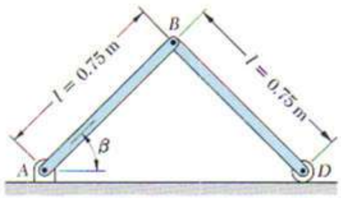
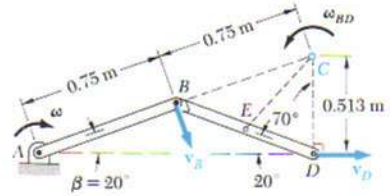
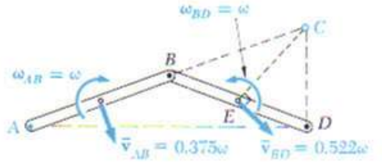


SAMPLE PROBLEM 17.5

Each of the two slender rods shown is 0.75 m long and has a mass of 6 kg. If the system is released from rest when $\beta = 60°$, determine (*a*) the angular velocity of rod AB when $\beta = 20°$, (*b*) the velocity of point D at the same instant.





Kinematics of Motion When $\beta = 20°$. Since $\mathbf{v}_B$ is perpendicular to the rod AB and $\mathbf{v}_D$ is horizontal, the instantaneous center of rotation of rod BD is located at C. Considering the geometry of the figure, we obtain

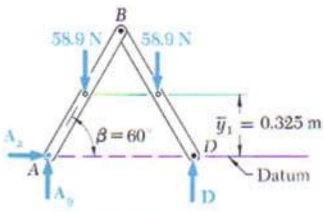
$$BC = 0.75 \text{ m} \qquad CD = 2(0.75 \text{ m}) \sin 20° = 0.513 \text{ m}$$

Applying the law of cosines to triangle CDE, where E is located at the mass center of rod BD, we find $EC = 0.522$ m. Denoting by ω the angular velocity of rod AB, we have

$$\bar{v}_{AB} = (0.375 \text{ m})\omega \qquad \bar{\mathbf{v}}_{AB} = 0.375\omega \searrow$$
$$v_B = (0.75 \text{ m})\omega \qquad \mathbf{v}_B = 0.75\omega \searrow$$

Since rod BD seems to rotate about point C, we may write

$$v_B = (BC)\omega_{BD} \qquad (0.75 \text{ m})\omega = (0.75 \text{ m})\omega_{BD} \qquad \omega_{BD} = \omega \circlearrowleft$$
$$\bar{v}_{BD} = (EC)\omega_{BD} = (0.522 \text{ m})\omega \qquad \bar{\mathbf{v}}_{BD} = 0.522\omega \searrow$$



Position 1

Position 1. ***Potential Energy.*** Choosing the datum as shown, and observing that $W = (6 \text{ kg})(9.81 \text{ m/s}^2) = 58.9$ N, we have

$$V_1 = 2W\bar{y}_1 = 2(58.9 \text{ N})(0.325 \text{ m}) = 38.3 \text{ J}$$

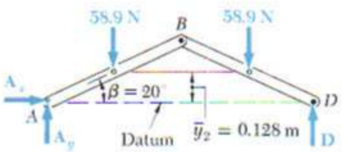
Kinetic Energy. Since the system is at rest, $T_1 = 0$.

Position 2. ***Potential Energy***

$$V_2 = 2W\bar{y}_2 = 2(58.9 \text{ N})(0.128 \text{ m}) = 15.1 \text{ J}$$

Kinetic Energy

$$\bar{I}_{AB} = \bar{I}_{BD} = \tfrac{1}{12}ml^2 = \tfrac{1}{12}(6 \text{ kg})(0.75 \text{ m})^2 = 0.281 \text{ kg} \cdot \text{m}^2$$
$$\begin{aligned} T_2 &= \tfrac{1}{2}m\bar{v}_{AB}^2 + \tfrac{1}{2}\bar{I}_{AB}\omega_{AB}^2 + \tfrac{1}{2}m\bar{v}_{BD}^2 + \tfrac{1}{2}\bar{I}_{BD}\omega_{BD}^2 \\ &= \tfrac{1}{2}(6)(0.375\omega)^2 + \tfrac{1}{2}(0.281)\omega^2 + \tfrac{1}{2}(6)(0.522\omega)^2 + \tfrac{1}{2}(0.281)\omega^2 \\ &= 1.520\omega^2 \end{aligned}$$



Position 2

Conservation of Energy

$$T_1 + V_1 = T_2 + V_2: \qquad 0 + 38.3 \text{ J} = 1.520\omega^2 + 15.1 \text{ J}$$
$$\omega = 3.91 \text{ rad/s} \qquad \omega_{AB} = 3.91 \text{ rad/s} \circlearrowright \blacktriangleleft$$

Velocity of Point D

$$v_D = (CD)\omega = (0.513 \text{ m})(3.91 \text{ rad/s}) = 2.01 \text{ m/s}$$
$$\mathbf{v}_D = 2.01 \text{ m/s} \rightarrow \blacktriangleleft$$

PROBLEMS

17.1 The rotor of a generator has an angular velocity of 3600 rpm when the generator is taken off line. The 150-kg rotor, which has a centroidal radius of gyration of 250 mm, then coasts to rest. Knowing that the kinetic friction of the rotor produces a couple of magnitude $2 \text{ N} \cdot \text{m}$, determine the number of revolutions that the rotor executes before coming to rest.

17.2 A large flywheel of mass 1800 kg has a radius of gyration of 0.75 m. It is observed that 2500 revolutions are required for the flywheel to coast from an angular velocity of 450 rpm to rest. Determine the average magnitude of the couple due to kinetic friction in the bearings.

17.3 Two disks of the same material are attached to a shaft as shown. Disk A is of radius r and has a thickness $3b$, while disk B is of radius nr and thickness b. A couple $\mathbf{M}$ of constant magnitude is applied when the system is at rest and is removed after the system has executed two revolutions. Determine the value of n which results in the largest final speed for a point on the rim of disk B.

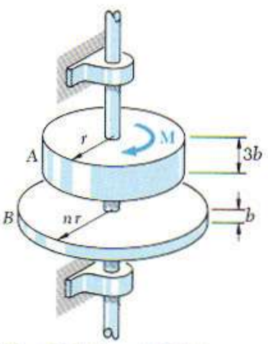


Fig. P17.3 and P17.4

17.4 Two disks of the same material are attached to a shaft as shown. Disk A weighs 30 lb and has a radius $r = 5$ in. Disk B is one-third as thick as disk A. A couple $\mathbf{M}$ of magnitude $10 \text{ lb} \cdot \text{ft}$ is applied to disk A when the system is at rest. Determine the radius nr of disk B if the angular velocity of the system is to be 450 rpm after 5 revolutions.

17.5 The flywheel of a small punch rotates at 240 rpm. It is known that $1500 \text{ ft} \cdot \text{lb}$ of work must be done each time a hole is punched. It is desired that the speed of the flywheel after one punching be not less than 90 percent of the original speed of 240 rpm. (*a*) Determine the required moment of inertia of the flywheel. (*b*) If a constant $20\text{-lb} \cdot \text{ft}$ couple is applied to the shaft of the flywheel, determine the number of revolutions which must occur between each punching, knowing that the initial velocity is to be 240 rpm at the start of each punching.

17.6 The flywheel of a punching machine weighs 600 lb and has a radius of gyration of 24 in. Each punching operation requires $1500 \text{ ft} \cdot \text{lb}$ of work. (*a*) Knowing that the speed of the flywheel is 300 rpm just before a punching, determine the speed immediately after the punching. (*b*) If a constant $15\text{-lb} \cdot \text{ft}$ couple is applied to the shaft of the flywheel, determine the number of revolutions executed before the speed is again 300 rpm.

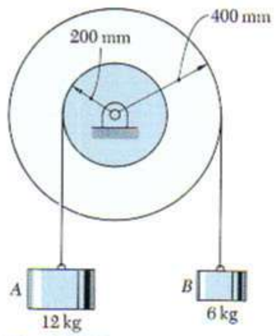


Fig. P17.7

17.7 Two cylinders are attached by cords to an 18-kg double pulley which has a radius of gyration of 300 mm. When the system is at rest and in equilibrium, a 3-kg collar is added to the 12-kg cylinder. Neglecting friction, determine the velocity of each cylinder after the pulley has completed one revolution.

17.8 Solve Prob. 17.7, assuming that the 3-kg collar is added to the 6-kg cylinder.

17.9 Using the principle of work and energy, solve Prob. 16.36*b*.

17.10 Using the principle of work and energy, solve Prob. 16.34*c*.

17.11 A disk of constant thickness and initially at rest is placed in contact with the belt, which moves with a constant velocity **v**. Denoting by μ the coefficient of friction between the disk and the belt, derive an expression for the number of revolutions executed by the disk before it reaches a constant angular velocity.

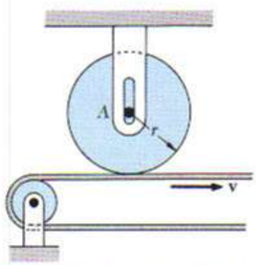


Fig. P17.11 and P17.12

17.12 Disk *A*, of weight 5 lb and radius $r = 3$ in., is at rest when it is placed in contact with the belt, which moves with a constant speed $v = 50$ ft/s. Knowing that $\mu = 0.20$ between the disk and the belt, determine the number of revolutions executed by the disk before it reaches a constant angular velocity.

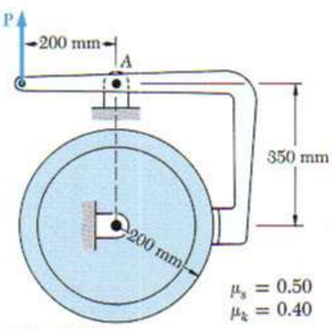


Fig. P17.13

17.13 The 200-mm-radius brake drum is attached to a larger flywheel which is not shown. The total mass moment of inertia of the flywheel and drum is 8 kg·m^2. Knowing that the initial angular velocity is 120 rpm clockwise, determine the force **P** which must be applied if the system is to come to rest in 8 revolutions.

17.14 Solve Prob. 17.13, assuming that the initial angular velocity of the flywheel is 120 rpm counterclockwise.

17.15 Each of the gears A and B has a mass of 2 kg and a radius of gyration of 70 mm, while gear C has a mass of 10 kg and a radius of gyration of 175 mm. A couple **M** of constant magnitude 12 N · m is applied to gear C. Determine (a) the number of revolutions required for the angular velocity of gear C to increase from 100 to 450 rpm, (b) the corresponding tangential force acting on gear A.

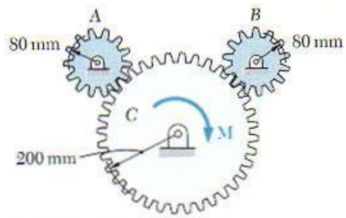


Fig. P17.15

17.16 Solve Prob. 17.15, assuming that the 12-N·m couple is applied to gear B.

17.17 A cord is wrapped around a cylinder of radius r and mass m as shown. If the cylinder is released from rest, determine the velocity of the center of the cylinder after it has moved through a distance h.

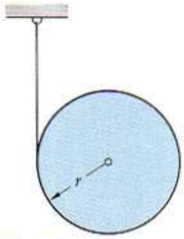


Fig. P17.17

17.18 Two 10-kg disks, each of radius $r = 0.3$ m, are connected by a cord. At the instant shown, the angular velocity of disk B is 20 rad/s clockwise. Determine how far disk A will rise before the angular velocity of disk B is reduced to 4 rad/s.

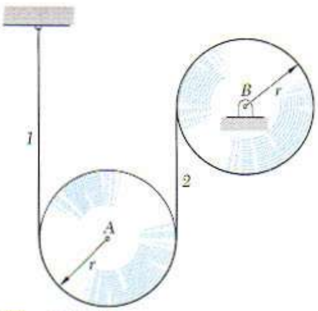


Fig. P17.18

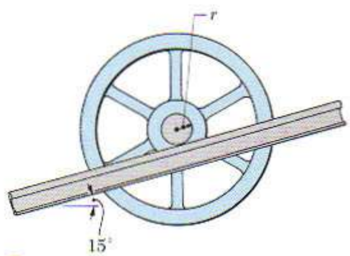


Fig. P17.19

17.19 A flywheel is rigidly attached to a $1\frac{1}{2}$-in.-radius shaft which rolls without sliding along parallel rails. The system is released from rest and attains a speed of 6 in./s after moving 75 in. along the rails. Determine the centroidal radius of gyration of the system.

17.20 A hemisphere of mass m and radius r is released from rest in the position shown. Assuming that the hemisphere rolls without sliding, determine (a) its angular velocity after it has rolled through 90°, (b) the normal reaction at the surface at the same instant. [*Hint.* Note that $GO = 3r/8$ and that, by the parallel-axis theorem, $\bar{I} = \frac{2}{5}mr^2 - m(GO)^2$.]

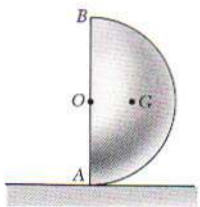


Fig. P17.20

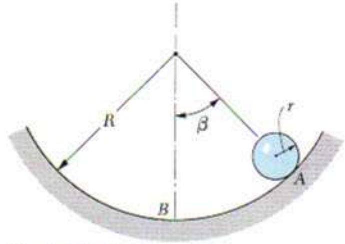


Fig. P17.21

17.21 A sphere of mass m and radius r rolls without slipping inside a curved surface of radius R. Knowing that the sphere is released from rest in the position shown, derive an expression (*a*) for the linear velocity of the sphere as it passes through B, (*b*) for the magnitude of the vertical reaction at that instant.

17.22 Solve Prob. 17.21, assuming that the sphere is replaced by a uniform cylinder of mass m and radius r.

17.23 A slender rod of length l and mass m is pivoted at one end as shown. It is released from rest in a horizontal position and swings freely. (*a*) Determine the angular velocity of the rod as it passes through a vertical position and the corresponding reaction at the pivot. (*b*) Solve part *a* for $m = 1.5$ kg and $l = 0.9$ m.

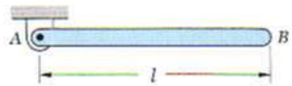


Fig. P17.23

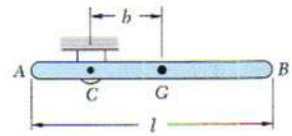


Fig. P17.24

17.24 A uniform rod of length l is pivoted about a point C located at a distance b from its center G. The rod is released from rest in a horizontal position. Determine (*a*) the distance b so that the angular velocity of the rod as it passes through a vertical position is maximum, (*b*) the value of the maximum angular velocity.

17.25 A 6- by 8-in. rectangular plate is suspended by two pins at A and B. The pin at B is removed and the plate swings about point A. Determine (*a*) the angular velocity of the plate after it has rotated through 90°, (*b*) the maximum angular velocity attained by the plate as it swings freely.

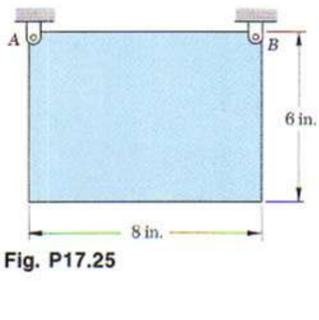


Fig. P17.25

17.26 and 17.27 Gear C weighs 6 lb and has a centroidal radius of gyration of 3 in. The uniform bar AB weighs 5 lb, and gear D is stationary. If the system is released from rest in the position shown, determine the velocity of point B after bar AB has rotated through 90°.

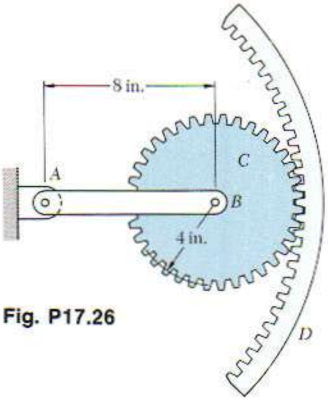


Fig. P17.26

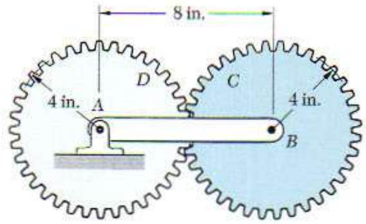


Fig. P17.27

17.28 The mass center G of a 1.5-kg wheel of radius $R = 150$ mm is located at a distance $r = 50$ mm from its geometric center C. The centroidal radius of gyration of the wheel is $\bar{k} = 75$ mm. As the wheel rolls without sliding, its angular velocity is observed to vary. Knowing that in position *1* the angular velocity is 10 rad/s, determine the angular velocity of the wheel (*a*) in position *2*, (*b*) in position *3*.

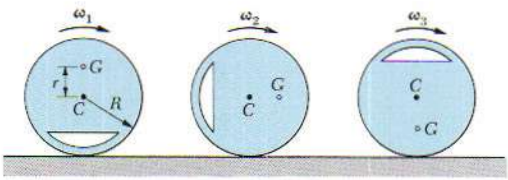


Fig. P17.28 and P17.29

17.29 The mass center G of a wheel of radius R is located at a distance r from its geometric center C. The centroidal radius of gyration of the wheel is denoted by $\bar{k}$. As the wheel rolls freely and without sliding on a horizontal plane, its angular velocity is observed to vary. Denoting by ω_1, ω_2, and ω_3, respectively, the angular velocity of the wheel when G is directly above C, level with C, and directly below C, show that ω_1, ω_2, and ω_3 satisfy the relation

$$\frac{\omega_2^2 - \omega_1^2}{\omega_3^2 - \omega_2^2} = \frac{g/R + \omega_1^2}{g/R + \omega_3^2}$$

17.30 and 17.31 The 12-lb carriage is supported as shown by two uniform disks, each of weight 8 lb and radius 3 in. Knowing that the system is initially at rest, determine the velocity of the carriage after it has moved 3 ft. Assume that the disks roll without sliding.

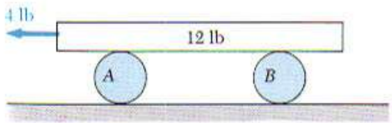


Fig. P17.30

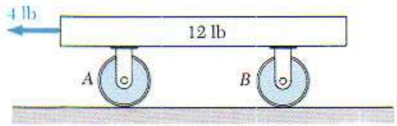


Fig. P17.31

17.32 The motion of the 240-mm rod AB is guided by pins at A and B which slide freely in the slots shown. If the rod is released from rest in position *1*, determine the velocity of A and B when the rod is (*a*) in position *2*, (*b*) in position *3*.

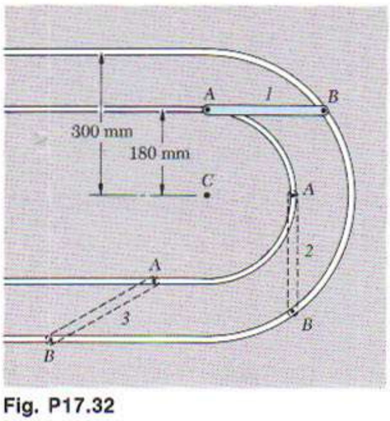


Fig. P17.32

17.33 The motion of a 0.6-m slender rod is guided by pins at A and B which slide freely in the slots shown. Knowing that the rod is released from rest when $\theta = 0$ and that end A is given a slight push to the right, determine (*a*) the angle θ for which the speed of end A is maximum, (*b*) the corresponding maximum speed of A.

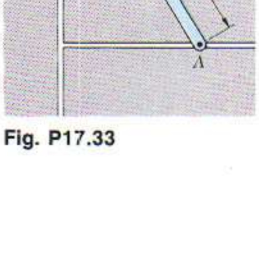


Fig. P17.33

17.34 In Prob. 17.33, determine the velocity of ends A and B (*a*) when $\theta = 30°$, (*b*) when $\theta = 90°$.

17.35 The ends of a 25-lb rod AB are constrained to move along the slots shown. A spring of constant 3 lb/in. is attached to end A. Knowing that the rod is released from rest when $\theta = 0$ and that the initial tension in the spring is zero, determine the maximum distance through which end A will move.

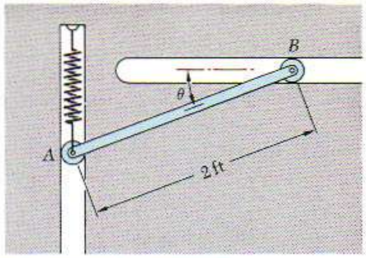


Fig. P17.35 and P17.36

17.36 The ends of a 25-lb rod AB are constrained to move along the slots shown. A spring of constant 10 lb/in. is attached to end A in such a way that its tension is zero when $\theta = 0$. If the rod is released from rest when $\theta = 60°$, determine the angular velocity of the rod when $\theta = 30°$.

17.37 Determine the velocity of pin B as the rods of Sample Prob. 17.5 strike the horizontal surface.

17.38 The uniform rods AB and BC are of mass 4.5 and 1.5 kg respectively. If the system is released from rest in the position shown, determine the angular velocity of rod BC as it passes through a vertical position.

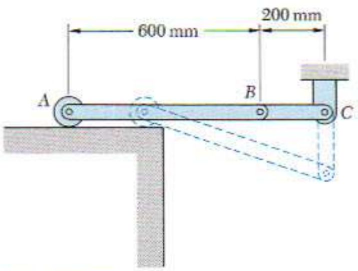


Fig. P17.38

17.39 In Prob. 17.38, determine the angular velocity of rod BC after it has rotated 45°.

***17.40** A small matchbox is placed on top of the rod AB. End B of the rod is given a slight horizontal push, causing it to slide on the horizontal floor. Assuming no friction and neglecting the weight of the matchbox, determine the angle θ through which the rod will have rotated when the matchbox loses contact with the rod.

Fig. P17.40

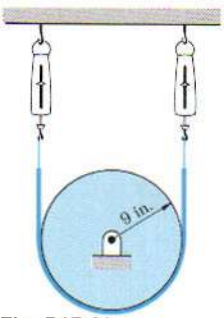


Fig. P17.41

17.41 The experimental setup shown is used to measure the power output of a small turbine. When the turbine is operating at 200 rpm, the readings of the two spring scales are 10 and 22 lb, respectively. Determine the power being developed by the turbine.

17.42 In Sample Prob. 17.2 determine the power being delivered to gear B at the instant when (*a*) the gear starts rotating, (*b*) the gear attains an angular velocity $\omega_B = 300$ rpm.

17.43 Knowing that the maximum allowable couple which can be applied to a shaft is $12\ \mathrm{kN \cdot m}$, determine the maximum power which can be transmitted by the shaft (*a*) at 100 rpm, (*b*) at 1000 rpm.

17.44 Determine the moment of the couple which must be exerted by a motor to develop $\frac{1}{4}$ hp at a speed of (*a*) 3600 rpm, (*b*) 720 rpm.

17.7. Principle of Impulse and Momentum for the Plane Motion of a Rigid Body. We shall now apply the principle of impulse and momentum to the analysis of the plane motion of rigid bodies and of systems of rigid bodies. As was pointed out in Chap. 13, the method of impulse and momentum is particularly well adapted to the solution of problems involving time and velocities. Moreover, the principle of impulse and momentum provides the only practicable method for the solution of problems involving impulsive motion or impact (Secs. 17.10 and 17.11).

Considering again a rigid body as made of a large number of particles P_i, we recall from Sec. 14.8 that the system formed by the momenta of the particles at time t_1 and the system of the impulses of the external forces applied from t_1 to t_2 are together equipollent to the system formed by the momenta of the particles at time t_2. Since the vectors associated with a rigid body may be considered as sliding vectors, it follows (Sec. 3.18) that the systems of vectors shown in Fig. 17.6 are not only equipollent but

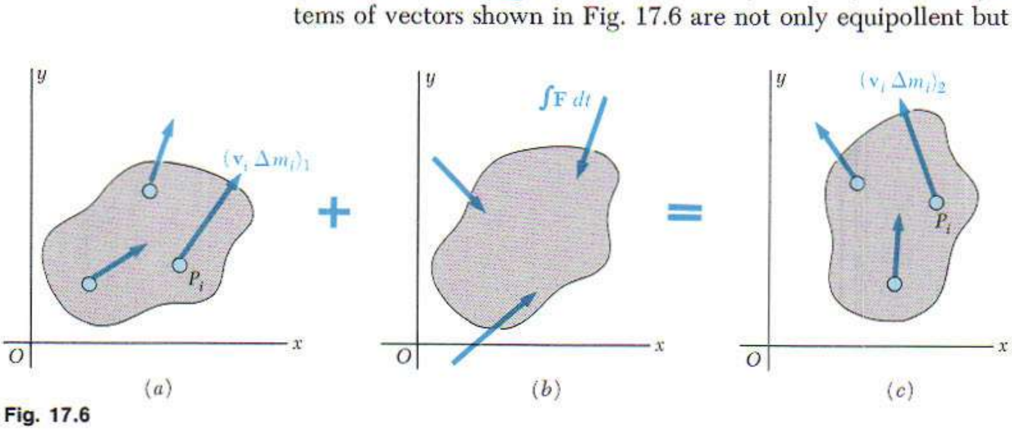


Fig. 17.6

truly *equivalent* in the sense that the vectors on the left-hand side of the equals sign may be transformed into the vectors on the right-hand side through the use of the fundamental operations listed in Sec. 3.12. We therefore write

$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1\to 2} = \textbf{Syst Momenta}_2 \quad (17.14)$$

But the momenta $\mathbf{v}_i\,\Delta m_i$ of the particles may be reduced to a vector attached at G, equal to their sum

$$\mathbf{L} = \sum_{i=1}^{n} \mathbf{v}_i\,\Delta m_i$$

and to a couple of moment equal to the sum of their moments about G

$$\mathbf{H}_G = \sum_{i=1}^{n} \mathbf{r}'_i \times \mathbf{v}_i\,\Delta m_i$$

We recall from Sec. 14.2 that $\mathbf{L}$ and $\mathbf{H}_G$ define, respectively, the linear momentum and the angular momentum about G of the system of particles forming the rigid body. We also note from Eq. (14.14) that $\mathbf{L} = m\overline{\mathbf{v}}$. On the other hand, restricting the present analysis to the plane motion of a rigid slab or of a rigid body symmetrical with respect to the reference plane, we recall from Eq. (16.4) that $\mathbf{H}_G = \bar{I}\boldsymbol{\omega}$. We thus conclude that the system of the momenta $\mathbf{v}_i\,\Delta m_i$ is equivalent to the *linear momentum vector* $m\overline{\mathbf{v}}$ attached at G and to the *angular momentum couple* $\bar{I}\boldsymbol{\omega}$ (Fig. 17.7). Observing that the system of momenta reduces to the vector $m\overline{\mathbf{v}}$ in the particular case of a translation ($\boldsymbol{\omega} = 0$) and to the couple $\bar{I}\boldsymbol{\omega}$ in the particular case of a centroidal rotation ($\overline{\mathbf{v}} = 0$), we verify once more that the plane motion of a rigid body symmetrical with respect to the reference plane may be resolved into a translation with the mass center G and a rotation about G.

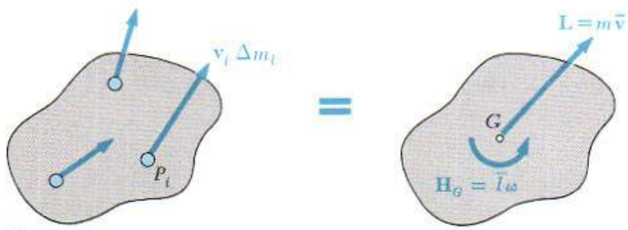


Fig. 17.7

Replacing the system of momenta in parts *a* amd *c* of Fig. 17.6 by the equivalent linear momentum vector and angular momentum couple, we obtain the three diagrams shown in Fig. 17.8.

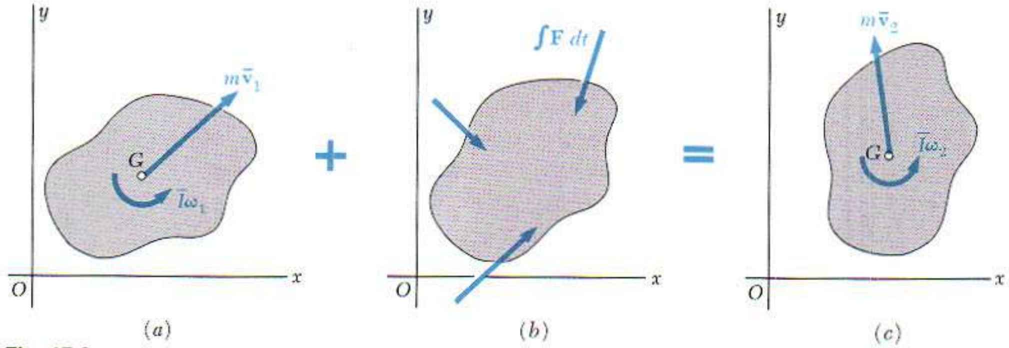


Fig. 17.8

This figure expresses graphically the fundamental relation (17.14) in the case of the plane motion of a rigid slab or of a rigid body symmetrical with respect to the reference plane.

Three equations of motion may be derived from Fig. 17.8. Two equations are obtained by summing and equating the x and y *components* of the momenta and impulses, and the third by summing and equating the *moments* of these vectors *about any given point.* The coordinate axes may be chosen fixed in space, or they may be allowed to move with the mass center of the body while maintaining a fixed direction. In either case, the point about which moments are taken should keep the same position relative to the coordinate axes during the interval of time considered.

In deriving the three equations of motion for a rigid body, care should be taken not to add indiscriminately linear and angular momenta. Confusion will be avoided if it is kept in mind that $m\bar{v}_x$ and $m\bar{v}_y$ represent the *components of a vector,* namely, the linear momentum vector $m\bar{\mathbf{v}}$, while $\bar{I}\omega$ represents the *magnitude of a couple,* namely, the angular momentum couple $\bar{I}\boldsymbol{\omega}$. Thus the quantity $\bar{I}\omega$ should be added only to the *moment* of the linear momentum $m\bar{\mathbf{v}}$, never to this vector itself nor to its components. All quantities involved will then be expressed in the same units, namely $\mathrm{N \cdot m \cdot s}$ or $\mathrm{lb \cdot ft \cdot s}$.

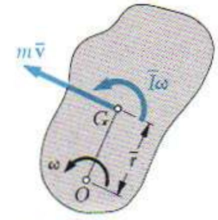


Fig. 17.9

Noncentroidal Rotation. In this particular case of plane motion, the magnitude of the velocity of the mass center of the body is $\bar{v} = \bar{r}\omega$, where $\bar{r}$ represents the distance from the mass center to the fixed axis of rotation and ω the angular velocity of the body at the instant considered; the magnitude of the momentum vector attached at G is thus $m\bar{v} = m\bar{r}\omega$. Summing the moments about O of the momentum vector and momentum couple (Fig. 17.9) and using the parallel-axis theorem for moments of inertia, we find that the angular momentum $\mathbf{H}_O$ of the body about O has the magnitude†

$$\bar{I}\omega + (m\bar{r}\omega)\bar{r} = (\bar{I} + m\bar{r}^2)\omega = I_O\omega \tag{17.15}$$

Equating the moments about O of the momenta and impulses in (17.14), we write

$$I_O\omega_1 + \sum \int_{t_1}^{t_2} M_O\, dt = I_O\omega_2 \tag{17.16}$$

In the general case of plane motion of a rigid body symmetrical with respect to the reference plane, Eq. (17.16) may be used with respect to the instantaneous axis of rotation under certain conditions. It is recommended, however, that all problems of plane motion be solved by the general method described earlier in this section.

17.8. Systems of Rigid Bodies. The motion of several rigid bodies may be analyzed by applying the principle of impulse and momentum to each body separately (Sample Prob. 17.6).

However, in solving problems involving no more than three unknowns (including the impulses of unknown reactions), it is often found convenient to apply the principle of impulse and momentum to the system as a whole. The momentum and impulse diagrams are drawn for the entire system of bodies. The diagrams of momenta should include a momentum vector, a momentum couple, or both, for each moving part of the system.

† Note that the sum $\mathbf{H}_A$ of the moments about an arbitrary point A of the momenta of the particles of a rigid slab is, in general, *not* equal to $I_A\omega$. (See Prob. 17.59.)

Impulses of forces internal to the system may be omitted from the impulse diagram since they occur in pairs of equal and opposite vectors. Summing and equating successively the x components, y components, and moments of all vectors involved, one obtains three relations which express that the momenta at time t_1 and the impulses of the external forces form a system equipollent to the system of the momenta at time t_2.† Again, care should be taken not to add indiscriminately linear and angular momenta; each equation should be checked to make sure that consistent units have been used. This approach has been used in Sample Prob. 17.8 and, further on, in Sample Probs. 17.9 and 17.10.

17.9. Conservation of Angular Momentum. When no external force acts on a rigid body or a system of rigid bodies, the impulses of the external forces are zero and the system of the momenta at time t_1 is equipollent to the system of the momenta at time t_2. Summing and equating successively the x components, y components, and moments of the momenta at times t_1 and t_2, we conclude that the total linear momentum of the system is conserved in any direction and that its total angular momentum is conserved about any point.

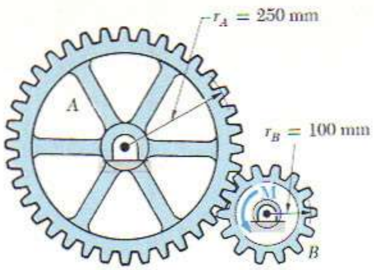
There are many engineering applications, however, in which *the linear momentum is not conserved*, yet in which *the angular momentum* $\mathbf{H}_O$ of the system about a given point O is *conserved:*

$$(\mathbf{H}_O)_1 = (\mathbf{H}_O)_2 \tag{17.17}$$

Such cases occur when the lines of action of all external forces pass through O or, more generally, when the sum of the angular impulses of the external forces about O is zero.

Problems involving *conservation of angular momentum* about a point O may be solved by the general method of impulse and momentum, i.e., by drawing momentum and impulse diagrams as described in Secs. 17.7 and 17.8. Equation (17.17) is then obtained by summing and equating moments about O (Sample Prob. 17.8). As we shall see later in Sample Prob. 17.9, two additional equations may be written by summing and equating x and y components; these equations may be used to determine two unknown linear impulses, such as the impulses of the reaction components at a fixed point.

† Note that, as in Sec. 16.7, we cannot speak of *equivalent* systems since we are not dealing with a single rigid body.



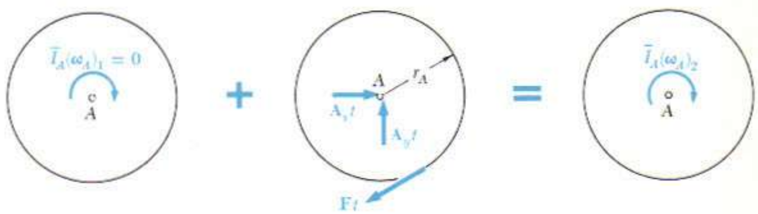
SAMPLE PROBLEM 17.6

Gear A has a mass of 10 kg and a radius of gyration of 200 mm, while gear B has a mass of 3 kg and a radius of gyration of 80 mm. The system is at rest when a couple $\mathbf{M}$ of magnitude $6\ \mathrm{N \cdot m}$ is applied to gear B. Neglecting friction, determine (*a*) the time required for the angular velocity of gear B to reach 600 rpm, (*b*) the tangential force which gear B exerts on gear A. These gears have been previously considered in Sample Prob. 17.2.

Solution. We apply the principle of impulse and momentum to each gear separately. Since all forces and the couple are constant, their impulses are obtained by multiplying them by the unknown time t. We recall from Sample Prob. 17.2 that the centroidal moments of inertia and the final angular velocities are

$$\bar{I}_A = 0.400\ \mathrm{kg \cdot m^2} \qquad \bar{I}_B = 0.0192\ \mathrm{kg \cdot m^2}$$
$$(\omega_A)_2 = 25.1\ \mathrm{rad/s} \qquad (\omega_B)_2 = 62.8\ \mathrm{rad/s}$$

Principle of Impulse and Momentum for Gear A. The systems of initial momenta, impulses, and final momenta are shown in three separate sketches.

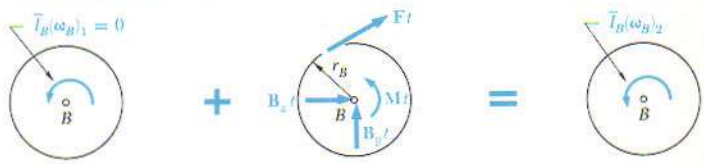


$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1\text{-}2} = \textbf{Syst Momenta}_2$$

$+\circlearrowright$ moments about A: $\qquad 0 - Ftr_A = -\bar{I}_A(\omega_A)_2$

$$Ft(0.250\ \mathrm{m}) = (0.400\ \mathrm{kg \cdot m^2})(25.1\ \mathrm{rad/s})$$
$$Ft = 40.2\ \mathrm{N \cdot s}$$

Principle of Impulse and Momentum for Gear B.



$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1\text{-}2} = \textbf{Syst Momenta}_2$$

$+\circlearrowright$ moments about B: $\qquad 0 + Mt - Ftr_B = \bar{I}_B(\omega_B)_2$

$$+(6\ \mathrm{N \cdot m})t - (40.2\ \mathrm{N \cdot s})(0.100\ \mathrm{m}) = (0.0192\ \mathrm{kg \cdot m^2})(62.8\ \mathrm{rad/s})$$

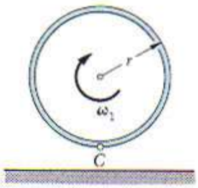
$t = 0.871\ \mathrm{s}$ ◄

Recalling that $Ft = 40.2\ \mathrm{N \cdot s}$, we write

$$F(0.871\ \mathrm{s}) = 40.2\ \mathrm{N \cdot s} \qquad F = +46.2\ \mathrm{N}$$

Thus, the force exerted by gear B on gear A is $\quad \mathbf{F} = 46.2\ \mathrm{N} \swarrow$ ◄

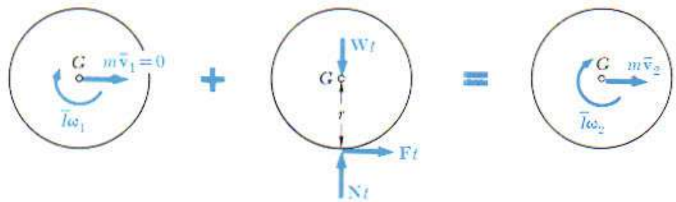
SAMPLE PROBLEM 17.7



A hoop of radius r and mass m is placed on a horizontal surface with no linear velocity but with a clockwise angular velocity $\boldsymbol{\omega}_1$. Denoting by μ the coefficient of friction between the hoop and the surface, determine (*a*) the time t_2 at which the hoop will start rolling without sliding, (*b*) the linear and angular velocities of the hoop at time t_2.

Solution. Since the entire mass m is located at a distance r from the center of the hoop, we have $\bar{I} = mr^2$. While the hoop is sliding relative to the surface, it is acted upon by the normal force **N**, the friction force **F**, and its weight **W** of magnitude $W = mg$.

Principle of Impulse and Momentum. We apply the principle of impulse and momentum to the hoop from the time t_1 when it is placed on the surface until the time t_2 when it starts rolling without sliding.



$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1\to2} = \textbf{Syst Momenta}_2$$

$+\uparrow$ y components:	$Nt - Wt = 0$	(1)
$\xrightarrow{+}$ x components:	$Ft = m\bar{v}_2$	(2)
$+\circlearrowright$ moments about G:	$-\bar{I}\omega_1 + Ftr = -\bar{I}\omega_2$	(3)

From (1) we obtain $N = W = mg$. For $t < t_2$, sliding occurs at point C and we have $F = \mu N = \mu mg$. Substituting for F into (2), we write

$$\mu mgt = m\bar{v}_2 \qquad \bar{v}_2 = \mu gt \tag{4}$$

Substituting $F = \mu mg$ and $\bar{I} = mr^2$ into (3),

$$-mr^2\omega_1 + \mu mgtr = -mr^2\omega_2 \qquad \omega_2 = \omega_1 - \frac{\mu g}{r}t \tag{5}$$

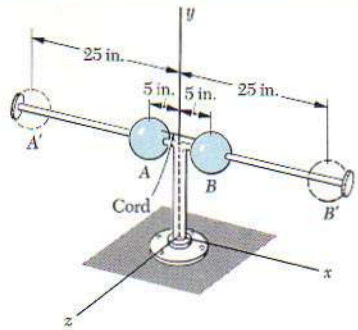
The hoop will start rolling without sliding when the velocity $\mathbf{v}_C$ of the point of contact is zero. At that time, $t = t_2$, point C becomes the instantaneous center of rotation, and we have $\bar{v}_2 = r\omega_2$. Substituting from (4) and (5), we write

$$\bar{v}_2 = r\omega_2 \qquad \mu g t_2 = r\left(\omega_1 - \frac{\mu g}{r}t_2\right) \qquad t_2 = \frac{r\omega_1}{2\mu g} \quad \blacktriangleleft$$

Substituting this expression for t_2 into (4),

$$\bar{v}_2 = \mu g t_2 = \mu g \frac{r\omega_1}{2\mu g} \qquad \bar{v}_2 = \tfrac{1}{2}r\omega_1 \qquad \bar{\mathbf{v}}_2 = \tfrac{1}{2}r\omega_1 \rightarrow \quad \blacktriangleleft$$

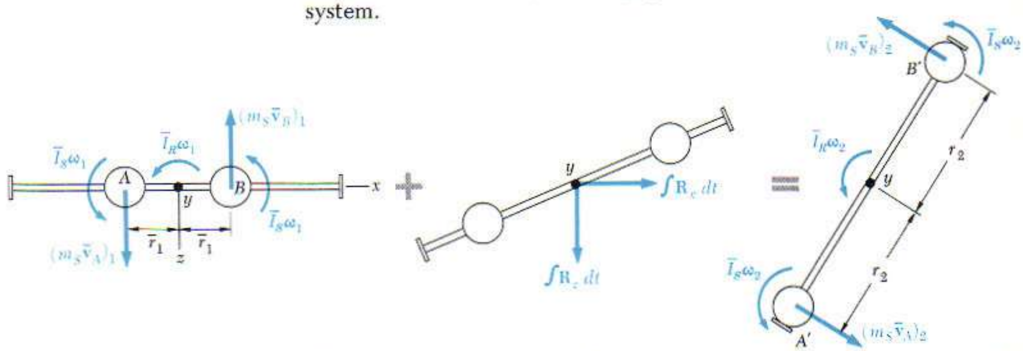
$$\omega_2 = \frac{\bar{v}_2}{r} \qquad \omega_2 = \tfrac{1}{2}\omega_1 \qquad \boldsymbol{\omega}_2 = \tfrac{1}{2}\omega_1 \circlearrowright \quad \blacktriangleleft$$



SAMPLE PROBLEM 17.8

Two solid spheres of radius 3 in., weighing 2 lb each, are mounted at A and B on the horizontal rod $A'B'$, which rotates freely about the vertical with a counterclockwise angular velocity of 6 rad/s. The spheres are held in position by a cord which is suddenly cut. Knowing that the centroidal moment of inertia of the rod and pivot is $\bar{I}_R = 0.25 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$, determine (*a*) the angular velocity of the rod after the spheres have moved to positions A' and B', (*b*) the energy lost due to the plastic impact of the spheres and the stops at A' and B'.

a. **Principle of Impulse and Momentum.** In order to determine the final angular velocity of the rod, we shall express that the initial momenta of the various parts of the system and the impulses of the external forces are together equipollent to the final momenta of the system.



Observing that the external forces consist of the weights and the reaction at the pivot, which have no moment about the y axis, and noting that $\bar{v}_A = \bar{v}_B = \bar{r}\omega$, we write

$+\circlearrowleft$ moments about y axis:

$$2(m_S\bar{r}_1\omega_1)\bar{r}_1 + 2\bar{I}_S\omega_1 + \bar{I}_R\omega_1 = 2(m_S\bar{r}_2\omega_2)\bar{r}_2 + 2\bar{I}_S\omega_2 + \bar{I}_R\omega_2$$

$$(2m_S\bar{r}_1^2 + 2\bar{I}_S + \bar{I}_R)\omega_1 = (2m_S\bar{r}_2^2 + 2\bar{I}_S + \bar{I}_R)\omega_2 \qquad (1)$$

which expresses that *the angular momentum of the system about the y axis is conserved.* We now compute

$$\bar{I}_S = \tfrac{2}{5}m_S a^2 = \frac{2}{5}\left(\frac{2\text{ lb}}{32.2\text{ ft/s}^2}\right)(\tfrac{3}{12}\text{ ft})^2 = 0.00155 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

$$m_S\bar{r}_1^2 = \frac{2}{32.2}\left(\frac{5}{12}\right)^2 = 0.0108 \qquad m_S\bar{r}_2^2 = \frac{2}{32.2}\left(\frac{25}{12}\right)^2 = 0.2696$$

Substituting these values and $\bar{I}_R = 0.25$, $\omega_1 = 6$ rad/s into (1):

$$0.275(6\text{ rad/s}) = 0.792\omega_2 \qquad \omega_2 = 2.08\text{ rad/s} \circlearrowleft \blacktriangleleft$$

b. **Energy Lost.** The kinetic energy of the system at any instant is

$$T = 2(\tfrac{1}{2}m_S\bar{v}^2 + \tfrac{1}{2}\bar{I}_S\omega^2) + \tfrac{1}{2}\bar{I}_R\omega^2 = \tfrac{1}{2}(2m_S\bar{r}^2 + 2\bar{I}_S + \bar{I}_R)\omega^2$$

Recalling the numerical values found above, we have

$$T_1 = \tfrac{1}{2}(0.275)(6)^2 = 4.95 \text{ ft} \cdot \text{lb} \qquad T_2 = \tfrac{1}{2}(0.792)(2.08)^2 = 1.713 \text{ ft} \cdot \text{lb}$$

$$\Delta T = T_2 - T_1 = 1.71 - 4.95 \qquad \Delta T = -3.24 \text{ ft} \cdot \text{lb} \blacktriangleleft$$

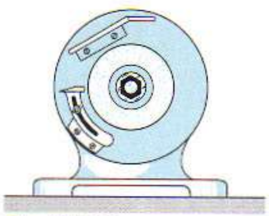


Fig. P17.45

PROBLEMS

17.45 A small grinding wheel is attached to the shaft of an electric motor which has a rated speed of 3600 rpm. When the power is turned off, the unit coasts to rest in 70 s. The grinding wheel and rotor have a combined weight of 6 lb and a combined radius of gyration of 2 in. Determine the average magnitude of the couple due to kinetic friction in the bearings of the motor.

17.46 A turbine-generator unit is shut off when its rotor is rotating at 3600 rpm; it is observed that the rotor coasts to rest in 7.10 min. Knowing that the 1850-kg rotor has a radius of gyration of 234 mm, determine the average magnitude of the couple due to bearing friction.

17.47 A bolt located 50 mm from the center of an automobile wheel is tightened by applying the couple shown for 0.1 s. Assuming that the wheel is free to rotate and is initially at rest, determine the resulting angular velocity of the wheel. The 20-kg wheel has a radius of gyration of 250 mm.

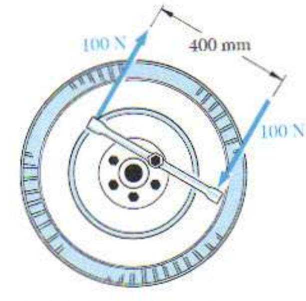


Fig. P17.47

17.48 Solve Prob. 17.3, assuming that the couple **M** is applied for a time t_0 and then removed.

17.49 A disk of constant thickness, initially at rest, is placed in contact with a belt which moves with a constant velocity **v**. Denoting by μ the coefficient of friction between the disk and the belt, derive an expression for the time required for the disk to reach a constant angular velocity.

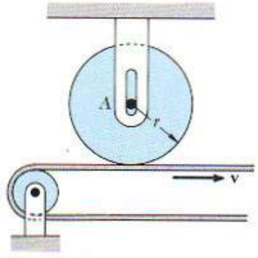


Fig. P17.49 and P17.50

17.50 Disk A, of weight 5 lb and radius $r = 3$ in., is at rest when it is placed in contact with the belt, which moves with a constant speed $v = 50$ ft/s. Knowing that $\mu = 0.20$ between the disk and the belt, determine the time required for the disk to reach a constant angular velocity.

17.51 Solve Prob. 12.17*b*, assuming that each pulley is of 8-in. radius and has a centroidal moment of inertia of 0.25 lb·ft·s^2.

17.52 Using the principle of impulse and momentum, solve Prob. 16.34*b*.

17.53 Disks *A* and *B* are of mass 5 and 1.8 kg, respectively. The disks are initially at rest and the coefficient of friction between them is 0.20. A couple **M** of magnitude 4 N · m is applied to disk *A* for 1.50 s and then removed. Determine (*a*) whether slipping occurs between the disks, (*b*) the final angular velocity of each disk.

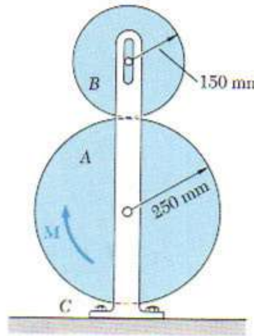


Fig. P17.53

17.54 In Prob. 17.53, determine (*a*) the largest couple **M** for which no slipping occurs, (*b*) the corresponding final angular velocity of each disk.

17.55 Two disks *A* and *B* are connected by a belt as shown. Each disk weighs 30 lb and has a radius of 1.5 ft. The shaft of disk *B* rests in a slotted bearing and is held by a spring which exerts a constant force of 15 lb. If a 20-lb·ft couple is applied to disk *A*, determine (*a*) the time required for the disks to attain a speed of 600 rpm, (*b*) the tension in both portions of the belt, (*c*) the minimum coefficient of friction if no slipping is to occur.

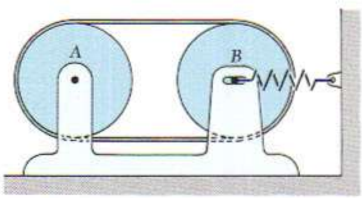


Fig. P17.55

17.56 Solve Prob. 17.55, assuming that disk *A* weighs 10 lb and disk *B* weighs 50 lb.

17.57 Show that the system of momenta for a rigid slab in plane motion reduces to a single vector, and express the distance from the mass center *G* to the line of action of this vector in terms of the centroidal radius of gyration $\bar{k}$ of the slab, the magnitude $\bar{v}$ of the velocity of *G*, and the angular velocity ω.

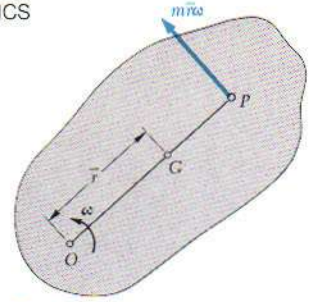


Fig. P17.58

17.58 Show that, when a rigid slab rotates about a fixed axis through O perpendicular to the slab, the system of momenta of its particles is equivalent to a single vector of magnitude $m\bar{r}\omega$, perpendicular to the line OG, and applied to a point P on this line, called the *center of percussion*, at a distance $GP = \bar{k}^2/\bar{r}$ from the mass center of the slab.

17.59 Show that the sum $\mathbf{H}_A$ of the moments about a point A of the momenta of the particles of a rigid slab in plane motion is equal to $I_A\omega$, where ω is the angular velocity of the slab at the instant considered and I_A the moment of inertia of the slab about A, *if and only if* one of the following conditions is satisfied: (*a*) A is the mass center of the slab, (*b*) A is the instantaneous center of rotation, (*c*) the velocity of A is directed along a line joining point A and the mass center G.

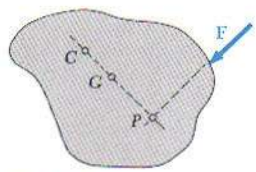


Fig. P17.60

17.60 Consider a rigid slab initially at rest and subjected to an impulsive force $\mathbf{F}$ contained in the plane of the slab. We define the *center of percussion* P as the point of intersection of the line of action of $\mathbf{F}$ with the perpendicular drawn from G. (*a*) Show that the instantaneous center of rotation C of the slab is located on line GP at a distance $GC = \bar{k}^2/GP$ on the opposite side of G. (*b*) Show that, if the center of percussion were located at C, the instantaneous center of rotation would be located at P.

17.61 A cord is wrapped around a solid cylinder of radius r and mass m as shown. If the cylinder is released from rest at time $t = 0$, determine the velocity of the center of the cylinder at a time t.

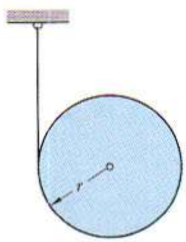


Fig. P17.61

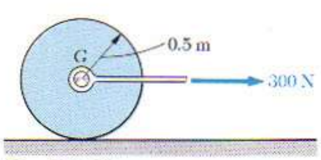


Fig. P17.62

17.62 A 100-kg cylindrical roller is initially at rest and is acted upon by a 300-N force as shown. Assuming that the body rolls without slipping, determine (*a*) the velocity of the center G after 6 s, (*b*) the friction force required to prevent slipping.

17.63 A section of thin-walled pipe of radius r is released from rest at time $t = 0$. Assuming that the pipe rolls without slipping, determine (*a*) the velocity of the center at time t, (*b*) the coefficient of friction required to prevent slipping.

Fig. P17.63

17.64 Two disks, each of weight 12 lb and radius 6 in., which roll without slipping, are connected by a drum of radius r and of negligible weight. A rope is wrapped around the drum and is pulled horizontally with a force **P** of magnitude 8 lb. Knowing that $r = 3$ in. and that the disks are initially at rest, determine (*a*) the velocity of the center G after 3 s, (*b*) the friction force required to prevent slipping.

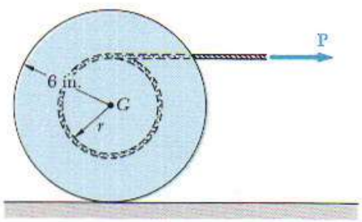


Fig. P17.64

17.65 In Prob. 17.64, determine the required value of r and the corresponding velocity after 3 s if the friction force is to be zero.

17.66 and 17.67 The 12-lb carriage is supported as shown by two uniform disks, each of weight 8 lb and radius 3 in. Knowing that the carriage is initially at rest, determine the velocity of the carriage 3 s after the 4-lb force is applied. Assume that the disks roll without sliding.

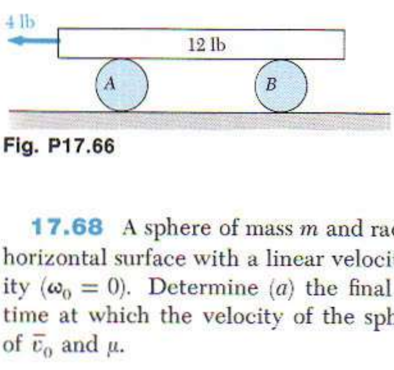


Fig. P17.66

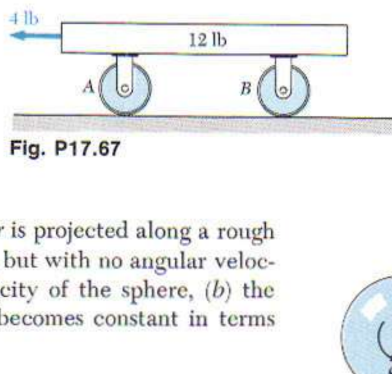


Fig. P17.67

17.68 A sphere of mass m and radius r is projected along a rough horizontal surface with a linear velocity $\bar{\mathbf{v}}_0$ but with no angular velocity ($\boldsymbol{\omega}_0 = 0$). Determine (*a*) the final velocity of the sphere, (*b*) the time at which the velocity of the sphere becomes constant in terms of $\bar{v}_0$ and μ.

17.69 A sphere of mass m and radius r is projected along a rough horizontal surface with the initial velocities indicated. If the final velocity of the sphere is to be zero, express (*a*) the required ω_0 in terms of $\bar{v}_0$ and r, (*b*) the time required for the sphere to come to rest in terms of $\bar{v}_0$ and μ.

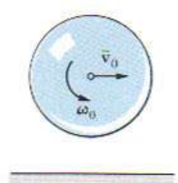


Fig. P17.68 and P17.69

17.70 Solve Sample Prob. 17.7, assuming that the hoop is replaced by a uniform sphere of radius r and mass m.

17.71 Solve Sample Prob. 17.8, assuming that, after the cord is cut, sphere B moves to position B' but that an obstruction prevents sphere A from moving.

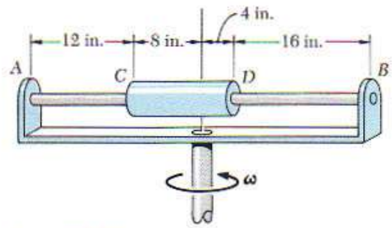


Fig. P17.72

17.72 An 8-lb tube CD may slide freely on rod AB, which in turn may rotate freely in a horizontal plane. At the instant shown, the assembly is rotating with an angular velocity of magnitude $\omega = 8$ rad/s and the tube is moving toward A with a speed of 5 ft/s relative to the rod. Knowing that the centroidal moment of inertia about a vertical axis is 0.022 lb·ft·s^2 for the tube and 0.400 lb·ft·s^2 for the rod and bracket, determine (a) the angular velocity of the assembly after the tube has moved to end A, (b) the energy lost due to the plastic impact at A.

17.73 Four rectangular panels, each of length b and height $\frac{1}{2}b$, are attached with hinges to a circular plate of diameter $\sqrt{2}b$ and held by a wire loop in the position shown. The plate and the panels are made of the same material and have the same thickness. The entire assembly is rotating with an angular velocity ω_0 when the wire breaks. Determine the angular velocity of the assembly after the panels have come to rest in a horizontal position.

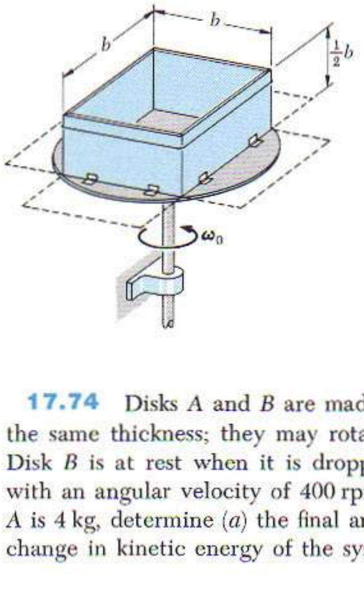


Fig. P17.73

Fig. P17.74

17.74 Disks A and B are made of the same material and are of the same thickness; they may rotate freely about the vertical shaft. Disk B is at rest when it is dropped onto disk A which is rotating with an angular velocity of 400 rpm. Knowing that the mass of disk A is 4 kg, determine (a) the final angular velocity of the disks, (b) the change in kinetic energy of the system.

17.75 In Prob. 17.74, show that if both disks are initially rotating, the change in kinetic energy ΔT of the system depends only upon the initial relative velocity $\omega_{B/A}$ of the disks, and derive an expression for ΔT in terms of $\omega_{B/A}$.

17.76 A small 250-g ball may slide in a slender tube of length 1 m and of mass 1 kg which rotates freely about a vertical axis passing through its center C. If the angular velocity of the tube is 10 rad/s as the ball passes through C, determine the angular velocity of the tube (*a*) just before the ball leaves the tube, (*b*) just after the ball has left the tube.

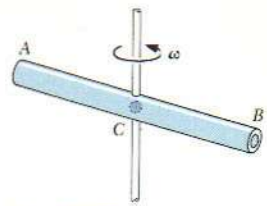


Fig. P17.76

17.77 The rod AB is of mass m and slides freely inside the tube CD which is also of mass m. The angular velocity of the assembly was ω_1 when the rod was entirely inside the tube ($x = 0$). Neglecting the effect of friction, determine the angular velocity of the assembly when $x = \frac{1}{2}L$.

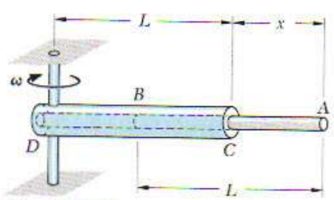


Fig. P17.77

17.78 In the helicopter shown, a vertical tail propeller is used to prevent rotation of the cab as the speed of the main blades is changed. Assuming that the tail propeller is not operating, determine the final angular velocity of the cab after the speed of the main blades has been changed from 180 to 240 rpm. The speed of the main blades is measured relative to the cab, which has a centroidal moment of inertia of 650 lb·ft·s^2. Each of the four main blades is assumed to be a 14-ft slender rod weighing 55 lb.

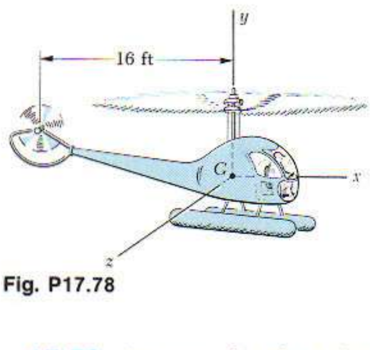


Fig. P17.78

17.79 Assuming that the tail propeller in Prob. 17.78 is operating and that the angular velocity of the cab remains zero, determine the final horizontal velocity of the cab when the speed of the main blades is changed from 180 to 240 rpm. The cab weighs 1250 lb and is initially at rest. Also determine the force exerted by the tail propeller if this change in speed takes places uniformly in 12 s.

17.80 The 5-kg disk is attached to the arm AB which is free to rotate about the vertical axle CD. The arm and motor unit has a moment of inertia of 0.03 kg·m^2 with respect to the axle CD, and the normal operating speed of the motor is 360 rpm. Knowing that the system is initially at rest, determine the angular velocities of the arm and of the disk when the motor reaches a speed of 360 rpm.

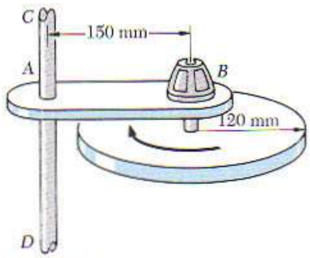


Fig. P17.80

17.81 In Prob. 17.77, determine the velocity of the rod relative to the tube when $x = \frac{1}{2}L$.

17.82 Knowing that in Prob. 17.76 the speed of the ball is 1.2 m/s as it passes through C, determine the radial and transverse components of the velocity of the ball as it leaves the tube at B.

Fig. P17.83

17.83 Collar B weighs 3 lb and may slide freely on rod OA which in turn may rotate freely in the horizontal plane. The assembly is rotating with an angular velocity $\omega = 1.5$ rad/s when a spring located between A and B is released, projecting the collar along the rod with an initial relative speed $v_r = 5$ ft/s. Knowing that the moment of inertia about O of the rod and spring is 0.15 lb · ft · s^2, determine (*a*) the minimum distance between the collar and point O in the ensuing motion, (*b*) the corresponding angular velocity of the assembly.

17.84 In Prob. 17.83, determine the required magnitude of the initial relative velocity $\mathbf{v}_r$ if during the ensuing motion the minimum distance between collar B and point O is to be 1 ft.

17.85 Solve Prob. 17.83, assuming that the initial relative speed of the collar is $v_r = 10$ ft/s.

17.10. Impulsive Motion. We saw in Chap. 13 that the method of impulse and momentum is the only practicable method for the solution of problems involving impulsive motion. Now we shall also find that, compared with the various problems considered in the preceding sections, problems involving impulsive motion are particularly well adapted to a solution by the method of impulse and momentum. The computation of linear impulses and angular impulses is quite simple, since, the time interval considered being very short, the bodies involved may be assumed to occupy the same position during that time interval.

17.11. Eccentric Impact. In Secs. 13.13 and 13.14, we learned to solve problems of *central impact*, i.e., problems in which the mass centers of the two colliding bodies are located on the line of impact. We shall now analyze the *eccentric impact* of two rigid bodies. Consider two bodies which collide, and denote by $\mathbf{v}_A$ and $\mathbf{v}_B$ the velocities before impact *of the two points of contact* A and B (Fig. 17.10*a*). Under the impact, the two bodies will *deform* and, at the end of the period of deformation, the velocities $\mathbf{u}_A$ and $\mathbf{u}_B$ of A and B will have equal components along the line of impact nn (Fig. 17.10*b*). A period of *restitution* will then take place, at the end of which A and B will have velocities $\mathbf{v}'_A$ and $\mathbf{v}'_B$ (Fig. 17.10*c*). Assuming the bodies frictionless, we find that the forces they exert on each other are directed along the line of impact. Denoting, respectively, by $\int P\,dt$ and $\int R\,dt$ the magnitude of the impulse of one of these forces during the period of deformation and during the period of restitution, we recall that the coefficient of restitution e is defined as the ratio

$$e = \frac{\int R\,dt}{\int P\,dt} \tag{17.18}$$

We propose to show that the relation established in Sec. 13.13 between the relative velocities of two particles before and after impact also holds between the components along the line of impact of the relative velocities of the two points of contact A and B. We propose to show, therefore, that

$$(v'_B)_n - (v'_A)_n = e[(v_A)_n - (v_B)_n] \tag{17.19}$$

We shall first assume that the motion of each of the two colliding bodies of Fig. 17.10 is unconstrained. Thus the only impulsive forces exerted on the bodies during the impact are applied at A and B respectively. Consider the body to which point A belongs and draw the three momentum and impulse

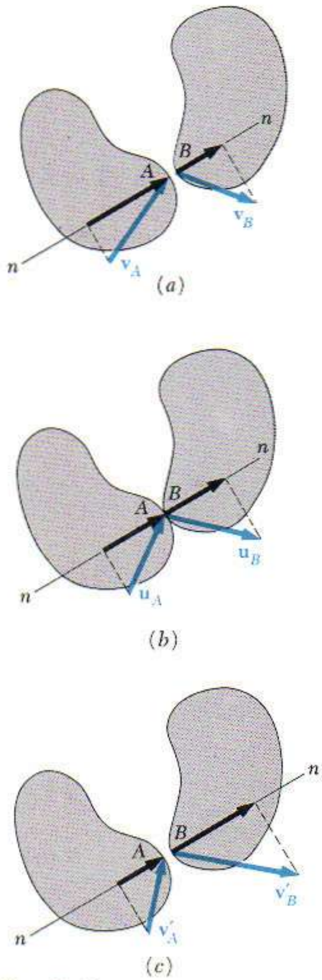


Fig. 17.10

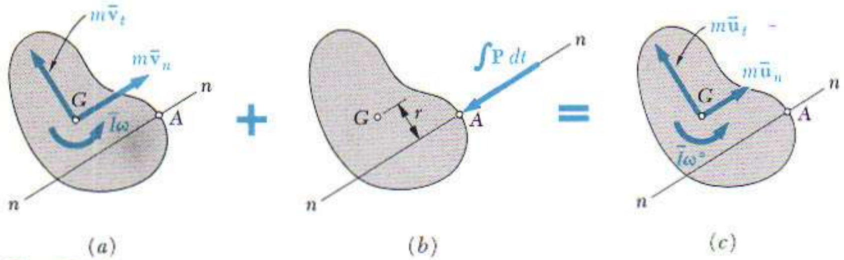


Fig. 17.11

diagrams corresponding to the period of deformation (Fig. 17.11). We denote by $\bar{\mathbf{v}}$ and $\bar{\mathbf{u}}$, respectively, the velocity of the mass center at the beginning and at the end of the period of deformation, and by $\boldsymbol{\omega}$ and $\boldsymbol{\omega}^*$ the angular velocity of the body at the same instants. Summing and equating the components of the momenta and impulses along the line of impact nn, we write

$$m\bar{v}_n - \int P\,dt = m\bar{u}_n \tag{17.20}$$

Summing and equating the moments about G of the momenta and impulses, we also write

$$\bar{I}\omega - r\int P\,dt = \bar{I}\omega^* \tag{17.21}$$

where r represents the perpendicular distance from G to the line of impact. Considering now the period of restitution, we obtain in a similar way

$$m\bar{u}_n - \int R\,dt = m\bar{v}'_n \tag{17.22}$$

$$\bar{I}\omega^* - r\int R\,dt = \bar{I}\omega' \tag{17.23}$$

where $\bar{\mathbf{v}}'$ and $\boldsymbol{\omega}'$ represent, respectively, the velocity of the mass center and the angular velocity of the body after impact. Solving (17.20) and (17.22) for the two impulses and substituting into (17.18), and then solving (17.21) and (17.23) for the same two impulses and substituting again into (17.18), we obtain the following two alternate expressions for the coefficient of restitution:

$$e = \frac{\bar{u}_n - \bar{v}'_n}{\bar{v}_n - \bar{u}_n} \qquad\qquad e = \frac{\omega^* - \omega'}{\omega - \omega^*} \tag{17.24}$$

Multiplying by r the numerator and denominator of the second expression obtained for e, and adding respectively to the numerator and denominator of the first expression, we have

$$e = \frac{\bar{u}_n + r\omega^* - (\bar{v}'_n + r\omega')}{\bar{v}_n + r\omega - (\bar{u}_n + r\omega^*)} \tag{17.25}$$

Observing that $\bar{v}_n + r\omega$ represents the component $(v_A)_n$ along nn of the velocity of the point of contact A and that, similarly,

$\bar{u}_n + r\omega^*$ and $\bar{v}'_n + r\omega'$ represent, respectively, the components $(u_A)_n$ and $(v'_A)_n$, we write

$$e = \frac{(u_A)_n - (v'_A)_n}{(v_A)_n - (u_A)_n} \tag{17.26}$$

The analysis of the motion of the second body leads to a similar expression for e in terms of the components along nn of the successive velocities of point B. Recalling that $(u_A)_n = (u_B)_n$, and eliminating these two velocity components by a manipulation similar to the one used in Sec. 13.13, we obtain relation (17.19).

If one or both of the colliding bodies is constrained to rotate about a fixed point O, as in the case of a compound pendulum (Fig. 17.12a), an impulsive reaction will be exerted at O (Fig. 17.12b). We shall verify that, while their derivation must be

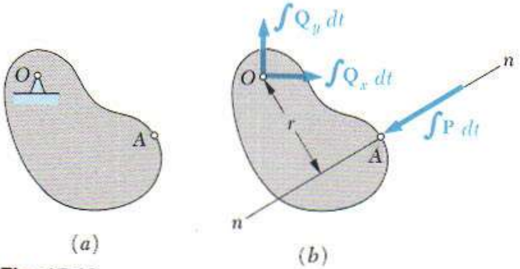


Fig. 17.12

modified, Eqs. (17.26) and (17.19) remain valid. Applying formula (17.16) to the period of deformation and to the period of restitution, we write

$$I_O\omega - r\int P\,dt = I_O\omega^* \tag{17.27}$$
$$I_O\omega^* - r\int R\,dt = I_O\omega' \tag{17.28}$$

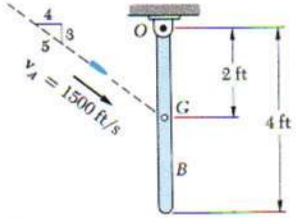
where r represents the perpendicular distance from the fixed point O to the line of impact. Solving (17.27) and (17.28) for the two impulses and substituting into (17.18), and then observing that $r\omega$, $r\omega^*$, and $r\omega'$ represent the components along nn of the successive velocities of point A, we write

$$e = \frac{\omega^* - \omega'}{\omega - \omega^*} = \frac{r\omega^* - r\omega'}{r\omega - r\omega^*} = \frac{(u_A)_n - (v'_A)_n}{(v_A)_n - (u_A)_n}$$

and check that Eq. (17.26) still holds. Thus Eq. (17.19) remains valid when one or both of the colliding bodies is constrained to rotate about a fixed point O.

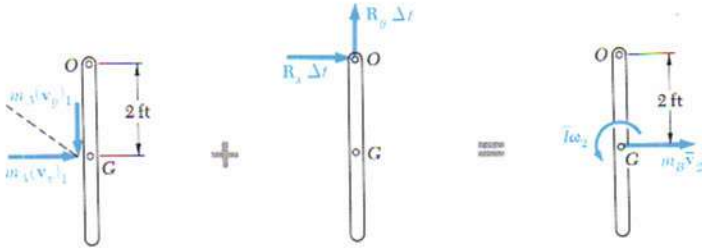
In order to determine the velocities of the two colliding bodies after impact, relation (17.19) should be used in conjunction with one or several other equations obtained by applying the principle of impulse and momentum (Sample Prob. 17.10).

