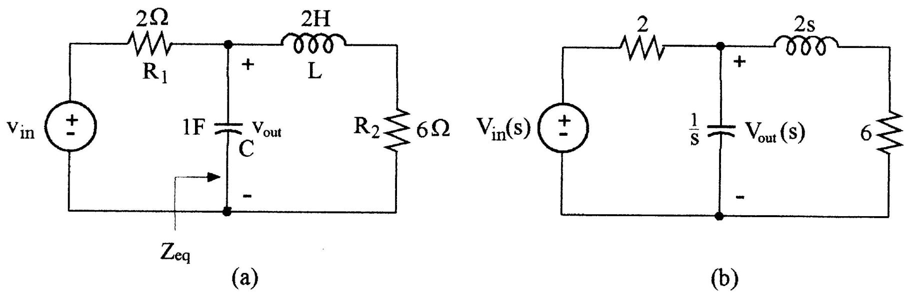


Figure 3.8 A circuit (a) and its Laplace transform model (b).

and the voltage

$$V_{out}(s) = V_{in}(s)\frac{Z_{eq}}{R_1 + Z_{eq}} = \frac{0.5s + 1.5}{s^2 + 3.5s + 2}.$$

Therefore, the transfer function is

$$H(s) = \frac{V_{out}}{V_{in}} = \frac{0.5s + 1.5}{s^2 + 3.5s + 2}. \tag{3.57}$$

3.6.1 Duhamel's integral

Suppose that the circuit response to a voltage unit-step function, $u(t) \leftrightarrow (1/s)$, is known and assigned as $g(t)$. Then, the Laplace transform of the input current might be found as

$$I_{in}(s) = \frac{V_{in}(s)}{Z_{in}(s)} = sV_{in}(s)\frac{1}{sZ_{in}(s)},$$

where

$$\frac{1}{sZ_{in}(s)} = G(s) \leftrightarrow g(t),$$

and

$$V_{in}(s) \leftrightarrow v_{in}(t).$$

Therefore, the Laplace transform of the current can be written as

$$sV(s)G(S).$$

Applying now the convolution theorem (equation 3.52a) and the differentiation properties (for the case when the initial values are zero) (equation 3.17a) we can obtain

$$[sV_{in}(s)][G(s)] \leftrightarrow v'_{in}(t) * g(t)$$

or

$$i_{in}(t) = \int_{0_-}^{t} v'_{in}(\tau) g(t-\tau) d\tau.$$

This integral is known as Duhamel's integral (one of its forms) or superposition integral, since the total response is obtained as superimposed responses to varying voltages delayed by $\Delta\tau$ (note that integration actually means summation).

When the initial values are none zero, we should subtract the initial voltage $v_{in}(0_-)$, in accordance with equation 3.17, from the first factor: $[sV_{in}(s)]$, which results in

$$I_{in}(s) = [sV_{in}(s) - v_{in}(0_-)][G(s)] + v_{in}(0_-)G(s).$$

This means that in the time domain the current is

$$i_{in}(t) = v_{in}(0_-)g(t) + \int_{0_-}^{t} v'_{in}(\tau) g(t-\tau) d\tau.$$

The other forms of Duhamel's integral in general notation are given in Table 3.2.

As a simple example of using Duhamel's integral, let us find the current for $t > T$ in the series RC circuit if the voltage forcing function is a triangular pulse, as shown in Fig. 3.7(b). The Laplace transform of the reaction to a unit-step function is

$$G(s) = \frac{1}{s\left(R + \frac{1}{sC}\right)} = \frac{1}{R\left(s + \frac{1}{RC}\right)},$$

which in time domain gives

$$g(t) = \frac{1}{R} e^{-\frac{t}{RC}}.$$

Using the first form of Duhamel's integral yields:

$$\begin{aligned} i(t) &= \int_{0_-}^{t} v'_{in}(\tau) g(t-\tau) d\tau \\ &= \int_{0}^{T} \frac{1}{TR} e^{-\frac{t-\tau}{RC}} d\tau = \frac{C}{T}\left(e^{\frac{T}{RC}} - 1\right) e^{-\frac{t}{RC}}. \end{aligned}$$

(The reader may be convinced that the method using Duhamel's integral is much simpler than the straightforward solution.)

3.7 INVERSE TRANSFORM AND PARTIAL FRACTION EXPANSIONS

The analysis of a circuit by Laplace transforms yields the transform expression (like equation 3.57, for example) of the desired variable. The next step, therefore, is to go from the Laplace transform back to the time function, i.e. from the frequency domain to the time domain.

This section will represent methods more useful in engineering for finding $f(t)$ when $F(s)$ is known, avoiding the complex integration of equation 3.4. These methods convert $F(s)$ into a sum of terms, each of which can be found in Table 3.1 (or in more complete tables of Laplace transforms, in suitable handbooks). It is typically the case that $F(s)$ is the ratio of polynomials:

$$F(s)=\frac{N(s)}{D(s)}=\frac{a_n s^n+a_{n-1}s^{n-1}+\cdots+a_1 s+a_0}{b_m s^m+b_{m-1}s^{m-1}+\cdots+b_1 s+b_0}. \tag{3.58}$$

If the degree of $N(s)$ is larger or equal to the degree of $D(s)$, the numerator can be divided by the denominator to obtain the quotient $Q(s)$ and the remainder $R(s)$. Hence

$$F(s)=Q(s)+\frac{R(s)}{D(s)}=Q(s)+P(s), \tag{3.59}$$

where $R(s)/D(s)$ is a proper fraction. Let us consider the following example.

Example 3.5

Find the quotient and the remainder of the given $F(s)$

$$F(s)=\frac{s^3+5s^2+8s+7}{s^2+4s+3}.$$

Solution

Dividing the numerator by the denominator

$$\begin{array}{l|l}
s^3+5s^2+8s+7 & s^2+4s+3 \\ \cline{2-2}
 & s+1 \\
\underline{s^3+4s^2+3s} & \\
\quad s^2+5s+7 & \\
\quad \underline{s^2+4s+3} & \\
\qquad s+4 &
\end{array}$$

yields

$$F(s)=s+1+\frac{s+4}{s^2+4s+3}. \tag{3.60}$$

Note that the time function, whose Laplace transform is the quotient polynomial, is obtained directly from

$$\mathbf{L}^{-1}\{q_{n-m}s^{n-m} + q_{n-m-1}s^{n-m-1} + \cdots + q_1 s + q_0\}$$
$$= q_{n-m}\delta^{(n-m)}(t) + \cdots + q_1\delta'(t) + q_0\delta, \quad (3.61)$$

where $\delta^{(n-m)}$, $\delta^{(n-m-1)}$, ... $\delta'(t)$ are derivatives of the unit impulse function.

For further treatment the proper fraction polynomials $R(s)$ and $D(s)$ have to be **coprime**, that is, any non-trivial common factor has to be cancelled out.

There are a few methods for expanding a proper fraction into partial fractions. We will discuss two of them: 1) equating coefficients and 2) Heaviside's expansion theorem. Each of them may be the best to use, depending on the situation.

3.7.1 Method of equating coefficients

(a) *Simple poles*

We first assume that the rational function $F(s)$ has simple poles, i.e. that the denominator of (3.58) has simple zeros. Then the proper fraction of equation 3.59 may be written as

$$P(s) = \frac{R(s)}{D(s)} = \frac{R(s)}{(s-p_1)(s-p_2)\ldots(s-p_m)} = \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \cdots + \frac{A_m}{s-p_m}. \quad (3.62)$$

The constants A_i are known in mathematics as residues of the appropriate pole p_i. The equating coefficients method for determining A_i is illustrated in Example 3.6.

Example 3.6

Find the time function $f(t)$ if its Laplace transform is given by equation 3.60 (see Example 3.5).

Solution

First, we have to find the zeros of the equation $s^2 + 4s + 3 = 0$, which will be the poles

$$p_{1,2} = -2 \pm \sqrt{4-3} = -1, -3.$$

Therefore, in accordance with equation 3.62, the partial fraction expansion of the proper fraction (equation 3.60) is

$$\frac{s+4}{(s+1)(s+3)} = \frac{A_1}{s+1} + \frac{A_2}{s+3}. \quad (3.63)$$

Now combining two terms on the right side of this equation (by finding a common denominator) yields

$$\frac{s+4}{(s+1)(s+3)} = \frac{(A_1+A_2)s + (3A_1+A_2)}{(s+1)(s+3)}.$$

The constants A_1 and A_2 are found by equating like coefficients in the numerators. Thus,

$$A_1 + A_2 = 1, \quad 3A_1 + A_2 = 4,$$

or

$$A_1 = \frac{3}{2}, \quad A_2 = -\frac{1}{2}.$$

Therefore $F(s)$ is given by

$$F(s) = s + 1 + \frac{3/2}{s+1} - \frac{1/2}{s+3}.$$

Using the table of Laplace transform we obtain

$$f(t) = \delta'(t) + \delta(t) + \left(\frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}\right)u(t).$$

(b) *Multiple poles*

The following example illustrates the case when the denominator has repeated roots.

Example 3.7

Find the time domain function of the Laplace transform, which is given by

$$F(s) = \frac{s+2}{(s+3)^2}.$$

In this case, the expansion is given in the form

$$\frac{s+2}{(s+3)^2} = \frac{A_1}{s+3} + \frac{A_2}{(s+3)^2}.$$

Combining terms on the right gives

$$\frac{s+2}{(s+3)^2} = \frac{A_1(s+3) + A_2}{(s+3)^2}.$$

Equating like coefficients in the numerators yields

$$A_1 = 1, \quad 3A_1 + A_2 = 2 \quad \text{and} \quad A_2 = -1.$$

Therefore,

$$F(s) = \frac{1}{s+3} - \frac{1}{(s+3)^2},$$

and

$$f(t) = (e^{-3t} - te^{-3t})u(t).$$

In general, the method of equating the coefficients produces m simultaneous equations for determination A_i $(i = 1, 2 \ldots m)$ constants. When m is large (usually more than three) and when the poles are complex, Heaviside's method is more appropriate.

3.7.2 Heaviside's expansion theorem

This method allows determining the unknown residue A_i by using one equation, which contains only one residue. To develop it we should again distinguish between different kinds of poles.

(a) *Simple poles*

Consider equation 3.62 and multiply both sides by $(s - p_1)$:

$$\frac{(s-p_1)R(s)}{(s-p_1)(s-p_2)\ldots(s-p_m)} = A_1 + \frac{(s-p_1)A_2}{(s-p_2)} + \cdots + \frac{(s-p_1)A_m}{(s-p_m)}.$$

Now we note that if $s = p_1$ then every term on the right side is zero, except A_1, while on the left side the $(s - p_1)$ terms in the numerator and denominator are cancelled. Therefore, A_1 can be evaluated as follows

$$A_1 = (s-p_1)\left.\frac{R(s)}{D(s)}\right|_{s=p_1} \tag{3.64}$$

or, in general,

$$A_k = (s-p_k)F(s)\Big|_{s=p_k} = \frac{N(p_k)}{\prod\limits_{\substack{i=1\\ i\neq k}}^{m}(p_k-p_i)}, \tag{3.64a}$$

where $F(s)$ is the Laplace transform function in proper fraction form and $N(s)$ is its numerator. Note that the substitution of the poles p_k for s in equation 3.64 and equation 3.64a has to be performed after canceling the term $(s - p_k)$ in the nominator and denominator.

Let us, for example, evaluate the residues of equation 3.63 from Example 3.6

$$F(s) = \frac{s+4}{(s+1)(s+3)},$$

using the general formula of equation 3.64. We have, since $p_1 = -1$ and $p_2 = -3$,

$$A_1 = (s+1)F(s) = \left.\frac{s+4}{s+3}\right|_{s=-1} = \frac{3}{2}$$

$$A_2 = (s+3)F(s) = \left.\frac{s+4}{s+1}\right|_{s=-3} = -\frac{1}{2},$$

which is, of course, identical to the results of Example 3.6.

Once the residues have been found, the Laplace transform, in accordance with partial fraction expansion, may be written as

$$F(s) = \frac{N(s)}{D(s)} = \sum_{k=1}^{m} \frac{A_k}{s-p_k}, \tag{3.65}$$

and the time domain function is

$$f(t) = \mathbf{L}^{-1}\{F(s)\} = \left(\sum_{k=1}^{m} A_k e^{p_k t}\right) u(t). \tag{3.66}$$

(b) *Multiple poles*

Suppose that the function $F(s)$ has a double pole at p_1 and the remaining poles are simple, which means that the denominator has a double zero at p_1 and hence

$$D(s) = (s-p_1)^2(s-p_2)\cdots(s-p_m).$$

Then the partial fraction expansion may be written as

$$F(s) = \frac{N(s)}{D(s)} = \frac{A_{11}}{s-p_1} + \frac{A_{12}}{(s-p_1)^2} + \sum_{k=2}^{m} \frac{A_k}{s-p_k}. \tag{3.67}$$

Multiplying both sides of equation 3.67 by $(s-p_1)^2$ and letting $s = p_1$, yields as before

$$A_{12} = \left.(s-p_1)^2 F(s)\right|_{s=p_1} = \frac{N(p_1)}{(p_1-p_2)(p_1-p_3)\ldots(p_1-p_m)}. \tag{3.68}$$

To find A_{11} we again multiply both sides of equation 3.67 by $(s-p_1)^2$

$$\left.(s-p_1)^2 F(s)\right|_{s=p_1} = A_{11}(s-p_1) + A_{12} + (s-p_1)^2 \sum_{k=2}^{m} \frac{A_k}{s-p_k}.$$

After differentiating the last expression with respect to s and letting $s = p_1$, we obtain

$$A_{11} = \left.\left(\frac{d}{ds}\left[(s-p_1)^2 F(s)\right]\right)\right|_{s=p_1}. \tag{3.69}$$

Note that the differentiation is performed after canceling the term $(s-p_1)^2$ in the bracketed expression, and substituting p_1 for s has to be done after differentiation.

In general, when $F(s)$ has a multiple pole p_q, which is repeated r times, i.e. the denominator of $F(s)$ contains a factor of $(s - p_q)^r$, the residues at the multiple pole are evaluated as

$$A_{q,i} = \frac{1}{(r-i)!}\left[\frac{d^{(r-i)}}{ds^{(r-i)}}(s-p_q)^r F(s)\right]_{s=p_q}. \tag{3.70}$$

The time domain expression corresponding to these terms can be obtained as

$$f(t) = \left[A_{q1} + A_{q2}t + A_{q3}\frac{t^2}{2!} + \cdots + A_{qr}\frac{t^{(t-r)}}{(r-1)!}\right]e^{p_q t}. \tag{3.71}$$

Example 3.8

Let $F(s) = \dfrac{s+2}{s^3(s+1)^2}$; find the time domain function.

Solution

The given Laplace transform has the poles $p_1 = 0$ repeated three times and $p_2 = -1$ repeated twice. Thus,

$$A_{11} = \left[\frac{1}{2!}\frac{d^2}{ds^2}\frac{s+2}{(s+1)^2}\right]_{s=0} = 4, \quad A_{12} = \left[\frac{1}{1!}\frac{d}{ds}\frac{s+2}{(s+1)^2}\right]_{s=0} = -3,$$

$$A_{13} = \left[\frac{s+2}{(s+1)^2}\right]_{s=0} = 2$$

and

$$A_{21} = \left[\frac{1}{1!}\frac{d}{ds}\frac{s+2}{s^3}\right]_{s=-1} = -4, \quad A_{22} = \left[\frac{s+2}{s^3}\right]_{s=-1} = -1.$$

Hence, in accordance with equation 3.71, the time domain function is

$$f(t) = [4 - 3t + t^2 - (4+t)e^{-t}]u(t).$$

(c) *Complex poles*

As we know from mathematics, polynomials with real coefficients may only have a pair of complex-conjugate poles, i.e. any complex pole in the denominator of $F(s)$ will be accompanied by its complex-conjugate pole. Then the corresponding residues are also a complex-conjugate pair, so only one of them must be found. Combining these two terms, yields to the appropriate time-domain fraction of the whole response.

Let the complex-conjugate pair be $p_1 = -\alpha \pm j\beta$ and the corresponding residues be A_1 and $\overset{*}{A}_1$, which are a complex-conjugate pair. Hence, using the exponential form

$$A_1 = |A_1|e^{j\psi_1} \quad \text{and} \quad \overset{*}{A}_1 = |A_1|e^{-j\psi_1}$$

we can write the expansion of $F(s)$, which is appropriate to complex poles, as

$$F_{c1} = \frac{|A_1|e^{j\psi_1}}{s+\alpha_1 - j\beta_1} + \frac{|A_1|e^{-j\psi_1}}{s+\alpha_1 + j\beta_1}, \tag{3.72}$$

and its time-domain inverse is

$$\begin{aligned}\mathbf{L}^{-1}\{F_{c1}(s)\} &= |A_1|e^{j\psi_1}e^{-(\alpha_1 - j\beta_1)t} + |A_1|e^{-j\psi_1}e^{-(\alpha_1 + j\beta_1)t} \\ &= |A_1|e^{-\alpha_1 t}[e^{j(\beta_1 t+\psi_1)} + e^{-j(\beta_1 t+\psi_1)}] \\ &= 2|A_1|e^{-a_1 t}\cos(\beta_1 t + \psi_1)u(t). \end{aligned} \tag{3.73}$$

This expression shows that the complex poles are associated with the time-domain response which is similar to the natural response of an underdamped second-order circuit.

Note that equation 3.73 can be simply obtained as a double real part of an inverse transform of only one of the fractions in equation 3.72. Indeed,

$$\begin{aligned}2\ \mathrm{Re}\left[\mathbf{L}^{-1}\left\{\frac{|A_1|e^{j\psi_1}}{s+\alpha_1 - j\beta_1}\right\}\right] &= 2\ \mathrm{Re}[|A_1|e^{j\psi_1}e^{-(\alpha_1 - j\beta_1)t}] \\ &= 2|A_1|e^{-\alpha t}\cos(\beta_1 t + \psi_1)u(t). \end{aligned} \tag{3.74}$$

Example 3.9

Find $f(t)$ if $F(s)$ is given by

$$F(s) = \frac{s+4}{s^2+2s+5}.$$

Solutions

First, we find the roots of the denominator, which are the poles of $F(s)$: $p_{1,2} = -1 \pm j2$. Hence,

$$F(s) = \frac{s+4}{(s+1-j2)(s+1+j2)}.$$

Now, in accordance with Heaviside's Expansion formula (equation 3.64), we have

$$A_1 = \left.\frac{s+4}{s+1+j2}\right|_{s=-1+j2} = \frac{3+j2}{j4} = \frac{1}{2} - j\frac{3}{4} = 0.9e^{-j32^\circ}.$$

Therefore,

$$f(t) = 2\ \mathrm{Re}[0.9e^{-j32^\circ}e^{-(1-j2)t}] = 1.8e^{-t}\cos(2t - 32^\circ)u(t).$$

The time-domain function of a complex-conjugate pair fraction can be obtained, using the complex residues in rectangular form $A_1 = a_{1r} + ja_{1i}$. Then the partial fraction, which is appropriate to the complex poles, will be

$$F_{c1}(s) = \frac{a_{1r} + ja_{1i}}{s + \alpha_1 - j\beta_1} + \frac{a_{1r} - ja_{1i}}{s + \alpha_1 + j\beta_1}, \tag{3.75}$$

and its inverse transform is

$$f_{c1}(t) = (a_{1r} + ja_{1i})e^{-(\alpha_1 - j\beta_1)t} + (a_{1r} - ja_{1i})e^{-(\alpha_1 + j\beta_1)t},$$

or, using Euler's formula,

$$f_{c1}(t) = e^{-\alpha_1 t} 2(a_{1r} \cos \beta_1 t - a_{1i} \sin \beta_1 t)u(t). \tag{3.76}$$

In conclusion, it is worthwhile giving one other notation of a residue evaluation formula (see equations 3.64 and 3.64a).

If the pole factor $(s - p_1)$ cannot be easily canceled (for example when the denominator $D(s)$ is not given in factored form), then the residue A_1 must be treated as limit:

$$A_1 = \lim_{s \to p_1} R(s) \frac{(s - p_1)}{D(s)},$$

which in accordance with l'Hopital's Rule gives

$$A_1 = R(s) \lim_{s \to p_1} \frac{\frac{d}{ds}(s - p_1)}{\frac{d}{ds}(D(s))} = R(s) \frac{1}{D'(s)}\bigg|_{s = p_1}. \tag{3.77}$$

3.8 CIRCUIT ANALYSIS WITH THE LAPLACE TRANSFORM

There are two basic approaches to using Laplace transforms to find the circuit complete response. The first is to write differential equations describing the circuit and then to solve them using the Laplace transform of the variable and its derivatives. The advantage of this approach is that the Laplace transform provides an algebraic method for solving differential equations. Taking the inverse transform gives the time domain solution.

A second method of finding a circuit response is based on a model that directly describes relationships between the Laplace transforms of the circuit variables and its elements. This Laplace model is in some way similar to the frequency-domain circuits developed earlier.

First, we will discuss the differential equations approach. Consider the second-order circuit shown in Fig. 3.9. Then the KVL equation around the loop and the differential equation for v_C are

$$L\frac{di}{dt} + v_C + Ri = v_s, \quad C\frac{dv_C}{dt} = i.$$

These two equations are sufficient to solve the two unknown variables. The

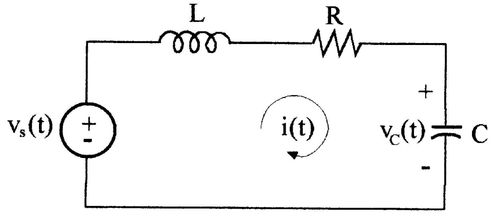


Figure 3.9 A second-order series connection *RLC* circuit.

Laplace transform of these two equations is

$$\begin{aligned} \mathbf{L}\,[sI(s) - i(0_-)] + V_C(s) + RI(s) &= V_{in}(s) \\ C[sV_C(s) - v_C(0_-)] &= I(s). \end{aligned} \tag{3.78}$$

Let the initial condition be $i(0_-) = i(0) = I_0$ and $v_C(0_-) = v_C(0) = V_{C0}$ and after rearranging the equations 3.78 we have

$$\begin{aligned} (R + sL)I(s) + V_C(s) &= V_{in}(s) + LI_0 \\ -I(s) + sCV_C(s) &= CV_{C0}. \end{aligned} \tag{3.79}$$

Solving equation 3.79 for $I(s)$ using *Cramer's rule*, we have

$$I(s) = \frac{sCV_{in}(s)}{LCs^2 + RCs + 1} + \frac{sC[LI_0 - (1/s)V_{C0}]}{LCs^2 + RCs + 1} \tag{3.80}$$

or

$$I(s) = Y_{in}(s)V_{in}(s) + Y_{in}(s)W_0(s), \tag{3.81}$$

where

$$Y_{in}(s) = \frac{sC}{LCs^2 + RCs + 1}$$

is the Laplace transform of the input impedance and $W_0(s) = LI_0 - (1/s)V_{C0}$ is the initial condition representation. Equation 3.81 shows that $I(s)$ is the sum of two terms: one **due to the input source** and the second **due to all the initial conditions**. Going back to the time-domain, we can say that circuit variables contain two terms: a zero-state response (when all the initial conditions are zero) and a zero-input response (when all the inputs are zero).

Example 3.10

Let the elements of the circuit in Fig. 3.9 be normalized and have the values $L = 1$ H, $R = 3\ \Omega$ and $C = 1/2$ F. Let the voltage input $v_s(t) = (10 \sin t)u(t)$ and the initial condition $I_0 = 2$ A and $V_{C0} = 5$ V. Find $i(t)$.

Solution

Since the Laplace transform of the input voltage is $V_{in}(s) = 10/(s^2 + 1)$, expression

(3.80) after substituting the numerical values yields

$$I(s) = \frac{s}{s^2+3s+2}\frac{10}{s^2+1} + \frac{2s-5}{s^2+3s+2}.$$

The roots of the denominators are $s_1 = -1$, $s_2 = -2$ and $s_{3,4} = \pm j$. Therefore, using partial fractions, we obtain

$$I(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2} + \frac{A_3}{s-j} + \frac{\overset{*}{A}_3}{s+j}.$$

Performing the computation, we obtain

$$A_1 = \left|\frac{10s}{(s+2)(s^2+1)} + \frac{2s-5}{s+2}\right|_{s=-1} = -12$$

$$A_2 = \left|\frac{10s}{(s+1)(s^2+1)} + \frac{2s-5}{s+1}\right|_{s=-2} = 13$$

$$A_3 = \left|\frac{10s}{(s^2+3s+2)(s+j)}\right|_{s=j} = \frac{j10}{(-1+3j+2)2j} = 1.58 \angle -71.6^\circ.$$

So,

$$i(t) = [-12e^{-t} + 13e^{2t} + 3.16\cos(t - 71.6^\circ)]u(t).$$

The second approach which leads to more simplicity in Laplace transform circuit analysis uses the Laplace circuit model, which can be analyzed by frequency-domain methods. In these models, all the elements are expressed in terms of their impedances (admittances) at a complex frequency s (see Table 3.3 at the end of the chapter) and the voltage/current sources – by their Laplace transforms, i.e. as a function of s. Then one of the known methods (KVL, KCL, nodal/mesh analysis, Thévenin-Norton's theorem, etc.) can be used for identifying the desired variable transform. Finally, the time-domain response may be found with the help of the inverse transform (partial fraction expansion).

In the next few paragraphs we will illustrate how this technique may be used for circuit analysis with Laplace transform, starting with networks without initial energy stored (zero initial conditions).

3.8.1 Zero initial conditions

As an example, let us examine the circuit shown in Fig. 3.10(a). First, we convert the circuit to frequency-domain (or to its Laplace transform representation) as shown in Fig. 3.10(b). Let the voltage across the capacitance C_2, which is output voltage, be of interest. It may be found by the node equation. The output voltage is the node 1 voltage V_1. Therefore

$$\frac{V_1 - V_{in,1}}{R_1} + \frac{V_1}{R_2 + 1/sC_1} + \frac{V_1 - V_{in,2}}{R_3} + sC_2V_1 = 0.$$

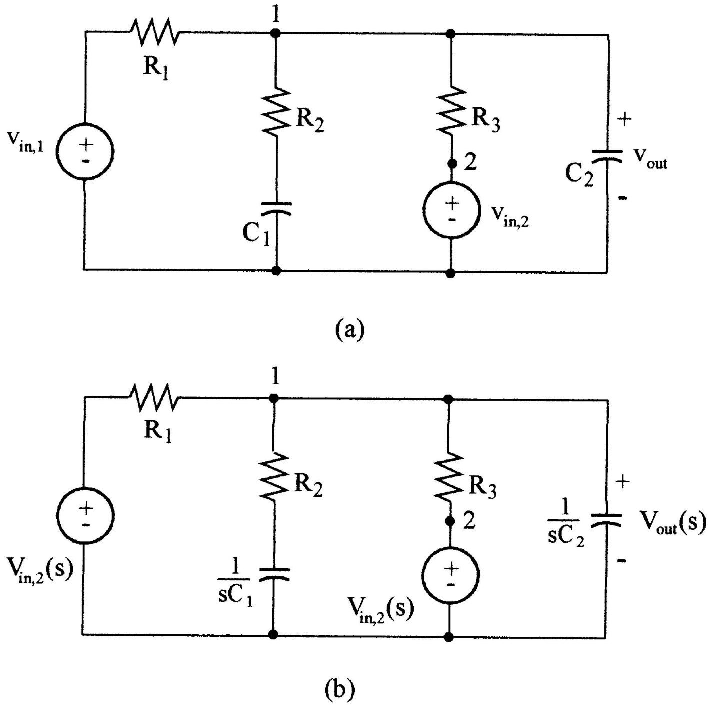


Figure 3.10 A two node circuit expressed in time-domain (a) and in the s-domain (b).

Solving for $V_1(V_{out})$ yields

$$V_{out}(s) = \frac{a_1 s + a_0}{b_2 s^2 + b_1 s + b_0}\left(\frac{V_{in,1}(s)}{R_1} + \frac{V_{in,2}(s)}{R_3}\right), \tag{3.82}$$

where

$$a_1 = R_1 R_2 R_3 C_1 \quad a_0 = R_1 R_3$$
$$b_1 = (R_1 R_2 + R_1 R_3 + R_2 R_3) C_1 \quad b_0 = R_1 + R_3$$
$$b_2 = R_1 R_2 R_3 C_1 C_2.$$

Now, in accordance with the transfer function concept

$$V_{out}(s) = H_1(s) V_{in,1}(s) + H_2(s) V_{in,2}(s), \tag{3.83}$$

and we can use the results in equation 3.83 for different inputs. It is obvious that

$$H_1(s) = (1/R_1) H_0(s), \quad H_2(s) = (1/R_3) H_0(s),$$

$$H_0(s) = \frac{a_1 s + a_0}{b_2 s^2 + b_1 s + b_0}.$$

To find $v_{out}(t)$ we need to evaluate the inverse transform of each term in equation 3.83

$$v_{out}(t) = \mathbf{L}^{-1}\{H_1(s)V_{in,1}(s)\} + \mathbf{L}^{-1}\{H_2(s)V_{in,2}(s)\}. \tag{3.84}$$

Example 3.11

Determine the voltage across the resistance R in the circuit shown in Fig. 3.11(a), which is already expressed in terms of the Laplace transform. The normalized elements are $L_1 = L_2 = 1$ H, $R = 1\ \Omega$, $v_1 = \cos tu(t)$, $v_2 = 1\delta(t)V$.

Solution

The first step is to convert the voltage sources to current sources and, after simplification, we obtain a simple circuit as shown in Fig. 3.11(b) and (c). Thus

$$I_0(s) = \frac{1}{L_1(s^2+1)} + \frac{1}{sL} = \frac{s^2+s+1}{s(s^2+1)}$$

$$Y(s) = \frac{1}{s} + \frac{1}{s} + \frac{1}{1} = \frac{s+2}{s},$$

and

$$v_R = I_0(s)\frac{1}{Y(s)} = \frac{s^2+s+1}{(s+2)(s^2+1)}.$$

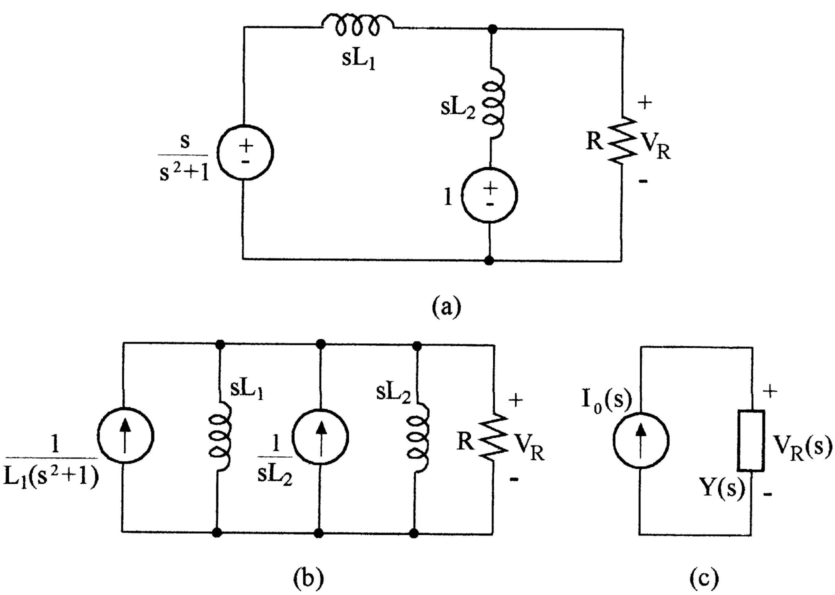


Figure 3.11 A circuit under study in Example 3.11: frequency-domain representation (a); circuit with current sources (b); final circuit (c).

Using the partial fraction expansion yields

$$V_R = \frac{A_1}{s+2} + \frac{A_2}{s-j} + \frac{\overset{*}{A}_2}{s+j}.$$

Therefore

$$A_1 = \left|\frac{s^2+s+1}{s^2+1}\right|_{s=-2} = 0.6$$

$$A_2 = \left|\frac{s^2+s+1}{(s+2)(s+j)}\right|_{s=j} = \frac{j}{(s+j)2j} = 0.2236 \angle -26.6^\circ.$$

Then the desirable voltage in time-domain is

$$V_R(t) = 0.6e^{-2t} + 0.447\cos(t - 26.6^\circ)\ \text{V} \quad \text{for} \quad t \geq 0.$$

3.8.2 Non-zero initial conditions

As noted in the beginning of this chapter, the important advantage of the Laplace transform method is taking "automatically" into account the initial conditions. In the Laplace model approach, it is done by the appropriate frequency-domain equivalent of an inductor L with initial current and a capacitor C with initial voltage.

First, consider the initially charged inductor shown in Fig. 3.12(a). Since

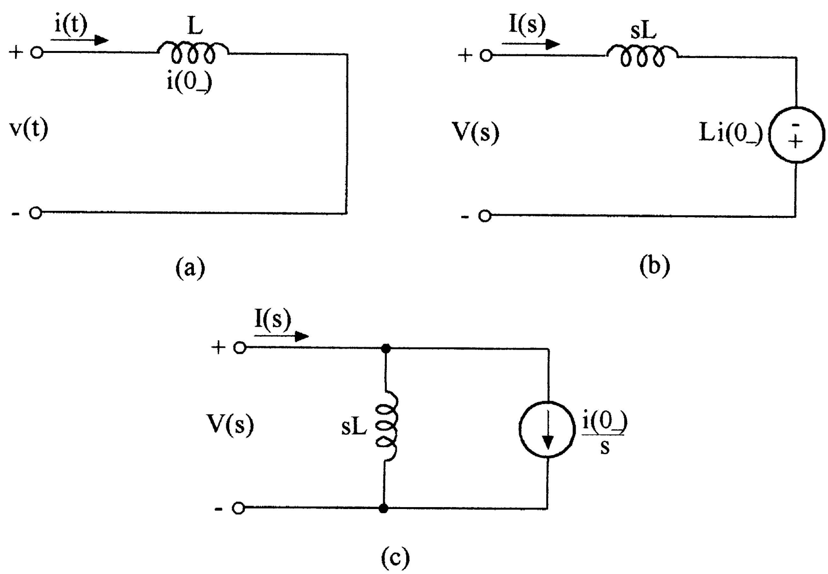


Figure 3.12 A Laplace model of an inductor with initial current: time-domain circuit (a); inductor representation in series with a voltage source (b); inductor representation in parallel with a current source (c).

$v(t) = L(di/dt)$, then

$$V(s) = sLI(s) - Li(0_-). \tag{3.85}$$

In accordance with this expression, the Laplace model for the inductor might be represented by a voltage source in series with an uncharged inductor, as shown in Fig. 3.12(b). An alternative Laplace model for the inductor can be obtained by converting the voltage source into the current source as shown in Fig. 3.12(c). Note that the voltage source in Fig. 3.12(b) is the transform of an impulse, while the current source in Fig. 3.12(c) is the transform of a step-function.

Considering the initially charged capacitor shown in Fig. 3.13(a) and in accordance with $i = C(dv/dt)$, yields

$$I(s) = sCV(s) - Cv(0_-). \tag{3.86}$$

The Laplace model of a capacitor with equation 3.86 is shown in Fig. 3.13(b). (It is the dual of the inductor model in Fig. 3.12(b).) By converting the current source in Fig. 3.13(b) into the voltage source, the second alternative of the capacitor model can be obtained as shown in Fig. 3.13(c). The voltage and current sources due to non-zero initial conditions, as represented above, are called **initial-condition generators**.

By using initial-condition generators, the Laplace transform circuit model is completed and can be analysed by frequency-domain methods when the initial conditions are not zero. The following examples illustrate these techniques.

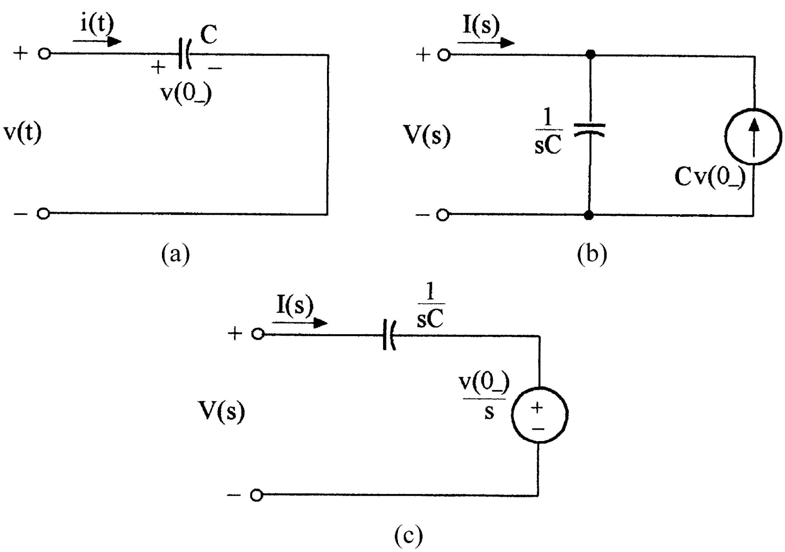


Figure 3.13 A Laplace model of initial charged capacitor: time-domain circuit (a); capacitor representation in parallel with a current source (b); capacitor representation in series with a voltage source (c).

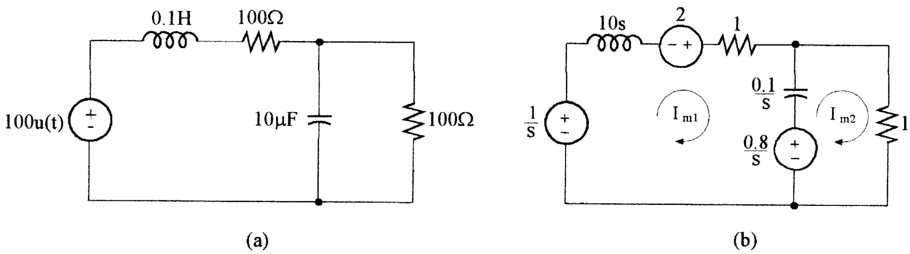


Figure 3.14 The given circuit of Example 3.12 (a); its normalized Laplace model (b).

Example 3.12

Find the complete response of the current $i(t)$ in the circuit shown in Fig. 3.14(a), if $i(0_-) = 0.2$ A and $v_C(0_-) = 80$ V.

Solution

To work with more convenient numbers, we first normalize them by choosing the impedance normalization factor K_m and frequency normalization factor K_f. Let $K_m = 10^{-2}$ and $K_f = 10^{-4}$, then $R_{new} = 10^{-2} R_{old} = 1\,\Omega$, $L_{new} = (10^{-2}/10^{-4}) L_{old} = 10$ H and $C_{new} = (1/10^{-2}10^{-4}) C_{old} = 0.1$ F. The Laplace model circuit with normalized elements is shown in Fig. 3.14(b). Note that, to keep the same currents, voltage sources are also normalized in accordance to K_m. Using mesh analysis, we have

$$\left(1 + 10s + \frac{0.1}{s}\right) I_{m1} - \frac{0.1}{s} I_{m2} = \frac{1}{s} + 2 - \frac{0.8}{s}$$

$$-\frac{0.1}{s} I_{m1} + \left(1 + \frac{0.1}{s}\right) I_{m2} = \frac{0.8}{s}.$$

Solving for I_{m1} gives

$$I_{m1}(s) = \frac{0.2s^2 + 0.04s + 0.01}{s(s^2 + 0.2s + 0.02)},$$

with the poles $p_1 = 0$ and $p_{2,3} = -0.1 \pm j0.1$. Using the partial fraction expansion yields

$$I_{m1}(s) = \frac{A_1}{s} + \frac{A_2}{s + 0.1 - j0.1} + \frac{\overset{*}{A}_2}{s + 0.1 + j0.1},$$

where

$$A_1 = \left| \frac{0.2s^2 + 0.04s + 0.01}{s^2 + 0.2s + 0.02} \right|_{s=0} = 0.5$$

$$A_2 = \left| \frac{0.2s^2 + 0.04s + 0.01}{s(s + 0.1 + j0.1)} \right|_{s=-0.1+j0.1} = 0.212 \angle 135°.$$

Then the current in time-domain is

$$i(t) = 0.5 + 0.424e^{-0.1t}\cos(0.1t + 135^\circ)\ \mathsf{A} \quad \text{for} \quad t \geq 0.$$

Returning to the original circuit, i.e. that the actual natural frequency of the circuit is

$$s_{old} = \frac{s_{new}}{K_f} = 10^4(-0.1 \pm j0.1) = 10^3 \pm j10^3,$$

then

$$i(t) = 0.5 + 0.424e^{-10^3 t}\cos(10^3 t + 135^\circ). \tag{3.87}$$

Inspection of the circuit in Fig. 3.14(a) shows that the steady-state value of the current is 0.5 A, which is in agreement with the above results. Also checking the initial value of the current gives

$$i(t) = 0.5 + 0.424\cos 135^\circ = 0.2\ \mathsf{A}.$$

The waveform of the current (equation 3.87) begins at a value of 0.2 A and approaches 0.5 A with decayed oscillation in approximately 5 ms.

Example 3.13

The circuit of Fig. 3.15(a) is in steady-state behavior. At $t = 0$ the second voltage source is applied in series with the capacitor. Find the transient response of the capacitor voltage $v_C(t)$.

Solution

Using the superposition approach, we construct the Laplace model circuit in which the second voltage source acts alone (Fig. 3.15(b)). Then the Laplace transform of a desirable voltage can be written as

$$V_{C2}(s) = -\frac{V_2}{s}\frac{1/sC}{Z_{in}(s)},$$

where

$$Z_{in}(s) = \frac{(10 + 0.1s)100}{110 + 0.1s} + \frac{10^4}{s} = \frac{10(s^2 + 200s + 11\cdot 10^4)}{s(0.1s + 110)}.$$

Therefore,

$$V_{C2}(s) = -\frac{(0.1s + 110)10^4}{(s^2 + 200s + 11\cdot 10^4)s} = \frac{10^4}{s}\frac{0.1s + 110}{(s + a)^2 + \omega^2}.$$

Using the method of equating the coefficients, this voltage can be obtained as

$$V_{C2}(s) = -\frac{10}{s} + 10\frac{s + a}{(s + a)^2 + \omega^2},$$

where $a = 100$ 1/s and $\omega = 316$ 1/s.

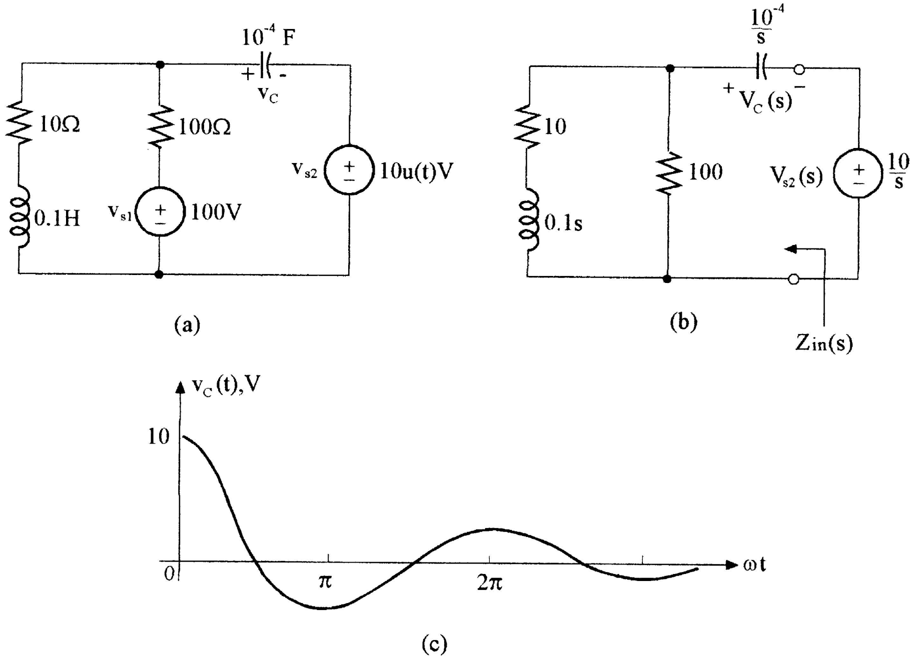


Figure 3.15 The given circuit of Example 3.13 (a); the Laplace model of the circuit driven by only the second source (b); the capacitor voltage waveform (c).

In accordance with the table of Laplace transform pairs, we obtain

$$v_{C2}(t) = 10(e^{-at}\cos\omega t - 1)u(t)\ \mathrm{V}.$$

Since the capacitor voltage v_{C1} caused by the first voltage source is 10 V, the entire capacitor voltage will be

$$v_C(t) = v_{C1} + v_{C2} = 10e^{-100t}\cos 316t\ \mathrm{V} \quad \text{for} \quad t \geq 0,$$

which is shown in Fig. 3.15(c).

3.8.3 Transient and steady-state responses

With the Laplace transform we can determine the transient and steady-state responses of the circuit variables. In order not to get involved in complicated notations, we will consider a simple example.

Let us say that the desired response of a circuit is described by a second-order differential equation

$$a_2\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0y(t) = b_1\frac{dw}{dt} + b_0w(t), \tag{3.88}$$

and the initial conditions are $y(0_-) = y_0,\ y'(0_-) = y'_0$. Taking the Laplace transform of both sides of equation 3.88 gives (remember that $w(t)$ applies for $t \geq 0$)

$$a_2[s^2 Y(s) - sy_0 - y'_0] + a_1[sY(s) - y_0] + a_0 Y(s) = b_1 sW(s) + b_0 W(s),$$

or

$$(a_2 s^2 + a_1 s + a_0)Y(s) + (-a_2 y_0 s - a_2 y'_0 - a_1) = (b_1 s + b_0)W(s).$$

Solving for the Laplace transform $Y(s)$ yields

$$Y(s) = W(s)\frac{b_1 s + b_0}{a_2 s^2 + a_1 s + a_0} + \frac{W_0(s)}{a_2 s^2 + a_1 s + a_0}, \tag{3.89}$$

where $W_0(s) = a_2 y_0 s + a_2 y'_0 + a_1 y_0$, i.e., includes all of the terms that involve the initial conditions of y and its derivatives.

Noting that the first expression includes the transfer function $H(s) = Y(s)/W(s)$ (since it is separated from the initial conditions) we can finally write

$$Y(s) = H(s)W(s) + B(s), \tag{3.90}$$

where $B(s) = W_0(s)/(a_2 s^2 + a_1 s + a_0)$.

Now, taking the inverse Laplace transform of equation 3.90 via partial fraction expansion, we pay attention that the term $H(s)W(s)$ has two groups of poles: due to $W(s)$ and $H(s)$, while the term $B(s)$ only has poles due to $H(s)$. Therefore, the time response of $y(t)$ can be grouped into two kinds of terms

$$y(t) = \mathbf{L}^{-1}\{H(s)W(s)\}|_{p_w} + \mathbf{L}^{-1}\{H(s)W(s) + B(s)\}|_{p_h}, \tag{3.91}$$

where the first term includes all the partial fractions which correspond to the pole p_w of $W(s)$, while the second one includes all the partial fractions which correspond to the poles p_h of $H(s)$. Now, if all the poles of $H(s)$ are strictly in the left half of the s-plane (LHP), which is the most practical case, the steady-state value of $y(t)$ is entirely due to the first term:

$$y_{ss}(t) = \lim \mathbf{L}^{-1}\{H(s)W(s)\}|_{p_w}, \tag{3.92a}$$

i.e., y_{ss} will be non-zero if and only if $W(s)$ has at least one pole on the j-axis or in the right half of the s-plane (RHP). It means that only the input sources determine the *steady-state response*.

The *natural response* is determined in accordance with the second term of equation 3.91:

$$y_{nat}(t) = \mathbf{L}^{-1}\{H(s)W(s) + B(s)\}|_{p_h} \tag{3.92b}$$

which is entirely due to the poles of $H(s)$ and, if all of them are in the LHP, the natural response must eventually die out. However, the transient response or *complete response*, which is given by (3.91), is obviously determined by the poles of the circuit ($H(s)$) and the poles of the input sources ($W(s)$). Note again that the transfer function denominator roots determine all the natural frequencies, i.e. the roots of $a_2 s^2 + a_1 s + a_0 = 0$ in the above example.

Example 3.14

Determine the forced and natural responses of the output voltage in the circuit of Fig. 3.16(a) assuming that the capacitor was pre-charged with $v_{C0} = 6\,\mathrm{V}$ and $v_g = 4e^{-t}u(t)\,\mathrm{V}$.

Solution

First, we construct the Laplace transform model, shown in Fig. 3.16(b), of the given circuit. Next, we write the nodal equation for this circuit model:

$$V_1 - \frac{4}{s+1} + \frac{V_1}{s} + \frac{V_1 - 6/s}{2/s} - 0.58\frac{V_1}{s} = 0,$$

or

$$\frac{V_1(0.5s^2 + s + 0.42)}{s} = \frac{4}{s+1} + 3.$$

Solving for V_1 yields

$$V_{out}(s) = V_1(s) = \frac{8s}{(s+1)(s^2+2s+0.84)} + \frac{6s}{s^2+2s+0.4}.$$

The natural frequencies are $p_{1h} = -0.6$, $p_{2h} = -1.4$ and the forced frequency is

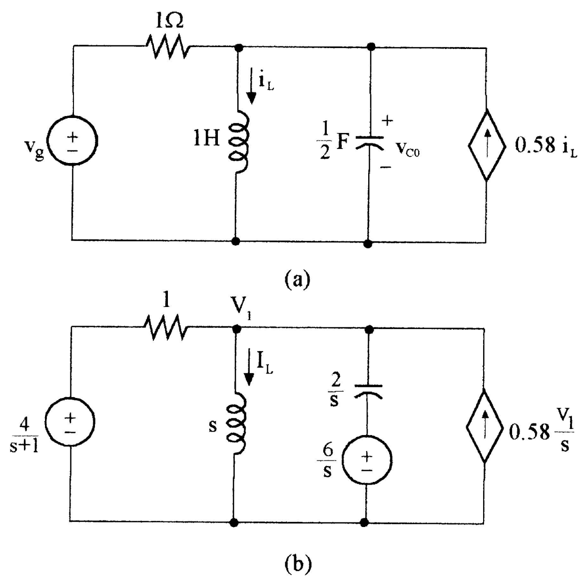


Figure 3.16 The circuit under study in Example 3.14 (a) and its Laplace model (b).

$p_w = -1$. Therefore, the residues of the first term are

$$A_{1h} = \left.\frac{8s}{(s+1)(s+1.4)}\right|_{s=-0.6} = -15, \quad A_{2h} = \left.\frac{8s}{(s+1)(s+0.6)}\right|_{s=-1.4} = -35,$$

$$A_w = \left.\frac{8s}{s^2+2s+0.4}\right|_{s=-1} = 50,$$

and the residues of the second term are

$$A'_{1h} = \left.\frac{6s}{s+1.4}\right|_{s=-0.6} = -4.5, \quad A'_{2h} = \left.\frac{6s}{s+0.6}\right|_{s=-1.4} = 10.5.$$

The time-domain responses are:

the forced response

$$v_{out,f} = 50e^{-t}u(t),$$

the natural response

$$\begin{aligned} v_{out,h} &= (-15e^{-0.6t} - 35e^{-1.4t} - 4.5e^{-0.6t} + 10.5e^{-1.4t})u(t) \\ &= (-19.5e^{-0.6t} - 24.5e^{-1.4t})u(t), \end{aligned}$$

and the complete transient response is

$$v_{out} = (50e^{-t} - 19.5e^{-0.6t} - 24.5e^{-1.4t})u(t)\ \mathrm{V},$$

which proves the initial voltage

$$v_{out}(0) = v_{C0} = 50 - 19.5 - 24.5 = 6\,\mathrm{V}.$$

3.8.4 Response to sinusoidal functions

Circuit responses to sinusoidal inputs are widely met in Power Systems Analysis. The transient analysis of such circuits by the Laplace transform might be simplified if the sinusoidal input function is taken as a complex function $\tilde{e}(t) = \tilde{E}e^{j\omega t}$ where $\tilde{E} = E_m e^{j\psi}$ is the phasor. The relation between the complex function and the actual input is given by

$$e(t) = \mathrm{Im}\{\tilde{e}(t)\} = \mathrm{Im}\{\tilde{E}e^{j\omega t}\} = E_m \sin(\omega t + \psi).$$

The Laplace transform of the complex input will be just $\tilde{E}(s) \leftrightarrow \tilde{E}/(s - j\omega)$. Then the Laplace transform of the output will be

$$\tilde{X}(s) = \tilde{E}(s)H(s) = \frac{E_m e^{j\psi}}{s - j\omega} H(s). \tag{3.93}$$

Taking the inverse transform yields

$$\tilde{x}(t) = E_m e^{j\psi} H(j\omega) e^{j\omega t} + E_m e^{j\psi} \sum_{k=1}^{n} \lim_{s \to p_k} \frac{(s - p_k)H(s)}{s - j\omega}. \tag{3.94a}$$

If the initial conditions are not zero, they have to be evaluated as imaginary quantities $I_L(0_-) = ji_L(0_-)$, $V_C(0_-) = jv_C(0_-)$ (since the imaginary part of the complex representation of phasors corresponds to the time-domain functions). Then, in accordance with equation 3.91, the complete complex response will be

$$\tilde{x}(t) = E_m e^{j\psi} H(j\omega) e^{j\omega t} + \sum_{k=1}^{n} \lim_{s \to p_k} \left[E_m e^{j\psi} \frac{(s - p_k) H(s)}{s - j\omega} + (s - p_k) B(s) \right] e^{p_k t}. \tag{3.94b}$$

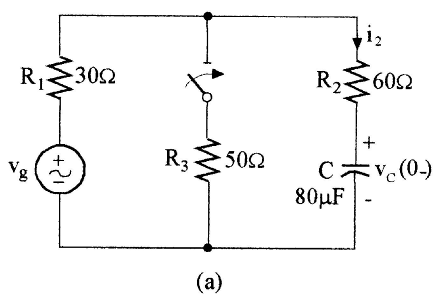
With the complex response in equation 3.94a or 3.94b the actual response will be

$$x(t) = \operatorname{Im}\{\tilde{x}(t)\}. \tag{3.95}$$

We will illustrate this method by the following example.

Example 3.15

In the circuit shown in Fig. 3.17(a), the switch closes at time $t = 0$ after having been opened for a long time. Find $i_2(t)$ assuming that the circuit is driven by the sinusoidal voltage source $v_g = 180 \sin(314t + 30°)$ V.



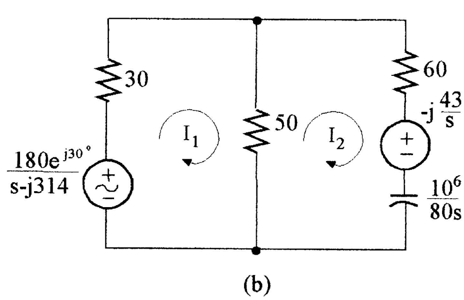


Figure 3.17 The circuit under study in Example 3.15 (a) and its Laplace model (b).

Solution

To determine the initial condition we must first calculate the capacitor steady-state voltage (before the switch is closed). The voltage source complex representation is $\tilde{v}_g = 180e^{j30^\circ}e^{j314t}$. So,

$$V_C(j\omega) = \frac{V_g(j\omega)}{R_1 + R_2 + 1/j\omega C}\frac{1}{j\omega C}$$
$$= \frac{180e^{j30^\circ}}{90 - j1/(314\cdot 80\cdot 10^{-6})}\frac{1}{j314\cdot 80\cdot 10^{-6}} = 72.3e^{-j36.2^\circ}\ \mathrm{V}.$$

Therefore, the voltage across the capacitor at $t = 0_-$

$$v_C(0_-) = 72.3\sin(-36.2) = -43.0\ \mathrm{V}.$$

Now we will construct the Laplace transform model circuit shown in Fig. 3.17(b). The Laplace transform of the voltage source, which is taken as a complex function, is $V_g(s) = 180e^{j30^\circ}/(s - j314)$. The initial capacitor voltage $v_C(0_-) = -43$ V is replaced by an initial-condition generator whose value is equal to the Laplace transform of this voltage multiplied by j:

$$V_{C0} = j\frac{(-43)}{s}.$$

In accordance with mesh analysis

$$80I_1(s) - 50I_2(s) = \frac{180e^{j30^\circ}}{s - j314}$$
$$-50I_1(s) + \left(110 + \frac{12.5\cdot 10^3}{s}\right)I_2(s) = j\frac{43}{s}.$$

Using Cramer's rule yields

$$I_2(s) = \frac{1.43e^{j30^\circ}s}{(s - j314)(s + 159)} + j\frac{0.546}{s + 159}.$$

Taking the inverse Laplace transform (with the help of the Laplace transform pairs, Table 3.1) we obtain

$$\tilde{i}_2(t) = 1.43e^{j30^\circ}\left[\frac{1}{-j314 - 159}(-j314e^{j314t} - 159e^{-159t})\right] + j0.546e^{-159t}$$
$$= 128e^{j(314t + 56.9^\circ)} + (0.646e^{-j33.2^\circ} + j0.546)e^{-159t}.$$

Finally, the imaginary part of the above expression gives the time-domain current

$$i_2(t) = 1.28\sin(314t + 56.9^\circ) + [0.646\sin(-33.2^\circ) + 0.548]e^{-159t}$$
$$= 1.28\sin(314t + 56.9^\circ) + 0.159e^{-159t}\ \text{A}.$$

3.8.5 Thévenin and Norton equivalent circuits

Thévenin/Norton's theorem can also be useful by circuit analysis via Laplace transform techniques. As we have already noted the Thévenin/Norton equivalent can be applied to the Laplace transform circuit model, using frequency-domain analysis. Here, the Thévenin/Norton equivalent can be especially helpful in reducing the initial state response to the zero-state response.

Consider any active one-port network shown in Fig. 3.18(a). The switch, after having been opened for a long time, closes at $t = 0$ (or any given time t_0) and the external branch ab is connected to the network. The Thévenin equivalent of the given network after closing the switch is shown in Fig. 3.18(b). It is obvious that this circuit is initially quiescent, so its response is ZSR (zero-state response). Therefore, the Laplace transform for the current I_{ab} is

$$I_{ab}(s) = \frac{V_{oc}(s)}{Z_{Th}(s) + Z_{ab}(s)}. \tag{3.96}$$

Anyway, it should be emphasized that the complete response of currents and voltages of the given network (except the branch ab) results from the superposition of two networks (a) and (b) in Fig. 3.18, i.e. the ZSR of currents and voltages has to be added to their previous steady-state values. The following example illustrates this technique.

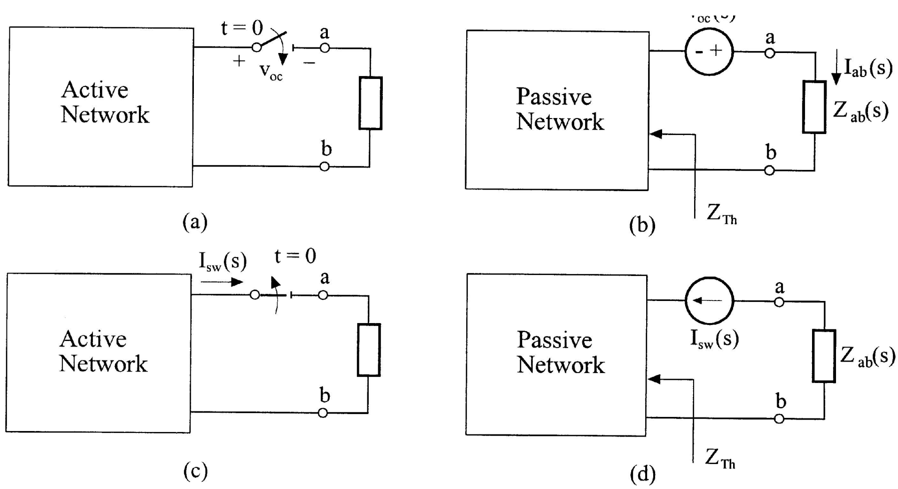


Figure 3.18 The illustration of Thévenin and Norton equivalents in Laplace transform representations: the active network with an open switch (a); the Thévenin equivalent with a voltage source (b); the active network with a closed switch (c); the Norton representation with a current source (d).

Example 3.16

The switch in the circuit shown in Fig. 3.19(a) closes after having been opened for a long time. Find the currents through the capacitor $i_C(t)$ and through the inductor $i_L(t)$.

Solution

The open circuit voltage across the switch is

$$V_{oc} = V_g \frac{R_2}{R_1 + R_2} = 200 \frac{10}{10 + 10} = 100 \text{ V}.$$

The Thévenin equivalent impedance of the circuit is

$$Z_{Th} = \frac{(R_1 + sL)R_2}{R_1 + R_2 + sL} = \frac{100 + s}{20 + 0.1s}.$$

The Thévenin equivalent of the Laplace transform circuit is shown in Fig. 3.19(b). Thus, the Laplace transform of the capacitor current is

$$I_C(s) = \frac{V_{oc}(s)}{Z_{Th} + Z_{ab}} = \frac{100}{s}\frac{1}{(100 + s)/(20 + 0.1s) + 10^3/s}$$
$$= \frac{100(0.1s + 20)}{s^2 + 200s + 20\cdot 10^3} = \frac{100(0.1s + 20)}{(s + 100 - j100)(s + 100 + j100)},$$

where roots of the denominator are $p_{1,2} = -100 \pm j100$. Therefore,

$$A_{1C} = \left|\frac{100(0.1s + 20)}{s + 100 + j100}\right|_{s=-100+j100} = 5 - j5 = 5\sqrt{2}e^{-j45^\circ},$$

and, in accordance with equation 3.73, the inverse Laplace transform will be

$$i_C(t) = 10\cdot\sqrt{2}e^{-100t}\cos(100t - 45^\circ) \text{ A}.$$

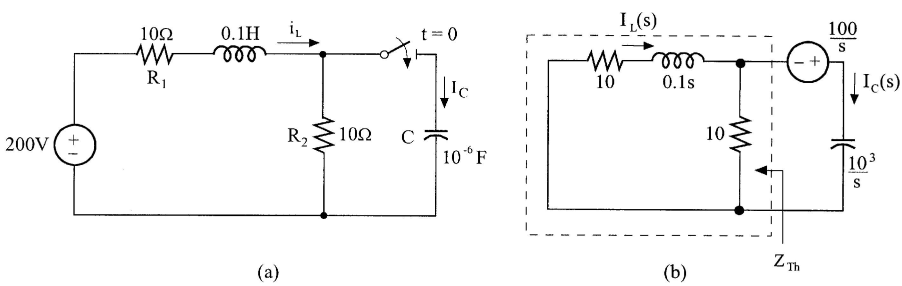


Figure 3.19 Circuit for Example 3.16 (a) and its Thévenin equivalent in *s*-domain (b).

