Complex representation of Fourier series

$$e^{jwt} = \cos\omega t + j\sin\omega t$$

Bertrand Russell called this equation "the most beautiful, profound and subtle expression in mathematics." Richard Feyman, the noble laureate said that it is "the most amazing equation in all of mathematics". In electrical engineering, this enigmatic equation is equivalent in importance to F = ma.

This perplexing looking equation was first developed by Euler (pronounced Oiler) in the early1800's. A student of Johann Bernoulli, Euler was the foremost scientist of his day. Born in Switzerland, he spent his later years at the University of St. Petersburg in Russia. He perfected plane and solid geometry, created the first comprehensive approach to complex numbers and is the father of modern calculus. He was the first to introduce the concept of log x and e^x as a function and it was his efforts that made the use of e, i and π the common language of mathematics. He derived the equation $e^x + 1 = 0$ and its more general form given above. Among his other contributions were the consistent use of the sin, and cos functions and the use of symbols for summation. A father of 13 children, he was a prolific man in all aspects, in languages, medicine, botany, geography and all physical sciences.

e^{jwt} in Euler's equation is a decidedly confusing concept. What exactly is the role of *j* in e^{jwt}? We know that it stands for $\sqrt{-1}$ but what is it doing here? Can we visualize this function?

The function $e^{j\omega t}$ goes by the name **complex exponential**. It is also called a **Cisoid** {(**cos** x + **j** **s**in x)us**oid**} from contraction of the names of its parts. This function is of the greatest importance in signal processing. We are going to look at it carefully so both its meaning and application are clear.

Let's ignore the complex exponential e^{jwt} on the LHS and concentrate on the RHS containing the sines and cosine waves.

$$e^{j\omega t} = \cos\omega t + j\sin\omega t \tag{1.1}$$

We can plot this function by fixing a value of ω, and then calculating for each value of t, both $\cos\omega t$ and $\sin\omega t$. The value of ω for most of what we are about to do remains constant. In computing the Fourier series trignometric coefficients in Chapter 1, we stepped through harmonic frequencies and they were treated as a constant for each set of coeffcient calculations. Same here. The three values, t, $\cos\omega t$ and $\sin\omega t$ create the three dimensional plot shown in Figure 1. The function plots out as a helix. In this figure we also plot the projections of the helix on *Real* and *Imaginary* axes. And not very surprisingly the projections are sine and cosine waves. But that's exactly what the above formula is telling us. The Real part of the complex exponential is the cosine and Imaginary part is the sine wave.

The presence of j in front of sine in (1.1) just means that the sine wave is 90° degrees shifted in the complex plane from the cosine wave. Conceptually it makes more sense to think of j as telling you where to plot this part rather than a multiplication by $\sqrt{-1}$. (The Matlab program used to create this figure is given at the back of the chapter.)

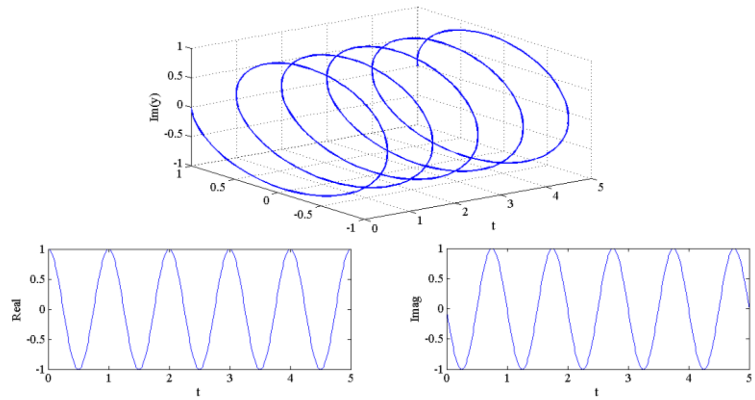


Figure 1 - $e^{j\omega t}$ is a helix. Its projection on the real axis is a cosine and on the imaginary axis is sine.

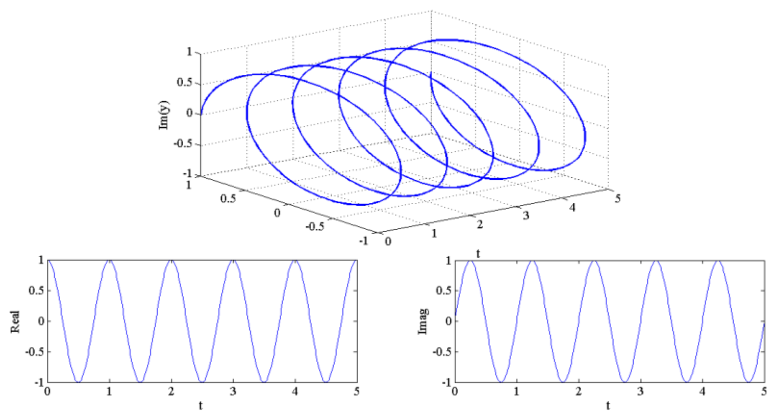


Figure 2 – $e^{-j\omega t}$ is a helix rotating in the opposite direction. The projections on the real axis is a cosine and on imaginary axis is a negative sine.

In Figure 2, we plot exponential $e^{-j\omega t}$ with negative exponent. The expression for the negative exponential is written as

$$e^{-j\omega t} = \cos \omega t - j \sin \omega t \tag{1.2}$$

The difference between $e^{j\omega t}$ and $e^{-j\omega t}$ can be seen in the figures in that the helix seems to be moving in the opposite direction when the sign of its exponent is changed. This is better seen in the imaginary projection which is a sine wave. For the negative exponent, it is flipped 180° degrees from the one we see for the positive exponential in Figure 1. Often this exponential is referred to as having a negative frequency. But there does not seem to be any obvious reason for doing that. We only see the effect of the negative sign on the imaginary projection, the sine wave.

The Real part of the negative exponential is the cosine wave.

$$\mathrm{Re}\left(e^{-j\omega t}\right) = \cos\omega t \tag{1.3}$$

The imaginary part is a negative sine wave.

$$\mathrm{Im}\left(e^{-j\omega t}\right) = -\sin\omega t \tag{1.4}$$

The imaginary part of the positive exponential is instead a positive sine wave.

$$\mathrm{Im}\left(e^{j\omega t}\right) = \sin\omega t \tag{1.5}$$

That's exactly what the plot is showing us. In Fig. 2, the negative exponential has as its imaginary part a negative sine wave. The positive exponential has a positive sine as its imaginary part. The real parts are same for both.