SAMPLE PROBLEM 17.9



A 0.125-lb bullet A is fired with an initial velocity of 1500 ft/s into a 50-lb wooden beam B which is suspended from a hinge at O. Knowing that the beam is initially at rest, determine (a) the angular velocity of the beam immediately after the bullet becomes embedded in the beam, (b) the impulsive reaction at the hinge, assuming that the bullet becomes embedded in 0.0002 s.

Solution. *Principle of Impulse and Momentum.* We consider the bullet and the beam as a single system and express that the initial momenta of the bullet and beam and the impulses of the external forces are together equipollent to the final momenta of the system. Since the time interval $\Delta t = 0.0002$ s is very short, we neglect all nonimpulsive forces and consider only the external impulses $\mathbf{R}_x \Delta t$ and $\mathbf{R}_y \Delta t$.



$$+\circlearrowright \text{ moments about } O: \quad m_A(v_x)_1(2\text{ ft}) + 0 = \bar{I}\omega_2 + m_B\bar{v}_2(2\text{ ft}) \qquad (1)$$

$$\xrightarrow{+} x \text{ components:} \quad m_A(v_x)_1 + R_x\,\Delta t = m_B\bar{v}_2 \qquad (2)$$

$$+\uparrow y \text{ components:} \quad -m_A(v_y)_1 + R_y\,\Delta t = 0 \qquad (3)$$

The components of the velocity of the bullet and the centroidal moment of inertia of the beam are

$$(v_x)_1 = \tfrac{4}{5}(1500\text{ ft/s}) = 1200\text{ ft/s} \qquad (v_y)_1 = \tfrac{3}{5}(1500\text{ ft/s}) = 900\text{ ft/s}$$

$$\bar{I} = \tfrac{1}{12}ml^2 = \frac{1}{12}\frac{50\text{ lb}}{32.2\text{ ft/s}^2}(4\text{ ft})^2 = 2.07\text{ lb}\cdot\text{ft}\cdot\text{s}^2$$

Substituting these values into (1) and noting that $\bar{v}_2 = (2\text{ ft})\omega_2$:

$$(0.125/32.2)(1200)(2) = 2.07\omega_2 + (50/32.2)(2\omega_2)(2)$$

$$\omega_2 = 1.125\text{ rad/s} \qquad \boldsymbol{\omega}_2 = 1.125\text{ rad/s}\circlearrowright \blacktriangleleft$$

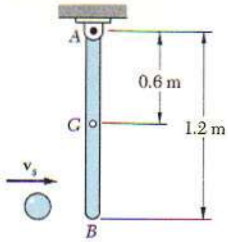
Substituting $\bar{v}_2 = (2\text{ ft})(1.125\text{ rad/s}) = 2.25$ ft/s into (2), we solve Eqs. (2) and (3) for R_x and R_y, respectively.

$$(0.125/32.2)(1200) + R_x(0.0002) = (50/32.2)(2.25)$$

$$R_x = -5820\text{ lb} \qquad \mathbf{R}_x = 5820\text{ lb}\leftarrow \blacktriangleleft$$

$$-(0.125/32.2)(900) + R_y(0.0002) = 0$$

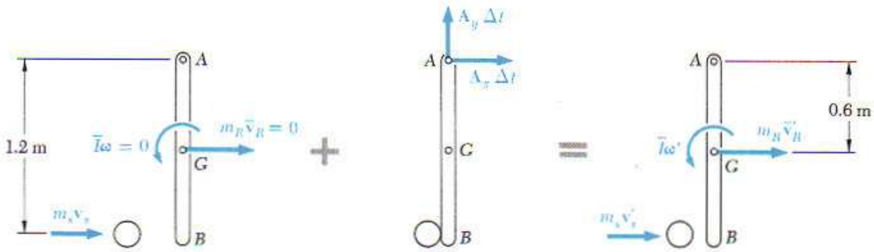
$$R_y = +17{,}470\text{ lb} \qquad \mathbf{R}_y = 17{,}470\text{ lb}\uparrow \blacktriangleleft$$



SAMPLE PROBLEM 17.10

A 2-kg sphere moving horizontally to the right with an initial velocity of 5 m/s strikes the lower end of an 8-kg rigid rod AB. The rod is suspended from a hinge at A and is initially at rest. Knowing that the coefficient of restitution between the rod and sphere is 0.80, determine the angular velocity of the rod and the velocity of the sphere immediately after the impact.

Principle of Impulse and Momentum. We consider the rod and sphere as a single system and express that the initial momenta of the rod and sphere and the impulses of the external forces are together equipollent to the final momenta of the system. We note that the only impulsive force external to the system is the impulsive reaction at A.



$+\circlearrowleft$ moments about A:

$$m_s v_s(1.2\text{ m}) = m_s v_s'(1.2\text{ m}) + m_R \bar{v}_R'(0.6\text{ m}) + \bar{I}\omega' \quad (1)$$

Since the rod rotates about A, we have $\bar{v}_R' = \bar{r}\omega' = (0.6\text{ m})\omega'$. Also,

$$\bar{I} = \tfrac{1}{12}mL^2 = \tfrac{1}{12}(8\text{ kg})(1.2\text{ m})^2 = 0.96\text{ kg}\cdot\text{m}^2$$

Substituting these values and the given data into Eq. (1), we have

$$(2\text{ kg})(5\text{ m/s})(1.2\text{ m}) = (2\text{ kg})v_s'(1.2\text{ m}) + (8\text{ kg})(0.6\text{ m})\omega'(0.6\text{ m}) + (0.96\text{ kg}\cdot\text{m}^2)\omega'$$

$$12 = 2.4v_s' + 3.84\omega' \quad (2)$$

Relative Velocities. Choosing positive to the right, we write

$$v_B' - v_s' = e(v_s - v_B)$$

Substituting $v_s = 5\text{ m/s}$, $v_B = 0$, and $e = 0.80$, we obtain

$$v_B' - v_s' = 0.80(5\text{ m/s}) \quad (3)$$

Again noting that the rod rotates about A, we write

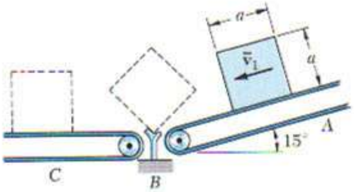
$$v_B' = (1.2\text{ m})\omega' \quad (4)$$

Solving Eqs. (2) to (4) simultaneously, we obtain

$\omega' = +3.21\text{ rad/s}$ $\qquad$ $\omega' = 3.21\text{ rad/s}\circlearrowleft$ ◄

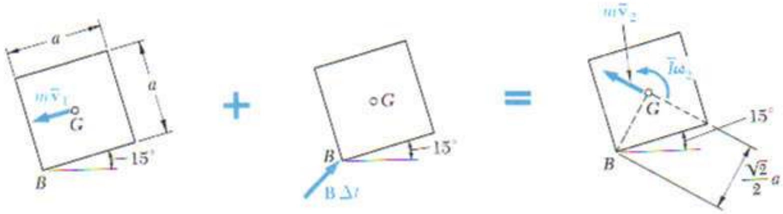
$v_s' = -0.143\text{ m/s}$ $\qquad$ $\mathbf{v}_s' = 0.143\text{ m/s}\leftarrow$ ◄

SAMPLE PROBLEM 17.11



A square package of side a and mass m moves down a conveyor belt A with a constant velocity $\bar{\mathbf{v}}_1$. At the end of the conveyor belt, the corner of the package strikes a rigid support at B. Assuming that the impact at B is perfectly plastic, derive an expression for the smallest magnitude of the velocity $\bar{\mathbf{v}}_1$ for which the package will rotate about B and reach conveyor belt C.

Principle of Impulse and Momentum. Since the impact between the package and the support is perfectly plastic, the package rotates about B during the impact. We apply the principle of impulse and momentum to the package and note that the only impulsive force external to the package is the impulsive reaction at B.



$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1\to2} = \textbf{Syst Momenta}_2$$

$+\circlearrowleft$ moments about B: $\quad (m\bar{v}_1)(\tfrac{1}{2}a) + 0 = (m\bar{v}_2)(\tfrac{1}{2}\sqrt{2}a) + \bar{I}\omega_2 \qquad (1)$

Since the package rotates about B, we have $\bar{v}_2 = (GB)\omega_2 = \frac{1}{2}\sqrt{2}a\omega_2$. We substitute this expression, together with $\bar{I} = \frac{1}{6}ma^2$, into Eq. (1):

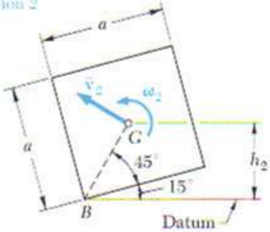
$$(m\bar{v}_1)(\tfrac{1}{2}a) = m(\tfrac{1}{2}\sqrt{2}a\omega_2)(\tfrac{1}{2}\sqrt{2}a) + \tfrac{1}{6}ma^2\omega_2 \qquad \bar{v}_1 = \tfrac{4}{3}a\omega_2 \quad (2)$$

Principle of Conservation of Energy. We apply the principle of conservation of energy between position 2 and position 3.

Position 2. $V_2 = Wh_2$. Recalling that $\bar{v}_2 = \frac{1}{2}\sqrt{2}a\omega_2$, we write

$$T_2 = \tfrac{1}{2}m\bar{v}_2^2 + \tfrac{1}{2}\bar{I}\omega_2^2 = \tfrac{1}{2}m(\tfrac{1}{2}\sqrt{2}a\omega_2)^2 + \tfrac{1}{2}(\tfrac{1}{6}ma^2)\omega_2^2 = \tfrac{1}{3}ma^2\omega_2^2$$

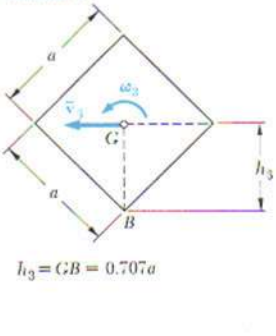
Position 2



$GB = \frac{1}{2}\sqrt{2}a = 0.707a$
$h_2 = GB\sin(45° + 15°)$
$= 0.612a$

Position 3. Since the package must reach conveyor belt B, it must pass through position 3 where G is directly above B. Also, since we wish to determine the smallest velocity for which the package will reach this position, we choose $\bar{v}_3 = \omega_3 = 0$. Therefore $T_3 = 0$ and $V_3 = Wh_3$.

Position 3



$h_3 = GB = 0.707a$

Conservation of Energy

$$T_2 + V_2 = T_3 + V_3$$
$$\tfrac{1}{3}ma^2\omega_2^2 + Wh_2 = 0 + Wh_3$$
$$\omega_2^2 = \frac{3W}{ma^2}(h_3 - h_2) = \frac{3g}{a^2}(h_3 - h_2) \qquad (3)$$

Substituting the computed values of h_2 and h_3 into Eq. (3), we obtain

$$\omega_2^2 = \frac{3g}{a^2}(0.707a - 0.612a) = \frac{3g}{a^2}(0.095a) \qquad \omega_2 = \sqrt{0.285g/a}$$

$$\bar{v}_1 = \tfrac{4}{3}a\omega_2 = \tfrac{4}{3}a\sqrt{0.285g/a} \qquad \bar{v}_1 = 0.712\sqrt{ga} \quad ◄$$

PROBLEMS

17.86 A 45-g bullet is fired with a horizontal velocity of 400 m/s into a 9-kg square panel of side $b = 200$ mm. Knowing that $h = 200$ mm and that the panel is initially at rest, determine (*a*) the velocity of the center of the panel immediately after the bullet becomes embedded, (*b*) the impulsive reaction at A, assuming that the bullet becomes embedded in 1 ms.

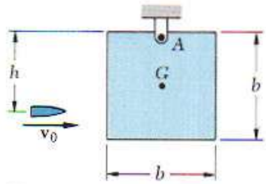


Fig. P17.86

17.87 In Prob. 17.86, determine (*a*) the required distance h if the impulsive reaction at A is to be zero, (*b*) the corresponding velocity of the center of the panel after the bullet becomes embedded.

17.88 A bullet weighing 0.08 lb is fired with a horizontal velocity of 1800 ft/s into the 15-lb wooden rod AB of length $L = 30$ in. The rod, which is initially at rest, is suspended by a cord of length $L = 30$ in. Knowing that $h = 6$ in., determine the velocity of each end of the rod immediately after the bullet becomes embedded.

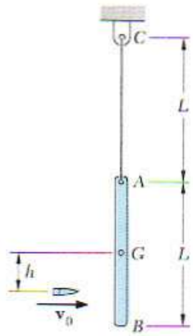


Fig. P17.88

17.89 In Prob. 17.88, determine the distance h for which, immediately after the bullet becomes embedded, the instantaneous center of rotation of the rod is point C.

17.90 A bullet of mass m is fired with a horizontal velocity $\mathbf{v}_0$ and at a height $h = \frac{1}{2}R$ into a wooden disk of much larger mass M and radius R. The disk rests on a horizontal plane and the coefficient of friction between the disk and the plane is finite. (*a*) Determine the linear velocity $\overline{\mathbf{v}}_1$ and the angular velocity $\boldsymbol{\omega}_1$ of the disk immediately after the bullet has penetrated the disk. (*b*) Describe the ensuing motion of the disk and determine its linear velocity after the motion has become uniform.

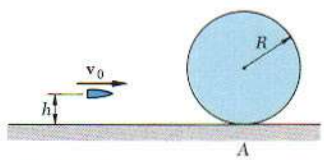


Fig. P17.90

17.91 Determine the height h at which the bullet of Prob. 17.90 should be fired (*a*) if the disk is to roll without sliding immediately after impact, (*b*) if the disk is to slide without rolling immediately after impact.

17.92 A uniform slender rod AB is equipped at both ends with the hooks shown and is supported by a frictionless horizontal table. Initially the rod is hooked at A to a fixed pin C about which it rotates with the constant angular velocity $\boldsymbol{\omega}_1$. Suddenly end B of the rod hits and gets hooked to the pin D, causing end A to be released. Determine the magnitude of the angular velocity $\boldsymbol{\omega}_2$ of the rod in its subsequent rotation about D.

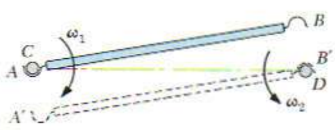


Fig. P17.92

17.93 A uniform disk of radius r and mass m is supported by a frictionless horizontal table. Initially the disk is spinning freely about its mass center G with a constant angular velocity ω_1. Suddenly a latch B is moved to the right and is struck by a small stop A welded to the edge of the disk. Assuming that the impact of A and B is perfectly plastic, determine the angular velocity of the disk and the velocity of its mass center immediately after impact.

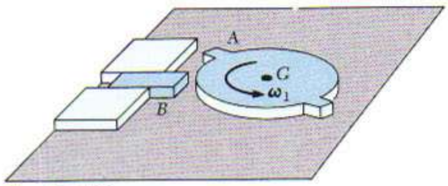


Fig. P17.93

17.94 Solve Prob. 17.93, assuming that the impact of A and B is perfectly elastic.

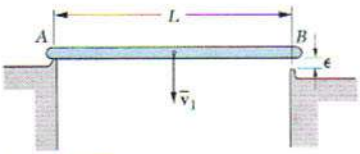


Fig. P17.95

17.95 A uniform slender rod of length L is dropped onto rigid supports at A and B. Immediately before striking A the velocity of the rod is $\bar{\mathbf{v}}_1$. Since support B is slightly lower than support A, the rod strikes A before it strikes B. Assuming perfectly elastic impact at both A and B, determine the angular velocity of the rod and the velocity of its mass center immediately after the rod (a) strikes support A, (b) strikes support B, (c) again strikes support A.

17.96 A square block of mass m moves along a frictionless horizontal surface and strikes a small obstruction at B. Assuming that the impact between corner A and the obstruction B is perfectly plastic, determine the angular velocity of the block and the velocity of its mass center G immediately after the impact.

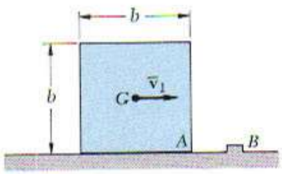


Fig. P17.96

17.97 Solve Prob. 17.96, assuming that the impact between corner A and the obstruction B is perfectly elastic.

17.98 A uniformly loaded square crate is released from rest with its corner D directly above A; it rotates about A until its corner B strikes the floor, and then rotates about B. The floor is sufficiently rough to prevent slipping and the impact at B is perfectly plastic. Denoting by ω_0 the angular velocity of the crate immediately before B strikes the floor, determine (*a*) the angular velocity of the crate immediately after B strikes the floor, (*b*) the fraction of the kinetic energy of the crate lost during the impact, (*c*) the angle θ through which the crate will rotate after B strikes the floor.

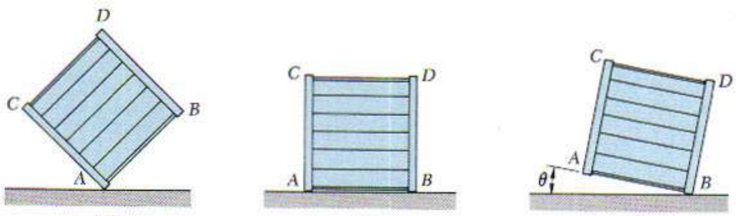


Fig. P17.98

17.99 A uniform sphere of radius r rolls without slipping down the incline shown. It hits the horizontal surface and, after slipping for a while, starts rolling again. Assuming that the sphere does not bounce as it hits the horizontal surface, determine its angular velocity and the velocity of its mass center after it has resumed rolling.

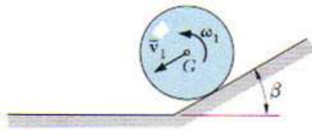


Fig. P17.99

17.100 A sphere A of mass m and radius r rolls without slipping with a velocity v_0 on a horizontal plane. It hits squarely an identical sphere B which is at rest. Denoting by μ the coefficient of friction between the spheres and the plane, neglecting the friction between the spheres, and assuming perfectly elastic impact ($e = 1$), determine (*a*) the linear and angular velocity of each sphere immediately after impact, (*b*) the velocity of each sphere after it has started rolling uniformly. (*c*) Discuss the special case when $\mu = 0$.

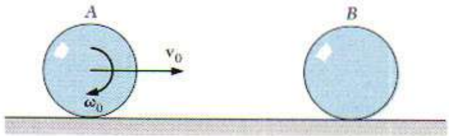


Fig. P17.100

17.101 A slender rod of length l strikes a frictionless floor at A with a vertical velocity $\bar{v}_1$ and no angular velocity. Assuming that the impact at A is perfectly elastic, derive an expression for the angular velocity of the rod immediately after impact.

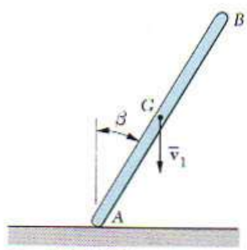


Fig. P17.101

17.102 Solve Prob. 17.101, assuming that the impact at A is perfectly plastic.

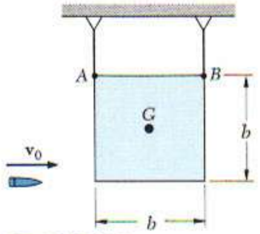


Fig. P17.103

17.103 A bullet of mass m is fired with a horizontal velocity $\mathbf{v}_0$ into the lower corner of a square panel of much larger mass M. The panel is held by two vertical wires as shown. Determine the velocity of the center G of the panel immediately after the bullet becomes embedded.

17.104 Two uniform rods, each of mass m, form the L-shaped rigid body ABC which is initially at rest on the frictionless horizontal surface when hook D of the carriage E engages a small pin at C. Knowing that the carriage is pulled to the right with a constant velocity $\mathbf{v}_0$, determine immediately after the impact (*a*) the angular velocity of the body, (*b*) the velocity of corner B. Assume that the velocity of the carriage is unchanged and that the impact is perfectly plastic.

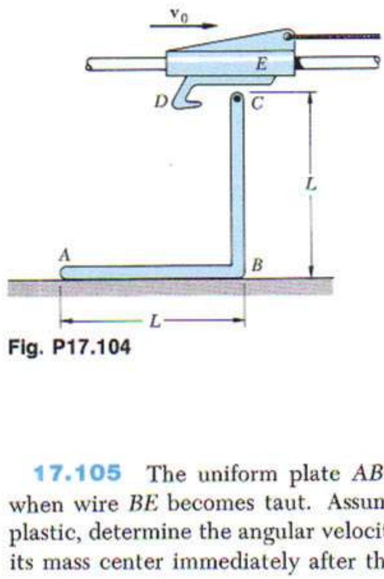


Fig. P17.104

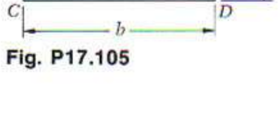


Fig. P17.105

17.105 The uniform plate $ABCD$ is falling with a velocity $\mathbf{v}_1$ when wire BE becomes taut. Assuming that the impact is perfectly plastic, determine the angular velocity of the plate and the velocity of its mass center immediately after the impact.

17.106 In Prob. 17.96, determine the line of action of the impulsive force exerted on the block by the obstruction at B.

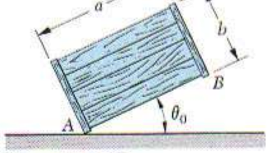


Fig. P17.107

17.107 A uniformly loaded rectangular crate is released from rest in the position shown. Assuming that the floor is sufficiently rough to prevent slipping and that the impact at B is perfectly plastic, determine the largest value of the ratio b/a for which corner A will remain in contact with the floor.

17.108 A slender rod of mass m and length l is held in the position shown. Roller B is given a slight push to the right and moves along the horizontal plane, while roller A is constrained to move vertically. Determine the magnitudes of the impulses exerted on the rollers A and B as roller A strikes the ground. Assume perfectly plastic impact.

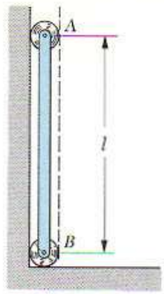


Fig. P17.108

17.109 In a game of billiards, ball A is rolling without slipping with a velocity $\mathbf{v}_0$ as it hits obliquely ball B which is at rest. Denoting by r the radius of each ball, by μ the coefficient of friction between the balls and the table, neglecting friction between the balls, and assuming perfectly elastic impact ($e = 1$), determine (a) the linear and angular velocity of each ball immediately after impact, (b) the velocity of B after it has started rolling uniformly.

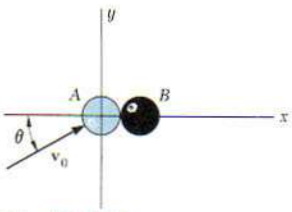


Fig. P17.109

17.110 In Prob. 17.109, determine the equation of the path described by the center of ball A while the ball is slipping.

17.111 For the billiard balls of Prob. 17.109, determine (a) the velocity of ball A after it has started rolling again without slipping, (b) the angle ϕ formed by the velocities of balls A and B after they have finished slipping. (Compare the result obtained here with the one obtained for the pucks of Prob. 13.139 when $e = 1$.)

17.112 A small rubber ball of radius r is thrown against a rough floor with a velocity $\mathbf{v}_A$ of magnitude v_0 and a "backspin" $\boldsymbol{\omega}_A$ of magnitude ω_0. It is observed that the ball bounces from A to B, then from B to A, then from A to B, etc. Assuming perfectly elastic impact, determine (a) the required magnitude ω_0 of the "backspin" in terms of v_0 and r, (b) the minimum required value of the coefficient of friction.

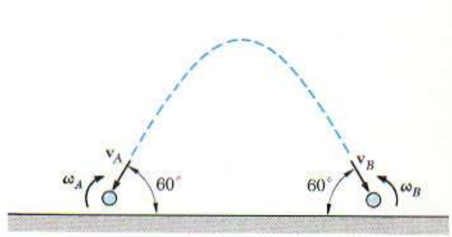


Fig. P17.112

17.113 Two identical rods AB and CD, each of length L, may move freely on a frictionless horizontal surface. Rod AB is rotating about its mass center with an angular velocity ω_0 when end B strikes end C of rod CD, which is at rest. Knowing that at the instant of impact the rods are parallel and assuming perfectly elastic impact ($e = 1$), determine the angular velocity of each rod and the velocity of its mass center immediately after impact.

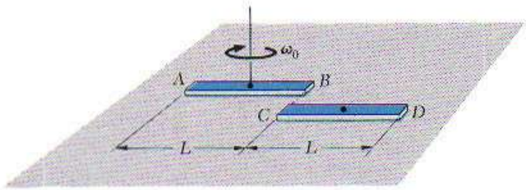


Fig. P17.113

17.114 Solve Prob. 17.113, assuming that the impact is perfectly plastic ($e = 0$).

REVIEW PROBLEMS

17.115 A small disk A is driven at a constant angular velocity of 1200 rpm and is pressed against disk B, which is initially at rest. The normal force between disks is 10 lb, and $\mu_k = 0.20$. Knowing that disk B weighs 50 lb, determine the number of revolutions executed by disk B before its speed reaches 120 rpm.

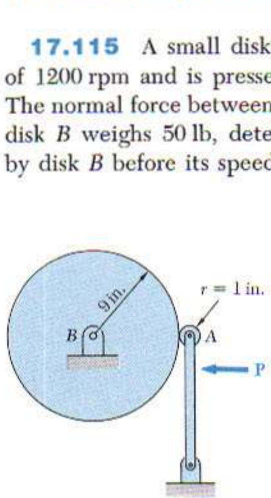


Fig. P17.115

17.116 A small collar of mass m is attached at B to the rim of a hoop of mass m and radius r. The hoop rolls without sliding on a horizontal plane. Find the angular velocity ω_1 of the hoop when B is directly above the center A in terms of g and r, knowing that the angular velocity of the hoop is $3\omega_1$ when B is directly below A.

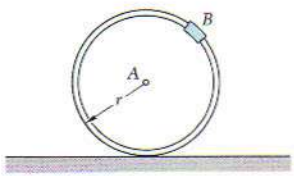


Fig. P17.116

17.117 The gear train shown consists of four gears of the same thickness and of the same material; two gears are of radius r, and the other two are of radius nr. The system is at rest when the couple $\mathbf{M}_0$ is applied to shaft C. Denoting by I_0 the moment of inertia of a gear of radius r, determine the angular velocity of shaft A if the couple $\mathbf{M}_0$ is applied (*a*) for one revolution of shaft C, (*b*) for t seconds.

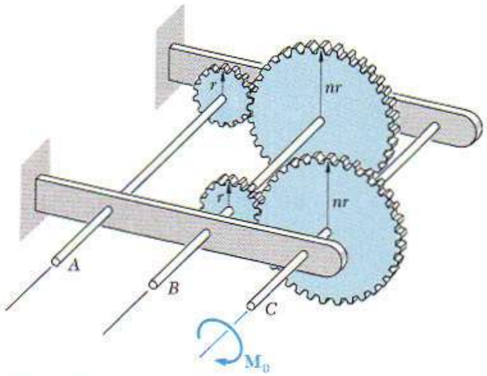


Fig. P17.117

17.118 The motion of a 16-kg sliding panel is guided by rollers at B and C. The counterweight A has a mass of 12 kg and is attached to a cable as shown. If the system is released from rest, determine for each case shown the velocity of the counterweight as it strikes the ground. Neglect the effect of friction.

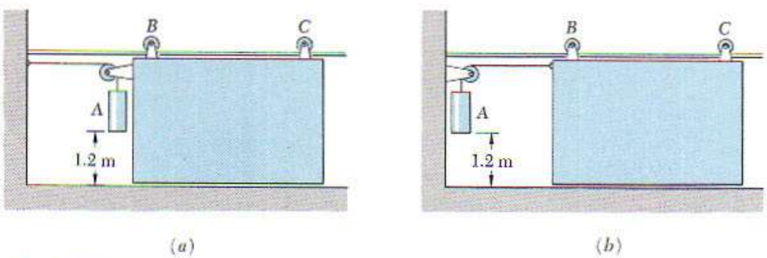


Fig. P17.118

17.119 A uniform rod of length L and weight W is attached to two wires, each of length b. The rod is released from rest when $\theta = 0$ and swings to the position $\theta = 90°$, at which time wire BD suddenly breaks. Determine the tension in wire AC (*a*) immediately before wire BD breaks, (*b*) immediately after wire BD breaks.

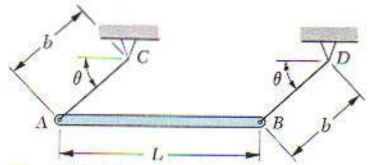


Fig. P17.119

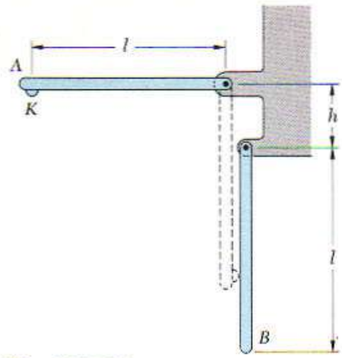


Fig. P17.120

17.120 Two identical slender rods may swing freely from the pivots shown. Rod A is released from rest in a horizontal position and swings to a vertical position, at which time the small knob K strikes rod B which was at rest. If $h = \frac{1}{2}l$ and $e = \frac{1}{2}$, determine (*a*) the angle through which rod B will swing, (*b*) the angle through which rod A will rebound.

17.121 Solve Prob. 17.120, assuming $e = 1$.

17.122 The motor shown runs a machine attached to the shaft at A. The motor develops 4 hp and runs at a constant speed of 300 rpm. Determine the magnitude of the couple exerted (*a*) by the shaft on pulley A, (*b*) by the motor on pulley B.

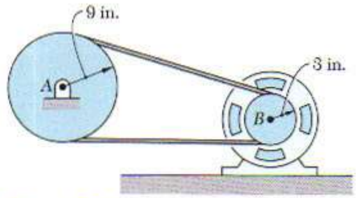


Fig. P17.122

17.123 The plank CDE of mass m_P rests on top of a small pivot at D. A gymnast A of mass m stands on the plank at end C; a second gymnast B of the same mass m jumps from a height h and strikes the plank at E. Assuming perfectly plastic impact, determine the height to which gymnast A will rise. (Assume that gymnast A stands completely rigid.)

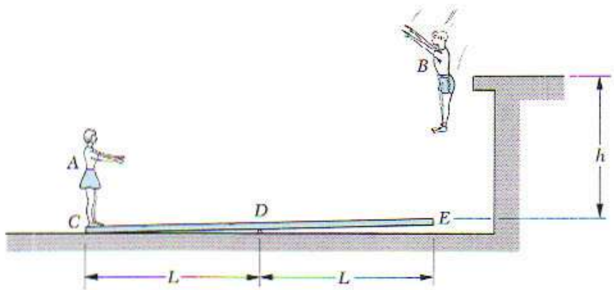


Fig. P17.123

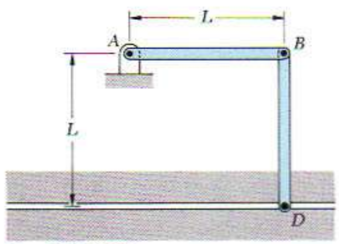


Fig. P17.124

17.124 Two uniform rods, each of mass m and length L, are connected to form the linkage shown. End D of rod BD may slide freely in the horizontal slot, while end A of rod AB is attached to a fixed pin support. If end D is moved slightly to the left and then released, determine its velocity (*a*) when D is directly below A, (*b*) when rod AB is vertical.

17.125 A rectangular slab of mass m_S moves across a series of rollers, each of which is equivalent to a uniform disk of mass m_R and is initially at rest. Since the length of the slab is slightly less than three times the distance b between two adjacent rollers, the slab leaves a roller just before it reaches another one. Each time a new roller enters into contact with the slab, slipping occurs between the roller and the slab for a short period of time (less than the time needed for the slab to move through the distance b). Denoting by $\mathbf{v}_0$ the velocity of the slab in the position shown, determine the velocity of the slab after it has moved (*a*) a distance b, (*b*) a distance nb.

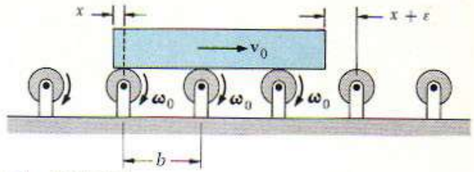


Fig. P17.125

17.126 Solve Prob. 17.125, assuming $v_0 = 5$ m/s, $m_S = 17$ kg, $m_R = 2$ kg, and $n = 5$.

CHAPTER 18 Kinetics of Rigid Bodies in Three Dimensions

***18.1. Introduction.** In Chaps. 16 and 17 we were concerned with the plane motion of rigid bodies and of systems of rigid bodies. In Chap. 16 and in the second half of Chap. 17 (momentum method), our study was further restricted to that of plane slabs and of bodies symmetrical with respect to the reference plane. However, many of the fundamental results obtained in these two chapters remain valid in the case of the motion of a rigid body in three dimensions.

For example, the two fundamental equations

$$\Sigma \mathbf{F} = m\overline{\mathbf{a}} \tag{18.1}$$
$$\Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G \tag{18.2}$$

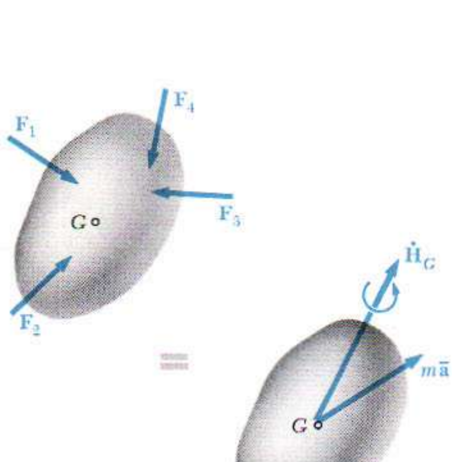


Fig. 18.1

on which the analysis of the plane motion of a rigid body was based, remain valid in the most general case of motion of a rigid body. As it was indicated in Sec. 16.2, these equations express that the system of the external forces is equipollent to the system consisting of the vector $m\overline{\mathbf{a}}$ attached at G and the couple of moment $\dot{\mathbf{H}}_G$ (Fig. 18.1). However, the relation $\mathbf{H}_G = \overline{I}\boldsymbol{\omega}$, which enabled us to determine the angular momentum of a rigid slab and which played an important part in the solution of problems involving the plane motion of slabs and bodies symmetrical with respect to the reference plane, ceases to be valid in the case of nonsymmetrical bodies or three-dimensional motion. It will thus be necessary for us to develop in Sec. 18.2 a more general method for the computation of the angular momentum $\mathbf{H}_G$ of a rigid body in three dimensions.

Similarly, the main feature of the impulse-momentum method discussed in Sec. 17.7, namely the reduction of the momenta of the particles of a rigid body to a linear momentum vector $m\overline{\mathbf{v}}$ attached at the mass center G of the body and an angular momentum couple $\mathbf{H}_G$, remains valid. Here again, however, the relation $\mathbf{H}_G = \bar{I}\boldsymbol{\omega}$ will have to be discarded and replaced by the more general relation to be developed in Sec. 18.2.

Finally, we note that the work-energy principle (Sec. 17.1) and the principle of conservation of energy (Sec. 17.5) still apply in the case of the motion of a rigid body in three dimensions. However, the expression obtained in Sec. 17.3 for the kinetic energy of a rigid body in plane motion will be replaced by a new expression to be developed in Sec. 18.4 for a rigid body in three-dimensional motion.

***18.2. Angular Momentum of a Rigid Body in Three Dimensions.** We shall see in this section how the angular momentum $\mathbf{H}_G$ of the body about its mass center G may be determined from the angular velocity $\boldsymbol{\omega}$ of the body in the case of three-dimensional motion.

According to Eq. (14.24), the angular momentum of the body about G may be expressed as

$$\mathbf{H}_G = \sum_{i=1}^{n} (\mathbf{r}'_i \times \mathbf{v}'_i \, \Delta m_i) \tag{18.3}$$

where $\mathbf{r}'_i$ and $\mathbf{v}'_i$ denote, respectively, the position vector and the velocity of the particle P_i, of mass Δm_i, relative to the centroidal frame $Gxyz$ (Fig. 18.2). But $\mathbf{v}'_i = \boldsymbol{\omega} \times \mathbf{r}'_i$, where $\boldsymbol{\omega}$ is the angular velocity of the body at the instant considered. Substituting into (18.3), we have

$$\mathbf{H}_G = \sum_{i=1}^{n} [\mathbf{r}'_i \times (\boldsymbol{\omega} \times \mathbf{r}'_i) \, \Delta m_i]$$

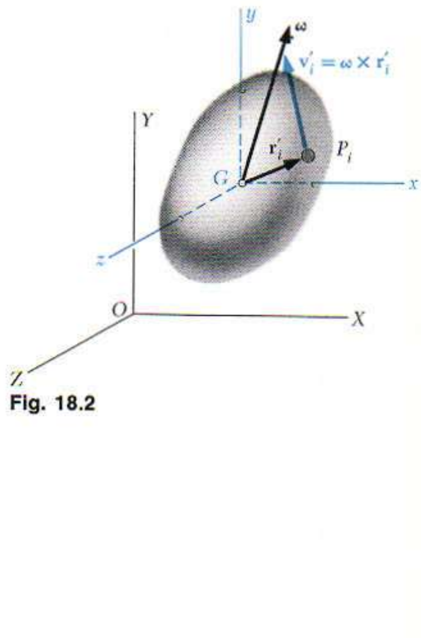


Fig. 18.2

Recalling the rule for determining the rectangular components of a vector product (Sec. 3.4), we obtain the following expression for the x component of the angular momentum:

$$\begin{aligned}
H_x &= \sum_{i=1}^{n} [\, y_i(\boldsymbol{\omega} \times \mathbf{r}'_i)_z - z_i(\boldsymbol{\omega} \times \mathbf{r}'_i)_y] \, \Delta m_i \\
&= \sum_{i=1}^{n} [\, y_i(\omega_x y_i - \omega_y x_i) - z_i(\omega_z x_i - \omega_x z_i)] \, \Delta m_i \\
&= \omega_x \sum_i (y_i^2 + z_i^2) \, \Delta m_i - \omega_y \sum_i x_i y_i \, \Delta m_i - \omega_z \sum_i z_i x_i \, \Delta m_i
\end{aligned}$$

Replacing the sums by integrals in this expression and in the two similar expressions which are obtained for H_y and H_z, we have

$$\begin{aligned} H_x &= \omega_x \int (y^2 + z^2)\, dm - \omega_y \int xy\, dm - \omega_z \int zx\, dm \\ H_y &= -\omega_x \int xy\, dm + \omega_y \int (z^2 + x^2)\, dm - \omega_z \int yz\, dm \\ H_z &= -\omega_x \int zx\, dm - \omega_y \int yz\, dm + \omega_z \int (x^2 + y^2)\, dm \end{aligned} \tag{18.4}$$

We note that the integrals containing squares represent the *centroidal mass moments of inertia* of the body about the x, y, and z axes, respectively (Sec. 9.10); we have

$$\bar{I}_x = \int (y^2 + z^2)\, dm \qquad \bar{I}_y = \int (z^2 + x^2)\, dm \qquad \bar{I}_z = \int (x^2 + y^2)\, dm \tag{18.5}$$

Similarly, the integrals containing products of coordinates represent the *centroidal mass products of inertia* of the body (Sec. 9.15); we have

$$\bar{P}_{xy} = \int xy\, dm \qquad \bar{P}_{yz} = \int yz\, dm \qquad \bar{P}_{zx} = \int zx\, dm \tag{18.6}$$

Substituting from (18.5) and (18.6) into (18.4), we obtain the components of the angular momentum $\mathbf{H}_G$ of the body about its mass center G:

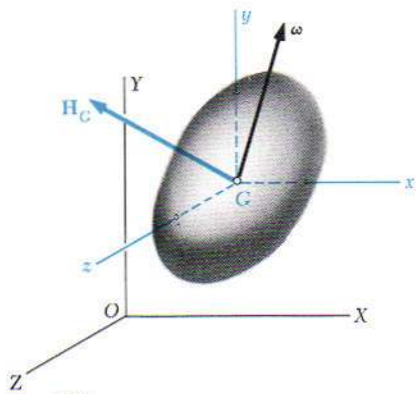


Fig. 18.3

$$\begin{aligned} H_x &= +\bar{I}_x\omega_x - \bar{P}_{xy}\omega_y - \bar{P}_{xz}\omega_z \\ H_y &= -\bar{P}_{yx}\omega_x + \bar{I}_y\omega_y - \bar{P}_{yz}\omega_z \\ H_z &= -\bar{P}_{zx}\omega_x - \bar{P}_{zy}\omega_y + \bar{I}_z\omega_z \end{aligned} \tag{18.7}$$

The relations (18.7) show that the operation which transforms the vector $\boldsymbol{\omega}$ into the vector $\mathbf{H}_G$ (Fig. 18.3) is characterized by the array of moments and products of inertia

$$\begin{pmatrix} \bar{I}_x & -\bar{P}_{xy} & -\bar{P}_{xz} \\ -\bar{P}_{yx} & \bar{I}_y & -\bar{P}_{yz} \\ -\bar{P}_{zx} & -\bar{P}_{zy} & \bar{I}_z \end{pmatrix} \tag{18.8}$$

The array (18.8) defines the *inertia tensor* of the body at its mass center G.† A new array of moments and products of inertia

† Setting $\bar{I}_x = I_{11}$, $\bar{I}_y = I_{22}$, $\bar{I}_z = I_{33}$, and $-\bar{P}_{xy} = I_{12}$, $-\bar{P}_{xz} = I_{13}$, etc., we may write the inertia tensor in the standard form

$$\begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}$$

Denoting by H_1, H_2, H_3 the components of the angular momentum $\mathbf{H}_G$ and by ω_1, ω_2, ω_3 the components of the angular velocity $\boldsymbol{\omega}$, we may write the relations (18.7) in the form

$$H_i = \sum_j I_{ij}\omega_j$$

where i and j take the values 1, 2, 3. The quantities I_{ij} are said to be the *components* of the inertia tensor. Since $I_{ij} = I_{ji}$, the inertia tensor is a *symmetric tensor of the second order.*

would be obtained if a different system of axes were used. The transformation characterized by this new array, however, would still be the same. Clearly, the angular momentum $\mathbf{H}_G$ corresponding to a given angular velocity $\boldsymbol{\omega}$ is independent of the choice of the coordinate axes. As it was shown in Sec. 9.16, it is always possible to select a system of axes $Gx'y'z'$, called *principal axes of inertia*, with respect to which all the products of inertia of a given body are zero. The array (18.8) takes then the diagonalized form

$$\begin{pmatrix} \bar{I}_{x'} & 0 & 0 \\ 0 & \bar{I}_{y'} & 0 \\ 0 & 0 & \bar{I}_{z'} \end{pmatrix} \tag{18.9}$$

where $\bar{I}_{x'}$, $\bar{I}_{y'}$, $\bar{I}_{z'}$ represent the *principal centroidal moments of inertia* of the body, and the relations (18.7) reduce to

$$H_{x'} = \bar{I}_{x'}\omega_{x'} \qquad H_{y'} = \bar{I}_{y'}\omega_{y'} \qquad H_{z'} = \bar{I}_{z'}\omega_{z'} \tag{18.10}$$

We note that, if the three principal centroidal moments of inertia $\bar{I}_{x'}$, $\bar{I}_{y'}$, $\bar{I}_{z'}$ are equal, the components $H_{x'}$, $H_{y'}$, $H_{z'}$ of the angular momentum about G are proportional to the components $\omega_{x'}$, $\omega_{y'}$, $\omega_{z'}$ of the angular velocity, and the vectors $\mathbf{H}_G$ and $\boldsymbol{\omega}$ are collinear. In general, however, the principal moments of inertia will be different, and the vectors $\mathbf{H}_G$ and $\boldsymbol{\omega}$ *will have different directions*, except when two of the three components of $\boldsymbol{\omega}$ happen to be zero, i.e., when $\boldsymbol{\omega}$ is directed along one of the coordinate axes. Thus, *the angular momentum* $\mathbf{H}_G$ *of a rigid body and its angular velocity* $\boldsymbol{\omega}$ *have the same direction if, and only if,* $\boldsymbol{\omega}$ *is directed along a principal axis of inertia.*† Since this condition is satisfied in the case of the plane motion of a rigid body symmetrical with respect to the reference plane, we were able in Secs. 16.3 and 17.7 to represent the angular momentum $\mathbf{H}_G$ of such a body by the vector $\bar{I}\boldsymbol{\omega}$. We must realize, however, that this result cannot be extended to the case of the plane motion of a nonsymmetrical body, or to the case of the three-dimensional motion of a rigid body. Except when $\boldsymbol{\omega}$ happens to be directed along a principal axis of inertia, the angular momentum and angular velocity of a rigid body have different directions, and the relation (18.7) or (18.10) must be used to determine $\mathbf{H}_G$ from $\boldsymbol{\omega}$.

† In the particular case when $\bar{I}_{x'} = \bar{I}_{y'} = \bar{I}_{z'}$, any line through G may be considered as a principal axis of inertia, and the vectors $\mathbf{H}_G$ and $\boldsymbol{\omega}$ are always collinear.

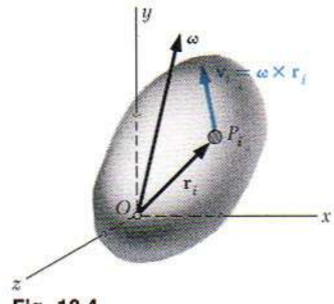


Fig. 18.4

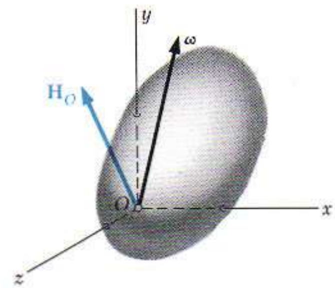


Fig. 18.5

Angular Momentum of a Rigid Body About a Fixed Point. In the particular case of a rigid body rotating in three-dimensional space about a fixed point O (Fig. 18.4), it is sometimes useful to determine the angular momentum $\mathbf{H}_O$ of the body about the fixed point O. Recalling Eq. (14.7), we write

$$\mathbf{H}_O = \sum_{i=1}^{n} (\mathbf{r}_i \times \mathbf{v}_i \, \Delta m_i) \tag{18.11}$$

where $\mathbf{r}_i$ and $\mathbf{v}_i$ denote, respectively, the position vector and the velocity of the particle P_i with respect to the fixed frame $Oxyz$. Substituting $\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i$, and after manipulations similar to the ones used above, we find that the components of the angular momentum $\mathbf{H}_O$ (Fig. 18.5) are given by the relations

$$\begin{aligned} H_x &= +\ I_x\omega_x - P_{xy}\omega_y - P_{xz}\omega_z \\ H_y &= -P_{yx}\omega_x +\ I_y\omega_y - P_{yz}\omega_z \\ H_z &= -P_{zx}\omega_x - P_{zy}\omega_y +\ I_z\omega_z \end{aligned} \tag{18.12}$$

where the moments of inertia I_x, I_y, I_z and the products of inertia P_{xy}, P_{yz}, P_{zx} are computed with respect to the frame $Oxyz$ centered at the fixed point O.

*18.3. Application of the Principle of Impulse and Momentum to the Three-dimensional Motion of a Rigid Body.

Before we can apply the fundamental equation (18.2) to the solution of problems involving the three-dimensional motion of a rigid body, we shall have to learn to compute the derivative of the vector $\mathbf{H}_G$. This will be done in Sec. 18.5. We may, however, immediately use the results obtained in the preceding section to solve problems by the impulse-momentum method.

Recalling from Sec. 17.7 that the system formed by the momenta of the particles of a rigid body reduces to a linear momentum vector $m\bar{\mathbf{v}}$ attached at the mass center G of the body and an angular momentum couple $\mathbf{H}_G$, we represent graphically the fundamental relation

$$\textbf{Syst Momenta}_1 + \textbf{Syst Ext Imp}_{1\to 2} = \textbf{Syst Momenta}_2 \tag{17.14}$$

by means of the three sketches shown in Fig. 18.6. To solve a given problem, we may use these sketches to write appropriate component and moment equations, keeping in mind that the components of the angular momentum $\mathbf{H}_G$ are related to the components of the angular velocity $\boldsymbol{\omega}$ by Eqs. (18.7) of the preceding section.

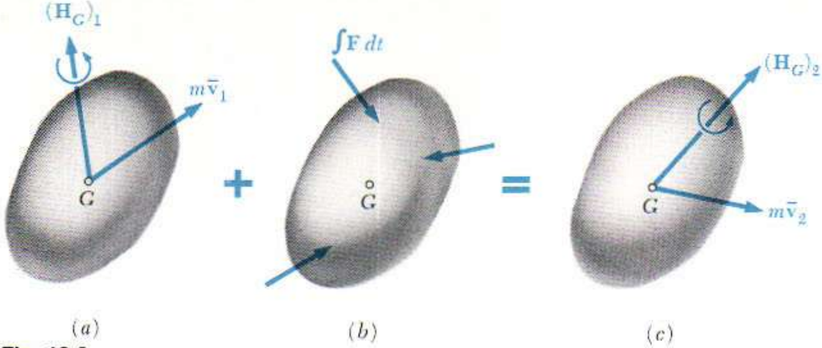


Fig. 18.6

In solving problems dealing with the motion of a body rotating about a fixed point O, it will be convenient to eliminate the impulse of the reaction at O by writing an equation involving the moments of the momenta and impulses about O. We note in this connection that the angular momentum $\mathbf{H}_O$ of the body about the fixed point O may be obtained directly from Eqs. (18.12).

*18.4. Kinetic Energy of a Rigid Body in Three Dimensions.

Consider a rigid body of mass m in three-dimensional motion. We recall from Sec. 14.6 that, if the absolute velocity $\mathbf{v}_i$ of each particle P_i of the body is expressed as the sum of the velocity $\bar{\mathbf{v}}$ of the mass center G of the body and of the velocity $\mathbf{v}'_i$ of the particle relative to a frame $Gxyz$ attached to G and of fixed orientation (Fig. 18.7), the kinetic energy of the system of particles forming the rigid body may be written in the form

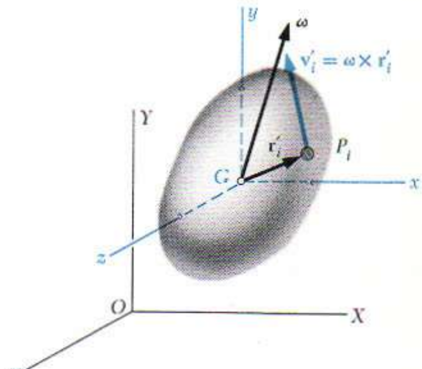


Fig. 18.7

$$T = \tfrac{1}{2}m\bar{v}^2 + \frac{1}{2}\sum_{i=1}^{n}(\Delta m_i)v_i'^2 \tag{18.13}$$

where the last term represents the kinetic energy T' of the body relative to the centroidal frame $Gxyz$. Since $\mathbf{v}'_i = \boldsymbol{\omega} \times \mathbf{r}'_i$, we write

$$T' = \frac{1}{2}\sum_{i=1}^{n}(\Delta m_i)v_i'^2 = \frac{1}{2}\sum_{i=1}^{n}(\boldsymbol{\omega} \times \mathbf{r}'_i)^2\,\Delta m_i \tag{18.14}$$

Expressing the square of the vector product in terms of its rectangular components, and replacing the sums by integrals, we have

$$\begin{aligned} T' &= \int[(\omega_x y - \omega_y x)^2 + (\omega_y z - \omega_z y)^2 + (\omega_z x - \omega_x z)^2]\,dm \\ &= \omega_x^2\int(y^2 + z^2)\,dm + \omega_y^2\int(z^2 + x^2)\,dm + \omega_z^2\int(x^2 + y^2)\,dm \\ &\qquad - 2\omega_x\omega_y\int xy\,dm - 2\omega_y\omega_z\int yz\,dm - 2\omega_z\omega_x\int zx\,dm \end{aligned}$$

or, recalling the relations (18.5) and (18.6),

$$T' = \tfrac{1}{2}(\bar{I}_x\omega_x^2 + \bar{I}_y\omega_y^2 + \bar{I}_z\omega_z^2 - 2\bar{P}_{xy}\omega_x\omega_y - 2\bar{P}_{yz}\omega_y\omega_z - 2\bar{P}_{zx}\omega_z\omega_x) \quad (18.15)$$

Substituting into (18.13) the expression (18.15) we have just obtained for the kinetic energy of the body relative to centroidal axes, we write

$$T = \tfrac{1}{2}m\bar{v}^2 + \tfrac{1}{2}(\bar{I}_x\omega_x^2 + \bar{I}_y\omega_y^2 + \bar{I}_z\omega_z^2 - 2\bar{P}_{xy}\omega_x\omega_y - 2\bar{P}_{yz}\omega_y\omega_z - 2\bar{P}_{zx}\omega_z\omega_x) \quad (18.16)$$

If the axes of coordinates are chosen so that they coincide at the instant considered with the principal axes x', y', z' of the body, the relation obtained reduces to

$$T = \tfrac{1}{2}m\bar{v}^2 + \tfrac{1}{2}(\bar{I}_{x'}\omega_{x'}^2 + \bar{I}_{y'}\omega_{y'}^2 + \bar{I}_{z'}\omega_{z'}^2) \quad (18.17)$$

where $\bar{\mathbf{v}}$ = velocity of mass center
$\boldsymbol{\omega}$ = angular velocity
m = mass of rigid body
$\bar{I}_{x'}, \bar{I}_{y'}, \bar{I}_{z'}$ = principal centroidal moments of inertia

The results we have obtained enable us to extend to the three-dimensional motion of a rigid body the application of the principle of work and energy (Sec. 17.1) and of the principle of conservation of energy (Sec. 17.5).

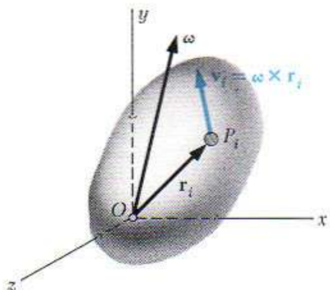


Fig. 18.8

Kinetic Energy of a Rigid Body With a Fixed Point. In the particular case of a rigid body rotating in three-dimensional space about a fixed point O, the kinetic energy of the body may be expressed in terms of its moments and products of inertia with respect to axes attached at O (Fig. 18.8). Recalling the definition of the kinetic energy of a system of particles, and substituting $\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i$, we write

$$T = \frac{1}{2}\sum_{i=1}^{n}(\Delta m_i)v_i^2 = \frac{1}{2}\sum_{i=1}^{n}(\boldsymbol{\omega} \times \mathbf{r}_i)^2\,\Delta m_i \quad (18.18)$$

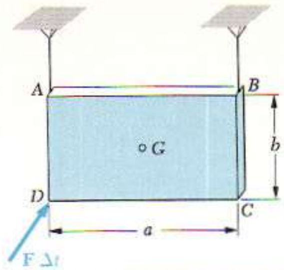
Manipulations similar to those used to derive (18.15) from (18.14) yield

$$T = \tfrac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 - 2P_{xy}\omega_x\omega_y - 2P_{yz}\omega_y\omega_z - 2P_{zx}\omega_z\omega_x) \quad (18.19)$$

or, if the principal axes x', y', z' of the body at the origin O are chosen as coordinate axes,

$$T = \tfrac{1}{2}(I_{x'}\omega_{x'}^2 + I_{y'}\omega_{y'}^2 + I_{z'}\omega_{z'}^2) \quad (18.20)$$

SAMPLE PROBLEM 18.1



A rectangular plate of mass m suspended from two wires at A and B is hit at D in a direction perpendicular to the plate. Denoting by $\mathbf{F}\,\Delta t$ the impulse applied at D, determine immediately after impact (a) the velocity of the mass center G, (b) the angular velocity of the plate.

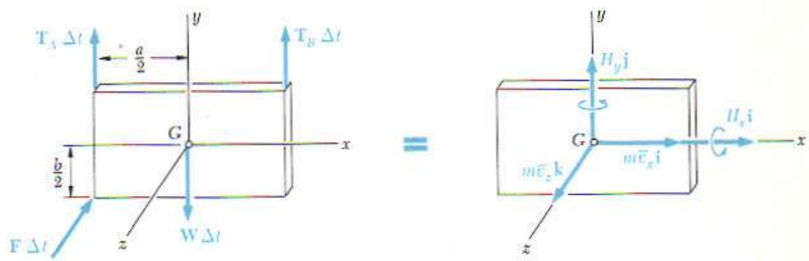
Solution. We shall assume that the wires remain taut and, thus, that the components $\bar{v}_y$ of $\bar{\mathbf{v}}$ and ω_z of $\boldsymbol{\omega}$ are zero after impact. We have therefore

$$\bar{\mathbf{v}} = \bar{v}_x\mathbf{i} + \bar{v}_z\mathbf{k} \qquad \boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j}$$

and, since the x, y, z axes are principal axes of inertia,

$$\mathbf{H}_G = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} \qquad \mathbf{H}_G = \tfrac{1}{12}mb^2\omega_x\mathbf{i} + \tfrac{1}{12}ma^2\omega_y\mathbf{j} \tag{1}$$

Principle of Impulse and Momentum. Since the initial momenta are zero, the system of the impulses must be equivalent to the system of the final momenta:



a. Velocity of Mass Center. Equating the components of the impulses and momenta in the x and z directions:

x comp.: $\quad 0 = m\bar{v}_x \qquad \bar{v}_x = 0$

z comp.: $\quad -F\,\Delta t = m\bar{v}_z \qquad \bar{v}_z = -F\,\Delta t/m$

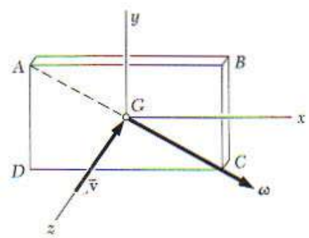
$$\bar{\mathbf{v}} = \bar{v}_x\mathbf{i} + \bar{v}_z\mathbf{k} \qquad \bar{\mathbf{v}} = -(F\,\Delta t/m)\mathbf{k} \quad ◄$$

b. Angular Velocity. Equating the moments of the impulses and momenta about the x and y axes:

About x axis: $\quad \tfrac{1}{2}bF\,\Delta t = H_x$

About y axis: $\quad -\tfrac{1}{2}aF\,\Delta t = H_y$

$$\mathbf{H}_G = H_x\mathbf{i} + H_y\mathbf{j} \qquad \mathbf{H}_G = \tfrac{1}{2}bF\,\Delta t\,\mathbf{i} - \tfrac{1}{2}aF\,\Delta t\,\mathbf{j} \tag{2}$$



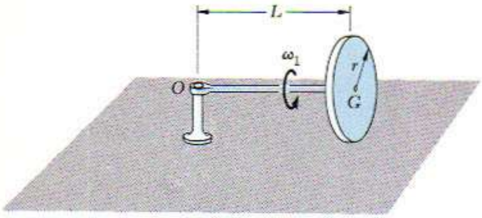
Comparing Eqs. (1) and (2), we conclude that

$$\omega_x = 6F\,\Delta t/mb \qquad \omega_y = -6F\,\Delta t/ma$$
$$\boldsymbol{\omega} = \omega_x\mathbf{i} + \omega_y\mathbf{j} \qquad \boldsymbol{\omega} = (6F\,\Delta t/mab)(a\mathbf{i} - b\mathbf{j}) \quad ◄$$

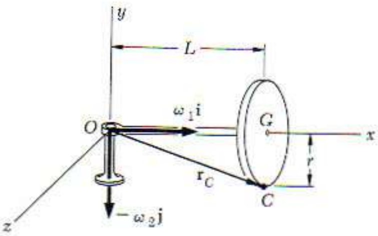
We note that $\boldsymbol{\omega}$ is directed along the diagonal AC.

Remark: Equating the y components of the impulses and momenta, and their moments about the z axis, we obtain two additional equations which yield $T_A = T_B = \tfrac{1}{2}W$. We thus verify that the wires remain taut and that our assumption was correct.

SAMPLE PROBLEM 18.2



A homogeneous disk of radius r and mass m is mounted on an axle OG of length L and negligible mass. The axle is pivoted at the fixed point O, and the disk is constrained to roll on a horizontal floor. Knowing that the disk rotates counterclockwise at the rate ω_1 about the axle OG, determine (*a*) the angular velocity of the disk, (*b*) its angular momentum about O, (*c*) its kinetic energy, (*d*) the vector and couple at G equivalent to the momenta of the particles of the disk.



a. Angular Velocity. As the disk rotates about the axle OG it also rotates with the axle about the y axis at a rate ω_2 clockwise. The total angular velocity of the disk is therefore

$$\boldsymbol{\omega} = \omega_1\mathbf{i} - \omega_2\mathbf{j} \tag{1}$$

To determine ω_2 we write that the velocity of C is zero:

$$\mathbf{v}_C = \boldsymbol{\omega} \times \mathbf{r}_C = 0$$
$$(\omega_1\mathbf{i} - \omega_2\mathbf{j}) \times (L\mathbf{i} - r\mathbf{j}) = 0$$
$$(L\omega_2 - r\omega_1)\mathbf{k} = 0 \qquad \omega_2 = r\omega_1/L$$

Substituting into (1) for ω_2:
$$\boldsymbol{\omega} = \omega_1\mathbf{i} - (r\omega_1/L)\mathbf{j} \quad ◄$$

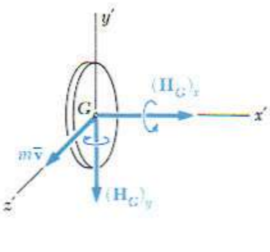
b. Angular Momentum About O. Assuming the axle to be part of the disk, we may consider the disk to have a fixed point at O. Since the x, y, and z axes are principal axes of inertia for the disk,

$$H_x = I_x\omega_x = (\tfrac{1}{2}mr^2)\omega_1$$
$$H_y = I_y\omega_y = (mL^2 + \tfrac{1}{4}mr^2)(-r\omega_1/L)$$
$$H_z = I_z\omega_z = (mL^2 + \tfrac{1}{4}mr^2)\,0 = 0$$
$$\mathbf{H}_O = \tfrac{1}{2}mr^2\omega_1\mathbf{i} - m(L^2 + \tfrac{1}{4}r^2)(r\omega_1/L)\mathbf{j} \quad ◄$$

c. Kinetic Energy. Using the values obtained for the moments of inertia and the components of $\boldsymbol{\omega}$, we have

$$T = \tfrac{1}{2}(I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2) = \tfrac{1}{2}[\tfrac{1}{2}mr^2\omega_1^2 + m(L^2 + \tfrac{1}{4}r^2)(-r\omega_1/L)^2]$$
$$T = \tfrac{1}{8}mr^2\left(6 + \frac{r^2}{L^2}\right)\omega_1^2 \quad ◄$$

d. Momentum Vector and Couple at G. The linear momentum vector $m\overline{\mathbf{v}}$ and the angular momentum couple $\mathbf{H}_G$ are



$$m\overline{\mathbf{v}} = mr\omega_1\mathbf{k} \quad ◄$$

and

$$\mathbf{H}_G = \overline{I}_{x'}\omega_x\mathbf{i} + \overline{I}_{y'}\omega_y\mathbf{j} + \overline{I}_{z'}\omega_z\mathbf{k} = \tfrac{1}{2}mr^2\omega_1\mathbf{i} + \tfrac{1}{4}mr^2(-r\omega_1/L)\mathbf{j}$$
$$\mathbf{H}_G = \tfrac{1}{2}mr^2\omega_1\left(\mathbf{i} - \frac{r}{2L}\mathbf{j}\right) \quad ◄$$

PROBLEMS

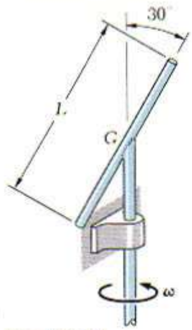


Fig. P18.1

18.1 A thin homogeneous rod of mass m and length L rotates with a constant angular velocity ω about a vertical axis through its mass center G. Determine the magnitude and direction of the angular momentum $\mathbf{H}_G$ of the rod about its mass center.

18.2 A thin homogeneous disk of mass m and radius r spins at the constant rate ω_2 about an axle held by a fork-ended horizontal rod which rotates at the constant rate ω_1. Determine the angular momentum of the disk about its mass center.

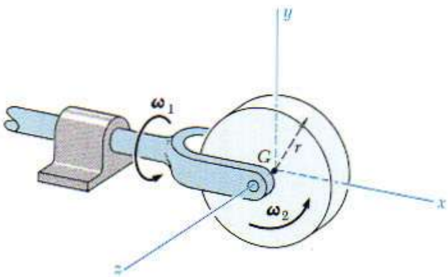


Fig. P18.2

18.3 A thin homogeneous disk of mass m and radius r is mounted on the vertical axle AB. The plane of the disk forms an angle $\beta = 30°$ with the horizontal. Knowing that the axle rotates with an angular velocity ω, determine the angle θ formed by the axle and the angular momentum of the disk about G.

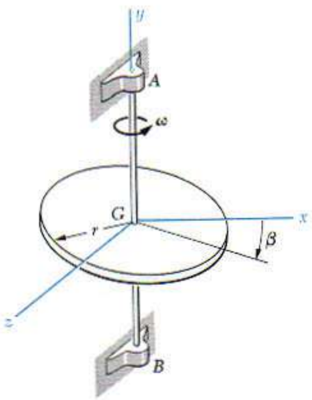


Fig. P18.3

18.4 A thin rectangular plate of mass 9 kg is attached to a shaft as shown. If the angular velocity ω of the plate is 4 rad/s at the instant shown, determine its angular momentum about its mass center G.

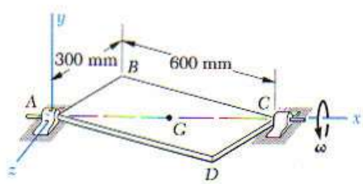


Fig. P18.4

18.5 In Prob. 18.3, determine the value of β for which the angle θ formed by the axle and the angular momentum $\mathbf{H}_G$ is maximum.

18.6 A thin homogeneous disk of mass 800 g and radius 100 mm rotates at a constant rate $\omega_2 = 20$ rad/s with respect to the arm ABC, which itself rotates at a constant rate $\omega_1 = 10$ rad/s about the x axis. Determine the angular momentum of the disk about point C.

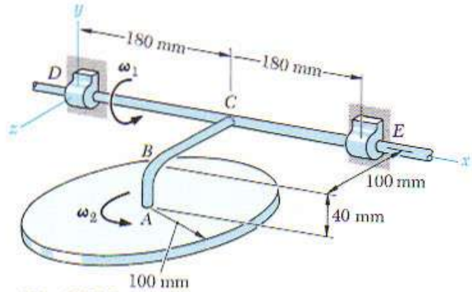


Fig. P18.6

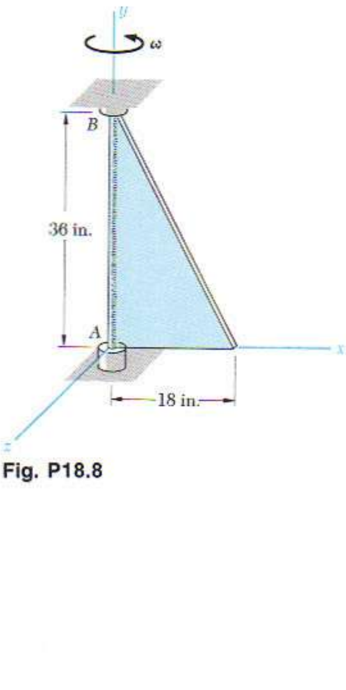


Fig. P18.8

18.7 Determine the angular momentum of the disk of Prob. 18.6 about point D.

18.8 A thin homogeneous triangular plate weighing 12 lb is welded to a light axle which can rotate freely in bearings at A and B. Knowing that the plate rotates at a constant rate $\omega = 5$ rad/s, determine its angular momentum about A.

18.9 Determine the angular momentum of the plate of Prob. 18.8 about its mass center.

18.10 Each element of the crankshaft shown is a homogeneous rod of mass m per unit length. Knowing that the crankshaft rotates with a constant angular velocity $\boldsymbol{\omega}$, determine (*a*) the angular momentum of the crankshaft about G, (*b*) the angle formed by the angular momentum and the axis AB.

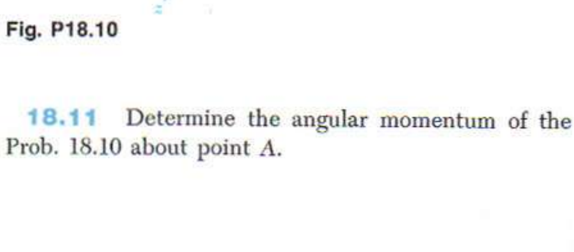


Fig. P18.10

18.11 Determine the angular momentum of the crankshaft of Prob. 18.10 about point A.

18.12 Show that, when a rigid body rotates about a fixed axis, its angular momentum is the same about any two points A and B on the fixed axis ($\mathbf{H}_A = \mathbf{H}_B$) if, and only if, the mass center G of the body is located on the fixed axis.

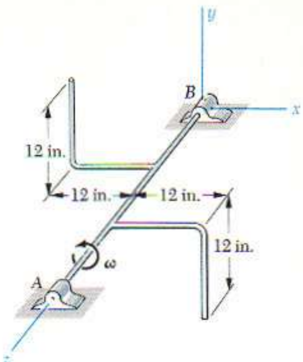


Fig. P18.13

18.13 Two L-shaped arms, each weighing 6 lb, are welded at the third points of the 3-ft shaft AB. Knowing that shaft AB rotates at a constant rate $\omega = 300$ rpm, determine (*a*) the angular momentum of the body about A, (*b*) the angle formed by the angular momentum and the shaft AB.

18.14 At a given instant during its flight, a launch vehicle has an angular velocity $\boldsymbol{\omega} = (0.3 \text{ rad/s})\mathbf{j} + (2 \text{ rad/s})\mathbf{k}$ and its mass center G has a velocity $\mathbf{v} = (6 \text{ m/s})\mathbf{i} + (9 \text{ m/s})\mathbf{j} + (1800 \text{ m/s})\mathbf{k}$, where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit vectors corresponding to the principal centroidal axes of inertia of the vehicle. Knowing that the vehicle has a mass of 40 Mg and that its centroidal radii of gyration are $\bar{k}_x = \bar{k}_y = 6$ m and $\bar{k}_z = 1.5$ m, determine (*a*) the linear momentum $m\bar{\mathbf{v}}$ and the angular momentum $\mathbf{H}_G$, (*b*) the angle between the vectors representing $m\bar{\mathbf{v}}$ and $\mathbf{H}_G$.

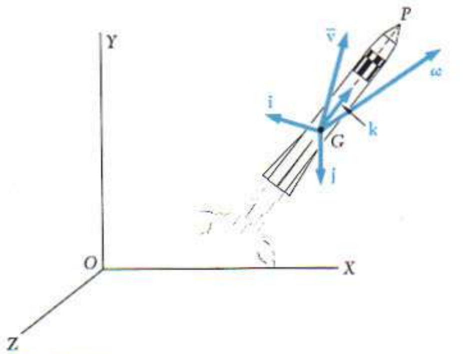


Fig. P18.14

18.15 For the launch vehicle of Prob. 18.14, determine the sum $\mathbf{H}_P$ of the moments about P of the momenta of the particles of the vehicle, knowing that the distance from G to P is 10 m.

18.16 A homogeneous wire, of weight 2 lb/ft, is used to form the wire figure shown, which is suspended from point A. If an impulse $\mathbf{F}\Delta t = -(10 \text{ lb}\cdot\text{s})\mathbf{k}$ is applied at point D of coordinates $x = 3$ ft, $y = 2$ ft, $z = 3$ ft, determine (*a*) the velocity of the mass center of the wire figure, (*b*) the angular velocity of the figure.

18.17 Solve Prob. 18.16, assuming that the impulse applied at point D is $\mathbf{F}\Delta t = (10 \text{ lb}\cdot\text{s})\mathbf{j}$.