To find the inductor current in circuit Fig. 3.19(b) we first use the current divider formula

$$I_L(s) = I_C(s)\frac{R_2}{R_1 + sL + R_2} = \frac{100(0.1s + 20)\cdot 10}{(s^2 + 200s + 20\cdot 10^3)(0.1s + 20)}$$

$$= \frac{1000}{(s + 100 - j100)(s + 100 + j100)},$$

which yields

$$A_{1L} = \left|\frac{1000}{s + 100 + j100}\right|_{s=-100+j100} = -j5 = 5e^{-j90^\circ},$$

and

$$i_L(t) = 10e^{-100t}\cos(100t - 90^\circ) = 10e^{-100t}\sin 100t \text{ A}.$$

The steady-state value of the inductor current in Fig. 3.19(a), i.e. before the switch is closed:

$$I_L(0\_) = \frac{V_g}{R_1 + R_2} = \frac{200}{10 + 10} = 10 \text{ A}.$$

Therefore, the complete response of the current is

$$i_L(t) = 10 + 10e^{-100t}\sin 100t \text{ A}.$$

Note that initial capacitance current $i_C(0) = 10\sqrt{2}\cos(-45^\circ) = 10$ A is in agreement with its value, which can also be obtained by inspection of the circuit in Fig. 3.19(a):

$$i_C(0) = I_L(0\_) = 10 \text{ A}.$$

This result may also be obtained by straightforward calculation of $i_L(0)$ in accordance with the above formula: $i_L(0) = 10 + 10\cdot e^0 \sin 0 = 10$ A.

When the switch in any branch is opened after having been closed for a long time, as shown in Fig. 3.18(c), the equivalent circuit can be constructed by using a current source insert instead of the switch as shown in Fig. 3.18(d). The value of the current source is equal, and its direction is opposite, to the current flowing through the closed switch (short circuit current) just before its opening. Therefore, the rest of the network is passive, i.e. all the network sources are killed and it can be represented by its Thévenin impedance, as shown in Fig. 3.18(d). It is obvious again that this circuit is having zero initial conditions. For getting the complete response, the ZSR of the circuit in Fig. 3.18(d) has to be superimposed on the previous steady-state regime of the circuit in Fig. 3.18(c).

Example 3.17

In the circuit shown in Fig. 3.20(a), the switch is opened at time $t_1 = 0.2$ s, while the whole circuit has been driven by the voltage source $v_g = 10u(t)$ V since $t =$

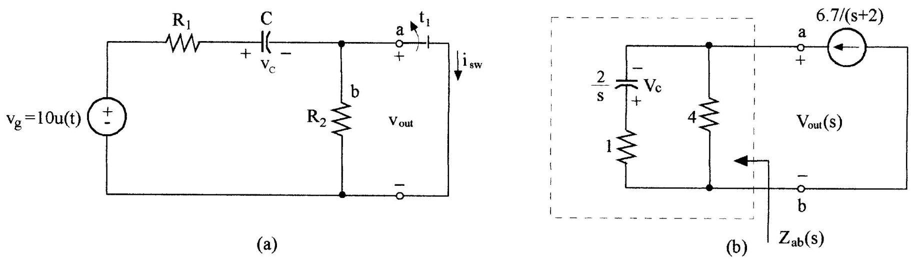


Figure 3.20 Circuit for Example 3.17 (a) and its Norton equivalent in s-domain (b).

0. Let $R_1 = 1\,\Omega$, $R_2 = 4\,\Omega$ and $C = 1/2\,\mathsf{F}$. Find the output voltage v_{out} and capacitance voltage v_C versus time.

Solution

First, we construct the Laplace transform circuit having zero initial conditions. For this purpose, we must find the current through the switch at the time $t = t_1$.

$$i_{sw}(t) = \frac{V_g}{R_1}\, e^{-at} = 10e^{-2t}, \quad t \geq 0,$$

since $a = 1/(R_1 C) = 2\,\mathsf{s}^{-1}$ and $i_{sw}(0) = (10/1) = 10\,\mathsf{A}$.

Changing the variable $t = t_1 + t'$ yields

$$i_{sw}(t') = 10e^{-2t_1}e^{-2t'} = 6.7e^{-2t'}, \quad t' \geq 0,$$

and the transformed current is

$$I_{sw}(s) = 6.7\,\frac{1}{s+2}.$$

Next we calculate the Laplace transform internal impedance measured at the *ab* terminals (see Fig. 3.20(b)).

$$Z_{ab}(s) = \frac{4(1 + 2/s)}{4 + 1 + 2/s} = 0.8\,\frac{s+2}{s+0.4}.$$

The Laplace transform of the output voltage is

$$V_{out}(s) = Z_{ab}(s) I_{sw}(s) = 5.36\,\frac{1}{s+0.4},$$

and taking the inverse transform we obtain

$$v_{out}(t') = 5.36e^{-0.4t'}\,\mathsf{V}, \quad t' \geq 0,$$

since, because the voltage before opening the switch was zero, the complete response is the same. Next, we use voltage division to obtain the expression for the transformed capacitor voltage in Fig. 3.20(b):

$$V_C(s) = -V_{out}(s)\frac{2/s}{1+2/s} = -10.72\frac{1}{(s+0.4)(s+2)}.$$

In accordance with the Laplace transform pairs (see Table 3.1) we have

$$v_{C(ZSR)}(t') = \frac{-10.72}{0.4-2}(e^{-2t'} - e^{-0.4t'}) = 67(e^{-2t'} - e^{-0.4t'})\ \mathrm{V}, \quad t \geq 0.$$

To get the complete response, we have to find the previous capacitor voltage, i.e., before the switch was opened (see circuit in Fig. 3.20(a))

$$v_{C(pr)}(t) = 10(1 - e^{-2t}) = (10 - 6.7e^{-2t'})\ \mathrm{V}.$$

Therefore, the complete response is

$$v_C(t') = v_{C(ZSR)} + v_{C(pr)} = 10 - 6.7e^{-0.4t'}\ \mathrm{V}, \quad t' \geq 0.$$

Note that, according to this expression, the capacitor voltage at $t' = 0$ is 3.3 V, which is equal to the capacitor voltage at the moment of the switch commutation in Fig. 3.20(a).

3.8.6 The transients in magnetically coupled circuits

The Laplace transform techniques are also very useful for the analysis of coupled circuits. Consider the magnetically coupled circuits shown in Fig. 3.21(a). The KVL mesh equations are

$$\begin{aligned} R_1 i_1 + L_1\frac{di_1}{dt} + M\frac{di_2}{dt} &= v_1(t) \\ R_2 i_2 + L_2\frac{di_2}{dt} + M\frac{di_1}{dt} &= v_2(t). \end{aligned} \tag{3.97}$$

Assuming non-zero initial conditions, the Laplace transform of equation 3.97

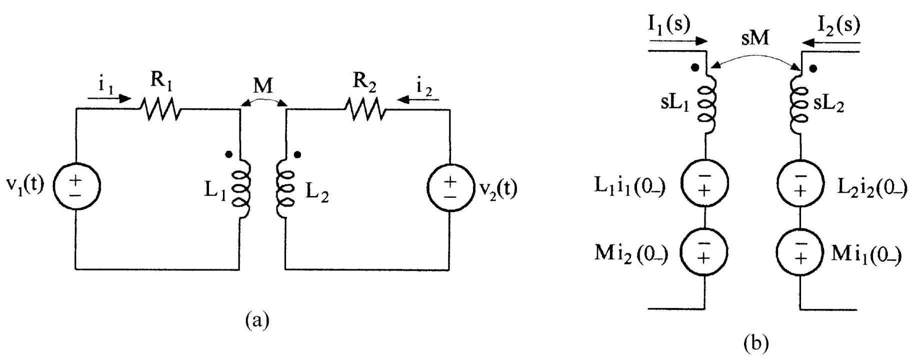


Figure 3.21 Magnetically coupled circuit (a) and its Laplace model (b).

gives

$$\begin{aligned} R_1I_1(s) + sL_1I_1(s) - L_1i_1(0_-) + sMI_2(s) - Mi_2(0_-) &= V_1(s) \\ R_2I_2(s) + sL_2I_2(s) - L_2i_2(0_-) + sMI_1(s) - Mi_1(0_-) &= V_2(s). \end{aligned} \tag{3.98}$$

Combining terms yields

$$\begin{aligned} Z_1(s)I_1(s) + sMI_2(s) &= V_1(s) + L_1i_1(0_-) + Mi_2(0_-) \\ sMI_1(s) + Z_2(s)I_2(s) &= V_2(s) + L_2i_2(0_-) + Mi_1(0_-). \end{aligned} \tag{3.99}$$

The Laplace transform circuit model in Fig. 3.21(b) represents equations equation 3.99 by the s-domain impedances and two voltage sources in each loop. The voltage sources $Mi_1(0_-)$ and $Mi_2(0_-)$ represent the time-domain effect of the initial stored energy in the mutual inductance due to the currents i_1 and i_2. Solving equation 3.99 for $I_1(s)$ and $I_2(s)$ gives

$$I_1(s) = \frac{(V_1(s) + B_1)(s + a_2) - (V_2(s) + B_2)(M/L_2)s}{L_1(1 - k^2)(s^2 + as + b)} \tag{3.100a}$$

$$I_2(s) = \frac{(V_2(s) + B_2)(s + a_1) - (V_1(s) + B_1)(M/L_1)s}{L_2(1 - k^2)(s^2 + as + b)}, \tag{3.100b}$$

where

$$\begin{gathered} a_1 = R_1/L_1, \quad a_2 = R_2/L_2, \quad k = \frac{M}{\sqrt{L_1L_2}}, \\ a = \frac{a_1 + a_2}{1 - k^2}, \quad b = \frac{a_1a_2}{1 - k^2}, \end{gathered} \tag{3.101}$$

and

$$\begin{aligned} B_1 &= L_1i_1(0_-) + Mi_2(0_-) \\ B_2 &= L_2i_2(0_-) + Mi_1(0_-), \end{aligned} \tag{3.102}$$

are initial-condition equivalent generators of the first and the second loops respectively. It is worthwhile noting that in accordance with equation 3.100 the mutual coupled circuit has a second order characteristic equation: $s^2 + as + b = 0$, i.e., the mutual inductance does not increase the order of the circuit response.

Example 3.18

The mutually coupled circuit in Fig. 3.22(a) has $V_{in} = 120\,\mathrm{V}$, $R = 60\,\Omega$, $L = 0.2\,\mathrm{H}$, $M = 0.1\,\mathrm{H}$. The switch is closed at $t = 0$ after having been opened for a long time. Find the currents $i_1(t)$ and $i_2(t)$ for $t \geq 0$.

Solution

First, we must find the initial conditions:

$$i_2(0_-) = 0 \quad \text{and} \quad i_1(0_-) = \frac{V}{R} = 2\,\mathrm{A}.$$

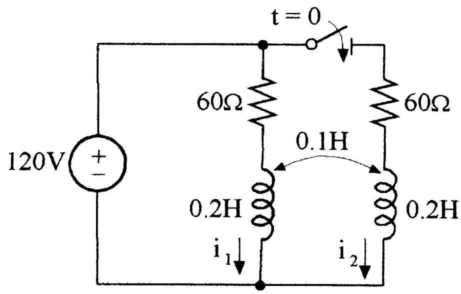


Figure 3.22 Circuit for Example 3.18.

In accordance with equations 3.101 and 3.102

$$a_1 = a_2 = \frac{R}{L} = 300\ \mathrm{s}^{-1}, \quad k = \frac{M}{L} = 0.5, \quad a = \frac{2a_1}{1-k^2} = \frac{600}{1-0.5^2} = 800\ \mathrm{s}^{-1},$$

$$b = \frac{a_1^2}{1-k^2} = \frac{300^2}{1-0.5^2} = 12\cdot 10^4\ \mathrm{s}^{-1}, \quad B_1 = Li_1(0_-) = 0.2\cdot 2 = 0.4,$$

$$B_2 = Mi_1(0_-) = 0.1\cdot 2 = 0.2.$$

With the help of equation 3.100, we obtain the transformed currents

$$I_1(s) = \frac{(120/s+0.4)(s+300) - (120/s+0.2)0.5s}{0.2\cdot 0.75(s^2+800s+12\cdot 10^4)} = \frac{2(s^2+600s+12\cdot 10^4)}{(s^2+800s+12\cdot 10^4)}$$

$$= \frac{2(s^2+600s+12\cdot 10^4)}{s(s+200)(s+600)}$$

$$I_2(s) = \frac{(120/s+0.2)(s+300) - (120/s+0.4)0.5s}{0.2\cdot 0.75(s^2+800s+12\cdot 10^4)} = \frac{800(s+300)}{(s^2+800s+12\cdot 10^4)}$$

$$= \frac{800(s+300)}{s(s+200)(s+600)},$$

i.e., the poles are $p_0 = 0$, $p_1 = -200$, $p_2 = -600\ 1/\mathrm{s}$. Therefore, the appropriate residues are:

$$A_{10} = \lim_{s\to 0} sI_1(s) = \frac{2\cdot 12\cdot 10^4}{12\cdot 10^4} = 2, \qquad A_{20} = 2,$$

$$A_{11} = \left.\frac{2(s^2+600s+12\cdot 10^4)}{s(s+600)}\right|_{s=-200} = -1, \quad A_{12} = -1,$$

$$A_{21} = \left.\frac{2(s^2+600s+12\cdot 10^4)}{s(s+200)}\right|_{s=-600} = -1, \quad A_{22} = -1,$$

which gives the time-domain currents

$$i_1(t) = (2 - e^{-200t} + e^{-600t})\ \mathrm{A}$$

$$i_2(t) = (2 - e^{-200t} + e^{-600t})\ \mathrm{A}.$$

In conclusion, it is worthwhile mentioning that the *Laplace transform technique is also widely used for solving electromechanical problems.* Consider, for example, the starting transients of a no-load shunt exciting d.c. motor (see Fig. 3.23). The torque equation is

$$T = mi = J\frac{d\omega}{dt},$$

where the motor torque T (Nm) is proportional to the current, J (kgm^2) is the moment of inertia and ω (rad/s) is the angular velocity.

The Kirchhoff's-law voltage equation for the motor is

$$V = Ri + L\frac{di}{dt} + k\omega,$$

where the term $k\omega$ is the generated, or back, voltage which is proportional to the angular velocity, and R, L are the resistance and the inductance of the armature winding. With zero-initial conditions the Laplace transform of these two equations will be

$$mI = Js\Omega$$

$$\frac{V}{s} = (R + sL)I + k\Omega,$$

where $\Omega(s)$ and $I(s)$ are the Laplace transform of the angular frequency and the current respectively. Solving the above equations for Ω and I yields

$$\Omega = V\frac{m}{JL}\frac{1}{s\left(s^2 + \frac{R}{L}s + \frac{km}{JL}\right)}$$

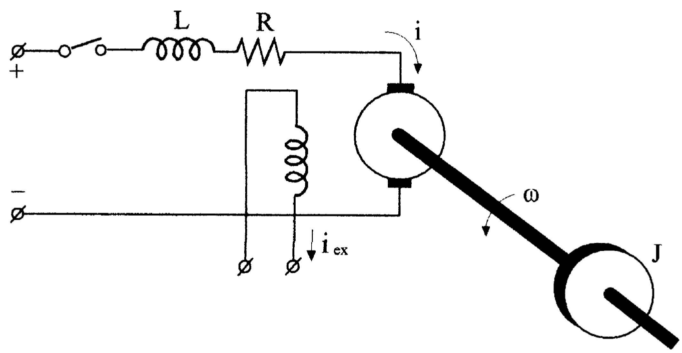


Figure 3.23 An electromechanical system with a shunt exciting d.c. motor.

and

$$I = V\frac{1}{L}\frac{1}{s^2 + \frac{R}{L}s + \frac{km}{JL}}.$$

The roots of the denominator are $s_{1,2} = -\alpha \pm \beta$, where $\alpha = R/2L$ and $\beta = \sqrt{\alpha^2 - (km/JL)}$. Thus, in accordance with the table of Laplace transform pairs, we obtain

$$\omega(t) = \frac{V}{k}\left[1 - \left(\cosh\beta t + \frac{\alpha}{\beta}\sinh\beta t\right)e^{-\alpha t}\right],$$

where $V/k = \omega_0$ is the no-load angular velocity, and

$$i(t) = \frac{V}{\beta L}e^{-\alpha t}\sinh\beta t,$$

where $i(0_-) = 0$ because of zero-initial conditions and $i(\infty) = 0$ since the motor is no-loaded and the losses in this example were neglected.

The condition of oscillations is $R^2 < 4k^2L/J$, and then $s_{1,2} = -\alpha \pm j\beta$.

Chapter #4

TRANSIENT ANALYSIS USING THE FOURIER TRANSFORM

4.1 INTRODUCTION

Like the Laplace transform, discussed in the previous chapter, the Fourier transform is very useful for transient analysis of electrical circuits. The Fourier transform, just as the Laplace transform, converts a function of time (time-domain function) into a function of frequency (frequency-domain function). However, in distinction to the Laplace transform, the Fourier transform transforms the time functions into a function of $j\omega$, a pure imaginary frequency, rather than a function of $s = c + j\omega$, which is a complex frequency. (More about the relation between Fourier and Laplace transforms further on.) From another point of view, the Fourier transform extends the Fourier series, which represents any **periodic** (but not **non-periodic)** function by an infinite sum of harmonics of different frequencies. The Fourier coefficients of such harmonics are functions of multiple $n\omega_0$ of a basic frequency ω_0, and are therefore discrete quantities corresponding to the integer n.

However, in circuit analysis there are many cases in which the forcing functions are non-periodic: such as different kinds of pulses and signals in communication engineering systems, or pulses resulting from lightning or some other strokes in power engineering systems. In these cases, as will be seen in this chapter, we may be able to find a Fourier transform $\mathbf{F}(j\omega) = |\mathbf{F}(j\omega)|e^{j\Psi(\omega)}$ of a **non-periodic** function, whose amplitude, $|\mathbf{F}(j\omega)|$ and phase $\Psi(\omega)$ spectra are continuous rather than discrete, i.e., they are functions of ω (but not of $n\omega$). The conversion of a non-periodic time function to a function of frequency ω allows us to analyze the transient behavior of any linear circuit by using their frequency characteristics, such as: impedance $Z(j\omega)$, admittance $Y(j\omega)$ and/or transfer coefficient $K(j\omega)$. This means that we will be able to use all the methods of steady-state analysis by applying them to the transient analysis, which again will reduce the integro-differential operating in the time domain to more simple algebraic operations in the frequency domain.

Thus, the Fourier transform extends the phasor concept, which has been developed for sinusoidal (periodic) functions to non-periodic functions, which are more general than just sinusoids.

4.2 THE INTER-RELATIONSHIP BETWEEN THE TRANSIENT BEHAVIOR OF ELECTRICAL CIRCUITS AND THEIR SPECTRAL PROPERTIES

The study of transient behavior of electrical circuits in the previous chapter shows that this behavior is largely related to their frequency characteristics. This was especially evident from applying the Laplace transform. Thus, if the voltage was applied to the input of an electric circuit, whose Laplace transform was given as

$$\mathbf{V}_1(s) \leftrightarrow v_1(t),$$

then the Laplace transform of its response, for example of the current, can be found as

$$\mathbf{I}_1(s) = \mathbf{V}_1(s)/\mathbf{Z}(s) = \mathbf{V}_1(s)\mathbf{Y}(s), \tag{4.1a}$$

where

$$\mathbf{Z}(s) = \mathbf{Z}(j\omega)|_{j\omega=s} \quad or \quad \mathbf{Y}(s) = \mathbf{Y}(j\omega)|_{j\omega=s}.$$

Here the complex impedance $\mathbf{Z}(j\omega)$ or the admittance $\mathbf{Y}(j\omega)$ are actually the frequency characteristics of the circuit. In exactly the same way, we can represent the transfer function

$$\mathbf{H}(s) = \mathbf{H}(j\omega)|_{j\omega=s},$$

and with this function the Laplace transform of the output voltage will be

$$\mathbf{V}_2(s) = \mathbf{H}(s)\mathbf{V}_1(s). \tag{4.1b}$$

In finding the response function of the circuit to any forcing function the properties of the circuit are completely determined by its frequency characteristic $\mathbf{Z}(j\omega)$, $\mathbf{Y}(j\omega)$ or $\mathbf{H}(j\omega)$. This relationship between the frequency characteristics of the system and its behavior in transients is obvious when taking into consideration the physical properties of the circuit elements. Thus the inductance, which prevents an abrupt change of current, is characterized by changing, in particular by increasing, its reactance (X) by increasing the frequency; and in the same way the capacitance, which prevents an abrupt change of the voltage, is characterized by changing, in particular by increasing, its susceptance (B) by increasing the frequency. The availability of resonant oscillations during the transients also depends on the characteristics of the system, i.e., its impedance/admittance.

Expressions like equation 4.1 are completely analogous to the phasor expressions for different harmonics in the steady-state analysis. Using the Fourier series in a non-sinusoidal analysis, we might write

$$\mathbf{I}_n = \mathbf{V}_n/\mathbf{Z}(j\omega) = \mathbf{Y}(j\omega)\mathbf{V}_n,$$

where $j\omega = jn\omega_0$ is the discrete frequency of different harmonics: $\mathbf{I}_n = \mathbf{I}(jn\omega_0)$ and $\mathbf{V}_n = \mathbf{V}(jn\omega_0)$. Thus, the discrete spectra of the current, $\mathbf{I}_n$ in accordance with the above expression, can be found if we are able to find the spectra of the forcing function, for instance, $\mathbf{V}_n$ by knowing its Laplace transform $\mathbf{V}(s)$ or straightforwardly by applying the Fourier series coefficient formulas.

However, if the forcing function is non-periodical, which happens in many cases where this function is exponential, rectangular or any kind of pulse, its spectra cannot be found by just replacing s by $j\omega$. (More precisely, as will be shown further on, in some special cases of the above functions, the frequency characteristics can be found anyway by replacing s by $j\omega$.) As is known, using the Fourier series this problem cannot be solved either, since the Fourier series is appropriate only for periodic functions.

Our goal in this chapter, therefore, is to develop a method which allows extending the phasor concept to non-periodic functions. The solution is the Fourier transform, which is, as we already mentioned, an extension of the Fourier series to non-periodic functions.

4.3 THE FOURIER TRANSFORM

4.3.1 The definition of the Fourier transform

Let us proceed to define the Fourier transform by first recalling the spectrum presentation of the periodic function. In the simplest case of one harmonic $A\sin(\omega t + \psi)$ (Fig. 4.1a) its amplitude and phase spectra will be as shown in Fig. 4.1b. Using the complex form of sinusoids,

$$\mathrm{A}\sin(\omega t + \psi) = \frac{A}{2j}\left[e^{j(\omega t+\psi)} - e^{-j(\omega t+\psi)}\right] = \frac{A}{2}\left[e^{j(\omega t+\psi-\pi/2)} + e^{-j(\omega t+\psi-\pi/2)}\right]^{(*)},$$

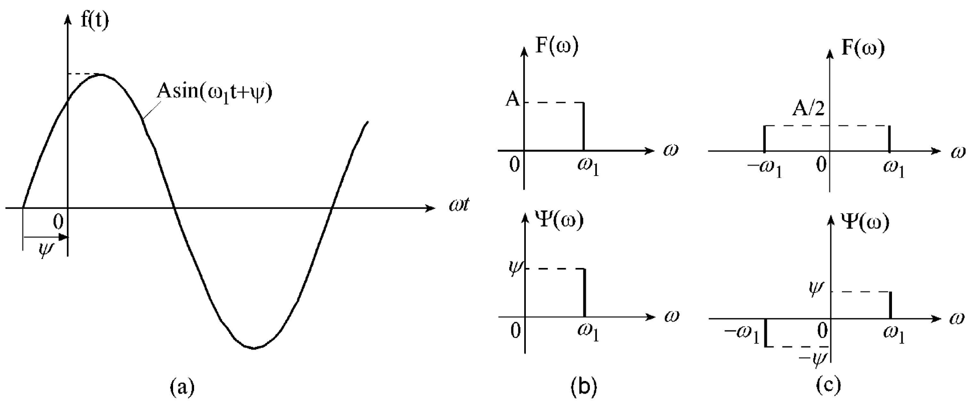


Figure 4.1 Sinusoidal faction (a) and its spectra: in real notation (b) and complex notation (c).

[(*)] Note that in this expression the additional angle $-\pi/2$ appears for a cosine presentation of the sinusoidal function. It should be noted, however, that in this book we are using sine presentation rather than cosine.

we may take into consideration also negative frequencies as in the second term, in the brackets, of this expression. (It has to be mentioned that the definition of a negative frequency has a purely mathematical meaning without any physical connection.) In this case, the amplitude and phase spectra will be as shown in Fig. 4.1(c). As can be seen, both spectra are presented by two ordinates correspondingly for positive and negative frequencies. The amplitude spectrum components are symmetrical about the vertical axis and the phase spectrum components are symmetrical about the origin.

As is known, if any current or voltage wave is not sinusoidal, but periodical, it may be represented by the *infinite Fourier series*. In trigonometrical form it will be:

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty} A_n \sin(n\omega_0 t+\psi_n),$$

where amplitudes A_n and phases ψ_n can be expressed as

$$A_n=\sqrt{a_n^2+b_n^2}, \quad \psi_n=\tan^{-1}\frac{a_n}{b_n},$$

and

$$\begin{aligned} a_0&=\frac{2}{T}\int_0^T f(t)dt,\\ a_n&=\frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\, dt, \\ b_n&=\frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\, dt, \quad n=0,1,2,\ldots. \end{aligned} \tag{4.2}$$

In accordance with the above expressions, the amplitude $A_n=f(n\omega_0)$ and phase $\psi_n=f(n\omega_0)$ spectra of a non-sinusoidal function may be sketched, as functions of $n\omega_0$. With a complex, or exponentional form of the Fourier series:

$$f(t)=\sum_{n=-\infty}^{\infty} \mathbf{C}_n e^{jn\omega_0 t}, \tag{4.3}$$

where

$$\mathbf{C}_n=\frac{1}{2}(a_n-jb_n), \quad \mathbf{C}_{-n}=\frac{1}{2}(a_n+jb_n), \quad \mathbf{C}_0=\frac{1}{2}a_0$$

and (4.3a)

$$\angle\mathbf{C}_n=-\tan^{-1}\frac{b_n}{a_n}; \quad \angle\mathbf{C}_{-n}=-\angle\mathbf{C}_n=\tan^{-1}\frac{b_n}{a_n}.$$

The amplitude $|\mathbf{C}_n|=f(n\omega_0)$ and phase $\angle\mathbf{C}_n=f(n\omega_0)$ spectra will be functions of both positive and negative frequencies: the amplitude spectrum will be symmetrical about the vertical axis, and the phase spectrum will be symmetrical about the origin. Note that both the amplitude and phase spectra are discrete functions of harmonic frequencies.

Coefficients $\mathbf{C}_n$ can be determined by substituting the expressions for a_n and b_n (equation 4.2) into equation 4.3a by changing the limits of integration, i.e.,

$$\mathbf{C}_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)(\cos n\omega_0 t - j\sin n\omega_0 t)dt = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt.$$

An example of such *discrete spectra of a periodic function* is a train of rectangular pulses, having duration d, period T and amplitude V_0, which is shown in Fig. 4.2(a) and its spectra are shown in Fig. 4.2(b) and (c). Here the magnitudes are $|\mathbf{C}_n| = \varepsilon V_0 Sa(n\varepsilon\pi)$, where $Sa(x)$ is called a **sampling function** or **sinc function**

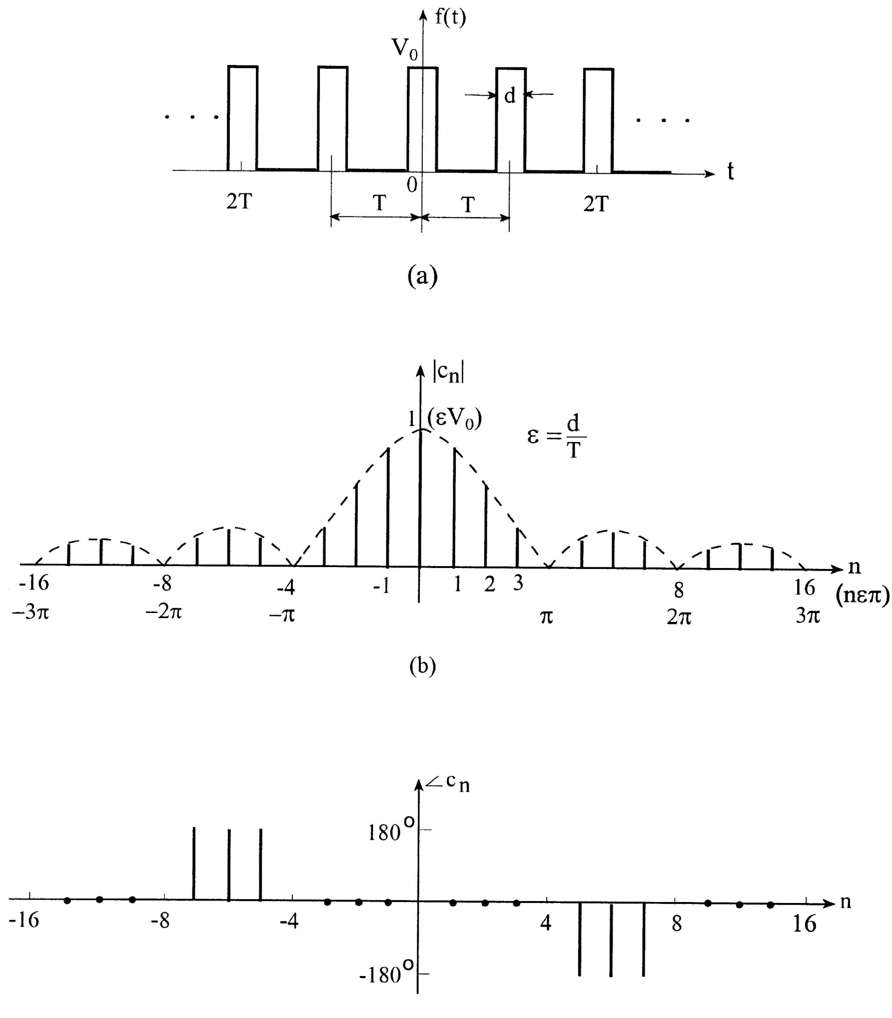


Figure 4.2 A train of rectangular pulses (a), its discrete amplitude (b) and phase (c) spectras.

(in mathematics: sinc $x = (\sin \pi x)/\pi x = \mathrm{Sa}(\pi x)$) and it might be calculated with most mathematical programs like MATHCAD, MATHLAB etc).

However, there are many important forcing functions that are not periodic functions, such as a single rectangular pulse, an impulse function, a step function and a rump function. Another example of a non-periodic function is an impulse voltage waveform, which appears in high-voltage transmission lines, when a stroke of lightning influences the line conductors(*).

Frequency spectra may also be obtained for such non-periodic functions; however, they will be continuous spectra, rather than discrete. These spectra can be obtained by using the **Fourier transform**, which is an extension of the **Fourier series** for non-periodic functions. With such spectra we will be able to extend the frequency analysis and the phasor concept to non-periodic functions.

Thus, the Fourier transform, in contrast to the Fourier series, is a function of the continuous frequency ω (but not of discrete frequency $n\omega$) and corresponds to the time-domain non-periodic function. To develop the Fourier transform technique we shall consider the non-periodic function $f(t)$, Fig. 4.3a, as defined on an infinite interval.

This function should satisfy the Dirichlet conditions: in any finite interval, $f(t)$ has at most a finite number of finite discontinuities, a finite number of maxima and minima and $\int_{-\infty}^{\infty} |f(t)|dt < \infty$, i.e., the integral converges. To be able to extend the use of the Fourier series to a non-periodic function we will define a new function $f_{per}(t)$ which is identical to $f(t)$ on the interval

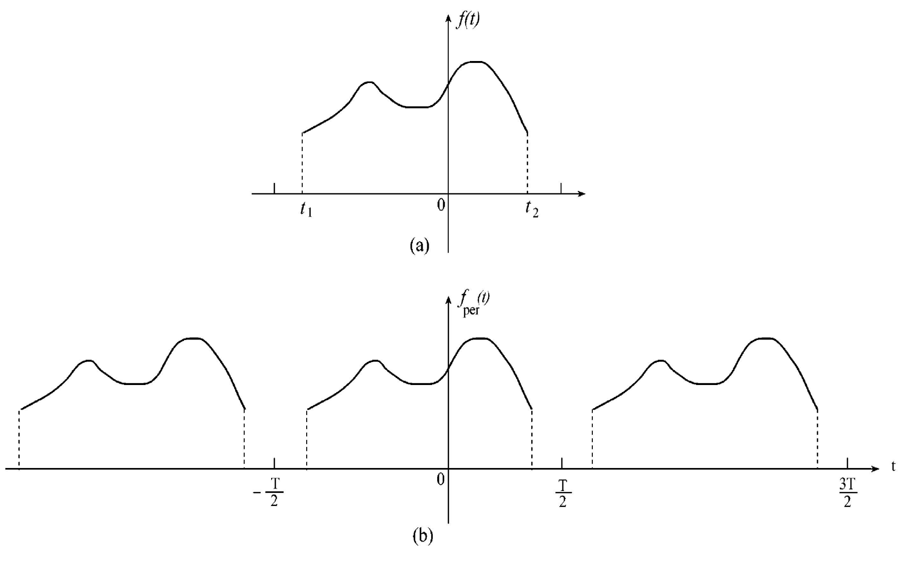


Figure 4.3 A non-periodic function (a) and its periodic extension (b).

(*)Gonen, T. (1988) *Electric Power Transmission System Engineering*. Wiley, New York, Chichester, Brisbane, Toronto, Singapore.

$-T/2 < t < T/2$ and is periodic of any period $T > t_2 - t_1$ as shown in Fig. 4.3b. Such a function $f_{per}(t)$ is said to be the *periodic extension of* $f(t)$ and might be represented by the Fourier series. The given non-periodic function $f(t)$, therefore, is also given by the same Fourier series, but only in the interval $(-T/2, T/2)$. Outside of this interval, this function cannot be represented by the Fourier series. Using the exponential form of the Fourier series for f_{per}(t) we will have

$$f_{per}(t) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \mathbf{C}_n e^{jn\omega_0 t}, \tag{4.5}$$

where

$$\mathbf{C}_n = \frac{1}{T} \int_{-T/2}^{T/2} f_{per}(t) e^{-jn\omega_0 t} dt. \tag{4.6}$$

Our intention is to let $T \to \infty$, in which case

$$f_{per}(t) \to f(t). \tag{4.7}$$

We will then have extended the Fourier series concept to the non-periodic function $f(t)$ by considering it to be periodic with an infinite period. Since now $T \to \infty$ and then $\omega_0 = 2\pi/T$ becomes vanishingly small, we may represent this limit by a differential, i.e., $\omega_0 \to d\omega$ so that:

$$\frac{1}{T} = \frac{\omega_0}{2\pi} \to \frac{d\omega}{2\pi} \quad (\text{when } T \to \infty). \tag{4.8}$$

Now, the harmonic discrete frequency $n\omega_0$ will approach the continuous frequency variable ω, since ω_0 becomes vanishingly small $(\omega_0 \to 0)$ and all the nearby frequencies approach a smoothly changing frequency ω_0. In other words, n tends to infinity as ω_0 approaches zero, such that the product is finite:

$$n\omega_0 \to \omega. \tag{4.9}$$

Substituting equation 4.6 into equation 4.5, and taking into consideration equations 4.7 and 4.9, we may obtain

$$f(t) = \frac{1}{2\pi} \sum \frac{2\pi}{T} \left[\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \right] e^{j\omega t}. \tag{4.10}$$

The inner integral (in brackets) is a function of $j\omega$ (not of t) and we assign it to $\mathbf{F}(j\omega)$, so that

$$\mathbf{F}(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt, \tag{4.11}$$

and it is the **Fourier transform** of $f(t)$. Then as

$$T \to \infty \left(\frac{1}{T} \to \frac{d\omega}{2\pi} \ (\text{equation 4.8}) \right) \quad \text{and} \quad \frac{2\pi}{T} \to d\omega,$$

the sum in equation 4.10 becomes an integral:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{F}(j\omega)e^{j\omega t}\,d\omega, \tag{4.12}$$

which is called the **inverse Fourier transform**.

These two expressions above are known as the Fourier transform pair

$$\begin{cases} \mathbf{F}(j\omega) = \displaystyle\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt & \text{(4.13a)} \\ f(t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \mathbf{F}(j\omega)e^{j\omega t}\,d\omega, & \text{(4.13b)} \end{cases}$$

which are also often stated symbolically as

$$\begin{cases} \mathbf{F}(j\omega) = \mathscr{F}[f(t)] & \text{(a)} \\ f(t) = \mathscr{F}^{-1}[\mathbf{F}(j\omega)], & \text{(b)} \end{cases} \tag{4.14}$$

where $\mathscr{F}$ denotes the operation of taking the Fourier transform. These two expressions in equation 4.14 may also be indicated as

$$f(t) \leftrightarrow \mathbf{F}(j\omega). \tag{4.15}$$

The Fourier transform as seen in equation 4.13a is a transformation of the function $f(t)$ from the *time domain* to the *frequency domain* and corresponds to the Fourier coefficient expressions in equation 4.3a. Equation 4.13b, the inverse transform, is an opposite transformation of the complex function $\mathbf{F}(j\omega)$ from the *frequency domain* into the *time domain* and is a direct analogy to the Fourier series (equation 4.3). Another explanation of these two analogies is to say that the Fourier transform is a *continuous* representation (with ω being a continuous variable) of a non-periodic function, whereas the Fourier series is a *discrete* representation (with $n\omega_0$ being a discrete variable) of a periodic function. Finally, it must be indicated that the Fourier transform-pair relationship is unique: for a given function $f(t)$ there is one specific $\mathbf{F}(j\omega)$ and for a given $\mathbf{F}(j\omega)$ there is one specific $f(t)$.

The following examples show how we can use the above-developed expressions to find the Fourier transform of a non-periodic function and its spectra.

Example 4.1

Let us find the Fourier transform of the exponential function

$$f(t) = e^{-at}u(t) \quad (a > 0). \tag{4.16}$$

Using equation 4.13a we have

$$\mathscr{F}[e^{-at}u(t)] = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt = \frac{1}{-(a+j\omega)}\,e^{-(a+j\omega)}\Big|_{0}^{\infty}.$$

Because $a > 0$, the upper limit results in zero (since the imaginary part $e^{-j\omega t}$ represents the rotation features of the exponential amplitude e^{-at} and is therefore bound while the exponential approaches 0)[(*)]. Thus, we have

$$\mathscr{F}[e^{-at}u(t)] = \frac{1}{a + j\omega}, \tag{4.17a}$$

or

$$e^{-at}u(t) \leftrightarrow \frac{1}{a + j\omega} \quad (a > 0), \tag{4.17b}$$

and

$$\mathbf{F}(j\omega) = \frac{1}{a + j\omega}. \tag{4.17c}$$

In accordance with the obtained expressions (14.17) the amplitude spectra of an exponential function will be

$$[\mathbf{F}(j\omega)] = \frac{1}{\sqrt{a^2 + \omega^2}}, \tag{4.18}$$

and its phase spectrum

$$\Psi(\omega) = -\tan^{-1}\frac{\omega}{a}. \tag{4.19}$$

The exponential function (14.16) and its spectra (14.18) and (14.19) are shown in Fig. 4.4a–c.

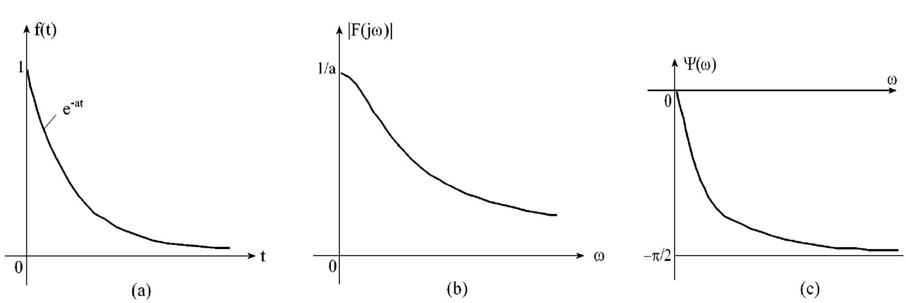


Figure 4.4 Exponential function (a) and its amplitude (b) and phase (c) spectra.

[(*)]Note that the function e^{-at} $(a > 0)$ does not have a Fourier transform, since the integral $\int_{-\infty}^{\infty} e^{-at}\, e^{-j\omega t}dt$ of its lower limit approaches infinity (infinite value).

Example 4.2

As another example, let us find the Fourier transform of the single *rectangular pulse.*

$$f(t)=\begin{cases} V_0 & -\dfrac{d}{2}<t<\dfrac{d}{2} \\ 0 & -\dfrac{d}{2}>t>\dfrac{d}{2}, \end{cases}$$

which is shown in Fig. 4.5(a). By the definition in equation 4.13a we have

$$\mathbf{F}(j\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = V_0\int_{-d/2}^{d/2} e^{-j\omega t}dt = \frac{V_0}{-j\omega}\left(e^{-j(\omega d/2)}-e^{j(\omega d/2)}\right), \quad (4.20a)$$

$$\mathbf{F}(j\omega)=\frac{2V_0}{\omega}\left(\frac{e^{j(\omega d/2)}-e^{-j(\omega d/2)}}{j2}\right)=\frac{2V_0}{\omega}\sin\frac{\omega d}{2}=V_0 d\,\frac{\sin(\omega d/2)}{\omega d/2}, \quad (4.20b)$$

or shortly

$$f(t)=\left\{\text{at time} \quad \begin{matrix} \dfrac{d}{2}<t<\dfrac{d}{2}: & V_0 \\ -\dfrac{d}{2}>t>\dfrac{d}{2}: & 0 \end{matrix}\right\} \leftrightarrow V_0 d\ \mathrm{Sa}\left(\frac{\omega d}{2}\right), \quad (4.20c)$$

where *Sa*($\omega d/2$) is the sampling function. This function yields the **continuous spectrum** of a rectangular pulse (Fig. 4.5) which is shown in Fig. 4.5(b). Whenever $\omega d/2 = k\pi$ ($k = 1, 2, \ldots$) or the frequencies $\omega = 2\pi/d, 4\pi/d, \ldots$ the above spectrum curve crosses the ω-*axis*, i.e., is zero. For $\omega d/2 = (\pi/2)(2k+1)$ the spectrum curve reaches the maximum points which are $\mathbf{F}(j\omega)|_{\max} = V_0 d/(\pi/2)(2k+1)$. Note that in this case the phase spectrum equals zero.

We should again emphasize that this spectrum is a continuous function of ω as opposed to the *discrete spectrum* of a periodic sequence of rectangular pulses (as shown in Fig. 4.2). Note also that the value of the continuous spectrum in

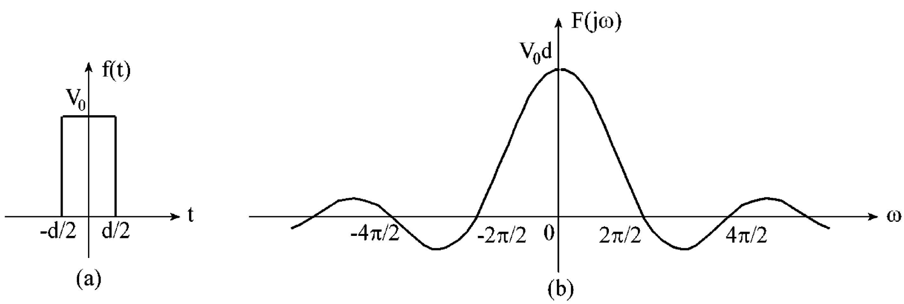


Figure 4.5 A rectangular pulse (a) and its spectrum (b)

equation 4.20 is as a product of the value of the pulse and the value of its duration, i.e., dimensionally it is indicated as "*volts times seconds*", or "*volts per unit frequency*". In the case of a discrete spectrum the value of its magnitude is given just in the same dimension as a periodic function (volts, or amperes). In order to better understand the above differences we should analyze more deeply the relationship between the Fourier transform and Fourier series and look into some of the properties of $\mathbf{F}(j\omega)$.

4.3.2 Relationship between a discrete and continuous spectra

To define the relationship between the discrete spectra of a non-sinusoidal periodic function and the continuous spectra of a non-periodic function we should use the complex form of the Fourier series. In Table 4.1 the basic formulas of the Fourier series and of the Fourier transform are given.

As was previously mentioned, any periodic function has discrete or line spectra for both its magnitude and phase. However, as the period increases, the lines of the discrete spectra become more dense with more lines. In Fig. 4.6 the changing of the magnitude spectrum of a train of rectangular pulses by increasing its period, which is the same as decreasing the scaled duration $\varepsilon = d/T$, is shown. The solid line, or the envelope of the magnitude spectrum, in this figure crosses the frequency axis at the $n\omega_0 d/2 = \pi k$, or $n\omega_0 = 2\pi k/d$ ($n = k/\varepsilon$, $k = \pm 1, \pm 2, \ldots$), which means that the zero point of the envelope does not depend on T, but only on the duration d. However, when period increases, the number of lines, N, between the origin and the first zero, which is the same as between two adjoining zeros, increases directly proportional to T. Note also that the product $n\omega_0$ remains the same as can be seen from Fig. 4.6.

This number of lines might be calculated as

$$N = n - 1 = \frac{2\pi}{\omega_0 d} - 1 = \frac{1}{\varepsilon} - 1. \tag{4.21}$$

The line magnitudes of the discrete spectrum are inversely proportional to the period or directly proportional to the scaled duration ε, as can be seen from

Table 4.1 Basic formulas of Fourier series and Fourier transform

Fourier series (periodic function)	Fourier transform (non-periodic function)
$f(t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$	$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{F}(j\omega)e^{j\omega t}d\omega$
$\mathbf{C}_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt$	$\mathbf{F}(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$
$\mathbf{C}_n = c_n e^{j\psi_n}$	$\mathbf{F}(j\omega) = \lvert\mathbf{F}(j\omega)\rvert e^{j\psi(\omega)}$

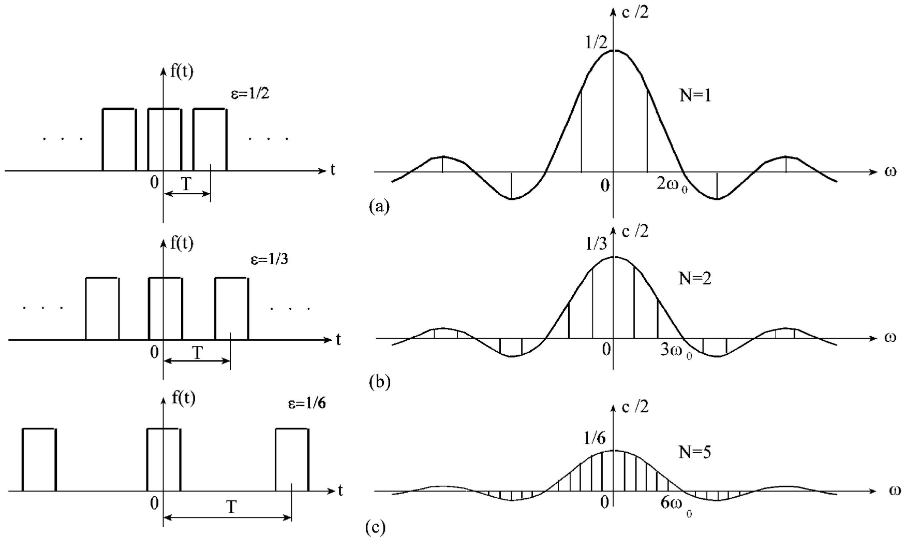


Figure 4.6 The changing of the spectrum as duration ε decreases.

expression (4.4) and Fig. 4.6, i.e. the broken-line curve is lower and approaches zero, as $T \to 0$, and finally coincides with the horizontal axis.

If we multiply the Fourier coefficients $1/2\mathbf{C}_n$ by period T and then let the period become infinite ($T \to \infty$), then the frequency interval $\omega_0 = 2\pi/T$ between the lines of the discrete spectrum approaches zero and the discrete spectrum will turn into a continuous spectrum of a non-periodic rectangular pulse of duration d. On the other hand the Fourier coefficients, being multiplied by period T, become non-dependent on the period and their magnitudes will not change, i.e. the envelope curve is not dependent on T and follows the expression

$$\mathbf{F}(j\omega) = \int_{-d/2}^{d/2} e^{-j\omega t} dt = d\,\frac{\sin(\omega d/2)}{\omega d/2}.$$

The above explanation is illustrated in Fig. 4.7.

It is obvious that the above example of a rectangular pulse can be generalized to any other kind of non-periodic function. In this case it is always possible to choose such a T, that

$$\mathbf{F}(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt = \int_{-T/2}^{T/2} f(t)e^{-j\omega t} dt. \tag{4.22}$$

The periodic function, which coincides with the above non-periodic function $f(t)$ in the interval $-T/2$, $T/2$ and is of period T will have the line spectrum in accordance with the equation

$$\frac{1}{2}\mathbf{C}_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t} dt. \tag{4.23}$$

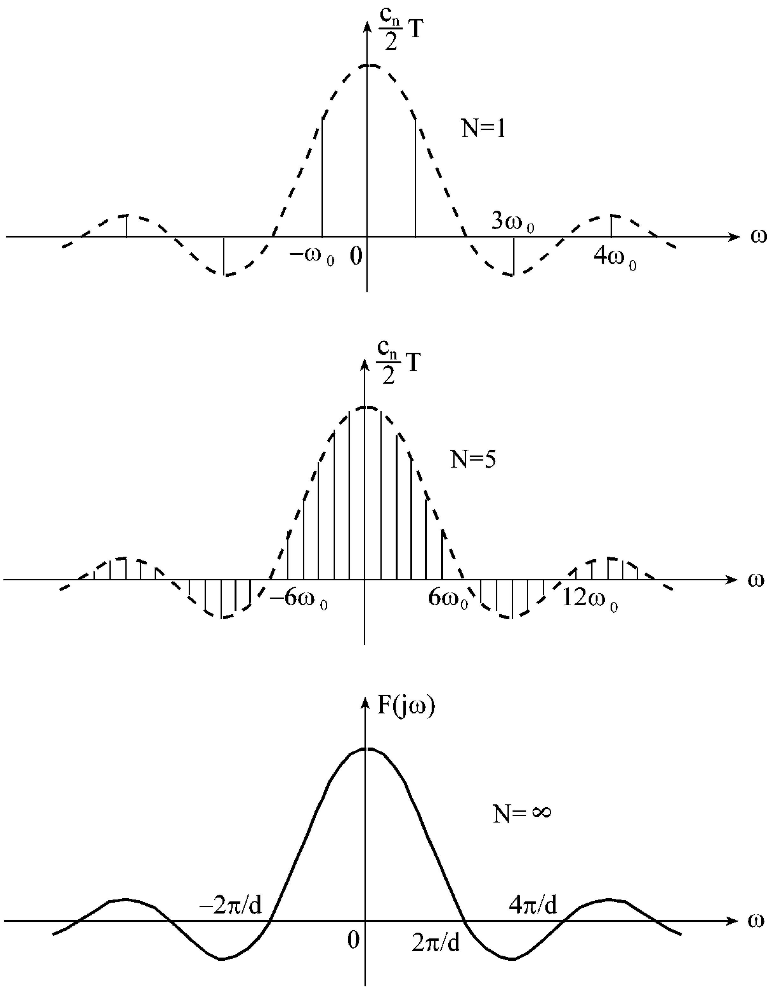


Figure 4.7 The transformation of a discrete spectrum into a continuous one.

By comparing these two equations 4.22 and 4.23 one can conclude that the continuous spectrum $\mathbf{F}(j\omega)$ (4.22) of the non-periodic function, being scaled by $1/T$, is identical to the envelope of the line spectrum (4.23) of the periodically repeated given function:

$$\frac{1}{2}\mathbf{C}_n \bigg/ \frac{1}{T} = \frac{T}{2}\mathbf{C}_n = \int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt = \mathbf{F}(j\omega)|_{\omega=n\omega_0}. \tag{4.24}$$

From the above it also follows that the phase spectrum:

$$\Psi_n = \Psi(\omega)|_{\omega=n\omega_0}. \tag{4.25}$$

Since the continuous spectrum is actually a line spectrum scaled by frequency

$1/T$, its measurement is in the unit of a function multiplied by a unit of time, as was previously mentioned.

4.3.3 Symmetry properties of the Fourier transform

Our objective in this section is to establish several of the symmetrical properties of the Fourier transform in order to use them in our further studies. Replacing $e^{-j\omega t}$ in equation 4.13a by trigonometric functions, using Euler's identity, we will get

$$\mathbf{F}(j\omega) = \int_{-\infty}^{\infty} f(t) \cos \omega t \, dt - j \int_{-\infty}^{\infty} f(t) \sin \omega t \, dt \tag{4.26}$$

All the functions, $\cos \omega t$ and $\sin \omega t$, are real functions of time, therefore both integrals in equation 4.26 are real functions of ω. Thus, we may write

$$\mathbf{F}(j\omega) = A(\omega) - jB(\omega) = |\mathbf{F}(j\omega)| e^{j\phi(\omega)}, \tag{4.27}$$

where

$$A(\omega) = \int_{-\infty}^{\infty} f(t) \cos \omega t \, dt \tag{4.27a}$$

$$B(\omega) = \int_{-\infty}^{\infty} f(t) \sin \omega t \, dt \tag{4.27b}$$

and

$$|\mathbf{F}(j\omega)| = \sqrt{A^2(\omega) + B^2(\omega)} \tag{4.27c}$$

$$\phi(\omega) = \tan^{-1} \frac{-B(\omega)}{A(\omega)}. \tag{4.27d}$$

Replacing ω by $(-\omega)$ shows that $A(\omega)$ and $|\mathbf{F}(j\omega)|$ are both even functions of ω, and $B(\omega)$ and $\phi(\omega)$ are both odd functions of ω. Let us now consider three cases.

(a) *Function* f(t) *is an even function of* t

As is known, an even function is symmetrical about the vertical axis, and an odd function is symmetrical about the origin. Since the *cosine* and *sine* are even and odd functions of t respectively, then $f(t) \cos \omega t$ is an even function and $f(t) \cos \omega t$ is an odd function of t. Therefore the integral of symmetrical limits in equation 4.27b is zero, i.e., $B(\omega) = 0$ and

$$\mathbf{F}(j\omega) = A(\omega) = \int_{-\infty}^{\infty} f(t) \cos \omega t \, dt = 2 \int_{0}^{\infty} f(t) \cos \omega t \, dt. \tag{4.28}$$

With those results we may conclude that the Fourier transform of an even function is a real even function of ω and the phase function in (4.27d) is zero or π for all ω. Replacing $e^{-j\omega t}$ in (4.13b) by trigonometrical functions, yields

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{F}(j\omega)\cos\omega t\,d\omega + j\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{F}(j\omega)\sin\omega t\,d\omega. \tag{4.29}$$

Since $\mathbf{F}(j\omega)$ is a real and even function of ω, the second integrand is an odd function of ω which results in a zero imaginary part of equation 4.29. Thus, in this case

$$f(t) = \frac{1}{\pi}\int_{0}^{\infty} A(\omega)\cos\omega t\,d\omega. \tag{4.30}$$

Comparing the equations 4.28 and 4.30, we may see that the arguments ω and t might be interchanged, i.e., considering

$$F(jt) = F(-jt)$$

(since it is an even function) as a function of t, then its spectrum should be $f(\omega) = f(-\omega)$ as shown in Fig. 4.8 (a and b).

(b) *Function* f(t) *is an odd function of* t

In this case the function $f(t)\cos\omega t$ is an odd function of t and $f(t)\sin\omega t$ is an even function of t. Therefore, the integral in equation 4.27a is zero, i.e. $A(\omega) = 0$ and

$$\mathbf{F}(j\omega) = -jB(j\omega) = -j2\int_{0}^{\infty} f(t)\sin\omega t\,dt, \tag{4.31}$$

i.e., $F(j\omega)$ is a pure imaginary and odd function of ω and therefore $\phi(\omega)$ is

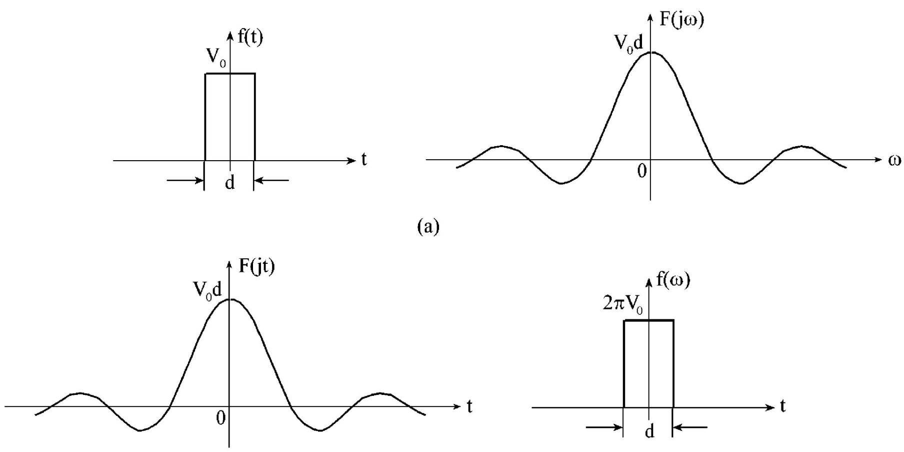


Figure 4.8 Interchange between the function and its spectrum: (a) a rectangular pulse $f(t)$ and its spectrum $F(j\omega)$, (b) a rectangular pulse spectrum $f(\omega)$ of the time sinc function.

$\pm\pi/2$. The function $F(j\omega)\cos\omega t$ is an odd function of ω, therefore the first integral in equation 4.29 turns into zero and

$$f(t) = \frac{1}{\pi}\int_0^{\infty} B(\omega)\sin\omega t\, dt. \tag{4.32}$$

The interchanging properties of the time function and its spectrum are applicable also in this case, i.e. consideration of the function

$$F(-jt) = -F(jt)$$

as a function of time yields its spectrum as $f(\omega)$.

(c) *Function* f(t) *is a non-symmetrical function, i.e., neither even nor odd*

Any non-symmetrical function can be presented as the sum of an even and odd function, i.e.,

$$f(t) = f_e(t) + f_o(t).$$

However,

$$f(-t) = f_e(t) - f_o(t),$$

which means that such a function does not obey either an even or odd function definition. Performing summation and subtraction of the above expression we obtain

$$f_e(t) = \frac{1}{2}[f(t) + f(-t)], \quad f_o(t) = \frac{1}{2}[f(t) - f(-t)].$$

With this result of splitting a non-symmetrical function into two subfunctions: even and odd, we may prove that in this case $\mathbf{F}(j\omega)$ is a complex function, whose real part is even while the imaginary part is an odd function of ω. Finally, we note that the replacement of ω by $-\omega$ in equation 4.27 gives the conjugate complex of $\mathbf{F}(j\omega)$, i.e.,

$$\mathbf{F}(-j\omega) = A(\omega) + jB(\omega) = \overset{*}{\mathbf{F}}(j\omega),$$

and we have

$$\mathbf{F}(j\omega)\mathbf{F}(-j\omega) = \mathbf{F}(j\omega)\overset{*}{\mathbf{F}}(j\omega) = |\mathbf{F}(j\omega)|^2 = A^2(\omega) + B^2(\omega). \tag{4.33}$$

4.3.4 Energy characteristics of a continuous spectrum

If $f(t)$ is a periodic function of either the voltage across or the current through a circuit, then $(1/T)\int_0^T f^2(t)dt$ is proportional to the average power delivered to this circuit. With a complex form of the Fourier series, applied to a non-sinusoidal function, we obtain

$$\frac{1}{T}\int_0^T f^2(t)dt = \sum_{n=-\infty}^{\infty}\left(\frac{\mathbf{C}_n}{2}\right)^2, \tag{4.34}$$

which might be interpreted as the sum of the powers of all the amplitude spectrum components of a given function(*). In accordance with the previously explained relationship between the discrete and continuous spectra (para. 4.3.2), we can easily obtain an expression similar to equation 4.34, but for the non-periodic function $f(t)$. For this purpose we first multiply equation 4.34 by T and replace $(1/2)\mathbf{C}_n$ by $(1/T)|\mathbf{F}(j\omega)|_{\omega=n\omega_0}$:

$$\int_0^T f^2(t)dt = \frac{1}{T}\sum |\mathbf{F}(j\omega)|^2_{\omega=n\omega_0}. \tag{4.35}$$

Now, when $T \to \infty$ and $n\omega_0 = \omega$, then

$$\frac{1}{T} = \frac{\omega_0}{2\pi} = \left.\frac{\omega/n}{2\pi}\right|_{n\to\infty} \to \frac{d\omega}{2\pi}$$

and replacing the sum in equation 4.35 by the integral, we obtain

$$\lim_{T\to\infty} \sum_{n=-\infty}^{\infty} \frac{1}{T}|\mathbf{F}(j\omega)|^2_{\omega=n\omega_0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathbf{F}(j\omega)|^2 d\omega. \tag{4.36}$$

or, finally

$$\int_{-\infty}^{\infty} f^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathbf{F}(j\omega)|^2 d\omega, \tag{4.37}$$

This equation is a very useful expression known as *Parseval's theorem*, which confirms the connection between the energy associated with $f(t)$ and its spectrum. In other words, equation 4.37 shows that *the energy of the signal can be calculated either by an integration over all the time of applying the signal in the time domain or by an integration over all the frequencies in the frequency domain.*

In accordance with this theorem, we are able to calculate the energy associated with any bandwidth of a given function by integrating $|\mathbf{F}(j\omega)|^2$ over an appropriate frequency interval, i.e., that portion of the total energy lying within the chosen interval, or *energy density* (J/Hz). In other words, the shape of $|\mathbf{F}(j\omega)|^2$ gives the "picture" of energy distribution in the spectrum of a non-periodic function, as is shown, for example, in Fig. 4.9. For instance, 90% of the total energy of a rectangular pulse is concentrated into the frequency interval from $\omega = 0$ to $\omega = 2\pi/d$. The narrower the pulse the wider the bandwidth interval, where most of the energy is concentrated.

