{–}28)\end{aligned}$$

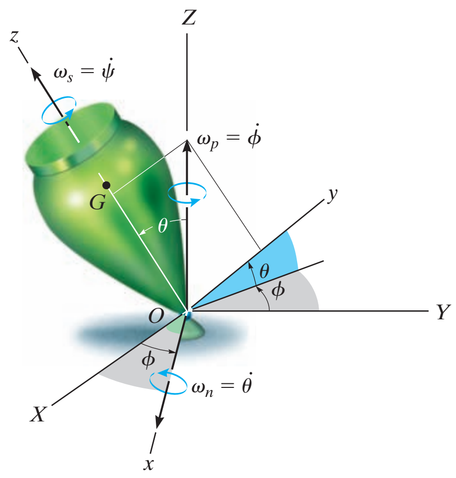


Fig. 21–16

Have the x, y, z axes represent principal axes of inertia for the top, and so the moments of inertia will be represented as $I_{xx} = I_{yy} = I$ and $I_{zz} = I_z$. Since $\boldsymbol{\Omega} \neq \boldsymbol{\omega}$, Eqs. 21–26 are used to establish the rotational equations of motion. Substituting into these equations the respective angular-velocity components defined by Eqs. 21–27 and 21–28, their corresponding time derivatives, and the moment of inertia components, yields

$$\begin{aligned}\Sigma M_x &= I(\ddot{\theta} - \dot{\phi}^2\sin\theta\cos\theta) + I_z\dot{\phi}\sin\theta(\dot{\phi}\cos\theta + \dot{\psi})\\ \Sigma M_y &= I(\ddot{\phi}\sin\theta + 2\dot{\phi}\dot{\theta}\cos\theta) - I_z\dot{\theta}(\dot{\phi}\cos\theta + \dot{\psi}) \qquad (21\text{–}29)\\ \Sigma M_z &= I_z(\ddot{\psi} + \ddot{\phi}\cos\theta - \dot{\phi}\dot{\theta}\sin\theta)\end{aligned}$$

Each moment summation applies only at the fixed point O or the center of mass G of the body. Since the equations represent a coupled set of nonlinear second-order differential equations, in general a closed-form solution may not be obtained. Instead, the Euler angles ϕ, θ, and ψ may be obtained graphically as functions of time using numerical analysis and computer techniques.

A special case, however, does exist for which simplification of Eqs. 21–29 is possible. Commonly referred to as *steady precession*, it occurs when the nutation angle θ, precession $\dot{\phi}$, and spin $\dot{\psi}$ all remain *constant*. Equations 21–29 then reduce to the form

$$\boxed{\Sigma M_x = -I\dot{\phi}^2\sin\theta\cos\theta + I_z\dot{\phi}\sin\theta(\dot{\phi}\cos\theta + \dot{\psi})} \qquad (21\text{–}30)$$

$$\Sigma M_y = 0$$

$$\Sigma M_z = 0$$

21

Equation 21–30 can be further simplified by noting that, from Eq. 21–27, $\omega_z = \dot{\phi}\cos\theta + \dot{\psi}$, so that

$$\Sigma M_x = -I\dot{\phi}^2 \sin\theta\cos\theta + I_z\dot{\phi}(\sin\theta)\omega_z$$

or

$$\boxed{\Sigma M_x = \dot{\phi}\sin\theta(I_z\omega_z - I\dot{\phi}\cos\theta)} \qquad (21\text{–}31)$$

It is interesting to note what effects the spin $\dot{\psi}$ has on the moment about the x axis. To show this, consider the spinning rotor in Fig. 21–17. Here $\theta = 90°$, in which case Eq. 21–30 reduces to the form

$$\Sigma M_x = I_z\dot{\phi}\dot{\psi}$$

or

$$\boxed{\Sigma M_x = I_z\Omega_y\omega_z} \qquad (21\text{–}32)$$

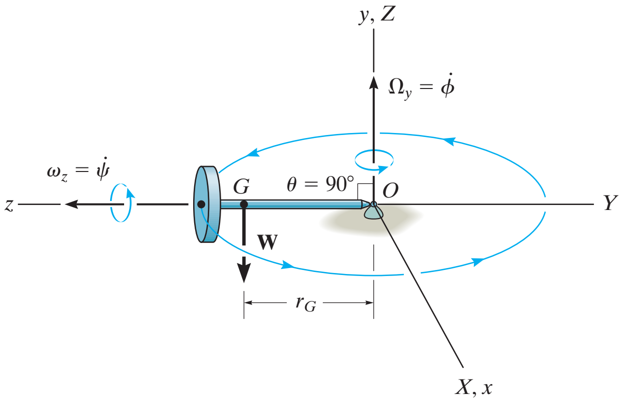


Fig. 21–17

21

From the figure it can be seen that $\mathbf{\Omega}_y$ and $\boldsymbol{\omega}_z$ act along their respective *positive axes* and therefore are mutually perpendicular. Instinctively, one would expect the rotor to fall down under the influence of gravity! However, this is not the case at all, provided the product $I_z\Omega_y\omega_z$ is correctly chosen to counterbalance the moment $\Sigma M_x = Wr_G$ of the rotor's weight about O. This unusual phenomenon of rigid-body motion is often referred to as the *gyroscopic effect*.

Perhaps a more intriguing demonstration of the gyroscopic effect comes from studying the action of a *gyroscope*, frequently referred to as a *gyro*. A gyro is a rotor which spins at a very high rate about its axis of symmetry. This rate of spin is considerably greater than its precessional rate of rotation about the vertical axis. Hence, for all practical purposes, the angular momentum of the gyro can be assumed directed along its axis of spin. Thus, for the gyro rotor shown in Fig. 21–18, $\omega_z \gg \Omega_y$, and the magnitude of the angular momentum about point O, as determined from Eqs. 21–11, reduces to the form $H_O = I_z\omega_z$. Since both the magnitude and direction of $\mathbf{H}_O$ are constant as observed from x, y, z, direct application of Eq. 21–22 yields

$$\boxed{\Sigma\mathbf{M}_x = \boldsymbol{\Omega}_y \times \mathbf{H}_O} \qquad (21\text{–}33)$$

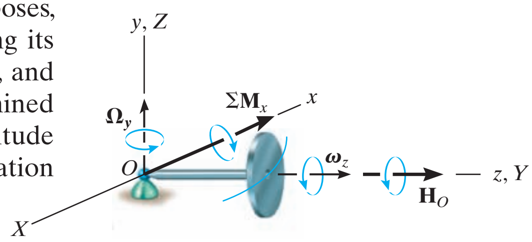


Fig. 21–18

Using the right-hand rule applied to the cross product, it can be seen that $\boldsymbol{\Omega}_y$ always swings $\mathbf{H}_O$ (or $\boldsymbol{\omega}_z$) toward the sense of $\Sigma\mathbf{M}_x$. In effect, the *change in direction* of the gyro's angular momentum, $d\mathbf{H}_O$, is equivalent to the angular impulse caused by the gyro's weight about O, i.e., $d\mathbf{H}_O = \Sigma\mathbf{M}_x\,dt$, Eq. 21–20. Also, since $H_O = I_z\omega_z$ and $\Sigma\mathbf{M}_x$, $\boldsymbol{\Omega}_y$, and $\mathbf{H}_O$ are mutually perpendicular, Eq. 21–33 reduces to Eq. 21–32.

When a gyro is mounted in gimbal rings, Fig. 21–19, it becomes *free* of external moments applied to its base. Thus, in theory, its angular momentum **H** will never precess but, instead, maintain its same fixed orientation along the axis of spin when the base is rotated. This type of gyroscope is called a *free gyro* and is useful as a gyrocompass when the spin axis of the gyro is directed north. In reality, the gimbal mechanism is never completely free of friction, so such a device is useful only for the local navigation of ships and aircraft. The gyroscopic effect is also useful as a means of stabilizing both the rolling motion of ships at sea and the trajectories of missiles and projectiles. Furthermore, this effect is of significant importance in the design of shafts and bearings for rotors which are subjected to forced precessions.

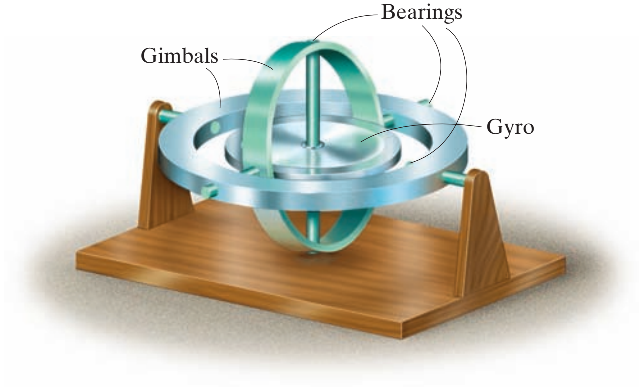
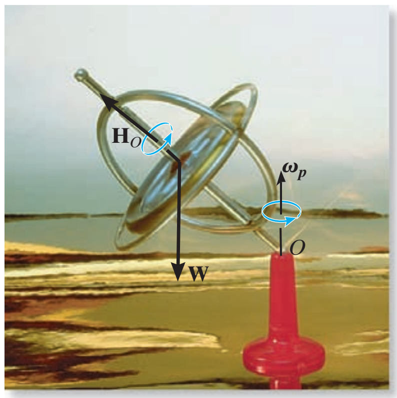


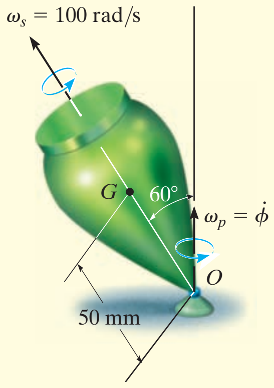
Fig. 21–19



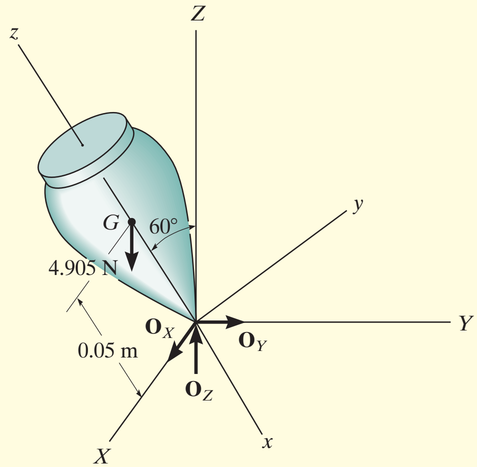
The spinning of the gyro within the frame of this toy gyroscope produces angular momentum $\mathbf{H}_O$, which is changing direction as the frame precesses $\boldsymbol{\omega}_p$ about the vertical axis. The gyroscope will not fall down since the moment of its weight **W** about the support is balanced by the change in the direction of $\mathbf{H}_O$. (© R.C. Hibbeler)

EXAMPLE 21.7

The top shown in Fig. 21–20*a* has a mass of 0.5 kg and is precessing about the vertical axis at a constant angle of $\theta = 60°$. If it spins with an angular velocity $\omega_s = 100$ rad/s, determine the precession $\boldsymbol{\omega}_p$. Assume that the axial and transverse moments of inertia of the top are $0.45(10^{-3})\ \text{kg}\cdot\text{m}^2$ and $1.20(10^{-3})\ \text{kg}\cdot\text{m}^2$, respectively, measured with respect to the fixed point O.



(a)



(b)

Fig. 21–20

SOLUTION

Equation 21–30 will be used for the solution since the motion is *steady precession*. As shown on the free-body diagram, Fig. 21–20*b*, the coordinate axes are established in the usual manner, that is, with the positive z axis in the direction of spin, the positive Z axis in the direction of precession, and the positive x axis in the direction of the moment ΣM_x (refer to Fig. 21–16). Thus,

$$\Sigma M_x = -I\dot{\phi}^2 \sin\theta\cos\theta + I_z\dot{\phi}\sin\theta(\dot{\phi}\cos\theta + \dot{\psi})$$

$$4.905\text{ N}(0.05\text{ m})\sin 60° = -[1.20(10^{-3})\ \text{kg}\cdot\text{m}^2\,\dot{\phi}^2]\sin 60°\cos 60° + [0.45(10^{-3})\ \text{kg}\cdot\text{m}^2]\dot{\phi}\sin 60°(\dot{\phi}\cos 60° + 100\text{ rad/s})$$

or

$$\dot{\phi}^2 - 120.0\dot{\phi} + 654.0 = 0 \qquad (1)$$

Solving this quadratic equation for the precession gives

$$\dot{\phi} = 114\text{ rad/s} \quad \text{(high precession)}$$ *Ans.*

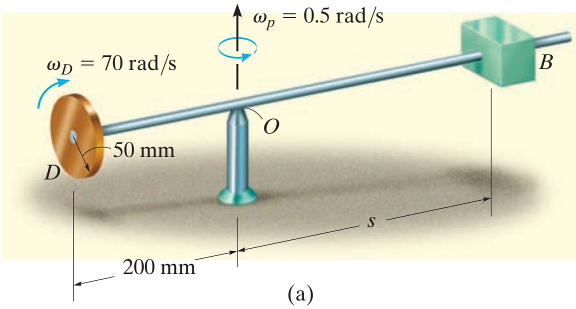
and

$$\dot{\phi} = 5.72\text{ rad/s} \quad \text{(low precession)}$$ *Ans.*

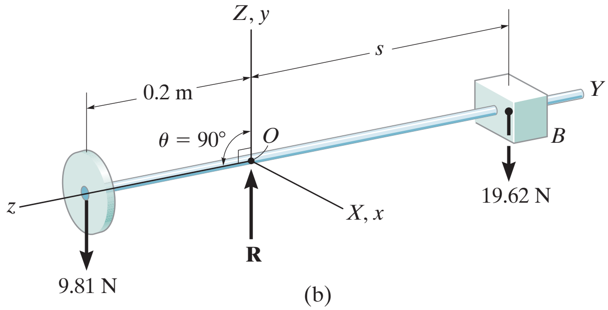
NOTE: In reality, low precession of the top would generally be observed, since high precession would require a larger kinetic energy.

21

The 1-kg disk shown in Fig. 21–21*a* spins about its axis with a constant angular velocity $\omega_D = 70$ rad/s. The block at B has a mass of 2 kg, and by adjusting its position s one can change the precession of the disk about its supporting pivot at O while the shaft remains horizontal. Determine the position s that will enable the disk to have a constant precession $\omega_p = 0.5$ rad/s about the pivot. Neglect the weight of the shaft.



(a)



(b)

Fig. 21–21

SOLUTION

The free-body diagram of the assembly is shown in Fig. 21–21*b*. The origin for both the x, y, z and X, Y, Z coordinate systems is located at the fixed point O. In the conventional sense, the Z axis is chosen along the axis of precession, and the z axis is along the axis of spin, so that $\theta = 90°$. Since the precession is *steady*, Eq. 21–32 can be used for the solution.

$$\Sigma M_x = I_z \Omega_y \omega_z$$

Substituting the required data gives

$$(9.81\text{ N})\,(0.2\text{ m}) - (19.62\text{ N})s = \left[\tfrac{1}{2}(1\text{ kg})(0.05\text{ m})^2\right]0.5\text{ rad/s}(-70\text{ rad/s})$$

$$s = 0.102\text{ m} = 102\text{ mm} \qquad \textit{Ans.}$$

21

21.6 Torque-Free Motion

When the only external force acting on a body is caused by gravity, the general motion of the body is referred to as *torque-free motion*. This type of motion is characteristic of planets, artificial satellites, and projectiles—provided air friction is neglected.

In order to describe the characteristics of this motion, the distribution of the body's mass will be assumed *axisymmetric*. The satellite shown in Fig. 21–22 is an example of such a body, where the z axis represents an axis of symmetry. The origin of the x, y, z coordinates is located at the mass center G, such that $I_{zz} = I_z$ and $I_{xx} = I_{yy} = I$. Since gravity is the only external force present, the summation of moments about the mass center is zero. From Eq. 21–21, this requires the angular momentum of the body to be constant, i.e.,

$$\mathbf{H}_G = \text{constant}$$

At the instant considered, it will be assumed that the inertial frame of reference is oriented so that the positive Z axis is directed along $\mathbf{H}_G$ and the y axis lies in the plane formed by the z and Z axes, Fig. 21–22. The Euler angle formed between Z and z is θ, and therefore, with this choice of axes the angular momentum can be expressed as

$$\mathbf{H}_G = H_G \sin\theta\,\mathbf{j} + H_G \cos\theta\,\mathbf{k}$$

Furthermore, using Eqs. 21–11, we have

$$\mathbf{H}_G = I\omega_x\mathbf{i} + I\omega_y\mathbf{j} + I_z\omega_z\mathbf{k}$$

Equating the respective $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ components of the above two equations yields

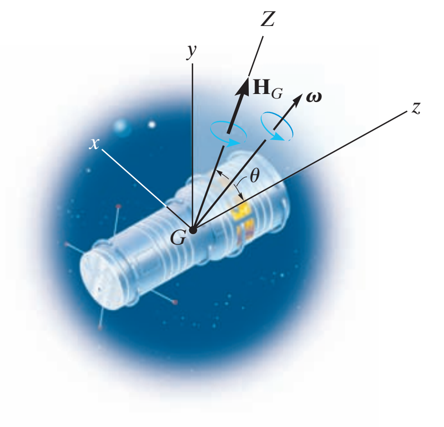


Fig. 21–22

$$\omega_x = 0 \quad \omega_y = \frac{H_G \sin\theta}{I} \quad \omega_z = \frac{H_G \cos\theta}{I_z} \tag{21–34}$$

or

$$\boxed{\boldsymbol{\omega} = \frac{H_G \sin\theta}{I}\mathbf{j} + \frac{H_G \cos\theta}{I_z}\mathbf{k}} \tag{21–35}$$

In a similar manner, equating the respective **i**, **j**, **k** components of Eq. 21–27 to those of Eq. 21–34, we obtain

$$\dot{\theta} = 0$$

$$\dot{\phi}\sin\theta = \frac{H_G \sin\theta}{I}$$

$$\dot{\phi}\cos\theta + \dot{\psi} = \frac{H_G \cos\theta}{I_z}$$

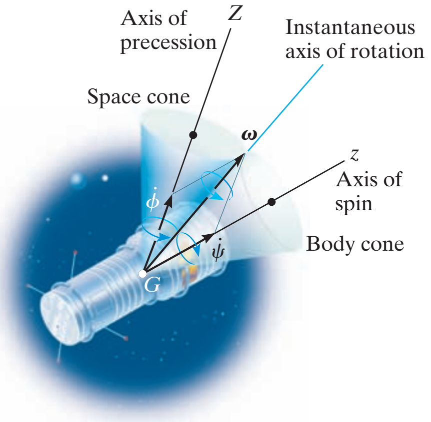
Solving, we get

$$\boxed{\begin{aligned} \theta &= \text{constant} \\ \dot{\phi} &= \frac{H_G}{I} \\ \dot{\psi} &= \frac{I - I_z}{I I_z} H_G \cos\theta \end{aligned}} \tag{21–36}$$

Thus, for torque-free motion of an axisymmetrical body, the angle θ formed between the angular-momentum vector and the spin of the body remains constant. Furthermore, the angular momentum $\mathbf{H}_G$, precession $\dot{\phi}$, and spin $\dot{\psi}$ for the body remain constant at all times during the motion.

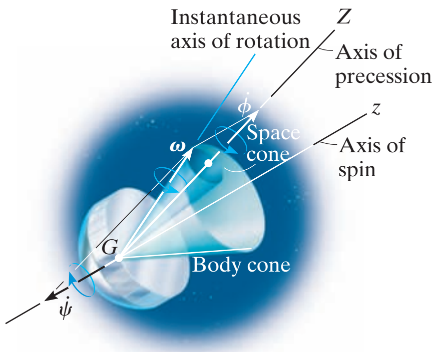
Eliminating H_G from the second and third of Eqs. 21–36 yields the following relation between the spin and precession:

$$\boxed{\dot{\psi} = \frac{I - I_z}{I_z}\dot{\phi}\cos\theta} \tag{21–37}$$



$I > I_z$

(a)



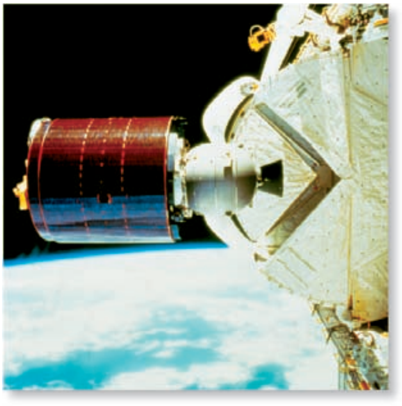
$I < I_z$

(b)

Fig. 21–23

These two components of angular motion can be studied by using the body and space cone models introduced in Sec. 20.1. The *space cone* defining the precession is fixed from rotating, since the precession has a fixed direction, while the outer surface of the *body cone* rolls on the space cone's outer surface. Try to imagine this motion in Fig. 21–23*a*. The interior angle of each cone is chosen such that the resultant angular velocity of the body is directed along the line of contact of the two cones. This line of contact represents the instantaneous axis of rotation for the body cone, and hence the angular velocity of both the body cone and the body must be directed along this line. Since the spin is a function of the moments of inertia I and I_z of the body, Eq. 21–36, the cone model in Fig. 21–23*a* is satisfactory for describing the motion, provided $I > I_z$. Torque-free motion which meets these requirements is called *regular precession*. If $I < I_z$, the spin is negative and the precession positive. This motion is represented by the satellite motion shown in Fig. 21–23*b* ($I < I_z$). The cone model can again be used to represent the motion; however, to preserve the correct vector addition of spin and precession to obtain the angular velocity $\boldsymbol{\omega}$, the inside surface of the body cone must roll on the outside surface of the (fixed) space cone. This motion is referred to as *retrograde precession*.

Satellites are often given a spin before they are launched. If their angular momentum is not collinear with the axis of spin, they will exhibit precession. In the photo on the left, regular precession will occur since $I > I_z$, and in the photo on the right, retrograde precession will occur since $I < I_z$.



(© R.C. Hibbeler)

(© R.C. Hibbeler)

21

The motion of a football is observed using a slow-motion projector. From the film, the spin of the football is seen to be directed 30° from the horizontal, as shown in Fig. 21–24*a*. Also, the football is precessing about the vertical axis at a rate $\dot{\phi} = 3$ rad/s. If the ratio of the axial to transverse moments of inertia of the football is $\frac{1}{3}$, measured with respect to the center of mass, determine the magnitude of the football's spin and its angular velocity. Neglect the effect of air resistance.

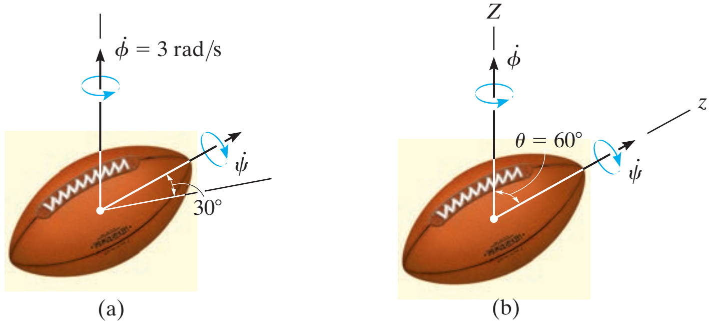


Fig. 21–24

SOLUTION

Since the weight of the football is the only force acting, the motion is torque-free. In the conventional sense, if the z axis is established along the axis of spin and the Z axis along the precession axis, as shown in Fig. 21–24*b*, then the angle $\theta = 60°$. Applying Eq. 21–37, the spin is

$$\dot{\psi} = \frac{I - I_z}{I_z}\dot{\phi}\cos\theta = \frac{I - \frac{1}{3}I}{\frac{1}{3}I}(3)\cos 60°$$
$$= 3 \text{ rad/s} \qquad \textit{Ans.}$$

Using Eqs. 21–34, where $H_G = \dot{\phi} I$ (Eq. 21–36), we have

$$\omega_x = 0$$
$$\omega_y = \frac{H_G \sin\theta}{I} = \frac{3I\sin 60°}{I} = 2.60 \text{ rad/s}$$
$$\omega_z = \frac{H_G \cos\theta}{I_z} = \frac{3I\cos 60°}{\frac{1}{3}I} = 4.50 \text{ rad/s}$$

Thus,

$$\omega = \sqrt{(\omega_x)^2 + (\omega_y)^2 + (\omega_z)^2}$$
$$= \sqrt{(0)^2 + (2.60)^2 + (4.50)^2}$$
$$= 5.20 \text{ rad/s} \qquad \textit{Ans.}$$

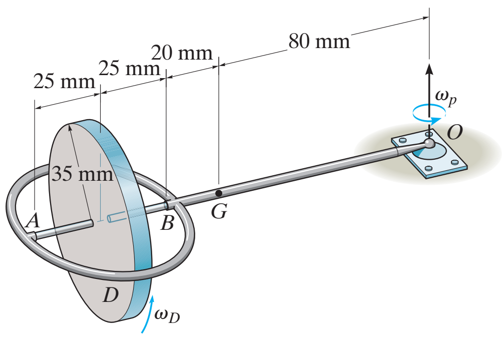
21

PROBLEMS

***21–60.** Show that the angular velocity of a body, in terms of Euler angles ϕ, θ, and ψ, can be expressed as $\omega = (\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\mathbf{i} + (\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)\mathbf{j} + (\dot{\phi}\cos\theta + \dot{\psi})\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are directed along the x, y, z axes as shown in Fig. 21–15*d*.

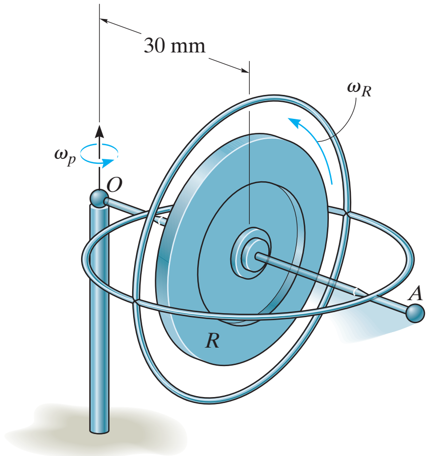
21–61. A thin rod is initially coincident with the Z axis when it is given three rotations defined by the Euler angles $\phi = 30°$, $\theta = 45°$, and $\psi = 60°$. If these rotations are given in the order stated, determine the coordinate direction angles α, β, γ of the axis of the rod with respect to the X, Y, and Z axes. Are these directions the same for any order of the rotations? Why?

21–62. The gyroscope consists of a uniform 450-g disk D which is attached to the axle AB of negligible mass. The supporting frame has a mass of 180 g and a center of mass at G. If the disk is rotating about the axle at $\omega_D = 90$ rad/s, determine the constant angular velocity ω_p at which the frame precesses about the pivot point O. The frame moves in the horizontal plane.



Prob. 21–62

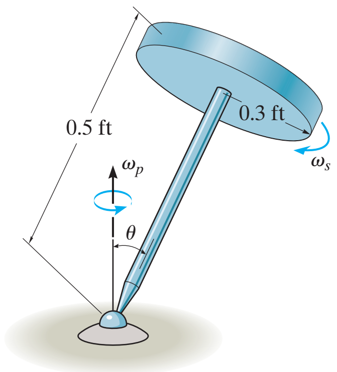
21–63. The toy gyroscope consists of a rotor R which is attached to the frame of negligible mass. If it is observed that the frame is precessing about the pivot point O at $\omega_p = 2$ rad/s, determine the angular velocity ω_R of the rotor. The stem OA moves in the horizontal plane. The rotor has a mass of 200 g and a radius of gyration $k_{OA} = 20$ mm about OA.



Prob. 21–63

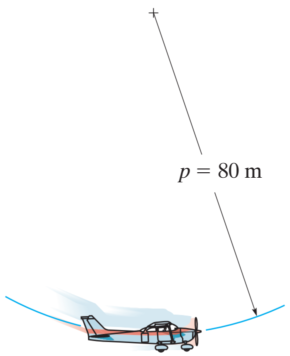
***21–64.** The top consists of a thin disk that has a weight of 8 lb and a radius of 0.3 ft. The rod has a negligible mass and a length of 0.5 ft. If the top is spinning with an angular velocity $\omega_s = 300$ rad/s, determine the steady-state precessional angular velocity ω_p of the rod when $\theta = 40°$.

21–65. Solve Prob. 21–64 when $\theta = 90°$.



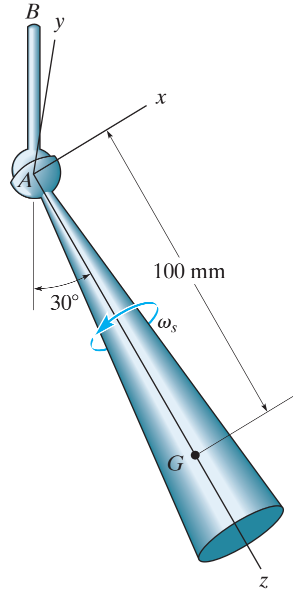
Probs. 21–64/65

21–66. The propeller on a single-engine airplane has a mass of 15 kg and a centroidal radius of gyration of 0.3 m computed about the axis of spin. When viewed from the front of the airplane, the propeller is turning clockwise at 350 rad/s about the spin axis. If the airplane enters a vertical curve having a radius of 80 m and is traveling at 200 km/h, determine the gyroscopic bending moment which the propeller exerts on the bearings of the engine when the airplane is in its lowest position.



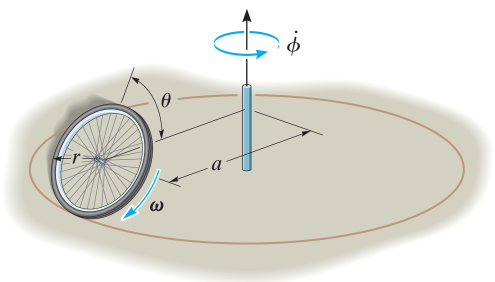
Prob. 21–66

***21–68.** The conical top has a mass of 0.8 kg, and the moments of inertia are $I_x = I_y = 3.5(10^{-3})\ \text{kg}\cdot\text{m}^2$ and $I_z = 0.8(10^{-3})\ \text{kg}\cdot\text{m}^2$. If it spins freely in the ball-and socket joint at A with an angular velocity $\omega_s = 750$ rad/s, compute the precession of the top about the axis of the shaft AB.



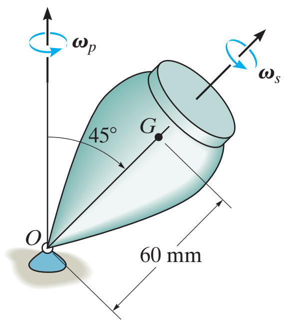
Prob. 21–68

21–67. A wheel of mass m and radius r rolls with constant spin $\boldsymbol{\omega}$ about a circular path having a radius a. If the angle of inclination is θ, determine the rate of precession. Treat the wheel as a thin ring. No slipping occurs.



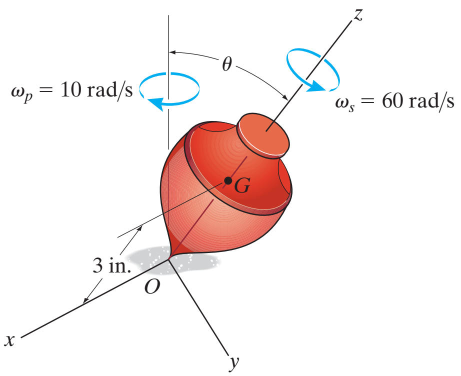
Prob. 21–67

21–69. The top has a mass of 90 g, a center of mass at G, and a radius of gyration $k = 18$ mm about its axis of symmetry. About any transverse axis acting through point O the radius of gyration is $k_t = 35$ mm. If the top is connected to a ball-and-socket joint at O and the precession is $\omega_p = 0.5$ rad/s, determine the spin $\boldsymbol{\omega}_s$.



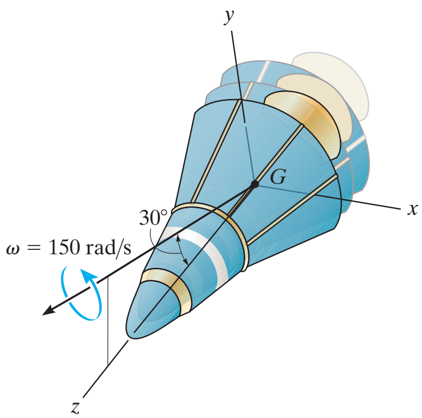
Prob. 21–69

21–70. The 1-lb top has a center of gravity at point G. If it spins about its axis of symmetry and precesses about the vertical axis at constant rates of $\omega_s = 60\text{ rad/s}$ and $\omega_p = 10\text{ rad/s}$, respectively, determine the steady state angle θ. The radius of gyration of the top about the z axis is $k_z = 1\text{ in.}$, and about the x and y axes it is $k_x = k_y = 4\text{ in.}$



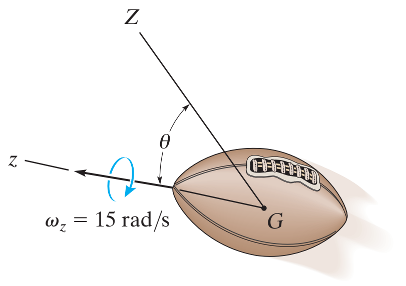
Prob. 21–70

21–71. The space capsule has a mass of 2 Mg, center of mass at G, and radii of gyration about its axis of symmetry (z axis) and its transverse axes (x or y axis) of $k_z = 2.75\text{ m}$ and $k_x = k_y = 5.5\text{ m}$, respectively. If the capsule has the angular velocity shown, determine its precession $\dot{\phi}$ and spin $\dot{\psi}$. Indicate whether the precession is regular or retrograde. Also, draw the space cone and body cone for the motion.



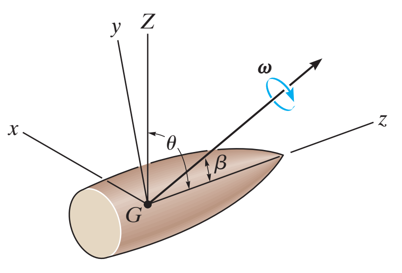
Prob. 21–71

***21–72.** The 0.25 kg football is spinning at $\omega_z = 15\text{ rad/s}$ as shown. If $\theta = 40^\circ$, determine the precession about the z axis. The radius of gyration about the spin axis is $k_z = 0.042\text{ m}$, and about a transverse axis is $k_y = 0.13\text{ m}$.



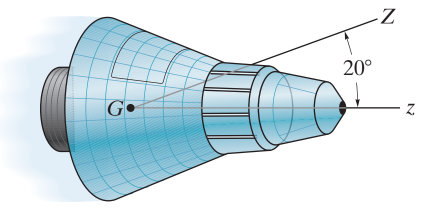
Prob. 21–72

21–73. The projectile shown is subjected to torque-free motion. The transverse and axial moments of inertia are I and I_z, respectively. If θ represents the angle between the precessional axis Z and the axis of symmetry z, and β is the angle between the angular velocity $\boldsymbol{\omega}$ and the z axis, show that β and θ are related by the equation $\tan\theta = (I/I_z)\tan\beta$.



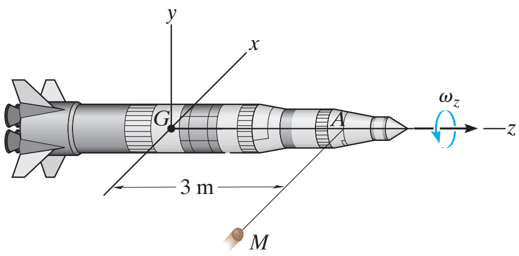
Prob. 21–73

21–74. The radius of gyration about an axis passing through the axis of symmetry of the 1.6-Mg space capsule is $k_z = 1.2\text{ m}$ and about any transverse axis passing through the center of mass G, $k_t = 1.8\text{ m}$. If the capsule has a known steady-state precession of two revolutions per hour about the Z axis, determine the rate of spin about the z axis.



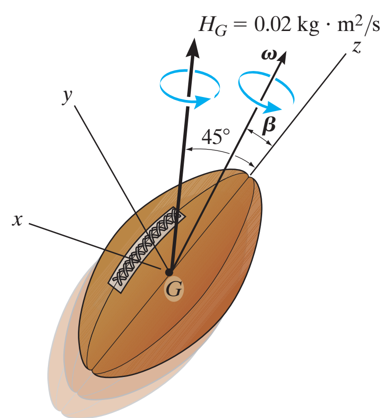
Prob. 21–74

21–75. The rocket has a mass of 4 Mg and radii of gyration $k_z = 0.85$ m and $k_x = k_y = 2.3$ m. It is initially spinning about the z axis at $\omega_z = 0.05$ rad/s when a meteoroid M strikes it at A and creates an impulse $\mathbf{I} = \{300\mathbf{i}\}$ N·s. Determine the axis of precession after the impact.



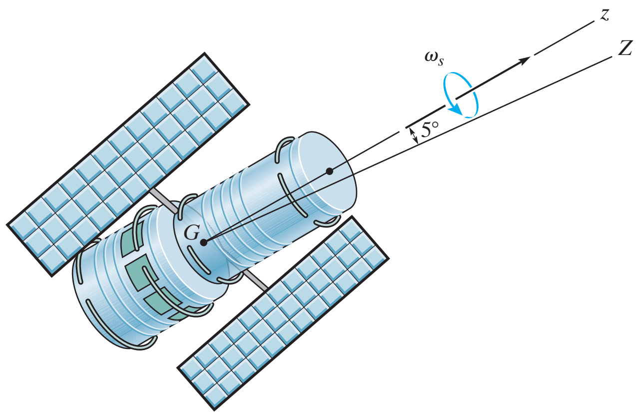
Prob. 21–75

***21–76.** The football has a mass of 450 g and radii of gyration about its axis of symmetry (z axis) and its transverse axes (x or y axis) of $k_z = 30$ mm and $k_x = k_y = 50$ mm, respectively. If the football has an angular momentum of $H_G = 0.02\ \text{kg}\cdot\text{m}^2/\text{s}$, determine its precession $\dot{\phi}$ and spin $\dot{\psi}$. Also, find the angle β that the angular velocity vector makes with the z axis.



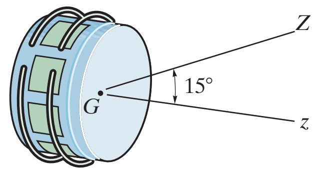
Prob. 21–76

21–77. The satellite has a mass of 1.8 Mg, and about axes passing through the mass center G the axial and transverse radii of gyration are $k_z = 0.8$ m and $k_t = 1.2$ m, respectively. If it is spinning at $\omega_s = 6$ rad/s when it is launched, determine its angular momentum. Precession occurs about the Z axis.



Prob. 21–77

21–78. The radius of gyration about an axis passing through the axis of symmetry of the 1.2-Mg satellite is $k_z = 1.4$ m, and about any transverse axis passing through the center of mass G, $k_t = 2.20$ m. If the satellite has a known spin of 2700 rev/h about the z axis, determine the steady-state precession about the z axis.



Prob. 21–78

CHAPTER REVIEW

Moments and Products of Inertia

A body has six components of inertia for any specified x, y, z axes. Three of these are moments of inertia about each of the axes, I_{xx}, I_{yy}, I_{zz}, and three are products of inertia, each defined from two orthogonal planes, I_{xy}, I_{yz}, I_{xz}. If either one or both of these planes are planes of symmetry, then the product of inertia with respect to these planes will be zero.

$$I_{xx} = \int_m r_x^2\,dm = \int_m (y^2 + z^2)\,dm \qquad I_{xy} = I_{yx} = \int_m xy\,dm$$

$$I_{yy} = \int_m r_y^2\,dm = \int_m (x^2 + z^2)\,dm \qquad I_{yz} = I_{zy} = \int_m yz\,dm$$

$$I_{zz} = \int_m r_z^2\,dm = \int_m (x^2 + y^2)\,dm \qquad I_{xz} = I_{zx} = \int_m xz\,dm$$

The moments and products of inertia can be determined by direct integration or by using tabulated values. If these quantities are to be determined with respect to axes or planes that do not pass through the mass center, then parallel-axis and parallel-plane theorems must be used.

Provided the six components of inertia are known, then the moment of inertia about any axis can be determined using the inertia transformation equation.

$$I_{Oa} = I_{xx}u_x^2 + I_{yy}u_y^2 + I_{zz}u_z^2 - 2I_{xy}u_xu_y - 2I_{yz}u_yu_z - 2I_{zx}u_zu_x$$

Principal Moments of Inertia

At any point on or off the body, the x, y, z axes can be oriented so that the products of inertia will be zero. The resulting moments of inertia are called the principal moments of inertia. In general, one will be a maximum and the other a minimum.

$$\begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}$$

Principle of Impulse and Momentum

The angular momentum for a body can be determined about any arbitrary point A.

Once the linear and angular momentum for the body have been formulated, then the principle of impulse and momentum can be used to solve problems that involve force, velocity, and time.

$$m(\mathbf{v}_G)_1 + \Sigma\int_{t_1}^{t_2} \mathbf{F}\,dt = m(\mathbf{v}_G)_2 \qquad (\mathbf{H}_O)_1 + \Sigma\int_{t_1}^{t_2} \mathbf{M}_O\,dt = (\mathbf{H}_O)_2$$

$$\mathbf{H}_O = \int_m \boldsymbol{\rho}_O \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_O)\,dm$$

Fixed Point O

$$\mathbf{H}_G = \int_m \boldsymbol{\rho}_G \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_G)\,dm$$

Center of Mass

$$\mathbf{H}_A = \boldsymbol{\rho}_{G/A} \times m\mathbf{v}_G + \mathbf{H}_G$$

Arbitrary Point

where

$$H_x = I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z$$
$$H_y = -I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z$$
$$H_z = -I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z$$

Principle of Work and Energy

The kinetic energy for a body is usually determined relative to a fixed point or the body's mass center.

$$T = \tfrac{1}{2}I_x\omega_x^2 + \tfrac{1}{2}I_y\omega_y^2 + \tfrac{1}{2}I_z\omega_z^2$$

Fixed Point

$$T = \tfrac{1}{2}mv_G^2 + \tfrac{1}{2}I_x\omega_x^2 + \tfrac{1}{2}I_y\omega_y^2 + \tfrac{1}{2}I_z\omega_z^2$$

Center of Mass

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These formulations can be used with the principle of work and energy to solve problems that involve force, velocity, and displacement.

$$T_1 + \Sigma U_{1-2} = T_2$$

Equations of Motion

There are three scalar equations of translational motion for a rigid body that moves in three dimensions.

$$\Sigma F_x = m(a_G)_x$$
$$\Sigma F_y = m(a_G)_y$$
$$\Sigma F_z = m(a_G)_z$$

The three scalar equations of rotational motion depend upon the motion of the x, y, z reference. Most often, these axes are oriented so that they are principal axes of inertia. If the axes are fixed in and move with the body so that $\boldsymbol{\Omega} = \boldsymbol{\omega}$, then the equations are referred to as the Euler equations of motion.

A free-body diagram should always accompany the application of the equations of motion.

$$\Sigma M_x = I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z$$
$$\Sigma M_y = I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x$$
$$\Sigma M_z = I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y$$

$$\boldsymbol{\Omega} = \boldsymbol{\omega}$$

$$\Sigma M_x = I_x\dot{\omega}_x - I_y\Omega_z\omega_y + I_z\Omega_y\omega_z$$
$$\Sigma M_y = I_y\dot{\omega}_y - I_z\Omega_x\omega_z + I_x\Omega_z\omega_x$$
$$\Sigma M_z = I_z\dot{\omega}_z - I_x\Omega_y\omega_x + I_y\Omega_x\omega_y$$

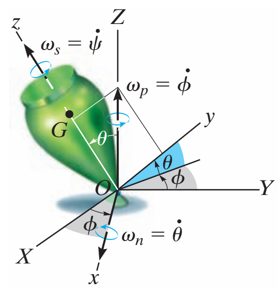
$$\boldsymbol{\Omega} \neq \boldsymbol{\omega}$$

Gyroscopic Motion

The angular motion of a gyroscope is best described using the three Euler angles ϕ, θ, and ψ. The angular velocity components are called the precession $\dot{\phi}$, the nutation $\dot{\theta}$, and the spin $\dot{\psi}$.

If $\dot{\theta} = 0$ and $\dot{\phi}$ and $\dot{\psi}$ are constant, then the motion is referred to as steady precession.

It is the spin of a gyro rotor that is responsible for holding a rotor from falling downward, and instead causing it to precess about a vertical axis. This phenomenon is called the gyroscopic effect.



$$\Sigma M_x = -I\dot{\phi}^2 \sin\theta\cos\theta + I_z\dot{\phi}\sin\theta(\dot{\phi}\cos\theta + \dot{\psi})$$

$$\Sigma M_y = 0, \ \Sigma M_z = 0$$

Torque-Free Motion

A body that is only subjected to a gravitational force will have no moments on it about its mass center, and so the motion is described as torque-free motion. The angular momentum for the body about its mass center will remain constant. This causes the body to have both a spin and a precession. The motion depends upon the magnitude of the moment of inertia of a symmetric body about the spin axis, I_z, versus that about a perpendicular axis, I.

$$\theta = \text{constant}$$

$$\dot{\phi} = \frac{H_G}{I}$$

$$\dot{\psi} = \frac{I - I_z}{I I_z} H_G \cos\theta$$

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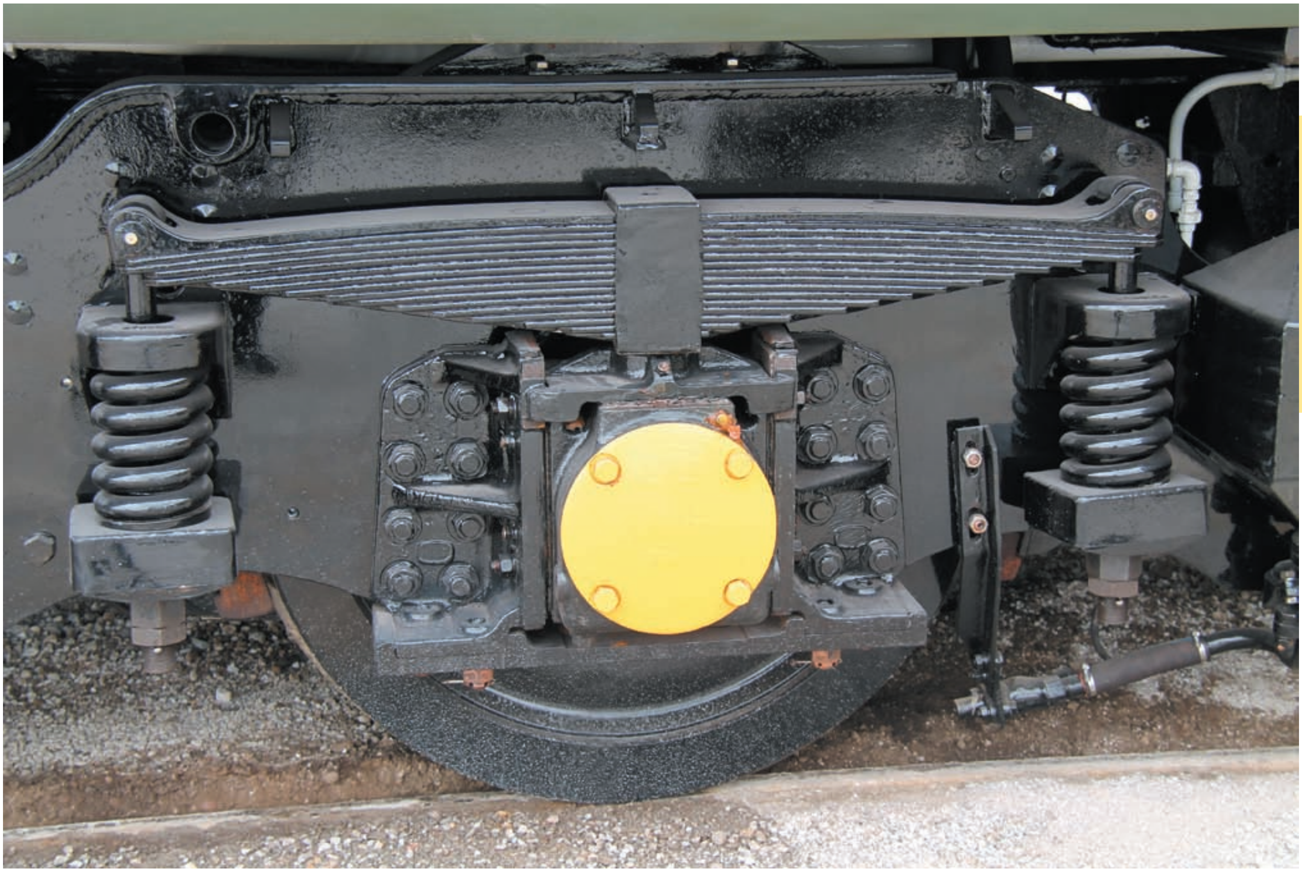
Chapter 22



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The analysis of vibrations plays an important role in the study of the behavior of structures subjected to earthquakes.

Vibrations

CHAPTER OBJECTIVES

- To discuss undamped one-degree-of-freedom vibration of a rigid body using the equation of motion and energy methods.
- To study the analysis of undamped forced vibration and viscous damped forced vibration.

*22.1 Undamped Free Vibration

A *vibration* is the oscillating motion of a body or system of connected bodies displaced from a position of equilibrium. In general, there are two types of vibration, free and forced. *Free vibration* occurs when the motion is maintained by gravitational or elastic restoring forces, such as the swinging motion of a pendulum or the vibration of an elastic rod. *Forced vibration* is caused by an external periodic or intermittent force applied to the system. Both of these types of vibration can either be damped or undamped. *Undamped* vibrations exclude frictional effects in the analysis. Since in reality both internal and external frictional forces are present, the motion of all vibrating bodies is actually *damped*.

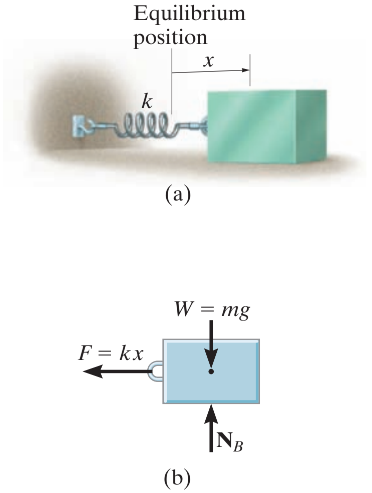


Fig. 22–1

The simplest type of vibrating motion is undamped free vibration, represented by the block and spring model shown in Fig. 22–1*a*. Vibrating motion occurs when the block is released from a displaced position x so that the spring pulls on the block. The block will attain a velocity such that it will proceed to move out of equilibrium when $x = 0$, and provided the supporting surface is smooth, the block will oscillate back and forth.

The time-dependent path of motion of the block can be determined by applying the equation of motion to the block when it is in the displaced position x. The free-body diagram is shown in Fig. 22–1*b*. The elastic restoring force $F = kx$ is always directed toward the equilibrium position, whereas the acceleration **a** is assumed to act in the direction of *positive displacement*. Since $a = d^2x/dt^2 = \ddot{x}$, we have

$$\overset{+}{\rightarrow} \Sigma F_x = ma_x; \qquad -kx = m\ddot{x}$$

Note that the acceleration is proportional to the block's displacement. Motion described in this manner is called *simple harmonic motion*. Rearranging the terms into a "standard form" gives

$$\ddot{x} + \omega_n^2 x = 0 \tag{22–1}$$

The constant ω_n, generally reported in rad/s, is called the *natural frequency*, and in this case

$$\boxed{\omega_n = \sqrt{\frac{k}{m}}} \tag{22–2}$$

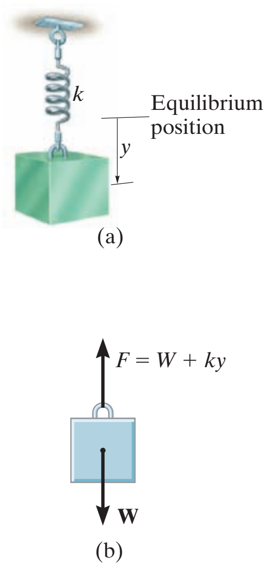


Fig. 22–2

Equation 22–1 can also be obtained by considering the block to be suspended so that the displacement y is measured from the block's *equilibrium position*, Fig. 22–2*a*. When the block is in equilibrium, the spring exerts an upward force of $F = W = mg$ on the block. Hence, when the block is displaced a distance y downward from this position, the magnitude of the spring force is $F = W + ky$, Fig. 22–2*b*. Applying the equation of motion gives

$$+\downarrow \Sigma F_y = ma_y; \qquad -W - ky + W = m\ddot{y}$$

or

$$\ddot{y} + \omega_n^2 y = 0$$

which is the same form as Eq. 22–1 and ω_n is defined by Eq. 22–2.

Equation 22–1 is a homogeneous, second-order, linear, differential equation with constant coefficients. It can be shown, using the methods of differential equations, that the general solution is

$$\boxed{x = A \sin \omega_n t + B \cos \omega_n t} \tag{22–3}$$

Here A and B represent two constants of integration. The block's velocity and acceleration are determined by taking successive time derivatives, which yields

$$v = \dot{x} = A\omega_n \cos \omega_n t - B\omega_n \sin \omega_n t \tag{22–4}$$

$$a = \ddot{x} = -A\omega_n^2 \sin \omega_n t - B\omega_n^2 \cos \omega_n t \tag{22–5}$$

When Eqs. 22–3 and 22–5 are substituted into Eq. 22–1, the differential equation will be satisfied, showing that Eq. 22–3 is indeed the solution to Eq. 22–1.

The integration constants in Eq. 22–3 are generally determined from the initial conditions of the problem. For example, suppose that the block in Fig. 22–1a has been displaced a distance x_1 to the right from its equilibrium position and given an initial (positive) velocity $\mathbf{v}_1$ directed to the right. Substituting $x = x_1$ when $t = 0$ into Eq. 22–3 yields $B = x_1$. And since $v = v_1$ when $t = 0$, using Eq. 22–4 we obtain $A = v_1/\omega_n$. If these values are substituted into Eq. 22–3, the equation describing the motion becomes

$$x = \frac{v_1}{\omega_n} \sin \omega_n t + x_1 \cos \omega_n t \tag{22–6}$$

Equation 22–3 may also be expressed in terms of simple sinusoidal motion. To show this, let

$$A = C \cos \phi \tag{22–7}$$

and

$$B = C \sin \phi \tag{22–8}$$

where C and ϕ are new constants to be determined in place of A and B. Substituting into Eq. 22–3 yields

$$x = C \cos \phi \sin \omega_n t + C \sin \phi \cos \omega_n t$$

And since $\sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi$, then

$$\boxed{x = C \sin(\omega_n t + \phi)} \tag{22–9}$$

If this equation is plotted on an x versus $\omega_n t$ axis, the graph shown in Fig. 22–3 is obtained. The maximum displacement of the block from its equilibrium position is defined as the *amplitude* of vibration. From either the figure or Eq. 22–9 the amplitude is C. The angle ϕ is called the *phase angle* since it represents the amount by which the curve is displaced from the origin when $t = 0$. We can relate these two constants to A and B using Eqs. 22–7 and 22–8. Squaring and adding these two equations, the amplitude becomes

$$C = \sqrt{A^2 + B^2} \tag{22–10}$$

If Eq. 22–8 is divided by Eq. 22–7, the phase angle is then

$$\phi = \tan^{-1}\frac{B}{A} \tag{22–11}$$

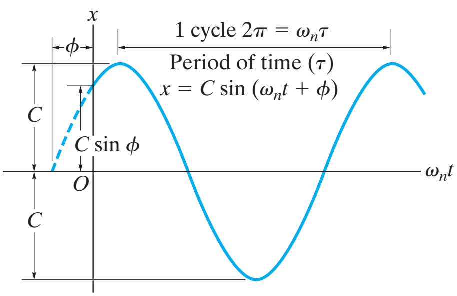


Fig. 22–3

Note that the sine curve, Eq. 22–9, completes one *cycle* in time $t = \tau$ (tau) when $\omega_n\tau = 2\pi$, or

$$\boxed{\tau = \frac{2\pi}{\omega_n}} \tag{22–12}$$

This time interval is called a *period*, Fig. 22–3. Using Eq. 22–2, the period can also be represented as

$$\tau = 2\pi\sqrt{\frac{m}{k}} \tag{22–13}$$

Finally, the *frequency* f is defined as the number of cycles completed per unit of time, which is the reciprocal of the period; that is,

$$\boxed{f = \frac{1}{\tau} = \frac{\omega_n}{2\pi}} \tag{22–14}$$

or

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{22–15}$$

The frequency is expressed in cycles/s. This ratio of units is called a *hertz* (Hz), where 1 Hz = 1 cycle/s = 2π rad/s.

When a body or system of connected bodies is given an initial displacement from its equilibrium position and released, it will vibrate with the *natural frequency*, ω_n. Provided the system has a single degree of freedom, that is, it requires only one coordinate to specify completely the position of the system at any time, then the vibrating motion will have the same characteristics as the simple harmonic motion of the block and spring just presented. Consequently, the motion is described by a differential equation of the same "standard form" as Eq. 22–1, i.e.,

$$\boxed{\ddot{x} + \omega_n^2 x = 0} \tag{22–16}$$

Hence, if the natural frequency ω_n is known, the period of vibration τ, frequency f, and other vibrating characteristics can be established using Eqs. 22–3 through 22–15.

Important Points

- Free vibration occurs when the motion is maintained by gravitational or elastic restoring forces.
- The amplitude is the maximum displacement of the body.
- The period is the time required to complete one cycle.
- The frequency is the number of cycles completed per unit of time, where 1 Hz = 1 cycle/s.
- Only one position coordinate is needed to describe the location of a one-degree-of-freedom system.

Procedure for Analysis

As in the case of the block and spring, the natural frequency ω_n of a body or system of connected bodies having a single degree of freedom can be determined using the following procedure:

Free-Body Diagram.

- Draw the free-body diagram of the body when the body is displaced a *small amount* from its equilibrium position.
- Locate the body with respect to its equilibrium position by using an appropriate *inertial coordinate* q. The acceleration of the body's mass center $\mathbf{a}_G$ or the body's angular acceleration $\boldsymbol{\alpha}$ should have an assumed sense of direction which is in the *positive direction* of the position coordinate.
- If the rotational equation of motion $\Sigma M_P = \Sigma(\mathcal{M}_k)_P$ is to be used, then it may be beneficial to also draw the kinetic diagram since it graphically accounts for the components $m(\mathbf{a}_G)_x$, $m(\mathbf{a}_G)_y$, and $I_G\boldsymbol{\alpha}$, and thereby makes it convenient for visualizing the terms needed in the moment sum $\Sigma(\mathcal{M}_k)_P$.

Equation of Motion.

- Apply the equation of motion to relate the elastic or gravitational *restoring* forces and couple moments acting on the body to the body's accelerated motion.

Kinematics.

- Using kinematics, express the body's accelerated motion in terms of the second time derivative of the position coordinate, $\ddot{q}$.
- Substitute the result into the equation of motion and determine ω_n by rearranging the terms so that the resulting equation is in the "standard form," $\ddot{q} + \omega_n^2 q = 0$.

EXAMPLE 22.1

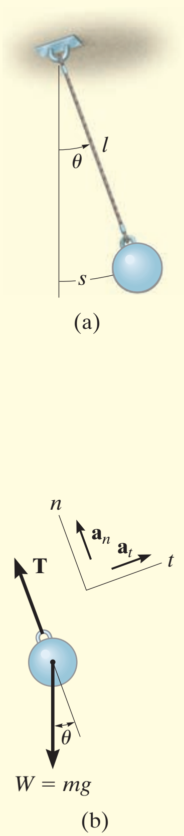


Fig. 22–4

Determine the period of oscillation for the simple pendulum shown in Fig. 22–4*a*. The bob has a mass m and is attached to a cord of length l. Neglect the size of the bob.

SOLUTION

Free-Body Diagram. Motion of the system will be related to the position coordinate ($q =$) θ, Fig. 22–4*b*. When the bob is displaced by a small angle θ, the *restoring force* acting on the bob is created by the tangential component of its weight, $mg \sin \theta$. Furthermore, $\mathbf{a}_t$ acts in the direction of *increasing* s (or θ).

Equation of Motion. Applying the equation of motion in the *tangential direction*, since it involves the restoring force, yields

$$+\nearrow \Sigma F_t = ma_t; \qquad -mg \sin \theta = ma_t \tag{1}$$

Kinematics. $a_t = d^2s/dt^2 = \ddot{s}$. Furthermore, s can be related to θ by the equation $s = l\theta$, so that $a_t = l\ddot{\theta}$. Hence, Eq. 1 reduces to

$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0 \tag{2}$$

The solution of this equation involves the use of an elliptic integral. For *small displacements*, however, $\sin \theta \approx \theta$, in which case

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \tag{3}$$

Comparing this equation with Eq. 22–16 ($\ddot{x} + \omega_n^2 x = 0$), it is seen that $\omega_n = \sqrt{g/l}$. From Eq. 22–12, the period of time required for the bob to make one complete swing is therefore

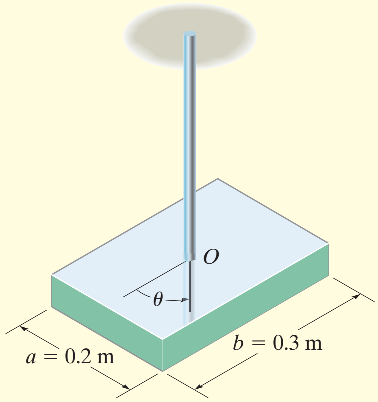
$$\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{l}{g}} \qquad \textit{Ans.}$$

This interesting result, originally discovered by Galileo Galilei through experiment, indicates that the period depends only on the length of the cord and not on the mass of the pendulum bob or the angle θ.

NOTE: The solution of Eq. 3 is given by Eq. 22–3, where $\omega_n = \sqrt{g/l}$ and θ is substituted for x. Like the block and spring, the constants A and B in this problem can be determined if, for example, one knows the displacement and velocity of the bob at a given instant.

EXAMPLE 22.2

The 10-kg rectangular plate shown in Fig. 22–5*a* is suspended at its center from a rod having a torsional stiffness $k = 1.5\ \text{N}\cdot\text{m/rad}$. Determine the natural period of vibration of the plate when it is given a small angular displacement θ in the plane of the plate.