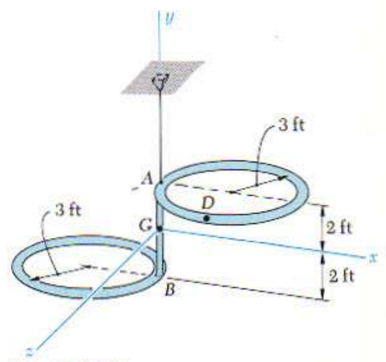


Fig. P18.16

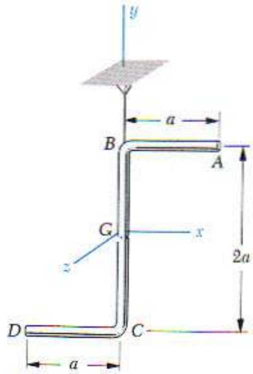


Fig. P18.18

18.18 A uniform rod of total mass m is bent into the shape shown and is suspended by a wire attached at B. The bent rod is hit at D in a direction perpendicular to the plane containing the rod (in the negative z direction). Denoting the corresponding impulse by $\mathbf{F}\Delta t$, determine (*a*) the velocity of the mass center of the rod, (*b*) the angular velocity of the rod.

18.19 Solve Prob. 18.18, assuming that the bent rod is hit at C.

18.20 Three slender homogeneous rods, each of mass m and length d, are welded together to form the assembly shown, which hangs from a wire at G. The assembly is hit at A in a vertical downward direction. Denoting the corresponding impulse by $\mathbf{F}\,\Delta t$, determine immediately after impact (*a*) the velocity of the mass center G, (*b*) the angular velocity of the assembly.

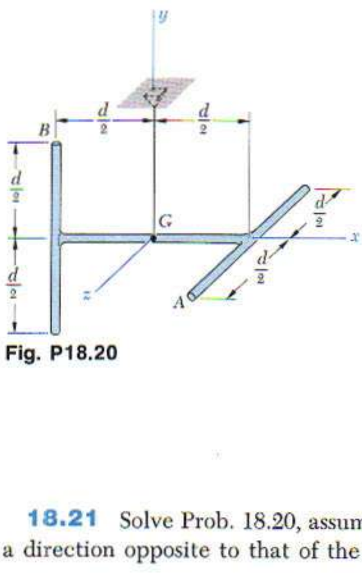


Fig. P18.20

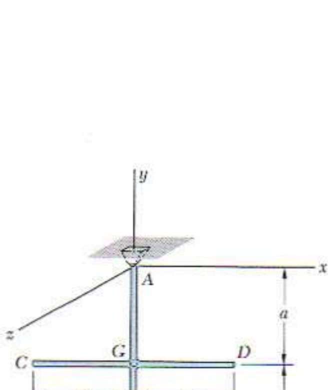


Fig. P18.22

18.21 Solve Prob. 18.20, assuming that the assembly is hit at B in a direction opposite to that of the z axis.

18.22 A cross of total mass m, made of two rods AB and CD, each of length $2a$ and welded together at G, is suspended from a ball-and-socket joint at A. The cross is hit at C in a direction perpendicular to its plane (in the negative z direction). Denoting the corresponding impulse by $\mathbf{F}\,\Delta t$, determine immediately after impact (*a*) the angular velocity of the cross, (*b*) its instantaneous axis of rotation.

18.23 A bullet of mass m_0 is fired with an initial velocity $\mathbf{v}_0$ into a heavy circular plate of mass m which is suspended from a ball-and-socket joint at O. Knowing that the bullet strikes point A and becomes embedded in the plate, determine immediately after impact (*a*) the angular velocity of the plate, (*b*) its instantaneous axis of rotation.

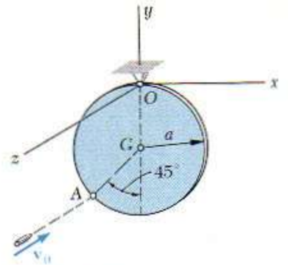


Fig. P18.23

18.24 A circular plate of radius a and mass m supported by a ball-and-socket joint at O was rotating about the y axis with a constant angular velocity $\boldsymbol{\omega} = \omega_0\mathbf{j}$ when an obstruction was suddenly introduced at A. Assuming that the impact at A is perfectly plastic, determine immediately after impact (*a*) the angular velocity of the plate, (*b*) the velocity of the mass center G.

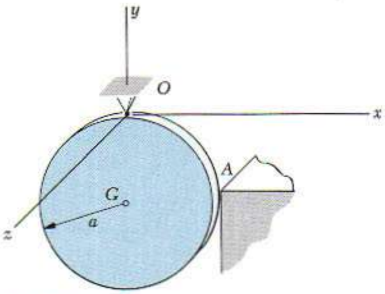


Fig. P18.24

18.25 Solve Prob. 18.24, assuming that, before the obstruction was introduced, the plate was rotating about the x axis with a constant angular velocity $\boldsymbol{\omega} = \omega_0\mathbf{i}$.

18.26 The angular velocity of a 1000-kg space capsule is $\boldsymbol{\omega} = (0.02 \text{ rad/s})\mathbf{i} + (0.10 \text{ rad/s})\mathbf{j}$ when two small jets are activated at A and B, each in a direction parallel to the z axis. Knowing that the radii of gyration of the capsule are $\bar{k}_x = \bar{k}_z = 1.00$ m and $\bar{k}_y = 1.25$ m, and that each jet produces a thrust of 50 N, determine (*a*) the required operating time of each jet if the angular velocity of the capsule is to be reduced to zero, (*b*) the resulting change in the velocity of the mass center G.

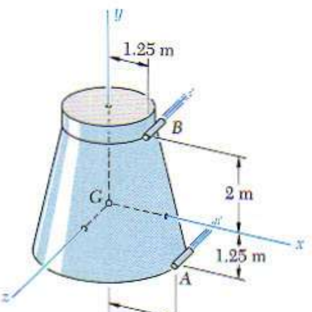


Fig. P18.26

18.27 If jet B in Prob. 18.26 is inoperative, determine (*a*) the required operating time of jet A to reduce the x component of the angular velocity $\boldsymbol{\omega}$ of the capsule to zero, (*b*) the resulting final angular velocity $\boldsymbol{\omega}$, (*c*) the resulting change in the velocity of the mass center G.

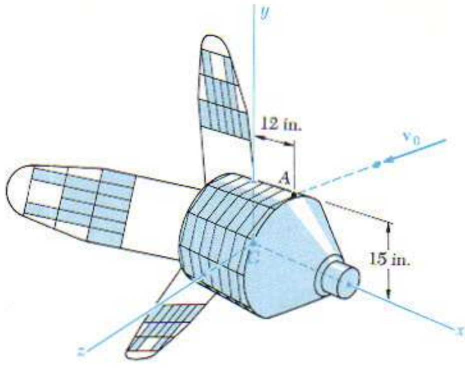


Fig. P18.28

18.28 A satellite weighing 320 lb has no angular velocity when it is struck at A by a 0.04-lb meteorite traveling with a velocity $\mathbf{v}_0 = -(2400 \text{ ft/s})\mathbf{i} - (1800 \text{ ft/s})\mathbf{j} + (4000 \text{ ft/s})\mathbf{k}$ relative to the satellite. Knowing that the radii of gyration of the satellite are $\bar{k}_x = 12$ in. and $\bar{k}_y = \bar{k}_z = 16$ in., determine the angular velocity of the satellite in rpm immediately after the meteorite has become imbedded.

18.29 Solve Prob. 18.28, assuming that, initially, the satellite was spinning about its axis of symmetry with an angular velocity of 12 rpm clockwise as viewed from the positive x axis.

Fig. P18.30

18.30 Show that the kinetic energy of a rigid body with a fixed point O may be expressed as

$$T = \tfrac{1}{2} I_{OL} \omega^2$$

where $\boldsymbol{\omega}$ is the instantaneous angular velocity of the body and I_{OL} its moment of inertia about the line of action OL of $\boldsymbol{\omega}$. Derive this expression (*a*) from Eqs. (9.46) and (18.19), (*b*) by considering T as the sum of the kinetic energies of particles P_i describing circles of radius ρ_i about line OL.

18.31 Denoting respectively by $\boldsymbol{\omega}$, $\mathbf{H}_O$, and T the angular velocity, the angular momentum, and the kinetic energy of a rigid body with a fixed point O, (*a*) prove that

$$\mathbf{H}_O \cdot \boldsymbol{\omega} = 2T$$

(*b*) show that the angle θ between $\boldsymbol{\omega}$ and $\mathbf{H}_O$ will always be acute.

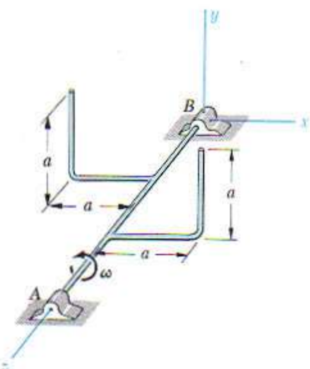


Fig. P18.32

18.32 The body shown is made of slender, homogeneous rods and may rotate freely in bearings at A and B. If the body is at rest when it is given a slight push, determine its angular velocity after it has rotated through 180°.

18.33 Determine the angular velocity of the body of Prob. 18.32 after it has rotated through 90°.

18.34 Determine the kinetic energy of the plate of Prob. 18.4.

18.35 Determine the kinetic energy of the disk of Prob. 18.3.

18.36 Determine the change in kinetic energy of the plate of Prob. 18.24 due to its impact with the obstruction.

18.37 Determine the change in kinetic energy of the plate of Prob. 18.25 due to its impact with the obstruction.

18.38 Determine the change in the kinetic energy of the satellite of Prob. 18.28 in its motion about its mass center due to the impact of the meteorite, knowing that before the impact the satellite was spinning about its axis of symmetry with an angular velocity of 12 rpm clockwise as viewed from the positive x axis.

18.39 Gear A rolls on the fixed gear B and rotates about the axle AD of length $L = 500$ mm which is rigidly attached at D to the vertical shaft DE. The shaft DE is made to rotate with a constant angular velocity $\boldsymbol{\omega}_1$ of magnitude 4 rad/s. Assuming that gear A can be approximated by a thin disk of mass 2 kg and radius $a = 100$ mm, and that $\beta = 30°$, determine (a) the angular momentum of gear A about point D, (b) the kinetic energy of gear A.

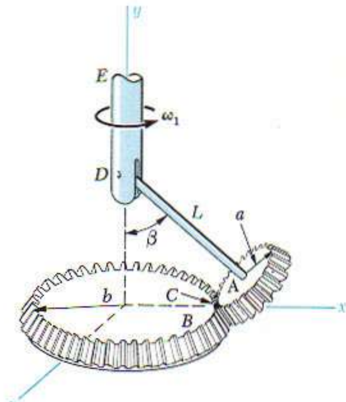


Fig. P18.39

*18.5. Motion of a Rigid Body in Three Dimensions.

As was indicated in Sec. 18.2, the fundamental equations

$$\Sigma \mathbf{F} = m\bar{\mathbf{a}} \tag{18.1}$$
$$\Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G \tag{18.2}$$

remain valid in the most general case of the motion of a rigid body. Before Eq. (18.2) could be applied to the three-dimensional motion of a rigid body, however, it was necessary to derive Eqs. (18.7), which relate the components of the angular momentum $\mathbf{H}_G$ and of the angular velocity $\boldsymbol{\omega}$. It still remains for us to find an effective and convenient way for computing the components of the derivative $\dot{\mathbf{H}}_G$ of the angular momentum.

Since $\mathbf{H}_G$ represents the angular momentum of the body in its motion relative to centroidal axes $GX'Y'Z'$ of fixed orientation (Fig. 18.9), and since $\dot{\mathbf{H}}_G$ represents the rate of change of $\mathbf{H}_G$ with respect to the same axes, it would seem natural to use components of $\boldsymbol{\omega}$ and $\mathbf{H}_G$ along the axes X', Y', Z' in writing the relations (18.7). But, since the body rotates, its moments and products of inertia would change continuously, and it would be necessary to determine their values as functions of the time. It is therefore more convenient to use axes x, y, z attached to the body, thus making sure that its moments and products of inertia will maintain the same values during the motion. This is permis-

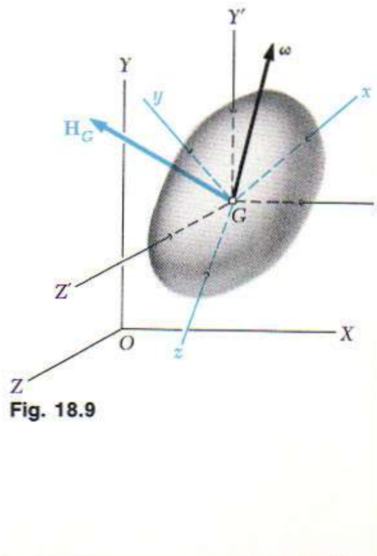


Fig. 18.9

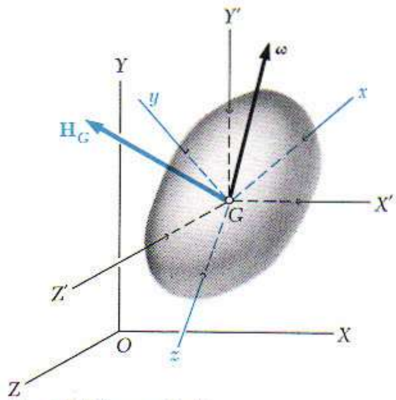


Fig. 18.9 (*repeated*)

sible since, as indicated earlier, the transformation of $\boldsymbol{\omega}$ into $\mathbf{H}_G$ is independent of the system of coordinate axes which has been selected. The angular velocity $\boldsymbol{\omega}$, however, should still be *defined* with respect to the frame $GX'Y'Z'$ of fixed orientation. The vector $\boldsymbol{\omega}$ may then be *resolved* into components along the rotating x, y, and z axes. Applying the relations (18.7), we obtain the *components* of the vector $\mathbf{H}_G$ along the rotating axes. The vector $\mathbf{H}_G$, however, represents the angular momentum about G of the body *in its motion relative to the frame $GX'Y'Z'$*.

Differentiating with respect to t the components of the angular momentum in (18.7), we define the rate of change of the vector $\mathbf{H}_G$ with respect to the rotating frame $Gxyz$:

$$(\dot{\mathbf{H}}_G)_{Gxyz} = \dot{H}_x\mathbf{i} + \dot{H}_y\mathbf{j} + \dot{H}_z\mathbf{k} \tag{18.21}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the unit vectors along the rotating axes. Recalling from Sec. 15.10 that the rate of change $\dot{\mathbf{H}}_G$ of the vector $\mathbf{H}_G$ with respect to the frame $GX'Y'Z'$ may be obtained by adding to $(\dot{\mathbf{H}}_G)_{Gxyz}$ the vector product $\boldsymbol{\Omega} \times \mathbf{H}_G$, where $\boldsymbol{\Omega}$ denotes the angular velocity of the rotating frame, we write

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega} \times \mathbf{H}_G \tag{18.22}$$

where $\mathbf{H}_G$ = angular momentum of the body with respect to the frame $GX'Y'Z'$ of fixed orientation

$(\dot{\mathbf{H}}_G)_{Gxyz}$ = rate of change of $\mathbf{H}_G$ with respect to the rotating frame $Gxyz$, to be computed from the relations (18.7) and (18.21)

$\boldsymbol{\Omega}$ = angular velocity of the rotating frame $Gxyz$

Substituting for $\dot{\mathbf{H}}_G$ from (18.22) into (18.2), we have

$$\Sigma\mathbf{M}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\Omega} \times \mathbf{H}_G \tag{18.23}$$

If, as it has been assumed in this discussion, the rotating frame is attached to the body, its angular velocity $\boldsymbol{\Omega}$ is identically equal to the angular velocity $\boldsymbol{\omega}$ of the body. There are many applications, however, where it is advantageous to use a frame of reference which is not actually attached to the body, but rotates in an independent manner. For example, if the body considered is axisymmetrical, as in Sample Prob. 18.5 or Sec. 18.9, it is possible to select a frame of reference with respect to which the moments and products of inertia of the body remain constant, but which rotates less than the body itself.† As a result, simpler expressions

† More specifically, the frame of reference will have no spin (see Sec. 18.9).

may be obtained for the angular velocity $\boldsymbol{\omega}$ and the angular momentum $\mathbf{H}_G$ of the body than would have been possible if the frame of reference had actually been attached to the body. It is clear that in such cases the angular velocity $\boldsymbol{\Omega}$ of the rotating frame and the angular velocity $\boldsymbol{\omega}$ of the body are different.

***18.6. Euler's Equations of Motion. Extension of D'Alembert's Principle to the Motion of a Rigid Body in Three Dimensions.** If the x, y, and z axes are chosen to coincide with the principal axes of inertia of the body, the simplified relations (18.10) may be used to determine the components of the angular momentum $\mathbf{H}_G$. Omitting the primes from the subscripts, we write

$$\mathbf{H}_G = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k} \tag{18.24}$$

where $\bar{I}_x$, $\bar{I}_y$, and $\bar{I}_z$ denote the principal centroidal moments of inertia of the body. Substituting for $\mathbf{H}_G$ from (18.24) into (18.23) and setting $\boldsymbol{\Omega} = \boldsymbol{\omega}$, we obtain the three scalar equations

$$\begin{aligned} \Sigma M_x &= \bar{I}_x\dot{\omega}_x - (\bar{I}_y - \bar{I}_z)\omega_y\omega_z \\ \Sigma M_y &= \bar{I}_y\dot{\omega}_y - (\bar{I}_z - \bar{I}_x)\omega_z\omega_x \\ \Sigma M_z &= \bar{I}_z\dot{\omega}_z - (\bar{I}_x - \bar{I}_y)\omega_x\omega_y \end{aligned} \tag{18.25}$$

These equations, called *Euler's equations of motion* after the Swiss mathematician Leonhard Euler (1707–1783), may be used to analyze the motion of a rigid body about its mass center. In the following sections, however, we shall use Eq. (18.23) in preference to Eqs. (18.25), since the former is more general, and the compact vectorial form in which it is expressed is easier to remember.

Writing Eq. (18.1) in scalar form, we obtain the three additional equations

$$\Sigma F_x = m\bar{a}_x \qquad \Sigma F_y = m\bar{a}_y \qquad \Sigma F_z = m\bar{a}_z \tag{18.26}$$

which, together with Euler's equations, form a system of six differential equations. Given appropriate initial conditions, these differential equations have a unique solution. Thus, the motion of a rigid body in three dimensions is completely defined by the resultant and the moment resultant of the external forces acting on it. This result will be recognized as a generalization of a similar result obtained in Sec. 16.4 in the case of the plane motion of a rigid slab. It follows that, in three as well as in two dimensions, two systems of forces which are equipollent are also equivalent; i.e., they have the same effect on a given rigid body.

Considering in particular the system of the external forces acting on a rigid body (Fig. 18.10a) and the system of the effective forces associated with the particles forming the rigid body (Fig. 18.10b), we may state that the two systems—which were shown in Sec. 14.1 to be equipollent—are also equivalent. This is the extension of D'Alembert's principle to the three-dimensional

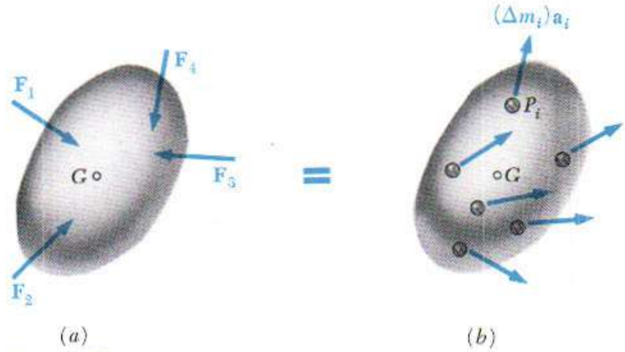


Fig. 18.10

motion of a rigid body. Replacing the effective forces in Fig. (18.10b) by an equivalent force-couple system, we verify that the system of the external forces acting on a rigid body in three-dimensional motion is equivalent to the system consisting of the vector $m\bar{\mathbf{a}}$ attached at the mass center G of the body and the couple of moment $\dot{\mathbf{H}}_G$ (Fig. 18.11), where $\dot{\mathbf{H}}_G$ is obtained

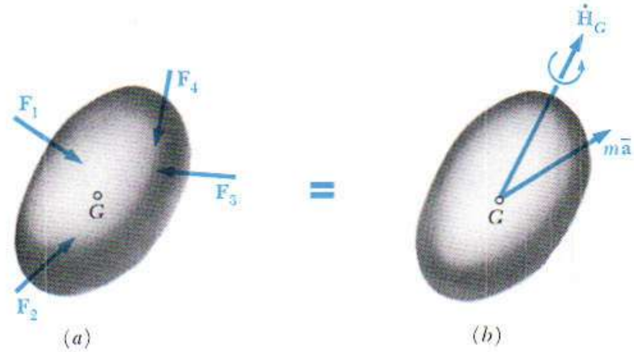


Fig. 18.11

from the relations (18.7) and (18.22). Problems involving the three-dimensional motion of a rigid body may be solved by drawing the two sketches shown in Fig. 18.11 and writing appropriate equations relating the components or moments of the external and effective forces (see Sample Prob. 18.3).

***18.7. Motion of a Rigid Body about a Fixed Point.** When a rigid body is constrained to rotate about a fixed point O, it is desirable to write an equation involving the moments about O of the external and effective forces, since this equation will not contain the unknown reaction at O. While such an equation may be obtained from Fig. 18.11, it may be more convenient to write it by considering the rate of change of the angular momentum $\mathbf{H}_O$ of the body about the fixed point O

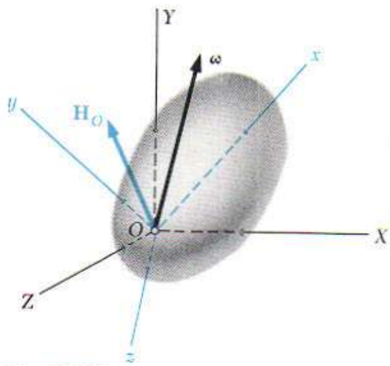


Fig. 18.12

(Fig. 18.12). Recalling Eq. (14.11), we write

$$\Sigma \mathbf{M}_O = \dot{\mathbf{H}}_O \tag{18.27}$$

where $\dot{\mathbf{H}}_O$ denotes the rate of change of the vector $\mathbf{H}_O$ with respect to the fixed frame $OXYZ$. A derivation similar to that used in Sec. 18.5 enables us to relate $\dot{\mathbf{H}}_O$ to the rate of change $(\dot{\mathbf{H}}_O)_{Oxyz}$ of $\mathbf{H}_O$ with respect to the rotating frame $Oxyz$. Substitution into (18.27) leads to the equation

$$\Sigma \mathbf{M}_O = (\dot{\mathbf{H}}_O)_{Oxyz} + \mathbf{\Omega} \times \mathbf{H}_O \tag{18.28}$$

where $\Sigma \mathbf{M}_O$ = sum of the moments about O of the forces applied to the rigid body
$\mathbf{H}_O$ = angular momentum of the body with respect to the fixed frame $OXYZ$
$(\dot{\mathbf{H}}_O)_{Oxyz}$ = rate of change of $\mathbf{H}_O$ with respect to the rotating frame $Oxyz$, to be computed from the relations (18.12)
$\mathbf{\Omega}$ = angular velocity of the rotating frame $Oxyz$†

† Read last paragraph of Sec. 18.5, replacing $\mathbf{H}_G$ by $\mathbf{H}_O$.

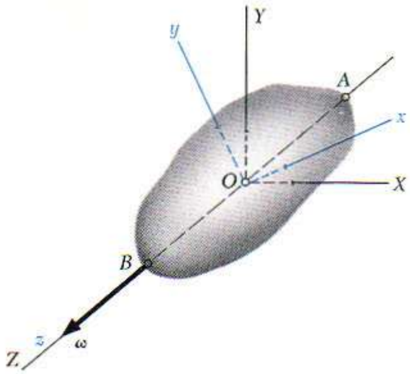


Fig. 18.13

***18.8 Rotation of a Rigid Body about a Fixed Axis.** We shall use Eq. (18.28), which was derived in the preceding section, to analyze the motion of a rigid body constrained to rotate about a fixed axis AB (Fig. 18.13). First, we note that the angular velocity of the body with respect to the fixed frame $OXYZ$ is represented by the vector $\boldsymbol{\omega}$ directed along the axis of rotation. Attaching the moving frame of reference $Oxyz$ to the body, with the z axis along AB, we have $\boldsymbol{\omega} = \omega\mathbf{k}$. Substituting $\omega_x = 0$, $\omega_y = 0$, $\omega_z = \omega$ into the relations (18.12), we obtain the components along the rotating axes of the angular momentum $\mathbf{H}_O$ of the body about O:

$$H_x = -P_{xz}\omega \qquad H_y = -P_{yz}\omega \qquad H_z = I_z\omega$$

Since the frame $Oxyz$ is attached to the body, we have $\boldsymbol{\Omega} = \boldsymbol{\omega}$ and Eq. (18.28) yields

$$\begin{aligned}\Sigma\mathbf{M}_O &= (\dot{\mathbf{H}}_O)_{Oxyz} + \boldsymbol{\omega} \times \mathbf{H}_O \\ &= (-P_{xz}\mathbf{i} - P_{yz}\mathbf{j} + I_z\mathbf{k})\dot{\omega} + \omega\mathbf{k} \times (-P_{xz}\mathbf{i} - P_{yz}\mathbf{j} + I_z\mathbf{k})\omega \\ &= (-P_{xz}\mathbf{i} - P_{yz}\mathbf{j} + I_z\mathbf{k})\alpha + (-P_{xz}\mathbf{j} + P_{yz}\mathbf{i})\omega^2\end{aligned}$$

The result obtained may be expressed by the three scalar equations

$$\begin{aligned}\Sigma M_x &= -P_{xz}\alpha + P_{yz}\omega^2 \\ \Sigma M_y &= -P_{yz}\alpha - P_{xz}\omega^2 \\ \Sigma M_z &= I_z\alpha\end{aligned} \tag{18.29}$$

When the forces applied to the body are known, the angular acceleration α may be obtained from the last of Eqs. (18.29). The angular velocity ω is then determined by integration and the values obtained for α and ω may be substitued into the first two equations (18.29). These equations, plus the three equations (18.26), which define the motion of the mass center of the body, may then be used to determine the reactions at the bearings A and B.

It should be noted that axes other than the ones shown in Fig. 18.12 may be selected to analyze the rotation of a rigid body about a fixed axis. In many cases, the principal axes of inertia of the body will be found more advantageous. It is wise, therefore, to revert to Eq. (18.28) and to select the system of axes which best fits the problem under consideration.

If the rotating body is symmetrical with respect to the xy plane, the products of inertia P_{xz} and P_{yz} are equal to zero and Eqs. (18.29) reduce to

$$\Sigma M_x = 0 \qquad \Sigma M_y = 0 \qquad \Sigma M_z = I_z\alpha \tag{18.30}$$

which is in accord with the results obtained in Chap. 16. If, on the other hand, the products of inertia P_{xz} and P_{yz} are different from zero, the sum of the moments of the external forces about the x and y axes will also be different from zero, even when the body rotates at a constant rate ω. Indeed, in the latter case, Eqs. (18.29) yield

$$\Sigma M_x = P_{yz}\omega^2 \qquad \Sigma M_y = -P_{xz}\omega^2 \qquad \Sigma M_z = 0 \quad (18.31)$$

This last observation leads us to discuss the *balancing of rotating shafts*. Consider, for instance, the crankshaft shown in Fig. 18.14a, which is symmetrical about its mass center G. We first observe that, when the crankshaft is at rest, it exerts no lateral thrust on its supports, since its center of gravity G is located directly above A. The shaft is said to be *statically balanced*. The reaction at A, often referred to as a *static reaction*, is vertical and its magnitude is equal to the weight W of the shaft. Let us now assume that the shaft rotates with a constant angular velocity $\boldsymbol{\omega}$. Attaching our frame of reference to the shaft, with its origin at G, the z axis along AB, and the y axis in the plane of symmetry of the shaft (Fig. 18.14b), we note that P_{xz} is zero and that P_{yz} is positive. According to Eqs. (18.31), the external forces must include a couple of moment $P_{yz}\omega^2\mathbf{i}$. Since this couple is formed by the reaction at B and the horizontal component of the reaction at A, we have

$$\mathbf{A}_y = \frac{P_{yz}\omega^2}{l}\mathbf{j} \qquad \mathbf{B} = -\frac{P_{yz}\omega^2}{l}\mathbf{j} \qquad (18.32)$$

Since the bearing reactions are proportional to ω^2, the shaft will have a tendency to tear away from its bearings when rotating at high speeds. Moreover, since the bearing reactions $\mathbf{A}_y$ and $\mathbf{B}$, called *dynamic reactions*, are contained in the yz plane, they rotate with the shaft and cause the structure supporting it to vibrate. These undesirable effects will be avoided if, by rearranging the distribution of mass around the shaft, or by adding corrective masses, we let P_{yz} become equal to zero. The dynamic reactions $\mathbf{A}_y$ and $\mathbf{B}$ will vanish and the reactions at the bearings will reduce to the static reaction $\mathbf{A}_z$, the direction of which is fixed. The shaft will then be *dynamically as well as statically balanced*.

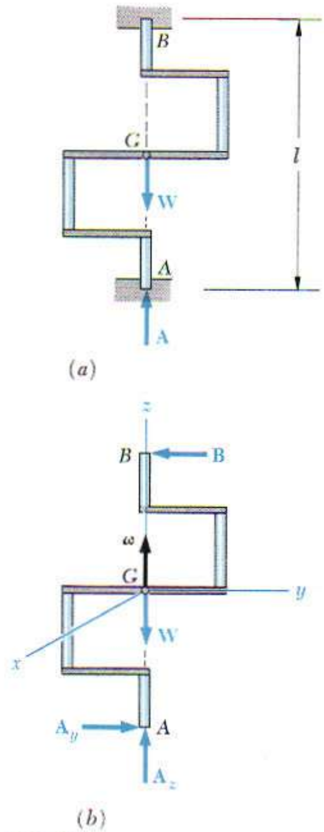
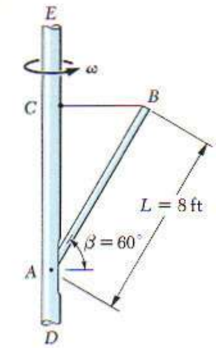


Fig. 18.14

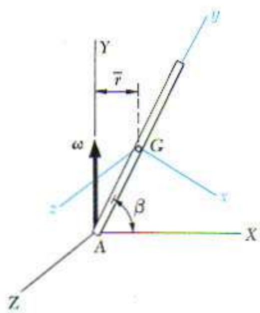
SAMPLE PROBLEM 18.3



A slender rod AB of length $L = 8$ ft and weight $W = 40$ lb is pinned at A to a vertical axle DE which rotates with a constant angular velocity ω of 15 rad/s. The rod is maintained in position by means of a horizontal wire BC attached to the axle and to the end B of the rod. Determine the tension in the wire and the reaction at A.

Solution. The effective forces reduce to the vector $m\bar{\mathbf{a}}$ attached at G and the couple $\dot{\mathbf{H}}_G$. Since G describes a horizontal circle of radius $\bar{r} = \frac{1}{2}L \cos\beta$ at the constant rate ω, we have

$$\bar{\mathbf{a}} = \mathbf{a}_n = -\bar{r}\omega^2\mathbf{I} = -(\tfrac{1}{2}L\cos\beta)\omega^2\mathbf{I} = -(450 \text{ ft/s}^2)\mathbf{I}$$
$$m\bar{\mathbf{a}} = \frac{40}{g}(-450\mathbf{I}) = -(559 \text{ lb})\mathbf{I}$$

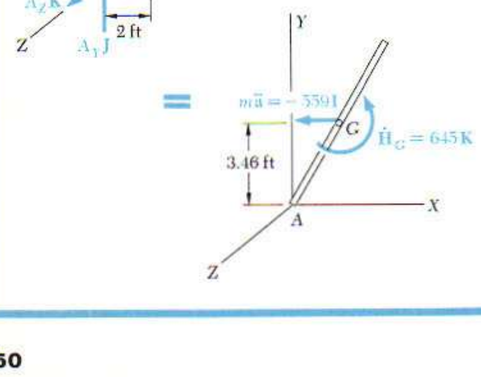


Determination of $\dot{\mathbf{H}}_G$. We first compute the angular momentum $\mathbf{H}_G$. Using the principal centroidal axes of inertia x, y, z, we write

$$\bar{I}_x = \tfrac{1}{12}mL^2 \qquad \bar{I}_y = 0 \qquad \bar{I}_z = \tfrac{1}{12}mL^2$$
$$\omega_x = -\omega\cos\beta \qquad \omega_y = \omega\sin\beta \qquad \omega_z = 0$$
$$\mathbf{H}_G = \bar{I}_x\omega_x\mathbf{i} + \bar{I}_y\omega_y\mathbf{j} + \bar{I}_z\omega_z\mathbf{k}$$
$$\mathbf{H}_G = -\tfrac{1}{12}mL^2\omega\cos\beta\,\mathbf{i}$$

The rate of change $\dot{\mathbf{H}}_G$ of $\mathbf{H}_G$ with respect to axes of fixed orientation is obtained from Eq. (18.22). Observing that the rate of change $(\dot{\mathbf{H}}_G)_{Gxyz}$ of $\mathbf{H}_G$ with respect to the rotating frame $Gxyz$ is zero, and that the angular velocity $\boldsymbol{\Omega}$ of that frame is equal to the angular velocity $\boldsymbol{\omega}$ of the rod, we have

$$\dot{\mathbf{H}}_G = (\dot{\mathbf{H}}_G)_{Gxyz} + \boldsymbol{\omega} \times \mathbf{H}_G$$
$$\dot{\mathbf{H}}_G = 0 + (-\omega\cos\beta\,\mathbf{i} + \omega\sin\beta\,\mathbf{j}) \times (-\tfrac{1}{12}mL^2\omega\cos\beta\,\mathbf{i})$$
$$\dot{\mathbf{H}}_G = \tfrac{1}{12}mL^2\omega^2\sin\beta\cos\beta\,\mathbf{k} = (645 \text{ lb}\cdot\text{ft})\mathbf{k}$$



Equations of Motion. Expressing that the system of the external forces is equivalent to the system of the effective forces, we write

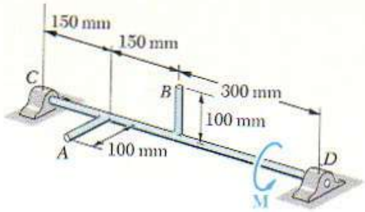
$\Sigma\mathbf{M}_A = \Sigma(\mathbf{M}_A)_{\text{eff}}$:

$$6.93\mathbf{J} \times (-T\mathbf{I}) + 2\mathbf{I} \times (-40\mathbf{J}) = 3.46\mathbf{J} \times (-559\mathbf{I}) + 645\mathbf{K}$$
$$(6.93T - 80)\mathbf{K} = (1934 + 645)\mathbf{K} \qquad T = 384 \text{ lb}$$

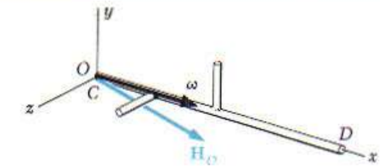
$$\Sigma\mathbf{F} = \Sigma\mathbf{F}_{\text{eff}}: \quad A_X\mathbf{I} + A_Y\mathbf{J} + A_Z\mathbf{K} - 384\mathbf{I} - 40\mathbf{J} = -559\mathbf{I}$$
$$\mathbf{A} = -(175 \text{ lb})\mathbf{I} + (40 \text{ lb})\mathbf{J}$$

Remark. The value of T could have been obtained from $\mathbf{H}_A$ and Eq. (18.28). However, the method used here also yields the reaction at A. Moreover, it draws attention to the effect of the asymmetry of the rod on the solution of the problem by clearly showing that both the vector $m\bar{\mathbf{a}}$ and the couple $\dot{\mathbf{H}}_G$ must be used to represent the effective forces.

SAMPLE PROBLEM 18.4



Two 100-mm rods A and B, each of mass 300 g, are welded to the shaft CD which is supported by bearings at C and D. If a couple $\mathbf{M}$ of magnitude equal to $6\ \mathrm{N\cdot m}$ is applied to the shaft, determine the components of the dynamic reactions at C and D at the instant when the shaft has reached an angular velocity of 1200 rpm. Neglect the moment of inertia of the shaft itself.

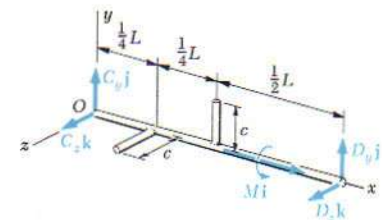


Angular Momentum about O. We attach to the body the frame of reference $Oxyz$ and note that the axes chosen are not principal axes of inertia for the body. Since the body rotates about the x axis, we have $\omega_x = \omega$ and $\omega_y = \omega_z = 0$. Substituting into Eqs. (18.12),

$$H_x = I_x\omega \qquad H_y = -P_{xy}\omega \qquad H_z = -P_{xz}\omega$$
$$\mathbf{H}_O = (I_x\mathbf{i} - P_{xy}\mathbf{j} - P_{xz}\mathbf{k})\omega$$

Moments of the External Forces about O. Since the frame of reference rotates with the angular velocity $\boldsymbol{\omega}$, Eq. (18.28) yields

$$\begin{aligned}\Sigma\mathbf{M}_O &= (\dot{\mathbf{H}}_O)_{Oxyz} + \boldsymbol{\omega}\times\mathbf{H}_O\\ &= (I_x\mathbf{i} - P_{xy}\mathbf{j} - P_{xz}\mathbf{k})\alpha + \omega\mathbf{i}\times(I_x\mathbf{i} - P_{xy}\mathbf{j} - P_{xz}\mathbf{k})\omega\\ &= I_x\alpha\mathbf{i} - (P_{xy}\alpha - P_{xz}\omega^2)\mathbf{j} - (P_{xz}\alpha + P_{xy}\omega^2)\mathbf{k} \qquad (1)\end{aligned}$$



Dynamic Reaction at D. The external forces consist of the weights of the shaft and rods, the couple $\mathbf{M}$, the static reactions at C and D, and the dynamic reactions at C and D. Since the weights and static reactions are balanced, the external forces reduce to the couple $\mathbf{M}$ and the dynamic reactions $\mathbf{C}$ and $\mathbf{D}$ as shown in the figure. Taking moments about O, we have

$$\Sigma\mathbf{M}_O = L\mathbf{i}\times(D_y\mathbf{j} + D_z\mathbf{k}) + M\mathbf{i} = M\mathbf{i} - D_zL\mathbf{j} + D_yL\mathbf{k} \qquad (2)$$

Equating the coefficients of the unit vector $\mathbf{i}$ in (1) and (2):

$$M = I_x\alpha \qquad M = 2(\tfrac{1}{3}mc^2)\alpha \qquad \alpha = 3M/2mc^2$$

Equating the coefficients of $\mathbf{k}$ and $\mathbf{j}$ in (1) and (2):

$$D_y = -(P_{xz}\alpha + P_{xy}\omega^2)/L \qquad D_z = (P_{xy}\alpha - P_{xz}\omega^2)/L \qquad (3)$$

Using the parallel-axis theorem, and noting that the product of inertia of each rod is zero with respect to centroidal axes, we have

$$P_{xy} = \Sigma m\bar{x}\bar{y} = m(\tfrac{1}{2}L)(\tfrac{1}{2}c) = \tfrac{1}{4}mLc$$
$$P_{xz} = \Sigma m\bar{x}\bar{z} = m(\tfrac{1}{4}L)(\tfrac{1}{2}c) = \tfrac{1}{8}mLc$$

Substituting into (3) the values found for P_{xy}, P_{xz}, and α:

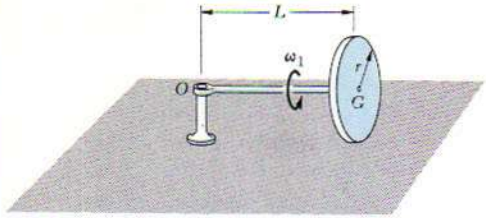
$$D_y = -\tfrac{3}{16}(M/c) - \tfrac{1}{4}mc\omega^2 \qquad D_z = \tfrac{3}{8}(M/c) - \tfrac{1}{8}mc\omega^2$$

Substituting $\omega = 1200\ \text{rpm} = 125.7\ \text{rad/s}$, $c = 0.100\ \text{m}$, $M = 6\ \mathrm{N\cdot m}$, and $m = 0.300\ \text{kg}$, we have

$$D_y = -129.8\ \text{N} \qquad D_z = -36.8\ \text{N} \quad ◄$$

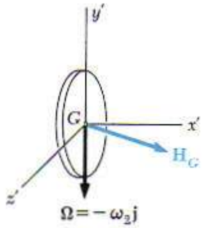
Dynamic Reaction at C. Using a frame of reference attached at D, we obtain equations similar to Eqs. (3), which yield

$$C_y = -152.2\ \text{N} \qquad C_z = -155.2\ \text{N} \quad ◄$$



SAMPLE PROBLEM 18.5

A homogeneous disk of radius r and mass m is mounted on an axle OG of length L and negligible mass. The axle is pivoted at the fixed point O and the disk is constrained to roll on a horizontal floor. Knowing that the disk rotates counterclockwise at the constant rate ω_1 about the axle, determine (*a*) the force (assumed vertical) exerted by the floor on the disk, (*b*) the reaction at the pivot O.



Solution. The effective forces reduce to the vector $m\overline{\mathbf{a}}$ attached at G and the couple $\dot{\mathbf{H}}_G$. Recalling from Sample Prob. 18.2 that the axle rotates about the y axis at the rate $\omega_2 = r\omega_1/L$, we write

$$m\overline{\mathbf{a}} = -mL\omega_2^2\mathbf{i} = -mL(r\omega_1/L)^2\mathbf{i} = -(mr^2\omega_1^2/L)\mathbf{i} \qquad (1)$$

Determination of $\dot{\mathbf{H}}_G$. We recall from Sample Prob. 18.2 that the angular momentum of the disk about G is

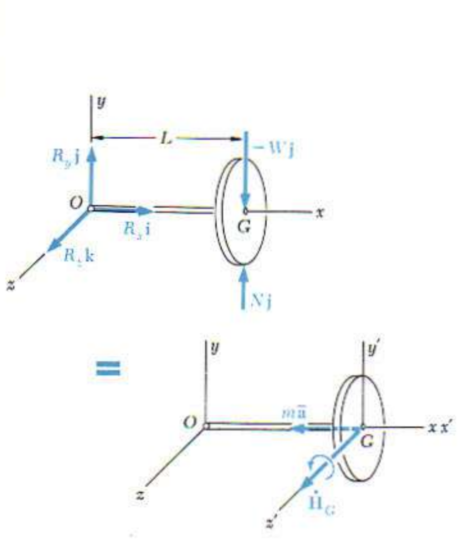
$$\mathbf{H}_G = \tfrac{1}{2}mr^2\omega_1\left(\mathbf{i} - \frac{r}{2L}\mathbf{j}\right)$$

where $\mathbf{H}_G$ is resolved into components along the rotating axes x', y', z', with x' along OG and y' vertical. The rate of change $\dot{\mathbf{H}}_G$ of $\mathbf{H}_G$ with respect to axes of fixed orientation is obtained from Eq. (18.22). Noting that the rate of change $(\dot{\mathbf{H}}_G)_{Gx'y'z'}$ of $\mathbf{H}_G$ with respect to the rotating frame is zero, and that the angular velocity $\mathbf{\Omega}$ of that frame is

$$\mathbf{\Omega} = -\omega_2\mathbf{j} = -(r\omega_1/L)\mathbf{j}$$

we have

$$\begin{aligned}\dot{\mathbf{H}}_G &= (\dot{\mathbf{H}}_G)_{Gx'y'z'} + \mathbf{\Omega} \times \mathbf{H}_G \\ &= 0 - \frac{r\omega_1}{L}\mathbf{j} \times \tfrac{1}{2}mr^2\omega_1\left(\mathbf{i} - \frac{r}{2L}\mathbf{j}\right) \\ &= \tfrac{1}{2}mr^2(r/L)\omega_1^2\mathbf{k} \qquad (2)\end{aligned}$$



Equations of Motion. Expressing that the system of the external forces is equivalent to the system of the effective forces, we write

$\Sigma\mathbf{M}_O = \Sigma(\mathbf{M}_O)_{\text{eff}}$: $\qquad L\mathbf{i} \times (N\mathbf{j} - W\mathbf{j}) = \dot{\mathbf{H}}_G$

$$(N - W)L\mathbf{k} = \tfrac{1}{2}mr^2(r/L)\omega_1^2\mathbf{k}$$

$$N = W + \tfrac{1}{2}mr(r/L)^2\omega_1^2 \qquad \mathbf{N} = [W + \tfrac{1}{2}mr(r/L)^2\omega_1^2]\mathbf{j} \quad (3) \blacktriangleleft$$

$\Sigma\mathbf{F} = \Sigma\mathbf{F}_{\text{eff}}$: $\qquad \mathbf{R} + N\mathbf{j} - W\mathbf{j} = m\overline{\mathbf{a}}$

Substituting for N from (3), for $m\overline{\mathbf{a}}$ from (1), and solving for $\mathbf{R}$:

$$\mathbf{R} = -(mr^2\omega_1^2/L)\mathbf{i} - \tfrac{1}{2}mr(r/L)^2\omega_1^2\mathbf{j}$$

$$\mathbf{R} = -\frac{mr^2\omega_1^2}{L}\left(\mathbf{i} + \frac{r}{2L}\mathbf{j}\right) \blacktriangleleft$$

PROBLEMS

18.40 Determine the rate of change $\dot{\mathbf{H}}_G$ of the angular momentum $\mathbf{H}_G$ of the disk of Prob. 18.2.

18.41 Determine the rate of change $\dot{\mathbf{H}}_G$ of the angular momentum $\mathbf{H}_G$ of the disk of Prob. 18.3, assuming that the angular velocity $\boldsymbol{\omega}$ of axle AB remains constant.

18.42 Determine the rate of change $\dot{\mathbf{H}}_G$ of the angular momentum $\mathbf{H}_G$ of the plate of Prob. 18.4, assuming that its angular velocity $\boldsymbol{\omega}$ remains constant.

18.43 Determine the rate of change $\dot{\mathbf{H}}_A$ of the angular momentum $\mathbf{H}_A$ of the disk of Prob. 18.6.

18.44 Determine the rate of change $\dot{\mathbf{H}}_G$ of the angular momentum $\mathbf{H}_G$ of the plate of Prob. 18.4 if, at the instant considered, the angular velocity $\boldsymbol{\omega}$ of the plate is 4 rad/s and is increasing at the rate of 8 rad/s^2.

18.45 Determine the rate of change $\dot{\mathbf{H}}_G$ of the angular momentum $\mathbf{H}_G$ of the disk of Prob. 18.3 if axle AB has an angular acceleration $\boldsymbol{\alpha}$.

18.46 Two 600-mm rods BE and CF, each of mass 4 kg, are attached to the shaft AD which rotates at a constant speed of 20 rad/s. Knowing that the two rods and the shaft lie in the same plane, determine the dynamic reactions at A and D.

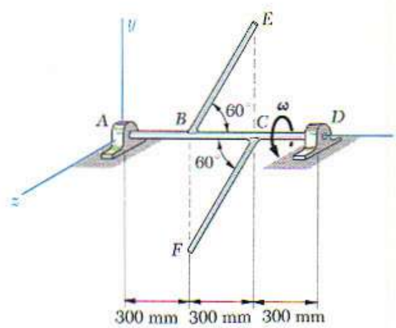


Fig. P18.46

18.47 Two triangular plates weighing 10 lb each are welded to a vertical shaft AB. Knowing that the system rotates at the constant rate $\omega = 6$ rad/s, determine the dynamic reactions at A and B.

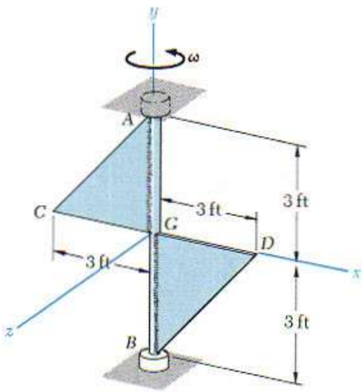


Fig. P18.47

18.48 Each element of the crankshaft shown is a homogeneous rod of weight w per unit length. Knowing that the crankshaft rotates with a constant angular velocity $\boldsymbol{\omega}$, determine the dynamic reactions at A and B.

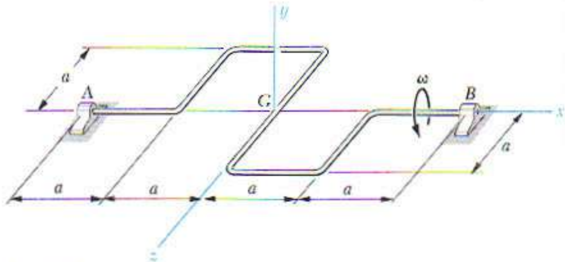


Fig. P18.48

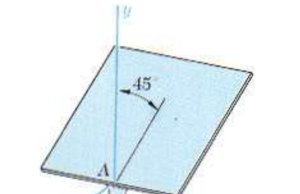


Fig. P18.49

18.49 A thin homogeneous square plate of mass m and side a is welded to a vertical shaft AB with which it forms an angle of 45°. Knowing that the shaft rotates with a constant angular velocity $\boldsymbol{\omega}$, determine the force-couple system representing the dynamic reaction at A.

18.50 The shaft of Prob. 18.48 is initially at rest ($\omega = 0$) and is accelerated at the rate $\alpha = \dot{\omega} = 100 \text{ rad/s}^2$. Knowing that $w = 4$ lb/ft and $a = 3$ in., determine (*a*) the couple **M** required to cause the acceleration, (*b*) the corresponding dynamic reactions at A and B.

18.51 The system of Prob. 18.47 is initially at rest ($\omega = 0$) and has an angular acceleration $\boldsymbol{\alpha} = (30 \text{ rad/s}^2)\mathbf{j}$. Determine (*a*) the couple **M** required to cause the acceleration, (*b*) the corresponding dynamic reactions at A and B.

18.52 The square plate of Prob. 18.49 is at rest ($\omega = 0$) when a couple of moment $M_0\mathbf{j}$ is applied to the shaft. Determine (*a*) the angular acceleration of the plate, (*b*) the force-couple system representing the dynamic reaction at A at that instant.

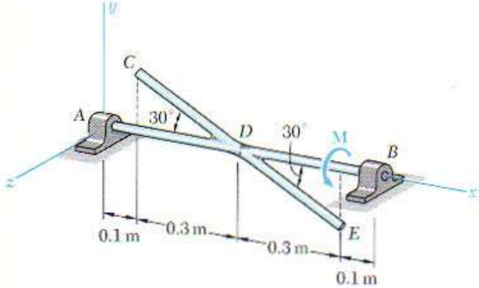


Fig. P18.53 and P18.54

18.53 Two uniform rods CD and DE, each of mass 2 kg, are welded to the shaft AB, which is at rest. If a couple **M** of magnitude 10 N·m is applied to the shaft, determine the dynamic reactions at A and B.

18.54 Two uniform rods CD and DE, each of mass 2 kg, are welded to the shaft AB. At the instant shown the angular velocity of the shaft is 15 rad/s and the angular acceleration is 100 rad/s², both counterclockwise when viewed from the positive x axis. Determine (*a*) the couple **M** which must be applied to the shaft, (*b*) the corresponding dynamic reactions at A and B.

18.55 Two L-shaped arms, each weighing 6 lb, are welded at the third points of the 3-ft shaft AB. A couple $\mathbf{M} = (15 \text{ lb}\cdot\text{ft})\mathbf{k}$ is applied to the shaft, which is initially at rest. Determine (*a*) the angular acceleration of the shaft, (*b*) the dynamic reactions at A and B as the shaft reaches an angular velocity of 10 rad/s.

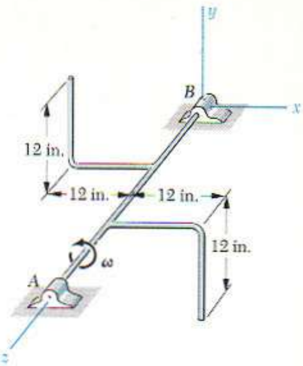


Fig. P18.55

18.56 The blade of a portable saw and the rotor of its motor have a combined mass of 1.2 kg and a radius of gyration of 35 mm. Determine the couple that a man must exert on the handle to rotate the saw about the y axis with a constant angular velocity of 3 rad/s clockwise, as viewed from above, when the blade rotates at the rate $\omega = 1800$ rpm as shown.

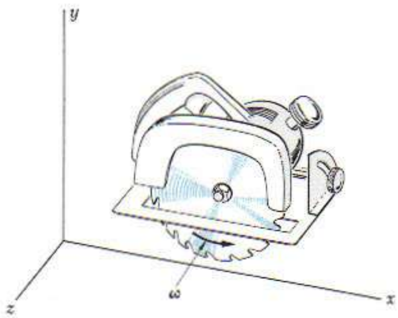


Fig. P18.56

18.57 A three-bladed airplane propeller has a mass of 120 kg and a radius of gyration of 900 mm. Knowing that the propeller rotates at 1500 rpm, determine the moment of the couple applied by the propeller to its shaft when the airplane travels in a circular path of 360-m radius at 600 km/h.

18.58 The flywheel of an automobile engine, which is mounted on the crankshaft, is equivalent to a 16-in.-diameter steel plate of $\frac{15}{16}$-in. thickness. At a time when the flywheel is rotating at 4000 rpm the automobile is traveling around a curve of 600-ft radius at a speed of 60 mi/h. Determine, at that time, the magnitude of the couple exerted by the flywheel on the horizontal crankshaft. (Specific weight of steel = 490 lb/ft^3.)

18.59 The essential structure of a certain type of aircraft turn indicator is shown. Springs AC and BD are initially stretched and exert equal vertical forces at A and B when the airplane is traveling in a straight path. Knowing that the disk weighs $\frac{1}{2}$ lb and spins at the rate of 10,000 rpm, determine the angle through which the yoke will rotate when the airplane executes a horizontal turn of radius 2500 ft at a speed of 500 mi/h. The constant of each spring is 2 lb/in.

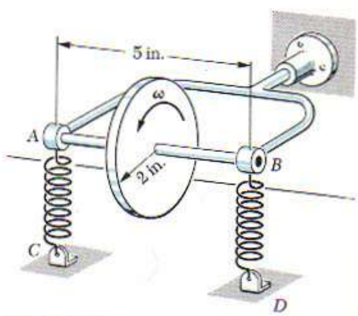


Fig. 18.59

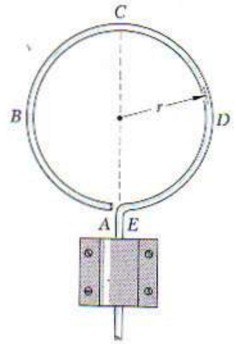


Fig. P18.60

18.60 A thin homogeneous wire, of mass m per unit length and in the shape of a circle of radius r, is made to rotate about a vertical shaft with a constant angular velocity $\boldsymbol{\omega}$. Determine the bending moment in the wire (*a*) at point C, (*b*) at point E, (*c*) at point B. (Neglect the effect of gravity.)

18.61 A thin homogeneous disk of mass m and radius r spins at the constant rate ω_2 about a horizontal axle held by a fork-ended vertical rod which rotates at the constant rate ω_1. Determine the couple $\mathbf{M}$ exerted by the rod on the disk.

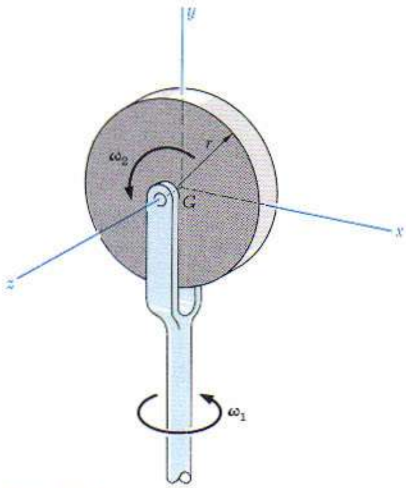


Fig. P18.61

18.62 A thin ring of radius a is attached by a collar at A to a vertical shaft which rotates with a constant angular velocity $\boldsymbol{\omega}$. Derive an expression (*a*) for the constant angle β that the plane of the ring forms with the vertical, (*b*) for the maximum value of ω for which the ring will remain vertical ($\beta = 0$).

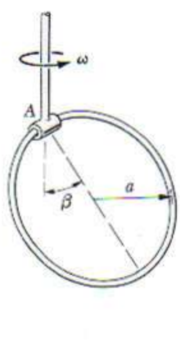


Fig. 18.62

18.63 A uniform disk of radius r is welded to a rod AB of negligible weight, which is attached to the pin of a clevis which rotates with a constant angular velocity $\boldsymbol{\omega}$. Derive an expression (*a*) for the constant angle β that the rod forms with the vertical, (*b*) for the maximum value of ω for which the rod will remain vertical ($\beta = 0$).

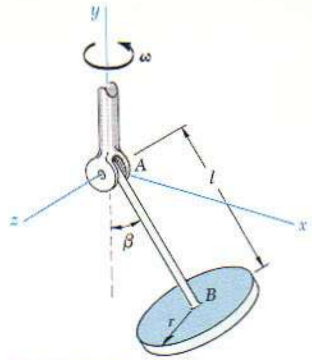


Fig. P18.63

18.64 A disk of mass m and radius r rotates at a constant rate ω_2 with respect to the arm OA, which itself rotates at a constant rate ω_1 about the y axis. Determine the force-couple system representing the dynamic reaction at O.

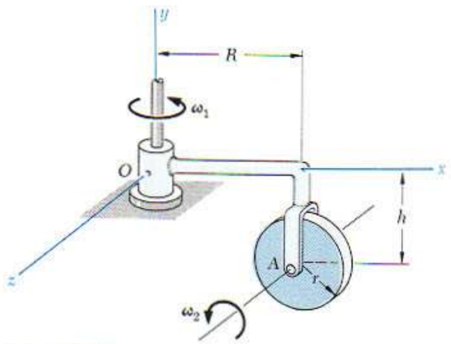


Fig. P18.64

18.65 Two disks, each of mass 5 kg and radius 300 mm, spin as shown at 1200 rpm about the rod AB, which is attached to shaft CD. The entire system is made to rotate about the z axis with an angular velocity $\boldsymbol{\Omega}$ of 60 rpm. (*a*) Determine the dynamic reactions at C and D as the system passes through the position shown. (*b*) Solve part *a* assuming that the direction of spin of disk B is reversed.

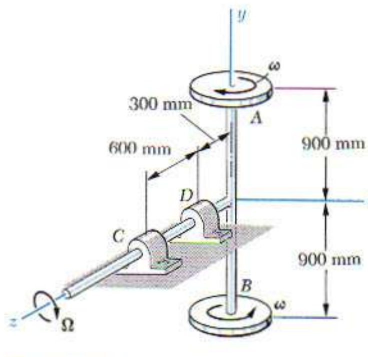


Fig. P18.65

18.66 A stationary horizontal plate is attached to the ceiling by means of a fixed vertical tube. A wheel of radius a and mass m is mounted on a light axle AC which is attached by means of a clevis at A to a rod AB fitted inside the vertical tube. The rod AB is made to rotate with a constant angular velocity $\boldsymbol{\Omega}$ causing the wheel to roll on the lower face of the stationary plate. Determine the minimum angular velocity Ω for which contact is maintained between the wheel and the plate. Consider the particular cases (*a*) when the mass of the wheel is concentrated in the rim, (*b*) when the wheel is equivalent to a thin disk of radius a.

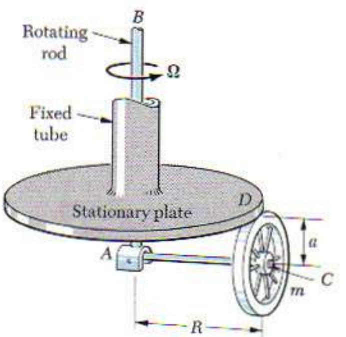


Fig. P18.66

18.67 Assuming that the wheel of Prob. 18.66 weighs 8 lb, has a radius $a = 4$ in. and a radius of gyration of 3 in., and that $R = 20$ in., determine the force exerted by the plate on the wheel when $\Omega = 25$ rad/s.

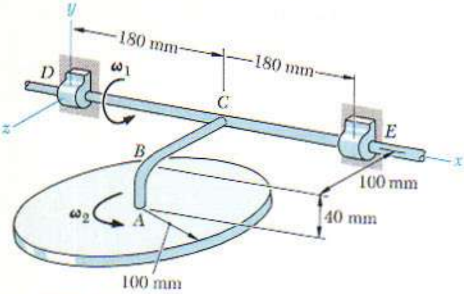


Fig. P18.68

18.68 A thin homogeneous disk of mass 800 g and radius 100 mm rotates at a constant rate $\omega_2 = 20$ rad/s with respect to the arm ABC, which itself rotates at a constant rate $\omega_1 = 10$ rad/s about the x axis. For the position shown, determine the dynamic reactions at the bearings D and E.

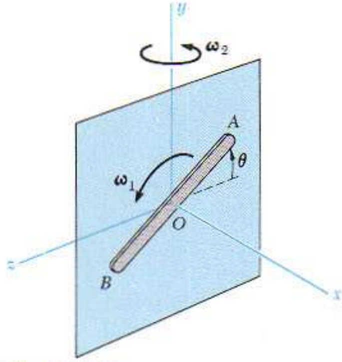


Fig. P18.69

18.69 A slender homogeneous rod AB of mass m and length L is made to rotate at the constant rate ω_1 about the horizontal x axis, while the vertical plane in which it rotates is made to rotate at the constant rate ω_2 about the vertical y axis. Express as a function of the angle θ (*a*) the couple $M_1\mathbf{i}$ required to maintain the rotation of the rod in the vertical plane, (*b*) the couple $M_2\,\mathbf{j}$ required to maintain the rotation of that plane.

*18.9. Motion of a Gyroscope. Eulerian Angles.

A *gyroscope* consists essentially of a rotor which may spin freely about its geometric axis. When mounted in a Cardan's suspension (Fig. 18.15), a gyroscope may assume any orientation, but its mass center must remain fixed in space. In order to define the position of a gyroscope at a given instant, we shall select a fixed frame of reference $OXYZ$, with the origin O located at the mass center of the gyroscope and the Z axis directed along the line defined by the bearings A and A' of the outer gimbal, and we shall consider a reference position of the gyroscope in which the two gimbals and a given diameter DD' of the rotor are located in the fixed YZ plane (Fig. 18.15*a*). The gyroscope may be brought from this reference position into any arbitrary position (Fig. 18.15*b*) by means of the following steps: (1) a rotation of the outer gimbal through an angle ϕ about the axis AA', (2) a rotation of the inner gimbal through θ about BB', (3) a rotation of the rotor through ψ about CC'. The angles ϕ, θ, and ψ are called the *Eulerian angles;* they completely characterize the position of the gyroscope at any given instant. Their derivatives $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$ define, respectively, the rate of *precession*, the rate of *nutation*, and the rate of *spin* of the gyroscope at the instant considered.

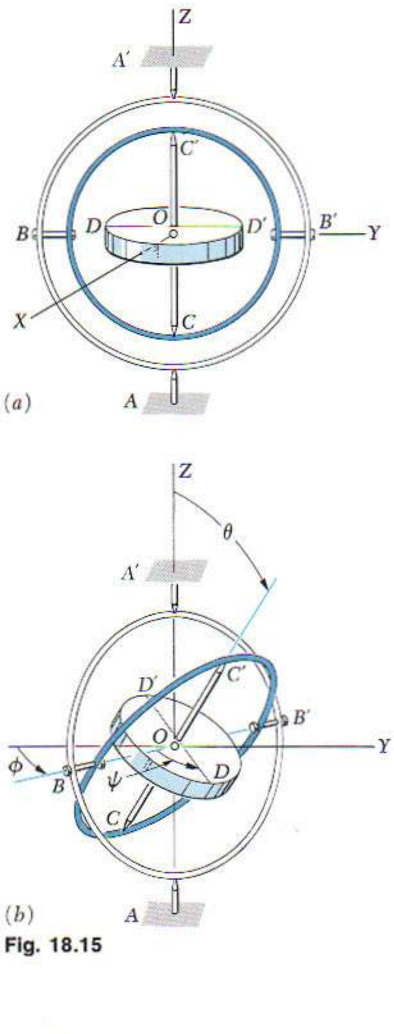


Fig. 18.15

In order to compute the components of the angular velocity and of the angular momentum of the gyroscope, we shall use a rotating system of axes $Oxyz$ *attached to the inner gimbal,* with the y axis along BB' and the z axis along CC' (Fig. 18.16). These axes are principal axes of inertia for the gyroscope but, while they follow it in its precession and nutation, they do not spin. For that reason, they are more convenient to use than axes actually attached to the gyroscope. We shall now express the angular velocity $\boldsymbol{\omega}$ of the gyroscope with respect to the fixed frame of reference $OXYZ$ as the sum of three partial angular velocities corresponding respectively to the precession, the nutation, and the spin of the gyroscope. Denoting by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ the unit vectors along the rotating axes, and by $\mathbf{K}$ the unit vector along the fixed Z axis, we have

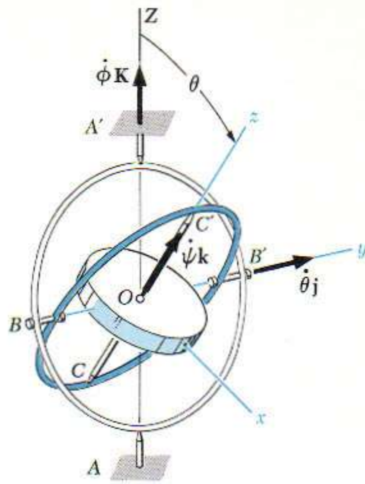


Fig. 18.16

$$\boldsymbol{\omega} = \dot{\phi}\mathbf{K} + \dot{\theta}\mathbf{j} + \dot{\psi}\mathbf{k} \tag{18.33}$$

Since the vector components obtained for $\boldsymbol{\omega}$ in (18.33) are not orthogonal (Fig. 18.16), we shall resolve the unit vector $\mathbf{K}$ into components along the x and z axes; we write

$$\mathbf{K} = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{k} \tag{18.34}$$

and, substituting for $\mathbf{K}$ into (18.33),

$$\boldsymbol{\omega} = -\dot{\phi}\sin\theta\,\mathbf{i} + \dot{\theta}\mathbf{j} + (\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{k} \tag{18.35}$$

Since the coordinate axes are principal axes of inertia, the components of the angular momentum $\mathbf{H}_O$ may be obtained by multiplying the components of $\boldsymbol{\omega}$ by the moments of inertia of the rotor about the x, y, and z axes, respectively. Denoting by I the moment of inertia of the rotor about its spin axis, by I' its moment of inertia about a transverse axis through O, and neglecting the mass of the gimbals, we write

$$\mathbf{H}_O = -I'\dot{\phi}\sin\theta\,\mathbf{i} + I'\dot{\theta}\mathbf{j} + I(\dot{\psi} + \dot{\phi}\cos\theta)\mathbf{k} \tag{18.36}$$

Recalling that the rotating axes are attached to the inner gimbal, and thus do not spin, we express their angular velocity as the sum

$$\boldsymbol{\Omega} = \dot{\phi}\mathbf{K} + \dot{\theta}\mathbf{j} \tag{18.37}$$

or, substituting for $\mathbf{K}$ from (18.34),

$$\boldsymbol{\Omega} = -\dot{\phi}\sin\theta\,\mathbf{i} + \dot{\theta}\mathbf{j} + \dot{\phi}\cos\theta\,\mathbf{k} \tag{18.38}$$

Substituting for $\mathbf{H}_O$ and $\boldsymbol{\Omega}$ from (18.36) and (18.38) into the equation

$$\Sigma \mathbf{M}_O = (\dot{\mathbf{H}}_O)_{Oxyz} + \boldsymbol{\Omega} \times \mathbf{H}_O \tag{18.28}$$

we obtain the three differential equations

$$\begin{aligned}
\Sigma M_x &= -I'(\ddot{\phi} \sin\theta + 2\dot{\theta}\dot{\phi}\cos\theta) + I\dot{\theta}(\dot{\psi} + \dot{\phi}\cos\theta) \\
\Sigma M_y &= I'(\ddot{\theta} - \dot{\phi}^2 \sin\theta\cos\theta) + I\dot{\phi}\sin\theta(\dot{\psi} + \dot{\phi}\cos\theta) \qquad (18.39) \\
\Sigma M_z &= I\frac{d}{dt}(\dot{\psi} + \dot{\phi}\cos\theta)
\end{aligned}$$

The equations (18.39) define the motion of a gyroscope subjected to a given system of forces when the mass of its gimbals is neglected. They may also be used to define the motion of an *axisymmetrical body* (or body of revolution) attached at a point on its axis of symmetry, or the motion of an axisymmetrical body about its mass center. While the gimbals of the gyroscope helped us visualize the Eulerian angles, it is clear that these angles may be used to define the position of any rigid body with respect to axes centered at a point of the body, regardless of the way in which the body is actually supported.

Since the equations (18.39) are nonlinear, it will not be possible, in general, to express the Eulerian angles ϕ, θ, and ψ as analytical functions of the time t, and numerical methods of solution may have to be used. However, as we shall see in the following sections, there are several particular cases of interest which may be analyzed easily.

***18.10. Steady Precession of a Gyroscope.** We shall consider in this section the particular case of gyroscopic motion in which the angle θ, the rate of precession $\dot{\phi}$, and the rate of spin $\dot{\psi}$ remain constant. We propose to determine the forces which must be applied to the gyroscope to maintain this motion, known as the *steady precession* of a gyroscope.

Instead of applying the general equations (18.39), we shall determine the sum of the moments of the required forces by computing the rate of change of the angular momentum of the gyroscope in the particular case considered. We first note that the angular velocity $\boldsymbol{\omega}$ of the gyroscope, its angular momentum $\mathbf{H}_O$, and the angular velocity $\boldsymbol{\Omega}$ of the rotating frame of reference (Fig. 18.17) reduce, respectively, to

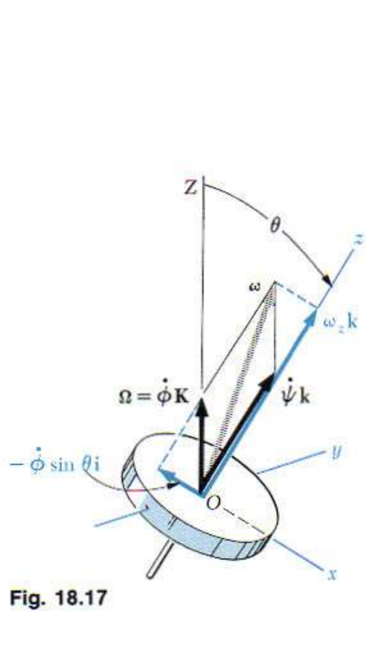


Fig. 18.17

$$\boldsymbol{\omega} = -\dot{\phi}\sin\theta\,\mathbf{i} + \omega_z\,\mathbf{k} \tag{18.40}$$
$$\mathbf{H}_O = -I'\dot{\phi}\sin\theta\,\mathbf{i} + I\omega_z\,\mathbf{k} \tag{18.41}$$
$$\boldsymbol{\Omega} = -\dot{\phi}\sin\theta\,\mathbf{i} + \dot{\phi}\cos\theta\,\mathbf{k} \tag{18.42}$$

where $\omega_z = \dot{\psi} + \dot{\phi}\cos\theta$ = component along the spin axis of the total angular velocity of the gyroscope

Since θ, $\dot{\phi}$, and $\dot{\psi}$ are constant, the vector $\mathbf{H}_O$ is constant in magnitude and direction with respect to the rotating frame of reference, and its rate of change $(\dot{\mathbf{H}}_O)_{Oxyz}$ with respect to that frame is zero. Thus Eq. (18.28) reduces to

$$\Sigma \mathbf{M}_O = \boldsymbol{\Omega} \times \mathbf{H}_O \tag{18.43}$$

which yields, after substitutions from (18.41) and (18.42),

$$\Sigma \mathbf{M}_O = (I\omega_z - I'\dot{\phi}\cos\theta)\dot{\phi}\sin\theta\,\mathbf{j} \tag{18.44}$$

Since the mass center of the gyroscope is fixed in space, we have, by (18.1), $\Sigma \mathbf{F} = 0$; thus, the forces which must be applied to the gyroscope to maintain its steady precession reduce to a couple of moment equal to the right-hand member of Eq. (18.44). We note that *this couple should be applied about an axis perpendicular to the precession axis and to the spin axis of the gyroscope* (Fig. 18.18).

