1F model

$$\begin{cases} r(t) = \theta(t) + x_1(t) \\ dx_1(t) = -\beta_1(t)\, x_1(t)\, dt + \sigma(t)\, dW_1(t) \end{cases}$$

$r(t)$: instantaneous interest short rate

$\beta_1(t)$, $\theta(t)$: deterministic functions used to fill the yield curve

$W_1(t)$: Standard Brownian Motion

$\sigma(t)$: Mean-reversion parameter

$x_1(t)$: an Itô Process

Let $f(t, y) := \exp\left(\int_0^t \beta_1(u)\, du \right) y$.

By Itô's lemma / Ito-Dorblin Formula:

$$df(t, x_1) = \frac{\partial f}{\partial t}(t, x_1)\, dt + \frac{\partial f}{\partial y}(t, x_1)\, dx_1 + \frac{1}{2} \cdot \frac{\partial^2 f}{\partial y^2}(t, x_1)\, d\langle x_1, x_1 \rangle_t$$

$$\Rightarrow d\left[x_1 \exp\left(\int_0^t \beta_1(u)\, du \right) \right] = \beta_1(t)\, x_1 \exp\left(\int_0^t \beta_1(u)\, du \right) dt$$

$$+ \exp\left(\int_0^t \beta_1(u)\, du \right) \left(-\beta_1(t)\, x_1\, dt + \sigma(t)\, dW_1(t) \right)$$

$$+ \frac{1}{2} \cdot 0 \cdot \sigma^2(t)\, dt$$

$$\Rightarrow d\left[x_1 \exp\left(\int_0^t \beta_1(u)\, du \right) \right] = \sigma(t) \exp\left(\int_0^t \beta_1(u)\, du \right) dW_1(t)$$

$$x_1(t) \exp\left(\int_0^t \beta_1(u)\, du \right) = x_1(0) + \int_0^t \sigma(s) \exp\left(\int_0^s \beta_1(u)\, du \right) dW_1(s)$$

$$r(t) = \theta(t) + \exp\left(-\int_0^t \beta_1(u)\, du \right) \left[x_1(0) + \int_0^t \sigma(s) \exp\left(\int_0^s \beta_1(u)\, du \right) dW_1(s) \right]$$

In Practice, as $\theta(t)$ can be modified to fit the yield curve, we can set $x_1(0) = 0$, then

$$r(t) = \theta(t) + \exp\left(-\int_0^t \beta_1(u)\,du\right) \int_0^t \sigma(s) \exp\left(\int_0^s \beta_1(u)\,du\right) dW_1(s)$$

CCY Swaption (Cross Currency Swaption)

FX swaption (Forward Exchange Interest Rate Swaption)