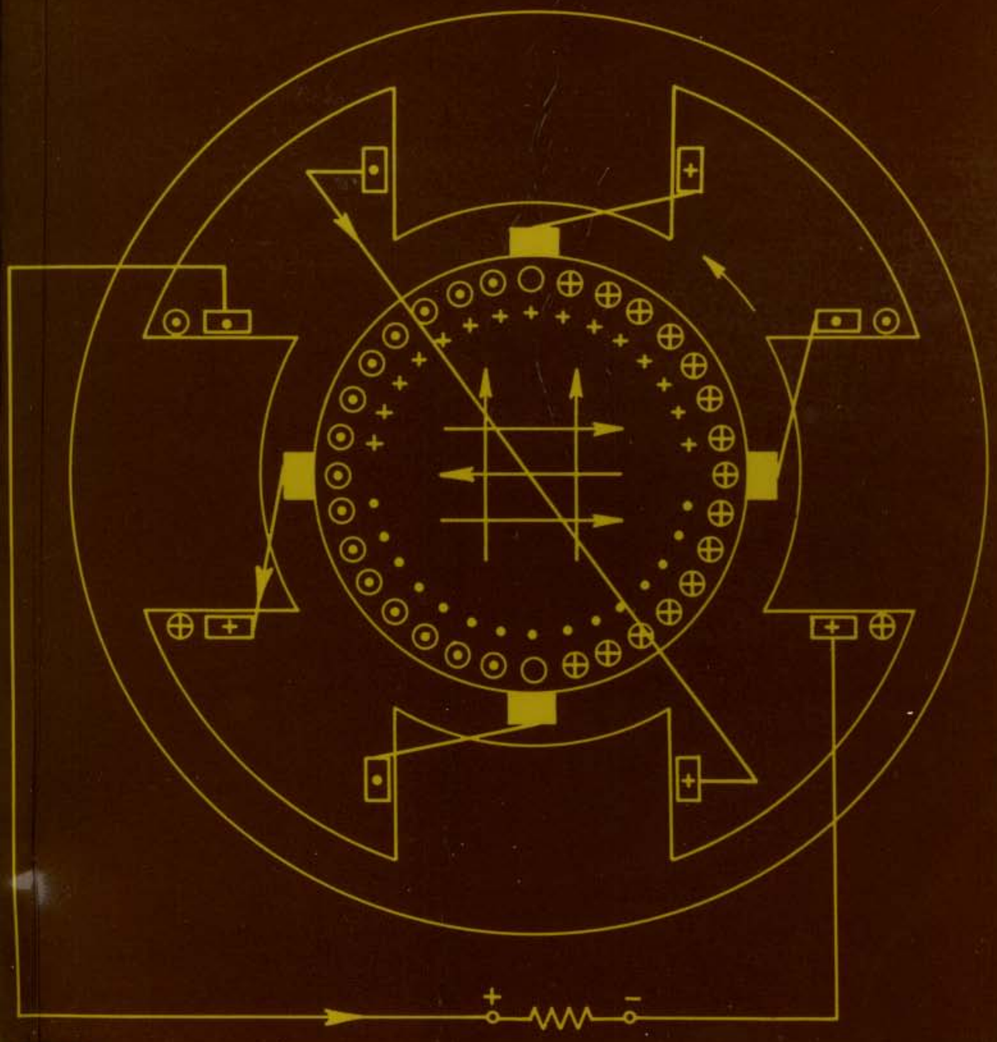
Higher Electrical Engineering

J SHEPHERD, A H MORTON & L F SPENCE



Higher Electrical Engineering

SHEPHERD
MORTON
SPENCE

PITMAN

HIGHER ELECTRICAL ENGINEERING

By the same authors
Basic Electrical Engineering

By A. H. Morton
Advanced Electrical Engineering

HIGHER ELECTRICAL ENGINEERING

By the same authors

Basic Electrical Engineering

By A. H. Morton

Advanced Electrical Engineering

Higher Electrical Engineering

J. SHEPHERD

B.Sc., C.Eng., F.I.E.E.
Senior Lecturer, Department of Electrical Engineering, University of Edinburgh

A. H. MORTON

B.Sc., C.Eng., F.I.E.E., F.I.E.R.E.
Head of Department of Electrical Engineering, Paisley College of Technology

L. F. SPENCE

B.Sc., B.Sc.(Econ.), C.Eng., F.I.E.E.
Principal Lecturer, Department of Electrical and Electronic Engineering, Polytechnic of the South Bank

SECOND EDITION

PITMAN PUBLISHING

Second edition 1970
Reprinted 1973
Reprinted 1975

PITMAN PUBLISHING LTD
Pitman House, 39 Parker Street, London WC2B 5PB, UK

PITMAN MEDICAL PUBLISHING CO LTD
42 Camden Road, Tunbridge Wells, Kent TN1 2QD, UK

FOCAL PRESS LTD
31 Fitzroy Square, London W1P 6BH, UK

PITMAN PUBLISHING CORPORATION
6 East 43 Street, New York, NY 10017, USA

FEARON PUBLISHERS INC
6 Davis Drive, Belmont, California 94002, USA

PITMAN PUBLISHING PTY LTD
Pitman House, 158 Bouverie Street, Carlton, Victoria 3053, Australia

PITMAN PUBLISHING
COPP CLARK PUBLISHING
517 Wellington Street West, Toronto M5V 1G1, Canada

SIR ISAAC PITMAN AND SONS LTD
Banda Street, PO Box 46038, Nairobi, Kenya

PITMAN PUBLISHING CO SA (PTY) LTD
Craighall Mews, Jan Smuts Avenue, Craighall Park, Johannesburg 2001, South Africa

ISBN 0 273 40063 0

Text set in 10/11 pt. Monotype Times New Roman, printed by letterpress, and bound in Great Britain at The Pitman Press, Bath

T1320:74

Preface

In preparing this second edition the basic aims of the first edition have not been changed. We have designed this volume to cover the A1 and A2 stages of the Higher National Certificate in Electrical and Electronic Engineering. It does not pretend to be exhaustive, nor specialist, but is designed for a broad background course, with stress laid on the fundamental aspects of each section. The contents will be found to correspond to much of the work in the recent Department of Education and Science outline syllabuses for H.N.C. Courses in England. It accords also with the Scottish H.N.C. scheme. In addition this volume should prove useful in undergraduate C.E.I. Part 1 and H.N.D. courses.

The text has been brought up to date, particularly in the sections on machines and electronics. An introduction to the generalized theory of machines has been given, but conventional treatments (substantially revised) of synchronous and induction machines have been retained. The positive convention for the sign of the voltage induced in a circuit due to a changing flux has been maintained, since further experience has shown that this concept has helped students (and staff) to a clearer understanding of the physical concept of induced e.m.f.

In electronics the shift of emphasis to semiconductor devices is reflected in the omission of valve circuits from large sections of the text, in favour of bipolar transistor circuits. The field-effect transistor is dealt with in a later chapter.

In an age where systems engineering is coming more into prominence it is important that electrical and electronic engineers have some knowledge of reliability. We are grateful to Mr. E. L. Topple of the Polytechnic of the South Bank for undertaking the task of preparing a chapter on this subject for us. A further chapter giving an introduction to logic has also been added.

As in the first edition we have included a large number of worked examples in the text. Problems (with answers) at the end of each chapter give the reader the opportunity of testing his understanding of the text as he proceeds. Thanks are due to the Senate of the University of London, and to the Scottish Association for National Certificates and Diplomas for their willingness to allow us to use examples from their examination papers (designated L.U. and H.N.C. respectively).

The new edition is larger than the authors had hoped and its preparation has not been entirely uncontroversial. Because of this the wife of one author has suggested that the book should be subtitled "War and Peace"!

We should also like to record our thanks to colleagues who read the manuscript and undertook the corrections at the proof stage. These include in particular Mr. T. Grassie (now of Strathclyde University), Mr. W. R. M. Craig and Mr. A. McKenzie of Paisley College of Technology and Mr. G. Heywood of the Polytechnic of the South Bank. Thanks are also due to those who contributed to the massive typing effort required.

PAISLEY
June, 1970

J.S.
A.H.M.
L.F.S.

Contents

Chapter 1

SYMBOLIC NOTATION

It is assumed that the reader is already familiar with the diagrams which are used to give the relationship between sinusoidal alternating currents and voltages in simple a.c. circuits. These *complexor diagrams* represent the magnitude of any quantity they depict as the length of a line, while the direction of the line gives phase information. Because of similarities in the manipulation of quantities represented in this way with vector methods of addition and subtraction these diagrams used to be called *vector diagrams*. In this book they will be called complexor diagrams, and the lines will be called complexors.*

Problems involving the manipulation of complexors may be solved by representing the complexors as algebraic expressions. The notation used for doing so is called the symbolic notation, and the advantage of the use of this notation is that the processes of manipulation become algebraic processes.

1.1 The Operator j

In complexor diagrams the direction of the X-axis is called the *reference direction*, since it is often used as the reference from which phase angles are measured. The direction of the Y-axis may be called the *quadrate direction*. Fig. 1.1 shows three typical complexors, V_1, V_2 and V_3. The lengths of the lines are proportional to the magnitudes of the quantities they represent. The phase angle of a complexor

* In recent years they have also been called *phasor diagrams* and *phasors*.

is represented as the angle turned through (anticlockwise) from the positive reference direction to the direction of the complexor. Note that the position of a complexor has no bearing on the magnitude

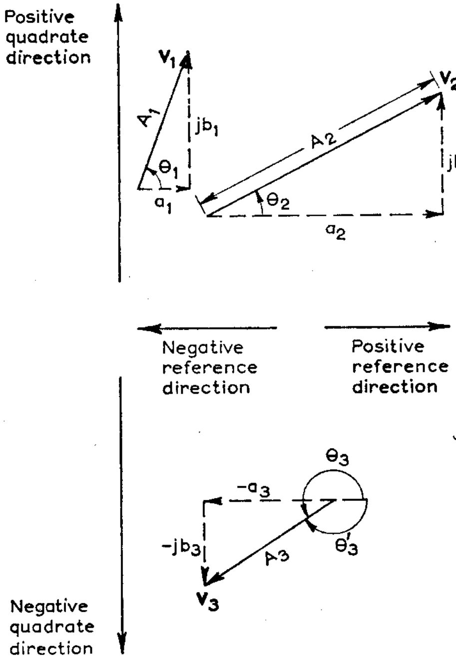


Fig. 1.1 TYPICAL COMPLEXORS

of the quantity it represents, so that in Fig. 1.1, for example, V_1 could be represented by any parallel line of the same length.

If the complexor diagram represents alternating currents or voltages the whole diagram may be assumed to rotate at a constant angular velocity. If the complexor is proportional to the peak value of the alternating quantity, its projection in a particular direction will give the instantaneous value of the quantity.

A complexor may be completely described by

(i) a statement of its magnitude with respect to a given scale unit; and
(ii) a statement of its phase with respect to a reference direction.

Thus in Fig. 1.1, $V_1 = A_1\underline{/\theta_1}$; $V_2 = A_2\underline{/\theta_2}$, etc. The magnitude is sometimes called the *modulus* and is represented by $|V|$ or V.

Note that the symbols for complexors are printed in **bold** type while those for magnitudes are printed in ordinary type.

If the phase angle is greater than 180°, the negative phase angle (i.e. the phase angle measured in a clockwise direction) is often stated for convenience. For example,

$$\mathbf{V}_3 = A_3\underline{/\theta_3} \quad \text{or} \quad A_3\underline{/-\theta_3'} \quad \text{or} \quad A_3\overline{/\theta_3'}$$

where $\theta_3' = 360° - \theta_3$

This method of describing a complexor is termed *polar notation.* A and $\underline{/\theta}$ are called *operators.*

A, the magnitude operator or modulus, is the number by which the scale unit should be multiplied to give the magnitude of the complexor. $\underline{/\theta}$, the phase operator, is the anticlockwise angle through which a complexor in the reference direction must be turned in order to take up the direction of the given complexor.

It will be realized that the operations -1, and $\underline{/180°}$ or $\underline{/-180°}$ are identical. Hence $-\mathbf{A}$ is a complexor in the negative reference direction.

The operation $\underline{/90°}$ or $\underline{/\pi/2}$ rad is found to occur frequently and is commonly represented by the symbol j:

$$j \equiv \underline{/90°} \tag{1.1}$$

i.e. j represents the operation of turning a complexor through 90° in an anticlockwise direction. Hence

$$jb = b\underline{/90°} \tag{1.2}$$

i.e. jb is a complexor of length b in the quadrate direction. In the same way,

$$-jb = -(jb) = -b\underline{/90°}$$

and is thus a complexor of length b in the negative quadrate direction.

It is very convenient to represent a complexor by the sum of two components, one of which is in either the positive or the negative reference direction, while the other is in either the positive or the negative quadrate direction. Thus, in Fig. 1.1,

$$\mathbf{V}_1 = A_1\underline{/\theta_1} = a_1 + jb_1$$

where $a_1 = A_1 \cos \theta_1$ and $b_1 = A_1 \sin \theta_1$, and

$$\mathbf{V}_3 = A_3\underline{/\theta_3} = -a_3 - jb_3$$

where

$$a_3 = -A_3 \cos \theta_3 = A_3 \cos (\pi - \theta_3)$$

and

$$b_3 = -A_3 \sin \theta_3 = -A_3 \sin (\pi - \theta_3)$$

The $(a + jb)$ method of describing a complexor is termed *rectangular notation.*

The above complexors may also be expressed in the form

$$V = A (\cos \theta + j \sin \theta) \tag{1.3}$$

this being termed the *trigonometric notation.*

From the geometry of the diagrams,

$$A = \sqrt{(a^2 + b^2)} \tag{1.4}$$

and

$$\theta = \tan^{-1} \frac{b}{a} \tag{1.5}$$

For example,

$$V_A = -3 + j2$$

may be expressed as

$$V_A = \sqrt{(3^2 + 2^2)} \underline{/\tan^{-1} \frac{2}{-3}} = 3{\cdot}61\underline{/180° - 33{\cdot}6°}$$

Conversely,

$$V_B = 12\underline{/-60°}$$

may be expressed as

$$V_B = 12[\cos(-60°) + j \sin(-60°)] = 6 - j10{\cdot}39$$

In the preceding paragraphs A, $\underline{/\theta}$, a, b and j are all operators of various types. The combined expressions $A\underline{/\theta}$ and $(a + jb)$ are called complex operators in polar and rectangular forms respectively. A complexor may be expressed as a complex operator when a particular scale unit and reference direction are given.

In the conversion from rectangular to polar form for a complex number, or operator, the square root of the sum of the squares must be calculated. This operation may be conveniently performed on an ordinary slide-rule which has the usual A, B, C, and D scales. For example, suppose it is desired to find $\sqrt{(3^2 + 7^2)}$. Set the smaller of the two numbers (in this case 3) on the C scale against unity on the D scale. Move the cursor to the higher of the two numbers on the C scale, and read off the corresponding figure on the A scale at the top of the rule (5·46). Add one to this figure (giving 6·46). Set the cursor at this new number on the A scale, and read off the desired result on the C scale (7·62). If the two numbers are such that the numerically smaller has a bigger initial figure (e.g. $\sqrt{(7^2 + 12^2)}$) the smaller number is set against 10 on the D scale, the rest of the method being the same.

For numbers which differ by more than a factor of ten it is usually sufficient to take the square root of the sum of the squares as being simply the larger of the two numbers. Thus

$$|10 + j1| = \sqrt{101} \approx 10{\cdot}05 \approx 10$$

SUCCESSIVE OPERATIONS BY j (Fig. 1.2)

Since j is defined as an operator which turns a complexor through $+90°$ without changing its size, two operations by j will turn a

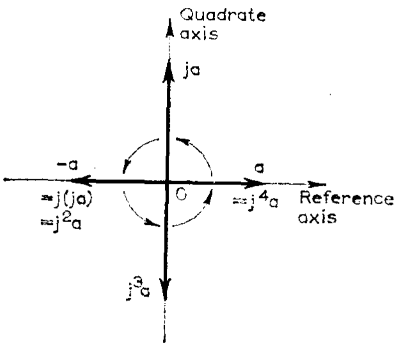


Fig. 1.2 SUCCESSIVE OPERATIONS BY j

complexor through a total of 180° from the original reference direction; i.e. the original direction is reversed. Thus

$$j(ja) = j^2a = -1 \times a \tag{1.6}$$

The operator j^2 is the 180° operator. It is convenient to think of j^2 as being algebraically the same as -1.

If now j^2a **is** operated on by j (written as $j(j^2a) = j^3a$) the original complexor $+a$ is turned through a total of 270°.

A further operation on j^3a by j brings the complexor back to its original position. Thus

$$j^4a = 1 \times a \tag{1.7}$$

and the operation of j^4 on a complexor leaves it unchanged in size and direction.

THE OPERATION $(-j)$

When a complexor is operated on by $-j$, the operation may be divided into two parts:

(i) operate on the complexor by -1; and
(ii) operate on the resulting complexor by j.

Thus the result is a positive rotation of 270°. Hence

$$-ja = j^3a$$

and the negative sign can be taken to mean that the rotation is clockwise. This conclusion is peculiar to the operator j, and does not apply to any other rotational operator.

1.2 Addition and Subtraction of Complex Operators

ADDITION

The complexors **OP** and **CQ** in Fig. 1.3 may be added graphically by

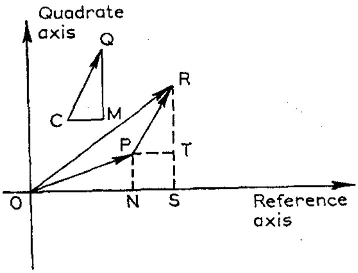


Fig. 1.3 ADDITION OF COMPLEXORS

placing them end to end in order. Let

$$\mathbf{ON} = a \qquad \mathbf{NP} = jb = \mathbf{ST}$$
$$\mathbf{CM} = c = \mathbf{PT} = \mathbf{NS} \qquad \mathbf{MQ} = jd = \mathbf{TR}$$

Then

$$\mathbf{OP} + \mathbf{CQ} = \mathbf{OS} + \mathbf{SR} = \mathbf{ON} + \mathbf{NS} + \mathbf{ST} + \mathbf{TR}$$

Therefore

$$(a + jb) + (c + jd) = a + c + jb + jd = (a + c) + j(b + d)$$

The rule for addition of complex operators is thus seen to be: *Add the reference and quadrate terms separately.* For example,

$$(7 + j9) + (8 - j12) = 15 - j3$$

and

$$(-3 + j7) + (-2 - j10) = -5 - j3$$

SUBTRACTION

If one complexor is to be subtracted from another the graphical method is to reverse the former and then add. Thus in Fig. 1.4,

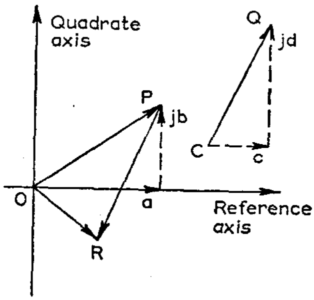


Fig. 1.4 SUBTRACTION OF COMPLEXORS

let **OP** $= a + jb$ and **CQ** $= c + jd$. Then

$$\mathbf{OP} - \mathbf{CQ} = \mathbf{OP} + \mathbf{PR} = (a + jb) + (-c - jd)$$
$$= (a - c) + j(b - d) = \mathbf{OR}$$

The rule is: *To subtract one complex operator from another, subtract the reference and the quadrate terms separately.*

It should be noted that for both addition and subtraction the normal algebraic rules for signs will operate. For example,

$$(-4 + j7) - (8 + j2) = -12 + j5$$

The polar form of the complex operator is not suitable for addition and subtraction, and complex operators which are expressed in polar form must first be changed to rectangular form if they are to be added or subtracted.

1.3 Multiplication and Division of Complex Operators

MULTIPLICATION

For multiplication in the rectangular form the normal rules of algebra apply, so that

$$(a + jb)(c + jd) = ac + jad + jbc + j^2bd$$
$$= (ac - bd) + j(ad + bc) \qquad (1.8)$$

since j^2 may be given the value -1. Particular attention should be paid to the following product:

$$(a + jb)(a - jb) = a^2 + jab - jab - j^2b^2$$
$$= a^2 + b^2 \qquad (1.9)$$

$(a + jb)$ and $(a - jb)$ are termed a pair of *conjugate* complex operators since their product contains no quadrate term. In the same way $(-a + jb)$ and $(-a - jb)$ are also conjugate complex operators.

In the trigonometric form,

$$A(\cos\theta + j\sin\theta) \times B(\cos\phi + j\sin\phi)$$
$$= AB(\cos\theta\cos\phi + j\sin\theta\cos\phi + j\sin\phi\cos\theta + j^2\sin\theta\sin\phi)$$
$$= AB\{(\cos\theta\cos\phi - \sin\theta\sin\phi) + j(\sin\theta\cos\phi + \sin\phi\cos\theta)\}$$
$$= AB(\cos(\theta + \phi) + j\sin(\theta + \phi)) \qquad (1.10)$$

i.e. in the trigonometric form the product of two complex operators is the product of their moduli taken with the sum of their phase angles.

Since the polar and trigonometric forms of a complex operator are really identical, the same results will hold for both forms. Hence,

$$A\underline{/\theta} \times B\underline{/\phi} = AB\underline{/\theta + \phi} \qquad (1.11)$$

In the same way,

$$A\underline{/\theta} \times \frac{1}{B}\underline{/-\phi} = \frac{A}{B}\underline{/\theta - \phi} \qquad (1.12)$$

It can also be shown that successive multiplication obeys the same rules. Thus

$$A\underline{/\theta} \times B\underline{/\phi} \times C\underline{/\psi} = ABC\underline{/\theta + \phi + \psi} \qquad (1.13)$$

It will be observed that multiplication in the polar form is much less tedious than in the rectangular form, and for this reason it is frequently convenient to convert rectangular operators into the polar form before multiplication.

DIVISION

Division of complex operators in the rectangular form is achieved by *rationalizing* the denominator, i.e. eliminating the quadrate term from the denominator, by multiplying both the numerator and the denominator by the conjugate of the denominator. Thus

$$\frac{a+jb}{c+jd}=\frac{(a+jb)(c-jd)}{(c+jd)(c-jd)}=\frac{ac+bd+j(bc-ad)}{c^2+d^2}$$
$$=\frac{ac+bd}{c^2+d^2}+j\frac{bc-ad}{c^2+d^2}$$

In the same way,

$$\frac{1}{-a+jb}=\frac{-a-jb}{(-a+jb)(-a-jb)}$$
$$=\frac{-a}{a^2+b^2}-j\frac{b}{a^2+b^2}$$

In polar form the conjugate of the operator $A\angle\theta$ is $A\angle-\theta$. This follows directly from the expression $A\angle\theta \times A\angle-\theta = A^2$. Similarly $B\angle\phi \times B\angle-\phi = B^2$.

Division of the operator $A\angle\theta$ by the operator $B\angle\phi$ is achieved by multiplying both numerator and denominator by the conjugate of the denominator. Thus

$$\frac{A\angle\theta}{B\angle\phi}=\frac{A\angle\theta}{B\angle\phi}\,\frac{B\angle-\phi}{B\angle-\phi}=\frac{A}{B}\angle\theta-\phi$$

i.e. the quotient of two complex operators in polar form is the quotient of their moduli taken with the difference of their phase angles.

Also

$$\frac{A\angle\theta}{B\angle-\phi}=\frac{A}{B}\angle\theta+\phi$$

EXAMPLE 1.1 Divide $(10-j10)$ by $(8{\cdot}66+j5)$.

Method (i)

$$\frac{10-j10}{8{\cdot}66+j5}=\frac{14{\cdot}14\angle-45^\circ}{10\angle30^\circ}=1{\cdot}414\angle-75^\circ=\underline{\underline{0{\cdot}366-j1{\cdot}37}}$$

Method (ii)

$$\frac{10-j10}{8{\cdot}66+j5}=\frac{(10-j10)(8{\cdot}66-j5)}{(8{\cdot}66+j5)(8{\cdot}66-j5)}$$
$$=\frac{86{\cdot}6-50-j50-j86{\cdot}6}{8{\cdot}66^2+5^2}$$
$$=\frac{36{\cdot}6}{100}-j\frac{136{\cdot}6}{100}=\underline{\underline{0{\cdot}366-j1{\cdot}37}}$$

POWERS AND ROOTS

Powers of complex operators simply represent successive multiplication. For numbers in polar form the procedure is to take the power of the modulus and to multiply the phase angle by the index. Thus

$$(A\underline{/\theta})^n = A^n\underline{/n\theta} \tag{1.14}$$

Roots may be dealt with in the same manner by taking the root of the modulus and dividing the phase angle by the root. Thus

$$\sqrt[n]{(B\underline{/\phi})} = \sqrt[n]{B}\,\underline{\Big/\frac{\phi}{n}} \tag{1.15}$$

It should be noted that in complex operator notation the nth root of any number, including pure or reference numbers, has always n possible values. For example,

$$\sqrt[3]{(8\underline{/60^\circ})} = 2\underline{/20^\circ}$$

or

$$2\underline{\Big/\frac{60^\circ + 360^\circ}{3}} = 2\underline{/140^\circ}$$

or

$$2\underline{\Big/\frac{60^\circ + 720^\circ}{3}} = 2\underline{/260^\circ}$$

since each of these operators to the power 3 will give $8\underline{/60^\circ}$ or its identicals $8\underline{/60^\circ + 360^\circ}$ and $8\underline{/60^\circ + 720^\circ}$.

In the same way,

$$\sqrt[4]{(16)} = 2\underline{/0^\circ} \quad \text{or} \quad 2\underline{/90^\circ} = j2 \quad \text{or} \quad 2\underline{/180^\circ} = -2$$

or

$$2\underline{/270^\circ} = -j2$$

EQUATIONS

Consider the equation

$$a + jb = c + jd$$

which relates two complex operators V and W, where $V = a + jb$ and $W = c + jd$. Since the operators are equal, their components along the reference axis must be equal, so that $a = c$. Also, for identity, the components along the quadrate axis must be equal, so that $b = d$.

In general, in any complex equation, the sum of the reference components on one side must be equal to the sum of the reference components on the other; and similarly the sum of the quadrate components must be equal on both sides of the equation.

Note that the above identity can be expressed as

$$\boldsymbol{V} = V\underline{/\theta} = \boldsymbol{W} = W\underline{/\phi}$$

where $V = \sqrt{(a^2 + b^2)}$; $W = \sqrt{(c^2 + d^2)}$; $\tan\theta = b/a$; and $\tan\phi = d/c$. For identity,

$$V = W \quad \text{and} \quad \theta = \phi$$

1.4 Simple Circuits—Impedance

The lines which represent alternating voltages or currents in a complexor diagram can be expressed as complex operators when suitable scale units and a reference direction have been chosen. The complexors may then be summed with or subtracted from other complexors which represent quantities expressed with respect to the same scale unit and reference direction.

Impedance is among those quantities which can be represented by a complex operator.

PURE RESISTANCE

Suppose a sinusoidal current represented by the complexor $\boldsymbol{I}$, is passed through a pure resistance R. The potential difference across R will be a sinusoidal voltage represented by the complexor $\boldsymbol{V}$, where $\boldsymbol{V}$ and $\boldsymbol{I}$ are in phase with one another, and where $|\boldsymbol{V}|/|\boldsymbol{I}|$ is equal to R. If $\boldsymbol{I}$ is chosen as the reference complexor, then

$$\boldsymbol{I} = |\boldsymbol{I}|\underline{/0^\circ} = I\underline{/0^\circ}$$

Hence

$$\boldsymbol{V} = |\boldsymbol{V}|\underline{/0^\circ} = V\underline{/0^\circ}$$

and the impedance is given by

$$\boldsymbol{Z} = \frac{\boldsymbol{V}}{\boldsymbol{I}} = \frac{V}{I}\underline{/0^\circ} = R \tag{1.16}$$

Thus the impedance of a pure resistance may be represented by the reference operator R.

PURE INDUCTANCE

Suppose that a sinusoidal current represented by the complexor $\boldsymbol{I}$ is passed through a pure inductance L. The p.d. across L will be a sinusoidal voltage represented by the complexor $\boldsymbol{V}$, where $\boldsymbol{V}$ leads $\boldsymbol{I}$ by 90° and $|\boldsymbol{V}|/|\boldsymbol{I}|$ is equal to ωL. If $\boldsymbol{I}$ is chosen as the reference complexor then $\boldsymbol{I} = |\boldsymbol{I}|\underline{/0°} = I\underline{/0°}$ and $\boldsymbol{V} = |\boldsymbol{V}|\underline{/90°} = j|\boldsymbol{V}| = V\underline{/90°}$.

Therefore the impedance is given by

$$\boldsymbol{Z} = \frac{\boldsymbol{V}}{\boldsymbol{I}} = \frac{V}{I}\underline{/90°} = j\omega L = jX_L \tag{1.17}$$

Thus the impedance of an inductive reactance may be represented by the quadrate operator $j\omega L$.

PURE CAPACITANCE

Suppose that a sinusoidal current represented by the complexor $\boldsymbol{I}$ is passed through a pure capacitance C. The p.d. across C will be a sinusoidal voltage represented by the complexor $\boldsymbol{V}$, where $\boldsymbol{V}$ lags $\boldsymbol{I}$ by 90°, and $|\boldsymbol{V}|/|\boldsymbol{I}|$ is equal to $1/\omega C$. If $\boldsymbol{I}$ is chosen as the reference complexor, then $\boldsymbol{I} = I\underline{/0°}$ and $\boldsymbol{V} = V\underline{/-90°} = -jV$.

Therefore the impedance is given by

$$\boldsymbol{Z} = \frac{\boldsymbol{V}}{\boldsymbol{I}} = \frac{V}{I}\underline{/-90°} = \frac{-j}{\omega C} = \frac{-j^2}{j\omega C} = \frac{1}{j\omega C} \tag{1.17}$$

i.e.

$$\boldsymbol{Z} = \frac{1}{j} X_C = -jX_C \tag{1.18}$$

Thus the impedance of a capacitive reactance may be represented by the negative quadrate operator $-j/\omega C$ or $1/j\omega C$.

SERIES CIRCUITS

If the resistances and reactances of a circuit are expressed as reference and quadrate operators then the total impedance of the circuit may be determined by the processes of complex algebra. Thus

$$\boldsymbol{Z} = R + j\omega L$$

represents the impedance of a circuit in which a resistance R is connected in series with an inductive reactance ωL across a sinusoidal supply of frequency $f = \omega/2\pi$ hertz*. For a resistance and capacitance in series the impedance is given by

$$\boldsymbol{Z} = R - j/\omega C$$

* The unit of frequency, the *cycle per second* (c/s) is known as the *hertz* (Hz).

For a circuit with R, L and C in series, the impedance in complex form will be

$$Z = R + j\omega L - \frac{j}{\omega C} \tag{1.19}$$

$$= R + j\left(\omega L - \frac{1}{\omega C}\right)$$

$$= R + j(X_L - X_C) \tag{1.19a}$$

For the series connexion of impedances $Z_1, Z_2, \ldots Z_N$, the total effective impedance is

$$Z_{eq} = Z_1 + Z_2 + \ldots + Z_N \tag{1.20}$$

For parallel connexion of N impedances the total effective impedance is given by

$$1/Z_{eq} = 1/Z_1 + 1/Z_2 + \ldots + 1/Z_N \tag{1.21}$$

For two impedances in parallel this reduces to

$$Z_{eq} = \frac{Z_1 Z_2}{Z_1 + Z_2} \tag{1.22}$$

EXAMPLE 1.2 Find the voltage which, when applied to a circuit consisting of a resistance of 120Ω in series with a capacitive reactance of 250Ω, causes a current of 0·9A to flow. Also find the voltage across each component and the overall power factor of the circuit.

For a series circuit the current is taken as the reference complexor (Fig. 1.5).

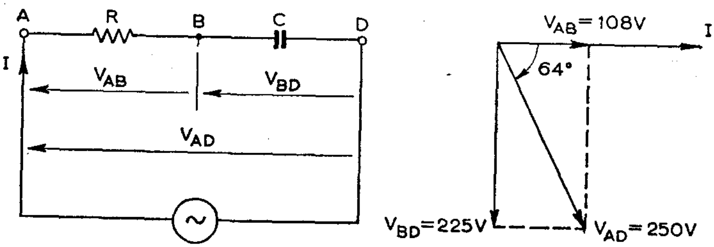


Fig. 1.5

Hence

$$I = 0{\cdot}9\underline{/0^\circ}\,\text{A}$$

The impedance, Z, expressed in complex form, is

$$Z = 120 - j250 = 278\underline{/-64^\circ}\,\Omega$$

Thus

$$V_{AD} = IZ = 0{\cdot}9\underline{/0^\circ} \times 278\underline{/-64^\circ} = \underline{\underline{250\underline{/-64^\circ}\,\text{V}}}$$

The power factor of the circuit is cos 64°, i.e. 0·432 leading.

Also

$$V_{AB} = IR = 0{\cdot}9\angle 0^\circ \times 120 = \underline{\underline{108\angle 0^\circ\,\mathrm{V}}}$$

and

$$V_{BD} = I \times jX_C = 0{\cdot}9\angle 0^\circ \times 250\angle -90^\circ = \underline{\underline{225\angle -90^\circ\,\mathrm{V}}}$$

EXAMPLE 1.3 A capacitor of 100 Ω reactance is connected in parallel with a coil of 70·7 Ω resistance and 70·7 Ω reactance to a 250 V a.c. supply. Find the current in each branch, and the total current taken from the supply. Also find the overall power factor.

For the parallel circuit it is convenient to take the voltage as the reference complexor. Thus

$$V = 250\angle 0^\circ\,\mathrm{V}$$

The coil impedance is

$$Z_L = 70{\cdot}7 + j70{\cdot}7 = 100\angle 45^\circ\,\Omega$$

Therefore the coil current is

$$I_L = \frac{V}{Z_L} = \frac{250\angle 0^\circ}{100\angle 45^\circ}$$

$$= \underline{\underline{2{\cdot}5\angle -45^\circ}} \qquad \text{or} \qquad (1{\cdot}77 - j1{\cdot}77)\,\mathrm{A}$$

$$\text{Impedance of capacitor} = -j100 = 100\angle -90^\circ\,\Omega$$

Hence

$$\text{Current in capacitor} = I_C = \frac{250\angle 0^\circ}{100\angle -90^\circ}$$

$$= \underline{\underline{2{\cdot}5\angle 90^\circ}} \quad \text{or} \quad j2{\cdot}5\,\mathrm{A}$$

The total current is

$$\begin{aligned} I &= I_L + I_C \\ &= 1{\cdot}77 - j1{\cdot}77 + j2{\cdot}5 \\ &= (1{\cdot}77 + j0{\cdot}73) \qquad \text{or} \qquad \underline{\underline{1{\cdot}91\angle 22{\cdot}5^\circ\,\mathrm{A}}} \end{aligned}$$

The power factor is cos 22·5°, i.e. $\underline{\underline{\text{0·92 leading.}}}$

1.5 Parallel Circuits—Admittance

The *admittance*, *Y*, of a circuit is defined as the r.m.s. current flowing per unit r.m.s. applied voltage. It is thus the reciprocal of the circuit impedance:

$$Y = \frac{I}{V} = \frac{1}{Z} \tag{1.23}$$

The admittance of a circuit may be represented by a complex operator in the same way as the impedance. The reference term of this complex operator is called the *conductance*, and the quadrate term is called the *susceptance*, the symbols for these terms being G and B respectively. Thus an admittance Y may be expressed as

$$Y = G + jB \tag{1.24}$$

If the circuit consists of a pure resistance, R, then

$$Z = R$$

and

$$Y = \frac{1}{Z} = \frac{1}{R} = G \tag{1.25}$$

For a purely inductive reactance, $j\omega L$, the admittance is

$$Y = \frac{1}{j\omega L} = \frac{-j}{\omega L} = -jB \tag{1.26}$$

For a purely capacitive reactance, $1/j\omega C$, the admittance will be

$$Y = \frac{1}{Z} = j\omega C = jB \tag{1.27}$$

From this it is seen that inductive susceptance is negative while capacitive susceptance is positive in the complex form.

If a circuit contains both resistance and reactance in series, the admittance may be derived as follows. Let

$$Z = R + jX$$

Then

$$Y = \frac{1}{Z} = \frac{1}{R + jX} = \frac{R - jX}{R^2 + X^2}$$

$$= \frac{R}{R^2 + X^2} - j\frac{X}{R^2 + X^2}$$

(Note the change of sign.)

The main advantage of the idea of admittance arises when dealing with parallel circuits. In this case the voltage across each element is the same, and the total current is the complexor sum of the branch currents. Thus for three admittances, Y_1, Y_2 and Y_3 in parallel,

$$I = I_1 + I_2 + I_3 = VY_1 + VY_2 + VY_3 = V(Y_1 + Y_2 + Y_3)$$

so that

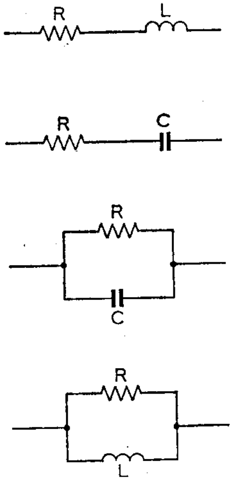
$$\frac{I}{V} = Y_{eq} = Y_1 + Y_2 + Y_3$$

In general, for N admittances in parallel,

$$Y_{eq} = \sum_{n=1}^{n=N} Y_n \tag{1.28}$$

The unit of admittance, conductance, and susceptance is the *siemen* (S).

The admittances and impedances of two simple series and of two simple parallel circuits are shown in Fig. 1.6. These show that a series circuit is more easily represented as an impedance, while



$Z = R + j\omega L$ $\qquad Y = \dfrac{R}{R^2 + \omega^2 L^2} - \dfrac{j\omega L}{R^2 + \omega^2 L^2}$

$Z = R - j/\omega C$ $\qquad Y = \dfrac{R}{R^2 + 1/\omega^2 C^2} + \dfrac{j/\omega C}{R^2 + 1/\omega^2 C^2}$

$Y = \dfrac{1}{R} + j\omega C$ $\qquad Z = \dfrac{R}{1 + \omega^2 C^2 R^2} - \dfrac{j\omega C R^2}{1 + \omega^2 C^2 R^2}$

$Y = 1/R - j/\omega L$ $\qquad Z = \dfrac{R\omega^2 L^2}{R^2 + \omega^2 L^2} + \dfrac{j\omega L R^2}{R^2 + \omega^2 L^2}$

Fig. 1.6 ADMITTANCE AND IMPEDANCE OF FOUR SIMPLE CIRCUITS

a parallel circuit is more easily represented as an admittance. For the case of a resistor R, in parallel with a capacitor C,

$$Y_{eq} = Y_R + Y_C = \frac{1}{R} + j\omega C$$

$$Z_{eq} = \frac{1}{\dfrac{1}{R} + j\omega C}$$

$$= \frac{\dfrac{1}{R} - j\omega C}{\left(\dfrac{1}{R} + j\omega C\right)\left(\dfrac{1}{R} - j\omega C\right)} = \frac{R}{1 + \omega^2 C^2 R^2} - j\frac{\omega C R^2}{1 + \omega^2 C^2 R^2} \tag{1.29}$$

It is left for the reader to verify the other results.

EXAMPLE 1.4 Three impedances of $(70\cdot7 + j70\cdot7)\Omega$, $(120 + j160)\Omega$ and $(120 + j90)\Omega$ are connected in parallel across a 250 V supply. Calculate the admittance of the combination and the total current taken. Also determine the value of the pure reactance which, when connected across the supply, will bring the overall power factor to unity, and find the new value of the total current.

The first step is to express the impedances as admittances in rectangular form, from which the equivalent admittance is easily obtained. Thus

$$\boldsymbol{Z}_1 = 70\cdot7 + j70\cdot7 = 100\underline{/45^\circ}\,\Omega$$

Therefore

$$\boldsymbol{Y}_1 = \frac{1}{100\underline{/45^\circ}} = 0\cdot01\underline{/-45^\circ} = (0\cdot00707 - j0\cdot00707)\,\mathrm{S}$$

$$\boldsymbol{Z}_2 = 120 + j160 = 200\underline{/53\cdot1^\circ}\,\Omega$$

Therefore

$$\boldsymbol{Y}_2 = \frac{1}{200\underline{/53\cdot1^\circ}} = 0\cdot005\underline{/-53\cdot1^\circ} = (0\cdot003 - j0\cdot004)\,\mathrm{S}$$

$$\boldsymbol{Z}_3 = 120 + j90 = 150\underline{/36\cdot9^\circ}\,\Omega$$

Therefore

$$\boldsymbol{Y}_3 = \frac{1}{150\underline{/36\cdot9^\circ}} = 0\cdot00667\underline{/-36\cdot9^\circ} = (0\cdot0053 - j0\cdot004)\,\mathrm{S}$$

Therefore

$$\begin{aligned}\boldsymbol{Y}_{eq} = \boldsymbol{Y}_1 + \boldsymbol{Y}_2 + \boldsymbol{Y}_3 &= 0\cdot0154 - j0\cdot015\\ &= \underline{\underline{0\cdot0215\underline{/-44\cdot3^\circ}\,\mathrm{S}}}\end{aligned}$$

With the voltage as the reference complexor the current will be

$$\boldsymbol{I} = \boldsymbol{V}\boldsymbol{Y}_{eq} = 250\underline{/0^\circ} \times 0\cdot0215\underline{/-44\cdot3^\circ} = \underline{\underline{5\cdot37\underline{/-44\cdot3^\circ}\,\mathrm{A}}}$$

i.e. the current is 5·37 A at a power factor of 0·71 lagging.

To bring the overall power factor to unity the susceptance required in parallel with the three given impedances must be such that there is no quadrate term in the expression for the resultant admittance. Thus

$$\text{Susceptance required} = +j0\cdot015\,\mathrm{S}$$

Therefore

$$\text{Pure reactance required} = \frac{1}{j0\cdot015} = \underline{\underline{-j66\cdot6\,\Omega}}$$

i.e. a capacitive reactance of 66·6 Ω.

With this reactance connected across the input, the total admittance will be 0·0154 S (pure conductance); hence new value of current is

$$250\underline{/0^\circ} \times 0\cdot0154\underline{/0^\circ} = \underline{\underline{3\cdot85\,\mathrm{A}}}$$

EXAMPLE 1.5 Find the parallel combination of resistance and capacitance which takes the same current at the same power factor from a 5 kHz supply as an impedance of $(17\cdot3 - j10)\Omega$.

The admittance of the series circuit is

$$Y = \frac{1}{Z} = \frac{1}{17{\cdot}3 - j10} = 0{\cdot}05\underline{/30^\circ} = (0{\cdot}0433 + j0{\cdot}025)\,\mathrm{S}$$

As well as representing the admittance of the series circuit this expression also gives the admittance of a parallel circuit consisting of a pure resistance of

$$R_P = \frac{1}{0{\cdot}0433} = \underline{\underline{23{\cdot}1\,\Omega}}$$

in parallel with a purely capacitive susceptance of

$$B_P = 0{\cdot}025\,\mathrm{S} = \omega C_P$$

Hence

$$C_P = \frac{B_P}{\omega} = \frac{0{\cdot}025}{10{,}000\pi}\,\mathrm{F} = \underline{\underline{0{\cdot}796\,\mu\mathrm{F}}}$$

Note that these circuits would not be equivalent at any other frequency.

1.6 Impedance and Admittance Diagrams

When an impedance is expressed as a complex operator it may be represented on a diagram which is similar to a complexor diagram, but with the two important differences:

(*a*) an impedance diagram has two mutually perpendicular axes, and

(*b*) position on an impedance diagram is important, while position (as distinct from direction) on a complexor diagram is not important. The electrical impedance diagram is equivalent to the mathematician's Argand diagram.

It should be noted that in an *impedance diagram*, such as is shown in Fig. 1.7, both axes must have the same scale, i.e. unit length must represent the same number of ohms on both axes. Pure resistance values are plotted in the reference direction (horizontal), while pure reactance values are plotted above or below the reference axis according to the inductive or capacitive nature of the reactance. For example, point B (Fig. 1.7) represents an impedance which is equivalent to a pure resistance of 40Ω in series with an inductive reactance of 10Ω. The magnitude of the impedance is given by the length OB, and the phase angle by ϕ_B. Also point C on the same diagram represents an impedance which is equivalent to a pure resistance of 20Ω in series with a capacitive reactance of −10Ω.

If the impedance is not constant then a line on the impedance diagram may be drawn to show all the possible values which the impedance may have. The diagram is then called an *impedance locus diagram*. Fig. 1.7 shows two such impedance loci. The line AB represents the locus for a circuit consisting of a fixed inductance of 10Ω in series with a resistance which can be varied between zero and 40Ω. The length OE gives the impedance of the circuit when the resistance is 20Ω, the phase angle then being ϕ_1. In the same way the line CD represents the locus for a circuit consisting of a resistance of 20Ω in series with a capacitive reactance which can be varied from −10Ω to −30Ω. The line OF gives the impedance of this circuit when the capacitive reactance is −20Ω.

In the same way admittance (which has been shown in Section 1.5 to be expressible as a complex operator of the form $Y = G \pm jB$)

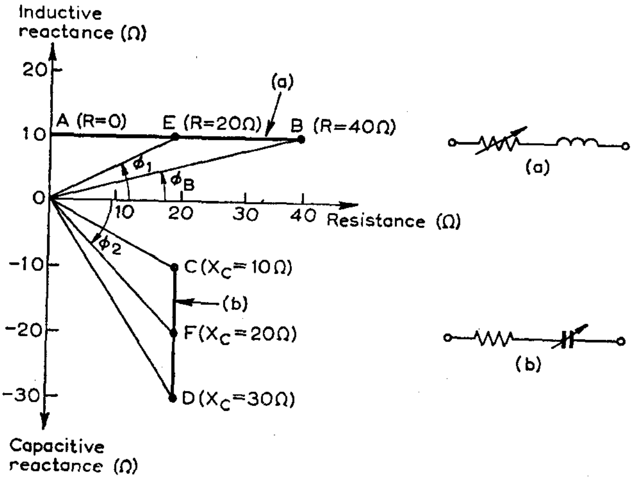


Fig. 1.7 IMPEDANCE LOCI FOR TWO SIMPLE CIRCUITS WITH ONE VARIABLE

may be represented on an admittance diagram in which the reference axis represents conductance and the quadrate axis represents susceptance. Thus if Fig. 1.7 were an admittance diagram the line AB would represent a fixed capacitive susceptance of 10 S in parallel with a resistance whose conductance varied from zero (open-circuit) to 40 S. Inductive susceptance (which is negative) is, of course, represented along the negative quadrate axis.

1.7 Current Locus Diagrams for Series Circuits

If a resistance R and a reactance jX are connected in series across a constant voltage supply of V volts, and the voltage V is taken to be in the reference direction, then the current I is given by the expression

$$I = \frac{V}{R + jX}$$

where jX may be either positive (inductive) or negative (capacitive). If the resistance is variable while the reactance is constant, and the above equation is rewritten as

$$I = \frac{\dfrac{V}{jX}}{1 + \dfrac{R}{jX}}$$

i.e.

$$I + \frac{1}{j}\frac{R}{X}I = \frac{V}{jX} = -j\frac{V}{X} \tag{1.30}$$

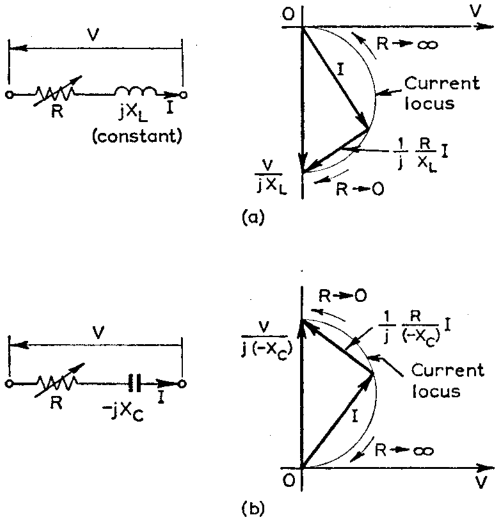


Fig. 1.8 CURRENT LOCUS DIAGRAMS FOR SIMPLE SERIES CIRCUITS WITH FIXED REACTANCE AND VARIABLE RESISTANCE
(*a*) Inductive reactance (*b*) Capacitance reactance

then the equation may be interpreted as follows. If the current complexor I is drawn from the origin and has the complexor $\frac{1}{j}\frac{R}{X}I$ added to it, then the sum is the constant complexor $\frac{V}{jX}$, and since I and $\frac{1}{j}\frac{R}{X}I$ are mutually perpendicular, the extremity of I must lie on a circle of diameter $\frac{V}{jX}$. The complexor loci for inductive and capacitive circuits with variable resistance are shown in Fig. 1.8.

If the resistance is fixed while the reactance varies, the basic equation is rewritten as

$$I = \frac{\dfrac{V}{R}}{1 + \dfrac{jX}{R}}$$

i.e.

$$I + j\frac{X}{R}I = \frac{V}{R} \tag{1.31}$$

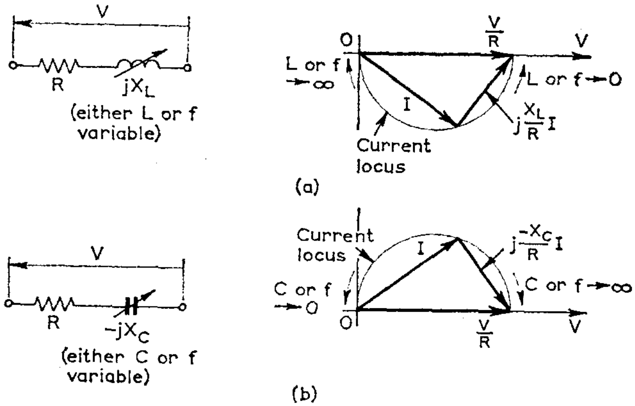


Fig. 1.9 CURRENT LOCI FOR SERIES CIRCUITS WITH VARIABLE REACTANCE
(*a*) Inductive reactance (*b*) Capacitance reactance

This, when interpreted as in the previous case, indicates that the extremity of the current complexor, drawn from the origin, lies on a circle of diameter V/R. The loci for the current in series circuits containing resistance and variable reactance are shown in Fig. 1.9.

EXAMPLE 1.6 A capacitive reactance of $0{\cdot}5\,\Omega$ is connected in series with a variable resistor R to a 2 V, 50 kHz supply. Draw the impedance locus, and the locus of current as the resistance varies between zero and $4\,\Omega$. From these loci find the current and its phase angle when the resistance is (*a*) $0{\cdot}2\,\Omega$, and (*b*) $2\,\Omega$. Also determine the maximum power input and the corresponding current, phase angle and resistance.

The impedance locus is shown as the line AB in Fig. 1.10. Since the reactance is constant, this line will be parallel to the reference axis and a distance below it representing the constant capacitive reactance of $0{\cdot}5\,\Omega$.

(*a*) OC represents the impedance when $R = 0{\cdot}2\,\Omega$. By measurement from the diagram OC $= 0{\cdot}55\,\Omega$ to scale.

The phase angle is $\phi_1 = -69°$

and the current is $\dfrac{2}{0{\cdot}55} = \underline{\underline{3{\cdot}64\,\text{A}}}$

(*b*) OD represents the impedance when $R = 2\Omega$. From the diagram OD $= 2{\cdot}09\Omega$ to scale.

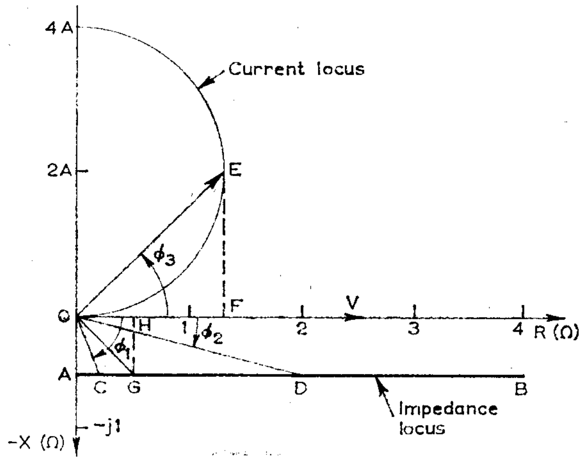


Fig. 1.10

The phase angle is $\phi_2 = -14°$

and the current is $\dfrac{2}{2{\cdot}09} = \underline{\underline{0{\cdot}957\,\text{A}}}$

From eqn. (1.30), the diameter of the current locus is

$$\frac{V}{-jX} = \frac{2}{-j0{\cdot}5} = \underline{\underline{j4\,\text{A}}}$$

If V is taken in the reference direction then the diameter of the current locus lies along the positive direction of the quadrate axis as shown in the diagram. The maximum power input occurs when the component of the current in the direction of V is a maximum, i.e. when the active component of the current is a maximum. Thus OE gives the current for maximum power and ϕ_3 gives the corresponding phase angle.

$$\text{OE} = 2\sqrt{2} = \underline{\underline{2{\cdot}83\,\text{A}}} \qquad \text{and} \qquad \phi_3 = 45°$$

$$\begin{aligned} \text{Maximum power} &= V \times \text{maximum value of active component of current} \\ &= V \times \text{OF} \\ &= 2 \times 2 = \underline{\underline{4\,\text{W}}} \end{aligned}$$

The corresponding value of resistance may be obtained by drawing the line OG, making an angle of $-45°$ with the reference axis, to cut the impedance locus at G. Then OG represents the impedance for maximum power, so that the corresponding resistance is OH to scale, i.e. $0{\cdot}5\Omega$.

1.8 Volt-ampere Calculations for Parallel Loads and Generators

If several loads are connected in parallel, the total current flowing is the complexor sum of the individual load currents, i.e.

$$\boldsymbol{I}_{\text{total}} = \boldsymbol{I}_1 + \boldsymbol{I}_2 + \boldsymbol{I}_3 + \ldots$$

The complexor diagram representing the above equation is shown in Fig. 1.11 (*a*), where the common system voltage is taken as the reference. If each of the current complexors is multiplied by the magnitude of the system voltage V, the above expression becomes

$$V\boldsymbol{I}_{\text{total}} = V\boldsymbol{I}_1 + V\boldsymbol{I}_2 + V\boldsymbol{I}_3 + \ldots$$

The corresponding complexor diagram is shown in Fig. 1.11 (*b*),

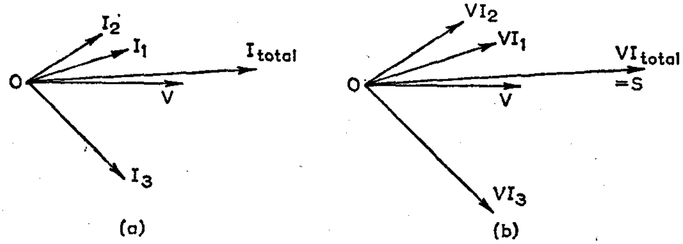


Fig. 1.11 CURRENT AND CORRESPONDING VOLT-AMPERE COMPLEXOR DIAGRAMS FOR PARALLEL CIRCUITS

which is a diagram of complexor volt-amperes. Volt-amperes will be represented by the symbol $\boldsymbol{S}$.

The reference component of the total volt-amperes is $S\cos\phi$, i.e. $VI_{\text{total}}\cos\phi$, or the total power P absorbed by the load. The quadrature component of the volt-amperes is $S\sin\phi$ or the reactive volt-amperes Q.

$$\begin{aligned}\boldsymbol{S} &= S\cos\phi \pm jS\sin\phi = S\underline{/\pm\phi} \\ &= P + jQ \qquad (1.32) \\ &= (P_1 + P_2 + P_3 + \ldots) + j(Q_1 + Q_2 + Q_3 + \ldots) \qquad (1.32a)\end{aligned}$$

Note that if the voltage applied to a device is represented by $\boldsymbol{V} = V\underline{/\theta}$ volts, while the corresponding current through the device is represented by $\boldsymbol{I} = I\underline{/\phi}$ amperes, then the product of these complexors will be $\boldsymbol{VI}\ (= VI\underline{/\theta + \phi})$, which is neither the power nor the total volt-amperes absorbed by the device. This product has no practical significance, the power absorbed always being given by

$$P = VI \cos \text{ (phase angle between } V \text{ and } I)$$

EXAMPLE 1.7 An alternator supplies a load of 200kVA at a power factor of 0·8 lagging, and a load of 50kVA at a power factor of 0·6 leading. Find the total kVA, power, and kVAr supplied, and determine the power factor of the alternator.

$$\begin{aligned} \text{Load 1} &= 200\underline{/-\cos^{-1} 0{\cdot}8} \text{ (i.e. 0·8 lag)} \\ &= 200(0{\cdot}8 - j0{\cdot}6) = 160 - j120\,\text{kVA} \\ \text{Load 2} &= 50\underline{/\cos^{-1} 0{\cdot}6} \text{ (lead)} \\ &= 50(0{\cdot}6 + j0{\cdot}8) = 30 + j40\,\text{kVA} \end{aligned}$$

Therefore

$$\text{Total load} = 190 - j80 = \underline{\underline{206\underline{/-22^\circ\ 50'}\,\text{kVA}}}$$

Also,

$$\text{Total power} = \underline{\underline{190\,\text{kW}}} \quad \text{and} \quad \text{Total kVAr} = \underline{\underline{80\,\text{kVAr lagging}}}$$

$$\text{Alternator power factor} = \cos 22^\circ\ 50', \text{ i.e. } \underline{\underline{0{\cdot}92 \text{ lagging}}}$$

1.9 Mutual Inductance in Networks

When two coils or circuits are linked by a mutual inductance M, then an alternating current I in one, will set up an alternating

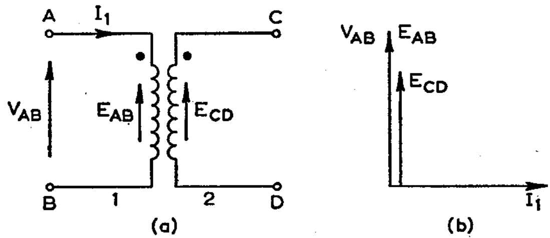


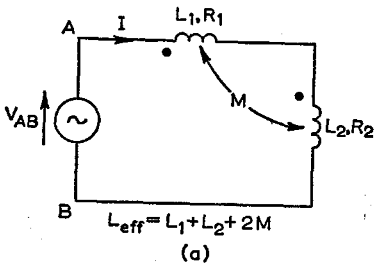
Fig. 1.12 MUTUAL INDUCTANCE
(*a*) Circuit (*b*) Open-circuit complexor diagram

e.m.f. of magnitude $I\omega M$ in the other. A difficulty arises over the direction of the mutually induced e.m.f. in the second coil, since this depends on the relative winding directions and the relative positions of the two coils. The dot notation will be used here to indicate the relative e.m.f. directions. In this notation a dot is placed at an arbitrary end of one coil, and a second dot is placed at the end of the second coil which has the same polarity as the dotted end of the first coil, when the current through either of the coils is changing. In a.c. circuits employing the dot notation for mutual inductance the mutually induced e.m.f. in each coil will be in such a direction as to give the same polarity to the dotted ends of the coils.

In the circuit of Fig. 1.12 a current represented by the complexor I_1 flows in the first coil. Hence, neglecting resistance,

$$V_{AB} = E_{AB} = j\omega L_1 I_1$$

where L_1 is the self-inductance of the first coil.



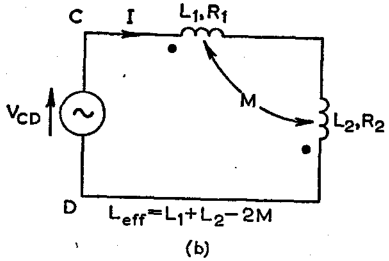


Fig. 1.13 MUTUAL INDUCTANCE BETWEEN PARTS OF THE SAME MESH

(*a*) Coils in series aiding (eqn. (1.35))
(*b*) Coils in series opposing (eqn. (1.36))

The e.m.f. induced by the mutual inductance M of the circuit is E_{CD}, which leads I_1 by 90°, and is given by

$$E_{CD} = j\omega M I_1 \tag{1.33}$$

and

$$\text{Mutual inductive reactance} = E_{CD}/I_1 = j\omega M \tag{1.34}$$

To illustrate the use of the dot notation consider the series circuits of Fig. 1.13. Applying Kirchhoff's laws to the series-aiding circuit at (*a*),

$$V_{AB} = IR_1 + Ij\omega L_1 + Ij\omega M + IR_2 + Ij\omega L_2 + Ij\omega M$$

since the mutual e.m.f. will have the same polarity as the self-induced e.m.f.

Thus the combined impedance is

$$Z = R_1 + R_2 + j\omega(L_1 + L_2 + 2M) \tag{1.35}$$

Similarly, for the series-opposing circuit at (*b*),

$$V_{CD} = IR_1 + Ij\omega L_1 - Ij\omega M + IR_2 + Ij\omega L_2 - Ij\omega M$$

since the mutual e.m.f. in each coil will have the opposite polarity to the self-induced e.m.f.

Therefore the combined impedance is

$$Z = R_1 + R_2 + j\omega(L_1 + L_2 - 2M) \tag{1.36}$$

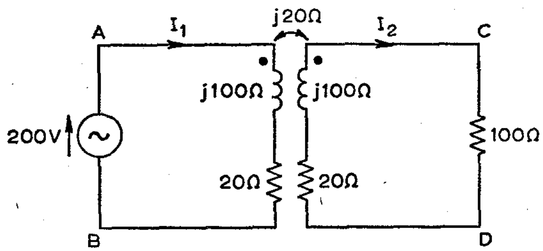


Fig. 1.14

EXAMPLE 1.8 Calculate the currents in each coil of the circuit of Fig. 1.14.

The mesh currents are inserted in arbitrary directions as shown.

For mesh 1,

$$200\underline{/0^\circ} = (20 + j100)I_1 - j20I_2 \tag{i}$$

Since the positive direction of I_2 is shown as entering the second coil at the undotted end, the e.m.f. of self-inductance will make the undotted end of this coil positive. The corresponding mutual e.m.f. in the first coil will make the undotted end of that coil positive; hence, since I_1 leaves the first coil at this end, the voltage drop due to the mutual inductance will be negative (i.e. $-j20I_2$).

Similarly for mesh 2,

$$0 = -j20I_1 + (120 + j100)I_2 \quad \text{i.e.} \quad I_1 = (5 - j6)I_2 \tag{ii}$$

Substituting in eqn. (i),

$$200 = (100 + j500 - j120 + 600 - j20)I_2$$

Hence

$$I_2 = \underline{\underline{0{\cdot}255\underline{/-27{\cdot}3^\circ}\text{A}}} \quad \text{and} \quad I_1 = \underline{\underline{1{\cdot}99\underline{/-77{\cdot}2^\circ}\text{A}}}$$

1.10 Equivalent Mutual Inductance Circuits

Consider the inductively coupled circuit shown in Fig. 1.15 (*a*), where Z_{11} is the total self-impedance of the primary loop (including the primary resistance and self-inductance of the mutual inductor M), and Z_{22} is similarly the total self-impedance of the secondary loop

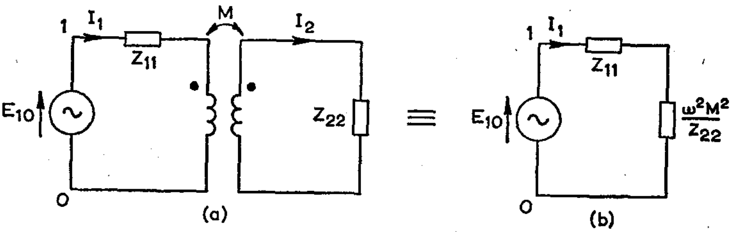


Fig. 1.15 EQUIVALENT CIRCUIT FOR INDUCTIVELY COUPLED NETWORK

(*a*) Network with pure mutual inductance
(*b*) Equivalent circuit

(including the load impedance). With the winding directions indicated, the mesh equations are

$$\text{Mesh 1} \qquad E_{10} = Z_{11}I_1 - j\omega MI_2 \qquad \text{(i)}$$
$$\text{Mesh 2} \qquad 0 = -j\omega MI_1 + Z_{22}I_2 \qquad \text{(ii)}$$

Hence from eqn. (i),

$$E_{10}Z_{22} = Z_{11}Z_{22}I_1 - j\omega MZ_{22}I_2$$

and from eqn. (ii),

$$0 = +\omega^2M^2I_1 + j\omega MZ_{22}I_2$$

Adding,

$$E_{10}Z_{22} = (Z_{11}Z_{22} + \omega^2M^2)I_1$$

Therefore

$$\frac{E_{10}}{I_1} = Z_{11} + \frac{\omega^2M^2}{Z_{22}} \qquad (1.37)$$

= Impedance of network looking into the primary circuit

Hence the circuit of Fig. 1.15 (*a*) may be replaced by a simple circuit, consisting of the self-impedance of mesh 1 in series with the term $\{\omega^2M^2/(\text{self-impedance of mesh 2})\}$ as shown at (*b*). This circuit, when viewed from the generator will always appear to be the same as the original circuit.

It may easily be shown that if the relative winding directions of the mutual inductor are reversed the same result will be obtained.

EXAMPLE 1.9 The coefficient of coupling between the primary and secondary of an air-cored transformer is 0·8. The primary winding has an inductance of 0·6mH and a resistance of 2Ω, while the secondary has an inductance of 5·5mH and a resistance of 20Ω. Calculate the primary current and the secondary terminal voltage when the primary is connected to a 10V, 50kHz supply and the load on the secondary has an impedance of 250Ω at a phase angle of 45° leading. (*L.U.*)

The circuit diagram is shown in Fig. 1.16.

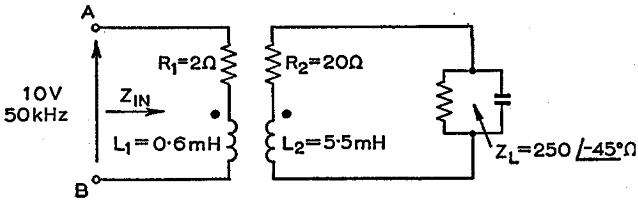


Fig. 1.16

$$\text{Coupling coefficient} = k = \frac{M}{\sqrt{(L_1L_2)}}$$

so that

$$M = 0{\cdot}8\sqrt{(0{\cdot}6 \times 5{\cdot}5)} = 1{\cdot}45\,\text{mH}$$

$$\text{Load impedance, } Z_L = 250\angle{-45^\circ} = (178 - j178)\,\Omega$$

$$\text{Impedance of secondary winding} = 20 + j100\pi \times 5{\cdot}5$$

Therefore

$$Z_{22} = (198 + j1{,}552)\,\Omega$$

$$\text{Impedance of primary circuit, } Z_{11} = (2 + j189)\,\Omega$$

Hence from eqn. (1.37),

$$Z_{in} = Z_{11} + \frac{\omega^2M^2}{Z_{22}}$$

$$= 2 + j189 + \frac{(100\pi)^2 \times 1{\cdot}45^2}{(198 + j1{,}552)}$$

$$= 18{\cdot}8 + j56 = 59\angle{71{\cdot}4^\circ}\,\Omega$$

Therefore

$$\text{Primary current} = \frac{10}{59}\angle{-71{\cdot}4^\circ} = \underline{\underline{0{\cdot}17\angle{-71{\cdot}4^\circ}\,\text{A}}}$$

$$\text{Load voltage} = j\frac{\omega MI_1}{Z_{22}}\,Z_L$$

Therefore

$$\text{Magnitude of load voltage} = \frac{100\pi \times 10^3 \times 1{\cdot}45 \times 10^{-3} \times 0{\cdot}17 \times 250}{1{,}565}$$
$$= \underline{\underline{12{\cdot}4\,\text{V}}}$$

1.11 Series Resonance: Q-factor

The general form of a series *RLC* circuit is shown in Fig. 1.17(*a*).

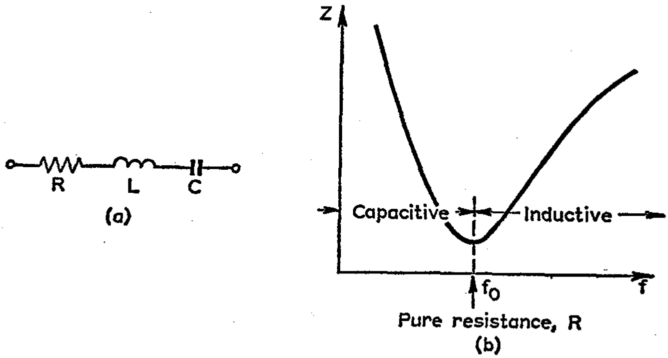


Fig. 1.17 SERIES RESONANCE

The impedance of the circuit is

$$Z = R + j\omega L + \frac{1}{j\omega C} \tag{1.38}$$

$$= R + j\omega L\left(1 - \frac{1}{\omega^2 LC}\right) \tag{1.38a}$$

At the frequency for which $\omega^2 LC = 1$ the quadrate term in this equation will be zero and the impedance will have a minimum value of $Z = R$. The frequency, f_0, at which this occurs is the resonant frequency of the circuit. Thus

$$f_0 = \frac{1}{2\pi\sqrt{(LC)}} \tag{1.39}$$

or

$$\omega_0 = \frac{1}{\sqrt{(LC)}} \tag{1.39a}$$

At resonance $\omega_0 L = 1/\omega_0 C$. At frequencies below the resonant frequency ωL is less than $1/\omega C$, and from eqn. (1.38) the circuit behaves as a capacitive reactance. Above f_0 the circuit behaves as an inductive reactance, since in this case ωL is greater than $1/\omega C$. The variation of impedance of the circuit with frequency is shown in Fig. 1.17(*b*).

It is useful at this point to consider a figure of merit for the coil, known as the coil *Q-factor*, or simply the Q of the coil. The Q-factor may be defined as the ratio of reactance to resistance of a coil:

$$Q = \frac{\text{Reactance}}{\text{Resistance}} = \frac{\omega L}{R} \tag{1.40}$$

Of particular importance is the Q-factor at the resonant frequency of the tuned circuit. This is

$$Q_0 = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR} = \frac{1}{R}\sqrt{\frac{L}{C}} \tag{1.41}$$

The expression for impedance given by eqn. (1.38*a*) can be rewritten in terms of Q and ω_0 as

$$\boldsymbol{Z} = R\left\{1 + jQ\left(1 - \frac{\omega_0^2}{\omega^2}\right)\right\}$$

since $\omega_0^2 = 1/LC$.

For frequencies near the resonant frequency this can be written

$$\begin{aligned}\boldsymbol{Z} &= R\left\{1 + jQ_0\left(1 - \frac{\omega_0^2}{\omega^2}\right)\right\}\\ &= R\left\{1 + jQ_0\left(\frac{\omega^2 - \omega_0^2}{\omega^2}\right)\right\}\\ &= R\left\{1 + jQ_0\frac{(\omega + \omega_0)(\omega - \omega_0)}{\omega^2}\right\}\end{aligned}$$

Again, at frequencies which are near resonance, $\omega \approx \omega_0$ and $\omega + \omega_0 \approx 2\omega$ so that

$$\begin{aligned}\boldsymbol{Z} &\approx R\left\{1 + jQ_0 2\frac{\omega - \omega_0}{\omega_0}\right\}\\ &\approx R\{1 + j2Q_0\delta\}\end{aligned} \tag{1.42}$$

where $\delta = (\omega - \omega_0)/\omega_0 = (f - f_0)/f_0$ is the *per-unit frequency deviation*, or the difference between the circuit frequency, f, and the resonant frequency f_0 expressed as a fraction of f_0. If $f < f_0$, δ is negative, if $f > f_0$, δ is positive.

A convenient way of defining the sharpness of the resonance curve (Fig. 1.17(*b*)) is to find the frequency, f_L, below the resonant frequency and the frequency f_H above it at which the circuit impedance increases to $\sqrt{2}$ of its value at resonance. This figure is chosen since if $Z = \sqrt{2}R$ then from eqn. (1.42),

$$|1 \pm j2Q_0\delta| = \sqrt{2} \qquad \text{so that} \qquad 2Q_0\delta = \pm 1$$

which is a convenient (but purely arbitrary) criterion.

From this

$$\frac{1}{Q_0} = \pm 2\delta = \frac{2(f_H - f_0)}{f_0} \qquad \text{or} \qquad \frac{1}{Q_0} = -2\delta = \frac{2(f_0 - f_L)}{f_0}$$

so that, adding the alternative expressions for $1/Q_0$,

$$\frac{1}{Q_0} = \frac{f_H - f_L}{f_0} \qquad \text{or} \qquad Q_0 = \frac{f_0}{f_H - f_L} \tag{1.43}$$

$f_H - f_L$ $(= f_0/Q)$ is often called the *bandwidth* of the tuned circuit.

Note that at the frequency f_L the circuit impedance is

$$Z_L = R(1 - j1) = \sqrt{2}R\underline{/-45^\circ}$$

while at the frequency f_H the impedance is

$$Z_H = R(1 + j1) = \sqrt{2}R\underline{/+45^\circ}$$

1.12 Parallel Resonance

The usual form of a parallel resonant circuit is shown in Fig. 1.18(a),

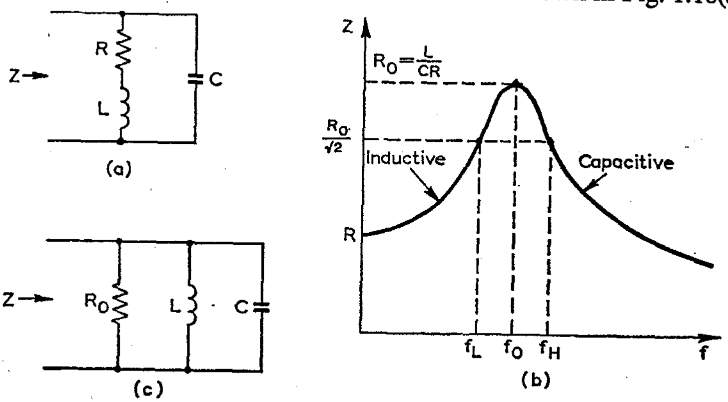


Fig. 1.18 PARALLEL RESONANCE

R being the resistance of the inductor L, and C being a pure capacitance. The impedance of the circuit at an angular frequency ω is

$$Z = \frac{(R + j\omega L)(1/j\omega C)}{R + j\omega L + 1/j\omega C} = \frac{R + j\omega L}{1 - \omega^2 LC + j\omega CR} \tag{1.44}$$

The impedance is R for $\omega = 0$ and zero for $\omega \to \infty$. Between these extremes Z will normally rise to some maximum value as shown at (*b*). Two cases will be considered, (A) when $R \ll \omega L$, and (B) when R is of the same order as ωL.

CASE (A): $R \ll \omega L$

In this case the impedance can be written

$$Z = \frac{L/C}{R + j\omega L(1 - 1/\omega^2 LC)}$$

The R in the denominator cannot be neglected since the term $\omega L(1 - 1/\omega^2 LC)$ may be small. Rewriting the expression for Z,

$$Z = \frac{L/CR}{1 + \dfrac{j\omega L}{R}\left(1 - \dfrac{1}{\omega^2 LC}\right)} \tag{1.45}$$

At the frequency, f_0, for which $\omega L = 1/\omega C$, the impedance will be resistive and will have a maximum value of L/CR. This is called the *dynamic resistance* of the circuit, R_0. The frequency f_0 is the resonant frequency. Thus

$$R_0 = L/CR \tag{1.46}$$

and

$$f_0 = 1/2\pi\surd(LC) \qquad \text{or} \qquad \omega_0 = 1/\surd(LC) \tag{1.47}$$

Using the Q-factor as defined in Section 1.11, and considering frequencies near resonance (for which $Q = Q_0 = \omega_0 L/R$), the expression for impedance given by eqn. (1.45) can be written

$$Z = \frac{R_0}{1 + jQ_0\left(1 - \dfrac{\omega_0^2}{\omega^2}\right)}$$

Defining $\delta\ (= (f - f_0)/f_0)$ as in Section 1.11, it is apparent that the expression for impedance can be reduced to

$$Z = \frac{R_0}{1 + j2Q_0\delta} \tag{1.48}$$

in the same manner as for the series circuit. Similarly the bandwidth can be defined in terms of the frequency f_L below f_0 at which the impedance falls to $1/\surd 2$ of its value, R_0, at resonance and the frequency f_H above f_0 at which this same fall in impedance occurs. In both cases therefore

$$|1 \pm j2Q_0\delta| = \surd 2$$

so that, as in the series-resonance case,

$$2Q_0\delta = \pm 1$$

It follows that

$$\frac{1}{Q_0} = \pm 2\delta = \frac{f_H - f_L}{f_0}$$

Note that because Q must always be positive the minus sign in the above expression is required since δ will be negative if $f < f_0$. Hence the bandwidth is

$$f_H - f_L = \frac{f_0}{Q_0} \tag{1.49}$$

as for the series circuit.

The circuit impedances at f_L and f_H are given by

$$Z_L = R_0/(1 - j1) = 0{\cdot}707R_0\underline{/45^\circ}$$

$$Z_H = R_0/(1 + j1) = 0{\cdot}707R_0\underline{/-45^\circ}$$

Notice that the impedance has an inductive reactive component at frequencies below f_0 while it has a capacitive component at frequencies above f_0.

If the tuned circuit is supplied from a constant-current generator, the voltage across it will fall to 0·707 of its value at resonance at both f_L and f_H. Since the power factor at these frequencies is 0·707, the power dissipated will be $0{\cdot}707^2$ (= 0·5) times the power at resonance. For this reason f_L and f_H are known as the *half-power frequencies*.

A useful equivalent of the parallel tuned circuit is shown in Fig. 1.18(*c*). From eqn. (1.45), the admittance of the circuit of Fig. 1.18(*a*) is

$$Y = \frac{1}{Z} = \frac{CR}{L} + j\omega C + \frac{1}{j\omega L} = \frac{1}{R_0} + j\omega C + \frac{1}{j\omega L} \tag{1.50}$$

i.e. the admittance is equivalent to that of a three-element parallel circuit consisting of a pure resistance, R_0, in parallel with a pure capacitance, C, and a pure inductance, L, as shown in Fig. 1.18(*c*).

EXAMPLE 1.10 The inductance and magnification factor* of a coil are 200μH and 70 respectively. If this coil is connected in parallel with a capacitor of 200pF, calculate the magnitude and phase angle of the impedance of the parallel circuit for a frequency 0·8 per cent below the resonant frequency and the half-power bandwidth. (*H.N.C.*)

* Magnification factor is an alternative name for Q-factor.

$$\text{Resonant angular frequency} = \omega_0 = \frac{1}{\sqrt{(LC)}} = \frac{10^9}{\sqrt{(200 \times 200)}}$$
$$= 5 \times 10^6 \text{ rad/s}$$

$$Q_0 = 70 = \frac{\omega_0 L}{R}$$

and

$$R_0 = \text{Resonant impedance} = \frac{L}{CR} = \frac{\omega_0 L}{\omega_0 CR} = \frac{Q_0}{\omega_0 C} = Q_0\omega_0 L$$

Therefore

$$R_0 = Q_0\omega_0 L = 70 \times 5 \times 10^6 \times 200 \times 10^{-6} = 70{,}000\,\Omega$$

At 0·8 per cent below resonance

$$\omega = 0{\cdot}992\omega_0$$

Thus

$$\text{Impedance, } Z = \frac{70{,}000}{1 + j70\left(1 - \left(\frac{1}{0{\cdot}992}\right)^2\right)}$$
$$= \frac{70{,}000}{1 + j70 \times \dfrac{(0{\cdot}992 - 1)(0{\cdot}992 + 1)}{0{\cdot}992^2}}$$
$$= \underline{\underline{46{,}400\angle 48{\cdot}6^\circ\,\Omega}}$$

The half-power bandwidth is given by

$$f_H - f_L = f_0/Q_0 = 5\times\ 10^6/2\pi \times 70 = \underline{\underline{11{\cdot}37\text{kHz}}}$$

CASE (B): TUNED CIRCUIT WITH APPRECIABLE RESISTANCE

If the tuned circuit of Fig. 1.18(a) has appreciable resistance, the simplifying assumptions made above cannot be used. The impedance, Z, is given by eqn. (1.44):

$$Z = \frac{(R + j\omega L)(1/j\omega C)}{R + j\omega L + 1/j\omega C} = \frac{R + j\omega L}{1 - \omega^2 LC + j\omega CR}$$

In this case there are three ways of defining the resonant frequency. These are (i) the frequency at which $\omega L = 1/\omega C$ (called the series resonant frequency), (ii) the frequency at which the circuit is purely resistive, (iii) the frequency at which the impedance is a maximum. Rationalizing the denominator of the expression for Z gives

$$Z = \frac{(R + j\omega L)(1 - \omega^2 LC - j\omega CR)}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}$$
$$= \frac{R + j\omega\{L(1 - \omega^2 LC) - CR^2\}}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}$$

This has a zero quadrate term when

$$L(1 - \omega^2 LC) - CR^2 = 0$$

i.e. when

$$\omega^2 = \frac{1}{LC} - \frac{R^2}{L^2}$$

or

$$\omega = \sqrt{\left(\frac{1}{LC} - \frac{R^2}{L^2}\right)} \tag{1.51}$$

This equation gives the frequency at which the circuit impedance is purely resistive. Notice that, when $R^2/L^2 \geqslant 1/LC$ (i.e. when $R \geqslant \sqrt{(L/C)}$, ω^2 becomes zero, and there will be no frequency at which the circuit is purely resistive.

A special case occurs when there is a *series* resistance of $\sqrt{(L/C)}$ in each branch of the circuit. The impedance is then

$$\begin{aligned} \boldsymbol{Z} &= \frac{\{\sqrt{(L/C)} + j\omega L\}\{\sqrt{(L/C)} + 1/j\omega C\}}{2\sqrt{(L/C)} + j\omega L + 1/j\omega C} \\ &= \frac{2L/C + j\sqrt{(L/C)}(\omega L - 1/\omega C)}{2\sqrt{(L/C)} + j(\omega L - 1/\omega C)} \\ &= \sqrt{\frac{L}{C}} \end{aligned} \tag{1.52}$$

i.e. the impedance is a pure resistance, $R = \sqrt{(L/C)}$ at all frequencies.

The impedance is a maximum at a frequency which lies between the series resonant frequency and that for unity power factor. The natural resonant frequency will be dealt with in Chapter 6.

1.13 Reactance/Frequency Graphs

When circuits contain several reactive elements, multiple resonances (both series and parallel) may occur. In many cases the operation of such circuits can be seen by sketching the graph of reactance to a base of frequency. Such graphs (and graphs giving the variation of susceptance with frequency) can be built up from the well-known form of reactance/frequency curves for a pure inductance and a pure capacitance.

For a pure inductance,

$$\text{Impedance} = jX_L = j\omega L$$

so that X_L varies linearly with frequency.

For a pure capacitance,

$$\text{Impedance} = -jX_C = \frac{-j}{\omega C}$$

so that X_C varies inversely with frequency, and may be considered to be negative. The form of these curves is shown in Fig. 1.19.

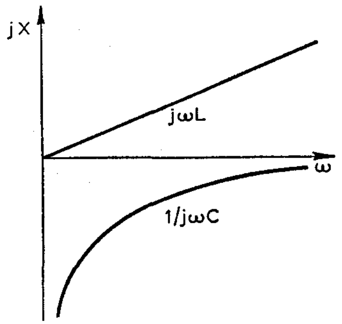


Fig. 1.19 REACTANCE/ANGULAR-FREQUENCY GRAPHS FOR PURE INDUCTANCE AND PURE CAPACITANCE

SIMPLE SERIES CIRCUIT

For a simple series circuit consisting of a pure inductance in series with a pure capacitance, the impedance is $Z = j\omega L - j/\omega C$.

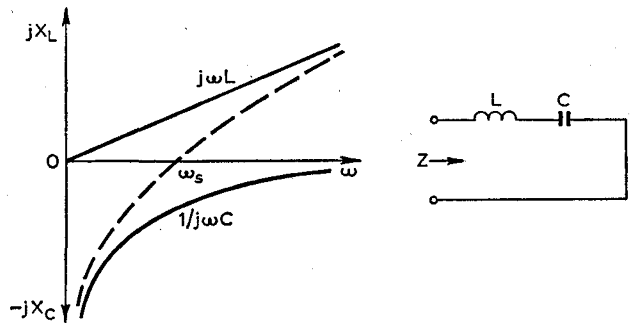


Fig. 1.20 SIMPLE *LC* SERIES CIRCUIT

The overall reactance graph is obtained by adding the individual graphs of $j\omega L$ and $-j/\omega C$ for each value of ω as shown in Fig. 1.20. The series resonant frequency is then obtained by the intersection of this resultant graph with the axis of ω at ω_s.

SIMPLE PARALLEL CIRCUIT

For a parallel circuit comprising a pure inductance and a pure capacitance, it is simpler to work initially in terms of admittance. Thus for the circuit of Fig. 1.21, the admittance is

$$Y = j\omega C - \frac{j}{\omega L}$$

The capacitive susceptance, $j\omega C$, is positive and varies linearly with ω, the inductive susceptance, $-j/\omega L$, is negative and varies inversely with ω, and the resultant graph of admittance, Y, crosses

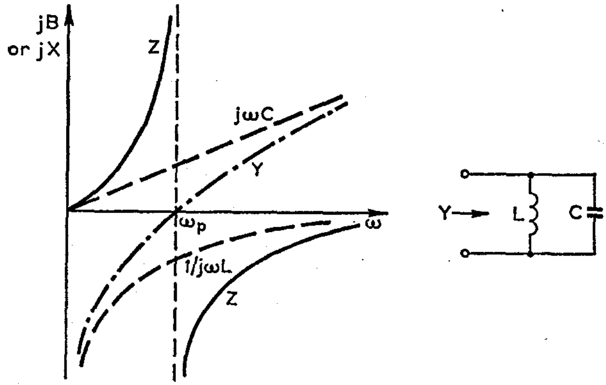


Fig. 1.21 SIMPLE LC PARALLEL CIRCUIT

the axis of ω at the parallel-resonant angular frequency, ω_p, as shown. The curve of impedance to a base of ω can now be sketched. Thus when $Y \to -j\infty$ (at the origin), the impedance tends to zero. Between $\omega = 0$ and $\omega = \omega_p$, the admittance is a negative susceptance, and hence the impedance must be a positive reactance which rises towards ∞ as ω approaches ω_p (since for $\omega = \omega_p$, $Y = 0$ and hence $Z = \pm j\infty$).

ω_p gives the parallel-resonant angular frequency as stated above.

Above $\omega = \omega_p$ the admittance is positive and rises towards infinity; hence the impedance is negative and rises from $-j\infty$ when $\omega = \omega_p$ (where $Y = 0$) towards zero as $\omega \to \infty$. The curves for Z are shown as the full lines in Fig. 1.21.

1.14 General Form of the Reactance/Frequency Graph

Several general points concerning reactance/frequency graphs are evident from the previous section. Thus these graphs will (i) always have a positive slope, (ii) always start at $X = 0$ or $X = -j\infty$ (since there is no reactance which has any other value for $\omega = 0$), (iii) will always finish either tending towards zero from $-j\infty$ (if the network reactance is capacitive as $\omega \to \infty$) or towards infinity (if the network reactance is inductive as $\omega \to \infty$), and (iv) will have a number of series resonances (where the graph cuts the frequency axis) and a number of parallel resonances (where the graph changes from $+j\infty$ to $-j\infty$) depending on the network configuration. These can easily be obtained by inspection of a given network.

The reactance/frequency graphs for complicated networks of purely inductive and purely capacitive elements can be built up by considering what happens when additional elements are added (*a*) in series or (*b*) in parallel with the original network.

ADDING SERIES REACTANCE

In Fig. 1.22(*a*) the full lines indicate the reactance/frequency graph

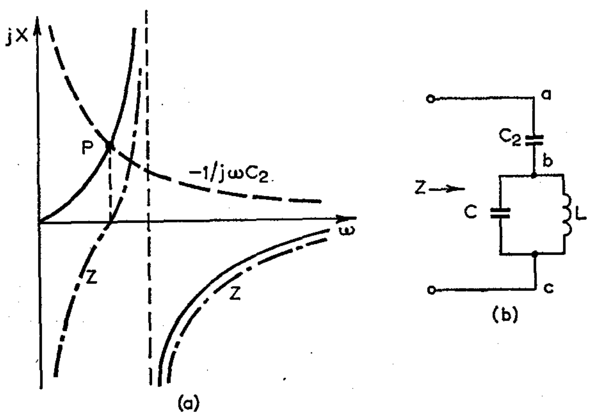


Fig. 1.22

for the parallel *CL* circuit shown at (*b*). The addition of series reactance cannot affect the parallel resonant frequency, but it will cause series resonance whenever the magnitude of the added reactance is equal to that of the original network but is of opposite sign. The angular frequency at which this occurs is readily obtained by plotting the *negative* of the added series reactance on the same graph. In the example shown, the curve of $-1/j\omega C_2$ $(= j/\omega C_2)$ is shown dashed at (*a*). The two curves cut at P, which gives the value of the additional series resonance, ω_s. The resultant reactance curve (shown chain dotted) is thus easily constructed. If the added series element is itself a parallel circuit the same rules apply, but in this case there will be a second parallel resonant frequency. Any series resonances in the original network will be modified by the added reactance.

ADDING PARALLEL REACTANCE

Adding reactance in parallel with a given *LC* network cannot affect the series resonances of the original network, since the network impedance is zero at these frequencies. It will, however, give rise to additional parallel resonances at those frequencies for which the added reactance is equal in magnitude but opposite in sign to that of the original network. The location of these new parallel resonances is readily obtained by superimposing the negative of the added reactance on the original reactance/frequency graph. The original parallel resonances will all be modified.

EXAMPLE 1.11 Plot the reactance/angular-frequency graph for the circuit shown in Fig. 1.23(*a*) over the range 0–3,000 rad/s. Obtain from the graph the angular frequencies at which (i) the reactance is zero, (ii) the reactance is infinite. Over what ranges is the reactance (*a*) inductive (*b*) capacitive? *(H.N.C.)*

Sketch the reactance/angular-frequency curve if a capacitor C_2 of $2\,\mu F$ is now connected in parallel with the whole circuit.

From the values given the following table can be constructed.

ω	0	500	1,000	2,000	3,000
$\omega C_1(\times 10^{-3})$	0	$j0{\cdot}5$	$j1$	$j2$	$j3$
$1/\omega L_1(\times 10^{-3})$	$-j\infty$	$-j2$	$-j1$	$-j0{\cdot}5$	$-j0{\cdot}33$
$Y_p(\times 10^{-3})$	$-j\infty$	$-j1{\cdot}5$	0	$j1{\cdot}5$	$j2{\cdot}67$
$Z_p = 1/Y_p$	0	$j667$	$\pm j\infty$	$-j667$	$-j375$
$j\omega L_2$	0	$j250$	$j500$	$j1{,}000$	$j1{,}500$
$1/j\omega C_2$	0	$-j1{,}000$	$-j500$	$-j250$	$-j167$

From this the reactance/angular-frequency graph can be plotted as shown at (*b*).

Following the rules for adding reactances in series, the values of $-j\omega L_2$ are also plotted on the above graph. This curve intersects the curve for Z_p at point X, and hence gives the series-resonant angular frequency as $\omega_s = 1{,}700$ rad/s. The parallel-resonant angular frequency is not affected by the added series element, and is (from the graph) $\omega_p = 1{,}000$ rad/s. The total impedance, Z_T, is found by adding $j\omega L_2$ to the parallel circuit impedance (chain dotted curve at (*b*)). Hence the reactance is zero when $\omega = 0$ or $\omega = \omega_s$ i.e. at 0 and 1,700 rad/s.

Also, the reactance is infinite when $\omega = \omega_p =$ 1,000 rad/s.

The circuit is inductive between $\omega = 0$ and $\omega = 1{,}000$ rad/s and between $\omega = 1{,}700$ rad/s and ∞, and is capacitive between $\omega = 1{,}000$ and $\omega = 1{,}700$ rad/s.

The curves for Z_T are replotted at (*c*), and the curve of $-1/j\omega C_2$ is superimposed as shown, to cut the Z_T curves at P and Q. These points therefore give the new parallel-resonant frequencies of the circuit when C_2 is connected in parallel with it. For $\omega = 0$, the inductances L_1 and L_2 form a short-circuit, so that the overall impedance is zero. As $\omega \to \infty$ the capacitor C_2 short-circuits the input and hence the impedance tends to zero. Hence the form of the reactance/angular-frequency curve is as shown chain dotted at (*c*). The series resonance point is, of course, unaffected by the added parallel capacitance, so that the resultant curve must still pass through ω_s.

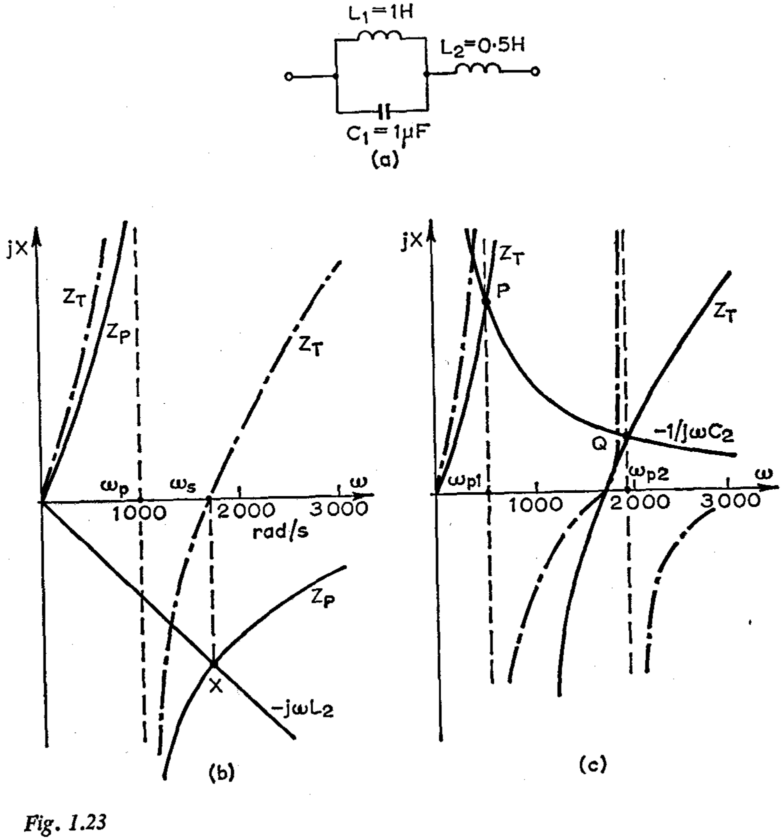


Fig. 1.23

PROBLEMS

The following complex numbers refer to Problems 1.1–1.5:

$A = 1 + j7$, $B = 3 + j3$, $C = -7 - j9$, $D = 5 - j12$, $E = 8 - j4$, $F = -7 + j2$, $G = -5 + j8$, $H = -10 - j10$, $J = 5 + j13$, $K = 6 + j2$.

1.1 Express these complex numbers in polar form.

Ans. $7{\cdot}07\underline{/81{\cdot}8^\circ}$; $4{\cdot}24\underline{/45^\circ}$; $11{\cdot}4\underline{/-127{\cdot}9^\circ}$; $13\underline{/-67{\cdot}3^\circ}$; $8{\cdot}95\underline{/-26{\cdot}6^\circ}$; $7{\cdot}28\underline{/164^\circ}$; $9{\cdot}45\underline{/122^\circ}$; $14{\cdot}14\underline{/-135^\circ}$; $13{\cdot}95\underline{/69^\circ}$; $6{\cdot}32\underline{/18{\cdot}4^\circ}$.

1.2 Find $A + B$, $C + D$, $E + F$, $G + H$, $J + K$, and express the answers in polar form.

Ans. $10{\cdot}78\underline{/68{\cdot}2^\circ}$; $21{\cdot}05\underline{/-95{\cdot}5^\circ}$; $2{\cdot}25\underline{/-63{\cdot}4^\circ}$; $15{\cdot}1\underline{/-172^\circ}$; $18{\cdot}6\underline{/53{\cdot}7^\circ}$.

1.3 Find $A - B$, $C - D$, $E - F$, $G - H$, $J - K$, and express the answers in polar form.

Ans. $4{\cdot}47\underline{/116{\cdot}6^\circ}$; $12{\cdot}38\underline{/165{\cdot}9^\circ}$; $16{\cdot}15\underline{/-21{\cdot}8^\circ}$; $18{\cdot}7\underline{/74{\cdot}4^\circ}$; $11{\cdot}04\underline{/95{\cdot}2^\circ}$.

1.4 Find AB, CD, EF, GH, JK, and express the answers in rectangular form.

Ans. $-18 + j24$; $-143 + j39$; $-48 + j44$; $130 - j30$; $4 + j88$.

1.5 Find A/B, C/D, E/F, G/H, J/K, and express the answers in rectangular form.

Ans. $1{\cdot}33 + j1$; $0{\cdot}432 - j0{\cdot}763$; $-1{\cdot}21 + j0{\cdot}226$; $-0{\cdot}15 - j0{\cdot}65$; $1{\cdot}4 + j1{\cdot}7$.

1.6 Two impedances $(5 + j7)\,\Omega$ and $(10 - j5)\,\Omega$ are connected (*a*) in series, (*b*) in parallel to a 200 V supply. Calculate in each case (*a*) the current drawn from the mains, (*b*) the power supplied, and (*c*) the power factor.

Ans. 13·2 A, 2,620 W, 0·991 lagging; 31·5 A, 5,900 W, 0·937 lagging.

1.7 Explain how alternating quantities can be represented by complex operators. If the potential difference across a circuit is represented by $(40 + j25)$ volts and the circuit consists of a resistance of 20 Ω in series with an inductance of 0·06 H and the frequency is 79·5 Hz find the complex operator representing the current in amperes. (*L.U.*)

Ans. $(1{\cdot}19 - j0{\cdot}54)$ A.

1.8 The impedance of a coil at a frequency of 1 MHz can be expressed as $(300 + j400)\,\Omega$. What do you understand by the symbols of the expression, and what information does this expression convey that is lacking from the simple statement that the impedance of the coil is 500 Ω?

This coil is connected in parallel with a capacitor of capacitance 159 pF. Calculate the impedance of the combined circuit, expressing it in the complex form $\boldsymbol{Z} = R + jX$. (*L.U.*)

Ans. $(667 + j333)\,\Omega$.

1.9 If the impedance of a circuit is expressed in the form $R + jX$, deduce an expression for the corresponding admittance. Two circuits of impedance $(9 - j12)\,\Omega$ and $(4 + j3)\,\Omega$ are connected (*a*) in series, and (*b*) in parallel across a supply of $V = (200 + j150)$ volts. Find the total current, power and power factor in each case.

Ans. 15·8 A, 3,250 W, 0·822 leading; 52·6 A, 12,500 W, 0·948 lagging.

1.10 The series portion of a series-parallel circuit consists of a coil P, the inductance of which is 0·05 H and the resistance 20 Ω. The parallel portion consists of two branches A and B. Branch A consists of a coil Q the inductance of which is 0·1 H and the resistance 30 Ω, branch B consists of a 100 μF capacitor in series with a 15 Ω resistor. Calculate the current and power factor from a 230 V a.c. supply when the frequency is 50 Hz. (*L.U.*)

Ans. 4·4 A; 0·996.

1.11 Two coils of resistance 9 Ω and 6 Ω and inductance 0·0159 H and 0·0382 H respectively are connected in parallel across a 200 V, 50 Hz supply. Calculate: (*a*) the conductance, susceptance and admittance of each coil and the entire circuit; (*b*) the current and power factor for each coil and for the total circuit; and (*c*) the total power taken from the supply.

Ans. (*a*) 0·085 S, 0·047 S, 0·097 S; 0·033 S, 0·066 S, 0·074 S; 0·118 S, 0·113 S, 0·164 S. (*b*) 19·4 A, 0·877 lagging; 14·8 A, 0·446 lagging; 32·8 A, 0·722 lagging. (*c*) 4,720 W.

1.12 Three circuits having impedances of $(10 + j30)$, $(20 + j0)$ and $(1 - j20)\,\Omega$ are connected in parallel across a 200V supply. Find the total current flowing and its phase angle.

Ans. 13·1 A; 17·6° leading.

1.13 Three impedances Z_1, Z_2, and Z_3 are connected in parallel to a 240 V, 50 Hz supply. If $Z_1 = (8 + j6)\,\Omega$ and $Z_2 = (12 + j20)\,\Omega$, determine the complex impedance of the third branch if the total current is 35 A at a power factor of 0·9 lagging.

Ans. $(15 - 16{\cdot}6)\,\Omega$.

1.14 Coils of impedances $(8 + j6)\,\Omega$ and $(15 + j10)\,\Omega$ are connected in parallel. In series with this combination is an impedance of $(20 - j31)\,\Omega$. The supply is $200\angle 0°$ V at 50 Hz. Calculate in polar form: (*a*) the total impedance, (*b*) the current in each branch and the total current, and (*c*) the power factor.

Ans. (*a*) $37{\cdot}1\angle -47{\cdot}2°\,\Omega$; (*b*) $3{\cdot}48\angle 46°$ A, $1{\cdot}93\angle 49{\cdot}2°$ A, $5{\cdot}4\angle 47{\cdot}2°$ A; (*c*) 0·681 leading.

1.15 Two impedances $Z_1 = (6 + j3)\,\Omega$ and $Z_2 = (5 - j8)\,\Omega$ are connected in parallel and an impedance $Z_3 = (4 + j6)\,\Omega$ is connected in series with them across a 2·5 V, 100 kHz supply.

Determine: (*a*) the complex expressions for the admittance of each section and of the whole circuit, (*b*) the current and phase angle of the whole circuit, and (*c*) the total power taken from the supply.

Ans. (*a*) $0{\cdot}149\angle -26{\cdot}6°$, $0{\cdot}106\angle 58°$, $0{\cdot}139\angle -56{\cdot}3°$, $0{\cdot}094\angle -30{\cdot}4°$ mho; (*b*) 0·24 A, 30·4° lagging; (*c*) 510 mW.

1.16 A series circuit consists of a coil of impedance Z_1 and two other impedances Z_2 and Z_3. A voltmeter V_1 is connected to read the p.d. across Z_1 and a second voltmeter V_2 reads the p.d. across Z_2.

When a 100 V, d.c. supply is applied to the circuit, the current is 2 A and V_1 and V_2 read 30 V and 50 V respectively. With a 210 V supply at 50 Hz applied the current in the same circuit is 3 A at a lagging power factor and V_1 and V_2 now read 60 V and 75 V respectively.

Find the complex expressions for the three impedances. If they are now connected in parallel to a supply of 250 V, 50 Hz, calculate for each branch the current and its phase angle with respect to the supply voltage. The voltmeters read correctly on both a.c. and d.c. supplies. Assume no iron losses and that Z_3 is inductive.

Ans. $Z_1 = 20\angle 41{\cdot}4°\,\Omega$, $Z_2 = 25\angle 0°\,\Omega$, $Z_3 = 37{\cdot}2\angle 74{\cdot}4\,\Omega$; $12{\cdot}5\angle -41{\cdot}42$ A, $10\angle 0°$ A, $6{\cdot}72\angle -74{\cdot}4°$ A.

1.17 A circuit having a constant resistance of 60Ω and a variable inductance of 0 to 0·4 mH is connected across a 5 V, 50 kHz supply. Derive from first principles the locus of the extremity of the current complexor.

Find (*a*) the power, and (*b*) the inductance when the power factor is 0·8.

Ans. (*a*) 265 mW; (*b*) 0·143 mH. (*L.U.*)

1.18 A variable capacitor and a resistance of 300Ω are connected in series across 240 V, 50 Hz mains.

Draw the complexor loci of impedance and current as the capacitance changes from 5 μF to 30 μF.

From the diagram find (*a*) capacitance to give current of 0·7 A, (*b*) current when capacitance is 10 μF. (*L.U.*)

Ans. 19·2 μF; 0·55 A.

1.19 A circuit consisting of an inductor L in series with a resistance r, which is variable between zero and infinity, is connected to a constant-voltage constant-frequency supply. Prove that the locus of the extremity of the current complexor is a semicircle.

If $V = 400\underline{/0^\circ}$ V, $f = 50$ Hz, $L = 31{\cdot}8$ mH, draw to scale the current locus. Calculate the maximum power input, the corresponding current and power factor, and the value of r for this condition. (*H.N.C.*)

Ans. 8,000 W; 28·28 A; 0·707; 10Ω.

1.20 Two coils have inductances of 250 μH and 100 μH. They are placed so that their mutual inductance is 50 μH. What will be their effective inductance: (*a*) in series aiding, (*b*) in series opposing, (*c*) in parallel aiding, and (*d*) in parallel opposing? Deduce the formula for the effective inductance in the four cases.

Ans. 450 μH; 250 μH; 90 μH; 50 μH.

1.21 The load on an alternator consists of:

(i) a lighting and heating load of 700 kW at unity p.f.,

(ii) a motor load of 709 kW which has an average efficiency of 0·9 and an overall power factor of 0·8 lagging,

(iii) a synchronous motor load absorbing 50 kW at a leading power factor of 0·6.

Calculate the minimum rating of the alternator, and determine the additional power which it could supply if the load power factors were improved to unity. (*H.N.C.*)

Ans. 1,625 kVA; 85 kW.

1.22 A single-phase cable has a maximum carrying capacity of 173 A. It supplies the following loads at 440 V:

(i) a 15 kW lighting load at unity p.f.,

(ii) a 30 kVA motor load at a power factor of 0·8 lagging.

Determine, graphically or otherwise, the maximum kVA rating of an additional load at 0·7 power factor lagging which could be connected to the cable. If the power factor of the additional load is improved to unity by apparatus which has a constant loss of 2 kW, determine the new maximum kW rating of the additional load which can be installed. (*H.N.C.*)

Ans. 35 kVA, 33 kW.

1.23 In the circuit shown in Fig. 1.24 a current of 2 A at 50 Hz is fed in at X

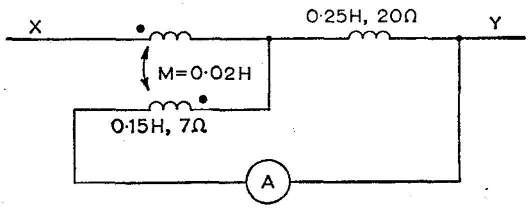


Fig. 1.24

and out at Y. Calculate the current in the ammeter A for the winding directions shown. The ammeter impedance may be neglected. (*L.U.*)

Ans. 1·17 A.

1.24 Two tuned circuits are coupled inductively, the coupling coefficient being 0·01. The primary consists of a coil of 4 mH and 20 Ω and a capacitor of 200 pF and the secondary has a coil of 1 mH and 10 Ω and is tuned to the resonant frequency of the primary. Calculate the magnitude and phase of the secondary current in terms of primary input voltage. If the primary were driven at half its natural frequency, calculate its input impedance. (*H.N.C.*)

Ans. $0{\cdot}0319 V_1\underline{/90^\circ}$ volts, 6,725 Ω capacitive reactance.

1.25 For a series *LCR* resonant circuit supplied at constant voltage, show that the bandwidth between the half-power (3 dB) points is given by

$$\Delta\omega = \omega_H - \omega_L = \frac{R}{L}$$

The *Q*-factor and resonant frequency of such a circuit are 100 and 51 kHz respectively. Find the bandwidth between the half-power points. (*H.N.C.*)

Ans. 510 Hz.

1.26 Plot the reactance/angular-frequency curve over the range 0–2.5 × 10^6 rad/s for a circuit comprising an inductance of 500 μH in parallel with a capacitance of 2,000 pF, all in series with a capacitance of 5,000 pF. If a second inductance of 250 μH is now connected in parallel with the whole circuit, sketch the resultant graph and determine the series- and parallel-resonant angular frequencies.

Ans. $\omega_s = 0{\cdot}6 \times 10^6$ rad/s; $\omega_p = 0{\cdot}55 \times 10^6$ and $1{\cdot}9 \times 10^6$ rad/s.

1.27 An inductor having negligible self-capacitance is connected in parallel with a loss-free variable air-capacitor and an electronic voltmeter whose input impedance is equivalent to 6 pF in parallel with 1 MΩ. When the circuit is energized from a constant current source, a maximum indication on the electronic voltmeter is obtained at a frequency of 0·8 MHz, with the capacitor set at 79 pF. Increasing or decreasing this capacitance by 2 pF reduces the indication on the voltmeter to 70·7 % of the maximum indication. Calculate the inductance and *Q*-factor of the inductor.

Ans. 466 mH, 47·2.

1.28 A coil, having negligible self-capacitance, has a resistance of 8 Ω at a frequency of 750 kHz, and a capacitor of 350 pF is required to produce a parallel resonant circuit at this frequency. Calculate the Q-factor of the coil and the resonant impedance of the circuit. What will be the Q-factor and bandwidth of the circuit if a 50 kΩ resistor is connected in parallel with the capacitor?

Ans. 75·7, 45·7 kΩ, 39·6, 19 kHz.

Chapter 2

CIRCUITS AND CIRCUIT THEOREMS

In any electric circuit the currents and voltages at any point may be found by applying Kirchhoff's laws. As the complexity of the circuit increases, however, the labour involved in the solution becomes multiplied, and several electrical circuit theorems have been developed which reduce the amount of work required for a solution.

The theorems considered here apply to linear circuits, i.e. to circuits with impedances which are independent of the direction and magnitude of the current. They apply to both a.c. and d.c. circuits, provided that in linear a.c. circuits the voltages and currents are expressed as complexors, and the impedances are represented by complex operators. Worked examples will serve to illustrate the method in which the theorems are applied.

In these circuits there are two types of source—the *constant-voltage* source and the *constant-current* source. A constant-voltage source is one which generates a constant predetermined e.m.f., $\boldsymbol{E_E}$, which may be alternating or direct, and has a series internal impedance $\boldsymbol{Z_i}$. A constant-current source is one which produces a constant predetermined internal current, $\boldsymbol{I_E}$, which may be alternating or direct and has a shunt internal impedance $\boldsymbol{Z_i}$. The graphical symbols for these sources are shown in Fig. 2.1.

In general, a practical source with linear characteristics (having a terminal voltage which falls in proportion to the current) may be represented in either of the above forms. If V_{oc} is the open-circuit terminal voltage and I_{sc} the terminal short-circuit current of a practical source, then the effective internal source impedance is $Z_i = V_{oc}/I_{sc}$. Such a source may be equally well represented by either (*a*) a constant-voltage source with e.m.f. $E_E = V_{oc}$ and series internal impedance Z_i, or (*b*) a constant-current source with $I_E = I_{sc}$

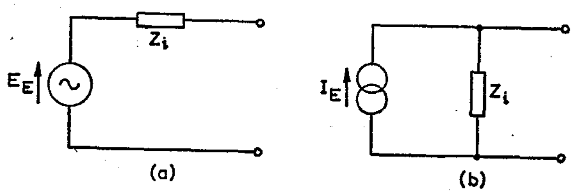


Fig. 2.1 REPRESENTATION OF LINEAR SOURCES

(*a*) Constant-voltage generator
(*b*) Constant-current generator

and shunt internal impedance Z_i. It is convenient in calculation to be able to choose either form of source to represent a practical source.

2.1 Multi-mesh Networks

Multi-mesh networks are made up of branches which form closed loops or *meshes*. The junction points of the impedances are known as *nodes*. Kirchhoff's laws in complex form can be used to solve for the currents and voltages in the network. These laws are

1. The complexor sum of the currents at any node in a network is zero.
2. The complexor sum of the e.m.f.s round any closed loop is equal to the complexor sum of the potential drops round the same loop.

Circuits involving multiple meshes may be solved by considering either the meshes (mesh analysis) or the junctions (node analysis). The method chosen will depend on whether a given network gives rise to fewer mesh equations than node equations, or vice versa.

MESH ANALYSIS

Using Maxwell's mesh-current method, each closed loop in the network is assumed to carry a mesh current. The actual current in any branch of the network is then the complexor sum of the mesh currents which flow through that branch. If the internal mesh currents are all assumed to circulate in the same sense, the mesh equations take a standard form. Thus for the circuit of Fig. 2.2,

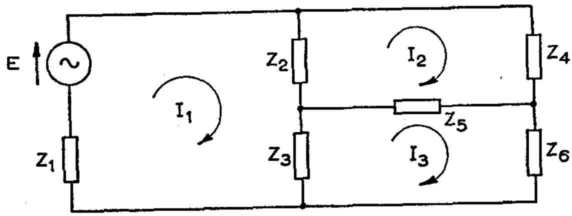


Fig. 2.2

$$\left.\begin{array}{ll} \text{Mesh 1} & E = I_1(Z_1 + Z_2 + Z_3) - I_2Z_2 - I_3Z_3 \\ \text{Mesh 2} & 0 = -I_1Z_2 + I_2(Z_2 + Z_4 + Z_5) - I_3Z_5 \\ \text{Mesh 3} & 0 = -I_1Z_3 - I_2Z_5 + I_3(Z_3 + Z_5 + Z_6) \end{array}\right\} \qquad (2.1)$$

These equations are now solved for the unknown currents, remembering that all the quantities are to be expressed in complex form. The same method may be applied to any other network configuration, but it is obvious that the labour of solution increases rapidly with complexity of the network.

The general form of the mesh equations may be seen by writing $Z_{11} = Z_1 + Z_2 + Z_3$ (the self-impedance of mesh 1), $Z_{12} = Z_2$ (the mutual impedance between meshes 1 and 2), etc., so that

$$\begin{aligned} E_1 &= I_1Z_{11} - I_2Z_{12} - I_3Z_{13} - \ldots \\ E_2 &= -I_1Z_{12} + I_2Z_{22} - I_3Z_{23} - \ldots \\ E_3 &= -I_1Z_{13} - I_2Z_{23} + I_3Z_{33} - \ldots \end{aligned}$$

where E_1, E_2 . . . are the mesh e.m.f.s.

NODE ANALYSIS

In node analysis, potentials V_1, V_2, etc., are assumed at the circuit nodes. If any sources of e.m.f. are present these are represented by the equivalent constant-current generators. If now the admittances (or, of course, impedances) of each branch are known the node potentials and branch currents can be found. Thus for the circuit of Fig. 2.3:

$$\begin{aligned} \text{Node 1} \quad I &= I_1 + I_2 + I_3 \\ &= (V_1 - V_4)Y_1 + (V_1 - V_2)Y_2 + (V_1 - V_3)Y_3 \\ &= V_1(Y_1 + Y_2 + Y_3) - V_2Y_2 - V_3Y_3 - V_4Y_1 \end{aligned} \qquad (2.2\text{(i)})$$

Node 2
$$\begin{aligned} 0 &= -I_2 - I_4 + I_6 \\ &= -(V_1 - V_2)Y_2 - (V_3 - V_2)Y_4 + (V_2 - V_4)Y_6 \\ &= -V_1Y_2 + V_2(Y_2 + Y_4 + Y_6) - V_3Y_4 - V_4Y_6 \end{aligned} \tag{2.2(ii)}$$

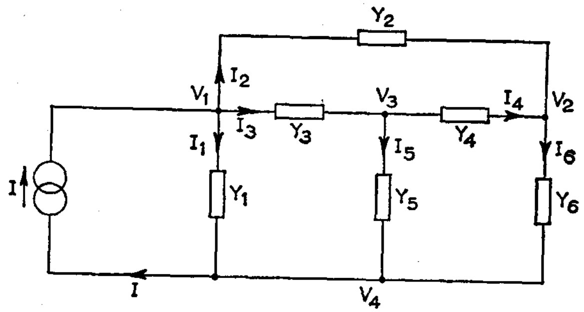


Fig. 2.3

Node 3
$$\begin{aligned} 0 &= -I_3 + I_4 + I_5 \\ &= -(V_1 - V_3)Y_3 + (V_3 - V_2)Y_4 + (V_3 - V_4)Y_5 \\ &= -V_1Y_3 - V_2Y_4 + V_3(Y_3 + Y_4 + Y_5) - V_4Y_5 \end{aligned} \tag{2.2(iii)}$$

Node 4
$$\begin{aligned} -I &= -I_1 - I_5 - I_6 \\ &= -(V_1 - V_4)Y_1 - (V_2 - V_4)Y_6 - (V_3 - Y_4)Y_5 \\ &= -V_1Y_1 - V_2Y_6 - V_3Y_5 + V_4(Y_1 + Y_5 + Y_6) \end{aligned} \tag{2.2(iv)}$$

The general form of the node equations is seen by writing $Y_{11} = Y_1 + Y_2 + Y_3 + \ldots$ as the total admittance at node 1, $Y_{12} = Y_{21}$ as the admittance between nodes 1 and 2, etc., so that

$$\left.\begin{aligned} I_A &= V_1Y_{11} - V_2Y_{12} - V_3Y_{13} - \ldots \\ I_B &= -V_1Y_{12} + V_2Y_{22} - V_3Y_{23} - \ldots \end{aligned}\right\} \tag{2.3}$$

where I_A, I_B, . . . are the currents fed in at nodes 1, 2, . . .

The reader should note that mutual inductance coupling between branches of a network is taken into account in mesh analysis by including appropriately directed e.m.f.s in the meshes. In node analysis the voltage between any two nodes is likewise altered by the e.m.f. induced in any coupled coils between these nodes.

EXAMPLE 2.1 In the circuit of Fig. 2.4, the two sources have the same frequency ($\omega = 5{,}000$ rad/s). Find the p.d. across resistor R_2 if $E_1 = 10\underline{/0°}$V, $E_2 = 10\underline{/90°}$V; $R_1 = 100\Omega$, $R_2 = 50\Omega$, $R_3 = 50\Omega$; $L_1 = 40$mH, $L_2 = 15$mH, and $M = 10$mH.

Using mesh analysis, the mesh equations are:

Mesh 1 $\quad E_1 - j\omega MI_2 = (R_1 + R_3 + j\omega L_1)I_1 - R_3I_2$

since the current I_2 entering the dotted end of L_2 gives an e.m.f. $j\omega MI_2$ directed out of the dotted end of L_1.

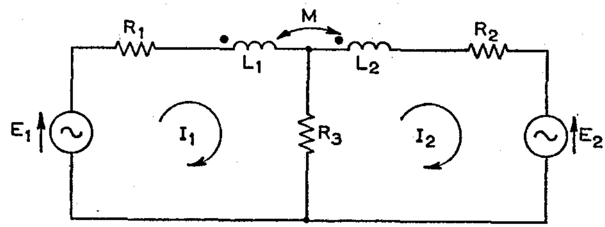


Fig. 2.4

Mesh 2 $\quad -E_2 - j\omega MI_1 = -R_3I_1 + (R_2 + R_3 + j\omega L_2)I_2$

Hence

$$10\underline{/0^\circ} - j50I_2 = (150 + j200)I_1 - 50I_2 \tag{i}$$

and

$$-j10 - j50I_1 = -50I_1 + (100 + j75)I_2 \tag{ii}$$

so that

$$I_1 = \frac{j10}{50 - j50} + \frac{100 + j75}{50 - j50}I_2$$

Substituting in eqn. (i),

$$10 = \frac{(150 + j200)j10}{50 - j50} + \frac{(150 + j200)(100 + j75)}{50 - j50}I_2 - (50 - j50)I_2$$

so that

$$I_2 = 0{\cdot}089\underline{/-129^\circ} = (0{\cdot}056 - j0{\cdot}07)\,\mathrm{A}$$

Using the current directions of Fig. 2.4, it follows that

$$V_{R2} = I_2R_2 = -2{\cdot}8 - j3{\cdot}5 = \underline{\underline{4{\cdot}5\underline{/231^\circ}\mathrm{V}}}$$

2.2 Superposition Theorem

"An e.m.f. acting on any linear network produces the same effect whether it acts alone or in conjunction with other e.m.f.s."

Hence a network containing many sources of e.m.f. may be analysed by considering the currents due to each e.m.f. in turn acting alone, the other e.m.f.s being suppressed and represented only by their internal source impedances.

