TRANSIENT ANALYSIS OF ELECTRIC POWER CIRCUITS HANDBOOK

Transient Analysis of Electric Power Circuits Handbook

by

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Springer

A C.I.P. Catalogue record for this book is available from the Library of Congress.

ISBN-10 0-387-28797-3 (HB)
ISBN-13 978-0-387-28797-3 (HB)
ISBN-10 0-387-28799-X (e-book)
ISBN-13 978-0-387-28799-7 (e-book)

Published by Springer,
P.O. Box 17, 3300 AA Dordrecht, The Netherlands.

www.springeronline.com

Printed on acid-free paper

Printed in the Netherlands.

This book is dedicated to my dearest wife Iris
and wonderful children
Daniel, Elana and Joella

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FOREWORD

Every now and then, a good book comes along and quite rightfully makes itself a distinguished place among the existing books of the electric power engineering literature. This book by Professor Arieh Shenkman is one of them.

Today, there are many excellent textbooks dealing with topics in power systems. Some of them are considered to be classics. However, many of them do not particularly address, nor concentrate on, topics dealing with transient analysis of electrical power systems.

Many of the fundamental facts concerning the transient behavior of electric circuits were well explored by Steinmetz and other early pioneers of electrical power engineering. Among others, *Electrical Transients in Power Systems* by Allan Greenwood is worth mentioning. Even though basic knowledge of transients may not have advanced in recent years at the same rate as before, there has been a tremendous proliferation in the techniques used to study transients. The application of computers to the study of transient phenomena has increased both the knowledge as well as the accuracy of calculations.

Furthermore, the importance of transients in power systems is receiving more and more attention in recent years as a result of various blackouts, brownouts, and recent collapses of some large power systems in the United States, and other parts of the world. As electric power consumption grows exponentially due to increasing population, modernization, and industrialization of the so-called third world, this topic will be even more important in the future than it is at the present time.

Professor Arieh Shenkman is to be congratulated for undertaking such an important task and writing this book that singularly concentrates on the topics related to the transient analysis of electric power systems. The book successfully fills the long-existing gap in such an important area.

Turan Gonen, Ph.D., Fellow IEEE
Professor and Director
Electric Power Educational Institute
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PREFACE

Most of the textbooks on electrical and electronic engineering only partially cover the topic of transients in simple *RL*, *RC* and *RLC* circuits and the study of this topic is primarily done from an electronic engineer's viewpoint, i.e., with an emphasis on low-current systems, rather than from an electrical engineer's viewpoint, whose interest lies in high-current, high-voltage power systems. In such systems a very clear differentiation between steady-state and transient behavior of circuits is made. Such a division is based on the concept that steady-state behavior is normal and transients arise from the faults. The operation of most electronic circuits (such as oscillators, switch capacitors, rectifiers, resonant circuits etc.) is based on their transient behavior, and therefore the transients here can be referred to as "desirable". The transients in power systems are characterized as completely "undesirable" and should be avoided; and subsequently, when they do occur, in some very critical situations, they may result in the electrical failure of large power systems and outages of big areas. Hence, the Institute of Electrical and Electronic Engineers (IEEE) has recently paid enormous attention to the importance of power engineering education in general, and transient analysis in particular.

It is with the belief that transient analysis of power systems is one of the most important topics in power engineering analysis that the author proudly presents this book, which is wholly dedicated to this topic.

Of course, there are many good books in this field, some of which are listed in the book; however they are written on a specific technical level or on a high theoretical level and are intended for top specialists. On the other hand, introductory courses, as was already mentioned, only give a superficial knowledge of transient analysis. So that there is a gap between introductory courses and the above books.

The present book is designed to fill this gap. It covers the topic of transient analysis from simple to complicated, and being on an intermediate level, this book therefore is a link between introductory courses and more specific technical books. In the book the most important methods of transient analysis, such as the classical method, Laplace and Fourier transforms and state variable analysis are presented; and of course, the emphasis on transients in three-phase systems and transmission lines is made.

The appropriate level and the concentration of all the topics under one cover make this book very special in the field under consideration. The author believes that this book will be very helpful for all those specializing in electrical engineering and power systems. It is recommended as a textbook for specialized undergraduate and graduate curriculum, and can also be used for master and doctoral studies. Engineers in the field may also find this book useful as a handbook and/or resource book that can be kept handy to review specific points. Theoreticians/researchers who are looking for the mathematical background of transients in electric circuits may also find this book helpful in their work.

The presentation of the covered material is geared to readers who are being exposed to (a) the basic concept of electric circuits based on their earlier study of physics and/or introductory courses in circuit analysis, and (b) basic mathematics, including differentiation and integration techniques

This book is composed of eight chapters. The study of transients, as mentioned, is presented from simple to complicated. Chapters 1 and 2 are dedicated to the classical method of transient analysis, which is traditional for many introductory courses. However, these two chapters cover much more material giving the mathematical as well as the physical view of transient behavior of electrical circuits. So-called incorrect initial conditions and two generalized commutation laws, which are important for a better understanding of the transient behavior of transformers and synchronous machines, are also discussed in Chapter 2.

Chapters 3 and 4 give the transform methods of transient analysis, introducing the Laplace as well as the Fourier transforms. What is common between these two methods and the differences are emphasized. The theoretical study of the transform methods is accompanied by many practical examples.

The state variable method is presented in Chapter 5. Although this method is not very commonly used in transient analysis, the author presumes that the topic of the book will not be complete without introducing this essential and interesting method. It should be noted that the state variable method in its matrix notation, which is given here, is very appropriate for transient analysis using computers.

Naturally, an emphasis and a great amount of material are dedicated to transients in three-phase circuits, which can be found in Chapter 6. As power systems are based on employing three-phase generators and transformers, the complete analysis of their behavior under short-circuit faults at both steady-state and first moment operations is given. The overvoltages following switching-off in power systems are also analyzed under the influence of the electric arc, which accompanies such switching.

In Chapter 7 the transient behavior of transmission lines is presented. The transmission line is presented as a network with distributed parameters and subsequently by partial differential equations. The transient analysis of such lines is done in two ways: as a method of traveling waves and by using the

Laplace transform. Different engineering approaches using both methods are discussed.

Finally, in Chapter 8 an overview of the static and dynamic stability of power systems is given. Analyzing system stability is done in traditional ways, i.e., by solving a swing equation and by using an equal area criterion.

Throughout the text, the theoretical discussions are accompanied by many worked-out examples, which will hopefully enable the reader to get a better understanding of the various concepts.

The author hopes that this book will be helpful to all readers studying and specializing in power system engineering, and of value to professors in the educational process and to engineers who are concerned with the design and R&D of power systems.

Last but not least, my sincere appreciation goes to my wife, Iris, who prodigiously supported and aided me throughout the writing of this book. I am also extremely grateful for her assistance in editing and typing in English.

Chapter #1

CLASSICAL APPROACH TO TRANSIENT ANALYSIS

1.1 INTRODUCTION

Transient analysis (or just transients) of electrical circuits is as important as steady-state analysis. When transients occur, the currents and voltages in some parts of the circuit may many times exceed those that exist in normal behavior and may destroy the circuit equipment in its proper operation. We may distinguish the transient behavior of an electrical circuit from its steady-state, in that during the transients all the quantities, such as currents, voltages, power and energy, are changed in time, while in steady-state they remain invariant, i.e. constant (in d.c. operation) or periodical (in a.c. operation) having constant amplitudes and phase angles.

The cause of transients is any kind of changing in circuit parameters and/or in circuit configuration, which usually occur as a result of switching (commutation), short, and/or open circuiting, change in the operation of sources etc. The changes of currents, voltages etc. during the transients are not instantaneous and take some time, even though they are extremely fast with a duration of milliseconds or even microseconds. These very fast changes, however, cannot be instantaneous (or abrupt) since the transient processes are attained by the interchange of energy, which is usually stored in the magnetic field of inductances or/and the electrical field of capacitances. Any change in energy cannot be abrupt otherwise it will result in infinite power (as the power is a derivative of energy, $p = dw/dt$), which is in contrast to physical reality. All transient changes, which are also called transient responses (or just responses), vanish and, after their disappearance, a new steady-state operation is established. In this respect, we may say that the transient describes the circuit behavior between two steady-states: an old one, which was prior to changes, and a new one, which arises after the changes.

A few methods of transient analysis are known: the classical method, The Cauchy-Heaviside (C-H) operational method, the Fourier transformation method and the Laplace transformation method. The C-H operational or symbolic (formal) method is based on replacing a derivative by symbol s ($(d/dt) \leftrightarrow s$) and an integral by $1/s$

$$\left(\int dt \leftrightarrow \frac{1}{s} \right).$$

Although these operations are also used in the Laplace transform method, the C-H operational method is not as systematic and as rigorous as the Laplace transform method, and therefore it has been abandoned in favor of the Laplace method. The two transformation methods, Laplace and Fourier, will be studied in the following chapters. Comparing the classical method and the transformation method it should be noted that the latter requires more knowledge of mathematics and is less related to the physical matter of transient behavior of electric circuits than the former.

This chapter is concerned with the classical method of transient analysis. This method is based on the determination of differential equations and splitting the solution into two components: natural and forced responses. The classical method is fairly complicated mathematically, but is simple in engineering practice. Thus, in our present study we will apply some known methods of steady-state analysis, which will allow us to simplify the classical approach of transient analysis.

1.2 APPEARANCE OF TRANSIENTS IN ELECTRICAL CIRCUITS

In the analysis of an electrical system (as in any physical system), we must distinguish between the stationary operation or steady-state and the dynamical operation or transient-state.

An electrical system is said to be in **steady-state** when the variables describing its behavior (voltages, currents, etc.) are either invariant with time (d.c. circuits) or are periodic functions of time (a.c. circuits). An electrical system is said to be in **transient-state** when the variables are changed non-periodically, i.e., when the system is not in steady-state. The transient-state vanishes with time and a new steady-state regime appears. Hence, we can say that the transient-state, or just transients, is usually the transmission state from one steady-state to another.

The parameters L and C are characterized by their ability to store energy: *magnetic energy* $w_L = \frac{1}{2}\psi i = \frac{1}{2}Li^2$ (since $\psi = Li$), in the magnetic field and *electric energy* $w_C = \frac{1}{2}qv = \frac{1}{2}Cv^2$ (since $q = Cv$), in the electric field of the circuit. The voltage and current sources are the elements through which the energy is supplied to the circuit. Thus, it may be said that an electrical circuit, as a physical system, is characterized by certain energy conditions in its steady-state behavior. Under steady-state conditions the energy stored in the various inductances and capacitances, and supplied by the sources in a d.c. circuit, are constant; whereas in an a.c. circuit the energy is being changed (transferred between the magnetic and electric fields and supplied by sources) *periodically*.

When any sudden change occurs in a circuit, there is usually a redistribution of energy between L-s and C-s, and a change in the energy status of the sources, which is required by the new conditions. **These energy redistributions cannot take place instantaneously, but during some period of time, which brings about the transient-state.**

The main reason for this statement is that an instantaneous change of energy would require infinite power, which is associated with inductors/capacitors. As previously mentioned, power is a derivative of energy and any abrupt change in energy will result in an infinite power. Since infinite power is not realizable in physical systems, the energy cannot change abruptly, but only within some period of time in which transients occur. Thus, from a physical point of view it may be said that the transient-state exists in physical systems while the energy conditions of one steady-state are being changed to those of another.

Our next conclusion is about the current and voltage. To change *magnetic energy* requires a change of current through inductances. Therefore, currents in inductive circuits, or inductive branches of the circuit, cannot change abruptly. From another point of view, the change of current in an inductor brings about the induced voltage of magnitude $L(di/dt)$. An instantaneous change of current would therefore require an infinite voltage, which is also unrealizable in practice. Since the induced voltage is also given as $d\psi/dt$, where ψ is a magnetic flux, the magnetic flux of a circuit cannot suddenly change.

Similarly, we may conclude that to change the *electric energy* requires a change in voltage across a capacitor, which is given by $v = q/C$, where q is the charge. Therefore, neither the voltage across a capacitor nor its charge can be abruptly changed. In addition, the rate of voltage change is $dv/dt = (1/C)\,dq/dt = i/C$, and the instantaneous change of voltage brings about infinite current, which is also unrealizable in practice. Therefore, we may summarize that **any change in an electrical circuit, which brings about a change in energy distribution, will result in a transient-state.**

In other words, by any switching, interrupting, short-circuiting as well as any rapid changes in the structure of an electric circuit, the transient phenomena will occur. Generally speaking, **every change of state leads to a temporary deviation from one regular, steady-state performance of the circuit to another one.** The redistribution of energy, following the above changes, i.e., the transient-state, theoretically takes infinite time. However, in reality the transient behavior of an electrical circuit continues a relatively very short period of time, after which the voltages and currents almost achieve their new steady-state values.

The change in the energy distribution during the transient behavior of electrical circuits is governed by the principle of energy conservation, i.e., the amount of supplied energy is equal to the amount of stored energy plus the energy dissipation. The rate of energy dissipation affects the time interval of the transients. The higher the energy dissipation, the shorter is the transient-state. Energy dissipation occurs in circuit resistances and its storage takes place in inductances and capacitances. In circuits, which consist of only resistances, and neither inductances nor capacitances, the transient-state will not occur at all and the change from one steady-state to another will take place instantaneously. However, since even resistive circuits contain some inductances and capacitances the transients will practically appear also in such circuits; but these transients are very short and not significant, so that they are usually neglected.

Transients in electrical circuits can be recognized as either desirable or undesirable. In power system networks, the transient phenomena are wholly undesirable as they may bring about an increase in the magnitude of the voltages and currents and in the density of the energy in some or in most parts of modern power systems. All of this might result in equipment distortion, thermal and/or electrodynamics' destruction, system stability interferences and in extreme cases an outage of the whole system.

In contrast to these unwanted transients, there are desirable and controlled transients, which exist in a great variety of electronic equipment in communication, control and computation systems whose normal operation is based on switching processes.

The transient phenomena occur in electric systems either by intentional switching processes consisting of the correct manipulation of the controlling apparatus, or by unintentional processes, which may arise from ground faults, short-circuits, a break of conductors and/or insulators, lightning strokes (particularly in high voltage and long distance systems) and similar inadvertent processes.

As was mentioned previously, there are a few methods of solving transient problems. The most widely known of these appears in all introductory textbooks and is used for solving simpler problems. **It is called the classical method.** Other useful methods are Laplace (see Chap. 3) and Fourier (see Chap. 4) transformation methods. These two methods are more general and are used for solving problems that are more complicated.

1.3 DIFFERENTIAL EQUATIONS DESCRIBING ELECTRICAL CIRCUITS

Circuit analysis, as a physical system, is completely described by *integrodifferential equations* written for voltages and/or currents, which characterize circuit behavior. For linear circuits these equations are called linear differential equations with constant coefficients, i.e. in which every term is of the first degree in the dependent variable or one of its derivatives. Thus, for example, for the circuit of three basic elements: R, L and C connected in series and driven by a voltage source $v(t)$, Fig. 1.1, we may apply Kirchhoff's voltage law

$$v_R + v_L + v_C = v(t),$$

in which

$$v_R = Ri$$

$$v_L = L\frac{di}{dt}$$

$$v_C = \int i\,dt,$$

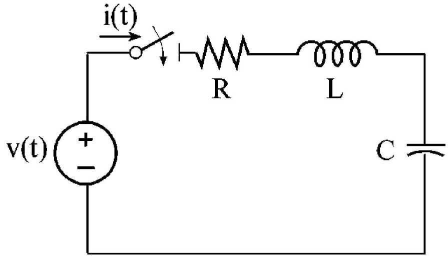

Figure 1.1 Series *RLC* circuit driven by a voltage source.

and then we have

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = v(t). \tag{1.1}$$

After the differentiation of both sides of equation 1.1 with respect to time, the result is a second order differential equation

$$L\frac{d^2t}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{dv}{dt}. \tag{1.2}$$

The same results may be obtained by writing two simultaneous first order differential equations for two unknowns, i and v_C:

$$\frac{dv_C}{dt} = \frac{1}{C}i \tag{1.3a}$$

$$Ri + L\frac{di}{dt} + v_C = v(t). \tag{1.3b}$$

After differentiation equation 1.3b and substituting dv_C/dt by equation 1.3a, we obtain the same (as equation 1.2) second order singular equation. The solution of differential equations can be completed only if the initial conditions are specified. It is obvious that in the same circuit under the same commutation, but with different initial conditions, its transient response will be different.

For more complicated circuits, built from a number of loops (nodes), we will have a set of differential equations, which should be written in accordance with Kirchhoff's two laws or with nodal and/or mesh analysis. For example, considering the circuit shown in Fig. 1.2, after switching, we will have a circuit, which consists of two loops and two nodes. By applying Kirchhoff's two laws, we may write three equations with three unknowns, i, i_L and v_C,

$$C\frac{dv_C}{dt} + i_L - i = 0 \tag{1.4a}$$

$$L\frac{di_L}{dt} + R_1 i_L + Ri = 0 \tag{1.4b}$$

$$L\frac{di_L}{dt} + R_1 i_L - v_C = 0 \tag{1.4c}$$

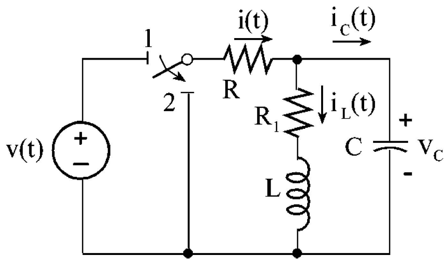

Figure 1.2 A two-loop circuit.

These three equations can then be redundantly transformed into a single second order equation. First, we differentiate the third equation of 1.4c once with respect to time and substitute dv_C/dt by taking it from the first one. After that, we have two equations with two unknowns, i_L and i. Solving these two equations for i_L (i.e. eliminating the current i) results in the second order homogeneous differential equation

$$LCR\frac{d^2i_L}{dt^2} + (L + CRR_1)\frac{di_L}{dt} + (R + R_1)i_L = 0. \tag{1.5}$$

As another example, let us consider the circuit in Fig. 1.3. Applying mesh analysis, we may write three *integro-differential equations* with three unknown mesh currents:

$$\begin{aligned} L\frac{di_1}{dt} - L\frac{di_2}{dt} + R_1 i_1 &= v(t) \\ L\frac{di_2}{dt} - L\frac{di_1}{dt} + (R_2 + R_3)i_2 - R_3 i_3 &= 0 \\ -R_3 i_2 + R_3 i_3 + \frac{1}{C}\int i_3 dt &= 0. \end{aligned} \tag{1.6}$$

In this case it is preferable to solve the problem by treating the whole set of equations 1.6 rather than reducing them to a single one (see further on).

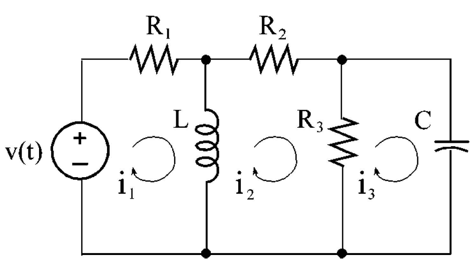

Figure 1.3 A three-loop circuit.

From mathematics, we know that there are a number of ways of solving differential equations. Our goal in this chapter is to analyze the transient behavior of electrical circuits from the physical point of view rather than applying complicated mathematical methods. (This will be discussed in the following chapters.) Such a way of transient analysis is in the formulation of differential equations in accordance with the properties of the circuit elements and in the *direct* solution of the obtained equations, using only the necessary mathematical rules. Such a method is called the **classical method or classical approach** in transient analysis. We believe that the classical method of solving problems enables the student to better understand the transient behavior of electrical circuits.

1.3.1 Exponential solution of a simple differential equation

Let us, therefore, begin our study of transient analysis by considering the simple series RC circuit, shown in Fig. 1.4. After switching we will get a source free circuit in which the precharged capacitor C will be discharged via the resistance R. To find the capacitor voltage we shall write a differential equation, which in accordance with Kirchhoff's voltage law becomes

$$Ri + v_C = 0, \quad \text{or} \quad RC\frac{dv_C}{dt} + v_C = 0. \tag{1.7}$$

A direct method of solving this equation is to write the equation in such a way that the variables are separated on both sides of the equation and then to integrate each of the sides. Multiplying by dt and dividing by v_C, we may arrange the variables to be separated.

$$\frac{dv_C}{v_C} = -\frac{1}{RC}dt. \tag{1.8}$$

The solution may be obtained by integrating each side of equation 1.8 and by adding a constant of integration:

$$\int\frac{dv_C}{v_C} = -\frac{1}{RC}\int dt + K,$$

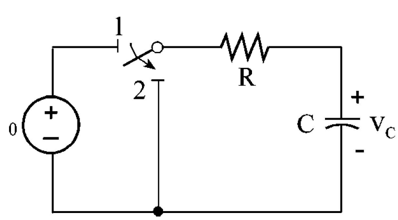

Figure 1.4 A series RC circuit.

and the integration yields

$$\ln v_C = -\frac{1}{RC}t + K \tag{1.9}$$

Since the constant can be of any kind, and we may designate $K = \ln D$, we have

$$\ln v_C = -\frac{1}{RC}t + \ln D,$$

then

$$v_C = De^{-\frac{t}{RC}}. \tag{1.10}$$

The constant D cannot be evaluated by substituting equation 1.10 into the original differential equation 1.7, since the identity $0 \equiv 0$ will result for any value of D (indeed: $D(-1/RC)RCe^{-t/RC} + De^{-r/RC} = 0$). The constant of integration must be selected to satisfy the initial condition $v_C(0) = V_0$, which is the initial voltage across the capacitance. Thus, the solution of equation 1.10 at $t = 0$ becomes $v_C(0) = D$, and we may conclude that $D = V_0$. Therefore, with this value of D we will obtain the desired response

$$v_C = V_0e^{-\frac{t}{RC}}. \tag{1.11}$$

We shall consider the nature of this response by analyzing the curve of the voltage change shown in Fig. 1.5. At zero time, the voltage is the assumed value V_0 and, as time increases, the voltage decreases and approaches zero, following the physical rule that any condenser shall finally be discharged and its final voltage therefore reduces to zero.

Let us now find the time that would be required for the voltage to drop to

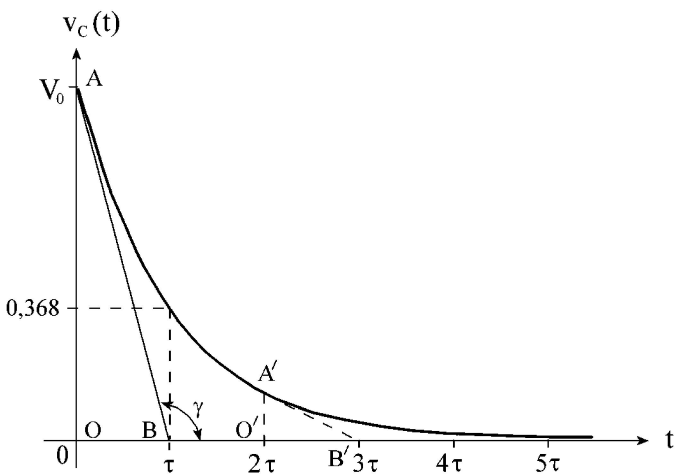

Figure 1.5 The exponential curve of the voltage changing.

zero if it continued to drop linearly at its initial rate. This value of time, usually designated by τ, is called the **time constant**. The value of τ can be found with the derivative of $v_C(t)$ at zero time, which is proportional to the angle γ between the tangent to the voltage curve at $t = 0$, and the t-axis, Fig. 1.5, i.e.,

$$\tan\lambda \propto -\frac{V_0}{\tau} = \frac{d}{dt}\left(V_0 e^{-\frac{t}{RC}}\right)_{t=0} = \frac{-V_0}{RC},$$

or

$$\tau = RC$$

and equation 1.11 might be written in the form

$$V_C = V_0 e^{-\frac{t}{\tau}}. \tag{1.12}$$

The units of the time constant are seconds ($[\tau] = [R][C] = \Omega\cdot\mathrm{F}$), so that the exponent t/RC is dimensionless, as it is supposed to be. The time constant may be easily found graphically from the response curve, as can be seen from Fig. 1.5: the interception point, B, of the tangent line $\overline{AB}$ with the time axis determine the time constant τ. This line segment $\overline{OB}$ is called under-tangent. It is interesting to note that the under-tangent remains the same no matter at which point the tangent to the curve is drawn (see under-tangent $\overline{O'B'}$).

Another interpretation of the time constant is obtained from the fact that in the time interval of one time constant the voltage drops relatively to its initial value, to the reciprocal of *e*; indeed, at $t = \tau$ we have $(v_C/V_0) = e^{-1} = 0.368$ (36.8%). At the end of the 5τ interval the voltage is less than one percent of its initial value. Thus, it is usual to presume that in the time interval of three to five time constants, the transient response declines to zero or, in other words, we may say that the duration of the transient response is about five time constants. Note again that, precisely speaking, the transient response declines to zero in infinite time, since $e^{-t} \to 0$, when $t \to \infty$.

Before we continue our discussion of a more general analysis of transient circuits, let us check the power and energy relationships during the *period of transient response*. The power being dissipated in the resistor *R*, or its reciprocal *G*, is

$$p_R = Gv_C^2 = GV_0^2 e^{-2t/RC}, \tag{1.13}$$

and the total dissipated energy (turned into heat) is found by integrating equation 1.13 from zero time to infinite time

$$w_R = \int_0^\infty p_R dt = V_0 G \int_0^\infty e^{-2t/RC} = -V_0^2 G \frac{RC}{2} e^{-2t/RC}\Big|_0^\infty = \frac{1}{2} CV_C^2.$$

This is actually the energy being stored in the capacitor at the beginning of the transient. This result means that all the initial energy, stored in the capacitor, dissipates in the circuit resistances during the transient period.

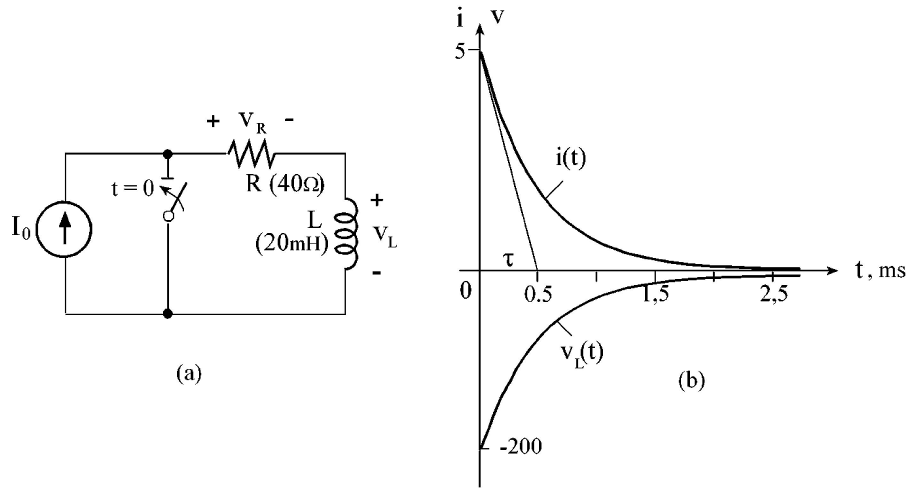

Figure 1.6 A circuit of Example 1.1 (a) and two plots of current and voltage (b).

Example 1.1

Consider a numerical example. The *RL* circuit in Fig. 1.6(a) is fed by a d.c. current source, $I_0 = 5\,\mathrm{A}$. At instant $t = 0$ the switch is closed and the circuit is short-circuited. Find: 1) the current after switching, by separating the variables and applying the *definite integrals*, 2) the voltage across the inductance.

Solution

1) First, we shall write the differential equation:

$$v_L + v_R = L\frac{di}{dt} + Ri = 0,$$

or after separating the variables

$$\frac{di}{i} = \frac{R}{L}\,dt.$$

Since the current changes from I_0 at the instant of switching to $i(t)$, at any instant of t, which means that the time changes from $t = 0$ to this instant, we may perform the integration of each side of the above equation between the corresponding limits

$$\int_{I_0}^{i(t)} \frac{di}{i} = \int_0^t -\frac{R}{L}\,dt.$$

Therefore,

$$\ln i|_{I_0}^{i(t)} = -\frac{R}{L}\,t|_0^t$$

and

$$\ln i(t) - \ln I_0 = \frac{R}{L}t, \quad \text{or} \quad \ln\frac{i(t)}{I_0} = -\frac{R}{L}t,$$

which results in

$$\frac{i(t)}{I_0} = e^{-\frac{R}{L}t}$$

Thus,

$$i(t) = I_0 e^{-\frac{R}{L}t} = 5e^{-2000t},$$

or

$$i(t) = I_0 e^{-\frac{t}{\tau}} = 5e^{-\frac{t}{0.5\cdot 10^{-3}}}.$$

where

$$R/L = \frac{40}{20\cdot 10^{-3}} = 2000\ \mathrm{s}^{-1},$$

which results in time constant

$$\tau = \frac{L}{R} = 0.5\ \mathrm{ms}.$$

Note that by applying the definite integrals we avoid the step of evaluating the constant of the integration.

2) The voltage across the inductance is

$$v_L = L\frac{di}{dt} = L\frac{d}{dt}(5e^{-2000t}) = 20\cdot 10^{-3}\cdot 5\cdot(-2000)e^{-2000t} = -200e^{-\frac{t}{0.5}},\ \mathrm{V}$$

(time in ms).

Note that the voltage across the resistance is $v_R = Ri = 40\cdot 5e^{-t/0.5} = 200e^{-t/0.5}$, i.e., it is equal in magnitude to the inductance voltage, but opposite in sign, so that the total voltage in the short-circuit is equal to zero. The plots of the current and voltage are shown in Fig. 1.6(b).

1.4 NATURAL AND FORCED RESPONSES

Our next goal is to introduce a general approach to solving differential equations by the classical method. Following the principles of mathematics we will consider the complete solution of any linear differential equation as composed of two parts: the complementary solution (or natural response in our study) and the particular solution (or forced response in our study). To understand these principles, let us consider a first order differential equation, which has already been derived in the previous section. In a more general form it is

$$\frac{dv}{dt} + P(t)v = Q(t). \tag{1.14}$$

Here $Q(t)$ is identified as a forcing function, which is generally a function of time (or constant, if a d.c. source is applied) and $P(t)$, is also generally a function of time, represents the circuit parameters. In our study, however, it will be a constant quantity, since the value of circuit elements does not change during the transients (indeed, the circuit parameters do change during the transients, but we may neglect this change as in many cases it is not significant).

A more general method of solving differential equations, such as equation 1.14, is to multiply both sides by a so-called *integrating factor*, so that each side becomes an exact differential, which afterwards can be integrated directly to obtain the solution. For the equation above (equation 1.14) the integrating factor is $e^{\int P\,dt}$ or e^{Pt}, since P is constant. We multiply each side of the equation by this integrating factor and by dt and obtain

$$e^{Pt}dv + vPe^{Pt}dt = Qe^{Pt}dt.$$

The left side is now the exact differential of ve^{Pt} (indeed, $d(ve^{Pt}) = e^{Pt}dv + vPe^{Pt}dt$), and thus

$$d(ve^{Pt}) = Qe^{Pt}dt.$$

Integrating each side yields

$$ve^{Pt} = \int Qe^{Pt}dt + A, \tag{1.15}$$

where A is a constant of integration. Finally, the multiplication of both sides of equation 1.15 by e^{-Pt} yields

$$v(t) = e^{-Pt}\int Qe^{Pt}dt + Ae^{-Pt}, \tag{1.16}$$

which is the solution of the above differential equation. As we can see, this complete solution is composed of two parts. The first one, which is dependent on the forcing function Q, is the *forced response* (it is also called the steady-state response or the particular solution or the particular integral). The second one, which does not depend on the forcing function, but only on the circuit parameters P (the types of elements, their values, interconnections, etc) and on the initial conditions A, i.e., on the "nature" of the circuit, is the *natural response.* It is also called the solution of the homogeneous equation, which does not include the source function and has anything but zero on its right side.

Following this rule, we will solve differential equations by finding natural and forced responses separately and combining them for a complete solution. This principle of dividing the solution of the differential equations into two components can also be understood by applying the superposition theorem. Since the differential equations, under study, are linear as well as the electrical circuits, we may assert that superposition is also applicable for the transient-state. Following this principle, we may subdivide, for instance, the current into two components

$$i = i' + i'',$$

and by substituting this into the set of differential equations, say of the form

$$\sum \left(L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt \right) = \sum v_s,$$

we obtain the following two sets of equations

$$\sum \left(L\frac{di'}{dt} + Ri' + \frac{1}{C}\int i'\,dt \right) = \sum v_s,$$

$$\sum \left(L\frac{di''}{dt} + Ri'' + \frac{1}{C}\int i''\,dt \right) = 0.$$

It is obvious that by summation (superimposition) of these two equations, the original equation will be achieved. This means that i'' is a natural response since it is the solution of a homogeneous equation with a zero on the right side and develops without any action of any source, and i' is a steady-state current as it develops under the action of the voltage sources v_s (which are presented on the right side of the equations).

The most difficult part in the classical method of solving differential equations is evaluating the particular integral in equation 1.16, especially when the forcing function is not a simple d.c. or exponential source. However, in circuit analysis we can use all the methods: node/mesh analysis, circuit theorems, the phasor method for a.c. circuits (which are all given in introductory courses on steady-state analysis) to find the forced response. In relation to the natural response, the most difficult part is to formulate the characteristic equation (see further on) and to find its roots. Here in circuit analysis we also have special methods for evaluating the characteristic equation simply by inspection of the analyzed circuit, avoiding the formulation of differential equations.

Finally, it is worthwhile to clarify the use of exponential functions as an integrating factor in solving linear differential equations. As we have seen in the previous section, such differential equations in general consist of the second (or higher) derivative, the first derivative and the function itself, each multiplied by a *constant factor*. If the sum of all these derivatives (the function itself might be treated as a derivative of order zero) achieves zero, it becomes a homogeneous equation. A function whose derivatives have the same form as the function itself is an exponential function, so it may satisfy these kinds of equations. Substituting this function into the differential equation, whose right side is zero (a homogeneous differential equation) the exponential factor in each member of the equation might be simply crossed out, so that the remaining equation's coefficients will be only circuit parameters. Such an equation is called a characteristic equation.

1.5 CHARACTERISTIC EQUATION AND ITS DETERMINATION

Let us start by considering the simple circuit of Fig. 1.7(a) in which an *RL* in series is switching on to a d.c. voltage source.

Let the desired response in this circuit be current $i(t)$. We shall first express it as the sum of the natural and forced currents

$$i = i_n + i_f.$$

The form of the natural response, as was shown, must be an exponential function, $i_n = Ae^{st}$(*). Substituting this response into the homogeneous differential equation, which is $L(di/dt) + Ri = 0$, we obtain $Ls\, e^{st} + R\, e^{st} = 0$, or

$$Ls + R = 0. \tag{1.17a}$$

This is a characteristic (or auxiliary) equation, in which the left side expresses the input impedance seen from the source terminals of the analyzed circuit.

$$Z_{in}(s) = Ls + R. \tag{1.17b}$$

We may treat s as the complex frequency $s = \sigma + j\omega$ (for more about complex frequencies see any introductory course to circuit analysis and further on in Chap. 3). Note that by equaling this expression of circuit impedance to zero, we obtain the *characteristic equation.* Solving this equation we have

$$s = -\frac{R}{L} \quad \text{and} \quad \tau = \frac{L}{R}. \tag{1.18}$$

Hence, the natural response is

$$i_n = Ae^{-\frac{R}{L}t}. \tag{1.19}$$

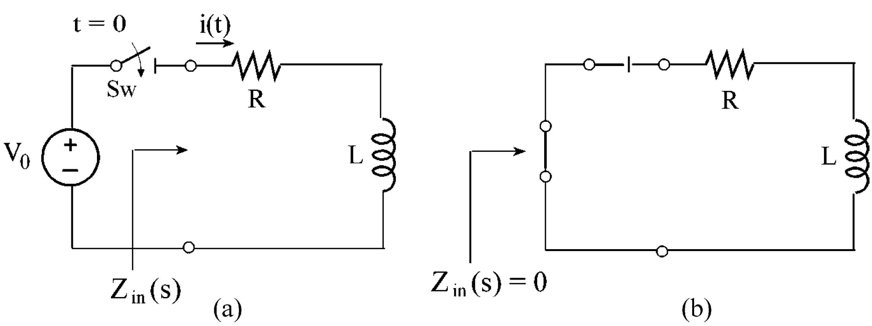

Figure 1.7 An *RL* circuit switching to a d.c. voltage source (a) and after "killing" the source (b).

(*) Here and in the future, we will use the letter s for the circuit parameters' dependent exponent.

Subsequently, the root of the characteristic equation defines the exponent of the natural response. The fact that the input impedance of the circuit should be equaled to zero can be explained from a physical point of view.(*) Since the natural response does not depend on the source, the latter should be "killed". i.e. short-circuited as shown in Fig. 1.7(b). This action results in short-circuiting the entire circuit, i.e. its input impedance.

Consider now a parallel *LR* circuit switching to a d.c. current source in which the desired response is $v_L(t)$, as shown in Fig. 1.8(a). Here, "killing" the current source results in open-circuiting, as shown in Fig. 1.8(b).

This means that the input admittance should be equaled to zero. Thus,

$$\frac{1}{R} + \frac{1}{sL} = 0,$$

or

$$sL + R = 0,$$

which however gives the same root

$$s = -\frac{R}{L} \quad \text{and} \quad \tau = \frac{L}{R}. \tag{1.20}$$

Next, we will consider a more complicated circuit, shown in Fig. 1.9(a). This circuit, after switching and short-circuiting the remaining voltage source, will be as shown in Fig. 1.9(b). The input impedance of this circuit "measured" at the switch (which is the same as seen from the "killed" source) is

$$Z_{in}(s) = R_1 + R_3//R_4//(R_2 + sL),$$

or

$$Z_{in}(s) = R_1 + \left(\frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_2 + sL}\right)^{-1}.$$

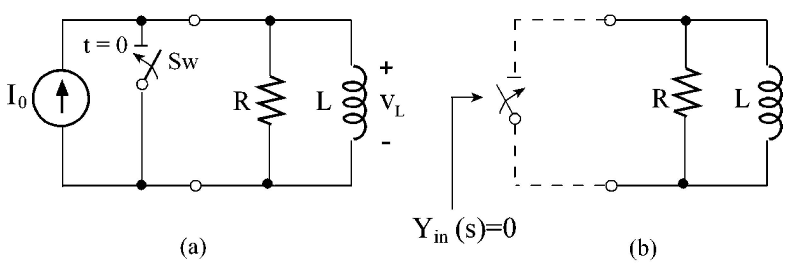

Figure 1.8 A parallel *RL* circuit switching to d.c. current source (a) and after " killing" the source (b).

(*)This fact is proven more correctly mathematically in Laplace transformation theory (see further on).

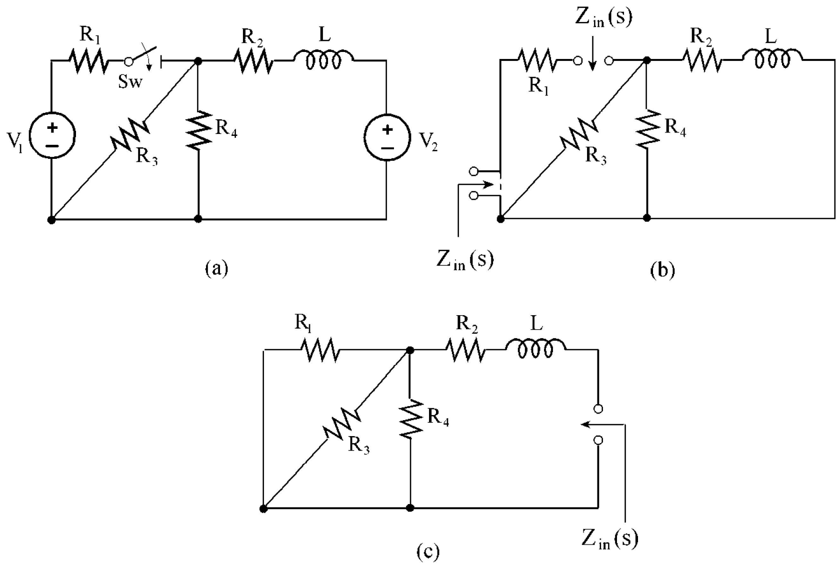

Figure 1.9 A given circuit (a), determining the input impedance as seen from the switch (b) and as seen from the inductance branch (c).

Evaluating this expression and equaling it to zero yields

$$(R_1R_3 + R_1R_4 + R_3R_4)(R_2 + sL) + R_1R_3R_4 = 0,$$

and the root is

$$s = -\frac{R_{eq}}{L}, \quad \text{where} \quad R_{eq} = \frac{R_1R_3R_4 + R_1R_2R_3 + R_1R_2R_4 + R_2R_3R_4}{R_1R_3 + R_1R_4 + R_3R_4}.$$

It is worthwhile to mention that the same results can be obtained if the input impedance is "measured" from the inductance branch, i.e. the energy-storing element, as is shown in Fig. 1.9(c).

The characteristic equation can also be determined by inspection of the differential equation or set of equations. Consider the second-order differential equation like in equation 1.2

$$L\frac{d^2i(t)}{dt} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = g(t). \tag{1.21}$$

Replacing each derivative by s^n, where n is the order of the derivative (the function by itself is considered as a zero-order derivative), we may obtain the characteristic equation:

$$Ls^2 + Rs^1 + \frac{1}{C}s^0 = 0, \quad \text{or} \quad s^2 + \frac{R}{L}s + \frac{1}{LC} = 0. \tag{1.22}$$

This characteristic equation is of the second order (in accordance with the second order differential equation) and it possesses two roots s_1 and s_2.

If any system is described by a set of integro-differential equations, like in equation 1.6, then we shall first rewrite it in a slightly different form as homogeneous equations

$$\begin{gathered}\left(L\frac{d}{dt}+R\right)i_1-L\frac{d}{dt}i_2+0\cdot i_3=0\\ -L_1\frac{d}{dt}i_1+\left(L_2\frac{d}{dt}+R_2+R_3\right)i_2-R_3i_3=0\\ 0\cdot i_1-R_3i_2+\left(\frac{1}{C}\int dt\right)i_3=0.\end{gathered}\tag{1.23}$$

Replacing the derivatives now by s^n and an integral by s^{-1} (since an integral is a counter version of a derivative) we have

$$\begin{gathered}(Ls+R_1)i_1-sLi_2+0\cdot i_3=0\\ -Lsi_1+(Ls+R_2+R_3)i_2-R_3=0\\ 0\cdot i_1-R_3i_2+\left(\frac{1}{sC}+R_3\right)i_3=0.\end{gathered}\tag{1.24}$$

We obtained a set of algebraic equations with the right side equal to zero. In the matrix form

$$\begin{bmatrix}Ls+R_1 & -sL & 0\\ -sL & Ls+R_2+R_3 & -R_3\\ 0 & -R_3 & \dfrac{1}{Cs}+R_3\end{bmatrix}\begin{bmatrix}i_1\\ i_2\\ i_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}\tag{1.24a}$$

With Cramer's rule the solution of this equation can be written as

$$i_{1,n}=\frac{\Delta_1}{\Delta}\quad i_{2,n}=\frac{\Delta_2}{\Delta}\quad i_{3,n}=\frac{\Delta_3}{\Delta},\tag{1.24b}$$

where Δ is the determinant of the system matrix and determinants Δ_1, Δ_2, Δ_3 are obtained from Δ, by replacing the appropriate column (in Δ_1 the first column is replaced, in Δ_2 the second column is replaced, and so forth), by the right side of the equation, i.e. by zeroes. As is known from mathematics such determinants are equal to zero and for the non-zero solution in equation 1.24 the determinant Δ in the denominator must also be zero. Thus, by equaling this determinant to zero, we get the characteristic equation:

$$\begin{vmatrix}sL+R_1 & -sL & 0\\ -sL & sL+R_2+R_3 & -R_3\\ 0 & -R_3 & \dfrac{1}{sC}+R_3\end{vmatrix}=0,$$

or

$$(sL + R_1)(sL + R_2 + R_3)\left(\frac{1}{sC} + R_3\right) - R_3^2(sL + R_1) - s^2L^2\left(\frac{1}{sC} + R_3\right) = 0$$

Simplifying this equation yields a second-order equation

$$s^2 + \left(\frac{R_{1,eq}}{L} + \frac{1}{R_{2,eq}C}\right)s + \frac{1}{LC}\xi = 0, \tag{1.25}$$

where

$$R_{1,eq} = \frac{R_1R_2}{R_1 + R_2} \quad R_{2,eq} = \frac{R_1 + R_2}{R_1/R_3 + R_2/R_3 + 1} \quad \xi = \frac{1 + R_2/R_3}{1 + R_2/R_1}.$$

We could have achieved the same results by inspecting the circuit in Fig. 1.3 and determining the input impedance (we leave this solution as an exercise for the reader). The characteristic equation 1.25 is of second order, since the circuit (Fig. 1.3) consists of two energy-storing elements (one inductance and one capacitance).

There is a more general rule, which states that the order of a characteristic equation is as high as the number of energy-storing elements. However, we should distinguish between the elements, which cannot be replaced by their equivalent and those which can be eliminated by simplifying the circuit. We therefore shall first combine the inductances and capacitances, which are connected in series and/or in parallel, or can be brought to such connections. For instance, in the circuit in Fig. 1.10(a) we may account for five L-s/C-s elements. However, after simplification their number is reduced to only two energy-storing elements, as shown in Fig. 1.10(b). Therefore, we may conclude that the given circuit and its characteristic equation are of second order only. Another example is the circuit in Fig. 1.10(c), which contains three inductive elements and two resistances (after switching). By inspection of this circuit, we may simplify it to only one equivalent inductance:

$$L_{eq} = L_1 + \frac{L_1L_2}{L_1 + L_2}.$$

Therefore, the circuit is of the first order. The equivalent resistance is $R_{eq} = R_1 + R_2$.

In such "reduced" circuits, the inductances and capacitances are associated with their currents (through inductances) and voltages (across capacitances), which at $t = 0$ define the independent initial conditions (see further on). The number of these initial conditions must comply with the order of the characteristic equation, so that we will be able to determine the integration constant, the number of which is also equal to the order of the characteristic equation.

In more complicated circuits we may find that a few, let us say k inductances are connected in a so-called "inductance" node, as shown in Fig. 1.11(a) and (b). Taking into consideration that, in accordance with KCL, the sum of the

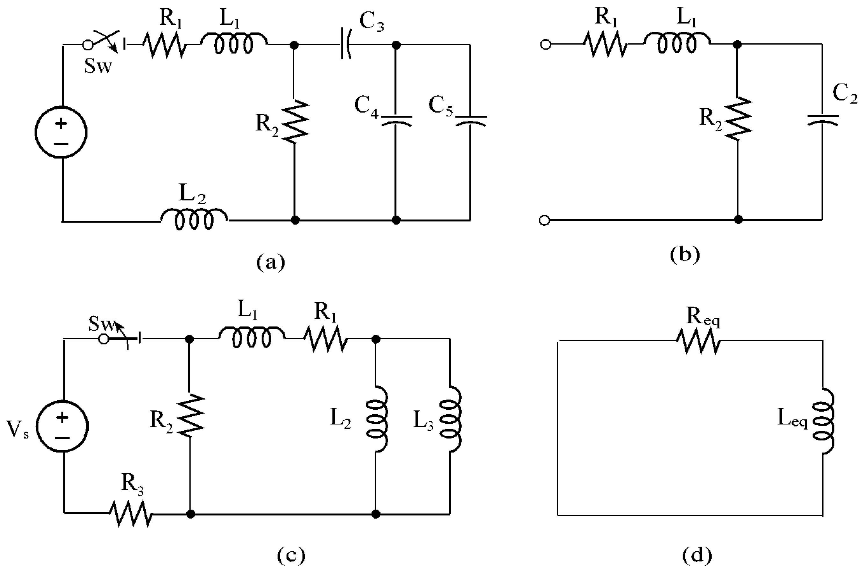

Figure 1.10 A given circuit of five L/C elements (a) and its equivalent of only two L/C elements (b), a circuit of three L elements (c) and its equivalent of only one L element.

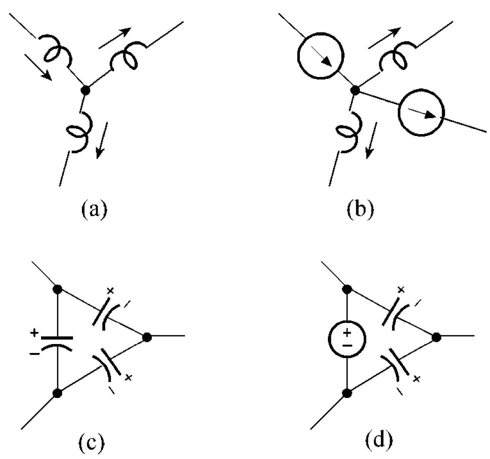

Figure 1.11 An "inductance" node of three inductances (a), an "inductance" node of two inductances and two current sources (b), a "capacitance" loop of three capacitances (c) and a "capacitance" loop of two capacitances and one voltage source.

currents in a node is zero, we may conclude that only $k - 1$ inductance currents are independent. This means that the contribution to the order of the characteristic equation, which will be made by the inductances, is one less than the number of inductances. The "*capacitance*" *loop*, Fig. 1.11(c) and (b) is a dual to the "inductance" node, so that the number of independent voltages across the capacitances in the loop will be one less than the number of capacitances. Thus, if the total number of inductances and capacitances is n_L and n_C respectively, and the number of "inductance" nodes and "capacitance" loops is m_L and m_C respectively, then *the order of the characteristic equation* is $n_s = n_L + n_C - m_L - m_C$. Finally, it must be mentioned that the mutual inductance does not influence the order of the characteristic equation.

By analyzing the circuits in their transient behavior and determining their characteristic equations, we should also take into consideration that the natural responses might be different depending on the kind of applied source: voltage or current. Actually, we have to distinguish between two cases:

1) If the voltage source, in its physical representation (i.e. with an inner resistance connected in series) is replaced by an equivalent current source (i.e. with the same resistance connected in parallel), the transient responses will not change. Indeed, as can be seen from Fig. 1.12, the same circuit A is connected in (a) to the voltage source and in (b) to the current source. By "killing" the sources (i.e. short-circuiting the voltage sources and opening the current sources) we are getting the same passive circuits, for which the impedances are the same. This means that the characteristic equations of both circuits will be the same and therefore the natural responses will have the same exponential functions.

2) However, if the *ideal voltage source is replaced by an ideal current source*, Fig. 1.13, the passive circuits in (a) and (b), i.e. after killing the sources, are different, having different input impedances and therefore different natural responses.

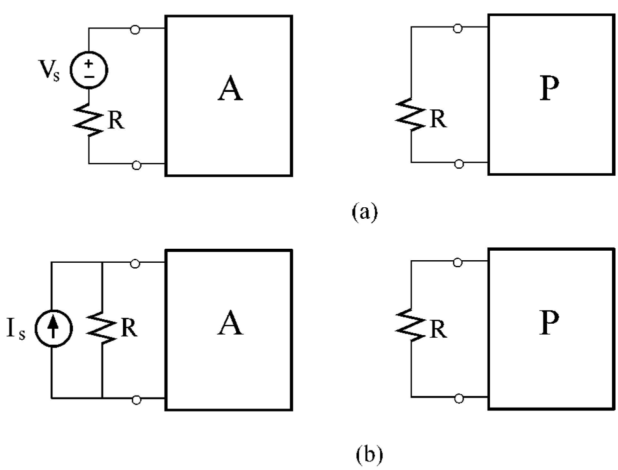

Figure 1.12. A circuit with an applied voltage source (a) and with a current source (b).

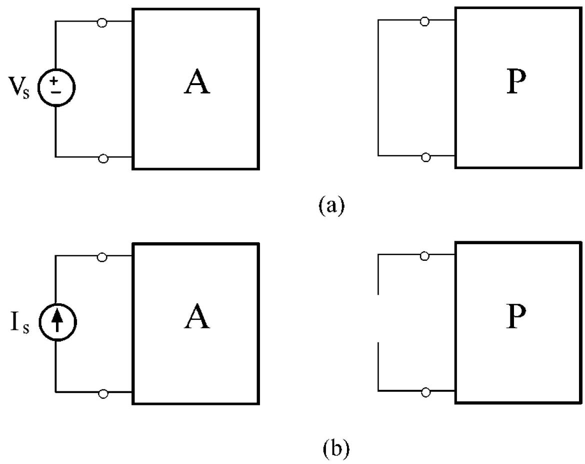

Figure 1.13 Circuit with an applied ideal voltage source (a) and an ideal current source (b).

1.6 ROOTS OF THE CHARACTERISTIC EQUATION AND DIFFERENT KINDS OF TRANSIENT RESPONSES

1.6.1 First-order characteristic equation

If an electrical circuit consists of only one energy-storing element (L or C) and a number of energy dissipation elements (R's), the characteristic equation will be of the first order:

For an RL circuit

$$Ls + R_{eq} = 0 \tag{1.26a}$$

and its root is

$$s = -\frac{R_{eq}}{L} = -\frac{1}{\tau}, \tag{1.26b}$$

where

$$\tau = \frac{L}{R_{eq}}$$

is a time constant.

For an RC circuit

$$\frac{1}{sC} + R_{eq} = 0 \tag{1.27a}$$

and its root is

$$s = -\frac{1}{R_{er}C} = -\frac{1}{\tau}, \tag{1.27b}$$

where $\tau = R_{eq}C$ is a time constant. In both cases the natural solution is

$$f_n(t) = A e^{st}, \tag{1.28a}$$

or

$$f_n(t) = A\, e^{-\frac{t}{\tau}}, \tag{1.28b}$$

which is a decreasing exponential, which approaches zero as the time increases without limit. However, as we have seen earlier (in Fig. 1.5), during the time interval of five times τ the difference between the exponential and zero is less than 1%, so that practically we may state that the duration of the transient response is about 5τ.

1.6.2 Second-order characteristic equation

If an electrical circuit consists of two energy-storing elements, then the characteristic equation will be of the second order. For an electrical circuit, which consists of an inductance, capacitance and several resistances this equation may look like equations 1.22, 1.25 or in a generalized form

$$s^2 + 2\alpha + \omega_d^2 = 0. \tag{1.29}$$

The coefficients in the above equation shall be introduced as follows: α as the exponential *damping coefficient* and ω_d as a *resonant frequency*. For a series *RLC* circuit $\alpha = R/2L$ and $\omega_d = \omega_0 = 1/\sqrt{LC}$. For a parallel *RLC* circuit $\alpha = 1/2RC$ and $\omega_d = \omega_0 1/\sqrt{LC}$, which is the same as in a series circuit. For more complicated circuits, as in Fig. 1.3, the above terms may look like $\alpha = \frac{1}{2}(R_{1,eq}/L + 1/R_{2,eq}C)$, which is actually combined from those coefficients for the series and parallel circuits and $\omega_d = \omega_0 \xi$, where ξ is a distortion coefficient, which influences the resonant/oscillatory frequency.