(a)

SOLUTION

Free-Body Diagram. Fig. 22–5*b*. Since the plate is displaced in its own plane, the torsional *restoring* moment created by the rod is $M = k\theta$. This moment acts in the direction opposite to the angular displacement θ. The angular acceleration $\ddot{\theta}$ acts in the direction of *positive* θ.

Equation of Motion.

$$\Sigma M_O = I_O\alpha; \qquad -k\theta = I_O\ddot{\theta}$$

or

$$\ddot{\theta} + \frac{k}{I_O}\theta = 0$$

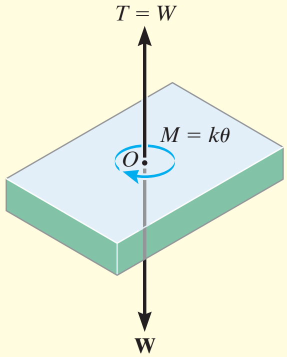
Since this equation is in the "standard form," the natural frequency is $\omega_n = \sqrt{k/I_O}$.

From the table on the inside back cover, the moment of inertia of the plate about an axis coincident with the rod is $I_O = \frac{1}{12}m(a^2 + b^2)$. Hence,

$$I_O = \frac{1}{12}(10\ \text{kg})\left[(0.2\ \text{m})^2 + (0.3\ \text{m})^2\right] = 0.1083\ \text{kg}\cdot\text{m}^2$$

The natural period of vibration is therefore,

$$\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{I_O}{k}} = 2\pi\sqrt{\frac{0.1083}{1.5}} = 1.69\ \text{s} \qquad \textit{Ans.}$$



(b)

Fig. 22–5

EXAMPLE 22.3

The bent rod shown in Fig. 22–6*a* has a negligible mass and supports a 5-kg collar at its end. If the rod is in the equilibrium position shown, determine the natural period of vibration for the system.

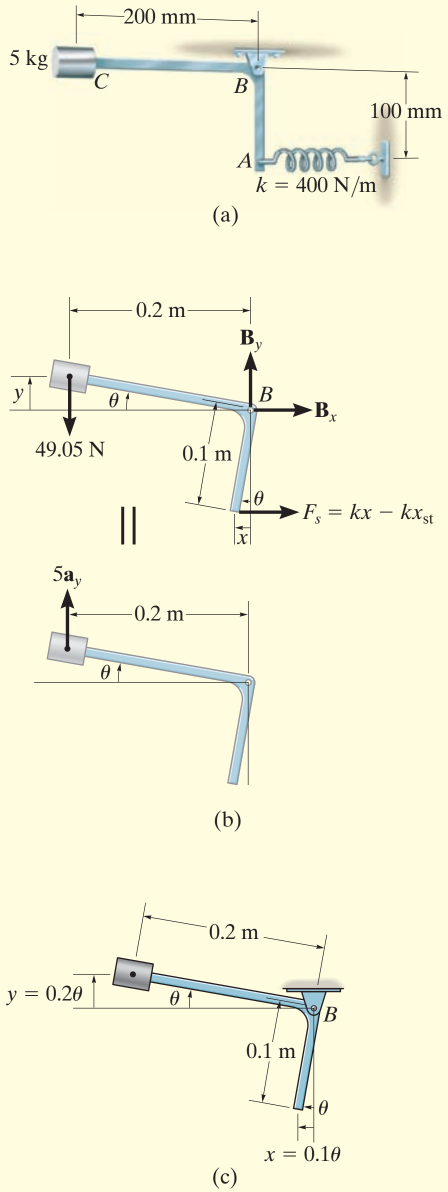


Fig. 22–6

SOLUTION

Free-Body and Kinetic Diagrams. Fig. 22–6*b*. Here the rod is displaced by a small angle θ from the equilibrium position. Since the spring is subjected to an initial compression of x_{st} for equilibrium, then when the displacement $x > x_{st}$ the spring exerts a force of $F_s = kx - kx_{st}$ on the rod. To obtain the "standard form," Eq. 22–16, $5\mathbf{a}_y$ must act *upward*, which is in accordance with positive θ displacement.

Equation of Motion. Moments will be summed about point B to eliminate the unknown reaction at this point. Since θ is small,

$$\zeta+\Sigma M_B = \Sigma(\mathcal{M}_k)_B;$$

$$kx(0.1 \text{ m}) - kx_{st}(0.1 \text{ m}) + 49.05 \text{ N}(0.2 \text{ m}) = -(5 \text{ kg})a_y(0.2 \text{ m})$$

The second term on the left side, $-kx_{st}(0.1 \text{ m})$, represents the moment created by the spring force which is necessary to hold the collar in *equilibrium*, i.e., at $x = 0$. Since this moment is equal and opposite to the moment 49.05 N(0.2 m) created by the weight of the collar, these two terms cancel in the above equation, so that

$$kx(0.1) = -5a_y(0.2) \qquad (1)$$

Kinematics. The deformation of the spring and the position of the collar can be related to the angle θ, Fig. 22–6*c*. Since θ is small, $x = (0.1 \text{ m})\theta$ and $y = (0.2 \text{ m})\theta$. Therefore, $a_y = \ddot{y} = 0.2\ddot{\theta}$. Substituting into Eq. 1 yields

$$400(0.1\theta)\,0.1 = -5(0.2\ddot{\theta})0.2$$

Rewriting this equation in the "standard form" gives

$$\ddot{\theta} + 20\theta = 0$$

Compared with $\ddot{x} + \omega_n^2 x = 0$ (Eq. 22–16), we have

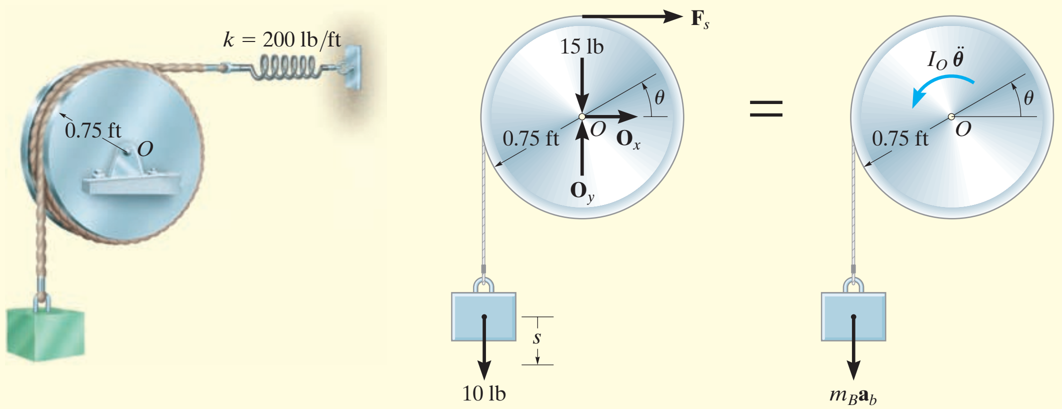
$$\omega_n^2 = 20 \qquad \omega_n = 4.47 \text{ rad/s}$$

The natural period of vibration is therefore

$$\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{4.47} = 1.40 \text{ s} \qquad \textit{Ans.}$$

EXAMPLE 22.4

A 10-lb block is suspended from a cord that passes over a 15-lb disk, as shown in Fig. 22–7*a*. The spring has a stiffness $k = 200 \text{ lb/ft}$. Determine the natural period of vibration for the system.



(a) (b)

SOLUTION

Free-Body and Kinetic Diagrams. Fig. 22–7*b*. The *system* consists of the disk, which undergoes a rotation defined by the angle θ, and the block, which translates by an amount s. The vector $I_O\ddot{\boldsymbol{\theta}}$ acts in the direction of *positive* θ, and consequently $m_B\mathbf{a}_b$ acts downward in the direction of *positive* s.

Equation of Motion. Summing moments about point O to eliminate the reactions $\mathbf{O}_x$ and $\mathbf{O}_y$, realizing that $I_O = \frac{1}{2}mr^2$, yields

$$\circlearrowleft + \Sigma M_O = \Sigma(\mathcal{M}_k)_O;$$

$$10 \text{ lb}(0.75 \text{ ft}) - F_s(0.75 \text{ ft})$$
$$= \frac{1}{2}\left(\frac{15 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(0.75 \text{ ft})^2\,\ddot{\theta} + \left(\frac{10 \text{ lb}}{32.2 \text{ ft/s}^2}\right)a_b(0.75 \text{ ft}) \quad (1)$$

Kinematics. As shown on the kinematic diagram in Fig. 22–7*c*, a small positive displacement θ of the disk causes the block to lower by an amount $s = 0.75\theta$; hence, $a_b = \ddot{s} = 0.75\ddot{\theta}$. When $\theta = 0°$, the spring force required for *equilibrium* of the disk is 10 lb, acting to the right. For position θ, the spring force is $F_s = (200 \text{ lb/ft})(0.75\theta \text{ ft}) + 10 \text{ lb}$. Substituting these results into Eq. 1 and simplifying yields

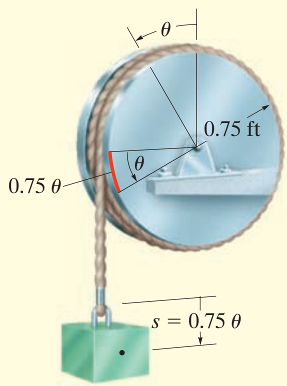
$$\ddot{\theta} + 368\theta = 0$$

Hence,

$$\omega_n^2 = 368 \qquad \omega_n = 19.18 \text{ rad/s}$$

Therefore, the natural period of vibration is

$$\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{19.18} = 0.328 \text{ s} \qquad \textit{Ans.}$$



(c)

Fig. 22–7

PROBLEMS

22–1. A spring is stretched 175 mm by an 8-kg block. If the block is displaced 100 mm downward from its equilibrium position and given a downward velocity of 1.50 m/s, determine the differential equation which describes the motion. Assume that positive displacement is downward. Also, determine the position of the block when $t = 0.22$ s.

22–2. A spring has a stiffness of 800 N/m. If a 2-kg block is attached to the spring, pushed 50 mm above its equilibrium position, and released from rest, determine the equation that describes the block's motion. Assume that positive displacement is downward.

22–3. A spring is stretched 200 mm by a 15-kg block. If the block is displaced 100 mm downward from its equilibrium position and given a downward velocity of 0.75 m/s, determine the equation which describes the motion. What is the phase angle? Assume that positive displacement is downward.

***22–4.** When a 20-lb weight is suspended from a spring, the spring is stretched a distance of 4 in. Determine the natural frequency and the period of vibration for a 10-lb weight attached to the same spring.

22–5. When a 3-kg block is suspended from a spring, the spring is stretched a distance of 60 mm. Determine the natural frequency and the period of vibration for a 0.2-kg block attached to the same spring.

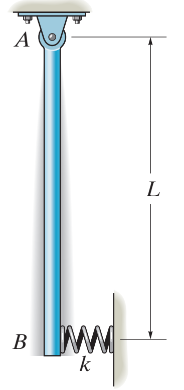
22–6. An 8-kg block is suspended from a spring having a stiffness $k = 80$ N/m. If the block is given an upward velocity of 0.4 m/s when it is 90 mm above its equilibrium position, determine the equation which describes the motion and the maximum upward displacement of the block measured from the equilibrium position. Assume that positive displacement is measured downward.

22–7. A 2-lb weight is suspended from a spring having a stiffness $k = 2$ lb/in. If the weight is pushed 1 in. upward from its equilibrium position and then released from rest, determine the equation which describes the motion. What is the amplitude and the natural frequency of the vibration?

***22–8.** A 6-lb weight is suspended from a spring having a stiffness $k = 3$ lb/in. If the weight is given an upward velocity of 20 ft/s when it is 2 in. above its equilibrium position, determine the equation which describes the motion and the maximum upward displacement of the weight, measured from the equilibrium position. Assume positive displacement is downward.

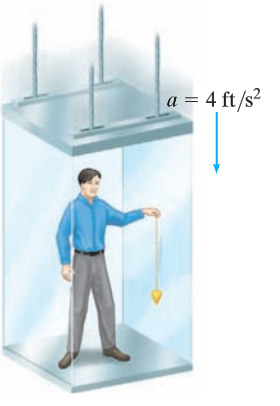
22–9. A 3-kg block is suspended from a spring having a stiffness of $k = 200$ N/m. If the block is pushed 50 mm upward from its equilibrium position and then released from rest, determine the equation that describes the motion. What are the amplitude and the natural frequency of the vibration? Assume that positive displacement is downward.

22–10. The uniform rod of mass m is supported by a pin at A and a spring at B. If B is given a small sideward displacement and released, determine the natural period of vibration.



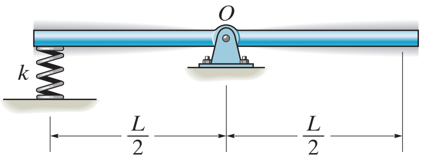
Prob. 22–10

22–11. While standing in an elevator, the man holds a pendulum which consists of an 18-in. cord and a 0.5-lb bob. If the elevator is descending with an acceleration $a = 4$ ft/s^2, determine the natural period of vibration for small amplitudes of swing.



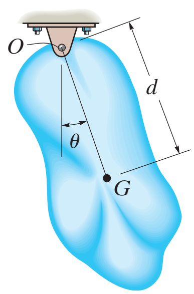
Prob. 22–11

***22–12.** Determine the natural period of vibration of the uniform bar of mass m when it is displaced downward slightly and released.



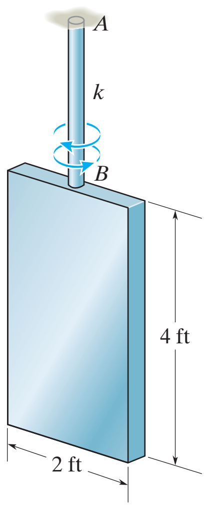
Prob. 22–12

22–13. The body of arbitrary shape has a mass m, mass center at G, and a radius of gyration about G of k_G. If it is displaced a slight amount θ from its equilibrium position and released, determine the natural period of vibration.



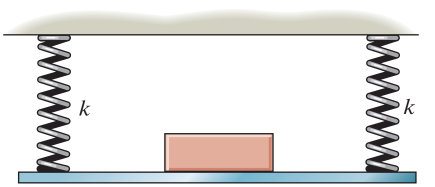
Prob. 22–13

22–14. The 20-lb rectangular plate has a natural period of vibration $\tau = 0.3$ s, as it oscillates around the axis of rod AB. Determine the torsional stiffness k, measured in lb · ft/rad, of the rod. Neglect the mass of the rod.



Prob. 22–14

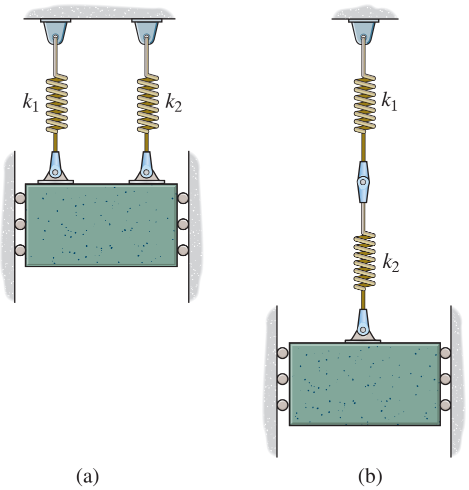
22–15. A platform, having an unknown mass, is supported by *four* springs, each having the same stiffness k. When nothing is on the platform, the period of vertical vibration is measured as 2.35 s; whereas if a 3-kg block is supported on the platform, the period of vertical vibration is 5.23 s. Determine the mass of a block placed on the (empty) platform which causes the platform to vibrate vertically with a period of 5.62 s. What is the stiffness k of each of the springs?



Prob. 22–15

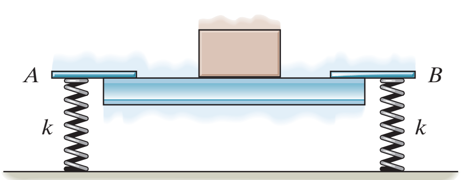
***22–16.** A block of mass m is suspended from two springs having a stiffness of k_1 and k_2, arranged a) parallel to each other, and b) as a series. Determine the equivalent stiffness of a single spring with the same oscillation characteristics and the period of oscillation for each case.

22–17. The 15-kg block is suspended from two springs having a different stiffness and arranged a) parallel to each other, and b) as a series. If the natural periods of oscillation of the parallel system and series system are observed to be 0.5 s and 1.5 s, respectively, determine the spring stiffnesses k_1 and k_2.



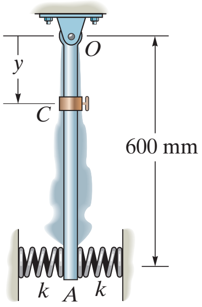
Probs. 22–16/17

22–18. The uniform beam is supported at its ends by two springs A and B, each having the same stiffness k. When nothing is supported on the beam, it has a period of vertical vibration of 0.83 s. If a 50-kg mass is placed at its center, the period of vertical vibration is 1.52 s. Compute the stiffness of each spring and the mass of the beam.



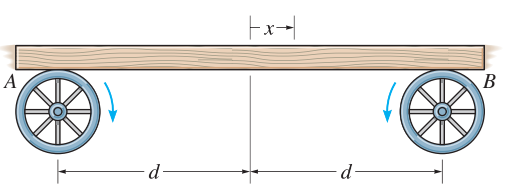
Prob. 22–18

22–19. The slender rod has a mass of 0.2 kg and is supported at O by a pin and at its end A by two springs, each having a stiffness $k = 4$ N/m. The period of vibration of the rod can be set by fixing the 0.5-kg collar C to the rod at an appropriate location along its length. If the springs are originally unstretched when the rod is vertical, determine the position y of the collar so that the natural period of vibration becomes $\tau = 1$ s. Neglect the size of the collar.



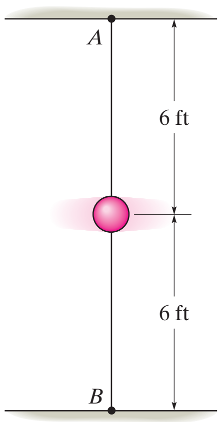
Prob. 22–19

***22–20.** A uniform board is supported on two wheels which rotate in opposite directions at a constant angular speed. If the coefficient of kinetic friction between the wheels and board is μ, determine the frequency of vibration of the board if it is displaced slightly, a distance x from the midpoint between the wheels, and released.



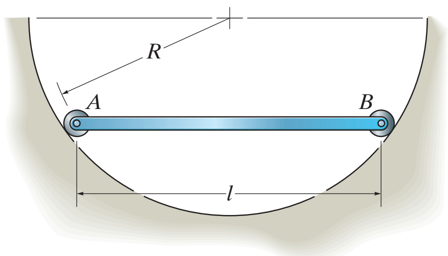
Prob. 22–20

22–21. If the wire AB is subjected to a tension of 20 lb, determine the equation which describes the motion when the 5-lb weight is displaced 2 in. horizontally and released from rest.



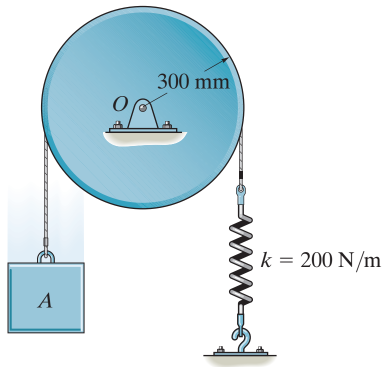
Prob. 22–21

22–22. The bar has a length l and mass m. It is supported at its ends by rollers of negligible mass. If it is given a small displacement and released, determine the natural frequency of vibration.



Prob. 22–22

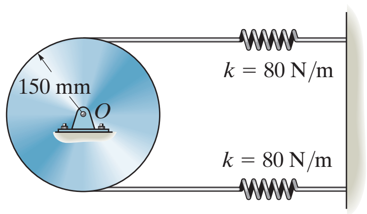
22–23. The 20-kg disk, is pinned at its mass center O and supports the 4-kg block A. If the belt which passes over the disk is not allowed to slip at its contacting surface, determine the natural period of vibration of the system.



Prob. 22–23

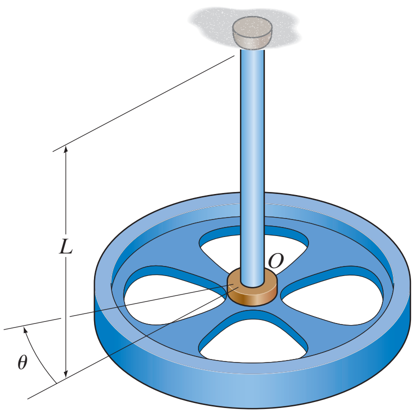
***22–24.** The 10-kg disk is pin connected at its mass center. Determine the natural period of vibration of the disk if the springs have sufficient tension in them to prevent the cord from slipping on the disk as it oscillates. *Hint:* Assume that the initial stretch in each spring is δ_O.

22–25. If the disk in Prob. 22–24 has a mass of 10 kg, determine the natural frequency of vibration. *Hint:* Assume that the initial stretch in each spring is δ_O.



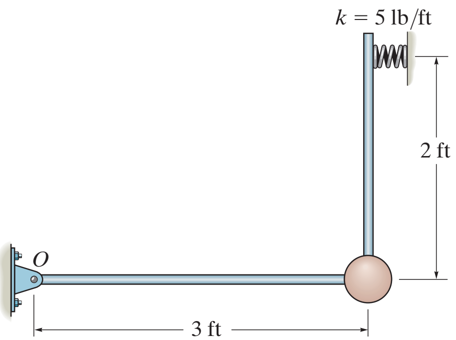
Probs. 22–24/25

22–26. A flywheel of mass m, which has a radius of gyration about its center of mass of k_O, is suspended from a circular shaft that has a torsional resistance of $M = C\theta$. If the flywheel is given a small angular displacement of θ and released, determine the natural period of oscillation.



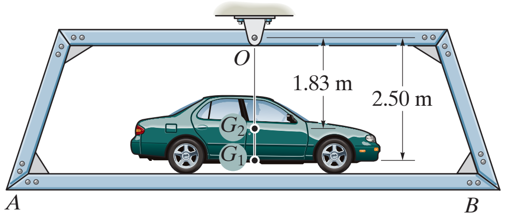
Prob. 22–26

22–27. The 6-lb weight is attached to the rods of negligible mass. Determine the natural frequency of vibration of the weight when it is displaced slightly from the equilibrium position and released.



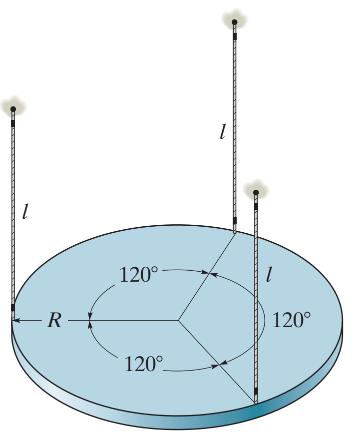
Prob. 22–27

***22–28.** The platform AB when empty has a mass of 400 kg, center of mass at G_1, and natural period of oscillation $\tau_1 = 2.38$ s. If a car, having a mass of 1.2 Mg and center of mass at G_2, is placed on the platform, the natural period of oscillation becomes $\tau_2 = 3.16$ s. Determine the moment of inertia of the car about an axis passing through G_2.



Prob. 22–28

22–29. The plate of mass m is supported by three symmetrically placed cords of length l as shown. If the plate is given a slight rotation about a vertical axis through its center and released, determine the natural period of oscillation.



Prob. 22–29

*22.2 Energy Methods

The simple harmonic motion of a body, discussed in the previous section, is due only to gravitational and elastic restoring forces acting on the body. Since these forces are *conservative*, it is also possible to use the conservation of energy equation to obtain the body's natural frequency or period of vibration. To show how to do this, consider again the block and spring model in Fig. 22–8. When the block is displaced x from the equilibrium position, the kinetic energy is $T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$ and the potential energy is $V = \frac{1}{2}kx^2$. Since energy is conserved, it is necessary that

$$T + V = \text{constant}$$

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \text{constant} \qquad (22\text{–}17)$$

The differential equation describing the *accelerated motion* of the block can be obtained by *differentiating* this equation with respect to time; i.e.,

$$m\dot{x}\ddot{x} + kx\dot{x} = 0$$

$$\dot{x}(m\ddot{x} + kx) = 0$$

Since the velocity $\dot{x}$ is not *always* zero in a vibrating system,

$$\ddot{x} + \omega_n^2 x = 0 \qquad \omega_n = \sqrt{k/m}$$

which is the same as Eq. 22–1.

If the conservation of energy equation is written for a *system of connected bodies*, the natural frequency or the equation of motion can also be determined by time differentiation. It is *not necessary* to dismember the system to account for the internal forces because they do no work.

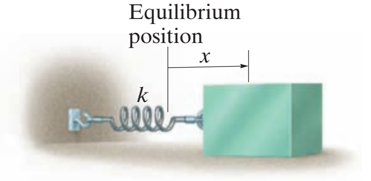


Fig. 22–8

The suspension of a railroad car consists of a set of springs which are mounted between the frame of the car and the wheel truck. This will give the car a natural frequency of vibration which can be determined. (© R.C. Hibbeler)

Procedure for Analysis

The natural frequency ω_n of a body or system of connected bodies can be determined by applying the conservation of energy equation using the following procedure.

Energy Equation.

- Draw the body when it is displaced by a *small amount* from its equilibrium position and define the location of the body from its equilibrium position by an appropriate position coordinate q.

- Formulate the conservation of energy for the body, $T + V =$ constant, in terms of the position coordinate.

- In general, the kinetic energy must account for both the body's translational and rotational motion, $T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$, Eq. 18–2.

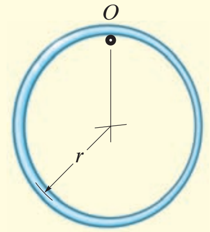
- The potential energy is the sum of the gravitational and elastic potential energies of the body, $V = V_g + V_e$, Eq. 18–17. In particular, V_g should be measured from a datum for which $q = 0$ (equilibrium position).

Time Derivative.

- Take the time derivative of the energy equation using the chain rule of calculus and factor out the common terms. The resulting differential equation represents the equation of motion for the system. The natural frequency of ω_n is obtained after rearranging the terms in the "standard form," $\ddot{q} + \omega_n^2 q = 0$.

EXAMPLE 22.5

The thin hoop shown in Fig. 22–9*a* is supported by the peg at O. Determine the natural period of oscillation for small amplitudes of swing. The hoop has a mass m.



(a)

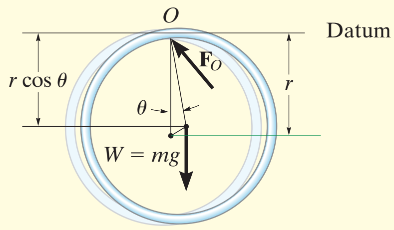
SOLUTION

Energy Equation. A diagram of the hoop when it is displaced a small amount $(q =)\,\theta$ from the equilibrium position is shown in Fig. 22–9*b*. Using the table on the inside back cover and the parallel-axis theorem to determine I_O, the kinetic energy is

$$T = \tfrac{1}{2}I_O\omega_n^2 = \tfrac{1}{2}[mr^2 + mr^2]\dot{\theta}^2 = mr^2\dot{\theta}^2$$

If a horizontal datum is placed through point O, then in the displaced position, the potential energy is

$$V = -mg(r\cos\theta)$$



(b)

Fig. 22–9

The total energy in the system is

$$T + V = mr^2\dot{\theta}^2 - mgr\cos\theta$$

Time Derivative.

$$mr^2(2\dot{\theta})\ddot{\theta} + mgr(\sin\theta)\dot{\theta} = 0$$

$$mr\dot{\theta}(2r\ddot{\theta} + g\sin\theta) = 0$$

Since $\dot{\theta}$ is not always equal to zero, from the terms in parentheses,

$$\ddot{\theta} + \frac{g}{2r}\sin\theta = 0$$

For small angle θ, $\sin\theta \approx \theta$.

$$\ddot{\theta} + \frac{g}{2r}\theta = 0$$

$$\omega_n = \sqrt{\frac{g}{2r}}$$

so that

$$\tau = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{2r}{g}} \qquad \textit{Ans.}$$

EXAMPLE 22.6

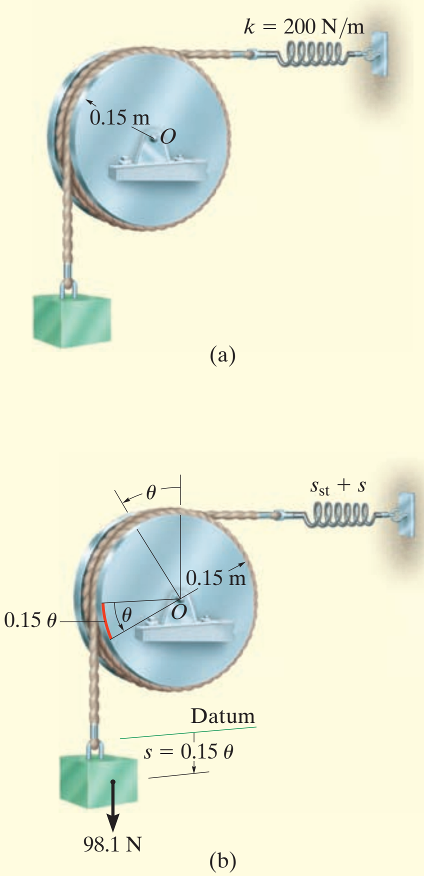


Fig. 22–10

A 10-kg block is suspended from a cord wrapped around a 5-kg disk, as shown in Fig. 22–10*a*. If the spring has a stiffness $k = 200\text{ N/m}$, determine the natural period of vibration for the system.

SOLUTION

Energy Equation. A diagram of the block and disk when they are displaced by respective amounts s and θ from the equilibrium position is shown in Fig. 22–10*b*. Since $s = (0.15\text{ m})\theta$, then $v_b \approx \dot{s} = (0.15\text{ m})\dot{\theta}$. Thus, the kinetic energy of the system is

$$\begin{aligned}T &= \tfrac{1}{2}m_b v_b^2 + \tfrac{1}{2}I_O\omega_d^2\\ &= \tfrac{1}{2}(10\text{ kg})[(0.15\text{ m})\dot{\theta}]^2 + \tfrac{1}{2}\left[\tfrac{1}{2}(5\text{ kg})(0.15\text{ m})^2\right](\dot{\theta})^2\\ &= 0.1406(\dot{\theta})^2\end{aligned}$$

Establishing the datum at the equilibrium position of the block and realizing that the spring stretches s_{st} for equilibrium, the potential energy is

$$\begin{aligned}V &= \tfrac{1}{2}k(s_{st} + s)^2 - Ws\\ &= \tfrac{1}{2}(200\text{ N/m})[s_{st} + (0.15\text{ m})\theta]^2 - 98.1\text{ N}[(0.15\text{ m})\theta]\end{aligned}$$

The total energy for the system is therefore,

$$T + V = 0.1406(\dot{\theta})^2 + 100(s_{st} + 0.15\theta)^2 - 14.715\theta$$

Time Derivative.

$$0.28125(\dot{\theta})\ddot{\theta} + 200(s_{st} + 0.15\theta)0.15\dot{\theta} - 14.72\dot{\theta} = 0$$

Since $s_{st} = 98.1/200 = 0.4905\text{ m}$, the above equation reduces to the "standard form"

$$\ddot{\theta} + 16\theta = 0$$

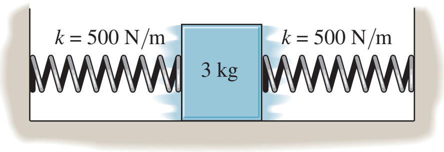
so that

$$\omega_n = \sqrt{16} = 4\text{ rad/s}$$

Thus,

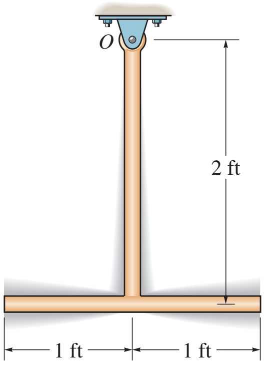
$$\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{4} = 1.57\text{ s} \qquad \textit{Ans.}$$

22–30. Determine the differential equation of motion of the 3-kg block when it is displaced slightly and released. The surface is smooth and the springs are originally unstretched.



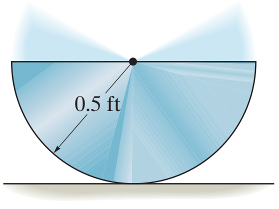
Prob. 22–30

22–31. Determine the natural period of vibration of the pendulum. Consider the two rods to be slender, each having a weight of 8 lb/ft.



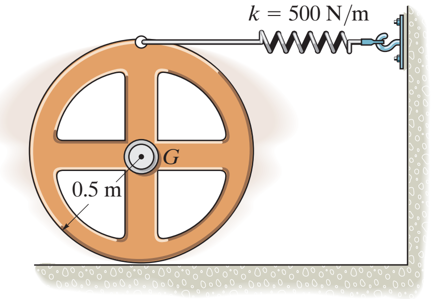
Prob. 22–31

***22–32.** Determine the natural period of vibration of the 10-lb semicircular disk.



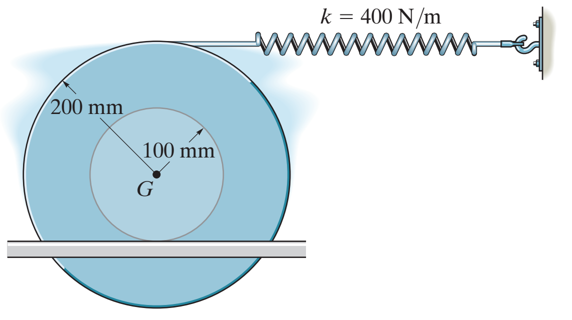
Prob. 22–32

22–33. If the 20-kg wheel is displaced a small amount and released, determine the natural period of vibration. The radius of gyration of the wheel is $k_G = 0.36$ m. The wheel rolls without slipping.



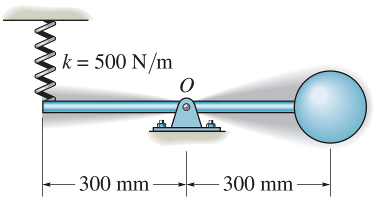
Prob. 22–33

22–34. Determine the differential equation of motion of the 3-kg spool. Assume that it does not slip at the surface of contact as it oscillates. The radius of gyration of the spool about its center of mass is $k_G = 125$ mm.



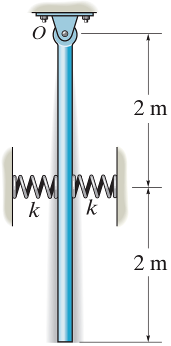
Prob. 22–34

22–35. Determine the natural period of vibration of the 3-kg sphere. Neglect the mass of the rod and the size of the sphere.



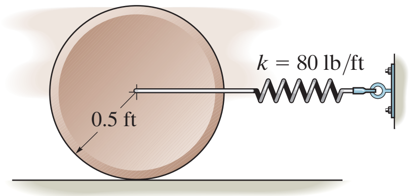
Prob. 22–35

*22–36. If the lower end of the 6-kg slender rod is displaced a small amount and released from rest, determine the natural frequency of vibration. Each spring has a stiffness of $k = 200$ N/m and is unstretched when the rod is hanging vertically.



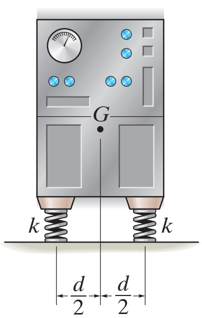
Prob. 22–36

22–37. The disk has a weight of 30 lb and rolls without slipping on the horizontal surface as it oscillates about its equilibrium position. If the disk is displaced, by rolling it counterclockwise 0.2 rad, determine the equation which describes its oscillatory motion and the natural period when it is released.



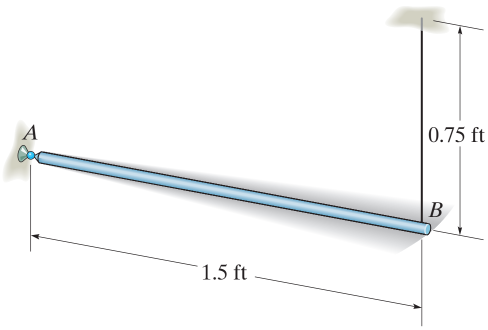
Prob. 22–37

22–38. The machine has a mass m and is uniformly supported by *four* springs, each having a stiffness k. Determine the natural period of vertical vibration.



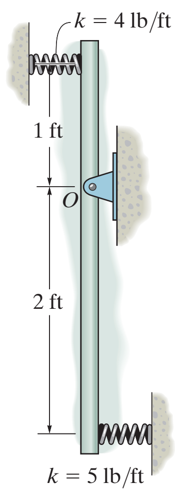
Prob. 22–38

22–39. The slender rod has a weight of 4 lb/ft. If it is supported in the horizontal plane by a ball-and-socket joint at A and a cable at B, determine the natural frequency of vibration when the end B is given a small horizontal displacement and then released.



Prob. 22–39

*22–40. If the slender rod has a weight of 5 lb, determine the natural frequency of vibration. The springs are originally unstretched.



Prob. 22–40

*22.3 Undamped Forced Vibration

Undamped forced vibration is considered to be one of the most important types of vibrating motion in engineering. Its principles can be used to describe the motion of many types of machines and structures.

Periodic Force. The block and spring shown in Fig. 22–11*a* provide a convenient model which represents the vibrational characteristics of a system subjected to a periodic force $F = F_0 \sin \omega_0 t$. This force has an amplitude of F_0 and a *forcing frequency* ω_0. The free-body diagram for the block when it is displaced a distance x is shown in Fig. 22–11*b*. Applying the equation of motion, we have

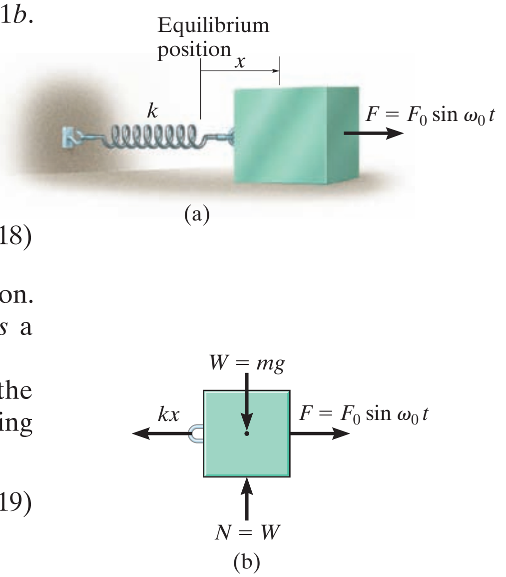


Fig. 22–11

$$\xrightarrow{+} \Sigma F_x = ma_x; \qquad F_0 \sin \omega_0 t - kx = m\ddot{x}$$

or

$$\ddot{x} + \frac{k}{m}x = \frac{F_0}{m}\sin \omega_0 t \tag{22–18}$$

This equation is a nonhomogeneous second-order differential equation. The general solution consists of a complementary solution, x_c, *plus* a particular solution, x_p.

The *complementary solution* is determined by setting the term on the right side of Eq. 22–18 equal to zero and solving the resulting homogeneous equation. The solution is defined by Eq. 22–9, i.e.,

$$x_c = C \sin(\omega_n t + \phi) \tag{22–19}$$

where ω_n is the natural frequency, $\omega_n = \sqrt{k/m}$, Eq. 22–2.

Since the motion is periodic, the *particular solution* of Eq. 22–18 can be determined by assuming a solution of the form

$$x_p = X \sin \omega_0 t \tag{22–20}$$

where X is a constant. Taking the second time derivative and substituting into Eq. 22–18 yields

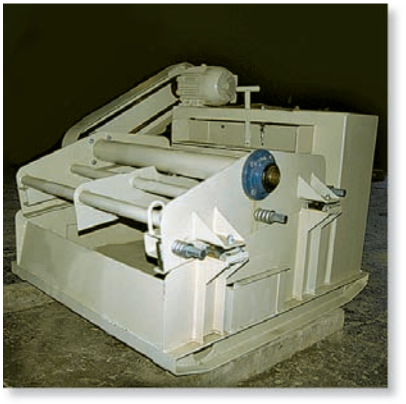
$$-X\omega_0^2 \sin \omega_0 t + \frac{k}{m}(X \sin \omega_0 t) = \frac{F_0}{m}\sin \omega_0 t$$

Factoring out $\sin \omega_0 t$ and solving for X gives

$$X = \frac{F_0/m}{(k/m) - \omega_0^2} = \frac{F_0/k}{1 - (\omega_0/\omega_n)^2} \tag{22–21}$$

Substituting into Eq. 22–20, we obtain the particular solution

$$\boxed{x_p = \frac{F_0/k}{1 - (\omega_0/\omega_n)^2}\sin \omega_0 t} \tag{22–22}$$



Shaker tables provide forced vibration and are used to separate out granular materials. (© R.C. Hibbeler)

The *general solution* is therefore the sum of two sine functions having different frequencies.

$$x = x_c + x_p = C \sin(\omega_n t + \phi) + \frac{F_0/k}{1 - (\omega_0/\omega_n)^2} \sin \omega_0 t \qquad (22\text{–}23)$$

The *complementary solution* x_c defines the *free vibration*, which depends on the natural frequency $\omega_n = \sqrt{k/m}$ and the constants C and ϕ. The *particular solution* x_p describes the *forced vibration* of the block caused by the applied force $F = F_0 \sin \omega_0 t$. Since all vibrating systems are subject to *friction*, the free vibration, x_c, will in time dampen out. For this reason the free vibration is referred to as *transient*, and the forced vibration is called *steady-state*, since it is the only vibration that remains.

From Eq. 22–21 it is seen that the *amplitude* of forced or steady-state vibration depends on the *frequency ratio* ω_0/ω_n. If the *magnification factor* MF is defined as the ratio of the amplitude of steady-state vibration, X, to the static deflection, F_0/k, which would be produced by the amplitude of the periodic force F_0, then, from Eq. 22–21,

The soil compactor operates by forced vibration developed by an internal motor. It is important that the forcing frequency not be close to the natural frequency of vibration of the compactor, which can be determined when the motor is turned off; otherwise resonance will occur and the machine will become uncontrollable. (© R.C. Hibbeler)

$$\text{MF} = \frac{X}{F_0/k} = \frac{1}{1 - (\omega_0/\omega_n)^2} \qquad (22\text{–}24)$$

This equation is graphed in Fig. 22–12. Note that if the force or displacement is applied with a frequency close to the natural frequency of the system, i.e., $\omega_0/\omega_n \approx 1$, the amplitude of vibration of the block becomes extremely large. This occurs because the force **F** is applied to the block so that it always follows the motion of the block. This condition is called *resonance*, and in practice, resonating vibrations can cause tremendous stress and rapid failure of parts.*

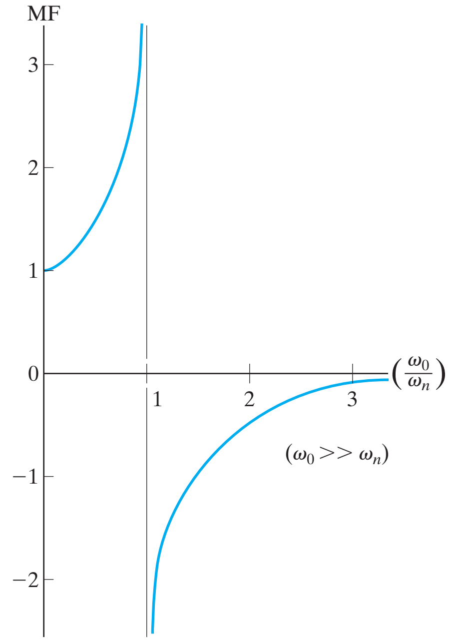


Fig. 22–12

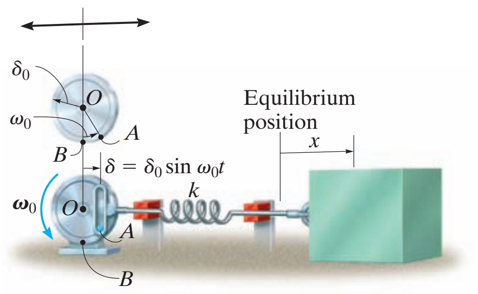
Periodic Support Displacement. Forced vibrations can also arise from the periodic excitation of the support of a system. The model shown in Fig. 22–13*a* represents the periodic vibration of a block which is caused by harmonic movement $\delta = \delta_0 \sin \omega_0 t$ of the support. The free-body diagram for the block in this case is shown in Fig. 22–13*b*. The displacement δ of the support is measured from the point of zero displacement, i.e., when the radial line OA coincides with OB. Therefore, general deformation of the spring is $(x - \delta_0 \sin \omega_0 t)$. Applying the equation of motion yields

$$\xrightarrow{+} \Sigma F_x = ma_x; \qquad -k(x - \delta_0 \sin \omega_0 t) = m\ddot{x}$$

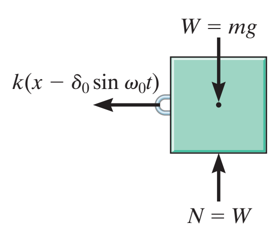
or

$$\ddot{x} + \frac{k}{m}x = \frac{k\delta_0}{m}\sin \omega_0 t \qquad (22\text{–}25)$$

By comparison, this equation is identical to the form of Eq. 22–18, *provided* F_0 is *replaced* by $k\delta_0$. If this substitution is made into the solutions defined by Eqs. 22–21 to 22–23, the results are appropriate for describing the motion of the block when subjected to the support displacement $\delta = \delta_0 \sin \omega_0 t$.



(a)



(b)

Fig. 22–13

*A swing has a natural period of vibration, as determined in Example 22.1. If someone pushes on the swing only when it reaches its highest point, neglecting drag or wind resistance, resonance will occur since the natural and forcing frequencies are the same.

EXAMPLE 22.7

The instrument shown in Fig. 22–14 is rigidly attached to a platform P, which in turn is supported by *four* springs, each having a stiffness $k = 800$ N/m. If the floor is subjected to a vertical displacement $\delta = 10 \sin(8t)$ mm, where t is in seconds, determine the amplitude of steady-state vibration. What is the frequency of the floor vibration required to cause resonance? The instrument and platform have a total mass of 20 kg.

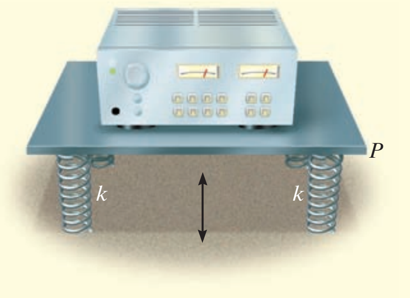


Fig. 22–14

SOLUTION

The natural frequency is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4(800 \text{ N/m})}{20 \text{ kg}}} = 12.65 \text{ rad/s}$$

The amplitude of steady-state vibration is found using Eq. 22–21, with $k\delta_0$ replacing F_0.

$$X = \frac{\delta_0}{1 - (\omega_0/\omega_n)^2} = \frac{10}{1 - [(8 \text{ rad/s})/(12.65 \text{ rad/s})]^2} = 16.7 \text{ mm} \qquad \textit{Ans.}$$

Resonance will occur when the amplitude of vibration X caused by the floor displacement approaches infinity. This requires

$$\omega_0 = \omega_n = 12.6 \text{ rad/s} \qquad \textit{Ans.}$$

*22.4 Viscous Damped Free Vibration

The vibration analysis considered thus far has not included the effects of friction or damping in the system, and as a result, the solutions obtained are only in close agreement with the actual motion. Since all vibrations die out in time, the presence of damping forces should be included in the analysis.

In many cases damping is attributed to the resistance created by the substance, such as water, oil, or air, in which the system vibrates. Provided the body moves slowly through this substance, the resistance to motion is directly proportional to the body's speed. The type of force developed under these conditions is called a *viscous damping force*. The magnitude of this force is expressed by an equation of the form

$$F = c\dot{x} \qquad (22\text{–}26)$$

where the constant c is called the *coefficient of viscous damping* and has units of $\mathrm{N \cdot s/m}$ or $\mathrm{lb \cdot s/ft}$.

The vibrating motion of a body or system having viscous damping can be characterized by the block and spring shown in Fig. 22–15*a*. The effect of damping is provided by the *dashpot* connected to the block on the right side. Damping occurs when the piston P moves to the right or left within the enclosed cylinder. The cylinder contains a fluid, and the motion of the piston is retarded since the fluid must flow around or through a small hole in the piston. The dashpot is assumed to have a coefficient of viscous damping c.

If the block is displaced a distance x from its equilibrium position, the resulting free-body diagram is shown in Fig. 22–15*b*. Both the spring and damping force oppose the forward motion of the block, so that applying the equation of motion yields

$$\overset{+}{\rightarrow} \Sigma F_x = ma_x; \qquad -kx - c\dot{x} = m\ddot{x}$$

or

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad (22\text{–}27)$$

This linear, second-order, homogeneous, differential equation has a solution of the form

$$x = e^{\lambda t}$$

where e is the base of the natural logarithm and λ (lambda) is a constant. The value of λ can be obtained by substituting this solution and its time derivatives into Eq. 22–27, which yields

$$m\lambda^2 e^{\lambda t} + c\lambda e^{\lambda t} + ke^{\lambda t} = 0$$

or

$$e^{\lambda t}(m\lambda^2 + c\lambda + k) = 0$$

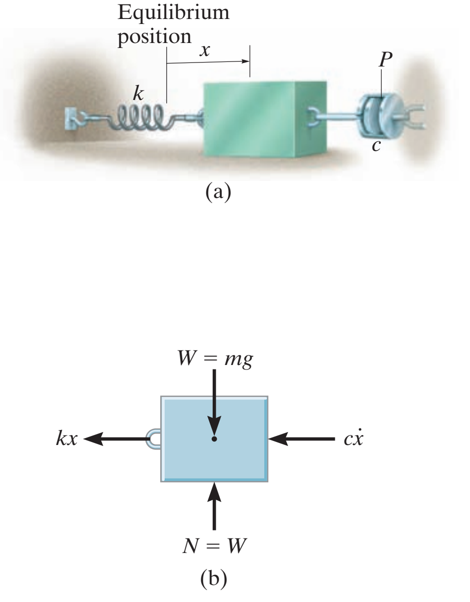


Fig. 22–15

Since $e^{\lambda t}$ can never be zero, a solution is possible provided

$$m\lambda^2 + c\lambda + k = 0$$

Hence, by the quadratic formula, the two values of λ are

$$\lambda_1 = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$
$$\lambda_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \qquad (22\text{–}28)$$

The general solution of Eq. 22–27 is therefore a combination of exponentials which involves both of these roots. There are three possible combinations of λ_1 and λ_2 which must be considered. Before discussing these combinations, however, we will first define the *critical damping coefficient* c_c as the value of c which makes the radical in Eqs. 22–28 equal to zero; i.e.,

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0$$

or

$$\boxed{c_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n} \qquad (22\text{–}29)$$

Overdamped System.

When $c > c_c$, the roots λ_1 and λ_2 are both real. The general solution of Eq. 22–27 can then be written as

$$x = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \qquad (22\text{–}30)$$

Motion corresponding to this solution is *nonvibrating*. The effect of damping is so strong that when the block is displaced and released, it simply creeps back to its original position without oscillating. The system is said to be *overdamped*.

Critically Damped System.

If $c = c_c$, then $\lambda_1 = \lambda_2 = -c_c/2m = -\omega_n$. This situation is known as *critical damping*, since it represents a condition where c has the smallest value necessary to cause the system to be nonvibrating. Using the methods of differential equations, it can be shown that the solution to Eq. 22–27 for critical damping is

$$x = (A + Bt)e^{-\omega_n t} \qquad (22\text{–}31)$$

Underdamped System. Most often $c < c_c$, in which case the system is referred to as *underdamped*. In this case the roots λ_1 and λ_2 are complex numbers, and it can be shown that the general solution of Eq. 22–27 can be written as

$$\boxed{x = D[e^{-(c/2m)t}\sin(\omega_d t + \phi)]} \tag{22–32}$$

where D and ϕ are constants generally determined from the initial conditions of the problem. The constant ω_d is called the *damped natural frequency* of the system. It has a value of

$$\boxed{\omega_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} = \omega_n\sqrt{1 - \left(\frac{c}{c_c}\right)^2}} \tag{22–33}$$

where the ratio c/c_c is called the *damping factor*.

The graph of Eq. 22–32 is shown in Fig. 22–16. The initial limit of motion, D, diminishes with each cycle of vibration, since motion is confined within the bounds of the exponential curve. Using the damped natural frequency ω_d, the period of damped vibration can be written as

$$\boxed{\tau_d = \frac{2\pi}{\omega_d}} \tag{22–34}$$

Since $\omega_d < \omega_n$, Eq. 22–33, the period of damped vibration, τ_d, will be greater than that of free vibration, $\tau = 2\pi/\omega_n$.

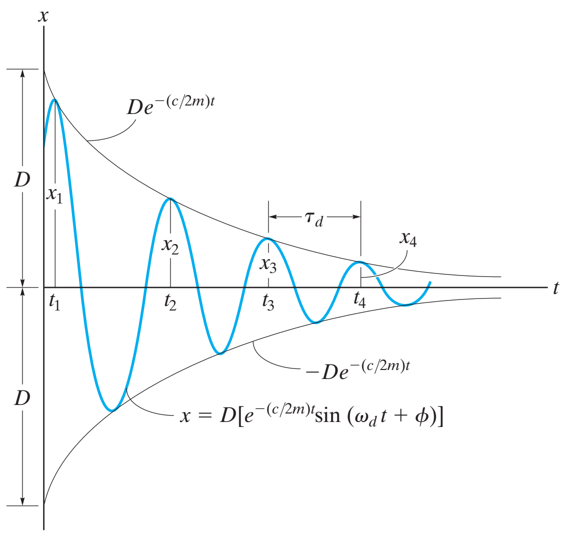


Fig. 22–16

*22.5 Viscous Damped Forced Vibration

The most general case of single-degree-of-freedom vibrating motion occurs when the system includes the effects of forced motion and induced damping. The analysis of this particular type of vibration is of practical value when applied to systems having significant damping characteristics.

If a dashpot is attached to the block and spring shown in Fig. 22–11*a*, the differential equation which describes the motion becomes

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega_0 t \qquad (22\text{–}35)$$

A similar equation can be written for a block and spring having a periodic support displacement, Fig. 22–13*a*, which includes the effects of damping. In that case, however, F_0 is replaced by $k\delta_0$. Since Eq. 22–35 is nonhomogeneous, the general solution is the sum of a complementary solution, x_c, and a particular solution, x_p. The complementary solution is determined by setting the right side of Eq. 22–35 equal to zero and solving the homogeneous equation, which is equivalent to Eq. 22–27. The solution is therefore given by Eq. 22–30, 22–31, or 22–32, depending on the values of λ_1 and λ_2. Because all systems are subjected to friction, then this solution will dampen out with time. Only the particular solution, which describes the *steady-state vibration* of the system, will remain. Since the applied forcing function is harmonic, the steady-state motion will also be harmonic. Consequently, the particular solution will be of the form

$$X_P = X' \sin(\omega_0 t - \phi') \qquad (22\text{–}36)$$

The constants X' and ϕ' are determined by taking the first and second time derivatives and substituting them into Eq. 22–35, which after simplification yields

$$-X'm\omega_0^2 \sin(\omega_0 t - \phi') + X'c\omega_0 \cos(\omega_0 t - \phi') + X'k \sin(\omega_0 t - \phi') = F_0 \sin \omega_0 t$$

Since this equation holds for all time, the constant coefficients can be obtained by setting $\omega_0 t - \phi' = 0$ and $\omega_0 t - \phi' = \pi/2$, which causes the above equation to become

$$X'c\omega_0 = F_0 \sin \phi'$$

$$-X'm\omega_0^2 + X'k = F_0 \cos \phi'$$

The amplitude is obtained by squaring these equations, adding the results, and using the identity $\sin^2\phi' + \cos^2\phi' = 1$, which gives

$$X' = \frac{F_0}{\sqrt{(k - m\omega_0^2)^2 + c^2\omega_0^2}} \tag{22–37}$$

Dividing the first equation by the second gives

$$\phi' = \tan^{-1}\left[\frac{c\omega_0}{k - m\omega_0^2}\right] \tag{22–38}$$

Since $\omega_n = \sqrt{k/m}$ and $c_c = 2m\omega_n$, then the above equations can also be written as

$$\boxed{X' = \frac{F_0/k}{\sqrt{[1 - (\omega_0/\omega_n)^2]^2 + [2(c/c_c)(\omega_0/\omega_n)]^2}}}$$

$$\boxed{\phi' = \tan^{-1}\left[\frac{2(c/c_c)(\omega_0/\omega_n)}{1 - (\omega_0/\omega_n)^2}\right]} \tag{22–39}$$

The angle ϕ' represents the phase difference between the applied force and the resulting steady-state vibration of the damped system.

The *magnification factor* MF has been defined in Sec. 22.3 as the ratio of the amplitude of deflection caused by the forced vibration to the deflection caused by a static force F_0. Thus,

$$\text{MF} = \frac{X'}{F_0/k} = \frac{1}{\sqrt{[1 - (\omega_0/\omega_n)^2]^2 + [2(c/c_c)(\omega_0/\omega_n)]^2}} \tag{22–40}$$

The MF is plotted in Fig. 22–17 versus the frequency ratio ω_0/ω_n for various values of the damping factor c/c_c. It can be seen from this graph that the magnification of the amplitude increases as the damping factor decreases. Resonance obviously occurs only when the damping factor is zero and the frequency ratio equals 1.

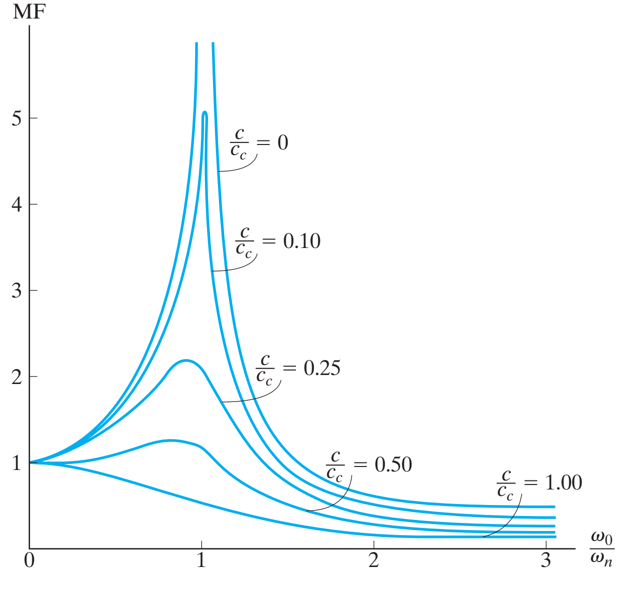


Fig. 22–17

EXAMPLE 22.8

The 30-kg electric motor shown in Fig. 22–18 is confined to move vertically, and is supported by *four* springs, each spring having a stiffness of 200 N/m. If the rotor is unbalanced such that its effect is equivalent to a 4-kg mass located 60 mm from the axis of rotation, determine the amplitude of vibration when the rotor is turning at $\omega_0 = 10$ rad/s. The damping factor is $c/c_c = 0.15$.

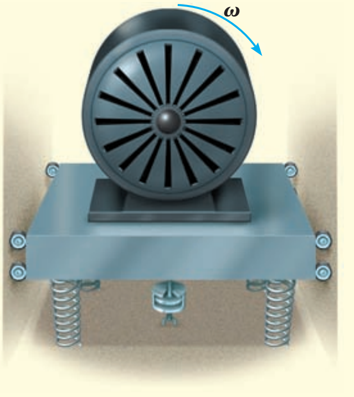


Fig. 22–18

SOLUTION

The periodic force which causes the motor to vibrate is the centrifugal force due to the unbalanced rotor. This force has a constant magnitude of

$$F_0 = ma_n = mr\omega_0^2 = 4\text{ kg}(0.06\text{ m})(10\text{ rad/s})^2 = 24\text{ N}$$

The stiffness of the entire system of four springs is $k = 4(200\text{ N/m}) = 800\text{ N/m}$. Therefore, the natural frequency of vibration is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{800\text{ N/m}}{30\text{ kg}}} = 5.164\text{ rad/s}$$

Since the damping factor is known, the steady-state amplitude can be determined from the first of Eqs. 22–39, i.e.,

$$X' = \frac{F_0/k}{\sqrt{[1-(\omega_0/\omega_n)^2]^2 + [2(c/c_c)(\omega_0/\omega_n)]^2}}$$

$$= \frac{24/800}{\sqrt{[1-(10/5.164)^2]^2 + [2(0.15)(10/5.164)]^2}}$$

$$= 0.0107\text{ m} = 10.7\text{ mm}$$ *Ans.*

*22.6 Electrical Circuit Analogs

The characteristics of a vibrating mechanical system can be represented by an electric circuit. Consider the circuit shown in Fig. 22–19*a*, which consists of an inductor L, a resistor R, and a capacitor C. When a voltage $E(t)$ is applied, it causes a current of magnitude i to flow through the circuit. As the current flows past the inductor the voltage drop is $L(di/dt)$, when it flows across the resistor the drop is Ri, and when it arrives at the capacitor the drop is $(1/C)\int i\,dt$. Since current cannot flow past a capacitor, it is only possible to measure the charge q acting on the capacitor. The charge can, however, be related to the current by the equation $i = dq/dt$. Thus, the voltage drops which occur across the inductor, resistor, and capacitor become $L\,d^2q/dt^2$, $R\,dq/dt$, and q/C, respectively. According to Kirchhoff's voltage law, the applied voltage balances the sum of the voltage drops around the circuit. Therefore,

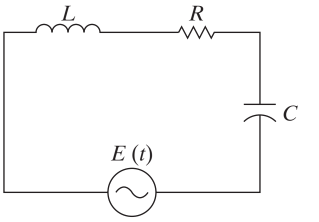
$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t) \tag{22–41}$$

Consider now the model of a single-degree-of-freedom mechanical system, Fig. 22–19*b*, which is subjected to both a general forcing function $F(t)$ and damping. The equation of motion for this system was established in the previous section and can be written as

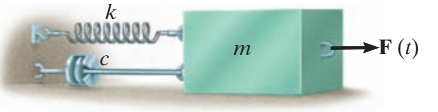
$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t) \tag{22–42}$$

By comparison, it is seen that Eqs. 22–41 and 22–42 have the same form, and hence mathematically the procedure of analyzing an electric circuit is the same as that of analyzing a vibrating mechanical system. The analogs between the two equations are given in Table 22–1.