EXAMPLE 2.2 Two a.c. sources, each of internal resistance 20Ω, are connected in parallel across a 10Ω pure resistance load. If the generated e.m.f.s are 50V and are 90° out of phase with each other, determine the current which will flow in the 10Ω resistor and in each generator.

The complete circuit is shown in Fig. 2.5(*a*). Consider the two e.m.f.s to be $E_1(=50\,V)$, and $E_2(=j50\,V)$. Then for E_1 acting alone (Fig. 2.5(*b*)),

$$I_1 = \frac{E_1}{R_{1eq}} = \frac{50}{20 + \dfrac{10 \times 20}{10 + 20}} = \frac{50}{26{\cdot}7} = 1{\cdot}88\,A$$

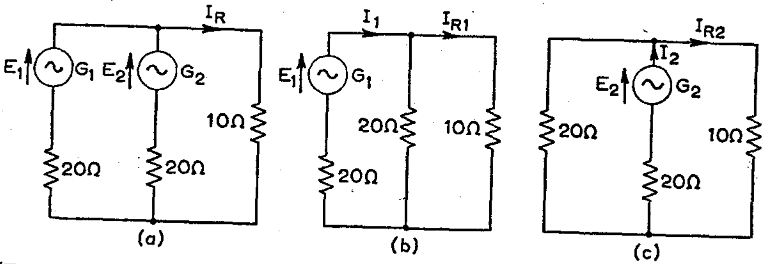


Fig. 2.5

Also,

$$I_{R1} = 1{\cdot}88\,\frac{20}{20 + 10} = 1{\cdot}25\,A$$

For E_2 acting alone (Fig. 2.5(*c*)),

$$I_2 = \frac{E_2}{R_{2eq}} = \frac{j50}{26{\cdot}7} = j1{\cdot}88\,A$$

and

$$I_{R2} = j1{\cdot}88\,\frac{20}{30} = j1{\cdot}25\,A$$

By the superposition theorem the total current through the 10 Ω resistor with both e.m.f.s acting at once will be

$$I_R = I_{R1} + I_{R2} = 1{\cdot}25 + j1{\cdot}25 = \underline{\underline{1{\cdot}77\angle 45°\,A}}$$

In the same way the current through the first generator G_1 when both e.m.f.s are active will be the algebraic sum of the currents through G_1 when each e.m.f. acts alone.

When E_1 acts alone, the current through G_1 is

$$I_1 = 1{\cdot}88\,A$$

and when E_2 acts alone, the current through G_1 is

$$I_2\,\frac{10}{20 + 10} = j0{\cdot}63\,A$$

Comparing Figs. 2.5(*b*) and (*c*) the above currents through G_1 are found to be oppositely directed. Therefore

$$\text{Total current through } G_1 = \underline{\underline{(1{\cdot}88 - j0{\cdot}63)\,A}}$$

Similarly,

Total current through $G_2 = \underline{\underline{(-0{\cdot}63 + j1{\cdot}88)\text{A}}}$

The superposition theorem may be restated in order to apply to distribution type networks as follows.

"The total current through any branch of a network is equal to the algebraic sum of the currents through the particular branch due to each load current alone, and the no-load current, if any."

EXAMPLE 2.3 A 500V d.c. generator supplies a load A of 500A through a 0·02Ω distributor, and a load B of 200A through a 0·015Ω distributor. If A and B are joined by a 0·03Ω interconnector, determine the interconnector current.

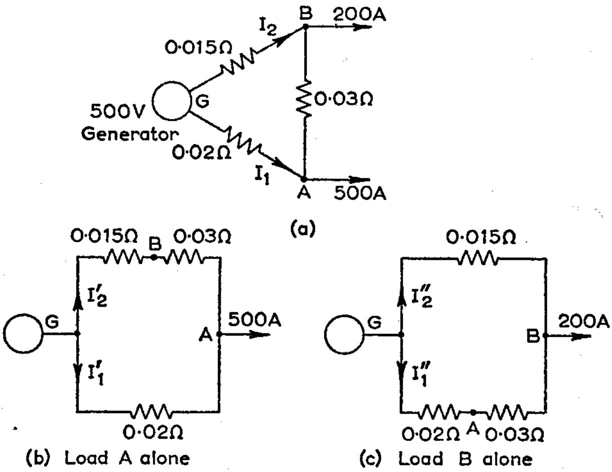


Fig. 2.6

The circuit is shown in Fig. 2.6(*a*). The resistances are assumed to be the total for both "go" and "return" conductors; since the return conductors are then assumed to be perfect conductors which will not give rise to voltage drops, they are omitted.

Applying the superposition theorem, the 500A load is taken first alone. The circuit then becomes that shown in Fig. 2.6(*b*).

$$I_1' = 500 \times \frac{0{\cdot}045}{(0{\cdot}045 + 0{\cdot}02)} = 346\text{A}$$

and

$$I_2' = 500 - 346 = 154\text{A}$$

Now the 200A load is taken alone (Fig. 2.6(*c*)). From the diagram,

$$I_2'' = 200 \times \frac{0{\cdot}05}{0{\cdot}065} = 152\text{A}$$

Therefore

$$I_1'' = 200 - 152 = 48\,\text{A}$$

By the superposition theorem,

$$\text{Total current, } I_1 = I_1' + I_1'' = 346 + 48 = 394\,\text{A}$$
$$\text{Total current, } I_2 = I_2' + I_2'' = 154 + 152 = 306\,\text{A}$$

The current through the interconnector is made up of 154 A ($=I_2'$) flowing from B to A (Fig. 2.6(b)) and 48 A ($=I_1''$) flowing from A to B (Fig. 2.6(c)). Therefore

Resultant current through interconnector = 106 A flowing from B to A.

2.3 Thévenin's Theorem and Norton's Theorem

The circuit theorems of Thévenin and Norton are extensions of the superposition principle of Section 2.2. Proofs of the theorems will be omitted. It must be understood that the equivalent circuits derived by the use of these theorems are valid only where all elements and actual sources are linear as discussed in the introduction to this chapter. The ideas are simple and of extreme value.

Thévenin: the linear network behind a pair of terminals may be replaced by a constant-voltage generator with an e.m.f. equal to the open-circuit voltage at the terminals and an internal impedance equal to the impedance seen at the actual terminals, with all internal sources removed and replaced by their internal impedances.

Norton: the linear network behind a pair of terminals may be replaced by a constant-current generator with a current equal to the short-circuit current at the terminals and an internal impedance equal to the internal impedance seen at the actual terminals, with all sources removed and replaced by their internal impedances.

Note that Norton's theorem follows directly from Thévenin's theorem and the equivalence of constant-current and constant-voltage sources.

Consider a complicated network of sources and impedances connected to two terminals A and B as in Fig. 2.7(a). Let the voltage across the terminals when they are open-circuited be V_T volts and the impedance measured at the terminals with all the sources suppressed and replaced only by their internal impedances be Z_I, Fig. 2.7(b). Then the circuit, as viewed from the terminals, is exactly equivalent to a generator of V_T volts and internal impedance Z_I ohms, Fig. 2.7(c). The current through the impedance Z connected across the terminals will therefore be

$$I = \frac{V_T}{Z_I + Z} \tag{2.4}$$

The circuit of Fig. 2.7(*c*) is called the *constant-voltage equivalent circuit*.

Alternatively, if I_{sc} is the short-circuit current from A to B in Fig. 2.7(*a*), then the circuit, as viewed from the terminals, is exactly equivalent to a generator of I_{sc} amperes and shunt internal impedance

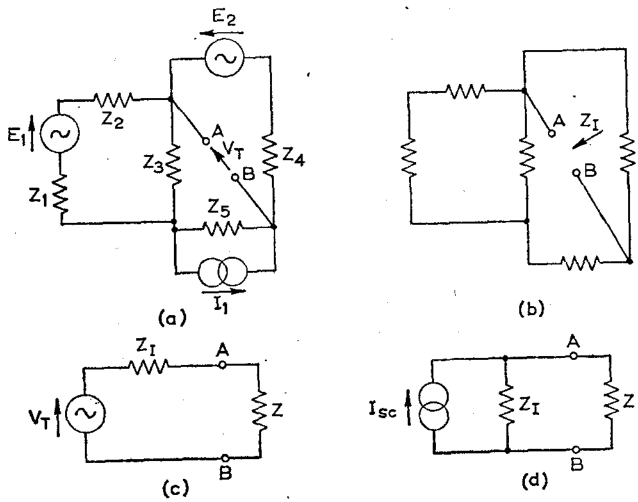


Fig. 2.7

Z_I ohms (Fig. 2.7(*d*)). The current through the impedance Z connected across the terminals will therefore be

$$I = I_{sc}\frac{Z_I}{Z_I + Z} \tag{2.5}$$

The circuit of Fig. 2.7(*d*) is called the *constant-current equivalent circuit.*

EXAMPLE 2.4 Solve Example 2.2 using Thévenin's theorem.

Let the circuit of Fig. 2.5(*a*) be broken just above the 10 Ω resistor. Then the voltage across the break is

$V_T = E_1$ − Voltage drop across impedance of source G1

$$\text{Circulating current} = \frac{E_1 - E_2}{40} = \frac{50 - j50}{40} = (1{\cdot}25 - j1{\cdot}25)\text{A}$$

Thus

$$V_T = 50 - (1{\cdot}25 - j1{\cdot}25) \times 20 = (25 + j25)\text{V}$$

The impedance looking into the break with the e.m.f.s suppressed is $10 + (20 \times 20)/(20 + 20) = 20\Omega$. Therefore

$$\text{Current through } 10\Omega \text{ resistor} = \frac{25 + j25}{20} = \underline{\underline{1{\cdot}25 + j1{\cdot}25\,\text{A}}}$$

The terminal voltage across the 10Ω load resistor is

$V_R = (12{\cdot}5 + j12{\cdot}5)\,\text{V}$

Therefore

$$\text{Current through } G_1 = \frac{E_1 - V_R}{20} = \frac{37{\cdot}5 - j12{\cdot}5}{20} = \underline{\underline{(1{\cdot}88 - j0{\cdot}63)\,\text{A}}}$$

and

$$\text{Current through } G_2 = \frac{E_2 - V_R}{20} = \frac{-12{\cdot}5 + j37{\cdot}5}{20} = \underline{\underline{(-0{\cdot}63 + j1{\cdot}88)\,\text{A}}}$$

For a circuit which has any number of internal sources and impedances, and has two free terminals the short-circuit current between the terminals is

$$I_{sc} = \frac{V_T}{Z_I} \tag{2.6}$$

where V_T is the open-circuit voltage between the terminals and Z_1 the internal impedance between them.

If I_{sc} and V_T can be measured, Z_I can be found from this equation.

EXAMPLE 2.5 Find the constant-voltage and constant-current equivalent circuits of the actual circuit of Fig. 2.8(a).

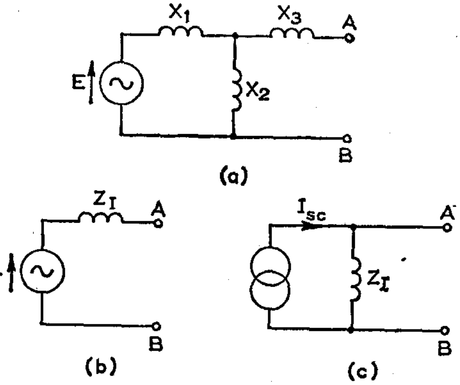


Fig. 2.8

$$\text{Open-circuit voltage, } V_T = E\frac{jX_2}{j(X_1 + X_2)} = \frac{EX_2}{X_1 + X_2}$$

$$\text{Internal impedance, } Z_I = jX_3 + \frac{jX_1 \times jX_2}{jX_1 + jX_2}$$

$$= j\left(X_3 + \frac{X_1X_2}{X_1 + X_2}\right)$$

The constant-voltage equivalent circuit is as in Fig. 2.8(*b*).

$$\text{Short-circuit current, } I_{sc} = \frac{E}{jX_1 + \dfrac{jX_2 jX_3}{jX_2 + jX_3}} \frac{jX_2}{jX_2 + jX_3}$$

$$= \frac{EX_2}{jX_1X_2 + jX_2X_3 + jX_1X_3}$$

The constant-current equivalent circuit is as in Fig. 2.8(*c*).

EXAMPLE 2.6 The frequency of the generator shown in Fig. 2.9(*a*) is the

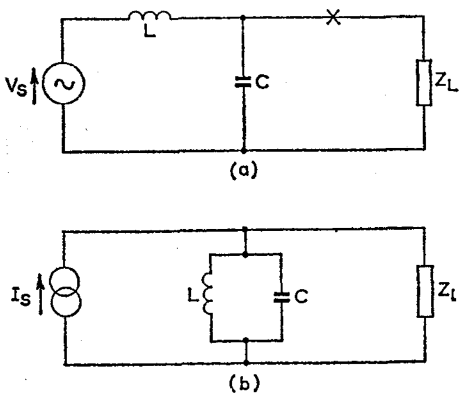


Fig. 2.9

series resonant frequency of L and C. Show, by application of Norton's theorem, that the current through the load Z_L is constant independent of the load.

Given that $V_S = 100$ V, $L = 10$ mH and $C = 2{,}000$ pF, find the value of the voltage across a load consisting of a resistance, R_L, of 1 kΩ in parallel with a capacitance, C_L, of 2,000 pF. *(H.N.C.)*

Since L and C form a resonant circuit, $\omega = 1/\sqrt{LC}$. If the load is short-circuited the current through the short-circuit is

$$I_S = V_S/\omega L = V_S\sqrt{\frac{C}{L}}$$

The impedance looking back into the circuit is infinite, since L and C are assumed to be pure circuit elements, and form a parallel tuned circuit when seen from the load terminals. Hence the constant-current equivalent circuit is as shown in Fig. 2.9(*b*), so that the load current is I_S, independent of the load.

For the values given,

$$I_S = 100\sqrt{\frac{2{,}000 \times 10^{-12}}{10 \times 10^{-3}}} = 45\,\text{mA}$$

The load impedance is

$$Z_L = \frac{R_L}{j\omega C_L(R_L + 1/j\omega C_L)} = \frac{R_L}{1 + j\omega C_L R_L} = \frac{R_L}{1 + j\dfrac{C_L R_L}{\sqrt{(LC)}}}$$

$$= \frac{1{,}000}{1 + j0{\cdot}45}$$

Therefore $Z_L = 1{,}000/1{\cdot}1 = 909\,\Omega$, so that

$$V_L = I_S|Z_L| = 45 \times 10^{-3} \times 909 = \underline{\underline{41\,\text{V}}}$$

It is left as an exercise for the reader to investigate the effect of a small resistance in series with L, and also to determine whether Thévenin's theorem would result in a suitable simplification of the problem. (The Thévenin impedance would be infinite!)

2.4 Maximum Power Transfer and Matching Theorems

1. A pure resistance load will abstract maximum power from a network when the load resistance is equal to the magnitude of the internal impedance of the network.

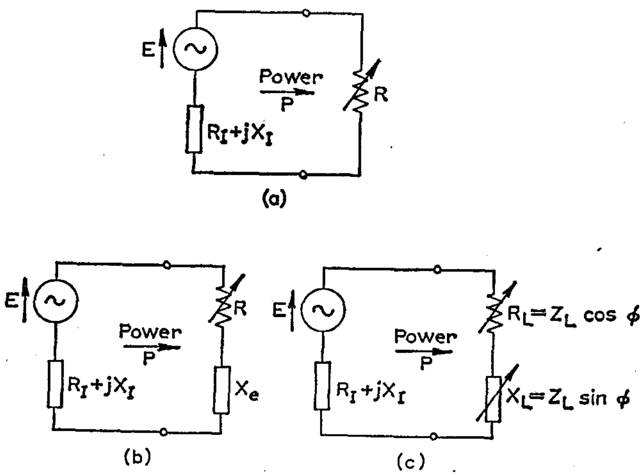


Fig. 2.10 RELATING TO MAXIMUM POWER TRANSFER

Assume that the network is represented by the constant-voltage equivalent circuit of Fig. 2.10(a):

$$\text{Load power} = I^2R = \frac{E^2}{(R + R_I)^2 + X_I^2}R$$

For maximum load power,

$$\frac{d}{dR}\left\{\frac{E^2R}{(R + R_I)^2 + X_I^2}\right\} = 0$$

whence

$$R = \sqrt{(R_I^2 + X_I^2)} \qquad (2.7)$$

i.e. for maximum power the load resistance should equal the magnitude of the internal impedance.

2. A constant-reactance variable-resistance load will abstract maximum power from a network when the resistance of the load is equal to the magnitude of the internal impedance of the network plus the reactance of the load.

The constant-voltage equivalent circuit of Fig. 2.10(*b*) shows that so far as power transfer is concerned X_e could be grouped with X_I. With this grouping the proof would correspond to the previous one.

3. A variable-impedance load of constant power factor will abstract maximum power from a network when the magnitudes of the load impedance and the internal impedance are equal.

It should be noticed that a constant-power-factor load would be one in which the resistance and reactance were varied in proportion.

Let ϕ be the constant load phase angle, while $Z_L(= \sqrt{(R_L^2 + X_L^2)})$ is the magnitude of the variable load impedance. Fig. 2.10(*c*) shows the equivalent constant-voltage circuit. As in the previous circuit,

$$\text{Load power} = \frac{E^2 Z_L \cos\phi}{(R_1 + Z_L \cos\phi)^2 + (X_I + Z_L \sin\phi)^2}$$

For maximum power,

$$\frac{d}{dZ_L}\{\text{load power}\} = 0$$

whence

$$R_I^2 + X_I^2 = Z_L^2 \qquad (2.8)$$

i.e. for maximum power the magnitude of the load impedance should equal the magnitude of the internal impedance of the generator.

4. If the load resistance and reactance are independently variable, maximum power will be abstracted when the load reactance equals the conjugate of the internal reactance and the load resistance equals the internal resistance.

Clearly, when the two reactances are equal in magnitude but of opposite sign (conjugate), the resultant reactance will be zero and the load resistance will absorb maximum power when it equals the internal resistance according to theorem 1.

Small transformers used in low-power circuits may usually be regarded as ideal: i.e. having a primary-to-secondary voltage ratio V_P/V_S equal to the turns ratio N_P/N_S, and a primary-to-secondary current ratio I_P/I_S equal to the reciprocal of the turns ratio N_S/N_P.

Fig. 2.11 shows a transformer feeding a load impedance Z_L. The

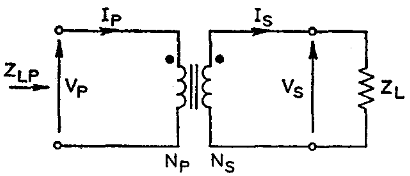


Fig. 2.11 MATCHING WITH AN IDEAL TRANSFORMER

primary input impedance, Z_{LP} (or impedance observed looking into the primary winding), is given by

$$Z_{LP} = \frac{V_P}{I_P} = V_S \frac{N_P}{N_S} \frac{1}{I_S} \frac{N_P}{N_S} = \frac{V_S}{I_S} \left(\frac{N_P}{N_S}\right)^2$$

$$= Z_L \left(\frac{N_P}{N_S}\right)^2 \tag{2.9}$$

Thus a transformer may, for circuit work, be regarded as a device which transforms impedance by the square of the turns ratio of the transformer.

A transformer is often used to obtain a maximum power transfer condition—the transformer so used being termed a *matching transformer*.

EXAMPLE 2.7 A variable-frequency generator is represented by an e.m.f., V_S, a resistance R_1 and an inductance L in series. It is to be connected, by an ideal matching transformer, to a load consisting of a resistor R_2 and a capacitor C in series. If $R_1 = 100\,\Omega$, $L = 0{\cdot}1\,\text{H}$, $R_2 = 1\,\text{k}\Omega$, $C = 1\,\mu\text{F}$ and $V_S = 10\,\text{V}$, calculate (*a*) the turns ratio of the transformer to give maximum power in the load, (*b*) the frequency at which this maximum power is obtained, and (*c*) the value of the maximum power.

(*a*) The turns ratio for maximum power transfer must be such that eqn. (2.9) is satisfied for the resistive terms. The frequency at which maximum power is obtained is such that with the above turns ratio the capacitive reactance reflected into the primary side is equal to the inductive reactance of the source. Thus

$$R_1 = R_2 \left(\frac{N_P}{N_S}\right)^2$$

or

$$\frac{N_P}{N_S} = \sqrt{\frac{R_1}{R_2}} = \sqrt{\frac{100}{1{,}000}} = \frac{1}{3{\cdot}16}$$

(*b*) Also the reflected capacitive reactance is

$$X_C = \frac{1}{\omega C}\frac{N_P{}^2}{N_S{}^2} = \frac{1}{10\omega C}$$

so that, for maximum power,

$$\frac{1}{10\omega C} = \omega L \qquad \text{or} \qquad \omega = \sqrt{\frac{1}{10LC}} = 1{,}000$$

and

$$f = \omega/2\pi = \underline{\underline{159\,\text{Hz}}}$$

(*c*) For the matched condition, the power delivered from the source is

$$P_{max} = \frac{V^2}{2R_1} = \frac{100}{200} = 0{\cdot}5\,\text{W}$$

Hence the maximum load power is $\underline{\underline{0{\cdot}25\,\text{W}}}$

2.5 Millman's Theorem

Problems in which interest is centred on one particular node of a circuit (such as in an unbalanced 3-phase star-connected load or an

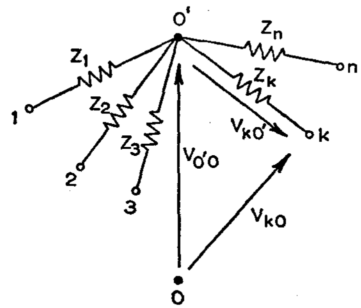


Fig. 2.12 RELATING TO MILLMAN'S THEOREM

electronic amplifier) may often be simplified by a circuit theorem due to J. E. Millman. This theorem (also called the parallel-generator theorem) states:

If any number of linear impedances $Z_1, Z_2, Z_3 \ldots$, etc., meet at a common point 0′, and the voltages from another point 0 to the free ends of these impedances are known, the voltage $V_{0'0}$ is given by

$$V_{0'0} = \frac{\sum_{k=1}^{n} V_{k0} Y_k}{\sum_{k=1}^{n} Y_k} \tag{2.10}$$

where V_{k0} = Voltage of point k with respect to point 0

Y_k = Admittance of Z_k

Proof. In Fig. 2.12, $0'$ is the common point of the impedances $Z_1, Z_2, \ldots Z_n$, and the potential differences between the point 0 and the ends ($1, 2, 3, \ldots n$) of the impedances are known. Then round the closed loop $00'k$, the sum of the p.d.s is zero. Thus

$$V_{0'0} + V_{k0'} + V_{0k} = 0$$

or

$$V_{0'0} + V_{k0'} - V_{k0} = 0$$

whence

$$V_{k0'} = V_{k0} - V_{0'0}$$

The current through Z_k is

$$I_{k0'} = \frac{V_{k0'}}{Z_k} = V_{k0'} Y_k = (V_{k0} - V_{0'0}) Y_k$$

By Kirchhoff's first law, the sum of the currents at $0'$ is zero:

$$I_{10'} + I_{20'} + \ldots + I_{k0'} + \ldots + I_{n0'} = 0$$

$$(V_{10} - V_{0'0}) Y_1 + (V_{20} - V_{0'0}) Y_2 + \ldots + (V_{k0} - V_{0'0}) Y_k + \ldots = 0$$

$$V_{10} Y_1 + V_{20} Y_2 + \ldots + V_{k0} Y_k + \ldots = V_{0'0}(Y_1 + Y_2 + \ldots + Y_k + \ldots)$$

Therefore

$$V_{0'0} = \frac{V_{10} Y_1 + V_{20} Y_2 + \ldots}{Y_1 + Y_2 + \ldots} = \frac{\sum_{k=1}^{n} V_{k0} Y_k}{\sum_{k=1}^{n} Y_k}$$

It should be noted that the impedances between the point 0 and points 1, 2, 3, etc., need not be known.

2.6 General Star–Mesh Transformation

It is always possible to transform a network of n admittances which are connected to a star point from n terminals into a corresponding mesh of admittances connecting each pair of terminals. It is, however, possible to find a unique transform from a mesh to a star only in the case of three elements (the delta–star transformation).

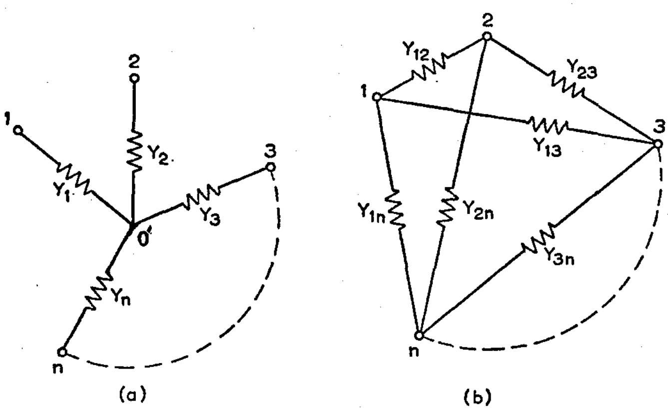


Fig. 2.13 GENERAL STAR–MESH TRANSFORMATION

Consider the star of admittances Y_1, Y_2, . . . Y_n shown in Fig. 2.13(*a*). If terminal 1 is taken as the reference point and 0′ as the star point then (noting that $V_{11} = 0$) Millman's theorem gives

$$V_{0'1} = \frac{\sum_{k=1}^{n} V_{k1} Y_k}{\sum_{k=1}^{n} Y_k} = \frac{V_{21}\, Y_2 + V_{31}\, Y_3 + \dots}{Y_1 + Y_2 + Y_3 + \dots}$$

Hence

$$I_1 = V_{0'1} Y_1 = \frac{V_{21} Y_1 Y_2 + V_{31} Y_1 Y_3 + \dots}{\Sigma Y_k}$$

This is the same current as would flow into terminal 1 if it were connected to terminal 2 by $Y_{12} = Y_1 Y_2/\Sigma Y_k)$, to terminal 3 by $Y_{13} = (Y_1 Y_3/\Sigma Y_k)$, etc., as shown in Fig. 2.13(*b*). In the same way it may be shown that the current into terminal 2 in the star connection is equivalent to the current which would flow if admittances of

$Y_{21} = (Y_2Y_1/\Sigma Y_k)$, $Y_{23} = (Y_2Y_3/\Sigma Y_k)$, etc., were connected between terminal 2 and terminals 1, 3, etc., respectively. Hence the element of the equivalent mesh between any two terminals p and q is

$$Y_{pq} = \frac{Y_pY_q}{\Sigma Y_k} \tag{2.11}$$

This is known as *Rosen's theorem.*

2.7 Star–Delta Transformations

The particular case of the transformation from a 3-element star to a delta is important and is shown in Fig. 2.14. For this circuit, and using eqn. (2.11),

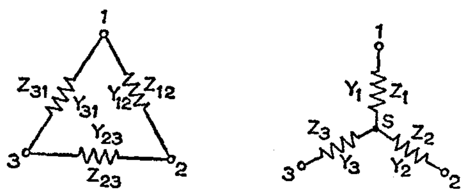


Fig. 2.14 DELTA–STAR OR STAR–DELTA TRANSFORMATION

$$Y_{12} = \frac{Y_1Y_2}{Y_1 + Y_2 + Y_3} \tag{2.12(i)}$$

$$Y_{23} = \frac{Y_2Y_3}{Y_1 + Y_2 + Y_3} \tag{2.12(ii)}$$

$$Y_{31} = \frac{Y_3Y_1}{Y_1 + Y_2 + Y_3} \tag{2.12(iii)}$$

Inverting these gives the impedance forms,

$$Z_{12} = \frac{1}{Y_{12}} = Z_1 + Z_2 + \frac{Z_1Z_2}{Z_3} \tag{2.13}$$

with similar expressions for Z_{23} and Z_{31}.

2.8 Delta–Star Transformation

From the general star–mesh transformation it will be seen that a 3-element star becomes a 3-branch mesh or delta, whereas a 4-element star forms a 6-branch mesh. In general there are more branches in the mesh than there are elements in the corresponding star. Thus any arbitrary mesh cannot be replaced by a star since there are a greater number of variables in a mesh than in a star. The 3-branch mesh or delta is exceptional, and here the inverse transform exists.

From eqn. (2.12(i)), writing $Y = Y_1 + Y_2 + Y_3$,

$$Y_{12}Y = Y_1 Y_2 \tag{i}$$

Similarly,

$$Y_{31}Y = Y_1 Y_3 \tag{ii}$$

and

$$Y_{23}Y = Y_2 Y_3 \tag{iii}$$

where $Y = Y_1 + Y_2 + Y_3$.

(iii)/(i) yields $Y_3 = Y_1 Y_{23}/Y_{12}$
(iii)/(ii) yields $Y_2 = Y_1 Y_{23}/Y_{31}$

Substituting for Y_3 and Y_2 in eqn. (2.12(i)) yields

$$Y_{12} = \frac{Y_1^2 Y_{23}/Y_{31}}{Y_1 + Y_1 Y_{23}/Y_{31} + Y_1 Y_{23}/Y_{12}}$$

so that

$$Y_1 = Y_{12} + Y_{31} + \frac{Y_{12}Y_{31}}{Y_{23}} \tag{2.14}$$

Inverting and simplifying this equation,

$$Z_1 = \frac{Z_{12}Z_{31}}{Z_{12} + Z_{23} + Z_{31}} \tag{2.15(i)}$$

Similarly,

$$Z_2 = \frac{Z_{23}Z_{12}}{Z_{12} + Z_{23} + Z_{31}} \tag{2.15(ii)}$$

and

$$Z_3 = \frac{Z_{23}Z_{31}}{Z_{12} + Z_{23} + Z_{31}} \tag{2.15(iii)}$$

EXAMPLE 2.8 Find the input impedance of the circuit shown in Fig. 2.15(*a*).

The circuit is first simplified by applying the Δ–Y transformation to the inductive reactances between terminals 1, 2 and 3, when the circuit becomes that shown in Fig. 2.14(*b*). Thus

$$Z_{10} = X_{L1} = \frac{j5 \times j10}{j30} = j1{\cdot}67\,\Omega$$

$$Z_{20} = X_{L2} = \frac{j5 \times j15}{j30} = j2{\cdot}5\Omega$$

$$Z_{30} = X_{L3} = \frac{j10 \times j15}{j30} = j5\Omega$$

Hence

$$Z_{024} = j2{\cdot}5 - j22{\cdot}5 = -j20 = 20\underline{/-90^\circ}\Omega$$

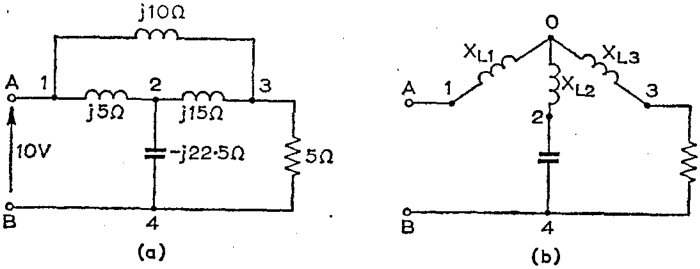


Fig. 2.15

Therefore

$$Y_{024} = \frac{1}{20\underline{/-90^\circ}} = 0{\cdot}05\underline{/90^\circ} = j0{\cdot}05\text{ S}$$

and

$$Z_{034} = 5 + j5 = 7{\cdot}07\underline{/45^\circ}\Omega$$

Therefore

$$Y_{034} = \frac{1}{7{\cdot}07\underline{/45}} = 0{\cdot}141\underline{/-45^\circ} = (0{\cdot}1 - j0{\cdot}1)\text{S}$$

so that

$$Y_{OB} = Y_{024} + Y_{034} = j0{\cdot}05 + 0{\cdot}1 - j0{\cdot}1 = 0{\cdot}112\underline{/-26^\circ\ 34'}\text{S}$$

Therefore

$$Z_{OB} = 8{\cdot}95\underline{/26^\circ\ 34'} = (8 + j4)\Omega$$

and

$$Z_{AB} = j1{\cdot}67 + (8 + j4) = 8 + j5{\cdot}67 = \underline{\underline{9{\cdot}8\underline{/35^\circ\ 45'}\Omega}}$$

2.9 Reciprocity Theorem

If an e.m.f. acting in one branch of a network causes a current I to flow in a second branch, the same e.m.f. acting in the second branch would produce the same current in the first branch.

An obvious conclusion from the theorem is that in a Wheatstone bridge network the source and the galvanometer may be interchanged. The principal application of the reciprocity theorem is to four-terminal and transmission-line networks.

2.10 Compensation Theorem

If a change, ΔZ say, is made to the impedance of any branch of a network where the current was originally I, then the change of current at any other point in the network may be calculated by assuming that an e.m.f. of $-I\Delta Z$ has been introduced into the changed branch, while all other sources have their e.m.f.s suppressed and are represented by their internal impedances only.

EXAMPLE 2.9 A 100 V battery supplies the current shown in Fig. 2.16(*a*).

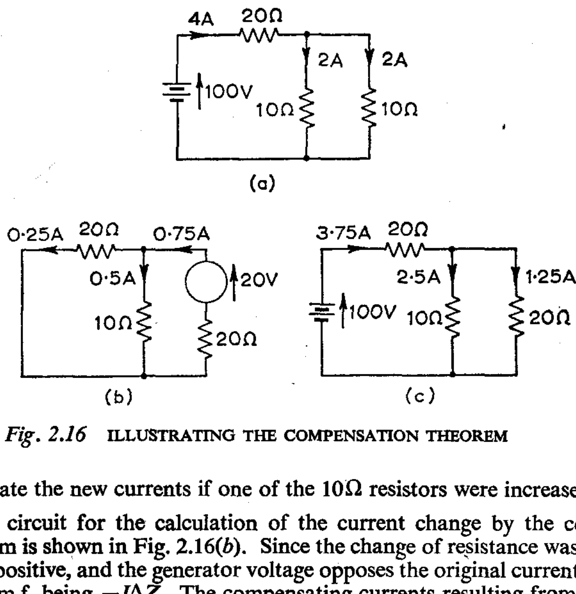


Fig. 2.16 ILLUSTRATING THE COMPENSATION THEOREM

Calculate the new currents if one of the 10Ω resistors were increased to 20Ω.

The circuit for the calculation of the current change by the compensation theorem is shown in Fig. 2.16(*b*). Since the change of resistance was an increase, ΔZ is positive, and the generator voltage opposes the original current, the equivalent e.m.f. being $-I\Delta Z$. The compensating currents resulting from the application of this e.m.f. are shown in Fig. 2.16(*b*). If these currents are added to the original currents in the corresponding limbs, the final current distribution will be as indicated at (*c*). In this simple case the result may readily be verified by a series-parallel method.

PROBLEMS

2.1 Three batteries A, B and C have their negative terminals connected together. The positive terminal of A is connected to the positive terminal of B by a resistance of 0·3 Ω, and the positive terminal of B is connected to the positive terminal of C by a resistance of 0·45 Ω. The respective e.m.f.s and resistances of the batteries are: battery A, 100 V, 0·25 Ω; battery B, 105 V, 0·2 Ω; and battery C, 95 V, 0·15 Ω.

Calculate the current in each of the external resistors and the p.d. across the battery B. (*C. & G. Inter.*)

Ans. 3·6 A, 11·65 A, 102 V.

2.2 In the circuit shown in Fig. 2.17 transform the star, ABC, to a delta and then apply Thévenin's theorem to find the voltage across the 30 Ω resistor. (*H.N.C. part question*)

Ans. 3·33 V.

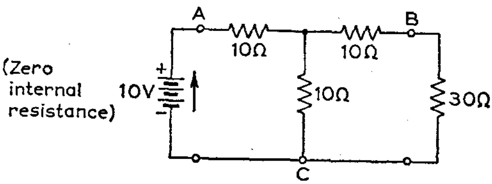


Fig. 2.17

2.3 A network is composed of the following resistances: AB = 0·1 Ω; BC = 0·2 Ω; CD = 0·1 Ω; DA = 0·1 Ω; AC = 0·2 Ω.

A current of 80 A is fed into the network at A and currents of 25 A, 35 A, and 20 A leave at the points B, C and D respectively. Calculate the current in AC. (*H.N.C.*)

Ans. 20 A.

2.4 A generator A supplies a load B of 50 A through a 0·1 Ω distributor, and a load C of 30 A through a 0·15 Ω distributor. B and C are joined by a 0·2 Ω interconnector. Find the magnitude and direction of the current in the interconnector by (*a*) Thévenin's theorem, (*b*) superposition theorem. (*H.N.C.*)

Ans. 1·11 A from C to B.

2.5 A constant-voltage generator has an internal resistance of 5,000 Ω, and the generated e.m.f. is 200 V at a frequency of 1 kHz. Deduce the equivalent constant-current generator. If the load on the generator consists of a resistance of 4,000 Ω in parallel with a capacitance of 0·1 μF determine, using Norton's theorem, the voltage across the capacitor.

Ans. 51·8 V.

2.6 A generator has an output impedance of $(600 + j50)\,\Omega$. Calculate the turns ratio of an ideal transformer necessary to match the generator to a load of $(65 + j30)\,\Omega$ for maximum transfer of power. Prove any formula used.

Ans. 2·9. (*L.U.*)

2.7 A Wheatstone bridge network has the following components: AB = 1 Ω; BC = 1·7 Ω; AD = 4 Ω; DC = 6 Ω.

A 10 V d.c. supply of internal impedance 2 Ω is connected across terminals A and C. Determine, using Thévenin's theorem or the compensation theorem, the current in a 100 Ω resistor connected between terminals B and D.

Ans. 1·49 mA.

2.8 Transform the star-connected impedances C, C, and R_2 to a delta in the bridged-T circuit shown in Fig. 2.18 and hence show that the voltage between D and E is zero when

$$R_1 = \frac{1}{R_2\omega^2C^2} \quad \text{and} \quad \omega L = \frac{2}{\omega C}$$

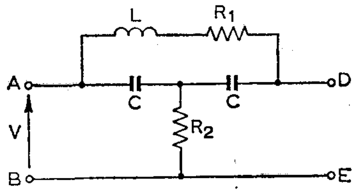


Fig. 2.18

2.9 A small transformer has primary and secondary inductances of 450μH and 35μH respectively and negligible resistance. The load on the secondary is a resistance of 15Ω, and the primary forms part of a series resonant circuit to which an e.m.f. of frequency $5/2\pi$ MHz is applied by a generator of internal resistance 20Ω. Find the mutual inductance between the primary and secondary windings, and the setting of the tuned circuit capacitor to make the power developed in the load a maximum. (*H.N.C.*)

Ans. 40·6μH; 99·4pF.

HINT. Convert the inductive coupling to an equivalent circuit as in Section 1.10.

2.10 A generating station A with a line voltage of 11kV supplies two substations B and C through two independent feeders, the substations also being interconnected by another feeder.

The impedances of the feeders are: A to B, $(2 + j4)\Omega$; A to C, $(2 + j3)\Omega$; B to C, $(3 + j5)\Omega$. The load at B is 100A at 0·8 power factor lagging, and at C, 70A at 0·9 lagging.

Calculate the current flowing in each feeder, and also the voltage between B and C if the feeder BC is removed. (*L.U.*)

Ans. 86A, 84A, 14·1A, 196·4V.

2.11 Show that the voltage $V_{o'o}$ in Fig. 2.19 is given by

$$V_{o'o} = \frac{E_1Y_1 + E_2Y_2 + E_3Y_3}{Y_1 + Y_2 + Y_3}$$

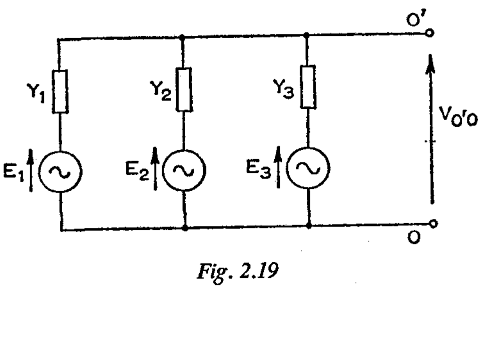


Fig. 2.19

where the generators marked E_1, E_2 and E_3 are all of the same frequency. If the instantaneous e.m.f.s are $e_1 = -10\sqrt{2}\sin 1{,}000t$, $e_2 = 20\sqrt{2}\cos 1{,}000t$ and $e_3 = 15\sqrt{2}\sin 1{,}000t$, and if $Y_1 = -j0{\cdot}1$, $Y_2 = j0{\cdot}05$ and $Y_3 = 0{\cdot}067$ S find the r.m.s. value of the magnitude of $V_{0'0}$ and its phase relationship to E_1.

(*H.N.C.*)

Ans. 12V, 127° lagging.

Chapter 3

MEASUREMENT CIRCUITS

The Wheatstone bridge network may be adapted for a.c. measurements by making the supply an alternating one of the frequency desired, and using a detector which is sensitive to alternating currents. The a.c. bridge network is used for comparison measurements of resistance, inductance and capacitance to a high degree of precision.

3.1 Standards

Primary electrical standards have values which may be determined with reference to the units of mass, length, time, and one arbitrarily chosen electrical constant, which is the unit of electric current, namely the ampere. A determination of any electrical quantity in the above terms is called an absolute measurement. Electrical units and standards form part of the internationally adopted Système Internationale d'Unités, or SI units, in which there are six fundamental units, the kilogramme, metre, second, ampere, kelvin and candela.*

Primary electrical standards of mutual inductance (the Campbell mutual inductance at the National Physical Laboratory) and self-inductance have been constructed so that the inductances are accurately known in terms of their physical dimensions and the ampere. The standard mutual inductor is used to calibrate variable

* The kelvin (formerly the degree kelvin) is the absolute unit of thermodynamic temperature. The candela is the absolute unit of luminous intensity.

laboratory standard mutual inductors which may then be used for further measurements. For instance, a resistor may be calibrated in terms of the product of a mutual inductance and a speed of rotation of a disc, as in the Lorenz method.

No primary reference standard of current is possible, but current can be measured in absolute terms by measuring the force exerted between two circuits carrying the same current. If the current is passed through a standard resistor, the resulting known p.d. may be used to calibrate a standard of e.m.f. (i.e. the standard cell).

The three laboratory reference standards are thus

(*a*) A standard variable mutual inductor (mutual inductometer) which usually has a range of up to 11·1 mH from zero.

(*b*) Standard resistors, calibrated as has already been indicated.

(*c*) A standard e.m.f. derived from a Weston cadmium cell.

Laboratory measurements can be made in terms of these reference standards.

It is customary also to have standards of capacitance available. Primary standards, in which the capacitance is measured in terms of the physical dimensions, are theoretically possible, but the difficulties involved are great and normally only capacitors whose values are determined by comparison measurements are used. These are called secondary standards, and their values are usually known to a very high degree of accuracy.

The following a.c. bridges will illustrate methods by which inductance and capacitance may be compared with mutual inductance and resistance, and further circuits will illustrate the use of the secondary standards of capacitance. Bridges employing self-inductance standards are uncommon.

3.2 Balance Conditions

In the same way as for the d.c. Wheatstone bridge network, an alternating-current bridge is said to be balanced when the current through the detector is zero. Fig. 3.1 shows a generalized bridge circuit, in which Z_1, Z_2, Z_3 and Z_4 are the impedances (in complex form) of the bridge arms. If the current through the detector is zero, then the current I_1 in Z_1 must also flow through Z_2, and the current I_4 in Z_4 must also flow through Z_3. Equating the voltage drops between A and B, and A and D, gives

$$I_1Z_1 = I_4Z_4$$

$$I_1Z_2 = I_4Z_3$$

whence

$$Z_1Z_3 = Z_2Z_4 \tag{3.1}$$

This equality represents the balance conditions of the bridge.

Normally one of the bridge arms contains the unknown impedance while the other arms contain known fixed or variable comparison

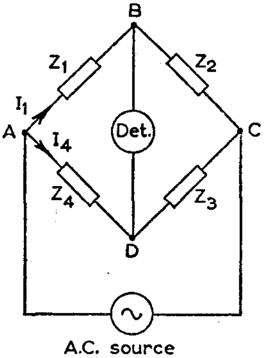


Fig. 3.1 GENERAL A.C. BRIDGE CIRCUIT

standards. The bridge is set up, and the current through the detector is reduced to zero by successive adjustments of the variable circuit-elements (usually only two elements in the bridge are variable). When this is achieved the unknown impedance may be expressed in terms of the comparison standards.

It should be noted that eqn. (3.1) will be in complex form, so that for balance the reference terms on each side must be equated, and also the quadrate terms. Balance will obviously be achieved quickly if there is one variable in the reference terms, and the other variable in the quadrate terms. If both variables appear in either of the resulting balance equations, balance will not be achieved quickly; the bridge is then said to be slow to converge.

3.3 Detectors

The detectors used to determine the balance point in an a.c. bridge vary with the type of bridge and with the frequency at which it is operated.

The cathode-ray tuning indicator can be used over a very wide range of frequencies.

For mains-operated bridges, a suitable detector is the vibration galvanometer. This consists essentially of a narrow moving coil which is suspended on a fine phosphor-bronze wire between the poles of a magnet. The mechanical resonant frequency of the suspension is made equal to the electrical frequency of the coil current, so that, when a current of the correct frequency flows through the coil, it is set into vibration. A small mirror attached to the coil reflects a spot of light on to a scale. When the coil is vibrating this spot appears as an extended band of light. Balance of the bridge is indicated when the band reduces to a spot.

The vibration galvanometer is insensitive to frequencies other than the one to which it is tuned. This tuning may be adjusted by altering the tension of the suspension. Vibration galvanometers are constructed for frequencies ranging from 10 to about 300 Hz.

For audio frequencies a telephone headset is often used as a bridge detector. Since the human ear is very sensitive in the 1,000 Hz frequency region, this is a common a.c. bridge frequency. The disadvantage of telephones is the fact that if other frequencies are present in the a.c. supply (harmonics) these will also be heard, and zero current may not be obtained when the bridge is balanced.

Frequency-sensitive heterodyne detectors are used where extreme sensitivity in detection is desired or when frequencies are above the audio range.

3.4 Owen Bridge for Inductance

The derivation of the balance conditions for an a.c. bridge are illustrated by the Owen bridge (Fig. 3.2), which measures the resistance, R_x, and inductance, L_x, of a coil in terms of fixed and variable

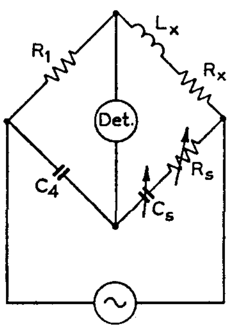


Fig. 3.2 OWEN BRIDGE

resistors and capacitors. By suitable choice of bridge elements a wide range of inductance can be measured. The bridge may be simply modified to measure the inductance and losses in iron-cored coils which are subject to both a.c. and d.c. magnetization.

The balance conditions are

$$(R_x + j\omega L_x)\frac{1}{j\omega C_4} = R_1\left(R_s + \frac{1}{j\omega C_s}\right)$$

Equating the quadrate terms,

$$R_x = R_1\frac{C_4}{C_s} \tag{3.2}$$

and, from the reference terms,

$$L_x = C_4R_1R_s \tag{3.3}$$

Note that the variable standards affect each balance condition independently, i.e. varying C_s will not affect the balance for L_x, and varying R_s will not affect the balance for R_x. This means that a very quick and accurate balance can be obtained.

3.5 Anderson Bridge for Inductance

This bridge may be used to measure inductances ranging from a few microhenrys up to the order of one henry. The schematic

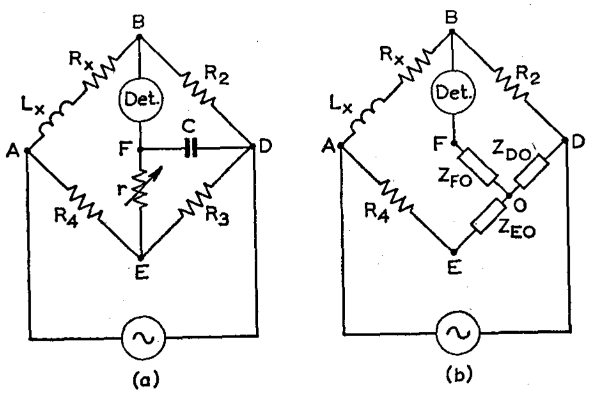


Fig. 3.3 ANDERSON BRIDGE

circuit is shown in Fig. 3.3(*a*). The arm AB contains the coil whose inductance, L_x, and resistance, R_x, it is desired to measure. The components of the other arms of the bridge are known standards. Balance may be obtained by variation of r and either R_2, R_3 or R_4.

To obtain the balance conditions the delta of impedances FDE Fig. 3.3(*a*)) is first transformed to an equivalent star. The bridge) then reduces to the circuit shown at (*b*). If O is the star point, then from eqn. (2.15),

$$Z_{EO} = \frac{rR_3}{r + R_3 + 1/j\omega C}$$

Also,

$$Z_{DO} = \frac{R_3/j\omega C}{r + R_3 + 1/j\omega C}$$

The impedance Z_{FO} does not affect the balance of the bridge since it is in series with the detector. Thus at balance,

$$(R_x + j\omega L_x)\frac{R_3}{j\omega C(r + R_3 + 1/j\omega C)} = R_2\left(R_4 + \frac{rR_3}{r + R_3 + 1/j\omega C}\right)$$

so that

$$R_x + j\omega L_x = \frac{R_2}{R_3} j\omega C\{R_4(r + R_3 + 1/j\omega C) + rR_3\}$$

Equating the reference terms,

$$R_x = \frac{R_2 R_4}{R_3} \tag{3.4}$$

and equating the quadrate terms,

$$L_x = R_2 C\left(\frac{R_4 r}{R_3} + R_4 + r\right) \tag{3.5}$$

3.6 Loss in Capacitors

In capacitors with solid dielectrics there is a power loss due to leakage currents, and also a dielectric heat loss (analogous to the hysteresis loss in magnetic circuits) when an alternating voltage is applied to the capacitor. The total loss may be represented as the loss in an additional resistance connected between the plates. The dielectric loss will normally exceed the leakage loss except in air dielectrics. The total alternating current passing through the capacitor will be made up of (*a*) the capacitive current plus (*b*) the loss current, and the equivalent circuit of such a capacitor will consist of a pure capacitance, C_p, in parallel with a high resistance R_p as shown in Fig. 3.4(*a*). The complexor diagram for the arrangement is shown at (*b*), where I_a is the loss current, and I_c is the capacitive current, the voltage across the capacitor being V.

Normally the loss current will be very much smaller than the capacitive current, so that the resultant current will lead the voltage by an angle which is nearly 90°. The difference between 90° and the actual phase angle of the capacitor is the angle δ in Fig. 3.4(*b*), and this is termed the *loss angle* of the capacitor.

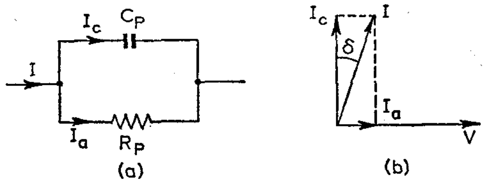


Fig. 3.4 IMPERFECT CAPACITOR
(*a*) Equivalent circuit (*b*) Complexor diagram

It is also possible to represent an imperfect capacitor by an equivalent series circuit, in which a capacitance C_s is connected in series with a low resistance R_s.

For the equivalent parallel circuit,

$$\tan \delta = I_a/I_c = 1/R_p\omega C_p \tag{3.6}$$

and the impedance is approximately $1/\omega C_p$, since the parallel loss resistance will be very much greater than the capacitive reactance. For the equivalent series circuit, $\tan \delta$ is $R_s\omega C_s$, and the impedance is approximately $1/\omega C_s$ since the series loss resistance will be very much smaller than the capacitive reactance. For equivalence between the series and the parallel circuits we have

$$C_s \approx C_p = C \tag{3.7}$$

and

$$R_s \approx 1/R_p\omega^2 C^2 \tag{3.8}$$

3.7 Modified Carey Foster Bridge for Capacitance

The circuit of the bridge is shown in Fig. 3.5. The unknown capacitor is represented by its equivalent series circuit, and a known variable resistor R_3 is connected in series with it to form the arm BD. The arm EA is a short-circuit. The mutual inductometer M must be connected with the winding directions indicated by the dot notation, or balance will be impossible.

Let L be the self-inductance of the secondary of the mutual inductometer, and let the resistance R_4 include the resistance of this winding. Then the total current I taken from the bridge supply will divide at point E, into I_A and I_B.

If the detector current is zero, the voltage drop between E and B must be zero giving

$$I_B(R_4 + j\omega L) - j\omega M(I_A + I_B) = 0$$

Therefore

$$j\omega M I_A = (R_4 + j\omega L - j\omega M)I_B \tag{3.9}$$

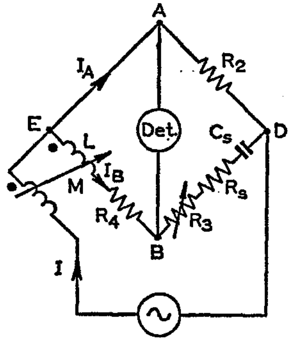


Fig. 3.5 MODIFIED CAREY FOSTER BRIDGE

In the same way the voltage drop from A to D must be equal to that from B to D, and further, the current I_A must flow through arm AD and the current I_B through BD. Hence

$$I_A R_2 = \left(R_3 + R_s - j\frac{1}{\omega C_s}\right) I_B \tag{3.10}$$

Dividing eqn. (3.9) by eqn. (3.10) and cross-multiplying,

$$j\omega M\{(R_3 + R_s) + 1/j\omega C_s\} = R_2\{R_4 + j\omega(L - M)\}$$

Equating the reference terms,

$$C_s = \frac{M}{R_2 R_4} \tag{3.11}$$

and equating the quadrate terms,

$$R_s = \frac{R_2}{M}(L - M) - R_3 \tag{3.12}$$

3.8 Schering Bridge for Capacitance

The bridges so far considered have operated with supply voltages of the order of 10 V. The Schering bridge was developed to measure the loss resistance of dielectrics, line insulators, cables and high-voltage capacitors under high-voltage conditions (up to 100kV). The bridge is shown in Fig. 3.6. The unknown capacitance is represented by the equivalent series circuit (C_s, R_s). C_1 is a fixed high-voltage air capacitor, whose value is of the order of 50pF. R_2 is a fixed non-inductive resistor, while the resistor R_3 and the capacitor C_2 (mica type) are the variable elements. R_2 and R_3 are normally of the order of a few hundred ohms, so that their impedance will be negligible compared with that of C_s or C_1. This means

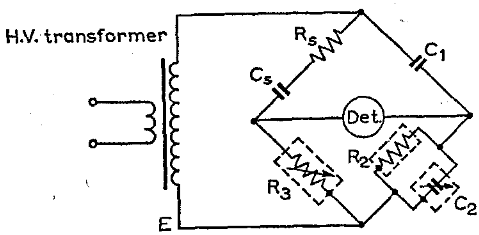


Fig. 3.6 HIGH-VOLTAGE SCHERING BRIDGE

that the voltage across the variable arms of the bridge (and the detector) will be only a very small fraction of the supply voltage, so that the bridge will be safe to operate despite the high voltage. Protective earthed screening is always employed to reduce the danger to the operator, and to stabilize stray leakage currents.

Substituting the appropriate values in the balance equation (3.1),

$$\frac{R_3}{j\omega C_1} = \left(R_s + \frac{1}{j\omega C_s}\right)\frac{1}{\frac{1}{R_2} + j\omega C_2}$$

Therefore

$$C_s = C_1\frac{R_2}{R_3} \tag{3.13}$$

and

$$R_s = R_3\frac{C_2}{C_1} \tag{3.14}$$

The loss angle of the test capacitor is

$$\delta = \tan^{-1} R_s\omega C_s = \tan^{-1}\omega C_2 R_2 \tag{3.15}$$

3.9 Campbell–Heaviside Equal-ratio Bridge

This bridge (Fig. 3.7) is used for the determination of small inductances in terms of a laboratory standard mutual inductance. R_1 and R_4 are two equal low-inductance standard resistors, while R_3 is a variable resistor whose value is of the same order as the resistance of the unknown coil (L_x and r_x). The mutual inductometer has mutual inductances M_x and M_y between the primary and the two halves of the secondary, and M_s is the mutual inductance between these halves themselves. The self-inductances of the secondary windings

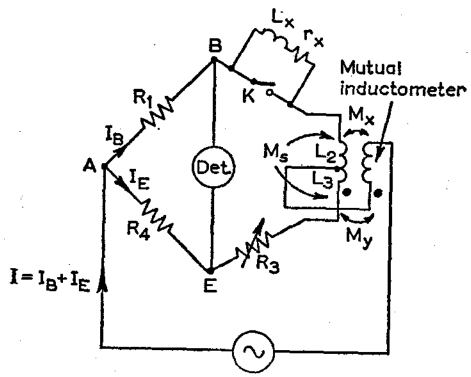


Fig. 3.7 CAMPBELL–HEAVISIDE EQUAL-RATIO BRIDGE

are L_2 and L_3. At balance there is no current through the detector, so that the total input current I will divide equally between paths AB and AE (since R_1 and R_4 are equal). Thus

$$I_B = I_E = \tfrac{1}{2}I \tag{3.16}$$

With key K open,

$$I_B(r_x + j\omega L_x) + I_B j\omega L_2 - Ij\omega M_x - I_E j\omega M_s$$
$$= I_E(R_3 + j\omega L_3) + Ij\omega M_y - I_B j\omega M_s$$

Substituting from eqn. (3.16),

$$\tfrac{1}{2}I\{r_x + j\omega(L_x + L_2)\} - Ij\omega(M_x + \tfrac{1}{2}M_s)$$
$$= \tfrac{1}{2}I(R_3 + j\omega L_3) + Ij\omega(M_y - \tfrac{1}{2}M_s)$$

Equating the reference terms,

$$r_x = R_3 \tag{3.17}$$

and equating the quadrate terms,

$$L_x + L_2 - L_3 = 2(M_x + M_y) \tag{3.18}$$

$(M_x + M_y)$ is the dial reading on the mutual inductometer. The method for small inductances is to balance the bridge first with key K open, and then with the key closed. The difference between the results gives the values of the unknown inductance and resistance, all residual errors (which would be important in this case) being cancelled out.

3.10 Frequency-dependent Bridges

In many forms of a.c. bridge the balance conditions are dependent on the frequency of the source. A bridge of this nature may therefore

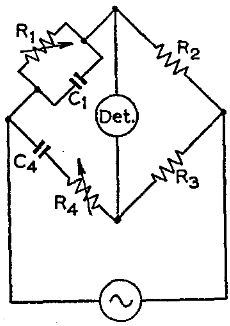


Fig. 3.8 WIEN PARALLEL BRIDGE

be used to measure the frequency of an a.c. supply. A simple bridge of this type has a series resonant circuit (with variable capacitance) as one arm, the other arms being pure resistors of suitable value. At resonance the resonant circuit has no reactance, and an adjustment of one of the resistors will give balance.

Another frequency-dependent bridge which does not rely on inductance is the Wien parallel bridge illustrated in Fig. 3.8. R_3 is a standard resistor of twice the value of R_2, and C_1 and C_4 are equal standard capacitors. R_1 and R_4 are resistors which can be varied together so that their values remain equal.

At balance,

$$R_3 \frac{1}{\dfrac{1}{R_1} + j\omega C_1} = R_2 \left(R_4 + \frac{1}{j\omega C_4} \right)$$

Hence

$$\frac{R_3}{R_2} = \frac{R_4}{R_1} + \frac{C_1}{C_4} + j \left(\omega C_1 R_4 - \frac{1}{\omega C_4 R_1} \right)$$

and

$$\omega^2 = \frac{1}{C_1 C_4 R_1 R_4}$$

Therefore

$$f = \frac{1}{2\pi CR} \tag{3.19}$$

where $C = C_1 = C_4$, and $R = R_1 = R_4$.

Note that with the values chosen for R_1, R_2, R_3, R_4, C_1 and C_4,

$$\frac{R_3}{R_2} = 2 = \frac{R_4}{R_1} + \frac{C_1}{C_4}$$

If f is known the bridge may be used to measure capacitance.

3.11 Accuracy of Bridges

In bridge measurements it is desirable to be able to calculate the accuracy of the measurement in terms of the accuracy of the known bridge elements, and the accuracy with which balance is achieved. The comparison standards used in bridges normally have an error of about ±0·02 per cent or less. The accuracy with which balance conditions are achieved varies from bridge to bridge, but the error should be less than ±0·5 per cent.

Suppose that the balance equation for a bridge is in the form

$$X = \frac{AB}{C}$$

Then

$$\log_e X = \log_e A + \log_e B - \log_e C$$

Therefore

$$\frac{dX}{X} = \frac{dA}{A} + \frac{dB}{B} - \frac{dC}{C} \tag{3.20}$$

If the percentage errors of A, B and C are known, then eqn. (3.20) shows that the percentage error with which X is determined is simply the *sum* of the percentage errors of A, B, and C. The negative sign in front of C does not count since errors are always given as plus or minus.

If the balance conditions result in the subtraction of two quantities, then the actual limits of each quantity must be determined (not the percentage error). These limits are then added to give the limits of the final result. If the two subtracted quantities are almost equal, the resultant error can be large.

EXAMPLE 3.1 The Schering bridge shown in Fig. 3.6 is used to measure the capacitance and loss resistance of a length of concentric cable. The supply voltage is 100 kV at 50 Hz. C_1 is an air capacitor of 40 pF, R_2 is fixed at $1{,}000/\pi$ ohms, R_3 is $122 \pm 0{\cdot}5\,\Omega$, and C_2 is $0{\cdot}921 \pm 0{\cdot}001\,\mu$F.

Determine (*a*) the cable capacitance, (*b*) the parallel loss resistance, (*c*) the loss angle of the cable, and (*d*) the power loss in the cable.

$$\text{Percentage error of } R_3 = \pm\frac{0{\cdot}5}{122} \times 100 = \pm 0{\cdot}4\%$$

$$\text{Percentage error of } C_2 = \frac{0{\cdot}001}{0{\cdot}921} \times 100 = 0{\cdot}108\%$$

(*a*) From the balance condition of eqn. (3.13),

$$C_s = 40 \times 10^{-12} \times \frac{1{,}000}{\pi(122 \pm 0{\cdot}4\%)} = \underline{\underline{104{\cdot}3\,\text{pF} \pm 0{\cdot}4\%}}$$

(*b*) From the balance condition of eqn. (3.14),

$$R_s = \frac{(122 \pm 0{\cdot}4\%)(0{\cdot}921 \pm 0{\cdot}108\%) \times 10^{-6}}{40 \times 10^{-12}} = 2{\cdot}81\,\text{M}\Omega \pm 0{\cdot}508\%$$

The parallel loss resistance of the cable is the equivalent parallel resistance corresponding to R_s, namely

$$R_p = \frac{1}{R_s\omega^2 C_s^2} = \frac{1}{\omega^2}\frac{C_1}{R_3 C_2}\left(\frac{R_3}{C_1 R_2}\right)^2 = \frac{1}{\omega^2}\frac{R_3}{C_1 C_2 R_2^2}$$

$$= \frac{1}{314^2}\frac{122 \pm 0{\cdot}4\%}{(40 \times 10^{-12})(0{\cdot}921 \pm 0{\cdot}108\%)10^{-6}(1{,}000/\pi)^2}$$

$$= \underline{\underline{332\,\text{M}\Omega \pm 0{\cdot}508\%}}$$

(*c*) From eqn. (3.15),

$$\delta = \tan^{-1}\left(100\pi \times 0{\cdot}921 \times \frac{1{,}000}{\pi} \times 10^{-6} \pm 0{\cdot}108\%\right)$$

$$= \underline{\underline{0{\cdot}0921\,\text{rad} \pm 0{\cdot}108\%}}$$

since for small angles $\tan\delta = \delta$.

(*d*) Since R_3 is so small compared with R_s, the whole supply voltage may be considered to be across C_s and R_s. Therefore

$$\text{Power loss} = \frac{V^2}{R_p} = \frac{10^{10}}{332 \times 10^6} = \underline{\underline{30{\cdot}1\,\text{W}}}$$

3.12 Stray Effects and Residuals

Where extreme accuracy is desired from a.c. bridges the effects of residual inductance and capacitance in the standard resistors, and losses in capacitors, should be allowed for by including appropriate terms in the balance equations. In addition to those effects there will be stray effects which are generally not calculable.

Stray electromagnetic coupling between components may be minimized by (*a*) using non-inductive resistors, (*b*) having all inductance elements in toroidal form so that there is little external

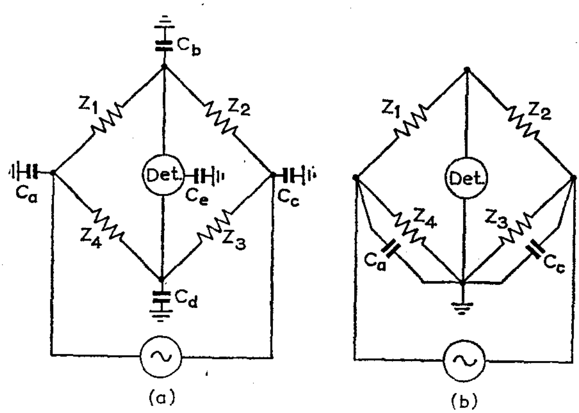


Fig. 3.9 STRAY CAPACITANCES IN A.C. BRIDGES

magnetic field, and (*c*) twisting the leads to the various components where possible.

Stray electrostatic coupling between components is minimized by ensuring adequate spacing between them.

More serious is the capacitance between components and earth. Where possible screened components are used, the capacitance between the components and the screen then being measurable and fixed. The stray earth capacitance of the detector will also cause a current through it which will make true balance impossible. This is termed "head effect" and can only be eliminated if the detector is at earth potential at balance.

Fig. 3.9(*a*) shows diagrammatically the stray earth capacitances in an a.c. bridge. The effect of connecting one end of the detector to earth is shown at (*b*). C_b at (*a*) simply shunts the detector, and so may be neglected. So also may C_e since the detector is at earth potential. C_a and C_c, however, now appear across the arms Z_4 and Z_3 so introducing an unknown factor into the balance conditions. For this reason direct earthing of the detector is not employed.

A device which overcomes this difficulty is the *Wagner earth*, shown in Fig. 3.10. Z_1 and Z_2 are preferably the fixed arms of the bridge, while Z_5 and Z_6 are additional elements which must be made to balance with Z_1 and Z_2. The junction point of Z_5 and Z_6 is solidly earthed so that the stray capacitances C_a and C_c are in parallel with them. An approximate balance is obtained with the switch in position 1. The switch is then moved to position 2, and a further balance is obtained by varying Z_5 and/or Z_6. When this is achieved the point A must be at earth potential. The switch is then moved back to position 1 and a further balance is obtained. When this is achieved the detector must still be at earth potential

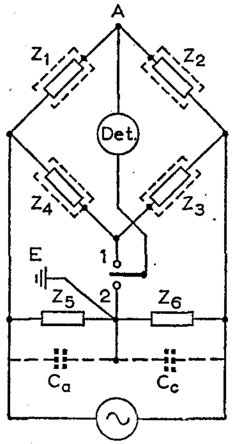


Fig. 3.10 WAGNER EARTHING DEVICE

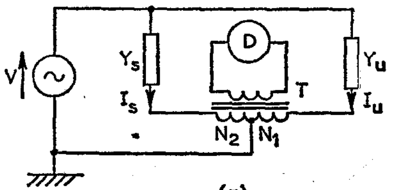
and so there will be no stray capacitance effects. The values of Z_5 and Z_6 need not, of course, be accurately known.

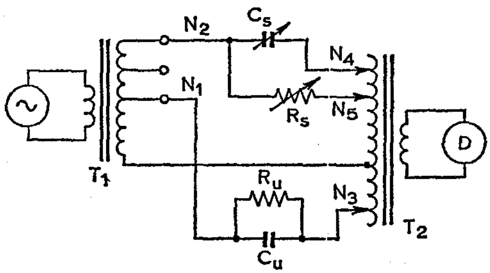
3.13 Transformer Ratio-arm Bridges

Unlike the bridge circuits considered so far, the transformer ratio-arm bridge depends on an ampere-turn balance in a transformer. Admittance measurements over a wide range of frequencies up to some 250 MHz are possible. The basic circuit is shown in Fig. 3.11(*a*), where Y_u is the unknown admittance and Y_s is a standard variable. Assuming ideal transformers the detector will indicate a null when there are no net ampere-turns in the output transformer T, i.e. When

$$I_u N_1 = I_s N_2$$

Neglecting leakage reactance and winding resistance there is





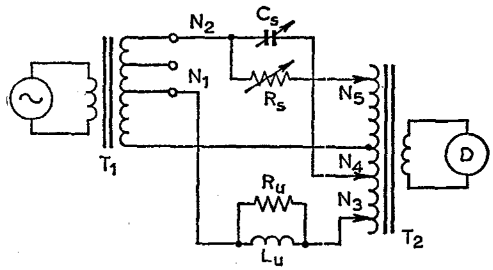


Fig. 3.11 TRANSFORMER RATIO-ARM BRIDGE

then no voltage drop across either input section of the transformer. Hence

$$I_u = VY_u \qquad \text{and} \qquad I_s = VY_s$$

so that

$$\frac{Y_u}{Y_s} = \frac{I_u}{V}\frac{V}{I_s} = \frac{N_2}{N_1}$$

or

$$Y_u = \frac{N_2}{N_1} Y_s \tag{3.21}$$

In practice the leakage reactances of the transformers can be made to balance out and resistances can be made negligible. An important feature of the bridge is that one terminal of both source and detector can be earthed. Further, if the centre tap of transformer T is earthed, then at balance one end of both the standard and the unknown will be earthed. Strays may readily be taken into account by setting the standard admittance to zero, and balancing without the unknown connected, by means of auxiliary variables in the standard arm which need not be calibrated. In this way the effect of long leads to the unknown may be eliminated.

A simplified set-up for the measurement of an unknown capacitance is shown in Fig. 3.11(*b*). Balancing components are used to eliminate stray effects, and the bridge is rearranged to increase its flexibility.

Taking resistive and reactive balance conditions separately,

$$R_u = \frac{N_1N_3}{N_2N_5}R_s \qquad \text{and} \qquad C_u = \frac{N_2N_4}{N_1N_3}C_s \tag{3.22}$$

To measure an unknown inductance, the standard capacitor is connected to the same side of transformer T_2 as the unknown, as shown in Fig. 3.11(*c*). At balance,

$$R_u = \frac{N_1N_3}{N_2N_5}R_s \qquad \text{and} \qquad \frac{1}{j\omega L_u} = -\frac{N_2N_4}{N_1N_3}j\omega C_s$$

or

$$L_u = \frac{N_1N_3}{N_2N_4}\frac{1}{\omega^2 C_s} \tag{3.23}$$

For this measurement the frequency must be accurately known.

In commercial bridges the turns-ratio terms are directly read as multipliers (usually decades) on dials.

Errors can be reduced to around 0·1 per cent in audio-frequency bridges, while from 50 to 250 MHz, accuracy within 1 or 2 per cent is still possible.

3.14 Bridged-T and Parallel-T Networks

These networks have the advantage over 4-arm bridges that one end of both source and detector may be solidly earthed. Such networks can be used for measurements at frequencies above the audio-frequency range.

Consider the bridged-T network shown in Fig. 3.12(*a*). At balance there is no current through the detector, and hence

$$I_2 = -I_4$$

Also point A must then be at earth potential, so that

$$I_4 = \frac{V}{Z_4} \quad \text{and} \quad I_2 = \frac{V}{\{Z_1 + Z_2Z_3/(Z_2 + Z_3)\}} \frac{Z_3}{Z_2 + Z_3}$$

$$= \frac{VZ_3}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3}$$

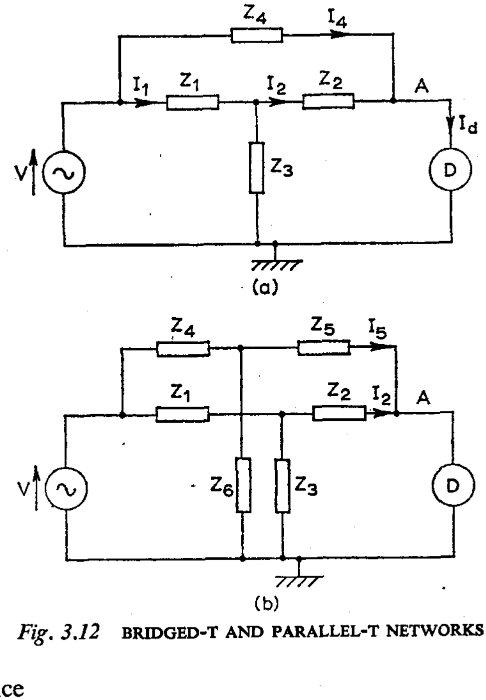


Fig. 3.12 BRIDGED-T AND PARALLEL-T NETWORKS

Hence

$$-Z_4 = Z_1 + Z_2 + Z_1Z_2/Z_3$$

or

$$Z_1 + Z_2 + Z_4 + Z_1Z_2/Z_3 = 0 \tag{3.24}$$

For the parallel-T circuit shown in Fig. 3.12(*b*), if the detector current is zero it follows that

$$I_2 = -I_5$$

Since point A is then at earth potential,

$$I_5 = \frac{VZ_6}{Z_4Z_5 + Z_4Z_6 + Z_5Z_6}$$

and

$$I_2 = \frac{VZ_3}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3}$$

Hence

$$Z_4 + Z_5 + Z_4Z_5/Z_6 = -(Z_1 + Z_2 + Z_1Z_2/Z_3) \tag{3.25}$$

represents the balance conditions.

EXAMPLE 3.2 The bridged-T circuit of Fig. 3.13 is used to measure the inductance L and series resistance R of a coil at 3·18 MHz. If $C = 45{\cdot}9$ pF and $R_3 = 10{\cdot}4$ kΩ find L and R.

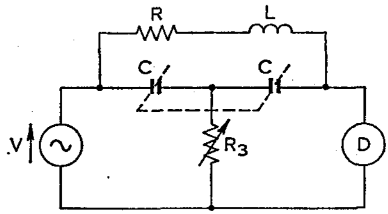


Fig. 3.13 TYPICAL BRIDGED-T CIRCUIT

In this circuit the two capacitors are ganged together to form one variable, while R_3 forms the other.

From eqn. (3.24), at balance

$$\frac{1}{j\omega C} + \frac{1}{j\omega C} - \frac{1}{\omega^2 C^2 R_3} + R + j\omega L = 0$$

Equating reference terms,

$$R = \frac{1}{\omega^2 C^2 R_3} = \frac{1}{4 \times 10^{14} \times 21 \times 10^{-22} \times 10{\cdot}4 \times 10^3} = \underline{\underline{115\,\Omega}}$$

and equating quadrate terms

$$L = \frac{2}{\omega^2 C} = \frac{2}{4 \times 10_{14} \times 45{\cdot}9 \times 10^{-12}} = \underline{\underline{109\,\mu\text{H}}}$$

3.15 The Q-meter

At radio frequencies it is often convenient to be able to measure the Q-factor of a coil directly. The Q-meter is designed specifically to do this, and can also be used to measure inductance and capacitance. The method is not a null but a resonance method, and since it depends on the calibration of a meter, may involve errors of 1 or 2 per cent. However, since the measurement is at maximum current the effect of strays is minimized.

The basic circuit of a Q-meter is shown in Fig. 3.14. A variable-frequency oscillator is loosely coupled to a very low resistance, r. The unknown coil is inserted in series with a calibrated standard variable air capacitor C_s to form a series resonant circuit. If r is much smaller than R, the voltage, V, applied to the tuned circuit will be constant and equal to Ir. The current I is measured by a thermocouple ammeter, A, and can be set at some standard known value by altering the coupling between the oscillator and the load

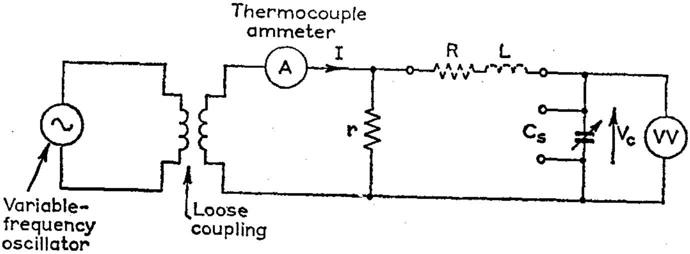


Fig. 3.14 THE Q-METER

circuit. With the oscillator set at the desired frequency f_0 ($= \omega_0/2\pi$), the capacitor C_s is tuned to give a maximum indication on the electronic voltmeter VV. In this condition,

$$V_c = \frac{I_c}{j\omega_0 C_s} = \frac{V}{Rj\omega_0 C_s} = \frac{Ir}{j\omega_0 C_s R}$$

Also, at resonance and assuming a reasonably large value of Q,

$$\omega L = 1/\omega C_s \tag{3.26}$$

so that the magnitude of V_c is given by

$$V_c = Ir\frac{\omega_0 L}{R} = \text{constant} \times Q_0 \tag{3.27}$$

The electronic voltmeter can be calibrated to read directly in units of Q. The range of Q measured can be altered by changing the current I to a new standard on the ammeter scale.

MEASUREMENT OF CAPACITANCE

The value of an unknown capacitance within the range of the standard C_s can be readily determined by the method of substitution.

A standard coil is used, and the circuit is tuned to resonance at a suitable frequency. The unknown capacitor is then connected in parallel with C_s and the circuit retuned to resonance by varying C_s. The decrease in C_s is then equal to the value of the unknown capacitor. By noting the change in Q for the two conditions the parallel loss resistance of the unknown capacitor can also be determined.

MEASUREMENT OF INDUCTANCE

The unknown inductance is connected in series with the standard capacitor, C_s, and the circuit is tuned to resonance. The unknown inductance, L_u, is then given by

$$\omega L_u = \frac{1}{\omega C_s}$$

SELF-CAPACITANCE OF A COIL

An equivalent circuit of a coil is shown in Fig. 3.15(*a*). At high frequencies the self-capacitance C_0, may materially affect the coil

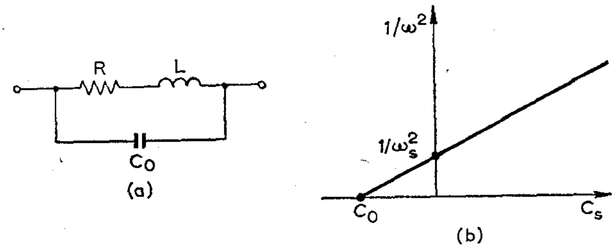


Fig. 3.15 REACTANCE VARIATION METHOD OF FINDING THE SELF-CAPACITANCE OF A COIL

impedance, and generally coils are not operated at more than one-third of their self-resonant frequency, f_0 ($= 1/2\pi\sqrt{(LC_0)}$). C_0 can be measured on the Q-meter by the method of reactance variation. The values of standard capacitor setting, C_s, to achieve resonance are noted for a range of frequencies. Since the resistance r (Fig. 3.14) is so low, the self-capacitance may be taken as being in parallel with C_s, and resonance occurs when

$$\omega L = \frac{1}{\omega(C_s + C_0)}$$

Hence

$$\frac{1}{\omega^2 L} = C_s + C_0 \tag{3.28}$$

If $1/\omega^2$ is plotted to a base of C_s, a straight-line graph is obtained, which may be extrapolated as shown in Fig. 3.15(*b*) to give C_0 and the self-resonant angular frequency $\omega_s = 1/\sqrt{(LC_0)}$.

PROBLEMS

3.1 An a.c. bridge network for the measurement of inductance consists of four impedances arranged as a closed loop ABCD, where AB is the unknown inductance coil; BC is a variable resistance Q, in series with a capacitor C_1; CD is a capacitor C_2; and DA is a resistance R.

State the conditions for balance and obtain expressions for the inductance and resistance of the coil.

Hence calculate the values of the inductance and resistance of a coil if balance is obtained when $Q = 250\Omega$, $C_1 = 5\mu F$, $C_2 = 2\mu F$, and $R = 1{,}000\Omega$. (*H.N.C.*)

Ans. 0·5 H, 400Ω.

3.2 A coil having an inductance of the order of 1 H is measured by an a.c. bridge method at a frequency of 1 kHz. In order to bring the impedance to be measured within the range of the bridge, the coil is shunted by a non-inductive resistance of 500Ω and the equivalent series impedance of the combination is measured, the values obtained being 6·35 mH and 487Ω.

Determine the inductance and resistance of the coil. Indicate the general effect of this procedure on the possible accuracy of the measurement. (*L.U.*)

Ans. 0·92 H, 1,420Ω.

3.3 A modified Carey Foster bridge is arranged as follows.

Arm AB is a non-inductive resistance of 10Ω.
Arm BC is a non-inductive resistance of 500Ω.
Arm CD is a variable resistor, R, in series with a 1·0μF capacitor.
Arm DA is the secondary of a variable mutual inductor the primary of which is connected between A and the source of supply, the other lead of the supply being taken to C. A detector is across BD. The secondary of the mutual inductor has a resistance of 15Ω and $\omega = 5{,}000$ rad/s. At balance $R = 185\Omega$.

Find the corresponding mutual inductance and the self-inductance of the secondary. Also calculate the current in each arm of the bridge assuming $V_{AC} = 3$ V.

Draw a complexor diagram representing the currents and p.d.s across the arms. (*L.U.*)

Ans. 5·65 mH, 6·94 mH, 5·88 mA, 10·8 mA.

3.4 The conditions at balance of a Schering bridge set up to measure the capacitance and loss angle of a paper-dielectric capacitor are as follows:

$f = 500$ Hz
Z_1 = a pure capacitance of 0·1μF
Z_2 = a resistance of 500Ω shunted by a capacitance of 0·0033μF
Z_3 = pure resistance of 163Ω
Z_4 = capacitor under test

Calculate the approximate values of the loss resistance of the capacitor assuming: (*a*) series loss resistance, and (*b*) shunt loss resistance. (L.U.)

Ans. 5·37Ω, 197kΩ.

3.5 An a.c. bridge network consists of the following four arms: AB—a fixed resistor R_1; BC—a variable resistor R_2 in series with a variable capacitor C; CD—a fixed resistor R_3; DA—a coil of unknown inductance L, and loss resistance R.

Derive expressions for L and R when the bridge is balanced at a frequency f. (This is Hay's bridge.) Evaluate L and R, with their limits of possible error if the values of the components when the bridge is balanced are

$$R_1 = 1{,}000\Omega \pm 1 \text{ part in } 10{,}000$$
$$R_2 = 2{,}370\Omega \pm 0{\cdot}1\Omega$$
$$C = 4{,}210\text{pF} \pm 1\text{pF}$$
$$R_3 = 1{,}000\Omega \pm 1 \text{ part in } 10{,}000$$

The frequency of the bridge supply is 1,595Hz to an accuracy of ±1Hz. (*L.U.*)

Ans. 4,170 ± 1·88μH, 4·26 ± 0·009Ω.

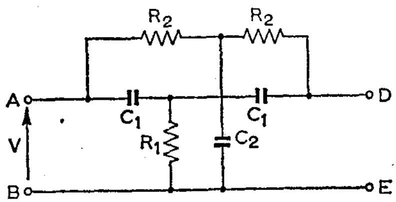


Fig. 3.16

3.6 Using the star-delta transformation show that the voltage between D and E in the parallel-T circuit shown in Fig. 3.16 is zero when

$$\frac{2}{\omega C_1} = R_2{}^2\omega C_2 \quad \text{and} \quad \frac{1}{R_1(\omega C_1)^2} = 2R_2$$

3.7 Solve Problem 3.6 by any other method.

Chapter 4

ADVANCED THREE-PHASE THEORY

Symbolic notation may be simply extended to cover 3-phase systems. It allows solutions to be obtained more easily for problems involving unbalanced loads, these problems being extremely awkward without symbolic methods.

4.1 The 120° Operator

It is important to maintain a conventional positive direction in which to measure voltages in a 3-phase system. To facilitate this the following double-subscript notation will be used. V_{RY} denotes the voltage of the red line with respect to the yellow line, V_{YB} denotes the voltage of the yellow line with respect to the blue line, and V_{BR} denotes the voltage of the blue line with respect to the red line. These directions are illustrated in Fig. 4.1(*a*), from which it is clear that $V_{YB} = -V_{BY}$, etc.

In any 3-phase system there are two possible sequences in which the voltages may pass through their maximum positive values, namely red → yellow → blue, or red → blue → yellow. By convention the first of these sequences is called the *positive sequence*, and the second is called the *negative sequence*. The conventional positive sequence is the one which is most common for electricity supply and will be assumed in the following sections unless specifically stated otherwise. In Fig. 4.1(*b*) the line voltage complexor diagrams are drawn for both positive and negative sequences.

In 3-phase systems, the voltage complexors are displaced from one another by 120°, so that it is convenient to have an operator which rotates a complexor through this angle. This operator is a.*

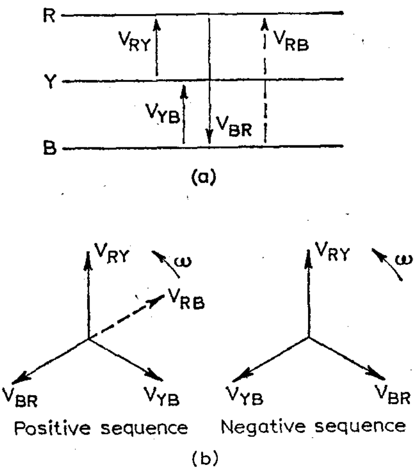


Fig. 4.1 VOLTAGE COMPLEXORS FOR SYMMETRICAL 3-PHASE SYSTEM

Any complexor when multiplied by a, remains unchanged in magnitude, and has 120° added to its phase angle. Thus

$$a = 1\underline{/120^\circ} \tag{4.1}$$

$$= -\tfrac{1}{2} + j\frac{\sqrt{3}}{2} \tag{4.2}$$

Also

$$a^2 = 1\underline{/120^\circ} \times 1\underline{/120^\circ} = 1\underline{/240^\circ} = -\tfrac{1}{2} - j\sqrt{3}/2$$

The operator a^2 will turn a complexor through 240° in an anti-clockwise direction. This is the same as turning it through 120° in a clockwise direction. Thus

$$a^2 = 1\underline{/-120^\circ}$$

In the same way,

$$a^3 = 1\underline{/360^\circ} = 1$$

It will be recalled that the operation $-j$ results in the complexor concerned being turned through an angle of $-90°$. It should be

* The symbol h may also be used to represent the 120° operator.

noted that the operation $-a$, however, does not turn a complexor through $-120°$. This can be seen as follows:

$$-a = a \times (-1) = a \times 1\underline{/180°} = 1\underline{/120°} \times 1\underline{/180°}$$
$$= 1\underline{/300°} = 1\underline{/-60°} \tag{4.3}$$

Thus $-a$ turns a complexor through 60° in a clockwise direction. From the rectangular forms for the operators, the following important identity may be verified:

$$a^2 + a + 1 = 0 \tag{4.4}$$

4.2 Four-wire Balanced Star

In the balanced 4-wire star-connected system shown in Fig. 4.2(*a*), the voltages are assumed to be symmetrical. The positive direction

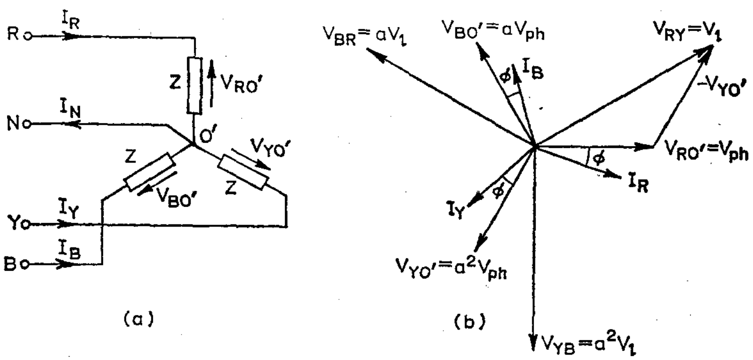


Fig. 4.2 BALANCED 4-WIRE STAR

of phase voltage at the load is assumed to be the potential of the line terminal with respect to the star point 0′. 0′ is in this case the neutral point of the system. Let the phase voltages have magnitude V_{ph}, and take the voltage between the neutral and the red line as the reference complexor, i.e.

$$V_{RO'} = V_{ph}\underline{/0°}$$

Then

$$V_{YO'} = V_{ph}\underline{/-120°} = a^2V_{ph}$$

and

$$V_{BO'} = V_{ph}\underline{/+120°} = aV_{ph}$$

The line voltage is the difference between the phase voltages concerned. Thus

$$V_{RY} = V_{RO'} - V_{YO'} \text{ (see Fig. 4.2(}b\text{))}$$
$$= V_{ph} - a^2V_{ph}$$
$$= V_{ph}\left\{1 - \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)\right\}$$
$$= V_{ph}\left\{\frac{3}{2} + j\frac{\sqrt{3}}{2}\right\} = \sqrt{3}V_{ph}\underline{/30^\circ} \tag{4.5}$$

In the same way,

$$V_{YB} = V_{YO'} - V_{BO'} = a^2V_{ph} - aV_{ph} = \sqrt{3}V_{ph}\underline{/-90^\circ}$$

and

$$V_{BR} = V_{BO'} - V_{RO'} = aV_{ph} - V_{ph} = \sqrt{3}V_{ph}\underline{/150^\circ}$$

In each case the line voltage is $\sqrt{3}$ times the phase voltage in magnitude, and leads the corresponding phase voltage by 30°.

In the same way, with I_R as the reference quantity,

$$I_R = I_{ph} = I_l = V_{ph}/Z_{ph}$$
$$I_Y = a^2I_l$$
$$I_B = aI_l$$

The current through the neutral line, by Kirchhoff's law, is

$$I_N = I_R + I_Y + I_B$$
$$= I_l + a^2I_l + aI_l$$
$$= (1 + a^2 + a)I_l$$
$$= 0 \text{ (from eqn. (4.4))}$$

4.3 Three-wire Balanced Star

Since there will be no neutral wire current in a 4-wire star with symmetrical supply voltages and balanced loads, the neutral wire may be removed, and the familiar 3-wire system is obtained.

4.4 Balanced Delta-connected Load

If the current I_1 through the load connected between the red and yellow lines of the delta-connected system shown in Fig. 4.3(*a*) is taken as the reference complexor at (*b*) then, since all the load impedances are equal, each load current will lag by the same angle behind its respective line voltage. Since the line voltages are assumed to be symmetrical, they will be 120° displaced from one another, and the load currents will thus be displaced by the same amount.

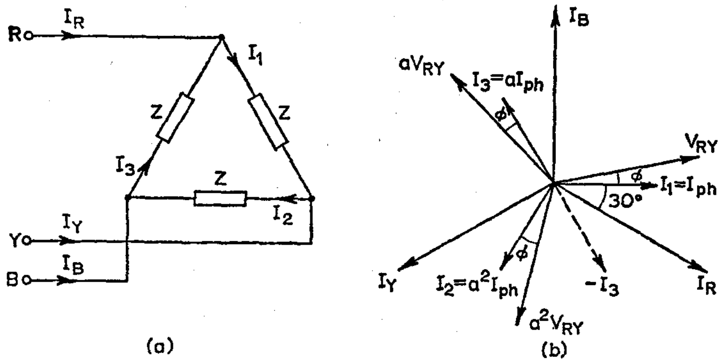


Fig. 4.3 BALANCED DELTA-CONNECTED LOAD

Let

$$I_1 = I_{ph} = \text{Reference complexor}$$

Then

$$I_2 = I_1\underline{/-120^\circ} = a^2 I_{ph}$$

and

$$I_3 = I_1\underline{/+120^\circ} = aI_{ph}$$

The line currents, by Kirchhoff's law, will be the difference between the phase currents at the corresponding terminals, i.e.

$$I_R = I_1 - I_3 = I_{ph} - aI_{ph}$$

$$= I_{ph}\left(\frac{3}{2} - j\frac{\sqrt{3}}{2}\right) = \sqrt{3}I_{ph}\underline{/-30^\circ}$$

In the same way,

$$I_Y = I_2 - I_1 = a^2 I_{ph} - I_{ph} = \sqrt{3}I_{ph}\underline{/-150^\circ}$$

and

$$I_B = I_3 - I_2 = aI_{ph} - a^2 I_{ph} = \sqrt{3}I_{ph}\underline{/90^\circ}$$

The sum of the three line currents is

$$I_R + I_Y + I_B = I_{ph}(1 - a) + I_{ph}(a^2 - 1) + I_{ph}(a - a^2) = 0$$

EXAMPLE 4.1 A symmetrical 3-phase 450 V system supplies a balanced delta-connected load of 12 kW at 0·8 p.f. lagging. Calculate (*a*) the phase currents, (*b*) the line currents, and (*c*) the effective impedance per phase.

The complexor diagram corresponds to that of Fig. 4.3(*b*).

$$\text{Power per phase} = \tfrac{1}{3} \times \text{total power} = 4\,\text{kW}$$

Therefore

$$\text{kVA per phase} = \frac{4}{0{\cdot}8} = 5\,\text{kVA}$$

and

$$\text{Current per phase} = \frac{5 \times 10^3}{450} = 11{\cdot}1\,\text{A at } 0{\cdot}8 \text{ power factor lagging}$$

(*a*) Take V_{RY} as the reference complexor; then the phase currents will be

$$I_1 = 11{\cdot}1\underline{/-\cos^{-1} 0{\cdot}8} = 11{\cdot}1\underline{/-36^\circ\ 52'}\,\text{A} = \underline{\underline{(8{\cdot}88 - j6{\cdot}66)\,\text{A}}}$$

$$I_2 = a^2 I_1 = 11{\cdot}1\underline{/-36^\circ\ 52'} \times 1\underline{/-120^\circ} = 11{\cdot}1\underline{/-156^\circ\ 52'}$$
$$= \underline{\underline{(-10{\cdot}2 - j4{\cdot}36)\,\text{A}}}$$

and

$$I_3 = aI_1 = 11{\cdot}1\underline{/-36^\circ\ 52'} \times 1\underline{/120^\circ} = 11{\cdot}1\underline{/83^\circ\ 8'}$$
$$= \underline{\underline{(1{\cdot}33 + j11{\cdot}0)\,\text{A}}}$$

(*b*) The line currents are found by subtraction:

$$I_R = I_1 - I_3 = 8{\cdot}88 - j6{\cdot}66 - 1{\cdot}33 - j11{\cdot}0 = 7{\cdot}55 - j17{\cdot}6$$
$$= \underline{\underline{19{\cdot}2\underline{/-66^\circ\ 52'}\,\text{A}}}$$

$$I_Y = I_2 - I_1 = -19{\cdot}1 + j2{\cdot}3 = \underline{\underline{19{\cdot}2\underline{/-186^\circ\ 52'}\,\text{A}}}$$

$$I_B = I_3 - I_2 = 11{\cdot}5 + j15{\cdot}4 = \underline{\underline{19{\cdot}2\underline{/53^\circ\ 8'}\,\text{A}}}$$

Note that, once I_R is found, I_Y and I_B follow for a balanced load by subtracting and adding 120° to the phase angle of I_R.

(*c*) The impedance per phase is given by

$$Z_{ph} = \frac{V_{RY}}{I_1} = \frac{450\underline{/0^\circ}}{11{\cdot}1\underline{/-36^\circ\ 52'}} = 40{\cdot}5\underline{/36^\circ\ 52'} = \underline{\underline{(32{\cdot}4 + j24{\cdot}3)\,\Omega}}$$

EXAMPLE 4.2 A short 3-phase transmission line has an effective resistance per conductor of 0·6 Ω and an effective inductive reactance per conductor of 0·8 Ω. Find the sending-end line voltage and power factor when the line supplies a balanced load of 1,800 kVA at 5·2 kV and 0·8 power factor lagging.

Since the load and transmission system are balanced, this problem may be treated in the same way as a single-phase problem.

$$\text{Receiving-end phase voltage, } V_{RN} = (5{,}200/\sqrt{3})\underline{/0^\circ} = 3{,}000\underline{/0^\circ}\,\text{V}$$

(This voltage is taken as the reference complexor.)

$$\text{Line current} = 1{,}800/(\sqrt{3} \times 5{\cdot}2) = 200\,\text{A}$$

With respect to the reference phase voltage this current may be expressed as $200\underline{/-36.9°}$ A. Thus

$$\text{Line voltage drop} = IZ = 200\underline{/-36.9°}(0.6 + j0.8) = (192 + j56)\,\text{V}$$

$$\text{Sending-end phase voltage} = 3{,}000 + 192 + j53 \approx 3{,}190\underline{/1°}\,\text{V}$$

and

$$\text{Sending-end line voltage} = \underline{\underline{5.51\,\text{kV}}}$$

Also,

$$\text{Sending-end power factor} = \underline{\underline{\cos 37.9°}}$$

4.5 Unbalanced Four-wire Star-connected Load on System of Negligible Line Impedance (Fig. 4.4)

This is the simplest case of an unbalanced load, and may be treated as three separate single-phase systems with a common return lead.

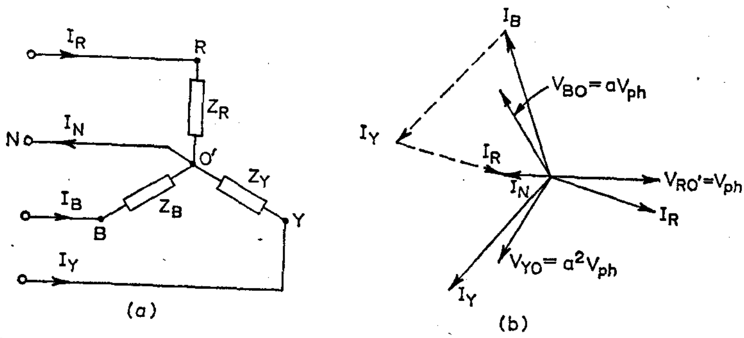


Fig. 4.4 UNBALANCED 4-WIRE STAR

The phase voltages will be equal in magnitude and displaced by 120° from each other because the voltage of the load star point is the same as that of the generator star point. The neutral wire current is the complexor sum of the three line currents.

If the voltage across the red phase is taken to be in the reference direction, then

$$V_{RO'} = V_{ph};\quad V_{YO'} = a^2 V_{ph} = V_{ph}\underline{/-120°}$$

and

$$V_{BO'} = aV_{ph} = V_{ph}\underline{/120°}$$

The corresponding phase currents are

$$I_R = \frac{V_{ph}}{Z_R} \qquad I_Y = \frac{V_{ph}\underline{/-120°}}{Z_Y} \qquad I_B = \frac{V_{ph}\underline{/120°}}{Z_B}$$

These currents will also be the line currents in the system. The current through the neutral wire will then be given by

$$I_N = I_R + I_Y + I_B = \frac{V_{ph}}{Z_R} + \frac{V_{ph}\angle -120^\circ}{Z_Y} + \frac{V_{ph}\angle 120^\circ}{Z_B} \qquad (4.6)$$

The impedances are, of course, in complex form. The currents are shown in Fig. 4.4(*b*).

This result may be obtained from the complexor diagram as follows.

1. Draw the three phase voltages, equal in magnitude and 120° apart.
2. Calculate, by single-phase theory, the current in each phase and the phase angle relative to the corresponding phase voltage, taking each voltage in turn as a reference quantity ($I = V_{ph}/Z$).
3. Draw these currents in the complexor diagram in the correct phase relationship to the corresponding phase voltages.
4. Find, by complexor addition, the sum of the three phase currents. This will give the neutral wire current in magnitude and phase.

The overall power factor of an unbalanced load is taken to be the ratio of total kW to total kVA.

EXAMPLE 4.3 The 440 V, 50 Hz, 3-phase 4-wire main to a workshop provides power for the following loads.

(*a*) Three 3 kW induction motors each 3-phase, 85 per cent efficient, and operating at a lagging power factor of 0·9.

(*b*) Two single-phase electric furnaces of 250 V rating each consuming 6 kW at unity power factor.

(*c*) A general lighting load of 3 kW, 250 V at unity power factor.

If the lighting load is connected between one phase and neutral, while the furnaces are connected one between each of the other phases and neutral, calculate the current in each line and the neutral current at full load. (*H.N.C.*)

$$\text{Total motor power input} = \frac{3 \times 3}{0{\cdot}85} = 10{\cdot}6\,\text{kW}$$

Therefore

$$\text{Motor kVA input} = \frac{\text{kW}}{\text{power factor}} = \frac{10{\cdot}6}{0{\cdot}9} = 11{\cdot}8\,\text{kVA}$$

Also

$$\text{Each motor kVAr} = \text{kVA} \times \sin(\cos^{-1} 0{\cdot}9) = 5{\cdot}1\,\text{kVAr}$$

$$\text{Current in each line due to motor load} = \frac{11{\cdot}8 \times 1{,}000}{\sqrt{3} \times 440} = 15{\cdot}4\,\text{A}$$

This current will lag behind the corresponding voltage by $\cos^{-1} 0{\cdot}9 = 25^\circ\,50'$. Thus the current through each line due to the motor, and with reference to each phase voltage, will be $15{\cdot}4\angle 25^\circ\,50' = (14 - j6{\cdot}7)$A. The line current for each furnace will be 6,000/250 = 24A at 0° phase angle (i.e. $24\underline{/0°}$, or $24 + j0$), with respect to the corresponding phase voltage. The line current for the lighting load will be 3,000/250 = 12A at unity p.f. (i.e. $12\underline{/0°}$ or $12 + j0$) with respect to the corresponding phase voltage. The total current in each furnace line will then be

$$14 - j6{\cdot}7 + 24 = 38 - j6{\cdot}7 = \underline{\underline{38{\cdot}5\underline{/-10°}\,\mathrm{A}}}$$

with respect to the corresponding phase voltage.

The current in the third line will be

$$14 - j6{\cdot}7 + 12 = 26 - j6{\cdot}7 = \underline{\underline{26{\cdot}8\underline{/-14°}\,\mathrm{A}}}$$

The complexor diagram corresponds to that of Fig. 4.4(*b*).

To find the neutral current, the three line current complexors may be added graphically or by the following method using symbolic notation.

To simplify the calculation, it should be noted that, since the motors form balanced loads, the motor currents will not give rise to a neutral current and may be neglected in the calculation of the neutral current.

Let the lighting load phase voltage be taken as the reference complexor. Then

Lighting load current $= (12 + j0)\mathrm{A}$

First heating load current $= (24 + j0)a^2 = (-12 - j20{\cdot}8)\mathrm{A}$

Second heating load current $= (24 + j0)a = (-12 + j20{\cdot}8)\mathrm{A}$

Therefore

$$\begin{aligned}\text{Neutral current} &= I_R + I_Y + I_B\\ &= (12 + j0) + (-12 - j20{\cdot}8) + (-12 + j20{\cdot}8)\\ &= -12 + j0 = \underline{\underline{12\underline{/180°}\,\mathrm{A}}}\end{aligned}$$

4.6 Unbalanced Delta-connected Load (Fig. 4.5)

In the case of the delta-connected unbalanced load with symmetrical line voltages, full line voltage will be across each load phase.

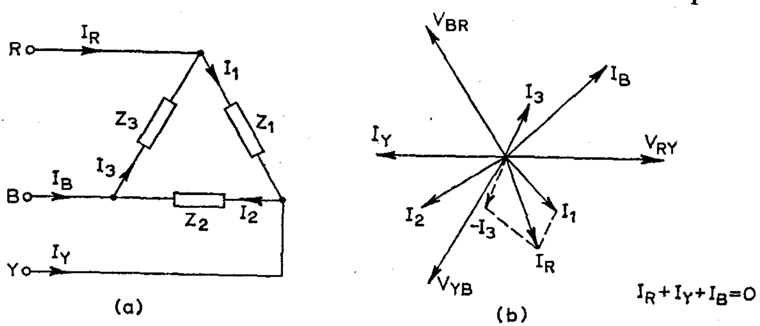


Fig. 4.5 UNBALANCED DELTA-CONNECTED LOAD

The problem thus resolves itself into three independent single-phase systems supplied with voltages which are 120° apart in phase.

In the analytical solution the line voltage V_{RY} will be taken as the reference complexor (Fig. 4.5(*b*)), so that

$$V_{RY} = V_l \qquad V_{YB} = V_l\underline{/-120^\circ} \qquad V_{BR} = V_l\underline{/120^\circ}$$

The complex impedances of the load are Z_1, Z_2 and Z_3, connected as shown in Fig. 4.5(*a*).

Then the phase currents are easily obtained from the equations

$$I_1 = \frac{V_l}{Z_1} \qquad I_2 = \frac{V_l\underline{/-120^\circ}}{Z_2} \qquad I_3 = \frac{V_l\underline{/120^\circ}}{Z_3}$$

By Kirchhoff's first law, the line currents will be the differences between the corresponding phase currents. Since there is no neutral wire, the complexor sum of these three line currents must be zero.

TRIANGULAR COMPLEXOR DIAGRAMS

In some cases it is convenient to represent the line voltage and line current complexor diagrams of 3-phase circuits in a triangular

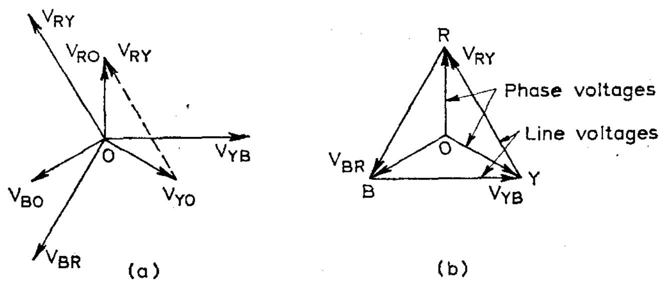


Fig. 4.6 PHASE- AND LINE-VOLTAGE COMPLEXOR DIAGRAMS IN TRIANGULAR FORM

form. In Fig. 4.6(*a*) the conventional complexor diagram for line and phase voltages is shown, while Fig. 4.6(*b*) shows the triangular diagram. The line voltage complexors form a closed triangle. These complexors correspond exactly in magnitude and direction to the line voltage complexors shown at (*a*).

For symmetrical line voltages, the line voltage triangle will be equilateral and 0 will be the centroid of the triangle.

In a 3-wire system, there is no resultant current so that the three line current complexors must form a closed triangle. Fig. 4.7 shows the diagram for a delta-connected system.

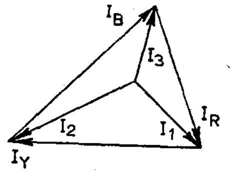


Fig. 4.7 LINE CURRENTS IN A 3-WIRE SYSTEM

EXAMPLE 4.4 Three impedances, $Z_1 = (10 + j10)\Omega$, $Z_2 = (8{\cdot}66 + j5)\Omega$, and $Z_3 = (12 + j16)\Omega$ are delta-connected to a 380V, 3-phase system. Determine the line currents and draw the complexor diagram.

The circuit is the same as that of Fig. 4.5(*a*). Take V_{RY} as the reference. Then

$$V_{YB} = 380\angle 240^\circ \text{V} \qquad \text{and} \qquad V_{BR} = 380\angle 120^\circ \text{V}$$

so that

$$I_1 = \frac{V_{RY}}{Z_1} = \frac{380\angle 0^\circ}{10 + j10} = \frac{380\angle 0^\circ}{14{\cdot}14\angle 45^\circ} = 26{\cdot}8\angle -45^\circ = (19 - j19)\,\text{A}$$

$$I_2 = \frac{V_{YB}}{Z_2} = \frac{380\angle 240^\circ}{8{\cdot}66 + j5} = \frac{380\angle 240^\circ}{10\angle 30^\circ} = 38\angle 210^\circ = (-32{\cdot}9 - j19)\,\text{A}$$

$$I_3 = \frac{V_{BR}}{Z_3} = \frac{380\angle 120^\circ}{12 + j16} = \frac{380\angle 120^\circ}{20\angle 53{\cdot}1^\circ} = 19\angle 66{\cdot}9^\circ = (7{\cdot}45 + j17{\cdot}5)\,\text{A}$$

Therefore

$$I_R = I_1 - I_3 = 11{\cdot}5 - j36{\cdot}5 = \underline{\underline{38{\cdot}2\angle -72{\cdot}5^\circ \text{A}}}$$

$$I_Y = I_2 - I_1 = -51{\cdot}9 - j0 = \underline{\underline{51{\cdot}9\angle 180^\circ \text{A}}}$$

$$I_B = I_3 - I_2 = 40{\cdot}4 + j36{\cdot}5 = \underline{\underline{54{\cdot}3\angle 42{\cdot}1^\circ \text{A}}}$$

The complexor diagram is that shown in Fig. 4.5(*b*).

4.7 Unbalanced Three-wire Star-connected Load

The unbalanced 3-wire star load is the most difficult unbalanced 3-phase load to deal with, but several methods are available. One method is to apply the star–mesh transformation to the load. The problem is then solved as a delta-connected system, and the line currents are obtained. A second method is to apply Maxwell's mesh equations to the system, using the complex notation for impedances, voltages and currents. Both of these methods involve a fairly large amount of arithmetical work, which, while not eliminated, is at least simplified by the use of Millman's theorem.

The circuit and complexor diagrams are shown in Fig. 4.8. In this case 0 is the star point of the generator or the neutral of the supply (normally zero potential). The voltages between 0 and the end points of Z_R, Z_Y and Z_B are the phase voltages of the supply. Hence, by Millman's theorem, the voltage of 0′ with respect to 0 is given by

$$V_{0'0} = \frac{V_{R0}Y_R + V_{Y0}Y_Y + V_{B0}Y_B}{Y_R + Y_Y + Y_B} \tag{4.7}$$

The voltage across each phase of the load is derived by considering that, for example, the voltage $V_{R0'}$ is the voltage of line R with respect to 0′. This is then the voltage of line R with respect to 0 less the voltage of 0′ with respect to 0. In symbols this gives

$$\left.\begin{aligned} V_{R0'} &= (V_{R0} - V_{0'0}) \\ V_{Y0'} &= (V_{Y0} - V_{0'0}) \\ V_{B0'} &= (V_{B0} - V_{0'0}) \end{aligned}\right\} \tag{4.8}$$

The line currents are then given by

$$\left.\begin{aligned} I_{R0'} &= (V_{R0} - V_{0'0})Y_R \\ I_{Y0'} &= (V_{Y0} - V_{0'0})Y_Y \\ I_{B0'} &= (V_{B0} - V_{0'0})Y_B \end{aligned}\right\} \tag{4.9}$$

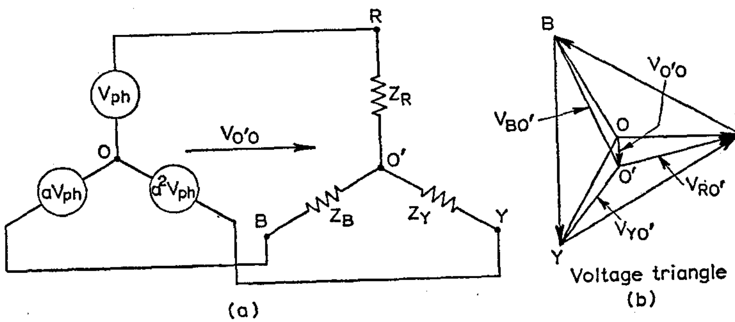


Fig. 4.8 UNBALANCED 3-WIRE STAR

EXAMPLE 4.5 Loads of 2, 2 + j2 and −j5-ohm elements are connected in star to a 433 V 50 Hz 3-phase 3-wire symmetrical system. Find (*a*) the potential of the load star point with respect to the supply neutral, (*b*) the load phase voltages, and (*c*) the line currents. Draw the complete complexor diagram.

The circuit diagram is shown in Fig. 4.8. V_{R0} is chosen as the reference complexor. The corresponding complexor diagram is shown at (*b*). Then

$$V_{R0} = V_{ph}\underline{/0^\circ} = \frac{433}{\sqrt{3}}\underline{/0^\circ} = 250\underline{/0^\circ}\text{ V}$$

The admittances of the load are

$$Y_R = \tfrac{1}{2} = 0{\cdot}5\,\text{S}$$

$$Y_Y = \frac{1}{2 + j2} = \frac{1}{2{\cdot}83\underline{/45^\circ}} = 0{\cdot}353\underline{/-45^\circ} = (0{\cdot}25 - j0{\cdot}25)\,\text{S}$$

$$Y_B = \frac{1}{-j5} = j0{\cdot}2 = 0{\cdot}2\underline{/90^\circ}\,\text{S}$$

(*a*) Applying Millman's theorem, eqn. (4.7),

$$V_{0'0} = \frac{(250\underline{/0^\circ} \times 0{\cdot}5) + (250\underline{/240^\circ} \times 0{\cdot}353\underline{/-45^\circ}) + (250\underline{/120^\circ} \times 0{\cdot}2\underline{/90^\circ})}{0{\cdot}5 + 0{\cdot}25 - j0{\cdot}25 + j0{\cdot}2}$$

This reduces to

$$V_{0'0} \approx \underline{\underline{-j64\,\text{V}}}$$

(*b*) The load phase voltages are then, by eqns (4.8),

$$V_{R0'} = V_{R0} - V_{0'0} = 250 + j64 = \underline{\underline{258\underline{/14^\circ\,21'}\,\text{V}}}$$

$$V_{Y0'} = V_{Y0} - V_{0'0} = a^2 250 + j64 = -125 - j152$$
$$= \underline{\underline{197\underline{/230^\circ\,34'}\,\text{V}}}$$

$$V_{B0'} = V_{B0} - V_{0'0} = a250 + j64 = -125 + j280$$
$$= \underline{\underline{307\underline{/114^\circ\,4'}\,\text{V}}}$$

(*c*) Having obtained the load phase voltages, the load currents follow simply from the expressions

$$I_{R0'} = V_{R0'}Y_1 = 258\underline{/14^\circ\,21'} \times 0{\cdot}5 = \underline{\underline{129\underline{/14^\circ\,21'}\,\text{A}}}$$

$$I_{Y0'} = V_{Y0'}Y_2 = 197\underline{/230^\circ\,34'} \times 0{\cdot}353\underline{/-45^\circ} = \underline{\underline{69{\cdot}4\underline{/185^\circ 34'}\,\text{A}}}$$

$$I_{B0'} = V_{B0'}Y_3 = 307\underline{/114^\circ\,4'} \times 0{\cdot}2\underline{/90^\circ} = \underline{\underline{61{\cdot}4\underline{/204^\circ 4'}\,\text{A}}}$$

Note that these three currents must form a closed complexor triangle.

4.8 Effect of Line Impedance

If the impedances of the lines connecting the generator to the load are appreciable, then for the 3-wire system with a star-connected load, the line impedances may be lumped with the load impedances to obtain the line currents. In the case of a 4-wire system, a solution may be readily effected by Millman's theorem. Fig. 4.9 shows a star-connected load, supplied through lines of impedance Z_L, and having a fourth wire of impedance Z_N. Then the total impedances in the lines are $(Z_L + Z_R)$, $(Z_L + Z_Y)$ and $(Z_L + Z_B)$. Let N be the end of the fourth wire so that 0 coincides with N; then

$$V_{0'0} = \frac{V_{R0}\dfrac{1}{Z_L + Z_R} + V_{Y0}\dfrac{1}{Z_L + Z_Y} + V_{B0}\dfrac{1}{Z_L + Z_B} + V_{N0}\dfrac{1}{Z_N}}{\dfrac{1}{Z_L + Z_R} + \dfrac{1}{Z_L + Z_Y} + \dfrac{1}{Z_L + Z_B} + \dfrac{1}{Z_N}}$$

$$= \frac{V_{R0}Y_1 + V_{Y0}Y_2 + V_{B0}Y_3}{Y_1 + Y_2 + Y_3 + Y_N} \tag{4.10}$$

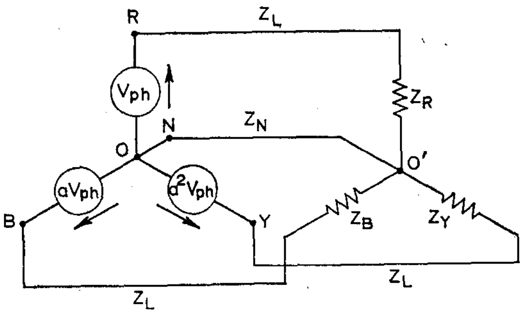


Fig. 4.9 UNBALANCED 4-WIRE STAR WITH APPRECIABLE LINE IMPEDANCES

since $V_{N0} = 0$.

Then

$$I_N = (V_{0'0})Y_N$$

and

$$I_R = (V_{R0} - V_{0'0})Y_1, \text{ etc.}$$

4.9 Power Measurement in General Three-phase Systems

The theoretically simplest method of measuring unbalanced 3-phase power in a 3-wire system is to insert a wattmeter in each line, with the voltage coils connected together to a common point (Fig. 4.10). The total power is then the sum of the three wattmeter readings. It can be shown that this method gives the correct result whether the wattmeters are identical or not.

There is a rather unexpected theorem related to polyphase power measurement, called *Blondel's theorem*. This states that the minimum number of wattmeters required to measure the power in a polyphase system is one less than the number of wires carrying current in the system.

Thus for a 3-phase 4-wire system three wattmeters are required, but for a 3-phase 3-wire system only two wattmeters are required. This two-wattmeter method will now be considered in detail.

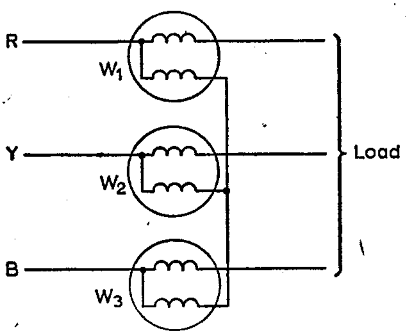


Fig. 4.10 THREE-WATTMETER POWER MEASUREMENT IN A 3-WIRE SYSTEM

The connexions for the two-wattmeter method are shown in Fig. 4.11. The wattmeters have their current coils connected in any two

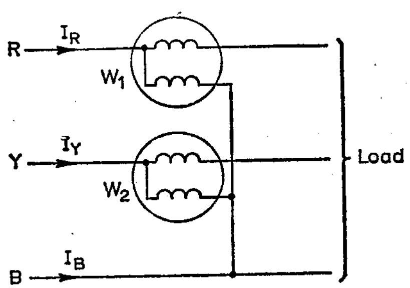


Fig. 4.11 TWO-WATTMETER METHOD

lines, while their voltage coils are connected between the corresponding lines and the third line. Then the sum of the two readings gives the total power irrespective of balance of load or waveform of supply.

Proof. Let P_1 and P_2 be the average value of the two wattmeter readings. Then

$$P_1 = \frac{1}{T}\int_0^T v_{RB} i_R \, dt \qquad \text{and} \qquad P_2 = \frac{1}{T}\int_0^T v_{YB} i_Y \, dt$$

Note that with both wattmeter voltage coils connected in the same way (i.e. corresponding ends joined to the blue line) the voltage across the voltage coil of W_2 is v_{YB} and not v_{BY}. Now,

$$v_{RB} = v_{RO'} - v_{BO'} \qquad \text{and} \qquad v_{YB} = v_{YO'} - v_{BO'}$$

Also, for a 3-wire system,

$$i_B = -(i_R + i_Y)$$

Therefore

$$P_1 + P_2 = \frac{1}{T}\int_0^T (v_{RO'}i_R - v_{BO'}i_R + v_{YO'}i_Y - v_{BO'}i_Y)\,dt$$

$$= \frac{1}{T}\int_0^T (v_{RO'}i_R + v_{YO'}i_Y + v_{BO'}i_B)\,dt$$

$$= \frac{1}{T}\int_0^T (\text{total instantaneous power})\,dt$$

$$= \text{Average total power}$$

In this analysis no assumptions have been made with regard to phase sequence, balance of load or waveform.

4.10 Special Case of Balanced Loads and Sine Waveforms

If the currents and voltages are sinusoidal they may be represented on a complexor diagram, this being done in Fig. 4.12 for the case of the balanced load and a phase sequence RYB. It is assumed that W_1 is the wattmeter in the leading line (e.g. if the wattmeters were in the R and B lines, then the wattmeter in the B line would be called W_1).