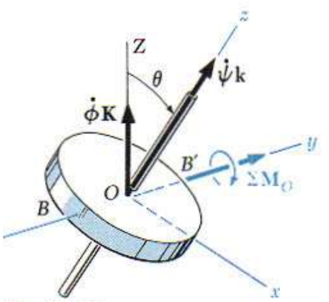


Fig. 18.18

In the particular case when the precession axis and the spin axis are at a right angle to each other, we have $\theta = 90°$ and Eq. (18.44) reduces to

$$\Sigma \mathbf{M}_O = I\dot{\psi}\dot{\phi}\mathbf{j} \tag{18.45}$$

Thus, if we apply to the gyroscope a couple $\mathbf{M}_O$ about an axis perpendicular to its axis of spin, the gyroscope will precess about an axis perpendicular to both the spin axis and the couple axis, in a sense such that the vectors representing respectively the spin, the couple, and the precession form a right-handed triad (Fig. 18.19).

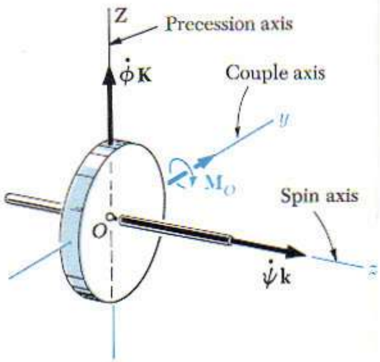


Fig. 18.19

Because of the relatively large couples required to change the orientation of their axles, gyroscopes are used as stabilizers in torpedoes and ships. Spinning bullets and shells remain tangent to their trajectory because of gyroscopic action. And a bicycle is easier to keep balanced at high speeds because of the stabilizing effect of its spinning wheels. However, gyroscopic action is not always welcome and must be taken into account in the design of bearings supporting rotating shafts subjected to forced precession. The reactions exerted by its propellers on an airplane which changes its direction of flight must also be taken into consideration and compensated for whenever possible.

∗18.11. Motion of an Axisymmetrical Body under No Force.

We shall consider in this section the motion about its mass center of an axisymmetrical body under no force, except its own weight. Examples of such a motion are furnished by projectiles, if air resistance is neglected, and by artificial satellites and space vehicles after burnout of their launching rockets.

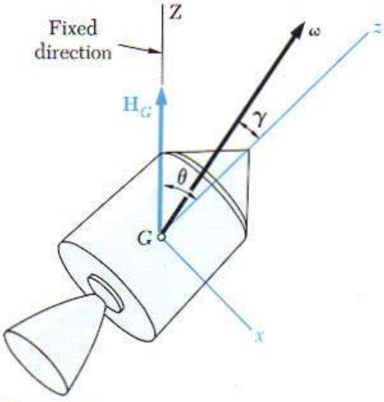


Fig. 18.20

Since the sum of the moments of the external forces about the mass center G of the body is zero, Eq. (18.2) yields $\dot{\mathbf{H}}_G = 0$. It follows that the angular momentum $\mathbf{H}_G$ of the body about G is constant. Thus, the direction of $\mathbf{H}_G$ is fixed in space and may be used to define the Z axis, or axis of precession (Fig. 18.20). Selecting a rotating system of axes $Gxyz$ with the z axis along the axis of symmetry of the body and the x axis in the plane defined by the Z and z axes, we have

$$H_x = -H_G \sin\theta \qquad H_y = 0 \qquad H_z = H_G \cos\theta \quad (18.46)$$

where θ represents the angle formed by the Z and z axes, and H_G denotes the constant magnitude of the angular momentum of the body about G. Since the x, y, and z axes are principal axes of inertia for the body considered, we may write

$$H_x = I'\omega_x \qquad H_y = I'\omega_y \qquad H_z = I\omega_z \qquad (18.47)$$

where I denotes the moment of inertia of the body about its axis of symmetry, and I' its moment of inertia about a transverse axis through G. It follows from Eqs. (18.46) and (18.47) that

$$\omega_x = -\frac{H_G \sin\theta}{I'} \qquad \omega_y = 0 \qquad \omega_z = \frac{H_G \cos\theta}{I} \quad (18.48)$$

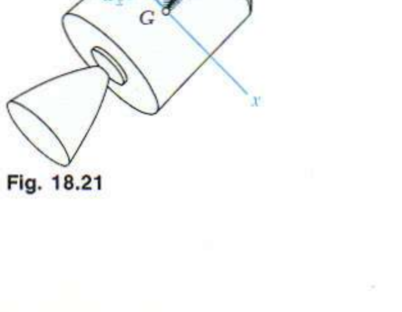


Fig. 18.21

The second of the relations obtained shows that the angular velocity $\boldsymbol{\omega}$ has no component along the y axis, i.e., along an axis perpendicular to the Zz plane. Thus, the angle θ formed by the Z and z axes remains constant and *the body is in steady precession about the Z axis.*

Dividing the first and third of the relations (18.48) member by member, and observing from Fig. 18.21 that $-\omega_x/\omega_z = \tan\gamma$, we obtain the following relation between the angles γ and θ that the vectors $\boldsymbol{\omega}$ and $\mathbf{H}_G$ respectively form with the axis of symmetry of the body:

$$\tan\gamma = \frac{I}{I'}\tan\theta \qquad (18.49)$$

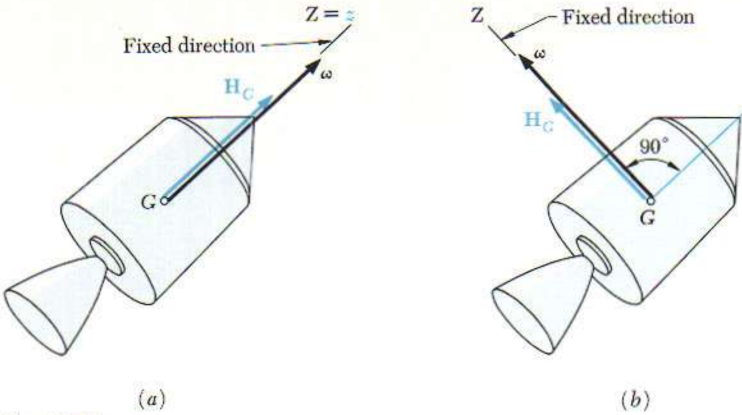


Fig. 18.22

There are two particular cases of motion of an axisymmetrical body under no force which involve no precession: (1) If the body is set to spin about its axis of symmetry, we have $\omega_x = 0$ and, by (18.47), $H_x = 0$; the vectors $\boldsymbol{\omega}$ and $\mathbf{H}_G$ have the same orientation and the body keeps spinning about its axis of symmetry (Fig. 18.22*a*). (2) If the body is set to spin about a transverse axis, we have $\omega_z = 0$ and, by (18.47), $H_z = 0$; again $\boldsymbol{\omega}$ and $\mathbf{H}_G$ have the same orientation and the body keeps spinning about the given transverse axis (Fig. 18.22*b*).

Considering now the general case represented in Fig. 18.21, we recall from Sec. 15.12 that the motion of a body about a fixed point—or about its mass center—may be represented by the motion of a body cone rolling on a space cone. In the case of steady precession, the two cones are circular, since the angles γ and $\theta - \gamma$ that the angular velocity $\boldsymbol{\omega}$ forms, respectively, with the axis of symmetry of the body and with the precession axis are constant. Two cases should be distinguished:

1. $I < I'$. This is the case of an elongated body, such as the space vehicle of Fig. 18.23. By (18.49) we have $\gamma < \theta$; the vector $\boldsymbol{\omega}$ lies inside the angle ZGz; the space cone and the body cone are tangent externally; the spin and the precession are both observed as counterclockwise from the positive z axis. The precession is said to be *direct*.
2. $I > I'$. This is the case of a flattened body, such as the satellite of Fig. 18.24. By (18.49) we have $\gamma > \theta$; since the vector $\boldsymbol{\omega}$ must lie outside the angle ZGz, the vector $\dot{\psi}\mathbf{k}$ has a sense opposite to that of the z axis; the space cone is inside the body cone; the precession and the spin have opposite senses; the precession is said to be *retrograde*.

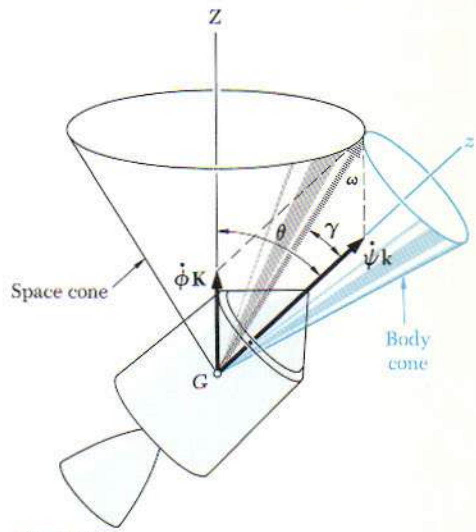


Fig. 18.23

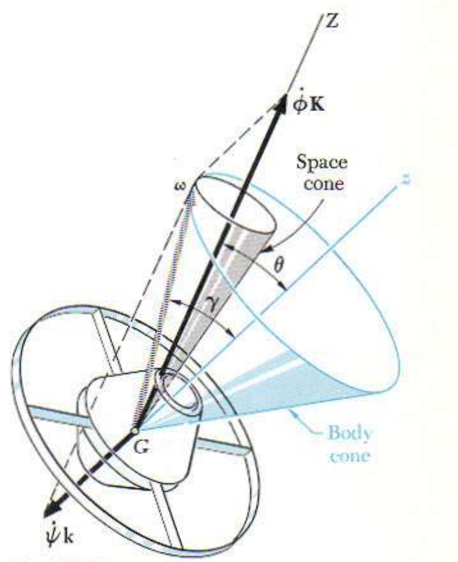
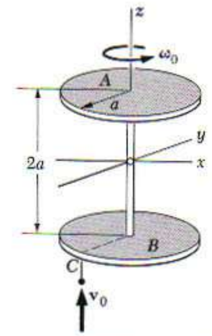
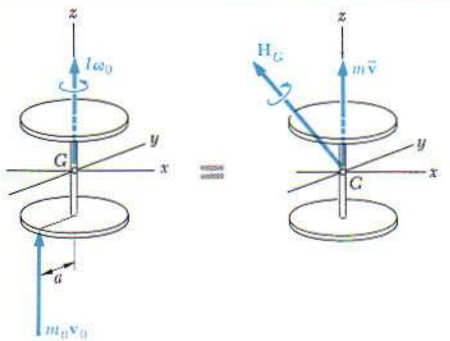


Fig. 18.24

SAMPLE PROBLEM 18.6



A space satellite of mass m is known to be dynamically equivalent to two thin disks of equal mass. The disks are of radius $a = 800$ mm and are rigidly connected by a light rod of length $2a$. Initially the satellite is spinning freely about its axis of symmetry at the rate $\omega_0 = 60$ rpm. A meteorite, of mass $m_0 = m/1000$ and traveling with a velocity $\mathbf{v}_0$ of 2000 m/s relative to the satellite, strikes the satellite and becomes embedded at C. Determine (a) the angular velocity of the satellite immediately after impact, (b) the precession axis of the ensuing motion, (c) the rates of precession and spin of the ensuing motion.



Solution. *Moments of Inertia.* We note that the axes shown are principal axes of inertia for the satellite and write

$$I = I_z = \tfrac{1}{2}ma^2 \qquad I' = I_x = I_y = 2[\tfrac{1}{4}(\tfrac{1}{2}m)a^2 + (\tfrac{1}{2}m)a^2] = \tfrac{5}{4}ma^2$$

Principle of Impulse and Momentum. We consider the satellite and the meteorite as a single system. Since no external force acts on this system, the momenta before and after impact are equipollent. Taking moments about G we write

$$-a\mathbf{j} \times m_0v_0\mathbf{k} + I\omega_0\mathbf{k} = \mathbf{H}_G$$

$$\mathbf{H}_G = -m_0v_0a\mathbf{i} + I\omega_0\mathbf{k} \tag{1}$$

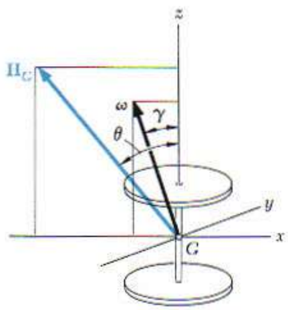
Angular Velocity after Impact. Substituting the values obtained for the components of $\mathbf{H}_G$ and for the moments of inertia into

$$H_x = I_x\omega_x \qquad H_y = I_y\omega_y \qquad H_z = I_z\omega_z$$

we write

$$-m_0v_0a = I'\omega_x = \tfrac{5}{4}ma^2\omega_x \qquad 0 = I'\omega_y \qquad I\omega_0 = I\omega_z$$

$$\omega_x = -\frac{4}{5}\frac{m_0v_0}{ma} \qquad \omega_y = 0 \qquad \omega_z = \omega_0 \tag{2}$$



For the satellite considered we have $\omega_0 = 60 \text{ rpm} = 6.28$ rad/s, $m_0/m = 1/1000$, $a = 0.800$ m, and $v_0 = 2000$ m/s; we find

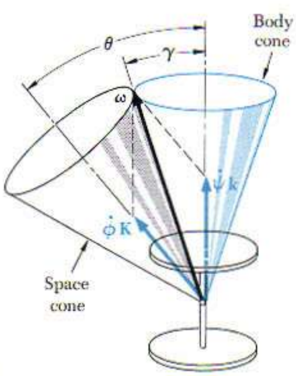
$$\omega_x = -2 \text{ rad/s} \qquad \omega_y = 0 \qquad \omega_z = 6.28 \text{ rad/s}$$

$$\omega = \sqrt{\omega_x^2 + \omega_z^2} = 6.59 \text{ rad/s} \qquad \tan\gamma = \frac{-\omega_x}{\omega_z} = +0.3185$$

$\omega = 63.0$ rpm $\quad \gamma = 17.7°$ ◄

Precession Axis. Since, in free motion, the direction of the angular momentum $\mathbf{H}_G$ is fixed in space, the satellite will precess about this direction. The angle θ formed by the precession axis and the z axis is

$$\tan\theta = \frac{-h_x}{h_z} = \frac{m_0v_0a}{I\omega_0} = \frac{2m_0v_0}{ma\omega_0} = 0.796 \qquad \theta = 38.5° \blacktriangleleft$$



Rates of Precession and Spin. We sketch the space and body cones for the free motion of the satellite. Using the law of sines, we compute the rates of precession and spin.

$$\frac{\omega}{\sin\theta} = \frac{\dot{\phi}}{\sin\gamma} = \frac{\dot{\psi}}{\sin(\theta - \gamma)}$$

$\dot{\phi} = 30.7$ rpm $\quad \dot{\psi} = 36.0$ rpm ◄

PROBLEMS

18.70 The rate of steady precession $\dot{\phi}$ of the cone shown about the vertical is observed to be 30 rpm. Knowing that $r = 75$ mm and $h = 300$ mm, determine the rate of spin $\dot{\psi}$ of the cone about its axis of symmetry if $\beta = 120°$.

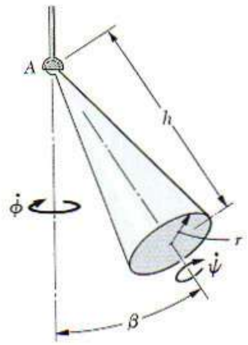


Fig. P18.70

18.71 Solve Prob. 18.70, assuming the same rate of steady precession and $\beta = 60°$.

18.72 A 5-lb disk of 9-in. diameter is attached to the end of a rod AB of negligible weight which is supported by a ball-and-socket joint at A. If the rate of steady precession $\dot{\phi}$ of the disk about the vertical is observed to be 24 rpm, determine the rate of spin $\dot{\psi}$ of the disk about AB when $\beta = 60°$.

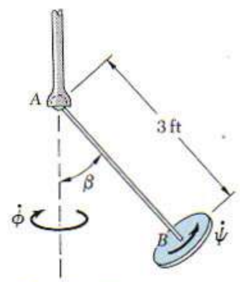


Fig. P18.72

18.73 Solve Prob. 18.72, assuming the same rate of steady precession and $\beta = 30°$.

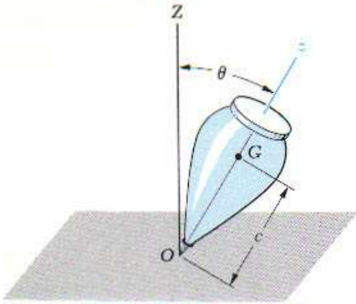


Fig. P18.74, P18.75, and 18.76

18.74 The top shown is supported at the fixed point O. Denoting by I and I', respectively, the moments of inertia of the top about its axis of symmetry and about a transverse axis through O, show that the condition for steady precession is

$$(I\omega_z - I'\dot{\phi}\cos\theta)\dot{\phi} = Wc$$

where $\dot{\phi}$ is the rate of precession and ω_z the component of the angular velocity along the axis of symmetry of the top.

18.75 Show that, if the rate of spin $\dot{\psi}$ of a top is very large compared to its rate of precession $\dot{\phi}$, the condition for steady precession is $I\dot{\psi}\dot{\phi} \approx Wc$.

18.76 The top shown weighs 0.2 lb and is supported at the fixed point O. The radii of gyration of the top with respect to its axis of symmetry and with respect to a transverse axis through O are 0.75 in. and 1.75 in., respectively. It is known that $c = 1.50$ in. and that the rate of spin of the top with respect to its axis of symmetry is 1600 rpm. (*a*) Using the relation of Prob. 18.74, determine the two possible rates of steady precession corresponding to $\theta = 30°$. (*b*) Determine the relative error introduced when the slower of the two rates obtained in part *a* is approximated by the relation of Prob. 18.75.

18.77 If the earth were a sphere, the gravitational attraction of the sun, moon, and planets would at all times be equivalent to a single force **R** acting at the mass center of the earth. However, the earth is actually an oblate spheroid and the gravitational system acting on the earth is equivalent to a force **R** and a couple **M**. Knowing that the effect of the couple **M** is to cause the axis of the earth to precess about the axis *GA* at the rate of one revolution in 25,800 years, determine the average magnitude of the couple **M** applied to the earth. Assume that the average density of the earth is 5.51, that the average radius of the earth is 3960 mi, and that $\bar{I} = \frac{2}{5}mR^2$. (*Note.* This forced precession is known as the precession of the equinoxes and is not to be confused with the free precession discussed in Prob. 18.85.)

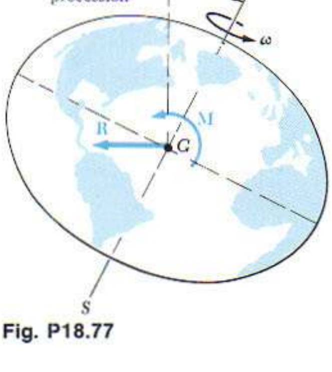


Fig. P18.77

18.78 A high-speed photographic record shows that a certain projectile was fired with a horizontal velocity $\overline{\mathbf{v}}$ of 2000 ft/s and with its axis of symmetry forming an angle $\beta = 3°$ with the horizontal. The rate of spin $\dot{\psi}$ of the projectile was 6000 rpm, and the atmospheric drag was equivalent to a force $\mathbf{D}$ of 25 lb acting at the center of pressure C_P located at a distance $c = 3$ in. from G. (*a*) Knowing that the projectile weighs 40 lb and has a radius of gyration of 2 in. with respect to its axis of symmetry, determine its approximate rate of steady precession. (*b*) If it is further known that the radius of gyration of the projectile with respect to a transverse axis through G is 8 in., determine the exact values of the two possible rates of precession.

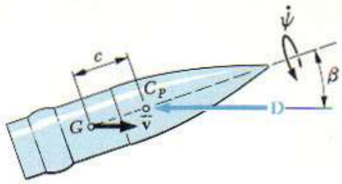


Fig. P18.78

18.79 The essential features of the gyrocompass are shown. The rotor spins at the rate $\dot{\psi}$ about an axis mounted in a single gimbal, which may rotate freely about the vertical axis AB. The angle formed by the axis of the rotor and the plane of the meridian is denoted by θ and the latitude of the position on the earth is denoted by λ. We note that the line OC is parallel to the axis of the earth and we denote by $\boldsymbol{\omega}_e$ the angular velocity of the earth about its axis.

(*a*) Show that the equations of motion of the gyrocompass are

$$I'\ddot{\theta} + I\omega_z\omega_e \cos\lambda \sin\theta - I'\omega_e^2 \cos^2\lambda \sin\theta \cos\theta = 0$$
$$I\dot{\omega}_z = 0$$

where ω_z is the component of the total angular velocity along the axis of the rotor, and I and I' are the moments of inertia of the rotor with respect to its axis of symmetry and a transverse axis through O, respectively.

(*b*) Neglecting the term containing ω_e^2, show that, for small values of θ, we have

$$\ddot{\theta} + \frac{I\omega_z\omega_e \cos\lambda}{I'}\theta = 0$$

and that the axis of the gyrocompass oscillates about the north-south direction.

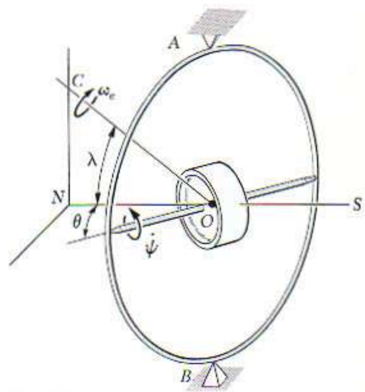


Fig. P18.79

18.80 Show that, for an axisymmetrical body under no force, the rates of precession and spin may be expressed, respectively, as

$$\dot{\phi} = \frac{H_G}{I'}$$

and

$$\dot{\psi} = \frac{H_G \cos\theta\,(I' - I)}{I\,I'}$$

where H_G is the constant value of the angular momentum of the body.

18.81 (*a*) Show that, for an axisymmetrical body under no force, the rate of precession may be expressed as

$$\dot{\phi} = \frac{I\omega_z}{I' \cos \theta}$$

where ω_z is the component of $\boldsymbol{\omega}$ along the axis of symmetry of the body. (*b*) Use this result to check that the condition (18.44) for steady precession is satisfied by an axisymmetrical body under no force.

18.82 Show that the angular velocity vector $\boldsymbol{\omega}$ of an axisymmetrical body under no force is observed from the body itself to rotate about the axis of symmetry at the constant rate

$$n = \frac{I' - I}{I'} \omega_z$$

where ω_z is the component of $\boldsymbol{\omega}$ along the axis of symmetry of the body.

18.83 For an axisymmetrical body under no force, prove (*a*) that the rate of retrograde precession can never be less than twice the rate of spin of the body about its axis of symmetry, (*b*) that in Fig. 18.24 the axis of symmetry of the body can never lie within the space cone.

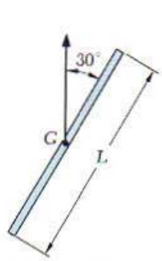


Fig. P18.84

18.84 Determine the precession axis and the rates of precession and spin of a rod which is given an initial angular velocity $\boldsymbol{\omega}$ of 12 rad/s in the direction shown.

18.85 Using the relation given in Prob. 18.82, determine the period of precession of the north pole of the earth about the axis of symmetry of the earth. The earth may be approximated by an oblate spheroid of axial moment of inertia I and of transverse moment of inertia $I' = 0.9967I$. (*Note.* Actual observations show a period of precession of the north pole of about 432.5 mean solar days; the difference between the observed and computed periods is due to the fact that the earth is not a perfectly rigid body. The free precession considered here should not be confused with the much slower precession of the equinoxes, which is a forced precession. See Prob. 18.77.)

18.86 Determine the precession axis and the rates of precession and spin of the satellite of Prob. 18.28 after the impact.

18.87 Determine the precession axis and the rates of precession and spin of the satellite of Prob. 18.28 knowing that, before impact, the angular velocity of the satellite was $\boldsymbol{\omega}_0 = -(12 \text{ rpm})\mathbf{i}$.

18.88 The space capsule has no angular velocity when the jet at A is activated for 1 s in a direction parallel to the x axis. Knowing that the capsule has a mass of 1000 kg, that its radii of gyration are $\bar{k}_x = \bar{k}_y = 1.00$ m and $\bar{k}_z = 1.25$ m, and that the jet at A produces a thrust of 50 N, determine the axis of precession and the rates of precession and spin after the jet has stopped.

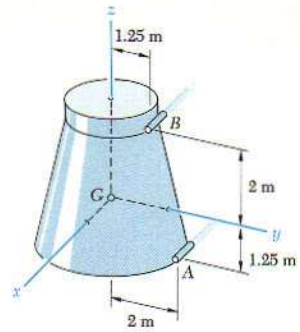


Fig. P18.88 and P18.89

18.89 The space capsule has an angular velocity $\boldsymbol{\omega} = (0.02 \text{ rad/s})\mathbf{j} + (0.10 \text{ rad/s})\mathbf{k}$ when the jet at B is activated for 1 s in a direction parallel to the x axis. Knowing that the capsule has a mass of 1000 kg, that its radii of gyration are $\bar{k}_x = \bar{k}_y = 1.00$ m and $\bar{k}_z = 1.25$ m, and that the jet at B produces a thrust of 50 N, determine the axis of precession and the rates of precession and spin after the jet has stopped.

18.90 The space station shown is known to precess about the fixed direction OC at the rate of one revolution per hour. Assuming that the station is dynamically equivalent to a homogeneous cylinder of length 100 ft and radius 10 ft, determine the rate of spin of the station about its axis of symmetry.

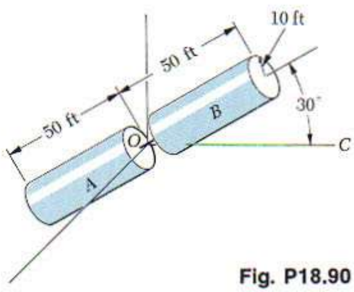


Fig. P18.90

18.91 The link connecting portions A and B of the space station of Prob. 18.90 may be severed to allow each portion to move freely. Each portion of the station is dynamically equivalent to a cylinder of length 50 ft and radius 10 ft. Knowing that, when the link is severed, the station is oriented as shown, determine for portion B the axis of precession, the rate of precession, and the rate of spin about the axis of symmetry.

18.92 Solve Sample Prob. 18.6, assuming that the meteorite strikes the satellite at C with a velocity $\mathbf{v}_0 = -(2000 \text{ m/s})\mathbf{i}$.

18.93 After the motion determined in Sample Prob. 18.6 has been established, the rod connecting disks A and B of the satellite breaks, and disk A moves freely as a separate body. Knowing that the rod and the z axis coincide when the rod breaks, determine the precession axis, the rate of precession, and the rate of spin for the ensuing motion of disk A.

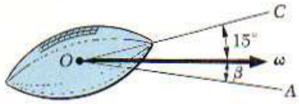


Fig. P18.94

18.94 The angular velocity vector of a football which has just been kicked is horizontal, and its axis of symmetry OC is oriented as shown. Knowing that the magnitude of the angular velocity is 180 rpm and that the ratio of the axial and transverse moments of inertia is $I/I' = 1/3$, determine (*a*) the orientation of the axis of precession OA, (*b*) the rates of precession and spin.

18.95 A slender homogeneous rod OA of mass m and length L is supported by a ball-and-socket joint at O and may swing freely under its own weight. If the rod is held in a horizontal position ($\theta = 90°$) and given an initial angular velocity $\dot{\phi}_0 = \sqrt{8g/L}$ about the vertical OB, determine (*a*) the smallest value of θ in the ensuing motion, (*b*) the corresponding value of the angular velocity $\dot{\phi}$ of the rod about OB. (*Hint.* Apply the principle of conservation of energy and the principle of impulse and momentum, observing that, since $\Sigma M_{OB} = 0$, the component of $\mathbf{H}_O$ along OB must be constant.)

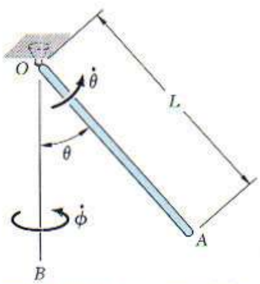


Fig. P18.95 and P18.96

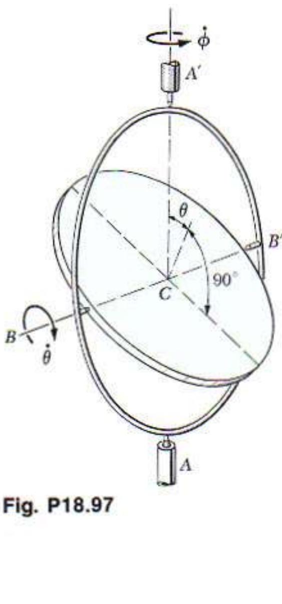


Fig. P18.97

18.96 A slender homogeneous rod OA of mass m and length L is supported by a ball-and-socket joint at O and may swing freely under its own weight. If the rod is held in a horizontal position ($\theta = 90°$), what initial angular velocity $\dot{\phi}_0$ should be given to the rod about the vertical OB if the smallest value of θ in the ensuing motion is to be 60°? (See hint of Prob. 18.95.)

18.97 The gimbal $ABA'B'$ is of negligible mass and may rotate freely about the vertical AA'. The uniform disk of radius a and mass m may rotate freely about its diameter BB', which is also the horizontal diameter of the gimbal. (*a*) Applying the principle of conservation of energy, and observing that, since $\Sigma M_{AA'} = 0$, the component of the angular momentum of the disk along the fixed axis AA' must be constant, write two first-order differential equations defining the motion of the disk. (*b*) Given the initial conditions $\theta_0 \neq 0$, $\dot{\phi}_0 \neq 0$, and $\dot{\theta}_0 = 0$, express the rate of nutation $\dot{\theta}$ as a function of θ. (*c*) Show that the angle θ will never be larger than θ_0 during the ensuing motion.

***18.98** The top shown is supported at the fixed point O. We denote by ϕ, θ, and ψ the Eulerian angles defining the position of the top with respect to a fixed frame of reference. We shall consider the general motion of the top in which all Eulerian angles vary.

(*a*) Observing that $\Sigma M_Z = 0$ and $\Sigma M_z = 0$, and denoting by I and I', respectively, the moments of inertia of the top about its axis of symmetry and about a transverse axis through O, derive the two first-order differential equations of motion

$$I'\dot{\phi}\sin^2\theta + I(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta = \alpha \qquad (1)$$
$$I(\dot{\psi} + \dot{\phi}\cos\theta) = \beta \qquad (2)$$

where α and β are constants depending upon the initial conditions. These equations express that the angular momentum of the top is conserved about both the Z and z axes, i.e., that the rectangular component of $\mathbf{H}_O$ along each of these axes is constant.

(*b*) Use Eqs. (1) and (2) to show that the component ω_z of the angular velocity of the top is constant and that the rate of precession $\dot{\phi}$ depends upon the value of the angle of nutation θ.

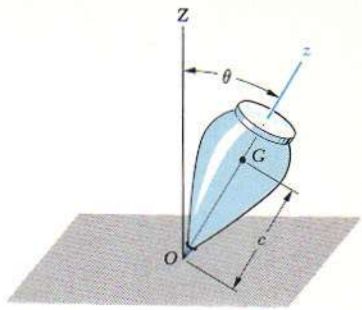


Fig. P18.98

***18.99** (*a*) Applying the principle of conservation of energy, derive a third differential equation for the general motion of the top of Prob. 18.98. (*b*) Eliminating the derivatives $\dot{\phi}$ and $\dot{\psi}$ from the equation obtained and from the two equations of Prob. 18.98, express the rate of nutation $\dot{\theta}$ as a function of the angle θ.

***18.100** A thin homogeneous disk of radius a and mass m is mounted on a light axle OA of length a which is held by a ball-and-socket support at O. The disk is released in the position $\beta = 0$ with a rate of spin $\dot{\psi}_0$, clockwise as viewed from O, and with no precession or nutation. Knowing that the largest value of β in the ensuing motion is 30°, determine in terms of $\dot{\psi}_0$ the rates of precession and spin of the disk when $\beta = 30°$. (*Hint.* The angular momentum of the disk is conserved about both the Z and z axes; see Prob. 18.98, part *a*.)

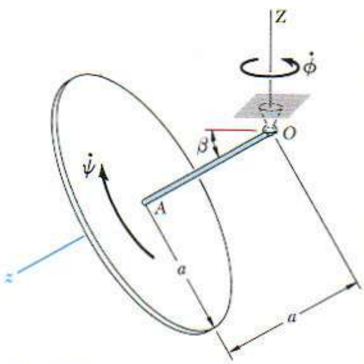


Fig. P18.100

***18.101** For the disk of Prob. 18.100, determine the initial value $\dot{\psi}_0$ of the spin, knowing that the largest value of β in the ensuing motion is 30°. (*Hint.* Use the principle of conservation of energy and the answers obtained for Prob. 18.100.)

***18.102** A solid homogeneous cone of mass m, radius a, and height $h = \frac{3}{2}a$, is held by a ball-and-socket support O. Initially the axis of symmetry of the cone is vertical ($\theta = 0$) with the cone spinning about it at the constant rate $\dot{\psi}_0$, counterclockwise as viewed from above. However, after being slightly disturbed, the cone starts falling and precessing. If the largest value of θ in the ensuing motion is 90°, determine (*a*) the rate of spin $\dot{\psi}_0$ of the cone in its initial vertical position, (*b*) the rates of precession and spin as the cone passes through its lowest position ($\theta = 90°$). (*Hint.* Use the principle of conservation of energy and the fact that the angular momentum of the cone is conserved about both the Z and z axes; see Prob. 18.98, part *a*.)

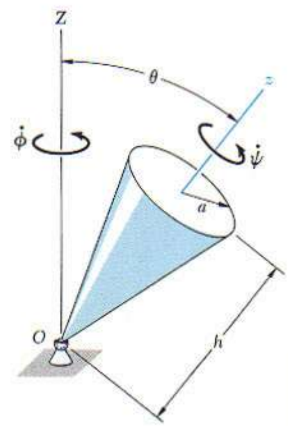


Fig. P18.102

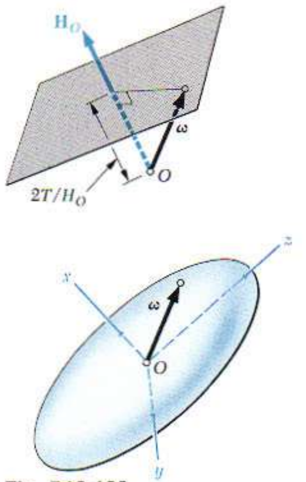


Fig. P18.103

$*$18.103 Consider a rigid body of arbitrary shape which is attached at its mass center O and subjected to no force other than its weight and the reaction of the support at O. (*a*) Prove that the angular momentum $\mathbf{H}_O$ of the body about the fixed point O is constant in magnitude and direction, that the kinetic energy T of the body is constant, and that the projection along $\mathbf{H}_O$ of the angular velocity $\boldsymbol{\omega}$ of the body is constant. (*b*) Show that the tip of the vector $\boldsymbol{\omega}$ describes a curve on a fixed plane in space (called the *invariable plane*), which is perpendicular to $\mathbf{H}_O$ and at a distance $2T/H_O$ from O. (*c*) Show that, with respect to a frame of reference attached to the body and coinciding with its principal axes of inertia, the tip of the vector $\boldsymbol{\omega}$ appears to describe a curve on an ellipsoid of equation

$$I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 = 2T = \text{constant}$$

This ellipsoid (called the *Poinsot ellipsoid*) is rigidly attached to the body and is of the same shape as the ellipsoid of inertia, but of a different size.

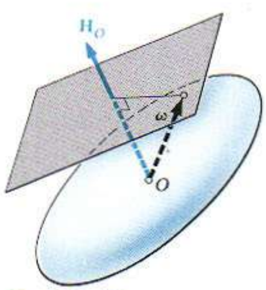


Fig. P18.104

$*$18.104 Referring to Prob. 18.103, (*a*) prove that the Poinsot ellipsoid is tangent to the invariable plane, (*b*) show that the motion of the rigid body must be such that the Poinsot ellipsoid appears to roll on the invariable plane. (*Hint.* In part *a*, show that the normal to the Poinsot ellipsoid at the tip of $\boldsymbol{\omega}$ is parallel to $\mathbf{H}_O$. It is recalled that the direction of the normal to a surface of equation $F(x,y,z)$ = constant at a point P is the same as that of **grad** F at point P.)

$*$18.105 Using the results obtained in Probs. 18.103 and 18.104, show that, for an axisymmetrical body attached at its mass center O and under no force other than its weight and the reaction at O, the Poinsot ellipsoid is an ellipsoid of revolution and the space and body cones are both circular and are tangent to each other. Further show that (*a*) the two cones are tangent externally, and the precession is direct, when $I < I'$, where I and I' denote, respectively, the axial and transverse moment of inertia of the body, (*b*) the space cone is inside the body cone, and the precession is retrograde, when $I > I'$.

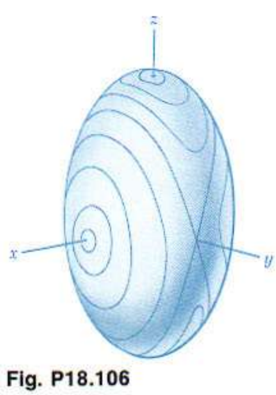


Fig. P18.106

$*$18.106 Referring to Probs. 18.103 and 18.104, (*a*) show that the curve (called *polhode*) described by the tip of the vector $\boldsymbol{\omega}$ with respect to a frame of reference coinciding with the principal axes of inertia of the rigid body is defined by the equations

$$I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2 = 2T = \text{constant} \tag{1}$$

$$I_x^2\omega_x^2 + I_y^2\omega_y^2 + I_z^2\omega_z^2 = H_O^2 = \text{constant} \tag{2}$$

and that this curve may, therefore, be obtained by intersecting the Poinsot ellipsoid with the ellipsoid defined by Eq. (2). (*b*) Further show, assuming $I_x > I_y > I_z$, that the polhodes obtained for various values of H_O have the shapes indicated in the figure. (*c*) Using the result obtained in part *b*, show that a rigid body under no force can rotate about a fixed centroidal axis if, and only if, that axis coincides with one of the principal axes of inertia of the body, and that the motion will be stable if the axis of rotation coincides with the major or minor axis of the Poinsot ellipsoid (z or x axis in the figure) and unstable if it coincides with the intermediate axis (y axis).

REVIEW PROBLEMS

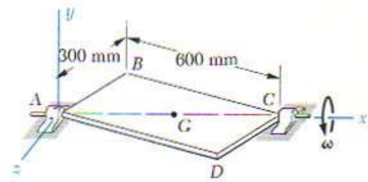


Fig. P18.107 and P18.108

18.107 A thin rectangular plate of mass 9 kg is attached to a shaft as shown. If at the instant shown the angular velocity $\boldsymbol{\omega}$ of the plate is 4 rad/s and is increasing at the rate of 8 rad/s^2, determine (*a*) the couple $\mathbf{M}$ which must be applied to the shaft, (*b*) the corresponding dynamic reactions at A and C.

18.108 A thin rectangular plate of mass 9 kg is attached to a shaft as shown. A couple of moment (3N · m)$\mathbf{i}$ is applied to the plate which is initially at rest. Determine (*a*) the angular acceleration of the plate, (*b*) the dynamic reactions at A and C as the plate reaches an angular velocity of 10 rad/s.

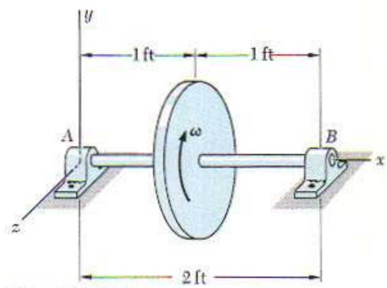


Fig. P18.109

18.109 The rotor of a given turbine may be approximated by a 50-lb disk of 12-in. radius. Knowing that the turbine rotates clockwise at 10,000 rpm as viewed from the positive x axis, determine the components due to gyroscopic action of the forces exerted by the bearings on axle AB if the instantaneous angular velocity of the turbine housing is 2 rad/s clockwise as viewed from (*a*) the positive y axis, (*b*) the positive x axis.

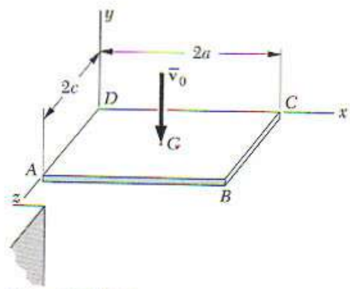


Fig. P18.110

18.110 The rectangular plate shown is falling with a velocity $\bar{\mathbf{v}}_0$ and no angular velocity when its corner A strikes an obstruction. Assuming the impact at A is perfectly plastic, determine immediately after impact (*a*) the angular velocity of the plate, (*b*) the velocity of the mass center G of the plate.

***18.111** Solve Prob. 18.110, assuming that the impact at A is perfectly elastic.

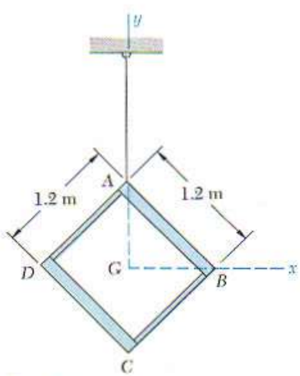


Fig. P18.112

18.112 A rigid square frame $ABCD$ consisting of four slender uniform bars, each 1.2 m long, is suspended by a wire attached at A. Bars AB and CD have each a mass of 25 kg, while bars AD and BC have each a mass of 5 kg. The frame is hit at B in a direction perpendicular to, and into the plane of, the frame. Knowing that the corresponding impulse applied to the frame is 75 N·s, determine immediately after the impact (*a*) the velocity of the mass center of the frame, (*b*) the angular velocity of the frame.

18.113 Solve Prob. 18.112, assuming that the frame is hit at corner C.

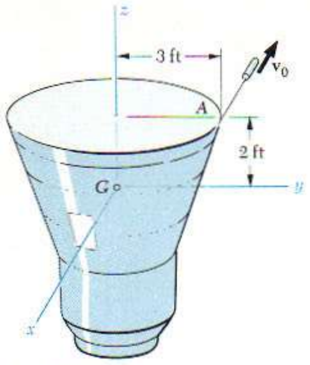


Fig. P18.114

18.114 The 800-lb space capsule is spinning with an angular velocity $\boldsymbol{\omega}_0 = (100 \text{ rpm})\mathbf{k}$ when a 1-lb projectile is fired from A in a direction parallel to the x axis and with a velocity $\mathbf{v}_0$ of 4000 ft/s. Knowing that the radii of gyration of the capsule are $\bar{k}_x = \bar{k}_y = 1.50$ ft and $\bar{k}_z = 2.00$ ft, determine immediately after the projectile has been fired (*a*) the angular velocity of the capsule, (*b*) the kinetic energy of the capsule.

18.115 Determine the precession axis and the rates of precession and spin of the capsule of Prob. 18.114 after the projectile has been fired.

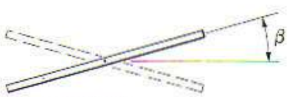


Fig. P18.116

18.116 A coin is tossed into the air. During the free motion the angle β between the plane of the coin and the horizontal is observed to be constant. (*a*) Derive an expression for the angle formed by the angular velocity of the coin and the vertical. (*b*) Denoting by $\dot{\psi}$ the rate of spin of the coin about its axis of symmetry, derive an expression for the rate of precession. (*c*) Solve parts *a* and *b* for the case $\beta = 10°$.

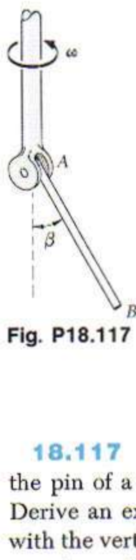


Fig. P18.117

18.117 A uniform rod AB of length l and mass m is attached to the pin of a clevis which rotates with a constant angular velocity ω. Derive an expression (*a*) for the constant angle β that the rod forms with the vertical, (*b*) for the maximum value of ω for which the rod will remain vertical ($\beta = 0$).

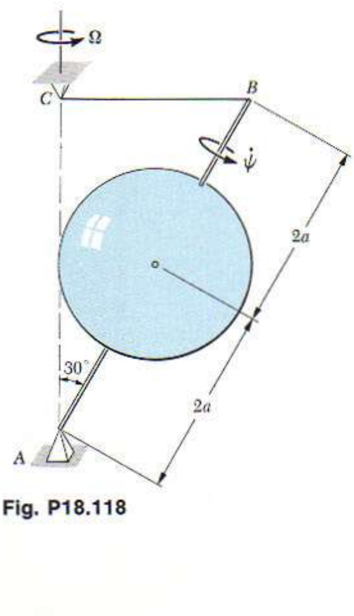


Fig. P18.118

18.118 A homogeneous sphere of radius a and mass m is attached to a light rod of length $4a$. The rod forms an angle of 30° with the vertical and rotates about AC at the constant rate $\Omega = \sqrt{g/a}$. (*a*) Assuming that the sphere does not spin about the rod ($\dot{\psi} = 0$), determine the tension in the cord BC and the kinetic energy of the sphere. (*b*) Determine the spin $\dot{\psi}$ (magnitude and sense) which should be given to the sphere if the tension in the cord BC is to be zero. What is the corresponding kinetic energy of the sphere?

Mechanical Vibrations

CHAPTER

19

19.1. Introduction. A *mechanical vibration* is the motion of a particle or a body which oscillates about a position of equilibrium. Most vibrations in machines and structures are undesirable because of the increased stresses and energy losses which accompany them. They should therefore be eliminated or reduced as much as possible by appropriate design. The analysis of vibrations has become increasingly important in recent years owing to the current trend toward higher-speed machines and lighter structures. There is every reason to expect that this trend will continue and that an even greater need for vibration analysis will develop in the future.

The analysis of vibrations is a very extensive subject to which entire texts have been devoted. We shall therefore limit our present study to the simpler types of vibrations, namely, the vibrations of a body or a system of bodies with one degree of freedom.

A mechanical vibration generally results when a system is displaced from a position of stable equilibrium. The system tends to return to this position under the action of restoring forces (either elastic forces, as in the case of a mass attached to a spring, or gravitational forces, as in the case of a pendulum). But the system generally reaches its original position with a certain acquired velocity which carries it beyond that position. Since the process can be repeated indefinitely, the system keeps moving back and forth across its position of equilibrium. The time interval required for the system to complete a full cycle of motion is called the *period* of the vibration. The number of cycles per unit time defines the *frequency*, and the maximum displacement of the system from its position of equilibrium is called the *amplitude* of the vibration.

When the motion is maintained by the restoring forces only, the vibration is said to be a *free vibration* (Secs. 19.2 to 19.6). When a periodic force is applied to the system, the resulting motion is described as a *forced vibration* (Sec. 19.7). When the effects of friction may be neglected, the vibrations are said to be *undamped*. However, all vibrations are actually *damped* to some degree. If a free vibration is only slightly damped, its amplitude slowly decreases until, after a certain time, the motion comes to a stop. But damping may be large enough to prevent any true vibration; the system then slowly regains its original position (Sec. 19.8). A damped forced vibration is maintained as long as the periodic force which produces the vibration is applied. The amplitude of the vibration, however, is affected by the magnitude of the damping forces (Sec. 19.9).

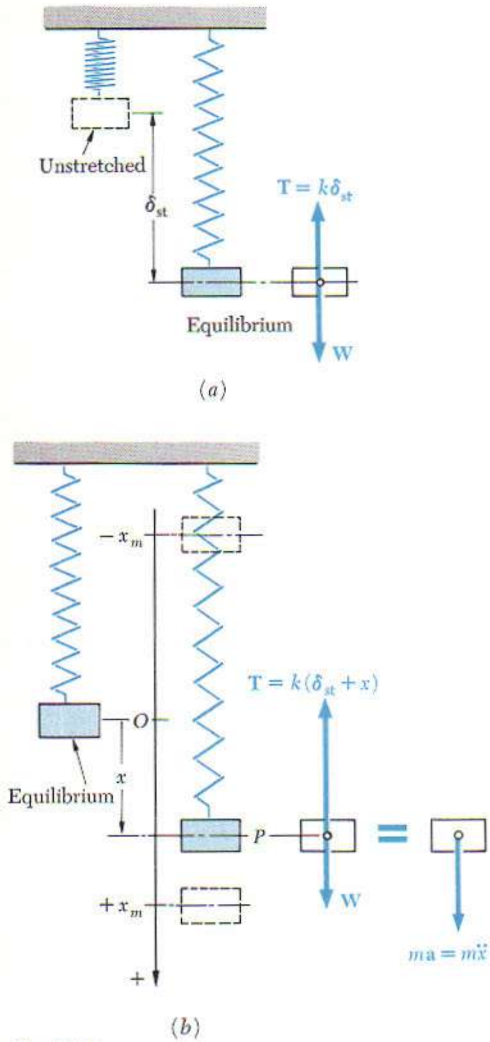


Fig. 19.1

VIBRATIONS WITHOUT DAMPING

19.2. Free Vibrations of Particles. Simple Harmonic Motion. Consider a body of mass m attached to a spring of constant k (Fig. 19.1a). Since, at the present time, we are concerned only with the motion of its mass center, we shall refer to this body as a particle. When the particle is in static equilibrium, the forces acting on it are its weight $\mathbf{W}$ and the force $\mathbf{T}$ exerted by the spring, of magnitude $T = k\delta_{st}$, where δ_{st} denotes the elongation of the spring. We have, therefore,

$$W = k\delta_{st} \tag{19.1}$$

Suppose now that the particle is displaced through a distance x_m from its equilibrium position and released with no initial velocity. If x_m has been chosen smaller than δ_{st}, the particle will move back and forth through its equilibrium position; a vibration of amplitude x_m has been generated. Note that the vibration may also be produced by imparting a certain initial velocity to the particle when it is in its equilibrium position $x = 0$ or, more generally, by starting the particle from any given position $x = x_0$ with a given initial velocity $\mathbf{v}_0$.

To analyze the vibration, we shall consider the particle in a position P at some arbitrary time t (Fig. 19.1b). Denoting by x the displacement OP measured from the equilibrium position

O (positive downward), we note that the forces acting on the particle are its weight $\mathbf{W}$ and the force $\mathbf{T}$ exerted by the spring which, in this position, has a magnitude $T = k(\delta_{st} + x)$. Recalling (19.1), we find that the magnitude of the resultant $\mathbf{F}$ of the two forces (positive downward) is

$$F = W - k(\delta_{st} + x) = -kx \tag{19.2}$$

Thus the *resultant* of the forces exerted on the particle is proportional to the displacement *OP measured from the equilibrium position*. Recalling the sign convention, we note that $\mathbf{F}$ is always directed *toward* the equilibrium position O. Substituting for F into the fundamental equation $F = ma$ and recalling that a is the second derivative $\ddot{x}$ of x with respect to t, we write

$$m\ddot{x} + kx = 0 \tag{19.3}$$

Note that the same sign convention should be used for the acceleration $\ddot{x}$ and for the displacement x, namely, positive downward.

Equation (19.3) is a linear differential equation of the second order. Setting

$$p^2 = \frac{k}{m} \tag{19.4}$$

we write (19.3) in the form

$$\ddot{x} + p^2 x = 0 \tag{19.5}$$

The motion defined by Eq. (19.5) is called *simple harmonic motion*. It is characterized by the fact that *the acceleration is proportional to the displacement and of opposite direction*. We note that each of the functions $x_1 = \sin pt$ and $x_2 = \cos pt$ satisfies (19.5). These functions, therefore, constitute two *particular solutions* of the differential equation (19.5). As we shall see presently, the *general solution* of (19.5) may be obtained by multiplying the two particular solutions by arbitrary constants A and B and adding. We write

$$x = Ax_1 + Bx_2 = A \sin pt + B \cos pt \tag{19.6}$$

Differentiating, we obtain successively the velocity and acceleration at time t,

$$v = \dot{x} = Ap \cos pt - Bp \sin pt \tag{19.7}$$

$$a = \ddot{x} = -Ap^2 \sin pt - Bp^2 \cos pt \tag{19.8}$$

Substituting from (19.6) and (19.8) into (19.5), we verify that the expression (19.6) provides a solution of the differential equation (19.5). Since this expression contains two arbitrary constants A

and B, the solution obtained is the general solution of the differential equation. The values of the constants A and B depend upon the *initial conditions* of the motion. For example, we have $A = 0$ if the particle is displaced from its equilibrium position and released at $t = 0$ with no initial velocity, and we have $B = 0$ if P is started from O at $t = 0$ with a certain initial velocity. In general, substituting $t = 0$ and the initial values x_0 and v_0 of the displacement and velocity into (19.6) and (19.7), we find $A = v_0/p$ and $B = x_0$.

The expressions obtained for the displacement, velocity, and acceleration of a particle may be written in a more compact form if we observe that (19.6) expresses that the displacement $x = OP$ is the sum of the x components of two vectors **A** and **B**, respectively of magnitude A and B, directed as shown in Fig. 19.2*a*. As t varies, both vectors rotate clockwise; we also note that the magnitude of their resultant $\overrightarrow{OQ}$ is equal to the maximum displacement x_m. The simple harmonic motion of P along the x axis may thus be obtained by projecting on this axis the motion of a point Q describing an *auxiliary circle* of radius x_m *with a constant angular velocity* p. Denoting by ϕ the angle formed by the vectors $\overrightarrow{OQ}$ and **A**, we write

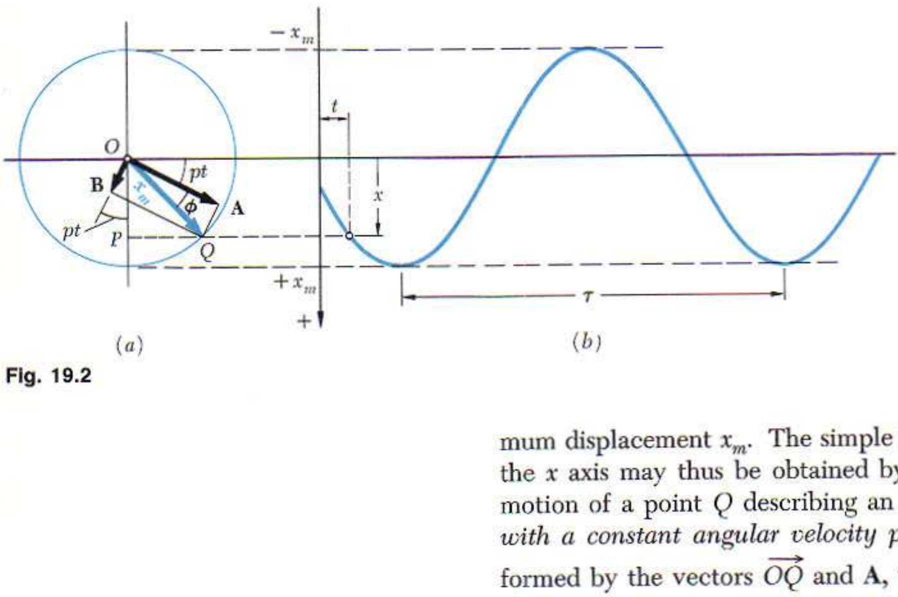


Fig. 19.2

$$OP = OQ \sin(pt + \phi) \tag{19.9}$$

which leads to new expressions for the displacement, velocity, and acceleration of P,

$$x = x_m \sin(pt + \phi) \tag{19.10}$$

$$v = \dot{x} = x_m p \cos(pt + \phi) \tag{19.11}$$

$$a = \ddot{x} = -x_m p^2 \sin(pt + \phi) \tag{19.12}$$

The displacement-time curve is represented by a sine curve (Fig. 19.2*b*), and the maximum value x_m of the displacement is called the *amplitude* of the vibration. The angular velocity p of the point Q which describes the auxiliary circle is known as the *circular frequency* of the vibration and is measured in rad/s, while the angle ϕ which defines the initial position of Q on the circle is called the *phase angle*. We note from Fig. 19.2 that a full *cycle* has been described after the angle pt has increased by 2π rad. The corresponding value of t, denoted by τ, is called the *period* of the vibration and is measured in seconds. We have

$$\text{Period} = \tau = \frac{2\pi}{p} \tag{19.13}$$

The number of cycles described per unit of time is denoted by f and is known as the *frequency* of the vibration. We write

$$\text{Frequency} = f = \frac{1}{\tau} = \frac{p}{2\pi} \tag{19.14}$$

The unit of frequency is a frequency of 1 cycle per second, corresponding to a period of 1 s. In terms of base units the unit of frequency is thus 1/s or s^{-1}. It is called a *hertz* (Hz) in the SI system of units. It also follows from Eq. (19.14) that a frequency of $1\ s^{-1}$ or 1 Hz corresponds to a circular frequency of 2π rad/s. In problems involving angular velocities expressed in revolutions per minute (rpm), we have $1\text{ rpm} = \frac{1}{60}\ s^{-1} = \frac{1}{60}$ Hz, or $1\text{ rpm} = (2\pi/60)$ rad/s.

Recalling that p was defined in (19.4) in terms of the constant k of the spring and the mass m of the particle, we observe that the period and the frequency are independent of the initial conditions and of the amplitude of the vibration. Note that τ and f depend on the *mass* rather than on the *weight* of the particle and thus are independent of the value of g.

The velocity-time and acceleration-time curves may be represented by sine curves of the same period as the displacement-time curve, but with different phase angles. From (19.11) and (19.12), we note that the maximum values of the magnitudes of the velocity and acceleration are

$$v_m = x_m p \qquad a_m = x_m p^2 \tag{19.15}$$

Since the point Q describes the auxiliary circle, of radius x_m, at the constant angular velocity p, its velocity and acceleration are equal, respectively, to the expressions (19.15). Recalling Eqs. (19.11) and (19.12), we find, therefore, that the velocity and acceleration of P may be obtained at any instant by projecting on the x axis vectors of magnitudes $v_m = x_m p$ and $a_m = x_m p^2$ representing respectively the velocity and acceleration of Q at the same instant (Fig. 19.3).

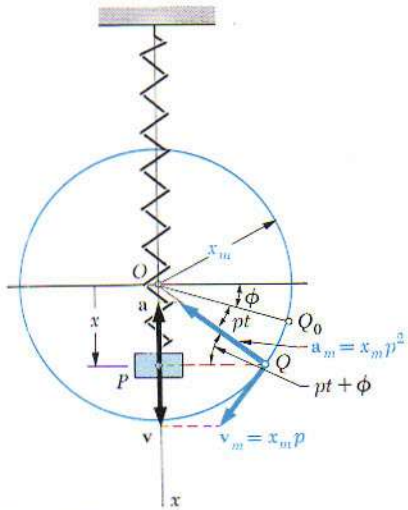


Fig. 19.3

The results obtained are not limited to the solution of the problem of a mass attached to a spring. They may be used to analyze the rectilinear motion of a particle *whenever the resultant* $\mathbf{F}$ *of the forces acting on the particle is proportional to the displacement* x *and directed toward* O. The fundamental equation of motion $F = ma$ may then be written in the form (19.5), which characterizes simple harmonic motion. Observing that the coefficient of x in (19.5) represents the square of the circular frequency p of the vibration, we easily obtain p and, after substitution into (19.13) and (19.14), the period τ and the frequency f of the vibration.

19.3. Simple Pendulum (Approximate Solution). Most of the vibrations encountered in engineering applications may be represented by a simple harmonic motion. Many others, although of a different type, may be *approximated* by a simple harmonic motion, provided that their amplitude remains small. Consider for example a *simple pendulum*, consisting of a bob of mass m attached to a cord of length l, which may oscillate in a vertical plane (Fig. 19.4*a*). At a given time t, the cord forms an angle θ with the vertical. The forces acting on the bob are its weight $\mathbf{W}$ and the force $\mathbf{T}$ exerted by the cord (Fig. 19.4*b*). Resolving the vector $m\mathbf{a}$ into tangential and normal components,

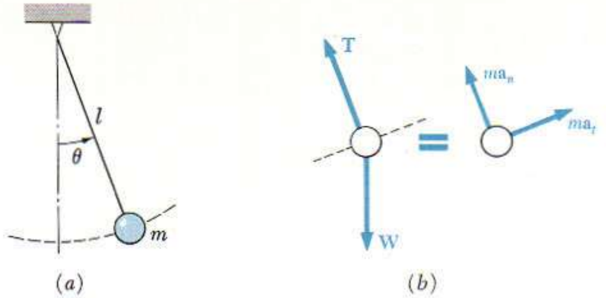


Fig. 19.4

with $m\mathbf{a}_t$ directed to the right, i.e., in the direction corresponding to increasing values of θ, and observing that $a_t = l\alpha = l\ddot{\theta}$, we write

$$\Sigma F_t = ma_t: \qquad -W \sin\theta = ml\ddot{\theta}$$

Noting that $W = mg$ and dividing through by ml, we obtain

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \qquad (19.16)$$

For oscillations of small amplitude, we may replace $\sin\theta$ by θ, expressed in radians, and write

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \qquad (19.17)$$

Comparison with (19.5) shows that the equation obtained is that of a simple harmonic motion and that the circular frequency p of the oscillations is equal to $(g/l)^{1/2}$. Substitution into (19.13) yields the period of the small oscillations of a pendulum of length l,

$$\tau = \frac{2\pi}{p} = 2\pi\sqrt{\frac{l}{g}} \qquad (19.18)$$

***19.4. Simple Pendulum (Exact Solution).** Formula (19.18) is only approximate. To obtain an exact expression for the period of the oscillations of a simple pendulum, we must return to (19.16). Multiplying both terms by $2\dot{\theta}$ and integrating from an initial position corresponding to the maximum deflection, that is, $\theta = \theta_m$ and $\dot{\theta} = 0$, we write

$$\dot{\theta}^2 = \frac{2g}{l}(\cos\theta - \cos\theta_m)$$

or

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{l}(\cos\theta - \cos\theta_m)$$

Replacing $\cos\theta$ by $1 - 2\sin^2(\theta/2)$ and $\cos\theta_m$ by a similar expression, solving for dt, and integrating over a quarter period from $t = 0$, $\theta = 0$ to $t = \tau/4$, $\theta = \theta_m$, we have

$$\tau = 2\sqrt{\frac{l}{g}}\int_0^{\theta_m}\frac{d\theta}{\sqrt{\sin^2(\theta_m/2) - \sin^2(\theta/2)}}$$

The integral in the right-hand member is known as an *elliptic integral;* it cannot be expressed in terms of the usual algebraic or trigonometric functions. However, setting

$$\sin(\theta/2) = \sin(\theta_m/2)\sin\phi$$

we may write

$$\tau = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1 - \sin^2(\theta_m/2)\sin^2\phi}} \qquad (19.19)$$

where the integral obtained, commonly denoted by K, may be found in *tables of elliptic integrals* for various values of $\theta_m/2$.† In order to compare the result just obtained with that of the preceding section, we write (19.19) in the form

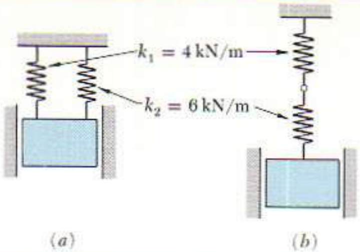
$$\tau = \frac{2K}{\pi}\left(2\pi\sqrt{\frac{l}{g}}\right) \qquad (19.20)$$

Formula (19.20) shows that the actual value of the period of a simple pendulum may be obtained by multiplying the approximate value (19.18) by the correction factor $2K/\pi$. Values of the correction factor are given in Table 19.1 for various values of the amplitude θ_m. We note that for ordinary engineering computations the correction factor may be omitted as long as the amplitude does not exceed 10°.

Table 19.1 Correction Factor for the Period of a Simple Pendulum

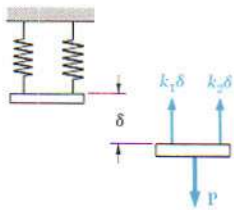
θ_m	0°	10°	20°	30°	60°	90°	120°	150°	180°
K	1.571	1.574	1.583	1.598	1.686	1.854	2.157	2.768	∞
$2K/\pi$	1.000	1.002	1.008	1.017	1.073	1.180	1.373	1.762	∞

† See, for example, Dwight, "Table of Integrals and Other Mathematical Data," The Macmillan Company, or Peirce, "A Short Table of Integrals," Ginn and Company.



SAMPLE PROBLEM 19.1

A 50-kg block moves between vertical guides as shown. The block is pulled 40 mm down from its equilibrium position and released. For each spring arrangement, determine the period of the vibration, the maximum velocity of the block, and the maximum acceleration of the block.



a. Springs Attached in Parallel. We first determine the constant k of a single spring equivalent to the two springs *by finding the magnitude of the force* **P** required to cause a given deflection δ. Since for a deflection δ the magnitudes of the forces exerted by the springs are, respectively, $k_1\delta$ and $k_2\delta$, we have

$$P = k_1\delta + k_2\delta = (k_1 + k_2)\delta$$

The constant k of the single equivalent spring is

$$k = \frac{P}{\delta} = k_1 + k_2 = 4\text{ kN/m} + 6\text{ kN/m} = 10\text{ kN/m} = 10^4\text{ N/m}$$

Period of Vibration: Since $m = 50$ kg, Eq. (19.4) yields

$$p^2 = \frac{k}{m} = \frac{10^4\text{ N/m}}{50\text{ kg}} \qquad p = 14.14\text{ rad/s}$$

$$\tau = 2\pi/p \qquad \tau = 0.444\text{ s} \blacktriangleleft$$

Maximum Velocity: $v_m = x_m p = (0.040\text{ m})(14.14\text{ rad/s})$

$$v_m = 0.566\text{ m/s} \qquad \mathbf{v}_m = 0.566\text{ m/s} \updownarrow \blacktriangleleft$$

Maximum Acceleration: $a_m = x_m p^2 = (0.040\text{ m})(14.14\text{ rad/s})^2$

$$a_m = 8.00\text{ m/s}^2 \qquad \mathbf{a}_m = 8.00\text{ m/s}^2 \updownarrow \blacktriangleleft$$

b. Springs Attached in Series. We first determine the constant k of a single spring equivalent to the two springs *by finding the total elongation* δ of the springs under a given static load **P**. To facilitate the computation, a static load of magnitude $P = 12$ kN is used.

$$\delta = \delta_1 + \delta_2 = \frac{P}{k_1} + \frac{P}{k_2} = \frac{12\text{ kN}}{4\text{ kN/m}} + \frac{12\text{ kN}}{6\text{ kN/m}} = 5\text{ m}$$

$$k = \frac{P}{\delta} = \frac{12\text{ kN}}{5\text{ m}} = 2.4\text{ kN/m} = 2400\text{ N/m}$$

Period of Vibration: $p^2 = \dfrac{k}{m} = \dfrac{2400\text{ N/m}}{50\text{ kg}} \qquad p = 6.93\text{ rad/s}$

$$\tau = \frac{2\pi}{p} \qquad \tau = 0.907\text{ s} \blacktriangleleft$$

Maximum Velocity: $v_m = x_m p = (0.040\text{ m})(6.93\text{ rad/s})$

$$v_m = 0.277\text{ m/s} \qquad \mathbf{v}_m = 0.277\text{ m/s} \updownarrow \blacktriangleleft$$

Maximum Acceleration: $a_m = x_m p^2 = (0.040\text{ m})(6.93\text{ rad/s})^2$

$$a_m = 1.920\text{ m/s}^2 \qquad \mathbf{a}_m = 1.920\text{ m/s}^2 \updownarrow \blacktriangleleft$$

PROBLEMS

19.1 A particle moves in simple harmonic motion with an amplitude of 4 in. and a period of 0.60 s. Find the maximum velocity and the maximum acceleration.

19.2 The analysis of the motion of a particle shows a maximum acceleration of $30\ \mathrm{m/s^2}$ and a frequency of 120 cycles per minute. Assuming that the motion is simple harmonic, determine (*a*) the amplitude, (*b*) the maximum velocity.

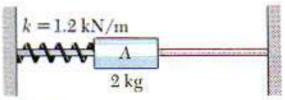


Fig. P19.3

19.3 Collar A is attached to the spring shown and may slide without friction on the horizontal rod. If the collar is moved 75 mm from its equilibrium position and released, determine the period, the maximum velocity, and the maximum acceleration of the resulting motion.

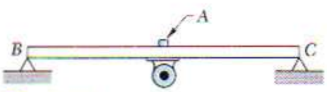


Fig. P19.4

19.4 A variable-speed motor is rigidly attached to the beam BC. The rotor is slightly unbalanced and causes the beam to vibrate with a circular frequency equal to the motor speed. When the speed of the motor is less than 600 rpm or more than 1200 rpm, a small object placed at A is observed to remain in contact with the beam. For speeds between 600 and 1200 rpm the object is observed to "dance" and actually to lose contact with the beam. Determine the amplitude of the motion of A when the speed of the motor is (*a*) 600 rpm, (*b*) 1200 rpm. Give answers in both SI and U.S. customary units.

19.5 The 6-lb collar rests on, but is not attached to, the spring shown. The collar is depressed 2 in. and released. If the ensuing motion is to be simple harmonic, determine (*a*) the largest permissible value of the spring constant k, (*b*) the corresponding frequency of the motion.

Fig. P19.5, P19.6, and P19.8

19.6 The 5-kg collar is attached to a spring of constant $k = 800$ N/m as shown. If the collar is given a displacement of 50 mm from its equilibrium position and released, determine for the ensuing motion (*a*) the period, (*b*) the maximum velocity of the collar, (*c*) the maximum acceleration of the collar.

19.7 In Prob. 19.6, determine the position, velocity, and acceleration of the collar 0.20 s after it has been released.

19.8 An 8-lb collar is attached to a spring of constant $k = 5$ lb/in. as shown. If the collar is given a displacement of 2 in. downward from its equilibrium position and released, determine (*a*) the time required for the collar to move 3 in. upward, (*b*) the corresponding velocity and acceleration of the collar.

19.9 and 19.10 A 35-kg block is supported by the spring arrangement shown. If the block is moved vertically downward from its equilibrium position and released, determine (*a*) the period and frequency of the resulting motion, (*b*) the maximum velocity and acceleration of the block if the amplitude of the motion is 20 mm.

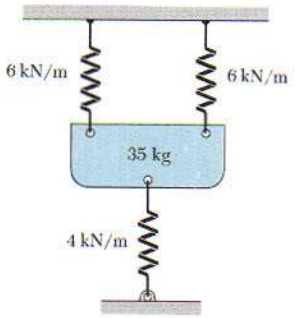


Fig. P19.9

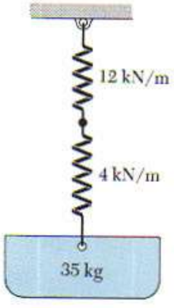


Fig. P19.10

19.11 Denoting by δ_{st} the static deflection of a beam under a given load, show that the frequency of vibration of the load is

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}}$$

Neglect the mass of the beam, and assume that the load remains in contact with the beam.

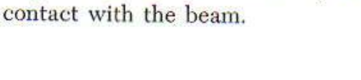


Fig. P19.11

19.12 The period of vibration of the system shown is observed to be 0.8 s. If block *A* is removed, the period is observed to be 0.7 s. Determine (*a*) the weight of block *C*, (*b*) the period of vibration when both blocks *A* and *B* have been removed.

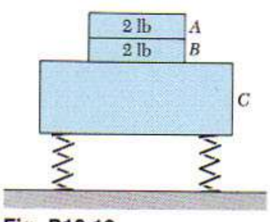


Fig. P19.12

19.13 A simple pendulum of length l is suspended in an elevator. A mass m is attached to a spring of constant k and is carried in the same elevator. Determine the period of vibration of both the pendulum and the mass if the elevator has an upward acceleration **a**.

19.14 Determine (*a*) the required length l of a simple pendulum if the period of small oscillations is to be 2 s, (*b*) the required amplitude of this pendulum if the maximum velocity of the bob is to be 200 mm/s.

19.15 A small bob is attached to a cord of length 4 ft and is released from rest when $\theta_A = 5°$. Knowing that $d = 2$ ft, determine (*a*) the time required for the bob to return to point *A*, (*b*) the amplitude θ_C.