Adding and subtracting the two complex exponentials, e^{jwt} and e^{-jwt}, we get following expressions for sine and a cosine wave.

$$\begin{aligned} \sin\omega t &= \frac{1}{2j}\left((\cos\omega t + j\sin\omega t) - (\cos\omega t - j\sin\omega t)\right) \\ &= \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right) \end{aligned} \tag{1.6}$$

$$\begin{aligned} \cos\omega t &= \frac{1}{2}\left((\cos\omega t + j\sin\omega t) + (\cos\omega t - j\sin\omega t)\right) \\ &= \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) \end{aligned} \tag{1.7}$$

If j in the denominator of (1.6) bothers you, then remember that division by j is same as multiplication by –j. We can show the equivalence as follows.

$$\begin{aligned} 2\sin wt &= -j(e^{jwt} - e^{-jwt}) \\ &= -j(\cancel{\cos\omega t} + j\sin\omega t \cancel{-\cos\omega t} + j\sin\omega t) \end{aligned}$$

Let's plot Eq. (1.6) and (1.7) to see what we get.

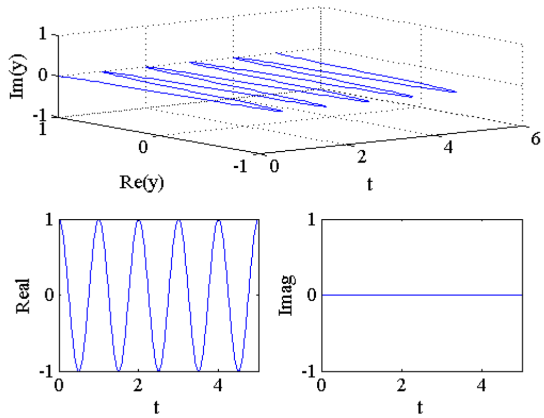


Figure 3 – Plotting $\left(e^{j\omega t}+e^{-j\omega t}\right)/2$ gives a a cosine wave with no projection on the imaginary axis.

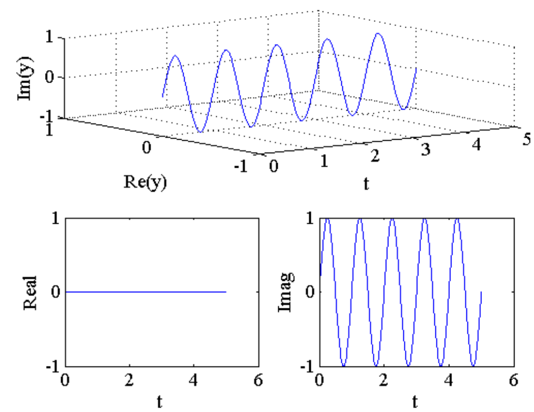


Figure 4 – Plotting $\left(e^{j\omega t}-e^{-j\omega t}\right)/2$ gives us a sine wave with no projection on the real axis.

One way to think of this relationship is to say that both sine and cosine are composed of complex exponentials. It takes two exponentials to create one real sinusoid. In case of cosine, the exponentials add and for sine they subtract. Two 3-D functions that create a 2-D sinusoid.

Unlike in the trigonometric domain where waves are specified by frequency, amplitude and time, complex exponentials (in the complex domain) are specified not just by the amplitude but also by the direction of the phase change that gives the wave a two dimensional quality, one of amplitude and the other of its direction of motion or phase change. In the complex domain, this results in the frequency to have an additional degree of freedom, giving it a positive or negative sense. But this happens only in complex domain. In real space we get by just fine with the value of frequency that is always positive and phase direction is not important.

So how did the Euler's equation come about and why is it so important to signal processing. We will try to answer that by first looking at Taylor series representations of the exponential e^x, sines and cosines. The Taylor series expansion for the two sinusoids is given by the infinite series as

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{1.8}$$

Note that each one of these series is composed of is exponential functions. Both sines and cosines can be seen to be already composed of exponentials. So when we say that they are also sums of complex exponentials, then the concept should not be too had to accept. Taylor series expansion for the exponential e^x gives this series.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{1.9}$$

All three of these series equations are straight forward concepts and can be plotted in on a Cartesian plot. And indeed if we plot these functions against x, we would get just what we are expecting, the exponential and the sinusoids. However how close they come to the continuous function depends on the number of terms that are included in the summation.

The similarity between the exponential and the sinusoids series in Eq. (1.8) and (1.9) shows clearly that there is a relationship here. Now let's change the exponent in (1.9) from x to $j\theta$. Note we will use the term θ here instead of ωt to keep the equation concise. Now we have by simple substitution, the expression for $e^{j\theta}$ as

$$e^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \cdots$$

We know that $j^2 = -1$ and $j^4 = 1$, $j^6 = -1$ etc., substituting these values, we rewrite this series as

$$e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2!} - \frac{j\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{j\theta^5}{3!} - \frac{\theta^2}{6!} - \frac{j\theta^7}{7!} + \cdots$$

We can separate out every other term with j as a coefficient to create a two-part series, one without the j and the other with

$$e^{j\theta} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^2}{6!} + \cdots \qquad \text{This is a cosine.}$$
$$+ j\theta - \frac{j\theta^3}{3!} + \frac{j\theta^5}{3!} - \frac{j\theta^7}{7!} + \cdots \qquad \text{This is } j \text{ times sine.}$$

We see that first part of the series is a cosine and the second part with j as its coefficient is the series for a sine wave. We get the expression we were trying to show.

We can now derive some interesting results like these. By setting $\theta = \pi/2$, we can show that

$$\begin{aligned} e^{j\pi/2} &= \cancel{\cos(\pi/2)} + j\sin(\pi/2) \\ &= 0 + j(1) \\ &= j \end{aligned}$$

By setting $\theta = 3\pi/2$, we can show that

$$\begin{aligned} e^{j3\pi/2} &= \cancel{\cos(3\pi/2)} + j\sin(3\pi/2) \\ &= 0 + j(-1) \\ &= -j \end{aligned}$$

And an another quite interesting result

$$\begin{aligned} &e^{j\pi} = \cos(\pi) + j\sin(\pi) = -1 \\ &e^{j\pi} + 1 = 0 \end{aligned}$$

The purpose of this exercise is to convince you that indeed the complex exponential is the sum of sines and cosines. The function still retains a mysterious quality. However, we need to get over our fear of this equation and learn to love it. The question now is why bring up Euler's equation in context of Fourier analysis? Why all this rigmarole about the complex exponential, why aren't sines and cosines not good enough?

In Fourier analysis, we computed the coefficients of sines and cosines (the harmonics) separately. We mentioned three forms of Fourier series, two of them in trigonometric form and one in complex exponential form. Fourier analysis using trigonometric form is not easy. Trig functions are easy to understand but hard to manipulate. Adding and multiplying is a pain. Doing math with exponentials is considerably easier. Now that we have shown that both sine and cosine are related to a single exponential, we can simplify the math by just working with one exponential of frequency ω . This is the main motivation for switching to complex exponentials. But complex exponentials bring with them conceptual difficulties. They are hard to visualize and confusing because we now claim they have negative frequencies.