This analogy has important application to experimental work, for it is much easier to simulate the vibration of a complex mechanical system using an electric circuit, which can be constructed on an analog computer, than to make an equivalent mechanical spring-and-dashpot model.



(a)



(b)

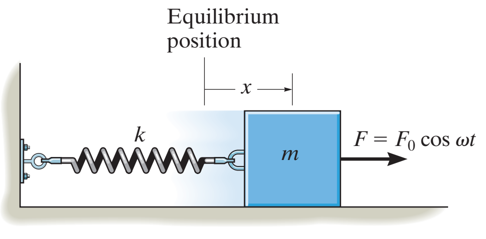
Fig. 22–19

TABLE 22–1
Electrical–Mechanical Analogs

Electrical		Mechanical	
Electric charge	q	Displacement	x
Electric current	i	Velocity	dx/dt
Voltage	$E(t)$	Applied force	$F(t)$
Inductance	L	Mass	m
Resistance	R	Viscous damping coefficient	c
Reciprocal of capacitance	$1/C$	Spring stiffness	k

PROBLEMS

22–41. If the block-and-spring model is subjected to the periodic force $F = F_0 \cos \omega t$, show that the differential equation of motion is $\ddot{x} + (k/m)x = (F_0/m)\cos \omega t$, where x is measured from the equilibrium position of the block. What is the general solution of this equation?



Prob. 22–41

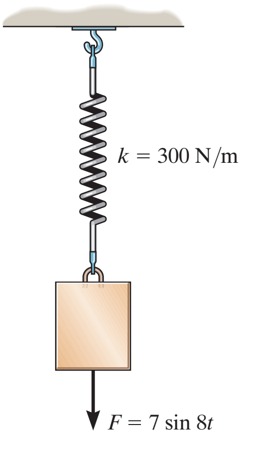
22–42. A block which has a mass m is suspended from a spring having a stiffness k. If an impressed downward vertical force $F = F_O$ acts on the weight, determine the equation which describes the position of the block as a function of time.

22–43. A 4-lb weight is attached to a spring having a stiffness $k = 10$ lb/ft. The weight is drawn downward a distance of 4 in. and released from rest. If the support moves with a vertical displacement $\delta = (0.5 \sin 4t)$ in., where t is in seconds, determine the equation which describes the position of the weight as a function of time.

***22–44.** A 4-kg block is suspended from a spring that has a stiffness of $k = 600$ N/m. The block is drawn downward 50 mm from the equilibrium position and released from rest when $t = 0$. If the support moves with an impressed displacement of $\delta = (10 \sin 4t)$ mm, where t is in seconds, determine the equation that describes the vertical motion of the block. Assume positive displacement is downward.

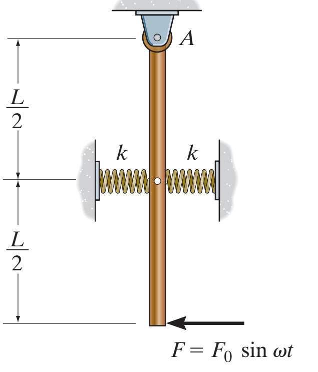
22–45. Use a block-and-spring model like that shown in Fig. 22–13a, but suspended from a vertical position and subjected to a periodic support displacement $\delta = \delta_0 \sin \omega_0 t$, determine the equation of motion for the system, and obtain its general solution. Define the displacement y measured from the static equilibrium position of the block when $t = 0$.

22–46. A 5-kg block is suspended from a spring having a stiffness of 300 N/m. If the block is acted upon by a vertical force $F = (7 \sin 8t)$ N, where t is in seconds, determine the equation which describes the motion of the block when it is pulled down 100 mm from the equilibrium position and released from rest at $t = 0$. Assume that positive displacement is downward.



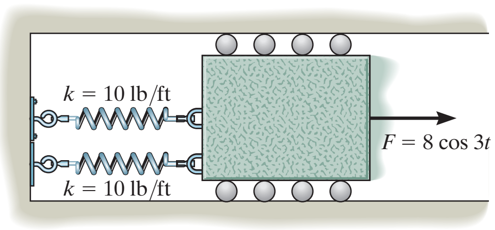
Prob. 22–46

22–47. The uniform rod has a mass of m. If it is acted upon by a periodic force of $F = F_0 \sin \omega t$, determine the amplitude of the steady-state vibration.



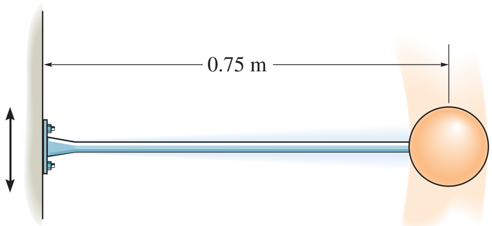
Prob. 22–47

*22–48. The 30-lb block is attached to two springs having a stiffness of 10 lb/ft. A periodic force $F = (8 \cos 3t)$ lb, where t is in seconds, is applied to the block. Determine the maximum speed of the block after frictional forces cause the free vibrations to dampen out.



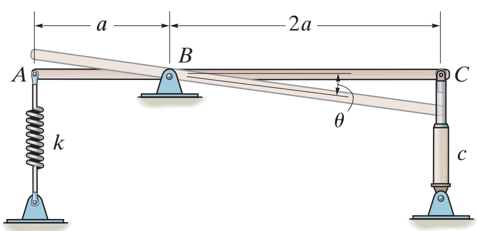
Prob. 22–48

22–49. The light elastic rod supports a 4-kg sphere. When an 18-N vertical force is applied to the sphere, the rod deflects 14 mm. If the wall oscillates with harmonic frequency of 2 Hz and has an amplitude of 15 mm, determine the amplitude of vibration for the sphere.



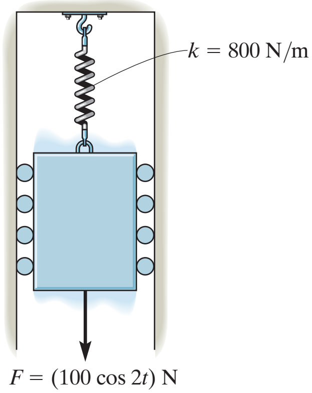
Prob. 22–49

22–50. Find the differential equation for small oscillations in terms of θ for the uniform rod of mass m. Also show that if $c < \sqrt{mk}/2$, then the system remains underdamped. The rod is in a horizontal position when it is in equilibrium.



Prob. 22–50

22–51. The 40-kg block is attached to a spring having a stiffness of 800 N/m. A force $F = (100 \cos 2t)$ N, where t is in seconds is applied to the block. Determine the maximum speed of the block for the steady-state vibration.



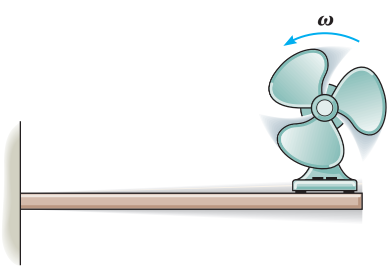
Prob. 22–51

*22–52. Using a block-and-spring model, like that shown in Fig. 22–13*a*, but suspended from a vertical position and subjected to a periodic support displacement of $\delta = \delta_0 \cos \omega_0 t$, determine the equation of motion for the system, and obtain its general solution. Define the displacement y measured from the static equilibrium position of the block when $t = 0$.

22–53. The fan has a mass of 25 kg and is fixed to the end of a horizontal beam that has a negligible mass. The fan blade is mounted eccentrically on the shaft such that it is equivalent to an unbalanced 3.5-kg mass located 100 mm from the axis of rotation. If the static deflection of the beam is 50 mm as a result of the weight of the fan, determine the angular velocity of the fan blade at which resonance will occur. *Hint*: See the first part of Example 22.8.

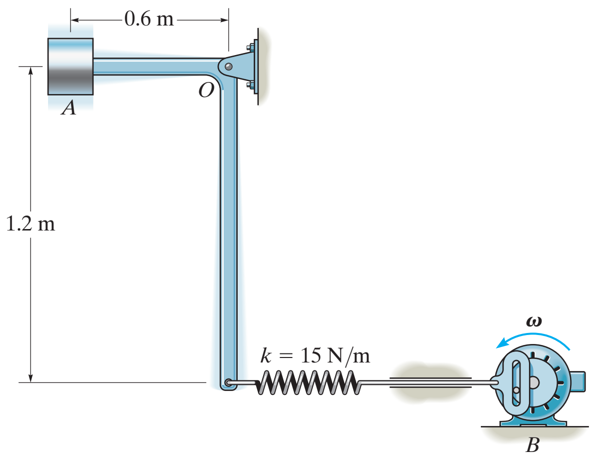
22–54. In Prob. 22–53, determine the amplitude of steady-state vibration of the fan if its angular velocity is 10 rad/s.

22–55. What will be the amplitude of steady-state vibration of the fan in Prob. 22–53 if the angular velocity of the fan blade is 18 rad/s? *Hint*: See the first part of Example 22.8.



Probs. 22–53/54/55

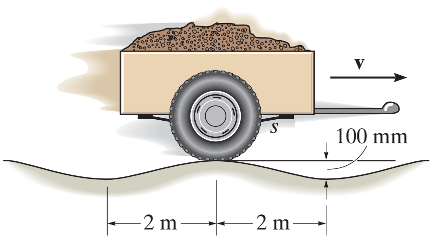
***22–56.** The small block at A has a mass of 4 kg and is mounted on the bent rod having negligible mass. If the rotor at B causes a harmonic movement $\delta_B = (0.1 \cos 15t)$ m, where t is in seconds, determine the steady-state amplitude of vibration of the block.



Prob. 22–56

***22–60.** The 450-kg trailer is pulled with a constant speed over the surface of a bumpy road, which may be approximated by a cosine curve having an amplitude of 50 mm and wave length of 4 m. If the two springs s which support the trailer each have a stiffness of 800 N/m, determine the speed v which will cause the greatest vibration (resonance) of the trailer. Neglect the weight of the wheels.

22–61. Determine the amplitude of vibration of the trailer in Prob. 22–60 if the speed $v = 15$ km/h.

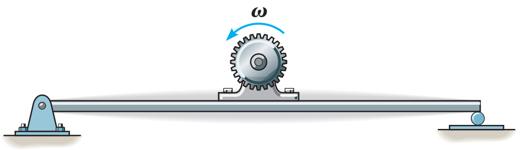


Probs. 22–60/61

22–57. The electric motor turns an eccentric flywheel which is equivalent to an unbalanced 0.25-lb weight located 10 in. from the axis of rotation. If the static deflection of the beam is 1 in. because of the weight of the motor, determine the angular velocity of the flywheel at which resonance will occur. The motor weighs 150 lb. Neglect the mass of the beam.

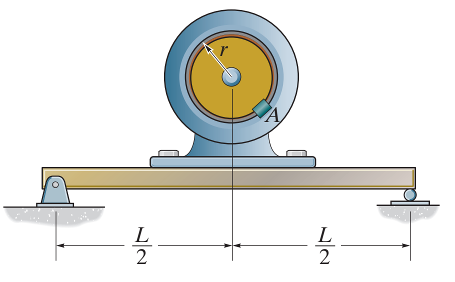
22–58. What will be the amplitude of steady-state vibration of the motor in Prob. 22–57 if the angular velocity of the flywheel is 20 rad/s?

22–59. Determine the angular velocity of the flywheel in Prob. 22–57 which will produce an amplitude of vibration of 0.25 in.



Probs. 22–57/58/59

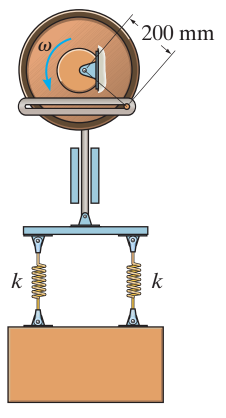
22–62. The motor of mass M is supported by a simply supported beam of negligible mass. If block A of mass m is clipped onto the rotor, which is turning at constant angular velocity of ω, determine the amplitude of the steady-state vibration. *Hint*: When the beam is subjected to a concentrated force of P at its mid-span, it deflects $\delta = PL^3/48EI$ at this point. Here E is Young's modulus of elasticity, a property of the material, and I is the moment of inertia of the beam's cross-sectional area.



Prob. 22–62

22–63. The spring system is connected to a crosshead that oscillates vertically when the wheel rotates with a constant angular velocity of $\boldsymbol{\omega}$. If the amplitude of the steady-state vibration is observed to be 400 mm, and the springs each have a stiffness of $k = 2500$ N/m, determine the two possible values of $\boldsymbol{\omega}$ at which the wheel must rotate. The block has a mass of 50 kg.

***22–64.** The spring system is connected to a crosshead that oscillates vertically when the wheel rotates with a constant angular velocity of $\omega = 5$ rad/s. If the amplitude of the steady-state vibration is observed to be 400 mm, determine the two possible values of the stiffness k of the springs. The block has a mass of 50 kg.



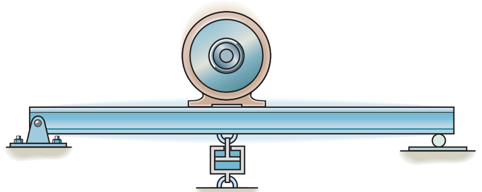
Probs. 22–63/64

22–65. A 7-lb block is suspended from a spring having a stiffness of $k = 75$ lb/ft. The support to which the spring is attached is given simple harmonic motion which may be expressed as $\delta = (0.15 \sin 2t)$ ft, where t is in seconds. If the damping factor is $c/c_c = 0.8$, determine the phase angle ϕ of forced vibration.

22–66. Determine the magnification factor of the block, spring, and dashpot combination in Prob. 22–65.

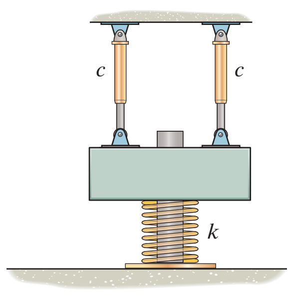
22–67. A block having a mass of 7 kg is suspended from a spring that has a stiffness $k = 600$ N/m. If the block is given an upward velocity of 0.6 m/s from its equilibrium position at $t = 0$, determine its position as a function of time. Assume that positive displacement of the block is downward and that motion takes place in a medium which furnishes a damping force $F = (50|v|)$ N, where v is in m/s.

***22–68.** The 200-lb electric motor is fastened to the midpoint of the simply supported beam. It is found that the beam deflects 2 in. when the motor is not running. The motor turns an eccentric flywheel which is equivalent to an unbalanced weight of 1 lb located 5 in. from the axis of rotation. If the motor is turning at 100 rpm, determine the amplitude of steady-state vibration. The damping factor is $c/c_c = 0.20$. Neglect the mass of the beam.



Prob. 22–68

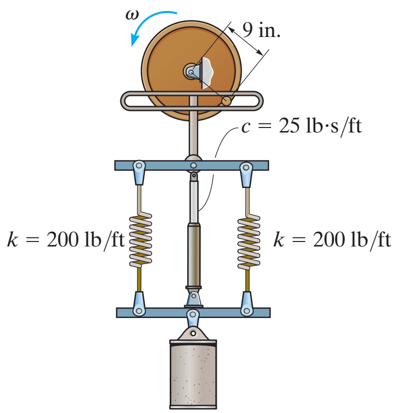
22–69. Two identical dashpots are arranged parallel to each other, as shown. Show that if the damping coefficient $c < \sqrt{mk}$, then the block of mass m will vibrate as an underdamped system.



Prob. 22–69

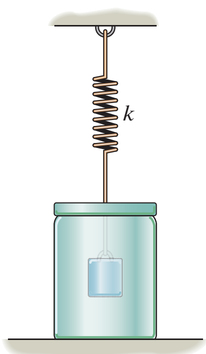
22–70. The damping factor, c/c_c, may be determined experimentally by measuring the successive amplitudes of vibrating motion of a system. If two of these maximum displacements can be approximated by x_1 and x_2, as shown in Fig. 22–16, show that $\ln (x_1/x_2) = 2\pi(c/c_c)/\sqrt{1-(c/c_c)^2}$. The quantity $\ln (x_1/x_2)$ is called the *logarithmic decrement*.

22–71. If the amplitude of the 50-lb cylinder's steady-state vibration is 6 in., determine the wheel's angular velocity ω.



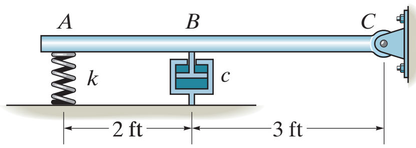
Prob. 22–71

***22–72.** The block, having a weight of 12 lb, is immersed in a liquid such that the damping force acting on the block has a magnitude of $F = (0.7|v|)$ lb, where v is in ft/s. If the block is pulled down 0.62 ft and released from rest, determine the position of the block as a function of time. The spring has a stiffness of $k = 53$ lb/ft. Assume that positive displacement is downward.



Prob. 22–72

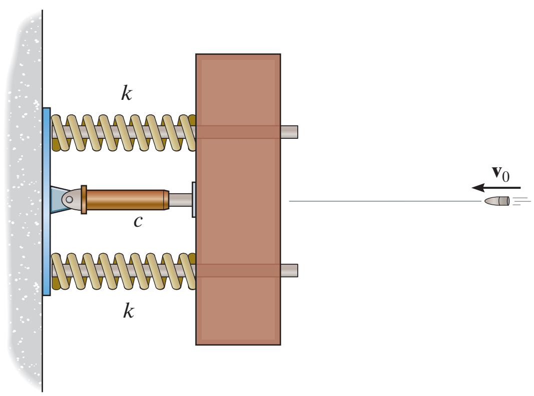
22–73. The bar has a weight of 6 lb. If the stiffness of the spring is $k = 8$ lb/ft and the dashpot has a damping coefficient $c = 60$ lb·s/ft, determine the differential equation which describes the motion in terms of the angle θ of the bar's rotation. Also, what should be the damping coefficient of the dashpot if the bar is to be critically damped?



Prob. 22–73

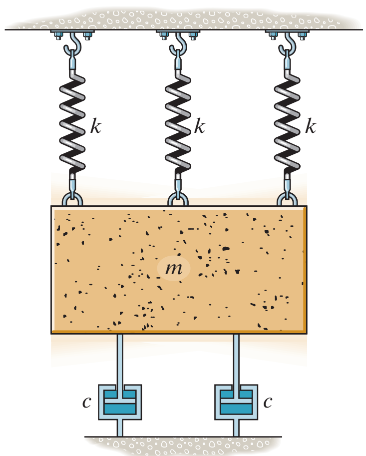
22–74. A bullet of mass m has a velocity of $\mathbf{v}_0$ just before it strikes the target of mass M. If the bullet embeds in the target, and the vibration is to be critically damped, determine the dashpot's critical damping coefficient, and the springs' maximum compression. The target is free to move along the two horizontal guides that are "nested" in the springs.

22–75. A bullet of mass m has a velocity $\mathbf{v}_0$ just before it strikes the target of mass M. If the bullet embeds in the target, and the dashpot's damping coefficient is $0 < c << c_c$, determine the springs' maximum compression. The target is free to move along the two horizontal guides that are "nested" in the springs.



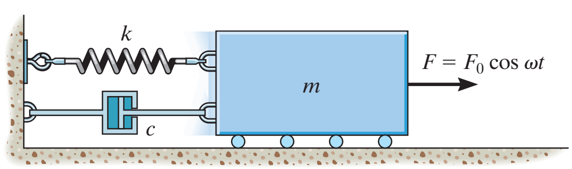
Probs. 22–74/75

***22–76.** Determine the differential equation of motion for the damped vibratory system shown. What type of motion occurs? Take $k = 100\ \text{N/m}$, $c = 200\ \text{N}\cdot\text{s/m}$, $m = 25\ \text{kg}$.



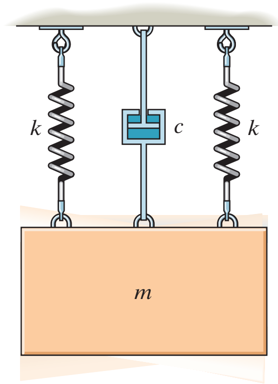
Prob. 22–76

22–77. Draw the electrical circuit that is equivalent to the mechanical system shown. Determine the differential equation which describes the charge q in the circuit.



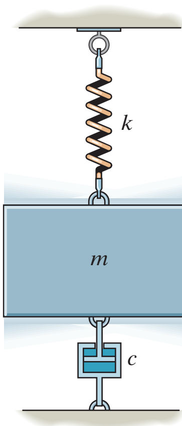
Prob. 22–77

22–78. Draw the electrical circuit that is equivalent to the mechanical system shown. What is the differential equation which describes the charge q in the circuit?



Prob. 22–78

22–79. Draw the electrical circuit that is equivalent to the mechanical system shown. Determine the differential equation which describes the charge q in the circuit.



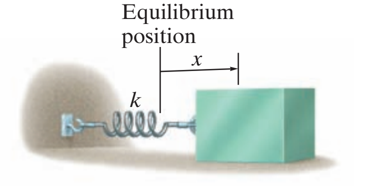
Prob. 22–79

CHAPTER REVIEW

Undamped Free Vibration

A body has free vibration when gravitational or elastic restoring forces cause the motion. This motion is undamped when friction forces are neglected. The periodic motion of an undamped, freely vibrating body can be studied by displacing the body from the equilibrium position and then applying the equation of motion along the path.

For a one-degree-of-freedom system, the resulting differential equation can be written in terms of its natural frequency ω_n.



$$\ddot{x} + \omega_n^2 x = 0 \qquad \tau = \frac{2\pi}{\omega_n} \qquad f = \frac{1}{\tau} = \frac{\omega_n}{2\pi}$$

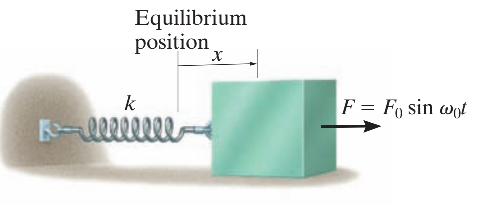
Energy Methods

Provided the restoring forces acting on the body are gravitational and elastic, then conservation of energy can also be used to determine its simple harmonic motion. To do this, the body is displaced a small amount from its equilibrium position, and an expression for its kinetic and potential energy is written. The time derivative of this equation can then be rearranged in the standard form $\ddot{x} + \omega_n^2 x = 0$.

Undamped Forced Vibration

When the equation of motion is applied to a body, which is subjected to a periodic force, or the support has a displacement with a frequency ω_0, then the solution of the differential equation consists of a complementary solution and a particular solution. The complementary solution is caused by the free vibration and can be neglected. The particular solution is caused by the forced vibration.

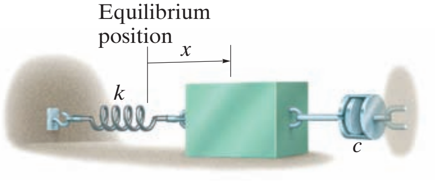
Resonance will occur if the natural frequency of vibration ω_n is equal to the forcing frequency ω_0. This should be avoided, since the motion will tend to become unbounded.



$$x_p = \frac{F_0/k}{1 - (\omega_0/\omega_n)^2} \sin \omega_0 t$$

Viscous Damped Free Vibration

A viscous damping force is caused by fluid drag on the system as it vibrates. If the motion is slow, this drag force will be proportional to the velocity, that is, $F = c\dot{x}$. Here c is the coefficient of viscous damping. By comparing its value to the critical damping coefficient $c_c = 2m\omega_n$, we can specify the type of vibration that occurs. If $c > c_c$, it is an overdamped system; if $c = c_c$, it is a critically damped system; if $c < c_c$, it is an underdamped system.



Viscous Damped Forced Vibration

The most general type of vibration for a one-degree-of-freedom system occurs when the system is damped and subjected to periodic forced motion. The solution provides insight as to how the damping factor, c/c_c, and the frequency ratio, ω_0/ω_n, influence the vibration.

Resonance is avoided provided $c/c_c \neq 0$ and $\omega_0/\omega_n \neq 1$.

Electrical Circuit Analogs

The vibrating motion of a complex mechanical system can be studied by modeling it as an electrical circuit. This is possible since the differential equations that govern the behavior of each system are the same.

APPENDIX

A Mathematical Expressions

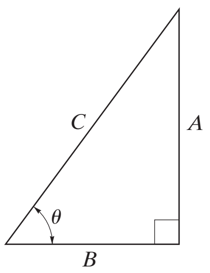
Quadratic Formula

If $ax^2 + bx + c = 0$, then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Hyperbolic Functions

$\sinh x = \dfrac{e^x - e^{-x}}{2}$, $\cosh x = \dfrac{e^x + e^{-x}}{2}$, $\tanh x = \dfrac{\sinh x}{\cosh x}$

Trigonometric Identities



$\sin\theta = \dfrac{A}{C}$, $\csc\theta = \dfrac{C}{A}$

$\cos\theta = \dfrac{B}{C}$, $\sec\theta = \dfrac{C}{B}$

$\tan\theta = \dfrac{A}{B}$, $\cot\theta = \dfrac{B}{A}$

$\sin^2\theta + \cos^2\theta = 1$

$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos\theta = \pm\sqrt{\dfrac{1 + \cos 2\theta}{2}}$, $\sin\theta = \pm\sqrt{\dfrac{1 - \cos 2\theta}{2}}$

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$1 + \tan^2\theta = \sec^2\theta \quad 1 + \cot^2\theta = \csc^2\theta$

Power-Series Expansions

$\sin x = x - \dfrac{x^3}{3!} + \cdots \qquad \sinh x = x + \dfrac{x^3}{3!} + \cdots$

$\cos x = 1 - \dfrac{x^2}{2!} + \cdots \qquad \cosh x = 1 + \dfrac{x^2}{2!} + \cdots$

Derivatives

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$\frac{d}{dx}(\cot u) = -\csc^2 u\frac{du}{dx}$$

$$\frac{d}{dx}(\sec u) = \tan u \sec u\frac{du}{dx}$$

$$\frac{d}{dx}(\csc u) = -\csc u \cot u\frac{du}{dx}$$

$$\frac{d}{dx}(\sin u) = \cos u\frac{du}{dx}$$

$$\frac{d}{dx}(\cos u) = -\sin u\frac{du}{dx}$$

$$\frac{d}{dx}(\tan u) = \sec^2 u\frac{du}{dx}$$

$$\frac{d}{dx}(\sinh u) = \cosh u\frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u\frac{du}{dx}$$

Integrals

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\; n \neq -1$$

$$\int \frac{dx}{a+bx} = \frac{1}{b}\ln(a+bx) + C$$

$$\int \frac{dx}{a+bx^2} = \frac{1}{2\sqrt{-ba}}\ln\left[\frac{a+x\sqrt{-ab}}{a-x\sqrt{-ab}}\right] + C,\; ab < 0$$

$$\int \frac{x\,dx}{a+bx^2} = \frac{1}{2b}\ln(bx^2+a) + C$$

$$\int \frac{x^2\,dx}{a+bx^2} = \frac{x}{b} - \frac{a}{b\sqrt{ab}}\tan^{-1}\frac{x\sqrt{ab}}{a} + C,\; ab > 0$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left[\frac{a+x}{a-x}\right] + C,\; a^2 > x^2$$

$$\int \sqrt{a+bx}\,dx = \frac{2}{3b}\sqrt{(a+bx)^3} + C$$

$$\int x\sqrt{a+bx}\,dx = \frac{-2(2a-3bx)\sqrt{(a+bx)^3}}{15b^2} + C$$

$$\int x^2\sqrt{a+bx}\,dx = \frac{2(8a^2-12abx+15b^2x^2)\sqrt{(a+bx)^3}}{105b^3} + C$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right] + C,\; a > 0$$

$$\int x\sqrt{x^2 \pm a^2}\,dx = \frac{1}{3}\sqrt{(x^2 \pm a^2)^3} + C$$

$$\int x^2\sqrt{a^2-x^2}\,dx = -\frac{x}{4}\sqrt{(a^2-x^2)^3} + \frac{a^2}{8}\left(x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right) + C,\; a > 0$$

$$\int \sqrt{x^2 \pm a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2 \pm a^2} \pm a^2\ln\left(x + \sqrt{x^2 \pm a^2}\right)\right] + C$$

$$\int x\sqrt{a^2-x^2}\,dx = -\frac{1}{3}\sqrt{(a^2-x^2)^3} + C$$

$$\int x^2\sqrt{x^2 \pm a^2}\,dx = \frac{x}{4}\sqrt{(x^2 \pm a^2)^3} \mp \frac{a^2}{8}x\sqrt{x^2 \pm a^2} - \frac{a^4}{8}\ln\left(x + \sqrt{x^2 \pm a^2}\right) + C$$

$$\int \frac{dx}{\sqrt{a+bx}} = \frac{2\sqrt{a+bx}}{b} + C$$

$$\int \frac{x\,dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2} + C$$

$$\int \frac{dx}{\sqrt{a+bx+cx^2}} = \frac{1}{\sqrt{c}}\ln\left[\sqrt{a+bx+cx^2} + x\sqrt{c} + \frac{b}{2\sqrt{c}}\right] + C,\; c > 0$$

$$= \frac{1}{\sqrt{-c}}\sin^{-1}\left(\frac{-2cx-b}{\sqrt{b^2-4ac}}\right) + C,\; c < 0$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int x\cos(ax)\,dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax) + C$$

$$\int x^2\cos(ax)\,dx = \frac{2x}{a^2}\cos(ax) + \frac{a^2x^2-2}{a^3}\sin(ax) + C$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1) + C$$

$$\int \sinh x\,dx = \cosh x + C$$

$$\int \cosh x\,dx = \sinh x + C$$

APPENDIX

B Vector Analysis

The following discussion provides a brief review of vector analysis. A more detailed treatment of these topics is given in *Engineering Mechanics: Statics*.

Vector. A vector, $\mathbf{A}$, is a quantity which has magnitude and direction, and adds according to the parallelogram law. As shown in Fig. B–1, $\mathbf{A} = \mathbf{B} + \mathbf{C}$, where $\mathbf{A}$ is the *resultant vector* and $\mathbf{B}$ and $\mathbf{C}$ are *component vectors*.

Unit Vector. A unit vector, $\mathbf{u}_A$, has a magnitude of one "dimensionless" unit and acts in the same direction as $\mathbf{A}$. It is determined by dividing $\mathbf{A}$ by its magnitude A, i.e,

$$\boxed{\mathbf{u}_A = \frac{\mathbf{A}}{A}} \tag{B–1}$$

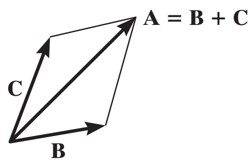


Fig. B–1

Cartesian Vector Notation. The directions of the positive *x, y, z* axes are defined by the Cartesian unit vectors **i**, **j**, **k**, respectively.

As shown in Fig. B–2, vector **A** is formulated by the addition of its *x, y, z* components as

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} \tag{B–2}$$

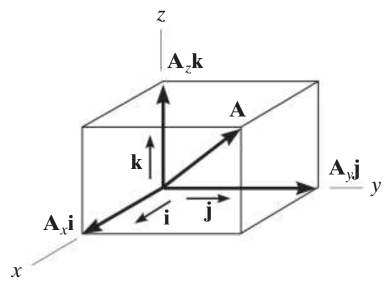


Fig. B–2

The *magnitude* of **A** is determined from

$$\boxed{A = \sqrt{A_x^2 + A_y^2 + A_z^2}} \tag{B–3}$$

The *direction* of **A** is defined in terms of its *coordinate direction angles*, α, β, γ, measured from the *tail* of **A** to the *positive x, y, z* axes, Fig. B–3. These angles are determined from the *direction cosines* which represent the **i**, **j**, **k** components of the unit vector $\mathbf{u}_A$; i.e., from Eqs. B–1 and B–2

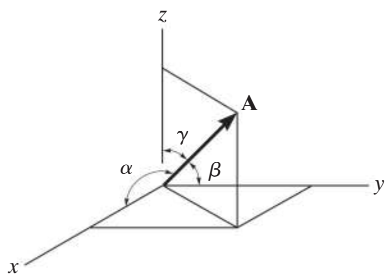


Fig. B–3

$$\boxed{\mathbf{u}_A = \frac{A_x}{A}\mathbf{i} + \frac{A_y}{A}\mathbf{j} + \frac{A_z}{A}\mathbf{k}} \tag{B–4}$$

so that the direction cosines are

$$\boxed{\cos\alpha = \frac{A_x}{A} \quad \cos\beta = \frac{A_y}{A} \quad \cos\gamma = \frac{A_z}{A}} \tag{B–5}$$

Hence, $\mathbf{u}_A = \cos\alpha\mathbf{i} + \cos\beta\mathbf{j} + \cos\gamma\mathbf{k}$, and using Eq. B–3, it is seen that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{B–6}$$

The Cross Product. The cross product of two vectors **A** and **B**, which yields the resultant vector **C**, is written as

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \tag{B–7}$$

and reads **C** equals **A** "cross" **B**. The *magnitude* of **C** is

$$C = AB\sin\theta \tag{B–8}$$

where θ is the angle made between the *tails* of **A** and **B** ($0° \le \theta \le 180°$). The *direction* of **C** is determined by the right-hand rule, whereby the fingers of the right hand are curled *from* **A** *to* **B** and the thumb points in the direction of **C**, Fig. B–4. This vector is perpendicular to the plane containing vectors **A** and **B**.

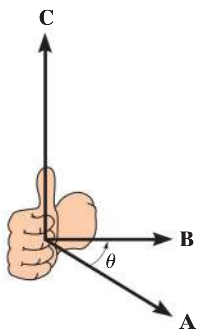


Fig. B–4

B

The vector cross product is *not* commutative, i.e., $\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$. Rather,

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{B–9}$$

The distributive law is valid; i.e.,

$$\mathbf{A} \times (\mathbf{B} + \mathbf{D}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{D} \tag{B–10}$$

And the cross product may be multiplied by a scalar m in any manner; i.e.,

$$m(\mathbf{A} \times \mathbf{B}) = (m\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (m\mathbf{B}) = (\mathbf{A} \times \mathbf{B})m \tag{B–11}$$

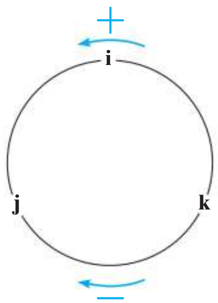


Fig. B–5

Equation B–7 can be used to find the cross product of any pair of Cartesian unit vectors. For example, to find $\mathbf{i} \times \mathbf{j}$, the magnitude is $(i)(j) \sin 90° = (1)(1)(1) = 1$, and its direction $+\mathbf{k}$ is determined from the right-hand rule, applied to $\mathbf{i} \times \mathbf{j}$, Fig. B–2. A simple scheme shown in Fig. B–5 may be helpful in obtaining this and other results when the need arises. If the circle is constructed as shown, then "crossing" two of the unit vectors in a *counterclockwise* fashion around the circle yields a *positive* third unit vector, e.g., $\mathbf{k} \times \mathbf{i} = \mathbf{j}$. Moving *clockwise*, a *negative* unit vector is obtained, e.g., $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

If $\mathbf{A}$ and $\mathbf{B}$ are expressed in Cartesian component form, then the cross product, Eq. B–7, may be evaluated by expanding the determinant

$$\boxed{\mathbf{C} = \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}} \tag{B–12}$$

which yields

$$\mathbf{C} = (A_yB_z - A_zB_y)\mathbf{i} - (A_xB_z - A_zB_x)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}$$

Recall that the cross product is used in statics to define the moment of a force $\mathbf{F}$ about point O, in which case

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} \tag{B–13}$$

where $\mathbf{r}$ is a position vector directed from point O to *any point* on the line of action of $\mathbf{F}$.

The Dot Product. The dot product of two vectors **A** and **B**, which yields a scalar, is defined as

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta \tag{B–14}$$

and reads **A** "dot" **B**. The angle θ is formed between the *tails* of **A** and **B** $(0^\circ \leq \theta \leq 180^\circ)$.

The dot product is commutative; i.e.,

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \tag{B–15}$$

The distributive law is valid; i.e.,

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{D}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{D} \tag{B–16}$$

And scalar multiplication can be performed in any manner, i.e.,

$$m(\mathbf{A} \cdot \mathbf{B}) = (m\mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot (m\mathbf{B}) = (\mathbf{A} \cdot \mathbf{B})m \tag{B–17}$$

Using Eq. B–14, the dot product between any two Cartesian vectors can be determined. For example, $\mathbf{i} \cdot \mathbf{i} = (1)(1) \cos 0^\circ = 1$ and $\mathbf{i} \cdot \mathbf{j} = (1)(1) \cos 90^\circ = 0$.

If **A** and **B** are expressed in Cartesian component form, then the dot product, Eq. C–14, can be determined from

$$\boxed{\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z} \tag{B–18}$$

The dot product may be used to determine the *angle θ formed between two vectors*. From Eq. B–14,

$$\theta = \cos^{-1}\left(\frac{\mathbf{A} \cdot \mathbf{B}}{AB}\right) \tag{B–19}$$

B

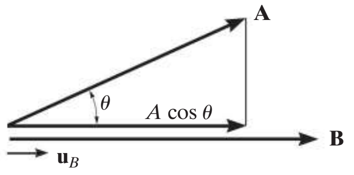


Fig. B–6

It is also possible to find the *component of a vector in a given direction* using the dot product. For example, the magnitude of the component (or projection) of vector **A** in the direction of **B**, Fig. B–6, is defined by $A \cos \theta$. From Eq. B–14, this magnitude is

$$A \cos \theta = \mathbf{A} \cdot \frac{\mathbf{B}}{B} = \mathbf{A} \cdot \mathbf{u}_B \qquad \text{(B–20)}$$

where $\mathbf{u}_B$ represents a unit vector acting in the direction of **B**, Fig. B–6.

Differentiation and Integration of Vector Functions. The rules for differentiation and integration of the sums and products of scalar functions also apply to vector functions. Consider, for example, the two vector functions $\mathbf{A}(s)$ and $\mathbf{B}(s)$. Provided these functions are smooth and continuous for all s, then

$$\frac{d}{ds}(\mathbf{A} + \mathbf{B}) = \frac{d\mathbf{A}}{ds} + \frac{d\mathbf{B}}{ds} \qquad \text{(B–21)}$$

$$\int (\mathbf{A} + \mathbf{B})\, ds = \int \mathbf{A}\, ds + \int \mathbf{B}\, ds \qquad \text{(B–22)}$$

For the cross product,

$$\frac{d}{ds}(\mathbf{A} \times \mathbf{B}) = \left(\frac{d\mathbf{A}}{ds} \times \mathbf{B}\right) + \left(\mathbf{A} \times \frac{d\mathbf{B}}{ds}\right) \qquad \text{(B–23)}$$

Similarly, for the dot product,

$$\frac{d}{ds}(\mathbf{A} \cdot \mathbf{B}) = \frac{d\mathbf{A}}{ds} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{d\mathbf{B}}{ds} \qquad \text{(B–24)}$$

The Chain Rule

APPENDIX

C

The chain rule of calculus can be used to determine the time derivative of a composite function. For example, if y is a function of x and x is a function of t, then we can find the derivative of y with respect to t as follows

$$\dot{y} = \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \tag{C–1}$$

In other words, to find $\dot{y}$ we take the ordinary derivative (dy/dx) and multiply it by the time derivative (dx/dt).

If several variables are functions of time and they are multiplied together, then the product rule $d(uv) = du\,v + u\,dv$ must be used along with the chain rule when taking the time derivatives. Here are some examples.

EXAMPLE C–1

If $y = x^3$ and $x = t^4$, find $\ddot{y}$, the second derivative of y with respect to time.

SOLUTION

Using the chain rule, Eq. C–1,

$$\dot{y} = 3x^2\dot{x}$$

To obtain the second time derivative we must use the product rule since x and $\dot{x}$ are both functions of time, and also, for $3x^2$ the chain rule must be applied. Thus, with $u = 3x^2$ and $v = \dot{x}$, we have

$$\ddot{y} = [6x\dot{x}]\dot{x} + 3x^2[\ddot{x}]$$

$$= 3x[2\dot{x}^2 + x\ddot{x}]$$

Since $x = t^4$, then $\dot{x} = 4t^3$ and $\ddot{x} = 12t^2$ so that

$$\ddot{y} = 3(t^4)[2(4t^3)^2 + t^4(12t^2)]$$

$$= 132t^{10}$$

Note that this result can also be obtained by combining the functions, then taking the time derivatives, that is,

$$y = x^3 = (t^4)^3 = t^{12}$$

$$\dot{y} = 12t^{11}$$

$$\ddot{y} = 132t^{10}$$

EXAMPLE C–2

If $y = xe^x$, find $\ddot{y}$.

SOLUTION

Since x and e^x are both functions of time the product and chain rules must be applied. Have $u = x$ and $v = e^x$.

$$\dot{y} = [\dot{x}]e^x + x[e^x\dot{x}]$$

The second time derivative also requires application of the product and chain rules. Note that the product rule applies to the three time variables in the last term, i.e., x, e^x, and $\dot{x}$.

$$\ddot{y} = \left\{[\ddot{x}]e^x + \dot{x}[e^x\dot{x}]\right\} + \left\{[\dot{x}]e^x\dot{x} + x[e^x\dot{x}]\dot{x} + xe^x[\ddot{x}]\right\}$$

$$= e^x[\ddot{x}(1 + x) + \dot{x}^2(2 + x)]$$

If $x = t^2$ then $\dot{x} = 2t$, $\ddot{x} = 2$ so that in terms in t, we have

$$\ddot{y} = e^{t^2}[2(1 + t^2) + 4t^2(2 + t^2)]$$

C

EXAMPLE C–3

If the path in radial coordinates is given as $r = 5\theta^2$, where θ is a known function of time, find $\ddot{r}$.

SOLUTION

First, using the chain rule then the chain and product rules where $u = 10\theta$ and $v = \dot{\theta}$, we have

$$
\begin{aligned}
r &= 5\theta^2 \\
\dot{r} &= 10\theta\dot{\theta} \\
\ddot{r} &= 10[(\dot{\theta})\dot{\theta} + \theta(\ddot{\theta})] \\
&= 10\dot{\theta}^2 + 10\theta\ddot{\theta}
\end{aligned}
$$

EXAMPLE C–4

If $r^2 = 6\theta^3$, find $\ddot{r}$.

SOLUTION

Here the chain and product rules are applied as follows.

$$
\begin{aligned}
r^2 &= 6\theta^3 \\
2r\dot{r} &= 18\theta^2\dot{\theta} \\
2[(\dot{r})\dot{r} + r(\ddot{r})] &= 18[(2\theta\dot{\theta})\dot{\theta} + \theta^2(\ddot{\theta})] \\
\dot{r}^2 + r\ddot{r} &= 9(2\theta\dot{\theta}^2 + \theta^2\ddot{\theta})
\end{aligned}
$$

To find $\ddot{r}$ at a specified value of θ which is a known function of time, we can first find $\dot{\theta}$ and $\ddot{\theta}$. Then using these values, evaluate r from the first equation, $\dot{r}$ from the second equation and $\ddot{r}$ using the last equation.

Fundamental Problems Partial Solutions And Answers

Chapter 12

F12–1. $v = v_0 + a_c t$
$10 = 35 + a_c(15)$
$a_c = -1.67\ \text{m/s}^2 = 1.67\ \text{m/s}^2 \leftarrow$ *Ans.*

F12–2. $s = s_0 + v_0 t + \frac{1}{2}a_c t^2$
$0 = 0 + 15t + \frac{1}{2}(-9.81)t^2$
$t = 3.06\ \text{s}$ *Ans.*

F12–3. $ds = v\,dt$
$$\int_0^s ds = \int_0^t \left(4t - 3t^2\right)dt$$
$s = \left(2t^2 - t^3\right)\text{m}$
$s = 2\left(4^2\right) - 4^3$
$= -32\ \text{m} = 32\ \text{m} \leftarrow$ *Ans.*

F12–4. $a = \frac{dv}{dt} = \frac{d}{dt}\left(0.5t^3 - 8t\right)$
$a = \left(1.5t^2 - 8\right)\ \text{m/s}^2$
When $t = 2$ s,
$a = 1.5\left(2^2\right) - 8 = -2\ \text{m/s}^2 = 2\ \text{m/s}^2 \leftarrow$ *Ans.*

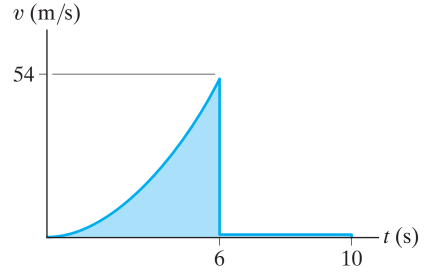
F12–5. $v = \frac{ds}{dt} = \frac{d}{dt}(2t^2 - 8t + 6) = (4t - 8)\ \text{m/s}$
$v = 0 = (4t - 8)$
$t = 2\ \text{s}$ *Ans.*
$s|_{t=0} = 2\left(0^2\right) - 8(0) + 6 = 6\ \text{m}$
$s|_{t=2} = 2\left(2^2\right) - 8(2) + 6 = -2\ \text{m}$
$s|_{t=3} = 2\left(3^2\right) - 8(3) + 6 = 0\ \text{m}$
$(\Delta s)_{\text{Tot}} = 8\ \text{m} + 2\ \text{m} = 10\ \text{m}$ *Ans.*

F12–6. $\int v\,dv = \int a\,ds$
$$\int_{5\ \text{m/s}}^{v} v\,dv = \int_0^s (10 - 0.2s)ds$$
$v = \left(\sqrt{20s - 0.2s^2 + 25}\right)\ \text{m/s}$
At $s = 10$ m,
$v = \sqrt{20(10) - 0.2(10^2) + 25}$
$= 14.3\ \text{m/s} \rightarrow$ *Ans.*

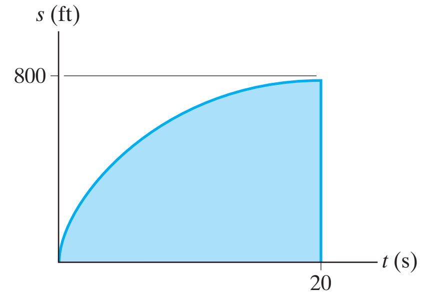
F12–7. $v = \int (4t^2 - 2)\,dt$
$v = \frac{4}{3}t^3 - 2t + C_1$
$s = \int \left(\frac{4}{3}t^3 - 2t + C_1\right)dt$
$s = \frac{1}{3}t^4 - t^2 + C_1 t + C_2$
$t = 0, s = -2, C_2 = -2$
$t = 2, s = -20, C_1 = -9.67$
$t = 4, s = 28.7\ \text{m}$ *Ans.*

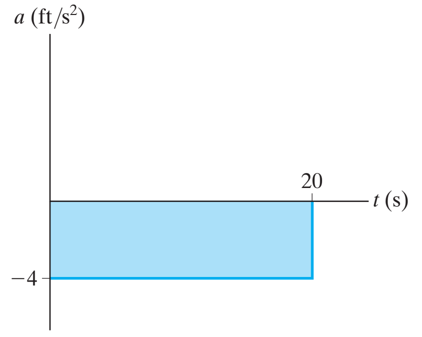
F12–8. $a = v\frac{dv}{ds}$
$= \left(20 - 0.05s^2\right)\left(-0.1s\right)$
At $s = 15$ m,
$a = -13.1\ \text{m/s}^2 = 13.1\ \text{m/s}^2 \leftarrow$ *Ans.*

F12–9. $v = \frac{ds}{dt} = \frac{d}{dt}\left(0.5t^3\right) = 1.5t^2$
$v = \frac{ds}{dt} = \frac{d}{dt}(108) = 0$ *Ans.*



F12–10. $ds = v\,dt$
$$\int_0^s ds = \int_0^t (-4t + 80)\,dt$$
$s = -2t^2 + 80t$
$a = \frac{dv}{dt} = \frac{d}{dt}(-4t + 80) = -4\ \text{ft/s}^2 = 4\ \text{ft/s}^2 \leftarrow$
Also,
$a = \frac{\Delta v}{\Delta t} = \frac{0 - 80\ \text{ft/s}}{20\ \text{s} - 0} = -4\ \text{ft/s}^2$

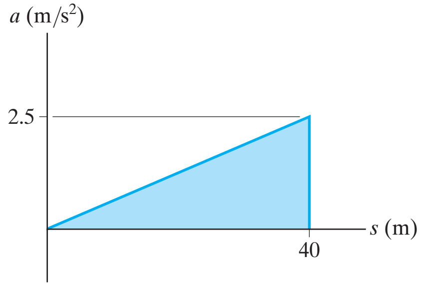




F12–11. $a\,ds = v\,dv$

$$a = v\frac{dv}{ds} = 0.25s\frac{d}{ds}(0.25s) = 0.0625s$$

$$a|_{s=40\text{ m}} = 0.0625(40\text{ m}) = 2.5\text{ m/s}^2 \rightarrow$$



F12–12. For $0 \le s \le 10$ m

$$a = s$$

$$\int_0^v v\,dv = \int_0^s s\,ds$$

$$v = s$$

at $s = 10$ m, $v = 10$ m

For $10\text{ m} \le s \le 15$

$$a = 10$$

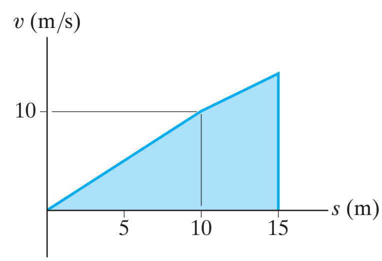
$$\int_{10}^v v\,dv = \int_{10}^s 10\,ds$$

$$\frac{1}{2}v^2 - 50 = 10s - 100$$

$$v = \sqrt{20s - 100}$$

at $s = 15$ m

$$v = 14.1\text{ m/s} \qquad \textit{Ans.}$$



F12–13. $0 \le t < 5$ s,

$$dv = a\,dt \qquad \int_0^v dv = \int_0^t 20\,dt$$

$$v = (20t)\text{ m/s}$$

$5\text{ s} < t \le t'$,

$$(\overset{+}{\rightarrow}) \quad dv = a\,dt \qquad \int_{100\text{ m/s}}^v dv = \int_{5\text{ s}}^t -10\,dt$$

$$v - 100 = (50 - 10t)\text{ m/s},$$

$$0 = 150 - 10t'$$

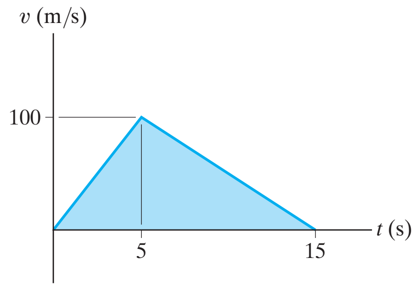
$$t' = 15\text{ s}$$

Also,

$\Delta v = 0 =$ Area under the $a{-}t$ graph

$$0 = (20\text{ m/s}^2)(5\text{ s}) + [-(10\text{ m/s})(t' - 5)\text{ s}]$$

$$t' = 15\text{ s}$$



F12–14. $0 \le t \le 5$ s,

$$ds = v\,dt \qquad \int_0^s ds = \int_0^t 30t\,dt$$

$$s|_0^s = 15t^2|_0^t$$

$$s = (15t^2)\text{ m}$$

$5\text{ s} < t \le 15$ s,

$$(\overset{+}{\rightarrow})\, ds = v\,dt; \qquad \int_{375\text{ m}}^s ds = \int_{5\text{ s}}^t (-15t + 225)dt$$

$$s = (-7.5t^2 + 225t - 562.5)\text{ m}$$

$$s = (-7.5)(15)^2 + 225(15) - 562.5\text{ m}$$

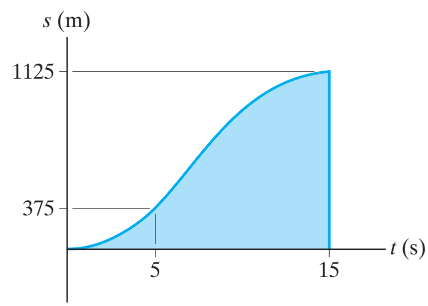
$$= 1125\text{ m} \qquad \textit{Ans.}$$

Also,

$\Delta s =$ Area under the $v{-}t$ graph

$$= \tfrac{1}{2}(150\text{ m/s})(15\text{ s})$$

$$= 1125\text{ m} \qquad \textit{Ans.}$$



F12–15. $\displaystyle\int_0^x dx = \int_0^t 32t\,dt$

$$x = \left(16t^2\right)\text{ m} \qquad (1)$$

$$\int_0^y dy = \int_0^t 8\,dt$$

$$t = \frac{y}{8} \qquad (2)$$

Substituting Eq. (2) into Eq. (1), get

$y = 2\sqrt{x}$ *Ans.*

F12–16. $y = 0.75(8t) = 6t$

$v_x = \dot{x} = \frac{dx}{dt} = \frac{d}{dt}(8t) = 8 \text{ m/s} \rightarrow$

$v_y = \dot{y} = \frac{dy}{dt} = \frac{d}{dt}(6t) = 6 \text{ m/s} \uparrow$

The magnitude of the particle's velocity is

$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(8 \text{ m/s})^2 + (6 \text{ m/s})^2}$

$= 10 \text{ m/s}$ *Ans.*

F12–17. $y = (4t^2)$ m

$v_x = \dot{x} = \frac{d}{dt}\left(4t^4\right) = \left(16t^3\right) \text{ m/s} \rightarrow$

$v_y = \dot{y} = \frac{d}{dt}\left(4t^2\right) = (8t) \text{ m/s} \uparrow$

When $t = 0.5$ s,

$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(2 \text{ m/s})^2 + (4 \text{ m/s})^2}$

$= 4.47 \text{ m/s}$ *Ans.*

$a_x = \dot{v}_x = \frac{d}{dt}\left(16t^3\right) = \left(48t^2\right) \text{ m/s}^2$

$a_y = \dot{v}_y = \frac{d}{dt}(8t) = 8 \text{ m/s}^2$

When $t = 0.5$ s,

$a = \sqrt{a_x^2 + a_y^2} = \sqrt{(12 \text{ m/s}^2)^2 + (8 \text{ m/s}^2)^2}$

$= 14.4 \text{ m/s}^2$ *Ans.*

F12–18. $y = 0.5x$

$\dot{y} = 0.5\dot{x}$

$v_y = t^2$

When $t = 4$ s,

$v_x = 32 \text{ m/s} \qquad v_y = 16 \text{ m/s}$

$v = \sqrt{v_x^2 + v_y^2} = 35.8 \text{ m/s}$ *Ans.*

$a_x = \dot{v}_x = 4t$

$a_y = \dot{v}_y = 2t$

When $t = 4$ s,

$a_x = 16 \text{ m/s}^2 \qquad a_y = 8 \text{ m/s}^2$

$a = \sqrt{a_x^2 + a_y^2} = \sqrt{16^2 + 8^2} = 17.9 \text{ m/s}^2$ *Ans.*

F12–19. $v_y = \dot{y} = 0.5\,x\,\dot{x} = 0.5(8)(8) = 32 \text{ m/s}$

Thus,

$v = \sqrt{v_x^2 + v_y^2} = 33.0 \text{ m/s}$ *Ans.*

$a_y = \dot{v}_y = 0.5\,\dot{x}^2 + 0.5\,x\ddot{x}$

$= 0.5(8)^2 + 0.5(8)(4)$

$= 48 \text{ m/s}^2$

$a_x = 4 \text{ m/s}^2$

Thus,

$a = \sqrt{a_x^2 + a_y^2} = \sqrt{4^2 + 48^2} = 48.2 \text{ m/s}^2$ *Ans.*

F12–20. $\dot{y} = 0.1x\dot{x}$

$v_y = 0.1(5)(-3) = -1.5 \text{ m/s} = 1.5 \text{ m/s} \downarrow$ *Ans.*

$\ddot{y} = 0.1\left[\dot{x}\dot{x} + x\ddot{x}\right]$

$a_y = 0.1\left[(-3)^2 + 5(-1.5)\right] = 0.15 \text{ m/s}^2 \uparrow$ *Ans.*

F12–21. $(v_B)_y^2 = (v_A)_y^2 + 2a_y(y_B - y_A)$

$0^2 = (5 \text{ m/s})^2 + 2(-9.81 \text{ m/s}^2)(h - 0)$

$h = 1.27$ m *Ans.*

F12–22. $y_C = y_A + (v_A)_y t_{AC} + \frac{1}{2}a_y t_{AC}^2$

$0 = 0 + (5 \text{ m/s})t_{AC} + \frac{1}{2}(-9.81 \text{ m/s}^2)t_{AC}^2$

$t_{AC} = 1.0194$ s

$(v_C)_y = (v_A)_y + a_y t_{AC}$

$(v_C)_y = 5 \text{ m/s} + (-9.81 \text{ m/s}^2)(1.0194 \text{ s})$

$= -5 \text{ m/s} = 5 \text{ m/s} \downarrow$

$v_C = \sqrt{(v_C)_x^2 + (v_C)_y^2}$

$= \sqrt{(8.660 \text{ m/s})^2 + (5 \text{ m/s})^2} = 10 \text{ m/s}$ *Ans.*

$R = x_A + (v_A)_x t_{AC} = 0 + (8.660 \text{ m/s})(1.0194 \text{ s})$

$= 8.83$ m *Ans.*

F12–23. $s = s_0 + v_0 t$

$10 = 0 + v_A \cos 30° t$

$s = s_0 + v_0 t + \frac{1}{2}a_c t^2$

$3 = 1.5 + v_A \sin 30° t + \frac{1}{2}(-9.81)t^2$

$t = 0.9334 \text{ s}, \quad v_A = 12.4 \text{ m/s}$ *Ans.*

F12–24. $s = s_0 + v_0 t$

$R\left(\frac{4}{5}\right) = 0 + 20\left(\frac{3}{5}\right)t$

$s = s_0 + v_0 t + \frac{1}{2}a_c t^2$

$-R\left(\frac{3}{5}\right) = 0 + 20\left(\frac{4}{5}\right)t + \frac{1}{2}(-9.81)t^2$

$t = 5.10$ s

$R = 76.5$ m *Ans.*

F12–25. $x_B = x_A + (v_A)_x t_{AB}$

$12 \text{ ft} = 0 + (0.8660\, v_A)t_{AB}$

$v_A t_{AB} = 13.856$ (1)

$y_B = y_A + (v_A)_y t_{AB} + \frac{1}{2}a_y t_{AB}^2$

$$(8-3)\text{ ft} = 0 + 0.5v_A t_{AB} + \tfrac{1}{2}(-32.2\text{ ft/s}^2)t_{AB}^2$$

Using Eq. (1),

$$5 = 0.5(13.856) - 16.1\,t_{AB}^2$$

$$t_{AB} = 0.3461\text{ s}$$

$$v_A = 40.0\text{ ft/s} \qquad \textit{Ans.}$$

F12–26. $y_B = y_A + (v_A)_y t_{AB} + \frac{1}{2}a_y t_{AB}^2$

$$-150\text{ m} = 0 + (90\text{ m/s})t_{AB} + \tfrac{1}{2}(-9.81\text{ m/s}^2)t_{AB}^2$$

$$t_{AB} = 19.89\text{ s}$$

$$x_B = x_A + (v_A)_x t_{AB}$$

$$R = 0 + 120\text{ m/s}(19.89\text{ s}) = 2386.37\text{ m}$$

$$= 2.39\text{ km} \qquad \textit{Ans.}$$

F12–27. $a_t = \dot{v} = \frac{dv}{dt} = \frac{d}{dt}(0.0625t^2) = (0.125t)\text{ m/s}^2\Big|_{t=10\text{ s}}$

$$= 1.25\text{ m/s}^2$$

$$a_n = \frac{v^2}{\rho} = \frac{(0.0625t^2)^2}{40\text{ m}} = \left[97.656(10^{-6})t^4\right]\text{ m/s}^2\Big|_{t=10\text{ s}}$$

$$= 0.9766\text{ m/s}^2$$

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{(1.25\text{ m/s}^2)^2 + (0.9766\text{ m/s}^2)^2}$$

$$= 1.59\text{ m/s}^2 \qquad \textit{Ans.}$$

F12–28. $v = 2s\,\Big|_{s=10} = 20\text{ m/s}$

$$a_n = \frac{v^2}{\rho} = \frac{(20\text{ m/s})^2}{50\text{ m}} = 8\text{ m/s}^2$$

$$a_t = v\frac{dv}{ds} = 4s\Big|_{s=10} = 40\text{ m/s}^2$$

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{\left(40\text{ m/s}^2\right)^2 + \left(8\text{ m/s}^2\right)^2}$$

$$= 40.8\text{ m/s}^2 \qquad \textit{Ans.}$$

F12–29. $v_C^2 = v_A^2 + 2a_t(s_C - s_A)$

$$(15\text{ m/s})^2 = (25\text{ m/s})^2 + 2a_t(300\text{ m} - 0)$$

$$a_t = -0.6667\text{ m/s}^2$$

$$v_B^2 = v_A^2 + 2a_t(s_B - s_A)$$

$$v_B^2 = (25\text{ m/s})^2 + 2(-0.6667\text{ m/s}^2)(250\text{ m} - 0)$$

$$v_B = 17.08\text{ m/s}$$

$$(a_B)_n = \frac{v_B^2}{\rho} = \frac{(17.08\text{ m/s})^2}{300\text{ m}} = 0.9722\text{ m/s}^2$$

$$a_B = \sqrt{(a_B)_t^2 + (a_B)_n^2}$$

$$= \sqrt{(-0.6667\text{ m/s}^2)^2 + (0.9722\text{ m/s}^2)^2}$$

$$= 1.18\text{ m/s}^2 \qquad \textit{Ans.}$$

F12–30. $\tan\theta = \frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{24}x^2\right) = \frac{1}{12}x$

$$\theta = \tan^{-1}\left(\tfrac{1}{12}x\right)\Big|_{x=10\text{ ft}}$$

$$= \tan^{-1}\left(\tfrac{10}{12}\right) = 39.81° = 39.8° \qquad \textit{Ans.}$$

$$\rho = \frac{\left[1 + (dy/dx)^2\right]^{3/2}}{|d^2y/dx^2|} = \frac{\left[1 + \left(\frac{1}{12}x\right)^2\right]^{3/2}}{\left|\frac{1}{12}\right|}\Bigg|_{x=10\text{ ft}}$$

$$= 26.468\text{ ft}$$

$$a_n = \frac{v^2}{\rho} = \frac{(20\text{ ft/s})^2}{26.468\text{ ft}} = 15.11\text{ ft/s}^2$$

$$a = \sqrt{(a_t)^2 + (a_n)^2} = \sqrt{(6\text{ ft/s}^2)^2 + (15.11\text{ ft/s}^2)^2}$$

$$= 16.3\text{ ft/s}^2 \qquad \textit{Ans.}$$

F12–31. $(a_B)_t = -0.001s = (-0.001)(300\text{ m})\left(\frac{\pi}{2}\text{ rad}\right)\text{ m/s}^2$

$$= -0.4712\text{ m/s}^2$$

$$v\,dv = a_t\,ds$$

$$\int_{25\text{ m/s}}^{v_B} v\,dv = \int_0^{150\pi\text{ m}} -0.001s\,ds$$

$$v_B = 20.07\text{ m/s}$$

$$(a_B)_n = \frac{v_B^2}{\rho} = \frac{(20.07\text{ m/s})^2}{300\text{ m}} = 1.343\text{ m/s}^2$$

$$a_B = \sqrt{(a_B)_t^2 + (a_B)_n^2}$$

$$= \sqrt{(-0.4712\text{ m/s}^2)^2 + (1.343\text{ m/s}^2)^2}$$

$$= 1.42\text{ m/s}^2 \qquad \textit{Ans.}$$

F12–32. $a_t\,ds = v\,dv$

$$a_t = v\,\frac{dv}{ds} = (0.2s)(0.2) = (0.04s)\text{ m/s}^2$$

$$a_t = 0.04(50\text{ m}) = 2\text{ m/s}^2$$

$$v = 0.2(50\text{ m}) = 10\text{ m/s}$$

$$a_n = \frac{v^2}{\rho} = \frac{(10\text{ m/s})^2}{500\text{ m}} = 0.2\text{ m/s}^2$$
$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{(2\text{ m/s}^2)^2 + (0.2\text{ m/s}^2)^2}$$
$$= 2.01\text{ m/s}^2 \qquad \textit{Ans.}$$