The two roots of a second order (quadratic) equation 1.29 are given as

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_d^2} \tag{1.30a}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_d^2}, \tag{1.30b}$$

and the natural response in this case is

$$f_n(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}. \tag{1.31}$$

Since each of these two exponentials is a solution of the given differential equation, it can be shown that the sum of the two solutions is also a solution (it can be shown, for example, by substituting equation 1.31 into the considered equation. The proof of it is left for the reader as an exercise.)

As is known from mathematics, the two roots of a quadratic equation can be one of three kinds:

1) negative real different, such as $|s_2| > |s_1|$, if $\alpha > \omega_d$;
2) negative real equal, such as $|s_2| = |s_1| = |s|$, if $\alpha = \omega_d$ and

3) complex conjugate, such as $s_{1,2} = -\alpha \pm j\omega_n$, if $\alpha < \omega_d$ and then $\omega_n = \sqrt{\omega_d^2 - \alpha^2}$ is the frequency of oscillation or **natural frequency** (see further on).

A detailed analysis of the natural response of all three cases will be given in the next chapter. Here, we will restrict ourselves to their short specification.

1) *Overdamping*. In this case, the natural response (equation 1.31) is given as the sum of two decreasing exponential forms, both of which approach zero as $t \to \infty$. However, since $|s_2| > |s_1|$, the term of s_2 has a more rapid rate of decrease so that the transients' time interval is defined by s_1 ($t_{tr} \approx 5(1/|s_1|)$). This response is shown in Fig. 1.14(a).

2) *Critical damping*. In this case, the natural response (equation 1.31) converts into the form

$$f(t) = (A_1 t + A_2)e^{-st}, \tag{1.32}$$

which is shown in Fig. 1.14(b).

3) *Underdamping*. In this case, the natural response becomes oscillatory, which may be imaged as a decaying alternating current (voltage)

$$f(t) = Be^{-\alpha t}\sin(\omega_n t + \beta), \tag{1.33}$$

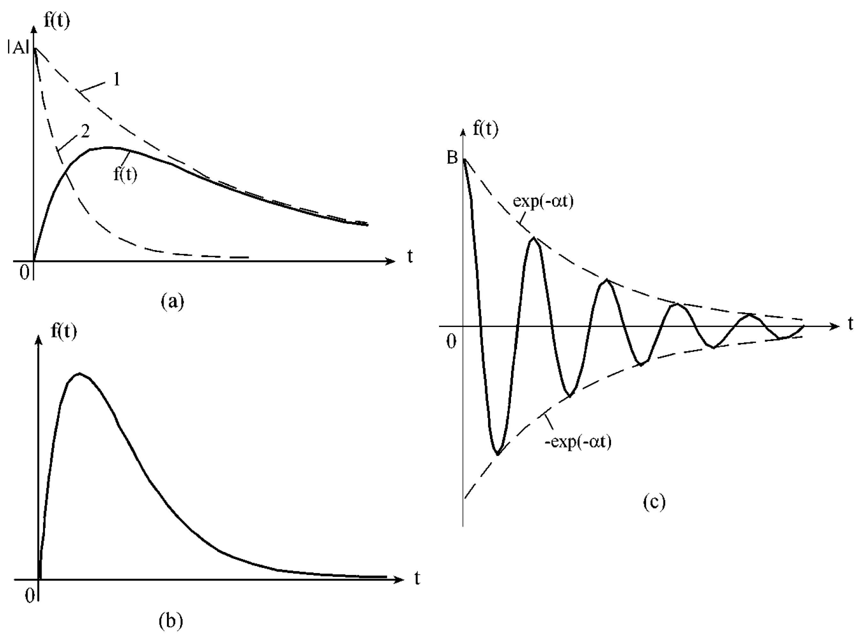

Figure 1.14 An overdamped response (a), a critical response (b) and an underdamped response (c).

which is shown in Fig. 1.14(c). Here term α is the rate of decay and ω_n is the angular frequency of the oscillations.

Now the critical damping may be interpreted as the boundary case between the overdamped and underdamped responses. It should be noted however that the critical damping is of a more theoretical than practical interest, since the exact satisfaction of the critical damping condition $\alpha = \omega_d$ in a circuit, which has a variety of parameters, is of very low probability. Therefore, the transient response in a second order circuit will always be of an exponential or oscillatory form. Let us now consider a numerical example.

Example 1.2

The circuit shown in Fig 1.15 represents an equivalent circuit of a one-phase transformer and has the following parameters: $L_1 = 0.06$ H, $L_2 = 0.02$ H, $M = 0.03$ H, $R_1 = 6\ \Omega$, $R_2 = 1\ \Omega$. If the transformer is loaded by an inductive load, whose parameters are $L_{ld} = 0.005$ H and $R_{ld} = 9\ \Omega$, a) determine the characteristic equation of a given circuit and b) find the roots and write the expression of a natural response.

Solution

Using mesh analysis, we may write a set of two algebraic equations (which represent two differential equations in operational form)

$$(R_1 + sL_1)i_1 - sM\, i_2 = 0$$

$$-sM\, i_1 + (R_2 + sL_2 + R_{ld} + sL_{ld})i_2 = 0.$$

The determinant of this set of two equations is

$$\det = \begin{vmatrix} R_1 + sL_1 & -sM \\ -sM & (R_2 + R_{ld}) + s(L_2 + L_{ld}) \end{vmatrix}$$

$$= (L_1 L_2' - M^2)s^2 + (R_1 L_2' + R_2' L_1)s + R_1 R_2',$$

where, to shorten the writing, we assigned $L_2' = L_2 + L_{ld}$ and $R_2' = R_2 + R_{ld}$.

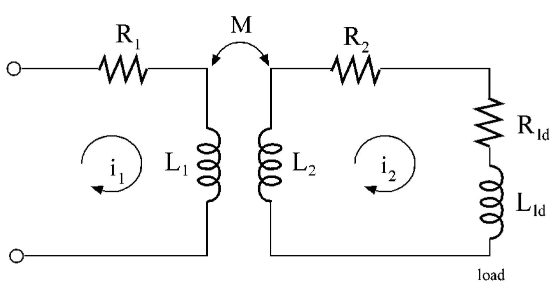

Figure 1.15 A given circuit for example 1.2.

Letting det = 0, we obtain the characteristic equation in the form

$$s^2 + \frac{R_1 L_2' + R_2' L_1}{L_1 L_2' - M^2} s + \frac{R_1 R_2'}{L_1 L_2' - M^2} = 0.$$

Substituting the given values, we have

$$s^2 + \frac{6\cdot 0.025 + 10\cdot 0.06}{0.06\cdot 0.025 - 0.03^2} s + \frac{6\cdot 10}{0.06\cdot 0.025 - 0.03^2} = 0,$$

or

$$s^2 + 12.5\cdot 10^2 s + 10\cdot 10^4 = 0.$$

The roots of this equation are:

$$s_1 = \left[-\frac{12.5}{2} + \sqrt{\left(\frac{12.5}{2}\right)^2 - 10} \right]\cdot 10^2 = -0.860\cdot 10^2\ \mathrm{s}^{-1}$$

$$s_2 = \left[-\frac{12.5}{2} - \sqrt{\left(\frac{12.5}{2}\right)^2 - 10} \right]\cdot 10^2 = -11.60\cdot 10^2\ \mathrm{s}^{-1},$$

which are two different negative real numbers. Therefore the natural response is:

$$i_n(t) = A_1 e^{-86t} + A_2 e^{-1160t},$$

which consists of two exponential functions and is of the overdamped kind.

It should be noted that in second order circuits, which contain two energy-storing elements of the same kind (two L-s, or two C-s), the transient response cannot be oscillatory and is always exponential overdamped. It is worthwhile to analyze the roots of the above characteristic equation. We may then obtain

$$s_{1,2} = \frac{1}{2(L_1 L_2' - M^2)} \left[(R_1 L_2' + R_2' L_1) \pm \sqrt{(R_1 L_2' + R_2' L_1)^2 - 4(L_1 L_2' - M^2) R_1 R_2'} \right] \tag{1.34}$$

The expression under the square root can be simplified to the form: $(R_1 L_2' + R_2' L_1)^2 + 4R_1 R_2' M^2 > 0$, which is always positive, i.e., both roots are negative real numbers and the transient response of the overdamped kind. These results once again show that in a circuit, which contains energy-storing elements of the same kind, the transient response cannot be oscillatory.

In conclusion, it is important to pay attention to the fact that all the real roots of the characteristic equations, under study, were negative as well as the real part of the complex roots. This very important fact follows the physical reality that the natural response and transient-state cannot exist in infinite time. As we already know, the natural response takes place in the circuit free of sources and must vanish due to the energy losses in the resistances. Thus, natural responses, as exponential functions e^{st}, must be of a negative power ($s < 0$) to decay with time.

1.7 INDEPENDENT AND DEPENDENT INITIAL CONDITIONS

From now on, we will use the term "switching" for any change or interruption in an electrical circuit, planned as well as unplanned, i.e. different kinds of faults or other sudden changes in energy distribution.

1.7.1 Two switching rules (laws)

The principle of a gradual change of energy in any physical system, and specifically in an electrical circuit, means that the energy stored in magnetic and electric fields cannot change instantaneously. Since the magnetic energy is related to the magnetic flux and the current through the inductances (i.e., $w_m = \lambda i_L/2$), both of them **must not be allowed to change instantaneously**. In transient analysis it is common to assume that the switching action takes place at an instant of time that is defined as $t = 0$ (or $t = t_0$) and occurs instantaneously, i.e. in **zero time**, which means **ideal switching**. Henceforth, we shall indicate two instants: the instant just prior to the switching by the use of the symbol 0_-, i.e. $t = 0_-$, and the instant just after the switching by the use of the symbol 0_+, i.e. $t = 0_+$, (or just 0), as shown in Fig. 1.16. Using mathematical language, the value of the function $f(0_-)$, is the "limit from the left", as t approaches zero from the left and the value of the function $f(0_+)$ is the "limit from the right", as t approaches zero from the right.

Keeping the above comments in mind, we may now formulate two switching rules.

(a) First switching law (or first switching rule)

The first switching rule/law determines that **the current (magnetic flux) in an inductance just after switching $i_L(0_+)$ is equal to the current (flux) in the same inductance just prior to switching**

$$i_L(0_+) = i_L(0_-) \tag{1.35a}$$

$$\lambda(0_+) = \lambda(0_-). \tag{1.35b}$$

Equation 1.35a determines the initial value of the inductance current and enables

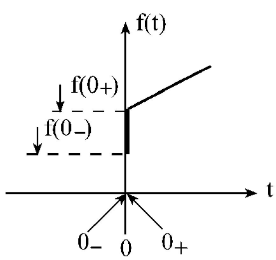

Figure 1.16 The instants: prior to switching (0_-), switching (0) and after switching (0_+).

us to find the integration constant of the natural response in circuits containing inductances. If the initial value of the inductance current is zero (zero initial conditions), the inductance at the instant $t = 0$ (and only at this instant) is equivalent to an open circuit (open switch) as shown in Fig. 1.17(a). If the initial value of the inductance current is not zero (non-zero initial conditions) the inductance is equivalent at the instant $t = 0$ (and only at this instant) to a current source whose value is the initial value of the inductance current $I_s = i_L(0)$, as shown in Fig. 1.17(b). Note that this equivalent, current source may represent the inductance in a most general way, i.e., also in the case of the zero initial current. In this case, the value of the current source is zero, and inner resistance is infinite (which means just an open circuit).

(*b*) *Second switching law* (*or second switching rule*)

The *second* switching rule/law determines that **the voltage (electric charge) in a capacitance just after switching $v_C(0_+)$ is equal to the voltage (electric charge) in the same capacitance just prior to switching**

$$v_C(0_+) = v_C(0_-) \tag{1.36a}$$

$$q(0_+) = q(0_-). \tag{1.36b}$$

Equation 1.36a determines the initial value of the capacitance voltage and

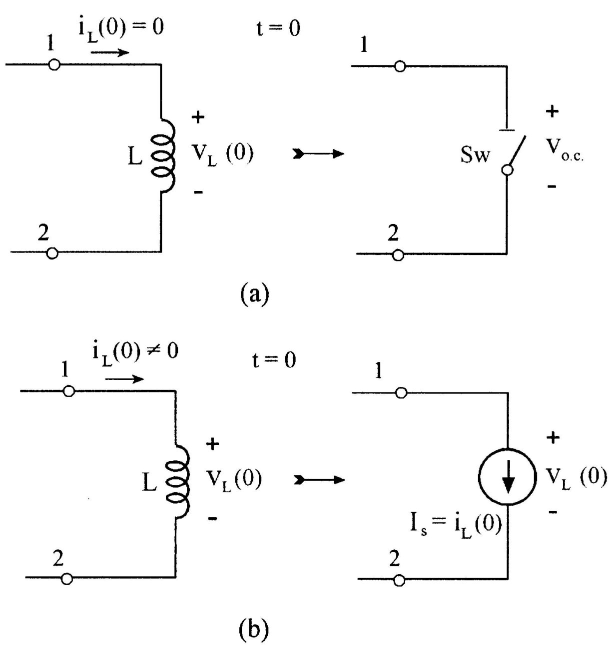

Figure 1.17 An equivalent circuit for an inductance at $t = 0$, with a zero initial current (a) and with current $i_L(0)$ (b).

enables us to find the integration constant of the natural response in circuits containing capacitances. If the initial value of the voltage across a capacitance is zero, zero initial conditions, the capacitance at the instant $t = 0$ (and only at this instant) is equivalent to a short-circuit (closed switch) as shown in Fig. 1.18(a). If the initial value of the capacitance voltage is not zero (non-zero initial conditions), the capacitance, at the instant $t = 0$ (and only at this instant), is equivalent to the voltage source whose value is the initial capacitance voltage $V_s = v_C(0)$, as shown in Fig. 1.18(b). Note that this equivalent, voltage source may represent the capacitance in a most general way, i.e., also in the case of the zero initial voltage. In this case, the value of the voltage source is zero, and inner resistance is zero (which means just a short-circuit).

In a similar way, as a current source may represent an inductance with a zero initial current, we can also use the voltage source as an equivalent of the capacitance with a zero initial voltage. Such a source will supply zero voltage, but its zero inner resistance will form a short-circuit.

If the initial conditions are zero, it means that the current through the inductances and the voltage across the capacitances will start from zero value, whereas if the initial conditions are non-zero, they will continue with the same values, which they possessed prior to switching.

The initial conditions, given by equations 1.35 and 1.36, i.e., the currents

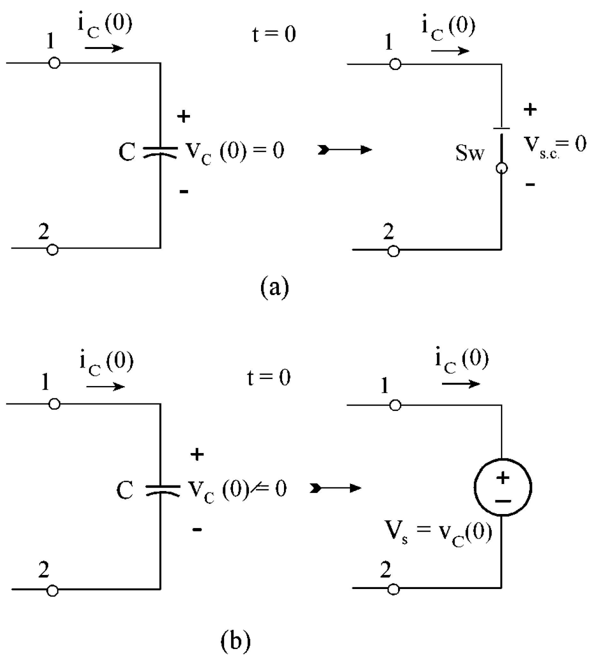

Figure 1.18 An equivalent circuit for a capacitance, at $t = 0$, with zero initial voltage (a) and with non-zero initial voltage $v_C(0)$ (b).

through the inductances and voltages across the capacitances, are called **independent initial conditions**, since they do not depend either on the circuit sources or on the status of the rest of the circuit elements. It does not matter how they had been set up, or what kind of switching or interruption took place in the circuit.

The rest of the quantities in the circuit, i.e., the currents and the voltages in the resistances, the voltages across the inductances and currents through the capacitances, can change abruptly and their values at the instant just after the switching ($t = 0_+$) are called **dependent initial conditions.** They depend on the independent initial conditions and on the status of the rest of the circuit elements. The determination of the dependent initial conditions is actually the most arduous part of the classical method. In the next sections, methods of determining the initial conditions will be introduced. We shall first, however, show how the independent initial conditions can be found.

1.7.2 Methods of finding independent initial conditions

For the determination of independent initial conditions the given circuit/network shall be inspected at its steady-state operation prior to the switching. Let us illustrate this procedure in the following examples.

Example 1.3

In the circuit in Fig. 1.19, a transient-state occurs due to the closing of the switch (*Sw*). Find the expressions of the independent initial values, if prior to the switching the circuit operated in a d.c. steady-state.

Solution

By inspection of the given circuit, we may easily determine 1) the current through the inductance and 2) the voltages across two capacitances.

1) Since the two capacitances in a d.c. steady-state are like an open switch the

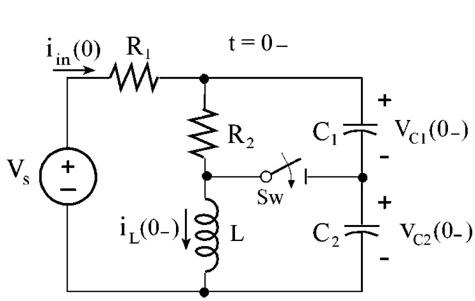

Figure 1.19 The circuit of example 1.3 at instant time $t = 0_-$.

inductance current is

$$i_L(0_-) = \frac{V_s}{R_1 + R_2}.$$

2) Since the voltage across the inductance in a d.c. steady-state is zero (the inductance provides a closed switch), the voltage across the capacitances is

$$v_C(0_-) = R_2 i_L(0_-).$$

This voltage is divided between two capacitors in inverse proportion to their values (which follows from the principle of their charge equality, i.e., $C_1 v_{C1} = C_2 v_{C2}$), which yields:

$$v_{C1}(0_-) = R_2 i_L(0_-) \frac{C_2}{C_1 + C_2}$$

$$v_{C2}(0_-) = R_2 i_L(0_-) \frac{C_1}{C_1 + C_2}.$$

Example 1.4

Find the independent initial conditions $i_L(0_-)$ and $v_C(0_-)$ in the circuit shown in Fig. 1.20, if prior to opening the switch, the circuit was under a d.c. steady-state operation.

Solution

1) First, we find the current i_4 with the current division formula (no current is flowing through the capacitance branch)

$$i_4 = I_s \frac{R_5}{R_5 + R_4 + R_3//R_1} = I_s \frac{R_5(R_1 + R_3)}{R_1R_3 + R_1R_4 + R_1R_5 + R_3R_4 + R_3R_5}.$$

Using once again the current division formula, we obtain the current through the inductance

$$i_L(0_-) = i_4 \frac{R_3}{R_3 + R_1} = i_s \frac{R_3R_5}{R_1R_3 + R_1R_4 + R_1R_5 + R_3R_4 + R_3R_5}.$$

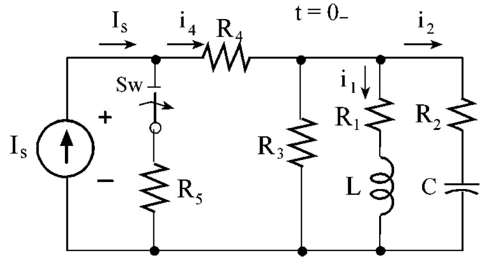

Figure 1.20 The circuit prior to the switching $t = 0_-$ of example 1.4.

2) The capacitance voltage can now be found as the voltage drop in resistance R_1

$$v_C(0_-) = R_1 i_L(0_-).$$

The examples given above show that in order to determine the independent initial conditions, i.e., the initial values of inductance currents and/or capacitance voltages, we must consider the circuit under study prior to the switching, i.e. at instant $t = 0_-$. It is usual to suppose that the previous switching took place a long time ago so that the transient response has vanished. We may apply all known methods for the analysis of circuits in their steady-state operation. Our goal is to choose the most appropriate method based on our experience in order to obtain the quickest answer for the quantities we are looking for.

1.7.3 Methods of finding dependent initial conditions

As already mentioned the currents and voltages in resistances, the voltages across inductances and the currents through capacitances can change abruptly at the instant of switching. Therefore, the initial values of these quantities should be found in the circuit just after switching, i.e., at instant $t = 0_+$. Their new values will depend on the new operational conditions of the circuit, which have been generated after switching, as well as on the values of the currents in the inductances and voltages of the capacitances. For this reason we will call them **dependent initial conditions**.

As we have already observed, the natural response in the circuit of the second order is, for instance, of form equation 1.31. Therefore, two arbitrary constants A_1 and A_2, **called integration constants**, have to be determined to satisfy the two initial conditions. One is the initial value of the function and the other one, as we know from mathematics, is the initial value of its first derivative. Thus, for circuits of the second order or higher the initial values of derivatives at $t = 0_+$ must also be found. We also consider the initial values of these derivatives as **dependent initial conditions**.

In order to find the dependent initial conditions we must consider the analyzed circuit, which has arisen after switching and in which all the inductances and capacitances are replaced by current and voltage sources (or, with zero initial conditions, by an open and/or short-circuit). Note that this circuit fits only at the instant $t = 0_+$. For finding the desirable quantities, we may use all the known methods of steady-state analysis. Let us introduce this technique by considering the following examples.

Example 1.5

Consider once again the circuit in Fig. 1.20. We now however need to find the initial value of current $i_2(0_+)$, which flows through the capacitance and therefore can be changed instantaneously.

Solution

We start the solution by drawing the equivalent circuit for instant $t = 0_+$, i.e. just after switching, Fig. 1.21. The inductance and capacitance in this circuit are replaced by the current and voltage sources, whose values have been found in Example 1.4 and are assigned as I_{L0} and V_{C0}.

The achieved circuit has two nodes and the most appropriate method for its solution is node analysis. Thus,

$$-I_s + G_3 V_{ab} + I_{L0} + i_2(0) = 0,$$

where $G_3 = 1/R_3$. Substituting $V_{ab} = V_{C0} + R_2 i_2(0)$ for V_{ab} we may obtain

$$i_2(0)(1 + G_3 R_2) = I_s - I_{L0} - G_3 V_{C0},$$

or

$$i_2(0) = \frac{I_s - I_{L0} - G_3 V_{C0}}{1 + G_3 R_2}.$$

Example 1.6

Let us say that we are interested in finding the initial value of the input current in the circuit of Example 1.3, shown in Figure 1.19.

Solution

Since the current we are looking for is a current in a resistance, which can change abruptly, we shall consider the circuit at instant $t = 0_+$. This circuit is shown in Fig. 1.22 where the inductance is replaced by a current source and the capacitances are replaced by voltage sources.

The quickest way to find $i_{in}(0_+)$ is by using the superposition principle. For this purpose, we shall consider two circuits: in the first one only the voltage sources are in action (circuit (b) in Fig. 1.22) and in the other one only the current source is in action (circuit (c) in Fig. 1.22). By inspection of the first circuit and by applying Kirchhoff's voltage law to the outer loop, we have

$$i'_{in}(0_+) = \frac{V_s - V_{C1} - V_{C2}}{R_1}.$$

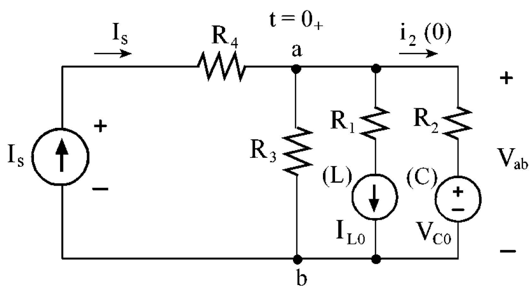

Figure 1.21 An equivalent circuit for Example 1.5.

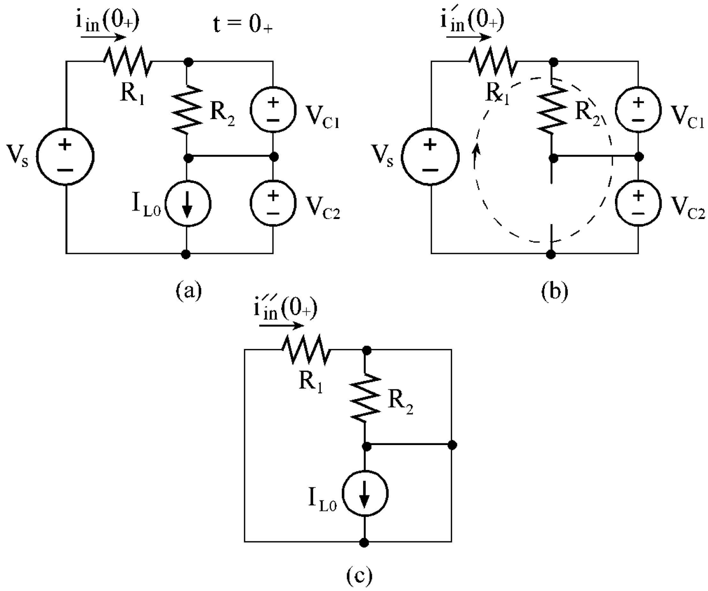

Figure 1.22 The circuit for finding $i_{in}(0_+)$ (a), the subcircuit with voltage sources (b) and the subcircuit with a current source (c).

By inspection of the second circuit in which the current source is short-circuited, we have

$$i''_{in}(0_+) = 0.$$

Therefore, finally

$$i_{in}(0_+) = i'_{in}(0_+) = \frac{V_s - V_{C1} - V_{C2}}{R_1}.$$

Example 1.7

As a numerical example, let us consider the circuit in Fig. 1.23. Suppose that we wish to find the initial value of the output voltage, just after switch *Sw* instantaneously changes its position from “1” to “2”. The circuit parameters are: $L = 0.1$ H, $C = 0.1$ mF, $R_1 = 10\ \Omega$, $R_2 = 20\ \Omega$, $R_{ld} = 100\ \Omega$, $V_{s1} = 110$ V and $V_{s2} = 60$ V.

Solution

In order to answer this question, we must first find the independent initial conditions, i.e., $i_L(0_+)$ and $v_C(0_+)$. By inspection of the circuit for instant $t = 0_-$, Fig. 1.23(a), we have

$$i_L(0_-) = \frac{V_{s1}}{R_1 + R_{ld}} = \frac{110}{100 + 10} = 1 \text{ A},$$

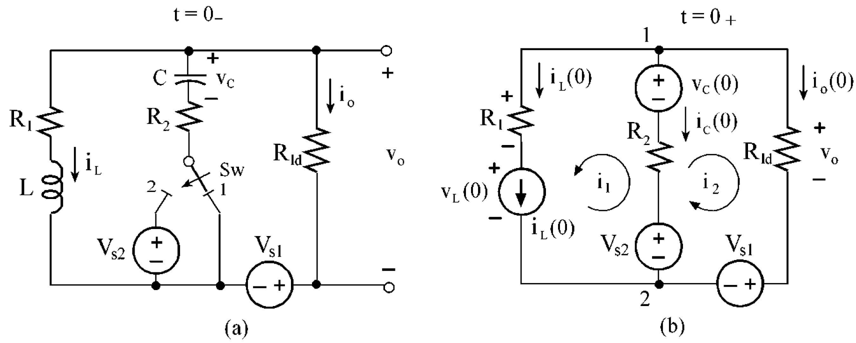

Figure 1.23 A given circuit for Example 1.7(a) and its equivalent at $t = 0_+$(b).

and

$$v_C(0_-) = V_{s1}\frac{R_1}{R_1 + R_{ld}} = 110\frac{10}{100 + 10} = 10\ \text{V}.$$

With two switching rules we have

$$i_L(0_+) = i_L(0_-) = 1\ \text{A}$$

$$v_C(0_+) = v_C(0_-) = 10\ \text{V},$$

and we can now draw the equivalent circuit for instant $t = 0_+$, Fig. 1.23(b). By inspection, using KCL (Kirchhoff's current law), we have

$$R_{ld}i_2 + R_2(i_2 + i_1) = -V_{s1} + V_{s2} + v_C(0). \tag{1.37}$$

Keeping in mind that $i_2 = i_o$ and $i_1 = i_L(0)$, we obtain

$$i_2(0) = \frac{-V_{s1} + V_{s2} + v_C(0) - R_2 i_L(0)}{R_2 + R_{ld}} = \frac{-110 + 60 + 10 - 20\cdot 1}{20 + 100} = -0.5\ \text{A}.$$

Thus the initial value of the output current is -0.5 A. Note that, prior to switching, the value of the output current was -1 A, therefore, with switching the current drops to half of its previous value.

The circuit of this example is of the second order and, as earlier mentioned, its natural response consists of two unknown constants of integration. Therefore, we shall also find the derivative of the output current at instant $t = 0_+$. By differentiating equation 1.37 with respect to time, and taking into consideration that V_{s1} and V_{s2} are constant, we have

$$(R_2 + R_{ld})\frac{di_o}{dt} + R_2\frac{di_L}{dt} = \frac{dv_C}{dt},$$

and, since $\dfrac{di_L}{dt} = \dfrac{1}{L}v_L$ and $\dfrac{dv_C}{dt} = \dfrac{1}{C}i_C$,

$$\left.\frac{di_o}{dt}\right|_{t=0} = \frac{1}{R_2 + R_{ld}}\left[\frac{1}{C}i_C(0) - \frac{R_2}{L}v_L(0)\right].$$

By inspection of the circuit in Fig. 1.23(b) once again, we may find

$$v_L(0) = V_{s1} + R_{ld} i_o(0) - R_1 i_L(0) = 110 + 100(-0.5) - 10 \cdot 1 = 40\,\text{V}.$$

$$i_C(0) = -i_o(0) - i_L(0) = 0.5 - 1 = -0.5\ \text{A}.$$

Thus,

$$\left.\frac{di_o}{dt}\right|_{t=0} = \frac{1}{20+100}\left(\frac{-0.5}{0.1 \cdot 10^{-3}} - \frac{10}{0.1}40\right) = -75\ \text{A s}^{-1}.$$

1.7.4 Generalized initial conditions

Our study of initial conditions would not be complete without mention of the so-called *incorrect initial conditions*, i.e. by which it looks as though the two switching laws are disproved.

(a) Circuits containing capacitances

As an example of such a "disproval", consider the circuit in Fig. 1.24(a). In this circuit, the voltage across the capacitance prior to switching is $v_C(0_-) = 0$ and after switching it should be $v_C(0_+) = V_s$, because of the voltage source. Thus,

$$v_C(0_+) \neq v_C(0_-),$$

and the second switching law is disproved.

This paradox can be explained by the fact that the circuit in Fig 1.24(a) is not a physical reality, but only a mathematical model, since it is built of two ideal elements: an ideal voltage source and an ideal capacitance. However, every electrical element in practice has some value of resistance, and generally speaking some value of inductance (but this inductance is very small and in our future discussion it will be neglected). Because, in a real switch, the switching process takes some time (even very small), during which the spark appears, the latter is also usually approximated by some value of resistance. By taking into consideration just the resistances of the connecting wires and/or the inner resistance of the source or the resistance of the spark, connected in series, and a resistance, which represents the capacitor insulation, connected in parallel, we obtain the

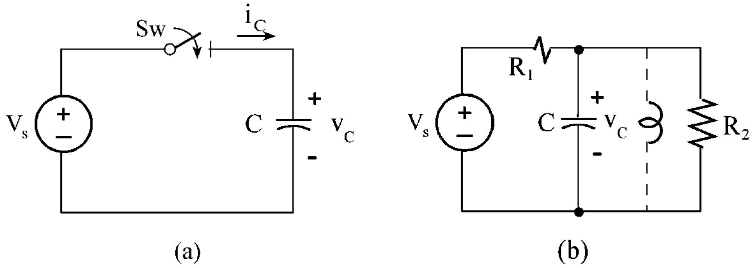

Figure 1.24 An incorrect circuit model of a source and a capacitor (a) and its corrected version (b).

circuit shown in Fig. 1.24(b). In this circuit, the second switching law is correct and we may write

$$v_C(0_+) = v_C(0_-).$$

Now, at the instant of switching, i.e., at $t = 0$, the magnitude of the voltage drop across this resistance will be as large as the source value. As a result the current of the first moment will be very large, however not unlimited, like it is supposed to be in Fig. 1.24(a). In order to illustrate the transient behavior in the circuit discussed, let us turn to a numerical example. Suppose that a 1.0 nF condenser is connected to a 100 V source and let the resistance of the connecting wires be about one hundredth of an ohm. In such a case, the "spike" of the current will be $I_\delta = 100/0.01 = 10{,}000$ A, which is a very large current in a 100 V source circuit (but it is not infinite). This current is able to charge the above condenser during the time period of about 10^{-11} s, since the required charge is $q = CV = 10^{-9}\cdot 10^2 = 10^{-7}$ C and $\Delta t \cong \Delta q/\Delta i = 10^{-7}/10^4 = 10^{-11}$ s. This period of time is actually equal to the time constant of the series RC circuit, $\tau = RC = 10^{-2}\cdot 10^{-9} = 10^{-11}$ s.

From another point of view, the amount of the charge, which is transferred by an exponentially decayed current, is equal to the product of its initial value, I_0 and the time constant. Indeed, from Fig. 1.25, we have

$$q = \int i\,dt = I_0 \int_0^\infty e^{-t/\tau} dt = I_0(-\tau)e^{-t/\tau}\Big|_0^\infty = I_0\tau, \qquad (1.38)$$

i.e., $q = 10{,}000\cdot 10^{-11} = 10^{-7}$ C, as estimated earlier. This result (equation 1.38) justifies using an impulse function δ (see further on) for representing very large (approaching infinity) magnitudes applying very short (approaching zero) time intervals, whereas their product stays finite, as shown in Fig. 1.25.

Note that the second resistance R_2 is very large (hundreds of mega ohms), so that the current through this resistance, being very small (less than a tenth of a microampere), can be neglected.

In conclusion, when a capacitance is connected to a voltage source, a very large current, tens of kiloamperes, charges the capacitance during a vanishing time interval, so that we may say that the capacitance voltage changes from zero to its final value, practically immediately. However, of course, none of the

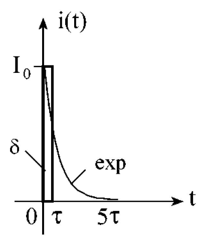

Figure 1.25 A large and fast decaying exponent and an equivalent impulse.

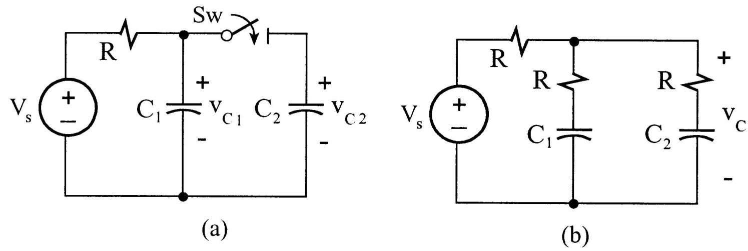

Figure 1.26 A circuit in which the second switching law is "disproved": prior switching (a) and after switching (b).

physical laws, neither the switching law nor the law of energy conservation, has been disproved.

As a second example, let us consider the circuit in Fig. 1.26(a). At first glance, applying the second switching law, we have

$$\begin{aligned} v_{C1}(0_+) &= v_{C1}(0_-) = V_s \\ v_{C2}(0_+) &= v_{C2}(0_-) = 0. \end{aligned} \tag{1.39}$$

But after switching, at $t = 0$, the capacitances are connected in parallel, Fig. 1.26(b), and it is obvious that

$$v_{C1}(0_+) = v_{C2}(0_-) \tag{1.40}$$

which is in contrast to equation 1.39.

To solve this problem we shall divide it into two stages. In the first one, the second capacitance is charged practically immediately in the same way that was explained in the previous example. During this process, part of the first capacitance charge is transferred by a current impulse to the second capacitance, so that the entire charge is distributed between the two capacitances in reciprocal proportion to their values. The common voltage of these two capacitances, connected in parallel, after the switching at instant $t = 0$, is reduced to a new value lower than the applied voltage V_s.

In the second stage of the transient process in this circuit, the two capacitances will be charged up so that the voltage across the two of them will increase up to the applied voltage V_s. To solve this second stage problem we have to know the new initial voltage in equation 1.40. We shall find it in accordance with equation 1.36b which, as was mentioned earlier, expresses the physical principal of continuous electrical charges, i.e. the latter cannot change instantaneously. This requirement is general but even more stringent than the requirement of continuous voltages, and therefore is called the **generalized second switching law**. Thus,

$$q_\Sigma(0_+) = q_\Sigma(0_-) = C_1 v_{C1}(0_-). \tag{1.41}$$

This law states that: **the total amount of charge in the circuit cannot change instantaneously and its value prior to switching is equal to its value just after the switching, i.e., the charge always changes gradually.**

Since the new equivalent capacitance after switching is $C_{eq} = C_1 + C_2$, we may write

$$q_\Sigma(0_+) = (C_1 + C_2)v_{C1}(0_+) = C_1 v_{C1}(0_-).$$

Since, in this example, $v_{C1}(0_-) = V_s$, we finally have

$$v_{C1}(0_+) = \frac{C_1}{C_1 + C_2} v_{C1}(0_-) = \frac{C_1}{C_1 + C_2} V_s. \tag{1.42}$$

With this initial condition, the integration constant can easily be found.

It is interesting to note that by taking into consideration the small resistances (wires, sparks, etc.) the circuit becomes of second order and its characteristic equations will have two roots (different real negative numbers). One of them will be very small, determining the first stage of transients, and the second one, relatively large, will determine the second stage.

Let us now check the energy relations in this scheme, Fig. 1.26, before and after switching. The energy stored in the electric field of the first capacitance (prior to switching) is $w_e(0_-) = \frac{1}{2}C_1 V_{C1}^2(0_-) = \frac{1}{2}C_1 V_s^2$ and the energy stored in the electric field of both capacitances (after switching) is $w_e(0_+) = \frac{1}{2}(\mathrm{C}_1 + \mathrm{C}_2)\mathrm{v}_C^2(0_+)$. Thus, the energy "lost" is

$$\Delta w_e = w_e(0_-) - w_e(0_+) = \frac{C_1 V_s^2}{2} - \frac{C_1 + C_2}{2}\left(\frac{C_1 V_s}{C_1 + C_2}\right)^2 = \frac{C_1 C_2 V_s^2}{2(C_1 + C_2)}. \tag{1.43}$$

This energy actually dissipates in the above-discussed resistances.

When two capacitances, connected in series, switch to the voltage source, as shown in Fig. 1.27(a), the transients will also consist of two stages. In the first stage, the current impulse will charge two capacitances equally to the same charge

$$q(0_+) = V_s \frac{C_1 C_2}{C_1 + C_2}, \tag{1.44}$$

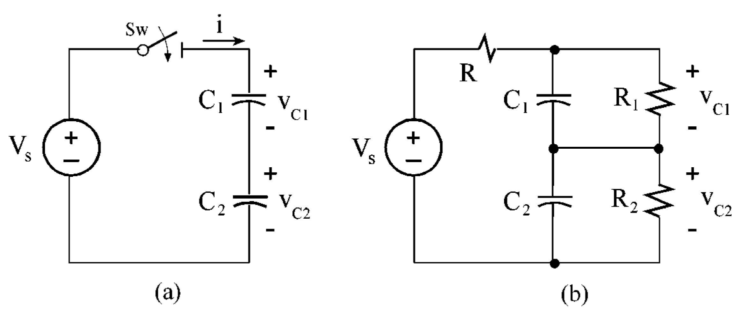

Figure 1.27 Two capacitances in series are connected to the voltage source: incorrect (a) and correct (b) circuits.

but to different voltages, in reciprocal proportion to their values:

$$v_{C1}(0_+) = V_s \frac{C_2}{C_1 + C_2}, \quad v_{C2}(0_+) = V_s \frac{C_1}{C_1 + C_2}. \tag{1.45}$$

However, in accordance with the correct equivalent circuit in Fig. 1.27(b), the final steady-state voltages (at $t \to \infty$) across two capacitances must be determined by the voltage division in proportion to their resistances:

$$v_{C1}(\infty) = V_s \frac{R_1}{R_1 + R_2}, \quad v_{C2}(\infty) = V_s \frac{R_2}{R_1 + R_2}. \tag{1.46}$$

This change in voltages, from equation 1.45 to equation 1.46, takes place during the second stage with the time constant $\tau = (C_1 + C_2)/(G_1 + G_2)$ (proof of this expression is left to the reader as an exercise).

Finally it should be noted that the very fast charging of the capacitances by the flow of *very large currents* (*current impulses*) results in relatively *small energy dissipation*, so that usually no damage is caused to the electrical equipment. Indeed, with the numerical data of our first example, we may calculate

$$w_d = RI_\delta^2 \int_0^\infty e^{-\frac{2t}{\tau}} = RI_\delta^2 \frac{\tau}{2} = 10^{-2}(10^4)^2 \cdot 10^{-11} \cdot 0.5 = 0.5 \cdot 10^{-5}\ \mathrm{J},$$

which is negligibly small. Checking the law of energy conservation, we may find that the energy being delivered by the source is

$$w_s = \int_0^\infty V_s i\, dt = V_s \int_0^\infty C \frac{dv_C}{dt} dt = CV_s \int_0^{V_s} dv_C = CV_s^2,$$

and the energy being stored into the capacitances is $w_e = \frac{1}{2} CV_s^2$, i.e., half of the energy delivered by the source is dissipated in the resistances. Calculating this energy yields

$$\Delta w_s = \frac{CV_s^2}{2} = \frac{10^{-9} \cdot 10^4}{2} = 0.5 \cdot 10^{-5}\ \mathrm{J},$$

as was previously calculated.

(*b*) *Circuits containing inductances*

We shall analyze the circuits containing inductances keeping in mind that such circuits are dual to those containing capacitances and using the results, which have been obtained in our previous discussion.

Consider the circuit shown in Fig. 1.28 in which the current prior to switching is $i_L(0_-) = I_0$ and after switching is supposed to be $i_L(0_+) = 0$, so that the first switching law is disproved

$$i_L(0_+) \neq i_L(0_-).$$

However, by taking into consideration the small parameters G, R_L, and C, we

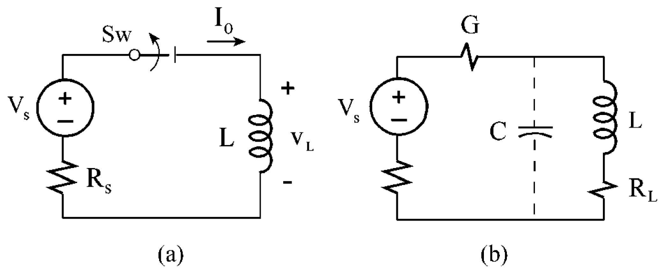

Figure 1.28 An incorrect circuit containing a disconnected inductance (a) and its improved equivalent (b).

may obtain the correct circuit, shown in Fig 1.28(b), in which all the physical laws are proven.

In this circuit, the open switch is replaced by a very small conductance G (very big resistance), so that we can now write $i_L(0_+) = i_L(0_-)$, but because of the vanishingly small time constant $\tau = GL$, the current decays almost instantaneously.

From another point of view the almost abrupt change of inductance current results in a very large voltage induced in inductance, $v_L = L(di/dt)$, which is applied practically all across the switch, and causes an arc, which appears between the opening contacts of the switch. Let us estimate the magnitude of such an overload across the coil in Fig. 1.28(a), having 0.1 H and 20 Ω;, which disconnects almost instantaneously from the voltage source, and the current through the coil prior to switching was 5 A. Assume that the time of switching is $\Delta t = 10$ μs (note that this time, during which the current changes from the initial value to zero, can be achieved if the switch is replaced by a resistor of at least 50 kΩ, as shown in Fig. 1.28(b)), then the overvoltage will be $V_{\max} \cong L(\Delta i/\Delta t) = 0.1 \cdot 5 \cdot 10^5 = 50$ kV.

Such a high voltage usually causes an arc, which appears between the opening contacts of the switch. This transient phenomenon is of great practical interest since in power system networks the load is mostly of the inductance kind and any disconnection of the load and/or short-circuited branch results in overvoltages and arcs. However, the capacitances associated with all the electric parts of power systems affect its transient behavior and usually result in reducing the overvoltages. (We will analyze this phenomenon in more detail also taking into consideration the capacitances, see Chapter 2).

Consider next the circuit in Fig. 1.29, which is dual to the circuit in Fig. 1.26. (It should be noted that the duality between the two circuits above, Figs 1.28 and 1.29, and the corresponding capacitance circuits, in Figs 1.24 and 1.26, is not full. For full duality the voltage sources must be replaced by current sources. However, the quantities, the formulas, and the transient behavior are similar.) In this circuit, prior to switching $i_{L1}(0_-) = I_0$ and $i_{L2}(0_-) = 0$. Applying the first

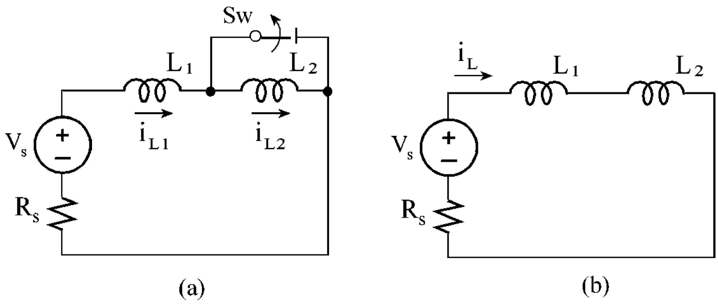

Figure 1.29 A circuit containing two inductances, in which the first switching law is "disproved": prior to switching (a) and after switching (b).

switching law we shall write

$$\begin{aligned} i_{L1}(0_+) &= i_{L1}(0_-) = I_0 \\ i_{L2}(0_+) &= i_{L2}(0_-) = 0. \end{aligned} \tag{1.47}$$

After switching the two inductances are connected in series, Fig 1.29(b), therefore

$$i_{L1}(0_+) = i_{L2}(0_-), \tag{1.48}$$

which is obviously contrary to equation 1.47. However, we may consider the transient response of this circuit as similar to that in capacitance and conclude that it is composed of two stages. In the first stage, the currents change almost instantaneously, in a very short period of time $\Delta t \to 0$, so that voltage impulses appear across the inductances. In the second stage, the current in both inductances changes gradually from its initial value up to its steady-state value. In order to find the initial value of the common current flowing through both inductances connected in series (just after switching and after accomplishing the first stage) we may apply the so-called **first generalized switching law** (equation 1.35b). This law states that: **the total flux linkage in the circuit cannot change instantaneously and its value prior to switching is equal to its value just after switching, i.e. the flux linkage always changes gradually.**

If an electrical circuit contains only one inductance element, then

$$Li_L(0_+) = Li_L(0_-) \quad \text{or} \quad i_L(0_-) = i_L(0_+),$$

and the first switching law regarding flux linkages (equation 1.35b) is reduced to a particular case with regard to the currents. For this reason the first switching law, regarding flux linkages, is more general.

Applying the first generalized law to the circuit in Fig. 1.29, we have

$$L_1 i_{L1}(0_-) + L_2 i_{L2}(0_-) = L_1 i_{L1}(0_+) + L_2 i_{L2}(0_+), \tag{1.49}$$

or since $i_{L1}(0_+) = i_{L2}(0_+) = i_L(0_+)$ we have

$$i_L(0_+) = \frac{L_1 i_{L1}(0_-) + L_2 i_{L2}(0_-)}{L_1 + L_2}.$$

Substituting $i_{L2}(0_-) = 0$ and $i_{L1}(0_-) = I_0$ the above expression becomes

$$i_L(0_+) = \frac{L_1}{L_1 + L_2} I_0. \tag{1.50}$$

This equation enables us to determine the initial condition of the inductance current in the second stage of a transient response.

The energy stored in the magnetic field of two inductances prior to switching is

$$w_m(0_-) = \frac{L_1 i_{L1}^2(0_-)}{2} + \frac{L_2 i_{L2}^2(0_-)}{2}, \tag{1.50a}$$

and after switching

$$w_m(0_+) = \frac{(L_1 + L_2) i_L^2(0_+)}{2}. \tag{1.50b}$$

Then the amount of energy dissipated in the first stage of the transients, i.e., in circuit resistances and in the arc, with equations 1.50a and 1.50b will be

$$\Delta w_m = w_m(0_-) - w_m(0_+) = \frac{1}{2} \frac{L_1 L_2}{L_1 + L_2} [i_{L1}(0_-) - i_{L2}(0_-)]^2. \tag{1.51}$$

(Developing this formula is left to the reader as an exercise.) For the circuit under consideration the above equation 1.51 becomes

$$\Delta w_m = \frac{1}{2} \frac{L_1 L_2}{L_1 + L_2} I_0^2. \tag{1.52}$$

It is interesting to note that this expression is similar to formula 1.43 for a capacitance circuit. Let us now consider a numerical example.

Example 1.8

In the circuit in Fig. 1.30(a) the switch opens at instant $t = 0$. Find the initial current $i(0_+)$ in the second stage of the transient response and the energy

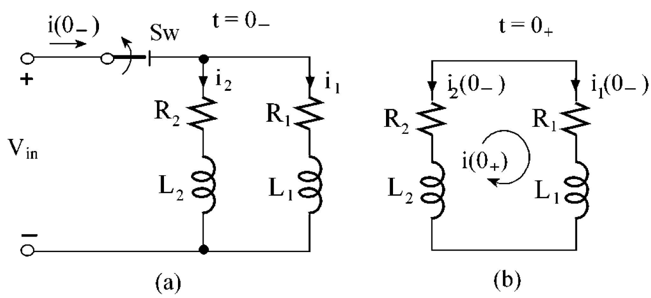

Figure 1.30 A circuit for Example 1.8: prior to switching (a) and after switching (b).

dissipated in the first stage if the parameters are: $R_1 = 50\,\Omega$, $R_2 = 40\,\Omega$, $L_1 = 160$ mH, $L_2 = 40$ mH, $V_{in} = 200$ V.

Solution

The values of the two currents in circuit (a) are

$$i_{L1}(0_-) = \frac{V_{in}}{R_1} = \frac{200}{50} = 4\ \text{A}$$

and

$$i_{L2}(0_-) = \frac{V_{in}}{R_2} = \frac{200}{40} = 5\ \text{A}.$$

Thus, the initial value of the current in circuit (b), in accordance with equation 1.49, is

$$i_L(0_+) = \frac{L_1 i_{L1}(0_-) - L_2 i_{L2}(0_-)}{L_1 + L_2} = \frac{160 \cdot 4 - 40 \cdot 5}{160 + 40} = 2.2\ \text{A}.$$

Note that for the calculation of the initial current $i(0_+)$ in circuit (b), we took into consideration that the current $i_{L2}(0_-)$ is negative since its direction is opposite to the direction of $i(0_+)$, which has been chosen as the positive direction. The dissipation of energy, in accordance with equation 1.51, is

$$\Delta w_m = \frac{L_1 L_2 [i_{L1}(0_-) - i_{L2}(0_-)]^2}{2(L_1 + L_2)} = \frac{160 \cdot 40 \cdot 10^{-3}(4+5)^2}{2(160+40)} \cong 1.3\ \text{J}.$$

As a final example, consider the circuit in Fig. 1.31. This circuit of two inductive branches in parallel to a current source is a complete dual to the circuit in Fig. 1.27, in which two capacitances in series are connected to a voltage source.

Prior to switching the inductances are short-circuited, so that both currents $i_{L1}(0_-)$ and $i_{L2}(0_-)$ are equal to zero. The current of the current source flows through the switch. (In the dual circuit, the voltages across the capacitances prior to switching are also zero.) At the instant of switching the currents through

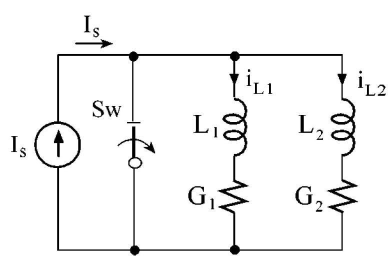

Figure 1.31 A circuit of two parallel inductances and a current source, which is a complete dual to the circuit with two series capacitances and a voltage source.

the inductances change almost instantaneously, so that their sum should be I_s. This abrupt change of currents results in a voltage impulse across the opening switch. Since this voltage is much larger than the voltage drop on the resistances, we may neglect these drops and assume that the inductances are connected in parallel. As we know, the current is divided between two parallel inductances in inverse proportion to the value of the inductances. Thus,

$$i_{L1}(0_+) = I_s \frac{L_2}{L_1 + L_2} \quad \text{and} \quad i_{L2}(0_+) = I_s \frac{L_1}{L_1 + L_2}. \tag{1.53}$$

These expressions enable us to determine the initial condition in the second stage of the transient response. The steady-state values of the inductance currents will be directly proportional to the conductances G_1 and G_2. Hence, the induced voltages across the inductances will be zero (the inductances are now short-circuited) and the resistive elements are in parallel (note that in the capacitance circuit of Fig. 1.27 the voltages across the capacitances in steady state are also directly proportional, but to the resistances, which are parallel to the capacitances). Thus,

$$i_{L1}(\infty) = I_s \frac{G_1}{G_1 + G_2} \quad \text{and} \quad i_{L2}(\infty) = I_s \frac{G_2}{G_1 + G_2}.$$

Knowing the initial and final values, the complete response can be easily obtained (see the next chapter).

1.8 METHODS OF FINDING INTEGRATION CONSTANTS

From our previous study, we know that the natural response is formed from a sum of exponential functions:

$$f_n(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + \cdots = \sum_1^n A_k e^{s_k t}. \tag{1.54}$$

where the number of exponents is equal to the number of roots of a characteristic equation. In order to determine the integration constants $A_1, A_2, \ldots A_n$ it is necessary to formulate n equations, which must obey the instant of switching, $t = 0$ (or $t = t_0$). By differentiation of the above expression $(n - 1)$ times, we may obtain

$$\begin{gathered} A_1 + A_2 + \cdots = \sum_1^n A_k = f_n(0) \\ s_1 A_1 + s_2 A_2 + \cdots = \sum_1^n s_k A_k = f'_n(0) \\ \cdots \\ s_1^{n-1} A_1 + s_2^{n-1} A_2 + \cdots = \sum_1^n s_k^{n-1} A_k = f_k^{(n-1)}(0), \end{gathered} \tag{1.55}$$

where it has been taken into consideration that

$$
\begin{aligned}
&A_k e^{s_k t}|_{t=0} = A_k \\
&\frac{d}{dt} A_k e^{s_k t}\bigg|_{t=0} = s_k A_k \\
&\cdots \\
&\frac{d^{(n-1)}}{dt^{(n-1)}} A_k e^{s_k t}\bigg|_{t=0} = s_k^{n-1} A_k.
\end{aligned}
\tag{1.56}
$$

The initial values of the natural responses are found as

$$
\begin{aligned}
&f_n(0) = f(0) - f_f(0) \\
&f'_n(0) = f'(0) - f'_f(0) \\
&\cdots \\
&f_n^{(n-1)}(0) = f^{(n-1)}(0) - f_f^{(n-1)}(0)
\end{aligned}
\tag{1.57}
$$

Thus, for the formulation in equation 1.55 of its left side quantities, we must know:

(1) the initial values of the complete transient response $f(0)$ and its $(n-1)$ derivatives, and
(2) the initial values of the force response $f_f(0)$ and its $(n-1)$ derivatives.