In complex domain instead of decomposing the wave into sines and cosines, we will say that this wave is equal to the sum of two exponentials of opposite signs.

$$\begin{aligned} &x(t) = A\cos(\omega t + \theta) \\ &= \frac{A}{2} e^{j(\omega t + \theta)} + \frac{A}{2} e^{-j(\omega t + \theta)} \\ &= \frac{A}{2} e^{j\omega t} e^{j\theta} + \frac{A}{2} e^{-j\omega t} e^{-j\theta} \end{aligned}$$

If we expand this expression into trigonometric domain using Euler's equation we see that indeed we get back the trigonometric cosine wave we started with.

$$
\begin{aligned}
&= A/2\left(\cos(\omega t+\theta) + \cancel{j\sin(\omega t+\theta)} + \cos(\omega t+\theta) - \cancel{j\sin(\omega t+\theta)}\right) \\
&= A\cos(\omega t+\theta)
\end{aligned}
$$

We can write the complex exponential form of the cosine wave in this fashion.

$$
\begin{aligned}
x(t) &= \frac{A}{2}e^{j\theta}e^{j\omega t} + \frac{A}{2}e^{-j\theta}e^{-j\omega t} \\
&= Q_{+}e^{j\omega t} + Q_{-}e^{-j\omega t}
\end{aligned}
\tag{1.10}
$$

Let's call the terms in blue, phasors that we mentioned earlier. The phasor contains the amplitude and the phase but no frequency term.

$$
Q_{+} = \frac{A}{2}e^{j\theta} \tag{1.11}
$$

A phasor is a **complex number**. It can be used to make all sorts of math easier. To fully define a wave in complex domain requires both a positive and a negative phasor. We just said that the phasor is independent of the frequency and yet we say there are positive and negative phasors. To keep things complex, here we are referring to not the carrier part, but to the sign of the phase change. And then instead of plotting phase, these get plotted as positive and negative frequencies making most of us scratch our heads. We don't like the idea of negative frequencies although it is not hard to convince oneself that such a thing is possible. It seems that although there is no physical evidence, we don't have any trouble imagining negative time. Nor do we have problem with negative distance and even negative people! Its all a matter of convention. Until one takes a DSP class, one never thinks of frequency as having a sign. But frequency is actually defined as rate of change of phase.

$$
f = \frac{d\theta}{dt} \tag{1.12}
$$

So by assigning a sign either to time or to movement of phase, we can easily get a positive or negative frequency. But this is not the reason why Fourier spectrum is plotted against negative frequency nor the whole story of negative frequency as used in Fourier analysis and Fourier Transforms. There is more. You will see shortly that all these explanations about what is a negative frequency miss the mark on what negative frequencies are doing in the Fourier analysis and why we plot spectrums that have negative frequencies.

Back to the Fourier Series—

Recall that Fourier series is a sum of sinusoids. In Fourier series, time is continuous but frequency is not. Frequency takes on distinct harmonic values. If the fundamental is ω, then each ω_n is a integer multiple of ω or $n\omega$.

$$f(t) = a_0 + \sum_{n=1}^{N} a_n \cos(\omega_n t) + \sum_{n=1}^{N} b_n \sin(\omega_n t) \tag{1.13}$$

The coefficients a_0, a_n and b_n (which we can call the trignometric coefficients) are defined as

$$a_0 = \frac{1}{T}\int_0^T f(t)dt \tag{1.14}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t\, dt \tag{1.15}$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t\, dt \tag{1.16}$$

The presence of the integral sign tells us the time is continuous. Now substitute Eq. (1.6), (1.7) as the definition of sine and cosine into Eq. (1.13), and we get

$$f(t) = a_0 + \sum_{n=1}^{N} \frac{a_n}{2}(e^{jn\omega t} + e^{-jn\omega t}) + \sum_{n=1}^{N} \frac{b_n}{2j}(e^{jn\omega t} - e^{-jn\omega t}) \tag{1.17}$$

Make the same substitution for sine and cosine trigonometric Fourier coefficients Eq. (1.15 and 1.16).

$$a_n = \frac{2}{T}\int_0^T f(t)\,\frac{1}{2}(e^{jn\omega t} + e^{jn\omega t})dt \tag{1.18}$$

$$b_n = \frac{2}{T}\int_0^T f(t)\,\frac{1}{2j}\left(e^{jn\omega t} - e^{-jn\omega t}\right)dt \tag{1.19}$$

Rearranging Eq. (1.17) so that the each exponential is separated, we get

$$f(t) = a_0 + \sum_{n=1}^{\infty} \frac{1}{2}(a_n - jb_n)e^{jn\omega t} + \sum_{n=1}^{\infty} \frac{1}{2}(a_n + jb_n)e^{-jn\omega t} \tag{1.20}$$

Recall division by j is same as multiplication by –j. The coefficients in Eq. (1.18) can also be expanded as follows.

$$a_n = \frac{1}{T}\int_0^T f(t)e^{jn\omega t}dt + \frac{1}{T}\int_0^T f(t)e^{-jn\omega t}dt \tag{1.21}$$

Look at this equation carefully. You see that the trigonometric coefficient will be split into two parts now, one for each of the exponentials. To make the new coefficients concise, let's redefine them as complex coefficients, A_n and B_n

$$A_n = \frac{1}{2}(a_n - jb_n) \tag{1.22}$$

$$B_n = \frac{1}{2}(a_n + jb_n) \tag{1.23}$$

Substituting these new definitions of the coefficients into the Eq. (1.21), we get a much simpler representation in the complex form

$$f(t) = a_0 + \sum_{n=1}^{\infty} A_n e^{jn\omega t} + \sum_{n=1}^{\infty} B_n e^{-jn\omega t} \tag{1.24}$$

$$\begin{aligned} A_n &= \frac{1}{T}\int_0^T f(t)e^{jn\omega t}dt \\ B_n &= \frac{1}{T}\int_0^T f(t)e^{-jn\omega t}dt \end{aligned} \tag{1.25}$$

It is clear from this equation that A_n can be thought of as the coefficient of the positive exponential and B_n the coefficient of the negative exponential. These coefficients are not the same as the ones we computed in the trigonometric form. They are complex combinations of the trigonometric coefficients a_n and b_n, just as are the complex exponentials.

The term a_0 stands for the DC. We generally do not like DC terms so we will remove it by expanding the range of the second term from 0 to ∞. Rewrite Eq. (1.24) as

$$f(t) = \sum_{n=0}^{\infty} A_n e^{jn\omega t} + \sum_{n=0}^{\infty} B_n e^{-jn\omega t} \tag{1.26}$$

The above equation can be simplified still further by extending the range of coefficients from -∞ to ∞. We can do this by changing the sign of the index which was one-sided because we had included both positive and negative exponentials explicitly. A positive index n allowed inclusion of both of these. Now both terms can be combined into one with a two-sided index to write a much more compact and elegant equation for the Fourier series. Now we do not need the negative exponential in the equation. The index takes care of that. And here is a much shorter equation for Fourier series in the complex domain.