F12–33. $v_r = \dot{r} = 0$
$$v_\theta = r\dot{\theta} = (400\dot{\theta})\text{ ft/s}$$
$$v = \sqrt{v_r^2 + v_\theta^2}$$
$$55\text{ ft/s} = \sqrt{0^2 + [(400\dot{\theta})\text{ ft/s}]^2}$$
$$\dot{\theta} = 0.1375\text{ rad/s} \qquad \textit{Ans.}$$

F12–34. $r = 0.1t^3\big|_{t=1.5\text{ s}} = 0.3375\text{ m}$
$$\dot{r} = 0.3t^2\big|_{t=1.5\text{ s}} = 0.675\text{ m/s}$$
$$\ddot{r} = 0.6t\big|_{t=1.5\text{ s}} = 0.900\text{ m/s}^2$$
$$\theta = 4t^{3/2}\big|_{t=1.5\text{ s}} = 7.348\text{ rad}$$
$$\dot{\theta} = 6t^{1/2}\big|_{t=1.5\text{ s}} = 7.348\text{ rad/s}$$
$$\ddot{\theta} = 3t^{-1/2}\big|_{t=1.5\text{ s}} = 2.449\text{ rad/s}^2$$
$$v_r = \dot{r} = 0.675\text{ m/s}$$
$$v_\theta = r\dot{\theta} = (0.3375\text{ m})(7.348\text{ rad/s}) = 2.480\text{ m/s}$$
$$a_r = \ddot{r} - r\dot{\theta}^2$$
$$= (0.900\text{ m/s}^2) - (0.3375\text{ m})(7.348\text{ rad/s})^2$$
$$= -17.325\text{ m/s}^2$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = (0.3375\text{ m})(2.449\text{ rad/s}^2)$$
$$+ 2(0.675\text{ m/s})(7.348\text{ rad/s}) = 10.747\text{ m/s}^2$$
$$v = \sqrt{v_r^2 + v_\theta^2}$$
$$= \sqrt{(0.675\text{ m/s})^2 + (2.480\text{ m/s})^2}$$
$$= 2.57\text{ m/s} \qquad \textit{Ans.}$$
$$a = \sqrt{a_r^2 + a_\theta^2}$$
$$= \sqrt{(-17.325\text{ m/s}^2)^2 + (10.747\text{ m/s}^2)^2}$$
$$= 20.4\text{ m/s}^2 \qquad \textit{Ans.}$$

F12–35. $r = 2\theta$
$$\dot{r} = 2\dot{\theta}$$
$$\ddot{r} = 2\ddot{\theta}$$
At $\theta = \pi/4$ rad,
$$r = 2\left(\tfrac{\pi}{4}\right) = \tfrac{\pi}{2}\text{ ft}$$
$$\dot{r} = 2(3\text{ rad/s}) = 6\text{ ft/s}$$
$$\ddot{r} = 2(1\text{ rad/s}) = 2\text{ ft/s}^2$$
$$a_r = \ddot{r} - r\dot{\theta}^2 = 2\text{ ft/s}^2 - \left(\tfrac{\pi}{2}\text{ ft}\right)(3\text{ rad/s})^2$$
$$= -12.14\text{ ft/s}^2$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$
$$= \left(\tfrac{\pi}{2}\text{ ft}\right)(1\text{ rad/s}^2) + 2(6\text{ ft/s})(3\text{ rad/s})$$
$$= 37.57\text{ ft/s}^2$$
$$a = \sqrt{a_r^2 + a_\theta^2}$$
$$= \sqrt{(-12.14\text{ ft/s}^2)^2 + (37.57\text{ ft/s}^2)^2}$$
$$= 39.5\text{ ft/s}^2 \qquad \textit{Ans.}$$

F12–36. $r = e^\theta$
$$\dot{r} = e^\theta\dot{\theta}$$
$$\ddot{r} = e^\theta\ddot{\theta} + e^\theta\dot{\theta}^2$$
$$a_r = \ddot{r} - r\dot{\theta}^2 = (e^\theta\ddot{\theta} + e^\theta\dot{\theta}^2) - e^\theta\dot{\theta}^2 = e^{\pi/4}(4)$$
$$= 8.77\text{ m/s}^2 \qquad \textit{Ans.}$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = (e^\theta\ddot{\theta}) + (2(e^\theta\dot{\theta})\dot{\theta}) = e^\theta(\ddot{\theta} + 2\dot{\theta}^2)$$
$$= e^{\pi/4}(4 + 2(2)^2)$$
$$= 26.3\text{ m/s}^2 \qquad \textit{Ans.}$$

F12–37. $r = [0.2(1 + \cos\theta)]\text{ m}\big|_{\theta=30°} = 0.3732\text{ m}$
$$\dot{r} = \left[-0.2\,(\sin\theta)\dot{\theta}\right]\text{ m/s}\big|_{\theta=30°}$$
$$= -0.2\sin 30°(3\text{ rad/s})$$
$$= -0.3\text{ m/s}$$
$$v_r = \dot{r} = -0.3\text{ m/s}$$
$$v_\theta = r\dot{\theta} = (0.3732\text{ m})(3\text{ rad/s}) = 1.120\text{ m/s}$$
$$v = \sqrt{v_r^2 + v_\theta^2} = \sqrt{(-0.3\text{ m/s})^2 + (1.120\text{ m/s})^2}$$
$$= 1.16\text{ m/s} \qquad \textit{Ans.}$$

F12–38. $30\text{ m} = r\sin\theta$
$$r = \left(\frac{30\text{ m}}{\sin\theta}\right) = (30\csc\theta)\text{ m}$$
$$r = (30\csc\theta)\big|_{\theta=45°} = 42.426\text{ m}$$
$$\dot{r} = -30\csc\theta\,\text{ctn}\,\theta\,\dot{\theta}\big|_{\theta=45°} = -\left(42.426\dot{\theta}\right)\text{ m/s}$$
$$v_r = \dot{r} = -(42.426\dot{\theta})\text{ m/s}$$
$$v_\theta = r\dot{\theta} = (42.426\dot{\theta})\text{ m/s}$$
$$v = \sqrt{v_r^2 + v_\theta^2}$$
$$2 = \sqrt{(-42.426\dot{\theta})^2 + (42.426\dot{\theta})^2}$$
$$\dot{\theta} = 0.0333\text{ rad/s} \qquad \textit{Ans.}$$

F12–39. $l_T = 3s_D + s_A$
$$0 = 3v_D + v_A$$
$$0 = 3v_D + 3\text{ m/s}$$
$$v_D = -1\text{ m/s} = 1\text{ m/s}\uparrow \qquad \textit{Ans.}$$

F12–40. $s_B + 2s_A + 2h = l$

$v_B + 2v_A = 0$

$6 + 2v_A = 0 \quad v_A = -3 \text{ m/s} = 3 \text{ m/s} \uparrow$ *Ans.*

F12–41. $3s_A + s_B = l$

$3v_A + v_B = 0$

$3v_A + 1.5 = 0 \quad v_A = -0.5 \text{ m/s} = 0.5 \text{ m/s} \uparrow$ *Ans.*

F12–42. $l_T = 4\, s_A + s_F$

$0 = 4\, v_A + v_F$

$0 = 4\, v_A + 3 \text{ m/s}$

$v_A = -0.75 \text{ m/s} = 0.75 \text{ m/s} \uparrow$ *Ans.*

F12–43. $s_A + 2(s_A - a) + (s_A - s_P) = l$

$4s_A - s_P = l + 2a$

$4v_A - v_P = 0$

$4v_A - (-4) = 0$

$4v_A + 4 = 0 \qquad v_A = -1 \text{ m/s} = 1 \text{ m/s} \nearrow$ *Ans.*

F12–44. $s_C + s_B = l_{CED}$ (1)

$(s_A - s_C) + (s_B - s_C) + s_B = l_{ACDF}$

$s_A + 2s_B - 2s_C = l_{ACDF}$ (2)

Thus

$v_C + v_B = 0$

$v_A + 2v_B - 2v_C = 0$

Eliminating v_C,

$v_A + 4v_B = 0$

Thus,

$4 \text{ ft/s} + 4v_B = 0$

$v_B = -1 \text{ ft/s} = 1 \text{ ft/s} \uparrow$ *Ans.*

F12–45. $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

$100\mathbf{i} = 80\mathbf{j} + \mathbf{v}_{B/A}$

$\mathbf{v}_{B/A} = 100\mathbf{i} - 80\mathbf{j}$

$\mathbf{v}_{B/A} = \sqrt{(v_{B/A})_x^2 + (v_{B/A})_y^2}$

$= \sqrt{(100 \text{ km/h})^2 + (-80 \text{ km/h})^2}$

$= 128 \text{ km/h}$ *Ans.*

$\theta = \tan^{-1}\left[\dfrac{(v_{B/A})_y}{(v_{B/A})_x}\right] = \tan^{-1}\left(\dfrac{80 \text{ km/h}}{100 \text{ km/h}}\right) = 38.7° \measuredangle$ *Ans.*

F12–46. $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

$(-400\mathbf{i} - 692.82\mathbf{j}) = (650\mathbf{i}) + \mathbf{v}_{B/A}$

$\mathbf{v}_{B/A} = [-1050\mathbf{i} - 692.82\mathbf{j}] \text{ km/h}$

$\mathbf{v}_{B/A} = \sqrt{(v_{B/A})_x^2 + (v_{B/A})_y^2}$

$= \sqrt{(1050 \text{ km/h})^2 + (692.82 \text{ km/h})^2}$

$= 1258 \text{ km/h}$ *Ans.*

$\theta = \tan^{-1}\left[\dfrac{(v_{B/A})_y}{(v_{B/A})_x}\right] = \tan^{-1}\left(\dfrac{692.82 \text{ km/h}}{1050 \text{ km/h}}\right) = 33.4° \measuredangle$ *Ans.*

F12–47. $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

$(5\mathbf{i} + 8.660\mathbf{j}) = (12.99\mathbf{i} + 7.5\mathbf{j}) + \mathbf{v}_{B/A}$

$\mathbf{v}_{B/A} = [-7.990\mathbf{i} + 1.160\mathbf{j}] \text{ m/s}$

$v_{B/A} = \sqrt{(-7.990 \text{ m/s})^2 + (1.160 \text{ m/s})^2}$

$= 8.074 \text{ m/s}$

$d_{AB} = v_{B/A}t = (8.074 \text{ m/s})(4 \text{ s}) = 32.3 \text{ m}$ *Ans.*

F12–48. $\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$

$-20 \cos 45°\mathbf{i} + 20 \sin 45°\mathbf{j} = 65\mathbf{i} + \mathbf{v}_{A/B}$

$\mathbf{v}_{A/B} = -79.14\mathbf{i} + 14.14\mathbf{j}$

$\mathbf{v}_{A/B} = \sqrt{(-79.14)^2 + (14.14)^2}$

$= 80.4 \text{ km/h}$ *Ans.*

$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$

$\dfrac{(20)^2}{0.1} \cos 45°\mathbf{i} + \dfrac{(20)^2}{0.1} \sin 45°\mathbf{j} = 1200\mathbf{i} + \mathbf{a}_{A/B}$

$\mathbf{a}_{A/B} = 1628\mathbf{i} + 2828\mathbf{j}$

$a_{A/B} = \sqrt{(1628)^2 + (2828)^2}$

$= 3.26(10^3) \text{ km/h}^2$ *Ans.*

Chapter 13

F13–1. $s = s_0 + v_0 t + \frac{1}{2} a_c t^2$

$6 \text{ m} = 0 + 0 + \frac{1}{2} a(3 \text{ s})^2$

$a = 1.333 \text{ m/s}^2$

$\Sigma F_y = ma_y; \quad N_A - 20(9.81) \text{ N} \cos 30° = 0$

$N_A = 169.91 \text{ N}$

$\Sigma F_x = ma_x; \quad T - 20(9.81) \text{ N} \sin 30°$

$- 0.3(169.91 \text{ N}) = (20 \text{ kg})(1.333 \text{ m/s}^2)$

$T = 176 \text{ N}$ *Ans.*

F13–2. $(F_f)_{max} = \mu_s N_A = 0.3(245.25 \text{ N}) = 73.575 \text{ N}.$
Since $F = 100 \text{ N} > (F_f)_{max}$ when $t = 0$, the crate will start to move immediately after $\mathbf{F}$ is applied.

$+\uparrow \Sigma F_y = ma_y; \quad N_A - 25(9.81) \text{ N} = 0$

$N_A = 245.25 \text{ N}$

$\overset{+}{\rightarrow} \Sigma F_x = ma_x;$

$10t^2 + 100 - 0.25(245.25 \text{ N}) = (25 \text{ kg})a$

$a = (0.4t^2 + 1.5475) \text{ m/s}^2$

$dv = a\, dt$

$\int_0^v dv = \int_0^{4 \text{ s}} \left(0.4t^2 + 1.5475\right) dt$

$v = 14.7 \text{ m/s} \rightarrow$ *Ans.*

F13–3. $\xrightarrow{+} \Sigma F_x = ma_x;$

$\left(\frac{4}{5}\right)500\text{ N} - (500s)\text{N} = (10\text{ kg})a$

$a = (40 - 50s)\text{ m/s}^2$

$v\,dv = a\,ds$

$\int_0^v v\,dv = \int_0^{0.5\text{ m}} (40 - 50s)\,ds$

$\frac{v^2}{2}\Big|_0^v = \left(40s - 25s^2\right)\Big|_0^{0.5\text{ m}}$

$v = 5.24\text{ m/s}$ *Ans.*

F13–4. $\xrightarrow{+} \Sigma F_x = ma_x \quad 100(s + 1)\text{ N} = (2000\text{ kg})a$

$a = (0.05(s + 1))\text{ m/s}^2$

$v\,dv = a\,ds$

$\int_0^v v\,dv = \int_0^{10\text{ m}} 0.05(s + 1)\,ds$

$v = 2.45\text{ m/s}$

F13–5. $F_{sp} = k(l - l_0) = (200\text{ N/m})(0.5\text{ m} - 0.3\text{ m})$

$= 40\text{ N}$

$\theta = \tan^{-1}\left(\frac{0.3\text{ m}}{0.4\text{ m}}\right) = 36.86°$

$\xrightarrow{+} \Sigma F_x = ma_x;$

$100\text{ N} - (40\text{ N})\cos 36.86° = (25\text{ kg})a$

$a = 2.72\text{ m/s}^2$

F13–6. Blocks A and B:

$\xrightarrow{+} \Sigma F_x = ma_x;\ 6 = \frac{70}{32.2}a;\ a = 2.76\text{ ft/s}^2$

Check if slipping occurs between A and B.

$\xrightarrow{+} \Sigma F_x = ma_x;\ 6 - F = \frac{20}{32.2}(2.76);$

$F = 4.29\text{ lb} < 0.4(20) = 8\text{ lb}$

$a_A = a_B = 2.76\text{ ft/s}^2$ *Ans.*

F13–7. $\Sigma F_n = m\frac{v^2}{\rho};\ (0.3)m(9.81) = m\frac{v^2}{2}$

$v = 2.43\text{ m/s}$ *Ans.*

F13–8. $+\downarrow \Sigma F_n = ma_n;\ m(32.2) = m\left(\frac{v^2}{250}\right)$

$v = 89.7\text{ ft/s}$ *Ans.*

F13–9. $+\downarrow \Sigma F_n = ma_n;\ 150 + N_p = \frac{150}{32.2}\left(\frac{(120)^2}{400}\right)$

$N_p = 17.7\text{ lb}$ *Ans.*

F13–10. $\xleftarrow{+} \Sigma F_n = ma_n;$

$N_c \sin 30° + 0.2\,N_c \cos 30° = m\frac{v^2}{500}$

$+\uparrow \Sigma F_b = 0;$

$N_c \cos 30° - 0.2N_c \sin 30° - m(32.2) = 0$

$v = 119\text{ ft/s}$ *Ans.*

F13–11. $\Sigma F_t = ma_t;\quad 10(9.81)\text{ N}\cos 45° = (10\text{ kg})a_t$

$a_t = 6.94\text{ m/s}^2$ *Ans.*

$\Sigma F_n = ma_n;$

$T - 10(9.81)\text{ N}\sin 45° = (10\text{ kg})\frac{(3\text{ m/s})^2}{2\text{ m}}$

$T = 114\text{ N}$ *Ans.*

F13–12. $\Sigma F_n = ma_n;$

$F_n = (500\text{ kg})\frac{(15\text{ m/s})^2}{200\text{ m}} = 562.5\text{ N}$

$\Sigma F_t = ma_t;$

$F_t = (500\text{ kg})(1.5\text{ m/s}^2) = 750\text{ N}$

$F = \sqrt{F_n^2 + F_t^2} = \sqrt{(562.5\text{ N})^2 + (750\text{ N})^2}$

$= 938\text{ N}$ *Ans.*

F13–13. $a_r = \ddot{r} - r\dot{\theta}^2 = 0 - (1.5\text{ m} + (8\text{ m})\sin 45°)\dot{\theta}^2$

$= (-7.157\,\dot{\theta}^2)\text{ m/s}^2$

$\Sigma F_z = ma_z;$

$T\cos 45° - m(9.81) = m(0) \quad T = 13.87\,m$

$\Sigma F_r = ma_r;$

$-(13.87m)\sin 45° = m(-7.157\,\dot{\theta}^2)$

$\dot{\theta} = 1.17\text{ rad/s}$ *Ans.*

F13–14. $\theta = \pi t^2\big|_{t=0.5\text{ s}} = (\pi/4)\text{ rad}$

$\dot{\theta} = 2\pi t\big|_{t=0.5\text{ s}} = \pi\text{ rad/s}$

$\ddot{\theta} = 2\pi\text{ rad/s}^2$

$r = 0.6\sin\theta\big|_{\theta=\pi/4\text{ rad}} = 0.4243\text{ m}$

$\dot{r} = 0.6(\cos\theta)\dot{\theta}\big|_{\theta=\pi/4\text{ rad}} = 1.3329\text{ m/s}$

$\ddot{r} = 0.6\left[(\cos\theta)\ddot{\theta} - (\sin\theta)\dot{\theta}^2\right]\big|_{\theta=\pi/4\text{ rad}} = -1.5216\text{ m/s}^2$

$a_r = \ddot{r} - r\dot{\theta}^2 = -1.5216\text{ m/s}^2 - (0.4243\text{ m})(\pi\text{ rad/s})^2$

$= -5.7089\text{ m/s}^2$

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.4243\text{ m}(2\pi\text{ rad/s}^2)$

$+ 2(1.3329\text{ m/s})(\pi\text{ rad/s})$

$= 11.0404\text{ m/s}^2$

$\Sigma F_r = ma_r;$

$F\cos 45° - N\cos 45° - 0.2(9.81)\cos 45°$

$= 0.2(-5.7089)$

$\Sigma F_\theta = ma_\theta;$

$F\sin 45° + N\sin 45° - 0.2(9.81)\sin 45°$

$= 0.2(11.0404)$

$N = 2.37\text{ N} \qquad F = 2.72\text{ N}$ *Ans.*

F13–15. $r = 50e^{2\theta}\big|_{\theta=\pi/6\text{ rad}} = \left[50e^{2(\pi/6)}\right]\text{ m} = 142.48\text{ m}$

$\dot{r} = 50\left(2e^{2\theta}\,\dot{\theta}\right) = 100e^{2\theta}\,\dot{\theta}\big|_{\theta=\pi/6\text{ rad}}$

$= \left[100e^{2(\pi/6)}(0.05)\right] = 14.248\text{ m/s}$

$\ddot{r} = 100\left((2e^{2\theta}\dot{\theta})\dot{\theta} + e^{2\theta}(\ddot{\theta})\right)\big|_{\theta=\pi/6\text{ rad}}$

$= 100\left[2e^{2(\pi/6)}(0.05^2) + e^{2(\pi/6)}(0.01)\right]$

$= 4.274\text{ m/s}^2$

$a_r = \ddot{r} - r\dot{\theta}^2 = 4.274\text{ m/s}^2 - 142.48\text{ m}(0.05\text{ rad/s})^2$

$= 3.918\text{ m/s}^2$

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 142.48\text{ m}(0.01\text{ rad/s}^2)$

$+ 2(14.248\text{ m/s})(0.05\text{ rad/s})$

$= 2.850\text{ m/s}^2$

$\Sigma F_r = ma_r;$

$F_r = (2000\text{ kg})(3.918\text{ m/s}^2) = 7836.55\text{ N}$

$\Sigma F_\theta = ma_\theta;$

$F_\theta = (2000\text{ kg})(2.850\text{ m/s}^2) = 5699.31\text{ N}$

$F = \sqrt{F_r^2 + F_\theta^2}$

$= \sqrt{(7836.55\text{ N})^2 + (5699.31\text{ N})^2}$

$= 9689.87\text{ N} = 9.69\text{ kN}$

F13–16. $r = (0.6\cos 2\theta)\text{ m}\big|_{\theta=0^\circ} = [0.6\cos 2(0^\circ)]\text{ m} = 0.6\text{ m}$

$\dot{r} = (-1.2\sin 2\theta\dot{\theta})\text{ m/s}\big|_{\theta=0^\circ}$

$= \left[-1.2\sin 2(0^\circ)(-3)\right]\text{ m/s} = 0$

$\ddot{r} = -1.2\left(\sin 2\theta\ddot{\theta} + 2\cos 2\theta\dot{\theta}^2\right)\text{ m/s}^2\big|_{\theta=0^\circ}$

$= -21.6\text{ m/s}^2$

Thus,

$a_r = \ddot{r} - r\dot{\theta}^2 = -21.6\text{ m/s}^2 - 0.6\text{ m}(-3\text{ rad/s})^2$

$= -27\text{ m/s}^2$

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.6\text{ m}(0) + 2(0)(-3\text{ rad/s}) = 0$

$\Sigma F_\theta = ma_\theta;\quad F - 0.2(9.81)\text{ N} = 0.2\text{ kg}(0)$

$F = 1.96\text{ N}\uparrow$ *Ans.*

Chapter 14

F14–1. $T_1 + \Sigma U_{1-2} = T_2$

$0 + \left(\tfrac{4}{5}\right)(500\text{ N})(0.5\text{ m}) - \tfrac{1}{2}(500\text{ N/m})(0.5\text{ m})^2$

$= \tfrac{1}{2}(10\text{ kg})v^2$

$v = 5.24\text{ m/s}$ *Ans.*

F14–2. $\Sigma F_y = ma_y;\ N_A - 20(9.81)\text{ N}\cos 30^\circ = 0$

$N_A = 169.91\text{ N}$

$T_1 + \Sigma U_{1-2} = T_2$

$0 + 300\text{ N}(10\text{ m}) - 0.3(169.91\text{ N})(10\text{ m})$

$- 20(9.81)\text{N}(10\text{ m})\sin 30^\circ$

$= \tfrac{1}{2}(20\text{ kg})v^2$

$v = 12.3\text{ m/s}$ *Ans.*

F14–3. $T_1 + \Sigma U_{1-2} = T_2$

$0 + 2\left[\int_0^{15\text{ m}}(600 + 2s^2)\text{ N}\,ds\right] - 100(9.81)\text{ N}(15\text{ m})$

$= \tfrac{1}{2}(100\text{ kg})v^2$

$v = 12.5\text{ m/s}$ *Ans.*

F14–4. $T_1 + \Sigma U_{1-2} = T_2$

$\tfrac{1}{2}(1800\text{ kg})(125\text{ m/s})^2 - \left[\frac{(50\,000\text{ N} + 20\,000\text{ N})}{2}(400\text{ m})\right]$

$= \tfrac{1}{2}(1800\text{ kg})v^2$

$v = 8.33\text{ m/s}$ *Ans.*

F14–5. $T_1 + \Sigma U_{1-2} = T_2$

$\tfrac{1}{2}(10\text{ kg})(5\text{ m/s})^2 + 100\text{ N}s' + [10(9.81)\text{ N}]\,s'\sin 30^\circ$

$-\tfrac{1}{2}(200\text{ N/m})(s')^2 = 0$

$s' = 2.09\text{ m}$

$s = 0.6\text{ m} + 2.09\text{ m} = 2.69\text{ m}$ *Ans.*

F14–6. $T_A + \Sigma U_{A-B} = T_B$

Consider difference in cord length $AC - BC$, which is distance F moves.

$0 + 10\text{ lb}\left(\sqrt{(3\text{ ft})^2 + (4\text{ ft})^2} - 3\text{ ft}\right)$

$= \tfrac{1}{2}\left(\tfrac{5}{32.2}\text{ slug}\right)v_B^2$

$v_B = 16.0\text{ ft/s}$ *Ans.*

F14–7. $\overset{+}{\rightarrow}\Sigma F_x = ma_x;$

$30\left(\tfrac{4}{5}\right) = 20a \quad a = 1.2\text{ m/s}^2 \rightarrow$

$v = v_0 + a_c t$

$v = 0 + 1.2(4) = 4.8\text{ m/s}$

$P = \mathbf{F}\cdot\mathbf{v} = F(\cos\theta)v$

$= 30\left(\tfrac{4}{5}\right)(4.8)$

$= 115\text{ W}$ *Ans.*

F14–8. $\overset{+}{\rightarrow}\Sigma F_x = ma_x;$

$10s = 20a \quad a = 0.5s\text{ m/s}^2 \rightarrow$

$v\,dv = a\,ds$

$\int_1^v v\,dv = \int_0^{5\text{ m}} 0.5\,s\,ds$

$v = 3.674\text{ m/s}$

$P = \mathbf{F}\cdot\mathbf{v} = [10(5)](3.674) = 184\text{ W}$ *Ans.*

F14–9. $(+\uparrow)\Sigma F_y = 0;$

$T_1 - 100 \text{ lb} = 0 \qquad T_1 = 100 \text{ lb}$

$(+\uparrow)\Sigma F_y = 0;$

$100 \text{ lb} + 100 \text{ lb} - T_2 = 0 \quad T_2 = 200 \text{ lb}$

$P_{\text{out}} = \mathbf{T}_B \cdot \mathbf{v}_B = (200 \text{ lb})(3 \text{ ft/s}) = 1.091 \text{ hp}$

$P_{\text{in}} = \dfrac{P_{\text{out}}}{\varepsilon} = \dfrac{1.091 \text{ hp}}{0.8} = 1.36 \text{ hp}$ *Ans.*

F14–10. $\Sigma F_{y'} = ma_{y'};\;\; N - 20(9.81)\cos 30^\circ = 20(0)$

$N = 169.91 \text{ N}$

$\Sigma F_{x'} = ma_{x'};$

$F - 20(9.81)\sin 30^\circ - 0.2(169.91) = 0$

$F = 132.08 \text{ N}$

$P = \mathbf{F}\cdot\mathbf{v} = 132.08(5) = 660 \text{ W}$ *Ans.*

F14–11. $+\uparrow\Sigma F_y = ma_y;$

$T - 50(9.81) = 50(0) \quad T = 490.5 \text{ N}$

$P_{\text{out}} = \mathbf{T}\cdot\mathbf{v} = 490.5(1.5) = 735.75 \text{ W}$

Also, for a point on the other cable

$P_{\text{out}} = \left(\dfrac{490.5}{2}\right)(1.5)(2) = 735.75 \text{ W}$

$P_{\text{in}} = \dfrac{P_{\text{out}}}{\varepsilon} = \dfrac{735.75}{0.8} = 920 \text{ W}$ *Ans.*

F14–12. $2s_A + s_P = l$

$2a_A + a_P = 0$

$2a_A + 6 = 0$

$a_A = -3 \text{ m/s}^2 = 3 \text{ m/s}^2 \uparrow$

$\Sigma F_y = ma_y;\quad T_A - 490.5 \text{ N} = (50 \text{ kg})(3 \text{ m/s}^2)$

$T_A = 640.5 \text{ N}$

$P_{\text{out}} = \mathbf{T}\cdot\mathbf{v} = (640.5 \text{ N}/2)(12) = 3843 \text{ W}$

$P_{\text{in}} = \dfrac{P_{\text{out}}}{\varepsilon} = \dfrac{3843}{0.8} = 4803.75 \text{ W} = 4.80 \text{ kW}$ *Ans.*

F14–13. $T_A + V_A = T_B + V_B$

$0 + 2(9.81)(1.5) = \frac{1}{2}(2)(v_B)^2 + 0$

$v_B = 5.42 \text{ m/s}$ *Ans.*

$+\uparrow\Sigma F_n = ma_n;\; T - 2(9.81) = 2\left(\dfrac{(5.42)^2}{1.5}\right)$

$T = 58.9 \text{ N}$ *Ans.*

F14–14. $T_A + V_A = T_B + V_B$

$\frac{1}{2}m_A v_A^2 + mgh_A = \frac{1}{2}m_B v_B^2 + mgh_B$

$\left[\frac{1}{2}(2 \text{ kg})(1 \text{ m/s})^2\right] + [2\,(9.81) \text{ N}(4 \text{ m})]$

$= \left[\frac{1}{2}(2 \text{ kg})v_B^2\right] + [0]$

$v_B = 8.915 \text{ m/s} = 8.92 \text{ m/s}$ *Ans.*

$+\uparrow\Sigma F_n = ma_n;\quad N_B - 2(9.81) \text{ N}$

$= (2 \text{ kg})\left(\dfrac{(8.915 \text{ m/s})^2}{2 \text{ m}}\right)$

$N_B = 99.1 \text{ N}$ *Ans.*

F14–15. $T_1 + V_1 = T_2 + V_2$

$\frac{1}{2}(2)(4)^2 + \frac{1}{2}(30)(2-1)^2$

$= \frac{1}{2}(2)(v)^2 - 2(9.81)(1) + \frac{1}{2}(30)(\sqrt{5}-1)^2$

$v = 5.26 \text{ m/s}$ *Ans.*

F14–16. $T_A + V_A = T_B + V_B$

$0 + \frac{1}{2}(4)(2.5-0.5)^2 + 5(2.5)$

$= \frac{1}{2}\left(\frac{5}{32.2}\right)v_B^2 + \frac{1}{2}(4)(1-0.5)^2$

$v_B = 16.0 \text{ ft/s}$ *Ans.*

F14–17. $T_1 + V_1 = T_2 + V_2$

$\frac{1}{2}mv_1^2 + mgy_1 + \frac{1}{2}ks_1^2$

$= \frac{1}{2}mv_2^2 + mgy_2 + \frac{1}{2}ks_2^2$

$[0] + [0] + [0] = [0] +$

$[-75 \text{ lb}(5 \text{ ft} + s)] + \Big[2\left(\frac{1}{2}(1000 \text{ lb/ft})s^2\right)$

$+ \frac{1}{2}(1500 \text{ lb/ft})(s - 0.25 \text{ ft})^2\Big]$

$s = s_A = s_C = 0.580 \text{ ft}$ *Ans.*

Also,

$s_B = 0.5803 \text{ ft} - 0.25 \text{ ft} = 0.330 \text{ ft}$ *Ans.*

F14–18. $T_A + V_A = T_B + V_B$

$\frac{1}{2}mv_A^2 + \left(\frac{1}{2}ks_A^2 + mgy_A\right)$

$= \frac{1}{2}mv_B^2 + \left(\frac{1}{2}ks_B^2 + mgy_B\right)$

$\frac{1}{2}(4 \text{ kg})(2 \text{ m/s})^2 + \frac{1}{2}(400 \text{ N/m})(0.1 \text{ m} - 0.2 \text{ m})^2 + 0$

$= \frac{1}{2}(4 \text{ kg})v_B^2 + \frac{1}{2}(400 \text{ N/m})(\sqrt{(0.4 \text{ m})^2 + (0.3 \text{ m})^2}$

$- 0.2 \text{ m})^2 + [4(9.81) \text{ N}](-(0.1 \text{ m} + 0.3 \text{ m}))$

$v_B = 1.962 \text{ m/s} = 1.96 \text{ m/s}$ *Ans.*

Chapter 15

F15–1. $(\overset{+}{\rightarrow})\quad m(v_1)_x + \Sigma\displaystyle\int_{t1}^{t2} F_x\,dt = m(v_2)_x$

$(0.5 \text{ kg})(25 \text{ m/s})\cos 45^\circ - \displaystyle\int F_x\,dt$

$= (0.5 \text{ kg})(10 \text{ m/s})\cos 30^\circ$

$I_x = \displaystyle\int F_x\,dt = 4.509 \text{ N}\cdot\text{s}$

$(+\uparrow)\quad m(v_1)_y + \Sigma\displaystyle\int_{t1}^{t2} F_y\,dt = m(v_2)_y$

$-(0.5 \text{ kg})(25 \text{ m/s})\sin 45^\circ + \displaystyle\int F_y\,dt$

$= (0.5 \text{ kg})(10 \text{ m/s})\sin 30^\circ$

$I_y = \displaystyle\int F_y\,dt = 11.339 \text{ N}\cdot\text{s}$

$I = \displaystyle\int F\,dt = \sqrt{(4.509 \text{ N}\cdot\text{s})^2 + (11.339 \text{ N}\cdot\text{s})^2}$

$= 12.2 \text{ N}\cdot\text{s}$ *Ans.*

F15–2. $(+\uparrow) \quad m(v_1)_y + \Sigma \int_{t1}^{t2} F_y\, dt = m(v_2)_y$

$0 + N(4\text{ s}) + (100\text{ lb})(4\text{ s})\sin 30^\circ$
$\qquad - (150\text{ lb})(4\text{ s}) = 0$

$N = 100\text{ lb}$

$(\overset{+}{\rightarrow}) \quad m(v_1)_x + \Sigma \int_{t1}^{t2} F_x\, dt = m(v_2)_x$

$0 + (100\text{ lb})(4\text{ s})\cos 30^\circ - 0.2(100\text{ lb})(4\text{ s})$
$\qquad = \left(\frac{150}{32.2}\text{ slug}\right)v$

$v = 57.2\text{ ft/s}$ *Ans.*

F15–3. Time to start motion,

$+\uparrow \Sigma F_y = 0; \quad N - 25(9.81)\text{ N} = 0 \quad N = 245.25\text{ N}$

$\overset{+}{\rightarrow} \Sigma F_x = 0; \quad 20t^2 - 0.3(245.25\text{ N}) = 0 \quad t = 1.918\text{ s}$

$(\overset{+}{\rightarrow}) \quad m(v_1)_x + \Sigma \int_{t1}^{t2} F_x\, dt = m(v_2)_x$

$0 + \int_{1.918\text{ s}}^{4\text{ s}} 20t^2\, dt - (0.25(245.25\text{ N}))(4\text{ s} - 1.918\text{ s})$
$\qquad = (25\text{ kg})v$

$v = 10.1\text{ m/s}$ *Ans.*

F15–4. $(\overset{+}{\rightarrow}) \quad m(v_1)_x + \Sigma \int_{t1}^{t2} F_x\, dt = m(v_2)_x$

$(1500\text{ kg})(0) + \left[\frac{1}{2}(6000\text{ N})(2\text{ s}) + (6000\text{ N})(6\text{ s} - 2\text{ s})\right]$
$\qquad = (1500\text{ kg})\, v$

$v = 20\text{ m/s}$ *Ans.*

F15–5. SUV and trailer,

$m(v_1)_x + \Sigma \int_{t1}^{t2} F_x\, dt = m(v_2)_x$

$0 + (9000\text{ N})(20\text{ s}) = (1500\text{ kg} + 2500\text{ kg})v$

$v = 45.0\text{ m/s}$ *Ans.*

Trailer,

$m(v_1)_x + \Sigma \int_{t1}^{t2} F_x\, dt = m(v_2)_x$

$0 + T(20\text{ s}) = (1500\text{ kg})(45.0\text{ m/s})$

$T = 3375\text{ N} = 3.375\text{ kN}$ *Ans.*

F15–6. Block B:

$(+\downarrow)\, mv_1 + \int F\, dt = mv_2$

$0 + 8(5) - T(5) = \frac{8}{32.2}(1)$

$T = 7.95\text{ lb}$ *Ans.*

Block A:

$(\overset{+}{\rightarrow})\, mv_1 + \int F\, dt = mv_2$

$0 + 7.95(5) - \mu_k(10)(5) = \frac{10}{32.2}(1)$

$\mu_k = 0.789$ *Ans.*

F15–7. $(\overset{+}{\rightarrow}) \quad m_A(v_A)_1 + m_B(v_B)_1 = m_A(v_A)_2 + m_B(v_B)_2$

$(20(10^3)\text{ kg})(3\text{ m/s}) + (15(10^3)\text{ kg})(-1.5\text{ m/s})$
$= (20(10^3)\text{ kg})(v_A)_2 + (15(10^3)\text{ kg})(2\text{ m/s})$

$(v_A)_2 = 0.375\text{ m/s} \rightarrow$ *Ans.*

$(\overset{+}{\rightarrow}) \quad m(v_B)_1 + \Sigma \int_{t1}^{t2} F\, dt = m(v_B)_2$

$(15(10^3)\text{ kg})(-1.5\text{ m/s}) + F_{avg}(0.5\text{ s})$
$\qquad = (15(10^3)\text{ kg})(2\text{ m/s})$

$F_{avg} = 105(10^3)\text{ N} = 105\text{ kN}$ *Ans.*

F15–8. $(\overset{+}{\rightarrow}) \quad m_p[(v_p)_1]_x + m_c[(v)_1]_x = (m_p + m_c)v_2$

$5\left[10\left(\frac{4}{5}\right)\right] + 0 = (5 + 20)v_2$

$v_2 = 1.6\text{ m/s}$ *Ans.*

F15–9. $T_1 + V_1 = T_2 + V_2$

$\frac{1}{2}m_A(v_A)_1^2 + (V_g)_1 = \frac{1}{2}m_A(v_A)_2^2 + (V_g)_2$

$\frac{1}{2}(5)(5)^2 + 5(9.81)(1.5) = \frac{1}{2}(5)(v_A)_2^2 + 0$

$(v_A)_2 = 7.378\text{ m/s}$

$(\overset{+}{\leftarrow}) \quad m_A(v_A)_2 + m_B(v_B)_2 = (m_A + m_B)v$

$5(7.378) + 0 = (5 + 8)v$

$v = 2.84\text{ m/s}$ *Ans.*

F15–10. $(\overset{+}{\rightarrow}) \quad m_A(v_A)_1 + m_B(v_B)_1 = m_A(v_A)_2 + m_B(v_B)_2$

$0 + 0 = 10(v_A)_2 + 15(v_B)_2 \qquad (1)$

$T_1 + V_1 = T_2 + V_2$

$\frac{1}{2}m_A(v_A)_1^2 + \frac{1}{2}m_B(v_B)_1^2 + (V_e)_1$
$= \frac{1}{2}m_A(v_A)_2^2 + \frac{1}{2}m_B(v_B)_2^2 + (V_e)_2$

$0 + 0 + \frac{1}{2}\left[5\left(10^3\right)\right]\left(0.2^2\right)$
$\qquad = \frac{1}{2}(10)(v_A)_2^2 + \frac{1}{2}(15)(v_B)_2^2 + 0$

$5(v_A)_2^2 + 7.5(v_B)_2^2 = 100 \qquad (2)$

Solving Eqs. (1) and (2),

$(v_B)_2 = 2.31\text{ m/s} \rightarrow$ *Ans.*

$(v_A)_2 = -3.464\text{ m/s} = 3.46\text{ m/s} \leftarrow$ *Ans.*

F15–11. $(\overset{+}{\leftarrow}) \quad m_A(v_A)_1 + m_B(v_B)_1 = (m_A + m_B)v_2$
$\qquad 0 + 10(15) = (15 + 10)v_2$

$v_2 = 6\text{ m/s}$

$T_1 + V_1 = T_2 + V_2$

$\frac{1}{2}(m_A + m_B)v_2^2 + (V_e)_2 = \frac{1}{2}(m_A + m_B)v_3^2 + (V_e)_3$

$\frac{1}{2}(15 + 10)\left(6^2\right) + 0 = 0 + \frac{1}{2}\left[10\left(10^3\right)\right]s_{max}^2$

$s_{max} = 0.3\text{ m} = 300\text{ mm}$ *Ans.*

F15–12. $(\xrightarrow{+})\quad 0 + 0 = m_p(v_p)_x - m_c v_c$

$0 = (20\text{ kg})\,(v_p)_x - (250\text{ kg})v_c$

$(v_p)_x = 12.5\, v_c \qquad (1)$

$\mathbf{v}_p = \mathbf{v}_c + \mathbf{v}_{p/c}$

$(v_p)_x\mathbf{i} + (v_p)_y\mathbf{j} = -v_c\mathbf{i} + [(400\text{ m/s})\cos 30°\mathbf{i} + (400\text{ m/s})\sin 30°\mathbf{j}]$

$(v_p)_x\mathbf{i} + (v_p)_y\mathbf{j} = (346.41 - v_c)\mathbf{i} + 200\mathbf{j}$

$(v_p)_x = 346.41 - v_c$

$(v_p)_y = 200\text{ m/s}$

$(v_p)_x = 320.75\text{ m/s} \quad v_c = 25.66\text{ m/s}$

$v_p = \sqrt{(v_p)_x^2 + (v_p)_y^2}$

$= \sqrt{(320.75\text{ m/s})^2 + (200\text{ m/s})^2}$

$= 378\text{ m/s}$ *Ans.*

F15–13. $(\xrightarrow{+})\quad e = \dfrac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}$

$= \dfrac{(9\text{ m/s}) - (1\text{ m/s})}{(8\text{ m/s}) - (-2\text{ m/s})} = 0.8$

F15–14. $(\xrightarrow{+})\quad m_A(v_A)_1 + m_B(v_B)_1 = m_A(v_A)_2 + m_B(v_B)_2$

$[15(10^3)\text{ kg}](5\text{ m/s}) + [25(10^3)](-7\text{ m/s})$

$= [15(10^3)\text{ kg}](v_A)_2 + [25(10^3)](v_B)_2$

$15(v_A)_2 + 25(v_B)_2 = -100 \qquad (1)$

Using the coefficient of restitution equation,

$(\xrightarrow{+})\quad e = \dfrac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}$

$0.6 = \dfrac{(v_B)_2 - (v_A)_2}{5\text{ m/s} - (-7\text{ m/s})}$

$(v_B)_2 - (v_A)_2 = 7.2 \qquad (2)$

Solving,

$(v_B)_2 = 0.2\text{ m/s} \rightarrow$ *Ans.*

$(v_A)_2 = -7\text{ m/s} = 7\text{ m/s} \leftarrow$ *Ans.*

F15–15. $T_1 + V_1 = T_2 + V_2$

$\frac{1}{2}m(v_A)_1^2 + mg(h_A)_1 = \frac{1}{2}m(v_A)_2^2 + mg(h_A)_2$

$\frac{1}{2}\left(\frac{30}{32.2}\text{ slug}\right)(5\text{ ft/s})^2 + (30\text{ lb})(10\text{ft})$

$= \frac{1}{2}\left(\frac{30}{32.2}\text{ slug}\right)(v_A)_2^2 + 0$

$(v_A)_2 = 25.87\text{ ft/s} \leftarrow$

$(\xleftarrow{+})\quad m_A(v_A)_2 + m_B(v_B)_2 = m_A(v_A)_3 + m_B(v_B)_3$

$\left(\frac{30}{32.2}\text{ slug}\right)(25.87\text{ ft/s}) + 0$

$= \left(\frac{30}{32.2}\text{ slug}\right)(v_A)_3 + \left(\frac{80}{32.2}\text{ slug}\right)(v_B)_3$

$30(v_A)_3 + 80(v_B)_3 = 775.95 \qquad (1)$

$(\xleftarrow{+})\quad e = \dfrac{(v_B)_3 - (v_A)_3}{(v_A)_2 - (v_B)_2}$

$0.6 = \dfrac{(v_B)_3 - (v_A)_3}{25.87\text{ ft/s} - 0}$

$(v_B)_3 - (v_A)_3 = 15.52 \qquad (2)$

Solving Eqs. (1) and (2), yields

$(v_B)_3 = 11.3\text{ ft/s} \leftarrow$

$(v_A)_3 = -4.23\text{ ft/s} = 4.23\text{ ft/s} \rightarrow$ *Ans.*

F15–16. $(+\uparrow)\quad m[(v_b)_1]_y = m[(v_b)_2]_y$

$[(v_b)_2]_y = [(v_b)_1]_y = (20\text{ m/s})\sin 30° = 10\text{ m/s} \uparrow$

$(\xrightarrow{+})\quad e = \dfrac{(v_w)_2 - [(v_b)_2]_x}{[(v_b)_1]_x - (v_w)_1}$

$0.75 = \dfrac{0 - [(v_b)_2]_x}{(20\text{ m/s})\cos 30° - 0}$

$[(v_b)_2]_x = -12.99\text{ m/s} = 12.99\text{ m/s} \leftarrow$

$(v_b)_2 = \sqrt{[(v_b)_2]_x^2 + [(v_b)_2]_y^2}$

$= \sqrt{(12.99\text{ m/s})^2 + (10\text{ m/s})^2}$

$= 16.4\text{ m/s}$ *Ans.*

$\theta = \tan^{-1}\left(\dfrac{[(v_b)_2]_y}{[(v_b)_2]_x}\right) = \tan^{-1}\left(\dfrac{10\text{ m/s}}{12.99\text{ m/s}}\right)$

$= 37.6°$ *Ans.*

F15–17. $\Sigma m(v_x)_1 = \Sigma m(v_x)_2$

$0 + 0 = 2\,(1) + 11\,(v_{Bx})_2$

$(v_{Bx})_2 = -0.1818\text{ m/s}$

$\Sigma m(v_y)_1 = \Sigma m(v_y)_2$

$2\,(3) + 0 = 0 + 11\,(v_{By})_2$

$(v_{By})_2 = 0.545\text{ m/s}$

$(v_B)_2 = \sqrt{(-0.1818)^2 + (0.545)^2}$

$= 0.575\text{ m/s}$ *Ans.*

F15–18. $+\nearrow\quad 1\,(3)\left(\frac{3}{5}\right) - 1\,(4)\left(\frac{4}{5}\right)$

$= 1\,(v_B)_{2x} + 1\,(v_A)_{2x}$

$+\nearrow\quad 0.5 = [(v_A)_{2x} - (v_B)_{2x}]/[(3)\left(\frac{3}{5}\right) - (-4)\left(\frac{4}{5}\right)]$

Solving,

$(v_A)_{2x} = 0.550\text{ m/s},\ (v_B)_{2x} = -1.95\text{ m/s}$

Disc A,

$+\nwarrow\quad -1\,(4)\left(\frac{3}{5}\right) = 1\,(v_A)_{2y}$

$(v_A)_{2y} = -2.40\text{ m/s}$

Disc B,

$$-1(3)\left(\tfrac{4}{5}\right) = 1(v_B)_{2y}$$
$$(v_B)_{2y} = -2.40 \text{ m/s}$$
$$(v_A)_2 = \sqrt{(0.550)^2 + (2.40)^2} = 2.46 \text{ m/s} \quad \textit{Ans.}$$
$$(v_B)_2 = \sqrt{(1.95)^2 + (2.40)^2} = 3.09 \text{ m/s} \quad \textit{Ans.}$$

F15–19. $H_O = \Sigma mvd;$
$$H_O = \left[2(10)\left(\tfrac{4}{5}\right)\right](4) - \left[2(10)\left(\tfrac{3}{5}\right)\right](3)$$
$$= 28 \text{ kg}\cdot\text{m}^2/\text{s}$$

F15–20. $H_P = \Sigma mvd;$
$$H_P = [2(15)\sin 30°](2) - [2(15)\cos 30°](5)$$
$$= -99.9 \text{ kg}\cdot\text{m}^2/\text{s} = 99.9 \text{ kg}\cdot\text{m}^2/\text{s} \circlearrowright$$

F15–21. $(H_z)_1 + \Sigma\int M_z\,dt = (H_z)_2$
$$5(2)(1.5) + 5(1.5)(3) = 5v(1.5)$$
$$v = 5 \text{ m/s} \quad \textit{Ans.}$$

F15–22. $(H_z)_1 + \Sigma\int M_z\,dt = (H_z)_2$
$$0 + \int_0^{4\text{ s}} (10t)\left(\tfrac{4}{5}\right)(1.5)dt = 5v(1.5)$$
$$v = 12.8 \text{ m/s} \quad \textit{Ans.}$$

F15–23. $(H_z)_1 + \Sigma\int M_z\,dt = (H_z)_2$
$$0 + \int_0^{5\text{ s}} 0.9t^2\,dt = 2v(0.6)$$
$$v = 31.2 \text{ m/s} \quad \textit{Ans.}$$

F15–24. $(H_z)_1 + \Sigma\int M_z\,dt = (H_z)_2$
$$0 + \int_0^{4\text{ s}} 8t\,dt + 2(10)(0.5)(4) = 2[10v(0.5)]$$
$$v = 10.4 \text{ m/s} \quad \textit{Ans.}$$

Chapter 16

F16–1. $\theta = (20 \text{ rev})\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right) = 40\pi \text{ rad}$
$$\omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$$
$$(30 \text{ rad/s})^2 = 0^2 + 2\alpha_c[(40\pi \text{ rad}) - 0]$$
$$\alpha_c = 3.581 \text{ rad/s}^2 = 3.58 \text{ rad/s}^2 \quad \textit{Ans.}$$
$$\omega = \omega_0 + \alpha_c t$$
$$30 \text{ rad/s} = 0 + (3.581 \text{ rad/s}^2)t$$
$$t = 8.38 \text{ s} \quad \textit{Ans.}$$

F16–2. $\frac{d\omega}{d\theta} = 2(0.005\theta) = (0.01\theta)$
$$\alpha = \omega\frac{d\omega}{d\theta} = \left(0.005\,\theta^2\right)(0.01\theta) = 50\left(10^{-6}\right)\theta^3 \text{ rad/s}^2$$
When $\theta = 20 \text{ rev}(2\pi \text{ rad}/1 \text{ rev}) = 40\pi \text{ rad}$,
$$\alpha = \left[50\left(10^{-6}\right)(40\pi)^3\right] \text{ rad/s}^2$$
$$= 99.22 \text{ rad/s}^2 = 99.2 \text{ rad/s}^2 \quad \textit{Ans.}$$

F16–3. $\omega = 4\theta^{1/2}$
$$150 \text{ rad/s} = 4\,\theta^{1/2}$$
$$\theta = 1406.25 \text{ rad}$$
$$dt = \frac{d\theta}{\omega}$$
$$\int_0^t dt = \int_{1\text{ rad}}^{\theta} \frac{d\theta}{4\theta^{1/2}}$$
$$t\Big|_0^t = \tfrac{1}{2}\,\theta^{1/2}\Big|_{1\text{ rad}}^{\theta}$$
$$t = \tfrac{1}{2}\,\theta^{1/2} - \tfrac{1}{2}$$
$$t = \tfrac{1}{2}(1406.25)^{1/2} - \tfrac{1}{2} = 18.25 \text{ s} \quad \textit{Ans.}$$

F16–4. $\omega = \frac{d\theta}{dt} = (1.5t^2 + 15) \text{ rad/s}$
$$\alpha = \frac{d\omega}{dt} = (3t) \text{ rad/s}^2$$
$$\omega = [1.5(3^2) + 15] \text{ rad/s} = 28.5 \text{ rad/s}$$
$$\alpha = 3(3) \text{ rad/s}^2 = 9 \text{ rad/s}^2$$
$$v = \omega r = (28.5 \text{ rad/s})(0.75 \text{ ft}) = 21.4 \text{ ft/s} \quad \textit{Ans.}$$
$$a = \alpha r = (9 \text{ rad/s}^2)(0.75 \text{ ft}) = 6.75 \text{ ft/s}^2 \quad \textit{Ans.}$$

F16–5. $\omega\,d\omega = \alpha\,d\theta$
$$\int_{2\text{ rad/s}}^{\omega} \omega\,d\omega = \int_0^{\theta} 0.5\theta\,d\theta$$
$$\frac{\omega^2}{2}\Big|_{2\text{ rad/s}}^{\omega} = 0.25\theta^2\Big|_0^{\theta}$$
$$\omega = (0.5\,\theta^2 + 4)^{1/2} \text{ rad/s}$$
When $\theta = 2 \text{ rev} = 4\pi \text{ rad}$,
$$\omega = [0.5(4\pi)^2 + 4]^{1/2} \text{ rad/s} = 9.108 \text{ rad/s}$$
$$v_P = \omega r = (9.108 \text{ rad/s})(0.2 \text{ m}) = 1.82 \text{ m/s} \quad \textit{Ans.}$$
$$(a_P)_t = \alpha r = (0.5\theta \text{ rad/s}^2)(0.2 \text{ m})\Big|_{\theta = 4\pi \text{ rad}}$$
$$= 1.257 \text{ m/s}^2$$
$$(a_P)_n = \omega^2 r = (9.108 \text{ rad/s})^2(0.2 \text{ m}) = 16.59 \text{ m/s}^2$$
$$a_p = \sqrt{(a_P)_t^{\,2} + (a_P)_n^{\,2}}$$
$$= \sqrt{(1.257 \text{ m/s}^2)^2 + (16.59 \text{ m/s}^2)^2}$$
$$= 16.6 \text{ m/s}^2 \quad \textit{Ans.}$$

F16–6. $\alpha_B = \alpha_A\left(\frac{r_A}{r_B}\right)$

$= (4.5\text{ rad/s}^2)\left(\frac{0.075\text{ m}}{0.225\text{ m}}\right) = 1.5\text{ rad/s}^2$

$\omega_B = (\omega_B)_0 + \alpha_B t$

$\omega_B = 0 + (1.5\text{ rad/s}^2)(3\text{ s}) = 4.5\text{ rad/s}$

$\theta_B = (\theta_B)_0 + (\omega_B)_0 t + \frac{1}{2}\alpha_B t^2$

$\theta_B = 0 + 0 + \frac{1}{2}(1.5\text{ rad/s}^2)(3\text{ s})^2$

$\theta_B = 6.75\text{ rad}$

$v_C = \omega_B r_D = (4.5\text{ rad/s})(0.125\text{ m})$

$= 0.5625\text{ m/s}$ *Ans.*

$s_C = \theta_B r_D = (6.75\text{ rad})(0.125\text{ m}) = 0.84375\text{ m}$

$= 844\text{ mm}$ *Ans.*

F16–7. Vector Analysis

$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$

$-v_B\,\mathbf{j} = (3\mathbf{i})\text{ m/s}$

$+ (\omega\mathbf{k}) \times (-1.5\cos 30^\circ\mathbf{i} + 1.5\sin 30^\circ\mathbf{j})$

$-v_B\,\mathbf{j} = [3 - \omega_{AB}(1.5\sin 30^\circ)]\mathbf{i} - \omega(1.5\cos 30^\circ)\mathbf{j}$

$0 = 3 - \omega(1.5\sin 30^\circ)$ (1)

$-v_B = 0 - \omega(1.5\cos 30^\circ)$ (2)

$\omega = 4\text{ rad/s} \qquad v_B = 5.20\text{ m/s}$ *Ans.*

Scalar Solution

$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

$\left[\downarrow v_B\right] = \left[\underset{\rightarrow}{3}\right] + \left[\omega(1.5)\measuredangle 30^\circ\right]$

This yields Eqs. (1) and (2).

F16–8. Vector Analysis

$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$

$(v_B)_x\mathbf{i} + (v_B)_y\mathbf{j} = \mathbf{0} + (-10\mathbf{k}) \times (-0.6\mathbf{i} + 0.6\mathbf{j})$

$(v_B)_x\mathbf{i} + (v_B)_y\mathbf{j} = 6\mathbf{i} + 6\mathbf{j}$

$(v_B)_x = 6\text{ m/s}$ and $(v_B)_y = 6\text{ m/s}$

$v_B = \sqrt{(v_B)_x^2 + (v_B)_y^2}$

$= \sqrt{(6\text{ m/s})^2 + (6\text{ m/s})^2}$

$= 8.49\text{ m/s}$ *Ans.*

Scalar Solution

$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

$\left[\underset{\rightarrow}{(v_B)_x}\right] + \left[(v_B)_y\uparrow\right] = \left[0\right] + \left[\measuredangle 45^\circ\ \ 10\left(\frac{0.6}{\cos 45^\circ}\right)\right]$

$\overset{+}{\rightarrow}\ (v_B)_x = 0 + 10(0.6/\cos 45^\circ)\cos 45^\circ = 6\text{ m/s}\rightarrow$

$+\uparrow (v_B)_y = 0 + 10(0.6/\cos 45^\circ)\sin 45^\circ = 6\text{ m/s}\uparrow$

F16–9. Vector Analysis

$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$

$(4\text{ ft/s})\mathbf{i} = (-2\text{ ft/s})\mathbf{i} + (-\omega\mathbf{k}) \times (3\text{ ft})\mathbf{j}$

$4\mathbf{i} = (-2 + 3\omega)\mathbf{i}$

$\omega = 2\text{ rad/s}$ *Ans.*

Scalar Solution

$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

$\left[\underset{\rightarrow}{4}\right] = \left[\underset{\leftarrow}{2}\right] + \left[\underset{\rightarrow}{\omega(3)}\right]$

$\overset{+}{\rightarrow}\quad 4 = -2 + \omega(3); \qquad \omega = 2\text{ rad/s}$

F16–10. Vector Analysis

$\mathbf{v}_A = \boldsymbol{\omega}_{OA} \times \mathbf{r}_A$

$= (12\text{ rad/s})\mathbf{k} \times (0.3\text{ m})\mathbf{j}$

$= [-3.6\mathbf{i}]\text{ m/s}$

$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{B/A}$

$v_B\,\mathbf{j} = (-3.6\text{ m/s})\mathbf{i}$

$+ (\omega_{AB}\,\mathbf{k}) \times (0.6\cos 30^\circ\mathbf{i} - 0.6\sin 30^\circ\mathbf{j})\text{ m}$

$v_B\,\mathbf{j} = [\omega_{AB}(0.6\sin 30^\circ) - 3.6]\mathbf{i} + \omega_{AB}(0.6\cos 30^\circ)\mathbf{j}$

$0 = \omega_{AB}(0.6\sin 30^\circ) - 3.6$ (1)

$v_B = \omega_{AB}(0.6\cos 30^\circ)$ (2)

$\omega_{AB} = 12\text{ rad/s} \quad v_B = 6.24\text{ m/s}\uparrow$ *Ans.*

Scalar Solution

$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

$\left[v_B\uparrow\right] = \left[\overset{\leftarrow}{12(0.3)}\right] + \left[\measuredangle 30^\circ\omega(0.6)\right]$

This yields Eqs. (1) and (2).

F16–11. Vector Analysis

$\mathbf{v}_C = \mathbf{v}_B + \boldsymbol{\omega}_{BC} \times \mathbf{r}_{C/B}$

$v_C\,\mathbf{j} = (-60\mathbf{i})\text{ ft/s}$

$+ (-\omega_{BC}\mathbf{k}) \times (-2.5\cos 30^\circ\mathbf{i} + 2.5\sin 30^\circ\mathbf{j})\text{ ft}$

$v_C\mathbf{j} = (-60)\mathbf{i} + 2.165\omega_{BC}\mathbf{j} + 1.25\omega_{BC}\,\mathbf{i}$

$0 = -60 + 1.25\omega_{BC}$ (1)

$v_C = 2.165\,\omega_{BC}$ (2)

$\omega_{BC} = 48\text{ rad/s}$ *Ans.*

$v_C = 104\text{ ft/s}$

Scalar Solution

$\mathbf{v}_C = \mathbf{v}_B + \mathbf{v}_{C/B}$

$\left[v_C\uparrow\right] = \left[\underset{\leftarrow}{v_B}\right] + \left[\measuredangle 30^\circ\,\omega\,(2.5)\right]$

This yields Eqs. (1) and (2).

F16–12. Vector Analysis

$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A}$

$-v_B \cos 30^\circ\,\mathbf{i} + v_B \sin 30^\circ\,\mathbf{j} = (-3\text{ m/s})\mathbf{j} + (-\omega\mathbf{k}) \times (-2\sin 45^\circ\mathbf{i} - 2\cos 45^\circ\mathbf{j})\text{ m}$

$-0.8660v_B\,\mathbf{i} + 0.5v_B\,\mathbf{j} = -1.4142\omega\mathbf{i} + (1.4142\omega - 3)\mathbf{j}$

$$-0.8660v_B = -1.4142\omega \quad (1)$$

$$0.5v_B = 1.4142\omega - 3 \quad (2)$$

$\omega = 5.02\text{ rad/s} \quad v_B = 8.20\text{ m/s}$ *Ans.*

Scalar Solution

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$

$$\left[\overset{30^\circ}{\nwarrow} v_B\right] = \left[\downarrow 3\right] + \left[\overset{45^\circ}{\nwarrow} \omega(2)\right]$$

This yields Eqs. (1) and (2).