The technique of finding the initial values of the complete transient response in (1) has been discussed in the previous section. In brief, according to this technique: a) we have to determine the independent initial condition (currents through the inductances at and voltages across the capacitances at $t = 0_-$), and b) by inspection of the equivalent circuit which arose after switching, i.e., at $t = 0$, we have to find all other quantities by using Kirchhoff's two laws and/or any known method of circuit analysis. For determining the initial values in (2), the forced response must also be found. Let us now introduce the procedure of finding integration constants in more detail.

Consider a first order transient response and assume, for instance, that the response we are looking for is a current response. Then its natural response is

$$i_n(t) = Ae^{st}.$$

Knowing the current initial value $i(0_+)$ and its force response $i_f(t)$ we may find

$$A = i(0_+) - i_f(0). \tag{1.58}$$

If the response is of the second order and the roots of the characteristic equation are real, then

$$i_n(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}, \tag{1.59}$$

and after differentiation, we obtain

$$i'_n(t) = s_1 A_1 e^{s_1 t} + s_2 A_2 e^{s_2 t}. \tag{1.59a}$$

Suppose that we found $i(0)$ and $i'(0)$, and also $i_f(0)$ and $i'_f(0)$, then with equation 1.57

$$\begin{aligned} i_n(0) &= i(0) - i_f(0) \\ i'_n(0) &= i'(0) - i'_f(0), \end{aligned} \tag{1.60}$$

and in accordance with equation 1.55 we have two equations for determining two unknowns: A_1 and A_2

$$\begin{aligned} A_1 + A_2 &= i_n(0) \\ s_1 A_1 + s_2 A_2 &= i'_n(0). \end{aligned} \tag{1.61}$$

The solution of equation 1.61 yields

$$\begin{aligned} A_1 &= \frac{i'_n(0) - s_2 i_n(0)}{s_1 - s_2} \\ A_2 &= \frac{i'_n(0) - s_1 i_n(0)}{s_2 - s_1}. \end{aligned} \tag{1.61a}$$

If the roots of the characteristic equation are complex-conjugate, $s_{1,2} = \alpha \pm j\omega_n$, then A_1 and A_2 are also complex-conjugate, $\mathbf{A}_{1,2} = Ae^{\pm j\vartheta}$ and the natural response (equation 1.59) may be written in the form

$$i_n(t) = Ae^{+j\vartheta}e^{-\alpha t}e^{+j\omega_n t} + Ae^{-j\vartheta}e^{-\alpha t}e^{-j\omega_n t} = Be^{-\alpha t}\sin(\omega_n t + \beta), \tag{1.62}$$

where $B = 2A$ and $\beta = \vartheta + 90°$. Taking a derivative of equation 1.62 we will have

$$i'_n(t) = -B\alpha\, e^{-\alpha t}\sin(\omega_n t + \beta) + B\omega_n e^{-\alpha t}\cos(\omega_n t + \beta). \tag{1.63}$$

Equations 1.62 and 1.63 for instant $t = 0$, with the known initial conditions (equation 1.60), yield

$$\begin{aligned} B\sin\beta &= i_n(0), \\ -B\alpha\sin\beta + B\omega_n\cos\beta &= i'_n(0). \end{aligned} \tag{1.64}$$

By division of the second equation by the first one, we have

$$\omega_n \cot\beta = \frac{i'_n(0)}{i_n(0)} + \alpha,$$

and the solution is

$$\beta = \tan^{-1}\left[\frac{\omega_n i_n(0)}{i'_n(0) + \alpha i_n(0)}\right] \tag{1.65a}$$

$$B = \frac{i_n(0)}{\sin\beta}. \tag{1.65b}$$

The natural response (equation 1.62) might be written in a different form (which is preferred in some textbooks)

$$i_n(0) = e^{-\alpha t}(M \sin \omega_n t + N \cos \omega_n t), \tag{1.66}$$

where

$$M = B \cos \beta \quad \text{and} \quad N = B \sin \beta. \tag{1.67}$$

Then, by differentiating equation 1.66 and with the known initial conditions, the two equations for determining two unknowns, M and N, may be written as

$$\begin{aligned} N &= i_n(0), \\ M\omega_n - \alpha N &= i_n'(0), \end{aligned} \tag{1.68a}$$

and

$$M = \frac{i_n'(0) + \alpha i_n(0)}{\omega_n}. \tag{1.68b}$$

Knowing M and N we can find B and β and vice versa. Thus for instance

$$\beta = \tan^{-1}\frac{N}{M} \quad \text{and} \quad B = \sqrt{M^2 + N^2}$$

(substituting M and N from equation 1.68 into these expressions yields equation 1.65).

If the characteristic equation is of an order higher than two, the higher derivatives shall be found and the solution shall be performed in accordance with equation 1.55.

Example 1.9

Using the results of Example 1.7 (Fig. 1.23), find the two integration constants of the natural response of current i_o. The circuit of Example 1.7 after switching is shown here in Fig. 1.32(a).

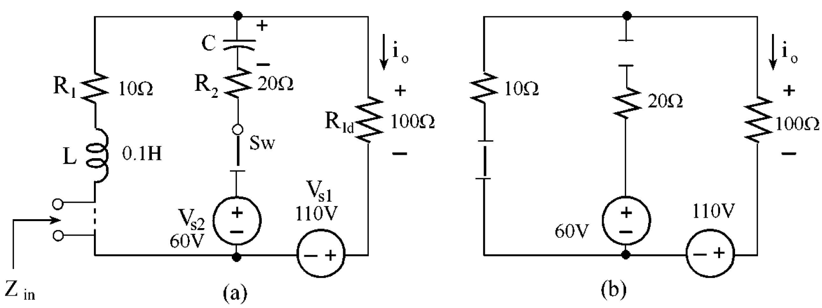

Figure 1.32 A given circuit for Example 1.9: prior to switching (a) and its equivalent in steady-state operation (b).

Solution

From Example 1.7 it is known that $i_o(0) = -0.5$ A and $i'_o(0) = -75$ A s^{-1}. To find the two constants of the integration we have to know: 1) the two roots of the second order characteristic equation and 2) the forced response.

1) In order to determine the characteristic equation we must short-circuit the voltage sources and find the input impedance by opening, for instance, the inductance branch, Fig. 1.32(a),

$$Z_{in} = R_1 + sL + \frac{(R_2 + 1/sC)R_{ld}}{R_2 + R_{ld} + 1/sC}.$$

Equaling zero and substituting the numerical values, we obtain the characteristic equation

$$s^2 = 350s + 9.17 \cdot 10^4 = 0,$$

and the roots are a complex-conjugate pair $s_{1,2} = -175 \pm j247\,\mathrm{s}^{-1}$.

2) By inspection of the circuit in the steady-state operation, Fig. 1.32(b), we have

$$i_{o,f} = \frac{-110}{100 + 10} = -1\ \mathrm{A}.$$

(Note that this current is negative, since it flows opposite to the positive direction, assigned by a solid arrow). Now we can find the initial values of the natural response. With equation 1.60 and noting that $i'_{o,f} = 0$, we have

$$i_{o,n}(0) = i_o(0) - i_{o,f}(0) = -0.5 - (-1) = 0.5\ \mathrm{A}$$

$$i'_{o,n}(0) = -75 - 0 = -75\ \mathrm{A\,s^{-1}}.$$

Since the roots are complex numbers, we shall use equation 1.65 (or equation 1.68):

$$\beta = \tan^{-1}\frac{0.5 \cdot 247}{-75 + 0.5 \cdot 175} = 84.2^\circ$$

$$B = \frac{0.5}{\sin 84.2^\circ} = 0.502.$$

(With equation 1.68 $N = i_n(0) = 0.5$ and $M = (-75 + 175 \cdot 0.5)/247 = 0.0506$ and $\beta = \tan^{-1}(0.5/0.0506) = 84.2^\circ$).

Chapter #2

TRANSIENT RESPONSE OF BASIC CIRCUITS

2.1 INTRODUCTION

In this chapter, we shall proceed with transient analysis and apply the classical approach technique, which was introduced in the previous chapter, for a further and intimate understanding of the transient behavior of different kinds of circuits. It will be shown that by applying the so-called **five-step solution** we may greatly simplify the transient analysis of any circuit, upon any interruption and under any supply, so that the determination of transient responses becomes a simple procedure.

Starting with relatively simple *RC* and *RL* circuits, we will progress to more complicated *RLC* circuits, wherein their transient analysis is done under both kinds of supplies, d.c. and a.c. The emphasis is made on the treatment of *RLC* circuits, in the sense that these circuits are more general and are more important when the power system networks are analyzed via different kinds of interruptions. All three kinds of transients in *RLC* circuit, overdamped, underdamped and critical damping, are analyzed in detail.

In power system networks, when interrupted, different kinds of resonances, on a fundamental or system frequency, as well as on higher or lower frequencies, may occur. Such resonances usually cause excess voltages and/or currents. Thus, the transients in an *RLC* circuit under this resonant behavior are also treated and the conditions for such overvoltages and overcurrents have been defined.

It is shown that using the superposition principle in transient analysis allows the simplification of the entire solution by bringing it to zero initial conditions and to only one supplied source. The theoretical material is accompanied by many numerical examples.

2.2 THE FIVE STEPS OF SOLVING PROBLEMS IN TRANSIENT ANALYSIS

As we have seen in our previous study of the classical method in transient analysis, there is no general answer, or ready-made formula, which can be applied to every kind of electrical circuit or transient problem. However, we can formulate a **five-step solution**, which will be applicable to any kind of circuit or problem. Following these five steps enables us to find the complete response in transient behavior of an electrical circuit after any kind of switching (turning on or off different kinds of sources, short and/or open-circuiting of circuit elements, changing the circuit configuration, etc.). We shall summarize the five-step procedure of solving transient problems by the classical approach as follows:

1) *Determination of a characteristic equation and evaluation of its roots.* Formulate the input impedance as a function of s by inspection of the circuit, which arises after switching, at instant $t=0_+$. Note that all the independent voltage sources should be short-circuited and the current sources should be open-circuited. Equate the expression of $Z_{in}(s)$ to zero to obtain the characteristic equation $Z_{in}(s)=0$. Solve the characteristic equation to evaluate the roots.

The input impedance can be determined in a few different ways: a) As seen from a *voltage source*; b) Via any branch, which includes one or more energy storing elements L and/or C (by opening this branch). The characteristic equation can also be obtained using: c) an input admittance as seen from a *current source* or d) with the determinant of a matrix (of circuit parameters) written in accordance with mesh or node analysis.

Knowing the roots s_k the expression of a natural response (for instance, of current) may be written as

$$i_n(t)=\sum_k A_k e^{s_k t}, \quad \text{for real roots (see 1.31)}$$

or

$$i_n(t)=\sum_k B_k \sin(\omega_{n,k}t+\beta_k), \quad \text{for complex roots (see 1.33)}$$

2) *Determination of the forced response.* Consider the circuit, which arises after switching, for the instant time $t\to\infty$, and find the steady-state solution for the response of interest. Note that any of the appropriate methods (which are usually studied in introductory courses) can be applied to evaluate the solution $i_f(t)$.

3) *Determination of the independent initial conditions.* Consider the circuit, which existed prior to switching at instant $t=0_-$. Assuming that the circuit is operating in steady state, find all the currents through the inductances $i_L(0_-)$ and all the voltages across the capacitances $v_C(0_-)$. By applying two switching laws (1.35) and (1.36), evaluate the independent initial conditions

$$i_L(0_+)=i_L(0_-), \quad v_C(0_+)=v_C(0_-). \tag{2.1}$$

4) *Determination of the dependent initial conditions.* When the desirable response is current or voltage, which can change abruptly, we need to find their initial values, i.e. at the first moment after switching. For this purpose the inductances

must be replaced by current sources, having the values of the currents through these inductances at the moment prior to switching $i_L(0_-)$ and the capacitances should be replaced by voltage sources, having the values of the voltages across these capacitances prior to switching $v_C(0_-)$. If the current through an inductance prior to switching was zero, this inductance should be replaced by an open circuit (i.e., open switch), and if the voltage across a capacitance prior to switching was zero, this capacitance should be replaced by a short circuit (i.e., closed switch). By inspecting and solving this equivalent circuit, the initial values of the desirable quantities can be found. If the characteristic equation is of the second or higher order, the initial values of the derivatives must also be found. This can be done by applying Kirchhoff's two laws and using the other known initial conditions.

5) *Determination of the integration constants.* With all the known initial conditions apply equations (1.58), (1.61) or (1.65), (1.68), and by solving them find the constants of the integration (see section 1.8). The number of constants must be the same as the order of the characteristic equation. For instance, if the characteristic equation is of the first order, then only one constant of integration has to be calculated as

$$A = i(0_+) - i_f(0), \tag{2.2a}$$

and the complete response will be

$$i(t) = i_f(t) + [i(0_+) - i_f(0)]e^{st}. \tag{2.2b}$$

Keeping the above-classified rules in mind, we shall analyze (in the following sections) the transient behavior of different circuits.

2.3 FIRST ORDER *RL* CIRCUITS

2.3.1 *RL* circuits under d.c. supply

Let us start with a simple *RL* series circuit, which is connected to a d.c. voltage source, to illustrate how to determine its complete response by using the 5-step solution method. This circuit, shown in Fig, 2.1(a), has been previously analyzed (in its short-circuiting behavior) by applying a mathematical approach.

1) Determining the input impedance and equating it to zero yields

$$Z_{in}(s) = R + sL = 0. \tag{2.3a}$$

The root of these equations is

$$s = -\frac{R}{L}. \tag{2.3b}$$

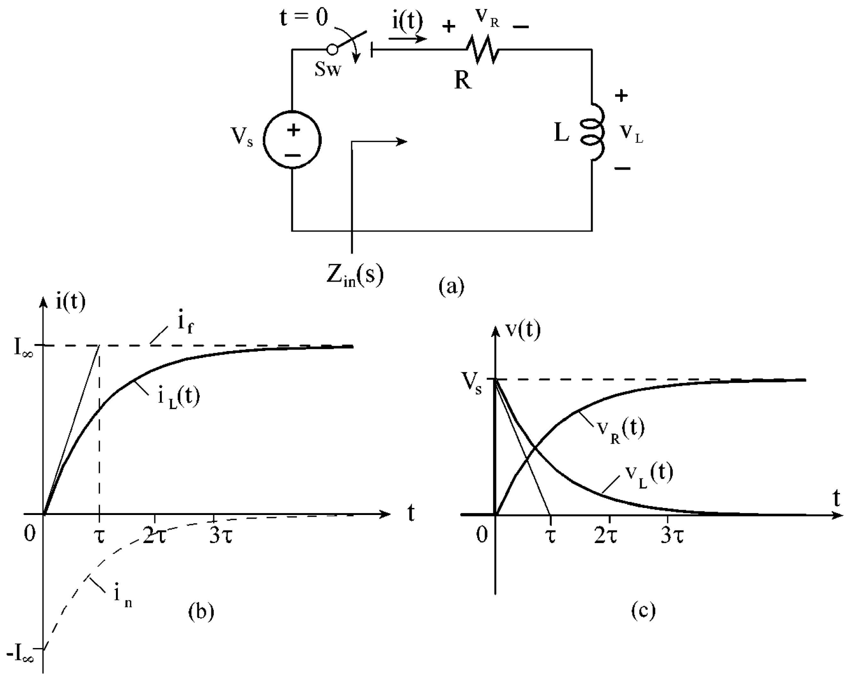

Figure 2.1 A series RL circuit switching at $t = 0$ (a), the current plot after switching (b) and the voltages $v_L(t)$ and $v_R(t)$ (c).

Thus, the natural response will be

$$i_n(t) = Ae^{-\frac{R}{L}t}. \tag{2.3c}$$

2) The forced response, i.e. the steady-state current (after the switch is closed, at $t \to \infty$, the inductance is equivalent to a short circuit) will be

$$i_f(t) = \frac{V_s}{R} = I_\infty. \tag{2.4}$$

3) Because the current through the inductance, prior to closing the switch, was zero, the independent initial condition is

$$i_L(0_+) = i_L(0_-) = 0.$$

4) Since no dependent initial conditions are required, we proceed straight to the 5th step.

5) With equation 2.2a we have

$$A = 0 - \frac{V_s}{R} = -I_\infty,$$

and

$$i(t) = I_\infty - I_\infty e^{-\frac{R}{L}t} = I_\infty\left(1 - e^{-\frac{R}{L}t}\right). \tag{2.5}$$

This complete response and its two components, natural and forced responses, are shown in Fig. 2.1(b). Note that the natural response, at $t = 0$, is exactly equal to the steady-state response, but is opposite in sign, so that the whole current at the first moment of the transient is zero (in accordance with the initial conditions). It should once again be emphasized that the natural response appears to insure the initial condition (at the beginning of the transients) and disappears at the steady state (at the end of the transients). It is logical therefore, to conclude that in a particular case, when the steady state, i.e., the forced response at $t = 0$, equals the initial condition, the natural response will not appear at all[*].

The time constant in this example is

$$\tau = \frac{L}{R} \quad \text{or in general} \quad \tau = \frac{1}{|s|}.$$

The time constant, in this example, is also found graphically as a line segment on the asymptote, i.e. on the line of a steady-state value, determined by the intercept of a tangent to the curve $i(t)$ at $t = 0$ and the asymptote, as shown in Fig. 2.1(b).

Knowing the current response, we can now easily find the voltages across the inductance, v_L and the resistance, v_R:

$$v_L = L\frac{di}{dt} = L\frac{d}{dt}[I_\infty(1 - e^{-(R/L)t})] = L\frac{V_s}{R}\left(-\frac{R}{L}\right)(-e^{-(R/L)t}) = V_s e^{-(R/L)t},$$

and

$$v_R = Ri = V_s(1 - e^{-(R/L)t}),$$

where $V_s = RI_\infty$.

Both these curves are shown in Fig 2.1(c). As we can see at the first moment the whole voltage is applied to the inductance and at the end of the transient it is applied to the resistance. This voltage exchange between two circuit elements occurs gradually during the transient.

Before we turn our attention to more complicated RL circuits, consider once again the circuit of Fig 1.8, which is presented here (for the reader's convenience) in Fig. 2.2(a). The time constant of this circuit has been found (see (1.20)) and is the same as in a series RL circuit. Therefore the natural response (step 1) is $Ae^{-(R/L)t}$. The forced response (step 2) here is $i_{L,f} = I_s$ and the initial value (step 3) is zero. Hence, the integration constant subsequently (step 5) is $A = 0 - I_s =$

[*]This statement is only true in first order circuits.

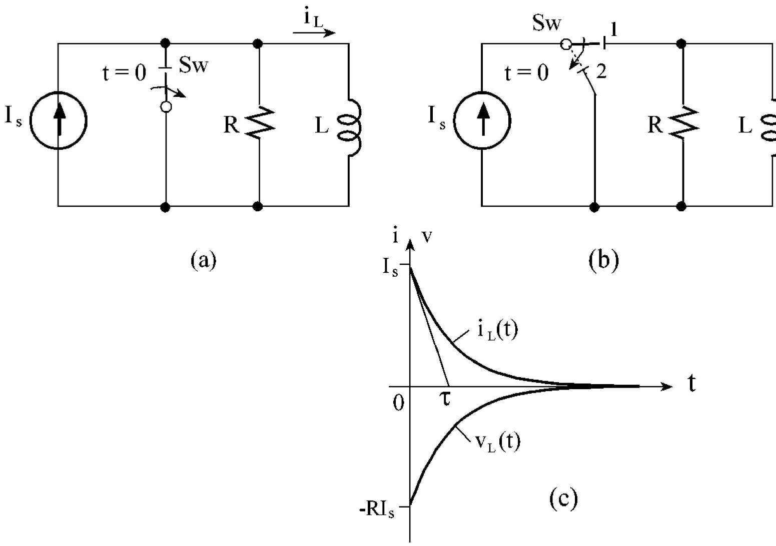

Figure 2.2 An *RL* parallel circuit (a), the circuit in which the inductance discharges through a resistance (b) and the plots of the discharging current and voltage (c).

$-I_s$. Thus, the complete response will be $i_L = I_s(1 - e^{-(R/L)t})$, which is in the same form as in the *RL* series circuit.

To complete our analysis of a simple *RL* series circuit, consider the circuit in Fig. 2.2(b), in which the switch changes its position from "1" to "2" instantaneously and the inductance "discharges" through the resistance. In this case, the natural response, obviously, is the same as in the circuit (a), but the forced response is zero. Therefore, we have $i_L = Ae^{-(R/L)t} = I_s e^{-(R/L)t}$, where $A = I_s$ since the initial value of the inductance current (prior to switching) is I_s. This response and the voltage across the inductance and the resistance are shown in Fig. 2.2(c). Verifying the voltage response is left to the reader.

Let us illustrate the 5-step method by considering more complicated circuits in the following numerical examples.

Example 2.1

In the circuit, Fig. 2.3(a), find current $i_2(t)$ after opening the switch. The circuit parameters are $V_1 = 20\ \text{V}$, $V_2 = 4\ \text{V}$, $R_1 = 8\ \Omega$, $R_2 = 2\ \Omega$, $R_3 = R_4 = 16\ \Omega$ and $L = 1\ \text{mH}$.

Solution

1) We start our solution by expressing the impedance *Z(s)* of the circuit that arises after switching, at the instant $t = 0_+$. We shall determine $Z_{in}(s)$ as seen from source V_2. (However, the impedance $Z_{in}(s)$ can be found in a few different ways, as will be shown further on.) By inspecting the circuit in Fig. 2.3(b) we

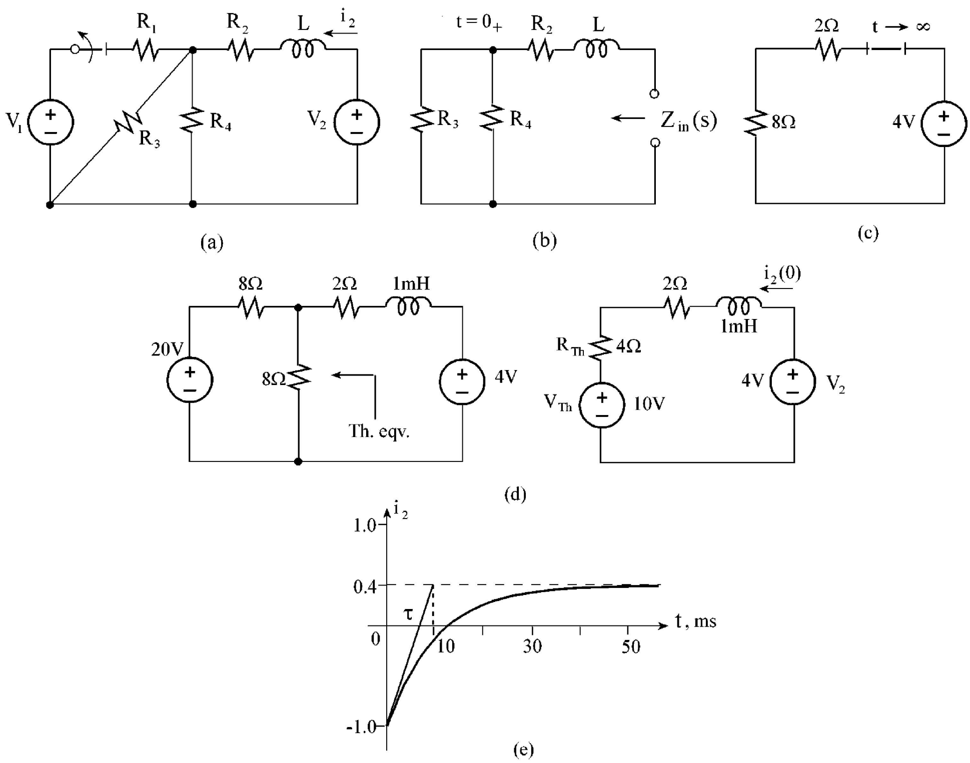

Figure 2.3 The given circuit (a), its equivalent for $t = 0$ (b), its equivalent for $t \to \infty$ (c), its equivalent for $t < 0$ (d) and the curve of current $i_2(t)$ (e).

have

$$Z_{in}(s) = sL + R_2 + \frac{R_3 R_4}{R_3 + R_4}.$$

Substituting the numerical values and equating the expression to zero yields

$$10^{-3}s + 2 + 8 = 0.$$

This equation has the root

$$s = -100\,\mathrm{s}^{-1} \quad \text{and} \quad \tau = 0.01\ \mathrm{s},$$

and the natural response will be

$$i_{2,n} = Ae^{-100t}.$$

2) The forced response, i.e., the steady-state current $i_{2,f}$, is found in the circuit, Fig. 2.2(c) that is derived from the given circuit after the switching, at $t \to \infty$,

while the inductance behaves as a short circuit

$$i_{2,f} = \frac{V_2}{R_{eq}} = \frac{4}{10} = 0.4 \text{ A}.$$

3) The independent initial condition, i.e., $i_L(0_-)$ is found in the circuit prior to switching, shown in Fig. 2.3(d). Using Thévenin's equivalent for the left part of the circuit, as shown in (d), we have

$$i_2(0_+) = i_2(0_-) = \frac{V_2 - V_{Th}}{R_2 + R_{Th}} = \frac{4 - 10}{2 + 4} = -1 \text{ A}.$$

4) None of the dependent initial conditions is needed.

5) In order to evaluate constant A, we use equation 2.2a: $A = i_2(0_+) - i_f(0) = -1 - 0.4 = -1.4$ A. Thus the complete response is $i_2(t) = 0.4 - 1.4e^{-100t}$ A, which is sketched in Fig. 2.3(e).

Example 2.2

For the circuit shown in Fig. 2.4(a) find the current response $i_1(t)$ after closing the switch. The circuit parameters are: $R_1 = R_2 = 20\ \Omega$, $L_1 = 0.1$ H, $L_2 = 0.4$ H, $V_s = 120$ V.

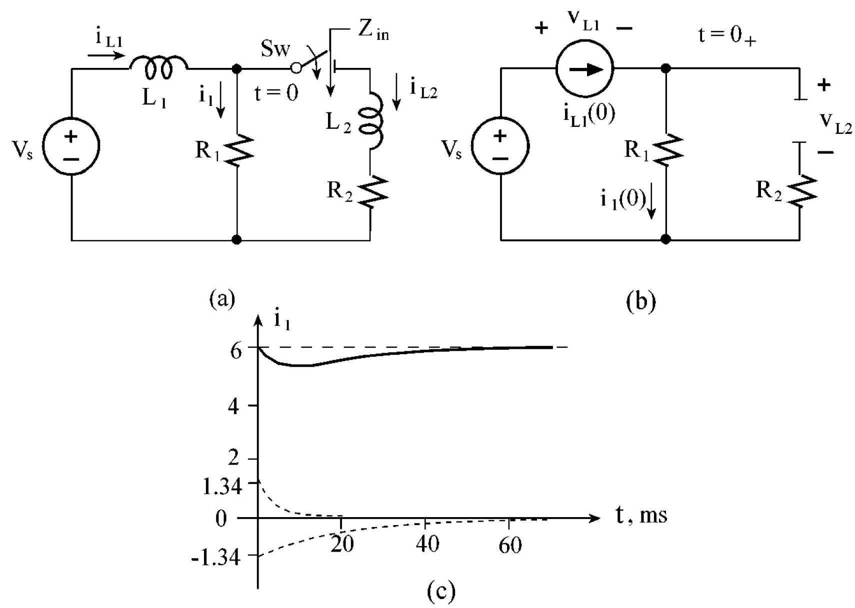

Figure 2.4 A given circuit for Example 2.2(a), its equivalent at time $t = 0_+$ (b) and the plot of current $i_1(t)$ and its components (c).

Solution

1) The input impedance is found as seen from the L_2 branch (we just "measure" it from the open switch point of view), with the voltage source short-circuited

$$Z_{in}(s) + sL_2 + R_2 + \frac{R_1 sL_1}{R_1 + sL_1}.$$

Equating this expression to zero and after simplification, we get the characteristic equation

$$s^2 + \frac{R_1 L_1 + R_2 L_1 + R_1 L_2}{L_1 L_2} + \frac{R_1 R_2}{L_1 L_2} = 0,$$

or by substituting the numerical data

$$s^2 + 3\cdot 10^2 s + 10^4 = 0.$$

Thus, the roots of this equation are

$$s_1 = -38.2\ \mathrm{s}^{-1}, \quad s_2 = -262\ \mathrm{s}^{-1},$$

and the natural response is

$$i_{1,n} = A_1 e^{-38.2t} + e^{-262t}.$$

2) By inspecting the circuit after the switch is closed, at $t \to \infty$, we may determine the forced response

$$i_{1,f} = \frac{V_s}{R_1} = \frac{120}{20} = 6\ \mathrm{A}.$$

3) By inspection of the circuit prior to switching we observe that $i_{L1}(0_-) = 120/20 = 6$ A and $i_{L2}(0_-) = 0$. Therefore, the independent initial conditions are

$$i_{L1}(0_+) = 6\ \mathrm{A}, \quad i_{L2}(0_+) = 0.$$

4) Since the characteristic equation is of the second order, and the desired response, which is a current through a resistance, can be changed abruptly, we need its two dependent initial conditions, namely:

$$i_1(0) \quad \text{and} \quad \left.\frac{di}{dt}\right|_{t=0}.$$

By inspection of the circuit in Fig. 2.4(b) for instant $t = 0_+$, we may find $t_1(0) = 6$ A. (Note that in this specific case the current i_1 does not change abruptly and, therefore, its initial value equals its steady-state value, but because the circuit is of the second order, the transient response of the current is expected.)

By applying KCL we have $i_1 = i_{L1} - i_{L2}$ and after the differentiation and evaluation of $t = 0$ we obtain

$$\left.\frac{di}{dt}\right|_{t=0} = \left.\frac{di_1}{dt}\right|_{t=0} - \left.\frac{di_2}{dt}\right|_{t=0} = \frac{1}{L_1} v_{L1}(0) - \frac{1}{L_2} v_{L2}(0).$$

Since, Fig. 2.4(b), $v_{R1}(0) = V_s$, then $v_{L1}(0) = 0$ and $v_{R1}(0) = v_{L2}(0) = 120$ V. Therefore, we have

$$\left.\frac{di}{dt}\right|_{t=0} = 0 - \frac{120}{0.4} = -300,$$

and we may obtain two equations

$$A_1 + A_2 = i(0) - i_f(0) = 6 - 6 = 0$$

$$s_1 A_1 + s_2 A_2 = \left.\frac{di}{dt}\right|_{t=0} - \left.\frac{di_f}{dt}\right|_{t=0} = -300 - 0 = -300.$$

Solving these two equations yields $A_1 = -1.34$, $A_2 = 1.34$ and the answer is

$$i_1(t) = 6 - 1.34e^{-38.2t} + 1.34e^{-262t} \text{ A}.$$

This current and its components are plotted in Fig. 2.4(c).

Example 2.3

Consider the circuit of the transformer of Example 1.2, which is shown here in Fig. 2.5 in a slightly different form. For measuring purposes, the transformer is connected to a 120 V d.c.-source. Find both current i_1 and i_2 responses.

1) The characteristic equation and its roots have been found in Example 1.2: $s_1 = -86 \text{ s}^{-1}$, $s_2 = -1160 \text{ s}^{-1}$. Therefore, the natural responses are

$$i_{1,n} = A_1 e^{-86t} + A_2 e^{-1160t}$$

$$i_{2,n} = B_1 e^{-86t} + B_2 e^{-1160t}.$$

2) The forced responses are found by inspection of the circuit after switching $(t \to \infty)$:

$$i_{1,f} = \frac{V_s}{R_1} = \frac{120}{6} = 20 \text{ A}, \quad i_{2,f} = 0.$$

3) The independent initial conditions are zero, since prior to switching no currents are flowing through the inductances: $i_1 = (0_+) = i_1(0_-) = 0$, $i_2 = (0_+) = i_2(0_-) = 0$.

4) In order to determine the integration constant we need to evaluate the current derivatives. By inspection of the circuit in Fig. 2.5(b), we have $v_{L1}(0) = 120$ V, $v_{L2}(0) = 0$, and

$$L_1 \left.\frac{di_1}{dt}\right|_{t=0} + M \left.\frac{di_2}{dt}\right|_{t=0} = 120$$

$$L_2 \left.\frac{di_2}{dt}\right|_{t=0} + M \left.\frac{di_1}{dt}\right|_{t=0} = 0.$$

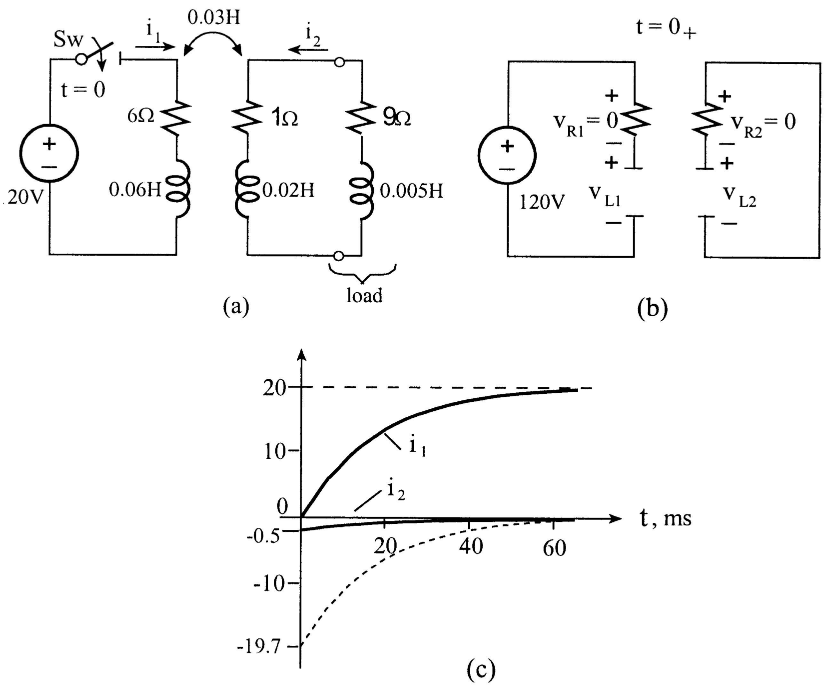

Figure 2.5 The circuit of a transformer (a), its equivalent for $t = 0_+$ (b) and the plots of two currents (c).

Solving these two relatively simple equations yields

$$\left.\frac{di_1}{dt}\right|_{t=0} = 5000, \quad \left.\frac{di_2}{dt}\right|_{t=0} = -6000.$$

5) With the initial value of $i_{1,n}(0) = i_1(0) - i_{1,f}(0) = 0 - 20 = -20$ and the initial value of its derivative

$$\left.\frac{di_{1,n}}{dt}\right|_{t=0} = \left.\frac{di_1}{dt}\right|_{t=0} - \left.\frac{di_{1,f}}{dt}\right|_{t=0} = 5000 - 0 = 5000,$$

we obtain two equations in the two integration constants of current i_1

$$A_1 + A_2 = -20$$
$$s_1 A_1 + s_2 A_2 = 5000,$$

for which the solution is: $A_1 = -19.7$, $A_2 = -0.3$. In a similar way, the two equations in the two integration constants of current i_2

$$B_1 + B_2 = 0$$
$$s_1 B_1 + s_2 B_2 = -6000,$$

for which the solution is $B_1 = -0.52$, $B_2 = 0.52$.

Therefore, the current responses are

$$i_1 = 20 - 19.7e^{-86t} - 0.3e^{-1160t}$$

$$i_2 = -0.52e^{-86t} + 0.52e^{-1160t}.$$

These two currents are sketched in Fig. 2.5(c). Note that the second exponential parts decay much faster than the first ones and are not shown in Fig. 2.5(c). Note also that the second exponential term in i_1 is relatively small and might be completely neglected.

Example 2.4

As a final example of inductive circuits let us consider the "inductance" node circuit, which is shown in Fig. 2.6(a). Find the currents i_1 and i_2 after switching, if the circuit parameters are: $L_1 = L_2 = 0.05$ H, $L_3 = 0.15$ H, $R_1 = R_2 = R_3 = 1\ \Omega$ and $V_s = 15$ V.

Solution

1) Let us determine the characteristic equation by using mesh analysis. The

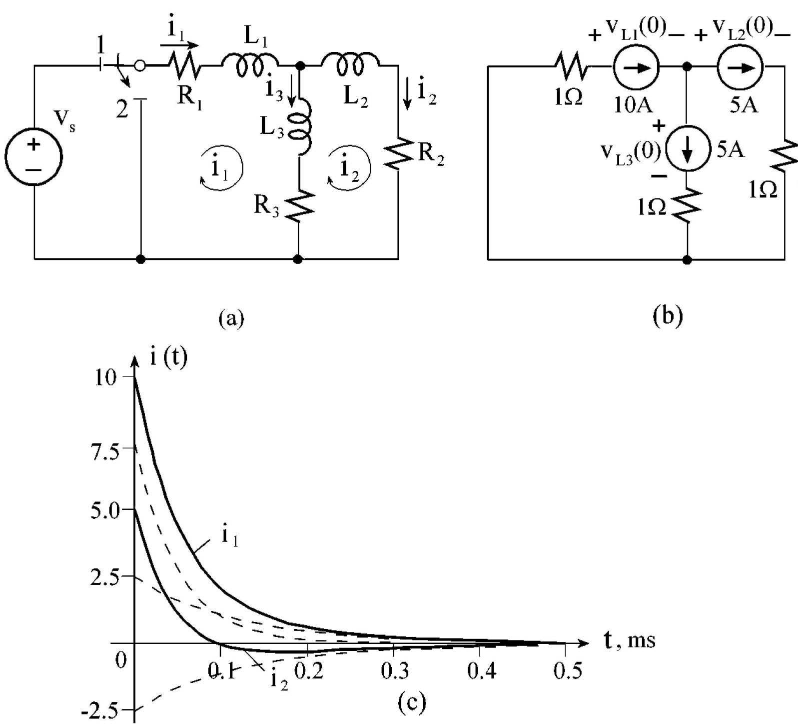

Figure 2.6 A circuit containing an "inductance node" (a), its equivalent at $t = 0$ (b) and the plots of the currents and their components (c).

impedance matrix is

$$\begin{bmatrix} s(L_1+L_3)+R_1+R_3 & -(sL_3+R_3) \\ -(sL_3+R_3) & s(L_2+L_3)+R_2+R_3 \end{bmatrix} = \begin{bmatrix} 0.2s+2 & -(0.15s+1) \\ -(0.15s+1) & 0.2s+2 \end{bmatrix}$$

Equating the determinant to zero and after simplification, we obtain the characteristic equation

$$0.0175s^2 + 0.5s + 3 = 0,$$

for which the roots are

$$s_1 = -8.6\ \mathrm{s}^{-1}, \quad s_2 = -20\ \mathrm{s}^{-1}.$$

Thus, the natural responses of the currents are

$$i_{1,n} = A_1 e^{-8.6t} + A_2 e^{-20t}$$
$$i_{2,n} = B_1 e^{-8.6t} + B_2 e^{-20t}.$$

2) The steady-state values of the currents are zero, since after switching the circuit is source free.

3) The independent initial conditions can be found by inspection of the circuit in Fig. 2.6(a) prior to switching and keeping in mind that all the inductances are short-circuited

$$i_1(0) = i_1(0_-) = \frac{V_s}{R_1 + R_2//R_3} = \frac{15}{1.5} = 10\ \mathrm{A}$$
$$i_2(0) = i_2(0_-) = \frac{10}{2} = 5\ \mathrm{A}.$$

Note that only two initial independent currents can be found (although the circuit contains three inductances), since the third current is dependent on two others. However, because the circuit is of the second order, the two initial values are enough for solving this problem.

4) Next, we have to find the initial values of the current derivatives for which we must find the voltage drops in the inductances $v_{L1}(0)$ and $v_{L2}(0)$ for the instant of switching, i.e., $t = 0$. By inspection of the circuits in Fig. 2.6(b), we have

$$v_{L1}(0) + v_{L2}(0) = -15, \quad v_{L1}(0) + v_{L3}(0) = -15, \tag{2.6a}$$
$$v_{L2}(0) = v_{L3}(0). \tag{2.6b}$$

With KCL we may write $i_1 = i_2 + i_3$ and by differentiation

$$\frac{di_1}{dt} = \frac{di_2}{dt} + \frac{di_3}{dt}, \quad \text{or} \quad \frac{v_{L1}}{L_1} = \frac{v_{L2}}{L_2} + \frac{v_{L3}}{L_3}.$$

With equation 2.6b we have

$$\frac{1}{L_1}v_{L1} = \left(\frac{1}{L_2} + \frac{1}{L_3}\right)v_{L2}, \quad \text{or} \quad v_{L2} = \frac{L_2 L_3}{(L_2 + L_3)L_1}v_{L1} = 60.75v_{L1},$$

and with equation 2.6a $v_{L1} = -8.57\ \mathrm{V}$ and $v_{L2} = -6.43\ \mathrm{V}$. Therefore,

$$\left.\frac{di_1}{dt}\right|_{t=0} = \frac{v_{L1}}{L_1} = -\frac{8.57}{0.05} = -171.4\ \mathrm{A\ s^{-1}}$$

$$\left.\frac{di_2}{dt}\right|_{t=0} = \frac{v_{L2}}{L_2} = -\frac{6.43}{0.05} = -128.6\ \mathrm{A\ s^{-1}}.$$

5) We may now obtain a set of equations to evaluate the integration constant

$$A_1 + A_2 = 10$$

$$s_1 A_1 + s_2 A_2 = -171.4.$$

for which the solution is $A_1 \cong 2.5$, $A_2 \cong 7.5$. In a similar way we can obtain

$$B_1 + B_2 = 5$$

$$s_1 B_1 + s_2 B_2 = -128.6,$$

and the solution is $B_1 \cong -2.5$, $B_2 \cong 7.5$. Therefore, two current responses are

$$i_1 = 2.5e^{-8.6t} + 7.5e^{-20t}$$

$$i_2 = -2.5e^{-8.6t} + 7.5e^{-20t}.$$

The plots of these currents and their components are shown in Fig. 2.6 (c).

2.3.2 *RL* circuits under a.c. supply

As we already know, the natural response does not depend on the source function, and therefore the first step of the solution, i.e. determining the characteristic equation and evaluating its roots, is the same as in previous cases. This is also understandable from the fact that the natural response arises from the solution of the homogeneous differential equation, which has zero on the right side. The forced response can be determined from the steady-state solution of the given circuit. The symbolic, or phasor, method should be used for this solution.

To illustrate the above principles, let us consider the circuit shown in Fig. 2.7. The solution will be completed by applying the five steps as previously done. In the first step, we have to determine the characteristic equation and its root. However, for such a simple circuit it is already known that $s = -R/L$. Therefore the natural response is

$$i_n = Ae^{-t/\tau}, \quad \text{where} \quad \tau = L/R. \tag{2.7}$$

In the next step, our attention turns to obtaining the steady-state current.

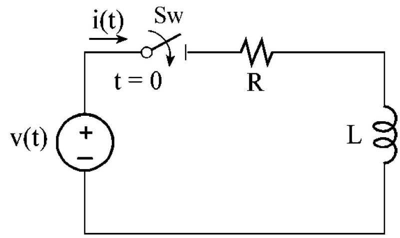

Figure 2.7 A series *RL* circuit switching to an a.c. source.

Applying the phasor method we have

$$\tilde{I}_m = \frac{\tilde{V}_m}{Z} = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} \angle (\psi_v - \varphi),$$

where $\tilde{V}_m = V_m e^{j\psi_v}$ and $\tilde{I}_m = I_m e^{j\psi_i}$ are voltage and current phasors respectively and $\varphi = \psi_v - \psi_i = \tan^{-1}(\omega L/R)$ is the phase angle difference between the voltage and current phasors. Thus,

$$i_f = I_m \sin(\omega t + \psi_i), \tag{2.8}$$

where

$$I_m = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}}, \quad \psi_i = \psi_v - \varphi.$$

In the next two steps, 3 and 4, we shall determine the only initial condition, which is necessary to find the current through the inductance. Since prior to switching this current was zero, we have $i(0_+) = i(0_-) = 0$. In the final step, with this initial value we may obtain the integration constant

$$A = i(0) - i_f(0) = -I_m \sin \psi_i. \tag{2.9}$$

Thus, the complete response of an *RL* circuit to applying an a.c. voltage source is

$$i = i_f + i_n = I_m \sin(\omega t + \psi_i) - I_m \sin \psi_i e^{-t/\tau}. \tag{2.10}$$

This current and its components are plotted in Fig. 2.8(a).

Note that the initial values of $i_f(0)$ and $i_n(0)$ are equal and opposite in sign, so that with zero initial conditions the actual current $i(t)$ always starts with the zero value. If switching occurs at the instant that $\psi_i = \pm\pi/2$, then the total response reaches its maximum value at the point of one-half a period. This extreme value of the current may increase up to twice that of the amplitude of the steady-state current and occurs if the time constant of the circuit is much greater than the period of the a.c. current so that the natural response current decays relatively slowly. Thus, if $\tau \gg T$, where T is a period of an a.c. current, then $i_{\max} \to 2I_m$. This is shown in Fig 2.8(b). If, however, the time constant of the circuit is small compared to the period of the a.c. current, the natural current

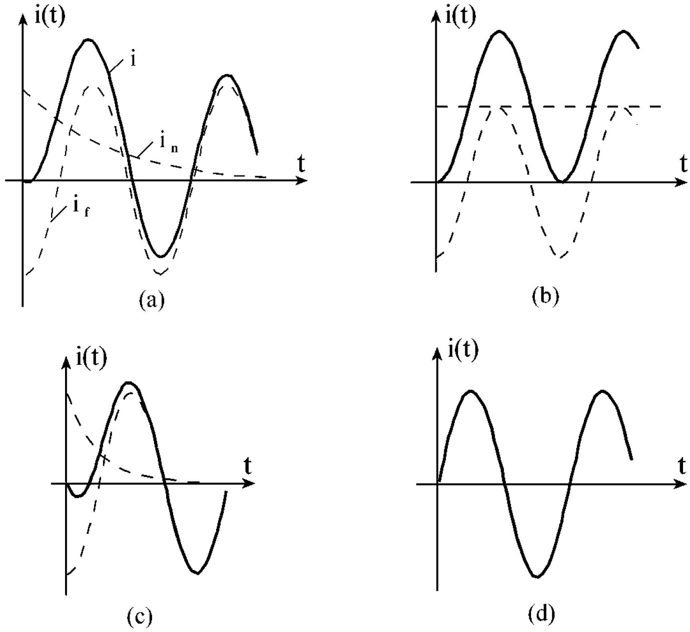

Figure 2.8 The transient response of a series RL circuit when switching to an a.c. voltage source (a) and maximal (b), minimal (c) and zero (d) responses.

decreases quickly during the first half period and no considerable excess current can develop, as shown in Fig. 2.8(c). If the phase angle ψ_i is zero, which means that the forced (steady-state) current passes through zero at the instant of switching, no transient current (equation 2.9) occurs, so that the a.c. current immediately starts in its normal way, Fig. 2.8(d).

In highly inductive circuits, which are common for industrial networks, the displacement angle between the voltage and current is nearly 90°. Thus the favorable case, Fig. 2.8(d), corresponds to the switching on at the maximum instantaneous voltage, which usually occurs in high voltage circuit breakers. The switching-on process in such breakers is initiated by a discharged spark between the breaker contacts, wherein the contacts approach each other relatively slowly compared with the a.c. frequency, and when the voltage passes its maximum.

We shall now illustrate the transients in a.c. circuits by the following numerical examples.

Example 2.5

In an RL circuit of Fig. 2.7, the switch closes at $t = 0$. Find the complete current response and sketch its plot, if $r = 10\,\Omega$, $L = 0.01$ H, and $v_s = 120\sqrt{2}\sin(1000t + 15°)$ V.

Solution

1) The time constant of the circuit is

$$\tau = \frac{L}{R} = \frac{0.01}{10} = 10^{-3} = 1\ \text{ms}$$

and the natural response is

$$i_n = Ae^{-1000t}.$$

2) The steady-state current is calculated by phasor analysis. The impedance of the circuit is $Z(j\omega) = R + j\omega L = 10 + j10 = \sqrt{2}10 \angle 45^\circ\ \Omega$, the voltage source phasor is $\tilde{V}_{s,m} = 100\sqrt{2}e^{j15^\circ}$. Thus, the current phasor will be

$$\tilde{I}_f = \frac{\tilde{V}_{s,m}}{Z} = \frac{100\sqrt{2} \angle 15^\circ}{10\sqrt{2} \angle 45^\circ} = 10 \angle -30^\circ\ \text{A},$$

and the current versus time is

$$i_f = 10\sin(1000t - 30^\circ)\ \text{A}.$$

3) The initial condition is zero, i.e., $i(0_+) = i(0_-)) = 0$.

4) Non-dependent initial conditions are needed.

5) The integration constant can now be found $A = i(0) - i_f(0) = 0 - 10\sin(-30^\circ) = 5$ and the complete response is $i = 10\sin(1000t - 30^\circ) + 5e^{-1000t}$ A, which is sketched in Fig. 2.9.

Example 2.6

At the receiving end of the transmission line in a no-load operation, a short-circuit fault occurs. The impedance of the line is $Z = (1 + j5)\ \Omega$ and the a.c.

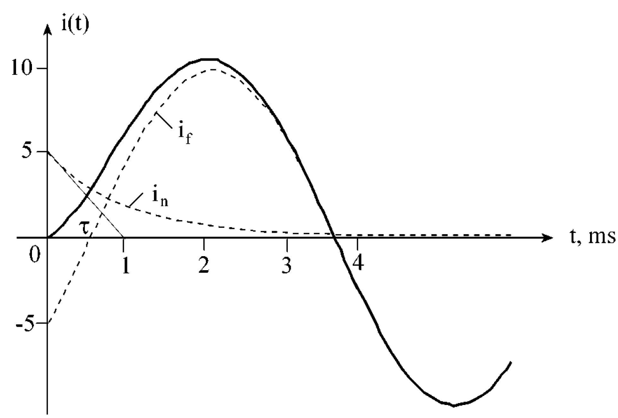

Figure 2.9 A current response in a series *RL*, of example 2.5, circuit switched to the a.c. source.

voltage at the sending end is 10 kV at 60 Hz. a) Find the transient short-circuit current if the instant of short-circuiting is when the voltage phase angle is 1) $-\pi/4+\varphi$; 2) $-\pi/2+\varphi$ and b) estimate the maximal short-circuit current and the applied voltage phase angle under the given conditions.

Solution

a) First we shall evaluate the line inductance $L = x/\omega = 5/2\pi 60 = 0.01326 \cong 13.3\ \mathrm{mH}$. The voltage at the sending end versus time is $v_s = 10\sqrt{2}\sin(\omega t + \psi_v)$.

1) The time constant of the line (which is represented by RL in series) is $\tau = L/R = 13.3/1 = 13.3\ \mathrm{ms}$ or $s = -1/\tau = -75.2\ \mathrm{s}^{-1}$ and the natural current is

$$i_n = Ae^{-75.2t}$$

2) The steady-state short current (r.m.s.) is found using phasor analysis:

$$\tilde{I}_f = \frac{10\angle\psi_v}{1+j5} = \frac{10\angle\psi_v}{5.1\angle 78.7^\circ} = 1.96\angle\psi_v - 78.7^\circ.$$

Thus

$$i_f = I_m \sin(377t + \psi_v - 78.7^\circ),$$

where

$$I_m = 1.96\sqrt{2}\ \mathrm{A} \quad \text{and} \quad \omega = 2\pi 60 = 377\ \mathrm{rad/s}.$$

3) Because of the zero initial condition, $i(0_+) = i(0_-) = 0$.

5) We omit step 4) (since no dependent initial conditions are needed) and evaluate constant A for two cases:

(1) $\psi_v = -180^\circ/4 + 78.7^\circ = 33.7^\circ$
and $A = i(0) - i_f(0) = 0 - I_m\sin(33.7^\circ - 78.7^\circ) = (\sqrt{2}/2)I_m$.

Therefore, the complete response is

$$I_{sc} = I_m\sin(\omega t - \pi/4) + (\sqrt{2}/2)I_m e^{-75.2t}.$$

(2) $\psi_v = -180^\circ/2 + 78.7^\circ = -11.3^\circ$
and $A = i(0) - i_f(0) = 0 - I_m\sin(-11.3^\circ - 78.7^\circ) = I_m$.

Therefore, the complete response is

$$i_{sc} = I_m\sin(\omega t - \pi/2) + I_m e^{-75.2t}.$$

b) The maximal value of the short-circuit current is dependent on the initial phase angle of the applied voltage and will appear if the natural response is the largest possible one as in (2), i.e., when $A = I_m$. The instant at which the current reaches its peak is about half of the period after switching. To find the exact time we have to equate the current derivative to zero. Thus,

$$\frac{di_{sc}}{dt}=\frac{di_f}{dt}+\frac{di_n}{dt}=0, \quad \text{or} \quad \frac{di_f}{dt}=-\frac{di_n}{dt}.$$

Performing this procedure we may find

$$I_m\omega\cos(\omega t+\psi_v-\varphi)=I_m\frac{1}{\tau}\sin(\psi_v-\varphi)e^{-\frac{t}{\tau}},$$

or in accordance with (2)

$$\cos(\omega t-\pi/2)=\frac{1}{\omega\tau}e^{-\frac{\omega t}{\omega\tau}}.$$

Taking into consideration that

$$\omega\tau=\frac{x}{L}\cdot\frac{L}{R}=5$$

we may solve the above transcendental equation finding

$$\omega t_{(\max)}\cong 3.03\,\text{rad}.$$

Therefore, the short-circuit current, of the form $i_{sc}=I_m\sin(\omega t-\pi/2)-I_m e^{-t/\tau}$, will reach its maximal value at $\omega t_{(\max)}\cong 3.03$ rad, Fig. 2.10, and this value will be

$$I_{\max}=I_m(\sin(3.03-\pi/2)+e^{-3.03/5})\cong 1.54 I_m.$$

Example 2.7

The switch in the circuit of Fig. 2.11 closes at $t=0$, after being open for a long

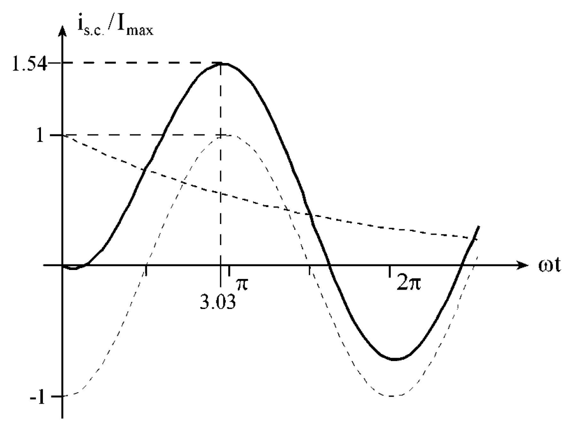

Figure 2.10 A plot of the short-circuit current in which it reaches its maximal value.