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega t} \tag{1.27}$$

This is called the exponential or the complex form of the Fourier series. It is rigorously related to the sinusoidal form but its coefficients C_n are usually complex.

Note that our index (or the frequency) was always positive in the trignometric form and hence the spectrum was one sided. The x-axis for the one-sided spectrum was plotted against frequency. Now the index goes from $-\infty$ to $+\infty$. The coefficients we are calculating are the coefficients of complex exponentials and not the frequency. What is being indexed in this case is the complex exponential index n. We start on the negative side with a negative n, go through calculations of all negative exponentials and then all the positive ones. Let's take a look again what it means for index n to be negative. It is not the frequency of the exponential that is negative but the sign in between the real and the imaginary parts. Sort of a mathematical way of saying, "Hey, keep that sine wave separate from the cosine and make its sign same as the index." The notation of exponential $e^{+jn\omega t}$ should be thought of as a shorthand for the sum (or difference) of the two sinusoids of frequency ω. Sort of like f(t) notation for a function of time.

$$e^{+jn\omega t} = \cos\omega t + j\sin\omega t$$
$$e^{-jn\omega t} = \cos\omega t - j\sin\omega t$$

The coefficients of these exponentials, for index n, from $-\infty$ to ∞, are the amplitudes *of these exponentials*, NOT *of the frequency*. But we are doing all this to create a spectrum of the signal, so how do we plot an exponential as we do frequency, which is as a scalar? We can't. So what do we do, we take frequency and use it as a placeholder for the complex exponential index. Then we go ahead and use the trigonometric Fourier series concept and name the x-axis frequency. It has two advantages, one it extends a concept that we know and two it feels right, albeit the negative frequency part looks strange. We want to see the distribution of power along a frequency line, which is the bandwidth of the signal. Using frequency as an alternate name for the complex exponential index allows us to plot the coefficients and make sense of them. In trigonometric domain, the index n goes from 0 up. We term that as positive frequency. In complex domain the index goes from $-\infty$ up, and for negative index, we call that negative frequency. Is it a negative frequency, I don't think so. But that is what we call it. We should instead say that in complex domain, the x-axis is not frequency but the complex exponential. But this is mouthful and ends up making things not much clearer, so we stick with frequency as the name of this axis. In the process creating a legion of people who hate DSP.

The complex coefficient values are half of what they are in the trignometric domain. (See the factor ½ in Eq. (1.22).) Now we make up another story that this is because the frequency as being split into two parts, a negative part and positive part. This is how most books try to explain the conundrum of positive and negative frequency in relation to Fourier analysis. But they are just trying to explain a plotting convention. The real story is that we do not know who to plot a complex exponential on a line, so we use signed frequency as a representation of it.

The coefficient C_n in Eq. (1.27) is given by

$$C_n = \frac{1}{2T}\int_{-T}^{T} f(t)e^{-jn\omega t}dt \tag{1.28}$$

Where C_n is equal to

$$C_n = A_n + jB_n \tag{1.29}$$

The magnitude and phase of C_n is defined by

$$\begin{aligned} C_n &= \sqrt{A_n^{\ 2} + B_n^{\ 2}} \\ \phi_n &= \tan^{-1}\left(\frac{B_n}{A_n}\right) \end{aligned} \tag{1.30}$$

A_n and B_n can be seen as the complex coefficients of the two exponentials.

Magnitude and Power spectrum of a signal derived from the Fourier coefficients

The plot of coefficients we draw from the Fourier series coefficients is called the amplitude spectrum. The coefficients are the amplitudes of the harmonics, so the plot is not the power spectrum. The amplitude spectrum can be converted to power spectrum by the Parseval's relationship.

$$P = \frac{1}{T}\int_{0}^{T} |x(t)|^2 \, dt \tag{1.31}$$

In time domain this can be written as

$$P = \sum_{k=-\infty}^{\infty} |x_k|^2 \tag{1.32}$$

If we have a discrete signal, then average power of the signal is the sum of the sample amplitudes squared divided by the period. The average power of a sinusoid is the square of its RMS value. For a sinusoid of amplitude A, its average power is equal to

$$P_{avg} = A^2/2$$

If the signal is constant, then its average power is just A^2. In frequency domain, we can calculate the power by this expression.

$$P_{avg} = \mathrm{a}_0^{\ 2} + \sum_{n=1}^{\infty}\left(\frac{1}{2}a_n^2 + \frac{1}{2}b_n^2\right) \tag{1.33}$$

In complex domain, the power is calculated by

$$= \sum_{n=-\infty}^{\infty} C_n C_{-n} \tag{1.34}$$

Here C_{-n} is the conjugate of C_n. If C_n is given by (1.29), then C_{-n} is equal $C_{-n} = A_n - jB_n$. The coefficients of cosines and sines a_n and b_n are related to the complex coefficients as follows.

$$\begin{aligned} |C_0|^2 &= \frac{1}{2}|a_0|^2 \\ |C_n|^2 = |C_{-n}|^2 &= \frac{1}{4}\left(a_n^2 + b_n^2\right) \end{aligned} \tag{1.35}$$

In looking at a spectrum you need to be careful of what is plotted. The Y-axis can be magnitudes (or amplitudes) or it can be either peak or average power. Most often though spectrums computed by Matlab and other programs are for peak power.

Example 1 – Compute complex coefficients of a cosine wave.

$$f(t) = A\cos\omega t$$

$$= \frac{A}{2}e^{j\omega t} + \frac{A}{2}e^{-j\omega t}$$

We get the trigonometric coefficients by looking at the coefficient of the cosine. It is simply A at frequency ω. We get the complex coefficients by looking at the coefficients of the two exponential in the second equation. They are A/2 located at $+\omega$ and $-\omega$. (Note they are really the coefficients of positive and negative exponentials, and not $+\omega$ and $-\omega$ but now we are following the conventional way to talking about complex frequency.) The x-axis is labeled as exponential frequency, but now you know what this means.