If the circuit current leads the phase voltage by an angle ϕ, and the wattmeters are connected as in Fig. 4.11, then

Power indicated by W_1 = (voltage across voltage coil) × (current in current coil) × cos (angle between them)

i.e.

$$P_1 = V_{RB}I_R\cos(\phi + 30°) \qquad \text{(from the complexor diagram)}$$

$$= V_lI_l\cos(\phi + 30°) \tag{4.11}$$

In the same way the power indicated by W_2 is

$$P_2 = V_{YB}I_Y\cos(\phi - 30°) = V_lI_l\cos(\phi - 30°) \tag{4.12}$$

For the case of a lagging phase angle, ϕ will be negative in the above equations.

For phase angles greater than 60° (either leading or lagging) one wattmeter will have a negative deflexion, in which case the connexions to the voltage coil of that wattmeter should be reversed to

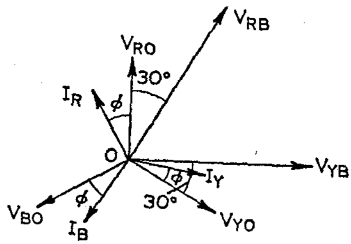


Fig. 4.12 COMPLEXOR DIAGRAM FOR TWO-WATTMETER METHOD WITH BALANCED LOADS

give an upscale reading. Such a reversed reading must always be subtracted to give the total power.

For balanced loads the power factor may be obtained from the wattmeter readings. Thus, if W_1 is in the leading line, then from eqn. (4.11) the reading is

$$P_1 = V_l I_l \cos(\phi + 30°)$$
$$= V_l I_l (\cos\phi \cos 30° - \sin\phi \sin 30°)$$
$$= V_l I_l \left(\frac{\sqrt{3}}{2}\cos\phi - \tfrac{1}{2}\sin\phi\right) \tag{4.13}$$

In the same way, from eqn. (4.12),

$$P_2 = V_l I_l \left(\frac{\sqrt{3}}{2}\cos\phi + \tfrac{1}{2}\sin\phi\right) \tag{4.14}$$

Therefore

$$P_1 + P_2 = \sqrt{3} V_l I_l \cos\phi$$

(i.e. the sum of readings gives the total circuit power)

Also

$$P_2 - P_1 = V_l I_l \sin\phi$$

Dividing this equation by the previous one,

$$\frac{P_2 - P_1}{P_1 + P_2} = \frac{1}{\sqrt{3}} \tan\phi \tag{4.15}$$

But

$$\text{Power factor, } \cos\phi = \frac{1}{\sec\phi} = \frac{1}{\sqrt{(1 + \tan^2\phi)}}$$

Therefore

$$\cos\phi = \frac{1}{\sqrt{\left\{1 + 3\left(\dfrac{P_2 - P_1}{P_1 + P_2}\right)^2\right\}}} \tag{4.16}$$

It should be noted that the total reactive volt-amperes, Q, may be obtained as

$$Q = \sqrt{3}V_l I_l \sin\phi = \sqrt{3}(P_2 - P_1) \tag{4.17}$$

The two-wattmeter method may be used with a single wattmeter if suitable switching is provided. The switches consist of make-before-break ammeter switches and a reversing switch.

In the polyphase wattmeter, two wattmeter elements are mounted in the same housing. They are screened and insulated from each other, but their moving coils are fixed to the same spindle and rotate against the same control spring. The meter indication is then the sum of the separate indications. The connexions are the same as for two separate meters.

EXAMPLE 4.6 A 3-phase induction motor develops 11·2 kW when running at 85 per cent efficiency and at a power factor of 0·45 lagging. Calculate the readings on each of two wattmeters connected to read the input power.

Let P_1 and P_2 be the readings on the "leading" and "lagging" phase wattmeters respectively.

$$\text{Total input power} = \frac{11{,}200}{0{\cdot}85} = 13{,}100\,\text{W}$$

i.e.

$$P_1 + P_2 = 13{,}100\,\text{W} \tag{i}$$

$$\text{Total input VAr} = 13{,}100 \times \tan\phi$$

$$= -13{,}100 \times \frac{\sqrt{(1 - 0{\cdot}45^2)}}{0{\cdot}45} = -26{,}000\,\text{VAr}$$

Therefore

$$P_2 - P_1 = \frac{-26{,}000}{\sqrt{3}} = -15{,}000\,\text{W} \tag{ii}$$

Adding eqns. (i) and (ii),

$$2P_2 = -1{,}900\,\text{W}$$

Therefore

$$P_2 = \underline{\underline{-950\,\text{W}}} \qquad \text{and} \qquad P_1 = \underline{\underline{14{,}050\,\text{W}}}$$

EXAMPLE 4.7 Two wattmeters W_1 and W_2 connected to read the input to a 3-phase induction motor running unloaded, indicate 3 kW and 1 kW respectively. On increasing the load, the reading on W_1 increases while that on W_2 decreases and eventually reverses.

Explain the above phenomenon and find the unloaded power and power factor of the motor. *(H.N.C.)*

When the load increases the power factor may be assumed to improve. If the unloaded power factor were less than 0·5 lagging (i.e. 60° lag), then reference to eqns. (4.11) and (4.12) shows that, if W_1 were in the leading line and W_2 in the lagging line, an increase in power would have the results described. This also assumes that the connexions to the voltage coil of W_2 have been reversed. As the phase angle decreases P_2 falls to zero and P_1 increases; any further decrease in phase angle causes P_2 to increase in the opposite sense.

Hence, assuming that in the unloaded condition W_2 reads a negative power,

$$\text{Input power} = 3 - 1 = \underline{\underline{2\text{kW}}}$$

$$\text{Input power factor} = \frac{1}{\sqrt{\{1 + 3(\frac{4}{2})^2\}}} = \frac{1}{\sqrt{13}} = \underline{\underline{0{\cdot}278 \text{ lagging}}}$$

4.11 Power Measurement in Balanced Three-phase Systems

If it is known that the load is balanced, then one wattmeter is sufficient to measure the power in a 3-phase system. There are two methods in which one wattmeter may be applied.

ARTIFICIAL-STAR METHOD (Fig. 4.13)

The wattmeter will read the phase power, and the total power will be

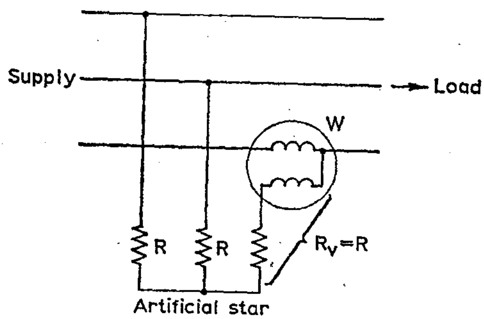


Fig. 4.13 ARTIFICIAL-STAR METHOD FOR BALANCED 3-PHASE POWER MEASUREMENT

three times the wattmeter reading. No indication is given of whether the power factor is leading or lagging.

DOUBLE-READING METHOD (Fig. 4.14)

Two readings are taken, with the wattmeter current coil in the same line for both readings and with the voltage coil connected across the current coil line and each of the other lines in turn. Suppose the current coil is inserted in the R line and the phase sequence is RYB. Let P_1 be the wattmeter reading when the voltage coil is

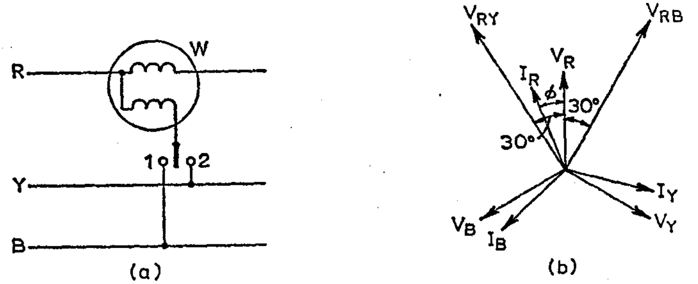


Fig. 4.14 DOUBLE-READING METHOD FOR BALANCED 3-PHASE POWER MEASUREMENT

connected to the leading line (B in this case) and P_2 be the wattmeter reading when the voltage coil is connected to the lagging line (Y in this case). Now,

Wattmeter reading = (voltage across voltage coil)
× (current through current coil)
× cos (phase angle between above current and voltage)

i.e.

$$P_1 = V_l I_l \cos(\phi + 30°) \qquad \text{(see Fig. 4.14}(b))$$

and

$$P_2 = V_l I_l \cos(\phi - 30°)$$

Thus

$$P_1 = V_l I_l (\cos\phi \cos 30° - \sin\phi \sin 30°)$$

and

$$P_2 = V_l I_l (\cos\phi \cos 30° + \sin\phi \sin 30°)$$

Therefore

$$P_1 + P_2 = \sqrt{3} V_l I_l \cos\phi = \text{total power}$$

$$P_2 - P_1 = V_l I_l \sin\phi = (1/\sqrt{3}) \times \text{total reactive volt-amperes}$$

and

$$\tan\phi = \frac{\sqrt{3}(P_2 - P_1)}{P_1 + P_2}$$

4.12 Phase Sequence Determination

Suppose that a 3-phase supply, with balanced line voltages, is brought out to three terminals marked A, B, C, but the sequence of the supply is unknown, e.g. the sequence might be A → B → C or A → C → B as illustrated by the complexor diagrams of Fig. 4.15(*a*) and (*b*). For many purposes it is essential to determine

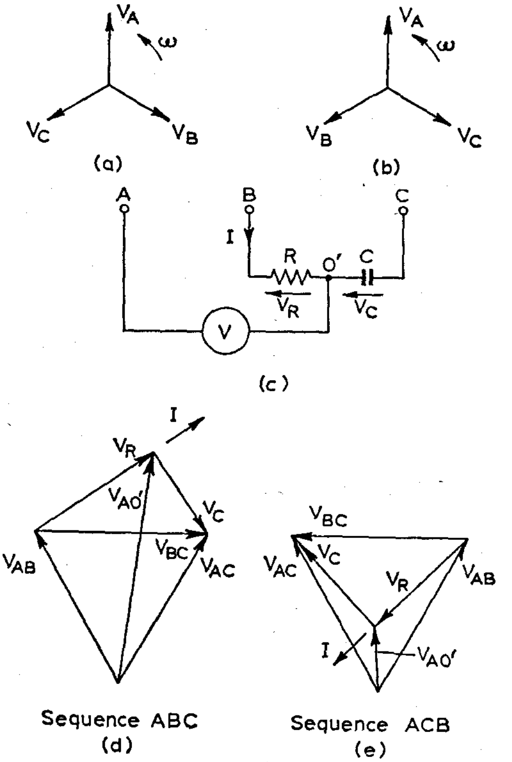


Fig. 4.15 PHASE SEQUENCE DETERMINATION

which of the two possible sequences the phase voltages follow. The determination may be carried out by various circuits, one of which is illustrated in Fig. 4.15(*c*).

The complexor diagram of Fig. 4.15(*d*) is drawn for the circuit when the phase sequence is A → B → C, and Fig. 4.15(*e*) applies when the phase sequence is A → C → B. It should be particularly noted that V_{BC} appears in anti-phase in the second case with respect to its direction in the first case. Since the circuit between terminals

B and C is capacitive, the current I through this circuit must lead the voltage V_{BC} across it.

Hence the current complexor may be drawn leading V_{BC} in both cases. V_R may then be drawn in the same direction as the current complexor. The voltage $V_{AO'}$ will then be represented by the complexor $V_{AO'}$ in each diagram. It is apparent that

(*a*) $V_{AO'}$ will exceed the line voltage for sequence A → B → C.
(*b*) $V_{AO'}$ will be less than the line voltage for sequence A → C → B.

The following points should be noted:

1. The voltmeter current has been neglected, so that a high-impedance voltmeter is necessary.
2. The clearest differentiation between the readings will be obtained when the resistance and the capacitive reactance are equal.

4.13 Symmetrical Components

Any unbalanced system of 3-phase currents may be represented by the superposition of a balanced system of 3-phase currents having *positive phase sequence*, a balanced system of 3-phase currents having the opposite or *negative phase sequence*, and a system of three currents equal in phase and magnitude and called the *zero phase sequence*. For example, Fig. 4.16(*a*) shows an unbalanced system of 3-phase currents, and Figs. 4.16(*b*), (*c*) and (*d*) show the positive, negative and zero phase-sequence components.

Adopting the nomenclature indicated in Fig. 4.16, where I_R, I_Y and I_B represent any unbalanced system of 3-phase currents,

$$I_R = I_{R+} + I_{R-} + I_{R0} \tag{4.18}$$

$$I_Y = I_{Y+} + I_{Y-} + I_{Y0} \tag{4.19}$$

$$I_B = I_{B+} + I_{B-} + I_{B0} \tag{4.20}$$

Evidently,

$$I_{Y+} = a^2 I_{R+} \tag{4.21}$$

$$I_{B+} = a I_{R+} \tag{4.22}$$

$$I_{Y-} = a I_{R-} \tag{4.23}$$

$$I_{B-} = a^2 I_{R-} \tag{4.24}$$

$$I_{R0} = I_{Y0} = I_{B0} \tag{4.25}$$

The original unbalanced currents may now be expressed in terms of red-phase symmetrical components only by substitution in eqns. (4.19) and (4.20). Repeating eqn. (4.18) for convenience, this gives

$$I_R = I_{R+} + I_{R-} + I_{R0} \tag{4.18}$$
$$I_Y = a^2 I_{R+} + a I_{R-} + I_{R0} \tag{4.26}$$
$$I_B = a I_{R+} + a^2 I_{R-} + I_{R0} \tag{4.27}$$

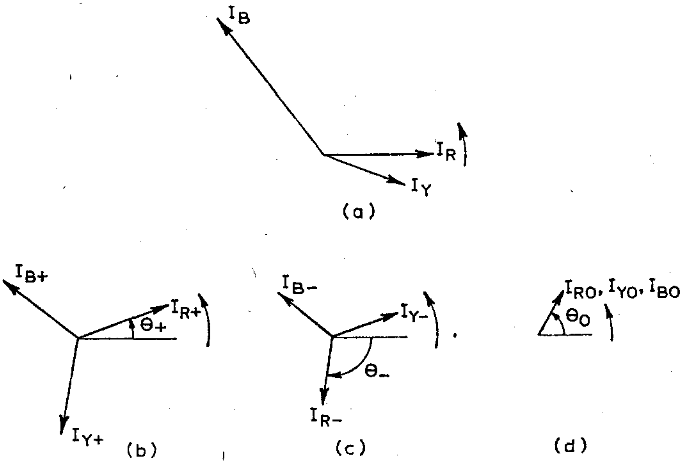


Fig. 4.16 SYMMETRICAL COMPONENTS
(*a*) Unbalanced 3-phase currents
(*b*) Positive phase-sequence currents
(*c*) Negative phase-sequence currents
(*d*) Zero phase-sequence currents

The symmetrical components I_{R+}, I_{R-} and I_{R0} may now be expressed in terms of the original unbalanced system I_R, I_Y, I_B.

Addition of eqns. (4.18), (4.26) and (4.27) gives

$$I_R + I_Y + I_B = (1 + a^2 + a)I_{R+} + (1 + a + a^2)I_{R-} + 3I_{R0}$$
$$I_{R0} = \tfrac{1}{3}(I_R + I_Y + I_B) \tag{4.28}$$

bearing in mind that

$$1 + a + a^2 = 0 \tag{4.4}$$

If eqn. (4.26) is multiplied by a and eqn. (4.27) by a^2, this gives

$$I_R = I_{R+} + I_{R-} + I_{R0} \tag{4.18}$$
$$aI_Y = a^3 I_{R+} + a^2 I_{R-} + a I_{R0} \tag{4.29}$$
$$a^2 I_B = a^3 I_{R+} + a^4 I_{R-} + a^2 I_{R0} \tag{4.30}$$

Addition of eqns. (4.18), (4.29) and (4.30) gives

$$I_{R+} = \tfrac{1}{3}(I_R + aI_Y + a^2I_B) \tag{4.31}$$

bearing in mind that $a^3 = 1$ and $a^4 = a$.

If eqn. (4.26) is multiplied by a^2 and eqn. (4.27) by a it can be shown that

$$I_{R-} = \tfrac{1}{3}(I_R + a^2I_Y + aI_B) \tag{4.32}$$

Eqns. (4.28), (4.31) and (4.32) show that it is possible to find zero, positive and negative phase-sequence components of current in terms of the original unbalanced system.

An unbalanced system of 3-phase voltages may similarly be represented by symmetrical components and the value of these components found by equations of the same form as (4.28), (4.31) and (4.32).

EXAMPLE 4.8 The currents flowing in an unbalanced 4-wire system are

$$I_R = 100\underline{/0^\circ}\,\text{A} \qquad I_Y = 200\underline{/-90^\circ}\,\text{A} \qquad I_B = 100\underline{/120^\circ}\,\text{A}$$

Find the positive, negative and zero phase-sequence components of these currents.

$$I_R = 100\underline{/0^\circ} = (100 + j0)\,\text{A}$$

$$I_Y = 200\underline{/-90^\circ} = (0 - j200)\,\text{A}$$

$$aI_Y = 200\underline{/-90^\circ} \times 1\underline{/120^\circ} = 200\underline{/30^\circ} = (173{\cdot}2 + j100)\,\text{A}$$

$$a^2I_Y = 200\underline{/-90^\circ} \times 1\underline{/-120^\circ} = 200\underline{/150^\circ} = (-173{\cdot}2 + j100)\,\text{A}$$

$$I_B = 100\underline{/120^\circ} = (-50 + j86{\cdot}6)\,\text{A}$$

$$aI_B = 100\underline{/120^\circ} \times 1\underline{/120^\circ} = 100\underline{/-120} = (-50 - j86{\cdot}6)\,\text{A}$$

$$a^2I_B = 100\underline{/120^\circ} \times 1\underline{/-120^\circ} = 100\underline{/0^\circ} = (100 + j0)\,\text{A}$$

$$I_{R+} = \tfrac{1}{3}(I_R + aI_Y + a^2I_B)$$

$$3I_{R+} = 373{\cdot}2 + j100$$

$$\underline{\underline{I_{R+} = 129\underline{/15^\circ}\,\text{A}}}$$

$$I_{R-} = \tfrac{1}{3}(I_R + a^2I_Y + aI_B)$$

$$3I_{R-} = -123{\cdot}2 + j13{\cdot}4$$

$$\underline{\underline{I_{R-} = 41{\cdot}3\underline{/174^\circ}\,\text{A}}}$$

$$I_{R0} = \tfrac{1}{3}(I_R + I_Y + I_B)$$

$$3I_{R0} = 50 - j113{\cdot}4$$

$$\underline{\underline{I_{R0} = 41{\cdot}3\underline{/-66{\cdot}2^\circ}\,\text{A}}}$$

A graphical solution is shown in Fig. 4.17. The complexors I_R, I_Y and I_B are first drawn using a suitable scale. The complexors aI_Y, a^2I_Y, aI_B and a^2I_B are then drawn 120° leading and lagging on I_Y and I_B respectively. The required

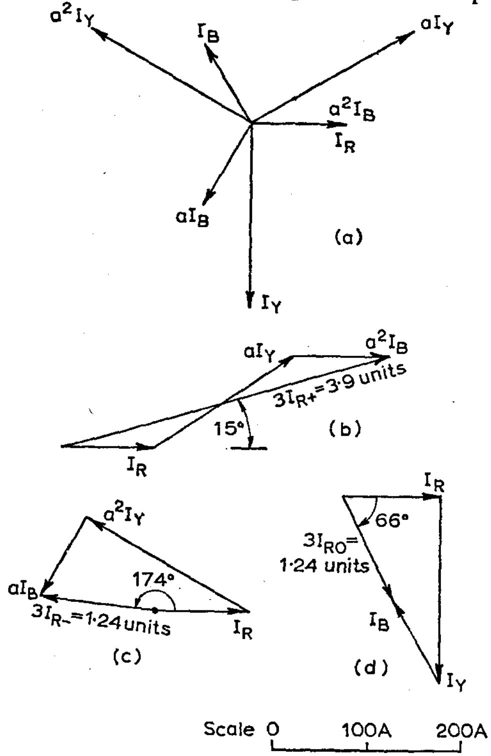


Fig. 4.17

$I_{R+} = 130\underline{/15^\circ}$ A $\quad I_{R-} = 41{\cdot}3\underline{/174^\circ}$ A $\quad I_{R0} = 41{\cdot}3\underline{/-66^\circ}$ A

complexor additions are then performed, and these are shown in Figs. 4.17(*b*), (*c*) and (*d*).

4.14 Applications of Symmetrical Components

The method of symmetrical components may be applied to the solution of all kinds of unbalanced 3-phase network problems such as those discussed in Sections 4.5, 4.6 and 4.7. However, when the source impedance is assumed to be negligible, the methods of solution adopted there are easier than the application of the method of symmetrical components. Where source impedance is taken into account the method of symmetrical components *must* be applied if the source is a synchronous machine or a transformer supplied by a synchronous machine. This is because the impedance of a synchronous machine to positive phase-sequence currents is different from its impedance to negative phase-sequence currents. The source impedance is likely to be of significance under fault conditions in 3-phase networks. Indeed for a fault at the generator terminals the generator impedance is the only impedance present. The main field of application of symmetrical components, therefore, is the analysis of 3-phase networks under asymmetrical fault conditions.

4.15 Synchronous Generator supplying an Unbalanced Load

Three-phase generators are specifically designed so that the phase

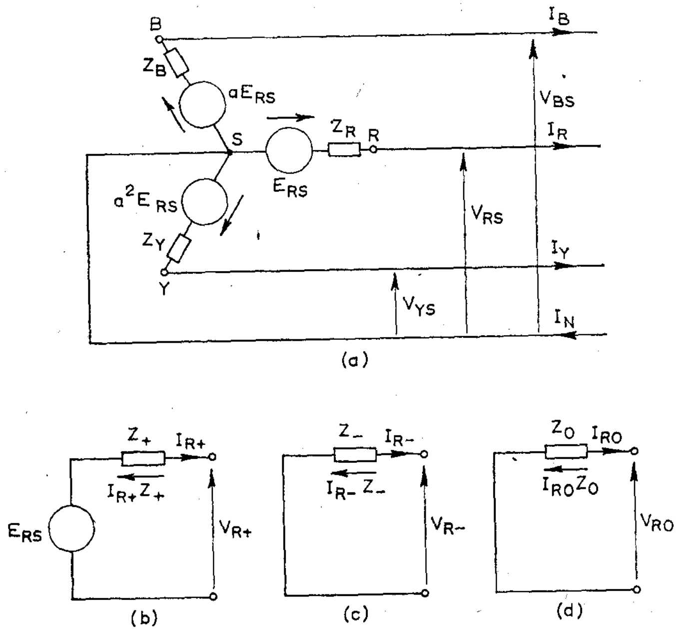


Fig. 4.18 SYNCHRONOUS GENERATOR SUPPLYING AN UNBALANCED LOAD

(*a*) Actual network
(*b*) Positive phase-sequence network
(*c*) Negative phase-sequence network
(*d*) Zero phase-sequence network

e.m.f.s form a balanced 3-phase system. It is assumed, therefore, that the e.m.f. of the synchronous generator shown in Fig. 4.18 is of positive phase sequence only. Then the generator phase e.m.f.s are

$$E_{RS},\ E_{YS}\ (= a^2E_{RS}),\ E_{BS}\ (= aE_{RS})$$

If the generator is connected by four wires to an unbalanced star-connected load (the most general case), the generator phase currents will be unbalanced, and as shown in Section 4.13, may be represented by the superposition of positive, negative and zero phase-sequence components of current. These components are:

$$I_{R+} = \tfrac{1}{3}(I_R + aI_Y + a^2I_B) \tag{4.33}$$

$$I_{Y+} = a^2I_R \tag{4.34}$$

$$I_{B+} = aI_R \tag{4.35}$$

$$I_{R-} = \tfrac{1}{3}(I_R + a^2I_Y + aI_B) \tag{4.36}$$

$$I_{Y-} = aI_{R-} \tag{4.37}$$

$$I_{B-} = a^2I_{R-} \tag{4.38}$$

$$I_{R0} = \tfrac{1}{3}(I_R + I_Y + I_B) \tag{4.39}$$

$$I_{Y0} = I_{R0} \tag{4.40}$$

$$I_{BO} = I_{R0} \tag{4.41}$$

Assuming, as is usual, that the impedances of the generator phases to positive phase-sequence current are equal, i.e.

$$Z_{R+} = Z_{Y+} = Z_{B+} = Z_+$$

then

Voltage drop in R-phase due to positive phase-sequence current $= I_{R+}Z_+$

Voltage drop in Y-phase due to positive phase-sequence current $= I_{Y+}Z_+ = a^2I_{R+}Z_+$

Voltage drop in B-phase due to positive phase-sequence current $= I_{B+}Z_+ = aI_{R+}Z_+$

That is, the voltage drops due to the positive phase-sequence currents are positive phase sequence only. This would not be the case if Z_{R+}, Z_{Y+} and Z_{B+} were different.

Similarly, if the impedances of the generator phases to negative phase-sequence current are equal, i.e.

$$Z_{R-} = Z_{Y-} = Z_{B-} = Z_-$$

then

Voltage drop in R-phase due to negative phase sequence current $= I_{R-}Z_-$

Voltage drop in Y-phase due to negative phase sequence current $= I_{Y-}Z_- = aI_{R-}Z_-$

Voltage drop in B-phase due to negative phase sequence current $I_{B-}Z_- = a^2I_{R-}Z_-$

That is, the voltage drops due to the negative phase-sequence currents are of negative phase sequence only. Again this result depends on Z_{R-}, Z_{Y-} and Z_{B-} being equal.

Also, if the impedances of the generator phases to zero phase-sequence current are equal, i.e.

$$Z_{R0} = Z_{Y0} = Z_{B0} = Z_0$$

then

Voltage drop in R-phase due to negative phase-sequence current $= I_{R0}Z_0$

Voltage drop in Y-phase due to negative phase-sequence current $= I_{Y0}Z_0 = I_{R0}Z_0$

Voltage drop in B-phase due to negative phase-sequence current $= I_{B0}Z_0 = I_{R0}Z_0$

E.M.F. induced in R-phase

= (Terminal voltage of R-phase)

+ (Internal voltage drop in R-phase)

$$E_{RS} = V_{RS} + I_{R+}Z_+ + I_{R-}Z_- + I_{R0}Z_0 \tag{4.42}$$

Similarly,

$$E_{YS} = a^2E_{RS} = V_{YS} + a^2I_{R+}Z_+ + aI_{R-}Z_- + I_{R0}Z_0 \tag{4.43}$$

and

$$E_{BS} = aE_{RS} = V_{BS} + aI_{R+}Z_+ + a^2I_{R-}Z_- + I_{R0}Z_0 \tag{4.44}$$

Adding these three equations,

$$0 = V_{RS} + V_{YS} + V_{BS} + 3I_{R0}Z_0$$

or $$0 = V_{R0} + I_{R0}Z_0 \tag{4.45}$$

where V_{R0} is the zero phase-sequence component of the generator terminal voltage V_{RS}, i.e.

$$V_{R0} = \tfrac{1}{3}(V_{RS} + V_{YS} + V_{BS})$$

If eqn. (4.43) is multiplied by a^2 and eqn. (4.44) by a,

$$E_{RS} = V_{RS} + I_{R+}Z_+ + I_{R-}Z_- + I_{R0}Z_0 \tag{4.42}$$

$$a^4E_{RS} = a^2V_{YS} + a^4I_{R+}Z_+ + a^3I_{R-}Z_- + a^2I_{R0}Z_0 \tag{4.46}$$

$$a^2E_{RS} = aV_{BS} + a^3I_{R+}Z_+ + a^3I_{E-}Z_- + aI_{R0}Z_0 \tag{4.47}$$

Adding these three equations,

$$0 = V_{RS} + a^2V_{YS} + aV_{BS} + 3I_{R-}Z_-$$

or $$0 = V_{R-} + I_{R-}Z_- \tag{4.48}$$

where V_{R-} is the negative phase-sequence component of the generator terminal voltage V_{RS}.

If eqn. (4.43) is multiplied by a and eqn. (4.44) by a^2,

$$E_{RS} = V_{RS} + I_{R+}Z_{+} + I_{R-}Z_{-} + I_{R0}Z_{0} \tag{4.42}$$

$$a^3E_{RS} = aV_{YS} + a^3I_{R+}Z_{+} + a^2I_{R-}Z_{-} + aI_{R0}Z_{0} \tag{4.49}$$

$$a^3E_{RS} = a^2V_{BS} + a^3I_{R+}Z_{+} + a^4I_{R-}Z_{-} + a^2I_{R0}Z_{0} \tag{4.50}$$

Adding the above equations,

$$3E_{RS} = V_{RS} + aV_{YS} + a^2V_{BS} + 3I_{R+}Z_{+}$$

$$E_{RS} = \tfrac{1}{3}(V_{RS} + aV_{YS} + a^2V_{BS}) + I_{R+}Z_{+}$$

$$E_{RS} = V_{R+} + I_{R+}Z_{+} \tag{4.51}$$

Examination of eqns. (4.45), (4.48) and (4.51) reveals that, when an unbalanced load is imposed on a system, the positive, negative and zero phase-sequence components may be considered separately if the impedance of each phase of the network is balanced. Under these conditions the positive, negative and zero phase-sequence networks, which are interpretations of eqns. (4.45), (4.48) and (4.51), may be drawn, as in Fig. 4.18. It should be noted particularly that negative and zero phase-sequence currents may flow even when there is no negative or zero phase-sequence e.m.f., since such current components may arise due to unbalanced loading.

4.16 Analysis of Asymmetrical Faults

In the following analysis it will be assumed that

1. The positive, negative and zero phase-sequence impedances of the generator and any interconnected plant are known.
2. The generator e.m.f. system is of positive phase sequence only.
3. No current flows in the network other than that due to the fault.
4. The network impedances in each phase are balanced.
5. The impedance of the fault is zero.

ONE-LINE-TO-EARTH FAULT

The circuit diagram of Fig. 4.19 shows the fault conditions. Evidently, $V_{RS} = 0$ since the R-phase terminal and the star point are earthed. Also

$$I_Y = 0 \qquad \text{and} \qquad I_B = 0$$

since only fault current is assumed to flow.

The phase sequence components of current are, from eqns. (4.31), (4.32) and (4.28),

$$I_{R+} = \tfrac{1}{3}(I_R + aI_Y + a^2I_B) = \tfrac{1}{3}I_R$$
$$I_{R-} = \tfrac{1}{3}(I_R + a^2I_Y + aI_B) = \tfrac{1}{3}I_R$$
$$I_{R0} = \tfrac{1}{3}(I_R + I_Y + I_B) = \tfrac{1}{3}I_R$$

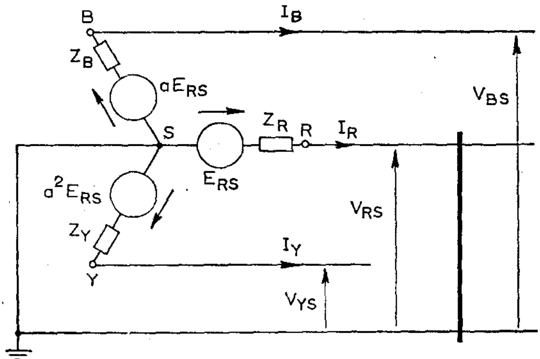


Fig. 4.19 ONE-LINE-TO-EARTH FAULT

Therefore

$$I_{R+} = I_{R-} = I_{R0} = \tfrac{1}{3}I_R \tag{4.52}$$

A complexor diagram of the phase-sequence components of current is given in Fig. 4.20.

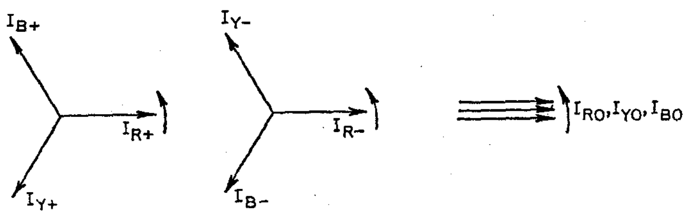


Fig. 4.20 SYMMETRICAL COMPONENTS OF ONE-LINE-TO-EARTH CURRENT

Since the generator e.m.f. system is of positive phase sequence only, and since the network impedances in the three phases are balanced so that each phase sequence may be considered separately as proved in Section 4.15, then

$$E_{RS} = V_{R+} + I_{R+}Z_+ \tag{4.51}$$
$$0 = V_{R-} + I_{R-}Z_- \tag{4.48}$$
$$0 = V_{R0} + I_{R0}Z_0 \tag{4.45}$$

Adding these three equations,

$$E_{RS} = V_{R+} + V_{R-} + V_{R0} + I_{R+}Z_+ + I_{R-}Z_- + I_{R0}Z_0$$

But $V_{R+} + V_{R-} + V_{R0} = \frac{1}{3}V_{RS} = 0$, and $I_{R+} + I_{R-} + I_{R0} = \frac{1}{3}I_R$, so that

$$E_{RS} = \tfrac{1}{3}I_R(Z_+ + Z_- + Z_0) \tag{4.53}$$

The fault current is

$$I_F = I_R = \frac{3E_{RS}}{Z_+ + Z_- + Z_0} \tag{4.54}$$

Examination of this equation shows that an equivalent circuit from which the fault current may be calculated is as given in Fig. 4.21(*a*). Since the positive, negative and zero phase-sequence

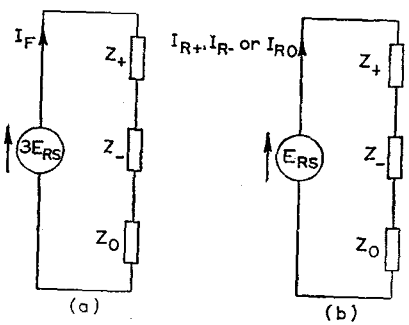


Fig. 4.21 EQUIVALENT CIRCUITS FOR ONE-LINE-TO-EARTH FAULT

components of current in the faulted phase are all equal and in phase, the equivalent circuit from which each phase-sequence component of current may be calculated is given in Fig. 4.21(*b*).

SHORT-CIRCUIT BETWEEN TWO LINES

The circuit diagram of Fig. 4.22 shows the fault conditions. Evidently

$$I_Y + I_B = 0 \qquad I_R = 0 \qquad \text{and} \qquad V_{YS} = V_{BS}$$

The phase-sequence components of current are, from eqns. (4.31), (4.32) and (4.28),

$$I_{R+} = \tfrac{1}{3}(I_R + aI_Y + a^2I_B) = \tfrac{1}{3}(a - a^2)I_Y$$
$$I_{R-} = \tfrac{1}{3}(I_R + a^2I_Y + aI_B) = \tfrac{1}{3}(a^2 - a)I_Y$$
$$I_{R0} = \tfrac{1}{3}(I_R + I_Y + I_B) = 0$$

The above equations show that

$$I_{R0} = 0 \qquad \text{and} \qquad I_{R+} = -I_{R-}$$

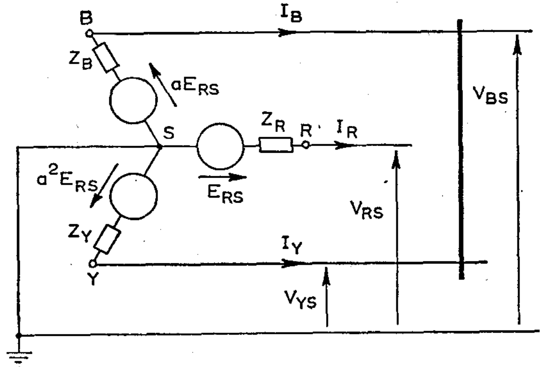


Fig. 4.22 SHORT-CIRCUIT BETWEEN TWO LINES

A complexor diagram of the phase-sequence components of current is given in Fig. 4.23.

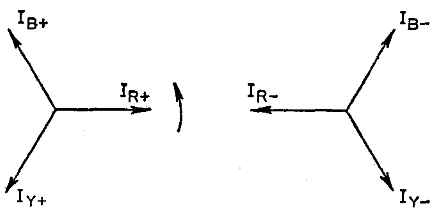


Fig. 4.23 SYMMETRICAL COMPONENTS OF FAULT CURRENT FOR A SHORT-CIRCUIT BETWEEN TWO LINES

Since the generator e.m.f. system is of positive phase sequence only and the network impedances in each phase are balanced,

$$E_{RS} = V_{R+} + I_{R+}Z_+ \tag{4.51}$$

$$0 = V_{R-} + I_{R-}Z_- \tag{4.48}$$

$$0 = V_{R0} + I_{R0}Z_0 \tag{4.45}$$

Also,

$$V_{R+} = \tfrac{1}{3}(V_{RS} + aV_{YS} + a^2V_{BS}) = \tfrac{1}{3}(V_{RS} + (a + a^2)V_{YS}) \tag{4.55}$$

$$V_{R-} = \tfrac{1}{3}(V_{RS} + a^2 V_{YS} + aV_{BS}) = \tfrac{1}{3}(V_{RS} + (a^2 + a)V_{YS}) \tag{4.56}$$

since $V_{YS} = V_{BS}$.

From eqns. (4.55) and (4.56),

$$V_{R+} = V_{R-}$$

Therefore, subtracting eqn. (4.48) from eqn. (4.51),

$$E_{RS} = V_{R+} - V_{R-} + I_{R+}Z_+ - I_{R-}Z_- = I_{R+}(Z_+ + Z_-) \tag{4.57}$$

since $V_{R+} = V_{R-}$ and $I_{R+} = -I_{R-}$.

From eqn. (4.57),

$$I_{R+} = \frac{E_{RS}}{Z_+ + Z_-} \tag{4.58}$$

Also,

$$I_{R-} = \frac{-E_{RS}}{Z_+ + Z_-} \tag{4.59}$$

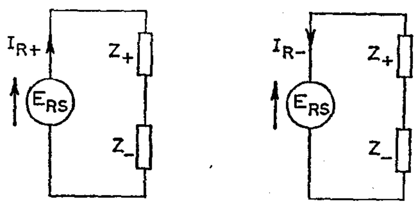


Fig. 4.24 EQUIVALENT CIRCUITS FOR A SHORT-CIRCUIT BETWEEN TWO LINES

Examination of these two equations shows that the equivalent circuits from which the positive and negative phase-sequence components of current may be calculated are as shown in Figs. 4.24(*a*) and (*b*).

The fault current is

$$\begin{aligned} I_F = I_Y &= I_{Y+} + I_{Y-} \\ &= a^2 I_{R+} + aI_{R-} \\ &= a^2 I_{R+} - aI_{R+} \\ &= \frac{E_{RS}}{Z_+ + Z_-}(a^2 - a) \\ &= \frac{-j\sqrt{3}E_{RS}}{Z_+ + Z_-} \end{aligned} \tag{4.60}$$

TWO-LINES-TO-EARTH FAULT

The circuit diagram of Fig. 4.25 shows the fault conditions. Evidently,

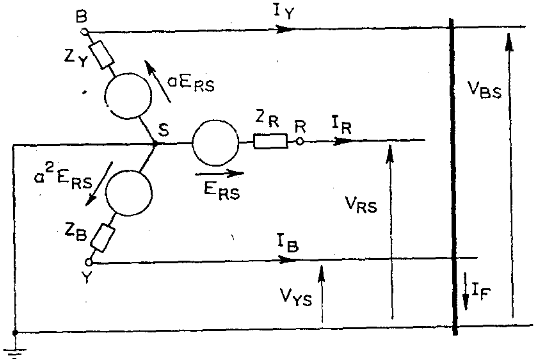


Fig. 4.25 TWO-LINES-TO-EARTH FAULT

$$I_R = 0$$
$$V_{YS} = V_{BS} = 0$$
$$I_F = I_Y + I_B$$
$$I_R = I_{R+} + I_{R-} + I_{R0} = 0 \tag{4.61}$$

The phase sequence components of the generator terminal voltage are

$$V_{R+} = \tfrac{1}{3}(V_{RS} + aV_{YS} + a^2V_{BS}) = \tfrac{1}{3}V_{RS}$$
$$V_{R-} = \tfrac{1}{3}(V_{RS} + a^2V_{YS} + aV_{BS}) = \tfrac{1}{3}V_{RS}$$
$$V_{R0} = \tfrac{1}{3}(V_{RS} + V_{YS} + V_{BS}) = \tfrac{1}{3}V_{RS}$$

Therefore

$$V_{R+} = V_{R-} = V_{R0} = \tfrac{1}{3}V_{RS} \tag{4.62}$$

Since the generator e.m.f. system is of positive sequence only, and since the network impedances in each phase are balanced,

$$E_{RS} = V_{R+} + I_{R+}Z_+ \tag{4.51}$$
$$0 = V_{R-} + I_{R-}Z_- \tag{4.48}$$
$$0 = V_{R0} + I_{R0}Z_0 \tag{4.45}$$

The phase-sequence components of the generator terminal voltage may be eliminated from these equations by subtraction:

$$(4.51) - (4.48) \qquad E_{RS} = I_{R+}Z_+ - I_{R-}Z_- \tag{4.63}$$

$$(4.51) - (4.45) \qquad E_{RS} = I_{R+}Z_+ - I_{R0}Z_0 \tag{4.64}$$

$$(4.48) - (4.45) \qquad 0 = I_{R-}Z_- - I_{R0}Z_0 \tag{4.65}$$

Substituting (from eqn. (4.61)) $-I_{R0} = I_{R+} + I_{R-}$ in eqn. (4.64),

$$E_{RS} = I_{R+}(Z_+ + Z_0) + I_{R-}Z_0 \tag{4.66}$$

I_{R-} can now be eliminated between eqns. (4.63) and (4.66) by multiplying eqn. (4.63) by Z_0 and eqn. (4.66) by Z_-; thus

$$E_{RS}Z_0 = I_{R+}Z_+Z_0 - I_{R-}Z_-Z_0$$

$$E_{RS}Z_- = I_{R+}(Z_+ + Z_0)Z_- + I_{R-}Z_-Z_0$$

Adding,

$$E_{RS}(Z_0 + Z_-) = I_{R+}(Z_+Z_0 + Z_+Z_- + Z_-Z_0)$$

Therefore,

$$I_{R+} = \frac{E_{RS}(Z_0 + Z_-)}{Z_+Z_0 + Z_+Z_- + Z_-Z_0} = \frac{E_{RS}}{Z_+ + \dfrac{Z_-Z_0}{Z_0 + Z_-}} \tag{4.67}$$

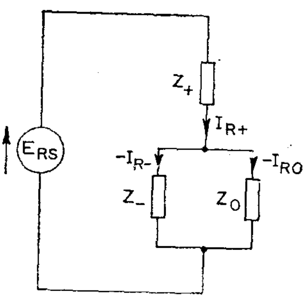


Fig. 4.26 EQUIVALENT CIRCUIT FOR TWO-LINES-TO-EARTH FAULT

Examination of this equation shows that the equivalent circuit from which the positive phase-sequence component of current may be determined is as shown in Fig. 4.26.

Further, from eqn. (4.63),

$$-I_{R-} = \frac{E_{RS} - I_{R+}Z_+}{Z_-} \tag{4.68}$$

$E_{RS} - I_{R+}Z_+$ is the voltage drop across the parallel branch of the equivalent circuit in Fig. 4.26. Therefore the current flowing in Z_- of this equivalent circuit is $-I_{R-}$.

Also, from eqn. (4.64),

$$-I_{R0} = \frac{E_{RS} - I_{R+}Z_+}{Z_0} \tag{4.69}$$

Therefore the current flowing in Z_0 of the equivalent circuit of Fig. 4.26 is $-I_{R0}$.

EXAMPLE 4.9 A 3-phase 75 MVA 0·8 p.f. (lagging) 11·8 kV star-connected alternator having its star point solidly earthed supplies a feeder. The relevant per-unit (p.u.)* impedances, based on the rated phase voltage and phase current of the alternator are as follows:

	Generator Z_G	*Feeder* Z_F
Positive sequence impedance (p.u.)	$j1·70$	$j0·10$
Negative sequence impedance (p.u.)	$j0·18$	$j0·10$
Zero sequence impedance (p.u.)	$j0·12$	$j0·30$

Determine the fault current and the line-to-neutral voltages at the generator terminals for a one-line-to-earth fault occurring at the distant end of the feeder. The generator e.m.f. per phase is of positive sequence only and is equal to the rated voltage per phase.

$$\text{Rated voltage per phase} = \frac{11·8 \times 10^3}{\sqrt{3}} = 6{,}820\,\text{V}$$

$$\text{Rated current per phase} = \frac{75 \times 10^6}{\sqrt{3} \times 11·8 \times 10^3} = 3{,}670\,\text{A}$$

The circuit diagram is shown in Fig. 4.27. The fault is assumed to occur on the red phase.

$$\text{Let } Z_{T+} = Z_{G+} + Z_{F+} = j1·80\,\text{p.u.}$$
$$Z_{T-} = Z_{G-} + Z_{F-} = j0·28\,\text{p.u.}$$
$$Z_{T0} = Z_{G0} + Z_{F0} = j0·42\,\text{p.u.}$$

Take E_{RS} as the reference complexor, i.e.

$$E_{RS} = 1\underline{/0°}\,\text{p.u.}$$

Then

$$I_{R+} = I_{R-} = I_{R0} = \frac{E_{RS}}{Z_{T+} + Z_{T-} + Z_{T0}} = \frac{1\underline{/0°}}{2·50\underline{/90°}} = 0·4\underline{/-90°}\,\text{p.u.}$$

and

$$\text{Fault current, } I_F = 3I_{R+} = 3 \times 0·4\underline{/-90°} = 1·2\underline{/-90°}\,\text{p.u.}$$

i.e.

$$I_F = 1·2 \times 3{,}670 = \underline{\underline{4{,}400\,\text{A}}}$$

* See Section 9.16.

The positive, negative and zero sequence networks are shown in Figs. 4.27(*c*), (*d*) and (*e*).

$$E_{RS} = (V_{rS})_+ + I_{R+}Z_{G+}$$
$$0 = (V_{rS})_- + I_{R-}Z_{G-}$$
$$0 = (V_{rS})_0 + I_{R0}Z_{G0}$$

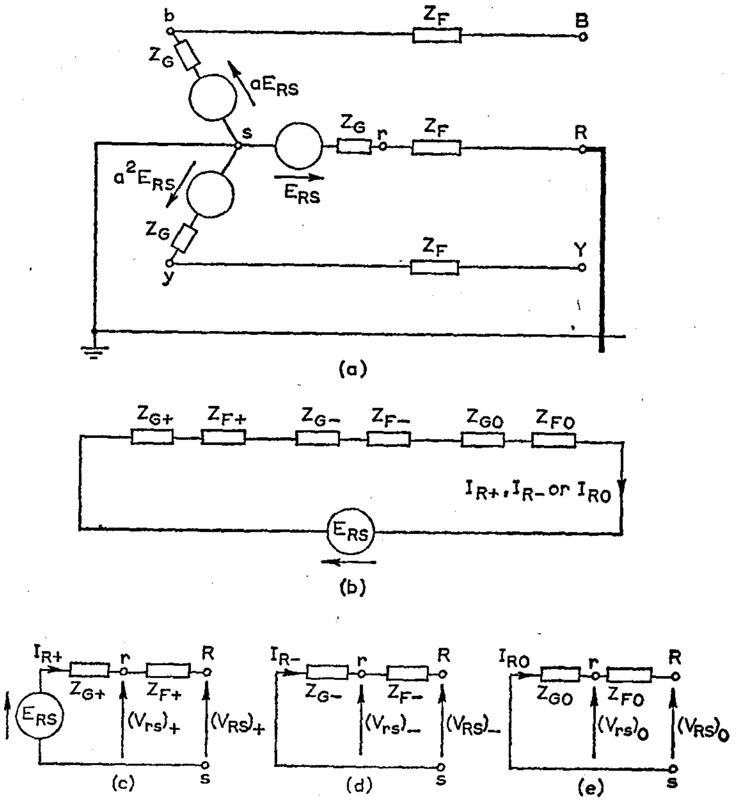


Fig. 4.27 CIRCUIT DIAGRAMS FOR EXAMPLE 4.9

(*a*) Actual network
(*b*) Equivalent circuit for phase-sequence components of current
(*c*) Positive phase-sequence network
(*d*) Negative phase-sequence network
(*e*) Zero phase-sequence network

where $(V_{rS})_+$, $(V_{rS})_-$ and $(V_{rS})_0$ are the positive, negative and zero phase-sequence components of the generator red-phase terminal voltage.

$$(V_{rS})_+ = E_{RS} - I_R Z_G = 1\angle 0^\circ - (0{\cdot}4\angle -90^\circ \times 1{\cdot}70\angle 90^\circ) = 0{\cdot}32\angle 0^\circ \text{ p.u.}$$
$$(V_{rS})_- = -I_{R-}Z_{G-} = -0{\cdot}4\angle -90^\circ \times 0{\cdot}18\angle 90^\circ = 0{\cdot}072\angle 180^\circ \text{ p.u.}$$
$$(V_{rS})_0 = -I_{R0}Z_{G0} = -0{\cdot}4\angle -90^\circ \times 0{\cdot}12\angle 90^\circ = 0{\cdot}048\angle 180^\circ \text{ p.u.}$$

$$V_{rS} = (V_{rS})_+ + (V_{rS})_- + (V_{rS})_0 = 0{\cdot}32 - 0{\cdot}072 - 0{\cdot}048 = 0{\cdot}20\underline{/0^\circ}\text{p.u.}$$

$$V_{yS} = a^2(V_{rS})_+ + a(V_{rS})_- + (V_{rS})_0$$
$$= (1\underline{/-120^\circ} \times 0{\cdot}32\underline{/0^\circ}) + (1\underline{/120^\circ} \times 0{\cdot}072\underline{/180^\circ}) + 0{\cdot}048\underline{/180^\circ}$$
$$= -0{\cdot}16 - j0{\cdot}277 + 0{\cdot}036 - j0{\cdot}0624 - 0{\cdot}048 = 0{\cdot}38\underline{/-116{\cdot}9^\circ}\text{p.u.}$$

$$V_{bS} = a(V_{rS})_+ + a^2(V_{rS})_- + (V_{rS})_0$$
$$= (1\underline{/120^\circ} \times 0{\cdot}32\underline{/0^\circ}) + (1\underline{/-120^\circ} \times 0{\cdot}072\underline{/180^\circ}) + 0{\cdot}048\underline{/180^\circ}$$
$$= -0{\cdot}16 + j0{\cdot}277 + 0{\cdot}036 + j0{\cdot}0624 - 0{\cdot}048 = 0{\cdot}38\underline{/-116{\cdot}9^\circ}\text{p.u.}$$

$$V_{rS} = 0{\cdot}20 \times 6{,}820 = \underline{\underline{1{,}364\,\text{V}}}$$

$$V_{yS} = V_{bS} = 0{\cdot}38 \times 6{,}820 = \underline{\underline{2{,}550\,\text{V}}}$$

PROBLEMS

4.1 A symmetrical 3-phase 400 V system supplies a balanced mesh-connected load. The RY branch current is $20\underline{/40^\circ}$ A. Find (*a*) the line currents, and (*b*) the total power.

Draw a complexor diagram showing the voltages and currents in the lines and phases. Assume V_{RY} is the reference complexor.

Ans. $34{\cdot}6\underline{/10^\circ}$ A, $34{\cdot}6\underline{/-110^\circ}$ A, $34{\cdot}6\underline{/130^\circ}$ A, 18·24 kW.

4.2 An unbalanced 4-wire star-connected load has balanced line voltages of 440 V.. The loads are

$$Z_R = 10\,\Omega \qquad Z_Y = (5 + j10)\,\Omega \qquad Z_B = (15 - j5)\,\Omega$$

Calculate the value of the current in the neutral wire and its phase relationship to the voltage across the red phase. The phase sequence is RYB.

Hence sketch the complexor diagram.

Ans. 15·2 A, 127·3° lagging.

4.3 A 3-phase, 4-wire, 440 V system is loaded as follows:

(i) An induction motor load of 350 kW at power factor 0·71 lagging.

(ii) Resistance loads of 150 kW, 250 kW and 400 kW connected between neutral and the R, Y and B lines respectively. Calculate (*a*) the line currents, (*b*) the current in the neutral, and (*c*) the power factor of the system. Phase sequence, RYB. (*H.N.C.*)

Ans. 1,145 A, 1,512 A, 2,080 A, 855 A, 0·957 lagging.

4.4 Three impedances Z_1, Z_2 and Z_3 are mesh connected to a symmetrical 3-phase 400 V 50 Hz supply of phase sequence RYB.

$Z_1 = (10 + j0)\,\Omega$ and is connected between lines R and Y
$Z_2 = (8 + j6)\,\Omega$ connected between lines Y and B
$Z_3 = (5 - j5)\,\Omega$ connected between lines B and R

Calculate the phase and line currents and the total power consumed. (*H.N.C.*)

Ans. 40 A, 40 A, 56·6 A; 95·7 A, 78·4 A, 35·2 A; 44·8 kW.

4.5 A symmetrical 3-phase supply, of which the line voltage is 380 V, feeds a mesh-connected load as follows:

Load A: 19 kVA at p.f. 0·5 lagging connected between lines R and Y
Load B: 30 kVA at p.f. 0·8 lagging connected between lines Y and B
Load C: 10 kVA at p.f. 0·9 leading connected between lines B and R

Determine the line currents and their phase angles. Phase sequence, RYB. *(H.N.C.)*

Ans. $74{\cdot}6\underline{/-51^\circ}$ A, $98{\cdot}6\underline{/173^\circ}$ A, $68{\cdot}3\underline{/41{\cdot}8}$ A.

4.6 Determine the line currents in an unbalanced star-connected load supplied from a symmetrical 3-phase 440 V 3-wire system. The branch impedances of the load are $Z_1 = 5\underline{/30^\circ}\Omega$, $Z_2 = 10\underline{/45^\circ}\Omega$ and $Z_3 = 10\underline{/60^\circ}\Omega$. The phase sequence is RYB. Draw the complexor diagram. *(H.N.C.)*

Ans. 35·7 A, 32·8 A, 27·7 A.

4.7 A 440 V symmetrical 3-phase supply feeds a star-connected load consisting of three non-inductive resistances of 10, 5 and 12 Ω connected to the R, Y and B phases respectively.

Calculate the line currents and the voltage across each resistor. Phase sequence, RYB. *(H.N.C.)*

Ans. 28·9 A, 36·5 A, 25·4 A; 290 V, 182 V, 304 V.

4.8 The power input to a 2,000 V 50 Hz 3-phase motor running on full load at an efficiency of 90 per cent is measured by two wattmeters which indicate 300 kW and 100 kW. Calculate (*a*) the input, (*b*) the power factor, (*c*) the line current, (*d*) the power output.

Ans. 400 kW, 0·756, 152 A, 360 kW.

4.9 The wattmeter readings in an induction motor circuit are 34·7 W and 4·7 W respectively, the latter reading being obtained after reversal of the connexions of the instrument voltage coil. Calculate the power factor at which the motor is working assuming normal two-wattmeter connexion of the wattmeters.

Ans. 0·4.

4.10 A 500 V 3-phase motor has an output of 37·8 kW and operates at a p.f. of 0·85 with an efficiency of 90 per cent. Calculate the reading on each of two wattmeters connected to measure the power input.

Ans. 28·2 kW, 13·3 kW.

4.11 Give the circuit arrangement and the theory of the two-wattmeter method of measuring power in a 3-phase 3-wire system.

A balanced star-connected load, each phase having a resistance of 10 Ω and an inductive reactance of 30 Ω is connected to a 400 V 50 Hz supply. The phase rotation is RYB. Wattmeters connected to read the total power have their current coils in the red and blue lines respectively. Calculate the reading of each wattmeter. *(L.U.)*

Ans. 2,190 W; −583 W.

4.12 Three impedances are mesh connected to a symmetrical 3-phase 440 V, 50 Hz supply of phase sequence RYB.

$Z_1 = (5 + j10)\Omega$ is connected between lines R and Y
$Z_2 = (5 + j5)\Omega$ is connected between lines Y and B
$Z_3 = (6 - j4)\Omega$ is connected between lines B and R

Two wattmeters connected to measure the power input have their current coils in lines R and Y respectively. Calculate the line currents and the wattmeter readings. *(H.N.C.)*

Ans. 95·5 A, 79·4 A, 43·3 A; 39·8 kW, 9·6 kW.

4.13 A 3-phase transmission line delivers a current at 33 kV to a balanced load having an equivalent impedance to neutral of $(240 + j320)\,\Omega$. The line has a resistance per conductor of $20\,\Omega$ and a reactance per conductor of $30\,\Omega$.

Calculate the voltage at the generator end.

If the load is made $(280 + j370)\,\Omega$, calculate the receiving-end voltage if the voltage at the generator end is unchanged. *(L.U.)*

Ans. 36 kV; 33·5 kV.

4.14 A 3-phase, 50 Hz, transmission line is 25 km long and delivers, 2,500 kW at 30 kV, 0·8 power factor lagging. Calculate the voltage at the sending end if each conductor has $R = 0{\cdot}8\,\Omega$ and $X_L = 1{\cdot}0\,\Omega$ per km. If an extra load consisting of capacitors having $C = 1{\cdot}5\,\mu\text{F}$ to neutral is connected at the middle of the line, calculate the voltage at the sending end. *(H.N.C.)*

Ans. 34·1 kV, 33·1 kV.

4.15 What are the advantages of overhead lines as compared with underground cables for transmission at very high voltages?

A 3-phase, 50 Hz transmission line is 100 km long and has the following constants:

Resistance/phase/km = $0{\cdot}2\,\Omega$
Inductance/phase/km = 2 mH
Capacitance (line-to-neutral) per km = $0{\cdot}015\,\mu\text{F}$

If the line supplies a load of 50 MW at 0·8 p.f. lagging and 132 kV, determine, using the nominal-T method, the sending-end voltage, current and power factor and the line efficiency. *(L.U.)*

Ans. 156 kV; 248 A; 0·8 lagging; 93·5 per cent.

4.16 Determine the voltage at the sending end, and the efficiency of a 3-phase transmission line given:

Output of line—250 A, 132 kV at 0·8 p.f. lagging
Line reactance—$42\,\Omega$ in each wire
Line resistance—$12\,\Omega$ in each wire
Line susceptance—$3{\cdot}75 \times 10^{-4}$ mho line-to-neutral
Line leakance—negligible

The capacitance may be assumed to be located wholly at the centre of the line. *(L.U.)*

Ans. 146 kV; 96 per cent.

4.17 The resistance, reactance and line-to-neutral susceptance of a 3-phase transmission line are $15\,\Omega$, $45\,\Omega$ and 4×10^{-4} S respectively, leakance being negligible. The load at the receiving end of the line is 50 MVA at 130 kV, 0·7 p.f. lagging.

Assuming the susceptance to be lumped half at each end of the line, find the sending-end voltage and current, and the efficiency of transmission. *(H.N.C.)*

Ans. 145 kV; 200 A; 94·6 per cent.

4.18 The three currents in a 3-phase system are

$$I_A = (120 + j60)\,\text{A},\ I_B = (120 - j120)\,\text{A and } I_C = (-150 + j100)\,\text{A}$$

Find the symmetrical components of these currents.

Ans. $I_{A+} = (109 + j100)\,\text{A};\ I_{A-} = (-18{\cdot}5 - j54{\cdot}7)\,\text{A};\ I_{A0} = (30 + j13{\cdot}3)\,\text{A}.$

4.19 A 3-phase 75 MVA 11·8 kV star-connected alternator with a solidly earthed star point has the following p.u. impedances based on rated phase voltage and rated phase current: positive phase sequence impedance, $j2{\cdot}0$ p.u.; negative phase sequence impedance, $j0{\cdot}16$ p.u.; zero phase sequence impedance, $j0{\cdot}08$ p.u.

Determine the steady-state fault current for the following: (*a*) a 3-phase symmetrical short-circuit, (*b*) a one-line-to-earth fault, and (*c*) a two-line-to-earth fault. The generator e.m.f. per phase is equal to the rated voltage.

Ans. 1,840 A; 4,920 A; 3,580 A.

4.20 Two similar 3-phase 50 MVA 11 kV star-connected alternators, A and B, are connected in parallel to 3-phase busbars to which an 11 kV 3-phase feeder is also connected. The star point of generator A is solidly earthed, but that of generator B is insulated from earth. The p.u. impedances of each generator and the feeder based on rated phase current and rated phase voltage are:

	Generator	*Feeder*
Positive phase-sequence impedance	$j2{\cdot}00$ p.u.	$j0{\cdot}20$ p.u.
Negative phase-sequence impedance	$j0{\cdot}16$ p.u.	$j0{\cdot}20$ p.u.
Zero phase-sequence impedance	$j0{\cdot}08$ p.u.	$j0{\cdot}60$ p.u.

Determine the total fault current in each line, the fault current to earth and the fault current supplied by each generator, for a two-line-to-earth fault at the distant end of the feeder. Each generator has an e.m.f. per phase equal to rated voltage. Assume the blue and yellow lines to be the faulted lines.

Notes

1. Since only zero phase sequence current flows in the earth, generator B, the star point of which is not earthed, cannot supply any zero phase sequence current.
2. In the equivalent circuit the generator positive and negative phase sequence impedances are in parallel and in series with that of the feeder. The zero phase sequence impedance of generator B does not appear in the equivalent circuits since it supplies no zero sequence current.

Ans. $I_R = 0;\ I_Y = 1{,}680\underline{/162{\cdot}8^\circ}\,\text{A};\ I_B = 1{,}680\underline{/17{\cdot}2^\circ}\,\text{A};\ I_E = 1{,}000\underline{/90^\circ}\,\text{A}$

Generator A: $I_R = 0;\ I_Y = 900\underline{/152{\cdot}9^\circ}\,\text{A};\ I_B = 900\underline{/27{\cdot}1^\circ}\,\text{A}$

Generator B: $I_R = 0;\ I_Y = 807\underline{/174{\cdot}2^\circ}\,\text{A};\ I_B = 807\underline{/5{\cdot}8^\circ}\,\text{A}$

Chapter 5

HARMONICS

Up to this stage it has been assumed that all alternating currents and voltages have been sinusoidal in waveform, this being by far

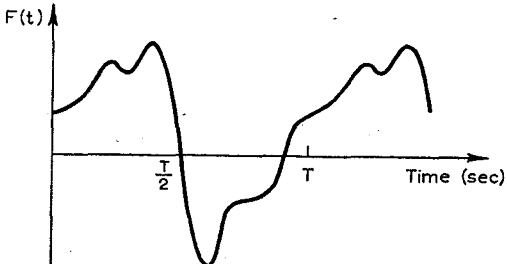


Fig. 5.1 A COMPLEX WAVEFORM

the most important form of alternating quantity which is met in electrical engineering. An alternating waveform which is not sinusoidal is said to be *complex*, and the complex wave (subject to certain mathematical conditions which are always complied with in electrical engineering applications) may be shown to be built up of a series of sinusoidal waves whose frequencies are integral multiples of the frequency of the *fundamental*, or basic, wave. The sinusoidal components of a complex wave are called the *harmonics*, the second harmonic having a frequency of twice the fundamental, the third harmonic three times the fundamental and so on.

Fig. 5.1 shows the graph to a base of time of a complex wave, whose period is T second, i.e. the fundamental frequency is $f = 1/T$ hertz (cycles per second).

Although it is desirable to have sinusoidal currents and voltages in a.c. systems, this is not always possible, and currents and voltages with complex waveforms do occur in practice. From the above, these complex voltages and currents may be represented by a series of sinusoidal waves naving frequencies which are integral multiples of the fundamental. The resultant effect of complex currents and/or voltages in a linear electric circuit is simply the sum of the individual effects of each harmonic by itself. This applies to instantaneous values and not to r.m.s. values, as will be seen later.

5.1 General Equation for a Complex Wave

Consider a complex voltage wave to be built up of a fundamental plus harmonic terms, each of which will have its own phase angle with respect to zero time. The instantaneous value of the resultant voltage will be given by the expression

$$v = V_{1m} \sin(\omega t + \psi_1) + V_{2m} \sin(2\omega t + \psi_2) + \dots + V_{nm} \sin(n\omega t + \psi_n) \tag{5.1}$$

where $f\ (= \omega/2\pi)$ is the frequency of the fundamental of the complex wave, and V_{nm} is the peak value of the nth harmonic.

In the same way the instantaneous value of a complex alternating current is given by

$$i = I_{1m} \sin(\omega t + \phi_1) + I_{2m} \sin(2\omega t + \phi_2) + \dots + I_{nm} \sin(n\omega t + \phi_n) \tag{5.2}$$

If eqns. (5.1) and (5.2) refer to the voltage across and the current through a given circuit, then the phase angles between harmonic currents and voltages are $(\psi_1 - \phi_1)$ for the fundamental, $(\psi_2 - \phi_2)$ for the second harmonic, and so on.

It should be noted that eqn. (5.2) may be rewritten in the form

$$i = I_{1m} \sin(\omega t + \phi_1') + I_{2m} \sin 2(\omega t + \phi_2') + \dots + I_{nm} \sin n(\omega t + \phi_n') \tag{5.2a}$$

In this equation the phase angle ϕ_1', ϕ_2', ϕ_n', etc., refer to the scale of the fundamental wave. A similar expression for the complex voltage wave of eqn. (5.1) may also be used.

5.2 Harmonic Synthesis

Figs. 5.2–5.5 show some complex waves which have been built up from simple harmonics. This synthesis is carried out by adding the instantaneous values of the fundamental and the harmonics for given instants in time.

In Fig. 5.2 a second harmonic, $E_{2m} \sin 2\omega t$, has been added to a fundamental $E_{1m} \sin \omega t$. Since E_{2m} is about 35 per cent of E_{1m}

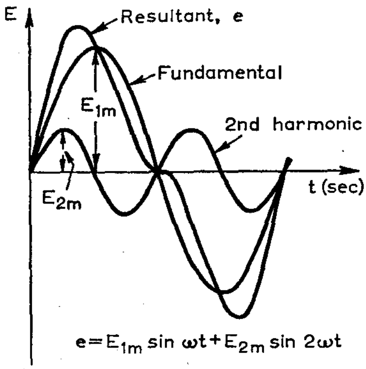


Fig. 5.2 SYNTHESIS OF FUNDAMENTAL AND SECOND HARMONIC

the resultant complex wave is said to contain 35 per cent second harmonic. The effect of a phase change of the harmonic with respect to the fundamental is shown in Fig. 5.3, where the added second

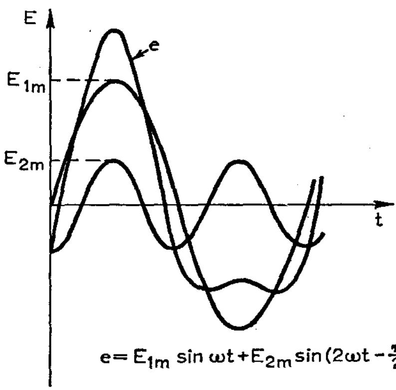


Fig. 5.3 EFFECT OF PHASE SHIFT OF SECOND HARMONIC ON RESULTANT WAVEFORM

harmonic is $E_{2m} \sin \left(2\omega t - \frac{\pi}{2}\right)$. It can be seen that the shape of the resultant wave has been completely changed, although the percentage of second harmonic remains the same.

Fig. 5.4 shows a fundamental with about 30 per cent third harmonic added, while Figs. 5.5(*a*) and (*b*) illustrate the effect of a phase shift of the harmonic with respect to the fundamental.

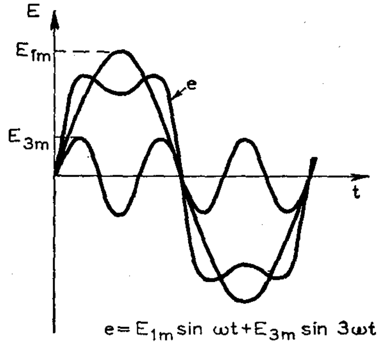


Fig. 5.4 SYNTHESIS OF FUNDAMENTAL AND THIRD HARMONIC

From these diagrams it can be seen that the complex wave produced by a fundamental and third harmonic has identical positive and negative half-cycles, while the wave produced by adding a

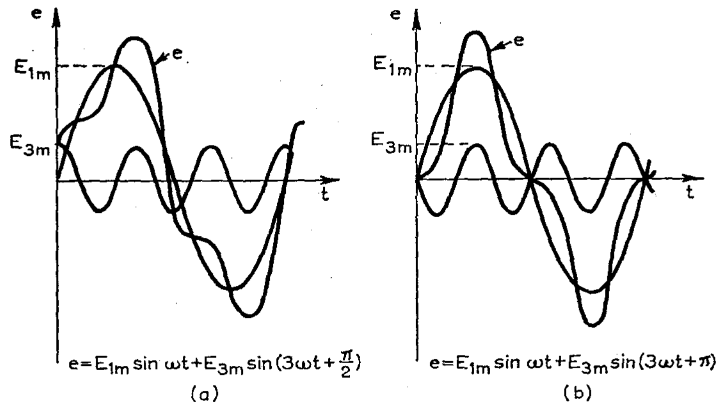


Fig. 5.5 EFFECT OF PHASE SHIFT OF THIRD HARMONIC ON RESULTANT WAVEFORM

second harmonic to the fundamental has dissimilar positive and negative half-cycles. In general a complex wave with identical positive and negative half-cycles can have no even-harmonic components. Since most rotating machines produce similar positive and negative half-cycles there will be only odd harmonics in their output waves. Non-linear circuit elements (i.e. circuit elements whose impedance varies with the direction and/or magnitude of the applied voltage, such as thermionic valves and rectifiers) will produce non-symmetrical current waves which must therefore contain even harmonics.

5.3 R.M.S. Value of a Complex Wave

Consider a current given by eqn. (5.2). The effective or r.m.s. value of this current is

$$I = \sqrt{(\text{Average value of } i^2)}$$

Now,

$$i^2 = I_{1m}^2 \sin^2(\omega t + \phi_1) + \ldots + I_{nm}^2 \sin^2(n\omega t + \phi_n) \\ + 2I_{1m}I_{2m} \sin(\omega t + \phi_1)\sin(2\omega t + \phi_2) + \ldots \tag{5.3}$$

The right-hand side of this equation is seen to consist of two types of term: (*a*) harmonic self-products of the form $I_{pm}^2 \sin^2(p\omega t + \phi_p)$ for the pth harmonic, and (*b*) products of different harmonics of the form $I_{pm}I_{qm} \sin(p\omega t + \phi_p)\sin(q\omega t + \phi_q)$ for the pth and qth harmonics. The average value of i^2 is the sum of the average values of each term in eqn. (5.3). For the general self-product,

Average value of $I_{pm}^2 \sin^2(p\omega t + \phi_p)$ over one cycle of the fundamental

$$= \frac{1}{2\pi}\int_0^{2\pi} I_{pm}^2 \sin^2(p\omega t + \phi_p)\,d(\omega t)$$

$$= \frac{I_{pm}^2}{2\pi}\int_0^{2\pi} \tfrac{1}{2}\{1 - \cos 2(p\omega t + \phi_p)\}\,d(\omega t)$$

$$= \frac{I_{pm}^2}{2} - 0 - \frac{I_{pm}^2}{4\pi}\left\{\frac{1}{2p}\sin 2(p\,.\,2\pi + \phi_p) - \frac{1}{2p}\sin(2\phi_p)\right\}$$

$$= \frac{I_{pm}^2}{2} \tag{5.4}$$

For the products of different harmonics,

Average value of $I_{pm}I_{qm}\sin(p\omega t + \phi_p)\sin(q\omega t + \phi_q)$

$$= \frac{1}{2\pi}\int_0^{2\pi} I_{pm}I_{qm}\sin(p\omega t + \phi_p)\sin(q\omega t + \phi_q)\,d(\omega t)$$

$$= \frac{I_{pm}I_{qm}}{2\pi}\int_0^{2\pi} \tfrac{1}{2}\{\cos((p - q)\omega t + \phi_p - \phi_q) \\ - \cos((p + q)\omega t + \phi_p + \phi_q)\}\,d(\omega t)$$

$$= \frac{I_{pm}I_{qm}}{4\pi}\left[\frac{1}{p - q}\sin\{(p - q)\omega t + \phi_p - \phi_q\} \\ - \frac{1}{p + q}\sin\{(p + q)\omega t + \phi_p + \phi_q\}\right]_0^{2\pi}$$

$$= 0 \tag{5.5}$$

i.e. the average value of the product of two different harmonic sinusoidal waves is zero. Therefore

$$\text{Average value of } i^2 = \frac{I_{1m}^2}{2} + \frac{I_{2m}^2}{2} + \ldots \frac{I_{nm}^2}{2}$$

so that

$$I = \sqrt{\left(\frac{I_{1m}^2}{2} + \frac{I_{2m}^2}{2} + \ldots \frac{I_{nm}^2}{2}\right)} \tag{5.6}$$

The r.m.s. values of each harmonic can be inserted in eqn. (5.6), which will then become

$$I = \sqrt{(I_1^2 + I_2^2 \ldots + I_n^2)} \tag{5.7}$$

where $I_1 = I_{1m}/\sqrt{2}$, etc.

Exactly similar expressions may be derived for the r.m.s. value of a complex voltage wave.

EXAMPLE 5.1 A complex voltage of r.m.s. value 240 V has 22 per cent third-harmonic content, and 5 per cent fifth-harmonic content. Find the r.m.s. values of the fundamental and of each harmonic.

The expression for voltage corresponding to eqn. (5.7) is

$$V = \sqrt{(V_1^2 + V_3^2 + V_5^2)}$$

But $V_3 = 0{\cdot}22V_1$, and $V_5 = 0{\cdot}05V_1$, so that

$$V = \sqrt{(V_1^2 + 0{\cdot}0484V_1^2 + 0{\cdot}0025V_1^2)} = 1{\cdot}03V_1$$

Therefore

$$V_1 = \frac{240}{1{\cdot}03} = \underline{\underline{233\,\text{V}}}$$

Hence

$$V_3 = 0{\cdot}22 \times 233 = \underline{\underline{51{\cdot}3\,\text{V}}}$$

and

$$V_5 = 0{\cdot}05 \times 233 = \underline{\underline{11{\cdot}7\,\text{V}}}$$

5.4 Power Associated with Complex Waves

Consider a voltage wave given by

$$v = V_{1m} \sin \omega t + V_{2m} \sin 2\omega t + V_{3m} \sin 3\omega t + \ldots$$

applied to a circuit and causing a current given by

$$i = I_{1m} \sin (\omega t - \phi_1) + I_{2m} \sin (2\omega t - \phi_2) + I_{3m} \sin (3\omega t - \phi_3) + \ldots$$

The power supplied to this circuit at any instant is the product vi. This product will involve multiplying every term in the voltage wave in turn by every term in the current wave. The average power supplied over a cycle will be the sum of the average values over one cycle of each individual product term. From eqn. (5.5) it can be seen that the average value of all product terms involving harmonics of different frequencies will be zero, so that only the products of current and voltage harmonics of the same frequency need be considered. For the general term of this nature, the average value of the product over one cycle of the fundamental is given by

$$P_n = \frac{1}{2\pi}\int_0^{2\pi} V_{nm}I_{nm} \sin n\omega t \sin(n\omega t - \phi_n)\, d(\omega t)$$

$$= \frac{V_{nm}I_{nm}}{2\pi}\int_0^{2\pi} \tfrac{1}{2}\{\cos\phi_n - \cos(2n\omega t - \phi_n)\}\, d(\omega t)$$

$$= V_n I_n \cos\phi_n \tag{5.8}$$

where V_n and I_n are r.m.s. values.

The total power supplied by complex voltages and currents is thus the sum of the powers supplied by each harmonic component acting independently. The average power supplied per cycle of the fundamental is

$$P = V_1I_1\cos\phi_1 + V_2I_2\cos\phi_2 + V_3I_3\cos\phi_3 + \ldots \tag{5.9}$$

When harmonics are present the overall power factor is defined as

$$\text{Overall power factor} = \frac{\text{Total power supplied}}{\text{Total r.m.s. voltage} \times \text{Total r.m.s. current}}$$

$$= \frac{V_1I_1\cos\phi_1 + V_2I_2\cos\phi_2 + \ldots}{VI} \tag{5.10}$$

Alternatively, if R_s is the equivalent series resistance of the circuit the total power is given by

$$P = I_1^2R_s + I_2^2R_s + I_3^2R_s + \ldots \tag{5.11}$$

$$= I^2R_s \tag{5.12}$$

If the effective parallel resistance of the whole circuit (R_p) is known, then the power supplied will be

$$P = \frac{V^2}{R_p} \tag{5.13}$$

where V is the r.m.s. value of the complex voltage wave.

EXAMPLE 5.2 A voltage given by

$$v = 50 \sin \omega t + 20 \sin (3\omega t + 30^\circ) + 10 \sin (5\omega t - 90^\circ) \text{ volts}$$

is applied to a circuit and the resulting current is found to be given by

$$i = 0{\cdot}5 \sin (\omega t - 37^\circ) + 0{\cdot}1 \sin (3\omega t - 15^\circ) + 0{\cdot}09 \sin (5\omega t - 150^\circ) \text{ amperes}$$

Find the total power supplied and the overall power factor.

The best method of tackling problems involving harmonics is to deal with each harmonic separately. Thus,

$$\text{Power at fundamental frequency} = \frac{V_{1m}I_{1m}}{2} \times \cos 37^\circ = \frac{50 \times 0{\cdot}5}{2} \times 0{\cdot}8$$

$$= 10\,\text{W}$$

$$\text{Power at third harmonic} = \frac{20 \times 0{\cdot}1}{2} \times \cos 45^\circ = 0{\cdot}707\,\text{W}$$

$$\text{Power at fifth harmonic} = \frac{10 \times 0{\cdot}09}{2} \times \cos 60^\circ = 0{\cdot}23\,\text{W}$$

Therefore

$$\text{Total power} = 10 + 0{\cdot}707 + 0{\cdot}23 = \underline{\underline{10{\cdot}9\,\text{W}}}$$

Also,

$$I = \sqrt{\left(\frac{0{\cdot}5^2}{2} + \frac{0{\cdot}1^2}{2} + \frac{0{\cdot}09^2}{2}\right)} = 0{\cdot}365\,\text{A}$$

and

$$V = \sqrt{\left(\frac{50^2}{2} + \frac{20^2}{2} + \frac{10^2}{2}\right)} = 38{\cdot}8\,\text{V}$$

Therefore

$$\text{Overall power factor} = \frac{10{\cdot}9}{38{\cdot}8 \times 0{\cdot}365} = \underline{\underline{0{\cdot}77}}$$

It should be noted that the overall power factor of a circuit when harmonics are present cannot be stated as leading or lagging; it is simply the ratio of the power to the product of r.m.s. voltage and current.

5.5 Harmonics in Single-phase Circuits

If an alternating voltage containing harmonics is applied to a single-phase circuit containing linear circuit elements, then the current which will result will also contain harmonics. By the superposition principle the effect of each voltage harmonic term may be considered separately. In the following paragraphs a voltage given by

$$v = V_{1m} \sin \omega t + V_{2m} \sin 2\omega t + V_{3m} \sin 3\omega t + \ldots \qquad (5.14)$$

will be applied to various circuits, and the resulting current will be determined.

PURE RESISTANCE

The impedance of a pure resistance is independent of frequency and the current and voltage will be in phase for each harmonic, so that the expression for the current will be

$$i = \frac{V_{1m}}{R} \sin \omega t + \frac{V_{2m}}{R} \sin 2\omega t + \frac{V_{3m}}{R} \sin 3\omega t + \ldots \qquad (5.15)$$

This equation shows that the percentage harmonic content in the current wave will be exactly the same as that in the voltage wave. The current and voltage waves will therefore be identical in shape.

PURE INDUCTANCE

If the voltage of eqn. (5.14) is applied to an inductance of L henrys, the inductive reactance $(2\pi fL)$ will vary with the harmonic frequency. For every harmonic term, however, the current will lag behind the voltage by 90°. Hence the general expression for the current is

$$i = \frac{V_{1m}}{\omega L} \sin (\omega t - 90°) + \frac{V_{2m}}{2\omega L} \sin (2\omega t - 90°) + \frac{V_{3m}}{3\omega L} \sin (3\omega t - 90°) + \ldots \qquad (5.16)$$

This shows that for the nth harmonic the percentage harmonic content in the current wave is only $1/n$ of the corresponding harmonic content in the voltage wave.

PURE CAPACITANCE

The capacitive reactance $1/2\pi fC$ of a capacitor C will vary with the harmonic frequency, but for every harmonic the current will lead the voltage by 90°. For the nth harmonic the reactance will be $1/n\omega C$, so that the peak current at this frequency will be

$$I_{nm} = V_{nm} n\omega C$$

Hence if the voltage of eqn. (5.14) is applied to a capacitor C, the current will be

$$i = V_{1m}\omega C \sin (\omega t + 90°) + V_{2m} \times 2\omega C \sin (2\omega t + 90°) + V_{3m} \times 3\omega C \sin (3\omega t + 90°) \qquad (5.17)$$

This shows that the percentage harmonic content of the current wave is larger than that of the voltage wave—for the nth harmonic, for instance, it will be n times larger.

EXAMPLE 5.3 A resistance of 10Ω is connected in series with a coil of inductance 6·36 mH. The supply voltage is given by

$$v = 300 \sin 314t + 50 \sin 942t + 40 \sin 1{,}570t \text{ volts}$$

Find (*a*) an expression for the instantaneous value of the current, and (*b*) the power dissipated.

For the fundamental, ω is 314 rad/s. The inductive reactance is therefore

$$X_1 = \omega L = 314 \times 6{\cdot}36 \times 10^{-3} = 2\Omega$$

The impedance at fundamental frequency is

$$Z_1 = \sqrt{(R^2 + X_1^2)} = \sqrt{(10^2 + 2^2)} = 10{\cdot}2\Omega$$

Also the phase angle is

$$\tan^{-1}\frac{X_1}{R} = \tan^{-1} 0{\cdot}2 = 11{\cdot}3^\circ \text{ lagging}$$

At the third-harmonic frequency, the inductive reactance, the impedance and the phase angle are

$$X_3 = 3\omega L = 3X_1 = 6\Omega$$

$$Z_3 = \sqrt{(10^2 + 6^2)} = 11{\cdot}7\Omega$$

$$\tan^{-1}\frac{X_3}{R} = \tan^{-1} 0{\cdot}6 = 31^\circ \text{ lagging}$$

At the fifth-harmonic frequency the reactance will have increased to $5X_1$, so that the impedance will now be

$$Z_5 = \sqrt{(10^2 + 10^2)} = 14{\cdot}1\,\Omega$$

and the phase angle will be

$$\tan^{-1}\frac{10}{10} = 45^\circ \text{ lagging}$$

(*a*) The resultant expression for the total current (amperes) will be

$$i = \frac{300}{10{\cdot}2}\sin(\omega t - 11{\cdot}3^\circ) + \frac{50}{11{\cdot}7}\sin(3\omega t - 31^\circ) + \frac{40}{14{\cdot}1}\sin(5\omega t - 45^\circ)$$

$$= \underline{\underline{29{\cdot}4 \sin(\omega t - 11{\cdot}3^\circ) + 4{\cdot}28 \sin(3\omega t - 31^\circ) + 2{\cdot}83 \sin(5\omega t - 45^\circ)}}$$

(*b*) In this case the power dissipated will be the product of the resistance and the square of the r.m.s. current.

$$I^2 = \frac{29{\cdot}4^2}{2} + \frac{4{\cdot}28^2}{2} + \frac{2{\cdot}83^2}{2} = 445{\cdot}4$$

Therefore

$$\text{Power dissipated} = I^2R_s = \underline{\underline{4{,}454\,\text{W}}}$$

EXAMPLE 5.4 A capacitor of 3·18μF capacitance is connected in parallel with a resistance of 1,000Ω, the combination being connected in series with a 1,000Ω resistor to a voltage given by

$$v = 350 \sin \omega t + 150 \sin (3\omega t + 30°) \text{ volts}$$

(*a*) Determine the power dissipated in the circuit if $\omega = 314$ rad/sec.
(*b*) Obtain an expression for the voltage across the series resistor.
(*c*) Determine the percentage harmonic content of the resultant current.

$$\text{Reactance of capacitor at fundamental frequency} = \frac{10^6}{314 \times 3{\cdot}18}$$
$$= 1{,}000\Omega$$

The complex impedance of the circuit at this frequency is

$$Z_1 = 1{,}000 + \frac{1{,}000(-j1{,}000)}{1{,}000 - j1{,}000} = 1{,}500 - j500 = 1{,}580\underline{/-18{\cdot}5°}\Omega$$

The reactance of the capacitor at the third-harmonic frequency will be one-third of that at the fundamental frequency. Therefore the complex impedance of the circuit at the third-harmonic frequency is

$$Z_3 = 1{,}000 + \frac{1{,}000(-j333)}{1{,}000 - j333} = 1{,}000 - j300 = 1{,}140\underline{/-15{\cdot}3°}\Omega$$

Thus

$$i = \frac{350}{1{,}580} \sin (\omega t + 18{\cdot}5°) + \frac{150}{1{,}140} \sin (3\omega t + 45{\cdot}3)$$
$$= 0{\cdot}222 \sin (\omega t + 18{\cdot}5°) + 0{\cdot}131 \sin (3\omega t + 45{\cdot}3°)$$

(*a*) From eqn. (5.9), the total power is

$$P = \frac{350}{\sqrt{2}} \times \frac{0{\cdot}222}{\sqrt{2}} \cos 18{\cdot}5° + \frac{150}{\sqrt{2}} \times \frac{0{\cdot}131}{\sqrt{2}} \cos 15{\cdot}3° = \underline{\underline{46{\cdot}3\,\text{W}}}$$

(*b*) The voltage across the series resistor is

$$v_r = iR$$
$$= \underline{\underline{222 \sin (\omega t + 18{\cdot}5°) + 131 \sin (3\omega t + 45{\cdot}3) \text{ volts}}}$$

(*c*) The percentage harmonic content of the current wave is

$$\frac{131}{222} \times 100 = \underline{\underline{59 \text{ per cent}}}$$

5.6 Selective Resonance

If a voltage which is represented by a complex wave is applied to a circuit containing both inductance and capacitance, the resulting current can be found by the methods previously described. It may happen that the circuit resonates at one of the harmonic frequencies of the applied voltage, and this effect is termed *selective resonance*. If series selective resonance occurs, then large currents at the resonant frequency may be produced, and in addition large harmonic voltages may appear across both the inductance and the capacitance. If parallel selective resonance occurs, on the other hand, the resultant current from the supply at the resonant frequency will be a minimum, but the current at this frequency through both the inductance and the capacitance will be large (i.e. current magnification).

The possibility of selective resonance is one reason why it is undesirable to have harmonics in a supply voltage. Selective resonance is used, however, in some wave analysers, which are instruments for determining the harmonic content of alternating waveforms. A simple form of analyser may consist of a series resonant circuit, which can be tuned over the range of harmonic frequencies to be measured. The voltage across the inductance or capacitance in the circuit will then give a measure of the size of the harmonic to which the circuit is tuned.

5.7 Effect of Harmonics on Single-phase Measurements

If measurements of impedance are made in circuits containing reactive elements, the presence of harmonics in the current and voltage waveforms may cause considerable errors unless they are allowed for.

Consider a capacitor C, across which the voltage is

$$V = V_{1m} \sin \omega t + V_{2m} \sin 2\omega t + V_{3m} \sin 3\omega t + \ldots$$

Then from eqn. (5.17) the current flowing through the capacitor will be

$$i = V_{1m}\omega C \sin (\omega t + 90°) + V_{2m}2\omega C \sin (2\omega t + 90°) + V_{3m}3\omega C \sin (3\omega t + 90°) + \ldots$$

If the r.m.s. value of the voltage is V, then

$$V = \frac{1}{\sqrt{2}} \sqrt{(V_{1m}^2 + V_{2m}^2 + V_{3m}^2 \ldots)}$$

and if the r.m.s. value of the current is I, then

$$I = \frac{\omega C}{\sqrt{2}} \sqrt{(V_{1m}^2 + 4V_{2m}^2 + 9V_{3m}^2 + \ldots)}$$

V and I will be the indications on r.m.s. measuring instruments if these are connected in circuit. From the above equations, the value of C will be given by

$$C = \frac{I}{\omega V} \sqrt{\left(\frac{V_{1m}^2 + V_{2m}^2 + V_{3m}^2 + \ldots}{V_{1m}^2 + 4V_{2m}^2 + 9V_{3m}^2 + \ldots}\right)} \tag{5.18}$$

If the effect of the harmonics were neglected, the value of the capacitance would appear to be $C' = I/\omega V$, from normal circuit theory. The true capacitance will be less than this.

In a similar manner it can be shown that the true inductance of a coil will be less than its apparent value in a circuit where harmonics are actually present but have been neglected.

If a wattmeter is included in a circuit when harmonics are present, the power indicated will be the true total power including the harmonic power. This follows since only sinusoidal currents of the same frequency in the fixed and moving coils will produce a resultant torque in the instrument.

This fact may be used to allow the wattmeter to be employed as a wave analyser. The current which is to be analysed is passed through the current coils of the wattmeter, while a variable-frequency sinusoidal generator supplies the voltage coil. The frequency of this generator is set successively at the fundamental and the harmonic frequencies, the voltage being maintained constant. The meter indication at each frequency setting will then be proportional to the magnitude of the component of the same frequency in the complex current wave. Hence the relative sizes of the fundamental and of each harmonic component may be obtained.

5.8 Superimposed Alternating and Direct Current

A special case of a complex wave is obtained when both an alternating and a direct current flow through the same circuit. The

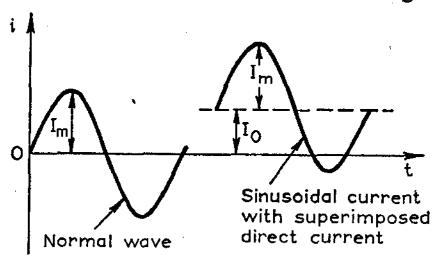


Fig. 5.6 ALTERNATING CURRENT WITH SUPERIMPOSED DIRECT CURRENT

effect of the d.c. component is to cause a shift of the whole a.c. wave by an amount equal to the magnitude of the direct current. In other words, the effective base line of the wave is moved relatively by an amount depending on the direct current (Fig. 5.6). This form of complex wave is found in electronic circuits and in saturable reactors. As far as calculations are concerned the d.c. component may be treated in the same way as any other harmonic term, provided that the following points are remembered.

(i) The r.m.s. value of a d.c. component is the actual value of the component, hence the r.m.s. value of a complex wave containing a d.c. term, I_0, is

$$I = \sqrt{\left(I_0^2 + \frac{I_{1m}^2}{2} + \frac{I_{2m}^2}{2} + \frac{I_{3m}^2}{2} + \ldots\right)} \tag{5.19}$$

$$= \sqrt{(I_0^2 + I_1^2 + I_2^2 + I_3^2 + \ldots)} \tag{5.20}$$

where I_{1m}, etc., are the peak values of the a.c. components, and I_1, etc., are the r.m.s. values of these components.

(ii) The steady voltage drop across a pure inductance due to a direct current is zero.

(iii) No direct current will flow through a capacitor.

EXAMPLE 5.5 A voltage given by

$$v = 30 + 25 \sin \omega t - 20 \sin 2\omega t \text{ volts}$$

is applied to the circuit shown in Fig. 5.7. Find the reading on each instrument

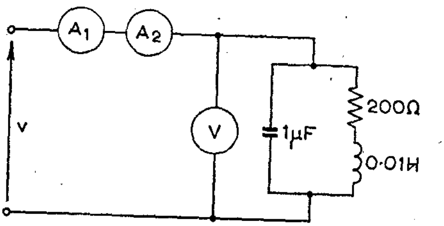


Fig. 5.7

if $\omega = 10{,}000$ rad/sec. A_1 is a thermocouple ammeter, A_2 a moving-coil ammeter, and V is an electrostatic voltmeter.

The thermocouple and electrostatic instruments will read r.m.s. values, while the moving-coil instrument will read average values, i.e. the moving-coil meter will record the d.c. component of current only. This d.c. component is given by

$$I_0 = \frac{V_{dc}}{200} = \frac{30}{200} = \underline{\underline{0{\cdot}15\,\text{A}}}$$

This follows since no direct current will flow through the capacitive arm of the network.

The equivalent impedance at the fundamental frequency is

$$Z_1 = \frac{(R + jX_{L1})(-jX_{C1})}{R + jX_{L1} - jX_{C1}} = \frac{(200 + j100)(-j100)}{200}$$

$$= 112\underline{/-63^\circ\, 26'}\,\Omega$$

The r.m.s. fundamental current is therefore

$$I_1 = \frac{25}{\sqrt{2} \times 112} = 0{\cdot}158\,\text{A}$$

The equivalent impedance at second harmonic is

$$Z_2 = \frac{(R + jX_{L2})(-jX_{C2})}{R + jX_{L2} - jX_{C2}} = \frac{(200 + j200)(-j50)}{200 + j150}$$
$$= 56{\cdot}6\underline{/-81^\circ\ 51'}\,\Omega$$

Therefore the r.m.s. second-harmonic current is

$$I_2 = \frac{20}{\sqrt{2} \times 56{\cdot}6} = 0{\cdot}25\,\text{A}$$

The reading on the thermocouple ammeter is, from eqn. (5.20),

$$I = \sqrt{(I_0^2 + I_1^2 + I_2^2)} = \sqrt{(0{\cdot}0225 + 0{\cdot}025 + 0{\cdot}0625)} = \underline{\underline{0{\cdot}332\,\text{A}}}$$

The voltmeter reading is, from eqn. (5.19),

$$V = \sqrt{\left(900 + \frac{625}{2} + \frac{400}{2}\right)} = \underline{\underline{37{\cdot}6\,\text{V}}}$$

5.9 Production of Harmonics

Harmonics may be produced in the output waveform of an a.c. generator, due to a non-sinusoidal air-gap flux distribution, or to *tooth ripple* which is caused by the effect of the slots which house the windings. In large supply systems, the greatest care is taken to ensure a sinusoidal output from the generators, but even in this case any non-linearity in the circuit will give rise to harmonics in the current waveform. Some non-linear circuit elements will be considered in the following sections.

RECTIFIERS

A rectifier is a circuit-element which has a low impedance to the flow of current in one direction, and a nearly infinite impedance to the flow of current in the opposite direction. Thus, when an alternating voltage is applied to the rectifier circuit, current will flow through it during the positive half-cycles only, being zero during the negative half-cycles. The current waveform is shown in Fig. 5.8. This waveform is seen to correspond roughly in shape to that shown in Fig. 5.3, but in this case the presence of a d.c. component brings the negative half-cycle up to the zero current position. Since the average value of a sine wave over one half-cycle is $(2/\pi) \times$ (maximum value), the average value taken over one cycle, with the negative half-cycle zero, will be $(1/\pi) \times$ (maximum value), and this then represents the d.c. component of the wave shown in Fig. 5.8. Also the wave must contain a large proportion of second harmonic.

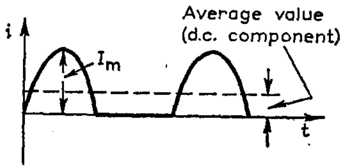


Fig. 5.8 CURRENT WAVEFORM FOR AN IDEAL RECTIFIER SUPPLIED WITH A SINUSOIDAL VOLTAGE

EXAMPLE 5.6 A battery charger is connected to a sinusoidal 220 V supply through a 20 Ω resistor. The equipment takes 5·5 A (r.m.s.) with 30 per cent second harmonic. Calculate the total circuit power, the overall power factor, and the power factor of the charging equipment alone. (*L.U.*)

The circuit is shown in Fig. 5.9(*a*). Since there is no second harmonic in the supply voltage, the battery charger may be regarded as a second-harmonic generator whose output is dissipated in the 20 Ω resistor. The circuit then becomes

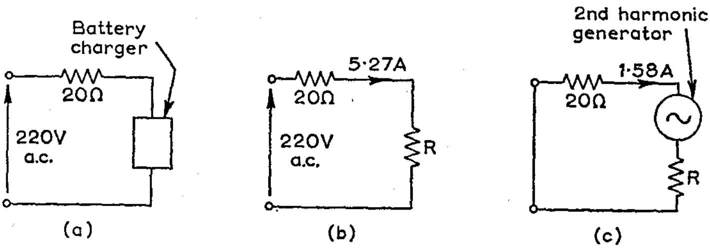


Fig. 5.9

that shown at (*b*) for the fundamental, and that shown at (*c*) for the second harmonic. R is the equivalent resistance of the charging unit.

Let I_1 = r.m.s. value of the fundamental current
I_2 = r.m.s. value of second-harmonic current
V_1 = r.m.s. value of fundamental supply voltage (=220 V)
V_2 = r.m.s. value of second-harmonic voltage across the 20 Ω resistor

It may be assumed that the charger will be connected to the supply circuit by a transformer so that there will be no direct current in the supply system.

Then $I_2 = 0{\cdot}3I_1$, and from eqn. (5.7),

$$5{\cdot}5 = \sqrt{(I_1^2 + (0{\cdot}3I_1)^2)} = 1{\cdot}044I_1$$

Therefore

$$I_1 = 5{\cdot}27\,\text{A} \quad \text{and} \quad I_2 = 1{\cdot}58\,\text{A}$$

The total power supplied must all be at the fundamental frequency since only a fundamental frequency voltage is applied. The value will be $V_1 I_1 \cos \phi_1$ watts. Since the circuit contains resistance only, $\cos \phi_1$ will be unity.

$$\text{Power supplied} = V_1 I_1 = 220 \times 5{\cdot}27 = \underline{\underline{1{,}160\,\text{W}}}$$

$$\text{Overall power factor (from eqn. (5.10))} = \frac{1{,}160}{220 \times 5{\cdot}5} = \underline{\underline{0{\cdot}96}}$$

The power supplied to the charger may be found by subtracting the power dissipated in the 20 Ω resistor from the total power; thus

$$\text{Power dissipated in } 20\,\Omega \text{ resistor} = I^2R = 5{\cdot}5^2 \times 20 = 605\,\text{W}$$

$$\text{Power supplied to charger} = 1{,}160 - 605 = 555\,\text{W}$$

Also, the fundamental-frequency voltage drop (r.m.s.) across the charger is

$$V_{C1} = 220 - (5{\cdot}27 \times 20) = 115\,\text{V} \qquad \text{(Fig. 5.9(}b\text{))}$$

and the second-harmonic voltage is

$$V_2 = 1{\cdot}58 \times 20 = 31{\cdot}6\,\text{V} \qquad \text{(Fig. 5.9(}c\text{))}$$

Thus

$$\text{R.M.S. value of charger voltage} = \sqrt{(V_{C1}^2 + V_2^2)} = 119\,\text{V}$$

Therefore

$$\text{Charger power factor} = \frac{555}{119 \times 5{\cdot}5} = \underline{\underline{0{\cdot}85}}$$

IRON-CORED COILS WITH SINUSOIDAL APPLIED VOLTAGE

Iron-cored coils are a source of harmonic generation in a.c. circuits owing to the non-linear character of the B/H curve and hysteresis loop, especially if saturation occurs. Consider a sinusoidal voltage, applied to an iron-cored coil of N turns and of cross-section A square metres. The instantaneous voltage is

$$v = N\frac{d\Phi}{dt}$$

where Φ is the flux produced in the iron core. If B is the core flux density,

$$v = NA\frac{dB}{dt}$$

Therefore

$$\int dB = \frac{1}{NA}\int v\,dt = \frac{1}{NA}\int V_m \sin\omega t\,dt \tag{5.21}$$

and

$$B = -\frac{V_m}{\omega NA}\cos\omega t = \frac{V_m}{\omega NA}\sin(\omega t - 90^\circ) \tag{5.22}$$

Hence, if the applied voltage is sinusoidal, the flux density in the iron core must also vary sinusoidally. Note that eqn. (5.21) leads to the familiar relation

$$B_m = \frac{V_m}{2\pi f N A} = \frac{V}{4{\cdot}44 f N A} \tag{5.23}$$

where V is the r.m.s. value of the applied voltage.

The hysteresis loop of a specimen of iron, for a given applied voltage, is shown in Fig. 5.10, the peak value of the flux density

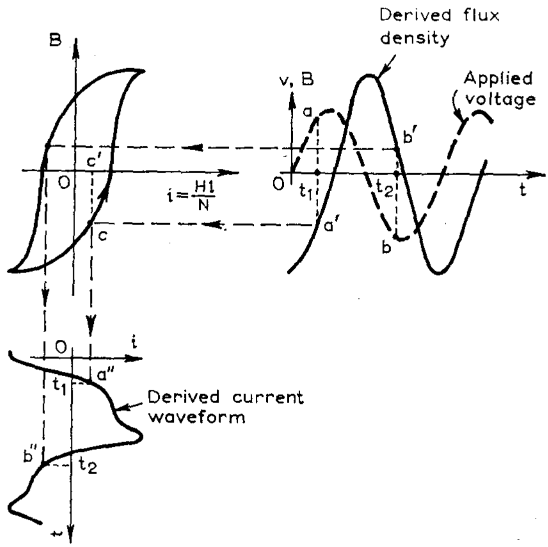


Fig. 5.10 CURRENT THROUGH AN IRON-CORED COIL WITH A SINUSOIDAL APPLIED VOLTAGE

being found from eqn. (5.23). The base is taken to a current scale $i = Hl/N$, where l is the length of the core. To derive the current waveform, sinusoidal voltage and flux density curves to a base of time are first drawn as shown. The current waveform is then derived by the point-by-point method indicated, care being taken to move round the hysteresis loop in the correct direction. Thus point a on the voltage curve corresponds to point a' on the flux density curve and to point c on the hysteresis loop, so that the current at this instant is $0c'$. This current is then plotted from the vertical time scale to give the derived point a'' on the current curve. By continuing this process for other points on the voltage curve, the current curve will be obtained.

It is seen that the current curve has identical positive and negative half-cycles, so that it contains no even harmonics. Comparison with Fig. 5.5(*a*) shows that there is a pronounced third-harmonic content.

FREE AND FORCED MAGNETIZATION

If the resistance of the circuit containing an iron-cored coil is high compared with the inductive reactance, then the current flowing from a sinusoidal supply will tend to be sinusoidal. This means

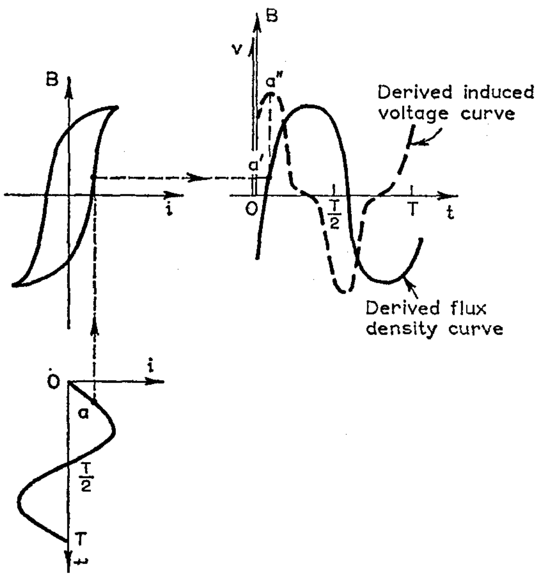


Fig. 5.11 WAVEFORMS OF CURRENT AND VOLTAGE UNDER FORCED MAGNETIZATION

that the core flux density cannot be sinusoidal, since it is related to the current by the hysteresis loop. In turn, this means that the induced voltage due to the alternating flux will not be sinusoidal. This condition is termed *forced magnetization.*

The condition of low circuit resistance relative to reactance gives sinusoidal flux from a sinusoidal supply voltage, and is called *free magnetization.*

To determine the shape of the induced voltage wave under forced magnetization, the hysteresis loop corresponding to the peak value of circuit current is drawn, and the flux density curve is derived from a sinusoidal current wave as shown in Fig. 5.11. The induced voltage is related to the flux density by the equation

$$v = NA\frac{dB}{dt}$$

so that the curve of flux density must be graphically differentiated and multiplied by the number of turns on the coil and the cross-sectional area of the iron to obtain the voltage curve. The result is shown by the dotted curve in the diagram. Comparing this curve with that of Fig. 5.5(*b*) it is seen that it contains a prominent third harmonic.

5.10 Harmonics in Three-phase Systems

Harmonics may be produced in 3-phase systems in the same way as in the single-phase systems already considered, and calculations are carried out similarly by considering each harmonic separately. As will be seen, however, care must be exercised when dealing with the third, and all multiples of the third harmonic (called the *triple-n* harmonics). It is unusual for even harmonics to be present.

PHASE E.M.F.S

Consider a 3-phase alternator with identical phase windings, in which harmonics are generated. The phase e.m.f.s will then be

$$e_R = E_{1m}\sin(\omega t + \psi_1) + E_{3m}\sin(3\omega t + \psi_3) + E_{5m}\sin(5\omega t + \psi_5) + \ldots \tag{5.24}$$

$$\begin{aligned} e_Y &= E_{1m}\sin\left(\omega t - \frac{2\pi}{3} + \psi_1\right) + E_{3m}\sin\left\{3\left(\omega t - \frac{2\pi}{3}\right) + \psi_3\right\} \\ &\quad + E_{5m}\sin\left\{5\left(\omega t - \frac{2\pi}{3}\right) + \psi_5\right\} + \ldots \\ &= E_{1m}\sin\left(\omega t - \frac{2\pi}{3} + \psi_1\right) + E_{3m}\sin(3\omega t - 2\pi + \psi_3) \\ &\quad + E_{5m}\sin\left(5\omega t - \frac{10\pi}{3} + \psi_5\right) + \ldots \\ &= E_{1m}\sin\left(\omega t - \frac{2\pi}{3} + \psi_1\right) + E_{3m}\sin(3\omega t + \psi_3) \\ &\quad + E_{5m}\sin\left(5\omega t - \frac{4\pi}{3} + \psi_5\right) + \ldots \end{aligned} \tag{5.25}$$

$$e_B = E_{1m} \sin\left(\omega t - \frac{4\pi}{3} + \psi_1\right) + E_{3m} \sin\left\{3\left(\omega t - \frac{4\pi}{3}\right) + \psi_3\right\}$$
$$+ E_{5m} \sin\left\{5\left(\omega t - \frac{4\pi}{3}\right) + \psi_5\right\} + \ldots$$
$$= E_{1m} \sin\left(\omega t - \frac{4\pi}{3} + \psi_1\right) + E_{3m} \sin(3\omega t + \psi_3)$$
$$+ E_{5m} \sin\left(5\omega t - \frac{2\pi}{3} + \psi_5\right) + \ldots \quad (5.26)$$

It can be seen from these expressions that all the third harmonics are in time phase, and that the fifth harmonics have a negative phase sequence, the fifth harmonic in the blue phase reaching its maximum value before that in the yellow phase. In the same way it can be shown that

(*a*) All triple-*n* harmonics are in phase (3rd, 9th, 15th, etc.).
(*b*) The 7th, 13th, 19th, etc., harmonics have a positive phase sequence.
(*c*) The 5th, 11th, 17th, etc., harmonics have a negative phase sequence.

LINE VOLTAGES FOR STAR CONNEXION

If the windings are star connected, the line voltages will be the difference between successive phase voltages, and hence will contain no third-harmonic terms, since these are identical in each phase. The fundamental will have a line value of $\sqrt{3}$ times the phase value and so too will the fifth harmonic.

It should be noted that the r.m.s. value of the line voltage in this case will be less than $\sqrt{3}$ times the r.m.s. value of the phase voltage, owing to the absence of the third-harmonic term from the expression for the line voltage.

LINE VOLTAGES FOR DELTA CONNEXION

If the alternator windings are delta connected, the resultant e.m.f. acting round the closed loop will be the sum of the phase e.m.f.s. This sum is zero for the fundamental, and for the 5th, 7th, 11th, etc., harmonics. All the third harmonics, however, are in phase, and hence there will be a resultant third-harmonic e.m.f. of three times the phase value acting round the mesh. This will cause a circulating current whose value will depend on the impedance of the windings at the third-harmonic frequency. Hence the third-harmonic e.m.f. is effectively short-circuited by the windings, so that there can be no third-harmonic voltage across the lines. The same applies to all triple-*n* harmonic terms. The line voltage will then be the phase voltage without the triple-*n* terms.

EXAMPLE 5.7 A 3-phase alternator has a generated e.m.f. per phase of 250V, with 10 per cent third and 6 per cent fifth harmonic content. Calculate (*a*) the r.m.s. line voltage for star connexion, and (*b*) the r.m.s. line voltage for mesh connexion.

Let V_1, V_3, V_5 be the r.m.s. values of the phase e.m.f.s. Then

$$V_3 = 0{\cdot}1V_1 \qquad \text{and} \qquad V_5 = 0{\cdot}06V_1$$

Hence, from eqn. (5.7),

$$250 = \sqrt{(V_1^2 + 0{\cdot}01V_1^2 + 0{\cdot}0036V_1^2)}$$

Therefore

$$V_1 = \frac{250}{\sqrt{1{\cdot}0136}} = 248\,\text{V} \qquad V_3 = 24{\cdot}8\,\text{V} \qquad \text{and} \qquad V_5 = 14{\cdot}9\,\text{V}$$

(*a*) R.M.S. value of fundamental line voltage $= \sqrt{3} \times 248 = 430\,\text{V}$

R.M.S. value of third-harmonic line voltage $= 0$

R.M.S. value of fifth-harmonic line voltage $= \sqrt{3} \times 14{\cdot}9 = 25{\cdot}8\,\text{V}$

Therefore

R.M.S. value of line voltage $= \sqrt{(430^2 + 25{\cdot}8^2)} = \underline{\underline{431\,\text{V}}}$

(*b*) In delta there is again no third harmonic component in the line voltage.

Thus

R.M.S. value of line voltage $= \sqrt{(248^2 + 14{\cdot}9^2)} = \underline{\underline{249\,\text{V}}}$

FOUR-WIRE SYSTEMS

In a 3-phase system there cannot be any third-harmonic term in the line voltage, as has already been seen. In a 4-wire system, however, each line-to-neutral voltage may contain a third-harmonic component, and if one is actually present a third-harmonic current will then flow in the star-connected load. If the load is balanced, the resulting third-harmonic currents will all be in phase, so that the neutral wire must carry three times the third-harmonic line current. There will be no neutral wire current at the fundamental frequency or at any harmonic frequency other than the triple-*n* frequencies.

5.11 Harmonics in Transformers

The flux density in transformer cores is usually fairly high to keep the volume of iron required to a minimum. It therefore follows that, owing to the non-linearity of the magnetization curve, there will be some third-harmonic distortion produced (Section 5.9(*b*)). There is usually a small percentage of fifth harmonic also. For single-phase transformers the conditions are the same as those already described for iron-cored coils with sinusoidal applied voltage, namely the magnetizing current will contain a proportion of mainly third harmonic depending on the size of the applied voltage, and the flux will be sinusoidal.

In 3-phase transformers the method of connexion and the type of construction will affect the production of harmonics, as the following cases will show.

PRIMARY WINDING IN DELTA

Each phase of the winding may be considered as separately connected across a sinusoidal supply. The flux will be sinusoidal, so that the magnetizing current will contain a third-harmonic component (in addition to other harmonics of higher order but of relatively small magnitude). These third-harmonic currents will be in phase in each winding, and will constitute a circulating current, so that there will be no third-harmonic component in the line current.

PRIMARY WINDING IN FOUR-WIRE STAR

Again in this case each primary phase may be considered as separately connected to a sinusoidal supply. The core flux will be sinusoidal, and hence the output voltage will also be sinusoidal. The magnetizing current will contain a third-harmonic component which is in phase in each winding and will therefore return through the neutral.

PRIMARY WINDING IN THREE-WIRE STAR

Since, in the absence of a neutral, there is no return path for the third-harmonic components of the magnetizing current, no such currents can flow, and a condition of forced magnetization must therefore exist. The core flux must then contain a third-harmonic component which is in phase in each limb (Section 5.9(*c*)). In the shell type of 3-phase transformer, or in the case of three single-phase units, there will be a magnetic path for these fluxes, but in the 3-limb core type of transformer the third-harmonic component of flux must return via the air (or through the steel tank in an oil-cooled transformer). The high reluctance of the magnetic path for the 3-limb core type of construction reduces the third-harmonic flux in this case to a fairly small value.

If the secondary is delta connected, then the third-harmonic flux component will give rise to a third-harmonic circulating current in the secondary winding. This current, by Lenz's law, tends to oppose the original effect which causes it, and hence minimizes the third-harmonic flux component. There is, of course, no circulating current at any but the triple-n frequencies.

If the secondary is star-connected, it is usual to have an additional delta-connected tertiary winding in which the third-harmonic currents can flow. This winding also preserves the magnetic equilibrium of the transformer on unbalanced loads. In this way the output voltage is kept reasonably sinusoidal.

5.12 Harmonic Analysis

It has been seen that a complex wave may be represented by an equation of the form

$$y = Y_{1m} \sin (\omega t + \psi_1) + Y_{2m} \sin (2\omega t + \psi_2) + \ldots \qquad (5.27)$$

If the shape of the complex wave is known, the process of harmonic analysis is one of finding the coefficients Y_{1m}, Y_{2m}, . . . etc., and the phase angles ψ_1, ψ_2, . . . etc., in this equation. If a mathematical expression for y is possible, then the analysis is the standard Fourier analysis found in appropriate advanced mathematics textbooks. Generally, however, the shape of a complex wave is readily obtainable, but the wave has no simple mathematical expression, and the following methods illustrate how complex waves may be analysed under such conditions.

SUPERPOSITION METHOD (WEDMORE'S METHOD)

This method is used mainly for the analysis of third-harmonic content, but may be extended to fifth and higher harmonics. It depends on the fact that, if a sine wave is divided into any number of equal parts and the parts are then superimposed, the sum of the ordinates at any point will be zero. Thus consider the sine wave given by

$$y = Y_{1m} \sin \omega t$$

The following table may then be drawn up:

ωt	0°	30°	60°	90°	120°	150°	
y	0	$0{\cdot}5\,Y_{1m}$	$\frac{\sqrt{3}}{2}\,Y_{1m}$	Y_{1m}	$\frac{\sqrt{3}}{2}\,Y_{1m}$	$0{\cdot}5\,Y_{1m}$	
ωt	180°	210°	240°	270°	300°	330°	360°
y	0	$-0{\cdot}5\,Y_{1m}$	$-\frac{\sqrt{3}}{2}\,Y_{1m}$	$-Y_{1m}$	$-\frac{\sqrt{3}}{2}\,Y_{1m}$	$-0{\cdot}5\,Y_{1m}$	0

Dividing this cycle into, say, three parts, and adding gives the following:

y(0–120°)	0	$0{\cdot}5\,Y_{1m}$	$\frac{\sqrt{3}}{2}\,Y_{1m}$	Y_{1m}	$\frac{\sqrt{3}}{2}\,Y_{1m}$
y(120°–240°)	$\frac{\sqrt{3}}{2}\,Y_{1m}$	$0{\cdot}5\,Y_{1m}$	0	$-0{\cdot}5\,Y_{1m}$	$-\frac{\sqrt{3}}{2}\,Y_{1m}$
y(240°–360°)	$-\frac{\sqrt{3}}{2}\,Y_{1m}$	$-Y_{1m}$	$-\frac{\sqrt{3}}{2}\,Y_{1m}$	$-0{\cdot}5\,Y_{1m}$	0
Sum	0	0	0	0	0

This is shown graphically in Fig. 5.12(*a*). If a third harmonic of this sine wave is treated in the same way (as shown at (*b*)), the resultant is three times the third harmonic, in the correct phase. Hence if a complex wave containing a third-harmonic component is divided into three equal parts and the ordinates are added, the resultant will be three times the third harmonic only. It may readily be verified that all harmonics other than the third (and multiples of the third) are absent from this resultant.

If the complex wave is divided into five equal sections and the ordinates are added, then it can be shown that the resultant will be five times the fifth harmonic, and will contain no fundamental or third-harmonic components. The method is not very suitable if the complex wave contains large percentages of harmonics higher than the fifth.

EXAMPLE 5.8 A complex current wave has the following shape over one half-cycle of the fundamental, the negative half-cycle being similar:

ωt	0	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
i	3	5·1	5·3	5·6	6	7·2	10	11·5	12	11	8	3	−3

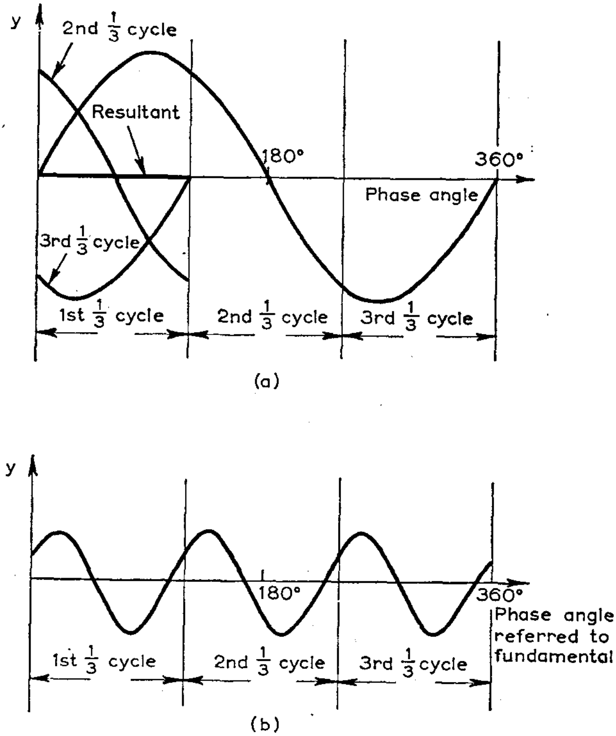


Fig. 5.12 SUPERPOSITION METHOD
(*a*) Sine wave (*b*) Third harmonic

Find the magnitude and phase angle of the third-harmonic component, and express this as a percentage of the fundamental.

One cycle of the wave is divided into three equal sections and the ordinates are added. The resultant wave is plotted in Fig. 5.13 from the figures obtained from the following table:

Abscissae	0°	15°	30°	45°	60°	75°	90°	105°	120°
i(0–120°)	3	5·1	5·3	5·6	6	7·2	10	11·5	12
i(120–240°)	12	11	8	3	−3	−5·1	−5·3	−5·6	−6
i(240–360°)	−6	−7·2	−10	−11·5	−12	−11	−8	−3	3
Sum	9	8·9	3·3	−2·9	−9	−8·9	−3·3	2·9	9

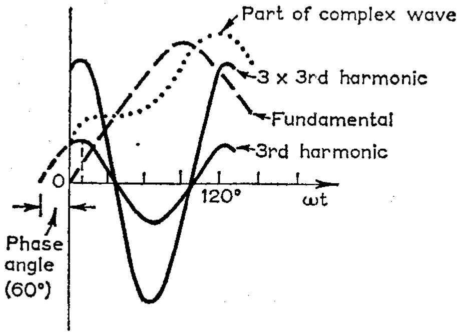


Fig. 5.13

This will represent three times the third harmonic so that the third harmonic itself must have the co-ordinates

Abscissae	0°	15°	30°	45°	60°	75°	90°	105°	120°
Ordinates	3	2·97	1·1	−0·97	−3	−2·97	−1·1	0·97	3

This wave is also plotted in Fig. 5.13, and from it the phase angle of the third harmonic is estimated to be 20° on the fundamental scale.

