1F model

$$\begin{cases} r(t) = \theta(t) + x_1(t) \\ dx_1(t) = -\beta_1(t)\, x_1(t)\, dt + \sigma(t)\, dW_1(t) \end{cases}$$

$r(t)$: instantaneous interest short rate

$\beta_1(t)$, $\theta(t)$: deterministic functions used to fill the yield curve

$W_1(t)$: Standard Brownian Motion

$\sigma(t)$: Mean-reversion parameter

$x_1(t)$: an Itô Process

Let $f(t,y) := \exp\left(\int_0^t \beta_1(u)\,du\right) y$.

By Itô's lemma / Ito-Doeblin Formula:

$$df(t,x_1) = \frac{\partial f}{\partial t}(t,x_1)\,dt + \frac{\partial f}{\partial y}(t,x_1)\,dx_1 + \frac{1}{2}\cdot\frac{\partial^2 f}{\partial y^2}(t,x_1)\,d\langle x_1,x_1\rangle_t$$

$$\Rightarrow d\left[x_1 \exp\left(\int_0^t \beta_1(u)\,du\right)\right] = \beta_1(t)\, x_1 \exp\left(\int_0^t \beta_1(u)\,du\right) dt$$

$$+ \exp\left(\int_0^t \beta_1(u)\,du\right)\left(-\beta_1(t)\, x_1\, dt + \sigma(t)\, dW_1(t)\right)$$

$$+ \frac{1}{2}\cdot 0 \cdot \sigma^2(t)\, dt$$

$$\Rightarrow d\left[x_1 \exp\left(\int_0^t \beta_1(u)\,du\right)\right] = \sigma(t) \exp\left(\int_0^t \beta_1(u)\,du\right) dW_1(t)$$

$$x_1(t) \exp\left(\int_0^t \beta_1(u)\,du\right) = x_1(0) + \int_0^t \sigma(s) \exp\left(\int_0^s \beta_1(u)\,du\right) dW_1(s)$$

$$r(t) = \theta(t) + \exp\left(-\int_0^t \beta_1(u)\,du\right)\left[x_1(0) + \int_0^t \sigma(s) \exp\left(\int_0^s \beta_1(u)\,du\right) dW_1(s)\right]$$

In Practice, as $\theta(t)$ can be modified to fit the yield curve,

we can set $x_1(0) = 0$, then

$$r(t) = \theta(t) + \exp\left(-\int_0^t \beta_1(u)\,du\right) \int_0^t \sigma(s) \exp\left(\int_0^s \beta_1(u)\,du\right) dW_1(s)$$

$$= \theta(t) + \int_0^t \sigma(s) \exp\left(-\int_s^t \beta_1(u)\,du\right) dW_1(s)$$

Two-Factor Model.

$$\begin{cases} r(t) = \theta(t) + x_1(t) + x_2(t) \\ dx_1(t) = -\beta_1(t)\,x_1(t)\,dt + w_1(t)\,\sigma(t)\,dW_1(t) \\ dx_2(t) = -\beta_2(t)\,x_2(t)\,dt + w_2(t)\,\sigma(t)\,dW_2(t) \\ dW_1(t)\,dW_2(t) = \rho(t)\,dt \end{cases}$$

$x_1(t)$, $x_2(t)$: Itô processes, set $x_1(0) = x_2(0) = 0$

$w_1(t)$, $w_2(t)$: weights, set $w_1(t) \equiv 1$ for all t

Denote a function $f(t, y_1, y_2)$,

By the two-dimensional Itô-Doeblin Formula,

$$df(t, x_1, x_2) = \frac{\partial f}{\partial t}(t, x_1, x_2)\,dt + \frac{\partial f}{\partial y_1}(t, x_1, x_2)\,dx_1$$

$$+ \frac{\partial f}{\partial y_2}(t, x_1, x_2)\,dx_2 + \frac{1}{2}\frac{\partial^2 f}{\partial y_1^2}(t, x_1, x_2)\,d\langle x_1, x_1\rangle_t$$

$$+ \frac{\partial^2 f}{\partial y_1 \partial y_2}(t, x_1, x_2)\,d\langle x_1, x_2\rangle_t + \frac{1}{2}\frac{\partial^2 f}{\partial y_2^2}(t, x_1, x_2)\,d\langle x_2, x_2\rangle_t$$

$$= \left(\frac{\partial f}{\partial t} - \beta_1 x_1 \frac{\partial f}{\partial y_1} - \beta_2 x_2 \frac{\partial f}{\partial y_2} + \frac{1}{2}\frac{\partial^2 f}{\partial y_1^2}\sigma^2 + \frac{\partial^2 f}{\partial y_1 \partial y_2} w_2 \sigma^2 \rho + \frac{1}{2}\frac{\partial^2 f}{\partial y_2^2} w_2^2 \sigma^2\right)$$

