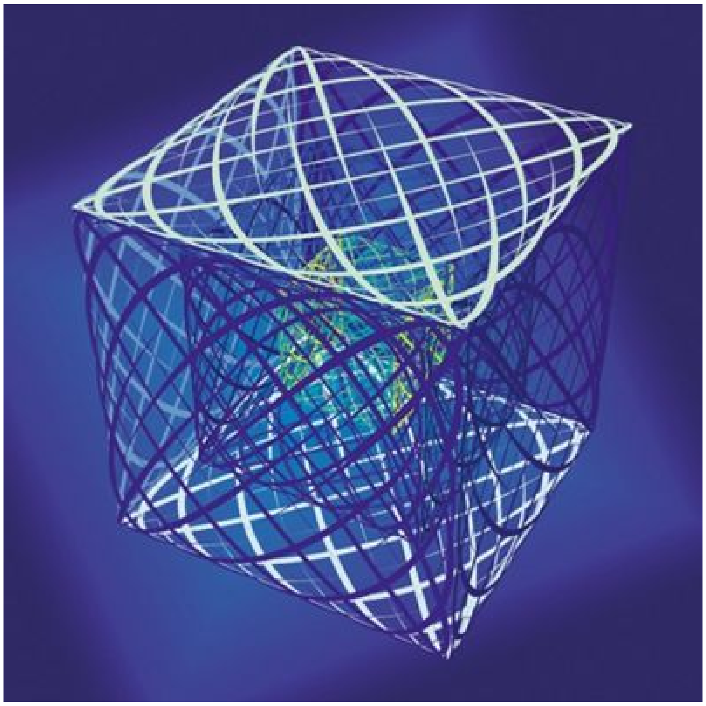
Lissajous Curve

Inprogress.png

Lissajous Box



Field: Geometry

Image Created By: Michael Trott

Website: www.wolfram.com

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Lissajous Box

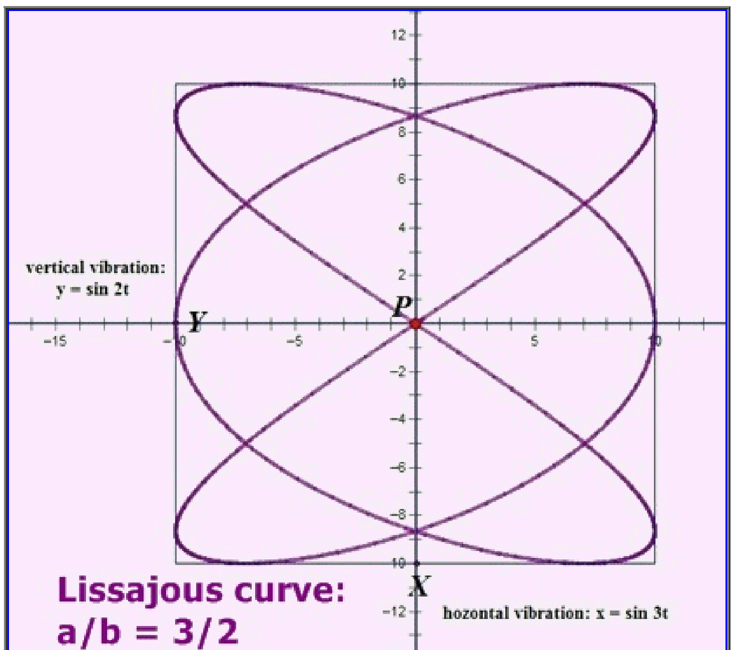
This is a beautiful Lissajous Box. The curves on its sides are Lissajous Curves with a frequency ratio of 10:7.

Contents

Basic Description

In physics, **harmonic vibration** is a type of periodic motion where the restoring force is proportional to the displacement. If you haven't seen harmonic vibrations before, please read through this helper page before proceeding, since this concept is crucial in our following discussion of Lissajous Curves.