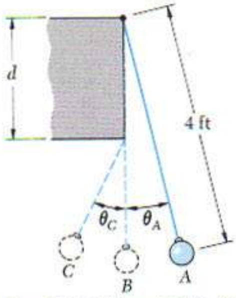


Fig. P19.15 and P19.16

19.16 A small bob is attached to a cord of length 4 ft and released from rest at *A* when $\theta_A = 4°$. Determine the distance *d* for which the bob will return to point *A* in 2 s.

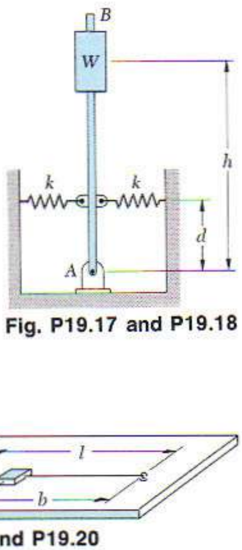


Fig. P19.17 and P19.18

19.17 The rod *AB* is attached to a hinge at *A* and to two springs each of constant *k*. When $h = 24$ in., $d = 10$ in., and $W = 50$ lb, determine the value of *k* for which the period of small oscillations is (*a*) 1 s, (*b*) infinite. Neglect the weight of the rod and assume that each spring can act in either tension or compression.

19.18 If $d = 16$ in., $h = 24$ in., and each spring has a constant $k = 4$ lb/in., determine the load *W* for which the period of small oscillations is (*a*) 0.50 s, (*b*) infinite. Neglect the weight of the rod and assume that each spring can act in either tension or compression.

19.19 A small block of mass *m* rests on a frictionless horizontal surface and is attached to a taut string. Denoting by *T* the tension in the string, determine the period and frequency of small oscillations of the block in a direction perpendicular to the string. Show that the longest period occurs when $a = b = \frac{1}{2}l$.

Fig. P19.19 and P19.20

19.20 A 3-lb block rests on a frictionless horizontal surface and is attached to a taut string. Knowing that the tension in the string is 8 lb, determine the frequency of small oscillations of the block when $a = 12$ in. and $b = 18$ in.

***19.21** A particle is placed with no initial velocity on a frictionless *plane* tangent to the surface of the earth. (*a*) Show that the particle will theoretically execute simple harmonic motion with a period of oscillation equal to that of a simple pendulum of length equal to the radius of the earth. (*b*) Compute the theoretical period of oscillation and show that it is equal to the periodic time of an earth satellite describing a low-altitude circular orbit. [*Hint.* See Eq. (12.44).]

***19.22** Expanding the integrand in (19.19) into a series of even powers of $\sin \phi$ and integrating, show that the period of a simple pendulum of length l may be approximated by the formula

$$\tau = 2\pi \sqrt{\frac{l}{g}}\left(1 + \tfrac{1}{4}\sin^2 \frac{\theta_m}{2}\right)$$

where θ_m is the amplitude of the oscillations.

***19.23** Using the data of Table 19.1, determine the period of a simple pendulum of length 800 mm (*a*) for small oscillations, (*b*) for oscillations of amplitude $\theta_m = 30°$, (*c*) for oscillations of amplitude $\theta_m = 90°$.

***19.24** Using the formula given in Prob. 19.22, determine the amplitude θ_m for which the period of a simple pendulum is $\frac{1}{2}$ percent longer than the period of the same pendulum for small oscillations.

***19.25** Using a table of elliptic integrals, determine the period of a simple pendulum of length $l = 800$ mm if the amplitude of the oscillations is $\theta_m = 40°$.

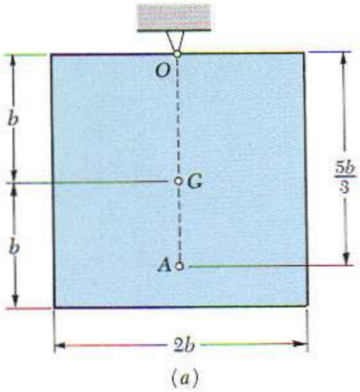
19.5. Free Vibrations of Rigid Bodies. The analysis of the vibrations of a rigid body or of a system of rigid bodies possessing a single degree of freedom is similar to the analysis of the vibrations of a particle. An appropriate variable, such as a distance x or an angle θ, is chosen to define the position of the body or system of bodies, and an equation relating this variable and its second derivative with respect to t is written. If the equation obtained is of the same form as (19.5), i.e., if we have

$$\ddot{x} + p^2 x = 0 \qquad \text{or} \qquad \ddot{\theta} + p^2\theta = 0 \tag{19.21}$$

the vibration considered is a simple harmonic motion. The period and frequency of the vibration may then be obtained by identifying p and substituting into (19.13) and (19.14).

In general, a simple way to obtain one of Eqs. (19.21) is to express that the system of the external forces is equivalent to the system of the effective forces by drawing a diagram of the body for an arbitrary value of the variable and writing the appropriate equation of motion. We recall that our goal should be *the determination of the coefficient* of the variable x or θ, *not* the determination of the variable itself or of the derivatives $\ddot{x}$ or $\ddot{\theta}$. Setting this coefficient equal to p^2, we obtain the circular frequency p, from which τ and f may be determined.

The method we have outlined may be used to analyze vibrations which are truly represented by a simple harmonic motion, or vibrations of small amplitude which can be *approximated* by a simple harmonic motion. As an example, we shall determine the period of the small oscillations of a square plate of side $2b$



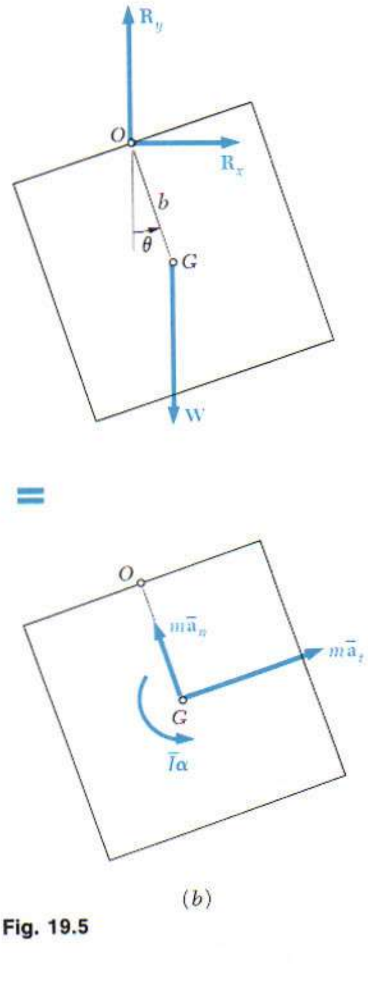


Fig. 19.5

which is suspended from the midpoint O of one side (Fig. 19.5*a*). We consider the plate in an arbitrary position defined by the angle θ that the line OG forms with the vertical and draw a diagram to express that the weight $\mathbf{W}$ of the plate and the components $\mathbf{R}_x$ and $\mathbf{R}_y$ of the reaction at O are equivalent to the vectors $m\bar{\mathbf{a}}_t$ and $m\bar{\mathbf{a}}_n$ and to the couple $\bar{I}\boldsymbol{\alpha}$ (Fig. 19.5*b*). Since the angular velocity and angular acceleration of the plate are equal, respectively, to $\dot{\theta}$ and $\ddot{\theta}$, the magnitudes of the two vectors are, respectively, $mb\ddot{\theta}$ and $mb\dot{\theta}^2$, while the moment of the couple is $\bar{I}\ddot{\theta}$. In previous applications of this method (Chap. 16), we tried whenever possible to assume the correct sense for the acceleration. Here, however, we must assume the same positive sense for θ and $\ddot{\theta}$ in order to obtain an equation of the form (19.21). Consequently, the angular acceleration $\ddot{\theta}$ will be assumed positive counterclockwise, even though this assumption is obviously unrealistic. Equating moments about O, we write

$$+\circlearrowleft \qquad -W(b\sin\theta) = (mb\ddot{\theta})b + \bar{I}\ddot{\theta}$$

Noting that $\bar{I} = \frac{1}{12}m[(2b)^2 + (2b)^2] = \frac{2}{3}mb^2$ and $W = mg$, we obtain

$$\ddot{\theta} + \frac{3}{5}\frac{g}{b}\sin\theta = 0 \tag{19.22}$$

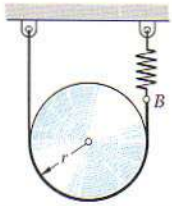
For oscillations of small amplitude, we may replace $\sin\theta$ by θ, expressed in radians, and write

$$\ddot{\theta} + \frac{3}{5}\frac{g}{b}\theta = 0 \tag{19.23}$$

Comparison with (19.21) shows that the equation obtained is that of a simple harmonic motion and that the circular frequency p of the oscillations is equal to $(3g/5b)^{1/2}$. Substituting into (19.13), we find that the period of the oscillations is

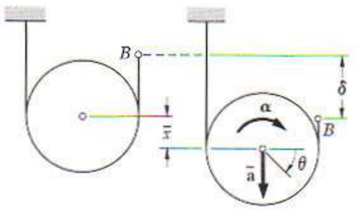
$$\tau = \frac{2\pi}{p} = 2\pi\sqrt{\frac{5b}{3g}} \tag{19.24}$$

The result obtained is valid only for oscillations of small amplitude. A more accurate description of the motion of the plate is obtained by comparing Eqs. (19.16) and (19.22). We note that the two equations are identical if we choose l equal to $5b/3$. This means that the plate will oscillate as a simple pendulum of length $l = 5b/3$, and the results of Sec. 19.4 may be used to correct the value of the period given in (19.24). The point A of the plate located on line OG at a distance $l = 5b/3$ from O is defined as the *center of oscillation* corresponding to O (Fig. 19.5*a*).



SAMPLE PROBLEM 19.2

A cylinder of weight W and radius r is suspended from a looped cord as shown. One end of the cord is attached directly to a rigid support, while the other end is attached to a spring of constant k. Determine the period and frequency of vibration of the cylinder.

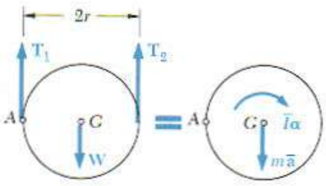


Kinematics of Motion. We express the linear displacement and the acceleration of the cylinder in terms of the angular displacement θ. Choosing the positive sense clockwise and measuring the displacements from the equilibrium position, we write

$$\bar{x} = r\theta \qquad \delta = 2\bar{x} = 2r\theta$$

$$\boldsymbol{\alpha} = \ddot{\theta} \downarrow \qquad \bar{a} = r\alpha = r\ddot{\theta} \qquad \bar{\mathbf{a}} = r\ddot{\theta} \downarrow \tag{1}$$

Equations of Motion. The system of external forces acting on the cylinder consists of the weight $\mathbf{W}$ and of the forces $\mathbf{T}_1$ and $\mathbf{T}_2$ exerted by the cord. We express that this system is equivalent to the system of effective forces represented by the vector $m\bar{\mathbf{a}}$ attached at G and the couple $\bar{I}\boldsymbol{\alpha}$.



$$+\downarrow \Sigma M_A = \Sigma (M_A)_{\text{eff}}: \qquad Wr - T_2(2r) = m\bar{a}r + \bar{I}\alpha \tag{2}$$

When the cylinder is in its position of equilibrium, the tension in the cord is $T_0 = \frac{1}{2}W$. We note that, for an angular displacement θ, the magnitude of $\mathbf{T}_2$ is

$$T_2 = T_0 + k\delta = \tfrac{1}{2}W + k\delta = \tfrac{1}{2}W + k(2r\theta) \tag{3}$$

Substituting from (1) and (3) into (2), and recalling that $\bar{I} = \frac{1}{2}mr^2$, we write

$$Wr - (\tfrac{1}{2}W + 2kr\theta)(2r) = m(r\ddot{\theta})r + \tfrac{1}{2}mr^2\ddot{\theta}$$

$$\ddot{\theta} + \frac{8}{3}\frac{k}{m}\theta = 0$$

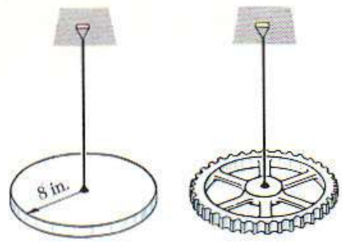
The motion is seen to be simple harmonic, and we have

$$p^2 = \frac{8}{3}\frac{k}{m} \qquad p = \sqrt{\frac{8}{3}\frac{k}{m}}$$

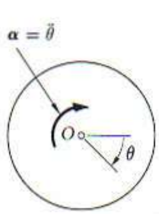
$$\tau = \frac{2\pi}{p} \qquad \tau = 2\pi\sqrt{\frac{3}{8}\frac{m}{k}} \blacktriangleleft$$

$$f = \frac{p}{2\pi} \qquad f = \frac{1}{2\pi}\sqrt{\frac{8}{3}\frac{k}{m}} \blacktriangleleft$$

SAMPLE PROBLEM 19.3



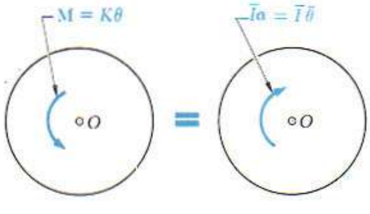
A circular disk, weighing 20 lb and of radius 8 in., is suspended from a wire as shown. The disk is rotated (thus twisting the wire) and then released; the period of the torsional vibration is observed to be 1.13 s. A gear is then suspended from the same wire, and the period of torsional vibration for the gear is observed to be 1.93 s. Assuming that the moment of the couple exerted by the wire is proportional to the angle of twist, determine (*a*) the torsional spring constant of the wire, (*b*) the centroidal moment of inertia of the gear, (*c*) the maximum angular velocity reached by the gear if it is rotated through 90° and released.



a. **Vibration of Disk.** Denoting by θ the angular displacement of the disk, we express that the magnitude of the couple exerted by the wire is $M = K\theta$, where K is the torsional spring constant of the wire. Since this couple must be equivalent to the couple $\bar{I}\boldsymbol{\alpha}$ representing the effective forces of the disk, we write

$$+\circlearrowleft \Sigma M_O = \Sigma(M_O)_{\text{eff}}: \qquad +K\theta = -\bar{I}\ddot{\theta}$$

$$\ddot{\theta} + \frac{K}{\bar{I}}\theta = 0$$



The motion is seen to be simple harmonic, and we have

$$p^2 = \frac{K}{\bar{I}} \qquad \tau = \frac{2\pi}{p} \qquad \tau = 2\pi\sqrt{\frac{\bar{I}}{K}} \tag{1}$$

For the disk, we have

$$\tau = 1.13 \text{ s} \qquad \bar{I} = \tfrac{1}{2}mr^2 = \frac{1}{2}\left(\frac{20 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(\tfrac{8}{12} \text{ ft})^2 = 0.138 \text{ lb} \cdot \text{ft} \cdot \text{s}^2$$

Substituting into (1), we obtain

$$1.13 = 2\pi\sqrt{\frac{0.138}{K}} \qquad K = 4.27 \text{ lb} \cdot \text{ft/rad} \blacktriangleleft$$

b. **Vibration of Gear.** Since the period of vibration of the gear is 1.93 s and $K = 4.27$ lb · ft/rad, Eq. (1) yields

$$1.93 = 2\pi\sqrt{\frac{\bar{I}}{4.27}} \qquad \bar{I}_{\text{gear}} = 0.403 \text{ lb} \cdot \text{ft} \cdot \text{s}^2 \blacktriangleleft$$

c. **Maximum Angular Velocity of the Gear.** Since the motion is simple harmonic, we have

$$\theta = \theta_m \sin pt \qquad \omega = \theta_m p \cos pt \qquad \omega_m = \theta_m p$$

Recalling that $\theta_m = 90° = 1.571$ rad and $\tau = 1.93$ s, we write

$$\omega_m = \theta_m p = \theta_m(2\pi/\tau) = (1.571 \text{ rad})(2\pi/1.93 \text{ s})$$

$$\omega_m = 5.11 \text{ rad/s} \blacktriangleleft$$

PROBLEMS

19.26 and 19.27 The uniform rod shown weighs 8 lb and is attached to a spring of constant $k = 2.5$ lb/in. If end A of the rod is depressed 2 in. and released, determine (*a*) the period of vibration, (*b*) the maximum velocity of end A.

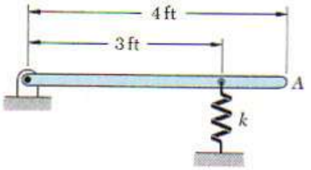


Fig. P19.26

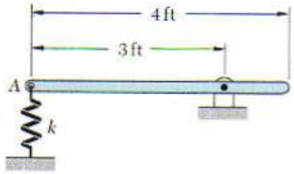


Fig. P19.27

19.28 A belt is placed over the rim of a 15-kg disk as shown and then attached to a 5-kg cylinder and to a spring of constant $k = 600$ N/m. If the cylinder is moved 50 mm down from its equilibrium position and released, determine (*a*) the period of vibration, (*b*) the maximum velocity of the cylinder. Assume friction is sufficient to prevent the belt from slipping on the rim.

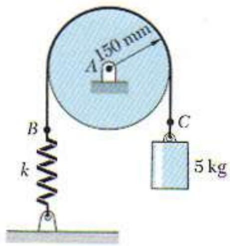


Fig. P19.28

19.29 In Prob. 19.28, determine (*a*) the frequency of vibration, (*b*) the maximum tension which occurs in the belt at B and at C.

19.30 A 600-lb flywheel has a diameter of 4 ft and a radius of gyration of 20 in. A belt is placed around the rim and attached to two springs, each of constant $k = 75$ lb/in. The initial tension in the belt is sufficient to prevent slipping. If the end C of the belt is pulled 1 in. down and released, determine (*a*) the period of vibration, (*b*) the maximum angular velocity of the flywheel.

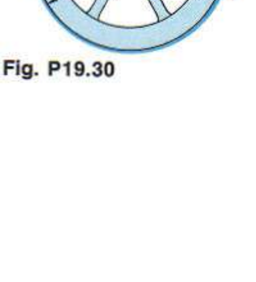


Fig. P19.30

19.31 A uniform square plate of mass m is supported in a horizontal plane by a vertical pin at B and is attached at A to a spring of constant k. If corner A is given a small displacement and released, determine the period of the resulting motion.

Fig. P19.31

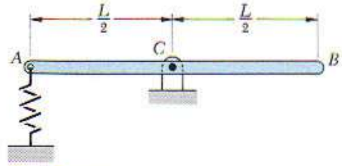


Fig. P19.32

19.32 A uniform rod of mass m is supported by a pin at its midpoint C and is attached to a spring of constant k. If end A is given a small displacement and released, determine the frequency of the resulting motion.

19.33 A uniform circular plate of mass m and radius r is held by four springs, each of constant k. Determine the frequency of the resulting vibration if the plate is (a) given a small vertical displacement and released, (b) rotated through a small angle about diameter AC and released, (c) rotated through a small angle about any other diameter and released.

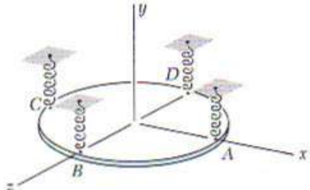


Fig. P19.33

19.34 A *compound pendulum* is defined as a rigid slab which oscillates about a fixed point O, called the center of suspension. Show that the period of oscillation of a compound pendulum is equal to the period of a simple pendulum of length OA, where the distance from A to the mass center G is $GA = \bar{k}^2/\bar{r}$. Point A is defined as the center of oscillation and coincides with the center of percussion defined in Prob. 17.58.

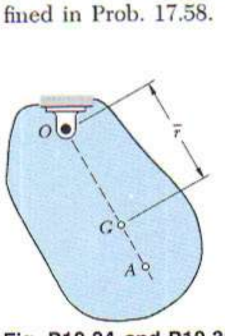


Fig. P19.34 and P19.36

19.35 Show that, if the compound pendulum of Prob. 19.34 is suspended from A instead of O, the period of oscillation is the same as before and the new center of oscillation is located at O.

19.36 A rigid slab oscillates about a fixed point O. Show that the smallest period of oscillation occurs when the distance $\bar{r}$ from point O to the mass center G is equal to $\bar{k}$.

19.37 A uniform bar of length l may oscillate about a hinge at A located a distance c from its mass center G. (*a*) Determine the frequency of small oscillations if $c = \frac{1}{2}l$. (*b*) Determine a second value of c for which the frequency of small oscillations is the same as that found in part *a*.

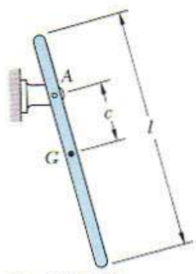


Fig. P19.37

19.38 For the rod of Prob. 19.37, determine (*a*) the distance c for which the frequency of oscillation is maximum, (*b*) the corresponding minimum period.

19.39 A thin hoop of radius r and mass m is suspended from a rough rod as shown. Determine the frequency of small oscillations of the hoop (*a*) in the plane of the hoop, (*b*) in a direction perpendicular to the plane of the hoop. Assume that μ is sufficiently large to prevent slipping at A.

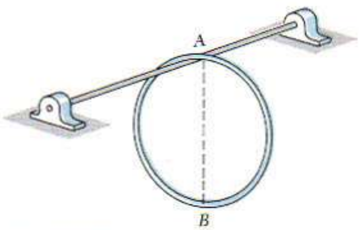


Fig. P19.39

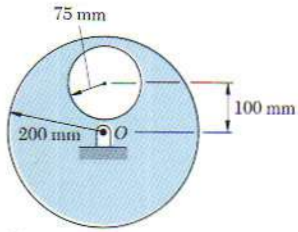


Fig. P19.40

19.40 A 75-mm-radius hole is cut in a 200-mm-radius uniform disk which is attached to a frictionless pin at its geometric center O. Determine (*a*) the period of small oscillations of the disk, (*b*) the length of a simple pendulum which has the same period.

19.41 A uniform rectangular plate is suspended from a pin located at the midpoint of one edge as shown. Considering the dimension b constant, determine (*a*) the ratio c/b for which the period of oscillation of the plate is minimum, (*b*) the ratio c/b for which the period of oscillation of the plate is the same as the period of a simple pendulum of length c.

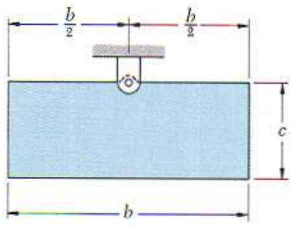


Fig. P19.41 and P19.42

19.42 A uniform rectangular plate is suspended from a pin located at the midpoint of one edge as shown. (*a*) Determine the period of small oscillations if $c = b$. (*b*) Considering the dimension b constant, determine a second value of c for which the period of oscillations is the same as that found in part *a*.

19.43 The period of small oscillations about A of a connecting rod is observed to be 1.12 s. Knowing that the distance r_a is 7.50 in., determine the centroidal radius of gyration of the connecting rod.

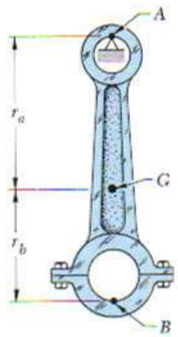


Fig. P19.43 and 19.44

19.44 A connecting rod is supported by a knife-edge at point A; the period of small oscillations is observed to be 0.945 s. The rod is then inverted and supported by a knife-edge at point B and the period of small oscillations is observed to be 0.850 s. Knowing that $r_a + r_b = 11.50$ in., determine (*a*) the location of the mass center G, (*b*) the centroidal radius of gyration $\bar{k}$.

19.45 A slender rod of length l is suspended from two vertical wires of length h, each located a distance $\frac{1}{2}b$ from the mass center G. Determine the period of oscillation when (*a*) the rod is rotated through a small angle about a vertical axis passing through G and released, (*b*) the rod is given a small horizontal translation along AB and released.

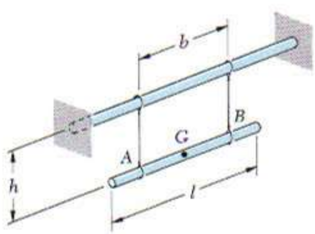


Fig. P19.45

19.46 A uniform disk of weight 5 lb is suspended from a steel wire which is known to have a torsional spring constant $K = 0.30$ lb·in./rad. If the disk is rotated through 360° about the vertical and then released, determine (*a*) the period of oscillation, (*b*) the maximum velocity of a point on the rim of the disk.

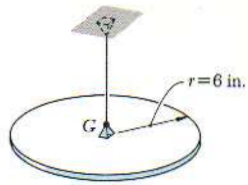


Fig. P19.46

19.47 A uniform disk of radius 200 mm and of mass 8 kg is attached to a vertical shaft which is rigidly held at *B*. It is known that the disk rotates through 3° when a 4 N · m static couple is applied to the disk. If the disk is rotated through 6° and then released, determine (*a*) the period of the resulting vibration, (*b*) the maximum velocity of a point on the rim of the disk.

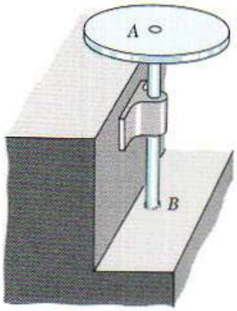


Fig. P19.47

19.48 A steel casting is rigidly bolted to the disk of Prob. 19.47. Knowing that the period of torsional vibration of the disk and casting is 0.650 s, determine the moment of inertia of the casting with respect to the shaft *AB*.

19.49 A torsion pendulum may be used to determine experimentally the moment of inertia of a given object. The horizontal platform *P* is held by several rigid bars which are connected to a vertical wire. The period of oscillation of the platform is found equal to τ_0 when the platform is empty and to τ_A when an object of known moment of inertia I_A is placed on the platform so that its mass center is directly above the center of the plate. (*a*) Show that the moment of inertia I_0 of the platform and its supports may be expressed as $I_0 = I_A\tau_0^2/(\tau_A^2 - \tau_0^2)$. (*b*) If a period of oscillation τ_B is observed when an object *B* of unknown moment of inertia I_B is placed on the platform, show that $I_B = I_A(\tau_B^2 - \tau_0^2)/(\tau_A^2 - \tau_0^2)$.

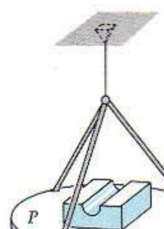


Fig P19.49

19.6. Application of the Principle of Conservation of Energy. We saw in Sec. 19.2 that, when a particle of mass m is in simple harmonic motion, the resultant $\mathbf{F}$ of the forces exerted on the particle has a magnitude proportional to the displacement x measured from the position of equilibrium O and is directed toward O; we write $F = -kx$. Referring to Sec. 13.6, we note that $\mathbf{F}$ is a *conservative force* and that the corresponding potential energy is $V = \frac{1}{2}kx^2$, where V is assumed equal to zero in the equilibrium position $x = 0$. Since the velocity of the particle is equal to $\dot{x}$, its kinetic energy is $T = \frac{1}{2}m\dot{x}^2$ and we may express that the total energy of the particle is conserved by writing

$$T + V = \text{constant} \qquad \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 = \text{constant}$$

Setting $p^2 = k/m$ as in (19.4), where p is the circular frequency of the vibration, we have

$$\dot{x}^2 + p^2x^2 = \text{constant} \tag{19.25}$$

Equation (19.25) is characteristic of simple harmonic motion; it may be obtained directly from (19.5) by multiplying both terms by $2\dot{x}$ and integrating.

The principle of conservation of energy provides a convenient way for determining the period of vibration of a rigid body or of a system of rigid bodies possessing a single degree of freedom, once it has been established that the motion of the system is a simple harmonic motion, or that it may be approximated by a simple harmonic motion. Choosing an appropriate variable, such as a distance x or an angle θ, we consider two particular positions of the system:

1. *The displacement of the system is maximum;* we have $T_1 = 0$, and V_1 may be expressed in terms of the amplitude x_m or θ_m (choosing $V = 0$ in the equilibrium position).
2. *The system passes through its equilibrium position;* we have $V_2 = 0$, and T_2 may be expressed in terms of the maximum velocity $\dot{x}_m$ or $\dot{\theta}_m$.

We then express that the total energy of the system is conserved and write $T_1 + V_1 = T_2 + V_2$. Recalling from (19.15) that for simple harmonic motion the maximum velocity is equal to the product of the amplitude and of the circular frequency p, we find that the equation obtained may be solved for p.

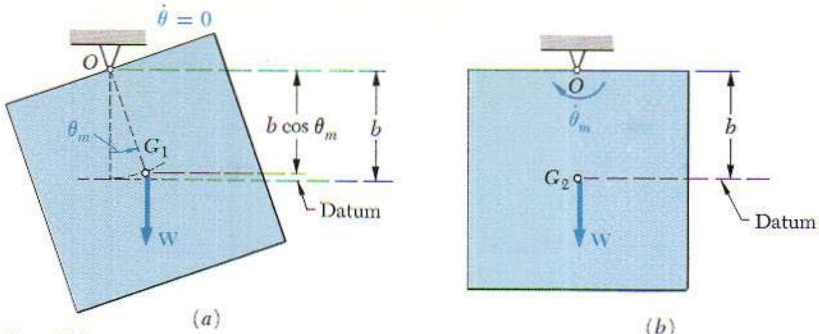


Fig. 19.6

As an example, we shall consider again the square plate of Sec. 19.5. In the position of maximum displacement (Fig. 19.6*a*), we have

$$T_1 = 0 \qquad V_1 = W(b - b\cos\theta_m) = Wb(1 - \cos\theta_m)$$

or, since $1 - \cos\theta_m = 2\sin^2(\theta_m/2) \approx 2(\theta_m/2)^2 = \theta_m^2/2$ for oscillations of small amplitude,

$$T_1 = 0 \qquad V_1 = \tfrac{1}{2}Wb\theta_m^2 \tag{19.26}$$

As the plate passes through its position of equilibrium (Fig. 19.6*b*), its velocity is maximum and we have

$$T_2 = \tfrac{1}{2}m\bar{v}_m^2 + \tfrac{1}{2}\bar{I}\omega_m^2 = \tfrac{1}{2}mb^2\dot{\theta}_m^2 + \tfrac{1}{2}\bar{I}\dot{\theta}_m^2 \qquad V_2 = 0$$

or, recalling from Sec. 19.5 that $\bar{I} = \tfrac{2}{3}mb^2$,

$$T_2 = \tfrac{1}{2}(\tfrac{5}{3}mb^2)\dot{\theta}_m^2 \qquad V_2 = 0 \tag{19.27}$$

Substituting from (19.26) and (19.27) into $T_1 + V_1 = T_2 + V_2$, and noting that the maximum velocity $\dot{\theta}_m$ is equal to the product $\theta_m p$, we write

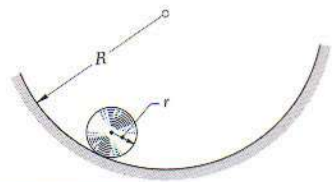
$$\tfrac{1}{2}Wb\theta_m^2 = \tfrac{1}{2}(\tfrac{5}{3}mb^2)\theta_m^2 p^2 \tag{19.28}$$

which yields $p^2 = 3g/5b$ and

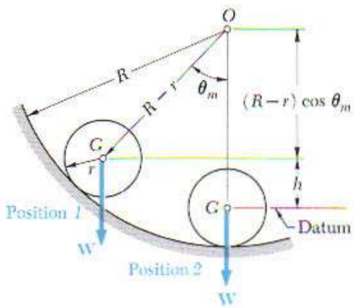
$$\tau = \frac{2\pi}{p} = 2\pi\sqrt{\frac{5b}{3g}} \tag{19.29}$$

as previously obtained.

SAMPLE PROBLEM 19.4



Determine the period of small oscillations of a cylinder of radius r which rolls without slipping inside a curved surface of radius R.



Solution. We denote by θ the angle which line OG forms with the vertical. Since the cylinder rolls without slipping, we may apply the principle of conservation of energy between position *1*, where $\theta = \theta_m$, and position 2, where $\theta = 0$.

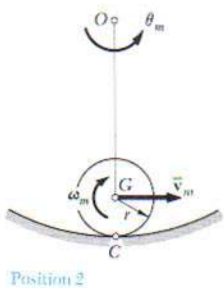
Position *1*. ***Kinetic Energy.*** Since the velocity of the cylinder is zero, we have $T_1 = 0$.

Potential Energy. Choosing a datum as shown and denoting by W the weight of the cylinder, we have

$$V_1 = Wh = W(R - r)(1 - \cos\theta)$$

Noting that for small oscillations $(1 - \cos\theta) = 2\sin^2(\theta/2) \approx \theta^2/2$, we have

$$V_1 = W(R - r)\frac{\theta_m^2}{2}$$



Position 2. Denoting by $\dot\theta_m$ the angular velocity of line OG as the cylinder passes through position 2, and observing that point C is the instantaneous center of rotation of the cylinder, we write

$$\bar{v}_m = (R - r)\dot\theta_m \qquad \omega_m = \frac{\bar{v}_m}{r} = \frac{R - r}{r}\dot\theta_m$$

Kinetic Energy:

$$\begin{aligned} T_2 &= \tfrac{1}{2}m\bar{v}_m^2 + \tfrac{1}{2}\bar{I}\omega_m^2 \\ &= \tfrac{1}{2}m(R - r)^2\dot\theta_m^2 + \tfrac{1}{2}(\tfrac{1}{2}mr^2)\left(\frac{R - r}{r}\right)^2\dot\theta_m^2 \\ &= \tfrac{3}{4}m(R - r)^2\dot\theta_m^2 \end{aligned}$$

Potential Energy: $V_2 = 0$

Conservation of Energy

$$T_1 + V_1 = T_2 + V_2$$

$$0 + W(R - r)\frac{\theta_m^2}{2} = \tfrac{3}{4}m(R - r)^2\dot\theta_m^2 + 0$$

Since $\dot\theta_m = p\theta_m$ and $W = mg$, we write

$$mg(R - r)\frac{\theta_m^2}{2} = \tfrac{3}{4}m(R - r)^2(p\theta_m)^2 \qquad p^2 = \frac{2}{3}\frac{g}{R - r}$$

$$\tau = \frac{2\pi}{p} \qquad \tau = 2\pi\sqrt{\frac{3}{2}\frac{R - r}{g}} \blacktriangleleft$$

PROBLEMS

Fig. P19.50

19.50 Using the method of Sec. 19.6, determine the period of a simple pendulum of length l.

19.51 The springs of an automobile are observed to expand 8 in. to an undeformed position as the body is lifted by several jacks. Assuming that each spring carries an equal portion of the weight of the automobile, determine the frequency of the free vertical vibrations of the body.

19.52 Using the method of Sec. 19.6, solve Prob. 19.6.

19.53 Using the method of Sec. 19.6, solve Prob. 19.9.

19.54 Using the method of Sec. 19.6, solve Prob. 19.10.

19.55 Neglecting fluid friction, determine the frequency of oscillation of the liquid in the U-tube manometer shown. Show that this frequency is independent of the density of the liquid and of the amplitude of the oscillation.

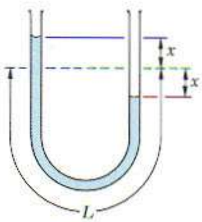


Fig. P19.55

19.56 Two collars, each of weight W, are attached as shown to a hoop of radius r and of negligible weight. (*a*) Show that for any value of β the period is $\tau = 2\pi\sqrt{2r/g}$. (*b*) Show that the same result is obtained if the weight of the hoop is not neglected.

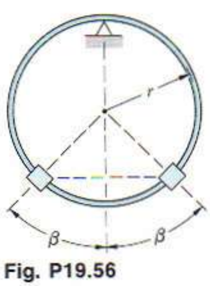


Fig. P19.56

19.57 A thin homogeneous wire is bent into the shape of a square of side l and suspended as shown. Determine the period of oscillation when the wire figure is given a small displacement to the right and released.

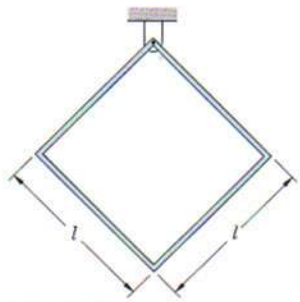


Fig. P19.57

19.58 Solve Prob. 19.57, assuming that the wire square is suspended from a pin located at the midpoint of one side.

19.59 Using the method of Sec. 19.6, solve Prob. 19.45.

19.60 Using the method of Sec. 19.6, solve Prob. 19.40.

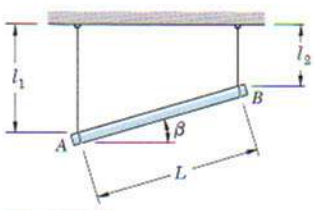


Fig. P19.61

19.61 A section of uniform pipe is suspended from two vertical cables attached at A and B. Determine the period of oscillation in terms of l_1 and l_2 when point B is given a small horizontal displacement to the right and released.

19.62 The motion of the uniform rod AB is guided by the cord BC and by the small roller at A. Determine the frequency of oscillation when the end B of the rod is given a small horizontal displacement and released.

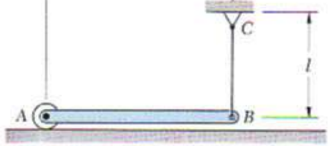


Fig. P19.62

19.63 The 20-lb rod AB is attached to 8-lb disks as shown. Knowing that the disks roll without sliding, determine the frequency of small oscillations of the system.

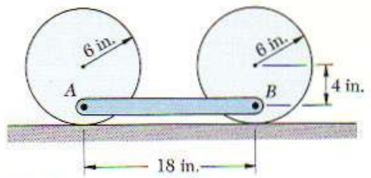


Fig. P19.63

19.64 Blade AB of the experimental wind-turbine generator shown is to be temporarily removed. Motion of the turbine generator about the y axis is prevented, but the remaining three blades may oscillate as a unit about the x axis. Assuming that each blade is equivalent to a 40-ft slender rod, determine the period of small oscillations of the blades.

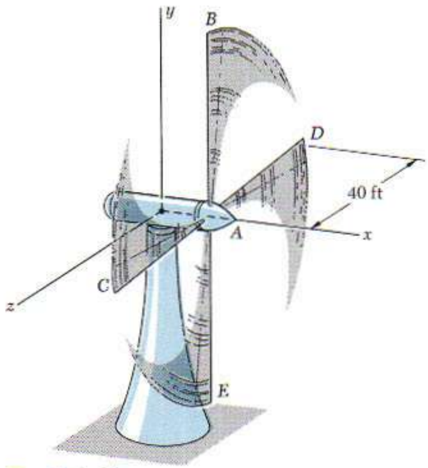


Fig. P19.64

19.65 The 8-kg rod AB is bolted to the 12-kg disk. Knowing that the disk rolls without sliding, determine the period of small oscillations of the system.

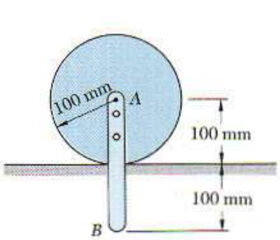


Fig. P19.65

19.66 Using the method of Sec. 19.6, solve Prob. 19.32.

19.67 Using the method of Sec. 19.6, solve Prob. 19.31.

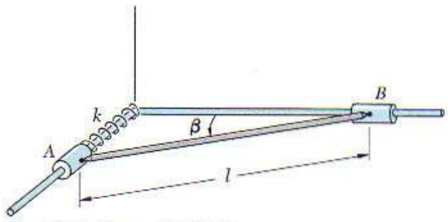


Fig. P19.68 and P19.70

19.68 The slender rod AB of mass m is attached to two collars of negligible mass. Knowing that the system lies in a horizontal plane and is in equilibrium in the position shown, determine the period of vibration if the collar A is given a small displacement and released.

19.69 Solve Prob. 19.68, assuming that rod AB is of mass m and that each collar is of mass m_c.

19.70 A slender rod AB of negligible mass connects two collars, each of mass m_c. Knowing that the system lies in a horizontal plane and is in equilibrium in the position shown, determine the period of vibration if the collar at A is given a small displacement and released.

19.71 Two uniform rods AB and CD, each of length l and weight W, are attached to two gears as shown. Neglecting the mass of the gears, determine for each of the positions shown the period of small oscillations of the system.

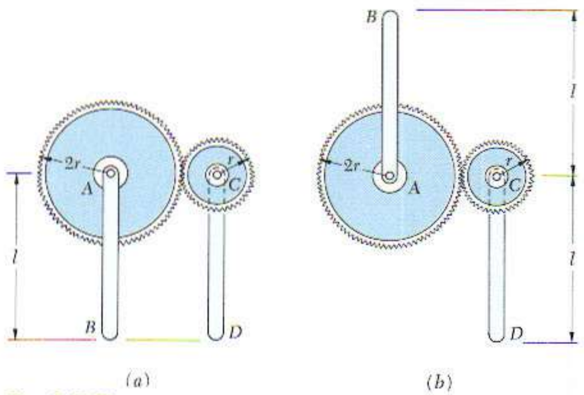


Fig. P19.71

19.72 Solve Prob. 19.71, assuming that $l = 18$ in., $W = 4$ lb, and $r = 3$ in.

19.73 A thin circular plate of radius r is suspended from three vertical wires of length h equally spaced around the perimeter of the plate. Determine the period of oscillation when (a) the plate is rotated through a small angle about a vertical axis passing through its mass center and released, (b) the plate is given a small horizontal translation and released.

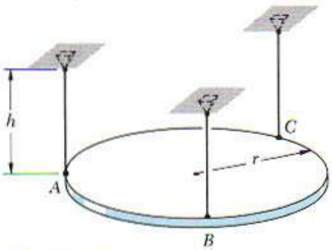


Fig. P19.73

19.74 Solve Prob. 19.73, assuming that $r = 500$ mm and $h = 300$ mm.

***19.75** As a submerged body moves through a fluid, the particles of the fluid flow around the body and thus acquire kinetic energy. In the case of a sphere moving in an ideal fluid, the total kinetic energy acquired by the fluid is $\frac{1}{4}\rho V v^2$, where ρ is the mass density of the fluid, V the volume of the sphere, and v the velocity of the sphere. Consider a 1-lb hollow spherical shell of radius 3 in. which is held submerged in a tank of water by a spring of constant 3 lb/in. (a) Neglecting fluid friction, determine the period of vibration of the shell when it is displaced vertically and then released. (b) Solve part a, assuming that the tank is accelerated upward at the constant rate of 10 ft/s^2.

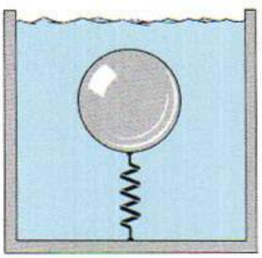


Fig. P19.75

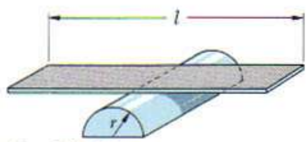


Fig. P19.76

***19.76** A thin plate of length l rests on a half cylinder of radius r. Derive an expression for the period of small oscillations of the plate.

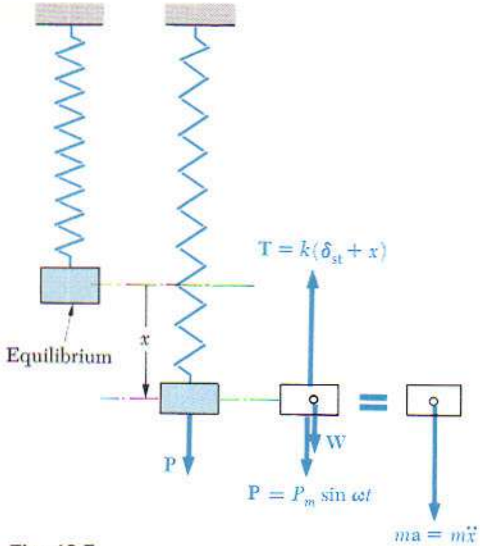


Fig. 19.7

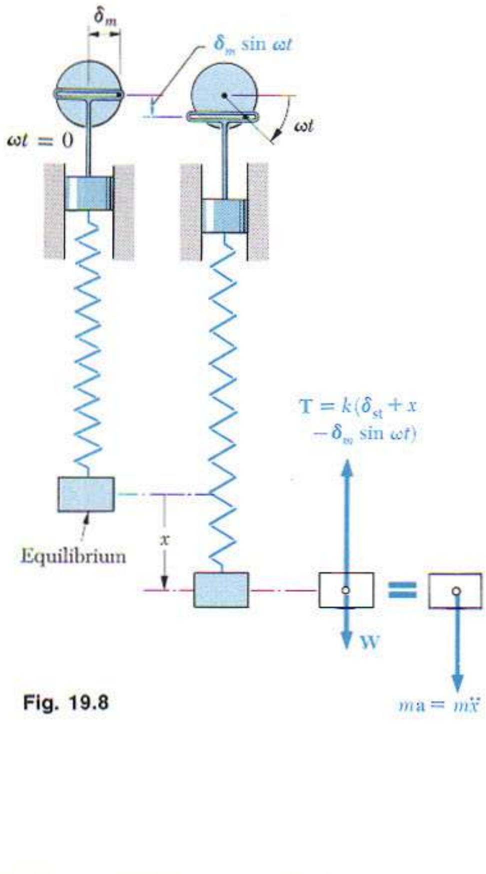


Fig. 19.8

19.7. Forced Vibrations. The most important vibrations from the point of view of engineering applications are the *forced vibrations* of a system. These vibrations occur when a system is subjected to a periodic force or when it is elastically connected to a support which has an alternating motion.

Consider first the case of a body of mass m suspended from a spring and subjected to a periodic force $\mathbf{P}$ of magnitude $P = P_m \sin \omega t$ (Fig. 19.7). This force may be an actual external force applied to the body, or it may be a centrifugal force produced by the rotation of some unbalanced part of the body. (See Sample Prob. 19.5.) Denoting by x the displacement of the body measured from its equilibrium position, we write the equation of motion,

$$+\downarrow \Sigma F = ma: \qquad P_m \sin \omega t + W - k(\delta_{st} + x) = m\ddot{x}$$

Recalling that $W = k\delta_{st}$, we have

$$m\ddot{x} + kx = P_m \sin \omega t \tag{19.30}$$

Next we consider the case of a body of mass m suspended from a spring attached to a moving support whose displacement δ is equal to $\delta_m \sin \omega t$ (Fig. 19.8). Measuring the displacement x of the body from the position of static equilibrium corresponding to $\omega t = 0$, we find that the total elongation of the spring at time t is $\delta_{st} + x - \delta_m \sin \omega t$. The equation of motion is thus

$$+\downarrow \Sigma F = ma: \qquad W - k(\delta_{st} + x - \delta_m \sin \omega t) = m\ddot{x}$$

Recalling that $W = k\delta_{st}$, we have

$$m\ddot{x} + kx = k\delta_m \sin \omega t \tag{19.31}$$

We note that Eqs. (19.30) and (19.31) are of the same form and that a solution of the first equation will satisfy the second if we set $P_m = k\delta_m$.

A differential equation like (19.30) or (19.31), possessing a right-hand member different from zero, is said to be *nonhomogeneous*. Its general solution is obtained by adding a particular solution of the given equation to the general solution of the corresponding *homogeneous* equation (with right-hand member equal to zero). A *particular solution* of (19.30) or (19.31) may be obtained by trying a solution of the form

$$x_{part} = x_m \sin \omega t \tag{19.32}$$

Substituting x_{part} for x into (19.30), we find

$$-m\omega^2 x_m \sin \omega t + kx_m \sin \omega t = P_m \sin \omega t$$

which may be solved for the amplitude,

$$x_m = \frac{P_m}{k - m\omega^2}$$

Recalling from (19.4) that $k/m = p^2$, where p is the circular frequency of the free vibration of the body, we write

$$x_m = \frac{P_m/k}{1 - (\omega/p)^2} \tag{19.33}$$

Substituting from (19.32) into (19.31), we obtain in a similar way

$$x_m = \frac{\delta_m}{1 - (\omega/p)^2} \tag{19.33'}$$

The homogeneous equation corresponding to (19.30) or (19.31) is Eq. (19.3), defining the free vibration of the body. Its general solution, called the *complementary function*, was found in Sec. 19.2,

$$x_{\text{comp}} = A \sin pt + B \cos pt \tag{19.34}$$

Adding the particular solution (19.32) and the complementary function (19.34), we obtain the *general solution* of Eqs. (19.30) and (19.31),

$$x = A \sin pt + B \cos pt + x_m \sin \omega t \tag{19.35}$$

We note that the vibration obtained consists of two superposed vibrations. The first two terms in (19.35) represent a free vibration of the system. The frequency of this vibration, called the *natural frequency* of the system, depends only upon the constant k of the spring and the mass m of the body, and the constants A and B may be determined from the initial conditions. This free vibration is also called a *transient* vibration since, in actual practice, it will soon be damped out by friction forces (Sec. 19.9).

The last term in (19.35) represents the *steady-state* vibration produced and maintained by the impressed force or impressed support movement. Its frequency is the *forced frequency* imposed by this force or movement, and its amplitude x_m, defined by (19.33) or (19.33′), depends upon the *frequency ratio* ω/p. The ratio of the amplitude x_m of the steady-state vibration to the static deflection P_m/k caused by a force P_m, or to the amplitude δ_m of the support movement, is called the *magnification factor.* From (19.33) and (19.33′), we obtain

$$\text{Magnification factor} = \frac{x_m}{P_m/k} = \frac{x_m}{\delta_m} = \frac{1}{1 - (\omega/p)^2} \qquad (19.36)$$

The magnification factor has been plotted in Fig. 19.9 against the frequency ratio ω/p. We note that, when $\omega = p$, the amplitude of the forced vibration becomes infinite. The impressed force or impressed support movement is said to be in *resonance* with the given system. Actually, the amplitude of the vibration remains finite because of damping forces (Sec. 19.9); nevertheless, such a situation should be avoided, and the forced frequency should not be chosen too close to the natural frequency of the system. We also note that for $\omega < p$ the coefficient of $\sin \omega t$ in (19.35) is positive, while for $\omega > p$ this coefficient is negative. In the first case the forced vibration is *in phase* with the impressed force or impressed support movement, while in the second case it is 180° *out of phase.*

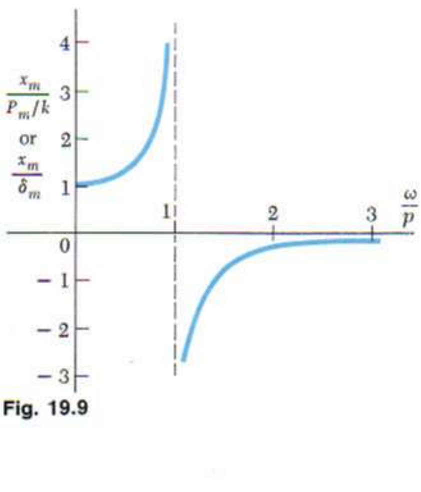
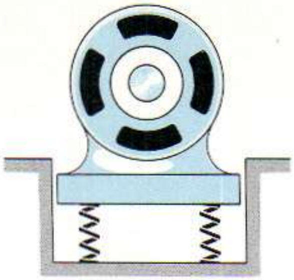


Fig. 19.9



SAMPLE PROBLEM 19.5

A motor weighing 350 lb is supported by four springs, each having a constant of 750 lb/in. The unbalance of the rotor is equivalent to a weight of 1 oz located 6 in. from the axis of rotation. Knowing that the motor is constrained to move vertically, determine (*a*) the speed in rpm at which resonance will occur, (*b*) the amplitude of the vibration of the motor at a speed of 1200 rpm.

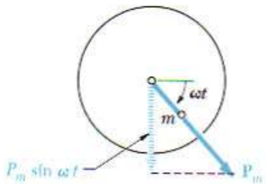
a. Resonance Speed. The resonance speed is equal to the circular frequency (in rpm) of the free vibration of the motor. The mass of the motor and the equivalent constant of the supporting springs are

$$m = \frac{350 \text{ lb}}{32.2 \text{ ft/s}^2} = 10.87 \text{ lb} \cdot \text{s}^2/\text{ft}$$

$$k = 4(750 \text{ lb/in.}) = 3000 \text{ lb/in.} = 36{,}000 \text{ lb/ft}$$

$$p = \sqrt{\frac{k}{m}} = \sqrt{\frac{36{,}000}{10.87}} = 57.5 \text{ rad/s} = 549 \text{ rpm}$$

Resonance speed = 549 rpm ◄



b. Amplitude of Vibration at 1200 rpm. The angular velocity of the motor and the mass of the equivalent 1-oz weight are

$$\omega = 1200 \text{ rpm} = 125.7 \text{ rad/s}$$

$$m = (1 \text{ oz})\frac{1 \text{ lb}}{16 \text{ oz}}\frac{1}{32.2 \text{ ft/s}^2} = 0.00194 \text{ lb} \cdot \text{s}^2/\text{ft}$$

The magnitude of the centrifugal force due to the unbalance of the rotor is

$$P_m = ma_n = mr\omega^2 = (0.00194 \text{ lb} \cdot \text{s}^2/\text{ft})(\tfrac{6}{12} \text{ ft})(125.7 \text{ rad/s})^2 = 15.3 \text{ lb}$$

The static deflection that would be caused by a constant load P_m is

$$\frac{P_m}{k} = \frac{15.3 \text{ lb}}{3000 \text{ lb/in.}} = 0.00510 \text{ in.}$$

Substituting the value of P_m/k together with the known values of ω and p into Eq. (19.33), we obtain

$$x_m = \frac{P_m/k}{1 - (\omega/p)^2} = \frac{0.00510 \text{ in.}}{1 - (125.7/57.5)^2}$$

$x_m = 0.00135$ in. ◄

Note. Since $\omega > p$, the vibration is 180° out of phase with the centrifugal force due to the unbalance of the rotor. For example, when the unbalanced mass is directly below the axis of rotation, the position of the motor is $x_m = 0.00135$ in. above the position of equilibrium.

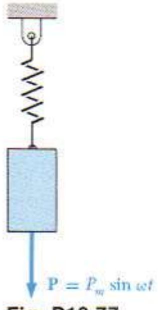


Fig. P19.77

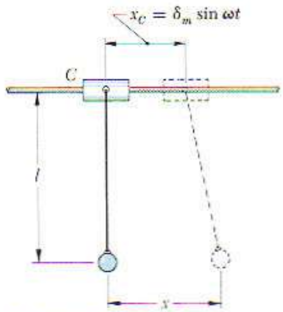


Fig. P19.79

PROBLEMS

19.77 A block of mass m is suspended from a spring of constant k and is acted upon by a vertical periodic force of magnitude $P = P_m \sin \omega t$. Determine the range of values of ω for which the amplitude of the vibration exceeds twice the static deflection caused by a constant force of magnitude P_m.

19.78 In Prob. 19.77, determine the range of values of ω for which the amplitude of the vibration is less than the static deflection caused by a constant force of magnitude P_m.

19.79 A simple pendulum of length l is suspended from a collar C which is forced to move horizontally according to the relation $x_C = \delta_m \sin \omega t$. Determine the range of values of ω for which the amplitude of the motion of the bob exceeds $2\delta_m$. (Assume δ_m is small compared to the length l of the pendulum.)

19.80 In Prob. 19.79, determine the range of values of ω for which the amplitude of the motion of the bob is less than δ_m.

19.81 A 500-lb motor is supported by a light horizontal beam. The unbalance of the rotor is equivalent to a weight of 1 oz located 10 in. from the axis of rotation. Knowing that the static deflection of the beam due to the weight of the motor is 0.220 in., determine (*a*) the speed (in rpm) at which resonance will occur, (*b*) the amplitude of the steady-state vibration of the motor at a speed of 800 rpm.

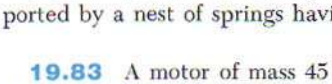


Fig. P19.81

19.82 Solve Prob. 19.81, assuming that the 500-lb motor is supported by a nest of springs having a total constant of 400 lb/in.

19.83 A motor of mass 45 kg is supported by four springs, each of constant 100 kN/m. The motor is constrained to move vertically, and the amplitude of its movement is observed to be 0.5 mm at a speed of 1200 rpm. Knowing that the mass of the rotor is 14 kg, determine the distance between the mass center of the rotor and the axis of the shaft.

19.84 In Prob. 19.83, determine the amplitude of the vertical movement of the motor at a speed of (*a*) 200 rpm, (*b*) 1600 rpm, (*c*) 900 rpm.

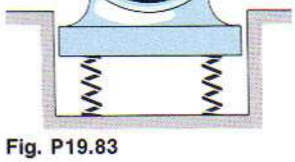


Fig. P19.83

19.85 Rod AB is rigidly attached to the frame of a motor running at a constant speed. When a collar of mass m is placed on the spring, it is observed to vibrate with an amplitude of 0.5 in. When two collars, each of mass m, are placed on the spring, the amplitude is observed to be 0.6 in. What amplitude of vibration should be expected when three collars, each of mass m, are placed on the spring? (Obtain two answers.)

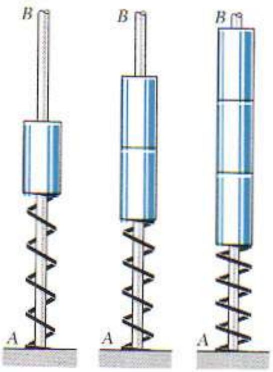


Fig. P19.85

19.86 Solve Prob. 19.85, assuming that the speed of the motor is changed and that one collar has an amplitude of 0.60 in. and two collars have an amplitude of 0.20 in.

19.87 A disk of mass m is attached to the midpoint of a vertical shaft which revolves at an angular velocity ω. Denoting by k the spring constant of the system for a horizontal movement of the disk and by e the eccentricity of the disk with respect to the shaft, show that the deflection of the center of the shaft may be written in the form

$$r = \frac{e(\omega/p)^2}{1 - (\omega/p)^2}$$

where $p = \sqrt{k/m}$.

Fig. P19.87 and P19.88

19.88 A disk of mass 30 kg is attached to the midpoint of a shaft. Knowing that a static force of 200 N will deflect the shaft 0.6 mm, determine the speed of the shaft in rpm at which resonance will occur.

19.89 Knowing that the disk of Prob. 19.88 is attached to the shaft with an eccentricity $e = 0.2$ mm, determine the deflection r of the shaft at a speed of 900 rpm.

19.90 A variable-speed motor is rigidly attached to the beam BC. When the speed of the motor is less than 1000 rpm or more than 2000 rpm, a small object placed at A is observed to remain in contact with the beam. For speeds between 1000 and 2000 rpm the object is observed to "dance" and actually to lose contact with the beam. Determine the speed at which resonance will occur.

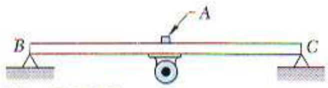


Fig. P19.90

19.91 As the speed of a spring-supported motor is slowly increased from 150 to 200 rpm, the amplitude of the vibration due to the unbalance of the rotor is observed to decrease continuously from 0.150 to 0.080 in. Determine the speed at which resonance will occur.

19.92 In Prob. 19.91, determine the speed for which the amplitude of the vibration is 0.200 in.

19.93 The amplitude of the motion of the pendulum bob shown is observed to be 3 in. when the amplitude of the motion of collar C is $\frac{3}{4}$ in. Knowing that the length of the pendulum is $l = 36$ in., determine the two possible values of the frequency of the horizontal movement of the collar C.

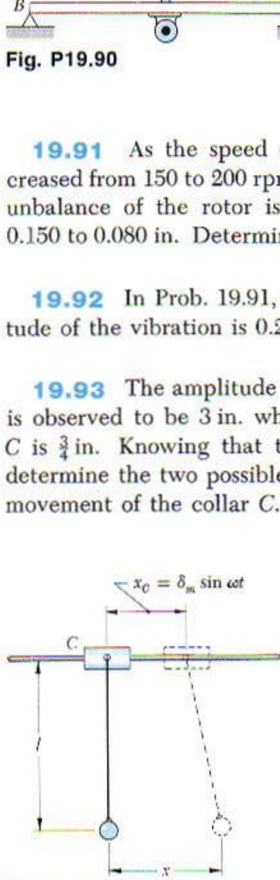


Fig. P19.93

19.94 A certain vibrometer used to measure vibration amplitudes consists essentially of a box containing a slender rod to which a mass m is attached; the natural frequency of the mass-rod system is known to be 5 Hz. When the box is rigidly attached to the casing of a motor rotating at 600 rpm, the mass is observed to vibrate with an amplitude of 1.6 mm relative to the box. Determine the amplitude of the vertical motion of the motor.

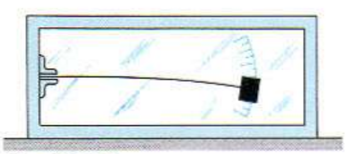


Fig. P19.94

19.95 A small trailer of mass 200 kg with its load is supported by two springs, each of constant 20 kN/m. The trailer is pulled over a road, the surface of which may be approximated by a sine curve of amplitude 30 mm and of period 5 m (i.e., the distance between two successive crests is 5 m, and the vertical distance from a crest to a trough is 60 mm). Determine (*a*) the speed at which resonance will occur, (*b*) the amplitude of the vibration of the trailer at a speed of 60 km/h.

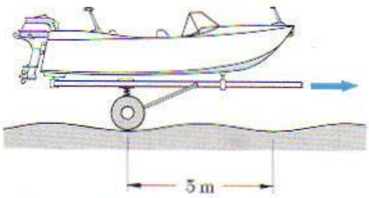


Fig. P19.95

19.96 Knowing that the amplitude of the vibration of the trailer of Prob. 19.95 is not to exceed 15 mm, determine the smallest speed at which the trailer can be pulled over the road.

DAMPED VIBRATIONS

***19.8. Damped Free Vibrations.** The vibrating systems considered in the first part of this chapter were assumed free of damping. Actually all vibrations are damped to some degree by friction forces. These forces may be caused by *dry friction*, or *Coulomb friction*, between rigid bodies, by *fluid friction* when a rigid body moves in a fluid, or by *internal friction* between the molecules of a seemingly elastic body.

A type of damping of special interest is the *viscous damping* caused by fluid friction at low and moderate speeds. Viscous damping is characterized by the fact that the friction force is *directly proportional to the speed* of the moving body. As an example, we shall consider again a body of mass m suspended from a spring of constant k, and we shall assume that the body is attached to the plunger of a dashpot (Fig. 19.10). The magnitude of the friction force exerted on the plunger by the surrounding fluid is equal to $c\dot{x}$, where the constant c, expressed in N · s/m or lb · s/ft and known as the *coefficient of viscous damping*, depends upon the physical properties of the fluid and the construction of the dashpot. The equation of motion is

$$+\downarrow \Sigma F = ma: \qquad W - k(\delta_{st} + x) - c\dot{x} = m\ddot{x}$$

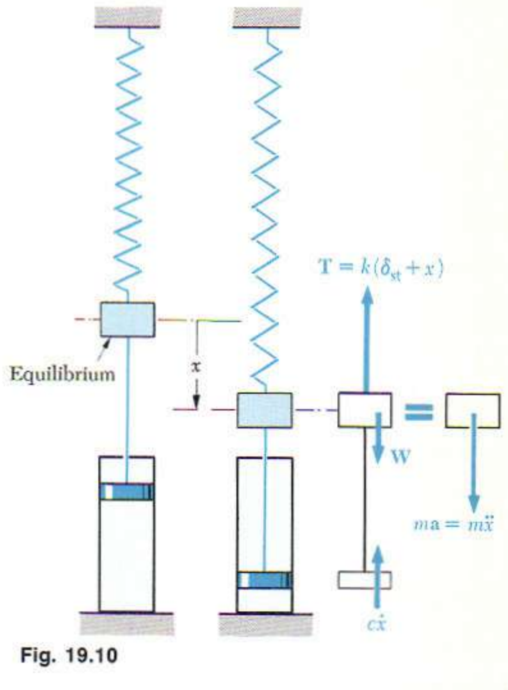


Fig. 19.10

Recalling that $W = k\delta_{st}$, we write

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{19.37}$$

Substituting $x = e^{\lambda t}$ into (19.37) and dividing through by $e^{\lambda t}$, we write the *characteristic equation*

$$m\lambda^2 + c\lambda + k = 0 \tag{19.38}$$

and obtain the roots

$$\lambda = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \tag{19.39}$$

Defining the *critical damping coefficient* c_c as the value of c which makes the radical in (19.39) equal to zero, we write

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0 \qquad c_c = 2m\sqrt{\frac{k}{m}} = 2mp \tag{19.40}$$

where p is the circular frequency of the system in the absence of damping. We may distinguish three different cases of damping, depending upon the value of the coefficient c.

1. *Heavy damping:* $c > c_c$. The roots λ_1 and λ_2 of the characteristic equation (19.38) are real and distinct, and the general solution of the differential equation (19.37) is

$$x = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \tag{19.41}$$

   This solution corresponds to a nonvibratory motion. Since λ_1 and λ_2 are both negative, x approaches zero as t increases indefinitely. However, the system actually regains its equilibrium position after a finite time.
2. *Critical damping:* $c = c_c$. The characteristic equation has a double root $\lambda = -c_c/2m = -p$, and the general solution of (19.37) is

$$x = (A + Bt)e^{-pt} \tag{19.42}$$

   The motion obtained is again nonvibratory. Critically damped systems are of special interest in engineering applications since they regain their equilibrium position in the shortest possible time without oscillation.
3. *Light damping:* $c < c_c$. The roots of (19.38) are complex and conjugate, and the general solution of (19.37) is of the form

$$x = e^{-(c/2m)t}(A \sin qt + B \cos qt) \tag{19.43}$$

where q is defined by the relation

$$q^2 = \frac{k}{m} - \left(\frac{c}{2m}\right)^2$$

Substituting $k/m = p^2$ and recalling (19.40), we write

$$q = p\sqrt{1 - \left(\frac{c}{c_c}\right)^2} \qquad (19.44)$$

where the constant c/c_c is known as the *damping factor.* A substitution similar to the one used in Sec. 19.2 enables us to write the general solution of (19.37) in the form

$$x = x_m e^{-(c/2m)t} \sin (qt + \phi) \qquad (19.45)$$

The motion defined by (19.45) is vibratory with diminishing amplitude (Fig. 19.11). Although this motion does not actually repeat itself, the time interval $\tau = 2\pi/q$, corresponding to two successive points where the curve (19.45) touches one of the limiting curves shown in Fig. 19.11, is commonly referred to as the period of the damped vibration. Recalling (19.44), we observe that τ is larger than the period of vibration of the corresponding undamped system.

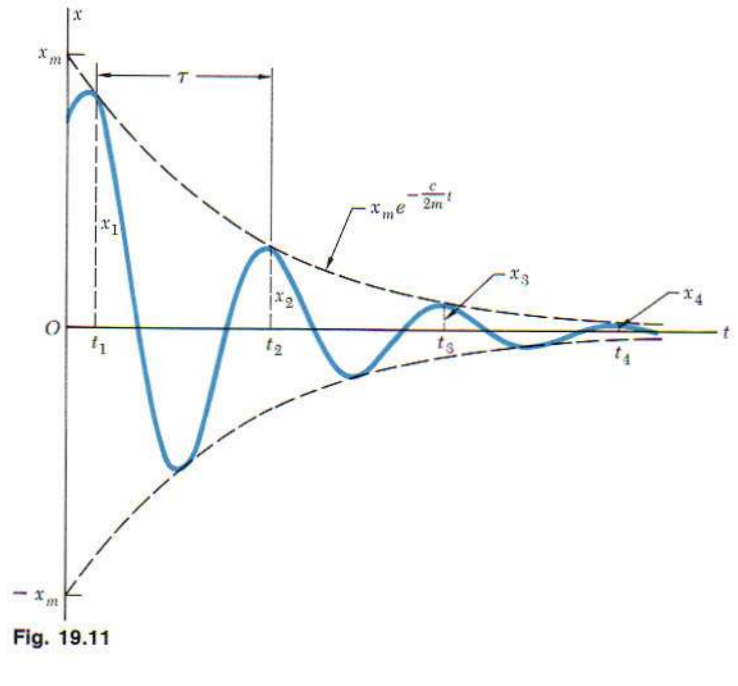


Fig. 19.11

***19.9. Damped Forced Vibrations.** If the system considered in the preceding section is subjected to a periodic force $\mathbf{P}$ of magnitude $P = P_m \sin \omega t$, the equation of motion becomes

$$m\ddot{x} + c\dot{x} + kx = P_m \sin \omega t \tag{19.46}$$

The general solution of (19.46) is obtained by adding a particular solution of (19.46) to the complementary function or general solution of the homogeneous equation (19.37). The complementary function is given by (19.41), (19.42), or (19.43), depending upon the type of damping considered. It represents a *transient* motion which is eventually damped out.

Our interest in this section is centered on the steady-state vibration represented by a particular solution of (19.46) of the form

$$x_{\text{part}} = x_m \sin (\omega t - \varphi) \tag{19.47}$$

Substituting x_{part} for x into (19.46), we obtain

$$-m\omega^2 x_m \sin(\omega t - \varphi) + c\omega x_m \cos(\omega t - \varphi) + kx_m \sin(\omega t - \varphi) = P_m \sin \omega t$$

Making $\omega t - \varphi$ successively equal to 0 and to $\pi/2$, we write

$$c\omega x_m = P_m \sin \varphi \tag{19.48}$$

$$(k - m\omega^2)x_m = P_m \cos \varphi \tag{19.49}$$

Squaring both members of (19.48) and (19.49) and adding, we have

$$[(k - m\omega^2)^2 + (c\omega)^2]x_m^2 = P_m^2 \tag{19.50}$$

Solving (19.50) for x_m and dividing (19.48) and (19.49) member by member, we obtain, respectively,

$$x_m = \frac{P_m}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \qquad \tan \varphi = \frac{c\omega}{k - m\omega^2} \tag{19.51}$$

Recalling from (19.4) that $k/m = p^2$, where p is the circular frequency of the undamped free vibration, and from (19.40) that $2mp = c_c$, where c_c is the critical damping coefficient of the system, we write

$$\frac{x_m}{P_m/k} = \frac{x_m}{\delta_m} = \frac{1}{\sqrt{[1 - (\omega/p)^2]^2 + [2(c/c_c)(\omega/p)]^2}} \tag{19.52}$$

$$\tan \varphi = \frac{2(c/c_c)(\omega/p)}{1 - (\omega/p)^2} \tag{19.53}$$

Formula (19.52) expresses the magnification factor in terms of the frequency ratio ω/p and damping factor c/c_c. It may be used to determine the amplitude of the steady-state vibration produced by an impressed force of magnitude $P = P_m \sin \omega t$ or by an impressed support movement $\delta = \delta_m \sin \omega t$. Formula (19.53) defines in terms of the same parameters the *phase difference* φ between the impressed force or impressed support movement and the resulting steady-state vibration of the damped system. The magnification factor has been plotted against the frequency ratio in Fig. 19.12 for various values of the damping factor. We observe that the amplitude of a forced vibration may be kept small by choosing a large coefficient of viscous damping c or by keeping the natural and forced frequencies far apart.

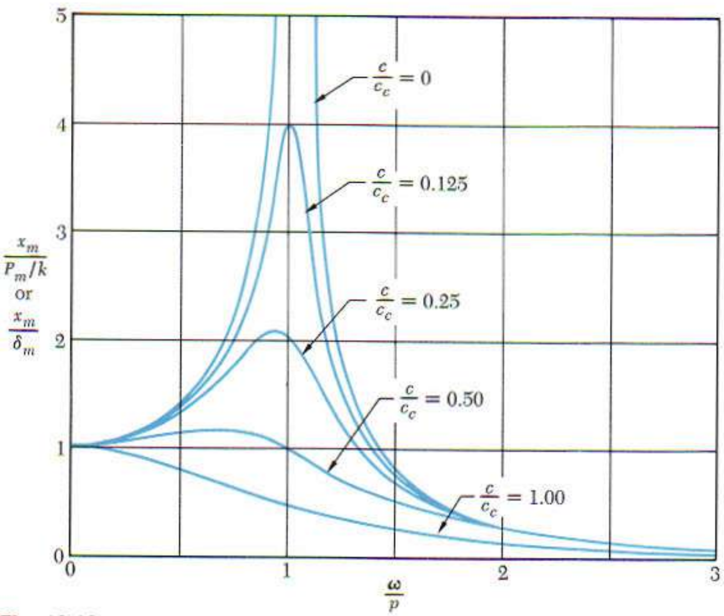


Fig. 19.12

*19.10. Electrical Analogues.

Oscillating electrical circuits are characterized by differential equations of the same type as those obtained in the preceding sections. Their analysis is therefore similar to that of a mechanical system, and the results obtained for a given vibrating system may be readily extended to the equivalent circuit. Conversely, any result obtained for an electrical circuit will also apply to the corresponding mechanical system.