$$(t, x_1, x_2)\,dt + \frac{\partial f}{\partial y_1}(t, x_1, x_2)\,\sigma\,dW_1 + \frac{\partial f}{\partial y_2}(t, x_1, x_2)\,w_2\,\sigma\,dW_2$$

we want to find the function $f(t, y_1, y_2)$ such that

$$\frac{\partial f}{\partial t} - \beta_1 y_1 \frac{\partial f}{\partial y_1} - \beta_2 y_2 \frac{\partial f}{\partial y_2} + \frac{1}{2}\frac{\partial^2 f}{\partial y_1^2}\sigma^2 + \frac{\partial^2 f}{\partial y_1 \partial y_2} w_2 \sigma^2 \rho + \frac{1}{2}\frac{\partial^2 f}{\partial y_2^2} w_2^2 \sigma^2 = 0$$

Set $f(t, y_1, y_2) = g(t)\, y_1 + k(t)\, y_2$, $\quad \frac{\partial^2 f}{\partial y_1^2} = \frac{\partial^2 f}{\partial y_1 \partial y_2} = \frac{\partial^2 f}{\partial y_2^2} = 0$

$$\Rightarrow \left(\frac{dg}{dt} y_1 + \frac{dk}{dt} y_2\right) - g(t)\beta_1 y_1 - k(t)\beta_2 y_2 = 0$$

$$y_1\left(\frac{dg}{dt} - g(t)\beta_1\right) + y_2\left(\frac{dk}{dt} - k(t)\beta_2\right) = 0$$

$g(t)$ and $k(t)$ are assumed to be independent of y_1 and y_2

$$\Rightarrow \begin{cases} \frac{dg}{dt} = g(t)\,\beta_1(t) \\ \frac{dk}{dt} = k(t)\,\beta_2(t) \end{cases},$$

Omit the trivial case that

$g(t) = k(t) = 0$

① $g(t) \neq 0$, $k(t) = 0$, $\Rightarrow g(t) = \exp\left(\int_0^t \beta_1(u)\, dt\right)$

and $\frac{\partial f}{\partial y_2} = 0$, $\Rightarrow df(t, x_1, x_2) = \frac{\partial f}{\partial y_1}(t, x_1, x_2)\,\sigma(t)\, dW_1(t)$

$$d\, g(t)\, x_1(t) = g(t)\,\sigma(t)\, dW_1(t).$$

$$\Rightarrow x_1(t) = \exp\left(-\int_0^t \beta_1(u)\,du\right)\int_0^t \sigma(s)\exp\left(\int_0^s \beta_1(u)\,du\right) dW_1(s)$$

$$= \int_0^t \sigma(s)\exp\left(-\int_s^t \beta_1(u)\,du\right) dW_1(s)$$

② $g(t) = 0$, $k(t) \neq 0$, $k(t) = \exp\left(\int_0^t \beta_2(u)\,du\right)$ similar to ①,

$$d\, k(t)\, x_2(t) = k(t)\, w_2(t)\,\sigma(t)\, dW_2(t)$$

$$x_2(t) = \int_0^t w_2(t)\,\sigma(s)\exp\left(-\int_s^t \beta_2(u)\,du\right) dW_2(s)$$

③ $g(t) \neq 0$, $k(t) \neq 0$, $\begin{cases} g(t) = \exp\left(\int_0^t \beta_1(u)\,du\right) \\ k(t) = \exp\left(\int_0^t \beta_2(u)\,du\right) \end{cases}$

$$df(t, x_1, x_2) = \frac{\partial f}{\partial y_1}(t, x_1, x_2)\, \sigma(t)\, dW_1(t) + \frac{\partial f}{\partial y_2}(t, x_1, x_2)\, w_2(t)\, \sigma(t)\, dW_2(t)$$

$$d\left(g(t) x_1 + k(t) x_2\right) = g(t)\, \sigma(t)\, dW_1(t) + k(t)\, w_2(t)\, \sigma(t)\, dW_2(t)$$

Observe that the SDE in ③ can be derived from the SDEs in ① and ②, so the general solution of this SDE is

$$\begin{cases} x_1(t) = \int_0^t \sigma(s) \exp\left(-\int_s^t \beta_1(u)\,du\right) dW_1(s) \\ x_2(t) = \int_0^t w_2(t)\, \sigma(s) \exp\left(-\int_s^t \beta_2(u)\,du\right) dW_2(s) \end{cases}$$

In conclusion, the solution is

$$\begin{cases} r_1(t) = \theta(t) + x_1(t) + x_2(t) \\ x_1(t) = \int_0^t \sigma(s) \exp\left(-\int_s^t \beta_1(u)\,du\right) dW_1(s) \\ x_2(t) = \int_0^t w_2(t)\, \sigma(s) \exp\left(-\int_s^t \beta_2(u)\,du\right) dW_2(s) \end{cases}$$

CCY Swaption (Cross Currency Swaption)

FX swaption (Forward Exchange Interest Rate Swaption)