In the physical world, we may find examples of this phenomenon. For instance, a lightning stroke, which is of very short duration, produces observable signal frequencies over the entire communication spectrum from the relatively low frequencies used in radio reception to the considerably higher ones used in television reception.

(*)The above equation is also known by the statement that the power delivered to the circuit by a non-sinusoidal function is equal to the algebraic sum of the powers of all the harmonics, which represent this function.

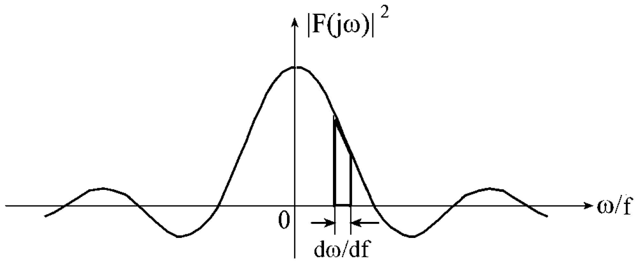


Figure 4.9 The amount of energy $|\mathbf{F}(j\omega t)|^2 d\omega$ associated with $f(t)$ lying in the bandwidth $d\omega$.

Example 4.3

As an example of using Parseval's theorem, let us assume that a 5 kV impulse of rectangular form, shown in Fig. 4.5(a), is applied to the input of an electrical circuit. Let us find the energy delivered to the circuit if $R_{in} = 1\ \Omega$ and the duration of the impulse $\tau = 2$ ms.

Solution

The Fourier transform of such an impulse in accordance with equation 4.20b is

$$\mathbf{F}(j\omega) = V_0\tau\ \mathrm{Sa}\left(\frac{\omega\tau}{2}\right) = V_0\tau\,\frac{\sin(\omega\tau/2)}{\omega\tau/2},$$

which in this case is pure a real function. Using Parseval's theorem, we have

$$W_{1\Omega} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j)|^2 d\omega = \frac{(V_0\tau)^2}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin(\omega\tau/2)}{\omega\tau/2}\right)^2 d\omega.$$

By changing the variable $x = \omega\tau/2$ we have $d\omega = (2/\tau)dx$ and

$$W_{1\Omega} = \frac{V_0^2\tau}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin x}{x}\right)^2 dx = \frac{25\cdot 10^6\cdot 2\cdot 10^{-3}}{\pi}\,3.142 = 50\ \mathrm{kJ}^{(*)}.$$

The same might be calculated straightforwardly

$$W_{1\Omega} = \int_{-\infty}^{\infty} v^2(t)dt = 25\cdot 10^6\cdot t\Big|_{-\tau/2}^{\tau/2} = 25\cdot 10^6\cdot 2\cdot 10^{-3} = 50\ \mathrm{kJ}.$$

.

(*)The value of the integral in this expression can be calculated with computer programs like MATLAB, MATCAD or tables of integrals.

4.3.5 The comparison between Fourier and Laplace transforms (similarities and differences)

As was shown in section 3.2 the one-sided Laplace transform is a function of s:

$$\mathbf{F}(s) = \int_0^{\infty} f(t)e^{-st}\,dt, \tag{4.38}$$

where s is a complex argument with a real part c and an imaginary part ω, i.e., $s = c + j\omega$. It also was shown that the Laplace transform exists only if the integral in equation 4.38 converges, i.e., the function $f(t)$ is restricted:

$$|f(t)| < Me^{\alpha t} \quad \text{while} \quad \alpha < c < \infty. \tag{4.38a}$$

The Fourier transformation is defined over the entire time and not just for the positive values of time. However, in the circuit analysis, as was previously mentioned, the forcing functions and their responses are usually initiated at $t = 0$. Therefore, for such functions the Fourier transform (equation 4.13a) might be written as

$$\mathbf{F}(j\omega) = \int_0^{\infty} f(t)e^{-j\omega t}\,dt. \tag{4.39}$$

Comparing the above two equations 4.38 and 4.39, we may find that, by assuming in the Laplace transform (equation 4.38) that $c = 0$ and $s = j\omega$, both transforms are quite similar. However, the integral in equation 4.39 converges, if

$$|f(t)| < Me^{\alpha t} \quad \text{while} \quad \alpha < 0. \tag{4.39a}$$

This restriction is stronger than equation 4.38a, and means that the given function $f(t)$ does not exceed some exponentially decreasing functions. Some of the functions useful in circuit analysis do not meet this condition. For instance, functions such as unit functions, ramp functions, increasing exponential functions, and periodic functions belong to this category. For the function which does possess condition (equation 4.39a) we may find the Fourier transform by just replacing s by $j\omega$ in the Laplace transform, i.e.,

$$\mathbf{F}(j\omega) = \mathbf{F}(s)|_{s=j\omega}. \tag{4.40}$$

This way of finding the Fourier transform or function spectra for most of the non-periodic functions is the simplest and most convenient one, i.e., for this purpose we can simply use the Table of Laplace transform pairs (see Table 3.1).

The inverse Fourier transform (equation 4.13b) is also similar to the inverse Laplace transform

$$f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} \mathbf{F}(s)e^{st}\,ds, \tag{4.41}$$

if we assume in equation 4.13b that $c = 0$ and $s = j\omega$, which means that the integration in equation 4.41 takes places on an imaginary axis. The restriction (equation 4.39a) also satisfies equation 4.38a, thus $\alpha < c$ in equation 4.38a also means $\alpha < 0$ in (4.39a), since $c = 0$. Therefore, for all the functions which meet condition of equation 4.39a, we may use all the rules for finding the inverse Fourier transform by applying those derived for the Laplace transform in Chap. 3. Finally we should note that, for functions that do not meet conditions of equation 4.39a, we still may calculate their Fourier transform, however not straightforwardly (see further on).

4.4 SOME PROPERTIES OF THE FOURIER TRANSFORM

Keeping in mind the similarity between Fourier and Laplace transformations, a brief account of the properties of the Fourier transform will be given here (the proof is similar to that given in section 3.4 for the Laplace transform).

(a) *Property of linearity*

If $f_1(t)$ and $f_2(t)$ have Fourier transforms $\mathbf{F}_1(j\omega)$ and $\mathbf{F}_2(j\omega)$ respectively, then

$$\mathscr{F}[f_1(t) \pm f_2(t)] = \mathbf{F}_1(j\omega) \pm \mathbf{F}_2(j\omega), \tag{4.42}$$

i.e., the Fourier transform of the sum (difference) of two (or more) time functions is equal to the sum (difference) of the transforms of the individual time functions and conversely:

$$\mathscr{F}^{-1}[\mathbf{F}_1(j\omega) \mp \mathbf{F}_2(j\omega)] = \mathscr{F}^{-1}[\mathbf{F}_1(j\omega)] \pm \mathscr{F}^{-1}[\mathbf{F}_2(j\omega)] = f_1(t) \pm f_2(t). \tag{4.43}$$

It is also obvious that for any constant K

$$Kf(t) \leftrightarrow K\mathbf{F}(j\omega). \tag{4.44}$$

The above properties are also known as *superposition* and *homogeneity* properties.

(b) *Differentiation properties*

Let us derive the transformation of the derivative of function $f(t)$. If $\mathbf{F}(j\omega)$ is the Fourier transform of $f(t)$, then

$$\mathscr{F}\left\{\frac{df}{dt}\right\} = \int_{-\infty}^{\infty} e^{-j\omega t}\frac{df}{dt}\,dt.$$

Its integration by parts, $u = e^{-j\omega t}$ and $dv = df$, gives

$$\mathscr{F}\left\{\frac{df}{dt}\right\} = f(t)e^{-j\omega t}\Big|_{-\infty}^{\infty} + j\omega\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = j\omega\mathbf{F}(j\omega), \tag{4.45}$$

since the first term in this expression gives zero for both limits $t = \pm\infty$ (note that the given function, having a Fourier transform, must vanish to zero when

$|t| \to \infty$). Thus, differentiating a time-domain function corresponds to the multiplication of a frequency-domain function $\mathbf{F}(j\omega)$ by the factor $j\omega$. So we may write

$$\mathscr{F}\left\{\frac{df}{dt}\right\} = j\omega\mathbf{F}(j\omega). \tag{4.46}$$

This result may be readily extended to the general case for derivatives of order n

$$\mathscr{F}\left\{\frac{d^n f}{dt^n}\right\} = (j\omega)^n\mathbf{F}(j\omega). \tag{4.46a}$$

For the one-sided Fourier transform in which the first term in equation 4.45 turns into $f(0)$, so in this case, we will have

$$\mathscr{F}\left\{\frac{df}{dt}\right\} = j\omega\mathbf{F}(j\omega) - f(0), \tag{4.46b}$$

which is similar to the differentiation property of the Laplace transform (it is obvious due to the similarity between the one-sided Fourier transform and Laplace transform).

(c) *Integration properties*

Let $G(j\omega)$ be a spectrum of an integral $g(t) = \int_{-\infty}^{t} f(\tau)d\tau$. In accordance with the differentiation theorem, we may find that the Fourier transform of the function $f(t) = dg/dt$ will be

$$\mathbf{F}(j\omega) = j\omega G(j\omega).$$

Thus,

$$G(j\omega) = \frac{\mathbf{F}(j\omega)}{j\omega},$$

or

$$\mathscr{F}\left\{\int_{-\infty}^{t} f(\tau)d\tau\right\} = \frac{\mathbf{F}(j\omega)}{j\omega}, \tag{4.47}$$

i.e., the integration in the time domain corresponds to the division by $j\omega$ in the frequency domain. For the one-sided Fourier transform this result will not be changed, since the integral from $-\infty$ to 0 turns into zero and the lower limit in equation 4.47 will simply be zero. However, in order for the function $g(t)$ to be transformable, $g(\infty)$ must be equal to 0 (in other words, this requires that $g(\infty) = \int_{-\infty}^{\infty} f(\tau)d\tau = F(0) = 0$). If this condition is not satisfied, then the more general result is

$$\mathscr{F}\left\{\int_{-\infty}^{t} f(\tau)d\tau\right\} = \frac{\mathbf{F}(j\omega)}{j\omega} + \pi\mathbf{F}(0)\delta(\omega). \tag{4.47a}$$

(More explanations and examples of using this result can be seen further on in the following sections.)

(d) *Scaling properties*

Next let us consider one of the most interesting properties of the Fourier transformation – the effect of changing the time scale of a function, i.e. replacing argument t by a new one at, where a is some positive constant. If the given function is $f(t)$, the time-scaled function becomes $f(at)$. Taking the Fourier transformation of such a function, we have

$$\mathscr{F}\{f(at)\} = \int_{-\infty}^{\infty} f(at)e^{-j\omega t}dt. \tag{4.48}$$

By changing the variable $\lambda = at$ the differential dt becomes $d\lambda/a$ and substituting this in equation 4.48, we obtain

$$\mathscr{F}\{f(at)\} = \frac{1}{a}\int_{-\infty}^{\infty} f(\lambda)e^{-j(\omega/a)\lambda}d\lambda = \frac{1}{a}\mathbf{F}\left(j\frac{\omega}{a}\right). \tag{4.49}$$

From this relation we may conclude that the scaling of the variable t in the time domain results in a reciprocal scaling of the variable ω in the frequency domain. In addition, there is a scaling of the spectrum magnitude $\mathbf{F}(j\omega)$ by $1/a$.

Scaling properties of the Fourier transform provide a mathematical justification for the phenomenon described in the preceding sections that shortening the duration of a pulse, i.e., expressing it in a larger scale ($a > 1$) as $f(at)$, results in an a times wider spectrum $\mathbf{F}(j\omega/a)$ being expressed in a smaller scale ω/a. Thus, for instance, a pulse $f(t)$ which occurs from 0 to 1 s after scaling by $a = 5$, transforms to a pulse of the same form which will occur from 0 to 1/5 s (since $f(t_1) = f[a(t_1/a)]$ where t_1/a is a new time after scaling). The frequency spectrum $\mathbf{F}(j(\omega/5))$ will be five times wider because of the new frequency scale.

(e) *Shifting properties*

As another significant property of the Fourier transform, let us consider the effect of shifting, or delaying, in the time domain. That is, let us find the transform of $f(t-\tau)$ where τ is a shifting constant. By defining a new variable of integration $\lambda = t - \tau$ in equation 4.13b, we have

$$f(t-\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{F}(j\omega)e^{j\omega(t-\tau)}d\omega = \frac{1}{2\pi}e^{-j\omega\tau}\int_{-\infty}^{\infty} \mathbf{F}(j\omega)e^{j\omega t}d\omega,$$

or

$$\mathscr{F}[f(t-\tau)] = e^{-j\omega t}\mathbf{F}(j\omega). \tag{4.50}$$

The physical meaning of this result is that a *delay* in the time domain (the function $f(t-\tau)$ is delayed τ seconds in respect to $f(t)$) corresponds to a phase shift by $-\omega\tau$ in the frequency domain.

(f) *Interchanging* t *and* ω *properties*

Finally, let us consider once again the property of interchanging t and ω in the Fourier transform pairs. In the discussion about the symmetrical properties (section 4.3.3), we have already considered such an interchanging. Now let us show that this property is general and can be applied to any function $f(t)$-symmetrical and non-symmetrical. To prove this statement, we first change the sign of ω in equation 4.13b and put the factor $1/2\pi$ inside the integral:

$$f(t)=\int_{-\infty}^{\infty}\frac{1}{2\pi}\mathbf{F}(-j\omega)e^{-j\omega t}d(-\omega)=\int_{-\infty}^{\infty}\frac{\mathbf{F}(-j\omega)}{2\pi}e^{-j\omega t}d\omega. \tag{4.51}$$

Secondly, we multiply both sides of equation 4.13a by $1/2\pi$ and change the sign of ω:

$$\frac{1}{2\pi}\mathbf{F}(-j\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(t)e^{j\omega t}dt. \tag{4.52}$$

Now, by interchanging t and ω in equations 4.51 and 4.52, we have

$$f(\omega)=\int_{-\infty}^{\infty}\frac{\mathbf{F}(-jt)}{2\pi}e^{-j\omega t}dt, \tag{4.53a}$$

$$\frac{1}{2\pi}\mathbf{F}(-jt)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(\omega)e^{j\omega t}d\omega, \tag{4.53b}$$

or in short

$$\frac{1}{2\pi}\mathbf{F}(-jt)\leftrightarrow f(\omega). \tag{4.54}$$

Comparing this expression with equation 4.15 we may state that, if the time function $f(t)$ has as its spectrum the function $\mathbf{F}(j\omega)$, then the time function $\mathbf{F}(-jt)$ will have as its spectrum the function $f(\omega)$.

With the help of these properties, we can get a new set of transform pairs by simply using known ones. For instance by applying equation 4.54, with the results of Example 4.2 given in equation 4.20, we have

$$\frac{1}{2\pi}\frac{2V_0}{-jt}\sin\frac{-jt\tau}{2}\leftrightarrow\begin{cases}V_0 & |\omega|<\dfrac{\tau}{2}\\[2ex] 0 & |\omega|>\dfrac{\tau}{2},\end{cases}$$

or after taking $-j$ out of sine:

$$V_0\tau\, Sa\frac{t\tau}{2}\leftrightarrow\begin{cases}2\pi V_0 & |\omega|<\dfrac{\tau}{2}\\[2ex] 0 & |\omega|>\dfrac{\tau}{2},\end{cases}$$

which means that the rectangular pulse in the frequency domain represents a spectrum of a sinc function in the time domain as shown in Fig. 4.8.

Other properties of the Fourier transform may be readily derived in a way and manner used in connection with the Laplace transform due to the similarity between both transforms. The above discussed Fourier transform properties and some other important ones are summarized in Table 4.2.

Table 4.2 Fourier transform operations

	Operation	$f(t)$	$\mathbf{F}(j\omega)$
1	Addition	$\sum_{n=1}^{n} f_i(t)$	$\sum_{n=1}^{n} \mathbf{F}_i(j\omega)$
2	Scalar multiplication	$Kf(t)$	$K\mathbf{F}(j\omega)$
3	Time differentiation:		
	(a) two-sided transform	$\frac{d}{dt}f(t)$	$j\omega\mathbf{F}(j\omega)$
	(b) one-sided transform	$\frac{d}{dt}f(t)$	$j\omega\mathbf{F}(j\omega) - f(0)$
4	Time integration		
	(a) $\int_{-\infty}^{\infty} f(\tau)d\tau = 0$	$\int_{-\infty}^{t} f(\tau)d\tau$	$\frac{\mathbf{F}(j\omega)}{j\omega}$
	(b) $\int_{-\infty}^{\infty} f(\tau)d\tau \neq 0$	$\int_{-\infty}^{t} f(\tau)d\tau$	$\frac{\mathbf{F}(j\omega)}{j\omega} + \pi\mathbf{F}(0)\delta(\omega)$
5	Time-shift	$f(t \pm a)$	$e^{\pm j\omega a}\mathbf{F}(j\omega)$
6	Frequency-shift	$f(t)e^{\pm j\omega_0 t}$	$\mathbf{F}[f(\omega \pm \omega_0)]$
7	Time-scaling	$f(at)$	$\frac{1}{a}\mathbf{F}\left(j\frac{\omega}{a}\right)$
8	Frequency differentiation	$(-jt)f(t)$	$\frac{d}{dt}\mathbf{F}(j\omega)$
9	Frequency integration	$\frac{f(t)}{(-jt)}$	$\int_{-\infty}^{\infty} \mathbf{F}(j\omega)d\omega$
10	Convolution in time domain	$f_1(t) * f_2(t)$	$\mathbf{F}_1(j\omega)\mathbf{F}_2(j\omega)$
11	Multiplication in time domain		
	(a) by sine	$f(t)\sin\omega_1 t$	$\frac{1}{2j}(\mathbf{F}[j(\omega-\omega_1)] - \mathbf{F}[j(\omega+\omega_1)])$
	(b) by cosine	$f(t)\cos\omega_1 t$	$\frac{1}{2}(\mathbf{F}[j(\omega-\omega_1)] + \mathbf{F}[j(\omega+\omega_1)])$
12	Parseval's theorem	$\int_{-\infty}^{\infty} f^2(t)dt$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert\mathbf{F}(j\omega)\rvert^2 d\omega$

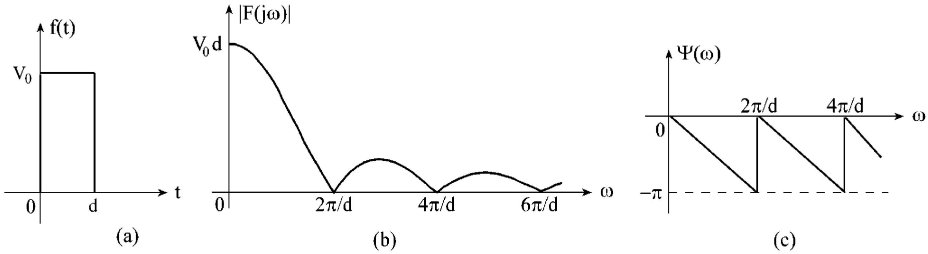


Figure 4.10 A pulse (a), its magnitude (b) and phase (c) spectra ($d = \tau$).

As an example of using Fourier transform properties let us derive the spectrum of the rectangular pulse shown in Fig. 4.10(a). This pulse is positioned at $0 < t < \tau$ and may be considered as shifting in respect to the pulse of Example 4.2 (note that $\tau = d$). Therefore, in order to obtain its spectrum we shall use the time shifting property. With the results of equation 4.20 in Example 4.2, and using equation 4.50, we have

$$\mathscr{F}\{f_{shift}(t)\} = \mathscr{F}\left\{F\left(t - \frac{\tau}{2}\right)\right\} = \frac{2V_0}{\omega}\sin\frac{\omega\tau}{2}\,e^{-j\frac{\omega\tau}{2}}$$

Thus, the magnitude spectrum (Fig. 4.10(b))

$$|\mathbf{F}(j\omega)| = V_0\tau \text{ Sa}\frac{\omega\tau}{2},$$

which is the same as in Example 4.2. The phase spectrum, however, will be

$$\Psi(\omega) = -\frac{\tau}{2}\omega,$$

which is declined lines changing from 0 to $-\pi$, as shown in Fig. 4.10(c), i.e., taking into consideration the sign of $\sin(\omega\tau/2)$, we have

$$\Psi(\omega) = -\frac{\tau}{2}\omega \qquad \text{for } 0 < \omega < \frac{2\pi}{\tau} \qquad \left(\sin\frac{\omega\tau}{2} > 0\right)$$

$$\Psi(\omega) = -\frac{\tau}{2}\omega + \pi \quad \text{for } \frac{2\pi}{\tau} < \omega < \frac{4\pi}{\tau} \quad \left(\sin\frac{\omega\tau}{2} < 0\right).$$

Our conclusion from this example is that time shifting does not influence the magnitude spectrum of the function, but changes its phase spectrum.

4.5 SOME IMPORTANT TRANSFORM PAIRS

For our future study of the Fourier transform technique, we shall develop the Fourier transform expression for those functions frequently used in circuit analysis. For this purpose we will do it either straightforwardly, using equations 4.13, or by applying the Fourier transform properties listed in Table 4.2.

4.5.1 Unit-impulse (delta) function

As we have already discussed in the previous chapter, the *unit-impulse* or *delta* function is defined as a time function which is zero when its argument is less or greater than zero and which is infinite when its argument is zero, while having a unit area, i.e.,

$$\delta(t-t_0)=0 \qquad t-t_0\neq 0\ (t\neq t_0) \tag{4.55a}$$

$$\int_{-\infty}^{\infty}\delta(t-t_0)=1 \quad t-t_0=0\ (t=t_0). \tag{4.55b}$$

If the switching operation occurs at $t=0$ (which always can be done by choosing $t_0=0$), we have

$$\delta(t)=0 \qquad t\neq 0 \tag{4.56a}$$

$$\int_{0_-}^{0_+}\delta(t-t_0)=1 \quad t=0. \tag{4.56b}$$

Multiplication of the delta function by a constant will not affect equation 4.55a and equation 4.56a, because the value of this function must still be zero when the argument is not zero and approaches infinity at $t=0$. However, this multiplication will change the integrals' value in equation 4.55b and equation 4.56b:

$$\int_{-\infty}^{\infty}A\delta(t)dt=\alpha. \tag{4.57}$$

This means that the area under the impulse is now equal to the multiplying factor, which is called the strength of the impulse. Following this rule we may interpret the multiplication of the delta function by any other function as follows:

$$\int_{-\infty}^{\infty}\mathrm{f(t)\delta(t)dt}=\mathrm{f(0)}, \quad \text{or} \tag{4.58a}$$

$$\int_{-\infty}^{\infty}f(t)\delta(t-t_0)dt=f(t_0). \tag{4.58b}$$

In this case, therefore, the strength of the impulse is the value of that function at the time for which the impulse argument is zero. For instance, the strength of the impulse multiplied by sine-function $f(t)\delta(t)=\sin(\omega t+60^\circ)\ \delta(0)$ is $\sqrt{3}/2$.

This property of a unit impulse function is sometimes called the **sampling property**. The graphical symbol for an impulse, used commonly, is an arrowhead line erected at the moment of the time when the impulse is applied (Fig. 4.11).

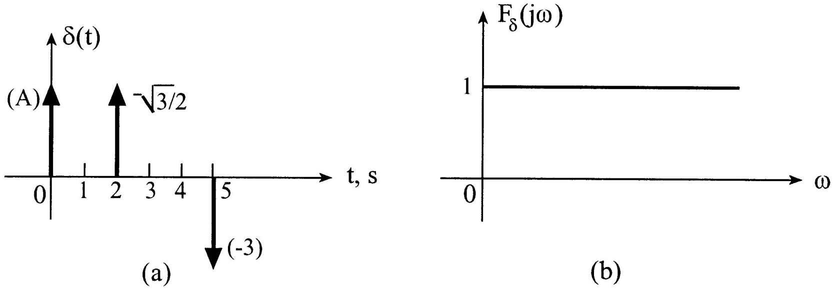


Figure 4.11 Positive and negative impulses of different strengths are plotted at the time of their appearances (a), a spectrum of impulse function (b).

The strength of the impulse is usually indicated by adjusting the arrow, as shown in Fig. 4.11(a).

Now bearing in mind the above properties of the impulse function and using the equation for finding the Fourier transform, we obtain

$$\mathscr{F}\{\delta(t-t_0)\} = \int_{-\infty}^{\infty} e^{-j\omega t}\delta(t-t_0) = e^{-j\omega t_0} \tag{4.59a}$$

and

$$\mathscr{F}\{\delta(t)\} = \int_{-\infty}^{\infty} e^{-j\omega t}\delta(t) = e^{-j\omega t}\bigg|_{t=0} = 1, \tag{4.59b}$$

or

$$\mathbf{F}_\delta(j\omega) = 1. \tag{4.60}$$

This function is shown in Fig. 4.11(b) as the straight line of a unit magnitude. Note that the spectrum of the impulse function is infinite, since it goes to infinity. The result of equation 4.59a may also be written as

$$\mathscr{F}\{\delta(t-t_0)\} = e^{-j\omega t_0} = \cos\omega t_0 - j\sin\omega t_0. \tag{4.61}$$

Therefore, the energy density of a delta function is unity:

$$|\mathscr{F}\{\delta(t-t_0)\}|^2 = \cos^2\omega t_0 + \sin^2\omega t_0 = 1. \tag{4.62}$$

This result states that the energy (released in a unit input resistance) per unit bandwidth is unity at all frequencies. Since the impulse function has an infinite bandwidth, the total energy in the unit impulse is infinitely large (note that a unit impulse function is only a mathematical model of real pulse source functions which are, of course, bound).

In order to find the reverse Fourier transform of a unit impulse spectrum, we shall use the property of the Fourier transform which states that there is a unique one-to-one correspondence between a time function and its Fourier transform. Therefore, we can say that the inverse Fourier transform of $e^{-j\omega t_0}$

is $\delta(t-t_0)$, thus

$$\mathscr{F}\{e^{-j\omega t_0}\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega t_0} e^{j\omega t} d\omega = \delta(t-t_0), \tag{4.63}$$

or in the symbolic way:

$$\delta(t-t_0) \leftrightarrow e^{-j\omega t_0}. \tag{4.64}$$

Next, by using the property of interchanging arguments t and ω in Fourier pairs, we may readily obtain from equation 4.64.

$$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega-\omega_0), \tag{4.65a}$$

which might be interpreted as a Fourier pair for a unit impulse in the frequency domain located at $\omega = \omega_0$. By changing the sign of the pulse location ω_0 to $-\omega_0$, we obtain

$$e^{-j\omega_0 t} \leftrightarrow 2\pi\delta(\omega+\omega_0). \tag{4.65b}$$

By letting $\omega_0 = 0$ we obtain

$$1 \leftrightarrow 2\pi\delta(\omega), \tag{4.66a}$$

from which it follows that

$$K \leftrightarrow 2\pi K\delta(\omega). \tag{4.66b}$$

Thus, the frequency spectrum of a constant K function in the time domain is a $2\pi K$ strength impulse in the frequency domain. An interpretation of this result is that a d.c. voltage or current forcing function, whose frequency is considered as zero, i.e., $\omega_0 = 0$, has its Fourier transform in accordance with equation 4.66.

Although the time functions in equation 4.65 are complex functions of time, which are not appropriate in the existing world of reality, with their help we can obtain in a very simple way the frequency spectra of such important functions as sine and cosine. Thus,

$$\mathscr{F}\{\cos\omega_0 t\} = \mathscr{F}\left\{\frac{1}{2}(e^{j\omega t} + e^{-j\omega t})\right\} = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0), \tag{4.67}$$

or

$$\cos\omega_0 t \leftrightarrow \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)], \tag{4.68a}$$

and similarly

$$\sin\omega_0 t \leftrightarrow j\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]. \tag{4.68b}$$

The above expressions indicate that the *frequency spectra of sinusoidal functions* are given as a pair of impulses, located at $\omega = \pm\omega_0$.

This result actually corresponds to the representation of a sinusoidal function by imaginary frequencies $s \pm j\omega_0$, which was used in our previous study of circuit analysis for instance, in the symbolic or complex method.

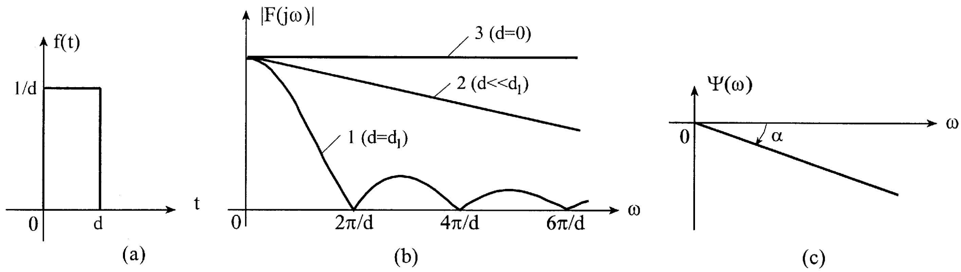


Figure 4.12 A short pulse (a) and its amplitude (b) and phase (c) spectra.

Example 4.4

Consider once again the rectangular pulse shown in Fig. 4.12(a). This pulse of a unit area (since $d(1/d) = 1$) approaches a unit impulse when $d \to 0$. In accordance with the result of Example 4.2 (see equation 4.20b) its spectrum is

$$\mathbf{F}(j\omega) = \frac{2}{\omega d} \sin \frac{\omega d}{2}. \tag{4.69}$$

By approaching $d \to 0$ in equation 4.69, we will obtain the Fourier transform of the unit impulse:

$$\mathbf{F}_\delta(j\omega) = \lim_{d \to 0} \frac{\sin(\omega d/2)}{\omega d/2} = 1. \tag{4.70}$$

Figure 4.12(b) shows the transformation of the spectrum (equation 4.69) into the spectrum (equation 4.70). The zero points of the spectrum, given for a sinc function (1) at $k(2\pi/d)$ ($k = 1, 2, \ldots$), move to the right along the frequency axis to higher frequencies (2) so that for $d = 0$, the whole spectrum approaches a straight line (3). Note that the phase spectrum of the impulse function, applied at $t_0 = 0$, is zero. However, the phase spectrum of the impulse, applied at the time t_0, will be in accordance with equation 4.64, $\mathbf{F}(j\omega) = e^{-j\omega t_0}$, which gives

$$\Psi(\omega) = -t_0 \omega. \tag{4.71}$$

Graphically it is a straight line having an angle of declination $\alpha \propto \tan^{-1}(-t_0)$ as shown in Fig. 4.12(c).

4.5.2 Unit-step function

Our next consideration will be the unit-step function $u(t)$. In the previous chapter we introduced this function, which usually indicates a switching or failure action. It is defined (Fig. 4.13) as

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0, \end{cases} \tag{4.72a}$$

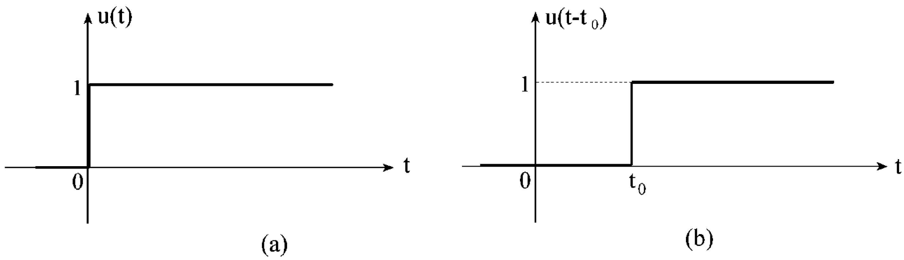


Figure 4.13 A unit-step function: at $t = 0$ (a) and at $t = t_0$ (b).

or

$$u(t - t_0) = \begin{cases} 0 & t < t_0 \\ 1 & t > t_0. \end{cases} \tag{4.72b}$$

Thus, the unit-step function is zero for all values of its argument (time) which are less than zero ($t < 0$) or less than $t_0 (t > t_0)$ (note that in both cases the argument (time) is just negative), and is unity for all positive values of its argument ($t > 0$) or ($t > t_0$). In order to find the Fourier transform of the unit-step function, we must indicate that this function is the kind of function whose transform cannot be obtained straightforwardly. This happens because the integral in equation 4.13 is unbound, which means that the unit-step function does not approach zero as t approaches infinity. One common way of achieving the Fourier transform of the unit-step function is by representing it as a sum of a constant and a *signum function* (Fig. 4.14):

$$u(t) = \frac{1}{2}[1 + \mathrm{sgn}(t)] = \frac{1}{2} + \frac{1}{2}\mathrm{sgn}(t). \tag{4.73}$$

In accordance with equation 4.66 the transform of the first member in equation 4.73 will be $\pi\delta(\omega)$. As is known the second member in equation 4.73, a **signum**

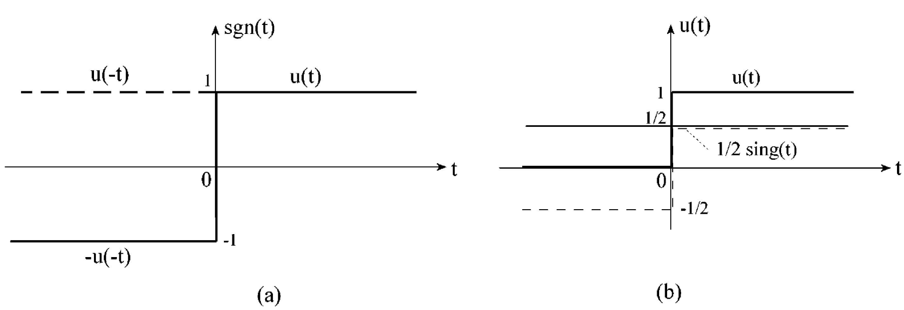


Figure 4.14 A signum function (a) and a representation of a unit function by the sum of a constant and a signum function (b).

function, is defined as

$$\operatorname{sgn}(t)\begin{cases}-1 & t<0\\ 1 & t>0.\end{cases} \tag{4.74a}$$

or

$$\operatorname{sgn}(t)=u(t)-u(-t). \tag{4.74b}$$

The signum function can also be written as

$$\operatorname{sgn}(t)=\lim_{c\to 0}\,[e^{-ct}u(t)-e^{ct}u(-t)].$$

Factor $e^{\pm ct}$ is used here (as a convergence factor) to insure the approaching of unit step zero, as t gets very large (i.e., when $t\to\infty$). On the other hand, by approaching $c\to 0$, we are getting back to the originally given signum function. Using the definition of the Fourier transform, we obtain

$$\begin{aligned}\mathscr{F}\{\operatorname{sgn}(t)\}&=\lim_{c\to 0}\left[\int_0^{\infty}e^{-ct}e^{-j\omega t}dt-\int_{-\infty}^{0}e^{ct}e^{j\omega t}dt\right]\\ &=\lim_{c\to 0}\left(\frac{-e^{-ct}}{c+j\omega}\bigg|_0^{\infty}-\frac{-e^{ct}}{c-j\omega}\bigg|_{-\infty}^{0}\right)=\lim_{c\to 0}\frac{-j2\omega}{c^2+\omega^2}=\frac{2}{j\omega}.\end{aligned}$$

Thus,

$$\operatorname{sgn}(t)\leftrightarrow\frac{2}{j\omega}, \tag{4.75}$$

and

$$\mathscr{F}\{u(t)\}=\mathscr{F}\left\{\frac{1}{2}\right\}+\mathscr{F}\left\{\frac{1}{2}\operatorname{sgn}(t)\right\}=\pi\delta(\omega)+\frac{1}{j\omega},$$

or

$$u(t)\leftrightarrow\pi\delta(\omega)+\frac{1}{j\omega}. \tag{4.76}$$

The first term represents an impulse, in the frequency domain, of strength π occurring at $\omega=0$. The second term is the same as the Laplace transform of a unit-step function in which s has been replaced by $j\omega$.

The magnitude and phase spectra of the unit-step function are shown in Fig. 4.15. Note that the magnitude spectrum of a unit-step contains the harmonics of all the frequencies, however the energy density at the low frequency harmonics is much higher. When $\omega\to 0$ the magnitude spectrum and its energy approach infinity. In general, any unbound signal is characterized by an infinite amount of energy.

Sudden spouts of d.c. or a.c. currents of industrial frequency (for instance by starting motors or short-circuiting) are similar to a unit-step function with a

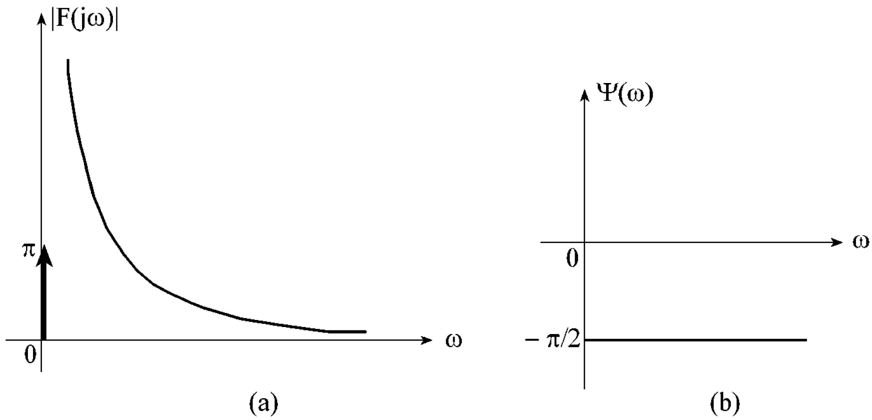


Figure 4.15 Magnitude (a) and phase (b) spectra of a unit-step function.

high energy density at low frequencies. This is the reason that most interference occurs on low-frequency radio broadcasts (long waves) and are almost invisible on high frequencies (short waves).

4.5.3 Decreasing sinusoid

Such a sine function is defined as

$$f(t) = e^{-at} \sin \omega_0 t \, u(t),$$

and its Fourier transform might be found as

$$\mathbf{F}(j\omega) = \int_0^{\infty} e^{-at} \sin \omega_0 t e^{-j\omega t} dt = \frac{\omega_0}{(a + j\omega)^2 + \omega_0^2}. \tag{4.77}$$

This results in magnitude spectrum

$$|\mathbf{F}(j\omega)| = \frac{\omega_0}{\sqrt{(a^2 + \omega_0^2 - \omega^2)^2 + 4a^2\omega^2}}, \tag{4.78}$$

and phase spectrum

$$\Psi(\omega) = -\tan^{-1} \frac{2a\omega}{a^2 + \omega_0^2 - \omega^2}. \tag{4.79}$$

The curves of $|\mathbf{F}(\mathbf{j}\omega)|$ and $\Psi_{(\omega)}$ are shown in Fig. 4.16, where $\omega_{(\max)} = \sqrt{\omega^2 + a^2}$.

4.5.4 Saw-tooth unit pulse

We can use the differentiation property to find the Fourier transform avoiding the straightforward integration of 4.13(b), which is in many cases extremely

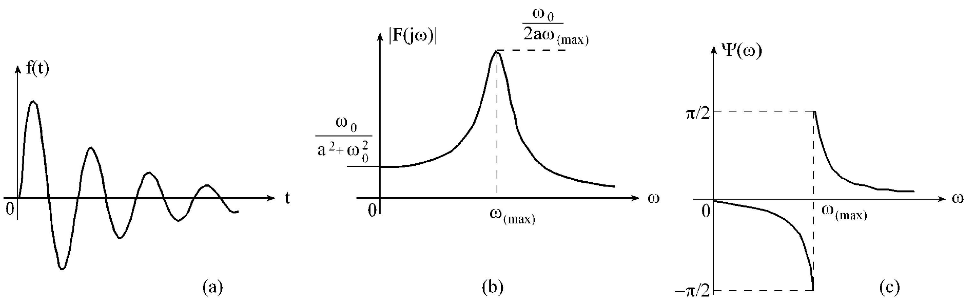


Figure 4.16 A decreasing sinusoidal function (a) and its magnitude (b) and phase (c) spectra.

difficult. Let us illustrate this method on the saw-tooth pulse, Fig. 4.17, where $\mathbf{F}(j\omega)$ represents the unknown spectrum of this pulse. After a single differentiation the saw-tooth pulse (a) takes the form (b). Now we add an equal and opposite impulse to the signal in (b) to cancel the appearing one. The result is the rectangular pulse remaining in (c). The second integration gives two impulses in (d), whose transform can be easily found, as $(1/a)(1 - e^{-j\omega a})$. Hence, by

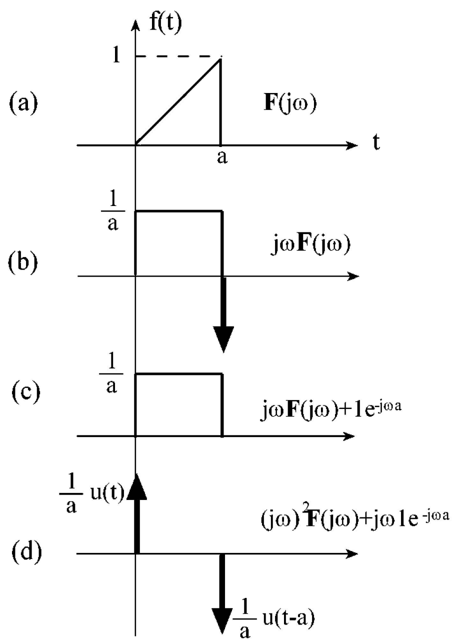


Figure 4.17 A unit saw-tooth pulse (a), its first differentiation (b), after adding a unit impulse (c) and after the second differentiation (d).

equaling:

$$\frac{1}{a}(1 - e^{-j\omega a}) = (j\omega)^2\mathbf{F}(\omega) + j\omega e^{-j\omega a},$$

we obtain

$$\mathbf{F}(\omega) = \frac{-1 + (1 + j\omega)e^{-j\omega a}}{a\omega^2}.$$

This method, actually, is generalized because of the fact that any signal may be approximated as a piecewise-linear, in which case the signal reduces to impulses after two (or three) differentiations.

4.5.5 The Fourier transform of a periodic time function

Here we face the same problem, which we had in section 4.5.2 looking for the Fourier transform of a unit-step function. Any periodic function is, obviously, unbound, since it does not approach zero, as t approaches infinity. In order to obtain the Fourier transform of a periodic function we should distinguish between two cases: *two-sided* and *one-sided transforms*. The two-sided Fourier transform of a sinusoidal function, as shown in Fig. 4.17(a), has already been found in section 4.5.1 (see equation 4.68)].

However, in circuit analysis, the most frequently used forcing periodic functions are sinusoidal functions applied at $t = 0$, shown in Fig. 4.18(b). In this case, we can define such a function as

$$f(t) = \sin \omega_0 t \; u(t). \tag{4.80}$$

Using the Fourier transform property of multiplication by sine/cosine in the time domain (see entry 11 in Table 4.2) we have

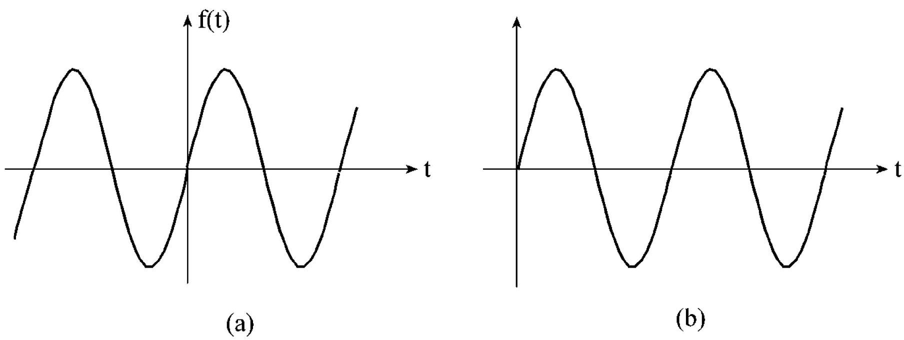


Figure 4.18 Sinusoidal function: (a) for both sides of the t-axis and (b) for only the positive side of the t-axis ($t > 0$).

$$F\{u(t)\sin\omega_0 t\} = \frac{1}{2j}(F_u[j(\omega-\omega_0)] - F_u[j(\omega+\omega_0)])$$
$$= \frac{1}{2j}\left(\pi\delta(\omega-\omega_0) + \frac{1}{j(\omega-\omega_0)} - \pi\delta(\omega+\omega_0) - \frac{1}{j(\omega+\omega_0)}\right)$$
$$= \frac{\pi}{2j}[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)] + \frac{\omega_0}{\omega_0^2-\omega^2}.$$

Thus

$$\sin\omega_0 t u(t) \leftrightarrow \frac{j\pi}{2}[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)] + \frac{\omega_0}{\omega_0^2-\omega^2}. \qquad (4.81)$$

Note that the second member on the right side of equation 4.81 might be readily obtained from the Laplace transform of the sinusoid by replacing s by $j\omega$:

$$\mathscr{F}\{\sin\omega_0 t\} = \mathscr{L}\{\sin\omega_0 t\}_{s=j\omega} = \left.\frac{\omega_0}{s^2+\omega_0^2}\right|_{s=j\omega} = \frac{\omega_0}{\omega_0^2-\omega^2} \qquad (4.82)$$

The first member on the right side of equation 4.81 represents the switching property of the unit-step function.

Table 4.3 gives the Fourier transform pairs for most of the familiar time functions encountered in circuit analysis. They may be used for finding the inverse transform of frequency domain functions, as was done for the Laplace transform method.

4.6 CONVOLUTION INTEGRAL IN THE TIME DOMAIN AND ITS FOURIER TRANSFORM

In the circuit analysis technique, applying Fourier transforms, the multiplication of two transforms (namely, the transform of the forcing function and the system function) is frequently used to obtain the transform of the response function. The inverse-transform operation must be performed to obtain the response function. In a similar way, as was shown in the previous chapter (with respect to the Laplace transform), we may state that the inverse transform of the product of two Fourier transforms is the *convolution integral*, i.e.,

$$f_{res}(t) = \mathscr{F}^{-1}\{\mathbf{F}_1(j\omega)\mathbf{F}_2(j\omega)\} = f_1(t)*f_2(t), \qquad (4.85a)$$

where

$$f_1(t)*f_2(t) = \int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau = \int_{-\infty}^{\infty} f_1(t-\tau)f_2(\tau)d\tau. \qquad (4.85b)$$

Two integrals given by equation 4.85b are the two forms of the convolution integral in a very general form. By using equation 4.85 we shall take into consideration the physically realizable properties of electrical systems. Thus, the

Table 4.3 Fourier transform pairs

	$f(t)$	$\mathbf{F}(j\omega)$
1	$\delta(t)$	1
2	$\delta(t-t_0)$	$e^{-j\omega t_0}$
3	1	$2\pi\delta(\omega)$
4	$u(t)$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$
5	$\mathrm{sgn}(t)$	$\dfrac{2}{j\omega}$
6	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
7	$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}$
8	$te^{-at}u(t)$	$\dfrac{1}{(a+j\omega)^2}$
9	$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$
10	$\cos\omega_0 t$	$\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$
11	$\sin\omega_0 t\, u(t)$	$\dfrac{j\pi}{2}[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]+\dfrac{\omega_0}{\omega_0^2-\omega^2}$
12	$\cos\omega_0 t\, u(t)$	$\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]+\dfrac{j\omega_0}{\omega_0^2-\omega^2}$
13	$e^{-at}\sin\omega_0 t\, u(t)$	$\dfrac{\omega_0}{(a+j\omega)^2+\omega_0^2}$
14	$e^{-at}\cos\omega_0 t\, u(t)$	$\dfrac{a+\omega_0}{(a+j\omega)^2+\omega_0^2}$
15	$u(t+\tau/2)-u(t-\tau/2)$	$\tau\dfrac{\sin\omega\tau/2}{\omega\tau/2}$

response of the system cannot begin before the forcing function is applied. Let us say that $f_2(t)\equiv h(t)$ is the response of the system, usually resulting from the application of a unit impulse at $t=0$; (see section 3.6). Therefore, $h(t)$ cannot exist for $t<0$ which means that in the second integral of equation 4.85b the integrand is zero when $\tau<0$ and the low limit of integration may be changed and the response function is

$$f_{res}=f_1(t)*f_2(t)=\int_0^{\infty} f_1(t-\tau)h(\tau)d\tau. \tag{4.86a}$$

For the same reason, in the first integral of equation 4.85b, $f_2(t-\tau)\equiv h(t-\tau)$ cannot exist for $t<\tau$, which means that the integrand is zero when $t-\tau$ is negative. The upper limit in this integral, therefore, may be changed and the response function is

$$f_{res} = f_1(t) * f_2(t) = \int_{-\infty}^{t} f_1(\tau) h(t - \tau) d\tau. \tag{4.86b}$$

Before continuing our discussion of applying the Fourier transformation method in circuit analysis, let us consider an example of using the convolution integral.

Example 4.5

Using the convolution integral, find the output voltage $v_o(t)$ in the series RL circuit, if the input $v_i(t)$ is a rectangular voltage pulse of 6 V in amplitude that starts at $t = 0$ and has a duration of 1 s (Fig. 4.19(a)). Assume that $L = 5$ H and $R = 4\ \Omega$.

Mathematically the input voltage may be written as $v_i(t) = u(t) - u(t - 1)$. The impulse response $h(t)$ for the given circuit (Fig. 4.19(a)) might be evaluated as follows. Using the phasor method for analyzing circuits in the frequency domain or the so-called symbolic method (see further on), we obtain

$$H(j\omega) = V_{o,\delta}(j\omega) = V_{i,\delta} \frac{R}{R + j\omega L} = 1 \frac{4}{4 + j\omega 5} = 0.8 \frac{1}{0.8 + j\omega},$$

or, with entry 7 in Table 4.3,

$$h(t) = 0.8 e^{-0.8t} u(t).$$

Now, applying the convolution integral yields

$$v_o(t) = v_i(t) * h(t) = \int_{-\infty}^{\infty} 6[u(t - \tau) - u(t - \tau - 1)][0.8 e^{-0.8\tau} u(\tau)]\, d\tau \tag{4.87}$$

or separating equation 4.87 into two integrals, we have

$$v_0(t) = 4.8 \int_{-\infty}^{\infty} u(t - \tau) e^{-0.8\tau} u(\tau) d\tau - 4.8 \int_{-\infty}^{\infty} u(t - 1 - \tau) e^{-0.8\tau} u(\tau) d\tau.$$

The first integral should be taken in the limits from 0 to t and the second in

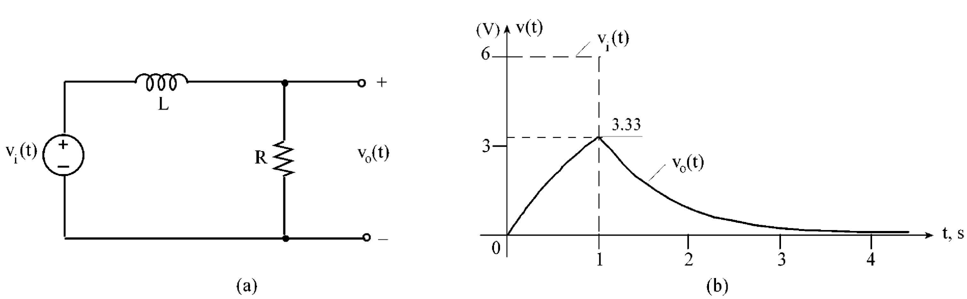


Figure 4.19 The given circuit (a) and the input $v_i(t)$ and output $v_o(t)$ voltages (b).

the limits from t to $t-1$. Thus,

$$v_0(t) = \begin{cases} 4.8 \displaystyle\int_0^t e^{-0.8\tau} d\tau = 6(1 - e^{-0.8t}) & 0 < t < 1 \\ -4.8 \displaystyle\int_t^{t-1} e^{-0.8\tau} d\tau = 6(e^{0.8} - 1)e^{-0.8t} = 7.35 e^{-0.8t} & 1 > t > \infty. \end{cases}$$

This function is shown in Fig. 4.19(b).

4.7 CIRCUIT ANALYSIS WITH THE FOURIER TRANSFORM

As we already know, the Fourier transform extends the Fourier series to a non-periodic function transforming the discrete spectra into continuous ones. Therefore, we can state that the Fourier transform represents the non-periodic function as an infinite sum of the harmonics, i.e. periodic functions possessing vanishingly small amplitudes. Therefore, we may apply the phasor concept and symbolic (complex) method used for steady-state analysis of the circuits driven by sinusoidal forcing functions.

Thus, considering the general circuit of Fig. 4.20 in the time domain we will obtain a differential equation, which describes the relation between the input (forcing) voltage $v_i(t)$ and the output (response) voltage $v_o(t)$:

$$a_0 v_0(t) + a_1 \frac{dv_o(t)}{dt} + a_2 \frac{d^2 v_o(t)}{dt^2} + \cdots = b_0 v_i(t) + b_1 \frac{dv_i(t)}{dt} + b_2 \frac{d^2 v_i(t)}{dt^2} + \cdots. \tag{4.88}$$

Taking the Fourier transform of both sides of equation 4.88 and using the differentiation and linearity properties, yields:

$$[a_0 + a_1(j\omega) + a_2(j\omega)^2 + \cdots]\mathbf{V}_o(j\omega) = [b_0 + b_1(j\omega) + b_2(j\omega)^2 + \cdots]\mathbf{V}_i(j\omega), \tag{4.89}$$

where $\mathbf{V}_i(j\omega)$ and $\mathbf{V}_o(j\omega)$ are the Fourier transforms of the input and output functions $v_i(t)$ and $v_o(t)$. From this result we may write

$$\mathbf{H}(j\omega) = \frac{\mathbf{V}_o(j\omega)}{\mathbf{V}_i(j\omega)} = \frac{b_0 + b_1(j\omega) + b_2(j\omega)^2 + \cdots}{a_0 + a_1(j\omega) + a_2(j\omega)^2 + \cdots} \tag{4.90}$$

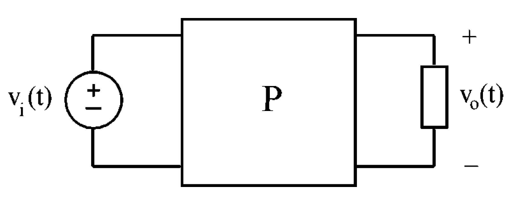


Figure 4.20 General passive circuit.

where $\mathbf{H}(j\omega)$ is identified as a network or system function (usually as an impulse response).

This is exactly the same result as would be obtained by the application of the phasor method and analyzing the circuit of Fig. 4.20 in the frequency domain. Note also that the same result could be achieved with the Laplace transform simply by replacing s by $j\omega$, in an expression like equation 3.58 in the previous chapter. Therefore, the above conclusion allows us to apply all the methods based on the phasor concept, using the impedances $\mathbf{Z}(j\omega)$ and admittances $\mathbf{Y}(j\omega)$ for finding the quantity $\mathbf{H}(j\omega)$ and solving other problems which relate to the Fourier transform. The only difference is that here the forcing functions, inputs, and the response functions, outputs, are Fourier transforms rather than phasors. This means that in the time domain the forcing functions and the responses are arbitrary (any) non-periodic functions rather than sinusoidal functions. In conclusion, just as the use of phasor (symbolic) transforms simplified the determination of the steady-state sinusoidal response, the use of Fourier transforms of various forcing functions can simplify the determination of the complete response of both the natural and forced components. The reason for this is quite simple: in both techniques, the differentiation in the time domain is represented in the frequency domain by multiplication by the factor $j\omega$; and similarly integration is related to division by the factor $j\omega$. By these means, relatively complicated differential and/or integral expressions are reduced to a relatively simple algebraic function of ω.

The next step in Fourier transform analysis is to find the time-domain description of the response transform for which we must evaluate an inverse Fourier transform technique. Some of the methods of this procedure will be developed in the following chapters. With the above remarks in mind, let us now consider some specific analysis problems.

Example 4.6

Let the decreasing exponential voltage $v_{in}(t) = e^{-5t}u(t)$ be applied to a given circuit and be related to the output voltage $v_o(t)$ by the equation

$$\frac{dv_o}{dt} + 3v_o = 3v_{in}.$$

Transforming the equation into the frequency domain using the Fourier technique, we have

$$(j\omega + 3)\mathbf{V}_o(j\omega) = 3\mathbf{V}_{in}(j\omega),$$

and the transfer function is

$$\mathbf{H}(j\omega) = \frac{\mathbf{V}_o(j\omega)}{\mathbf{V}_{in}(j\omega)} = \frac{3}{3 + j\omega}.$$

According to Table 4.3 the Fourier transform of the applied function will be

$$\mathbf{V}_{in}(j\omega) = \frac{1}{5 + j\omega}.$$

Therefore, the transform of the output voltage is

$$\mathbf{V}_o(j\omega) = \mathbf{H}(j\omega)\mathbf{V}_{in}(j\omega) = \frac{3}{(3 + j\omega)(5 + j\omega)}.$$

By partial fraction expansion (see section 3.7) we obtain

$$\mathbf{V}_o(j\omega) = 4.5\frac{1}{3 + j\omega} - 4.5\frac{1}{5 + j\omega}.$$

By using the linearity property of the Fourier transform and the table of Fourier transform pairs, we have

$$v_o(t) = 4.5(e^{-3t} - e^{-5t})u(t).$$

4.7.1 Ohm's and Kirchhoff's laws with the Fourier transform

Supposing that $\mathbf{V}_{in}(j\omega)$ is the Fourier transform of voltage $v_{in}(t)$, applied to the one-port circuit having the impedance $Z(j\omega)$, we may find the Fourier transform of the input current as

$$I_{in}(j\omega) = \frac{\mathbf{V}_{in}(j\omega)}{Z(j\omega)} = Y(j\omega)\mathbf{V}_{in}(j\omega). \tag{4.91}$$

This expression may be presented as Ohm's law in Fourier transform form.

For two-port circuits the input/output quantities might be found if the spectral characteristics of the transform coefficient $K(j\omega)$ or the transfer admittance/impedance $Y_{21}(j\omega)/Z_{21}(j\omega)$ are known.

Then the transform of the output voltage will be

$$\mathbf{V}_2(j\omega) = K_{21}(j\omega)\mathbf{V}_1(j\omega), \tag{4.92a}$$

or the output current will be

$$\mathbf{I}_2(j\omega) = Y_{21}(j\omega)\mathbf{V}_1(j\omega). \tag{4.92b}$$

Note that these two expressions are similar to Ohm's law in Fourier transform form.

In a similar way, using the phasor method, Kirchhoff's laws' equations can be written and analyzed.

4.7.2 Inversion of the Fourier transform using the residues of complex functions

The inverse Fourier transform for the above expressions can be found with the help of the residues of complex functions. Thus, if the Fourier transform of the given function is of the form (like in 4.91) we have

$$\frac{\mathbf{F}_1(j\omega)}{\mathbf{F}_2(j\omega)} \leftrightarrow \sum_{k=1}^{n} \frac{\mathbf{F}_1(j\omega_k)e^{j\omega_k t}}{\left[\frac{d}{d_j\omega}\mathbf{F}_2(j\omega)\right]_{\omega=\omega_k}} = j\sum_{k=1}^{n} \frac{\mathbf{F}_1(j\omega_k)e^{j\omega_k t}}{\left[\frac{d}{d\omega}\mathbf{F}_2(j\omega)\right]_{\omega=\omega_k}}, \tag{4.93a}$$

where ω_k are the roots of the equation $F_2(j\omega) = 0$. If the expression in the denominator is of the form $\mathbf{F}_2(j\omega) = j\omega\mathbf{F}_3(j\omega)$, which means that $\mathbf{F}_2(j\omega)$ has a zero root $\omega_0 = 0$, then the inverse Fourier transform will be

$$\frac{\mathbf{F}_1(j\omega)}{j\omega\mathbf{F}_3(j\omega)} \leftrightarrow \frac{\mathbf{F}_1(0)}{\mathbf{F}_2(0)} + \sum_{k=1}^{n-1} \frac{\mathbf{F}_1(j\omega_k)e^{j\omega_k t}}{\omega_k \left[\dfrac{d}{d\omega}\mathbf{F}_3(j\omega)\right]_{\omega=\omega_k}}. \tag{4.93b}$$

These formulas (equation 4.93) are useful for cases in which a voltage/current source is applied (at $t = 0$) to the circuit with zero initial conditions. Note that for such circuits all the voltages/currents for $t < 0$ are zero, which means that a one-sided Fourier transform is used:

$$F(j\omega) = \int_0^{\infty} f(t)e^{-j\omega t}\,dt.$$

Formulas such as those in equation 4.93 are sometimes called "switching formulas" since they are used when the circuits are switched to different sources, i.e. for $t > 0$.

Example 4.7

The *T*-circuit, shown in Fig. 4.21(a), is connected to the d.c. voltage source at $t = 0$. Find the current $i_2(t)$ using a switching formula.

