

Assignment 3

PHSGCOR04T

Sound

Model Questions & Answers

Lissajous Figures & Beat

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60 Lectures \equiv 4 Credits

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1. Compositions of Two SHMs in a Straight line: Motions having Equal periods:

When a particle is acted upon simultaneously by two SHMs in the same straight line, its displacement at any instant is the algebraic sum of the individual displacements for the two separate motions. Let the two SHMs of the same period be represented by

$$x = a \sin(\omega t + \phi) \quad (1)$$

and

$$y = b \sin(\omega t + \psi) \quad (2)$$

where a and b are the amplitudes. The parameters ϕ and ψ are initial phases of the two motions. The period of each is $\frac{2\pi}{\omega}$. The resultant displacement z at any instant t is then given by

$$z = x + y \quad (3)$$

$$= a \sin(\omega t + \phi) + b \sin(\omega t + \psi) \quad (4)$$

Using the identity

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

we find that

$$\begin{aligned} z &= a (\sin \omega t \cos \phi + \cos \omega t \sin \phi) + b (\sin \omega t \cos \psi + \cos \omega t \sin \psi) \\ z &= a \sin \omega t \cos \phi + a \cos \omega t \sin \phi + b \sin \omega t \cos \psi + b \cos \omega t \sin \psi \\ z &= \sin \omega t (a \cos \phi + b \cos \psi) + \cos \omega t (a \sin \phi + b \sin \psi) \end{aligned} \quad (5)$$

Let us make a change in constants a, b, ϕ and ψ by putting

$$a \cos \phi + b \cos \psi = r \cos \theta, \quad (6)$$

$$a \sin \phi + b \sin \psi = r \sin \theta, \quad (7)$$

where r and θ are new constants. This gives

$$z = \sin \omega t (r \cos \theta) + \cos \omega t (r \sin \theta) \quad (8)$$

$$z = r (\sin \omega t \cos \theta + \cos \omega t \sin \theta) = r \sin(\omega t + \theta). \quad (9)$$

This equation is of the same type as Eq. (1) and Eq. (2) which represent SHMs. Hence the resultant motions of the particle is also simple harmonic with the same period $\frac{2\pi}{\omega}$ as the individual motions, but with a different amplitude r and a different initial phase θ .

To obtain r , let us square and add Eq. (6) and Eq. (7), when we get

$$\begin{aligned} r^2(\cos^2 \theta + \sin^2 \theta) &= (a \cos \phi + b \cos \psi)^2 + (a \sin \phi + b \sin \psi)^2 \\ r^2 &= a^2 \cos^2 \phi + b^2 \cos^2 \psi + 2ab \cos \phi \cos \psi + a^2 \sin^2 \phi + b^2 \sin^2 \psi + \\ &\quad 2ab \sin \phi \sin \psi \\ r^2 &= a^2(\cos^2 \phi + \sin^2 \phi) + b^2(\cos^2 \psi + \sin^2 \psi) \\ &\quad + 2ab(\cos \phi \cos \psi + \sin \phi \sin \psi) \end{aligned}$$

2. Using the identity

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

We get

$$r^2 = a^2 + b^2 + 2ab \cos(\phi - \psi)$$

Therefore the resultant amplitude r is given by

$$r = \sqrt{a^2 + b^2 + 2ab \cos(\phi - \psi)}. \quad (10)$$

To obtain the resultant phase θ , let us divide Eq. (7) by Eq. (6), we get

$$\tan \theta = \frac{a \sin \phi + b \sin \psi}{a \cos \phi + b \cos \psi}.$$

There are two special cases:

Case I: When the phase difference between two individual motions is zero, or any any integral multiple of 2π , i.e.

$$\phi - \psi = 2n\pi,$$

where $n = 0, 1, 2, \dots$, then Eq. (10) gives

$$r = \sqrt{a^2 + b^2 + 2ab \cos(2n\pi)}$$

Using the identity $\cos 2n\pi = 1$, we find

$$r = \sqrt{a^2 + b^2 + 2ab} = \sqrt{(a + b)^2} = a + b$$

Thus the resultant amplitude is equal to the sum of the amplitudes of the individual motions.