Consider an electrical circuit consisting of an inductor of inductance L, a resistor of resistance R, and a capacitor of

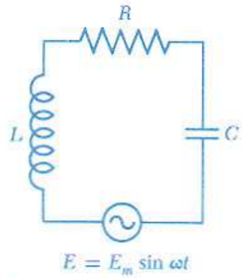


Fig. 19.13

capacitance C, connected in series with a source of alternating voltage $E = E_m \sin \omega t$ (Fig. 19.13). It is recalled from elementary electromagnetic theory† that, if i denotes the current in the circuit and q the electric charge on the capacitor, the drop in potential is $L(di/dt)$ across the inductor, Ri across the resistor, and q/C across the capacitor. Expressing that the algebraic sum of the applied voltage and of the drops in potential around the circuit loop is zero, we write

$$E_m \sin \omega t - L\frac{di}{dt} - Ri - \frac{q}{C} = 0 \qquad (19.54)$$

Rearranging the terms and recalling that, at any instant, the current i is equal to the rate of change $\dot{q}$ of the charge q, we have

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = E_m \sin \omega t \qquad (19.55)$$

We verify that Eq. (19.55), which defines the oscillations of the electrical circuit of Fig. 19.13, is of the same type as Eq. (19.46), which characterizes the damped forced vibrations of the mechanical system of Fig. 19.10. By comparing the two equations, we may construct a table of the analogous mechanical and electrical expressions.

Table 19.2 may be used to extend to their electrical analogues the results obtained in the preceding sections for various mechanical systems. For instance, the amplitude i_m of the current in the circuit of Fig. 19.13 may be obtained by noting that it corresponds to the maximum value v_m of the velocity in the analogous mechanical system. Recalling that $v_m = \omega x_m$, substituting for x_m from Eq. (19.51), and replacing the constants of

Table 19.2 Characteristics of a Mechanical System and of Its Electrical Analogue

Mechanical System		**Electrical Circuit**	
m	mass	L	Inductance
c	Coefficient of viscous damping	R	Resistance
k	Spring constant	$1/C$	Reciprocal of capacitance
x	Displacement	q	Charge
v	Velocity	i	Current
P	Applied force	E	Applied voltage

† See Hammond, "Electrical Engineering," McGraw-Hill Book Company, or Smith, "Circuits, Devices, and Systems," John Wiley & Sons.

the mechanical system by the corresponding electrical expressions, we have

$$i_m = \frac{\omega E_m}{\sqrt{\left(\frac{1}{C} - L\omega^2\right)^2 + (R\omega)^2}}$$

$$i_m = \frac{E_m}{\sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}} \tag{19.56}$$

The radical in the expression obtained is known as the *impedance* of the electrical circuit.

The analogy between mechanical systems and electrical circuits holds for transient as well as steady-state oscillations. The oscillations of the circuit shown in Fig. 19.14, for instance, are analogous to the damped free vibrations of the system of Fig. 19.10. As far as the initial conditions are concerned, we may note that closing the switch S when the charge on the capacitor is $q = q_0$ is equivalent to releasing the mass of the mechanical system with no initial velocity from the position $x = x_0$. We should also observe that, if a battery of constant voltage E is introduced in the electrical circuit of Fig. 19.14, closing the switch S will be equivalent to suddenly applying a force of constant magnitude P to the mass of the mechanical system of Fig. 19.10.

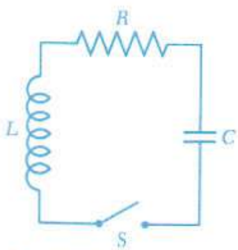


Fig. 19.14

The above discussion would be of questionable value if its only result were to make it possible for mechanics students to analyze electrical circuits without learning the elements of electromagnetism. It is hoped, rather, that this discussion will encourage the students to apply to the solution of problems in mechanical vibrations the mathematical techniques they may learn in later courses in electrical circuits theory. The chief value of the concept of electrical analogue, however, resides in its application to *experimental methods* for the determination of the characteristics of a given mechanical system. Indeed, an electrical circuit is much more easily constructed than a mechanical model, and the fact that its characteristics may be modified by varying the inductance, resistance, or capacitance of its various components makes the use of the electrical analogue particularly convenient.

To determine the electrical analogue of a given mechanical system, we shall focus our attention on each moving mass in the system and observe which springs, dashpots, or external forces are applied directly to it. An equivalent electrical loop may then be constructed to match each of the mechanical units

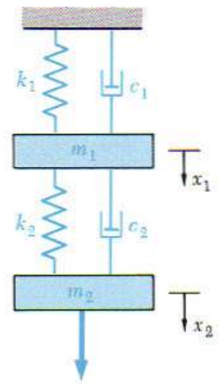


Fig. 19.15

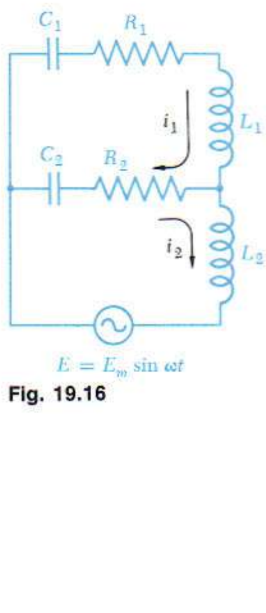


Fig. 19.16

thus defined; the various loops obtained in that way will form together the desired circuit. Consider, for instance, the mechanical system of Fig. 19.15. We observe that the mass m_1 is acted upon by two springs of constants k_1 and k_2 and by two dashpots characterized by the coefficients of viscous damping c_1 and c_2. The electrical circuit should therefore include a loop consisting of an inductor of inductance L_1 proportional to m_1, of two capacitors of capacitance C_1 and C_2 inversely proportional to k_1 and k_2, respectively, and of two resistors of resistance R_1 and R_2, proportional to c_1 and c_2, respectively. Since the mass m_2 is acted upon by the spring k_2 and the dashpot c_2, as well as by the force $P = P_m \sin \omega t$, the circuit should also include a loop containing the capacitor C_2, the resistor R_2, the new inductor L_2, and the voltage source $E = E_m \sin \omega t$ (Fig. 19.16).

To check that the mechanical system of Fig. 19.15 and the electrical circuit of Fig. 19.16 actually satisfy the same differential equations, we shall first derive the equations of motion for m_1 and m_2. Denoting respectively by x_1 and x_2 the displacements of m_1 and m_2 from their equilibrium positions, we observe that the elongation of the spring k_1 (measured from the equilibrium position) is equal to x_1, while the elongation of the spring k_2 is equal to the relative displacement $x_2 - x_1$ of m_2 with respect to m_1. The equations of motion for m_1 and m_2 are therefore

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + c_2(\dot{x}_1 - \dot{x}_2) + k_1x_1 + k_2(x_1 - x_2) = 0 \quad (19.57)$$

$$m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = P_m \sin \omega t \quad (19.58)$$

Consider now the electrical circuit of Fig. 19.16; we denote respectively by i_1 and i_2 the current in the first and second loops, and by q_1 and q_2 the integrals $\int i_1\,dt$ and $\int i_2\,dt$. Noting that the charge on the capacitor C_1 is q_1, while the charge on C_2 is $q_1 - q_2$, we express that the sum of the potential differences in each loop is zero:

$$L_1\ddot{q}_1 + R_1\dot{q}_1 + R_2(\dot{q}_1 - \dot{q}_2) + \frac{q_1}{C_1} + \frac{q_1 - q_2}{C_2} = 0 \quad (19.59)$$

$$L_2\ddot{q}_2 + R_2(\dot{q}_2 - \dot{q}_1) + \frac{q_2 - q_1}{C_2} = E_m \sin \omega t \quad (19.60)$$

We easily check that Eqs. (19.59) and (19.60) reduce to (19.57) and (19.58), respectively, when the substitutions indicated in Table 19.2 are performed.

PROBLEMS

19.97 Show that, in the case of heavy damping ($c > c_c$), a body never passes through its position of equilibrium O (*a*) if it is released with no initial velocity from an arbitrary position or (*b*) if it is started from O with an arbitrary initial velocity.

19.98 Show that, in the case of heavy damping ($c > c_c$), a body released from an arbitrary position with an arbitrary initial velocity cannot pass more than once through its equilibrium position.

19.99 In the case of light damping, the displacements x_1, x_2, x_3, etc., shown in Fig. 19.11 may be assumed equal to the maximum displacements. Show that the ratio of any two successive maximum displacements x_n and x_{n+1} is a constant and that the natural logarithm of this ratio, called the *logarithmic decrement*, is

$$\ln \frac{x_n}{x_{n+1}} = \frac{2\pi(c/c_c)}{\sqrt{1 - (c/c_c)^2}}$$

19.100 In practice, it is often difficult to determine the logarithmic decrement defined in Prob. 19.99 by measuring two successive maximum displacements. Show that the logarithmic decrement may also be expressed as $(1/n) \ln (x_1/x_{n+1})$, where n is the number of cycles between readings of the maximum displacement.

19.101 Successive maximum displacements of a spring-mass-dashpot system similar to that shown in Fig. 19.10 are 75, 60, 48, and 38.4 mm. Knowing that $m = 20$ kg and $k = 800$ N/m, determine (*a*) the damping factor c/c_c, (*b*) the value of the coefficient of viscous damping c. (*Hint.* See Probs. 19.99 and 19.100.)

19.102 In a system with light damping ($c < c_c$), the period of vibration is commonly defined as the time interval $\tau = 2\pi/q$ corresponding to two successive points where the displacement-time curve touches one of the limiting curves shown in Fig. 19.11. Show that the interval of time (*a*) between a maximum positive displacement and the following maximum negative displacement is $\frac{1}{2}\tau$, (*b*) between two successive zero displacements is $\frac{1}{2}\tau$, (*c*) between a maximum positive displacement and the following zero displacement is greater than $\frac{1}{4}\tau$.

19.103 The barrel of a field gun weighs 1200 lb and is returned into firing position after recoil by a recuperator of constant $k = 8000$ lb/ft. Determine the value of the coefficient of damping of the recoil mechanism which causes the barrel to return into firing position in the shortest possible time without oscillation.

19.104 A critically damped system is released from rest at an arbitrary position x_0 when $t = 0$. (*a*) Determine the position of the system at any time t. (*b*) Apply the result obtained in part *a* to the barrel of the gun of Prob. 19.103, and determine the time at which the barrel is halfway back to its firing position.

19.105 Assuming that the barrel of the gun of Prob. 19.103 is modified, with a resulting increase in weight of 300 lb, determine the constant k of the recuperator which should be used if the recoil mechanism is to remain critically damped.

19.106 In the case of the forced vibration of a system with a given damping factor c/c_c, determine the frequency ratio ω/p for which the amplitude of the vibration is maximum.

19.107 Show that for a small value of the damping factor c/c_c, (*a*) the maximum amplitude of a forced vibration occurs when $\omega \approx p$, (*b*) the corresponding value of the magnification factor is approximately $\frac{1}{2}(c_c/c)$.

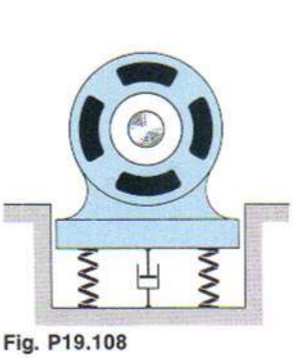


Fig. P19.108

19.108 A motor of mass 25 kg is supported by four springs, each having a constant of 200 kN/m. The unbalance of the rotor is equivalent to a mass of 30 g located 125 mm from the axis of rotation. Knowing that the motor is constrained to move vertically, determine the amplitude of the steady-state vibration of the motor at a speed of 1800 rpm, assuming (*a*) that no damping is present, (*b*) that the damping factor c/c_c is equal to 0.125.

19.109 Assume that the 25-kg motor of Prob. 19.108 is directly supported by a light horizontal beam. The static deflection of the beam due to the weight of the motor is observed to be 5.75 mm, and the amplitude of the vibration of the motor is 0.5 mm at a speed of 400 rpm. Determine (*a*) the damping factor c/c_c, (*b*) the coefficient of damping c.

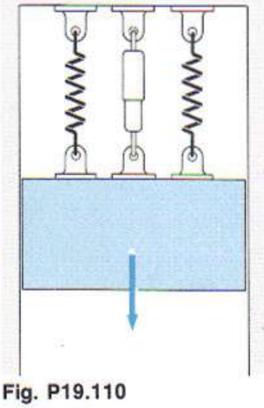


Fig. P19.110

19.110 A machine element weighing 800 lb is supported by two springs, each having a constant of 200 lb/in. A periodic force of maximum value 30 lb is applied to the element with a frequency of 2.5 cycles per second. Knowing that the coefficient of damping is 8 lb · s/in., determine the amplitude of the steady-state vibration of the element.

19.111 In Prob. 19.110, determine the required value of the coefficient of damping if the amplitude of the steady-state vibration of the element is to be 0.15 in.

19.112 A platform of mass 100 kg, supported by a set of springs equivalent to a single spring of constant $k = 80$ kN/m, is subjected to a periodic force of maximum magnitude 500 N. Knowing that the coefficient of damping is $2 \text{ kN} \cdot \text{s/m}$, determine (*a*) the natural frequency in rpm of the platform *if* there were no damping, (*b*) the frequency in rpm of the periodic force corresponding to the maximum value of the magnification factor, assuming damping, (*c*) the amplitude of the *actual* motion of the platform for each of the frequencies found in parts *a* and *b*.

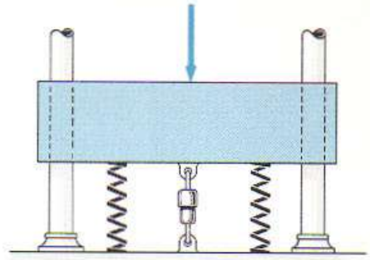


Fig. P19.112

***19.113** The suspension of an automobile may be approximated by the simplified spring-and-dashpot system shown. (*a*) Write the differential equation defining the absolute motion of the mass m when the system moves at a speed v over a road of sinusoidal cross section as shown. (*b*) Derive an expression for the amplitude of the absolute motion of m.

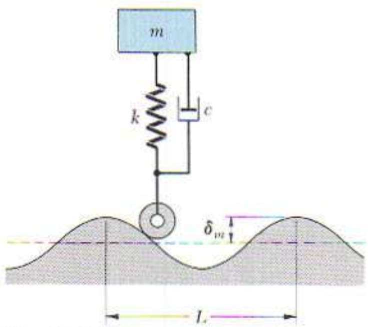


Fig. P19.113

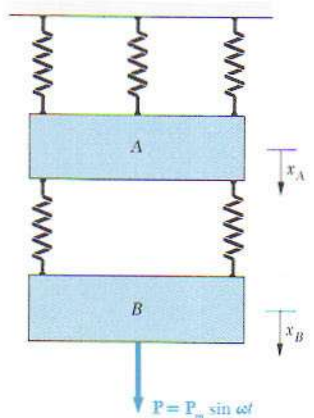


Fig. P19.114

***19.114** Two loads A and B, each of mass m, are suspended as shown by means of five springs of the same constant k. Load B is subjected to a force of magnitude $P = P_m \sin \omega t$. Write the differential equations defining the displacements x_A and x_B of the two loads from their equilibrium positions.

19.115 Determine the range of values of the resistance R for which oscillations will take place in the circuit shown when the switch S is closed.

19.116 Consider the circuit of Prob. 19.115 when the capacitance C is equal to zero. If the switch S is closed at time $t = 0$, determine (*a*) the final value of the current in the circuit, (*b*) the time t at which the current will have reached $(1 - 1/e)$ times its final value. (The desired value of t is known as the *time constant* of the circuit.)

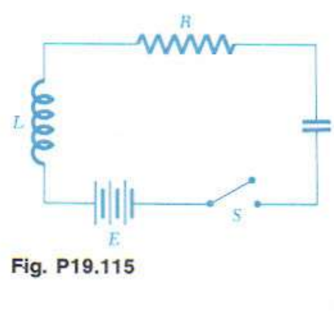


Fig. P19.115

19.117 through 19.120 Draw the electrical analogue of the mechanical system shown. (*Hint.* In Probs. 19.117 and 19.118, draw the loops corresponding to the free bodies m and A.)

19.121 and 19.122 Write the differential equations defining (a) the displacements of mass m and point A, (b) the currents in the corresponding loops of the electrical analogue.

19.123 and 19.124 Write the differential equations defining (a) the displacements of the masses m_1 and m_2, (b) the currents in the corresponding loops of the electrical analogue.

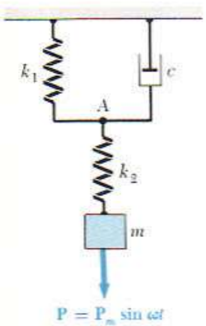


Fig. P19.117 and P19.121

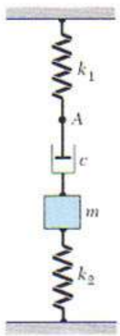


Fig. P19.118 and P19.122

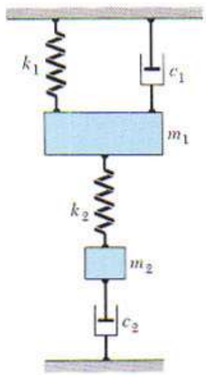


Fig. P19.119 and P19.123

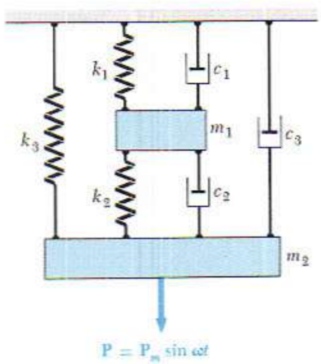


Fig. P19.120 and P19.124

REVIEW PROBLEMS

19.125 A homogeneous wire of length $2l$ is bent as shown and allowed to oscillate about a frictionless pin at B. Denoting by τ_0 the period of small oscillations when $\beta = 0$, determine the angle β for which the period of small oscillations is $2\tau_0$.

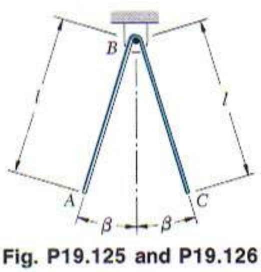


Fig. P19.125 and P19.126

19.126 Knowing that $l = 0.6$ m and $\beta = 30°$, determine the period of oscillation of the bent homogeneous wire shown.

19.127 A period of 4.10 s is observed for the angular oscillations of a 1-lb gyroscope rotor suspended from a wire as shown. Knowing that a period of 6.20 s is obtained when a 2-in.-diameter steel sphere is suspended in the same fashion, determine the centroidal radius of gyration of the rotor. (Specific weight of steel = 490 lb/ft^3.)

Fig. P19.127

19.128 An automobile wheel-and-tire assembly of total weight 47 lb is attached to a mounting plate of negligible weight which is suspended from a steel wire. The torsional spring constant of the wire is known to be $K = 0.40$ lb · in./rad. The wheel is rotated through 90° about the vertical and then released. Knowing that the period of oscillation is observed to be 30 s, determine the centroidal mass moment of inertia and the centroidal radius of gyration of the wheel-and-tire assembly.

Fig. P19.128

19.129 A homogeneous wire is bent to form a square of side l which is supported by a ball-and-socket joint at A. Determine the period of small oscillations of the square (a) in the plane of the square, (b) in a direction perpendicular to the square.

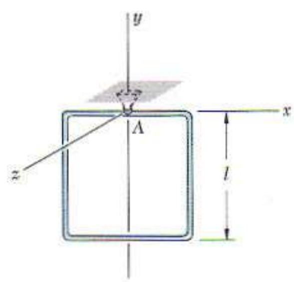


Fig. P19.129

19.130 A 150-kg electromagnet is at rest and is holding 100 kg of scrap steel when the current is turned off and the steel is dropped. Knowing that the cable and the supporting crane have a total stiffness equivalent to a spring of constant 200 kN/m, determine (a) the frequency, the amplitude, and the maximum velocity of the resulting motion, (b) the minimum tension which will occur in the cable during the motion, (c) the velocity of the magnet 0.03 s after the current is turned off.

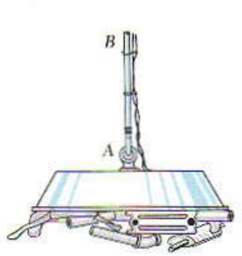


Fig. P19.130

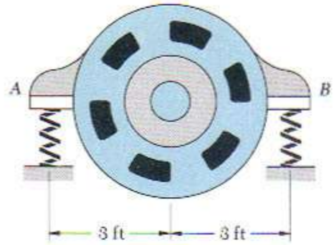


Fig. P19.131

19.131 During the normal operation of a single-phase generator, the transmission of undesirable vibrations is prevented by four springs mounted as shown (two springs at A and two springs at B). Knowing that the stator of the generator weighs 400 lb and has a centroidal radius of gyration of 24 in., determine the required constant of each spring if the frequency of the free *angular* vibration of the stator is to be 15 cycles per second.

19.132 The *rotor* of the generator of Prob. 19.131 weighs 300 lb and has a centroidal radius of gyration of 18 in. Knowing that the springs have been chosen so that the angular frequency of the *stator* alone is 15 cycles per second, determine the frequency of the *angular* vibration of the generator if the bearings are frozen so that the rotor and stator move as a single rigid body.

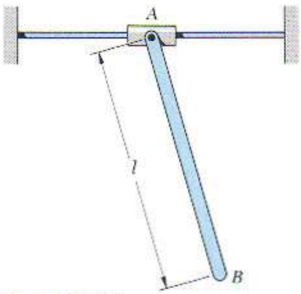


Fig. P19.133

19.133 A slender bar of length l is attached by a smooth pin at A to a collar of negligible mass. Determine the period of small oscillations of the bar, assuming that the coefficient of friction between the collar and the horizontal rod (a) is sufficient to prevent any movement of the collar, (b) is zero.

19.134 A 2-kg instrument is spring-mounted on the casing of a motor rotating at 1800 rpm. The motor is unbalanced and the amplitude of the motion of its casing is 0.5 mm. Knowing that $k = 9000$ N/m, determine the amplitude of the motion of the instrument.

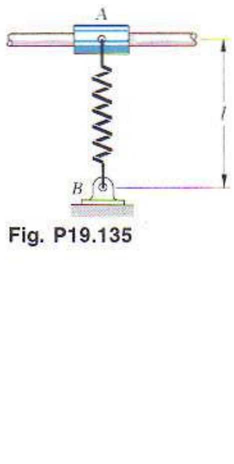


Fig. P19.135

19.135 A collar of mass m slides without friction on a horizontal rod and is attached to a spring AB of constant k. (a) If the unstretched length of the spring is just equal to l, show that the collar does not execute simple harmonic motion even when the amplitude of the oscillations is small. (b) If the unstretched length of the spring is less than l, show that the motion may be approximated by a simple harmonic motion for small oscillations.

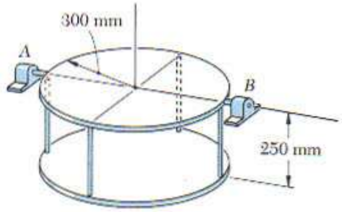


Fig. P19.136

19.136 The assembly shown consists of two thin disks, each of mass 3 kg, and four slender rods, each of mass 0.4 kg. Determine the period of oscillation of the assembly.

Some Useful Definitions and Properties of Vector Algebra

APPENDIX

A

The following definitions and properties of vector algebra were discussed fully in Chaps. 2 and 3 of *Vector Mechanics for Engineers: Statics*. They are summarized here for the convenience of the reader, with references to the appropriate sections of the *Statics* volume. Equation and illustration numbers are those used in the original presentation.

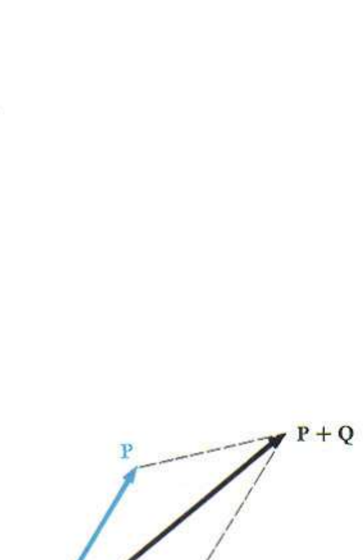


Fig. 2.6

A.1. Addition of Vectors (Secs. 2.2 and 2.3). Vectors are defined as *mathematical expressions possessing magnitude and direction, which add according to the parallelogram law.* Thus the sum of two vectors **P** and **Q** is obtained by attaching the two vectors to the same point A and constructing a parallelogram, using **P** and **Q** as two sides of the parallelogram (Fig. 2.6). The diagonal that passes through A represents the sum of the vectors **P** and **Q**, and this sum is denoted by $\mathbf{P} + \mathbf{Q}$. Vector addition is *associative* and *commutative.*

The *negative vector* of a given vector **P** is defined as a vector having the same magnitude P and a direction opposite to that of **P** (Fig. 2.5); the negative of the vector **P** is denoted by $-\mathbf{P}$. Clearly, we have

$$\mathbf{P} + (-\mathbf{P}) = 0$$

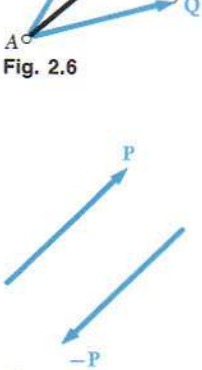


Fig. 2.5

A.2. Product of a Scalar and a Vector (Sec. 2.3). The product $k\mathbf{P}$ of a scalar k and a vector $\mathbf{P}$ is defined as a vector having the same direction as $\mathbf{P}$ (if k is positive), or a direction opposite to that of $\mathbf{P}$ (if k is negative), and a magnitude equal to the product of the magnitude P and of the absolute value of k (Fig. 2.13).

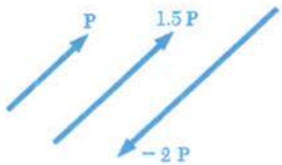


Fig. 2.13

A.3. Unit Vectors. Resolution of a Vector into Rectangular Components (Secs. 2.6 and 2.11). The vectors **i**, **j**, and **k**, called *unit vectors*, are defined as vectors of magnitude 1, directed respectively along the positive x, y, and z axes (Fig. 2.32).

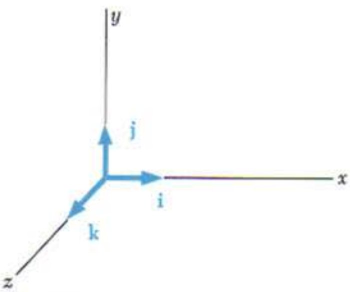


Fig. 2.32

Denoting by F_x, F_y, and F_z the scalar components of a vector **F**, we have (Fig. 2.33)

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \tag{2.20}$$

In the particular case of a unit vector $\boldsymbol{\lambda}$ directed along a line forming angles θ_x, θ_y, and θ_z with the coordinate axes, we have

$$\boldsymbol{\lambda} = \cos\theta_x\mathbf{i} + \cos\theta_y\mathbf{j} + \cos\theta_z\mathbf{k} \tag{2.22}$$

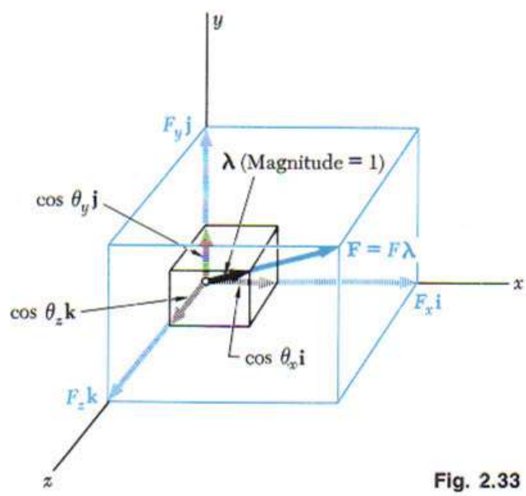


Fig. 2.33

A.4. Vector Product of Two Vectors (Secs. 3.3 and 3.4). The vector product, or *cross product,* of two vectors **P** and **Q** is defined as the vector

$$\mathbf{V} = \mathbf{P} \times \mathbf{Q}$$

which satisfies the following conditions:

1. The line of action of **V** is perpendicular to the plane containing **P** and **Q** (Fig. 3.6).
2. The magnitude of **V** is the product of the magnitudes of **P** and **Q** and of the sine of the angle θ formed by **P** and **Q** (the measure of which will always be 180° or less); we thus have

$$V = PQ \sin\theta \tag{3.1}$$

3. The sense of **V** is such that a man located at the tip of **V** will observe as counterclockwise the rotation through θ which brings the vector **P** in line with the vector **Q**; note that if **P** and **Q** do not have a common point of application, they should first be redrawn from the same point. The three vectors **P**, **Q**, and **V**—taken in that order—form a *right-handed triad.*

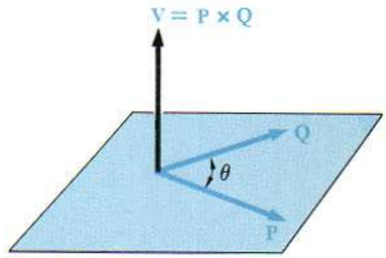


Fig. 3.6

Vector products are *distributive*, but they are *not commutative.* We have

$$\mathbf{Q} \times \mathbf{P} = -(\mathbf{P} \times \mathbf{Q}) \tag{3.4}$$

Vector Products of Unit Vectors. It follows from the definition of the vector product of two vectors that

$$\begin{array}{lll} \mathbf{i} \times \mathbf{i} = 0 & \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{j} = 0 & \mathbf{k} \times \mathbf{j} = -\mathbf{i} \\ \mathbf{i} \times \mathbf{k} = -\mathbf{j} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{k} = 0 \end{array} \tag{3.7}$$

Rectangular Components of Vector Product. Resolving the vectors **P** and **Q** into rectangular components, we obtain the following expressions for the components of their vector product **V**:

$$\begin{aligned} V_x &= P_y Q_z - P_z Q_y \\ V_y &= P_z Q_x - P_x Q_z \\ V_z &= P_x Q_y - P_y Q_x \end{aligned} \tag{3.9}$$

In determinant form, we have

$$\mathbf{V} = \mathbf{P} \times \mathbf{Q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} \tag{3.10}$$

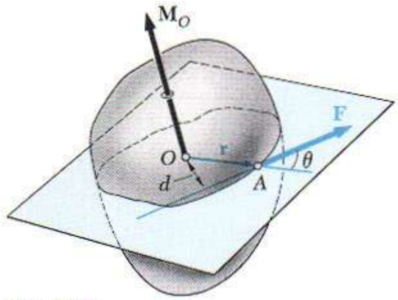


Fig. 3.12

A.5. Moment of a Force about a Point (Secs. 3.5 and 3.7). The moment of a force **F** (or, more generally, of a vector **F**) about a point O is defined as the vector product

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} \tag{3.11}$$

where **r** denotes the *position vector* of the point of application A of **F** (Fig. 3.12).

Rectangular Components of the Moment of a Force. Denoting by x, y, and z the coordinates of the point of application A of **F**, we obtain the following expressions for the components of the moment $\mathbf{M}_O$ of **F**:

$$\begin{aligned} M_x &= yF_z - zF_y \\ M_y &= zF_x - xF_z \\ M_z &= xF_y - yF_x \end{aligned} \tag{3.18}$$

In determinant form, we have

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} \tag{3.19}$$

To compute the moment $\mathbf{M}_B$ about an arbitrary point B of a force **F** applied at A, we must use the vector $\Delta\mathbf{r} = \mathbf{r}_A - \mathbf{r}_B$ instead of the vector **r**. We write

$$\mathbf{M}_B = \Delta\mathbf{r} \times \mathbf{F} = (\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F} \tag{3.20}$$

or, using the determinant form,

$$\mathbf{M}_B = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & \Delta y & \Delta z \\ F_x & F_y & F_z \end{vmatrix} \tag{3.21}$$

where Δx, Δy, Δz are the components of the vector $\Delta\mathbf{r}$ joining A and B:

$$\Delta x = x_A - x_B \qquad \Delta y = y_A - y_B \qquad \Delta z = z_A - z_B$$

A.6. Scalar Product of Two Vectors (Sec. 3.8). The scalar product, or *dot product*, of two vectors **P** and **Q** is defined as the product of the magnitudes of **P** and **Q** and of the cosine of the angle θ formed by **P** and **Q** (Fig. 3.19). The scalar product of **P** and **Q** is denoted by $\mathbf{P} \cdot \mathbf{Q}$. We write

$$\mathbf{P} \cdot \mathbf{Q} = PQ\cos\theta \tag{3.24}$$

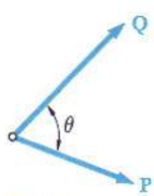


Fig. 3.19

Scalar products are *commutative* and *distributive*.

Scalar Products of Unit Vectors. It follows from the definition of the scalar product of two vectors that

$$\begin{matrix} \mathbf{i}\cdot\mathbf{i} = 1 & \mathbf{j}\cdot\mathbf{j} = 1 & \mathbf{k}\cdot\mathbf{k} = 1 \\ \mathbf{i}\cdot\mathbf{j} = 0 & \mathbf{j}\cdot\mathbf{k} = 0 & \mathbf{k}\cdot\mathbf{i} = 0 \end{matrix} \qquad (3.29)$$

Scalar Product Expressed in Terms of Rectangular Components. Resolving the vectors **P** and **Q** into rectangular components, we obtain

$$\mathbf{P}\cdot\mathbf{Q} = P_xQ_x + P_yQ_y + P_zQ_z \qquad (3.30)$$

Angle Formed by Two Vectors. It follows from (3.24) and (3.29) that

$$\cos\theta = \frac{\mathbf{P}\cdot\mathbf{Q}}{PQ} = \frac{P_xQ_x + P_yQ_y + P_zQ_z}{PQ} \qquad (3.32)$$

Projection of a Vector on a Given Axis. The projection of a vector **P** on the axis OL defined by the unit vector $\boldsymbol{\lambda}$ (Fig. 3.23) is

$$P_{OL} = OA = \mathbf{P}\cdot\boldsymbol{\lambda} \qquad (3.36)$$

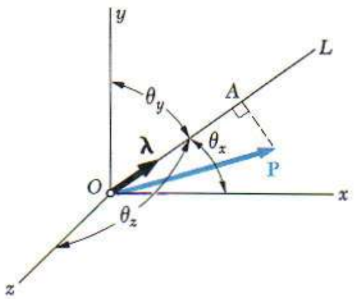


Fig. 3.23

A.7. Mixed Triple Product of Three Vectors (Sec. 3.9).

The mixed triple product of the three vectors **S**, **P**, and **Q** is defined as the scalar expression

$$\mathbf{S}\cdot(\mathbf{P}\times\mathbf{Q}) \qquad (3.38)$$

obtained by forming the scalar product of **S** with the vector product of **P** and **Q**. Mixed triple products are invariant under *circular permutations*, but change sign under any other permutation:

$$\begin{aligned} \mathbf{S}\cdot(\mathbf{P}\times\mathbf{Q}) &= \mathbf{P}\cdot(\mathbf{Q}\times\mathbf{S}) = \mathbf{Q}\cdot(\mathbf{S}\times\mathbf{P}) \\ &= -\mathbf{S}\cdot(\mathbf{Q}\times\mathbf{P}) = -\mathbf{P}\cdot(\mathbf{S}\times\mathbf{Q}) = -\mathbf{Q}\cdot(\mathbf{P}\times\mathbf{S}) \end{aligned} \qquad (3.39)$$

Mixed Triple Product Expressed in Terms of Rectangular Components. The mixed triple product of **S**, **P**, and **Q** may be expressed in the form of a determinant:

$$\mathbf{S}\cdot(\mathbf{P}\times\mathbf{Q}) = \begin{vmatrix} S_x & S_y & S_z \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} \qquad (3.41)$$

The mixed triple product $\mathbf{S}\cdot(\mathbf{P}\times\mathbf{Q})$ measures the volume of the parallelepiped having the vectors **S**, **P**, and **Q** for sides (Fig. 3.25).

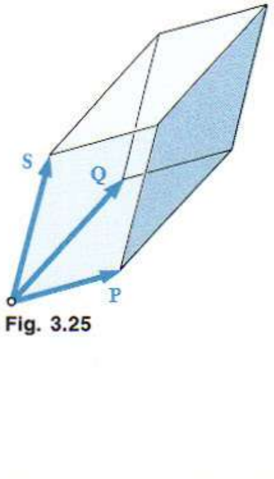


Fig. 3.25

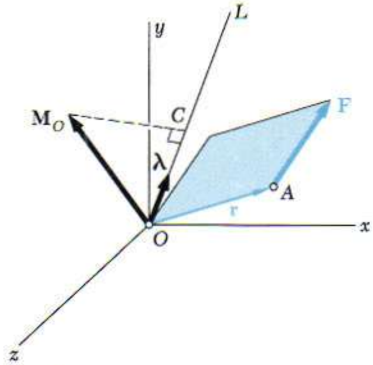


Fig. 3.27

A.8. Moment of a Force about a Given Axis (Sec. 3.10). The moment M_{OL} of a force $\mathbf{F}$ (or, more generally, of a vector $\mathbf{F}$) about an axis OL is defined as the projection OC on the axis OL of the moment $\mathbf{M}_O$ of $\mathbf{F}$ about O (Fig. 3.27). Denoting by $\boldsymbol{\lambda}$ the unit vector along OL, we have

$$M_{OL} = \boldsymbol{\lambda} \cdot \mathbf{M}_O = \boldsymbol{\lambda} \cdot (\mathbf{r} \times \mathbf{F}) \tag{3.42}$$

or, in determinant form,

$$M_{OL} = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} \tag{3.43}$$

where $\lambda_x, \lambda_y, \lambda_z$ = direction cosines of axis OL

x, y, z = coordinates of point of application of $\mathbf{F}$

F_x, F_y, F_z = components of force $\mathbf{F}$

The moments of the force $\mathbf{F}$ about the three coordinate axes are given by the expressions (3.18) obtained earlier for the rectangular components of the moment $\mathbf{M}_O$ of $\mathbf{F}$ about O:

$$\begin{aligned} M_x &= yF_z - zF_y \\ M_y &= zF_x - xF_z \\ M_z &= xF_y - yF_x \end{aligned} \tag{3.18}$$

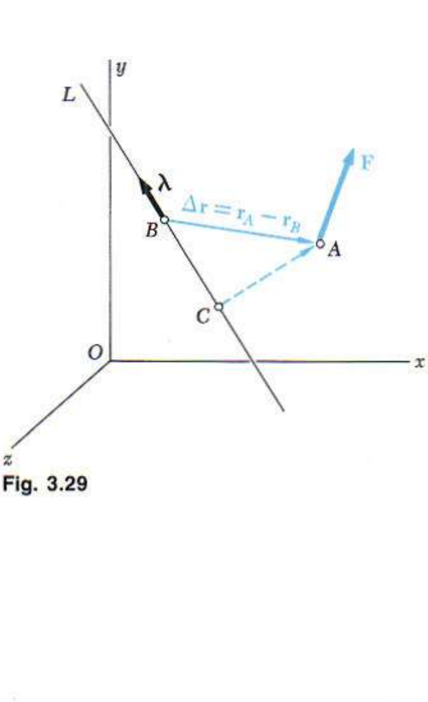


Fig. 3.29

More generally, the moment of a force $\mathbf{F}$ applied at A about an axis which does not pass through the origin is obtained by choosing an arbitrary point B on the axis (Fig. 3.29) and determining the projection on the axis BL of the moment $\mathbf{M}_B$ of $\mathbf{F}$ about B. We write

$$M_{BL} = \boldsymbol{\lambda} \cdot \mathbf{M}_B = \boldsymbol{\lambda} \cdot (\Delta\mathbf{r} \times \mathbf{F}) \tag{3.45}$$

where $\Delta\mathbf{r} = \mathbf{r}_A - \mathbf{r}_B$ represents the vector joining B and A. Expressing M_{BL} in the form of a determinant, we have

$$M_{BL} = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ \Delta x & \Delta y & \Delta z \\ F_x & F_y & F_z \end{vmatrix} \tag{3.46}$$

where $\lambda_x, \lambda_y, \lambda_z$ = direction cosines of axis BL

$\Delta x = x_A - x_B$, $\Delta y = y_A - y_B$, $\Delta_z = z_A - z_B$

F_x, F_y, F_z = components of force $\mathbf{F}$

It should be noted that the result obtained is independent of the choice of the point B on the given axis; the same result would have been obtained if point C had been chosen instead of B.

APPENDIX

B

Moments of Inertia of Masses

MOMENTS OF INERTIA OF MASSES*

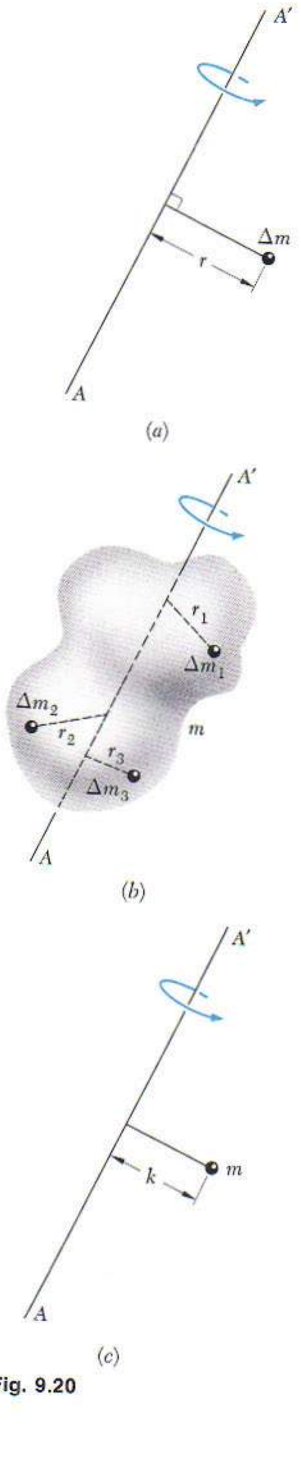


Fig. 9.20

9.10. Moment of Inertia of a Mass. Consider a small mass Δm mounted on a rod of negligible mass which may rotate freely about an axis AA' (Fig. 9.20*a*). If a couple is applied to the system, the rod and mass, assumed initially at rest, will start rotating about AA'. The details of this motion will be studied later in dynamics. At present, we wish only to indicate that the time required for the system to reach a given speed of rotation is proportional to the mass Δm and to the square of the distance r. The product $r^2 \Delta m$ provides, therefore, a measure of the *inertia* of the system, i.e., of the resistance the system offers when we try to set it in motion. For this reason, the product $r^2 \Delta m$ is called the *moment of inertia* of the mass Δm with respect to the axis AA'.

Consider now a body of mass m which is to be rotated about an axis AA' (Fig. 9.20*b*). Dividing the body into elements of mass Δm_1, Δm_2, etc., we find that the resistance offered by the body is measured by the sum $r_1^2 \Delta m_1 + r_2^2 \Delta m_2 + \cdots$. This sum defines, therefore, the moment of inertia of the body with respect to the axis AA'. Increasing the number of elements, we find that the moment of inertia is equal, at the limit, to the integral

$$I = \int r^2 \, dm \tag{9.28}$$

The *radius of gyration* k of the body with respect to the axis AA' is defined by the relation

$$I = k^2 m \qquad \text{or} \qquad k = \sqrt{\frac{I}{m}} \tag{9.29}$$

The radius of gyration k represents, therefore, the distance at which the entire mass of the body should be concentrated if its moment of inertia with respect to AA' is to remain unchanged (Fig. 9.20*c*). Whether it is kept in its original shape (Fig. 9.20*b*) or whether it is concentrated as shown in Fig. 9.20*c*, the mass m will react in the same way to a rotation, or *gyration*, about AA'.

If SI units are used, the radius of gyration k is expressed in meters and the mass m in kilograms. The moment of inertia of a mass, therefore, will be expressed in $\text{kg} \cdot \text{m}^2$. If U.S. customary units are used, the radius of gyration is expressed in feet

*This repeats Secs. 9.10 through 9.16 of the volume on statics.

and the mass in slugs, i.e., in lb · s^2/ft. The moment of inertia of a mass, then, will be expressed in lb · ft · s^2.†

The moment of inertia of a body with respect to a coordinate axis may easily be expressed in terms of the coordinates x, y, z of the element of mass dm (Fig. 9.21). Noting, for example, that the square of the distance r from the element dm to the y axis is $z^2 + x^2$, we express the moment of inertia of the body with respect to the y axis as

$$I_y = \int r^2\, dm = \int (z^2 + x^2)\, dm$$

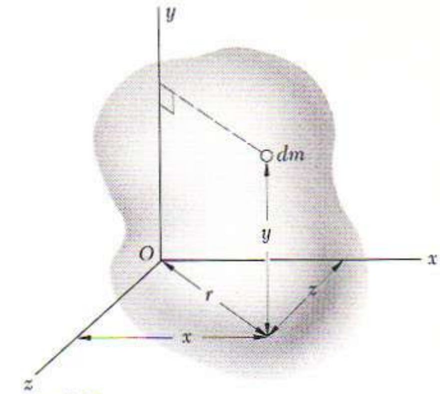


Fig. 9.21

Similar expressions may be obtained for the moments of inertia with respect to the x and z axes. We write

$$\begin{aligned} I_x &= \int (y^2 + z^2)\, dm \\ I_y &= \int (z^2 + x^2)\, dm \\ I_z &= \int (x^2 + y^2)\, dm \end{aligned} \qquad (9.30)$$

9.11. Parallel-Axis Theorem. Consider a body of mass m. Let $Oxyz$ be a system of rectangular coordinates with origin at an arbitrary point O, and $Gx'y'z'$ a system of parallel *centroidal axes*, i.e., a system with origin at the center of gravity G of the body‡ and with axes x', y', z', respectively parallel to x, y, z (Fig. 9.22). Denoting by $\bar{x}$, $\bar{y}$, $\bar{z}$ the coordinates of G with respect to $Oxyz$, we write the following relations between the coordinates x, y, z of the element dm with respect to $Oxyz$ and its coordinates x', y', z' with respect to the centroidal axes $Gx'y'z'$:

$$x = x' + \bar{x} \qquad y = y' + \bar{y} \qquad z = z' + \bar{z} \qquad (9.31)$$

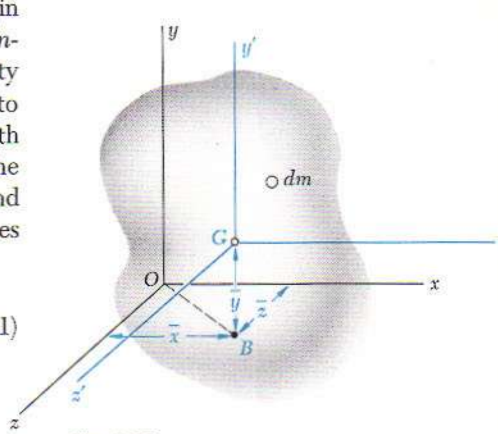


Fig. 9.22

†It should be kept in mind, when converting the moment of inertia of a mass from U.S. customary units to SI units, that the base unit pound used in the derived unit lb · ft · s^2 is a unit of force (*not* of mass) and, therefore, should be converted into newtons. We have

$$1 \text{ lb}\cdot\text{ft}\cdot\text{s}^2 = (4.45 \text{ N})(0.3048 \text{ m})(1 \text{ s})^2 = 1.356 \text{ N}\cdot\text{m}\cdot\text{s}^2$$

or, since N = kg · m/s^2,

$$1 \text{ lb}\cdot\text{ft}\cdot\text{s}^2 = 1.356 \text{ kg}\cdot\text{m}^2$$

‡Note that the term centroidal is used to define an axis passing through the center of gravity G of the body, whether or not G coincides with the centroid of the volume of the body.

Referring to Eqs. (9.30), we may express the moment of inertia of the body with respect to the x axis as follows:

$$\begin{aligned} I_x &= \int(y^2 + z^2)\,dm = \int[(y' + \bar{y})^2 + (z' + \bar{z})^2]\,dm \\ &= \int(y'^2 + z'^2)\,dm + 2\bar{y}\int y'\,dm + 2\bar{z}\int z'\,dm + (\bar{y}^2 + \bar{z}^2)\int dm \end{aligned}$$

The first integral in the expression obtained represents the moment of inertia $\bar{I}_{x'}$ of the body with respect to the centroidal axis x'; the second and third integrals represent the first moment of the body with respect to the $z'x'$ and $x'y'$ planes, respectively, and, since both planes contain G, the two integrals are zero; the last integral is equal to the total mass m of the body. We write, therefore,

$$I_x = \bar{I}_{x'} + m(\bar{y}^2 + \bar{z}^2) \tag{9.32}$$

and, similarly,

$$I_y = \bar{I}_{y'} + m(\bar{z}^2 + \bar{x}^2) \qquad I_z = \bar{I}_{z'} + m(\bar{x}^2 + \bar{y}^2) \tag{9.32$'$}$$

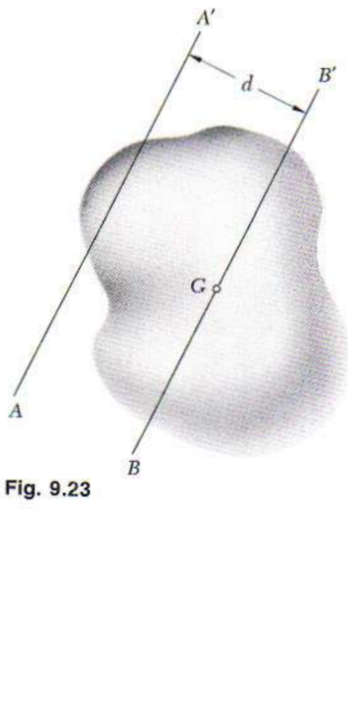


Fig. 9.23

We easily verify from Fig. 9.22 that the sum $\bar{z}^2 + \bar{x}^2$ represents the square of the distance OB between the y and y' axis. Similarly, $\bar{y}^2 + \bar{z}^2$ and $\bar{x}^2 + \bar{y}^2$ represent the squares of the distances between the x and x' axes, and the z and z' axes, respectively. Denoting by d the distance between an arbitrary axis AA' and a parallel centroidal axis BB' (Fig. 9.23), we may, therefore, write the following general relation between the moment of inertia I of the body with respect to AA' and its moment of inertia $\bar{I}$ with respect to BB':

$$I = \bar{I} + md^2 \tag{9.33}$$

Expressing the moments of inertia in terms of the corresponding radii of gyration, we may also write

$$k^2 = \bar{k}^2 + d^2 \tag{9.34}$$

where k and $\bar{k}$ represent the radii of gyration about AA' and BB', respectively.

9.12. Moments of Inertia of Thin Plates. Consider a thin plate of uniform thickness t, made of a homogeneous material of density ρ (density = mass per unit volume). The mass moment of inertia of the plate with respect to an axis AA' *contained in the plane* of the plate (Fig. 9.24a) is

$$I_{AA',\text{mass}} = \int r^2\, dm$$

Since $dm = \rho t\, dA$, we write

$$I_{AA',\text{mass}} = \rho t \int r^2\, dA$$

But r represents the distance of the element of area dA to the axis AA'; the integral is therefore equal to the moment of inertia of the area of the plate with respect to AA'. We have

$$I_{AA',\text{mass}} = \rho t I_{AA',\text{area}} \tag{9.35}$$

Similarly, we have with respect to an axis BB' perpendicular to AA' (Fig. 9.24b)

$$I_{BB',\text{mass}} = \rho t I_{BB',\text{area}} \tag{9.36}$$

Considering now the axis CC' *perpendicular* to the plate through the point of intersection C of AA' and BB' (Fig. 9.24c), we write

$$I_{CC',\text{ mass}} = \rho t J_{C,\text{ area}} \tag{9.37}$$

where J_C is the *polar* moment of inertia of the area of the plate with respect to point C.

Recalling the relation $J_C = I_{AA'} + I_{BB'}$ existing between polar and rectangular moments of inertia of an area, we write the following relation between the mass moments of inertia of a thin plate:

$$I_{CC'} = I_{AA'} + I_{BB'} \tag{9.38}$$

Rectangular Plate. In the case of a rectangular plate of sides a and b (Fig. 9.25), we obtain the following mass moments of inertia with respect to axes through the center of gravity of the plate:

$$I_{AA',\text{mass}} = \rho t I_{AA',\text{area}} = \rho t(\tfrac{1}{12}a^3 b)$$
$$I_{BB',\text{mass}} = \rho t I_{BB',\text{area}} = \rho t(\tfrac{1}{12}ab^3)$$

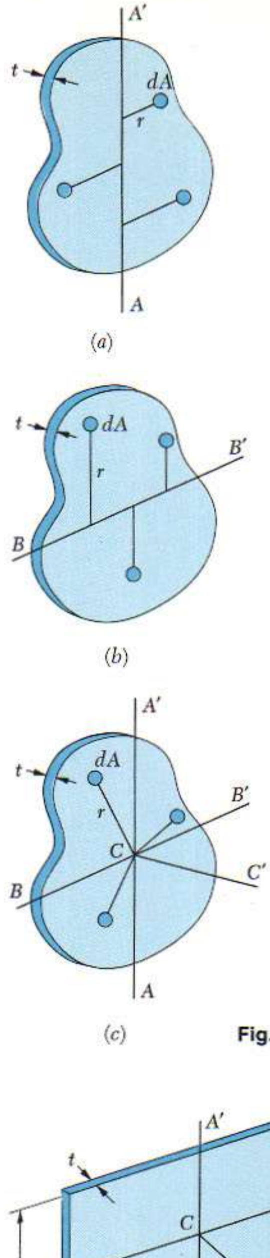


Fig. 9.24

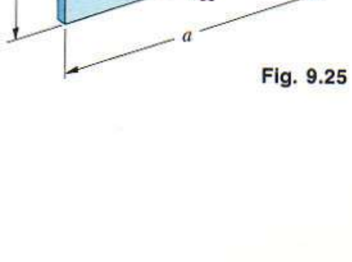


Fig. 9.25

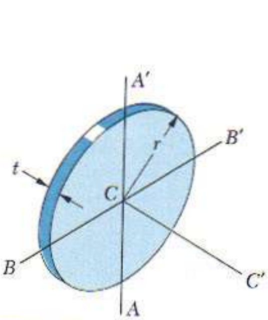


Fig. 9.26

Observing that the product ρabt is equal to the mass m of the plate, we write the mass moments of inertia of a thin rectangular plate as follows:

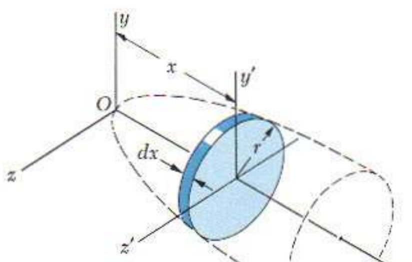
$$I_{AA'} = \tfrac{1}{12}ma^2 \qquad I_{BB'} = \tfrac{1}{12}mb^2 \tag{9.39}$$
$$I_{CC'} = I_{AA'} + I_{BB'} = \tfrac{1}{12}m(a^2 + b^2) \tag{9.40}$$

Circular Plate. In the case of a circular plate, or disk, of radius r (Fig. 9.26), we write

$$I_{AA',\text{mass}} = \rho t I_{AA',\text{area}} = \rho t(\tfrac{1}{4}\pi r^4)$$

Observing that the product $\rho\pi r^2 t$ is equal to the mass m of the plate and that $I_{AA'} = I_{BB'}$, we write the mass moments of inertia of a circular plate as follows:

$$I_{AA'} = I_{BB'} = \tfrac{1}{4}mr^2 \tag{9.41}$$
$$I_{CC'} = I_{AA'} + I_{BB'} = \tfrac{1}{2}mr^2 \tag{9.42}$$



$dm = \rho\pi r^2\,dx$
$dI_x = \frac{1}{2}r^2\,dm$
$dI_y = dI_{y'} + x^2\,dm = (\frac{1}{4}r^2 + x^2)\,dm$
$dI_z = dI_{z'} + x^2\,dm = (\frac{1}{4}r^2 + x^2)\,dm$

Fig. 9.27 Determination of the moment of inertia of a body of revolution.

9.13. Determination of the Moment of Inertia of a Three-dimensional Body by Integration. The moment of inertia of a three-dimensional body is obtained by computing the integral $I = \int r^2\,dm$. If the body is made of a homogeneous material of density ρ, we have $dm = \rho\,dV$ and write $I = \rho\int r^2\,dV$. This integral depends only upon the shape of the body. In order to compute it, it will generally be necessary to perform a triple, or at least a double, integration.

However, if the body possesses two planes of symmetry, it is usually possible to determine its moment of inertia through a single integration by choosing as an element of mass dm the mass of a thin slab perpendicular to the planes of symmetry. In the case of bodies of revolution, for example, the element of mass should be a thin disk (Fig. 9.27). Using formula (9.42), the moment of inertia of the disk with respect to the axis of revolution may be readily expressed as indicated in Fig. 9.27. Its moment of inertia with respect to each of the other two axes of coordinates will be obtained by using formula (9.41) and the parallel-axis theorem. Integration of the expressions obtained will yield the desired moments of inertia of the body of revolution.

9.14. Moments of Inertia of Composite Bodies. The moments of inertia of a few common shapes are shown in Fig. 9.28. The moment of inertia with respect to a given axis of a body made of several of these simple shapes may be obtained by computing the moments of inertia of its component parts about the desired axis and adding them together. We should note, as we already have noted in the case of areas, that the radius of gyration of a composite body *cannot* be obtained by adding the radii of gyration of its component parts.

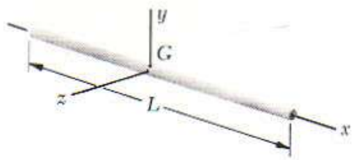
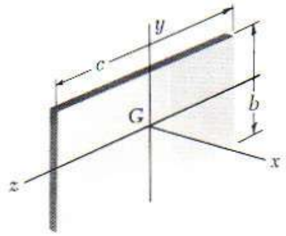
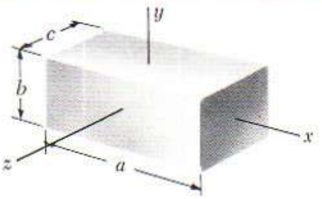
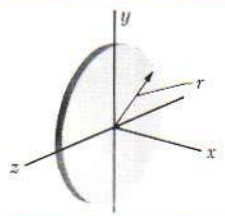
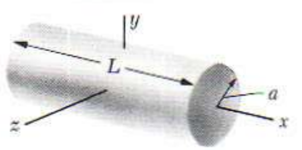
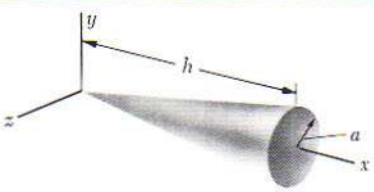
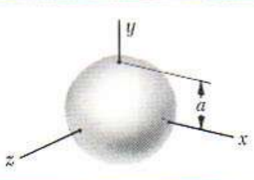
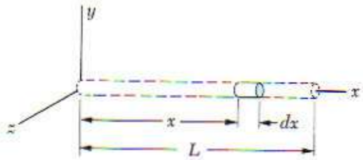
Slender rod		$I_y = I_z = \frac{1}{12}mL^2$
Thin rectangular plate		$I_x = \frac{1}{12}m(b^2 + c^2)$ $I_y = \frac{1}{12}mc^2$ $I_z = \frac{1}{12}mb^2$
Rectangular prism		$I_x = \frac{1}{12}m(b^2 + c^2)$ $I_y = \frac{1}{12}m(c^2 + a^2)$ $I_z = \frac{1}{12}m(a^2 + b^2)$
Thin disk		$I_x = \frac{1}{2}mr^2$ $I_y = I_z = \frac{1}{4}mr^2$
Circular cylinder		$I_x = \frac{1}{2}ma^2$ $I_y = I_z = \frac{1}{12}m(3a^2 + L^2)$
Circular cone		$I_x = \frac{3}{10}ma^2$ $I_y = I_z = \frac{3}{5}m(\frac{1}{4}a^2 + h^2)$
Sphere		$I_x = I_y = I_z = \frac{2}{5}ma^2$

Fig. 9.28 Mass moments of inertia of common geometric shapes

SAMPLE PROBLEM 9.9

Determine the mass moment of inertia of a slender rod of length L and mass m with respect to an axis perpendicular to the rod and passing through one end of the rod.

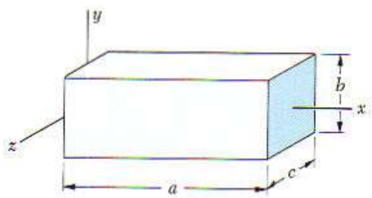


Solution. Choosing the differential element of mass shown, we write

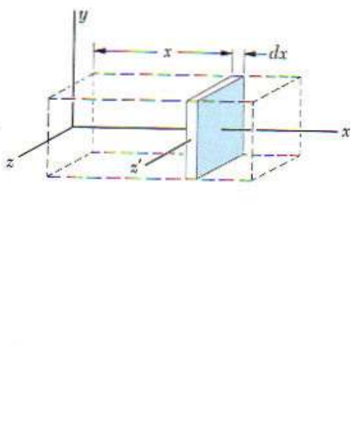
$$dm = \frac{m}{L}\,dx$$

$$I_y = \int x^2\,dm = \int_0^L x^2 \frac{m}{L}\,dx = \left[\frac{m}{L}\frac{x^3}{3}\right]_0^L \qquad I_y = \frac{mL^2}{3} \blacktriangleleft$$

SAMPLE PROBLEM 9.10



Determine the mass moment of inertia of the homogeneous rectangular prism shown with respect to the z axis.



Solution. We choose as a differential element of mass the thin slab shown for which

$$dm = \rho bc\,dx$$

Referring to Sec. 9.12, we find that the moment of inertia of the element with respect to the z' axis is

$$dI_{z'} = \tfrac{1}{12}b^2\,dm$$

Applying the parallel-axis theorem, we obtain the mass moment of inertia of the slab with respect to the z axis.

$$dI_z = dI_{z'} + x^2\,dm = \tfrac{1}{12}b^2\,dm + x^2\,dm = (\tfrac{1}{12}b^2 + x^2)\,\rho bc\,dx$$

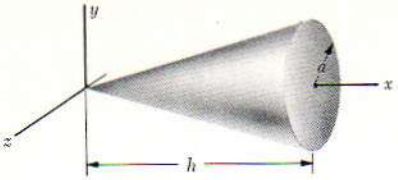
Integrating from $x = 0$ to $x = a$, we obtain

$$I_z = \int dI_z = \int_0^a (\tfrac{1}{12}b^2 + x^2)\,\rho bc\,dx = \rho abc\,(\tfrac{1}{12}b^2 + \tfrac{1}{3}a^2)$$

Since the total mass of the prism is $m = \rho abc$, we may write

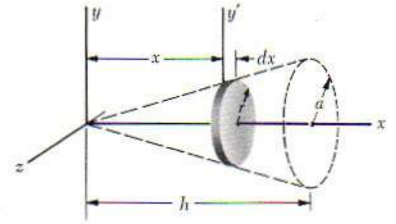
$$I_z = m(\tfrac{1}{12}b^2 + \tfrac{1}{3}a^2) \qquad I_z = \tfrac{1}{12}m(4a^2 + b^2) \blacktriangleleft$$

We note that if the prism is slender, b is small compared to a and the expression for I_z reduces to $ma^2/3$, which is the result obtained in Sample Prob. 9.9 when $L = a$.



SAMPLE PROBLEM 9.11

Determine the mass moment of inertia of a right circular cone with respect to (*a*) its longitudinal axis, (*b*) an axis through the apex of the cone and perpendicular to its longitudinal axis, (*c*) an axis through the centroid of the cone and perpendicular to its longitudinal axis.



Solution. We choose the differential element of mass shown.

$$r = a\frac{x}{h} \qquad dm = \rho\pi r^2\,dx = \rho\pi\frac{a^2}{h^2}x^2\,dx$$

a. Moment of Inertia I_x. Using the expression derived in Sec. 9.12 for a thin disk, we compute the mass moment of inertia of the differential element with respect to the x axis.

$$dI_x = \tfrac{1}{2}r^2\,dm = \tfrac{1}{2}\left(a\frac{x}{h}\right)^2\left(\rho\pi\frac{a^2}{h^2}x^2\,dx\right) = \tfrac{1}{2}\rho\pi\frac{a^4}{h^4}x^4\,dx$$

Integrating from $x = 0$ to $x = h$, we obtain

$$I_x = \int dI_x = \int_0^h \tfrac{1}{2}\rho\pi\frac{a^4}{h^4}x^4\,dx = \tfrac{1}{2}\rho\pi\frac{a^4}{h^4}\frac{h^5}{5} = \tfrac{1}{10}\rho\pi a^4 h$$

Since the total mass of the cone is $m = \frac{1}{3}\rho\pi a^2 h$, we may write

$$I_x = \tfrac{1}{10}\rho\pi a^4 h = \tfrac{3}{10}a^2(\tfrac{1}{3}\rho\pi a^2 h) = \tfrac{3}{10}ma^2 \qquad I_x = \tfrac{3}{10}ma^2 \blacktriangleleft$$

b. Moment of Inertia I_y. The same differential element will be used. Applying the parallel-axis theorem and using the expression derived in Sec. 9.12 for a thin disk, we write

$$dI_y = dI_{y'} + x^2\,dm = \tfrac{1}{4}r^2\,dm + x^2\,dm = (\tfrac{1}{4}r^2 + x^2)\,dm$$

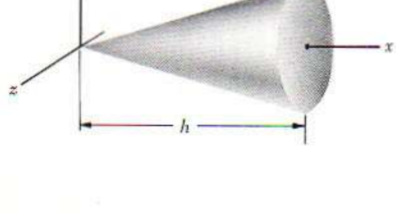
Substituting the expressions for r and dm, we obtain

$$dI_y = \left(\frac{1}{4}\frac{a^2}{h^2}x^2 + x^2\right)\left(\rho\pi\frac{a^2}{h^2}x^2\,dx\right) = \rho\pi\frac{a^2}{h^2}\left(\frac{a^2}{4h^2} + 1\right)x^4\,dx$$

$$I_y = \int dI_y = \int_0^h \rho\pi\frac{a^2}{h^2}\left(\frac{a^2}{4h^2} + 1\right)x^4\,dx = \rho\pi\frac{a^2}{h^2}\left(\frac{a^2}{4h^2} + 1\right)\frac{h^5}{5}$$

Introducing the total mass of the cone m, we rewrite I_y as follows:

$$I_y = \tfrac{3}{5}(\tfrac{1}{4}a^2 + h^2)\tfrac{1}{3}\rho\pi a^2 h \qquad I_y = \tfrac{3}{5}m(\tfrac{1}{4}a^2 + h^2) \blacktriangleleft$$



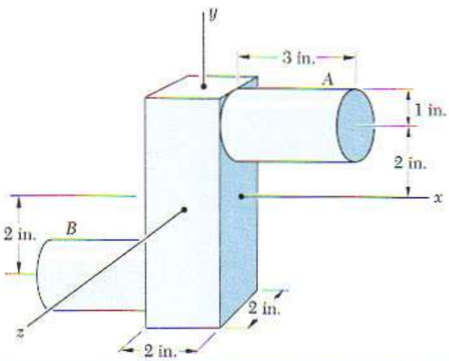
c. Moment of Inertia $\bar{I}_{y''}$. We apply the parallel-axis theorem and write

$$I_y = \bar{I}_{y''} + m\bar{x}^2$$

Solving for $\bar{I}_{y''}$ and recalling that $\bar{x} = \frac{3}{4}h$, we have

$$\bar{I}_{y''} = I_y - m\bar{x}^2 = \tfrac{3}{5}m(\tfrac{1}{4}a^2 + h^2) - m(\tfrac{3}{4}h)^2$$

$$\bar{I}_{y''} = \tfrac{3}{20}m(a^2 + \tfrac{1}{4}h^2) \blacktriangleleft$$



SAMPLE PROBLEM 9.12

A steel forging consists of a rectangular prism 6 by 2 by 2 in. and of two cylinders of diameter 2 in. and length 3 in., as shown. Determine the mass moments of inertia with respect to the coordinate axes. (Specific weight of steel = 490 lb/ft³.)

Computation of Masses

Prism

$$V = 24\ \text{in}^3 \qquad W = \frac{(24\ \text{in}^3)(490\ \text{lb/ft}^3)}{1728\ \text{in}^3/\text{ft}^3} = 6.81\ \text{lb}$$

$$m = \frac{6.81\ \text{lb}}{32.2\ \text{ft/s}^2} = 0.211\ \text{lb}\cdot\text{s}^2/\text{ft}$$

Each Cylinder

$$V = \pi(1\ \text{in.})^2(3\ \text{in.}) = 9.42\ \text{in}^3$$

$$W = \frac{(9.42\ \text{in}^3)(490\ \text{lb/ft}^3)}{1728\ \text{in}^3/\text{ft}^3} = 2.67\ \text{lb}$$

$$m = \frac{2.67\ \text{lb}}{32.2\ \text{ft/s}^2} = 0.0829\ \text{lb}\cdot\text{s}^2/\text{ft}$$

Mass Moments of Inertia. The mass moments of inertia of each component are computed from Fig. 9.28, using the parallel-axis theorem when necessary. Note that all lengths should be expressed in feet.

Prism

$$I_x = I_z = \tfrac{1}{12}(0.211\ \text{lb}\cdot\text{s}^2/\text{ft})[(\tfrac{6}{12}\ \text{ft})^2 + (\tfrac{2}{12}\ \text{ft})^2] = 4.88 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$
$$I_y = \tfrac{1}{12}(0.211\ \text{lb}\cdot\text{s}^2/\text{ft})[(\tfrac{2}{12}\ \text{ft})^2 + (\tfrac{2}{12}\ \text{ft})^2] = 0.977 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

Each cylinder

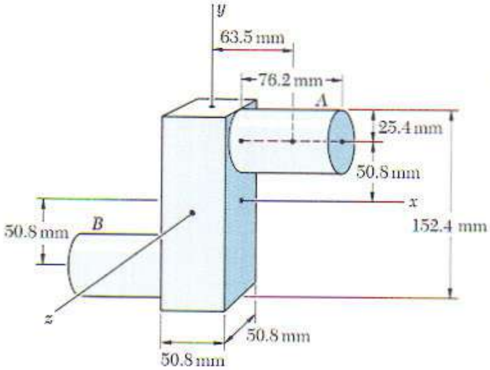
$$I_x = \tfrac{1}{2}ma^2 + m\bar{y}^2 = \tfrac{1}{2}(0.0829\ \text{lb}\cdot\text{s}^2/\text{ft})(\tfrac{1}{12}\ \text{ft})^2 + (0.0829\ \text{lb}\cdot\text{s}^2/\text{ft})(\tfrac{2}{12}\ \text{ft})^2 = 2.59 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$
$$I_y = \tfrac{1}{12}m(3a^2 + L^2) + m\bar{x}^2 = \tfrac{1}{12}(0.0829\ \text{lb}\cdot\text{s}^2/\text{ft})[3(\tfrac{1}{12}\ \text{ft})^2 + (\tfrac{3}{12}\ \text{ft})^2] + (0.0829\ \text{lb}\cdot\text{s}^2/\text{ft})(\tfrac{2.5}{12}\ \text{ft})^2 = 4.17 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$
$$I_z = \tfrac{1}{12}m(3a^2 + L^2) + m(\bar{x}^2 + \bar{y}^2) = \tfrac{1}{12}(0.0829)[3(\tfrac{1}{12})^2 + (\tfrac{3}{12})^2] + (0.0829)[(\tfrac{2.5}{12})^2 + (\tfrac{2}{12})^2] = 6.48 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2$$

Entire Body. Adding the values obtained:

$$I_x = 4.88 \times 10^{-3} + 2(2.59 \times 10^{-3})$$
$$I_x = 10.06 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \quad ◄$$
$$I_y = 0.977 \times 10^{-3} + 2(4.17 \times 10^{-3})$$
$$I_y = 9.32 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \quad ◄$$
$$I_z = 4.88 \times 10^{-3} + 2(6.48 \times 10^{-3})$$
$$I_z = 17.84 \times 10^{-3}\ \text{lb}\cdot\text{ft}\cdot\text{s}^2 \quad ◄$$

SAMPLE PROBLEM 9.13

Solve Sample Prob. 9.12 using SI units.