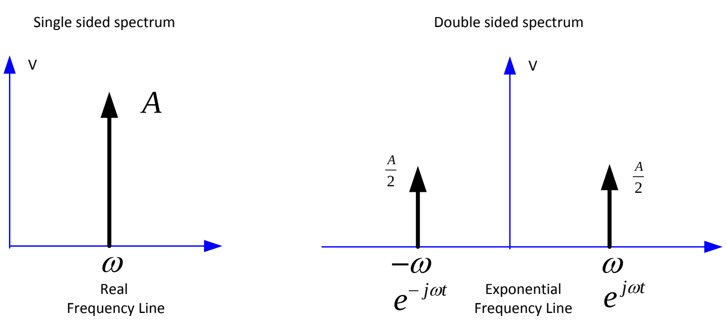


Figure 5 – Amplitude spectrum of $A\cos\omega t$

Example 2 – Compute complex coefficients of a sine wave

$$f(t) = A\sin\omega t$$

$$= \frac{A}{2j}e^{j\omega t} - \frac{A}{2j}e^{-j\omega t}$$

We get the trigonometric coefficient by looking at the first equation. It is simply A and it is at ω = 1 just as it was for the cosine wave above. We get the complex coefficients by looking at the coefficients of the two exponentials in the second equation. They are A/2 and -A/2 located at + ω and - ω. Presence of j tells us that the coefficients are on the Imaginary axis.

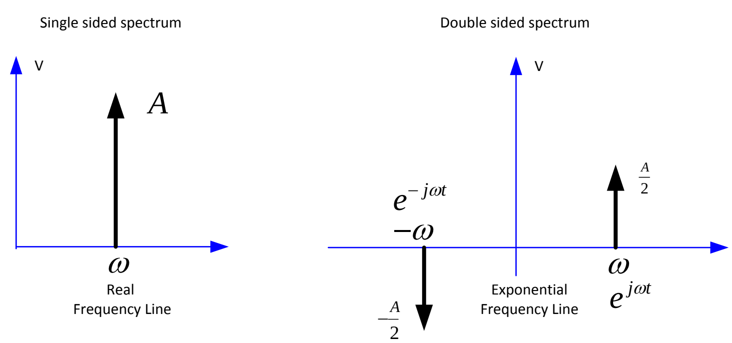


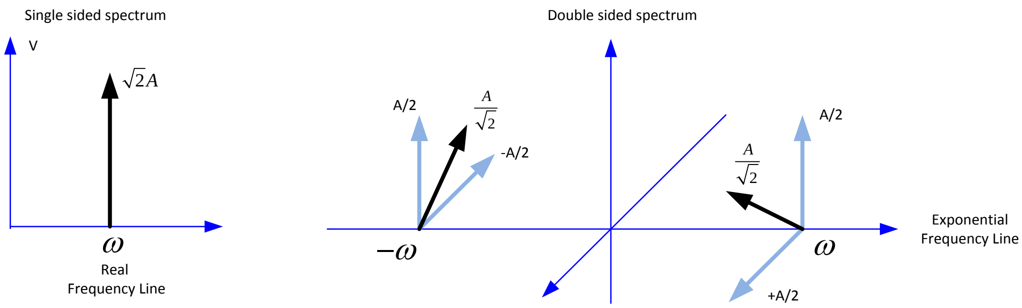
Figure 6 – Amplitude spectrum of $A\sin\omega t$

Example 3 – Compute coefficients of f(t) = A(cos wt + sin wt)

$$f(t) = A(\cos\omega t + \sin\omega t) \tag{1.36}$$

$$= \frac{A}{2}e^{j\omega t} + \frac{A}{2}e^{-j\omega t} + \frac{A}{2j}e^{j\omega t} - \frac{A}{2j}e^{-j\omega t}$$

We get the trigonometric coefficients from looking at the first equation. It is simply A for the cosine and A for sine with magnitude equal to square root of $\sqrt{2}A$ located at ω = 1.We get the complex coefficients by looking at the coefficients of the two exponentials in the second equation. e^{jwt} exponential has two coefficients at 90 degrees to each other, each A/2. The vector sum of these is $A/\sqrt{2}$. Same for the negative exponential, except the amplitude contribution from the sine is negative. However, the vector sum for both is the same. This is shown in second figure drawn in a more conventional style showing only the vector sum. Note that the one-sided spectrum is twice that of complex.



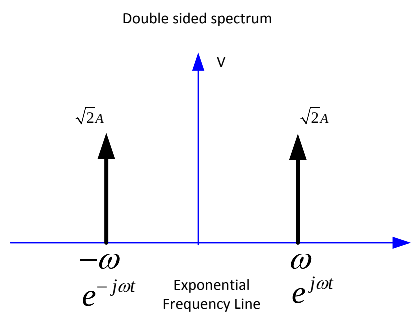


Figure 7 – Amplitude spectrum of $A\sin\omega t + A\cos\omega t$

Up to now the signals have been real. A signal is real if its coefficient is equal to its complex conjugate. The signal in Example 4 is not a real signal. Its complex coefficients are not symmetrical.

Example 4 - Compute coefficients of the complex signal $f(t) = A\cos\omega t + jA\sin\omega t$

$$f(t) = A(\cos w + j\sin wt)$$

$$= \frac{A}{2}e^{j\omega t} + \frac{A}{2}e^{-j\omega t} + \frac{A}{2}e^{j\omega t} - \frac{A}{2}e^{-j\omega t}$$
$$= Ae^{j\omega t}$$

Now here we see something different. The coefficients from sine and cosine for the negative exponential cancel. On the positive side, the contribution from sine and cosine are coincident and add. So we see a single value at positive frequency only. For this signal both the single and double-sided spectrums are identical. This is a surprising and perhaps a counter-intuitive result. Only real signals have symmetrical spectrums. Complex signals do not.

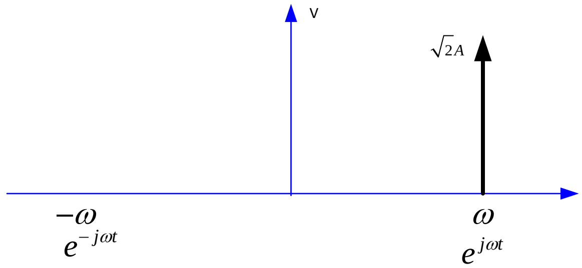


Figure 8 – Double-sided spectrum of $A\cos\omega t + jA\sin\omega t$

Example 5 – Compute coefficients of a constant signal, A.

We can write the function A as an exponential of zero frequency.

$$f(t) = A\cos(\omega = 0)t$$
$$= \frac{A}{2}e^{j\omega t} + \frac{A}{2}e^{-j\omega t} = A$$

From the first representation, we get the trigonometric coefficient = A at ω = 0. From the complex representation we get the two complex coefficients, A/2 and A/2 but both are at ω = 0 so their sum is A which is exactly the same as in the trigonometric representation. The function f(t) is a non-changing function of time and we classify it as a DC signal. The DC component, if any, always shows up at the origin for this reason. The single and double sided spectrums here are same as well.