F16–13. $\omega_{AB} = \dfrac{v_A}{r_{A/IC}} = \dfrac{6}{3} = 2\text{ rad/s}$ *Ans.*

$\phi = \tan^{-1}\left(\frac{2}{1.5}\right) = 53.13^\circ$

$r_{C/IC} = \sqrt{(3)^2 + (2.5)^2 - 2(3)(2.5)\cos 53.13^\circ} = 2.5\text{ m}$

$v_C = \omega_{AB}\, r_{C/IC} = 2(2.5) = 5\text{ m/s}$ *Ans.*

$\theta = 90^\circ - \phi = 90^\circ - 53.13^\circ = 36.9^\circ$ *Ans.*

F16–14. $v_B = \omega_{AB}\, r_{B/A} = 12(0.6) = 7.2\text{ m/s} \downarrow$

$v_C = 0$ *Ans.*

$\omega_{BC} = \dfrac{v_B}{r_{B/IC}} = \dfrac{7.2}{1.2} = 6\text{ rad/s}$ *Ans.*

F16–15. $\omega = \dfrac{v_O}{r_{O/IC}} = \dfrac{6}{0.3} = 20\text{ rad/s}$ *Ans.*

$r_{A/IC} = \sqrt{0.3^2 + 0.6^2} = 0.6708\text{ m}$

$\phi = \tan^{-1}\left(\frac{0.3}{0.6}\right) = 26.57^\circ$

$v_A = \omega r_{A/IC} = 20(0.6708) = 13.4\text{ m/s}$ *Ans.*

$\theta = 90^\circ - \phi = 90^\circ - 26.57^\circ = 63.4^\circ$ *Ans.*

F16–16. The location of IC can be determined using similar triangles.

$\dfrac{0.5 - r_{C/IC}}{3} = \dfrac{r_{C/IC}}{1.5} \qquad r_{C/IC} = 0.1667\text{ m}$

$\omega = \dfrac{v_C}{r_{C/IC}} = \dfrac{1.5}{0.1667} = 9\text{ rad/s}$ *Ans.*

Also, $r_{O/IC} = 0.3 - r_{C/IC} = 0.3 - 0.1667 = 0.1333\text{ m}$.

$v_O = \omega r_{O/IC} = 9(0.1333) = 1.20\text{ m/s}$ *Ans.*

F16–17. $v_B = \omega r_{B/A} = 6(0.2) = 1.2\text{ m/s}$

$r_{B/IC} = 0.8\tan 60^\circ = 1.3856\text{ m}$

$r_{C/IC} = \frac{0.8}{\cos 60^\circ} = 1.6\text{ m}$

$\omega_{BC} = \dfrac{v_B}{r_{B/IC}} = \dfrac{1.2}{1.3856} = 0.8660\text{ rad/s} = 0.866\text{ rad/s}$ *Ans.*

Then,

$v_C = \omega_{BC}\, r_{C/IC} = 0.8660(1.6) = 1.39\text{ m/s}$ *Ans.*

F16–18. $v_B = \omega_{AB}\, r_{B/A} = 10(0.2) = 2\text{ m/s}$

$v_C = \omega_{CD}\, r_{C/D} = \omega_{CD}\,(0.2) \rightarrow$

$r_{B/IC} = \frac{0.4}{\cos 30^\circ} = 0.4619\text{ m}$

$r_{C/IC} = 0.4\tan 30^\circ = 0.2309\text{ m}$

$\omega_{BC} = \dfrac{v_B}{r_{B/IC}} = \dfrac{2}{0.4619} = 4.330\text{ rad/s} = 4.33\text{ rad/s}$ *Ans.*

$v_C = \omega_{BC}\, r_{C/IC}$

$\omega_{CD}\,(0.2) = 4.330(0.2309)$

$\omega_{CD} = 5\text{ rad/s}$ *Ans.*

F16–19. $\omega = \dfrac{v_A}{r_{A/IC}} = \dfrac{6}{3} = 2\text{ rad/s}$

Vector Analysis

$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2\,\mathbf{r}_{B/A}$

$a_B\mathbf{i} = -5\mathbf{j} + (\alpha\mathbf{k}) \times (3\mathbf{i} - 4\mathbf{j}) - 2^2(3\mathbf{i} - 4\mathbf{j})$

$a_B\mathbf{i} = (4\alpha - 12)\mathbf{i} + (3\alpha + 11)\mathbf{j}$

$$a_B = 4\alpha - 12 \quad (1)$$

$$0 = 3\alpha + 11 \quad (2)$$

$\alpha = -3.67\text{ rad/s}^2$ *Ans.*

$a_B = -26.7\text{ m/s}^2$ *Ans.*

Scalar Solution

$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$

$$\left[\underset{\rightarrow}{a_B}\right] = \left[\downarrow 5\right] + \left[\alpha\,(5)\ \overset{5}{\nearrow}{}^{3}_{4}\right] + \left[{}^{4}\overset{5}{\nwarrow}{}_{3}\,(2)^2(5)\right]$$

This yields Eqs. (1) and (2).

F16–20. Vector Analysis

$$\mathbf{a}_A = \mathbf{a}_O + \boldsymbol{\alpha} \times \mathbf{r}_{A/O} - \omega^2 \mathbf{r}_{A/O}$$
$$= 1.8\mathbf{i} + (-6\mathbf{k}) \times (0.3\mathbf{j}) - 12^2 (0.3\mathbf{j})$$
$$= \{3.6\mathbf{i} - 43.2\mathbf{j}\} \text{m/s}^2 \qquad \textit{Ans.}$$

Scalar Analysis

$$\mathbf{a}_A = \mathbf{a}_O + \mathbf{a}_{A/O}$$
$$\begin{bmatrix} (a_A)_x \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (a_A)_y \uparrow \end{bmatrix} = \begin{bmatrix} (6)(0.3) \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (6)(0.3) \\ \rightarrow \end{bmatrix} + \begin{bmatrix} \downarrow (12)^2(0.3) \end{bmatrix}$$
$$\overset{+}{\rightarrow} \qquad (a_A)_x = 1.8 + 1.8 = 3.6 \text{ m/s}^2 \rightarrow$$
$$+\uparrow \qquad (a_A)_y = -43.2 \text{ m/s}^2$$

F16–21. Using

$$v_O = \omega r; \qquad 6 = \omega(0.3)$$
$$\omega = 20 \text{ rad/s}$$
$$a_O = \alpha r; \qquad 3 = \alpha(0.3)$$
$$\alpha = 10 \text{ rad/s}^2 \qquad \textit{Ans.}$$

Vector Analysis

$$\mathbf{a}_A = \mathbf{a}_O + \boldsymbol{\alpha} \times \mathbf{r}_{A/O} - \omega^2 \mathbf{r}_{A/O}$$
$$= 3\mathbf{i} + (-10\mathbf{k}) \times (-0.6\mathbf{i}) - 20^2(-0.6\mathbf{i})$$
$$= \{243\mathbf{i} + 6\mathbf{j}\} \text{ m/s}^2 \qquad \textit{Ans.}$$

Scalar Analysis

$$\mathbf{a}_A = \mathbf{a}_O + \mathbf{a}_{A/O}$$
$$\begin{bmatrix} (a_A)_x \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (a_A)_y \\ \uparrow \end{bmatrix} = \begin{bmatrix} 3 \\ \rightarrow \end{bmatrix} + \begin{bmatrix} 10(0.6) \\ \uparrow \end{bmatrix} + \begin{bmatrix} (20)^2(0.6) \\ \rightarrow \end{bmatrix}$$
$$\overset{+}{\rightarrow} \qquad (a_A)_x = 3 + 240 = 243 \text{ m/s}^2$$
$$+\uparrow \qquad (a_A)_y = 10(0.6) = 6 \text{ m/s}^2 \uparrow$$

F16–22. $\dfrac{r_{A/IC}}{3} = \dfrac{0.5 - r_{A/IC}}{1.5}; \qquad r_{A/IC} = 0.3333 \text{ m}$

$$\omega = \frac{v_A}{r_{A/IC}} = \frac{3}{0.3333} = 9 \text{ rad/s}$$

Vector Analysis

$$\mathbf{a}_A = \mathbf{a}_C + \boldsymbol{\alpha} \times \mathbf{r}_{A/C} - \omega^2 \mathbf{r}_{A/C}$$
$$1.5\mathbf{i} - (a_A)_n \mathbf{j} = -0.75\mathbf{i} + (a_C)_n \mathbf{j} + (-\alpha\mathbf{k}) \times 0.5\mathbf{j} - 9^2 (0.5\mathbf{j})$$
$$1.5\mathbf{i} - (a_A)_n \mathbf{j} = (0.5\alpha - 0.75)\mathbf{i} + \left[(a_C)_n - 40.5\right]\mathbf{j}$$
$$1.5 = 0.5\alpha - 0.75$$
$$\alpha = 4.5 \text{ rad/s}^2 \qquad \textit{Ans.}$$

Scalar Analysis

$$\mathbf{a}_A = \mathbf{a}_C + \mathbf{a}_{A/C}$$
$$\begin{bmatrix} 1.5 \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (a_A)_n \\ \downarrow \end{bmatrix} = \begin{bmatrix} 0.75 \\ \leftarrow \end{bmatrix} + \begin{bmatrix} (a_C)_n \\ \uparrow \end{bmatrix} + \begin{bmatrix} \alpha(0.5) \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (9)^2(0.5) \\ \downarrow \end{bmatrix}$$
$$\overset{+}{\rightarrow} \qquad 1.5 = -0.75 + \alpha(0.5)$$
$$\alpha = 4.5 \text{ rad/s}^2$$

F16–23. $v_B = \omega r_{B/A} = 12(0.3) = 3.6 \text{ m/s}$

$$\omega_{BC} = \frac{v_B}{r_{B/IC}} = \frac{3.6}{1.2} = 3 \text{ rad/s}$$

Vector Analysis

$$\mathbf{a}_B = \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{B/A}$$
$$= (-6\mathbf{k}) \times (0.3\mathbf{i}) - 12^2(0.3\mathbf{i})$$
$$= \{-43.2\mathbf{i} - 1.8\mathbf{j}\} \text{ m/s}^2$$
$$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}^2 \mathbf{r}_{C/B}$$
$$a_C \mathbf{i} = (-43.2\mathbf{i} - 1.8\mathbf{j}) + (\alpha_{BC} \mathbf{k}) \times (1.2\mathbf{i}) - 3^2(1.2\mathbf{i})$$
$$a_C \mathbf{i} = -54\mathbf{i} + (1.2\alpha_{BC} - 1.8)\mathbf{j}$$
$$a_C = -54 \text{ m/s}^2 = 54 \text{ m/s}^2 \leftarrow \qquad \textit{Ans.}$$
$$0 = 1.2\alpha_{BC} - 1.8 \quad \alpha_{BC} = 1.5 \text{ rad/s}^2 \qquad \textit{Ans.}$$

Scalar Analysis

$$\mathbf{a}_C = \mathbf{a}_B + \mathbf{a}_{C/B}$$
$$\begin{bmatrix} a_C \\ \leftarrow \end{bmatrix} = \begin{bmatrix} 6(0.3) \\ \downarrow \end{bmatrix} + \begin{bmatrix} (12)^2(0.3) \\ \leftarrow \end{bmatrix} + \begin{bmatrix} \alpha_{BC}(1.2) \\ \uparrow \end{bmatrix} + \begin{bmatrix} (3)^2(1.2) \\ \leftarrow \end{bmatrix}$$
$$\overset{+}{\leftarrow} \qquad a_C = 43.2 + 10.8 = 54 \text{ m/s}^2 \leftarrow$$
$$+\uparrow \qquad 0 = -6(0.3) + 1.2\alpha_{BC}$$
$$\alpha_{BC} = 1.5 \text{ rad/s}^2$$

F16–24. $v_B = \omega\, r_{B/A} = 6(0.2) = 1.2 \text{ m/s} \rightarrow$

$$r_{B/IC} = 0.8 \tan 60^\circ = 1.3856 \text{ m}$$
$$\omega_{BC} = \frac{v_B}{r_{B/IC}} = \frac{1.2}{1.3856} = 0.8660 \text{ rad/s}$$

Vector Analysis

$$\mathbf{a}_B = \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{B/A}$$
$$= (-3\mathbf{k}) \times (0.2\mathbf{j}) - 6^2 (0.2\mathbf{j})$$
$$= [0.6\mathbf{i} - 7.2\mathbf{j}] \text{ m/s}$$
$$\mathbf{a}_C = \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega^2 \mathbf{r}_{C/B}$$
$$a_C \cos 30^\circ \mathbf{i} + a_C \sin 30^\circ \mathbf{j}$$
$$= (0.6\mathbf{i} - 7.2\mathbf{j}) + (\alpha_{BC}\, \mathbf{k} \times 0.8\mathbf{i}) - 0.8660^2(0.8\mathbf{i})$$

$0.8660a_C\,\mathbf{i} + 0.5a_C\,\mathbf{j} = (0.8\alpha_{BC} - 7.2)\mathbf{j}$

$0.8660a_C = 0$ (1)

$0.5a_C = 0.8\alpha_{BC} - 7.2$ (2)

$a_C = 0 \qquad \alpha_{BC} = 9\ \text{rad/s}^2$ *Ans.*

Scalar Analysis

$\mathbf{a}_C = \mathbf{a}_B + \mathbf{a}_{C/B}$

$$\begin{bmatrix} a_C \\ \measuredangle 30^\circ \end{bmatrix} = \begin{bmatrix} 3(0.2) \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (6)^2(0.2) \\ \downarrow \end{bmatrix} + \begin{bmatrix} \alpha_{BC}(0.8) \\ \uparrow \end{bmatrix} + \begin{bmatrix} (0.8660)^2(0.8) \\ \leftarrow \end{bmatrix}$$

This yields Eqs. (1) and (2).

Chapter 17

F17–1. $\overset{+}{\rightarrow} \Sigma F_x = m(a_G)_x;\ 100\left(\frac{4}{5}\right) = 100a$

$a = 0.8\ \text{m/s}^2 \rightarrow$ *Ans.*

$+\uparrow \Sigma F_y = m(a_G)_y;$

$N_A + N_B - 100\left(\frac{3}{5}\right) - 100(9.81) = 0$ (1)

$\circlearrowleft + \Sigma M_G = 0;$

$N_A(0.6) + 100\left(\frac{3}{5}\right)(0.7)$

$\qquad - N_B(0.4) - 100\left(\frac{4}{5}\right)(0.7) = 0$ (2)

$N_A = 430.4\ \text{N} = 430\ \text{N}$ *Ans.*

$N_B = 610.6\ \text{N} = 611\ \text{N}$ *Ans.*

F17–2. $\Sigma F_{x'} = m(a_G)_{x'};\quad 80(9.81)\sin 15^\circ = 80a$

$a = 2.54\ \text{m/s}^2$ *Ans.*

$\Sigma F_{y'} = m(a_G)_{y'};$

$N_A + N_B - 80(9.81)\cos 15^\circ = 0$ (1)

$\circlearrowleft + \Sigma M_G = 0;$

$N_A(0.5) - N_B(0.5) = 0$ (2)

$N_A = N_B = 379\ \text{N}$ *Ans.*

F17–3. $\circlearrowleft + \Sigma M_A = \Sigma(\mathcal{M}_k)_A;\quad 10\left(\frac{3}{5}\right)(7) = \frac{20}{32.2}a(3.5)$

$a = 19.3\ \text{ft/s}^2$ *Ans.*

$\overset{+}{\rightarrow} \Sigma F_x = m(a_G)_x;\quad A_x + 10\left(\frac{3}{5}\right) = \frac{20}{32.2}(19.32)$

$A_x = 6\ \text{lb}$ *Ans.*

$+\uparrow \Sigma F_y = m(a_G)_y;\quad A_y - 20 + 10\left(\frac{4}{5}\right) = 0$

$A_y = 12\ \text{lb}$ *Ans.*

F17–4. $F_A = \mu_s N_A = 0.2N_A \quad F_B = \mu_s N_B = 0.2N_B$

$\overset{+}{\rightarrow} \Sigma F_x = m(a_G)_x;$

$0.2N_A + 0.2N_B = 100a$ (1)

$+\uparrow \Sigma F_y = m(a_G)_y;$

$N_A + N_B - 100(9.81) = 0$ (2)

$\circlearrowleft + \Sigma M_G = 0;$

$0.2N_A(0.75) + N_A(0.9) + 0.2N_B(0.75)$

$\qquad - N_B(0.6) = 0$ (3)

Solving Eqs. (1), (2), and (3),

$N_A = 294.3\ \text{N} = 294\ \text{N}$

$N_B = 686.7\ \text{N} = 687\ \text{N}$

$a = 1.96\ \text{m/s}^2$ *Ans.*

Since N_A is positive, the table will indeed slide before it tips.

F17–5. $(a_G)_t = \alpha r = \alpha(1.5\ \text{m})$

$(a_G)_n = \omega^2 r = (5\ \text{rad/s})^2(1.5\ \text{m}) = 37.5\ \text{m/s}^2$

$\Sigma F_t = m(a_G)_t;\qquad 100\ \text{N} = 50\ \text{kg}[\alpha(1.5\ \text{m})]$

$\alpha = 1.33\ \text{rad/s}^2$ *Ans.*

$\Sigma F_n = m(a_G)_n;\qquad T_{AB} + T_{CD} - 50(9.81)\ \text{N}$

$\qquad = 50\ \text{kg}(37.5\ \text{m/s}^2)$

$T_{AB} + T_{CD} = 2365.5$

$\circlearrowleft + \Sigma M_G = 0;\quad T_{CD}(1\ \text{m}) - T_{AB}(1\ \text{m}) = 0$

$T_{AB} = T_{CD} = 1182.75\ \text{N} = 1.18\ \text{kN}$ *Ans.*

F17–6. $\circlearrowleft + \Sigma M_C = 0;$

$\mathbf{a}_G = \mathbf{a}_D = \mathbf{a}_B$

$D_y(0.6) - 450 = 0 \quad D_y = 750\ \text{N}$ *Ans.*

$(a_G)_n = \omega^2 r = 6^2(0.6) = 21.6\ \text{m/s}^2$

$(a_G)_t = \alpha r = \alpha(0.6)$

$+\uparrow \Sigma F_t = m(a_G)_t;$

$750 - 50(9.81) = 50[\alpha(0.6)]$

$\alpha = 8.65\ \text{rad/s}^2$ *Ans.*

$\overset{+}{\rightarrow} \Sigma F_n = m(a_G)_n;$

$F_{AB} + D_x = 50(21.6)$ (1)

$\circlearrowleft + \Sigma M_G = 0;$

$D_x(0.4) + 750(0.1) - F_{AB}(0.4) = 0$ (2)

$D_x = 446.25\ \text{N} = 446\ \text{N}$ *Ans.*

$F_{AB} = 633.75\ \text{N} = 634\ \text{N}$ *Ans.*

F17–7. $I_O = mk_O^2 = 100(0.5^2) = 25\ \text{kg}\cdot\text{m}^2$

$\circlearrowleft + \Sigma M_O = I_O\alpha; \qquad -100(0.6) = -25\alpha$

$\alpha = 2.4\ \text{rad/s}^2$

$\omega = \omega_0 + \alpha_c t$

$\omega = 0 + 2.4(3) = 7.2\ \text{rad/s}$ *Ans.*

F17–8. $I_O = \frac{1}{2}mr^2 = \frac{1}{2}(50)(0.3^2) = 2.25\ \text{kg}\cdot\text{m}^2$

$\circlearrowleft + \Sigma M_O = I_O\alpha;$

$-9t = -2.25\alpha \qquad \alpha = (4t)\ \text{rad/s}^2$

$d\omega = \alpha\, dt$

$\int_0^\omega d\omega = \int_0^t 4t\, dt$

$\omega = (2t^2)\ \text{rad/s}$

$\omega = 2(4^2) = 32\ \text{rad/s}$ *Ans.*

F17–9. $(a_G)_t = \alpha r_G = \alpha(0.15)$

$(a_G)_n = \omega^2 r_G = 6^2(0.15) = 5.4\ \text{m/s}^2$

$I_O = I_G + md^2 = \frac{1}{12}(30)(0.9^2) + 30(0.15^2)$

$= 2.7\ \text{kg}\cdot\text{m}^2$

$\circlearrowleft + \Sigma M_O = I_O\alpha; \quad 60 - 30(9.81)(0.15) = 2.7\alpha$

$\alpha = 5.872\ \text{rad/s}^2 = 5.87\ \text{rad/s}^2$ *Ans.*

$\xleftarrow{+} \Sigma F_n = m(a_G)_n; \quad O_n = 30(5.4) = 162\ \text{N}$ *Ans.*

$+\uparrow \Sigma F_t = m(a_G)_t;$

$O_t - 30(9.81) = 30[5.872(0.15)]$

$O_t = 320.725\ \text{N} = 321\ \text{N}$ *Ans.*

F17–10. $(a_G)_t = \alpha r_G = \alpha(0.3)$

$(a_G)_n = \omega^2 r_G = 10^2(0.3) = 30\ \text{m/s}^2$

$I_O = I_G + md^2 = \frac{1}{2}(30)(0.3^2) + 30(0.3^2)$

$= 4.05\ \text{kg}\cdot\text{m}^2$

$\circlearrowleft + \Sigma M_O = I_O\alpha;$

$50\left(\frac{3}{5}\right)(0.3) + 50\left(\frac{4}{5}\right)(0.3) = 4.05\alpha$

$\alpha = 5.185\ \text{rad/s}^2 = 5.19\ \text{rad/s}^2$ *Ans.*

$+\uparrow \Sigma F_n = m(a_G)_n;$

$O_n + 50\left(\frac{3}{5}\right) - 30(9.81) = 30(30)$

$O_n = 1164.3\ \text{N} = 1.16\text{kN}$ *Ans.*

$\xrightarrow{+} \Sigma F_t = m(a_G)_t;$

$O_t + 50\left(\frac{4}{5}\right) = 30[5.185(0.3)]$

$O_t = 6.67\ \text{N}$ *Ans.*

F17–11. $I_G = \frac{1}{12}ml^2 = \frac{1}{12}(15\ \text{kg})(0.9\ \text{m})^2 = 1.0125\ \text{kg}\cdot\text{m}^2$

$(a_G)_n = \omega^2 r_G = 0$

$(a_G)_t = \alpha(0.15\ \text{m})$

$I_O = I_G + md_{OG}^2$

$= 1.0125\ \text{kg}\cdot\text{m}^2 + 15\ \text{kg}(0.15\ \text{m})^2$

$= 1.35\ \text{kg}\cdot\text{m}^2$

$\circlearrowleft + \Sigma M_O = I_O\alpha;$

$[15(9.81)\ \text{N}](0.15\ \text{m}) = (1.35\ \text{kg}\cdot\text{m}^2)\alpha$

$\alpha = 16.35\ \text{rad/s}^2$ *Ans.*

$+\downarrow \Sigma F_t = m(a_G)_t; \quad -O_t + 15(9.81)\text{N}$

$= (15\ \text{kg})[16.35\ \text{rad/s}^2(0.15\ \text{m})]$

$O_t = 110.36\ \text{N} = 110\ \text{N}$ *Ans.*

$\xrightarrow{+} \Sigma F_n = m(a_G)_n; \qquad O_n = 0$ *Ans.*

F17–12. $(a_G)_t = \alpha r_G = \alpha(0.45)$

$(a_G)_n = \omega^2 r_G = 6^2(0.45) = 16.2\ \text{m/s}^2$

$I_O = \frac{1}{3}ml^2 = \frac{1}{3}(30)(0.9^2) = 8.1\ \text{kg}\cdot\text{m}^2$

$\circlearrowleft + \Sigma M_O = I_O\alpha;$

$300\left(\frac{4}{5}\right)(0.6) - 30(9.81)(0.45) = 8.1\alpha$

$\alpha = 1.428\ \text{rad/s}^2 = 1.43\ \text{rad/s}^2$ *Ans.*

$\xleftarrow{+} \Sigma F_n = m(a_G)_n; \quad O_n + 300\left(\frac{3}{5}\right) = 30(16.2)$

$O_n = 306\ \text{N}$ *Ans.*

$+\uparrow \Sigma F_t = m(a_G)_t; \quad O_t + 300\left(\frac{4}{5}\right) - 30(9.81)$

$= 30[1.428(0.45)]$

$O_t = 73.58\ \text{N} = 73.6\ \text{N}$ *Ans.*

F17–13. $I_G = \frac{1}{12}ml^2 = \frac{1}{12}(60)(3^2) = 45\ \text{kg}\cdot\text{m}^2$

$+\uparrow \Sigma F_y = m(a_G)_y;$

$80 - 20 = 60a_G \quad a_G = 1\ \text{m/s}^2\uparrow$

$\circlearrowleft + \Sigma M_G = I_G\alpha; \qquad 80(1) + 20(0.75) = 45\alpha$

$\alpha = 2.11\ \text{rad/s}^2$ *Ans.*

F17–14. $\circlearrowleft + \Sigma M_A = (\mathcal{M}_k)_A;$

$-200(0.3) = -100a_G(0.3) - 4.5\alpha$

$30a_G + 4.5\alpha = 60 \qquad (1)$

$a_G = \alpha r = \alpha(0.3) \qquad (2)$

$\alpha = 4.44\ \text{rad/s}^2 \quad a_G = 1.33\ \text{m/s}^2 \rightarrow$ *Ans.*

F17–15. $+\uparrow \Sigma F_y = m(a_G)_y;$

$N - 20(9.81) = 0 \quad N = 196.2\ \text{N}$

$\xrightarrow{+} \Sigma F_x = m(a_G)_x; \quad 0.5(196.2) = 20a_O$

$a_O = 4.905\ \text{m/s}^2 \rightarrow$ *Ans.*

$\circlearrowleft + \Sigma M_O = I_O\alpha;$

$0.5(196.2)(0.4) - 100 = -1.8\alpha$

$\alpha = 33.8 \text{ rad/s}^2$ *Ans.*

F17–16. Sphere $I_G = \frac{2}{5}(20)(0.15)^2 = 0.18 \text{ kg}\cdot\text{m}^2$

$\circlearrowleft + \Sigma M_{IC} = (\mathcal{M}_k)_{IC};$

$20(9.81)\sin 30°(0.15) = 0.18\alpha + (20a_G)(0.15)$

$0.18\alpha + 3a_G = 14.715$

$a_G = \alpha r = \alpha(0.15)$

$\alpha = 23.36 \text{ rad/s}^2 = 23.4 \text{ rad/s}^2$ *Ans.*

$a_G = 3.504 \text{ m/s}^2 = 3.50 \text{ m/s}^2$ *Ans.*

F17–17. $+\uparrow \Sigma F_y = m(a_G)_y;$

$N - 200(9.81) = 0 \quad N = 1962 \text{ N}$

$\xrightarrow{+} \Sigma F_x = m(a_G)_x;$

$T - 0.2(1962) = 200a_G \qquad (1)$

$\circlearrowleft + \Sigma M_A = (\mathcal{M}_k)_A; \quad 450 - 0.2(1962)(1)$

$= 18\alpha + 200a_G(0.4) \qquad (2)$

$(a_A)_t = 0 \qquad a_A = (a_A)_n$

$\mathbf{a}_G = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{G/A} - \omega^2\mathbf{r}_{G/A}$

$a_G\mathbf{i} = -a_A\mathbf{j} + \alpha\mathbf{k} \times (-0.4\mathbf{j}) - \omega^2(-0.4\mathbf{j})$

$a_G\mathbf{i} = 0.4\alpha\mathbf{i} + (0.4\omega^2 - a_A)\mathbf{j}$

$a_G = 0.4\alpha \qquad (3)$

Solving Eqs. (1), (2), and (3),

$\alpha = 1.15 \text{ rad/s}^2 \quad a_G = 0.461 \text{ m/s}^2$

$T = 485 \text{ N}$ *Ans.*

F17–18. $\xrightarrow{+} \Sigma F_x = m(a_G)_x; \quad 0 = 12(a_G)_x \quad (a_G)_x = 0$

$\circlearrowleft + \Sigma M_A = (\mathcal{M}_k)_A$

$-12(9.81)(0.3) = 12(a_G)_y(0.3) - \frac{1}{12}(12)(0.6)^2\alpha$

$0.36\alpha - 3.6(a_G)_y = 35.316 \qquad (1)$

$\omega = 0$

$\mathbf{a}_G = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{G/A} - \omega^2\mathbf{r}_{G/A}$

$(a_G)_y\,\mathbf{j} = a_A\mathbf{i} + (-\alpha\mathbf{k}) \times (0.3\mathbf{i}) - \mathbf{0}$

$(a_G)_y\,\mathbf{j} = (a_A)\mathbf{i} - 0.3\,\mathbf{j}$

$a_A = 0$ *Ans.*

$(a_G)_y = -0.3\alpha \qquad (2)$

Solving Eqs. (1) and (2)

$\alpha = 24.5 \text{ rad/s}^2$

$(a_G)_y = -7.36 \text{ m/s}^2 = 7.36 \text{ m/s}^2\downarrow$ *Ans.*

Chapter 18

F18–1. $I_O = mk_O^2 = 80\left(0.4^2\right) = 12.8 \text{ kg}\cdot\text{m}^2$

$T_1 = 0$

$T_2 = \frac{1}{2}I_O\omega^2 = \frac{1}{2}(12.8)\omega^2 = 6.4\omega^2$

$s = \theta r = 20(2\pi)(0.6) = 24\pi \text{ m}$

$T_1 + \Sigma U_{1-2} = T_2$

$0 + 50(24\pi) = 6.4\omega^2$

$\omega = 24.3 \text{ rad/s}$ *Ans.*

F18–2. $T_1 = 0$

$T_2 = \frac{1}{2}m(v_G)_2^2 + \frac{1}{2}I_G\omega_2^2$

$= \frac{1}{2}\left(\frac{50}{32.2}\text{ slug}\right)(2.5\omega_2)^2$

$+ \frac{1}{2}\left[\frac{1}{12}\left(\frac{50}{32.2}\text{ slug}\right)(5\text{ ft})^2\right]\omega_2^2$

$T_2 = 6.4700\omega_2^2$

Or,

$I_O = \frac{1}{3}ml^2 = \frac{1}{3}\left(\frac{50}{32.2}\text{ slug}\right)(5\text{ ft})^2$

$= 12.9400 \text{ slug}\cdot\text{ft}^2$

So that

$T_2 = \frac{1}{2}I_O\omega_2^2 = \frac{1}{2}(12.9400 \text{ slug}\cdot\text{ft}^2)\omega_2^2$

$= 6.4700\omega_2^2$

$T_1 + \Sigma U_{1-2} = T_2$

$T_1 + [-Wy_G + M\theta] = T_2$

$0 + \left[-(50\text{ lb})(2.5\text{ ft}) + (100\text{ lb}\cdot\text{ft})\left(\frac{\pi}{2}\right)\right]$

$= 6.4700{\omega_2}^2$

$\omega_2 = 2.23 \text{ rad/s}$ *Ans.*

F18–3. $(v_G)_2 = \omega_2 r_{G/IC} = \omega_2(2.5)$

$I_G = \frac{1}{12}ml^2 = \frac{1}{12}(50)\left(5^2\right) = 104.17 \text{ kg}\cdot\text{m}^2$

$T_1 = 0$

$T_2 = \frac{1}{2}m(v_G)_2^2 + \frac{1}{2}I_G\omega_2^2$

$= \frac{1}{2}(50)\left[\omega_2(2.5)\right]^2 + \frac{1}{2}(104.17)\omega_2^2 = 208.33\omega_2^2$

$U_P = Ps_P = 600(3) = 1800 \text{ J}$

$U_W = -Wh = -50(9.81)(2.5 - 2) = -245.25 \text{ J}$

$T_1 + \Sigma U_{1-2} = T_2$

$0 + 1800 + (-245.25) = 208.33\omega_2^2$

$\omega_2 = 2.732 \text{ rad/s} = 2.73 \text{ rad/s}$ *Ans.*

F18–4. $T = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2$

$= \frac{1}{2}(50\text{ kg})(0.4\omega)^2 + \frac{1}{2}\left[50\text{ kg}(0.3\text{ m})^2\right]\omega^2$

$= 6.25\omega^2\text{ J}$

Or,

$T = \frac{1}{2} I_{IC}\omega^2$

$= \frac{1}{2}\left[50\text{ kg}(0.3\text{ m})^2 + 50\text{ kg}(0.4\text{ m})^2\right]\omega^2$

$= 6.25\omega^2\text{ J}$

$s_G = \theta r = 10(2\pi\text{ rad})(0.4\text{ m}) = 8\pi\text{ m}$

$T_1 + \Sigma U_{1-2} = T_2$

$T_1 + P\cos 30^\circ\, s_G = T_2$

$0 + (50\text{ N})\cos 30^\circ(8\pi\text{ m}) = 6.25\omega^2\text{ J}$

$\omega = 13.2\text{ rad/s}$ *Ans.*

F18–5. $I_G = \frac{1}{12} m l^2 = \frac{1}{12}(30)\left(3^2\right) = 22.5\text{ kg}\cdot\text{m}^2$

$T_1 = 0$

$T_2 = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G\omega^2$

$= \frac{1}{2}(30)[\omega(0.5)]^2 + \frac{1}{2}(22.5)\omega^2 = 15\omega^2$

Or,

$I_O = I_G + md^2 = \frac{1}{12}(30)\left(3^2\right) + 30\left(0.5^2\right)$

$= 30\text{ kg}\cdot\text{m}^2$

$T_2 = \frac{1}{2} I_O\omega^2 = \frac{1}{2}(30)\omega^2 = 15\omega^2$

$s_1 = \theta r_1 = 8\pi(0.5) = 4\pi\text{ m}$

$s_2 = \theta r_2 = 8\pi(1.5) = 12\pi\text{ m}$

$U_{P_1} = P_1 s_1 = 30(4\pi) = 120\pi\text{ J}$

$U_{P_2} = P_2 s_2 = 20(12\pi) = 240\pi\text{ J}$

$U_M = M\theta = 20[4(2\pi)] = 160\pi\text{ J}$

$U_W =$ (0 bar returns to same position)

$T_1 + \Sigma U_{1-2} = T_2$

$0 + 120\pi + 240\pi + 160\pi = 15\omega^2$

$\omega = 10.44\text{ rad/s} = 10.4\text{ rad/s}$ *Ans.*

F18–6. $v_G = \omega r = \omega(0.4)$

$I_G = m k_G^2 = 20\left(0.3^2\right) = 1.8\text{ kg}\cdot\text{m}^2$

$T_1 = 0$

$T_2 = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G\omega^2$

$= \frac{1}{2}(20)\left[\omega(0.4)\right]^2 + \frac{1}{2}(1.8)\omega^2$

$= 2.5\omega^2$

$U_M = M\theta = M\left(\frac{s_O}{r}\right) = 50\left(\frac{20}{0.4}\right) = 2500\text{ J}$

$T_1 + \Sigma U_{1-2} = T_2$

$0 + 2500 = 2.5\omega^2$

$\omega = 31.62\text{ rad/s} = 31.6\text{ rad/s}$ *Ans.*

F18–7. $v_G = \omega r = \omega(0.3)$

$I_G = \frac{1}{2} m r^2 = \frac{1}{2}(30)\left(0.3^2\right) = 1.35\text{ kg}\cdot\text{m}^2$

$T_1 = 0$

$T_2 = \frac{1}{2} m (v_G)_2^2 + \frac{1}{2} I_G\omega_2^2$

$= \frac{1}{2}(30)[\omega_2(0.3)]^2 + \frac{1}{2}(1.35)\omega_2^2 = 2.025\omega_2^2$

$\left(V_g\right)_1 = W y_1 = 0$

$\left(V_g\right)_2 = -W y_2 = -30(9.81)(0.3) = -88.29\text{ J}$

$T_1 + V_1 = T_2 + V_2$

$0 + 0 = 2.025\omega_2^2 + (-88.29)$

$\omega_2 = 6.603\text{ rad/s} = 6.60\text{ rad/s}$ *Ans.*

F18–8. $v_O = \omega r_{O/IC} = \omega(0.2)$

$I_O = m k_O^2 = 50\left(0.3^2\right) = 4.5\text{ kg}\cdot\text{m}^2$

$T_1 = 0$

$T_2 = \frac{1}{2} m (v_O)_2^2 + \frac{1}{2} I_O\omega_2^2$

$= \frac{1}{2}(50)\left[\omega_2(0.2)\right]^2 + \frac{1}{2}(4.5)\omega_2^2$

$= 3.25\omega_2^2$

$\left(V_g\right)_1 = W y_1 = 0$

$\left(V_g\right)_2 = -W y_2 = -50(9.81)(6\sin 30^\circ)$

$= -1471.5\text{J}$

$T_1 + V_1 = T_2 + V_2$

$0 + 0 = 3.25\omega_2^2 + (-1471.5)$

$\omega_2 = 21.28\text{ rad/s} = 21.3\text{ rad/s}$ *Ans.*

F18–9. $v_G = \omega r_G = \omega(1.5)$

$I_G = \frac{1}{12}(60)\left(3^2\right) = 45\text{ kg}\cdot\text{m}^2$

$T_1 = 0$

$T_2 = \frac{1}{2} m (v_G)_2^2 + \frac{1}{2} I_G\omega_2^2$

$= \frac{1}{2}(60)[\omega_2(1.5)]^2 + \frac{1}{2}(45)\omega_2^2$

$= 90\omega_2^2$

Or,

$T_2 = \frac{1}{2} I_O\omega_2^2 = \frac{1}{2}\left[45 + 60\left(1.5^2\right)\right]\omega_2^2 = 90\omega_2^2$

$\left(V_g\right)_1 = W y_1 = 0$

$\left(V_g\right)_2 = -W y_2 = -60(9.81)(1.5\sin 45^\circ)$

$= -624.30\text{ J}$

$\left(V_e\right)_1 = \frac{1}{2} k s_1^2 = 0$

$\left(V_e\right)_2 = \frac{1}{2} k s_2^2 = \frac{1}{2}(150)(3\sin 45^\circ)^2 = 337.5\text{ J}$

$T_1 + V_1 = T_2 + V_2$

$0 + 0 = 90\omega_2^2 + [-624.30 + 337.5]$

$\omega_2 = 1.785\text{ rad/s} = 1.79\text{ rad/s}$ *Ans.*

F18–10. $v_G = \omega r_G = \omega(0.75)$

$I_G = \frac{1}{12}(30)\left(1.5^2\right) = 5.625\text{ kg}\cdot\text{m}^2$

$T_1 = 0$

$$T_2 = \tfrac{1}{2} m(v_G)_2^2 + \tfrac{1}{2} I_G\omega_2^2$$
$$= \tfrac{1}{2}(30)[\omega(0.75)]^2 + \tfrac{1}{2}(5.625)\omega_2^2 = 11.25\omega_2^2$$

Or,

$$T_2 = \tfrac{1}{2} I_O\omega_2^2 = \tfrac{1}{2}\left[5.625 + 30\left(0.75^2\right)\right]\omega_2^2$$
$$= 11.25\omega_2^2$$
$$\left(V_g\right)_1 = Wy_1 = 0$$
$$\left(V_g\right)_2 = -Wy_2 = -30(9.81)(0.75)$$
$$= -220.725 \text{ J}$$
$$(V_e)_1 = \tfrac{1}{2}ks_1^2 = 0$$
$$(V_e)_2 = \tfrac{1}{2}ks_2^2 = \tfrac{1}{2}(80)\left(\sqrt{2^2 + 1.5^2} - 0.5\right)^2 = 160 \text{ J}$$
$$T_1 + V_1 = T_2 + V_2$$
$$0 + 0 = 11.25\omega_2^2 + (-220.725 + 160)$$
$$\omega_2 = 2.323 \text{ rad/s} = 2.32 \text{ rad/s} \qquad \textit{Ans.}$$

F18–11. $(v_G)_2 = \omega_2 r_{G/IC} = \omega_2(0.75)$

$$I_G = \tfrac{1}{12}(30)\left(1.5^2\right) = 5.625 \text{ kg}\cdot\text{m}^2$$
$$T_1 = 0$$
$$T_2 = \tfrac{1}{2}m(v_G)_2^2 + \tfrac{1}{2}I_G\omega_2^2$$
$$= \tfrac{1}{2}(30)[\omega_2(0.75)]^2 + \tfrac{1}{2}(5.625)\omega_2^2 = 11.25\omega_2^2$$
$$\left(V_g\right)_1 = Wy_1 = 30(9.81)(0.75 \sin 45^\circ) = 156.08 \text{ J}$$
$$\left(V_g\right)_2 = -Wy_2 = 0$$
$$\left(V_e\right)_1 = \tfrac{1}{2}ks_1^2 = 0$$
$$(V_e)_2 = \tfrac{1}{2}ks_2^2 = \tfrac{1}{2}(300)(1.5 - 1.5\cos 45^\circ)^2$$
$$= 28.95 \text{ J}$$
$$T_1 + V_1 = T_2 + V_2$$
$$0 + (156.08 + 0) = 11.25\omega_2^2 + (0 + 28.95)$$
$$\omega_2 = 3.362 \text{ rad/s} = 3.36 \text{ rad/s} \qquad \textit{Ans.}$$

F18–12. $(V_g)_1 = -Wy_1 = -[20(9.81) \text{ N}](1 \text{ m}) = -196.2 \text{ J}$

$$(V_g)_2 = 0$$
$$(V_e)_1 = \tfrac{1}{2}ks_1^2$$
$$= \tfrac{1}{2}(100 \text{ N/m})\left(\sqrt{(3 \text{ m})^2 + (2 \text{ m})^2} - 0.5 \text{ m}\right)^2$$
$$= 482.22 \text{ J}$$
$$(V_e)_2 = \tfrac{1}{2}ks_2^2 = \tfrac{1}{2}(100 \text{ N/m})(1 \text{ m} - 0.5 \text{ m})^2$$
$$= 12.5 \text{ J}$$
$$T_1 = 0$$
$$T_2 = \tfrac{1}{2}I_A\omega^2 = \tfrac{1}{2}\left[\tfrac{1}{3}(20 \text{ kg})(2 \text{ m})^2\right]\omega^2$$
$$= 13.3333\omega^2$$
$$T_1 + V_1 = T_2 + V_2$$
$$0 + [-196.2 \text{ J} + 482.22 \text{ J}]$$
$$= 13.3333\omega_2^2 + [0 + 12.5 \text{ J}]$$
$$\omega_2 = 4.53 \text{ rad/s} \qquad \textit{Ans.}$$

Chapter 19

F19–1. $\circlearrowleft + I_O\omega_1 + \Sigma\int_{t_1}^{t_2} M_O\,dt = I_O\omega_2$

$$0 + \int_0^{4\text{ s}} 3t^2\,dt = \left[60\left(0.3\right)^2\right]\omega_2$$
$$\omega_2 = 11.85 \text{ rad/s} = 11.9 \text{ rad/s} \qquad \textit{Ans.}$$

F19–2. $\circlearrowleft + (H_A)_1 + \Sigma\int_{t_1}^{t_2} M_A dt = (H_A)_2$

$$0 + 300(6) = 300\left(0.4^2\right)\omega_2 + 300[\omega(0.6)](0.6)$$
$$\omega_2 = 11.54 \text{ rad/s} = 11.5 \text{ rad/s} \qquad \textit{Ans.}$$
$$\overset{+}{\rightarrow} \quad m(v_1)_x + \Sigma\int_{t_1}^{t_2} F_x dt = m(v_2)_x$$
$$0 + F_f(6) = 300[11.54(0.6)]$$
$$F_f = 346 \text{ N} \qquad \textit{Ans.}$$

F19–3. $v_A = \omega_A r_{A/IC} = \omega_A(0.15)$

$$\circlearrowleft + \Sigma M_O = 0; \quad 9 - A_t(0.45) = 0 \quad A_t = 20 \text{ N}$$
$$\circlearrowleft + (H_C)_1 + \Sigma\int_{t_1}^{t_2} M_C\,dt = (H_C)_2$$
$$0 + [20(5)](0.15)$$
$$= 10[\omega_A(0.15)](0.15)$$
$$+ \left[10\left(0.1^2\right)\right]\omega_A$$
$$\omega_A = 46.2 \text{ rad/s} \qquad \textit{Ans.}$$

F19–4. $I_A = mk_A^2 = 10\left(0.08^2\right) = 0.064 \text{ kg}\cdot\text{m}^2$

$$I_B = mk_B^2 = 50\left(0.15^2\right) = 1.125 \text{ kg}\cdot\text{m}^2$$
$$\omega_A = \left(\frac{r_B}{r_A}\right)\omega_B = \left(\frac{0.2}{0.1}\right)\omega_B = 2\omega_B$$
$$\circlearrowleft + I_A(\omega_A)_1 + \Sigma\int_{t_1}^{t_2} M_A\,dt = I_A(\omega_A)_2$$
$$0 + 10(5) - \int_0^{5\text{ s}} F(0.1)dt = 0.064[2(\omega_B)_2]$$
$$\int_0^{5\text{ s}} Fdt = 500 - 1.28(\omega_B)_2 \qquad (1)$$
$$\circlearrowleft + I_B(\omega_B)_1 + \Sigma\int_{t_1}^{t_2} M_B\,dt = I_B(\omega_B)_2$$
$$0 + \int_0^{5\text{ s}} F(0.2)dt = 1.125(\omega_B)_2$$
$$\int_0^{5\text{ s}} Fdt = 5.625(\omega_B)_2 \qquad (2)$$

Equating Eqs. (1) and (2),

$$500 - 1.28(\omega_B)_2 = 5.625(\omega_B)_2$$
$$(\omega_B)_2 = 72.41 \text{ rad/s} = 72.4 \text{ rad/s} \qquad \textit{Ans.}$$

F19–5. $(\overset{+}{\rightarrow})\quad m\left[(v_O)_x\right]_1 + \Sigma\int F_x\,dt = m\left[(v_O)_x\right]_2$

$$0 + (150\text{ N})(3\text{ s}) + F_A(3\text{ s})$$
$$= (50\text{ kg})(0.3\omega_2)$$
$$\circlearrowleft + I_G\omega_1 + \Sigma\int M_G\,dt = I_G\omega_2$$
$$0 + (150\text{ N})(0.2\text{ m})(3\text{ s}) - F_A(0.3\text{ m})(3\text{ s})$$
$$= [(50\text{ kg})(0.175\text{ m})^2]\,\omega_2$$
$$\omega_2 = 37.3\text{ rad/s} \qquad \textit{Ans.}$$
$$F_A = 36.53\text{ N}$$

Also,

$$I_{IC}\omega_1 + \Sigma\int M_{IC}\,dt = I_{IC}\omega_2$$
$$0 + [(150\text{ N})(0.2 + 0.3)\text{ m}](3\text{ s})$$
$$= [(50\text{ kg})(0.175\text{ m})^2 + (50\text{ kg})(0.3\text{ m})^2]\omega_2$$
$$\omega_2 = 37.3\text{ rad/s} \qquad \textit{Ans.}$$

F19–6. $(+\uparrow)\ m\left[(v_G)_1\right]_y + \Sigma\int F_y\,dt = m\left[(v_G)_2\right]_y$

$$0 + N_A(3\text{ s}) - (150\text{ lb})(3\text{ s}) = 0$$
$$N_A = 150\text{ lb}$$
$$\circlearrowleft + (H_{IC})_1 + \Sigma\int M_{IC}\,dt = (H_{IC})_2$$
$$0 + (25\text{ lb}\cdot\text{ft})(3\text{ s}) - [0.15(150\text{ lb})(3\text{ s})](0.5\text{ ft})$$
$$= \left[\tfrac{150}{32.2}\text{ slug}(1.25\text{ ft})^2\right]\omega_2 + \left(\tfrac{150}{32.2}\text{ slug}\right)\left[\omega_2(1\text{ ft})\right](1\text{ ft})$$
$$\omega_2 = 3.46\text{ rad/s} \qquad \textit{Ans.}$$

Preliminary Problems Dynamics Solutions

Chapter 12

P12–1. a) $v = \dfrac{ds}{dt} = \dfrac{d}{dt}(2t^3) = 6t^2\Big|_{t=2\text{ s}} = 24 \text{ m/s}$

b) $a\,ds = v\,dv, \quad v = 5s, \quad dv = 5\,ds$

$a\,ds = (5s)\,5\,ds$

$a = 25s\Big|_{s=1\text{ m}} = 25 \text{ m/s}^2$

c) $a = \dfrac{dv}{dt} = \dfrac{d}{dt}(4t + 5) = 4 \text{ m/s}^2$

d) $v = v_0 + a_c t$

$v = 0 + 2(2) = 4 \text{ m/s}$

e) $v^2 = v_0^2 + 2a_c(s - s_0)$

$v^2 = (3)^2 + 2(2)(4 - 0)$

$v = 5 \text{ m/s}$

f) $a\,ds = v\,dv$

$$\int_{s_1}^{s_2} s\,ds = \int_0^v v\,dv$$

$$s^2\Big|_4^5 = v^2\Big|_0^v$$

$25 - 16 = v^2$

$v = 3 \text{ m/s}$

g) $s = s_0 + v_0 t + \dfrac{1}{2}a_c t^2$

$s = 2 + 2(3) + \dfrac{1}{2}(4)(3)^2 = 26 \text{ m}$

h) $dv = a\,dt$

$$\int_0^v dv = \int_0^1 (8t^2)\,dt$$

$$v = 2.67t^3\Big|_0^1 = 2.67 \text{ m/s}$$

i) $v = \dfrac{ds}{dt} = \dfrac{d}{dt}(3t^2 + 2) = 6t\Big|_{t=2\text{ s}} = 12 \text{ m/s}$

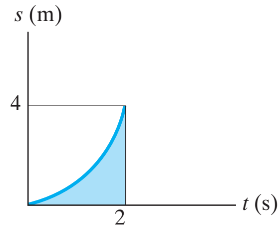
j) $v_{\text{avg}} = \dfrac{\Delta s}{\Delta t} = \dfrac{6 \text{ m} - (-1 \text{ m})}{10 \text{ s} - 0} = 0.7 \text{ m/s} \rightarrow$

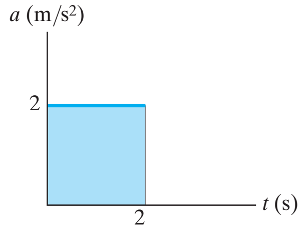
$(v_{\text{sp}})_{\text{avg}} = \dfrac{s_T}{\Delta t} = \dfrac{7 \text{ m} + 14 \text{ m}}{10 \text{ s} - 0} = 2.1 \text{ m/s}$

P12–2. a) $v = 2t$

$s = t^2$

$a = 2$

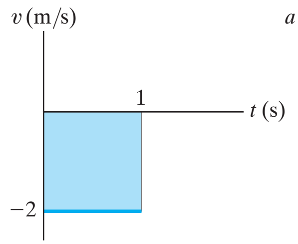


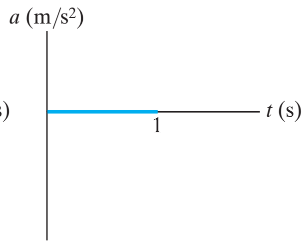


b) $s = -2t + 2$

$v = -2$

$a = 0$

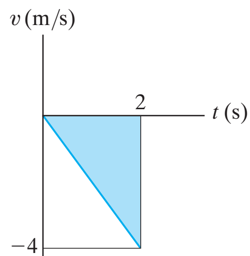


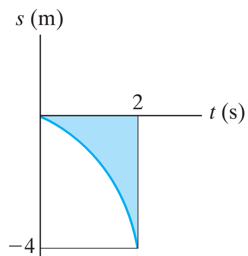


c) $a = -2$

$v = -2t$

$s = -t^2$





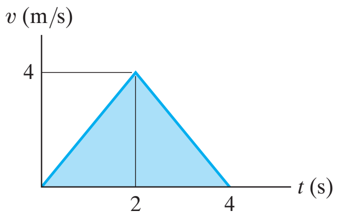
d)

$$\Delta s = \int_0^3 v\,dt = \text{Area} = \frac{1}{2}(2)(2) + 2(3 - 2) = 4 \text{ m}$$

$s - 0 = 4 \text{ m}, \qquad s = 4 \text{ m}$

$a = \dfrac{dv}{dt} = \text{slope at } t = 3 \text{ s}, a = 0$

e) For $a = 2$,
$v = 2t$
When $t = 2$ s, $v = 4$ m/s.
For $a = -2$,
$$\int_4^v dv = \int_2^t -2\,dt$$
$v - 4 = -2t + 4$
$v = -2t + 8$



f) $$\int_1^v v\,dv = \int_0^2 a\,ds = \text{Area}$$
$$\frac{1}{2}v^2 - \frac{1}{2}(1)^2 = \frac{1}{2}(2)(4)$$
$v = 3$ m/s

g) $v\,dv = a\,ds$ At $s = 1$ m, $v = 2$ m/s.
$$a = v\frac{dv}{ds} = v(\text{slope}) = 2(-2) = -4 \text{ m/s}$$

P12–3. a) $y = 4x^2$
$\dot{y} = 8x\dot{x}$
$\ddot{y} = (8\dot{x})\dot{x} + 8x(\ddot{x})$
b) $y = 3e^x$
$\dot{y} = 3e^x\dot{x}$
$\ddot{y} = (3e^x\dot{x})\dot{x} + 3e^x(\ddot{x})$
c) $y = 6\sin x$
$\dot{y} = (6\cos x)\dot{x}$
$\ddot{y} = [(-6\sin x)\dot{x}]\,\dot{x} + (6\cos x)(\ddot{x})$

P12–4. $y_A, t_{AB}, (v_B)_y$
$20 = 0 + 40t_{AB}$
$$0 = y_A + 0 + \frac{1}{2}(-9.81)(t_{AB})^2$$
$(v_B)_y^2 = 0^2 + 2(-9.81)(0 - y_A)$

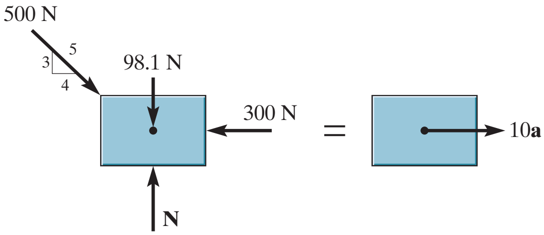
P12–5. $x_B, t_{AB}, (v_B)_y$
$x_B = 0 + (10\cos 30°)(t_{AB})$
$$0 = 8 + (10\sin 30°)t_{AB} + \frac{1}{2}(-9.81)t_{AB}^2$$
$(v_B)_y^2 = 0^2 + 2(-9.81)(0 - 8)$

P12–6. $x_B, y_B, (v_B)_y$
$x_B = 0 + (60\cos 20°)(5)$
$$y_B = 0 + (60\sin 20°)(5) + \frac{1}{2}(-9.81)(5)^2$$
$(v_B)_y = 60\sin 20° + (-9.81)(5)$

P12–7. a) $a_t = \dot{v} = 3 \text{ m/s}^2$
$$a_n = \frac{v^2}{\rho} = \frac{(2)^2}{1} = 4 \text{ m/s}^2$$
$$a = \sqrt{(3)^2 + (4)^2} = 5 \text{ m/s}^2$$
b) $a_t = \dot{v} = 4 \text{ m/s}^2$
$v^2 = v_0^2 + 2a_c(s - s_0)$
$v^2 = 0 + 2(4)(2 - 0)$
$v = 4$ m/s
$$a_n = \frac{v^2}{\rho} = \frac{(4)^2}{2} = 8 \text{ m/s}^2$$
c) $a_t = 0$
$$\rho = \left.\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}\right|_{x=0} = \frac{1+0}{4} = \frac{1}{4}$$
$$a_n = \frac{v^2}{\rho} = \frac{(2)^2}{\frac{1}{4}} = 16 \text{ m/s}^2$$
$$a = \sqrt{(0)^2 + (16)^2} = 16 \text{ m/s}^2$$
d) $a_t\,ds = v\,dv$
$a_t\,ds = (4s + 1)(4\,ds)$
$a_t = (16s + 4)\big|_{s=0} = 4 \text{ m/s}^2$
$$a_n = \frac{v^2}{\rho} = \frac{(4(0) + 1)^2}{2} = 0.5 \text{ m/s}^2$$
e) $a_t\,ds = v\,dv$
$$\int_0^s 2s\,ds = \int_1^v v\,dv$$
$$s^2 = \frac{1}{2}(v^2 - 1)$$
$$v = \left.\sqrt{2s^2 + 1}\right|_{s = 2\text{ m}} = 3 \text{ m/s}$$
$a_t = \dot{v} = 2(2) = 4 \text{ m/s}^2$
$$a_n = \frac{v^2}{\rho} = \frac{(3)^2}{3} = 3 \text{ m/s}^2$$
$$a = \sqrt{(4)^2 + (3)^2} = 5 \text{ m/s}^2$$
f) $$a_t = \dot{v} = 8t\Big|_{t=1} = 8 \text{ m/s}^2$$
$$a_n = \frac{v^2}{\rho} = \frac{(4(1)^2 + 2)^2}{6} = 6 \text{ m/s}^2$$
$$a = \sqrt{(8)^2 + (6)^2} = 10 \text{ m/s}^2$$

Chapter 13

P13–1. a)

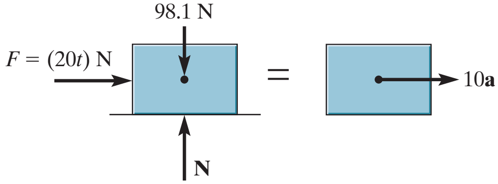


$$\overset{+}{\rightarrow} \Sigma F_x = ma_x; \quad \left(\frac{4}{5}\right)(500 \text{ N}) - 300 \text{ N} = 10a$$

$$a = 10 \text{ m/s}^2$$

$$\overset{+}{\rightarrow} v = v_0 + a_c t; \quad v = 0 + 10(2) = 20 \text{ m/s}$$

b)



$$\overset{+}{\rightarrow} \Sigma F_x = ma_x; \quad 20t = 10a$$

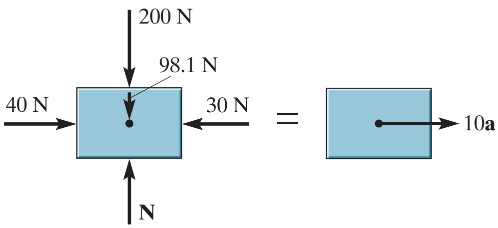
$$a = 2t$$

$$dv = a\,dt; \quad \int_0^v dv = \int_0^2 2t\,dt$$

$$v = 4 \text{ m/s}$$

P13–2.

a)



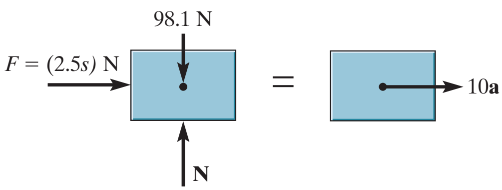
$$\overset{+}{\rightarrow} \Sigma F_x = ma_x; \quad 40 \text{ N} - 30 \text{ N} = 10a$$

$$a = 1 \text{ m/s}^2$$

$$\overset{+}{\rightarrow} v^2 = v_0^2 + 2a_c(s - s_0); \quad v^2 = (3)^2 + 2(1)(8 - 0)$$

$$v = 5 \text{ m/s}$$

b)



$$\overset{+}{\rightarrow} \Sigma F_x = ma_x; \quad 2.5s = 10a$$

$$a = 2.5s$$

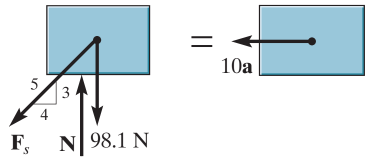
$$v\,dv = a\,ds$$

$$\int_3^v v\,dv = \int_0^8 2.5s\,ds$$

$$v^2 - (3)^2 = 2.5(8 - 0)^2$$

$$v = 13 \text{ m/s}$$

P13–3.