Solution

The transfer admittance of the circuit is

$$Y_{21}(j\omega) = \frac{1}{R + R//\dfrac{1}{j\omega C}} \cdot \frac{1/j\omega C}{R + \dfrac{1}{j\omega C}} = \frac{1}{2R + j\omega R^2 C}.$$

Therefore, since the Fourier transform of the input voltage is $V/j\omega$, we have

$$I_2(j\omega) = Y_{21}(j\omega)V(j\omega) = \frac{V}{j\omega(2R + j\omega R^2 C)}.$$

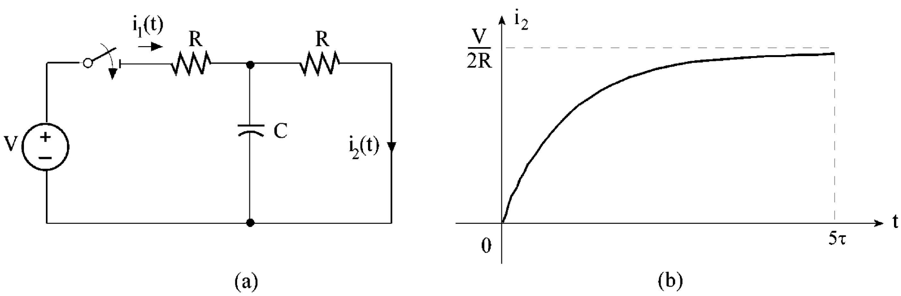


Figure 4.21 *T*-circuit (a) and the waveform of the current (b).

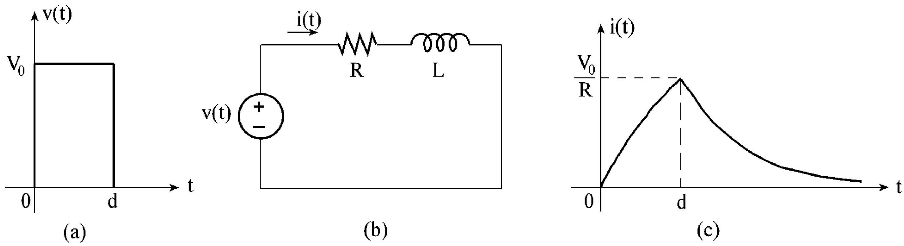


Figure 4.22 Input voltage waveform (a), a given circuit (b), and an input current waveform (c).

In this expression, in accordance to equation 4.93b, $F_3(j\omega) = 2R + j\omega R^2 C$, $F_3'(j\omega) = jR^2 C$ and the root $\omega_1 = j2/RC$. Thus,

$$i_2(t) = \frac{V}{2R} + \frac{Ve^{-\frac{2}{RC}t}}{j(2/RC)(jR^2C)} = \frac{V}{2R}\left(1 - e^{-\frac{2}{RC}t}\right).$$

This waveform of the current $i_2(t)$ is shown in Fig. 4.21(b).

Example 4.8

A rectangular pulse of voltage, Fig. 4.22(a), is applied to the series RL circuit shown in Fig. 4.22(b). Find the circuit current.

Solution

The transform of the given waveform of the applied voltage may be found by using equation 4.20a for the rectangular pulse in the interval 0–d, i.e.

$$V(j\omega) = \frac{V_0}{j\omega}(1 - e^{-j\omega d}).$$

The transform of the circuit impedance is simply $Z(j\omega) = R + j\omega L = jL(\omega - j\xi)$, where $\xi = R/L$ then, with Ohm's law, for the Fourier transforms in equation 4.91 we have

$$I(j\omega) = \frac{V_0(1 - e^{-j\omega d})}{j^2\omega L(\omega - j\xi)} = -\frac{V_0}{L}\frac{1 - e^{-j\omega d}}{\omega(\omega - j\xi)}.$$

or

$$I(j\omega) = -\frac{V_0}{L}\frac{1}{\omega(\omega - j\xi)} + \frac{V_0}{L}\frac{e^{-j\omega d}}{\omega(\omega - j\xi)} = I'(j\omega) + I''(j\omega).$$

The first part of the time-domain function $i(t)$ is found by using the partial fraction expansion (first multiplying the given fraction by $(j\xi)^2$):

$$\frac{-\xi^2}{(j\xi\omega)[j\xi(\omega - j\xi)]} = \frac{1}{j\xi\omega} + \frac{1}{j\xi(\omega - j\xi)} = -\frac{1}{\xi}\frac{1}{j\omega} + \frac{1}{\xi}\frac{1}{\xi + j\omega}.$$

In accordance with Table 4.3 (the 5th and 7th entries) and taking into consideration that $u(t) = \frac{1}{2}\,\mathrm{sgn}(t) + \frac{1}{2}$, we have:

$$\frac{1}{j\omega} \leftrightarrow -\frac{1}{2}\,\mathrm{sgn}(t) = -u(t) + \frac{1}{2}, \quad \frac{1}{\xi + j\omega} \leftrightarrow e^{-\xi t}u(t).$$

Therefore,

$$i'(t) = -\frac{V_0}{L\xi}\left[-\left(u(t) - \frac{1}{2}\right) + e^{-\xi t}u(t)\right] = \frac{V_0}{R}(1 - e^{-\xi t}) - \frac{1}{2}\frac{V_0}{R}.$$

The second part of the current $i(t)$ differs from the first one by the sign and the shifting factor $e^{-j\omega\tau}$, therefore,

$$i''(t) = -\frac{V_0}{R}(1 - e^{-\xi(t-d)})u(t-d) + \frac{1}{2}\frac{V_0}{R},$$

and finally

$$i(t) = i'(t) + i''(t) = \frac{V_0}{R}[(1 - e^{-\xi t})u(t) - (1 - e^{-\xi(t-d)})u(t-d)].$$

The same results might be obtained by using the switching formula of equation 4.93b. We may write the first part of $I(j\omega)$ as

$$I'(j\omega) = -\frac{V_0}{L}\frac{j}{j\omega(\omega - j\xi)},$$

where $F_1(j\omega) = j$, $F_3(j\omega) = \omega - j\xi$ and $\omega_1 = j\xi$. Then,

$$i'(t) = -\frac{V_0}{L}\left(\frac{j}{-j\xi} + \frac{je^{-\xi t}}{j\xi}\right) = \frac{V_0}{R}(1 - e^{-\xi t}),$$

and for the second part we have

$$i''(t) = -\frac{V_0}{R}(1 - e^{-\xi(t-d)}).$$

Therefore,

$$i(t) = \frac{V_0}{R}[(1 - e^{-\xi t}) - (1 - e^{-\xi(t-d)})], \quad \text{for} \quad t > 0.$$

A plot of this waveform is shown in Fig. 4.22(c).

In general, the time domain current according to equation 4.91 may be found as an inverse Fourier formula

$$i(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\omega)V_{in}(j\omega)e^{j\omega t}\,d\omega. \tag{4.94}$$

Let us consider, for instance, applying a constant voltage source V_0 to any one-port circuit having the admittance

$$Y(j\omega) = G(\omega) - jB(\omega) = \frac{\cos\varphi}{|Z(j\omega)|} - j\frac{\sin\varphi}{|Z(j\omega)|}. \tag{4.95}$$

Using for the input voltage the integral notation of a unit function[(*)]

$$u(t) = \frac{1}{2} + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin\omega t}{\omega}\, d\omega \tag{4.96}$$

we have

$$\begin{aligned} i(t) &= \frac{V_0}{2Z(0)} + \frac{V_0}{\pi}\int_{-\infty}^{\infty} \frac{\sin(\omega t - \varphi)}{\omega|Z(j\omega)|}\, d\omega \\ &= \frac{V_0}{2Z(0)} + \frac{V_0}{\pi}\int_{0}^{\infty} \frac{\cos\varphi \sin\omega t}{\omega|Z(j\omega)|} - \frac{V_0}{\pi}\int_{0}^{\infty} \frac{\sin\varphi \cos\omega t}{\omega|Z(j\omega)|}\, d\omega, \end{aligned}$$

or with equation 4.95

$$i(t) = \frac{G(0)V_0}{2} + \frac{V_0}{\pi}\int_{0}^{\infty} G(\omega)\frac{\sin\omega t}{\omega}\, d\omega - \frac{V_0}{\pi}\int_{0}^{\infty} B(\omega)\frac{\cos\omega t}{\omega}\, d\omega. \tag{4.97}$$

This expression is valid for any instant of time; however, the current in the given circuit should be zero for $t < 0$. Then for $t > 0$ $i(-t)$ should be zero, which results in

$$\frac{G(0)V_0}{2} - \frac{V_0}{\pi}\int_{0}^{\infty} G(\omega)\frac{\sin\omega t}{\omega}\, d\omega - \frac{V_0}{\pi}\int_{0}^{\infty} B(\omega)\frac{\cos\omega t}{\omega}\, d\omega = 0. \tag{4.98}$$

By subtracting equation 4.98 from equation 4.97, we finally have

$$i_{tot} = i(t) - i(-t) = \frac{2V_0}{\pi}\int_{0}^{\infty} G(\omega)\frac{\sin\omega t}{\omega}\, d\omega \quad (\text{for } t > 0). \tag{4.99}$$

This formula can be used for finding the one-port current when only the resistive (active) spectrum of the one-port impedance is known. Thus, if the resistive spectrum of the circuit is given, the most efficient method of calculating the input current is the Fourier transform technique.

Example 4.9

As an example of using this method, let us examine a simple circuit shown in Fig. 4.23(a). (This circuit may be considered as a first moment simplified equivalent circuit of a power transformer, which is a capacitor, C, connected to a

[(*)]This presentation of a unit function is based on the known integral

$$\int_{0}^{\infty} \frac{\sin ax}{x}\, dx = \begin{cases} \pi/2 & \text{for } a > 0 \\ -\pi/2 & \text{for } a < 0 \end{cases}$$

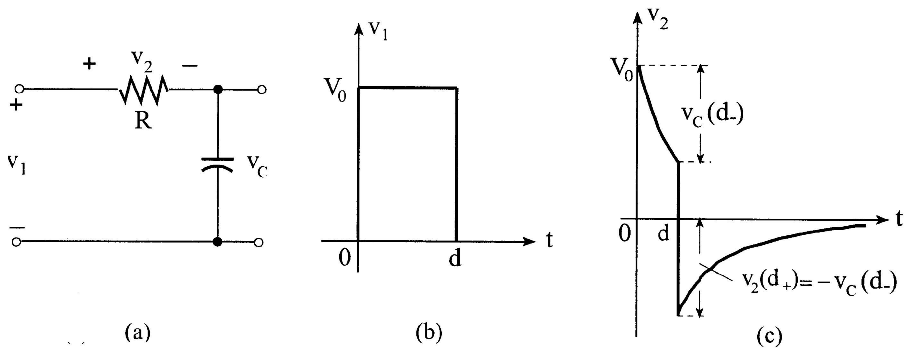


Figure 4.23 A given circuit (a), an applied voltage (b) and the waveform of the voltage across the resistance (c).

cable transmission line, represented by its characteristic resistance, R (see further on in Chapter 7).) Assume that a pulse voltage of rectangular form, 4.23(b), is applied to this circuit and find the voltage across the cable.

Solution

We first should find the real part of the transmission coefficient for $v_2(t)$

$$K_R(\omega) = \mathrm{Re}\left[\frac{R}{R + 1/j\omega C}\right] = \mathrm{Re}\left[\frac{j\omega CR}{1 + j\omega CR}\right] = \frac{(\omega CR)^2}{1 + (\omega CR)^2} = \frac{(\omega\tau)^2}{1 + (\omega\tau)^2},$$

where $\tau = RC$ (time constant). By treating the voltage pulse as two constant voltages shifted by time interval τ, we will have for the first voltage applied at $t = 0$

$$v_2'(t) = \frac{2V_0}{\pi}\int_0^\infty K_R(\omega)\frac{\sin\omega t}{\omega}\,d\omega = \frac{2V_0}{\pi}\int_0^\infty \frac{(\omega\tau)^2}{1 + (\omega\tau)^2}\frac{\sin\omega t}{\omega}\,d\omega.$$

By assigning $x = \omega\tau$ we have $\omega = x/\tau$, $d\omega = (1/\tau)dx$ and

$$v_2'(t) = \frac{2V_0}{\pi}\int_0^\infty \frac{x\sin(x/\tau)t}{1 + x^2}\,dx = \frac{2V_0}{\pi}\frac{\pi}{2}e^{-\frac{t}{\tau}} = V_0 e^{-\frac{t}{\tau}}.^{(*)}$$

The second part of the voltage differs from the first one by the sign and the shifting factor $e^{-j\omega d}$. Therefore,

$$v_2''(t) = -V_0 e^{-\frac{t-d}{\tau}}u(t - d).$$

(*)The integral in this expression is tabulated integral: $\int_0^\infty \frac{x\sin ax}{b^2 + x^2} = \frac{\pi}{2}e^{-(ab)}$.

Finally, we have

$$v_2(t) = v_2'(t) + v_2''(t) = V_0\left[e^{-\frac{t}{\tau}}u(t) - e^{-\frac{t-d}{\tau}}u(t-d)\right].$$

A plot of this waveform is shown in Fig.4.23(c).

4.7.3 Approximate transient analysis with the Fourier transform

In previous paragraphs, we have introduced how to use the Fourier transform for solving problems in circuit transient analysis; but as we have seen, only simple problems can be analyzed using the Fourier method straightforwardly. The main difficulty is in the evolution of the inverse transform integral.

However, the main significance of using the Fourier transform is in the fact that any impulse (such as signals in communication or lightning strokes in power systems) may be presented by its spectra and with the frequency characteristic of the circuit or system function (which usually is known) we can find the spectra of the system input or output response. Since there is a direct connection between Fourier transform techniques and sinusoidal steady-state analysis, the ratio of the phasor response to the phasor forcing function presents the transfer function or the system function

$$\frac{\mathbf{F}_o(j\omega)}{\mathbf{F}_{in}(j\omega)} = \mathbf{K}_{oi}(j\omega) = \frac{B}{A}e^{j(\beta-\alpha)},$$

where A and B are the magnitudes and α and β the phase angles of the input and output phasor for each value of ω. Moreover, we may conclude that the phasor analysis of linear circuits, which is presented in introductory courses, is but a special case of the more general techniques of Fourier transform analysis being studied here. As it was previously shown, the use of Fourier transforms and system functions enables us to handle non-sinusoidal, non-periodic forcing functions and responses. In many cases, when the analytical expression of a system function is not known, there is the possibility of achieving it experimentally. In both cases, the system function is given either analytically or experimentally. To find the time-domain response, we must apply the inverse Fourier transform

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{F}(j\omega)e^{j\omega t}d\omega, \tag{4.100}$$

where $\mathbf{F}(j\omega)$ may be presented, for instance, as a product of a forcing function $\mathbf{V}(j\omega)$ and a system function $\mathbf{K}(j\omega)$:

$$\mathbf{F}(j\omega) = \mathbf{V}_{in}(j\omega)\mathbf{K}_{oi}(j\omega),$$

However, in most practical cases, when the function is fairly complicated, the evaluation of an inverse Fourier transform can be extremely difficult. To find the time-domain description of the response function in such cases, we may apply approximate methods.

(a) *Method of trapezoids*

One of these methods is known as the method of **trapezoids**. To use this method only the real part of the integrand function in equation 4.100 is necessary. To show this, we must first simplify the inverse Fourier transform expression of equation 4.100. Let us assume

$$\mathbf{F}(j\omega) = G(\omega) - jB(\omega) \quad \text{and} \quad e^{j\omega t} = \cos\omega t + j\sin\omega t.$$

Then the integral in equation 4.100 will be

$$f(t) = \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty}[G(\omega)\cos\omega t + B(\omega)\sin(\omega t)]\,d\omega + j\int_{-\infty}^{\infty}[G(\omega)\sin\omega t - B(\omega)\cos(\omega t)]\,d\omega\right\}.$$

The second integral in the above expression should be equal to zero, since the real-time function $f(t)$ cannot include an imaginary part. This decision also follows from the fact that the integrand is an odd function of ω ($G(\omega)$ is even and $B(\omega)$ is odd, therefore $G(\omega)\sin(\omega t)$ is odd and so is $B(\omega)\cos(\omega t)$). With the same consideration, we may conclude that the integrand of the first integral is an even function of ω. Therefore the first integral may be replaced by a double quantity of the same integral, but in limits of 0 and ∞:

$$f(t) = \frac{1}{\pi}\int_{0}^{\infty}[G(\omega)\cos\omega t + B(\omega)\sin(\omega t)]\,d\omega. \tag{4.101a}$$

Furthermore, for the functions, which are zero, i.e. $f(t) = 0$, for $t < 0$ by changing the sign of t, we have

$$f(-t) = \frac{1}{\pi}\int_{0}^{\infty}[G(\omega)\cos\omega t + B(\omega)\sin\omega t]\,d\omega. \tag{4.101b}$$

By adding equation 4.101b to equation 4.101a we obtain a simple expression for the inverse Fourier transform

$$f(t) = \frac{2}{\pi}\int_{0}^{\infty}G(\omega)\cos\omega t\,d\omega \quad \text{for} \quad f(t) = 0|_{t<0}. \tag{4.102}$$

Usually function $G(\omega)$ is finite for $t = 0$ and $G(\omega) \to 0$ for $t \to \infty$, then we can provide the integration of equation 4.102 by parts:

$$f(t) = \frac{2}{\pi}\int_{0}^{\infty}\frac{1}{t}G(\omega)d\sin\omega t = \frac{2}{\pi t}\left\{G(\omega)\sin\omega t\Big|_{\omega=0}^{\omega=\infty} - \int_{0}^{\infty}\sin\omega t\frac{dG(\omega)}{d\omega}d\omega\right\},$$

which finally gives

$$f(t) = -\frac{2}{\pi t}\int_{0}^{\infty}\frac{dG(\omega)}{d\omega}\sin\omega t\,d\omega. \tag{4.103}$$

With this expression, we may find the approximate time-domain response, if the frequency response $G(\omega)$ is known.

Suppose the analytical or experimental curve of $G(\omega)$ is known, as it is shown, for example, in Fig. 4.24(a). We then approximate the given curve $G(\omega)$ by the piecewise-linear curve $\overline{G}(\omega)$ so that a series of trapezoids can be built, whose bases are parallel to the ω axis, one side is perpendicular and the other is at an angle to the ω axis. In such a way, we have obtained in the above example three trapezoids g_1, g_2 and g_3 as shown in Fig. 4.24(b), which by their summation give the approximate curve $\overline{G}(\omega)$:

$$G(\omega) \cong \overline{G}(\omega) = \sum_{i=1}^{n} g_i(\omega).$$

Consider now a single trapezoid $g_{i(\omega)}$, which is shown in Fig. 4.25. For such a

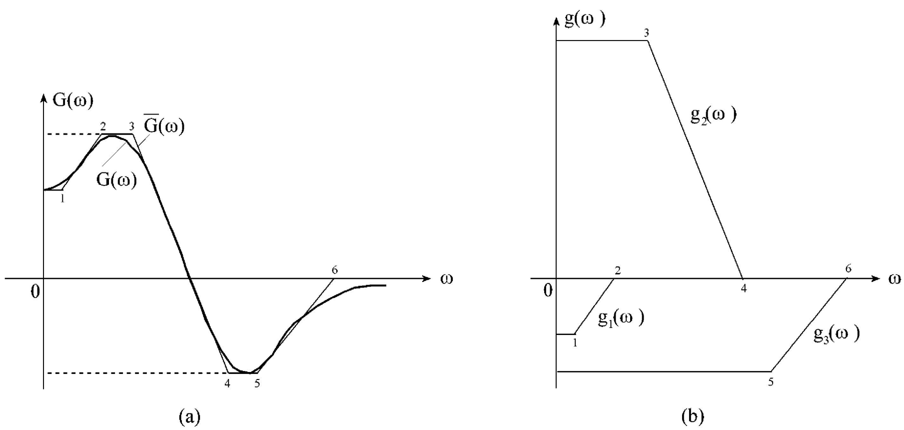


Figure 4.24 Given curve $G(\omega)$ (a) and its approximating trapezoids (b).

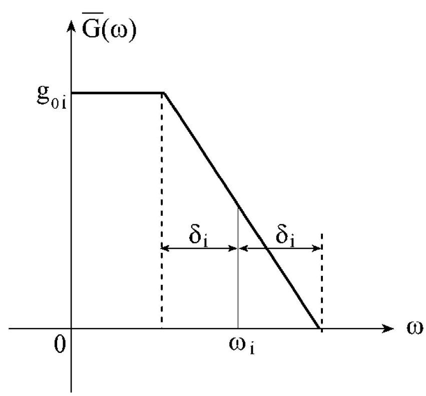


Figure 4.25 A single trapezoid of approximation curve $\overline{G}(\omega)$.

trapezoid its derivative will be:

$$\frac{dG(\omega)}{d\omega} = \begin{cases} 0 & \text{for } 0 < \omega < \omega_i - \delta_i \\ -g_{0i}/2\delta_i & \text{for } \omega_i - \delta_i < \omega < \omega_i + \delta_i. \end{cases}$$

Then the formula in equation 4.103 yields

$$f(t) = \frac{2}{\pi t}\frac{g_{0i}}{2\delta_i}\int_{\omega_i-\delta_i}^{\omega_i+\delta_i} \sin\omega t\, d\omega = -\frac{g_{0i}}{\pi\delta_i t^2}\left[\cos(\omega_i+\delta_i)t - \cos(\omega_i-\delta_i)t\right]$$

$$= 2\frac{g_{0i}\omega_i}{\pi}\frac{\sin\omega_i t}{\omega_i t}\frac{\sin\delta_i t}{\delta_i t},$$

and

$$f(t) = \sum f_i(t) = \frac{2}{\pi}\sum g_{0i}\omega_i \mathrm{Sa}(\omega_i t)\,\mathrm{Sa}(\delta_i t). \tag{4.104}$$

The time response (equation 4.104) may be calculated using the tables of sinc function or with an appropriate computer program. It should be noted that the approximation of $G(\omega)$ by several trapezoids gives in many practical cases good results. The method of trapezoids, actually, is a generalized method because of the fact that any signal may by approximated as a piecewise-linear, in which case the signal reduces to impulses after two (or three) differentiations.

Example 4.10

As an example of using this method let us assume that, at the time $t = 0$, an exponential pulse $V_0 e^{-at}$, shown in Fig. 4.26(a), is applied to RL equivalent circuit, Fig. 4.26(b). Our goal is to find the voltage across the inductance, i.e., an output voltage. The Fourier transform of the output voltage may be found as

$$\mathbf{V}_o(j\omega) = \mathbf{V}_{in}(j\omega)\mathbf{K}_{oi}(j\omega).$$

Here:

$$\mathbf{V}_{in}(j\omega) = \frac{V_0}{a + j\omega}$$

(see 7th entry in Table 4.3) and

$$\mathbf{K}_{oi}(j\omega) = \frac{j\omega L}{R + j\omega L} = \frac{j\omega\tau}{1 + j\omega\tau},$$

where $\tau = L/R$. The real part of $V_o(j\omega)$ then will be

$$G(\omega) = V_0\left|\frac{j\omega\tau}{(a+j\omega)(1+j\omega\tau)}\right| = V_0\frac{-\omega^2\tau(1+a\tau)}{(a-\omega^2\tau)^2 + (\omega(1+a\tau))^2}.$$

The positive plot of this function, for $a = 1$ ms and $\tau = 5$ ms, is shown in

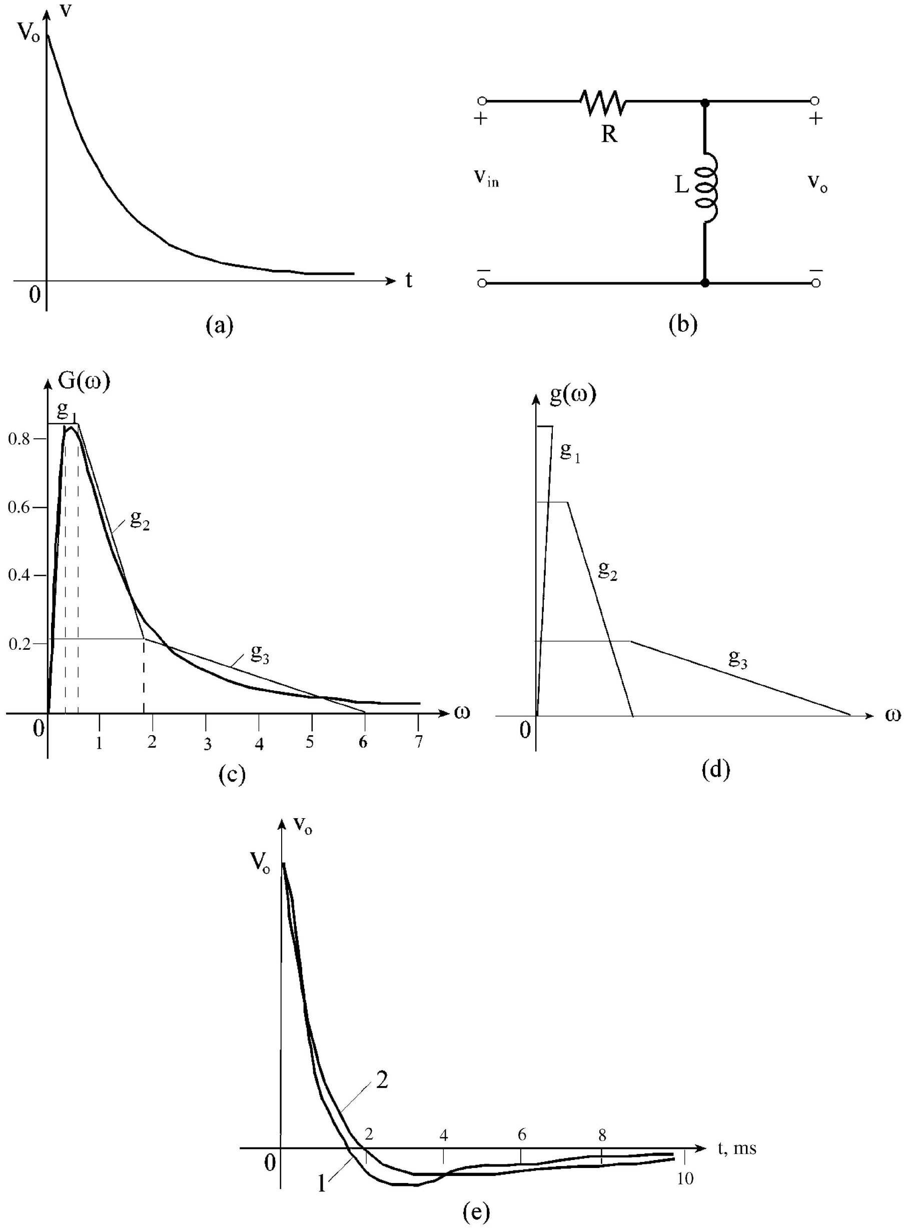


Figure 4.26 An exponential pulse (a), *RL* circuit (b), a positive plot of $|\mathbf{V}_o(j\omega)|$ (c), the obtained trapezoids (d) and the resulting curves of the output voltage $v_o(t)$.

Fig. 4.26(c). This plot might be divided into 4 trapezoids, as shown in Fig. 4.26(d). Then in accordance with equation 4.104 and the data obtained from Fig. 4.26(d) the time-domain response of the output voltage can be calculated, and the result is shown in Fig. 4.26(c), curve 1. Note that at the first moment the whole voltage applied to the circuit is transferred to the output: $v_o(0) = v_{in}(0)$, since the current, $i(0)$, is equal to zero.

This example is, of course, simple enough to use approximate methods and can be easily solved analytically, for instance with switching formula in equation 4.93. (The result is $0.25 \cdot (5e^{-t} - e^{-0.2t})$, which is also shown in Fig. 4.26(d), curve 2. However, we brought this example to illustrate the above method, which can be used for solving complicated problems using appropriate computer programs.

Chapter #5

TRANSIENT ANALYSIS USING STATE VARIABLES

5.1 INTRODUCTION

When the dynamic behavior of a circuit is under consideration, the equations representing the circuit, say in node or mesh analysis, are generally integro-differential. They can then be transformed into one scalar differential equation of the second or higher order. However, the differential equations of a circuit may also be written as a set of first-order differential equations, or when expressed in matrix form it results in a first-order vector differential equation of the form

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{w}, t),$$

where $\mathbf{x}$ is a vector of unknown variables called *state variables*, $\mathbf{w}$ represents the set of inputs and t is the time.

The set of first-order differential equations written in such a form is called a *state equation* and the vector $\mathbf{x}$ represents the *state* of the network. State equations play an important role in the study of the dynamic behavior of a circuit. There are three basic advantages in using the state equations in this form. (1) There is an enormous amount of mathematical knowledge for solving such equations while the equations by themselves can be derived from formal topological properties of the circuit, using the matrix approach. (2) It can be easily and naturally extended to nonlinear and time-varying or switched networks and is, in fact, the approach most often used in characterizing such networks and (3) it is easily programmed for and solved by computers.

In this chapter, we shall formulate, derive and solve first-order vector differential equations, i.e. state equations. As before, we shall be limited here to linear, time-invariant circuits that may be reciprocal or nonreciprocal. On the other hand, this approach is applicable to circuits of any complicity, especially with computer-aided analysis. In this study, when using a computer is suggested, we are referring to the MATHCAD or MATHLAB programs which are also suitable for symbolic computation.

5.2 THE CONCEPT OF STATE VARIABLES

Two general methods of circuit analysis are usually studied in-depth in introductory courses in circuit analysis[(*)], namely nodal analysis and mesh analysis. Both of these methods are very useful for resistive d.c. and *RLC* a.c. circuits in their steady-state behavior. The basic variables in these two kinds of circuits, node voltages and mesh currents, were constant quantities, i.e. with no variation in time. Thus, the nodal and mesh equations in such circuits happen to be algebraic equations, without derivatives and integrals. However, node voltages or mesh currents when used as basic variables in *transient analysis* are expressed as a function of time. Therefore, the node and loop equations here are in general integro-differential equations of the second order.

Consider, as an example, the circuit in Fig. 5.1, in which the inductor current and two capacitor currents may be expressed as

$$i_{L2} = \frac{1}{L_2}\int_0^t (v_{n1} - v_{n2})d\tau + I_0, \tag{5.1a}$$

$$\begin{aligned} i_4 &= C_4\frac{dv_{C4}}{dt} = C_4\frac{dv_{n2}}{dt} \\ i_5 &= C_5\frac{dv_{C5}}{dt} = C_5\frac{dv_{n3}}{dt} \end{aligned} \tag{5.1b}$$

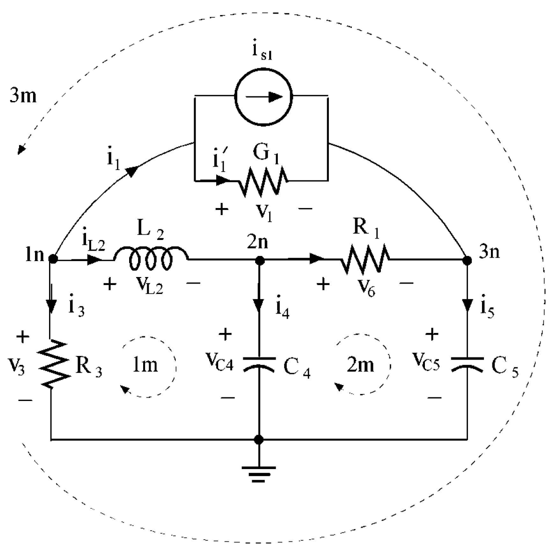


Figure 5.1 Circuit of the example for writing node and mesh equations.

[(*)]See for example W. H. Hayt and J. E. Kemmerly (1998) *Engineering Circuit Analysis*, McGraw-Hill.

Then the node equations may be written by inspection of the circuit as:

$$(G_1+G_2)v_{n1}+\frac{1}{L_2}\int_0^t v_{n1}d\tau-\frac{1}{L_2}\int_0^t v_{n2}d\tau-G_1v_{n3}=-i_{s1}-I_0$$

$$-\frac{1}{L_2}\int_0^t v_{n1}d\tau+G_6v_{n2}+C_4\frac{dv_{n2}}{dt}+\frac{1}{L_2}\int_0^t v_{n2}d\tau-G_6v_{n3}=I_0$$

$$-G_1v_{n1}-G_6v_{n3}+C_5\frac{dv_{n3}}{dt}=i_{s1}. \tag{5.2}$$

Once these equations are solved for the node voltages v_{n1}, v_{n2} and v_{n3}, the remaining variables are easily obtained.

However, the presence of the integrals of unknowns in node equations 5.2 causes some difficulties in the solution. The integrals can be eliminated by differentiating the equations in which they appear, but this will increase the order of the derivatives. An easier way of analyzing would be if we avoid the appearance of the integrals altogether. We note that an integral appears in the present example of node equations when the current of an inductor is eliminated by using equation 5.1a. In a similar way, the integrals appear in mesh equations when the voltages of the capacitors are eliminated by substituting their v–i relationship. Therefore these integrals will not appear if we leave both the capacitor voltages and inductor currents as variables using a mixed set of equations, i.e. based on Kirchhoff's laws.

Let us illustrate this idea of using capacitor voltages and inductor currents as unknown variables in the same example of the circuit in Fig. 5.1. We may write three independent KCL equations for the nodes $1n$, $2n$ and $3n$, and three KVL equations for loops (meshes) indicated by the dashed arrows:

$$\begin{aligned}i_1'+i_{L2}+i_3&=-i_{s1},\\-i_{L2}+i_4+i_6&=0,\\-i_1'+i_5-i_6&=i_{s1},\end{aligned} \tag{5.3a}$$

$$\begin{aligned}v_{L2}+v_{C4}-v_3&=0,\\-v_{C4}+v_6+v_{C5}&=0,\\v_3-v_{C5}-v_1&=0.\end{aligned} \tag{5.5b}$$

Substituting equation 5.1b for i_4 and i_5, taking into consideration that $L_2(di_{L2}/dt)=v_{L2}$ and eliminating all branch voltages except for the capacitor voltages by using the v–i relationships, and after rearranging the terms, yields

$$\begin{aligned}C_4\frac{dv_{C4}}{dt}&=i_{L2}=i_6,\\C_5\frac{dv_{C5}}{dt}&=i_1'+i_6+i_{s1},\\L_2\frac{di_{L2}}{dt}&=-v_{C4}+R_3i_3\end{aligned} \tag{5.4}$$

$$R_6 i_6 = v_{C5} - v_{C4} \tag{5.5a}$$

$$\begin{aligned} i_1' + i_3 &= i_{L2} - i_{s1} \\ R_1 i_1' - R_3 i_3 &= v_{C5}. \end{aligned} \tag{5.b}$$

These are six equations in six unknowns. However, we can reduce the number of equations that must be solved simultaneously. We note that equations 5.5a and 5.5b are algebraic, i.e., they contain no derivatives or integrals. They can be used to eliminate the rest of the unknown variables in (5.4) except v_{C4}, v_{C5} and i_{L2}, whose derivatives are involved in these equations. The algebraic equations 5.5a and 5.5b can be easily solved (the first one trivially) to yield

$$\begin{aligned} i_6 &= -\frac{1}{R_6} v_{C4} + \frac{1}{R_6} v_{C5} \\ i_1' &= \frac{1}{R_1 + R_3} v_{C5} + \frac{R_3}{R_1 + R_3} i_{L2} - \frac{R_3}{R_1 + R_2} i_{s1} \\ i_3 &= -\frac{1}{R_1 + R_3} v_{C5} + \frac{R_3}{R_1 + R_3} i_{L2} - \frac{R_1}{R_1 + R_3} i_{s1}. \end{aligned} \tag{5.6}$$

Finally, these equations can be substituted into equation 5.4 to yield, after rearrangement,

$$\begin{aligned} C_4 \frac{dv_{C4}}{dt} &= \frac{1}{R_6} v_{C4} - \frac{1}{R_6} v_{C5} + i_{L2} \\ C_5 \frac{dv_{C5}}{dt} &= -\frac{1}{R_6} v_{C4} - \frac{R_1 + R_3 + R_6}{R_6(R_1 + R_3)} v_{C5} + \frac{R_3}{R_1 + R_3} i_{L2} + i_{s1} \\ L_2 \frac{di_{L2}}{dt} &= -v_{C4} - \frac{R_3}{R_1 + R_3} v_{C5} + \frac{R_3 R_1}{R_1 + R_3} i_{L2} - \frac{R_1 R_3}{R_1 + R_3} i_{s1}, \end{aligned} \tag{5.7a}$$

or in matrix form, after dividing by the coefficients on the left,

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} v_{C4} \\ v_{C5} \\ i_{L2} \end{bmatrix} &= \begin{bmatrix} \dfrac{1}{C_4 R_6} & -\dfrac{1}{C_4 R_6} & \dfrac{1}{C_4} \\ -\dfrac{1}{C_5 R_6} & \dfrac{R_1 + R_3 + R_6}{C_5 R_6 (R_1 + R_3)} & \dfrac{R_3}{C_5(R_1 + R_3)} \\ -\dfrac{1}{L_2} & -\dfrac{R_3}{L_2(R_1 + R_3)} & \dfrac{R_1 R_3}{L_2(R_1 + R_3)} \end{bmatrix} \begin{bmatrix} v_{C4} \\ v_{C5} \\ i_{L2} \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ \dfrac{1}{C_5} \\ -\dfrac{R_1 R_3}{L_2(R_1 + R_3)} \end{bmatrix} i_{s1}. \end{aligned} \tag{5.7b}$$

The resulting *matrix equation* 5.7b represents three first-order differential equations in three unknowns. It is called the **state equation** and the variables v_{C4}, v_{C5} and i_{L2} are called the **state variables**.

As can be seen, the advantage of this method is that no integrals appear, and subsequently no second derivatives occur as a result of the differentiation. The initial conditions, or **initial state** of the circuit, are the initial values of the capacitor voltages and inductor currents, which usually can be independently specified in the circuit, i.e. their values just after t_0 are determined by their values just before t_0. This is the second reason for choosing capacitor voltages and inductor currents as unknown variables.

Further advantages in describing the network by first-order differential equations are:

1) A simple systematic method for writing such equations can be formulated by using the graph theory.
2) A systematic matrix solution may be applied for solving these first-order differential equations. It may be easily programmed for a numerical and symbolic solution with appropriate computer software.
3) It is quite easy to extend the state-variable representation to time-varying and nonlinear networks.

The concept of *state variables*, or just **state**, satisfies two basic conditions of circuit analysis:

a) If at any time, say t_0, the state is known (which is the initial condition or initial state), then the state equations uniquely determine the state at any time $t > t_0$ for any given input. In other words, given the state of the circuit at time t_0 and all the inputs, the behavior of the circuit is completely determined for all $t > t_0$.

b) The state and the input uniquely determine the value of the remaining circuit variables.

Proof a) From the theory of differential equations we know that the initial values of the variables uniquely define, by differential equations, such as 5.7, the value of the variables for all $t \geq t_0$. In other words, the state $(v_C(t), i_L(t))$ can be expressed by the state equations in terms of the initial state.

Proof b) We may use the substitution (or compensation) principle, which states that in any linear circuit any voltage drop across a passive element, say the capacitance, may be substituted by an independent voltage source equal to this drop. In addition, any current through a passive element, say the inductance, may be substituted by an independent current source equal to this current. Hence, we will replace all the inductors by independent current sources whose values $i_L(t)$ are given by the found state variables and all the capacitors by independent voltage sources whose values are equal to the found state variables $v_C(t)$. As a result, we will obtain a pure resistive network in which any variable can be determined by any well-known method of resistive circuit analysis.

For example, let the desired output quantities be v_3 and v_6 in the circuit being considered in Fig. 5.1. Since $v_3 = R_3 i_3$ and $v_6 = R_6 i_6$, by multiplying the third and the first equations of 5.6 correspondingly by R_3 and R_6, we have

$$v_3 = -\frac{R_3}{R_1 + R_3} v_{C5} + \frac{R_1 R_3}{R_1 + R_3} i_{L2} - \frac{R_1 R_3}{R_1 + R_3} i_{s1}$$

$$v_6 = -v_{C4} + v_{C5},$$

where v_{C4}, v_{C5} and i_{L2} represent the voltage and current sources, which substitute the elements C_4, C_5 and L_2 subsequently. The above expressions in matrix form are

$$\begin{bmatrix} v_3 \\ v_6 \end{bmatrix} = \begin{bmatrix} 0 & -\dfrac{R_3}{R_1 + R_3} & \dfrac{R_1 R_3}{R_1 + R_3} \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_{C4} \\ v_{C5} \\ i_{L2} \end{bmatrix} + \begin{bmatrix} -\dfrac{R_1 R_3}{R_1 + R_3} \\ 0 \end{bmatrix} [i_{s1}]. \quad (5.8)$$

This matrix equation is called an output equation.

Both the state equation 5.7b and the output equation 5.8 equations may be written in compact matrix notation as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\mathbf{w} \quad (5.9a)$$

$$\mathbf{y} = \mathbf{c}\mathbf{x} + \mathbf{d}\mathbf{w}, \quad (5.9b)$$

where **x** is the state vector, **w** is the input and **y** is the output vector. The meanings of matrixes, **A**, **b**, **c** and **d**, which are dependent upon circuit elements, are obvious from equations 5.7b and 5.8.

Next, we shall consider the number of independent state variables that represent the transient behavior of a network.

5.3 ORDER OF COMPLEXITY OF A NETWORK

As is known, node-voltage, mesh-currents, and mixed variable equations (based on Kirchhoff's two laws) completely represent any electrical circuit. Recall that the number of independent node-voltage equations, i.e., number of independent Kirchhoff's current law (KCL) equations, is $B-(N-1)$, where B is the number of branches and N is the number of nodes. These numbers are determined only by the graph of the circuit and not by the types of the branches, i.e. they would not be influenced if the branches were all resistors, or if some were capacitors and/or inductors. However, in resistive circuits driven by d.c. sources the node or mesh equations are algebraic, with no variation in time. On the other hand, when capacitors or inductors are present, the equations will be integro-differential. Hence, the question is how many independent variables represent the circuit in its transient (dynamic) behavior. We know that each capacitor and each inductor introduces a variable in such behavior since the v-i characteristic of each contains a derivative or integral. We also know that, for a unique solution of differential equations, the arbitrary constants have to be determined.

The number of these constants is equal to the number of independent initial conditions that can be specified in a circuit. It is also known that the number of initial conditions is related to the energy-storing elements, capacitors and inductors, and in general is equal to the number of such elements in the circuit. The exceptions are the, so-called, **all-capacitor loops** and **all-inductor cut-sets**. Consider the circuit shown in Fig. 5.2. There are five energy-storing elements, but in this circuit there is an all-capacitor loop, consisting of two capacitors C_1 and C_2 and a voltage source, and an all-inductor cut-set (see dashed line in Fig. 5.2) consisting of three inductors L_3, L_4 and L_5. In this case, the capacitor voltages and inductor currents will be restricted by KVL and KCL, namely

$$v_{C1} + v_{C2} = v_{s8} \tag{5.10a}$$

$$i_{L4} + i_{L5} = i_{L3}, \tag{5.10b}$$

which means that one of the voltages and one of the currents can be determined if the other is known. This also means that the initial values of both v_{C1} and v_{C2} cannot be prescribed independently, nor can the initial values of all three currents i_{L3}, i_{L4} and i_{L5}. Therefore, each of the constraint relationships, such as equations 5.10a and 5.10b, reduce the number of independent variables.

In other words, the order of complexity of any network equals the total number of *energy-storing elements minus the number of all-capacitor loops and the number of all-inductor cut-sets*. Thus, the order of complexity of the circuit of Fig. 5.2 is $5 - 1 - 1 = 3$. Note that (1) all-capacitor loops may also consist of ideal voltage sources and all-inductor cut-sets may also include ideal current sources, and (2) only independent all-capacitor loops and all-inductor cut-sets are taken into account(*).

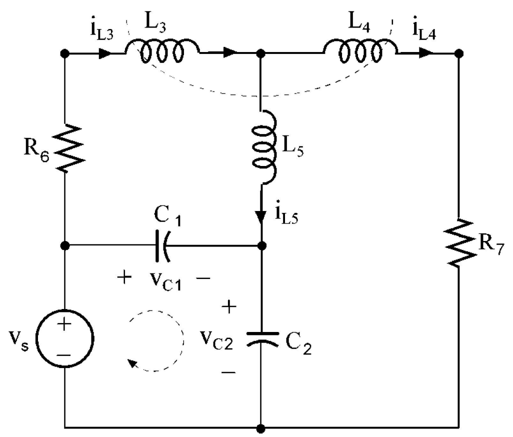


Figure 5.2 Circuit with an all-capacitor loop and an all-inductor cut-set.

(*)The opposite situation, when the circuit consists of all-inductor loops and all-capacitor cut-sets, does not influence the order of complexity, but it influences the values of the natural frequencies, namely $s = 0$. For more about all-capacitor loops/cut-sets and all-inductor cut-sets/loops see in Balabanian, N. and Bickart T. A. (1969) *Electrical Network Theory*, John Wiley & Sons.

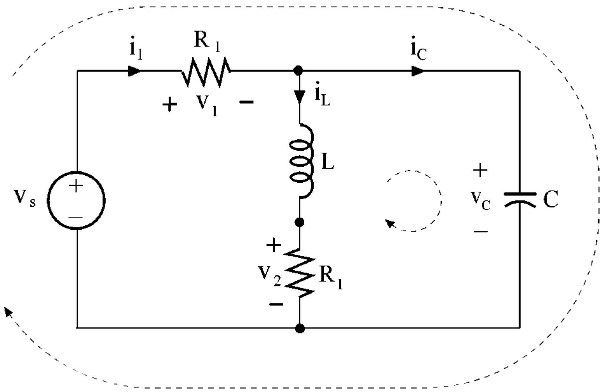


Figure 5.3 Second order circuit.

5.4 STATE EQUATIONS AND TRAJECTORY

Consider the circuit in Fig. 5.3. Let us use capacitor voltage v_C and inductor current i_L as state variables. Applying KCL to node $1n$ and KVL to the right loop and outer loop, we obtain

$$C\frac{dv_C}{dt} = -i_L + i_1, \quad L\frac{di_L}{dt} = v_C - R_2 i_L \tag{5.11}$$

$$R_1 i_1 + v_C = v_s, \tag{5.12}$$

Eliminating the non-desirable variable i_1 from equation 5.12 and substituting it into equation 5.11, after rearranging the terms, gives the state equations

$$\begin{aligned} \frac{dv_C}{dt} &= -\frac{1}{CR_1}v_C - \frac{1}{C}i_L + \frac{1}{CR_1}v_s, \\ \frac{di_L}{dt} &= \frac{1}{L}v_C - \frac{R_2}{L}i_L, \end{aligned} \tag{5.13}$$

or in *matrix form*

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\mathbf{w}(t), \tag{5.14}$$

where:

$\mathbf{x}(t) = \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}$ is a *vector of state variables,*

$\mathbf{A} = \begin{bmatrix} -\frac{1}{CR_1} & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R_2}{L} \end{bmatrix}$ is a *constant* 2×2 *matrix,*

$\mathbf{b} = \begin{bmatrix} -\dfrac{1}{R_1} \\ 0 \end{bmatrix}$ is a *constant vector*,

$\mathbf{w}(t) = v_s(t)$ is the *scalar input*, or *input vector*.

For solving equation 5.14, the initial conditions of the inductor current and of the capacitor voltage have to be known. Thus, the pair $i_L(0) = I_0$ and $v_C(0) = V_0$ is called the initial state

$$\mathbf{x}_0 = \begin{bmatrix} I_0 \\ V_0 \end{bmatrix} \tag{5.15}$$

The zero input response, i.e., circuit response when $\mathbf{w}(t) = 0$,

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) \tag{5.16}$$

is completely determined by the initial state equation 5.15. Thus, if we consider $[i_L(t), v_C(t)]$ as the coordinates of a point on the $i_L - v_C$ plane, then as t increases from 0 to ∞ the point $[i_L(t), v_C(t)]$ will trace a curve, which is called the *state-space trajectory* and the plane $i_L - v_C$ is called the *state-space* of the circuit. It is obvious that the trajectory curve starts at the initial point (I_0, V_0) and ends at the origin $(0, 0)$ when $t = \infty$. Since $v_C(t)$ and $i_L(t)$ are the components of the state vector $\mathbf{x}(t)$, the trajectory defines it in the state space. The velocity of the trajectory $(di_L/dt, dv_C/dt)$ can be obtained from the state equation 5.16. In other words, the trajectory of the state vector in a two-dimensional space characterizes the behavior of a second order circuit, i.e., for every t, the corresponding point of the trajectory specifies $i_L(t)$ and $v_C(t)$.

As an example, three different kinds of trajectory, for: a) overdamped, b) underdamped and 3) loss-less, are shown in Fig. 5.4(d). Note, that in the first case, the trajectory starts at $(0.7, 0.9)$, when $t = 0$, and ends at the origin $(0, 0)$, when $t = \infty$. In the second case, the trajectory is a shrinking spiral starting at the same point and terminating at the origin. Finally, when the circuit is loss-less (which of course is an ideal circuit) the trajectory is an ellipse centered at the origin whose semi-axes depend on the circuit parameters L and C and the initial state $[i_L(0), v_C(0)]$. The ellipse shape trajectory indicates that the response is oscillatory.

For suitably chosen different initial states (usually uniformly spaced points) in the $i_L - v_C$ plane we obtain a family of trajectories, called a *phase portrait*, as shown in Fig. 5.5(a).

As we have already mentioned, the state equations in matrix representation may be easily programmed to a numerical solution. Let us illustrate the approximate method for the calculation of the trajectory. We start at the initial point, determined by the initial state $\mathbf{x}_0[v_C(0), i_L(0)]^T$, and step forward a small interval of time to find an estimate of $\mathbf{x}$ at this new time. From this point we step

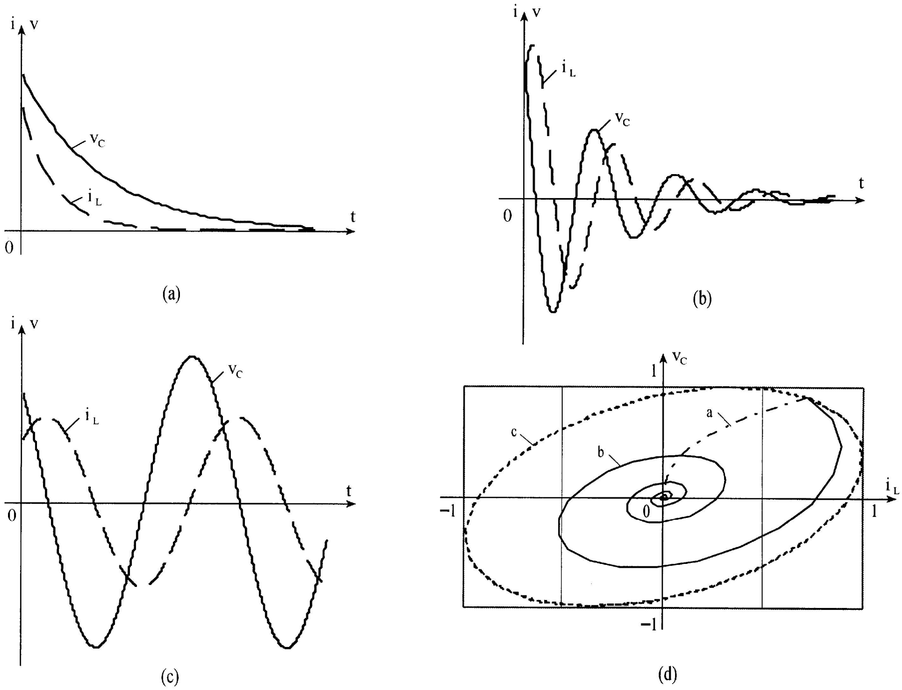


Figure 5.4 Waveforms for i_L and v_C in the second order circuits of an overdamped response (a), underdamped response (b), loss-less response (c) and state trajectories (d).

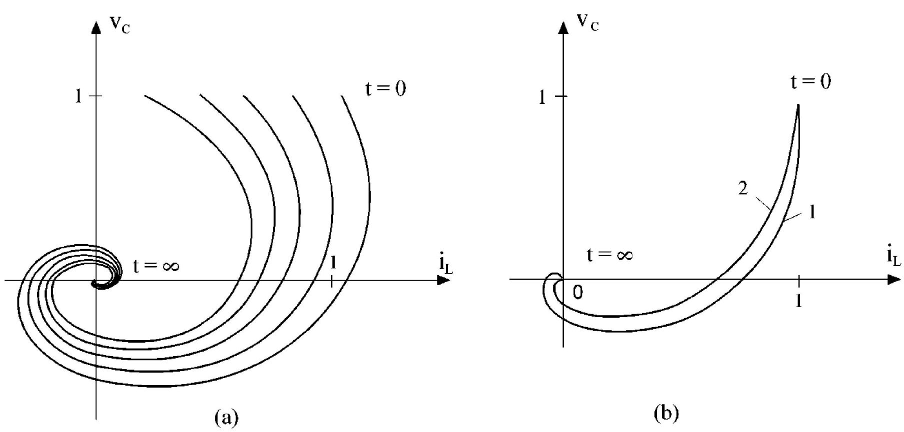


Figure 5.5 State trajectories: phase portrait (a) and for Example 5.1 (b): 1) an approximation with $\Delta t = 0.2$ s and 2) an exact trajectory.

forward again and estimate $\mathbf{x}$ after another short interval of time and so on. The estimate of $\mathbf{x}$ at the new time is found by evaluating $d\mathbf{x}/dt$ at the old time using the differential equation 5.16 and estimating the new value of $\mathbf{x}$ by the formula

$$\mathbf{x}_{new} = \mathbf{x}_{old} + \Delta t \left(\frac{d\mathbf{x}}{dt}\right)_{old}, \tag{5.17}$$

where Δt is the "step length". This step-by-step method is known as Euler's method.

Essentially, we are using a straight-line approximation to the function in each interval. In other words, this method is based on the assumption that if a sufficiently small interval of time Δt is chosen, then during that interval the trajectory velocity $d\mathbf{x}/dt$ is approximately constant. Thus, the straight-line segment, which approximates the trajectory on each step of calculation, is

$$\Delta\mathbf{x} = \left(\frac{d\mathbf{x}}{dt}\right)_{const} \Delta t.$$

It is obvious that the approximation calculated in this manner reaches the exact trajectory when Δt approaches zero. In practice, the value of Δt that should be selected depends primarily on the accuracy required and on the length of the time interval over which the trajectory is calculated. Once the trajectory is computed, the response of the circuit is easily obtained by plotting each of the state variables v_C, i_L versus time.

Example 5.1

Let us employ Euler's (first-order) method to calculate the state trajectory and capacitor voltage versus the time of the circuit shown in Fig. 5.3.

Solution

Let the values of the circuit elements be $R_1 = 1\ \Omega$, $R_2 = 1\ \Omega$, $L = 1\ \mathsf{H}$, $C = 1\ \mathsf{F}$ and the initial state be $I_0 = 1\ \mathsf{A}$ and $V_0 = 1\ \mathsf{V}$.

Then, substituting the above parameters in the matrix $\mathbf{A}$, we have the state equation 5.16 as

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \mathbf{x},$$

and the initial state is

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let us pick $\Delta t = 0.1$ s. Using equation 5.17 yields the state at 0.1 s:

$$\mathbf{x}(0.1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0.1 \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 1 \end{bmatrix}.$$

Next, we can obtain the state at $t = 2\Delta t = 0.2$ s:

$$\mathbf{x}(0.2) = \begin{bmatrix} 0.8 \\ 1 \end{bmatrix} + 0.1 \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0.8 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.62 \\ 0.98 \end{bmatrix}.$$

From these two steps, we can write the state at $(k+1)\Delta t$ in terms of the state at $k\Delta t$

$$\mathbf{x}[(k+1)0.1] = \left(1 + 0.1 \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}\right) \mathbf{x}(k\Delta t) = \begin{bmatrix} 0.9 & -0.1 \\ 0.1 & 0.9 \end{bmatrix} \mathbf{x}(k\Delta t).$$

In accordance with this formula the computer-aided calculation results are shown in Fig. 5.5(b). If we use $\Delta t = 0.01$, the resulting trajectory will coincide with the exact trajectory.

In conclusion, the general recurrence formula for approximating the trajectory may be written as(*)

$$\mathbf{x}[(k+1)\Delta t] = (1 + \Delta t\, \mathbf{A})\mathbf{x}(k\Delta t). \tag{5.18}$$

5.5 BASIC CONSIDERATIONS IN WRITING STATE EQUATIONS

In this section, we shall introduce a systematic method for writing state equations. This method is based on the topological properties of the network and is called the "proper tree" method. However, we must first consider KCL and KVL equations based on a cut-set and loop analysis.

5.5.1 Fundamental cut-set and loop matrixes

As is known from matrix analysis, the matrix formulation of independent KCL equations is given by using the reduced incident matrix **A**. Recall that for any connected graph, having N nodes and B branches, **A** has $N-1$ rows and B columns. Thus, the set of $N-1$ linearly independent KCL equations, written on the node basis, has the matrix form

$$\mathbf{Ai} = \mathbf{0}. \tag{5.19}$$

However, equation 5.19 is not the only way of writing KCL equations. It may also be done on the cut-set basis. A cut-set is defined as a set of k branches with the property that if all k branches are removed from the graph, it is separated into two parts. As an example, consider the graph shown in Fig. 5.6.

(*)For a more accurate approximation of the state-space trajectory, the Runge-Kutta fourth-order method can be used (see, for example in Bajpai, A. C., et al. (1974) *Engineering Mathematics*, John Wiley & Sons.

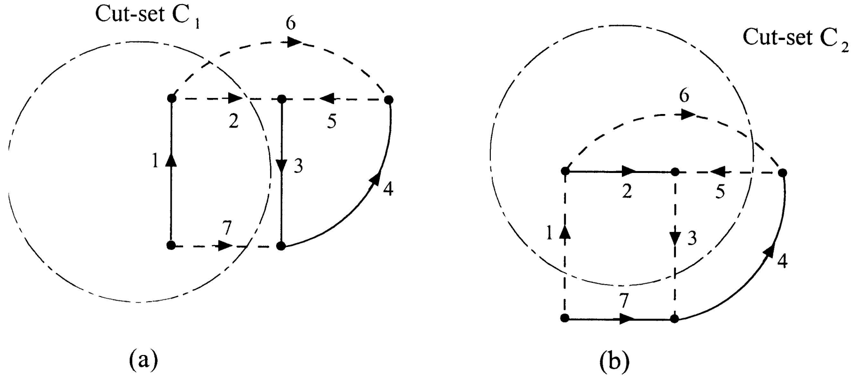


Figure 5.6 Two distinct cut-sets indicated by dashed lines.

Two distinct cut-sets are shown by dashed lines, namely $C_1 = (b_2, b_6, b_7)$ and $C_2 = (b_1, b_3, b_5, b_6)$. Recall now the generalized version of the KCL. By enclosing one of the cut parts of the circuit in the balloon-shaped surface, (see the dotted-dash line in Fig. 5.6(b)) we can write a KCL equation for this particular cut-set

$$-i_1 + i_3 - i_4 + i_5 = 0.$$

The number of such KCL equations is obviously equal to the number of distinct cut-sets. However, as we know, the number of independent KCL equations is $N - 1$, where N is the number of nodes in the graph/circuit. Naturally, we are interested in writing linearly independent cut-set equations. For this purpose, we shall introduce the so-called **fundamental cut-set**. Choosing any tree in the graph, we define a fundamental cut-set as that associated with the tree branch, i.e. every tree branch together with some links constitutes a **unique cut-set** of the graph. Such a cut-set is shown, for example, in Fig. 5.7. As can be seen, removing the tree branch t_3 separates the tree into two parts T_1 and T_2. Then the links ℓ_a and ℓ_b together with twig t_3 constitute a unique cut-set. Indeed, removing any of the remaining links, even all of them (thin lines), cannot

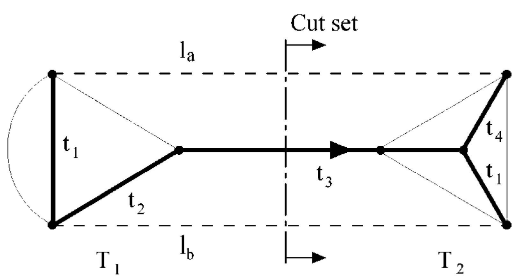


Figure 5.7 An example of a graph, tree and fundamental cut-set.

separate either T_1 or T_2 into two parts. Therefore, the above cut-set is unique. Obviously, each of the fundamental cut-sets is independent of any other, because each of them contains one and only one twig. Since the number of twigs in any tree is $N-1$, we can write $N-1$ linearly independent KCL equations following $N-1$ fundamental cut-sets. Note that the orientation of each fundamental cut-set is defined by the direction of the associated twig as shown in Fig. 5.7.

We will next consider the oriented graph of Fig. 5.8(a). A chosen tree is shown by heavy lines, and four fundamental cut-sets associated with four twigs (since a given graph has five nodes) are marked by dashed lines. For the sake of convenience, we first number the twigs from 1 to 4 and the links from 5 to 7, and adopt a reference direction for the cut-set, which agrees with the tree branch defining the cut-set. Applying KCL to the four cut-sets, we obtain

$$
\begin{array}{lllllllll}
\text{cut-set 1:} & i_1 & & & & & & +\, i_7 & = 0 \\
\text{cut-set 2:} & & i_2 & & & & +\, i_6 & +\, i_7 & = 0 \\
\text{cut-set 3:} & & & i_3 & & -\, i_5 & +\, i_6 & -\, i_7 & = 0 \\
\text{cut-set 4:} & & & & i_4 & -\, i_5 & +\, i_6 & & = 0,
\end{array}
$$

or in matrix form

$$
\begin{array}{cc}
\textit{cut sets} & \begin{array}{cc} \overbrace{\begin{array}{cccc}1 & 2 & 3 & 4\end{array}}^{\textit{twigs}} & \overbrace{\begin{array}{ccc}5 & 6 & 7\end{array}}^{\textit{links}} \end{array} \\
\begin{array}{c}1\\2\\3\\4\end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \end{bmatrix}
\end{array}
\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.20}
$$

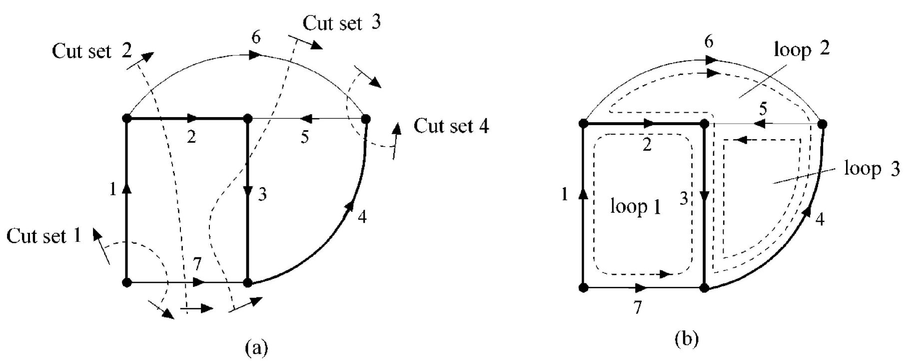


Figure 5.8 Fundamental cut-sets for the chosen tree (dashed lines) (a) and fundamental loops (dashed lines) (b).