Case II: When the phase difference between two individual motions is an odd multiple of π , i.e.

$$\phi - \psi = (2n + 1)\pi,$$

where $n = 0, 1, 2, \dots$, then Eq. (10) gives

$$r = \sqrt{a^2 + b^2 + 2ab \cos(2n + 1)\pi}$$

Using the identity $\cos(2n + 1)\pi = -1$, we find

$$r = \sqrt{a^2 + b^2 - 2ab} = \sqrt{(a - b)^2} = a - b$$

Thus the resultant amplitude is equal to the difference of the amplitudes of the individual motions. In this special case, if $a = b$, then $r = a - b = 0$ that means if a particle is acted upon simultaneously by two simple harmonic motions of equal amplitudes and periods but opposite in phase in the same straight line, it will remain at rest.

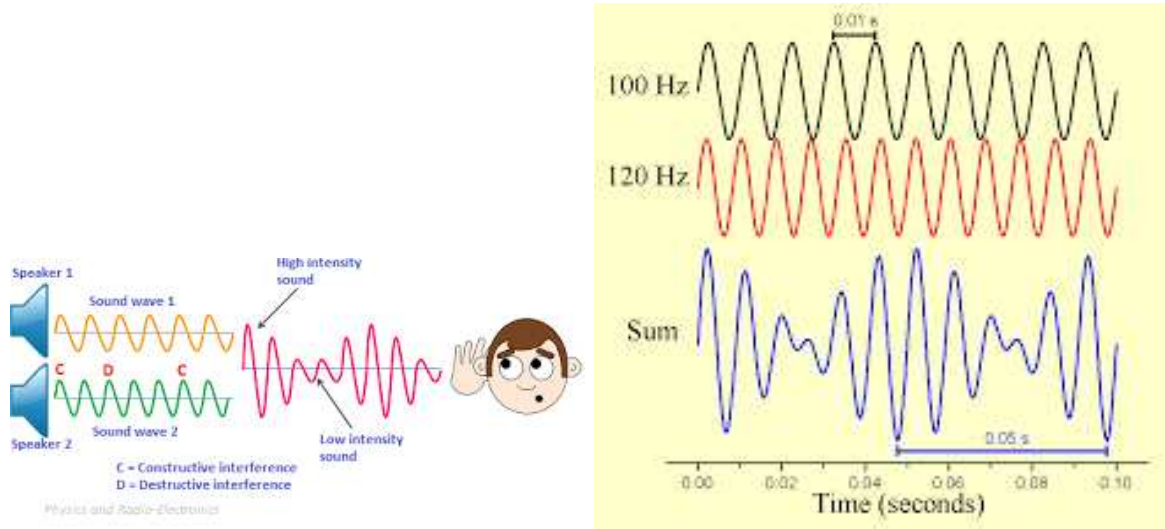


Figure 1:

3. What is Beat? Derive analytical expression.

When two simple harmonic waves of same amplitude but different frequency (or wavelength) are superposed on each other there is a regular rhythm of sound or throbbing sound could be heard. This is called beats. Let us now consider what happens when two SHMs of equal amplitudes but different periods (or different frequencies) are superposed. Let us express them as

$$x = a \sin \omega_1 t = a \sin 2\pi n_1 t \quad (11)$$

$$y = a \sin \omega_2 t = a \sin 2\pi n_2 t, \quad (12)$$

where n_1 and n_2 are the frequencies. The resultant displacement is given by

$$z = x + y = a(\sin 2\pi n_1 t + \sin 2\pi n_2 t) \quad (13)$$

Using the identity

$$\sin A + \sin B = 2 \sin \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right)$$