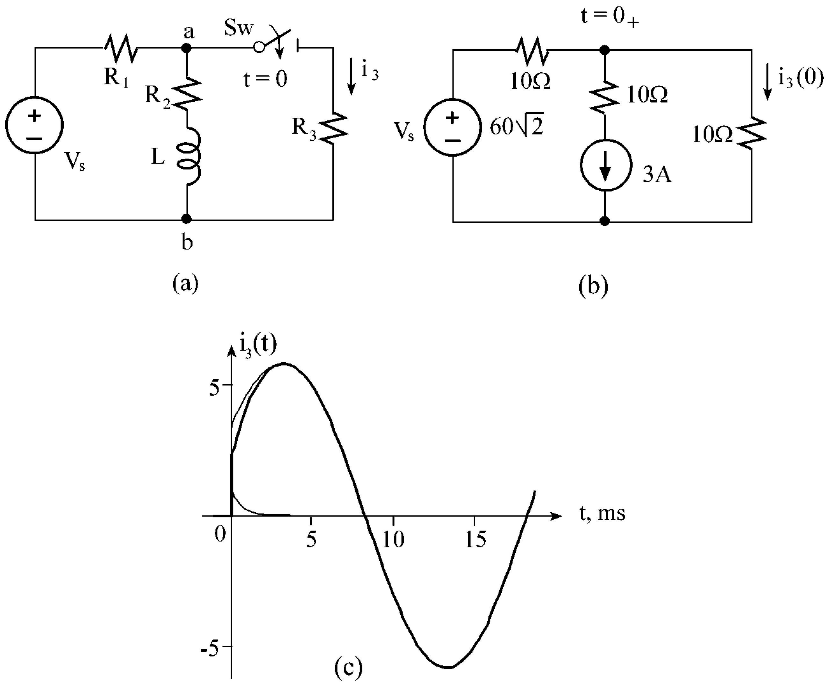

Figure 2.11 A given circuit of Example 2.7 (a), its equivalent at $t = 0$ (b) and the current plot (c).

time. Find the transient current $i_3(t)$, if $R_1 = R_2 = R_3 = 10\ \Omega$, $L = 0.01$ H and $V_{sm} = 120\sqrt{2}$ V at $f = 50$ Hz and $\psi_v = 30°$.

Solution

1) The simplest way to determine the characteristic equation is by observing it from the inductive branch

$$Z(s) = sL + R_2 + R_1//R_2 = 0.$$

With the given data we have

$$0.01s + 15 = 0, \quad \text{or} \quad s = -1500\ \mathrm{s}^{-1},$$

and

$$i_{3,n} = Ae^{-1500t}.$$

2) The forced response of the current will be found by nodal analysis

$$\tilde{V}_a = \frac{\tilde{V}_s}{R_1\left[\dfrac{1}{R_1} + \dfrac{1}{R_2 + jx_L} + \dfrac{1}{R_3}\right]} = \frac{120\angle 30°}{2 + \dfrac{1}{1 + j0.314}} = 41.1\angle 35.6°,$$

where $x_L = \omega t = 314 \cdot 0.01 = 3.14\ \Omega$. Thus

$$\tilde{I}_3 = \frac{\tilde{V}_a}{R_3} = 4.11 \angle 35.6^\circ \quad \text{and} \quad i_{3,f} = 4.11\sqrt{2}\sin(\omega t + 35.6^\circ).$$

3) The independent initial condition may be obtained from the circuit prior to switching:

$$\tilde{I}_L = \frac{\overline{V}_s}{R_1 + R_2 + jx_L} = 5.92 \angle 21.1^\circ.$$

Therefore, $i_L(0_-) = 5.92\sqrt{2}\sin 21.1^\circ = 3.0$ A.

4) With the superposition principle being applied to the circuit in Fig. 2.11(b), we obtain

$$i_3(0) = i_3'(0) + i_3''(0) = \frac{60\sqrt{2}}{20} - \frac{3}{2} = 2.74 \text{ A}.$$

Note that the current i_3 is a resistance current and it changes abruptly.

5) The integration constant is now found as

$$A = i_3(0) - i_{3,f}(0) = 2.74 - 4.11\sqrt{2}\sin 35.6^\circ = -0.64.$$

Therefore,

$$i_3(t) = 4.11\sqrt{2}\sin(\omega t + 35.6^\circ) - 0.64e^{-1500t} \text{ A},$$

which is plotted in Fig. 2.11(c).

Example 2.8

As our next example consider the circuit in Fig. 2.12 and find the current through the switch, which closes at $t = 0$ after being open for a long time. The circuit parameters are: $R_1 = 2\ \Omega$, $x_1 = 10\ \Omega$, $R_2 = 20\ \Omega$, $x_2 = 50\ \Omega$ and $V_m = 15$ V at $f = 50$ Hz and $\psi_v = -15^\circ$.

Solution

1) After short-circuiting, the circuit is divided into two parts, so that each of

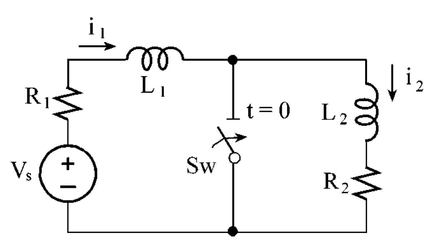

Figure 2.12 A given circuit for Example 2.8.

them has two different time constants:

$$\tau_1 = \frac{L_1}{R_1} = \frac{x_1}{\omega R_1} = \frac{10}{314 \cdot 2} = 15.9\ \mathrm{ms}, \quad \text{or} \quad s_1 = -1/\tau_1 = -62.9\ \mathrm{s}^{-1},$$

$$\tau_2 = \frac{L_2}{R_2} = \frac{x_2}{\omega R_2} = \frac{50}{314 \cdot 20} = 7.96\ \mathrm{ms}, \quad \text{or} \quad s_2 = -1/\tau_2 = -125\ \mathrm{s}^{-1}.$$

Thus, the natural response of the current contains two parts:

$$i_{sw,n} = A_1 e^{-62.9t} + A_2 e^{-125t}.$$

2) The right loop of the circuit is free of sources, so that only the left side current will contain the forced response:

$$i_{1,f} = \frac{15}{\sqrt{2^2 + 10^2}} \sin(314t - 15^\circ - \tan^{-1} 10/2) = 1.47 \sin(314t - 93.7^\circ)\ \mathrm{A}.$$

3) The independent initial conditions, i.e., the currents into two inductances prior to switching, are the same:

$$i_L(0_+) = i_{L1}(0_-) = i_{L2}(0_-) = \frac{15}{\sqrt{22^2 + 60^2}} \sin(-15^\circ - \tan^{-1} 60/22) = -0.234\ \mathrm{A}.$$

4–5) Since non-dependent initial conditions are required, we may now evaluate the integration constants:

$$A_1 = i_L(0) - i_{1,f}(0) = -0.234 - 1.47 \sin(-93.7) = 1.23,$$

$$A_2 = i_L(0) - 0 = -0.234.$$

Therefore, the answer is:

$$i_{sw} = i_1 - i_2 = 1.47 \sin(314t - 93.7^\circ) + 1.23 e^{-62.9t} + 0.234 e^{-125t}\ \mathrm{A}.$$

Example 2.9

Our final example of *RL* circuits will be the circuit shown in Fig. 2.13, in which both kinds of sources, d.c. and a.c., are presented. Consider the above circuit and find the transient current through resistance R_1. The circuit parameters are: $R_1 = R_2 = 5\ \Omega$, $L = 0.01\ \mathrm{H}$, $I_s = 4\ \mathrm{A}$ d.c. and $v_s(t) = 100\sqrt{2} \sin(1000t + 15^\circ)\ \mathrm{V}$.

Solution

1) The characteristic equation for this circuit may be determined as

$$Z(s) = R_1 + R_2 + sL = 0 \quad \text{or} \quad 0.01s + 10 = 0,$$

which gives

$$s = -1000\ \mathrm{s}^{-1} \quad \text{or} \quad \tau = 1\ \mathrm{ms}.$$

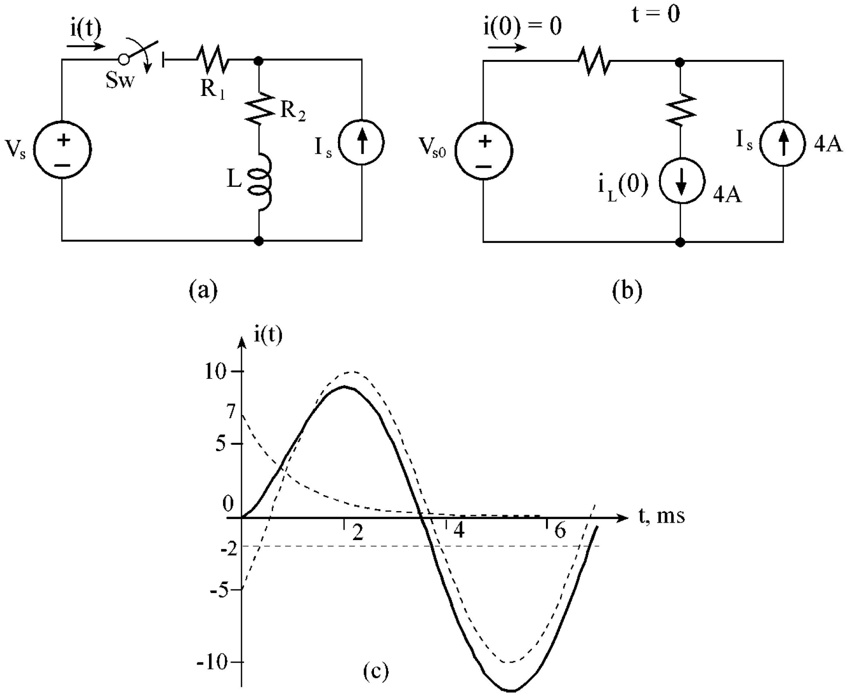

Figure 2.13 A given circuit of Example 2.9 (a), its equivalent for $t = 0$ (b) and the plots of the current and its components (c).

Thus,

$$i_{1,n} = Ae^{-1000t}.$$

2) The forced response (using the superposition principle) is

$$\begin{aligned} i_f &= i_{(Is)} + i_{(vs)} = -2 + \frac{100\sqrt{2}}{\sqrt{10^2 + 10^2}} \sin(1000t + 15^\circ - 45^\circ) \\ &= -2 + 10\sin(1000t - 30^\circ)\ \text{A}. \end{aligned}$$

3) The inductance current prior to (and after) switching is $i_L(0) = i_L(0_-) = I_s = 4$ A.

4) The initial value of the current through R_1 (the dependent initial condition) is found in the circuit of Fig. 2.13(b). By inspection of this circuit, we shall conclude that this current is zero (since both branches with current sources, which possess an infinite inner resistance, behave as an open circuit for the voltage source, and the two equal current sources are connected in the right loop in series without sending any current to the left loop). Thus, $i_1(0) = 0$.

5) The integration constant, therefore, is obtained as $A = i_1(0) - i_{1,f}(0) = 0 + 2 - 10\sin(-30°) = 7$ A. Hence,

$$i_1(t) = -2 + 10\sin(1000t - 30°) + 7e^{-1000t}\ \text{A}.$$

This current is plotted in Fig. 2.13(c).

2.3.3 Applying the continuous flux linkage law to *L*-circuits

As we have observed earlier (see Section 1.7.4), when an *RL* circuit is disconnected from a source, say for instance a d.c. source, by the rapid opening of a switch a very high voltage appears across the switch, which may result in a breakdown of the circuit insulation. In this section, we shall review this phenomenon by introducing a number of examples in which the problem is solved using the continuous flux linkage principle. Let us consider the circuit in Fig. 2.14(a). The current prior to switching is $i(0_-) = I_0 = V_0/R$ and, according to the switching law, at the first moment after switching it remains the same

$$i(0_+) = I_0.$$

Because the resistance of an open switch is infinite $R_{sw} \to \infty$, the voltage across the switch will also be infinite $v_{sw} \to \infty$. In reality an infinite voltage will not be reached, since the resistance of the actual switches in the open position is very high, but not infinite. Another reason that the voltage cannot reach infinity is that the spark appears between the switch contacts, and the stored energy is

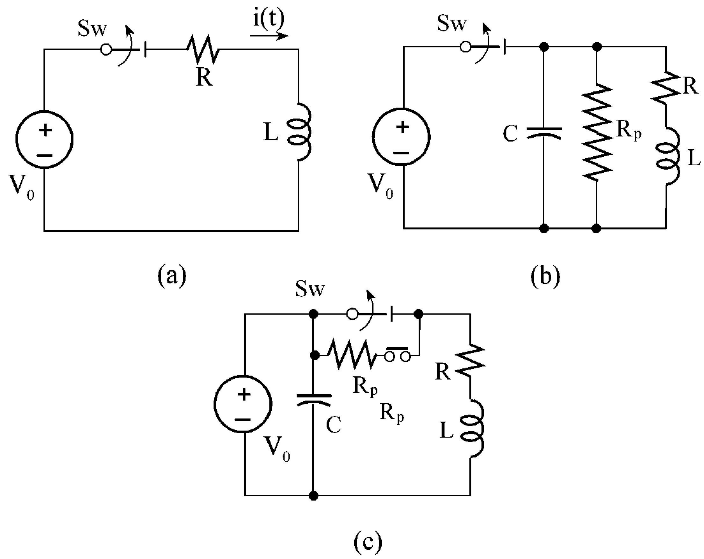

Figure 2.14 A series *RL* circuit switched off instantaneously (a), an *RL* circuit with a parallel resistance and capacitance (b) and an *RL* circuit with a resistance and capacitance parallel to the switch (c).

dissipated in ionizing the air surrounding the contacts. This phenomenon is used in special inductance coils for generating high voltage peaks (for instance, in the ignition system of an automobile such a coil is used to initialize the arcs across the spark plugs to ignite the gasoline in the engine cylinders).

In power circuits, such excess voltages are detrimental and must be avoided. It is useful to connect a substantial resistance parallel to the circuit, Fig. 2.14(b), or, which is even better, parallel to the switch (or breaker), Fig. 2.14(c). In these figures, C represents the stray capacitance shunting the breaker. The presence of an inductance and capacitance raises the differential equation to one of the second order, which will be examined in the following sections. Let us next consider a few examples of the switching phenomenon in first order RL circuits.

Example 2.10

In the circuit of Fig. 2.15, which contains two coils, the switch opens almost instantaneously and coil L_2, whose current prior to switching was different from that of coil L_1, is connected in series with coil L_1. (a) Find the transient current and (b) Estimate the voltage drop between the switch contacts, if the estimated switching time is $\Delta t \approx 10\,\mu\text{s}$. The circuit parameters are: $L_1 = 20$ mH, $L_2 = 80$ mH, $R_1 = 2\,\Omega$, $R_2 = R_3 = 4\,\Omega$ and $V_s = 12$ V (see also Example 1.8).

Solution (a)

1) By inspection of the circuit after switching, we observe that two coils are connected in series, thus

$$s = \frac{R_1 + R_2}{L_1 + L_2} = \frac{6}{(20 + 80)10^{-3}} = 60\ \text{s}^{-1},$$

and the natural response is

$$i_n = Ae^{-60}.$$

2) The forced response is:

$$i_L = \frac{V_s}{R_1 + R_2} = \frac{12}{2 + 4} = 2\ \text{A}.$$

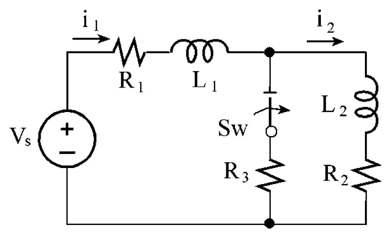

Figure 2.15 A given circuit for Example 2.10.

3) The currents in each coil prior to switching are:

$$i_1(0_-) = \frac{V_s}{R_1 + R_3//R_2} = \frac{12}{2+2} = 3\ \text{A} \quad \text{and} \quad i_2(0_-) = \frac{1}{2}\, i_1(0_-) = 1.5\ \text{A}.$$

4) Using the first generalized switching law regarding flux linkages (1.35b), we may write

$$L_1 i_1(0_-) + L_2 i_2(0_-) = (L_1 + L_2) i(0_+).$$

Therefore the common current $i(0_+)$ in both coils after switching is

$$i(0_+) = \frac{20\cdot 3 + 80\cdot 1.5}{20 + 80} = 1.8\ \text{A}.$$

5) The integrating constant is

$$A = i(0_+) - i_f(0) = 1.8 - 2 = -0.2,$$

and the complete constant is

$$i_f + i_n = 2 - 0.2e^{-60t}\ \text{A}.$$

Solution (b)

To approximate the voltage drop we use the expression

$$v_{sw}(0) \cong L_2 \frac{|\Delta i_2|}{\Delta t}.$$

Since the current rise is $\Delta i_2 = i_2(0_-) - i(0_+) = 1.5 - 1.8 = -0.3$ A, therefore

$$v_{sw} \cong 80\cdot 10^{-3} \frac{0.3}{10\cdot 10^{-6}} \cong 2.4\cdot 10^3 = 2.4\ \text{kV}.$$

Example 2.11

In the circuit of Fig. 2.16, with $L_1 = L_2 = 24$ mH, $M = 12$ mH and $R_1 = R_2 = 1\ \Omega$, the switch opens practically instantaneously after being closed for a long time. Find the current i_2 and estimate the voltage drop in the switch, if $\Delta t_{sw} \cong 1\ \mu$s.

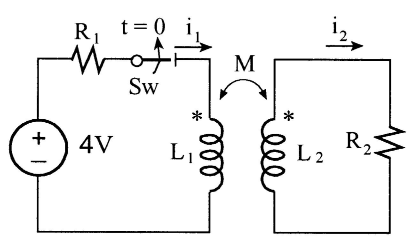

Figure 2.16 A given circuit of Example 2.11.

Solution

1) The time constant of the secondary circuit is

$$\tau = \frac{L_2}{R_2} = \frac{24 \cdot 10^{-3}}{1} = 24\ \text{ms} \quad \text{or} \quad s = -41.7\ \text{s}^{-1}$$

and the natural response is

$$i_{2,n} = Ae^{-41.7t}.$$

2) Since the circuit after switching is source free, the forced response is zero: $i_{2,f} = 0$.

3) The initial value of the current in the transformer secondary may be found in accordance with the principle of flux linkage continuance (first generalized switching law), i.e.,

$$(\lambda_L + \lambda_M)_{t=0_+} = (\lambda_L + \lambda_M)_{t=0_-},$$

or

$$L_1 i_1(0_+) - Mi_2(0_+) + L_2 i_2(0_+) - Mi_1(0_+) = L_1 i_1(0_-) - Mi_2(0_-) + L_2 i_2(0_-) - Mi_1(0_-).$$

Since $i_2(0_-) = 0$, $i_1(0_-) = 4\ \text{A}$ and $i_1(0_+) = 0$, we have $(L_2 - M)i_2(0_+) = (L_1 - M)i_1(0_-)$ and since $L_1 = L_2$, we have $i_2(0_+) = i_1(0_-) = 4\ \text{A}$.

5) Omitting step 4 (since non-dependent initial values are needed) we obtain

$$A = i_2(0) - i_{2,f}(0) = 4,$$

and

$$i_2 = 4e^{-41.7t}.$$

The voltage drop across the switch will be

$$v_{sw} = \left| L\frac{-\Delta i_1}{\Delta t} - M\frac{\Delta i_2}{\Delta t} \right| = \frac{(24 \cdot 4 + 12 \cdot 4)10^{-3}}{10^{-6}} = 144 \cdot 10^3 = 144\ \text{kV}.$$

Checking the energy preservation, we may find:

The energy prior to switching:

$$\frac{L_1 i_1^2(0_-)}{2} + \frac{L_2 i_2^2(0_-)}{2} + Mi_1(0_-)i_2(0_-) = \frac{L_1 i_1^2(0_-)}{2} = \frac{24 \cdot 10^{-3} \cdot 4^2}{2} = 192\ \text{mJ},$$

and the energy after switching:

$$\frac{L_1 i_1^2(0_+)}{2} + \frac{L_2 i_2^2(0_+)}{2} + Mi_1(0_+)i_2(0_+) = \frac{L_2 i_2^2(0_+)}{2} = \frac{24 \cdot 10^{-3} \cdot 4^2}{2} = 192\ \text{mJ},$$

which are the same.

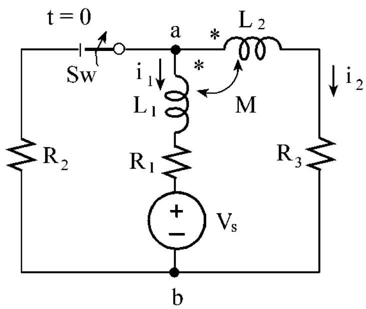

Figure 2.17 A given circuit for Example 2.12.

Example 2.12

In the circuit of Fig. 2.17 containing the mutual inductance, the switch opens practically instantaneously, after being closed for a long time. Find the transient response of current i_1 for two cases: (a) Both dotted terminals are connected to the common node "a" and (b) Only one dotted terminal is connected to the common node "a". The circuit parameters are $R_1 = 5\ \Omega$, $R_2 = R_3 = 10\ \Omega$, $L_1 = 0.1$ H, $L_2 = 0.2$ H, $M = 0.05$ H and $V_s = 60$ V.

Solution (a)

1) The characteristic equation will be determined by writing the KVL equation for the right loop and equating it to zero (note that after switching $i_1 = i_2 = i$ and all the elements are connected in series):

$$[R_1 + R_2 + s(L_1 + L_2 - 2M)]i = 0.$$

Thus,

$$(0.1 + 0.2 - 0.1)s + 5 + 10 = 0 \quad \text{and} \quad s = -75\ \text{s}^{-1}.$$

Therefore, the natural response is

$$i_n = Ae^{-75t}.$$

2) The steady-state current is

$$i_f = \frac{V_s}{R_1 + R_2} = \frac{60}{5 + 10} = 4\ \text{A}.$$

3) The initial value of the current $i(0_+)$ shall now be found using the first generalized law, i.e.,

$$i_1(0_+)(L_1 + L_2 - 2M) = i_1(0_-)L_1 + i_2(0_-)L_2 - [i_1(0_-) + i_2(0_-)]M,$$

or

$$i_1(0_+) = \frac{i_1(0_-)L_1 + i_2(0_-)L_2 - [i_1(0_-) + i_2(0_-)]M}{(L_1 + L_2 - 2M)}$$

$$= \frac{6 \cdot 0.1 + 3 \cdot 0.2 - 9 \cdot 0.05}{0.1 + 0.2 - 0.1} = 3.75 \text{ A},$$

where the currents prior to switching are (by inspection of the circuit in Fig. 2.17): $i_1(0_-) = 6$ A and $i_2(0_-) = 3$ A.

4–5) The integration constant can now be evaluated as

$$A = i(0) - i_f(0) = 3.75 - 4 = -0.25,$$

and the complete response is

$$i(t) = 4 - 0.25e^{-75t} \text{ A}.$$

Solution (b)

1) The exchange of the position of the dotted terminals results in a positive sign connection of the mutual inductance. Therefore,

$$[R_1 + R_2 + s(L_1 + L_2 + 2M)]i = 0,$$

or

$$15 + 0.4s = 0 \quad \text{and} \quad s = -37.5 \text{ s}^{-1}.$$

Thus,

$$i_n = Ae^{-37.5t}.$$

2) The forced response is not influenced by the dotted terminal exchange and remains the same $i_f = 4$ A.

3) The initial condition is now found as

$$i_1(0_+) = \frac{i_1(0_-)L_1 + i_2(0_-)L_2 + [i_1(0_-) + i_2(0_-)]M}{(L_1 + L_2 + 2M)}$$

$$= \frac{6 \cdot 0.1 + 3 \cdot 0.2 + 9 \cdot 0.05}{0.1 + 0.2 + 0.1} = 4.125 \text{ A}.$$

4–5) The integration constant is

$$A = 4.125 - 4 = 0.125,$$

and the complete response in this case is

$$i(t) = 4 + 0.125e^{-37.5t} \text{ A}.$$

Example 2.13

Our last example in this section will be the circuit shown in Fig. 2.18(a). This

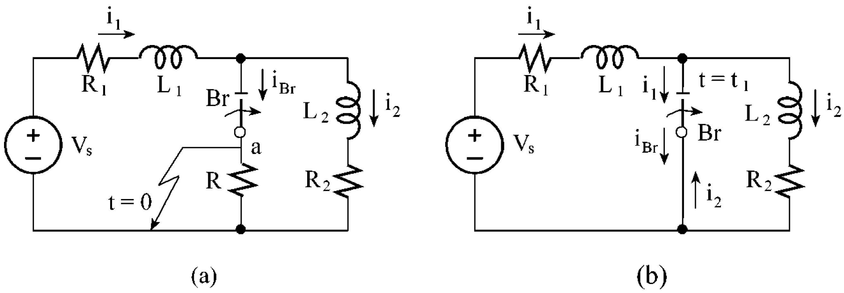

Figure 2.18 A given circuit for Example 2.13(a) and the circuit after the short-circuiting for the second stage of the transients (b).

circuit represents the equivalent of a d.c. supply network. At the instant of time $t = 0$, the short-circuit fault occurs at node "a" and when the short-circuit current i_{sc} through the breaker reaches the value $I = 500$ A, the circuit breaker opens practically instantaneously. Find the transient response of current i_2 after the fault. The circuit parameters are $R_1 = 1\ \Omega$, $R = R_2 = 9\ \Omega$, $L_1 = 0.01$ H, $L_2 = 0.45$ H and $V_s = 1100$ V.

Solution

First stage (the period between a short circuit $t = 0$ and opening the circuit breaker, BR, $t = t_1$).

1) Since the circuit is divided into two sub circuits: the left one with current i_1 and the right one with current i_2, we shall obtain two time constants and two natural responses:

$$(1)\quad \tau_1 = \frac{L_1}{R_1} = \frac{0.01}{1} = 0.01\ \text{s}, \quad \text{or} \quad s_1 = -100\ \text{s}^{-1} \quad \text{and} \quad i_{1,n} = A_1 e^{-100t},$$

$$(2)\quad \tau_2 = \frac{L_2}{R_2} = \frac{0.45}{9} = 0.05\ \text{s}, \quad \text{or} \quad s_2 = -20\ \text{s}^{-1} \quad \text{and} \quad i_{2,n} = A_2 e^{-20t}.$$

2) The forced responses in these circuits are:

$$(1)\quad i_{1,f} = \frac{V_s}{R_1} = \frac{1100}{1} = 1100\ \text{A}, \qquad (2)\quad i_{2,f} = 0.$$

3) The initial conditions of the above two currents may be obtained by inspection of the given circuit prior to short-circuiting

$$(1)\quad i_1(0_-) = \frac{V_s}{R_1 + R_2 // R_3} = \frac{1100}{1 + 4.5} = 200\ \text{A}, \qquad (2)\quad i_2(0_-) = \frac{1}{2} i_1(0_-) = 100\ \text{A}.$$

4–5) The integration constants are

$$(1) \quad A_1 = i_1(0) - i_{1,f} = 200 - 1100 = -900,$$

$$(2) \quad A_2 = i_2(0) - i_{2,f} = 100 - 0 = 100.$$

The complete response in each of the circuits is

$$(1) \quad i_1(t) = 1100 - 900e^{-100t}\ \mathsf{A}, \qquad (2) \quad i_2(t) = 100e^{-20t}\ \mathsf{A}.$$

In order to determine the instant of time, at which the breaker opens, we must solve the equation

$$i_{Br} = i_1 - i_2|_{t=t_1} = 500,$$

or

$$1100 - 900e^{-100t_1} - 100e^{-20t_1} = 500.$$

This transcendental equation can now be solved by the iteration approach. Since the time constant of the second circuit is relatively large, we assume that current i_2 is a constant. Thus, the first estimation of time t_1 will be found as

$$900e^{-100t_1} = 1100 - 600, \quad \text{and} \quad -100t_1 = \ln\frac{500}{900} \quad \text{or} \quad t_1^{(1)} = 5.6\ \mathsf{ms}$$

For the second estimation we assume that current $i_2 = 100e^{-20\cdot 5.6\cdot 10^{-3}} = 89.4\ \mathsf{A}$, therefore, now

$$900e^{-100t_1} = 511,$$

to which the solution is

$$t_1^{(2)} = -\frac{\ln(511/900)}{100} = 0.566\cdot 10^{-2} \cong 5.7\ \mathsf{ms}.$$

Since this result is very close to the previous one, no more estimations are needed, and the value $t_1 = 5.7$ ms is taken as the answer.

Second stage (the period of time after the breaker opens $t > 5.7$ ms)

1) In this stage the circuit consists of only one loop, whose characteristic equation is

$$R_1 + R_2 + s(L_1 + L_2) = 0.$$

.

Upon substitution of the numerical data the time constant becomes

$$\tau = \frac{0.46}{10} = 0.046\ \mathsf{s}, \quad \text{or} \quad s = 21.7\ \mathsf{s}^{-1}.$$

2) The forced response is

$$i_{2,f} = \frac{1100}{1+9} = 110\ \mathsf{A}.$$

3) The current values prior to switching are

$$i_1(0_-) = i_1(t_1) = 1100 - 900e^{-100\cdot 5.7\cdot 10^{-3}} = 591\ \mathrm{A},$$
$$i_2(0_-) = i_2(t_1) = 100e^{-20\cdot 5.7\cdot 10^{-3}} = 89\ \mathrm{A}.$$

The initial value of i_2 after switching (note that both currents are now equal), in accordance with the first generalized law, is

$$i_2(0_+) = \frac{i_1(0_-)L_1 + i_2(0_-)L_2}{L_1 + L_2} = \frac{591\cdot 0.01 + 89\cdot 0.45}{0.46} \cong 100\ \mathrm{A}.$$

4–5) The integration constant can now be found $A = i_2(0) - i_f = 100 - 110 = -10$.

Therefore, the complete response of current i_2 after the short-circuit fault is

$$i_2(t) = 100e^{-20t} \quad \text{for} \quad 0 < t < 5.7$$
$$i_2(t) = 110 - 10e^{-21.7(t-t_1)} \quad \text{for} \quad 5.7 < t < \infty.$$

Note that at the moment $t = 5.7$ ms the current changes rapidly (however, the total magnetic flux of both inductances remains the same).

2.4 *RC* CIRCUITS

We shall approach the transient analysis of *RC* circuits keeping in mind the principle of duality. As we have noted the *RC* circuit is dual to the *RL* circuit. This means that we may use all the achievements and results we obtained in the previous section regarding the inductive circuit for capacitance circuit analysis. For instance, the time constant of a simple *RL* circuit has been obtained as $\tau_L = L/R$, for a simple *RC* circuit it must be $\tau_C = C/G$ (i.e., L is replaced by C and R by G, which are dual elements). Since $G = 1/R$, the time constant of an *RC* circuit can, of course, be written as $\tau_C = RC$. In the following sections, more examples of such duality will be presented.

2.4.1 Discharging and charging a capacitor

Consider once again the *RC* circuit (also see section 1.3.1) shown in Fig. 2.19(a), in which R and C are connected in parallel. Prior to switching the capacitance was charged up to the voltage of the source V_s. After opening the switch, the capacitance discharges through the resistance. The time constant of the circuits is $\tau = RC$ and the initial value of the capacitance voltage is $V_{C0} = V_s$. The forced response component of the capacitance voltage is zero, since the circuit after switching is source free. Thus,

$$v_C(t) = V_{C0}e^{-\frac{t}{RC}}. \tag{2.11}$$

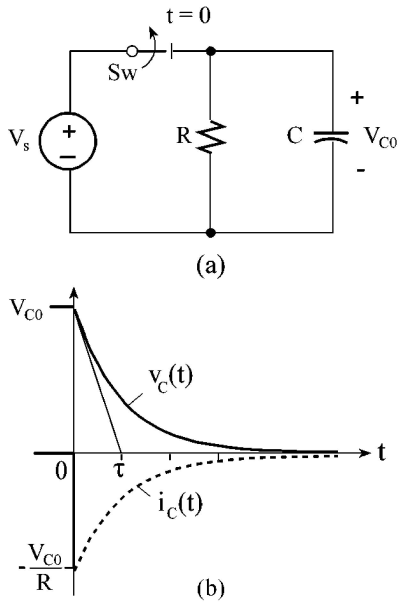

Figure 2.19 A circuit of a parallel connection of resistance and capacitance (a) and the plots of the discharging voltage and current (b).

The current response will be

$$i_C = C\frac{dv_C}{dt} = -\frac{V_{C0}}{R}e^{-\frac{t}{RC}}. \tag{2.12}$$

Note that 1) the current changes abruptly at $t = 0$ from zero (prior to switching) to V_{C0}/R and 2) its direction is opposite to the charging current. This current and the capacitance voltage are plotted in Fig. 2.19(b). Also note that the voltage curve in Fig. 2.19(b) is similar to the current curve in the RL circuit, and inversely the current curve is similar to the voltage curve in the RL circuit, as shown in Fig. 2.2(c). This fact is actually another example of duality.

Let us now show that the energy stored in the electric field of the capacitance completely dissipates in the resistance, converging into heat, during the transients. The energy stored is

$$w_e = \frac{CV_{C0}^2}{2}. \tag{2.13}$$

The energy dissipated is

$$w_R = \int_0^\infty \frac{v_C^2}{R}dt = \frac{V_{C0}^2}{R}\int_0^\infty e^{-\frac{2t}{RC}}dt = -\frac{RCV_{C0}^2}{2R}e^{-\frac{2t}{RC}}\bigg|_0^\infty = \frac{CV_{C0}^2}{2}. \tag{2.14}$$

Hence, the energy conservation law has been conformed to.

Consider next the circuit of Fig. 2.20, in which the capacitance is charging

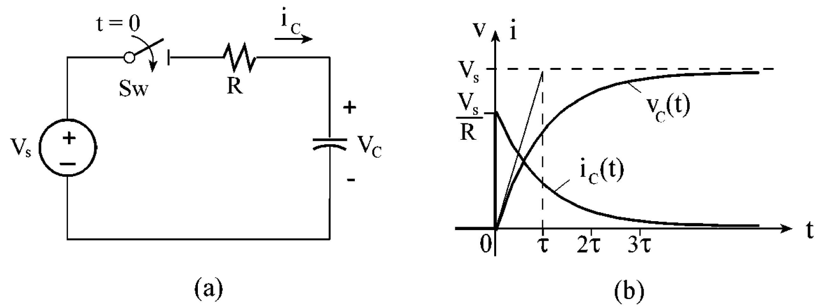

Figure 2.20 An *RC* circuit in which the capacitance is charging (a) and the plots of the voltage and current responses (b).

through the resistance after closing the switch. The natural response of this circuit is similar to the previous circuit, i.e.,

$$v_{C,n} = A e^{-t/RC}.$$

However, because of the presence of a voltage source, the forced response (step 2) will be $v_{C,f} = V_s$, since in the steady-state operation the current is zero (the capacitance is fully charged), and the voltage across the capacitance is equal to the source voltage.

Next, we realize that the initial value of the capacitance voltage, prior to switching (step 3), is zero, and the constant of integration (step 5) is obtained as $A = 0 - V_s = -V_s$.

The complete response, therefore is

$$v_C = V_s - V_s e^{-t/RC} = V_s(1 - e^{-t/RC}). \tag{2.15}$$

The current response can now be found as

$$i_C = C\frac{dv_C}{dt} = \frac{V_s}{R} e^{-t/RC}. \tag{2.16}$$

Both responses, voltage and current, are plotted in Fig. 2.20(b). Note again that these two curves are similar to the current and voltage response curves respectively in the *RL* circuit as shown in Fig. 2.1(b) and (c).

2.4.2 *RC* circuits under d.c. supply

Let us now consider more complicated *RC* circuits, fed by a d.c. source. If, for instance, in such circuits a few resistances are connected in series/parallel, we may simplify the solution by determining R_{eq} and reducing the circuit to a simple *RC*-series, or *RC*-parallel circuit. An example of this follows.

Example 2.14

Consider the circuit of Fig. 2.21 with $R_1 = R_2 = R_3 = R_4 = 50\ \Omega$, $C = 100\ \mu F$ and

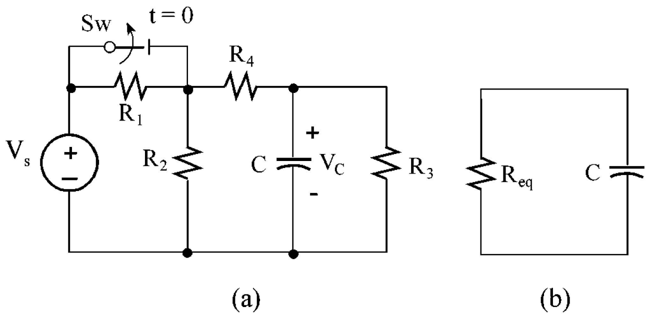

Figure 2.21 A given circuit of Example 2.14 (a) and its simplified equivalent (b).

$V_s = 250$ V. Find the voltage across the capacitance after the switch opens at $t = 0$.

Solution

After the voltage source is "killed" (short-circuited), we may determine the equivalent resistance, which is in series/parallel to the capacitance, Fig. 2.21(b): $R_{eq} = (R_1//R_2 + R_4)//R_3$, which, upon substituting the numerical values, results in $R_{eq} = 30\ \Omega$. Thus, the time constant (step 1) is

$$\tau = R_{eq}C = 30 \cdot 100 \cdot 10^{-6} = 3 \text{ ms}, \quad \text{and} \quad v_{C,n} = Ae^{-t/3}, \quad (t \text{ is in ms}).$$

By inspection of the circuit in its steady-state operation ($t \to \infty$) the voltage across the capacitor (the forced response) can readily be found (step 2): $v_{C,f} = 50$ V. The initial value of the capacitance voltage (step 3) must be determined prior to switching:

$$v_C(0_+) = v_C(0_-) = V_s \frac{R_3}{R_3 + R_4} = 250 \frac{50}{100} = 125 \text{ V}.$$

Hence, the integration constant (step 5) is found to be $A = v_C(0) - v_{C,f} = 125 - 50 = 75$, and the complete response is

$$v_C(t) = 50 + 75e^{-t/3}.$$

With the above expression of the integration constant (see step 5), the complete response in the first order circuit can be written in accordance with the following formula (given here in its general notation, for either voltage or current):

$$f(t) = f_f + f_n = f_f + (f_0 - f_{f,0})e^{-t/\tau}, \tag{2.17}$$

where f_0 and $f_{f,0}$ are the initial values of the complete and the forced responses respectively. Or in the form

$$f(t) = f_f(1 - e^{-t/\tau}) + f_0 e^{-t/\tau}, \tag{2.18}$$

and for zero initial conditions ($f_0 = 0$)

$$f(t) = f_f(1 - e^{-t/\tau}). \tag{2.19}$$

In the following examples, we shall consider more complicated *RC* circuits.

Example 2.15

At the instant $t = 0$ the capacitance is interswitched between two voltage sources, as shown in Fig. 2.22(a). The circuit parameters are $R_1 = 20\ \Omega$, $R_2 = 10\ \Omega$, $R_3 = R_4 = 100\ \Omega$, $C = 0.01$ F, and the voltage sources are $V_{s1} = 60$ V and $V_{s2} = 120$ V. Find voltage $v_C(t)$ and current $i_2(t)$ for $t > 0$.

Solution

1) The input impedance, Fig. 2.22(b), is:

$$Z_{in}(s) = \frac{1}{sC} + R_2 // R_3 // R_4.$$

Upon substitution of the numerical data and equating it to zero yields

$$\frac{1}{s}10^2 + \frac{50}{6} = 0, \quad \text{or} \quad s = -12\ \text{s}^{-1},$$

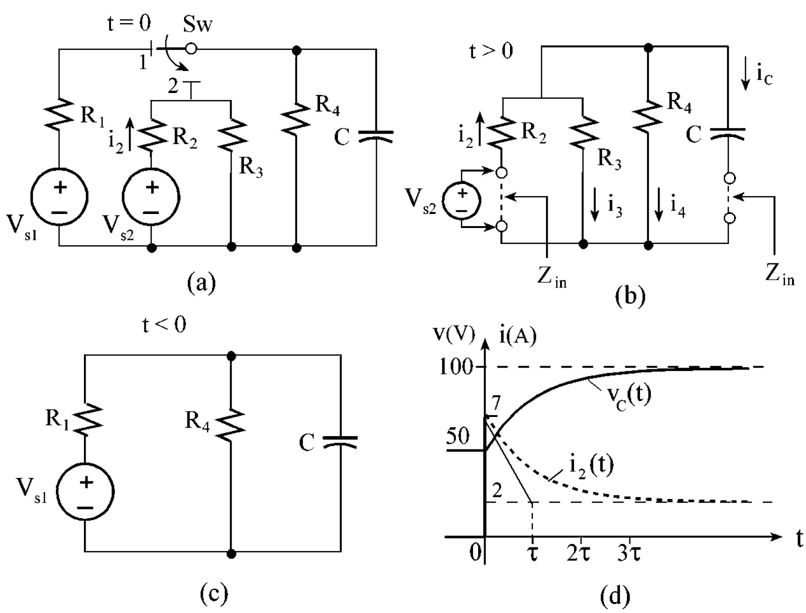

Figure 2.22 A given *RC* circuit of Example 2.15 (a), a circuit for determining the input impedance and the forced response (b), a circuit for determining the initial value (c) and the curves of the voltage and current responses (d).

and the natural response becomes

$$v_{C,n} = Ae^{-12t}.$$

2) The forced response is found as the voltage drop in two parallel resistances $R_{3,4} = 50\ \Omega$. With the voltage division formula, we obtain

$$v_{C,f} = V_s \frac{R_{3,4}}{R_2 + R_{3,4}} = 120 \frac{50}{10 + 50} = 100\ \mathsf{V}.$$

3) The initial value of the capacitance voltage must be determined from the circuit prior to switching, as shown in Fig. 2.22(c). Applying the voltage division once again, we have

$$v_C(0_+) = v_C(0_-) = 60 \frac{100}{20 + 100} = 50\ \mathsf{V}.$$

5) (Step 4 is omitted, as it is unnecessary). In accordance with equation 2.17 we obtain

$$v_C(t) = 100 + (50 - 100)e^{-12t} = 100 - 50e^{-12t}\ \mathsf{V}.$$

Current i_2 can now be easily found as, Fig. 2.22(b),

$$\begin{aligned} i_2(t) = i_R + i_C = \frac{v_C}{R_{3,4}} + C\frac{dv_C}{dt} &= 2 - 1e^{-12t} + 0.01(-50)(-12)e^{-12t} \\ = 2 + 5e^{-12t}\ \mathsf{A}. \end{aligned}$$

Both curves, of v_C and i_2, are plotted in Fig. 2.22(d). Note that the current i_2 changes abruptly from zero to 7 A. Our next example will be a second order *RC* circuit.

Example 2.16

Consider the second order *RC* circuit shown in Fig. 2.23(a), having $R_1 = R_3 = 200\ \Omega$, $R_2 = R_4 = 100\ \Omega$, $C_1 = C_2 = 100\ \mu\mathsf{F}$ and two sources $V_s = 300\ \mathsf{V}$ and $I_s = 1\ \mathsf{A}$. The switch opens at $t = 0$ after having been closed for a long time. Find current $i_2(t)$ for $t > 0$.

Solution

1) We shall determine the characteristic equation by using mesh analysis for the circuit in Fig. 2.23(a) after opening the switch and with “killed” sources

$$\left(\frac{1}{sC_1} + \frac{1}{sC_2} + R_1 + R_2\right) i_2 - \frac{1}{sC_2} i_3 = 0$$

$$-\frac{1}{sC_2} i_2 + \left(\frac{1}{sC_2} + R_3\right) i_3 = 0.$$

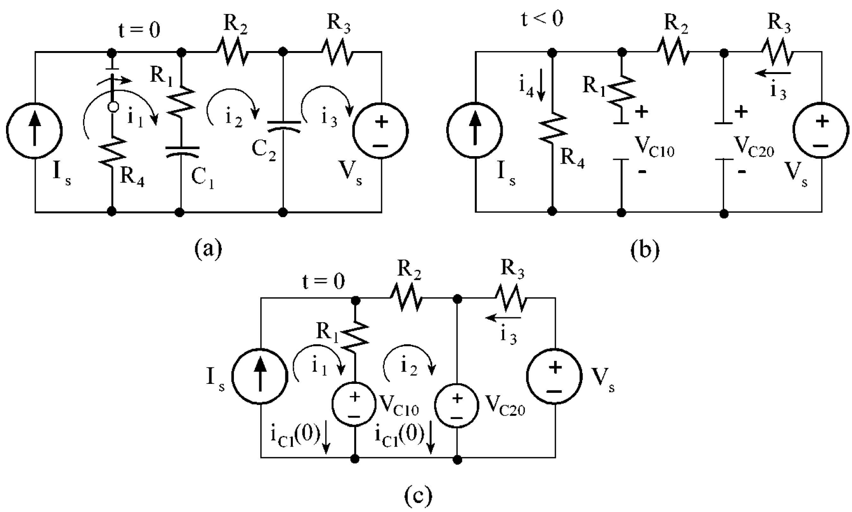

Figure 2.23 A second order *RC* circuit of Example 2.16 (a), an equivalent circuit for the calculation of the independent initial conditions (b) and an equivalent circuit for the calculation of the dependent initial conditions.

Equating the determinant for this set of equations to zero, we may obtain the characteristic equation (note that $C_1 = C_2 = C$)

$$\left(\frac{2}{sC} + R_1 + R_2\right)\left(\frac{1}{sC} + R_3\right) - \left(\frac{1}{sC}\right)^2 = 0.$$

Upon substituting the numerical data the above becomes $6s^2 + 700s + 10^4 = 0$, and the roots are $s_1 = -16.7\ \mathrm{s}^{-1}$ and $s_2 = -100\ \mathrm{s}^{-1}$. Therefore, the natural response becomes

$$i_{2,n} = A_1 e^{-16.7t} + A_2 e^{-100t}\ \mathrm{A}.$$

2) By inspection of the circuit in Fig. 2.23(a), in its steady-state operation (after the switch had been open for a long time), we may conclude that the only current flowing through resistance R_2 is the current of the current source, i.e., $i_{2,f} = I_s = 1\ \mathrm{A}$.

3) In order to determine the independent initial condition, i.e. the capacitance voltages at $t = 0$, we shall consider the circuit equivalent for this instant of time, shown in Fig. 2.23(b). Using the superposition principle, we may find the current through resistance R_3 as

$$i_3 = \frac{V_3}{R_2 + R_3 + R_4} - I_s \frac{R_4}{R_2 + R_3 + R_4} = \frac{300}{400} - 1\,\frac{100}{400} = 0.5\ \mathrm{A},$$

and the voltage across capacitance C_2 as $v_{C2} = V_{C20} = V_s - R_3 i_3 = 300 -$

$200 \cdot 0.5 = 200$ V. In a similar way

$$i_4 = \frac{V_3}{R_2 + R_3 + R_4} + I_s \frac{R_2 + R_3}{R_2 + R_3 + R_4} = \frac{300}{400} + 1\,\frac{100 + 200}{400} = 1.5 \text{ A},$$

and

$$v_{C1}(0) = V_{C10} = R_4 i_4 = 100 \cdot 1.5 = 150 \text{ V}.$$

4) Since the response that we are looking for is the current in a resistance, it can change abruptly. For this reason, and also since the response is of the second order, we must determine the dependent initial conditions, namely $i_2(0)$ and $di_2/dt|_{t=0}$. This step usually has an abundance of calculations. (This is actually the reason why the transformation methods, in which there is no need to determine the dependent initial conditions, are preferable). However, let us now perform these calculations in order to complete the classical approach.

In order to determine $i_2(0)$ we must consider the equivalent circuit, which fits instant $t = 0$, Fig. 2.23(c). With the mesh analysis we have $R_1[i_2(0) - I_s] + R_2 i_2(0) = V_{C10} - V_{C20}$, or

$$i_2(0) = \frac{V_{C10} - V_{C20} + R_1 I_s}{R_1 + R_2} = \frac{150 - 200 + 200 \cdot 1}{200 + 100} = 0.5 \text{ A}.$$

For the following calculations, we also need the currents through the capacitances, i.e., through the voltage sources, which represent the capacitances. First, we find current i_3:

$$i_3 = (V_s - V_{C20})/R_3 = (300 - 200)/200 = 0.5 \text{ A},$$

then

$$i_{C1}(0) = I_s - i_2(0) = 1 - 0.5 = 0.5 \text{ A}$$

$$i_{C2}(0) = i_2(0) + i_3(0) = 0.5 + 0.5 = 1.0 \text{ A}.$$

In order to determine the derivative of i_2, we shall write the KVL equation for the middle loop (Fig. 2.23(c)):

$$-v_{C1} - R_1(I_s - i_2) + R_2 i_2 + v_{C2} = 0.$$

After differentiation we have

$$(R_1 + R_2)\frac{di_2}{dt} = \frac{dv_{C1}}{dt} - \frac{dv_{C2}}{dt} = \frac{1}{C}(i_{C1} - i_{C2}),$$

or

$$\left.\frac{di_2}{dt}\right|_{t=0} = \frac{1}{(R_1 + R_2)C}(i_{C1} - i_{C2}) = \frac{1}{300 \cdot 10^{-4}}(0.5 - 1) = -16.7.$$

5) In accordance with equation 1.61 we can now find the integration constants

$$A_1 + A_2 = i_2(0) - i_{2,f}(0) = 0.5 - 1$$

$$s_1 A_1 + s_2 A_2 = \left.\frac{di_2}{dt}\right|_{t=0} - \left.\frac{di_{2,f}}{dt}\right|_{t=0} = -16.7 - 0,$$

or

$$A_1 + A_2 = -0.5$$
$$-16.7A_1 - 100A_2 = -16.7,$$

to which the solution is

$$A_1 = \frac{-0.5(-100) + 16.7}{-100 + 16.7} = -0.8$$
$$A_2 = -0.5 - A_1 = -0.5 + 0.8 = 0.3.$$

Thus the complete response is

$$i_2(t) = 1 - 0.8e^{-16.7t} + 0.3e^{-100t}\ \text{A}.$$

2.4.3 *RC* circuits under a.c. supply

If the capacitive branch switches to the a.c. supply of the form $v_s = V_{sm}\sin(\omega t + \psi_v)$, as shown in Fig, 2.24(a), the forced response of the capacitance

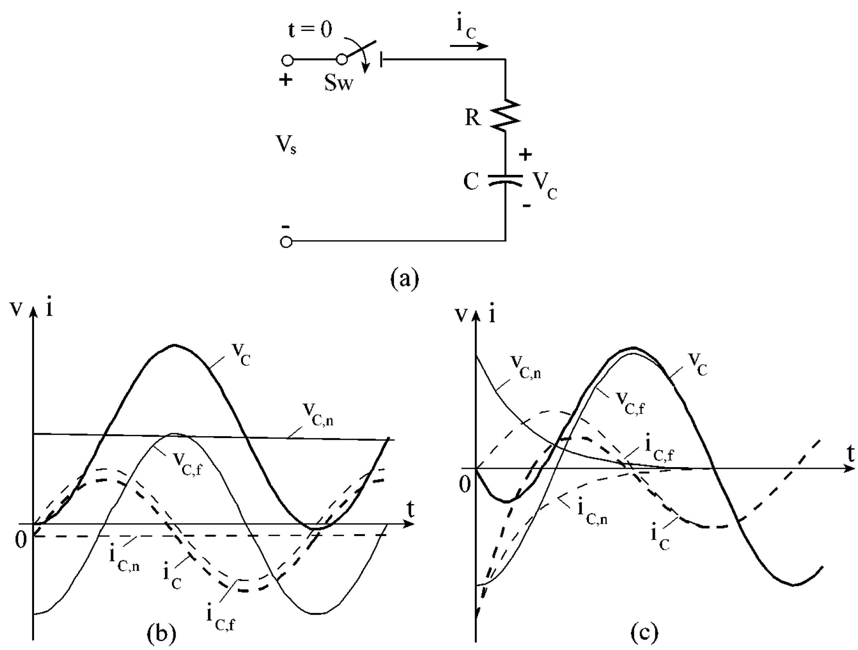

Figure 2.24 An *RC* circuit under an a.c. supply (a), the transient response when the overvoltage occurs (b) and the transient response with the current peak (c).

voltage will be

$$v_{C,f} = V_{Cm}\sin(\omega t + \psi_v - \varphi - \pi/2). \tag{2.20}$$

Here phase angle ψ_v (switching angle), is appropriate to the instant of switching $t = 0$

$$V_{cm} = \frac{1}{\omega C}\frac{V_{sm}}{\sqrt{R^2 + (1/\omega C)^2}} \tag{2.21a}$$

and

$$\varphi = \tan^{-1}(-1/R\omega C) \tag{2.21b}$$

Since the natural response does not depend on the source, it is

$$v_{C,f} = Ae^{-t/RC}.$$

With zero initial conditions, i.e., $v_C(0) = 0$, the integration constant becomes

$$A = v_C(0) - v_{C,f}(0) = -V_{Cm}\sin(\psi_v - \varphi - \pi/2). \tag{2.22}$$

Thus, the complete response of the capacitance voltage will be

$$v_C(t) = V_{Cm}[\sin(\omega t + \psi_v - \varphi - \pi/2) - \sin(\psi_v - \varphi - \pi/2)e^{-t/RC}], \tag{2.23}$$

and of the current

$$i_C(t) = C\frac{dv_C}{dt} = I_m\left[\sin(\omega t + \psi_v - \varphi) + \frac{1}{\omega RC}\sin(\psi_v - \varphi - \pi/2)e^{-t/RC}\right], \tag{2.24}$$

where

$$I_m = \omega C V_{Cm} = \frac{V_{sm}}{R\sqrt{1 + (1/\omega RC)^2}} \tag{2.25}$$

and

$$A = \frac{I_m}{\omega RC}\sin(\psi_v - \varphi - \pi/2). \tag{2.26}$$

Since, during the transient behavior, the natural response is added to the forced response of the voltage and current, it may happen that the complete responses will exceed their rated amplitudes. The maximal values of overvoltages and current peaks depend on the switching angle and time constant. If switching occurs at the moment when the forced voltage equals its amplitude value, i.e. when the switching angle $\psi_v = \varphi$ and with a large time constant, the overvoltage may reach the value of an almost double amplitude, $2V_{Cm}$. This is shown in Fig. 2.24(b). It should be noted that the current in this case will almost be its regular value, since at the switching moment its forced response equals zero, and the initial value of the natural response (equation 2.26) is small because of the large resistance due to the large time constant, Fig. 2.24(b). (Compare with

Figs. 2.9 and 2.10 of the current response in an RL circuit under an a.c. supply). On the other hand, if the time constant is small due to the small resistance R, the current peak, at $t=0$, may reach a very high level, many times that of its rated amplitude, Fig. 2.24(c). However the overvoltage will not occur.