In general, the KCL equations based on the fundamental cut-sets may be written in the short form:

$$\mathbf{Qi} = \mathbf{0}, \tag{5.21}$$

where $\mathbf{Q}$ is the *fundamental cut-set matrix* associated with the tree. The order of the $\mathbf{Q}$ matrix is $(N-1) \times B$, and its jk-th element is defined as follows:

$$q_{jk} \begin{cases} 1 & \text{if branch } k \text{ belongs to cut-set } j \text{ and has the same direction} \\ -1 & \text{if branch } k \text{ belongs to cut-set } j \text{ and has the opposite direction} \\ 0 & \text{if branch } k \text{ does not belong to cut-set } j. \end{cases}$$

Note that the fundamental cut-set matrix in equation 5.20 includes a unit sub-matrix of order $(N-1)$, which is the number of fundamental cut-sets and the number of twigs. Therefore,

$$\mathbf{Q} = [\mathbf{1}_t \quad \mathbf{Q}_\ell], \tag{5.22}$$

where $\mathbf{Q}_\ell$ is a sub-matrix of the order $(N-1) \times \ell$, i.e. it consists of $(N-1)$ rows and of ℓ (number of links) columns. The fundamental cut-set matrix $\mathbf{Q}$ will always have the form of equation 5.22 because each fundamental cut-set contains one and only one twig and its orientation agrees with the reference direction of the cut-set, by definition.

Next, we shall introduce the loop matrix. Mesh analysis, which is commonly studied in introductory courses in circuit analysis, is not the only method of writing a set of independent equations based on KVL. Another and actually more flexible method, which allows us to derive independent KVL equations, is based on the so-called **fundamental loop**. Every link of a co-tree (complement of the tree) together with some twigs, which are connected to the link, constitutes a unique loop associated with the link. Indeed, there cannot be any other path between two nodes of the tree, to which the link is connected. If there were two or more paths between two nodes of the tree, they will form a loop; this contradicts the main property of a tree. The set of fundamental loops is independent, since each of them contains one and only one link, i.e. every loop differs from another by at least one branch. Therefore, each link uniquely defines a fundamental loop. Hence, the number of fundamental loops is equal to the number of links, i.e. $B-(N-1)$. Each fundamental loop has a reference direction, which is defined by the direction of its associated link, as shown in Fig. 5.8(b).

So we use the fundamental loops to define $B-(N-1)$ linearly independent KVL equations. For the graph in Fig. 5.8(b), we may write the following three independent KVL equations:

$$\begin{array}{lllllll} \text{Loop 1:} & & v_3 + v_4 + v_5 & & & = 0 \\ \text{Loop 2:} & & -v_2 - v_3 - v_4 & & + v_6 & & = 0 \\ \text{Loop 3:} & -v_1 - v_2 - v_3 & & & & + v_7 & = 0 \end{array}$$

or in matrix form

$$\begin{array}{cc} \underset{\downarrow}{\textit{loops}} & \overbrace{\begin{array}{cccc}1 & 2 & 3 & 4\end{array}}^{\textit{twigs}}\ \overbrace{\begin{array}{ccc}5 & 6 & 7\end{array}}^{\textit{links}} \\ \begin{array}{c}1\\2\\3\end{array} & \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 & 0 \\ -1 & -1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.23}$$

In general, the KVL equations based on fundamental loops may be written in the short form:

$$\mathbf{Bv} = \mathbf{0}, \tag{5.24}$$

where $\mathbf{B}$ is the *fundamental loop matrix* associated with the tree. The order of the $\mathbf{B}$ matrix is $\ell \times B$, where ℓ is the number of loops, and its jk-th element is defined as follows:

$$b_{jk} \begin{cases} 1 & \text{if branch } k \text{ belongs to loop } j \text{ and has the same direction as the loop} \\ -1 & \text{if branch } k \text{ is in loop } j \text{ and has the opposite direction} \\ 0 & \text{if branch } k \text{ is not in loop } j. \end{cases}$$

Note that the fundamental loop matrix in equation 5.23 includes a unit sub-matrix of order ℓ, which is the number of fundamental loops and also the number of links. Therefore, we can express $\mathbf{B}$ in the form

$$\mathbf{B} = [\mathbf{B}_t \quad \mathbf{1}_\ell], \tag{5.25}$$

where $\mathbf{B}_t$ is a sub-matrix of $\ell \times (N-1)$, i.e. it consists of ℓ (number of links) rows and of $t = N - 1$ (number of twigs) columns. The unit matrix in $\mathbf{B}$ results from the fact that each fundamental loop contains one and only one link and by convention the reference directions of the fundamental loops are the same as that of the associated links.

Let us think that twig voltages are a set of the basic independent variables. Since each fundamental loop is formed from twigs and only one link, the link voltage can always be expressed in terms of twig voltages. Therefore, the branch voltages in any circuit can be determined by twig voltages, when the latter ones are used as independent variables. Indeed, in accordance with equations 5.24 and 5.25

$$[\mathbf{B}_t \quad \mathbf{1}_\ell] \begin{bmatrix} \mathbf{v}_t \\ \mathbf{v}_\ell \end{bmatrix} = \mathbf{0}, \tag{5.26}$$

where the branch voltage vector $\mathbf{v}$ is partitioned into two sub-vectors: $\mathbf{v}_t$ and

$\mathbf{v}_\ell$, which are, respectively, the twig-voltage sub-vector and link-voltage sub-vector. Performing the multiplication yields

$$\mathbf{B}_t \mathbf{v}_t + \mathbf{v}_\ell = \mathbf{0},$$

or

$$\mathbf{v}_\ell = -\mathbf{B}_t \mathbf{v}_t. \tag{5.27}$$

This means that link voltages are determined by twig voltages. Obviously, we can write the twig branch-voltage sub-vector as

$$\mathbf{v}_t = \mathbf{1}_t \mathbf{v}_t. \tag{5.28}$$

Combining equations 5.27 and 5.28, we have

$$\begin{bmatrix} \mathbf{v}_t \\ \mathbf{v}_\ell \end{bmatrix} = \begin{bmatrix} \mathbf{1}_t \\ -\mathbf{B}_t \end{bmatrix} \mathbf{v}_t, \tag{5.29}$$

or simply

$$\mathbf{v} = \begin{bmatrix} \mathbf{1}_t \\ -\mathbf{B}_t \end{bmatrix} \mathbf{v}_t, \tag{5.30}$$

which states that all the branch voltages in any circuit can be expressed in terms of twig voltages.

Now, let us again examine the fundamental cut-sets. Since each fundamental cut-set is formed from links and only one twig, we can express the twig-currents in terms of link-currents. Therefore, using the link-currents as basic independent variables, we can always determine the all branch currents by the independent variables. After partitioning the branch currents into twig-currents and link-currents, with equations 5.21 and 5.22, we have

$$[\mathbf{1}_t \quad \mathbf{Q}_\ell] \begin{bmatrix} \mathbf{i}_t \\ \mathbf{i}_\ell \end{bmatrix} = \mathbf{0}, \tag{5.31}$$

where $\mathbf{i}_t$ and $\mathbf{i}_\ell$ are, respectively, the twig-current and link-current sub-vectors. Then two matrixes in equation 5.31 can be multiplied to yield

$$\mathbf{i}_t + \mathbf{Q}_\ell \mathbf{i}_\ell = \mathbf{0},$$

or

$$\mathbf{i}_t = -\mathbf{Q}_\ell \mathbf{i}_\ell. \tag{5.32}$$

Combining equation 5.32 and the identity $\mathbf{i}_\ell = \mathbf{1}_\ell \mathbf{i}_\ell$, yields

$$\begin{bmatrix} \mathbf{i}_t \\ \mathbf{i}_\ell \end{bmatrix} = \begin{bmatrix} -\mathbf{Q}_\ell \\ \mathbf{1}_\ell \end{bmatrix} \mathbf{i}_\ell, \tag{5.33}$$

or

$$\mathbf{i} = \begin{bmatrix} -\mathbf{Q}_\ell \\ \mathbf{1}_\ell \end{bmatrix} \mathbf{i}_\ell, \tag{5.34}$$

which again states that all branch currents in any circuit can be expressed in terms of link currents.

A useful relation between two matrixes $\mathbf{Q}$ and $\mathbf{B}$ can now be determined. Recall *Tellegen's theorem* in the form

$$\mathbf{v}^T\mathbf{i} = \mathbf{0}. \tag{5.35}$$

By taking the transpose of $\boldsymbol{v}$ (equation 5.30), we obtain

$$\mathbf{v}^T = \left(\begin{bmatrix} \mathbf{1}_t \\ -\mathbf{B}_t \end{bmatrix}\mathbf{v}_t\right)^T = \mathbf{v}_t^T\begin{bmatrix} \mathbf{1}_t \\ -\mathbf{B}_t \end{bmatrix}^T = \mathbf{v}^T[\mathbf{1}_t - \mathbf{B}_t^T]. \tag{5.36}$$

After substituting equations 5.36 and 5.34 into equation 5.35 we have

$$\mathbf{v}_t^T[\mathbf{1}_t - \mathbf{B}_t^T]\begin{bmatrix} -\mathbf{Q}_t \\ \mathbf{1}_\ell \end{bmatrix}\mathbf{i}_\ell = \mathbf{0}, \quad \text{for all } \mathbf{v}_t \text{ and all } \mathbf{i}_\ell. \tag{5.37}$$

Since $\mathbf{v}_t^T \neq \mathbf{0}$ and $\mathbf{i}_\ell \neq \mathbf{0}$ then

$$[\mathbf{1}_t - \mathbf{B}_t^T]\begin{bmatrix} -\mathbf{Q}_t \\ \mathbf{1}_\ell \end{bmatrix} = \mathbf{0}. \tag{5.38}$$

Performing the multiplication, we obtain the identities

$$\mathbf{Q}_\ell = -\mathbf{B}_t^T \tag{5.39a}$$

and

$$\mathbf{B}_t = -\mathbf{Q}_\ell^T. \tag{5.39b}$$

This relationship between two sub-matrixes $\mathbf{Q}_\ell$ and $\mathbf{B}_t$ results from the fact that both fundamental cut-set matrix $\mathbf{Q}_\ell$ and fundamental loop matrix $\mathbf{B}_t$ give the topological relation between graph branches and fundamental cut-sets and fundamental loops respectively. Also, note that both matrixes $\mathbf{Q}_\ell$ and $\mathbf{B}_t$ come from the same tree.

Replacing $-\mathbf{B}_t$ by $\mathbf{Q}_\ell^T$ in equation 5.30, we obtain

$$\mathbf{v} = \begin{bmatrix} \mathbf{1}_t \\ \mathbf{Q}_\ell^T \end{bmatrix}\mathbf{v}_t = \mathbf{Q}^T\mathbf{v}_t, \tag{5.40}$$

which can be interpreted as a matrix transformation of twig-voltages into branch voltages. Similarly, replacing $-\mathbf{Q}_\ell$ by $\mathbf{B}_t^T$ in equation 5.34, we obtain

$$\mathbf{i} = \begin{bmatrix} \mathbf{B}_t^T \\ \mathbf{1}_\ell \end{bmatrix}\mathbf{i}_\ell = \mathbf{B}^T\mathbf{i}_\ell, \tag{5.41}$$

which is a matrix transformation of link-currents into branch currents.

Finally, substituting equations 5.40 and 5.41 into Tellengen's theorem (equation 5.35), we have

$$\mathbf{v}_t^T\mathbf{Q}\mathbf{B}^T\mathbf{i}_\ell = \mathbf{0}, \quad \text{for all } \mathbf{v}_t \text{ and } \mathbf{i}_\ell, \tag{5.42}$$

which can be reduced to the following relation between the matrixes

$$\mathbf{QB}^T = \mathbf{0}. \tag{5.43}$$

In conclusion, the following comments on loop and cut-set matrixes have to be made. The methods of circuit analysis based on loop and cut-set matrixes are more flexible, allowing more possible applications than the node and mesh analyses. So, as we remember, the mesh analysis based on mesh matrix **M** is restricted to the planar graph only, whereas the fundamental loop matrix **B**, based on tree, is applicable to any graph including non-planar, by means of allowing us to write a maximal number of linearly independent KVL equations.

The concept of duality is usually applied (in introductory courses) to planar graphs and planar circuits by means of node and mesh terms. By now, we may extend this concept to fundamental matrixes **B** and **Q**, pertaining to non-planar graphs and circuits. So, the listing of dual terms can be extended as follows:

Twig	– Link,
Fundamental cut-set	– Fundamental loop,
Twig voltage, v_t	– Link current, i_ℓ,
Fundamental cut-set matrix, **Q**	– Fundamental loop matrix, **B.**

Thus, two graphs, G_1 and G_2 having the same number of branches, are dual if the number of fundamental cut-sets of one of them is equal to the number of fundamental loops of the second and their **Q** and **B** matrixes are identical, namely

$$\mathbf{Q}_1 = \mathbf{B}_2.$$

5.5.2 "Proper tree" method for writing state equations

Our aim now is to write the state and output equations in the form of equation 5.9

$$\dot{\mathbf{x}}(t) = \mathbf{Ax}(t) + \mathbf{bw}(t) \tag{5.44a}$$

$$\mathbf{y}(t) = \mathbf{cx}(t) + \mathbf{dw}(t), \tag{5.44b}$$

where **x** is the state vector containing all the capacitor voltages and all the inductor currents, **w** is the input vector containing all the independent voltage and current sources, driving the circuit and **y** is the desired output vector. **A**, **b**, **c** and **d** are constant matrixes whose elements depend on circuit parameters. Equation 5.44a is a first order matrix differential equation with constant matrix coefficients. $\dot{\mathbf{x}}$ is the first derivative of the state vector **x**, i.e. it consists of the derivatives of the state variables dv_C/dt and di_L/dt. We note that these quantities are given by currents in the capacitors $C(dv_C/dt)$ and voltages across inductors $L(di_L/dt)$. To evaluate capacitor currents in terms of other currents, we must write cut-set equations and to evaluate inductor voltages in terms of other voltages we must write loop equations. Therefore, it turns out that we could do this if, using the concept of cut-set and loop analysis, we chose a tree which

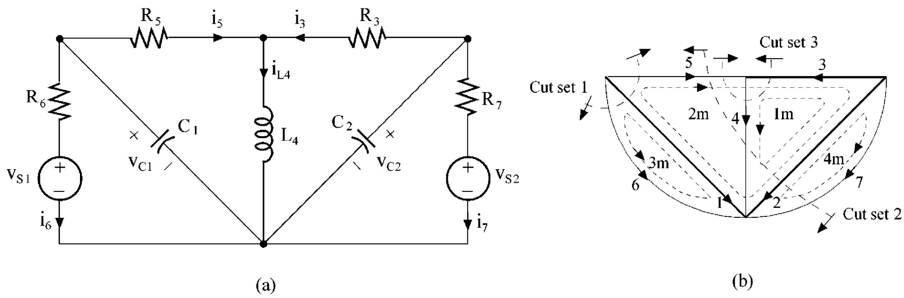


Figure 5.9 A circuit of the example for writing state equations (a), the oriented graph and proper tree (b).

includes all the capacitors but no inductors. Such a tree is called a *proper tree*(*) We can complete the proper tree if the number of twigs is larger than the number of capacitors by adding resistors and voltage sources. Thus, the inductors, the remaining resistors and possibly the current sources will constitute the co-tree links.

Following this method, we may write a fundamental cut-set equation for each capacitor-twig, in which the capacitor current $C(dv_C/dt)$ is expressed in terms of other currents. We may write a fundamental loop equation as well for each link inductor in which the inductor voltage $L(di_L/dt)$ is expressed in terms of other voltages. We shall also take into consideration that the basic variables in cut-set/loop analysis are twig voltages and link currents. Hence, we shall use the appropriate v–i relationships for resistive and active elements. Thus for twig resistors we use the form $v_t = Ri$ and for the link resistors $i_\ell = Gv$. For the same reason we put the voltage sources into the twigs and the current sources into the links. (To fulfill these requirements, we can use a source transformation and shifting techniques.) At this point, let us illustrate the above description by the following example. For the sake of generality, we will divide the solution procedure into five steps. Consider the circuit shown in Fig. 5.9(a).

Step 1 Choosing the state variables

The circuit contains two capacitors and one inductor. Therefore, the state variables are v_{C1}, v_{C2} and i_{L4}, and the state vector is

$$\mathbf{x} = \begin{bmatrix} v_{C1} \\ v_{C2} \\ i_{L4} \end{bmatrix}. \tag{5.45}$$

Step 2 Choosing the proper tree

(*)If a circuit contains an all-capacitor loop or an all-inductor cut-set, a proper tree does not exist. For such cases see in Balabanian, N. and Bickart, T. A. (1969) *Electrical Network Theory*, John Wiley & Sons.

The proper tree picked for the circuit, shown in Fig. 5.9(b), includes two capacitors and resistor R_3.

Step 3 Writing the fundamental cut-set equations

These equations are written in such a way that the capacitor currents are defined by other link currents and/or current sources (if such are present), and the remaining currents are written in terms of inductor currents and/or current sources.

$$\text{cut-set 1:} \quad C_1 \frac{dv_{C1}}{dt} = -i_5 - i_6$$
$$\text{cut-set 2:} \quad C_2 \frac{dv_{C2}}{dt} = -i_{L4} + i_5 - i_7 \tag{5.46}$$

$$\text{cut-set 3:} \quad G_3 v_3 + i_5 = i_{L4}. \tag{5.47}$$

Step 4 Writing the fundamental loop equations

The loop equations are written in such a way that the inductor voltages are defined by other twig voltages and/or voltage sources (if such are present), and the remaining voltages are written in terms of capacitor voltages and/or voltage sources

$$\text{Loop 1:} \quad L_4 \frac{di_{L4}}{dt} = v_{C2} - v_3 \tag{5.48}$$

$$\text{Loop 2:} \quad -v_3 + R_5 i_5 = v_{C1} - v_{C2} \tag{5.49}$$

$$\left.\begin{array}{ll} \text{Loop 3:} & R_6 i_6 = v_{C1} - v_{s1} \\ \text{Loop 4:} & R_7 i_7 = v_{C2} - v_{s2} \end{array}\right\}. \tag{5.50}$$

The last two steps lead to state equations

$$\begin{aligned} C_1 \frac{dv_{C1}}{dt} &= -i_5 - i_6 \\ C_2 \frac{dv_{C2}}{dt} &= -i_{L4} + i_5 - i_7 \\ L_4 \frac{di_{L4}}{dt} &= v_{C2} - v_3. \end{aligned} \tag{5.51}$$

Step 5 Expressing the right-hand side of the state equations in terms of state variables and/or inputs. In this example, currents i_5, i_6, i_7 and voltage v_3 have to be expressed in terms of the capacitor voltages v_{C1}, v_{C2} and the inductor current i_{L4}. By solving equations 5.50, we have

$$i_6 = \frac{1}{R_6} v_{C1} - \frac{1}{R_6} v_{s1}, \quad i_7 = \frac{1}{R_7} v_{C2} - \frac{1}{R_7} v_{s2}, \tag{5.52}$$

equations 5.47 and 5.49 form a set of two algebraic equations of two unknowns:

$$\begin{bmatrix} -1 & R_5 \\ G_3 & 1 \end{bmatrix}\begin{bmatrix} v_3 \\ i_5 \end{bmatrix} = \begin{bmatrix} v_{C1} - v_{C2} \\ i_{L4} \end{bmatrix} \tag{5.53}$$

Solving equation 5.53 yields

$$\begin{aligned} v_3 &= -\frac{1}{1+R_5G_3}v_{C1} + \frac{1}{1+R_5G_3}v_{C2} + \frac{R_5}{1+R_5G_3}i_{L4} \\ i_5 &= \frac{G_3}{1+R_5G_3}v_{C1} - \frac{G_3}{1+R_5G_3}v_{C2} + \frac{1}{1+R_5G_3}i_{L4}. \end{aligned} \tag{5.54}$$

Finally, equations 5.52 and 5.54 can be substituted into equation 5.51 to yield, after rearrangement and dividing through the equations by appropriate C_1, C_2, L_4,

$$\frac{d}{dt}\begin{bmatrix} v_{C1} \\ v_{C2} \\ i_{L4} \end{bmatrix} = \begin{bmatrix} -\dfrac{1+aR_6G_3}{R_6C_1} & \dfrac{aG_3}{C_1} & -\dfrac{a}{C_1} \\ \dfrac{aG_3}{C_2} & -\dfrac{1+aR_7G_3}{R_7C_2} & -\dfrac{1-a}{C_2} \\ \dfrac{a}{L_4} & \dfrac{1-a}{L_4} & -\dfrac{aR_5}{L_4} \end{bmatrix}\begin{bmatrix} v_{C1} \\ v_{C2} \\ i_{L4} \end{bmatrix} + \begin{bmatrix} \dfrac{1}{R_6C_1} & 0 \\ 0 & \dfrac{1}{R_7C_2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_{s1} \\ v_{s2} \end{bmatrix}, \tag{5.55}$$

where $a = 1/(1+R_5G_3)$.

Note that state equations here are written in the matrix form of equation 5.44a where the input vector (in this example) is $\mathbf{w} = [v_{s1} \;\; v_{s2}]^T$ and the meanings of matrixes $\mathbf{A}$ and $\mathbf{b}$ are obvious.

Suppose now that the remaining branch variables, i.e. v_3, i_5, i_6 and i_7 are a desired output. Then, using equations 5.54 and 5.52, we can express the output in terms of the state variables and the input as

$$\begin{bmatrix} v_3 \\ i_5 \\ i_6 \\ i_7 \end{bmatrix} = \begin{bmatrix} -a & a & aR_5 \\ aG_3 & -aG_3 & a \\ 1/R_6 & 0 & 0 \\ 0 & 1/R_7 & 0 \end{bmatrix}\begin{bmatrix} v_{C1} \\ v_{C2} \\ i_{L4} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1/R_6 & 0 \\ 0 & -1/R_7 \end{bmatrix}\begin{bmatrix} v_{s1} \\ v_{s2} \end{bmatrix}. \tag{5.56}$$

This is an output equation in the form of equation 5.44b, where the output vector is $\mathbf{y} = [v_3 \;\; i_5 \;\; i_6 \;\; i_7]^T$ and the meanings of the constant matrixes are obvious.

Remark. The capacitor charges and the inductor fluxes can also be used as state variables. Then in the above example the state vector will be

$$\mathbf{x} = [q_1 \quad q_2 \quad \lambda_4]^T,$$

where $q_1 = C_1 v_{C1}$, $q_2 = C_2 v_{C2}$ and $\lambda_4 = L_4 i_{L4}$.

Substituting $v_{C1} = q_1/C_1$, $v_{C2} = q_2/C_2$ and $i_4 = \lambda_4/L_4$ in equation 5.55, and after simplification, we obtain

$$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} -\dfrac{1 + aR_6G_3}{R_6C_1} & \dfrac{aG_3}{C_2} & -\dfrac{a}{L_4} \\ \dfrac{aG_3}{C_1} & -\dfrac{1 + aR_7G_3}{R_7C_2} & -\dfrac{1+a}{L_4} \\ \dfrac{a}{C_1} & \dfrac{1-a}{C_2} & -\dfrac{aR_5}{L_4} \end{bmatrix}\begin{bmatrix} q_1 \\ q_{C2} \\ \lambda_4 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{R_6} & 0 \\ 0 & \dfrac{1}{R_7} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_{s1} \\ v_{s2} \end{bmatrix} \tag{5.57}$$

which is the state equation using the charges and fluxes as state variables.

It is worthwhile mentioning that some other variables in the circuit may be used as state variables. For example, a current through a resistor in parallel with a capacitor or voltage across a resistor in series with an inductor can be treated as state variables. Also any linear combination of capacitor voltages or inductor currents may be used as state variables. This can be helpful in writing state equations when the circuit consists of all-capacitor loops or all-inductor cut-sets. The next step would be to solve the state equations. However, before doing so, we shall consider the general approach for deriving state equations in matrix form.

5.6 A SYSTEMATIC METHOD FOR WRITING A STATE EQUATION BASED ON CIRCUIT MATRIX REPRESENTATION

Consider a network whose elements are inductors, capacitors, resistors and independent sources. As stated, we assume that capacitors do not form a loop and inductors do not form a cut-set. We also assume that the network graph is connected and as a first step we will pick a *proper tree*. We can obviously include all capacitors into the tree branches, since they do not form any loop. Usually, it might be necessary to add some resistors and/or voltage sources in order to complete the tree. Then all the inductors will be assigned to the links. In the next step we shall partition the circuit branches into four sub-sets: the capacitive twigs, the resistive twigs, the inductive links and the resistive links. For the sake of specifics, we shall use an example to illustrate this procedure.

Consider again the circuit shown in Fig. 5.9(a). The circuit graph and the proper tree are shown in Fig. 5.9(b). The KCL equations for the fundamental cut-sets, in accordance with equation 5.31, are

$$[\mathbf{1}_t \quad \mathbf{Q}_\ell] \begin{bmatrix} \mathbf{i}_C \\ \mathbf{i}_G \\ -- \\ \mathbf{i}_L \\ \mathbf{i}_R \end{bmatrix} = \mathbf{0}, \tag{5.58}$$

where subvectors of twig and link currents are

$$\mathbf{i}_t = \begin{bmatrix} \mathbf{i}_C \\ \mathbf{i}_G \end{bmatrix}, \quad \mathbf{i}_\ell = \begin{bmatrix} \mathbf{i}_L \\ \mathbf{i}_R \end{bmatrix}$$

and $\mathbf{i}_C$, $\mathbf{i}_G$, $\mathbf{i}_L$ and $\mathbf{i}_R$ are in turn subvectors representing currents in capacitive and resistive (conductive) twigs and inductive and resistive links, respectively. In our example, these four subvectors are

$$\mathbf{i}_C = \begin{bmatrix} i_{C1} \\ i_{C2} \end{bmatrix}, \quad \mathbf{i}_G = [i_{G3}], \quad \mathbf{i}_L = [i_{L4}], \quad \mathbf{i}_R = \begin{bmatrix} i_{R5} \\ i_{R6} \\ i_{R7} \end{bmatrix} \tag{5.59}$$

and the equation 5.58 becomes

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\ \ \underbrace{\begin{array}{c:ccc} \overbrace{0}^{\mathbf{Q}_{CL}} & \multicolumn{3}{c}{\overbrace{1\quad 1\quad 0}^{\mathbf{Q}_{CR}}} \\ 1 & -1 & 0 & 1 \\ \hdashline \underbrace{-1}_{\mathbf{Q}_{GL}} & \multicolumn{3}{c}{\underbrace{1\quad 0\quad 0}_{\mathbf{Q}_{GR}}} \end{array}}_{\mathbf{Q}_\ell}\right] \begin{bmatrix} i_{C1} \\ i_{C2} \\ i_{C3} \\ \hdashline i_{L4} \\ i_{R5} \\ i_{R6} \\ i_{R7} \end{bmatrix} = \mathbf{0} \tag{5.60}$$

The KVL equations may be written in the form (see equation 5.26)

$$[\mathbf{B}_t \quad \mathbf{1}_\ell] \begin{bmatrix} \mathbf{v}_C \\ \mathbf{v}_G \\ -- \\ \mathbf{v}_L \\ \mathbf{v}_R \end{bmatrix} = \mathbf{0}, \tag{5.61}$$

where

$$\mathbf{v}_t = \begin{bmatrix} \mathbf{v}_C \\ \mathbf{v}_G \end{bmatrix}, \quad \mathbf{v}_\ell = \begin{bmatrix} \mathbf{v}_L \\ \mathbf{v}_R \end{bmatrix}$$

are subvectors of twig and link voltages and $\mathbf{v}_C$, $\mathbf{v}_G$, $\mathbf{v}_L$, $\mathbf{v}_R$ are in turn subvectors representing voltages across the capacitive and resistive (conductive) twigs and inductive and resistive links, respectively. For the circuit in Fig. 5.9(a) the voltage subvectors are

$$\mathbf{v}_C = \begin{bmatrix} v_{C1} \\ v_{C2} \end{bmatrix}, \quad \mathbf{v}_G = [v_{G3}], \quad \mathbf{v}_L = [v_{L4}], \quad \mathbf{v}_R = \begin{bmatrix} v_{R5} \\ v_{R6} - v_{sR6} \\ v_{R7} - v_{sR7} \end{bmatrix} = \begin{bmatrix} v_\ell 5 \\ v_6 \\ v_7 \end{bmatrix} \tag{5.62}$$

where v_{sR6} represents v_{s1} and v_{sR7} represents v_{s2}. The KVL equation 5.61 becomes

$$\underbrace{\begin{bmatrix} \overbrace{\begin{matrix} 0 & -1 \end{matrix}}^{\mathbf{B}_{LC}} & \overbrace{1}^{\mathbf{B}_{LG}} & \begin{matrix} 1 & 0 & 0 & 0 \end{matrix} \\ \underbrace{\begin{matrix} -1 & 1 \\ -1 & 0 \\ 0 & -1 \end{matrix}}_{\mathbf{B}_{RC}} & \underbrace{\begin{matrix} -1 \\ 0 \\ 0 \end{matrix}}_{\mathbf{B}_{RG}} & \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \end{bmatrix}}_{\mathbf{B}_t} \begin{bmatrix} v_{C1} \\ v_{C2} \\ v_{G3} \\ \hdashline v_{L4} \\ v_{R5} \\ v_6 \\ v_7 \end{bmatrix} = \mathbf{0}. \tag{5.63}$$

Note that $\mathbf{B}_t = -\mathbf{Q}_\ell^T$.

Next we shall use the v-i, or i-v characteristics to introduce branch equations. We will employ the concept of a generalized branch, i.e. combining passive and active elements together. However, we must now take into consideration four different branches: two for twigs and two for links, as shown in Fig. 5.10. As was mentioned earlier, we shall assume that the voltage sources are located in the link branches and the current sources are located in the twig branches. Therefore, in matrix form we have:

$$\begin{aligned} &\textit{capacitor twigs} \quad \mathbf{i}_C = \mathbf{C}\frac{d}{dt}\mathbf{v}_C + \mathbf{i}_{sC} \\ &\textit{inductor links} \quad \mathbf{v}_L = \mathbf{L}\frac{d}{dt}\mathbf{i}_L + \mathbf{v}_{sL} \end{aligned} \tag{5.64}$$

$$\begin{aligned} &\textit{resistor twigs} \quad \mathbf{i}_G = \mathbf{G}\mathbf{v}_G + \mathbf{i}_{sG} \\ &\textit{resistor links} \quad \mathbf{v}_R = \mathbf{R}\mathbf{i}_R + \mathbf{v}_{sR} \end{aligned} \tag{5.65}$$

where the matrixes **C**, **L**, **G** and **R** are the branch parameter matrixes; namely,

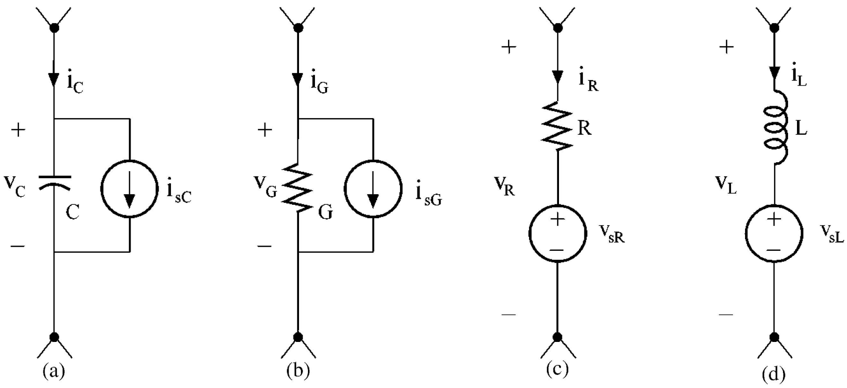


Figure 5.10 Generalized branches with independent sources: twig capacitor (a), twig resistor (b), link resistor (c) and link inductor (d).

the twig capacitance matrix, the link inductance matrix, the twig conductance matrix and the link resistance matrix, respectively. Note that **C**, **L**, **G** and **R** are square diagonal matrixes, but if the circuit consists of coupled elements (mutual inductances and/or dependent sources), **L**, **G** and **R** might not be diagonal any more. For the example in Fig. 5.9

$$\mathbf{C} = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}, \quad \mathbf{L} = [L_4] \tag{5.66}$$

$$\mathbf{G} = [G_3], \quad \mathbf{R} = \begin{bmatrix} R_5 & & 0 \\ & R_6 & \\ 0 & & R_7 \end{bmatrix}. \tag{5.67}$$

The vectors $\mathbf{v}_{sR}$, $\mathbf{v}_{sL}$ and $\mathbf{i}_{sG}$, $\mathbf{i}_{sC}$ represent the independent voltage and current sources, which in the present example are

$$\mathbf{v}_{sR} = \begin{bmatrix} 0 \\ v_{s1} \\ v_{s2} \end{bmatrix}, \quad \mathbf{v}_{sL} = \mathbf{0}, \quad \mathbf{i}_{sG} = \mathbf{0}, \quad \mathbf{i}_{sC} = \mathbf{0}. \tag{5.68}$$

Equation 5.64 can be rewritten to yield

$$\mathbf{C}\frac{d}{dt}\mathbf{v}_C = \mathbf{i}_C - \mathbf{i}_{sC}, \quad \mathbf{L}\frac{d}{dt}\mathbf{i}_L = \mathbf{v}_L - \mathbf{v}_{sL}. \tag{5.69}$$

To bring these equations to the form of state equations, we must eliminate the variables. For this purpose, we shall solve the KCL equation 5.58 and KVL equation 5.61 equations together with the branch equations 5.64 and 5.65.

Equations 5.58 and 5.61 can be rewritten as

$$\begin{bmatrix} \mathbf{i}_C \\ \mathbf{i}_G \end{bmatrix} = -\mathbf{Q}_\ell \begin{bmatrix} \mathbf{i}_L \\ \mathbf{i}_R \end{bmatrix} = -\begin{bmatrix} \mathbf{Q}_{CL} & \mathbf{Q}_{CR} \\ \mathbf{Q}_{GL} & \mathbf{Q}_{GR} \end{bmatrix} \begin{bmatrix} \mathbf{i}_L \\ \mathbf{i}_R \end{bmatrix} \tag{5.70a}$$

and

$$\begin{bmatrix} \mathbf{v}_L \\ \mathbf{v}_R \end{bmatrix} = -\mathbf{B}_t \begin{bmatrix} \mathbf{v}_C \\ \mathbf{v}_G \end{bmatrix} = -\begin{bmatrix} \mathbf{B}_{LC} & \mathbf{B}_{LG} \\ \mathbf{B}_{RC} & \mathbf{B}_{RG} \end{bmatrix} \begin{bmatrix} \mathbf{v}_C \\ \mathbf{v}_G \end{bmatrix} \tag{5.70b}$$

where in the following solution matrixes $\mathbf{Q}_\ell$ and $\mathbf{B}_t$ are partitioned into submatrixes. The order of each of the submatrixes in equations 5.70 is determined by the number of twigs (which is the number of rows) and by the number of corresponding links (which is the number of columns) in equation 5.70a and vice versa in equation 5.70b. For example, the number of rows in $\mathbf{Q}_{CL}$ (equation 5.70a) is equal to the number of capacitor currents in $\mathbf{i}_C$ (capacitor twigs) and the number of its columns is equal to the number of inductor currents in $\mathbf{i}_L$ (inductor links). It can also be shown that there are simple relations between $\mathbf{Q}_\ell$ and $\mathbf{B}_t$ submatrixes, namely

$$\mathbf{B}_{LC} = -\mathbf{Q}_{CL}^T, \quad \mathrm{B}_{LG} = -\mathbf{Q}_{GL}^T, \quad \mathbf{B}_{RC} = -\mathbf{Q}_{CR}^T, \quad \mathbf{B}_{RG} = -\mathbf{Q}_{GR}^T. \tag{5.71}$$

The undesirable variables $\mathbf{i}_C$ and $\mathbf{v}_L$ in equation 5.69 can now be expressed from equation 5.70 to yield

$$\mathbf{i}_C = -\mathbf{Q}_{CL}\mathbf{i}_L - \mathbf{Q}_{CR}\mathbf{i}_R \tag{5.72a}$$

$$\mathbf{v}_L = -\mathbf{B}_{LC}\mathbf{v}_C - \mathbf{B}_{LG}\mathbf{v}_G, \tag{5.72b}$$

and after substituting these two expressions into equation 5.69, we obtain

$$\begin{aligned} \mathbf{C}\frac{d}{dt}\mathbf{v}_C &= -\mathbf{Q}_{CL}\mathbf{i}_L - \mathbf{Q}_{CR}\mathbf{i}_R - \mathbf{i}_{sC} \\ \mathbf{L}\frac{d}{dt}\mathbf{i}_L &= -\mathbf{B}_{LC}\mathbf{v}_C - \mathbf{B}_{LG}\mathbf{v}_G - \mathbf{v}_{sL}. \end{aligned} \tag{5.73}$$

However, we still need to eliminate $\mathbf{i}_R$ and $\mathbf{v}_G$. Substituting $\mathbf{i}_G$ and $\mathbf{v}_R$ from equation 5.70 into equation 5.65, and after rearrangement, results in two simultaneous matrix equations in two unknowns $\mathbf{i}_R$ and $\mathbf{v}_G$,

$$\mathbf{R}\mathbf{i}_R + \mathbf{B}_{RG}\mathbf{v}_G = \mathbf{M} \tag{5.73a}$$

$$\mathbf{Q}_{GR}\mathbf{i}_R + \mathbf{G}\mathbf{v}_G = \mathbf{N}, \tag{5.73b}$$

where

$$\mathbf{M} = -\mathbf{B}_{RC}\mathbf{v}_C - \mathbf{v}_{sR} \quad \text{and} \quad \mathbf{N} = -\mathbf{Q}_{GL}\mathbf{i}_L - \mathbf{i}_{sG} \tag{5.74}$$

Solving these two equations by the substitution method yields

$$\mathbf{i}_R = \mathbf{R}_{eq}^{-1}(-\mathbf{B}_{RG}\mathbf{G}^{-1}\mathbf{N} + \mathbf{M}) \tag{5.75a}$$

$$\mathbf{v}_G = \mathbf{G}_{eq}^{-1}(-\mathbf{Q}_{GR}\mathbf{R}^{-1}\mathbf{M} + \mathbf{N}), \tag{5.75b}$$

where

$$\mathbf{R}_{eq} = \mathbf{R} - \mathbf{B}_{RG}\mathbf{G}^{-1}\mathbf{Q}_{GR} \tag{5.76a}$$

$$\mathbf{G}_{eq} = \mathbf{G} - \mathbf{Q}_{GR}\mathbf{R}^{-1}\mathbf{B}_{RG}. \tag{5.76b}$$

Finally, we substitute equation 5.75 with equation 5.74 in equation 5.73 to obtain, after rearrangement, the state representation is follows

$$\frac{d}{dt}\begin{bmatrix}\mathbf{v}_C \\ \mathbf{i}_L\end{bmatrix} = \underbrace{\begin{bmatrix}\mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}\end{bmatrix}^{-1}}_{\mathbf{A}} \underbrace{\begin{bmatrix}\mathbf{A}_{11}^1 & \mathbf{A}_{12}^1 \\ \mathbf{A}_{21}^1 & \mathbf{A}_{22}^1\end{bmatrix}}_{\mathbf{A}^1}\begin{bmatrix}\mathbf{v}_C \\ \mathbf{i}_L\end{bmatrix}$$

$$+ \underbrace{\begin{bmatrix}\mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}\end{bmatrix}^{-1}}_{\mathbf{b}} \underbrace{\begin{bmatrix}\mathbf{b}_{11}^1 & \mathbf{b}_{12}^1 & \mathbf{b}_{13}^1 & \mathbf{b}_{14}^1 \\ \mathbf{b}_{21}^1 & \mathbf{b}_{22}^1 & \mathbf{b}_{23}^1 & \mathbf{b}_{24}^1\end{bmatrix}}_{\mathbf{b}^1}\begin{bmatrix}\mathbf{i}_{sC} \\ \mathbf{i}_{sG} \\ \mathbf{v}_{sL} \\ \mathbf{v}_{sR}\end{bmatrix} \tag{5.77}$$

where the matrix terms are

$$\begin{aligned}
&\mathbf{A}_{11}^1 = \mathbf{Q}_{CR}\mathbf{R}_{eq}^{-1}\mathbf{B}_{RC} \quad && \mathbf{A}_{12}^1 = -\mathbf{Q}_{CL} - \mathbf{Q}_{CR}\mathbf{R}_{eq}^{-1}\mathbf{B}_{RG}\mathbf{G}^{-1}\mathbf{Q}_{GL} \\
&\mathbf{A}_{22}^1 = \mathbf{B}_{LG}\mathbf{G}_{eq}^{-1}\mathbf{Q}_{GL} \quad && \mathbf{A}_{21}^1 = -\mathbf{B}_{LC} - \mathbf{B}_{LG}\mathbf{G}_{eq}^{-1}\mathbf{Q}_{GR}\mathbf{R}^{-1}\mathbf{B}_{RC}
\end{aligned} \tag{5.78}$$

$$\begin{aligned}
&\mathbf{b}_{11}^1 = -\mathbf{1} \quad && \mathbf{b}_{12}^1 = -\mathbf{Q}_{CR}\mathbf{R}_{eq}^{-1}\mathbf{B}_{RG}\mathbf{G}^{-1} \quad && \mathbf{b}_{13}^1 = \mathbf{0} \quad && \mathbf{b}_{14}^1 = \mathbf{Q}_{CR}\mathbf{R}_{eq}^{-1} \\
&\mathbf{b}_{21}^1 = \mathbf{0} && \mathbf{b}_{22}^1 = \mathbf{B}_{LG}\mathbf{G}_{eq}^{-1} && \mathbf{b}_{23}^1 = -\mathbf{1} && \mathbf{b}_{24}^1 = -\mathbf{B}_{LG}\mathbf{G}_{eq}^{-1}\mathbf{Q}_{GR}\mathbf{R}^{-1}.
\end{aligned} \tag{5.79}$$

Let us now use the above expressions to calculate the **A** and **b** matrixes in our example.

First we determine the submatrixes of the $\mathbf{Q}_\ell$ matrix

$$\mathbf{Q}_\ell\begin{bmatrix}\mathbf{Q}_{CL} & \mathbf{Q}_{CR} \\ \mathbf{Q}_{GL} & \mathbf{Q}_{GR}\end{bmatrix} = \left[\begin{array}{c:ccc} 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ \hdashline -1 & 1 & 0 & 0 \end{array}\right].$$

Then with equation 5.76 and equation 5.71 we have

$$\mathbf{R}_{eq}\begin{bmatrix}R_5 & & \mathbf{0} \\ & R_6 & \\ \mathbf{0} & & R_7\end{bmatrix} - \begin{bmatrix}1 \\ 0 \\ 0\end{bmatrix}\left[\frac{1}{G_3}\right][1 \quad 0 \quad 0] = \begin{bmatrix}\dfrac{1+R_5G_3}{G_3} & 0 & 0 \\ 0 & R_6 & 0 \\ 0 & 0 & R_7\end{bmatrix}$$

$$\mathbf{R}_{eq}^{-1} = \begin{bmatrix}aG_3 & 0 & 0 \\ 0 & 1/R_6 & 0 \\ 0 & 0 & 1/R_7\end{bmatrix}, \quad \text{where again } a = \frac{1}{1+R_5G_3}$$

$$\mathbf{G}_{eq} = [G_3] + [1 \quad 0 \quad 0]\begin{bmatrix} 1/R_5 & 0 & 0 \\ 0 & 1/R_6 & 0 \\ 0 & 0 & 1/R_7 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \left[\frac{1 + R_5 G_3}{R_5}\right] = \left[\frac{1}{aR_5}\right]$$

$$\mathbf{G}_{eq}^{-1} = [aR_5]$$

$$\mathbf{A}_{11}^1 = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} aG_3 & & 0 \\ & 1/R_6 & \\ 0 & & 1/R_7 \end{bmatrix}\begin{bmatrix} -1 & +1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= -\begin{bmatrix} (aG_3 + 1/R_6) & -aG_3 \\ -aG_3 & (aG_3 + 1/R_7) \end{bmatrix}$$

$$\mathbf{A}_{22}^1 = [1][aR_5][-1] = -[aR_5]$$

$$\mathbf{A}_{12}^1 = -\begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} aG_3 & 0 & 0 \\ 0 & 1/R_6 & 0 \\ 0 & 0 & 1/R_7 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}[1/G_3][-1]$$

$$= -\begin{bmatrix} 1 \\ 1 - a \end{bmatrix}$$

$$\mathbf{A}_{21}^1 = -[0 \quad -1] - [1][aR_5][1 \quad 0 \quad 0]\begin{bmatrix} 1/R_5 & 0 & 0 \\ 0 & 1/R_6 & 0 \\ 0 & 0 & 1/R_7 \end{bmatrix}\begin{bmatrix} -1 & +1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= [a \quad (1 - a)]$$

$$\begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \end{bmatrix}^{-1} = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & L_2 \end{bmatrix}^{-1} = \begin{bmatrix} 1/C_1 & 0 & 0 \\ 0 & 1/C_2 & 0 \\ 0 & 0 & 1/L_4 \end{bmatrix}$$

Therefore the **A** matrix is

$$\mathbf{A} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \end{bmatrix}^{-1}\begin{bmatrix} \mathbf{A}_{11}^1 & \mathbf{A}_{12}^1 \\ \mathbf{A}_{21}^1 & \mathbf{A}_{22}^1 \end{bmatrix} = \left[\begin{array}{cc|c} -\dfrac{1 + R_6 aG_3}{R_6 C_1} & \dfrac{aG_3}{C_1} & -\dfrac{a}{C_1} \\ \dfrac{aG_3}{C_2} & -\dfrac{1 + R_7 aG_3}{R_7 C_2} & -\dfrac{1-a}{C_2} \\ \hline \dfrac{a}{L_4} & \dfrac{1-a}{L_4} & -\dfrac{aR_5}{L_4} \end{array}\right],$$

which agrees with the results previously obtained (see equation 5.55).

To find the **b** matrix we will calculate equation 5.79. Since only the $\mathbf{v}_{sR}$ vector is present we need only two elements of **b**:

$$\mathbf{b}_{14}^{1} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} aG_3 & 0 & 0 \\ 0 & 1/R_6 & 0 \\ 0 & 0 & 1/R_7 \end{bmatrix} = \begin{bmatrix} aG_3 & 1/R_6 & 0 \\ -aG_3 & 0 & 1/R_7 \end{bmatrix}$$

$$\mathbf{b}_{24}^{1} = -[1][aR_5][1 \quad 0 \quad 0] \begin{bmatrix} 1/R_5 & 0 & 0 \\ 0 & 1/R_6 & 0 \\ 0 & 0 & 1/R_7 \end{bmatrix} = -[a \quad 0 \quad 0]$$

Therefore, the reduced **b** matrix is

$$\mathbf{b} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{b}_{14}^{1} \\ \mathbf{b}_{24}^{1} \end{bmatrix} = \begin{bmatrix} aG_3/C_1 & 1/R_6C_1 & 0 \\ -aG_3/C_2 & 0 & 1/R_7C_2 \\ -a/L_4 & 0 & 0 \end{bmatrix}$$

which also agrees with the results in equation 5.55. Note that a voltage source in link 5 is absent ($v_{sR5} = 0$), therefore the above matrix can be reduced even more, namely

$$\mathbf{b} = \begin{bmatrix} 1/R_6C_1 & 0 \\ 0 & 1/R_7C_2 \\ 0 & 0 \end{bmatrix}$$

which is exactly the same as in equation 5.55.

Comparing the systematic method for writing state equations with the intuitive approach, which we first presented in the previous sections, we may conclude that it is rather complicated. In many practical instances, the final results can be arrived at much easier and faster by following the intuitive approach. However, the systematic method has an appreciable advantage for computer-aided analysis, since it can be easily programmed.

5.7 COMPLETE SOLUTION OF THE STATE MATRIX EQUATION

We will now turn to the solution of the state equation of the form of equation 5.44a, repeated here for convenience:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\mathbf{w}(t). \tag{5.80}$$

5.7.1 The natural solution

We will begin by considering the natural or zero-input (non-forced) solution; that is $\mathbf{w}(t) = 0$. Equation 5.80 then simplifies to

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \quad \text{or} \quad \dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{0}. \tag{5.81}$$

It is customary to compare a vector problem with its scalar version. In this case, the scalar version of equation 5.81 is

$$\frac{dx(t)}{dt} = ax(t). \tag{5.82}$$

The solution of equation 5.82, that satisfies the initial condition $x(0)$, is

$$x(t) = e^{at}x(0).$$

Suppose we try the same form for the solution of equation 5.81, that is

$$\mathbf{x}(t) = \mathbf{e}^{\mathbf{A}t}\mathbf{x}(0). \tag{5.83}$$

where $\mathbf{e}^{\mathbf{A}t}$ is called the *matrix exponential* and is an example of a function of matrix **A**.

5.7.2 Matrix exponential

In mathematics the matrix exponential is defined similarly to a scalar exponential (or complex exponential), i.e. in terms of the power series expansion:

$$\mathbf{e}^{\mathbf{A}t} = \mathbf{1} + \frac{t}{1!}\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \cdots + \frac{t^k}{k!}\mathbf{A}^k + \cdots = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{A}^k. \tag{5.84}$$

Since **A** is a square matrix of order n, the matrix exponential $\mathbf{e}^{\mathbf{A}t}$ is also a square matrix of order n.

Example 5.2

As an example, let us take the matrix of Example 5.1, namely

$$\mathbf{A} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

then

$$\mathbf{A}^2 = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \quad \mathbf{A}^3 = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$$

and

$$\begin{aligned} \mathbf{e}^{\mathbf{A}t} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} + \frac{t^3}{6}\begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} + \cdots \\ &= \begin{bmatrix} 1 - t + \dfrac{t^3}{3} + \cdots & -t + t^2 - \dfrac{t^3}{3} + \cdots \\ t - t^2 + \dfrac{t^3}{3} + \cdots & 1 - t + \dfrac{t^3}{3} + \cdots \end{bmatrix}. \end{aligned} \tag{5.85}$$

As can be seen from equation 5.85, each of the elements of the matrix $e^{\mathbf{A}t}$ is a continuous function of t. Term-by-term differentiation of the matrix exponential (equation 5.84) results in

$$\begin{aligned}\frac{d}{dt}(\mathbf{e}^{\mathbf{A}t}) &= \mathbf{A} + t\mathbf{A}^2 + \frac{t^2}{2!}\mathbf{A}^3 + \frac{t^3}{3!}\mathbf{A}^4 + \cdots \\ &= \mathbf{A}\left(\mathbf{1} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \frac{t^3}{3!}\mathbf{A}^3 + \cdots\right) = \mathbf{A}\mathbf{e}^{\mathbf{A}t},\end{aligned} \tag{5.86}$$

i.e., the formula for the derivative of a matrix exponential is the same as it is for a scalar exponential. Substituting equation 5.83 into the matrix differential equation 5.81, results in identity:

$$\mathbf{A}\mathbf{e}^{\mathbf{A}t}\mathbf{x}(0) = \mathbf{A}\mathbf{e}^{\mathbf{A}t}\mathbf{x}(0).$$

Thus, we have established that equation 5.83 is indeed the solution to equation 5.81.

We must now show that the inverse of a matrix exponential exists and equals $(\mathbf{e}^{\mathbf{A}t})^{-1} = \mathbf{e}^{-\mathbf{A}t}$. For the latter we can write

$$\mathbf{e}^{-\mathbf{A}t} = \mathbf{1} - \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2!} - \mathbf{A}^3\frac{t^3}{3!} + \cdots + (-1)^k\mathbf{A}^k\frac{t^k}{k!} + \cdots.$$

Now let this series be multiplied by the series for the positive exponential in equation 5.84. This term-by-term multiplication results in **1** since all other terms are cancelled. Thus,

$$\mathbf{e}^{\mathbf{A}t}\mathbf{e}^{-\mathbf{A}t} = \mathbf{1}.$$

This result tells us that the matrix $\mathbf{e}^{-\mathbf{A}t}$ is the inverse of $\mathbf{e}^{\mathbf{A}t}$, since by definition the product of the matrix by its inverse gives a unit matrix. This result can be used, first of all, to show that in general if the initial vector $\mathbf{x}(0)$ is known for some time, for instance t_0, namely $\mathbf{x}_{nat}(t_0)$ then the solution will be

$$\mathbf{x}_n(t) = \mathbf{e}^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0). \tag{5.87}$$

Indeed, substituting $t = t_0$, results in identity:

$$\mathbf{x}_n(t_0) = \mathbf{e}^{\mathbf{A}t_0}\mathbf{e}^{-\mathbf{A}t_0}\mathbf{x}(t_0) = \mathbf{1}\mathbf{x}(t_0),$$

where we have used

$$\mathbf{e}^{\mathbf{A}+\mathbf{B}} = \mathbf{e}^{\mathbf{A}}\cdot\mathbf{e}^{\mathbf{B}}.$$

(This can be verified by using equation 5.84 for both sides of equality.)

5.7.3 The particular solution

To find the complete solution to equation 5.80, we must now find the particular solution to the differential equation, i.e. the forced response. For this purpose, assume a solution of the form

$$\mathbf{x}_p(t) = \mathbf{e}^{\mathbf{A}t}\mathbf{q}(t), \tag{5.88}$$

where $\mathbf{q}(t)$ is an unknown function to be determined. In order to be a solution, equation 5.88 has to satisfy the differential equation. Substituting equation 5.88 in equation 5.80 gives

$$\frac{d}{dt}[\mathbf{e}^{\mathbf{A}t}\mathbf{q}(t)] = \mathbf{A}\mathbf{e}^{\mathbf{A}t}\mathbf{q}(t) + \mathbf{b}\mathbf{w}(t),$$

or

$$\mathbf{A}\mathbf{e}^{\mathbf{A}t}\mathbf{q}(t) + \mathbf{e}^{\mathbf{A}t}\frac{d\mathbf{q}(t)}{dt} = \mathbf{A}\mathbf{e}^{\mathbf{A}t}\mathbf{q}(t) + \mathbf{b}\mathbf{w}(t).$$

Thus

$$\frac{d\mathbf{q}(t)}{dt} = \mathbf{e}^{-\mathbf{A}t}\mathbf{b}\mathbf{w}(t). \tag{5.89}$$

Integrating, we obtain

$$\mathbf{q}(t) = \mathbf{q}(t_0) + \int_{t_0}^{t} \mathbf{e}^{-\mathbf{A}\tau}\mathbf{b}\mathbf{w}(\tau)d\tau.$$

Thus, the particular solution is

$$\mathbf{x}_p(t) = \mathbf{e}^{\mathbf{A}t}\mathbf{q}(t) = \mathbf{e}^{\mathbf{A}t}\mathbf{q}(t_0) + \int_{t_0}^{t} \mathbf{e}^{\mathbf{A}(t-\tau)}\mathbf{b}\mathbf{w}(\tau)d\tau.$$

To evaluate $\mathbf{q}(t_0)$, we use the complete solution being evaluated at t_0

$$\mathbf{x}(t)|_{t=t_0} = \mathbf{x}_n(t) + \mathbf{x}_p(t) = \mathbf{e}^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \mathbf{e}^{\mathbf{A}t}\mathbf{q}(t_0) + \left.\int_{t_0}^{t} \mathbf{e}^{\mathbf{A}(t-\tau)}\mathbf{b}\mathbf{w}(\tau)d\tau\right|_{t=t_0},$$

or

$$\mathbf{x}(t_0) = \mathbf{x}(t_0) + \mathbf{e}^{\mathbf{A}t_0}\mathbf{q}(t_0) + \mathbf{0},$$

which implies that $\mathbf{q}(t_0) = \mathbf{0}$.

Hence, finally the complete solution of the state equation 5.80 is

$$\mathbf{x}(t) = \mathbf{e}^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{e}^{\mathbf{A}(t-\tau)}\mathbf{b}\mathbf{w}(\tau)d\tau. \tag{5.90}$$

To evaluate this solution the basic calculation is a determination of the matrix exponential $\mathbf{e}^{\mathbf{A}t}$. This will be discussed in the next subsection.

5.8 BASIC CONSIDERATIONS IN DETERMINING FUNCTIONS OF A MATRIX

In this section, we shall examine two methods of computing $\mathbf{e}^{\mathbf{A}t}$ in closed form. This matrix exponential is a particular function of a matrix. The simplest functions of a matrix are powers of a matrix and polynomials. As we have seen, the matrix exponential can be represented by an infinite series of such functions. The matrix polynomial has the form

$$f(\mathbf{A}) = \mathbf{A}^n + a_{n-1}\mathbf{A}^{n-1} + \cdots + a_1\mathbf{A} + a_0\mathbf{1}. \tag{5.91}$$

The generalization of polynomials is an infinite series:

$$f(\mathbf{A}) = a_0\mathbf{1} + a_0\mathbf{A} + a_2\mathbf{A}^2 + \cdots + a_k\mathbf{A}^k + \cdots = \sum_{k=0}^{\infty} a_k\mathbf{A}^k. \tag{5.92}$$

The function $f(\mathbf{A})$ is itself a matrix, and in the last case each of the matrix elements is an infinite series. This matrix series is said to converge if each of the element series converges.

We will begin with a brief description of some of the properties of matrixes that will be useful in our studies.

5.8.1 Characteristic equation and eigenvalues

An algebraic equation that often appears in network transient analysis is

$$\lambda\mathbf{x} = \mathbf{A}\mathbf{x}, \tag{5.93}$$

where $\mathbf{A}$ is a square matrix of order n. The problem is to find scalars λ and vectors $\mathbf{x}$ that satisfy this equation. A value of λ for which a nontrivial solution of $\mathbf{x}$ exists, is called an eigenvalue, or characteristic value of $\mathbf{A}$. The corresponding vector $\mathbf{x}$ is called an eigenvector, or characteristic vector, of $\mathbf{A}$. After collecting the terms on the left-hand side, we have

$$[\lambda\mathbf{1} - \mathbf{A}]\mathbf{x} = \mathbf{0}. \tag{5.94}$$

This equation will have a nontrivial solution for $\mathbf{x}$ only if the matrix $[\lambda\mathbf{1} - \mathbf{A}]$ is singular, i.e.,

$$\det[\lambda\mathbf{1} - \mathbf{A}] = \mathbf{0}. \tag{5.95}$$

This equation is known as the characteristic equation associated with $\mathbf{A}$. It is also closely related to the auxiliary (characteristic) equation of the corresponding differential equation of order n for the system. The determinant on the left-hand side of equation 5.95 is actually a polynomial of degree n in λ and is called the characteristic polynomial of $\mathbf{A}$. For each value of λ that satisfies the characteristic equation, a nontrivial solution of equation 5.94 can be found. To illustrate this procedure, consider the following example.

Example 5.3

Let us find the eigenvalues and eigenvectors of a matrix of the second order

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}.$$

The characteristic polynomial is also of order two:

$$\det\left\{\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}2 & 1\\ 3 & 4\end{bmatrix}\right\}=\det\begin{bmatrix}\lambda-2 & -1\\ -3 & \lambda-4\end{bmatrix}=\lambda^2-6\lambda+5$$

$$=(\lambda-5)(\lambda-1)=g(\lambda).$$

Thus, $\lambda^2-6\lambda+5=0$ is the characteristic equation of the matrix. The roots of the characteristic equation, or the eigenvalues, are

$$\lambda_1=5 \quad \text{and} \quad \lambda_2=1.$$

To obtain the eigenvector corresponding to the eigenvalue $\lambda_1=5$, we solve equation 5.94 by using the given matrix **A**. Thus

$$\left\{\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}-\begin{bmatrix}2 & 1\\ 3 & 4\end{bmatrix}\right\}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

or

$$\begin{bmatrix}3 & -1\\ -3 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \quad \text{and} \quad x_2=3x_1.$$

Therefore

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}x_1\\ 3x_1\end{bmatrix}=\begin{bmatrix}1\\ 3\end{bmatrix}[x_1] \quad \text{for any value of } x_1.$$

The eigenvector corresponding to the eigenvalue $\lambda_2=1$ is obtained similarly.

$$\begin{bmatrix}-1 & -1\\ -3 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

from which

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}x_1\\ -x_1\end{bmatrix}=\begin{bmatrix}1\\ -1\end{bmatrix}[x_1] \quad \text{for any value of } x_1.$$

The first method to be discussed for finding functions of a matrix is based on the Caley-Hamilton theorem.

5.8.2 The Caley-Hamilton theorem

This theorem states that *every square matrix satisfies its own characteristic equation*. For example, if we substitute **A** for λ in the characteristic equation of Example 5, we obtain the matrix equation

$$g(\mathbf{A})=\mathbf{A}^2-6\mathbf{A}+5\cdot\mathbf{1}=\mathbf{0},$$

where, again, **1** is an identity matrix and **0** is a matrix whose elements are all zero. Thus,

$$\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} - 6\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 18 & 19 \end{bmatrix} - \begin{bmatrix} 12 & 6 \\ 18 & 24 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The equation is certainly satisfied in this example.