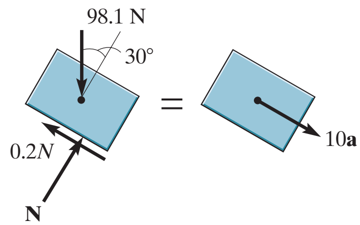


$$F_s = kx = (10 \text{ N/m})(5 \text{ m} - 1 \text{ m}) = 40 \text{ N}$$

$$\overset{+}{\leftarrow} \Sigma F_x = ma_x; \quad \frac{4}{5}(40 \text{ N}) = 10a$$

$$a = 3.2 \text{ m/s}^2$$

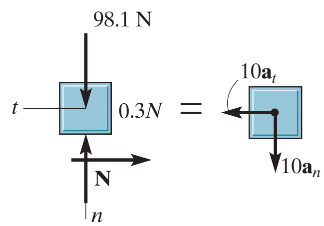
P13–4.



$$\searrow^{+} \Sigma F_x = ma_x; \quad 98.1 \sin 30° - 0.2N = 10a$$

$$^{+}\nearrow \Sigma F_y = ma_y; \quad N - 98.1 \cos 30° = 0$$

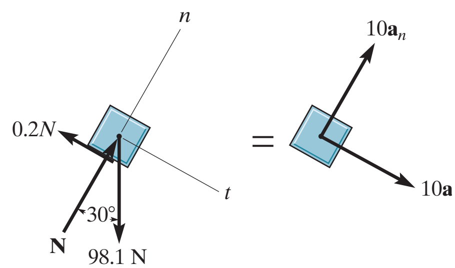
P13–5. a)



$$\overset{+}{\leftarrow} \Sigma F_t = ma_t; \quad -0.3N = 10a_t$$

$$+\downarrow \Sigma F_n = ma_n; \quad 98.1 - N = 10\left(\frac{(6)^2}{10}\right)$$

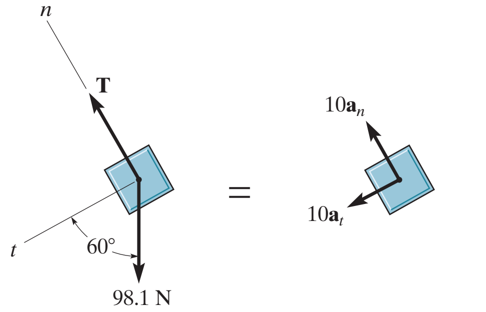
b)



$\overset{+}{\searrow}\Sigma F_t = ma_t;\ 98.1 \sin 30° - 0.2N = 10a_t$

$\overset{+}{\nearrow}\Sigma F_n = ma_n; \quad N - 98.1 \cos 30° = 10\left(\frac{(4)^2}{5}\right)$

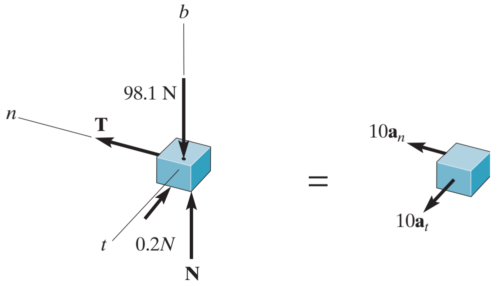
c)



$\overset{+}{\swarrow}\Sigma F_t = ma_t; \qquad 98.1 \cos 60° = 10a_t$

$\overset{+}{\nwarrow}\Sigma F_n = ma_n; \quad T - 98.1 \sin 60° = 10\left(\frac{8^2}{6}\right)$

P13–6. a)

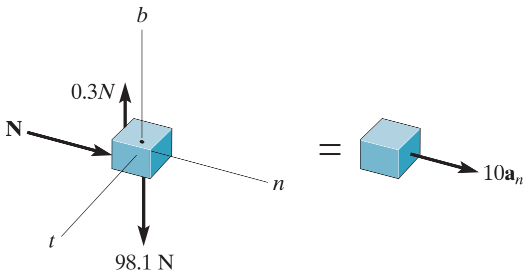


$\Sigma F_b = 0; \qquad N - 98.1 = 0$

$\Sigma F_t = ma_t; \qquad -0.2N = 10a_t$

$\Sigma F_n = ma_n; \qquad T = 10\,\frac{(8)^2}{4}$

b)



$\Sigma F_b = 0; \qquad 0.3N - 98.1 = 0$

$\Sigma F_t = ma_t; \qquad 0 = 0$

$\Sigma F_n = ma_n; \qquad N = 10\,\frac{v^2}{2}$

Chapter 14

P14–1. a) $U = \frac{3}{5}(500\text{ N})(2\text{ m}) = 600\text{ J}$

b) $U = 0$

c) $U = \int_0^2 6s^2\,ds = 2(2)^3 = 16\text{ J}$

d) $U = 100\text{ N}\left(\frac{3}{5}(2\text{ m})\right) = \frac{3}{5}(100\text{ N})(2\text{ m}) = 120\text{ J}$

e) $U = \frac{4}{5}(\text{Area}) = \frac{4}{5}\left[\frac{1}{2}(1)(20) + (1)(20)\right] = 24\text{ J}$

f) $U = \frac{1}{2}(10\text{ N/m})((3\text{ m})^2 - (1\text{ m})^2) = 40\text{ J}$

g) $U = -\left(\frac{4}{5}\right)(100\text{ N})(2\text{ m}) = -160\text{ J}$

P14–2. a) $T = \frac{1}{2}(10\text{ kg})(2\text{ m/s})^2 = 20\text{ J}$

b) $T = \frac{1}{2}(10\text{ kg})(6\text{ m/s})^2 = 180\text{ J}$

P14–3. a) $V = (100\text{ N})(2\text{ m}) = 200\text{ J}$

b) $V = (100\text{ N})(3\text{ m}) = 300\text{ J}$

c) $V = 0$

P14–4. a) $V = \frac{1}{2}(10\text{ N/m})(5\text{ m} - 4\text{ m})^2 = 5\text{ J}$

b) $V = \frac{1}{2}(10\text{ N/m})(10\text{ m} - 4\text{ m})^2 = 180\text{ J}$

c) $V = \frac{1}{2}(10\text{ N/m})(5\text{ m} - 4\text{ m})^2 = 5\text{ J}$

Chapter 15

P15–1. a) $I = (100\text{ N})(2\text{ s}) = 200\text{ N}\cdot\text{s} \swarrow$

b) $I = (200\text{ N})(2\text{ s}) = 400\text{ N}\cdot\text{s}\downarrow$

c) $I = \int_0^2 6t\,dt = 3(2)^2 = 12\text{ N}\cdot\text{s} \searrow$

d) $I = \text{Area} = \frac{1}{2}(1)(20) + (2)(20) = 50\text{ N}\cdot\text{s} \nearrow$

e) $I = (80\text{ N})(2\text{ s}) = 160\text{ N}\cdot\text{s} \rightarrow$

f) $I = (60\text{ N})(2\text{ s}) = 120\text{ N}\cdot\text{s} \nearrow$

P15–2. a) $L = (10\text{ kg})(10\text{ m/s}) = 100\text{ kg}\cdot\text{m/s}\searrow$

b) $L = (10\text{ kg})(2\text{ m/s}) = 20\text{ kg}\cdot\text{m/s} \swarrow$

c) $L = (10\text{ kg})(3\text{ m/s}) = 30\text{ kg}\cdot\text{m/s} \rightarrow$

Chapter 16

P16–1. a) $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A\,(\text{pin})}$

$$v_B = 18\text{ m/s} + 2\omega$$

60°

Also,

$$-v_B\mathbf{j} = -18\mathbf{j} + (-\omega\mathbf{k}) \times (-2\cos 60°\mathbf{i}) - 2\sin 60°\mathbf{j})$$

b) $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A\,(\text{pin})}$

$$(v_B)_x + (v_B)_y = 4(0.5)\text{ m/s} + 4(0.5)\text{ m/s}$$

30°

Also,

$$(v_B)_x\mathbf{i} + (v_B)_y\mathbf{j} = 2\mathbf{i} + (-4\mathbf{k}) \times (-0.5\cos 30°\mathbf{i} + \mathbf{0.5}\sin 30°\mathbf{j})$$

c) $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A\,(\text{pin})}$

$$v_B = 6\text{ m/s} + \omega\,(5)$$

45° 30°

Also,

$$v_B\cos 45°\mathbf{i} + v_B\sin 45°\mathbf{j} = 6\mathbf{i} + (\omega\mathbf{k}) \times (4\mathbf{i} - 3\mathbf{j})$$

d) $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A\,(\text{pin})}$

$$v_B = 6\text{ m/s} + \omega\,(3)$$

30°

Also,

$$v_B\mathbf{i} = 6\cos 30°\mathbf{i} + 6\sin 30°\mathbf{j} + (\omega\mathbf{k}) \times (3\mathbf{i})$$

e) $v_A = 12\text{ m/s} = \omega\,(0.5\text{ m}) \quad \omega = 24\text{ rad/s}$

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A\,(\text{pin})}$$

$$(v_B)_x + (v_B)_y = 12\text{ m/s} + (24)(0.5)$$

Also,

$$(v_B)_x\mathbf{i} + (v_B)_y\mathbf{j} = 12\mathbf{j} + (24\mathbf{k}) \times (0.5\mathbf{j})$$

f) $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A\,(\text{pin})}$

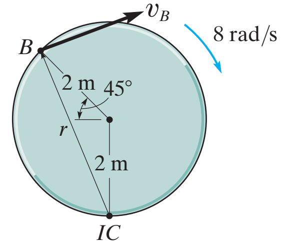
$$v_B = 6\text{ m/s} + \omega(5)$$

4 5 3

Also,

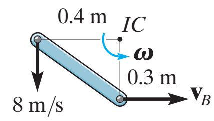
$$v_B\mathbf{i} = 6\mathbf{i} + (\omega\mathbf{k}) \times (4\mathbf{i} + 3\mathbf{j})$$

P16–2. a)

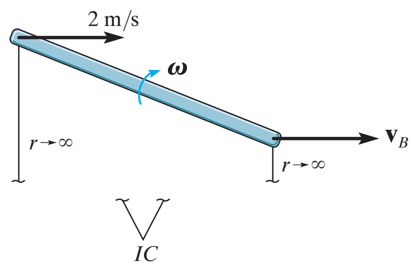


$$r = \sqrt{(2\cos 45°)^2 + (2 + 2\sin 45°)^2}$$

b)

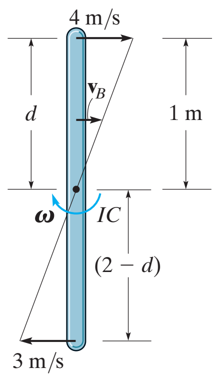


c)

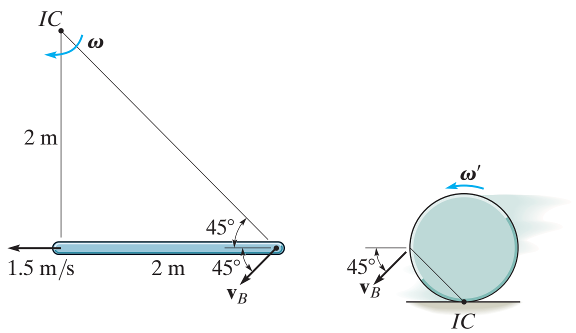


$$v_B = 2\text{ m/s}, \omega = 0$$

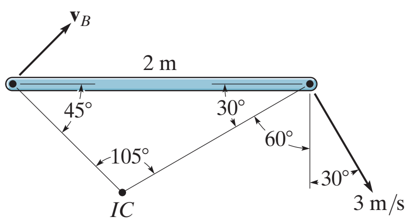
d)



e)



f)



P16–3. a)

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A\,(\text{pin})}$$

$$\underset{\downarrow}{a_B} = \underset{\leftarrow}{2\text{ m/s}^2} + \underset{\uparrow}{\frac{(3)^2}{3}} + \underset{\searrow 45^\circ}{2\alpha} + \underset{45^\circ \swarrow}{(2.12)^2(2)}$$

Also,

$$-a_B\mathbf{j} = -2\mathbf{i} + 3\mathbf{j} + (-\alpha\mathbf{k}) \times (2\sin 45^\circ\mathbf{i} + 2\cos 45^\circ\mathbf{j}) - (2.12)^2(2\sin 45^\circ\mathbf{i} + 2\cos 45^\circ\mathbf{j})$$

b)

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A\,(\text{pin})}$$

$$\underset{\rightarrow}{(a_B)_x} + \underset{\uparrow}{(a_B)_y} = \underset{\rightarrow}{(2)(2)\text{ m/s}^2} + \underset{\nearrow 45^\circ}{\alpha(2)} + \underset{\searrow 45^\circ}{(4)^2(2)}$$

Also,

$$(a_B)_x\mathbf{i} + (a_B)_y\mathbf{j} = 4\mathbf{i} + (-\alpha\mathbf{k}) \times (-2\cos 45^\circ\mathbf{i} + 2\sin 45^\circ\mathbf{j}) - (4)^2(-2\cos 45^\circ\mathbf{i} + 2\sin 45^\circ\mathbf{j})$$

c)

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A\,(\text{pin})}$$

$$\underset{\rightarrow}{(a_B)_x} + \underset{\downarrow}{(6)^2(1)} = \underset{\rightarrow}{2(2)} + \underset{\downarrow}{(3)^2(2)} + \underset{\downarrow}{\alpha(4)}$$

Also,

$$(a_B)_x\mathbf{i} - 36\mathbf{j} = 4\mathbf{i} - 18\mathbf{j} + (-\alpha\mathbf{k}) \times (4\mathbf{i})$$

d)

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A\,(\text{pin})}$$

$$\underset{\rightarrow}{a_B} = \underset{60^\circ \swarrow}{6} + \underset{\uparrow}{\alpha(2)} + \underset{\rightarrow}{(3)^2(2)}$$

Also,

$$a_B\mathbf{i} = -6\cos 60^\circ\mathbf{i} - 6\sin 60^\circ\mathbf{j} + (-\alpha\mathbf{k}) \times (-2\mathbf{i}) - (3)^2(-2\mathbf{i})$$

e)

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A\,(\text{pin})}$$

$$\underset{\leftarrow}{a_B} = \underset{\downarrow}{8(0.5)} + \underset{\rightarrow}{(4)^2(0.5)} + \underset{\nwarrow 30^\circ}{\alpha(2)} + \underset{\nearrow 30^\circ}{(1.15)^2(2)}$$

Also,

$$-a_B\mathbf{i} = -4\mathbf{j} + 8\mathbf{i} + (-\alpha\mathbf{k}) \times (-2\cos 30^\circ\mathbf{i} - 2\sin 30^\circ\mathbf{j}) - (1.15)^2(-2\cos 30^\circ\mathbf{i} - 2\sin 30^\circ\mathbf{j})$$

f)

$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A\,(\text{pin})}$$

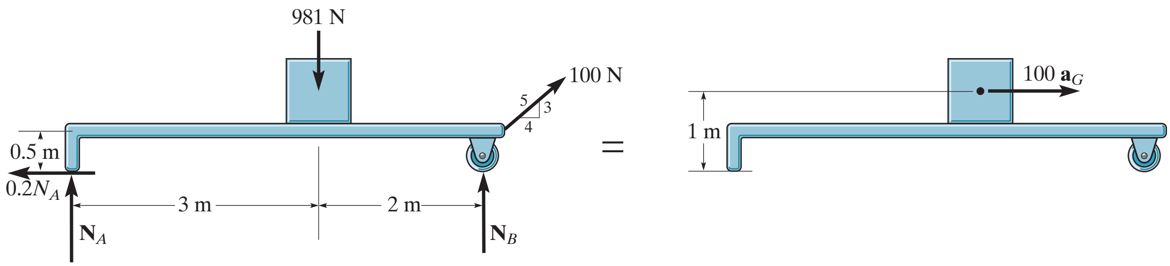
$$\underset{\rightarrow}{(a_B)_x} + \underset{\uparrow}{(a_B)_y} = \underset{\downarrow}{2(0.5)} + \underset{\rightarrow}{2(0.5)} + \underset{\downarrow}{(4)^2(0.5)}$$

Also,

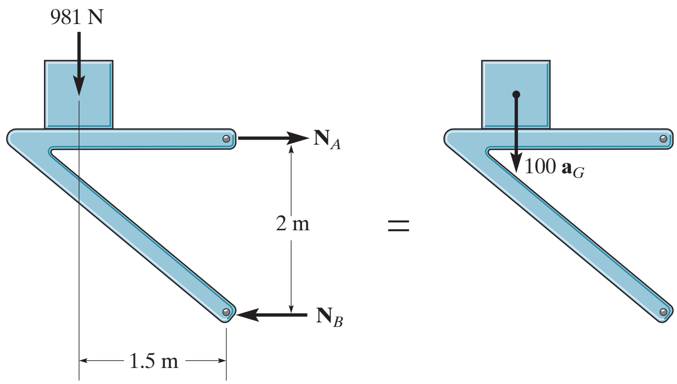
$$(a_B)_x\mathbf{i} + (a_B)_y\mathbf{j} = -1\mathbf{j} + (-2\mathbf{k}) \times (0.5\mathbf{j}) - (4)^2(0.5\mathbf{j})$$

Chapter 17

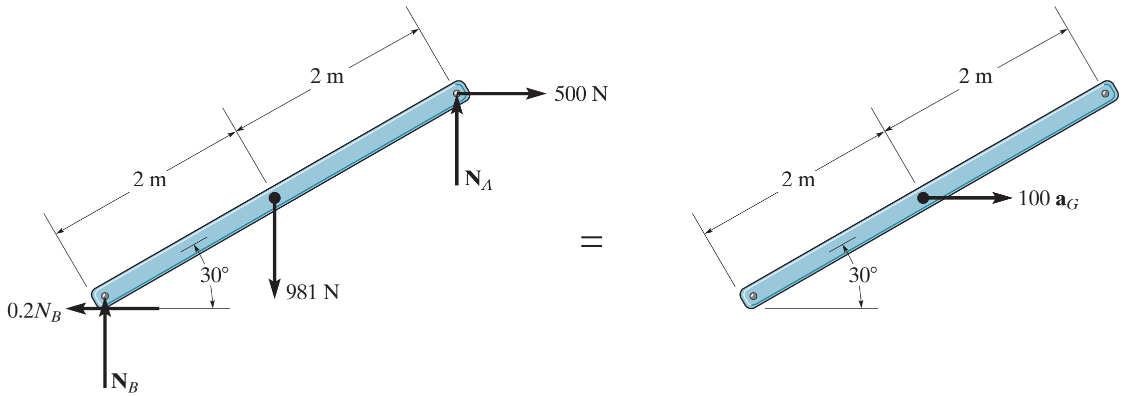
P17–1. a)



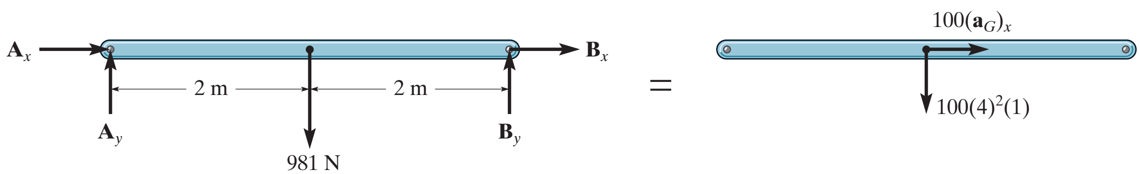
b)



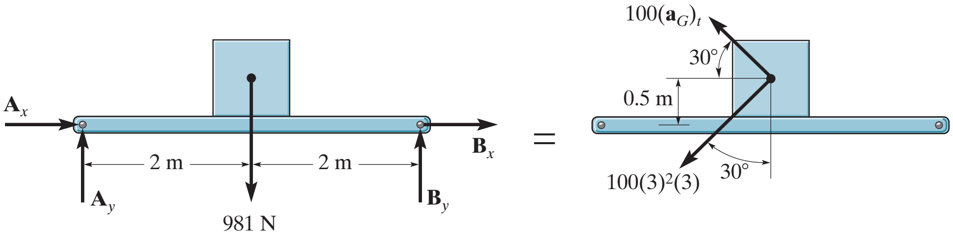
c)



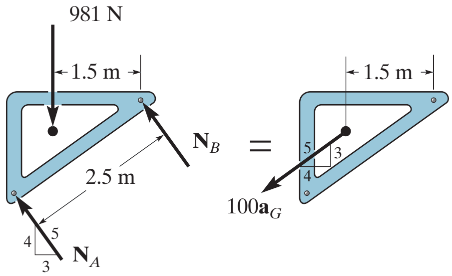
d)



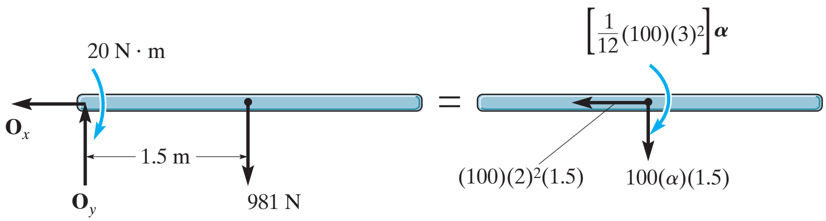
e)


A_x
A_y
2 m
2 m
981 N
B_x
B_y
=
$100(a_G)_t$
30°
0.5 m
$100(3)^2(3)$
30°

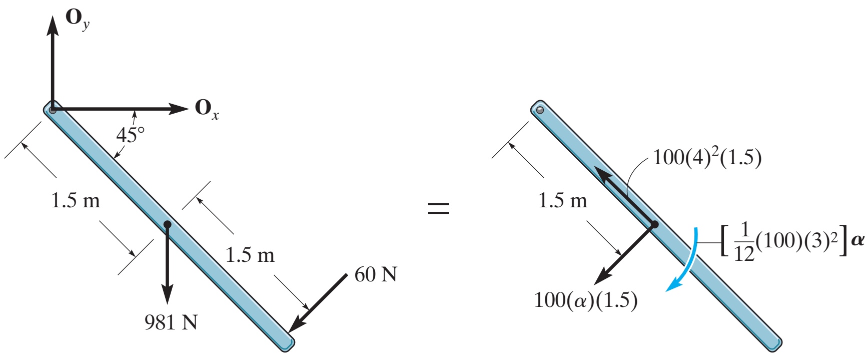
f)


981 N
1.5 m
2.5 m
N_B
N_A
5
4
3
=
1.5 m
5
3
4
$100a_G$

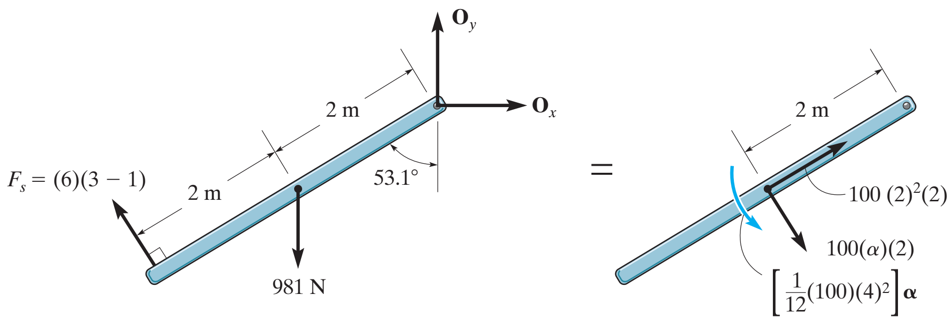
P17–2. a)


20 N · m
O_x
O_y
1.5 m
981 N
=
$\left[\frac{1}{12}(100)(3)^2\right]\alpha$
$(100)(2)^2(1.5)$
$100(\alpha)(1.5)$

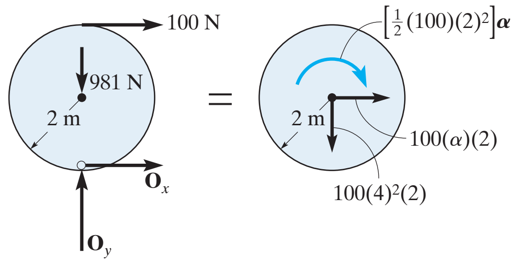
b)


O_y
O_x
45°
1.5 m
1.5 m
981 N
60 N
=
1.5 m
$100(4)^2(1.5)$
$\left[\frac{1}{12}(100)(3)^2\right]\alpha$
$100(\alpha)(1.5)$

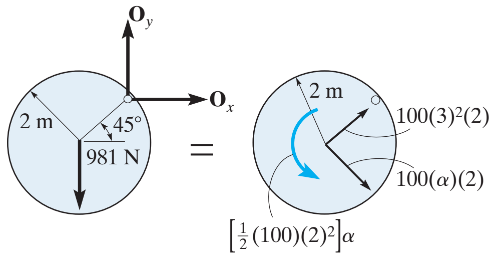
c)


O_y
O_x
2 m
2 m
53.1°
$F_s = (6)(3 - 1)$
981 N
=
2 m
$100\ (2)^2(2)$
$100(\alpha)(2)$
$\left[\frac{1}{12}(100)(4)^2\right]\alpha$

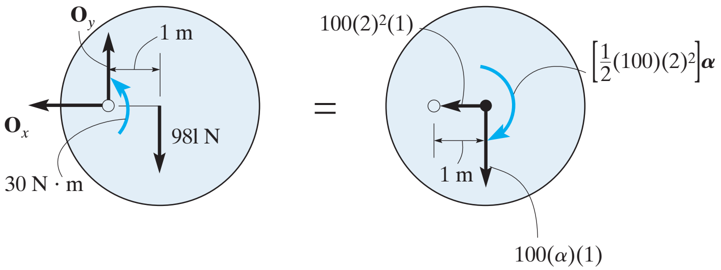
d)



e)



f)



Chapter 18

P18–1. a) $T = \frac{1}{2}\left[\frac{100(2)^2}{2}\right](3)^2 = 900 \text{ J}$

b) $T = \frac{1}{2}(100)[2(1)]^2 + \frac{1}{2}\left[\frac{1}{12}(100)(6)^2\right](2)^2$

$= 800 \text{ J}$

Also,

$T = \frac{1}{2}\left[\frac{1}{12}(100)(6)^2 + 100(1)^2\right](2)^2 = 800 \text{ J}$

c) $T = \frac{1}{2}(100)[2(2)]^2 + \frac{1}{2}\left[\frac{1}{2}(100)(2)^2\right](2)^2$

$= 1200 \text{ J}$

Also,

$T = \frac{1}{2}\left[\frac{1}{2}(100)(2)^2 + 100(2)^2\right](2)^2$

$= 1200 \text{ J}$

d) $T = \frac{1}{2}(100)[2(1.5)]^2 + \frac{1}{2}\left[\frac{1}{12}(100)(3)^2\right](2)^2$

$= 600 \text{ J}$

Also,

$T = \frac{1}{2}\left[\frac{1}{12}(100)(3)^2 + 100(1.5)^2\right](2)^2$

$= 600 \text{ J}$

e) $T = \frac{1}{2}(100)[4(2)]^2 + \frac{1}{2}\left[\frac{1}{2}(100)(2)^2\right](4)^2$

$= 4800 \text{ J}$

Also,

$T = \frac{1}{2}\left[\frac{1}{2}(100)(2)^2 + 100(2)^2\right](4)^2$

$= 4800 \text{ J}$

f) $T = \frac{1}{2}(100)[(4)(2)]^2 = 3200 \text{ J}$

Chapter 19

P19–1. a) $H_G = \left[\frac{1}{2}(100)(2)^2\right](3) = 600 \text{ kg}\cdot\text{m}^2/\text{s} \circlearrowright$

$H_O = \left[\frac{1}{2}(100)(2)^2 + 100(2)^2\right](3)$

$= 1800 \text{ kg}\cdot\text{m}^2/\text{s} \circlearrowright$

b) $H_G = \left[\frac{1}{12}(100)(3)^2\right](4) = 300\ \text{kg}\cdot\text{m}^2/\text{s}$ ↻

$H_O = \left[\frac{1}{12}(100)(3)^2 + (100)(1.5)^2\right](4)$

$= 1200\ \text{kg}\cdot\text{m}^2/\text{s}$ ↻

c) $H_G = \left[\frac{1}{2}(100)(2)^2\right](4) = 800\ \text{kg}\cdot\text{m}^2/\text{s}$ ↻

$H_O = \left[\frac{1}{2}(100)(2)^2 + (100)(2)^2\right](4)$

$= 2400\ \text{kg}\cdot\text{m}^2/\text{s}$ ↻

d) $H_G = \left[\frac{1}{12}(100)(4)^2\right]3 = 400\ \text{kg}\cdot\text{m}^2/\text{s}$ ↺

$H_O = \left[\frac{1}{12}(100)(4)^2 + (100)(1)^2\right]3$

$= 700\ \text{kg}\cdot\text{m}^2/\text{s}$ ↺

P19–2. a) $\int M_O\,dt = \left(\frac{4}{5}\right)(500)(2)(3) = 2400\ \text{N}\cdot\text{s}\cdot\text{m}$ ↻

b) $\int M_O\,dt = \left[2(20) + \frac{1}{2}(3-2)(20)\right]4$

$= 200\ \text{N}\cdot\text{s}\cdot\text{m}$ ↻

c) $\int M_O\,dt = \frac{3}{5}\int_0^3 4(2t+2)dt = 36\ \text{N}\cdot\text{s}\cdot\text{m}$ ↻

d) $\int M_O\,dt = \int_0^3 (30t^2)dt = 270$ ↻

Review Problem Solutions

Chapter 12

R12–1. $s = t^3 - 9t^2 + 15t$

$$v = \frac{ds}{dt} = 3t^2 - 18t + 15$$

$$a = \frac{dv}{dt} = 6t - 18$$

a_{max} occurs at $t = 10$ s.

$a_{max} = 6(10) - 18 = 42 \text{ ft/s}^2$ *Ans.*

v_{max} occurs when $t = 10$ s

$v_{max} = 3(10)^2 - 18(10) + 15 = 135 \text{ ft/s}$ *Ans.*

R12–2. $(\overset{+}{\rightarrow})$ $\quad s = s_0 + v_0 t + \frac{1}{2} a_c t^2$

$$s = 0 + 12(10) + \frac{1}{2}(-2)(10)^2$$

$s = 20.0$ ft *Ans.*

R12–3. $v = \dfrac{ds}{dt} = 1800(1 - e^{-0.3t})$

$$\int_0^x ds = \int_0^t 1800(1 - e^{-0.3t})\, dt$$

$$s = 1800\left(t + \frac{1}{0.3} e^{-0.3t}\right) - 6000$$

Thus, in $t = 3$ s

$$s = 1800\left(3 + \frac{1}{0.3} e^{-0.3(3)}\right) - 6000$$

$s = 1839.4 \text{ mm} = 1.84 \text{ m}$ *Ans.*

R12–4. $0 \le t \le 5 \quad a = \dfrac{\Delta v}{\Delta t} = \dfrac{20}{5} = 4 \text{ m/s}^2$ *Ans.*

$5 \le t \le 20 \quad a = \dfrac{\Delta v}{\Delta t} = \dfrac{20 - 20}{20 - 5} = 0 \text{ m/s}^2$ *Ans.*

$20 \le t \le 30 \quad a = \dfrac{\Delta v}{\Delta t} = \dfrac{0 - 20}{30 - 20} = -2 \text{ m/s}^2$ *Ans.*

At $t_1 = 5$ s, $t_2 = 20$ s, and $t_3 = 30$ s,

$s_1 = A_1 = \frac{1}{2}(5)(20) = 50 \text{ m}$ *Ans.*

$s_2 = A_1 + A_2 = 50 + 20(20 - 5) = 350 \text{ m}$ *Ans.*

$s_3 = A_1 + A_2 + A_3 = 350 + \frac{1}{2}(30 - 20)(20) = 450 \text{ m}$ *Ans.*

R12–5. $v_A = 20\mathbf{i}$

$v_B = 21.21\mathbf{i} + 21.21\mathbf{j}$

$v_C = 40\mathbf{i}$

$$\mathbf{a}_{AB} = \frac{\Delta v}{\Delta t} = \frac{21.21\mathbf{i} + 21.21\mathbf{j} - 20\mathbf{i}}{3}$$

$\mathbf{a}_{AB} = \{0.404\mathbf{i} + 7.07\mathbf{j}\} \text{ m/s}^2$ *Ans.*

$$\mathbf{a}_{AC} = \frac{\Delta v}{\Delta t} = \frac{40\mathbf{i} - 20\mathbf{i}}{8}$$

$\mathbf{a}_{AC} = \{2.50\mathbf{i}\} \text{ m/s}^2$ *Ans.*

R12–6. $(\overset{+}{\rightarrow})$ $\quad s = s_0 + v_0 t$

$$126 = 0 + (v_0)_x(3.6)$$

$$(v_0)_x = 35 \text{ ft/s}$$

$(+\uparrow)$ $\quad s = s_0 + v_0 t + \frac{1}{2} a_c t^2$

$$O = 0 + (v_0)_y(3.6) + \frac{1}{2}(-32.2)(3.6)^2$$

$$(v_0)_y = 57.96 \text{ ft/s}$$

$v_0 = \sqrt{(35)^2 + (57.96)^2} = 67.7 \text{ ft/s}$ *Ans.*

$\theta = \tan^{-1}\left(\dfrac{57.96}{35}\right) = 58.9°$ *Ans.*

R12–7. $v\, dv = a_t\, ds$

$$\int_4^v v\, dv = \int_0^{10} 0.05s\, ds$$

$$0.5v^2 - 8 = \frac{0.05}{2}(10)^2$$

$v = 4.583 = 4.58 \text{ m/s}$ *Ans.*

$$a_n = \frac{v^2}{\rho} = \frac{(4.583)^2}{50} = 0.420 \text{ m/s}^2$$

$$a_t = 0.05(10) = 0.5 \text{ m/s}^2$$

$a = \sqrt{(0.420)^2 + (0.5)^2} = 0.653 \text{ m/s}^2$ *Ans.*

R12–8. $dv = a\, dt$

$$\int_0^v dv = \int_0^t 0.5e^t\, dt$$

$$v = 0.5\left(e^t - 1\right)$$

When $t = 2$ s, $v = 0.5\left(e^2 - 1\right) = 3.195 \text{ m/s}$
$= 3.19 \text{ m/s}$ *Ans.*

When $t = 2$ s $a_t = 0.5e^2 = 3.695\ \text{m/s}^2$

$$a_n = \frac{v^2}{\rho} = \frac{3.195^2}{5} = 2.041\ \text{m/s}^2$$

$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{3.695^2 + 2.041^2}$$

$$= 4.22\ \text{m/s}^2 \qquad \textit{Ans.}$$

R12–9. $r = 2$ m $\qquad \theta = 5t^2$

$\dot{r} = 0 \qquad \dot{\theta} = 10t$

$\ddot{r} = 0 \qquad \ddot{\theta} = 10$

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{u}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta$$

$$= \left[0 - 2(10t)^2\right]\mathbf{u}_r + [2(10) + 0]\mathbf{u}_\theta$$

$$= \{-200t^2\mathbf{u}_r + 20\mathbf{u}_\theta\}\ \text{m/s}^2$$

When $\theta = 30° = 30\left(\dfrac{\pi}{180}\right) = 0.524$ rad

$$0.524 = 5t^2$$

$$t = 0.324\ \text{s}$$

$$\mathbf{a} = \left[-200(0.324)^2\right]\mathbf{u}_r + 20\mathbf{u}_\theta$$

$$= \{-20.9\mathbf{u}_r + 20\mathbf{u}_\theta\}\ \text{m/s}^2$$

$$a = \sqrt{(-20.9)^2 + (20)^2} = 29.0\ \text{m/s}^2 \qquad \textit{Ans.}$$

R12–10. $4s_B + s_A = l$

$$4v_B = -v_A$$

$$4a_B = -a_A$$

$$4a_B = -0.2$$

$$a_B = -0.05\ \text{m/s}^2$$

$(+\downarrow)$ $\quad v_B = (v_B)_0 + a_B t$

$$-8 = 0 - (0.05)(t)$$

$$t = 160\ \text{s} \qquad \textit{Ans.}$$

R12–11. $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$

$$[500 \leftarrow] = [600 \ \overset{75°}{\nwarrow\theta}\ ] + v_{B/A}$$

$(\overset{+}{\leftarrow})$ $\quad 500 = -600\cos 75° + (v_{B/A})_x$

$$(v_{B/A})_x = 655.29 \leftarrow$$

$(+\uparrow)$ $\quad 0 = -600\sin 75° + (v_{B/A})_y$

$$(v_{B/A})_y = 579.56 \uparrow$$

$$(v_{B/A}) = \sqrt{(655.29)^2 + (579.56)^2}$$

$$v_{B/A} = 875\ \text{km/h} \qquad \textit{Ans.}$$

$$\theta = \tan^{-1}\left(\frac{579.56}{655.29}\right) = 41.5° \qquad \textit{Ans.}$$

Chapter 13

R13–1. $20\ \text{km/h} = \dfrac{20(10)^3}{3600} = 5.556\ \text{m/s}$

$(\overset{+}{\leftarrow})$ $\quad v^2 = v_0^2 + 2a_c(s - s_0)$

$$a = -0.3429\ \text{m/s}^2 = 0.3429\ \text{m/s}^2 \rightarrow$$

$\overset{+}{\rightarrow}\Sigma F_x = ma_x;\quad F = 250(0.3429) = 85.7\ \text{N} \qquad$ *Ans.*

R13–2. $\nwarrow + \Sigma F_y = ma_y;\quad N_C - 50(9.81)\cos 30° = 0$

$$N_C = 424.79$$

$\nearrow + \Sigma F_x = ma_x;\quad 3T - 0.3(424.79) - 50(9.81)\sin 30° = 50a_C \qquad$ **(1)**

Kinematics, $2s_C + (s_C - s_p) = l$

Taking two time derivatives, yields

$$3a_C = a_p$$

Thus, $a_C = \dfrac{6}{3} = 2$

Substituting into Eq. (1) and solving,

$$T = 158\ \text{N} \qquad \textit{Ans.}$$

R13–3. Suppose the two blocks move together.

Then

$$50\ \text{lb} = \frac{50 + 20}{32.2}a$$

$$a = 23\ \text{m/s}^2$$

Then the friction force on block B is

$$F_B = \frac{50}{32.2}(23) = 35.7\ \text{lb}$$

The maximum friction force between blocks A and B is

$$F_{\max} = 0.4(20) = 8\ \text{lb} < 35.7\ \text{1b}$$

The blocks have different accelerations.

Block A:

$\overset{+}{\rightarrow}\Sigma F_x = ma_x;\quad 20(0.3) = \dfrac{20}{32.2}a_A$

$$a_A = 70.8\ \text{ft/s}^2 \qquad \textit{Ans.}$$

Block B:

$\overset{+}{\rightarrow}\Sigma F_x = ma_x;\quad 20(0.3) = \dfrac{50}{32.2}a_B$

$$a_B = 3.86\ \text{ft/s}^2 \qquad \textit{Ans.}$$

R13–4. ***Kinematics:*** Since the motion of the crate is known, its acceleration **a** will be determined first.

$$a = v\frac{dv}{ds} = (0.05s^{3/2})\left[(0.05)\left(\frac{3}{2}\right)s^{1/2}\right]$$

$$= 0.00375s^2\ \text{m/s}^2$$

When $s = 10$ m,

$$a = 0.00375(10^2) = 0.375\ \text{m/s}^2 \rightarrow$$

Free-Body Diagram: The kinetic friction $F_f = \mu_k N = 0.2N$ must act to the left to oppose the motion of the crate which is to the right.

Equations of Motion: Here, $a_y = 0$. Thus,

$$+\uparrow \Sigma F_y = ma_y; \quad N - 20(9.81) = 20(0)$$
$$N = 196.2 \text{ N}$$

Using the results of **N** and **a**,

$$\overset{+}{\rightarrow} \Sigma F_x = ma_x; \quad T - 0.2(196.2) = 20(0.375)$$
$$T = 46.7 \text{ N} \qquad \textit{Ans.}$$

R13–5. $+\nwarrow \Sigma F_n = ma_n; \quad T - 30(9.81)\cos\theta = 30\left(\frac{v^2}{4}\right)$

$$+\nearrow \Sigma F_t = ma_t; \quad -30(9.81)\sin\theta = 30a_t$$
$$a_t = -9.81 \sin\theta$$

$a_t\, ds = v\, dv$ Since $ds = 4\, d\theta$, then

$$-9.81\int_0^\theta \sin\theta\,(4\, d\theta) = \int_4^v v dv$$

$$9.81(4)\cos\theta\Big|_0^\theta = \frac{1}{2}(v)^2 - \frac{1}{2}(4)^2$$

$$39.24(\cos\theta - 1) + 8 = \frac{1}{2}v^2$$

At $\theta = 20°$

$v = 3.357$ m/s

$a_t = -3.36 \text{ m/s}^2 = 3.36 \text{ m/s}^2 \ \swarrow$ *Ans.*

$T = 361$ N *Ans.*

R13–6. $\Sigma F_z = ma_z; \quad N_z - mg = 0 \quad N_z = mg$

$$\Sigma F_x = ma_n; \quad 0.3(mg) = m\left(\frac{v^2}{r}\right)$$

$$v = \sqrt{0.3gr} = \sqrt{0.3(32.2)(3)} = 5.38 \text{ ft/s} \qquad \textit{Ans.}$$

R13–7. $v = \frac{1}{8}x^2$

$$\frac{dy}{dx} = \tan\theta = \frac{1}{4}x\Big|_{x=-6} = -1.5 \quad \theta = -56.31°$$

$$\frac{d^2y}{dx^2} = \frac{1}{4}$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{\left[1 + (-1.5)^2\right]^{\frac{3}{2}}}{\left|\frac{1}{4}\right|} = 23.436 \text{ ft}$$

$$+\nearrow \Sigma F_n = ma_n; \quad N - 10\cos 56.31° = \left(\frac{10}{32.2}\right)\left(\frac{(5)^2}{23.436}\right)$$

$$N = 5.8783 = 5.88 \text{ lb} \qquad \textit{Ans.}$$

$$+\searrow \Sigma F_t = ma_t; \quad -0.2(5.8783) + 10\sin 56.31° = \left(\frac{10}{32.2}\right)a_t$$

$$a_t = 23.0 \text{ ft/s}^2 \qquad \textit{Ans.}$$

R13–8. $r = 0.5$ m

$\dot{r} = 3$ m/s $\quad \dot{\theta} = 6$ rad/s

$\ddot{r} = 1 \text{ m/s}^2 \quad \ddot{\theta} = 2$ rad/s

$$a_r = \ddot{r} - r\dot{\theta}^2 = 1 - 0.5(6)^2 = -17$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.5(2) + 2(3)(6) = 37$$

$$\Sigma F_r = ma_r; \quad F_r = 4(-17) = -68 \text{ N}$$
$$\Sigma F_\theta = ma_\theta; \quad N_\theta = 4(37) = 148 \text{ N}$$
$$\Sigma F_z = ma_z; \quad N_z = 4(9.81) = 0$$
$$N_z = 39.24 \text{ N}$$
$$F_r = -68 \text{ N} \qquad \textit{Ans.}$$

$$N = \sqrt{(148)^2 + (39.24)^2} = 153 \text{ N} \qquad \textit{Ans.}$$

Chapter 14

R14–1. $+\nwarrow \Sigma F_y = 0; \quad N_C - 150\cos 30° = 0$

$$N_C = 129.9 \text{ lb}$$

$$T_1 + \Sigma U_{1-2} = T_2$$

$$0 + 150\sin 30°(30) - (0.3)129.9(30) = \frac{1}{2}\left(\frac{150}{32.2}\right)v_2^2$$

$$v_2 = 21.5 \text{ ft/s} \qquad \textit{Ans.}$$

R14–2. $r_{AB} = r_B - r_A = -4\mathbf{i} + 8\mathbf{j} - 9\mathbf{k}$

$$T_1 + \Sigma\int Fds = T_2$$

$$0 + 2(10 - 1) + \int_4^0 10dx + \int_0^8 6y\, dy + \int_{10}^1 2z\, dz = \frac{1}{2}\left(\frac{2}{32.2}\right)v^2n$$

$$v_B = 47.8 \text{ ft/s} \qquad \textit{Ans.}$$

R14–3. $T_1 + V_1 = T_2 + V_2$

$$0 + 1.5(10) = \frac{1}{2}\left(\frac{1.5}{32.2}\right)v_B^2$$

$$v_B = 25.4 \text{ ft/s} \qquad \textit{Ans.}$$

R14–4. The work done by F depends upon the difference in the cord length AC–BC.

$$T_A + \Sigma U_{A-B} = T_B$$

$$0 + F\left[\sqrt{(0.3)^2 + (0.3)^2} - \sqrt{(0.3)^2 + (0.3 - 0.15)^2}\right] - 0.5(9.81)(0.15)$$

$$-\frac{1}{2}(100)(0.15)^2 = \frac{1}{2}(0.5)(2.5)^2$$

$$F(0.0889) = 3.423$$

$$F = 38.5 \text{ N} \qquad \textit{Ans.}$$

R14–5. $(+\uparrow)$ $\quad v^2 = v_0^2 + 2a_c(s - s_0)$

$$(12)^2 = 0 + 2a_c(10 - 0)$$

$$a_c = 7.20 \text{ ft/s}^2$$

$$+\uparrow\Sigma F_y = ma_y; \quad 2T - 50 = \frac{50}{32.2}(7.20)$$

$$T = 30.6 \text{ lb}$$

$$s_C + (s_C - s_M) = l$$

$$v_M = 2v_C$$

$$v_M = 2(12) = 24 \text{ ft/s}$$

$$P_0 = \mathbf{T}\cdot\mathbf{v} = 30.6(24) = 734.2 \text{ lb}\cdot\text{ft/s}$$

$$P_i = \frac{734.2}{0.74} = 992.1 \text{ lb}\cdot\text{ft/s} = 1.80 \text{ hp} \qquad \textit{Ans.}$$

R14–6. $+\uparrow\Sigma F_y = m\,a_y; \quad 2(30) - 50 = \dfrac{50}{32.2}a_B$

$$a_B = 6.44 \text{ m/s}^2$$

$$(+\uparrow) \quad v^2 = v_0^2 + 2a_c(s - s_0)$$

$$v_B^2 = 0 + 2(6.44)(10 - 0)$$

$$v_B = 11.349 \text{ ft/s}$$

$$2s_B + s_M = l$$

$$2v_B = -v_M$$

$$v_M = -2(11.349) = 22.698 \text{ ft/s}$$

$$P_o = \mathbf{F}\cdot\mathbf{v} = 30(22.698) = 680.94 \text{ ft}\cdot\text{lb/s}$$

$$P_i = \frac{680.94}{0.76} = 895.97 \text{ ft}\cdot\text{lb/s}$$

$$P_i = 1.63 \text{ hp} \qquad \textit{Ans.}$$

R14–7. $T_A + V_A = T_B + V_B$

$$0 + (0.25)(9.81)(0.6) + \frac{1}{2}(150)(0.6 - 0.1)^2$$

$$= \frac{1}{2}(0.25)(v_B)^2 + \frac{1}{2}(150)(0.4 - 0.1)^2$$

$$v_B = 10.4 \text{ m/s} \qquad \textit{Ans.}$$

R14–8. $\dfrac{6}{z} = \dfrac{\sqrt{15^2 + 2^2}}{15}$

$$z = 5.95 \text{ ft}$$

$$T_1 + V_1 = T_2 + V_2$$

$$0 + 0 = \frac{1}{2}\left(\frac{10}{32.2}\right)v_2^2 + \frac{1}{2}\left(\frac{30}{32.2}\right)v_2^2 + 10(5.95) - 30(5.95)$$

$$v_2 = 13.8 \text{ ft/s} \qquad \textit{Ans.}$$

Chapter 15

R15–1. $(+\uparrow) \quad m(v_1)_y + \Sigma\displaystyle\int F_y\,dt = m(v_2)_y$

$$0 + N_p(t) - 58.86(t) = 0$$

$$N_p = 58.86 \text{ N}$$

$$(\overset{+}{\rightarrow}) \quad m(v_1)_x + \Sigma\int F_x\,dt = m(v_2)_x$$

$$6(3) - 0.2(58.86)(t) = 6(1)$$

$$t = 1.02 \text{ s} \qquad \textit{Ans.}$$

R15–2. $+\nwarrow\Sigma F_x = 0; \quad N_B - 50(9.81)\cos 30° = 0$

$$N_B = 424.79 \text{ N}$$

$$(+\nearrow) \quad m(v_x)_1 + \Sigma\int F_x\,dt = m(v_x)_2$$

$$50(2) + \int_0^2\left(300 + 120\sqrt{t}\right)dt - 0.4(424.79)(2) - 50(9.81)\sin 30°(2) = 50v_2$$

$$v_2 = 1.92 \text{ m/s} \qquad \textit{Ans.}$$

R15–3. The crate starts moving when

$$F = F_r = 0.6(196.2) = 117.72 \text{ N}$$

From the graph since

$$F = \frac{200}{5}t. \quad 0 \le t \le 5 \text{ s}$$

The time needed for the crate to start moving is

$$t = \frac{5}{200}(117.72) = 2.943 \text{ s}$$

Hence, the impulse due to F is equal to the area under the curve from $2.943 \text{ s} \le t \le 10 \text{ s}$

$$\overset{+}{\rightarrow} \quad m(v_x)_1 + \Sigma\int F_x\,dt = m(v_x)_2$$

$$0 + \int_{2.943}^{5}\frac{200}{5}t\,dt + \int_5^{10}200\,dt - (0.5)196.2(10 - 2.943) = 20v_2$$

$$40\left(\frac{1}{2}t^2\right)\Big|_{2.943}^{5} + 200(10 - 5) - 692.292 = 20v_2$$

$634.483 = 20v_2$

$v_2 = 31.7 \text{ m/s}$ *Ans.*

R15–4. $(v_A)_1 = \left[20(10^3)\frac{\text{m}}{\text{h}}\right]\left(\frac{1\text{ h}}{3600\text{ s}}\right) = 5.556 \text{ m/s}$

$(v_B)_1 = \left[5(10^3)\frac{\text{m}}{\text{h}}\right]\left(\frac{1\text{ h}}{3600\text{ s}}\right) = 1.389 \text{ m/s},$

and $(v_C)_1 = \left[25(10^3)\frac{\text{m}}{\text{h}}\right]\left(\frac{1\text{ h}}{3600\text{ s}}\right) = 6.944 \text{ m/s}$

For the first case,

$(\overset{+}{\rightarrow})$ $m_A(v_A)_1 + m_B(v_B)_1 = (m_A + m_B)v_2$

$10000(5.556) + 5000(1.389) = (10000 + 5000)v_{AB}$

$v_{AB} = 4.167 \text{ m/s} \rightarrow$

Using the result of v_{AB} and considering the second case,

$(\overset{+}{\rightarrow})$ $(m_A + m_B)v_{AB} + m_C(v_C)_1 = (m_A + m_B + m_C)v_{ABC}$

$(10000 + 5000)(4.167) + [-20000(6.944)] = (10000 + 5000 + 20000)v_{ABC}$

$v_{ABC} = -2.183 \text{ m/s} = 2.18 \text{ m/s} \leftarrow$ *Ans.*

R15–5. $(\overset{+}{\rightarrow})$ $m_P(v_P)_1 + m_B(v_B)_1 = m_P(v_p)_2 + m_B(v_B)_2$

$0.2(900) + 15(0) = 0.2(300) + 15(v_B)_2$

$(v_B)_2 = 8 \text{ m/s} \rightarrow$ *Ans.*

$(+\uparrow)$ $m(v_1)_y + \Sigma\int_{t_1}^{t_2} F_y\,dt = m(v_2)_y$

$15(0) + N(t) - 15(9.81)(t) = 15(0)$

$N = 147.15 \text{ N}$

$(\overset{+}{\rightarrow})$ $m(v_1)_x + \Sigma\int_{t_1}^{t_2} F_x\,dt = m(v_2)_x$

$15(8) + [-0.2(147.15)(t)] = 15(0)$

$t = 4.077 \text{ s} = 4.08 \text{ s}$ *Ans.*

R15–6. $(\overset{+}{\rightarrow})$ $\Sigma mv_1 = \Sigma mv_2$

$3(2) + 0 = 3(v_A)_2 + 2(v_B)_2$

$(\overset{+}{\rightarrow})$ $e = \dfrac{(v_B)_2 - (v_A)_2)}{(v_A)_1 - (v_B)_1}$

$1 = \dfrac{(v_B)_2 - (v_A)_2}{2 - 0}$

Solving

$(v_A)_2 = 0.400 \text{ m/s} \rightarrow$ *Ans.*

$(v_B)_2 = 2.40 \text{ m/s} \rightarrow$ *Ans.*

Block A:

$T_1 + \Sigma U_{1-2} = T_2$

$\frac{1}{2}(3)(0.400)^2 - 3(9.81)(0.3)d_A = 0$

$d_A = 0.0272 \text{ m}$

Block B:

$T_1 + \Sigma U_{1-2} = T_2$

$\frac{1}{2}(2)(2.40)^2 - 2(9.81)(0.3)d_B = 0$

$d_B = 0.9786 \text{ m}$

$d = d_B - d_A = 0.951 \text{ m}$ *Ans.*

R15–7. $(v_A)_{x_1} = -2\cos 40° = -1.532 \text{ m/s}$

$(v_A)_{y_1} = -2\sin 40° = -1.285 \text{ m/s}$

$(\overset{+}{\rightarrow})$ $m_A(v_A)_{x_1} + m_B(v_B)_{x_1} = m_A(v_A)_{x_2} + m_B(v_B)_{x_2}$

$-2(1.532) + 0 = 0.2(v_A)_{x_2} + 0.2(v_B)_{x_2}$ **(1)**

$(\overset{+}{\rightarrow})$ $e = \dfrac{(v_{ref})_2}{(v_{ref})_1}$

$0.75 = \dfrac{(v_A)_{x_2} - (v_B)_{x_1}}{1.532}$ **(2)**

Solving Eqs. (1) and (2)

$(v_A)_{x_2} = -0.1915 \text{ m/s}$

$(v_B)_{x_2} = -1.3405 \text{ m/s}$

For A:

$(+\downarrow)$ $m_A(v_A)_{y_1} = m_A(v_A)_{y_2}$

$(v_A)_{y_2} = 1.285 \text{ m/s}$

For B:

$(+\uparrow)$ $m_B(v_B)_{y_1} = m_B(v_B)_{y_2}$

$(v_B)_{y_2} = 0$

Hence $(v_B)_2 = (v_B)_{x_2} = 1.34 \text{ m/s} \leftarrow$ *Ans.*

$(v_A)_2 = \sqrt{(-0.1915)^2 + (1.285)^2} = 1.30 \text{ m/s}$ *Ans.*

$(\theta_A)_2 = \tan^{-1}\left(\dfrac{0.1915}{1.285}\right) = 8.47°$ *Ans.*

R15–8. $(H_z)_1 + \Sigma\int M_z\,dt = (H_z)_2$

$$(10)(2)(0.75) + 60(2)\left(\frac{3}{5}\right)(0.75) + \int_0^2 (8t^2 + 5)\,dt = 10v(0.75)$$

$$69 + \left[\frac{8}{3}t^3 + 5t\right]_0^2 = 7.5v$$

$v = 13.4 \text{ m/s}$ *Ans.*

Chapter 16

R16–1. $(\omega_A)_O = 60$ rad/s

$\alpha_A = -1$ rad/s^2

$\omega_A = (\omega_A)_O + \alpha_A t$

$\omega_A = 60 + (-1)(3) = 57$ rad/s

$v_A = r\omega_A = (1)(57) = 57$ ft/s $= v_B$

$\omega_B = \dfrac{v_B}{r} = 57/2 = 28.5$ rad/s

$v_W = r_C\omega_C = (0.5)(28.5) = 14.2$ ft/s *Ans.*

$\alpha_A = 1$

$a_{A_t} = l(1) = 1$ ft/s^2

$\alpha_B = \dfrac{1}{2} = 0.5$ rad/s^2

$a_W = r\alpha_B = (0.5)(0.5) = 0.25$ ft/s^2 *Ans.*

R16–2. $\alpha_a = 0.6\theta_A$

$\theta_C = \dfrac{0.5}{0.075} = 6.667$ rad

$\theta_A(0.05) = (6.667)(0.15)$

$\theta_A = 20$ rad

$\alpha d\theta = \omega d\omega$

$$\int_0^{20} 0.6\theta_A d\theta_A = \int_3^{\omega_A} \omega_A d\omega_A$$

$$0.3\theta_A^2\Big|_0^{20} = \frac{1}{2}\omega_A^2\Big|_3^{\omega_A}$$

$120 = \dfrac{1}{2}\omega_A^2 - 4.5$

$\omega_A = 15.780$ rad/s

$15.780(0.05) = \omega_C(0.15)$

$\omega_C = 5.260$ rad/s

$v_B = 5.260(0.075) = 0.394$ m/s *Ans.*

R16–3. A point on the drum which is in contact with the board has a tangential acceleration of

$a_t = 0.5$ m/s^2

$a^2 = a_t^2 + a_n^2$

$(3)^2 = (0.5)^2 + a_n^2$

$a_n = 2.96$ m/s^2

$a_n = \omega^2 r, \qquad \omega = \sqrt{\dfrac{2.96}{0.25}} = 3.44$ rad/s

$v_B = \omega r = 3.44(0.25) = 0.860$ m/s *Ans.*

R16–4. $\mathbf{v}_B = \omega_{AB} \times \mathbf{r}_{B/A}$

$\mathbf{v}_C = \mathbf{v}_B + \omega \times \mathbf{r}_{C/B}$

$v_C\mathbf{i} = (6\mathbf{k}) \times (0.2\cos 45°\mathbf{i} + 0.2\sin 45°\mathbf{j}) + (\omega\mathbf{k}) \times (0.5\cos 30°\mathbf{i} - 0.5\sin 30°\mathbf{j})$

$v_C = -0.8485 + \omega(0.25)$

$0 = 0.8485 + 0.433\,\omega$

Solving

$\omega = 1.96$ rad/s ↻

$v_C = 1.34$ m/s *Ans.*

R16–5. $\omega = \dfrac{2}{0.08} = 25$ rad/s

$\alpha = \dfrac{4}{0.08} = 50$ rad/s^2

$\mathbf{a}_C = \mathbf{a}_A + (\mathbf{a}_{C/A})_n + (\mathbf{a}_{C/A})_t$

$\mathbf{a}_C = 4\mathbf{j} + (25)^2(0.08)\mathbf{i} + 50(0.08)\mathbf{j}$

$\xrightarrow{+}\quad a_C\cos\theta = 0 + 50$

$+\uparrow\quad a_C\sin\theta = 4 + 0 + 4$

Solving, $a_C = 50.6$ m/s^2 *Ans.*

$\theta = 9.09°$ ∠θ *Ans.*

The cylinder moves up with an acceleration

$a_B = (a_C)_t = 50.6\sin 9.09° = 8.00$ m/s$^2\uparrow$ *Ans.*

R16–6. $\mathbf{a}_C = \mathbf{a}_B + \mathbf{a}_{C/B}$

$$\underset{\rightarrow}{2.057} + \underset{\downarrow}{(a_C)_t} = \underset{\downarrow}{1.8} + \underset{\leftarrow}{1.2} + \underset{\theta\ 30°}{\alpha_{CB}(0.5)}$$

$(\xrightarrow{+})\quad 2.057 = -1.2 + \alpha_{CB}(0.5)\cos 30°$

$(+\downarrow)\quad (a_C)_t = 1.8 + \alpha_{CB}(0.5)\sin 30°$

$\alpha_{CB} = 7.52$ rad/s^2 *Ans.*

$(a_C)_t = 3.68$ m/s^2

$a_C = \sqrt{(3.68)^2 + (2.057)^2} = 4.22$ m/s^2 *Ans.*

$\theta = \tan^{-1}\left(\dfrac{3.68}{2.057}\right) = 60.8°$ *Ans.*

Also,

$\mathbf{a}_C = \mathbf{a}_B + \alpha_{CB} \times \mathbf{r}_{C/B} - \omega^2\mathbf{r}_{C/B}$

$-(a_C)_t\mathbf{j} + \dfrac{(0.6)^2}{0.175}\mathbf{i} = -(2)^2(0.3)\mathbf{i} - 6(0.3)\mathbf{j} + (\alpha_{CB}\mathbf{k}) \times (-0.5\cos 60°\mathbf{i} - 0.5\sin 60°\mathbf{j}) - \mathbf{0}$

$2.057 = -1.20 + \alpha_{CB}(0.433)$

$-(a_C)_t = -1.8 - \alpha_{CB}(0.250)$

$\alpha_{CB} = 7.52$ rad/s^2 *Ans.*

$a_t = 3.68$ m/s^2

$a_C = \sqrt{(3.68)^2 + (2.057)^2} = 4.22$ m/s^2 *Ans.*

$\theta = \tan^{-1}\left(\dfrac{3.68}{2.057}\right) = 60.8°$ θ *Ans.*

R16–7. $a_C = 0.5(8) = 4\ \text{m/s}^2$

$\mathbf{a}_B = \mathbf{a}_C + \mathbf{a}_{B/C}$

$$\mathbf{a}_B = \begin{bmatrix} 4 \\ \leftarrow \end{bmatrix} + \begin{bmatrix} (3)^2(0.5) \\ \angle 30^\circ \end{bmatrix} + \begin{bmatrix} (0.5)(8) \\ 30^\circ \end{bmatrix}$$

$(\xrightarrow{+})\quad (a_B)_x = -4 + 4.5\cos 30^\circ + 4\sin 30^\circ$

$= 1.897\ \text{m/s}^2$

$(+\uparrow)\quad (a_B)_y = 0 + 4.5\sin 30^\circ - 4\cos 30^\circ$

$= -1.214\ \text{m/s}^2$

$a_B = \sqrt{(1.897)^2 + (-1.214)^2}$

$= 2.25\ \text{m/s}^2$ *Ans.*

$\theta = \tan^{-1}\left(\dfrac{1.214}{1.897}\right) = 32.6^\circ$ *Ans.*

Also,

$\mathbf{a}_B = \mathbf{a}_C + \alpha \times \mathbf{r}_{B/C} - \omega^2\mathbf{r}_{B/C}$

$(a_B)_x\mathbf{i} + (a_B)_y\mathbf{j} = -4\mathbf{i} + (8\mathbf{k}) \times (-0.5\cos 30^\circ\mathbf{i}$

$-0.5\sin 30^\circ\mathbf{j}) - (3)^2(-0.5\cos 30^\circ\mathbf{i} - 0.5\sin 30^\circ\mathbf{j})$

$(\xrightarrow{+})\ (a_B)_x = -4 + 8(0.5\sin 30^\circ) + (3)^2(0.5\cos 30^\circ)$

$= 1.897\ \text{m/s}^2$

$(+\uparrow)\ (a_B)_y = 0 - 8(0.5\cos 30^\circ) + (3)^2(0.5\sin 30^\circ)$

$= -1.214\ \text{m/s}^2$

$\theta = \tan^{-1}\left(\dfrac{1.214}{1.897}\right) = 32.6^\circ$ *Ans.*

$a_B = \sqrt{(1.897)^2 + (-1.214)^2} = 2.25\ \text{m/s}^2$ *Ans.*

R16–8. $v_B = 3(7) = 21\ \text{in./s} \leftarrow$

$\mathbf{v}_C = \mathbf{v}_B + \omega \times \mathbf{r}_{C/B}$

$-v_C\left(\dfrac{4}{5}\right)\mathbf{i} - v_C\left(\dfrac{3}{5}\right)\mathbf{j} = -21\mathbf{i} + \omega\mathbf{k} \times (-5\mathbf{i} - 12\mathbf{j})$

$(\xrightarrow{+})\quad -0.8v_C = -21 + 12\omega$

$(+\uparrow)\quad -0.6v_C = -5\omega$

Solving:

$\omega = 1.125\ \text{rad/s}$

$v_C = 9.375\ \text{in./s} = 9.38\ \text{in./s}$ *Ans.*

$(a_B)_n = (3)^2(7) = 63\ \text{in./s}^2 \downarrow$

$(a_B)_t = (2)(7) = 14\ \text{in./s}^2 \leftarrow$

$\mathbf{a}_C = a_B + \alpha \times \mathbf{r}_{C/B} - \omega^2\mathbf{r}_{C/B}$

$-a_C\left(\dfrac{4}{5}\right)\mathbf{i} - a_C\left(\dfrac{3}{5}\right)\mathbf{j} = -14\mathbf{i} - 63\mathbf{j} + (\alpha\mathbf{k})$