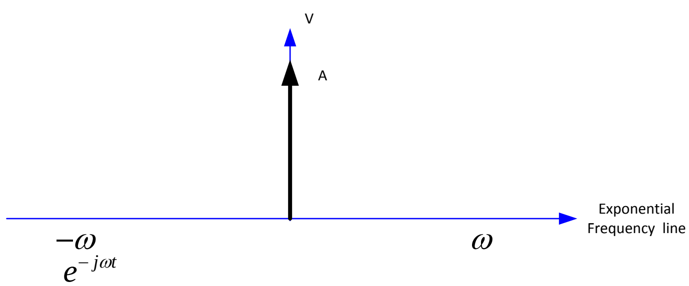


Figure 9 – Double-sided spectrum of A

Example 6 – Compute coefficients of $x(t) = 2\cos^2(\omega t)$

We can express this function in complex form as

$$= 2\left(\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right)^2$$
$$= 1 + \frac{1}{2}e^{j2\omega t} + \frac{1}{2}e^{-j2\omega t}$$

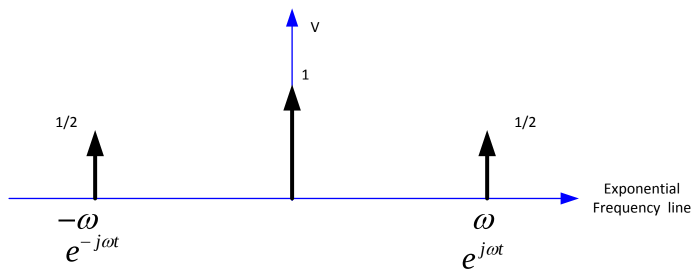


Figure 10 – Double-sided amplitude spectrum of $2\cos^2(\omega t)$

Note that the function has a large value at DC.

Example 7 – Compute coefficients of a

$$x(t) = 2\cos(\omega t)\cos(2\omega t)$$

We can express this in complex form as

$$= \cos(\omega t) + \cos(3\omega t)$$
$$= \frac{1}{2}e^{j\omega t} + \frac{1}{2}e^{-j\omega t} + \frac{1}{2}e^{j3\omega t} + \frac{1}{2}e^{-j3\omega t}$$

Noting how easily we were able to convert this function to a complex form and get the coefficients, tells you that it is much easier it is to work in complex form. From this we draw the spectrum as

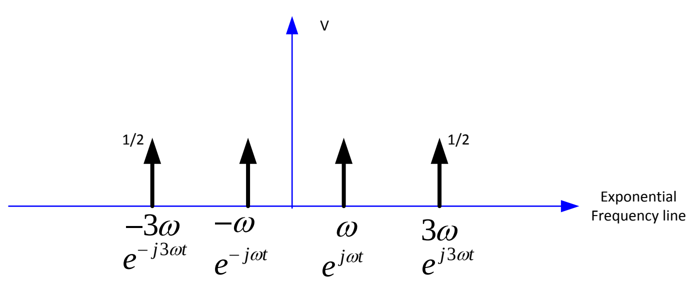


Figure 11 – Double-sided amplitude spectrum of $2\cos(\omega t)\cos(2\omega t)$

Example 8 – Compute complex coefficients of a real signal. Also compute its power spectrum.

$$f(t) = .8\cos 2\pi t - .6\sin 2\pi t + .8\cos 4\pi t + .3\sin 7\pi t$$

The signal has three harmonics, at $\omega = 2\pi$, 4π and 7π. We can write down the trigonometric coefficients for each harmonic just by looking at the equation. We can write this equation is complex form as

$$= \frac{0.8}{2}e^{j2\pi t} + \frac{0.8}{2}e^{-j2\pi t} - \frac{0.6}{2j}e^{j2\pi t} + \frac{0.6}{2j}e^{-j2\pi t} + \frac{0.8}{2}e^{j4\pi t} + \frac{0.8}{2}e^{-j4\pi t} + \frac{0.3}{2j}e^{j7\pi t} - \frac{0.3}{2j}e^{-j7\pi t}$$

Here we have contributions from both sine and cosine at $\omega = 2\pi$, so these have to be vector summed. The contributions at $\omega = 4\pi$ comes only from a cosine and at $\omega = 7\pi$ only from a sine. Note we plot these on the same line at full amplitude as if j does not exist in the equation.

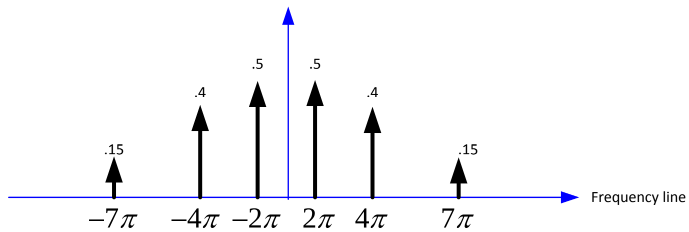


Figure 12 - Two-sided spectrum

The peak power spectrum of this signal is same as the above except each quantity is squared. If spectrum of the average power is desired, then the amplitude should be divided by $\sqrt{2}$ first before squaring, which is same as one half of the peak power.

Example 9 – Compute complex coefficients of another real signal with variable phase

$$x(t) = 3 + 6\cos(4\pi t + 2) + j4\sin(4\pi t + 3) - j6\sin(10\pi t + 1.5)$$

We convert this to the complex form as

$$\begin{aligned} x(t) &= 3 + \left(3e^{4\pi t}e^{2} + 3e^{-4\pi t}e^{-2}\right) + \left(2e^{4\pi t}e^{3} - 2e^{-4\pi t}e^{-3}\right) + \left(3e^{10\pi t}e^{1.5} - 3e^{-10\pi t}e^{-1.5}\right) \\ &= 3 + e^{j4\pi t}\left(3e^{2j} + 2e^{3j}\right) + e^{-j4\pi t}\left(3e^{-2j} - 2e^{-3j}\right) + 3e^{j10\pi t}\left(e^{j1.5}\right) + 3e^{-j10\pi t}\left(e^{-j1.5}\right) \end{aligned}$$

The magnitudes of the exponential come from the phasors in parenthesis. To add them we need to convert them first to rectangular form as follows. (See Appendix)

$$e^{j4\pi t}\left(3e^{2j}+2e^{3j}\right)$$
$$=\sqrt{\left(3\cos(2)+2\cos(3)\right)^2+\left(3\sin(2)+2\sin(3)\right)^2}$$
$$=6.16e^{j4\pi t}$$

Similarly, the coefficient of the negative exponential is

$$e^{-j4\pi t}\left(3e^{-2j}-2e^{-3j}\right)$$
$$=\sqrt{\left(3\cos(2)-2\cos(3)\right)^2+\left(3\sin(2)-2\sin(3)\right)^2}$$
$$=3.46e^{-j4\pi t}$$

We draw the spectrum as in Figure below and note that the spectrum is not symmetric because the signal is not real.