The Caley-Hamilton theorem permits us to reduce the order of a matrix polynomial of any higher order to be of an order no greater than $n-1$, where n is the order of the matrix. For example, if $\mathbf{A}$ is a square matrix of order 3, then its characteristic equation is

$$g(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0, \tag{5.96}$$

and by the Caley-Hamilton theorem we have

$$\mathbf{A}^3 + a_2\mathbf{A}^2 + a_1\mathbf{A} + a_0\mathbf{1} = \mathbf{0}.$$

Then

$$\mathbf{A}^3 = -a_2\mathbf{A}^2 - a_1\mathbf{A} - a_0\mathbf{1}. \tag{5.97}$$

Thus, $\mathbf{A}^3$ may be expressed in terms of the matrixes of an order not higher than 2 and identity matrix. Hence, the given polynomial of order 3 is reduced to a polynomial of order 2. To extend these results to polynomials of an even higher order, we multiply equation 5.97 throughout by $\mathbf{A}$ to obtain

$$\mathbf{A}^4 = -a_2\mathbf{A}^3 - a_1\mathbf{A}^2 - a_0\mathbf{A}. \tag{5.98}$$

Substituting equation 5.97 for $\mathbf{A}^4$, we obtain

$$\mathbf{A}^4 = (a_2^2 - a_1)\mathbf{A}^2 + (a_2a_1 - a_0)\mathbf{A} + a_2a_0\mathbf{1}. \tag{5.99a}$$

To generalize these results, let us develop an iterative formula for expressing higher powers of $\mathbf{A}$. We assign the obtained coefficients in equation 5.99 by upper script, as follows

$$\mathbf{A}^4 = a_2^{(1)}\mathbf{A}^2 + a_1^{(1)}\mathbf{A} + a_0^{(1)}\mathbf{1}. \tag{5.99b}$$

Multiplying this expression throughout by $\mathbf{A}$, and collecting like terms, yields

$$\mathbf{A}^5 = (-a_2a_2^{(1)} + a_1^{(1)})\mathbf{A}^2 + (-a_1a_2^{(1)} + a_0^{(1)})\mathbf{A} + (-a_0a_2^{(1)})\mathbf{1} = a_2^{(2)}\mathbf{A}^2 + a_1^{(2)}\mathbf{A} + a_0^{(2)}\mathbf{1},$$

where again $a_2^{(2)}, a_1^{(2)}, a_0^{(2)}$ are the new coefficients and a_2, a_1, a_0 are as before the coefficients of the characteristic equation 5.96. Now the iterative formula for this case, $n = 3$, can be written as

$$\begin{aligned}\mathbf{A}^{3+k} &= (-a_2a_2^{(k-1)} + a_1^{(k-1)})\mathbf{A}^2 + (-a_1a_2^{(k-1)} + a_0^{(k-1)})\mathbf{A} + (-a_0a_2^{(k-1)})\mathbf{1} \\ &= a_2^{(k)}\mathbf{A}^2 + a_1^{(k)}\mathbf{A} + a_0^{(k)}1.\end{aligned} \tag{5.100}$$

Note that this formula also works fine for the first calculation of $\mathbf{A}^4$ (equation

5.99) if the coefficients in equation 5.97 are assigned as $a_2^{(0)} = -a_2$, $a_1^{(0)} = -a_1$ and $a_0^{(0)} = -a_0$. Generalizing this result (equation 5.100) for any matrix of order n, we can write

$$\mathbf{A}^{n+k} = (-a_{n-1}a_{n-1}^{(k-1)} + a_{n-2}^{(k-1)})\mathbf{A}^{n-1} + (-a_{n-2}a_{n-1}^{(k-1)} + a_{n-3}^{(k-1)})\mathbf{A}^{n-2} + \cdots + (-a_0 a_{n-1}^{(k-1)})\mathbf{1}. \qquad (5.101)$$

This gives us an expression for $\mathbf{A}^{n+k}$, $k = 0, 1, 2, \ldots$, in terms of $\mathbf{A}^{n-1}$, $\mathbf{A}^{n-2}, \ldots, \mathbf{A}$ and $\mathbf{1}$.

Continuing this process, we see that any power of $\mathbf{A}$ can be represented as a weighted polynomial in $\mathbf{A}$ of an order, at most $n-1$. Hence, functions of matrixes, including $\mathbf{e}^{\mathbf{A}t}$, that can be expressed as a polynomial(*)

$$f(\mathbf{A}) = \alpha_0\mathbf{1} + \alpha_1\mathbf{A} + \cdots + \alpha_k\mathbf{A}^k + \cdots = \sum_{k=0}^{\infty} \alpha_k\mathbf{A}^k, \qquad (5.102)$$

may be reduced to the expression

$$f(\mathbf{A}) = \beta_0\mathbf{1} + \beta_1\mathbf{A} + \cdots + \beta_{n-1}\mathbf{A}^{n-1} = \sum_{k=0}^{n-1} \beta_k\mathbf{A}^k. \qquad (5.103)$$

Here, the coefficients $\beta_0, \beta_1, \ldots, \beta_{n-1}$ are functions of $a_0, a_1, \ldots, a_{n-1}$ and $\alpha_0, \alpha_1, \ldots$. Their approximate calculation can be carried out by the iterative method used in the calculation of higher powers of $\mathbf{A}$ in equation 5.101 and by using equation 5.102. However this straightforward method can be lengthy.

Example 5.4

(a) Let us first calculate a simple matrix function $f(\mathbf{A}) = \mathbf{A}^4$, where $\mathbf{A}$ is the matrix of the previous example. Since the characteristic equation of $\mathbf{A}$ is $\lambda^2 - 6\lambda + 5 = 0$, we have

$$\mathbf{A}^2 = 6\mathbf{A} - 5\cdot\mathbf{1},$$

where $a_1 = -6$ and $a_0 = 5$. Using an iterative formula, and noting that in the first calculation $a_1^{(0)} = -a_1$ and $a_0^{(0)} = -a_0$, yields

$$\begin{aligned}\mathbf{A}^3 &= [-a_1 a_1^{(0)} + a_0^{(0)}]\mathbf{A} + (-a_0 a_1^{(0)})\mathbf{1} \\ &= [6\cdot 6 - 5]\mathbf{A} + (-5\cdot 6)\mathbf{1} = 31\mathbf{A} - 30\,\mathbf{1},\end{aligned}$$

where $a_1^{(1)} = 31$ and $a_0^{(1)} = -30$. Hence,

$$\mathbf{A}^4 = [(6)(31) - 30]\mathbf{A} - 5\cdot 31\,\mathbf{1} = 156\mathbf{A} - 155\,\mathbf{1},$$

and finally

$$\mathbf{A}^4 = 156\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 155 & 0 \\ 0 & 155 \end{bmatrix} = \begin{bmatrix} 157 & 156 \\ 468 & 469 \end{bmatrix}$$

(*) In general, any analytic function of matrix $\mathbf{A}$ can be expressed as a polynomial in $\mathbf{A}$ of an order no greater than one less than the order of $\mathbf{A}$. For proof see N. Balabanian and T. A. Bickart (1969) *Electrical Network Theory*, John Wiley & Sons.

(b) As a second example, let us calculate a matrix potential $f(\mathbf{A}) = \mathbf{e}^{\mathbf{A}t}$ for $t =$ 1 s, using the approximation up to fifth term:

$$\begin{aligned}\mathbf{e}^{\mathbf{A}} &\cong \mathbf{1} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \frac{1}{3!}\mathbf{A}^3 + \frac{1}{4!}\mathbf{A}^4 \\ &= \mathbf{1} + \mathbf{A} + \frac{1}{2}(-5\cdot\mathbf{1} + 6\mathbf{A}) + \frac{1}{6}(-30\cdot\mathbf{1} + 31\mathbf{A}) + \frac{1}{24}(155\cdot\mathbf{1} + 156\mathbf{A}) \\ &= -12.96\cdot\mathbf{1} + 15.67\mathbf{A}\end{aligned}$$

and finally

$$\mathbf{e}^{\mathbf{A}} \cong \begin{bmatrix} -13 & 0 \\ 0 & -13 \end{bmatrix} + 15.7\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 18.4 & 15.7 \\ 47.1 & 49.8 \end{bmatrix}.$$

We shall next develop an easier, one-step method for finding β-coefficients in the function of matrix expression (equation 5.103). Let us return to the characteristic equation of matrix **A**

$$g(\lambda) = |\lambda\mathbf{1} - \mathbf{A}| = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0. \tag{5.104}$$

The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, which are the roots of the characteristic equation 5.104, obviously satisfy the equation 5.104 as well as matrix **A** (in accordance with the Caleg-Hamilton theorem). Therefore, using the same procedure as before, we can derive an expression similar to equation 5.103 for the eigenvalues instead of the matrix by itself, namely:

$$f(\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2 + \cdots + \beta_{n-1}\lambda^{n-1} = \sum_{k=0}^{n-1} \beta_k\lambda^k. \tag{5.105}$$

It is understandable that this expression holds for any λ that is a solution of the characteristic equation 5.104, that is for any eigenvalue of the matrix **A**.

(a) *Distinct eigenvalues*

Assume first that the eigenvalues are *distinct*; that is, that none is repeated. Substituting $\lambda_1, \lambda_2, \ldots, \lambda_n$ in equation 5.105 gives n equations in n unknown β's:

$$\begin{aligned} &\beta_0 + \beta_1\lambda_1 + \beta_1\lambda_1^2 + \cdots + \beta_{n-1}\lambda_1^{n-1} = f(\lambda_1) \\ &\beta_0 + \beta_1\lambda_1 + \beta_2\lambda_2^2 + \cdots + \beta_{n-1}\lambda_2^{n-1} = f(\lambda_2) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\beta_0 + \beta_1\lambda_n + \beta_2\lambda_n^2 + \cdots + \beta_{n-1}\lambda_1^{n-1} = f(\lambda_n). \end{aligned} \tag{5.106}$$

The coefficients $\beta_0, \beta_1, \ldots, \beta_{n-1}$ can then be obtained as the solution to this linear system of scalar equations, i.e. the inversion of the set of equations 5.106 gives the solution. With the known β-coefficients, the function of the matrix representation problem is solved:

$$f(\mathbf{A}) = \sum_{k=0}^{n-1} \beta_k \mathbf{A}^k. \tag{5.107}$$

Example 5.5

Let us illustrate this process with the same simple example (as in Example 5.4):

(a) Find $f(\mathbf{A}) = \mathbf{A}^4$, if $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

The characteristic equation is (see Example 5.3)

$$g(\lambda) = \lambda^2 - 6\lambda + 5 = 0.$$

Thus, the eigenvalues are

$$\lambda_1 = 5, \quad \lambda_2 = 1.$$

In accordance with equation 5.106, we have

$$\beta_0 + \beta_1 5 = 5^4,$$
$$\beta_0 + \beta_1 1 = 1^4.$$

Solving these simple equations for unknowns β_0 and β_1, gives

$$\beta_1 = 156, \quad \beta_0 = -155.$$

The solution for $\mathbf{A}^4$ is found by using equation 5.107

$$f(\mathbf{A}) = \mathbf{A}^4 = -155 \cdot \mathbf{1} + 156 \cdot \mathbf{A}$$

which is the same as the results obtained in the previous example.

(b) Find $f(\mathbf{A}) = \mathbf{e}^{\mathbf{A}t}$ for the same matrix $\mathbf{A}$

The equations for unknowns β_0 and β_1 in this case will be

$$\beta_0 + 5\beta_1 = e^{5t},$$
$$\beta_0 + \beta_1 = e^t.$$

Solving this equation gives

$$\beta_1 = \tfrac{1}{4}e^{5t} - \tfrac{1}{4}e^t, \quad \beta_0 = -\tfrac{1}{4}e^{5t} + \tfrac{5}{4}e^t.$$

Thus, the matrix exponential is

$$\begin{aligned} \mathbf{e}^{\mathbf{A}t} &= (-\tfrac{1}{4}e^{5t} + \tfrac{5}{4}e^t)\mathbf{1} + (\tfrac{1}{4}e^{5t} - \tfrac{1}{4}e^t)\mathbf{A} \\ &= (-\tfrac{1}{4}e^{5t} + \tfrac{5}{4}e^t)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (\tfrac{1}{4}e^{5t} - \tfrac{1}{4}e^t)\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}. \end{aligned}$$

By an obvious rearrangement, this becomes

$$\mathbf{e}^{\mathbf{A}t} = \begin{bmatrix} \frac{1}{4}e^{5t} + \frac{3}{4}e^{t} & \frac{1}{4}e^{5t} - \frac{1}{4}e^{t} \\ \frac{3}{4}e^{5t} - \frac{3}{4}e^{t} & \frac{3}{4}e^{5t} + \frac{1}{4}e^{t} \end{bmatrix}. \tag{5.108}$$

It is interesting to compare these results with those obtained in the previous example. The approximate, up to fifth term, evaluation of the exponents e^5 and e^1 ($t = 1$ s) gives

$$e^5 \cong 1 + 5 + \frac{1}{2!}5^2 + \frac{1}{3!}5^3 + \frac{1}{4!}5^4 = 65.4$$

$$e^1 \cong 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} = 2.71.$$

Substituting these results in equation 5.108 yields

$$\mathbf{e}^{\mathbf{A}} \cong \begin{bmatrix} 18.4 & 15.6 \\ 47.0 & 49.7 \end{bmatrix}$$

which agrees with the previous results.

Therefore, the series form of the exponential may permit some approximate numerical results; it does not lead to a closed form. However, with the help of the Caley-Hamilton theorem, we obtained the closed-form equivalent for the exponential $\mathbf{e}^{\mathbf{A}t}$ (equation 5.107). We shall now return our consideration to the complete solution of the state equation in the form of equation 5.90, repeated here for convenience:

$$\mathbf{x}(t) = \mathbf{e}^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} \mathbf{e}^{\mathbf{A}(t-\tau)}\mathbf{b}\mathbf{w}(\tau)d\tau. \tag{5.109}$$

The following example illustrates this computation.

Example 5.6

Find the complete solution of the state equation describing the circuit in Fig. 5.9, considered before. For the sake of convenience, it is redrawn here again in Fig. 5.11(a). Let the circuit element values be $C_1 = 1$ F, $C_2 = 2$ F, $L_4 = 1$ H, $G_3 = 1$ S, $R_5 = 1\ \Omega$, $R_6 = 2/7\ \Omega$, $R_7 = 1/3\ \Omega$.

Solution

Substituting these parameters into equation 5.55, we obtain the following **A** matrix

$$\mathbf{A} = \begin{bmatrix} -4 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{7}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}. \tag{5.110}$$

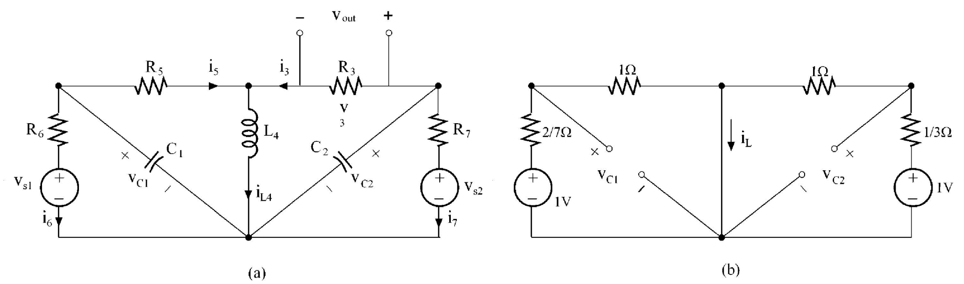


Figure 5.11 A circuit of Example 5.6 (a) and its steady-state equivalent (b).

The characteristic equation is

$$g(\lambda) = |\lambda \cdot \mathbf{1} - \mathbf{A}| = \begin{vmatrix} \lambda + 4 & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{4} & \lambda + \frac{7}{4} & \frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda + \frac{1}{2} \end{vmatrix} = 0.$$

Thus,

$$g(\lambda) = (\lambda + 4)[(\lambda + \tfrac{7}{4})(\lambda + \tfrac{1}{2}) + \tfrac{1}{4}] = 0.$$

Simplifying yields

$$(\lambda + 4)(\lambda^2 + \tfrac{9}{4}\lambda + \tfrac{9}{8}) = 0. \tag{5.111}$$

Thus, the eigenvalues of **A** are

$$\lambda_{1,2} = -\frac{9}{8} \pm \sqrt{\left(\frac{9^2}{8^2} - \frac{9}{8}\right)} = -1.125 \pm 0.375$$

or

$$\lambda_1 = -0.75, \ \lambda_2 = -1.5, \ \lambda_3 = -4.$$

Using the results of equation 5.106, we can evaluate β_0, β_1, and β_2 from the equations

$$\begin{aligned} \beta_0 - 0.75\beta_1 + (-0.75)^2\beta_2 &= e^{-0.75t} \\ \beta_0 - 1.5\beta_1 + (-1.5)^2\beta_2 &= e^{-1.5t} \\ \beta_0 - 4\beta_1 + (-4)^2\beta_2 &= e^{-4t}, \end{aligned}$$

which in the matrix form are

$$\begin{bmatrix} 1 & -0.75 & 0.5625 \\ 1 & -1.5 & 2.25 \\ 1 & -4 & 16 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} e^{-0.75t} \\ e^{-1.5t} \\ e^{-4t} \end{bmatrix}. \tag{5.113}$$

The solution for β's is found by inversion, as

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 & -0.75 & 0.5625 \\ 1 & -1.5 & 2.25 \\ 1 & -4 & 16 \end{bmatrix}^{-1} \begin{bmatrix} e^{-0.75t} \\ e^{-1.5t} \\ e^{-4t} \end{bmatrix}$$

$$= \begin{bmatrix} 2.462 & -1.6 & 0.1385 \\ 2.256 & -2.533 & 0.2769 \\ 0.4103 & -0.5333 & 0.1231 \end{bmatrix} \begin{bmatrix} e^{-0.75t} \\ e^{-1.5t} \\ e^{-4t} \end{bmatrix}$$

$$= \begin{bmatrix} 2.462e^{-0.75t} & -1.6e^{-1.5t} & 0.1385e^{-4t} \\ 2.256e^{-0.75t} & -2.533e^{-1.5t} & 0.2769e^{-4t} \\ 0.4103e^{-0.75t} & -0.5333e^{-1.5t} & 0.1231e^{-4t} \end{bmatrix}. \tag{5.114}$$

With β's now known, matrix $\mathbf{e^{At}}$ will be

$$\mathbf{e^{At}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \beta_0 + \begin{bmatrix} -4 & 0.5 & -0.5 \\ 0.25 & -1.75 & -0.25 \\ 0.5 & 0.5 & -0.5 \end{bmatrix} \beta_1$$

$$+ \begin{bmatrix} 15.87 & -3.125 & 2.125 \\ -1.563 & 3.063 & 0.438 \\ -2.125 & -0.875 & -0.125 \end{bmatrix} \beta_2.$$

Substituting equation 5.114 for β's and collecting like terms yields the final results

$$\mathbf{e^{At}} = \begin{bmatrix} -0.048 & -0.154 & -0.256 \\ -0.077 & -0.229 & -0.384 \\ 0.256 & 0.769 & 1.283 \end{bmatrix} e^{-0.75t} + \begin{bmatrix} 0.066 & 0.4 & 0.133 \\ 0.2 & 1.2 & 0.4 \\ -0.133 & -0.8 & -0.267 \end{bmatrix} e^{-1.5t}$$

$$+ \begin{bmatrix} 0.985 & -0.246 & 0.123 \\ -0.123 & 0.031 & -0.015 \\ -0.123 & 0.031 & -0.015 \end{bmatrix} e^{-4t}. \tag{5.115}$$

Now suppose that the initial state vector at $t_0 = 0$ is $\mathbf{x}(0) = [0.5 \;\; 1.5 \;\; 1]^T$, then the natural solution (for $\mathbf{w}(t) = \mathbf{0}$) in equation 5.109 is

$$\mathbf{x}_{nat}(t) = \mathbf{e^{At}}\mathbf{x}(0) = \begin{bmatrix} -0.511e^{-0.75t} & +0.767e^{-1.5t} & +0.246e^{-4t} \\ -0.766e^{-0.75t} & +2.30e^{-1.5t} & -0.031e^{-4t} \\ 2.564e^{-0.75t} & -1.534e^{-1.5t} & -0.031e^{-4t} \end{bmatrix}.$$

(5.116)

The next step is to find the particular or forced solution of the state equation. Let the input vector $\mathbf{w}(t) = [1 \ \ 1]^T$. Substituting the circuit parameters into matrix $\mathbf{b}$ in equation 5.55, we obtain

$$\mathbf{b} = \begin{bmatrix} 3.5 & 0 \\ 0 & 1.5 \\ 0 & 0 \end{bmatrix}. \tag{5.117}$$

Since the input is a constant (d.c.), evaluating the integral in equation 5.55 results, for $t_0 = 0$, in

$$\int_0^t \mathbf{e}^{\mathbf{A}(t-\tau)}\mathbf{bw}\, d\tau = -\mathbf{A}^{-1}\mathbf{e}^{\mathbf{A}(t-\tau)}\mathbf{bw}|_0^t = \mathbf{A}^{-1}[\mathbf{e}^{\mathbf{A}t} - \mathbf{1}]\mathbf{bw}, \tag{5.118}$$

where the inverse of the $\mathbf{A}$ matrix is found as follows

$$\mathbf{A}^{-1} = \begin{bmatrix} -4 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{7}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}^{-1} = \begin{bmatrix} -0.222 & 0 & 0.222 \\ 0 & -0.5 & 0.25 \\ -0.222 & -0.5 & -1.528 \end{bmatrix}. \tag{5.119}$$

Performing now, all the calculations in equation 5.118, with equations 5.119, 5.115, 5.117 and $\mathbf{w} = [1 \ \ 1]^T$, we obtain the particular solution

$$\mathbf{x}_{par}(t) = \begin{bmatrix} 0.547e^{-0.75t} - 0.556e^{-1.5t} - 0.769e^{-4t} + 0.778 \\ 0.821e^{-0.75t} - 1.667e^{-1.5t} + 0.096e^{-4t} + 0.750 \\ -2.735e^{-0.75t} + 1.111e^{-1.5t} + 0.096e^{-1.5t} + 1.528 \end{bmatrix}. \tag{5.120}$$

The final result of the complete solution is simply obtained by combining the above two solutions: the natural (equation 5.116) and the particular (equation 5.120), which leads to

$$\mathbf{x}(t) = \mathbf{x}_{nat} + \mathbf{x}_{par} = \begin{bmatrix} 0.034e^{-0.75t} + 0.211e^{-1.5t} - 0.523e^{-4t} + 0.778 \\ 0.052e^{-0.75t} + 0.633e^{-1.5t} + 0.065e^{-4t} + 0.750 \\ -0.171e^{-0.75t} - 0.423e^{-1.5t} + 0.065e^{-4t} + 1.528 \end{bmatrix} \begin{bmatrix} v_{c1} \\ v_{c2} \\ i_{L4} \end{bmatrix}. \tag{5.121}$$

Figure 5.12 shows the state variables v_{c1}, v_{c2}, i_{L4} behavior versus time.

The computer calculation of the state variables in the above example, using the MATHCAD program is shown in Appendix I. (Note that the computing results are slightly different from those obtained above.)

To complete this example, suppose that voltage v_3 is of interest. Then the

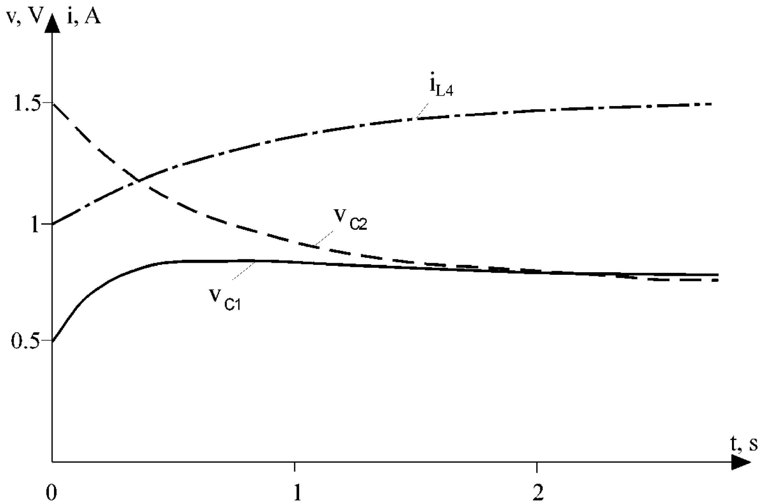


Figure 5.12 Two capacitor voltages and inductor current curves versus time of Example 5.6.

output equation 5.56 simplifies to

$$v_3(t) = [-a \quad a \quad aR_5]\mathbf{x}(t) = [-\tfrac{1}{2} \quad \tfrac{1}{2} \quad \tfrac{1}{2}]\begin{bmatrix} v_{c1} \\ v_{c2} \\ i_{L4} \end{bmatrix}.$$

Thus, the output voltage is

$$\begin{aligned} v_{out}(t) &= v_3 = \tfrac{1}{2}(-v_{c1} + v_{c2} + i_{L4}) \\ &= -0.077e^{-0.75t} - 0.0005e^{-1.5t} + 0.327e^{-4t} + 0.750\ \mathrm{V}. \end{aligned} \qquad (5.122)$$

Note that by inspection of the given circuit in its d.c. steady-state behavior, i.e. the capacitors are open-circuited and the inductor is short-circuited as shown in Fig. 5.11(b), we may find

$$v_{c1}(\infty) = \frac{v_{s1}}{R_5 + R_6} R_5 = \frac{1}{1 + 2/7} \cdot 1 = 0.778\ \mathrm{V}$$

$$v_{c2}(\infty) = \frac{v_{s2}}{R_3 + R_7} R_3 = \frac{1}{1 + 1/3} \cdot 1 = 0.75\ \mathrm{V}$$

$$i_L(\infty) = v_{c1}/R_5 + v_{c2}/R_3 = 0.778 + 0.75 = 1.528\ \mathrm{A},$$

which is in agreement with the final results in equation 5.121.

(b) *Multiple eigenvalues*

If some of the eigenvalues of **A** (roots of the characteristic equation $g(\lambda) \neq 0$)

are not distinct and there are repeated values (for example $\lambda_1 = \lambda_2$), then in this case, the number of independent equations in 5.106 would be fewer than n unknown coefficients β. The following theorem allows us to extend the solution for finding all β's to the case of repeated eigenvalues.

Theorem:(*) Let **A** be the $n \times n$ matrix with n_0 distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{n0}$ and m multiple eigenvalues ($n_0 < n$, if no eigenvalue is repeated, then $n_0 = n$). Let the eigenvalue λ_i occur with multiplicity r_i, and define the polynomials

$$\mathbf{P}(\mathbf{A}) = \sum_{k=0}^{n-1} \beta_k \mathbf{A}, \tag{5.123}$$

and

$$P(\lambda) = \sum_{k=0}^{n-1} \beta_k \lambda^k. \tag{5.124}$$

Then the matrix function $f(\mathbf{A})$ is identical to the matrix polynomial $\mathbf{P}(\mathbf{A})$ (see 5.107) if the following conditions are obeyed:

for each distinct eigenvalue

$$f(\lambda_i) = P(\lambda_i) \quad i = 1, 2, \ldots, n_0 \tag{5.125a}$$

for each multiple eigenvalue

$$\frac{d^q}{d\lambda^q} f(\lambda)|_{\lambda=\lambda_i} = \frac{d^q}{d\lambda^q} P(\lambda)|_{\lambda=\lambda_i},$$

$$i = n_{0+1}, n_{0+2}, \ldots, n_{0+m}, \quad q = 0, 1, 2, \ldots, r_i - 1 \tag{5.125b}$$

that the first condition (equation 5.125a) gives us only n_0 ($n_0 < n$) independent equations for finding n unknown β-coefficients. However, the second condition (equation 5.125b) yields the remaining equations needed to solve for $\beta_0, \beta_1, \ldots, \beta_{n-1}$. For this purpose equation 5.125b shall be rewritten in terms of the unknown β's

$$\frac{d^q}{d\lambda^q} f(\lambda)|_{\lambda=\lambda_i} = \frac{d^q}{d\lambda^q} \sum_{k=0}^{n-1} \beta_k \lambda^k|_{\lambda=\lambda_i} = \sum_{k=q}^{k-1} k(k-1)\cdots(k-q+1)\beta_k \lambda_i^{k-q},$$

$$i = n_{0+1}, n_{0+2}, \ldots, n_{0+m}, \quad q = 0, 1, 2, \ldots, r-1 \tag{5.126}$$

The total number of independent equations, therefore, will be

$$n_0 + \sum_1^m r_i = n.$$

Example 5.7

As an example of the determination of a matrix function when **A** has multiple

(*)The proof can be found in the book by Balabanian N. and Bickart T. A. (1969) *Electrical Network Theory*, John Wiley & Sons.

eigenvalues, let us consider the same circuit in Fig. 5.11 of the previous example with slightly different parameters, namely: $R_6 = 1/3\,\Omega$, $R_7 = 2/5\,\Omega$ (the rest of the parameters are the same). Suppose we wish to find $\mathbf{e}^{\mathbf{A}t}$.

Solution

The **A** matrix in this case will be

$$\mathbf{A} = \begin{bmatrix} -\frac{7}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{3}{2} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

which yields the characteristic equation

$$g(\lambda) = \begin{bmatrix} \lambda + \frac{7}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{4} & \lambda + \frac{3}{2} & \frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda + \frac{1}{2} \end{bmatrix}$$
$$= (\lambda + \tfrac{7}{2})(\lambda + \tfrac{3}{2})(\lambda + \tfrac{1}{2}) + \tfrac{1}{4}\lambda + \tfrac{3}{8} = (\lambda + \tfrac{7}{2})(\lambda^2 + 2\lambda + 1) = 0.$$

Thus, the eigenvalues are $\lambda_1 = -\frac{7}{2}$ and double $\lambda_2 = -1$, i.e. the multiplicity $r = 2$. Therefore, for the first distinct eigenvalue, in accordance with equation 5.125a, we have

$$\beta_0 + \beta_1(-\tfrac{7}{2}) + \beta_2(-\tfrac{7}{2})^2 = e^{-(7/2)t},$$

and for the double eigenvalue, in accordance with equation 5.125b we have

$$\beta_0 + \beta_1(-1) + \beta_2(-1)^2 = e^{-t}, \quad q = 0$$
$$\beta_1 + 2\beta_2(-1) = te^{-t}, \quad q = 1.$$

Since

$$\left.\frac{df(\lambda_2)}{d\lambda}\right|_{\lambda_2 = -1} = \left.\frac{d}{d\lambda_2}(e^{\lambda_2 t})\right|_{\lambda_2 = -1} = te^{-t},$$

the above equations in the matrix form are

$$\begin{bmatrix} 1 & -7/2 & 49/4 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} e^{-3.5t} \\ e^{-t} \\ te^{-t} \end{bmatrix}.$$

The solution for β's gives

$$\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 0.16e^{-3.5t} + 0.84e^{-t} + 1.4te^{-t} \\ 0.32e^{-3.5t} - 0.32e^{-t} + 1.8te^{-t} \\ 0.16e^{-3.5t} - 0.16e^{-t} + 0.4te^{-t} \end{bmatrix}.$$

With β's known, the desired matrix is

$$\mathbf{e}^{\mathbf{A}t} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} \beta_0 + \begin{bmatrix} -3.5 & 0.5 & -0.5 \\ 0.25 & -1.5 & -0.25 \\ 0.5 & 0.5 & -0.5 \end{bmatrix} \beta_1$$

$$+ \begin{bmatrix} 12.125 & -2.75 & 1.875 \\ -1.375 & 2.25 & 0.375 \\ -1.875 & -0.75 & -0.125 \end{bmatrix} \beta_2.$$

Substituting the β's from the previous solution, and after simplifying, we obtain

$$\mathbf{e}^{\mathbf{A}t} = \begin{bmatrix} 0.98e^{-3.5t} + 0.02e^{-t} - 0.05te^{-t} & -0.28e^{-3.5t} + 0.28e^{-t} - 0.2te^{-t} & 0.14e^{-3.5t} - 0.14e^{-t} - 0.15te^{-t} \\ -0.14e^{-3.5t} + 0.14e^{-t} - 0.1te^{-t} & 0.04e^{-3.5t} + 0.96e^{-t} - 0.4te^{-t} & -0.02e^{-3.5t} + 0.02e^{-t} - 0.3te^{-t} \\ -0.14e^{-3.5t} + 0.14e^{-t} + 0.15te^{-t} & 0.04e^{-3.5t} - 0.04e^{-t} + 0.6te^{-t} & 1.02e^{-3.5t} - 0.02e^{-t} + 0.45te^{-t} \end{bmatrix}.$$

(c) *Complex eigenvalues*

We shall illustrate the computation of a matrix exponential when some of the roots of the characteristic equation are complex quantities, considering the following example.

Example 5.8

Let the circuit in Fig. 5.11 (of the previous example) have the same parameters, excluding $R_6 = 2/5\ \Omega$ and $R_7 = 1/2\ \Omega$. Our purpose is again to compute $\mathbf{e}^{\mathbf{A}t}$.

Solution

We substitute the above parameters into the **A** matrix of equation 5.55 to yield

$$\mathbf{A} = \begin{bmatrix} -3 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{5}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

Thus, the characteristic equation of **A** is

$$g(\lambda) = (\lambda + 3)(\lambda + \tfrac{5}{4})(\lambda + \tfrac{1}{2}) + \tfrac{1}{4}\lambda + \tfrac{3}{4} = 0,$$

or after a rearrangement of terms

$$(\lambda + 3)(\lambda^2 + \tfrac{7}{4}\lambda + \tfrac{7}{8}) = 0,$$

Therefore, the eigenvalues are

$$\lambda_1 = -3, \quad \lambda_{2,3} = -\tfrac{7}{8} \pm \sqrt{\tfrac{49-56}{64}} = -0.875 \pm j0.331.$$

Note that two complex eigenvalues are a conjugate pair. Thus, in accordance with equation 5.106, we have

$$\beta_0 + \beta_1(-3) + \beta_2(-3)^2 = e^{-3t}$$
$$\beta_0 + \beta_1(-0.875 + j0.331) + \beta_2(-0.875 + j0.331)^2 = e^{-0.875t}\, e^{j0.331t}$$
$$\beta_0 + \beta_1(-0.875 - j0.331) + \beta_2(-0.875 - j0.331)^2 = e^{-0.875t}\, e^{-j0.331t}.$$

Next, we solve these equations to yield for β's:

$$\beta_0 = 0.819e^{-3t} + e^{-0.875t}(3.86 \sin 0.331t + 0.811 \cos 0.331t)$$
$$\beta_1 = 0.378e^{-3t} + e^{-0.875t}(5.46 \sin 0.331t - 0.378 \cos 0.331t)$$
$$\beta_2 = 0.216e^{-3t} + e^{-0.875t}(1.39 \sin 0.331t - 0.216 \cos 0.331t).$$

Hence, matrix $\mathbf{e}^{\mathbf{A}t}$ will be

$$\mathbf{e}^{\mathbf{A}t} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} \beta_0 + \begin{bmatrix} -3 & 0.5 & -0.5 \\ 0.25 & -1.25 & -0.25 \\ 0.5 & 0.5 & -0.5 \end{bmatrix} \beta_1$$
$$+ \begin{bmatrix} 8.875 & -2.375 & 1.625 \\ -1.187 & 1.563 & 0.313 \\ -1.625 & -0.625 & -1.125 \end{bmatrix} \beta_2.$$

Finally, substituting the above results for β's, after simplifying, we obtain

$$\mathbf{e}^{\mathbf{A}t} = \begin{bmatrix} 0.973e^{-3t} - 0.174\zeta_1 + 0.027\zeta_2 & -0324e^{-3t} - 0.572\zeta_1 + 0.324\zeta_2 & 0.162e^{-3t} - 0.470\zeta_1 - 0.162\zeta_2 \\ -0.162e^{-3t} - 0.280\zeta_1 + 0.162\zeta_2 & 0.054e^{-3t} - 0.787\zeta_1 + 0.946\zeta_2 & -0.027e^{-3t} - 0.930\zeta_1 + 0.027\zeta_2 \\ -0.162e^{-3t} + 0.470\zeta_1 + 0.162\zeta_2 & -0.054e^{-3t} + 1.86\zeta_1 - 0.054\zeta_2 & -0.027e^{-3t} + 0.960\zeta_1 + 1.027\zeta_2 \end{bmatrix}$$

where $\zeta_1 = e^{-0.875t} \sin 0.331t$, $\zeta_2 = e^{-0.875t} \cos 0.331t$.

Suppose we now wish to know the zero input response of the circuit to the initial vector, $\mathbf{x}(0) = [1 \;\; 1 \;\; 0]^T$, i.e. the capacitors are initially charged to 1 V each. Then,

$$\mathbf{x}_{nat}(t) = \mathbf{e}^{\mathbf{A}t}[1 \quad 1 \quad 0]^T = \begin{bmatrix} v_{c1} \\ v_{c2} \\ i_{L4} \end{bmatrix}$$
$$= \begin{bmatrix} 0.649e^{-3t} + e^{-0.875t}(-0.746 \sin 0.331t + 0.351 \cos 0.331t) \\ -0.108e^{-3t} + e^{-0.875t}(-1.073 \sin 0.331t + 1.108 \cos 0.331t) \\ -0.108e^{-3t} + e^{-0.875t}(2.329 \sin 0.331t + 0.108 \cos 0.331t) \end{bmatrix}.$$

These two voltage curves and one current curve versus time are shown in Fig. 5.13.

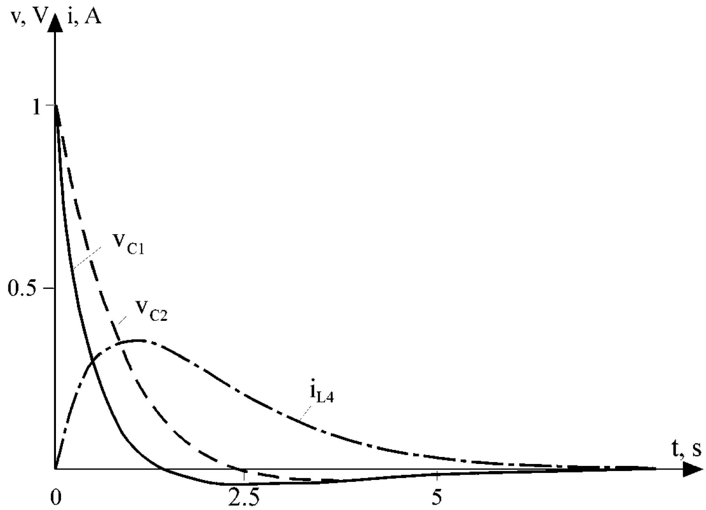


Figure 5.13 Two capacitor voltages and inductor current curves versus time of Example 5.8 in the case of complex-conjugate eigenvalues.

5.8.3 Lagrange interpolation formula

One other method of computing functions of a matrix is based on the Lagrange interpolation formula (this formula is also known as the Silvestre formula). Thus, knowing the eigenvalues λ's of matrix **A**, any function of **A** may be determined as:

$$f(\mathbf{A}) = \sum_{i=1}^{n} \left(\prod_{\substack{k=1 \\ k \neq 1}}^{n} \frac{\mathbf{A} - \lambda_k \mathbf{1}}{\lambda_i - \lambda_k} \right) f(\lambda_i), \tag{5.127}$$

where $\prod_{\substack{k=1 \\ k \neq 1}}^{n}$ means the product of terms $\dfrac{\mathbf{A} - \lambda_k \mathbf{1}}{\lambda_i - \lambda_k}$ where k takes the values $1, 2, \ldots, n$ but excluding $k = i$. For example, using the data of Example 5.6, equation 5.127 implies that

$$\mathbf{e}^{\mathbf{A}t} = \frac{(\mathbf{A} + 1.5\cdot\mathbf{1})(\mathbf{A} + 4\cdot\mathbf{1})}{(-0.75 + 1.5)(-0.75 + 4)} e^{-0.75t} + \frac{(\mathbf{A} + 0.75\cdot\mathbf{1})(\mathbf{A} + 4\cdot\mathbf{1})}{(-1.5 + 0.75)(-1.5 + 4)} e^{-1.5t}$$
$$+ \frac{(\mathbf{A} + 0.75\cdot\mathbf{1})(\mathbf{A} + 1.5\cdot\mathbf{1})}{(-4 + 0.75)(-4 + 1.5)} e^{-4t}.$$

Substituting matrix **A** (equation 5.110) and performing all the arithmetic, leads to

$$\mathbf{e}^{\mathbf{A}t} = \begin{bmatrix} -0.050 & -0.154 & -0.256 \\ -0.077 & -0.230 & -0.385 \\ 0.256 & 0.769 & 1.282 \end{bmatrix} e^{-0.75t} + \begin{bmatrix} 0.067 & 0.4 & 0.133 \\ 0.2 & 1.2 & 0.4 \\ -0.133 & -0.8 & -0.267 \end{bmatrix} e^{-1.5t}$$
$$+ \begin{bmatrix} 0.985 & -0.246 & 0.123 \\ -0.123 & 0.031 & -0.015 \\ -0.123 & 0.031 & -0.015 \end{bmatrix} e^{-4t}$$

which agrees with the previous results obtained in equation 5.115.

The Lagrange interpolation formula can be easily programmed, which is an advantage in computer-aided calculations.

5.9 EVALUATING THE MATRIX EXPONENTIAL BY LAPLACE TRANSFORM

In conclusion, let us introduce the Laplace transform application for solving the matrix differential equation. To simplify the procedure, we first apply the Laplace transform to the homogeneous equation (see equation 5.81):

$$\frac{d}{dt}\mathbf{x}(t) - \mathbf{A}\mathbf{x}(t) = \mathbf{0}. \tag{5.128}$$

Applying the Laplace transform to equation 5.128, we get

$$s\mathbf{X}(s) - \mathbf{X}(0) - \mathbf{A}\mathbf{X}(s) = \mathbf{0}, \tag{5.129}$$

where $\mathbf{X}(s)$ is the Laplace transform of $\mathbf{x}(t)$. Supposing that $\mathbf{X}(0) = 1$ (equation 5.129) can be written as follows:

$$(s\cdot\mathbf{1} - \mathbf{A})\mathbf{X}(s) = \mathbf{1}, \tag{5.130}$$

or

$$\mathbf{X}(s) = (s\cdot\mathbf{1} - \mathbf{A})^{-1}. \tag{5.131}$$

Now, we take the inverse transform to get $\mathbf{x}(t)$

$$\mathbf{x}(t) = L^{-1}\{(s\cdot\mathbf{1} - \mathbf{A})^{-1}\} = \mathbf{e}^{\mathbf{A}t}. \tag{5.132}$$

As can be seen, since we have taken $\mathbf{X}(0) = \mathbf{1}$, this expression is also equal to the matrix exponential $\mathbf{e}^{\mathbf{A}t}$.

Example 5.9

Let us apply this result to the simple circuit shown in Fig. 5.14, where the proper tree branches are emphasized.

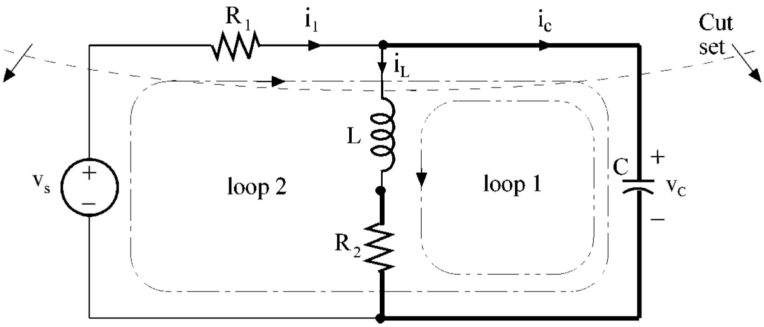


Figure 5.14 A circuit of Example 5.9.

Solution

The capacitor voltage v_C and the inductor current i_L are the state variables in this case. The fundamental cut-set equation and two fundamental loop equations yield

$$C\frac{dv_C}{dt} = -i_L + i_1$$

$$L\frac{di_L}{dt} = v_C - R_2 i_L$$

$$R_1 i_1 = -v_C + v_s \quad \text{or} \quad i_1 = -\frac{1}{R_1}v_C + \frac{1}{R_1}v_s.$$

To eliminate a non-desirable variable, i_1, in the first equation, in this simple case, the third equation shall be inserted into the first one for i_1. Thus, the state equations are

$$\frac{dv_C}{dt} = -\frac{1}{R_1 C}v_C - i_L + \frac{1}{R_1}v_s$$

$$\frac{di_L}{dt} = \frac{1}{L}v_C - \frac{R_2}{L}i_L,$$

or in the matrix form

$$\frac{d}{dt}\begin{bmatrix} v_C \\ i_L \end{bmatrix} = \begin{bmatrix} -1/R_1 C & -1 \\ 1/L & -R_2/L \end{bmatrix}\begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 1/R_1 \\ 0 \end{bmatrix}[v_s]. \tag{5.133}$$

Let the element values be $C = 1.0$ F, $L = 4/3$ H, $R_1 = 2/5\ \Omega$, $R_2 = 2/3\ \Omega$ and $v_s =$ 1 V. This yields the coefficient matrixes **A** and **b**

$$\mathbf{A} = \begin{bmatrix} -5/2 & -1 \\ 3/4 & -1/2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5/2 \\ 0 \end{bmatrix} \tag{5.134}$$

and the input matrix $\mathbf{w} = [v_s] = [1]$. Next, we find the matrix $[s\mathbf{1} - \mathbf{A}]$ and its determinant

$$s\mathbf{1} - \mathbf{A} = \begin{bmatrix} s + \frac{5}{2} & 1 \\ -\frac{3}{4} & s + \frac{1}{2} \end{bmatrix}$$

$$\det(s\mathbf{1} - \mathbf{A}) = (s + \tfrac{5}{2})(s + \tfrac{1}{2}) + \tfrac{3}{4} = s^2 + 3s + 2 = (s+1)(s+2).$$

The inverse matrix $[s{\cdot}\mathbf{1} - \mathbf{A}]^{-1}$ is now easily obtained as

$$[s{\cdot}\mathbf{1} - \mathbf{A}] = \begin{bmatrix} \dfrac{s + \frac{1}{2}}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ -\dfrac{\frac{3}{4}}{(s+1)(s+2)} & \dfrac{s + \frac{5}{2}}{(s+1)(s+2)} \end{bmatrix}$$

$$= \begin{bmatrix} -\dfrac{\frac{1}{2}}{s+1} + \dfrac{\frac{3}{2}}{s+2} & \dfrac{1}{s+1} - \dfrac{1}{s+2} \\ -\dfrac{\frac{3}{4}}{s+1} + \dfrac{\frac{3}{4}}{s+2} & \dfrac{\frac{3}{2}}{s+1} - \dfrac{\frac{1}{2}}{s+2} \end{bmatrix}.$$

A partial-fraction expansion was performed in the last step. The inverse Laplace transform of this expression is

$$L^{-1}[s{\cdot}\mathbf{1} - \mathbf{A}]^{-1} = \begin{bmatrix} -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-2t} & e^{-t} - e^{-2t} \\ -\frac{3}{4}e^{-t} + \frac{3}{4}e^{-2t} & \frac{3}{2}e^{-t} - \frac{1}{2}e^{-t} \end{bmatrix} = \mathbf{e}^{\mathbf{A}t}. \tag{5.135}$$

(It is left as an exercise for the reader to verify this result using one of the above given methods for determining a matrix exponential.)

Suppose that the initial conditions are $v_C = 1$ V and $i_L(0) = 0$, and then the natural response will be

$$\mathbf{x}_n(t) = \begin{bmatrix} v_{C,n} \\ i_{L,n} \end{bmatrix} = \mathbf{e}^{\mathbf{A}t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-2t} \\ -\frac{3}{4}e^{-t} + \frac{3}{4}e^{-2t} \end{bmatrix}. \tag{5.136}$$

Note that the verification of equation 5.136 at $t = 0$ yields the initial values of $v_C(0)$ and $i_L(0)$. The particular solution of equation 5.133 may also be obtained with equation 5.135 using, for example, equation 5.118. Thus,

$$\mathbf{x}_p(t) = \mathbf{A}^{-1}[\mathbf{e}^{\mathbf{A}t} - \mathbf{1}]\mathbf{bw} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} \\ -\frac{3}{8} & -\frac{5}{4} \end{bmatrix}\begin{bmatrix} -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-2t} & e^{-t} - e^{-2t} \\ -\frac{3}{4}e^{-t} + \frac{3}{4}e^{-2t} & \frac{3}{2}e^{-t} - \frac{1}{2}e^{-2t} \end{bmatrix}\begin{bmatrix} \frac{5}{2} \\ 0 \end{bmatrix}$$

or after performing all the calculations

$$\mathbf{x}_{part}(t)\begin{bmatrix} v_{C,p} \\ i_{L,p} \end{bmatrix} = \begin{bmatrix} \frac{5}{4}e^{-t} - \frac{15}{8}e^{-2t} + \frac{5}{8} \\ -\frac{15}{8}e^{-t} + \frac{15}{16}e^{-2t} + \frac{15}{16} \end{bmatrix}.$$

By inspection (see the circuit in Fig. 5.13) it can be easily verified that the steady-state values of the capacitor voltage and the inductor current agree with those found below:

$$v_{C,p(\infty)} = \tfrac{5}{8}\,\mathrm{V} \quad \text{and} \quad i_{L,p(\infty)} = \tfrac{15}{16}\,\mathrm{A}.$$

The Laplace transform is one of the ways of evaluating the matrix exponential. However, if we are going to use the Laplace transform for circuit analysis, we may do it straightforwardly using the methods described in Chapter 3. The methods of matrix function evaluation, considered in this chapter, are the most general and suitable for computer-aided computation.

Chapter #6

TRANSIENTS IN THREE-PHASE SYSTEMS

6.1 INTRODUCTION

In the previous chapters we have discussed transients in single-phase circuits. However, practically all-electric power is generated, transmitted, distributed and utilized in three-phase systems. Three-phase networks are generally more complicated than single-phase circuits. The complication arises from the interconnection and displacement angle between phases, the triplicate number of components and the branches introduced by the three phases and, also, because of the need to sometimes consider mutual coupling between phases. Naturally, we started our study of transient analysis with single-phase circuits, while establishing the principles and different methods, and gaining experience in techniques of solving problems. Our continued analysis of transients in three-phase networks, therefore, will be based on our previous study.

There are two basic methods for the analysis and calculation of transients in three-phase circuits: 1) to extend the single-phase approach and 2) to use symmetrical components. The first approach is based on the use of the generalized current/voltage phasor of the three-phase system and the two axes representation of a synchronous machine. The single-phase approach, hence, considers the three-phase system as one entity and that a disturbance occurring at one point affects the whole system, and that the transient components excited are not symmetrical and do not obey the three-phase relationships like in steady-state behavior. The method of symmetrical components has been used for many years to calculate the steady-state behavior of three-phase networks when some part of the network happens to run under unbalanced conditions (primarily with an unbalanced load). The method may also be used to analyze unsymmetrical faults, such as: the single-phase to earth fault, the phase-to-phase short circuit, etc. The method of symmetrical components, actually, removes the unsymmetrical conditions and allows the computation to proceed much the same as for symmetrical three-phase short-circuit conditions, with, of course, some extra complications of the whole procedure.

In this chapter we will discuss the short-circuit faults (symmetrical as well as unsymmetrical) at different points of a three-phase system and the transient overvoltages. The emphasis will be placed on the analysis of the terminal short circuits of power transformers and generators.

6.2 SHORT-CIRCUIT TRANSIENTS IN POWER SYSTEMS

The dominant causes of disturbance of the normal operation of power systems are short-circuits. Short-circuit currents are generally of a magnitude many times that of their rated values. In consequence, high dynamic and thermal stresses are generated, which affect the electrical equipment. In the case of short circuit to earth, unacceptable contact potentials arise, which can lead to damage to the equipment and personal danger. Hence, in planning and designing electric power networks the highest consideration must be given to short-circuit analysis and short-circuit current estimations. Knowing the value of short-circuit currents and their flow is also necessary for the specification of protective devices. The following sections are dedicated to short-circuit transient analysis and different methods of calculating short-circuit currents.

In three-phase systems a distinction is made between the following kinds of short-circuits:

a) **Three-pole short-circuit**, in which the three voltages at the short-circuit point are all zero, and the three conductors are symmetrically loaded by the short-circuit currents, as shown in Fig. 6.1(a). Hence this kind of short-circuit fault is called symmetrical and the analysis of this kind of short circuit is performed on a single-phase representation. It should be noted that this kind of short circuit is relatively rare, but it is usually the most dangerous since the short-circuit currents developed in this fault are of the highest magnitude. They are

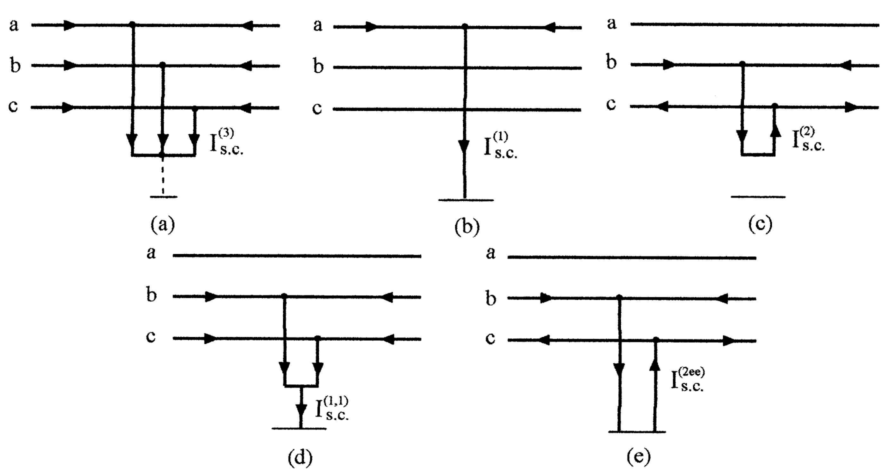


Figure 6.1 Designation of short-circuit faults: three-pole short circuit (a), a single-pole-ground short circuit (b), two-pole-ground fault (d) and double-earth fault (e).

important for specifying the equipment under the short-circuit fault. Since the three-phase voltage at this kind of fault drops to zero, stability problems arise and non-static loads, such as induction motors, run down to stand still (see further on in Chapter 8).

The other four kinds of short-circuits are entirely unsymmetrical conditions. In particular, the voltages at the short-circuit point are not all zero. As a result of the unsymmetrical conditions, mutual couplings are introduced between the phase conductor and the neutral conductor, if present.

b) **The single-pole short-circuit between one of the phases and earth**, Fig. 6.1(b). This kind of fault is the most frequently encountered. Sometimes when the network possesses a low neutral earth impedance, the fault current can even exceed the largest currents produced by a three-pole short-circuit.

c) **The two-pole short-circuit without an earth fault**, Fig. 6.1(c), in which only two phase voltages at the short-circuit point are zero. In this kind of short-circuiting the short-circuit currents are usually less than those produced by a three-pole short circuit. However, if the short-circuit location is close to the generator, the subsequent short-circuit current can become greater than in the three-pole case.

d) **The two-pole short-circuit with an earth fault**, Fig. 6.1(d). This kind of fault may occur in a system with a grounded neutral and has similar characteristics to the previous one.

e) **The double earth fault**, which occurs in a system with an isolated neutral, Fig. 6.1(e). The short-circuit currents in this case may not exceed the rated values, but are significant with regard to the determination of the contact potential and dimension of the earthing systems.

6.2.1 Base quantities and per-unit conversion in three-phase circuits

In the analysis of power networks it is common to use a so-called **"per unit"** system (denoted p.u.) for expressing network quantities rather than a system of actual units (Ω, A, V, etc.). According to this system all the quantities are expressed as fractions of reference quantities, or **base values**, such as base apparent power S_b (VA), base voltage, V_b, and/or base current, I_b. It is obvious that it is enough to choose only two of these quantities since they are related by the expression

$$S_b = \sqrt{3} V_b I_b. \tag{6.1}$$

Usually the base voltage is chosen, in addition to the base power, and the base current is calculated as

$$I_b = \frac{S_b}{\sqrt{3} V_b}, \tag{6.1a}$$

where V_b and I_b are the line quantities.

Hence, the p.u. quantities will be:

$$V_{\mathrm{pu}} = \frac{V}{V_{\mathrm{b}}} \qquad \text{(p.u. voltage)} \tag{6.2a}$$

$$I_{\mathrm{pu}} = \frac{I}{I_{\mathrm{b}}} \qquad \text{(p.u. current)} \tag{6.2b}$$

$$S_{\mathrm{pu}} = \frac{S}{S_{\mathrm{b}}}, \quad P_{\mathrm{pu}} = \frac{P}{S_{\mathrm{b}}}, \quad Q_{\mathrm{pu}} = \frac{Q}{S_{\mathrm{b}}} \qquad \text{(p.u. power)} \tag{6.2c}$$

and the most important p.u. quantity, the p.u. impedance and its components:

$$Z_{\mathrm{pu}} = \frac{Z_{\Omega}}{Z_{\mathrm{b}}}, \quad R_{\mathrm{pu}} = \frac{R_{\Omega}}{Z_{\mathrm{b}}}, \quad X_{\mathrm{pu}} = \frac{X_{\Omega}}{Z_{\mathrm{b}}}. \tag{6.3}$$

Here the base impedance, Z_{b}, is established with Ohm's Law as

$$Z_{\mathrm{b}} = \frac{V_{\mathrm{b}}}{\sqrt{3} I_{\mathrm{b}}} = \frac{V_{\mathrm{b}}^2}{S_{\mathrm{b}}}. \tag{6.4}$$

With equation 6.4 we can write

$$Z_{\mathrm{pu}} = Z_{\Omega} \frac{S_{\mathrm{b}}}{V_{\mathrm{b}}^2}, \quad R_{\mathrm{pu}} = R_{\Omega} \frac{S_{\mathrm{b}}}{V_{\mathrm{b}}^2}, \quad X_{\mathrm{pu}} = X_{\Omega} \frac{S_{\mathrm{b}}}{V_{\mathrm{b}}^2}. \tag{6.5}$$

Note that in expressions (equations 6.3–6.5) the impedances and their components are per-phase quantities. It should also be denoted that all the expressions (equations 6.1–6.5) are proper for a one-phase network. In such a case the $\sqrt{3}$ must be omitted, and all the quantities are phase or just circuit values. With the known p.u. value, the actual value can be obtained as

$$Z_{\Omega} = Z_{\mathrm{pu}} \frac{V_{\mathrm{b}}^2}{S_{\mathrm{b}}} = Z_{\mathrm{pu}} \frac{V_{\mathrm{b}}}{\sqrt{3} I_{\mathrm{b}}}. \tag{6.6}$$

The p.u. system is widely used in "Electric machine and transformer" courses, where the parameters of electric machines and transformers and their characteristics are usually expressed in per-unit quantities. It stands to reason, therefore, that the p.u. system is used in "Power system" courses, since power systems consist, primarily, of synchronous generators, transformers and motors. All such equipment varies widely in size, power, voltages etc. However, for equipment of the same type the p.u. impedances, voltage drops and losses are in the same order, regardless of size.

For example, if the primary winding reactance of a 50 kVA, 6.6 kV single-phase transformer is $X_1 = 38.5\ \Omega$, then this reactance measured in p.u. will be

$$X_{1,\mathrm{pu}} = \frac{X_{1,\Omega}}{Z_{\mathrm{b}}} = \frac{38.5}{871} = 0.044,$$

where the base impedance is

$$Z_b = \frac{V_r^2}{S_r} = \frac{6.6^2 \cdot 10^3}{50} = 871\ \Omega.$$

Per-unit quantities are often expressed as a percentage. Percent quantities differ from per-unit by a factor of 100. Hence, the above p.u. reactance, in percent, will be $X_{1,\%} = 4.4\%$. All the transformers of the same series as the above transformer will have about the same percent reactance regardless of their power.

The p.u. values of different items of apparatus by themselves, such as transformers, synchronous generators, motors etc. are given in terms of their own kVA/MVA power and voltage ratings. Hence, for any power system in which several pieces of equipment are involved, it is necessary to refer all the given p.u. values to the system base values: base MVA power and base voltage. Thus, if $Z_{pu}^{(r)}$ is the per-unit impedance (reactance) for rated values, the same impedance (reactance) referred to the base values, will be

$$Z_{pu}^{(b)} = Z_{pu}^{(r)} \frac{S_b V_r^2}{S_r V_b^2}, \tag{6.7}$$

which shows that the "new" p.u. value is directly proportional to the ratio of powers and inversely proportional to the ratio of the squared voltages. If $V_r = V_b$, then

$$Z_{pu}^{(b)} = Z_{pu}^{(r)} \frac{S_b}{S_r}. \tag{6.7a}$$

As already mentioned, in a three-phase system X_{pu} is a per-phase reactance, $S_b(S_r)$ is a three-phase power and $V_b(V_r)$ is a line voltage.

The single base power chosen is to be relatively large, at least equal to, or larger than, the highest power source in the network. All the system impedances will then be related to this base power. The base voltages, however, differ in the dependence on the level of transformation. As we know, these voltages are intended for supplying the transmission and distribution lines over a range from a few thousand volts to a million volts. Hence, the entire power network may have many different voltage levels. By analyzing such a network, all the impedances must be referred to one voltage level. Since all voltages and currents are related directly or inversely as the turn ratio of transformers in any part of power systems, all voltages, currents, volt-amperes and impedances will have the same per-unit values regardless of where they appear in the system. Applying the per-unit values allows the elimination of different voltage levels and represents the entire network on a single voltage level. This is another reason for using a per-unit system of representing the power system quantities.

Let us discuss this topic in more detail. If some particular device is located on the voltage level, which differs from the base voltage level, which is chosen as a main or system base voltage (sb), its base quantities should be calculated as

$$V^{(\mathrm{b})} = \frac{1}{n_1 n_2 \ldots n_k} V^{(\mathrm{sb})}, \quad I^{(\mathrm{b})} = (n_1 n_2 \ldots n_k) I^{(\mathrm{sb})}, \tag{6.8}$$

where $n_1, n_2, \ldots, n_k$ are the turn ratios of the transformers, which are connected in series between the location of the device and the main base level (the turn ratios must be taken in the direction of the main voltage level towards the level of the device location).

