Question

⊘ 问题 question

If $x_n o x$ for $x \in \mathbb{R}$, prove that

$$\lim_{n o\infty}rac{inom{n}{0}x_0+inom{n}{1}x_1+\cdots+inom{n}{n}x_n}{2^n}=x$$

Solution

The following is my solution.

分段估计 (Piecewise estimation)

Assume for $n>N_{arepsilon}\in\mathbb{N}$, $|x_n-x|<rac{arepsilon}{2}$,

Denote that for any $n \in \mathbb{N}$,

$$S_n:=2^{-n}\sum_{k=0}^n inom{n}{k}|x_n-x|$$

So consider $S_{N_{arepsilon}}$ as the first part, and $S_n-S_{N_{arepsilon}}$ as the second part:

1. For the first part, use a lemma:

// Lemma

For any $m,n\in\mathbb{N}$, and $m\leq n$, the sequence

$$y = \binom{2n}{m}$$

is increasing on $0 \le m \le n-1$.

And also since

$$2^n = (1+1)^n = \sum_{k=0}^n \binom{n}{k}$$

and

$$egin{pmatrix} 2n \ m \end{pmatrix} = egin{pmatrix} 2n \ (2n-m) \end{pmatrix}$$

, and denote $z^* = \max_{0 \leq m \leq N_{arepsilon}} |x_m - x|$, it can be found that for

$$egin{aligned} orall n > N^* := \left\lceil rac{2z^*N_arepsilon}{arepsilon}
ight
ceil \ & S_{N_arepsilon} = 2^{-n} \sum_{k=0}^{N_arepsilon} inom{n}{k} |x_n - x| \leq 2^{-n} z^* \sum_{k=0}^{N_arepsilon} inom{n}{k} \ & \leq z^* rac{N_arepsilon}{N^*} \leq z^* rac{N_arepsilon arepsilon}{2z^*N_arepsilon} = rac{arepsilon}{2} \end{aligned}$$

2. For the second part, for $n>N_{\varepsilon}$

$$|S_n-S_{N_arepsilon}| = 2^{-n} \sum_{k=N_arepsilon+1}^n inom{n}{k} |x_k-x| \leq 2^{-n} \sum_{k=N_arepsilon+1}^n inom{n}{k} rac{arepsilon}{2} \leq rac{arepsilon}{2}.$$

Therefore, for $n>\max\{N^*,N_{arepsilon}\}$, $S_n o 0$, so the original statement is proven.

Proof of the Lemma

In the part of <u>Solution</u>, a lemma is used. The proof of the lemma is an induction:

1. Base Step: For m=0,

$$egin{pmatrix} 2n \ 0 \end{pmatrix} = 1 \leq 2n = egin{pmatrix} 2n \ 1 \end{pmatrix}$$

2. Induction Hypothesis:

Assume that for $m \leq n-2$, $\forall 0 \leq k \leq m$,

$$egin{pmatrix} 2n \ k \end{pmatrix} \leq egin{pmatrix} 2n \ m+1 \end{pmatrix}$$

3. Induction Step:

For m + 1 < n - 1,

$$egin{aligned} rac{\binom{2n}{m+2}}{\binom{2n}{m+1}} &= rac{(2n)!}{(m+2)! \ (2n-m-2)!} \cdot rac{(m+1)! \ (2n-m-1)!}{(2n)!} \ &= rac{2n-m-1}{m+2} \geq rac{2n-n-1}{n} > 1 \end{aligned}$$

Toeplitz Theorem

Reference



Let a_n be a real sequence convergent to $a \in \mathbb{R}$. Let $c_{k,n}$ (where $1 \le k \le n$) be a sequence such that:

$$orall k \lim_{n o \infty} c_{k,n} = 0$$
 (i)

$$\lim_{n\to\infty}\sum_{k=1}^n c_{k,n}=1 \tag{ii}$$

$$\exists M>0: orall n, \quad \sum_{k=1}^n |c_{k,n}| \leq M$$
 (iii)

Then $\lim_{n\to\infty} s_n = a$, where

$$s_n \equiv \sum_{k=1}^n c_{k,n} \cdot a_k$$
 (iv)

Proof

If all the terms of the sequence $\{a_n\}$ are equal to a, then by (ii),

$$s_n = a \sum_{k=1}^n c_{k,n}, \quad \lim_{n o \infty} s_n = a \cdot 1 = a$$

Thus it is enough to consider the case where the sequence converges to zero. Then, for any m>1 and $n\geq m$,

$$egin{align} |s_n - 0| &= \left| \sum_{k=1}^n c_{k,n} a_k
ight| \ &\leq \sum_{k=1}^{m-1} |c_{k,n}| \cdot |a_k| + \sum_{k=m}^n |c_{k,n}| \cdot |a_k| \end{cases}$$

The convergence to zero of $\{a_n\}$ implies that for a given $\varepsilon>0$ there exists n_1 such that

$$|a_n|<rac{arepsilon}{2M} \quad ext{for} \quad n\geq n_1$$

Of course, the sequence $\{a_n\}$ is bounded, say by D>0. It follows from (i) that there exists n_2 such that for $n\geq n_2$, (because n_1 is a constant)

$$\sum_{k=1}^{n_1-1} |c_{k,n}| < \frac{\varepsilon}{2D}.$$

Next putting $m = n_1$ in (*), we get

$$|s_n| \leq D \sum_{k=1}^{n_1-1} |c_{k,n}| + rac{arepsilon}{2M} \sum_{k=n_1}^n |c_{k,n}| < rac{arepsilon}{2} + rac{arepsilon}{2} = arepsilon$$

for $n \geq \max\{n_1,n_2\}$. Hence $\lim_{n \to \infty} s_n = 0$