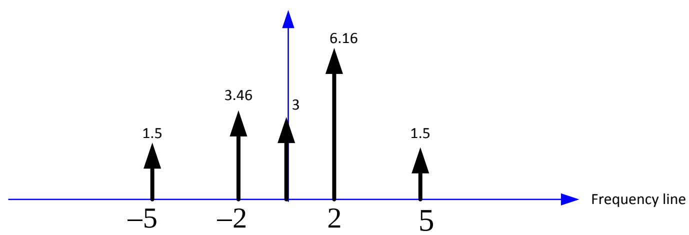


Figure 13 - Two-sided spectrum of a complex signal

Try other functions in the Matlab Program no. 2 at the end of the chapter.

In this section, we covered the complex form of Fourier series as a prelude to the next topic which is Fourier Transform. We see that even though the time domain function is a continuous function, the Fourier series coefficients and hence the spectrum developed is discrete. In most real applications the signals are sampled and hence not continuous. We will see how that affects the Fourier spectrum.

A little bit about complex numbers

In DSP, we use complex numbers to denote quantities that have more than one parameters associated with them. A point in a plane is one example. It has a y coordinate and a x coordinate. Another example is a sine wave, it has a frequency and a phase. The two parts of a complex number are denoted by the terms Real and Imaginary, but the Imaginary pat is just as real as the Real part. Both are equally important because they are needed to nail down a physical signal. It often said that the signals traveling through the air are real signals and that it is only in the hardware that the processing is done in the complex domain. There is a very real analogy that will make this clear. When you hear any sound, the processing is done by our brains with two orthogonally placed receivers, the ears. The ears hear the sound in slightly different phase and time delay. The received signal is different by the two ears and from this our brains can derive fair amount of information about the direction, amplitude and frequency of the sound. So although yes, most signals are real, the processing needs to be done in complex plane if we are to drive maximum information.

The concept of complex numbers starts with real numbers as points on a line. Multiplication of a number by -1 rotates that point 180° about the origin on the number line. If a point is 3, then multiplication by -1 makes it -3 and it is now located 180° from +3. Multiplication by -1 can be seen as 180° shift. Multiplying this rotated number again by -1, gives the original number back, we can say adding another 180° shift. So multiplication by $(-1)^2$ results in a 360° shift. What do we have to do to shift a number off the line, say by 90°? This is where j comes in. Multiply 3 by j, so it becomes 3j. Where do we plot it now? Herein lies our answer to what multiplication with j does.

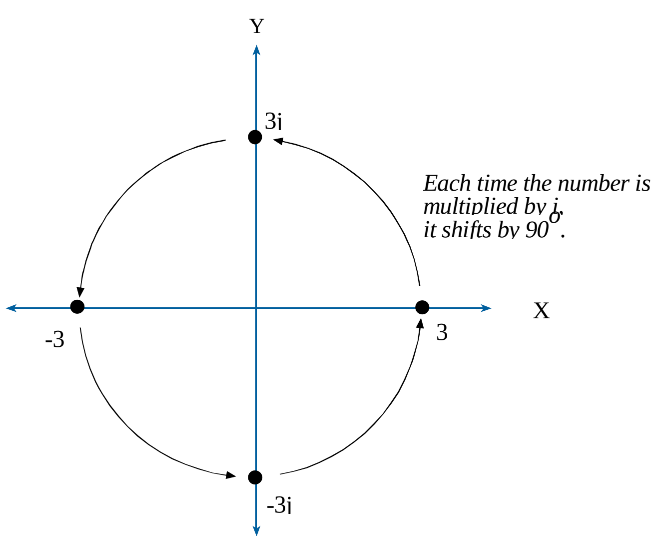


Figure 14 – Multiplication by j shifts the point by 90°.

The magnitude of the number stays exactly the same, 3j is the same as 3, except that multiplication with j shifts the angle of this number by +90°. So instead of an X-axis number, it becomes a Y-axis number. It is no longer located on the real number line where it was. Each subsequent multiplication by j rotates it further by 90° in anti-clockwise manner in the X-Y plane as shown in Figure 1. 3 become 3j, then -3 and then -3j and back to 3 doing a complete 360° degree turn.

Question: What does division by j mean?
Answer: It is same as multiplying by –j.

$$\frac{x}{j} \times \frac{j}{j} = \frac{jx}{-1} = -jx$$

This is essentially the concept of complex numbers. Complex numbers often thought of as "complicated numbers" follow all of the common rules of mathematics. Perhaps a better name for complex numbers would have been 2D numbers. To further complicate matters, the axes, which were called X and Y in Cartesian mathematics are now called respectively *Real* and *Imaginary*. Why so? Is the quantity 3j any less *real* than 3? This semantic confusion is the unfortunate result of the naming convention of complex numbers and helps to make them confusing, complicated and of course complex.

Now let's compare how a number is represented in the complex plane.

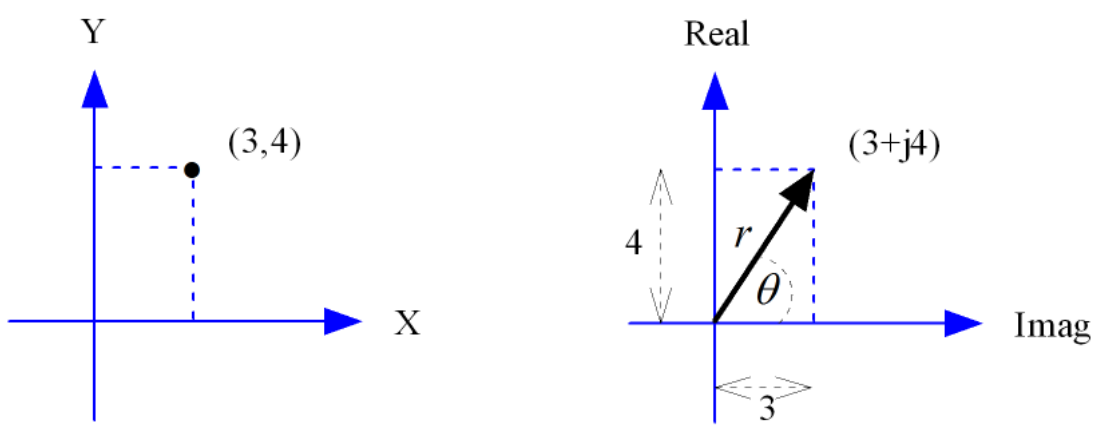


Figure 15 – a. A point is space on a Cartesian diagram

b. Plotting a complex function on a complex plane

Plot a complex number, 3 + j4. In a Cartesian plot we have the usual X-Y axes and we write this number as (3, 4) indicating 3 units on the X-axis and 4 units in the Y-axis. We can represent this number in a complex plane in two ways. One form is called the **rectangular form** and is given as

$$z = x + jy$$

The part with the j is called the **imaginary part** (although of course it is a real number) and the one without is called the **real part**. Here 3 is the Real part of z and 4 is the Imaginary part. Both are real numbers of course, I add that just to confuse you some more. Note when we refer to the imaginary part, we do not include j. The symbol j is there to remind you that this part (the imaginary part) lies on a different axis.

$$\operatorname{Re}(z) = x$$
$$\operatorname{Im}(z) = y$$

Alternate form of a complex number is the **polar form**.

$$z = M\angle\theta$$

Where M is the magnitude and θ its angle with the real axis.