To obtain the fundamental, the third harmonic is subtracted from the complex wave, assuming there is no higher harmonic. Thus:

Angle ωt	0°	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
Complex wave	3	5·1	5·3	5·6	6	7·2	10	11·5	12	11	8	3	−3
Third harmonic	3	2·97	1·1	−0·97	−3	−2·97	−1·1	0·97	3	2·97	1·1	−0·97	3
Difference	0	2·13	4·2	6·57	9	10·17	11·1	10·53	9	9·03	6·9	3·97	0

When this wave is plotted it is seen to be almost sinusoidal in shape, indicating that the complex wave contained mainly fundamental and third-harmonic frequencies. From the diagram, the peak value of the fundamental (neglecting higher harmonics) is 11·1, and that of the third harmonic is 3·3. Therefore

$$\text{Percentage third harmonic} = \frac{3{\cdot}3}{11{\cdot}1} \times 100 = \underline{\underline{29{\cdot}7 \text{ per cent}}}$$

The equation of the complex wave is

$$i = 11{\cdot}1 \sin \omega t + 3{\cdot}3 \sin 3(\omega t + 20°) = 11{\cdot}1 \sin \omega t + 3{\cdot}3 \sin (3\omega t + 60°)$$

TWENTY-FOUR ORDINATE METHOD

The complex wave given by eqn. (5.27) may be expressed in a slightly different form by expanding the sine terms. Thus, for the fundamental term,

$$\begin{aligned} y_1 &= Y_{1m} \sin(\omega t + \psi_1) \\ &= Y_{1m} \cos\psi_1 \sin\omega t + Y_{1m} \sin\psi_1 \cos\omega t \\ &= A_1 \sin\omega t + B_1 \cos\omega t \end{aligned}$$

where $A_1 = Y_{1m} \cos\psi_1$ and $B_1 = Y_{1m} \sin\psi_1$, and hence

$$\psi_1 = \tan^{-1}\frac{B}{A} \qquad Y_{1m} = \sqrt{(A_1^2 + B_1^2)}$$

Thus

$$\begin{aligned} y = A_1 \sin\omega t + A_2 \sin 2\omega t + \ldots \\ + B_1 \cos\omega t + B_2 \cos 2\omega t + \ldots \end{aligned} \tag{5.28}$$

In the evaluation of $A_1, A_2, \ldots B_1, B_2, \ldots$ etc., the following results of integral calculus will be used:

(a) $$\int_0^{2\pi} \sin^2 m\omega t\, d(\omega t) = \pi = \int_0^{2\pi} \cos^2 m\omega t\, d(\omega t) \tag{5.29}$$

(b) $$\int_0^{2\pi} \sin m\omega t \times \sin n\omega t\, d(\omega t) = 0 \qquad \text{for } n \neq m \tag{5.30}$$

(c) $$\int_0^{2\pi} \cos m\omega t \times \cos n\omega t\, d(\omega t) = 0 \qquad \text{for } n \neq m \tag{5.31}$$

(d) $$\int_0^{2\pi} \sin m\omega t \times \cos n\omega t\, d(\omega t) = 0 \tag{5.32}$$

To evaluate A_n, each term in eqn. (5.28) is multiplied by $\sin n\omega t$ and integrated between 0 and 2π, giving

$$\begin{aligned} \int_0^{2\pi} y \sin n\omega t\, d(\omega t) = \int_0^{2\pi} \{ & A_1 \sin\omega t \sin n\omega t + A_2 \sin 2\omega t \sin n\omega t \\ & + \ldots + A_n \sin^2 n\omega t + \ldots \\ & + B_1 \cos\omega t \sin n\omega t + B_2 \cos 2\omega t \sin n\omega t \\ & + \ldots + B_n \cos n\omega t \sin n\omega t + \ldots\}\, d(\omega t) \\ = \pi A_n & \end{aligned}$$

Therefore

$$A_n = 2 \times \frac{1}{2\pi}\int_0^{2\pi} y \sin n\omega t\, d(\omega t)$$

Thus A_n is twice the average value of $y \sin n\omega t$ over one cycle of the fundamental, and in the same way it follows that

$$A_1 = 2 \times \text{average value of } y \sin \omega t \text{ over one cycle}$$

$$A_2 = 2 \times \text{average value of } y \sin 2\omega t \text{ over one cycle}$$

$$B_1 = 2 \times \text{average value of } y \cos \omega t \text{ over one cycle} \ldots \text{etc.}$$

If the wave is known to contain odd harmonics only, the analysis need only be carried out over one half-cycle, since the positive and negative half-cycles are identical. The integrals (*b*)–(*d*) are true over one half-cycle, and the integral (*a*) has a value of $\pi/2$ for a half-cycle. The coefficients are then $2 \times$ average value of $y \sin n\omega t$ or $y \cos n\omega t$ over one half-cycle.

To obtain the average value of $y \sin n\omega t$ or $y \cos n\omega t$ the complex wave is divided into 24 equal parts (or more if increased accuracy is desired) and 24 ordinates are erected, one at each division starting from zero. Each ordinate is then multiplied by $\sin n\omega t$ (or $\cos n\omega t$) and the sum of these terms is divided by 24. Thus,

$$A_1 = \tfrac{2}{24}(y_0 \sin 0° + y_{15} \sin 15° + y_{30} \sin 30° + \ldots y_{345} \sin 345°) \qquad (5.33)$$

where $y_0, y_{15}, y_{30} \ldots$ etc., are the ordinates erected at 0°, 15°, 30°, etc. (to give 24 intervals), and

$$A_3 = \tfrac{2}{24}\{y_0 \sin (3 \times 0°) + y_{15} \sin (3 \times 15°) + y_{30} \sin (3 \times 30°)\} \qquad (5.34)$$

For symmetrical waves, twelve ordinates are erected over one half-cycle, and the coefficients are then of the form

$$A_n = \tfrac{2}{12}\{y_0 \sin (n \times 0°) + y_{15} \sin (n \times 15°) + \ldots y_{165} \sin (n \times 165°)\} \qquad (5.35)$$

EXAMPLE 5.9 Analyse the wave given in Example 5.8 by the 24-ordinate method, assuming only third and fifth harmonics to be present.

The best approach is to use a tabular method. In this case, since the positive and negative half-cycles are identical, the average over one half-cycle only need be considered.

SINE TERMS

ωt	i	$\sin \omega t$	$i \sin \omega t$	$\sin 3\omega t$	$i \sin 3\omega t$	$\sin 5\omega t$	$i \sin \omega t$
0	3·0	0·0	0·0	0·0	0·0	0·0	0·0
15°	5·1	0·26	1·32	0·707	3·6	0·97	4·95
30°	5·3	0·5	2·65	1·0	5·3	0·5	2·65
45°	5·6	0·707	3·9	0·707	3·96	−0·707	−3·96
60°	6·0	0·87	5·2	0·0	0·0	−0·87	−5·2
75°	7·2	0·97	7·0	−0·707	−5·1	0·26	1·87
90°	10·0	1·0	10·0	−1.0	−10·0	1·0	10·0
105°	11·5	0·97	11·15	−0·707	−8·15	0·26	3·0
120°	12·0	0·87	10·45	0·0	0·0	−0·87	10·45
135°	11·0	0·707	7·8	0·707	7·8	−0·707	−7·8
150°	8·0	0·5	4·0	1·0	8·0	0·5	4·0
165°	3·0	0·26	0·78	0·707	2·12	0·97	2·9
	Sum		64·25		7·53		1·96
	$\frac{2}{12}$ × Sum		$A_1 = 10{\cdot}74$		$A_3 = 1{\cdot}25$		$A_5 = 0{\cdot}327$

COSINE TERMS

ωt	i	$\cos \omega t$	$i \cos \omega t$	$\cos 3\omega t$	$i \cos 3\omega t$	$\cos 5\omega t$	$i \cos 5\omega t$
0	3·0	1·0	3·0	1·0	3·0	1·0	3·0
15°	5·1	0·97	4·95	0·707	3·6	0·26	1·32
30°	5·3	0·87	4·6	0·0	0·0	−0·87	−4·6
45°	5·6	0·707	3·96	−0·707	−3·96	−0·707	−3·96
60°	6·0	0·5	3·0	−1·0	−6·0	0·5	3·0
75°	7·2	0·26	1·87	−0·707	−5·1	0·97	7·0
90°	10·0	0·0	0·0	0·0	0·0	0·0	0·0
105°	11·5	−0·26	−3·0	0·707	8·15	−0·97	−11·15
120°	12·0	−0·5	−6·0	1·0	12·0	−0·5	−6·0
135°	11·0	−0·707	−7·8	0·707	7·8	0·707	7·8
150°	8·0	−0·87	−6·95	0·0	0·0	0·87	6·95
165°	3·0	−0·97	−2·9	−0·707	−2·12	−0·26	−0·78
	Sum		−5·27		17·37		2·58
	$\frac{2}{12}$ × Sum		$B_1 = -0{\cdot}88$		$B_3 = 2{\cdot}90$		$B_5 = 0{\cdot}43$

From these results

$$I_{1m} = \sqrt{(A_1^2 + B_1^2)} = 10{\cdot}8$$
$$I_{3m} = \sqrt{(A_3^2 + B_3^2)} = 3{\cdot}18$$
$$I_{5m} = \sqrt{(A_5^2 + B_5^2)} = 0{\cdot}54$$

Also

$$\Psi_1 = \tan^{-1} - \frac{0{\cdot}88}{10{\cdot}74} = -5^\circ$$

$$\Psi_3 = \tan^{-1} \frac{2{\cdot}90}{1{\cdot}25} = 66{\cdot}7^\circ$$

$$\Psi_5 = \tan^{-1} \frac{0{\cdot}43}{0{\cdot}327} = 53{\cdot}7^\circ$$

Thus

$$i = 10{\cdot}8 \sin(\omega t - 5^\circ) + 3{\cdot}18 \sin(3\omega t + 66{\cdot}7^\circ) + 0{\cdot}43 \sin(5\omega t + 53{\cdot}7^\circ)$$

This is a more accurate result than that obtained by the superposition method, and clearly shows the presence of a small fifth harmonic. The results for the fundamental and third harmonic compare favourably in each case.

5.13 Form Factor

The form factor, k_f, of any alternating waveform may be defined as

$$k_f = \frac{\text{R.M.S. value}}{\text{Full-wave-rectified mean value}} \tag{5.36}$$

The full-wave-rectified waveform has its negative-going portions inverted, as shown in Fig. 5.14. Its mean value is found by integrating over the period of the wave. If the wave has identical positive and negative half-cycles, the integration may be performed over one half-cycle, between zero values.

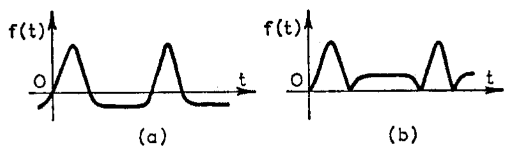


Fig. 5.14

For a full sine wave, $i = I_m \sin \omega t$, and since positive and negative half-cycles are identical, and zeros occur at times $t = 0$ and $t = T/2$, the r.m.s. value is

$$I = \sqrt{\left\{\frac{1}{T/2}\int_0^{T/2} I_m^2 \sin^2 \omega t\, dt\right\}} = \frac{I_m}{\sqrt{2}}$$

and the full-wave-rectified mean is

$$I_{av} = \frac{1}{T/2}\int_0^{T/2} I_m \sin\omega t\, dt = \frac{2}{\pi} I_m$$

so that the form factor is

$$k_f = \frac{I_m/\sqrt{2}}{I_m 2/\pi} = 1{\cdot}11 \tag{5.37}$$

For a square wave with no d.c. component the r.m.s. and mean values are both equal to the peak value and hence the form factor is unity.

The form factors of complex waves whose positive and negative half-cycles are identical (and hence contain odd harmonics only), and which have only two zero values per cycle can be readily evaluated. The r.m.s. value is obtained as in Section 5.3 as the square root of the sum of the squares of the r.m.s. values of each harmonic. The mean value is obtained by integrating between two zeros (i.e. the half-cycle mean). Thus consider the sine series

$$i = I_{1m}\sin\omega t + I_{3m}\sin 3\omega t + I_{5m}\sin 5\omega t + \ldots$$

The r.m.s. value is (from eqn. (5.6))

$$I = \sqrt{\left(\frac{I_{1m}^2}{2} + \frac{I_{3m}^2}{2} + \frac{I_{5m}^2}{2} + \ldots\right)}$$

while the mean value is

$$I_{av} = \frac{1}{T/2}\int_0^{T/2} i\, dt$$

since zeros occur when $t = 0$ and when $t = T/2$. Hence

$$I_{av} = \frac{2}{T}\left(\frac{2}{\omega} I_{1m} + \frac{2}{3\omega} I_{3m} + \frac{2}{5\omega} I_{5m} + \ldots\right)$$

$$= \frac{2}{\pi}\left(I_{1m} + \frac{I_{3m}}{3} + \frac{I_{5m}}{5} + \ldots\right) \tag{5.38}$$

where $\omega = 2\pi/T$. It follows that the form factor is

$$k_f = \frac{\pi}{2\sqrt{2}}\frac{\sqrt{(I_{1m}^2 + I_{3m}^2 + I_{5m}^2 + \ldots)}}{(I_{1m} + I_{3m}/3 + I_{5m}/5 + \ldots)} \tag{5.39}$$

In a similar way a cosine series of odd harmonics yields an easily evaluated form factor. Let

$$i = I_{1m}\cos\omega t + I_{3m}\cos 3\omega t + I_{5m}\cos 5\omega t + \ldots$$

The r.m.s. value is the same as for the sine series which has just been evaluated. Zeros occur at $t = -T/4$ and $t = +T/4$, so that the mean value is

$$
\begin{aligned}
I_{av} &= \frac{1}{T/2}\int_{-T/4}^{T/4} i\,dt \\
&= \frac{2}{T}\left(\frac{2}{\omega}I_{1m} - \frac{2}{3\omega}I_{3m} + \frac{2}{5\omega}I_{5m} - \ldots\right) \\
&= \frac{2}{\pi}\left(I_{1m} - \frac{I_{3m}}{3} + \frac{I_{5m}}{5} - \ldots\right) \qquad (5.40)
\end{aligned}
$$

and the form factor is

$$k_f = \frac{\pi}{2\sqrt{2}}\frac{\sqrt{(I_{1m}^2 + I_{3m}^2 + I_{5m}^2 + \ldots)}}{(I_{1m} - I_{3m}/3 + I_{5m}/5 - \ldots)} \qquad (5.41)$$

In the general case it is necessary to determine the instants in the cycle at which the wave has zero values, and integrate separately over each period of time between zeros, adding the integrals so obtained arithmetically, and dividing the result by the period of the wave. This can be a cumbersome process.

The form factor of a complex wave can sometimes cause errors in instrument readings. Moving-iron, electrodynamic and electrostatic instruments will always read true r.m.s. values independent of form factor. Moving-coil rectifier instruments, on the other hand, have a deflexion which is proportional to the full-wave-rectified mean current which flows in the moving coil. These instruments are normally calibrated in r.m.s. values assuming sine wave inputs (i.e. form factors of 1·11). The readings on such instruments are really the mean values multiplied by 1·11. Hence if the form factor of a wave measured on a rectifier instrument is k_f, the mean value is the instrument reading divided by 1·11, and

$$\text{True r.m.s. value} = \frac{\text{Instrument reading}}{1{\cdot}11}k_f \qquad (5.42)$$

It follows that the form factor of a complex wave can be found experimentally by measuring its value on a true r.m.s. instrument and on a rectifier instrument. The form factor is then

$$k_f = \frac{\text{R.M.S. instrument reading}}{\text{Rectifier instrument reading}} \times 1{\cdot}11 \qquad (5.43)$$

One further important application of form factor in complex waves occurs when iron-cored coils are excited into saturation by an alternating current. It has already been seen in Section 5.9(*c*) that the flux waveform in this case is flat-topped, and analysis shows it to be represented by a sine series of odd harmonics

$$\Phi = \Phi_{1m} \sin \omega t + \Phi_{3m} \sin 3\omega t + \ldots$$

The induced e.m.f. in a coil of N turns linking this flux is given by

$$e = N\frac{d\Phi}{dt} = N\omega\Phi_{1m} \cos \omega t + 3N\omega\Phi_{3m} \cos 3\omega t + \ldots$$

From eqn. (5.40) the average e.m.f. is therefore

$$\begin{aligned} E_{av} &= \frac{2}{\pi}(N\omega\Phi_{1m} - N\omega\Phi_{3m} + \ldots) \\ &= 4fN(\Phi_{1m} - \Phi_{3m} + \ldots) \\ &= 4fN\Phi_{max} \end{aligned}$$

since in this instance the peak flux (Φ_{max}) is given by

$$\Phi_{max} = \Phi_{1m} - \Phi_{3m} + \Phi_{5m} - \ldots$$

If the form factor of the e.m.f. wave if k_f then the r.m.s. value is

$$E = 4k_f fN\Phi_{max} \tag{5.44}$$

PROBLEMS

5.1 Show that the current wave through a capacitor contains a larger percentage of harmonics than the voltage wave across it.

A voltage of 200 V (r.m.s.), containing 20 per cent third harmonic is applied to a circuit containing a resistor and a capacitor in series. The current is 3 A (r.m.s.) with 30 per cent harmonic. Determine the resistance and capacitance of the circuit, and the overall power factor. (The fundamental frequency is 50 Hz.)
(*H.N.C.*)

Ans. 41·8 Ω, 59 μF, 0·626.

5.2 The magnetization curve for a ferromagnetic material is given in the following table:

B (T)*	0	0·42	0·8	0·97	1·08	1·15
H (At/m)	0	100	200	300	400	500

The material is used for a transformer working from a 250 V 50 Hz supply. The supply waveform may be assumed sinusoidal and the resistance of the transformer primary negligible. The net iron cross-sectional area of the transformer is 0·004 m² and there are 250 turns on the primary. Deduce the waveform of the magnetizing current, neglecting hysteresis.

5.3 Using Wedmore's method find the third-harmonic component of the magnetizing current in the above example.

Ans. 15 per cent.

* See page 221.

5.4 A voltage wave is given by the following expression:

$$v = 30(\sin 314t + \tfrac{1}{3}\sin 942t + \tfrac{1}{5}\sin 1{,}570t) \text{ volts}$$

It is applied to a circuit consisting of a 32·7Ω resistor in series with a parallel combination of a 100 mH pure inductance and a 4·06 μF capacitor.

Calculate the total power delivered to the circuit and also the total r.m.s. current through the capacitance.

Ans. 6·95 W; 0·032 A.

5.5 A 3-phase 50 Hz alternator has a phase voltage

$$v = 100 \sin \omega t + 10 \sin 3\omega t + 5 \sin 5\omega t \text{ volts}$$

What are the line voltages if the alternator is (*a*) star connected, (*b*) mesh connected?

Three similar star-connected coils of 50Ω resistance and 0·1 H inductance are supplied from the alternator, which is star connected. Calculate the line current and the current through the neutral when the neutral is connected.

Ans. 122·6 V; 70·8 V; 1·2 A; 0·196 A.

5.6 A series circuit consists of a coil of inductance 0·1 H and resistance 25Ω and a variable capacitor. Across this circuit is applied a voltage whose instantaneous value is given by

$$v = 100 \sin \omega t + 20 \sin (3\omega t + 45°) + 5 \sin (5\omega t - 30°) \text{ volts}$$

where $\omega = 314$ rad/s.

Determine the value of C which will produce resonance at the third-harmonic frequency, and with this value of C find (*a*) an expression for the current in the circuit, (*b*) the r.m.s. value of this current, (*c*) the total power absorbed. (*H.N.C.*)

Ans. 11·25 μF: (*a*) $i = 0·398 \sin (\omega t + 84·3°) + 0·8 \sin (3\omega t + 45°) + 0·0485 \sin (5\omega t + 106°)$, (*b*) 0·633 A, (*c*) 10 W.

5.7 With the aid of clear diagrams, explain the anticipated waveform of:

(*a*) The current in a reactor with negligible resistance when a sinusoidal voltage, sufficient to saturate the reactor, is applied.
(*b*) The e.m.f. in a reactor with negligible resistance when a sinusoidal current, sufficient to saturate, flows through the reactor.
(*c*) The current when a sinusoidal voltage is applied to a rectifier with a forward resistance of 1,000Ω and a back resistance of 100,000Ω.

What general type of harmonic would you expect to be present or absent in each case?

In sections (*a*) and (*b*) a *B*/*H* curve, neglecting hysteresis, may be considered. (*H.N.C.*)

5.8 Derive an expression for the r.m.s. value of the complex voltage wave represented by the equation

$$v = V_0 + V_{1m} \sin (\omega t + \phi_1) + V_{3m} \sin (3\omega t + \phi_3)$$

A voltage $v = 200 \sin 314t + 50 \sin (942t + 45°)$ volts is applied to a circuit consisting of a resistance of 20Ω, an inductance of 20 mH and a capacitance of 56·3 μF all connected in series.

Calculate the r.m.s. values of the applied voltage and the current.

Find also the total power absorbed by the circuit. (*H.N.C.*)

Ans. 146 V; 3·16 A; 200 W.

5.9 The e.m.f. of one phase of a 3-phase mesh-connected alternator is represented by the following expression:

$$e = 500 \sin \theta + 60 \sin 3\theta - 40 \sin 5\theta \text{ volts}$$

The fundamental frequency is 50 Hz and each phase of the windings has a resistance of 3 Ω and an inductance of 0·01 H. Calculate the r.m.s. value of (*a*) the current circulating in the windings, and (*b*) the current through a 100 μF capacitor connected across a pair of line wires. (*L.U.*)

Ans. 4·28 A; 14·7 A.

(*Hint.* Do not neglect internal impedance.)

5.10 If the voltage applied to a circuit be represented by

$$V_1 \sin \omega t + V_n \sin n\omega t$$

and if the current is

$$I_1 \sin (\omega t - \phi_1) + I_n \sin (n\omega t - \phi_n)$$

derive an expression for the average power in the circuit.

A voltage represented by 250 sin ωt volts is applied to a circuit consisting of a non-inductive resistance of 30 Ω in series with an iron-cored inductance. The corresponding current is represented approximately by

$$3 \sin \left(\omega t - \frac{\pi}{3}\right) + 1{\cdot}2 \sin \left(3\omega t - \frac{\pi}{2}\right) \text{ amperes}$$

Calculate (*a*) the power absorbed by the resistance, (*b*) the effective value of the voltage across the inductance, and (*c*) the power factor of the whole circuit.

Draw to scale the waveform of the fundamental and third-harmonic currents showing their phase relation.

Explain why accurately calibrated rectifier and dynamometer [electrodynamic] ammeters would read differently when placed in the above circuit. (*L.U.*)

Ans. 156 W; 157 V; 0·47.

5.11 A p.d. of the form $v = 400 \sin \omega t + 30 \sin 3\omega t$ volts is applied to a rectifier having a resistance of 50 Ω in one direction and 200 Ω in the reverse direction. Find the average and effective values of the current and the p.f. of the circuit.

(*L.U.*)

Ans. 1·96 A; 4·1 A; 0·51.

5.12 The following table gives the characteristics of each of the four elements of a copper-oxide bridge rectifier.

Voltage (V)	0·1	0·15	0·2	0·24	0·28	0·34	0·38
Current (mA)	0·2	0·4	1	2	4	8	12

This bridge is connected directly across a supply voltage represented by $(1{\cdot}0 \sin \theta + 0{\cdot}1 \sin 3\theta)$ volts, and a milliammeter having a resistance of 20 Ω is connected across appropriate points of the bridge. Determine the reading on the milliammeter, assuming the reverse current to be negligible. (*L.U.*)

Ans. 5·4 mA.

Chapter 6

TRANSIENTS

The steady direct current which flows in a circuit connected to a battery or a d.c. generator may easily be calculated. Similarly the alternating current, which flows in a circuit connected to an alternator, may be calculated by the methods previously discussed. These are called *steady-state* currents, for it is assumed that the components in the circuits are unvarying and have been previously connected to the generator for so long that any peculiar disturbance, associated with the initial connexion or switching on of the apparatus, has had time to resolve itself.

In most cases the connexion and disconnexion of apparatus causes a disturbance which dies out in a short time, i.e. a *transient* disturbance. In this chapter the effect of suddenly switching on and off various circuits will be considered. In each case the resultant current is assumed to be the steady-state or normal current, with a transient or disturbance current superimposed.

The transient currents are found to be associated with the changes in stored energy in inductors and capacitors. Since there is no stored energy in a resistor there will be no transient currents in a pure-resistance circuit, i.e. the steady-state direct or alternating current will be attained immediately when the supply is connected.

6.1 Inductive Circuits

At any instant in a series circuit containing resistance R and inductance L, the applied voltage v is equal to the sum of the voltage drops across the resistance and the inductance, i.e.

$$v = iR + L\frac{di}{dt} \tag{6.1}$$

where i is the instantaneous value of the current.

The circuit current i is composed, as has already been noted, of two parts, i_s, the steady-state current, and i_t the transient current, i.e.

$$i = i_s + i_t$$

When the transient current has ceased, the steady-state current must still satisfy eqn. (6.1); therefore

$$v = i_sR + L\frac{di_s}{dt} \tag{6.2}$$

During the period in which the transient exists,

$$v = (i_s + i_t)R + L\frac{d}{dt}(i_s + i_t)$$

$$= i_sR + L\frac{di_s}{dt} + i_tR + L\frac{di_t}{dt} \tag{6.3}$$

whence

$$i_tR + L\frac{di_t}{dt} = 0 \tag{6.4}$$

The current i_s is obtained mathematically by solving eqn. (6.2). Since this is done implicitly when normal circuit theory is applied, the formal mathematical solution need not be given here. It therefore remains only to solve eqn. (6.4) for i_t. Thus, rearranging the terms,

$$\frac{di_t}{i_t} = -\frac{R}{L}dt$$

so that

$$\int\frac{di_t}{i_t} = -\frac{R}{L}\int dt$$

and

$$\log_e i_t + \log_e A = -\frac{R}{L}t$$

where $\log_e A$ is the constant of integration. Continuing,

$$\log_e Ai_t = -\frac{R}{L}t$$

so that

$$Ai_t = e^{-(R/L)t}$$

and

$$i_t = Be^{-(R/L)t} \tag{6.5}$$

where $B = 1/A$ = constant.

The complete solution is then

$$i = i_s + Be^{-(R/L)t} \tag{6.6}$$

The current i_s is found from simple circuit theory, and the constant B is then determined by substituting a known set of values for i and t in eqn. (6.6). These known values are normally the initial conditions in the circuit.

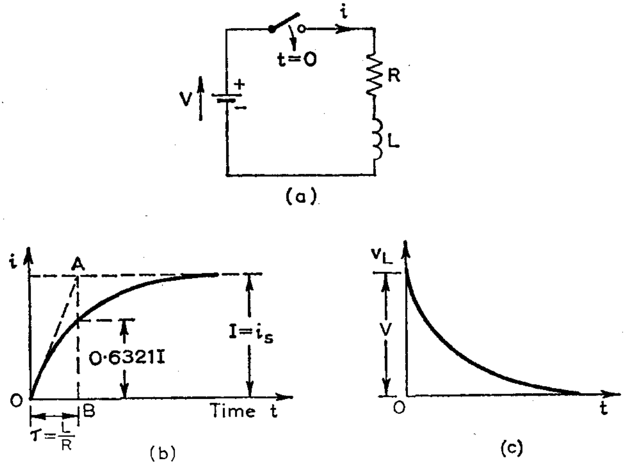


Fig. 6.1 GROWTH OF CURRENT IN AN INDUCTIVE D.C. CIRCUIT

6.2 Growth of Current in an Inductive Circuit (D.C.)

Suppose the switch in the circuit shown in Fig. 6.1(*a*) is closed at a datum time, taken as $t = 0$. Before the switch is closed the current is obviously zero. At the moment when the switch is closed the current will remain at an instantaneous value of zero since the current through an inductor cannot immediately change (Lenz's law). Thus in the present problem at $t = 0$, $i = 0$.

Also, from d.c. theory, $i_s = V/R = I$. Substituting in eqn. (6.6) at the instant $t = 0$,

$$0 = \frac{V}{R} + Be^0 \qquad \text{whence} \qquad B = -\frac{V}{R}$$

Again from eqn. (6.6),

$$i = \frac{V}{R} - \frac{V}{R}e^{-(R/L)t}$$

$$= \frac{V}{R}(1 - e^{-(R/L)t})$$

$$= I(1 - e^{-(R/L)t}) \tag{6.7}$$

The curve of i plotted to a base of time is shown in Fig. 6.1(*b*). It is called an *exponential-growth curve.*

The rate of change of current is found by differentiating eqn. (6.7). Thus

$$\frac{di}{dt} = \frac{V}{R}\left(\frac{R}{L}e^{-(R/L)t}\right) = \frac{V}{L}e^{-(R/L)t}$$

The initial rate of change of current is then

$$\left.\frac{di}{dt}\right|_{t=0} = \frac{V}{L} \text{ amperes/second} \tag{6.8}$$

Consider now the value of the current when $t = L/R$ seconds. Then from eqn. (6.7),

$$i = I(1 - e^{-1}) = 0{\cdot}6321I$$

L/R is called the *time constant*, τ (tau), of the RL circuit. It may easily be verified that this would be the time required for the current to reach its final value if the initial rate of increase were continued (i.e. point A on Fig. 6.1(*b*)). The time constant is defined as the time required for the current to reach 63·21 per cent (i.e. approximately $\frac{2}{3}$) of its final value.

The voltage across the resistor R is easily obtained as

$$v_R = iR = I(1 - e^{-(R/L)t})R = V(1 - e^{-(R/L)t}) \tag{6.9}$$

The voltage across the inductor at any instant is then,

$$v_L = V - v_R = Ve^{-(R/L)t} \tag{6.10}$$

This is an exponential decay curve, and is shown in Fig. 6.1(*c*).

It should be noted that the general form for the curve of exponential growth is $y = Y(1 - e^{-t/\tau})$ where τ is the time constant. The corresponding expression for an exponential decay curve is $y = Ye^{-t/\tau}$.

The energy relations in the circuit may be derived as follows. The energy supplied by the battery in time dt is $Vidt$ joules. Hence the total energy supplied in t seconds is $\int_0^t Vidt$. This energy is partly dissipated as heat in the resistor R, and is partly stored in the magnetic field of the coil. If the coil current at a given instant is i amperes, then the stored energy at the same instant is $\frac{1}{2}Li^2$ joules. The energy dissipated in the resistor R up to an instant t is $\int_0^t i^2Rdt$, this being easily calculated as the total energy supplied minus the energy stored in the magnetic field.

6.3 Decay of Current in an Inductive Circuit

Consider the circuit shown in Fig. 6.2(a). At the datum time $t = 0$, the switch is opened, disconnecting the inductor L and

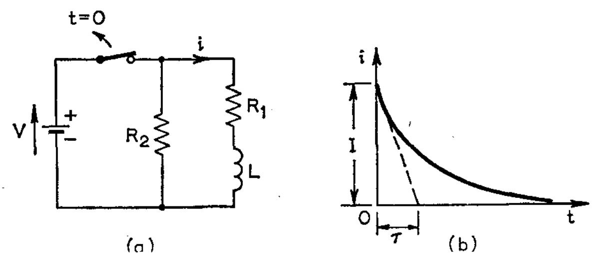


Fig. 6.2 DECAY OF CURRENT IN AN INDUCTIVE D.C. CIRCUIT

resistor R_1 from the supply. Since there is no continuous source of e.m.f. in the circuit formed by L, R_1 and R_2, the steady-state current will be zero. Hence eqn. (6.6) becomes,

$$i = Be^{-(R/L)t} \tag{6.11}$$

where R = total circuit resistance $= R_1 + R_2$.

As before, at the instant of switching the current through the inductor remains momentarily unchanged, i.e. it has the same value (V/R_1 amperes) as it had before the switch was operated. Hence at $t = 0$, $i = V/R_1 = I$. Therefore from eqn. (6.11),

$$\frac{V}{R_1} = B$$

Substituting in eqn. (6.11),

$$i = \frac{V}{R_1} e^{-(R/L)t} = Ie^{(-R/L)t} \tag{6.12}$$

This is the exponential decay curve shown in Fig. 6.2(*b*). The time constant for this circuit is L/R, i.e. $L/(R_1 + R_2)$ seconds. The total energy available is $\frac{1}{2}LI^2$ joules; all of this energy is eventually dissipated as heat in the resistances of the circuit.

It should be noted that if R_2 is omitted the energy stored in the magnetic field will cause a spark at the switch contacts, or will destroy the insulation of the coil (owing to the large induced e.m.f.).

EXAMPLE 6.1 A coil of 10H inductance, and 5Ω resistance is connected in parallel with a 20Ω resistor across a 100V d.c. supply which is suddenly disconnected. Find:

(*a*) The initial rate of change of current after switching.
(*b*) The voltage across the 20Ω resistor initially, and after 0·3s.
(*c*) The voltage across the switch contacts at the instant of separation.
(*d*) The rate at which the coil is losing stored energy 0·3s after switching.

(*H.N.C.*)

(*a*) The steady-state current is zero; hence

$$i = Be^{-(R/L)t}$$

where R is the total circuit resistance after switching (25Ω).

At $t = 0$, the current is 100/5 = 20A, i.e. the current through the coil immediately prior to the opening of the switch is 20A. Thus

$$20 = Be^0 = B$$

whence

$$i = 20e^{-2 \cdot 5t}$$

$$\text{Initial rate of change of current} = \left.\frac{di}{dt}\right|_{t=0} = -20 \times 2{\cdot}5e^{-2\cdot5t}\Big|_{t=0}$$
$$= \underline{\underline{-50\,\text{A/s}}}$$

The negative sign indicates that the current is decreasing.

(*b*) The current through the 20Ω resistor after the supply has been disconnected is i ampere.

$$\text{Initial voltage across 20}\Omega\text{ resistor} = (\text{current at } t = 0) \times 20$$
$$= 20 \times 20 = \underline{\underline{400\,\text{V}}}$$

$$\text{Current after 0·3 sec} = 20e^{-0\cdot75} = 9{\cdot}44\,\text{A}$$

Therefore

$$\text{Voltage across 20}\,\Omega\text{ resistor after 0·3 s} = 9{\cdot}44 \times 20 = \underline{\underline{188\,\text{V}}}$$

(*c*) Since the e.m.f. induced in the inductor tends to maintain the current through it in the original direction, the direction of the current through R_2 (20Ω) will be upwards. The voltage across the switch contacts will therefore be the supply voltage plus the voltage across R_2. Therefore

Initial voltage across contacts $= 100 + 400 = \underline{\underline{500\,\text{V}}}$

(*d*) Rate at which coil loses stored energy = Power
= Coil e.m.f. × Current

At 0·3 s, $i = 9{\cdot}44$ A.
The rate of change of current at $t = 0{\cdot}3$ is

$$\left.\frac{di}{dt}\right|_{t=0\cdot3} = -20 \times 2{\cdot}5 \times e^{-0\cdot75} = -23{\cdot}6\,\text{A/s}$$

The rate at which the coil loses stored energy is

$$L\frac{di}{dt} \times i = -10 \times 23{\cdot}6 \times 9{\cdot}44 = \underline{\underline{-2{,}230\,\text{J/s}}}$$

The negative sign indicates a decrease in stored energy.

6.4 Growth of Current in an Inductive Circuit (A.C.) (Fig. 6.3)

Let a voltage given by $v = V_m \sin(\omega t + \psi)$ be suddenly applied to an *RL* series inductive circuit at the instant $t = 0$, i.e. the voltage

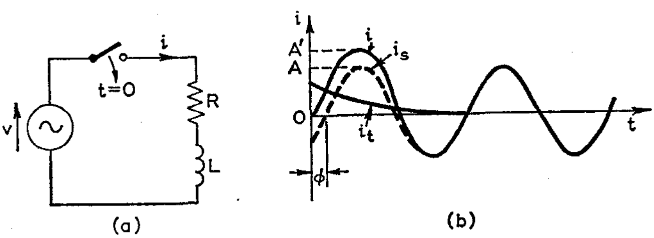


Fig. 6.3 A.C. SWITCHING TRANSIENTS

is suddenly applied when it is passing through the value $V_m \sin \psi$. Since the contacts may close at any instant in the cycle, ψ may have any value from zero to 2π radians. The voltage equation is then

$$V_m \sin(\omega t + \psi) = iR + L\frac{di}{dt} \tag{6.13}$$

As has already been explained, the steady state-current i_s is easily found by normal circuit theory. If the circuit impedance is $\sqrt{(R^2 + \omega^2L^2)}$, the peak steady-state current is

$$I_m = \frac{V_m}{\sqrt{(R^2 + \omega^2L^2)}}$$

This current lags behind the applied voltage by ϕ radians, where

$\phi = \tan^{-1} \omega L/R$. The expression for the instantaneous value of the steady-state current is therefore

$$i_s = \frac{V_m}{\sqrt{(R^2 + \omega^2 L^2)}} \sin \left(\omega t + \psi - \tan^{-1} \frac{\omega L}{R}\right)$$
$$= I_m \sin (\omega t + \psi - \phi) \tag{6.14}$$

The transient current has already been derived from eqn. (6.4) as

$$i_t = Be^{-(R/L)t}$$

Hence

$$i = i_s + i_t = I_m \sin (\omega t + \psi - \phi) + Be^{-(R/L)t} \tag{6.15}$$

As before, the condition that the current though an inductor is instantaneously the same before and after switching is used to evaluate the constant B. In this case at $t = 0$, $i = 0$. Substituting in eqn. (6.15),

$$0 = I_m \sin (\psi - \phi) + B$$

whence

$$B = -I_m \sin (\psi - \phi)$$

Therefore

$$i = I_m \sin (\omega t + \psi - \phi) - I_m \sin (\psi - \phi)e^{-(R/L)t} \tag{6.15a}$$

From this it will be seen that the value of B, and hence the size of the switching transient, depends on the value of ψ, i.e. on the instant in the cycle at which the contacts close. Three cases will be investigated.

Case 1. At $t = 0$, the voltage is passing through zero and is positive going, i.e. $\psi = 0$.

$$i = I_m \sin (\omega t - \phi) - I_m \sin (-\phi) e^{-(R/L)t}$$
$$= I_m (\sin (\omega t - \phi) + \sin \phi e^{-(R/L)t})$$

The curve of i to a base of ωt is shown in Fig. 6.3(b). This shows that the maximum instantaneous peak current (OA′) may be larger than the normal peak current (OA).

Case 2. At $t = 0$, the voltage is passing through $V_m \sin \phi$, i.e. $\psi = \phi$, and $(\psi - \phi) = 0$.

In this case, $B = 0$, and there is no switching transient ($i_t = 0$). This corresponds to the contacts closing at the instant when the steady-state current will itself be zero.

Case 3. At $t = 0$, the voltage is passing through $V_m \sin \left(\phi \pm \frac{\pi}{2}\right)$, i.e.

$$\psi = \phi \pm \frac{\pi}{2} \qquad \text{and} \qquad (\psi - \phi) = \pm \frac{\pi}{2} \tag{6.16}$$

The transient term in this case is

$$i_t = -I_m \sin\left(\pm \frac{\pi}{2}\right) e^{-(R/L)t}$$

$$= \pm I_m e^{-(R/L)t}$$

i.e. the transient now has its maximum possible initial value.

EXAMPLE 6.2 A 50 Hz alternating voltage of peak value 300 V is suddenly applied to a circuit which has a resistance of 0·1 Ω and an inductance of 3·18 mH. Determine the first peak value of the resultant current when the transient term has a maximum value.

Inductive reactance, $X_L = 2\pi fL = 2\pi \times 50 \times 0{\cdot}00318 = 1\,\Omega$

Therefore

Circuit impedance $= 0{\cdot}1 + j1 \approx 1\underline{/84{\cdot}3^\circ}$

whence

Peak steady-state current $= 300/1 = 300\,\text{A}$

If $v = 300 \sin(\omega t + \psi)$, then the maximum transient will occur when $\psi = \phi \pm \pi/2$, where ϕ is the phase angle of the current with respect to the voltage (i.e. 84·3°). Therefore

$\psi = 84{\cdot}3 \pm 90 = -5{\cdot}7^\circ$ (choosing the negative value)

Hence

$$i = 300 \sin(\omega t - 90^\circ) + Be^{-31{\cdot}4t} \tag{6.15}$$

At $t = 0$, $i = 0$. Therefore

$$0 = 300 \sin(-90^\circ) + B$$

so that $B = 300$, and

$$i = 300 \sin(\omega t - 90^\circ) + 300e^{-31{\cdot}4t}$$

To obtain an exact solution for the first peak of the current, the above expression must be differentiated, equated to zero, and the resulting expression solved graphically for t. It is usually sufficiently accurate to determine the instant at which the steady-state term reaches its first maximum positive value, and to add the value of the transient term at this instant to the peak value of the steady-state term. Thus the first maximum positive value of the steady-state term occurs when

$$(\omega t - 90^\circ) = 90^\circ = \pi/2\,\text{rad}$$

i.e. when $t = 0{\cdot}01$ sec. At this time $i_t = 300e^{-0{\cdot}314} = 219\,\text{A}$. Therefore

Resultant current at this instant $= 300 + 219 = \underline{\underline{519\,\text{A}}}$

6.5 Capacitive Circuits

The voltage equation for a circuit consisting of a capacitor C in series with a resistor R is

$$v = iR + \frac{q}{C} \tag{6.17}$$

where q is the instantaneous charge on the capacitor.

As for the inductive circuit the current i is expressed as the sum of the steady-state current i_s, and the transient current i_t. The transient current is the solution of the equation

$$i_t R + \frac{q_t}{C} = 0$$

Differentiating,

$$R\frac{di_t}{dt} + \frac{1}{C}\frac{dq_t}{dt} = 0$$

i.e.

$$\frac{1}{C}i_t + R\frac{di_t}{dt} = 0$$

since $i = dq/dt$.

This equation has the same form as eqn. (6.4) for the case of the inductive circuit, and the solution for i_t will follow exactly that previously derived, if $1/C$ is substituted for R in eqn. (6.4) and R is substituted for L. Hence

$$i_t = Be^{-(1/CR)t} \tag{6.18}$$

The complete solution for i is therefore

$$i = i_s + Be^{-(1/CR)t} \tag{6.19}$$

where i_s is found from normal circuit theory, and B is a constant obtained by substituting known values in eqn. (6.19).

The initial condition, which is used to determine the constant B in eqn. (6.19), is that the charge on a capacitor cannot instantaneously change since an instantaneous change of charge would require an infinite current and hence an infinite rate of change of voltage. In effect, since the capacitance is constant, this means that the voltage across a capacitor is momentarily the same before and after any sudden change in the circuit conditions.

6.6 Charging of a Capacitor through a Resistor

Consider a circuit consisting of a resistor R in series with a capacitor C, connected to a battery of voltage V, at time $t = 0$ (Fig. 6.4(a)).

The steady-state current in this case is obviously zero, since no current will flow from a d.c. supply through a capacitor. Hence $i_s = 0$, and eqn. (6.19) becomes

$$i = Be^{-(1/CR)t} \tag{6.20}$$

When the switch is closed there will be momentarily no voltage across the capacitor, so that the battery voltage V must all appear across the resistor R. Hence the initial current from the battery must be $i = V/R$ amperes, i.e. at $t = 0$, $i = V/R = I$, say. Therefore

$$i = \frac{V}{R}e^{-(1/CR)t} = Ie^{-(1/CR)t} \tag{6.21}$$

This equation represents the exponential decay curve drawn in Fig. 6.4(*b*).

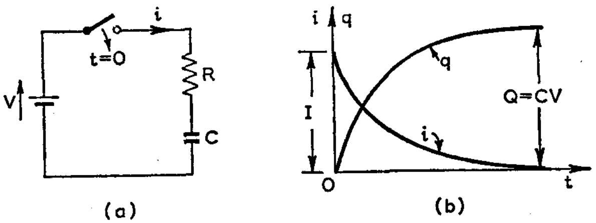


Fig. 6.4 CHARGING A CAPACITOR THROUGH A RESISTOR

The voltage v_R across the resistor at any instant is

$$v_R = iR = IRe^{-(1/CR)t} = Ve^{-(1/CR)t} \tag{6.22}$$

From this, the voltage v_c across the capacitor is

$$v_c = V - v_R = V(1 - e^{-(1/CR)t}) \tag{6.23}$$

This equation represents an exponential growth to a final value of V volts. From it, the charge q on the capacitor may be found at any instant. Thus

$$q = v_cC = VC(1 - e^{-(1/CR)t}) = Q(1 - e^{-(1/CR)t}) \tag{6.24}$$

where $Q = VC$ = final charge on the capacitor.

The time constant τ of the CR circuit is defined as the time required for the charge on the capacitor to attain 63·21 per cent of its final value, i.e. the index $(1/CR)t$ must be unity when $t = \tau$. Therefore

$$\tau = CR \quad \text{seconds} \tag{6.25}$$

Energy will only be supplied from the battery during the time required to charge the capacitor. Some of this energy will be dissipated as heat in the resistor, and some will be stored in the electric field of the capacitor.

Total energy from the battery in t seconds $= \int_0^t Vi\,dt$ joules

Energy stored in electric field in t seconds $= \frac{1}{2}Cv_c^2$

where v_c is the voltage across the capacitor after t seconds. The energy dissipated in the resistor will be the difference between the total energy taken from the battery and that stored in the capacitor.

6.7 Discharge of a Capacitor through a Resistor

Suppose that a capacitor which is originally charged to V_c volts is discharged through a resistor of R ohms (Fig. 6.5(a)). Since there is

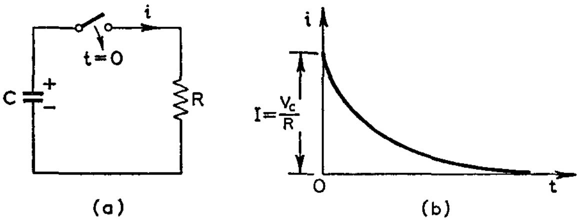


Fig. 6.5 DISCHARGING A CAPACITOR THROUGH A RESISTOR

no generator in the circuit the steady-state current must be zero, so that, from eqn. (6.19), the general equation for the circuit current is

$$i = B\mathrm{e}^{-(1/CR)t}$$

The initial condition is that the voltage across the capacitor must be the same after the switch is closed as it was before, i.e. V_c volts. Hence at $t = 0$ the voltage across R is V_c volts, and the current through it is $i = V_c/R = I$, say. The general equation for the current must therefore be

$$i = \frac{V_c}{R}\mathrm{e}^{-(1/CR)t} = I\mathrm{e}^{-(1/CR)t} \tag{6.26}$$

This is the exponential decay curve shown in Fig. 6.5(b).

The voltage at any instant is the same across both the capacitor and the resistor. Let this voltage be v. Then

$$v = iR = V_c\mathrm{e}^{-(1/CR)t} \tag{6.27}$$

The charge on the capacitor at any instant is

$$q = Cv$$
$$= CV_c e^{-(1/CR)t}$$
$$= Q e^{-(1/CR)t} \text{ coulombs} \qquad (6.28)$$

where Q ($= CV_c$) is the initial charge on the capacitor.

EXAMPLE 6.3 A simple sawtooth voltage generator consists of a 10,000 Ω resistor in series with a 0·25 μF capacitor across a 200 V d.c. supply. A gas discharge tube with a striking voltage of 120 V is connected across the capacitor. Determine (*a*) the frequency of the oscillation, and (*b*) the average power required from the d.c. supply.

Also sketch three cycles of the output voltage.

The circuit is shown in Fig. 6.6(*a*). The operation of the circuit is briefly as follows. The capacitor charges up exponentially through the resistor until the

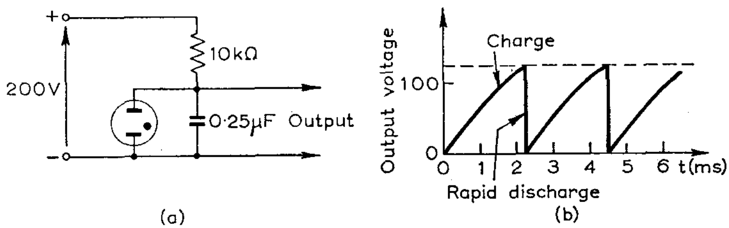


Fig. 6.6

voltage across it is 120 V. Up to this point the gas discharge tube is inoperative, but at this voltage the gas becomes conducting, and the capacitor discharges quickly through it. The voltage across the capacitor falls rapidly to zero, and the discharge tube again becomes non-conducting, so allowing the capacitor to charge up once more, and so on. The voltage across the capacitor is thus the saw-tooth wave shown in Fig. 6.6(*b*), the discharge time being so small that it may be taken as instantaneous.

(*a*) In order to determine the repetition frequency, the time taken to charge the capacitor to 120 V must be found.

$$V_c = 120 = 200(1 - e^{-(100/0\cdot25)t})$$

$$200\, e^{-400t} = 80$$

$$e^{400t} = \frac{200}{80} = 2\cdot5$$

$$400t = \log_e 2\cdot5 = 0\cdot916$$

Therefore $t = 0\cdot00229$ s, so that the repetition frequency f is

$$f = \frac{1}{t} = 436\,\text{Hz}$$

(*b*) The energy taken from the d.c. supply per cycle is $\int_0^{0 \cdot 00229} Vi\,dt$, where i is the instantaneous current in the circuit. This current is given by

$$i = \frac{V}{R} e^{-(1/CR)t} = \frac{200}{10{,}000} e^{-400t} = 0 \cdot 02e^{-400t}$$

Thus

$$\text{Energy per cycle} = \int_0^{0 \cdot 00229} 200 \times 0 \cdot 02e^{-400t}\,dt$$

$$= 4\left[-\frac{1}{400} e^{-400t}\right]_0^{0 \cdot 00229}$$

$$= 0 \cdot 006\,\text{J}$$

Hence the average power taken from the supply is $0 \cdot 006/0 \cdot 00229$ J/s, or $\underline{\underline{2 \cdot 62\,\text{W}}}$.

6.8 Transients in Capacitive A.C. Circuits

When an alternating voltage is applied to a capacitive circuit the resultant current may be determined by a method similar to that employed in Section 6.4 for inductive a.c. transients. An expression for the instantaneous value of the steady-state term is found from normal circuit theory. This is substituted in eqn. (6.19), and the constant of integration is then found from the known initial conditions.

6.9 Thermal Transients

In electrical apparatus the losses cause a rise in temperature with a final value determined by the magnitude of the losses and the rate of cooling. The rate of cooling approximately obeys *Newton's law of cooling*, which states that the rate of loss of heat from a body is proportional to the temperature rise, θ, of the body above the ambient temperature (i.e. the temperature of the surroundings). If the loss power is P watts, then in dt seconds the energy supplied as heat is $P\,dt$ joules. Suppose that in this time the temperature of the body rises from a value θ above ambient to $(\theta + d\theta)$, and that the heat stored in the body per deg C rise in temperature is H joules. Then

$$\begin{matrix}\text{Energy supplied as heat} \\ \text{in } dt \text{ seconds}\end{matrix} = \begin{bmatrix}\text{Heat stored in} \\ dt \text{ seconds}\end{bmatrix} + \begin{bmatrix}\text{Heat lost in} \\ dt \text{ seconds}\end{bmatrix}$$

or

$$P\,dt = H\,d\theta + K\theta\,dt$$

since $K\theta$ is, by Newton's law, the rate of heat loss, where K is a constant which depends on the convection, conduction and radiation heat loss. Hence

$$P = H\frac{d\theta}{dt} + K\theta \tag{6.29}$$

Comparing this with eqn. (6.1), it will be seen that the solution consists of a steady-state term, θ_s, and a transient term, θ_t, so that, solving for θ as in section 6.1,

$$\theta = \theta_s + B\mathrm{e}^{-(K/H)t} = \theta_s + B\mathrm{e}^{-t/\tau} \tag{6.30}$$

where B is a constant which is determined from known conditions, and τ $(= H/K)$ is the thermal time-constant.

HEATING CURVE

Consider a constant loss power, P, existing in an electrical apparatus. When a steady temperature has been reached (i.e. when the rate of heat loss is just equal to the rate of heat supplied), the value of $d\theta/dt$ will be zero, so that, from eqn. (6.29), the steady temperature rise, θ_s, attained will be

$$\theta_s = \frac{P}{K} \text{ kelvins or degrees Celsius above ambient}$$

At the instant when the apparatus is switched on its temperature will be simply the ambient temperature. This can be expressed mathematically by writing

$$\text{at } t = 0,\ \theta = 0$$

Substituting this in eqn. (6.30) gives a particular solution from which B can be evaluated. Thus

$$0 = \theta_s + B\mathrm{e}^0 \qquad \text{or} \qquad B = -\theta_s$$

The complete solution for θ is obtained by substituting this value of B back in eqn. (6.30) to give

$$\theta = \theta_s(1 - \mathrm{e}^{-t/\tau}) \tag{6.31}$$

or

$$\theta = \frac{P}{K}(1 - \mathrm{e}^{-(K/H)t}) \tag{6.31a}$$

This is an exponential growth curve, as shown in Fig. 6.7(a).

It should be noted that, in order to reduce θ_s, the loss power P must be reduced or the cooling constant K must be increased.

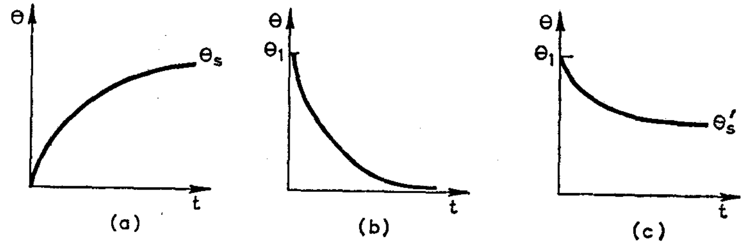


Fig. 6.7 THERMAL TRANSIENTS

COOLING CURVE

If a piece of apparatus which has attained a temperature θ_1 is allowed to cool, by switching off the supply, then $P = 0$ and the final temperature rise is zero (i.e. $\theta_s = 0$). Also at $t = 0$, $\theta = \theta_1$ and for this value of t, eqn. (6.30) becomes

$$\theta_1 = Be^0 \qquad \text{or} \qquad B = \theta_1$$

Hence the general expression for θ is

$$\theta = \theta_1 e^{-t/\tau} \tag{6.32}$$

This is the equation to the cooling curve of Fig. 6.7(*b*).

Suppose, however, that when the apparatus has attained some temperature θ_1, the loss power is reduced to some lower value P'. As before, the steady-state temperature rise is obtained from eqn. (6.29) by putting $d\theta/dt = 0$, so that now

$$\theta'_s = \frac{P'}{K}$$

Also at $t = 0$ the temperature rise is θ_1, and substituting this condition in eqn. (6.30) gives

$$\theta_1 = \theta'_s + Be^0 \qquad \text{or} \qquad B = (\theta_1 - \theta'_s)$$

so that the general solution is

$$\theta = \theta'_s + (\theta_1 - \theta'_s)e^{-t/\tau} \tag{6.33}$$

as is shown in Fig. 6.7(*c*)

6.10 Double-energy Transients

Circuits containing both inductance and capacitance involve both electromagnetic and electrostatic stored energies, and hence any sudden change in the circuit conditions will involve the redistribution of two forms of stored energy. The transient currents resulting from this redistribution are called double-energy transients.

Consider the general series circuit of resistance R, inductance L and capacitance C. If v_R, v_L, v_C are the instantaneous voltages across R, L and C respectively, then the supply voltage, v, is given by

$$\begin{aligned} v &= v_R + v_L + v_C \\ &= iR + L\frac{di}{dt} + \frac{q}{C} \end{aligned} \tag{6.34}$$

where i is the instantaneous circuit current and q is the instantaneous charge on the capacitor. As before, the complete solution of this equation will have two parts—the steady-state current i_s, and the transient current i_t. The steady-state current may readily be obtained from normal circuit theory, while the transient current is the solution of the equation

$$i_tR + L\frac{di_t}{dt} + \frac{q_t}{C} = 0 \tag{6.35}$$

Differentiating,

$$L\frac{d^2i_t}{dt^2} + R\frac{di_t}{dt} + \frac{i_t}{C} = 0$$

since $dq_t/dt = i_t$. On rearranging this equation it becomes

$$\frac{d^2i_t}{dt^2} + \frac{R}{L}\frac{di_t}{dt} + \frac{i_t}{LC} = 0 \tag{6.36}$$

The solution of this equation for i_t may be obtained in several ways. An operational method will be used here, the operator p standing for d/dt and the operator p^2 standing for d^2/dt^2, i.e.

$$pi_t \equiv \frac{di_t}{dt} \qquad \text{and} \qquad p^2i_t \equiv \frac{d^2i_t}{dt^2}$$

Use will be made of the following operational relationships which may be verified by differentiation.

$$(a) \qquad (p - m)Ce^{mt} = \frac{d}{dt}(Ce^{mt}) - mCe^{mt} = 0 \tag{6.37}$$

$$\begin{aligned} (b) \qquad (p - m)^2(B + Ct)e^{mt} &= \frac{d^2}{dt^2}(B + Ct)e^{mt} \\ &\quad - 2m\frac{d}{dt}(B + Ct)e^{mt} + m^2(B + Ct)e^{mt} = 0 \end{aligned} \tag{6.38}$$

(c) $$(p - m_1)(p - m_2)(Be^{m_1t} + Ce^{m_2t})$$

$$= \frac{d^2}{dt^2}(Be^{m_1t} + Ce^{m_2t}) - (m_1 + m_2)\frac{d}{dt}(Be^{m_1t} + Ce^{m_2t})$$

$$+ m_1m_2(Be^{m_1t} + Ce^{m_2t}) = 0 \quad (6.39)$$

where B and C are constants.

In the operational notation eqn. (6.36) becomes,

$$p^2i_t + \frac{R}{L}pi_t + \frac{1}{LC}i_t = 0$$

i.e.

$$\left(p^2 + \frac{R}{L}p + \frac{1}{LC}\right)i_t = 0$$

The expression in brackets may be factorized to $(p - m_1)(p - m_2)$, where

$$m_1 = \frac{-R/L + \sqrt{(R^2/L_2 - 4/LC)}}{2} = \frac{-R}{2L} + \sqrt{\left(\frac{R^2}{4L^2} - \frac{1}{LC}\right)} \quad (6.40)$$

and

$$m_2 = \frac{-R/L - \sqrt{(R^2/L^2 - 4/LC)}}{2} = \frac{-R}{2L} - \sqrt{\left(\frac{R^2}{4L^2} - \frac{1}{LC}\right)} \quad (6.41)$$

Hence

$$(p - m_1)(p - m_2)i_t = 0 \quad (6.42)$$

Comparing eqns. (6.42) and (6.39),

$$i_t = Be^{m_1t} + Ce^{m_2t} \quad (6.43)$$

excepting where $m_1 = m_2 = m$, say, when eqn. (6.42) becomes

$$(p - m)^2i_t = 0 \quad (6.44)$$

and hence, by comparison with eqn. (6.38),

$$i_t = (B + Ct)e^{mt} \quad (6.45)$$

According to the values of m_1 and m_2 four different conditions of the circuit are distinguishable.

CASE 1: LOSSLESS CIRCUIT: $R = 0$, i.e. UNDAMPED

In this case,

$$m_1 = \sqrt{\left(-\frac{1}{LC}\right)} = j\frac{1}{\sqrt{(LC)}} = j\omega' \quad \text{from eqn. (6.40)}$$

and

$$m_2 = -\sqrt{\left(-\frac{1}{LC}\right)} = -j\frac{1}{\sqrt{(LC)}} = -j\omega' \quad \text{from eqn. (6.41)}$$

Eqn. (6.43) gives the solution for i_t as

$$i_t = Be^{j\omega' t} + Ce^{-j\omega' t} \tag{6.46}$$

But

$$e^{j\omega' t} = \cos \omega' t + j \sin \omega' t$$

and

$$e^{-j\omega' t} = \cos \omega' t - j \sin \omega' t$$

Therefore

$$i_t = D \cos \omega' t + E \sin \omega' t \tag{6.47}$$

where

$$D = (B + C) \qquad \text{and} \qquad E = j(B - C)$$

Eqn. (6.47) may be still further reduced to

$$i_t = I_m \sin(\omega' t + \psi) \tag{6.47a}$$

where $I_m = \sqrt{(D^2 + E^2)}$ and $\psi = \tan^{-1} D/E$. Hence the transient current in this case is a sine wave of constant peak value, and of frequency $f' = \dfrac{1}{2\pi\sqrt{(LC)}}$ as shown in Fig. 6.8(*a*). It will be observed that the solution contains two constant terms, namely I_m and ψ, which must be determined in any particular case from a knowledge of *two* initial circuit conditions. These conditions are

(*a*) The initial current in the inductance.
(*b*) The initial voltage across the capacitance.

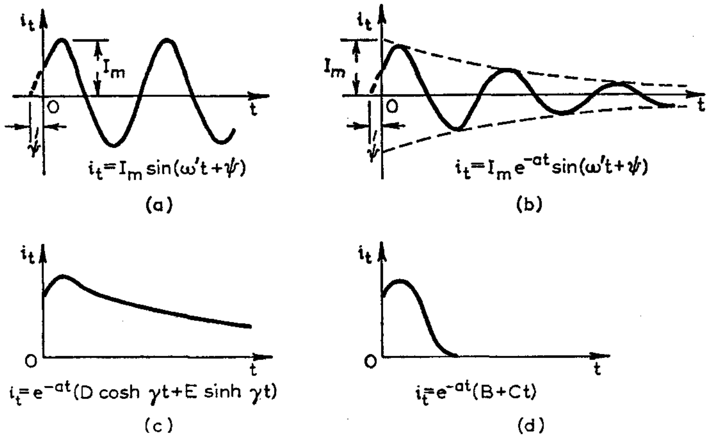


Fig. 6.8 TYPICAL DOUBLE-ENERGY TRANSIENT WAVEFORMS
(*a*) Undamped
(*b*) Underdamped
(*c*) Overdamped
(*d*) Critically damped

CASE 2: LOW-LOSS CIRCUIT: $R^2/4L^2 < 1/LC$, i.e. UNDERDAMPED

In this case, as in the theoretical Case 1, the term under the square root sign in eqns. (6.40) and (6.41) is negative, so that m_1 and m_2 will be conjugate complex numbers. Let

$$m_1 = -a + j\omega'$$

where

$$a = \frac{R}{2L} \qquad \text{and} \qquad \omega' = \sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}$$

Then

$$m_2 = -a - j\omega'$$

Substituting these values in eqn. (6.43),

$$i_t = Be^{(-a+j\omega')t} + Ce^{(-a-j\omega')t} = e^{-at}(Be^{j\omega' t} + Ce^{-j\omega' t})$$

This may be reduced to

$$i_t = I_m e^{-at} \sin(\omega' t + \psi) \tag{6.48}$$

where I_m and ψ are constants. This is the equation of a damped oscillation as shown in Fig. 6.8(*b*).

The factor e^{-at}, which accounts for the decay of the oscillation, is called the *damping factor*. The ratio between successive positive (or negative) peak values of the oscillation is $1 : e^{-a\tau}$, where τ is the period of the oscillation.

The frequency of the damped oscillation is

$$f' = \frac{1}{2\pi}\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}$$

and is called the *natural frequency* of the circuit. In a great many cases $\frac{R^2}{4L^2} \ll \frac{1}{LC}$, and with quite sufficient accuracy $f' = \frac{1}{2\pi\sqrt{(LC)}}$

CASE 3. HIGH-LOSS CIRCUIT: $R^2/4L^2 > 1/LC$: i.e. OVERDAMPED

If $R^2/4L^2$ is greater than $1/LC$, then the term under the square root sign in eqns. (6.40) and (6.41) will be positive, so that m_1 and m_2 will be pure numbers. Let

$$m_1 = -a + \gamma'$$

where

$$a = \frac{R}{2L} \quad \text{and} \quad \gamma' = \sqrt{\left(\frac{R^2}{4L^2} - \frac{1}{LC}\right)}$$

Then

$$m_2 = -a - \gamma'$$

Therefore

$$\begin{aligned} i_t &= Be^{(-a+\gamma')t} + Ce^{(-a-\gamma')t} \quad \text{(from eqn. (6.43))} \\ &= e^{-at}(Be^{\gamma' t} + Ce^{-\gamma' t}) \end{aligned}$$

But

$$e^{\gamma' t} = \sinh \gamma' t + \cosh \gamma' t$$

and

$$e^{-\gamma' t} = \cosh \gamma' t - \sinh \gamma' t$$

Therefore

$$\begin{aligned} i_t &= e^{-at}\{(B + C)\cosh \gamma' t + (B - C)\sinh \gamma' t\} \\ &= e^{-at}\{D \cosh \gamma' t + E \sinh \gamma' t\} \end{aligned} \tag{6.49}$$

A typical curve of this equation is shown in Fig. 6.8(*c*).

CASE 4. CRITICAL DAMPING: $R^2/4L^2 = 1/LC$

When $R^2/4L^2$ is equal to $1/LC$, m_1 and m_2 become equal, each having a value of $-R/2L$. Hence, from eqn. (6.45),

$$i_t = e^{-(R/2L)t}(B + Ct) \tag{6.50}$$

In this case i_t reduces to almost zero in the shortest possible time. A typical curve for i_t is shown in Fig. 6.8(*d*).

To summarize,

$$\text{Transient term is oscillatory if } R < 2\sqrt{\frac{L}{C}} \tag{6.51}$$

$$\text{Transient term is non-oscillatory if } R \geqslant 2\sqrt{\frac{L}{C}} \tag{6.52}$$

$$\text{Critical damping occurs when } R = 2\sqrt{\frac{L}{C}} \tag{6.53}$$

EXAMPLE 6.4 A 4μF capacitor is discharged suddenly through a coil of inductance 1 H and resistance 100Ω. If the initial voltage on the capacitor is 10V, derive an expression for the resulting current, and find the additional resistance required to give critical damping.

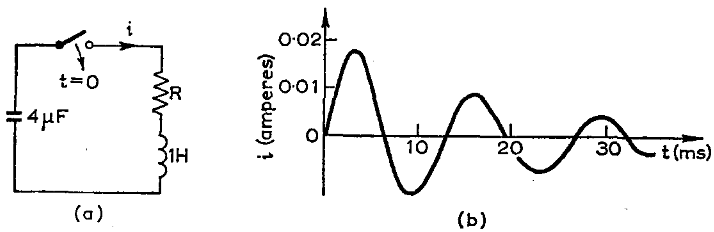


Fig. 6.9

The circuit is shown in Fig. 6.9(*a*). Since there is no generator in the circuit the steady-state current must be zero, so that the resultant current is simply the transient current.

The value of $2\sqrt{(L/C)}$ is 1,000, and hence from the inequality (6.51) the circuit is originally oscillatory. The transient current is therefore

$$i_t = I_m e^{-at} \sin(\omega' t + \psi)$$

where

$$a = \frac{R}{2L} = \frac{100}{2} = 50$$

and

$$\omega' = \sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)} = \sqrt{(250{,}000 - 2{,}500)} = 497\,\text{rad/s}$$

Therefore

$$i_t = I_m e^{-50t} \sin(497t + \psi) = i \tag{6.54}$$

The two known initial conditions are (*a*) at $t = 0$, $i = 0$, and (*b*) at $t = 0$, capacitor voltage, $v_C = 10$V. Applying condition (*a*) to equation (6.54) gives

$$0 = I_m \sin \psi$$

whence

$$\psi = 0$$

$$i = I_m e^{-50t} \sin 497t \tag{6.55}$$

At time $t = 0$, the voltage across the inductor must be 10 V (from condition (*b*)), since the current through the resistor is zero, i.e.

$$L\frac{di}{dt}\bigg|_{t=0} = 10\,\text{V}$$

Therefore

$$\frac{di}{dt}\bigg|_{t=0} = \frac{10}{L} = 10\,\text{A/s} \tag{6.56}$$

But from eqn. (6.55),

$$\frac{di}{dt} = -50I_m e^{-50t} \sin 497t + 497I_m e^{-50t} \cos 497t$$

At $t = 0$, this becomes

$$\frac{d}{dt}\bigg|_{t=0} = 497I_m = 10\,\text{A/s (from eqn. (6.56))}$$

Therefore

$$I_m = \frac{10}{497} = 0{\cdot}0201\,\text{A}$$

Hence the general expression for the current is

$$i = \underline{\underline{0{\cdot}0201 e^{-50t} \sin 497t \text{ amperes}}}$$

The first few cycles of this current are shown in Fig. 6.9(*b*).

From eqn. (6.53) the total resistance required for critical damping is

$$R = 2\sqrt{\frac{L}{C}} = 1{,}000\,\Omega$$

Therefore the additional resistance required is $\underline{\underline{900\,\Omega}}$.

EXAMPLE 6.5 A damped oscillation is given by the equation

$$i = 100e^{-10t} \sin (500t) \text{ amperes}$$

Determine the number of oscillations which will occur before the amplitude decays to $\frac{1}{10}$th of its undamped value.

The decay of the peak of the oscillations is given by the term $100e^{-10t}$. Thus

$$\tfrac{1}{10} \times 100 = 100e^{-10t_1}$$

where t_1 is the time required for the oscillation to die to $\frac{1}{10}$th of its undamped value.

$$e^{10t_1} = 10$$

$$10t_1 = \log_e 10 = 2{\cdot}303$$

$$t_1 = 0{\cdot}2303\,\text{s}$$

$$\text{Frequency of oscillation} = \frac{500}{2\pi}\,\text{Hz}$$

Therefore the number of oscillations before decay to $\frac{1}{10}$th amplitude is

$$n = 0{\cdot}2303 \times \frac{500}{2\pi} = \underline{\underline{18{\cdot}4}}$$

6.11 Energy Transformations

With the underdamped or oscillatory transients discussed in Section 6.10 it will be noticed that at some instants the current is zero. At these instants there will be no energy stored in the magnetic field However, at succeeding instants current does flow and there must be an associated magnetic energy. This energy must be stored as electrostatic energy during the instants when the current is zero, so that there is a continuous transformation of energy from an electrostatic to an electromagnetic form, and vice versa. Consider the circuit illustrated in Fig. 6.10. It is assumed for simplicity that

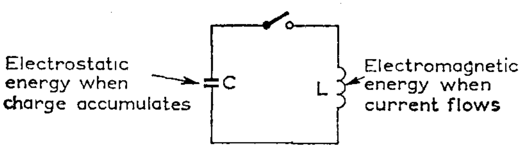


Fig. 6.10 ENERGY TRANSFORMATION

the circuit is lossless ($R = 0$). Initially the capacitor is charged to a potential V volts and the switch is closed at the instant $t = 0$. The steady-state current must be zero, and

$$i_t = I_m \sin(\omega' t + \psi) \quad \text{(from eqn. (6.47))}$$

At $t = 0$, $i = 0$ due to the inductor action; therefore $\psi = 0$. Hence

$$i = I_m \sin \omega' t \tag{6.57}$$

At $t = 0$, $i = 0$, magnetic energy $= 0$, electrostatic energy $= \frac{1}{2}CV^2$.

At $t = \pi/2\omega'$, $i = I_m$, magnetic energy $= \frac{1}{2}LI_m{}^2$, electrostatic energy $= 0$.

At $t = \pi/\omega'$, $i = 0$, magnetic energy $= 0$, electrostatic energy $= \frac{1}{2}CV^2$.

Since there is assumed to be no energy loss from this circuit and no energy supplied after $t = 0$, the peak stored magnetic energy must equal the peak stored electrostatic energy:

$$\tfrac{1}{2}CV^2 = \tfrac{1}{2}LI_m{}^2 \tag{6.58}$$

If the circuit is of the low-loss type rather than the imaginary lossless type, then at each interchange of energy there is a small loss of energy from the system. Many low-loss circuits may be regarded as obeying eqn. (6.58) for one energy cycle.

Example 6.6 If a break occurs at the point X in the circuit of Fig. 6.11, determine the voltage across the break. It may be assumed that, prior to the break, the circuit current had reached a steady-state value.

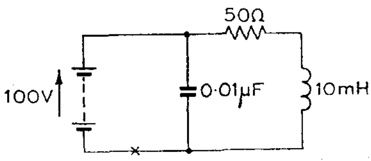


Fig. 6.11

$$\text{Before the break, inductor current} = 2\,\text{A}$$

$$\text{Energy initially stored in inductor} = \tfrac{1}{2}LI^2$$
$$= \tfrac{1}{2} \times 10^{-2} \times 4 = 2 \times 10^{-2}\,\text{J}$$

$$\text{Energy initially stored in capacitor} = \tfrac{1}{2}CV^2$$
$$= \tfrac{1}{2} \times 10^{-8} \times 10^4\,\text{J} \qquad \text{(negligible)}$$

After the break occurs the energy initially stored in the magnetic field of the inductor will be transferred to the capacitor. Neglecting energy loss in the first transfer,

$$\text{Maximum energy stored in capacitor} = 2 \times 10^{-2}\,\text{J}$$

Therefore

$$\tfrac{1}{2}CV_m{}^2 = 2 \times 10^{-2}$$

and

$$\text{Peak voltage across capacitor} = V_m = \sqrt{\left(\frac{2 \times 10^{-2} \times 2}{10^{-8}}\right)} = 2{,}000\,\text{V}$$

Therefore

$$\text{Maximum voltage across break} = 2{,}000 + 100 = 2{,}100\,\text{V}$$

The voltage will be oscillatory as the energy alternates between the inductor and the capacitor.

$$\text{Frequency of voltage oscillations} = f = \frac{1}{2\pi\sqrt{(LC)}}$$
$$= \frac{1}{2\pi\sqrt{(10^{-2} \times 10^{-8})}} = 15{,}900\,\text{Hz}$$

$$\text{Damping factor} = e^{-(R/2L)t} = e^{-(50/2\times10^{-2})t} = e^{-2{,}500t}$$

Therefore

$$\text{Voltage across break} = \underline{\underline{(2{,}000\,e^{-2{,}500t} \sin 10^5 t + 100)\,\text{volts}}}$$

PROBLEMS

6.1 The field circuit of an alternator has an effective inductance of 100 H and a resistance of 10 Ω. Calculate the time required to increase the excitation current from I to 99 per cent of $2I$ amperes if the supply voltage is doubled.

Ans. 39·1 s.

6.2 Deduce from first principles an expression for the current growth in an inductive circuit.

A 15 H inductance coil of 10 Ω resistance is suddenly connected to a 20 V d.c. supply. Calculate

(*a*) the initial rate of change of current,
(*b*) the current after 2 s,
(*c*) the rate of change of current after 2 s,
(*d*) the energy stored in the magnetic field in this time,
(*e*) the energy lost as heat in this time, and
(*f*) the time constant. (*H.N.C.*)

Ans. 1·33 A/s; 1·47 A; 0·352 A/s; 16·3 J; 19·5 J; 1·5 s.

6.3 Derive an expression for the value of the current in a circuit of resistance R ohms and inductance L henrys, t seconds after the sudden application of a constant voltage V to the circuit.

A constant voltage of 100 V is suddenly applied to a circuit of resistance 2 Ω and inductance 10 H. After 7·5 s the voltage is suddenly increased to 200 V. What will be the value of the current after a further 2 s?

Sketch the approximate shape of the current/time graph. (*H.N.C.*)

Ans. 59·0 A.

6.4 A coil of resistance r ohms and inductance L henrys is connected in parallel with an R-ohm resistor to a d.c. supply of V volts. After a "long period" the supply is suddenly disconnected; derive an expression for the current t seconds later.

6.5 Sketch the shape of the graph showing the growth of current in an inductive circuit when a steady voltage is applied.

Explain the desirability of a discharge resistance when such a circuit is switched off.

What must be the greatest permissible value of the suppressor resistance used in conjunction with a 500 V field circuit having a resistance of 50 Ω in order that the voltage across the terminals of the field winding shall not exceed 750 V when the circuit is opened?

(*C. & G. Inter.*)

Ans. 75 Ω.

6.6 A 12 μF capacitor is allowed to discharge through its own leakage resistance, and a fall of p.d. from 120 V to 100 V is recorded in 300 s by an electrostatic voltmeter. Calculate the leakage resistance of the capacitor. Prove any formula used. (*L.U. Part I*)

Ans. 137 MΩ.

6.7 A 2 μF capacitor is charged to 100 V. It is then discharged through a 1 MΩ resistor in parallel with a 1 μF capacitor. Find the voltage across the resistor 2 s after connexion, and also determine the energy dissipated up to this time.

Ans. 34·2 V, 0·00825 J.

6.8 A $1\,\mu$F capacitor is charged from a 2V, d.c. supply and is then discharged through a $10\,\text{M}\Omega$ resistor. After 5s of discharge the capacitor is connected across a ballistic galvanometer of negligible resistance, and causes a deflection of 1·2 divisions. Calculate

(*a*) the voltage across the capacitor after the 5s of slow discharge,
(*b*) the sensitivity of the galvanometer in microcoulombs per division, and
(*c*) the energy expended in heating the $10\,\text{M}\Omega$ resistor during the 5s of discharge. (*H.N.C.*)

Ans. (*a*) 1·21V; (*b*) $1\,\mu$C/div; (*c*) $1{\cdot}27 \times 10^{-6}$ J.

6.9 A single-phase 50Hz transformer fed from an "infinite" supply has an equivalent impedance of $(1 + j10)$ ohms referred to the secondary. The open-circuit secondary voltage is 200V. Find

(*a*) the steady-state secondary short-circuit current,
(*b*) the transient secondary short-circuit current assuming that the short-circuit occurs at the instant when the voltage is passing through zero going positive.
(*c*) the total short-circuit current under the same conditions.

Plot these curves to a base of time for the period of three cycles from the instant when the short-circuit occurs. Neglect saturation. (*H.N.C.*)

Ans. $28{\cdot}3 \sin(314t - 84{\cdot}3°)$; $28{\cdot}3\, e^{-31{\cdot}4t} \sin 84{\cdot}3°$; $28{\cdot}3(e^{-31{\cdot}4t} + \sin(314t - 84{\cdot}3°))$.

6.10 A single-phase 11,000/1,100V 50Hz transformer is supplied from 11kV "infinite" busbars. The transformer leakage impedance referred to the low-voltage side is $0{\cdot}08\,\Omega$ resistance and $0{\cdot}8\,\Omega$ reactance. Calculate

(*a*) the r.m.s. steady-state short-circuit current which might develop on the secondary side for a short-circuit at the secondary terminals,
(*b*) the corresponding initial transient short-circuit current assuming the short-circuit to occur at the "worst" instant in the voltage cycle,
(*c*) the instantaneous total current magnitude at the instant 0·04s after the short-circuit has occurred.

What are the "worst" and "best" instants in the voltage cycle? (*H.N.C.*)

Ans. (*a*) 1,370A; (*b*) ±1,930A, (*c*) ±1,380A.

6.11 A coil having a resistance *R* ohms and an inductance *L* henrys is suddenly connected to a voltage of constant r.m.s. value *V* and varying in time according to the law $v = V_m \sin 2\pi ft$. Deduce an expression for the current at any instant.

In the case of an alternator suddenly short-circuited explain how the expression for the current would differ from the above and enumerate the factors causing the difference. (*L.U.*)

6.12 A $4\,\mu$F capacitor is initially charged to 300V. It is discharged through a 10mH inductance and a resistor in series. Find

(*a*) the frequency of the discharge if the resistance is zero,
(*b*) how many cycles at the above frequency would occur before the discharge oscillation decays to $\frac{1}{10}$th of its initial value if the resistance is $1\,\Omega$,
(*c*) the value of the resistance which would just prevent oscillation. (*H.N.C.*)

Ans. 796Hz; 36·6; $100\,\Omega$.

6.13 Derive an expression for the instantaneous current in a circuit consisting of a resistance of *R* ohms in series with a capacitance of *C* farads at a time *t* seconds after applying a sinusoidal voltage, the switch being closed t_1 seconds after the voltage had passed through its zero value.

If the values of R and C are 1,000Ω and 10μF respectively and if the voltage is 200V at 50Hz, calculate the value of the voltage at the instant of closing the switch such that no transient current is set up. (*L.U.*)

Ans. 269V.

6.14 A capacitor C, initially charged to 350V, is discharged through a coil of inductance 8mH and resistance R ohms. The amplitude of the resulting oscillation dies away to 0·1 of its initial value in 2·3ms. If the value of resistance for critical damping is 113·2Ω, calculate the natural frequency and the actual value of R. (*H.N.C.*)

Ans. 1,130Hz; 1·6Ω.

6.15 An *RLC* series circuit has $R = 5\Omega$, $L = 10$mH and $C = 400\mu$F. Show that the current immediately after switch closure on to a direct-voltage source is oscillatory and of gradually decreasing amplitude. Calculate the frequency of oscillation and the damping factor. (Part question, *H.N.C.*)

Ans. 69Hz; e^{-250t}.

6.16 A coil having a resistance of R ohms and an inductance of L henrys is connected in series with a resistance of R_1 ohms to a d.c. supply of V volts. After the current has reached a steady value, R_1 is short-circuited by a switch. Deduce, from first principles, an expression for the current in the coil at a time t seconds after the switch is closed.

If $L = 3$H, $R = 50\Omega$ and $R_1 = 30\Omega$, calculate the time taken for the current to increase by 10 per cent after the switch is closed. (*H.N.C.*)

Ans. 10·9ms.

6.17 Discuss the factors which determine the kVA rating of a transformer.

A transformer has a heating time constant of 4 hours and its temperature rises by 18°C from ambient after 1 hour on full load. The winding loss (proportional to I^2) is twice the core loss (constant) at full load. Estimate the final temperature rise in service after a consecutive loading of 1·5 hours at full load, 0·5 hour at one-half full load, and 1 hour at 25 per cent overload. (*H.N.C.*)

Ans. 46°C.

Chapter 7

ELECTRIC AND MAGNETIC FIELD THEORY

Electrostatic fields, magnetic fields and conduction fields exhibit similar characteristics, and all may be analysed by similar processes. In this chapter some linear fields of each type will be dealt with, linear fields being those which exist in materials which have constant electrical properties.

7.1 Streamlines and Current Tubes in Conduction Fields

A *conduction field* is the region in which an electric current flows,

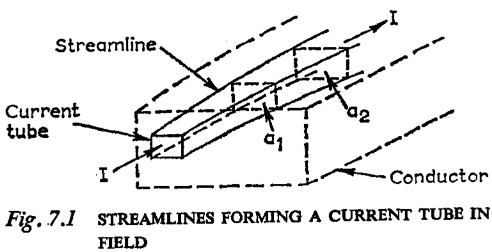


Fig. 7.1 STREAMLINES FORMING A CURRENT TUBE IN A CONDUCTION FIELD

and a *streamline* is a line drawn in such a field with a direction which is everywhere parallel to the direction of current flow; current will never flow across a streamline. If a series of streamlines is taken to enclose a tube of current as in Fig. 7.1, then the total current across any cross-section of the tube will be the same. The most important cross-sections are those normal to the direction of current flow (such as a_1 and a_2 in the diagram), and these will be assumed to be taken unless otherwise stated.

Suppose the total current enclosed by the tube in Fig. 7.1 is I amperes. Then the current density at any point inside the tube is

$$J = \frac{I}{a} \quad \text{amperes/metre}^2$$

where a is the cross-sectional area of the current tube at the particular point. It is assumed that the current density within the tube is sufficiently uniform to be taken as constant at each cross-section. This will be the case if the tube is relatively small, or, in the limit, where $a \to 0$.

The current density at a point is a vector quantity having magnitude J (usually measured in amperes/metre^2), and having the same direction as the current or streamline at the point.

7.2 Equipotential Surfaces

The *potential* of a point in a conduction field is the work done in moving a unit charge from a specified point of zero potential (usually

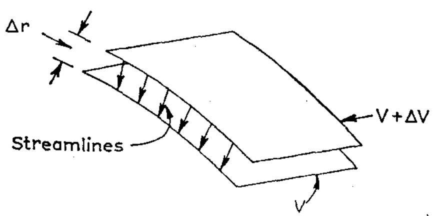


Fig. 7.2 RELATING TO ELECTRIC FIELD STRENGTH

the "earthed" point) to the point in question. There will in general be a large number of points with the same potential and the surface which contains these points is called an *equipotential surface*.

Since all points on an equipotential surface have, by definition, the same potential it follows that there will be no current flow between any points on the surface. Thus a streamline (which is in the direction of current flow) must intersect an equipotential surface at right angles. If it did not, there would then be at least a component of current flow along the equipotential surface which contradicts the previous statements. Thus the surfaces a_1 and a_2 in Fig. 7.1 are equipotential surfaces.

7.3 Electric Field Strength

Consider two equipotential surfaces at potentials V and $V + \Delta V$, where ΔV is a small potential step. The two surfaces lie a small distance Δr apart (Fig. 7.2). Since streamlines cross equipotentials normally, and since here the equipotentials are very close together, the streamlines may be taken to cross the intervening space normally.

Work done in moving a unit charge, or coulomb, across inter-space

$$= \Delta V \text{ volts} = -E \times \Delta r \quad \text{newton-metres/coulomb}$$

where E is the force per coulomb in the direction of a streamline. This is called the *electric field strength*.

$$\text{Electric field strength} = E \quad \text{newtons/coulomb}$$

$$= -\frac{\Delta V}{\Delta r} \quad \text{volts/metre}$$

As $\Delta r \to 0$,

$$E = -\frac{dV}{dr} \quad \text{volts/metre or newtons/coulomb} \tag{7.1}$$

The negative sign is included, since if ΔV is a positive increase in potential, the direction of the force on the charge will be towards the lower potential surface.

7.4 Relationship between Field Strength and Current Density

At a point in a conduction field both the field strength E and the current density J are vector quantities with the same direction. Then the conductivity σ at the point is given by the equation,

$$\sigma = \frac{J}{E} \quad \frac{\text{amperes/metre}^2}{\text{volts/metre}} \quad \text{or siemens/metre}$$

Also

$$\sigma = \frac{1}{\rho}$$

where ρ is the resistivity of the conductor. By Ohm's law, σ is a constant which is independent of J and E for most materials.

Consider a uniform conductor of overall length l metres and cross-sectional area a metres². If a voltage V across the conductor gives rise to a current I through it, then

$$\text{Current density in conductor, } J = \frac{I}{a} \quad \text{amperes/metre}^2$$

and

Potential gradient throughout the conductor, $E = \dfrac{V}{l}$ volts/metre

Therefore

$$\sigma = \frac{J}{E} = \frac{I}{V}\frac{l}{a} = \frac{l}{Ra} \tag{7.2}$$

where $R \left(= \rho \dfrac{l}{a}\right)$ is the total resistance of the uniform conductor.

7.5 Boundary Conditions

In almost every case a conductor has a well-defined edge. Outside this edge, or boundary, there is an insulating material whose conductivity is negligible compared with that of the conductor. In effect, then, no current will pass across the edge of the conductor, and hence the edge must be a streamline. Also, since equipotential surfaces intersect streamlines at right angles, it follows that these surfaces must cross the conductor edges normally.

7.6 Field Plotting Methods

If the boundaries of the field cannot be simply expressed mathematically, an approximate estimate of the conductance may be obtained by a "mapping" method. This applies to plane fields, i.e. fields whose variations may be represented on a flat plane (e.g. the field of two long parallel conductors). The field between two spheres could not be tackled simply by this method since there are variations in all planes.

The basis of the method is the division of the plane of the field into a number of squares formed between adjacent streamlines and adjacent equipotentials. Since the streamlines and equipotentials will in general be curved lines rather than straight lines, true squares will not be formed. However, since the streamlines and equipotentials intersect normally, "square-like" figures are formed—these are usually called *near*, or *curvilinear*, squares. The test for a given figure being a near square is that it should be capable of subdivision into smaller squares which tend to be true squares with equal numbers of the true squares along each side of the original square.

Fig. 7.3 shows a pair of adjacent equipotentials and a pair of adjacent streamlines forming a large square-like figure (90° corners).

The square-like figure is a near square since on successive sub-division by equal numbers of intermediate streamlines and equi-potentials the smaller figures are seen to approach the true square form.

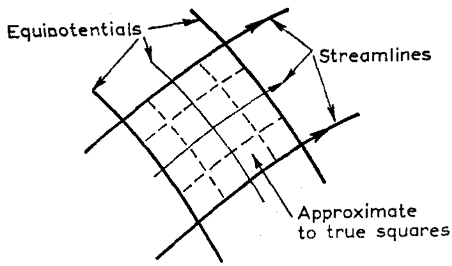


Fig. 7.3 FORMATION OF CURVILINEAR SQUARES

Consider a true square formed between an adjacent pair of streamlines and an adjacent pair of equipotentials (Fig. 7.4). Let this square be the end of d metres depth of the field so that the portion of the field behind the square forms a current tube between the

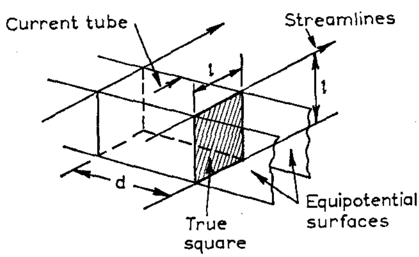


Fig. 7.4 CONDUCTANCE OF A CURRENT TUBE WITH SQUARE CROSS-SECTION

adjacent equipotential surfaces. Let l be the length of a side of the square, and σ be the conductivity of the medium in which the current tube is situated.

$$\text{Conductance of the current tube} = \sigma \times \frac{\text{area}}{\text{length}} = \sigma \times \frac{d \times l}{l}$$

$$= \sigma d \text{ siemens}$$

Therefore the conductance of a tube whose end is a true square is σd mhos, independent of the size of the end square.

In Fig. 7.5 a rectangular block carrying a uniform current has been mapped into a number of tubes whose ends are true squares. The conductance of each tube is σd siemens, and

$$\text{Total conductance} = \sigma d \times \frac{m}{n} \tag{7.3}$$

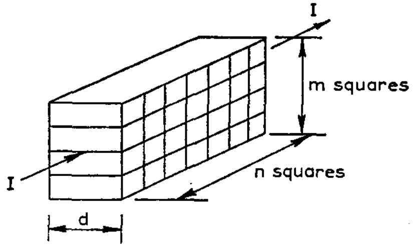


Fig. 7.5 CONDUCTANCE OF A RECTANGULAR BLOCK

where m is the number of parallel squares (across the direction of current flow), and n is the number of series squares (i.e. along the direction of current flow). Neither n nor m need be an integer.

In other cases the field will not be reducible to true squares but to near squares—the conductance per near-square-ended tube still being σd, and the total conductance still being given by eqn. (7.3). This follows since the definition of a near square is that it should be capable of subdivision to form small figures which approach to being true squares each corresponding to a conductance σd mhos (independent of size). Thus the conductance of any plane field may be estimated by a field plot which divides the field into a number of curvilinear squares. The plotting may be performed by eye as in succeeding examples. Alternatively, the electrolytic trough or the rubber membrane method may be used.

EXAMPLE 7.1 A 6×3 cm sheet of high-resistance conducting material of uniform depth is soldered to massive copper blocks at either end. Find the fractional increase in resistance when a thin slot is cut halfway across the sheet, as shown in Fig. 7.6.

Since the end pieces are of copper and of large section, voltage drops in them should be negligible compared with voltage drops in the material of the sheet, so that the ends of the sheet may be taken as equipotentials.

The vertical centre-line is, by symmetry in this particular case, an equipotential, and the conduction field on each side should be symmetrical. The edges are, of course, streamlines.

To solve the problem the area must be mapped by a series of equipotentials and streamlines forming good near squares. This may be performed by successive graphical approximation, as follows.

1. Draw in the middle streamline which divides the total current in half; this will start a little lower than mid-point at one edge (say at A), and cut the central equipotential a little above mid-point (say at B). C is similarly placed to A.

2. Draw in the two quarter streamlines DEF and GHI; note that XH < HB < BE < EY and RF < FC < CI < IT. The increases are judged by eye.

3. Commence drawing equipotentials cutting the above streamlines orthogonally, and as far as possible completing near squares. In this case start drawing the equipotentials either at one of the ends or at the central equipotential. The right-hand side of the figure shows the plot at this stage. Manifestly all the areas are not near squares.

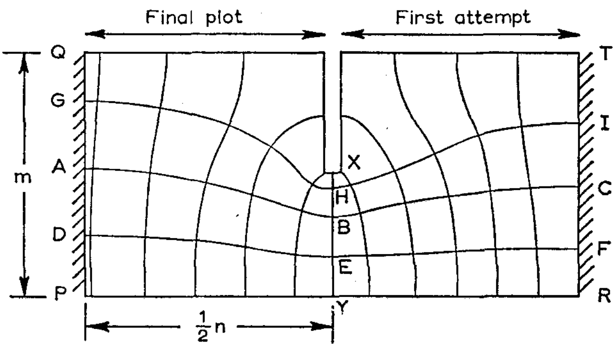


Fig. 7.6

4. Either rub out the previously drawn streamlines, leaving the equipotentials, or better, take a tracing of the figure with equipotentials but not streamlines. Now redraw the streamlines cutting the equipotentials orthogonally and forming near squares. It will be unlikely that an integral number of near squares will fit into the total area. The resultant trace at this stage is probably fairly accurate and sufficiently good for most purposes. It is shown on the left-hand side of the plot. If, however, greater accuracy is required, the figure should be retraced excluding the equipotentials but including the new streamlines and so on. It must be emphasized that, while one retracing provides fair accuracy after some practice, the result without the retracing operation has a very poor accuracy and is unsatisfactory except for the roughest estimations. Less than 5 per cent error may often be attained with one retracing only.

Suppose there are n squares in the current direction and m squares normal to the current direction where m and n are not necessarily integral numbers.

The conductance per square element is σd siemens, where σ is the conductivity of the material and d is the uniform depth. Then

$$\text{Total conductance} = \sigma d \times \frac{m}{n} \text{ siemens} \tag{7.3}$$

From the field plot, $m = 4$ and $n = 2 \times 5{\cdot}1 = 10{\cdot}2$. Had the sheet been without the slot, the number of squares in the current direction would have been proportional to the length l and the number normal to the current direction proportional to the breadth b. Then

$$\text{Total conductance without slot} = \sigma d \times \frac{b}{l} \quad \text{siemens}$$

$$\frac{\text{Conductance with slot}}{\text{Conductance without slot}} = \frac{ml}{nb} = \frac{4}{10{\cdot}2} \times \frac{6}{3} = 0{\cdot}78$$

Therefore

$$\frac{\text{Resistance with slot}}{\text{Resistance without slot}} = \frac{1}{0{\cdot}78} = \underline{\underline{1{\cdot}28}}$$

The symmetry of the figure determines in each particular case whether it is better to start the field plot by drawing the streamlines or the equipotentials. The number of such lines must also be judged for each case on its own. If too few lines are chosen, then the result will be inaccurate owing to the difficulty of estimating near squares. If too many lines are chosen, the plot becomes too confused to be useful.

It will be noticed in the above example that a five-sided near square appears at the corners of the slot. This is admissible, since on further subdivision the figure will (in the limit) give true squares except for one square.

7.7 Streamlines and Tubes of Electric Flux in Electrostatic Fields

A streamline in an electrostatic field is a line so drawn that its direction is everywhere parallel to the direction of the electrostatic flux. It is also a line of force and has the same properties as a streamline in a conduction field. Several streamlines may be taken as enclosing a *tube of electric flux*. Let the total flux through a tube be Ψ coulombs; then the *electric flux density* at a point in the tube is

$$D = \frac{\Psi}{a} \quad \text{coulombs/metre}^2 \tag{7.4}$$

where a square metres is the cross-section of the tube at the particular point and it is assumed that the flux density within the tube is sufficiently uniform to be taken as constant over that area.

7.8 Equipotential Surfaces and Electric Field Strength

Equipotential surfaces have the same definition and properties in electrostatic as in conduction fields. Hence the electric field strength in an electrostatic field is given by

$$E = -\frac{dV}{dr} \quad \text{volts/metre} \tag{7.1}$$

where V is the potential at the point, and r is in the direction in which E is measured.*

* Electric field strength is also often called *electric intensity*, *electric stress* or *potential gradient*.

7.9 Relationship between Field Strength and Flux Density

At all points in an electrostatic field both the field strength, E, and the electric flux density, D, are vector quantities in the direction of the streamline through the point. From electrostatic theory,

$$D = \epsilon E$$

where ϵ is the permittivity of the dielectric material, and

$$\epsilon = \epsilon_0 \epsilon_r = \frac{1}{36\pi \times 10^9} \epsilon_r$$

ϵ_r being the relative permittivity of the material. D is measured in coulombs per square metre, and E is measured in volts per metre (or newtons per coulomb).

7.10 Boundary Conditions

Usually an electrostatic field is set up in the insulating medium between two good conductors. The conductors have such a high conductivity that any voltage drop within the conductors (except at very high frequencies) is negligible compared with the potential differences across the insulator. All points in the conductors are therefore at the same potential so that the conductors form the boundary equipotentials for the electrostatic field.

Streamlines, which must cut all equipotentials at right angles, will leave one boundary at right angles, pass across the field, and enter the other boundary at right angles. Curvilinear squares may again be formed between the streamlines and the equipotentials.

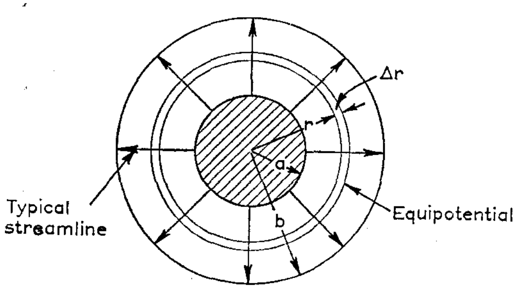


Fig. 7.7 CAPACITANCE BETWEEN CONCENTRIC CYLINDERS

7.11 Capacitance between Concentric Cylinders

The electrostatic field between two concentric conducting cylinders is illustrated in Fig. 7.7. The boundary equipotentials are concentric cylinders of radii a and b, and the streamlines are radial lines cutting the equipotentials normally.

Let Q' be the charge per unit length (coulombs per metre run) of the inner conductor. Then the total electric flux, Ψ, across the dielectric per unit length is Q' coulombs per metre. This total flux will pass through the elemental cylinder of thickness Δr at radius r and a distance of 1 metre into the plane of the paper. Thus

$$\text{Flux density at radius } r = D = \frac{\Psi}{A} = \frac{Q'}{2\pi r \times 1}$$

Therefore

$$\text{Electric stress at radius } r = E = \frac{D}{\epsilon} = \frac{Q'}{2\pi\epsilon r} \tag{7.5}$$

Also

$$\text{Potential difference across element} = \Delta V$$

Therefore

$$\Delta V = -E\Delta r = -\frac{Q'}{2\pi\epsilon r}\Delta r$$

$$\text{Total potential difference between boundaries, } V = -\frac{Q'}{2\pi\epsilon}\int_b^a \frac{dr}{r}$$

Therefore

$$V = \frac{Q'}{2\pi\epsilon}\log_e\frac{b}{a} \quad \text{volts} \tag{7.6}$$

But

$$\text{Capacitance per unit length} = C' = \frac{\text{Charge per unit length}}{\text{Potential difference}}$$

Therefore

$$C' = \frac{Q'}{V} = \frac{2\pi\epsilon}{\log_e\dfrac{b}{a}} \quad \text{farads/metre} \tag{7.7}$$

7.12 Electric Stress in a Single-core Cable

A single-core cable with a metal sheath has the same electrostatic field as a pair of concentric cylinders. From eqn. (7.6),

$$\frac{Q'}{2\pi\epsilon} = \frac{V}{\log_e\dfrac{b}{a}}$$

Substituting this expression in eqn. (7.5),

$$E = \frac{V}{r \log_e \dfrac{b}{a}} \tag{7.8}$$

Thus the stress at any point in the dielectric varies inversely as r, and will have a maximum value at the minimum radius, i.e. when $r = a$.

Thus

$$E_{max} = \frac{V}{a \log_e \dfrac{b}{a}} \tag{7.9}$$

When designing a cable it is important to obtain the most economical dimensions. The greater the value of the permissible maximum stress, E_{max}, the smaller the cable may be for a given voltage V. The maximum permissible stress, however, is limited to the safe working stress for the dielectric material. With V and E_{max} both fixed, the relationship between b and a will be given by

$$a \log_e \frac{b}{a} = \frac{V}{E_{max}} = R$$

where R is a constant. Therefore

$$\log_e \frac{b}{a} = \frac{R}{a}$$

i.e.

$$b = a\mathrm{e}^{R/a} \tag{i}$$

For the most economical cable, b will be a minimum, and hence

$$\frac{db}{da} = 0 = \mathrm{e}^{R/a} + a\left(-\frac{R}{a^2}\right)\mathrm{e}^{R/a}$$

i.e.

$$a = R = V/E_{max} \tag{7.10}$$

and from (i)

$$b = a\mathrm{e} = 2{\cdot}718a \tag{7.11}$$

The core diameter given by eqn. (7.10) is usually found to give a larger conductor cross-sectional area than is necessary for the economical transmission of current through a high-voltage cable. The high cost of the unnecessary copper may be reduced by (*a*) making the core of hollow construction, or (*b*) having a hemp-cord centre for the core, or (*c*) constructing the centre of the core of a cheaper metal.

EXAMPLE 7.2 A single-core concentric cable is to be manufactured for a 100 kV 50 Hz transmission system. The paper used has a maximum permissible safe stress of 10^7 V/m (r.m.s.) and a dielectric constant (relative permittivity) of 4. Calculate the dimensions for the most economical cable, and the charging current per kilometre run with this cable.

By eqn. (7.10),

$$\text{Core radius} = a = \frac{V}{E_{max}} = \frac{10^5}{10^7} = 10^{-2}\,\text{m} = 1\,\text{cm}$$

By eqn. (7.11),

$$\text{Internal sheath radius} = b = ea = \underline{\underline{2{\cdot}718\,\text{cm}}}$$

By eqn. (7.7),

$$\text{Capacitance per metre} = C' = \frac{2\pi \times 4}{36\pi \times 10^9}\ \text{F}$$

since $\log_e (b/a) = \log_e e = 1$. Thus

$$C' = 0{\cdot}222 \times 10^{-3}\mu\text{F/m}$$

Therefore

$$\text{Capacitance for } 1\,\text{km} = C = 0{\cdot}222\mu\text{F}$$

and

$$\text{Charging current per km} = V\omega C = 10^5 \times 314 \times 0{\cdot}222 \times 10^{-6} = \underline{\underline{7\,\text{A}}}$$

7.13 Capacitance of an Isolated Twin Line

Fig. 7.8(*a*) shows the actual field distribution round a pair of oppositely charged long conductors each of radius a. The conductors are spaced so that the distance between their centres is D, where a and D are measured in the same units and $D \gg a$. Fig. 7.8(*b*) shows the field of each conductor separately, this being the field assumed for calculation purposes.

Let conductor A carry a charge of $+q'$ coulombs per metre length while conductor B is uncharged. Consider a cylindrical element of radius r about conductor A, of depth 1 metre and of thickness Δr.

$$\text{Total electric flux through element, } \Psi' = q'$$

whence

$$\text{Electric flux density at element, } D' = \frac{\Psi'}{\text{Area of element}} = \frac{q'}{2\pi r \times 1}$$

Thus

$$\text{Field strength at element, } E = \frac{D'}{\epsilon} = \frac{q'}{2\pi\epsilon r}$$

and

$$\text{Voltage drop across element} = -E\Delta r = -\frac{q'\Delta r}{2\pi\epsilon r}$$

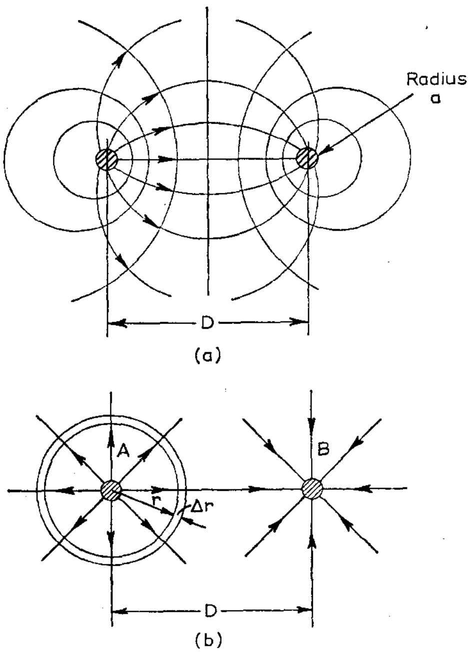


Fig. 7.8 FIELD BETWEEN PARALLEL CONDUCTORS

(*a*) Actual field
(*b*) Field of each conductor alone

Suppose that at a large radius R, the potential may be considered zero. Then

$$\text{Potential of conductor A above zero} = -\frac{q'}{2\pi\epsilon}\int_R^a \frac{dr}{r}$$

$$= \frac{q'}{2\pi\epsilon}\log_e \frac{R}{a} \text{ volts} \quad (7.12)$$

and

$$\text{Potential at conductor B above zero} = -\frac{q'}{2\pi\epsilon}\int_R^D \frac{dr}{r}$$

$$= \frac{q'}{2\pi\epsilon}\log_e \frac{R}{D}$$

since conductor B lies in the electrostatic field of conductor A.

Let conductor B carry a charge of $-q'$ coulombs per metre length while conductor A is uncharged. Then

$$\text{Potential of conductor B below zero} = -\frac{q'}{2\pi\epsilon}\log_e \frac{R}{a}$$

and

$$\left.\begin{array}{l}\text{Potential at conductor A below zero}\\ \quad\text{due to the charge on conductor B}\end{array}\right\} = -\frac{q'}{2\pi\epsilon}\log_e \frac{R}{D}$$

When both conductors are charged simultaneously,

$$\text{Total potential of A above zero} = \frac{q'}{2\pi\epsilon}\left(\log_e \frac{R}{a} - \log_e \frac{R}{D}\right)$$

$$= \frac{q'}{2\pi\epsilon}\log_e \frac{D}{a} \tag{7.13}$$

and

$$\text{Total potential of B below zero} = -\frac{q'}{2\pi\epsilon}\log_e \frac{D}{a}$$

Therefore

$$\text{Potential difference between A and B} = 2\frac{q'}{2\pi\epsilon}\log_e \frac{D}{a}$$

Capacitance between A and B per metre length

$$= C' = \frac{q'}{\text{P.D. between A and B}}$$

Therefore

$$C' = \frac{1}{2}\frac{2\pi\epsilon}{\log_e \dfrac{D}{a}} \quad \text{farads/metre} \tag{7.14}$$

7.14 Potential and Electric Field Strength for an Isolated Twin Line

Consider a twin line where the potential difference between conductors is V volts and the system is balanced to earth so that the potential of conductor A with respect to earth is $\frac{1}{2}V$ and the potential of conductor B with respect to earth is $-\frac{1}{2}V$ (Fig. 7.9).

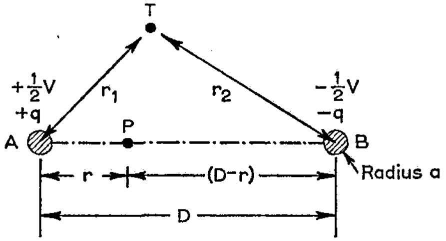


Fig. 7.9 ISOLATED TWIN LINE

By eqn. (7.13),

$$\text{Potential of conductor A above earth} = \tfrac{1}{2}V = \frac{q'}{2\pi\epsilon}\log_e\frac{D}{a}$$

Therefore

$$\frac{q'}{2\pi\epsilon} = \frac{V}{2\log_e\dfrac{D}{a}}$$

Also, the potential at a point T, distant r_1 from conductor A, is

$$V_{r1} = -\frac{q'}{2\pi\epsilon}\int_R^{r_1}\frac{dr_1}{r_1} = \frac{q'}{2\pi\epsilon}\log_e\frac{R}{r_1} = \frac{V}{2\log_e\dfrac{D}{a}}\log_e\frac{R}{r_1}$$

due to the change on conductor A. The point T will also have a potential due to the charge on conductor B. Suppose T is distant r_2 from conductor B; then

$$\text{Potential at T due to charge on B} = \frac{-V}{2\log_e\dfrac{D}{a}}\log_e\frac{R}{r_2}$$

The total potential at T will be the sum of the potentials due to A and B separately. Hence

$$\text{Potential at T} = \frac{V}{2\log_e\dfrac{D}{a}}\log_e\frac{r_2}{r_1} \tag{7.15}$$

The electric field strength has its highest value along the line joining the conductor centres, since on this line the forces due to each line charge will be acting in the same direction. At the point P, which is distant r from A, the potential will be

$$V_p = \frac{V}{2 \log_e \frac{D}{a}} \log_e \left(\frac{D-r}{r}\right)$$

$$\text{Electric field strength at } P,\ E_p = -\frac{dV}{dr} \tag{7.1}$$

$$= \frac{-V}{2 \log_e D/a} \frac{r}{D-r} \frac{\{-r-(D-r)\}}{r^2}$$

$$= \frac{V}{2 \log_e D/a} \frac{D}{r(D-r)} \quad \text{volts/metre} \tag{7.16}$$

From eqn. (7.16) the field strength E_p will have a maximum value when either r or $(D - r)$ has a minimum value. This occurs when $r = a$ or when $(D - r) = a$, i.e. at each conductor surface. The maximum field strength is almost independent of the conductor spacing, and is almost inversely proportional to the conductor radius. High-voltage conductors should not be of a small radius (even if the current is small) or the stress at the conductor surface will exceed the breakdown strength of the surrounding air.

7.15 Lines Above a Conducting Earth

In Section 7.10 it was seen that a good conductor would be an equipotential boundary for an electrostatic field. If a conductor is introduced into an electrostatic field in a random manner and with a random potential, then the field will be greatly altered, but if a conductor of negligible thickness and at the correct potential is introduced into a field so that it lies entirely in the corresponding equipotential surface then the field will be unaltered.

Fig. 7.10(*a*) shows the field between two long isolated parallel conductors, which have potentials of +1,000 V and −1,000V respectively. The edges of several equipotential surfaces between the conductors are drawn in. If a cylindrical conductor at a potential of 500 V were inserted in the field to coincide with the 500 V equipotential surface, then the resultant field would not be changed. It would, however, now be divided into two separate parts, screened from each other by the conducting cylinder, so that either the enclosed or the outer conductor could be removed without affecting the field in the opposite part.

The zero equipotential surface is a plane, and if a plane conducting sheet at zero potential is inserted in the field to coincide with this equipotential surface the field will be divided, as above, into two separate parts, each independent of the other. The negative conductor could therefore be removed and replaced by any other charged system, or by a solid conductor, without affecting the shape

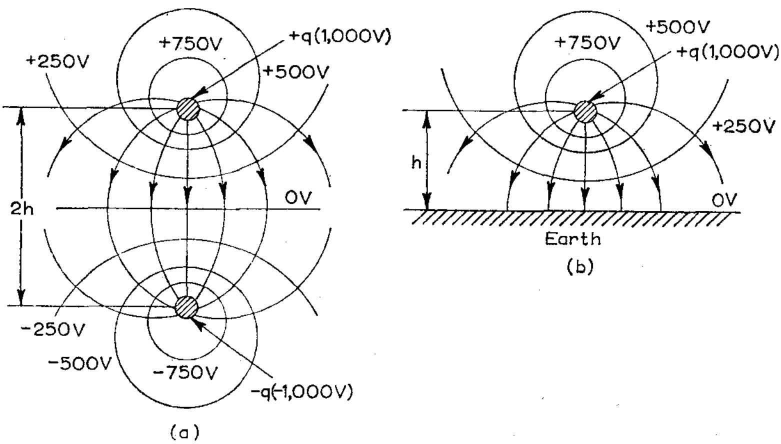


Fig. 7.10 FIELD BETWEEN A CONDUCTOR AND A PARALLEL PLANE

of the field between the positive conductor and the zero equipotential surface.

Fig. 7.10(*b*) shows one conductor at a height h metres above an infinite conducting earth. The shape of the field between the conductor and the earth must, by the preceding argument, be the same as the shape of the field which would exist above the zero-potential surface in an isolated twin-line system where the conductor spacing is $2h$.

Thus, to analyse the field of a single charged conductor (or any other system) above a conducting earth, the earth is replaced by an image system, carrying the opposite charge to the real system and placed the same distance below earth as the real system is above earth. Of course, only the field above earth actually exists.

Consider the single wire of radius a, suspended at height h above a uniform conducting earth, and having a line charge of q' coulombs per metre run. The earth effect is represented by the image conductor with a line charge $-q'$ as shown in Fig. 7.11. Then the potential of the actual wire to earth due to its own charge and to that on the image conductor is given by eqn. (7.13) as

$$V = \frac{q'}{2\pi\epsilon} \log_e \frac{2h}{a}$$

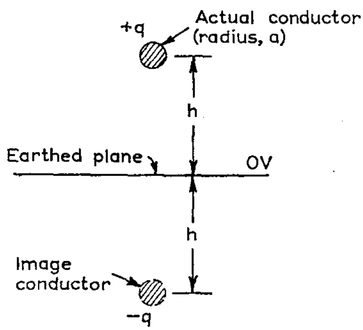


Fig. 7.11 RELATING TO CAPACITANCE BETWEEN A CONDUCTOR AND A PARALLEL PLANE

where $2h$ is the distance between the wire and its image. Thus

$$\text{Capacitance to earth, } C' = \frac{q'}{V} = \frac{2\pi\epsilon}{\log_e \frac{2h}{a}} \quad \text{farads/metre} \tag{7.17}$$

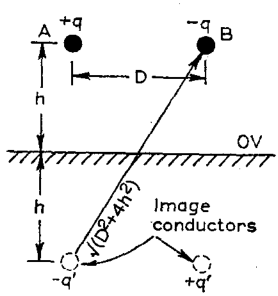


Fig. 7.12 RELATING TO CAPACITANCE OF A TWIN LINE ABOVE A CONDUCTING PLANE

To find the capacitance for a twin line with each conductor of radius a at a height h above earth and with a line spacing of D, the earth effect is represented by the image pair as shown in Fig. 7.12. Let R be the large radius at which the potential is zero. Let a potential above zero be represented as positive and a potential below zero as negative. Then the total potential of conductor A, due to its own charge and to the charges on B and the image pair is

$$V_A = \frac{q'}{2\pi\epsilon}\log_e\frac{R}{a} - \frac{q'}{2\pi\epsilon}\log_e\frac{R}{D} - \frac{q'}{2\pi\epsilon}\log_e\frac{R}{2h} + \frac{q'}{2\pi\epsilon}\log_e\frac{R}{\sqrt{(D^2+4h^2)}}$$

$$= \frac{q'}{2\pi\epsilon}\log_e\left\{\frac{D}{a}\frac{2h}{\sqrt{(D^2+4h^2)}}\right\}$$

Conductor B will be at a similar potential below zero, so that the p.d. between A and B will be twice V_A.

$$\text{Capacitance per metre, } C' = \frac{q'}{2V_A}$$

$$= \frac{1}{2}\frac{2\pi\epsilon}{\log_e\left\{\frac{D}{a}\frac{2h}{\sqrt{(D^2+4h^2)}}\right\}} \text{ farads/metre} \qquad (7.18)$$

7.16 Equivalent Phase Capacitance of an Isolated Three-phase Line

Fig. 7.13(a) shows the cross-section of a typical 3-phase line in which the conductors all have equal radii a, but different spacings D_{12}, D_{23} and D_{31}. There will actually be capacitances C_{12}', C_{23}' and C_{31}' per metre length between each pair of lines, but it is more convenient to represent these as equivalent phase capacitances C_1', C_2' and C_3' per metre length as shown in Fig. 7.13(b). With an irregular spacing C_1', C_2' and C_3' will be unequal.

Assuming that the line is part of a 3-wire system, then

$$i_1 + i_2 + i_3 = 0$$

where i_1, i_2 and i_3 are the instantaneous charging currents.

Then $\int(i_1 + i_2 + i_3)dt = \text{constant} = 0$, since in an a.c. system there are no constant charges. Thus

$$q_1 + q_2 + q_3 = 0 \text{ (total charge)}$$

or

$$q_1' + q_2' + q_3' = 0 \text{ (charge per metre)}$$

Therefore

$$q_2' + q_3' = -q_1 \qquad (7.19)$$

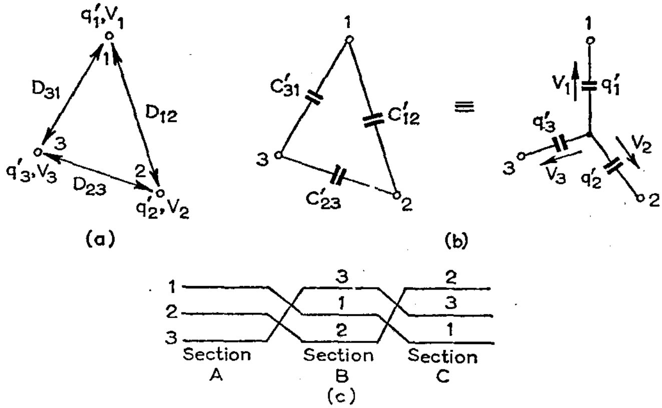


Fig. 7.13 CAPACITANCE OF A 3-PHASE LINE: UNIFORM TRANSPOSITION

The total potential of line 1 due to its own charge and to the charges on the other lines is

$$V_1 = \frac{q_1'}{2\pi\epsilon}\log_e\frac{R}{a} + \frac{q_2'}{2\pi\epsilon}\log_e\frac{R}{D_{12}} + \frac{q_3'}{2\pi\epsilon}\log_e\frac{R}{D_{31}}$$

$$= \frac{q_1'}{2\pi\epsilon}\log_e\frac{R}{a} + \frac{q_2'}{2\pi\epsilon}\log_e\frac{R}{D_{12}} - \frac{q_1'+q_2'}{2\pi\epsilon}\log_e\frac{R}{D_{31}}$$

i.e.

$$V_1 = \frac{q_1'}{2\pi\epsilon}\log_e\frac{D_{31}}{a} + \frac{q_2'}{2\pi\epsilon}\log_e\frac{D_{31}}{D_{12}}$$

In the same way, the total potential of line 2 due to its own charge and to the charges on the other lines is

$$V_2 = \frac{q_2'}{2\pi\epsilon}\log_e\frac{D_{23}}{a} + \frac{q_1'}{2\pi\epsilon}\log_e\frac{D_{23}}{D_{12}}$$

Therefore

$$V_1 - V_2 = \frac{q_1'}{2\pi\epsilon}\log_e\frac{D_{12}D_{31}}{aD_{23}} - \frac{q_2'}{2\pi\epsilon}\log_e\frac{D_{12}D_{23}}{aD_{31}} \tag{7.20}$$

Also, from the equivalent circuit of star-connected capacitances of Fig. 7.13(*b*),

$$V_1 - V_2 = \frac{q_1'}{C_1'} - \frac{q_2'}{C_2'} \tag{7.21}$$

From eqns. (7.20) and (7.21),

$$C_1' = \frac{2\pi\epsilon}{\log_e \dfrac{D_{12}D_{31}}{aD_{23}}} \text{ farads/metre} \tag{7.22}$$

$$C_2' = \frac{2\pi\epsilon}{\log_e \dfrac{D_{23}D_{12}}{aD_{31}}} \text{ farads/metre} \tag{7.23}$$

Similarly it can be shown that

$$C_3' = \frac{2\pi\epsilon}{\log_e \dfrac{D_{31}D_{23}}{aD_{12}}} \text{ farads/metre} \tag{7.24}$$

Fig. 7.13(*c*) illustrates a uniformly transposed line where line 1 runs for one-third of its length in position 1, one-third of its length in position 2 and one-third of its length in position 3. Conductors 2 and 3 are similarly transposed. Transposition of this nature is often adopted for practical lines since there is the obvious advantage of equalizing phase capacitances of the line, and also the advantage of minimizing stray potentials induced in parallel telephone lines or other conductors.

An approximate solution for the effective phase capacitance of a uniformly transposed line may be obtained by assuming that the charge per unit length of line is uniform despite the transpositions. The potential of a line must then change at each transposition. The approximate value of capacitance is based on finding the average value for the potential of a line. The method is approximate since each line is, in fact, equipotential. Successive transposed sections could have different potentials only if they were insulated from each other.