Lissajous Curves, or Lissajous Figures, are beautiful patterns formed when two harmonic vibrations along perpendicular lines are superimposed. The following animation tells us how to generate one Lissajous Curve:



Click to stop animation.

Figure 1

How to generate a Lissajous curve

In the animation above, points X and Y are simple harmonic oscillators in x and y directions. They have the same magnitude of 10, but their angular frequencies are different. As we can see in the animation, the x - vibrator completes 3 cycles from the beginning to the end, while the y - vibrator completes only 2. In fact, these vibrators follow the equations of motion $x = \sin(3t)$, and $y = \sin(2t)$, respectively.

Now, we will try to get the superposition of these two vibrations, which is what we really care about. To get this superposition, we can draw from X a line perpendicular to x-axis, and from Y a line perpendicular to y-axis, and locate their intersection P. By simple geometry, P will have the same x-coordinate as X, and y-coordinate as Y, so it combines the motion of X and Y. As we can see in Figure 1, the trace of P turns out to be a complicated and beautiful curve, which we refer to as the "Lissajous Curve". More specifically, it's one Lissajous Curve in a big family, since we can easily generate more Lissajous Curves with other angular frequencies and phases using the same mechanism.

Mathematically speaking, since the motion of point P consists of two component vibrations, whose equations of motions are already known to us, we can easily get the parametric equations of P 's motion:

$$\begin{aligned} x &= A\sin(at+\phi) \\ y &= B\sin(bt) \end{aligned}$$

in which A and B are magnitudes of two harmonic vibrations, a and b are their angular frequencies, and φ is their phase difference. If you are unfamiliar with these terms, please refer to this helper page.

The Lissajous Curve in Figure 1 has $A = B = 10$, $a = 3$, $b = 2$, and $\varphi = 0$. As we have stated before, we can get more Lissajous Curves by changing these parameters. The following images show some of these figures:

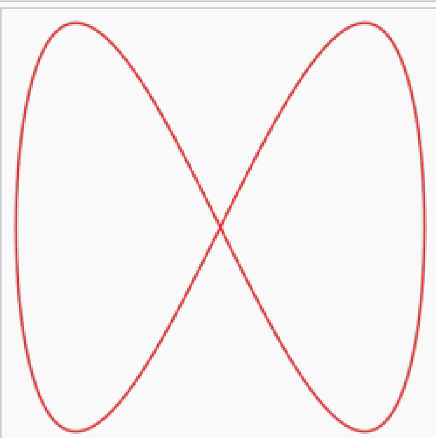


Figure 2-a
Lissajous Curve: $a = 1$, $b = 2$

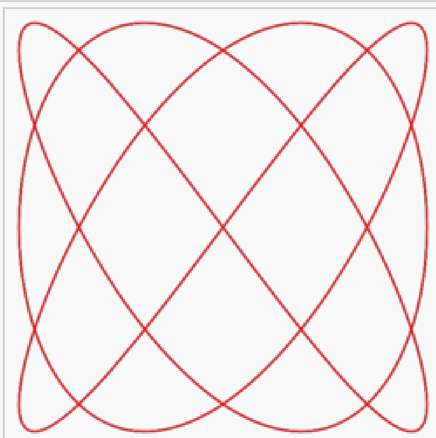


Figure 2-b
Lissajous Curve: $a = 3$, $b = 4$

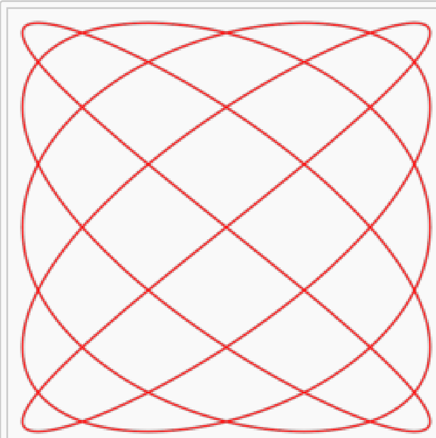


Figure 2-c
Lissajous Curve: $a = 5$, $b = 4$

A Dip Into the History

Lissajous Curves were named after French mathematician Jules Antoine Lissajous (http://en.wikipedia.org/wiki/Jules_Antoine_Lissajous) (1822–1880)[1], who devised a simple optical method to study compound vibrations. Lissajous entered the Ecole Normale Superieure in 1841, and later became a professor of physics at the Lycee Saint-Louis in Paris, where he studied vibrations and sound.

