来源于 b 站 up PiKaChu345 8月22日发布的每日一题

Question

问题 question

If $x_n \to x$ for $x \in \mathbb{R}$, prove that

$$\lim_{n\to\infty} \frac{\binom{n}{0}x_0 + \binom{n}{1}x_1 + \cdots + \binom{n}{n}x_n}{2^n} = x$$

Solution

The following is my solution.

分段估计（Piecewise estimation)
Assume for $n > N_\varepsilon \in \mathbb{N}$, $|x_n - x| < \frac{\varepsilon}{2}$,
Denote that for any $n \in \mathbb{N}$,

$$S_n := 2^{-n} \sum_{k=0}^{n} \binom{n}{k} |x_n - x|$$

So consider S_{N_ε} as the first part, and $S_n - S_{N_\varepsilon}$ as the second part:

1. For the first part, use a lemma:

   Lemma
   For any $m, n \in \mathbb{N}$, and $m \le n$, the sequence

   $$y = \binom{2n}{m}$$

   is increasing on $0 \le m \le n-1$.

And also since

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k}$$

and

$$\binom{2n}{m} = \binom{2n}{(2n-m)}$$

, and denote $z^* = \max_{0\le m\le N_\varepsilon} |x_m - x|$, it can be found that for

$$\forall n > N^* := \left\lceil \frac{2z^* N_\varepsilon}{\varepsilon} \right\rceil$$

$$\begin{aligned} S_{N_\varepsilon} &= 2^{-n}\sum_{k=0}^{N_\varepsilon}\binom{n}{k}|x_n - x| \le 2^{-n}z^*\sum_{k=0}^{N_\varepsilon}\binom{n}{k} \\ &\le z^*\frac{N_\varepsilon}{N^*} \le z^*\frac{N_\varepsilon \varepsilon}{2z^*N_\varepsilon} = \frac{\varepsilon}{2} \end{aligned}$$

2. For the second part, for $n > N_\varepsilon$

$$S_n - S_{N_\varepsilon} = 2^{-n}\sum_{k=N_\varepsilon+1}^{n}\binom{n}{k}|x_k - x| \le 2^{-n}\sum_{k=N_\varepsilon+1}^{n}\binom{n}{k}\frac{\varepsilon}{2} \le \frac{\varepsilon}{2}$$

Therefore, for $n > \max\{N^*, N_\varepsilon\}$, $S_n \to 0$, so the original statement is proven.

Proof of the Lemma

In the part of Solution, a lemma is used. The proof of the lemma is an induction:

1. **Base Step**: For $m = 0$,

$$\binom{2n}{0} = 1 \le 2n = \binom{2n}{1}$$

2. **Induction Hypothesis**:
   Assume that for $m \le n-2$, $\forall 0 \le k \le m$,

$$\binom{2n}{k} \le \binom{2n}{m+1}$$

3. **Induction Step**:
   For $m+1 < n-1$,

$$\begin{aligned} \frac{\binom{2n}{m+2}}{\binom{2n}{m+1}} &= \frac{(2n)!}{(m+2)!\,(2n-m-2)!}\cdot\frac{(m+1)!\,(2n-m-1)!}{(2n)!} \\ &= \frac{2n-m-1}{m+2} \ge \frac{2n-n-1}{n} > 1 \end{aligned}$$

Toeplitz Theorem

Reference

Theorem

Let a_n be a real sequence convergent to $a \in \mathbb{R}$. Let $c_{k,n}$ (where $1 \le k \le n$) be a sequence such that:

$$\forall k \lim_{n\to\infty} c_{k,n} = 0 \tag{i}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} c_{k,n} = 1 \tag{ii}$$

$$\exists M > 0 : \forall n, \quad \sum_{k=1}^{n} |c_{k,n}| \le M \tag{iii}$$

Then $\lim_{n\to\infty} s_n = a$, where

$$s_n \equiv \sum_{k=1}^{n} c_{k,n} \cdot a_k \tag{iv}$$

Proof

If all the terms of the sequence $\{a_n\}$ are equal to a, then by (ii),

$$s_n = a \sum_{k=1}^{n} c_{k,n}, \quad \lim_{n\to\infty} s_n = a \cdot 1 = a$$

Thus it is enough to consider the case where the sequence converges to zero. Then, for any $m > 1$ and $n \ge m$,

$$\begin{aligned} |s_n - 0| &= \left| \sum_{k=1}^{n} c_{k,n} a_k \right| \\ &\le \sum_{k=1}^{m-1} |c_{k,n}| \cdot |a_k| + \sum_{k=m}^{n} |c_{k,n}| \cdot |a_k| \end{aligned} \tag{*}$$

The convergence to zero of $\{a_n\}$ implies that for a given $\varepsilon > 0$ there exists n_1 such that

$$|a_n| < \frac{\varepsilon}{2M} \quad \text{for} \quad n \ge n_1$$

Of course, the sequence $\{a_n\}$ is bounded, say by $D > 0$. It follows from (i) that there exists n_2 such that for $n \ge n_2$, (because n_1 is a constant)

$$\sum_{k=1}^{n_1 - 1} |c_{k,n}| < \frac{\varepsilon}{2D}.$$

Next putting $m = n_1$ in $(*)$, we get

$$|s_n| \le D \sum_{k=1}^{n_1-1} |c_{k,n}| + \frac{\varepsilon}{2M} \sum_{k=n_1}^{n} |c_{k,n}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for $n \ge \max\{n_1, n_2\}$. Hence $\lim_{n\to\infty} s_n = 0$