For section A and Fig. 7.13(*c*), the total potential of line 1 is

$$V_{1A} = \frac{q_1'}{2\pi\epsilon}\log_e\frac{R}{a} + \frac{q_2'}{2\pi\epsilon}\log_e\frac{R}{D_{12}} + \frac{q_3'}{2\pi\epsilon}\log_e\frac{R}{D_{31}}$$

For section B the total potential of line 1 is

$$V_{1B} = \frac{q_1'}{2\pi\epsilon}\log_e\frac{R}{a} + \frac{q_2'}{2\pi\epsilon}\log_e\frac{R}{D_{23}} + \frac{q_3'}{2\pi\epsilon}\log_e\frac{R}{D_{12}}$$

For section C the total potential of line 1 is

$$V_{1C} = \frac{q_1'}{2\pi\epsilon}\log_e\frac{R}{a} + \frac{q_2'}{2\pi\epsilon}\log_e\frac{R}{D_{31}} + \frac{q_3'}{2\pi\epsilon}\log_e\frac{R}{D_{23}}$$

The average potential of line 1 is

$$V_1 = \tfrac{1}{3}(V_{1A} + V_{1B} + V_{1C})$$

$$= \frac{1}{3}\left(\frac{3q_1'}{2\pi\epsilon}\log_e \frac{R}{a} + \frac{q_2'}{2\pi\epsilon}\log_e \frac{R^3}{D_{12}D_{23}D_{31}} + \frac{q_3'}{2\pi\epsilon}\log_e \frac{R^3}{D_{12}D_{23}D_{31}}\right)$$

$$= \frac{q_1'}{2\pi\epsilon}\log_e \frac{R}{a} + \frac{q_2' + q_3'}{2\pi\epsilon}\log_e \frac{R}{\sqrt[3]{(D_{12}D_{23}D_{31})}}$$

Since $q_2' + q_3' = -q_1'$,

$$V_1 = \frac{q_1'}{2\pi\epsilon}\log_e \frac{\sqrt[3]{(D_{12}D_{23}D_{31})}}{a}$$

The average capacitance of each line is

$$C' = C_1' = \frac{q_1'}{V_1} = \frac{2\pi\epsilon}{\log_e \dfrac{\sqrt[3]{(D_{12}D_{23}D_{31})}}{a}} \quad \text{farads/metre} \qquad (7.25)$$

7.17 Electrostatic Field Plotting Methods

The mapping or field plotting methods which were developed for conduction fields in Section 7.6 are equally applicable to plane electrostatic fields. The electrostatic field should be divided by streamlines and equipotentials into a number of curvilinear squares.

For unit depth of field behind each curvilinear square the capacitance of the flux tube formed will be ϵ farads, where ϵ is the permittivity of the dielectric material. The capacitance of the flux tube is independent of the size of the curvilinear square. The total capacitance of the field will be given by

$$C = \epsilon d \times \frac{m}{n} \quad \text{farads} \qquad (7.26)$$

where d = Depth of field in metres

m = Number of "parallel" squares measured along each equipotential

n = Number of "series" squares measured along each streamline

EXAMPLE 7.3 A cross-section of a parallel-strip transmission line is shown in Fig. 7.14(a); the conductors have each a breadth of $2c$ metres and are spaced

c metres apart. Compare the values of capacitance per metre length found by (*a*) neglecting fringing at the edges, and (*b*) estimating the capacitance by a field plotting method.

Comment on the electric stress distribution.

(*a*) *Neglecting fringing*

$$\text{Capacitance/metre length} = \frac{\epsilon \times \text{Area of plates}}{\text{Separation}}$$

$$= \epsilon \times \frac{2c \times 1}{c} = \underline{\underline{2\epsilon \text{ farads/metre}}}$$

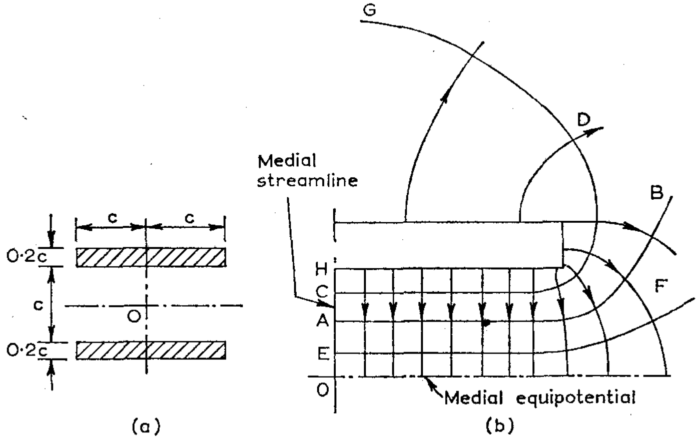


Fig. 7.14

(*b*) *Mapping*. The centre-line joining the two plates will be the medial streamline, by the symmetry of the arrangement; also the centre-line between the two plates will be the medial equipotential. Between the medial equipotential and the medial streamline the whole field may be divided into four separate symmetrical parts. One such part is enlarged for mapping in Fig. 7.14(*b*).

The mapping may be as follows.

1. Estimate the position of the equipotential AB which has the mean potential between that of the plate and that of the medial equipotential. The end B is not taken too far, since if it were, the exact position of the line would be very difficult to estimate. Point A will lie slightly closer to H than to O.
2. Estimate the positions of the intermediate equipotentials CD and EF, stopping these curves when the position which they should occupy becomes difficult to estimate.
3. Draw in a series of streamlines to cut the equipotentials normally, and to form, as far as possible, curvilinear squares.
4. Continue the equipotential CD to the point G, making the curve normal to the vertical at G, and forming squares with the streamlines.
5. Erase the equipotentials and redraw to fit the streamlines.
6. Repeat this procedure as necessary.

In this case it should be noted that between the plates the field is almost uniform, giving a field plot consisting of true squares in this region. At the corner of the plate the squares are smaller—hence there is a greater electric stress here. On the top of the plate the squares become extremely large, indicating that the main field exists between the plates.

From Fig. 7.14(*b*),

$$\text{Total capacitance per unit depth} = \epsilon \times \frac{\text{Number of parallel squares}}{\text{Number of series squares}}$$

$$= \frac{\epsilon \times 2 \times 13{\cdot}2}{2 \times 4}$$

$$= \underline{\underline{3{\cdot}3\epsilon \text{ farads/metre}}}$$

From the field plot it is possible to estimate the electric stress at any point in the dielectric except just at a sharp edge where the field is greatly affected by the "sharpness" of the edge. Since there is the same potential difference across each curvilinear square and since the electric stress over one square is approximately uniform, the electric stress at any point is given approximately by

$$\frac{\text{Potential drop across a square}}{\text{Length of one side of an adjacent square}}$$

Thus the electric stress is inversely proportional to the length of the sides of a curvilinear square, and hence the stress is highest where the squares are smallest.

7.18 Streamlines and Tubes of Magnetic Flux

A streamline in a magnetic field is a line so drawn that its direction is everywhere parallel to the direction of the magnetic flux. It is also a line of *magnetic field strength* and has the same properties as a streamline in a conduction field. Several streamlines may be taken as enclosing a *tube of magnetic flux.*

Let the total flux through a tube be Φ webers. Then the *magnetic flux density* at a point in the tube is

$$B = \frac{\Phi}{a} \text{ webers/metre}^2 \text{ or teslas*}$$

where a square metres is the cross-section of the tube at the particular point and it is assumed that the flux density within the tube is sufficiently uniform to be taken as constant over the area a.

7.19 Equipotential Surfaces and Magnetic Field Strength

The term "potential" is less frequently used with respect to magnetic fields than with respect to conduction and electrostatic fields, probably owing to the difficulty in fixing a zero or reference potential. Fig. 7.15 shows a typical magnetic field system where the magnetic

* The *tesla* (T) is the SI unit of magnetic flux density. Some writers prefer to retain the *weber per square metre* (Wb/m^2) since this conveys the idea of surface density.

flux is partly in air and partly in iron. An equipotential surface in this field will be a surface over which a *magnetic pole* could be moved without the expenditure of work or energy. Various lines are drawn across the flux path of Fig. 7.15 to show the edges of typical equipotential surfaces. Any one of the equipotential surfaces can be taken as the zero or reference potential surface. The work done in moving a *unit magnetic pole* from the zero-potential surface

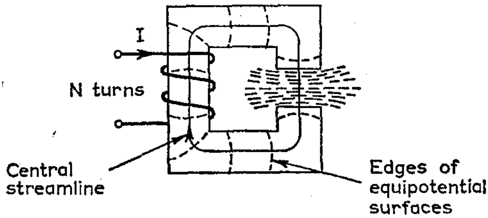


Fig. 7.15 MAGNETIC EQUIPOTENTIALS IN A TYPICAL MAGNETIC FIELD SYSTEM

to a second equipotential will give the *magnetic potential* of the second surface.

If the unit pole is moved completely round the magnetic circuit so that its completed path links the total magnetizing current, I amperes, in the N turns of the magnetizing coil, then the total work done is IN joules per pole, or IN ampere-turns. Thus the zero magnetic potential surface may also be considered to have a magnetic potential of IN ampere-turns; in the same way all points in a magnetic field may be considered to have many potential values, but only the basic value denoting the movement of a unit pole round a fraction of the magnetic circuit from the reference surface need be considered. Thus the magnetic potential of a point (F ampere-turns) is equal to the work done in moving a unit pole from the arbitrary zero potential surface to the particular point.

Total magnetic p.d. round a complete loop = IN ampere-turns

Let H ampere-turns per metre (or newtons per unit pole) be the field strength in any given direction r, at a point in a magnetic field where the magnetic potential is F ampere-turns. Then,

$$H = -\frac{dF}{dr} \quad \text{ampere-turns/metre} \tag{7.27}$$

In the same way as for the conduction and electrostatic fields, the minus sign is included since H acts in the opposite direction with respect to r to that in which F increases.

At all points in a magnetic field both the field strength H and the flux density B are vector quantities in the direction of the streamline through the point. They are related by the equation

$$B = \mu H$$

where μ is the permeability of the material at the point ($= \mu_0\mu_r$); μ_0 being the permeability of space ($= 4\pi \times 10^{-7}$), and μ_r the permeability of the material relative to the permeability of space.

7.20 Boundary Conditions

The magnetic field in the air gap between two iron surfaces may be considered as bounded by two equipotentials which follow the

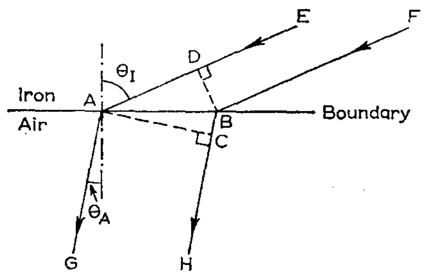


Fig. 7.16 REFRACTION OF MAGNETIC FLUX AT AN IRON-AIR BOUNDARY

iron surfaces. This follows from the following proof, which shows that at an iron-air boundary a line of force will emerge from the iron in a direction almost normal to the iron-air boundary.

Consider unit depth of an iron-air boundary as depicted in Fig. 7.16. Let AE and BF be two adjacent streamlines (or lines of force) in the iron which strike the boundary at an angle θ_I to the normal at the boundary. The streamlines continue in the air as AG and BH at an angle θ_A to the boundary normal.

For unit depth at the boundary a tube of magnetic flux is formed between the streamlines EAG and FBH. Let Φ webers be the flux in this tube.

$$\text{Flux density in iron, } B_I = \frac{\Phi}{\text{DB} \times 1} = \frac{\Phi}{\text{AB} \cos \theta_I}$$

$$\text{Flux density in air, } B_A = \frac{\Phi}{\text{AC} \times 1} = \frac{\Phi}{\text{AB} \cos \theta_A}$$

Therefore

$$\text{Field strength in iron, } H_I = \frac{B_I}{\mu_I} = \frac{\Phi}{\mu_I \text{AB} \cos \theta_I}$$

and

$$\text{Field strength in air, } H_A = \frac{B_A}{\mu_A} = \frac{\Phi}{\mu_A \text{AB} \cos \theta_A}$$

Consider that a unit pole is moved round the closed path ACBDA—the total work done must be zero since no current is linked. No work will be done in the movements along AC and DB since these paths are normal to the field direction, i.e. equipotentials. Thus

Work done in movement CB = Work done in movement DA

i.e.

$$H_A \times \text{CB} = H_I \times \text{AD}$$

or

$$\frac{\Phi}{\mu_A \text{AB} \cos \theta_A} \text{AB} \sin \theta_A = \frac{\Phi}{\mu_I \text{AB} \cos \theta_I} \text{AB} \sin \theta_I$$

Therefore

$$\frac{\tan \theta_A}{\mu_A} = \frac{\tan \theta_I}{\mu_I}$$

and

$$\theta_A = \tan^{-1} \left(\frac{\mu_A}{\mu_I} \tan \theta_I \right)$$

Since $\mu_A \ll \mu_I$, $(\mu_A/\mu_I) \tan \theta_I$ is very small, and hence θ_A must be nearly zero. Thus a streamline will cross an iron-air boundary almost normally.

A non-varying magnetic field (i.e. one that is set up by a permanent-magnet or a direct current) will be unaffected by the presence of a conductor provided that the conductor is non-magnetic. A permanent field, e.g. the earth's magnetic field, will, for instance, penetrate a block of brass. If the field is varying (i.e. set up by an alternating current), then in general it will not penetrate a conducting material since the eddy currents which would be set up within the material would oppose the magnetic field. The actual depth to which a magnetic field will effectively penetrate a conductor decreases as the frequency increases. The depth of penetration also depends on the resistivity and permeability of the conductor.

If the depth of penetration of flux is small, then the eddy currents in the conductor must be effectively neutralizing any flux which is tending to enter the conductor. Thus all the flux must be parallel to the conductor surface, i.e. the streamlines will be parallel to the conducting surface, and the magnetic equipotentials will intersect the conducting surface normally.

7.21 Shielding for Static Magnetic Fields

The object of shielding (or screening) is to prevent a magnetic field from existing at some particular point. For steady (or static) fields, the only method of achieving shielding is to provide a low-reluctance magnetic path for the stray flux, in such a way that this flux bypasses the shielded point. Since no material has infinite

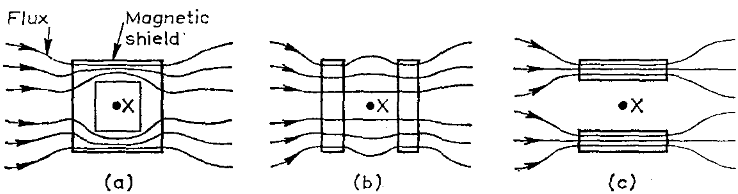


Fig. 7.17 PRINCIPLE OF SCREENING FROM STATIC FIELDS
(*a*) Very good screening
(*b*) Negligible screening
(*c*) Good screening

permeability, perfect shielding is not possible, but shields made of materials which have high permeabilities at low flux densities give satisfactory results.

The best type of shield is indicated in Fig. 7.17(*a*), where the shielded point (X) is entirely surrounded by a magnetic container.

It should be noted that for weak fields the thickness of the shield is relatively unimportant, since the flux tends to follow the outside edge of the shield. Indeed better shielding is usually provided by two thin shields with an air gap between them, than by one thick shield.

In Fig. 7.17(*b*) the effect of removing the sides of the shield which are parallel to the magnetic field is shown. The field at the shielded point X is actually stronger than if no shielding materials were present, owing to the concentration of flux in the magnetic material. If, however, the other sides of the shield are removed (as in Fig. 7.17(*c*)), it will be seen that the shielding is still fairly effective, since the interfering field bypasses the point X through the magnetic material.

7.22 Shielding for Alternating Magnetic Fields

Conducting sheets are used to restrict the extent of alternating magnetic fields. The thickness of the shielding plate should be greater than the depth of penetration of the field at the operating frequency, though considerable shielding may be obtained with plates of smaller thickness.

When shielding an alternating magnetic field (e.g. that of a coil) it is important to arrange that the eddy currents in the shield do not give a high power loss. This usually means that the shield should not be too small.

Sometimes sufficient shielding can be obtained by a few short-circuited copper turns, placed round the object to be shielded in

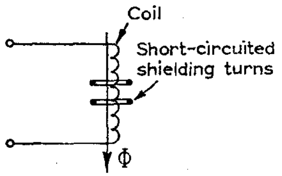


Fig. 7.18 USE OF SHORT-CIRCUITED TURNS FOR SHIELDING FROM ALTERNATING FIELDS

such a direction that the axis of the turns is in the direction of the magnetic field. Fig. 7.18 shows such an arrangement, which may be used either to prevent an external field from affecting the coil, or to restrict the field of the coil itself. An external field linking the coil would also link the short-circuited turns and induce eddy currents in them. These eddy currents would set up an m.m.f. opposing the flux. Similarly the field set up by a current in the coil would be restricted to the region within the short-circuited turns, since any flux beyond this limit would link the short-circuited turns and be opposed by the m.m.f. of the eddy-currents in the turns.

7.23 Skin Effect

A direct current flowing in a uniform conductor will distribute itself uniformly over the cross-section of the conductor. An alternating current on the other hand always tends to flow at the surface of a conductor. This is called *skin effect*. The effect is more pronounced at high frequencies and with conductors of large cross-section.

Consider the round conductor of Fig. 7.19 and suppose that a direct current is flowing into the plane of the paper. There will be a magnetic field outside the conductor and there will also be a magnetic field inside the conductor, as shown by the dotted line. This inner field is produced by the current at the centre of the conductor. The portion of the conductor inside the dotted line may be regarded as a separate conductor in parallel with the portion of the conductor outside the dotted line. The inner conductor is linked by the magnetic field from the dotted line outwards, while the outer conductor is only linked by the magnetic field from the conductor surface outwards. Thus the inner conductor may be regarded as having a larger inductance than the outer conductor, since the

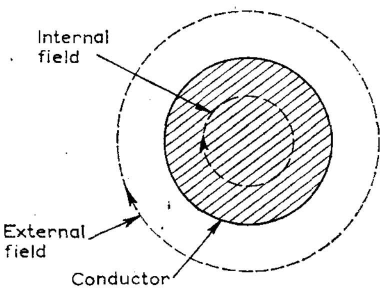


Fig. 7.19 PERTAINING TO SKIN EFFECT

former links a larger magnetic field than the latter. The larger inductance does not affect direct currents, but obviously the inner conductor will have a larger impedance to alternating currents than the outer, and hence the current and current density will tend to be greater in the outer conductor. The effect increases with frequency, until at high frequencies the current is almost entirely in the "outer skin" of the conductor.

Exact analysis shows that the depth of penetration of the current is given by the same equation as is the depth of penetration of magnetic flux.

7.24 Inductance due to Low-frequency Internal Linkages

For a conductor used at high frequencies (i.e. where the depth of penetration is small compared with the conductor cross-section), the internal linkages are negligible and the circuit inductance is simply the inductance due to the fields in the surrounding space. At very low frequencies (i.e. where the current distribution may be considered uniform over the section of the conductor), the inductance due to internal linkages has a maximum value. At other frequencies this inductance will have a value between this maximum value and zero.

Consider a conductor of radius a carrying a total current I amperes uniformly distributed over the conductor cross-section (Fig. 7.20).

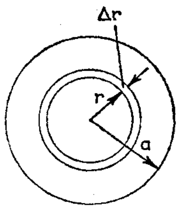


Fig. 7.20 PERTAINING TO LOW-FREQUENCY INTERNAL LINKAGES IN A CONDUCTOR

$$\text{Current density at all points on the cross-section} = \frac{I}{\pi a^2}$$

Consider a unit magnetic pole moved round a path of radius r within the conductor.

Current enclosed by path of radius r

$$= \text{Current density} \times \text{Area enclosed}$$

$$= \frac{I}{\pi a^2} \times \pi r^2 = \frac{Ir^2}{a^2}$$

By the work law,

Work done in moving unit pole round closed path

$$= \text{Ampere-turns linked}$$

Thus

$$H_r \times 2\pi r = \frac{Ir^2}{a^2} \times 1 \qquad \text{and} \qquad H_r = \frac{Ir}{2\pi a^2}$$

where H_r is the field strength at radius r and there is only one turn.

$$\text{Flux density at radius } r = B_r = \mu H_r = \frac{\mu Ir}{2\pi a^2}$$

For 1 metre depth of the conductor,

$$\text{Flux within element of thickness } \Delta r = \frac{\mu I r \Delta r}{2\pi a^2}$$

The flux in the element links the portion r^2/a^2 of the total conductor. Therefore

$$\text{Linkages due to flux in element} = \frac{\mu I r^3}{2\pi a^4}\,\Delta r$$

and

Total linkages per metre due to flux in conductor

$$= \frac{\mu I}{2\pi a^4}\int_0^a r^3\,dr = \frac{1}{4}\frac{\mu I}{2\pi}$$

Inductance per metre due to internal flux

$$= \text{Internal flux linkages per ampere}$$

$$= \frac{1}{4}\frac{\mu}{2\pi}\ \text{henrys/metre} \tag{7.28}$$

It is notable that this inductance is independent of the conductor radius.

7.25 Inductance of a Concentric Cable

Assume that the core of a concentric cable carries a current of I amperes in one direction while the sheath carries the same current in the opposite direction.

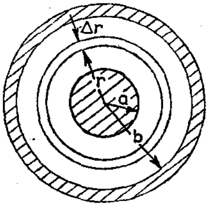


Fig. 7.21 INDUCTANCE OF A CONCENTRIC CABLE

Consider a unit pole moved once round a path of radius r in the interspace between core and sheath, i.e. so that it links the core current (Fig. 7.21). By the work law,

$$H_r \times 2\pi r = I$$

or

$$H_r = \frac{I}{2\pi r}$$

where H_r is the field strength at radius r.

Flux density at radius $r = B_r = \mu H_r$

Consider unit depth of the element of thickness Δr at radius r.

$$\text{Flux within unit depth of the element} = B_r \Delta r \times 1$$

$$= \frac{\mu I}{2\pi r} \Delta r$$

This flux links the core of the cable, i.e. links the loop of the cable formed by the core and the sheath.

$$\text{Flux linkages due to element flux per metre of cable} = \frac{\mu I \Delta r}{2\pi r}$$

$$\text{Total flux linkages per metre} = \frac{\mu I}{2\pi} \int_a^b \frac{dr}{r} = \frac{\mu I}{2\pi} \log_e \frac{b}{a}$$

Therefore

$$\text{Inductance per metre} = \mu \frac{\log_e \frac{b}{a}}{2\pi} \text{ henrys/metre} \tag{7.29}$$

To this should be added terms representing the inductances due to internal core linkages and internal sheath linkages. Both of these are negligible at high frequencies, and the sheath linkages are also negligible (to a good accuracy) at low frequencies since the sheath is usually relatively thin and has a weak magnetic field over most of its cross-section.

Total inductance per metre at low frequencies

$$= \frac{1}{4} \frac{\mu}{2\pi} + \mu \frac{\log_e \frac{b}{a}}{2\pi} \quad \text{henrys/metre} \tag{7.30}$$

7.26 Inductance of a Twin Line

Consider two isolated, long, straight, parallel conductors of radius a metres, spaced D metres apart, and each carrying a current of I amperes in opposite directions. D is assumed large compared with a. The magnetic field surrounding the conductors is shown in Fig. 7.22(*a*). The field is most easily analysed by considering each conductor alone in turn.

Consider conductor A only carrying current (Fig. 7.22(*b*)).

$$\text{Magnetic field strength at radius } r = H_r = \frac{I}{2\pi r}$$

$$\text{Flux density at radius } r = B_r = \frac{\mu I}{2\pi r}$$

Therefore

$$\text{Total flux in 1 metre depth of element} = B_r \Delta r \times 1 = \frac{\mu I}{2\pi r} \Delta r$$

This flux links conductor A once.

$$\text{Linkage with conductor A due to flux in element} = \frac{\mu I}{2\pi} \frac{\Delta r}{r}$$

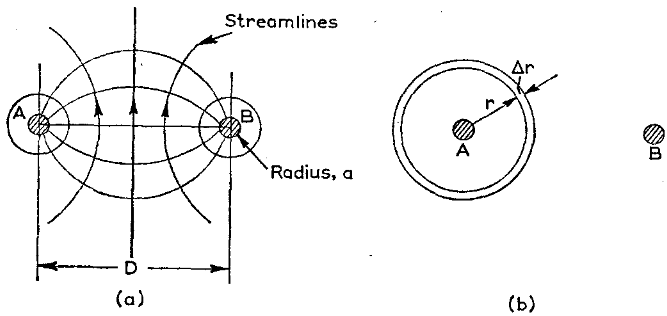


Fig. 7.22 INDUCTANCE OF A TWIN LINE

Therefore

Total linkages with conductor A due to current in conductor A

$$= \frac{\mu I}{2\pi} \int_a^R \frac{dr}{r} = I \frac{\mu}{2\pi} \log_e \frac{R}{a}$$

where R is a very large radius at which the magnetic field strength may be regarded as zero. Similarly,

Total linkages with conductor B due to current in A

$$= \frac{\mu I}{2\pi} \int_D^R \frac{dr}{r} = I \frac{\mu}{2\pi} \log_e \frac{R}{D}$$

since magnetic flux at a radius greater than D about A will link B.

Similarly when only B carries current of $-I$ amperes,

$$\text{Total linkages with B due to current in B} = -I \frac{\mu}{2\pi} \log_e \frac{R}{a}$$

and

$$\text{Total linkages with A due to current in B} = -I \frac{\mu}{2\pi} \log_e \frac{R}{D}$$

Therefore

$$\text{Total linkages with A} = I\frac{\mu}{2\pi}\log_e\frac{R}{a} - I\frac{\mu}{2\pi}\log_e\frac{R}{D}$$

$$= I\frac{\mu}{2\pi}\log_e\frac{D}{a}\ \text{weber-turns/metre}$$

and similarly,

$$\text{Total linkages with B} = I\frac{\mu}{2\pi}\log_e\frac{D}{a}$$

For 1 metre length of twin lines,

$$\text{Total inductance} = \text{Flux linkages/ampere}$$

$$= 2\mu\frac{\log_e\dfrac{D}{a}}{2\pi}\quad\text{henrys/metre} \tag{7.31}$$

This does not include linkages internal to each line. If these are to be included, the inductance at low frequency is

Total inductance per loop metre

$$= \frac{1}{2}\frac{\mu}{2\pi} + 2\mu\frac{\log_e\dfrac{D}{a}}{2\pi}\quad\text{henrys/metre} \tag{7.32}$$

7.27 Inductance of a Single Line above a Conducting Plane

It has been shown (Section 7.20) that an alternating flux cannot penetrate (beyond a certain depth) a conducting sheet; thus, if a "thick" conducting sheet is to be introduced into an alternating magnetic field without affecting the field, the sheet must be so introduced that its surface is everywhere along streamlines. In Fig. 7.22(*a*) an infinite flat sheet could be introduced along the medial streamline between the two conducting wires without affecting the field. If the frequency were such that the sheet thickness was considerably greater than the penetration depth, then the sheet would completely divide the field into two independent parts. Either line could then be removed without affecting the field around the other line.

Applying these deductions in reverse to the actual system of Fig. 7.23, the field between the actual wire and the conducting sheet will be the same as it would be if the conducting sheet were replaced by an *image conductor* carrying the same current as the actual conductor (but in the opposite direction), at a depth h below the surface of the conducting sheet. The calculation of the inductance of the actual conductor system may be made on this basis.

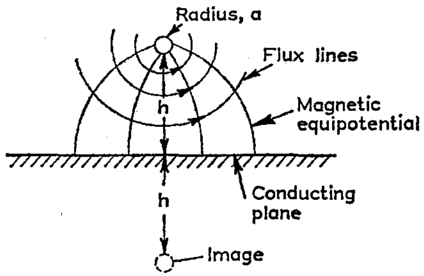


Fig. 7.23 INDUCTANCE OF A WIRE ABOVE A PARALLEL CONDUCTING PLANE

Linkages with actual conductor due to its own current

$$= \frac{\mu I}{2\pi} \log_e \frac{R}{a}$$

Linkages with actual conductor due to image current

$$= -\frac{\mu I}{2\pi} \log_e \frac{R}{2h}$$

where R is a very large distance, according to the method of the previous section. Therefore

$$\text{Total linkages with actual conductor per metre} = \frac{\mu I}{2\pi} \log_e \frac{2h}{a}$$

Inductance of actual conductor per metre

$$= \frac{\mu}{2\pi} \log_e \frac{2h}{a} \quad \text{henrys/metre} \qquad (7.33a)$$

To this must be added the inductance due to the internal flux linkages within the conductor, giving, at low frequency only,

$$L_{eff} = \frac{1}{4}\frac{\mu}{2\pi} + \mu \frac{\log_e \dfrac{2h}{a}}{2\pi} \quad \text{henrys/metre} \qquad (7.33b)$$

7.28 Equivalent Phase Inductance of a Three-phase Line

A simple expression may be deduced for the equivalent phase inductance of an isolated 3-phase 3-wire line if the line is uniformly transposed and the line spacings are considerably greater than the line diameters. By *equivalent phase inductance* is meant the total linkages with a given wire under 3-phase conditions—this inductance may be considered as a series inductance in each line.

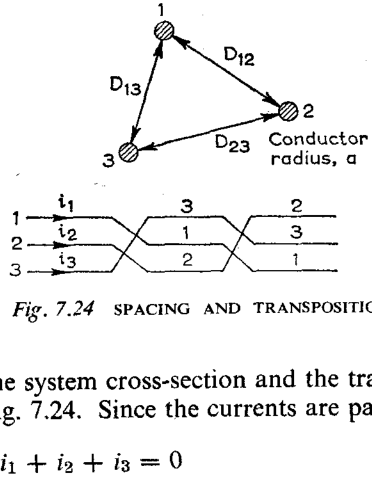


Fig. 7.24 SPACING AND TRANSPOSITION OF A 3-PHASE LINE

The system cross-section and the transposition scheme are shown in Fig. 7.24. Since the currents are part of a three-wire system,

$$i_1 + i_2 + i_3 = 0$$

and $(i_2 + i_3) = -i_1$ at all instants.

Following the method of Section 7.26,

Total linkages with conductor 1 in first position

$$= \frac{\mu i_1}{2\pi} \log_e \frac{R}{a} + \frac{\mu i_2}{2\pi} \log_e \frac{R}{D_{12}} + \frac{\mu i_3}{2\pi} \log_e \frac{R}{D_{13}}$$

Total linkages with conductor 1 in second position

$$= \frac{\mu i_1}{2\pi} \log_e \frac{R}{a} + \frac{\mu i_2}{2\pi} \log_e \frac{R}{D_{23}} + \frac{\mu i_3}{2\pi} \log_e \frac{R}{D_{12}}$$

and

Total linkages with conductor 1 in third position

$$= \frac{\mu i_1}{2\pi} \log_e \frac{R}{a} + \frac{\mu i_2}{2\pi} \log_e \frac{R}{D_{13}} + \frac{\mu i_3}{2\pi} \log_e \frac{R}{D_{23}}$$

Therefore

Average linkage with conductor 1

$$= \frac{1}{3}\left(\frac{3\mu i_1}{2\pi}\log_e\frac{R}{a} + \frac{\mu i_2}{2\pi}\log_e\frac{R^3}{D_{12}D_{23}D_{31}} + \frac{\mu i_3}{2\pi}\log_e\frac{R^3}{D_{12}D_{23}D_{31}}\right)$$

$$= \frac{\mu i_1}{2\pi}\log_e\frac{R}{a} + \frac{\mu(i_2+i_3)}{2\pi}\log_e\frac{R}{\sqrt[3]{(D_{12}D_{23}D_{31})}}$$

$$= \frac{\mu i_1}{2\pi}\log_e\frac{R}{a} - \frac{\mu i_1}{2\pi}\log_e\frac{R}{\sqrt[3]{(D_{12}D_{23}D_{31})}}$$

$$= \frac{\mu i_1}{2\pi}\log_e\frac{\sqrt[3]{(D_{12}D_{23}D_{31})}}{a}$$

Thus

Average equivalent inductance/phase/metre

$$= \mu\,\frac{\log_e\dfrac{\sqrt[3]{(D_{12}D_{23}D_{31})}}{a}}{2\pi} \text{ henrys/metre} \tag{7.34}$$

This does not include the inductance due to internal linkages.

Total low-frequency equivalent inductance/phase/metre

$$= \frac{1}{4}\frac{\mu}{2\pi} + \mu\,\frac{\log_e\dfrac{\sqrt[3]{(D_{12}D_{23}D_{31})}}{a}}{2\pi} \text{ henrys/metre} \tag{7.35}$$

7.29 Determination of Air-gap Permeance by a Mapping Method

The *reluctance* of an air gap is commonly used in elementary treatments, but to treat magnetic fields in the same manner as electrostatic and conduction fields it is preferable to deal with its reciprocal, namely *permeance*:

$$\text{Permeance} = \frac{1}{\text{Reluctance}}$$

or, in symbols,

$$\Lambda = \frac{1}{S}$$

For a uniform field of area a square metres and length l metres,

$$\Lambda = \frac{1}{S} = \frac{\mu a}{l}$$

Supposing a magnetic field is mapped into a number of curvilinear squares, and a flux tube is considered of d metres depth and with curvilinear squares for its ends, as in Fig. 7.4. Then

$$\text{Permeance of tube} = \mu \times \frac{\text{area}}{\text{length}} = \mu \frac{dl}{l} = \mu d$$

where l is the length of one side of the curvilinear square. Therefore

$$\text{Total permeance} = \mu d \frac{m}{n} \quad \text{webers/ampere-turn} \qquad (7.36)$$

where m is the number of parallel squares, i.e. the number of squares across the flux direction, and n is the number of series squares, i.e. the number of squares along the flux direction.

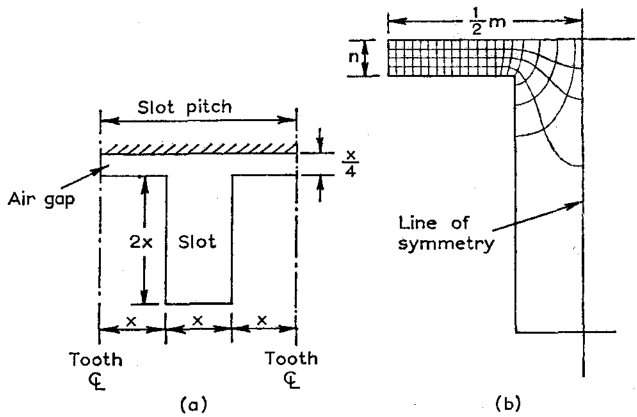


Fig. 7.25

EXAMPLE 7.4 The arrangement of Fig. 7.25(*a*) resembles a slot pitch in an electrical machine. Estimate the reluctance of the slot pitch if the rotor length is 20 cm. The field may be considered uniform over this length.

Assuming that there are similar slots on either side of the one depicted, the teeth centre-lines will be streamlines. Also, by symmetry, the slot centre-line will be a streamline dividing the slot pitch into two symmetrical parts. One part of the slot pitch is enlarged for mapping in Fig. 7.25(*b*).

The iron surfaces are boundary equipotentials which the streamlines intersect normally (by Section 7.20). The division into curvilinear squares is carried out by the same method as previously (Examples 7.1 and 7.3). By eqn. (7.36),

$$\text{Total permance per slot pitch} = \mu_0 d \times \frac{m}{n} = \mu_0 \times \frac{20}{100} \times \frac{2 \times 20{\cdot}1}{4}$$

$$= 2{\cdot}52 \times 10^{-6}\,\text{Wb/At}$$

Therefore

$$\text{Reluctance per slot pitch} = \frac{1}{2{\cdot}52} \times 10^{6} = \underline{\underline{397{,}000\,\text{At/Wb}}}$$

7.30 Determination of Inductance by a Mapping Method

In simple magnetic fields (e.g. fields without iron magnetic circuits) it is generally the inductance of the arrangement rather than the reluctance or permeance which is required. The inductance may be simply calculated from a field plot, provided that all the magnetic flux may be assumed to link all the turns of the conductor arrangement producing the field.

$$\text{Flux, } \Phi = \frac{\text{M.M.F.}}{\text{Reluctance}} = \frac{IN}{S} = IN\Lambda$$

But

$$\text{Inductance, } L = \text{Flux linkages/ampere} = \frac{\Phi N}{I} = N^2\Lambda$$

i.e.

$$L = N^2\mu d\frac{m}{n} \quad \text{henrys} \tag{7.37}$$

EXAMPLE 7.5 For an isolated twin-wire line with a spacing-to-diameter ratio of 10, calculate the inductance per metre of the line by (*a*) direct calculation, and (*b*) a field plot. Neglect linkages within the conductors.

$$\text{Inductance per metre} \approx 2\mu_0 \frac{\log_e \dfrac{D}{a}}{2\pi}$$

$$= \frac{2 \times 4\pi \times 10^{-7} \times \log_e 20}{2\pi} = \underline{\underline{1{\cdot}2 \times 10^{-6}\,\text{H/m}}}$$

Fig. 7.26(*a*) on the next page shows the conductor system. The centre-line through the conductors is a symmetrical equipotential and the centre-line between the conductors is a symmetrical streamline. The field is thus divided into four symmetrical parts, and one of these is enlarged in Fig. 7.26(*b*) for field plotting. From the field plot,

$$\text{Inductance per metre} = \mu_0 \times 1 \times \frac{m}{n} = 4\pi \times 10^{-7} \times \frac{2 \times 4{\cdot}5}{2 \times 5{\cdot}1}$$

$$= \underline{\underline{1{\cdot}11 \times 10^{-6}\,\text{H/m}}}$$

PROBLEMS

7.1 A concentric cable has a core diameter of 1cm and a sheath diameter of 5cm. If the core is displaced from the true centre of the cable by 0·75cm, calculate the capacitance per metre run by field plotting. ϵ_r for the dielectric is 3. From the same field plot write down the inductance per metre run of cable. (*H.N.C.*)

Ans. 0·00012μF, 0·28μH.

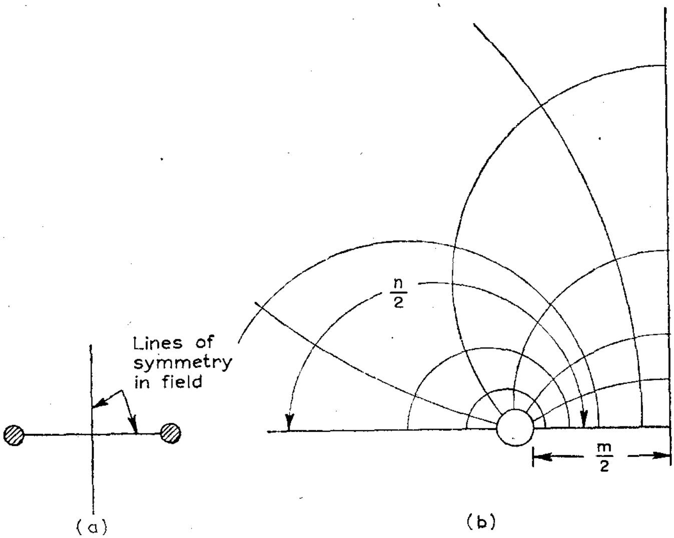


Fig. 7.26
(*a*) Actual spacing (*b*) Enlarged spacing for field plot

7.2 The plan view of a poor electrolytic plating bath is shown in Fig. 7.27. A is the anode plate and the central uniform cylinder is the object to be plated,

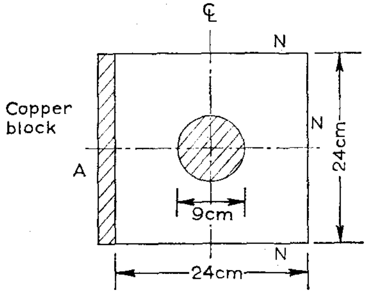


Fig. 7.27

while N is the non-conducting wall of the container. Determine the ohmic resistance of the bath if the depth of the electrolyte is 6 cm and its resistivity is 100 Ω-cm. Draw also a polar diagram showing the variation in thickness of plating round the circumference of the cylinder.

Ans. 7·2 Ω.

7.3 Fig. 7.28 represents very approximately a conductor lying in an open slot. Plot the electrostatic field for the system and estimate the capacitance per metre length of conductor. Explain the dependence of the electric stress on the shape of the corners.

Ans. 85 pF.

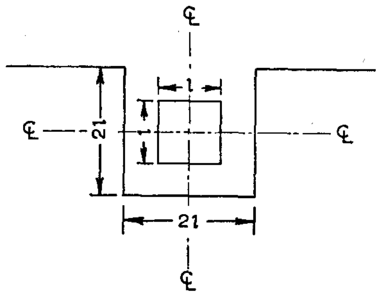


Fig. 7.28

7.4 Explain why magnetic flux may be assumed to emerge from an iron boundary into air at right angles to the surface.

Use field plotting to determine the reluctance per unit depth of the air gap, shown in Fig. 7.29, between two long flat iron plates. Determine the percentage

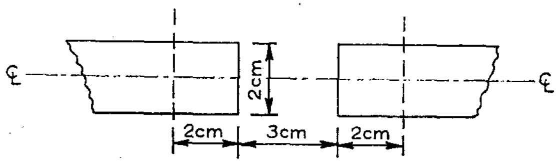


Fig. 7.29

error which is caused by neglecting fringing. Neglect the field beyond the dashed lines. (*H.N.C.*)

Ans. 450,000 At/Wb, 160 per cent.

7.5 Two parallel infinitely long straight conductors of diameter 0·2 cm are spaced 10 cm apart and relatively far from conducting objects. Draw the shape of the field surrounding these conductors and from it estimate the inductance and capacitance per unit length of the conductors. Check the result by calculation on the usual basis.

Ans. 1·84 μH/m, 6·03 pF/m.

7.6 From Ohm's law in its usual circuit form derive an expression for Ohm's law applicable to a point in an electrical conductor (i.e. the conduction field form).

Explain the meaning of the terms (*a*) flowline, (*b*) equipotential, with reference to a conductor. Show that these must necessarily intersect at right angles.

By a field-plotting method estimate the percentage increase in the resistance of the busbar length shown in Fig. 7.30 due to the four holes. Assume the busbar thickness uniform and the current distribution uniform at each end.

(*H.N.C.*)

Ans. 20 per cent.

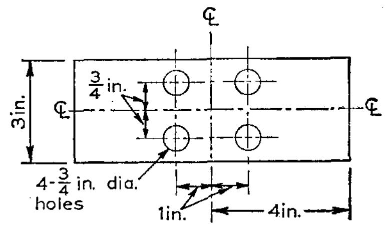


Fig. 7.30

7.7 If the electromagnetic field in the interspace between two long parallel concentric conductors is "mapped" by a number of "curvilinear squares" derive an expression for the inductance per metre run of the conductors in terms of numbers of squares and the permeability of the interspace material. (Neglect inductance due to flux linkages within the conductors.)

A conductor of cross-section 1 in. × ½ in. is enclosed in a tube of square cross-section whose internal side is 2 in. (Fig. 7.31). Draw (several times full

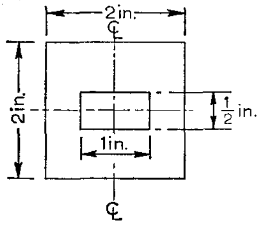


Fig. 7.31

size) a symmetrical part of this cross-section and hence estimate the inductance of the arrangement per metre run. (Assume a non-magnetic interspace and neglect flux linkages within the conductors.)

(*H.N.C.*)

Ans. $0{\cdot}157\,\mu\text{H/m}$.

7.8 A liquid rheostat is to be constructed with its outer electrode a hollow steel cylinder of 1·5 m internal diameter; this cylinder is also the liquid container. The inner electrode is a rod of 0·5 m diameter. Calculate the resistance between electrodes per metre length of the rheostat when the liquid has a resistivity 100 Ω-m and the centre-line of the inner electrode is coincident with the centre-line of the outer cylinder.

By means of a field plot find the resistance when the centre-line of the rod is parallel to the centre-line of the cylinder but spaced 0·25 m from it. (*H.N.C.*)

Ans. 17·5Ω, 15Ω.

7.9 Calculate the loop inductance and capacitance of a 1 km length of single-phase line having conductors of diameter 1 cm and spaced 72 cm apart. If this line is to be converted to a 3-phase line by the addition of a third conductor of the same cross-section as the originals, calculate the phase-to-neutral inductance and capacitance when the third conductor (*a*) is in line with the first two and 72 cm from the nearest, (*b*) forms an equilateral triangle with the first two. Assume regular transposition and neglect earth effects.

Ans. 0·00212 H/loop km; 0·00111 H/km; 0·00106 H/km; 0·0056 μF/km; 0·0104 μF/km; 0·0112 μF/km.

7.10 Derive an expression for the inductance per metre of a coaxial cable of core diameter *d* metres and internal sheath diameter *D* metres. (The sheath thickness may be neglected.)

A coaxial cable 8·05 km long has a core 1 cm diameter and a sheath 3 cm diameter of negligible thickness. Calculate the inductance and capacitance of the cable, assuming non-magnetic materials and a dielectric of relative permittivity 4.

For this cable calculate also (*a*) the charging current when connected to a 66 kV, 50 Hz supply (neglect inductance), (*b*) the surge impedance of the cable,* (*c*) the surge velocity for the cable.*

Ans. $21{\cdot}7 \times 10^{-4}$ H, 1·63 μF, (*a*) 33·8 A, (*b*) 33Ω, (*c*) $1{\cdot}5 \times 10^{8}$ m/s.

7.11 A single-core lead-sheathed cable has a conductor of 10 mm diameter and two layers of insulating material each 10 mm thick. The permittivities are 4 and 2·5, and the layers are placed to allow the greater applied voltage. Calculate the potential gradient at the surface of the conductor when the potential difference between the conductor and the lead sheathing is 60 kV. (*H.N.C.*)

Ans. 6·29 kV/mm.

7.12 A single-core cable has a conductor diameter *d* and an inside sheath diameter *D*. Show that the maximum voltage which can be applied so as not to exceed the permissible electric stress *E* is given by $\frac{Ed}{2} \times \log_e \frac{D}{d}$.

Find this voltage for a cable in which $D = 8$ cm, $d = 1$ cm and $E = 50$ kV/cm (r.m.s.).

If *D* is fixed at 8 cm, find the most suitable value for *d* so that the greatest voltage can be applied to the cable. (*H.N.C.*)

Ans. 52 kV, 2·95 cm.

* *See* Chapter 16.

Chapter 8

TWO-PORT NETWORKS

Networks in which electrical energy is fed in at one pair of terminals and taken out at a second pair of terminals are called *two-port*

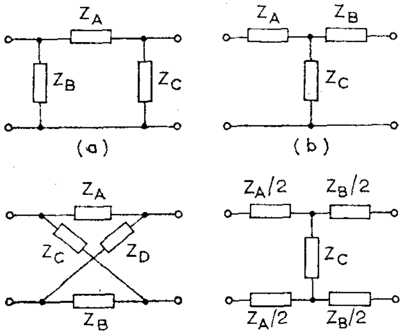


Fig. 8.1 BASIC PASSIVE TWO-PORT NETWORKS

(*a*) π-network
(*b*) T-network
(*c*) Lattice network
(*d*) Balanced T-network

networks. Thus a transmission line (whether for power or communications) is a form of two-port network. So too are attenuators, electric wave filters, electronic amplifiers, transformers, etc. The network between the input port and the output port is a transmission network for which a known relationship exists between the input and output voltages and currents.

Two-port networks are said to be *passive* if they contain only passive circuit-elements, and *active* if they contain sources of e.m.f. Thus an electronic amplifier is an active two-port. Passive two-port networks are symmetrical if they look identical from both the input and output ports. Fig. 8.1 shows some typical passive two-port networks. The π-section shown at (*a*) is symmetrical if $Z_B = Z_C$. At (*b*) is shown a T-section, which is symmetrical if $Z_A = Z_B$. The lattice section at (*c*) is symmetrical and balanced if $Z_A = Z_B$ and $Z_C = Z_D$. This network is simply a rearranged Wheatstone bridge. The π- and T-networks shown at (*a*) and (*b*) have one common terminal, which may be earthed, and are therefore said to be *unbalanced*. The *balanced* form of the T-circuit is shown at (*d*).

In this simple introduction to two-port theory, only passive symmetrical circuits will be considered.

8.1 Characteristic Impedance

The input impedance of a network is the complex ratio of voltage to current at the input terminals. For a two-port network the input

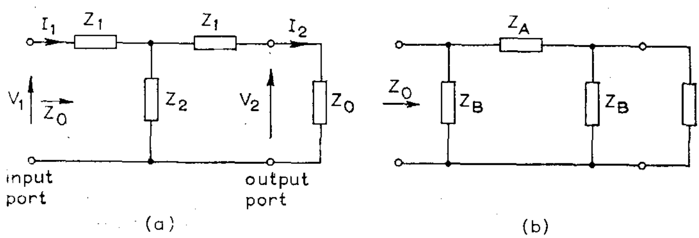


Fig. 8.2 ILLUSTRATING THE CONCEPT OF CHARACTERISTIC IMPEDANCE

impedance will normally vary according to the load impedance across the output terminals. It is found that for any passive two-port network a load impedance can always be found which will give rise to an input impedance which is the same as the load impedance. For an unsymmetrical network this is called the *iterative impedance*, and its value will depend upon which port (i.e. which pair of terminals) is taken to be the input and which the output port. For a symmetrical network there is only one value for iterative impedance, and this is called the *characteristic impedance*, Z_0, of the symmetrical two-port network.

Consider the symmetrical T-circuit of Fig. 8.2(*a*) terminated in an impedance Z_0 such that

$$\frac{V_1}{I_1} = Z_0 = \frac{V_2}{I_2}$$

From simple circuit theory the impedance seen looking into the input port is

$$\frac{V_1}{I_1} = Z_1 + \frac{Z_2(Z_1 + Z_0)}{Z_1 + Z_2 + Z_0}$$

$$= \frac{Z_1^2 + Z_1Z_2 + Z_1Z_0 + Z_2Z_1 + Z_2Z_0}{Z_1 + Z_2 + Z_0}$$

If this has to be equal to Z_0, then

$$\begin{aligned} Z_1^2 + 2Z_1Z_2 + Z_0(Z_1 + Z_2) &= Z_0(Z_1 + Z_2 + Z_0) \\ &= Z_0(Z_1 + Z_2) + Z_0^2 \end{aligned}$$

or

$$Z_0 = \sqrt{\{Z_1(Z_1 + 2Z_2)\}} \tag{8.1}$$

A useful general expression for Z_0 can be deduced by investigating the input impedance of the network when the output terminals are (*a*) short-circuited, and (*b*) open-circuited. Thus the short-circuit input impedance is

$$Z_{sc} = Z_1 + \frac{Z_1Z_2}{Z_1 + Z_2} = \frac{Z_1(Z_1 + 2Z_2)}{Z_1 + Z_2}$$

and the open-circuit input impedance is

$$Z_{oc} = Z_1 + Z_2$$

Hence

$$Z_{sc}Z_{oc} = Z_1(Z_1 + 2Z_2)$$

Comparison of this with eqn. (8.1) yields the important general relation

$$Z_0 = \sqrt{(Z_{sc}Z_{oc})} \tag{8.2}$$

In the same way, for the symmetrical-π circuit of Fig. 8.2 (*b*), if the input impedance when the network is terminated by Z_0 is equal to Z_0, then

$$Z_0 = \frac{Z_B(Z_A + Z_BZ_0/(Z_B + Z_0))}{Z_B + Z_A + Z_BZ_0/(Z_B + Z_0)}$$

whence, after some manipulation,

$$Z_0 = \sqrt{\frac{Z_AZ_B^2}{Z_A + 2Z_B}} \tag{8.3}$$

It is not difficult to show that eqn. (8.2) applies in this case also. Hence for any passive symmetrical network,

$$Z_0 = \sqrt{(Z_{sc}Z_{oc})} \tag{8.2a}$$

since by suitable manipulation all such circuits can be represented by an equivalent π- or T-section (note, however, that the equivalent sections may contain unrealizable circuit-elements such as negative resistance).

The concept of characteristic impedance is important since it facilitates the design of networks with specific transmission properties. It is important to realize that terminating a network in its characteristic impedance does not mean that the input and output voltages (or currents) are equal—only that $V_1/I_1 = V_2/I_2$.

8.2 Logarithmic Ratios: the Decibel

It is often convenient to express the ratio of two powers P_1 and P_2 in logarithmic form. If natural (Napierian) logarithms are used, the ratio is said to be in *nepers*. Thus

$$\text{Power ratio in nepers (Np)} = \tfrac{1}{2} \log_e \frac{P_1}{P_2} \tag{8.4}$$

If logarithms to base 10 are used, then the ratio is said to be in *bels*. The bel is rather a large unit, and one-tenth of a bel, or *decibel* (dB) is more commonly used. Thus

$$\text{Power ratio in decibels} = 10 \log_{10} \frac{P_1}{P_2} \tag{8.5}$$

The decibel is roughly the smallest difference in power level between two sound waves which is detectable as a change in volume by the human ear.

If the powers P_1 and P_2 refer to power developed in *two equal resistors*, R, then

$$P_1 = \frac{V_1^2}{R} \qquad \text{and} \qquad P_2 = \frac{V_2^2}{R}$$

so that the ratio can be expressed as

$$\text{Ratio in dB} = 10 \log_{10} \frac{V_1^2/R}{V_2^2/R} = 20 \log_{10} \frac{V_1}{V_2} \tag{8.6}$$

This is called the *logarithmic voltage ratio*, but it is really a power ratio. Strictly it should not be applied to the ratio of two voltages across different resistances, but this is sometimes done.

Similarly, if currents I_1 and I_2 in two equal resistors, R, give powers P_1 and P_2, then

$$\text{Ratio in dB} = 10 \log_{10} \frac{I_1^2 R}{I_2^2 R} = 20 \log_{10} \frac{I_1}{I_2} \tag{8.6a}$$

It should be particularly noted that the decibel notation applies to the *sizes* of voltages and currents—it gives no phase information.

For example, if a two-port network is terminated in its characteristic impedance and if the input voltage is $\sqrt{2}$ times the output voltage, then since the input and the output voltages refer to the equal impedances, the voltage ratio is given by

$$\text{Input/output ratio in dB} = 20 \log_{10} \frac{V_1}{V_2} = 20 \log_{10} \sqrt{2} \approx 3\,\text{dB}$$

Note that the output/input voltage ratio can be expressed in logarithmic units in the same manner. Thus in the above case,

$$\text{Output/input ratio in dB} = 20 \log_{10} \frac{V_2}{V_1} = 20 \log_{10} \frac{1}{\sqrt{2}}$$

$$= 20(\log_{10} 1 - \log_{10}\sqrt{2}) \approx -3\text{dB}$$

i.e. the minus sign indicates that the ratio denotes a power loss, while the positive sign indicates a power gain.

8.3 Insertion Loss

When a two-port network is inserted between a generator and a load, the voltage and current at the load will generally be less than

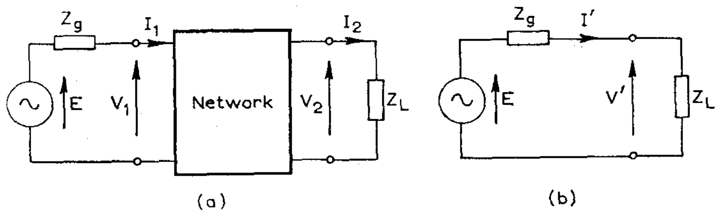


Fig. 8.3 PERTAINING TO INSERTION LOSS

those which would arise if the load were connected directly to the generator. The *insertion loss ratio* of a two-port network is defined as the ratio of the voltage across the load when it is connected directly to the generator to the voltage across the load when the two-port network is inserted. Referring to Fig. 8.3,

$$\text{Insertion loss ratio, } A_L = \frac{V'}{V_2} = \frac{I'}{I_2} \tag{8.7}$$

Since V' and V_2 both refer to voltages across the same impedance, Z_L, it is permissible to express the insertion loss ratio as

$$\text{Insertion loss} = 20 \log_{10} \frac{V'}{V_2} \quad \text{decibels} \tag{8.8}$$

$$= 20 \log_{10} \frac{I'}{I_2} \quad \text{decibels} \tag{8.9}$$

Of particular importance is the case when the network is terminated in its characteristic impedance, Z_0. The network is said to be matched, and in this case the input impedance is also Z_0 so that the insertion loss is simply the ratio of input to output voltage (or current), i.e.

$$\text{Insertion loss} = 20 \log_{10} \frac{V_1}{V_2} \text{ (for network terminated in } Z_0) \tag{8.10}$$

$$= 20 \log_{10} \frac{I_1}{I_2} \tag{8.11}$$

It will be seen that in the matched condition, the impedance of the generator does not affect the insertion loss.

EXAMPLE 8.1 For the attenuator pad shown in Fig. 8.4 determine the characteristic impedance and the insertion loss when the attenuator feeds a matched load. $R_1 = 312\Omega$ and $R_2 = 423\Omega$.

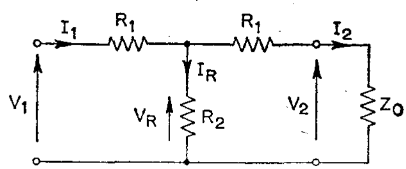


Fig. 8.4 SYMMETRICAL T-NETWORK

From eqn. (8.2) the characteristic impedance is

$$Z_0 = \sqrt{(Z_{sc}Z_{oc})} = \sqrt{\left\{\left(R_1 + \frac{R_1R_2}{R_1 + R_2}\right)(R_1 + R_2)\right\}}$$

$$= \sqrt{(R_1^2 + 2R_1R_2)} = \sqrt{\{312^2 + (2 \times 312 \times 423)\}}$$

$$= \underline{\underline{600\Omega}}$$

The ratio of V_1/V_2 is obtained as follows. Since $V_2 = I_2Z_0$, it follows that

$$A_L = \frac{V_1}{V_2} = \frac{I_1Z_0}{I_2Z_0} = \frac{I_1}{I_1 \dfrac{R_2}{R_1 + Z_0 + R_2}} = \frac{R_1 + R_2 + Z_0}{R_2}$$

$$= \frac{1{,}335}{423} = \underline{\underline{3{\cdot}16}}$$

or

$$\text{Insertion loss} = 20 \log_{10} 3{\cdot}16 = \underline{\underline{10\,\text{dB}}}$$

EXAMPLE 8.2 For the simple π-section shown in Fig. 8.5 determine the range of frequencies over which the insertion loss is zero, when the section is terminated in its characteristic impedance.

It is apparent that when the frequency is zero, there is no insertion loss (since the inductor behaves as a short-circuit), and that when the frequency is infinite the capacitors act as a short-circuit and the insertion loss is infinite. The network is a *low-pass filter* section, i.e. it permits signals up to a certain *cut-off frequency*

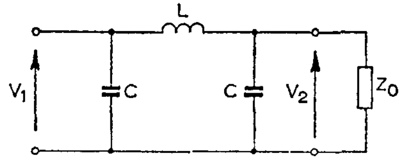


Fig. 8.5 SIMPLE LOW-PASS FILTER

to pass through unattenuated, while attenuating all signals above the cut-off frequency.

The insertion loss ratio is

$$A_L = \frac{V_1}{V_2} = \frac{j\omega L + \dfrac{Z_0/j\omega C}{Z_0 + 1/j\omega C}}{\dfrac{Z_0/j\omega C}{Z_0 + 1/j\omega C}}$$

$$= \frac{\left(j\omega L Z_0 + \dfrac{L}{C} + \dfrac{Z_0}{j\omega C}\right) j\omega C}{Z_0} = 1 - \omega^2 LC + \frac{j\omega L}{Z_0}$$

From eqn. (8.3),

$$Z_0 = \sqrt{\frac{j\omega L}{2j\omega C - j\omega^3 LC^2}} = \sqrt{\frac{L}{2C - \omega^2 LC^2}}$$

Note that Z_0 will vary with frequency, and will be infinite when $\omega^2 LC = 2$. At low frequencies Z_0 will be approximately constant and equal to $\sqrt{(L/2C)}$. It is, however, a reference quantity and will therefore represent a pure resistance provided that

$$2C > \omega^2 LC^2 \qquad \text{or} \qquad \omega < \sqrt{\frac{2}{LC}}$$

For this condition,

$$A_L = 1 - \omega^2 LC + \frac{j\omega L\sqrt{(2C - \omega^2 LC^2)}}{\sqrt{L}}$$

so that

$$A_L = \sqrt{\{(1 - \omega^2 LC)^2 + \omega^2 L(2C - \omega^2 LC^2)\}} = 1$$

This means that for all frequencies below that for which $\omega = \sqrt{(2/LC)}$ the insertion loss ratio is unity, i.e. there is no attenuation. This gives the cut-off frequency of the filter.

EXAMPLE 8.3 A resistance voltage divider with a total resistance of 5,000Ω is connected across the output of a generator of internal resistance 600Ω.

Determine the insertion loss ratio if a load of 1,000Ω is connected across the divider at a tapping of (*a*) 2,500 Ω, (*b*) 1,250 Ω. The circuit is shown in Fig. 8.6(*a*).

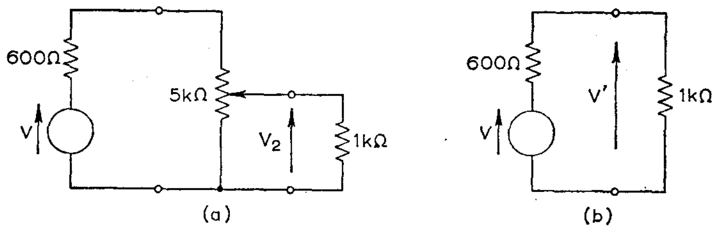


Fig. 8.6 VOLTAGE DIVIDER AS A TWO-PORT NETWORK

Without the voltage divider the voltage across the 1 kΩ load resistor is

$$V' = \frac{V}{1{,}600} \times 1{,}000 = 0{\cdot}625 \text{ V}$$

where V is the generator e.m.f.

(*a*) With the 2·5 kΩ tapping,

$$V_2 = \frac{V}{600 + 2{,}500 + \dfrac{2{,}500 \times 1{,}000}{3{,}500}} \times \frac{2{,}500 \times 1{,}000}{3{,}500} = 0{\cdot}187 \text{ V}$$

Hence the insertion loss ratio is

$$A_L = \frac{V'}{V_2} = \frac{0{\cdot}625}{0{\cdot}187} = \underline{\underline{3{\cdot}34}}$$

(*b*) With the 1·25 kΩ tapping the voltage across the load is

$$V_2 = \frac{V}{600 + 3{,}750 + \dfrac{1{,}250 \times 1{,}000}{2{,}250}} \times \frac{1{,}250 \times 1{,}000}{2{,}250} = 0{\cdot}113 \text{ V}$$

Hence the insertion loss ratio is

$$A_L = \frac{0{\cdot}625}{0{\cdot}113} = \underline{\underline{5{\cdot}5}}$$

Note that the insertion loss is not doubled by halving the tapping.

8.4 Equivalent Circuit of a Short Transmission Line

A single-phase a.c. transmission line consists of two conductors, conventionally called the "go" and "return" conductors. These have series resistance due to the finite resistivity of both conductors. They also have

(*a*) a distributed leakage resistance between the conductors which depends on the conductivity of the insulation,

(*b*) a distributed inductance since they form a current-carrying loop which sets up and links a magnetic field, and
(*c*) a distributed shunt capacitance, since the conductors form the electrodes of an electric field.

The line is a two-port network.

In an overhead line not exceeding 100 km, or a cable not exceeding 20 km, at a frequency of 50 Hz (and proportionately shorter at higher frequencies), the shunt capacitance and leakage may normally be neglected, and the equivalent circuit of the line will then

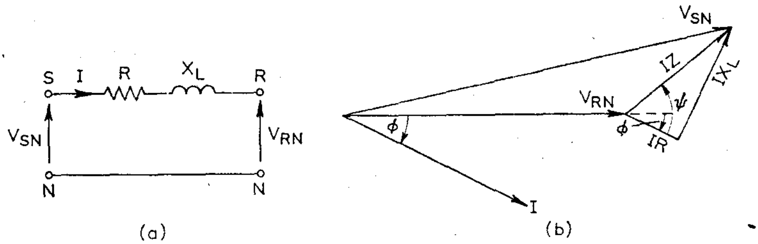


Fig. 8.7 SHORT TRANSMISSION LINE SUPPLYING A LOAD AT A LAGGING PHASE ANGLE ϕ
(*a*) Equivalent circuit (*b*) Complexor diagram

consist of a resistance in series with an inductive reactance. For convenience the impedance of both "go" and "return" conductors are lumped together in one line as shown in Fig. 8.7.

The sending-end voltage V_{SN} is the complexor sum of the receiving-end voltage V_{RN} and the line voltage drop $\boldsymbol{IZ}$, where $\boldsymbol{Z} = R + jX_L$, i.e.

$$V_{SN} = V_{RN} + \boldsymbol{IZ}$$

The *regulation* of the transmission line is defined as the rise in voltage at the receiving end of the line when the load is removed, with the sending-end voltage remaining constant. Hence

$$\text{Regulation} = \frac{V_{SN} - V_{RN}}{V_{RN}} \quad \text{per unit} \tag{8.12}$$

Since V_{RN} will depend on the load, the regulation will also depend on the load current and power factor.

8.5 Medium-length Lines

When 50 Hz lines exceed 100 km in length (and about 20 km in the case of cables), the capacitance can no longer be neglected. For the purposes of calculation the line may then be approximately represented by a nominal-π or nominal-T network, as shown in

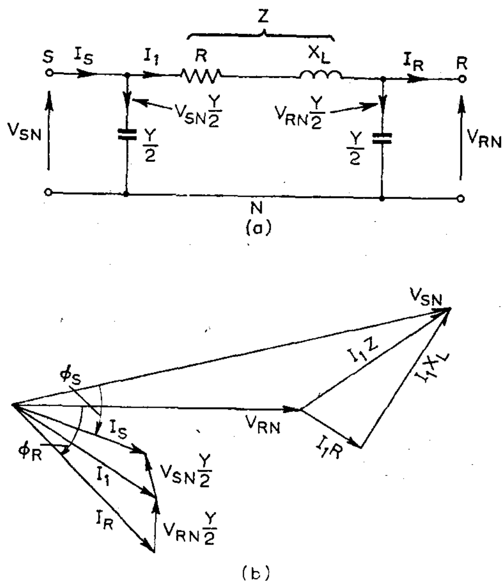


Fig. 8.8 NOMINAL-π EQUIVALENT CIRCUIT AND COMPLEXOR DIAGRAM (LAGGING LOAD)

Figs. 8.8 and 8.9 respectively. These approximate representations do not hold for 50 Hz lines of over 350 km, or for lines operating at frequencies higher than normal power frequencies.

Consider the nominal-π circuit of Fig. 8.8. Let the "go" and "return" resistance of the line be R ohms, the loop inductance L henrys, and the capacitance between the two lines C farads. The series impedance of the line is

$$Z = R + j\omega L = R + jX_L$$

The total admittance between conductors is

$$Y = j\omega C$$

Each "leg" of the π-circuit will have half this admittance.

The complexor diagram shown at (*b*) is constructed taking the receiving-end voltage, V_{RN}, as the reference complexor, and is self-explanatory.

In the nominal-T circuit the resistance and inductance are divided into two, and the capacitance is considered to be concentrated at the centre of the line as shown in Fig. 8.9(*a*).

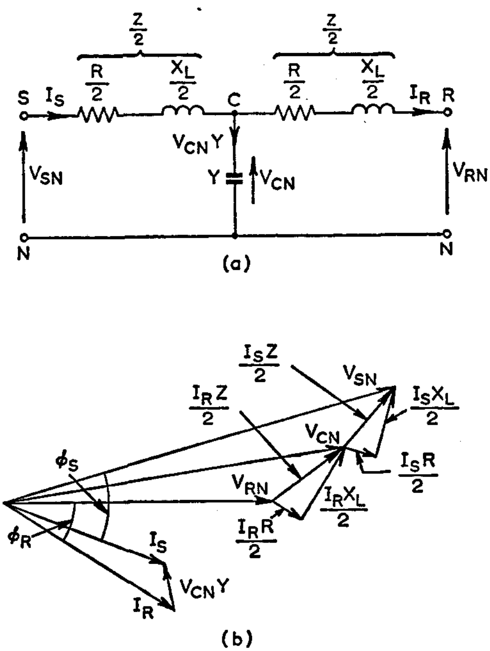


Fig. 8.9 NOMINAL-T EQUIVALENT CIRCUIT AND COMPLEXOR DIAGRAM (LAGGING LOAD)

EXAMPLE 8.4 A single-phase 50 Hz transmission line has the following line constants—resistance 25 Ω; inductance 200 mH; capacitance 1·4 μF. Calculate the sending-end voltage, current and power factor when the load at the receiving-end is 273 A at 76·2 kV with a power factor of 0·8 lagging, using (*a*) a nominal-π circuit, and (*b*) a nominal-T circuit, to represent the line.

Series impedance, $\boldsymbol{Z} = R + j\omega L = 25 + j62{\cdot}8 = 67{\cdot}6\underline{/68{\cdot}3^\circ}\,\Omega$

Shunt admittance, $\boldsymbol{Y} = j\omega C = 0{\cdot}44 \times 10^{-3}\underline{/+90^\circ}\,\mathrm{S}$

Receiving-end voltage, $\boldsymbol{V}_{RN} = 76{\cdot}2 \times 10^{3}\underline{/0^\circ}\,\mathrm{V}$ (reference complexor)

and

Receiving-end current, $\boldsymbol{I}_R = 273\underline{/-36{\cdot}9^\circ}\,\mathrm{A}$

(*a*) Using the nominal-π circuit (Fig. 8.8):

$$\begin{aligned}
\text{Mid-section current} &= I_1 = \text{receiving-end current} \\
&\qquad + \text{current through receiving-end half of } Y \\
&= I_R + V_{RN}Y/2 \\
&= 273\underline{/-36{\cdot}9^\circ} + (72{\cdot}6 \times 10^3\underline{/0^\circ} \times 0{\cdot}22 \times 10^{-3}\underline{/90^\circ}) \\
&= 264\underline{/-33{\cdot}9^\circ}\,\text{A}
\end{aligned}$$

$$\begin{aligned}
\text{Sending-end voltage} &= V_{SN} = V_{RN} + I_1 Z \\
&= (76{\cdot}2 \times 10^3\underline{/0^\circ}) + (264\underline{/-33{\cdot}9^\circ} \times 67{\cdot}6\underline{/68{\cdot}3^\circ}) \\
&= (91 + j10{\cdot}1) \times 10^3 = \underline{\underline{91{,}500\underline{/6{\cdot}6^\circ}\,\text{V}}}
\end{aligned}$$

$$\begin{aligned}
\text{Sending-end current} &= I_S = I_1 + V_{SN}Y/2 \\
&= (219 - j147) + (91{\cdot}5 \times 10^3\underline{/6{\cdot}6^\circ} \times 0{\cdot}22 \times 10^{-3}\underline{/90^\circ}) \\
&= \underline{\underline{251\underline{/-30{\cdot}3^\circ}\,\text{A}}}
\end{aligned}$$

$$\text{Sending-end power factor} = \cos\,(6{\cdot}6 - (-30{\cdot}3))^\circ = \underline{\underline{0{\cdot}8 \text{ lagging}}}$$

(*b*) Using the nominal-T circuit (Fig. 8.9):

$$\begin{aligned}
\text{Mid-section voltage} &= V_{CN} = V_{RN} + I_R Z/2 \\
&= (76{\cdot}2 \times 10^3\underline{/0^\circ}) + (273\underline{/-36{\cdot}9^\circ} \times 33{\cdot}8\underline{/68{\cdot}3^\circ}) \\
&= (84{\cdot}1 + j4{\cdot}8) \times 10^3 = 84{\cdot}1 \times 10^3\underline{/3^\circ}\,\text{V}
\end{aligned}$$

$$\begin{aligned}
\text{Sending-end current} &= I_S = \text{Receiving-end current} + \text{current in mid-section} \\
&= (219 - j164) + (84{\cdot}1 \times 10^3\underline{/3^\circ} \times 0{\cdot}44 \times 10^{-3}\underline{/(90^\circ)}) \\
&= \underline{\underline{252\underline{/-30{\cdot}3^\circ}\,\text{A}}}
\end{aligned}$$

Sending-end voltage

$$\begin{aligned}
&= V_{SN} = V_{CN} + I_S Z/2 \\
&= (84{\cdot}1 + j4{\cdot}8) \times (10^3 + 252\underline{/30{\cdot}3^\circ} \times 33{\cdot}8\underline{/68{\cdot}3^\circ}) \\
&= \underline{\underline{91{\cdot}3 \times 10^3\underline{/6{\cdot}5^\circ}\,\text{V}}}
\end{aligned}$$

$$\text{Sending-end power factor} = \cos\,(6{\cdot}5 - (-30{\cdot}3))^\circ = \underline{\underline{0{\cdot}8 \text{ lagging}}}$$

8.6 ABCD Constants

For any linear passive two-port network there will be a linear relationship between the input voltage and current (V_1, I_1) and the output voltage and current (V_2, I_2). This relationship can be expressed in the form

$$\left.\begin{aligned} V_1 &= \mathbf{A}V_2 + \mathbf{B}I_2 \\ I_1 &= \mathbf{C}V_2 + \mathbf{D}I_2 \end{aligned}\right\} \tag{8.13}$$

where **A**, **B**, **C** and **D** are constants. It is obvious from the form of the equations that **A** and **D** are dimensionless, **B** has the dimensions of an impedance, and **C** has the dimensions of an admittance.

For passive networks there is a fixed relationship between the ABCD constants. Consider any passive network represented by the "black box" in Fig. 8.10(*a*) with its output terminals short-circuited (and hence with $V_2 = 0$). Then,

$$\left.\begin{aligned} V_1 &= \mathbf{B} I_{2sc} \\ I_{1sc} &= \mathbf{D} I_{2sc} \end{aligned}\right\} \tag{8.14}$$

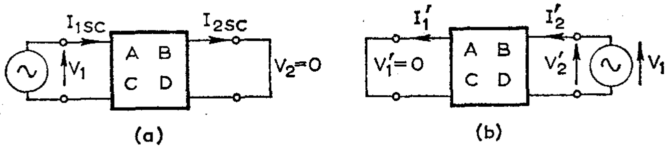


Fig. 8.10

If now the generator is connected across the output terminals, and the input terminals are short-circuited as at (*b*), then, taking into account the current directions, the fact that V_1' is zero, and choosing $V_2' = V_1$,

$$0 = \mathbf{A}V_1 - \mathbf{B}I_2' \qquad \text{or} \qquad I_2' = \frac{\mathbf{A}V_1}{\mathbf{B}} \tag{8.15}$$

and

$$-I_1' = CV_1 - DI_2' \tag{8.16}$$

Since the network is passive, the reciprocity theorem (Section 2.9) applies, so that $I_1' = I_{2sc}$ and hence from eqns. (8.15) and (8.16),

$$-I_{2sc} = \mathbf{C}V_1 - \frac{\mathbf{DA}V_1}{\mathbf{B}}$$

But $I_{2sc} = V_1/\mathbf{B}$ (from eqn. (8.14)), so that

$$-\frac{V_1}{\mathbf{B}} = \mathbf{C}V_1 - \frac{\mathbf{DA}V_1}{\mathbf{B}}$$

or

$$\mathbf{AD} - \mathbf{BC} = 1 \tag{8.17}$$

This important relationship can be simplified for symmetrical networks, since in this case the ratio of V/I at any pair of terminals with the other pair short-circuited must result in the same value (the network looks the same from either port). For the left-hand port,

$$\frac{V_1}{I_{1sc}} = \frac{\mathbf{B}}{\mathbf{D}}$$

For the right hand port,

$$\frac{V_2'}{I_2'} = \frac{V_1}{I_2'} = \frac{\mathbf{B}}{\mathbf{A}} \quad \text{(from eqn. (8.15))}$$

Since these two expressions must be equal it follows that $\mathbf{A} = \mathbf{D}$ for a symmetrical network, and hence eqn. (8.17) becomes

$$\mathbf{A}^2 - \mathbf{BC} = 1 \tag{8.18}$$

8.7 Evaluation of ABCD Constants

The ABCD constants of a passive network can be found from measurements of input and output currents and voltages under open- and short-circuit conditions. If the actual configuration of a network is known, the ABCD constants can be found from the circuit by inspection (or short calculation).

Thus if the output port is open-circuited it follows that $I_2 = 0$, and from eqn. (8.13),

$$V_{1oc} = \mathbf{A}V_{2oc}$$

or

$$\mathbf{A} = \left.\frac{V_1}{V_2}\right|_{oc} \tag{8.19}$$

and

$$I_{1oc} = \mathbf{C}V_{2oc}$$

or

$$\mathbf{C} = \left.\frac{I_1}{V_2}\right|_{oc} \tag{8.20}$$

If the output is short-circuited and the input current and voltage and the output current are measured, then, since $V_2 = 0$,

$$V_{1sc} = \mathbf{B}I_{2sc}$$

or

$$\mathbf{B} = \left.\frac{V_1}{I_2}\right|_{sc} \tag{8.21}$$

and

$$I_{1sc} = \mathbf{D}I_{2sc}$$

or

$$\mathbf{D} = \left.\frac{I_1}{I_2}\right|_{sc} \tag{8.22}$$

EXAMPLE 8.5 Determine the ABCD constants for the symmetrical-T circuit shown in Fig. 8.11.

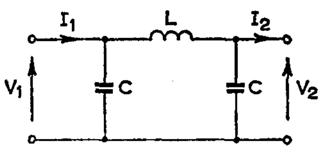


Fig. 8.11 SYMMETRICAL-T TWO-PORT NETWORK

First imagine the output to be open-circuited so that $I_2 = 0$. Then for an input voltage V_1, eqn. (8.19) gives

$$\mathbf{A} = \left.\frac{V_1}{V_2}\right|_{oc} = \frac{V_1}{\dfrac{V_1}{j\omega L + 1/j\omega C}\ \dfrac{1}{j\omega C}} \text{ by inspection}$$

$$= j\omega C(j\omega L + 1/j\omega C) = \underline{\underline{1 - \omega^2 LC}}$$

Since the network is symmetrical $\mathbf{A} = \mathbf{D} = 1 - \omega^2 LC$.

If the output is now short-circuited, then from eqn. (8.21),

$$\mathbf{B} = \left.\frac{V_1}{I_2}\right|_{sc} = \frac{V_1}{V_1/j\omega L} = \underline{\underline{j\omega L}}$$

The constant $\mathbf{C}$ can be found either from eqn. (8.20) or by using the relation for a passive symmetrical two-port network, $\mathbf{A}^2 - \mathbf{BC} = 1$. Using the latter method,

$$\mathbf{C} = \frac{\mathbf{A}^2 - 1}{\mathbf{B}} = \frac{\omega^4 L^2 C^2 - 2\omega^2 LC}{j\omega L} = \underline{\underline{2\omega C - j\omega^3 LC^2}}$$

8.8 Characteristic Impedance in terms of ABCD Constants

It has been seen that the characteristic impedance, Z_0, of a symmetrical network is obtained from the relation

$$Z_0 = \sqrt{(Z_{sc}Z_{oc})}$$

The input impedances under short- and open-circuit *output* conditions can be obtained from eqns. (8.13). Thus for short-circuited output, $V_2 = 0$, and

$$Z_{in\,sc} = \left.\frac{V_1}{I_1}\right|_{sc} = \frac{\mathbf{B}}{\mathbf{D}} \left(= \frac{\mathbf{B}}{\mathbf{A}} \text{ for a symmetrical network} \right)$$

Similarly for an open-circuited output, $I_2 = 0$, and

$$Z_{in\,oc} = \left.\frac{V_1}{I_1}\right|_{oc} = \frac{\mathbf{A}}{\mathbf{C}}$$

It follows directly that, for a symmetrical network,

$$Z_0 = \sqrt{\frac{\mathbf{B}}{\mathbf{A}}\frac{\mathbf{A}}{\mathbf{C}}} = \sqrt{\frac{\mathbf{B}}{\mathbf{C}}} \tag{8.23}$$

EXAMPLE 8.6 A symmetrical two-port network has $\mathbf{A} = \mathbf{D} = 0{\cdot}8\underline{/30°}$; $\mathbf{B} = 100\underline{/60°}\,\Omega$; $\mathbf{C} = 5 \times 10^{-3}\underline{/90°}$ S. Determine (*a*) the characteristic impedance, and (*b*) the output voltage and current when the input voltage is $120\underline{/0°}$ V and the output load is a pure resistance of 141 Ω. The circuit is shown in Fig. 8.12.

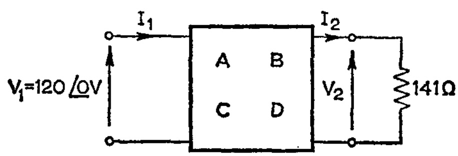


Fig. 8.12

The characteristic impedance, Z_0, is readily obtained from eqn. (8.23) as

$$Z_0 = \sqrt{\frac{\mathbf{B}}{\mathbf{C}}} = \sqrt{\frac{100\underline{/60°}}{5 \times 10^{-3}\underline{/90°}}} = \underline{\underline{141\underline{/-15°}\,\Omega}}$$

From Fig. 8.12,

$$\frac{V_2}{I_2} = 141\underline{/0°} \qquad \text{or} \qquad V_2 = 141 I_2 \tag{i}$$

Hence, from eqn. (8.13),

$$V_1 = \mathbf{A} \times 141 I_2 + \mathbf{B} I_2$$

or

$$I_2 = \frac{120\underline{/0°}}{(0{\cdot}8\underline{/30°} \times 141) + 100\underline{/60°}} = 0{\cdot}58\underline{/-44°}\,\text{A}$$

and from (i),

$$V_2 = 141 \times 0{\cdot}58\underline{/-44°} = \underline{\underline{82\underline{/-44°}\,\text{V}}}$$

8.9 Two-port Networks in Cascade

Very frequently two-port networks are connected in cascade as shown in Fig. 8.13. Thus an attenuator or a filter may consist of several cascaded sections in order to achieve a given desired overall performance. If the chain of networks is designed on an iterative basis, then each section will have the same characteristic impedance (assuming symmetrical sections), and the last network will be terminated in Z_0. It follows that each network will have a matched

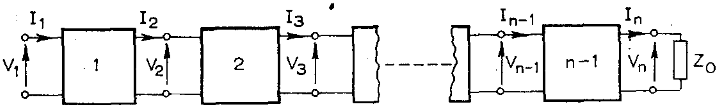


Fig. 8.13 TWO-PORT NETWORKS IN CASCADE

termination, and hence the insertion loss in decibels of section 1 is $m_1 = 20 \log_{10}(V_1/V_2)$; of section 2 is $m_2 = 20 \log_{10}(V_2/V_3)$; etc.

The overall insertion loss will be

$$\begin{aligned} m &= 20 \log_{10}(V_1/V_n) \\ &= 20 \log_{10}\left(\frac{V_1}{V_2} \times \frac{V_2}{V_3} \times \frac{V_3}{V_4} \cdots \frac{V_{n-1}}{V_n}\right) \\ &= 20 \log_{10}\frac{V_1}{V_2} + 20 \log_{10}\frac{V_2}{V_3} + \ldots + 20 \log_{10}\frac{V_{n-1}}{V_n} \\ &= m_1 + m_2 + \ldots + m_{n-1} \end{aligned} \tag{8.24}$$

This illustrates the great convenience of the use of matched sections in cascade—the overall insertion loss is simply the sum of the insertion losses of the sections. If the sections do not have the same characteristic impedance or are not terminated in Z_0, eqn. (8.24) is no longer valid, and the calculation of insertion loss may be very tedious.

PROBLEMS

8.1 In a symmetrical-T attenuator pad each series arm has a resistance of 30Ω and the shunt arm has a resistance of 100Ω. Determine the characteristic impedance, and the insertion loss when feeding a matched load.

Ans. 83Ω; 6·6dB.

8.2 A symmetrical-π attenuator pad has a series arm of 2kΩ resistance, and each shunt arm of 1kΩ resistance. Determine the characteristic impedance, and the insertion loss when feeding a matched load.

Ans. 707Ω; 15·4dB.

8.3 A 10:1 voltage divider has a total resistance of 1,000Ω. Calculate from first principles the insertion loss when the divider is connected between a generator of internal resistance 600 Ω and a load of resistance 200 Ω. Repeat for a generator internal resistance of 5 Ω.

Ans. 5·9 (15·4 dB); 14·2 (23 dB).

8.4 A symmetrical attenuator pad has a characteristic impedance of 75 Ω and an insertion loss of 20 dB. It feeds a load of 75 Ω resistance. Calculate the load voltage when the attenuator is connected to a generator of e.m.f. 20 V and internal resistance (*a*) 75 Ω, (*b*) 600 Ω.

Ans. 1 V; 0·22 V.

8.5 A symmetrical-T section has each series arm of 100 mH pure inductance and a shunt arm of 0·1 μF capacitance. Determine (*a*) the frequency at which $Z_0 = 0$; (*b*) the insertion loss of the network below this frequency, when correctly matched. Plot the variation of Z_0 with frequency from zero up to the frequency found for (*a*).

Ans. 2·25 kHz; 1.

8.6 A single-phase 50 Hz line, 5 km long, supplies 5,000 kW at a p.f. of 0·71 lagging. The line has a resistance of 0·0345 Ω per km for each wire, and a loop inductance per km of 1·5 mH. The receiving-end voltage is 10 kV. A capacitor is connected across the load to raise its p.f. to 0·9 lagging. Calculate (*a*) the capacitance of the capacitor, (*b*) the sending-end voltage with the capacitor in use and out of use, and (*c*) the efficiency of transmission in each case.*

Ans. 82·4 μF; 10·8 kV, 11·4 kV; 98 per cent, 96 per cent.

8.7 A single-phase transmission line 50 miles long, delivers 4,000 kW at a voltage of 38 kV and a power factor of 0·85 lagging. Find the sending-end voltage, current, and power factor by the nominal-T method. The resistance, reactance and susceptance per mile are 0·3 Ω, 0·7 Ω and 12×10^{-6} S.

Ans. 41·5 kV, 112·5 A, 0·9 lagging.

8.8 Repeat Problem 8.7 using the nominal-π equivalent circuit.

8.9 Calculate the **ABCD** constants of the line in Problem 8.6.
Ans. 1; $0·17 + j2·36$; 0; 1.

8.10 Calculate the **ABCD** constants of the line in Problem 8.7.
Ans. 1; $15 + j35\,\Omega$; $j6 \times 10^{-4}$ S; 1.

8.11 A 3-phase transmission line has the following constants:

$$\mathbf{A} = \mathbf{D} = 1\underline{/0^\circ} \qquad \mathbf{B} = 50\underline{/60^\circ} \qquad \mathbf{C} = 10^{-3}\underline{/90^\circ}$$

Determine the input voltage, current and power factor when the output current is 100 A lagging behind the output phase voltage of 20 kV by 37°.
Ans. 39 kV; 90 A; 0·79.

* Efficiency of transmission $= \dfrac{\text{Output power}}{\text{Input power}}$.

Chapter 9

TRANSFORMERS

Two circuits are said to be *mutually inductive* when some of the magnetic flux caused by the excitation of one circuit links with some or all of the turns of the second circuit. Transformers are mutual inductors designed so that the magnetic coupling between the two circuits is *tight*, i.e. the greater part of the flux linking each circuit is mutual. This is achieved by providing a low-reluctance path for the mutual flux.

Equivalent circuits consisting of resistive and self-inductive (not mutually inductive) elements can be derived for mutual inductors or transformers. The self-inductive elements may be expressed either in terms of the total circuit inductances and the mutual inductance or in terms of leakage inductances and magnetizing inductances.

9.1 Winding Inductances

Fig. 9.1 shows a pair of magnetically coupled coils. At (*a*) is shown the flux distribution when only the winding denoted by subscript 1 carries current; at (*b*) is shown the flux distribution when only the winding denoted by subscript 2 carries current, while the fluxes when both windings carry current are shown at (*c*). The dot notation is used to indicate the relative coil polarities; thus currents entering the dotted ends of the coils produce aiding mutual fluxes. At (*c*), current is shown entering the dotted end of one winding and leaving the dotted end of the other. When both windings carry current simultaneously these current directions are the most common as they represent power transmission through the mutual coupling.

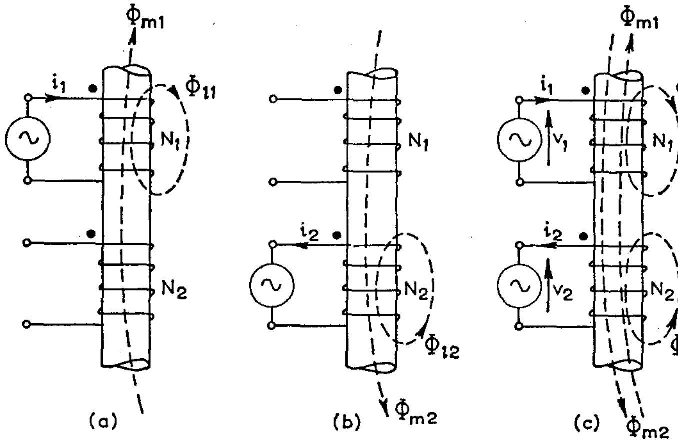


Fig. 9.1 MAGNETIZING COUPLED COILS

(*a*) Primary winding carrying current
(*b*) Secondary winding carrying current
(*c*) Both windings carrying current

The diagrams show the total flux linkage of the coils to be due to four component fluxes, namely

Φ_{m1}, a component, due to the current in winding 1, which links all the turns of both windings.

Φ_{l1}, a component, due to the current in winding 1, which links all the turns of winding 1 but none of those of winding 2 (this is the primary leakage flux).

Φ_{m2}, a component, due to the current in winding 2, which links all the turns of both windings.

Φ_{l2}, a component, due to the current in winding 2, which links only the turns of winding 2 (this is the secondary leakage flux).

These component fluxes may be used to define various inductances. It is assumed that the magnetic circuit is unsaturated so that the defined inductances are constant.

$$\text{Primary self-inductance, } L_{11} = \frac{(\Phi_{m1} + \Phi_{l1})N_1}{i_1} \tag{9.1}$$

$$\text{Secondary self-inductance, } L_{22} = \frac{(\Phi_{m2} + \Phi_{l2})N_2}{i_2} \tag{9.2}$$

$$\text{Mutual inductance, } L_{12} = \frac{\Phi_{m1}N_2}{i_1} \tag{9.3}$$

Alternatively the mutual inductance is

$$L_{21} = \frac{\Phi_{m2}N_1}{i_2} \tag{9.4}$$

These mutual inductances are equal,

$$\text{i.e.}\quad L_{12} = L_{21} \tag{9.5}$$

The coupling coefficient, k, of magnetically coupled coils is defined as

$$k = \frac{L_{12}}{\sqrt{(L_{11}L_{22})}} \tag{9.6}$$

k has a maximum value of 1, corresponding to the case when no leakage flux links either coil.

Equivalent circuits for magnetically coupled coils may be obtained by representing the flux linkage in terms of L_{11}, L_{22} and L_{12}, and this is undertaken in the following section. Equivalent circuits may also be obtained by representing the flux linkage in terms of leakage and magnetizing inductances, and these inductances are now defined.

$$\text{Primary leakage inductance, } L_{l1} = \frac{\Phi_{l1}N_1}{i_1} \tag{9.7}$$

$$\text{Secondary leakage inductance, } L_{l2} = \frac{\Phi_{l2}N_2}{i_2} \tag{9.8}$$

$$\text{Primary magnetising inductance, } L_{m1} = \frac{\Phi_{m1}N_1}{i_1} \tag{9.9}$$

$$\text{Secondary magnetising inductance, } L_{m2} = \frac{\Phi_{m2}N_2}{i_2} \tag{9.10}$$

9.2 Equivalent Circuits for Magnetically Coupled Coils

Applying Kirchhoff's second law to the magnetically coupled coils shown in Fig. 9.1(c) gives the equations

$$v_1 = R_1i_1 + N_1\frac{d}{dt}(\Phi_{l1} + \Phi_{m1} - \Phi_{m2}) \tag{9.11}$$

$$v_2 = -R_2i_2 + N_2\frac{d}{dt}(\Phi_{m1} - \Phi_{m2} - \Phi_{l2}) \tag{9.12}$$

where R_1 and R_2 are the resistances of the primary and secondary coils respectively. The primary and secondary flux linkages may now be expressed in terms of the self- and mutual inductances according to eqns. (9.1)–(9.3). Rearranging eqns. (9.11) and (9.12),

$$v_1 = R_1 i_1 + N_1 \frac{d}{dt}(\Phi_{l1} + \Phi_{m1}) - N_1 \frac{d}{dt}\Phi_{m2}$$

$$v_2 = -R_2 i_2 + N_2 \frac{d}{dt}\Phi_{m1} - N_2 \frac{d}{dt}(\Phi_{m2} + \Phi_{l2})$$

Therefore,

$$v_1 = R_1 i_1 + L_{11}\frac{di_1}{dt} - L_{12}\frac{di_2}{dt} \tag{9.13}$$

$$v_2 = -R_2 i_2 - L_{22}\frac{di_2}{dt} + L_{12}\frac{di_1}{dt} \tag{9.14}$$

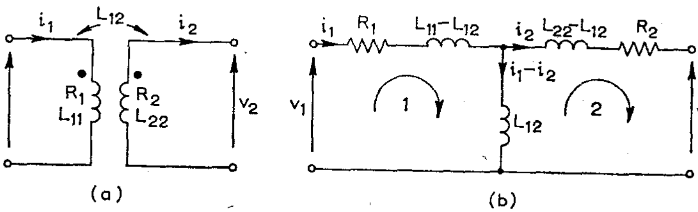


Fig. 9.2 EQUIVALENT T-CIRCUIT FOR MAGNETICALLY COUPLED COILS
(*a*) Mutually inductive coils
(*b*) Equivalent T-circuit

If the direction of i_2 had been chosen as entering the dotted end of the secondary, then all fluxes would have been additive and no minus signs would have occurred in eqns. (9.13) and (9.14).

Fig. 9.2(*a*) shows the circuit representation for mutually inductive coils. Eqns. (9.13) and (9.14) may be rewritten

$$v_1 = R_1 i_1 + L_{11}\frac{di_1}{dt} - L_{12}\frac{di_1}{dt} + L_{12}\frac{di_1}{dt} - L_{12}\frac{di_2}{dt} \tag{9.15}$$

adding and subtracting $L_{12}(di_1/dt)$ in eqn. (9.13); and

$$v_2 = -R_2 i_2 - L_{22}\frac{di_2}{dt} + L_{12}\frac{di_2}{dt} + L_{12}\frac{di_1}{dt} - L_{12}\frac{di_2}{dt} \tag{9.16}$$

adding and subtracting $L_{12}(di_2/dt)$ in eqn. (9.14).

When Kirchhoff's second law is applied to meshes 1 and 2 of Fig. 9.2(*b*), eqns. (9.15) and (9.16) respectively are obtained. Fig. 9.2(*b*) is therefore the equivalent T-circuit of the mutually inductive coils. This equivalent circuit is convenient for the consideration of networks containing mutual inductance since it consists of resistive and self-inductive elements only.

9.3 Mutual Inductance in Networks

Eqns. (9.13) and (9.14) may be made to refer to steady-state sinusoidal operation by

(*a*) Replacing all the time-varying quantities by complexor quantities, putting i.e. V_1 for v_1, I_1 for i_1, etc.
(*b*) Replacing d/dt operators by $j\omega$.

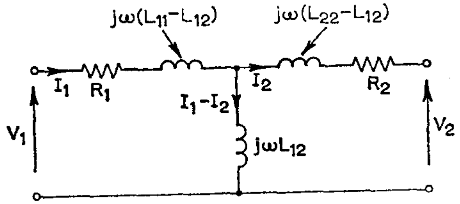


Fig. 9.3 EQUIVALENT T-CIRCUIT FOR STEADY-STATE A.C. OPERATION

Eqns. (9.13) and (9.14) then become

$$V_1 = R_1 I_1 + j\omega L_{11} I_1 - j\omega L_{12} I_2 \tag{9.17}$$

$$V_2 = -R_2 I_2 - j\omega L_{22} I_2 + j\omega L_{12} I_1 \tag{9.18}$$

The corresponding equivalent circuit is shown in Fig. 9.3.

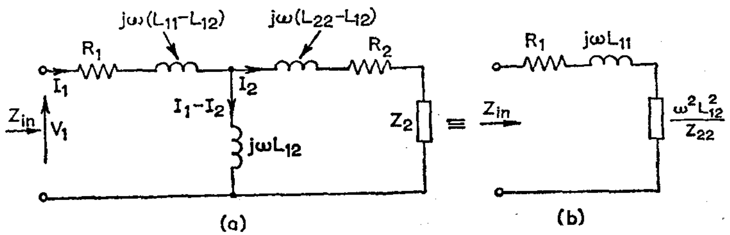


Fig. 9.4 CALCULATION OF INPUT IMPEDANCE

(*a*) Equivalent T-circuit
(*b*) Equivalent series circuit: $Z_{22} = R_2 + j\omega L_{22} + Z_2$

Fig. 9.4(*a*) shows the equivalent T-circuit with a secondary load of impedance Z_2. Fig. 9.4(*b*) shows the series equivalent circuit. Since the equivalent T-circuit does not contain any mutual coupling (but only self-inductances whose values are expressed in terms of the mutual inductance), the input impedance may be calculated using series-parallel circuit theory. Thus the input impedance measured at the primary terminals is

$$Z_{in} = R_1 + j\omega(L_{11} - L_{12}) + \frac{j\omega L_{12}\{R_2 + j\omega(L_{22} - L_{12}) + Z_2\}}{j\omega L_{12} + R_2 + j\omega(L_{22} - L_{12}) + Z_2} \quad (9.19)$$

Putting $R_2 + j\omega L_{22} + Z_2 = Z_{22}$ gives, after simplification,

$$Z_{in} = R_1 + j\omega L_{11} + \frac{\omega^2 L_{12}{}^2}{Z_{22}} \quad (9.20)$$

This confirms the result found in Section 1.10.

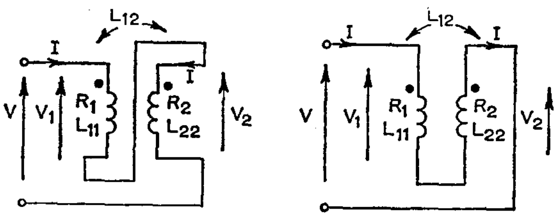


Fig. 9.5 IMPEDANCE OF INDUCTIVELY COUPLED COILS

(*a*) Series aiding: $Z = R_1 + R_2 + jw(L_{11} + L_{22} + 2L_{12})$
(*b*) Series opposing: $Z = R_1 + R_2 + jw(L_{11} + L_{22} - 2L_{12})$

In Fig. 9.5(*a*) two mutually coupled coils are connected in series. The connexion is called *series aiding*, since current enters the dotted ends of the coils, which thus produce aiding fluxes. Eqns. (9.17) and (9.18) will apply with $I_1 = -I_2$ ($= I$, say), since the current in the second coil is reversed compared with the direction assumed when these equations were established. This gives

$$V_1 = R_1 I + j\omega L_{11} I + j\omega L_{12} I$$

$$V_2 = R_2 I + j\omega L_{22} I + j\omega L_{12} I$$

$$V = V_1 + V_2 = (R_1 + R_2)I + j\omega(L_{11} + L_{22} + 2L_{12})I$$

i.e.

$$Z = \frac{V}{I} = R_1 + R_2 + j\omega(L_{11} + L_{22} + 2L_{12}) \quad (9.21)$$

Fig. 9.5(*b*) shows a *series-opposing* connexion of mutually coupled coils. In this case the current enters the dotted end of coil 1 and leaves the dotted end of coil 2. Eqns. (9.17) and (9.18) again apply with $\boldsymbol{I}_1 = \boldsymbol{I}$ and $\boldsymbol{I}_2 = \boldsymbol{I}$. This gives

$$\boldsymbol{V}_1 = R_1\boldsymbol{I} + j\omega L_{11}\boldsymbol{I} - j\omega L_{12}\boldsymbol{I}$$

$$\boldsymbol{V}_2 = -R_2\boldsymbol{I} - j\omega L_{22}\boldsymbol{I} + j\omega L_{12}\boldsymbol{I}$$

$$\boldsymbol{V} = \boldsymbol{V}_1 - \boldsymbol{V}_2 = (R_1 + R_2)\boldsymbol{I} + j\omega(L_{11} + L_{22} - 2L_{12})\boldsymbol{I}$$

i.e.

$$\boldsymbol{Z} = \frac{\boldsymbol{V}}{\boldsymbol{I}} = R_1 + R_2 + j\omega(L_{11} + L_{22} - 2L_{12}) \qquad (9.22)$$

Eqns. (9.21) and (9.22) confirm the results found in Section 1.10.

9.4 The Ideal Transformer

The terminal voltages of a pair of magnetically coupled coils as shown in Fig. 9.1(c) are, from eqns. (9.11) and (9.12),

$$v_1 = R_1 i_1 + N_1 \frac{d}{dt}(\Phi_{l1} + \Phi_{m1} - \Phi_{m2}) \qquad (9.11)$$

$$v_2 = -R_2 i_2 + N_2 \frac{d}{dt}(\Phi_{m1} - \Phi_{m2} - \Phi_{l2}) \qquad (9.12)$$

Let e_1, e_2 be the voltages induced in the primary and secondary coils respectively due to the mutual flux only; then

$$e_1 = N_1 \frac{d}{dt}(\Phi_{m1} - \Phi_{m2}) \qquad (9.23)$$

$$e_2 = N_2 \frac{d}{dt}(\Phi_{m1} - \Phi_{m2}) \qquad (9.24)$$

Dividing eqn. (9.23) by eqn. (9.24),

$$\frac{e_1}{e_2} = \frac{N_1}{N_2} \qquad (9.25)$$

This equation defines an ideal voltage transformation. Examination of eqns. (9.11) and (9.12) shows that an ideal voltage transformation would be obtained from a pair of magnetically coupled coils in which the winding resistances and leakage fluxes were zero. In practice there must always be some leakage flux, but a close approximation to the ideal is obtained if the two windings are placed physically close to each other on a common magnetic core.

The mutual flux linking an ideal voltage transformation is

$$\Phi = \Phi_{m1} - \Phi_{m2} = \Lambda(i_1 N_1 - i_2 N_2) \tag{9.26}$$

where Λ represents the permeance of the magnetic circuit carrying the mutual flux Φ, and $i_1N_1 - i_2N_2$ is the net m.m.f. For the circuit of Fig. 9.1(*c*) the net m.m.f. is as stated since i_1 is shown entering the dotted end of the primary winding and i_2 is shown leaving the dotted end of the secondary winding.

For a magnetic circuit of finite permeance, i_1N_1 must be greater than i_2N_2 for there to be resultant magnetization. For a core of infinite permeance, however,

$$i_1N_1 - i_2N_2 = 0 \qquad \text{or} \qquad i_1N_1 = i_2N_2 \tag{9.27}$$

This equation defines the ideal current transformation that would be obtained from a pair of magnetically coupled coils linked by a magnetic circuit which had infinite permeance. In practice the magnetic circuit must have a finite permeance, but a close approximation to the ideal is obtained by making the magnetic circuit of a high-permeability material.

A pair of magnetically coupled coils which provides both an ideal voltage transformation and an ideal current transformation is called an *ideal transformer*. Eqns. (9.25) and (9.27) both apply to such a device:

$$\frac{e_1}{e_2} = \frac{N_1}{N_2} \tag{9.25}$$

$$i_1N_1 = i_2N_2 \tag{9.27}$$

Substituting for N_1/N_2 in eqn. (9.25) in terms of the winding currents,

$$\frac{e_1}{e_2} = \frac{i_2}{i_1} \qquad \text{or} \qquad e_1i_1 = e_2i_2 \tag{9.28}$$

Therefore, for an ideal transformer,

$$\left.\begin{array}{c}\text{Instantaneous power absorbed}\\ \text{by primary winding}\end{array}\right\} = \left\{\begin{array}{c}\text{Instantaneous power deliver-}\\ \text{ed by secondary winding}\end{array}\right.$$

9.5 Transformer Equivalent Circuit

In Section 9.2 an equivalent circuit for magnetically coupled coils is obtained by representing the flux linkage in terms of the coil self-inductances L_{11} and L_{22} and the mutual inductance L_{12}. An alternative approach is to represent the flux linkage in terms of the coil leakage and magnetizing inductances. This approach leads to an equivalent circuit which will be called the transformer equivalent circuit.

The terminal voltages of a pair of magnetically coupled coils as shown in Fig. 9.1(c) are, from eqns. (9.11) and (9.12),

$$v_1 = R_1 i_1 + N_1 \frac{d}{dt}(\Phi_{l1} + \Phi_{m1} - \Phi_{m2}) \tag{9.11}$$

$$v_2 = -R_2 i_2 + N_2 \frac{d}{dt}(\Phi_{m1} - \Phi_{m2} - \Phi_{l2}) \tag{9.12}$$

From eqn. (9.7),

Primary leakage flux linkage, $N_1\Phi_{l1} = L_{l1}i_1$

From eqn. (9.8),

Secondary leakage flux linkage, $N_2\Phi_{l2} = L_{l2}i_2$

Substituting for these flux linkages in eqns. (9.11) and (9.12),

$$v_1 = R_1 i_1 + L_{l1}\frac{di_1}{dt} + N_1 \frac{d}{dt}(\Phi_{m1} - \Phi_{m2}) \tag{9.29}$$

$$v_2 = -R_2 i_2 - L_{l2}\frac{di_2}{dt} + N_2 \frac{d}{dt}(\Phi_{m1} - \Phi_{m2}) \tag{9.30}$$

Substituting e_1 and e_2 in terms of eqns. (9.23) and (9.24) in the above equations gives

$$v_1 = R_1 i_1 + L_{l1}\frac{di_1}{dt} + e_1 \tag{9.31}$$

$$v_2 = -R_2 i_2 - L_{l2}\frac{di_2}{dt} + e_2 \tag{9.32}$$

Fig. 9.6 shows an equivalent circuit which is consistent with eqns. (9.31) and (9.32), in which the resistance and leakage inductance of the two windings are shown as series elements external to the windings. The windings are represented as wound on a common core in close physical proximity to give an ideal voltage transformation element.

In an actual transformer the core does not have infinite permeance and it is necessary to represent the effects of finite core permeance in the equivalent circuit. The core permeance is

$$\Lambda = \frac{\text{Flux}}{\text{M.M.F.}} = \frac{\Phi_{m1}}{i_1 N_1} \tag{9.33}$$

Also, from eqn. (9.9),

$$\frac{\Phi_{m1}}{i_1} = \frac{L_{m1}}{N_1}$$

Substituting for Φ_{m1}/i_1 in eqn. (9.33),

$$\Lambda = \frac{L_{m1}}{N_1^2} \tag{9.34}$$

or

$$L_{m1} = \Lambda N_1^2 \tag{9.35}$$

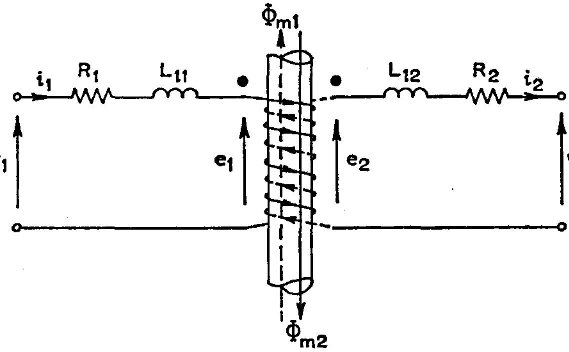


Fig. 9.6 TRANSFORMER EQUIVALENT CIRCUIT INCORPORATING IDEAL VOLTAGE TRANSFORMATION

In a similar manner it can be shown that

$$L_{m2} = \Lambda N_2^2 \tag{9.36}$$

Dividing eqn. (9.35) by eqn. (9.36),

$$\frac{L_{m1}}{L_{m2}} = \frac{N_1^2}{N_2^2} \tag{9.37}$$

Substituting for Λ from eqn. (9.34) in eqn. (9.26),

$$\Phi_{m1} - \Phi_{m2} = \frac{L_{m1}}{N_1^2}(i_1N_1 - i_2N_2)$$

i.e.

$$(\Phi_{m1} - \Phi_{m2})N_1 = L_{m1}\left(i_1 - i_2\frac{N_2}{N_1}\right)$$

or

$$(\Phi_{m1} - \Phi_{m2})N_1 = L_{m1}(i_1 - i_2') \tag{9.38}$$

where

$$i_2' = i_2 \frac{N_2}{N_1}$$

i_2' is called the *secondary current referred to the primary.* The effect of finite core permeance may therefore be represented in the equivalent circuit by the inclusion of an element of inductance L_{m1} carrying a current $i_1 - i_2'$. The voltage v across the element is, from eqn. (9.38),

$$v = L_{m1} \frac{d}{dt}(i_1 - i_2') = N_1 \frac{d}{dt}(\Phi_{m1} - \Phi_{m2}) = e_1$$

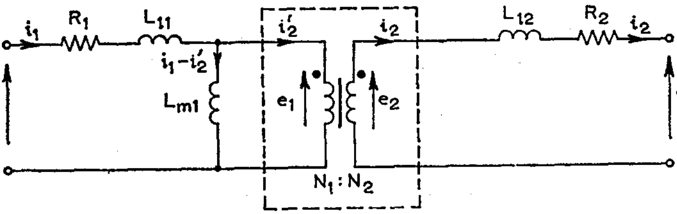


Fig. 9.7 TRANSFORMER EQUIVALENT CIRCUIT INCORPORATING AN IDEAL TRANSFORMER

Fig. 9.7 shows the transformer equivalent circuit with the circuit-element so connected that the voltage across its ends is e_1. The current through L_{m1} will be $i_1 - i_2'$ provided that the element in the "box" is an ideal transformer ($e_1/e_2 = N_1/N_2$ and $i_2'N_1 = i_2N_2$).

Fig. 9.2(*b*), which is repeated in Fig. 9.8(*a*), shows the T-equivalent circuit for a pair of magnetically coupled coils the inductive circuit elements of which are expressed in terms of L_{11}, L_{22} and L_{12}. Fig. 9.7 shows the transformer equivalent circuit in which the inductive elements are expressed in terms of L_{l1}, L_{l2} and L_{m1}. This latter equivalent circuit differs from that of Fig. 9.8(*a*) in that it includes an ideal transformer of turns ratio N_1/N_2.

When the equivalent circuit of Fig. 9.8(*a*) is rearranged by including in it at section XX the ideal transformer shown in Fig. 9.8(*b*), the equivalent circuit shown in Fig. 9.8(*c*) is obtained. This is identical with the transformer equivalent circuit of Fig. 9.7. Using eqns. (9.1)–(9.9), it is easy to show that

$$L_{l1} = L_{11} - \frac{N_1}{N_2} L_{12}$$

$$L_{l2} = L_{22} - \frac{N_2}{N_1} L_{12} \qquad \text{and} \qquad L_{m1} = \frac{N_1}{N_2} L_{12}$$

For identity between the equivalent circuits Figs. 9.7 and 9.8(*c*), $v_1' = e_1$, so that the shunt element of Fig. 9.8(*b*) is $(N_1/N_2)L_{12}$. Comparison of the mesh equations of Figs. 9.8(*a*) and (*c*) shows that these circuits are equivalent.

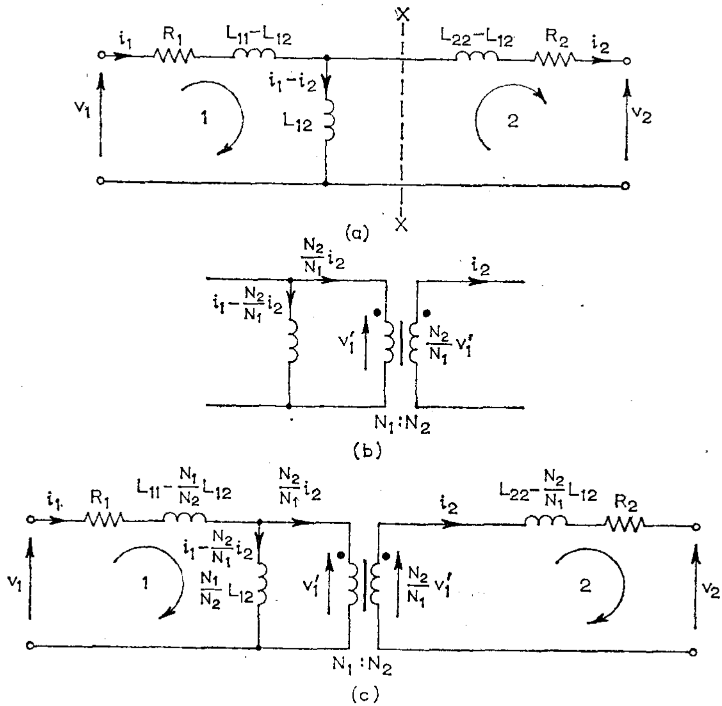


Fig. 9.8 TRANSFORMER EQUIVALENT T-CIRCUIT WITH IDEAL N_1/N_2 TRANSFORMATION

EXAMPLE 9.1 Measurements on an air-cored mutual inductor gave the following results:

Input impedance of primary (secondary open-circuited) .	$(34.0 + j413)\,\Omega$
Input impedance of secondary (primary open-circuited) .	$(40{\cdot}8 + j334)\,\Omega$
Input impedance when windings are joined in series aiding	$(74{\cdot}8 + j1{,}109)\,\Omega$

The measurements were carried out at a frequency of 1,592 Hz ($\omega = 10{,}000$ rad/s). The primary winding has 1,200 turns and the secondary 1,000 turns.

Determine the self-inductance of each winding, the mutual inductance and the coupling coefficient. Draw the equivalent T-circuit.

Determine also the primary and secondary leakage and magnetizing inductances and draw the transformer equivalent circuit.

Determine the input impedance to the primary for a secondary load of $(200 + j0)\Omega$, and the input impedance to the secondary when the same load is connected to the primary terminals. The angular frequency in both cases is 10,000 rad/s.

$$L_{11} = \frac{X_1}{\omega} = \frac{413 \times 10^3}{10^4} = \underline{\underline{41{\cdot}3\,\text{mH}}}$$

$$L_{22} = \frac{X_2}{\omega} = \frac{334 \times 10^3}{10^4} = \underline{\underline{33{\cdot}4\,\text{mH}}}$$

From eqn. (9.21), the total inductance for a series-aiding connection is

$$L_T = L_{11} + L_{22} + 2L_{12}$$

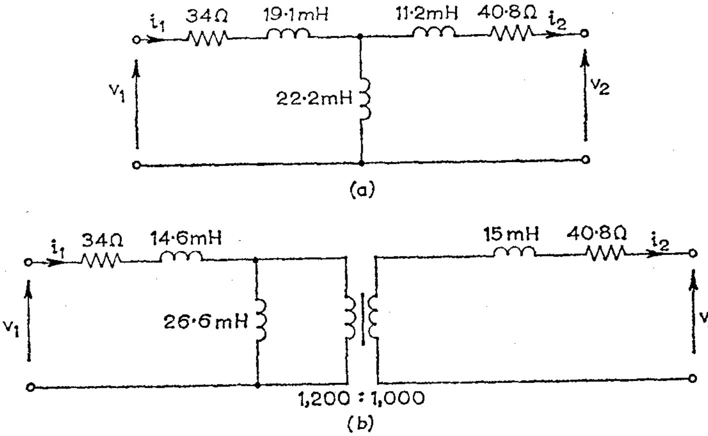


Fig. 9.9

Thus

$$L_T = \frac{1{,}190}{\omega} = \frac{1{,}190 \times 10^3}{10^4} = 119\,\text{mH}$$

and the mutual inductance is

$$L_{12} = \frac{L_T - L_{11} - L_{22}}{2} = \frac{119 - 41{\cdot}3 - 33{\cdot}4}{2} = \underline{\underline{22{\cdot}2\,\text{mH}}}$$

$$\text{Coupling coefficient, } k = \frac{L_{12}}{\sqrt{(L_{11}L_{22})}} = \frac{22{\cdot}2}{\sqrt{41{\cdot}3 \times 33{\cdot}4}} = \underline{\underline{0{\cdot}537}}$$

$$L_{11} - L_{12} = 41{\cdot}3 - 22{\cdot}2 = 19{\cdot}1\,\text{mH}$$

$$L_{22} - L_{12} = 33{\cdot}4 - 22{\cdot}2 = 11{\cdot}2\,\text{mH}$$

The equivalent-T circuit is shown in Fig. 9.9(*a*).

Primary leakage inductance, $L_{l1} = L_{11} - \dfrac{N_1 L_{12}}{N_2}$

$$= 41{\cdot}3 - \frac{1{,}200}{1{,}000} \times 22{\cdot}2 = \underline{\underline{14{\cdot}6\text{mH}}}$$

Secondary leakage inductance, $L_{l2} = L_{22} - \dfrac{N_2 L_{12}}{N_1}$

$$= 33{\cdot}4 - \frac{1{,}000}{1{,}200} \times 22{\cdot}2 = \underline{\underline{15{\cdot}0\text{mH}}}$$

Primary magnetizing inductance, $L_{m1} = \dfrac{N_1 L_{12}}{N_2} = \dfrac{1{,}200}{1{,}000} \times 22{\cdot}2 = \underline{\underline{26{\cdot}6\text{mH}}}$

Secondary magnetizing inductance, $L_{m2} = \dfrac{N_2 L_{12}}{N_1} = \dfrac{1{,}000}{1{,}200} \times 22{\cdot}2$

$$= \underline{\underline{18{\cdot}5\text{mH}}}$$

The transformer equivalent circuit is shown in Fig. 9.9(*b*). From eqn. (9.20) the input impedance for a secondary load impedance Z_2 is

$$Z_{in} = R_1 + j\omega L_{11} + \frac{\omega^2 L_{12}{}^2}{Z_{22}}$$

where

$$Z_{22} = R_2 + j\omega L_{22} + Z_2$$

Thus

$$Z_{in} = 34 + j413 + \frac{10^8 \times 22{\cdot}2^2 \times 10^{-6}}{40{\cdot}8 + j334 + 200 + j0}$$

$$= 34 + j413 + \frac{22{\cdot}2^2 \times 10^2}{412^2}(241 - j334)$$

$$= 104 + j306 = \underline{\underline{321\angle 71{\cdot}2^\circ\,\Omega}}$$

By analogy with eqn. (9.20), the secondary input impedance for a primary load Z_1 is

$$Z_{in} = R_2 + j\omega L_{22} + \frac{\omega^2 L_{12}{}^2}{Z_{11}}$$

where

$$Z_{11} = R_1 + j\omega L_{11} + Z_1$$

Thus

$$Z_{in} = 40{\cdot}8 + j334 + \frac{10^8 \times 22{\cdot}2^2 \times 10^{-6}}{34 + j413 + 200 + j0}$$

$$= 40{\cdot}8 + j334 + \frac{22{\cdot}2^2 \times 10^2}{475^2}(234 - j413)$$

$$= 91{\cdot}8 + j244 = \underline{\underline{260\angle 69{\cdot}4^\circ\,\Omega}}$$

9.6 Power Transformers

Power transformers are normally operated in a.c. circuits under approximately constant voltage and constant frequency conditions. The remainder of this chapter will refer mainly to power transformers used in this way.

The transformation element enclosed by the broken lines in Fig. 9.7 is ideal and satisfies not only the conditions both for ideal voltage transformation and ideal current transformation but also, from eqn. (9.28),

$$e_1 i_1 = e_2 i_2 \tag{9.39}$$

i.e. the instantaneous power input is equal to the instantaneous power output.

An actual power transformer will tend towards this ideal if the terminal voltages and currents as defined in Fig. 9.7 tend to conform to eqns. (9.25), (9.26) and (9.28) so that

$$\frac{v_1}{v_2} = \frac{N_1}{N_2} \tag{9.40}$$

$$i_1 N_1 = i_2 N_2 \tag{9.41}$$

$$v_1 i_1 = v_2 i_2 \tag{9.42}$$

To make a transformer for which eqn. (9.40) is approximately true the voltage drops in the series elements must be small compared with the respective terminal voltages. This can be realized practically by making the winding resistances, R_1 and R_2, and the leakage inductances, L_{l1} and L_{l2} small. To obtain low values of leakage inductance the coupling between the windings must be tight. In practice this is achieved by using sandwich or concentric coils (see page 277).