During that age, people were enthusiastic about standardization in science. And the science of acoustics was no exception, since musicians and instrument makers were crying out for a standard in pitches. In response to their demand, Lissajous invented the Lissajous Tuning Forks (http://www.hps.cam.ac.uk/whipple/explore/acoustics/lissajoustuningforks/), which turned out to be a great success since they not only allowed people to visualize and analyse sound vibrations, but also showed the beauty of math through interesting patterns.

Figure 3-a
Photograph of Joules Lissajous. Year and Photographer Unknown

The structure and usage of Lissajous Tuning Forks are shown in Figure 3-b. Each tuning fork is manufactured with a small piece of mirror attached to one prong, and a small metal ball attached to the other as counterweight. Two tuning forks like this are placed besides each other, oriented in perpendicular directions. A beam of light is bounced off the two mirrors in turn and directed to a screen. If we put a magnifying glass between the second tuning fork and the screen (to make the small deflections of light beam visible to human eyes), we can actually see Lissajous Curves forming on the screen.

The idea of visualizing sound vibrations may not be surprising nowadays, but it was ground-breakingly new in Lissajous' age. Moreover, as we are going to see in the More Mathematical Explanation section, the appearances of Lissajous Curves are extremely sensitive to the frequency ratio of tuning forks. The most stable and beautiful patterns only appear when the two forks vibrate at frequencies of simple ratios, such as 2:1 or 3:2. These frequency ratios correspond to the musical intervals of the octave and perfect fifth, respectively. So, by observing the Lissajous Curve formed by an unadjusted fork and a standard fork of known frequency, people were able to make tuning adjustments far more accurately than tuning by ear.

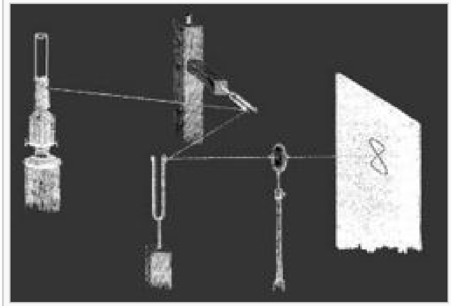


Figure 3-b
Demonstration of Lissajous Tuning Forks

Because of his contributions to acoustic science, Lissajous was honored as member of a musical science commission set up by the French Government in 1858, which also featured great composers such as Hector Berlioz (1803-1869) and Gioachino Rossini (1792-1868).

Acknowledgement: Most of the historical information in this section comes from this website: click here (http://www.hps.cam.ac.uk/whipple/explore/acoustics/lissajoustuningforks/), and *Trigonometric Delights*, by Eli Maor[2][3].

A More Mathematical Explanation

[Click to view A More Mathematical Explanation]

In previous sections, we have encountered this question for many times:

- What determines the appea **[...]**

Why It's Interesting

As a family of beautiful figures, Lissajous Curves are themselves an interesting subject to study. Moreover, they also have some practical applications, including oscilloscopes and harmonographs.