The resultant displacement is thus

$$z = 2a \sin 2\pi \left(\frac{n_1 + n_2}{2} \right) t \cos 2\pi \left(\frac{n_1 - n_2}{2} \right) t \quad (14)$$

$$= \left[2a \cos 2\pi \left(\frac{n_1 - n_2}{2} \right) t \right] \sin 2\pi \left(\frac{n_1 + n_2}{2} \right) t \quad (15)$$

$$= A \sin 2\pi \left(\frac{n_1 + n_2}{2} \right) t \quad (16)$$

where

$$A = 2a \cos 2\pi \left(\frac{n_1 - n_2}{2} \right) t$$

The above Eq. (16) represents a SHM of frequency

$$\left(\frac{n_1 + n_2}{2} \right)$$

and of amplitude

$$A = 2a \cos 2\pi \left(\frac{n_1 - n_2}{2} \right) t$$

which is a function of time t .

- This shows that the resultant amplitude of motion varies periodically between $2a$ and zero under the influence of cosine term.
- This amplitude A will change, therefore, as the time passes on.
- Sometimes the amplitude will be maximum and sometimes minimum.
- Consequently the intensity of the sound which is proportional to the square of the amplitude, will be maximum at one time and minimum at the other. In other words, beats will be produced

4. What is Beat frequency?

- The amplitude A becomes **maximum** when

$$\cos 2\pi \left(\frac{n_1 - n_2}{2} \right) t = \pm 1$$

or

$$2\pi \left(\frac{n_1 - n_2}{2} \right) t = 0, \pi, 2\pi, \dots, m\pi \quad [m = 0, 1, 2, \dots \text{etc.}]$$

or

$$t = 0, \left(\frac{1}{n_1 - n_2} \right), \left(\frac{2}{n_1 - n_2} \right) \dots \left(\frac{m}{n_1 - n_2} \right).$$

- Thus, the amplitude of the resultant wave will be **maximum** and equal to $2a$ at times

$$t = 0, t = \left(\frac{1}{n_1 - n_2} \right), t = \left(\frac{2}{n_1 - n_2} \right) \dots$$

- Hence the time interval between two maxima or two waxings

$$= \left(\frac{1}{n_1 - n_2} \right) \tag{17}$$

- Again, the amplitude A becomes **minimum** when

$$\cos 2\pi \left(\frac{n_1 - n_2}{2} \right) t = 0$$

or

$$2\pi \left(\frac{n_1 - n_2}{2} \right) t = \frac{\pi}{2}, \frac{3\pi}{2}, \dots, (2m + 1)\frac{\pi}{2} \quad [m = 0, 1, 2, \dots \text{etc.}]$$

or

$$t = \frac{1}{2(n_1 - n_2)}, \frac{3}{2(n_1 - n_2)} \dots \frac{(2m + 1)}{2(n_1 - n_2)}.$$

- Thus, the amplitude of the resultant wave will be **minimum** and equal to 0 at times