The device's actual impedance, which referred (reflected) to the main voltage level, will be

$$Z_{\Omega}^{(\mathrm{sb})} = n_1^2 n_2^2 \cdots n_k^2 Z_{\Omega}^{(\mathrm{b})} = n_{\mathrm{eq}}^{(2)} Z_{\Omega}^{(\mathrm{b})},$$

where

$$n_{\mathrm{eq}} \cong V_{\mathrm{sb}}/V_{\mathrm{b}} \tag{6.9}$$

With n_{eq} the p.u. value of the impedance is

$$Z_{\mathrm{pu}}^{(\mathrm{sb})} = Z_{\Omega}^{(\mathrm{sb})} \frac{S_{\mathrm{b}}}{V_{\mathrm{sb}}^2} = n_{\mathrm{eq}}^2 Z_{\Omega}^{(\mathrm{b})} \frac{S_{\mathrm{b}}}{V_{\mathrm{sb}}^2} = Z_{\Omega}^{(\mathrm{b})} \frac{S_{\mathrm{b}}}{V_{\mathrm{b}}^2} = Z_{\mathrm{pu}}^{(\mathrm{b})}. \tag{6.10}$$

This important result shows that the p.u. impedance value referred to the system (main) base voltage can be calculated with the same expression (equation 6.5) as has been referred to the base voltage of the equipment location, regardless of which system (main) base voltage is chosen.

It is important to note that for the same reason the p.u. impedance of a transformer is the same whether it is referred to the primary or secondary side. Indeed, let us assume that the p.u. impedance, which referred to the primary (step-down transformer), is Z_1, and that which referred to the secondary is $Z_2 = Z_1/n^2$, where n is the turn ratio ($n = N_1/N_2 = V_{1\mathrm{r}}/V_{2\mathrm{r}}$). then

$$Z_{1\mathrm{pu}} = Z_1 \frac{S_{\mathrm{r}}}{V_{1\mathrm{r}}^2},$$

and

$$Z_{2\mathrm{pu}} = Z_2 \frac{S_{\mathrm{r}}}{V_{2\mathrm{r}}^2} = Z_1 \frac{S_{\mathrm{r}}}{n^2 V_{2\mathrm{r}}^2} = Z_1 \frac{S_{\mathrm{r}}}{V_{1\mathrm{r}}^2} = Z_{1\mathrm{pu}}.$$

Hence, the result is the same as the p.u. impedance, which is referred to the primary.

It shall be noted that, since the voltages at the sending V_1 and receiving V_2 ends of a transmission line are different (because of the voltage drop), the line rated voltage is usually taken as an average value

$$V_{\ell,\mathrm{r}} = \frac{V_1 + V_2}{2} \tag{6.11}$$

The average values of the voltages are taken as base voltages for each of the voltage levels in the network[*].

The turn ratio, i.e., the ratio of rated voltages, of the power network transformers may not be the same as the ratio of the average voltages of different levels, so that the impedance referring can be done in two ways: approximate or exact. The referring in accordance to the base-average voltages is approximate. In this case the turns ratio of the transformers (or the ratio of their rated voltages) is taken equal to the ratio of the level voltages. If the base voltages are related by the turn ratios of the transformers, the referring is accounted as an exact one. Let us illustrate these two approaches of expressing p.u. impedances in the following example.

Example 6.1

Consider the three-phase network whose one-line diagram is shown in Fig. 6.2. The rating values and p.u. reactances of the generator and the transformers as well as the parameters of the transmission line and current-limiting reactor are indicated in this diagram. For the calculation of a short-circuit current draw the equivalent circuit and find all the p.u. reactances, which are referred to the generator voltage level in two ways: 1) approximately and 2) exactly.

Solution

We first have to specify the base volt-ampere power, which for a given network it is reasonable to choose a value of 100 MVA.

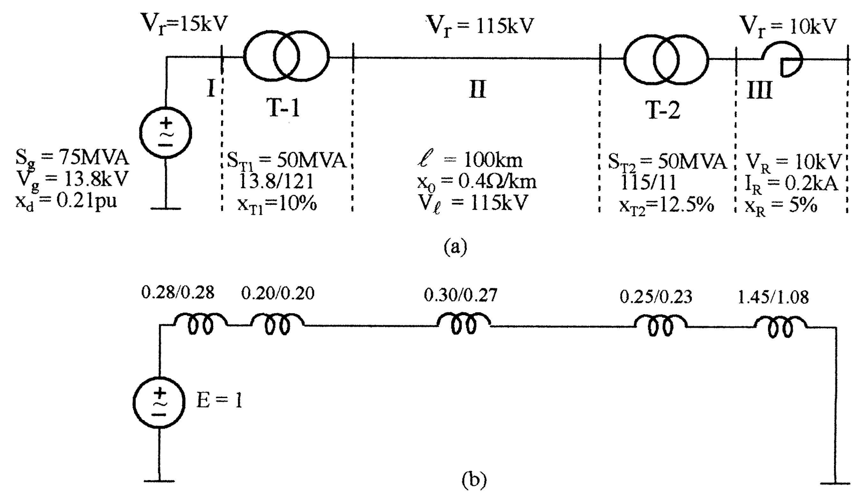


Figure 6.2 A one-line diagram of a given network (a) and its equivalent circuit in terms of p.u. (b).

[*]The average voltages are usually in accordance with those recommended by electric companies or general standards.

1) *Approximate evaluation.* The rated voltages on each level will be as base values, i.e., on the generator level: $V_{bI} = 13.8$ kV, on the line level: $V_{bII} = 115$ kV and on the distribution level: $V_{bIII} = 10$ kV.

The base currents in accordance with equation 6.1a are

$$I_{bI} = \frac{100}{\sqrt{3}\cdot 13.8} = 4.18 \text{ kA}, \quad I_{bII} = \frac{100}{\sqrt{3}\cdot 115} = 0.5 \text{ kA}, \quad I_{bIII} = \frac{100}{\sqrt{3}\cdot 10} = 5.77 \text{ kA}.$$

Then the p.u. reactances are obtained in accordance with equation 6.7a as:

for the generator $$X_g = 0.21\,\frac{100}{75} = 0.28 \text{ pu},$$

for the sending end transformer

$$X_{T1} = 0.1\,\frac{100}{50} = 0.2 \text{ pu},$$

for the receiving and transformer

$$X_{T2} = 0.125\,\frac{100}{50} = 0.25 \text{ pu},$$

for the transmission line in accordance with equation 6.5

$$X_\ell = 0.4\cdot 100\,\frac{100}{115^2} = 0.30 \text{ pu},$$

for the current-limiting reactor

$$X_{rct} = 0.05\,\frac{100}{3.46} = 1.45 \text{ pu},$$

where $S_{re} = \sqrt{3}\cdot 10\cdot 0.2 = 3.46$ MVA is the reactor rating apparent power.

2) *Exact evaluation.* The base voltage on the generator level, as in the previous calculation, will be $V_{bI} = 13.8$ V. The base voltages on the line level and on the distribution level, in accordance to the turn ratio of the transformers, will be (equation 6.8),

$$V_{bII} = \frac{1}{13.8/121} = 121 \text{ kV} \quad \text{and} \quad V_{bIII} = \frac{1}{115/11} = 11.6 \text{ kV}$$

The base currents are (equation 6.1a)

$$I_{bI} = \frac{100}{\sqrt{3}\cdot 13.8} = 4.18 \text{ kA}, \quad I_{bII} = \frac{100}{\sqrt{3}\cdot 121} = 0.48 \text{ kA}, \quad I_{bIII} = \frac{100}{\sqrt{3}\cdot 11.6} = 4.98 \text{ kA}.$$

The per-unit reactances are obtained as:

for the generator (equation 6.7a)

$$X_g = 0.21\,\frac{100}{75} = 0.28 \text{ pu},$$

i.e., the same as in the previous calculation (since the base voltage on the generator level did not change),

for the sending and transformer

$$X_{\mathrm{T1}} = 0.1\,\frac{100}{50} = 0.2\ \mathrm{pu},$$

i.e., it did not change either for the same reason,

for the receiving end transformer (equation 6.7)

$$X_{\mathrm{T2}} = 0.125\,\frac{100}{50}\left(\frac{115}{121}\right)^2 = 0.23\ \mathrm{pu}$$

(since the base voltage and the rated voltage are not equal),

for the transmission line (equation 6.5)

$$X_\ell = 0.4\cdot 100\,\frac{100}{121^2} = 0.27\ \mathrm{pu}$$

for the current-limiting reactor (equation 6.7)

$$X_{\mathrm{re}} = 0.05\,\frac{100}{3.46}\left(\frac{10}{11.6}\right)^2 = 1.08\ \mathrm{pu}.$$

Finally, it might be good to point out that using per-unit quantities in short-circuit fault analysis simplifies to a great extent the numerical calculations manually and/or by using computers.

6.2.2 Equivalent circuits and their simplification

As the equivalent circuit of the power system network in per-unit quantities is established, the next step in short-circuit calculation is to simplify the network. Using the known methods of circuit analysis we may, in most cases, simplify the network so that only a single equivalent generator will feed the short-circuit fault through an equivalent impedance. The following will remind the reader of the most useful of these methods.

(a) *Series and parallel connections*

We start with the series and parallel connections, simplifying them by well-known formulas. Thus, if we have a few generators operating in parallel (usually at the same power station), as shown in Fig. 6.3, we may integrate them into a single one by using this formula (sometimes called Millman's formula).

$$E_{\mathrm{eq}} = \frac{E_1 Y_1 + E_2 Y_2 + \cdots + E_n Y_n}{Y_1 + Y_2 + \cdots + Y_n} = \frac{\sum^n EY}{\sum^n Y}, \quad X_{\mathrm{eq}} = \frac{1}{\sum^n Y}, \tag{6.12}$$

where

$$Y_1 = \frac{1}{X_1},\ Y_2 = \frac{1}{X_2},\ \cdots\ Y_n = \frac{1}{X_n}.$$

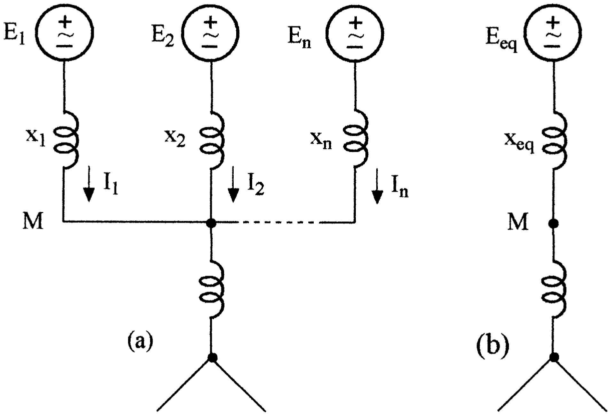


Figure 6.3 Three generators in parallel (a) and the equivalent circuit (b).

For only two generators the above formula will be

$$E_{\mathrm{eq}} = \frac{E_1X_1 + E_2X_2}{X_1 + X_2}, \quad X_{\mathrm{eq}} = \frac{X_1X_2}{X_1 + X_2}. \tag{6.13}$$

These formulas are valid for any value of E's (EMF's) including zero. In particular, the load may be treated as a main generator having zero EMF ($E = 0$). Then such a generator can be combined with others, instead of connecting the zero potential point of the load with the point of the short-circuit fault, as shown in Fig. 6.4. This consideration of the load is approximate; however, it allows us to easily simplify the network. As can be seen in Fig. 6.4(b) the generators can be gradually integrated all together into one single generator, as shown in Fig. 6.4(c). With two more steps, as shown in Figs. 6.4(c) and (d) the given network is simplified to a single generator and a single reactance. In contrast to the above procedure, the connection of zero potential points, as shown in Fig. 6.4(a) (see the dashed line) gives rise to a more complicated circuit, which includes two loops.

(b) *Delta-star (and vice-versa) transformation*

The delta-star transformation can also be useful for the simplification of networks having a short-circuit fault. For introducing this technique, let us consider the network shown in Fig. 6.5(a). In the first step the star $X_3 - X_4 - X_5$ is replaced by delta (shown by dash lines) whose reactances are calculated by the following formulas

$$X_8 = X_3 + X_4 + \frac{X_3X_4}{X_5}, \quad X_9 = X_4 + X_5 + \frac{X_4X_5}{X_3}, \quad X_{10} = X_3 + X_5 + \frac{X_3X_5}{X_4}.$$

Replacing the parallel connecting reactances with their equivalents, we obtain the circuit in Fig. 6.5(b). In the next step we transform the delta $X_8 - X_{11} - X_{12}$

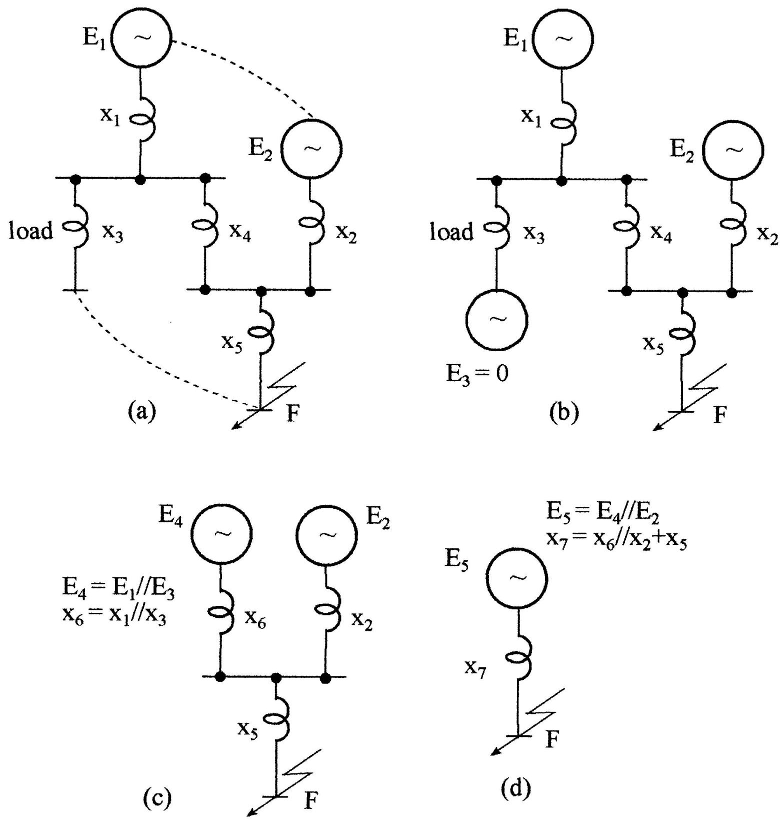


Figure 6.4 A network containing a load (a), the load has been replaced by a generator having zero EMF (b), two steps of simplifying the circuit (c) and (d).

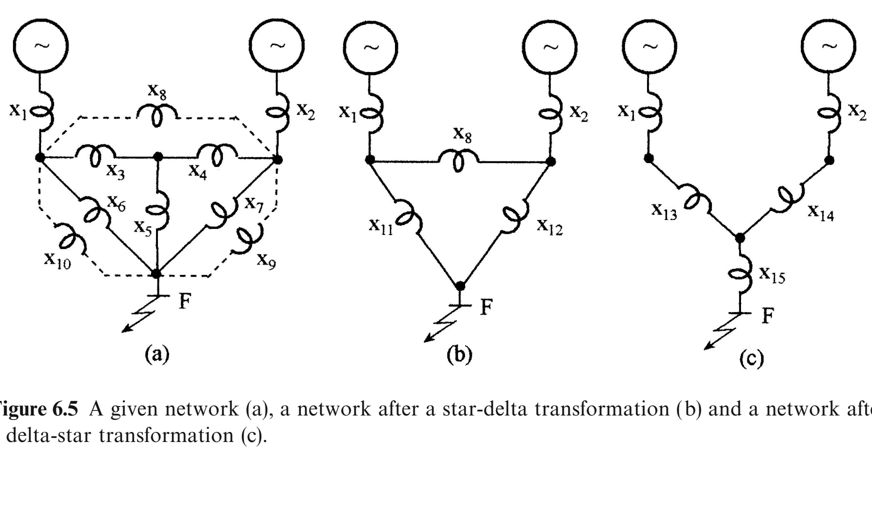


Figure 6.5 A given network (a), a network after a star-delta transformation (b) and a network after a delta-star transformation (c).

into a star by using the formulas

$$X_{13}=\frac{X_8X_{11}}{X_8+X_{11}+X_{12}},\quad X_{14}=\frac{X_8X_{12}}{X_8+X_{11}+X_{12}},\quad X_{15}=\frac{X_{11}X_{12}}{X_8+X_{11}+X_{12}}.$$

The obtained circuit, Fig. 6.5(c), can now be simplified, as was previously done, into one having a single generator and a single impedance.

(c) *Using symmetrical properties of a network*

We may use symmetrical properties to simplify a given network. Consider the network shown in Fig. 6.6(a). If the rating values of transformers, reactors and cables are identical, the entire network is symmetrical relative to the fault point and can be simplified as shown in Fig. 6.6(b). The rest of the elements may not be included in this circuit, since the fault current will not flow through them. The obtained scheme has two parallel branches and can be easily simplified to a single reactance.

6.2.3 The superposition principle in transient analysis

By neglecting the magnetic saturation in synchronous machines and transformers (which is common practice in the transient analysis of power systems), the power network may be treated as a linear system. Hence, the principle of superposition can be applied to its analysis. As was shown in section 2.6, to find the short-circuit current at the fault point, we may superimpose two regimes:

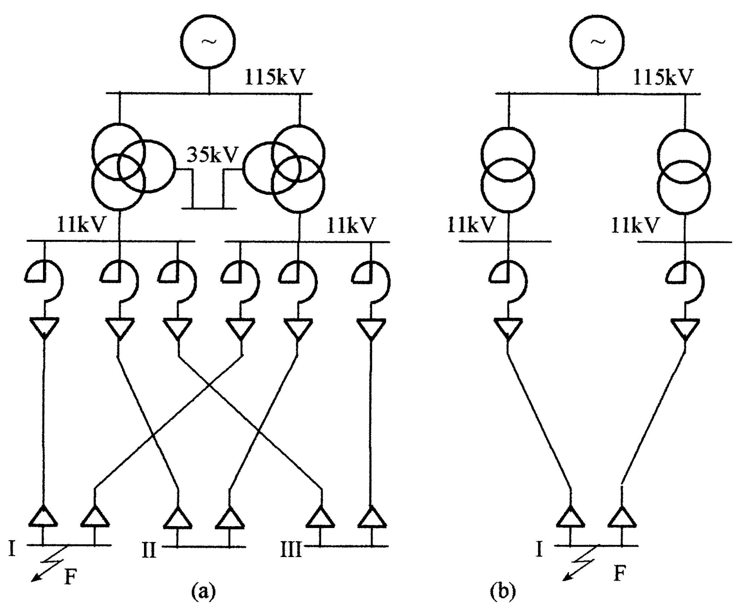


Figure 6.6 A symmetrical network (a) and its simplification (b).

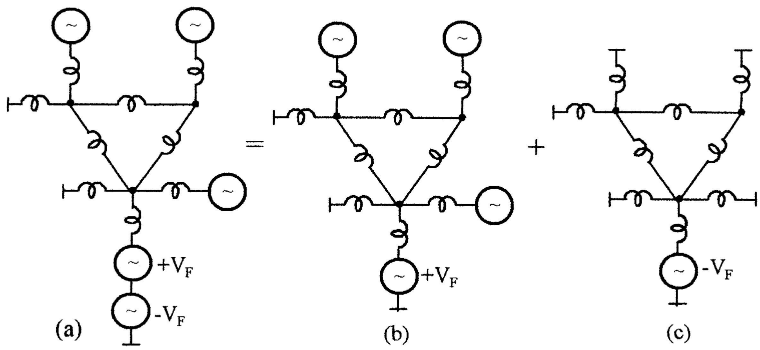


Figure 6.7 A given network (a), the network of a previous regime (b) and the network due to the fault.

1) a previous one, i.e. prior to fault and 2) the additional one, which arises due to the fault. To illustrate this technique, consider the network shown in Fig. 6.7. It is obvious that the fault conditions will not change, if we insert in the fault point two voltage sources equal in magnitude, but opposite in sign, as shown in this figure. The magnitude of these sources should be chosen equal to the voltage value at the fault prior to the fault (if this voltage is not known, the rated value can be used). Following the superposition principle the network in Fig. 6.7(a) can now be represented as two separate networks.

The first one, shown in Fig. 6.7(b), is actually the network of a normal operation, prior to the fault occurring. The second one, shown in Fig. 6.7(c), is the network of the fault regime. Usually the operational conditions (the voltages at the nodes and the branch currents) are known, so that only the network in figure (c) must be analyzed. This network is simpler than the given one, since it has only one source, and therefore might be easier to simplify to a single reactance. The total currents will be found by the summation of the normal condition currents and the fault currents found in the circuit of Fig. 6.7(c).

Example 6.2

The equivalent circuit of part of a power system is shown in Fig. 6.8. The p.u. impedances of the generators, transformers and transmission lines, as well as the generators' EMF's, are indicated on the scheme. (The one-line diagram of the network and the calculations of the p.u. values are given in Appendix II.) Simplify this network up to a single source and single impedance.

Solution

As a first step we replace two parallel EMF's, E_1 and E_5, by their equivalent one (since all the values are in per unit quantities, the indication p.u. is omitted)

$$E_{\mathrm{eq1}} = \frac{1.25/(0.64 + 0.19)}{1/0.83 + 1/4.55} = 1.06,$$

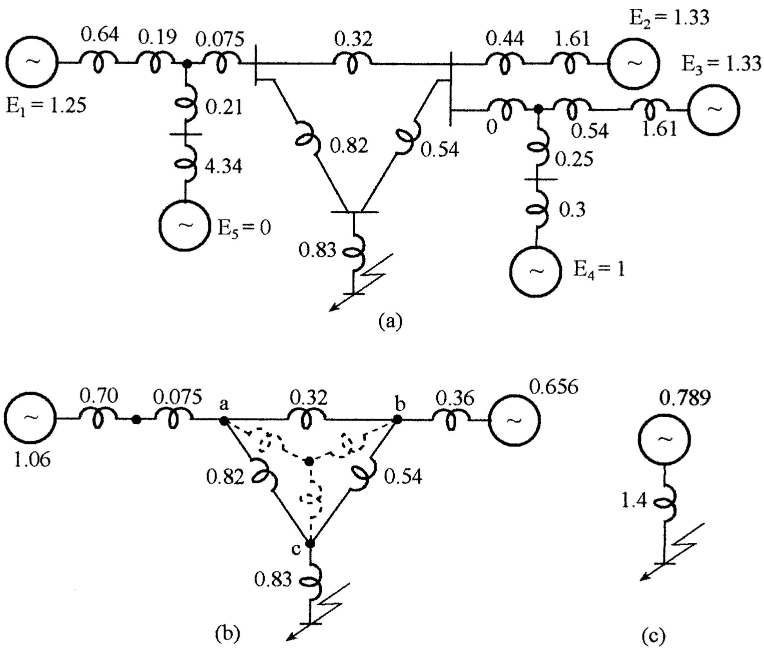


Figure 6.8 A given network (a), after the first step of simplification (b) and the resulting circuit (c).

and

$$X_{\mathrm{eq1}} = \frac{1}{1/0.83 + 1/4.55} = 0.70.$$

In the same way we replace three parallel EMF's E_1, E_3 and E_4 by E_{eq2}

$$E_{\mathrm{eq2}} = \frac{1.33/2.05 + 1.33/2.15 + 1/0.55}{1/2.05 + 1/2.15 + 1/0.55} = \frac{1.818}{2.771} = 0.656,$$

and

$$X_{\mathrm{eq2}} = 1/2.771 = 0.360.$$

As a result we obtain the circuit shown in Fig. 6.8(b).

The next step is the delta-star transformation and replacing two EMF's by a total one

$$X_a = \frac{0.32 \cdot 0.82}{0.32 + 0.82 + 0.84} = \frac{0.262}{1.68} = 0.160,$$

$$X_b = \frac{0.32 \cdot 0.54}{1.68} = 0.100,$$

$$X_c = \frac{0.54 \cdot 0.82}{1.68} = 0.260.$$

Now, the total EMF is obtained as

$$E_{\text{tot}} = \frac{1.06 \cdot 0.46 + 0.656 \cdot 0.94}{0.46 + 0.94} = \frac{1.104}{1.4} = 0.789,$$

$$X_{\text{tot}} = \frac{0.46 \cdot 0.94}{1.4} + 0.26 + 0.83 = 1.40.$$

The resulting circuit is shown in Fig. 6.8(c).

6.3 SHORT-CIRCUITING IN A SIMPLE CIRCUIT

As we have already mentioned, in the majority of the fault situations, such as short-circuiting a single conductor to ground or earth (a one-phase short-circuit) or short-circuiting between two conductors (a two-phase short-circuit), the power system network becomes unsymmetrical. However, we shall start our study of transients in three-phase systems with a symmetrical three-phase fault, where all three conductors touch each other or fall to ground. Although this kind of fault occurs in only a very small percentage of cases, it is very severe for the system and its devices. The very extreme magnitudes of the fault currents in such faults give engineers the ratings of the circuit breakers and other equipment of the power network to be used.

In the case of a symmetrical three-phase fault in a symmetrical system, we can use a single-phase approach, which simplifies to a great degree the calculation of the short-circuit currents and performance of the transient analysis. By simplifying the system network, as was discussed in the previous sections, we may reduce it to the simplest circuit including a single source and a single impedance.

In the case of unsymmetrical faults, the most common method of analysis is to use symmetrical components (see further on), in which we attempt to find the symmetrical components of the voltages and the currents at the point of unbalance and connect the sequence networks, which are, in fact, symmetrical circuits. Hence, the following analysis can be made by again using a single-phase representation.

For a better understanding of the short-circuit phenomena in a three-phase system let us first consider the simple circuit, shown in Fig. 6.9, in which a

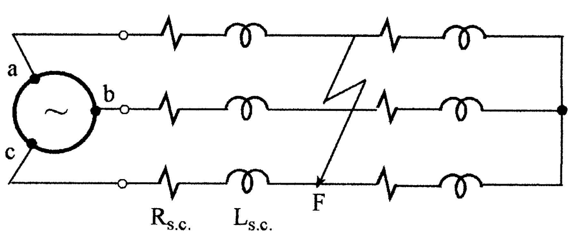


Figure 6.9 A simple three-phase circuit under a symmetrical three-phase fault.

symmetrical three-phase fault occurs. Following the classical approach in transient analysis (see Chaps. 1 and 2) we may represent the total fault current, say in phase "a", as the sum of a forced and a natural response

$$i_{sc} = i_f + i_n = I_{m,f}\sin(\omega t + \psi_v - \varphi_{sc}) + A e^{-(R_{sc}/L_{sc})t}, \tag{6.14}$$

where $I_{m,f} = V_m/Z_{sc}$ is an amplitude of the forced response, which is a steady-state short-circuit current,

$$Z_{sc} = \sqrt{R_{sc}^2 + (\omega L_{sc})^2}, \quad \varphi_{sc} = \tan^{-1}\frac{\omega L_{sc}}{R_{sc}}$$

are the magnitude and the angle of the total impedance up to the fault point F and ψ_v is an applied voltage phase angle at the moment of the short-circuiting.

Suppose that the current prior to short-circuiting was

$$i_{ld} = I_{m,ld}\sin(\omega t + \psi_v - \varphi_{ld}), \tag{6.15}$$

where $I_{m.ld} = V_m/Z_{ld}$ is the amplitude of the current under normal load conditions, just prior to short-circuiting,

$$Z_{ld} = \sqrt{R_{ld}^2 + (\omega L_{ld})^2} \quad \text{and} \quad \varphi_{ld} = \tan^{-1}\frac{\omega L_{ld}}{R_{ld}}$$

are the total impedance and the angle of a total impedance of the load and the system under normal operation. Then the integrating constant is

$$A = i_{n0} = i_{ld}(0) - i_f(0) = I_{m,ld}\sin(\psi_v - \varphi_{ld}) - I_{m,f}\sin(\psi_v - \varphi_{sc}), \tag{6.16}$$

and the time constant of the exponential term is

$$\tau = \frac{L_{sc}}{R_{sc}}. \tag{6.17}$$

Finally, we have the expressions of the natural and total responses:

$$i_n = A e^{-t/\tau} = [I_{m,ld}\sin(\psi_v - \varphi_{ld}) - I_{m,f}\sin(\psi_v - \varphi_{sc})]\, e^{-t/\tau}, \tag{6.18}$$

and

$$i_{sc} = I_{m,f}\sin(\omega t + \psi_v - \varphi_{sc}) - i_{n0} e^{-t/\tau}. \tag{6.19}$$

Since the current in phase "a" is determined, the rest of the currents in phases "b" and "c" may be found by replacing the current of phase "a" by -120° for the current of phase "b" and by 120° for the current of phase "c". In Fig. 6.10 the three-phase phasor diagram of all three currents is given.

In accordance with the phasor concept, the phasors on the phasor diagram are vectors rotated in a counterclockwise direction at an angular velocity of ω, rad/s, and their projections on axis "t" (or on an axis of imaginary numbers) give the instantaneous values of the currents/voltages. Hence, the differences of two phasors ($I_{m,ld} - I_{m,f}$) in each of three phases (dashed phasors) represent the vectorized values of the integration constants, and their projection on axis t

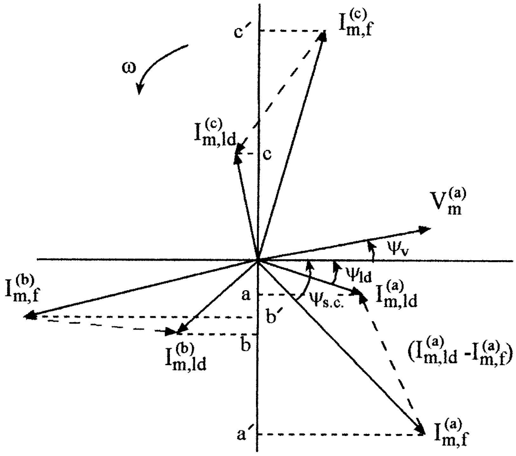


Figure 6.10 The phasor diagram of three-phase currents in a simple circuit at the three-phase short-circuit fault.

gives the initial values of the natural responses in the corresponding phase *a*, *b* and *c*. Such a representation clearly shows that the initial value of a natural response may vary from its maximal value, when the vector $(I_{m,ld} - I_{m,f})$, i.e., the dashed line, is parallel to axis *t*, to zero, when this vector is perpendicular to axis *t*. The position of this vector on the diagram is dependent on the applied voltage phase angle ψ_v at the moment of fault. In the latter case the exponential term is absent, which means that the forced current at the instant of switching is equal to the current prior to switching and no transient response takes place at all. It is obvious that such conditions may occur only in one of the phases. For the conditions of the phasor diagram, shown in Fig. 6.10, the short-circuit currents versus time in all three phases are shown in Fig. 6.11.

As can be seen from the current plots in Fig. 6.11, the transient currents in three phases, due to the aperiodic term, are different. Hence, we shall say that even the three-phase short circuit is not symmetrical. In one of the phases the instantaneous current might be much larger than in the others. However, after the aperiodical term decays, the short-circuit current becomes symmetrical.

The exponential term can be separated from the short-circuit current oscillogram, as shown in Fig. 6.11(c). As can be seen, the exponential term is a medium line in between two envelopes: an envelope of positive amplitudes and an envelope of negative amplitudes. We may also say that the exponential term represents the curve axis of a short-circuit current causing the current to be unsymmetrical.

The initial value of the exponential term also depends on the previous regime. It is easy to see that the largest value of the integration constant may be achieved, if in the previous regime the current was *leading* (Fig. 6.12(b)). Since the capacitance load in power systems is uncommon, the most severe case may occur if, prior to the fault, the network was under no load operation, Fig. 6.12(c).

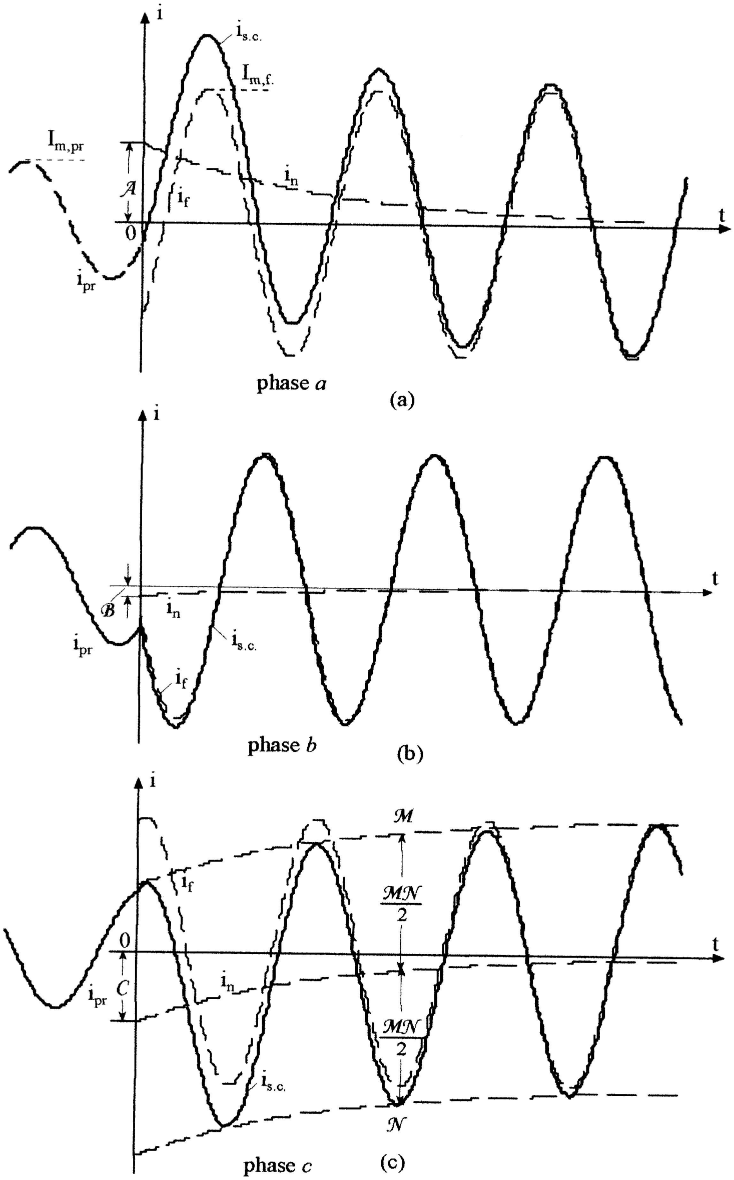


Figure 6.11 The short-circuit currents in a simple three-phase circuit.

The maximal value of the short-circuit current in the latter case will appear if the forced response current, at the instant of the fault, passes its maximum (positive or negative), so that $i_{n0} \cong I_{m,f}$ For the short-circuited network, which is primarily of inductive impedance, this takes place when the applied voltage

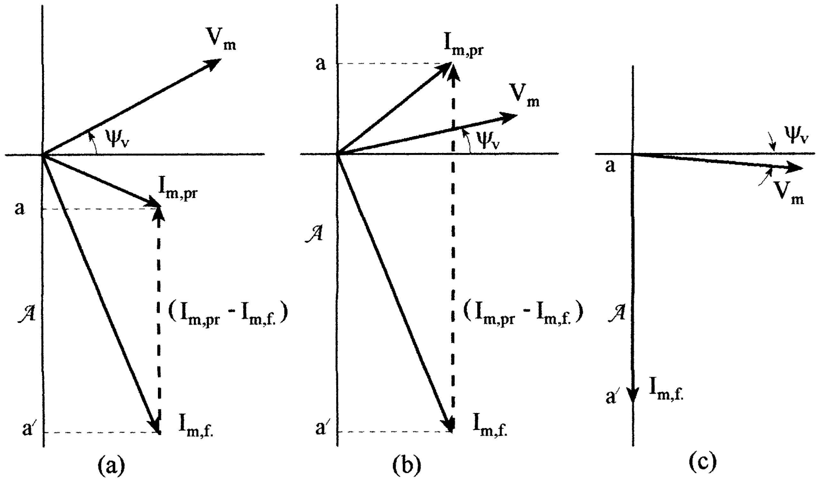


Figure 6.12 The most unfavorable conditions for the largest value of an aperiodic term to appear: 1) the lagging load (a), 2) the leading load (b) and 3) no-load operation (c).

passes its zero point. The plot of the short-circuit current under such conditions is shown in Fig. 6.13.

Note that the time constant T_a may be found experimentally from the short-circuit oscillogram, as shown in Fig. 6.13 (also refer to section 1.3.1). The time constant here is measured as an under-tangent, T_a, along axis t. To achieve good precision, using this method, point g must be taken at the beginning (the highest) part of the exponential curve.

We may estimate the highest peak (or just "peak") of a short-circuit current by using the "*peak-coefficient*". Since the highest peak is found to occur at about

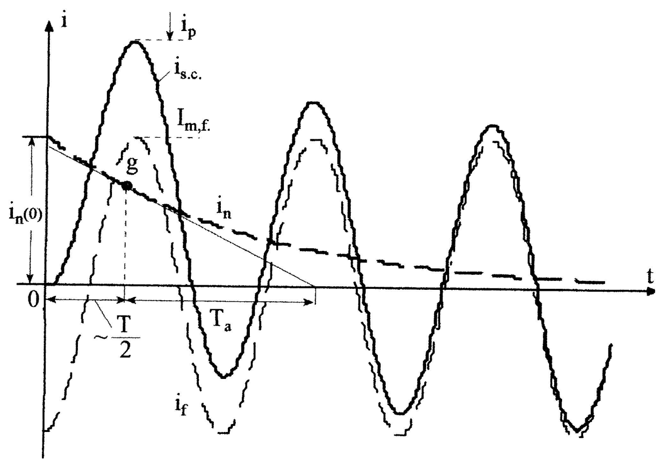


Figure 6.13 A plot of a short-circuit current having a maximal exponential component.

$t \cong T/2$ (i.e. at 50 Hz in 10 ms and at 60 Hz in 8 ms) after the incidence of the short-circuit, we have

$$i_{pk} = I_{m,f} + i_{n0}e^{-(T/2\tau)} = (1 + e^{-(T/2\tau)})I_{m,f} = k_p I_{m,f}$$

where k_{pk} is the peak coefficient. Thus,

$$k_{pk} = 1 + e^{-(T/2\tau)}. \tag{6.20a}$$

The time constant, τ, changes between zero ($L = 0$) to infinity ($R = 0$), therefore the peak coefficient lies in the range

$$1 < k_p < 2 \tag{6.20b}$$

(except for the much less common case of the leading current, shown in Fig. 6.12(b)).

Due to the resistivity of the short-circuit network, the exponential term finally vanishes. Usually the time constant of power system networks is relatively large ($\tau = 0.01$–0.2 s), so that it takes a few periods for the exponential term to decay.

To check the *thermal stability* of electrical equipment under short-circuit fault conditions, the r.m.s value of the short-circuit current in its initial stage has to be estimated. Since this current is unsymmetrical, i.e., consisting of two components: sinusoidal, or a.c., and exponential, or d.c., we may calculate its r.m.s. value as

$$I_{sc} = \sqrt{I_f^2 + I_{\mathrm{exp}}^2} \tag{6.21}$$

where $I_f = I_{m,f}/\sqrt{2}$ is an r.m.s. value of a.c. and I_{exp} is an r.m.s. value of the exponential term. The *r.m.s. value of the exponential term* may be estimated as its average value in the interval of one period T or approximately, as the value in the middle point of the period, as shown in Fig. 6.14.

The highest *r.m.s. value of a short-circuit current* will appear at the first period

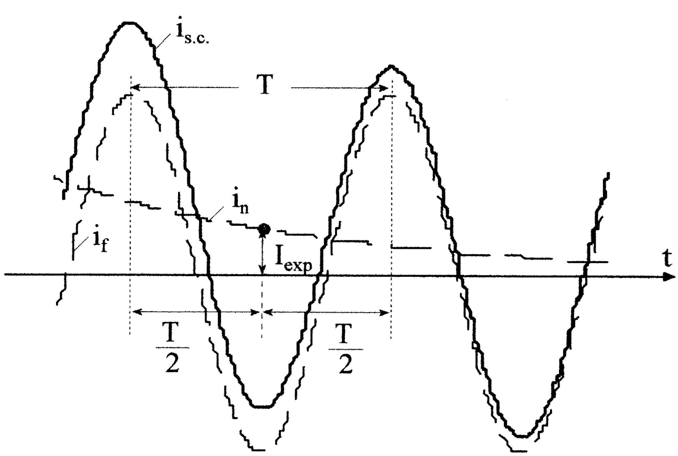


Figure 6.14 Calculation of an average value of the exponential term.

after the instant of fault. Thus, with equation 6.20a and in accordance with equation 6.21 we obtain

$$I_{sc,pk} = \sqrt{I_f^2 + [(k_{pk} - 1)\sqrt{2}I_f]^2} = I_f\sqrt{1 + 2(k_{pk} - 1)^2} \tag{6.22a}$$

and with equation 6.20b, the range limits of $I_{sc,pk}$ are

$$1 < \frac{I_{sc,pk}}{I_f} < \sqrt{3}. \tag{6.22b}$$

The value of i_{pk} is used by project engineers for checking the electrodynamic stability of electrical equipment under short-circuit fault conditions.

6.4 SWITCHING TRANSFORMERS

6.4.1 Short-circuiting of power transformers

The short-circuit phenomenon in any transformer must be analyzed as a transient response in mutual (magnetically interlinked) elements. Considering a three-phase transformer as a symmetrical element (which is an approximation of a three-phase core type transformer) we may reduce it to a single-phase circuit, as shown in Fig. 6.15. In this equivalent circuit a transformer is represented as two identical circuits. The resistance and inductance of the secondary winding are referred to the primary winding. Note that, as previously shown, p.u. impedances, resistances and inductances of a transformer are the same regardless of which winding they are referred to. This means that both the

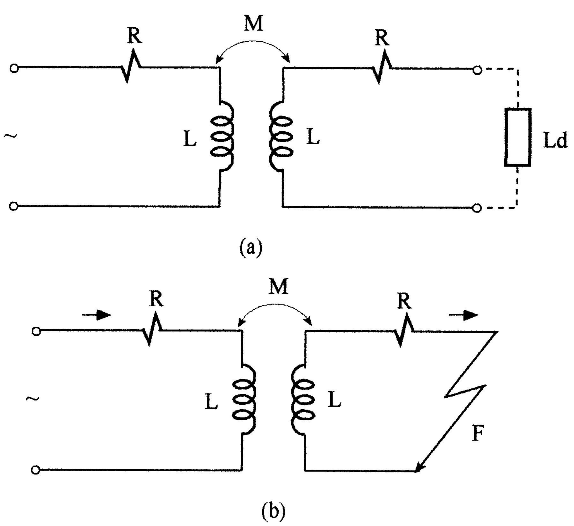


Figure 6.15 An equivalent single-phase transformer under the load (a) and under the short-circuit fault (b).

primary and secondary circuits have identical p.u. parameters. It should also be noted that the ratio of the winding inductance to its resistance is equal in both windings as is usual in power transformers (since the amounts of copper in the primary and secondary windings are nearly equal). Hence we may assume that the parameters of the secondary winding, which are referred to the primary, are about of the same values as the primary, and represent the transformer by two similar circuits, Fig. 6.15. Also note that this equivalent transformer has a unit turns ratio. We may also say that when the analysis is done in p.u. quantities, the actual values of the primary and secondary circuits may be obtained by simply multiplying the p.u. value of each current by an appropriate rated value.

By using the superposition properties discussed previously, we may separate the previous, i.e., the prior to short-circuiting, operation of the transformer and its transient behavior having zero initial conditions. To find its natural response we shall solve two homogeneous equations

$$\begin{aligned} L\frac{di_{1,n}}{dt} + Ri_{1,n} + M\frac{di_{2,n}}{dt} &= 0 \\ L\frac{di_{2,n}}{dt} + Ri_{2,n} + M\frac{di_{1,n}}{dt} &= 0. \end{aligned} \tag{6.23}$$

The characteristic equation has been developed in Example 1.2 (Chapter 1) and it roots are given by equation 1.34, which under the given conditions (that $L_1 = L_2 = L$ and $R_1 = R_2 = R$) yields

$$s_{1,2} = \frac{1}{L^2 - M^2}\left[RL \pm \sqrt{(RL)^2 - R^2(L^2 - M^2)}\right] = \frac{R(L \mp M)}{L^2 - M^2}, \tag{6.24a}$$

or

$$s_1 = -\frac{R}{L+M} = -\frac{1}{\tau_m}, \quad s_2 = -\frac{R}{L-M} = -\frac{1}{\tau_\ell}, \tag{6.24b}$$

and the time constants are

$$\tau_m = \frac{L+M}{R}, \quad \tau_\ell = \frac{L-M}{R}. \tag{6.24c}$$

Hence, the natural currents are

$$i_{1,n} = A_1 e^{-t/\tau_m} + A_2 e^{-t/\tau_\ell}, \quad i_{2,n} = B_1 e^{-t/\tau_m} + B_2 e^{-t/\tau_\ell}. \tag{6.25}$$

The transformer's equivalent circuit (Fig. 6.15) is of the second order and therefore both currents consist of two exponential terms, having two different time constants. The larger one τ_m is determined by the sum of the winding inductance L and the mutual inductance M and is related to the main magnetic flux linked to both windings. The smaller one τ_ℓ is determined by the difference between the inductances L and M and is related to the leakage flux. As is known from the power transformer theory, the difference between L and M represents the leakage inductance of the transformer windings and usually has a relatively small value. Thus,

$$L_\ell = L - M.$$

In the next step (step 2 of the classical approach) we shall find the forced response, i.e. the steady-state short-circuit current of a transformer. By neglecting the resistances and using the phasor approach: $i = Ie^{j\omega t}$ and $v = Ve^{j\omega t}$, for the transformer in Fig. 6.15(b) we may write

$$\begin{aligned} j\omega L I_1 + j\omega M I_2 &= V_s \\ j\omega M I_1 + j\omega L I_2 &= 0. \end{aligned} \tag{6.26}$$

From the second equation we have

$$I_2 = -\frac{M}{L} I_1. \tag{6.27a}$$

Substituting this in the first equation (equation 6.26) yields (for the magnitudes)

$$I_1 = \frac{V_s L}{\omega(L^2 - M^2)} = \frac{V_s L}{\omega(L+M)(L-M)}. \tag{6.27b}$$

Because of the small leakage we may neglect in the sum $(L+M)$ the difference between inductance L and mutual inductance M $(L \cong M)$. Then the above expression simplifies to

$$I_{1,f} = I_{sc} = \frac{V_s}{\omega 2(L-M)} = \frac{V_s}{2\omega L_\ell} = \frac{V_s}{X_\ell}, \tag{6.27c}$$

where L_ℓ is the leakage inductance of one winding and X_ℓ is the leakage reactance of a transformer. The p.u. value of the steady-state short-circuit current, therefore, is

$$\frac{I_{sc}}{I_\mathrm{r}} = \frac{V_s}{X_\ell I_\mathrm{r}} = \frac{V_\mathrm{r}}{V_{sc}},$$

i.e., as a ratio of the system voltage, which is usually the same as a rated voltage, and the voltage drop of the transformer caused by the short-circuit current (the voltage at the short-circuit test). Thus if, for instance, a relatively low power distribution transformer has a 4% short-circuit voltage, it will develop a steady-state short-circuit current

$$\frac{I_{sc}}{I_\mathrm{r}} = \frac{100}{4} = 25,$$

i.e., 25 times the normal current in either of the transformer windings.

The next step is finding the independent initial conditions, i.e. the value of both currents at the instant of switching. For this reason we have to take into consideration that prior to switching the transformer carried a *magnetizing, or exciting, current* (the current at no-load), which is obtained from the first equation (equation 6.26). At zero secondary, it yields

$$I_M = \frac{V_s}{\omega L}. \tag{6.28}$$

The p.u. value of the magnetizing current for power transformers lies in the 0.5–3% range; the first number is appropriate for very large transmission transformers (200–300 MVA) and the second one is appropriate for relatively small distribution transformers. The magnetizing current, which is an open-circuit current, relates to the short-circuit current, with equation 6.27c and equation 6.28, as

$$\frac{I_M}{I_{sc}} = \frac{V_s}{\omega l} \Bigg/ \frac{V_s}{2\omega L_\ell} = \frac{2L_\ell}{L} \cong \frac{2L_\ell}{M}. \tag{6.29}$$

It is worthwhile to mention that the same results can be obtained by inspection of the equivalent circuit of a transformer with a cancelled mutual inductance, Fig. 6.16(a), and its common approximation with the magnetized branch moved to the transformer input, Fig. 6.16(b). As can be seen from Fig. 6.16(b), after neglecting the resistances and assuming $L \cong M$, the magnetized current I_M and short-circuit current I_{sc} become the expressions as in equations 6.27c and 6.28.

Let us consider the most unfavorable instant of the short-circuiting, when the steady-state primary current $i_{1,f}$ passes through its maximum $I_{1,f}$ (equation 6.27). Since both currents, the magnetizing and the short-circuit current, are

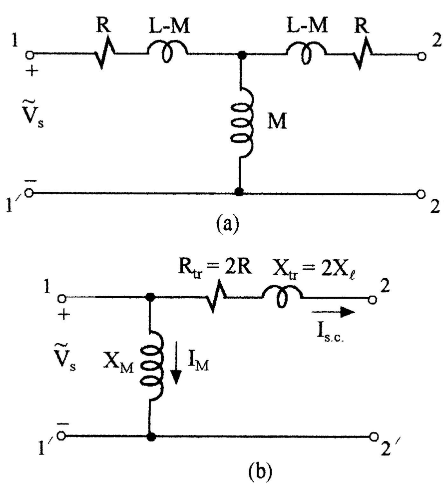


Figure 6.16 An equivalent circuit of a transformer with a cancelled mutual inductance (a) and an approximate circuit with the magnetized branch moved to the input (b).

almost purely inductive, and thus having nearly the same phase angle, we may write

$$i_{1,n}(0) = i_1(0) - i_{1,f}(0) = I_M - I_{1,f}, \tag{6.29a}$$

and

$$i_{2,n}(0) = 0 - i_{2,f}(0) = \frac{M}{L} i_{1,f}(0) = \frac{M}{L} I_{1,f}. \tag{6.30}$$

Next we shall find the dependent initial conditions, i.e., the derivatives of both natural currents at $t = 0$. The equations (equation 6.23) may be rewritten as

$$L \frac{di_{1,n}}{dt}\bigg|_{t=0} + M \frac{di_{2,n}}{dt}\bigg|_{t=0} = Ri_{1,n}(0) = I_m - I_{1,f}$$

$$M \frac{di_{1,n}}{dt}\bigg|_{t=0} + L \frac{di_{2,n}}{dt}\bigg|_{t=0} = Ri_{2,n}(0) = \frac{M}{N} I_{1,f}.$$

Solving these two equations for each of the derivatives yields

$$\begin{aligned}\frac{di_{1,n}}{dt}\bigg|_{t=0} &= -R\left[\frac{L}{L^2 - M^2} I_m - \frac{L^2 + M^2}{L(L^2 - M^2)} I_{1,f}\right]\\ \frac{di_{2,n}}{dt}\bigg|_{t=0} &= -R\left[\frac{-M}{L^2 - M^2} I_m + \frac{2M}{L^2 - M^2} I_{1,f}\right].\end{aligned} \tag{6.31}$$

We can now obtain the integration constant by solving two simultaneous equations (see equation 1.55 and Example 2.3).

For the primary current $i_{1,n}$:

$$\begin{cases} A_1 + A_2 = I_m - I_{1,f} \\ \dfrac{-R}{L+M} A_1 + \dfrac{-R}{L-M} A_2 = -R\left[\dfrac{L}{L^2 - M^2} I_m - \dfrac{L^2 + M^2}{L(L^2 - M^2)} I_{1,f}\right], \end{cases}$$

which yields

$$\begin{aligned} A_1 &= \frac{1}{2} I_m - \frac{L-M}{2L} I_{1,f} = \frac{1}{2} I_m - \frac{1}{2}\frac{L_\ell}{L} I_{1,f}, \\ A_2 &= \frac{1}{2} I_m - \frac{L+M}{2L} I_{1,f} \cong \frac{1}{2} I_m - I_{1,f}. \end{aligned} \tag{6.32}$$

For the secondary current, $i_{2,n}$:

$$\begin{cases} B_1 + B_2 = \dfrac{M}{L} I_{1,f} \\ \dfrac{-R}{L+M} B_1 + \dfrac{-R}{L-M} B_2 = -R\left[\dfrac{M}{L^2 - M^2} I_m + \dfrac{2M}{L^2 - M^2} I_{1,f}\right], \end{cases}$$

which yields

$$B_1 = \frac{1}{2} I_m - \frac{L - M}{2L} I_{1,f} = \frac{1}{2} I_m - \frac{1}{2} \frac{L_\ell}{L} I_{1,f},$$
$$A_2 = -\frac{1}{2} I_m + \frac{L + M}{2L} I_{1,f} \cong -\frac{1}{2} I_m + I_{1,f}. \tag{6.33}$$

These results actually show that $B_1 = A_1$ and $B_2 = -A_2$. The expressions for $A_1(B_1)$ and $A_2(B_2)$ may be simplified: with equation 6.29 we have

$$A_1 = B_1 = \frac{1}{2} I_m - \frac{1}{2} \frac{L_\ell}{L} \frac{L}{2L_\ell} I_m = \frac{1}{4} I_m.$$

And, since I_m is negligibly small relative to I_{sc},

$$A_2 = -B_2 \cong -I_{1,f} = -I_{sc}.$$

Finally,

$$i_{1,n} = \frac{1}{4} I_m e^{-t/\tau_m} - I_{sc} e^{-t/\tau_\ell}, \quad i_{2,n} = \frac{1}{4} I_m e^{-t/\tau_m} + I_{sc} e^{-t/\tau_\ell}. \tag{6.34}$$

These expressions show that the short-circuiting of the transformer results in the appearance in both windings of two exponential (aperiodic) currents, which superimpose with the steady-state short-circuit currents. The first one decays relatively slowly with the large time constant τ_m, however it is insignificantly small and can be neglected. The second one decays much faster with the smaller time constant τ_ℓ, but its initial value is as large as the amplitude of the steady-state short-circuit current. Half a cycle after short-circuiting, the exponential term is added to the steady-state short-circuit current, which results in an almost double amplitude value. This means that a transformer having a leakage inductance in the order of 4–10% will develop a maximal short-circuit current of 50–20 times the rated value. A typical curve of such a short-circuit current versus time is shown in Fig. 6.17,

It should be noted that by neglecting the very small effect of the transient

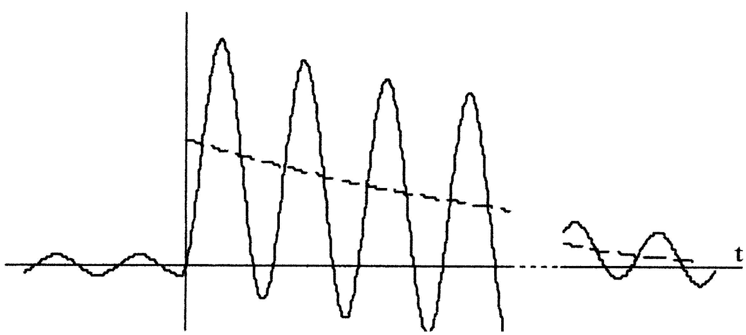


Figure 6.17 A typical waveform of a transformer's short-circuit current.

magnetizing currents in equation 6.34, the transient response to short-circuiting a transformer is similar that in the simple *RL* circuit having an inductance as a total leakage inductance of the transformer and the total resistance of both its windings. Hence, for the analysis of the short-circuit phenomena in any power network, we may replace every transformer by a single inductance in series with a resistance, both referred either to the high- or low-voltage side.

6.4.2 Current inrush by switching on transformers

Upon switching on a power transformer, an inrush of a magnetizing (exciting) current may initially reach a very high level of eight times the rated current, even under no-load conditions. From our previous study, we know that in *linear RL* circuits, even under the most unfavorable conditions, the transient current may not exceed the double value of its forced response. However, the magnetizing circuit of the transformer is non-linear due to its iron core. Hence, to analyze the transient phenomenon in the transformer we have to take into consideration the saturation of its *magnetizing characteristic*, i.e. $B = f(H)$.

The inrush is most severe when the transformer is switched on at the instant the voltage goes through zero with such polarity that the flux increases in the direction of the residual flux. For these conditions, we may write

$$v_s = \sqrt{2}V_s \sin \omega t = \frac{d\lambda}{dt} = N\frac{d\phi}{dt}.$$

The value of the flux is then found by integration:

$$\phi = \frac{\sqrt{2}V_s}{N}\int_0^t \sin \omega t \, dt + \phi(0), \tag{6.35a}$$

where $\phi_{(0)} = \Phi_0$ is the residual flux. Thus

$$\phi = \frac{\sqrt{2}V_s}{\omega N}(1 - \cos \omega t) + \Phi_0 = -\Phi_m \cos \omega t + \Phi_m + \Phi_0. \tag{6.35b}$$

Since we neglected all the resistances (representing the winding and core losses), the aperiodic (d.c.) component, $\Phi_m + \Phi_0$, is obtained as a constant quantity. However, due to these losses, the aperiodic term decays very slowly according to the large time constant of the magnetizing circuit. Then, at $\omega t = \pi$ (half a period after switching) the instantaneous flux will be

$$\phi_{\max} = 2\Phi_m + \Phi_0.$$

The magnetic flux density under steady-state conditions is $B_m \cong 1.3$ T. If Φ_0 is assumed to equal $0.6\Phi_m$, then the maximal flux density, which in a transformer is directly proportional to the flux, will be

$$B_{\max} = (2 + 0.6)\cdot 1.3 \cong 3.4\ \mathrm{T}.$$

This value is far beyond the rated range and according to the magnetizing

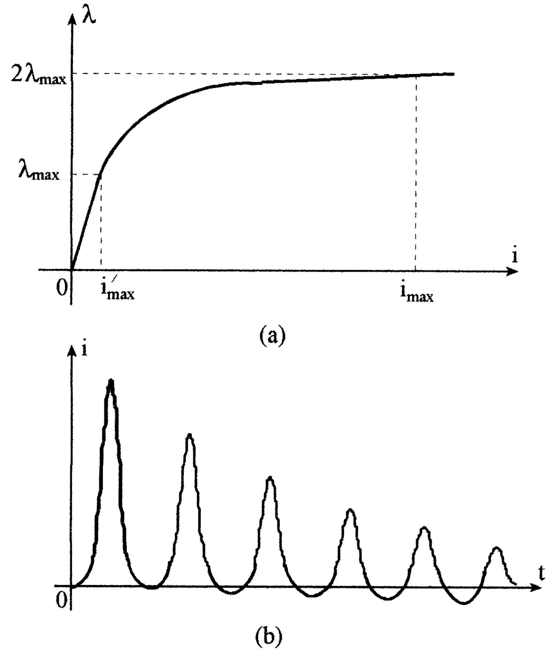


Figure 6.18 A magnetizing curve (a) and an inrush current of transformer (b).

curve, shown in Fig. 6.18(a), the magnetizing force, *H* (which is directly proportional to the magnetizing current) may reach as large a value as 8–10 times its rated value. The typical curve of an inrush current for a transformer switched at zero instantaneous voltage is shown in Fig. 6.18(b). Note that the waveform of this current is not sinusoidal due to the presence of high harmonics (as a result of the non-linearity of a transformer magnetize characteristic).

6.5 SHORT-CIRCUITING OF SYNCHRONOUS MACHINES

High-magnitude transient currents, or short-circuit currents, in the stator windings of a synchronous generator occur, particularly if the voltage at its terminals is suddenly changed by a considerable amount. This may happen as a result of a faulty switching operation, or by any other fault, which brings about short-circuiting, such as a result of bad synchronizing in the faulty position of the poles, by energizing a rotating machine by sudden connection to full voltage, etc. In such cases the transient currents may be much greater than the normal operating currents of the machine. Depending on the design of the machine and the process of switching excess currents, up to ten times the normal current may develop in the windings. In view of the large size of most modern generators, this would release an enormous amount of energy in the network, which might be dangerous for the normal operation of the network equipment.

The stator and rotor windings of a synchronous machine are mutually coupled, but in distinction to a transformer, due to the rotation of the rotor, they continuously change their relative position in the space. As a result of that, their mutual inductances are not constant, but vary in time. This leads to differential equations with variable coefficients, whose analysis and solution are very cumbersome.(*)

To simplify the practical approach to the calculation of short-circuit currents we shall make a few common assumptions. It should be noted that by any sudden change of the operation conditions of a synchronous machine, its revolution is disturbed and its angular velocity changes, which gives rise to mechanical oscillations. Obviously, the detailed analysis of the transient behavior of the synchronous generator becomes even more complicated. Thus, the first assumption is that the revolution of the generator does not change and remains constant during the transients.

As previously, we shall neglect the resistance of the generator windings and the short-circuit impedances of the generator will be considered approximately as an inductive reactance. The resistances will then be taken into consideration by determining the damping coefficients of decaying the transient currents.

To simplify the entire calculation of transients in a three-phase system and reduce it to a one-phase presentation, the generalized phasor of three-phase system currents will be introduced as well as the two-phase model of the synchronous machine.

6.5.1 Two-axis representation of a synchronous generator

Three-phase synchronous generators fall into two general classifications: 1) *cylindrical* (or *round*) *rotor* (high-speed turbogenerators) or 2) *salient-pole rotor* (low-speed hydrogenerators). While the air gap in the cylindrical rotor construction is practically of uniform length that of the salient-pole rotor is much longer in between the poles, Fig. 6.19.

We shall review here the two-axis representation of synchronous generators using the salient-rotor generator as an example rather than the cylindrical one, since the latter constitutes a particular case of the former. In Fig. 6.20 the schematic cross-section of a salient-rotor generator is given. Here the rotor has two axes: the *direct axis d*, which is in the direction of the magnetizing, or field flux, Φ_d and the *quadrature axis q*, which is perpendicular to axis *d* midway between the poles. Accordingly, the generator is represented by two reactances X_d and X_q and two EMF's, E_d and E_q in the direct axis and the quadrature axis respectively. The above two components of the EMF can always be combined in **one phasor** of a total generated EMF (or terminated voltage), E_{af}.