Application to Oscilloscopes

Oscilloscope is a type of electronic instrument in physics that allows observation of constantly varying signal voltages. The following image shows the simplified structure of a typical Cathode Ray Oscilloscope:

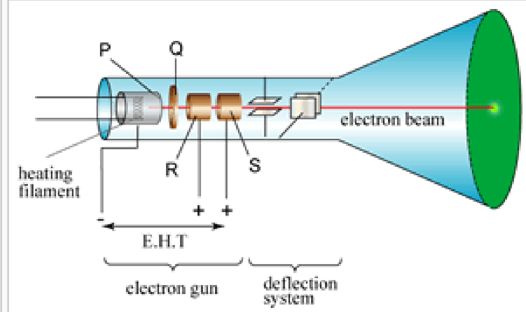


Figure 7-a
Structure of oscilloscope

In Figure 7-1, the electron gun at left generates a beam of electrons when heated, which is then directed through a deflecting system. The deflecting system is made of two sets of parallel metal plates, one for deflection in x - direction, and the other for y - direction. A signal voltage applied to the X-plates gives them an electronic potential difference, generates a uniform electronic field between them, and makes the electron beam deflect in x - direction. Same for the y - plates. The angle of deflection is proportional to the voltage applied.

After passing the deflecting system, the electron beam is then directed to a screen, which is covered by fluorescent material so that we can see green light on the places hit by electrons. If there is no voltage applied to the deflecting system, then the electron beam hits the screen right at the center. If there is voltage applied, then the electrons will hit somewhere else. So the oscilloscope makes signal voltages visible to us.

Now if we apply a sinusoidal signal on each set of the plates, then both the X-plates and the Y-plates will have varying electronic fields between them, and the electron beam will oscillate in both directions. As a result, the trace on the screen should be the superposition of these two oscillations. As we have discussed before, this is a Lissajous Curve. The following images show some of the Lissajous Curves achieved on oscilloscopes:

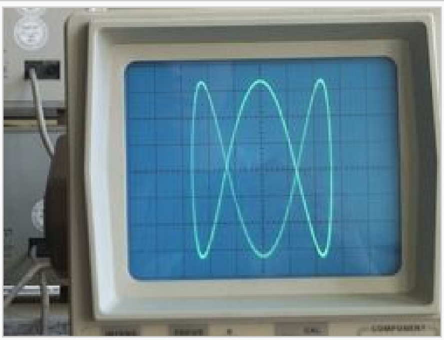


Figure 7-b

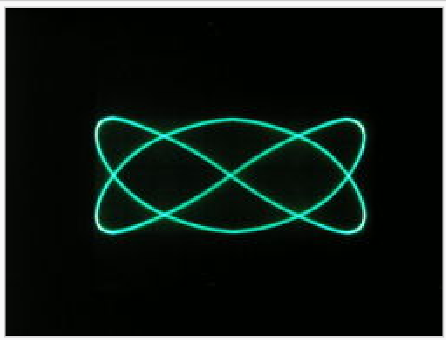


Figure 7-c

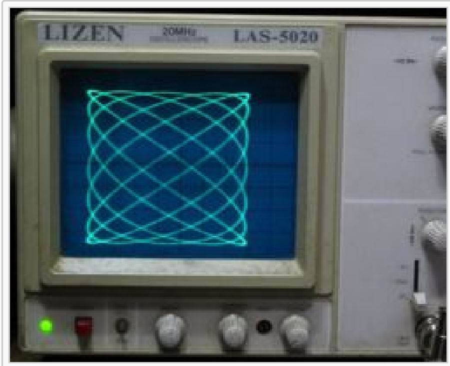
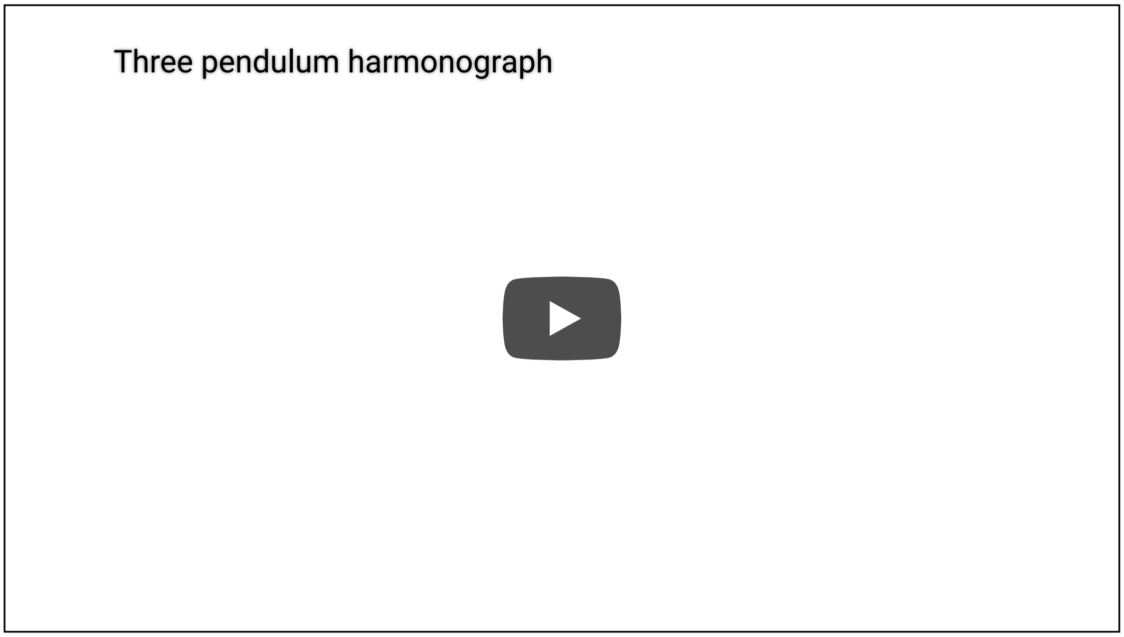


Figure 7-d

Similar to Lissajous Tuning Forks, Lissajous Figures on the oscilloscope can give us some information about the two component vibrations. For example, just by looking at the Lissajous Curve in Figure 7-b, experienced observers can tell that the frequency ratio between its two component vibrations is 1:3, and the phase difference is $\frac{\pi}{2}$. Engineers and physicists often use this method to analyze signals and waves.

Application to Harmonographs

A harmonograph is a mechanical apparatus that employs pendulums to create geometric images. The drawings created are typically Lissajous curves, or related drawings of greater complexity. See the following video the get a sense of how it works[8]:



As we can see in the video, a typical three pendulum rotary harmonograph consists of a table, a drawing board, a pen, and 3 pendulums. Two of them are linear pendulums oriented in perpendicular directions, and they control the motion of the pen. The third pendulum is free to swing in both directions, and it's connected to the drawing board.

Harmonographs can be used to draw Lissajous Curves. We only need to fix the pendulum connecting to the drawing board, and assume that there is no friction in the other two pendulums. In mechanics, it is a known fact that motion of a frictionless pendulum can be viewed as simple harmonic motion, provided that the swinging angle is small[9]. So the pen's motion is the superposition of two perpendicular harmonic vibrations, which is a Lissajous Curve by definition.

However, in practice, friction cannot be completely eliminated. So the two linear pendulums are actually doing damped oscillations, rather than simple harmonic motion. Physicists give us the following equation of motion for damped harmonic oscillations[10]:

$$x(t) = Ae^{-\gamma t}\sin(\omega t + \phi)$$

in which γ is called the damping constant. The larger γ is, the more heavily this oscillator is damped, and the faster its magnitude decreases.

Since both linear pendulums are doing damped harmonic oscillations, the pen should have the following equation of motion:

$$\begin{aligned} x &= Ae^{-\gamma_1 t}\sin(\omega_1 t + \phi) \\ y &= Be^{-\gamma_2 t}\sin(\omega_2 t) \end{aligned}$$

If γ_1 = γ_2, then the common factor $e^{-\gamma t}$ can be extracted from the equations above, and we are left with the parametric equations of a Lissajous Curve:

$$(x(t), y(t)) = e^{-\gamma t}(A\sin(\omega_1 t + \phi), B\sin(\omega_2 t))$$

which gives us a Lissajous Curve with exponentially decreasing magnitudes. For example, see the following computer simulations of the harmonograph with A = B =10, ω_1 = 3, ω_2 = 2, and $\varphi = \pi/2$:

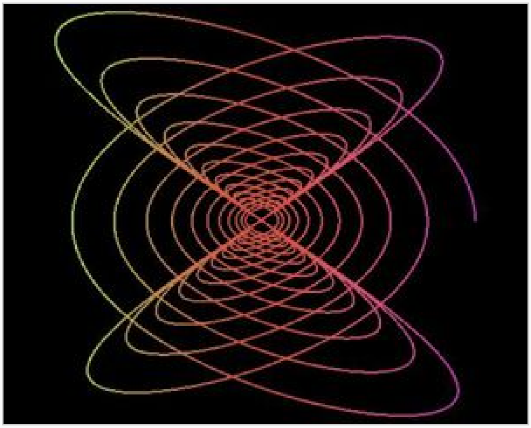


Figure 8-a
Lissajous Curve with decreasing magnitudes

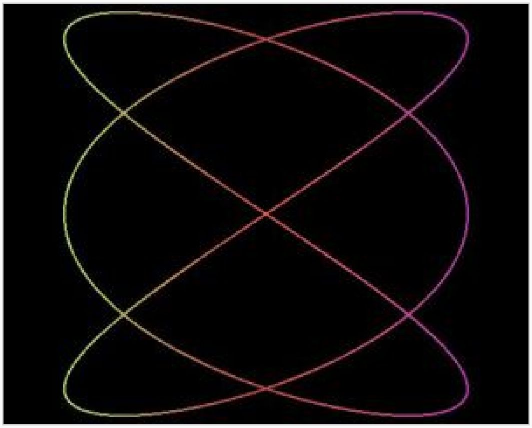


Figure 8-b
The corresponding frictionless case

Figure 8-a shows the damped case with γ = 0.04, and Figure 8-b shows the corresponding frictionless motion with γ = 0. One can clearly see that they have similar shapes, except the magnitude of the curve in Figure 8-a decreases a little bit after each cycle, which is exactly what we mean by damping.

If $\gamma_1 \neq \gamma_2$, then things get more complicated, because the shape of the curve is distorted during the damping process. For example, see the following images:

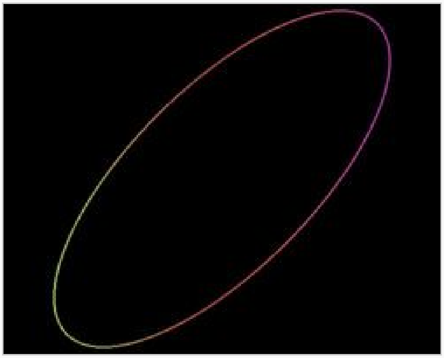


Figure 8-c
The frictionless case

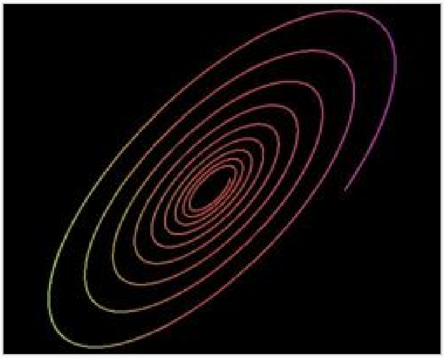


Figure 8-d
$\gamma_1 = \gamma_2 = 0.04$

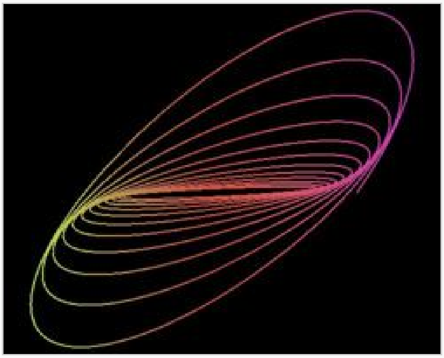
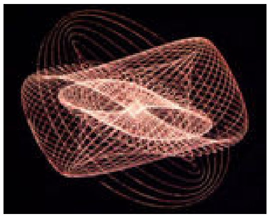
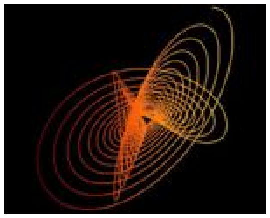
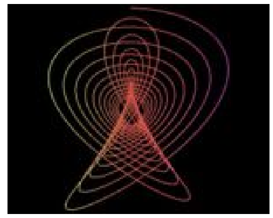
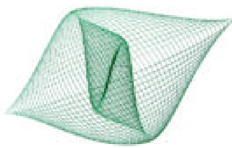
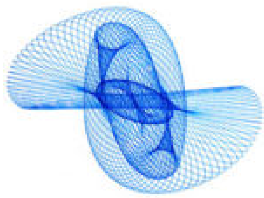
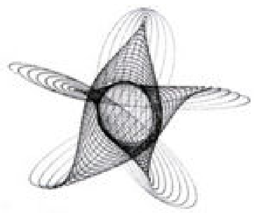


Figure 8-e
$\gamma_1 = 0.01$, $\gamma_2 = 0.04$

The three curves above all have $A = B$ =10, $\omega_1 = \omega_2 = 1$, and $\varphi = \pi/4$. Figure 8-c shows the frictionless case, which is a Lissajous Curve we have discussed before. Figure 8-d shows the damped case with equal damping constants, and one can see that the motion decreases uniformly in both directions. Figure 5-e shows the damped case with unequal damping constants. Since the motion in y - direction is damped much more heavily than in x - direction, the shape of this curve is distorted towards x - axis during the damping process.

If we add more complexity by releasing the free pendulum connecting to the drawing board, then the curve will be the superposition of all these motions. We are not going to study the math behind this, since it's way too complicated with more than 10 variable parameters. However, as we have seen in the video, more complexity also gives us more beautiful and interesting images. The following images are works created by harmonographs. Some of them are computer simulations, others are real pictures from harmonograph makers:

Harmonograph Gallery

Teaching Materials

There are currently no teaching materials for this page. Add teaching materials.

References

1. Jules Antoine Lissajous (http://en.wikipedia.org/wiki/Jules_Antoine_Lissajous), from Wikipedia. This is a biography of Jules Lissajous, discoverer of Lissajous Curves.
2. Lissajous tuning forks: the standardization of musical sound (http://www.hps.cam.ac.uk/whipple/explore/acoustics/lissajoustuningforks/) from Wipple Collections. This is a brief introduction to Lissajous' Tuning Forks and his contribution in acoustic science.
3. *Trigometric Delights*, by Eli Maor, Princeton Press. Pg. 145 - 149: Jules Lissajous and his figures.
4. List of Trigonometric identities (http://en.wikipedia.org/wiki/List_of_trigonometric_identities), from Wikipedia. This page lists some of the trigonometric formula we used the derive the shape of Lissajous curves.
5. Polynomial (http://en.wikipedia.org/wiki/Polynomial#Solving_polynomial_equations), from Wikipedia. This briefly explains why we can't find a general solution for equations of powers higher than 5.
6. Interval (music) (http://en.wikipedia.org/wiki/Interval_(music)), from Wikipedia. This article explains more about musical notes and their frequency intervals.
7. Animated Lissajous figures. (http://ibiblio.org/e-notes/Lis/Lissa.htm)This is the source of the embedded java applet.
8. Three Pendulum Harmonograph (http://www.youtube.com/watch?v=xIl0cqioahs), from youtube. This is the source of the embedded video.
9. Pendulum (http://en.wikipedia.org/wiki/Pendulum), from Wikipedia. This article explains the physics behind pendulums.

10. Damping (http://en.wikipedia.org/wiki/Damping), from Wikipedia. This article explains how we derive the equation of motion for damped oscillators.

Future Directions for this Page

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