Solution. First, the dimensions are converted into millimeters (1 in. = 25.4 mm). Next, the density of steel ρ (mass per unit volume) is determined in SI units. Recalling that 1 ft = 0.3048 m and that the mass of a block weighing 1 lb is 0.454 kg, we have

$$\rho = (490 \text{ lb/ft}^3)\left(\frac{0.454 \text{ kg}}{1 \text{ lb}}\right)\left(\frac{1 \text{ ft}}{0.3048 \text{ m}}\right)^3 = 7850 \text{ kg/m}^3$$

Computation of Masses

Prism. $V = (50.8 \text{ mm})^2(152.4 \text{ mm}) = 0.393 \times 10^6 \text{ mm}^3$

or, since $1 \text{ mm}^3 = (10^{-3} \text{ m})^3 = 10^{-9} \text{ m}^3$,

$$V = 0.393 \times 10^6 \times 10^{-9} \text{ m}^3 = 0.393 \times 10^{-3} \text{ m}^3$$
$$m = \rho V = (7.85 \times 10^3 \text{ kg/m}^3)(0.393 \times 10^{-3} \text{ m}^3) = 3.09 \text{ kg}$$

Each Cylinder

$$V = \pi r^2 h = \pi(25.4 \text{ mm})^2(76.2 \text{ mm}) = 0.1544 \times 10^6 \text{ mm}^3 = 0.1544 \times 10^{-3} \text{ m}^3$$
$$m = \rho V = (7.85 \times 10^3 \text{ kg/m}^3)(0.1544 \times 10^{-3} \text{ m}^3) = 1.212 \text{ kg}$$

Mass Moments of Inertia. The mass moments of inertia of each component are computed from Fig. 9.28, using the parallel-axis theorem when necessary. Note that all lengths should be expressed in millimeters.

Prism

$$I_x = I_z = \tfrac{1}{12}(3.09 \text{ kg})[(152.4 \text{ mm})^2 + (50.8 \text{ mm})^2] = 6640 \text{ kg}\cdot\text{mm}^2$$
$$I_y = \tfrac{1}{12}(3.09 \text{ kg})[(50.8 \text{ mm})^2 + (50.8 \text{ mm})^2] = 1329 \text{ kg}\cdot\text{mm}^2$$

Each Cylinder

$$I_x = \tfrac{1}{2}ma^2 + m\bar{y}^2 = \tfrac{1}{2}(1.212 \text{ kg})(25.4 \text{ mm})^2 + (1.212 \text{ kg})(50.8 \text{ mm})^2 = 3520 \text{ kg}\cdot\text{mm}^2$$
$$I_y = \tfrac{1}{12}m(3a^2 + L^2) + m\bar{x}^2 = \tfrac{1}{12}(1.212 \text{ kg})[3(25.4 \text{ mm})^2 + (76.2 \text{ mm})^2] + (1.212 \text{ kg})(63.5 \text{ mm})^2 = 5670 \text{ kg}\cdot\text{mm}^2$$
$$I_z = \tfrac{1}{12}m(3a^2 + L^2) + m(\bar{x}^2 + \bar{y}^2) = \tfrac{1}{12}(1.212 \text{ kg})[3(25.4 \text{ mm})^2 + (76.2 \text{ mm})^2] + (1.212 \text{ kg})[(63.5 \text{ mm})^2 + (50.8 \text{ mm})^2] = 8800 \text{ kg}\cdot\text{mm}^2$$

Entire Body. Adding the values obtained, and observing that $1 \text{ mm}^2 = (10^{-3} \text{ m})^2 = 10^{-6} \text{ m}^2$, we have

$$I_x = 6640 \text{ kg}\cdot\text{mm}^2 + 2(3520 \text{ kg}\cdot\text{mm}^2) = 13.68 \times 10^3 \text{ kg}\cdot\text{mm}^2$$
$$I_x = 13.68 \times 10^{-3} \text{ kg}\cdot\text{m}^2 \quad ◄$$
$$I_y = 1329 \text{ kg}\cdot\text{mm}^2 + 2(5670 \text{ kg}\cdot\text{mm}^2) = 12.67 \times 10^3 \text{ kg}\cdot\text{mm}^2$$
$$I_y = 12.67 \times 10^{-3} \text{ kg}\cdot\text{m}^2 \quad ◄$$
$$I_z = 6640 \text{ kg}\cdot\text{mm}^2 + 2(8800 \text{ kg}\cdot\text{mm}^2) = 24.2 \times 10^3 \text{ kg}\cdot\text{mm}^2$$
$$I_z = 24.2 \times 10^{-3} \text{ kg}\cdot\text{m}^2 \quad ◄$$

Recalling that $1 \text{ lb}\cdot\text{ft}\cdot\text{s}^2 = 1.356 \text{ kg}\cdot\text{m}^2$ (see footnote, page 383) we may check these answers against the values obtained in Sample Prob. 9.12.

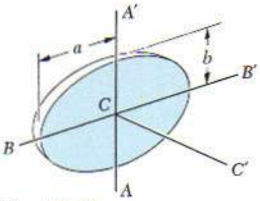


Fig. P9.72

PROBLEMS

9.72 Determine the mass moment of inertia of a thin elliptical plate of mass m with respect to (a) the axes AA' and BB' of the ellipse, (b) the axis CC' perpendicular to the plate.

9.73 Determine the mass moment of inertia of a ring of mass m, cut from a thin uniform plate, with respect to (a) the diameter AA' of the ring, (b) the axis CC' perpendicular to the plane of the ring.

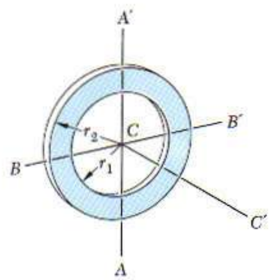


Fig. P9.73

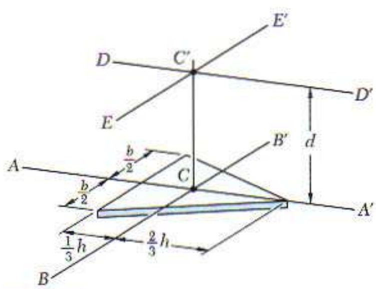


Fig. P9.74

9.74 A thin plate of mass m is cut in the shape of an isosceles triangle of base b and height h. Determine the mass moment of inertia of the plate with respect to (a) the centroidal axes AA' and BB' in the plane of the plate, (b) the centroidal axis CC' perpendicular to the plate.

9.75 Determine the mass moments of inertia of the plate of Prob. 9.74 with respect to the axes DD' and EE' parallel to the centroidal axes AA' and BB' respectively.

9.76 Determine by direct integration the mass moment of inertia with respect to the y axis of the right circular cylinder shown, assuming a uniform density and a mass m.

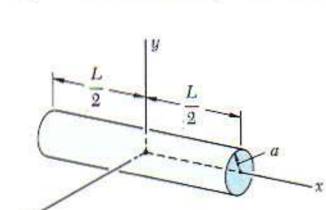


Fig. P9.76

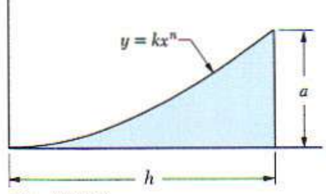


Fig. P9.77

9.77 The area shown is revolved about the x axis to form a homogeneous solid of revolution of mass m. Express the mass moment of inertia of the solid with respect to the x axis in terms of m, a, and n. The expression obtained may be used to verify (a) the value given in Fig. 9.28 for a cone (with $n = 1$), (b) the answer to Prob. 9.78 (with $n = \frac{1}{2}$), (c) the answer to Prob. 9.80 (with $n = 2$).

9.78 Determine by direct integration the mass moment of inertia and the radius of gyration with respect to the x axis of the paraboloid shown, assuming a uniform density and a mass m.

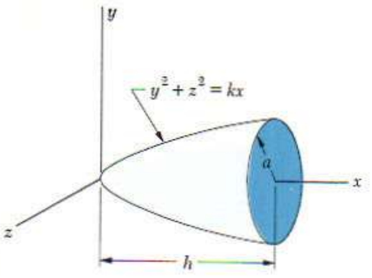


Fig. P9.78 and P9.79

9.79 Determine by direct integration the mass moment of inertia and the radius of gyration with respect to the y axis of the paraboloid shown, assuming a uniform density and a mass m.

9.80 The homogeneous solid shown was obtained by rotating the area of Prob. 9.77, with $n = 2$, through 360° about the x axis. Determine the mass moment of inertia $\bar{I}_x$ in terms of m and a.

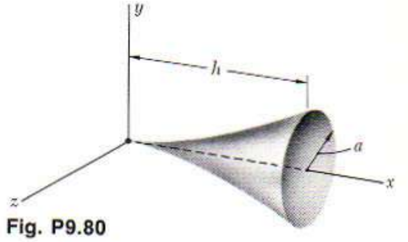


Fig. P9.80

9.81 Determine in terms of m and a the mass moment of inertia and the radius of gyration of the homogeneous solid of Prob. 9.80 with respect to the y axis.

9.82 Determine by direct integration the mass moment of inertia with respect to the x axis of the pyramid shown, assuming a uniform density and a mass m.

9.83 Determine by direct integration the mass moment of inertia with respect to the y axis of the pyramid shown, assuming a uniform density and a mass m.

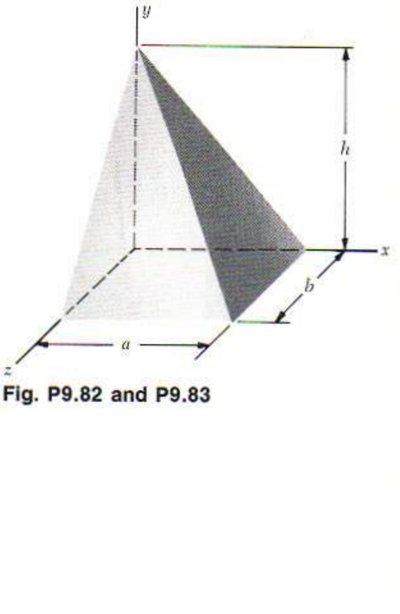


Fig. P9.82 and P9.83

9.84 Knowing that the thin hemispherical shell shown is of mass m and thickness t, determine the mass moment of inertia of the shell with respect to the x axis. (*Hint.* Consider the shell as formed by removing a hemisphere of radius r from a hemisphere of radius $r + t$; then neglect the terms containing t^2 and t^3, and keep those terms containing t.)

Fig. P9.84

Fig. P9.85

9.85 Determine the mass moment of inertia of the frustum of a right circular cone of mass m with respect to its axis of symmetry.

9.86 Determine the mass moment of inertia and the radius of gyration of the steel flywheel shown with respect to the axis of rotation. The web of the flywheel consists of a solid plate 25 mm thick. (Density of steel = 7850 kg/m^3.)

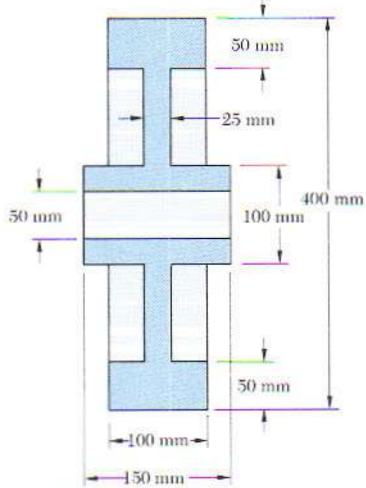


Fig. P9.86

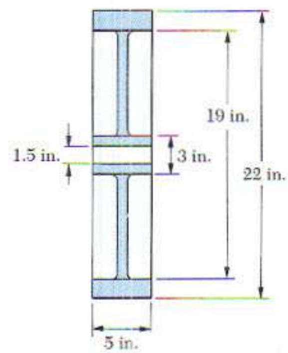


Fig. P9.87

9.87 The cross section of a small flywheel is shown. The rim and hub are connected by eight spokes (two of which are shown in the cross section). Each spoke has a cross-sectional area of 0.400 in^2. Determine the mass moment of inertia and radius of gyration of the flywheel with respect to the axis of rotation. (Specific weight of steel = 490 lb/ft^3.)

9.88 Three slender homogeneous rods are welded together as shown. Denoting the mass of each rod by m, determine the mass moment of inertia and the radius of gyration of the assembly with respect to (a) the x axis, (b) the y axis, (c) the z axis.

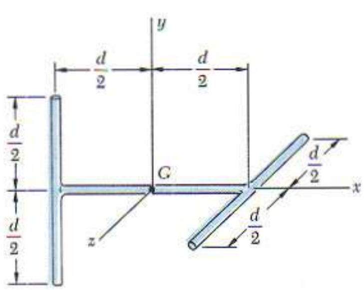


Fig. P9.88

9.89 In using the parallel-axis theorem, the error introduced by neglecting the centroidal moment of inertia is sometimes small. For a homogeneous sphere of radius a and mass m, (a) determine the mass moment of inertia with respect to an axis AA' at a distance R from the center of the sphere, (b) express as a function of a/R the relative error introduced by neglecting the centroidal moment of inertia, (c) determine the distance R in terms of a for which the relative error is 0.4 percent.

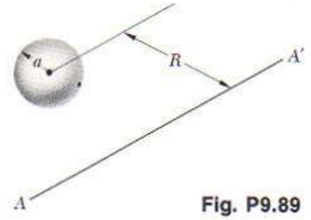


Fig. P9.89

9.90 A section of sheet steel, 2 mm thick, is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3, determine the mass moment of inertia of the component with respect to (a) the x axis, (b) the y axis, (c) the z axis.

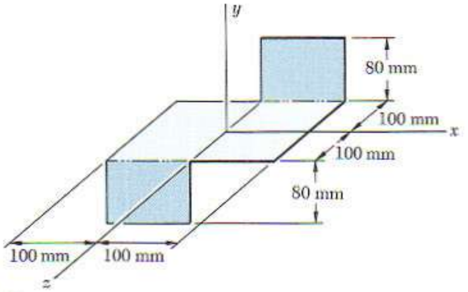


Fig. P9.90

9.91 Twelve uniform slender rods, each of length l, are welded together to form the cubical figure shown. Denoting by m the total mass of the twelve rods, determine the mass moment of inertia of the figure about the x axis.

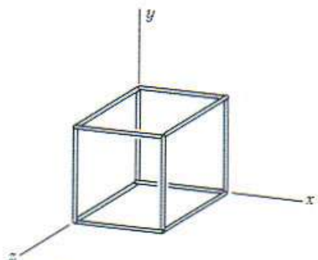


Fig. P9.91

9.92 and 9.93 Determine the mass moment of inertia and the radius of gyration of the steel machine element shown with respect to the x axis. (Specific weight of steel = 490 lb/ft^3; density of steel = 7850 kg/m^3.)

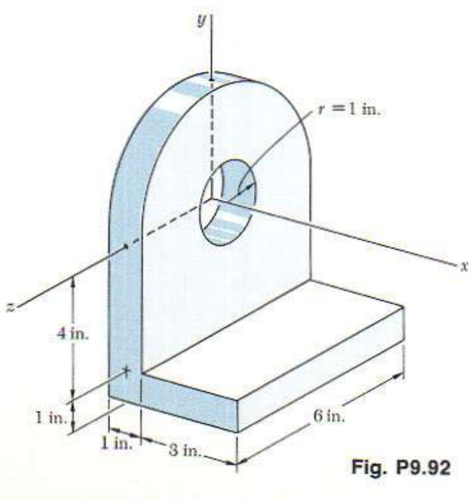


Fig. P9.92

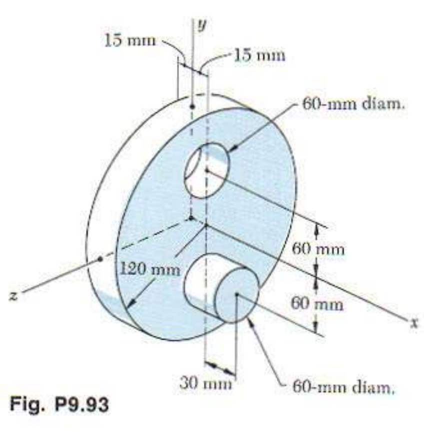


Fig. P9.93

9.94 A homogeneous wire, of weight 2 lb/ft, is used to form the figure shown. Determine the mass moment of inertia of the wire figure with respect to (*a*) the *x* axis, (*b*) the *y* axis, (*c*) the *z* axis.

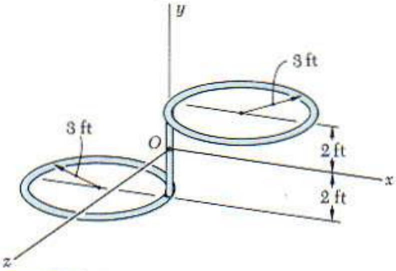


Fig. P9.94

9.95 Two holes, each of diameter 50 mm, are drilled through the steel block shown. Determine the mass moment of inertia of the body with respect to the axis of either of the holes. (Density of steel = 7850 kg/m^3.)

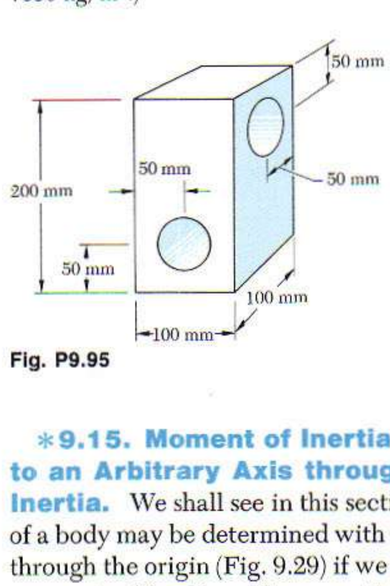


Fig. P9.95

***9.15. Moment of Inertia of a Body with Respect to an Arbitrary Axis through *O*. Mass Products of Inertia.** We shall see in this section how the moment of inertia of a body may be determined with respect to an arbitrary axis *OL* through the origin (Fig. 9.29) if we have computed beforehand its moments of inertia with respect to the three coordinate axes, as well as certain other quantities to be defined below.

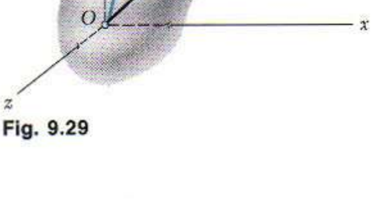


Fig. 9.29

The moment of inertia of the body with respect to *OL* is represented by the integral $I_{OL} = \int p^2\, dm$, where p denotes the perpendicular distance from the element of mass dm to the axis *OL*. But, denoting by $\boldsymbol{\lambda}$ the unit vector along *OL* and by $\mathbf{r}$ the position vector of the element dm, we observe that the perpendicular distance p is equal to the magnitude $r \sin\theta$ of the vector product $\boldsymbol{\lambda} \times \mathbf{r}$. We write therefore

$$I_{OL} = \int p^2\, dm = \int (\boldsymbol{\lambda} \times \mathbf{r})^2\, dm \qquad (9.43)$$

Expressing the square of the vector product in terms of its rectangular components, we have

$$I_{OL} = \int[(\lambda_x y - \lambda_y x)^2 + (\lambda_y z - \lambda_z y)^2 + (\lambda_z x - \lambda_x z)^2]\, dm$$

where the components λ_x, λ_y, λ_z of the unit vector $\boldsymbol{\lambda}$ represent the direction cosines of the axis OL, and the components x, y, z of $\mathbf{r}$ represent the coordinates of the element of mass dm. Expanding the squares in the expression obtained and rearranging the terms, we write

$$\begin{aligned} I_{OL} = \lambda_x^2 \int (y^2 + z^2)\, dm + \lambda_y^2 \int (z^2 + x^2)\, dm + \lambda_z^2 \int (x^2 + y^2)\, dm \\ - 2\lambda_x\lambda_y \int xy\, dm - 2\lambda_y\lambda_z \int yz\, dm - 2\lambda_z\lambda_x \int zx\, dm \quad (9.44) \end{aligned}$$

Referring to Eqs. (9.30), we note that the first three integrals in (9.44) represent, respectively, the moments of inertia I_x, I_y, and I_z of the body with respect to the coordinate axes. The last three integrals in (9.44), which involve products of coordinates, are called the *products of inertia* of the body with respect to the x and y axes, the y and z axes, and the z and x axes, respectively. We write

$$P_{xy} = \int xy\, dm \qquad P_{yz} = \int yz\, dm \qquad P_{zx} = \int zx\, dm \qquad (9.45)$$

Substituting for the various integrals from (9.30) and (9.45) into (9.44), we have

$$I_{OL} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2P_{xy}\lambda_x\lambda_y - 2P_{yz}\lambda_y\lambda_z - 2P_{zx}\lambda_z\lambda_x \qquad (9.46)$$

We note that the definition of the products of inertia of a mass given in Eqs. (9.45) is an extension of the definition of the product of inertia of an area (Sec. 9.7). Mass products of inertia reduce to zero under the same conditions of symmetry as products of inertia of areas, and the parallel-axis theorem for mass products of inertia is expressed by relations similar to the formula derived for the product of inertia of an area. Substituting for x, y, z from Eqs. (9.31) into Eqs. (9.45), we verify that

$$\begin{aligned} P_{xy} &= \overline{P}_{x'y'} + m\bar{x}\bar{y} \\ P_{yz} &= \overline{P}_{y'z'} + m\bar{y}\bar{z} \\ P_{zx} &= \overline{P}_{z'x'} + m\bar{z}\bar{x} \end{aligned} \qquad (9.47)$$

where $\bar{x}$, $\bar{y}$, $\bar{z}$ are the coordinates of the center of gravity G of the body, and $\overline{P}_{x'y'}$, $\overline{P}_{y'z'}$, $\overline{P}_{z'x'}$ denote the products of inertia with respect to the centroidal axes x', y', z' (Fig. 9.22).

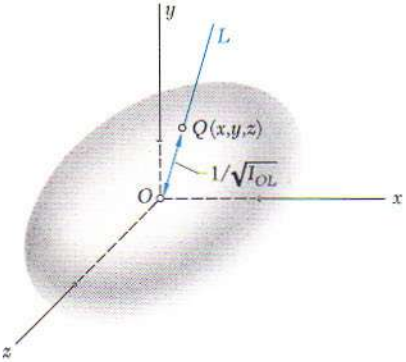


Fig. 9.30

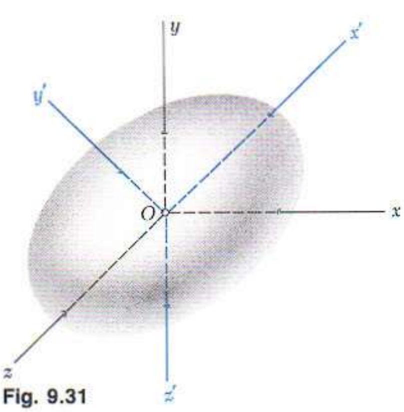


Fig. 9.31

***9.16. Ellipsoid of Inertia. Principal Axes of Inertia.** Let us assume that the moment of inertia of the body considered in the preceding section has been determined with respect to a large number of axes OL through the fixed point O, and that a point Q has been plotted on each axis OL at a distance $OQ = 1/\sqrt{I_{OL}}$ from O. The locus of the points Q thus obtained forms a surface (Fig. 9.30). The equation of that surface may be obtained by substituting $1/(OQ)^2$ for I_{OL} in (9.46) and multiplying both sides of the equation by $(OQ)^2$. Observing that

$$(OQ)\lambda_x = x \qquad (OQ)\lambda_y = y \qquad (OQ)\lambda_z = z$$

where x, y, z denote the rectangular coordinates of a point Q of the surface, we write

$$I_x x^2 + I_y y^2 + I_z z^2 - 2P_{xy}xy - 2P_{yz}yz - 2P_{zx}zx = 1 \quad (9.48)$$

The equation obtained is that of a *quadric*. Since the moment of inertia I_{OL} is different from zero for every axis OL, no point Q may be at an infinite distance from O. Thus, the quadric obtained is an *ellipsoid*. This ellipsoid, which defines the moment of inertia of the body with respect to any axis through O, is known as the *ellipsoid of inertia* of the body at O.

We observe that, if the axes in Fig. 9.30 are rotated, the coefficients of the equation defining the ellipsoid change, since these are equal to the moments and products of inertia of the body with respect to the rotated coordinate axes. However, the *ellipsoid itself remains unaffected,* since its shape depends only upon the distribution of mass in the body considered. Suppose now that we choose as coordinate axes the principal axes x', y', z' of the ellipsoid of inertia (Fig. 9.31). The equation of the ellipsoid with respect to these coordinate axes will be of the form

$$I_{x'}x'^2 + I_{y'}y'^2 + I_{z'}z'^2 = 1 \quad (9.49)$$

which does not contain any product of coordinates. Thus, the products of inertia of the body with respect to the x', y', z' axes are zero. The x', y', z' axes are known as the *principal axes of inertia* of the body at O, and the coefficients $I_{x'}$, $I_{y'}$, $I_{z'}$ as the *principal moments of inertia* of the body at O. Note that, given a body of arbitrary shape and a point O, it is always possible to find axes which are the principal axes of inertia of the body at O, i.e., axes with respect to which the products of inertia of the body are zero. Indeed, no matter how odd or irregular the shape of the body may be, the moments of inertia of the body with respect to axes through O will define an ellipsoid, and this ellipsoid will have principal axes which, by definition, are the principal axes of the body at O.

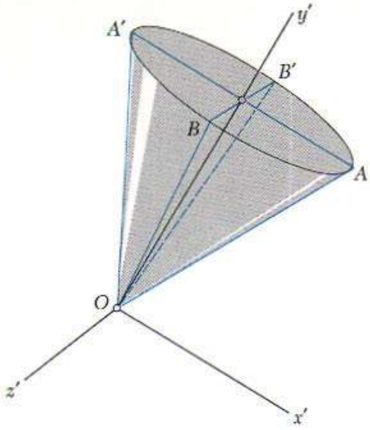


Fig. 9.32

If the principal axes of inertia x', y', z' are used as coordinate axes, the expression obtained in Eq. (9.46) for the moment of inertia of a body with respect to an arbitrary axis through O reduces to

$$I_{OL} = I_{x'}\lambda_{x'}^2 + I_{y'}\lambda_{y'}^2 + I_{z'}\lambda_{z'}^2 \tag{9.50}$$

While the determination of the principal axes of inertia of a body of arbitrary shape is somewhat involved and requires solving a cubic equation,† there are many cases when these axes may be spotted immediately. Consider, for instance, the homogeneous cone of elliptical base shown in Fig. 9.32; this cone possesses two mutually perpendicular planes of symmetry OAA' and OBB'. We check from the definition (9.45) that, if the $x'y'$ and $y'z'$ planes are chosen to coincide with the two planes of symmetry, all the products of inertia are zero. The x', y', and z' axes thus selected are therefore the principal axes of inertia of the cone at O. In the case of the homogeneous regular tetrahedron $OABC$ shown in Fig. 9.33, the line joining the corner O to the center D of the opposite face is a principal axis of inertia at O and any line through O perpendicular to OD is also a principal axis of inertia at O. This property may be recognized if we observe that a rotation through 120° about OD leaves the shape and the mass distribution of the tetrahedron unchanged. It follows that the ellipsoid of inertia at O also remains unchanged under this rotation. The ellipsoid, therefore, is of revolution about OD, and the line OD, as well as any perpendicular line through O, must be a principal axis of the ellipsoid.

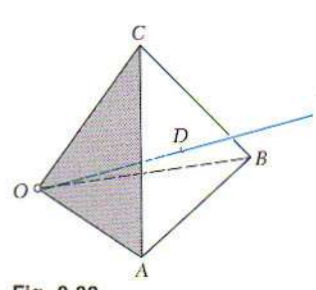
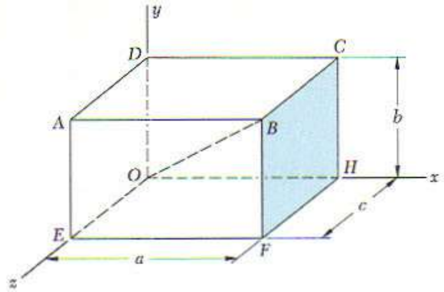


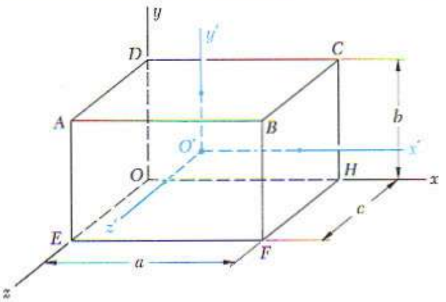
Fig. 9.33

† Cf. Synge and Griffith, *Principles of Mechanics*, McGraw-Hill Book Company, sec. 11.3.



SAMPLE PROBLEM 9.14

Consider a rectangular prism of mass m and sides a, b, c. Determine (a) the mass moments and products of inertia of the prism with respect to the coordinate axes shown, (b) its moment of inertia with respect to the diagonal OB.



a. **Moments and Products of Inertia with Respect to the Coordinate Axes.** *Moments of Inertia.* Introducing the centroidal axes x', y', z', with respect to which the moments of inertia are given in Fig. 9.28, we apply the parallel-axis theorem:

$$I_x = \bar{I}_{x'} + m(\bar{y}^2 + \bar{z}^2) = \tfrac{1}{12}m(b^2 + c^2) + m(\tfrac{1}{4}b^2 + \tfrac{1}{4}c^2)$$

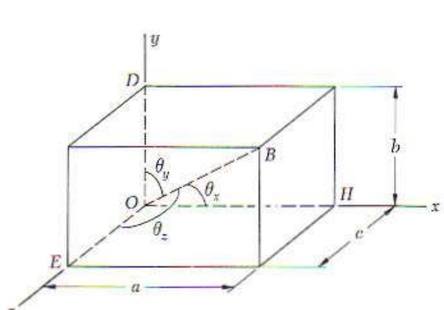
$$I_x = \tfrac{1}{3}m(b^2 + c^2) \quad \blacktriangleleft$$

Similarly: $$I_y = \tfrac{1}{3}m(c^2 + a^2) \qquad I_z = \tfrac{1}{3}m(a^2 + b^2) \quad \blacktriangleleft$$

Products of Inertia. Because of symmetry, the products of inertia with respect to the centroidal axes x', y', z' are zero and these axes are principal axes of inertia. Using the parallel-axis theorem, we have

$$P_{xy} = \bar{P}_{x'y'} + m\bar{x}\,\bar{y} = 0 + m(\tfrac{1}{2}a)(\tfrac{1}{2}b) \qquad P_{xy} = \tfrac{1}{4}mab \quad \blacktriangleleft$$

Similarly: $$P_{yz} = \tfrac{1}{4}mbc \qquad P_{zx} = \tfrac{1}{4}mca \quad \blacktriangleleft$$



b. **Moment of Inertia with Respect to OB.** We recall Eq. (9.46):

$$I_{OB} = I_x\lambda_x^2 + I_y\lambda_y^2 + I_z\lambda_z^2 - 2P_{xy}\lambda_x\lambda_y - 2P_{yz}\lambda_y\lambda_z - 2P_{zx}\lambda_z\lambda_x$$

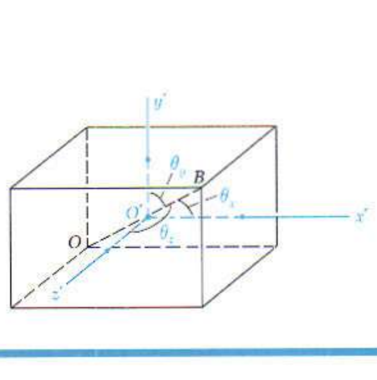
where the direction cosines of OB are

$$\lambda_x = \cos\theta_x = (OH)/(OB) = a/(a^2 + b^2 + c^2)^{1/2}$$
$$\lambda_y = b/(a^2 + b^2 + c^2)^{1/2} \qquad \lambda_z = c/(a^2 + b^2 + c^2)^{1/2}$$

Substituting the values obtained for the moments and products of inertia and for the direction cosines:

$$I_{OB} = \frac{1}{a^2 + b^2 + c^2}[\tfrac{1}{3}m(b^2 + c^2)a^2 + \tfrac{1}{3}m(c^2 + a^2)b^2 + \tfrac{1}{3}m(a^2 + b^2)c^2$$
$$- \tfrac{1}{2}ma^2b^2 - \tfrac{1}{2}mb^2c^2 - \tfrac{1}{2}mc^2a^2]$$

$$I_{OB} = \frac{m}{6}\frac{a^2b^2 + b^2c^2 + c^2a^2}{a^2 + b^2 + c^2} \quad \blacktriangleleft$$



Alternate Solution. The moment of inertia I_{OB} may be obtained directly from the principal moments of inertia $\bar{I}_{x'}$, $\bar{I}_{y'}$, $\bar{I}_{z'}$, since the line OB passes through the centroid O'. Since the x', y', z' axes are principal axes of inertia, we use Eq. (9.50) and write

$$I_{OB} = \bar{I}_{x'}\lambda_x^2 + \bar{I}_{y'}\lambda_y^2 + \bar{I}_{z'}\lambda_z^2$$
$$= \frac{1}{a^2 + b^2 + c^2}\left[\frac{m}{12}(b^2 + c^2)a^2 + \frac{m}{12}(c^2 + a^2)b^2 + \frac{m}{12}(a^2 + b^2)c^2\right]$$

$$I_{OB} = \frac{m}{6}\frac{a^2b^2 + b^2c^2 + c^2a^2}{a^2 + b^2 + c^2} \quad \blacktriangleleft$$

PROBLEMS

9.96 and 9.97 Determine the mass products of inertia P_{xy}, P_{yz}, and P_{zx} of the steel machine element shown. (Specific weight of steel = 490 lb/ft^3; density of steel = 7850 kg/m^3.)

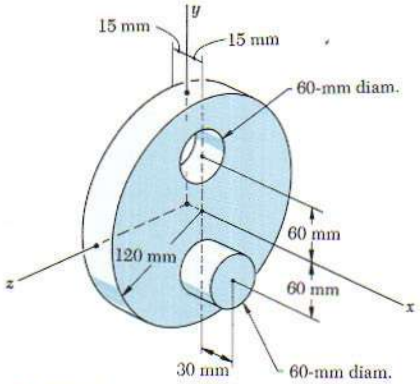


Fig. P9.96

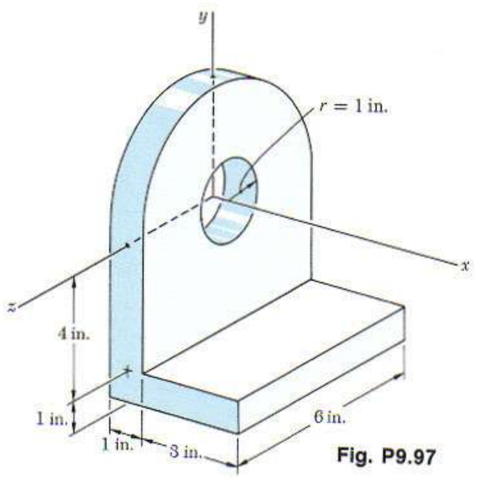


Fig. P9.97

9.98 A homogeneous wire, of weight 2 lb/ft, is used to form the figure shown. Determine the mass products of inertia P_{xy}, P_{yz}, and P_{zx} of the wire figure.

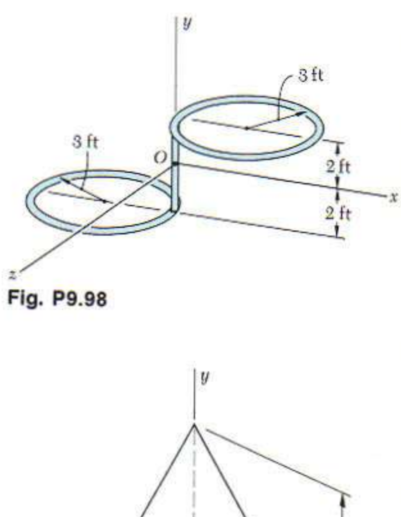


Fig. P9.98

9.99 A section of sheet steel, 2 mm thick, is cut and bent into the machine component shown. Knowing that the density of steel is 7850 kg/m^3, determine the mass products of inertia P_{xy}, P_{yz}, and P_{zx} of the component.

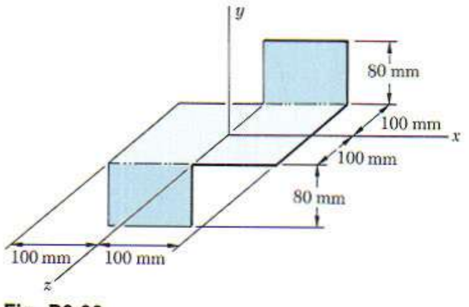


Fig. P9.99

9.100 For the homogeneous tetrahedron of mass m which is shown, (*a*) determine by direct integration the mass product of inertia P_{zx}, (*b*) deduce P_{yz} and P_{xy} from the result obtained in part *a*.

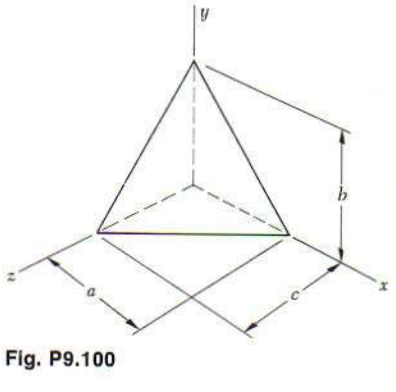


Fig. P9.100

9.101 Complete the derivation of Eqs. (9.47), which express the parallel-axis theorem for mass products of inertia.

9.102 Determine the mass moment of inertia of the right circular cone of Sample Prob. 9.11 with respect to a generator of the cone.

9.103 Determine the mass moment of inertia of the rectangular prism of Sample Prob. 9.14 with respect to the diagonal OF of its base.

9.104 Determine the mass moment of inertia of the bent wire of Probs. 9.94 and 9.98 with respect to the axis through O which forms equal angles with the x, y, and z axes.

9.105 Determine the mass moment of inertia of the forging of Sample Prob. 9.12 with respect to an axis through O characterized by the unit vector $\boldsymbol{\lambda} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$.

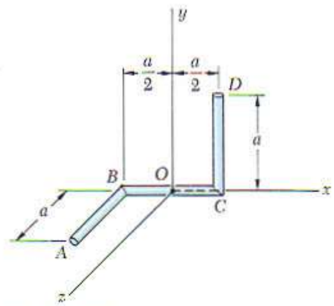


Fig. P9.106

9.106 Three uniform rods, each of mass m, are welded together as shown. Determine (a) the mass moments of inertia and the mass products of inertia with respect to the coordinate axes, (b) the mass moment of inertia with respect to a line joining the origin O and point D.

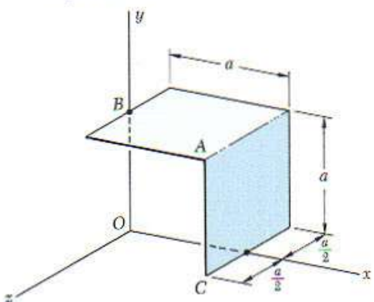


Fig. P9.107 and P9.108

9.107 The thin bent plate shown is of uniform density and mass m. Determine its mass moment of inertia with respect to a line joining the origin O and point A.

9.108 The thin bent plate shown is of uniform density and mass m. Determine its mass moment of inertia with respect to a line joining points B and C.

9.109 Consider a homogeneous circular cylinder of radius a and length L. Determine the value of the ratio a/L for which the ellipsoid of inertia of the cylinder is a sphere when computed (a) at the centroid of the cylinder, (b) at the center of one of its bases.

9.110 Determine the value of the ratio a/h for which the ellipsoid of inertia of the right circular cone of Sample Prob. 9.11 is a sphere when computed (a) at the apex of the cone, (b) at the centroid of the cone.

9.111 Given an arbitrary solid and three rectangular axes x, y, and z, prove that the mass moment of inertia of the solid with respect to any one of the three axes cannot be larger than the sum of the moments of inertia of the solid with respect to the other two axes; i.e., prove that the inequality $I_x \leq I_y + I_z$ is satisfied, as well as two similar inequalities. Further prove that, if the solid is homogeneous and of revolution, and if x is the axis of revolution and y a transverse axis, then $I_y \geq \frac{1}{2}I_x$.

9.112 Given a homogeneous solid of mass m and of arbitrary shape, and three rectangular axes x, y, and z of origin O, prove that the sum $I_x + I_y + I_z$ of the mass moments of inertia of the solid cannot be smaller than the similar sum computed for a sphere of the same mass and same material centered at O. Further prove, using the result of Prob. 9.111, that, if the solid is of revolution and if x is the axis of revolution, then its moment of inertia I_y about a transverse axis y must satisfy the inequality

$$I_y \geq \frac{3}{10}ma^2$$

where a is the radius of the sphere of the same mass and same material.

9.113 Consider a cube of mass m and side a. (*a*) Show that the ellipsoid of inertia at the center of the cube is a sphere, and use this property to determine the mass moment of inertia of the cube with respect to one of its diagonals. (*b*) Show that the ellipsoid of inertia at one of the corners of the cube is an ellipsoid of revolution, and determine the principal moments of inertia of the cube at that point.

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Answers to Even-numbered Problems

CHAPTER 11

11.2 $t = 0$, $x = 12$ in., $a = -18$ in./s^2; $t = 3$ s, $x = -15$ in., $a = 18$ in./s^2. SI: $t = 0$, $x = 0.305$ m, $a = -0.457$ m/s^2; $t = 3$ s, $x = -0.381$ m, $a = 0.457$ m/s^2.

11.4 (*a*) 2 s, 4 s. (*b*) 8 m, 7.33 m.

11.6 -4 m/s; 12 m; 20 m.

11.8 (*a*) 3 s. (*b*) 116 in., -56 in./s. (*c*) 65 in. SI: (*a*) 3 s. (*b*) 2.95 m, 1.422 m/s. (*c*) 1.651 m.

11.10 25 s^{-2}.

11.12 (*a*) 384 in^3/s^2. (*b*) 13.86 in./s. SI: (*a*) 6.29×10^{-3} m^3/s^2. (*b*) 0.352 m/s.

11.14 (*a*) 55.5 m. (*b*) Infinite.

11.16 142.7 ft/s. SI: 43.5 m/s.

11.18 (*a*) $v = \dfrac{kT}{\pi}\left(1 - \cos\dfrac{\pi t}{T}\right)$; $x = \dfrac{kT^2}{\pi^2}\left(\dfrac{\pi t}{T} - \sin\dfrac{\pi t}{T}\right)$. (*b*) $2kT/\pi$. (*c*) $2kT^2/\pi$. (*d*) kT/π.

11.20 (*a*) 15,540 ft. (*b*) 318 mi. (*c*) Infinite. SI: (*a*) 4740 m. (*b*) 511 km. (*c*) Infinite.

11.22 (*a*) 6.90 m/s. (*b*) Infinite.

11.24 (*a*) 5 m/s. (*b*) 11 m/s. (*c*) 60 m.

11.26 (*a*) 30.7 ft/s. (*b*) 98.2 ft/s. SI: (*a*) 9.34 m/s. (*b*) 29.9 m/s.

11.28 $t = 15$ s; $x = 450$ ft. SI: $t = 15$ s; $x = 137.2$ m.

11.30 (*a*) 17.10 s; 171.0 m. (*b*) 81.5 km/h.

11.32 (*a*) 36 ft/s $\downarrow$. (*b*) 18 ft/s $\downarrow$. (*c*) 54 ft/s $\downarrow$. (*d*) 36 ft/s $\downarrow$. SI: (*a*) 10.97 m/s $\downarrow$. (*b*) 5.49 m/s $\downarrow$. (*c*) 16.46 m/s $\downarrow$. (*d*) 10.97 m/s $\downarrow$.

11.34 (*a*) 200 mm/s $\leftarrow$. (*b*) 200 mm/s $\leftarrow$; 400 mm/s $\leftarrow$. (*c*) 100 mm/s $\leftarrow$. (*d*) 200 mm/s $\rightarrow$.

11.36 (*a*) 3 s. (*b*) 3.38 in. $\uparrow$. SI: (*a*) 3 s. (*b*) 85.7 mm $\uparrow$.

11.38 $v_A = 120$ mm/s $\downarrow$; $v_B = 40$ mm/s $\downarrow$; $v_C = 80$ mm/s $\uparrow$.

11.40 $v_A = 200$ mm/s $\downarrow$; $v_B = 40$ mm/s $\downarrow$; $v_C = 120$ mm/s $\uparrow$.

11.42 (*a*) 32 ft/s. (*b*) 192 ft. SI: (*a*) 9.75 m/s. (*b*) 58.5 m.

11.44 (*a*) 48 m. (*b*) 6 s, 13.75 s, 16.25 s.

11.46 19 s.

11.48 (*a*) 2.67 ft/s^2. (*b*) 23.2 mi/h. SI: (*a*) 0.813 m/s^2. (*b*) 37.3 km/h.

11.50 11 s; 70 m.

11.52 8.54 s; 58.3 mi/h. SI: 8.54 s; 93.8 km/h.

11.54 (*a*) 8.57 s. (*b*) 1.867 m/s^2; 1.400 m/s^2. (*c*) 68.6 m; 51.4 m.
11.56 (*a*) 150 in./s. (*b*) 800 in. (*c*) 100 in./s. SI: (*a*) 3.81 m/s. (*b*) 20.3 m. (*c*) 2.54 m/s.
11.58 (*a*) 10.0 m/s; 27.4 m. (*b*) 13.9 m/s; 51.5 m.
11.60 (*a*) -756 in./s^2. (*b*) -880 in./s^2. SI: (*a*) -19.20 m/s^2. (*b*) -22.4 m/s^2.
11.64 (*a*) 2.7 s. (*b*) 48.6 ft. SI: (*a*) 2.7 s. (*b*) 14.81 m.
11.66 (*a*) 12 m. (*b*) 48 m.
11.68 (*a*) 2 s. (*b*) 2 m/s $\leftarrow$; 2.24 m/s^2 $\measuredangle$ 26.6°.
11.70 $\mathbf{v} = 2.22$ ft/s $\measuredangle$ 34.2°; $\mathbf{a} = 2.22$ ft/s^2 $\measuredangle$ 34.2°. SI: $\mathbf{v} = 0.678$ m/s $\measuredangle$ 34.2°; $\mathbf{a} = 0.678$ m/s^2 $\measuredangle$ 34.2°.
11.74 $v = \sqrt{c^2 + R^2p^2}$; $a = Rp^2$.
11.76 4.20 m/s $\leqslant v_0 \leqslant$ 6.64 m/s.
11.78 44.0 ft/s; 38.1 ft/s. SI: 13.40 m/s; 11.61 m/s.
11.80 12.43 ft. SI: 3.79 m.
11.82 26.6° or 63.4°.
11.84 15° or 75°.
11.86 14.83 ft. SI: 4.52 m.
11.88 23.2 mi $\measuredangle$ 17.8°. SI: 37.3 km $\measuredangle$ 17.8°.
11.90 (*a*) 56.3° from rear of truck. (*b*) 16.63 m/s.
11.92 9.98 m/s $\measuredangle$ 81.2°.
11.94 22.4 mi/h from 63.4° east of north. SI: 36.0 km/h from 63.4° east of north.
11.96 10.18 m/s $\measuredangle$ 10.8°; 9.81 m/s^2 $\downarrow$.
11.98 (*a*) 2.08 m/s^2. (*b*) 63.6 km/h.
11.100 29.6×10^3 ft/s^2. SI: 9.02×10^3 m/s^2.
11.102 8.51 ft/s^2. SI: 2.59 m/s^2
11.104 1.2 m/s^2.
11.106 3810 m.
11.108 22,800 ft; 58,100 ft. SI: 6.96 km; 17.71 km.
11.110 $\rho = R + c^2/Rp^2$.
11.112 17,060 mi/h. SI: 27 400 km/h.
11.114 84.4 min.
11.116 (*a*) $\mathbf{v} = -(180\text{ mm/s})\mathbf{i}_r$; $\mathbf{a} = -(240\text{ mm/s}^2)\mathbf{i}_r - (4320\text{ mm/s}^2)\mathbf{i}_\theta$.
11.118 (*a*) $\mathbf{v} = -4\pi b\mathbf{i}_r + 4\pi b\mathbf{i}_\theta$; $\mathbf{a} = -8\pi^2 b\mathbf{i}_r - 16\pi^2 b\mathbf{i}_\theta$. (*b*) $\mathbf{v} = 0$; $\mathbf{a} = 8\pi^2 b\mathbf{i}_r$.
11.120 $v = b\sec^2\theta\,\dot{\theta}$.
11.122 (*a*) $\mathbf{v} = bk\,\mathbf{i}_\theta$; $\mathbf{a} = -\frac{1}{2}bk^2\mathbf{i}_r$. (*b*) $\mathbf{v} = 2bk\,\mathbf{i}_r + 2bk\,\mathbf{i}_\theta$; $\mathbf{a} = 2bk^2\,\mathbf{i}_r + 4bk^2\,\mathbf{i}_\theta$.
11.124 $v = 2\pi\sqrt{A^2 + B^2n^2\cos^2 2\pi nt}$; $a = 4\pi^2\sqrt{A^2 + B^2n^4\sin^2 2\pi nt}$.
11.126 $v = h\tan\beta\sqrt{4\pi^2t^2 + \csc^2 2\pi t}$. $a = 4\pi h\tan\beta\sqrt{1 + \pi^2t^2}$.
11.128 $\tan^{-1}(Rp/c)$.
11.130 (*a*) $(\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z})/(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}$

(*b*) $\left[\dfrac{(\dot{x}\ddot{y} - \dot{y}\ddot{x})^2 + (\dot{y}\ddot{z} - \dot{z}\ddot{y})^2 + (\dot{z}\ddot{x} - \dot{x}\ddot{z})^2}{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\right]^{1/2}$

(*c*) $\dfrac{[\dot{x}^2 + \dot{y}^2 + \dot{z}^2]^{3/2}}{[(\dot{x}\ddot{y} - \dot{y}\ddot{x})^2 + (\dot{y}\ddot{z} - \dot{z}\ddot{y})^2 + (\dot{z}\ddot{x} - \dot{x}\ddot{z})^2]^{1/2}}$.

11.132 4.28 m/s; 0.188 m.
11.134 (*a*) 10 m. (*b*) 0.0693 s. (*c*) -1000 m/s^2.
11.136 398 m.
11.138 1609 ft. SI: 490 m.
11.140 (*a*) 60 s; 960 m. (*b*) 240 s; 5280 m.
11.142 0.816 s, later; 276 ft below ground. SI: 0.816 s, later; 84.0 m below ground.

CHAPTER 12

12.2 $W = 16.49$ lb; $m = 100.00$ lb; $m = 3.11$ lb · s^2/ft. SI: $W = 73.4$ N; $m = 45.36$ kg.
12.4 3.22 ft/s^2; 20 lb. SI: 0.981 m/s^2; 9.07 kg.
12.6 (*a*) 65.9 ft/s. (*b*) 2.73 s. SI: (*a*) 20.1 m/s. (*b*) 2.73 s.
12.8 (*a*) 3.37 m/s. (*b*) 10.28 m.
12.10 23.9 N.
12.12 (*a*) $\mathbf{a}_A = \mathbf{a}_B = 2.42$ ft/s^2 $\swarrow$. (*b*) 1.160 lb $\nearrow$. SI: (*a*) $\mathbf{a}_A = \mathbf{a}_B = 0.739$ m/s^2 $\swarrow$. (*b*) 5.16 N $\nearrow$.
12.14 (*a*) 180 N. (*b*) 26.4 kg.
12.16 (*a*) 0.956 m. (*b*) 1.064 m.
12.18 (*a*) 302 N. (*b*) 6.79 m/s $\uparrow$. (*c*) 1.346 m/s $\downarrow$.
12.20 (*a*) 24.9 lb $\rightarrow$. (*b*) 7.96 lb. SI: (*a*) 110.6 N $\rightarrow$. (*b*) 35.4 N.
12.22 (*a*) $\mathbf{a}_A = 8.92$ ft/s^2 $\leftarrow$; $\mathbf{a}_B = 5.94$ ft/s^2 $\leftarrow$. (*b*) 3.08 lb. SI: (*a*) $\mathbf{a}_A = 2.72$ m/s^2 $\leftarrow$; $\mathbf{a}_B = 1.811$ m/s^2 $\leftarrow$. (*b*) 13.69 N.
12.24 (*a*) 9.56 ft/s^2. (*b*) 19.68 ft/s^2. SI: (*a*) 2.91 m/s^2. (*b*) 6.00 m/s^2.
12.26 1.905 m/s.
12.28 (*a*) 10.73 ft/s^2 $\leftarrow$. (*b*) 18.67 lb. SI: (*a*) 3.27 m/s^2 $\leftarrow$. (*b*) 83.0 N.

12.30 (a) $4.56 \text{ m/s}^2 \leftarrow$. (b) $1.962 \text{ m/s}^2 \leftarrow$. (c) $2.60 \text{ m/s}^2 \rightarrow$.
12.32 $a = (P/m)e^{-kt/m}$; $v = (P/k)[1 - e^{-kt/m}]$.
12.34 $a = -(kx/m)[1 - l/\sqrt{x^2 + l^2}]$.
12.36 $\mathbf{a}_A = 13.26 \text{ ft/s}^2 \uparrow$; $\mathbf{a}_B = 1.894 \text{ ft/s}^2 \downarrow$; $\mathbf{a}_C = 9.47 \text{ ft/s}^2 \downarrow$. Block C strikes ground first. SI: $\mathbf{a}_A = 4.04 \text{ m/s}^2 \uparrow$; $\mathbf{a}_B = 0.577 \text{ m/s}^2 \downarrow$; $\mathbf{a}_C = 2.89 \text{ m/s}^2 \downarrow$.
12.38 $\mathbf{a}_A = 0.577 \text{ m/s}^2 \downarrow$; $\mathbf{a}_B = 2.89 \text{ m/s}^2 \downarrow$; $\mathbf{a}_C = 4.04 \text{ m/s}^2 \uparrow$.
12.40 $\mathbf{a}_A = 4.91 \text{ m/s}^2 \uparrow$; $\mathbf{a}_B = 2.45 \text{ m/s}^2 \downarrow$; $\mathbf{a}_C = 0$.
12.42 (a) 5.51 m/s. (b) 60.6°.
12.44 (a) 10.56 ft/s. (b) 7.32 lb. SI: (a) 3.22 m/s. (b) 32.6 N.
12.46 (a) $g \sin\theta$. (b) $\sqrt{2gl(\cos\theta - \cos\theta_0)}$. (c) $W(3 - 2\cos\theta_0)$. (d) 60°.
12.48 2.71 m/s.
12.50 A: 12.86 ft/s^2. B: 25.8 ft/s^2. C: 19.32 ft/s^2. SI: A: 3.92 m/s^2. B: 7.86 m/s^2. C: 5.89 m/s^2.
12.52 $v_{\min} = \sqrt{gr\tan(\theta - \phi)}$; $v_{\max} = \sqrt{gr\tan(\theta + \phi)}$.
12.54 22.5°.
12.56 10.36 ft/s. SI: 3.16 m/s.
12.58 $y = hx^2/b^2$.
12.60 $\delta = eVlL/mv_0^2 d$.
12.62 $\sqrt{eV/mv_0^2}$.
12.64 (a) $F_r = 4$ N, $F_\theta = 0$. (b) $F_r = -21.3$ N, $F_\theta = 21.3$ N.
12.66 (a) $F_r = -73.6$ lb, $F_\theta = 0$. (b) $F_r = -24.5$ lb, $F_\theta = -49.0$ lb. SI: (a) $F_r = -327$ N, $F_\theta = 0$. (b) $F_r = -109.1$ N, $F_\theta = -218$ N.
12.68 (a) 11.93 lb. (b) 2.98 lb. SI: (a) 53.0 N. (b) 13.26 N.
12.70 12.96 N.
12.72 $n = 0$: uniform circular motion; $n = 1$: uniform rectilinear motion.
12.74 (a) 24 in./s. (b) $\rho_A = \frac{2}{3}$ in., $\rho_B = 18$ in. SI: (a) 0.610 m/s. (b) $\rho_A = 16.93$ mm, $\rho_B = 457$ mm.
12.76 $409 \times 10^{21} \text{ lb} \cdot \text{s}^2/\text{ft}$ or 13.17×10^{24} lb. SI: 5.97×10^{24} kg.
12.78 (a) 35 770 km or 22,230 mi. (b) 3070 m/s or 10,080 ft/s.
12.80 (a) 7.50 in./s. (b) Straight line. SI: (a) 0.1905 m/s.
12.82 2640 mi/h. SI: 4250 km/h.
12.84 (a) 6350 km/h. (b) 5940 km/h.
12.86 (a) $l_1^3 \sin^3\theta_1 \tan\theta_1 = l_2^3 \sin^3\theta_2 \tan\theta_2$. (b) 240 mm.
12.88 (a) 7910 ft/s. (b) 4800 ft/s. SI: (a) 2410 m/s. (b) 1462 m/s.
12.90 −30.4 m/s.
12.92 (a) 1537 km. (b) 4070 m/s. (c) 1.536.
12.94 (a) 5560 ft/s. (b) 61 ft/s. SI: (a) 1695 m/s. (b) 18.6 m/s.
12.96 45 h 30 min.
12.98 5 h 17 min.
12.100 79.7°.
12.102 197 ft/s. SI: 60 m/s.
12.106 (a) $\frac{1}{3}v_0$. (b) 75.9°.
12.108 (a) $v = R\sqrt{2g/r_0}\cos\frac{\theta}{2}$. (b) $\phi = \frac{1}{2}(\pi - \theta)$.
12.110 3.32 m.
12.112 (a) $9.91 \text{ ft/s}^2 \downarrow$. (b) $32.2 \text{ ft/s}^2 \downarrow$. SI: (a) $3.02 \text{ m/s}^2 \downarrow$. (b) $9.81 \text{ m/s}^2 \downarrow$.
12.114 $3.39 \text{ m/s}^2 \measuredangle 60°$.
12.116 $0.1438v_t^2/g$.
12.118 (a) 35 200 km/h. (b) 5150 km/h.
12.120 (a) $\mathbf{a}_A = 0$; $\mathbf{a}_B = 1.591 \text{ m/s}^2 \swarrow$. (b) $\mathbf{a}_A = \mathbf{a}_B = 0.643 \text{ m/s}^2 \swarrow$.

CHAPTER 13

13.2 2.37 GJ.
13.4 (a) $3.37 \text{ m/s} \searrow$. (b) 10.28 m.
13.6 8.72 ft/s. SI: 2.66 m/s.
13.8 14.40 N.
13.10 12.67 ft/s. SI: 3.86 m/s.
13.12 1.981 m/s.
13.14 (a) 9.27 ft/s. (b) 9.33 ft. SI: (a) 2.82 m/s. (b) 2.84 m.
13.16 10.99 ft/s. SI: 3.35 m/s.
13.22 34 in. SI: 0.86 m.
13.24 (a) 0.801 m/s. (b) 98.1 N.
13.26 19.67 in. SI: 0.500 m.
13.28 (a) $2.08 \text{ lb} \measuredangle 30°$. (b) $2.83 \text{ lb} \uparrow$. SI: (a) $9.27 \text{ N} \measuredangle 30°$. (b) $12.60 \text{ N} \uparrow$.
13.30 Loop 1: (a) $\sqrt{5gr} \leftarrow$. (b) $3W \rightarrow$. Loop 2: (a) $\sqrt{4gr} \leftarrow$. (b) $2W \rightarrow$.
13.32 1315 lb · in. SI: 148.5 J.
13.34 25,950 ft/s. SI: 7905 m/s.
13.36 14.13×10^3 km/h.
13.38 549 W; 628 W.
13.40 (a) 25.0 kW. (b) 6.13 kW.
13.42 (a) 8.18 hp. (b) 10.09 hp. SI: (a) 6.10 kW. (b) 7.52 kW.

13.44 (*a*) 55.2 kW. (*b*) 260 kW.
13.46 (*a*) 20.5 s; 701 ft. (*b*) 34.2 s; 1904 ft. SI: (*a*) 20.5 s; 214 m. (*b*) 34.2 s; 580 m.
13.48 (*a*) 278 kW. (*b*) 6.43 km/h.
13.50 (*a*) $2\,kl^2\,(1-\cos\theta)^2$. (*b*) $-mgl\sin\theta$.
13.54 (*b*) $V = -(x^2+y^2+z^2)^{-1/2}$.
13.56 46.6 ft/s. SI: 14.21 m/s.
13.58 2.45 m/s.
13.60 (*a*) 4.71 m/s. (*b*) 4.03 m/s.
13.62 7.05 ft/s. SI: 2.15 m/s.
13.64 23.7 m/s.
13.66 104.9 N.
13.68 (*a*) 22.7 ft/s. (*b*) 7.75 ft. SI: (*a*) 6.92 m/s. (*b*) 2.36 m.
13.70 6 *mg*.
13.72 1.600 in.; 24 lb. SI: 40.6 mm; 7.32 N.
13.74 36,700 ft/s. SI: 11.18 km/s.
13.76 (*a*) 0.943×10^6 ft · lb/lb. (*b*) 0.447×10^6 ft · lb/lb. SI: (*a*) 2.82 MJ/kg. (*b*) 1.336 MJ/kg.
13.80 (*a*) 1.155 m. (*b*) 5.20 m/s.
13.82 (*a*) 15.54 ft/s. (*b*) 5.18 ft/s. (*c*) 0.125 ft. SI: (*a*) 4.74 m/s. (*b*) 1.579 m/s. (*c*) 38.1 mm.
13.84 (*a*) 25.3 in. (*b*) 7.58 ft/s. SI: (*a*) 0.643 m. (*b*) 2.31 m/s.
13.90 10,780 ft/s. SI: 3285 m/s.
13.92 8420 m/s; 74.4°.
13.94 5160 ft/s; 79.9°. SI: 1572 m/s; 79.9°.
13.96 $65.7° \leq \phi_0 \leq 114.3°$.
13.102 380 mi. SI: 610 km.
13.104 (*b*) $\frac{1}{3}\sqrt{6}\,v_{esc}$, $\frac{1}{2}\sqrt{2}\,v_{esc}$.
13.106 (*a*) and (*b*) 6 min 4 s.
13.108 (*a*) 11.42 s. (*b*) $\mathbf{v} = -(125.5\text{ m/s})\mathbf{j} - (194.5\text{ m/s})\mathbf{k}$.
13.110 (*a*) 2.80 s. (*b*) 5.60 s.
13.114 (*a*) 38.9 s. (*b*) 10.71 kN *T*.
13.116 (*a*) 10.06 ft/s; 1.5 s. (*b*) 3 s. SI: (*a*) 3.07 m/s; 1.5 s. (*b*) 3 s.
13.118 (*a*) 9.03 m/s. (*b*) 0.
13.120 (*a*) and (*b*) 111.1 kN.
13.122 48.4 lb ←, 188.0 lb ↓. SI: 215 N ←, 836 N ↓.
13.124 9.38 ft/s. SI: 2.86 m/s.
13.126 (*a*) 2 m/s ←. (*b*) $T_A = 3$J. $T_B = 9$J.
13.128 (*a*) 0.6 mi/h. (*b*) 4370 lb. SI: (*a*) 0.966 km/h. (*b*) 19.45 kN.
13.130 0.742 m/s →.
13.132 (*a*) 135.6 N · s. (*b*) 108.5 N · s. (*c*) 368 J; 294 J.
13.134 (*a*) $\mathbf{v}_A = 1.125$ ft/s ←; $\mathbf{v}_B = 13.875$ ft/s →. (*b*) 5.10 ft · lb. SI: (*a*) $\mathbf{v}_A = 0.343$ m/s ←; $\mathbf{v}_B = 4.23$ m/s →. (*b*) 6.91 J.
13.136 (*a*) $\mathbf{v}_A = 2.30$ m/s ←; $\mathbf{v}_B = 2.20$ m/s →. (*b*) 2.84 J.
13.138 $\mathbf{v}_A = 3.50$ m/s ∠ 60°; $\mathbf{v}_B = 4.03$ m/s ∠ 21.7°.
13.140 (*a*) 0.571 $\mathbf{v}_0$. (*b*) 1.333 $\mathbf{v}_0$.
13.144 (*a*) 0.943. (*b*) 28.4 in.; 15.08 in. SI: (*a*) 0.943. (*b*) 0.722 m; 0.383 m.
13.146 (*a*) 0.883. (*b*) 11.30 in. SI: (*a*) 0.883. (*b*) 0.287 m.
13.148 $\mathbf{v}_A = 0.721\,v_0$ ∠ 16.1°; $\mathbf{v}_B = 0.693\,v_0$ ←.
13.150 69.9°.
13.152 (*a*) 8.29 ft/s →. (*b*) 6.85 lb. (*c*) 1.068 ft. SI: (*a*) 2.53 m/s →. (*b*) 30.5 N. (*c*) 0.326 m.
13.154 2.57 in. SI: 65.3 mm.
13.156 (*a*) 34.7 mm. (*b*) 8.18 J.
13.158 (*a*) 8890 J. (*b*) 24 km/h.
13.160 (*a*) 9.32 ft · lb. (*b*) 8.10 ft · lb. SI: (*a*) 12.63 J. (*b*) 10.98 J.
13.162 (*a*) Five. (*b*) 2 m/s →. (*c*) Same as original.
13.164 4.47 in. SI: 113.6 mm.
13.166 Impact at *A*: $\mathbf{v}_i = 1.333$ m/s →, $\mathbf{v}_f = 0.333$ m/s →; impact at *B*: $\mathbf{v}_i = 0$, $\mathbf{v}_f = 1$ m/s →.
13.168 5.29 m/s →.
13.170 4.89 ft. SI: 1.491 m.
13.172 317 N/m.

CHAPTER 14

14.2 (*a*) 5.20 km/h →. (*b*) 3.90 km/h →.
14.4 (*a*) 1670 ft/s →. (*b*) 1158 ft/s →. SI: (*a*) 509 m/s →. (*b*) 353 m/s →.
14.6 (*a*) $v_x = 19.50$ ft/s, $v_y = 16.00$ ft/s. (*b*) $-(1.118\text{ ft}\cdot\text{lb}\cdot\text{s})\mathbf{i}$. SI: (*a*) $v_x = 5.94$ m/s, $v_y = 4.88$ m/s. (*b*) $-(1.516\text{ kg}\cdot\text{m}^2/\text{s})\mathbf{i}$.
14.8 (*a*) $(3\text{ m})\mathbf{i} + (1.5\text{ m})\mathbf{j} + (1.5\text{ m})\mathbf{k}$. (*b*) $(17\text{ kg}\cdot\text{m/s})\mathbf{i} + (19\text{ kg}\cdot\text{m/s})\mathbf{j} - (5\text{ kg}\cdot\text{m/s})\mathbf{k}$. (*c*) $(2\text{ kg}\cdot\text{m}^2/\text{s})\mathbf{i} + (24.5\text{ kg}\cdot\text{m}^2/\text{s})\mathbf{j} + (25.5\text{ kg}\cdot\text{m}^2/\text{s})\mathbf{k}$.
14.10 $x = 780$ ft, $y = 17.55$ ft, $z = -21.0$ ft. SI: $x = 238$ m, $y = 5.35$ m, $z = 6.40$ m.
14.12 $x = 100$ m, $y = -40.7$ m, $z = 16$ m.

14.14 $v_B = 4.57$ ft/s; $v_C = 5.78$ ft/s.
SI: $v_B = 1.393$ m/s; $v_C = 1.761$ m/s.

14.16 $v_A = 919$ m/s; $v_B = 717$ m/s;
$v_C = 619$ m/s.

14.22 9.55%.

14.24 0.201%.

14.26 (*a*) $mv_0\mathbf{i}$; $\frac{3}{4}mlv_0\mathbf{k}$. (*b*) $\mathbf{v}_A = \frac{1}{4}v_0\mathbf{i} + \frac{3}{4}v_0\mathbf{j}$;
$\mathbf{v}_B = \frac{1}{4}v_0\mathbf{i} - \frac{1}{4}v_0\mathbf{j}$. (*c*) $\mathbf{v}_A = -\frac{1}{2}v_0\mathbf{i}$;
$\mathbf{v}_B = \frac{1}{2}v_0\mathbf{i}$.

14.28 $x = 181.7$ mm, $y = 0$, $z = 139.4$ mm.

14.30 $v_A = 1.500$ m/s; $v_B = 1.299$ m/s;
$v_C = 2.25$ m/s.

14.32 $\mathbf{v}_A = 34.3$ ft/s $\angle 29.7°$;
$\mathbf{v}_B = 17.59$ ft/s $\angle 40.1°$.
SI: $\mathbf{v}_A = 10.46$ m/s $\angle 29.7°$;
$\mathbf{v}_B = 5.36$ m/s $\angle 40.1°$.

14.34 (*a*) 13.00 ft/s $\rightarrow$. (*b*) 10.82 ft/s $\angle 33.7°$.
(*c*) $b = 8.33$ ft. SI: (*a*) 3.96 m/s $\rightarrow$.
(*b*) 3.30 m/s $\angle 33.7°$. (*c*) $b = 2.54$ m.

14.36 (*a*) 3 m/s $\angle 36.9°$.
(*b*) $\mathbf{H}_G = (4.80 \text{ kg} \cdot \text{m}^2/\text{s})\mathbf{k}$; $T' = 48.0$ J.
(*c*) 0.600 m. (*d*) 20 rad/s.

14.38 $P_x = 800$ N; $P_y = 800$ N.

14.40 $\mathbf{B} = mv_0 \rightarrow$; $\mathbf{C} = m\sqrt{2gh} \angle 30°$.

14.42 $Q_1 = \frac{1}{2}Q(1 - \sin\theta)$; $Q_2 = \frac{1}{2}Q(1 + \sin\theta)$.

14.44 $\mathbf{C}_x = 83.3$ lb $\rightarrow$, $\mathbf{C}_y = 30.3$ lb $\downarrow$,
$\mathbf{M}_C = 496$ lb $\cdot$ in. $\circlearrowright$. SI: $\mathbf{C}_x = 370$ N $\rightarrow$,
$\mathbf{C}_y = 134.8$ N $\downarrow$, $\mathbf{M}_C = 56.1$ N $\cdot$ m $\circlearrowright$.

14.46 $\mathbf{C}_x = 475$ N $\leftarrow$, $\mathbf{C}_y = 675$ N $\uparrow$;
$\mathbf{D} = 865$ N $\rightarrow$.

14.48 $\mathbf{C} = 321$ lb $\uparrow$; $\mathbf{D} = 479$ lb $\uparrow$.
SI: $\mathbf{C} = 1428$ N $\uparrow$; $\mathbf{D} = 2130$ N $\uparrow$.

14.50 (*a*) 10,250 lb. (*b*) 16,400 hp.
(*c*) 28,700 hp. SI: (*a*) 45.6 kN.
(*b*) 12.23 MW. (*c*) 21.4 MW.

14.52 (*a*) 26.4 kN. (*b*) 830 km/h.

14.54 (*a*) 1500 N. (*b*) 2500 N.

14.56 43.2 ft/s. SI: 13.17 m/s.

14.58 23.8 N.

14.60 24 rad/s $\circlearrowleft$. (*b*) 0.400 N $\cdot$ m $\circlearrowright$.

14.62 216 rpm.

14.64 $\sin\theta = v^2/gL$.

14.66 $P = mv(v + gt)$.

14.68 qv.

14.70 $v = m_0v_0/(m_0 + qt)$;
$a = -m_0v_0q/(m_0 + qt)^2$.

14.72 (*a*) 40.3 lb/s. (*b*) 10.06 lb/s.
SI: (*a*) 18.26 kg/s. (*b*) 4.56 kg/s.

14.74 (*a*) 240 ft/s^2. (*b*) 960 ft/s^2.
SI: (*a*) 73.2 m/s^2. (*b*) 293 m/s^2.

14.76 (*a*) 6820 kg. (*b*) 341 s.

14.78 18,480 mi/h. SI: 29.7×10^3 km/h.

14.80 452,000 ft. SI: 137.9 km.

14.84 (*a*) 0.855 m/s^2. (*b*) 987 km/h.

14.86 (*a*) 5.54 ft/s. (*b*) 0.641 ft from B.
SI: (*a*) 1.687 m/s. (*b*) 0.1955 m from B.