$\times (-5\mathbf{i} - 12\mathbf{j}) - (1.125)^2(-5\mathbf{i} - 12\mathbf{j})$

$(\xrightarrow{+})\quad -0.8a_C = -14 + 12\alpha + 6.328$

$(+\uparrow)\quad -0.6a_C = -63 - 5\alpha + 15.1875$

$a_C = 54.7\ \text{in./s}^2$ *Ans.*

$\alpha = -3.00\ \text{rad/s}^2$

Chapter 17

R17–1. $\xleftarrow{+}\Sigma F_x = ma_x;\quad 50\cos 60^\circ = 200a_G$ **(1)**

$+\uparrow\Sigma F_y = ma_y;\quad N_A + N_B - 200(9.81)$

$-50\sin 60^\circ = 0$ **(2)**

$\circlearrowleft +\Sigma M_G = 0;\quad -N_A(0.3) + N_B(0.2) +$

$50\cos 60^\circ(0.3)$

$-50\sin 60^\circ(0.6) = 0$ **(3)**

Solving,

$a_G = 0.125\ \text{m/s}^2$

$N_A = 765.2\ \text{N}$

$N_B = 1240\ \text{N}$

At each wheel

$N'_A = \dfrac{N_A}{2} = 383\ \text{N}$ *Ans.*

$N'_B = \dfrac{N_B}{2} = 620\ \text{N}$ *Ans.*

R17–2. ***Curvilinear Translation***:

$(a_G)_t = 8(3) = 24\ \text{ft/s}^2$

$(a_G)_n = (5)^2(3) = 75\ \text{ft/s}^2$

$\bar{x} = \dfrac{\Sigma\bar{x}m}{\Sigma m} = \dfrac{1(3) + 2(3)}{6} = 1.5\ \text{ft}$

$+\downarrow\Sigma F_y = m(a_G)_y;\quad E_y + 6 = \dfrac{6}{32.2}(24)\cos 30^\circ$

$+ \dfrac{6}{32.2}(75)\sin 30^\circ$

$\xrightarrow{+}\Sigma F_x = m(a_G)_x;\quad E_x = \dfrac{6}{32.2}(75)\cos 30^\circ$

$- \dfrac{6}{32.2}(24)\sin 30^\circ$

$\circlearrowleft +\Sigma M_G = 0;\quad M_E - E_y(1.5) = 0$

$E_x = 9.87\ \text{lb}$ *Ans.*

$E_y = 4.86\ \text{lb}$ *Ans.*

$M_E = 7.29\ \text{lb}\cdot\text{ft}$ *Ans.*

R17–3. (a) Rear wheel drive

Equations of motion:

$\xrightarrow{+}\Sigma F_x = m(a_G)_x;\quad 0.3N_B = 1.5(10)^3a_G$ **(1)**

$\circlearrowright +\Sigma M_A = \Sigma(M_k)_A;\quad 1.5(10)^3(9.81)(1.3)$

$$-N_B(2.9) = -1.5(10)^3 a_G(0.4) \qquad \textbf{(2)}$$

Solving Eqs. (1) and (2) yields:

$$N_B = 6881 \text{ N} = 6.88 \text{ kN}$$

$$a_G = 1.38 \text{ m/s}^2 \qquad \textit{Ans.}$$

R17–4. $\xrightarrow{+} \Sigma F_x = m(a_G)_x;\ 40 \sin 60° + N_C - \left(\frac{5}{13}\right)T = 0$

$$+\uparrow \Sigma F_y = m(a_G)_y;\ -40 \cos 60° + 0.3N_C - 20(9.81) + \frac{12}{13}T = 0$$

$$\circlearrowleft + \Sigma M_A = I_A \alpha; \quad 40(0.120) - 0.3N_C(0.120) = \left[\frac{1}{2}(20)(0.120)^2\right]\alpha$$

Solving,

$T = 218$ N *Ans.*

$N_C = 49.28$ N

$\alpha = 21.0 \text{ rad/s}^2$ *Ans.*

R17–5. $(a_G)_t = 4\alpha$

$$\xleftarrow{+} \Sigma F_t = m(a_G)_x; \quad F + 20 - 5 = \frac{30}{32.2}(4\alpha)$$

$$\circlearrowright + \Sigma M_O = I_O\alpha; \quad 20(3) + F(6) = \frac{1}{3}\left(\frac{30}{32.2}\right)(8)^2\alpha$$

Solving,

$\alpha = 12.1 \text{ rad/s}^2$ *Ans.*

$F = 30.0$ lb *Ans.*

R17–6. $I_O = \frac{2}{5}\left(\frac{30}{32.2}\right)(1)^2 + \left(\frac{30}{32.2}\right)(3)^2 + \frac{1}{3}\left(\frac{10}{32.2}\right)(2)^2 = 9.17 \text{ slug}\cdot\text{ft}^2$

$$\bar{x} = \frac{30(3) + 10(1)}{30 + 10} = 2.5 \text{ ft}$$

$$\xrightarrow{+} \Sigma F_n = ma_n; \quad O_x = 0$$

$$+\downarrow \Sigma F_t = ma_t; \quad 40 - O_y = \frac{40}{32.2}a_G$$

$$\circlearrowleft + \Sigma M_O = I_O\alpha; \quad 40(2.5) = 9.17\alpha$$

Kinematics

$a_G = 2.5\alpha$

Solving,

$\alpha = 10.90 \text{ rad/s}^2$

$a_G = 27.3 \text{ ft/s}^2$

$O_x = 0$

$O_y = 6.14$ lb

Thus:

$F_o = 6.14 \text{ lb} \rightarrow$ *Ans.*

R17–7. $+\uparrow \Sigma F_y = m(a_G)_y; \quad N_B - 20(9.81) = 0$

$$N_B = 196.2 \text{ N}$$

$$F_B = 0.1(196.2) = 19.62 \text{ N}$$

$$\circlearrowleft + \Sigma M_{IC} = \Sigma(M_k)_{IC}; \quad 30 - 19.62(0.6) = 20(0.2\alpha)(0.2) + [20(0.25)^2]\alpha$$

$\alpha = 8.89 \text{ rad/s}^2$ *Ans.*

R17–8. $\xleftarrow{+} \Sigma F_x = m(a_G)_x; \quad 0.3N_A = \frac{20}{32.2}a_G$

$$+\uparrow \Sigma F_y = m(a_G)_y; \quad N_A - 20 = 0$$

$$\circlearrowright + \Sigma M_G = I_G\alpha; \quad 0.3N_A(0.5) = \left[\frac{2}{5}\left(\frac{20}{32.2}\right)(0.5)^2\right]\alpha$$

Solving,

$N_A = 20$ lb

$a_G = 9.66 \text{ ft/s}^2$

$\alpha = 48.3 \text{ rad/s}^2$

$(\circlearrowleft +) \quad \omega = \omega_0 + \alpha_c t$

$0 = \omega_1 - 48.3t$

$\omega_1 = 48.3t$

$(\xrightarrow{+}) \quad v = v_0 + a_c t$

$0 = 20 - 9.66\left(\frac{\omega}{48.3}\right)$

$\omega = 100$ rad/s *Ans.*

Chapter 18

R18–1. $T_1 + \Sigma U_{1-2} = T_2$

$$0 + (50)(9.81)(1.25) = \frac{1}{2}\left[(50)(1.75)^2\right]\omega_2^2$$

$\omega_2 = 2.83$ rad/s *Ans.*

R18–2. ***Kinetic Energy and Work***: The mass moment inertia of the flywheel about its mass center is $I_O = mk_O^2 = 50(0.2^2) = 2 \text{ kg}\cdot\text{m}^2$. Thus,

$$T = \frac{1}{2}I_O\omega^2 = \frac{1}{2}(2)\omega^2 = \omega^2$$

Since the wheel is initially at rest, $T_1 = 0$. $\mathbf{W}$, $\mathbf{O}_x$, and $\mathbf{O}_y$ do no work while $\mathbf{M}$ does positive work. When the wheel rotates

$\theta = (5 \text{ rev})\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right) = 10\pi$, the work done by M is

$$U_M = \int M d\theta = \int_0^{10\pi} (9\theta^{1/2} + 1) d\theta$$
$$= (6\theta^{3/2} + \theta)\Big|_0^{10\pi}$$
$$= 1087.93 \text{ J}$$

Principle of Work and Energy:

$$T_1 + \Sigma U_{1-2} = T_2$$
$$0 + 1087.93 = \omega^2$$
$$\omega = 33.0 \text{ rad/s} \qquad \textit{Ans.}$$

R18–3. Before braking:

$$T_1 + \Sigma U_{1-2} = T_2$$
$$0 + 15(9.81)(3) = \frac{1}{2}(15)v_B^2 + \frac{1}{2}\left[50(0.23)^2\right]\left(\frac{v_B}{0.15}\right)^2$$
$$v_B = 2.58 \text{ m/s} \qquad \textit{Ans.}$$
$$\frac{s_B}{0.15} = \frac{s_C}{0.25}$$

Set $s_B = 3$ m, then $s_C = 5$ m.

$$T_1 + \Sigma U_{1-2} = T_2$$
$$0 - F(5) + 15(9.81)(6) = 0$$
$$F = 176.6 \text{ N}$$
$$N = \frac{176.6}{0.5} = 353.2 \text{ N}$$

Brake arm:

$$\circlearrowleft + \Sigma M_A = 0; \qquad -353.2(0.5) + P(1.25) = 0$$
$$P = 141 \text{ N} \qquad \textit{Ans.}$$

R18–4.

$$\frac{s_G}{0.3} = \frac{s_A}{(0.5 - 0.3)}$$
$$s_A = 0.6667 s_G$$
$$+\nwarrow \Sigma F_y = 0; \qquad N_A - 60(9.81)\cos 30^\circ = 0$$
$$N_A = 509.7 \text{ N}$$
$$T_1 + \Sigma U_{1-2} = T_2$$
$$0 + 60(9.81)\sin 30^\circ(s_G) - 0.2(509.7)(0.6667 s_G) = \frac{1}{2}\left[60(0.3)^2\right](6)^2 + \frac{1}{2}(60)\left[(0.3)(6)\right]^2$$
$$s_G = 0.859 \text{ m} \qquad \textit{Ans.}$$

R18–5. ***Conservation of Energy***: Originally, both gears are rotating with an angular velocity of $\omega_1 = \frac{2}{0.05} = 40$ rad/s. After the rack has traveled $s = 600$ mm, both gears rotate with an angular velocity of $\omega_2 = \frac{v_2}{0.05}$, where v_2 is the speed of the rack at that moment.

$$T_1 + V_1 = T_2 + V_2$$
$$\frac{1}{2}(6)(2)^2 + 2\left\{\frac{1}{2}\left[4(0.03)^2\right](40)^2\right\} + 0 = \left\{\frac{1}{2}\left[4(0.03)^2\right]\left(\frac{v_2}{0.05}\right)^2\right\} - 6(9.81)(0.6)$$
$$v_2 = 3.46 \text{ m/s} \qquad \textit{Ans.}$$

R18–6. Datum through A.

$$T_1 + V_1 = T_2 + V_2$$
$$\frac{1}{2}\left[\frac{1}{3}\left(\frac{50}{32.2}\right)(6)^2\right](2)^2 + \frac{1}{2}(6)(4-2)^2 = \frac{1}{2}\left[\frac{1}{3}\left(\frac{50}{32.2}\right)(6)^2\right]\omega^2 + \frac{1}{2}(6)(7-2)^2 - 50(1.5)$$
$$\omega = 2.30 \text{ rad/s} \qquad \textit{Ans.}$$

R18–7.

$$T_1 + V_1 = T_2 + V_2$$
$$0 + 4(1.5\sin 45^\circ) + 1(3\sin 45^\circ) = \frac{1}{2}\left[\frac{1}{3}\left(\frac{4}{32.2}\right)(3)^2\right]\left(\frac{v_C}{3}\right)^2 + \frac{1}{2}\left(\frac{1}{32.2}\right)(v_C)^2 + 0$$
$$v_C = 13.3 \text{ ft/s} \qquad \textit{Ans.}$$

R18–8. Datum at lowest point.

$$T_1 + V_1 = T_2 + V_2$$
$$\frac{1}{2}\left[\frac{1}{2}(40)(0.3)^2\right]\left(\frac{4}{0.3}\right)^2 + \frac{1}{2}(40)(4)^2 + 40(9.81)d\sin 30^\circ = 0 + \frac{1}{2}(200)d^2$$
$$100d^2 - 196.2d - 480 = 0$$

Solving for the positive root,

$$d = 3.38 \text{ m} \qquad \textit{Ans.}$$

Chapter 19

R19–1. $I_O = mk_O^2 = \frac{150}{32.2}(1.25)^2 = 7.279 \text{ slug}\cdot\text{ft}^2$

$$I_O\omega_1 + \Sigma\int_{t_1}^{t_2} M_O dt = I_O\omega_2$$
$$0 - \int_0^{3\text{ s}} 10t^2(1)\, dt = 7.279\omega_2$$
$$\frac{10t^3}{3}\Big|_0^{3\text{ s}} = 7.279\omega_2$$
$$\omega_2 = 12.4 \text{ rad/s} \qquad \textit{Ans.}$$

R19–3. $+\swarrow m(v_G)_1 + \Sigma\int F\,dt = m(v_G)_2$

$$0 + 9(9.81)(\sin 30°)(3) - \int_0^3 F\,dt = 9(v_G)_2 \quad \textbf{(1)}$$

$$\circlearrowleft +(H_G)_1 + \Sigma\int M_G dt = (H_G)_2$$

$$0 + \left(\int_0^3 F\,dt\right)(0.3) = \left[9(0.225)^2\right]\omega_2 \quad \textbf{(2)}$$

Since $(v_G)_2 = 0.3\omega_2$,

Eliminating $\int_0^3 F\,dt$ from Eqs. (1) and (2) and solving for $(v_G)_2$ yields.

$$(v_G)_2 = 9.42 \text{ m/s} \qquad \textit{Ans.}$$

Also,

$$\circlearrowleft +(H_A)_1 + \Sigma\int M_A dt = (H_A)_2$$

$$0 + 9(9.81)\sin 30°(3)(0.3) = \left[9(0.225)^2 + 9(0.3)^2\right]\omega$$

$$\omega = 31.39 \text{ rad/s}$$

$$v = 0.3(31.39) = 9.42 \text{ m/s} \qquad \textit{Ans.}$$

R19–4. $\overset{+}{\leftarrow}$ $m(v_x)_1 + \Sigma\int_{t_1}^{t_2} F_x dt = m(v_x)_2$

$$0 + 200(3) = 100(v_O)_2$$

$$(v_O)_2 = 6 \text{ m/s} \qquad \textit{Ans.}$$

and

$$I_z\omega_1 + \Sigma\int_{t_1}^{t_2} M_z dt = I_z\omega_2$$

$$0 - [200(0.4)(3)] = -9\omega_2$$

$$\omega_2 = 26.7 \text{ rad/s} \qquad \textit{Ans.}$$

R19–5. $(+\uparrow)$ $mv_1 + \Sigma\int F dt = mv_2$

$$0 + T(3) - 30(3) + 40(3) = \frac{30}{32.2}v_o$$

$$(\circlearrowleft +) \quad (H_O)_1 + \Sigma\int M_O dt = (H_O)_2$$

$$-T(0.5)3 + 40(1)3 = \left[\frac{30}{32.2}(0.65)^2\right]\omega$$

Kinematics,

$$v_o = 0.5\omega$$

Solving,

$$T = 23.5 \text{ lb}$$

$$\omega = 215 \text{ rad/s} \qquad \textit{Ans.}$$

$$v_O = 108 \text{ ft/s}$$

Also,

$$(\circlearrowleft +) \quad (H_{IC})_1 + \Sigma\int M_{IC}dt = (H_{IC})_2$$

$$0 - 30(0.5)(3) + 40(1.5)(3)$$

$$= \left[\frac{30}{32.2}(0.65)^2 + \frac{30}{32.2}(0.5)^2\right]\omega$$

$$\omega = 215 \text{ rad/s} \qquad \textit{Ans.}$$

R19–6. $\circlearrowright +$ $(H_A)_1 + \Sigma\int M_A dt = (H_A)_2$

$$\left[\frac{30}{32.2}(0.8)^2\right](6) - \int T\,dt(1.2) = \left[\frac{30}{32.2}(0.8)^2\right]\omega_A$$

$$\circlearrowright + \quad (H_B)_1 + \Sigma\int M_B dt = (H_B)_2$$

$$0 + \int T\,dt(0.4) = \left[\frac{15}{32.2}(0.6)^2\right]\omega_B$$

Kinematics:

$$1.2\omega_A = 0.4\omega_B$$

$$\omega_B = 3\omega_A$$

Thus,

$$\omega_A = 1.70 \text{ rad/s} \qquad \textit{Ans.}$$

$$\omega_B = 5.10 \text{ rad/s} \qquad \textit{Ans.}$$

R19–7. $H_1 = H_2$

$$\left(\frac{1}{2}mr^2\right)\omega_1 = \left[\frac{1}{2}mr^2 + mr^2\right]\omega_2$$

$$\omega_2 = \frac{1}{3}\omega_1 \qquad \textit{Ans.}$$

R19–8. $H_1 = H_2$

$$(0.940)(0.5) + (4)\left[\frac{1}{12}(20)\left((0.75)^2 + (0.2)^2\right) + (20)(0.375 + 0.2)^2\right](0.5)$$

$$= (0.940)(\omega) + 4\left[\frac{1}{12}(20)(0.2)^2 + (20)(0.2)^2\right]\omega$$

$$\omega = 3.56 \text{ rad/s} \qquad \textit{Ans.}$$

Answers to Selected Problems

Chapter 12

12–1. $s = 80.7$ m

12–2. $s = 20$ ft

12–3. $a = -24\ \text{m/s}^2$, $\Delta s = -880$ m, $s_T = 912$ m

12–5. $s_T = 8$ m, $v_{avg} = 2.67$ m/s

12–6. $s|_{t=6\text{ s}} = -27.0$ ft, $s_{tot} = 69.0$ ft

12–7. $v_{avg} = 0$, $(v_{sp})_{avg} = 3$ m/s, $a\Big|_{t=6\text{ s}} = 2\ \text{m/s}^2$

12–9. $v = 32$ m/s, $s = 67$ m, $d = 66$ m

12–10. $v = 1.29$ m/s

12–11. $v_{avg} = 0.222$ m/s, $(v_{sp})_{avg} = 2.22$ m/s

12–13. Normal: $d = 517$ ft, drunk: $d = 616$ ft

12–14. $v = 165$ ft/s, $a = 48\ \text{ft/s}^2$, $s_T = 450$ ft, $v_{avg} = 25.0$ ft/s, $(v_{sp})_{avg} = 45.0$ ft/s

12–15. $v = \left(2kt + \dfrac{1}{v_0^2}\right)^{-1/2}$, $s = \dfrac{1}{k}\left[\left(2kt + \dfrac{1}{v_0^2}\right)^{1/2} - \dfrac{1}{v_0}\right]$

12–17. $d = 16.9$ ft

12–18. $t = 5.62$ s

12–19. $s = 28.4$ km

12–21. $s = 123$ ft, $a = 2.99\ \text{ft/s}^2$

12–22. $h = 314$ m, $v = 72.5$ m/s

12–23. $v = (20e^{-2t})$ m/s, $a = (-40e^{-2t})\ \text{m/s}^2$, $s = 10(1 - e^{-2t})$ m

12–25. (a) $v = 45.5$ m/s, (b) $v_{max} = 100$ m/s

12–26. (a) $s = -30.5$ m,
(b) $s_{Tot} = 56.0$ m,
(c) $v = 10$ m/s

12–27. $t = 0.549\left(\dfrac{v_f}{g}\right)$

12–29. $h = 20.4$ m, $t = 2$ s

12–30. $s = 54.0$ m

12–31. $s = \dfrac{v_0}{k}(1 - e^{-kt})$, $a = -kv_0e^{-kt}$

12–33. $v = 11.2$ km/s

12–34. $v = -R\sqrt{\dfrac{2g_0(y_0 - y)}{(R + y)(R + y_0)}}$, $v_{imp} = 3.02$ km/s

12–35. $t' = 27.3$ s.
When $t = 27.3$ s, $v = 13.7$ ft/s.

12–37. $\Delta s = 1.11$ km

12–38. $a|_{t=0} = -4\ \text{m/s}^2$, $a|_{t=2\text{ s}} = 0$, $a|_{t=4\text{ s}} = 4\ \text{m/s}^2$, $v|_{t=0} = 3$ m/s, $v|_{t=2\text{ s}} = -1$ m/s, $v|_{t=4\text{ s}} = 3$ m/s

12–39. $s = 2\sin\left(\dfrac{\pi}{5}t\right) + 4$, $v = \dfrac{2\pi}{5}\cos\left(\dfrac{\pi}{5}t\right)$,
$a = -\dfrac{2\pi^2}{25}\sin\left(\dfrac{\pi}{5}t\right)$

12–41. $t = 7.48$ s. When $t = 2.14$ s, $v = v_{max} = 10.7$ ft/s, $h = 11.4$ ft.

12–42. $s = 600$ m. For $0 \le t < 40$ s, $a = 0$.
For $40\text{ s} < t \le 80$ s, $a = -0.250\ \text{m/s}^2$.

12–43. $t' = 35$ s
For $0 \le t < 10$ s, $s = \{300t\}$ ft, $v = 300$ ft/s
For $10\text{ s} < t < 20$ s,
$s = \left\{\dfrac{1}{6}t^3 - 15t^2 + 550t - 1167\right\}$ ft
$v = \left\{\dfrac{1}{2}t^2 - 30t + 550\right\}$ ft/s
For $20\text{ s} < t \le 35$ s,
$s = \{-5t^2 + 350t + 167\}$ ft
$v = (-10t + 350)$ ft/s

12–45. When $t = 0.1$ s, $s = 0.5$ m and a changes from $100\ \text{m/s}^2$ to $-100\ \text{m/s}^2$. When $t = 0.2$ s, $s = 1$ m.

12–46. $v\Big|_{s=75\text{ ft}} = 27.4$ ft/s, $v\Big|_{s=125\text{ ft}} = 37.4$ ft/s

12–47. For $0 \le t < 30$ s, $v = \left\{\dfrac{1}{5}t^2\right\}$ m/s, $s = \left\{\dfrac{1}{15}t^3\right\}$ m
For $30 \le t \le 60$ s, $v = \{24t - 540\}$ m/s, $s = \{12t^2 - 540t + 7200\}$ m

12–49. $v_{max} = 100$ m/s, $t' = 40$ s

12–50. For $0 \le s < 300$ ft, $v = \{4.90\, s^{1/2}\}$ m/s.
For $300\text{ ft} < s \le 450$ ft,
$v = \{(-0.04s^2 + 48s - 3600)^{1/2}\}$ m/s.
$s = 200$ ft when $t = 5.77$ s.

12–51. For $0 \le t < 60$ s, $s = \left\{\dfrac{1}{20}t^2\right\}$ m, $a = 0.1\ \text{m/s}^2$.
For $60\text{ s} < t < 120$ s, $s = \{6t - 180\}$ m, $a = 0$.
For $120\text{ s} < t \le 180$ s, $s = \left\{\dfrac{1}{30}t^2 - 2t + 300\right\}$ m, $a = 0.0667\ \text{m/s}^2$.

12–53. At $t = 8$ s, $a = 0$ and $s = 30$ m.
At $t = 12$ s, $a = -1\ \text{m/s}^2$ and $s = 48$ m.

12–54. For $0 \le t < 5$ s, $s = \{0.2t^3\}$ m, $a = \{1.2t\}\ \text{m/s}^2$
For $5\text{ s} < t \le 15$ s, $s = \left\{\dfrac{1}{4}(90t - 3t^2 - 275)\right\}$ m
$a = -1.5\ \text{m/s}^2$,
At $t = 15$ s, $s = 100$ m, $v_{avg} = 6.67$ m/s

12–55. $t' = 33.3$ s, $s|_{t=5\text{ s}} = 550$ ft, $s|_{t=15\text{ s}} = 1500$ ft, $s|_{t=20\text{ s}} = 1800$ ft, $s|_{t=33.3\text{ s}} = 2067$ ft

12–57. For $0 \le s < 100$ ft, $v = \left\{\sqrt{\dfrac{1}{50}(800s - s^2)}\right\}$ ft/s
For $100\text{ ft} < s \le 150$ ft,
$v = \left\{\dfrac{1}{5}\sqrt{-3s^2 + 900s - 25\,000}\right\}$ ft/s

12–58. For $0 \le t < 15$ s, $v = \left\{\frac{1}{2}t^2\right\}$ m/s, $s = \left\{\frac{1}{6}t^3\right\}$ m.
For 15 s $< t \le 40$ s,
$v = \{20t - 187.5$ m/s$\}$,
$s = \{10t^2 - 187.5t + 1125\}$ m

12–59. $s_T = 980$ m

12–61. When $t = 5$ s, $s_B = 62.5$ m.
When $t = 10$ s, $v_A = (v_A)_{\max} = 40$ m/s and $s_A = 200$ m.
When $t = 15$ s, $s_A = 400$ m and $s_B = 312.5$ m.
$\Delta s = s_A - s_B = 87.5$ m

12–62. $v = \{5 - 6t\}$ ft/s, $a = -6$ ft/s^2

12–63. For $0 \le t < 5$ s, $s = \{2t^2\}$ m and $a = 4$ m/s^2.
For 5 s $< t < 20$ s, $s = \{20t - 50\}$ m and $a = 0$.
For 20 s $< t \le 30$ s, $s = \{2t^2 - 60t + 750\}$ m and $a = 4$ m/s^2.

12–65. $v = 354$ ft/s, $t = 5.32$ s

12–66. When $s = 100$ m, $t = 10$ s.
When $s = 400$ m, $t = 16.9$ s.
$a|_{s=100\text{ m}} = 4$ m/s^2, $a|_{s=400\text{ m}} = 16$ m/s^2

12–67. At $s = 100$ s, a changes from $a_{\max} = 1.5$ ft/s^2 to $a_{\min} = -0.6$ ft/s^2.

12–69. $a = 5.31$ m/s^2, $\alpha = 53.0°$
$\beta = 37.0°$, $\gamma = 90.0°$

12–70. $\Delta\mathbf{r} = \{6\mathbf{i} + 4\mathbf{j}\}$ m

12–71. (4 ft, 2 ft, 6 ft)

12–73. (5.15 ft, 1.33 ft)

12–74. $\mathbf{r} = \{11\mathbf{i} + 2\mathbf{j} + 21\mathbf{k}\}$ ft

12–75. $(v_{sp})_{avg} = 4.28$ m/s

12–77. $v = 8.55$ ft/s, $a = 5.82$ m/s^2

12–78. $v = 1003$ m/s, $a = 103$ m/s^2

12–79. $d = 4.00$ ft, $a = 37.8$ ft/s^2

12–81. $(\mathbf{v}_{BC})_{avg} = \{3.88\mathbf{i} + 6.72\mathbf{j}\}$ m/s

12–82. $v = \sqrt{c^2k^2 + b^2}$, $a = ck^2$

12–83. $v = 10.4$ m/s, $a = 38.5$ m/s^2

12–85. $d = 204$ m, $v = 41.8$ m/s $a = 4.66$ m/s^2

12–86. $\theta = 58.3°$, $(v_0)_{\min} = 9.76$ m/s

12–87. $\theta = 76.0°$, $v_A = 49.8$ ft/s, $h = 39.7$ ft

12–89. $R_{\max} = 10.2$ m, $\theta = 45°$

12–90. $R = 8.83$ m

12–91. (13.3 ft, -7.09 ft)

12–93. $d = 166$ ft

12–94. $t = 3.57$ s, $v_B = 67.4$ ft/s

12–95. $v_A = 36.7$ ft/s, $h = 11.5$ ft

12–97. $v_A = 19.4$ m/s, $v_B = 40.4$ m/s

12–98. $v_A = 39.7$ ft/s, $s = 6.11$ ft

12–99. $v_B = 160$ m/s, $h_B = 427$ m,
$h_C = 1.08$ km, $R = 2.98$ km

12–101. $v_{\min} = 0.838$ m/s, $v_{\max} = 1.76$ m/s

12–102. $\theta_A = 11.6°$, $t = 0.408$ s, $\theta_B = 11.6°$

12–103. $\theta_A = 78.4°$, $t = 2.00$ s, $\theta_B = 78.4°$

12–105. $t_A = 0.553$ s, $x = 3.46$ m

12–106. $R = 19.0$ m, $t = 2.48$ s

12–107. $\theta_1 = 24.9°$, $\theta_2 = 85.2°$

12–109. $\theta = 76.0°$, $v_A = 49.8$ ft/s, $h = 39.7$ ft

12–110. $v = 63.2$ ft/s

12–111. $v = 38.7$ m/s

12–113. $v = 4.40$ m/s, $a_t = 5.04$ m/s^2, $a_n = 1.39$ m/s^2

12–114. $a_t = 8.66$ ft/s^2, $\rho = 1280$ ft

12–115. $v = 97.2$ ft/s, $a = 42.6$ ft/s^2

12–117. When cars A and B are side by side, $t = 55.7$ s.
When cars A and B are 90° apart, $t = 27.8$ s.

12–118. $t = 66.4$ s

12–119. $h = 5.99$ Mm

12–121. $a = 2.75$ m/s^2

12–122. $a = 1.68$ m/s^2

12–123. $v = 1.5$ m/s, $a = 0.117$ m/s^2

12–125. $v = 43.0$ m/s, $a = 6.52$ m/s^2

12–126. $v = 105$ ft/s, $a = 22.7$ ft/s^2

12–127. $a_t = 3.62$ m/s^2, $\rho = 29.6$ m

12–129. $t = 7.00$ s, $s = 98.0$ m

12–130. $a = 7.42$ ft/s^2

12–131. $a = 2.36$ m/s^2

12–133. $a = 3.05$ m/s^2

12–134. $a = 0.763$ m/s^2

12–135. $a = 0.952$ m/s^2

12–137. $y = -0.0766x^2$, $v = 8.37$ m/s,
$a_n = 9.38$ m/s^2, $a_t = 2.88$ m/s^2

12–138. $v_B = 19.1$ m/s, $a = 8.22$ m/s^2, $\phi = 17.3°$ up from negative$-t$ axis

12–139. $a_{\min} = 3.09$ m/s^2

12–141. $(a_n)_A = g = 32.2$ ft/s^2, $(a_t)_A = 0$,
$\rho_A = 699$ ft, $(a_n)_B = 14.0$ ft/s^2,
$(a_t)_B = 29.0$ ft/s^2, $\rho_B = 8.51(10^3)$ ft

12–142. $t = 1.21$ s

12–143. $a_{\max} = \dfrac{v^2 a}{b^2}$

12–145. $d = 11.0$ m, $a_A = 19.0$ m/s^2, $a_B = 12.8$ m/s^2

12–146. $t = 2.51$ s, $a_A = 22.2$ m/s^2, $a_B = 65.1$ m/s^2

12–147. $\theta = 10.6°$

12–149. $a = 0.511$ m/s^2

12–150. $a = 0.309$ m/s^2

12–151. $a = 322$ mm/s^2, $\theta = 26.6°$

12–153. $v_n = 0$, $v_t = 7.21$ m/s,
$a_n = 0.555$ m/s^2, $a_t = 2.77$ m/s^2

12–154. $a = 7.48$ ft/s^2

12–155. $a = 14.3$ in./s^2

12–157. $v_r = 5.44$ ft/s, $v_\theta = 87.0$ ft/s,
$a_r = -1386$ ft/s^2, $a_\theta = 261$ ft/s^2

12–158. $v = 464$ ft/s, $a = 43.2(10^3)$ ft/s^2

12–159. $\mathbf{v} = \{-14.2\mathbf{u}_r - 24.0\mathbf{u}_z\}$ m/s
$\mathbf{a} = \{-3.61\mathbf{u}_r - 6.00\mathbf{u}_z\}$ m/s^2

12–161. $v_r = -2\sin t,\ v_\theta = \cos t,$
$a_r = -\frac{5}{2}\cos t,\ a_\theta = -2\sin t$
12–162. $v_r = ae^{at},\ v_\theta = e^{at},$
$a_r = e^{at}(a^2-1),\ a_\theta = 2ae^{at}$
12–163. $v_r = 0,\ v_\theta = 10\ \text{ft/s},$
$a_r = -0.25\ \text{ft/s}^2,\ a_\theta = -3.20\ \text{ft/s}^2$
12–165. $\dot{\mathbf{a}} = (\dddot{r} - 3\dot{r}\dot{\theta}^2 - 3r\dot{\theta}\ddot{\theta})\mathbf{u}_r$
$+ (3\dot{r}\ddot{\theta} + \dot{r}\ddot{\theta} + 3\ddot{r}\dot{\theta} - r\dot{\theta}^3)\mathbf{u}_\theta + (\dddot{z})\mathbf{u}_z$
12–166. $a = 48.3\ \text{in./s}^2$
12–167. $v_r = 1.20\ \text{m/s},\ v_\theta = 1.26\ \text{m/s},$
$a_r = -3.77\ \text{m/s}^2,\ a_\theta = 7.20\ \text{m/s}^2$
12–169. $v_r = 1.20\ \text{m/s},\ v_\theta = 1.50\ \text{m/s},$
$a_r = -4.50\ \text{m/s}^2,\ a_\theta = 7.20\ \text{m/s}^2$
12–170. $v_r = 16.0\ \text{ft/s},\ v_\theta = 1.94\ \text{ft/s},$
$a_r = 7.76\ \text{ft/s}^2,\ a_\theta = 1.94\ \text{ft/s}^2$
12–171. $v = 4.24\ \text{m/s},\ a = 17.6\ \text{m/s}^2$
12–173. $a = 27.8\ \text{m/s}^2$
12–174. $v_r = 0,\ v_\theta = 12\ \text{ft/s},$
$a_r = -216\ \text{ft/s}^2,\ a_\theta = 0$
12–175. $v = 12.6\ \text{m/s},\ a = 83.2\ \text{m/s}^2$
12–177. $v_r = -1.84\ \text{m/s},\ v_\theta = 19.1\ \text{m/s},$
$a_r = -2.29\ \text{m/s}^2,\ a_\theta = 4.60\ \text{m/s}^2$
12–178. $v_r = -24.2\ \text{ft/s},\ v_\theta = 25.3\ \text{ft/s}$
12–179. $v_r = 0,\ v_\theta = 4.80\ \text{ft/s},$
$v_z = -0.664\ \text{ft/s},\ a_r = -2.88\ \text{ft/s}^2$
$a_\theta = 0,\ a_z = -0.365\ \text{ft/s}^2$
12–181. $v = 10.7\ \text{ft/s},\ a = 24.6\ \text{ft/s}^2$
12–182. $v = 10.7\ \text{ft/s},\ a = 40.6\ \text{ft/s}^2$
12–183. $\dot{\theta} = 0.333\ \text{rad/s},\ a = 6.67\ \text{m/s}^2$
12–185. $v = 1.32\ \text{m/s}$
12–186. $a = 8.66\ \text{m/s}^2$
12–187. $\dot{\theta} = 0.0178\ \text{rad/s}$
12–189. $v_r = 32.0\ \text{ft/s},\ v_\theta = 50.3\ \text{ft/s},$
$a_r = -201\ \text{ft/s}^2,\ a_\theta = 256\ \text{ft/s}^2$
12–190. $v_r = 32.0\ \text{ft/s},\ v_\theta = 50.3\ \text{ft/s},$
$a_r = -161\ \text{ft/s}^2,\ a_\theta = 319\ \text{ft/s}^2$
12–191. $v = 5.95\ \text{ft/s},\ a = 3.44\ \text{ft/s}^2$
12–193. $v_r = 0.242\ \text{m/s},\ v_\theta = 0.943\ \text{m/s},$
$a_r = -2.33\ \text{m/s}^2,\ a_\theta = 1.74\ \text{m/s}^2$
12–194. $\dot{\theta} = 10.0\ \text{rad/s}$
12–195. $v_B = 0.5\ \text{m/s}$
12–197. $v = 24\ \text{ft/s}$
12–198. $v_B = 1.67\ \text{m/s}$
12–199. $\Delta s_B = 1.33\ \text{ft} \rightarrow$
12–201. $t = 3.83\ \text{s}$
12–202. $v_B = 0.75\ \text{m/s}$
12–203. $t = 5.00\ \text{s}$
12–205. $v_{B/C} = 39\ \text{ft/s} \uparrow$
12–206. $v_B = 1.50\ \text{m/s}$
12–207. $v_A = 1.33\ \text{m/s}$
12–209. $v_B = 8\ \text{ft/s}\downarrow,\ a_B = 6.80\ \text{ft/s}^2 \uparrow$
12–210. $v_A = 2.5\ \text{ft/s}\uparrow,\ a_A = 2.44\ \text{ft/s}^2\uparrow$
12–211. $v_B = 2.40\ \text{m/s} \uparrow,\ a_B = 3.25\ \text{m/s}^2 \uparrow$
12–213. $v_A = 4\ \text{ft/s}$
12–214. $v_{A/B} = 13.4\ \text{m/s},\ \theta_v = 31.7^\circ$
$a_{A/B} = 4.32\ \text{m/s}^2,\ \theta_a = 79.0^\circ$
12–215. $v_A = 10.0\ \text{m/s} \leftarrow,\ a_A = 46.0\ \text{m/s}^2 \leftarrow$
12–217. $v_C = 1.2\ \text{m/s}\uparrow,\ a_C = 0.512\ \text{m/s}^2\uparrow$
12–218. $v_{B/A} = 1044\ \text{km/h},\ \theta = 54.5^\circ$
12–219. $v_{B/A} = 28.5\ \text{mi/h},\ \theta_v = 44.5^\circ,$
$a_{B/A} = 3418\ \text{mi/h}^2,\ \theta_a = 80.6^\circ$
12–221. $v_B = 13.5\ \text{ft/s},\ \theta = 84.8^\circ,\ t = 1.85\ \text{min}$
12–222. $v_w = 58.3\ \text{km/h},\ \theta = 59.0^\circ$
12–223. $v_{A/B} = 15.7\ \text{m/s},\ \theta = 7.11^\circ,\ t = 38.1\ \text{s}$
12–225. $v_{A/B} = 98.4\ \text{ft/s},\ \theta_v = 67.6^\circ,$
$a_{A/B} = 19.8\ \text{ft/s}^2,\ \theta_a = 57.4^\circ$
12–226. $v_{r/m} = 16.6\ \text{km/h},\ \theta = 25.0^\circ$
12–227. $v_{B/A} = 20.5\ \text{m/s},\ \theta_v = 43.1^\circ$
$a_{B/A} = 4.92\ \text{m/s}^2,\ \theta_a = 6.04^\circ$
12–229. $v_r = 34.6\ \text{km/h}\downarrow$
12–230. $v_m = 4.87\ \text{ft/s},\ t = 10.3\ \text{s}$
12–231. $v_{w/s} = 19.9\ \text{m/s},\ \theta = 74.0^\circ$
12–233. Yes, he can catch the ball.
12–234. $v_B = 5.75\ \text{m/s},\ v_{C/B} = 17.8\ \text{m/s},$
$\theta = 76.2^\circ,\ a_{C/B} = 9.81\ \text{m/s}^2\downarrow$
12–235. $v_{B/A} = 11.2\ \text{m/s},\ \theta = 50.3^\circ$

Chapter 13

13–1. $s = 97.4\ \text{ft}$
13–2. $T = 5.98\ \text{kip}$
13–3. $v = 3.36\ \text{m/s},\ s = 5.04\ \text{m}$
13–5. $F = 6.37\ \text{N}$
13–6. $v = 59.8\ \text{ft/s}$
13–7. $v = 60.7\ \text{ft/s}$
13–9. $t = 2.04\ \text{s}$
13–10. $s = 8.49\ \text{m}$
13–11. $t = 0.249\ \text{s}$
13–13. $a_A = 9.66\ \text{ft/s}^2 \leftarrow,\ a_B = 15.0\ \text{ft/s}^2 \rightarrow$
13–14. $T = 11.25\ \text{kN},\ F = 33.75\ \text{kN}$
13–15. $A_x = 685\ \text{N},\ A_y = 1.19\ \text{kN},\ M_A = 4.74\ \text{kN}\cdot\text{m}$
13–17. $a = \frac{1}{2}(1 - \mu_k)\,g$
13–18. $R = 5.30\ \text{ft},\ t_{AC} = 1.82\ \text{s}$
13–19. $R = 5.08\ \text{ft},\ t_{AC} = 1.48\ \text{s}$
13–21. $\theta = 22.6^\circ$
13–22. $v_B = 5.70\ \text{m/s} \uparrow$
13–23. $v = 3.62\ \text{m/s}\uparrow$
13–25. $R = 2.45\ \text{m},\ t_{AB} = 1.72\ \text{s}$
13–26. $R = \{150t\}\ \text{N}$
13–27. $t = 2.11\ \text{s}$
13–29. $v = 2.01\ \text{ft/s}$

13–30. $v = 0.301\text{ m/s}$
13–31. $T = 1.63\text{ kN}$
13–33. $P = 2mg\left(\dfrac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta}\right)$, $a = \left(\dfrac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta}\right)g$
13–34. $v = 2.19\text{ m/s}$
13–35. $t = 5.66\text{ s}$
13–37. $t = 0.519\text{ s}$
13–38. $s = 16.7\text{ m}$
13–39. $v = \dfrac{1}{m}\sqrt{1.09F_0^2t^2 + 2F_0tmv_0 + m^2v_0^2}$, $x = \dfrac{y}{0.3} + v_0\left(\sqrt{\dfrac{2m}{0.3F_0}}\right)y^{1/2}$
13–41. $x = d$, $v = \sqrt{\dfrac{kd^2}{m_A + m_B}}$
13–42. $x = d$ for separation.
13–43. $v = \sqrt{\dfrac{mg}{k}}\left[\dfrac{e^{2t\sqrt{mg/k}} - 1}{e^{2t\sqrt{mg/k}} + 1}\right]$, $v_t = \sqrt{\dfrac{mg}{k}}$
13–45. $v = 32.2\text{ ft/s}$
13–46. $P = 2mg\tan\theta$
13–47. $P = 2mg\left(\dfrac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta}\right)$
13–49. $a_B = 7.59\text{ ft/s}^2$
13–50. $v = 5.13\text{ m/s}$
13–51. $d = \dfrac{(m_A + m_B)g}{k}$
13–53. $r = 1.36\text{ m}$
13–54. $v = 10.5\text{ m/s}$
13–55. $N = 6.18\text{ kN}$
13–57. $v = 1.63\text{ m/s}$, $N = 7.36\text{ N}$
13–58. $v = 0.969\text{ m/s}$
13–59. $v = 1.48\text{ m/s}$
13–61. $v = 9.29\text{ ft/s}$, $T = 38.0\text{ lb}$
13–62. $v = 2.10\text{ m/s}$
13–63. $T = 0$, $T = 10.6\text{ lb}$
13–65. $v = 6.30\text{ m/s}$, $F_n = 283\text{ N}$, $F_t = 0$, $F_b = 490\text{ N}$
13–66. $v = 22.1\text{ m/s}$
13–67. $\theta = 26.7°$
13–69. $F_f = 1.11\text{ kN}$, $N = 6.73\text{ kN}$
13–70. $v_C = 19.9\text{ ft/s}$, $N_C = 7.91\text{ lb}$, $v_B = 21.0\text{ ft/s}$
13–71. $N = 277\text{ lb}$, $F = 13.4\text{ lb}$
13–73. $v = \sqrt{gr}$, $N = 2mg$
13–74. $v = 49.5\text{ m/s}$
13–75. $a_t = g\left(\dfrac{x}{\sqrt{1 + x^2}}\right)$, $v = \sqrt{v_0^2 + gx^2}$, $N = \dfrac{m}{\sqrt{1 + x^2}}\left[g - \dfrac{v_0^2 + gx^2}{1 + x^2}\right]$
13–77. $F_s = 4.90\text{ lb}$
13–78. $v = 40.1\text{ ft/s}$
13–79. $N_P = 2.65\text{ kN}$, $\rho = 68.3\text{ m}$
13–81. $\theta = 37.7°$
13–82. $N_B = 80.4\text{ N}$, $a_t = 1.92\text{ m/s}^2$
13–85. $F_A = 4.46\text{ lb}$
13–86. $F = 210\text{ N}$
13–87. $F = 1.60\text{ lb}$
13–89. $F_r = -29.4\text{ N}$, $F_\theta = 0$, $F_z = 392\text{ N}$
13–90. $F_r = 102\text{ N}$, $F_z = 375\text{ N}$, $F_\theta = 79.7\text{ N}$
13–91. $N = 4.90\text{ N}$, $F = 4.17\text{ N}$
13–93. $F_{OA} = 12.0\text{ lb}$
13–94. $F = 5.07\text{ kN}$, $N = 2.74\text{ kN}$
13–95. $F = 17.0\text{ N}$
13–97. $(N)_{\max} = 36.0\text{ N}$, $(N)_{\min} = 4.00\text{ N}$
13–98. $N_s = 3.72\text{ N}$, $F_r = 7.44\text{ N}$
13–99. $F_r = -900\text{ N}$, $F_\theta = -200\text{ N}$, $F_z = 1.96\text{ kN}$
13–101. $\theta = \tan^{-1}\left(\dfrac{4r_c\dot{\theta}_0^2}{g}\right)$
13–102. $N = 0.883\text{ N}$, $F = 3.92\text{ N}$
13–103. $N = 2.95\text{ N}$
13–105. $F_r = 1.78\text{ N}$, $N_s = 5.79\text{ N}$
13–106. $F_r = 2.93\text{ N}$, $N_s = 6.37\text{ N}$
13–107. $F = 0.163\text{ lb}$
13–109. $F_r = 25.6\text{ N}$, $F_{OA} = 0$
13–110. $F_r = 20.7\text{ N}$, $F_{OA} = 0$
13–111. $r = 0.198\text{ m}$
13–113. $v_o = 30.4\text{ km/s}$, $\dfrac{1}{r} = 0.348\,(10^{-12})\cos\theta + 6.74\,(10^{-12})$
13–114. $h = 35.9\text{ mm}$, $v_s = 3.07\text{ km/s}$
13–115. $v_0 = 7.45\text{ km/s}$
13–118. $v_B = 7.71\text{ km/s}$, $v_A = 4.63\text{ km/s}$
13–119. $v_A = 6.67(10^3)\text{ m/s}$, $v_B = 2.77(10^3)\text{ m/s}$
13–121. $v_A = 7.47\text{ km/s}$
13–122. $r_0 = 11.1\text{ Mm}$, $\Delta v_A = 814\text{ m/s}$
13–123. $(v_A)_C = 5.27(10^3)\text{ m/s}$, $\Delta v = 684\text{ m/s}$
13–125. (a) $r = 194\,(10^3)\text{ mi}$
(b) $r = 392\,(10^3)\text{ mi}$
(c) $194\,(10^3)\text{ mi} < r < 392\,(10^3)\text{ mi}$
(d) $r > 392\,(10^3)\text{ mi}$
13–126. $v_A = 4.89(10^3)\text{ m/s}$, $v_B = 3.26(10^3)\text{ m/s}$
13–127. $v_A = 11.5\text{ Mm/h}$, $d = 27.3\text{ Mm}$
13–129. $v_A = 2.01(10^3)\text{ m/s}$
13–130. $v_{A'} = 521\text{ m/s}$, $t = 21.8\text{ h}$
13–131. $v_A = 7.01(10^3)\text{ m/s}$

Chapter 14

14–1. $v = 10.7\text{ m/s}$
14–2. $x_{\max} = 3.24\text{ ft}$

14–3. $s = 1.35 \text{ m}$
14–5. $h = 39.3 \text{ m}, \rho = 26.2 \text{ m}$
14–6. $d = 12 \text{ m}$
14–7. Observer A: $v_2 = 6.08 \text{ m/s}$, Observer B: $v_2 = 4.08 \text{ m/s}$
14–9. $x_{\max} = 0.173 \text{ m}$
14–10. $s = 20.5 \text{ m}$
14–11. $v = 4.08 \text{ m/s}$
14–13. $v_B = 31.5 \text{ ft/s}, d = 22.6 \text{ ft}, v_C = 54.1 \text{ ft/s}$
14–14. $v_A = 7.18 \text{ ft/s}$
14–15. $v_A = 3.52 \text{ ft/s}$
14–17. $v_B = 27.8 \text{ ft/s}$
14–18. $y = 3.81 \text{ ft}$
14–19. $v_B = 3.34 \text{ m/s}$
14–21. $v_A = 0.771 \text{ ft/s}$
14–22. $s_{\text{Tot}} = 3.88 \text{ ft}$
14–23. $x = 0.688 \text{ m}$
14–25. $s = 0.0735 \text{ ft}$
14–26. $v_A = 28.3 \text{ m/s}$
14–27. $v_B = 18.0 \text{ m/s}, N_B = 12.5 \text{ kN}$
14–29. $s = 0.730 \text{ m}$
14–30. $s = 3.33 \text{ ft}$
14–31. $R = 2.83 \text{ m}, v_C = 7.67 \text{ m/s}$
14–33. $d = 36.2 \text{ ft}$
14–34. $s = 1.90 \text{ ft}$
14–35. $v_B = 42.2 \text{ ft/s}, N = 50.6 \text{ lb}, a_t = 26.2 \text{ ft/s}^2$
14–37. $h_A = 22.5 \text{ m}, h_C = 12.5 \text{ m}$
14–38. $v_B = 14.9 \text{ m/s}, N = 1.25 \text{ kN}$
14–39. $v_B = 5.42 \text{ m/s}$
14–41. $l_0 = 2.77 \text{ ft}$
14–42. $\theta = 47.2^\circ$
14–43. $P_i = 4.20 \text{ hp}$
14–45. $P = 8.32\,(10^3) \text{ hp}$
14–46. $t = 46.2 \text{ min}$
14–47. $P = 12.6 \text{ kW}$
14–49. $P_{\max} = 113 \text{ kW}, P_{\text{avg}} = 56.5 \text{ kW}$
14–50. $P_o = 4.36 \text{ hp}$
14–51. $P = 92.2 \text{ hp}$
14–53. $P_i = 483 \text{ kW}$
14–54. $P_i = 622 \text{ kW}$
14–55. $P_i = 22.2 \text{ kW}$
14–57. $P = 0.0364 \text{ hp}$
14–58. $P = 0.231 \text{ hp}$
14–59. $P = 12.6 \text{ kW}$
14–61. $P = \{400(10^3)t\} \text{ W}$
14–62. $P = \{160\,t - 533t^2\} \text{ kW}, U = 1.69 \text{ kJ}$
14–63. $P_{\max} = 10.7 \text{ kW}$
14–65. $P = 58.1 \text{ kW}$
14–66. $F = 227 \text{ N}$
14–67. $h = 133 \text{ in.}$
14–69. $N = 694 \text{ N}$
14–70. $\theta = 48.2^\circ$
14–71. $v_C = 17.7 \text{ ft/s}$
14–73. $N_B = 0, h = 18.75 \text{ m}, N_C = 17.2 \text{ kN}$
14–74. $v_A = 1.54 \text{ m/s}, v_B = 4.62 \text{ m/s}$
14–75. $s_B = 5.70 \text{ m}$
14–77. $h = 23.75 \text{ m}, v_C = 21.6 \text{ m/s}$
14–78. $v_B = 15.5 \text{ m/s}$
14–79. $l = 2.77 \text{ ft}$
14–81. $\theta = 118^\circ$
14–83. $F = GM_e m\left(\dfrac{1}{r_1} - \dfrac{1}{r_2}\right)$
14–85. $v_B = 34.8 \text{ Mm/h}$
14–86. $s = 130 \text{ m}$
14–87. $s_B = 0.638 \text{ m}, s_A = 1.02 \text{ m}$
14–89. $\theta = 22.3^\circ, s = 0.587 \text{ m}$
14–90. $N = 78.6 \text{ N}$
14–91. $y = 5.10 \text{ m}, N = 15.3 \text{ N}, a = 9.32 \text{ m/s}^2 \searrow$
14–93. $v = 1.68 \text{ m/s}$
14–94. $v_2 = \sqrt{\dfrac{2}{\pi}(\pi - 2)gr}$
14–95. $v = 6.97 \text{ m/s}$
14–97. $d = 1.34 \text{ m}$

Chapter 15

15–1. $v = 1.75 \text{ N}\cdot\text{s}$
15–2. $v = 29.4 \text{ ft/s}$
15–3. $F = 24.8 \text{ kN}$
15–5. $I = 5.68 \text{ N}\cdot\text{s}$
15–6. $F = 19.4 \text{ kN}, T = 12.5 \text{ kN}$
15–7. $F_{AB} = 16.7 \text{ lb}, v = 13.4 \text{ ft/s}$
15–9. $v = 6.62 \text{ m/s}$
15–10. $P = 205 \text{ N}$
15–11. $v = 60.0 \text{ m/s}$
15–13. $\mu_k = 0.340$
15–14. $I = 15 \text{ kN}\cdot\text{s}$ in both cases.
15–15. $v = 4.05 \text{ m/s}$
15–17. $v = 8.81 \text{ m/s}, s = 24.8 \text{ m}$
15–18. $v|_{t=3\text{ s}} = 5.68 \text{ m/s} \downarrow, v|_{t=6\text{ s}} = 21.1 \text{ m/s} \uparrow$
15–19. $v = 4.00 \text{ m/s}$
15–21. $T = 14.9 \text{ kN}, F = 24.8 \text{ kN}$
15–22. $v_{\max} = 108 \text{ m/s}, s = 1.83 \text{ km}$
15–23. $v = 10.1 \text{ ft/s}$
15–25. $v = 7.21 \text{ m/s} \uparrow$
15–26. Observer A: $v = 7.40 \text{ m/s}$, Observer B: $v = 5.40 \text{ m/s}$
15–27. $v = 5.07 \text{ m/s}$
15–29. $t = 1.02 \text{ s}, I = 162 \text{ N}\cdot\text{s}$
15–30. $v = 16.1 \text{ m/s}$
15–31. $(v_A)_2 = 10.5 \text{ ft/s} \rightarrow$
15–33. $v = 7.65 \text{ m/s}$
15–34. $v = 0.6 \text{ ft/s} \leftarrow$
15–35. $v = 18.6 \text{ m/s} \rightarrow$

15–37. $v = 5.21\ \text{m/s} \leftarrow,\ \Delta T = -32.6\ \text{kJ}$

15–38. $v = 0.5\ \text{m/s},\ \Delta T = -16.9\ \text{kJ}$

15–39. $v = 733\ \text{m/s}$

15–41. $v_B = 3.48\ \text{ft/s},\ d = 0.376\ \text{ft}$

15–42. $v_B = 3.48\ \text{ft/s},\ N_{\text{avg}} = 504\ \text{lb},\ t = 0.216\ \text{s}$

15–43. $s = 4.00\ \text{m}$

15–45. $v_2 = \sqrt{v_1^2 + 2gh},\ \theta_2 = \sin^{-1}\left(\dfrac{v_1 \sin\theta}{\sqrt{v_1^2 + 2gh}}\right)$

15–46. $\theta = \phi = 9.52^\circ$

15–47. $s_{\text{max}} = 481\ \text{mm}$

15–49. $x = 0.364\ \text{ft} \leftarrow$

15–50. $x = 1.58\ \text{ft} \rightarrow$

15–51. $s_B = 6.67\ \text{m} \rightarrow$

15–53. $s_B = 71.4\ \text{mm} \rightarrow$

15–54. $s_B = 71.4\ \text{mm} \rightarrow$

15–55. $v_c = 5.04\ \text{m/s} \leftarrow$

15–57. $d = 6.87\ \text{mm}$

15–59. $e = 0.75,\ \Delta T = -9.65\ \text{kJ}$

15–61. $x_{\text{max}} = 0.839\ \text{m}$

15–63. $v_C = 0.1875v \rightarrow,\ v_D = 0.5625v \rightarrow,$
$v_B = 0.8125v \rightarrow,\ v_A = 0.4375v \rightarrow$

15–65. $t = 0.226\ \text{s}$

15–66. $(v_B)_2 = \dfrac{1}{3}\sqrt{2gh}(1 + e)$

15–67. $(v_A)_2 = 1.04\ \text{ft/s},\ (v_B)_3 = 0.964\ \text{ft/s},$
$(v_C)_3 = 11.9\ \text{ft/s}$

15–69. $v'_B = 22.2\ \text{m/s},\ \theta = 13.0^\circ$

15–70. $(v_B)_2 = \dfrac{e(1 + e)}{2} v_0$

15–71. $v_A = 29.3\ \text{ft/s},\ v_{B2} = 33.1\ \text{ft/s},\ \theta = 27.7^\circ$

15–73. $v_A = 1.35\ \text{m/s} \rightarrow,\ v_B = 5.89\ \text{m/s},\ \theta = 32.9^\circ$

15–74. $e = 0.0113$

15–75. $h = 1.57\ \text{m}$

15–77. $(v_B)_3 = 3.24\ \text{m/s},\ \theta = 43.9^\circ$

15–78. $v'_B = 31.8\ \text{ft/s}$

15–79. $(v_A)_2 = 3.80\ \text{m/s} \leftarrow,$
$(v_B)_2 = 6.51\ \text{m/s},\ (\theta_B)_2 = 68.6^\circ$

15–81. (a) $(v_B)_1 = 8.81\ \text{m/s},\ \theta = 10.5^\circ$,
(b) $(v_B)_2 = 4.62\ \text{m/s},\ \phi = 20.3^\circ$,
(c) $s = 3.96\ \text{m}$

15–82. $s = 0.456\ \text{ft}$

15–83. $(v_A)_2 = 42.8\ \text{ft/s} \leftarrow,\ F = 2.49\ \text{kip}$

15–85. $\mu_k = 0.25$

15–86. $(v_B)_2 = 1.06\ \text{m/s} \leftarrow,\ (v_A)_2 = 0.968\ \text{m/s},$
$(\theta_A)_2 = 5.11^\circ$

15–87. $(v_A)_2 = 4.06\ \text{ft/s},\ (v_B)_2 = 6.24\ \text{ft/s}$

15–89. $(v_A)_2 = 12.1\ \text{m/s},\ (v_B)_2 = 12.4\ \text{m/s}$

15–90. $d = 1.15\ \text{ft},\ h = 0.770\ \text{ft}$

15–91. $(v_B)_3 = 1.50\ \text{m/s}$

15–93. $(v_A)_2 = 8.19\ \text{m/s},\ (v_B)_2 = 9.38\ \text{m/s}$

15–94. $\{-9.17\mathbf{i} - 6.12\mathbf{k}\}\ \text{slug}\cdot\text{ft}^2/\text{s}$

15–95. $\{-9.17\mathbf{i} + 4.08\mathbf{j} - 2.72\mathbf{k}\}\ \text{slug}\cdot\text{ft}^2/\text{s}$

15–97. $(H_A)_P = \{-52.8\mathbf{k}\}\ \text{kg}\cdot\text{m}^2/\text{s},$
$(H_B)_P = \{-118\mathbf{k}\}\ \text{kg}\cdot\text{m}^2/\text{s}$

15–98. $\{-21.5\mathbf{i} + 21.5\mathbf{j} + 37.6\}\ \text{kg}\cdot\text{m}^2/\text{s}$

15–99. $\{21.5\mathbf{i} + 21.5\mathbf{j} + 59.1\mathbf{k}\}\ \text{kg}\cdot\text{m}^2/\text{s}$

15–101. $v = 20.2\ \text{ft/s},\ h = 6.36\ \text{ft}$

15–102. $t = 11.9\ \text{s}$

15–103. $v_2 = 9.22\ \text{ft/s},\ \Sigma U_{1-2} = 3.04\ \text{ft}\cdot\text{lb}$

15–105. $v = 9.50\ \text{m/s}$

15–107. $v = 3.33\ \text{m/s}$

15–109. $v_C = 44.0\ \text{ft/s},\ H_A = 8.19\ \text{slug}\cdot\text{ft}^2/\text{s}.$
The cord will not unstretch.