We shall now consider a few numerical examples.

Example 2.17

In the circuit of Fig. 2.25(a), with $R_1 = R_2 = 5\ \Omega$, $C = 500\ \mu\text{F}$ and $v_s = 100\sqrt{2}\sin(\omega t + \pi/2)$, find current $i(t)$ after switching.

Solution

There are two ways of finding the current: 1) straightforwardly and 2) first to find the capacitance voltage and then to perform the differentiation $i = C(dv_C/dt)$. We will present both ways.

1) The time constant (step 1) is $\tau = RC = 5\cdot 500\cdot 10^{-6} = 2.5\cdot 10^{-3}$, therefore $s = -1/\tau = -400\ \text{s}^{-1}$ and the natural response is $i_n = Ae^{-400t}$. The forced response (step 2) is

$$i_f = I_m \sin(\omega t + \pi/2 - \varphi) = 17.5 \sin(\omega t + 141.8^\circ),$$

where

$$I_m = \frac{100\sqrt{2}}{\sqrt{5^2 + (1/314\cdot 5\cdot 10^{-4})^2}} = 17.5\ \text{A}$$

and

$$\varphi = \tan^{-1}\frac{-1/(314\cdot 5\cdot 10^{-4})}{5} = -51.8^\circ.$$

The initial value of the capacitance voltage (the initial independent condition, step 3) must be found in the circuit of Fig. 2.25(a) prior to switching

$$v_C(0_-) = \frac{100\sqrt{2}\cdot 6.37}{\sqrt{10^2 + 6.37^2}} \sin\left(\frac{\pi}{2} - \tan^{-1}\frac{-6.37}{10} - \frac{\pi}{2}\right) = 40.8\ \text{V},$$

where

$$x_C = \frac{1}{314\cdot 5\cdot 10^{-4}} = 6.37\ \Omega.$$

The initial value of the current, which is the dependent initial condition (step 4) may be found from the equivalent circuit, for the instant of switching, $t=0$, which is shown in Fig. 2.25(b):

$$i(0) = \frac{100\sqrt{2} - 40.8}{5} = 20.1\ \text{A}.$$

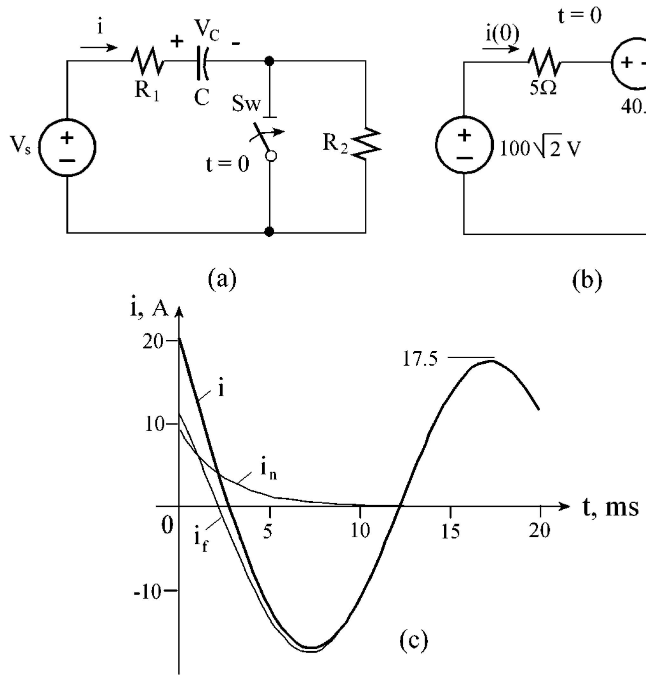

Figure 2.25 A given circuit for Example 2.15 (a), its equivalent for calculating $i(0)$ (b) and the plot of the current (c).

The integration constant and complete response (step 5) will then be

$$A_i = i(0) - i_f(0) = 20.1 - 17.5 \sin 141.8^\circ = 9.3 \text{ A},$$

and

$$i(t) = 17.5 \sin(314t + 141.8^\circ) + 9.3e^{-400t} \text{ A}.$$

2) The difference in the calculation according to way 2) is that we do not need Step 4. Step 1 is the same; therefore, the natural response of the capacitance voltage is $v_{C,n} = Ae^{-400t}$, and we continue with Step 2:

$$v_{C,f} = \frac{100\sqrt{2}\cdot 6.37}{\sqrt{5^2 + 6.37^2}} \sin(314t + \pi/2 + 51.8^\circ - \pi/2) = 111.3 \sin(314t + 51.8^\circ) \text{ V}.$$

Step 3 has already been performed so we can calculate the complete response as

$$v_C(t) = 111.3 \sin(314t + 51.8^\circ) - 46.7e^{-400t} \text{ V},$$

where

$$A_v = v_C(0) - v_{C,f}(0) = 40.8 - 111.3 \sin 51.8^\circ = -46.7.$$

The current can now be evaluated as

$$i = C\frac{dv_C}{dt} = 17.5\sin(314t + 51.8^\circ + \pi/2) + 9.3e^{-400t}\ \mathrm{A},$$

where

$$I_m = 5\cdot 10^{-4}\cdot 314\cdot 111.3 = 17.5 \quad \text{and} \quad A_i = 5\cdot 10^{-4}(-400)(-46.7) = 9.3\ \mathrm{A},$$

which is the same as previously obtained. The plot of current i is shown in Fig. 2.25(c).

Example 2.18

In the circuit of Fig. 2.26(a), the switch closes at $t = 0$. Find the current in the switching resistance R_3. The circuit parameters are: $R_1 = R_2 = R_3 = 10\ \Omega$, $C = 250\ \mu\mathrm{F}$ and $v_s = 100\sqrt{2}\sin(\omega t + \psi_v)$ at $f = 60$ Hz. To determine the switching angle ψ_v, assume that at the instant of switching $v_s = 0$ and its derivative is positive.

Solution

The voltage is zero if ψ_v is 0° or 180°. Since the derivative of the sine wave at 0° is positive (and at 180° it is negative), we should choose $\psi_v = 0^\circ$.

1) To determine the time constant (step 1) we shall first find the equivalent resistance $R_{eq} = R_2 + R_1//R_3 = 10 + 5 = 15\ \Omega$. Thus, $\tau = R_{eq}C = 15\cdot 250\cdot 10^{-6} = 3.75$ ms and $s = -1/\tau = -267\ \mathrm{s}^{-1}$. Therefore, the natural response is

$$i_{3,n} = Ae^{-267t}\ \mathrm{A}.$$

2) The forced response shall be found by using node analysis

$$\frac{\tilde{V}_a - \tilde{V}_s}{R_1} + \frac{\tilde{V}_a}{R_2 - jX_C} + \frac{\tilde{V}_a}{R_3} = 0.$$

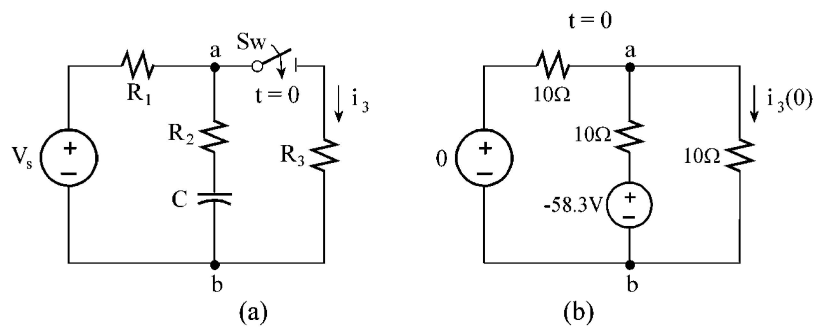

Figure 2.26 A given *RC* circuit for Example 2.16 (a) and its equivalent for determining the initial value of the current $i_3(0)$ (b).

Upon substituting $1/377 \cdot 2.5 \cdot 10^{-4} = 10.6$ for x_C, $141 \angle 0^\circ$ for $\tilde{V}_s$ and 10 for R_3 and R_1 the above equation becomes

$$\frac{\tilde{V}_a - 141}{10} + \frac{\tilde{V}_a}{10 - j10.6} + \frac{\tilde{V}_a}{10} = 0,$$

to which the solution is

$$\tilde{V}_a = 55.9 \angle -11.42^\circ \quad \text{and} \quad \tilde{I}_3 = \frac{\tilde{V}_a}{R_3} = \frac{55.9 \angle -11.42^\circ}{10} = 5.59 \angle -11.42^\circ.$$

The forced response, therefore, is

$$i_{3,f} = 5.59 \sin(377t - 11.42^\circ) \text{ A}.$$

3) The initial value of the capacitance voltage is found by inspection of the circuit prior to switching. By using the voltage division formula we have

$$\tilde{V}_C = \frac{\tilde{V}_s(-jx_C)}{R_1 + R_2 - jx_C} = \frac{141(-j10.6)}{20 - j10.6} = 66.0 \angle -62.07^\circ.$$

Therefore,

$$v_C(0) = 66.0 \sin(-62.07^\circ) = -58.3 \text{ V}.$$

4) The initial value of the current may now be found by inspection of the circuit in Fig. 2.26(b), which fits the instant of switching, $t = 0$. At this moment, the value of the voltage source is $v_s(0) = 0$ and the capacitance voltage is $v_C(0) = -58.3$ V. Using nodal analysis again, we have

$$\frac{V_a}{10} + \frac{V_a + 58.3}{10} + \frac{V_a}{10} = 0,$$

to which the solution is $V_a = -19.4$ V and the initial value of current is

$$i_3(0) = \frac{V_a}{R_3} = \frac{-19.4}{10} = -1.94 \text{ A}.$$

5) The integration constant will be $A = i_3(0) - i_{3,f}(0) = -1.94 - 5.59 \sin(-11.42^\circ) = -0.83$, and the complete response is

$$i_3 = 5.59 \sin(377t - 11.42^\circ) - 0.83 e^{-267t} \text{ A}.$$

Example 2.19

As a last example in this section, consider the circuit in Fig. 2.27(a), in which $R = 100\ \Omega$, $C = 10\ \mu\text{F}$ and two sources are $v_s = 1000\sqrt{2} \sin(1000t + 45^\circ)$ V and $I_s = 4$ A d.c. Find the response of the current through the voltage source after opening the switch and sketch it.

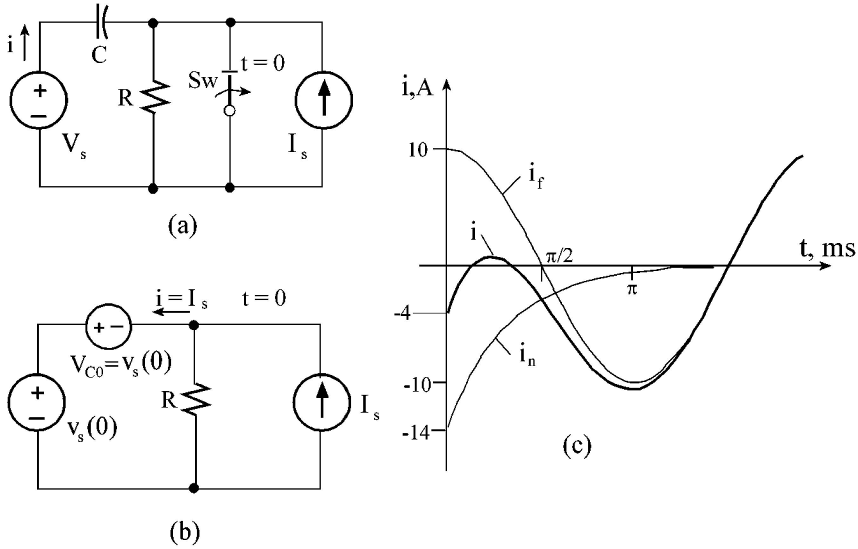

Figure 2.27 A given *RC* circuit of Example 2.17 (a), its equivalent for instant $t = 0$ (b) and the plot of current *i* (c).

Solution

The time constant (step 1) is $\tau = RC = 100 \cdot 10^{-5} = 10^{-3} = 1\ \mathrm{ms}$ or $s = -1000\ \mathrm{s}^{-1}$ and $i_n = Ae^{-1000t}$. The forced current (step 2) is found as a steady-state current in Fig. 2.27(a) after opening the switch

$$\tilde{I} = \frac{\tilde{V}_s}{R + jx_C} = \frac{1000\sqrt{2}\angle 45^\circ}{100 - j100} = 10 \angle 90^\circ$$

in which

$$x_C = 1/\omega C = 1/10^3 \cdot 10^{-5} = 100\ \Omega.$$

Thus,

$$i_f = 10 \sin(1000t + 90^\circ)\ \mathrm{A}.$$

The initial value of the capacitance voltage (step 3) must be evaluated in the circuit 2.27(a) prior to switching. By inspecting this circuit, and noting that the resistance and the current source are short-circuited, we may conclude that this voltage is equal to source voltage $v_C(0) = v_s(0)$.

By inspection of the circuit in Fig. 2.27(b), we shall find the initial value of current i (step 4), which is equal to the current source flowing in a negative direction, i.e., $i(0) = -4\ \mathrm{A}$. (Note that two voltage sources are equal and opposed to each other.)

Finally the complete response (step 5) in accordance with equation 2.17 will be:

$$i = i_f + [i(0) - i_f(0)]e^{st} = 10\sin(1000t + 90^\circ) - 14e^{-1000t}\ \mathrm{A},$$

where $i_f(0) = 10 \sin 90^\circ = 10$ A. The plot of this current is shown in Fig. 2.27(c). Note that the period of the forced current is

$$T = \frac{2\pi}{1000} = 2\pi 10^{-3}\ \mathrm{s} \quad \text{or} \quad T = 2\pi\ \mathrm{ms}.$$

2.4.4 Applying a continuous charge law to C-circuits

As we have observed earlier (see section 1.7.4) switching on circuits containing capacitances may result in very high pulses of current. (This phenomenon is dual to overvoltages in circuits containing inductances when switching them off as studied in section 2.3.3). When trying to solve these kinds of circuits the second switching law for capacitance voltages is usually disproved. However, as we already know, the problem might be solved by the principle of physics that electric charges are always continuous and cannot be abruptly changed. In this section we shall continue analyzing these kinds of circuits by introducing more numerical examples.

Example 2.20

Consider the circuit shown in Fig. 2.28(a), in which capacitance C_3 switches on in parallel to capacitance C_2. The resistances of the wires are very low and are presented by two resistors $R_2 = R_3 \approx 0.1\ \Omega$. The rest of the parameters are $R_1 = 40\ \mathrm{k\Omega}$, $C_1 = 4\ \mu\mathrm{F}$, $C_2 = 1\ \mu\mathrm{F}$, $C_3 = 3\ \mu\mathrm{F}$ and $V_s = 100$ V. (a) Assuming that the voltage change of two capacitances C_2 and C_3 occurs abruptly, find the charging current i, and the voltage v_2 across the capacitances, connected in parallel, in the second stage of transients. (b) Find the time and the charge interchanging between these two capacitances and the current pulse.

Solution

(a) After switching, capacitances C_2 and C_3 are connected in parallel with $C_{2,3} = C_2 + C_3 = 4\ \mu\mathrm{F}$, as shown in the circuit of Fig. 2.28(b). (The resistances R_1 and R_2 are neglected in comparison with R_1). The time constant of this circuit (step 1) is $\tau = R_1 C_{eq} = 40 \cdot 10^3 \cdot 2 \cdot 10^{-6} = 80 \cdot 10^{-3} = 80$ ms, where

$$C_{eq} = \frac{C_1 C_{2,3}}{C_1 + C_{2,3}} = \frac{4 \cdot 4}{4 + 4} = 2\ \mu\mathrm{F},$$

and the root of the characteristic equation is $s = -1/\tau = -12.5\ \mathrm{s}^{-1}$. The natural responses of the current and voltage will be:

$$i_n = Ae^{-12.5t}$$

$$v_2 = Be^{-12.5t}.$$

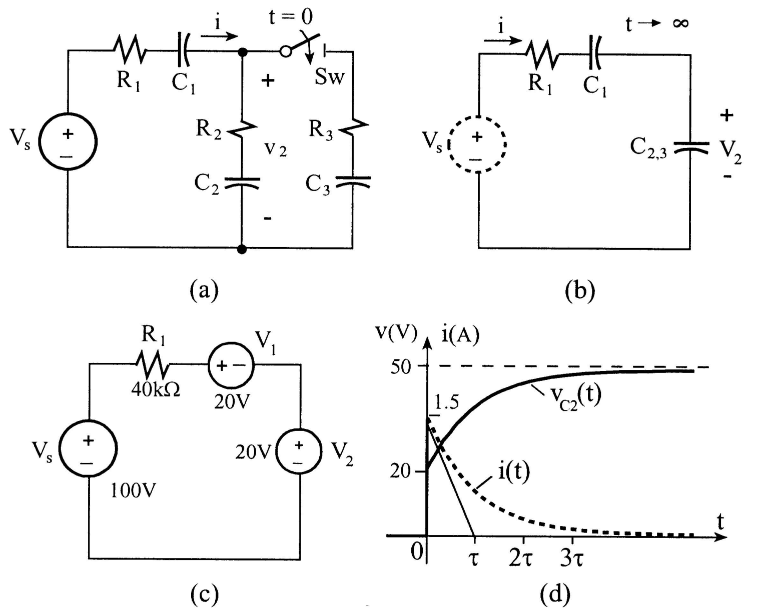

Figure 2.28 A given circuit for Example 2.20 (a), a circuit of the steady-state operation ($t \to \infty$) (b), an equivalent circuit of the instant of switching $t = 0$ (c) and the plots of the current and voltage (d).

The forced response (step 2) of the current is equal to zero as a steady-state current through the capacitance at the d.c. supply. However, the forced response of the capacitance voltages becomes half of the supply voltages, as divided between two equal capacitances C_1 and $C_{2,3}$. Thus, $i_f = 0$ and $v_{2,f} = 50\,\text{V}$.

Next, we shall find the initial value of the voltage of the two capacitances in parallel (step 3). With the generalized second switching law (1.36b), or the principle of continuous charges, we have

$$v_2(0) = \frac{C_2 v_{C2}(0_-) + C_3 v_{C3}(0_-)}{C_2 + C_3}.$$

Here $v_{C3}(0_-)$ should be zero and $v_{C2}(0_-)$ can be found with the voltage division formula

$$v_{C2}(0_-) = V_s \frac{C_1}{C_1 + C_2} = 100 \frac{4}{4+1} = 80\text{ V}.$$

Thus,

$$v_2(0) = \frac{1 \cdot 80 + 3 \cdot 0}{1+4} = 20\text{ V}.$$

In the next step (step 4) we shall find the initial value of the current, as the dependent initial condition. By inspection of the equivalent circuit fitting instant

$t=0$, Fig. 2.28(c), we obtain

$$i(0)=\frac{V_s-v_{C1}(0)-v_{C2}(0)}{R_1}=\frac{100-20-20}{40\cdot 10^3}=1.5\ \mathsf{mA}.$$

The integration constants (step 5) are

$$A=i(0)-i_f(0)=1.5-0=1.5\ \mathsf{mA}$$

$$B=v_2(0)-v_{2,f}(0)=20-50=-30\ \mathsf{V}.$$

The complete response of the current and capacitance voltage can now be written

$$i(t)=1.5e^{-12.5t}\ \mathsf{A}$$

$$v_2(t)=50-30e^{-12.5t}\ \mathsf{V}.$$

Both curves are sketched in Fig. 2.28(d).

(b) In order to find the time of the first stage of the transients we must take into consideration the wire resistances. Thus, after switching, the time constant of the right loop of the circuit, in which the first stage of the transients takes place, may be estimated as $\tau=2R_1C_{eq}=0.2\cdot 0.75\cdot 10^{-6}=0.15\ \mu\mathsf{s}$. Here:

$$C_{eq}=\frac{C_2C_3}{C_2+C_3}=\frac{1\cdot 3}{1+3}=0.75\ \mu\mathsf{F}.$$

The time of the first stage is estimated as $T\cong 5\tau=0.75\ \mu\mathsf{s}$, which is about $2\cdot 10^{-6}$ times shorter than the second stage.

The charge of C_2, $q_2(0_-)=C_2v_2(0_-)=1\cdot 10^{-6}\cdot 80=80\ \mu\mathsf{C}$, prior to switching decreases, during the first stage to $q_2(0_+)=C_2v_2(0_+)=1\cdot 10^{-6}\cdot 20=20\ \mu\mathsf{C}$. Thus, the interchange of the charges between two capacitances is $\Delta q=80-20=60\ \mu\mathsf{C}$. The current peak will be $I_\delta=v_{C2}(0)/2R_2=80/0.2=400\ \mathsf{A}$, which results in transferring the charge $\Delta q=I_\delta\tau=400\cdot 0.15\cdot 10^{-6}=60\ \mu\mathsf{C}$, as previously calculated.

Example 2.21

In the circuit of Fig. 2.29(a) the capacitance C_2 has been charged prior to switching up to voltage -6 V. Find current i and voltage v_C after switching, if $R_1=300\ \Omega$, $R_2=600\ \Omega$, $C_1=300\ \mu\mathsf{F}$, $C_2=200\ \mu\mathsf{F}$ and $V_s=36$ V.

Solution

In order to find the time constant and the root of the characteristic equation (step 1), we must find the equivalent resistance and capacitance:

$$R_{eq}=R_1//R_2=\frac{300\cdot 600}{300+600}=200\ \Omega,$$

$$C_{eq}=C_1//C_2=C_1+C_2=300+200=500\ \mu\mathsf{F}.$$

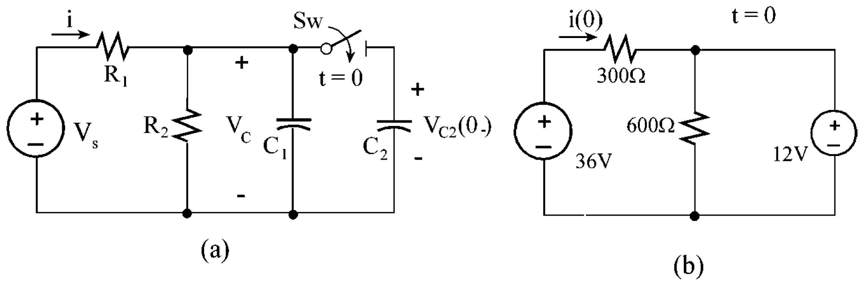

Figure 2.29 A circuit of Example 2.21 (a) and its equivalent at instant $t = 0$ (b).

Thus, $\tau = R_{eq}C_{eq} = 200 \cdot 500 \cdot 10^{-6} = 0.1$ s and $s = -1/\tau = -10\ \mathrm{s}^{-1}$, and the natural responses are

$$i_n = Ae^{-10t}, \quad v_{C,n} = Be^{-10t}.$$

The forced responses (step 2) are:

$$i_f = \frac{V_s}{R_1 + R_2} = \frac{36}{300 + 600} = 0.04\ \mathrm{A} = 40\ \mathrm{mA},$$

$$v_{C,f} = V_s \frac{R_2}{R_1 + R_2} = 36 \frac{600}{300 + 600} = 24\ \mathrm{V}.$$

The initial value of the voltage of these two capacitances (step 3) shall be found using the second generalized law and by taking into consideration that (in the circuit prior to switching) $v_{C2}(0_-)$ is negative, i.e., $v_{C2}(0_-) = -6$ V, and $V_{C1}(0_-) = 24$ V. Hence,

$$v_C(0) = \frac{C_1 v_{C1}(0) + C_2 v_{C2}(0)}{C_1 + C_2} = \frac{300 \cdot 24 + 200(-6)}{300 + 200} = 12\ \mathrm{V}.$$

The initial value of the current, which is a dependent initial condition (step 4), is found in the circuit of Fig. 2.29(b) for instant $t = 0$. By inspection, we find:

$$i(0) = \frac{V_s - v_{C2}}{R_1} = \frac{36 - 12}{300} = 0.08\ \mathrm{A} = 80\ \mathrm{mA}.$$

Now the integration constants (step 5) can be found

$$A = i(0) - i_f(0) = 80 - 40 = 40\ \mathrm{mA}$$

$$B = v_c(0) - v_{C,f}(0) = 12 - 24 = -12\ \mathrm{V}.$$

Thus, the complete responses are

$$i(t) = 40 + 40e^{-10t} = 40(1 + e^{-10t})\ \mathrm{mA}$$

$$v_C(0) = 24 - 12e^{-10t}\ \mathrm{V}.$$

Example 2.22

As a last example for this section, consider the circuit shown in Fig. 2.30(a), in

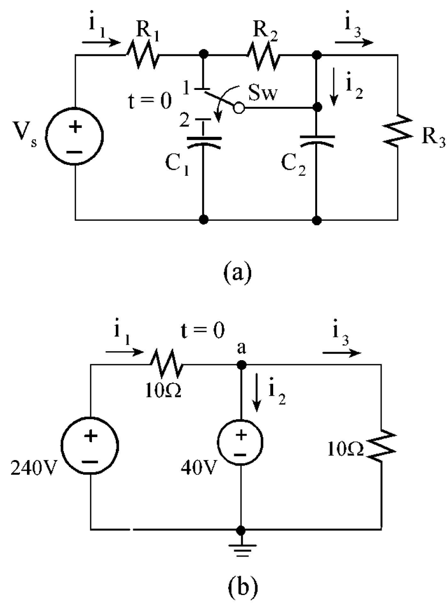

Figure 2.30 A given circuit of Example 2.22 (a) and its equivalent at the instant of switching, $t = 0$ (b).

which upon switching, the configuration of the circuit has been changed, namely, resistor R_2 connects in series to resistor R_1 and capacitance C_1 being uncharged connects in parallel to capacitance C_2. Assume that the switching occurs instantaneously and find all the current responses, $i_1(t)$, $i_2(t)$ and $i_3(t)$ after switching. The circuit parameters are: $R_1 = R_2 = 5\ \Omega$, $R_3 = 10\ \Omega$, $C_1 = 750\ \mu\text{F}$, $C_2 = 250\ \mu\text{F}$ and $V_s = 240\ \text{V}$.

Solution

The time constant (step 1) may be easily found after determining the equivalent resistance and capacitance:

$$R_{eq} = (R_1 + R_2)//R_3 = (5 + 5)//10 = 5\ \Omega,$$

$$C_{eq} = C_1 + C_2 = 250 + 750 = 1000\ \mu\text{F}.$$

Therefore, the time constant is $\tau = R_{eq}C_{eq} = 5 \cdot 10^{-3} = 5\ \text{ms}$ and $s = -1/\tau = -200\ \text{s}^{-1}$. The natural responses of the currents, therefore, are

$$i_{1,n} = A_1 e^{-200t}, \quad i_{3,n} = A_3 e^{-200t}$$

and

$$i_2 = i_1 - i_3 = (A_1 - A_3)e^{-200t}.$$

The forced responses (step 2) are found in the circuit after switching in its steady-state operation:

$$i_{1,f} = i_{3,f} = \frac{V_s}{R_1 + R_2 + R_3} = \frac{240}{5+5+10} = 12\ \text{A},$$

$$i_{2,f} = i_C = 0.$$

The initial value of the voltage across the two capacitances (in parallel) (step 3) may be found using the principle of continuous charge (the second generalized law):

$$v_C(0) = \frac{C_1 v_{C1}(0_-) + C_2 v_{C2}(0_-)}{C_1 + C_2} = \frac{0 + 250 \cdot 160}{750 + 250} = 40\ \text{V},$$

where $v_{C1}(0_-) = 0$ and $v_{C2}(0_-) = 240 \cdot 10/(5+10) = 160$ V. The initial values of the currents, which are dependent initial conditions (step 4), can be obtained in the equivalent circuit of Fig. 2.30(b). Since the potential of node "a" is 40 V, we have:

$$i_1(0) = \frac{240 - 40}{10} = 20\ \text{A},$$

$$i_3(0) = \frac{40}{10} = 4\ \text{A}, \quad i_2(0) = i_1(0) - i_3(0) = 20 - 4 = 16\ \text{A}.$$

The integration constants (step 5) are:

$$A_1 = i_1(0) - i_{1,f}(0) = 20 - 12 = 8\ \text{A}$$

$$A_2 = i_2(0) - i_{2,f}(0) = 16 - 0 = 16\ \text{A}$$

$$A_3 = i_3(0) - i_{3,f}(0) = 4 - 12 = -8\ \text{A},$$

and the complete responses of the three currents are:

$$i_1(t) = 12 + 8e^{-200t}\ \text{A}$$

$$i_2(t) = 16e^{-200t}\ \text{A}$$

$$i_3(t) = 12 - 8e^{-200t}\ \text{A}.$$

Note that current $i_2(t)$ might also be found as the difference between i_1 and i_3, i.e., $i_2(t) = i_1(t) - i_3(t) = 16e^{-200t}$ A, which is the same as was found earlier.

It is worthwhile to calculate current i_2, which is actually the current through two parallel capacitances, also as $i_2 = C_{eq}(dv_C/dt)$. In order to do this we first have to find the capacitance voltage. Since its forced value is $240 \cdot 10/(5+5+10) = 120$ V, we have

$$v_C(t) = 120 + (40 - 120)e^{-200t} = 120 - 80e^{-200t}\ \text{V},$$

and

$$i_2(t) = C_{eq}\frac{dv_C}{dt} = 10^{-3}(-80)(-200)e^{-200t} = 16e^{-200t}\ \text{A},$$

which again is the same as was calculated earlier.

2.5 THE APPLICATION OF THE UNIT-STEP FORCING FUNCTION

The reason that any transient responses at all appear in electrical circuits is because of the discontinuity or switching actions which take place at an instant of time that is defined as $t = 0$ (or $t = t_0$). In studying transient responses, it is convenient, in many cases, to use a special function, which represents this kind of discontinuous or switching action, and is called a **unit-step function**. Thus, the operation of a switch in series with a voltage source is equivalent to a forcing function, which is zero up to the instant that the switch is closed and is equal to the value of the voltage source thereafter. This change of voltage occurs abruptly (since we are considering the switch as an ideal device working instantaneously), expressing a discontinuity of the voltage at the instant the switch is closed. Such kinds of functions, which are discontinuous or have discontinuous derivatives, are called **singularity functions**. The two most important of them are the unit-step function and the unit-impulse function (see further on). Thus, the mathematical definition of the unit-step forcing function is

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0. \end{cases} \tag{2.27a}$$

or

$$u(t - t_0) = \begin{cases} 0 & t < t_0 \\ 1 & t > t_0. \end{cases} \tag{2.27b}$$

Therefore, the unit-step function is zero for all negative values of its argument ($t < 0$) and is unity for all positive values ($t > 0$). This is shown in Fig. 2.31. Note that at the instant of time $t = 0$ is not defined: but it is zero as a left limit and unity as a right limit. In accordance with equation 2.27a, a switching action takes place at the instant $t = 0$ and in accordance with equation 2.27b at instant $t = t_0$ ($t_0 \neq 0$, since, if $t_0 = 0$ we get equation 2.27a). To indicate that any voltage source of the value V is switching at $t = 0$ (or $t = t_0$) to a general network, we write $v(t) = Vu(t)$, which is illustrated in Fig. 2.32(a). Such representation of switching on sources by using a unit forcing function instead of a switch by itself is common and useful in transient analysis. (Note that the unit-step function is itself dimensionless.)

Because of the wide use of the unit-step function in transient analysis, we

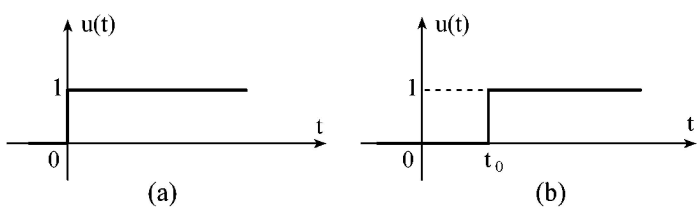

Figure 2.31 A unit-step function applied at $t = 0$ (a) and applied at $t = t_0$ (b).

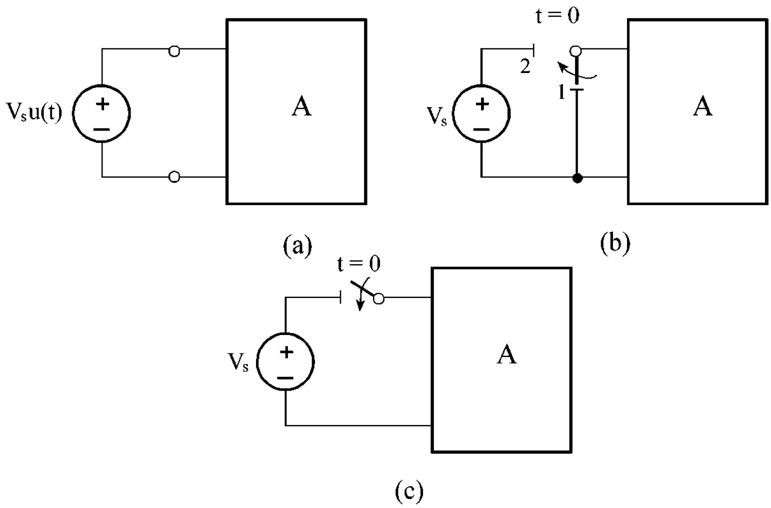

Figure 2.32 A circuit in which the voltage source is applied at $t = 0$ (a) and its switching equivalent drawn correctly (b) and incorrectly (c).

shall explain the features of this function in more detail. In the circuit of Fig. 2.32(a) the ideal voltage source possesses a zero internal impedance, so that circuit A is short-circuited the entire time, also prior to $t = 0$, even when the applied voltage equals zero. We have the same conditions in the circuit (b), which is therefore the correct equivalent of the circuit with the discontinuous forcing function (a). (Note that the switch in this circuit is an ideal instantaneously operating switch.) The circuit in (c) cannot be the correct equivalent of (a) since prior to switching circuit A is open-circuited. However, after switching, $t \geq 0$, the circuits in (c) and (a) are equivalent, and if this is the only time interval we are interested in, and if the initial currents which flow from the two circuits, A in (a) and in (c), are identical at $t = 0$, then Fig. 2.32(c) becomes a useful equivalent of Fig. 2.32(a).

The circuit with a discontinuous current source is a dual of the circuit with a discontinuous voltage source and is shown in Fig. 2.33. The above explanation regarding the voltage source may be easily understood from this figure. Using two unit step functions, we can obtain the rectangular pulse of a forcing function, as is shown in Fig. 2.34.

To show an application of the unit-step function in transient analysis, let us consider a numerical example in which a pulse current is applied.

Example 2.23

In the circuit, shown in Fig. 2.35(a), find the output voltage, if the current pulse of amplitude $I = 2$ A and duration $t_0 = 0.01$ s, shown in Fig. 2.35(b), is applied to this circuit.

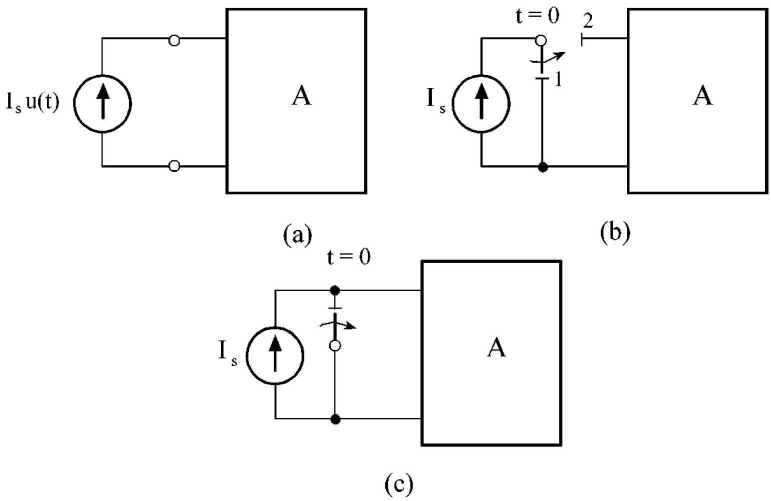

Figure 2.33 A circuit in which the current source is applied at $t = 0$ (a) and its switching equivalent drawn correctly (b) and incorrectly (c).

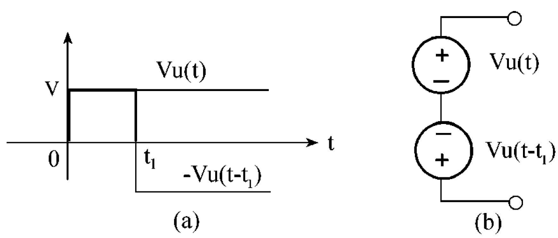

Figure 2.34 A rectangular forcing function (a) and a combined source, which yields the rectangular pulse (b).

Solution

The output voltage can be represented as

$$v_o(t) = v_o'(t) + v_o''(t),$$

where $v_o'(t)$ is the part of the total response due to the positive current source acting alone and $v_o''(t)$ is the part due to the negative current source acting alone. Starting with the first part of the output voltage and following the five steps, we must do as follows:

1) To obtain the characteristic equation we shall equal the input admittance to zero, since the current source possesses an infinite impedance (an open circuit):

$$Y_{in} = \frac{1}{15} + \frac{1}{10 + 1/(1/100 + 1/2s)} = \frac{1}{15} + \frac{100 + 2s}{1000 + 220s},$$

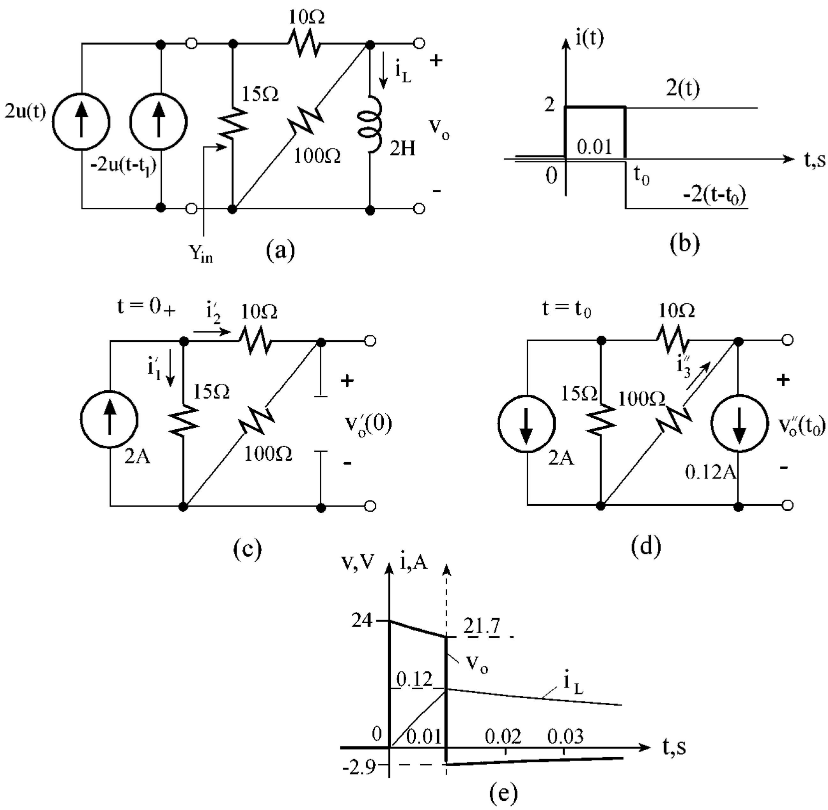

Figure 2.35 A circuit of Example 2.23 (a), the input current pulse (b), an equivalent circuit for determining $v_o'(0)$ (c), an equivalent circuit for determining $v_o''(t_0)$ (d) and the output voltage response (e).

or

$$250s + 2500 = 0 \quad \text{and} \quad s = -10\,\text{s}^{-1}.$$

(Alternatively, we may obtain the characteristic equation by equaling the admittance, seen from the inductance branch, to zero, which is left for the reader as an exercise.) Thus,

$$v_o'(t) = A'e^{-10t}\,\text{V}.$$

2) The forced response is zero as a voltage drop across an inductance in a d.c. circuit, thus, $v_{o,f} = 0$.

3) Inspecting the circuit for $t < 0$, we have $i_L(0_-) = 0$, so that $i_L(0_+) = 0$.

4) In the circuit drawn for $t = 0$, Fig. 2.35(c), we have

$$v_o'(0) = 100 i_2'(0) = 100 \cdot 2 \frac{15}{110 + 15} = 24\,\text{V}.$$

5) The integration constant, therefore, is $A' = v_o'(0) - v_{o,f}' = 24 - 0 = 24$. and the first part of the voltage response is $v_o'(t) = 24e^{-10t}\,\mathsf{V}$.

To find the second part of the voltage $v_o''(t)$ we start from step 3, since the root of the characteristic equation and the forced response have already been found, i.e., $s = -10\,\mathsf{s}^{-1}$ and $v_{o,f}'' = v_{o,f}' = 0$.

3) The independent initial condition for the inductance current at the instance of second commutation, t_0, is

$$i_L''(t_{0-}) = \frac{1}{L}\int_0^{t_0} v_o'\,dt = \frac{24}{2}\int_0^{0.01} e^{-10t} = \frac{12}{-10}e^{-10t}\bigg|_0^{0.01} = 0.12\,\mathsf{A}.$$

4) To find the initial condition of v_o'' in the second transient interval we must consider the given circuit for $t = t_0$, in which the inductance is represented by a current source of 0.12 A.

$$i_3'' = 2\frac{15}{110+15} + 0.12\frac{25}{100+25} = 0.264\,\mathsf{A}$$

and $\quad v_o''(t_{0+}) = -100\cdot 0.264 = -26.4\,\mathsf{V}$.

5) We can now find the constant of integration: $A'' = v_o''(0) - v_{o,f} = -24.6 - 0 = -24.6\,\mathsf{V}$, and

$$v_o''(t) = -24.6e^{-10t}\,\mathsf{V} \quad \text{for} \quad t > t_0.$$

Then

$$v_o = v_o' + v_o'' = 24e^{-10t} - 24.6e^{-10(t-t_0)}\,\mathsf{V} \quad \text{for} \quad t > t_0.$$

To simplify this expression we designate $t' = t - t_0$ or $t = t' + t_0$, then

$$v_o = 24e^{-10t_0}e^{-10t'} - 24.6e^{-10t'} = -2.9e^{-10t'},$$

which means that the y-axis has been moved to the new origin at t_0, i.e., now $t_0 = 0$.

The output voltage and inductance current are shown in Fig. 2.35(e). Note that the output voltage form is almost a rectangular pulse, i.e. similar to the input current pulse. In other words, the current pulse is transferred to the voltage pulse. Note also that this is correct in the case that $t_0 \ll \tau$ or $t_0/\tau \ll 1$, where $\tau = L/R_{eq}$.

2.6 SUPERPOSITION PRINCIPLE IN TRANSIENT ANALYSIS

In this section, we shall show how the property of superposition can be used for solving problems in transient analysis. Suppose that a new branch connects to a general active network A after closing the switch and, suppose that we are looking for the transient current in any other branch of the network, say i_1, as shown in Fig. 2.36(a). The transient behavior of the entire circuit can be written as a superposition of two regimes: 1) the previous one, which existed prior to switching and 2) an additional one, which is a result of the switching. Therefore, the unknown current i_1 will be the sum of the two currents. The first one, i_1' is the current which flowed in branch "1" before switching, figure (b), and the second one, i_1'', is the additional current which appears in circuit *P*, figure (c). This circuit arises from the original circuit *A*, in which all the sources have been "killed" and the switch is replaced by a voltage source, which is oppositely equal to the voltage across the open switch in circuit *A*, as shown in Fig. 2.36(c). (Remember that "to kill" a source means that the source is replaced by its inner resistance/impedance, or that the ideal voltage source is simply short-circuited and the current source is simply open-circuited.) Hence,

$$i_1 = i_1' + i_1'' \quad \text{or} \quad i_1 = i_{1pr} + i_{1ad}, \tag{2.28}$$

where i_1' is the previous current and i_1'' is the additional one. It is very important to note that, in the additional circuit, the independent conditions are zero.

If any branch in a general network is disconnected, as shown in Fig. 2.37(a), we may apply the principle of duality, which means that the switch in the passive circuit must be replaced by a *current source* that is oppositely equal to the current through the closed switch in circuit *A*, as shown in Fig. 2.37(c). The required current i_1 will then again be the sum of the previous current, which flows in the circuit with the switch closed, Fig. 2.37(b), and the additional current, which will flow in the passive circuit with the source current, Fig. 2.37(c), as indicated in equation 2.28. The above technique is illustrated in the following examples.

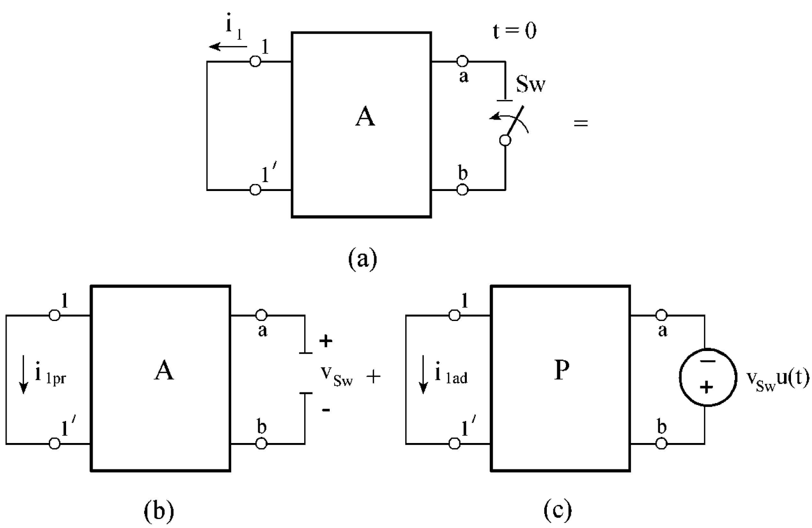

Figure 2.36 A given circuit (a), a previous circuit prior to switching (b) and the additional passive circuit with a voltage source (c).

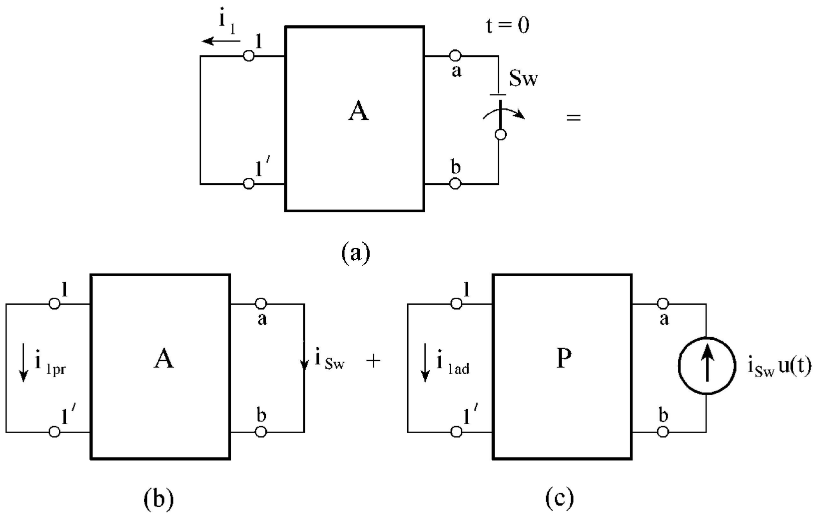

Figure 2.37 A given circuit (a), a previous circuit prior to switching (b) and the additional passive circuit with a current source (c).

Example 2.24

In the circuit, shown in Fig. 2.38(a), find current i after opening the switch, using the principle of superposition. The parameters of the circuit are: $v_s = 100 \sin \omega t$ at $f = 60$ Hz, $\omega L = 10\ \Omega$ and $R_1 = R_2 = 10\ \Omega$.

Solution

First, we find the currents in the circuit of Fig. 2.38(a) prior to switching

$$i' = \frac{100}{\sqrt{5^2 + 10^2}} \sin(\omega t + \psi_i) = 8.94 \sin(\omega t - 63.4°)\ \mathrm{A},$$

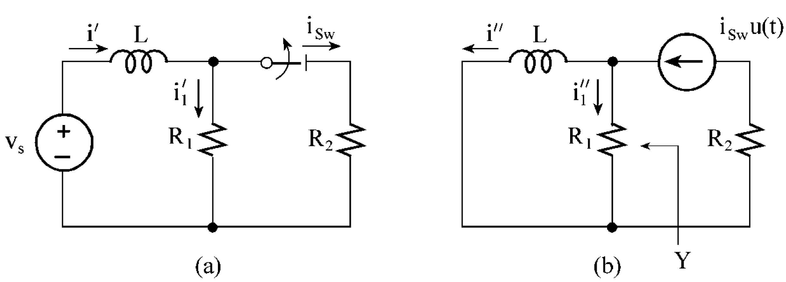

Figure 2.38 A circuit prior to switching (a) and the additional circuit (b).

where the current phase angle is $\psi_i = 0 - \tan^{-1}(10/5) = -63.4°$, and

$$i_{sw} = \frac{i'}{2} = 4.47\sin(\omega t - 63.4°).$$

Now we shall find the transient current in the circuit of Fig. 2.38(b), in which the initial value of the inductance current is zero. The characteristic equation is

$$Y = \frac{1}{sL} + \frac{1}{R_1} = 0 \quad \text{or} \quad sL + R_1 = 0,$$

and its root is

$$s = \frac{R_1}{L} = -\frac{10}{10/\omega} = -377\ \mathrm{s}^{-1}.$$

Hence, the natural response is

$$i''_n = Ae^{-377t}.$$

The forced response of current i''_f is found with phasor analysis:

$$\tilde{I}'_f = \tilde{I}_{sw}\frac{R_1}{R_1 + j\omega L} = (4.47\angle -63.4°)\frac{10}{10 + j10} = 3.16\angle -108.4°.$$

Therefore,

$$i''_f(t) = 3.16\sin(\omega t - 108.4°)\ \mathrm{A}.$$

Since the initial value of this current is zero (zero initial conditions), we have

$$A = 0 - i''_f(0) = -3.16\sin(-108.4°) = 3.0\ \mathrm{A}.$$

Thus, the total response of current i is

$$i = i' - i'' = 8.49\sin(\omega t - 63.4°) - 3.16\sin(\omega t - 108.4°) - 3e^{-377t}\ \mathrm{A}. \quad \text{(a)}$$

The initial value of this current at $t = 0$ is $i(0) = -8$ A, which is the current through the inductance prior to switching ($i'(0) = 8.94\sin(-63.4°) \cong -8$ A).

Note that the same current can be found as current i_1 through resistance R_1, since in the original circuit both currents are equal. The forced response of this current is determined as

$$\tilde{I}''_{1,f} = \tilde{I}_{sw}\frac{j\omega L}{R_1 + j\omega L} = 4.47\angle -63.4°\,\frac{j10}{10 + j10} = 3.16\angle -18.4°,$$

or, as versus time,

$$i''_{1,f} = 3.16\sin(\omega t - 18.4°).$$

Since $i''_L(0) = 0$, the initial value of the current through resistance R_1 is

$$i''_1(0) = 4.47\sin(-63.4°) = -4\ \mathrm{A},$$

and the integration constant for this current is $A = -4 - 3.16\sin(-18.4°) = -3$. Hence the total current is

$$i_1 = i_1' + i_1'' = 4.47\sin(\omega t - 63.4°) + 3.16\sin(\omega t - 18.4°) - 3e^{-377t}\ \mathrm{A}. \tag{b}$$

This current at $t = 0$ yields $i_1(0) = -8\ \mathrm{A}$, which is again the value of the inductance current prior to switching. Both results (a) and (b) can be simplified to the same expression

$$i(t) = 5\sqrt{2}\sin(\omega t - 45°) - 3e^{-377t}\ \mathrm{A}.$$

Example 2.25

In the circuit, having all R's of $10\ \Omega$, $C = 1\ \mu\mathrm{F}$ and $V_s = 60\ \mathrm{V}$, shown in Fig. 2.39(a), the switch closes at time $t = 0$. Find current i_1 using the superposition theorem.

Solution

First, we shall find the previous current i_1' and the voltage across the open switch. By inspection of the circuit in Fig. 2.39(a), we may find

$$i_1' = \frac{V_s}{R_1 + R_3 + R_4} = \frac{60}{3 \cdot 10} = 2\ \mathrm{A} \quad \text{and} \quad V_{sw} = R_4 i_1' = 10 \cdot 2 = 20\ \mathrm{V}.$$

To find the time constant of the circuit in Fig. 2.39(b), we must determine R_{eq}

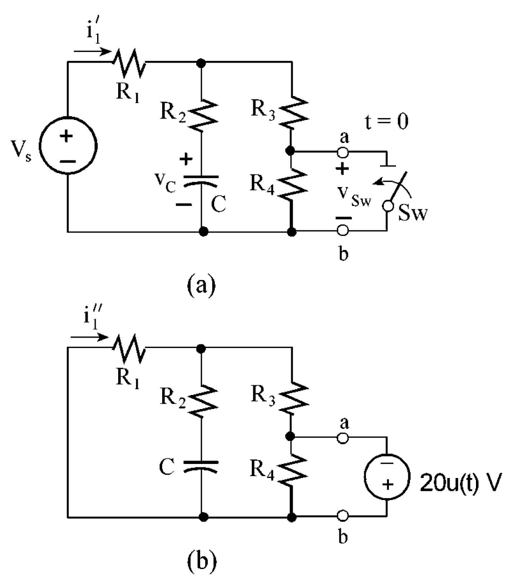

Figure 2.39 A circuit of Example 2.25 prior to switching (a) and an additional circuit for finding the transient response (b).

seen from the capacitance (note that R_4 is short-circuited by the voltage source): $R_{eq} = R_2 + R_1//R_3 = 15\,\Omega$ and $\tau = R_{eq}C = 15\cdot 10^{-6}\,\mathrm{s}$, or $s = -1/s = -66.7\cdot 10^3\,\mathrm{s}^{-1}$.