$$t = 0, t = \frac{1}{2(n_1 - n_2)}, t = \frac{3}{2(n_1 - n_2)} \dots$$

Beats: Examples

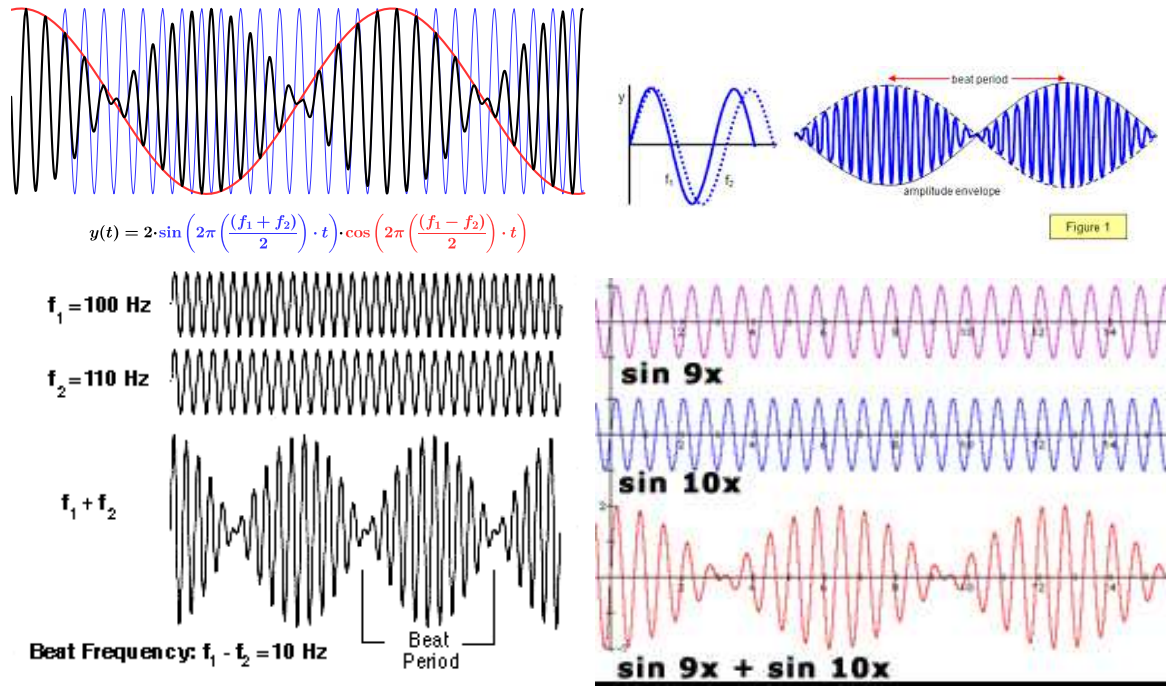


Figure 2:

- Hence, the time interval between two minima or two wanings of sound

$$= \left(\frac{1}{n_1 - n_2} \right) \quad (18)$$

- Eq. (17) and Eq. (18) clearly show that maxima and minima are equally spaced with respect to time and one minimum occurs between two successive maxima.
- Further, time interval between successive maxima or successive minima

$$= \left(\frac{1}{n_1 - n_2} \right) \quad (19)$$

- So, the number of waxings or waning of sound in a second

$$= n_1 - n_2.$$

- Therefore Beat frequency = $n_1 - n_2$ = the differences of frequencies of the tuning forks.

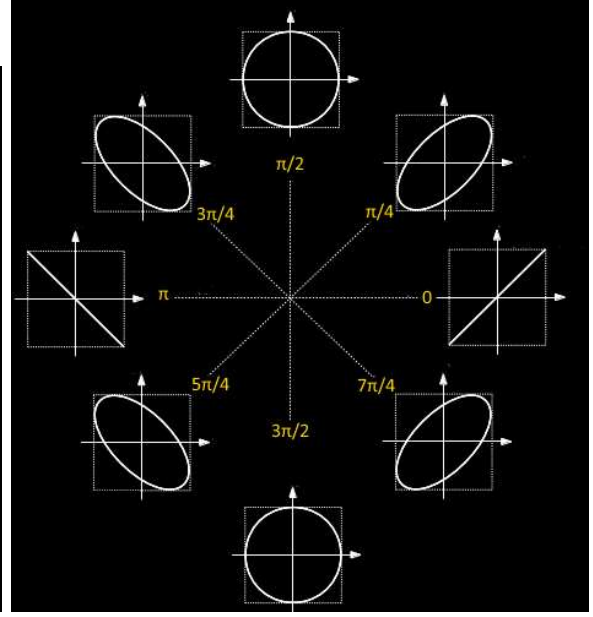
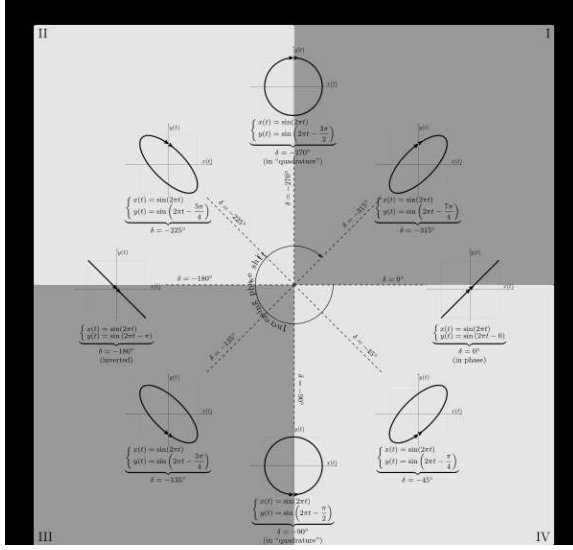
5. What is Lissajous Figures? Derive mathematical expression.