15–110. $v_2 = 4.03\ \text{m/s},\ \Sigma U_{1-2} = 725\ \text{J}$

15–111. $v_B = 10.8\ \text{ft/s},\ U_{AB} = 11.3\ \text{ft}\cdot\text{lb}$

15–113. $v_B = 10.2\ \text{km/s},\ r_B = 13.8\ \text{Mm}$

15–114. $T = 40.1\ \text{kN}$

15–115. $C_x = 4.97\ \text{kN},\ D_x = 2.23\ \text{kN},\ D_y = 7.20\ \text{kN}$

15–117. $F = 303\ \text{lb}$

15–118. $F = 50.0\ \text{lb}$

15–119. $F_x = 9.87\ \text{lb},\ F_y = 4.93\ \text{lb}$

15–121. $F_x = 19.5\ \text{lb},\ F_y = 1.96\ \text{lb}$

15–122. $F = 20.0\ \text{lb}$

15–123. $F = 22.4\ \text{lb}$

15–125. $T = 82.8\ \text{N},\ N = 396\ \text{N}$

15–126. $F = 6.24\ \text{N},\ P = 3.12\ \text{N}$

15–127. $d = 2.56\ \text{ft}$

15–129. $C_x = 4.26\ \text{kN},\ C_y = 2.12\ \text{kN},\ M_C = 5.16\ \text{kN}\cdot\text{m}$

15–130. $v = \left\{\dfrac{8000}{2000 + 50t}\right\}\ \text{m/s}$

15–131. $A_y = 4.18\ \text{kN},\ B_x = 65.0\ \text{N} \rightarrow,$
$B_y = 3.72\ \text{kN} \uparrow$

15–133. $a = 0.125\ \text{m/s}^2,\ v = 4.05\ \text{m/s}$

15–134. $v_{\text{max}} = 2.07\ (10^3)\ \text{ft/s}$

15–135. 452 Pa

15–137. $R = \{20t + 2.48\}\ \text{lb}$

15–138. $a_i = 133\ \text{ft/s}^2,\ a_f = 200\ \text{ft/s}^2$

15–139. $v_{\text{max}} = 580\ \text{ft/s}$

15–141. $v = \sqrt{\dfrac{2}{3} g\left(\dfrac{y^3 - h^3}{y^2}\right)}$

15–142. $F_D = 11.5\ \text{kN}$

15–143. $a = 37.5\ \text{ft/s}^2$

15–145. $a = 0.0476\ \text{m/s}^2$

15–146. $v_{\text{max}} = 2.07(10^3)\ \text{ft/s}$

15–147. $F = \{7.85t + 0.320\}\ \text{N}$

15–149. $F = m' v^2$

Chapter 16

16–1. $v_A = 2.60\ \text{m/s},\ a_A = 9.35\ \text{m/s}^2$

16–2. $v_A = 22.0\ \text{m/s},$
$(a_A)_t = 12.0\ \text{m/s}^2,\ (a_A)_n = 968\ \text{m/s}^2$

16–3. $v_A = 26.0\ \text{m/s},$
$(a_A)_t = 10.0\ \text{m/s}^2,\ (a_A)_n = 1352\ \text{m/s}^2$

16–5. $\theta = 5443$ rev, $\omega = 740$ rad/s, $\alpha = 8$ rad/s^2
16–6. $\theta = 3.32$ rev, $t = 1.67$ s
16–7. $t = 6.98$ s, $\theta_D = 34.9$ rev
16–9. $a_B = 29.0$ m/s^2
16–10. $a_B = 16.5$ m/s^2
16–11. $\alpha = 60$ rad/s^2, $\omega = 90.0$ rad/s, $\theta = 90.0$ rad
16–13. $\omega_B = 180$ rad/s, $\omega_C = 360$ rad/s
16–14. $\omega = 42.7$ rad/s, $\theta = 42.7$ rad
16–15. $a_t = 2.83$ m/s^2, $a_n = 35.6$ m/s^2
16–17. $\omega_B = 21.9$ rad/s $\circlearrowleft$
16–18. $\omega_B = 31.7$ rad/s $\circlearrowleft$
16–19. $\omega_B = 156$ rad/s
16–21. $v_A = 8.10$ m/s,
$(a_A)_t = 4.95$ m/s^2, $(a_A)_n = 437$ m/s^2
16–22. $\omega_D = 4.00$ rad/s, $\alpha_D = 0.400$ rad/s^2
16–23. $\omega_D = 12.0$ rad/s, $\alpha_D = 0.600$ rad/s^2
16–25. $v_P = 2.42$ ft/s, $a_P = 34.4$ ft/s^2
16–26. $\omega_C = 1.68$ rad/s, $\theta_C = 1.68$ rad
16–27. $\omega = 148$ rad/s
16–29. $r_A = 31.8$ mm, $r_B = 31.8$ mm,
1.91 canisters per minute
16–30. $(\omega_B)_{\max} = 8.49$ rad/s, $(v_C)_{\max} = 0.6$ m/s
16–31. $s_W = 2.89$ m
16–33. $\omega_B = 312$ rad/s, $\alpha_B = 176$ rad/s^2
16–34. $v_E = 3$ m/s,
$(a_E)_t = 2.70$ m/s^2, $(a_E)_n = 600$ m/s^2
16–35. $\mathbf{v}_C = \{-4.8\mathbf{i} - 3.6\mathbf{j} - 1.2\mathbf{k}\}$ m/s,
$\mathbf{a}_C = \{38.4\mathbf{i} - 64.8\mathbf{j} + 40.8\mathbf{k}\}$ m/s^2
16–37. $v_C = 2.50$ m/s, $a_C = 13.1$ m/s^2
16–38. $v = 7.21$ ft/s, $a = 91.2$ ft/s^2
16–39. $\omega = \dfrac{rv_A}{y\sqrt{y^2 - r^2}}, \ \alpha = \dfrac{rv_A^2(2y^2 - r^2)}{y^2(y^2 - r^2)^{3/2}}$
16–41. $\omega = 8.70$ rad/s, $\alpha = -50.5$ rad/s^2
16–42. $\omega = -19.2$ rad/s, $\alpha = -183$ rad/s^2
16–43. $\omega_{AB} = 0$
16–45. $v = -\left(\dfrac{r_1^2\omega \sin 2\theta}{2\sqrt{r_1^2\cos^2\theta + r_2^2 + 2r_1r_2}} + r_1\omega\sin\theta\right)$
16–46. $v = \omega d\left(\sin\theta + \dfrac{d\sin 2\theta}{2\sqrt{(R + r)^2 - d^2\sin^2\theta}}\right)$
16–47. $v = -r\omega\sin\theta$
16–49. $v_C = L\omega\uparrow$, $a_C = 0.577\,L\omega^2\uparrow$
16–50. $\omega = \dfrac{2v_0}{r}\sin^2\theta/2, \ \alpha = \dfrac{2v_0^2}{r^2}(\sin\theta)(\sin^2\theta/2)$
16–51. $v_B = \left(\dfrac{h}{d}\right)v_A$
16–53. $\dot{\theta} = \dfrac{v\sin\phi}{L\cos(\phi - \theta)}$
16–54. $\omega = \dfrac{v}{2r}$
16–55. $\omega' = \dfrac{(R + r)\omega}{r}, \ \alpha' = \dfrac{(R + r)\alpha}{r}$
16–57. $v_B = 12.6$ in./s, $65.7°$
16–58. $\omega_{AB} = 2.00$ rad/s
16–59. $v_C = 1.06$ m/s $\leftarrow$, $\omega_{BC} = 0.707$ rad/s $\circlearrowright$
16–61. $\omega_{BC} = 2.31$ rad/s $\circlearrowleft$, $\omega_{AB} = 3.46$ rad/s $\circlearrowleft$
16–62. $\omega_A = 32.0$ rad/s
16–63. $\omega_{CB} = 2.45$ rad/s $\circlearrowleft$, $v_C = 2.20$ ft/s $\leftarrow$
16–65. $\omega = 20$ rad/s, $v_A = 2$ ft/s $\rightarrow$
16–66. $\omega = 3.11$ rad/s, $v_O = 0.667$ ft/s $\rightarrow$
16–67. $v_A = 5.16$ ft/s, $\theta = 39.8°$
16–69. $v_C = 24.6$ m/s $\downarrow$
16–70. $\omega_{BC} = 10.6$ rad/s $\circlearrowleft$, $v_C = 29.0$ m/s $\rightarrow$
16–71. $v_P = 4.88$ m/s $\leftarrow$
16–73. $v_E = 4.00$ m/s, $\theta = 52.7°$
16–74. $\omega_B = 90$ rad/s $\circlearrowright$, $\omega_A = 180$ rad/s $\circlearrowleft$
16–75. $\omega_{CD} = 4.03$ rad/s
16–77. $\omega_P = 5$ rad/s, $\omega_A = 1.67$ rad/s
16–78. $\omega_D = 105$ rad/s $\circlearrowright$
16–79. $v_D = 7.07$ m/s
16–82. $\omega_{AB} = 1.24$ rad/s
16–83. $\omega_{BC} = 6.79$ rad/s
16–85. $v_A = 2$ ft/s $\rightarrow$, $v_B = 10$ ft/s $\leftarrow$.
The cylinder slips.
16–86. $v_B = 14$ in./s $\downarrow$,
$v_A = 10.8$ in./s, $\theta = 21.8°$
16–87. $\omega_{BC} = 8.66$ rad/s $\circlearrowleft$, $\omega_{AB} = 4.00$ rad/s $\circlearrowright$
16–89. $v_A = \omega(r_2 - r_1)$
16–90. $v_C = 2.50$ ft/s $\leftarrow$,
$v_D = 9.43$ ft/s, $\theta = 55.8°$
16–91. $v_C = 2.50$ ft/s $\leftarrow$,
$v_E = 7.91$ ft/s, $\theta = 18.4°$
16–93. $\omega_{BPD} = 3.00$ rad/s $\circlearrowright$, $v_P = 1.79$ m/s $\leftarrow$
16–94. $\omega_B = 6.67$ rad/s
16–95. $v_A = 60.0$ ft/s $\rightarrow$, $v_C = 220$ ft/s $\leftarrow$,
$v_B = 161$ ft/s, $\theta = 60.3°$
16–97. $\omega_S = 57.5$ rad/s $\circlearrowleft$, $\omega_{OA} = 10.6$ rad/s $\circlearrowleft$
16–98. $\omega_S = 15.0$ rad/s, $\omega_R = 3.00$ rad/s
16–99. $\omega_{CD} = 57.7$ rad/s $\circlearrowleft$
16–101. $\omega_R = 4$ rad/s
16–102. $\omega_R = 4$ rad/s
16–103. $v_C = 3.86$ m/s $\leftarrow$, $a_C = 17.7$ m/s^2 $\leftarrow$
16–105. $\alpha = 0.0962$ rad/s^2 $\circlearrowright$, $a_A = 0.385$ ft/s^2 $\rightarrow$
16–106. $a_C = 13.0$ m/s^2 $\swarrow$, $\alpha_{BC} = 12.4$ rad/s^2 $\circlearrowright$
16–107. $\omega = 6.67$ rad/s $\circlearrowleft$, $v_B = 4.00$ m/s $\searrow$
$\alpha = 15.7$ rad/s^2 $\circlearrowright$, $a_B = 24.8$ m/s^2 $\nwarrow$
16–109. $\omega_{BC} = 0$, $\omega_{CD} = 4.00$ rad/s $\circlearrowright$,
$\alpha_{BC} = 6.16$ rad/s^2 $\circlearrowright$, $\alpha_{CD} = 21.9$ rad/s^2 $\circlearrowright$
16–110. $\omega_C = 20.0$ rad/s $\circlearrowleft$, $\alpha_C = 127$ rad/s $\circlearrowright$
16–111. $\alpha_{AB} = 4.62$ rad/s^2 $\circlearrowleft$,
$a_B = 13.3$ m/s^2, $\theta = 37.0°$
16–113. $v_A = 0.424$ m/s, $\theta_v = 45°$,
$a_A = 0.806$ m/s^2, $\theta_a = 7.13°$

16–114. $v_B = 0.6 \text{ m/s} \downarrow$,
$a_B = 1.84 \text{ m/s}^2$, $\theta = 60.6°$

16–115. $v_B = 4v \rightarrow$,
$v_A = 2\sqrt{2}v$, $\theta = 45°$,
$a_B = \dfrac{2v^2}{r}\downarrow$, $a_A = \dfrac{2v^2}{r}\rightarrow$

16–117. $a_C = 10.0 \text{ m/s}^2$, $\theta = 2.02°$

16–118. $\alpha = 40.0 \text{ rad/s}^2$, $a_A = 2.00 \text{ m/s}^2 \leftarrow$

16–119. $v_B = 1.58\omega a$, $a_B = 1.58\alpha a - 1.77\omega^2 a$

16–121. $\omega_{AC} = 0$, $\omega_F = 10.7 \text{ rad/s} \circlearrowright$,
$\alpha_{AC} = 28.7 \text{ rad/s}^2 \circlearrowright$

16–122. $\omega_{CD} = 7.79 \text{ rad/s} \circlearrowleft$, $\alpha_{CD} = 136 \text{ rad/s}^2 \circlearrowright$

16–123. $v_C = 1.56 \text{ m/s} \leftarrow$,
$a_C = 29.7 \text{ m/s}^2$, $\theta = 24.1°$

16–125. $\omega = 4.73 \text{ rad/s} \circlearrowleft$, $\alpha = 131 \text{ rad/s}^2 \circlearrowright$

16–126. $\omega_{AB} = 7.17 \text{ rad/s} \circlearrowright$, $\alpha_{AB} = 23.1 \text{ rad/s}^2 \circlearrowleft$

16–127. $\alpha_{AB} = 3.70 \text{ rad/s}^2 \circlearrowright$

16–129. $\mathbf{v}_B = \{0.6\mathbf{i} + 2.4\mathbf{j}\} \text{ m/s}$,
$\mathbf{a}_B = \{-14.2\mathbf{i} + 8.40\mathbf{j}\} \text{ m/s}^2$

16–130. $v_B = 1.30 \text{ ft/s}$, $a_B = 0.6204 \text{ ft/s}^2$

16–131. $\mathbf{v}_m = \{7.5\mathbf{i} - 5\mathbf{j}\} \text{ ft/s}$, $\mathbf{a}_m = \{5\mathbf{i} + 3.75\mathbf{j}\} \text{ ft/s}^2$

16–133. $\mathbf{v}_A = \{-17.2\mathbf{i} + 12.5\mathbf{j}\} \text{ m/s}$,
$\mathbf{a}_A = \{349\mathbf{i} + 597\mathbf{j}\} \text{ m/s}^2$

16–134. $\mathbf{a}_A = \{-5.60\mathbf{i} - 16\mathbf{j}\} \text{ m/s}^2$

16–135. $v_C = 2.40 \text{ m/s}$, $\theta = 60°$

16–137. $(\mathbf{v}_{B/A})_{xyz} = \{31.0\mathbf{i}\} \text{ m/s}$,
$(\mathbf{a}_{B/A})_{xyz} = \{-14.0\mathbf{i} - 206\mathbf{j}\} \text{ m/s}^2$

16–138. $v_B = 7.7 \text{ m/s}$, $a_B = 201 \text{ m/s}^2$

16–139. $\omega_{CB} = 1.33 \text{ rad/s} \circlearrowleft$, $\alpha_{CD} = 3.08 \text{ rad/s}^2 \circlearrowright$

16–141. $\omega_{CD} = 3.00 \text{ rad/s} \circlearrowright$, $\alpha_{CD} = 12.0 \text{ rad/s}^2 \circlearrowright$

16–142. $\mathbf{v}_C = \{-0.944\mathbf{i} + 2.02\mathbf{j}\} \text{ m/s}$,
$\mathbf{a}_C = \{-11.2\mathbf{i} - 4.15\mathbf{j}\} \text{ m/s}^2$

16–143. $\omega_{AB} = 5 \text{ rad/s} \circlearrowright$, $\alpha_{AB} = 2.5 \text{ rad/s}^2 \circlearrowright$

16–145. $\mathbf{v}_C = \{-7.00\mathbf{i} + 17.3\mathbf{j}\} \text{ ft/s}$,
$\mathbf{a}_C = \{-34.6\mathbf{i} - 15.5\mathbf{j}\} \text{ ft/s}^2$

16–146. $\mathbf{v}_C = \{-7.00\mathbf{i} + 17.3\mathbf{j}\} \text{ ft/s}$,
$\mathbf{a}_C = \{-38.8\mathbf{i} - 6.84\mathbf{j}\} \text{ ft/s}^2$

16–147. $\omega_{AB} = 0.667 \text{ rad/s} \circlearrowleft$, $\alpha_{AB} = 3.08 \text{ rad/s}^2 \circlearrowright$

16–149. $(\mathbf{v}_{\text{rel}})_{xyz} = \mathbf{0}$, $(\mathbf{a}_{\text{rel}})_{xyz} = \{1\mathbf{i}\} \text{ m/s}^2$

16–150. $\omega_{DC} = 2.96 \text{ rad/s} \circlearrowright$

16–151. $\omega_{AC} = 0$, $\alpha_{AC} = 14.4 \text{ rad/s}^2 \circlearrowright$

Chapter 17

17–1. $I_y = \dfrac{1}{3}ml^2$

17–2. $m = \pi h R^2\left(k + \dfrac{aR^2}{2}\right)$, $I_z = \dfrac{\pi h R^4}{2}\left[k + \dfrac{2aR^2}{3}\right]$

17–3. $I_z = mR^2$

17–5. $k_x = 1.20 \text{ in.}$

17–6. $I_x = \dfrac{2}{5}mr^2$

17–7. $I_x = \dfrac{93}{70}mb^2$

17–9. $I_y = \dfrac{m}{6}(a^2 + h^2)$

17–10. $k_O = 2.17 \text{ m}$

17–11. $I_O = 1.36 \text{ kg}\cdot\text{m}^2$

17–13. $I_A = 7.67 \text{ kg}\cdot\text{m}^2$

17–14. $I_A = 222 \text{ slug}\cdot\text{ft}^2$

17–15. $I_O = 6.23 \text{ kg}\cdot\text{m}^2$

17–17. $I_G = 0.230 \text{ kg}\cdot\text{m}^2$

17–18. $I_O = 0.560 \text{ kg}\cdot\text{m}^2$

17–19. $I_G = 118 \text{ slug}\cdot\text{ft}^2$

17–21. $\bar{y} = 1.78 \text{ m}$, $I_G = 4.45 \text{ kg}\cdot\text{m}^2$

17–22. $I_x = 3.25 \text{ g}\cdot\text{m}^2$

17–23. $I_{x'} = 7.19 \text{ g}\cdot\text{m}^2$

17–25. $F = 5.96 \text{ lb}$, $N_B = 99.0 \text{ lb}$, $N_A = 101 \text{ lb}$

17–26. $A_y = 72.6 \text{ kN}$, $B_y = 71.6 \text{ kN}$, $a_G = 0.250 \text{ m/s}^2$

17–27. $N_A = 1393 \text{ lb}$, $N_B = 857 \text{ lb}$, $t = 2.72 \text{ s}$

17–29. $a = 2.74 \text{ m/s}^2$, $T = 25.1 \text{ kN}$

17–30. $N = 29.6 \text{ kN}$, $V = 0$, $M = 51.2 \text{ kN}\cdot\text{m}$

17–31. $h = 3.12 \text{ ft}$

17–33. $P = 579 \text{ N}$

17–34. $a = 4 \text{ m/s}^2 \rightarrow$, $N_B = 1.14 \text{ kN}$, $N_A = 327 \text{ N}$

17–35. $a_G = 13.3 \text{ ft/s}^2$

17–37. $P = 785 \text{ N}$

17–38. $P = 314 \text{ N}$

17–39. $N = 0.433wx$, $V = 0.25wx$, $M = 0.125wx^2$

17–41. $B_x = 73.9 \text{ lb}$, $B_y = 69.7 \text{ lb}$, $N_A = 120 \text{ lb}$

17–42. $a = 2.01 \text{ m/s}^2$.
The crate slips.

17–43. $a = 2.68 \text{ ft/s}^2$, $N_A = 26.9 \text{ lb}$, $N_B = 123 \text{ lb}$

17–45. $T = 15.7 \text{ kN}$, $C_x = 8.92 \text{ kN}$, $C_y = 16.3 \text{ kN}$

17–46. $a = 9.81 \text{ m/s}^2$, $C_x = 12.3 \text{ kN}$, $C_y = 12.3 \text{ kN}$

17–47. $h_{\max} = 3.16 \text{ ft}$, $F_A = 248 \text{ lb}$, $N_A = 400 \text{ lb}$

17–49. $F_{AB} = 112 \text{ N}$, $C_x = 26.2 \text{ N}$, $C_y = 49.8 \text{ N}$

17–50. $P = 765 \text{ N}$

17–51. $T = 1.52 \text{ kN}$, $\theta = 18.6°$

17–53. $\alpha = 9.67 \text{ rad/s}^2$

17–54. $F_C = 16.1 \text{ lb}$, $N_C = 159 \text{ lb}$

17–55. $\alpha = 2.62 \text{ rad/s}^2$

17–57. $\omega = 56.2 \text{ rad/s}$, $A_x = 0$, $A_y = 98.1 \text{ N}$

17–58. $\alpha = 14.7 \text{ rad/s}^2$, $A_x = 88.3 \text{ N}$, $A_y = 147 \text{ N}$

17–59. $F_A = \dfrac{3}{2}mg$

17–61. $\alpha = 0.694 \text{ rad/s}^2$

17–62. $\omega = 10.9 \text{ rad/s}$

17–63. $\omega = 9.45 \text{ rad/s}$

17–65. $M = 0.233 \text{ lb}\cdot\text{ft}$

17–67. $\alpha = 8.68 \text{ rad/s}^2$, $A_n = 0$, $A_t = 106 \text{ N}$

17–69. $\alpha = 7.28 \text{ rad/s}^2$

17–70. $F = 22.1 \text{ N}$

17–71. $\omega = 0.474 \text{ rad/s}$

17–73. $t = 6.71$ s
17–74. $\alpha = 14.2$ rad/s^2
17–75. $A_x = 89.2$ N, $A_y = 66.9$ N, $t = 1.25$ s
17–77. $t = 1.09$ s
17–78. $v = 4.88$ ft/s
17–79. $a = 2.97$ m/s^2
17–81. $A_x = 0$, $A_y = 289$ N, $\alpha = 23.1$ rad/s^2
17–82. $N_A = 177$ kN, $V_A = 5.86$ kN, $M_A = 50.7$ kN·m
17–83. $M = 0.3gml$
17–85. $N = wx\left[\frac{\omega^2}{g}\left(L - \frac{x}{2}\right) + \cos\theta\right]$,
$V = wx\sin\theta$, $M = \frac{1}{2}wx^2\sin\theta$
17–86. $\alpha = 12.5$ rad/s $\circlearrowright$, $a_G = 18.75$ m/s$^2\downarrow$
17–87. $N_B = 2.89$ kN,
$A_x = 0$, $A_y = 2.89$ kN
17–89. $\omega = 800$ rad/s
17–91. $\alpha = 5.62$ rad/s^2, $T = 196$ N
17–93. $\alpha = 2.45$ rad/s^2 $\circlearrowright$, $N_B = 2.23$ N, $N_A = 33.3$ N
17–94. $\alpha = 4.32$ rad/s^2
17–95. $\theta = 46.9°$
17–97. $\alpha = 0.500$ rad/s^2
17–98. $\alpha = 15.6$ rad/s^2
17–99. $a_A = 26.7$ m/s$^2\rightarrow$
17–101. $F = 42.3$ N
17–102. $\alpha = 4.01$ rad/s^2
17–103. $A_y = 15.0$ lb, $A_x = 0.776$ lb, $\alpha = 1.67$ rad/s^2
17–105. $\alpha = 18.9$ rad/s^2, $P = 76.4$ lb
17–106. $\alpha = \frac{6P}{mL}$, $a_B = \frac{2P}{m}$
17–107. $\alpha = \frac{6(P - \mu_k mg)}{mL}$, $a_B = \frac{2(P - \mu_k mg)}{m}$
17–109. $\alpha = 3$ rad/s^2
17–110. $\alpha = 14.5$ rad/s^2, $t = 0.406$ s
17–111. The disk does not slip.
17–113. $a_G = \mu_k g \leftarrow$, $\alpha = \frac{2\mu_k g}{r}$ $\circlearrowright$
17–114. $\omega = \frac{1}{3}\omega_0$, $t = \frac{\omega_0 r}{3\mu_k g}$
17–115. $\alpha_A = 43.6$ rad/s^2 $\circlearrowright$, $\alpha_B = 43.6$ rad/s^2 $\circlearrowleft$, $T = 19.6$ N
17–117. $T_A = \frac{4}{7}W$
17–118. $\alpha = 23.4$ rad/s^2, $B_y = 9.62$ lb
17–119. $\alpha = \frac{10g}{13\sqrt{2}\,r}$

Chapter 18

18–2. $\omega = 14.0$ rad/s
18–3. $\omega = 14.1$ rad/s
18–5. $\omega = 2.02$ rad/s
18–6. $\omega = 1.78$ rad/s
18–7. $T = 283$ ft·lb
18–9. $\omega = 21.5$ rad/s
18–10. $s = 5.16$ m, $T = 78.5$ N
18–11. $\omega = 14.9$ rad/s
18–13. $\omega = 6.11$ rad/s
18–14. $\omega = 8.64$ rad/s
18–15. $\omega = 3.16$ rad/s
18–17. $\omega = \sqrt{\omega_0^2 + \frac{g}{r^2}s\sin\theta}$
18–18. $v_C = 7.49$ m/s
18–19. $\omega = 6.92$ rad/s
18–21. $s = 0.304$ ft
18–22. $v_C = 19.6$ ft/s
18–23. $\theta = 0.445$ rev
18–25. $s_G = 1.60$ m
18–26. $\omega_2 = 5.37$ rad/s
18–27. $\omega = 44.6$ rad/s
18–29. $v_G = 11.9$ ft/s
18–30. $\omega = 2.50$ rad/s
18–31. $\omega = 5.40$ rad/s
18–33. $\theta = 0.891$ rev, regardless of orientation
18–34. $\omega = 5.74$ rad/s
18–35. $\omega_{AB} = 5.92$ rad/s
18–37. $s_C = 78.0$ mm
18–38. $s = 0.301$ m, $T = 163$ N
18–39. $v_A = 1.29$ m/s
18–41. $s_b = 242$ mm, $T = 67.8$ N
18–42. $v_b = 2.52$ m/s
18–43. $\theta = 48.2°$
18–45. $\omega = 3.78$ rad/s
18–46. $\omega = 3.75$ rad/s
18–47. $\omega = 3.28$ rad/s
18–49. $(\omega_{AB})_2 = (\omega_{BC})_2 = 1.12$ rad/s
18–50. $v_A = 1.40$ m/s
18–51. $\theta_0 = 8.94$ rev
18–53. $\omega_{BC} = 1.34$ rad/s
18–54. $v_b = 15.5$ ft/s
18–55. $v_A = 4.00$ m/s
18–57. $\omega = 12.8$ rad/s
18–58. $k = 18.4$ N/m
18–59. $\omega = 2.67$ rad/s
18–61. $\omega_{AB} = 3.70$ rad/s
18–62. $\omega = 1.80$ rad/s
18–63. $v_A = 21.0$ ft/s
18–65. $\omega = 2.71$ rad/s
18–66. $k = 100$ lb/ft
18–67. $(v_A)_2 = 7.24$ m/s

Chapter 19

19–5. $\int M\,dt = 0.833\ \text{kg}\cdot\text{m}^2/\text{s}$
19–6. $\omega = 0.0178$ rad/s
19–7. $v_B = 24.1$ m/s
19–9. $\omega_2 = 103$ rad/s
19–10. $t = 0.6125$ s
19–11. $\omega_2 = 53.7$ rad/s
19–13. $y = \frac{2}{3}l$
19–14. $d = \frac{2}{3}l$
19–15. (a) $\omega_{BC} = 68.7$ rad/s, (b) $\omega_{BC} = 66.8$ rad/s, (c) $\omega_{BC} = 68.7$ rad/s
19–17. $v_G = 26.8$ ft/s
19–18. $v_G = 2$ m/s, $\omega = 3.90$ rad/s
19–19. $v_A = 24.1$ m/s
19–21. $\omega = 12.7$ rad/s
19–22. $\omega_A = 47.3$ rad/s
19–23. $t = 1.32$ s
19–25. $t = 1.04$ s
19–26. $\omega = 9$ rad/s
19–27. $v_B = 1.59$ m/s
19–29. $\omega = 1.91$ rad/s
19–30. $\omega_2 = 0.656$ rad/s, $\theta = 18.8°$
19–31. $\omega_2 = 0.577$ rad/s, $\theta = 15.8°$
19–33. $\omega_2 = 2.55$ rev/s
19–34. $\omega = 0.190$ rad/s
19–35. $\omega = 0.0906$ rad/s
19–37. $\omega = 22.7$ rad/s
19–38. $h_C = 0.500$ ft
19–39. $\omega_2 = 1.01$ rad/s
19–41. $\theta = 66.9°$
19–42. $\omega_2 = 57$ rad/s, $U_F = 367$ J
19–43. $\omega_2 = 3.47$ rad/s
19–45. $v = 5.96$ ft/s
19–46. $h = \frac{7}{5}r$
19–47. $\theta = 50.2°$
19–49. $(v_D)_3 = 1.54$ m/s, $\omega_3 = 0.934$ rad/s
19–50. $\omega_1 = 7.17$ rad/s
19–51. $\theta = \tan^{-1}\left(\sqrt{\frac{7}{5}e}\right)$
19–53. $\omega_3 = 2.73$ rad/s
19–54. $\omega = \sqrt{7.5\,\frac{g}{L}}$
19–55. $h_B = 0.980$ ft
19–57. $(v_G)_{y2} = e(v_G)_{y1}\uparrow$, $(v_G)_{x2} = \frac{5}{7}\left((v_G)_{x1} - \frac{2}{5}\omega_1 r\right)\leftarrow$
19–58. $\theta_1 = 39.8°$

Chapter 20

20–1. (a) $\boldsymbol{\alpha} = \omega_s\omega_t\,\mathbf{j}$, (b) $\boldsymbol{\alpha} = -\omega_s\omega_t\,\mathbf{k}$
20–2. $\mathbf{v}_A = \{-0.225\mathbf{i}\}$ m/s, $\mathbf{a}_A = \{-0.1125\mathbf{j} - 0.130\mathbf{k}\}$ m/s^2
20–3. $\mathbf{v}_A = \{-5.20\mathbf{i} - 12\mathbf{j} + 20.8\mathbf{k}\}$ ft/s, $\mathbf{a}_A = \{-24.1\mathbf{i} - 13.3\mathbf{j} - 7.20\mathbf{k}\}$ ft/s^2
20–5. $(\omega_C)_{DE} = 40$ rad/s, $(\omega_{DE})_y = 5$ rad/s
20–6. $\boldsymbol{\omega} = \{-8.24\mathbf{j}\}$ rad/s, $\boldsymbol{\alpha} = \{24.7\mathbf{i} - 5.49\mathbf{j}\}$ rad/s^2
20–7. $\mathbf{v}_A = \{-7.79\mathbf{i} - 2.25\mathbf{j} + 3.90\mathbf{k}\}$ ft/s, $\mathbf{a}_A = \{8.30\mathbf{i} - 35.2\mathbf{j} + 7.02\mathbf{k}\}$ ft/s^2
20–9. $\mathbf{v}_B = \{-0.4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}\}$ m/s, $\mathbf{a}_B = \{-8.20\mathbf{i} + 40.6\mathbf{j} - \mathbf{k}\}$ rad/s^2
20–10. $\boldsymbol{\omega} = \{42.4\mathbf{j} + 43.4\mathbf{k}\}$ rad/s, $\boldsymbol{\alpha} = \{-42.4\mathbf{i}\}$ rad/s^2
20–11. $\boldsymbol{\omega} = \{2\mathbf{i} + 42.4\mathbf{j} + 43.4\mathbf{k}\}$ rad/s, $\boldsymbol{\alpha} = \{-42.4\mathbf{i} - 82.9\mathbf{j} + 84.9\mathbf{k}\}$ rad/s^2
20–13. $v_B = 0$, $v_C = 0.283$ m/s, $a_B = 1.13$ m/s^2, $a_C = 1.60$ m/s^2
20–14. $\mathbf{v}_C = \{1.8\mathbf{j} - 1.5\mathbf{k}\}$ m/s, $\mathbf{a}_C = \{-36.6\mathbf{i} + 0.45\mathbf{j} - 0.9\mathbf{k}\}$ m/s^2
20–15. $\mathbf{v}_A = \{-8.66\mathbf{i} + 8.00\mathbf{j} - 13.9\mathbf{k}\}$ ft/s, $\mathbf{a}_A = \{-24.8\mathbf{i} + 8.29\mathbf{j} - 30.9\mathbf{k}\}$ ft/s^2
20–17. $\mathbf{v}_A = \{-1.80\mathbf{i}\}$ ft/s, $\mathbf{a}_A = \{-0.750\mathbf{i} - 0.720\mathbf{j} - 0.831\mathbf{k}\}$ ft/s^2
20–18. $\boldsymbol{\omega}_P = \{-40\mathbf{j}\}$ rad/s, $\boldsymbol{\alpha}_B = \{-6400\mathbf{i}\}$ rad/s^2
20–19. $\boldsymbol{\omega} = \{4.35\mathbf{i} + 12.7\mathbf{j}\}$ rad/s, $\boldsymbol{\alpha} = \{-26.1\mathbf{k}\}$ rad/s^2
20–21. $\boldsymbol{\omega} = \{30\mathbf{j} - 5\mathbf{k}\}$ rad/s, $\boldsymbol{\alpha} = \{150\mathbf{i}\}$ rad/s^2
20–22. $\mathbf{v}_A = \{10\mathbf{i} + 14.7\mathbf{j} - 19.6\mathbf{k}\}$ ft/s, $\mathbf{a}_A = \{-6.12\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}\}$ ft/s^2
20–23. $\omega_A = 47.8$ rad/s, $\omega_B = 7.78$ rad/s
20–25. $\boldsymbol{\omega}_{BC} = \{0.204\mathbf{i} - 0.612\mathbf{j} + 1.36\mathbf{k}\}$ rad/s, $\mathbf{v}_B = \{-0.333\mathbf{j}\}$ m/s
20–26. $\boldsymbol{\omega}_{AB} = \{-1.00\mathbf{i} - 0.500\mathbf{j} + 2.50\mathbf{k}\}$ rad/s, $\mathbf{v}_B = \{-2.50\mathbf{j} - 2.50\mathbf{k}\}$ m/s
20–27. $\boldsymbol{\alpha}_{AB} = \{-7.9\mathbf{i} - 3.95\mathbf{j} + 4.75\mathbf{k}\}$ rad/s^2, $\mathbf{a}_B = \{-19.75\mathbf{j} - 19.75\mathbf{k}\}$ m/s^2
20–29. $\mathbf{a}_B = \{-37.6\mathbf{j}\}$ ft/s^2
20–30. $\mathbf{v}_B = \{-1.92\mathbf{j} + 2.56\mathbf{k}\}$ m/s
20–31. $v_B = 5.00$ m/s, $\boldsymbol{\omega}_{AB} = \{-4.00\mathbf{i} - 0.600\mathbf{j} - 1.20\mathbf{k}\}$ rad/s
20–33. $\boldsymbol{\omega}_{BD} = \{-1.20\mathbf{j}\}$ rad/s

20–34. $\boldsymbol{\alpha}_{BD} = \{-8.00\mathbf{j}\}\ \text{rad/s}^2$
20–35. $\boldsymbol{\omega}_{AB} = \{-0.500\mathbf{i} + 0.667\mathbf{j} - 1.00\mathbf{k}\}\ \text{rad/s}$
$\mathbf{v}_B = \{-7.50\mathbf{j}\}\ \text{ft/s}$
20–37. $\mathbf{v}_C = \{-1.00\mathbf{i} + 5.00\mathbf{j} + 0.800\mathbf{k}\}\ \text{m/s}$,
$\mathbf{a}_C = \{-28.8\mathbf{i} - 5.45\mathbf{j} + 32.3\mathbf{k}\}\ \text{m/s}^2$
20–38. $\mathbf{v}_C = \{-1\mathbf{i} + 5\mathbf{j} + 0.8\mathbf{k}\}\ \text{m/s}$,
$\mathbf{a}_C = \{-28.2\mathbf{i} - 5.45\mathbf{j} + 32.3\mathbf{k}\}\ \text{m/s}^2$
20–39. $\mathbf{v}_B = \{-2.75\mathbf{i} - 2.50\mathbf{j} + 3.17\mathbf{k}\}\ \text{m/s}$,
$\mathbf{a}_B = \{2.50\mathbf{i} - 2.24\mathbf{j} - 0.00389\mathbf{k}\}\ \text{ft/s}^2$
20–41. $\mathbf{v}_C = \{3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}\}\ \text{m/s}$,
$\mathbf{a}_C = \{-13.0\mathbf{i} + 28.5\mathbf{j} - 10.2\mathbf{k}\}\ \text{m/s}^2$
20–42. $\mathbf{v}_B = \{-17.8\mathbf{i} - 3\mathbf{j} + 5.20\mathbf{k}\}\ \text{m/s}$,
$\mathbf{a}_B = \{9\mathbf{i} - 29.4\mathbf{j} - 1.5\mathbf{k}\}\ \text{m/s}^2$
20–43. $\mathbf{v}_B = \{-17.8\mathbf{i} - 3\mathbf{j} + 5.20\mathbf{k}\}\ \text{m/s}$,
$\mathbf{a}_B = \{3.05\mathbf{i} - 30.9\mathbf{j} + 1.10\mathbf{k}\}\ \text{m/s}^2$
20–45. $\mathbf{v}_P = \{-0.849\mathbf{i} + 0.849\mathbf{j} + 0.566\mathbf{k}\}\ \text{m/s}$,
$\mathbf{a}_P = \{-5.09\mathbf{i} - 7.35\mathbf{j} + 6.79\mathbf{k}\}\ \text{m/s}^2$
20–46. $\mathbf{v}_A = \{-8.66\mathbf{i} + 2.26\mathbf{j} + 2.26\mathbf{k}\}\ \text{m/s}$,
$\mathbf{a}_A = \{-22.6\mathbf{i} - 47.8\mathbf{j} + 45.3\mathbf{k}\}\ \text{m/s}^2$
20–47. $\mathbf{v}_A = \{-8.66\mathbf{i} + 2.26\mathbf{j} + 2.26\mathbf{k}\}\ \text{m/s}$,
$\mathbf{a}_A = \{-26.1\mathbf{i} - 44.4\mathbf{j} + 7.92\mathbf{k}\}\ \text{m/s}^2$
20–49. $\mathbf{v}_P = \{-9.80\mathbf{i} + 14.4\mathbf{j} + 48.0\mathbf{k}\}\ \text{ft/s}$,
$\mathbf{a}_P = \{-160\mathbf{i} + 5.16\mathbf{j} - 13\mathbf{k}\}\ \text{ft/s}^2$
20–50. $\mathbf{v}_P = \{-25.5\mathbf{i} - 13.4\mathbf{j} + 20.5\mathbf{k}\}\ \text{ft/s}$,
$\mathbf{a}_P = \{161\mathbf{i} - 249\mathbf{j} - 39.6\mathbf{k}\}\ \text{ft/s}^2$
20–51. $\mathbf{v}_P = \{-25.5\mathbf{i} - 13.4\mathbf{j} + 20.5\mathbf{k}\}\ \text{ft/s}$,
$\mathbf{a}_P = \{161\mathbf{i} - 243\mathbf{j} - 33.9\mathbf{k}\}\ \text{ft/s}^2$
20–53. $\mathbf{v}_A = \{-8.66\mathbf{i} + 8\mathbf{j} - 13.9\mathbf{k}\}\ \text{ft/s}$,
$\mathbf{a}_A = \{-17.9\mathbf{i} + 8.29\mathbf{j} - 30.9\mathbf{k}\}\ \text{ft/s}^2$,
20–54. $\mathbf{v}_C = \{-1.73\mathbf{i} - 5.77\mathbf{j} + 7.06\mathbf{k}\}\ \text{ft/s}$,
$\mathbf{a}_C = \{9.88\mathbf{i} - 72.8\mathbf{j} + 0.365\mathbf{k}\}\ \text{ft/s}^2$

Chapter 21

21–2. $I_{\bar{y}} = \dfrac{3m}{80}(h^2 + 4a^2),\ I_{y'} = \dfrac{m}{20}(2h^2 + 3a^2)$
21–3. $I_y = 2614\ \text{slug}\cdot\text{ft}^2$
21–5. $I_{yz} = \dfrac{m}{6}ah$
21–6. $I_{xy} = \dfrac{m}{12}a^2$
21–7. $I_{xy} = 636\rho$
21–9. $I_{z'z'} = 0.0961\ \text{slug}\cdot\text{ft}^2$
21–10. $k_y = 2.35\ \text{ft},\ k_x = 1.80\ \text{ft}$
21–11. $I_{aa} = \dfrac{m}{12}(3a^2 + 4h^2)$
21–13. $I_{yz} = 0$
21–14. $I_{xy} = 0.32\ \text{kg}\cdot\text{m}^2,\ I_{yz} = 0.08\ \text{kg}\cdot\text{m}^2,\ I_{xz} = 0$
21–15. $I_{z'} = 0.0595\ \text{kg}\cdot\text{m}^2$
21–17. $\bar{y} = 0.5\ \text{ft},\ \bar{x} = -0.667\ \text{ft},\ I_{x'} = 0.0272\ \text{slug}\cdot\text{ft}^2$,
$I_{y'} = 0.0155\ \text{slug}\cdot\text{ft}^2,\ I_{z'} = 0.0427\ \text{slug}\cdot\text{ft}^2$
21–18. $I_x = 0.455\ \text{slug}\cdot\text{ft}^2$
21–19. $I_{aa} = 1.13\ \text{slug}\cdot\text{ft}^2$
21–21. $I_z = 0.0880\ \text{slug}\cdot\text{ft}^2$
21–25. $\mathbf{H} = \{-477(10^{-6})\,\mathbf{i} + 198(10^{-6})\,\mathbf{j} + 0.169\,\mathbf{k}\}\ \text{kg}\cdot\text{m}^2/\text{s}$
21–26. $\omega_2 = 61.7\ \text{rad/s}$
21–27. $\omega_2 = 87.2\ \text{rad/s}$
21–29. $\omega_x = 19.7\ \text{rad/s}$
21–30. $h = 2.24\ \text{in.}$
21–31. $T = 0.0920\ \text{ft}\cdot\text{lb}$
21–33. $\omega_p = 4.82\ \text{rad/s}$
21–34. $\mathbf{H}_A = \{-2000\mathbf{i} - 55\,000\mathbf{j} + 22\,500\mathbf{k}\}\ \text{kg}\cdot\text{m}^2/\text{s}$
21–35. $T = 37.0\ \text{MJ}$
21–37. $\boldsymbol{\omega} = \{-0.750\mathbf{j} + 1.00\mathbf{k}\}\ \text{rad/s}$
21–38. $T = 1.14\ \text{J}$
21–39. $H_z = 0.4575\ \text{kg}\cdot\text{m}^2/\text{s}$
21–41. $\Sigma M_x = (I_x\dot{\omega}_x - I_{xy}\dot{\omega}_y - I_{xz}\dot{\omega}_z)$,
$-\ \Omega_z(I_y\omega_y - I_{yz}\omega_z - I_{yx}\omega_x)$,
$+\ \Omega_y(I_z\omega_z - I_{zx}\omega_x - I_{zy}\omega_y)$
Similarly for ΣM_y and ΣM_z.
21–43. $B_z = 4\ \text{lb},\ A_x = -2.00\ \text{lb},\ A_y = 0.627\ \text{lb}$,
$B_x = 2.00\ \text{lb},\ B_y = -1.37\ \text{lb}$
21–45. $A_Z = 1.46\ \text{lb},\ B_Z = 13.5\ \text{lb},\ A_X = A_Y = B_X = 0$,
21–46. $\dot{\omega}_x = -14.7\ \text{rad/s}^2,\ B_z = 77.7\ \text{N},\ B_y = 3.33\ \text{N}$,
$A_x = 0,\ A_y = 6.67\ \text{N},\ A_z = 81.75\ \text{N}$
21–47. $\dot{\omega}_x = 9.285\ \text{rad/s}^2,\ B_z = 97.7\ \text{N},\ B_y = 3.33\ \text{N}$,
$A_x = 0,\ A_y = 6.67\ \text{N},\ A_z = 122\ \text{N}$
21–49. $\dot{\omega}_z = 200\ \text{rad/s}^2,\ D_y = -12.9\ \text{N},\ D_x = -37.5\ \text{N}$,
$C_x = -37.5\ \text{N},\ C_y = -11.1\ \text{N},\ C_z = 36.8\ \text{N}$
21–50. $T_B = 47.1\ \text{lb},\ M_y = 0,\ M_z = 0,\ A_x = 0$,
$A_y = -93.2\ \text{lb},\ A_z = 57.1\ \text{lb}$
21–51. $\dot{\omega}_y = -102\ \text{rad/s}^2,\ A_x = B_x = 0,\ A_y = 0$,
$A_z = 297\ \text{N},\ B_z = -143\ \text{N}$
21–53. $M_z = 0,\ A_x = 0,\ M_y = 0,\ \theta = 64.1°$,
$A_y = 1.30\ \text{lb},\ A_z = 20.2\ \text{lb}$
21–54. $N = 148\ \text{N},\ F_f = 0$
21–55. $(M_0)_x = 72.0\ \text{N}\cdot\text{m},\ (M_0)_z = 0$
21–57. $M_x = -\dfrac{4}{3}ml^2\,\omega_s\omega_p\cos\theta$,
$M_y = \dfrac{1}{3}ml^2\,\omega_p{}^2\sin 2\theta,\ M_z = 0$
21–58. $B_x = 0,\ B_y = -3.90\ \text{lb},\ A_y = -1.69\ \text{lb}$,
$A_z = B_z = 7.5\ \text{lb}$
21–59. $\Sigma M_x = 0,\ \Sigma M_y = (-0.036\sin\theta)\ \text{N}\cdot\text{m}$,
$\Sigma M_z = (0.003\sin 2\theta)\ \text{N}\cdot\text{m}$
21–61. $\alpha = 69.3°,\ \beta = 128°,\ \gamma = 45°$. No, the orientation will not be the same for any order. Finite rotations are not vectors.
21–62. $\omega_P = 27.9\ \text{rad/s}$
21–63. $\omega_R = 368\ \text{rad/s}$
21–65. $\omega_P = 1.19\ \text{rad/s}$
21–66. $M_x = 328\ \text{N}\cdot\text{m}$

21–67. $\dot{\phi} = \left(\dfrac{2g\cos\theta}{a + r\cos\theta}\right)^{1/2}$

21–69. $\omega_s = 3.63(10^3)$ rad/s

21–70. $\theta = 68.1°$

21–71. $\dot{\phi} = 81.7$ rad/s, $\dot{\psi} = 212$ rad/s, regular precession

21–74. $\dot{\psi} = 2.35$ rev/h

21–75. $\alpha = 90°$, $\beta = 9.12°$, $\gamma = 80.9°$

21–77. $H_G = 12.5\ \text{Mg}\cdot\text{m}^2/\text{s}$

21–78. $\dot{\phi} = 3.32$ rad/s

Chapter 22

22–1. $\ddot{y} + 56.1\,y = 0$, $y|_{t=0.22\text{ s}} = 0.192$ m

22–2. $x = -0.05\cos(20t)$

22–3. $y = 0.107\sin(7.00t) + 0.100\cos(7.00t)$, $\phi = 43.0°$

22–5. $\omega_n = 49.5$ rad/s, $\tau = 0.127$ s

22–6. $x = \{-0.126\sin(3.16t) - 0.09\cos(3.16t)\}$ m, $C = 0.155$ m

22–7. $\omega_n = 19.7$ rad/s, $C = 1$ in. $y = (0.0833\cos 19.7t)$ ft

22–9. $\omega_n = 8.16$ rad/s, $x = -0.05\cos(8.16t)$, $C = 50$ mm

22–10. $\tau = 2\pi\sqrt{\dfrac{2mL}{3mg + 6kL}}$

22–11. $\tau = 1.45$ s

22–13. $\tau = 2\pi\sqrt{\dfrac{k_G^2 + d^2}{gd}}$

22–14. $k = 90.8\ \text{lb}\cdot\text{ft/rad}$

22–15. $k = 1.36$ N/m, $m_B = 3.58$ kg

22–17. $k_1 = 2067$ N/m, $k_2 = 302$ N/m, or vice versa

22–18. $m_B = 21.2$ kg, $k = 609$ N/m

22–19. $y = 503$ mm

22–21. $x = 0.167\cos 6.55\,t$

22–22. $\omega_n = \sqrt{\dfrac{3g(4R^2 - l^2)^{1/2}}{6R^2 - l^2}}$

22–23. $\tau = 1.66$ s

22–25. $f = 0.900$ Hz

22–26. $\tau = 2\pi k_O\sqrt{\dfrac{m}{C}}$

22–27. $\omega_n = 3.45$ rad/s

22–29. $\tau = 2\pi\sqrt{\dfrac{l}{2g}}$

22–30. $\ddot{x} + 333x = 0$

22–31. $\tau = 1.52$ s

22–33. $\tau = 0.774$ s

22–34. $\ddot{\theta} + 468\theta = 0$

22–35. $\tau = 0.487$ s

22–37. $E = 0.175\dot{\theta}^2 + 10\,\theta^2$, $\tau = 0.830$ s

22–38. $\tau = \pi\sqrt{\dfrac{m}{k}}$

22–39. $f = 1.28$ Hz

22–41. $x = A\sin\omega_n t + B\cos\omega_n t + \ + \dfrac{F_0/k}{1 - (\omega/\omega_n)^2}\cos\omega t$

22–42. $y = A\sin\omega_n t + B\cos\omega_n t + \dfrac{F_O}{k}$

22–43. $y = \{-0.0232\sin 8.97\,t + 0.333\cos 8.97\,t + 0.0520\sin 4t\}$ ft

22–45. $y = A\sin\omega_n t + B\cos\omega_n t + \dfrac{\delta_0}{1 - (\omega/\omega_n)^2}\sin\omega t$

22–46. $y = \big(361\sin 7.75t + 100\cos 7.75t, - 350\sin 8t\big)$ mm

22–47. $C = \dfrac{3F_O}{\frac{3}{2}(mg + Lk) - mL\omega^2}$

22–49. $(x_p)_{\max} = 29.5$ mm

22–50. $\ddot{\theta} + \dfrac{4c}{m}\dot{\theta} + \dfrac{k}{m}\dot{\theta} = 0$

22–51. $(v_p)_{\max} = 0.3125$ m/s

22–53. $\omega = 14.0$ rad/s

22–54. $(x_p)_{\max} = 14.6$ mm

22–55. $(x_p)_{\max} = 35.5$ mm

22–57. $\omega = 19.7$ rad/s

22–58. $C = 0.490$ in.

22–59. $\omega = 19.0$ rad/s

22–61. $(x_p)_{\max} = 4.53$ mm

22–62. $Y = \dfrac{mr\omega^2L^3}{48EI - M\omega^2L^3}$

22–63. $\omega = 12.2$ rad/s, $\omega = 7.07$ rad/s

22–65. $\phi' = 9.89°$

22–66. MF $= 0.997$

22–67. $y = \{-0.0702\,e^{-3.57t}\sin(8.540)\}$ m

22–69. $F = 2c\dot{y}$, $c_c = 2m\sqrt{\dfrac{k}{m}}$, $c < \sqrt{mk}$

22–71. $\omega = 21.1$ rad/s

22–73. $1.55\ddot{\theta} + 540\dot{\theta} + 200\theta = 0$, $(c_{dp})_c = 3.92\ \text{lb}\cdot\text{s/ft}$

22–74. $c_c = \sqrt{8(m + M)k}$, $x_{\max} = \left[\dfrac{m}{e}\sqrt{\dfrac{1}{2k(m + M)}}\right]v_0$

22–75. $x_{\max} = \dfrac{2mv_0}{\sqrt{8k(m + M) - c^2}}\,e^{-\pi c/(2\sqrt{8k(m+M)-c^2})}$

22–77. $Lq + Rq + \left(\dfrac{1}{C}\right)q = E_0\cos\omega t$

22–78. $L\ddot{q} + R\dot{q} + \left(\dfrac{2}{C}\right)q = 0$

22–79. $L\ddot{q} + R\dot{q} + \dfrac{1}{C}q = 0$

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Fundamental Equations of Dynamics

KINEMATICS

Particle Rectilinear Motion

Variable a	*Constant $a = a_c$*
$a = \dfrac{dv}{dt}$	$v = v_0 + a_c t$
$v = \dfrac{ds}{dt}$	$s = s_0 + v_0 t + \frac{1}{2} a_c t^2$
$a\,ds = v\,dv$	$v^2 = v_0^2 + 2a_c(s - s_0)$

Particle Curvilinear Motion

x, y, z Coordinates		*r, θ, z Coordinates*	
$v_x = \dot{x}$	$a_x = \ddot{x}$	$v_r = \dot{r}$	$a_r = \ddot{r} - r\dot{\theta}^2$
$v_y = \dot{y}$	$a_y = \ddot{y}$	$v_\theta = r\dot{\theta}$	$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$
$v_z = \dot{z}$	$a_z = \ddot{z}$	$v_z = \dot{z}$	$a_z = \ddot{z}$

n, t, b Coordinates

$v = \dot{s}$	$a_t = \dot{v} = v\dfrac{dv}{ds}$
	$a_n = \dfrac{v^2}{\rho} \quad \rho = \dfrac{[1 + (dy/dx)^2]^{3/2}}{\lvert d^2y/dx^2 \rvert}$

Relative Motion

$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$ $\quad$ $\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$

Rigid Body Motion About a Fixed Axis

Variable α	*Constant $\alpha = \alpha_c$*
$\alpha = \dfrac{d\omega}{dt}$	$\omega = \omega_0 + \alpha_c t$
$\omega = \dfrac{d\theta}{dt}$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha_c t^2$
$\omega\,d\omega = \alpha\,d\theta$	$\omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$

For Point P

$s = \theta r \quad v = \omega r \quad a_t = \alpha r \quad a_n = \omega^2 r$

Relative General Plane Motion—Translating Axes

$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A(\text{pin})} \quad \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A(\text{pin})}$

Relative General Plane Motion—Trans. and Rot. Axis

$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\Omega} \times \mathbf{r}_{B/A} + (\mathbf{v}_{B/A})_{xyz}$

$$\mathbf{a}_B = \mathbf{a}_A + \dot{\boldsymbol{\Omega}} \times \mathbf{r}_{B/A} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}_{B/A}) + 2\boldsymbol{\Omega} \times (\mathbf{v}_{B/A})_{xyz} + (\mathbf{a}_{B/A})_{xyz}$$

KINETICS

Mass Moment of Inertia $\quad I = \int r^2\,dm$

Parallel-Axis Theorem $\quad I = I_G + md^2$

Radius of Gyration $\quad k = \sqrt{\dfrac{I}{m}}$

Equations of Motion

Particle	$\Sigma \mathbf{F} = m\mathbf{a}$
Rigid Body (Plane Motion)	$\Sigma F_x = m(a_G)_x$ $\Sigma F_y = m(a_G)_y$ $\Sigma M_G = I_G\alpha$ or $\Sigma M_P = \Sigma(\mathcal{M}_k)_P$

Principle of Work and Energy

$T_1 + \Sigma U_{1-2} = T_2$

Kinetic Energy

Particle	$T = \frac{1}{2}mv^2$
Rigid Body (Plane Motion)	$T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$

Work

Variable force	$U_F = \int F\cos\theta\,ds$
Constant force	$U_F = (F_c\cos\theta)\,\Delta s$
Weight	$U_W = -W\,\Delta y$
Spring	$U_s = -\left(\frac{1}{2}ks_2^2 - \frac{1}{2}ks_1^2\right)$
Couple moment	$U_M = M\Delta\theta$

Power and Efficiency

$P = \dfrac{dU}{dt} = \mathbf{F}\cdot\mathbf{v} \qquad \varepsilon = \dfrac{P_{\text{out}}}{P_{\text{in}}} = \dfrac{U_{\text{out}}}{U_{\text{in}}}$

Conservation of Energy Theorem

$T_1 + V_1 = T_2 + V_2$

Potential Energy

$V = V_g + V_e$, where $V_g = \pm Wy$, $V_e = +\frac{1}{2}ks^2$

Principle of Linear Impulse and Momentum

Particle	$m\mathbf{v}_1 + \Sigma\int \mathbf{F}\,dt = m\mathbf{v}_2$
Rigid Body	$m(\mathbf{v}_G)_1 + \Sigma\int \mathbf{F}\,dt = m(\mathbf{v}_G)_2$

Conservation of Linear Momentum

$\Sigma(\text{syst. } m\mathbf{v})_1 = \Sigma(\text{syst. } m\mathbf{v})_2$

Coefficient of Restitution

$e = \dfrac{(v_B)_2 - (v_A)_2}{(v_A)_1 - (v_B)_1}$

Principle of Angular Impulse and Momentum

Particle	$(\mathbf{H}_O)_1 + \Sigma\int \mathbf{M}_O\,dt = (\mathbf{H}_O)_2$ where $H_O = (d)(mv)$
Rigid Body (Plane motion)	$(\mathbf{H}_G)_1 + \Sigma\int \mathbf{M}_G\,dt = (\mathbf{H}_G)_2$ where $H_G = I_G\omega$ $(\mathbf{H}_O)_1 + \Sigma\int \mathbf{M}_O\,dt = (\mathbf{H}_O)_2$ where $H_O = I_O\omega$

Conservation of Angular Momentum

$\Sigma(\text{syst. } \mathbf{H})_1 = \Sigma(\text{syst. } \mathbf{H})_2$

SI Prefixes

Multiple	*Exponential Form*	*Prefix*	*SI Symbol*
1 000 000 000	10^9	giga	G
1 000 000	10^6	mega	M
1 000	10^3	kilo	k
Submultiple			
0.001	10^{-3}	milli	m
0.000 001	10^{-6}	micro	μ
0.000 000 001	10^{-9}	nano	n

Conversion Factors (FPS) to (SI)

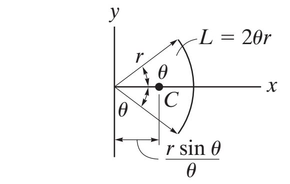
Quantity	*Unit of Measurement (FPS)*	*Equals*	*Unit of Measurement (SI)*
Force	lb		4.448 N
Mass	slug		14.59 kg
Length	ft		0.3048 m

Conversion Factors (FPS)

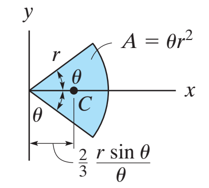
1 ft = 12 in. (inches)
1 mi. (mile) = 5280 ft
1 kip (kilopound) = 1000 lb
1 ton = 2000 lb

Geometric Properties of Line and Area Elements

Centroid Location	Centroid Location	Area Moment of Inertia



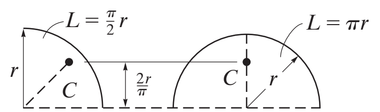
Circular arc segment



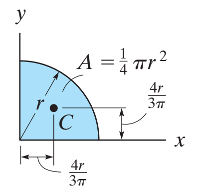
Circular sector area

$$I_x = \frac{1}{4} r^4 \left(\theta - \frac{1}{2} \sin 2\theta\right)$$

$$I_y = \frac{1}{4} r^4 \left(\theta + \frac{1}{2} \sin 2\theta\right)$$



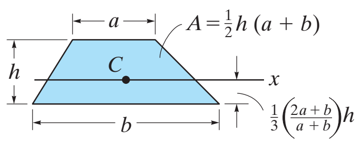
Quarter and semicircle arcs



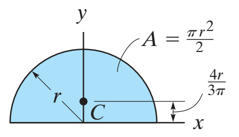
Quarter circle area

$$I_x = \frac{1}{16}\pi r^4$$

$$I_y = \frac{1}{16}\pi r^4$$



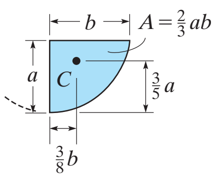
Trapezoidal area



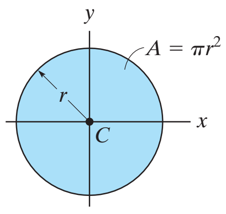
Semicircular area

$$I_x = \frac{1}{8}\pi r^4$$

$$I_y = \frac{1}{8}\pi r^4$$



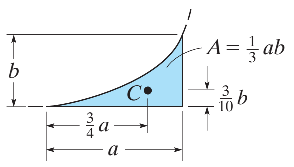
Semiparabolic area



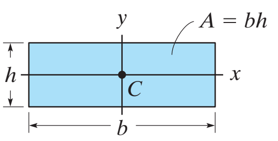
Circular area

$$I_x = \frac{1}{4}\pi r^4$$

$$I_y = \frac{1}{4}\pi r^4$$



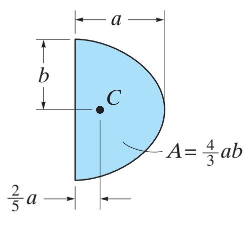
Exparabolic area



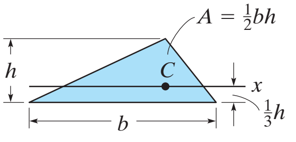
Rectangular area

$$I_x = \frac{1}{12}bh^3$$

$$I_y = \frac{1}{12}hb^3$$



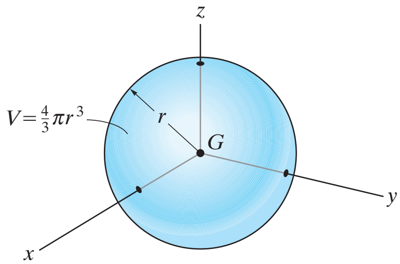
Parabolic area



Triangular area

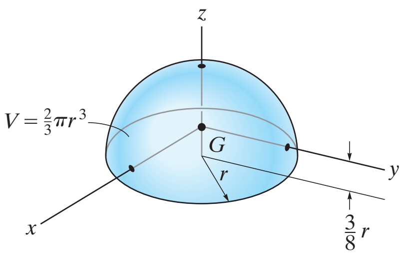
$$I_x = \frac{1}{36}bh^3$$

Center of Gravity and Mass Moment of Inertia of Homogeneous Solids



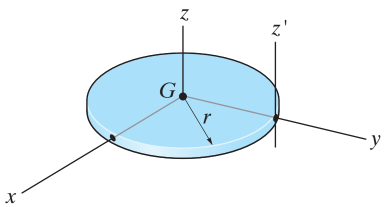
Sphere

$$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5} mr^2$$



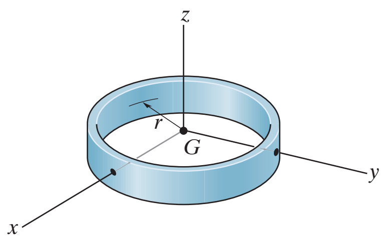
Hemisphere

$$I_{xx} = I_{yy} = 0.259\, mr^2 \quad I_{zz} = \frac{2}{5} mr^2$$



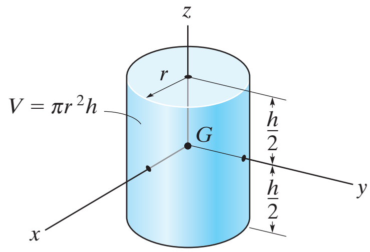
Thin Circular disk

$$I_{xx} = I_{yy} = \frac{1}{4} mr^2 \quad I_{zz} = \frac{1}{2} mr^2 \quad I_{z'z'} = \frac{3}{2} mr^2$$



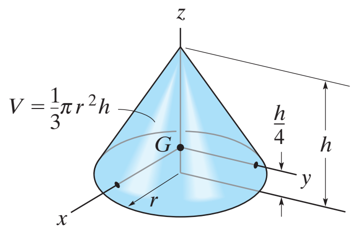
Thin ring

$$I_{xx} = I_{yy} = \frac{1}{2} mr^2 \quad I_{zz} = mr^2$$



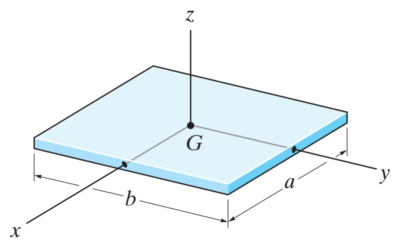
Cylinder

$$I_{xx} = I_{yy} = \frac{1}{12} m(3r^2 + h^2) \quad I_{zz} = \frac{1}{2} mr^2$$



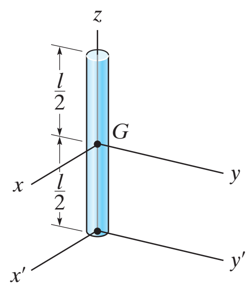
Cone

$$I_{xx} = I_{yy} = \frac{3}{80} m(4r^2 + h^2) \quad I_{zz} = \frac{3}{10} mr^2$$



Thin plate

$$I_{xx} = \frac{1}{12} mb^2 \quad I_{yy} = \frac{1}{12} ma^2 \quad I_{zz} = \frac{1}{12} m(a^2 + b^2)$$



Slender Rod

$$I_{xx} = I_{yy} = \frac{1}{12} ml^2 \quad I_{x'x'} = I_{y'y'} = \frac{1}{3} ml^2 \quad I_{z'z'} = 0$$