The forced response of the current in Fig. 2.39(b) will be found as:

$$i''_{1,f} = \frac{V_{sw}}{R_1 + R_3} = \frac{20}{10 + 10} = 1\,\mathrm{A}.$$

The initial value of current i''_1, since the capacitance voltage is zero (which means that the capacitance is short-circuited, i.e., zero initial conditions), will be

$$i'_1(0) = \frac{1}{2}\frac{V_{sw}}{R_3 + R_1//R_2} = \frac{1}{2}\frac{20}{10 + 5} = 0.667\,\mathrm{A}.$$

Therefore, the arbitrary constant will be: $A = i''_1(0) - i''_{1,f} = 0.667 - 1 = -0.133\,\mathrm{A}$. The additional current now is

$$i''_1 = 1 - 0.333e^{-66.7\cdot 10^3 t}\,\mathrm{A},$$

and the total current will be:

$$i_1 = i'_1 + i''_1 = 3 - 0.333e^{-66.6\cdot 10^3 t}\,\mathrm{A}.$$

2.7 *RLC* CIRCUITS

This section is devoted to analyzing very important circuits containing three basic circuit elements: *R*, *L*, and *C*. These circuits are considered important because the networks involved in many practical transient problems in power systems can be reduced to one or to a number of simple circuits made up of these three elements. In particular, the most important are series or parallel *RLC* circuits, with which we shall start our analysis.

From our preceding study, we already know that the transient response of a second order circuit contains two exponential terms and the natural component of the complete response might be of three different kinds: overdamped, underdamped or critical damping. The kind of response depends on the roots of the characteristic equation, which in this case is a quadratic equation. We also know that in order to determine *two* arbitrary integration constants, A_1 and A_2, we must find two initial conditions: 1) the value of the function at the instant of switching, $f(0)$, and 2) the value of its derivative, $(df/dt)|_{t=0}$.

In the following section, we shall deepen our knowledge of the transient analysis of second order circuits in their practical behavior and by solving several practical examples

2.7.1 *RLC* circuits under d.c. supply

We shall start our practical study of transients in second order circuits by considering examples in which the d.c. sources are applied. At the same time, we must remember that only the forced response is dependent on the sources. Natural responses on the other hand depend only on the circuit configuration and its parameter and do not depend on the sources. Therefore, by determining the natural responses we are actually practicing solving problems for both kinds of sources, d.c. and a.c. However, it should be mentioned that the natural response depends on from which source the circuit is fed: the voltage source or the current source. These two sources possess different inner resistances (impedances) and therefore they determine whether the source branch is short-circuited or open, which influences of course the equivalent circuit.

In our next example, we shall elaborate on the methods of determining characteristic equations and show how the kind of source (voltage or current) and the way it is connected may influence the characteristic equation. Let us determine the characteristic equation of the circuit, shown in Fig. 2.40, depending on the kind of source: voltage source or current source and on the place of its connection: (1) in series with resistance R_1, (2) in series with resistance R_2, (3) between nodes *m–n*.

(1) *Source connected in series with resistance* R_1

If a voltage source is connected in series with resistance R_1, Fig. 2.40(b), we may use the input impedance method for determining the characteristic equation. This impedance as seen from the source is

$$Z(s) = R_1 + (R_2 + 1/sC)//(R_3 + sL).$$

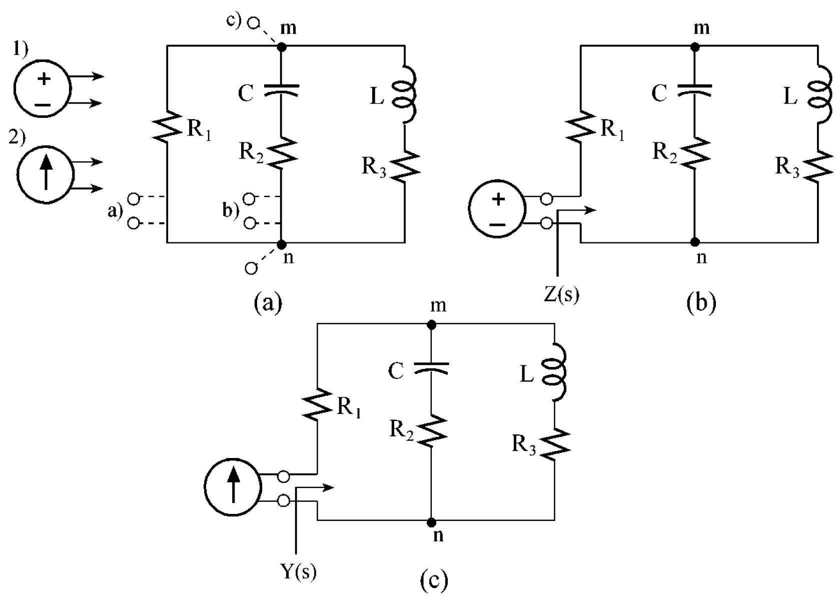

Figure 2.40 A given circuit (a), a circuit in which a voltage source is connected to the branch of R_1 (b) and a circuit in which a current source is connected to the branch of R_1 (c).

Performing the above operation and upon simplification and equating $Z(s)$ to zero we obtain

$$(R_1 + R_2)LCs^2 + \left(\sum R_iR_jC + L\right)s + (R_1 + R_3) = 0. \tag{2.29}$$

where $\sum R_iR_j = R_1R_2 + R_1R_3 + R_2R_3$, and the roots of (2.29) are

$$s_{1,2} = -\frac{1}{2}\left(\frac{R_{eq}}{L} + \frac{1}{R_{12}C}\right) \pm \sqrt{\frac{1}{4}\left(\frac{R_{eq}}{L} + \frac{1}{R_{12}C}\right)^2 - \varepsilon\frac{1}{LC}},$$

where

$$R_{eq} = \frac{\sum R_iR_j}{R_1 + R_2}, \quad R_{12} = R_2 + R_2 \quad \text{and} \quad \varepsilon = \frac{R_1 + R_3}{R_1 + R_2}.$$

If a current source is connected in series with resistance R_1 we may use the input admittance method. By inspection of Fig. 2.40(c), and noting that the branch with resistance R_1 is opened ($Y_1 = 0$), we have

$$Y(s) = 0 + \frac{1}{R_2 + 1/sC} + \frac{1}{R_3 + sL} = 0,$$

or, after simplification,

$$LCs^2 + (R_2 + R_3)Cs + 1 = 0, \tag{2.30}$$

and the roots of (2.30) are

$$s_{1,2} = -\frac{1}{2}\frac{R_{23}}{L} \pm \sqrt{\frac{1}{4}\left(\frac{R_{23}}{L}\right)^2 - \frac{1}{LC}}, \quad \text{where} \quad R_{23} = R_2 + R_3.$$

Since the characteristic equations 2.29 and 2.30 are completely different, and therefore their roots are also different, we may conclude that the transient response in the same circuit, but upon applying different kinds of sources, will be different.

(2) We leave this case to the reader to solve as an exercise.

(3) Source is connected between nodes *m–n*.

If a voltage source is connected between nodes m–n, the circuit is separated into three independent branches: 1) a branch with resistance R_1, in which no transients occur at all; 2) a branch with R_2 and C in series, for which the characteristic equation is $R_2Cs + 1 = 0$; and 3) a branch with R_3 and L in series, for which the characteristic equation is $Ls + R_3 = 0$.

If a current source is connected between nodes m–n, by using the rule $Y_{in}(s) = 0$ we may obtain

$$Y_{mn} = \frac{1}{R_1} + \frac{1}{R_2 + 1/sC} + \frac{1}{R_3 + sL} = 0.$$

Performing the above operations and upon simplification, we obtain

$$(R_1 + R_2)LCs^2 + \left(\sum R_i R_j C + L\right)s + (R_1 + R_3) = 0, \tag{2.31}$$

where $\sum R_i R_j$ is like in equation 2.29. Note that this equation (2.31) is the same as (2.29), which can be explained by the fact that connecting the sources in these two cases does not influence the configuration of the circuit: the voltage source in (1) keeps the branch short-circuited and the current source in (3) keeps the entire circuit open-circuited. In all the other cases the sources change the circuit configuration.

In the following analysis we shall discuss three different kinds of responses: overdamped, underdamped, and critical damping, which may occur in *RLC* circuits. Let us start with a free source simple *RLC* circuit.

(a) Series connected RLC circuits

Consider the circuit shown in Fig. 2.41. At the instant $t = 0$ the switch is moved from position "1" to "2", so that the capacitor, which is precharged to the initial voltage V_0, discharges through the resistance and inductance. Let us find the transient responses of $v_C(t)$, $i(t)$ and $v_L(t)$. The characteristic equation is

$$R + sL + \frac{1}{sC} = 0, \quad \text{or} \quad s^2 + \frac{R}{L}s + \frac{1}{LC} = 0. \tag{2.32}$$

The roots of this equation are

$$s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \tag{2.33a}$$

or as previously assigned (see section 1.6.2)

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_d^2}, \tag{2.33b}$$

where $\alpha = R/2L$ is the exponential damping coefficient and $\omega_d = 1/\sqrt{LC}$ is the resonant frequency of the circuit.

An overdamped response: Assume that the roots (equation 2.32) are real (or more precisely negative real) numbers, i.e., $\alpha > \omega_d$ or $R > 2\sqrt{L/C}$. The natural

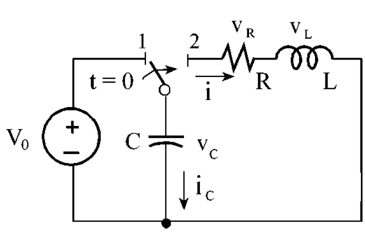

Figure 2.41 A series connected *RLC* circuit.

response will be the sum of two decreasing exponential terms. For the capacitance voltage it will be

$$v_{C,n} = A_1 e^{s_1 t} + A_2 e^{s_2 t}.$$

Since the absolute value of s_2 is larger that that of s_1, the second term, containing this exponent, has the more rapid rate of decrease.

The circuit in Fig. 2.41 after switching becomes source free; therefore, no forced response will occur and we continue with the evaluation of the initial conditions. For the second order differential equation, we need two initial conditions. The first one, an independent initial condition, is the initial capacitance voltage, which is V_0. The second initial condition, a dependent one, is the derivative dv_C/dt, which can be expressed as a capacitance current divided by C

$$\left.\frac{dv_{C,n}}{dt}\right|_{t=0} = \frac{1}{C} i_{C,n}(0) = 0. \tag{2.34}$$

This derivative equals zero, since in a series connection $i_C(0) = i_L(0)$ and the current through an inductance prior to switching is zero. Now we have two equations for determining two arbitrary constants

$$\begin{aligned} A_1 + A_2 &= V_0, \\ s_1 A_1 + s_2 A_2 &= 0. \end{aligned} \tag{2.35}$$

The simultaneous solution of equations 2.35 yields

$$A_1 = \frac{V_0 s_2}{s_2 - s_1} \quad \text{and} \quad A_2 = \frac{V_0 s_1}{s_1 - s_2}. \tag{2.36}$$

Therefore, the natural response of the capacitance voltage is

$$v_{C,n} = \frac{V_0}{s_2 - s_1}(s_2 e^{s_1 t} - s_1 e^{s_2 t}). \tag{2.37}$$

The current may now be obtained by a simple differentiation of the capacitance voltage, which results in

$$i_n(t) = C\frac{dv_C}{dt} = CV_0 \frac{s_1 s_2}{s_2 - s_1}(e^{s_1 t} - e^{s_2 t}) = \frac{V_0}{L(s_2 - s_1)}(e^{s_1 t} - e^{s_2 t}). \tag{2.38}$$

(The reader can easily convince himself that $s_1 s_2 = 1/LC$.) Finally, the inductance voltage is found as

$$v_{L,n}(t) = L\frac{di_n}{dt} = \frac{V_0}{s_2 - s_1}(s_1 e^{s_1 t} - s_2 e^{s_2 t}). \tag{2.39}$$

The plots of $v_{C,n}$, i_n, and $v_{L,n}$ are shown in Fig. 2.42(a). As can be seen from the inductance voltage plot, at $t = 0$ it abruptly changes from zero to $-V_0$, at the instant $t = t_1$ it equals zero and after that, the inductance voltage remains

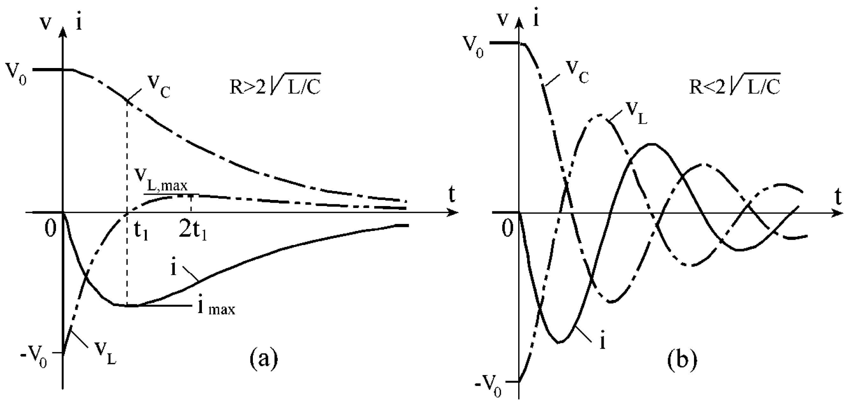

Figure 2.42 The natural responses of $v_{C,n}$, $v_{L,n}$ and i_n in a series connected *RLC* circuit: overdamped (a), and underdamped (b).

positive. The instant of t_1 can be found from the equation $s_1e^{s_1t_1} - s_2e^{s_2t_1} = 0$, to which the solution is

$$t_1 = \frac{\ln(s_2/s_1)}{s_1 - s_2}. \tag{2.40}$$

At this instant of time, the current reaches its maximum. By equating $dv_{L,n}/dt$ to zero it can be readily found that $v_{L,n}$ has its maximal value at $t = 2t_1$. The overdamped response is also called an *aperiodical response*. The energy exchange in such a response can be explained as follows. The energy initially stored in the capacitance decreases continuously with the decrease of the capacitance voltage. This energy is stored in the inductance throughout the period that the current increases. After $t = t_1$, the current decreases and the energy stored in the inductance decreases. Throughout the entire transient response, all the energy dissipates into resistance, converting into heat.

An underdamped response: Assume now that the roots of equation 2.32 are complex conjugate numbers, i.e., $\alpha < \omega_d$ or $R < 2\sqrt{L/C}$, and $s_{1,2} = -\alpha \pm j\omega_n$, where $\omega_n = \sqrt{\omega_d^2 - \alpha^2}$ is the frequency of the natural response, or *natural frequency*, and $\alpha = R/2L$ is, as previously, the exponential *damping coefficient*. As we have observed earlier (see section 1.6.2), the natural response of, for instance, the capacitance voltage in this case becomes a damped sinusoidal function of the form (1.33):

$$v_{C,n}(t) = Be^{-\alpha t}\sin(\omega_n t + \beta), \tag{2.41a}$$

where the arbitrary constants B and β can be found as was previously by solving two simultaneous equations

$$B\sin\beta = V_0,$$

$$-\alpha\sin\beta + \omega_n\cos\beta = 0,$$

to which the solution is (also see (1.65)):

$$B = \frac{V_0}{\sin\beta} \quad \text{and} \quad \beta = \tan^{-1}\frac{\omega_n}{\alpha}.$$

By using trigonometrical identities we may also obtain:

$$\sin\beta = \frac{\tan\beta}{\sqrt{1+\tan^2\beta}} = \frac{\omega_n}{\sqrt{\alpha^2+\omega_n^2}}$$

$$B = \frac{V_0\sqrt{\alpha^2+\omega_n^2}}{\omega_n} = V_0\sqrt{\frac{\alpha^2}{\omega_n^2}+1} \quad \text{or} \quad B = V_0\frac{\omega_d}{\omega_n},$$

where

$$\omega_d = \sqrt{\alpha^2+\omega_n^2}.$$

We may also look for the above response in the form of two sinusoids as in (1.66):

$$v_{C,n}(t) = e^{-\alpha t}(M\sin\omega_n t + N\cos\omega_n t). \tag{2.41b}$$

In this case, the arbitrary constants can be found, as in (1.68), with

$$(dv_{C,n}/dt)|_{t=0} = 0$$

and $v_{C,n}(0) = V_0$:

$$N = v_{C,n}(0) = V_0$$

$$M = \frac{\alpha}{\omega_n}V_0.$$

This results in

$$\beta = \tan^{-1}\frac{N}{M} = \frac{\omega_n}{\alpha} \quad \text{and} \quad B = \sqrt{M^2+N^2} = V_0\sqrt{\frac{\alpha^2}{\omega_n^2}+1} = V_0\frac{\omega_d}{\omega_n},$$

which is as was previously found. Therefore,

$$v_{C,n}(t) = e^{-\alpha t}\left(\frac{\alpha}{\omega_n}V_0\sin\omega_n t + V_0\cos\omega_n t\right), \tag{2.42a}$$

or

$$v_{C,n}(t) = V_0\frac{\omega_d}{\omega_n}e^{-\alpha t}\sin(\omega_n t+\beta). \tag{2.42b}$$

The current becomes

$$\begin{aligned} i_n(t) &= C\frac{dv_{C,n}}{dt} = V_0\frac{\omega_d}{\omega_n}Ce^{-\alpha t}[-\alpha\sin(\omega_n t+\beta)+\omega_n\cos(\omega_n t+\beta)] \\ &= \frac{V_0}{\omega_n L}e^{-\alpha t}\sin(\omega_n t+\beta+\nu), \end{aligned}$$

where $\tan \nu = \omega_n/(-\alpha)$ and, since $\tan \beta = \omega_n/\alpha$, $\beta + \nu = 180°$. Therefore,

$$i_n(t) = -\frac{V_0}{\omega_n L} e^{-\alpha t} \sin \omega_n t. \tag{2.43}$$

The inductance voltage may now be found as

$$\begin{aligned} v_{L,n}(t) = L\frac{di_n}{dt} &= -\frac{V_0}{\omega_n} e^{-\alpha t}[-\alpha \sin \omega_n t + \omega_n \cos \omega_n t] \\ &= -\frac{V_0 \omega_d}{\omega_n} e^{-\alpha t} \sin(\omega_n t + \nu) = \frac{V_0 \omega_d}{\omega_n} e^{-\alpha t} \sin(\omega_n t - \beta). \end{aligned} \tag{2.44}$$

The plots of $v_{C,n}$, i_n, and $v_{L,n}$ are shown in Fig. 2.42(b). This kind of response is also called *an oscillatory or periodical response.*

The energy, initially stored in the capacitance, during this response is interchanged between the capacitance and inductance and is accompanied by energy dissipation into the resistance. The transients will finish, when the entire capacitance energy $CV_0/2$ is completely dissipated.

Critical damping response: If the value of a resistance is close to $2\sqrt{L/C}$, i.e., $R \to 2\sqrt{L/C}$, the natural frequency $\omega_n = \sqrt{1/LC - R^2/4L^2} \to 0$ and the ratio in equation 2.43 $\sin \omega_n t/\omega_n \to \frac{0}{0}$ is indefinite. Applying l'Hopital's rule, gives

$$\lim_{\omega_n \to 0}\left(\frac{\sin \omega_n t}{\omega_n}\right) = \left.\frac{d/d\omega_n(\sin \omega_n t)}{d/d\omega_n(\omega_n)}\right|_{\omega_n \to 0} = \left.\frac{t \cos \omega_n t}{1}\right|_{\omega_n \to 0} = t.$$

Therefore in this critical response the current will be

$$i_n(t) = -\frac{V_0}{L} t e^{-\alpha t}, \tag{2.45}$$

which is also aperiodical. The capacitance voltage can now be found as

$$v_{C,n}(t) = \frac{1}{C}\int i_n(t)dt = \frac{1}{C}\left(-\frac{V_0}{L}\right)\frac{1}{\alpha^2} e^{-\alpha t}(-\alpha t - 1),$$

or since $\alpha^2 = 1/LC$,

$$v_{C,n} = V_0(1 + \alpha t)e^{-\alpha t}. \tag{2.46}$$

Finally, the inductive voltage is

$$v_{L,n}(t) = L\frac{di_n}{dt} = -V_0(e^{-\alpha t} - \alpha t e^{-\alpha t}) = -V_0(1 - \alpha t)e^{-\alpha t}. \tag{2.47}$$

It is also worthwhile to introduce here the graphical representation of the roots of the characteristic equation. On the complex plane the roots, which

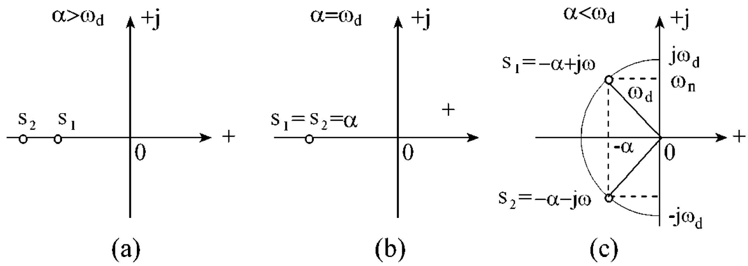

Figure 2.43 The location of the roots of the characteristic equation on the complex plane: overdamped response (a), critical damping response (b), and underdamped response (c).

define the three different cases of the transient response, are located as shown in Fig. 2.43.

The position of the roots on the complex plane, Fig. 2.43 (in other words the dependency of a specific kind of natural response on the relationship between the circuit parameters), is related to the quality factor of a resonant *RLC* circuit. Indeed, by rewriting the critical damping condition as $R/2L = 1/\sqrt{LC}$ we have $\frac{1}{2} = \sqrt{L/C}/R = Q$, this in terms of the resonant circuit is the quality factor. (In our future study, we shall call $Z_c = \sqrt{L/C}$ a surge or natural impedance.) Hence, if $Q < 1/2$ (the position of the roots is shown in (a)), the natural response is overdamped, if $Q > 1/2$ it is underdamped (c) and if $Q = 1/2$ the response is critical damping (b). Hence, the natural response becomes an underdamped oscillatory response, if the resistance of the *RLC* circuit is relatively low compared to the natural impedance.

In (a), two negative real roots are located on the negative axis (in the left half of the complex plane), which indicates the *overdamped* response. Note that $|s_2| > |s_1|$ and therefore $e^{s_2 t}$ decreases faster than $e^{s_1 t}$. In (b), two equal negative roots $s_1 = s_2 = -\alpha$, which indicate the *critical* damping, are still located on the real axis at the boundary point, i.e., no real roots are possible to the right of this point. In (c), the two roots become complex-conjugate numbers, located on the left half circle whose radius is the resonant frequency ω_d. This case indicates an *underdamped* response, having an oscillatory waveform of natural frequency. Note that the two frequencies $\pm j\omega_d$ represent a dissipation-free oscillatory response since the damping coefficient α is zero. This is, of course, a theoretical response: however there are very low resistive circuits in which the natural response could be very close to the theoretical one. Finally, in Fig. 2.44 the change of the form of the natural response with regard to decreasing the damping coefficient is shown.

(b) Parallel connected RLC *circuits*

The circuit containing an *RLC* in parallel is shown in Fig. 2.45. At the instant of $t = 0$ the switch is moved from position "1" to position "2", so that the initial value of the inductance current is I_0. In such a way, this circuit is a full dual

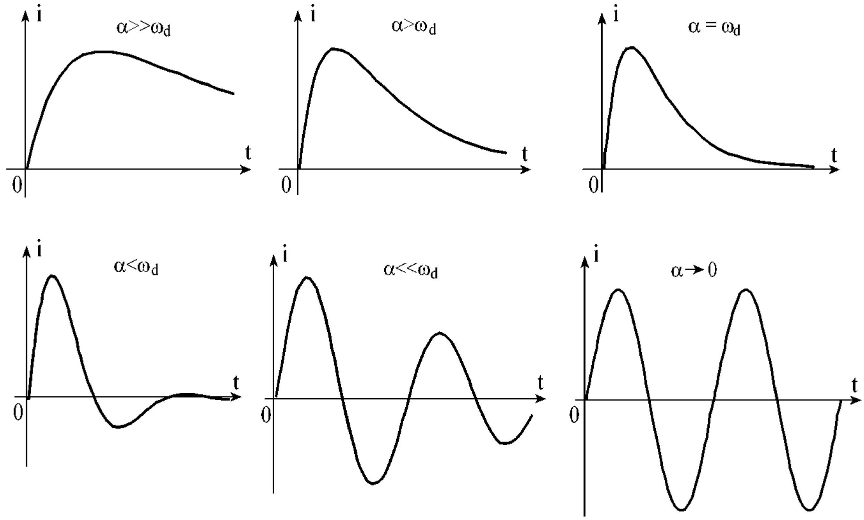

Figure 2.44 The transformation of the natural response in an *RLC* circuit by decreasing the damping coefficient α.

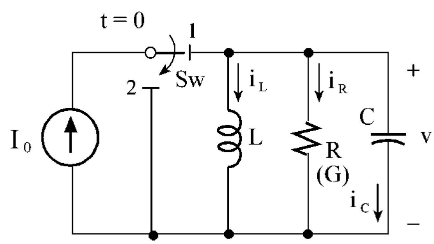

Figure 2.45 A parallel-connected *RLC* circuit.

of the circuit containing an *RLC* in series with an initial capacitance voltage. In order to perform the transient analysis of this circuit we shall apply the principle of duality. As a reminder of the principle of duality: the mathematical results for *RLC* in series are appropriate for *RLC* in parallel after interchanging between the dual parameters ($R \to G$, $L \to C$, $C \to L$), and then the solutions for currents are appropriate for voltages and vice versa. The roots of the characteristic equation will be of the same form: $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_d^2}$, but the meaning of α is different: $\alpha = G/2C$ (instead of $\alpha = R/2L$ for a series circuit), however, it is more common to write the above expression as $\alpha = 1/2RC$. The resonant frequency $\omega_d = 1/\sqrt{LC}$ remains the same, since the interchange between L and C does not change the expression.

Underdamped response: The common voltage of all three elements is appropriate to the common current in the series circuit, therefore (see equation 2.38).

$$v_n(t) = \frac{I_0}{C(s_2 - s_1)}(e^{s_1 t} - e^{s_2 t}). \tag{2.48}$$

The inductor current is appropriate to the capacitor voltage in the series circuit, therefore (see (2.37))

$$i_{L,n}(t) = \frac{I_0}{s_2 - s_1}\left(s_2 e^{s_1 t} - s_1 e^{s_2 t}\right). \tag{2.49}$$

In a similar way, we shall conclude that the capacitor current is appropriate to the inductance voltage (see equation 2.39)

$$i_{C,n}(t) = \frac{I_0}{s_2 - s_1}\left(s_1 e^{s_1 t} - s_2 e^{s_2 t}\right). \tag{2.50}$$

In order to check these results we shall apply the KCL for the common node of the parallel connection and by noting that $i_{R,n}(t) = v_n(t)/R$, we may obtain

$$i_{L,n} + i_{C,n} + i_{R,n} = 0,$$

or

$$\frac{I_0}{s_2 - s_1}\left(s_2 e^{s_1 t} - s_1 e^{s_2 t} + s_1 e^{s_1 t} - s_2 e^{s_2 t} + \frac{1}{RC} e^{s_1 t} - \frac{1}{RC} e^{s_2 t}\right)$$
$$= \frac{I_0}{s_2 - s_1}\left(s_2 + s_1 + \frac{1}{RC}\right)\left(e^{s_1 t} - e^{s_2 t}\right) = 0,$$

since $s_2 + s_1 = -2\alpha = -1/RC$.

Overdamped response: The analysis of the overdamped response in a parallel circuit can be performed in a similar way to an underdamped response, i.e., by using the principle of duality. This is left for the reader as an exercise.

(c) Natural response by two nonzero initial conditions

Our next approach in the transient analysis of an *RLC* circuit shall be the more general case in which both energy-storing elements *C* and *L* are previously charged. For this reason, let us consider the current in Fig. 2.46. In this circuit prior to switching, the capacitance is charged to voltage V_{C0} and there is current I_{L0} flowing through the inductance. Therefore, this circuit differs from the one

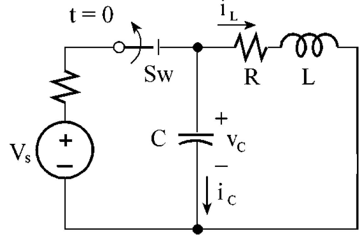

Figure 2.46 *RLC* circuit with a non-zero initial condition.

in Fig. 2.41 in that the initial condition of the inductor current is now $i_L(0_-) = I_{L0}$, but not zero. The capacitance current is now, after switching, $i_C(0) = -i_{L0}(0) = -I_{L0}$. By determining the initial value of the capacitance voltage derivative in equation 2.34, we must substitute $-I_{L0}$ for $i_C(0)$. Therefore,

$$\left.\frac{dv_C}{dt}\right|_{t=0} = -\frac{1}{C} I_{L0},$$

and the set of equations for determining the constants of integration becomes

$$\begin{aligned} A_1 + A_2 &= V_{C0} \\ s_1 A_1 + s_2 A_2 &= -(1/C) I_{L0}, \end{aligned} \tag{2.51}$$

to which the solution is

$$A_1 = \left(V_{C0} + \frac{I_{L0}}{s_2 C}\right)\frac{s_2}{s_2 - s_1} \quad \text{and} \quad A_2 = \left(V_{C0} + \frac{I_{L0}}{s_1 C}\right)\frac{s_1}{s_1 - s_2}. \tag{2.52}$$

The natural responses of an *RLC* circuit will now be

$$v_{C,n}(t) = \left(V_{C0} + \frac{1}{s_2 C} I_{L0}\right)\frac{s_2}{s_2 - s_1} e^{s_1 t} + \left(V_{C0} + \frac{1}{s_1 C} I_{L0}\right)\frac{s_1}{s_1 - s_2} e^{s_2 t}, \tag{2.53}$$

or in a slightly different way

$$v_{C,n}(t) = \frac{V_{C0}}{s_2 - s_1}(s_2 e^{s_1 t} - s_1 e^{s_2 t}) + \frac{I_{L0}}{C(s_2 - s_1)}(e^{s_1 t} - e^{s_2 t}), \tag{2.54}$$

which differs from equation 2.37 by the additional term due to the initial value of the current I_{L0}.

The current response will now be

$$i_n(t) = \frac{V_{C0}}{L(s_2 - s_1)}(e^{s_1 t} - e^{s_2 t}) + \frac{I_{L0}}{s_2 - s_1}(s_1 e^{s_1 t} - s_2 e^{s_2 t}), \tag{2.55}$$

and the inductance voltage

$$v_{L,n}(t) = \frac{V_{C0}}{s_2 - s_1}(s_1 e^{s_1 t} - s_2 e^{s_2 t}) + \frac{L I_{L0}}{s_2 - s_1}(s_1^2 e^{s_1 t} - s_2^2 e^{s_2 t}). \tag{2.56}$$

The above equations 2.54–2.56 can also be *written in terms of hyperbolical functions*. Such expressions are used for transient analysis in some professional books.(*) We shall first write roots s_1 and s_2 in a slightly different form

$$s_{1,2} = -\alpha \pm \gamma, \quad \text{where} \quad \gamma = \sqrt{\alpha^2 - \omega_d^2}, \tag{2.57a}$$

(*) Greenwood, A. (1991) *Electrical Transients in Power Systems.* Wiley, New York, Chichester, Brisbane, Toronto, Singapore.

then

$$s_2 - s_1 = -2\gamma, \quad s_1 s_2 = \alpha^2 - \gamma^2 = \omega_d^2 = 1/LC,$$

and

$$e^{s_{1,2}t} = e^{-\alpha t} e^{\pm \gamma t} = e^{-\alpha t}(e^{\gamma t} + e^{-\gamma t}) = e^{-\alpha t}[\cosh \gamma t \pm \sinh \gamma t]. \qquad (2.57b)$$

With the substitution of equation 2.57(a) for $s_{1,2}$ and taking into account the above relationships, after a simple mathematical rearrangement, one can readily obtain

$$v_{C,n}(t) = \left[V_{C0}\left(\cosh \gamma t + \frac{\alpha}{\gamma} \sinh \gamma t \right) + \frac{I_{L0}}{\gamma C} \sinh \gamma t \right] e^{-\alpha t}, \qquad (2.58)$$

and

$$i_n(t) = \left[-\frac{V_{C0}}{\gamma L} \sinh \gamma t + I_{L0}\left(\cosh \gamma t - \frac{\alpha}{\gamma} \sinh \gamma t \right) \right] e^{-\alpha t}. \qquad (2.59)$$

It should be noted that $1/\gamma C$ and γL (like $1/\omega C$ and ωL) are some kinds of resistances in units of Ohms.

For the overdamped response

$$s_{1,2} = -\alpha \pm j\omega,$$

which means that γ must be substituted by $j\omega$ and the hyperbolic *sine* and *cosine* turn into trigonometric ones

$$v_{C,n}(t) = \left[V_{C0}\left(\cos \omega_n t + \frac{\alpha}{\omega_n} \sin \omega_n t \right) + \frac{I_{L0}}{\omega_n C} \sin \omega_n t \right] e^{-\alpha t},$$

or

$$v_{C,n}(t) = e^{-\alpha t} \left[\left(\frac{I_{L0}}{\omega_n C} + \frac{V_{C0}\alpha}{\omega_n} \right) \sin \omega_n t + V_{C0} \cos \omega_n t \right]. \qquad (2.60)$$

(Which, by assumption $I_{L0} = 0$, turns into the previously obtained one in equation 2.42a.)

At this point we shall once more turn our attention to the *energy relations* in the *RLC* circuit upon its natural response. As we have already observed, the energy is stored in the magnetic and electric fields of the inductances and capacitances, and dissipates in the resistance. To obtain the relation between these processes in a general form we shall start with a differential equation describing the above circuit:

$$L\frac{di}{dt} + v_C + Ri = 0.$$

Multiplying all the terms of the equation by $i = C(dv_C/dt)$, we obtain

$$Li\frac{di}{dt} + Cv_C\frac{dv_C}{dt} + Ri^2 = 0.$$

Taking into consideration that

$$f\frac{df}{dt} = \frac{1}{2}\frac{d}{dt}(f^2)$$

we may rewrite

$$\frac{d}{dt}\left(\frac{Li^2}{2}\right) + \frac{d}{dt}\left(\frac{Cv_C^2}{2}\right) + Ri^2 = 0,$$

or

$$\frac{d}{dt}\left(\frac{Li^2}{2} + \frac{Cv_C^2}{2}\right) = -Ri^2. \tag{2.61}$$

The term inside the parentheses gives the sum of the stored energy and, therefore, the derivative of this energy is always negative (if, of course, $i \neq 0$), or, in other words, the total stored energy changes by decreasing. The change of each of the terms inside the parentheses can be either positive or negative (when the energy is exchanged between the inductance and capacitance), but it is impossible for both of them to change positively or increase. This means that the total stored energy decreases during the transients and the rate of decreasing is equal to the rate of its dissipating into resistance (Ri^2).

At this point, we will continue our study of transients in *RLC* circuits by solving numerical examples.

Example 2.26

In the circuit of Fig. 2.47 the switch is changed instantaneously from position "1" to "2". The circuit parameters are: $R_1 = 2\,\Omega$, $R_2 = 10\,\Omega$, $L = 0.1$ H, $C = 0.8$ mF and $V_s = 120$ V. Find the transient response of the inductive current.

Solution

The given circuit is slightly different from the previously studied circuit in that the additional resistance is in series with the parallel-connected inductance and capacitance branches.

In order to determine the characteristic equation and its roots (step 1), we

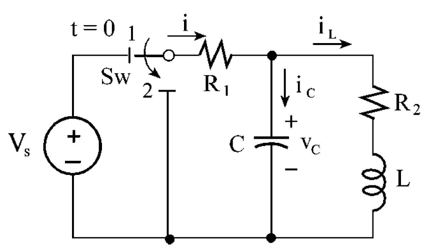

Figure 2.47 A given circuit for Example 2.26.

must indicate the input impedance (seen from the inductance branch)

$$Z(s) = (R_2 + sL) + R_1//(1/sC),$$

which results in

$$s^2 + \left(\frac{R_2}{L} + \frac{1}{R_1 C}\right)s + \frac{R_1 + R_2}{R_1}\frac{1}{LC} = 0,$$

or

$$s^2 + 725s + 7.5\cdot 10^4 = 0,$$

where

$$2\alpha = \left(\frac{R_2}{L} + \frac{1}{R_1 C}\right) = \frac{10}{0.1} + \frac{10^3}{2\cdot 0.8} = 725\ \mathrm{s}^{-1}$$

$$\omega_d = \frac{R_1 + R_2}{R_1}\frac{1}{LC} = \frac{2+10}{2}\frac{10^3}{0.1\cdot 0.8} = 7.5\cdot 10^4\ \mathrm{rad/s}.$$

Thus,

$$s_{1,2} = (-3.625 \pm \sqrt{3.625^2 - 7.5})\cdot 10^2 = -125,\ -600\ \mathrm{s}^{-1}$$

and

$$v_{C,n}(t) = A_1 e^{-125t} + A_2 e^{-600t}.$$

Since the circuit after switching is source free, no forced response (step 2) is expected.

The initial conditions (step 3) are:

$$v_C(0) = v_C(0_-) = V_s\frac{R_2}{R_1 + R_2} = 120\frac{10}{2+10} = 100\ \mathrm{V}$$

$$i_L(0) = i_L(0_-) = \frac{V_s}{R_1 + R_2} = \frac{120}{2+10} = 10\ \mathrm{A}.$$

The initial value of the current derivative (step 4) is found as

$$\left.\frac{di_L}{dt}\right|_{t=0} = \frac{v_L(0)}{L} = \frac{v_C(0) - R_2 i_L(0)}{L} = \frac{100 - 10\cdot 10}{L} = 0.$$

By solving the two equations below (step 5)

$$A_1 + A_2 = 10,$$

$$s_1 A_1 + s_2 A_2 = 0,$$

we have (see equation 2.36)

$$A_1 = \frac{I_{L0}s_2}{s_2 - s_1} = \frac{10(-600)}{-600+125} = 12.6, \quad A_2 = \frac{I_{L0}s_1}{s_1 - s_2} = \frac{10(-125)}{-125+600} = -2.6.$$

Thus,

$$i_L(t) = 12.6e^{-125t} - 2.6e^{-600t}\ \mathrm{A}.$$

In the next example, we will consider an *RLC* circuit, having a zero independent initial condition, which is connected to a d.c. power supply.

Example 2.27

In the circuit with $R = 100\ \Omega$, $R_1 = 5\ \Omega$, $R_2 = 3\ \Omega$, $L = 0.1\ \mathrm{H}$, $C = 100\ \mu\mathrm{F}$ and $V_s = 100\ \mathrm{V}$, shown in Fig. 2.48, find current $i_L(t)$ for $t > 0$. The voltage source is applied at $t = 0$, due to the unit forcing function $u(t)$.

Solution

The input impedance seen from the inductive branch is

$$Z_{12}(s) = R_1 + sL + \left(R_2 + \frac{1}{sC}\right) // R,$$

or, after performing the algebraic operations and equating it to zero, we obtain

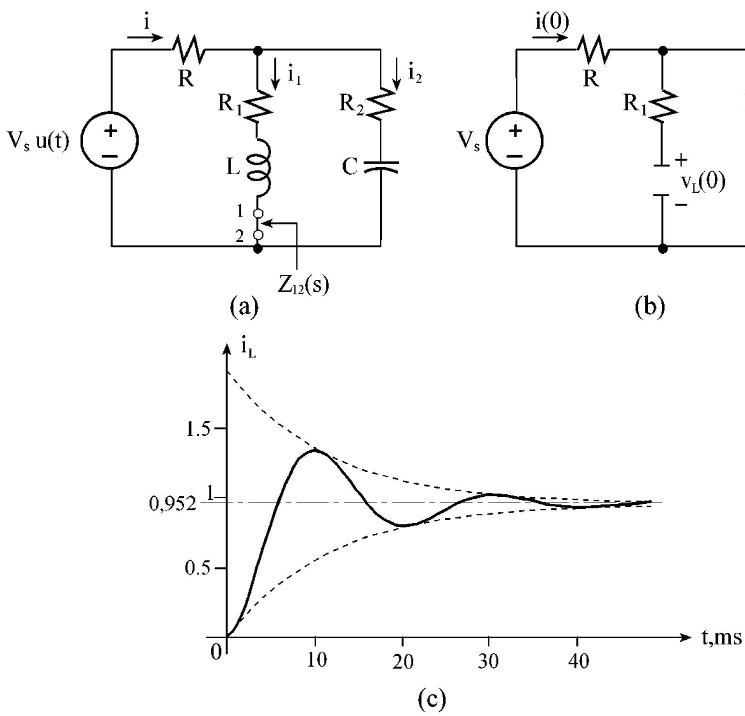

Figure 2.48 A given circuit for Example 2.27 (a), an equivalent circuit for instant $t = 0$ (b) and the current plot (c).

the characteristic equation

$$s^2 + \left(\frac{R_{eq}}{L} + \frac{1}{(R+R_2)C}\right)s + \frac{R+R_1}{R+R_2}\frac{1}{LC} = 0,$$

where

$$R_{eq} = (RR_1 + RR_2 + R_1R_2)/(R+R_2).$$

Substituting the numerical values yields

$$s^2 + 176.2s + 10.2\cdot 10^4 = 0,$$

to which the roots are:

$$s_{1,2} = -88.1 \pm j307\ \mathrm{s}^{-1}.$$

Since the roots are complex numbers, the natural response is

$$i_{L,n}(t) = Be^{-88.1t}\sin(307t+\beta).$$

The forced response is

$$i_{L,f} = \frac{V_s}{R+R_1} = \frac{100}{100+5} = 0.952\ \mathrm{A}.$$

The independent initial conditions are zero, therefore

$$v_C(0) = 0 \quad \text{and} \quad i_L(0) = 0.$$

The dependent initial condition is found in circuit (b), which is appropriate to the instant of switching $t=0$:

$$\left.\frac{di_L}{dt}\right|_{t=0} = \frac{v_L(0)}{L} = \frac{i(0)R_2}{L} = \frac{V_sR_2}{(R+R_2)L} = \frac{100\cdot 3}{(100+3)0.1} = 29.2.$$

The integration constant can now be found from

$$B\sin\beta = i(0) - i_f(0) = 0 - 0.952 = -0.952$$

$$-\alpha B\sin\beta + \omega_n B\cos\beta = \left.\frac{di}{dt}\right|_{t=0} - \left.\frac{di_f}{dt}\right|_{t=0} = 29.2 - 0 = 29.2,$$

to which the solution is

$$\beta = \tan^{-1}\frac{307}{-(29.2/0.952)+88} = 79.4^\circ \quad \text{and} \quad B = -\frac{0.952}{\sin 79.4^\circ} = -0.968.$$

Therefore, the complete response is

$$i_L = i_{L,f} + i_{L,n} = 0.952 - 0.968e^{-88t}\sin(307t + 79.4^\circ)\ \mathrm{A}.$$

To plot this curve we have to estimate the time constant of the exponent, $\tau = 1/88 \cong 11$ ms, and the period of sine, $T = 2\pi/307 \cong 20$ ms. The plot of the current is shown in Fig. 2.48(c).

Example 2.28

In the circuit with $R_1 = R_2 = 10\,\Omega$, $L = 5\,\mathrm{mH}$, $C = 10\,\mu\mathrm{F}$ and $V_s = 100\,\mathrm{V}$, in Fig. 2.49, find current $i_2(t)$ after the switch closes.

Solution

The input impedance seen from the source is $Z_{in}(s) = R_1 + sL + R_2//(1/sC)$. Then the characteristic equation becomes

$$s^2 + \left(\frac{R_1}{L} + \frac{1}{R_2C}\right)s + \frac{R_1 + R_2}{R_2}\frac{1}{LC} = 0.$$

Substituting the numerical values and solving this characteristic equation, we obtain the roots:

$$s_{1,2} = (-6 \pm j2)10^3\ \mathrm{s}^{-1}.$$

The natural response becomes

$$i_{2,n} = Be^{-6\cdot 10^3 t}\sin(2\cdot 10^3 t + \beta).$$

The forced response is

$$i_{2,f} = \frac{V_s}{R_1 + R_2} = \frac{100}{10 + 10} = 5\ \mathrm{A}.$$

The independent initial conditions are

$$i_1(0) = i_L(0) = \frac{V_s}{R_1 + R_2} = \frac{100}{10 + 10} = 5\ \mathrm{A} \quad \text{and} \quad v_C(0) = 0.$$

In order to determine the initial conditions for current i_2, which can change abruptly, we must consider the given circuit at the moment of $t = 0$. Since the capacitance at this moment is a short-circuit, the current in R_2 drops to zero, i.e., $i_2 = 0$. With the KVL for the right loop we have

$$R_2 i_2 - v_C = 0 \quad \text{or} \quad R_2 i_2 = v_C,$$

and

$$\left.\frac{di_2}{dt}\right|_{t=0} = \frac{1}{R_2}\frac{1}{C}i_3(0) = \frac{1}{10\cdot 10^{-5}}5 = 5\cdot 10^4.$$

Here $i_3(0) = i_1(0) = 5\ \mathrm{A}$, since $i_2(0) = 0$.

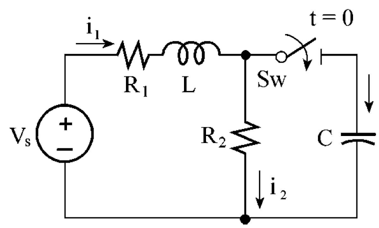

Figure 2.49 A given circuit for Example 2.28.

Our last step is to find the integration constants. We have

$$B \sin \beta = i_2(0) - i_{2,f}(0) = -5$$

$$-6 \cdot 10^3 \sin \beta + 2 \cdot 10^3 \cos \beta = \left. \frac{di_2}{dt} \right|_{t=0} = 5 \cdot 10^4,$$

to which the solution is

$$\beta = \tan^{-1} \frac{-5 \cdot 2 \cdot 10^3}{50 \cdot 10^3 - 30 \cdot 10^3} = -26.6^\circ, \quad B = \frac{-5}{\sin(-26.6^\circ)} = 11.2.$$

Therefore, the complete response is

$$i_2 = 5 + 11.2 e^{-6000t} \sin(2000t - 26.6^\circ)\ \mathrm{A}.$$

Example 2.29

Consider once again the circuit shown in Fig. 2.40, which is redrawn here, Fig. 2.50. This circuit has been previously analyzed and it was shown that the natural response is dependent on the kind of applied source, voltage or current. We will now complete this analysis and find the transient response a) of the current $i(t)$ when a voltage source of 100 V is connected between nodes m–n, Fig. 2.50(a); and (b) of the voltage $v(t)$ when a current source of 11 A is connected between nodes m–n, Fig. 2.50(b). The circuit parameters are $R_1 = R_2 = 100\ \Omega$, $R_3 = 10\ \Omega$, $L = 20$ mH and $C = 2\ \mu$F.

Solution

(a) In this case an ideal voltage source is connected between nodes m and n.

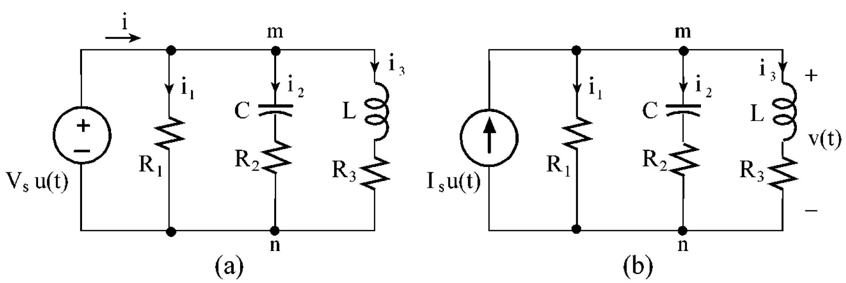

Figure 2.50 A circuit for Example 2.29 driven by a voltage source (a) and by a current source (b).

Therefore each of the three branches operates independently, and we may find each current very simply.

$$i_{1,f} = \frac{V_s}{R_1} = \frac{100}{100} = 1\ \mathsf{A} \quad \text{(no natural response)}$$

$$i_2 = i_{2,f} + i_{2,n} = 0 + \frac{V_s}{R_2} e^{s_2 t} = 1e^{-5000t}\ \mathsf{A},$$

where $s_2 = -\dfrac{1}{RC} = \dfrac{1}{100\cdot 2\cdot 10^{-6}} = 5000\ \mathsf{s}^{-1}$,

$$i_3 = i_{3,f} + i_{3,n} = \frac{V_s}{R_3} - \frac{V_s}{R_3} e^{s_3 t} = 10 - 10e^{-500t}\ \mathsf{A},$$

where $s_3 = -\dfrac{R_3}{L} = \dfrac{10}{20\cdot 10^{-3}} = -500\ \mathsf{s}^{-1}$.

Therefore, the total current is

$$i = i_1 + i_2 + i_3 = 11 + e^{-5000t} - 10e^{-500t}\ \mathsf{A}.$$

(b) In this case, in order to find the transient response we shall, as usual, apply the five-step solution. The characteristic equation (step 1) for this circuit has already been determined in equation 2.29. With its simplification, we have

$$s^2 + \left[\frac{R_{eq}}{L} + \frac{1}{(R_1 + R_2)C}\right] s + \frac{R_1 + R_3}{R_1 + R_2}\frac{1}{\sqrt{LC}} = 0,$$

where $R_{eq} = (R_1 R_2 + R_1 R_3 + R_2 R_3)/(R + R_2)$.

Upon substituting the numerical data the solution is

$$s_{1,2} = (-2.75 \pm j2.5)10^3\ \mathsf{s}^{-1}.$$

Thus the natural response will be

$$v_n = Be^{-2.75\cdot 10^3 t} \sin(2.5\cdot 10^3 t + \beta)\ \mathsf{V}.$$

The forced response (step 2) is

$$v_f = I_s \frac{R_1 R_3}{R_1 + R_3} = 11\,\frac{100\cdot 10}{100 + 10} = 100\ \mathsf{V}.$$

The independent initial conditions (step 3) are zero, i.e., $v_C(0) = 0$, $i_L(0) = 0$.

Next (step 4) we shall find the dependent initial condition, which will be used to determine the voltage derivative:

the voltage drop in the inductance, which is open circuited

$$v_L(0) = I_s(R_1 // R_2) = 11\cdot 50 = 550\ \mathsf{V};$$

the capacitance current, since the capacitance is short-circuited

$$i_C(0) = I_s \frac{R_1}{R_1 + R} = 11 \frac{100}{200} = 5.5 \text{ A};$$

the initial value of the node voltage (which is the voltage across the inductance) $v(0) = v_L(0) = 550$ V. In order to determine the voltage derivative we shall apply Kirchhoff's two laws

$$i_R + i_C + i_L = I_s, \quad v = R_1 i_R = R_2 i_C + v_C,$$

and, after differentiation, we have

$$\frac{di_R}{dt} + \frac{di_C}{dt} + \frac{di_L}{dt} = 0$$

$$\frac{dv}{dt} = R_2 \frac{di_C}{dt} + \frac{dv_C}{dt}.$$

By taking into consideration that

$$\left.\frac{dv_C}{dt}\right|_{t=0} = \frac{i_C(0)}{C}, \quad \left.\frac{di_L}{dt}\right|_{t=0} = \frac{v_L(0)}{L} \quad \text{and} \quad i_R = \frac{v}{R_1},$$

the solution for $(dv/dt)|_{t=0}$ becomes

$$\left.\frac{dv}{dt}\right|_{t=0} = \frac{R_1}{R_1 + R_2}\left(\frac{i_C(0)}{C} - \frac{R_2}{L} v_L(0)\right),$$

which, upon substitution of the data, results in $(dv/dt)|_{t=0} = 0$.

The integration constant, can now be found by solving the following set of equations

$$B \sin\beta = v(0) - v_f(0) = 550 - 100 = 450$$

$$-2.75 \cdot 10^3 \sin\beta + 2.5 \cdot 10^3 \cos\beta = \left.\frac{dv}{dt}\right|_{t=0} - \left.\frac{dv_f}{dt}\right|_{t=0} = 0.$$

The solution is

$$\beta = \tan^{-1}\frac{2 \cdot 5}{2.75} = 42.3^\circ$$

$$B = \frac{450}{\sin 42.3^\circ} = 669.$$

Therefore, the complete response is

$$v(t) = 100 + 669 e^{-2.75 \cdot 10^3 t} \sin(2.5 \cdot 10^3 t + 42.3^\circ) \text{ V}.$$

Note that this response is completely different from the one achieved in circuit (a). However, the forced response here, i.e., the node voltage, is 100 V, which is the same as the node voltage in circuit (a) due to the 100 V voltage source.

2.7.2 *RLC* circuits under a.c. supply

The analysis of an *RLC* circuit under a.c. supply does not differ very much from one under d.c. supply, since the natural response does not depend on the source and the five-step solution may again be applied. However, the evaluation of the forced response is different and somehow more labor consuming, since phasor analysis (based on using complex numbers) must be applied. Let us now illustrate this approach by solving numerical examples.

Example 2.30

Let us return to the circuit shown in Fig. 2.51 of Example 2.26 and suppose that the switch is moved from position "2" to "1", connecting this circuit to the a.c. supply: $v_s = V_m \sin(\omega t + \psi_v)$, having $V_m = 540$ V at $f = 50$ Hz and $\psi_v = 0°$. Find the current of the inductive branch, i_L, assuming that the circuit parameters are: $R_1 = 2\ \Omega$, $R_2 = 10\ \Omega$, $L = 0.1$ H and the capacitance $C = 100\ \mu$F, whose value is chosen to improve the power factor.