To make a transformer for which eqn. (9.41) is approximately true the current flowing in the magnetizing inductance L_{m1} must be small. This is achieved by winding the transformer coils on a core of high permeance as explained in Section 9.4. A large magnetizing flux per ampere is thus produced in the core, the magnetizing current is kept low and the magnetizing inductance is high.

In short, an actual power transformer tends towards the ideal when the series resistances and inductances in its equivalent circuit have low values and the shunt resistances and inductances have high values.

The power transformer is normally connected to a supply voltage which varies sinusoidally with time. As a result the magnetizing flux in the core varies sinusoidally with time (see page 288) and eddy-current and hysteresis losses occur in the core.

The core losses are primarily dependent on the maximum flux density attained and on the frequency of the alternating flux. In a power transformer operating in a constant-voltage constant-frequency system, the core loss will be substantially constant, and may be represented approximately by introducing into the equivalent

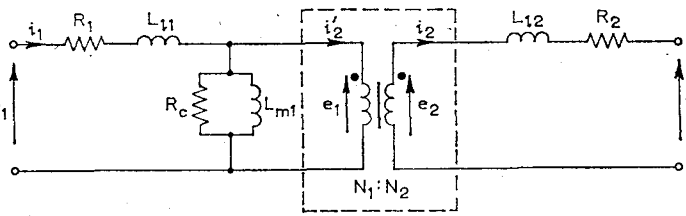


Fig. 9.10 TRANSFORMER EQUIVALENT CIRCUIT TAKING ACCOUNT OF CORE LOSS

circuit of Fig. 9.7 a resistance R_c the power dissipation in which, e_1^2/R_c, is equal to the core loss. Such an equivalent circuit is shown in Fig. 9.10.

For an efficient transformer the internal losses must be small compared with the input or output power at any instant. The internal losses consist of winding I^2R losses and core losses. Winding losses may be made small by making the winding resistances R_1 and R_2 small. The core hysteresis loss can be made small by using a core material which has a relatively narrow hysteresis loop. The core eddy-current loss is made small by laminating the core (see page 277).

9.7 Construction

MAGNETIC CIRCUIT

There are three distinct types of construction: core type, shell type and Berry type. In the core type (Fig. 9.11(*a*)), half of the primary winding and half of the secondary winding are placed round each limb. This reduces the effect of flux leakage which would seriously affect the operation if the primary and secondary were each wound separately on different limbs. The limbs are joined together by an iron yoke.

In the shell type (Fig. 9.11(*b*)), both windings are placed round a central limb, the two outer limbs acting simply as a low-reluctance flux path.

In the Berry type of construction (Fig. 9.11(*c*)), the core may be considered to be placed round the windings. It is essentially a shell type of construction with the magnetic paths distributed evenly

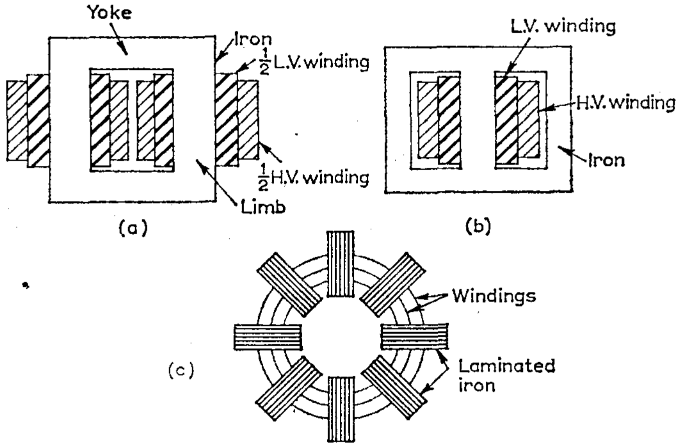


Fig. 9.11 CONSTRUCTION OF SINGLE-PHASE TRANSFORMERS
(*a*) Core type (*b*) Shell type (*c*) Berry type (plan view)

round the windings. Owing to constructional difficulties it is not so common as the first two.

Three-phase transformers are usually of the unsymmetrical core type (Fig. 9.12(*a*)). It will be seen that the flux path for the phase

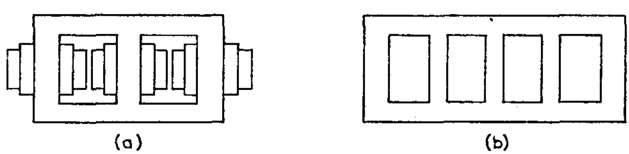


Fig. 9.12 THREE-PHASE TRANSFORMERS
(*a*) Core type (*b*) Shell type

which is wound on the central limb has a lower reluctance, or higher permeance, than that of a phase which is wound on an outer limb, but the effect is normally small. The shell type (Fig. 9.12(*b*)) has five limbs, the central three of which carry the windings. The cross-section of the yoke in this case can be a little less than that required for the core type.

Solid magnetic cores are inadmissible, since the core material is an electrical conductor, and the core would, in effect, form a single short-circuited turn, which would carry large induced eddy currents (Fig. 9.13(*a*)). Eddy currents are reduced by using high-resistivity silicon steel, made up in insulated laminations which are usually 0·355 mm. (0·014 in) thick. The length and the resistance of

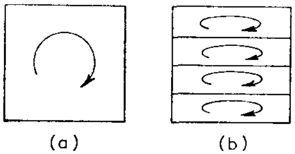


Fig. 9.13 USE OF LAMINATIONS TO REDUCE EDDY CURRENTS

the eddy-current paths are increased by this means, so that the heat loss due to the eddy currents is reduced to small proportions (Fig. 9.13(*b*)).

WINDINGS

There are two main forms of winding: the concentric cylinder (Fig. 9.14(*a*)) and the sandwich (Fig. 9.14(*b*)). The coils are made

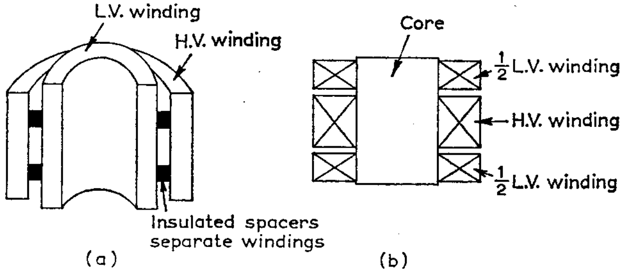


Fig. 9.14 TYPES OF TRANSFORMER WINDING
(*a*) Concentric (*b*) Sandwich

of varnished-cotton or paper-covered wire or strip and are circular in shape to prevent high mechanical stresses at corners. Due to the problems of insulation, the low-voltage windings are normally placed nearest the core in the concentric cylinder winding. For small transformers each layer is separated from the others by a thin paper, and the high-voltage winding is kept well insulated by special insulating sheets. In the larger transformers each winding is on a separate former, the two windings being well separated by insulating distance pieces.

In the sandwich winding, which has the advantage of reducing the leakage flux, the windings are arranged in layers, with a half-layer of the low-voltage winding at the top and bottom.

In all forms of winding for transformers used in power supply systems the end turns are specially insulated. This is necessary to prevent the destruction of the transformer when voltage surges arise in the system. Surges are described in a later chapter.

COOLING

Heat is produced in a transformer by the eddy-current and hysteresis losses in the core, and by the I^2R loss in the windings. To prevent undue temperature rise this heat is removed by cooling. In small transformers natural air cooling is usually employed. For larger transformers oil cooling is needed, especially where high voltages are in use. Oil-cooled transformers must be enclosed in steel tanks. Oil has the following advantages over air as a cooling medium:

(i) It has a larger specific heat than air, so that it will absorb larger quantities of heat for the same temperature rise.
(ii) It has a greater heat conductivity than air, and so enables the heat to be transferred to the oil more quickly.
(iii) It has about six times the breakdown strength of air—ensuring increased reliability at high voltages.

9.8 E.M.F. Equation

The e.m.f.s induced in the primary and secondary winding due to the mutual flux in a transformer are given by eqns. (9.23) and (9.24). The e.m.f.s induced due to the leakage fluxes are accounted for in the equivalent circuit by the inclusion of leakage-inductance elements.

Suppose the voltage applied to the primary varies sinusoidally with time, i.e.

$$v_1 = V_{1m} \sin \omega t$$

If the voltage drop in the primary resistance and leakage inductance is small enough to be neglected compared with the applied voltage, the induced primary e.m.f. at any instant is equal to the applied voltage, i.e.

$$e_1 = v_1 = V_{1m} \sin \omega t \tag{9.43}$$

Substituting this expression for e_1 in eqn. (9.23),

$$V_{1m} \sin \omega t = N_1 \frac{d}{dt} (\Phi_{m1} - \Phi_{m2})$$

Therefore the mutual flux at any instant is

$$\Phi = \Phi_{m1} - \Phi_{m2} = \frac{1}{N_1}\int V_{1m} \sin \omega t\, dt$$

$$= -\frac{V_{1m}}{N_1\omega} \cos \omega t + \text{constant}$$

Since there is no d.c. flux component the constant of integration is zero.

The peak value of the mutual flux is

$$\Phi_m = \frac{V_{1m}}{N_1\omega}$$

so that the peak value of the applied voltage is

$$V_{1m} = \omega\Phi_m N_1$$

Substituting this expression for V_{1m} in eqn. (9.43),

$$e_1 = \omega\Phi_m N_1 \sin \omega t$$

The r.m.s. value of the induced e.m.f. in the primary is

$$E_1 = \frac{\omega\Phi_m N_1}{\sqrt{2}} \tag{9.44}$$

Similarly the induced e.m.f. in the secondary is

$$E_2 = \frac{\omega\Phi_m N_2}{\sqrt{2}} \tag{9.45}$$

The above equations are often written in the following form:

$$E_1 = \frac{2\pi f}{\sqrt{2}}\Phi_m N_1 = 4{\cdot}44 f\Phi_m N_1 \tag{9.44a}$$

Similarly,

$$E_2 = 4{\cdot}44 f\Phi_m N_2 \tag{9.45a}$$

Substituting $\Phi_m = B_m A$ (where A is the cross-sectional area of the core) in eqn. (9.44*a*),

$$E_1 = 4{\cdot}44 f B_m A N_1 \tag{9.46}$$

9.9 Primary Current Waveform

The equivalent circuits derived for the transformer have assumed that magnetic saturation is absent and that as a result the various inductances are constant. In practice, transformers are nearly always designed so that their magnetic circuits are driven into saturation to reduce the magnetic cross-section required and so produce the most economical design. In these circumstances the inductances are not constant but vary with the degree of magnetic saturation present. Since power transformers are operated at approximately constant voltage and frequency the degree of saturation at maximum flux density does not vary greatly, and appropriate average values may be found for the inductances to give reasonably accurate predictions.

Another result of saturation of the magnetic core is that the magnetizing current becomes non-sinusoidal. The method of deriving the current waveform is given in Section 5.9.

The no-load current is considerably distorted in transformers of normal design. It may be represented by its *sine-wave equivalent*—the current having the same r.m.s. value as the actual no-load current and producing the same mean power. It is this current which is displayed on complexor diagrams and used in calculations based on the indicated value of no-load current.

9.10 Approximate Equivalent Circuit

In a power transformer the current taken by the shunt arm of the equivalent circuit, as given in Fig. 9.10, is small compared with the

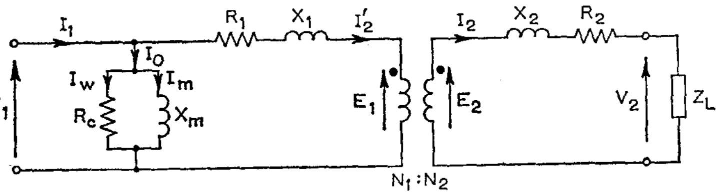


Fig. 9.15 APPROXIMATE TRANSFORMER EQUIVALENT CIRCUIT

rated primary current. Further, the rated primary current will cause only a small voltage drop in the series elements so that the drop in these elements due to the current in the shunt elements will be negligible. The shunt elements may therefore be placed at the input terminals without serious error. This results in a simplification of the equivalent circuit which is shown in Fig. 9.15, where the instantaneous currents and voltages are replaced by the corresponding complexor quantities, and the inductances by their corresponding reactances. The new circuit is convenient for predicting the steady-state response of the transformer to an input voltage varying sinusoidally with time.

9.11 Equivalent Circuit referred to Primary

For steady-state a.c. operation the instantaneous values for the voltage and current relations in eqns. (9.25) and (9.27) may be replaced by the corresponding r.m.s. values:

$$\frac{E_1}{E_2} = \frac{N_1}{N_2} \tag{9.47}$$

and

$$I_2' N_1 = I_2 N_2 \tag{9.48}$$

Applying Kirchhoff's law to the primary and secondary meshes of the equivalent circuit of Fig. 9.15,

$$V_1 = (R_1 + jX_1)I_2' + E_1 \tag{9.49}$$

$$E_2 = (R_2 + jX_2)I_2 + V_2 \tag{9.50}$$

If all the terms of eqn. (9.50) are multiplied by N_1/N_2 this gives

$$\frac{N_1}{N_2} E_2 = (R_2 + jX_2) \frac{N_1}{N_2} I_2 + \frac{N_1}{N_2} V_2$$

Substituting $I_2 = \dfrac{N_1}{N_2} I_2'$ gives

$$\frac{N_1}{N_2} E_2 = (R_2 + jX_2) \left(\frac{N_1}{N_2}\right)^2 I_2' + \frac{N_1}{N_2} V_2$$

or

$$E_1 = (R_2' + jX_2')I_2' + V_2' \tag{9.51}$$

where

$$R_2' = \left(\frac{N_1}{N_2}\right)^2 R_2 \tag{9.52}$$

$$X_2' = \left(\frac{N_1}{N_2}\right)^2 X_2 \tag{9.53}$$

$$V_2' = \frac{N_1}{N_2} V_2 \tag{9.54}$$

R_2' is called the *secondary resistance referred to the primary*;
X_2' is the *secondary reactance referred to the primary*; and
V_2' is the *secondary load voltage referred to the primary*.

Eqns. (9.49) and (9.50) are consistent with the equivalent circuit of Fig. 9.16(*a*), which is the transformer equivalent circuit referred to the primary. The load impedance, Z_L, may also be referred to the primary if desired to give an equivalent primary impedance, Z_L', where

$$Z_L' = \frac{V_2'}{I_2'} = \frac{\dfrac{N_1}{N_2} V_2}{\dfrac{N_2}{N_1} I_2} = \left(\frac{N_1}{N_2}\right)^2 \frac{V_2}{I_2} = \left(\frac{N_1}{N_2}\right)^2 Z_L \tag{9.55}$$

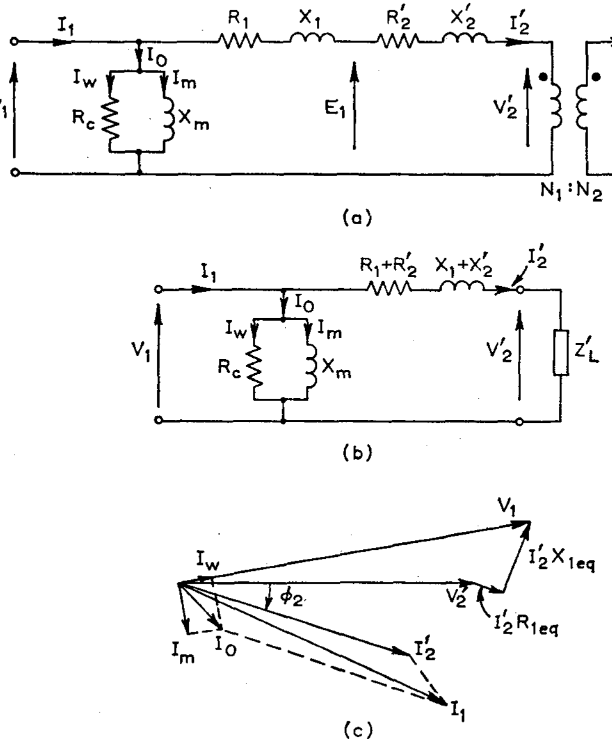


Fig. 9.16 TRANSFORMER EQUIVALENT CIRCUIT REFERRED TO PRIMARY

The output quantities now become V_2' and I_2', and the ideal transformer element may be omitted. This equivalent circuit is shown in Fig. 9.16(*b*). Thus the total equivalent series impedance referred to the primary is

$$Z_{1eq} = R_{1eq} + jX_{1eq} \tag{9.56}$$

where

$$R_{1eq} = R_1 + R_2' = R_1 + \left(\frac{N_1}{N_2}\right)^2 R_2 \tag{9.57}$$

and

$$X_{1eq} = X_1 + X_2' = X_1 + \left(\frac{N_1}{N_2}\right)^2 X_2 \tag{9.58}$$

Fig. 9.16(*c*) is the complexor diagram corresponding to the equivalent circuit shown at (*b*). The referred value of load voltage, V_2', is chosen as the reference complexor. The referred value of load current, I_2', is shown lagging behind V_2' by a phase angle ϕ_2. For a given value of V_2', both I_2' and ϕ_2 are determined by the applied load. The voltage drop $I_2'R_{1eq}$ is in phase with I_2', and the voltage drop $I_2'X_{1eq}$ leads I_2' by 90°. When these voltage drops are added to V_2' the input voltage, V_1, is obtained.

I_w is in phase with V_1 (V_1I_w is approximately equal to the core loss); I_m, the magnetizing current, lags behind V_1 by 90°. The complexor sum of I_w and I_m is I_0. I_0 will flow even when the secondary terminals are open-circuited and is therefore called the *no-load current*. The complexor sum of I_0 and I_2' is the input current, I_1. For the sake of clarity the size of I_0 relative to I_1 is exaggerated in the diagram.

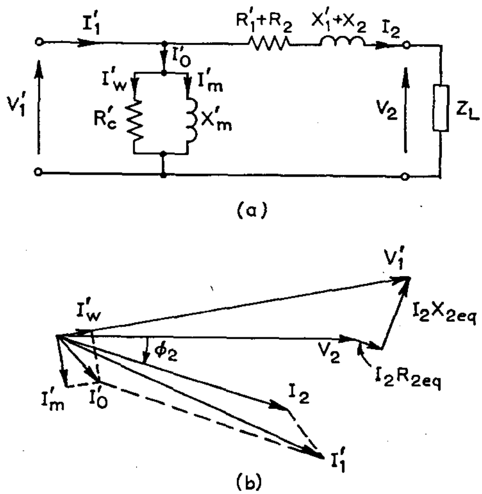


Fig. 9.17 TRANSFORMER EQUIVALENT CIRCUIT REFERRED TO SECONDARY

9.12 Equivalent Circuit referred to Secondary

A method similar to that adopted in Section 9.11 to obtain the transformer equivalent circuit referred to the primary may be used to obtain an equivalent circuit referred to the secondary. This is shown in Fig. 9.17(*a*), and the corresponding complexor diagram is shown at (*b*).

In Fig. 9.17(*a*) the primed symbols represent primary quantities referred to the secondary:

$$V_1' = \frac{N_2}{N_1} V_1 \tag{9.59}$$

$$I_1' = \frac{N_1}{N_2} I_1 \tag{9.60}$$

Any primary impedance is referred to the secondary by multiplying its value in the primary by $(N_2/N_1)^2$.

EXAMPLE 9.2 A 4,000/400 V 10 kVA transformer has primary and secondary winding resistances of 13 Ω and 0·15 Ω respectively. The leakage reactance referred to the primary is 45 Ω. The magnetizing impedance referred to the primary is 6 kΩ, and the resistance corresponding to the core loss is 12 kΩ.

Determine the total resistance referred to the primary and the values of all impedances referred to the secondary. Determine the input current (*a*) when the secondary terminals are open-circuited, and (*b*) when the secondary load current is 25 A at a power factor of 0·8 lagging.

$$\text{Turns ratio, } \frac{N_1}{N_2} = \frac{4{,}000}{400} = \underline{\underline{10}}$$

$$R_{1eq} = R_1 + R_2' = R_1 + R_2\left(\frac{N_1}{N_2}\right)^2 = 13 + 0{\cdot}15 \times 10^2 = \underline{\underline{28\,\Omega}}$$

$$R_{2eq} = R_{1eq}\left(\frac{N_2}{N_1}\right)^2 = 28 \times 0{\cdot}1^2 = \underline{\underline{0{\cdot}28\,\Omega}}$$

$$X_{2eq} = X_{1eq}\left(\frac{N_2}{N_1}\right)^2 = 45 \times 0{\cdot}1^2 = \underline{\underline{0{\cdot}45\,\Omega}}$$

$$R_c' = R_c\left(\frac{N_2}{N_1}\right)^2 = 12{,}000 \times 0{\cdot}1^2 = \underline{\underline{120\,\Omega}}$$

$$X_m' = X_m\left(\frac{N_2}{N_1}\right)^2 = 6{,}000 \times 0{\cdot}1^2 = \underline{\underline{60\,\Omega}}$$

$$\text{Core-loss component of current, } I_w = \frac{V}{R_c} = \frac{4{,}000}{12{,}000} = \underline{\underline{0{\cdot}333\,\text{A}}}$$

$$\text{Magnetizing current, } I_m = \frac{V}{X_m} = \frac{4{,}000}{6{,}000} = \underline{\underline{0{\cdot}667\,\text{A}}}$$

$$\text{Input current on no-load, } I_0 = 0{\cdot}333 - j0{\cdot}667 = \underline{\underline{0{\cdot}745\angle{-63{\cdot}5^\circ}\,\text{A}}}$$

When the secondary load current is 25 A at a power factor of 0·8 lagging,

$$I_2 = 25\angle{-36{\cdot}9^\circ}$$

$$I_2' = 2{\cdot}5\angle{-36{\cdot}9} = (2{\cdot}0 - j1{\cdot}5)\,\text{A}$$

$$\text{Input current, } I_1 = I_2' + I_0 \doteq (2{\cdot}0 - j1{\cdot}5) + (0{\cdot}333 - j0{\cdot}667)$$

$$= \underline{\underline{3{\cdot}18\angle{-43^\circ}\,\text{A}}}$$

9.13 Regulation

The *voltage regulation* of a transformer is the change in the terminal voltage between no load and full load at a given power factor. This is often expressed as a percentage of the rated voltage or in per-unit form, using rated voltage as base (see Section 9.16).

Consider the equivalent circuit referred to the secondary, Fig. 9.17. On no-load the secondary terminal voltage will be

$$V_2 = V_1' = V_1 \frac{N_2}{N_1}$$

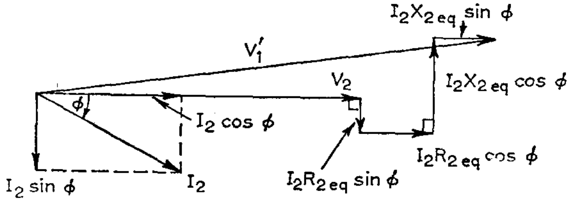


Fig. 9.18 REGULATOR OF A TRANSFORMER WITH LAGGING POWER FACTOR

When the load is applied the current flowing through the equivalent impedance produces a voltage drop (IZ), and for a fixed value of V_1 this will cause V_2 to change. The equivalent secondary complexor diagram is shown in Fig. 9.18.

The load current I_2 may be represented by its in-phase component $I_2 \cos \phi$, and its quadrature component $I_2 \sin \phi$, and these components each give rise to voltage drops in the total equivalent resistance, R_{2eq}, and the total equivalent reactance, X_{2eq}. These voltage drops are shown in Fig. 9.18, from which it will be seen that, for a lagging power factor,

$$V_1' = \sqrt{\{(V_2 + I_2R_{2eq}\cos\phi + I_2X_{2eq}\sin\phi)^2 + (I_2X_{2eq}\cos\phi - I_2R_{2eq}\sin\phi)^2\}} \quad (9.61)$$

This exact expression for V_1' is rather cumbersome and it is usually sufficient to adopt the following approximate expression for regulations of less than 20 per cent (or 0·2 p.u.):

$$V_1' = V_2 + I_2R_{2eq}\cos\phi + I_2X_{2eq}\sin\phi \quad (9.62)$$

The justification for this expression will be clear from Fig. 9.18, where it is apparent that the voltage drops represented by quadrate complexors have little effect on the total magnitude of V_1'.

$$\text{Voltage regulation} = V_1' - V_2 = I_2R_{2eq}\cos\phi + I_2X_{2eq}\sin\phi \quad (9.63)$$

If the circuit is referred to the primary, then V_1 will remain constant and V_2' will vary as the load changes. It can be shown, in the same manner as before, that for lagging power factors,

$$V_1 - V_2' \approx I_1(R_1 + R_2')\cos\phi + I_1(X_1 + X_2')\sin\phi \qquad (9.64)$$

The regulation of the transformer is then this voltage multiplied by N_2/N_1.

If the power factor is leading, the regulation is

$$V_1' - V_2 \approx I_2 R_{2\,eq}\cos\phi - I_2 X_{2\,eq}\sin\phi \qquad (9.65)$$

For a given load kVA, the current I_2 will be approximately constant. Hence the regulation may be differentiated with respect to the load phase angle ϕ to determine the way in which the power factor affects the regulation. Thus, from eqn. (9.63),

$$\frac{d}{d\phi}(V_1' - V_2) = -I_2 R_{2\,eq}\sin\phi + I_2 X_{2\,eq}\cos\phi$$

This will be zero when

$$I_2 R_{2\,eq}\sin\phi = I_2 X_{2\,eq}\cos\phi$$

i.e. when

$$\tan\phi = \frac{X_{2\,eq}}{R_{2\,eq}} \qquad \text{or} \qquad \phi = \tan^{-1}\frac{X_{2\,eq}}{R_{2\,eq}} \qquad (9.66)$$

Hence the regulation has a maximum value at that value of load phase angle which is equal to the internal total phase angle of the transformer itself.

Also from eqn. (9.63) the regulation will be zero when

$$I_2 R_{2\,eq}\cos\phi = -I_2 X_{2\,eq}\sin\phi$$

i.e. when

$$\phi = \tan^{-1}\left(-\frac{R_{2\,eq}}{X_{2\,eq}}\right) \qquad (9.67)$$

The minus sign denotes a leading power factor. For leading power factors smaller than this the regulation will be negative, denoting a voltage rise on load.

EXAMPLE 9.3 The primary and secondary windings of a 40kVA 6,600/250V single-phase transformer have resistances of 10Ω and 0·02Ω respectively. The leakage reactance of the transformer referred to the primary is 35Ω. Calculate (*a*) the primary voltage required to circulate full-load current when the secondary is short-circuited, (*b*) the full-load regulation at (i) unity (ii) 0·8 lagging power factor. Neglect the no-load current. (*H.N.C.*)

It is obviously convenient to refer impedances to the primary side, as shown in Fig. 9.16. Then, from eqn. (9.52),

$$R_2' = R_2 \left(\frac{N_1}{N_2}\right)^2 = 0{\cdot}02 \times \left(\frac{6{,}600}{250}\right)^2 = 13{\cdot}9\,\Omega$$

since the voltage ratio quoted is equal to the transformation ratio.

The full-load primary current is

$$I_{fl} = \frac{40{,}000}{6{,}600} = 6{\cdot}06\,\text{A}$$

(*a*) If the secondary is short-circuited, the impedance of the load reflected in the primary is zero, so that the total impedance is simply

$$Z_1 = R_1 + R_2' + jX_{1eq}$$

Hence the primary voltage required to circulate full-load current is

$$V_{sc} = 6{\cdot}06\sqrt{(23{\cdot}9^2 + 35^2)} = \underline{\underline{256\,\text{V}}}$$

(*b*) In this problem the parallel magnetizing circuit is neglected on full load, to simplify the solution. At unity p.f. and full load, from eqn. (9.64),

$$V_1 - V_2' = (6{\cdot}06 \times 23{\cdot}9 \times 1) + 0 = 145\,\text{V}$$

Therefore

$$\text{Actual regulation} = 145 \times \frac{250}{6{,}600} = 5{\cdot}5\,\text{V}$$

On no-load $V_2 = 250$ V. Therefore

$$\text{Regulation} = \frac{5{\cdot}5}{250} \times 100 = \underline{\underline{2{\cdot}2 \text{ per cent}}}$$

At 0·8 p.f. lagging, from eqn. (9.64),

$$V_1 = V_2' = (6{\cdot}06 \times (23{\cdot}9) \times 0{\cdot}8) + 6{\cdot}06 \times 35 \times 0{\cdot}6 = 244\,\text{V}$$

Therefore

$$\text{Actual regulation} = 244 \times \frac{250}{6{,}600} = 9{\cdot}25$$

and

$$\text{Percentage regulation} = \frac{9{\cdot}25}{250} \times 100 = \underline{\underline{3{\cdot}7 \text{ per cent}}}$$

9.14 Transformer Losses and Efficiency

Transformer losses may be divided into two main parts, (*a*) the losses which vary with the load current, and (*b*) the losses which vary with the core flux. Since the load current is not constant during normal operation the winding I^2R losses will vary. Under normal conditions, however, the core flux will remain approximately constant, so that the losses which vary with the core flux (core losses) will be approximately constant, independent of the load. These losses include stray losses due to e.m.f.s induced by stray fields in adjacent conductors. A further small source of loss is dielectric loss in the insulation, but this will be neglected.

There will be a loss due to the resistance of each winding, the total winding loss being

$$P_c = I_1^2R_1 + I_2^2R_2 = I_2^2(R_2 + R_1') \quad \text{watts} \tag{9.68}$$

The core losses are divided into two parts, the hysteresis loss, and the eddy-current loss.

HYSTERESIS LOSS (P_h)

In a specimen of steel subject to an alternating flux, the hysteresis loss per cycle is proportional to the area of the hysteresis loop, and therefore to B_m^n:

$$\text{Hysteresis loss} = k_h f B_m^n \quad \text{watts/metre}^3 \tag{9.69}$$

where B_m is the maximum flux density in teslas, and k_h is a constant. The coefficient n is empirically found to be in the range 1·6 to 2.

EDDY-CURRENT LOSS (P_e)

This is due to the flow of eddy currents in the core. Thin high-resistivity laminations effectively reduce the eddy-current loss to small proportions. Since essentially a transformer action is involved (considering each lamination as a single short-circuited secondary), the induced e.m.f. in the core will be proportional to fB_m (since $E = 4{\cdot}44fN\Phi_m$). This causes the flow of an eddy current,

$$I_e = \frac{\text{E.M.F.}}{\text{Impedance of core path}}$$

The impedance of the core path may be assumed constant and independent of frequency for low power frequencies and thin laminations. Thus

$$\text{Eddy current loss} \propto I_e^2 \propto f^2B_m^2 = k_e f^2 B_m^2 \quad \text{watts/m}^3 \tag{9.70}$$

where k_e is a constant. The eddy current loss is proportional to the square of the frequency. The total core loss, P_i, is

$$P_i = P_h + P_e = k_h f B_m^n + k_e f^2 B_m^2 \tag{9.71}$$

Provided that B_m and f are constant the core losses should be constant. In order that B_m shall be constant, the magnetizing current and hence the applied voltage must be constant.

EFFICIENCY

The rating of a transformer is an output rating, and hence the efficiency is calculated in terms of the output in kilowatts.

$$\text{Efficiency, } \eta = \frac{\text{Output}}{\text{Input}} = \frac{\text{Output}}{\text{Output} + \text{Losses}} = 1 - \frac{\text{Losses}}{\text{Input}}$$

$$= \frac{V_2 I_2 \cos\phi}{V_2 I_2 \cos\phi + I_2{}^2 R_{2eq} + P_i} \quad (9.72)$$

where $R_{2eq} = R_2 + R_1' = R_2 + R_1 \left(\dfrac{N_2}{N_1}\right)^2$, and P_i is the total constant core loss. Therefore

$$\eta = \frac{V_2 \cos\phi}{V_2 \cos\phi + I_2 R_{2eq} + \dfrac{P_i}{I_2}}$$

For operation at constant voltage, constant power factor and variable load current, the efficiency will be a maximum when $I_2 R_{2eq} + \dfrac{P_i}{I_2}$ is a minimum. This will happen when

$$\frac{d}{dI_2}\left\{I_2 R_{2eq} + \frac{P_i}{I_2}\right\} = 0 = R_{2eq} - \frac{P_i}{I_2{}^2}$$

i.e. when

$$P_i = I_2{}^2 R_{2eq}$$

Thus the efficiency at any given power factor is a maximum when the load is such that the I^2R losses are equal to the constant core losses.

EXAMPLE 9.4 The required no-load voltage ratio in a 150kVA single-phase 50Hz core-type transformer is 5,000/250V.

(*a*) Find the number of turns in each winding for a core flux of about 0·06Wb.
(*b*) Calculate the efficiency at half-rated kVA and unity power factor.
(*c*) Determine the efficiency at full load and 0·8p.f. lagging. The full-load I^2R loss is 1,800W, and the core loss is 1,500W.
(*d*) Find the load kVA for maximum efficiency.

(*a*) From eqn. (9.45*a*),

$$E_2 = 4{\cdot}44 f N_2 \Phi_m$$

Therefore

$$N_2 = \frac{250}{4{\cdot}44 \times 50 \times 0{\cdot}06} = 18{\cdot}8, \text{ or say } \underline{\underline{19 \text{ turns}}}$$

Since only a whole number of turns is possible it is most common, as above, to calculate the number of turns on the low-voltage winding first. Then the number of turns on the high-voltage winding is found to give the correct ratio. Thus

$$N_1 = 19 \times \frac{5{,}000}{250} = \underline{\underline{380 \text{ turns}}}$$

(*b*) At half-rated kVA, unity p.f. (i.e. at half full-load current).

Winding loss $= (\frac{1}{2})^2 \times 1{,}800 = 450\,\text{W} = 0{\cdot}45\,\text{kW}$

Core loss = constant $= P_i = 1{,}500\,\text{W} = 1{\cdot}5\,\text{kW}$

Power output $= \frac{1}{2} \times 150 = 75\,\text{kW}$

Therefore

$$\text{Efficiency} = 1 - \frac{0{\cdot}45 + 1{\cdot}5}{75 + 0{\cdot}45 + 1{\cdot}5} = \underline{\underline{97{\cdot}2 \text{ per cent}}}$$

(*c*) At full-load kVA, 0·8 p.f. lagging.

Power output $= 150 \times 0{\cdot}8 = 120\,\text{kW}$

and

$$\text{Efficiency} = 1 - \frac{1{\cdot}8 + 1{\cdot}5}{120 + 1{\cdot}8 + 1{\cdot}5} = \underline{\underline{97{\cdot}3 \text{ per cent}}}$$

(*d*) Maximum efficiency.

Let x be the fraction of the full-load kVA at which maximum efficiency occurs. For maximum efficiency the core loss is equal to the I^2R loss, i.e. $P_i = x^2P_c$, whence

$$x = \sqrt{\frac{P_i}{P_c}} = \sqrt{\frac{1{,}500}{1{,}800}} = 0{\cdot}913$$

Therefore

Load kVA for maximum efficiency $= 0{\cdot}913 \times 150 = \underline{\underline{137\,\text{kVA}}}$

9.15 Transformer Tests

OPEN-CIRCUIT TEST

The circuit is shown in Fig. 9.19. The transformer is connected

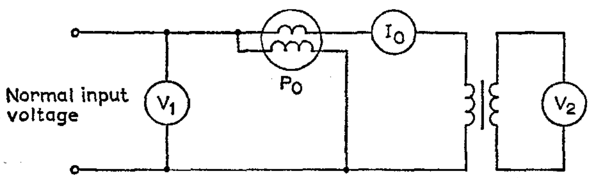


Fig. 9.19 OPEN-CIRCUIT TEST

(in its tank) and the normal rated voltage V_1 is applied. The secondary voltage V_2, the current I_0 and the power P_0 are measured. Then, since the I^2R loss on open-circuit may be neglected (or allowed for, since I_0 is known), the no-load input will be the normal core loss.

The values of R_c and X_m in the parallel exciting circuit can then be calculated. The open-circuit power factor is $\cos \phi_0 = P_0/V_1 I_0$, so that

$$I_w = I_0 \cos\phi_0 \qquad \text{and} \qquad I_m = I_0 \sin\phi_0$$

Then

$$R_c = \frac{V}{I_w} \tag{9.73}$$

and

$$X_m = \frac{V}{I_m} \tag{9.74}$$

These are shown on the diagram of Fig. 9.16. The open-circuit ratio V_1/V_2 will be almost equal to the turns ratio N_1/N_2.

SHORT-CIRCUIT TEST (Fig. 9.20)

One winding is short-circuited and the voltage applied to the other is gradually raised from zero until full-load current flows.

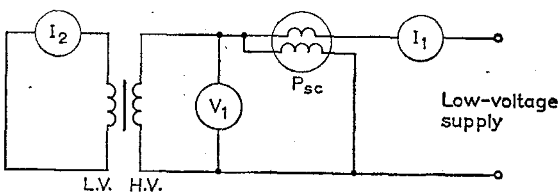


Fig. 9.20 SHORT-CIRCUIT TEST

The readings of the wattmeter (P_{sc}), ammeter (I_1), and voltmeter (V_1) are noted. Since the exciting voltage is small the core flux will be small, and the core losses will be negligible. If the impedance is referred to the primary, then

$$R_{1eq} = R_1 + R_2' \qquad \text{and} \qquad X_{1eq} = X_1 + X_2'$$

and these values may be found from the instrument readings.

The short-circuit power factor is

$$\cos \phi_{sc} = \frac{P_{sc}}{V_1 I_1}$$

and

$$Z_{1eq} = \frac{V_1}{I_1} = \frac{V_1}{I_1} \cos \phi_{sc} + j \frac{V_1}{I_1} \sin \phi_{sc} \tag{9.75}$$

$$= (R_1 + R_2') + j(X_1 + X_2') \tag{9.76}$$

The resistances of the windings can be measured separately by a d.c. test. It is, however, impossible to separate the leakage reactances.

Note that P_{sc} gives the full-load I^2R loss P_c, so that the efficiency at any load may be calculated from the open- and short-circuit tests.

EXAMPLE 9.5 A 10kVA 200/400V 50Hz single-phase transformer gave the following test results. O.C. test: 200V, 1·3A, 120W, on l.v. side. S.C. test: 22V, 30A, 200W on h.v. side.

(*a*) Calculate the magnetizing current and the component corresponding to core loss at normal frequency and voltage.

(*b*) Calculate the magnetizing-branch impedances.

(*c*) Find the percentage regulation when supplying full load at 0·8 p.f. leading.

(*d*) Determine the load which gives maximum efficiency, and find the value of this efficiency at unity p.f.

(*a*) O.C. test

$$\text{Open-circuit power factor} = \frac{120}{200 \times 1{\cdot}3} = 0{\cdot}462 = \cos\phi_0$$

Therefore

$$\text{Magnetizing current} = I_0 \sin\phi_0 = 1{\cdot}3 \times 0{\cdot}886 = \underline{\underline{1{\cdot}15\,\text{A}}}$$

and

$$\begin{aligned}\text{Component of current corresponding to core loss} &= I_0\cos\phi_0 \\ &= 1{\cdot}3 \times 0{\cdot}462 = \underline{\underline{0{\cdot}6\,\text{A}}}\end{aligned}$$

(*b*)

$$R_c = \frac{V_1}{I_0\cos\phi_0} = \frac{200}{0{\cdot}6} = \underline{\underline{333\,\Omega}}$$

$$X_m = \frac{V_1}{I_0\sin\phi_0} = \frac{200}{1{\cdot}15} = \underline{\underline{174\,\Omega}}$$

(*c*) Percentage regulation at 0·8 leading p.f.

$$\text{Total impedance referred to h.v. side, } Z_{2eq} = \frac{22}{30} = 0{\cdot}733\,\Omega$$

$$\text{Total resistance referred to h.v. side, } R_{2eq} = \frac{200}{30^2} = 0{\cdot}222\,\Omega$$

$$\text{Total reactance referred to h.v. side, } X_{2eq} = \sqrt{(0{\cdot}733^2 - 0{\cdot}222^2)} = 0{\cdot}698\,\Omega$$

$$\text{Full load current on h.v. side} = \frac{10{,}000}{400} = 25\,\text{A}$$

Thus

Regulation at 0·8 leading

$$\begin{aligned}&\approx R_{2eq}I_2\cos\phi - X_{2eq}I_2\sin\phi \qquad \text{(from eqn. (9.65))} \\ &= (0{\cdot}222 \times 25 \times 0{\cdot}8) - (0{\cdot}698 \times 25 \times 0{\cdot}6) \\ &= -6{\cdot}0\,\text{V} \qquad \text{(voltage rise due to leading power factor)}\end{aligned}$$

Therefore

$$\text{Regulation} = -\frac{6{\cdot}0}{400} \times 100 = \underline{\underline{-1{\cdot}5 \text{ per cent}}}$$

(*d*) Maximum efficiency

From o.c. test, full-voltage core loss, $P_i = 120\,\text{W}$

$$\text{From s.c. test, full-load } I^2R \text{ loss, } P_c = 200 \times \left(\frac{25}{30}\right)^2 = 140\,\text{W}$$

Let x be the fraction of full-load kVA at which maximum efficiency occurs. At maximum efficiency the I^2R loss is equal to the core loss, so that

$$x^2P_c = P_i$$

Therefore

$$x = \sqrt{\frac{P_i}{P_c}} = \sqrt{\frac{120}{140}} = 0{\cdot}925$$

and

$$\text{kVA for maximum efficiency} = 0{\cdot}925 \times 10 = 9{\cdot}25\,\text{kVA}$$

When the load has unity power factor,

$$\text{Load for maximum efficiency} = \underline{\underline{9{\cdot}25\,\text{kW}}}$$

$$\text{Maximum efficiency} = \frac{9{,}250}{9{,}250 + 120 + 120} \times 100 = \underline{\underline{97{\cdot}4 \text{ per cent}}}$$

SEPARATION OF HYSTERESIS AND EDDY-CURRENT LOSSES

From eqn. (9.71) it can be seen that, with sinusoidal flux in the core, the hysteresis loss increases with the frequency, while the eddy-current loss increases as the square of the frequency. This is the

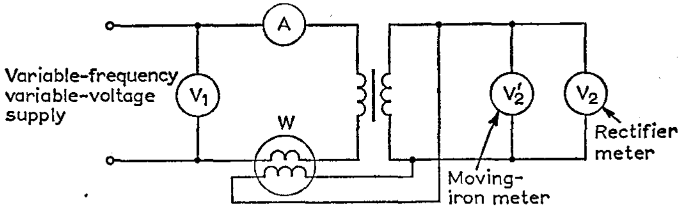


Fig. 9.21 CIRCUIT FOR SEPARATION OF CORE LOSS

basis of one method whereby these losses may be separated (Fig. 9.21). The transformer primary is connected to a variable-frequency and variable-voltage sinusoidal supply. Since it is necessary to ensure a sinusoidal core flux, the secondary is connected to a moving-iron voltmeter (measuring r.m.s. values), with a rectifier voltmeter in parallel (measuring average values × 1·11). If the flux is sinusoidal, the form factor is 1·11 and the instruments will record the same voltage. If the flux is not sinusoidal, the readings will differ, so that for this test it is essential that both instruments should give the same reading. This reading will be $4{\cdot}44fN_2\Phi_m$ volts, where N_2 is the number of secondary turns.

The hysteresis and eddy-current losses are separated by their different variations with frequency. The total loss should be measured at various frequencies while the other factors upon which the core losses depend are maintained constant. Thus it is necessary to maintain Φ_m constant as the frequency is varied. Now,

$$\Phi_m \propto \frac{V_1}{f} \quad \text{(from eqn. (9.44(}a\text{))}$$

so that as the supply frequency is varied the supply voltage must also be varied in such a way that the ratio V_1/f, and hence Φ_m, is maintained constant throughout the test.

Since the voltage coil of the wattmeter is energized from the transformer secondary, the primary I^2R loss is eliminated from the reading. The core loss is obtained by multiplying the wattmeter reading by the turns ratio, and subtracting from this the power loss in the voltmeters and in the voltage coil of the wattmeter.

$$\text{Hysteresis loss} = Af \text{ watts, where } A \text{ is a constant}$$

$$\text{Eddy current loss} = Cf^2 \text{ watts, where } C \text{ is a constant}$$

$$\text{Total core loss} = P_i = Af + Cf^2$$

whence

$$\frac{P_i}{f} = A + Cf$$

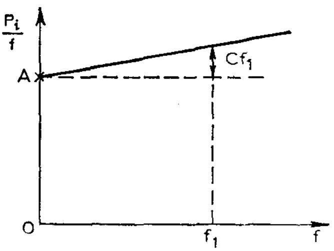


Fig. 9.22 SEPARATION OF CORE LOSSES

If a graph of P_i/f is plotted to a base of frequency, the graph will be a straight line intersecting the P_i/f axis at A and with a slope C (Fig. 9.22). Hence A and C, and the hysteresis and eddy-current losses at a given frequency and peak core flux density may be found.

9.16 The Per-unit System

It is often convenient to carry out calculations relating to transformers and other electrical plant using the *per-unit system.* In this method all the relevant quantities are expressed in per-unit form, i.e. as fractions of chosen base values.

There are two main advantages in using the per-unit system. First, the constants of transformers and other plant fall within narrow limits when expressed in per-unit form based on their rating. Second, in computations involving transformers the need to refer quantities from one side of the transformer to the other is eliminated.

The per-unit value of any quantity may be defined as

$$\text{Quantity in per-unit form} = \frac{\text{Actual quantity}}{\text{Base value of quantity}} \tag{9.77}$$

For any quantity A,

$$A_{pu} = \frac{A}{A_{base}} \tag{9.78}$$

The quantity A may be voltage, current, volt-amperes, impedance, admittance or any electrical quantity.

The base values of voltage and current, V_{base}, I_{base}, may be chosen arbitrarily though they will usually be chosen to correspond to *rated voltage* and *rated current.* Once these have been chosen the base values of all other electrical quantities are automatically fixed. For a single-phase system, the power, reactive volt-ampere and volt-ampere bases are

$$P_{base},\ Q_{base} \quad \text{and} \quad S_{base} = V_{base}I_{base} \tag{9.79}$$

The resistance, reactance and impedance bases are

$$R_{base},\ X_{base} \quad \text{and} \quad Z_{base} = \frac{V_{base}}{I_{base}} \tag{9.80}$$

The conductance, susceptance and admittance bases are

$$G_{base},\ B_{base} \quad \text{and} \quad Y_{base} = \frac{I_{base}}{V_{base}} \tag{9.81}$$

In practice it is more usual to choose the voltage base and the volt-ampere base and then to determine the current base from eqn. (9.79).

EXAMPLE 9.6 A 500V 10kVA single-phase generator has an open-circuit voltage of 500V. When the load current is 25A at a certain power factor the terminal voltage falls to 480V. Determine in per-unit form (*a*) the output voltage, (*b*) the output current, (*c*) the output volt-amperes, and (*d*) the voltage regulation.

Choosing rated voltage and rated volt-amperes as bases, and using the subscript B to represent base values,

$$V_B = 500\,\text{V} \quad \text{and} \quad S_B = 10{,}000\,\text{VA}$$

From eqn. (9.79),

$$I_B = \frac{S_B}{V_B} = \frac{10{,}000}{500} = 20\,\text{A}$$

$$\text{Output voltage p.u.} = \frac{480}{500} = \underline{\underline{0{\cdot}96}}$$

$$\text{Output current p.u.} = \frac{25}{20} = \underline{\underline{1{\cdot}25}}$$

(i.e. an overload since full load is represented by 1 p.u.)

$$\text{Output VA p.u.} = \frac{480 \times 25}{10{,}000} = 1{\cdot}2$$

or, more directly,

$$\text{Output VA p.u.} = V_{pu}I_{pu} = 0{\cdot}96 \times 1{\cdot}25 = \underline{\underline{1{\cdot}2}}$$

$$\text{Voltage regulation p.u.} = \frac{500 - 480}{500} = \underline{\underline{0{\cdot}04}}$$

or, more directly,

$$\text{Voltage regulation p.u.} = 1 - 0{\cdot}96 = \underline{\underline{0{\cdot}04}}$$

With transformers, provided that rated primary voltage is used as base with primary referred impedances, and rated secondary voltage with secondary referred impedances, the same per-unit values of impedance are obtained. As a result calculations in per-unit form are the same for the primary and the secondary. The actual primary or secondary values are obtained by multiplying the per-unit values by the appropriate base quantities.

EXAMPLE 9.7 A 5 kVA 200/400 V 50 Hz single-phase transformer has an equivalent circuit consisting of shunt admittance $(1{\cdot}5 \times 10^{-3} - j3{\cdot}15 \times 10^{-3})$ S mho and series leakage impedance $(0{\cdot}12 + j0{\cdot}32)\,\Omega$, both referred to the low-voltage side. Determine the per-unit values of the shunt admittance and the leakage impedance using first the primary referred values and then the secondary referred values. The voltage base is to be the rated value.

Determine also the per-unit value of the core loss when the transformer is excited at rated voltage, the full-load winding loss and the full-load voltage regulation if the load power factor is 0·8 lagging, and express each of these values in actual quantities.

(*a*) Per-unit values using primary data

Base VA, $S_B = 5{,}000\,\text{VA}$

Base voltage, V_{B1} = Rated primary voltage = 200 V

$$\text{Base current, } I_{B1} = \frac{S_B}{V_{B1}} = \frac{5{,}000}{200} = 25\,\text{A}$$

$$\text{Base impedance, } Z_{B1} = \frac{V_B}{I_B} = \frac{200}{25} = 8\,\Omega$$

$$\text{Base admittance, } Y_{B1} = \frac{I_B}{V_B} = \frac{25}{200} = 0{\cdot}125\text{S}$$

$$\text{Shunt admittance, } Y_{pu} = \frac{Y_{1eq}}{Y_B} = \frac{1{\cdot}5 \times 10^{-3} - j3{\cdot}15 \times 10^{-3}}{0{\cdot}125}$$

$$= \underline{\underline{(12 \times 10^{-3} - 25{\cdot}2 \times 10^{-3})\text{p.u.}}}$$

$$\text{Leakage impedance, } Z_{pu} = \frac{0{\cdot}12 + j0{\cdot}32}{8}$$

$$= \underline{\underline{(0{\cdot}015 + j0{\cdot}04)\text{p.u.}}}$$

(*b*) Actual referred values of equivalent circuit constants

$$Y_{2eq} = Y_{1eq}\left(\frac{200}{400}\right)^2 = \frac{1{\cdot}5 \times 10^{-3} - j3{\cdot}15 \times 10^{-3}}{4}$$

$$= (0{\cdot}375 \times 10^{-3} - j0{\cdot}787 \times 10^{-3})\text{ S}$$

$$Z_{2eq} = Z_{1eq}\left(\frac{400}{200}\right)^2 = 4(0{\cdot}12 + j0{\cdot}32) = (0{\cdot}48 + j1{\cdot}28)\Omega$$

(*c*) Per-unit values using secondary data

Base VA, $S_B = 5{,}000\text{VA}$

Base voltage, V_{B2} = Rated secondary voltage = 400V

$$\text{Base current, } I_{B2} = \frac{5{,}000}{400} = 12{\cdot}5\text{A}$$

$$\text{Base impedance, } Z_{B2} = \frac{I_{B2}}{V_{B2}} = \frac{400}{12{\cdot}5} = 32\Omega$$

$$\text{Base admittance, } Y_{B2} = \frac{I_{B2}}{V_{B2}} = \frac{12{\cdot}5}{400} = 0{\cdot}03125\text{S}$$

$$\text{Shunt admittance, } Y_{pu} = \frac{Y_{2eq}}{Y_{B2}} = \frac{0{\cdot}375 \times 10^{-3} - (j0{\cdot}787 \times 10^{-3})}{0{\cdot}03125}$$

$$= \underline{\underline{(12 \times 10^{-3} - j25{\cdot}2 \times 10^{-3})\text{p.u.}}}$$

$$\text{Leakage impedance, } Z_{pu} = \frac{Z_{2eq}}{Z_{B2}} = \frac{0{\cdot}48 + j1{\cdot}28}{32}$$

$$= \underline{\underline{(0{\cdot}015 + j0{\cdot}04)\text{p.u.}}}$$

It will be noted that, where the rated voltage and volt-amperes are used as bases, the per-unit values of the constants of the transformer equivalent circuit are the same whether primary or secondary data are used.

In the transformer equivalent circuit the core loss is represented by power dissipated in G_c the real part of the shunt admittance. Since the transformer is excited at rated voltage, in per-unit values the voltage will be unity.

$$\text{Core loss p.u., } P_{i\,pu} = V_{pu}^2 G_{pu} = 1^2 \times 12 \times 10^{-3} = \underline{\underline{12 \times 10^{-3}\text{p.u.}}}$$

It will be noted that the reference part of the per-unit shunt admittance is numerically equal to the per-unit core loss at normal voltage.

Base power, $P_B = V_{B1}I_{B1} = V_{B2}I_{B2} = 5{,}000$
Actual core loss, $P_i = P_{ipu}P_B = 12 \times 10^{-3} \times 5{,}000 = \underline{\underline{60\,\text{W}}}$

The winding loss may be calculated by determining the power dissipated in the reference part of the leakage impedance. Since the winding loss on full load is required the per unit current will be unity.

Winding loss p.u. $P_{cpu} = I_{pu}{}^2R_{pu} = 1^2 \times 0{\cdot}015 = \underline{\underline{0{\cdot}015\,\text{p.u.}}}$

It will be noted that the reference part of the per-unit leakage impedance is numerically equal to the full-load winding loss.

Actual winding loss, $P_c = P_{cpu}P_B = 0{\cdot}015 \times 5{,}000 = 75\,\text{W}$

The actual voltage regulation is given by eqn. (9.63).

Voltage regulation p.u. $= I_{2pu}R_{pu} \cos \phi + I_{2pu}X_{pu} \sin \phi$

since the full-load voltage regulation is required $I_{2pu} = 1$.

Voltage regulation $= (1 \times 0{\cdot}015 \times 0{\cdot}8) + (0{\cdot}04 \times 0{\cdot}6) = 0{\cdot}036\,\text{p.u.}$
Actual voltage regulation $= 0{\cdot}036V_B = 0{\cdot}036 \times 400 = \underline{\underline{14{\cdot}4\,\text{V}}}$

Since the secondary voltage base is used this gives the actual change in voltage at the secondary terminals.

This last result may be checked by direct substitution into eqn (9.63), which gives

Voltage regulation $= (12{\cdot}5 \times 0{\cdot}48 \times 0{\cdot}8) + (12{\cdot}5 \times 1{\cdot}28 \times 0{\cdot}6) = \underline{\underline{14{\cdot}4\,\text{V}}}$

If calculations relating to two or more transformers, or other plant, of different ratings are to be undertaken, then the per-unit values must all be referred to the same voltage and volt-ampere bases. In such a situation the base values chosen will not be the rated value of volt-amperes of some of the plant involved.

Following eqn. (9.77), the per-unit value of any quantity A to A_{base1} is

$$A_{pu1} = \frac{A}{A_{base1}} \tag{9.77(a)}$$

Similarily the per-unit value of A to a second base value is

$$A_{pu2} = \frac{A}{A_{base2}} \tag{9.77(b)}$$

Combining eqns. (9.77(*a*)) and (9.77(*b*)) gives

$$A = A_{pu1}\,A_{base1} = A_{pu2}\,A_{base2}$$

i.e.

$$A_{pu2} = A_{pu1} \times \frac{A_{base1}}{A_{base2}} \tag{9.82}$$

Eqn. (9.82) in combination with eqns. (9.79)–(9.81) may be used to change the base to which any electrical quantity is referred.

For 3-phase plant the rated phase voltage and rated volt-amperes per phase would normally be chosen as base values, and after this has been done eqns. (9.90), (9.91) and (9.92) apply and the problem may be treated as a single-phase problem.

EXAMPLE 9.8 A 60 MVA 3phase 33/11 kV mesh/star-connected transformer supplies a 10 MVA feeder. The leakage impedance per phase of the transformer is $(0{\cdot}015 + j0{\cdot}04)$ p.u. and the impedance per phase of the feeder is $(0{\cdot}06 + j0{\cdot}07)$ p.u. The p.u. impedances are based on the nominal ratings per phase of the transformer and feeder respectively. When the load on the distant end of the feeder is 10 MVA at a power factor of 0·8 lagging and the load voltage is 11 kV determine:

(*a*) The line current in the feeder.
(*b*) The transformer secondary phase current.
(*c*) The transformer primary phase current.
(*d*) The transformer primary line current.
(*e*) The transformer output line voltage.
(*f*) The transformer input line voltage.

The per-unit impedances of the feeder and transformer are based on their respective nominal ratings. It will be necessary to express these relative to a common base, and the nominal rating of the transformer is chosen as the common base.

From eqn. (9.82), the per-unit impedance of the feeder on the new base is

$$Z_{pu\,2} = Z_{pu\,1}\frac{Z_{base\,1}}{Z_{base\,2}}$$

$$= Z_{pu\,1}\frac{S_{base\,2}}{S_{base\,1}}$$

since the impedance base is inversely proportional to the volt-ampere base. The feeder impedance referred to this base is

$$Z_F = (0{\cdot}06 + j0{\cdot}07)\frac{60}{10} = (0{\cdot}36 + j0{\cdot}42)\text{p.u.} = 0{\cdot}551\angle 49{\cdot}4^\circ\text{p.u.}$$

The base values for the feeder and the l.v. side of the transformer are

$$S_B = \frac{60 \times 10^6}{3} = 20 \times 10^6\,\text{VA}$$

$$V_B = \frac{11 \times 10^3}{\sqrt{3}} = 6{\cdot}35 \times 10^3\,\text{V}$$

$$I_B = \frac{20 \times 10^6}{6{\cdot}35 \times 10^3} = 3{\cdot}15 \times 10^3\,\text{A}$$

The base values for the h.v. side of the transformer are

$$S_B = \frac{60 \times 10^6}{3} = 20 \times 10^6\,\text{VA}$$

$$V_B = 33 \times 10^3\,\text{V}$$

$$I_B = \frac{20 \times 10^6}{33 \times 10^3} = 0{\cdot}605 \times 10^3\,\text{A}$$

The actual load on the feeder is 10/3 MVA per phase at a power factor of 0·8 lagging and a phase voltage of $6{\cdot}35 \times 10^3$ V.

$$\text{Load VA, } S_{pu} = \frac{\text{Actual VA/phase}}{\text{Base VA}} = \frac{10/3}{20} = 0{\cdot}167\,\text{p.u.}$$

$$\text{Load voltage, } V_{pu} = \frac{6{\cdot}35 \times 10^3}{6{\cdot}35 \times 10^3} = 1\,\text{p.u.}$$

$$\text{Load current, } I_{pu} = \frac{S_{pu}}{V_{pu}} = 0{\cdot}167\,\text{p.u.}$$

(*a*) Feeder current $= I_{pu} \times I_{base} = 0{\cdot}167 \times 3{\cdot}15 \times 10^3 = \underline{\underline{525\,\text{A}}}$

(*b*) Transformer secondary phase current $= \underline{\underline{525\,\text{A}}}$

(*c*) Transformer primary phase current $= 0{\cdot}167 \times 0{\cdot}605 \times 10^3 = \underline{\underline{101\,\text{A}}}$

(*d*) Transformer primary line current $= \sqrt{3} \times 101 = \underline{\underline{175\,\text{A}}}$

The transformer secondary voltage is

$$V_2 = V + IZ_F$$

where V = load voltage and I = feeder current.

$$\begin{aligned} V_{pu\,2} &= 1\underline{/0^\circ} + (0{\cdot}167\underline{/-36{\cdot}9^\circ} \times 0{\cdot}551\underline{/49{\cdot}4^\circ}) \\ &= (1{\cdot}09 + j0{\cdot}0199) = 1{\cdot}09\underline{/1{\cdot}04^\circ}\,\text{p.u.} \end{aligned}$$

Transformer secondary phase voltage $= 1{\cdot}09 \times 6{\cdot}35 \times 10^3 = 6{,}920\,\text{V}$

(*e*) Transformer secondary line voltage $= \sqrt{3} \times 6{,}920 = 12{,}000\,\text{V}$

$$\begin{aligned} \text{Total series impedance, } Z_{pu\,T} &= (0{\cdot}015 + j0{\cdot}04) + (0{\cdot}36 + j0{\cdot}42) \\ &= 0{\cdot}375 + j0{\cdot}46 = 0{\cdot}594\underline{/50{\cdot}8^\circ}\,\text{p.u.} \end{aligned}$$

The transformer input line voltage is

$$\begin{aligned} V_{pu\,1} &= V_{pu} + I_{pu}Z_{pu\,T} \\ &= 1\underline{/0^\circ} + (0{\cdot}167\underline{/-36{\cdot}9^\circ} \times 0{\cdot}594\underline{/50{\cdot}8^\circ} \\ &= 1{\cdot}0963 + j0{\cdot}0238 = 1{\cdot}096\underline{/1{\cdot}24^\circ}\,\text{p.u.} \end{aligned}$$

(*f*) Transformer primary line voltage $= 1{\cdot}096 \times 33 \times 10^3 = \underline{\underline{36{,}200\,\text{V}}}$

9.17 Three-phase Transformers

For 3-phase working it is possible to have either a bank of three single-phase transformers, or a single 3-phase unit (Fig. 9.12). Single-phase construction has the advantage that where single units are concerned only one spare single-phase transformer is needed as a standby, instead of a complete spare 3-phase transformer.

The single 3-phase unit takes up less space and is somewhat cheaper. Technically the difference between the single 3-phase unit and the three single-phase units lies in the fact that there is a direct magnetic coupling between the phases in the first case but not in the second. Star, delta or zigzag windings are possible in

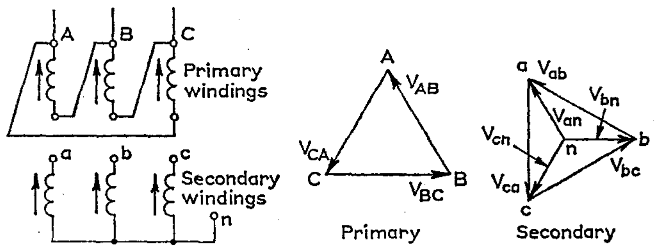


Fig. 9.23 DELTA-STAR 3-PHASE TRANSFORMER

both primary and secondary, giving many possible pairs of connexion. The complexor diagram for any connexion is drawn by observing that the e.m.f.s induced in all windings on the same limb are in phase and in direct ratio to the numbers of turns.

Two cases will be considered in detail, the others being easily followed by similar methods. In Fig. 9.23 the connexions and

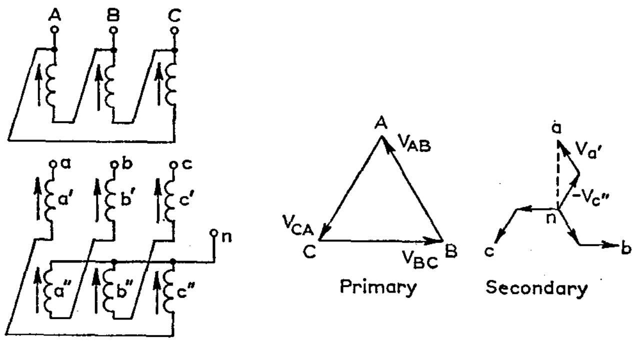


Fig. 9.24 DELTA-ZIGZAG 3-PHASE TRANSFORMER

complexor diagrams for a Δ-Y transformer are shown. The primary complexor diagram is shown on the left, with primary line voltages V_{AB}, V_{BC} and V_{CA}. The secondary is Y-connected, so that the phase voltages V_{an} will be in phase with the primary line voltage V_{AB}, and the ratio V_{an}/V_{AB} will be almost equal to the turns ratio N_2/N_1, etc. The secondary line voltages V_{ab}, etc., will be the complexor differences between successive phase voltages, as shown in the right-hand complexor diagram. In this case the secondary line voltages will lead the primary line voltages by 30°, and will be $\sqrt{3}V_p(N_2/N_1)$ in magnitude, where V_p is the primary line voltage.

In Fig. 9.24 are shown the circuit and complexor diagrams of a Δ-zigzag connexion. The secondary is divided into two equal halves on each limb, the top half on one limb being connected in opposition to the lower half on the preceding limb. This connexion is used if the load on the secondary is far out of balance, since each secondary phase is divided between two primary windings. Each secondary phase voltage is thus the difference between the e.m.f.s induced in windings on successive limbs. Thus

$$V_{an} = V_{a'} - V_{c''}$$

where $V_{a'}$ is in phase with V_{AB}, and $V_{c''}$ is in phase with Z_{CA}. The secondary line voltages in this case are in phase with the primary line voltages. If the magnitude of these line voltages is V_l, then

$$V_{a'} = \frac{V_l}{2}\frac{N_2}{N_1}$$

where N_2 is the total number of secondary turns on each limb, and

$$V_{c''} = \frac{V_l}{2}\frac{N_2}{N_1}$$

Hence, from the complexor diagram,

$$V_{an} = \frac{\sqrt{3}}{2}V_l\frac{N_2}{N_1}$$

and the secondary line voltages are

$$\frac{3}{2}V_l\frac{N_2}{N_1}$$

9.18 Star-star Connexion

It is possible to use a 3-phase transformer with both the primary and the secondary connected in star as follows.

THREE SINGLE-PHASE UNITS (Fig. 9.25)

If the primaries of a bank of three single-phase transformers are star connected to a 3-phase, 4-wire system, then a constant voltage is applied to each primary. The three transformers act independently of one another, the load on each secondary phase being reflected into the corresponding primary phase. This is a perfectly practical connexion, but has the disadvantage that there is no tendency for the primary currents to be balanced when the load on the secondary is unbalanced.

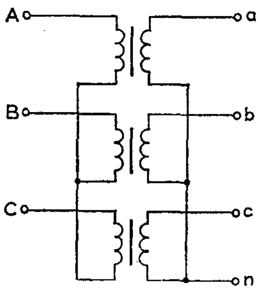


Fig. 9.25 STAR-STAR CONNEXION OF THREE SINGLE-PHASE TRANSFORMERS

If the primaries are connected to a 3-wire system, the primary line voltages are constant, but the primary phase voltages may be unbalanced, since the star-point potential is not fixed. The most extrême case is when one secondary phase only is loaded. The loaded transformer will have a low input impedance while the two unloaded transformers will act as high-impedance chokes. The voltage across the loaded phase will fall to a low value, while the voltages across the other two phases will rise almost to the line voltage value. This connexion is therefore unsuitable if there is any possibility of unbalanced loads.

FIVE-LIMB CORE-TYPE THREE-PHASE TRANSFORMER (Fig. 9.26)

If the primary windings are star connected to a 4-wire system, the primary phase voltages are constant, so that the primary phase e.m.f.s must also be approximately constant. Hence the flux in each core must be approximately constant, and the primary ampere-turns will balance the secondary ampere-turns on each limb. Thus the output will remain approximately constant, and the primary current in any line will be a reflection of the secondary current in the same line.

If there is no primary neutral connexion, the primary line voltages (but not necessarily the primary phase voltages) must remain constant. Hence between line terminals the primary ampere-turns must balance the secondary ampere-turns, but there need not necessarily be ampere-turn balance on each core. Consider, for simplicity, the 1:1 turns-ratio 3-wire star-star transformer which has only one secondary phase loaded, as shown in Fig. 9.26. Let the primary phase currents be I_A, I_B, and I_C, where I_B is the primary current on the loaded limb. By Kirchhoff's first law,

$$I_A + I_B + I_C = 0 \tag{9.83}$$

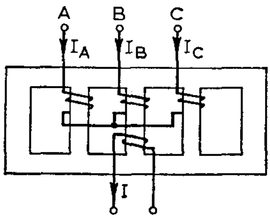


Fig. 9.26 STAR-STAR CONNEXION OF 3-PHASE TRANSFORMER

Suppose that there are N turns on each phase winding; then between terminals A and B:

$$\begin{aligned} \text{Primary ampere-turns} &= I_A N - I_B N \\ &= \text{Secondary ampere-turns} = IN \end{aligned} \tag{9.84}$$

Also between terminals A and C,

$$\begin{aligned} \text{Primary ampere-turns} &= I_A N - I_C N \\ &= \text{Secondary ampere-turns} = 0 \end{aligned}$$

Therefore $I_A = I_C$, and

$$I_B = -(I_A + I_C) = -2I_A \tag{9.85}$$

Hence, from eqn. (9.84),

$$I_A + 2I_A = I$$

Thus

$$I_A = I_C I/3, \text{ and } I_B = -2I/3.$$

Examination of the cores will show that on each there are resultant unbalanced ampere-turns of $IN/3$. Further, all these unbalanced ampere-turns are in phase and set up a flux through each of the wound cores in parallel with the path completed through the unwound cores. This flux linking the phase windings will unbalance the phase e.m.f.s with the result that the voltage across the loaded phase tends to fall while the voltages across the two other phases tend to rise. The result is similar to that with three single-phase transformers.

THREE-LIMB CORE-TYPE THREE-PHASE TRANSFORMER

Basically the 3-limb transformer will behave in the same way as the 5-limb transformer. When a 3-limb transformer is supplied from a 3-wire system and feeds an unbalanced load, the phase voltages will remain approximately balanced since there is no low-reluctance path through which the unbalanced ampere-turns on each core can set up a flux. The actual flux path is completed through the air or through the steel tank of enclosed transformers, in which case undesirable heating may occur.

Star-star connexion is not generally satisfactory and should not be used with unbalanced loads, unless an additional delta-connected winding is provided. This *tertiary winding* does not usually feed any load, but if the secondary load is unbalanced, the out-of-balance flux will give rise to a circulating current in the closed tertiary winding, whose ampere-turns will then cancel out the unbalanced ampere-turns due to the load. The phase voltages will then tend to remain balanced.

9.19 Other Three-phase Transformer Connexions

The following brief notes illustrate the applications and limitations of some of the other possible connexions for three-phase transformers.

DELTA-DELTA

This is useful for high-current low-voltage transformers, and can supply large unbalanced loads without disturbing the magnetic equilibrium. There is, however, no available star point.

DELTA-STAR

This arrangement is used to supply large powers at high voltage, and for distribution at low voltages. A large unbalanced load does not

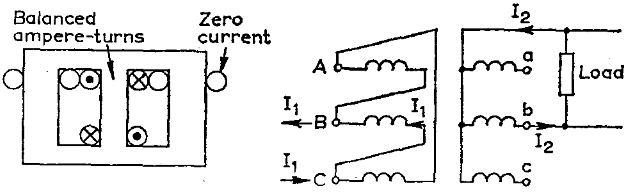


Fig. 9.27 DELTA-STAR CONNEXION WITH UNBALANCED LOAD

disturb the magnitude equilibrium since the primary current will flow in the corresponding winding only, and the primary and secondary ampere-turns will be balanced in each limb (Fig. 9.27).

STAR-DELTA

Used for substation transformers supplied from the grid but not for distribution purposes owing to the absence of a neutral.

DELTA-ZIGZAG

Used for supplying smaller powers with large out-of-balance neutral currents. The zigzag winding establishes magnetic equilibrium.

9.20 Scott Connexion

The Scott connexion is a method of connecting two single-phase transformers to give a 3-phase to 2-phase conversion. The method

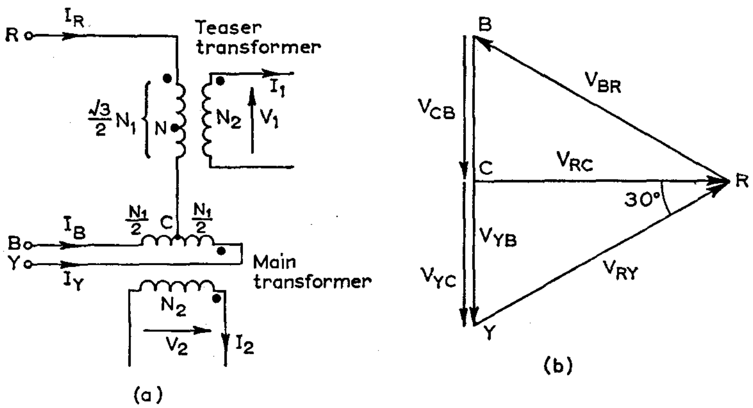


Fig. 9.28 SCOTT CONNEXION

is frequently used to obtain, from a 3-phase supply, a balanced 2-phase supply for a.c. control systems.

Referring to Fig. 9.28(*a*), the centre-tapped primary of the *main transformer* has the line voltage V_{YB} applied to its terminals. The secondary terminal voltage of the main transformer is

$$V_2 = \frac{N_2}{N_1} V_{YB} = \frac{N_2}{N_1} V_l$$

Fig. 9.28(*b*) is the relevant complexor diagram. The line voltages of the 3-phase system, V_{RY}, V_{YB} and V_{BR}, which are balanced, are shown on the complexor diagram as a closed equilateral triangle. The voltages across the two halves of the centre-tapped primary of the main transformer, V_{CB} and V_{YC}, are equal and in phase with

V_{YB}. Evidently V_{RC} leads V_{YB} by 90°. This voltage is applied to the primary of the *teaser transformer*, and so the secondary voltage of the teaser transformer, V_1, will lead the secondary terminal voltage of the main transformer by 90°. However,

$$V_{RC} = V_{RY} \cos 30° = \frac{\sqrt{3}}{2} V_L$$

To make V_1 equal in magnitude to V_2 the primary of the teaser transformer requires to have $\frac{\sqrt{3}}{2} N_1$ turns. Then

$$V_1 = \frac{N_2}{\frac{\sqrt{3}}{2} N_1} V_{RC} = \frac{N_2}{\frac{\sqrt{3}}{2} N_1} \frac{\sqrt{3}\, V_L}{2} = \frac{N_2}{N_1} V_L = V_2$$

The voltages V_1 and V_2 thus constitute what is usually regarded as a balanced 2-phase system comprising two voltages of equal magnitude having a phase difference of 90°.

The primaries of the two transformers may have a 4-wire connexion to the 3-phase supply if a tapping point N is provided on the primary of the teaser transformer such that $V_{RN} = V_l/\sqrt{3}$.

If n is the number of turns in the section RN of the primary winding of the teaser transformer,

$$\frac{n}{\frac{\sqrt{3}}{2} N_1} = \frac{V_l/\sqrt{3}}{\frac{\sqrt{3}}{2} V_l}$$

Therefore

$$n = \frac{2}{3} \frac{\sqrt{3}}{2} N_1 = 0{\cdot}577\, N_1$$

$$\left.\begin{array}{r}\text{Number of turns in section NC}\\ \text{of teaser transformer primary}\end{array}\right\} \begin{array}{l} = 0{\cdot}866\, N_1 - 0{\cdot}577\, N_1 \\ = 0{\cdot}299\, N_1 \end{array}$$

Frequently identical interchangeable transformers are used for the Scott connexion, in which case each transformer has a primary winding of N_1 turns and is provided with tapping points at $0{\cdot}299\, N_1$, $0{\cdot}5\, N_1$ and $0{\cdot}866\, N_1$.

If the 2-phase currents are balanced, the 3-phase currents are also balanced (neglecting magnetizing currents). This may be shown as follows.

Let I_1 be the reference complexor so that

$$I_1 = I\underline{/0°} \qquad \text{and} \qquad I_2 = I\underline{/-90°} = -jI$$

For m.m.f. balance of the teaser transformer,

$$I_R \frac{\sqrt{3}}{2} N_1 = I N_2 \underline{/0^\circ}$$

so that

$$I_R = \frac{2}{\sqrt{3}} \frac{N_2}{N_1} I \underline{/0^\circ} \tag{9.86}$$

For m.m.f. balance for the main transformer,

$$I_Y \frac{N_1}{2} - I_B \frac{N_1}{2} = -jIN_2$$

or

$$I_Y - I_B = -j \frac{2N_2}{N_1} I \tag{9.87}$$

For 3-wire connexion,

$$I_R + I_Y + I_B = 0 \tag{9.88}$$

Substituting for I_R from eqn. (9.86) in eqn. (9.88),

$$I_B + I_Y = \frac{-2}{\sqrt{3}} \frac{N_2}{N_1} I \tag{9.89}$$

Adding eqns. (9.87) and (9.89),

$$2I_Y = \frac{-2}{\sqrt{3}} \frac{N_2}{N_1} I - j2 \frac{N_2}{N_1} I$$

or

$$I_Y = \frac{2}{\sqrt{3}} \frac{N_2}{N_1} I \underline{/-120^\circ} \tag{9.90}$$

Similarly,

$$I_B = \frac{2}{\sqrt{3}} \frac{N_2}{N_1} I \underline{/+120^\circ} \tag{9.91}$$

That is, the 3-phase currents are balanced if the 2-phase currents are balanced, neglecting the effect of magnetizing current. Since the 3-wire connexion gives balanced currents there will be no neutral current if a neutral wire is connected.

9.21 Transformer Types

POWER TRANSFORMERS

These have a high *utilization factor*, i.e. it is arranged that they run with an almost constant load which is equal to their rating.

The maximum efficiency is designed to be at full load. This means that the full-load winding losses must be equal to the core losses.

DISTRIBUTION TRANSFORMERS

These have an intermittent and variable load which is usually considerably less than the full-load rating. They are therefore designed to have their maximum efficiency at between $\frac{1}{2}$ and $\frac{3}{4}$ of full load.

AUTO-TRANSFORMERS

Consider a single winding AC, on a magnetic core as shown in Fig. 9.29. If this winding is tapped at a point B, and a load is connected between B and C, then a current will flow, under the influence of the e.m.f. E_2, between the two points. The current I_2 will produce an m.m.f. in the core which will be balanced by a current I_1 flowing in the complete winding. This is called the auto-transformer action, and has the advantage that it effects a saving in winding material (copper or aluminium), since the secondary winding is now merged into the primary. The disadvantages of the auto-connexion are:

1. There is a direct connexion between the primary and secondary.
2. Should an open-circuit develop between points B and C, the full mains voltage would be applied to the secondary.
3. The short-circuit current is much larger than for the normal two-winding transformer.

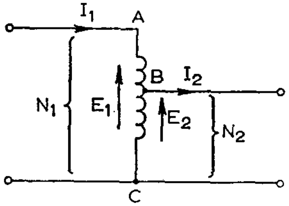


Fig. 9.29 AUTO-TRANSFORMER (STEP DOWN)

It can be seen from Fig. 9.29 that a short-circuited secondary causes part of the primary also to be short-circuited, reducing the effective resistance and reactance.

Applications are—(*a*) Boosting or bucking of a supply voltage by a small amount. (The smaller the difference between the output and input voltages the greater is the saving of winding material.) (*b*) Starting of a.c. machines, where the voltage is raised in two or more steps from a small value to the full supply voltage. (*c*) Continuously variable a.c. supply voltages. (*d*) Production of very high voltages. Auto-transformers are used in the 275 kV and 400 kV grid systems.

The connexion for increasing (boosting) the output voltage by auto-connexion is shown in Fig. 9.30. The operation will be better

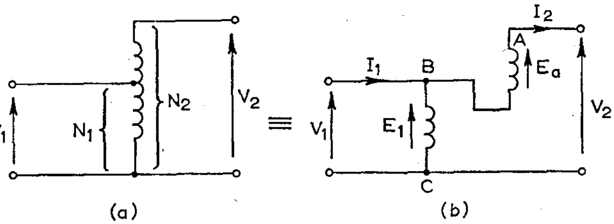


Fig. 9.30 AUTO-TRANSFORMER FOR BOOSTING THE OUTPUT VOLTAGE

understood by considering Fig. 9.30(*b*). Neglecting losses, the output voltage E_2 will be the sum of the input voltage E_1 and the e.m.f. E_a induced in the additional winding.

Continuously variable auto-transformers are constructed by arranging for one of the output terminals to be connected to a sliding contact which moves over the whole range of a single-layer winding. The transformer is usually overwound so that voltages in excess of the supply voltage may be obtained.

INSTRUMENT TRANSFORMERS

Current (series) and voltage (shunt) transformers are used for extending the range of a.c. instruments in preference to shunts and series resistors for the following reasons: (i) to eliminate errors due to stray inductance and capacitance in shunts, multipliers and their leads; (ii) the measuring circuit is isolated from the mains by the transformer and may be earthed; (iii) the length of connecting leads from the transformer to the instrument is of lesser importance, and the leads may be of small cross-sectional area; (iv) the instrument ranges may be standardized (usually 1 A or 5 A for ammeters and 110 V for voltmeters); (v) by using a clip-on type of transformer core the current in a heavy-current conductor can be measured without breaking the circuit.

The current transformer has the secondary effectively short-circuited through the low impedance of the ammeter (Fig. 9.31(*a*)).

The voltage across the primary terminals will thus be very small, so that there will only be a very small flux in the core (since $E \propto f\Phi N$). This means that both the magnetizing and the core-loss components of the primary current will be small. Also, the exact value of the secondary load (called the *burden*) will have a negligible effect on the primary current which is to be measured. The current transformation ratio I_p/I_s will not be quite equal to N_s/N_p, and will depend on the ratio of magnetizing current to ammeter current.

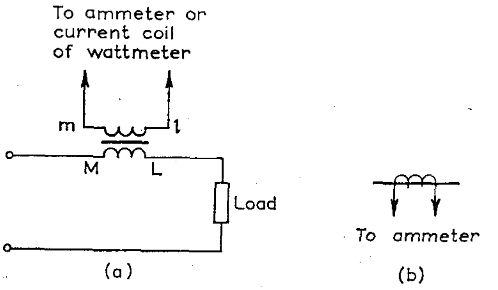


Fig. 9.31 CURRENT TRANSFORMERS

(*a*) With wound primary (for currents up to about 1,000 A)
(*b*) With primary consisting of either a single bar on the line itself (for larger currents)

Normally the correction is small. The presence of the magnetizing and loss components of current introduces a phase angle error, due to the fact that the secondary current is not exactly in phase with the primary current. This is of importance only when the transformer is to be used in conjunction with a wattmeter.

The current transformer must never be operated on open-circuit, for two reasons. Firstly, there will be no secondary demagnetizing ampere-turns, and since the primary current is fixed, the core flux will increase enormously. This will cause large eddy-current and hysteresis losses, and the resulting temperature rise may damage the insulation. Even in the absence of evident damage the core may be left with a high value of remanent magnetism which can lead to a large undetected error in subsequent use. Secondly, a very high voltage will be induced in the multi-turn secondary, being dangerous both to life and to the insulation.

Voltage transformers operate with their primaries at the full supply voltage and their secondaries connected to the high impedance of a voltmeter or the voltage coil of a wattmeter (giving a secondary phase angle of almost unity). The secondary current will be very small (of the same order as the magnetizing current) so that transformer may be regarded in the same way as a power transformer on no load. The voltage ratio is effectively the turns ratio, and the phase angle error (due to the fact that the secondary voltage will not be quite in phase with the primary voltage) will generally be negligible.

9.22 Short Transmission Lines in Parallel

Preliminary to the consideration of the operation of two transformers in parallel the simpler case of two transmission lines in parallel will be first considered.

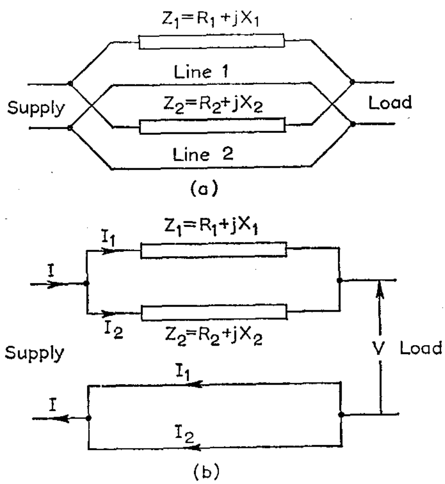


Fig. 9.32 TRANSMISSION LINES IN PARALLEL

Fig. 9.32(*a*) shows two single-phase short transmission lines connected in parallel. The total series impedance of each pair has been equivalently concentrated in one conductor of each. Fig. 9.32(*b*) is a simple redraft of the system. It is evident that the voltage drops in the two transmission lines are identical. If the total current $\boldsymbol{I}$ divides between the two lines so that $\boldsymbol{I}_1$ flows through line 1 (of impedance $\boldsymbol{Z}_1$) and $\boldsymbol{I}_2$ flows through line 2 (of impedance $\boldsymbol{Z}_2$), then

$$\boldsymbol{I}_1\boldsymbol{Z}_1 = \boldsymbol{I}_2\boldsymbol{Z}_2 \tag{9.92}$$

Also

$$\boldsymbol{I} = \boldsymbol{I}_1 + \boldsymbol{I}_2 \tag{9.93}$$

Therefore

$$I_1 = \frac{Z_2}{Z_1 + Z_2} I \tag{9.94}$$

Similarly,

$$I_2 = \frac{Z_1}{Z_1 + Z_2} I \tag{9.95}$$

This result will also apply to two balanced 3-phase systems operating in parallel if Z_1 and Z_2 are the equivalent impedances per phase of the 3-phase transmission lines.

In many cases the division of the load between the two lines is required. This may be determined as follows.

Let V be the receiving-end voltage. Multiplying eqns. (9.94) and (9.95) by V for a single-phase system or by $\sqrt{3}\,V$ for a 3-phase system,

$$S_1 = \frac{Z_2}{Z_1 + Z_2} S_T \tag{9.96}$$

and

$$S_2 = \frac{Z_1}{Z_1 + Z_2} S_T \tag{9.97}$$

where the total volt-amperes (S_T) and the volt-amperes delivered by each line (S_1 and S_2) are in complexor form, with the system voltage V as reference.

Eqns. (9.96) and (9.97) may be expressed using impedance in per-unit form.

$$S_1 = \frac{Z_{2pu}}{Z_{1pu} + Z_{2pu}} S_T \tag{9.98}$$

$$S_2 = \frac{Z_{1pu}}{Z_{1pu} + Z_{2pu}} S_T \tag{9.99}$$

If the lines have different volt-ampere ratings the p.u. impedances for both lines will have to be based on the volt-ampere rating of one line. If the impedances of the lines are given in per-unit form, each based on the individual volt-ampere rating of the line, the per-unit impedance of one line may be converted to a new base in accordance with eqn. (9.82).

EXAMPLE 9.9 A 3-phase cable A supplies a load of 2,000 kW at 6,600 V and p.f. 0·8 lagging. A second cable B of impedance $(3 + j4{\cdot}5)\,\Omega$/phase is connected in parallel with A, and it is found that for the same load as before, A carries 140 A and delivers 1,200 kW at a lagging p.f. What is the impedance of cable A?

$$\text{Total load, } \boldsymbol{S_T} = \frac{2{,}000}{0\cdot8}\underline{/-\cos^{-1}0\cdot8} = 2{,}500\underline{/-36\cdot9^\circ}\text{kVA}$$

Power delivered by A when in parallel with B = 1,200 kW
kVA delivered by A when in parallel with B

$$= \frac{\sqrt{3}\times6{,}600\times140}{1{,}000}\underline{\Big/-\cos^{-1}\frac{1{,}200}{1{,}600}} = 1{,}600\underline{/-41\cdot4^\circ}$$

By eqn. (9.96),

$$\frac{\boldsymbol{Z_B}}{\boldsymbol{Z_A}+\boldsymbol{Z_B}} = \frac{\boldsymbol{S_A}}{\boldsymbol{S_T}} = \frac{1{,}600\underline{/-41\cdot4^\circ}}{2{,}500\underline{/-36\cdot9^\circ}} = 0\cdot64\underline{/-4\cdot5^\circ}$$

and

$$\frac{\boldsymbol{Z_A}+\boldsymbol{Z_B}}{\boldsymbol{Z_B}} = 1\cdot56\underline{/4\cdot5^\circ} = 1\cdot56 + j0\cdot123$$

Therefore

$$\frac{\boldsymbol{Z_A}}{\boldsymbol{Z_B}} = 0\cdot56 + j0\cdot123 = 0\cdot574\underline{/12\cdot4^\circ}$$

and

$$\boldsymbol{Z_A} = \boldsymbol{Z_B}\times0\cdot574\underline{/12\cdot4^\circ} = 5\cdot41\underline{/56\cdot3^\circ}\times0\cdot574\underline{/12\cdot4^\circ} = \underline{\underline{3\cdot11\underline{/68\cdot7^\circ}\,\Omega}}$$

9.23 Single-phase Equal-ratio Transformers in Parallel

The correct method of connecting two single-phase transformers in parallel is shown in Fig. 9.33(*a*). The wrong method is shown at (*b*). At (*a*) it will be seen that round the loop formed by the secondaries, $\boldsymbol{E_1}$ and $\boldsymbol{E_2}$ oppose and there will be no circulating current, while at (*b*) it will be seen that round the loop formed by the two secondaries, $\boldsymbol{E_1}$ and $\boldsymbol{E_2}$ are additive, and will give rise to a short-circuit current.

Fig. 9.33(*c*) shows the two transformer equivalent circuits with the leakage impedances referred to the secondary sides. The two ideal transformers must now have identical secondary e.m.f.s since they have the same turns ratio and have their primaries connected to the same supply. The potentials at A and C and at B and D must then be identical so that these pairs of points may be joined without affecting the circuit. The imaginary joining of these points is shown in Fig. 9.33(*d*). From this diagram it is clear that two equal-ratio transformers connected in parallel will share the total load in the same way as two short transmission lines; all the previous equations are therefore applicable.

It is noteworthy that the per-unit impedance of a transformer is the same whether the actual impedance is referred to the primary or secondary.

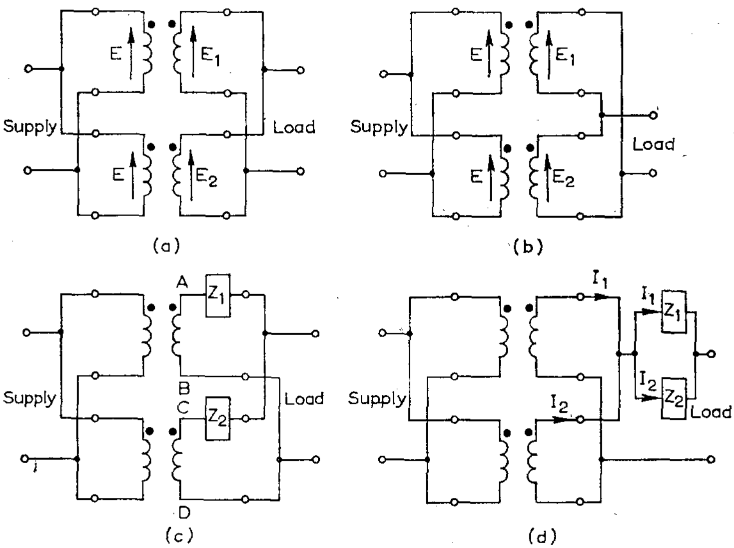


Fig. 9.33 EQUAL-RATIO TRANSFORMERS IN PARALLEL

From Fig. 9.33(*d*) it is evident that

$$I_1 Z_1 = I_2 Z_2 \tag{9.100}$$

and

$$\frac{I_1}{I_2} = \frac{Z_2}{Z_1}$$

The transformer currents (and hence the volt-ampere loads) are in the inverse ratio of the transformer impedances. This is important for parallel operation for it is usually desirable that both transformers be fully loaded simultaneously. If the full-load volt-ampere ratings are S_{fl1} and S_{fl2}, then this condition will be fulfilled if

$$\frac{S_{fl1}}{S_{fl2}} = \frac{Z_2}{Z_1}$$

Thus it is desirable that two transformers for parallel operation should have their rated full-load volt-amperes in the inverse ratio to their impedances. It is also desirable that the two transformers should operate at the same power factor to give the largest resultant volt-ampere rating. To achieve this the two impedances should have the same phase angle.

9.24 Single-phase Transformers with Unequal Ratios in Parallel

If the single-phase transformers connected as in Fig. 9.33(*a*) have unequal ratios, then E_1 and E_2 will be unequal and there will be a circulating current given by

$$I_c = \frac{E_1 - E_2}{Z_1 + Z_2} \tag{9.101}$$

in the secondary loop.

Since Z_1 and Z_2 will be small, the difference $E_1 - E_2$ must be small or the circulating current will be very large. It is possible

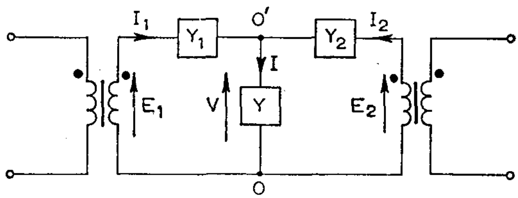


Fig. 9.34 UNEQUAL-RATIO TRANSFORMERS IN PARALLEL

to operate the transformers in parallel satisfactorily only if E_1 and E_2 are very nearly equal. If E_1 and E_2 are the secondary e.m.f.s when the impedances are referred to the secondary, they will be in phase with each other, since they are the e.m.f.s of ideal transformers connected to the same supply.

When the e.m.f.s are unequal they have an important bearing on the load sharing between the two transformers. Fig. 9.34 shows the usual equivalent circuit for the analysis in this case. E_1 and E_2 are in phase; Y_1 and Y_2 are the equivalent secondary admittances and Y is the load admittance. Then by Millman's theorem the voltage $V_{0'0}$ across the load is

$$V_{0'0} = \frac{\Sigma V_{k0} Y_k}{\Sigma Y_k} = \frac{E_1 Y_1 + E_2 Y_2}{Y_1 + Y_2 + Y} = V \tag{9.102}$$

Hence the load current is

$$I = VY = \frac{E_1 Y_1 Y + E_2 Y_2 Y}{Y_1 + Y_2 + Y} = \frac{E_1 Z_2 + E_2 Z_1}{ZZ_1 + Z_1 Z_2 + ZZ_2} \tag{9.103}$$

where $Z = 1/Y$, etc.

Also the current through transformer 1 is

$$I_1 = (E_1 - V)Y_1 = \frac{(E_1 - E_2)Y_2Y_1 + E_1YY_1}{Y_1 + Y_2 + Y} \tag{9.104}$$

and that through transformer 2 is

$$I_2 = (E_2 - V)Y_2 = \frac{(E_2 - E_1)Y_2Y_1 + E_2YY_2}{Y_1 + Y_2 + Y} \tag{9.105}$$

9.25 Three-phase Transformers in Parallel

In order that 3-phase transformers may operate in parallel the following conditions must be strictly observed:

(*a*) The secondaries must have the same phase sequence.
(*b*) All corresponding secondary line voltages must be in phase.
(*c*) The secondaries must give the same magnitude of line voltage.

In addition it is desirable that

(*d*) The impedances of each transformer, referred to its own rating, should be the same, i.e. each transformer should have the same per-unit resistance and per-unit reactance.

If conditions (*a*), (*b*) and (*c*) are not complied with, the secondaries will simply short-circuit one another and no output will be possible. If condition (*d*) is not complied with the transformers will not share the total load in proportion to their ratings, and one transformer will become overloaded before the total output reaches the sum of the individual ratings. It is difficult to ensure that transformers in parallel have identical per-unit impedances, and this affects the load sharing in the same manner as was indicated for single-phase transformers in parallel.

It is relatively simple to ensure that the phase sequence of all transformer secondaries is the same before connecting them in parallel. In transformers constructed in accordance with B.S. 171 the terminals of the h.v. and l.v. sides are labelled for the conventional positive phase sequence. It is then only necessary to ensure that correspondingly lettered terminals are connected together.

The main difficulty arising from the parallel connexion of 3-phase transformers is to ensure that condition (*b*) is satisfied. This is because of the phase shift which is possible between primary and secondary line voltages in such transformers.

Three-phase transformers are divided into four groups according to the phase displacement between the primary and secondary line voltages. These groups are

1. No phase displacement
2. 180° phase displacement
3. −30° phase displacement
4. +30° phase displacement

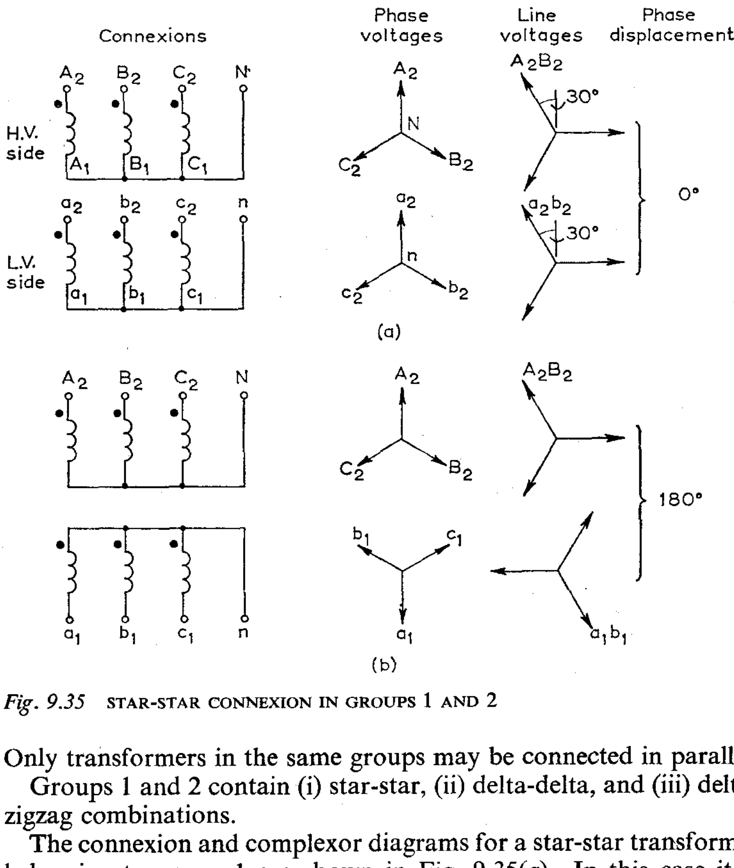


Fig. 9.35 STAR-STAR CONNEXION IN GROUPS 1 AND 2

Only transformers in the same groups may be connected in parallel.

Groups 1 and 2 contain (i) star-star, (ii) delta-delta, and (iii) delta-zigzag combinations.

The connexion and complexor diagrams for a star-star transformer belonging to group 1 are shown in Fig. 9.35(*a*). In this case it is immediately obvious that there is no phase displacement between the primary and secondary phase and line voltages. The essential point to observe is that all windings on the same limb of a transformer must give voltages which are either in phase or in antiphase, according to the relative winding directions. The line voltages are derived from the three phase voltages in the usual manner.

The effect of reversing the connexions to the l.v. winding is shown in Fig. 9.35(*b*). The directions of the phase e.m.f.s in the secondary are reversed, so that there is now 180° phase displacement between the primary and secondary line voltages. This connexion therefore belongs to group 2.

Fig. 9.36 shows diagrams for the delta-delta connexion in group 1, the complexor diagrams indicating that there is no phase shift

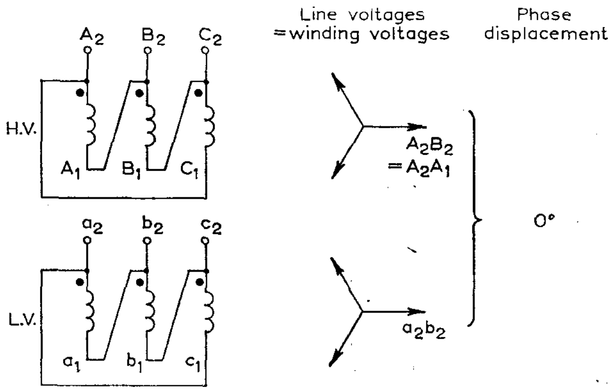


Fig. 9.36 DELTA-DELTA CONNEXION IN GROUP 1

between primary and secondary line voltages. If the connexions to either the primary or the secondary windings are reversed, there will be a 180° phase shift. The delta-zigzag connexion can similarly be shown to belong to either group 1 or group 2.

The connexions in groups 3 and 4, giving $-30°$ and $+30°$ phase displacements respectively, are: (i) delta-star, (ii) star-delta and (iii) star-zigzag. Fig. 9.37(*a*) shows the delta-star connexion which will give $-30°$ phase shift. It should be noted that in this case the line voltage between terminals A_2 and B_2 is actually in anti-phase to the voltage across the h.v. B-winding, so that the voltage induced in the *b*-phase of the l.v. winding will also be in antiphase to the line voltage A_2B_2. If the connexions to the h.v. side are reversed, as shown in Fig. 9.37(*b*), then the phase shift produced will be $+30°$. Similar connexion and complexor diagrams may be constructed for the star-delta and star-zigzag connexions.

The load-sharing properties of two 3-phase transformers with equal voltage ratios are governed by the same equations (9.98 and 9.99), as single-phase transformers, when the impedances are expressed as per-unit impedances, i.e. irrespective of the methods of connexion used for the transformers.

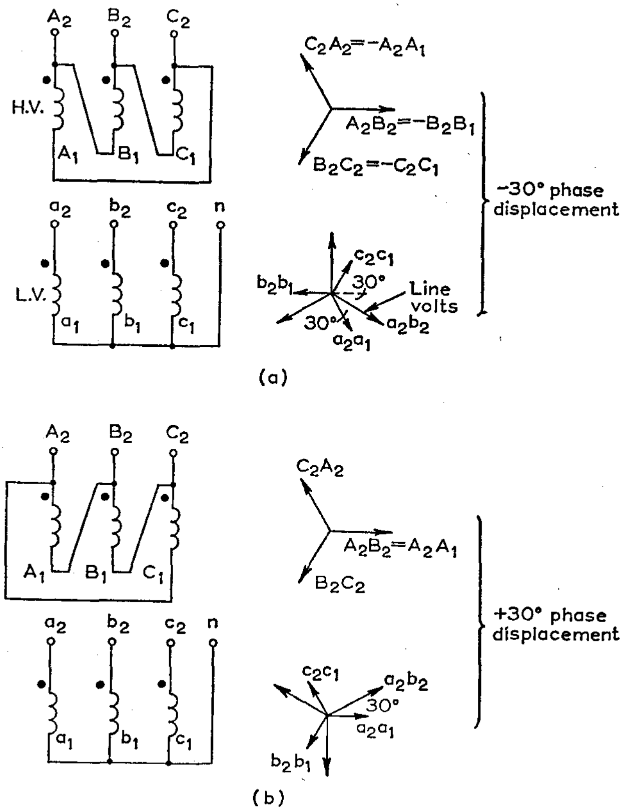


Fig. 9.37 DELTA-STAR CONNEXION IN GROUPS 3 AND 4

9.26 Transformers for High Frequencies

At radio frequencies and above, steel laminations cannot be used for transformers and coils because of the excessive eddy-current and hysteresis losses. One solution to this problem is to form cores of iron dust in an insulating binder. Such dust cores can be used up to radio frequencies, and have an effective relative permeability of about 10.

An alternative solution has become available with the development of homogeneous non-metallic materials called *ferrites*. These have the form $XO.Fe_2O_3$, where X stands for a divalent metallic atom. Ferrites crystallize in a cubic spinel structure, and are characterized by a high initial relative permeability (between 10 and 3,000) and a very high resistivity (typically $10^6\,\Omega$-m compared to about $10^{-7}\,\Omega$-m for iron). Owing to the high resistivity, eddy-current losses are virtually non-existent, so that ferrites can be used up to frequencies in excess of 10^9 Hz. They are not suitable, however, for power-frequency applications owing to their relatively high cost and fairly low saturation flux density (about 0·2 T). Their mechanical properties are similar to those of insulating ceramics—they are hard and brittle and not amenable to mechanical working. It is interesting to note that a naturally occurring ferrite known as lodestone or magnetite was the first material in which magnetic effects were observed.

The magnetic properties of ferrites depend on the metallic atom that occupies the position X in the ferrite formula. In magnetite this happens to be a divalent iron atom, so that magnetite is a double oxide of iron ($FeO.Fe_2O_3$). The manufactured ferrites are generally mixed crystals of two or more single ferrites.

Manganese zinc ferrite ($MnO.Fe_2O_3$, $ZnO.Fe_2O_3$) and nickel zinc ferrite ($NiO.Fe_2O_3$, $ZnO.Fe_2O_3$) have very narrow hysteresis loops and are suitable for high-Q coils, wideband transformers, radio-frequency and pulse transformers and aerial rods. The material is supplied in the form of extrusions or preformed rings. Various grades are available depending on the application and frequency range required.

Magnesium manganese ferrite ($MgO.Fe_2O_3$, $MnO.Fe_2O_3$) exhibits a relatively square hysteresis loop which makes the material suitable for switching and storage applications. A typical B/H characteristic is shown in Fig. 9.38(a).

For use as a storage element, the ferrite is formed in a ring, as shown at (b). With no currents in the windings the magnetic state of the core will be represented by either point X or point Y in (a)—i.e. with the residual flux directed either clockwise or anti-clockwise round the ring. Clockwise flux may arbitrarily designated as the "1" condition and anti-clockwise flux as the "0" condition, so that the core can be used as a storage element for "ones" or "zeros", i.e. as an element in a binary store. Note that no energy is required to maintain the core in either state.

In order to change the state of a core from "1" to "0" it is necessary momentarily to supply a current I in a winding in such a direction as to give rise to demagnetizing ampere-turns. The flux density then changes from X to Z, and falls back to the residual value Y when the current is removed, so changing the core state from "1" to "0". Similarly, if the initial state is "0", then supplying the appropriate magnetizing ampere-turns will cause a change to the "1" state.

In order to determine the state of a core (i.e. to "read" the core) it is also necessary to pass a current I through a winding in a standard direction. Thus, suppose the core state is represented by X at (*a*). The "read" signal may be chosen in the demagnetizing direction so that when a pulse of "read" current, *I*, passes, the core changes state. An output pulse will then appear in a further winding ("sense" winding) on the core due to the e.m.f. induced by the changing flux.

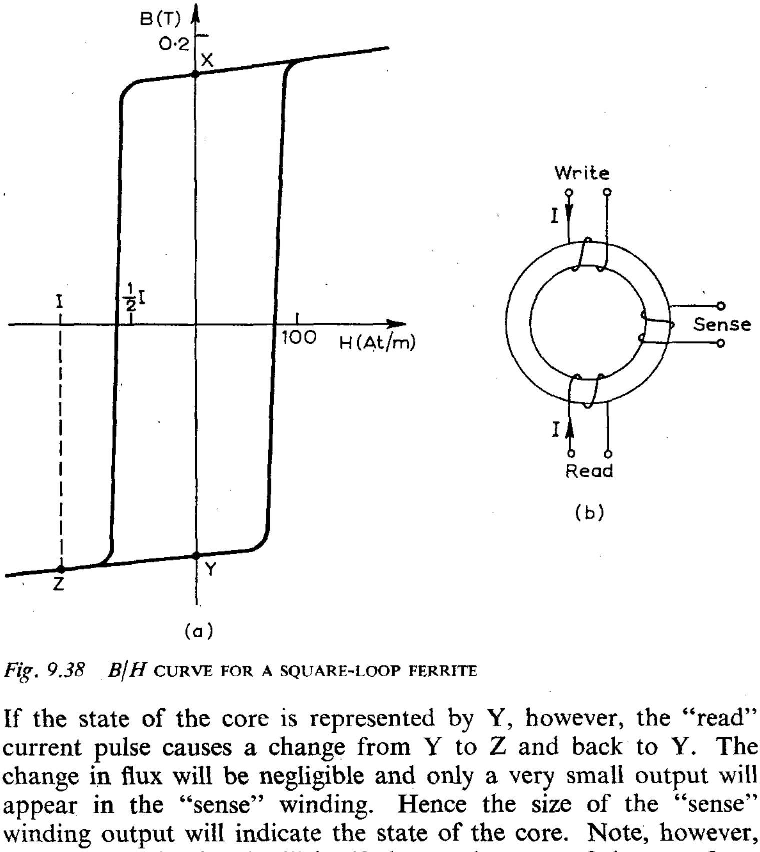


Fig. 9.38 *B/H* CURVE FOR A SQUARE-LOOP FERRITE

If the state of the core is represented by Y, however, the "read" current pulse causes a change from Y to Z and back to Y. The change in flux will be negligible and only a very small output will appear in the "sense" winding. Hence the size of the "sense" winding output will indicate the state of the core. Note, however, that the "read" signal will itself change the state of the core from "1" to "0", and hence a "rewrite" circuit is required if the "read" signal is not to destroy the "information" stored (i.e. the state of the core).

PROBLEMS

9.1 An air-cored mutual inductor has the following constants:

Resistance of primary winding, $R_1 = 48\,\Omega$
Self-inductance of primary winding, $L_{11} = 80\,\text{mH}$

Resistance of secondary winding, $R_2 = 58\,\Omega$
Self-inductance of secondary winding, $L_{22} = 60\,\text{mH}$
Coupling coefficient, $k = 0{\cdot}60$

Determine for $\omega = 10{,}000\,\text{rad/s}$ the input impedance to the primary when the secondary terminals are (*a*) open-circuited, and (*b*) short-circuited. Find also for $\omega = 10{,}000\,\text{rad/s}$ the input impedance to the coils when they are joined (*a*) in series aiding, and (*b*) in series opposing.

Ans. $(48 + j800)\,\Omega$; $(73{\cdot}4 + j515)\,\Omega$; $(106 + j2{,}230)\,\Omega$; $(106 + j568)\,\Omega$.

9.2 The primary winding of an air-cored mutual inductor has 1,500 turns and the secondary winding 2,000 turns. When measurements were made at an angular frequency of 5,000 rad/s the following results were obtained:

Input impedance to primary (secondary open-circuited)	. $(40 + j325)\,\Omega$
Input impedance to primary (secondary short-circuited)	. $(60 + j165)\,\Omega$
Input impedance to secondary (primary open-circuited)	. $(80 + j650)\,\Omega$

Determine the self-inductance of each winding, the mutual inductances and the coupling coefficient. Draw the equivalent-T circuit.

Determine also the primary and secondary leakage and magnetizing inductances and draw the transformer equivalent circuit.

What is the input impedance to the primary for a secondary load of $(200 + j200)\,\Omega$ at an angular frequency $\omega = 5{,}000\,\text{rad/s}$?

Ans. 65 mH; 130 mH; 65 mH; 0·707; 16·2 mH; 43·3 mH; 48·8 mH; 86·7 mH; $(77 + j213)\,\Omega$

9.3 Deduce an expression for the cross-sectional area of a transformer core in terms of the primary voltage, turns, frequency and flux density. A 50 Hz 3-phase core-type transformer is to be built for a 10,000/400 V ratio connected star-mesh. The cores are to have a square section. Assuming a maximum flux density of 1·1 T and an induced e.m.f. of 10 V per turn, determine the cross-sectional dimensions of the core and the number of turns per phase in each winding. (*H.N.C.*)

Ans. 20·2 cm × 20·2 cm; 578 turns/phase; 40 turns/phase.

9.4 A 3,200/400 V single-phase transformer has winding resistances and reactances of $3\,\Omega$ and $13\,\Omega$ respectively in the primary and $0{\cdot}02\,\Omega$ and $0{\cdot}065\,\Omega$ in the secondary. Express these in terms of (*a*) primary alone, (*b*) secondary alone.

Ans. $4{\cdot}28\,\Omega$, $17{\cdot}16\,\Omega$; $0{\cdot}067\,\Omega$, $0{\cdot}268\,\Omega$.

9.5 Explain with a diagram, how a transformer can be represented by an equivalent circuit. Derive an expression for the equivalent resistance and reactance referred to the primary winding.

A 50 Hz single-phase transformer has a turns ratio of 6. The resistances are $0{\cdot}9\,\Omega$ and $0{\cdot}03\,\Omega$ and the reactances $5\,\Omega$ and $0{\cdot}13\,\Omega$ for high-voltage and low-voltage windings respectively.

Find (*a*) the voltage to be applied to the high-voltage side to obtain full-load current of 200 A in the low-voltage winding on short-circuit, (*b*) the power factor on short-circuit. (*H.N.C.*)

Ans. 330 V; 0·2 lagging.

9.6 The primary and secondary windings of a 30 kVA 6,000/230 V transformer have resistances of $10\,\Omega$ and $0{\cdot}016\,\Omega$ respectively. The reactance of the transformer referred to the primary is $34\,\Omega$. Calculate (*a*) the primary voltage required to circulate full-load current when the secondary is short-circuited, (*b*) the percentage voltage regulation of the transformer for a load of 30 kVA having a p.f. of 0·8 lagging. *(L.U.)*

Ans. 200 V; 3·1 per cent.

9.7 Calculate (*a*) the full-load efficiency at unity power factor, and (*b*) the secondary terminal voltage when supplying full-load secondary current at power factors (i) 0·8 lagging, (ii) 0·8 leading for the 4 kVA 200/400 V 50 Hz single-phase transformer, of which the following are the test figures:

Open-circuit with 200 V applied to the primary winding—power 60 W. Short-circuit with 16 V applied to the high-voltage winding—current 8 A, power 40 W.

Show a complexor diagram in both cases. *(H.N.C.)*

Ans. 0·97; 383 V, 406 V.

9.8 A 12 kVA 220/440 V 50 Hz single-phase transformer gave the following test figures:

No-load: primary data—220 V, 2 A, 165 W.
Short-circuit: secondary data—12 V, 15 A, 60 W.

Draw the equivalent circuit, considered from the low-voltage side, and insert appropriate values. Find the secondary terminal voltage on full load at a power factor of 0·8 lagging. *(H.N.C.)*

Ans. 422 V.

9.9 The following results were obtained from a 125 kVA 2,000/400 V 50 Hz single-phase transformer:

No-load test h.v. data—2,000 V, 1 A, 1,000 W.
Short-circuit tests l.v. data—13 V, 200 A, 750 W.

Calculate:

(*a*) the magnetizing current and the component corresponding to core loss at normal voltage and frequency;
(*b*) the efficiency on full load at p.f.s of unity, 0·8 lagging, and 0·8 leading;
(*c*) the secondary voltage on full load at the above p.f.s. *(H.N.C.)*

Ans. 0·866 A; 0·5 A; 0·98; 0·976; 0·976; 394 V; 384 V; 406 V.

9.10 A 5 kVA 200/400 V 50 Hz single-phase transformer gave the following results:

Open-circuit test: 200 V, 0·7 A, 60 W—low-voltage side.
Short-circuit test: 22 V, 16 A, 120 W—high-voltage side.

(*a*) Find the percentage regulation when supplying full load at 0·9 power factor lagging.
(*b*) Determine the load which gives maximum efficiency and find the value of this efficiency at unity power factor. *(H.N.C.)*

Ans. 3·08 per cent, 4·54 kVA, 0·974.

9.11 Enumerate the losses in a transformer and explain how each loss varies with the load when the supply voltage and frequency are constant. Describe how the components of the losses at no load may be determined.

A transformer having a rated output of 100 kVA has an efficiency of 98 per cent at full-load unity p.f. and maximum efficiency occurs at $\frac{2}{3}$ full load (unity p.f.).

Calculate (i) the core losses, and (ii) the maximum efficiency. *(L.U.)*

Ans. 0·62 kW; 98·4 per cent.

9.12 Calculate the efficiencies at half-full, full, and $1\frac{1}{4}$ full load of a 100kVA transformer for power factors of (*a*) unity, (*b*) 0·8. The winding loss is 1,000W at full load and the core loss is 1,000W.

Ans. $\frac{1}{2}$ full, 0·975, 0·969; full, 0·98, 0·975; $1\frac{1}{4}$ full, 0·979, 0·974.

9.13 A 2-phase 240V supply is to be obtained from a 3-phase 3-wire 440V supply by means of a pair of Scott-connected single-phase transformers. Determine the turns ratios of the main and teaser transformers.

Find the input current in each of the 3-phase lines (*a*) when each of the 2-phase currents is 1A lagging behind the respective phase voltage by 36·9°, and (*b*) when the secondary phase on the main transformer is open-circuited the other secondary phase being loaded as in (*a*). Magnetizing current may be neglected.

Ans. 1·83; 1·59; 0·63A; $I_R = 0{\cdot}63\underline{/0^\circ}$A; $I_Y = I_B = 0{\cdot}315\underline{/180^\circ}$A.

9.14 Two transmission lines of impedance $(1 + j2)\Omega$ and $(2 + j2)\Omega$ respectively feed in parallel a load of 7,500kW at 0·8p.f. lagging.

Determine the power output of each line and its power factor.

Ans. 3,750kW, 0·707 lagging; 3,750kW, 0·894 lagging.

9.15 A 400kVA transformer of 0·01 per unit resistance and 0·05 per unit reactance is connected in parallel with a 200kVA transformer of 0·012 per unit resistance and 0·04 per unit reactance. Find how they share a load of 600kVA at 0·8p.f. lagging.

Ans. $373\underline{/-39^\circ}$kVA; $227\underline{/-33{\cdot}6^\circ}$kVA.

9.16 Two 3-phase transformers operating in parallel deliver 500A at a p.f. of 0·8 lagging. The resistances and reactances of the transformers are $R_1 = 0{\cdot}02\Omega$, $X_1 = 0{\cdot}2\Omega$; $R_2 = 0{\cdot}03\Omega$, $X_2 = 0{\cdot}3\Omega$. Calculate the current delivered by the first transformer and its phase angle with respect to the common terminal voltage.

In this example $R_1/X_1 = R_2/X_2$. Discuss, with reasons, whether or not this is desirable for parallel operation. (*H.N.C.*)

Ans. 300A, 0·8 lagging.

9.17 A small 3-phase substation receives power from a station some distance away by two feeders which follow different routes. The impedances per phase of the feeders are (*a*) cable $(3 + j2)\Omega$ and (*b*) overhead line $(2 + j6)\Omega$. If the power delivered by the line is 4,000kW at 11kV and p.f. 0·8 lagging, find the total power delivered by the cable and its phase angle. (*H.N.C.*)

Ans. 8,760kW; 1° leading.

9.18 Two transformers A and B are connected in parallel to supply a load having an impedance of $(2 + j1{\cdot}5)\Omega$. The equivalent impedances referred to the secondary windings are $(0{\cdot}15 + j0{\cdot}5)\Omega$ and $(0{\cdot}1 + j0{\cdot}6)\Omega$ respectively. The open-circuit e.m.f. of A is 207V and of B is 205V. Calculate (i) the voltage at the load, (ii) the power supplied to the load, (iii) the power output of each transformer, and (iv) the kVA input to each transformer.

Ans. (i) $189\underline{/-3{\cdot}8^\circ}$V, (ii) 11·5kW, (iii) 6·5kW, 4·95kW, (iv) 8·7kVA, 6·87kVA.

9.19 Explain clearly the essential conditions to be satisfied when two 3-phase transformers are connected in parallel. Give two sets of possible connexions, explaining how these are satisfactory or unsatisfactory.

Two transformers of equal voltage ratios but with the following ratings and impedances,

Transformer A—1,000 kVA, 1 per cent resistance, 5 per cent reactance,
Transformer B—1,500 kVA, 1·5 per cent resistance, 4 per cent reactance,

are connected in parallel to feed a load of 1,000 kW at 0·8 p.f. lagging. Determine the kVA in each transformer and its power factor. (*H.N.C.*)
(*Note*. Impedances may be expressed in per-unit form by dividing the percentage impedances by 100.)

Ans. A: 448 kVA, 0·73 lagging. B: 804 kVA, 0·834 lagging.

9.20 Two single-phase transformers work in parallel on a load of 750 A at 0·8 p.f. lagging. Determine the secondary voltage and the output and power factor of each transformer.

Test data are:

Open-circuit: 11,000 V/3,300 V for each transformer
Short-circuit with h.v. winding short-circuited:
Transformer A: secondary input 200 V, 400 A, 15 kW
Transformer B: secondary input 100 V, 400 A, 20 kW (*L.U.*)

Ans. 3,190 V; A: 807 kVA, 0·65 lagging. B: 1,615 kVA, 0·86 lagging.

Chapter 10

GENERAL PRINCIPLES OF ROTATING MACHINES

Rotating machines vary greatly in size, ranging from a few watts to 600 MW and above—a ratio of power outputs of over 10^7. They also vary greatly in type depending on the number and interconnexion of their windings and the nature of electrical supply to which they are to be connected. Despite these differences of size and type their general principles of operation are the same, and it is the purpose of this chapter to examine these common principles. Three succeeding chapters give a more detailed treatment of particular types of machine.