When a particle has superimposed upon it two mutually perpendicular SHMs simultaneously, the resultant path of the particle is known as a “Lissajous figure”. The form of the figure depends upon the ratio of the frequencies (or periods), the individual amplitudes and the relative phases of the two component motions. For example, if the two motions are of equal frequencies, the Lissajous’ figure is a straight line, ellipse or circle depending upon the amplitudes and phases difference of the motions.

Lissajous curves are the family of curves described by the parametric equations

$$x = a \sin(\omega t + \delta) \quad \text{and} \quad y = b \sin \omega t$$

They are sometimes known as Bowditch curves after Nathaniel Bowditch, who studied them in 1815. They were studied in more detail (independently) by Jules-Antoine Lissajous in 1857.



Lissajous curves have applications in physics, astronomy, and other sciences.

Analytical treatment of Lissajous figure

Let two SHMs of amplitudes a and b and of phase-difference δ act at right angles to each other. They are represented by the following equations

$$x = a \sin(\omega t + \delta) \quad (20)$$

$$y = b \sin \omega t \quad (21)$$

After superpositions, the resultant motion can be obtained by eliminating t from Eq. (20) and Eq. (21). Now expanding Eq. (20), we find

$$\frac{x}{a} = \sin \omega t \cos \delta + \cos \omega t \sin \delta. \quad (22)$$

with the aid of identity

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

Now from Eq. (21), we get

$$\sin \omega t = \frac{y}{b}, \Rightarrow \cos \omega t = \sqrt{1 - \sin^2 \omega t} = \sqrt{1 - \frac{y^2}{b^2}}$$

Therefore from Eq. (22), we get

$$\frac{x}{a} = \frac{y}{b} \cos \delta + \sqrt{1 - \frac{y^2}{b^2}} \sin \delta. \quad (23)$$

$$\Rightarrow \frac{x}{a} - \frac{y}{b} \cos \delta = \sqrt{1 - \frac{y^2}{b^2}} \sin \delta. \quad (24)$$

$$\Rightarrow \left(\frac{x}{a} - \frac{y}{b} \cos \delta \right)^2 = \left(1 - \frac{y^2}{b^2} \right) \sin^2 \delta. \quad (25)$$

$$\Rightarrow \frac{x^2}{a^2} - 2 \frac{xy}{ab} \cos \delta + \frac{y^2}{b^2} \cos^2 \delta = \left(1 - \frac{y^2}{b^2} \right) \sin^2 \delta. \quad (26)$$

$$\Rightarrow \frac{x^2}{a^2} - 2\frac{xy}{ab} \cos \delta + \frac{y^2}{b^2} (\cos^2 \delta + \sin^2 \delta) = \sin^2 \delta. \quad (27)$$

$$\Rightarrow \frac{x^2}{a^2} - 2\frac{xy}{ab} \cos \delta + \frac{y^2}{b^2} = \sin^2 \delta. \quad (28)$$

where we have used the identity

$$\cos^2 \delta + \sin^2 \delta = 1$$

This is the general equation of an ellipse enclosed inside a rectangle whose sides are $2a$ and $2b$. Hence the resultant vibration will be elliptical. It also depends on the individual amplitudes a , b and phase difference δ . Thus in general the Lissajous' figure (path of the particle) is an ellipse. There are, however, a number of special cases:

- (i) If $\delta = 0^\circ$, Eq. (28) becomes

$$\frac{x^2}{a^2} - 2\frac{xy}{ab} + \frac{y^2}{b^2} = 0. \quad [\text{Since } \sin 0^\circ = 0, \cos 0^\circ = 1] \quad (29)$$

$$\Rightarrow \left(\frac{x}{a} - \frac{y}{b}\right)^2 = 0 \Rightarrow y = \frac{b}{a}x$$

This represents two coincident straight lines coinciding with the diagonal of the rectangle.

- (ii) If $\delta = \frac{\pi}{4}$, Eq. (28) becomes

$$\frac{x^2}{a^2} - 2\frac{xy}{ab} \left(\frac{1}{\sqrt{2}}\right) + \frac{y^2}{b^2} = \left(\frac{1}{\sqrt{2}}\right)^2. \quad (30)$$

using the value of $\sin(\frac{\pi}{4}) = \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$.

- This represents an oblique ellipse.

- (iii) If $\delta = \frac{\pi}{2}$, Eq. (28) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad (31)$$

which is an ellipse whose axes coincide with the coordinate axes. If in addition $a = b$, the path of the particle becomes a circle, $x^2 + y^2 = a^2$.

- (iv) If $\delta = \frac{3\pi}{4}$, Eq. (28) becomes

$$\frac{x^2}{a^2} - 2\frac{xy}{ab} \left(\frac{1}{\sqrt{2}}\right) + \frac{y^2}{b^2} = \left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}. \quad (32)$$

using the value of $\sin(\frac{3\pi}{4}) = \frac{1}{\sqrt{2}}$ and $\cos(\frac{3\pi}{4}) = -\frac{1}{\sqrt{2}}$.

- This again represents an oblique ellipse.

- (v) If $\delta = \pi$, Eq. (28) becomes

$$\frac{x^2}{a^2} + 2\frac{xy}{ab} + \frac{y^2}{b^2} = 0. \quad [\sin \pi = 0, \cos \pi = -1] \quad (33)$$

$$\Rightarrow \left(\frac{x}{a} + \frac{y}{b}\right)^2 = 0 \Rightarrow y = -\left(\frac{b}{a}\right)x$$

This represents two coincident straight lines coinciding with the other diagonal of the rectangle.

- When the frequencies of the two SHMs are exactly equal, the elliptic path of the particle remains perfectly steady.
- If, however, there is a slight difference between the frequencies, the relative phase δ of the two component motion slowly changes and accordingly the form of the ellipse changes.
- Starting with $\delta = 0$, the ellipse coincides with the diagonal $y = \left(\frac{b}{a}\right)x$ of the rectangle of sides $2a$ and $2b$.
- As δ increases from 0 to $\frac{\pi}{2}$, the ellipse opens out to the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, afterwards closing up again and ultimately coinciding with the other diagonal $y = -\left(\frac{b}{a}\right)x$, as δ increases from $\frac{\pi}{2}$ to π .
- As δ changes from π to 2π , the reverse process takes place untill the ellipse again coincides with the first diagonal.
- Problem 1: *Find the resultant motion when two SHMs having amplitude a and b along the axis x and y respectively. The angular frequencies are 2ω and ω along the said direction. The phase difference is δ .*
- Hints: Let

$$x = a \sin(2\omega t + \delta) \quad (34)$$

$$y = b \sin \omega t \quad (35)$$