Solution

The characteristic equation of the circuit has been found in Example 2.26, in

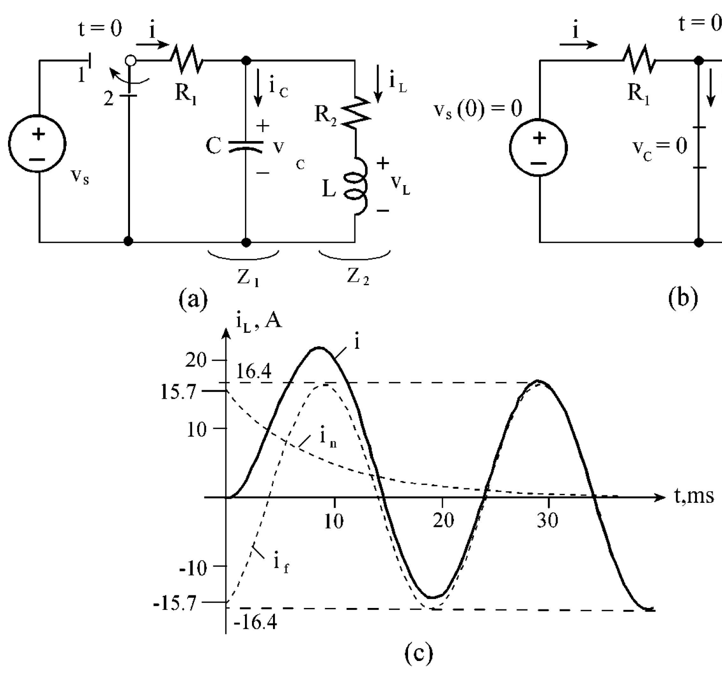

Figure 2.51 A given circuit of Example 2.30 (a), circuit equivalent at $t = 0$ (b) and the plot of current i_L (c).

which

$$\alpha = \frac{1}{2}\left(\frac{R_2}{L} + \frac{1}{R_1C}\right) = \frac{1}{2}\left(\frac{10}{0.1} + \frac{10^4}{2}\right) = 2.55\cdot 10^3$$

and

$$\omega_d^2 = \frac{R_1 + R_2}{R_1}\frac{1}{LC} = \frac{2+10}{2}\frac{10^4}{0.1} = 0.6\cdot 10^6.$$

Thus, the roots are

$$s_{1,2} = (-2.55 \pm \sqrt{2.55^2 - 0.6})10^3 \cong (-0.12, -5.0)10^3\ \mathrm{s}^{-1}$$

and the natural response is

$$i_{L,n} = A_1 e^{-120t} + A_2 e^{-5000t}.$$

The next step is to find the forced response. By using the phasor analysis method we have

$$\tilde{I}_{L,m} = \frac{\tilde{V}}{Z_{in}}\frac{Z_1}{Z_2 + Z_1} = \frac{540}{105\angle -17.4^\circ}\frac{31.8\angle -90^\circ}{10} = 16.4\angle -72.6^\circ\ \mathrm{A},$$

where

$$Z_2 = R_2 + j\omega L = 10 + j31.4 = 32.9\angle 72.3^\circ, \quad Z_1 = -j1/\omega C = -j31.8$$

and

$$Z_{in} = R_1 + Z_2//Z_1 = 105\angle -17.4^\circ.$$

The forced response is

$$i_{L,f} = 16.4\sin(314t - 72.6^\circ).$$

Since no initial energy is stored either in the capacitance or in the inductance, the initial conditions are zero: $v_C(0) = 0$ and $i_L(0) = 0$. By inspection of the circuit for the instant $t = 0$, Fig. 2.51(b), in which the capacitance is short-circuited, the inductance is open-circuited and the instant value of the voltage source is zero, we may conclude that $v_L(0) = 0$. Therefore, the second initial condition for determining the integration constant is

$$\left.\frac{di_L}{dt}\right|_{t=0} = \frac{v_L(0)}{L} = 0.$$

Thus, we have

$$A_1 + A_2 = i_L(0) - i_{L,f}(0) = 0 - 16.4\sin(-72.6^\circ) = 15.65$$

$$s_1A_1 + s_2A_2 = \left.\frac{di_L}{dt}\right|_{t=0} - \left.\frac{di_{L,f}}{dt}\right|_{t=0} = 0 - 16.4\cdot 314\cos(-72.6^\circ) = -1540,$$

and

$$A_1 = \frac{(-5\cdot 15.65 + 1.54)10^3}{(-5+0.12)10^3} = 15.72, \quad A_2 = 15.65, \quad A_1 = -0.07 \cong 0.$$

Therefore, the complete response is

$$i_L(t) = 16.4\sin(314t - 72.6^\circ) + 15.7e^{-120t}\ \mathsf{A}.$$

This response is plotted in Fig. 2.51(c), whereby the time constant of the exponential term is $\tau = 1/120 = 8.3$ ms.

Example 2.31

A capacitance of 200 μF is switched on at the end of a 1000 V, 60 Hz transmission line with $R = 10\ \Omega$ and load $R_1 = 30\ \Omega$ and $L = 0.1$ H, Fig. 2.52. Find the transient current i and sketch it, if the instant of switching the voltage phase angle is zero, $\psi_v = 0$.

Solution

The characteristic equation is obtained by equating the input impedance to zero

$$s^2 + 2\alpha s + \omega_d^2 = 0,$$

Here

$$\alpha = \frac{1}{2}\left(\frac{R_1}{L} + \frac{1}{RC}\right) = \frac{1}{2}\left(\frac{30}{0.1} + \frac{10^4}{10\cdot 2}\right) = 4\cdot 10^2$$

and

$$\omega_d^2 = \frac{R+R_1}{R}\frac{1}{LC} = \frac{10+30}{10}\frac{10^4}{0.1\cdot 2} = 20\cdot 10^4,$$

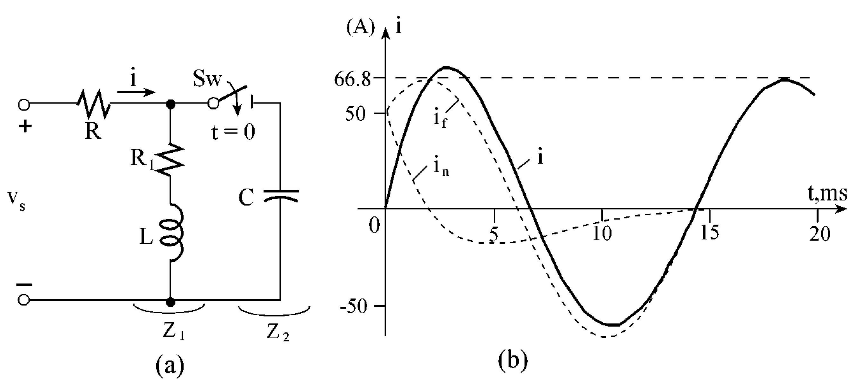

Figure 2.52 A given circuit of Example 2.31 (a) and the current plot (b).

which results in the roots

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_d^2} = (-4 \pm \sqrt{16-20})10^2 = (-4 \pm j2)10^2 \text{ s}^{-1}.$$

Thus, the natural response is

$$i_n(t) = Be^{-400t}\sin(200t + \beta).$$

The forced response is found by phasor analysis

$$\tilde{I} = \frac{\tilde{V}_s}{Z_{in}} = \frac{1000}{21.2\angle -50^\circ} = 47.2\angle 50^\circ,$$

where

$$Z_1 = R_1 + j\omega L = 30 + j37.7 = 48.2\angle 51.5^\circ,$$

$$Z_2 = -j\frac{1}{\omega C} = -j\frac{10}{377\cdot 2} = -j13.3$$

and

$$Z_{in} = R + \frac{Z_1 Z_2}{Z_1 + Z_2} = 10 + \frac{48.2\angle 51.5^\circ \cdot 13.3\angle -90^\circ}{30 + j37.7 - j13.3} = 21.2\angle -50^\circ.$$

Therefore,

$$i_f = 47.2\sqrt{2}\sin(377t + 50^\circ) = 66.8\sin(377t + 50^\circ)\text{ A}.$$

The independent initial conditions are $v_C(0) = 0$, $i_L(0) = 25.7\sin 43.3^\circ = 17.6$ A, since prior to switching:

$$\tilde{I}_{L,m} = \frac{V_s\sqrt{2}}{\sqrt{(R+R_1)^2 + x_L^2}} = \frac{1000\sqrt{2}}{\sqrt{40^2 + 37.7^2}} = 25.7 \quad \text{and} \quad \varphi = \tan^{-1}\frac{37.7}{40} = 43.3^\circ.$$

The next step is to determine the initial values of $i(0)$ and $(di/dt)|_{t=0}$. Since the input voltage at $t = 0$ is zero and the capacitance voltage is zero, we have $i(0) = [v_s(0) - v_C(0)]/R = 0$. The initial value of the current derivative is found with Kirchhoff's voltage law applied to the outer loop

$$-v_s + Ri + v_C = 0,$$

and, after differentiation, we have

$$\left.\frac{di}{dt}\right|_{t=0} = \frac{1}{R}\left(\frac{dv_s}{dt} - \frac{dv_C}{dt}\right)_{t=0} = \frac{1}{10}[533 - (-88)]10^3 = 62.1\cdot 10^3.$$

Here

$$\left.\frac{dv_s}{dt}\right|_{t=0} = 1000\sqrt{2}\cdot 377\cos\psi_v = 533\cdot 10^3$$

and

$$\left.\frac{dv_C}{dt}\right|_{t=0} = \frac{1}{C}\,i_C = \frac{10^4}{2}(-17.6) = -88 \cdot 10^3,$$

because $i_C(0) = -i_L(0) = -17.6$ (note that $i(0) = 0$). Hence, we now have two simultaneous equations for finding the integration constants

$$B \sin\beta = i(0) - i_f(0) = 0 - 47.2\sqrt{2}\sin 50^\circ = -51.1$$

$$-4 \cdot 10^2 B \sin\beta + 2 \cdot 10^2 B \cos\beta = \left.\frac{di}{dt}\right|_{t=0} - \left.\frac{di_f}{dt}\right|_{t=0}$$

$$= 62.1 \cdot 10^3 - 47.2\sqrt{2} \cdot 377 \cos 50^\circ = 45.9 \cdot 10^3,$$

for which the solution is

$$\beta = \tan^{-1}\frac{2 \cdot 10^2}{45.9 \cdot 10^3/(-51.1) + 4 \cdot 10^2} = 158.2^\circ \quad \text{and} \quad B = \frac{-51.1}{\sin 158.2^\circ} = -137.6.$$

Thus, the complete response is

$$i = 66.8\sin(377t + 50^\circ) - 137.6e^{-400t}\sin(200t + 158.2^\circ).$$

The plot of this curve can be seen in Fig. 2.52(c).

2.7.3 Transients in *RLC* resonant circuits

An *RLC* circuit whose quality factor Q is high (at least as large as 1/2) is considered a resonant circuit and, when interrupted, the transient response will be oscillatory. If the natural frequency of such oscillations is equal or close to some of the harmonics inherent in the system voltages or currents, then the resonant conditions may occur. In power system networks, the resonant circuit may arise in many cases of its operation.

In transmission and distribution networks, resonance may occur if an extended underground cable (having preponderant capacitance) is connected to an overhead line or transformer (having preponderant induction). The natural frequency of such a system may be close to the lower harmonics of the generating voltage. When feeder cables of high capacitances are protected against short-circuit currents by series reactors of high inductances, the resonance phenomenon may also arise. Banks of condensers, used, for example, for power factor correction, and directly connected under full voltage with the feeding transformer, may form a resonance circuit, i.e., where no sufficient damping resistance is present. Such circuits contain relatively small inductances and thus the frequency of the transient oscillation is extremely high.

Very large networks of high voltage may have such a great capacitance of the transmission lines and the inductance of the transformers, that their natural frequency approaches the system frequency. This may happen due to line-to-ground fault and would lead to significant overvoltages of fundamental frequency. More generally, it is certain that, for every alteration in the circuits and/or variation of the load, the capacitances and inductances of an actual network change substantially. In practice it is found, therefore, that the resonance during the transients in power systems, occur if and when the natural system frequency is equal or close to one of the generalized frequencies. During the resonance some harmonic voltages or currents, inherent in the source or in the load, might be amplified and cause dangerous overvoltages and/or overcurrents.

It should be noted that in symmetrical three-phase systems all higher harmonics of a mode divisible by 2 or 3 vanish, the fifth and seventh harmonics are the most significant ones due to the generated voltages and the eleventh and the thirteenth are sometimes noticeable due to the load containing electronic converters.

We shall consider the transients in the *RLC* resonant circuit in more detail assuming that the resistances in such circuits are relatively low, so that $R \ll Z_C$, where $Z_C = \sqrt{L/C}$, which is called a *natural or characteristic* impedance (or resistance); it is also sometimes called a *surge impedance.*

(a) *Switching on a resonant* RLC *circuit to an a.c. source*

The natural response of the current in such a circuit, Fig. 2.53 (see sections 1.62 and 2.72) may be written as

$$i_n = I_n e^{-\alpha t} \sin(\omega_n t + \beta). \tag{2.62}$$

The natural response of the capacitance voltage will then be

$$v_{C,n} = \frac{1}{C}\int i_n dt = I_n \frac{e^{-\alpha t}}{C(\alpha^2 + \omega_n^2)}[-\alpha \sin(\omega_n t + \beta) - \omega_n \cos(\omega_n t + \beta)],$$

upon simplification, combining the sine and cosine terms to a common sine term with the phase angle $(90^\circ + \delta)$,

$$v_{C,n} = V_{C,n} e^{-\alpha t} \sin[\omega_n t + \beta - (90^\circ + \delta)], \tag{2.63}$$

where

$$V_{C,n} = I_n \sqrt{\frac{L}{C}}, \tag{2.64a}$$

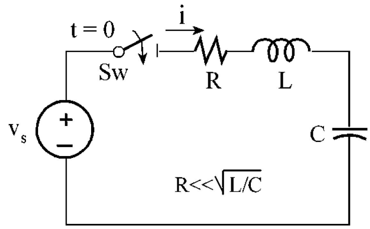

Figure 2.53 A resonant *RLC* circuit.

$$\delta = \tan^{-1}\left(\frac{\alpha}{\omega_n}\right), \tag{2.64b}$$

and

$$(\alpha^2 + \omega_n^2) = \omega_d^2 + \frac{1}{LC}. \tag{2.65}$$

The natural response of the inductive voltage may be found simply by differentiation:

$$v_{L,n} = L\frac{di_{i,n}}{dt} = LI_n e^{-\alpha t}\left[-\alpha \sin(\omega_n t + \beta) + \omega_n \cos(\omega_n t + \beta)\right],$$

or after simplification, as was previously done, we obtain

$$v_{L,n} = I_n\sqrt{\frac{L}{C}}e^{-\alpha t}\sin[\omega_n t + \beta + (90^\circ + \delta)]. \tag{2.66}$$

It is worthwhile to note here that by observing equation 2.63 and equation 2.66 we realize that $v_{C,n}$ is lagging slightly more and $v_{L,n}$ is leading slightly more than 90° with respect to the current. This is in contrast to the steady-state operation of the *RLC* circuit, in which the inductive and capacitive voltages are displaced by exactly $\pm 90^\circ$ with respect to the current. The difference, which is expressed by the angle δ, is due to the exponential damping. This angle is analytically given by equation 2.64b and indicates the *deviation* of the *displacement angle* between the current and the inductive/capacitance voltage from 90°. Since the resistance of the resonant circuits is relatively small, we may approximate

$$\omega_n = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \cong \frac{1}{\sqrt{LC}} \quad \text{and} \quad \tan\delta \cong R/2\sqrt{L/C}. \tag{2.67}$$

For most of the parts of the power system networks resistance R is much smaller than the natural impedance $\sqrt{L/C}$ so that the angle δ is usually small and can be neglected.

By switching the *RLC* circuit, Fig. 2.53, to the voltage source

$$v_s = V_m \sin(\omega t + \psi_v) \tag{2.68}$$

the steady-state current will be

$$i_f = I_f \sin(\omega t + \psi_i), \tag{2.69}$$

the amplitude of which is

$$I_f = \frac{V_m}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}, \tag{2.70}$$

and the phase angle is

$$\psi_i = \psi_v - \varphi, \quad \varphi = \tan^{-1}\frac{\omega L - 1/\omega C}{R}. \tag{2.71}$$

The steady-state capacitance voltage is

$$v_{C,f} = \frac{I_f}{\omega C}\sin(\omega t + \psi_i - 90^\circ). \tag{2.72}$$

For the termination of the arbitrary constant, β, we shall solve a set of equations, written for $i_n(0)$ and $v_{C,n}(0)$ in the form:

$$i_n(0) = i(0) - i_f(0)$$
$$v_{C,n}(0) = v_C(0) - v_{C,f}(0).$$

Since the independent initial conditions for current and capacitance voltage are zero and the initial values of the forced current and capacitance voltage are $i_f(0) = I_f \sin\psi_i$, and $v_{C,f}(0) = -(I_f/\omega C)\cos\psi_i$ we have

$$\begin{aligned} I_n \sin\beta &= 0 - I_f \sin\psi_i \\ -I_n\sqrt{\frac{L}{C}}\cos\beta &= 0 + \frac{I_f}{\omega C}\cos\psi_i. \end{aligned} \tag{2.73}$$

The simultaneous solution of these two equations, by dividing the first one by the second one, results in

$$\tan\beta = \frac{\omega}{\omega_n}\tan\psi_i. \tag{2.74}$$

Whereby the phase angle β of the natural current can be determined and, with its value, the first equation in 2.73 give the initial amplitude of the transient current

$$I_n = -I_f\frac{\sin\psi_i}{\sin\beta} = -I_f\sin\psi_i\sqrt{1 + \frac{1}{\tan^2\psi_i}\left(\frac{\omega_n}{\omega}\right)^2},$$

or

$$I_n = -I_f\sqrt{\sin^2\psi_i + \left(\frac{\omega_n}{\omega}\right)^2\cos^2\psi_i}. \tag{2.75}$$

The initial amplitude of the transient capacitance voltage can also be found with equation 2.64(a)

$$V_{C,n} = I_n\sqrt{\frac{L}{C}} = -I_f\sqrt{\frac{L}{C}}\sqrt{\sin^2\psi_i + \left(\frac{\omega_n}{\omega}\right)^2\cos^2\psi_i},$$

or, with the expression $V_{C,f} = I_f/\omega C$ (see equation 2.72), we may obtain

$$V_{C,n} = -V_{C,f}\sqrt{\left(\frac{\omega}{\omega_n}\right)^2 \sin^2\psi_i + \cos^2\psi_i}. \tag{2.76}$$

From the obtained equations 2.74, 2.75 and 2.76 we can understand that the phase angle, β, and the amplitudes, I_n and $V_{C,n}$, of the transient current and capacitance voltage depend on two parameters, namely, the instant of switching, given by the phase angle ψ_i of the steady-state current and the ratio of the natural, ω_n to the a.c. source frequency, ω. Using the obtained results let us now discuss a couple of practical cases.

(b) *Resonance at the fundamental (first) harmonic*

In this case, with $\omega_n = \omega$ the above relationships become very simple. According to equation 2.74

$$\tan\beta = \tan\psi_i \quad \text{and} \quad \beta = \psi_i, \tag{2.77}$$

i.e., the initial phase angles of the natural and forced currents are equal. According to equations 2.75 and 2.76

$$I_n = -I_f \quad \text{and} \quad V_{C,n} = -V_{C,f}, \tag{2.78}$$

which means that the amplitudes of the natural current and capacitance voltages are negatively equal (in other words they are in the opposite phase) to their steady-state values. Since the frequencies ω and ω_n are equal, we can combine the sine function of the forced response (the steady-state value) and the natural response, and therefore the complete response becomes

$$\begin{aligned} i &= I_f(1 - e^{-\alpha t})\sin(\omega t + \psi_i) \\ v_C &= V_{C,f}(1 - e^{-\alpha t})\sin(\omega t + \psi_i - 90^\circ). \end{aligned} \tag{2.79}$$

The plot of the transient current (equation 2.79) is shown in Fig. 2.54. As can be seen, in a resonant circuit the current along with the voltages reach their maximal values during transients after a period of 3–5 times the time constant of the exponential term. Since the time constant here is relatively low, due to

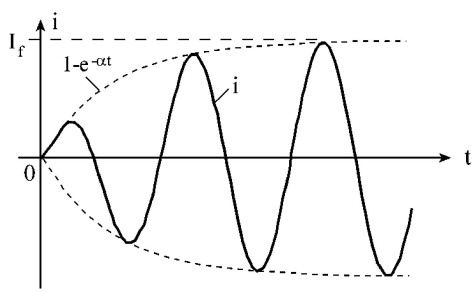

Figure 2.54 A current plot after switching in a resonant circuit.

the small resistances of the resonant circuits the current and voltages reach their final values after very many cycles. It should be noted that these values of current and voltages at resonance here, are much larger than in a regular operation.

(c) *Frequency deviation in resonant circuits*

In this case, equations 2.77 and 2.78 can still be considered as approximately true. However since the natural and the system frequencies are only approximately (and not exactly) equal, we can no longer combine the natural and steady-state harmonic functions and the complete response will be of the form

$$\begin{aligned} i &= I_f[\sin(\omega t + \psi_i) - e^{-\alpha t}\sin(\omega_n t + \psi_i)] \\ v_C &= V_{C,f}[\sin(\omega t + \psi_i - 90^\circ) - e^{-\alpha t}\sin(\omega_n t + \psi_i - 90^\circ)]. \end{aligned} \tag{2.80}$$

Since the natural current/capacitance voltage now has a slightly different frequency from the steady-state current/capacitance voltage, they will be displaced in time soon after the switching instant. Therefore, they will no longer subtract as in equation 2.79, but will gradually shift into such a position that they will either add to each other or subtract, as shown in Fig. 2.55. As can be seen with increasing time the addition and subtraction of the two components occur periodically, so that *beats* of the total current/voltage appear. These beats then diminish gradually and are decayed after the period of the 3–5 time constant, τ. It should also be noted that, as seen in Fig. 2.55, the current/capacitance voltage soon after switching rises up to nearly twice its large final value; so that in this case switching the circuit to an a.c. supply will be more dangerous than in the case of resonance proper. By combining the trigonometric functions in equation 2.80 (after omitting the damping factor $e^{-\alpha t}$ and the phase angle ψ_i,

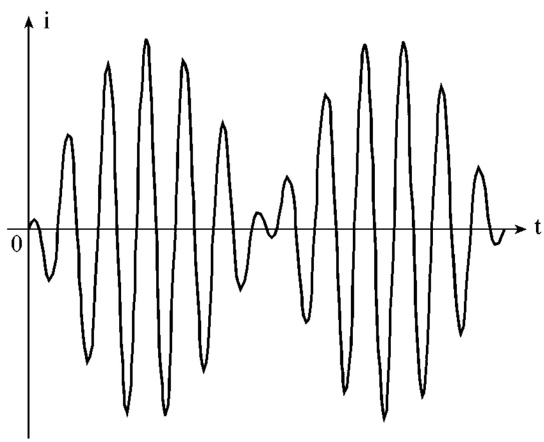

Figure 2.55 A plot of the current when the natural and fundamental frequencies are approximately equal.

i.e., supposing that the switching occurs at $\psi_i = 0$) we may obtain

$$
\begin{aligned}
i &= 2I_f \sin\frac{\omega - \omega_n}{2} t \cdot \cos\frac{\omega + \omega_n}{2} t \\
v_C &= -2V_{C,f} \sin\frac{\omega - \omega_n}{2} t \cdot \cos\frac{\omega + \omega_n}{2} t.
\end{aligned}
\tag{2.81}
$$

These expressions represent, however, the circuit behavior only a short time after the switching-on, as long as the damping effect is small. In accordance with the above expressions, and by observing the current change in Fig. 2.55, we can conclude that two oscillations are presented in the above current curve. One is a rapid oscillation of high frequency, which is a mean value of ω and ω_n, and the second one is a sinusoidal variation of the amplitude of a much lower frequency, which is the difference between ω and ω_n, and represents the beat frequency.

(d) *Resonance at multiple frequencies*

In this case, the transient phenomena are largely dependent on the instant of switching, i.e. on the angle ψ_i. Two extreme cases are of particular interest: 1) $\psi_i = 0$ and 2) $\psi_i = 90^\circ$.

If the switching occurs the moment at which $\psi_i = 90^\circ$, i.e., at the instant at which the steady-state current is maximal, while the capacitance voltage passes through zero, then the natural phase angle (equation 2.74) will also be

$$\beta = 90^\circ.$$

Then, see equations 2.75 and 2.76,

$$I_n = -I_f \quad \text{and} \quad V_{C,n} = -\frac{\omega}{\omega_n} V_{C,f}, \tag{2.82}$$

and the total current and voltage become

$$
\begin{aligned}
i &= I_f[\sin(\omega t + 90^\circ) - e^{-\alpha t}\sin(\omega_n t + 90^\circ)] \\
v_C &= V_{C,f}\left(\sin\omega t - \frac{\omega}{\omega_n} e^{-\alpha t}\sin\omega_n t\right).
\end{aligned}
\tag{2.83}
$$

For the cases in which the natural frequency ω_n is higher than the forced frequency ω, the current rises, at the instant half a cycle after the instant of switching, to almost twice the amount of the steady-state current, which is shown in Fig. 2.56. The excess capacitance voltage in this case, however, is relatively small due to the small ratio of the frequencies in the second term of the capacitance voltage, which lowers its natural response.

If the switching occurs, the moment at which $\psi_i = 0$, i.e., at the instant the steady-state current passes through zero and the capacitance voltage reaches its maximum, the natural current phase angle (equation 2.74) will also be zero

$$\beta = 0,$$

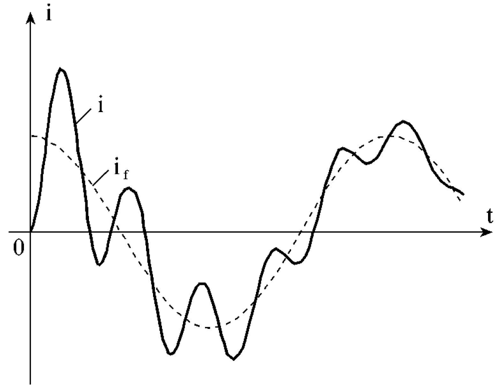

Figure 2.56 A plot of the current when $\psi_i = 90°$.

and, in accordance with equations 2.75 and 2.76,

$$I_n = -\frac{\omega_n}{\omega} I_f \quad \text{and} \quad V_{C,n} = -V_{C,f}. \tag{2.84}$$

Now the total current waveform and the total capacitance voltage waveform become

$$i = I_f \left(\sin \omega t - e^{-\alpha t} \frac{\omega_n}{\omega} \sin \omega_n t \right)$$

$$v_C = V_{C,f} [\sin(\omega t - 90°) - e^{-\alpha t} \sin(\omega_n t - 90°)],$$

which is almost inversely what is was in the former case. The plots of both the current and voltage are shown in Fig. 2.57. As can be seen, half a natural period after the switching moment the capacitance voltage is nearly doubled. The total current in this case may reach enormously high values due to the large ratio of

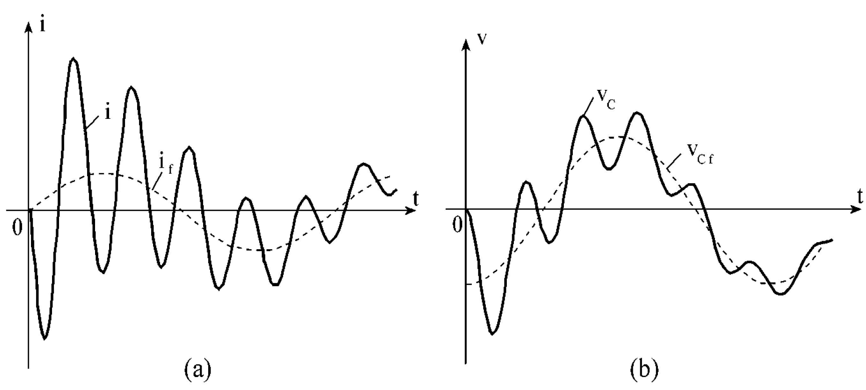

Figure 2.57 The plots of the current (a) and the capacitance voltage (b) for the case where $\psi_i = 0$.

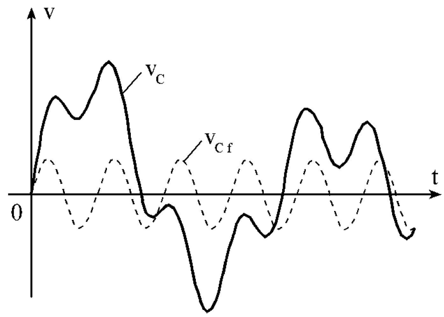

Figure 2.58 The plot of the capacitance voltage for $\psi_i = 90^\circ$ and $\omega_n < \omega$.

the frequencies, which determine the natural component initial amplitude, when the natural frequency is higher than the system frequency.

For cases in which the natural frequency is lower than the force frequency, the transient phenomenon is significantly changed. Hence, here the most dangerous case is where the switching on occurs at the initial phase $\psi_i = 90^\circ$ and the capacitance voltage (equation 2.83) rises to almost as much as the ratio of the frequencies ω/ω_n times the amount of its final value, Fig. 2.58.

In conclusion, as was previously mentioned, some parts of power system networks, particularly the windings of electrical machines and transformers, predominantly possess inductances, while other parts, particularly underground cables and high-voltage overhead lines, predominantly possess capacitances. Hence, the possibility of resonant conditions always exists, and by switching-on in such circuits the resonant phenomena may appear. The magnitude of the transient currents and voltages is dependent on the natural frequency and its ratio to the forced frequency as well as the instant of switching. Since it can never be predicted at what exact instant the switching occurs, we must always expect and analyze the most unfavorable cases.

2.7.4 Switching-off in *RLC* circuits

We have seen in sections 1.74 and 2.3.3 that very high voltages may develop if a current is suddenly interrupted. However, the presence of capacitances, which are associated with all electric circuit elements, as shown in Fig. 2.14, may change the transient behavior of such circuits. Thus, the raised voltages charge all these capacitances and thereby the actual voltages will be lower. To show this, consider a very simple approximation of such an arrangement by the parallel connection of L and C, as shown in Fig. 2.59. After instantaneously opening the switch, the current of the inductance flows through the capacitance charging it up to the voltage of V_C. The magnetic energy stored in the inductance, $W_m = \frac{1}{2}LI_L^2$, where I_L is the current through the inductance prior to switching, will be changed into the electric energy of the capacitance $W_e = \frac{1}{2}CV_C^2$. Since both amounts of energy, at the first moment after switching,

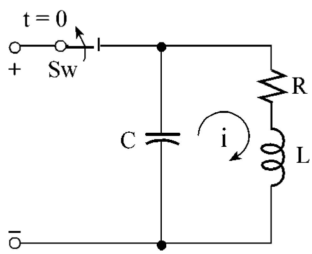

Figure 2.59 A circuit in which a coil with a parallel capacitance is disconnected from the voltage source.

are equal (by neglecting the energy dissipation due to low resistances), we have

$$\frac{CV_C^2}{2} = \frac{LI_L^2}{2},$$

and the maximal transient overvoltage appearing across the switch is

$$V_C = \sqrt{\frac{L}{C}}\, I_L. \tag{2.85}$$

Recalling from section 1.74, Fig. 1.28(a), that the overvoltage, by interrupting the coil of 0.1 H with the current of 5 A, was 50 kV, we can now estimate it more precisely. Assuming that the equivalent capacitance of the coil and the connecting cable is $C = 6$ nF, and is connected in parallel to the coil, as shown in Fig. 2.59,

$$V_C = \sqrt{\frac{0.1}{6\cdot 10^{-9}}}\, 5 = 20.4 \text{ kV}.$$

Hence, for reducing the overvoltages, capacitances should be used. Subsequently, by connecting the additional condensers of large magnitudes, the overvoltage might be reduced to moderate values.

For a more exact calculation, we shall now also consider the circuit resistances. By using the results obtained in the previous section, we shall take into consideration that when the circuit is disconnected, the forced response is absent. However, the independent initial values are not zero, hence the initial value of the transient (natural) current through the inductive branch is found as

$$I_0 = i_L(0) - 0, \tag{2.86a}$$

and similarly for the capacitance voltage

$$V_0 = v_C(0) - 0. \tag{2.86b}$$

With the current derivatives

$$\left.\frac{di_L}{dt}\right|_{t=0} = (1/L)v_L(0) = \frac{V_0 - Ri_L(0)}{L} \quad \text{and} \quad \left.\frac{di_{L,n}}{dt}\right|_{t=0} = \left.\frac{di_L}{dt}\right|_{t=0} - 0,$$

we have two equations for determining two integration constants

$$I_n \sin\beta = I_0, \tag{2.87a}$$

$$I_n(-\alpha\sin\beta + \omega_n\cos\beta) = \frac{V_0 - RI_0}{L}. \tag{2.87b}$$

By dividing equation 2.87b by equation 2.87a, and substituting $R/2L$ for α and

$$\sqrt{1/LC - (R/2L)^2}$$

for ω_n upon simplification we obtain

$$\tan\beta = \frac{\sqrt{L/C - (R/2)^2}}{V_0/I_0 - R/2}. \tag{2.88a}$$

For circuits having small resistances, namely if $R/2 \ll \sqrt{L/C}$, the above equation becomes

$$\tan\beta = \frac{\sqrt{L/C}}{V_0/I_0 - R/2}. \tag{2.88b}$$

Using equation 2.88 with equation 2.87a, we may obtain (the details of this computation are left for the reader to convince himself of the obtained results)

$$I_n = \sqrt{I_0^2 + \frac{(V_0 - RI_0/2)^2}{L/C - (R/2)^2}} \cong \sqrt{I_0^2 + \frac{C}{L}(V_0 - RI_0/2)^2}, \tag{2.89}$$

and with equation 2.64a

$$V_{C,n} = \sqrt{\frac{L}{C}}\,I_n = \sqrt{\frac{L}{C}I_0^2 + \frac{(V_0 - RI_0/2)^2}{1 - (C/L)(R/2)^2}} \cong \sqrt{\frac{L}{C}I_0^2 + \left(V_0 - \frac{1}{2}RI_0\right)^2} \tag{2.90}$$

The above relationships express, in an exact and approximate way, the amplitudes of transient oscillations of the current and capacitance voltage. They are valid for switching-off in any d.c. as well as in any a.c. circuit.

Example 2.32

Assume that, for reducing the overvoltage, which arises across the switch, by disconnecting the previously considered coil of 0.1 H inductance and 20 Ω inner resistance, the additional capacitance of 0.1 μF is connected in parallel to the coil, Fig. 2.59. Find the transient voltage across the switch, if the applied voltage is 100 V dc.

Solution

We shall first find the current phase angle. Since $(R/2 = 10) \ll (\sqrt{L/C} = 10^3)$, using equation 2.88b and taking into consideration that $V_0 = V_s$ and $I_0 = V_s/R$, we have

$$\tan\beta = \frac{\sqrt{L/C}}{R - R/2} = \frac{\sqrt{1/LC}\,L}{R/2} = \frac{\omega_n}{\alpha}.$$

The damping coefficient and the natural frequency are

$$\alpha = \frac{R}{2L} = \frac{20}{2\cdot 0.1} = 100\ \mathrm{s}^{-1}, \quad \omega_n = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.1\cdot 0.1\cdot 10^{-6}}} = 10^4\,\frac{\mathrm{rad}}{\mathrm{s}},$$

therefore,

$$\beta = \tan^{-1}\frac{\omega_n}{\alpha} = \tan^{-1}100 = 89.4^\circ.$$

In accordance with the approximate expression (equation 2.90), we have

$$V_{C,n} = \sqrt{\frac{L}{C}\frac{V_s^2}{R^2} - \left(V_s - \frac{1}{2}V_s\right)^2} \cong V_s\frac{\sqrt{L/C}}{R} = 100\,\frac{1000}{20} = 5\cdot 10^3\ \mathrm{V}.$$

(Note that this value is less than the previous estimation.) The capacitance voltage versus time (equation 2.63) therefore, is

$$v_{C,n}(t) = -V_{C,n}e^{-\alpha t}\sin(\omega t + \beta - \delta - 90^\circ) \cong -5e^{-100t}\sin(10^4 t - 2\delta)\ \mathrm{kV}.$$

Where δ is a *displacement angle* (equation 2.64b): $\delta = \tan^{-1}(\alpha/\omega_n) \cong 0.6^\circ$ (note that $\beta \cong 90^\circ - \delta = 89.4^\circ$ as calculated above). The negative sign of the capacitance voltage indicates the discharging process.

The voltage across the switch can now be found as the difference between the voltages of the source and the capacitance. Thus,

$$v_{sw} = V_s - v_C(t) = 100 + 5\cdot 10^3 e^{-100t}\sin(10^{4t} - 1.2^\circ)\ \mathrm{V},$$

which for $t = 0$ gives v_{sw} zero. Instead, $v_{sw}(0) = 100 + 5\cdot 10^3\sin(-1.2^\circ) \cong 0$.

The plot of v_{sw} is shown in Fig. 2.60 (the source voltage here is unproportionally enlarged relative to the transient voltage to clarify the relation between

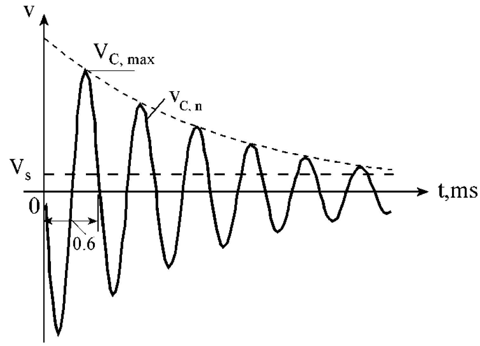

Figure 2.60 A plot of the voltage across the switch in the circuit of Fig. 2.59 after opening the switch.

these two voltages). As can be seen this voltage does not suddenly jump to its maximal value, but rises as a sinusoidal and reaches the peak after one-quarter of the natural period (which in this example is about 1.57 ms). Within this time, the contacts of the switch (circuit breaker) must have separated enough to avoid any sparking or an arc formation.

The circuit in Fig. 2.61(a) represents a very special resonant circuit, in which $R_1 = R_2 = \sqrt{L/C}$. As is known, the resonant frequency of such a circuit may be any frequency, i.e., the resonance conditions take place in this circuit, when it is connected to an a.c. source of any frequency. If such a circuit is interrupted, for instance by being switched off, the two currents i_C and i_L are always oppositely equal. In addition, since the time constants of each branch are equal ($\tau_L = L/R_1 = R_2C = \tau_C$), both currents decay equally, as shown in Fig. 2.61(b). Therefore, no current will flow through the switch when interrupted, providing its sparkless operation.

(a) *Interruptions in a resonant circuit fed from an a.c. source*

Finally, consider a resonant *RLC* circuit when disconnected from an a.c. source. The initial condition in such a circuit may be found from its steady-state operation prior to switching. Let the driving voltage be $v_s = V_m \sin(\omega t + \psi_v)$, then the current and the capacitance voltage (see, for example, the circuit in Fig. 2.59) are

$$\begin{aligned} i &= I_m \sin(\omega t + \psi_i) \\ v_C &= V_m \sin(\omega t + \psi_v), \end{aligned} \tag{2.91}$$

where

$$I_m = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} \quad \text{and} \quad \varphi = \tan^{-1}\frac{\omega L}{R} \quad (\psi_i = \psi_v - \varphi). \tag{2.92}$$

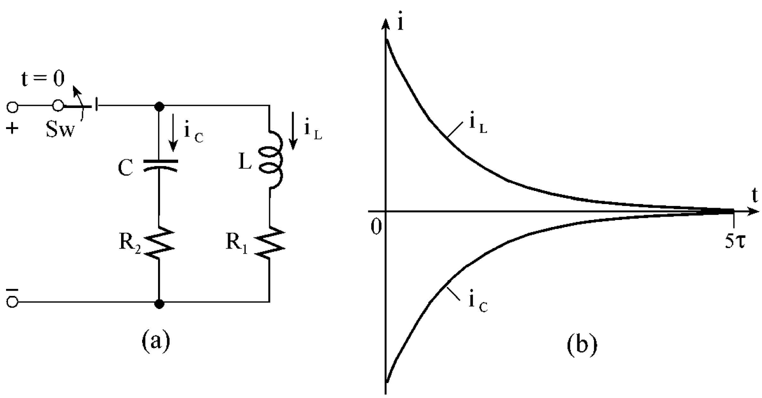

Figure 2.61 A special resonant circuit (a) and two transient currents after switching (b).

The initial conditions may now be found as

$$\begin{aligned} i(0) &= I_m \sin\psi_i = I_0 \\ v_C(0) &= V_m \sin\psi_v = V_0. \end{aligned} \tag{2.93}$$

Since the forced response in the switched-off circuit is zero, the initial values (equation 2.93) are used as the initial conditions for determining the integration constants in equation 2.87. Therefore, by substituting equation 2.93 in equations 2.88–2.90, and upon simplification and approximation for very small resistances, we obtain

$$\tan\beta = \sqrt{\frac{L}{C}}\frac{I_m \sin\psi_i}{V_m \sin\psi_v} = \frac{\omega_n}{\omega}\frac{\sin\psi_i}{\sin\psi_v \sin\varphi}, \tag{2.94}$$

where it is taking into account that the ratio

$$\frac{V_m}{I_m} = \frac{\omega L}{\sin\varphi},$$

and

$$I_n = I_m\sqrt{\sin^2\psi_i + \left(\frac{\omega}{\omega_n \sin\varphi}\right)^2 \sin\psi_v} \cong I_m\sqrt{\sin^2\psi_i + \left(\frac{\omega}{\omega_n}\right)^2 \cos^2\psi_i} \tag{2.95}$$

$$V_{C,n} = V_s\sqrt{\left(\frac{\omega_n \sin\varphi}{\omega}\right)^2 \sin^2\psi_i + \sin^2\psi_v} \cong V_s\sqrt{\left(\frac{\omega_n}{\omega}\right)^2 \sin^2\psi_i + \cos^2\psi_i}, \tag{2.96}$$

where the second approximation (the right hand term) is done for $\varphi \cong 90^\circ$, i.e., $\sin\varphi \cong 1$ and $\sin\psi_v = \sin(\psi_i + 90^\circ) = \cos\psi_i$.

As can be seen from the above expressions, the natural current and capacitance voltage magnitudes are dependent on the phase displacement angle φ (or the power factor of the circuit), on the ratio of the natural frequency ω_n and the system frequency ω, and on the current phase angle ψ_i, which is given by the instant of switching. Therefore, in *RLC* circuits with a natural frequency higher than the system frequency (which usually happens in power networks), the transient voltage across the capacitance may attain its maximal value, which is as large as the ratio of the frequencies. This occurs in highly inductive circuits with $\varphi \cong 90^\circ$ due to the interruption of the current while passing through its amplitude, i.e., when $\psi_i = 90^\circ$. However, the switching-off practically occurs at the zero passage of the current, i.e., when $\psi_i = 0$. In this very favorable case the transient voltage amplitude, with equation 2.96, will now be equal to the voltage before the interruption. The voltage across the switch contacts reaches a maximum, which, with small damping, is twice the value of the source amplitude

$$v_{sw,\max} = 2V_s,$$

and then decays gradually. The initial angle of the transient response in this case, with equation 2.94, will be

$$\beta \cong 0.$$

Suppose that the circuit in Fig. 2.59, which has been analyzed, represents, for instance, the interruption at the sending end of the underground cable or overhead line having a significant capacitance to earth, while the circuit in Fig. 2.62 may represent the interruption at the receiving end of such a cable or overhead line. One of such interruptions could be a short-circuit fault and its following switching-off. The analysis of this circuit is rather similar to the previous one. The difference, though, is that here the initial capacitance voltage is zero and the forced response is present. Therefore, the initial conditions for the transient (natural) response will be

$$\begin{aligned} i_{L,n}(0) &= i_L(0) - i_f(0) = I_0 \\ v_{C,n}(0) &= 0 - v_{C,f}(0) = V_0, \end{aligned} \tag{2.97}$$

and for the current derivative, we have

$$\left.\frac{di_{L,n}}{dt}\right|_{t=0} = \frac{1}{L} v_L(0) - \left.\frac{di_f}{dt}\right|_{t=0} = \frac{v_s(0) - Ri_L(0)}{L} - \left.\frac{di_f}{dt}\right|_{t=0}.$$

The current through the inductance prior to switching might be found as a short-circuit current

$$i_{sc} = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} \sin(\omega t + \psi_v - \varphi_{sc}), \tag{2.98a}$$

where

$$\varphi_{sc} = \tan^{-1}\frac{\omega L}{R}, \tag{2.98b}$$

and ψ_v is a voltage source phase angle at switching instant $t = 0$.

Since switching in a.c. circuits usually occurs at the moment when the current passes zero, we shall assume that $I_0 = 0$ and $\psi_i(0) = \psi_v - \varphi_{sc} = 0$ (or the voltage phase angle at the switching moment is equal to the short-circuit phase angle).

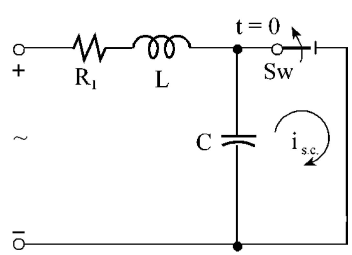

Figure 2.62 An *RLC* circuit, which arises after having been short-circuited.

Thus,

$$I_0 = 0 \quad \text{and} \quad \psi_v = \varphi_{sc}. \tag{2.99}$$

The forced response of the current and the capacitance voltage are found in the circuit after the disconnection of the short-circuit current, i.e. in the open-circuit, in which the cable or the line is disconnected (no load operation). In this regime the entire circuit is highly capacitive ($1/\omega C \gg \omega L$). Therefore, we have

$$I_f \cong V_m \omega C \quad \text{and} \quad \varphi_f \cong -90^\circ. \tag{2.100}$$

Now, the two equations for finding the integration constant are

$$\begin{aligned} I_n \sin\beta &= 0 - i_f(0) \cong -I_f \sin(\psi_v + 90^\circ) = -I_f \cos\psi_v \\ I_n(-\alpha \sin\beta + \omega_n \cos\beta) &= \frac{V_m \sin\psi_v}{L} - \omega_n I_f \sin\psi_v, \end{aligned} \tag{2.101}$$

for which the solution is

$$\tan\beta = \frac{-\omega_n \omega}{(\omega_n^2 + \omega^2)\tan\psi_v + \alpha\omega}. \tag{2.102}$$

Since in power system circuits the natural frequency usually is much higher than the system frequency, the above expression might be simplified for low resistive circuits to

$$\tan\beta \cong -\frac{\omega}{\omega_n \tan\psi_v}. \tag{2.102a}$$

Thus, the oscillation amplitudes of the natural current and capacitance voltage are

$$I_n = \frac{I_f \cos\psi_v}{\sin\beta} = I_f\sqrt{1 + \cot^2\beta} \cong I_f \frac{\omega_n}{\omega} \sin\psi_v, \tag{2.103}$$

$$V_{C,n} = \sqrt{\frac{L}{C}}\, I_n \cong \sqrt{\frac{L}{C}}\, I_f \frac{\omega_n}{\omega} \sin\psi_v = V_m \sin\psi_v, \tag{2.104}$$

where $I_f = \omega C V_m$. Let us illustrate this case in the following example.

Example 2.33

Determine the maximum voltage across the breaker and the transient current after it opens, disconnecting the system's short-circuit fault, as shown in Fig. 2.63(a). The system is fed by an underground cable, through a reactor (whose purpose is to reduce the short-circuit current). The parameters of the reactor and the cable are $L_1 = 6.13$ mH, $R_0 = 0.2\ \Omega$/km, $L_0 = 0.318$ mH/km and $C_0 = 0.267\ \mu$F/km. The system voltage is 10 kV (rms) at 60 Hz and the fault occurs at 13.5 km from the sending end. Suppose that the arc, which appears

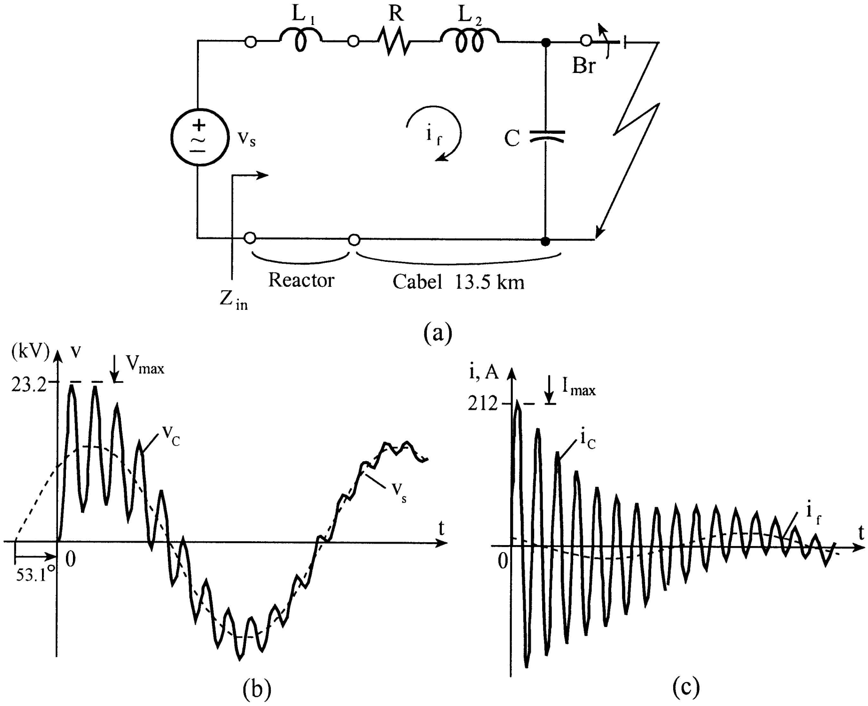

Figure 2.63 A given circuit for Example 2.33 (a) and the plots of the capacitance voltage (b) and transient current (c).

at the first moment of switching, is extinguished at a current pause, i.e. at its zero value.

Solution

The total circuit parameters are: $L = L_1 + L_0 l = 6.13 + 0.318 \cdot 13.5 = 10.4$ mH, $C = C_0 l = 0.267 \cdot 13.5 = 3.6$ μF, $R = R_0 l = 0.2 \cdot 13.5 = 2.7\ \Omega$.

The natural frequency and damping coefficient are

$$\omega_n \cong \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10.4 \cdot 10^{-3} \cdot 3.6 \cdot 10^{-9}}} = 5.17 \cdot 10^3 \text{ rad/s}$$

$$\alpha = \frac{R}{2L} = \frac{2.7}{2 \cdot 10.4 \cdot 10^{-3}} = 130 \text{ s}^{-1},$$

and the characteristic impedance is

$$R_c = \sqrt{\frac{L}{C}} = \sqrt{\frac{10.4 \cdot 10^{-3}}{3.6 \cdot 10^{-6}}} = 53.7\ \Omega.$$

The forced current amplitude and phase angle are

$$I_f = \frac{V_s}{Z_{in}} = \frac{10\sqrt{2}}{733} = 13.6\sqrt{2}\ \mathsf{A} \quad \text{and} \quad \varphi_{in} \cong -90^\circ,$$

where

$$\begin{aligned} Z_{in} &= \sqrt{R^2 + (\omega L - 1/\omega C)^2} \\ &= \sqrt{2.7^2 + (377 \cdot 10.4 \cdot 10^{-3} - 1/377 \cdot 3.6 \cdot 10^{-6})^2} \cong 733\ \Omega. \end{aligned}$$

Since the short current switching-off occurs at $\psi_i = 0$, the forced voltage initial angle should be (2.99)

$$\psi_v = \varphi_{sc} = \tan^{-1}\frac{\omega L}{R} = \tan^{-1}\frac{3.9}{2.7} = 53.1^\circ,$$

and the forced current phase angle will be

$$\psi_i = \psi_v - \varphi_{in} = 53.1^\circ - (-90^\circ) = 143.1^\circ.$$

Now we can find the phase angle of the natural current (equation 2.102a)

$$\tan\beta \cong \frac{-\omega}{\omega_n \tan\psi_v} = \frac{-377}{5.17 \cdot 10^3 \tan 53.1^\circ} = -54.7 \cdot 10^{-3} \quad \text{and} \quad \beta = -3.13^\circ.$$

The magnitude of the transient capacitance voltage is (equation 2.104)

$$V_{C,n} \cong V_m \sin\psi_v = 10\sqrt{2}\sin 53.1^\circ = 8.0\sqrt{2}\ \mathsf{kV},$$

and the complete capacitance voltage is

$$v_C(t) = 10\sqrt{2}\sin(\omega t + 53.1^\circ) + 8.0\sqrt{2}e^{-130t}\sin(5.17 \cdot 10^3 t - 93.1^\circ)\ \mathsf{kV}.$$

The transient current oscillation amplitude (equation 2.103) is

$$I_n \cong I_f \frac{\omega_n}{\omega} \sin\psi_v = 13.6\sqrt{2}\,\frac{5.17 \cdot 10^3}{377}\sin 53.1^\circ = 149.2\sqrt{2}\ \mathsf{A},$$

and the complete current response is

$$i(t) = 13.6\sqrt{2}\sin(\omega t + 143.1^\circ) + 149.2\sqrt{2}e^{-130t}\sin(5.17 \cdot 10^3 t - 3.13^\circ)\ \mathsf{A}.$$

Checking for $t = 0$ yields

$$i(0) = 13.6\sqrt{2}\sin(143.1^\circ) + 149.2\sqrt{2}\sin(-3.13^\circ) \cong 0$$

(since the switching occurs at the zero current), and

$$v_C(0) = 10\sqrt{2}\sin 53.1^\circ + 8.0\sqrt{2}\sin(-93.1^\circ) \cong 0$$

(since the cable was short-circuited prior to switching).

The voltage across the breaker is equal to the capacitance voltage and its maximum will occur at the moment when the forced response reaches its first maximum and the natural response is positive. Thus,

$$t_{\max,v} = \frac{(90^\circ - 53.1^\circ)/57.3^\circ}{\omega} = \frac{0.644}{377} \cong 1.71 \text{ ms},$$

(note that 1 rad = 57.3°) and the maximum voltage is

$$V_{sw,\max} \cong 10\sqrt{2} + 8.0\sqrt{2}e^{-130t_{\max,v}} = 16.4\sqrt{2} = 23.2 \text{ kV}.$$

The current maximum will occur at the moment when the natural response reaches its first maximum, i.e., at the time

$$t_{\max,i} = \frac{(90^\circ + 3.13^\circ)/57.3^\circ}{5.17 \cdot 10^{-3}} = 0.314 \text{ ms},$$

and the maximum current is

$$I_{\max} \cong [13.6 \sin(\omega t_{\max,i} + 143.1^\circ) + 149.2 \cdot e^{-130t_{\max,i}})\sqrt{2} = 212 \text{ A}.$$

The plots of the capacitance voltage and the current are shown in Fig. 2.63(b) and (c).

Chapter #3

TRANSIENT ANALYSIS USING THE LAPLACE TRANSFORM TECHNIQUES

3.1 INTRODUCTION

In the introductory courses of circuit analysis the transient response is usually examined for relatively simple circuits of one or two energy storage elements. This analysis is based on general (or classical) techniques, involves writing the differential equations for the network, and proceeds to use them to obtain the differential equation in terms of one variable. Then the complete solution, including the natural and forced responses, has to be obtained. The tedium and complexity of using this technique is in determining the initial conditions of the unknown variables and their derivatives and then evaluating the arbitrary constants by utilizing those initial conditions. This procedure usually requires a great amount of work, which increases with the complexity of the network. Therefore, we now focus our attention on more effective methods of transient analysis.