10.1 Modes of Operation

There are three distinguishable ways or modes of operation of rotating machines and these are illustrated in the block diagrams of Fig. 10.1. The three modes, motoring, generating and braking, are specified below.

MOTORING MODE

Electrical energy is supplied to the main or *armature winding* of the machine and a mechanical energy output is available at a rotating shaft. This mode of operation is illustrated in Fig. 10.1(*a*), which takes the form of a 2-port representation of a machine, one port being electrical and the other mechanical.

An externally applied voltage v drives a current i through the armature winding against an internally induced e.m.f. e. The process of induction of e.m.f. is discussed in Section 10.4. The winding is thus enabled to absorb electrical energy at the rate ei. At least some of this energy is available for conversion (some may be stored in associated magnetic fields). The armature winding gives rise to an instantaneous torque T_A'* which drives the rotating member of the machine (the *rotor*) at an angular velocity ω_r, and mechanical energy is delivered at the rate of $\omega_r T_A'$. The process of torque production is discussed in Section 10.5. An externally applied load torque

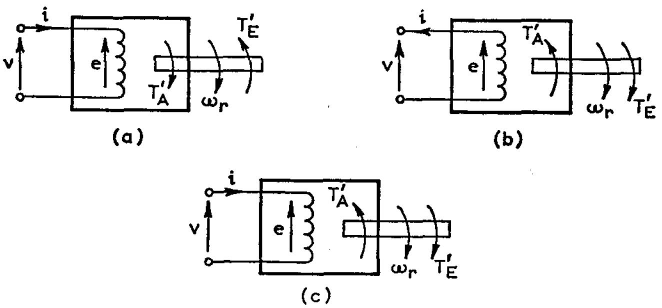


Fig. 10.1 MODES OF OPERATION OF ROTATING MACHINES
(*a*) Motoring mode
(*b*) Generating mode
(*c*) Braking mode

T_E' acting in a direction opposite to that of rotation enables the load to absorb mechanical energy:

$$T_A' - T_E' = J\frac{d\omega_r}{dt} \tag{10.1}$$

J being the moment of inertia of the rotor and its mechanical load.

If the windage and friction torque is included in the external applied load torque, then, following eqn. (10.1), if T_A' and T_E' are equal and opposite, $d\omega_r/dt = 0$ and the machine will rotate at a steady speed.

When steady-state operation prevails, provided a sufficient period of time is considered,

$$(\omega_r T_A')_{mean} = (ei)_{mean} \tag{10.2}$$

Since the armature winding must develop torque and have an e.m.f. induced in it, a magnetic field is required. In very small

* To avoid confusion with t for time, instantaneous torque will be represented by T'.

machines this may be provided by permanent magnets, but in most machines it is provided electromagnetically.

Some machines have a separate *field winding* to produce the required magnetic field. For such machines the block diagram of Fig. 10.1(*a*) would require a second electrical port. For the sake of simplicity this has been omitted. The energy fed to the field winding is either dissipated as loss in the field winding or is stored in the associated magnetic field and does not enter into the conversion process.

GENERATING MODE

Mechanical energy is supplied to the shaft of the machine by a prime mover and an electrical energy output is available at the armature-winding terminals. This mode of operation is illustrated in Fig. 10.1(*b*). The shaft of the machine is driven at an angular velocity ω_r in the direction of the applied external instantaneous torque T_E' and in opposition to the torque T_A' due to the armature winding, enabling the machine to absorb mechanical energy. The armature winding has an e.m.f. e induced in it which drives a current through an external load of terminal voltage v. Eqns. (10.1) and (10.2) apply equally to generator action.

BRAKING MODE

In this mode of operation the machine has both a mechanical energy input and an electrical energy input. The total energy input is dissipated as loss in the machine. This mode is of limited practical application but occurs sometimes in the operation of induction and other machines.

10.2 Rotating Machine Structures

Rotating electrical machines have two members, a stationary member called the *stator* and a rotating member called the *rotor*. The stator and rotor together constitute the magnetic circuit or core of the machine and both are made of magnetic material so that magnetic flux is obtained for moderate values of m.m.f. The rotor is basically a cylinder and the stator a hollow cylinder. The rotor and stator are separated by a small air-gap as shown in Fig. 10.2. Compared with the rotor diameter the radial air-gap length is small. The stator and rotor magnetic cores are usually, but not invariably, built up from laminations (typically 0·35 mm thick) in order to reduce eddy-current loss.

If the rotor is to rotate, a mutual torque has to be sustained between the rotor and stator. A winding capable of carrying current and of sustaining torque is required on at least one member and usually, but not always, on both. One method of arranging windings in a rotating machine is to place coils in uniformly distributed slots on both the stator and the rotor. This method is illustrated in Fig.

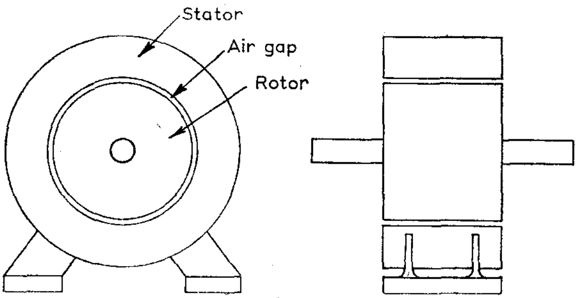


Fig. 10.2 BASIC ARRANGEMENT OF A ROTATING MACHINE

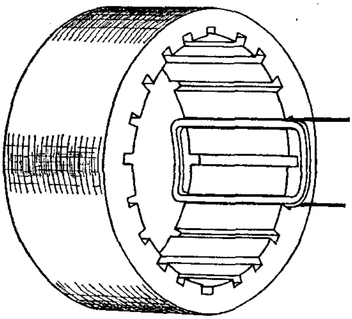


Fig. 10.3 STATOR AND ONE OF ITS COILS

10.3, where, for clarity, only the stator is shown. An arrangement of this sort is commonly used in induction machines. The distance between the coil sides is usually about one pole pitch.

To make a complete winding, similar coils are placed in other pairs of slots and all the coils are then connected together in groups. The groups of coils may then be connected in series or in parallel, and in 3-phase machines in star or mesh.

Some windings may be double-layer windings. In such windings each slot contains two coil sides, one at the top and the other at the bottom of a slot. Each coil has one coil side at the top of the slot and the other at the bottom.

An alternative arrangement to having uniform slotting on both sides of the air-gap is to have salient poles around which are wound concentrated coils to provide the field winding. The salient poles may be on either the stator or the rotor, and such arrangements are illustrated in Figs. 10.4 and 10.5(*a*).

The salient-pole stator arrangement is commonly used for direct-current machines and occasionally for small sizes of synchronous

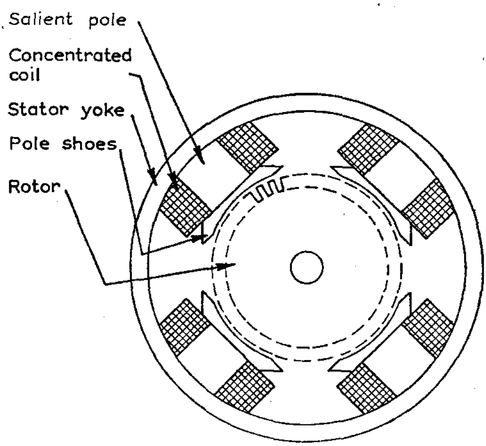


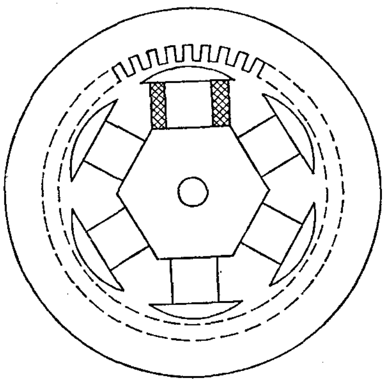
Fig. 10.4 SALIENT-POLE STATOR

machine. As far as d.c. machines are concerned the stator is most often referred to as the field and the rotor as the armature. The main winding in such a machine is on the rotor and is called the armature winding.

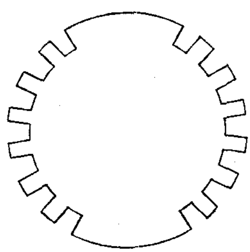
The salient pole rotor arrangement is most often used for synchronous machines. In such machines the main winding is on the stator but it is often called the armature winding.

In general, rotating machines can have any even number of poles. The concentrated coil windings surrounding the poles are excited so as to make successive poles of alternate north and south polarity.

The salient-pole rotor structure is unsuitable for large high-speed turbo-alternators used in the supply industry because of the high stress in the rotor due to centrifugal force. In such machines a cylindrical rotor is used as shown in Fig. 10.5(*b*). Uniform slotting occupies two thirds of the rotor surface, the remaining third being unslotted. Such rotors are usually solid steel forgings.



(a)



(b)

Fig. 10.5 ROTORS

(*a*) Salient pole (*b*) Cylindrical

10.3 Self- and Mutual Inductance of Stator and Rotor Windings

The simplest rotating machine structure is a 2-pole machine with a uniform air-gap as shown in Fig. 10.2 which does not exhibit "saliency" (i.e. does not have salient poles) on either side of the air-gap. In this and all succeeding sections of this chapter only 2-pole machines will be considered.

Fig. 10.6(*a*) shows such a machine. The stator winding axis is chosen to correspond with a horizontal angular reference axis called the *direct axis* (d-axis) at which $\theta = 0$.

A convention for positive current in a coil must be established and this is done in the following way. Consider a winding such as the stator winding of Fig. 10.6(*a*) whose axis corresponds with the d-axis. Positive current is taken to produce an m.m.f. acting in the positive direction of the d-axis. Thus the stator winding in Fig. 10.6(*a*) is excited by positive current. The angular position, θ, of a

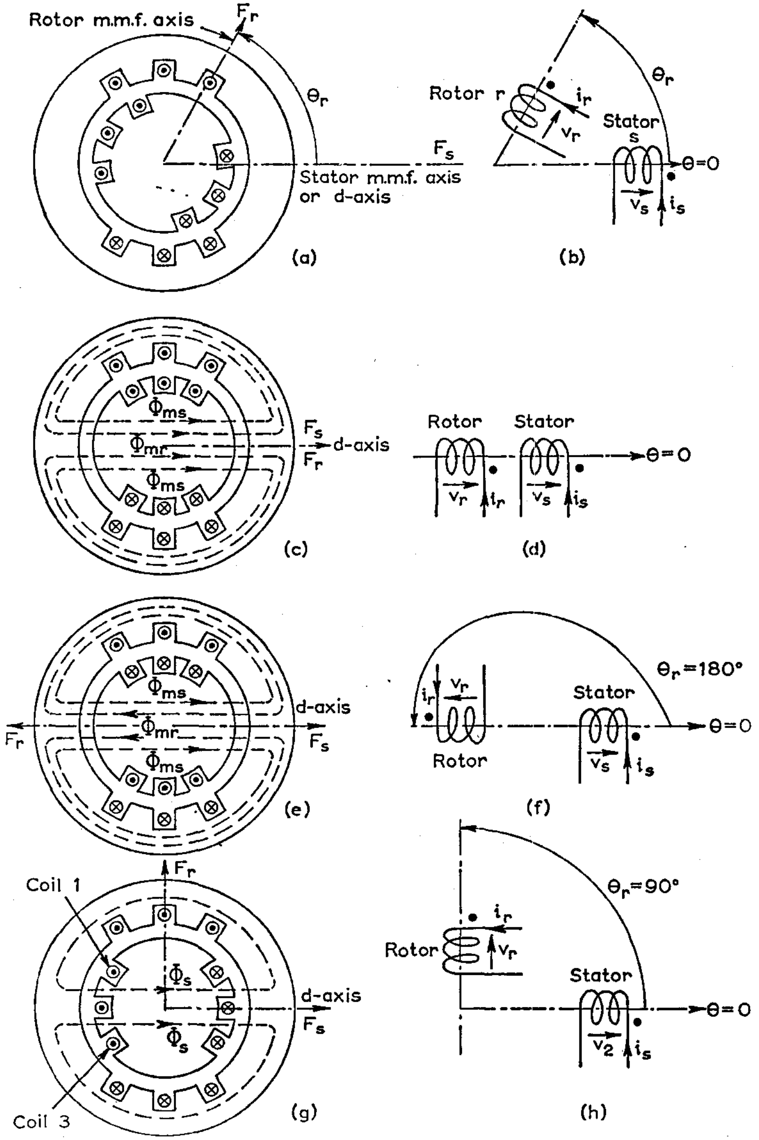


Fig. 10.6 MUTUAL COUPLING OF STATOR AND ROTOR COILS

winding whose axis does not correspond with the d-axis is the magnitude of the clockwise angle through which the coil must be rotated so that its m.m.f. acts along the positive direction of the d-axis.

Fig. 10.6(a) shows a rotor winding at an angle θ_r to the d-axis. If the current in this coil were reversed, represented in the diagram by interchange of the dots and crosses of the rotor winding, the winding position would be taken as $180° + \theta_r$.

Fig. 10.6(b) is a circuit representation of the configuration shown at (a) which uses the dot notation. The dot notation for coils capable of rotation can be expressed as "currents entering the dotted end of a winding give rise to an m.m.f. which acts towards the dotted coil end".

Evidently the mutual inductance between the stator and rotor windings is a positive maximum in the configuration of Fig. 10.6(c), where $\theta_r = 0$, and a negative maximum for that of Fig. 10.6(e), where $\theta_r = 180°$. Further, in the configuration of Fig. 10.6(g), where the winding axes are at right angles, the mutual inductance between the windings is zero. If the rotor is considered to have diametral coils (i.e. coil sides in diametrically opposite slots) then the current of coil 1 links the stator flux in the opposite direction to that of the current in coil 3 so that the net current-flux linkage is zero.

The mutual coupling between the stator and rotor coils depends on the angular separation of their m.m.f. axes θ_r. When an inductance is a function of θ in this way it will be denoted by the symbol $\mathscr{L}$. Where an inductance is not a function of θ it is written L.

The mutual coupling between the stator and rotor windings is evidently a cosine-like or even function of the form

$$\mathscr{L}_{sr} = L_{sr}(\cos\theta_r + k_3\cos 3\theta_r + k_5\cos 5\theta_r \dots) \tag{10.3}$$

If all terms except the fundamental are ignored,

$$\mathscr{L}_{sr} = L_{sr}\cos\theta_r \tag{10.4}$$

Since the air-gap is uniform, the permeance of the stator and rotor magnetic circuits is unaffected by rotor position and the stator and rotor winding self-inductances are constants.

If the stator has salient poles, the mutual inductance between the stator and rotor windings is still given by eqn. (10.3) though the space-harmonic coefficients, k_3, k_5, k_7, etc. will be different. Since the space harmonics are ignored, the mutual inductance is given by eqn. (10.4). The self-inductance of the stator winding will be a constant independent of θ_r, but the self-inductance of the rotor coil will depend on the rotor position (see Fig. 10.7). When the rotor m.m.f. axis is lined up with the d-axis, its self-inductance will be a maximum, L_{dd}, say, but when it is lined up with an axis at right angles to the d-axis, the *quadrature axis* (q-axis), the self-inductance will have fallen to a minimum value L_{qq}, say, because of the much lower permeance of the magnetic circuit centred on this axis.

After the rotor has turned through 180° the rotor m.m.f. axis again corresponds with the d-axis so that its self-inductance again

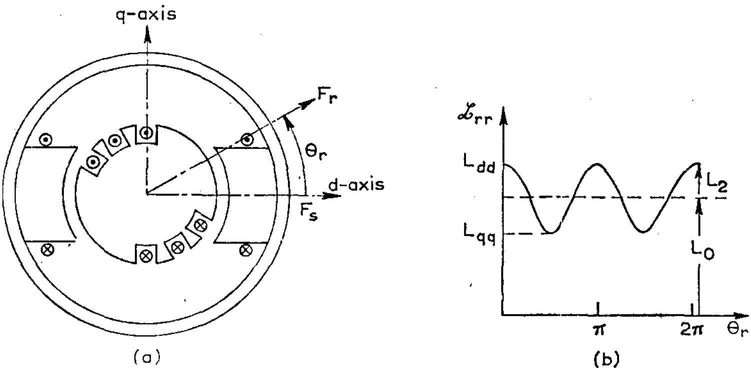


Fig. 10.7 VARIATION OF SELF-INDUCTANCE DUE TO SALIENCY

has the maximum value L_{dd}. Fig. 10.7(*b*) shows $\mathscr{L}_{rr}$ to a base of θ_r, from which approximately

$$\mathscr{L}_{rr} = L_0 + L_2 \cos 2\theta_r \tag{10.5}$$

or, from Fig. 10.7(*b*),

$$\mathscr{L}_{rr} = \tfrac{1}{2}(L_{dd} + L_{qq}) + \tfrac{1}{2}(L_{dd} - L_{qq}) \cos 2\theta_r \tag{10.6}$$

When the saliency occurs on the rotor, on the other hand, the rotor self-inductance becomes constant and the stator self-inductance changes. In such a case the d- and q-axis are deemed to rotate, the d-axis coinciding with the salient-pole axis and the q-axis being at right angles to the d-axis. Eqn. (10.6) then gives the variation of stator-winding self-inductance without modification as

$$\mathscr{L}_{ss} = \tfrac{1}{2}(L_{dd} + L_{qq}) + \tfrac{1}{2}(L_{dd} - L_{qq}) \cos 2\theta_r \tag{10.7}$$

10.4 General Expression for Induced E.M.F.

Consider a stator winding s, and a rotor winding r that rotates at a steady angular velocity $\omega_r = d\theta_r/dt$ with respect to the stator winding, as shown in Fig. 10.8. Let the windings carry instantaneous currents i_s, i_r which, in general, will be functions of time. The mutual inductance of the windings is a function of θ_r as also will be one of the self-inductances if saliency exists on either side of the air-gap. Since the position of the rotor winding θ_r is a function of time,

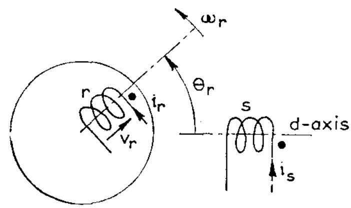


Fig. 10.8 INDUCED E.M.F. AND TORQUE IN A ROTATING MACHINE

such inductances are implicitly functions of time. The e.m.f. induced in the rotor winding is

$$e_r = \frac{d}{dt}(\mathscr{L}_{rr} i_r + \mathscr{L}_{sr} i_s)$$

Differentiating each term as a product,

$$e_r = \frac{\partial \mathscr{L}_{rr}}{\partial \theta_r}\frac{d\theta_r}{dt} i_r + \mathscr{L}_{rr}\frac{di_r}{dt} + \frac{\partial \mathscr{L}_{sr}}{\partial \theta_r}\frac{d\theta_r}{dt} i_s + \mathscr{L}_{sr}\frac{di_s}{dt}$$

$$e_r = \underbrace{\mathscr{L}_{rr}'\omega_r i_r + \mathscr{L}_{sr}'\omega_r i_s}_{\text{Rotational voltages}} + \underbrace{\mathscr{L}_{rr}\frac{di_r}{dt} + \mathscr{L}_{sr}\frac{di_s}{dt}}_{\text{Transformer voltages}} \tag{10.8}$$

where $\mathscr{L}_{rr}' = \partial\mathscr{L}_{rr}/\partial\theta$ etc. and $\omega_r = d\theta_r/dt$.

It will be seen that the expression for e_r contains terms of two distinct types: (*a*) voltages proportional to the rotor angular velocity and called *rotational voltages*, and (*b*) voltages proportional to the rate of change of the winding currents. These latter are often called *transformer voltages*.

Including the voltage drop in the rotor winding resistance the voltage applied to that winding is

$$v_r = r_r i_r + \mathscr{L}_{rr}'\omega_r i_r + \mathscr{L}_{sr}'\omega_r i_s + \mathscr{L}_{rr}\frac{di_r}{dt} + \mathscr{L}_{sr}\frac{di_s}{dt} \tag{10.9}$$

Similarly, the voltage applied to the stator winding is

$$v_s = r_s i_s + \mathscr{L}_{ss}'\omega_r i_s + \mathscr{L}_{sr}'\omega_r i_r + \mathscr{L}_{ss}\frac{di_s}{dt} + \mathscr{L}_{sr}\frac{di_r}{dt} \tag{10.10}$$

In general, for n coupled windings the voltage applied to the jth winding is

$$v_j = r_j i_j + \sum_{k=1}^{k=n} \frac{d}{dt}(\mathscr{L}_{jk} i_k) \tag{10.11}$$

When there are additional stator or rotor windings the expression for a winding voltage may usually be inferred by extension of eqn. (10.9) or eqn. (10.10).

10.5 General Expression for Torque

Consider again a machine consisting of a stator winding s and a rotor winding r rotating at a steady angular velocity ω_r, as shown in Fig. 10.8. The total instantaneous power fed into the machine is

$$p_e = v_r i_r + v_s i_s$$

Substituting for v_s and v_r in terms of eqns. (10.9) and (10.10),

$$p_e = r_r i_r^2 + \mathscr{L}_{rr}' \omega_r i_r^2 + \mathscr{L}_{sr}' \omega_r i_r i_s + \mathscr{L}_{rr} i_r \frac{di_r}{dt} + \mathscr{L}_{sr} i_r \frac{di_s}{dt}$$

$$+ r_s i_s^2 + \mathscr{L}_{ss}' \omega_r i_s^2 + \mathscr{L}_{sr}' \omega_r i_r i_s + \mathscr{L}_{ss} i_s \frac{di_s}{dt} + \mathscr{L}_{sr} i_s \frac{di_r}{dt} \tag{10.12}$$

In this equation the terms $r_r i_r^2$ and $r_s i_s^2$ represent power loss in the winding resistances.

The energy stored in the magnetic fields associated with the two coils is

$$w_f = \tfrac{1}{2}\mathscr{L}_{rr} i_r^2 + \tfrac{1}{2}\mathscr{L}_{ss} i_s^2 + \mathscr{L}_{sr} i_s i_r$$

The rate at which energy is stored in the magnetic field is

$$\frac{dw_f}{dt} = \tfrac{1}{2} i_r^2 \mathscr{L}_{rr}' \omega_r + \mathscr{L}_{rr} i_r \frac{di_r}{dt} + \tfrac{1}{2} i_s^2 \mathscr{L}_{ss}' \omega_r + \mathscr{L}_{ss} i_s \frac{di_s}{dt}$$

$$+ i_s i_r \mathscr{L}_{sr}' \omega_r + \mathscr{L}_{sr} i_s \frac{di_r}{dt} + \mathscr{L}_{sr} i_r \frac{di_s}{dt} \tag{10.13}$$

where again $\mathscr{L}_{rr}' = \partial \mathscr{L}_{rr}/\partial \theta_r$, etc., and $\omega_r = d\theta_r/dt$.

There is an instantaneous mechanical power output corresponding to that portion of the instantaneous electrical power input which is neither dissipated in the winding resistances nor used to store energy in the magnetic field. If the instantaneous torque on the rotor is T', then

$$p_m = \omega_r T' = p_e - r_r i_r^2 - r_s i_s^2 - \frac{dw_f}{dt}$$

$$= \tfrac{1}{2}\mathscr{L}_{rr}'\omega_r i_r^2 + \mathscr{L}_{sr}'\omega_r i_r i_s + \tfrac{1}{2}\mathscr{L}_{ss}'\omega_r i_s^2$$

or

$$T' = \tfrac{1}{2}\mathscr{L}_{rr}' i_r^2 + \mathscr{L}_{sr}' i_r i_s + \tfrac{1}{2}\mathscr{L}_{ss}' i_s^2 \qquad (10.14)$$

When there are additional stator or rotor windings the total torque acting on the rotor or stator may be inferred by extension of eqn. (10.14).

EXAMPLE 10.1 A torque motor has a uniform air-gap. The stator and rotor each carry windings and the axis of the rotor coil may rotate relative to that of the stator coil. The mutual inductance between the coils is such that

$$\mathscr{L}_{sr} = L_{sr} \cos \theta_r$$

(*a*) Show that, when the axes of the coils are lined up on the *d*-axis and each coil carries conventionally positive current, the coils are in a position of stable equilibrium.
(*b*) If with the coils so aligned the current in either coil is reversed show that the position is one of unstable equilibrium.
(*c*) In such an arrangement the rotor and stator coils are in series, the rotor coil axis is at $\theta_r = 135°$ and the maximum mutual inductance is 1 H. Calculate the coil currents if the mutual torque on the rotor is to be 100 N-m in the $-\theta$ direction and the coils are excited with direct current.

The instantaneous torque is given by eqn. (10.14) as

$$T' = \tfrac{1}{2}\mathscr{L}_{rr}' i_r^2 + \mathscr{L}_{sr}' i_r i_s + \tfrac{1}{2}\mathscr{L}_{ss}' i_s^2 \qquad (10.14)$$

Since the air-gap is uniform the rotor and stator winding self-inductances are constants and their angular rates of change are zero, i.e. $\mathscr{L}_{rr}' = 0$ and $\mathscr{L}_{ss}' = 0$.

$$\mathscr{L}_{sr}' = \frac{d}{d\theta_r}(\mathscr{L}_{sr}) = \frac{d}{d\theta_r}(L_{sr}\cos\theta_r) = -L_{sr}\sin\theta_r$$

Substituting these conditions into the expression for instantaneous torque,

$$T' = -L_{sr} i_r i_s \sin\theta_r \qquad (10.15)$$

With the rotor coil axis aligned with the stator coil axis $\theta_r = 0$ and the torque is zero. If the rotor coil is given a small deflection $+\delta\theta_r$ from this position, with positive currents flowing in both windings the torque takes on a negative value and acts in the $-\theta_r$ direction to restore the rotor to its initial position. Similarly, if the rotor is given a small deflection $-\delta\theta_r$ the torque takes on a positive value and acts in the $+\theta_r$ direction to restore the rotor to its initial position.

If one of the coil currents is reversed, however, the opposite result occurs and a small deflection in either direction leads to a torque acting so as to increase the deflection. If the rotor is free to move in this case it will take up an equilibrium position at $\theta_r = 180°$, where the torque is again zero.

It will be noted that in both cases the tendency is for the coils to align themselves in the positive maximum mutual inductance configuration.

From eqn. (10.15) for $i_s = i_r = I$, the steady torque when the axis of the rotor winding is at any angle θ_r is

$$T = -L_{sr}I^2 \sin \theta_r$$

As the torque on the rotor is to act in the $-\theta$ direction, $T = -100$ N-m. This gives

$$I = \sqrt{\frac{100}{1 \times \sin 135^\circ}} = \underline{\underline{11{\cdot}9\,\text{A}}}$$

EXAMPLE 10.2 An electrodynamic wattmeter has a fixed current coil and a rotatable voltage coil. The magnetic circuit of the device does not exhibit saliency. The following are details of a particular wattmeter:

Full-scale deflection	110°
Control-spring constant	10^{-7} N-m/deg
Maximum current-coil current (r.m.s.)	10 A
Maximum voltage-coil voltage (r.m.s.)	60 V
Voltage-coil resistance	600 Ω

The mutual inductance between the coils varies cosinusoidally with the angle of separation of the coil axes. The zero on the instrument corresponds to a voltage-coil position of $\theta_r = 145°$.

(*a*) Determine the direct current flowing in the current coil when a direct voltage of 60 V is applied to the voltage coil and the angular deflection is 100° from the instrument scale zero.
(*b*) For a sinusoidally varying current-coil current of 6 A (r.m.s.) and a voltage-coil voltage of 60 V (r.m.s.) of the same frequency as the current, determine the phase angle by which the current lags the voltage when the voltage-coil deflection is 60° from the instrument scale zero.

The reactance of the voltage coil is negligible compared with its resistance. A diagram of the arrangement is given in Fig. 10.9.

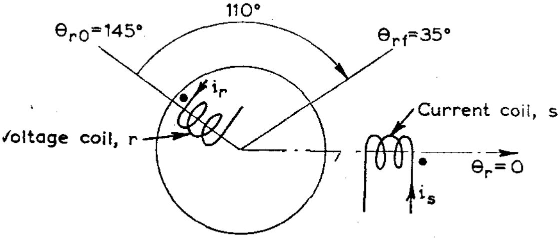


Fig. 10.9

Since there is no saliency and $\mathscr{L}_{sr} = L_{sr} \cos \theta_r$, the expression for the instantaneous torque is the same as that found in Example 10.2, namely

$$T' = -L_{sr}i_r i_s \sin \theta_r \tag{10.15}$$

If the conditions for full-scale deflection are substituted the value of L_{sr}, the maximum possible mutual inductance between the coils, is found. Since the deflection is in the $-\theta_r$ direction the torque is negative.

$$T = -110 \times 10^{-7}\,\text{N-m},\ i_r = \frac{60}{600} = 0{\cdot}1\,\text{A},\ i_s = 10\,\text{A},\ \theta_{rf} = 145° - 110° = 35°$$

whence

$$L_{sr} = \frac{110 \times 10^{-7}}{0{\cdot}1 \times 10 \times \sin 35°} = 192 \times 10^{-7}\,\text{H}$$

(*a*) When the angular deflection is 100° the position of the voltage coil is $\theta_r = 145 - 100 = 45°$. This gives, in eqn. (10.15),

$$-100 \times 10^{-7} = -192 \times 10^{-7} \times 0{\cdot}1 \times I_s \sin 45°$$

so that

$$I_s = \frac{100}{192 \times 0{\cdot}1 \times 0{\cdot}707} = \underline{\underline{7{\cdot}37\,\text{A}}}$$

(*b*) If the voltage-coil current is taken as reference, the current-coil current is

$$i_s = I_{sm} \cos(\omega t - \phi)$$

The instantaneous torque is, from eqn. (10.15),

$$\begin{aligned} T' &= -L_{sr} I_{rm} \cos \omega t I_{sm} \cos(\omega t - \phi) \sin \theta_r \\ &= -\tfrac{1}{2} L_{sr} I_{rm} I_{sm} \sin \theta_r [\cos \phi + \cos(2\omega t - \phi)] \end{aligned}$$

This expression shows that the instantaneous torque consists of two components: (*a*) a steady component, and (*b*) an alternating component which oscillates at twice the frequency of the currents in the two coils. The inertia of the rotating system will prevent its responding to the alternating component. The average torque is therefore

$$T = -\tfrac{1}{2} L_{sr} I_{rm} I_{sm} \sin \theta_r \cos\phi$$

whence

$$\cos \phi = -\frac{2T}{L_{sr} I_{rm} I_{sm} \sin \theta_r}$$

When the angular deflection is 60° the position of the voltage coil is $\theta_r = 145 - 60 = 85°$:

$$\cos \phi = -\frac{-2 \times 60 \times 10^{-7}}{192 \times 10^{-7} \times \dfrac{\sqrt{2} \times 60}{600} \times \sqrt{2} \times 6 \times 0{\cdot}995} = 0{\cdot}525$$

so that

$$\phi = \underline{\underline{58{\cdot}3°}}$$

10.6 The Alignment Principle

Example 10.1 has shown that the torque acting on the rotor of a simple rotating machine structure consisting of a stator and a rotor coil is such as to tend to align the coils in their maximum positive mutual inductance position. The mutual torque on the system is then zero. If a continuously rotating machine is to be produced, therefore, some method must be found of maintaining a constant angular displacement of the axes of the rotor and stator winding m.m.f.s under steady conditions despite the rotation of the rotor and its winding. Several different methods exist for achieving this; the particular method chosen determines the type of machine. The rest of the chapter is devoted to considering how this constant angular displacement of the axes of the winding m.m.f.s is brought about in some common types of machine.

10.7 The Commutator

In some types of machine the stator winding is excited with direct current and so the axis of the stator m.m.f., F_s, is fixed. If a constant angle is to be maintained between the axes of the stator and rotor winding m.m.f.s the rotor winding m.m.f. must be stationary despite

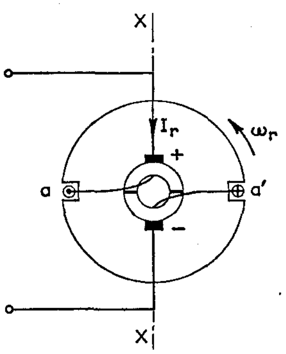


Fig. 10.10 A TWO-SEGMENT COMMUTATOR

the rotation of that winding. This may be achieved by exciting the rotor winding with direct current supplied through a commutator.

Fig. 10.10 shows a simple 2-segment commutator which consists essentially of a hollow cylinder of copper split in half, each being insulated from the other and from the shaft. One end of a rotor coil is joined to each commutator segment. Two brushes, fixed in space, make alternate contact with each segment of the commutator as it rotates. Although the current in the coil reverses twice in each revolution, it will be seen that whichever of the coil sides, *a* or *a′*, lies to the left of the brush axis XX will carry current in the direction indicated by ⊙, whereas whichever coil side lies to the right of XX carries current in the direction indicated by ⊕.

A rotor winding consisting of many coils wound into uniformly distributed slots may also be supplied through a commutator. Each of the two ends of each coil is connected to two different commutator segments. The rotor coils are connected in series, the ends of successive coils being joined at the commutator as shown in Fig. 10.11(*a*). Such windings are double-layer windings.

Fig. 10.11(*b*) is a conventional representation of such a winding where the commutator is not shown and the brushes are thought

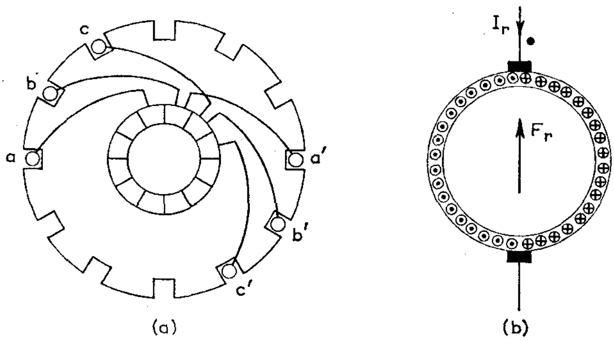


Fig. 10.11 A MULTI-SEGMENT COMMUTATOR

of as bearing directly on the conductors. All the conductors to the left of the brush axis carry current in the direction ⊙ and all those to its right carry current in the direction ⊕. It will be seen, therefore, that supplying the rotor winding through a commutator has the effect of fixing a certain current pattern in space despite rotation of the winding. As a result the axis of the rotor winding m.m.f., F_r, is fixed in space and coincides with the brush axis. The positive brush at which the current enters the winding corresponds to the dotted end of the winding.

10.8 Separately Excited D.C. Machine

The d.c. machine has almost invariably a salient pole structure on the stator and a non-salient pole rotor. The stator has a concentrated coil winding; the rotor winding is distributed in slots. Fig. 10.4 shows the structure commonly adopted for the d.c. machine. The stator winding is excited with direct current, and the rotor winding is supplied with direct current through a commutator, thus maintaining a constant angular displacement between the axes of the stator and rotor winding m.m.f.s as is required for torque maintenance.

Since the rotor is not salient pole the stator winding self-inductance does not vary with the angular position of the rotor as explained in Section 10.3, i.e.

$$\mathscr{L}_{ss} = L_{ss} \tag{10.16}$$

so that

$$\mathscr{L}_{ss}' = \frac{\partial \mathscr{L}_{ss}}{\partial \theta} = 0 \tag{10.17}$$

The mutual inductance between the stator and rotor windings is

$$\mathscr{L}_{sr} = L_{sr} \cos \theta_r \tag{10.4}$$

so that

$$\mathscr{L}_{sr}' = -L_{sr} \sin \theta_r \tag{10.18}$$

Since there is saliency on the stator the rotor self-inductance varies with the angular position of the rotor and is given by eqn. (10.6) as

$$\mathscr{L}_{rr} = \tfrac{1}{2}(L_{dd} + L_{qq}) + \tfrac{1}{2}(L_{dd} - L_{qq}) \cos 2\theta_r \tag{10.6}$$

so that

$$\mathscr{L}_{rr}' = -(L_{dd} - L_{qq}) \sin 2\theta_r \tag{10.19}$$

Eqn. (10.9) gives the voltage to the rotor winding as

$$v_r = r_r i_r + \mathscr{L}_{rr}' \omega_r i_r + \mathscr{L}_{sr}' \omega_r i_s + \mathscr{L}_{rr} \frac{di_r}{dt} + \mathscr{L}_{sr} \frac{di_s}{dt} \tag{10.9}$$

Consider a separately excited d.c. machine with steady, direct voltages, V_s and V_r, applied to the stator and rotor windings. Let the steady, direct currents in the stator and rotor windings be I_s and I_r. The time rates of change of these steady currents are zero so that the voltage applied to the rotor winding is

$$V_r = r_r I_r + \mathscr{L}_{rr}' \omega_r I_r + \mathscr{L}_{sr}' \omega_r I_s \tag{10.20}$$

or

$$V_r = r_r I_r - (L_{dd} - L_{qq}) \sin 2\theta_r \omega_r I_r - L_{sr} \sin \theta_r \omega_r I_s \tag{10.21}$$

In this equation the terms involving ω_r are rotational voltages, and the larger these terms are, for a given angular velocity and for given winding currents, the more effective the machine will be as an energy convertor. For the commutation of the rotor winding current to take place without sparking between brushes and commutator, the brush axis must be approximately at right angles to the stator winding m.m.f. This condition is represented by substituting the value $\theta_r = -\pi/2$ in eqn. (10.21) and has the effect of making the

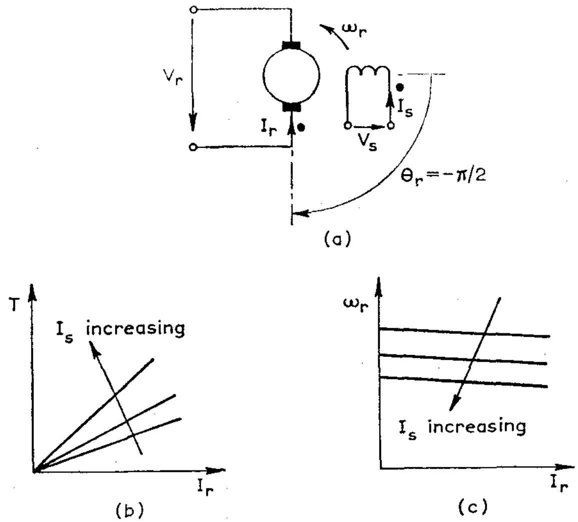


Fig. 10.12 SEPARATELY EXCITED D.C. MOTOR

rotational voltage involving the term $(L_{dd} - L_{qq})$ zero but the rotational voltage involving the the term L_{sr} a maximum. Substituting $\theta_r = -\pi/2$ in eqn. (10.21) has the advantage of removing from the equations minus signs which could be a source of confusion. Assigning the value $\theta_r = -\pi/2$ means that the dotted end of the rotor winding (i.e. the positive commutator brush) is placed at

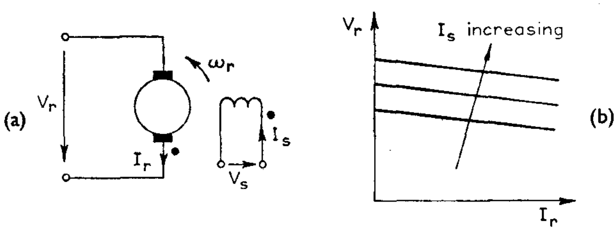


Fig. 10.13 SEPARATELY EXCITED D.C. GENERATOR

$\theta_r = -\pi/2$, as shown in Figs. 10.12, 10.13 and 10.14. Carrying out this substitution gives

$$V_r = r_r I_r + L_{sr}\omega_r I_s \tag{10.22}$$

The machine configuration is shown in Fig. 10.12(*a*).

The voltage $L_{sr}\omega_r I_s$ is a steady, direct rotational voltage due to the rotation of the rotor winding in a magnetic field set up by the stator winding. The fact that it is a direct voltage is due to the effect of the commutator.

When attention is directed to the voltage applied to the stator winding it will be realized that no rotational voltage will appear in the winding. This is so because, in spite of rotor rotation, the rotor winding m.m.f. axis is fixed in space by the action of the commutator, and so the rotor flux does not change its linkage with the stator winding even when the rotor rotates. The voltage applied to the stator winding is therefore given by eqn. (10.10):

$$V_s = r_s I_s \tag{10.23}$$

The instantaneous torque developed is given by eqn. (10.14) as

$$T' = \tfrac{1}{2}\mathscr{L}_{rr}' i_r^2 + \mathscr{L}_{sr}' i_r i_s + \tfrac{1}{2}\mathscr{L}_{ss}' i_s^2 \tag{10.14}$$

As previously noted,

$$\mathscr{L}_{ss}' = 0 \tag{10.17}$$

$$\mathscr{L}_{sr}' = -L_{sr}\sin\theta_r \tag{10.18}$$

$$\mathscr{L}_{rr}' = -(L_{dd} - L_{qq})\sin 2\theta_r \tag{10.19}$$

The positive rotor brush or terminal is located at $\theta_r = -\pi/2$, and the rotor and stator windings carry steady currents I_r and I_s. The steady torque T is then given by substitution in eqn. (10.14) as

$$\begin{aligned} T &= -\tfrac{1}{2}(L_{dd} - L_{qq})\sin(-\pi)I_r^2 - L_{sr}\sin(-\pi/2)I_r I_s \\ &= L_{sr} I_r I_s \end{aligned} \tag{10.24}$$

The torque/rotor-winding-current characteristic is shown in Fig. 10.12(*b*).

The result shown in eqn. (10.24) may be confirmed by multiplying eqn. (10.22) by I_r, which gives

$$V_r I_r = r_r I_r^2 + L_{sr}\omega_r I_r I_s$$

The term $V_r I_r$ represents the input power to the rotor winding, and $L_{sr}\omega_r I_r I_s$ the portion which is available for conversion to mechanical power. Therefore

$$\omega_r T = L_{sr}\omega_r I_r I_s$$

or the torque on the rotor is

$$T = L_{sr} I_r I_s \tag{10.24}$$

From eqn. (10.22),

$$\omega_r = \frac{V_r - r_r I_r}{L_{sr} I_s} \tag{10.25}$$

Eqns. (10.22), (10.23) and (10.24) have been set up for conventionally positive current entering the windings corresponding to electrical power input and therefore motoring mode operation. Eqn. (10.24) shows the torque developed as positive, i.e. acting in the $+\theta_r$ direction. For motor operation it is to be expected that rotation will take place in the same direction as that in which the torque acts, and this is confirmed by the positive sign of ω_r given by eqn. (10.25).

For a constant applied rotor-winding voltage and a constant stator current, eqn. (10.25) shows that the speed of the separately excited d.c. machine operating in the motoring mode will remain almost constant as the rotor winding current varies with load, since the internal voltage drop $r_r I_r$ will be small compared with V_r in any efficient machine. The speed/rotor-winding-current characteristic is shown in Fig. 10.12(*c*).

Eqns. (10.22), (10.23) and (10.24) apply equally to generator action. In this I_r will be taken to emerge from the dotted end of the rotor winding and will be negative. As a result the torque due to the rotor winding will be negative and will act in the $-\theta_r$ direction. Therefore the rotor must be assumed to be driven in the $+\theta_r$ direction by the prime mover, to be consistent with this assumed current direction. Changing the sign of I_r in eqn. (10.22),

$$V_r = -r_r I_r + L_{sr}\omega_r I_s \tag{10.26}$$

The V_r/I_r characteristic of the separately excited d.c. generator is shown in Fig. 10.13(*b*).

EXAMPLE 10.3 A separately excited d.c. machine is rotated at 500 rev/min by a prime mover. When the field (stator winding) current is 1 A the armature (rotor winding) generated voltage is 125 V with the armature open-circuited. The armature resistance is 0·1 Ω and the field resistance is 250 Ω. Determine:

(*a*) The rotational voltage coefficient, $L_{sr}\omega_r$.
(*b*) The maximum mutual inductance between the stator and rotor windings.
(*c*) The armature terminal voltage if the machine acts as a generator delivering a current of 200 A at a speed of 1,000 rev/min and the field current is 2 A.
(*d*) The input current and speed if the machine acts as a motor and develops a gross torque of 1,000 N-m. The armature and field windings are each excited from a 500 V supply.

Neglect iron loss and the effect of magnetic saturation.

Using the assumptions previously made the steady-state operating equations are

$$V_r = r_r I_r + L_{sr}\omega_r I_s \tag{10.22}$$

$$V_s = r_s I_s \tag{10.23}$$

$$T = L_{sr} I_r I_s \tag{10.24}$$

(*a*) Adhering to the previous sign conventions and considering the armature winding open-circuit test, $V_r = 125\text{V}$, $I_r = 0$, $I_s = 1\text{A}$. Substituting in eqn. (10.22),

$$125 = (0{\cdot}1 \times 0) + (L_{sr}\omega_{r1} \times 1)$$

Therefore

$$L_{sr}\omega_{r1} = \underline{\underline{125\,\text{V/A}}}$$

(*b*) $$L_{sr} = \frac{125}{\omega_{r1}} = \frac{125}{2\pi 500/60} = \underline{\underline{2{\cdot}39\,\text{H}}}$$

(*c*) When the speed of the machine is doubled, this will double the value of the voltage coefficient. Substituting the given data for generator action in (10.22),

$$V_r = \{0{\cdot}1 \times (-200)\} + \left\{125 \times \frac{1{,}000}{500} \times 2\right\} = \underline{\underline{480\,\text{V}}}$$

Note that $I_r = -200\text{A}$ for generator action.

(*d*) From eqn. (10.25),

$$I_s = \frac{V_s}{r_s} = \frac{500}{250} = 2\,\text{A}$$

In eqn. (10.24),

$$1{,}000 = 2{\cdot}39 \times I_r \times 2 \qquad \text{so that} \qquad I_r = \frac{1{,}000}{2{\cdot}39 \times 2} = \underline{\underline{209\,\text{A}}}$$

If ω_{r2} is the new angular velocity, then from (10.22) the new voltage coefficient, $L_{sr}\omega_{r2}$, is

$$L_{sr}\omega_{r2} = \frac{V_r - r_r I_r}{I_s} = \frac{500 - (0{\cdot}1 \times 209)}{2} = 240\,\text{V/A}$$

The new speed is

$$n_2 = n_1\left(\frac{L_{sr}\omega_{r2}}{L_{sr}\omega_{r1}}\right) = 500 \times \frac{240}{125} = \underline{\underline{960\,\text{rev/min}}}$$

10.9 Shunt and Series D.C. Machines

The stator winding of a d.c. machine is usually excited from the same supply as the rotor winding. The stator winding may be connected in parallel with the rotor winding across the supply to form a d.c. *shunt machine* or in series with the rotor winding to form a d.c. *series machine*.

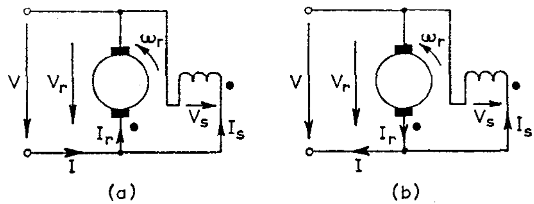


Fig. 10.14 D.C. SHUNT MACHINE

Fig. 10.14(*a*) shows the connexion diagram for a d.c. shunt machine operating in the motoring mode, and Fig. 10.14(*b*) shows it operating in the generating mode. The operating equations for the shunt machine may be obtained from those of the separately excited machine. In eqns. (10.22) and (10.23), putting $V_r = V$ and $V_s = V$ gives

$$V = r_r I_r + L_{rs}\omega_r I_r \tag{10.27}$$

$$V = r_s I_s \tag{10.28}$$

The torque equation remains unchanged as

$$T = L_{sr} I_r I_s \tag{10.24}$$

Referring to Fig. 10.14(*a*) for motor-mode operation,

$$I = I_r + I_s \tag{10.29}$$

The equations for generating action are obtained by putting $I_r = -I_r$ in eqns. (10.27) and (10.24). In addition, for generator action,

$$I_r = I + I_s \tag{10.30}$$

The characteristics of the shunt machine are similar to those of the separately excited machine shown in Figs. 10.12 and 10.13. The establishment of a stable output voltage for shunt generator operation requires some saturation of the magnetic circuit.

Fig. 10.15(*a*) shows the connexion diagram for a d.c. series machine operating in the motoring mode. The operating equations for the series machine may also be obtained from those of the separately excited machine. Substituting $I_r = I$ and $I_s = I$ in eqns. (10.22), (10.23) and (10.24) gives

$$V_r = r_r I + L_{sr}\omega_r I \tag{10.31}$$

$$V_s = r_s I \tag{10.32}$$

$$T = L_{sr} I^2 \tag{10.33}$$

The torque/current characteristic of the d.c. series motor is shown in Fig. 10.15(*b*). From Fig. 10.15(*a*),

$$V = V_r + V_s = r_r I + L_{sr}\omega_r I + r_s I$$

$$= (r_r + r_s)I + L_{sr}\omega_r I \tag{10.34}$$

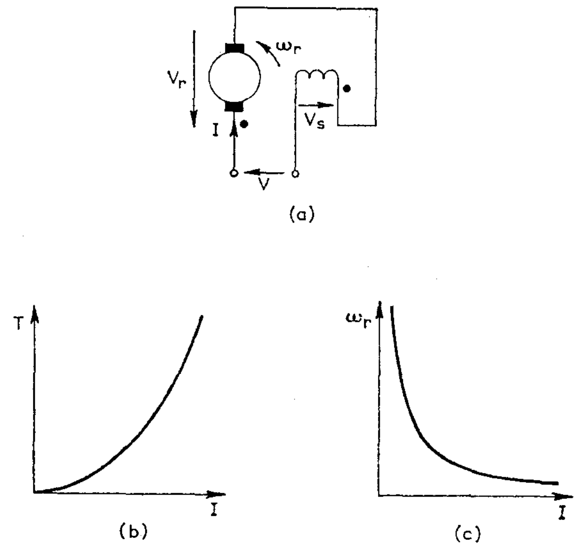


Fig. 10.15 D.C. SERIES MOTOR

From this equation,

$$\omega_r = \frac{V - (r_r + r_s)I}{L_{sr}I} \tag{10.35}$$

Since $(r_r + r_s)I$ is very much smaller than V, the speed of the d.c. series motor is approximately inversely proportional to the input current. Therefore, on light loads dangerously high speeds could be reached. In practical applications of the motor, protective devices are used to guard against this contingency. The speed/current characteristic is shown in Fig. 10.15(*c*).

The output voltage of a d.c. series generator is approximately proportional to the output current. The establishment of this output voltage also is dependent upon there being some saturation of the magnetic circuit.

10.10 Universal Motor

The universal motor is a series connected motor suitable for operation on either a.c. or d.c. supplies.

As previously, the inductance coefficients are

$$\mathscr{L}_{ss} = L_{ss} \tag{10.16}$$

$$\mathscr{L}_{sr} = L_{sr} \cos \theta_r \tag{10.4}$$

$$\mathscr{L}_{rr} = \tfrac{1}{2}(L_{dd} + L_{qq}) + \tfrac{1}{2}(L_{dd} - L_{qq}) \cos 2\theta_r \tag{10.6}$$

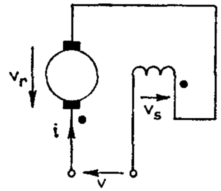


Fig. 10.16 A.C. SERIES MOTOR

To consider a.c. operation of the motor using the series connexion shown in Fig. 10.16, let the supply current be

$$i = i_r = i_s = I_m \cos \omega t$$

Following eqn. (10.9), the instantaneous rotor-winding voltage is

$$v_r = r_r i + \mathscr{L}_{rr}' \omega_r i + \mathscr{L}_{sr}' \omega_r i + \mathscr{L}_{rr} \frac{di}{dt} + \mathscr{L}_{sr} \frac{di}{dt} \tag{10.9}$$

The dotted end of the rotor winding is at $\theta_r = -\pi/2$; when substituted in the above equations this gives $\mathscr{L}_{rr}' = 0$, $\mathscr{L}_{sr}' = L_{sr}$, $\mathscr{L}_{rr} = \frac{1}{2}(L_{dd} + L_{qq}) = L_{rr}$, say, and $\mathscr{L}_{sr} = 0$. Eqn. (10.9) then becomes

$$v_r = r_r i + L_{sr}\omega_r i + L_{rr} \frac{di}{dt} \tag{10.36}$$

This equation may be written in complexor form. Let $\boldsymbol{V}_r$ be the complexor corresponding to v_r and $\boldsymbol{I}$ the complexor corresponding to i. Then

$$\boldsymbol{V}_r = r_r \boldsymbol{I} + L_{sr}\omega_r \boldsymbol{I} + j\omega L_{rr} \boldsymbol{I} \tag{10.37}$$

Due to the action of the commutator in fixing the axis of the rotor-winding m.m.f., no rotational voltages appear in the stator winding whether operation is from a d.c. or an a.c. supply. The instantaneous stator-winding voltage is therefore, from eqn. (10.10),

$$v_s = r_s i + \mathscr{L}_{ss}\frac{di}{dt} + \mathscr{L}_{sr}\frac{di}{dt} \tag{10.38}$$

$\mathscr{L}_{ss} = L_{ss}$, and with the rotor winding at $\theta_r = -\pi/2$, $\mathscr{L}_{sr} = 0$. This gives

$$v^s = r_s i + L_{ss}\frac{di}{dt} \tag{10.39}$$

Eqn. (10.38) written in complexor form gives

$$V_s = r_s I + j\omega L_{ss} I \tag{10.40}$$

If V is the complexor representing the supply voltage, then

$$V = V_r + V_s$$

i.e.

$$V = L_{sr}\omega_r I + [(r_r + r_s) + j\omega(L_{rr} + L_{ss})]I \tag{10.41}$$

The instantaneous torque developed is, from eqn. (10.14),

$$T' = \tfrac{1}{2}\mathscr{L}_{rr}'i_r^2 + \mathscr{L}_{sr}'i_r i_s + \tfrac{1}{2}\mathscr{L}_{ss}'i_s^2 \tag{10.14}$$

Substituting the conditions previously found ($\mathscr{L}_{rr}' = 0$, $\mathscr{L}_{sr}' = L_{sr}$, $\mathscr{L}_{ss}' = 0$, $i_r = i_s = i$) in eqn. 10.14 gives

$$T' = L_{sr}i^2 = L_{sr}I_m^2\cos^2\omega t = L_{sr}\frac{I_m^2}{2}(\cos 2\omega t + 1) \tag{10.42}$$

This equation therefore shows that for a.c. operation the torque developed by the machine consists of two components, a steady torque and one that pulsates at twice the supply frequency. The average torque is

$$T = L_{sr}I^2 \tag{10.43}$$

where I is the r.m.s. value of i. The torque/current characteristic is therefore the same as for d.c. operation.

Universal motors usually have a compensating winding on the stator with its m.m.f. axis coinciding with the rotor brush axis. The compensating winding is connected in series opposition with the rotor winding and serves to reduce the voltage drop in the internal reactance as well as assisting commutation. It has been neglected in the above analysis. A laminated stator construction is essential for a.c. operation.

EXAMPLE 10.4 A 0·1 kW series motor has the following constants:

Armature resistance	r_r	$= 12\Omega$
Series field resistance	r_s	$= 36\Omega$
Effective armature inductance	L_{rr}	$= 0{\cdot}3\,\text{H}$*
Series field inductance	L_{ss}	$= 0{\cdot}34\,\text{H}$
Maximum mutual inductance between rotor and stator windings	L_{sr}	$= 0{\cdot}71\,\text{H}$

Determine the input current and speed when the load torque applied to the motor is 0·18 N-m (*a*) when connected to a 200 V d.c. supply, and (*b*) when connected to a 200 V 50 Hz a.c. supply.

Neglect windage and friction and all core losses.

(*a*) Considering first d.c. operation, the applied voltage is

$$V = (r_r + r_s)I + L_{sr}\omega_r I \tag{10.34}$$

and the torque developed is

$$T = L_{sr}I^2 \tag{10.33}$$

From eqn. (10.33),

$$I = \sqrt{\frac{T}{L_{sr}}} = \sqrt{\frac{0{\cdot}18}{0{\cdot}71}} = \underline{\underline{0{\cdot}504\,\text{A}}}$$

From eqn. (10.34),

$$\omega_r = \frac{V - (r_r + r_s)I}{L_{sr}I} = \frac{200 - (48 \times 0{\cdot}504)}{0{\cdot}71 \times 0{\cdot}504} = 492\,\text{rad/s}$$

$$n_r = \frac{\omega_r}{2\pi} \times 60 = \frac{492}{2\pi} \times 60 = \underline{\underline{4{,}700\,\text{rev/min}}}$$

(*b*) Considering a.c. operation, the r.m.s. current is, from eqn. (10.43),

$$I = \sqrt{\frac{T}{L_{sr}}} = \sqrt{\frac{0{\cdot}18}{0{\cdot}71}} = \underline{\underline{0{\cdot}504\,\text{A}}}$$

The applied voltage is

$$V = L_{sr}\omega_r I + [(r_r + r_s) + j\omega(L_{rr} + L_{ss})]I$$

Taking I as the reference complexor,

$$200\underline{/\theta} = 0{\cdot}71 \times 0{\cdot}504\omega_r\underline{/0^\circ} + [48 + j2\pi \times 50(0{\cdot}3 + 0{\cdot}34)]0{\cdot}504\underline{/0^\circ}$$

$$200\cos\theta + j200\sin\theta = 0{\cdot}358\omega_r + 24{\cdot}2 + j101$$

Equating quadrate parts in this equation,

$$200\sin\theta = 101$$

whence $\sin\theta = 0{\cdot}505$, $\cos\theta = 0{\cdot}864$.

Equating reference parts,

$$200\cos\theta = 0{\cdot}358\omega_r + 24$$

$$\omega_r = \frac{200 \times 0{\cdot}864 - 24{\cdot}2}{0{\cdot}358} = 415\,\text{rad/s}$$

$$n_r = \frac{415}{2\pi} \times 60 = \underline{\underline{3{,}970\,\text{rev/min}}}$$

* The actual armature inductance $\approx L^2_{sr}/L_{ss} \approx 1{\cdot}48$ H. The effective value is reduced to 0·3 H due to the effect of a compensating winding connected in series opposition with the armature.

10.11 Rotating Field due to a Three-phase Winding

Fig. 10.17 shows a stator winding with three diametral coils aa', bb' and cc', each having N_s turns. The dots and crosses indicate the direction of conventionally positive current in each coil as explained

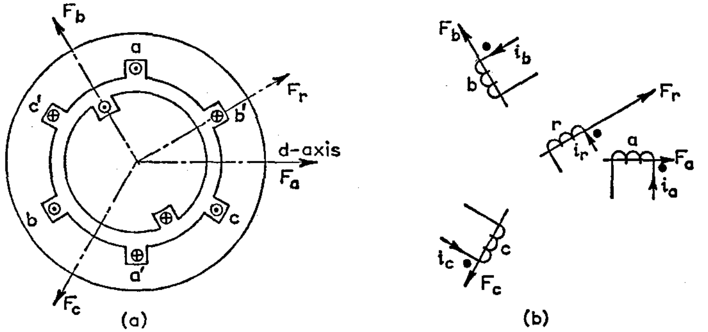


Fig. 10.17 M.M.F. DUE TO A 3-PHASE WINDING

in Section 10.3. The axes of the coil m.m.f.s are therefore mutually displaced by $2\pi/3$ radians, as shown in Fig. 10.17.

Suppose the three coils are supplied with balanced 3-phase currents, i_a, i_b and i_c, such that

$$i_a = I_{sm} \cos \omega t = \frac{I_{sm}}{2} (e^{j\omega t} + e^{-j\omega t}) \tag{10.44}$$

$$i_b = I_{sm} \cos (\omega t - 2\pi/3) = \frac{I_{sm}}{2} (e^{j(\omega t - 2\pi/3)} + e^{-j(\omega t - 2\pi/3)}) \tag{10.45}$$

$$i_c = I_{sm} \cos (\omega t + 2\pi/3) = \frac{I_{sm}}{2} (e^{j(\omega t + 2\pi/3)} + e^{-j(\omega t + 2\pi/3)}) \tag{10.46}$$

The m.m.f. of coil a is directed in the reference direction when i_a is positive. The instantaneous value of this m.m.f. is therefore

$$F_a' = \frac{I_{sm}N_s}{2} (e^{j\omega t} + e^{-j\omega t})e^{j0} \tag{10.47}^*$$

This expression has been multiplied by e^{j0} (= 1) to indicate that it acts in the space reference direction.

* To avoid confusion with f for frequency, instantaneous m.m.f. will be represented by F'.

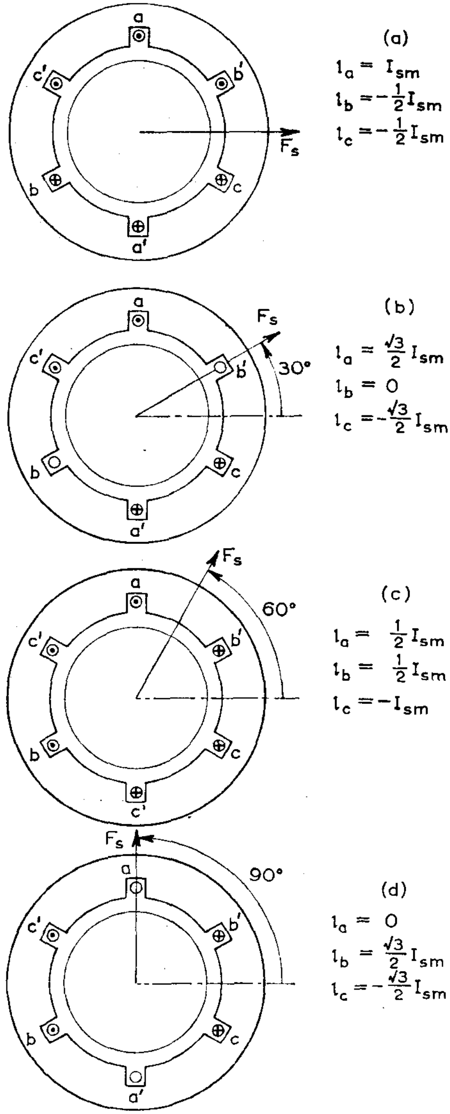


Fig. 10.18 M.M.F. DUE TO A 3-PHASE WINDING AT DIFFERENT INSTANTS

The m.m.f. of coil b is directed along an axis $+2\pi/3$ radians from the reference direction when i_b is positive. The instantaneous value of this m.m.f. is therefore

$$F_b' = \frac{I_{sm}N_s}{2}(e^{j(\omega t-2\pi/3)} + e^{-j(\omega t-2\pi/3)})e^{j2\pi/3}$$

$$= \frac{I_{sm}N_s}{2}(e^{j\omega t} + e^{-j(\omega t-4\pi/3)}) \tag{10.48}$$

Similarly the m.m.f. due to coil c at any instant is

$$F_c' = \frac{I_{sm}N_s}{2}(e^{j(\omega t+2\pi/3)} + e^{-j(\omega t+2\pi/3)})e^{-j2\pi/3}$$

$$= \frac{I_{sm}N_s}{2}(e^{j\omega t} + e^{-j(\omega t+4\pi/3)}) \tag{10.49}$$

The resultant stator m.m.f. due to all three coils is

$$F_s' = F_a' + F_b' + F_c'$$

$$= \frac{I_{sm}N_s}{2}[e^{j\omega t} + e^{-j\omega t} + e^{j\omega t} + e^{-j(\omega t-4\pi/3)} + e^{j\omega t} + e^{-j(\omega t+4\pi/3)}]$$

Since $e^{-j\omega t} + e^{-j(\omega t-4\pi/3)} + e^{-j(\omega t+4\pi/3)} = 0$,

$$F_s' = \tfrac{3}{2} I_{sm}N_s e^{j\omega t} \tag{10.50}$$

This equation shows that, when three coils are so positioned that their m.m.f. axes are mutually displaced by $2\pi/3$ radians and are then supplied with balanced 3-phase currents, an m.m.f. of constant magnitude results and the m.m.f. axis rotates at an angular velocity of ω radians per second.

For the coil configuration and phase sequence chosen the direction of rotation is in the $+\theta$ direction. It will be found that, if the phase sequence is reversed, the direction of rotation of the resultant m.m.f. axis is also reversed.

Fig. 10.18 shows the m.m.f. due to a 3-phase winding supplied with balanced 3-phase currents for a number of different instants. At (*a*) the current in phase a is positive maximum value and the currents in the two other phases are half the negative maximum value. The negative currents are indicated by showing the current in the cross direction in coil sides b and c, and in the dot direction in coil sides b' and c'. F_s is shown acting along the stator m.m.f. axis.

Figs. 10.18(*b*), (*c*) and (*d*) show successive instants in the 3-phase cycle corresponding to 30° rotations of the complexor diagram.

It will be seen that the axis of the stator m.m.f. is also displaced by successive steps of 30° in the $+\theta$ direction, so that F_s completes one revolution in each cycle and thus must rotate with an angular velocity of ω radians per second. This is in agreement with eqn. (10.50).

Eqn. (10.50) also shows that the m.m.f. due to a 3-phase winding when excited by balanced 3-phase currents could be represented as the m.m.f. of a single winding of N_s turns and excited with a direct current of value $\frac{3}{2}I_{sm}$, where the winding is considered to rotate at an angular velocity ω and N_s represents the number of turns of each stator phase.

In Chapter 11 the resultant m.m.f. due to 3-phase distributed windings is considered, and the effect of space harmonics is discussed. These are ignored in the present treatment.

10.12 Three-phase Synchronous Machine

In the previous section it has been shown that, when three coils have their m.m.f. axes mutually displaced by $2\pi/3$ and are then supplied with balanced 3-phase currents, an m.m.f. of constant magnitude results, the m.m.f. axis rotating at ω radians per second. If a constant angular displacement is to be maintained between the resultant stator and rotor m.m.f.s, as is required for the continuous production of torque, the rotor m.m.f. must also rotate at ω in the same direction as the stator m.m.f.

This rotation of the rotor m.m.f. may be brought about in a number of different ways. In the synchronous machine the rotor winding is excited with direct current supplied through slip rings. The axis of the rotor m.m.f. then rotates at the same speed as the rotor itself, so that the condition for continuous torque production is that the rotor should rotate at ω in the same direction as the resultant stator m.m.f. axis.

The rotors of synchronous machines are often of the salient-pole type shown in Fig. 10.5(*a*), but for simplicity only the non-salient pole type of rotor as shown in Fig. 10.5(*b*) will be considered. Instead of the stator phase windings consisting of the single coils considered in Section 10.11, the phase windings consist of several coils distributed in slots and occupying the whole stator periphery as shown in Fig. 10.19. The effect of this distribution of the winding is to introduce constants called *distribution factors* into equations relating to the operation of the machine. These constants are ignored here; in most practical cases they have numerical values close to unity.

The coupling between the d.c. excited rotor winding r (see Fig. 10.17) and the stator reference phase a is a cosine-like or even function which, ignoring space harmonics and taking θ_r as the instantaneous angle of the axis of the rotor winding with respect to phase a, is

$$\mathscr{L}_{ar} = L_{sr} \cos \theta_r \tag{10.51}$$

Since it is assumed that there are no salient poles, all the self-inductances are constant and their angular rates of change are zero.

As explained in the previous section, the 3-phase stator winding may be considered to be replaced by a representative stator windings of N_s turns, excited by a direct current of $\frac{3}{2}$ I_{sm} and rotating at an angular velocity of ω radians per second. The axes of both the stator

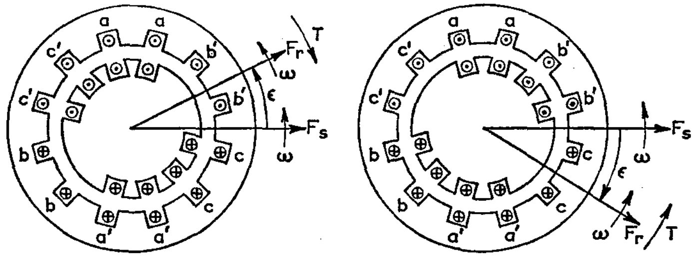


Fig. 10.19 SYNCHRONOUS MACHINE
(*a*) Generating (*b*) Motoring

and rotor windings therefore rotate at ω. If the angular displacement between these axes is ϵ as shown in Fig. 10.19, then, from eqn. (10.51),

$$\mathscr{L}_{sr} = L_{sr} \cos \epsilon \tag{10.52}$$

When ϵ is positive the rotor m.m.f. axis is displaced anticlockwise from the stator m.m.f. axis. The angular rate of change of this mutual inductance is

$$\mathscr{L}_{sr}' = -L_{sr} \sin \epsilon \tag{10.53}$$

The instantaneous torque on the rotor is given by eqn. (10.14) as

$$T' = \tfrac{1}{2}\mathscr{L}_{rr}' i_r^2 + \mathscr{L}_{sr}' i_r i_s + \tfrac{1}{2}\mathscr{L}_{ss}' i_s^2 \tag{10.14}$$

The rotor winding current i_r is I_r, a steady, direct current, and the representative stator winding carries a current of $\frac{3}{2}I_{sm}$, where I_{sm} is the maximum current per phase in the actual 3-phase winding. Substituting for these currents and for the angular rates of change of inductance in eqn. (10.14),

$$T' = -\tfrac{3}{2}L_{sr} I_r I_{sm} \sin \epsilon \tag{10.54}$$

For a steady angular displacement ϵ_s between the axes of the stator and rotor m.m.f.s, the mean torque on the rotor is

$$T = -\tfrac{3}{2}L_{sr}\, I_r I_{sm} \sin \epsilon_s \tag{10.55}$$

It is to be noted that under steady conditions the 3-phase machine, unlike the single-phase machine, does not produce an oscillating component of torque.

At starting, as a motor, the rotor angular velocity ω_r is zero, so that the displacement between the rotor and stator m.m.f. axes is $\epsilon = \omega t$, and the instantaneous torque on the rotor given by eqn. (10.54) is

$$T' = -\tfrac{3}{2}L_{sr} I_r I_{sm} \sin \omega t \tag{10.56}$$

The mean value of the torque given by this equation is zero, and since the inertia of the rotating system is too large to allow the rotor to respond to a torque which oscillates at mains frequency, the synchronous motor is not self-starting.

The currents in the actual 3-phase stator winding may be taken to be the same as those of Section 10.11:

$$i_a = I_{sm} \cos \omega t \tag{10.44}$$

$$i_b = I_{sm} \cos (\omega t - 2\pi/3) \tag{10.45}$$

$$i_c = I_{sm} \cos (\omega t + 2\pi/3) \tag{10.46}$$

For the stator phase currents so chosen, the stator m.m.f. axis at $t = 0$ is along the positive direction of the d-axis, and therefore ϵ_s is the angle of separation of the rotor and stator m.m.f. axes as shown in Fig. 10.19. When ϵ_s is positive the rotor m.m.f. axis is displaced anticlockwise from the stator m.m.f. axis, and as shown by eqn. (10.55), the torque on the rotor acts in the $-\theta$ direction, i.e. in the direction opposite to rotation. Under such circumstances the machine acts as a generator, the rotor being driven against the direction of the torque developed on it by a prime mover. When ϵ_s is negative the torque acts in the $+\theta$ direction, i.e. in the same direction as rotation, and the machine acts as a motor.

When the machine is unloaded $\epsilon_s = 0$, corresponding to the alignment of the rotor and stator m.m.f. axes. As load is imposed the value of ϵ_s increases, the rotor and stator m.m.f. axes are displaced and the appropriate torque is developed.

The operation of the synchronous machine is illustrated in Figs. 10.19(a) and (b). In both diagrams the stator current distribution is drawn for the instant in the 3-phase cycle when $i_a = I_{sm}$ and $i_b = i_c = -\tfrac{1}{2}I_{sm}$, so that the axis of the stator m.m.f. is along the positive direction of the d-axis. Fig. 10.19(a) illustrates generator action and Fig. 10.19(b) motor action.

If the stator phases b and c are assumed to be open-circuited, then the voltage applied to stator phase a may be obtained by adapting the subscripts of eqn. (10.10) as

$$v_a = ri_a + \mathscr{L}_{ar}'\omega_r i_r + \mathscr{L}_{aa}\frac{di_a}{dt} + \mathscr{L}_{ar}\frac{di_r}{dt} \tag{10.57}$$

$$\mathscr{L}_{ar} = L_{sr}\cos\theta_r \tag{10.51}$$

$$\mathscr{L}_{ar}' = -L_{sr}\sin\theta_r = -L_{sr}\sin(\omega_r t + \epsilon) \tag{10.58}$$

where ϵ is the position of the rotor winding axis at $t = 0$. Since the rotor winding is excited with direct current, $i_r = I_r$ and $di_r/dt = 0$. The current in phase a is, from eqn. (10.44), $i_a = I_{sm}\cos\omega t$. Substituting in eqn. (10.57),

$$\begin{aligned} v_a &= rI_{sm}\cos\omega t - \omega_r L_{sr} I_r \sin(\omega_r t + \epsilon) + L_{aa}\frac{d}{dt}(I_{sm}\cos\omega t) \\ &= rI_{sm}\cos\omega t + \omega_r L_{sr} I_r \cos(\omega_r t + \epsilon + \pi/2) \\ &\qquad + L_{aa}\frac{d}{dt}(I_{sm}\cos\omega t) \end{aligned} \tag{10.59}$$

Under normal operating conditions all three phases carry current, and under balanced conditions this has the effect of increasing the effective inductance per phase by approximately 50 per cent because of the mutual inductance between phases. If the effective inductance per phase is L_{ss}, eqn. (10.59) becomes

$$\begin{aligned} v_a &= rI_{sm}\cos\omega t + \omega_r L_{sr} I_r \cos(\omega_r t + \epsilon + \pi/2) \\ &\qquad + L_{ss}\frac{d}{dt}(I_{sm}\cos\omega t) \end{aligned} \tag{10.60}$$

In complexor form eqn. (10.60) becomes

$$V_s e^{j\phi} = \frac{\omega_r L_{sr} I_r}{\sqrt{2}} e^{j(\epsilon + \pi/2)} + (r + j\omega L_{ss})\frac{I_{sm}}{\sqrt{2}} e^{j0} \tag{10.61}$$

or

$$\boldsymbol{V}_s = \boldsymbol{E}_s + \boldsymbol{Z}_s\boldsymbol{I} \tag{10.62}$$

EXAMPLE 10.5 A 2-pole 1,000 V 50 Hz synchronous machine has a 3-phase star-connected stator winding each phase of which has an effective inductance of 0·01 H and negligible resistance. The maximum mutual inductance between the rotor winding and a stator phase is 0·4 H.

(a) Determine the developed torque, the stator phase current, the rotor winding current, the angle between the stator and rotor m.m.f. axes and the induced rotational voltage per phase when the machine acts as a motor with an output power of 224 kW. The input power factor is unity and the stator line voltage is 1,000 V. Neglect all losses.

(*b*) Determine the load current and output power when the machine acts as a generator if the rotor current is 12 A, the output power factor is 0·8 lagging and the stator line terminal voltage is 1,000 V. Find also the angle between the rotor and stator m.m.f. axes and the phase angle between the stator phase terminal voltage and the stator phase induced rotational voltage.

Neglect the effects of distribution of the windings and of magnetic saturation.

(*a*) Rotor angular velocity, $\omega_r = 2\pi f = 2\pi \times 50 = 314$ rad/s.

$$\text{Developed torque, } T = \frac{P}{\omega_r} = \frac{224{,}000}{314} = \underline{\underline{712\,\text{N-m}}}$$

Neglecting all losses,

$$\sqrt{3} V_L I_L \cos\phi = \text{Power output}$$

For the star connexion,

$$I_p = I_L = \frac{224{,}000}{\sqrt{3} \times 1{,}000 \times 1} = \underline{\underline{129\,\text{A}}}$$

From eqn. (10.61) the impedance per phase is

$$Z_s = r + j\omega L_{ss} = 0 + (j314 \times 0{\cdot}01) = j3{\cdot}14\,\Omega$$

From eqn. (10.62),

$$V = E + ZI$$

Taking the stator current as the reference complexor, and remembering that the input power factor is unity,

$$\frac{1{,}000}{\sqrt{3}}\underline{/0^\circ} = E\underline{/\epsilon + 90^\circ} + (j3{\cdot}14 \times 129)\underline{/0^\circ}$$

Therefore

$$E\underline{/\epsilon + 90^\circ} = 577 - j405 = 706\underline{/-35^\circ}$$

Thus the magnitude of the induced rotational voltage per phase is $\underline{\underline{706\,\text{V}}}$

The angle between the stator and rotor m.m.f. axes is

$$\epsilon = -35 - 90 = \underline{\underline{-125^\circ}}$$

From eqn. (10.55) the mean torque is

$$T = -\tfrac{3}{2} L_{sr} I_r I_{sm} \sin\epsilon \tag{10.55}$$

Thus the rotor current is

$$I_r = -\frac{712}{\frac{3}{2} \times 0{\cdot}4 \times \sqrt{2} \times 129 \sin(-125^\circ)} = \underline{\underline{7{\cdot}95\,\text{A}}}$$

As a check, from eqn. (10.73),

$$E = \frac{L_{sr}\omega_r I_r}{\sqrt{2}} = \frac{0{\cdot}4 \times 314 \times 7{\cdot}95}{\sqrt{2}} = 706\,\text{V}$$

(*b*) From eqn. (10.62),

$$I = \frac{V - E}{Z}$$

Taking the stator current as the reference complexor, then for an output power factor of 0·8 lagging, the phase voltage will *lead* the current by $\cos^{-1} 0{\cdot}8 = 36{\cdot}9°$. Therefore

$$I\underline{/0°} = \frac{\dfrac{577\underline{/+36{\cdot}9°} + 0{\cdot}4 \times 314 \times 12\underline{/\epsilon + 90°}}{\sqrt{2}}}{j3{\cdot}14}$$

and

$$I = 184\underline{/-53{\cdot}1°} - 339\underline{/\epsilon}$$
$$= 110 - j147 - (339 \cos \epsilon + j339 \sin \epsilon)$$

The quadrate part of the complex expression for the current is zero; hence

$$-147 - 339 \sin \epsilon = 0$$

$$\sin \epsilon = -\frac{147}{339} = -0{\cdot}433 \quad \epsilon = -25{\cdot}7° \quad \text{and} \quad \cos \epsilon = 0{\cdot}901$$

Thus

$$I = 110 - (339 \times 0{\cdot}901) = \underline{\underline{-196\,\text{A}}}$$

and

$$I = \underline{\underline{196\underline{/180°}\,\text{A}}}$$

The negative value of current corresponds to generating action and the reversal of current with respect to the terminal voltage. The axis of the stator m.m.f. at $t = 0$ is therefore at 180° whereas that of the rotor m.m.f. is at $\epsilon = -25{\cdot}7°$. The axis of the rotor m.m.f. is thus displaced from that of the stator m.m.f. by $180 - 25{\cdot}7 = 154{\cdot}3°$ in the $+\theta$ direction as is to be expected for generator action.

$$\text{Output power} = \frac{3 \times 577 \times 196 \times 0{\cdot}8}{1{,}000} = \underline{\underline{271\,\text{kW}}}$$

The phase angle between the induced rotational voltage per phase and the terminal voltage is

$$90 - 25{\cdot}7 - 36{\cdot}9 = \underline{\underline{27{\cdot}4°}}$$

10.13 Three-phase Induction Machine

The 3-phase induction machine has a uniformly slotted stator and rotor. The stator has a 3-phase winding like that of the synchronous machine. Unlike that of the synchronous machine the rotor winding is not excited with direct current and may be supposed to consist of short-circuited coils. The effects of distribution are again neglected, and the three stator phase windings and three rotor phase windings are each treated as if they were single, concentrated coils. The arrangement is shown in Fig. 10.20(*a*).

The stator winding is excited with balanced 3-phase currents, which, as explained in Section 10.11, set up an m.m.f. of constant magnitude the axis of which rotates at ω radians per second in the $+\theta$ direction for the 2-pole configuration shown in Fig. 10.20(*a*) and for the stator phase currents given by

$$i_a = I_{sm} \cos \omega t \tag{10.44}$$

$$i_b = I_{sm} \cos (\omega t - 2\pi/3) \tag{10.45}$$

$$i_c = I_{sm} \cos (\omega t + 2\pi/3) \tag{10.46}$$

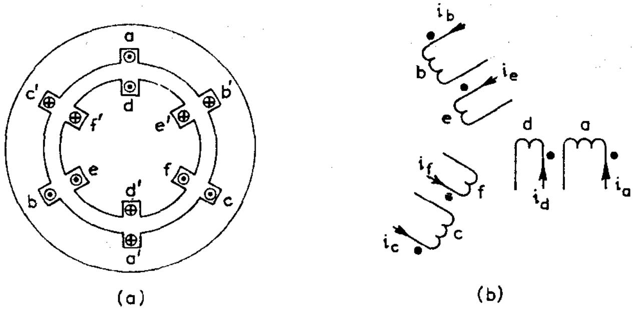


Fig. 10.20 MUTUAL INDUCTANCES OF THE 3-PHASE INDUCTION MACHINE

If the machine is to produce torque there must be a rotor winding current. Since the resultant stator m.m.f. is constant in magnitude and rotates at constant speed, a rotor winding current will only be obtained if the angular velocity of the rotor, ω_r, differs from that of the stator m.m.f., ω, because only in this way can a rotational voltage be obtained in the rotor winding.

The magnitude and frequency of the induced currents in the rotor windings are clearly proportional to $\omega - \omega_r$, the rotor angular velocity relative to that of the axis of the stator m.m.f. It is usual to analyse the action of induction machines in terms of the *per-unit slip*, *s*, which is defined as

$$s = \frac{\omega - \omega_r}{\omega} \tag{10.63}$$

The voltage applied to any winding is

$$v_j = r_j i_j + \sum_{k=1}^{k=n} \frac{d}{dt} (\mathscr{L}_{jk} i_k) \tag{10.11}$$

Each of the rotor coils is short-circuited so that the above equation may be used to obtain expressions for the rotor currents. For the configuration of Fig. 10.20, however, there are six self-inductances and nine mutual inductances. The resulting algebraic work though not difficult is extremely tedious. More sophisticated methods of analysis exist which greatly simplify the algebraic manipulation but are outside the scope of the present chapter.

Without carrying out the actual algebraic work, however, it is not difficult to anticipate the form of the expression for the current in any rotor phase.

Rotor current/phase $\propto$ Induced rotor e.m.f. per phase

Rotor e.m.f. per phase $\propto$ Relative angular velocity of stator m.m.f., $s\omega$
$\propto$ Maximum mutual inductance between a rotor and stator phase winding, L_{sr}
$\propto$ Maximum stator current per phase, I_{sm}.

The angular frequency of the rotor current per phase is also directly proportional to the relative angular velocity of the stator m.m.f., $s\omega$.

The current in phase d of the rotor winding is thus of the form

$$i_d = \frac{ks\omega L_{sr} I_{sm}}{Z_r} \cos\,(s\omega t - \phi_r + \alpha) \tag{10.64}$$

where k and α are constants, and Z_r is the rotor impedance per phase:

$$Z_r = r_r + js\omega L_{rr} \tag{10.65}$$

$$\phi_r = \tan^{-1} \frac{s\omega L_{rr}}{r_r} \tag{10.66}$$

L_{rr} is the total effective rotor inductance per phase. This includes a contribution due to mutual coupling with the two other rotor phases.

Analysis shows that $k = \frac{3}{2}$ and that if rotor phase d is at $\theta_r = 0$ at $t = 0$, then $\alpha = -\pi/2$. Therefore

$$i_d = \frac{\frac{3}{2} s\omega L_{sr} I_{sm}}{Z_r} \cos\,(s\omega t - \phi_r - \pi/2) \tag{10.67}$$

The currents in the two other rotor phases are such as to form, with that in rotor phase d, a balanced 3-phase system of currents. Therefore, as shown by eqn. (10.50), the 3-phase rotor winding will give rise to an m.m.f. of constant magnitude rotating at an angular velocity $s\omega$ in the $+\theta$ direction. Following eqn. (10.50), the rotor m.m.f. is

$$F_r = \tfrac{3}{2} I_{rm} N_r e^{j(s\omega t - \phi_r - \pi/2)} \tag{10.68}$$

where I_{rm} is the maximum rotor current per phase and N_r is the number of turns per rotor phase:

$$I_{rm} = \frac{\tfrac{3}{2} s\omega L_{sr} I_{sm}}{Z_r} \tag{10.69}$$

Since the rotor winding itself is rotating at $\omega_r = (1 - s)\omega$, the axis of the rotor m.m.f. has an angular velocity in space given by

Absolute angular velocity of rotor m.m.f.
$$= (1 - s)\omega + s\omega = \omega$$

i.e. the angular velocity of the rotor m.m.f. is constant and independent of rotor speed. The axes of the stator and rotor m.m.f.s both rotate at the same angular velocity with an angular displacement $-\phi - \pi/2$ (obtained by comparing eqns. (10.50) and (10.68)). The machine will therefore produce a steady torque.

Just as the m.m.f. due to a 3-phase winding when excited by balanced 3-phase currents can be represented as the m.m.f. of a single winding of N_s turns excited by a direct current of $3I_{sm}$, where the winding is considered to rotate at ω radians per second, so also may the m.m.f. due to a 3-phase rotor winding when excited by balanced 3-phase currents be represented as the m.m.f. of a winding of N_r turns carrying a direct current $\tfrac{3}{2}I_{rm}$ and rotating at an angular velocity ω. The mutual inductance between a stator phase a and a rotor phase r is

$$\mathscr{L}_{ar} = L_{sr} \cos \theta_r$$

The mutual inductance between the two equivalent windings of N_s and N_r turns carrying direct currents $\tfrac{3}{2}I_{sm}$ and $\tfrac{3}{2}I_{rm}$ respectively is

$$\mathscr{L}_{sr} = L_{sr} \cos \epsilon \qquad \text{whence} \qquad \mathscr{L}_{sr}' = -L_{sr} \sin \epsilon$$

The angular displacement of the axes of the stator and rotor m.m.f.s is

$$\epsilon = -\phi_r - \pi/2$$

The instantaneous torque developed on the rotor may be obtained from an application of eqn. (10.14):

$$\begin{aligned} T' &= \tfrac{1}{2}\mathscr{L}_{rr}' i_r^2 + \mathscr{L}_{sr}' i_r i_s + \tfrac{1}{2}\mathscr{L}_{ss}' i_s^2 \\ &= -L_{sr} \sin(-\phi_r - \pi/2) \tfrac{3}{2} I_{rm} \tfrac{3}{2} I_{sm} \end{aligned} \tag{10.14}$$

Substituting for I_{rm} from eqn. (10.69),

$$T' = -\left(\frac{3}{2}\right)^3 \frac{s\omega L_{sr}{}^2 I_{sm}{}^2}{Z_r} \sin(-\phi_r - \pi/2)$$

Since none of these terms varies with time the mean torque is

$$T = \left(\frac{3}{2}\right)^3 \frac{s\omega L_{sr}{}^2 I_{sm}{}^2 \cos\phi_r}{Z_r} \tag{10.70}$$

The per-unit slip, s, is positive when $\omega > \omega_r$, and when this condition obtains the torque developed on the rotor is positive and acts in the $+\theta$ direction, the direction of assumed rotor rotation. The machine therefore acts in the motoring mode for positive values of s. If the rotor is coupled to a prime mover and driven so that $\omega_r > \omega$ the slip and torque become negative and the machine acts in the generating mode.

Eqn. (10.70) shows that, like the 3-phase synchronous motor, the 3-phase induction motor produces a non-oscillatory torque. Unlike the 3-phase synchronous motor, however, the 3-phase induction motor is self-starting. At starting $\omega_r = 0$, and from eqn. (10.63),

$$s = \frac{\omega - 0}{\omega} = 1$$

Eqn. (10.70) shows that a torque will be developed on the rotor for this value of s.

PROBLEMS

10.1 An electrodynamic ammeter consists of a fixed coil and a moving coil connected in series. The self-inductance of the fixed coil is 400μH and that of the moving coil 200μH. The mutual inductance between the coils is

$$\mathscr{L}_{sr} = 100 \times 10^{-6} \cos\theta_r \quad \text{henry}$$

where θ_r is the position of the axis of the moving coil relative to that of the fixed one. The zero on the instrument scale corresponds to a position of the axis of the moving coil $\theta_r = 145°$. Full-scale deflection is 110° from the scale zero. The control constant is $5{\cdot}22 \times 10^{-5}$ N-m per degree of deflection. Determine the direct current required for full-scale deflection.

If an alternating current of 5 A r.m.s. and of frequency 50 Hz passes through the ammeter coils, what is the voltage drop across the instrument terminals? The resistance of the windings may be neglected.

Ans. 10 A; 0·91 V. (The angular deflection of the moving coil is approximately 48° from the instrument zero.)

10.2 A rotating relay consists of a stator coil of self-inductance 2·0 H and a rotor coil of self-inductance 1·0 H. The axes of the rotor and stator coils are displaced by 30°, and in this configuration the mutual inductance of the coils is 1·0 H. Neither stator nor rotor has salient poles. A current $i_s = 14{\cdot}14 \cos 314t$

amperes is passed through the stator coil. The rotor coil is short-circuited. Draw a diagram to show the relative directions of the stator and rotor coil currents and calculate the r.m.s. value of the rotor current. Determine also the r.m.s. value of voltage applied to the stator coil. Describe the direction in which the torque acts. The resistance of each winding can be neglected.

Ans. 10 A; 3,140 V. The torque acts so as to tend to align the coils.

10.3 A 2-pole d.c. machine has a field (stator) winding resistance of 200 Ω and an armature (rotor) resistance of 0·1 Ω. When operating as a generator the output voltage is 240 V when the armature winding current is 100 A, the field winding current 2 A, and the speed 500 rev/min.

Determine the armature current and speed when the machine is connected as a shunt motor to a 400 V d.c. supply and the total load torque imposed on the motor is 1,000 N-m. Assume that the machine is linear.

Ans. 209 A; 758 rev/min.

10.4 A 93 W 2-pole series motor has the following constants:

Armature resistance	r_r	= 12 Ω
Series field resistance	r_s	= 36 Ω
Effective armature inductance	L_{rr}	= 0·3 H
Series field inductance	L_{ss}	= 0·34 H
Maximum mutual inductance between rotor and stator windings	L_{sr}	= 0·71 H

Determine the speed, output power, input current and power factor when the motor is connected to a 200 V 50 Hz supply and the load torque applied is 0·25 N-m. Neglect windage and friction and iron losses.

Ans. 3,000 rev/min; 78·5 W; 0·594 A; 0·802 lagging.

10.5 A 100 V 3-phase 9 kVA 2-pole 50 Hz star-connected alternator has a total effective self-inductance per phase of 1·5 mH. The maximum mutual inductance between the rotor (field winding) and a stator phase winding is 90 mH. Determine the field current required to give an open-circuit line voltage of 100 V when the machine is driven at 3,000 rev/min. Find also, for this speed and field current, the terminal line voltage when the synchronous generator delivers rated full load current at (*a*) a power factor of 0·8 lagging; (*b*) unity power factor; (*c*) a power factor of 0·8 leading. The armature resistance per phase is negligible.

Ans. 2·89 A; 68·6 V; 89·1 V; 123 V.

10.6 The synchronous machine of Problem 10·5 is run as a motor connected to 3-phase 100 V 50 Hz busbars. Determine the field current required if the input power factor is to be unity when the gross torque imposed is 20 N-m. Neglect all losses.

Ans. 3·01 A.

Chapter 11

THREE-PHASE WINDINGS AND FIELDS

In an a.c. machine the armature (or main) winding may be either on the stator (i.e. the stationary part of the machine) or on the rotor, the same form of winding being used in each case. The simplest form of 3-phase winding has concentrated coils each spanning one pole pitch, and with the starts of each spaced 120° (electrical) apart on the stator or rotor. These coils may be connected in star or delta as required.

In most machines the coils are not concentrated but are distributed in slots over the surface of the stator or rotor, and it is this type of winding which will now be considered. The same type of winding is common to both synchronous and asynchronous (induction) machines.

11.1 Flux Density Distributions

In all a.c. machines an attempt is made to secure a sinusoidal flux density distribution in the air-gap. This may be achieved approximately by the distribution of the winding in slots round the air-gap or by using salient poles with shaped pole shoes.

In Fig. 11.1(*a*) a section of a multipolar machine is shown. If the flux density in the air-gap is to be sinusoidally distributed, the flux density must be zero on the inter-polar axes such as OA, OC and OE, and maximum on the polar axes OB and OD. Since successive poles are of alternate north and south polarities, the maximum flux densities along OB and OD are oppositely directed. Thus a complete cycle of variation of the flux density takes place in a

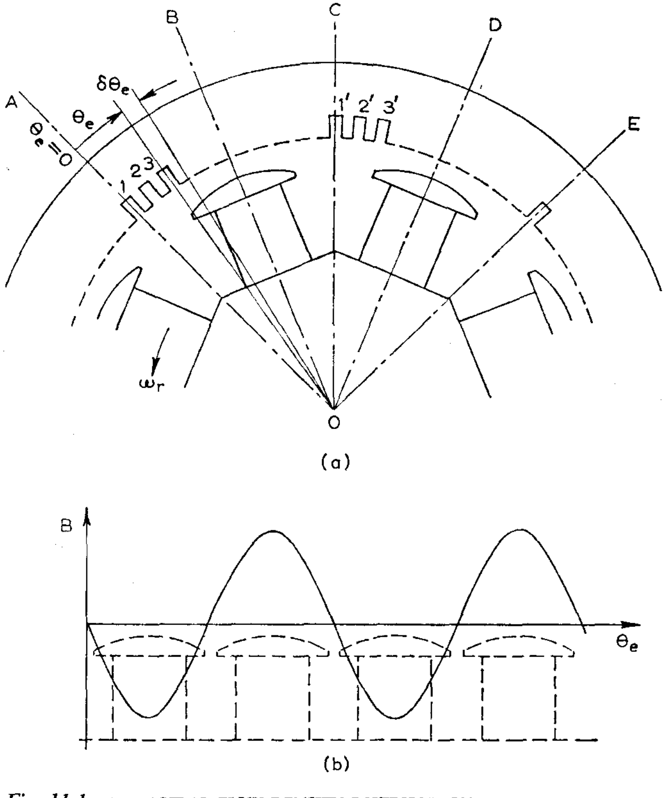


Fig. 11.1 SINUSOIDAL FLUX DENSITY DISTRIBUTION

double pole pitch from the axis OA to the axis OE. This is shown in Fig. 11.1(*b*).

Taking axis OA as the datum for angular measurements, the flux density at any point in the air-gap is

$$B = B_m \sin \theta_e \tag{11.1}$$

where θ_e is the angle from the origin measured in electrical radians or electrical degrees. Since one cycle of variation of the flux density occurs in a double pole pitch,

1 double pole pitch $\equiv 2\pi$ electrical radians or 360 electrical degrees

If the machine has $2p$ poles or p double pole pitches,

$$\theta_e = p\theta_m \qquad (11.2)$$

where θ_m is the angular measure in mechanical radians or degrees.

11.2 Three-phase Single-layer Concentric Windings

The two sides of an armature coil must be placed in slots which are approximately a pole pitch (180 electrical degrees) apart so that the e.m.f.s in the coil sides are cumulative. In addition, in 3-phase machines the starts of each phase winding must be 120 electrical degrees apart.

In single-layer windings one coil side occupies the whole of a slot. As a result, difficulty is experienced in arranging the end connectors, or overhangs. In concentric and split-concentric windings differently shaped coils having different spans are necessary. To preserve e.m.f. balance in each of the phases, each phase must contain the same number of each shape of coil.

Fig. 11.2(*a*) represents a developed stator with 24 stator slots, and it is desired to place a 4-pole 3-phase concentric winding in them:

$$\text{Number of slots per pole} = \frac{24}{4} = 6$$

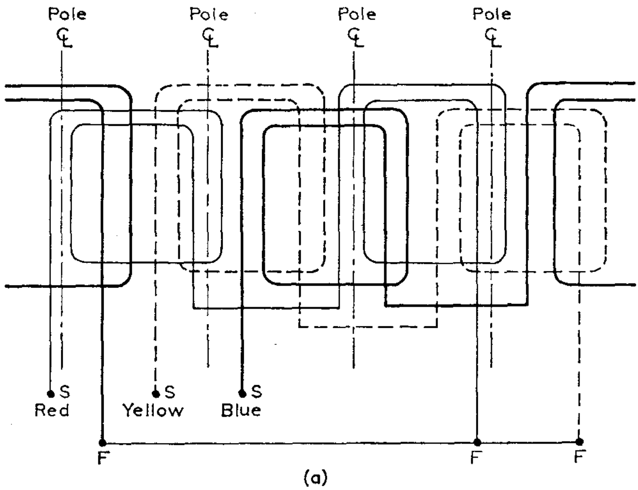
$$\text{Number of slots per pole and phase} = \frac{24}{4 \times 3} = 2$$

Fig. 11.2(*a*) shows the coil arrangement for the red phase as a thin full line. The start and finish (marked S and F respectively) of the phase winding are brought out, all the coils in the one phase being connected in series. For a phase sequence RYB, the yellow phase (shown dotted) must start 120 electrical degrees after the red phase. One pole pitch contains six slots and is equivalent to 180 electrical degrees. Hence a slot pitch is equivalent, in this case, to 30 electrical degrees.

The red phase starts in slot 1 and therefore the yellow phase must start in slot 5. In the same way the blue phase is 240 electrical degrees out of space phase with the red phase. The blue phase must therefore start in slot 9.

In Fig. 11.2 the finishes of the three phases have been commoned, making a star-connected winding. It would have been equally correct to common the three starts. The winding might also have been mesh-connected, in which case the finish of the red phase would have been connected to the start of the yellow phase, the finish of the yellow to the start of the blue, the finish of the blue to the start of the red, three connectors to the three junctions being brought out to terminals.

It will be observed that each phase has coils of each of the four different sizes used, thus maintaining balance between the phases.



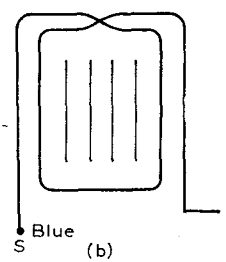


Fig. 11.2 FOUR-POLE 3-PHASE SINGLE-LAYER CONCENTRIC WINDING

It will also be seen that a coil group of any one phase consists of two coils per double pole pitch, one coil being greater than a pole pitch by one slot pitch and the other being less than a pole pitch by the same amount. If the end connexions of these two coils were crossed over as shown in Fig. 11.2(*b*) two full-pitch coils (i.e. having a span of exactly one-pole pitch) would be formed. Therefore each such coil group is the equivalent, electrically, of two full-pitch coils joined in series. All single-layer windings are effectively composed of full-pitch coils.

11.3 Three-phase Single-layer Mush Winding

Fig. 11.3 shows a 4-pole 3-phase single-layer mush winding. The distinctive feature of the mush winding is the utilization of constant-span coils. The overhangs are arranged in a similar manner to those of a conventional double-layer winding.

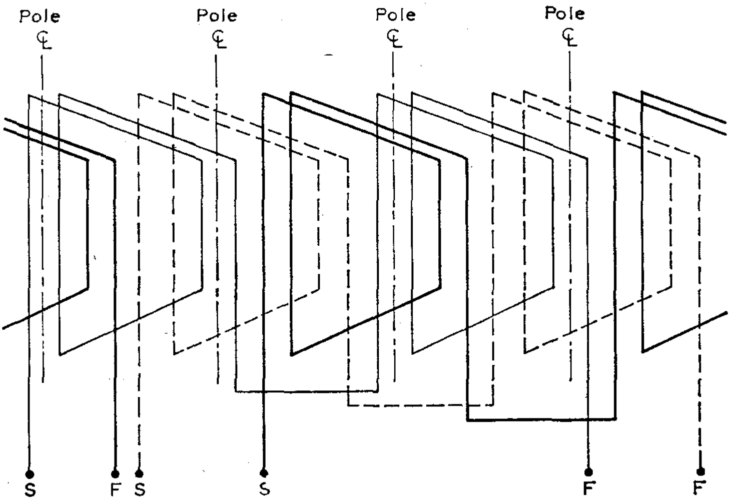


Fig. 11.3 FOUR-POLE 3-PHASE SINGLE-LAYER MUSH WINDING

11.4 Three-phase Double-layer Windings

The double-layer windings used in 3-phase machines are essentially similar to those used in d.c. machines except that no connexions to a commutator are required.

Since each phase must be balanced, all must contain equal numbers of coils and the starts of each phase must be displaced by 120 electrical degrees. If a number of groups of coils are to be connected in parallel, then similar parts in the winding at equal potentials must be available, a condition obtainable only in machines having a number of poles divisible by three when a wave winding is used.

On the other hand, tooth ripple, which arises where there are an integral number of slots per pole, resulting in the same relative positions of equivalent slots under each pole, may be avoided in double-layer windings by the use of winding pitches different from the pole pitch, thus giving a fractional number of slots per pole. A further advantage of the double-layer winding is the possibility of using constant-span coils. Only single-layer windings are considered in the rest of this chapter.

11.5 E.M.F. Induced in a Full-pitch Coil

Consider a full-pitch coil C with coil sides lying in slots 3 and 3′ as shown in Fig. 11.1. Let the coil side in slot 3 lie at θ_e so that the coil side in slot 3′ lies at $\theta_e + \pi$ electrical radians, where θ_e is measured from the interpolar axis OA. Let the stator diameter be D and the effective stator length L. Assume that the flux density distribution is sinusoidal, i.e. that

$$B = B_m \sin \theta_e \tag{11.1}$$

The flux in the stator segment between θ_e and $\theta_e + \delta\theta_e$ is

$$\delta\phi = BL\frac{D}{2}\,\delta\theta_m = B_m \sin\theta_e\, L\frac{D}{2}\frac{\delta\theta_e}{p}$$

The total flux linked with coil C is

$$\phi = \frac{B_m LD}{2p}\int_{\theta_e}^{\theta_e+\pi} \sin\theta_e\, d\theta_e$$

$$= +\frac{B_m LD}{2p}\,2\cos\theta_e \tag{11.3}$$

If a coil lies with its sides on the interpolar axes, as, for example, the coil lying in slots 1 and 1′ of Fig. 11.1, then the coil links the total flux per pole, Φ:

$$\Phi = \frac{B_m LD}{2p}\int_0^{\pi} \sin\theta_e\, d\theta_e$$

$$= +\frac{B_m LD}{2p}\,2 \tag{11.4}$$

The flux linked with coil C is therefore, by substitution in eqn. (11.3),

$$\phi = \Phi\cos\theta_e \tag{11.5}$$

Suppose the pole system rotates in the direction shown at a uniform angular velocity

$$\omega_r = 2\pi n_0 \quad \text{radians/second} \tag{11.6}$$

where n_0 is the rotor speed in revolutions per second. The position of any coil such as C at any instant, in electrical radians, is

$$\theta_e = \omega t + \theta_0$$

where θ_0 is the position of the coil at $t = 0$, and

$$\omega = p\omega_r = 2\pi n_0 p \quad \text{electrical radians/second} \tag{11.7}$$

Substituting for θ_e in eqn. (11.5), the flux linking any coil such as C at any time t is

$$\phi = \Phi \cos(\omega t + \theta_0) \tag{11.8}$$

The e.m.f. induced in any coil of N_c turns is

$$e = N_c \frac{d\phi}{dt}$$

$$= N_c \frac{d}{dt} \{\Phi \cos(\omega t + \theta_0)\}$$

$$= -\omega \Phi N_c \sin(\omega t + \theta_0)$$

The r.m.s. coil e.m.f. is therefore

$$E_c = \frac{\omega \Phi N_c}{\sqrt{2}} \tag{11.9}$$

Thus a sinusoidal flux density distribution in space may give rise to an e.m.f. induced in a coil which varies sinusoidally with time. This is achieved by giving the coil and the flux density distribution a constant relative angular velocity.

The frequency of the induced e.m.f. is

$$f = \frac{\omega}{2\pi} = \frac{2\pi n_0 p}{2\pi} = n_0 p \tag{11.10}$$

n_0 is called the *synchronous speed.* In this equation it is measured in revolutions per second.

11.6 Distribution (or Breadth) Factor and E.M.F. Equation

Suppose that under each pole pair each phase of the winding has g coils connected in series, each coil side being in a separate slot. The e.m.f. per phase and pole pair is the complexor sum of the coil voltages. These will not be in time phase with one another since successive coils are displaced round the armature, and hence will not be linked by the same value of flux at the same instant. $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, $\boldsymbol{E}_3 \ldots \boldsymbol{E}_g$ (as shown in Fig. 11.4(*a*)) represent the r.m.s. values of the e.m.f.s in successive coils. The phase displacement between successive e.m.f.s is ψ, which depends on the electrical angular displacement between successive slots on the armature.

Suppose the machine has a total of S slots and $2p$ poles. Then

$$\text{Number of slots per pole} = \frac{S}{2p}$$

The slot pitch (electrical angle between slot centre lines) is

$$\psi = \frac{180°_e}{S/2p} \quad \text{(since 1 pole pitch} = 180°_e) \tag{11.11}$$

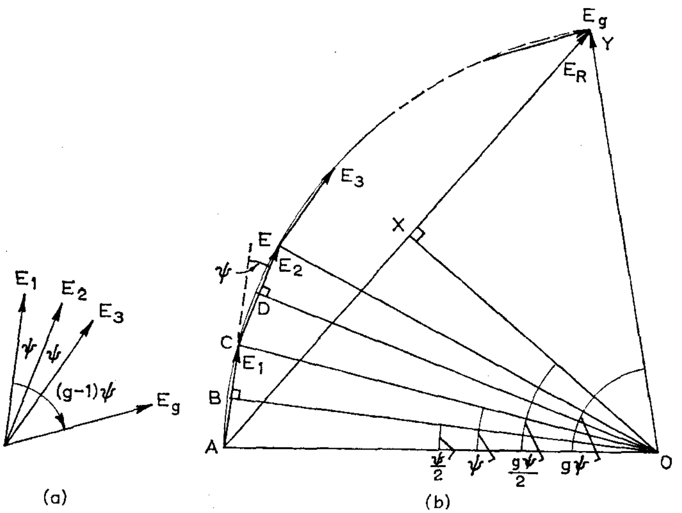


Fig. 11.4 DERIVATION OF DISTRIBUTION FACTOR

(*a*) Complexor diagram of slot e.m.f.s
(*b*) Resultant of slot e.m.f.s

The e.m.f. complexors $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, $\boldsymbol{E}_3$ . . . $\boldsymbol{E}_g$ are placed end to end in order in Fig. 11.4(*b*). The resultant complexor $\boldsymbol{E}_R$, represents the complexor sum of the e.m.f.s of the g coils connected in series.

Since the complexors $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, $\boldsymbol{E}_3$ . . . $\boldsymbol{E}_g$ are all of the same length and are displaced from one another by the same angle, they must be successive chords of the circle whose centre is O in Fig. 11.4(*b*). The complexor sum AY may be found as follows.

Join OA, OC, OE, etc., draw the perpendicular bisectors of each chord (i.e. OB, OD, etc.) and also the perpendicular bisector OX of the chord AY.

In the triangle AOX,

$$\text{AX} = \text{AO} \sin \text{AOX} = \text{AO} \sin g\frac{\psi}{2}$$

Therefore

$$\text{AY} = 2\text{AO} \sin g \frac{\psi}{2}$$

In the triangle AOB,

$$\text{AB} = \text{AO} \sin \text{AOB} = \text{AO} \sin \frac{\psi}{2}$$

$$\text{AC} = 2\text{AB} = 2\text{AO} \sin \frac{\psi}{2}$$

Therefore

$$\frac{\text{AY}}{\text{AC}} = \frac{E_R}{E_1} = \frac{\sin g \dfrac{\psi}{2}}{\sin \dfrac{\psi}{2}}$$

Thus the *distribution factor* is

$$K_d = \frac{\text{Complexor sum of coil e.m.f.s}}{\text{Arithmetic sum of coil e.m.f.s}}$$

$$= \frac{E_R}{gE_1} = \frac{\sin g \dfrac{\psi}{2}}{g \sin \dfrac{\psi}{2}} \qquad (11.12)$$

The product $g\psi$ represents the electrical angle over which the conductors of one phase are spread under any one pole and is referred to as the *phase spread.* In a 3-phase single-layer winding each phase has two phase spreads under each pole pair. Therefore, for a single-layer 3-phase winding,

$$g\psi = \frac{360}{2 \times 3} = 60°_e \qquad \text{or} \qquad \pi/3 \text{ electrical radians}$$

Clearly the highest value which the distribution factor K_d can have is unity, corresponding to a situation where there is one coil per pole pair and phase. A lower limit for the value of K_d also exists. Thus, if the number of separate slots g in the phase spread $g\psi$ is considered to increase without limit, then

$$\psi \to 0 \qquad \text{and} \qquad \sin \frac{\psi}{2} \to \frac{\psi}{2}$$

A 3-phase winding with a phase spread of $60°_e$ is said to be *narrow spread.*

For a narrow-spread 3-phase winding ($g\psi = \pi/3$),

$$\lim_{\psi\to 0} K_d = \frac{\sin\dfrac{g\psi}{2}}{g\dfrac{\psi}{2}} = \frac{\sin \pi/6}{\pi/6} = \frac{3}{\pi} \tag{11.13}$$

A winding having this limiting condition is called a *uniform winding*, and in such winding the phase spreads may be thought of as current sheets with the effect of the slotting eliminated.

The lower limit of K_d for a 3-phase narrow-spread winding ($3/\pi = 0{\cdot}955$), corresponding to a very large number of slots per pole and phase, shows that the distribution of the winding will have little effect on the magnitude of the fundamental e.m.f. per phase.

Ideally the flux density distribution linking the winding should be sinusoidal. In practice this ideal is not usually achieved; the air-gap flux density distribution is then of the form

$$B = B_{m1} \sin \theta_e + B_{m3} \sin (3\theta_e + \epsilon_3) + \ldots B_{mn} \sin (n\theta_e + \epsilon_n) \tag{11.14}$$

In this expression the first term on the right-hand side is called the *fundamental space distribution.* The other terms are referred to as *space harmonics.* The nth space harmonic goes through n cycles of variation for one cycle of variation of the fundamental. Only odd space harmonics are present since the flux density distribution repeats itself under each pole and is therefore symmetrical.

Just as the fundamental flux density gives rise to a fundamental e.m.f. induced in a coil, so the nth space harmonic in the flux density distribution will give rise to an nth time harmonic in the coil e.m.f. The distribution factor for the nth harmonic is

$$K_{dn} = \frac{\sin\dfrac{gn\psi}{2}}{g\sin\dfrac{n\psi}{2}} \tag{11.15}$$

Although the distribution of the winding has little effect on the magnitude of the fundamental, it may cause considerable reduction in the magnitude of harmonic e.m.f.s compared with those occurring in a winding for which $g = 1$, i.e. one coil per pole pair and phase.

11.7 Coil-span Factor

The e.m.f. equation of Section 11.5 has been deduced on the assumption of full-pitch coils, i.e. coils whose sides are separated by one pole pitch. As has been pointed out, the coils in double-layer windings are often made either slightly more or slightly less than a pole pitch. Fig. 11.5 illustrates coils with various pitches.

If the coil has a pitch of exactly one pole pitch, it will at some instant link the entire flux of a rotor pole. If the coil pitch is less than one pole pitch, it will never link the entire flux of a rotor pole and the maximum coil e.m.f. will be reduced. If the coil pitch is greater than one pole pitch, the coil must always be linking flux

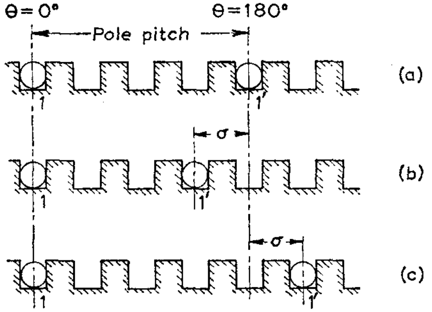


Fig. 11.5 COIL SPANS

(*a*) Full pitch
(*b*) Short pitch
(*c*) Over-full pitch

from at least two adjacent rotor poles so that the net flux linked will be less than the flux of one pole and the maximum coil e.m.f. will again be reduced.

The factor by which the e.m.f. per coil is reduced is called the *coil span factor*, K_s:

$$K_s = \frac{\text{E.M.F. in the short or long coil}}{\text{E.M.F. in a full-pitched coil}} \tag{11.16}$$

The magnitude of the coil span factor may most readily be obtained by considering the e.m.f. induced in each coil side, namely

$$e = Blv \text{ volts}$$

where B = air-gap flux density, l = active conductor length and v = conductor velocity at right angles to the direction of B.

This e.m.f. will have the same waveform as the flux density in the air-gap, since l and v are constant, and hence if the flux density is sinusoidally distributed the e.m.f. in each conductor will be sinusoidal so that the resultant coil e.m.f. will also be sinusoidal. If the pitch is short or long by an electrical angle σ, then, assuming a sinusoidal flux density distribution, the e.m.f.s in each side of the coil will differ in phase by σ but will have the same r.m.s. value. The resultant coil e.m.f. will be the complexor sum of the e.m.f.s in each coil side, as shown in Fig. 11.6.

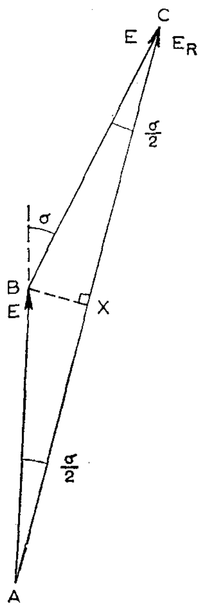


Fig. 11.6 DERIVATION OF COIL SPAN FACTOR

$$\text{Resultant e.m.f.} = \text{AC} = 2\text{AB} \cos \frac{\sigma}{2}$$

$$\text{E.M.F. for a full-pitch coil} = 2\text{AB}$$

Therefore

$$K_s = \frac{2\text{AB} \cos \frac{\sigma}{2}}{2\text{AB}} = \cos \frac{\sigma}{2} \tag{11.17}$$

If the flux density distribution contains space harmonics, the coil span factor for the *n*th harmonic e.m.f. is

$$K_{sn} = \cos \frac{n\sigma}{2} \tag{11.18}$$

All single-layer windings are effectively made up of full-pitch coils, but double-layer windings usually have short-pitched or short-chorded coils. The nth harmonic coil e.m.f. is reduced to zero if the *chording angle*, σ, is such that

$$\cos\frac{n\sigma}{2} = 0$$

or

$$\frac{n\sigma}{2} = 90°_e \tag{11.19}$$

This enables windings to be designed which will not permit specified harmonics to be generated (e.g. if $\sigma = 60°_e$ there can be no third-harmonic generation).

11.8 E.M.F. Induced per Phase of a Three-phase Winding

Following eqn. (11.9) the r.m.s. e.m.f. induced in a full-pitch coil of N_c turns due to its angular velocity relative to the pole system is

$$E_c = \frac{\omega\Phi N_c}{\sqrt{2}} \tag{11.9}$$

For a coil-span factor, K_s, due to chording,

$$E_c = K_s\frac{\omega\Phi N_c}{\sqrt{2}}$$

Further, if there are g coils in a phase group under a pole pair the resultant complexor sum is

$$E_g = K_d g E_c = K_d K_s g\frac{\omega\Phi N_c}{\sqrt{2}}$$

Assuming that the e.m.f.s of coil groups of the same phase under successive pole pairs are in phase and connected in series, the e.m.f. per phase is

$$E_p = pEg = pK_d K_s g\frac{\omega\Phi N_c}{\sqrt{2}}$$

or

$$E_p = K_d K_s\frac{\omega\Phi N_p}{\sqrt{2}} \tag{11.20}$$

where the number of turns per phase, N_p, is pgN_c.

This equation is sometimes written in the form

$$E_p = 4{\cdot}44\, K_d K_s f\Phi N_p \tag{11.21}$$

since $\omega = 2\pi f$ and $2\pi/\sqrt{2} = 4{\cdot}44$.

Sometimes the conductors per phase rather than the turns per phase are specified, in which case eqn. (11.21) becomes

$$E_p = 2{\cdot}22\ K_s K_d f \Phi Z_p \tag{11.22}$$

since $N_p = Z_p/2$.

The line voltage will depend on whether the winding is star or delta connected.

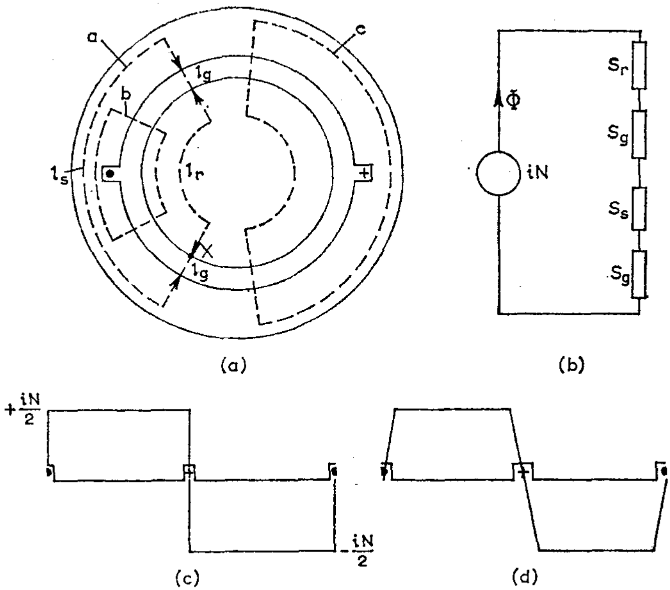


Fig. 11.7 M.M.F. DUE TO A FULL-PITCH COIL

11.9 M.M.F. due to a Full-pitch Coil

Fig. 11.7(*a*) shows a stator and rotor separated by a uniform air-gap, i.e. one whose radial length l_g is constant. The stator has two diametrically opposite slots in which one stator coil of N turns carries a current i. The slot opening is assumed to be very small compared with the internal circumference of the stator.

Consider the closed path a of Fig. 11.7(*a*) and let distances be measured from point X on this path. Now,

$$\text{M.M.F.} = iN = \oint H\,dl$$

Assuming that the reluctance of the rotor and stator core paths is zero, and that the magnetic field strength in the gap, H_g, is constant along the radial length l_g, then

$$iN = 2H_g l_g$$

or

$$H_g l_g = \frac{iN}{2} \tag{11.23}$$

Fig. 11.7(*b*) shows the equivalent magnetic circuit with path reluctances S_r (rotor), S_s (stator), S_g (air-gap). It will readily be confirmed that, if $S_r = 0$ and $S_s = 0$, the magnetic potential difference across each of the two equal reluctances S_g is $\frac{1}{2}iN$.

It is clear that, adhering to the assumptions of zero reluctance in the rotor and stator core and constant field strength in the air gap, the same result as that of eqn. (11.23) is obtained for other paths of integration such as *b* or *c* in Fig. 11.7(*a*). Indeed the magnetic potential drop across the air gap is $\frac{1}{2}iN$ at all points. The magnetic potential drop across the air-gap is, for the direction of coil current chosen, directed from rotor to stator for the upper half of the stator, and from stator to rotor for the lower half of the stator in this case.

Fig. 11.7(*c*) shows a graph of air-gap magnetic potential difference plotted to a base of the developed stator surface. The magnetic potential difference has been arbitrarily assumed positive when it is directed from rotor to stator and shown above the datum line. It is therefore taken to be negative when directed from stator to rotor and shown below the datum line. The magnetic potential difference is shown as changing abruptly from $+\frac{1}{2}iN$ to $-\frac{1}{2}iN$ opposite the slot opening. This corresponds to the situation where the slot is extremely thin.

Althongh Fig. 11.7(*c*) is properly described as showing the variation of air-gap magnetic potential difference to a base of the developed stator surface, such a diagram is often called an m.m.f. wave diagram, and the quantity $\frac{1}{2}iN$ is often called the m.m.f. per pole.

Where the width of the slot opening is not negligible the m.m.f. wave for a coil may be taken to be trapezoidal as shown in Fig. 11.7(*d*).