The polar form which looks like a vector and in essence it is, is called a phasor in signal processing. This form comes from circuit analysis and is very useful is that realm. We also use it in signal processing but it seems to cause some conceptual difficulty here. Mainly because unlike in circuit analysis, in signal processing time is important. We are interested in signals in time domain and the phasor which a time-less concept is confusing. The phase as the term is used in signal processing is kind of the initial value of phase, where it is an angle in vector terminology.

Question: If $z = Ae^{j\omega t}$ then what is its rectangular form?

Answer: $z = A\cos\omega t + Aj\sin\omega t$. We just substituted the Euler's equation for the complex exponential $e^{j\omega t}$. Think of $e^{j\omega t}$ as a shorthand functional notation for the expression $\cos\omega t + j\sin\omega t$. The real and imaginary parts of z are given by

$$\operatorname{Re}\{z\} = A\cos\omega t$$
$$\operatorname{Im}\{z\} = A\sin\omega t$$

Converting forms

Rule:

1. Given a rectangular form $z = x + jy$ then its polar form is equal to

$$\begin{aligned} M\angle\theta &= \sqrt{x^2+y^2}\angle\tan^{-1}(y/x) \\ &= \sqrt{x^2+y^2}\angle\left(\tan^{-1}(y/x)+\pi\right) \quad \text{if } x<0 \end{aligned}$$

2. Given a polar form $z = M\angle\theta$ then its rectangular form is given by

$$x + jy = M\cos\theta + jM\sin\theta$$

Example: Convert $z = 5\angle.927$ to rectangular form

Real part = $5\cos(.927) = 3$

Imaginary part = $5\sin(.927) = 4$

Z = 3 + j4

Example: Convert $z = -1 - j$ to rectangular form

$$M = \sqrt{(-1)^2 + (-j)^2} = \sqrt{2}$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

$$= \arctan\left(\frac{y}{x}\right) + \pi \quad \text{If x < 0}$$

$$= \arctan\left(\frac{-1}{-1}\right) + \pi = 3\pi / 4$$

$z = \sqrt{2}\angle 3\pi / 4$

Example: convert z = 1+j to polar form

Z = $\sqrt{1^2 + j^2} = \sqrt{2}$ and $\theta = \arctan(1) = \pi / 4 = .785$

Adding and multiplying

Add in rectangular form, multiply in polar. Its easier this way.

Rule: Given $z_1 = a + jb$ and $z_2 = c + jd$ then $z_1 + z_2 = (a+c) + j(b+d)$.

Rule: Given $z_1 = M_1\angle\theta_1$ and $z_2 = M_2\angle\theta_2$ then

$z_3 = z_1 \times z_2 = M_1 M_2 \angle(\theta_1 + \theta_2)$

Example: Add $z_1 = \sqrt{2}\angle.785$ and $z_2 = 5\angle.927$

Convert both to rectangular form.

$z_1 = 1 + j$ and $z_2 = 3 + 4j$

$z_3 = (1+3) + j(1+4) = 4 + j5$

Example: Multiply $z_1 = 1 + j$ and $z_2 = 3 + 4j$

First convert to polar form and then multiply. Although multiplying these two complex numbers in rectangular format looks easy in general that is not the case. Polar form is better for multiplication and division.

$z_1 = \sqrt{2}\angle.785 \times z_2 = 5\angle.927 = 5\sqrt{2}\angle 1.71$

Example: Divide $z_1 = 1 + j$ and $z_2 = 3 + 4j$

$z_2 = 5\angle.927 \div z_1 = \sqrt{2}\angle.785$

then $z_3 = \dfrac{z_2}{z_1} = \dfrac{5}{\sqrt{2}}\angle(.927 - .785)$

Conjugation

The conjugate for a complex number z, is given by $z^* = x - jy$.

For a complex exponential $e^{j\omega t}$ is the complex conjugate of $e^{-j\omega t}$. In polar format the complex conjugate is same phasor but rotating in the opposite direction.

$$z = M\angle\theta$$

$$z^* = M\angle -\theta$$

Useful properties of complex conjugates

$$|z|^2 = zz^*$$

This relationship is used to compute the power of the signal. The magnitude of the signal can be computed by half the sum of the signal and its complex conjugate. Note the imaginary part cancels out in this sum.

$$|z| = \frac{1}{2}\left(z + z^*\right)$$

Matlab Program no. 1 for plotting the complex exponential

```
%comexp produces the complex exponential diagram in Chapter 1.
%Try changing target function to see effect on signal.
t = 0:0.01:5;
y=5*exp(-(j*(2*pi))*t); % change this equation for different cases
subplot(2,2,1);
plot3(t,real(y),imag(y));
grid
xlabel('t'),ylabel('Re(y)'),zlabel('Im(y)');
title('3-D plot of a Complex Exponential');

subplot(2,2,3),plot(t,real(y)),xlabel('t'),
ylabel('Magnitude'),title('Re(y(t))');

subplot(2,2,4),plot(t,imag(y)),xlabel('t'),
ylabel('Angle'),title('Im(y(t))');
```

Matlab Program no. 2 – Compute double sided DFT of a signal

```
clc;close all; clear all;
%Generate the signal
fs=128; % Sampling rate should be at least 16 times higher frequency to get a good picture.
N=1024; % FFT size.
t=0:1/fs:((N/fs)-(1/fs)); %Time it takes to creat N points.
x=3+  6*cos(8*pi*t+2) +j*8*sin(8*pi*t+3) - j*6*sin(30*pi*t+1); %The target signal.
figure(1) ;
plot(t, x); %Plot signal, all points.
%fplot('x', [0,1]) % want to plot the first 1 second only.
xlabel('Time (Hz)');ylabel('Signal in time domain ' ) ;

figure(2) ;
xf1=abs(fft(x) )/N;% Compute the Double sided amplitude spectrum
xf=fftshift(xf1);
P=xf.*xf; %compute the power spectrum
% Map the frequency bin to the frequency (hz)
f=LINSPACE(0, fs, N);
f3=[-fs/2:fs/2:1024];%fk=k fs/N where k=0,1,2,...N-1
f2 = linspace(-fs/2, fs/2, 1024);
% now we will plot the DFT spectrums
plot(f2,xf);grid
xlabel('Frequency (Hz)');ylabel('Double-sided Amplitude Spectrum (DFT) ' ) ;
```