14.88 (*a*) $\frac{2}{3}v$; (BC) 33.3%. (*b*) $\frac{2}{3}v$; (BC) 25%,
(AB) 8.33%.

14.90 (*a*) $\frac{1}{2}v_A \rightarrow$. (*b*) $\frac{1}{4}A\rho(1 - \cos\theta)v_A^3$.
(*c*) $2(V/v_A)[1 - (V/v_A)](1 - \cos\theta)$.

14.92 $\mathbf{C} = 89.3$ N $\downarrow$; $\mathbf{D} = 138.4$ N $\uparrow$.

14.94 (*a*) 10,560 lb; 1.922 ft below B.
(*b*) 8300 lb; 4.89 ft below B.
SI: (*a*) 47.0 kN; 0.586 m below B.
(*b*) 36.9 kN; 1.490 m below B.

CHAPTER 15

15.2 (*a*) -2.51 rad/s^2. (*b*) 18 000 rev.

15.4 (*a*) -2.42 rad/s^2. (*b*) 52 s.

15.6 $\mathbf{v}_H = -(32 \text{ in./s})\mathbf{i} + (56 \text{ in./s})\mathbf{k}$;
$\mathbf{a}_H = -(368 \text{ in./s}^2)\mathbf{i} - (1040 \text{ in./s}^2)\mathbf{j}$
$- (396 \text{ in./s}^2)\mathbf{k}$.
SI: $\mathbf{v}_H = -(0.813 \text{ m/s})\mathbf{i} + (1.422 \text{ m/s})\mathbf{k}$;
$\mathbf{a}_H = -(9.35 \text{ m/s}^2)\mathbf{i} - (26.4 \text{ m/s}^2)\mathbf{j}$
$- (10.06 \text{ m/s}^2)\mathbf{k}$.

15.8 $\mathbf{v}_E = (0.14 \text{ m/s})\mathbf{i} - (0.48 \text{ m/s})\mathbf{j}$
$- (0.96 \text{ m/s})\mathbf{k}$; $\mathbf{a}_E = -(0.644 \text{ m/s}^2)\mathbf{i}$
$+ (2.21 \text{ m/s}^2)\mathbf{j} - (7.30 \text{ m/s}^2)\mathbf{k}$.

15.10 $\mathbf{v}_C = -(1.8 \text{ m/s})\mathbf{j} - (1.2 \text{ m/s})\mathbf{k}$;
$\mathbf{a}_C = (7.8 \text{ m/s}^2)\mathbf{i} - (12.6 \text{ m/s}^2)\mathbf{j}$
$+ (7.2 \text{ m/s}^2)\mathbf{k}$.

15.12 66,700 mi/h; 0.01947 ft/s^2.
SI: 107.3×10^3 km/h; 5.93 mm/s^2.

15.14 1.174 s; 7.05 rad/s.

15.16 (*a*) 2 rad/s $\circlearrowright$; 3 rad/s^2 $\circlearrowleft$.
(*b*) 20 in./s^2 $\angle 36.9°$. SI: (*a*) 2 rad/s $\circlearrowright$;
3 rad/s^2 $\circlearrowleft$. (*b*) 508 mm/s^2 $\angle 36.9°$.

15.18 (*a*) 10 rad/s. (*b*) $\mathbf{a}_B = 18$ m/s^2 $\downarrow$;
$\mathbf{a}_C = 6$ m/s^2 $\downarrow$.

15.20 3.49 s; 6.98 s; 13.96 s.

15.22 (*a*) $\boldsymbol{\alpha}_A = 4.19$ rad/s^2 $\circlearrowleft$;
$\boldsymbol{\alpha}_B = 6.98$ rad/s^2 $\circlearrowleft$. (*b*) 4.50 s.

15.24 $\alpha = bv^2/2\pi r^3$.

15.26 (*a*) 1.25 rad/s $\circlearrowright$. (*b*) 25 in./s $\angle 60°$.
SI: (*a*) 1.25 rad/s $\circlearrowright$.
(*b*) 635 mm/s $\angle 60°$.

15.28 (*a*) 2 rad/s^2 $\circlearrowright$.
(*b*) $\mathbf{v}_A = (80 \text{ mm/s})\mathbf{i} + (440 \text{ mm/s})\mathbf{j}$.

15.30 (*a*) $\mathbf{v}_B = -(160\text{ mm/s})\mathbf{i} + (200\text{ mm/s})\mathbf{j}$. (*b*) $x = 220$ mm, $y = 80$ mm.

15.32 (*a*) $\boldsymbol{\omega}_A = \boldsymbol{\omega}_B = v/r \circlearrowright$; $\boldsymbol{\omega}_C = v/2r \circlearrowleft$. (*b*) $\mathbf{v}_D = 2v \rightarrow$; $\mathbf{v}_E = 0$; $\mathbf{v}_F = \sqrt{2}v \measuredangle 45^\circ$.

15.34 (*a*) 180 rpm $\circlearrowright$. (*b*) 2.83 m/s $\swarrow$.

15.36 (*a*) $\mathbf{v}_P = 0$; $\boldsymbol{\omega}_{BD} = 39.3$ rad/s $\circlearrowleft$. (*b*) $\mathbf{v}_P = 6.28$ m/s $\downarrow$; $\boldsymbol{\omega}_{BD} = 0$. (*c*) $\mathbf{v}_P = 0$; $\boldsymbol{\omega}_{BD} = 39.3$ rad/s $\circlearrowright$.

15.38 $\boldsymbol{\omega}_{BD} = 2.94$ rad/s $\circlearrowleft$; $\mathbf{v}_D = 31.8$ in./s $\leftarrow$. SI: $\boldsymbol{\omega}_{BD} = 2.94$ rad/s $\circlearrowleft$; $\mathbf{v}_D = 0.807$ m/s $\leftarrow$.

15.40 $\boldsymbol{\omega}_{BD} = 1$ rad/s $\circlearrowleft$; $\boldsymbol{\omega}_{DE} = 3$ rad/s $\circlearrowleft$.

15.42 $\boldsymbol{\omega}_{BD} = 3.75$ rad/s $\circlearrowright$; $\boldsymbol{\omega}_{DE} = 2.25$ rad/s $\circlearrowleft$.

15.44 (*a*) $\mathbf{v}_A = 10$ in./s $\rightarrow$; $\boldsymbol{\omega}_{AC} = 0$. (*b*) $\mathbf{v}_A = 45.4$ in./s $\rightarrow$; $\boldsymbol{\omega}_{AC} = 0.566$ rad/s $\circlearrowright$. SI: (*a*) $\mathbf{v}_A = 0.254$ m/s $\rightarrow$; $\boldsymbol{\omega}_{AC} = 0$. (*b*) $\mathbf{v}_A = 1.152$ m/s $\rightarrow$; $\boldsymbol{\omega}_{AC} = 0.566$ rad/s $\circlearrowright$.

15.48 Vertical line intersecting zx plane at $x = 0$, $z = 9.34$ ft. SI: $x = 0$, $z = 2.85$ m.

15.50 (*a*) 2 rad/s $\circlearrowleft$. (*b*) 12 in./s $\leftarrow$. (*c*) 9 in./s, wound. SI: (*a*) 2 rad/s $\circlearrowleft$. (*b*) 0.305 m/s $\leftarrow$. (*c*) 0.229 m/s, wound.

15.52 (*a*) 0.6 rad/s $\circlearrowright$. (*b*) 24 mm/s $\rightarrow$.

15.54 (*a*) 4 rad/s $\circlearrowleft$. (*b*) 86.5 in./s $\measuredangle 16.1^\circ$. SI: (*a*) 4 rad/s $\circlearrowleft$. (*b*) 2.20 m/s $\measuredangle 16.1^\circ$.

15.56 (*a*) 6.67 rad/s $\circlearrowright$. (*b*) 2 m/s $\leftarrow$. (*c*) 1.250 m/s $\measuredangle 36.9^\circ$.

15.58 $\cos^3\theta = b/l$.

15.60 (*a*) 2 rad/s $\circlearrowleft$. (*b*) 18.33 in./s $\measuredangle 19.1^\circ$. SI: (*a*) 2 rad/s $\circlearrowleft$. (*b*) 0.466 m/s $\measuredangle 19.1^\circ$.

15.62 (*a*) 3 rad/s $\circlearrowleft$. (*b*) On or inside a 2-in.-radius circle centered at a point 1.928 in. below G. SI: (*a*) 3 rad/s $\circlearrowleft$. (*b*) On or inside a 50.8-mm-radius circle centered at a point 49.0 mm below *G*.

15.64 (*a*) 0.9 rad/s $\circlearrowright$. (*b*) 144 mm/s $\leftarrow$.

15.66 Space centrode: Circle of 12-in. radius with center at intersection of tracks. Body centrode: Circle of 6-in. radius with center on rod at point equidistant from *A* and *B*.

15.76 (*a*) 0.5 rad/s^2 $\circlearrowright$. (*b*) 5.5 ft/s^2 $\uparrow$. SI: (*a*) 0.5 rad/s^2 $\circlearrowright$. (*b*) 1.676 m/s^2 $\uparrow$.

15.78 (*a*) 0.4 m/s^2 $\leftarrow$. (*b*) 0.2 m/s^2 $\rightarrow$.

15.80 $\mathbf{a}_C = 316$ m/s^2 $\uparrow$; $\mathbf{a}_D = 316$ m/s^2 $\measuredangle 60^\circ$.

15.82 (*a*) 10 in./s^2 $\uparrow$. (*b*) 18.36 in./s^2 $\measuredangle 60.6^\circ$. (*c*) 42.2 in./s^2 $\measuredangle 22.3^\circ$. SI: (*a*) 0.254 m/s^2 $\uparrow$. (*b*) 0.466 m/s^2 $\measuredangle 60.6^\circ$. (*c*) 1.071 m/s^2 $\measuredangle 22.3^\circ$.

15.84 $\boldsymbol{\omega}_{AB} = 0$; $\boldsymbol{\alpha}_{AB} = \frac{3}{2}\omega_0^2 \circlearrowleft$; $\boldsymbol{\omega}_{BC} = \frac{1}{2}\omega_0 \circlearrowleft$, $\boldsymbol{\alpha}_{BC} = 0$.

15.86 (*a*) 157.0 m/s^2 $\uparrow$. (*b*) 592 m/s^2 $\downarrow$.

15.88 71.1 m/s^2 $\searrow$.

15.90 (*a*) 0. (*b*) 2.67 rad/s^2 $\circlearrowright$.

15.92 (*a*) 3.46 rad/s^2 $\circlearrowright$. (*b*) 15.59 in./s^2 $\measuredangle 30^\circ$. SI: (*a*) 3.46 rad/s^2 $\circlearrowright$. (*b*) 0.396 m/s^2 $\measuredangle 30^\circ$.

15.94 (*a*) 3.46 rad/s^2 $\circlearrowright$. (*b*) 19.30 in./s^2 $\measuredangle 45.6^\circ$. SI: (*a*) 3.46 rad/s^2 $\circlearrowright$. (*b*) 0.490 m/s^2 $\measuredangle 45.6^\circ$.

15.96 1.814 m/s^2 $\measuredangle 60.3^\circ$.

15.98 $\omega = (v_B \sin\beta)/(l\cos\theta)$

15.100 (*a*) $v_B = r\omega\cos\theta$. (*b*) $a_B = r\alpha\cos\theta - r\omega^2\sin\theta$.

15.102 $v_D = -2l\omega\sin\theta$; $a_D = -2l\alpha\sin\theta - 2l\omega^2\cos\theta$.

15.104 $v_B = R\omega\sec^2\theta$; $a_B = R\sec^2\theta(\alpha + 2\omega^2\tan\theta)$.

15.106 $\omega_{AB} = r\omega(a^2 + l^2 - 2al\cos\theta)^{1/2}/al\sin\theta$.

15.108 (*a*) $\omega = (v_A/b)\cos^2\theta$. (*b*) $(\mathbf{v}_B)_x = (v_A L/b)\sin\theta\cos^2\theta \leftarrow$, $(\mathbf{v}_B)_y = v_A[(L/b)\cos^3\theta - 1] \downarrow$.

15.110 (*a*) 2.58 rad/s $\circlearrowright$. (*b*) 19.75 in./s $\measuredangle 50^\circ$. SI: (*a*) 2.58 rad/s $\circlearrowright$. (*b*) 0.502 m/s $\measuredangle 50^\circ$.

15.112 $\boldsymbol{\omega}_{AP} = 1.958$ rad/s $\circlearrowleft$; $\boldsymbol{\omega}_{BD} = 3.80$ rad/s $\circlearrowleft$.

15.114 (*a*) $\boldsymbol{\omega}_{BD} = \omega \circlearrowleft$; $\mathbf{v}_{P/AH} = 0$; $\mathbf{v}_{P/BD} = l\omega \uparrow$. (*b*) $\boldsymbol{\omega}_{BD} = \omega \circlearrowleft$; $\mathbf{v}_{P/AH} = 0.299\, l\omega \measuredangle 15^\circ$; $\mathbf{v}_{P/BD} = 1.115\, l\omega \measuredangle 75^\circ$.

15.116 $\mathbf{a}_1 = r\omega^2\mathbf{i} + 2u\omega\mathbf{j}$; $\mathbf{a}_2 = -2u\omega\mathbf{i} - r\omega^2\mathbf{j}$; $\mathbf{a}_3 = (-r\omega^2 - u^2/r + 2u\omega)\mathbf{i}$; $\mathbf{a}_4 = (r\omega^2 - 2u\omega)\mathbf{j}$.

15.118 (*a*) $\mathbf{v}_B = 735$ mm/s $\measuredangle 71.8^\circ$. (*b*) $\mathbf{a}_B = 62.4$ mm/s $\measuredangle 7.4^\circ$.

15.120 (*a*) $\mathbf{a}_B = (10.9\text{ m/s}^2)\mathbf{j}$. (*b*) $\mathbf{a}_D = -(0.1\text{ m/s}^2)\mathbf{i} + (10.8\text{ m/s}^2)\mathbf{j}$. (*c*) $\mathbf{a}_E = (10.7\text{ m/s}^2)\mathbf{j}$.

15.122 (*a*) 0.00582 ft/s^2 west. (*b*) and (*c*) 0.00446 ft/s^2 west. SI: (*a*) 1.773 mm/s^2 west. (*b*) and (*c*) 1.358 mm/s^2 west.

15.124 $11.05 \text{ rad/s}^2 \circlearrowleft$.

15.126 (*a*) 476 ft/s^2. (*b*) 307 ft/s^2.
SI: (*a*) 145.1 m/s^2. (*b*) 93.7 m/s^2.

15.128 (*a*) $\boldsymbol{\omega}_{BD} = 2.4 \text{ rad/s} \circlearrowright$;
$\boldsymbol{\alpha}_{BD} = 34.6 \text{ rad/s}^2 \circlearrowright$.
(*b*) $\mathbf{v} = 1.342 \text{ m/s} \angle 63.4^\circ$;
$\mathbf{a} = 9.11 \text{ m/s}^2 \angle 18.4^\circ$.

15.130 (*a*) -120 mm/s.
(*b*) $\mathbf{v}_B = -(40 \text{ mm/s})\mathbf{i} - (100 \text{ mm/s})\mathbf{j} - (80 \text{ mm/s})\mathbf{k}$.

15.132 (*a*) $\boldsymbol{\omega} = (2 \text{ rad/s})\mathbf{i} + (4 \text{ rad/s})\mathbf{j} + (3 \text{ rad/s})\mathbf{k}$.
(*b*) $\mathbf{v}_B = -(3 \text{ in./s})\mathbf{i} - (6 \text{ in./s})\mathbf{j} + (10 \text{ in./s})\mathbf{k}$.
SI: (*b*) $\mathbf{v}_B = -(76.2 \text{ mm/s})\mathbf{i} - (152.4 \text{ mm/s})\mathbf{j} + (254 \text{ mm/s})\mathbf{k}$.

15.134 $\boldsymbol{\alpha} = (237 \text{ rad/s}^2)\mathbf{k}$.

15.136 $\boldsymbol{\alpha} = -(565 \text{ rad/s}^2)\mathbf{i} - (5 \text{ rad/s}^2)\mathbf{j}$.

15.138 (*a*) $\boldsymbol{\omega} = -(R\omega_1/r)\mathbf{i} + \omega_1\mathbf{j}$.
(*b*) $\boldsymbol{\alpha} = (R\omega_1^2/r)\mathbf{k}$.

15.140 $\omega_1 \cos 30^\circ$.

15.142 (*a*) $\boldsymbol{\alpha} = (3 \text{ rad/s}^2)\mathbf{i} + (2.5 \text{ rad/s}^2)\mathbf{k}$.
(*b*) $\mathbf{a}_A = -(125 \text{ in./s}^2)\mathbf{i} + (50 \text{ in./s}^2)\mathbf{j} + (67.5 \text{ in./s}^2)\mathbf{k}$;
$\mathbf{a}_B = -(50 \text{ in./s}^2)\mathbf{i} + (170 \text{ in./s}^2)\mathbf{j} - (180 \text{ in./s}^2)\mathbf{k}$.
SI: (*b*) $\mathbf{a}_A = -(3.18 \text{ m/s}^2)\mathbf{i} + (1.270 \text{ m/s}^2)\mathbf{j} + (1.715 \text{ m/s}^2)\mathbf{k}$;
$\mathbf{a}_B = -(1.270 \text{ m/s}^2)\mathbf{i} + (4.32 \text{ m/s}^2)\mathbf{j} - (4.57 \text{ m/s}^2)\mathbf{k}$.

15.144 (*a*) $\boldsymbol{\omega} = -(4 \text{ rad/s})\mathbf{j} + (1.6 \text{ rad/s})\mathbf{k}$.
(*b*) $\boldsymbol{\alpha} = -(6.4 \text{ rad/s}^2)\mathbf{i}$.
(*c*) $\mathbf{v}_P = -(0.4 \text{ m/s})\mathbf{i} + (0.693 \text{ m/s})\mathbf{j} + (1.732 \text{ m/s})\mathbf{k}$.
$\mathbf{a}_P = -(8.04 \text{ m/s}^2)\mathbf{i} - (0.64 \text{ m/s}^2)\mathbf{j} - (3.2 \text{ m/s}^2)\mathbf{k}$.

15.146 (*a*) $\boldsymbol{\alpha} = -\omega_1\omega_2\mathbf{j}$.
(*b*) $\mathbf{a}_P = -r\omega_2^2\mathbf{i} + 2r\omega_1\omega_2\mathbf{k}$.
(*c*) $\mathbf{a}_P = -r(\omega_1^2 + \omega_2^2)\mathbf{j}$.

15.148 (*a*) $\boldsymbol{\alpha} = -(150 \text{ rad/s}^2)\mathbf{k}$.
(*b*) $\mathbf{a} = -(225 \text{ in./s}^2)\mathbf{i} - (2400 \text{ in./s}^2)\mathbf{j}$.
SI: (*a*) $\boldsymbol{\alpha} = -(150 \text{ rad/s}^2)\mathbf{k}$.
(*b*) $\mathbf{a} = -(5.72 \text{ m/s}^2)\mathbf{i} - (61.0 \text{ m/s}^2)\mathbf{k}$.

15.150 (*a*) $\boldsymbol{\alpha} = -(8 \text{ rad/s}^2)\mathbf{k}$.
(*b*) $\mathbf{a}_C = (3.2 \text{ m/s}^2)\mathbf{i} - (0.8 \text{ m/s}^2)\mathbf{j}$.

15.152 $\mathbf{v}_B = (54 \text{ mm/s})\mathbf{i}$.

15.154 $\mathbf{v}_C = (32 \text{ in./s})\mathbf{j}$. SI: $\mathbf{v}_C = (0.813 \text{ m/s})\mathbf{j}$.

15.156 (*a*) $\boldsymbol{\omega} = (1.6 \text{ rad/s})\mathbf{i} + (15.2 \text{ rad/s})\mathbf{j} - (3.2 \text{ rad/s})\mathbf{k}$.
(*b*) $\mathbf{v}_C = (32 \text{ in./s})\mathbf{j}$.
SI: (*b*) $\mathbf{v}_C = (0.813 \text{ m/s})\mathbf{j}$.

15.158 $\mathbf{v}_B = -(14.41 \text{ in./s})\mathbf{i} - (4.32 \text{ in./s})\mathbf{j}$.
SI: $\mathbf{v}_B = -(0.366 \text{ m/s})\mathbf{i} - (0.1098 \text{ m/s})\mathbf{j}$.

15.160 $\mathbf{a}_B = -(49.2 \text{ mm/s}^2)\mathbf{i}$.

15.162 $\mathbf{a}_C = (1162 \text{ in./s}^2)\mathbf{j}$.
SI: $\mathbf{a}_C = (29.5 \text{ m/s}^2)\mathbf{j}$.

15.164 (*a*) $\mathbf{v}_D = (0.6 \text{ m/s})\mathbf{i} - (0.6 \text{ m/s})\mathbf{j} + (0.25 \text{ m/s})\mathbf{k}$.
(*b*) $\mathbf{a}_D = (3 \text{ m/s}^2)\mathbf{i} - (3.6 \text{ m/s}^2)\mathbf{k}$.

15.166 (*a*) $\mathbf{v}_D = -(20 \text{ in./s})\mathbf{i} - (34.6 \text{ in./s})\mathbf{j} - (46.8 \text{ in./s})\mathbf{k}$. (*b*) $\mathbf{a}_D = -(652 \text{ in./s}^2)\mathbf{i} + (133.3 \text{ in./s}^2)\mathbf{j} + (360 \text{ in./s}^2)\mathbf{k}$.
SI: (*a*) $\mathbf{v}_D = -(0.508 \text{ m/s})\mathbf{i} - (0.880 \text{ m/s})\mathbf{j} - (1.188 \text{ m/s})\mathbf{k}$.
(*b*) $\mathbf{a}_D = -(16.56 \text{ m/s}^2)\mathbf{i} + (3.39 \text{ m/s}^2)\mathbf{j} + (9.14 \text{ m/s}^2)\mathbf{k}$.

15.168 (*a*) $\mathbf{v}_D = (0.8 \text{ m/s})\mathbf{i} - (0.72 \text{ m/s})\mathbf{j} + (0.3 \text{ m/s})\mathbf{k}$. (*b*) $\mathbf{a}_D = (3 \text{ m/s}^2)\mathbf{i} + (2.4 \text{ m/s}^2)\mathbf{j} - (7.4 \text{ m/s}^2)\mathbf{k}$.

15.170 (*a*) $\mathbf{v}_P = -(1.701 \text{ m/s})\mathbf{i} + (5.95 \text{ m/s})\mathbf{j} - (3.12 \text{ m/s})\mathbf{k}$.
(*b*) $\mathbf{a}_P = -(4.29 \text{ m/s}^2)\mathbf{i} - (0.201 \text{ m/s}^2)\mathbf{j} + (1.021 \text{ m/s}^2)\mathbf{k}$.

15.172 (*a*) $\boldsymbol{\omega} = \omega_1\mathbf{j} + \omega_2\mathbf{k}$; $\boldsymbol{\alpha} = \omega_1\omega_2\mathbf{i}$.
(*b*) $\mathbf{v}_B = r\omega_2\mathbf{j} - (R + r)\omega_1\mathbf{k}$;
$\mathbf{a}_B = -[(R + r)\omega_1^2 + r\omega_2^2]\mathbf{i}$.

15.174 (*a*) $\boldsymbol{\alpha} = -(0.314 \text{ rad/s}^2)\mathbf{k}$.
(*b*) $\mathbf{v}_B = (124.7 \text{ ft/s})\mathbf{k}$;
$\mathbf{a}_B = (25.0 \text{ ft/s}^2)\mathbf{i} - (395 \text{ ft/s}^2)\mathbf{j}$.
SI: (*b*) $\mathbf{v}_B = (38.0 \text{ m/s})\mathbf{k}$;
$\mathbf{a}_B = (7.63 \text{ m/s}^2)\mathbf{i} - (120.3 \text{ m/s}^2)\mathbf{j}$.

15.176 (*a*) $\boldsymbol{\alpha} = (200 \text{ rad/s}^2)\mathbf{k}$.
(*b*) $\mathbf{v}_D = -(1 \text{ m/s})\mathbf{j} - (2.4 \text{ m/s})\mathbf{k}$;
$\mathbf{a}_D = -(40 \text{ m/s}^2)\mathbf{i} + (44 \text{ m/s}^2)\mathbf{j} - (10 \text{ m/s}^2)\mathbf{k}$.

15.178 $\mathbf{v}_A = -(18 \text{ in./s})\mathbf{j} + (160 \text{ in./s})\mathbf{k}$;
$\mathbf{v}_B = -(90 \text{ in./s})\mathbf{j} + (64 \text{ in./s})\mathbf{k}$;
$\mathbf{a}_A = -(1600 \text{ in./s}^2)\mathbf{j} - (360 \text{ in./s}^2)\mathbf{k}$;
$\mathbf{a}_B = -(880 \text{ in./s}^2)\mathbf{j} - (1000 \text{ in./s}^2)\mathbf{k}$.
SI: $\mathbf{v}_A = -(0.457 \text{ m/s})\mathbf{j} + (4.06 \text{ m/s})\mathbf{k}$;
$\mathbf{v}_B = -(2.29 \text{ m/s})\mathbf{j} + (1.626 \text{ m/s})\mathbf{k}$;
$\mathbf{a}_A = -(40.6 \text{ m/s}^2)\mathbf{j} - (9.14 \text{ m/s}^2)\mathbf{k}$;
$\mathbf{a}_B = -(22.4 \text{ m/s}^2)\mathbf{j} - (25.4 \text{ m/s}^2)\mathbf{k}$.

15.180 $\mathbf{v}_A = -(160 \text{ in./s})\mathbf{i} - (18 \text{ in./s})\mathbf{j}$;
$\mathbf{v}_B = -(40 \text{ in./s})\mathbf{i} + (24 \text{ in./s})\mathbf{k}$;
$\mathbf{a}_A = (360 \text{ in./s}^2)\mathbf{i} - (1600 \text{ in./s}^2)\mathbf{j}$;
$\mathbf{a}_B = -(400 \text{ in./s}^2)\mathbf{j} - (100 \text{ in./s}^2)\mathbf{k}$.
SI: $\mathbf{v}_A = -(4.06 \text{ m/s})\mathbf{i} - (0.457 \text{ m/s})\mathbf{j}$;
$\mathbf{v}_B = -(1.016 \text{ m/s})\mathbf{i} + (0.610 \text{ m/s})\mathbf{k}$;
$\mathbf{a}_A = (9.14 \text{ m/s}^2)\mathbf{i} - (40.6 \text{ m/s}^2)\mathbf{j}$;
$\mathbf{a}_B = -(10.16 \text{ m/s}^2)\mathbf{j} - (2.54 \text{ m/s}^2)\mathbf{k}$.

15.182 (*a*) $\mathbf{a}_B = (0.45\ \text{m/s}^2)\mathbf{j} - (1.979\ \text{m/s}^2)\mathbf{k}$.
(*b*) $\mathbf{a}_B = -(2.34\ \text{m/s}^2)\mathbf{i} + (0.346\ \text{m/s}^2)\mathbf{k}$.
(*c*) $\mathbf{a}_B = -(0.45\ \text{m/s}^2)\mathbf{j} + (2.67\ \text{m/s}^2)\mathbf{k}$.
15.184 $\mathbf{a}_B = -(3.03\ \text{m/s}^2)\mathbf{i} - (0.454\ \text{m/s}^2)\mathbf{k}$.
15.186 $\boldsymbol{\omega}_B = 40$ rpm ↻; $\boldsymbol{\omega}_C = 20$ rpm ↺.
15.188 (*a*) $\boldsymbol{\alpha}_{AB} = \boldsymbol{\alpha}_{BC} = 0$;
$\boldsymbol{\alpha}_{DB} = 1.333\ \text{rad/s}^2$ ↻.
(*b*) $\mathbf{a}_A = 0.8\ \text{m/s}^2 \downarrow$; $\mathbf{a}_B = 0.4\ \text{m/s}^2 \downarrow$.
15.190 (*a*) $\mathbf{a}_1 = -(302\ \text{ft/s}^2)\mathbf{i} - (66.6\ \text{ft/s}^2)\mathbf{j}$.
(*b*) $\mathbf{a}_2 = -(59.2\ \text{ft/s}^2)\mathbf{i} + (190.6\ \text{ft/s}^2)\mathbf{j}$.
SI: (*a*) $\mathbf{a}_1 = -(91.9\ \text{m/s}^2)\mathbf{i} - (20.3\ \text{m/s}^2)\mathbf{j}$.
(*b*) $\mathbf{a}_2 = -(18.05\ \text{m/s}^2)\mathbf{i} + (58.1\ \text{m/s}^2)\mathbf{j}$.
15.192 (*a*) $\boldsymbol{\omega} = 1.996$ rad/s ↺;
$\boldsymbol{\alpha} = 1.068\ \text{rad/s}^2$ ↻.
(*b*) $\mathbf{v}_B = 5.63$ m/s $\angle 40°$;
$\mathbf{a}_B = 8.25\ \text{m/s}^2\ \angle 40°$.
15.194 $\boldsymbol{\omega} = 2.25$ rad/s ↺. $\boldsymbol{\alpha} = 23.3\ \text{rad/s}^2$ ↻.
15.196 $\mathbf{v}_B = 7.85$ ft/s $\leftarrow$; $\mathbf{a}_B = 92.7\ \text{ft/s}^2 \rightarrow$.
SI: $\mathbf{v}_B = 2.39$ m/s $\leftarrow$; $\mathbf{a}_B = 28.3\ \text{m/s}^2 \rightarrow$.

CHAPTER 16

16.2 (*a*) $5\ \text{m/s}^2 \leftarrow$. (*b*) $\mathbf{B} = 41.6\ \text{N} \uparrow$;
$\mathbf{C} = 36.9\ \text{N} \uparrow$.
16.4 (*a*) 3.75 lb. (*b*) $\mathbf{A} = 1.194\ \text{lb} \rightarrow$;
$\mathbf{B} = 1.194\ \text{lb} \leftarrow$.
SI: (*a*) 16.68 N. (*b*) $\mathbf{A} = 5.31\ \text{N} \rightarrow$;
$\mathbf{B} = 5.31\ \text{N} \leftarrow$.
16.6 (*a*) $0.297g$. (*b*) 5.
16.8 (*a*) $2.55\ \text{m/s}^2 \rightarrow$. (*b*) $h \leqslant 1.047$ m.
16.10 (*a*) 3710 N $\uparrow$. (*b*) 1411 N $\uparrow$.
16.12 (*a*) $25.8\ \text{ft/s}^2$. (*b*) $12.27\ \text{ft/s}^2$.
(*c*) $13.32\ \text{ft/s}^2$.
SI: (*a*) $7.85\ \text{m/s}^2$. (*b*) $3.74\ \text{m/s}^2$.
(*c*) $4.06\ \text{m/s}^2$.
16.14 (*a*) 43.2 kN. (*b*) $8.38\ \text{m/s}^2 \nwarrow$.
16.16 (*a*) $7.99\ \text{ft/s}^2$. (*b*) $N_B = 101.8$ lb,
$F_B = 10.18$ lb.
SI: (*a*) $2.43\ \text{m/s}^2$. (*b*) $N_B = 453$ N,
$F_B = 45.3$ N.
16.18 $F_{CE} = 8.72$ N *C*; $F_{DF} = 15.80$ N *C*.
16.20 $\mathbf{A} = 10.77$ lb $\angle 30°$;
$\mathbf{B} = 0.774$ lb $\angle 30°$.
SI: $\mathbf{A} = 47.9$ N $\angle 30°$;
$\mathbf{B} = 3.44$ N $\angle 30°$.
16.22 1381 N.
16.24 (*a*) $0.500g\ \angle 30°$. (*b*) Platform:
$1.250g\ \angle 30°$; block: $0.625g \downarrow$.
16.26 $V_B = -6.13$ N; $M_B = -3.07$ N · m.
16.30 89.6 N · m.
16.32 $45.1\ \text{rad/s}^2$ ↻.
16.34 (1) $19.62\ \text{rad/s}^2$ ↺; 39.2 rad/s ↺;
19.81 rad/s ↺. (2) $14.01\ \text{rad/s}^2$ ↺;
28.0 rad/s ↺; 16.74 rad/s ↺.
(3) $6.54\ \text{rad/s}^2$ ↺; 13.08 rad/s ↺;
11.44 rad/s ↺. (4) $10.90\ \text{rad/s}^2$ ↺;
21.8 rad/s ↺; 10.44 rad/s ↺.
16.36 (*a*) $5.66\ \text{ft/s}^2 \downarrow$. (*b*) 8.24 ft/s $\downarrow$.
SI: (*a*) $1.725\ \text{m/s}^2 \downarrow$. (*b*) 2.51 m/s $\downarrow$.
16.38 $4.56\ \text{rad/s}^2$.
16.40 73.1 lb. SI: 325 N.
16.42 9.44 N.
16.44 (*a*) $\boldsymbol{\alpha}_A = 8.48\ \text{rad/s}^2$ ↺;
$\boldsymbol{\alpha}_B = 39.2\ \text{rad/s}^2$ ↺. (*b*) $\mathbf{C} = 66.7\ \text{N} \uparrow$,
$\mathbf{M}_C = 2.12$ N · m ↺.
16.46 (*a*) $\boldsymbol{\alpha}_A = 12.36\ \text{rad/s}^2$ ↺;
$\boldsymbol{\alpha}_B = 51.5\ \text{rad/s}^2$ ↺.
(*b*) $\boldsymbol{\omega}_A = 206$ rpm ↻; $\boldsymbol{\omega}_B = 343$ rpm ↺.
16.48 $I_R = \left(n + \dfrac{1}{n}\right)^2 I_0 + n^4 I_C$.
16.52 (*a*) $16.10\ \text{rad/s}^2$ ↺. (*b*) $8.05\ \text{ft/s}^2 \rightarrow$.
(*c*) 12 in. from *B*.
SI: (*a*) $16.10\ \text{rad/s}^2$ ↺. (*b*) $2.45\ \text{m/s}^2 \rightarrow$.
(*c*) 0.305 m from *B*.
16.54 (*a*) $-(2.37\ \text{rad/s}^2)\mathbf{j}$; 0.
(*b*) $-(1.778\ \text{rad/s}^2)\mathbf{j}$; $-(0.200\ \text{m/s}^2)\mathbf{i}$.
16.56 $T_A = 359$ lb; $T_B = 312$ lb.
SI: $T_A = 1595$ N; $T_B = 1388$ N.
16.58 $T_A = 1348$ N; $T_B = 1138$ N.
16.60 (*a*) $12\ \text{m/s}^2 \uparrow$. (*b*) $48\ \text{rad/s}^2$ ↻.
(*c*) $36\ \text{m/s}^2 \uparrow$.
16.62 (*a*) *W*. (*b*) $rg/\bar{k}^2$ ↺.
16.64 (*a*) $3g/L$ ↻. (*b*) $g \uparrow$. (*c*) $2g \downarrow$.
16.66 (*a*) $\bar{\mathbf{a}} = \mu g \leftarrow$; $\boldsymbol{\alpha} = 5\mu g/2r$ ↻.
(*b*) $2v_0/7\mu g$. (*c*) $12v_0^2/49\mu g$.
(*d*) $\bar{\mathbf{v}} = 5v_0/7 \rightarrow$; $\boldsymbol{\omega} = 5v_0/7r$ ↻.
16.68 $P = 4\mu W/\sqrt{58}$.
16.72 (*a*) 150 mm. (*b*) $125.0\ \text{rad/s}^2$ ↻.
16.74 (*a*) $3Pg/WL$ ↻. (*b*) $\mathbf{A}_x = \frac{1}{2}P \leftarrow$,
$\mathbf{A}_y = W \uparrow$.
16.76 $\frac{1}{2}(m/l)\omega^2(l^2 - x^2)$.
16.78 (*a*) 1529 kg. (*b*) 2.90 mm.
16.80 (*a*) $4W/7 \uparrow$. (*b*) $3g/7 \uparrow$.
16.82 (*a*) $0.750g/l$ ↺. (*b*) $0.275g/l$ ↺.
16.84 (*a*) $20.6\ \text{rad/s}^2$ ↻. (*b*) $\mathbf{A}_x = 48.3\ \text{N} \leftarrow$,
$\mathbf{A}_y = 39.3\ \text{N} \uparrow$.
16.86 (*a*) $34.8\ \text{rad/s}^2$ ↺.
(*b*) $\mathbf{A} = 66.6$ lb $\angle 60.9°$.
SI: (*b*) $\mathbf{A} = 296$ N $\angle 60.9°$.

16.88 (*a*) $3g/4L \circlearrowright$. (*b*) $\mathbf{N} = 13W/16 \uparrow$; $\mathbf{F} = 3\sqrt{3}W/16 \rightarrow$. (*c*) 0.400.
16.92 2.91 ft. SI: 0.887 m.
16.94 1.266 m.
16.96 (*a*) 24 rad/s$^2 \circlearrowright$; 3.84 m/s$^2 \rightarrow$. (*b*) 0.016.
16.98 (*a*) 8 rad/s$^2 \circlearrowleft$; 1.280 m/s$^2 \leftarrow$. (*b*) 0.220.
16.100 (*a*) Does not slide. (*b*) 23.2 rad/s$^2 \circlearrowright$; 15.46 ft/s$^2 \rightarrow$. SI: (*b*) 23.2 rad/s$^2 \circlearrowright$; 4.71 m/s$^2 \rightarrow$.
16.102 (*a*) Slides. (*b*) 12.88 rad/s$^2 \circlearrowleft$; 3.22 ft/s$^2 \leftarrow$. SI: (*b*) 12.88 rad/s$^2 \circlearrowleft$; 0.981 m/s$^2 \leftarrow$.
16.104 3.58 ft/s^2. SI: 1.091 m/s^2.
16.106 (*a*) $g/4r$. (*b*) $g\sqrt{2}/4 \measuredangle 45°$.
16.108 23.6 rad/s$^2 \circlearrowleft$.
16.110 (*a*) 28.0 N. (*b*) $\mathbf{A} = 9.25$ N $\leftarrow$; $\mathbf{B} = 50.0$ N $\uparrow$.
16.112 (*a*) 8.18 rad/s$^2 \circlearrowright$. (*b*) $\mathbf{A} = 12.74$ N $\leftarrow$; $\mathbf{B} = 31.9$ N $\uparrow$.
16.114 (*a*) 13.23 rad/s$^2 \circlearrowright$. (*b*) $\mathbf{A} = 1.375$ lb $\uparrow$; $\mathbf{B} = 1.460$ lb $\measuredangle 30°$. SI: (*b*) $\mathbf{A} = 6.12$ N $\uparrow$; $\mathbf{B} = 6.50$ N $\measuredangle 30°$.
16.118 $\mathbf{A} = 105.9$ lb $\leftarrow$; $\mathbf{B} = 200$ lb $\rightarrow$. SI: $\mathbf{A} = 471$ N $\leftarrow$; $\mathbf{B} = 890$ N $\rightarrow$.
16.120 (*a*) $\boldsymbol{\alpha}_{AB} = 3.77$ rad/s$^2 \circlearrowright$; $\boldsymbol{\alpha}_{BC} = 3.77$ rad/s$^2 \circlearrowleft$. (*b*) $\mathbf{A}_x = 15.68$ N $\rightarrow$, $\mathbf{A}_y = 43.8$ N $\uparrow$; $\mathbf{C} = 30.2$ N $\uparrow$.
16.122 $\mathbf{A}_x = \frac{1}{4}mr^2\omega_0^2 \leftarrow$, $\mathbf{A}_y = 2mgr \uparrow$; $\mathbf{B}_x = 0$, $\mathbf{B}_y = mgr \downarrow$.
16.124 (1*a*) $a/3 \rightarrow$. (1*b*) $3d/2$. (2*a*) $2a/7 \rightarrow$. (2*b*) $7d/5$.
16.126 (*a*) 74.4 rad/s$^2 \circlearrowleft$. (*b*) 24.8 ft/s$^2 \downarrow$. SI: (*b*) 7.56 m/s$^2 \downarrow$.
16.128 13.82 N $\measuredangle 26.6°$.
16.130 (*a*) 12.14 rad/s$^2 \circlearrowright$. (*b*) 11.21 m/s$^2 \measuredangle 30°$. (*c*) 14.56 N $\measuredangle 60°$.
16.132 $\boldsymbol{\alpha}_{AB} = 24.5$ rad/s$^2 \circlearrowright$; $\boldsymbol{\alpha}_{BC} = 122.7$ rad/s$^2 \circlearrowleft$.
16.134 (*a*) $\frac{1}{38}g \uparrow$. (*b*) $\frac{5}{38}g \downarrow$.
16.136 (On *AB*) 2.25 lb · ft $\circlearrowleft$. SI: (On *AB*) 3.05 N · m $\circlearrowleft$.
16.138 $V_{\max} = \frac{1}{3}mg$ at *A*; $M_{\max} = 4mgL/81$ at $\frac{1}{3}L$ to right of *A*.
16.140 (*a*) $\mathbf{a}_x = 0.3g \leftarrow$, $\mathbf{a}_y = 0.6g \downarrow$. (*b*) $\mathbf{a} = 0.630g \downarrow$.
16.142 (*a*) 21.5 ft/s$^2 \rightarrow$. (*b*) 15.95 ft/s$^2 \rightarrow$. SI: (*a*) 6.55 m/s$^2 \rightarrow$. (*b*) 4.86 m/s$^2 \rightarrow$.
16.144 (*a*) 1.634. (*b*) $0.1925m\omega^2r^3$.
16.146 6.33 in. SI: 160.8 mm.
16.148 $(v_0^2/2g)(\mu - \tan\theta)/\cos\theta\,(\frac{7}{2}\mu - \tan\theta)^2$.
16.150 (*a*) $\mathbf{a}_A = 2g/5 \leftarrow$; $\mathbf{a}_B = 2g/5 \downarrow$. (*b*) $\mathbf{a}_A = 2g/7 \leftarrow$; $(\mathbf{a}_B)_x = 2g/7 \leftarrow$, $(\mathbf{a}_B)_y = 2g/7 \downarrow$.

CHAPTER 17

17.2 71.6 N · m.
17.4 8.27 in. SI: 210 mm.
17.6 (*a*) 294 rpm. (*b*) 15.92 rev.
17.8 $\mathbf{v}_A = 1.293$ m/s $\uparrow$; $\mathbf{v}_B = 2.59$ m/s $\downarrow$.
17.12 61.8 rev.
17.14 338 N $\uparrow$.
17.16 (*a*) 2.40 rev. (*b*) 21.4 N $\swarrow$.
17.18 1.541 m.
17.20 (*a*) $1.074\sqrt{g/r}$. (*b*) $1.433mg \uparrow$.
17.22 (*a*) $\sqrt{\frac{4}{3}g(R - r)(1 - \cos\beta)}$. (*b*) $mg(7 - 4\cos\beta)/3$.
17.24 (*a*) $l/\sqrt{12}$. (*b*) $1.861\sqrt{g/l}$.
17.26 5.75 ft/s $\leftarrow$. SI: 1.752 m/s $\leftarrow$.
17.28 (*a*) 13.45 rad/s. (*b*) 20.4 rad/s.
17.30 6.55 ft/s $\leftarrow$. SI: 1.997 m/s $\leftarrow$.
17.32 (*a*) $\mathbf{v}_A = 1.922$ m/s $\downarrow$; $\mathbf{v}_B = 3.20$ m/s $\measuredangle 36.9°$. (*b*) $\mathbf{v}_A = \mathbf{v}_B = 2.87$ m/s $\leftarrow$.
17.34 (*a*) $\mathbf{v}_A = 1.332$ m/s $\rightarrow$; $\mathbf{v}_B = 0.769$ m/s $\downarrow$. (*b*) $\mathbf{v}_A = 0$; $\mathbf{v}_B = 4.20$ m/s $\downarrow$.
17.36 14.63 rad/s $\circlearrowright$.
17.38 7.67 rad/s $\circlearrowleft$.
17.40 36.4°.
17.42 (*a*) Zero. (*b*) 188.5 W.
17.44 (*a*) 0.365 lb · ft. (*b*) 1.824 lb · ft. SI: (*a*) 0.495 N · m. (*b*) 2.47 N · m.
17.46 89.7 N · m.
17.48 1.000.
17.50 3.88 s.
17.54 (*a*) 3.33 N · m. (*b*) $\boldsymbol{\omega}_A = 23.5$ rad/s $\circlearrowright$; $\boldsymbol{\omega}_B = 39.2$ rad/s $\circlearrowleft$.
17.56 (*a*) 6.59 s. (*b*) 13.06 lb; 1.944 lb. (*c*) 0.61. SI: (*b*) 58.1 N; 8.65 N.
17.62 (*a*) 12 m/s $\rightarrow$. (*b*) 100 N $\leftarrow$.
17.64 (*a*) 32.2 ft/s $\rightarrow$. (*b*) Zero. SI: (*a*) 9.81 m/s $\rightarrow$. (*b*) Zero.
17.66 21.5 ft/s $\leftarrow$. SI: 6.54 m/s $\leftarrow$.
17.68 (*a*) $5\bar{v}_0/7 \rightarrow$. (*b*) $2\bar{v}_0/7\mu g$.
17.70 (*a*) $t_2 = 2r\omega_1/7\mu g$. (*b*) $\bar{\mathbf{v}}_2 = 2r\omega_1/7 \rightarrow$. $\boldsymbol{\omega}_2 = 2\omega_1/7 \circlearrowright$.
17.72 (*a*) 4.51 rad/s. (*b*) 9.09 ft · lb. SI: (*a*) 4.51 rad/s. (*b*) 12.32 J.

17.74 (*a*) 334 rpm. (*b*) −6.51 J.
17.76 (*a*) and (*b*) 5.71 rad/s.
17.78 24.4 rpm.
17.80 *Disk:* 287.4 rpm; *Arm:* 72.6 rpm.
17.82 $v_r = 3.97$ m/s; $v_\theta = 2.86$ m/s.
17.84 3.82 ft/s. SI: 1.164 m/s.
17.86 (*a*) 2.4 m/s $\rightarrow$. (*b*) 3.6 kN $\rightarrow$.
17.88 $\mathbf{v}_A = 1.920$ ft/s $\leftarrow$; $\mathbf{v}_B = 21.12$ ft/s $\rightarrow$.
SI: $\mathbf{v}_A = 0.585$ m/s $\leftarrow$; $\mathbf{v}_B = 6.44$ m/s $\rightarrow$.
17.90 (*a*) $\bar{\mathbf{v}}_1 = mv_0/M \rightarrow$; $\boldsymbol{\omega}_1 = mv_0/MR \circlearrowleft$.
(*b*) $mv_0/3M \rightarrow$.
17.92 $\frac{1}{2}\omega_1$.
17.94 $\boldsymbol{\omega}_2 = \frac{1}{3}\omega_1 \circlearrowleft$; $\bar{\mathbf{v}}_2 = \frac{2}{3}r\omega_1 \uparrow$.
17.96 $\boldsymbol{\omega} = \frac{3}{4}\bar{v}_1/b \circlearrowright$; $\bar{\mathbf{v}} = \frac{3}{8}\sqrt{2}\,\bar{v}_1 \measuredangle 45°$.
17.98 (*a*) $\boldsymbol{\omega} = \frac{1}{4}\omega_0 \circlearrowright$. (*b*) $\frac{15}{16}$. (*c*) 1.5°.
17.100 (*a*) $\mathbf{v}_A = 0$, $\boldsymbol{\omega}_A = \omega_0 \circlearrowright$; $\mathbf{v}_B = v_0 \rightarrow$,
$\boldsymbol{\omega}_B = 0$. (*b*) $\mathbf{v}_A = 2v_0/7 \rightarrow$,
$\boldsymbol{\omega}_A = 2\omega_0/7 \circlearrowright$; $\mathbf{v}_B = 5v_0/7 \rightarrow$,
$\boldsymbol{\omega}_B = 5\omega_0/7 \circlearrowright$. (*c*) The motion of part *a* is the final motion.
17.102 $\boldsymbol{\omega}_2 = \dfrac{\bar{v}_1}{l}\dfrac{6\sin\beta}{3\sin^2\beta + 1} \circlearrowright$.
17.104 (*a*) $0.9v_0/l \circlearrowright$. (*b*) $0.1v_0 \rightarrow$.
17.106 31.0° $\measuredangle$.
17.108 $\mathbf{A}\,\Delta t = m\sqrt{gl/3}$; $\mathbf{B}\,\Delta t = m\sqrt{gl/12}$.
17.110 $y^2 = (2v_0^2 \sin^2\theta/\mu g)x$.
17.112 (*a*) $\frac{5}{4}v_A/r$. (*b*) $1/\sqrt{3}$.
17.114 $\boldsymbol{\omega}_{AB} = \frac{5}{8}\omega_0 \circlearrowright$, $\bar{\mathbf{v}}_{AB} = \frac{1}{16}\omega_0 L \uparrow$;
$\boldsymbol{\omega}_{CD} = \frac{3}{8}\omega_0 \circlearrowleft$, $\bar{\mathbf{v}}_{CD} = \frac{1}{16}\omega_0 L \downarrow$.
17.116 $\sqrt{g/3r}$.
17.118 (*a*) 3.76 m/s $\measuredangle$ 45°. (*b*) 3.18 m/s $\downarrow$.
17.120 (*a*) 50.2°. (*b*) 16.3°.
17.122 (*a*) 210 lb · ft. (*b*) 70.0 lb · ft.
SI: (*a*) 285 N · m. (*b*) 94.9 N · m.
17.124 (*a*) $0.926\sqrt{gL} \leftarrow$. (*b*) $1.225\sqrt{gL} \leftarrow$.
17.126 (*a*) 4.75 m/s $\rightarrow$. (*b*) 3.87 m/s $\rightarrow$.

CHAPTER 18

18.2 $\frac{1}{4}mr^2(\omega_1\mathbf{i} + 2\omega_2\mathbf{k})$.
18.4 $(0.432\ \text{kg}\cdot\text{m}^2/\text{s})\mathbf{i} - (0.324\ \text{kg}\cdot\text{m}^2/\text{s})\mathbf{k}$.
18.6 $(112.8\ \text{g}\cdot\text{m}^2/\text{s})\mathbf{i} + (80\ \text{g}\cdot\text{m}^2/\text{s})\mathbf{j}$.
18.8 $-(0.699\ \text{ft}\cdot\text{lb}\cdot\text{s})\mathbf{i} + (0.699\ \text{ft}\cdot\text{lb}\cdot\text{s})\mathbf{j}$.
SI: $-(0.947\ \text{kg}\cdot\text{m}^2/\text{s})\mathbf{i} + (0.947\ \text{kg}\cdot\text{m}^2/\text{s})\mathbf{j}$.
18.10 (*a*) $\frac{2}{3}ma^3\omega(5\mathbf{i} - 3\mathbf{k})$. (*b*) 31.0°.
18.14 (*a*) $m\mathbf{v} = (240\ \text{Mg}\cdot\text{m/s})\mathbf{i} + (360\ \text{Mg}\cdot\text{m/s})\mathbf{j} + (72 \times 10^3\ \text{Mg}\cdot\text{m/s})\mathbf{k}$;
$\mathbf{H}_G = (432\ \text{Mg}\cdot\text{m}^2/\text{s})\mathbf{j} + (180\ \text{Mg}\cdot\text{m}^2/\text{s})\mathbf{k}$.
(*b*) 67.1°.
18.16 (*a*) $-(3.86\ \text{ft/s})\mathbf{k}$; SI: $-(1.177\ \text{m/s})\mathbf{k}$.
(*b*) $-(0.643\ \text{rad/s})\mathbf{i} + (0.497\ \text{rad/s})\mathbf{j}$.
18.18 (*a*) $-(F\,\Delta t/m)\mathbf{k}$.
(*b*) $(12F\,\Delta t/7ma)(-\mathbf{i} - 5\mathbf{j})$.
18.20 (*a*) 0. (*b*) $(3F\,\Delta t/md)(\mathbf{i} - \frac{1}{4}\mathbf{k})$.
18.22 (*a*) $(6F\,\Delta t/7ma)(\mathbf{i} - 7\mathbf{j})$. (*b*) Axis through *A*, in *xy* plane, forming $\measuredangle$81.9° with *x* axis.
18.24 (*a*) $\frac{1}{6}\omega_0(-\mathbf{i} + \mathbf{j})$. (*b*) $\frac{1}{6}\omega_0 a\mathbf{k}$.
18.26 (*a*) $\Delta t_A = 1.213$ s; $\Delta t_B = 0.558$ s.
(*b*) $\Delta\bar{\mathbf{v}} = (0.0886\ \text{m/s})\mathbf{k}$.
18.28 $(5.97\ \text{rpm})\mathbf{i} - (2.69\ \text{rpm})\mathbf{j} + (0.806\ \text{rpm})\mathbf{k}$.
18.32 $\sqrt{6g/5a}$.
18.34 0.864 J.
18.36 $-5ma^2\omega_0^2/48$.
18.38 −5.10 ft · lb. SI: −6.92 J.
18.40 $-\frac{1}{2}mr^2\omega_1\omega_2\mathbf{j}$.
18.42 $(1.296\ \text{N}\cdot\text{m})\mathbf{j}$.
18.44 $(0.864\ \text{N}\cdot\text{m})\mathbf{i} + (1.296\ \text{N}\cdot\text{m})\mathbf{j} - (0.648\ \text{N}\cdot\text{m})\mathbf{k}$.
18.46 $\mathbf{A} = (46.2\ \text{N})\mathbf{j}$; $\mathbf{D} = -(46.2\ \text{N})\mathbf{j}$.
18.48 $\mathbf{A} = \frac{1}{2}(w/g)a^2\omega^2\mathbf{k}$; $\mathbf{B} = -\mathbf{A}$.
18.50 (*a*) $\mathbf{M} = (0.647\ \text{lb}\cdot\text{ft})\mathbf{i}$.
(*b*) $\mathbf{A} = (0.388\ \text{lb})\mathbf{j}$; $\mathbf{B} = -(0.388\ \text{lb})\mathbf{j}$.
SI: (*a*) $\mathbf{M} = (0.877\ \text{N}\cdot\text{m})\mathbf{i}$.
(*b*) $\mathbf{A} = (1.727\ \text{N})\mathbf{j}$; $\mathbf{B} = -(1.727\ \text{N})\mathbf{j}$.
18.52 (*a*) $(4M_0/ma^2)\mathbf{j}$. (*b*) $\mathbf{R}_A = -(M_0\sqrt{2}/a)\mathbf{i}$;
$\mathbf{M}_A = \frac{2}{3}M_0\mathbf{k}$.
18.54 (*a*) $\mathbf{M} = (4.00\ \text{N}\cdot\text{m})\mathbf{i}$.
(*b*) $\mathbf{A} = -(19.49\ \text{N})\mathbf{j} + (8.66\ \text{N})\mathbf{k}$;
$\mathbf{B} = -\mathbf{A}$.
18.56 $(0.831\ \text{N}\cdot\text{m})\mathbf{i}$.
18.58 22.7 lb · ft.
SI: 30.7 N · m.
18.60 (*a*) $2mr^3\omega^2$. (*b*) 0. (*c*) $\frac{1}{2}mr^3\omega^2$.
18.62 (*a*) $\cos\beta = 2g/3a\omega^2$. (*b*) $\sqrt{2g/3a}$.
18.64 $\mathbf{F} = -mR\omega_1^2\mathbf{i}$;
$\mathbf{M}_0 = \frac{1}{2}mr^2\omega_1\omega_2\mathbf{i} - mRh\omega_1^2\mathbf{k}$.
18.66 (*a*) $\sqrt{g/a}$. (*b*) $\sqrt{2g/a}$.
18.68 $\mathbf{D} = -(0.622\ \text{N})\mathbf{j} - (4.00\ \text{N})\mathbf{k}$.
$\mathbf{E} = (3.82\ \text{N})\mathbf{j} - (4.00\ \text{N})\mathbf{k}$.
18.70 4450 rpm.
18.72 3690 rpm.
18.76 (*a*) 36.1 rad/s; 7.40 rad/s. (*b*) −0.169.
18.78 (*a*) 2.75 rpm. (*b*) 2.77 rpm; 397 rpm.
18.84 Precession axis: $\measuredangle$30°; precession, 6.00 rad/s; spin, 10.39 rad/s.
18.86 Precession axis: $\theta_x = 39.9°$, $\theta_y = 127.9°$, $\theta_z = 79.4°$; precession, 4.38 rpm; spin, 2.61 rpm.

18.88 Precession axis: $\theta_x = 90°$, $\theta_y = 58.0°$, $\theta_z = 32.0°$; precession, 1.126 rpm (retrograde); spin, 0.343 rpm.
18.90 14.01 rev/h.
18.92 (*a*) $\omega = 22.7$ rpm, $\gamma = 57.3°$. (*b*) $\theta = 75.6°$. (*c*) Precession, 19.72 rpm; spin, 7.36 rpm.
18.94 (*a*) $\beta = 23.8°$. (*b*) Precession, 74.3 rpm; spin, 115.9 rpm.
18.96 $3\sqrt{g/2L}$.
18.100 $\dot\phi = 4\dot\psi_0/15$; $\dot\psi = 17\dot\psi_0/15$.
18.102 (*a*) $5\sqrt{3g/2a}$. (*b*) $\dot\phi = \sqrt{3g/2a}$; $\dot\psi = 5\sqrt{3g/2a}$.
18.108 (*a*) 27.8 rad/s^2. (*b*) $\mathbf{A} = (3.35\text{ N})\mathbf{j} + (12.08\text{ N})\mathbf{k}$; $\mathbf{C} = -\mathbf{A}$.
18.110 (*a*) $-\frac{3}{7}\bar{v}_0\left(\frac{1}{c}\mathbf{i} + \frac{1}{a}\mathbf{k}\right)$. (*b*) $-\frac{6}{7}\bar{v}_0\mathbf{j}$.
18.112 (*a*) $-(1.250\text{ m/s})\mathbf{k}$. (*b*) $(1.657\text{ rad/s})(\mathbf{i} + 3\mathbf{j})$.
18.114 (*a*) $(42.4\text{ rpm})\mathbf{j} + (64.2\text{ rpm})\mathbf{k}$. (*b*) 2800 ft · lb. SI: (*b*) 3790 J.
18.116 (*a*) Tangent of angle $= \dfrac{\tan\beta}{1 + 2\tan^2\beta}$. (*b*) $-2\dot\psi\sec\beta$. (*c*) 9.4°; $-2.03\dot\psi$.
18.118 (*a*) $F_{BC} = 0.789mg$; $T = 0.700mga$. (*b*) $\dot\psi = 13.66\sqrt{g/a}$; $T = 42.8mga$.

CHAPTER 19

19.2 (*a*) 0.1900 m. (*b*) 2.39 m/s.
19.4 (*a*) 2.49 mm; 0.0979 in. (*b*) 0.621 mm; 0.0245 in.
19.6 (*a*) 0.497 s. (*b*) 0.632 m/s. (*c*) 8.00 m/s^2.
19.8 (*a*) 0.1348 s. (*b*) 2.24 ft/s ↑; 20.1 ft/s^2 ↓. SI: (*b*) 0.683 m/s ↑; 6.13 m/s^2 ↓.
19.10 (*a*) 0.679 s; 1.473 Hz. (*b*) 0.1852 m/s; 1.714 m/s^2.
19.12 (*a*) 4.53 lb. (*b*) 0.583 s. SI: (*a*) 2.06 kg (mass). (*b*) 0.583 s.
19.14 (*a*) 0.994 m. (*b*) 3.67°.
19.16 1.400 ft. SI: 0.427 m.
19.18 (*a*) 7.90 lb. (*b*) 85.3 lb. SI: (*a*) 3.58 kg. (*b*) 38.7 kg.
19.20 1.904 Hz.
19.24 16.3°.
19.26 (*a*) 0.440 s. (*b*) 2.38 ft/s. SI: (*a*) 0.440 s. (*b*) 0.725 m/s.
19.28 (*a*) 0.907 s. (*b*) 0.346 m/s.
19.30 (*a*) 0.533 s. (*b*) 0.491 rad/s.
19.32 $f = (1/2\pi)\sqrt{3k/m}$.
19.38 (*a*) $l/\sqrt{12}$. (*b*) $4.77\sqrt{l/g}$.
19.40 (*a*) 2.28 s. (*b*) 1.294 m.
19.42 (*a*) $\tau = 2\pi\sqrt{5b/6g}$. (*b*) $c = \frac{1}{4}b$.
19.44 (*a*) $r_a = 7.09$ in. (*b*) 3.42 in. SI: (*a*) $r_a = 180.0$ mm. (*b*) 86.9 mm.
19.46 (*a*) 5.54 s. (*b*) 3.57 ft/s. SI: (*a*) 5.54 s. (*b*) 1.087 m/s.
19.48 0.658 kg · m^2.
19.50 $\tau = 2\pi\sqrt{l/g}$.
19.58 $\tau = 2\pi\sqrt{7l/6g}$.
19.62 $f = (1/2\pi)\sqrt{g/2l}$.
19.64 9.90 s.
19.68 $\tau = 2\pi\sqrt{m/3k\cos^2\beta}$.
19.70 $\tau = 2\pi\sqrt{m_c/k\cos^2\beta}$.
19.72 (*a*) 1.107 s. (*b*) 1.429 s.
19.74 (*a*) 0.777 s. (*b*) 1.099 s.
19.76 $\tau = \pi l/\sqrt{3gr}$.
19.78 $\omega > \sqrt{2k/m}$.
19.80 $\omega > \sqrt{2g/l}$.
19.82 (*a*) 168.0 rpm. (*b*) 0.00131 in. SI: (*a*) 168.0 rpm. (*b*) 33.3 μm.
19.84 (*a*) 11.38 μm. (*b*) 320 μm. (*c*) ∞.
19.86 0.0857 in. or 0.120 in. SI: 2.18 mm or 3.05 mm
19.88 1007 rpm.
19.90 1085 rpm.
19.92 109.5 rpm.
19.94 1.200 mm.
19.96 70.1 km/h.
19.104 (*a*) $x = x_0 e^{-pt}(1 + pt)$. (*b*) 0.1147 s.
19.106 $\sqrt{1 - 2(c/c_c)^2}$.
19.108 (*a*) 1.509 mm. (*b*) 0.583 mm.
19.110 0.1791 in. SI: 4.55 mm.
19.112 (*a*) 270 rpm. (*b*) 234 rpm. (*c*) 8.84 mm; 9.45 mm.
19.114 $m\ddot{x}_A + 5kx_A - 2kx_B = 0$; $m\ddot{x}_B - 2kx_A + 2kx_B = P_m \sin\omega t$.
19.116 (*a*) E/R. (*b*) L/R.
19.122 (*a*) $m\ddot{x}_m + k_2x_m + c(\dot{x}_m - \dot{x}_A) = 0$; $c(\dot{x}_A - \dot{x}_m) + k_1x_A = 0$

(*b*) $L\ddot{q}_m + \dfrac{q_m}{C_2} + R(\dot{q}_m - \dot{q}_A) = 0$;

$R(\dot{q}_A - \dot{q}_m) + \dfrac{q_A}{C_1} = 0.$

19.124 (a) $m_1\ddot{x}_1 + c_1\dot{x}_1 + c_2(\dot{x}_1 - \dot{x}_2) + k_1x_1 + k_2(x_1 - x_2) = 0;$
$m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + c_3\dot{x}_2 + k_2(x_2 - x_1) + k_3x_2 = P_m \sin \omega t.$
(b) $L_1\ddot{q}_1 + R_1\dot{q}_1 + R_2(\dot{q}_1 - \dot{q}_2) + \dfrac{q_1}{C_1} + \dfrac{q_1 - q_2}{C_2} = 0;$
$L_2\ddot{q}_2 + R_2(\dot{q}_2 - \dot{q}_1) + R_3\dot{q}_2 + \dfrac{(q_2 - q_1)}{C_2} + \dfrac{q_2}{C_3} = E_m \sin \omega t.$

19.126 1.363 s.

19.128 0.760 lb · ft · s^2; 8.66 in. SI: 1.030 kg · m^2; 0.220 m.

19.130 (a) 5.81 Hz; 4.91 mm; 179.2 mm/s. (b) 491 N. (c) 159.2 mm/s ↑ .

19.132 12.58 Hz.

19.134 72.5 μm.

19.136 1.346 s.

APPENDIX B

9.72 (a) $\frac{1}{4}ma^2$, $\frac{1}{4}mb^2$. (b) $\frac{1}{4}m(a^2 + b^2)$.

9.74 (a) $I_{AA'} = mb^2/24$; $I_{BB'} = mh^2/18$. (b) $I_{CC'} = m(3b^2 + 4h^2)/72$.

9.76 $m(3a^2 + L^2)/12$.

9.78 $\frac{1}{3}ma^2$; $a/\sqrt{3}$.

9.80 $5ma^2/18$

9.82 $m(2b^2 + h^2)/10$.

9.84 $2mr^2/3$.

9.86 1.514 kg · m^2; 155.7 mm.

9.88 (a) $md^2/6$. (b) $2md^2/3$. (c) $2md^2/3$.

9.90 (a) 5.14×10^{-3} kg · m^2. (b) 7.54×10^{-3} kg · m^2. (c) 3.47×10^{-3} kg · m^2.

9.92 0.0503 lb · ft · s^2; 3.73 in. SI: 0.0682 kg · m^2; 94.8 mm.

9.94 (a) 20.2 lb · ft · s^2. (b) 42.1 lb · ft · s^2. (c) 41.3 lb · ft · s^2. SI: (a) 27.4 kg · m^2. (b) 57.1 kg · m^2. (c) 56.0 kg · m^2.

9.96 $P_{xy} = -0.001199$ kg · m^2, $P_{yz} = P_{zx} = 0$.

9.98 $P_{xy} = 7.02$ lb · ft · s^2, $P_{yz} = P_{zx} = 0$. SI: $P_{xy} = 9.52$ kg · m^2, $P_{yz} = P_{zx} = 0$.

9.100 (a) $P_{zx} = mca/20$. (b) $P_{xy} = mab/20$; $P_{yz} = mbc/20$.

9.102 $3ma^2(a^2 + 6h^2)/20(a^2 + h^2)$.

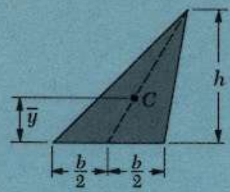
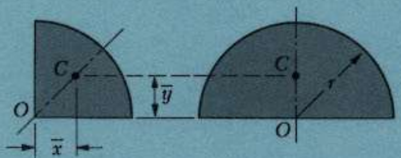
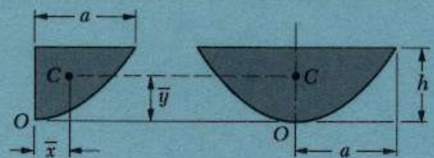
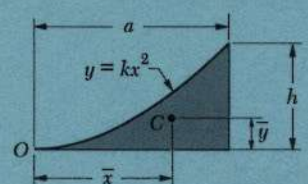
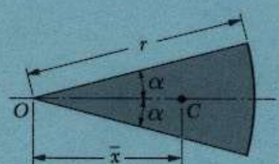
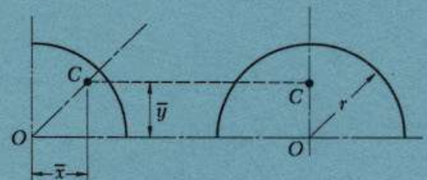
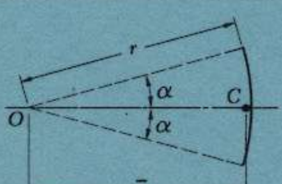
9.104 29.9 lb · ft · s^2. SI: 40.5 kg · m^2.

9.106 (a) $I_x = 2ma^2/3$, $I_y = I_z = 11ma^2/12$, $P_{xy} = ma^2/4$, $P_{yz} = 0$, $P_{zx} = -ma^2/4$. (b) $2ma^2/3$.

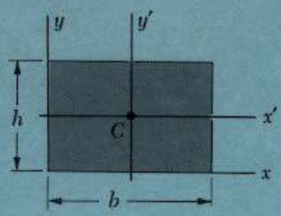
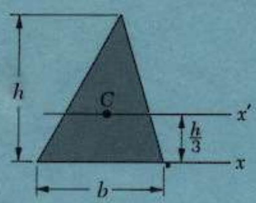
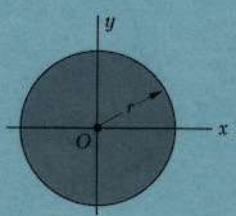
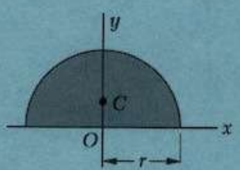
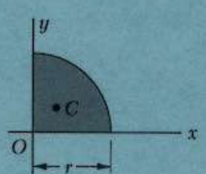
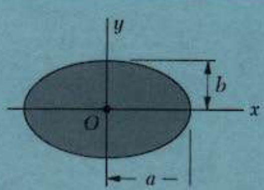
9.108 $0.426\,ma^2$.

9.110 (a) 2. (b) 0.5.

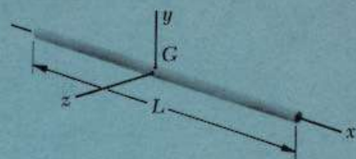
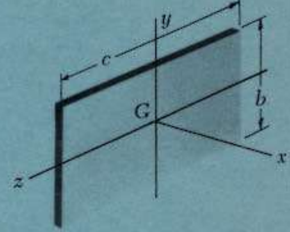
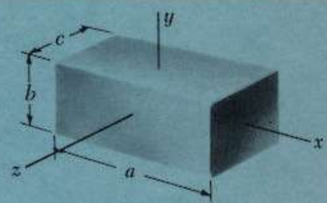
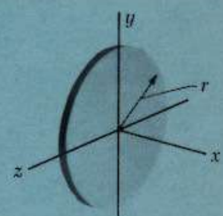
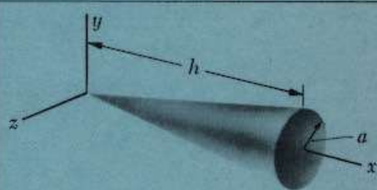
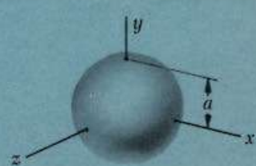
Centroids of Common Shapes of Areas and Lines

Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
Circular sector		$\frac{2r \sin \alpha}{3\alpha}$	0	αr^2
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r \sin \alpha}{\alpha}$	0	$2\alpha r$

Moments of Inertia of Common Geometric Shapes

Rectangle $\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$	
Triangle $\bar{I}_{x'} = \frac{1}{36}bh^3$ $I_x = \frac{1}{12}bh^3$	
Circle $\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_O = \frac{1}{2}\pi r^4$	
Semicircle $I_x = I_y = \frac{1}{8}\pi r^4$ $J_O = \frac{1}{4}\pi r^4$	
Quarter circle $I_x = I_y = \frac{1}{16}\pi r^4$ $J_O = \frac{1}{8}\pi r^4$	
Ellipse $\bar{I}_x = \frac{1}{4}\pi ab^3$ $\bar{I}_y = \frac{1}{4}\pi a^3b$ $J_O = \frac{1}{4}\pi ab(a^2 + b^2)$	

Mass Moments of Inertia of Common Geometric Shapes

Slender rod $I_y = I_z = \frac{1}{12}mL^2$	
Thin rectangular plate $I_x = \frac{1}{12}m(b^2 + c^2)$ $I_y = \frac{1}{12}mc^2$ $I_z = \frac{1}{12}mb^2$	
Rectangular prism $I_x = \frac{1}{12}m(b^2 + c^2)$ $I_y = \frac{1}{12}m(c^2 + a^2)$ $I_z = \frac{1}{12}m(a^2 + b^2)$	
Thin disk $I_x = \frac{1}{2}mr^2$ $I_y = I_z = \frac{1}{4}mr^2$	
Circular cylinder $I_x = \frac{1}{2}ma^2$ $I_y = I_z = \frac{1}{12}m(3a^2 + L^2)$	
Circular cone $I_x = \frac{3}{10}ma^2$ $I_y = I_z = \frac{3}{5}m(\frac{1}{4}a^2 + h^2)$	
Sphere $I_x = I_y = I_z = \frac{2}{5}ma^2$	

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