A simplification of solving different problems can be achieved by using mathematical transformation. We are already familiar with one kind of mathematical transformation: the phasor transform technique, which allows simplifying the solution of the circuit steady-state response to sinusoidal sources. As we have seen, this very useful technique transforms the trigonometrical equations describing a circuit in the time domain into the algebraic equations in the frequency domain. Then the solution for the desirable variable (being actually manipulated by complex numbers) is transformed back to the time domain.

In this chapter a very powerful tool for the transient analysis of circuits, i.e., the *Laplace transform techniques*, will be introduced. This method enables us to convert the set of integro-differential equations describing a circuit in its transient behavior in the time domain to the set of linear algebraic equations in the complex frequency domain. Then using an algebraic operation, one may solve them for the variables of interest. Finally, with the help of the inverse transform, the desired solution can be expressed in terms of time. The paramount benefit of applying the Laplace transform to circuit analysis is in "automatically" taking the initial conditions into account: they appear when a derivative or integral is transformed.

Moreover, the concept of the frequency-domain equivalent circuit, based on the Laplace transform analysis, will be introduced. These circuits can be analyzed using techniques such as nodal and mesh analysis, Thévenin's and Norton's theorems, source transformations and so on, as described in earlier chapters.

So, the transform method in general can be represented by the expression

$$f(t) \leftrightarrow F(s),$$

which shows the one-to-one correspondence between the time-domain function $f(t)$ and its frequency domain transform $F(s)$, where $s = \sigma + j\omega$ is the *complex frequency*.

3.2 DEFINITION OF THE LAPLACE TRANSFORM

The so called *two-sided or bilateral Laplace transform* of $F(t)$ is defined as(*)

$$F(s) = \int_{-\infty}^{\infty} e^{-st} f(t)dt. \tag{3.1}$$

In circuit analysis problems the forcing and response functions do not usually exist endlessly in time, but rather they are initiated at some specific instant of time selected as $t = 0$. Thus, such functions that do not exist for $t < 0$ can be described with the help of unit step functions as $f(t)u(t)$ (see sections 2.5 and 3.3.1). For these functions the Laplace transform defining integral is taken with the lower limit at $t = 0_-$(**):

$$F(s) = \int_{-\infty}^{\infty} e^{-st} f(t)u(t)dt = \int_{0_-}^{\infty} e^{-st} f(t)dt. \tag{3.2}$$

The latter integral defines the *one-sided or unilateral Laplace transform*, or simply the Laplace transform of $f(t)$. The lower limit $t = 0_-$ (as distinguished from $t = 0$ or $t = 0_+$) in a one-sided Laplace transform is taken in order to include the effect of any discontinuity at $t = 0$, such as an impulse function and independent initial conditions such as currents in inductances $i_L(0_-)$ and voltages across capacitances $v_C(0_-)$.

The direct Laplace transform (3.2) may also be indicated as $\mathbf{L}\{f(t)\} = F(s)$ so that $\mathbf{L}$ implies the Laplace transform and means that once the integral in equation 3.2 has been evaluated, $f(t)$, which is a time domain function, is transformed to $F(s)$, which is a frequency domain function.

(*)The terms "two-sided" or bilateral are used to emphasize the fact that both positive and negative times are included in the range of integration.

(**)In transient analysis of electric circuits $t = 0_-$ is denoted as the time just before the switching action, and $t = 0_+$ as the time just after the switching action, representing radically different states of the circuit. Mathematically, $f(0_-)$ is the limit of $f(t)$ as t approaches zero through negative values ($t < 0$), or the limit from the right, and $f(0_+)$ is the limit as t approaches zero through positive values ($t > 0$), or the limit from the left.

However, the Laplace transform of a function $f(t)$ exists only if the integral (3.2) converges, or

$$\int_0^{\infty} |f(t)| e^{-\sigma_1 t} dt < \infty, \quad \text{where} \quad \sigma_1 = \text{Re}(s).$$

This means that if the magnitude of $f(t)$ is restricted, or increases not faster than the exponential, i.e.,

$$|f(t)| < M e^{\alpha t} \tag{3.3}$$

for all positive t the integral will converge and the region of convergence is given by $\alpha < \sigma_1 < \infty$, as shown in Fig. 3.1(a). A function $f(t)$ which fits this condition is shown in Fig. 3.1(b). The physically possible functions of time, or functions which are common in practice, always have a Laplace transform. (An example of the function, which does not satisfy conditions of equation 3.3, is e^{t^2}, but not t^n or n^t.)

If we have a transformation $\mathbf{L}\{f(t)\}$, then we must have an inverse transformation $\mathbf{L}^{-1}\{F(s)\} = f(t)$, which is mathematically defined as

$$f(t) = \frac{1}{2\pi j} \int_{c - j\infty}^{c + j\infty} e^{st} F(s) ds. \tag{3.4}$$

3.3 LAPLACE TRANSFORM OF SOME SIMPLE TIME FUNCTIONS

For a better understanding of Laplace transformations, we shall begin by using this technique to determine the Laplace transforms for those time functions most frequently encountered in circuit analysis.

3.3.1 Unit-step function

As was mentioned already in Chapter 2 (see section 2.5), very often in circuit analysis a switching action takes place at an instant that is defined as $t = 0$ (or

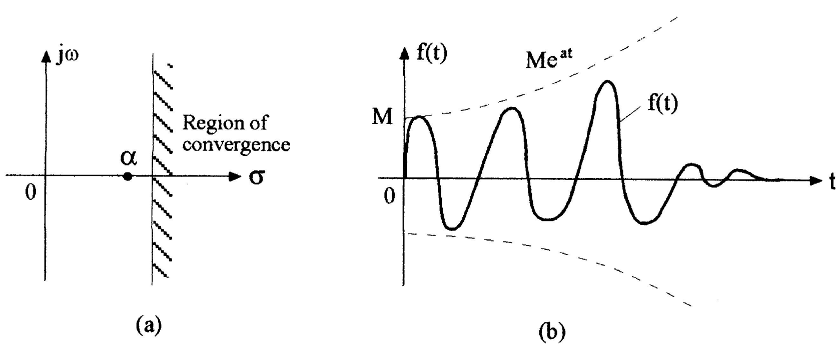

Figure 3.1 The illustration of the region of convergence in the Laplace transform definition (a); the function increasing (b).

$t_0 = 0$). We may indicate this action by using a *unit-step function*, which is

$$u(t) = \begin{cases} 0 & t < 0 \quad (t < t_0) \\ 1 & t > 0 \quad (t > t_0), \end{cases}$$

as shown in Fig. 3.2.

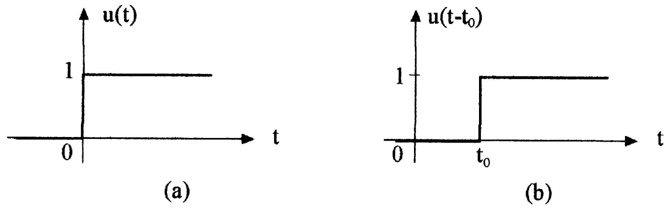

Figure 3.2 The unit-step function: $u(t)$ (a) and $u(t - t_0)$ (b).

Thus, the unit-step function is zero for all values of its argument (time), which are less than zero (or negative in the case of $(t - t_0)$) and which is unity for all positive values of its argument. By multiplying, for example, the voltage source value V_s by the unit-step function: $v(t) = V_s u(t)$, we indicate that this voltage source is connected to the network at the moment of time $t = 0$ (or if $v(t) = V_s u(t - t_0)$, at the time $t - t_0$).

In accordance with the Laplace transform definition (equation 3.2), we may write

$$\mathbf{L}\{u(t)\} = \int_{0_-}^{\infty} e^{-st} u(t) dt = \int_{0}^{\infty} e^{-st} dt = -\frac{1}{s} e^{-st}\Big|_0^{\infty} = \frac{1}{s}$$

for $\mathrm{Re}[s] = \sigma > 0$, i.e., that the region of convergence is the right half of the s-plane, except for the j-axis. Therefore,

$$u(t) \leftrightarrow \frac{1}{s}. \tag{3.5}$$

3.3.2 Unit-impulse function

Another singularity function, which is often used for circuit analysis, is the *unit-impulse function.* As was stated earlier, the impulse function is defined as

$$\delta(t) = 0 \quad \text{for} \quad t \neq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t) dt = 1.$$

Therefore, we have for any function $f(t)\delta(t) = f(0)\delta(t)$ since $\delta(t) = 0$ for $t \neq 0$. Now, by definition of Laplace transform

$$\mathbf{L}\{\delta(t)\} = \int_{0_-}^{\infty} e^{-st}\delta(t) dt = \int_{0_-}^{0_+} e^{0}\delta(t) dt = \int_{0_-}^{0_+} \delta(t) dt = 1.$$

Thus,

$$\delta(t) \leftrightarrow 1. \tag{3.6}$$

3.3.3 Exponential function

The next function of great interest is the *exponential function* $f(t) = e^{at}$ with a real, positive or negative, i.e.,

$$\mathbf{L}\{e^{at}u(t)\} = \int_{0_-}^{\infty} e^{-st}e^{at}\,dt = \int_{0_-}^{\infty} e^{-(s-a)t}\,dt = -\frac{1}{s-a}\,e^{-(s-a)}\Big|_0^{\infty} = \frac{1}{s-a}. \tag{3.7}$$

For both positive and negative a, the converge conditions are $\mathrm{Re}[s] > a$, then $s - a > 0$, and $e^{-(s-a)t} \to 0$ as $t \to \infty$. Thus,

$$e^{\pm at}u(t) \leftrightarrow \frac{1}{s \mp a}, \tag{3.7a}$$

where a is always positive. Considering a imaginary quantity $a = \pm j\omega$ yields

$$e^{\pm j\omega t}u(t) \leftrightarrow \frac{1}{s \mp j\omega} \tag{3.7b}$$

for $\mathrm{Re}[s] > 0$, since $|e^{\pm j\omega t}| = e^0 = 1$.

3.3.4 Ramp function

As an additional example, let us consider the *ramp function* $tu(t)$:

$$\mathbf{L}\{tu(t)\} = \int_{0_-}^{\infty} t\,e^{st}\,dt.$$

By a straightforward integration by parts $[u = t,\ v = -(1/s)e^{-st}]$:

$$\mathbf{L}\{tu(t)\} = -t\frac{1}{s}\,e^{-st}\Big|_0^{\infty} - \int_0^{\infty} -\frac{1}{s}\,e^{-st}\,dt = 0 - \frac{1}{s^2}\,e^{-st}\Big|_0^{\infty} = \frac{1}{s^2}.$$

Therefore,

$$tu(t) \leftrightarrow \frac{1}{s^2}. \tag{3.8}$$

3.4 BASIC THEOREMS OF THE LAPLACE TRANSFORM

For further evaluation of Laplace transform techniques, several basic theorems will be introduced.

3.4.1 Linearity theorem

This theorem is based on *linearity properties* of integrals: if $f_1(t)$ and $f_2(t)$ have Laplace transforms $F_1(s)$ and $F_2(s)$ respectively, then

$$\mathbf{L}\{f_1(t)+f_2(t)\}=F_1(s)+F_2(s), \tag{3.9}$$

i.e., the Laplace transform of the sum of two (or more) time functions is equal to the sum of the transforms of the individual time functions, and conversely

$$\mathbf{L}^{-1}\{F_1(s)+F_2(s)\}=\mathbf{L}^{-1}\{F_1(s)\}+\mathbf{L}^{-1}\{F_2(s)\}=f_1(t)+f_2(t). \tag{3.10}$$

It is also obvious that for any constant K

$$Kf(t)\leftrightarrow KF(s). \tag{3.11}$$

From this it follows that the Laplace transform of a constant (for example, of a constant voltage/current source) for $t\geq 0$, is its value divided by s:

$$V_0u(t)\leftrightarrow \frac{V_0}{s}. \tag{3.12}$$

As an example of the use of the linearity theorem, we will show the easiest way of obtaining the Laplace transform of the sinusoidal function $\sin \omega t$. Since

$$\sin \omega t=\frac{1}{2j}(e^{j\omega t}-e^{-j\omega t}),$$

in accordance with equation 3.7b, we have

$$\mathbf{L}\{\sin \omega t\}=\frac{1}{2j}\left(\frac{1}{s-j\omega}-\frac{1}{s+j\omega}\right)=\frac{(s+j\omega)-(s-j\omega)}{2j(s^2+\omega^2)}=\frac{\omega}{s^2+\omega^2}. \tag{3.13}$$

As a second example, let us consider the exponential of the form $(1-e^{-at})$ which is often met in circuit analysis:

$$\mathbf{L}\{(1-e^{-at})u(t)\}=\frac{1}{s}-\frac{1}{s+a}=\frac{a}{s(s+a)}. \tag{3.14}$$

As an example of using the opposite relationship (equation 3.10), let us determine the inverse Laplace transform of

$$F(s)=\frac{1}{(s+a)(s+b)}. \tag{3.15}$$

Using the partial-fraction expansion (see further on), we can split equation 3.15 into two parts:

$$F(s)=\frac{1}{(b-a)(s+a)}-\frac{1}{(b-a)(s+b)},$$

whose identity to equation 3.15 can be easily verified. In accordance with equation 3.10, we have

$$f(t) = \frac{1}{b-a}\, e^{-at}u(t) - \frac{1}{b-a}\, e^{-bt}u(t) = \frac{1}{b-a}\,(e^{-at} - e^{-bt})u(t).$$

Thus,

$$\frac{1}{b-a}\,(e^{-at} - e^{-bt})u(t) \leftrightarrow \frac{1}{(s+a)(s+b)}. \tag{3.16}$$

3.4.2 Time differentiation theorem

Time differentiation and integration (see further on) are the main theorems of Laplace transform techniques, which allow us to transform the derivatives and integrals appearing in the time-domain circuit equations.

Let $F(s)$ be the known transform of a time function $f(t)$, then

$$\mathbf{L}\left\{\frac{df}{dt}\right\} = \int_{0_-}^{\infty} e^{-st}\,\frac{df}{dt}\,dt,$$

and its integration by parts: $u = e^{-st}$ and $dv = (df/dt)dt$ gives

$$\mathbf{L}\left\{\frac{df}{dt}\right\} = f(t)e^{-st}\Big|_{0_-}^{\infty} - \int_{0_-}^{\infty} f(t)(-s)e^{-st}dt$$

$$= \lim_{t\to\infty} f(t)e^{-st} - f(0_-) + s\int_{0_-}^{\infty} f(t)e^{-st}dt.$$

The first limit must approach zero (since $F(s)$ exists) and the last integral is $F(s)$. Thus,

$$\mathbf{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0_-). \tag{3.17}$$

When the initial value of a function is zero, we simply have

$$\mathbf{L}\left\{\frac{df}{dt}\right\} = sF(s). \tag{3.17a}$$

By taking the derivative of a derivative, it may be shown that the differentiation properties for higher-order derivatives are

$$\mathbf{L}\left\{\frac{d^2f}{dt}\right\} = s^2F(s) - sf(0_-) - f'(0_-) \tag{3.18a}$$

$$\mathbf{L}\left\{\frac{d^3f}{dt}\right\} = s^3F(s) - s^2f(0_-) - sf'(0_-) - f''(0_-). \tag{3.18b}$$

In conclusion, when all initial conditions are zero, differentiating once with respect to t in the time domain corresponds to one multiplication by s in the frequency domain; differentiating twice in the time domain corresponds to multiplication by s^2 in the frequency domain and so on. Therefore, differentiation in the time domain is equivalent to multiplication by operands, which, of course, results in a substantial simplification. Note that when the initial conditions are not zero, by applying the differentiation theorem their presence is taken into account.

To demonstrate the use of the differential properties of the Laplace transform, let us consider the following example.

Example 3.1

Using Laplace transform techniques, find the current $i(t)$ in the series RL circuit driven by a constant voltage source, Fig. 3.3(a). Assume $L = 5$ H, $R = 4\ \Omega$, $v_s(t) = 6u(t)$ V and the initial value of the current is 4 A.

Solution

In accordance with KVL the loop equation is

$$5\frac{di}{dt} + 4i = 6u(t). \tag{3.19}$$

Assuming that the Laplace transform of the current is $I(s)$ and using the Laplace transform rules, with which we are already familiar, we transform the time domain equation 3.19 into the frequency domain.

$$5[sI(s) - 4] + 4I(s) = \frac{6}{s}. \tag{3.20}$$

Solving equation 3.20 for $I(s)$ yields

$$I(s) = 1.5\frac{0.8}{s(s+0.8)} + \frac{4}{s+0.8}, \tag{3.20a}$$

and with equations 3.7 and 3.14 we obtain

$$i(t) = 1.5(1 - e^{-0.8t})u(t) + 4e^{-0.8t}u(t) = (1.5 + 2.5e^{-0.8t})u(t)\ \text{A}. \tag{3.20b}$$

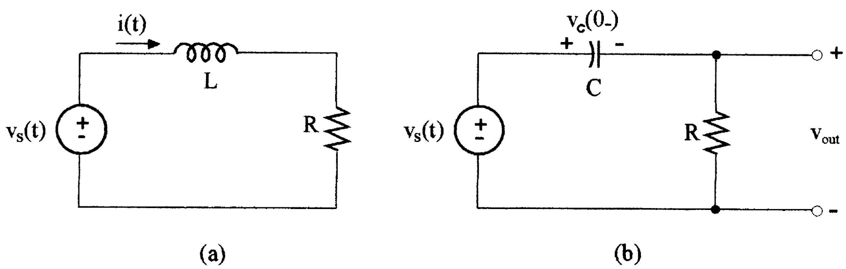

Figure 3.3 A circuit under study in Example 3.1 (a); circuit under study in Example 3.2 (b).

Note that instead of solving a differential equation 3.19, we actually solved the algebraic equation 3.20.

The time differentiation theorem also helps us to establish additional Laplace transform pairs. For example, consider $\mathbf{L}\{\cos \omega t\, u(t)\}$. Using equation 3.17a yields

$$\mathbf{L}\{\cos \omega t\, u(t)\} = \mathbf{L}\left\{\frac{1}{\omega}\frac{d}{dt}(\sin \omega t)u(t)\right\} = \frac{1}{\omega}s\frac{\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^2},$$

i.e.,

$$\cos \omega t\, u(t) \leftrightarrow \frac{s}{s^2+\omega^2}. \tag{3.21}$$

3.4.3 Time integration theorem

Let $F(s)$ be the known transform of a time function $f(t)$; then the Laplace transform of an integral as the time function can be determined in accordance with the definition (equation 3.2)

$$\mathbf{L}\left\{\int_{0_-}^{t} f(\tau)d\tau\right\} = \int_{0}^{\infty} e^{-st}\left[\int_{0}^{t} f(\tau)d\tau\right]dt.$$

Integrating by parts: $u = \int_{0_-}^{t} f(\tau)d\tau$ and $dv = e^{-st}$, yields

$$\begin{aligned}\mathbf{L}\left\{\int_{0_-}^{t} f(\tau)d\tau\right\} &= \left[\int_{0_-}^{t} f(\tau)d\tau\right]\left[-\frac{1}{s}e^{-st}\right]\Bigg|_{0_-}^{\infty} - \int_{0_-}^{\infty} -\frac{1}{s}e^{st}f(t)dt\\ &= \int_{0_-}^{0_-} f(t)dt\left(-\frac{1}{s}e^{0}\right) + \int_{0_-}^{\infty} f(\tau)d\tau - \left(-\frac{1}{s}e^{-\infty}\right)\\ &\quad + \frac{1}{s}\int_{0_-}^{\infty} e^{-st}f(t)dt.\end{aligned}$$

Since the first two terms on the right have vanished (note again that Re(s) is sufficiently large so $f(t)e^{-st} \to 0$ as $t \to \infty$) and the last integral is the Laplace transform of $f(t)$, we obtain

$$\int_{0_-}^{t} f(\tau)d\tau \leftrightarrow \frac{F(s)}{s} \tag{3.22}$$

which means that the integration in the time domain corresponds to the division by s in the frequency domain. In some cases, when the integral in equation 3.21 is taken for the low limit not zero but any positive or negative quantity a (for example when the capacitance in the electric circuit was precharged; thus the voltage across the capacitance is

$$v_C = \frac{1}{C}\int_{-\infty}^{t} i_C dt,$$

dividing the whole integral into two integrals, we obtain

$$\mathbf{L}\left\{\int_{-\infty}^{t} f(\tau)d\tau\right\} = \mathbf{L}\left\{\int_{-\infty}^{0_-} f(\tau)d\tau + \int_{0_-}^{t} f(\tau)d\tau\right\} = \mathbf{L}\{F_0\} + \mathbf{L}\left\{\int_{0_-}^{t} f(\tau)d\tau\right\}$$

$$= \frac{F(s)}{s} + \frac{F_0}{s}, \tag{3.23}$$

where F_0 is the value of the first integral (initial capacitance voltage) and $F(s)$ is the Laplace transform of the considered function $f(t)$. To demonstrate how the integration theorem helps us in circuit analysis, we shall consider the following example.

Example 3.2

Using Laplace transform techniques, find the output voltage $v_{out}(t)$ in the series RC circuit shown in Fig. 3.3(b). Assume $R = 5\ \Omega$, $C = 0.5$ F, with an initial voltage $v_C = 3$ V and $v_s(t) = 12u(t)$ V.

Solution

The voltage loop equation in the time domain is

$$12u(t) = 2\int_{-\infty}^{t} i(t)dt + 5i(t). \tag{3.24}$$

Taking the Laplace transform of both sides of equation 3.24 and since $v_C(0_-) = 3$ V, we obtain

$$\frac{12}{s} = \frac{3}{s} + \frac{2}{s}I(s) + 5I(s). \tag{3.25}$$

Solving equation 3.25 for $I(s)$ yields

$$I(s) = \frac{1.8}{s + 0.4}.$$

Since $v_{out} = 5i(t)$, its Laplace transform is

$$V_{out}(s) = 5\,\frac{1.8}{s + 0.4} = \frac{9}{s + 0.4},$$

which immediately gives $v_{out}(t)u(t) = 9e^{-0.4t}$ V.

It should be emphasized that if the time functions are zero at $t = 0$ (zero initial conditions) the linearity, differentiation and integration rules for phasor transform are identical to those for Laplace transform (only $j\omega$ has to be replaced by s). Consequently, the phasor impedance treatment of electric circuits and the Laplace transform impedance (see further on) analysis are identical. (Of course, we have to remember that these two techniques have different meanings:

the phasor analysis gives the *sinusoidal steady-state response*, while the Laplace transform relates to *zero-state response* to any Laplace transformable function.)

In conclusion, consider the complex exponential function $e^{(-\sigma+j\omega)t}$ and its transform equivalent

$$e^{(-\sigma+j\omega)t}u(t)\leftrightarrow\frac{1}{s+(\sigma+j\omega)}. \tag{3.26}$$

After separating real and imaginary parts of both sides of equation 3.26 and using linearity properties, we obtain two additional transform pairs

$$e^{-\sigma t}\cos\omega t\, u(t)\leftrightarrow\frac{s+\sigma}{(s+\sigma)^2+\omega^2}, \tag{3.26a}$$

$$e^{-\sigma t}\sin\omega t\, u(t)\leftrightarrow\frac{\omega}{(s+\sigma)^2+\omega^2}. \tag{3.26b}$$

Now let V and $\overset{*}{V}$ be a complex conjugate pair, then using linearity properties again, we obtain

$$\begin{aligned}\mathbf{L}^{-1}\left\{\frac{V}{s+(\sigma+j\omega)}+\frac{\overset{*}{V}}{s+(\sigma-j\omega)}\right\}&=Ve^{-(\sigma+j\omega)t}+\overset{*}{V}e^{-(\sigma-j\omega)t}\\&=2|V|e^{-\sigma t}\cos(\omega t+\psi)u(t),\end{aligned} \tag{3.27}$$

where $\psi=\angle V$.

Table 3.1 summarizes some of the more useful transform pairs (some of them were obtained above).

3.4.4 Time-shift theorem

Consider the transform of a time function shifted τ seconds in time as shown in Fig. 3.4. Using the definition of the Laplace transform, we obtain

$$\mathbf{L}\{f(t-\tau)u(t-\tau)\}=\int_{0_-}^{\infty}f(t-\tau)u(t-\tau)e^{-st}dt=\int_{\tau}^{\infty}f(t-\tau)e^{-st}dt.$$

Let $t-\tau=\theta$, then

$$\int_{\tau}^{\infty}f(t-\tau)e^{-st}dt=\int_{0_-}^{\infty}f(\theta)e^{-s(\tau+\theta)}d\theta=e^{-s\tau}\int_{0_-}^{\infty}f(\theta)e^{-s\theta}d\theta=e^{-s\tau}F(s).$$

Thus,

$$f(t-\tau)u(t-\tau)\leftrightarrow e^{-s\tau}F(s), \tag{3.28}$$

i.e., shifting by τ seconds in the time domain results in multiplication by $e^{-s\tau}$ in the frequency domain.

As an example of the application of this theorem, consider half a period of a sinusoidal function, as shown in Fig. 3.5(a). It can be represented as the sum of

Table 3.1 Laplace transform pairs

	$F(s)=\mathbf{L}\{f(t)\}$	$f(t)=\mathbf{L}^{-1}\{F(s)\}$
1	1	$\delta(t)$
2	$\frac{1}{s}$	$u(t)$
3	$\frac{1}{s^2}$	$tu(t)$
4	$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}u(t) \quad n=1, 2, \ldots$
5	$\frac{1}{s+a}$	$e^{-at}u(t)$
6	$\frac{1}{s(s+a)}$	$\frac{1}{a}(1-e^{-at})u(t)$
7	$\frac{1}{s^2(s+a)}$	$\frac{1}{a^2}[at-(1-e^{at})]u(t)$
8	$\frac{1}{(s+a)^2}$	$te^{-at}u(t)$
9	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}u(t)$
10	$\frac{1}{(s+a)^n}$	$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \quad n=1, 2, \ldots$
11	$\frac{1}{(s+a)(s+b)}$	$\frac{1}{b-a}(e^{-at}-e^{-bt})u(t)$
12	$\frac{1}{s(s+a)(s+b)}$	$\frac{1}{ab}\left[1+\frac{1}{a-b}(be^{-at}-ae^{-bt})\right]u(t)$
13	$\frac{s}{(s+a)(s+b)}$	$\frac{1}{a-b}(ae^{-at}-be^{bt})u(t)$
14	$\frac{s}{s^2+\omega^2}$	$\frac{1}{\omega}\sin\omega t\, u(t)$
15	$\frac{s}{s^2+\omega^2}$	$\cos\omega t\, u(t)$
16	$\frac{1}{s(s^2+\omega^2)}$	$\frac{1}{a^2}(1-\cos\omega t)u(t)$
17	$\frac{s\sin\psi+\omega\cos\psi}{s^2+\omega^2}$	$sin(\omega t+\psi)u(t)$
18	$\frac{s\cos\psi-\omega\sin\psi}{s^2+\omega^2}$	$\cos(\omega t+\psi)u(t)$
19	$\frac{1}{(s+a)^2+\omega^2}$	$\frac{1}{\omega}e^{-at}\sin\omega t\, u(t)$
20	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos\omega t\, u(t)$

Table 3.1 (Continued)

21	$\dfrac{1}{\mathrm{s}[(\mathrm{s}+\mathrm{a})^2+\omega^2]}$	$\dfrac{1}{a^2+\omega^2}\left[1-e^{-at}\left(\cos\omega t+\dfrac{a}{\omega}\sin\omega t\right)\right]u(t)$
22	$\dfrac{(s+a)\sin\psi+\omega\cos\psi}{(s+a)^2+\omega^2}$	$e^{-at}\sin(\omega t+\psi)u(t)$
23	$\dfrac{(s+a)\cos\psi-\omega\sin\psi}{(s+a)^2+\omega^2}$	$e^{-at}\cos(\omega t+\psi)u(t)$
24	$\dfrac{s}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega}t\sin\omega t\,u(t)$
25	$\dfrac{s^2-\omega^2}{(s^2-\omega^2)^2}$	$t\cos\omega t\,u(t)$
26	$\dfrac{1}{(s^2+\omega^2)^2}$	$\left[\dfrac{1}{s\omega^3}\sin\omega t-\dfrac{1}{2\omega^2}t\cos\omega t\right]u(t)$
27	$\dfrac{1}{s^2-a^2}$	$\dfrac{1}{a}\sinh(at)u(t)$
28	$\dfrac{s}{s^2-a^2}$	$\cos(at)u(t)$
29	$\dfrac{s}{(s^2-a^2)^2}$	$\dfrac{1}{2a}t\sinh(at)u(t)$
30	$\dfrac{1}{(s+a)^2-b^2}$	$\dfrac{1}{b}e^{-at}\sinh(bt)u(t)$
31	$\dfrac{s+a}{(s+a)^2-b^2}$	$e^{-at}\cosh(bt)u(t)$
32	$\dfrac{1}{s[(s+a)^2-b^2]}$	$\dfrac{1}{a^2-b^2}\left[1-e^{-at}\left(\cosh bt+\dfrac{a}{b}\sinh bt\right)\right]u(t)$
33	$\dfrac{s}{(s+a)^2-b^2}$	$e^{-at}\left(\cosh bt-\dfrac{a}{b}\sinh bt\right)u(t)$
34	$\dfrac{1}{\sqrt{s}}$	$\dfrac{1}{\sqrt{\pi t}}$
35	$\dfrac{1}{s\sqrt{s}}$	$2\sqrt{\dfrac{t}{\pi}}$

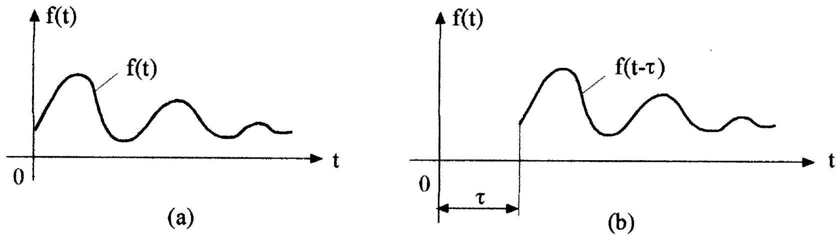

Figure 3.4 A function of time, $f(t)$, (a) and the same function delayed by τ (b).

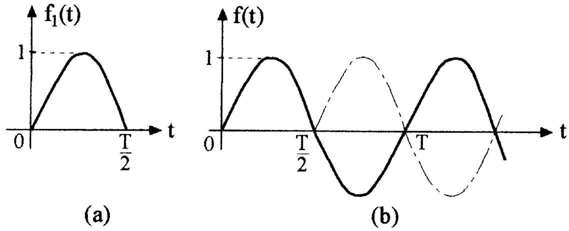

Figure 3.5 The positive half period of the sinusoidal function (a); shifted sinusoidal function (b).

two sinusoidal functions while the second one is delayed by half a period $T/2$, as shown in Fig. 3.5(b). Thus, $f_1(t)$ can be written as

$$f_1(t) = \sin \omega t + \sin \omega\left(t - \frac{T}{2}\right) u\left(t - \frac{T}{2}\right),$$

then in accordance with equations 3.13 and 3.28

$$\mathbf{L}\{f_1(t)\} = \frac{\omega(1 + e^{-sT/2})}{s^2 + \omega^2}. \tag{3.29}$$

The time-shift theorem is also useful in evaluating the Laplace transform of periodic time functions. Suppose that $f(t)$ is a periodic function (for $t \geq 0$) with period T, and $F_1(s)$ is the known transform of only the first period $f_1(t)$. Then the original $f(t)$ can be represented as the infinite sum of $f_1(t)$, delayed by an integer multiplied by T:

$$f(t) = \sum_{n=0}^{\infty} f_1(t - nT).$$

With the linearity and time-shift properties, the transform of $f(t)$ will be

$$F(s) = \sum_{n=0}^{\infty} e^{-nTs} F_1(s) = F_1(s) \sum_{n=0}^{\infty} e^{-nTs}. \tag{3.30}$$

The last sum in equation 3.30 is an infinitely decreasing geometric progression of the ratio e^{-Ts}, hence its sum is given by the formula $1/(1 - e^{-Ts})$. Therefore,

$$F(s) = \frac{F_1(s)}{1 - e^{-Ts}}, \tag{3.31}$$

where

$$F_1(s) = \mathbf{L}\{f_1(t)] = \int_{0_-}^{T} e^{-st} f(t) dt.$$

To illustrate the use of this transform theorem, let us apply it to the rectified sinus shown in Fig. 3.6. In accordance with equation 3.29 and using equation

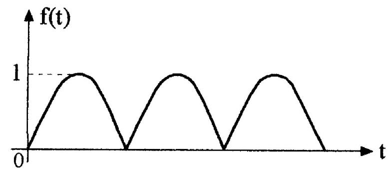

Figure 3.6 The sinusoidal shape function in full-wave rectification.

3.31, we obtain the transform of a periodic sinusoidal in full-wave rectification:

$$F(s) = \frac{\omega(1 + e^{-sT/2})}{(s^2 + \omega^2)(1 - e^{-Ts})}. \tag{3.32}$$

In another example of applying the time-shift theorem, let us find the Laplace transform of a triangular pulse train, Fig. 3.7(a). We first obtain the Laplace transform of the triangular pulse as the sum of ramp "1", shifted ramp "2" and shifted step functions "3" as shown in Fig. 3.7(b). Therefore,

$$F_1(s) = \frac{1}{Ts^2} - \frac{1}{Ts^2} e^{-sT} - \frac{1}{s} e^{-sT} = \frac{1}{Ts^2}(1 - e^{-sT}) - \frac{1}{s} e^{-sT}. \tag{3.33}$$

Now, to obtain the transform of a periodic pulse train, we divide equation 3.33 by $(1 - e^{-sT})$:

$$F(s) = \frac{1}{Ts^2} - \frac{e^{-sT}}{s(1 - e^{-sT})}. \tag{3.34}$$

3.4.5 Complex frequency-shift property

Shifting the origin of the transform in the frequency domain by s_0 has the same effect as multiplying the function $f(t)$ by $e^{-s_0 t}$ in the time domain. Indeed,

$$\mathbf{L}\{f(t)e^{-s_0 t}\} = \int_{0_-}^{\infty} f(t)e^{-s_0 t}e^{-st}dt = \int_{0_-}^{\infty} f(t)e^{-(s+s_0)t} = F(s + s_0). \tag{3.35}$$

This property of Laplace transform is especially useful in generating additional transforms.

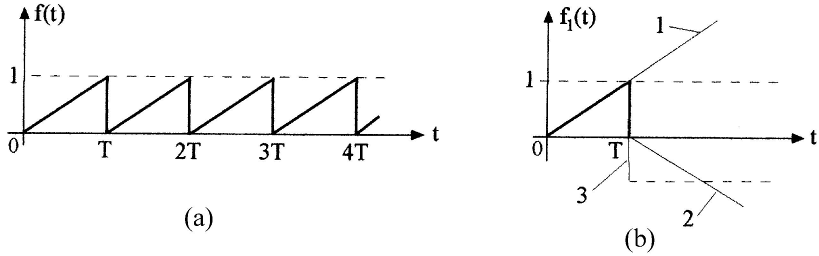

Figure 3.7 Triangular pulse train (a) and pulse representation (b).

For example, we can use the frequency-shift property to find the Laplace transform of $f(t) = e^{-\alpha t} \cos \omega_0 t\, u(t)$. Using the Laplace transform of $\cos \omega_0 t\, u(t)$ (equation 3.21), we have

$$e^{-\alpha t} \cos \omega_0 t\, u(t) \leftrightarrow \frac{s+\alpha}{(s+\alpha)^2 + \omega_0^2}, \tag{3.36}$$

which is the same as equation 3.26a.

As a second example, let us find the Laplace transform of $f(t) = te^{-s_0 t}u(t)$. With the help of (3.8), we have

$$te^{-s_0 t}u(t) \leftrightarrow \frac{1}{(s+s_0)^2}. \tag{3.37}$$

3.4.6 Scaling in the frequency domain

Scaling in the frequency domain, i.e. replacing s by s/a and dividing the transform by a, has the same effect as multiplying t by a in time domain. If

$$\mathbf{L}\{f(at)\} = \int_{0_-}^{\infty} f(at)e^{-st}\,dt,$$

then changing the variable $\lambda = at$, yields

$$\int_{0_-}^{\infty} f(\lambda)e^{-(s/a)\lambda}\left(\frac{1}{a}\right)d\lambda = \frac{1}{a}F\left(\frac{s}{a}\right).$$

Therefore,

$$f(at) \leftrightarrow \frac{1}{a}F\left(\frac{s}{a}\right), \tag{3.38a}$$

which is the same as

$$F(as) \leftrightarrow \frac{1}{a}f\left(\frac{t}{a}\right). \tag{3.38b}$$

This property also can be useful in obtaining additional transforms.

Example 3.3

Find the Laplace transform of the function $f_1(5t)$, if the Laplace transform of $f(t)$ is $F(s) = 1/(s^3 + 4)$.

Solution

In accordance with equation 3.38a

$$F_1(s) = \frac{1}{5}F\left(\frac{s}{5}\right) = \frac{1}{5}\frac{1}{(s/5)^3 + 4} = \frac{25}{s^3 + 500}.$$

3.4.7 Differentiation and integration in the frequency domain

Another property of interest will be obtained after examining the derivative of $F(s)$ with respect to s:

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_{0_-}^{\infty} e^{-st}f(t)dt.$$

Providing the differentiation of the integrand with respect to s gives the results:

$$\int_{0_-}^{\infty} -te^{-st}f(t)dt = \int_{0_-}^{\infty} [-tf(t)]e^{-st}dt,$$

which is simply the Laplace transform of $[-tf(t)]$. This means that *differentiation in the frequency domain* results in multiplication by $-t$ in the time domain:

$$-tf(t) \leftrightarrow \frac{d}{ds}F(s). \tag{3.39}$$

To illustrate the use of this rule, let us find the Laplace transform of higher powers of t. Noting that $tu(t) \leftrightarrow 1/s^2$, we apply the frequency domain differentiation theorem as follows:

$$\mathbf{L}\{-t^2u(t)\} = \frac{d}{ds}\frac{1}{s^2} = -2\frac{1}{s^3},$$

or

$$\frac{t^2u(t)}{2} \leftrightarrow \frac{1}{s^3}. \tag{3.40}$$

Continuing with the same procedure, we find

$$\frac{t^3}{3!}u(t) \leftrightarrow \frac{1}{s^4}, \tag{3.41}$$

and in general

$$\frac{t^{(n-1)}}{(n-1)!}u(t) \leftrightarrow \frac{1}{s^n}. \tag{3.42}$$

Next, let us examine the integration of $F(s)$ with respect to s and with the lower limit $s = \infty$:

$$\int_{\infty}^{s} F(s)ds = \int_{\infty}^{s}\left[\int_{0_-}^{\infty} f(t)e^{-st}dt\right]ds.$$

Interchanging the order of integration yields

$$\int_{\infty}^{s} F(s)ds = \int_{0_-}^{\infty}\left[\int_{\infty}^{s} e^{-st}ds\right]f(t)dt = \int_{0_-}^{\infty}\left[-\frac{1}{t}e^{-st}\right]f(t)dt$$
$$= \int_{0_-}^{\infty} -\frac{f(t)}{t}e^{-st}dt,$$

which is the Laplace transform of $f(t)/(-t)$. Thus, the *integration in the frequency domain* results in division by $-t$ in the time domain

$$\frac{f(t)}{-t} \leftrightarrow \int_{\infty}^{s} F(s)ds, \tag{3.43}$$

or by changing the limits in the integral

$$\frac{f(t)}{t} \leftrightarrow \int_{s}^{\infty} F(s)ds. \tag{3.43a}$$

For example, we have already obtained the pair (equation 3.14):

$$(1-e^{-at}) \leftrightarrow \frac{a}{s(s+a)}, \quad \text{for} \quad t \geq 0.$$

With the frequency integration theorem

$$\mathbf{L}\left\{\frac{1-e^{-at}}{t}\right\} = \int_{s}^{\infty}\frac{a}{s(s+a)}ds.$$

.

In accordance with the integral tables, the last integral is

$$\int_{s}^{\infty}\frac{a}{s(s+a)}ds = -\ln\frac{s+a}{s}\bigg|_{s}^{\infty} = \ln\frac{s+a}{s}.$$

Therefore

$$\frac{1-e^{-at}}{t} \leftrightarrow \ln\frac{s+a}{s}, \quad \text{for} \quad t \geq 0. \tag{3.44}$$

The Laplace transform theorems and some properties which have been discussed here are summarized in Table 3.2.

3.5 THE INITIAL-VALUE AND FINAL-VALUE THEOREMS

These two fundamental theorems enable us to evaluate $f(0_+)$ and $f(\infty)$ by examining the limiting values of the transform $F(s)$.

Table 3.2 Laplace transform operations

Operation	$f(t),\ t \geq 0$	$F(s)$
Addition	$\sum_{i=1}^{n} f_i(t)$	$\sum_{i=1}^{n} F_i(t)$
Scalar multiplication	$af(t)$	$aF(s)$
Time differentiation, where $f(0_-), f'(0_-)$ are the initial conditions	$\frac{df}{dt}$	$sF(s)-f(0_-)$
	$\frac{d^2f}{dt^2}$	$s^2F(s)-sf(0_-)-f'(0_-)$
Time integration, where $\int_{-\infty}^{0_-} f(t)dt$ is the initial condition	$\int_{0_-}^{t} f(t)dt$	$\frac{1}{s}F(s)$
	$\int_{-\infty}^{t} f(t)dt$	$\frac{1}{s}F(s)+\frac{1}{s}\int_{-\infty}^{0_-} f(t)dt$
Time shift	$f(t-a),\ a \geq 0$	$e^{-ad}F(s)$
Frequency shift	$f(t)e^{\mp at}$	$F(s \pm a)$
Frequency differentiation	$-tf(t)$	$\frac{dF(s)}{ds}$
Frequency integration	$\frac{f(t)}{t}$	$\int_{s}^{\infty} F(s)ds$
Scaling	$f(at),\ a \geq 0$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
Initial value	$f(0_+)$	$\lim_{s\to\infty} sF(s)$
Final value, where all poles of $sF(s)$ lie in LHP	$f(\infty)$	$\lim_{s\to 0} sF(s)$
sin or cos multiplication in the time domain	$f(t)\sin(\omega t)$	$\frac{1}{2j}[F(s-j\omega)-F(s+j\omega)]$
	$f(t)\cos(\omega t)$	$\frac{1}{2}[F(s-j\omega)+F(s+j\omega)]$
Convolution	$f_1(t)*f_2(t)$	$F_1(s)F_2(s)$
Du Hamel integral	$\frac{d}{dt}[f_1(t)*f_2(t)]$ $=f_1(0)f_2(t)+\int_{0_-}^{t} f'_1(\tau)f_2(t-\tau)d\tau$ $=f_1(t)f_2(0)+\int_{0_-}^{t} f_1(\tau)f'_2(t-\tau)d\tau$	$sF_1(s)F_2(s)$
Time periodicity:		
(1) the transform of the first period	$\int_{0_-}^{T} f(t)e^{-st}dt$	$F_1(s)$
(2) the transform of periodical function	$f(t)=f(t+nT)$	$\frac{F_1(s)}{1-e^{-Ts}}$

The initial-value theorem: Consider the Laplace transform of the derivative (equation 3.17)

$$\mathbf{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0_-) = \int_{0_-}^{\infty} \frac{df}{dt} e^{-st} dt. \tag{3.45}$$

By breaking the integral into two parts and approaching s infinity, we obtain

$$\lim_{s\to\infty}\left(\int_{0_-}^{0_+} e^0 \frac{df}{dt} dt + \int_{0_+}^{\infty} e^{-st} \frac{df}{dt} dt\right) = \lim_{s\to\infty} \int_{0_-}^{0_+} df = f(0_+) - f(0_-), \tag{3.46}$$

since the second integral approaches zero with $s \to \infty$.

Now taking the limit of both sides of equation 3.45 and applying the results of equation 3.46 yields

$$\lim_{s\to\infty} [sF(s) - f(0_-)] = f(0_+) - f(0_-),$$

or, after removing $f(0_-)$ from the limit, we obtain

$$\lim_{s\to\infty} [sF(s)] = f(0_+).$$

Therefore, in general

$$\lim_{t\to 0_+} f(t) = \lim_{s\to\infty} [sF(s)], \tag{3.47}$$

i.e., the initial value of the time function $f(t)$ can be obtained from its Laplace transform by multiplying the transform by s and evaluating the limit of $sF(s)$ by letting s approach infinity. It should be noted that if $f(t)$ is discontinuous at $t = 0$, then the initial value is the limit as $t \to 0_+$, i.e., the limit from the right.

The initial value theorem is useful in checking the results of a transformation or an inverse transformation. Thus in Example 3.1 we obtained the transform of the current (equation 3.20a)

$$I(s) = \frac{1.2}{s(s+0.8)} + \frac{4}{s+0.8}. \tag{3.48}$$

Applying the initial-value theorem yields

$$i(0) = \lim_{s\to\infty} [sI(s)] = \lim_{s\to\infty}\left(\frac{1.2}{s+0.8} + \frac{s4}{s+0.8}\right) = 4\ \mathrm{A},$$

which is in agreement with the initial condition given.

The final-value theorem: To prove the final value theorem, let us again consider the Laplace transform of the derivative *df*/*dt*

$$\int_{0_-}^{\infty} \frac{df}{dt} e^{-st} dt = sF(s) - f(0_-), \tag{3.49}$$

and take the limit as $s \to 0$ for both sides of equation 3.49. Taking the limit for the left side of equation 3.49 yields

$$\lim_{s\to\infty}\int_{0_-}^{\infty}\frac{df}{dt}e^{-st}dt=\int_{0_-}^{\infty}\frac{df}{dt}dt=f(\infty)-f(0_-),$$

and for the right side

$$\lim_{s\to\infty}[sF(s)-f(0_-)]=\lim_{s\to\infty}[sF(s)]-f(0_-).$$

Equating these two results, we have

$$f(\infty)=\lim_{s\to\infty}[sF(s)],$$

or in general

$$\lim_{s\to\infty}f(t)=\lim_{s\to\infty}[sF(s)] \tag{3.50}$$

which is known as the final value theorem.

Of course, we can apply this theorem only if the limit of $f(t)$, as t becomes infinite, exists. In other words, this requires that all the poles of $F(s)^{(*)}$, except one simple pole at the origin (which gives the constant value of $f(t)$), lie within the left half of the s plane.

Considering again, for example, the transform for current (equation 3.48) from Example 3.1 and applying the final-value theorem yields

$$i(\infty)=\lim_{s\to 0}[sI(s)]=\lim_{s\to 0}\left(\frac{1.2}{s+0.8}+\frac{s4}{s+0.8}\right)=1.5\ \mathrm{A},$$

which is evident by inspection of the circuit in Fig. 3.3(a) in its steady-state behavior, i.e. at $t\to\infty$.

It is interesting to check the final value of the sinusoidal function. In accordance with (3.50), we obtain

$$\lim_{s\to 0}[sF(s)]=\lim_{s\to 0}\frac{s\omega}{s^2+\omega^2}=0.$$

However, it is evident that the sinusoidal function: $f(t)=\sin\omega t$ does not have a final value. Looking again at

$$F(s)=\frac{\omega}{s^2+\omega^2}=\frac{\omega}{(s+j\omega)(s-j\omega)}$$

we can conclude that this transform fails the requirement that all the poles (except one) lie within the left half of the s plane, i.e. that $\mathrm{Re}[s_k]<0$ (here $\mathrm{Re}[s_{1,2}=0]$).

$^{(*)}$The roots of the denominator of $F(s)$ are considered as the poles of $F(s)$.

3.6 THE CONVOLUTION THEOREM

The convolution of two functions is defined as[(*)]

$$f_1(t) * f_2(t) = \int_{0_-}^{t} f(\tau)f(t-\tau)dt, \tag{3.51}$$

and its Laplace transform is given by

$$\mathbf{L}\{f_1(t) * f_2(t)\} = F_1(s)F_2(s). \tag{3.52}$$

Thus, the operation of convolution in the time domain is equivalent to multiplication in the frequency domain. Or, the inverse transform of the product of the transforms is the convolution of the individual inverse transforms. It is this property, among others, which makes the Laplace transform so useful in circuit analysis, especially since digital computers can be used for evaluating the convolution integral.

To prove the convolution theorem, let us calculate

$$J \triangleq \mathrm{L}\{\mathrm{f}_1(\mathrm{t}) * \mathrm{f}_2(\mathrm{t})\} = \int_{0_-}^{\infty} \left[\int_{0_-}^{t} \mathrm{f}_1(\tau)\mathrm{f}_2(\mathrm{t}-\tau)\mathrm{d}\tau\right] \mathrm{e}^{-st}dt,$$

and since $f(t-\tau)=0$ for all $\tau > t$, we may replace the upper limit "t" in the internal integral by "∞" and then interchange the order of integration:

$$J = \int_{0_-}^{\infty} f_1(\tau)\left[\int_{0_-}^{\infty} f_2(t-\tau)e^{-st}dt\right]d\tau.$$

Now in the inside integral we make the substitution $t' = t - \tau$ and $dt = dt'$ (note that the lower limit remained 0_-, since only for $t \geq 0_-$ does the function $f_2(t') \neq 0$). Thus,

$$J = \int_{0_-}^{\infty} f_1(\tau)\left[\int_{0_-}^{\infty} f_2(t')e^{-st'}dt'\right]e^{-s\tau}d\tau.$$

The bracketed term is $F_2(s)$, which is not a function of τ and can be pulled out of the integral, so we have

$$J = F_1(s)F_2(s).$$

Thus

$$f_1(t) * f_2(t) \leftrightarrow F_1(s)F_2(s). \tag{3.52a}$$

Since the right-hand side of equation 3.52a does not depend on the order of multiplication F_1 and F_2, consequently we can again conclude that the convolution is commutative.

As a simple example of the use of the convolution theorem, let us find the

---

[(*)]The lower limit in the convolution integral is taken here as $t = 0_-$, like in a one-sided Laplace transform, in order to include the effect of any discontinuity at $t = 0$.

convolution of $f_1(t) = t$ and $f_2(t) = e^{-\alpha t}$ for $t > 0$:

$$f_1(t) * f_2(t) = \mathbf{L}^{-1}\{F_1(s)F_2(s)\} = \mathbf{L}^{-1}\left\{\frac{1}{s^2}\frac{1}{s+a}\right\}. \tag{3.53}$$

The inverse transform of equation 3.53 can be obtained by the partial fraction expansion (see further on), so

$$\mathbf{L}^{-1}\left\{\frac{1}{s^2}\frac{1}{s+a}\right\} = \mathbf{L}^{-1}\left\{\frac{1}{as^2} - \frac{1}{a^2 s} + \frac{1}{a^2(s+a)}\right\} = \frac{1}{a}t - \frac{1}{a^2}(1 - e^{-at}), \quad t \geq 0.$$

Therefore,

$$t * e^{-at} = \left[\frac{1}{a}t - \frac{1}{a^2}(1 - e^{-at})\right]u(t). \tag{3.54}$$

The convolution theorem can be used for finding the Laplace transform of the functions which include square roots: $\sqrt{t}$. Indeed, if $F_1(s) \leftrightarrow f_1(t)$ then

$$F_1^2(s) \leftrightarrow f(t) = \int_0^t f_1(\tau) f_1(t-\tau)\,d\tau.$$

Changing the variable $\sigma = \tau - t/2$ and the integral limits respectively yields

$$f(t) = \int_{-t/2}^{t/2} f_1\left(\frac{t}{2} + \sigma\right) f_1\left(\frac{t}{2} - \sigma\right) d\sigma.$$

Now for $f_1(t) = 1/\sqrt{t}$ we obtain

$$f(t) = \int_{-t/2}^{t/2} \frac{d\sigma}{\sqrt{t/2+\sigma}\sqrt{t/2-\sigma}} = \int_{-t/2}^{t/2} \frac{d\sigma}{\sqrt{(t/2)^2 - \sigma^2}} = \sin^{-1}\frac{\sigma}{t/2}\bigg|_{\sigma=-t/2}^{\sigma=t/2} = \pi.$$

By taking the Laplace transform of both sides of $f(t) = \pi$ we have

$$F_1^2(s) = \frac{\pi}{s} \quad \text{or} \quad F_1 = \sqrt{\frac{\pi}{s}}.$$

Therefore,

$$\frac{1}{\sqrt{t}} \leftrightarrow \sqrt{\frac{\pi}{s}}.$$

Taking the integral of $f_1(t)$

$$\int_0^t \frac{1}{\sqrt{t}}\,dt = 2\sqrt{t},$$

and using the integration theorem, we finally have

$$\sqrt{t} \leftrightarrow \frac{\sqrt{\pi}}{2s\sqrt{s}}.$$

It is known, from basic circuit analysis, that the output voltage $v_{out}(t)$ at some point in a linear circuit driven by the input $v_{in}(t)$ can be obtained by convolving $v_{in}(t)$ with the impulse response $h(t)$ (response on a unit impulse at $t=0$ with initial conditions zero)

$$v_{out}(t) = v_{in}(t) * h(t). \tag{3.55}$$

Taking the Laplace transform of both sides of equation 3.55 yields

$$V_{out}(s) = V_{in}(s)H(s),$$

where $H(s)$ is the transform of the impulse response, so

$$\mathbf{L}\{h(t)\} = H(s) = \frac{V_{out}(s)}{V_{in}(s)}. \tag{3.56}$$

The ratio (equation 3.56) was termed as the transfer function. Since the same rules are used by Laplace transform derivative and integral representations (with zero initial conditions) and by complex frequency analysis (see Table 3.3), there is considerable similarity between the transfer function and the Laplace transform of impulse response (equation 3.56). This is an important fact that will be used in Laplace transform techniques to analyze the transient behavior of some circuits.

Example 3.4

Find the transfer function $H(s) = V_{out}(s)/V_{in}(s)$ of the circuit shown in Fig. 3.8(a).

Solution

First we represent the circuit elements in the frequency domain as shown in Fig. 3.8(b). Then we find Z_{eq} of the parallel connection C and $(L+R_2)$:

$$Z_{eq}(s) = \frac{(2s+6)\frac{1}{s}}{2s+6+\frac{1}{s}} = \frac{s+3}{s^2+3s+0.5},$$

Table 3.3 Laplace transform impedances of R, L, C elements

Element	Time-domain relationship $i(t)$	s-domain relationship $I(s)=Ie^{st}$	Laplace transform with $i_L(0_-)=0$, $v_C(0_-)=0$	Impedance $Z(s)=\frac{V(s)}{I(s)}$
R	$v=Ri$	$V(s)=RIe^{st}$	$V(s)=RI(s)$	R
L	$v=L\frac{di}{dt}$	$V(s)=sLIe^{st}$	$V(s)=sLI(s)$	sL
C	$v=\frac{1}{C}\int idt$	$V(s)=\frac{1}{sC}I^{st}$	$V(s)=\frac{1}{sC}I(s)$	$\frac{1}{sC}$