11.10 M.M.F. due to One Phase of a Three-phase Winding

Fig. 11.8(*a*) shows the coil for a double pole pitch of one phase of a 3-phase concentric winding of the type illustrated in Fig. 11.2. The position of the stator slots and coils is indicated on a developed diagram of the stator slotting. The start of the red phase winding is shown with a current emerging from the start end of the winding. This is taken as a conventionally positive current for generator

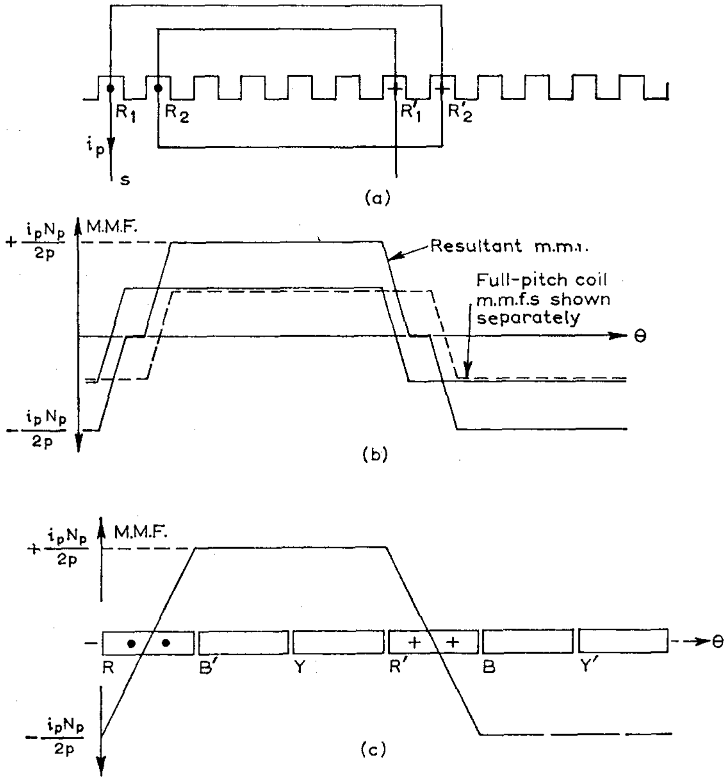


Fig. 11.8 M.M.F. DUE TO ONE PHASE OF A 3-PHASE WINDING

action. Conventionally positive current for motor action enters the start end of the winding.

As has been explained previously, the concentric coil arrangement shown is equivalent to an arrangement of two full-pitch coils where the coil sides in slots R_1 and R_1' and the coil sides in R_2 and R_2' are joined. Fig. 11.8(*b*) shows the m.m.f. for each of such coils separately and also their resultant obtained by adding together the two separate m.m.f. waves. The convention regarding positive m.m.f. explained in the previous section has been adhered to.

The resultant m.m.f. shown in Fig. 11.8(*b*) is stepped, owing to the effect of the discrete coils. Fig. 11.8(*c*) shows the m.m.f. per phase when the effect of discrete coils is ignored. The rectangular blocks represent the phase spreads, and these are considered to extend over both the regions previously occupied by slots and by teeth. The phase spread containing the start of the phase winding is identified by the unprimed letter R. The other phase spread of the same phase is marked R′. The phase current is considered to be uniformly distributed in the block representing the phase spread. Such a winding is a uniformly distributed winding as described in Section 11.6, and the m.m.f. per phase for such a winding is of the trapezoidal shape shown.

The maximum value of the m.m.f. wave at any instant is the m.m.f. per pole for the phase considered. For N_p total turns per phase the m.m.f. per phase and pole is $i_pN_p/2p$.

If a sinusoidal alternating current $i_p = I_{pm} \sin \omega t$ flows in the phase winding, the maximum value of the m.m.f. wave will vary sinusoidally:

$$\frac{i_pN_p}{2p} = \frac{I_{pm}N_p \sin \omega t}{2p}$$

In subsequent work the m.m.f. due to uniform windings only will be considered.

11.11 M.M.F. due to a Three-phase Winding (graphical treatment)

Fig. 11.9 shows the m.m.f.s for each phase of a 3-phase winding carrying balanced 3-phase currents for two different instants in the current cycle. The resultant m.m.f., due to the combined action of the separate phases, is also shown in each diagram.

Fig. 11.9(*a*) is drawn for the instant when the instantaneous currents in the three phases are

$$i_r = I_{pm}$$
$$i_y = -\tfrac{1}{2}I_{pm}$$
$$i_b = -\tfrac{1}{2}I_{pm}$$

The current in the red phase is positive, so according to the convention for positive current explained in Section 11.10, phase spread R has the current direction indicated by a dot and phase spread R′ has the current direction indicated by a cross. The red phase m.m.f., F_r, therefore has the trapezoidal distribution shown having a maximum value of

$$\frac{i_pN_p}{2p} = \frac{I_{pm}N_p}{2p} = F_{pm}$$

where F_{pm} is the maximum m.m.f. per phase and pole.

The currents in the yellow and blue phases are both negative so that the Y and B phase spreads have crosses, and the phase spreads Y′ and B′ have dots, to show the current direction. The m.m.f.

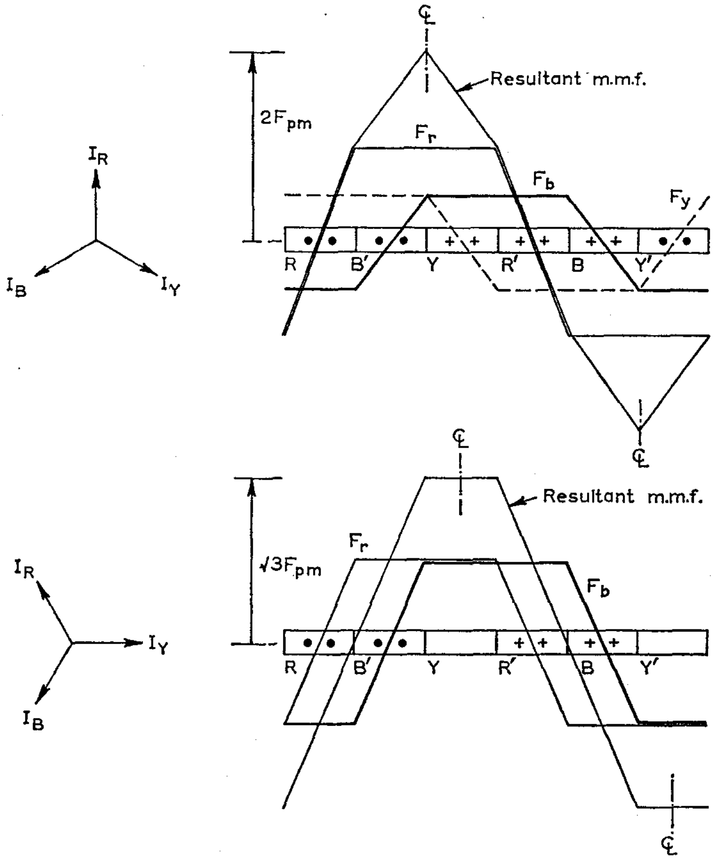


Fig. 11.9 M.M.F. DUE TO A 3-PHASE WINDING (GRAPHICAL TREATMENT)

waves for these phases, F_y and F_b, are also trapezoidal and at the instant shown in the diagram have a maximum value of

$$\frac{i_p N_p}{2p} = \frac{1}{2}\frac{I_{pm} N_p}{2p} = \tfrac{1}{2}F_{pm}$$

The resultant stator m.m.f., F_A, is obtained by finding the sum of the separate phase m.m.f.s, F_r, F_y and F_b. This resultant m.m.f. has a maximum value of $2F_{pm}$.

Fig. 11.9(*b*) has been drawn for an instant $\frac{1}{12}$th of a cycle later than Fig. 11.9(*a*). The instantaneous phase currents are

$$i_r = \frac{\sqrt{3}}{2} I_{pm}$$

$$i_y = 0$$

$$i_b = -\frac{\sqrt{3}}{2} I_{pm}$$

The red and blue phase m.m.f.s occupy the same positions as in Fig. 11.9(*a*), but the maximum value of the red phase m.m.f., F_r, has fallen to $(\sqrt{3}/2)F_{pm}$, whereas the maximum value of the blue phase m.m.f., F_b, has risen to this value. Since the yellow phase current is zero, the yellow phase m.m.f., F_y, is zero.

The resultant stator m.m.f. is the sum of F_r and F_b, and has a maximum value of $\sqrt{3}F_{pm}$.

Comparing the resultant m.m.f.s of Figs. 11.9(*a*) and (*b*) it will be seen that the centre-line of the resultant m.m.f. has moved $30°_e$ in the $+\theta$ direction, and that the shape of the distribution has become trapezoidal. The maximum value of the resultant m.m.f. has fallen slightly from $2F_{pm}$ to $\sqrt{3}_{pm}$.

After the next $\frac{1}{12}$th cycle the waveshape will be found to be the same as in Fig. 11.9(*a*), but displaced a further $30°_e$ round the armature. Hence the following points may be noted.

1. The m.m.f. wave is continually changing shape between the limits of the peaked wave of Fig. 11.9(*a*) and the flat-topped wave of Fig. 11.9(*b*).
2. The wave may be approximated to by a sinusoidal wave of constant maximum value. It is shown in Section 11.12 that this value is $\frac{18}{\pi^2} F_{pm}$.
3. The m.m.f. wave moves past the coils as the alternating currents vary throughout their cycle.
4. The m.m.f. wave moves by $\frac{1}{12}$th of one pole pair in $\frac{1}{12}$th cycle, i.e. the m.m.f. wave moves through one pole pair in one cycle.

If the frequency of the 3-phase currents is f and the speed of rotation of the field is n revolutions per second,

$$\text{Time to move through 1 pole pair} = \frac{1}{f} = \frac{1}{np}$$

Therefore

$$n = \frac{f}{p} = n_0$$

i.e. the field rotates at synchronous speed as defined by eqn. (11.10).

Summarizing these points it may be said that a 3-phase current in a 3-phase winding produces a rotating magnetic field in the air-gap of the machine, the speed of rotation being the synchronous speed for the frequency of the currents and the number of pole pairs in the machine.

The production of the rotating field is the significant difference between a 3-phase and a single-phase machine. Due to its rotating field a 3-phase machine gives a constant, non-pulsating torque in a direction independent of any subsidiary gear or auxiliary windings.

EXAMPLE 11.1 Compare the e.m.f.s at 50 Hz of the following 20-pole alternator windings wound in identical stators having 180 slots:

(*a*) a single-phase winding with 5 adjacents slots per pole wound, the remaining slots being unwound,
(*b*) a single-phase winding with all slots wound,
(*c*) a 3-phase star-connected winding with all slots wound.

All the coils in each phase are connected in series, and each slot accommodates 6 conductors. The total flux per pole is 0·025 Wb.

Assuming a single-layer winding with full-pitch coils there will be 6 turns per coil and the coil span factor will be unity.

There are 9 slots per pole, and thus the slot pitch, ψ, is given by

$$\psi = \frac{180}{9} = 20°_e$$

(*a*) Number of coils per pole pair and phase, $g = 5$

$$\text{Distribution factor} = \frac{\sin 5 \times \frac{20°}{2}}{5 \sin \frac{20°}{2}} = \frac{0{\cdot}766}{0{\cdot}868} = 0{\cdot}883$$

$$\text{E.M.F. per phase} = 4{\cdot}44 K_d K_s f \Phi N_p \tag{11.21}$$

$$= 4{\cdot}44 \times 0{\cdot}883 \times 1 \times 50 \times 0{\cdot}025 \times 5 \times 6 \times 10$$

$$= \underline{\underline{1{,}470\,\text{V}}}$$

(*b*) Number of coils per pole pair and phase, $g = 9$

$$\text{Distribution factor} = \frac{\sin 9 \times \frac{20°}{2}}{9 \sin \frac{20°}{2}} = \frac{1}{1{\cdot}563} = 0{\cdot}64$$

E.M.F. per phase $= 4{\cdot}44K_dK_sf\Phi N_p$
$= 4{\cdot}44 \times 0{\cdot}64 \times 1 \times 50 \times 0{\cdot}025 \times 9 \times 6 \times 1($
$= \underline{\underline{1{,}920\,\text{V}}}$

(c) Number of coils per pole pair and phase, $g = 3$

$$\text{Distribution factor} = \frac{\sin 3 \times \frac{20^\circ}{2}}{3 \sin \frac{20^\circ}{2}} = \frac{0{\cdot}5}{0{\cdot}521} = 0{\cdot}96$$

E.M.F. per phase $= 4{\cdot}44K_dK_sf\Phi N_p$
$= 4{\cdot}44 \times 0{\cdot}96 \times 1 \times 50 \times 0{\cdot}025 \times 3 \times 6 \times 10$
$= \underline{\underline{960\,\text{V}}}$

(Line voltage for star connexion = 1,600 V)

Comparing (a) and (b) it will be seen that the e.m.f. in case (b) is only 30 pei cent greater than that in case (a), while the amount of winding material is 80 per cent greater.

The winding losses for the same current would also be 80 per cent greater for case (b). Thus it is common practice to omit some coils in each pole pair in a single-phase winding.

Supposing that with the above e.m.f.s there is a current of I amperes in the coils.

In case (a), armature power $= 1{,}470I$ watts
In case (b), armature power $= 1{,}920I$ watts
In case (c), armature power $= \sqrt{3} \times 1{,}660I = 2{,}800I$ watts

Comparing (b) and (c) above it will be realized that for the same frame size with the same winding and core losses the output from a 3-phase machine is about 1·5 times greater than that from a single-phase machine.

11.12 M.M.F. due to a Three-phase winding (analytical treatment)

In Fig. 11.10 the m.m.f. due to one phase acting separately is shown as a trapezoidal wave. This trapezoidal wave can be represented by using an appropriate Fourier series consisting of a fundamental and a series of space harmonics. In the analysis below all space harmonics are neglected, and the m.m.f. due to each phase acting separately is assumed to be of sinusoidal form having a maximum value equal to the maximum value of the fundamental in the Fourier series.

In Fig. 11.10 the axis $\theta = 0$ is the centre-line of the positive half-wave of the m.m.f., F_r', due to the red phase only when the red phase carries conventionally positive current (i.e. emerging from the start end of the winding). The Fourier series of a trapezoidal wave having this origin is

$$F(\theta) = \frac{8A}{\pi(\pi - 2\beta)} \sum_n \frac{1}{n^2} \cos n\beta \cos n\theta \quad (n \text{ is odd}) \tag{11.24}$$

where A and β are as indicated on Fig. 11.10.

A is the maximum value of the trapezoid, and since the red phase is excited by alternating current, this maximum value varies sinusoidally with time so that

$$A = F_{pm} \cos \omega t \qquad \left(F_{pm} = \frac{I_{pm}N_p}{2p}\right)$$

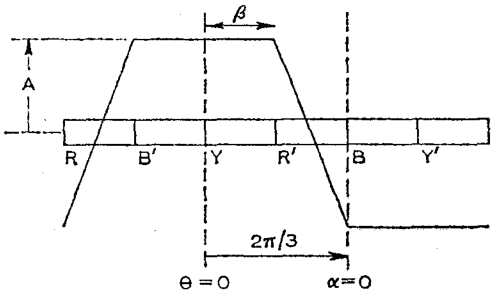


Fig. 11.10 M.M.F. DUE TO A 3-PHASE WINDING (ANALYTICAL TREATMENT)

The angle β corresponds to a phase spread, so that $\beta = \pi/3$. Substituting in eqn. (11.24), the red phase m.m.f. at any time t and at any position θ is

$$F_r' = \frac{8F_{pm} \cos \omega t}{\pi(\pi - 2\pi/3)} \cos \frac{\pi}{3} \cos \theta$$

$$= \frac{12F_{pm}}{\pi^2} \cos \omega t \cos \theta \tag{11.25}$$

In Fig. 11.10 the axis $\alpha = 0$ is the centre-line of the positive half-wave of the m.m.f. due to the yellow phase. Since the yellow phase current lags in time behind the red phase current by $2\pi/3$ radians, the variation of the maximum value of the yellow phase m.m.f. will lag behind that of the red phase m.m.f. by the same amount.

The yellow phase m.m.f. is

$$F_y' = \frac{8F_{pm} \cos (\omega t - 2\pi/3)}{\pi(\pi - 2\pi/3)} \cos \frac{\pi}{3} \cos \alpha$$

Evidently

$$\theta = \alpha + 2\pi/3 \qquad \text{so that} \qquad \alpha = \theta - 2\pi/3$$

and

$$F_y' = \frac{12F_{pm}}{\pi^2} \cos \left(\omega t - \frac{2\pi}{3}\right) \cos \left(\theta - \frac{2\pi}{3}\right) \tag{11.26}$$

Similarly the blue phase m.m.f. is

$$F_b' = \frac{12F_{pm}}{\pi^2} \cos\left(\omega t + \frac{2\pi}{3}\right) \cos\left(\theta + \frac{2\pi}{3}\right) \tag{11.27}$$

The resultant m.m.f. of the 3-phase winding is

$$F_A' = F_r' + F_y' + F_b'$$

$$= \frac{12F_{pm}}{\pi^2} \left\{ \begin{aligned} &\cos \omega t \cos \theta + \cos (\omega t - 2\pi/3) \cos (\theta - 2\pi/3) \\ &\qquad + \cos (\omega t + 2\pi/3) \cos (\theta + 2\pi/3) \end{aligned} \right\}$$

On simplifying this gives

$$F_A' = \frac{18}{\pi^2} F_{pm} \cos (\theta - \omega t) \tag{11.28}$$

This is the fundamental in the space distribution of m.m.f. It has a maximum value at $\theta - \omega t = 0$, i.e. when $\theta = \omega t$. That is, the position of the maximum value travels in the $+\theta$ direction with an angular velocity ω radians per second.

Thus an m.m.f. of the form $F = F_m \cos (\theta - \omega t)$ represents an m.m.f. wave cosinusoidally distributed in space and travelling in the $+\theta$ direction at ω radians per second. The above equation is therefore the equation of a *travelling wave* and is referred to as a *retarded function.*

11.13 Three-phase Rotating Field Torques

Consider a rotating field, derived from the rotor, say, linking a 3-phase winding in which a 3-phase current is flowing; i.e. the rotating field is due to the m.m.f. of rotating poles, not to the 3-phase current. Suppose the speed of the rotating field is such that the frequency of the e.m.f. induced in the 3-phase winding is the same as that of the currents in the 3-phase winding. Unless this is the case there will be no mean torque since the direction of the torque will be alternating.

Let E_{ph} = R.M.S. value of e.m.f. induced in each phase of 3-phase winding

ϕ = Phase angle between induced e.m.f. and winding current

I_{ph} = R.M.S. phase current

The machine may be acting as either a generator or a motor:

Mean phase power $= E_{ph}I_{ph} \cos \phi$ watts

This, by the law of conservation of energy and neglecting losses, must be the mechanical power required to drive the rotor if the machine is acting as a generator, or the mechanical power developed if the machine is acting as a motor.

$$\text{Total mechanical power developed} = 3E_{ph}I_{ph}\cos\phi = 2\pi n_0 T$$

where T is the total torque developed (newton-metres) and n_0 is the speed of the rotor (f/p revolutions per second). Therefore

$$T = \frac{3E_{ph}I_{ph}\cos\phi}{2\pi n_0} \tag{11.29}$$

$$= \frac{3}{2\pi\dfrac{f}{p}}\,\frac{2\pi f N_{ph}\Phi_m K_d K_s}{\sqrt{2}}\,I_{ph}\cos\phi$$

$$= \frac{3p}{\sqrt{2}}\,N_{ph}\Phi_m K_d K_s I_{ph}\cos\phi \quad \text{newton-metres} \tag{11.30}$$

When the machine is motoring, the torque will act on the rotor in the direction of rotation and react on the stator in the opposite direction. These directions will interchange when the machine is generating.

11.14 Non-pulsating Nature of the Torque in a Three-phase Machine

It has been shown that the 3-phase currents in the stator of a 3-phase machine produce a magnetic field of effectively constant amplitude rotating round the air-gap at synchronous speed. The torque developed is due to the magnetic forces between the rotor poles and the rotating field, so that so long as the rotor poles move at synchronous speed there will be a constant magnetic force between stator and rotor. Hence the 3-phase machine will develop a constant torque which does not pulsate in magnitude. (Note that this differs from the case of the single-phase machine.)

The above conclusion may also be derived by considering that the total power delivered to a balanced 3-phase load is non-pulsating, so that, if the load is a machine which is running at a constant speed, the torque developed must also be non-pulsating. On this basis the single-phase machine has a pulsating torque since the power supplied pulsates at twice the supply frequency.

PROBLEMS

11.1 Derive an expression for the e.m.f. induced in a full-pitched coil in an alternator winding, assuming a sinusoidal distribution of flux in the air-gap.

Show how the voltage of a group of such coils, connected in series, may be found.

Calculate the speed and open-circuit line and phase voltages of a 4-pole 3-phase 50 Hz star-connected alternator with 36 slots and 30 conductors per slot. The flux per pole is 0·0496 Wb, sinusoidally distributed.

Ans. 1,500 rev/min, 3,300 V, 1,910 V.

11.2 Derive the expression for the voltage in a group of m full-pitch coils each having an electrical displacement of ψ.

An 8-pole 3-phase star-connected alternator has 9 slots per pole and 12 conductors per slot. Calculate the necessary flux per pole to generate 1,500 V at 50 Hz on open-circuit. The coil span is one pole pitch.

With the same flux per pole and speed, what would be the e.m.f. when the armature is wound as a single-phase alternator using two-thirds of the slots? (*H.N.C.*)

Ans. 0·0283 Wb; 1,480 V.

11.3 A 6-pole machine has an armature of 90 slots and 8 conductors per slot and revolves at 1,000 rev/min, the flux per pole being 5×10^{-2} Wb. Calculate the e.m.f. generated (*a*) as a d.c. machine if the winding is lap-connected; (*b*) as a 3-phase star-connected machine if the winding factor is 0·96 and all the conductors in each phase are in series. Deduce the expression used in each case. (*L.U.*)

Ans. 600 V, 2,200 V.

11.4 Derive an expression for the e.m.f. induced in each phase of a single-layer distributed polyphase winding assuming the flux density distribution to be sinusoidal.

A 4-pole 3-phase 50 Hz star-connected alternator has a single-layer armature winding in 36 slots with 30 conductor per slot. The flux per pole is 0·05 Wb. Determine the speed of rotation. Draw, to scale, the complexor diagram of the phase e.m.f.s,

1. when the phase windings are symmetrically star-connected,
2. when the phase windings are asymmetrical star-connected, the yellow phase winding being reversed with respect to the red and blue phase windings.

Give the numerical values of all the line voltages in each case. The phase sequence for the symmetrical star connection is RYB.

Ans. 1,500 rev/min; $V_{RY} = V_{YB} = B_{BR} = 3{,}320$ V; $V_{RY} = 1{,}920$ V; $V_{YB} = 1{,}920$ V; $V_{BR} = 3{,}320$ V.

11.5 An 8-pole rotor, excited to give a steady flux per pole of 0·01 Wb, is rotated at 1,200 rev/min in a stator containing 72 slots. Two 100-turn coils A and B are accommodated in the stator slotting as follows:

Coil A. Coil sides lie in slots 1 and 11,
Coil B. Coil sides lie in slots 2 and 10.

Calculate the resultant e.m.f. of the two coils when they are joined (*a*) in series aiding and (*b*) in series opposing. Assume the flux density distribution to be sinusoidal.

Ans. 700 V; 0 V.

11.6 A rotor having a d.c. excited field winding is rotated at n revolutions per second in a stator having uniformly distributed slots. If the air-gap flux density is sinusoidally distributed, derive an expression for the e.m.f. induced in a coil of N turns the sides of which lie in any two slots.

In such an arrangement the flux per pole is 0·01 Wb, the rotor speed is 1,800 rev/min, the stator has 36 slots and the rotor has 4 poles. Calculate the frequency and r.m.s. value of the induced e.m.f. in the coil if the coil sides lie in slot 1 and slot 9 and the coil has 100 turns.

An exactly similar coil is now placed with its coil sides in slots adjacent to the first coil. Determine the resultant e.m.f.s when the coils are connected in series.

Ans 60 Hz; 263 V; 518 V or 91·3 V.

Chapter 12

THE THREE-PHASE SYNCHRONOUS MACHINE

A synchronous machine is an a.c. machine in which the rotor moves at a speed which bears a constant relationship to the frequency of the current in the armature winding. As a motor, the shaft speed must remain constant irrespective of the load, provided that the supply frequency remains constant. As a generator, the speed must remain constant if the frequency of the output is not to vary. The field of a synchronous machine is a steady one. In very small machines this field may be produced by permanent magnets, but in most cases the field is excited by a direct current obtained from an auxiliary generator which is mechanically coupled to the shaft of the main machine.

12.1 Types of Synchronous Machine

The armature or main winding of a synchronous machine may be on either the stator or the rotor. The difficulties of passing relatively large currents at high voltages across moving contacts have made the stator-wound armature the common choice for large machines. When the armature winding is on the rotor, the stator carries a salient-pole field winding excited by direct current and very similar to the stator of a d.c. machine (Fig. 12.1(*a*)). Stator-wound armature machines fall into two classes: (*a*) salient-pole rotor machines, and (*b*) non-salient-pole, or cylindrical-rotor, machines

(Fig. 12.1(*b*) and (*c*)). The salient-pole machine has concentrated field windings and generally is cheaper than the cylindrical-rotor machine when the speed is low (less than 1,500 rev/min). Salient-pole alternators are generally used when the prime mover is a water

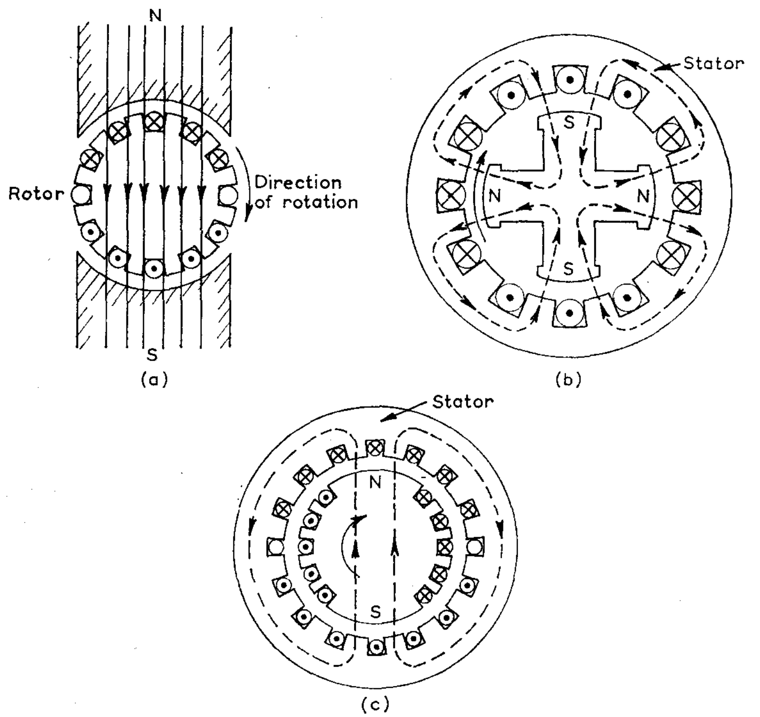


Fig. 12.1 SYNCHRONOUS MACHINES

(*a*) Armature on rotor
(*b*) Salient-pole rotor (4 poles)
(*c*) Cylindrical rotor (2 poles)

turbine or a reciprocating engine. The cylindrical rotor has a distributed winding in the rotor slots and is most suitable for high speeds: steam-turbine-driven alternators are generally high-speed machines (3,000 rev/min) and have the cylindrical-rotor construction.

For an alternator to generate a 50 Hz voltage the speed n_0 will be given by eqn. (11.10) as

$$n_0 = \frac{60 \times 50}{p} = \frac{3{,}000}{p} \quad \text{rev/min}$$

where p is the number of pole-pairs on the machine. n_0 would also be the speed of a synchronous motor with p pole-pairs operating from a 50 Hz supply.

12.2 M.M.F. Wave Diagrams of the Synchronous Generator

The operation of a synchronous machine may be understood by a consideration of its m.m.f. waves. There are three m.m.f. waves to be considered: that due to the field winding, F_F, which is separately excited with direct current; that due to the 3-phase armature winding, F_A; and their resultant, F_R.

In the first stages of the explanation the following assumptions will be made.

1. Magnetic saturation is absent so that the machine is linear.
2. The field and armature m.m.f.s are sinusoidally distributed.
3. The air-gap is uniform, i.e. the machine does not exhibit saliency on either side of the air-gap.
4. The reluctance of the magnetic paths in the stator and rotor is negligible.
5. The armature-winding leakage inductance and resistance are negligible.

The last assumption will be removed at a convenient stage in the development. The machine considered will be a cylindrical-rotor machine with the 3-phase winding on the stator and the d.c.-excited field winding on the rotor. The generating mode of action will first be considered.

The method of drawing the m.m.f. waves will be that used in Chapter 11, which should be read in conjunction with this chapter. The same conventions for positive m.m.f. and positive current are adopted. Positive current is assumed to emerge from the start ends of the phase windings, so that the unprimed phase spreads R,Y,B are dotted when the current is positive, and the primed phase spreads R′Y′B′ are crossed when the current is positive. Positive e.m.f. may now be defined in the same way. Positive m.m.f. is assumed to be directed from rotor to stator and is shown above the θ axis in the m.m.f. wave diagrams. These are shown superimposed on a representation of a double pole-pitch of the stator winding and of the rotor.

GENERATING-MODE OPERATION ON OPEN-CIRCUIT

Fig. 12.2(a) shows the relevant m.m.f. wave for the synchronous generator on open-circuit. The instant chosen in the e.m.f. cycle, indicated by the complexor diagram, is such that

$$E_R = E_{pm} \qquad E_Y = -\tfrac{1}{2}E_{pm} \qquad E_B = -\tfrac{1}{2}E_{pm}$$

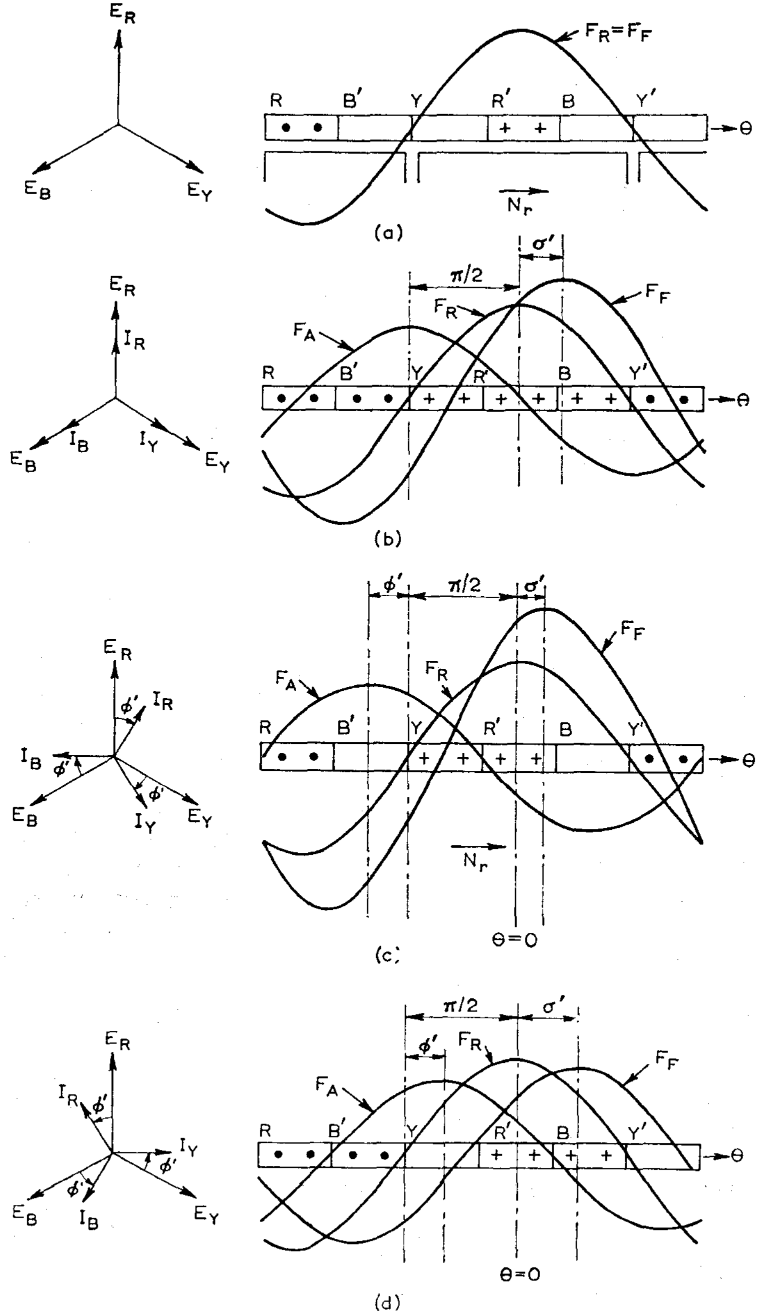


Fig. 12.2 M.M.F. WAVE DIAGRAMS FOR SYNCHRONOUS GENERATOR

The phase spreads of the red phase, R and R', have been dotted and crossed in accordance with the convention for positive e.m.f. The rotor rotates in the direction shown; the direction of the stator conductors relative to the rotor is in the opposite direction, and this latter direction must be used in applying the right-hand rule to find the direction of the field flux and m.m.f. This is as shown in Fig. 12.2(*a*), the maximum rotor m.m.f.s occurring opposite the centres of the red phase spreads, since this phase has maximum e.m.f. induced in it. The field m.m.f., F_F, shown in Fig. 12.2(*a*) is also the resultant m.m.f., F_R, since on open-circuit there is no armature current and consequently F_A is zero at all times and at all points in the air-gap.

The field m.m.f. is stationary with respect to the rotor winding, which is excited with direct current but moves, with the rotor, at synchronous speed past the stator winding.

GENERATING-MODE OPERATION AT UNITY POWER FACTOR

Fig. 12.2(*b*) represents the m.m.f. waves when the armature winding is supplying current at unity power factor.

To obtain comparability between Figs. 12.2(*a*) and (*b*), both diagrams have been drawn for the same instant in the e.m.f. cycle, and the magnitudes of the e.m.f.s are the same in each case as indicated by the e.m.f. complexor diagram. Since the e.m.f. is caused by the resultant m.m.f., F_R, this will have the same magnitude and position in Fig. 12.2(*b*) as it has in Fig. 12.2(*a*).

However, since in this case armature current flows, there will be an armature m.m.f., F_A. The resultant m.m.f., F_R, is the sum of F_A and F_F, so in this case F_F and F_R are different.

The instant in the current cycle is such that

$$i_R = I_{pm} \qquad i_Y = -\tfrac{1}{2}I_{pm} \qquad i_B = -\tfrac{1}{2}I_{pm}$$

This is the instant in the 3-phase cycle for which Fig. 11.9(*a*) was drawn. The armature m.m.f. in Fig. 12.2(*b*), therefore, is in the same position as the armature m.m.f. in Fig. 11.9(*a*) and lags behind the resultant m.m.f. wave by $\pi/2$ radians, but the space harmonics which give the armature m.m.f. wave its distinctive peaked shape are ignored, and this m.m.f. is shown as a sine distributed wave. The armature m.m.f. moves at synchronous speed, so that the m.m.f.s F_A and F_F and their resultant F_R all move at the same speed and in the same direction under steady conditions.

At any time and at any point in the air-gap,

$$F_R = F_A + F_F \tag{12.1}$$

Therefore $F_A = F_R - F_F$.

The field m.m.f., F_F, in Fig. 12.2(*b*) is therefore obtained by point-by-point subtraction of the resultant and armature m.m.f. waves. Comparing the field m.m.f. at Fig. 12.2(*b*) with that for Fig. 12.2(*a*) two changes may be observed:

1. To maintain the e.m.f. constant, the separate excitation has had to be increased in value as shown by the higher maximum value of F_F. The effect of armature m.m.f. is therefore the same as that of an internal voltage drop.
2. The axis of the field m.m.f. has been displaced by an angle σ' in the direction of rotation, and as a result a torque is exerted on the rotor in the direction opposite to that of rotation. The rotor must be driven by a prime mover against this torque, so that machine absorbs mechanical energy and is therefore able to deliver electrical energy.

GENERATING-MODE OPERATION AT POWER FACTORS OTHER THAN UNITY

Fig. 12.2(*c*) shows the m.m.f. waves for the same instant in the e.m.f. cycle but with the phase currents lagging behind their respective e.m.f.s by a phase angle ϕ'. As compared with its position in Fig. 12.2(*b*), therefore, the armature m.m.f. wave is displaced by an angle ϕ' in the direction opposite to that of rotation as compared with its unity-power-factor position. This change in the relative position of the armature m.m.f. wave brings it more into opposition with the field m.m.f., so that the latter must be further increased to maintain the resultant m.m.f. and e.m.f. constant.

Fig. 12.2(*d*) shows the m.m.f. waves for the same instant in the e.m.f. cycle but with the phase currents leading their respective e.m.f.s by a phase angle ϕ'. As compared with its position for unity power factor, the armature m.m.f. wave is displaced in the direction of rotation by the angle ϕ'. In this case the change in the relative position of the armature m.m.f. wave gives it a component which aids the field m.m.f., which must then be reduced for a constant resultant m.m.f. and e.m.f.

If the power factor is zero lagging, the armature and field m.m.f. waves are in direct opposition, whereas if the power factor is zero leading, the armature and field m.m.f. waves are directly aiding.

MOTORING-MODE OPERATION

Positive current in a motor conventionally enters the positive terminal and circulates in the windings in the direction opposite to that in which the e.m.f. acts. In Fig. 12.3 the phase of the current with

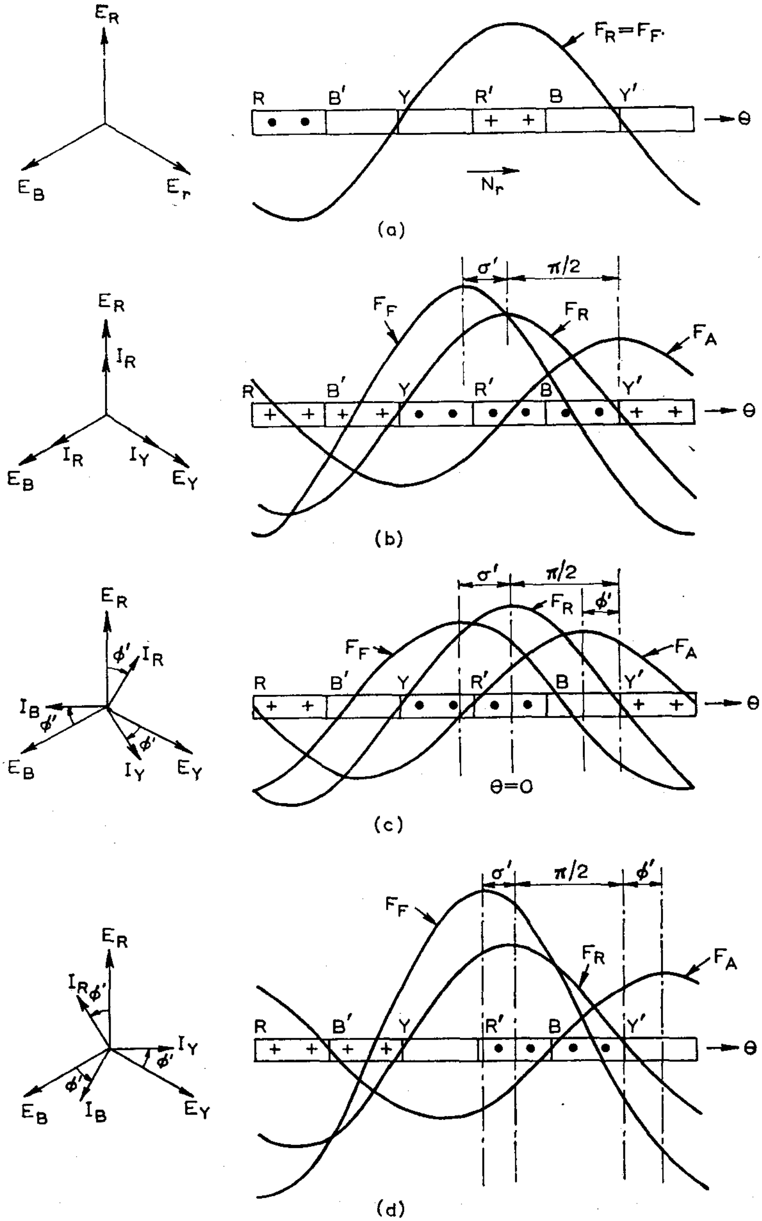


Fig. 12.3 M.M.F. WAVE DIAGRAMS FOR SYNCHRONOUS MOTOR

respect to the induced e.m.f. is kept the same as for generator action, but the opposite convention for positive current is used in drawing the m.m.f. wave; i.e. when the phase current is positive it is assumed to enter the start ends of the phase windings, so that the unprimed phase spreads are crossed when the current is positive, and the primed phase spreads are dotted when the current is positive. The convention for e.m.f. is the same as that used for generator action in the construction of Fig. 12.2.

Fig. 12.3(*a*) shows the m.m.f. waves for the synchronous motor on no-load. It has been assumed that the no-load current is negligible and the armature m.m.f. zero. The field and resultant m.m.f.s are then identical as shown, and Fig. 12.3(*a*) corresponds exactly to Fig. 12.2(*a*).

Fig. 12.3(*b*) shows the m.m.f. wave diagram corresponding to motor operation at unity power factor. The armature m.m.f. is reversed at all points compared with the corresponding armature m.m.f. wave in Fig. 12.2(*b*). The field m.m.f. is found by carrying out the subtraction $F_R - F_A$ point by point round the air-gap. It will be noted that, compared with the no-load condition, the field m.m.f. is displaced by an angle σ' in the direction opposite to rotation, thus giving rise to a torque on the rotor acting in the direction of rotation. The machine thus delivers mechanical power, having absorbed electrical power from the supply.

Figs. 12.3(*c*) and (*d*) show the m.m.f. for operation at lagging and leading power factors respectively. Synchronous motors are operated from a constant-voltage supply, so that variation in the field excitation cannot affect the machine e.m.f. It follows that the power factor of the armature current must alter with variation of field excitation (this effect also occurs in generators connected in large constant-voltage systems).

From Fig. 12.3(*c*) it will be seen that the input power factor becomes leading when the excitation is increased above the unity p.f. condition. Similarly when the excitation is reduced the p.f. becomes lagging (Fig. 12.3(*d*)).

12.3 M.M.F. Travelling-wave Equations

Figs. 12.2 and 12.3 are the m.m.f. wave diagrams of a synchronous machine for both the generating and motoring modes of operation at various power factors. Each of the waves travels in the $+\theta$ direction at synchronous speed. Therefore, each of these m.m.f.s may be represented by a travelling-wave equation, or retarded function, as explained in Section 11.12.

For example, the equation of the cosinusoidally distributed m.m.f.

shown in Fig. 12.4, which is travelling in the $+\theta$ direction at ω electrical radians per second is

$$F' = F_m \cos(\omega t - \theta) \tag{12.2}$$ *

Considering generator action at a lagging power factor, the resultant m.m.f. (Fig. 12.2(*c*)) is

$$F_R' = F_{Rm} \cos(\omega t - \theta) \tag{12.3}$$

The field m.m.f. is

$$F_F' = F_{Fm} \cos(\omega t - \theta + \sigma') \tag{12.4}$$

and the armature m.m.f. is

$$F_A' = F_{Am} \cos(\omega t - \theta - \pi/2 - \phi') \tag{12.5}$$

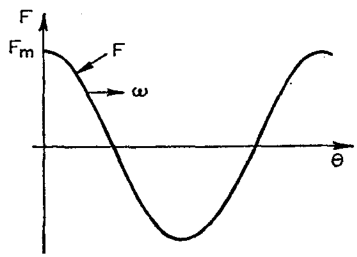


Fig. 12.4 TRAVELLING-WAVE M.M.F.

When the generator works at a leading power factor, eqns. (12.3) and (12.4) still apply for the resultant and separate field m.m.f.s. It will be seen, by examination of Fig. 12.2(*d*), that the equation for the armature m.m.f. becomes

$$F_A' = F_{Am} \cos(\omega t - \theta - \pi/2 + \phi') \tag{12.6}$$

Considering now the motoring mode at a lagging power factor, to which Fig. 12.3(*c*) refers, the resultant m.m.f. is the same as for generator action, i.e.

$$F_R' = F_{Rm} \cos(\omega t - \theta) \tag{12.7}$$

The separate field m.m.f. is

$$F_F' = F_{Fm} \cos(\omega t - \theta - \sigma') \tag{12.8}$$

and the armature m.m.f. is

$$F_A' = F_{Am} \cos(\omega t - \theta + \pi/2 - \phi') \tag{12.9}$$

* A prime (′) is used to indicate instantaneous values of m.m.f.

When the power factor is leading, the armature m.m.f. becomes

$$F_A' = F_{Am}\cos(\omega t - \theta + \pi/2 + \phi') \tag{12.10}$$

These results may be summarized as follows: The resultant m.m.f. for any mode of operation is

$$F_R' = F_{Rm}\cos(\omega t - \theta) \tag{12.3}$$

The separate field m.m.f. is

$$F_F' = F_{Fm}\cos(\omega t - \theta \pm \sigma') \tag{12.8}$$

where $+\sigma'$ is used for the generating mode and $-\sigma'$ for the motoring mode.

The armature m.m.f. is

$$F_A' = F_{Am}\cos(\omega t - \theta - \pi/2 \pm \phi') \tag{12.11}$$

for the generating mode and

$$F_A' = F_{Am}\cos(\omega t - \theta + \pi/2 \pm \phi') \tag{12.12}$$

for the motoring mode. For both cases $+\phi'$ refers to operation at a leading power factor and $-\phi'$ to operation at a lagging power factor.

12.4 M.M.F. Complexor Diagrams

The sum (or difference) of two sinusoidally space-distributed m.m.f.s may be found using the same methods as are used for time-varying sinusoidal quantities. Although the sinusoidally space-distributed m.m.f.s of the synchronous machine are all travelling at synchronous speed (in a stator-wound machine), they may still be dealt with by means of complexor diagrams, since, under steady-state conditions, the relative positions of the waves do not alter.

The m.m.f. complexor diagrams may be deduced either directly from the m.m.f. wave diagrams of Figs. 12.2 and 12.3 or from the travelling-wave equations derived in Section 12.3.

Adopting the latter method for generator mode operation at a lagging power the travelling-wave equations are

$$F_R' = F_{Rm}\cos(\omega t - \theta) \tag{12.2}$$

$$F_F' = F_{Fm}\cos(\omega t - \theta + \sigma') \tag{12.4}$$

$$F_A' = F_{Am}\cos(\omega t - \theta - \pi/2 - \phi') \tag{12.5}$$

At any particular point in the air-gap denoted by $\theta = \theta_0$ the m.m.f.s are

$$F_R' = F_{Rm}\cos(\omega t - \theta_0) \tag{12.13}$$

$$F_F' = F_{Fm}\cos(\omega t - \theta_0 + \sigma') \tag{12.14}$$

$$F_A' = F_{Am}\cos(\omega t - \theta_0 - \pi/2 - \phi') \tag{12.15}$$

Since θ_0 is a particular value of θ and therefore not a variable, the above equations represent, not travelling waves, but quantities varying sinusoidally with time.

The corresponding complexor diagram is shown in Fig. 12.5(*a*), where the m.m.fs. F_R', F_F' and F_A' are represented by the complexors

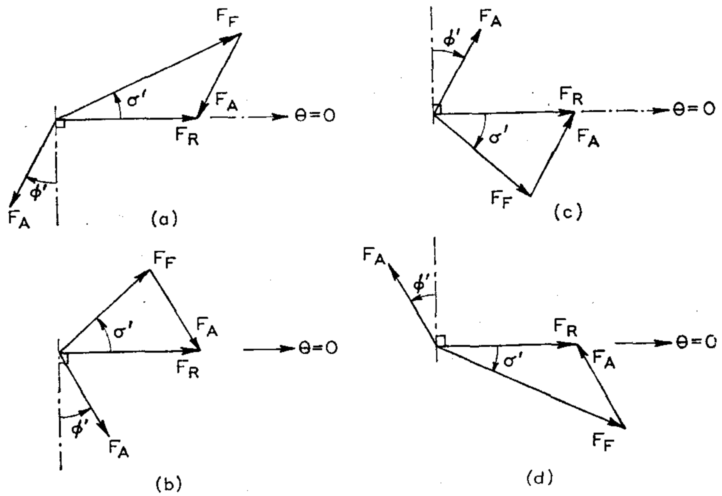


Fig. 12.5 M.M.F. COMPLEXOR DIAGRAMS FOR THE SYNCHRONOUS MACHINE

(*a*) Generator operation at a lagging power factor
(*b*) Generator operation at a leading power factor
(*c*) Motor operation at a lagging power factor
(*d*) Motor operation at a leading power factor

$\boldsymbol{F_R}$, $\boldsymbol{F_F}$ and $\boldsymbol{F_A}$. For simplicity the diagram has been drawn for the air-gap position $\theta_0 = 0$. The chain-dotted line indicates the unity-power-factor position of the m.m.f. $\boldsymbol{F_A}$. Figs. 12.5(*b*), (*c*) and (*d*) are similar diagrams for different power factors and different operating modes.

12.5 E.M.F. Complexor Diagram

Assuming that the reluctance of the magnetic paths in the stator and rotor is negligible and that the air-gap is uniform, the sinusoidally distributed travelling-wave m.m.f.s F_F and F_A and their resultant F_R may be assumed to give rise to separate sinusoidally distributed flux densities. Each of these flux density distributions will travel at synchronous speed, its maximum value occurring at the same place as that of the corresponding m.m.f. and travelling with it at synchronous speed. The relative motion between these flux density distributions and the phase windings will induce e.m.f.s in the windings.

Since only the air-gap reluctance is taken into account, the magnetic circuit is linear and the principle of superposition may be applied. For a particular phase winding let

$\boldsymbol{E_F}$ = E.M.F. due to field m.m.f., $\boldsymbol{F_F}$
$\boldsymbol{E_A}$ = E.M.F. due to armature m.m.f., $\boldsymbol{F_A}$
$\boldsymbol{E_R}$ = Resultant e.m.f. due to the resultant m.m.f., $\boldsymbol{F_R}$

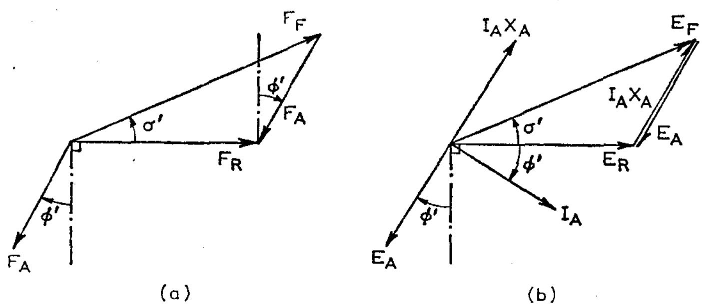


Fig. 12.6 M.M.F. AND E.M.F. COMPLEXOR DIAGRAMS FOR THE SYNCHRONOUS GENERATOR

The relative phase angles of $\boldsymbol{E_F}$, $\boldsymbol{E_A}$ and $\boldsymbol{E_R}$ will be the same as those of $\boldsymbol{F_F}$, $\boldsymbol{F_A}$ and $\boldsymbol{F_R}$.

The m.m.f. complexor diagram, as explained in Section 12.4, is drawn for a particular position round the air-gap $\theta_0 = 0$.

The m.m.f. and e.m.f. complexor diagrams are shown in Figs. 12.6(*a*) and (*b*) respectively for the case of a generator working at a lagging power factor. The resultant e.m.f. $\boldsymbol{E_R}$ is shown as the complexor sum of $\boldsymbol{E_F}$ and $\boldsymbol{E_A}$:

$$\boldsymbol{E_R} = \boldsymbol{E_F} + \boldsymbol{E_A} \tag{12.16}$$

For a fixed value of separate excitation, E_F will be constant. An examination of Fig. 12.6(*b*) reveals that any change of either the armature current or the load power factor would alter the resultant e.m.f. E_R. It is more convenient to work with a constant-voltage source in an equivalent circuit, so that E_F is customarily regarded as the e.m.f. since it does not alter with load and is also the terminal voltage on open-circuit (when $E_A = 0$). The effect of the armature m.m.f. is treated, not as a contribution to the available e.m.f. but as an internal voltage drop, and the phase opposite of $\boldsymbol{E_A}$ is subtracted from $\boldsymbol{E_F}$.

Examination of Fig. 12.6(*b*) shows that $\boldsymbol{E_A}$ will always lag $\boldsymbol{I_A}$ by 90° irrespective of the power-factor angle. The phase opposite of $\boldsymbol{E_A}$, namely $-\boldsymbol{E_A}$, leads $\boldsymbol{I_A}$ by 90° for all conditions.

The peak value of the e.m.f. due to the armature m.m.f. is E_{Am}:

$$E_{Am} \propto F_{Am} \propto I_{Am}$$

Therefore

$$\frac{E_{Am}}{I_{Am}} = \frac{E_A}{I_A} = \text{constant}$$

Since the quotient of E_A and I_A is a constant, and since the phase opposite of $\boldsymbol{E_A}$ leads $\boldsymbol{I_A}$ by 90°, this voltage may be represented as an inductive voltage drop, and the quotient as an inductive reactance, i.e.

$$-\boldsymbol{E_A} = \boldsymbol{I_A} X_A$$

where

$$X_A = \frac{E_A}{I_A} \tag{12.17}$$

Substituting for $\boldsymbol{E_A}$ in eqn. (12.16) and rearranging,

$$\boldsymbol{E_F} = \boldsymbol{E_R} + \boldsymbol{I_A} X_A \tag{12.18}$$

This complexor summation is shown in Fig. 12.6(*b*).

In Section 12.7 an expression for X_A is found in terms of the physical dimensions of the machine.

12.6 Equivalent Circuit of the Synchronous Machine

The preceding section has shown that the equivalent circuit of a synchronous machine must contain a voltage source $\boldsymbol{E_F}$ which is constant for a constant excitation current I_F and a series-connected reactance X_A. In addition, an actual machine winding will have resistance R and (in the same way as a transformer) leakage reactance X_L.

Fig. 12.7(*a*) shows the full equivalent circuit of the synchronous machine in which the current flows in the conventionally positive direction for generator-mode operation (a source), i.e. emerging from the positive terminal. Applying Kirchhoff's law to this circuit,

$$\boldsymbol{E_F} = \boldsymbol{V} + \boldsymbol{I}R + j\boldsymbol{I}X_L + j\boldsymbol{I}X_A \tag{12.19}$$

Fig. 12.7(*b*) is the corresponding complexor diagram. The resultant e.m.f. $\boldsymbol{E_R}$ is shown for the sake of completeness but will be omitted in subsequent diagrams.

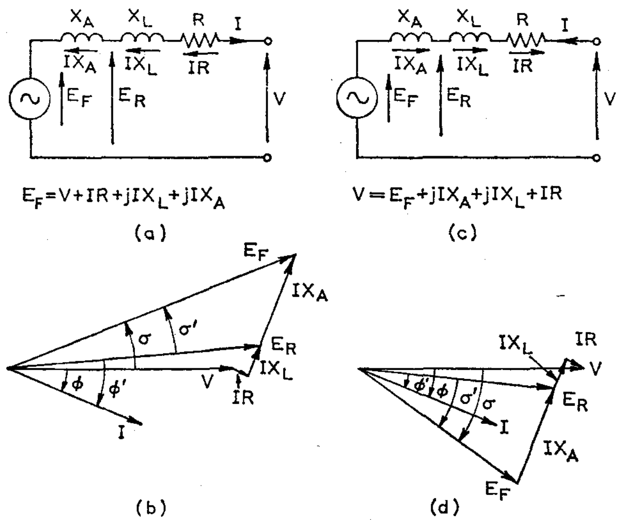


Fig. 12.7 EQUIVALENT CIRCUITS AND FULL COMPLEXOR DIAGRAMS FOR THE SYNCHRONOUS MACHINE

(*a*), (*b*) Generator (*c*), (*d*) Motor

Eqn. (12.19) may be rewritten as

$$\boldsymbol{E_F} = \boldsymbol{V} + \boldsymbol{IZ_s} \qquad (12.20)$$

where $\boldsymbol{Z_s}$ is the *synchronous impedance*.

$$\boldsymbol{Z_s} = R + j(X_L + X_A) \qquad (12.21)$$

or

$$\boldsymbol{Z_s} = R + jX_s \qquad (12.22)$$

where X_s is the *synchronous reactance*:

$$X_s = X_L + X_A \qquad (12.23)$$

In polar form the synchronous impedance is

$$\boldsymbol{Z_s} = Z_s\underline{/\psi} \qquad (12.24)$$

where

$$\psi = \tan^{-1}\frac{X_s}{R} \qquad (12.25)$$

and

$$Z_s = \sqrt{(R^2 + X_s^2)} \qquad (12.26)$$

Frequently in synchronous machines $X_s \gg R$, in which case eqn. (12.26) becomes

$$Z_s = X_s\underline{/90^\circ} = jX_s \tag{12.27}$$

Fig. 12.7(*c*) shows the full equivalent circuit of the synchronous motor, in which the current flows in the conventionally positive direction for motor-mode operation (a load), i.e. entering the position terminal. Applying Kirchhoff's law to this circuit gives

$$V = E_F + jIX_A + jIX_s + IR \tag{12.28}$$

Fig. 12.7(*d*) is the corresponding complexor diagram.

EXAMPLE 12.1 A 3-phase 11·8 kV 75 MVA, 50 Hz 2-pole star-connected synchronous generator requires a separate field m.m.f. having a maximum value of $3{\cdot}0 \times 10^4$ At/pole to give normal rated voltage on open-circuit. The flux per pole on open-circuit is approximately 5·3 Wb.

Determine (*a*) the maximum armature m.m.f. per pole corresponding to rated full-load current, and (*b*) the synchronous reactance if the leakage reactance of the armature winding is 0·18 Ω. Find also the p.u. value of the synchronous reactance.

Neglect the effect of space harmonics in the field and armature m.m.f.s. Assume the flux per pole to be proportional to the m.m.f. and the armature winding to be uniform and narrow spread.

The e.m.f. per phase, from eqn. (11.20), is

$$E_p = \frac{K_d K_s \omega \Phi N_p}{\sqrt{2}}$$

and the distribution factor for a uniform narrow-spread winding is

$$K_d = \frac{3}{\pi} = 0{\cdot}955 \tag{11.13}$$

$$E_p = \frac{11{\cdot}8 \times 10^3}{\sqrt{3}} = 6{,}800\,\text{V}$$

Taking $K_s = 1$, the number of turns per phase is

$$N_p = \frac{\sqrt{2}E_p}{K_s K_d \omega \Phi} = \frac{2 \times 6{,}800}{0{\cdot}955 \times 2\pi 50 \times 5{\cdot}3} = 6{\cdot}05$$

Since the number of turns per phase must be an integer, take $N_p = 6$.

The maximum armature m.m.f., from eqn. (11.28), is

$$F_{Am} = \frac{18 F_{pm}}{\pi^2} = \frac{18}{\pi^2}\,\frac{\sqrt{2}I_p N_p}{2}$$

$$\text{Rated current per phase} = \frac{75 \times 10^6}{3 \times 6{,}800} = 3{,}680\,\text{A}$$

so that

$$F_{Am} = \frac{18}{\pi^2} \times \sqrt{2} \times \frac{3{,}680 \times 6}{2} = \underline{\underline{2{\cdot}85 \times 10^4\,\text{At/pole}}}$$

Since the flux per pole is proportional to the m.m.f.,

$$\frac{E_A}{E_F} = \frac{F_{Am}}{F_{Fm}}$$

$$E_A = 6{,}800 \times \frac{2\cdot85 \times 10^4}{3\cdot0 \times 10^4} = 6{,}450\,\text{V}$$

$$X_A = \frac{E_A}{I_A} = \frac{6{,}450}{3{,}680} = 1\cdot75\,\Omega$$

$$X_s = X_A + X_L = 1\cdot75 + 0\cdot18 = \underline{\underline{1\cdot93\,\Omega}}$$

Taking rated phase voltage and current as bases,

$$X_{spu} = \frac{3{,}680 \times 1\cdot93}{6{,}800} = \underline{\underline{1\cdot04\,\text{p.u.}}}$$

12.7 Synchronous Reactance in Terms of Main Dimensions

It is assumed that (*a*) magnetic saturation is absent; (*b*) the armature m.m.f. is sinusoidally distributed; (*c*) the air-gap is uniform; and (*d*) the reluctance of the magnetic paths in the stator and rotor is negligible.

Let D = Internal stator diameter
L = effective stator (or core) length
l_g = radial gap length

The synchronous reactance X_s is

$$X_s = X_L + X_A \tag{12.23}$$

The reactance X_A is

$$X_A = \frac{E_A}{I_A} \tag{12.17}$$

where E_A is the armature e.m.f. per phase due to the armature m.m.f. F_A, and I_A is the armature current per phase.

The value of E_A may be found by using eqn. (11.20), which gives the e.m.f. per phase of a polyphase winding:

$$E_A = K_d K_s \frac{\omega \Phi_A N_p}{\sqrt{2}} \tag{12.30}$$

where the distribution factor for a narrow-spread uniform winding is, from eqn. (11.13), $K_d = 3/\pi$; the coil span factor, $k_s = 1$; and Φ_A is the flux per pole due to the armature m.m.f. F_A.

From eqn. (11.28), and assuming that the armature m.m.f. is sinusoidally distributed,

$$\left.\begin{array}{l}\text{Maximum armature}\\ \text{m.m.f. per pole}\end{array}\right\} F_{Am} = \frac{18F_{pm}}{\pi^2} = \frac{18}{\pi^2}\frac{\sqrt{2}I_A N_p}{2p}$$

$$\left.\begin{array}{l}\text{Maximum air-gap}\\ \text{field strength}\end{array}\right\} H_{gm} = \frac{F_{Am}}{l_g} = \frac{1}{l_g}\frac{18}{\pi^2}\frac{\sqrt{2}I_A N_p}{2p}$$

$$\left.\begin{array}{l}\text{Maximum air-gap}\\ \text{flux density}\end{array}\right\} B_{gm} = \mu_0 H_{gm} = \frac{\mu_0}{l_g}\frac{18}{\pi^2}\frac{\sqrt{2}I_A N_p}{2p}$$

Since the air-gap flux density is sinusoidally distributed,

$$\left.\begin{array}{l}\text{Average air-gap}\\ \text{flux density}\end{array}\right\} B_{av} = \frac{2}{\pi} B_{gm}$$

$$\left.\begin{array}{l}\text{Flux per pole due}\\ \text{to armature m.m.f.}\end{array}\right\} \Phi_A = B_{av} \times \text{Pole area}$$

$$= \frac{2}{\pi}\frac{\mu_0}{l_g}\frac{18}{\pi^2}\frac{\sqrt{2}\, I_A N_p}{2p}\frac{\pi DL}{2p}$$

Substituting for K_d, K_s and Φ_A in eqn. (12.30), and then substituting the resulting expression for E_A in eqn. (12.17),

$$X_A = \frac{E_A}{I_A} = \frac{\frac{3}{\pi}\omega\frac{2}{\pi}\frac{\mu_0}{l_g}\frac{18}{\pi^2}\frac{\pi DL}{2p}\frac{\sqrt{2}I_A N_p}{2p}N_p}{\sqrt{2}I_A}$$

$$= \omega\left(\frac{18}{\pi^2}\right)^2\frac{\mu_0}{3l_g}\pi DL\left(\frac{N_p}{2p}\right)^2 \qquad (12.31)$$

To obtain the synchronous reactance, an allowance for the leakage reactance X_L must be added to X_A. The leakage reactance is mainly due to (*a*) slot leakage flux, which links individual slots and is not, therefore, part of the main flux, and (*b*) end-turn leakage flux, which links the end turns of the stator winding following mainly air paths. The evaluation of these leakage fluxes, particularly the latter, presents some difficulties and is beyond the scope of the present volume.

The e.m.f. per phase, E_A, is due to the armature current itself and is therefore an e.m.f. of self-induction. The reactance X_A is therefore a *magnetizing reactance*.

Eqn. (12.31) shows that the value of this reactance may be reduced by increasing the gap length l_g.

Looked at in another way, for a given machine rating the rated armature current is fixed as is also, as a consequence, the maximum armature m.m.f. per pole. This fixed value of armature m.m.f. has a progressively smaller effect as the air-gap is lengthened.

EXAMPLE 12.2 A 3-phase 13·8 kV 100 MVA 50 Hz 2-pole star-connected cylindrical-rotor synchronous generator has an internal stator diameter of 1·08 m and an effective core length of 4·6 m. The machine has a synchronous reactance of 2 p.u. and a leakage reactance of 0·16 p.u. The average flux density over the pole area is approximately 0·6 Wb/m². Estimate the gap length.

Assume that the radial air-gap is constant and the armature winding uniform. Neglect the reluctance of the iron core and the space harmonics in the armature m.m.f.

With the above assumptions the reactance X_A is

$$X_A = \omega \left(\frac{18}{\pi^2}\right)^2 \frac{\mu_0}{3l_g} \pi DL \left(\frac{N_p}{2p}\right)^2 \tag{12.31}$$

$$\text{Base voltage, } V_B = V_p = \frac{13{\cdot}8 \times 10^3}{\sqrt{3}} = 7{,}960\,\text{V}$$

$$\text{Base current, } I_B = \frac{\text{VA/phase}}{V_B} = \frac{100 \times 10^6}{3 \times 7{,}960} = 4{,}180\,\text{A}$$

$$\text{Base impedance, } Z_B = \frac{V_B}{I_B} = \frac{7{,}960}{4{,}180} = 1{\cdot}91\,\Omega$$

$$X_{Apu} = X_{spu} - X_{Lpu} = 2{\cdot}00 - 0{\cdot}16 = 1{\cdot}84\,\text{p.u.}$$

$$X_A = X_{Apu} Z_B = 1{\cdot}84 \times 1{\cdot}91 = 3{\cdot}52\,\Omega$$

$$\text{Flux per pole, } B_{av} \times \text{Pole area} = B_{av}\frac{\pi DL}{2} = \frac{0{\cdot}6 \times \pi \times 1{\cdot}08 \times 4{\cdot}6}{2}$$

$$= 4{\cdot}68\,\text{Wb}$$

$$E_p = K_d K_s \frac{\omega \Phi N_p}{\sqrt{2}} \tag{11.20}$$

For a uniform winding, $K_d = 3/\pi$ and $K_s = 1$, so that

$$N_p = \frac{\sqrt{2}E_p}{K_d K_s \omega \Phi} = \frac{\sqrt{2} \times 7{,}960}{3/\pi \times 2\pi \times 50 \times 4{\cdot}68} = 8{\cdot}02$$

The number of turns per phase must be an integer, say 8. This will require a slightly higher flux per pole and average value of flux density. From eqn. (12.31),

$$l_g = 2\pi \times 50 \times \left(\frac{18}{\pi^2}\right)^2 \times \frac{4\pi \times 10^{-7}}{3 \times 3{\cdot}52} \times \pi \times 1{\cdot}08 \times 4{\cdot}6 \times \left(\frac{8}{2}\right)^2$$

$$= \underline{\underline{3{\cdot}10 \times 10^{-2}\,\text{m}}}$$

12.8 Determination of Synchronous Impedance

The ohmic value of the synchronous impedance, at a given value of excitation may be determined by open-circuit and short-circuit tests (Fig. 12.8).

On open-circuit the terminal voltage depends on the field excitation and the magnetic characteristics of the machine. Fig. 12.9 includes a

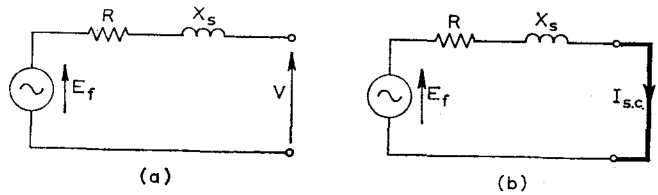


Fig. 12.8 DETERMINATION OF SYCHRONOUS IMPEDANCE
(*a*) Open-circuit test (*b*) Short-circuit test

typical open-circuit characteristic showing the usual initial linear portion and subsequent saturation portion of a magnetization curve.

On short-circuit the current in an alternator winding will normally lag behind the induced voltage by approximately 90° since the leakage

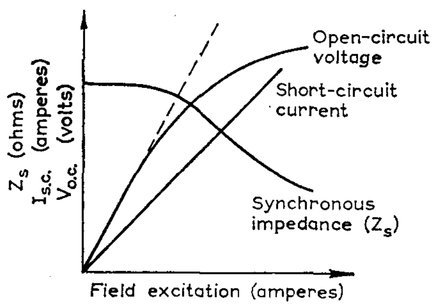


Fig. 12.9 VARIATION OF SYNCHRONOUS IMPEDANCE WITH EXCITATION

reactance of the winding is normally much greater than the winding resistance. The complexor diagram for short-circuit conditions is shown in Fig. 12.10. It is found that the armature and field m.m.f.s

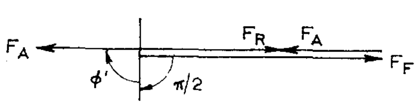


Fig. 12.10 COMPLEXOR DIAGRAM FOR SHORT-CIRCUIT CONDITIONS

are directly in opposition, so that a surprisingly large excitation is required to give full-load short-circuit current in the windings. The resultant m.m.f. and flux are small since the induced voltage is only required to overcome resistance and leakage reactance voltage drops in the windings. Since the flux is small, saturation effects will be negligible and the short-circuit characteristic is almost straight.

The synchronous impedance Z_s may be found by dividing the open-circuit voltage by the short-circuit current at any particular value of field excitation (Fig. 12.9). Over the range of values where the open-circuit characteristic is linear the synchronous impedance is constant, but when the open-circuit characteristic departs from linearity the value of the synchronous impedance falls. It is often difficult to estimate the most appropriate value of Z_s to use for a particular calculation.

12.9 Voltage Regulation

The voltage regulation of an alternator is normally defined as the rise in terminal voltage when a given load is thrown off. Thus, if E_F is the induced voltage on open-circuit and V is the terminal voltage at a given load, the voltage regulation is given by

$$\text{Per-unit regulation} = \frac{E_F - V}{V} \tag{12.32}$$

There are a number of methods of predicting the voltage regulation of an alternator. None are completely accurate. Only the synchronous impedance method is considered here.

12.10 Synchronous Impedance Method

Using a suitable value for Z_s,

$$\mathbf{E_F} = \mathbf{V} + \mathbf{IZ_s} \tag{12.20}$$

EXAMPLE 12.3 A 3-phase star-connected alternator has a resistance of 0·5Ω and a synchronous reactance of 5Ω per phase. It is excited to give 6,600V (line) on open circuit. Determine the terminal voltage and per-unit voltage regulation on full-load current of 130A when the load power factor is (*a*) 0·8 lagging, (*b*) 0·6 leading.

It is best to take the phase terminal voltage V as the reference complexor since the phase angle of the current is referred to this voltage. (The magnitude of V is, however, not known): i.e.

$$\text{Phase terminal voltage, } \mathbf{V} = V\underline{/0^\circ}$$

The magnitude of the e.m.f $\mathbf{E_F}$ is known but not its phase with respect to $\mathbf{V}$; i.e.

$$\mathbf{E_F} = E_F\underline{/\sigma^\circ} = \frac{6{,}600}{\sqrt{3}}\underline{/\sigma^\circ} = 3{,}810\underline{/\sigma^\circ}$$

where σ° is the phase of $\mathbf{E_F}$ with respect to $\mathbf{V}$ as reference.

(*a*) The phase current I lags behind V by a phase angle corresponding to a power factor of 0·8 lagging, i.e.

$$I = 130\underline{/-\cos^{-1} 0{\cdot}8} = 130\underline{/-36{\cdot}9^\circ}\,\text{A}$$

The synchronous impedance per phase is

$$Z_s = (0{\cdot}5 + j5)\,\Omega = 5{\cdot}02\underline{/84{\cdot}3^\circ}\,\Omega$$

In eqn. (12.20),

$$3{,}810\underline{/\sigma^\circ} = V\underline{/0^\circ} + (130\underline{/-36{\cdot}9^\circ} \times 5{\cdot}02\underline{/84{\cdot}3^\circ})$$

$$= V\underline{/0^\circ} + 653\underline{/47{\cdot}4^\circ}$$

Expressing all the terms in rectangular form,

$$3{,}810 \cos \sigma + j\,3{,}810 \sin \sigma = V + j0 + 442 + j482$$

Equating quadrate parts,

$$3{,}810 \sin \sigma = 482$$

whence $\sin \sigma = 0{\cdot}127$ and $\cos \sigma = 0{\cdot}992$

Equating reference parts,

$$3{,}810 \cos \sigma = V + 442$$

$$V = (3{,}810 \times 0{\cdot}992) - 442 = \underline{\underline{3{,}340\,\text{V}}}$$

and

$$\text{Per-unit regulation} = \frac{3{,}810 - 3{,}340}{3{,}340} = \underline{\underline{0{\cdot}141}}$$

(*b*) Phase current $= 130\,\text{A}$ at 0·6 leading with respect to V

$$= 130\underline{/+53{\cdot}1^\circ}$$

Following the same procedure as in part (*a*) it will be found that there is an on-load phase terminal voltage of $\underline{\underline{4{,}260\,\text{V}}}$. Hence the per-unit regulation, since there is a voltage rise, is given by

$$\frac{3810 - 4260}{4{,}260} = \underline{\underline{-0{\cdot}106\,\text{p.u.}}}$$

12.11 Synchronous Machines connected to Large Supply Systems

In Britain, electrical energy is supplied to consumers from approximately 200 generating stations. These stations vary considerably in size, the installed capacity of the largest exceeding 2,000 MW. About one-quarter of the stations have a rating of less than 50 MW and supply less than 2½ per cent of the electrical energy demanded from the public supply.

The generating stations do not operate as isolated units but are interconnected by the national *grid*, which consists of almost 10,000 miles of main transmission line, for the most part overhead lines operating at 132, 275 and 400 kV. The total generating capacity interconnected by the grid system is over 40,000 MW. The output of any single machine is therefore small compared with the total interconnected capacity. The biggest single generator has a rating of 500 MW. For this reason the performance of a single machine is unlikely to affect appreciably the voltage and frequency of the whole system. A machine connected to such a system, where the capacity of any one machine is small compared with the total interconnected capacity, is often said to be connected to *infinite busbars*. The outstanding electrical characteristics of such busbars are that they are constant-voltage constant-frequency busbars.

When the machine is connected to the infinite busbars the terminal voltage and frequency becomes fixed at the values maintained by the rest of the system. Unless the machine is grossly overloaded or under-excited, no change in the mechanical power supply, load or excitation will alter the terminal voltage or frequency. If the machine is acting as a generator and the mechanical driving power is increased the power output from the machine to the busbars must increase, assuming that the efficiency does not greatly change. In the same way, a decrease of mechanical driving power or the application of a mechanical load (motoring) will produce a decrease in output power or the absorption of power from the busbars.

12.12 Synchronizing

The method of connecting an incoming alternator to the live busbars will now be considered. This is called *synchronizing*.

A stationary alternator must not be connected to live busbars, or, since the induced e.m.f. is zero at standstill, a short-circuit will result. The alternator induced e.m.f. will prevent dangerously high switching currents only if the following conditions are almost exactly complied with:

1. The frequency of the induced voltages in the incoming machine must equal the frequency of the voltages of the live busbars.
2. The induced voltages in the incoming machine must equal the live busbar voltages in magnitude and phase.
3. The phase sequence of the busbar voltages and the incoming-machine voltages must be the same.

In modern power stations alternators are synchronized automatically. The principles may be illustrated by the three-lamp method, which, along with a voltmeter, may be used for synchronizing low-power machines.

Fig. 12.11(*a*) is the connexion diagram from which it will be noted that one lamp is connected between corresponding phases while the two others are cross-connected between the other two phases. In the complexor diagram at (*b*) the machine induced voltages are represented by $\boldsymbol{E}_R$, $\boldsymbol{E}_Y$ and $\boldsymbol{E}_B$, while the live supply voltages are represented by V_R, V_Y, and V_B. The lamp symbols have been added to the complexor diagram to indicate the instantaneous lamp voltages. It will be realized that the speed of rotation of the complexors will correspond to the frequencies of the supply and the machine—if these are the same then the lamp brilliancies will be constant. The speed of the machine should be adjusted until the machine frequency is nearly that of the supply, but exact equality is inconvenient for there would then be, in all probability, a permanent phase difference between corresponding voltages. The machine excitation should now be varied until the two sets of voltages

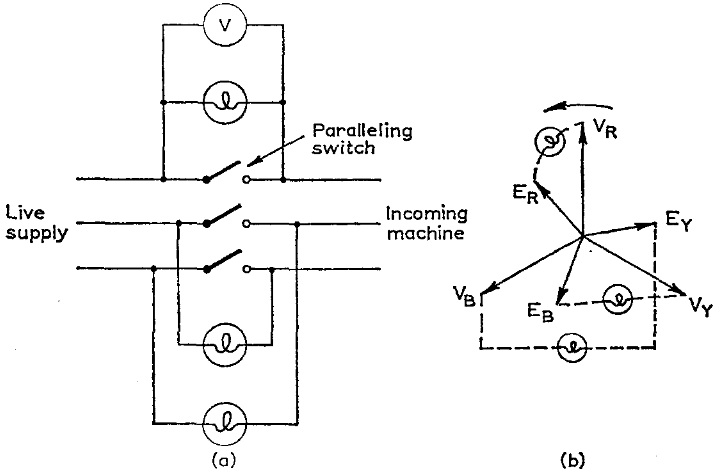


Fig. 12.11 SYNCHRONIZING BY CROSS-CONNECTED LAMP METHOD

are equal in magnitude. The correct conditions will be obtained at an instant when the straight-connected lamp is dark and the cross-connected lamps are equally bright. If the phase sequence is incorrect no such instant will occur as the cross-connected lamps will, in effect, be straight-connected and all the lamps will be dark simultaneously. In this event the direction of rotation of the incoming machine should be reversed or two lines of the machine should be interchanged. Since the dark range of a lamp extends over a considerable voltage range it is advisable to connect a voltmeter across the straight-connected lamp and to close the paralleling switch when the voltmeter reading is zero. It should be noted that the lamps and the voltmeter must be able to withstand twice the normal phase voltage.

12.13 Effect of Variation of Excitation of Synchronous Machine connected to Infinite Busbars

Consider an alternator connected to infinite busbars. Fig. 12.12(*a*) is the complexor diagram of such a machine when operating at unity power factor. The voltage drop $\boldsymbol{I}_a R$ is in phase with $\boldsymbol{I}_a$ and the voltage drop $\boldsymbol{I}_a X_s$ leads $\boldsymbol{I}_a$ by 90°. The complexor sum of $\boldsymbol{I}_a R$ and $\boldsymbol{I}_a X_s$ is $\boldsymbol{I}_a \boldsymbol{Z}_s$, and the e.m.f. ($\boldsymbol{E}_F$) of the machine is obtained by adding $\boldsymbol{I}_a \boldsymbol{Z}_s$ to $\boldsymbol{V}$, the constant busbar voltage. R, X_s and Z_s refer to the winding resistance, reactance and impedance respectively. $\boldsymbol{I}_a \boldsymbol{Z}_s$ makes an angle $\psi = \tan^{-1}(X_s/R)$ with $\boldsymbol{V}$.

Suppose that the excitation of the alternator is reduced while its power input is not altered. The power output will thus remain unchanged. As a result the active component of current, and the voltage drop $\boldsymbol{I}_a \boldsymbol{Z}_s$ due to this current, will remain unchanged. $\boldsymbol{I}_a \boldsymbol{Z}_s$ is shown separately in Fig. 12.12(*b*).

However, when the excitation of the alternator is reduced the e.m.f. ($\boldsymbol{E}_F$) of the machine must fall, so there must be a difference between this new, lower value of $\boldsymbol{E}_F$ and the complexor sum of $\boldsymbol{V}$ and $\boldsymbol{I}_a \boldsymbol{Z}_s$, both of which remain unchanged. This difference is made up by a leading reactive current (which contributes nothing to the power output of the alternator) which sets up the voltage drop $\boldsymbol{I}_r \boldsymbol{Z}_s$ which leads $\boldsymbol{I}_a \boldsymbol{Z}_s$ by 90°. The complexor diagram illustrating this condition is shown in Fig. 12.12(*b*). If it is said that the alternator is normally excited when it is working at unity power factor, then when the alternator is under-excited it will work at a leading power factor.

In a similar way, if the excitation of the alternator is increased from the normally excited unity-power-factor condition, the e.m.f. ($\boldsymbol{E}_F$) of the machine will increase, so that there must be a difference between this new, higher value of $\boldsymbol{E}_F$ and the complexor sum of $\boldsymbol{V}$ and $\boldsymbol{I}_a \boldsymbol{Z}_s$, both of which remain unchanged. This difference is made up by a lagging reactive current (which contributes nothing to the power output of the alternator) which sets up the voltage drop $\boldsymbol{I}_r \boldsymbol{Z}_s$ lagging behind $\boldsymbol{I}_a \boldsymbol{Z}_s$ by 90°. The complexor diagram illustrating this condition is shown in Fig. 12.12(*c*). Thus when the alternator is over-excited it will work with a lagging power factor.

Fig. 12.12(*d*), (*e*) and (*f*), give the corresponding diagrams for the synchronous motor connected to infinite busbars. These are essentially similar to those of the alternator. It will be noted that when the motor is under-excited it works with a lagging power factor, whereas the alternator under similar conditions of excitation works with a leading power factor; and that when the synchronous motor is over-excited it works with a leading power factor, whereas the alternator works with a lagging power factor.

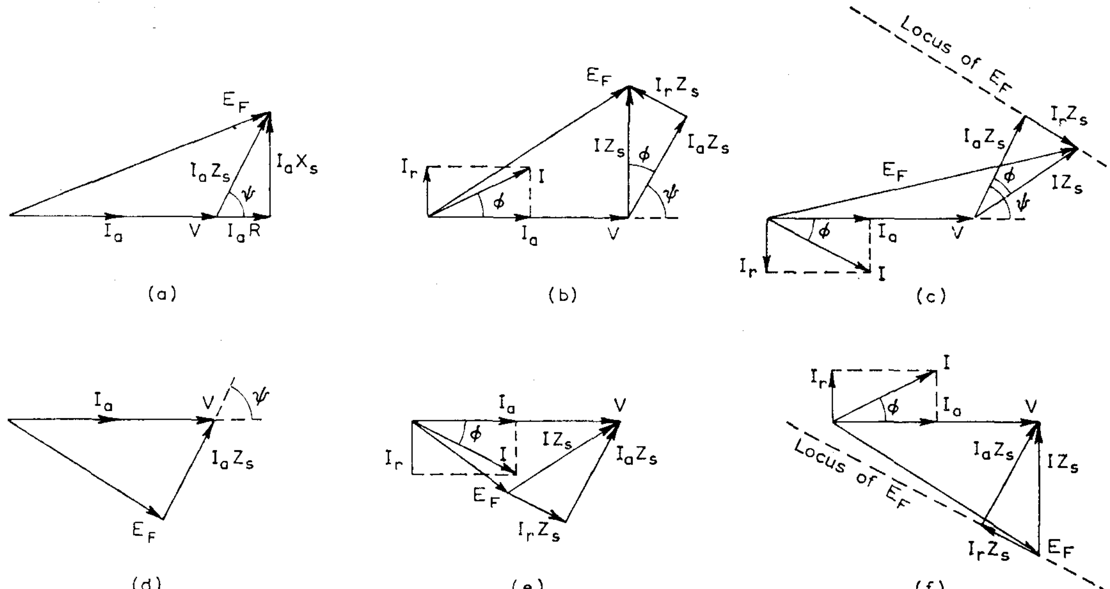


Fig. 12.12 SYNCHRONOUS GENERATOR DELIVERING CONSTANT POWER TO INFINITE BUSBARS

For a constant power output as a generator the voltage drop $\boldsymbol{I_a Z_s}$ will be constant. If the excitation is now varied, $\boldsymbol{I_a}$ remains unchanged but $\boldsymbol{I_r}$, and hence the $\boldsymbol{I_r Z_s}$ drop, will change in size. This voltage drop is always directed at right angles to the $\boldsymbol{I_a Z_s}$ voltage drop, however, so that the locus of $\boldsymbol{E_F}$ as the excitation varies at constant power is the straight line perpendicular to $\boldsymbol{I_a Z_s}$, as shown in Fig. 12.12(*c*).

In the same way the locus of $\boldsymbol{E_F}$ for constant load in a motor is the straight line perpendicular to $\boldsymbol{I_a Z_s}$ as shown in Fig. 12.12(*f*).

In each case illustrated in Fig. 12.12, the line joining the end point of the complexor $\boldsymbol{V}$ to the end point of the complexor $\boldsymbol{E_F}$ represents the overall internal voltage drop in the machine, i.e. $\boldsymbol{IZ_s}$. It should be noted that the angle between complexors $\boldsymbol{I_a Z_s}$ and $\boldsymbol{IZ_s}$ is in every case the phase angle, ϕ, of the resultant load current.

12.14 General Load Diagram

Fig. 12.13 shows the general load diagram of a synchronous machine connected to infinite busbars. In the diagram OV, represents the constant busbar voltage; VC, displaced from OV by the angle $\psi = \tan^{-1}(X_s/R)$, the phase angle of the synchronous impedance of the machine, represents a voltage drop, $\boldsymbol{I_a Z_s}$, caused by the full (100 per cent) load active component of current when the machine acts as a generator. If the machine is working at unity power factor, the e.m.f. $\boldsymbol{E_F}$, is represented by OC. If the machine works at other than unity power factor, the voltage drop caused by the reactive component of current ($\boldsymbol{I_r Z_s}$) must be at right angles to VC, since $\boldsymbol{I_r}$ is at right angles to $\boldsymbol{I_a}$. Hence the locus of $\boldsymbol{E_F}$ for constant full (100 per cent) load is AD, drawn at right angles to VC, passing through C.

The line ZZ is VC produced in both directions.

At 50 per cent load the active component of current will be half its value at 100 per cent load. The $\boldsymbol{I_a Z_s}$ drop at 50 per cent load is therefore half the value corresponding to 100 per cent load. Thus the locus of $\boldsymbol{E_F}$ for constant 50 per cent load is the straight line drawn parallel to AD and passing through the mid-point of VC. A series of constant-power lines for operation both as a generator and a motor may be drawn using the same principle and are shown in Fig. 12.13. The no-load line is the one which passes through the extremity of vector OV as shown.

If the machine is working at unity power factor there is no reactive current and hence no $\boldsymbol{I_r Z_s}$ voltage drop. The line ZZ therefore represents the locus of $\boldsymbol{E_F}$ for unity power factor at any load. Z_1Z_1 is a similar locus for power factor 0·866 leading, and Z_2Z_2 for power factor 0·866 lagging.

The locus of E_F for constant excitation is evidently a semicircle with centre O. Loci for 50 per cent, 100 per cent, 150 per cent and 200 per cent normal excitation are shown.

Referring to Fig. 12.13, suppose the machine has 100 per cent excitation and is on no-load; E_F is then coincident with V. Suppose the mechanical power input to the machine is increased; it must now act as a generator delivering electrical power to the busbars.

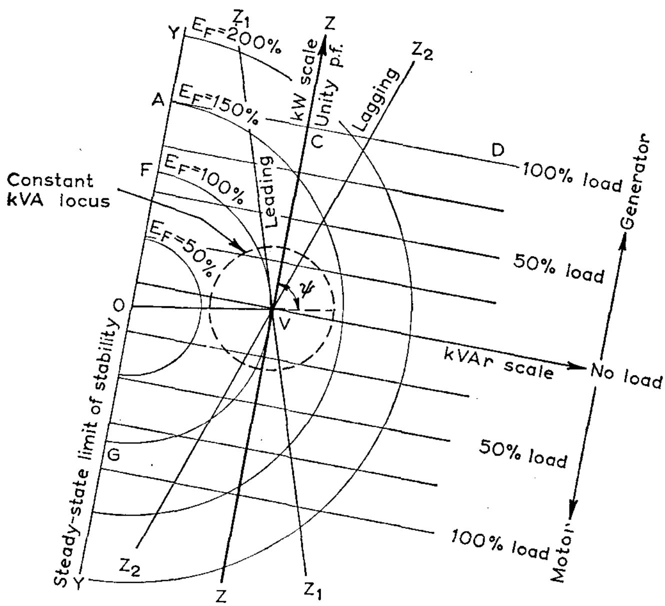


Fig. 12.13 GENERAL LOAD DIAGRAM FOR A SYNCHRONOUS MACHINE CONNECTED TO INFINITE BUSBARS

Physically the rotor is displaced slightly forward in the direction of rotation with respect to the instantaneous pole-centres of the stator field. In other words, E_F advances in phase on V along the circular locus marked "100 per cent excitation". As the mechanical power input is increased, there is an increasing phase displacement between E_F and V until a stage is reached where E_F has the position along the line YY, which is parallel to ZZ through O. If the mechanical power input were further increased E_F would tend to swing beyond YY. However, the extremity of E_F would then approach the 50 per cent load line instead of moving further away from it and the electrical power fed to the busbars would tend to decrease. The excess mechanical energy fed into the machine must then be absorbed by the machine increasing its speed and breaking from synchronism with the constant-frequency system. YY therefore represents the steady-state limit of stability of the machine. It gives the maximum load which the machine may deliver for a given excitation when the load is very gradually applied. For suddenly applied loads the limit is somewhat lower (see Section 15.6).

When the machine acts as a motor, the rotor poles are displaced backwards against the direction of rotation and $\boldsymbol{E_F}$ lags behind $\boldsymbol{V}$ when the load is increased. Thus the steady-state limit of stability is reached at OG along YY when the excitation is 100 per cent.

The general load diagram is based on the assumption that the synchronous impedance Z_s is a constant. Since the value of Z_s is affected by saturation, numerical results obtained from the diagram are only approximate.

The broken circle (Fig. 12.13) whose centre is the end point of V and whose radius is the internal voltage drop ($\boldsymbol{IZ_s}$ to scale) will represent the locus of $\boldsymbol{E_F}$ for a constant volt-ampere value.

EXAMPLE 12.4 An 11 kV 3-phase star-connected turbo-alternator delivers 200 A at unity power factor when connected to constant-voltage constant-frequency busbars.

(*a*) Determine the armature current and power factor at which the machine works when the mechanical input to the machine is increased by 100 per cent, the excitation remaining unchanged.

(*b*) Determine the armature current and power factor at which the machine works when the excitation is raised by 20 per cent, the power input remaining doubled.

(*c*) Determine the maximum power output and corresponding armature current and power factor at this new value of excitation, i.e. as in (*b*).

The armature resistance is 0·4 Ω/phase and the synchronous reactance is 8 Ω/phase.

Assume that the efficiency remains constant.

The problem is best solved graphically; an analytical solution is tedious in this case where the armature resistance is not neglected. The graphical solution is shown in Fig. 12.14.

Using phase values,

$$V_{ph} = \frac{11{,}000}{\sqrt{3}} = 6{,}350\,\text{V}$$

OV is drawn as reference complexor 6,350 V in length to a suitable scale.

$$\boldsymbol{Z_s} = 0{\cdot}4 + j8 = 8{\cdot}00\underline{/87{\cdot}1^\circ}\,\Omega$$

V*b* is drawn making an angle of 87·1° with OV as shown. This is the unity-power-factor line.

$$I_a Z_s = 200 \times 8 = 1{,}600\,\text{V}$$

Va is cut off along the unity-power-factor line 1,600 V in length to scale. Oa represents the e.m.f. under the initial conditions stated.

$E_{F1} = Oa = 6{,}600\,\text{V}$ to scale

The line at right angles to Va passing through a is the constant-power line corresponding to an active component of 200 A.

(*a*) When the mechanical power input is doubled the power output must be doubled and therefore the active component of current, I_a', is doubled.

$I_a'Z_s = 400 \times 8 = 3{,}200\,\text{V}$

Vb is cut off along the unity-power-factor line 3,200 V in length to scale.

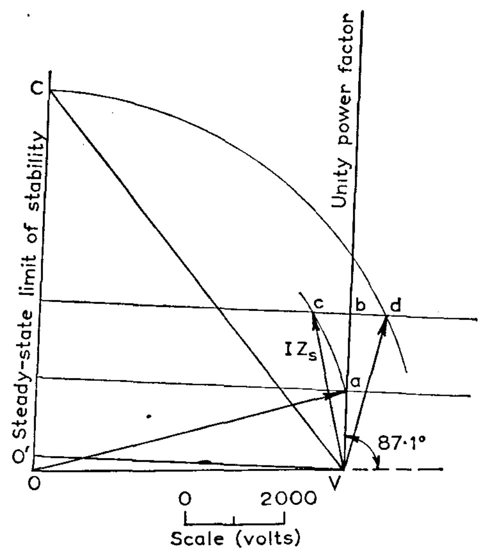


Fig. 12.14

A line at right angles to Vb passing through b is drawn. This is the constant-power line. With centre O and radius Oa an arc is drawn to intersect this constant-power line in c. Let the resultant current be I. Then

$IZ_s = Vc = 3{,}300\,\text{V}$ to scale

Thus

$$I = \frac{3{,}300}{8} = \underline{\underline{412\,\text{A}}}$$

$$\text{Power factor, } \cos\phi = \frac{I_a'Z_s}{IZ_s} = \frac{3{,}200}{3{,}300} = \underline{\underline{0{\cdot}97 \text{ leading}}}$$

(*b*) The new value of e.m.f. is $1{\cdot}2 \times 6{,}600 = 7{,}920\,\text{V}$.

With centre O and radius $E_{F2} = 7{,}920\,\text{V}$ to scale an arc is struck to cut the constant-power line in d, and OC, the steady-state limit of stability line, in C.

Vd = 3,300 V (by measurement)

New current, $I_3 = \dfrac{3{,}300}{8} = \underline{\underline{412\,\text{A}}}$

Power factor, $\cos\phi = \dfrac{I_a'Z_s}{I_3Z_s} = \dfrac{3{,}200}{3{,}300} = \underline{\underline{0{\cdot}97 \text{ lagging}}}$

(*c*) At the steady-state limit of stability for this excitation (point C), the current will be I_4, where $I_4Z_s = \text{CV} = 9{,}900\,\text{V}$. Thus

$I_4 = \dfrac{9{,}900}{8} = \underline{\underline{1{,}235\,\text{A}}}$

$I_{a4}Z_s = \text{CO}' = 7{,}600\,\text{V}$

Power factor, $\cos\phi = \dfrac{I_{a4}Z_s}{I_4Z_s} = \dfrac{7{,}600}{9{,}900} = \underline{\underline{0{\cdot}768 \text{ leading}}}$

Maximum power output $= \sqrt{3}\,VI\cos\phi$

$= \dfrac{\sqrt{3} \times 11{,}000 \times 1{,}235 \times 0{\cdot}768}{1{,}000} = \underline{\underline{18{,}100\text{kW}}}$

12.15 Power/Angle Characteristic of a Synchronous Machine

Fig. 12.15(*a*) is part of the general load diagram for a synchronous machine and shows the complexor diagram corresponding to generation into infinite busbars at a lagging power factor. Fig. 12.15(*b*) is

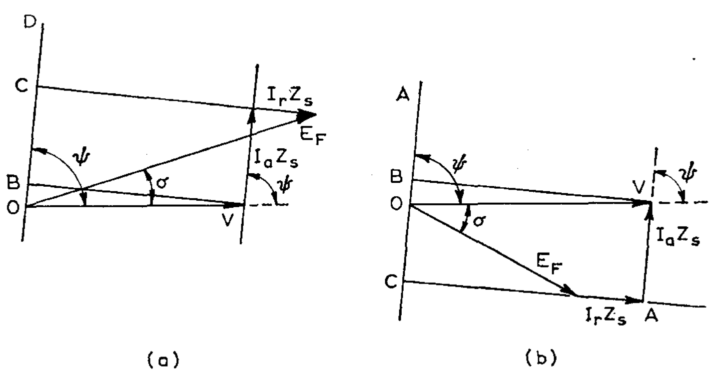


Fig. 12.15 POWER TRANSFER FOR A SYNCHRONOUS MACHINE
(*a*) Generator (*b*) Motor

the corresponding complexor diagram for motor operation also at a lagging power factor. The power transfer is

$$P = 3VI\cos\phi = 3VI_a \tag{12.33}$$

where V is the phase voltage and I is the phase current.

The projection of the complexors of Fig. 12.15(*a*) on the steady-state limit of stability line OD gives

$$I_a Z_s = E_F \cos(\psi - \sigma) - V\cos\psi \tag{12.34}$$

Substituting the expression for I_a obtained from eqn. (12.34) in eqn. (12.33) gives

$$P = \frac{3V}{Z_s}\{E_F \cos(\psi - \sigma) - V\cos\psi\} \tag{12.35}$$

Following the same procedure for motor action and using Fig. 12.15 (*b*) the power transfer is found to be

$$P = \frac{3V}{Zs}\{V\cos\psi - E_F \cos(\psi + \sigma)\} \tag{12.36}$$

Evidently eqn. (12.36) will cover both generator action and motor action if the power transfer P and the load angle σ are taken, conventionally, to be positive for generator action and negative for motor action.

Since, for steady-state operation, the speed of a synchronous machine is constant, the torque developed is

$$T = \frac{P}{2\pi n_0} = \frac{3}{2\pi n_0}\frac{V}{Z_s}\{E_F \cos(\psi - \sigma) - V\cos\psi\} \tag{12.37}$$

In many synchronous machines $X_s \gg R$, in which case $Z_s/\underline{\psi} \approx X_s/\underline{90^\circ}$. When this approximation is permissible eqn. (12.35) becomes

$$P = \frac{3V}{Z_s}\{E_F \cos(90^\circ - \sigma) - V\cos 90^\circ\}$$

$$= \frac{3VE_F}{X_s}\sin\sigma \tag{12.38}$$

Similarly eqn. (12.37) becomes

$$T = \frac{3}{2\pi n_0}\frac{VE_F}{X_s}\sin\sigma \tag{12.39}$$

The power/load-angle (or torque/load-angle) characteristic is shown in Fig. 12.16. The dotted parts of this characteristic refer to operation beyond the steady-state limit of stability. Usually stable operation cannot be obtained beyond this limit, so that if the load angle exceeds $\pm 90°$ the operation is dynamic with the machine either

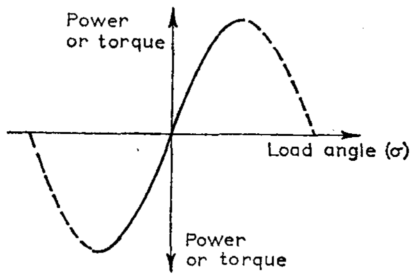


Fig. 12.16 POWER/LOAD-ANGLE AND TORQUE/LOAD-ANGLE CHARACTERISTICS OF A SYNCHRONOUS MACHINE CONNECTED TO INFINITE BUSBARS

accelerating or decelerating. In this case eqns. (12.38) and (12.39) are only approximately true.

12.16 Synchronizing Power and Synchronizing Torque Coefficients

A synchronous machine, whether a generator or a motor, when synchronized to infinite busbars has an inherent tendency to remain synchronized.

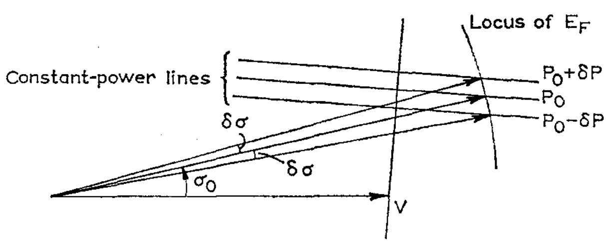


Fig. 12.17 DETERMINATION OF SYNCHRONIZING POWER COEFFICIENT

In Fig. 12.17, which applies to generator operation at a lagging power factor, the complexor diagram is part of the general load diagram. At a steady load angle σ_0 the steady power transfer is P_0.

Suppose that, due to a transient disturbance, the rotor of the machine accelerates, so that the load angle increases by $\delta\sigma$. This alters the operating point of the machine to a new constant-power line and the load on the machine increases to $P_0 + \delta P$. Since the steady power input remains unchanged, this additional load retards the machine and brings it back to synchronism.

Similarly, if owing to a transient disturbance, the rotor decelerates so that the load angle decreases, the load on the machine is thereby reduced to $P_0 - \delta P$. This reduction in load causes the rotor to accelerate and the machine is again brought back to synchronism.

Clearly the effectiveness of this inherent correcting action depends on the extent of the change in power transfer for a given change in load angle. A measure of this effectiveness is given by the *synchronizing power coefficient*, which is defined as

$$P_s = \frac{dP}{d\sigma} \tag{12.40}$$

From eqn. (12.35),

$$P = \frac{3V}{Z_s}\{E_F \cos(\psi - \sigma) - V\cos\psi\} \tag{12.35}$$

so that

$$P_s = \frac{dP}{d\sigma} = \frac{3VE_F}{Z_s}\sin(\psi - \sigma) \tag{12.41}$$

Similarly the synchronizing torque coefficient is defined as

$$T_s = \frac{dT}{d\sigma} = \frac{1}{2\pi n_0}\frac{dP}{d\sigma} \tag{12.42}$$

From eqn. (12.42), therefore,

$$T_s = \frac{3}{2\pi n_0}\frac{VE_F}{Z_s}\sin(\psi - \sigma) \tag{12.43}$$

In many synchronous machines $X_s \gg R$, in which case eqns. (12.42) and (12.43) become

$$P_s = \frac{3VE_F}{X_s}\cos\sigma \tag{12.44}$$

$$T_s = \frac{3}{2\pi n_0}\frac{VE_F}{X_s}\cos\sigma \tag{12.45}$$

Eqns. (12.44) and (12.45) show that the restoring action is greatest when $\sigma = 0$, i.e. on no-load. The restoring action is zero when $\sigma = \pm 90°$. At these values of load angle the machine would be at the steady-state limit of stability and in a condition of unstable equilibrium. It is impossible, therefore, to run a machine at the steady-state limit of stability since its ability to resist small changes is zero unless the machine is provided with a special fast-acting excitation system.

EXAMPLE 12.5 A 2 MVA 3-phase 8-pole alternator is connected to 6,000 V 50 Hz busbars and has a synchronous reactance of 4 Ω per phase. Calculate the synchronizing power and synchronizing torque per mechanical degree of rotor displacement at no-load. (Assume normal excitation.)

The synchronizing power coefficient is

$$P_s = \frac{3VE_F}{X_s}\cos\sigma \tag{12.44}$$

On no-load the load angle $\sigma = 0$.

Since there are 4 pole-pairs, 1 mechanical degree of displacement is equivalent to 4 electrical degrees; therefore

$$P_s = 3 \times \frac{6{,}000}{\sqrt{3}} \times \frac{6{,}000}{\sqrt{3} \times 4} \times \frac{4}{1{,}000} \times \frac{\pi}{180} = \underline{\underline{627\,\text{kW/mech. deg}}}$$

$$\text{Synchronous speed of alternator, } n_0 = \frac{f}{p} = 12{\cdot}5 \text{ rev/s}$$

Thus

$$2\pi n_0 T_s = 627 \times 10^3$$

and

$$\text{Synchronizing torque, } T_s = \underline{\underline{8{,}000\,\text{N-m/mech. deg}}}$$

12.17 Oscillation of Synchronous Machines

In the previous sections, transient accelerations or decelerations of an alternator rotor were assumed in order to investigate the synchronizing power and synchronizing torque. Such transients may be caused by irregularities in the driving torque of the prime mover or, in the case of a motor, by irregularities in the load torque, or by irregularities in other machines connected in parallel, or by sudden changes in load.

Normally the inherent stability of alternators when running in parallel quickly restores the steady-state condition, but if the effect is sufficiently marked, the machine may break from synchronism. Moreover, if the disturbance is cyclic in effect, recurring at regular intervals, it will produce forced oscillations in the machine rotor. If the frequency of this cyclic disturbance approaches the value of the natural frequency of the rotor, when connected to the busbar system, the rotor may be subject to continuous oscillation and may eventually break from synchronism. This continuous oscillation of the rotor (periods of acceleration and deceleration) is sometimes known as *phase swinging* or *hunting*.

Fig. 12.18 shows the torque/load-angle characteristic of a synchronous generator. The steady input torque is T_0, corresponding to a steady-state load angle σ_0. Suppose a transient disturbance occurs such as to make the rotor depart from the steady state by σ'. Let σ' be sufficiently small to assume that the synchronizing

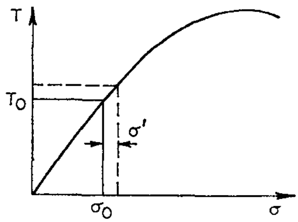


Fig. 12.18 OSCILLATION OF A SYNCHRONOUS MACHINE CONNECTED TO INFINITE BUSBARS

torque is constant; i.e. the torque/load-angle characteristic is assumed to be linear over the range of σ' considered.

Let T_s = Synchronizing torque coefficient (*N-m*/mech. rad)
σ' = Load angle deviation from steady-state position (mech. rad)
J = Moment of inertia of rotating system (*kg-m*^{2})

Assuming that there is no damping,

$$J\frac{d^2\sigma'}{dt^2} = -T_s\sigma' \tag{12.46}$$

The solution of this differential equation is

$$\sigma' = \sigma_m' \sin\left(\sqrt{\frac{T_s}{J}}\,t + \psi\right) \tag{12.47}$$

From eqn. (12.47), the frequency of undamped oscillation is

$$f = \frac{1}{2\pi}\sqrt{\frac{T_s}{J}} \tag{12.48}$$

Synchronous machines intended for operation on infinite busbars are provided with damping windings in order to prevent the sustained oscillations predicted by eqn. (12.48).

In salient-pole machines the damping winding takes the form of a short-circuited cage consisting of copper bars of relatively large cross-section embedded in the rotor pole-face. In cylindrical-rotor machines the solid rotor provides considerable damping, but a cage winding may also be provided. This consists of copper fingers inserted in the rotor slots below the slot wedges and joined together at each end of the rotor. The currents induced in the damping bars give a damping torque which prevents continuous oscillation of the rotor.

EXAMPLE 12.6 A 3-phase 3·3 kV 2-pole 3,000 rev/min 934 kW synchronous motor has an efficiency of 0·95 p.u. and delivers full-load torque with its excitation adjusted so that the input power factor is unity. The moment of inertia of the motor and its load is 30 kg-m², and its synchronous impedance is $(0 + j11{\cdot}1)\,\Omega$. Determine the period of undamped oscillation on full-load for small changes in load angle.

$$\text{Input current, } I = \frac{934 \times 10^3}{\sqrt{3} \times 3{\cdot}3 \times 10^3 \times 0{\cdot}95} = 172\,\text{A}$$

Taking the phase voltage as reference,

$$E_F = V - IX_s$$

$$= \frac{3{\cdot}3 \times 10^3}{\sqrt{3}}\underline{/0^\circ} - (172\underline{/0^\circ} \times 11{\cdot}1\underline{/90^\circ}) = 2{,}700\underline{/-45^\circ}\,\text{V}$$

The synchronizing torque coefficient is

$$T_s = \frac{3}{2\pi n_0}\frac{VE_F}{X_s}\cos\sigma \tag{12.45}$$

$$= \frac{3}{2\pi 50} \times \frac{3{\cdot}3 \times 10^3}{\sqrt{3}} \times \frac{2{,}700}{11{\cdot}1} \times 0{\cdot}707 = 3{\cdot}14 \times 10^3\,\text{N-m/rad}$$

The undamped frequency of oscillation is

$$f = \frac{1}{2\pi}\sqrt{\frac{T_s}{J}}$$

The period of oscillation is

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{30}{3{\cdot}14 \times 10^3}} = \underline{\underline{0{\cdot}612\,\text{s}}}$$

12.18 Synchronous Motors

A synchronous motor will not develop a driving torque unless it is running at synchronous speed, since at any other speed the field poles will alternately be acting on the effective N and S poles of the rotating field and only a pulsating torque will be produced. For starting either (*a*) the induction motor principle or (*b*) a separate starting motor must be used. If the latter method is used the machine must be run up to synchronous speed and synchronized as an alternator. To obviate this trouble, synchronous motors are usually started as induction motors, and have a squirrel-cage winding embedded in the rotor pole faces to give the required action. When the machine has run up to almost synchronous speed the d.c. excitation is switched on to the rotor, and it then pulls into synchronism. The induction motor action then ceases (see Chapter 13).

The starting difficulties of a synchronous motor severely limit its usefulness—it may only be used where the load may be reduced for starting and where starting is infrequent. Once started, the motor has the advantage of running at a constant speed with any desired power factor. Typical applications of synchronous motors are the driving of ventilation or pumping machinery where the machines run almost continuously. Synchronous motors are often run with no load to utilize their leading power factor characteristic for power factor correction or voltage control. In these applications the machine is called a synchronous phase modifier.

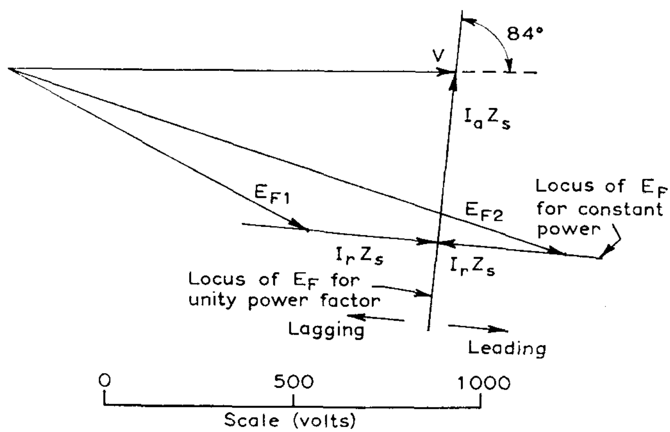


Fig. 12.19

EXAMPLE 12.7 A 2,000 V 3-phase 4-pole star-connected synchronous machine has resistance and synchronous reactance per phase of 0·2 Ω and 1·9 Ω respectively.

Calculate the e.m.f. and the rotor displacement when the machine acts as a motor with an input of 800 kW at power factors of 0·8 lagging and leading.

If a field current of 40 A is required to produce an e.m.f. per phase equal to rated phase voltage, determine also the field current for each condition.

Synchronous impedance, $Z_s = 0{\cdot}2 + j1{\cdot}9 = 1{\cdot}91\underline{/84^\circ}\,\Omega/\text{phase}$

Constant phase terminal voltage, $V = \dfrac{2{,}000}{\sqrt{3}} = 1{,}150\,\text{V}$

Total phase current in both cases $= \dfrac{800 \times 10^3}{\sqrt{3} \times 2{,}000 \times 0{\cdot}8} = 288\,\text{A}$

Active component of current in both cases, $I_a = 288 \times 0{\cdot}8 = 230\,\text{A}$

Reactive component of current in both cases, $I_r = 288 \times 0{\cdot}6 = 173\,\text{A}$

$I_aZ_s = 230 \times 1{\cdot}91 = 440\,\text{V}$

$I_rZ_s = 173 \times 1{\cdot}91 = 330\,\text{V}$

Fig. 12.19 is now drawn to scale for the motoring condition.

At the lagging power factor the excitation voltage is measured from the complexor diagram as $E_{F1} = \underline{\underline{880\,\text{V/phase.}}}$

$$\text{Field current required, } I_{F1} = 40 \times \frac{880}{1{,}150} = 30{\cdot}5\,\text{A}$$

The rotor displacement is the phase angle between $\boldsymbol{E}_{F1}$ and $\boldsymbol{V}$ with the rotor lagging for motor action as previously described. Therefore at the lagging p.f.

Rotor angle $= 27°_e = \underline{\underline{13{\cdot}5°_m}}$ for a 4-pole machine

At the leading p.f. the excitation voltage, $E_{F2} = \underline{\underline{1{,}520\,\text{V/phase}}}$

Field current required, $I_{F2} = \underline{\underline{52{\cdot}9\,\text{A}}}$

Rotor angle $= 17°_e = \underline{\underline{8{\cdot}5°_m}}$

EXAMPLE 12.8 A 2,000 V 3-phase 4-pole star-connected synchronous motor runs at 1,500 rev/min. The excitation is constant and gives an e.m.f. per phase of 1,150 V. The resistance is negligible compared with the synchronous reactance of 3 Ω per phase.

Determine the power input, power factor and torque developed for an armature current of 200 A.

Synchronous impedance $= j3 = 3\underline{/90°}\,\Omega/\text{phase}$

$$\text{Phase voltage, } V = \frac{2{,}000}{\sqrt{3}} = 1{,}150\,\text{V}$$

E.M.F./phase, $E_F = 1{,}150\,\text{V}$

$IZ_s = 200 \times 3 = 600\,\text{V}$

In Fig. 12.20 $\boldsymbol{V}$ represents the phase voltage taken as reference complexor.

A circular arc whose radius represents the open-circuit voltage of 1,150 V is the locus of $\boldsymbol{E}_F$ for constant excitation.

AB is the locus of $\boldsymbol{E}_F$ for unity power factor operation; in this case AB is perpendicular to $\boldsymbol{V}$ since the phase angle of $\boldsymbol{Z}$ is 90°.

A circle whose radius represents 600 V is the locus of $\boldsymbol{E}_F$ for constant kVA operation. For the actual operating conditions $\boldsymbol{E}_F$ must lie at the intersection of the two circles.

From the diagram,

$I_a Z_s = 580\,\text{V}$

Active component of current, $I_a = 193\,\text{A}$

Therefore

$$\text{Total power input} = \frac{3VI_a}{1{,}000} = \frac{3 \times 1{,}150 \times 193}{1{,}000} = \underline{\underline{666\,\text{kW}}}$$

$$\text{Operating power factor} = \frac{I_a}{I} = \frac{193}{200} = \underline{\underline{0{\cdot}96 \text{ lagging}}}$$

$$\text{Torque developed, } T = \frac{3VI_a}{2\pi n_0}$$

$$= \frac{3 \times 1150 \times 193}{2\pi \times 1{,}500} \times 60$$

$$= \underline{\underline{4{,}250 \text{ N-m}}}$$

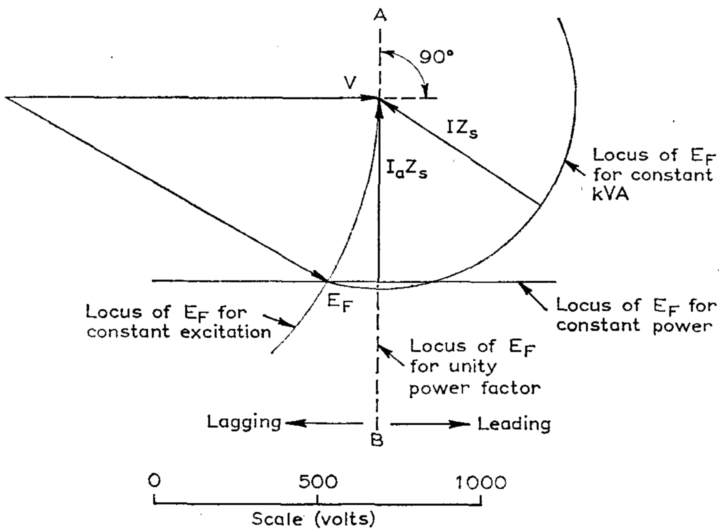


Fig. 12.20

PROBLEMS

12.1 A 3-phase 11 kV star-connected alternator has an effective armature resistance of 1 Ω and a synchronous reactance of 20 Ω per phase. Calculate the percentage regulation for a load of 1,500 kW at p.f.s of (*a*) 0·8 lagging, (*b*) unity, (*c*) 0·8 leading.

Ans. 22 per cent, 4·25 per cent, −13·4 per cent.

12.2 Describe the tests carried out in order that the synchronous impedance of an alternator can be obtained. By means of diagrams show how the synchronous impedance can be used to determine the regulation of an alternator at a particular load and power factor.

A 6,600 V 3-phase star-connected alternator has a synchronous impedance of $(0{\cdot}4 + j6)\,\Omega$/phase. Determine the percentage regulation of the machine when supplying a load of 1,000 kW at normal voltage and p.f. (i) 0·866 lagging, (ii) unity, (iii) 0·866 leading, giving complexor diagrams in each case. (*H.N.C.*)

Ans. 9·7 per cent, 1·84 per cent, −6·03 per cent.

12.3 Explain, with the aid of complexor diagrams, the effect of varying the excitation of a synchronous motor driving a constant-torque load.

A 3-phase 4-pole 50 Hz 2,200 V 1,870 kW star-connected synchronous motor has a synchronous impedance of $(0{\cdot}06 + j0{\cdot}6\Omega)$/phase. The motor is to be run in parallel with an inductive load of 1,000 kVA having a power factor of 0·707 lagging, and is to be so excited that the power factor of the combined loads is 0·9 lagging.

If the motor output is 1,870 kW and its efficiency is 0·9 p.u., determine (*a*) the kVAr, kW and kVA input to the motor, (*b*) the input current and power factor to the motor, (*c*) the load angle of the motor in mechanical degrees, and (*d*) the field current of the motor.

Ans. 64·8 kVAr; 2,080 kW; 2,170 kVA; 570 A; 0·955 lagging; 7·78 mechanical degrees.

12.4 A 2,200 V 3-phase star-connected synchronous motor has a resistance of $0{\cdot}6\Omega$/phase and a synchronous reactance of 6Ω/phase. Find graphically or otherwise the generated e.m.f. and the angular retardation of the rotor when the input is 200 kW at (*a*) a power factor of unity, (*b*) a power factor of 0·8 leading. (*C. & G.*)

Ans. 2,200 V, $15°_e$; 2,640 V, $13{\cdot}5°_e$.

12.5 A 400 V 3-phase 50 Hz star-connected synchronous motor has a synchronous impedance per phase of $(1 + j5)\Omega$. It takes a line current of 10 A at unity power factor when operating with a certain field current. If the load torque is increased until the line current is 40 A, the field current remaining unchanged, find the new power factor and the gross output power. (*H.N.C.*)

Ans. 0·957 lagging, 25 kW.

12.6 A 150 kW 3-phase induction motor has a full-load efficiency and power factor of 0·91 and 0·89 respectively. A 3-phase star-connected synchronous motor, connected to the same mains, is to be over-excited in order to improve the resultant power factor to unity. The synchronous motor also drives a constant load, its power input being 100 kW. The line voltage is 415 V and the synchronous reactance per phase of the synchronous motor is $0{\cdot}5\,\Omega$, the resistance being negligible. Determine the induced e.m.f. per phase of the synchronous motor. (*H.N.C.*)

Ans. 306 V.

12.7 A synchronous generator operates on constant-voltage constant-frequency busbars. Explain the effect of variation of (*a*) excitation and (*b*) steam supply on power output, power factor, armature current and load angle of the machine.

An 11 kV 3-phase star-connected synchronous generator delivers 4,000 kVA at unity power factor when running on constant-voltage constant-frequency busbars. If the excitation is raised by 20 per cent determine the kVA and power factor at which the machine now works. The steam supply is constant and the synchronous reactance is 30Ω/phase. Neglect power losses and assume the magnetic circuit to be unsaturated. (*L.U.*)

Ans. 4,280 kVA; 0·935 lagging.

12.8 Describe briefly the procedure for synchronizing and connecting a 3-phase alternator to constant-voltage constant-frequency busbars. How is the output of the machine adjusted?

A single-phase alternator operates on 10 kV 50 Hz busbars. The winding resistance is $1\,\Omega$ and the synchronous impedance 10Ω. If the excitation is adjusted to give an open-circuit e.m.f. of 12 kV, what is the maximum power