The stator three-phase winding carries three currents, which are displaced by 120° relative to each other. Following the idea of a one-phase representation of

(*) For a detailed discussion of this problem see, for example, C. Concordia, *Synchronous Machines*, John Wiley & Sons, New York, 1957.

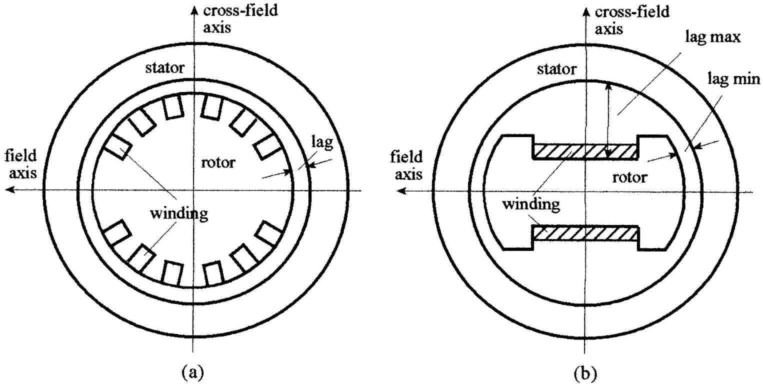


Figure 6.19 Two kinds of rotors: cylindrical rotor (a) and salient-pole rotor (b).

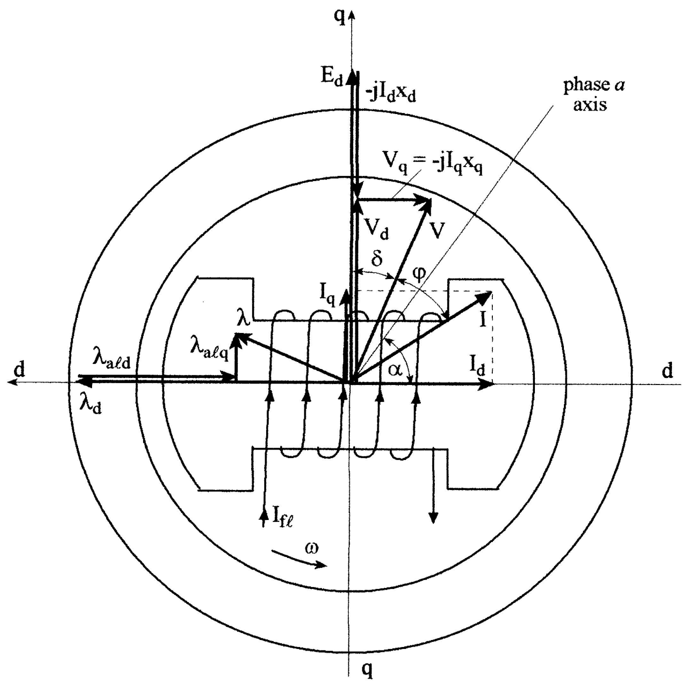


Figure 6.20 Salient rotor and phasor diagram in a two-axis representation.

a synchronous machine we shall transform the stator three-phase current system into one generalized current phasor I.

Consider the usual representation of a three-phase current by three phasors, as shown in Fig. 6.21a. The three instantaneous currents i_a, i_b and i_c can then be obtained as the projections of the three phasors on the time axis t, while the star of phasors is rotating with an angular velocity ω.

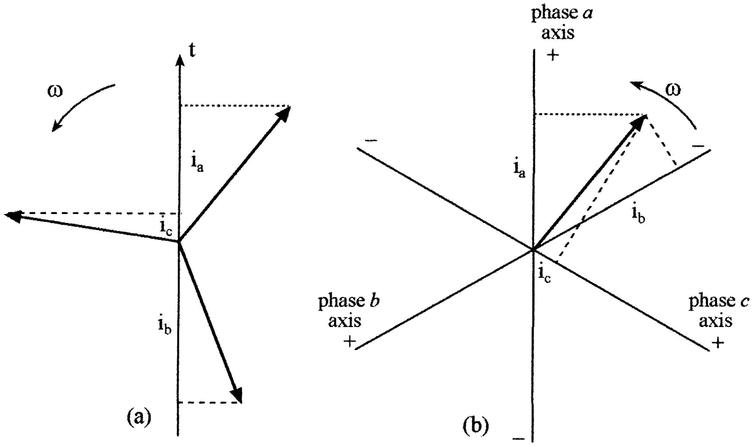


Figure 6.21 Determining instantaneous currents in a three-phase system: with three-phase phasors (a) and with one generalized phasor (b).

The same results may be derived using only one **rotating phasor**, or a so called **generalized phasor**, but with its projection on three time axes, which coincide with the axes of a three-phase stator winding, as shown in Fig. 6.21(b). If the generalized current phasor is rotated in the same direction as the phase-phasors, the sequence of the time axes should be taken as opposite to those of the phase-phasors, i.e., $a \to c \to b$.

Now, this single phasor can be expanded into two quadrature components, according to two rotor axes, I_d and I_q, as shown in Fig. 6.20. Here, λ_d is the flux linkage, produced by the field current, λ_{ald} and λ_{alq} are the armature reaction and stator winding leakage fluxes, produced by the currents I_d and I_q respectively and λ is the resultant flux linkage, which induces the terminal voltage V. In accordance with the phasor diagram for EMF's we may write

$$\tilde{V}_d = E_d - jX_d I_d, \quad \tilde{V}_q = -jX_q I_q, \tag{6.36a}$$

and

$$\tilde{V} = V_d + V_q \quad \text{or} \quad |V| = \sqrt{V_d^2 + V_q^2}, \tag{6.36b}$$

where X_d and X_q are the generator direct-axis and quadrature-axis reactances.

Finally, if the phasors I_d and I_q are known, and taking into consideration that $\tilde{I}_a + \tilde{I}_b + \tilde{I}_c = 0$, the phase-phasors can be expressed as

$$\begin{aligned} I_a &= I_d \cos\alpha + I_q \sin\alpha \\ I_b &= I_d \cos(\alpha + 2\pi/3) + I_q \sin(\alpha + 2\pi/3) \\ I_c &= I_d \cos(\alpha - 2\pi/3) + I_q \sin(\alpha - 2\pi/3), \end{aligned} \qquad (6.38a)^{(*)}$$

(*) If the sum of the phase current phasors is not equal to zero, then each phase current consists of a zero sequence term.

where α is the angle between the rotor direct axis and the axis of the phase a winding.

In turn, the two components of a generalized current can be expressed by the phase currents:

$$I_d = \frac{2}{3}\left[I_a \cos\alpha + I_b \cos(\alpha + 2\pi/3) + I_c \cos(\alpha - 2\pi/3)\right]$$
$$I_d = \frac{2}{3}\left[I_a \sin\alpha + I_b \sin(\alpha + 2\pi/3) + I_c \sin(\alpha - 2\pi/3)\right]. \tag{6.38b}$$

Thus, the generalized current completely represents the three-phase stator currents and allows for the reduction of a three-phase generator to a one-phase machine, having constant mutual inductances between the stator and rotor, which is

$$M_{eq} = \frac{3}{2} M,$$

where M is the mutual inductance between the phase winding of the stator and rotor winding, when the axis of the stator winding coincides with the direct axis of the rotor.

As was previously mentioned, the cylindrical-rotor generator is a particular case of a salient-pole rotor. Thus, since the air gap lengths of both the d and q axes of the cylindrical rotor are the same, we have $X_d \cong X_q$ and all the expressions obtained for a salient-pole generator are valid for a cylindrical rotor generator.

6.5.2 Steady-state short-circuit of synchronous machines

As we know the steady-state regime, or the forced response, takes place after the natural responses decay, i.e., a few seconds after the moment of short-circuiting. However, for the sake of protecting all kinds of electrical equipment and providing the dynamic stability of synchronous generators operating in parallel, the short-circuit fault in present-day power systems is disconnected very fast (by means of modern relay protection and switch gears), Therefore, steady-state short-circuit conditions are very uncommon. We shall, however, start our analysis of the synchronous generators' behavior under short-circuit conditions with the steady-state short-circuit. In order to get the total response and estimate the maximal magnitudes of short-circuit currents in the first moments of the fault, we must know the forced responses, i.e., the steady-state short-circuit currents. In addition, the study of steady-state short-circuit behavior of a synchronous generator contributes largely to a better understanding of the whole process.

The steady-state short-circuit behavior of a synchronous generator depends to a greater degree on *the automatic voltage regulator* (AVR). Excitation of a synchronous generator is derived from a d.c. supply with a variable voltage.

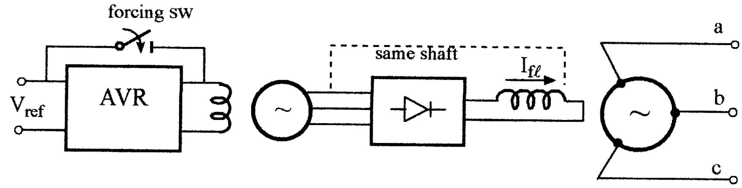


Figure 6.22 An excitation arrangement for a synchronous generator with AVR.

Originally, the main exciter consisted of an a.c. exciter with an integral diode or thyristor rectifiers rotating on the rotor (main) shaft, thus avoiding any brush gear, Fig. 6.22. In general, the AVR's are set out to control the output voltage of the synchronous generator, by controlling the exciter. The other important function of such regulators is to force the field current usually up to its maximal value at the event of a short-circuit fault, which requires a very fast-acting regulator. As a result of the AVR action, the steady-state short-circuit current might be larger than during the transients and even at the first moment of switching.

(a) *Short-circuit ratio* (*SCR*) *of a synchronous generator*

When short-circuiting occurs across the terminals of the generators or nearby, the magnetic saturation of their characteristics must be taken into consideration since the values of the voltages and of the inductances substantially depend on the magnetic saturation. The *open-circuit* (no-load) *characteristic* (OCC), or the magnetic curve, is the graph of the generated voltage against the *field current*, I_{fl}, of the machine on open circuit and running at synchronous speed. The typical OCC of turbo- and hydro-generators in p.u. are shown in Fig. 6.23. The air-gap line represents the linear part of the open-circuit characteristic and ignores saturation (Fig. 6.24).

For our further consideration, we will also need the *short-circuit characteristic* (SCC), which is the graph of a stator current against a field current with the terminals short-circuited. Both OCC and SCC are shown in Fig. 6.24.

With these two characteristics we may calculate, first of all, both the unsaturated and saturated (its approximate value) synchronous reactances of the generator. The p.u. **unsaturated reactance** is obtained with the air-gap (unsaturated) line as the ratio of the open-circuit voltage (length $\overline{ac}$) and the short-circuit current (length $\overline{ad}$), both produced by the same field current ($\overline{0a}$). Thus,

$$X_{du} = \frac{\overline{ac}}{\overline{ad}} \text{ pu.} \tag{6.39a}$$

With the saturated air-gap line (of), also called the modified air gap line, and by the same procedure we may obtain the **saturated reactance**:

$$X_d = \frac{\overline{bc'}}{\overline{be}} \text{ pu.} \tag{6.39b}$$

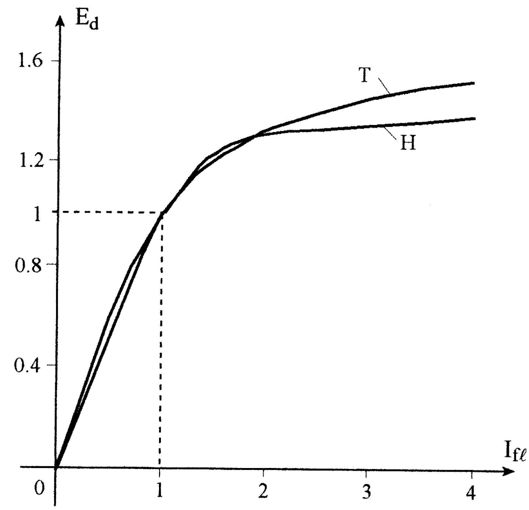


Figure 6.23 Typical open-circuit characteristics of turbogenerator (T) and hydrogenerator (H).

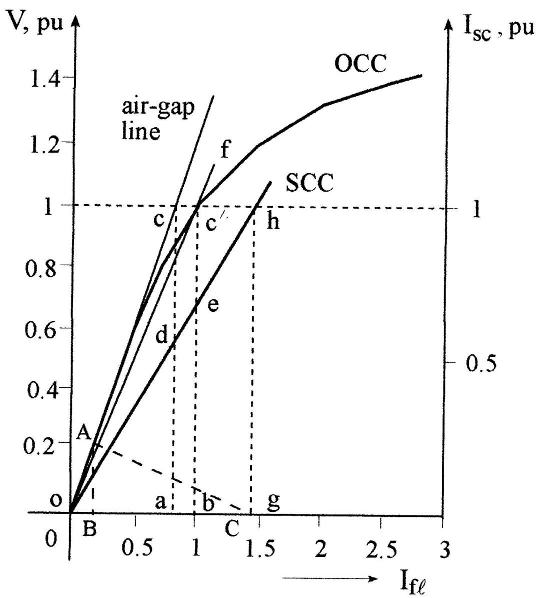


Figure 6.24 Open- and short-circuit characteristics of a synchronous generator.

As can be seen, the saturated reactance is less than the unsaturated reactance, and is usually taken as the active value of the synchronous reactance of a generator.[(*)] It is important to understand this feature of all magnetic circuits. The reactance is reciprocally proportional to the reluctance (or magnetic conductivity), which is a function of the permeability (μ) of the magnetic material. Since the permeability by saturation is getting larger, the reluctance subsequently decreases, which results in a lower reactance.

The second important parameter of a synchronous generator, which is obtained by the above two characteristics is the **short-circuit ratio** (SCR). It is defined as the ratio between the field current required for nominal open-circuit voltage and that required to circulate the full-load current in the armature winding when short-circuited. Thus with Fig. 6.24

$$SCR = \frac{\overline{ob}}{\overline{og}}. \tag{6.40}$$

With SCR the p.u. steady-state short-circuit current at the generator terminals will be

$$I_{sc,\infty} = SCR\ I_{fl}, \tag{6.41a}$$

where I_{fl} is a known *magnetizing, or field, current* in p.u. The steady-state short-circuit current in natural units (i.e., in amperes) will be

$$I_{sc,\infty} = SCR\ I_{fl} I_{\mathrm{r}}, \mathrm{A}$$

where I_{r} is a generator rated (nominal) current. The value of SCR in accordance with the OCC, and SCC in Fig. 6.24 is 0.67 (this value is typical for turbogenerators; for hydrogenerators it can be taken as 1.1).

Comparing triangles Δ ohg and Δ oeb and noting that $gh = bc'$ we have

$$\frac{gh}{be} = \frac{og}{ob} \quad \text{or} \quad \frac{bc'}{be} = \frac{1}{ob/og},$$

i.e.,

$$X_{d,pu} = \frac{1}{SCR}. \tag{6.42}$$

The *direct-axis synchronous reactance* X_d of a synchronous generator (it is often replaced by the so-called *synchronous reactance* X_s) includes the combined effect of the leakage reactance X_l and the *reactance* X_{ad} of the *armature reaction.* The value of the leakage reactances is usually in the range of 0.1–0.15 (for turbogenerators), 0.15–0.25 (for hydrogenerators).

As a reminder of the basic conditions at the terminal short-circuit (s.c.), the phasor diagram and the Potier triangle are shown in Fig. 6.25. The rotor current

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[(*)]For more about the effect of saturation and calculation of the saturation value of X_{d}, see, for example, in McPerson, G. and Laramore, R. D. (1990) *Electrical Machines and Transformers*, Wiley & Sons.

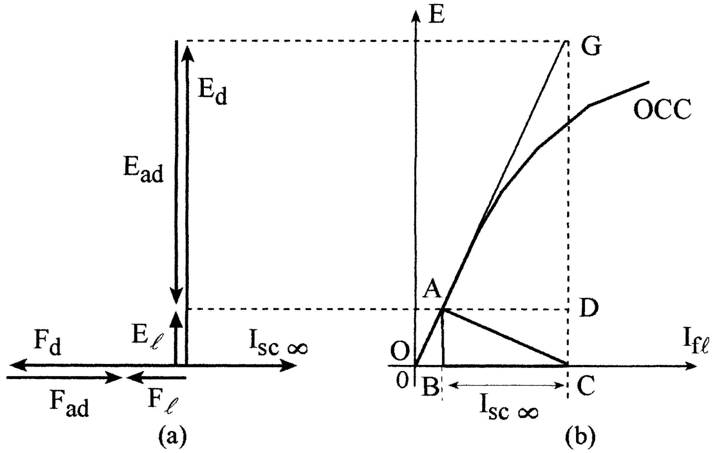

Figure 6.25 The phasor diagram (a) and OCC with a Potier triangle (b).

I_{fl} produces a *magnetomotive force* (MMF) F_d, which induces in the stator winding an *electromotive force* (EMF) E_d. Since the terminal voltage at short-circuiting is zero, this voltage is required to overcome the armature reaction $E_{ad} = jX_{ad}I_{sc}$ and the leakage reactance voltage drop $E_l = jX_lI_{sc}$, Fig. 6.25a. The corresponding fractions of F_d are also shown on the phasor diagram as leading the appropriate EMF by 90°. Note that the armature reaction F_{ad}, is in phase with F_d, since the short-circuit current is actually a zero-power-factor, or pure reactive, current.

With the known E_l point A, which is the upper vertex of the **Potier triangle** on the OCC, is determined. Point C, which is determined by the field current, required to produce a rated short-circuit current, gives the second vertex of the triangle. Point B, which is determined by the perpendicular drawn from vertex A to the abscissa, gives the third vertex. Length $\overline{BC}$ is the component of the field current required to overcome the MMF of the armature reaction and, therefore, is proportional to the stator current. The other component $\overline{OB}$ produces F_l, required for inducing E_l to overcome the leakage reactance voltage drop. Note that since point A is located on the linear part of OCC, the quantities E_d, E_{ad} and X_d are appropriate for an unsaturated generator.

Example 6.3

Use the open-circuit and short-circuit characteristics, shown in Fig. 6.26, for a 133.5 MVA three-phase 13.8 kV 60 Hz generator, to: a) find the unsaturated and saturated synchronous reactances in ohms and in p.u.; b) determine SCR; and c) draw the Potier triangle, if the leakage reactance is 0.145, and determine the scale of the stator current on the axis of the field current (abscissa).

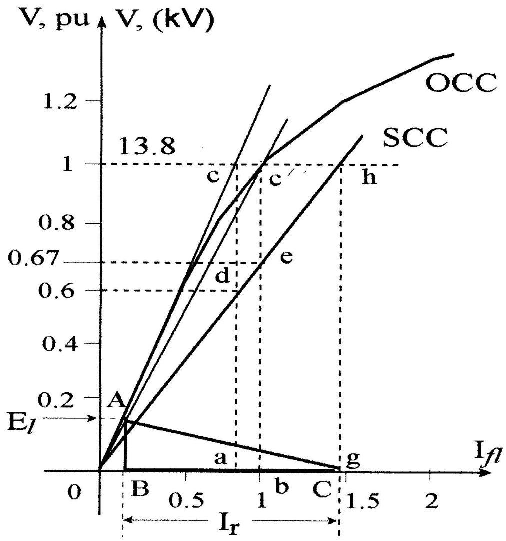

Figure 6.26 The OCC, SCC and Potier triangle for Example 6.3.

Solution

a) The unsaturated synchronous reactance (equation 6.39a) is (see Fig. 6.26)

$$X_{du} = \frac{\overline{ac}}{\overline{ad}} = \frac{1}{0.6} = 1.67 \text{ pu}$$

The rated impedance of the generator (equation 6.4) is

$$Z_{\text{r}} = \frac{13.8^2 \cdot 10^6}{133.5 \cdot 10^6} = 1.43\ \Omega.$$

Thus, the unsaturated reactance in ohms is

$$X_{du} = Z_{\text{r}} X_{du} = 1.43 \cdot 1.67 \cong 2.93\ \Omega.$$

The saturated reactance (equation 6.39b) is (see Fig. 6.26)

$$X_d = \frac{\overline{bc'}}{\overline{be}} = \frac{1}{0.67} = 1.49 \text{ pu}$$

or in ohms

$$X_d = Z_{\text{r}} X_d = 1.43 \cdot 1.49 \cong 2.13\ \Omega.$$

b) The short-circuit ratio (equation 6.40) with Fig. 6.21 is

$$SCR = \frac{\overline{ob}}{\overline{og}} = \frac{1}{1.49} \cong 0.67,$$

which is the reciprocal of X_d.

c) The vertex A is determined by the ordinate $E_{1,pu} = 0.145$ and the vertex C by the abscissa $\overline{OC} = 1.45$, which is the p.u. field current required to produce a rated short-circuit current (see Fig. 6.26). The rated current of the generator is

$$I_r = \frac{S_r}{\sqrt{3}\cdot 13.8\cdot 10^3} = 5580\ \mathrm{A}.$$

Since the length $\overline{BC}$ determines the portion of the field current, which produces F_{ad} to overcome the armature current reaction ($F_{ad} = X_{ad} I_{sc}$), and therefore it is proportional to this current, we may determine the scale of the stator current on the abscissa as

$$m_I = \frac{I_r}{\overline{BC}} = \frac{5580}{1.35} \cong 4130\ \mathrm{A/cm}.$$

The Potier triangle is shown in Fig. 6.26.

With the known p.u. field (magnetizing) current, I_{fl}, the steady-state short-circuit (s.c.) current can easily be found as

$$I_{sc,\infty} = SCR\ I_{fl} I_r.$$

However, this value of an s.c. current is valid only for an unsaturated generator, or for a linear OCC, i.e., in accordance with an air-gap line, which is, of course, only a rough approximation. For a more precise calculation of an s.c. current the graphical solution shall be introduced.

(b) *Graphical solution*

We shall start the graphical solution representation with a simple case of a short-circuit fault occurring on the main line fed by a single generator, as shown in Fig. 6.27. The generator is represented by the OCC and the leakage reactance X_l; the terminal voltage is V and X_F is the reactance of the external network (the resistances, as usual, are neglected).

The EMF of the generator required to overcome the leakage voltage and the terminal voltage is

$$E_g = E_l + V = (X_l + X_F)I_{sc} = X_{eq} I_{sc}. \tag{6.43}$$

This expression (since X_l and X_F are constants) can be represented graphically

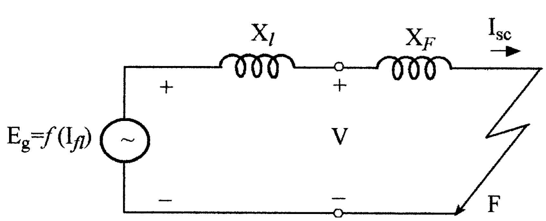

Figure 6.27 An equivalent circuit of a short-circuited synchronous generator through an external reactance.

as a straight line, as shown in Fig. 6.28. As has already been shown, the abscissa of the OCC can also be used as an axis of an armature current with an origin in point C and its positive direction opposes the positive direction of the axis of the field current. Hence, the straight line of $E_g = f(I_{sc})$ should be plotted from this point C with the slope angle

$$\alpha = \tan^{-1}(X_{l,pu} + X_{F,pu}),$$

i.e., line CM in Fig. 6.28.

The EMF of the generator is dependent on two quantities: 1) a magnetizing current I_{fl} (in accordance with its OCC) and 2) an s.c. current I_{sc} (in accordance with equation 6.43. Hence, the actual EMF will be given by the intersection, point M, of the two characteristics: the OCC and the straight line CM, as shown in Fig. 6.28.

The actual s.c. current will be determined by point N and can be expressed by the length $\overline{ON'}$ according to the scale of the axis I_{sc} (see Example 6.2). Note that this method of determining I_{sc} is actually a graphical solution of two equations (one of them is the OCC given as a curve and the second one is a straight line given by expression 6.43) on two unknowns: E_g and I_{sc}, i.e.,

$$E = f(I_{fl}), \quad E = X_{eq} I_{sc}. \tag{6.44}$$

(Also note that there is a relationship between I_{fl} and I_{sc}, e.g., given by equation 6.41a when the fault occurs at the generator terminals.)

Next we may separate the total EMF, induced by the stator winding, into two parts (in accordance with the circuit in Fig. 6.27): the leakage voltage drop E_l and the terminal voltage V. For this purpose we shall draw the Potier triangle

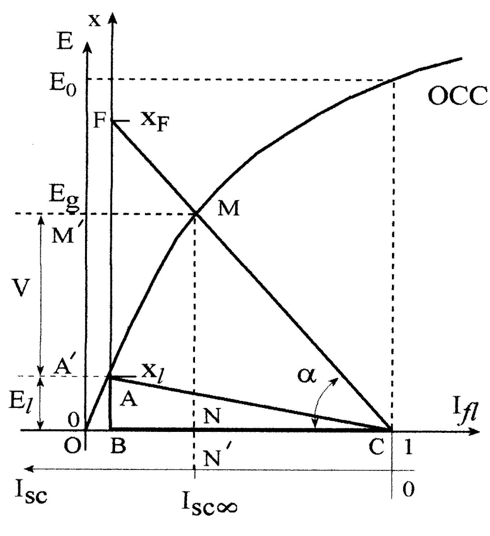

Figure 6.28 The graphical representation of two functions: 1) $E_g = f(I_{fl})$, which is the OCC and 2) $E_g = f(I_{sc})$ in accordance with equation 6.43).

ABC: vertex C is determined by the field current required for a rated s.c. current, i.e., $\overline{OC} = 1/SCR$ and vertex A (the triangle altitude) is determined by E_l.

We then plot the reactance axis as a continuation of the triangle leg AB (Fig. 6.28). The length $\overline{AB}$, which represents the leakage voltage E_l, is proportional to X_l ($E_l = X_l I_{sc}$) and therefore it determines the scale of reactance:

$$m_x = \frac{X_l}{\overline{AB}}, \text{ pu/cm.}$$

Then, length $\overline{AF}$ on the reactance axis will give X_F in the same scale, while the lengths $\overline{OA'}$ and $\overline{A'M'}$ on the voltage axis will give the leakage voltage E_l and the terminal voltage V respectively.

So far, in the above solution, the generator, previously to short-circuiting, was running under no-load conditions. Usually, short-circuits do not occur under no-load, but under the full operation of the power plant. Thus, the generators will carry a considerable current prior to the occurrence of a short-circuit, and in order to compensate for the armature reaction of the load current, the generator should be excited by a substantially higher field current than by the no-load field current in Fig. 6.28, which is often by a multiple of this value. If the field current under full load is known, we start the solution by indicating point C_1 according to the value of this current. Then we move the Potier triangle with the reactance axis, toward point C_1 so that its vertex C coincides with point C_1, as shown in Fig. 6.29.

Now, as in the previous case, we shall determine point F_1, in accordance with the value of X_F and plot the line $C_1 M_1$ through F_1. The projection of M_1 on the I_{sc} axis, point N_1, gives the value of the steady-state short-circuit current, $I_{sc,\infty}$. To find the remaining terminal voltage V we must extend the hypotenuse

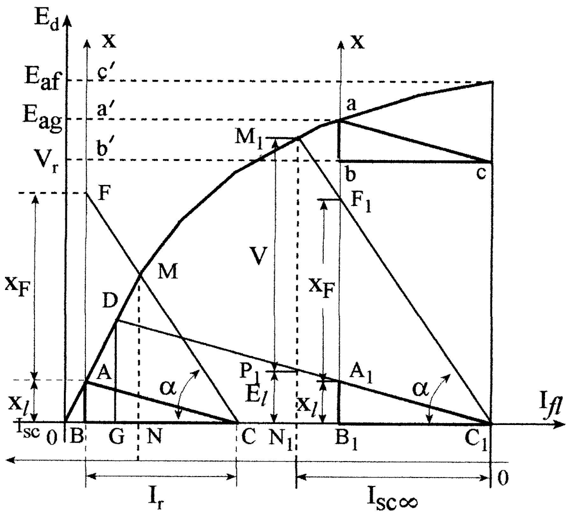

Figure 6.29 The graphical solution for finding the steady-state value of a short-circuit current.

A_1C_1 of the Potier triangle up to the OCC, point D. Then length $\overline{M_1P_1}$ will give the p.u. value of the generator terminal voltage. It is obvious that the above procedure can be performed for any given field (magnetizing) current and for any external reactance. However, if the field current is not known, we may determine it as follows. The leakage voltage E_l as length $\overline{ab} = \overline{AB}$ should be added to the rated voltage V_r level (dashed line $b'b$) so that point A reaches the OCC, as shown in Fig. 6.29. Then the length $\overline{bc} = 1/SCR$ should be plotted on the same line $b'b$ and the obtained triangle $\Delta\, abc$ is the Potier triangle. Point c, projected on the abscissa, as point C_1, will determine the required field current. On the OCC in Fig. 6.29: E_{ag} is the air-gap EMF and E_{af} is the total EMF generated by the field current I_{fl}. Let us now introduce the graphical solution in the following example.

Example 6.4

The synchronous generator, prior to short-circuiting, is operated under full load. Use the parameters and OCC of Example 6.3 to find the s.c. current and the terminal voltage of the generator if the fault is placed at the external reactances 1) 0.3 and 2) 0.9.

Solution

In Fig. 6.30 the OCC and the Potier triangle ABC from Example 6.3 are given. First we move the Potier triangle into the position of $\Delta\, abc$, so that $ab = AB$.

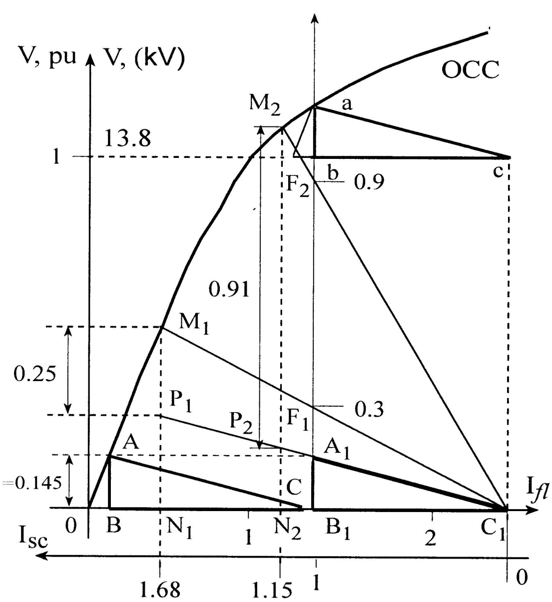

Figure 6.30 The graphical solution of Example 6.4.

Then, the field current of the generator under full load is given by point C_1, which is the projection of c on the abscissa, and the new position of the Potier triangle is $\Delta\, A_1B_1C_1$.

1) On the X-axis, which is an extension of A_1B_1, we determine point F_1 in accordance with the value of the external reactance 0.3. The intersection of the straight line, drawn from C_1 through F_1 to OCC, gives point M_1, which is the graphical solution of our problem. The length $\overline{C_1N_1}$, measured as 1.68, is the p.u. value of the s.c. current. Thus,

$$I_{sc,\infty} = 1.68I_\mathrm{r} = 1.65 \cdot 5580 \cong 9.4\ \mathrm{kA}.$$

The length $\overline{M_1P_1}$, measured as 0.25, is the p.u. value of the terminal voltage:

$$V = 0.25 \cdot 13.8 = 3.45\ \mathrm{kV}.$$

2) *On the* X-axis we determine point F_2 in accordance with the second value 0.9. Then the intersection point M_2 gives the solution of the s.c. current (length $\overline{CN_2}$):

$$I_{sc,\infty} = 1.15I_\mathrm{r} = 1.15 \cdot 5580 \cong 6.4\ \mathrm{kA},$$

and of the terminal voltage (length $\overline{M_2P_2}$):

$$V = 0.91V_\mathrm{r} = 0.91 \cdot 13.8 \cong 12.6\ \mathrm{kV}.$$

Note that, in the first case, the intersection point M_1 lies on the straight part of the OCC. Therefore, the s.c. current can be found with the unsaturated reactance. Indeed, the total reactance up to the fault is

$$X_{tot} = (X_{du} + X_F)X_\mathrm{r} = (1.67 + 0.3) \cdot 1.43 = 2.82\ \Omega,$$

and

$$I_{sc,\infty} = 13.8/2.82 = 4.89\ \mathrm{kA}.$$

Since the generator prior to fault was under full load operation, its field current was about twice as large as under no-load (see the diagram in Fig. 6.30), the actual s.c. should be

$$I_{sc,\infty} = 4.89 \cdot 2.00 \cong 9.8\ \mathrm{kA},$$

which is pretty close to the s.c estimated graphically.

It should also be noted that the actual PF of the generator load prior to the short-circuiting has not been taken into consideration, i.e. the armature reaction is considered as a pure reactive. However this approximation does not significantly change the final results.

In the above solution the field current has been kept constant, regardless of the distance to the fault (i.e. the value of the external reactance) and the level of the terminal voltage. However, as has already been mentioned, nowadays synchronous generators are equipped with an automatic voltage regulation system (AVR), which endeavors to hold the terminal voltage constant by changing the field current. Thus, if the short-circuit occurs far away from the power station, i.e. the external reactance is large enough so that the decrease in the terminal voltage will be unsubstantial, then the response of the AVR in increasing the field current will be low. On the other hand, if the short-circuit fault occurs close to the generator terminals, the drop in its voltage will be significant and the AVR response in increasing the field current will be very strong. It is also possible that the field current will reach its maximal value, but despite that the terminal voltage will remain lower than its normal level. Hence, we shall distinguish between two possible regimes:

a) the *maximal field current regime*, in which $I_{fl} = I_{fl,\max}$ and $V_\infty \leq V_r$; and

b) the *rated* (*nominal*) *voltage regime* $V_\infty = V_r$ and $I_{fl} \leq I_{fl,\max}$.

In order to determine in which of the two regimes the generator is operating, and to perform the graphical solution in these cases, let us consider the diagram shown in Fig. 6.31.

In this diagram $\overline{OC_m}$ represents the maximal field current $I_{fl,\max}$ and as previously $B_m a$ is a reactance axis. We shall now find the maximal value of X_F, in which the voltage drop, in the case of the maximal field current, will be equal to the rated voltage, $V_r = 1$. For this purpose we must plot a line from point R, which is positioned on the voltage axis at $V_r = 1$, parallel to the hypotenuse of the Potier triangle up to the intersection point K on the OCC. By connecting K with the origin C_m we obtain point k, on the x-axis, so that the length $\overline{A_m k}$ gives the desired reactance $X_{F,cr}$, which is called the *critical reactance*. Indeed, from the plotted diagram, it can be seen that the air-gap EMF, E_{ag}, at any

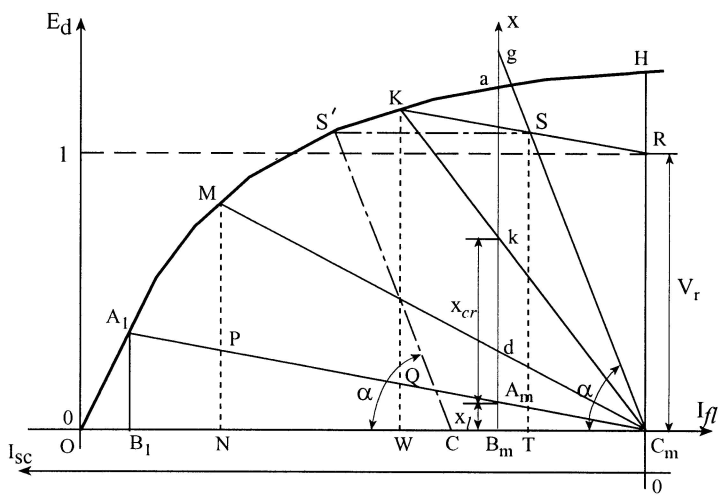

Figure 6.31 The graphical solution when the field current is not constant.

point on the OCC, which is lower than point K, less the leakage voltage E_l (at this point length $\overline{QW}$), will be smaller than unity (lower than at point R). At point K we have

$$E_{ag,k} - E_{l,Q} = \overline{QK} = 1.$$

For typical synchronous generators the critical fault reactance can be approximated as $X_{F,cr} \cong 0.5$ and therefore, $I_{\infty,cr} \cong 2$. As soon as $X_{F,cr}$ is found we may conclude that:

a) if $X_F \leq X_{F,cr}$ the regime of the maximal field current takes place,

b) if $X_F \geq X_{F,cr}$ the regime of the rated voltage takes place.

It is obvious that if $X_F = X_{F,cr}$, both regimes take place at the same time.

In the first case the graphical solution is held in the same way, which has been previously explained for the constant field current taking the maximal field current as a constant. The second case requires some additional discussion. Since the terminal voltage in this regime is the straight line KR, which represents this voltage as a function of the field current, it can be treated as an extension of the OCC (instead of the curve KH). Then, the s.c. current will be determined by point S on the intersection of $C_m g$ (g is given by X_F) and KR. The projection of RS on the abscissa, i.e., I_{sc}-axis, length $\overline{C_m T}$, gives the p.u. value of the steady-state s.c. current. The field current in this regime is smaller than $I_{fl,\max}$. To find its value we have to project point S on the OCC as point S^1 and to plot line CS^1 in parallel to $C_m S$. Then length $\overline{OC}$ will determine the actual field current.

Example 6.5

The generator of the previous example is equipped with the AVR, which ensures increasing the field current under the fault conditions up to $I_{fi,\max} = 4$ pu. Find $I_{sc,\infty}$ and the generator terminal voltage V_g, if the short-circuit fault occurred at 1) $X_F = 0.3$ and 2) $X_F = 0.9$. Determine the kind of regime: $I_{fl,\max}$ or $V_{g,\mathrm{r}}$, for both cases.

Solution

In Fig. 6.32 the OCC of the generator and the Potier triangle are redrawn. Since the maximal field current at the full operation of AVR is $I_{fl,\max} = 4$ pu, the Potier triangle is moved to position $A_m B_m C_m$. Next we plot lines KR (point R is at the rated voltage V_r) and $C_m K$. The intersection of line $C_m K$ with the x-axis at point k gives the critical reactance X_{cr}, which is 0.68 pu. Thus the fault critical reactance is

$$X_{F,cr} = X_{cr} - X_l = 0.78 - 0.145 = 0.64.$$

1) Hence, at the fault of $X_F = 0.3$, which is less than critical, the generator operates under the regime of the first kind, i.e., the maximal field current. To find the s.c. in this case we determine the total reactance $X_{tot} = 0.145 + 0.3 \cong 0.45$

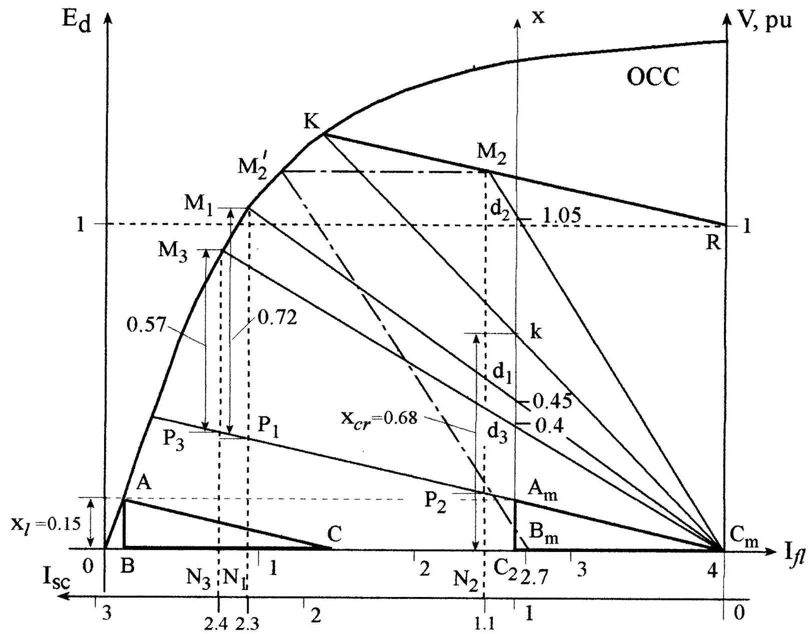

Figure 6.32 The graphical solution of Example 6.5.

on the x-axis at the point d_1 and plot line C_mM_1 through this point. The projection of point M_1 on the abscissa, i.e., point N_1, indicates the s.c. of the generator

$$I_{sc,\infty} = \overline{C_mN_1} \cdot I_r = 2.3 \cdot 5.58 = 12.8 \text{ kA},$$

and length $\overline{M_1P_1}$ gives the terminal voltage

$$V_\infty = \overline{M_1P_1} \cdot V_r = 0.72 \cdot 13.8 = 9.9 \text{ kV}.$$

2) The total fault reactance in this case is $X_{tot} = 0.145 + 0.9 = 1.05$, i.e., larger than the critical reactance and therefore the generator operates under the second kind of regime, in which the terminal voltage is of the rated value. The solution will be given by line C_mM_2 plotted through point d_2 on the x-axis at the value of 1.05. The s.c. current is determined by N_2, which is the projection of M_2. Thus,

$$I_{sc,\infty} = \overline{C_mN_2} \cdot I_r = 1.1 \cdot 5.58 = 6.2 \text{ kA},$$

or, since the terminal voltage is unity,

$$I_{sc,\infty} = \frac{1}{0.9} 5.58 = 6.2 \text{ kA},$$

and the terminal voltage

$$V_\infty = V_r = 13.8 \text{ kV}.$$

To find the field current at this regime we have to plot line M_2M_2' in parallel to the abscissa and line $M_2'C_2$ in parallel to M_2C_m. Point C_2 will indicate the field current, which, therefore, is $I_{fl} = 2.7$ pu and less than the maximal.

(c) *Influence of the load*

The load of power systems, especially induction motors that compose 50–70% of the entire load, largely influence the transient behavior of the synchronous generators under short-circuit faults. Generally speaking, any load connected to the same node as the short-circuit line, Fig. 6.33, changes the current values and their flow in the affected network. Thus, by simplifying the network to get an equivalent circuit, we simply connect the load branch in parallel to the short-circuited branch, as shown in Fig. 6.33(b). This results in lowering the total fault reactance and consequently in decreasing the generators' voltages, which in turn results in decreasing the s.c. currents and changes their distribution in the whole network. Hence, the load connections must be taken into consideration by the short-circuit fault analysis. On the other hand the exact consideration of the load presents a lot of difficulties. The most typical kinds of loads: lightning, heating and mechanical operating (primarily induction motors), are not constant, but vary as a function of the voltage power ($V^{1.6}$ in the case of a lightning load and V^2 in the case of heating and induction motors). Furthermore, induction motors stop operating (their rotor speed reduces to zero), when the voltage is decreased 70%; and the motor turns into a short-circuited branch (this situation is very dangerous for induction motors and they would be disconnected by means of the protection relays).

Generally speaking, all kinds of loads also depend on frequency. However, information regarding the characteristics of composite loads with frequency is scarce. With the small frequency changes during most of the short-circuit faults, this effect is neglected in calculations.

A detailed analysis of the different ways of load considerations (which is not given here as it is beyond the scope of this book) shows that for the purpose

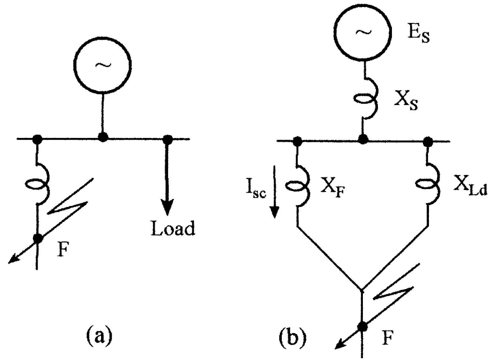

Figure 6.33 A simple network for illustrating the influence of the load on the short-circuit analysis.

of s.c. current calculations a good approximation can be achieved by considering the composite loads as a constant reactance of the 1.2 pu value.

Example 6.6

Suppose that the load of the generator of Example 6.5 is connected to its terminals as shown in Fig. 6.33. Find the generator's s.c. current if the value of the load is 85% of the generator rated power and the fault occurs at the p.u. reactance of 0.3 (the generator is equipped with AVR).

Solution

In accordance with the above recommendation, we shall represent the load as a 1.2 reactance. Hence, the reactance referred to the generator power is

$$X_{ld} = 1.2\frac{1}{0.85} = 1.41.$$

The equivalent fault reactance in this case, Fig. 6.33(b), will be

$$X_{eq} = 0.3//1.41 = 0.25,$$

and

$$X_{tot} = 0.25 + 0.145 \cong 0.4.$$

Since this reactance is less than critical, the regime of the generator is of maximal field current. Determining the above value on the x-axis, in Fig. 6.32, point d_3, and plotting line $C_m M_3$ through this point, we obtain the solution at point N_3. Thus

$$I_{sc,\infty} = \overline{C_m N_3} \cdot I_{\mathrm{r}} = 2.4 \cdot 5.58 = 13.4\ \mathrm{kA},$$

and

$$V_{\infty} = \overline{M_3 P_3} \cdot V_{\mathrm{r}} = 0.57 \cdot 13.8 \cong 7.87\ \mathrm{kV}.$$

As expected, consideration of the load results in increasing the s.c. current of the generator and in decreasing its terminal voltage (compare with the results of Example 6.5 for $X_F = 0.3$). Note that decreasing the terminal voltage results in decreasing the short-circuit current in the fault branch.

(d) *Approximate solution by linearization of the OCC*

A disadvantage of the graphical method is that its accuracy depends on the scale of the draft and experience of the performer of the graphical calculations. From this standpoint analytical methods are always preferable. However, to perform an analytical approximation of the short-circuit fault of a synchronous generator taking into consideration the saturation of its magnetic circuit, we need to know the analytical approximation of its OCC. The simplest one is a linearization of a given curve with a single straight line. It is obvious, however, that replacing the whole curve of the OCC with one straight line will give a very bad approximation. So, usually, only a specific part of the curve, which is considered as a working part, is replaced by a straight line. For generators having AVR (nowadays most synchronous generators are equipped with a voltage regulation system) the working part of the OCC, in accordance with Fig. 6.34, is A_mK' (note that the continuation of the generator characteristic in this case is also a straight line KR). This part of the OCC may be approximated by the straight line A_mN, which for a typical OCC is expressed as

$$E_g = 0.20 + 0.8I_{fl}. \tag{6.45}$$

(For different OCCs the numerical parameters in this expression may be different.)

The synchronous reactance of the generator, which is represented by a linear OCC, can also be estimated as a constant quantity. We shall obtain this value by considering a short-circuit fault at generator terminals. We then have

$$X_s = \frac{E_g}{I_{sc,\infty}} = \frac{E_g}{SCR\ I_{fl}}, \tag{6.46}$$

where E_g is in accordance with equation 6.45.

Since the position of point K' is different from those of point K on an actual OCC, we shall find a new value of the critical reactance $X'_{F,cr}$ for the linear characteristic. At point K' the terminal voltage is unity, the field current is still

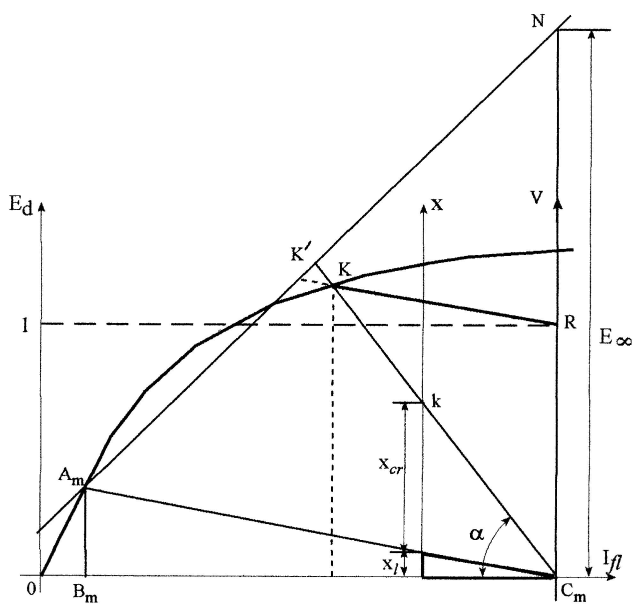

Figure 6.34 The linear approximation of a typical OCC.

maximal and therefore the induced EMF is also maximal. Hence

$$V_\infty = E_{g,\max} - X_s I_{\infty,cr} = 1,$$

which gives the critical short-circuit current

$$I_{\infty,cr} = \frac{E_{g,\max} - 1}{X_s}, \tag{6.47}$$

where

$$E_{g,\max} = 0.20 + 0.8 I_{fl,\max} \quad \text{and} \quad X_s = \frac{E_{g,\max}}{SCR\ I_{fl,\max}}.$$

With equation 6.44 the *critical reactance* is

$$X_{F,cr} = \frac{1}{I_{\infty,cr}}. \tag{6.48}$$

If the generator is operating under maximal field current, i.e., $X_F \le X_{F,cr}$, then

$$I_\infty = \frac{E_{g,\max}}{X_s + X_F} \ge I_{\infty,cr}, \tag{6.49a}$$

and

$$V = X_F I_\infty \le 1. \tag{6.49b}$$

If $X_F \ge X_{F,cr}$, which means that the generator operates under rated voltage, then

$$I_\infty = \frac{1}{X_F} \le I_{\infty,cr} \quad \text{and} \quad V_\infty = 1. \tag{6.50}$$

Example 6.7

For the generator of Example 6.5 find the s.c. current using the linearization method.

Solution

First we shall estimate the critical reactance. With equation 6.45 through equation 6.48 we have (in p.u.)

$$E_{g,\max} = 0.2 + 0.8\cdot 4 = 3.4, \quad X_s = \frac{3.4}{0.67\cdot 4} = 1.27,$$

and

$$I_{\infty,cr} = \frac{3.4 - 1}{1.27} = 1.89, \quad X_{F,cr} = \frac{1}{1.89} = 0.53.$$

1) Since the fault reactance in the first case, $X_{F1} = 0.3$ pu, is less than the critical

reactance, by using equation 6.49, we have

$$I_\infty = \frac{3.4}{1.27 + 0.3} = 2.16 \quad \text{and} \quad V_\infty = 0.3 \cdot 2.16 = 0.648$$

or in natural units

$$I_\infty = 2.16 \cdot 5.58 = 12.1\,\mathrm{kA} \quad \text{and} \quad V_\infty = 0.648 \cdot 13.8 = 8.94\,\mathrm{kV}.$$

The generator in this case is operated under maximal field current.

2) Since the fault reactance in the second case $X_{F2} = 0.9$ pu, which is greater than critical, we use equation 6.50. Thus,

$$I_\infty = \frac{1}{0.9} = 1.11 \quad \text{and} \quad V_\infty = 1,$$

or in natural units

$$I_\infty = 1.11 \cdot 5.58 = 6.2\,\mathrm{kA} \quad \text{and} \quad V_\infty = 1 \cdot 13.8 = 13.8\,\mathrm{kV}.$$

The generator in this case is operated under the nominal terminal voltage with less than maximal field current. The latter one may be estimated as

$$I_{fl} = \frac{I_\infty}{SCR} = \frac{1.11}{0.67} = 1.66.$$

Comparing the obtained results, for both cases of operation, with those of Example 6.5, we may conclude that the difference between them is less than 10% (note that the accuracy of all the engineering calculations is between 5–10%).

(e) *Calculation of steady-state short-circuit currents in complicated power networks*

As has been previously mentioned, most of the synchronous generators in a modern power system are equipped with AVR and, therefore, may operate under a short-circuit fault in one of two regimes: 1) maximal field current or 2) rated, i.e. normal terminal voltage. Depending on the kind of regime, each of the generators has to be represented by a different equivalent circuit: 1) in the first regime – with the OCC and X_l (using the graphical method) or with $E_{g,\max}$ and $X_{s,\max}$ (using the linearization method) and 2) in the second regime – as an ideal voltage source, i.e. with $E_g = 1$ and $X_s = 0$ (in both the graphical and linearization methods).

The determination of the kind of regime is made by comparing the actual short-circuit current of each of the generators with its critical value, $I_{\infty,cr}$, or the external reactance up to the fault with its critical value, $X_{F,cr}$. However, the s.c. currents of each of the generators are the goal of our solution and are not known at the first stage of the analysis, i.e. determining the equivalent circuit. To overcome this difficulty the iteration method, or method of successive approximations, may be applied. In accordance with this method, in the first calculation, as a starting point, the generators are represented by one of the two regimes, i.e. just by inspection of their location relative to the fault point. Those generators which are relatively "close" to the fault (by means of the estimated value of the reactance from the generators up to the fault) should be represented by an equivalent circuit as they operate under the regime of maximal field current, and those which are relatively "far" from the fault – as they operate under the regime of a normal voltage. Then, the results of this calculation, i.e., the first iterate, shall be compared with the critical ones, and the generator representation, which has been incorrectly chosen, should be changed. The calculation will be repeated and the results, i.e. the second iterate, shall be checked again and so on. In the final iterate all the generators will be represented in accordance with their actual behavior.

A straightforward method of s.c. fault analysis can also be applied to a complicated network, by means of a computer-aided calculation. With the superposition principle we may represent the s.c. current as a sum of the partial (or individual) currents caused by each generator acting alone:

$$I_{sc,\infty} = \sum_i I_{sc,i} = \sum_i B_{F,i} E_i, \tag{6.51}$$

where $B_{F,i}$ are the transfer susceptances (reciprocal of reactances) between a fault branch and each of the generator branches. These susceptances can be found by means of matrix analysis:

$$B_{F,i} = \frac{\Delta_{F,i}}{\Delta},$$

where Δ is the determinant of the network reactances' matrix, written in accordance with mesh analysis, and $\Delta_{F,i}$ is its appropriate cofactor. With these results an *equivalent circuit*, in which every generator is individually connected to the fault point, as shown in Fig. 6.35, may be obtained. Here, each generator is connected to the fault with the reactance $X_{F,i} = 1/B_{F,i}$. Now each of the reactances and/or currents (equation 6.51) can be compared with the critical reactance and/or critical currents and the correct representation of each generator

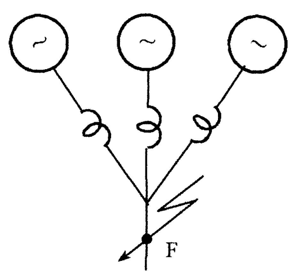

Figure 6.35 An equivalent circuit of a complicated network obtained by using the superposition principle.

will be chosen. An example of a short-circuit fault calculation in a complicated power network is given in Appendix III.

6.5.3 Transient performance of a synchronous generator

Mutually coupled stator and rotor windings of electrical machines, in distinction to the transformers, are in motion with respect to each other. The d.c. winding of the rotor of the synchronous generator moves with respect to the a.c. three-phase stator winding so that the mutual inductances between these two windings, and even between different phases of the stator winding, change with time. This leads to differential equations with variable coefficients, which results in a very cumbersome analysis and difficult understanding of the whole transient process. However, we may describe and analyze the transient behavior of the synchronous generator using an artifice. We shall use the generalized current phasor for the stator windings and the two-axis representation of a synchronous machine (see section 6.5.1), which reduce the three-phase system to a single one. Furthermore, we will make a couple of common assumptions, which allow us to not only simplify the analysis, but also to obtain results that are still close to the actual ones.

Firstly, we assume that the rotor angular speed ω stays constant during the whole transient process. For the machine with a damper winding in the rotor poles, we assume that the influence of the damper currents on the transient process can be obtained by superimposing the appropriate calculations on the transient results obtained first for the generator without damper windings. In the first stages of the transients we shall also neglect the winding resistances, being very small compared to their reactances (less than 10% for the stator winding, even when including the external network, and about 1% for the rotor winding). The influence of the resistances on the entire process will be taken into consideration as the cause of decaying all the natural responses. Transient analysis of synchronous generators will be given for a salient pole generator, as a more general case. For the round rotor generator the results may then be obtained by equaling the reactances on both axes.

(a) *Transient EMF, transient reactance and time constant*

It should be noted that the transient equivalent circuit of a generator differs from those representing it in the steady-state regime. This is shown in Fig. 6.36(a) and its simplification in Fig. 6.36(b). Here E_d is the EMF, induced by the magnetizing or field current, and

$$X_d = X_{ad} + X_l \tag{6.52}$$

is the *synchronous reactance* of a generator (sometimes designated as X_s). The value of E_d is obtained by OCC in accordance with I_{fl} (or I_μ).

However, since by the sudden interruption of a synchronous machine, the total magnetic flux has to be considered and kept constant, the rotor leakage

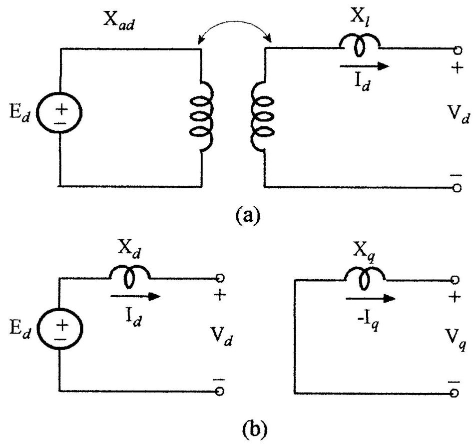

Figure 6.36 An equivalent circuit of a synchronous generator for steady-state operation (a) and its simplification in two axes (b).

reactance must be taken into account and added to the equivalent circuit (just like the leakage reactance of a transformer primary).

The equivalent circuit of the synchronous generator for the transient behavior is shown in Fig. 6.37(a). In this circuit the rotor and stator windings on the direct axes including the rotor winding leakage reactance are shown. Since the rotor and the stator magnetic fields are rotated with the same speed (as we have previously assumed), i.e., they remain fixed with respect to one another; we may treat this circuit as a transformer. Here, as in the previous circuit, X_{ad} is an *armature reaction reactance* (sometimes it is called the magnetizing reactance), and X_{rl} and X_{sl} are the *leakage reactances* of the rotor and the stator respectively. Note that in p.u. notation the magnetizing reactance expresses the relation between the rotor and stator currents:

$$I_d = \frac{I_{fl}}{X_{ad}}. \tag{6.53}$$

This circuit can be transformed into those shown in Fig. 6.37(b), in which the mutual inductance is illuminated. (Note that the p.u. values of the inductances and their corresponding reactances are equal and, therefore, the reactances can be used in place of inductances and vice versa.) We may now apply the Thévenin theorem to get the circuit in (c) and finally a very simple circuit including a voltage source and a single reactance, as shown in (d):

$$E'_d = E_{Th} = E_{fl}\frac{X_{ad}}{X_{rl} + X_{ad}}, \tag{6.54a}$$

and

$$X'_d = X_{Th} + X_{sl} = \frac{X_{rl}X_{ad}}{X_{rl} + X_{ad}} + X_l, \tag{6.54b}$$

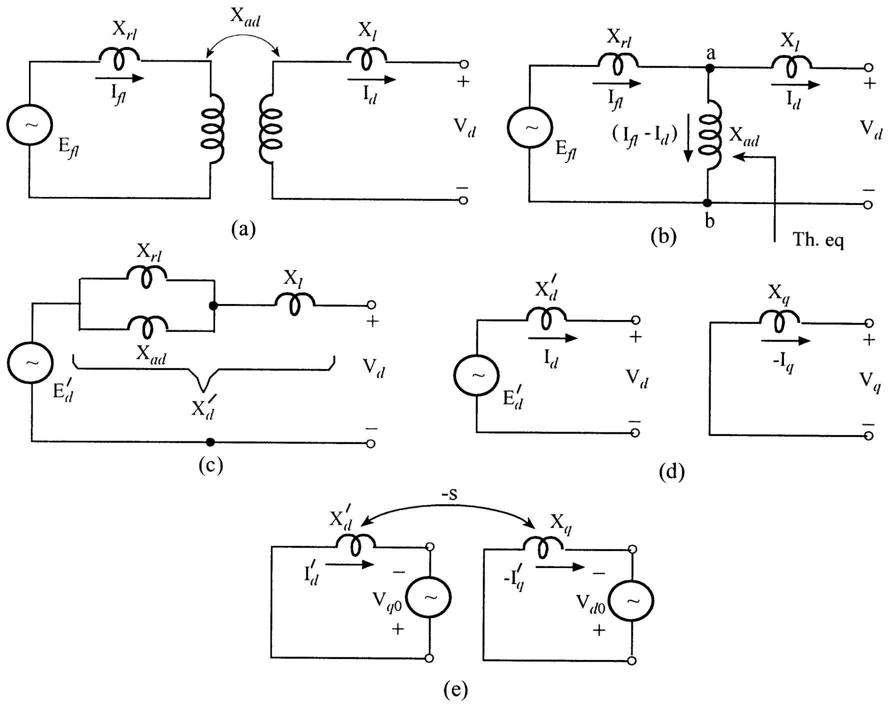

Figure 6.37 The equivalent circuit of the synchronous generator for transient behavior: in the *d*-axis as for a two-winding transformer (a), its simplification (b), after applying the Thévenin theorem (c), its Laplace transform equivalent (d) and the interconnection between the equivalent circuits in the d- and q-axes (e).

where E'_d and X'_d are the *transient EMF* (generated voltage) and the *transient reactance.*

In some technical books the transient reactance X'_d is given in the form

$$X'_d = X_d - \frac{X_{ad}^2}{X_{rl} + X_{ad}} = X_d - (1 - \sigma_{fd})X_{ad},$$

where $\sigma_{fd} = X_{rl}/(X_{rl} + X_{ad})$ is the leakage coefficient of the rotor winding. Then

$$X'_d = X_d - X_{ad} + \sigma_{fd}X_{ad} = X_l + \frac{X_{rl}X_{ad}}{X_{rl} + X_{ad}},$$

which is as was previously obtained.

Recall that similar results were obtained for power transformers (see section 6.4.1), i.e. the entire magnetic circuit of a transformer can be represented by only a single reactance, which incorporates all the magnetic fluxes of both windings in a total flux. In accordance with the principle of a constant flux linkage, this total flux must be kept constant at the instant of switching. Hence, the equivalent circuit in Fig. 6.37(d) represents the synchronous generator at